Arithmetic constructions of thin representations in higher rank Teichmüller spaces

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Arithmetic constructions of thin representations in higher rank Teichmüller spaces

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C'est ça la réalité de nos hôpitaux

O.V.

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Abstract

In this thesis, we construct Zariski-dense surface subgroups in some higher rank lattices. More precisely, we exhibit infinitely many commensurability classes of both uniform and non-uniform lattices of $\mathrm{SL}(n,\mathbb{R})$, $\mathrm{SO}(k+1,k)$, $\mathrm{Sp}(2n,\mathbb{R})$ and G_2 that contain images of representations in higher rank Teichmüller spaces.

In the first part, we construct Hitchin representations with image in lattices. One of our main results is the classification of lattices that contain the image of a *Fuchsian representation*. The proof relies on the classification of forms of algebraic groups using non-abelian Galois cohomology. A "bending" deformation then allows us to construct Zariski-dense surface subgroups in those lattices, and using the Strong Approximation Theorem we show that these deformations give infinitely many different subgroups. As a corollary, combined with [LT20] who deal with the case $\mathrm{SL}(p,\mathbb{Z})$, we obtain that all non-uniform lattices of $\mathrm{SL}(p,\mathbb{R})$, where $p \neq 2$ is a prime, admit Zariski-dense Hitchin representations. Furthermore, every lattice of $\mathrm{Sp}(2n,\mathbb{R})$, except $\mathrm{Sp}(2n,\mathbb{Z})$, contains Zariski-dense Hitchin representations. Together with [LT18] for the case $\mathrm{Sp}(4,\mathbb{Z})$, this implies that all lattices of $\mathrm{Sp}(4,\mathbb{R})$ admit Zariski-dense surface subgroups. Similar results hold for $\mathrm{SO}(k+1,k)$ and G_2.

In the second part, we apply the above ideas to maximal representations in $\mathrm{Sp}(2n,\mathbb{R})$ and classify lattices that contain the image of a *diagonal maximal representation*. We then show that all lattices of $\mathrm{Sp}(2n,\mathbb{R})$, except $\mathrm{Sp}(2n,\mathbb{Z})$ when n is odd, contain infinitely many mapping class group orbits of Zariski-dense maximal representations that are not Hitchin. In particular, $\mathrm{Sp}(4k,\mathbb{Z})$ admits Zariski-dense surface subgroups for all $k \geq 1$.

Résumé

Dans cette thèse, nous construisons des sous-groupes de surface Zariski-denses dans certains réseaux en rang supérieurs. Plus précisément, nous exhibons une infinité de classes de commensurabilité de réseaux à la fois uniformes et non-uniformes de $\mathrm{SL}(n,\mathbb{R})$, $\mathrm{SO}(k+1,k)$, $\mathrm{Sp}(2n,\mathbb{R})$ et G_2 qui contiennent les images de représentations dans des espaces de Teichmüller de rang supérieur.

Dans la première partie, nous construisons des représentations Hitchin dont l'image est dans un réseau. Un de nos résultats principaux est la classification des réseaux qui contiennent l'image d'une *représentation Fuchsienne*. La preuve repose sur la classification des formes des groupes algébriques via la cohomologie de Galois non-abélienne. Une déformation par "pliage" nous permet ensuite de construire des sous-groupes de surface Zariski-dense dans ces réseaux, et en utilisant le Théorème d'Approximation Forte nous montrons que ces déformations donnent une infinité de sous-groupes différents. Comme corollaire, combiné avec [LT20] qui traîte le cas $\mathrm{SL}(p,\mathbb{Z})$, nous obtenons que tous les réseaux non-uniformes de $\mathrm{SL}(p,\mathbb{R})$, où $p \neq 2$ est premier, admettent des représentations Hitchin Zariski-denses. En outre, tout réseau de $\mathrm{Sp}(2n,\mathbb{R})$, excepté $\mathrm{Sp}(2n,\mathbb{Z})$, contient des représentations Hitchin Zariski-denses. Avec [LT18] pour le cas $\mathrm{Sp}(4,\mathbb{Z})$, cela implique que tout réseau de $\mathrm{Sp}(4,\mathbb{Z})$ contient des sous-groupes de surface Zariski-denses. Des résultats similaires sont valables pour $\mathrm{SO}(k+1,k)$ et G_2.

Dans la deuxième partie, nous appliquons les idées ci-dessus aux représentations maximales dans $\mathrm{Sp}(2n,\mathbb{R})$ et classifions les réseaux qui contiennent l'image d'une *représentation diagonale maximale*. Nous montrons ensuite que tout réseau de $\mathrm{Sp}(2n,\mathbb{R})$, excepté $\mathrm{Sp}(2n,\mathbb{Z})$ quand n est impair, contient une infinité d'orbites par le mapping class group de représentations maximales Zariski-denses qui ne sont pas Hitchin. En particulier, $\mathrm{Sp}(4k,\mathbb{Z})$ admet des sous-groupes de surface Zariski-denses pour tout $k \geq 1$.

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Introduction

Thin groups

Let G be a semisimple Lie group. A *lattice* is a discrete subgroup Λ of G such that $\Lambda\backslash G$ has finite volume for the Haar measure of G. Lattices have been proven to be a central object in Riemannian geometry as they model finite-volume manifolds with interesting geometric properties. There is a systematic way to construct lattices using arithmetic. A major breakthrough in the 70's was made by Margulis who showed that in higher rank, all lattices come from this construction [Mar91], allowing one to classify them. However, subgroups of lattice are far less understood and are a very active field of research.

Borel's Density Theorem (Theorem 2.1.8 below) states that, under some mild assumptions, lattices are Zariski-dense in G. Being Zariski-dense is a generic property for subgroups of G. In particular, Zariski-dense discrete subgroups of Lie groups receive a lot of interest.

However, if a Zariski-dense subgroup Γ is contained in an arithmetic lattice, numerous number-theoretic questions arise. For instance, the reduction of Γ makes sense and it satisfies the Strong Approximation Theorem, which we states in the simple case of $\mathrm{SL}(n,\mathbb{Z})$ but which holds more generally, see Theorem 2.5.5.

Theorem 0.1 (Matthews−Vaserstein−Weisfeiler [MVW84])**.** *Let $\Gamma < \mathrm{SL}(n,\mathbb{Z})$ be a Zariski-dense subgroup. Then for all but finitely many primes p, Γ surjects onto $\mathrm{SL}(n,\mathbb{F}_p)$.*

More importantly, it satisfies the Super-strong Approximation Theorem [GV12] which has numerous applications in Diophantine analysis. For example, the Apollonian group is a Zariski-dense subgroup of an arithmetic lattice of $\mathrm{SO}(3,1)$ and its number-theoretic properties are related to conjectures on circle packings, see [Kon13].

If Λ is a lattice, its finite-index subgroups are still lattices, hence Zariski-dense. However, there exists subgroups of Λ that are Zariski-dense but not lattices themselves. These subgroups are said to be *thin* and attracted a lot of attention recently [KLLR19].

Definition 0.2 (Sarnak [Sar14])**.** A subgroup of Λ is said to be *thin* if it is infinite index and Zariski-dense in G.

A theorem of Tits ([Tit72, Theorem 3]) states that Λ contains thin non-abelian free subgroups. Among Gromov-hyperbolic groups, the next step after free groups are *surface groups*: fundamental groups of a closed connected orientable surfaces of genus at least 2. Surface subgroups of lattices have been proven to be of major importance in topology. The study of 3-manifolds, for instance, has lead to the investigation of surface subgroups of lattices of $\mathrm{SO}(3,1)$. In their celebrated work [KM12], Kahn and Markovic exhibit surface subgroups in all uniform lattices of $\mathrm{SO}(3,1)$, using the dynamical properties of lattices.

Theorem 0.3 (Kahn−Markovic [KM12])**.** *Let M be a closed connected 3-dimensional hyperbolic manifold. Then $\pi_1(M)$ contains a surface subgroup.*

Theorem 0.3 was a major tool in the proof of the Virtual Haken conjecture by Agol [Ago13]. In an attempt to generalize this theorem, the following question has been the subject of numerous researches.

Question 0.4. *Let Λ be a lattice of G. Does Λ contain a thin surface subgroup?*

It is generally believed that, outside of some obvious restrictions, the answer to Question 0.4 should be positive. Using tools from [KM12], Hamenstädt proved that when G is a simple non-compact rank 1 Lie group not isomorphic to $\mathrm{SO}(2n,1)$, all uniform lattices of G contain a surface subgroup [Ham15], see also [Dou22a]. Kahn, Labourie and Mozes [KLM18] extended this result for several other semisimple Lie groups, notably the complex ones. However, as explained in §1.2 of [KLM18], their constructions do not work when G is *real split*, i.e. when the real rank of G is equal to its complex rank.

The case we are interested in is the one of lattices of split real Lie groups. In [LRT11], Long, Reid and Thistlethwaite first constructed explicit examples of thin surface subgroups in $\mathrm{SL}(3,\mathbb{Z})$. Long and Thistlethwaite then generalized this to $\mathrm{SL}(4,\mathbb{Z})$ and $\mathrm{Sp}(4,\mathbb{Z})$. More recently, they showed that for all $k \geq 1$, $\mathrm{SL}(2k+1,\mathbb{Z})$ contains a thin surface subgroup [LT20]. These results are proven by means of *Hitchin representations*, see below. It is still unknown whether $\mathrm{SL}(2k,\mathbb{Z})$ has thin surface subgroups for $k \geq 3$. However, we prove that $\mathrm{Sp}(4k,\mathbb{Z})$ does for $k \geq 1$, see Theorem F.

In this thesis, we construct thin surface subgroups in some lattices of split real Lie groups, see below for precise statements. The lattices that appear in our theorems were all known to contain surface subgroups. However these subgroups are not even irreducible.

Finally, we mention that other groups have been proven to be thin subgroups of lattices. See for instance Ballas and Long [BL20] and [Bal20] for fundamental groups of hyperbolic manifolds inside $\mathrm{SL}(n,\mathbb{R})$ and Douba [Dou22b] for right-angled Coxeter groups in $\mathrm{O}(p,q)$.

The Hitchin component

For any $g \geq 2$, denote by S_g a closed connected orientable surface of genus $g \geq 2$. The *Teichmüller space* of S_g is the space of marked hyperbolic structures on the surface up to isotopy. This space is a the intersection of various mathematical fields and has been studied extensively. In [Gol82], Goldman showed that the Teichmüller space of S_g is naturally isomorphic to a connected component of $\mathrm{Hom}(\pi_1(S_g),\mathrm{PSL}(2,\mathbb{R}))/\mathrm{PSL}(2,\mathbb{R})$ where all representations are faithful and with discrete image.

Whenever G is a Lie group, we can endow $\mathrm{Hom}(\pi_1(S_g),G)$ with the compact-open topology. We denote by

$$\mathfrak{X}(S_g,G) = \mathrm{Hom}(\pi_1(S_g),G)/G$$

the quotient by the action of G by conjugation and call it the *character variety*. As analogy with the Teichmüller space of S_g, one can look for "special" connected components of the character variety and try to understand their geometric meaning. This lead to the following definition.

Definition 0.5 (Wienhard [Wie18])**.** A *higher rank Teichmüller space* is a connected component of $\mathfrak{X}(S_g,G)$ that consists only of discrete and faithful representations.

In [Hit92], Hitchin showed that the character variety $\mathfrak{X}(S_g,\mathrm{PSL}(n,\mathbb{R}))$ has 3 connected components for $n \geq 3$ odd and 6 connected components for $n \geq 4$ even. One of them is homeomorphic to $\mathbb{R}^{2(n^2-1)(g-1)}$: this is the *Hitchin component*. More precisely, for n is even, there are two Hitchin components.

For any $n \geq 2$ denote by $\tau_n : \mathrm{SL}(2,\mathbb{C}) \to \mathrm{SL}(n,\mathbb{C})$ the representation where

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{C})$$

acts on the space of homogeneous polynomials in two variables X and Y of degree $n-1$ as

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} X^{n-i-1}Y^i = (aX+cY)^{n-i-1}(bX+dY)^i$$

for every $0 \leq i \leq n-1$. We call τ_n *the irreducible representation*. Up to conjugation, this is the only n-dimensional representation of $\mathrm{SL}(2,\mathbb{C})$. Denote also by $\tau_n : \mathrm{PGL}(2,\mathbb{C}) \to \mathrm{PGL}(n,\mathbb{C})$ the induced representation.

Definition 0.6. A *Fuchsian representation* is a representation of the form $\tau_n \circ \rho$ where $\rho : \pi_1(S_g) \to \mathrm{PSL}(2,\mathbb{R})$ is discrete and faithful.

The Hitchin component is a connected component of $\mathfrak{X}(S_g, \mathrm{PSL}(n,\mathbb{R}))$ that contains a Fuchsian representation.

When $n = 3$, Goldman and Choi proved in [Gol90] and [CG93] that the Hitchin component parametrizes marked convex projective structures on S_g. This geometric interpretation of the Hitchin component implies that it consists only of discrete and faithful representations. Using τ_3, the Teichmüller space sits naturally in the Hitchin component of $\mathrm{PSL}(3,\mathbb{R})$. This reflects the fact that hyperbolic structures are particular cases of convex projective strutures.

More generally, the Hitchin component is the prototypical example of a higher rank Teichmüller space, as shown by Labourie and Fock-Goncharov.

Theorem 0.7 (Labourie [Lab06] Fock-Goncharov [FG06])**.** *Hitchin representations are discrete and faithful.*

The geometric interpretation of the Hitchin component is still investigated. Guichard and Wienhard [GW08] proved that when $n = 4$ the Hitchin component parametrizes properly convex foliated projective structures on the unit tangent bundle of S_g. Furthermore, it is conjectured that for all n the Hitchin component corresponds to higher complex structures on the surface, see [FT21] and [Nol22].

Hitchin representations in lattices

The major result of this thesis is the construction of Zariski-dense Hitchin representations in lattices of some split real Lie groups. The proof technique is completely different then the one presented in Kahn and Markovic [KM12] since our construction is arithmetic.

In the first part, we classify the lattices that contain the image of a Fuchsian representation. It is stated in the following theorem. We say that two subgroups of G are *widely commensurable*, abbreviated *w.c.*, if one is commensurable with a conjugate of the other.

Theorem A. *For every lattice Λ of G listed in Table 1, there exists $g \geq 2$ such that Λ contains the image of a Fuchsian representation of $\pi_1(S_g)$. Furthermore, up to wide commensurability, these are the only lattices of G that contain the image of a Fuchsian representation.*

G	n or k	Λ
$\mathrm{SO}(k+1,k)$	$k \geq 2$, $k \equiv 1,2[4]$	Every lattice not w.c. with $\mathrm{SO}(q_0,\mathbb{Z})$
$\mathrm{SO}(k+1,k)$	$k \geq 3$, $k \equiv 0,3[4]$	$\mathrm{SO}(q_F,\mathcal{O}_F)$
$\mathbf{G}_2(\mathbb{R})$		Every lattice
$\mathrm{Sp}(2n,\mathbb{R})$	$n \geq 2$	Every lattice not w.c. with $\mathrm{Sp}(2n,\mathbb{Z})$
$\mathrm{SL}(2k+1,\mathbb{R})$	$k \geq 1$	$\mathrm{SL}(2k+1,\mathbb{Z})$ or $\mathrm{SU}(I_{2k+1},\sigma;\mathcal{O}_F[\sqrt{d}])$
$\mathrm{SL}(2n,\mathbb{R})$	$n \geq 2$	$\mathrm{SL}(n,\mathcal{O})$ or $\mathrm{SU}(I_n,{}^{-}\otimes\sigma;\mathcal{O}\otimes\mathcal{O}_F[\sqrt{d}])$

Table 1: Here F is any totally real number field, $\mathcal{O}_F$ its ring of integers and d any element of $\mathcal{O}_F$ which is not a square and is positive at exactly one real place of F, say ι. Moreover $\mathcal{O}$ is any order of any quaternion division algebra over F which splits exactly at ι. If $k \equiv 0,3[4]$, let q_F be a non-degenerate quadratic form of rank $2k+1$ over F that has trivial discriminant, trivial Hasse invariant at each finite place of F, which is positive definite at all real places of F except one where it has signature equal to $(k+1,k)$ if $k \equiv 0[4]$ or to $(k,k+1)$ if $k \equiv 3[4]$. If $k \equiv 1,2[4]$, let q_0 be a non-degenerate quadratic form of rank $2k+1$ over $\mathbb{Q}$ that has trivial discriminant, trivial Hasse invariant at each odd prime, non-trivial Hasse invariant at 2, signature equal to $(k,k+1)$ if $k \equiv 1[4]$ or to $(k+1,k)$ if $k \equiv 2[4]$. See Section 2.3 for definitions.

See Example 2.3.8 for a uniform lattice of $\mathrm{SL}(3,\mathbb{R})$ that is not in Table 1. This Theorem implies that, up to finite cover, the corresponding locally symmetric space $K\backslash G/\Lambda$, where K is a maximal compact subgroup of G, contains a totally geodesic surface of *irreducible type*, i.e. such that the copy of the hyperbolic plane in $K\backslash G$ comes from an irreducible embedding of $\mathrm{SL}(2,\mathbb{R})$ in G.

Theorem A is proven using *non-abelian Galois cohomology*. This cohomology theory allows one to reformulate the question in number theoretic terms. When $G = \mathrm{SO}(k+1,k)$ for instance, it relates to the classification of quadratic forms over number fields. For $G = \mathrm{Sp}(2n,\mathbb{R})$, it reduces to classifying certain $^-$*-Hermitian forms* over *quaternion algebras*. With $\mathbf{G}_2(\mathbb{R})$ one needs to study *octonion algebras* over number fields. Etc.

Let Λ be a lattice in G as in Table 1. Following Theorem A, it contains a Fuchsian representation ρ. Its image is not Zariski-dense since it lies in a principal $\mathrm{SL}(2,\mathbb{R})$. To construct Zariski-dense Hitchin representations in Λ, we deform ρ using a "bending" deformation. The bending technique was introduced by Johnson and Millson [JM86] and has already been used to construct thin subgroups, see [LT20] or [BL20]. To bend, we need to find a simple closed curve on the surface which has a big enough centralizer in Λ. For $\Lambda = \mathrm{SL}(2k+1,\mathbb{Z})$ or $\Lambda = \mathrm{SL}(n,\mathcal{O})$ for an order $\mathcal{O}$, this is not possible. Since the deformed representations lie in the Hitchin component, they are still faithful and their image is thus a surface subgroup. To control their Zariski-closure, we make use of the following classification.

Theorem 0.8 (Guichard [Gui], see also Sambarino [Sam20])**.** *Let $\rho : \pi_1(S) \to \mathrm{PSL}(n,\mathbb{R})$ be a Hitchin representation. Then the Zariski-closure of ρ is either $\mathrm{PSL}(n,\mathbb{R})$, a principal $\mathrm{SL}(2,\mathbb{R})$ or conjugated to one of the following:*

- $\mathrm{PSp}(2k,\mathbb{R})$ *if* $n = 2k$,
- $\mathrm{SO}(k+1,k)$ *if* $n = 2k+1$,
- $\mathbf{G}_2(\mathbb{R})$ *if* $n = 7$.

Denote by $\mathrm{MCG}(S_g)$ the mapping class group of S_g, i.e. the group of orientation preserving homeomorphisms of the surface up to isotopy. Homeomorphisms of the surface induce automorphisms of the fundamental group. Hence $\mathrm{MCG}(S_g)$ acts naturally on the character variety by precomposition. The action of $\mathrm{MCG}(S_g)$ on the Hitchin component is properly discontinuous [Lab08]. Given a Hitchin representation in a lattice, one can use the action of the mapping class group to produce infinitely many other Hitchin representations in this lattice. It is more involved to construct representations that are not in the same orbit. Using the Strong Approximation Theorem (Theorem 0.1) we prove that the bending procedure gives rise to infinitely many mapping class group orbits of Hitchin representations.

Theorem B. *Let Λ be a lattice of G listed in Table 1. Suppose that Λ is not widely commensurable with $\mathrm{SL}(2k+1,\mathbb{Z})$ or $\mathrm{SL}(n,\mathcal{O})$ for any order $\mathcal{O}$. Then there exists $g \geq 2$ such that Λ contains the image of infinitely many $\mathrm{MCG}(S_g)$-orbits of Zariski-dense Hitchin representations of $\pi_1(S_g)$.*

For a given lattice Λ, the genus g in Theorem B agrees with the genus g in Theorem A. Theorem B should be compared to the result of Borel which states that for all $g \geq 2$, there are only finitely many Teichmüller representations of S_g in arithmetic subgroups of $\mathrm{PSL}(2,\mathbb{R})$ up to $\mathrm{MCG}(S_g)$ (Theorem 11.3.1 in [MR03]).

Note that surface subgroups have to be of infinite index in the lattices considered here because the virtual cohomological dimension of a surface group is 2 and the virtual cohomological dimension of the lattices is at least 3, see Aramayona, Degrijse, Martínez-Pérez and Souto [ADMPS17]. Hence the image of representations constructed in Theorem B are thin surface subgroups.

The lattice $\mathrm{SL}(2k+1,\mathbb{Z})$ has been shown to contain infinitely many conjugacy classes of thin Hitchin representations by Long and Thistlethwaite [LT20]. This result was also announced by Burger, Labourie and Wienhard, see Theorem 24 in [Wie18] and [Bur15].[1] Together with Theorem B, this implies the following.

Corollary C. *Let $p \neq 2$ be a prime. All non-uniform lattices of $\mathrm{SL}(p,\mathbb{R})$ contain Zariski-dense Hitchin representations.*

This corollary is a generalization of a theorem of Long, Reid and Thistlethwaite that states that all non-uniform lattices of $\mathrm{SL}(3,\mathbb{R})$ contain infinitely many conjugacy classes of Zariski-dense Hitchin representations, see [LRT11] and [LR14]. Theorem B together with Long and Thistlethwaite [LT18] for $\mathrm{Sp}(4,\mathbb{Z})$ give the following corollary.

[1] Precisely, they announced that for $k \geq 2$, $\mathrm{SL}(2k+1,\mathbb{Z})$ contains infinitely many $\mathrm{MCG}(S_g)$-orbits of Zariski-dense Hitchin representations for all $g \geq 2$.

Corollary D. *All lattices of* $\mathrm{Sp}(4,\mathbb{R})$ *contain Zariski-dense Hitchin representations.*

Maximal representations in $\mathrm{Sp}(2n,\mathbb{R})$

Endow the symmetric space $\mathrm{Sp}(2n,\mathbb{R})/\mathrm{U}(n)$ with its unique Hermitian metric ω of minimal holomorphic sectional curvature -1. Let $\rho : \pi_1(S_g) \to \mathrm{Sp}(2n,\mathbb{R})$ be a representation. There exists a smooth ρ-equivariant map $f : \tilde{S}_g \to \mathrm{Sp}(2n,\mathbb{R})/\mathrm{U}(n)$. The form $f^*\omega$ is invariant under the covering action of $\pi_1(S_g)$ and thus defines a form on S_g. Define the *Toledo invariant* of ρ as

$$\mathrm{T}(\rho) = \frac{1}{2\pi}\int_{S_g} \overline{f^*\omega}.$$

It is an integer and satisfies the following inequality

$$(2-2g)n \leq \mathrm{T}(\rho) \leq (2g-2)n$$

see [Tur84]. A representation is said to be *maximal* if $\mathrm{T}(\rho) = \pm(2g-2)n$. For instance, Hitchin representations with image in $\mathrm{Sp}(2n,\mathbb{R})$ are maximal. More generally, we have the following.

Theorem 0.9 (Burger−Iozzi−Wienhard [BIW10])**.** *Maximal representations are discrete and faithful.*

Consider the *diagonal embedding*

$$\begin{aligned} \phi_n : &\mathrm{SL}(2,\mathbb{C}) \to \mathrm{Sp}(2n,\mathbb{C}) \\ A \mapsto &\begin{pmatrix} A & & \\ & \ddots & \\ & & A \end{pmatrix}. \end{aligned}$$

We denote also by ϕ_n the induced embedding of $\mathrm{PGL}(2,\mathbb{C})$ in $\mathrm{PSp}(2n,\mathbb{C})$. If $\rho : \pi_1(S_g) \to \mathrm{SL}(2,\mathbb{R})$ is a discrete and faithful representation, $\phi_n \circ \rho$ is a maximal representation. In this thesis, we call such a representation a *maximal diagonal representation*. Since the Toledo invariant is constant on connected components, any continuous deformation of a maximal diagonal representation is also maximal. Hence it lies in a higher rank Teichmüller space.

In the last part of this thesis, we construct Zariski-dense maximal representations that are continuous deformations of maximal diagonal representations. The following proposition is analogous to Theorem A.

Proposition E. *Every lattice of* $\mathrm{Sp}(2n,\mathbb{R})$*, not widely commensurable with* $\mathrm{Sp}(2n,\mathbb{Z})$ *when* n *is odd, contains the image of a maximal diagonal representation of some genus. Conversely, when* n *is odd,* $\mathrm{Sp}(2n,\mathbb{Z})$ *does not contain the image of a maximal diagonal representations of any genus.*

See Proposition G for a more precise version of this proposition. The situation here is quite different to the one of Theorem A because the image of ϕ_n has a big centralizer in the symplectic group while the image of τ_n has a finite centralizer. Because of this, the classification is more subtle than in the Hitchin setting. Deforming those maximal diagonal representations, we prove the following.

Theorem F. *Let* $n \geq 2$ *and* Λ *be a lattice of* $\mathrm{Sp}(2n,\mathbb{R})$*, not widely commensurable with* $\mathrm{Sp}(2n,\mathbb{Z})$ *when* n *is odd. Then there exists* $g \geq 2$ *such that* Λ *contains infinitely many* $\mathrm{MCG}(S_g)$*-orbits of Zariski-dense maximal representations of* $\pi_1(S_g)$*. Furthermore these representations are continuous deformations of maximal diagonal representations.*

In particular, for all $k \geq 1$, $\mathrm{Sp}(4k,\mathbb{Z})$ contains infinitely many non-conjugate Zariski-dense surface subgroups. The proof of Theorem F is similar to the one of Theorem B, with the necessary adaptations. In the proof of Theorem B the trace of the bend representations is sufficient to distinguish the mapping class group orbits. However, this is no longer true for diagonal maximal representations and one need a finer control on the reduction of the bend representations to finite fields.

Method

Theorem A and Proposition E are established by considering representations of lattices of $\mathrm{SL}(2,\mathbb{R})$ that are *arithmetic*, i.e. lattices that are the "integer points" of an algebraic group over $\mathbb{Q}$. Given an arithmetic lattice Γ of $\mathrm{SL}(2,\mathbb{R})$, we classify which lattices contains the image of $\tau_n(\Gamma)$ or $\phi_n(\Gamma)$. For the case of $\phi_n(\Gamma)$, the classification is described in the following proposition. It is a more precise version of Proposition E which takes into account the arithmetic data given by Γ.

Proposition G. *Let Γ be an arithmetic lattice in $\mathrm{SL}(2,\mathbb{R})$ which is commensurable with the elements of norm 1 in an order $\mathcal{O}_\Gamma$ of a quaternion algebra over a totally real number field F.*

If n is even, $\phi_n(\Gamma) < \mathrm{SU}(I_n, {}^-; \mathcal{O})$, up to wide commensurability, for any order $\mathcal{O}$ in any quaternion algebra over F that splits at exactly one place of F. Furthermore, up to wide commensurability, these are the only lattices of $\mathrm{Sp}(2n,\mathbb{R})$ that contain $\phi_n(\Gamma)$.

If n is odd, $\phi_n(\Gamma) < \mathrm{SU}(I_n, {}^-; \mathcal{O}_\Gamma)$, up to wide commensurability. Furthermore, up to wide commensurability, this is the only lattice of $\mathrm{Sp}(2n,\mathbb{R})$ that contains $\phi_n(\Gamma)$.

To illustrate this proposition, let $p \equiv 3[4]$ be a prime. Then

$$\Gamma_p = \left\{ \begin{pmatrix} x_0 + x_1\sqrt{p} & x_2\sqrt{p} + px_3 \\ x_2\sqrt{p} - px_3 & x_0 - x_1\sqrt{p} \end{pmatrix} \;\middle|\; x_i \in \mathbb{Z},\ x_0^2 - px_1^2 - px_2^2 + p^2x_3^2 = 1 \right\}$$

is an arithmetic lattice of $\mathrm{SL}(2,\mathbb{R})$ which is a surface group (it is build out of the quaternion division algebra $(p,p)_{\mathbb{Q}}$, see Example 2.3.4). Even thought $\phi_n(\Gamma_p)$ lies in $\mathrm{Sp}(2n,\mathbb{Z}[\sqrt{p}])$, Proposition G implies that it can be conjugated within $\mathrm{Sp}(2n,\mathbb{Z})$ if and only if n is even.

This proposition has analogues in the Hitchin setting, see Propositions 3.2.2 for $\mathrm{SO}(k+1,k)$, 3.3.1 for G_2, 3.4.1 for $\mathrm{SL}(2k+1,\mathbb{R})$, 3.6.1 for $\mathrm{Sp}(2n,\mathbb{R})$ and 3.7.1 for $\mathrm{SL}(2n,\mathbb{R})$. To prove Proposition G, or any of the aforementioned propositions, we reformulate the problem in terms of algebraic groups using the following fact. If F is a number field and $f : \mathbb{G} \to \mathbb{H}$ is a F-morphism, then $f(\mathbb{G}(\mathcal{O}_F)$ lies inside $\mathbb{H}(\mathcal{O}_F)$ up to finite index, see Proposition 2.2.6.

We can then appeal to non-abelian Galois cohomology which classifies algebraic groups. More precisely, for a number field F, the first Galois cohomology set $\mathrm{H}^1(\mathrm{Gal}(\overline{\mathbb{Q}}/F), \mathrm{Aut}(\mathbb{G}(\overline{\mathbb{Q}}))$ is in bijection with isomorphism classes of *F-forms*, i.e. F-algebraic groups that become isomorphic to $\mathbb{G}$ over $\overline{\mathbb{Q}}$ (Proposition 1.6.9 below).

Let Γ be an arithmetic subgroup of $\mathrm{SL}(2,\mathbb{R})$. It is the "integer points" of a F-form of SL_2 for a totally real number field F. The latter corresponds to a 1-cocycle $\xi \in \mathrm{H}^1(\mathrm{Gal}(\overline{\mathbb{Q}}/F), \mathrm{Aut}(\mathrm{SL}(2,\overline{\mathbb{Q}})))$. The goal is to classify 1-cocycle of $\mathrm{H}^1(\mathrm{Gal}(\overline{\mathbb{Q}}/F), \mathrm{Aut}(\mathrm{Sp}(2n,\overline{\mathbb{Q}})))$ such that the corresponding F-form contains $\phi_n(\Gamma)$. Note that there is at least the 1-cocycle $\phi_n \circ \xi$. For n odd, this is the only one, but for n even there are many others. We then have to determine the F-forms of Sp_{2n} associated to such 1-cocycles.

Let $\mathbb{G}$ be an F-form of Sp_{2n} as above. As explained above, $\phi_n(\Gamma)$ is contained in $\mathbb{G}(\mathcal{O}_F)$ up to finite index. For $\mathbb{G}(\mathcal{O}_F)$ to be a lattice of $\mathrm{Sp}(2n,\mathbb{R})$, we need that the $\mathbb{R}$-points $\mathbb{G}(\mathbb{R}^\sigma)$ are compact for all embeddings $\sigma : F \hookrightarrow \mathbb{R}$ except for one which satisfies $\mathbb{G}(\mathbb{R}^\sigma) \simeq \mathrm{Sp}(2n,\mathbb{R})$. This is checked case by case.

Organization of the thesis

The first two chapters provide the necessary background. In Chapter 1, we present the number theoretic tools that will be used throughout all this manuscript: *number fields*, *quaternion algebras*, *linear algebraic groups*. We end this chapter by presenting *non-abelian Galois cohomology*. The latter will be of major importance in this thesis. In Chapter 2 we define the notion of *arithmetic subgroup* of a Lie group. They are particular examples of *lattices* and as such play a major role in geometry. We give several examples of arithmetic groups that were mentioned in Table 1. Finally, we present some classification results for arithmetic subgroups.

The next two chapters present a construction of Hitchin representations in lattices. We establish Theorem A in Chapter 3. For each Lie group of Table 1, we give a more precise version of the theorem:

Proposition 3.2.2 for $\mathrm{SO}(k+1,k)$, Proposition 3.3.1 for $\mathbf{G}_2(\mathbb{R})$, Proposition 3.4.1 for $\mathrm{SL}(2k+1,\mathbb{R})$, Proposition 3.6.1 for $\mathrm{Sp}(2n,\mathbb{R})$ and Proposition 3.7.1 for $\mathrm{SL}(2n,\mathbb{R})$. The proof relies mainly on number theoretic considerations. The main goal of Chapter 4 is to prove Theorem B. We first deal with the Zariski-closure of Hitchin representations. This requires us to determine the centralizer of a given simple closed curve in the given lattice. This is done using Galois cohomology, or for uniform lattices using a geometric argument. Lastly, we use the Strong Approximation Theorem to guarantee that our construction provides infinitely many mapping class group orbits of Hitchin representations. Corollaries C and D are proven in this chapter.

Chapter 5 is dedicated to constructing maximal representations in lattices of $\mathrm{Sp}(2n,\mathbb{R})$ that are deformation of a maximal diagonal representation. Using Galois cohomology, we first prove Proposition E, see Proposition G for a finer version. We then adapt the considerations of the previous chapter to this setting. We finish this thesis with the proof of Theorem F.

Part I

Preliminaries

Chapter 1

Forms of algebraic groups

In this chapter we present the number theoretic background necessary to construct and classify arithmetic subgroups. After *number fields*, we define *quaternion algebras* and more generally *central simple algebras*. These allow us to give examples of *linear algebraic groups*. We then deal with exceptional groups of type G_2. Finally, we introduce *Galois cohomology* and how it classifies forms of algebraic groups. The main reference here is Platonov and Rapinchuck's book [PR94].

1.1 Number fields

Definition 1.1.1. A *number field* F is a field that is a finite dimensional $\mathbb{Q}$-vector space. The *degree* of F is the dimension of F as a $\mathbb{Q}$-vector space.

The field $\mathbb{Q}$ is the only number field of degree 1. Other examples of number fields include $\mathbb{Q}(\sqrt{2})$ which has degree 2 or $\mathbb{Q}(\sqrt[3]{2})$ which has degree 3. On the opposite, $\overline{\mathbb{Q}}$, the field of algebraic numbers, is not a number field since it is infinite dimensional over $\mathbb{Q}$.

Definition 1.1.2. The *ring of integers* of a number field F is

$$\mathcal{O}_F = \{\alpha \in F \mid \exists P \in \mathbb{Z}[\mathrm{X}] \text{ monic such that } P(\alpha) = 0\}.$$

Examples 1.1.3.
- $\mathbb{Q}$ has for ring of integers $\mathbb{Z}$
- $\mathbb{Q}(\sqrt{2})$ has for ring of integers $\mathbb{Z}[\sqrt{2}]$
- $\mathbb{Q}(i\sqrt{3})$ has for ring of integers $\mathbb{Z}[\frac{1+i\sqrt{3}}{2}]$
- $\mathbb{Q}(\sqrt[3]{2})$ has for ring of integers $\mathbb{Z}[\sqrt[3]{2}]$

Definition 1.1.4. Let F be a number field. A field homomorphism $\sigma : F \hookrightarrow \mathbb{C}$ is called an *embedding* in $\mathbb{C}$. It is said to be *real* if $\sigma(F) \subset \mathbb{R}$, *complex* otherwise.

If $\sigma : F \hookrightarrow \mathbb{C}$ is a complex embedding, its conjugate $\overline{\sigma} : F \hookrightarrow \mathbb{C}$ is another complex embedding. Note that a field homomorphism is always injective.

Examples 1.1.5.
- $\mathbb{Q}$ has one real embedding
- $\mathbb{Q}(\sqrt{2})$ has two real embeddings which are $\sigma_1(a + b\sqrt{2}) = a + b\sqrt{2}$ and $\sigma_2(a + b\sqrt{2}) = a - b\sqrt{2}$
- $\mathbb{Q}(i\sqrt{3})$ has two complex embeddings that are conjugate
- $\mathbb{Q}(\sqrt[3]{2})$ has one real embedding and two complex embedding that are conjugate

Proposition 1.1.6. *The degree of a number field F is equal to the number of embeddings of F in $\mathbb{C}$.*

Definition 1.1.7. A number field is said to be *totally real* if all its embeddings in $\mathbb{C}$ are real.

For instance, $\mathbb{Q}$ and $\mathbb{Q}(\sqrt{2})$ are a totally real number fields while $\mathbb{Q}(i\sqrt{3})$ and $\mathbb{Q}(\sqrt[3]{2})$ are not. Note that there exists totally real number field of every degree.

Definition 1.1.8. A *place* of a number field is either an embedding in $\mathbb{C}$ or a prime ideal of its ring of integers.

Definition 1.1.9. Let $\mathcal{P}$ be a prime ideal of $\mathcal{O}_F$. The *valuation* associated to $\mathcal{P}$ is

$$\nu_{\mathcal{P}} : F \to \mathbb{Z},\ \frac{a}{b} \mapsto \frac{\max\{m \in \mathbb{Z} \mid a \in \mathcal{P}^m\}}{\max\{m \in \mathbb{Z} \mid b \in \mathcal{P}^m\}}$$

where a and b are in $\mathcal{O}_F$. The *absolute value* associated to $\mathcal{P}$ is

$$|x|_{\mathcal{P}} = N^{-\nu_{\mathcal{P}}(x)}$$

where $N = \left|\mathcal{O}_F/\mathcal{P}\right|$.

The number field F is not complete for the absolute value $|.|_{\mathcal{P}}$. We denote by $F_{\mathcal{P}}$ the completion of F for this absolute value. It is a field extension of F.

Example 1.1.10. We present a description of the completions of $\mathbb{Q}$. The prime ideals of $\mathbb{Z}$ are of the form (p) for a prime number p. The valuation of $\mathbb{Q}$ associated to p is

$$\nu_p : \mathbb{Q} \to \mathbb{Z},\ \frac{a}{b} \mapsto \frac{\max\{m \in \mathbb{Z} \mid p^m | a\}}{\max\{m \in \mathbb{Z} \mid p^m | b\}}.$$

The absolute value is $|x|_p = p^{-\nu_p(x)}$. It satisfies

$$|x + y|_p \leq \max(|x|_p, |y|_p)$$

for all $x, y \in F$. For this absolute value, the sequence p^n for $n \in \mathbb{N}$ converges to 0. The completion of $\mathbb{Q}$ for this absolute value is denoted by $\mathbb{Q}_p$. It is a locally compact field.

1.2 Quaternion algebras

Definition 1.2.1. Let K be a field of characteristic different from 2. A *quaternion algebra* over K is a 4-dimensional K-algebra which admits a basis $\{1, i, j, k\}$ such that 1 is the identity element and

$$i^2 = a1,\ j^2 = b1 \text{ and } ij = -ji = k \tag{1.1}$$

with $a, b \in K^\times$. We denote such an algebra by $(a, b)_K$.

A quaternion algebra is necessarily non commutative. It is implicit in the definition that it is associative. Note that the center of a quaternion algebra is the subspace spanned by 1 which can be identified with K.

For any choice of $a, b \in K$ there exists a unique quaternion algebra satisfying (1.1), up to isomorphism. Indeed, one can always define a quaternion algebra by considering the formal vector space

$$K \oplus Ki \oplus Kj \oplus Kk,$$

defining a multiplicative structure on it following (1.1) and extending it linearly. However, for a given quaternion algebra A there are several $a, b \in K^\times$ such that $A \simeq (a, b)_K$. For instance, one always have

$$(a, b)_K \simeq (b, a)_K \simeq (a, -ab)_K \simeq (ax^2, by^2)_K \tag{1.2}$$

for any $x, y \in K^\times$.

Examples 1.2.2.

- For any field K, $\mathrm{M}_2(K)$ is a quaternion algebra. The following basis satisfies (1.1) with $a = b = 1$:

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ i = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ j = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ and } k = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

- The Hamiltonian quaternions $\mathcal{H}$ is a quaternion algebra over $\mathbb{R}$. It is defined as $\mathcal{H} = \mathbb{R} \oplus \mathbb{R}i \oplus \mathbb{R}j \oplus \mathbb{R}k$ where
$$i^2 = j^2 = k^2 = ijk = -1.$$

Proposition 1.2.3. *If K is algebraically closed, the only quaternion algebra over K is $\mathrm{M}_2(K)$.*

Proof. Let $(a,b)_K$ be another quaternion algebra. Since a and b are squares in K, (1.2) implies that $(a,b)_K \simeq (1,1)_K$ which is isomorphic to $\mathrm{M}_2(K)$. □

Proposition 1.2.4. *Let $a, b \in \mathbb{R}^\times$. Then*
$$(a,b)_{\mathbb{R}} \simeq \begin{cases} \mathrm{M}_2(\mathbb{R}) & \text{if } a \text{ or } b \text{ is positive} \\ \mathcal{H} & \text{if both } a \text{ and } b \text{ are negative.} \end{cases}$$

Proof. Let $(a,b)_{\mathbb{R}}$ be a quaternion algebra. If both a and b are positive, they are squares in $\mathbb{R}$. Hence $(a,b)_{\mathbb{R}} \simeq (1,1)_{\mathbb{R}} \simeq \mathrm{M}_2(\mathbb{R})$. If a is positive and b is negative then $(a,b)_{\mathbb{R}} \simeq (a,-ab)_{\mathbb{R}}$ which is isomorphic to $\mathrm{M}_2(\mathbb{R})$ as we previously said. If both of them are negative then their opposite are squares so that $(a,b)_{\mathbb{R}} \simeq \mathcal{H}$. □

In the Hamiltonian quaternions $\mathcal{H}$, every non zero element is invertible. Indeed, the inverse of an element $x = x_0 + x_1 i + x_2 j + x_3 k$ is
$$\frac{x}{x_0^2 + x_1^2 + x_2^2 + x_3^2}.$$

Definition 1.2.5. We say that a quaternion algebra is a *division quaternion algebra* if every non zero element is invertible.

Theorem 1.2.6. *If A is a quaternion algebra over K then either A is $\mathrm{M}_2(K)$ or A is a division algebra.*

It is a consequence of Theorem 1.3.4 which we will see later.

Definition 1.2.7. Let $(a,b)_K$ be a quaternion algebra over K. We define the *conjugation* on $(a,b)_K$ as the map
$$x = x_0 + x_1 i + x_2 j + x_3 k \mapsto \overline{x} = x_0 - x_1 i - x_2 j - x_3 k.$$
We define the *reduced norm* of a quaternion as $\mathrm{Nrd}(x) = x\overline{x} \in K$. We denote by A^1 the elements of A of reduced norm equal to 1.

The conjugation does not depend on the choice of the basis $\{1, i, j, k\}$ of the quaternion algebra A. Indeed, the subspace of A spanned by i, j and k can be characterized as the set of element that are not in the center of A but which squares are in the center of A.

Example 1.2.8. In $\mathrm{M}_2(K)$ the conjugate of a matrix is the transpose of its cofactor matrix and the reduced norm is the determinant.

Definition 1.2.9. Let L be a field extension of K. We say that A *splits* over L if $A \otimes_K L \simeq \mathrm{M}_2(K)$. Otherwise we say that A *ramifies* over L.

To establish that two quaternion algebras over $\mathbb{Q}$ are non isomorphic, one can show that they are not isomorphic over $\mathbb{R}$ or over $\mathbb{Q}_p$ for a certain prime p.

Theorem 1.2.10 ([MR03, Theorem 2.6.6])**.** *Let F be a number field and $\mathcal{P}$ a prime ideal of $\mathcal{O}_F$ that does not contain 2. Let $a, b \in \mathcal{O}_F$. Then*

- *if $a, b \notin \mathcal{P}$ then $(a,b)_F$ splits over $F_{\mathcal{P}}$,*
- *if $a \notin \mathcal{P}$ and $b \in \mathcal{P} \setminus \mathcal{P}^2$ then $(a,b)_F$ splits over $F_{\mathcal{P}}$ if and only if a is a square modulo $\mathcal{P}$,*
- *if $a, b \in \mathcal{P} \setminus \mathcal{P}^2$ then $(a,b)_F$ splits over $F_{\mathcal{P}}$ if and only if $-a^{-1}b$ is a square modulo $\mathcal{P}$.*

Example 1.2.11. Let $p \equiv 3[4]$ be a prime and consider the quaternion algebra $(-1,p)_{\mathbb{Q}}$. Since -1 is not a square modulo p, $(-1,p)_{\mathbb{Q}}$ ramifies over $\mathbb{Q}_p$ and does not ramify over $\mathbb{Q}_l$ for $l \neq p$ odd. Hence the quaternion algebras $(-1,p)_{\mathbb{Q}}$ are pairwise non isomorphic.

Using local invariants, on e can classify quaternion algebras over number fields.

Theorem 1.2.12 ([MR03, Theorem 7.3.6])**.** *Let F be a number field.*

- *The set of places of F where A ramifies is finite of even cardinality.*
- *Two quaternion algebras over F are isomorphic if and only if they ramify at the same places.*
- *For any set S of places of F with even cardinality containing no complex embedding, there exists a quaternion algebra over F that ramifies exactly at S.*

1.3 Central simple algebras

Definition 1.3.1. Let K be a field of characteristic different from 2. A *central simple algebra* is a finite dimensional associative unital algebra which center is $K1$ and which has no proper two-sided ideals. We say that it is a *division algebra* if every non zero element is invertible.

Remark 1.3.2. Central simple algebras are not necessarily commutative.

Examples 1.3.3.

- The field K itself is a central simple algebra.
- The matrix algebra $\mathrm{M}_n(K)$ is a central simple algebra of dimension n^2.
- Quaternion algebras over K are central simple algebras of dimension 4. Indeed, if I is a two-sided ideal of a quaternion algebra K and if $\overline{K}$ is the algebraic closure of K, $I \otimes_K \overline{K}$ is a two-sided ideal of $A \otimes_K \overline{K} \simeq \mathrm{M}_2(\overline{K})$. Conversely, since K is not of characteristic 2, any central simple algebras of dimension 4 is a quaternion algebra (see Theorem 2.1.8 of [MR03]).
- If A is a quaternion algebra over K, then $\mathrm{M}_n(A)$ is a central simple algebra of dimension $4n^2$ over K.
- Let $\alpha = 2\cos(\frac{2\pi}{7})$. It satisfies $\alpha^3+\alpha^2-2\alpha-1$. Hence $\mathbb{Q}(\alpha)$ is a number field of degree 3. Denote by σ the field automorphism of $\mathbb{Q}(\alpha)$ defined by $\sigma(\alpha) = 2\cos(\frac{4\pi}{7})$. We let

  $$D = \mathbb{Q}(\alpha) \oplus \mathbb{Q}(\alpha)v \oplus \mathbb{Q}(\alpha)v^2$$

  and define multiplication by $v^3 = 5$ and $vx = \sigma(x)v$ for all $x \in \mathbb{Q}(\alpha)$. It is a division algebra of dimension 9 over $\mathbb{Q}$.

Theorem 1.3.4 (Wedderburn [MR03, Theorem 2.9.6])**.** *Any central simple algebra over K is isomorphic to $\mathrm{M}_n(D)$ with D a division algebra over K.*

Remark 1.3.5. The integer n and the division algebra D are uniquely determined by the central simple algebra.

Proposition 1.3.6. *If K is algebraically closed, the only central simple algebras over K are $\mathrm{M}_n(K)$ for $n \geq 1$.*

Proof. By Wedderburn's Theorem, Theorem 1.3.4, we need to prove that the only division algebra over K is K itself. Let D be a division algebra over K. Let $x \in D$ and consider the left multiplication by x. Denote by $\mu_x(X) = \prod(X-\lambda_i)$ its minimal polynomial with $\lambda_i \in F$. Then

$$\prod(x-\lambda_i) = 0$$

and since D is a division algebra, $x = \lambda_i$ for some i. Finally, $x \in K$. □

If A is a central simple algebra over K and if L is a field extension of K, $A \otimes_K L$ is a central simple algebra over L. Applying this to the algebraic closure of K, we deduce that the dimension of any central simple algebra over K is a square.

Definition 1.3.7. The *degree* of a central simple algebra A over K is the square root of its dimension.

Definition 1.3.8. Let A be a central simple algebra over K and denote by $\overline{K}$ an algebraic closure of K. Pick an isomorphism $\phi : A \otimes_K \overline{K} \to \mathrm{M}_n(\overline{K})$. We define the *reduced norm* as

$$\mathrm{Nrd} : A \to K, \ x \mapsto \det(\phi(x)).$$

It has image in K since is image is fixed under Galois conjugation. Moreover, it does not depend on the isomorphism ϕ since any two such isomorphism are conjugated, see Theorem 2.9.8 in [MR03].

Definition 1.3.9. Let A_1 and A_2 be central simple algebras over K, not necessarily of the same dimension. Suppose that $A_1 \simeq \mathrm{M}_{n_1}(D_1)$ and $A_2 \simeq \mathrm{M}_{n_2}(D_2)$ with D_1 and D_2 division algebras over K. We say that A_1 and A_2 are *Brauer equivalent* if $D_1 \simeq D_2$.

Example 1.3.10. For any central simple algebra A, $\mathrm{M}_n(A)$ is Brauer equivalent to A.

Definition 1.3.11. The *Brauer group of* K is the group of Brauer equivalence classes of central simple algebras over K with the law induced by the tensor product. We denote it $\mathrm{Br}(K)$.

The neutral element is the class of K. The inverse of the class of A is the class of the opposite algebra of A. As the tensor product is symmetric, $\mathrm{Br}(K)$ is an abelian group. One can show that every element of $\mathrm{Br}(K)$ has finite order (Corollary 4.4.8 in [GS17]).

Examples 1.3.12.

- If K is algebraically closed, $\mathrm{Br}(K)$ is trivial.
- The Brauer group of $\mathbb{R}$ is a group of order 2 with the non trivial element being represented by the class of the Hamiltonian quaternions $\mathcal{H}$ (Theorem 4.3 in Chapter VII of [Per82]).
- Over a finite field K, the only division algebra is K itself (Theorem 4.9 in Chapter III of [Per82]). Hence $\mathrm{Br}(K)$ is trivial.
- Since $(-1, p)_{\mathbb{Q}}$ for $p \equiv 3[4]$ are pairwise non isomorphic division quaternion algebras, see Example 1.2.11, $\mathrm{Br}(\mathbb{Q})$ is infinite. In fact, the Brauer group of $\mathbb{Q}$ can is described using an exact sequence of the form
$$1 \to \mathrm{Br}(\mathbb{Q}) \to \bigoplus_p \mathrm{Br}(\mathbb{Q}_p) \to \mathbb{Q}/\mathbb{Z} \to 1$$
see §1.5 in [PR94].

Since $\mathrm{Br}(K)$ is abelian, its elements of 2-torsion form a group. Quaternion algebras are elements of 2-torsion in the Brauer group. In general, there are other elements of 2-torsion. However, for number fields, on has the following.

Proposition 1.3.13 ([Alb61, Theorem 20 in Chapter X])**.** *Let F be a number field. Every element of order 2 in $\mathrm{Br}(F)$ is Brauer equivalent to a quaternion algebra.*

1.4 Linear algebraic groups

In this section, all algebras are taken to be commutative, unless they are central simple algebras.

Definition 1.4.1. Let K be a field. A (*linear*) *algebraic group* over K is a functor

$$\mathbb{G} : K\text{-algebra} \to \text{group}$$

such that there exists a K-algebra A which is finitely generated and such that

$$\mathbb{G}(\cdot) \simeq \mathrm{Hom}_{K\text{-alg}}(\mathrm{A}, \cdot)$$

as functors to sets.

Note that field extensions of K are K-algebras. As we can see in the following examples, the K-algebra A encodes the fact that $\mathbb{G}$ is defined by polynomial equations with coefficients in K.

Examples 1.4.2.

- $\mathrm{SL}_2 : \mathrm{R} \to \mathrm{SL}(2,\mathrm{R}) = \{M \in \mathrm{M}_2(\mathrm{R}) | \det(M) = 1\}$ is a $\mathbb{Q}$-algebraic group with
$$\mathrm{A} = \mathbb{Q}[a,b,c,d]/(ad - bc - 1).$$
- The following functor $\mathbb{G}$
$$\mathrm{R} \to \left\{ \begin{pmatrix} x_0 + \sqrt{3}x_1 & \sqrt{3}x_2 + 3x_3 \\ \sqrt{3}x_2 - 3x_3 & x_0 - \sqrt{3}x_1 \end{pmatrix} \mid x_0^2 - 3x_1^2 - 3x_2^2 + 9x_3^2 = 1, x_i \in \mathrm{R} \right\}$$
is a $\mathbb{Q}$-algebraic group with
$$\mathrm{A} = \mathbb{Q}[x_0,x_1,x_2,x_3]/(x_0^2 - 3x_1^2 - 3x_2^2 + 9x_3^2 - 1).$$
In fact, $\mathbb{G}(\mathrm{R})$ is isomorphic to the group of norm 1 elements of the non-commutative algebra $(3,3)_{\mathbb{Q}} \otimes \mathrm{R}$, where the reduced norm is defined as in Definition 1.2.7 by extending it R-linearly.

This generalizes in the following way.

Proposition 1.4.3 (§2.3.1 in [PR94])**.** *For any central simple algebra A over a field K, the functor*
$$\mathrm{R} \to \{x \in A \otimes_K \mathrm{R} \mid \mathrm{Nrd}(x) = 1\}$$
is a linear algebraic group.

At first sight, one might think that Definition 1.4.1 is too general to exclude any meaningful example. We include the following proposition to show the opposite.

Proposition 1.4.4. *The functor*
$$\begin{aligned} \mathrm{PSL}_2 : \mathbb{Q}\text{-}&\textit{algebra} \to \textit{group} \\ \mathrm{R} &\to \mathrm{PSL}_2(\mathrm{R}) = \mathrm{SL}(2,\mathrm{R})/Z(\mathrm{SL}(2,\mathrm{R})) \end{aligned}$$
is not a linear algebraic group.

Proof. Suppose that there exists a $\mathbb{Q}$-algebra A such that $\mathrm{PSL}_2 \simeq \mathrm{Hom}(\mathrm{A},\cdot)$ as functors to sets. Denote by $\sigma : \mathbb{C} \to \mathbb{C}$ the complex conjugation. It induces a morphism
$$f_\sigma : \mathrm{Hom}(\mathrm{A},\mathbb{C}) \to \mathrm{Hom}(\mathrm{A},\mathbb{C})$$
which is the post-composition by σ. The morphisms that are fixed by f_σ are the elements of $\mathrm{Hom}(\mathrm{A},\mathbb{R}) \simeq \mathrm{PSL}_2(\mathbb{R})$. However, the matrix
$$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$
is fixed by σ in $\mathrm{PSL}_2(\mathbb{C})$ but does not lie in $\mathrm{PSL}_2(\mathbb{R})$. □

Definition 1.4.5. Let K be a field. A *morphism of algebraic groups* $\phi : \mathbb{G}_1 \to \mathbb{G}_2$ is the data of a morphisms of groups
$$\phi(\mathrm{R}) : \mathbb{G}_1(\mathrm{R}) \to \mathbb{G}_2(\mathrm{R})$$
for every K-algebra R such that for all morphisms of K-algebras $f : \mathrm{R} \to \mathrm{R}'$ we have
$$\phi(\mathrm{R}') \circ \mathbb{G}_1(f) = \mathbb{G}_2(f) \circ \phi(\mathrm{R}).$$

Definition 1.4.6. Let K be a field and L an extension of K. Let $\mathbb{G}$ be a linear algebraic group over K. We denote by $\mathbb{G}_L$ the L-linear algebraic group defined by $\mathbb{G}_L(\mathrm{R}) = \mathbb{G}(\mathrm{R})$ for any L-algebra R.

Definition 1.4.7. Let L/K be a field extension. Let $\mathbb{G}$ be an L-algebraic group. A K-algebraic group $\mathbb{H}$ is said to be an *L/K-form of* $\mathbb{G}$ if $\mathbb{H}_L \simeq \mathbb{G}$. If L is the separable closure of K, we call L/K-forms of $\mathbb{G}$ simply K-forms of $\mathbb{G}$.

Example 1.4.8. The $\mathbb{Q}$-algebraic group $\mathbb{G}$ defined in Examples 1.4.2 is not isomorphic to the $\mathbb{Q}$-algebraic group SL_2 yet $\mathbb{G}_{\mathbb{R}}$ is isomorphic to the $\mathbb{R}$-algebraic group SL_2. Hence $\mathbb{G}$ is a $\mathbb{Q}$-form of SL_2.

More generally, central simple algebras provide a way to construct forms of SL_n.

Example 1.4.9. Let A be a central simple algebra of dimension n^2 over a number field F. Denote by

$$\mathbb{G} : \mathrm{R} \to (A \otimes_K \mathrm{R})^1 := \{x \in A \otimes_K \mathrm{R} \mid \mathrm{Nrd}(x) = 1\}$$

the associated linear F-algebraic group. Since $A \otimes_F \overline{\mathbb{Q}} \simeq \mathrm{M}_n(\overline{\mathbb{Q}})$, $\mathbb{G}$ is an F-form of SL_n.

Not all forms of SL_n are of this type.

Example 1.4.10. Let $d \in \mathbb{Q}$ which is not a square. Denote by σ the automorphism of $\mathbb{Q}$ defined by $\sigma(\sqrt{d}) = -\sqrt{d}$. Let $\mathbb{G}_d$ be the $\mathbb{Q}$-algebraic group

$$\mathrm{R} \to \mathrm{SU}(I_3, \sigma, \mathbb{Q}(\sqrt{d}) \otimes_{\mathbb{Q}} \mathrm{R}) = \{M \in \mathrm{SL}(3, \mathbb{Q}(\sqrt{d}) \otimes_{\mathbb{Q}} \mathrm{R}) \mid \sigma(M)^\top M = I_3\}$$

where we extend σ by R-linearity. Then $\mathbb{G}_d$ is a $\mathbb{Q}$-form of SL_n. Indeed, if R is a $\overline{\mathbb{Q}}$-algebra then

$$\mathbb{Q}(\sqrt{d}) \otimes_{\mathbb{Q}} \mathrm{R} \simeq \mathbb{Q}(\sqrt{d}) \otimes_{\mathbb{Q}} \overline{\mathbb{Q}} \otimes_{\overline{\mathbb{Q}}} \mathrm{R} \simeq \overline{\mathbb{Q}}(\sqrt{d}) \otimes_{\overline{\mathbb{Q}}} \mathrm{R} \simeq (\overline{\mathbb{Q}} \oplus \overline{\mathbb{Q}}) \otimes_{\overline{\mathbb{Q}}} \mathrm{R} \simeq \mathrm{R} \oplus \mathrm{R}$$

such that σ corresponds to the involution $(x, y) \mapsto (y, x)$ of $\mathrm{R} \oplus \mathrm{R}$. Thus

$$\begin{aligned}
\mathbb{G}_d(\mathrm{R}) &\simeq \{M \in \mathrm{SL}(3, \mathrm{R} \oplus \mathrm{R}) \mid \sigma(M)^\top M = I_3\} \\
&\simeq \{(M_1, M_2) \in \mathrm{SL}(3, \mathrm{R})^2 \mid (M_2, M_1)^\top (M_1, M_2) = (I_3, I_3)\} \\
&\simeq \{(M_1, M_1^{-\top}) \mid M_1 \in \mathrm{SL}(3, \mathrm{R})\} \\
&\simeq \mathrm{SL}(3, \mathrm{R}).
\end{aligned}$$

1.5 Exceptional groups of type G_2

Definition 1.5.1. An *octonion algebra* over a field K of characteristic not equal to 2 is a unital but non-associative and non-commutative algebra of dimension 8 over K, such that there exists a nondegenerate quadratic form N on the algebra satisfying

$$N(xy) = N(x)N(y)$$

for all x, y. An octonion algebra is said to be *split* if N is isotropic.

Example 1.5.2. Let A be a quaternion algebra over K with conjugation $^-$ and $a \in K^\times$. Then $A \oplus A$ with the product

$$(x, y)(z, w) = (xz + a\overline{w}y, wx + y\overline{z})$$

is an octonion algebra over K. Then norm is defined by $N((x, y)) = \mathrm{Nrd}(x) - a\mathrm{Nrd}(y)$. Reciprocally, all octonion algebras over K are of this form, see Theorem 1.6.2 in [SV00].

Theorem 1.5.3 ([SV00, Theorem 1.8.1])**.** *For any field K of characteristic not equal to 2, there is a unique split octonion algebra over K up to isomorphism.*

Example 1.5.4.

- Over $\mathbb{C}$ all quadratic forms in 8 variables are isotropic. Hence there is only one octonion algebra: the split one. Its automorphism group is isomorphic to the complex Lie group of type G_2.
- Over $\mathbb{R}$, there are two isomorphism classes of octonion algebras. The non-split one has a positive define quadratic form. Its automorphism group is the compact Lie group of type G_2. The split one has a quadratic form of signature $(4, 4)$ and its automorphism group is the centerless split real Lie group of type G_2. Note that the latter is not simply connected and has a universal cover of degree 2. Its universal cover is not the automorphism group of an octonion algebra.

Octonion algebras give rise to linear algebraic groups as follows.

Theorem 1.5.5 ([SV00, Theorem 2.3.5]). *Let $\mathbb{O}$ be an octonion algebra over K. Then the functor*

$$\begin{aligned} \mathrm{Aut}_{\mathbb{O}} : K\text{-algebra} &\to \text{group} \\ \mathrm{R} &\to \mathrm{Aut}(\mathbb{O} \otimes_K R) \end{aligned}$$

is a K-algebraic group of type G_2.

Remark 1.5.6. Let $\mathbb{G}$ be the linear algebraic group associated to the unique octonion algebra over $\overline{K}$. If $\mathbb{O}$ is any octonion algebra over K then $\mathrm{Aut}_{\mathbb{O}}$ is a K-form of $\mathbb{G}$. Conversely, any K-form of $\mathbb{G}$ is of this form.

We now give an alternative description of the linear algebraic groups associated to the unique split octonion algebra over $\mathbb{Q}$. To do so, we introduce the following seven dimensional cross product, which could also be defined in terms of quaternion algebras.

Definition 1.5.7. Let R be a ring. We denote by $\times : \mathrm{R}^7 \times \mathrm{R}^7 \to \mathrm{R}^7$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \\ y_7 \end{pmatrix} \mapsto \begin{pmatrix} 3(x_1y_4 - x_4y_1) - 2(x_2y_3 - x_3y_2) \\ 12(x_1y_5 - x_5y_1) - 3(x_2y_4 - x_4y_2) \\ 30(x_1y_6 - x_6y_1) - 3(x_3y_4 - x_4y_3) \\ 60(x_1y_7 - x_7y_1) + 10(x_2y_6 - x_6y_2) - 4(x_3y_5 - x_5y_3) \\ 30(x_2y_7 - x_7y_2) - 3(x_4y_5 - x_5y_4) \\ 12(x_3y_7 - x_7y_3) - 3(x_4y_6 - x_6y_4) \\ 3(x_4y_7 - x_7y_4) - 2(x_5y_6 - x_6y_5) \end{pmatrix}.$$

The choice of the coefficients in the above definition is justified by the following proposition that will be useful later on. Recall that τ_7 is the seven dimensional irreducible representation of $\mathrm{SL}(2, \mathbb{C})$ defined in the introduction.

Proposition 1.5.8. *For any $M \in \mathrm{SL}(2, \mathbb{C})$ and any $x, y \in \mathbb{C}^7$,*

$$\tau_7(M)(x \times y) = \tau_7(M)x \times \tau_7(M)y.$$

Proof. Since $\mathrm{SL}(2, \mathbb{Z})$ is Zariski-dense in $\mathrm{SL}(2, \mathbb{C})$, it suffices to prove the result for $M \in \mathrm{SL}(2, \mathbb{Z})$. The latter is proven with tedious computations using explicit generators of $\mathrm{SL}(2, \mathbb{Z})$. □

We can now define the linear algebraic group $\mathbf{G}_2$.

Definition 1.5.9. Define

$$\begin{aligned} \mathbf{G}_2 : \mathbb{Q}\text{-algebra} &\to \text{group} \\ \mathrm{R} &\to \{M \in \mathrm{SO}(J_7, \mathrm{R}) | M(x \times y) = Mx \times My,\ \forall x, y \in \mathrm{R}^7\} \end{aligned}$$

where $J_7 = \mathrm{Diag}(720, -120, 48, -36, 48, -120, 720)$.

Proposition 1.5.10. $\mathbf{G}_2 \simeq \mathrm{Aut}_{\mathbb{O}}$ *where $\mathbb{O}$ is the unique split octonion algebra over $\mathbb{Q}$.*

Proof. Consider the 8-dimensional algebra over $\mathbb{Q}$ defined as $\mathbb{O} = \mathbb{Q} \oplus \mathbb{Q}^7$ where the product is

$$(t, v)(s, w) = (ts - v^\top J_7 w, tw + sv + v \times w)$$

for all $t, s \in K$ and $v, w \in \mathbb{Q}^7$. The unit is $(1, 0)$. The quadratic form

$$N : (t, v) \mapsto t^2 + v^\top J_7 v$$

is nondegenerate. We can check that

$$v^\top J_7 (v \times w) = 0$$

and

$$(v \times w)^\top J_7(v \times w) = (v^\top J_7 v)(w^\top J_7 w) - (v^\top J_7 w)^2$$

for all $v, w \in \mathbb{Q}^7$. These two properties together imply that $N(xy) = N(x)N(y)$. Hence $\mathbb{O}$ is an octonion algebra over $\mathbb{Q}$. Since N is isotropic (Corollary 2 in §3 of Chapter 4 in [Ser73]) $\mathbb{O}$ is split.

Let R be a $\mathbb{Q}$-algebra. Pick $M \in \mathbf{G}_2(\mathrm{R})$ and define its action on $\mathbb{O} \otimes_{\mathbb{Q}} \mathrm{R}$ by $M(t, v) = (t, Mv)$ for all $(t, v) \in \mathbb{O} \otimes_{\mathbb{Q}} \mathrm{R}$. Then for all x, y $M(xy) = M(x)M(y)$. Thus M defines an automorphism of $\mathbb{O} \otimes_{\mathbb{Q}} \mathrm{R}$. Reciprocally let $\phi \in \mathrm{Aut}(\mathbb{O} \otimes_{\mathbb{Q}} \mathrm{R})$. Then $\phi(1, 0) = (1, 0)$ and thus ϕ fixes $\mathrm{R} \oplus \{0\}$. Since

$$N(t, v) = (t, v)(t, -v)$$

$N(\phi(x)) = N(x)$ for all x. This implies that ϕ has to preserve the orthogonal of $\mathrm{R} \oplus \{0\}$ for N, i.e. ϕ preserves $\{0\} \oplus \mathrm{R}^7$. For any $v, w \in \mathrm{R}^7$ we have

$$\begin{aligned}
&\phi(0, v)\phi(0, w) = \phi((0, v)(0, w)) \\
\implies &(0, \phi_{|\mathrm{R}^7}(v))(0, \phi_{|\mathrm{R}^7}(w)) = \phi(-v^\top J_7 w, v \times w) \\
\implies &(-\phi_{|\mathrm{R}^7}(v)^\top J_7 \phi_{|\mathrm{R}^7}(w), \phi_{|\mathrm{R}^7}(v) \times \phi_{|\mathrm{R}^7}(w)) = (-v^\top J_7 w, \phi_{|\mathrm{R}^7}(v \times w))
\end{aligned}$$

which imply that $\phi_{|\mathrm{R}^7} \in \mathbf{G}_2(\mathrm{R})$. Finally, we showed that the restriction induces an isomorphism $\mathbf{G}_2(\mathrm{R}) \simeq \mathrm{Aut}(\mathbb{O} \otimes_{\mathbb{Q}} \mathrm{R})$. One can check that this isomorphism give rise to an isomorphism of algebraic groups. □

1.6 Galois cohomology

Let $\mathcal{G}$ be a topological group acting continuously by automorphism on a discrete group M which is not necessarily abelian.

Definition 1.6.1. A *1-cocycle* is a continuous map $\zeta : \mathcal{G} \to M$ that satisfies

$$\zeta(st) = \zeta(s)s(\zeta(t))$$

for all $s, t \in \mathcal{G}$.

Note that if the action of $\mathcal{G}$ on M is trivial, 1-cocycles are just the continuous homomorphisms.

Examples 1.6.2.

- The map $\zeta : \mathcal{G} \to M$ which associates to every element of $\mathcal{G}$ the neutral element of M is a 1-cocycle. We call it the *trivial cocycle*.
- The map $\xi : \mathrm{Gal}(\mathbb{C}/\mathbb{R}) \to \mathrm{PGL}(2, \mathbb{C})$

$$\sigma \mapsto \begin{cases} I_2 & \text{if } \sigma \text{ is the identity,} \\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & \text{if } \sigma \text{ is the complex conjugation} \end{cases}$$

  is a 1-cocycle.
- If $\mathcal{G}$ acts on M, it also acts on $\mathrm{Aut}(M)$ by conjugation. One can see that a 1-cocycle $\zeta : G \to \mathrm{Aut}(M)$ define actions of G on M by $g \cdot m = \zeta(g)(g(m))$.

Definition 1.6.3. We say that two 1-cocycles ζ and ζ' are *equivalent* if there exists an element $m \in M$ such that

$$\zeta'(s) = m^{-1}\zeta(s)s(m)$$

for all $s \in \mathcal{G}$. We denote by $\mathrm{H}^1(\mathcal{G}, M)$ the set of equivalence classes of 1-cocycles and call it *the first cohomology set of $\mathcal{G}$ (with coefficient in M).*

The first cohomology set is not a group in general. It is a set with distinguished element, the distinguished element being the trivial cocycle. When M is an abelian group however, it has a canonical group structure and we recover usual group cohomology.

Proposition 1.6.4. *Let N be a subgroup of the center of M. Then there is a long exact sequence of sets with distinguished elements*

$$\mathrm{H}^1(\mathcal{G}, N) \to \mathrm{H}^1(\mathcal{G}, M) \to \mathrm{H}^1(\mathcal{G}, M/N) \xrightarrow{\delta} \mathrm{H}^2(\mathcal{G}, N)$$

where $\mathrm{H}^2(\mathcal{G}, N)$ is defined via the usual group cohomology.

The morphism of sets with distinguished element δ is called a *connecting morphism*. It is defined in the following way. Pick a 1-cocycle $\zeta : \mathcal{G} \to M/N$. For each $\sigma \in \mathcal{G}$ choose a lift $m_\sigma \in M$ of $\zeta(\sigma)$. Then

$$\delta(\zeta)_{\sigma,\tau} = m_\sigma \sigma(m_\tau) m_{\sigma\tau}^{-1} \in N$$

for all $\sigma, \tau \in \mathcal{G}$.

We are mainly interested in the setting where $\mathcal{G}$ is a Galois group and M is a group of automorphisms of algebraic structures. The following important result is a major tool to compute Galois cohomology sets. Recall that an algebraic field extension L/K is *Galois* if the field fixed by $\mathrm{Aut}(L/K)$ is K.

Theorem 1.6.5 (Hilbert 90)**.** *Let L/K be a Galois extension. For any $n \geq 1$*

$$\mathrm{H}^1(\mathrm{Gal}(L/K), \mathrm{GL}_n(L)) = \{1\}.$$

In other words, for every 1-cocycle $\zeta : \mathrm{Gal}(L/K) \to \mathrm{GL}_n(L)$ there exists a matrix $S \in \mathrm{GL}_n(L)$ such that for all $\sigma \in \mathrm{Gal}(L/K)$ one has $\zeta(\sigma) = S^{-1}\sigma(S)$. It will of crucial importance to be able to compute S later on. Fortunately, the proof provides an algorithm to compute such S. Note that the matrix S is not unique.

Proof. Suppose that L/K is a finite extension. Define

$$f : L^n \to L^n,\ x \mapsto \sum_{\sigma \in \mathrm{Gal}(L/K)} \zeta(\sigma)\sigma(x).$$

We show that f is surjective. Let ϕ be a L-linear form on L^n such that ϕ vanishes on $\mathrm{Im}(f)$. Then for all $\lambda \in L$ and $x \in L^n$

$$\begin{aligned} 0 = \phi(f(\lambda x)) &= \sum_{\sigma \in \mathrm{Gal}(L/K)} \phi(\zeta(\sigma)\sigma(\lambda x)) \\ &= \sum_{\sigma \in \mathrm{Gal}(L/K)} \sigma(\lambda)\phi(\zeta(\sigma)\sigma(x)). \end{aligned}$$

It follows from Dedekind's lemma, Lemma 5.1 in [Coh89], that for any $x \in L^n$ and any $\sigma \in \mathrm{Gal}(L/K)$ $\phi(\zeta(\sigma)\sigma(x)) = 0$, i.e. $\phi = 0$. Hence there exist $v_1, \ldots, v_n \in L^n$ such that $(f(v_1), \ldots, f(v_n))$ is a basis of L^n. Denote $S = (f(v_1)|\ldots|f(v_n))^{-1}$. Since $\sigma(f(v)) = \zeta(\sigma)^{-1} f(v)$, we have $S^{-1}\sigma(S) = \zeta(\sigma)$.

If L/K is an infinite Galois extension,

$$\mathrm{H}^1(\mathrm{Gal}(L/K), \mathrm{GL}(n, L)) = \varinjlim \mathrm{H}^1(\mathrm{Gal}(F/K), \mathrm{GL}(n, F)) = \{1\}$$

where F runs through all finite Galois extensions F of K. This concludes the proof. □

The Brauer group of a field (recall Definition 1.3.11) has a cohomological interpretation.

Theorem 1.6.6 ([GS17, Theorem 4.4.7])**.** *Let K be a field with separable closure L. Then* $\mathrm{Br}(K) \simeq \mathrm{H}^2(\mathrm{Gal}(L/K), L^\times)$.

The isomorphism is constructed in the following way. From an equivalence class $[(a_{s,t})_{s,t}] \in \mathrm{H}^2(\mathrm{Gal}(L/K), L^\times)$ we construct a central simple algebra as follows. Since

$$\mathrm{H}^2(\mathrm{Gal}(L/K), L^\times) = \varinjlim \mathrm{H}^2(\mathrm{Gal}(F/K), F^\times)$$

where F runs through all finite Galois extensions of K, there exists such an F and a element $[(b_{s,t})_{s,t}] \in \mathrm{H}^2(\mathrm{Gal}(F/K), F^\times)$ such that $[(a_{s,t})_{s,t}]$ is the image of $[(b_{s,t})_{s,t}]$. We then consider the central simple algebra over K whose underlying vector space is

$$\bigoplus_{\sigma \in \mathrm{Gal}(F/K)} F v_\sigma$$

and multiplication defined by

$$(x v_\sigma)(y v_\tau) = x\sigma(y) b_{\sigma,\tau} v_{\sigma\tau} \text{ for all } \sigma, \tau \in \mathrm{Gal}(F/K).$$

Remark 1.6.7. Note that if K is of characteristic 0 then the separable closure of K is exactly the algebraic closure of K.

We now show how these theorems help computing Galois cohomology sets in a specific example.

Example 1.6.8. The first cohomology set $\mathrm{H}^1(\mathrm{Gal}(\mathbb{C}/\mathbb{R}), \mathrm{PGL}(2,\mathbb{C}))$ has only two elements: the class of the trivial cocycle and the class of ξ as defined in Examples 1.6.2. Those two are not equivalent since the trivial cocycle lifts to a cocycle in $\mathrm{GL}(2,\mathbb{C})$ while ξ does not. To show that these are the only 1-cocycle up to equivalence consider the following exact sequence, see Proposition 1.6.4,

$$\mathrm{H}^1(\mathrm{Gal}(\mathbb{C}/\mathbb{R}), \mathrm{GL}(2,\mathbb{C})) \to \mathrm{H}^1(\mathrm{Gal}(\mathbb{C}/\mathbb{R}), \mathrm{PGL}(2,\mathbb{C})) \xrightarrow{\delta} \mathrm{H}^2(\mathrm{Gal}(\mathbb{C}/\mathbb{R}), \mathbb{C}^\times).$$

Theorem 1.6.5 states that $\mathrm{H}^1(\mathrm{Gal}(\mathbb{C}/\mathbb{R}), \mathrm{GL}(2,\mathbb{C}))$ is trivial. Using this together with a "twisting" argument, see §1.3.2 in [PR94], one can show that δ is injective. It follows that $\mathrm{H}^1(\mathrm{Gal}(\mathbb{C}/\mathbb{R}), \mathrm{PGL}(2,\mathbb{C}))$ cannot have more elements than $\mathrm{H}^2(\mathrm{Gal}(\mathbb{C}/\mathbb{R}), \mathbb{C}^\times) \simeq \mathrm{Br}(\mathbb{R})$ which has two elements, see Examples 1.3.12.

Galois cohomology sets classify forms of algebraic structures. Let L/K be a Galois field extension and $\mathbb{G}$ be a K-algebraic group. As Proposition 1.6.9 below asserts, there is a one-to-one correspondence between $\mathrm{H}^1(\mathrm{Gal}(L/K), \mathrm{Aut}(\mathbb{G}(L)))$ and L/K-forms of $\mathbb{G}_L$. By $\mathrm{Aut}(\mathbb{G}(L))$ we mean the group of automorphism of $\mathbb{G}(L)$ that come from algebraic automorphisms of $\mathbb{G}_L$. Let $\mathbb{H}$ be a L/K-form of $\mathbb{G}_L$ and $\phi : \mathbb{H}_L \to \mathbb{G}_L$ be an isomorphism, then the map

$$\begin{aligned} \mathrm{Gal}(L/K) &\to \mathrm{Aut}(\mathbb{G}(L)) \\ \sigma &\mapsto \phi(L) \circ \sigma \circ \phi(L)^{-1} \circ \sigma^{-1} \end{aligned}$$

is a 1-cocycle. This gives one direction of the following proposition.

Proposition 1.6.9 ([Ser02])**.** *Assuming L/K is Galois, the map that associates to a L/K-form of $\mathbb{G}_L$ a 1-cocycle defines a bijection between the set of isomorphism classes of L/K-forms of $\mathbb{G}_L$ and $\mathrm{H}^1(\mathrm{Gal}(L/K), \mathrm{Aut}(\mathbb{G}(L)))$.*

For $\zeta \in \mathrm{H}^1(\mathrm{Gal}(L/K), \mathrm{Aut}(\mathbb{G}(L)))$, we denote by ${}_\zeta\mathbb{G}$ the corresponding L/K-form. Concretely, for any K-algebra R

$${}_\zeta\mathbb{G}(\mathrm{R}) = \{g \in \mathbb{G}(L \otimes_K \mathrm{R}) \mid \zeta(\sigma)(\sigma(g)) = g \ \forall \sigma \in \mathrm{Gal}(L/K)\}.$$

If a cocycle has values in $\mathrm{Inn}(\mathbb{G}(L))$, then we say this cocycle is *inner*.

Example 1.6.10. Since $\mathrm{Aut}(\mathrm{SL}_2(\mathbb{C})) = \mathrm{PGL}(2,\mathbb{C})$, Example 1.6.8 implies that there are only two real forms of the $\mathbb{C}$-algebraic group SL_2. The real form corresponding to the trivial cocycle is the $\mathbb{R}$-algebraic group SL_2. The other real form is ${}_\xi\mathrm{SL}_2$ whose real points are

$$\begin{aligned} {}_\xi\mathrm{SL}_2(\mathbb{R}) &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C}) \;\middle|\; \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \overline{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right\} \\ &= \left\{ \begin{pmatrix} a & b \\ -\overline{b} & \overline{a} \end{pmatrix} \;\middle|\; a, b \in \mathbb{C},\ a\overline{a} + b\overline{b} = 1 \right\} \simeq \mathrm{SU}(2). \end{aligned}$$

Note that $\mathrm{SU}(2)$ is the norm 1 elements of Hamiltonian quaternions $\mathcal{H}$. Hence any $\mathbb{R}$-form of SL_2 is the norm 1 elements of a quaternion algebra.

One can generalize the argument of Example 1.6.8 and Example 1.6.10 ot prove the following.

Proposition 1.6.11. *Let K be field. Then K-forms of SL_2 are isomorphic to*

$$\mathrm{R} \to (A \otimes_K \mathrm{R})^1$$

for a quaternion algebra A over K.

In this thesis, we will be mainly interested in $\mathbb{R}/F$-forms of algebraic groups where F is a number field. Unfortunately, the extension $\mathbb{R}/F$ is not Galois, nor is $\mathbb{C}/F$, since its is not algebraic. One can still use non-abelian Galois cohomology to classify its $\mathbb{R}/F$-forms thanks to the following proposition.

Proposition 1.6.12 ([Ser02, Proposition 2, Chapter III, §1])**.** *Let V be a finite dimensional vector space over F and T be a tensor of V. Consider the following F-algebraic group*

$$\mathbb{G} : \mathrm{R} \to \mathrm{Aut}(V \otimes_F \mathrm{R}, T).$$

Then $\mathbb{R}/F$-forms of $\mathbb{G}$ are $\overline{\mathbb{Q}}/F$-forms of $\mathbb{G}$.

Note that all algebraic groups that will be considered in this thesis are the automorphism group of some tensor. If T is a non-degenerate bilinear form, its automorphism group is an orthogonal group. Similarly, tensors of type $(1, 2)$ define an algebra structure on V.

Chapter 2

Arithmetic subgroups

In this chapter, we introduce a specific class of discrete subgroups of Lie groups: *lattices*. Thanks to Borel−Harish-Chandra's Theorem, Theorem 2.2.1 below, there is an arithmetic construction that provides explicit examples of lattices. Subgroups that arise from this construction are called *arithmetic subgroups*. Margulis Arithmeticity Theorem, Theorem 2.2.14, states that for higher rank simple Lie groups, all lattices come from this construction. This allows to classify lattices in higher rank. Finally, we discuss the Strong Approximation Theorem which describe the reductions of a Zariski-dense subgroup of an arithmetic groups to finite fields. The main reference is Witt Morris' book [Mor15].

2.1 Lattices

Proposition 2.1.1. *Let G be a Lie group. There exists a unique, up to scalar, Borel measure σ-finite on G which is invariant under left-translation and finite on compact sets.*

Definition 2.1.2. A subgroup $\Lambda < G$ of a Lie group is a *lattice* if Λ is discrete and $\Lambda\backslash G$ has finite volume, i.e. there is a fundamental domain for the action of Λ on G with finite volume. If $\Lambda\backslash G$ is compact then Λ is said to be *cocompact* or *uniform*.

Examples 2.1.3.

- $\mathrm{SL}(n,\mathbb{Z}) < \mathrm{SL}(n,\mathbb{R})$ is a non-uniform lattice
- $\mathrm{Sp}(2n,\mathbb{Z}) < \mathrm{Sp}(2n,\mathbb{R})$ is a non-uniform lattice
- $\pi_1(\mathbb{S}^3\backslash K_8) < \mathrm{PSL}(2,\mathbb{C})$ discrete with K_8 the figure 8 knot is a non-uniform lattice

If $\Gamma < \Lambda < G$ with Γ of finite index in Λ then Γ is a lattice if and only if Λ is a lattice. If $\Lambda < G$ is a lattice then any of its conjugate is a lattice in G. Using the classification of topological surfaces, one can describe the group structures of lattices of $\mathrm{SL}(2,\mathbb{R})$ easily. A property of a group is *virtually* true if it is true for one of its finite index subgroup.

Proposition 2.1.4. *Let $\Lambda < \mathrm{SL}(2,\mathbb{R})$ be a lattice. If Λ is non-uniform then it is virtually a non-abelian free group. If Λ is uniform then it is virtually a surface group.*

Let $\Lambda < \mathrm{SO}(n,1)$ be a discrete subgroup. It acts properly discontinuously on the real hyperbolic space $\mathbb{H}^n$. Then Λ is a lattice if and only if the orbifold $\Lambda\backslash\mathbb{H}^n$ has finite volume. It is uniform if and only if $\Lambda\backslash\mathbb{H}^n$ is compact. Hence fundamental groups of compact hyperbolic manifolds are lattices of $\mathrm{SO}(n,1)$. Conversely, any uniform lattice of $\mathrm{SO}(n,1)$ is virtually the fundamental group of a compact hyperbolic manifold of dimension n. Indeed, all lattice contain a finite index torsion free subgroup. This is the content of Selberg's Lemma, stated as follows.

Theorem 2.1.5 (Selberg [Mor15, Theorem 4.8.2])**.** *Let Λ be a lattice of a linear Lie group. Then Λ admits a finite index subgroup that is torsion free.*

We now present a simple criterion to determine when a lattice is uniform.

Theorem 2.1.6 (Každan−Margulis [KM68, Theorem 2])**.** *Let G be a semisimple Lie group and $\Lambda < G$ be a lattice. Then Λ is non-uniform if and only if Λ contains non-trivial unipotent elements.*

Example 2.1.7. The element

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})$$

is a non-trivial unipotent element. Hence $\mathrm{SL}(2,\mathbb{Z})$ is non-uniform.

The following theorem is called Borel's Density Theorem.

Theorem 2.1.8 (Borel [Bor60])**.** *Let $G < \mathrm{GL}(n,\mathbb{R})$ be a connected semisimple Lie group without compact factors. Let $\Lambda < G$ be a lattice. Then Λ is Zariski-dense in G, i.e. if $P \in \mathbb{R}[x_{11},...,x_{nn}]$ satisfies $P(\Lambda) = 0$ then $P(G) = 0$.*

2.2 Definition

Theorem 2.2.1 (Borel−Harish-Chandra [BHC62, Theorem 9.4])**.** *Let $\mathbb{G}$ be a linear semisimple $\mathbb{Q}$-algebraic group. Then $\mathbb{G}(\mathbb{Z})$ is a lattice of $\mathbb{G}(\mathbb{R})$.*

Since $\mathbb{Z}$ is not a $\mathbb{Q}$-algebra, $\mathbb{G}(\mathbb{Z})$ doesn't make sense a priori. To define it we pick an embedding $\rho : \mathbb{G} \hookrightarrow \mathrm{GL}_n$ and define $\mathbb{G}(\mathbb{Z}) = \mathbb{G}(\mathbb{Q}) \cap \rho^{-1}(\mathrm{GL}_n(\mathbb{Z}))$. The group $\mathbb{G}(\mathbb{Z})$ does depend on ρ, but two different embeddings yield commensurable subgroups.

Corollary 2.2.2 (Borel [Bor63])**.** *Let G be a connected semisimple Lie group. Then G admits a lattice.*

As an example, consider the case $G = \mathrm{SO}(n,1)$. We define the $\mathbb{Q}$-algebraic group $\mathbb{G} = \mathrm{SO}(x_1^2 + \ldots + x_n^2 - x_{n+1}^2, \cdot)$ so that $\mathbb{G}(\mathbb{R}) \simeq G$. Hence $\mathbb{G}(\mathbb{Z})$ is a lattice in G. The Borel−Harish-Chandra Theorem can be generalized to algebraic groups over a number field in the following way.

Theorem 2.2.3 (Borel−Harish-Chandra [BHC62, Theorem 12.3])**.** *Let $\mathbb{G}$ be a linear semisimple algebraic group over a number field F. Then $\mathbb{G}(\mathcal{O}_F)$ is a lattice in*

$$\prod_{\sigma : F \hookrightarrow \mathbb{R}} \mathbb{G}(\mathbb{R}^\sigma) \times \prod_{\substack{\{\sigma;\overline{\sigma}\} \\ \sigma : F \hookrightarrow \mathbb{C} \\ \sigma(F) \not\subset \mathbb{R}}} \mathbb{G}(\mathbb{C}^\sigma).$$

Remarks 2.2.4.

- The notation $\{\sigma, \overline{\sigma}\}$ means that for each pair of conjugate complex embeddings, there is only one factor $\mathbb{G}(\mathbb{C}^\sigma)$ where we pick σ to be any of the two embeddings.
- There are various way to see $\mathbb{R}$ as a F-algebra, depending on the embedding σ chosen to embed F in $\mathbb{R}$. In the previous theorem, we denote $\mathbb{R}^\sigma$ instead of $\mathbb{R}$ to keep track of this F-algebra structure on $\mathbb{R}$. For two different embeddings of F in $\mathbb{R}$ the group $\mathbb{G}(\mathbb{R}^\sigma)$ may differ (see Example 2.2.9).
- The group $\mathbb{G}(\mathcal{O}_F)$ is embedded in the following way. Since $\mathbb{G}$ is a functor, for every embedding $\sigma : F \hookrightarrow \mathbb{C}$ we get an embedding $\mathbb{G}(F) \hookrightarrow \mathbb{G}(\mathbb{C}^\sigma)$. Informally, it consists of applying σ to the entries of the matrix $g \in \mathbb{G}(F)$. We then consider the product of all those embeddings.

Examples 2.2.5.

- $\mathrm{SL}(2, \mathbb{Z}[\frac{-1+i\sqrt{3}}{2}]) < \mathrm{SL}(2,\mathbb{C})$ is a non-uniform lattice. Hence $\mathrm{PSL}(2, \mathbb{Z}[\frac{-1+i\sqrt{3}}{2}])$ is a lattice of $\mathrm{PSL}(2,\mathbb{C})$. It contains $\pi_1(\mathbb{S}^3 \setminus K_8)$ as an index 12 subgroup, see [MR03] §1.4.3.
- $\mathrm{SL}_n(\mathbb{Z}[\sqrt[3]{2}]) < \mathrm{SL}_n(\mathbb{R}) \times \mathrm{SL}_n(\mathbb{C})$ is a lattice

The following proposition states that "integer-points" of $\mathbb{Q}$-algebraic groups behave well under morphisms. It will be used several times in this thesis.

Proposition 2.2.6 ([Mil13, Proposition 5.2 Appendix A])**.** *Let $f : \mathbb{G} \to \mathbb{H}$ be a homomorphism of algebraic groups over a number field F. Up to commensurability, $f(\mathbb{G}(\mathcal{O}_F)) < \mathbb{H}(\mathcal{O}_F)$.*

Proposition 2.2.7. *Let F be a number field and $\mathbb{G}$ be a semisimple F-algebraic group. If there is an embedding $\sigma : F \hookrightarrow \mathbb{R}$ such that $\mathbb{G}(\mathbb{R}^\sigma)$ is compact then $\mathbb{G}(\mathcal{O}_F)$ is uniform.*

Proof. Let $\sigma : F \hookrightarrow \mathbb{R}$ be an embedding such that $\mathbb{G}(\mathbb{R}^\sigma)$ is compact. Let $g \in \mathbb{G}(\mathcal{O}_F)$ be a unipotent element. Then $\sigma(g)$ is unipotent of a compact group. Hence it is trivial. By Theorem 2.1.6, $\mathbb{G}(\mathcal{O}_F)$ is uniform. □

The following lemma gives us a way to extend the construction of the Borel−Harish-Chandra Theorem.

Lemma 2.2.8 ([MR03, Lemma 8.1.3])**.** *Let $\Lambda < G_1 \times G_2$ be a lattice with G_2 compact. Then the projection of Λ on G_1 is a lattice.*

Example 2.2.9. Consider the $\mathbb{Q}(\sqrt{7})$-algebraic group

$$\mathbb{G} = \mathrm{SO}(x^2 + y^2 - \sqrt{7}z^2, \cdot).$$

Borel-Harish-Chandra Theorem states that $\mathrm{SO}(x^2 + y^2 - \sqrt{7}z^2, \mathbb{Z}[\sqrt{7}])$ is a lattice of

$$\mathrm{SO}(x^2 + y^2 - \sqrt{7}z^2, \mathbb{R}) \times \mathrm{SO}(x^2 + y^2 + \sqrt{7}z^2, \mathbb{R}) \simeq \mathrm{SO}(2,1) \times \mathrm{SO}(3).$$

Since $\mathrm{SO}(3)$ is compact, it is also a lattice of $\mathrm{SO}(2,1)$. By Corollary 2.2.7, it is virtually a surface group.

We can finally define what is an "arithmetic" subgroup of G. Informally, it is a lattice build by arithmetic techniques. We want the definition to be as general as possible. In particular the definition should not depend on the presence of compact factors in G.

Definition 2.2.10. Let $\mathbb{H}$ be a semisimple $\mathbb{R}$-algebraic group. A subgroup $\Lambda < \mathbb{H}(\mathbb{R})$ is *arithmetic* if there exist a number field F, a semisimple algebraic group $\mathbb{G}$ over F and a surjective Lie group morphism

$$\prod_{\sigma : F \hookrightarrow \mathbb{R}} \mathbb{G}(\mathbb{R}^\sigma) \times \prod_{\substack{\{\sigma;\overline{\sigma}\} \\ \sigma : F \hookrightarrow \mathbb{C} \\ \sigma(F) \not\subset \mathbb{R}}} \mathbb{G}(\mathbb{C}^\sigma) \xrightarrow{\phi} \mathbb{H}(\mathbb{R})$$

such that $\ker(\phi)$ is compact and Λ is commensurable with $\phi(\mathbb{G}(\mathcal{O}_F))$.

Remark 2.2.11. Thanks to Theorem 2.2.3 and Lemma 2.2.8, arithmetic subgroups of $\mathbb{H}(\mathbb{R})$ are lattices of $\mathbb{H}(\mathbb{R})$.

Examples 2.2.12.

- $\mathrm{SL}(n, \mathbb{Z})$ is an arithmetic subgroup of $\mathrm{SL}(n, \mathbb{R})$
- $\mathrm{SO}(x^2 + y^2 - \sqrt{7}z^2, \mathbb{Z}[\sqrt{7}])$ is an arithmetic subgroup of $\mathrm{SO}(2,1)$, see Example 2.2.9
- More generally, if p is a prime number, $\mathrm{SO}(x_1^2 + \cdots + x_n^2 - \sqrt{p}x_{n+1}^2, \mathbb{Z}[\sqrt{p}])$ is a uniform arithmetic subgroup of $\mathrm{SO}(n,1)$. For distinct primes p, these lattices are not commensurable. Hence there exists infinitely many compact hyperbolic manifold of dimension n without common covering.

Almost all semisimple Lie groups that we will discuss in this thesis are the $\mathbb{R}$-points of an $\mathbb{R}$-algebraic group. However, $\mathrm{PSL}(n, \mathbb{R})$ or the connected component of the identity of $\mathrm{SO}(p,q)$ are not. The definition can be generalized in this cases easily

Margulis gave a criterion to determine when a lattice is arithmetic. It is called the Margulis commensurability criterion.

Theorem 2.2.13 (Margulis [Mar91, Chapter 9 Theorem 1.12])**.** *Let G be a connected semisimple Lie group with no compact factors which is the $\mathbb{R}$-points of an $\mathbb{R}$-algebraic group. Let $\Lambda < G$ be a lattice. Then Λ is arithmetic if and only if*

$$\{g \in G \mid g\Lambda g^{-1} \text{ is commensurable with } \Lambda\}$$

is dense in G.

Margulis Arithmeticity Theorem will be of fundamental importance in this thesis. It allows one to classify lattices in higher rank simple Lie groups. We give instances of this classification in section 2.4.

Theorem 2.2.14 (Margulis [Mar91, Chapter 9 Theorem 1.11]). *Let G be a simple Lie group which is the $\mathbb{R}$-points of a $\mathbb{R}$-algebraic group. Assume that the real rank of G is at least 2. Then any lattice of G is arithmetic.*

On the other hand, lattices of $\mathrm{SL}(2,\mathbb{R})$ are very rarely arithmetic. For instance, there is a continuum of lattices of $\mathrm{SL}(2,\mathbb{R})$ isomorphic to $\pi_1(S_g)$ for a fixed $g \geq 2$ but, up to conjugation, only finitely many are arithmetic.

The connected simple Lie groups of rank 1 are, up to cover, $\mathrm{SO}_0(n,1)$, $\mathrm{SU}(n,1)$, $\mathrm{Sp}(n,1)$ and the exceptional group F_4^{-20}. For the latter two, Gromov and Schoen extended Margulis Arithmeticity Theorem.

Theorem 2.2.15 (Gromov−Schoen [GS92]). *All lattices of $\mathrm{Sp}(n,1)$, $n \geq 2$ and F_4^{-20} are arithmetic.*

However in $\mathrm{SO}(n,1)$, Gromov and Piatetski-Shapiro constructed non-arithmetic lattices for all $n \geq 2$, [GPS87]. To the author's knowledge, non-arithmetic lattices are known in $\mathrm{SU}(2,1)$ and $\mathrm{SU}(3,1)$ [Mos80], [DM93], [DPP16], [DPP21] and [Der20]. It is not known whether or not there exist non-arithmetic lattices in $\mathrm{SU}(n,1)$ for $n \geq 4$.

2.3 Examples

Definition 2.3.1. Let F be a number field and A a central simple algebra over F. An *order* of A is a finitely generated $\mathcal{O}_F$-submodule of A containing 1 which generates A as a vector space and which is a subring of A.

Example 2.3.2. Let $a, b \in \mathcal{O}_F$ be non-zero. Then $\mathcal{O}_F[1,i,j,k]$ is an order of $(a,b)_F$.

Let $\mathcal{O}$ be an order of a quaternion algebra A over a totally real number field F. Suppose that there exists exactly one real place σ of F over which A splits. Then

$$\mathcal{O}^1 := \{x \in \mathcal{O} \mid \mathrm{Nrd}(x) = 1\} < (A \otimes_F \mathbb{R}^\sigma)^1 \simeq \mathrm{SL}(2,\mathbb{R})$$

is an arithmetic subgroup.

Proposition 2.3.3. *The lattice $\mathcal{O}^1$ is uniform if and only if A is a division algebra.*

Proof. Suppose that A is a division algebra. By the multiplicative properties of the reduced norm, every non-zero element of A has norm non-zero. Let $x \in \mathcal{O}^1$ be a unipotent element. Then there exists $l \geq 1$ such that $(x-1)^l = 0$. Hence $0 = \mathrm{Nrd}((x-1)^l) = \mathrm{Nrd}(x-1)^l$ so that $\mathrm{Nrd}(x-1) = 0$ which implies that $x = 1$.

Conversely, if A is not a division algebra then $A \simeq \mathrm{M}_2(F)$. Since A is ramified at all real places of F except one, we must have $F = \mathbb{Q}$. Finally, $\mathcal{O}^1$ is commensurable with $\mathrm{SL}(2,\mathbb{Z})$ which is not uniform. □

Example 2.3.4. Let $a, b \in \mathbb{Z}$ be non-zero and consider the quaternion algebra $(a,b)_\mathbb{Q}$. Let $\{1,i,j,k\}$ be a basis satisfying $i^2 = a$, j^2 and $ij = -ij = k$. We can embed $(a,b)_\mathbb{Q}$ in $\mathrm{M}_2(\overline{\mathbb{Q}})$ as

$$\left\{ \begin{pmatrix} x_0 + \sqrt{a}x_1 & \sqrt{b}x_2 + \sqrt{ab}x_3 \\ \sqrt{b}x_2 - \sqrt{ab}x_3 & x_0 - \sqrt{a}x_1 \end{pmatrix} \middle| \, x_i \in \mathbb{Q} \right\}. \tag{2.1}$$

The subring $\mathbb{Z}[1,i,j,k]$ is an order in $(a,b)_\mathbb{Q}$. Its elements of norm 1 are

$$\left\{ \begin{pmatrix} x_0 + \sqrt{a}x_1 & \sqrt{b}x_2 + \sqrt{ab}x_3 \\ \sqrt{b}x_2 - \sqrt{ab}x_3 & x_0 - \sqrt{a}x_1 \end{pmatrix} \middle| \, x_i \in \mathbb{Z},\ \det = 1 \right\}$$

which is thus an arithmetic lattice of $\mathrm{SL}(2,\mathbb{R})$. It is uniform if and only if $(a,b)_\mathbb{Q} \not\simeq \mathrm{M}_2(\mathbb{Q})$, i.e. the equation $ax^2 + by^2 = 1$ has no solution $(x,y) \in \mathbb{Q}^2$, see Theorem 8.2.7 in [MR03].

Let $n \geq 2$. Recall that there is a well defined norm Nrd on $\mathrm{M}_n(A)$ that is defined as the composition of $\mathrm{M}_n(A) \hookrightarrow \mathrm{M}_{2n}(\overline{\mathbb{Q}})$ with the determinant. We denote by

$$\mathrm{SL}(n, \mathcal{O}) = \{M \in \mathrm{M}_n(\mathcal{O}) \mid \mathrm{Nrd}(M) = 1\}.$$

It is a lattice of $\mathrm{SL}(2n, \mathbb{R})$ which is uniform if and only if $F \neq \mathbb{Q}$.

Definition 2.3.5. Let A be a central simple algebra. An *anti-involution* of A is a map $\partial : A \to A$ such that for all $a, b \in A$

$$\partial(a+b) = \partial(a) + \partial(b),\ \partial(ab) = \partial(b)\partial(a) \text{ and } \partial^2(a) = a.$$

Note that an anti-involution can be non-trivial on the center of A, and thus is not necessarily F-linear. For instance, the conjugation on a quaternion algebra is an anti-involution.

Definition 2.3.6. Given an anti-involution $\partial : A \to A$ of a central simple algebra, we say that a matrix $B \in \mathrm{M}_n(A)$ is ∂-*Hermitian* if $\partial(B)^\top = B$. In this case, we define

$$\mathrm{SU}(B, \partial; A) = \{M \in \mathrm{SL}(n, A) \mid \partial(M)^\top B M = B\}.$$

Further more, if $\mathcal{O}$ is an order of A, define $\mathrm{SU}(B, \partial; \mathcal{O}) = \mathrm{SU}(B, \partial; A) \cap \mathrm{SL}(n, \mathcal{O})$.

Examples 2.3.7.

- Let A be a quaternion algebra over a totally real number field F that splits at exactly one real place of F. Let $\mathcal{O}$ be an order of A. Pick $B \in \mathrm{GL}(n, A)$, $n \geq 2$, a $^-$-Hermitian matrix. Then $\mathrm{SU}(B, ^-; \mathcal{O})$ is an arithmetic subgroup of $\mathrm{Sp}(2n, \mathbb{R})$. It is uniform if and only if $F \neq \mathbb{Q}$.
- Let $d \in \mathcal{O}_F$ not a square which is positive at exactly one real place of F. Let $\sigma \in \mathrm{Gal}(F(\sqrt{d})/F)$ be the non-trivial element. Pick $B \in \mathrm{GL}(n, F)$ a σ-Hermitian matrix for $n \geq 3$. Then $\mathrm{SU}(B, \sigma; \mathcal{O}_F[\sqrt{d}])$ is an arithmetic subgroup of $\mathrm{SL}(n, \mathbb{R})$. It is uniform if and only if $F \neq \mathbb{Q}$.
- Suppose that A splits at the place where d is positive. Denote by

$$\begin{aligned} ^- \otimes \sigma : A \otimes_F F(\sqrt{d}) &\to A \otimes_F F(\sqrt{d}) \\ x \otimes \lambda &\to \overline{x} \otimes \sigma(\lambda). \end{aligned}$$

  It is an anti-involution of $A \otimes_F F(\sqrt{d})$. Pick $B \in \mathrm{GL}(n, A \otimes_F F(\sqrt{d}))$, $n \geq 2$, a $^- \otimes \sigma$-Hermitian matrix. Then $\mathrm{SU}(B, ^- \otimes \sigma; \mathcal{O} \otimes_{\mathcal{O}_F} \mathcal{O}_F[\sqrt{d}])$ is an arithmetic subgroup of $\mathrm{SL}(2n, \mathbb{R})$. It is uniform if and only if $F \neq \mathbb{Q}$.

One can also construct lattices from orders of central simple algebras of higher degree.

Example 2.3.8. Recall that we defined in Example 1.3.3 a division algebra D of degree 3 over $\mathbb{Q}$ as $D = \mathbb{Q}(\alpha) \oplus \mathbb{Q}(\alpha)v \oplus \mathbb{Q}(\alpha)v^2$ for $\alpha = 2\cos(\frac{2\pi}{7})$. Let $\mathcal{O}$ be an order of D. Then $\mathcal{O}^1$ is a uniform lattice of $\mathrm{SL}(3, \mathbb{R})$. Concretely, $\mathcal{O}^1$ is commensurable with

$$\left\{ \begin{pmatrix} x & y & z \\ 5\sigma(z) & \sigma(x) & \sigma(y) \\ 5\sigma^2(y) & 5\sigma^2(z) & \sigma^2(x) \end{pmatrix} \middle| \ x, y, z \in \mathbb{Z}[\alpha],\ \det = 1 \right\}$$

where σ is the automorphism of $\mathbb{Q}(\alpha)$ defined by $\sigma(\alpha) = 2\cos(\frac{4\pi}{7})$.

2.4 Classification

In this section, we present the classification of lattices for some simple real Lie groups. Thanks to the following proposition, it suffices to classify algebraic groups over totally real number fields F that are isomorphic to G over some real place of F. For a number field F, denote by V_F the set of its embeddings in $\mathbb{C}$ where we identify a complex embedding and its conjugate.

Proposition 2.4.1 ([Mor15, Corollary 5.5.16]). *Let G be a non-compact simple Lie group and let Λ be an arithmetic subgroup of G. Then there exists a number field F and an F-algebraic group $\mathbb{H}$ such that*

- *there is $v \in V_F$ and an isogeny*
$$\phi : \mathbb{H}(F_v)_0 \to \mathrm{Ad}(G_0)$$
such that $\phi(\mathbb{H}(\mathcal{O}_F) \cap \mathbb{H}(F_v)_0)$ is commensurable with $\mathrm{Ad}(\Lambda \cap G_0)$ and
- *for all $w \in V_F$ not v, $\mathbb{H}(F_w)$ is compact with the euclidean topology.*

The classification of forms of classical algebraic groups is stated in [Mor15] §18.5. We include it here for completeness.

Theorem 2.4.2 ([Mor15, §18.5]). *Let F be a number field which admits a real embedding. Fix a real embedding $\sigma : F \hookrightarrow \mathbb{R}$.*

- *The $\mathbb{R}/F$-forms of SL_n are either of type $\mathrm{SL}(m, D)$ for D a division algebra over F of degree d such that $n = md$ and D splits over σ, or of type $\mathrm{SU}(B, \partial; D)$ for a division algebra D over a quadratic extension L of F such that L is real over σ and $D \otimes_F \mathbb{R}^\sigma \simeq \mathrm{M}_n(\mathbb{R}) \oplus \mathrm{M}_n(\mathbb{R})$, ∂ an anti-involution of D that induces a non-trivial automorphism on L and B a non-degenerate ∂-Hermitian form of rank n over D.*
- *The $\mathbb{R}/F$-forms of $\mathrm{SO}(p, q)$ are of type $\mathrm{SO}(B, F)$ for B a quadratic form of signature over F such that $\sigma(B)$ has signature (p, q).*
- *The $\mathbb{R}/F$-forms of Sp_{2n} are of type $\mathrm{SU}(B, {}^-; A)$ for A a quaternion algebra over F that splits over σ and B a non-degenerate ${}^-$-Hermitian form of rank n over A.*

We will need a more precise version of this classification. We provide proofs when needed.

Proposition 2.4.3. *Let $p \neq 2$ be prime. Non-uniform lattices of $\mathrm{SL}(p, \mathbb{R})$ are widely commensurable with $\mathrm{SL}(p, \mathbb{Z})$ or with $\mathrm{SU}(I_p, \sigma; \mathbb{Z}[\sqrt{d}])$ for $d \in \mathbb{N}$ not a square and $\sigma \in \mathrm{Gal}(\mathbb{Q}(\sqrt{d})/\mathbb{Q})$ non-trivial.*

To prove Proposition 2.4.3, we need the classification of σ-Hermitian forms.

Proposition 2.4.4 (See Lewis [Lew82, §4] or Milnor [Mil69, Example 5 of §1]). *Let F be a number field and $d \in \mathcal{O}_F$ be not a square. Let $\sigma \in \mathrm{Gal}(F(\sqrt{d})/F)$ be the non-trivial element. Then σ-Hermitian forms on $F(\sqrt{d})$ are classified by their rank, their discriminant and their signatures at each real place v of F satisfying $F(\sqrt{d}) \otimes_F F_v \simeq \mathbb{C}$.*

Proof of Proposition 2.4.3. Let Λ be a non-uniform lattice of $\mathrm{SL}(p, \mathbb{R})$. By Margulis' Arithmeticity Theorem (Theorem 2.2.14) Λ is arithmetic. Since Λ is non-uniform, Proposition 2.2.7 implies that it is commensurable with the integer points a $\mathbb{R}/\mathbb{Q}$-form of SL_p. From the classification of $\mathbb{R}/\mathbb{Q}$-forms of SL_p (Theorem 2.4.2) Λ can only be of two kinds.

Firstly, Λ can be widely commensurable with $\mathrm{SL}(m, \mathcal{O})$ for $\mathcal{O}$ an order of a division algebra D of degree r over $\mathbb{Q}$ satisfying $D \otimes_{\mathbb{Q}} \mathbb{R} \simeq \mathrm{M}_r(\mathbb{R})$. Here we must have $rm = p$. If $(r, m) = (1, p)$ then Λ is widely commensurable with $\mathrm{SL}(p, \mathbb{Z})$. If $(r, m) = (p, 1)$ then Λ is widely commensurable with $\mathrm{SL}(1, \mathcal{O})$. However, the latter is a uniform lattice (see §18.5 in [Mor15]).

Secondly they can be widely commensurable with $\mathrm{SU}(B, \partial; \mathcal{O})$ for $\mathcal{O}$ an order of a division algebra D of degree r over a real quadratic extension L of $\mathbb{Q}$, satisfying $D \otimes_{\mathbb{Q}} L \simeq \mathrm{M}_r(\mathbb{R}) \oplus \mathrm{M}_r(\mathbb{R})$, ∂ an anti-involution of D whose restriction to L is the non-trivial element $\sigma \in \mathrm{Gal}(L/\mathbb{Q})$ and $B \in \mathrm{GL}_m(D)$ a ∂-Hermitian matrix. Once again we must have $mr = p$. If $(r, m) = (1, p)$ then Λ is widely commensurable with $\mathrm{SU}(B, \sigma; \mathbb{Z}[\sqrt{d}])$ where $L = \mathbb{Q}(\sqrt{d})$. If $(r, m) = (p, 1)$ then Λ is widely commensurable with $\mathrm{SU}(B, \partial; \mathcal{O})$, but this is a uniform lattice (see §18.5 in [Mor15]).

It remains to show that all subgroups of the form $\mathrm{SU}(B, \sigma; \mathbb{Z}[\sqrt{d}])$ widely commensurable with $\mathrm{SU}(I_n, \sigma; \mathbb{Z}[\sqrt{d}])$. Since $d > 0$, Proposition 2.4.4 implies that, up to scalar multiplication, there is only one non-degenerate σ-Hermitian form on $\mathbb{Q}(\sqrt{d})^p$. Hence $\mathrm{SU}(B, \sigma; \mathbb{Z}[\sqrt{d}])$ is conjugate to $\mathrm{SU}(I_n, \sigma; \mathbb{Z}[\sqrt{d}])$. □

To state the precise version of the classification of lattices of $\mathrm{SO}(k+1,k)$ or of $\mathrm{Sp}(2n,\mathbb{R})$, we need the following lemma. It follows from the fact that F is dense in the product of its archimedean completions. We provide a proof of the statement we need for completeness.

Lemma 2.4.5. *Let F be a totally real number field. Fix a subset $V \subset V_F$. There exists $\lambda \in F$ such that for all $\iota \in V$, $\iota(\lambda) < 0$ and for all $\iota \notin V$ $\iota(\lambda) > 0$.*

Proof. First suppose that $V = \{\sigma\}$. We use σ to identify F with a subfield of $\mathbb{R}$. There is an element $\alpha_0 \in \overline{\mathbb{Q}}$ such that $F = \mathbb{Q}(\alpha_0)$. Denote $\alpha_1, .., \alpha_m$ its conjugates. Up to relabeling, we can suppose that

$$\alpha_0 < \alpha_1 < ... < \alpha_m.$$

Let $r \in \mathbb{Q}$ such that $\alpha_0 < r < \alpha_1$. Then we let $\lambda = \alpha_0 - r$. Then $\lambda < 0$ and the conjugates of λ are $\alpha_i - r$ which are all positive.

Let V be any subset of V_F. For all $\sigma \in V$ let λ_σ be an element of F such that $\sigma(\lambda_\sigma) < 0$ and for all $\iota \notin V_F \setminus \{\sigma\}$, $\iota(\lambda_\sigma) > 0$. Let

$$\lambda = \prod_{\sigma \in V} \lambda_\sigma.$$

Then for all $\iota \in V$ we have $\iota(\lambda) < 0$ and for all $\iota \notin V$ $\iota(\lambda) > 0$. □

Proposition 2.4.6. *Let $n = 2k+1 \geq 5$ be odd. Lattices of $\mathrm{SO}(k+1,k)$ are widely commensurable with $\mathrm{SO}(B, \mathcal{O}_F)$ for $\mathcal{O}_F$ the ring of integers of a totally real number field F and $B \in \mathrm{SL}(n,F)$ a symmetric matrix satisfying:*

- *the signature of B at one real place of F is $(k+1,k)$ if k is even and $(k,k+1)$ if k is odd, and*
- *B is positive definite at all other real places of F.*

Furthermore, they are non-uniform if and only if $F = \mathbb{Q}$.

Proof. Margulis' Arithmeticity Theorem (Theorem 2.2.14) implies that all lattices of $\mathrm{SO}(k+1,k)$ are arithmetic. From the classification of $\mathbb{R}/F$-forms of $\mathrm{SO}(k+1,k)$ (Theorem 2.4.2) lattices of $\mathrm{SO}(k+1,k)$ are widely commensurable with $\mathrm{SO}(B,\mathcal{O}_F)$ for $\mathcal{O}_F$ the ring of integers of a totally real number field F and $B \in \mathrm{GL}(n,F)$ a symmetric matrix with signature $(k+1,k)$ at one real place of F and such that B is positive or negative definite over all other real places of F.

Let $V \subset V_F$ be the set of real places of F where B is negative definite. By Lemma 2.4.5, there exists $\lambda \in F$ such that for all $\iota \in V$ $\iota(\lambda) < 0$ and for all $\iota \notin V$ $\iota(\lambda) > 0$. Then λB is positive definite for all real places except one where it has signature $(k+1,k)$ and $\mathrm{SO}(B,\mathcal{O}_F) = \mathrm{SO}(\lambda B, \mathcal{O}_F)$. Up to replacing B with λB we can assume that V is empty.

Up to replacing B with $\det(B)B$, we can assume that $\det(B)$ is a square. Then up to congruence, we can assume B has determinant 1.

We now prove the last statement. If $F = \mathbb{Q}$, Meyer's Theorem (Corollary 2 in §3 of Chapter 4 in [Ser73]) implies that B is isotropic over $\mathbb{Q}$ since $n \geq 5$. Thus $\mathrm{SO}(B,\mathbb{Z})$ is non-uniform, see §18.5 in [Mor15]. If $F \neq \mathbb{Q}$, Corollary 2.2.7 implies that $\mathrm{SO}(B,\mathcal{O}_F)$ is uniform. □

Proposition 2.4.7. *Let $n \geq 2$. Lattices of $\mathrm{Sp}(2n,\mathbb{R})$ are widely commensurable with $\mathrm{SU}(I_n, {}^-; \mathcal{O})$ for $\mathcal{O}$ an order of a quaternion algebra A over a totally real number field F such that A splits at exactly one real place of F. Furthermore, they are non-uniform if and only $F = \mathbb{Q}$.*

Note that

$$\begin{aligned}\mathrm{SU}(I_n, {}^-; \mathrm{M}_2(\mathbb{Z})) &= \{M \in \mathrm{SL}_n(\mathrm{M}_2(\mathbb{Z})) \mid \overline{M}^\top M = I_n\} \\ &= \{M \in \mathrm{SL}_{2n}(\mathbb{Z}) \mid K_n^{-1} M^\top K_n M = I_n\} = \mathrm{Sp}(K_n, \mathbb{Z})\end{aligned}$$

where K_n is the block-diagonal anti-symmetric matrix with blocks equal to $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Proof. Margulis' Arithmeticity Theorem (Theorem 2.2.14) implies that all lattices of $\mathrm{Sp}(2n,\mathbb{R})$ are arithmetic. From the classification of $\mathbb{R}/F$-forms of Sp_{2n} (Theorem 2.4.2) lattices of $\mathrm{Sp}(2n,\mathbb{R})$ are widely commensurable with $\mathrm{SU}(B,^{-};\mathcal{O})$, where $\mathcal{O}$ is an order of a quaternion algebra A over a totally real number field F, A ramifies at all $\iota \in V_F \setminus \{\sigma\}$ and B is a non-degenerate $^{-}$-Hermitian matrix over A such that for all $\iota \in V_F \setminus \{\sigma\}$ $\iota(B) \in \mathrm{GL}(\mathcal{H})$ has signature $(n,0)$ or $(0,n)$. Denote by

$$V_F^+ = \{\iota \in V_F \mid \iota(B) \text{ has signature } (n,0)\}.$$

Let $\lambda \in F$ such that $\iota(\lambda) > 0$ for all $\iota \in V_F^+$ and $\iota(\lambda) < 0$ for all $\iota \notin V_F^+$. It exists by Lemma 2.4.5. Then the $^{-}$-Hermitian matrix λI_n has the same signature as B for all real embeddings of F, except maybe σ. As explained in [Lew82] §5, non-degenerate $^{-}$-Hermitian forms on A are classified by the set of signatures at all real embeddings of F except σ. Hence $\mathrm{SU}(B,^{-};\mathcal{O})$ is conjugate to $\mathrm{SU}(\lambda I_n,^{-};\mathcal{O}) = \mathrm{SU}(I_n,^{-};\mathcal{O})$.

If $F = \mathbb{Q}$ then $\mathrm{SU}(I_n,^{-};\mathcal{O})$ is non-uniform since $^{-}$-Hermitian forms over quaternion algebras are always isotropic, see Lemma 6.2.3 in [Knu91]. If $F \neq \mathbb{Q}$, Corollary 2.2.7 implies that $\mathrm{SU}(I_n,^{-};\mathcal{O})$ is uniform. □

In order to give the classification of arithmetic subgroups of $\mathbf{G}_2(\mathbb{R})$, we introduce some notations.

Definition 2.4.8. Let R be a ring and let $a, b \in \mathrm{R}$ be non-zero. Denote by $\times : \mathrm{R}^7 \times \mathrm{R}^7 \to \mathrm{R}^7$

$$\begin{pmatrix} x_1\\x_2\\x_3\\x_4\\x_5\\x_6\\x_7 \end{pmatrix}, \begin{pmatrix} y_1\\y_2\\y_3\\y_4\\y_5\\y_6\\y_7 \end{pmatrix} \mapsto \begin{pmatrix} 3a(x_7y_4 - x_4y_7) - 2(x_2y_3 - x_3y_2) - 2a(x_6y_5 - x_5y_6)\\ 12b(x_3y_1 - x_1y_3) + 12ab(x_7y_5 - x_5y_7) - 3a(x_6y_4 - x_4y_6)\\ 30(x_2y_1 - x_1y_2) + 30a(x_7y_6 - x_6y_7) - 3a(x_5y_4 - x_4y_5)\\ 120b(x_1y_7 - x_7y_1) + 20(x_2y_6 - x_6y_2) - 8b(x_3y_5 - x_5y_3)\\ 30(x_1y_6 - x_6y_1) - 30(x_7y_2 - x_2y_7) - 3(x_3y_4 - x_4y_3)\\ 12b(x_3y_7 - x_7y_3) + 12b(x_1y_5 - x_5y_1) - 3(x_2y_4 - x_4y_2)\\ 3(x_1y_4 - x_4y_1) - 2(x_2y_5 - x_5y_2) - 2(x_6y_3 - x_3y_6) \end{pmatrix}.$$

Definition 2.4.9. Let F be a number field and $a, b \in F^\times$. Define

$$\begin{aligned} \mathbf{G}_2^{a,b} &: F\text{-algebra} \to \text{group}\\ &\mathrm{R} \to \{M \in \mathrm{SO}(J_7^{a,b}, \mathrm{R}) | M(x \times y) = Mx \times My,\ \forall x, y \in \mathrm{R}^7\} \end{aligned}$$

where $J_7^{a,b} = \mathrm{Diag}(-1440b, 240, -96b, -36a, 96ab, -240a, 1440ab)$.

The F-algebraic group $\mathbf{G}_2^{a,b}$ is simple of type G_2. Indeed if

$$S = \begin{pmatrix} \sqrt{b} & & & & & & \sqrt{ab}\\ & 1 & & & & \sqrt{a} & \\ & & \sqrt{b} & & \sqrt{ab} & & \\ & & & \sqrt{a} & & & \\ & & -\sqrt{b} & & \sqrt{ab} & & \\ & -1 & & & & \sqrt{a} & \\ -\sqrt{b} & & & & & & \sqrt{ab} \end{pmatrix}^{-1},$$

then $S^{-1}\mathbf{G}_2^{a,b}(\mathbb{C})S = \mathbf{G}_2(\mathbb{C})$. For any $a, b \in \mathbb{R}^\times$ which are both negative $\mathbf{G}_2^{a,b}(\mathbb{R})$ is isomorphic to the compact Lie group of type G_2. If either a or b is positive, $S^{-1}\mathbf{G}_2^{a,b}(\mathbb{R})S = \mathbf{G}_2(\mathbb{R})$, recall Definition 1.5.9.

Suppose that F is a totally real number field and fix an embedding $\sigma : F \hookrightarrow \mathbb{R}$. Let $a, b \in F^\times$ which are both negative at all real embeddings of F except σ. Denote by $\Lambda_F = S^{-1}\mathbf{G}_2^{a,b}(\mathcal{O}_F)S$. It is a lattice of $\mathbf{G}_2(\mathbb{R})$.

Lemma 2.4.10. *Up to wide commensurability, Λ_F does not depend on the choice of a and b.*

Proof. Wide commensurability classes of lattices of $\mathbf{G}_2(\mathbb{R})$ are in bijection with isomorphism classes of $\overline{\mathbb{Q}}/F$-forms of $\mathbf{G}_2$ that are compact over all real embeddings of F except σ. This is in bijection with isomorphism classes of octonion algebras over F that do not split at any real embedding of F except σ. As explained in §1.10 of [SV00], there is a unique such octonion algebra over F. Hence, different choice of a and b yields isomorphic $\overline{\mathbb{Q}}/F$-forms of $\mathbf{G}_2$ and widely commensurable arithmetic subgroups. □

Proposition 2.4.11. *Let Λ be lattice in $\mathbf{G}_2(\mathbb{R})$. Then Λ is widely commensurable with Λ_F for some totally real number field F.*

Proof. By Margulis' Arithmeticity Theorem, Theorem 2.2.14, Λ is arithmetic. By Proposition 2.4.1, there exists a semisimple F-algebraic group $\mathbb{H}$ such that $\mathbb{H}(F_v)$ is compact for all archimedean places v except one, denoted v_0, where we have an isogeny

$$\phi : \mathbb{H}(F_{v_0}) \to \mathbf{G}_2(\mathbb{R})$$

satisfying that $\phi(\mathbb{H}(\mathcal{O}_F))$ is commensurable with Λ. The group $\mathbb{H}(F_{v_0})$ cannot be the universal cover of $\mathbf{G}_2(\mathbb{R})$ since the latter is not algebraic. Hence ϕ is an isomorphism and $\mathbb{H}$ is an $\mathbb{R}/F$-form of $\mathbf{G}_2$. Those are classified by $\mathbb{R}/F$-forms of the split octonion algebra over $\mathbb{R}$ which are determined by their norm, see Theorem 1.7.1 in [SV00]. The norm associated to $\mathbb{H}$ is positive definite over all real places of F except one. So up to scalar multiplication, it is equivalent to $J_7^{a,b}$ for some $a, b \in F^\times$ which are both negative at all real places of F except one. Thus the octonion algebras associated to $\mathbb{H}$ and $\mathbf{G}_2^{a,b}$ are isomorphic which implies that $\mathbb{H} \simeq \mathbf{G}_2^{a,b}$. Finally Λ is widely commensurable with Λ_F. □

2.5 Strong Approximation

In this section we discuss the Strong Approximation Theorem. It will appear later in the proof of Theorem B and Theorem F. Since its statement requires new definitions, we start by describing for $\mathrm{SL}(n, \mathbb{Z})$.

Let $n \geq 2$ and p be a prime. One can show that the reduction

$$r_p : \mathrm{SL}(n, \mathbb{Z}) \to \mathrm{SL}(n, \mathbb{F}_p)$$

is surjective by considering a suitable set of generators of $\mathrm{SL}(n, \mathbb{F}_p)$. Thus given a subgroup $\Gamma < \mathrm{SL}(n, \mathbb{Z})$, we can ask for which primes p does $r_p(\Gamma) = \mathrm{SL}(n, \mathbb{F}_p)$. This question arises naturally when one tries to measure the density of Γ in the profinite completion of $\mathrm{SL}(n, \mathbb{Z})$.

Example 2.5.1. For $k \geq 1$ consider the subgroup of $\mathrm{SL}(2, \mathbb{Z})$

$$\Gamma_k = \left\langle \begin{pmatrix} 1 & 2^k \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2^k & 1 \end{pmatrix} \right\rangle.$$

For $k \geq 2$, Γ_k is Zariski-dense but infinite index in $\mathrm{SL}(2, \mathbb{Z})$. Via r_p, Γ_k surjects $\mathrm{SL}(2, \mathbb{F}_p)$ for any prime $p \neq 2$. Indeed, pick $x \in \mathbb{F}_p$. By the Chinese Remainder Theorem there exists $m \in \mathbb{Z}$ such that $m \equiv x \mod p$ and $m \equiv 0 \mod 2^l$. Then

$$r_p\left(\begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}\right) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}.$$

By the same argument for the transpose, Γ_k surjects $\mathrm{SL}(2, \mathbb{F}_p)$ since

$$\mathrm{SL}(2, \mathbb{Z}) = \left\langle \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \middle| \, x \in \mathbb{F}_p \right\rangle.$$

These examples are instances of the Strong Approximation Theorem which we stated in the Introduction.

Theorem 0.1 (Matthews−Vaserstein−Weisfeiler [MVW84])**.** *Let $\Gamma < \mathrm{SL}(n, \mathbb{Z})$ be a Zariski-dense subgroup. Then for all but finitely many primes p, Γ surjects onto $\mathrm{SL}(n, \mathbb{F}_p)$.*

Theorem 0.1 holds more generally for arithmetic subgroups commensurable with the integer points of connected almost simple and simply connected $\mathbb{Q}$-algebraic groups, see [MVW84].

Definition 2.5.2. Let K be a subfield of $\mathbb{C}$ and $\mathbb{G}$ be a linear algebraic group $\mathbb{G}$ over K. We say that

- $\mathbb{G}$ is *almost simple* if $\mathbb{G}(\mathbb{C})/Z(\mathbb{G}(\mathbb{C})$ is simple,
- $\mathbb{G}$ is *connected* if $\mathbb{G}(\mathbb{C})$ is connected with the euclidean topology and
- $\mathbb{G}$ is *simply connected* if $\mathbb{G}(\mathbb{C})$ is simply connected with the euclidean topology.

For instance, the $\mathbb{Q}$-algebraic group Sp_{2n} is connected, almost simple and simply connected. However, for a quadratic form of rank $n \geq 3$ over $\mathbb{Q}$, the $\mathbb{Q}$-algebraic group $\mathrm{SO}(q)$ is connected, almost simple (except if $n = 4$) but not simply connected. Note that if $\mathbb{G}$ has any of the above properties, so does any K-form of $\mathbb{G}$.

Weisfeiler extended Theorem 0.1 to other number fields. Note that the naive analogue of Theorem 0.1 does not hold over number fields, as shown by the following example.

Example 2.5.3. The ideal (3) is prime in $\mathbb{Z}[\sqrt{2}]$ and has for residue field $\mathbb{F}_9$. Consider the reduction $r_{(3)} : \mathrm{SL}(n, \mathbb{Z}[\sqrt{2}]) \to \mathrm{SL}(n, \mathbb{F}_9)$. Then $r_{(3)}(\mathrm{SL}(n, \mathbb{Z})) = \mathrm{SL}(n, \mathbb{F}_3)$ yet $\mathrm{SL}(n, \mathbb{Z})$ is Zariski-dense in $\mathrm{SL}(n, \mathbb{Z}[\sqrt{2}])$.

Weisfeiler showed that the right replacement for $\mathbb{Z}$ is not the ring of integers of F but the ring generated by the traces of Γ. To state Weisfeiler's result, we need the following definition.

Definition 2.5.4. Let R be a ring. A (*linear*) *group scheme* over R is a functor

$$\mathbb{G} : \text{R-algebra} \to \text{group}$$

such that there exists a R-algebra A which is finitely generated and such that

$$\mathbb{G}(\cdot) \simeq \mathrm{Hom}_{\text{R-alg}}(\mathrm{A}, \cdot)$$

as functors to sets.

For example, the functor

$$\mathrm{SL}_n : \mathbb{Z}\text{-algebra} \to \text{group}, \ \mathrm{R} \to \{M \in \mathrm{M}_n(\mathrm{R}) \mid \det(M) = 1\}$$

is a linear group scheme with $\mathrm{A} = \mathbb{Z}[x_{11}, x_{12}, \ldots, x_{nn}]/(\det - 1)$.

If R is a ring and $a \in \mathrm{R} \setminus \{0\}$ is a non-nilpotent element, then we denote by R_a the localization of R at the multiplicative set $\{a^n | n \in \mathbb{N}\}$. The ideals in R_a are in bijective correspondence with ideals of R that do not contain any power of a.

Theorem 2.5.5 (Strong Approximation, Weisfeiler [Wei84])**.** *Let $\mathbb{G}$ be a connected $\overline{\mathbb{Q}}$-algebraic group which is almost simple and simply connected. Let Γ be a finitely generated Zariski-dense subgroup of $\mathbb{G}(\overline{\mathbb{Q}})$. Denote by*

$$\mathrm{R} = \mathbb{Z}[\mathrm{Tr}(\mathrm{Ad}(\Gamma))].$$

Then there exists $a \in \mathrm{R}$, a finite index subgroup $\Gamma' < \Gamma$ and a structure of a group scheme $\mathbb{G}_0$ over R_a that become isomorphic to $\mathbb{G}$ over $\overline{\mathbb{Q}}$ such that $\Gamma' < \mathbb{G}_0(\mathrm{R}_a)$ and Γ' is dense in

$$\varprojlim_{|\mathrm{R}_a/I|<\infty} \mathbb{G}_0\big(\mathrm{R}_a/I\big),$$

where the projective limit is over ideals of R_a.

Remark 2.5.6. In the context of Theorem 2.5.5, let I be an ideal of R_a such that R_a/I is finite and denote by

$$\pi : \mathbb{G}_0(\mathrm{R}_a) \to \mathbb{G}_0\big(\mathrm{R}_a/I\big)$$

the canonical map. The theorem implies that $\pi(\Gamma') = \mathbb{G}_0(\mathrm{R}_a/I)$.

Note that Theorem 2.5.5 requires $\mathbb{G}$ to be simply connected. For groups that are not simply connected, such the orthogonal group, Nori proved that one still has control on the reduction of Zariski-dense subgroup. For a subgroup G of $\mathrm{GL}(n, \mathbb{F}_p)$ denote by G^+ the subgroup generated by the elements of G with order p.

Theorem 2.5.7 (Nori [Nor87])**.** *Let $\mathbb{G}$ be a linear $\mathbb{Z}$-group scheme such that $\mathbb{G}(\mathbb{C})$ is connected with the euclidean topology. For a prime p, denote by $r_p : \mathbb{G}(\mathbb{Z}) \to \mathbb{G}(\mathbb{F}_p)$ the reduction map. Let $\Gamma < \mathbb{G}(\mathbb{Z})$ be a Zariski-dense subgroup. Then for all but finitely many primes p*

$$\mathbb{G}(\mathbb{F}_p)^+ < r_p(\Gamma) < \mathbb{G}(\mathbb{F}_p).$$

Part II

Hitchin representations

Chapter 3

Fuchsian representations in lattices

In this chapter, we establish Theorem A. Recall that a Fuchsian representation is a representation of the form $\tau_n \circ \rho$ for $\rho : \pi_1(S_g) \to \mathrm{PSL}(2,\mathbb{R})$ discrete and faithful.

Theorem A. *For every lattice Λ of G listed in Table 1, there exists $g \geq 2$ such that Λ contains the image of a Fuchsian representation of $\pi_1(S_g)$. Furthermore, up to wide commensurability, these are the only lattices of G that contain the image of a Fuchsian representation.*

G	n or k	Λ
$\mathrm{SO}(k+1,k)$	$k \geq 2$, $k \equiv 1,2[4]$	Every lattice not w.c. with $\mathrm{SO}(q_0,\mathbb{Z})$
$\mathrm{SO}(k+1,k)$	$k \geq 3$, $k \equiv 0,3[4]$	$\mathrm{SO}(q_F,\mathcal{O}_F)$
$\mathbf{G}_2(\mathbb{R})$		Every lattice
$\mathrm{Sp}(2n,\mathbb{R})$	$n \geq 2$	Every lattice not w.c. with $\mathrm{Sp}(2n,\mathbb{Z})$
$\mathrm{SL}(2k+1,\mathbb{R})$	$k \geq 1$	$\mathrm{SL}(2k+1,\mathbb{Z})$ or $\mathrm{SU}(I_{2k+1},\sigma;\mathcal{O}_F[\sqrt{d}])$
$\mathrm{SL}(2n,\mathbb{R})$	$n \geq 2$	$\mathrm{SL}(n,\mathcal{O})$ or $\mathrm{SU}(I_n,{}^{-}\otimes\sigma;\mathcal{O}\otimes\mathcal{O}_F[\sqrt{d}])$

Table 3.1: Here F is any totally real number field, $\mathcal{O}_F$ its ring of integers and d any element of $\mathcal{O}_F$ which is not a square and is positive at exactly one real place of F, say ι. Moreover $\mathcal{O}$ is any order of any quaternion division algebra over F which splits exactly at ι. If $k \equiv 0,3[4]$, let q_F be a non-degenerate quadratic form of rank $2k+1$ over F that has trivial discriminant, trivial Hasse invariant at each finite place of F, which is positive definite at all real places of F except one where it has signature equal to $(k+1,k)$ if $k \equiv 0[4]$ or to $(k,k+1)$ if $k \equiv 3[4]$. If $k \equiv 1,2[4]$, let q_0 be a non-degenerate quadratic form of rank $2k+1$ over $\mathbb{Q}$ that has trivial discriminant, trivial Hasse invariant at each odd prime, non-trivial Hasse invariant at 2, signature equal to $(k,k+1)$ if $k \equiv 1[4]$ or to $(k+1,k)$ if $k \equiv 2[4]$. See Section 2.3 for definitions.

The proof is essentially contained in [Aud22] and [Aud23]. Using Proposition, 2.2.6, the problem reduces to classifying F-forms of SL_n, Sp_{2n}, SO or $\mathbf{G}_2$ that contain the image of an F-form of SL_2 under τ_n, for all totally real number fields F. With the help of Galois cohomology, we start by reformulating the problem in terms of *τ_n-compatible* cocycles. We classify these cocycles. We then determine, case by case, the associated F-form. This requires to compute the arithmetic invariants of the cocycles, such as the Hasse invariant for cocycles in the orthogonal group.

3.1 Compatible cocycles

Let F be a totally real number field. Any F-form of SL_2 is of the isomorphic to

$$\mathrm{R} \to (A \otimes_F \mathrm{R})^1$$

for A a quaternion algebra over F, see Proposition 1.6.11. Let $a, b \in F$ such that $A \simeq (a,b)_F$. Pick square roots of a and b in $\overline{\mathbb{Q}}$. We can embed A in $\mathrm{M}_2(\overline{\mathbb{Q}})$ with the following isomorphism

$$A \simeq \left\{ \begin{pmatrix} x_0 + \sqrt{a}x_1 & \sqrt{b}x_2 + \sqrt{ab}x_3 \\ \sqrt{b}x_2 - \sqrt{ab}x_3 & x_0 - \sqrt{a}x_1 \end{pmatrix} \middle| \; x_i \in F \right\}.$$

Since $\mathrm{Aut}(\mathrm{SL}_2(\overline{\mathbb{Q}})) = \mathrm{PGL}_2(\overline{\mathbb{Q}})$, F-forms of SL_2 are classified by elements of the first Galois cohomology set $\mathrm{H}^1(\mathrm{Gal}(\overline{\mathbb{Q}}/F), \mathrm{PGL}_2(\overline{\mathbb{Q}}))$ (Proposition 1.6.9). Explicit computation shows that the 1-cocycle associated to A^1 is

$$T^{a,b} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{PGL}(2, \overline{\mathbb{Q}})$$
$$\sigma \mapsto \begin{cases} I_2 & \text{if } \sigma(\sqrt{a}) = \sqrt{a} \text{ and } \sigma(\sqrt{b}) = \sqrt{b} \\ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & \text{if } \sigma(\sqrt{a}) = \sqrt{a} \text{ and } \sigma(\sqrt{b}) = -\sqrt{b} \\ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & \text{if } \sigma(\sqrt{a}) = -\sqrt{a} \text{ and } \sigma(\sqrt{b}) = \sqrt{b} \\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & \text{if } \sigma(\sqrt{a}) = -\sqrt{a} \text{ and } \sigma(\sqrt{b}) = -\sqrt{b}. \end{cases}$$

For simplicity we will often write T_σ or $T^{a,b}_\sigma$ instead of $T^{a,b}(\sigma)$. Note that we identify $\mathrm{PGL}(2, \overline{\mathbb{Q}})$ with $\mathrm{Aut}(\mathrm{SL}(2, \overline{\mathbb{Q}}))$ via the map $M \mapsto (X \mapsto MXM^{-1})$.

Hence we have the following.

Lemma 3.1.1. *For every 1-cocycle $\xi : \mathrm{Gal}(\overline{\mathbb{Q}}/F) \to \mathrm{PGL}(2, \overline{\mathbb{Q}})$, there exist $a, b \in F$ such that ξ is equivalent to $T^{a,b}$, i.e. there exists $P \in \mathrm{SL}(2, \overline{\mathbb{Q}})$ such that $\xi(\sigma) = P^{-1}T^{a,b}_\sigma \sigma(P)$ for every $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/F)$.*

Let $\xi : \mathrm{Gal}(\overline{\mathbb{Q}}/F) \to \mathrm{Aut}(\mathrm{SL}(2, \overline{\mathbb{Q}}))$ be a 1-cocycle.

Definition 3.1.2. Let $n \geq 3$ and let $\mathbb{G}$ be an F-algebraic subgroup of SL_n. We say that a 1-cocycle $\zeta : \mathrm{Gal}(\overline{\mathbb{Q}}/F) \to \mathrm{Aut}(\mathbb{G}(\overline{\mathbb{Q}}))$ is *τ_n-compatible* with ξ if

$$\tau_n({}_\xi\mathrm{SL}_2(F)) < {}_\zeta\mathbb{G}(F).$$

Equivalently, for any matrix $M \in \mathrm{SL}(2, \overline{\mathbb{Q}})$ that satisfies $\xi(\sigma)(\sigma(M)) = M$ for all $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/F)$, one has $\zeta(\sigma)\sigma(\tau_n(M)) = \tau_n(M)$ for all $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/F)$.

To describe the 1-cocycles τ_n-compatible with ξ, we introduce the following bilinear form.

Definition 3.1.3. For all $n \geq 3$ define

$$J_n = \begin{pmatrix} & & & & (n-1)! \\ & & & \iddots & \\ & & (-1)^{i-1}(n-i)!(i-1)! & & \\ & \iddots & & & \\ (-1)^{n-1}(n-1)! & & & & \end{pmatrix}.$$

It satisfies $\tau_n(M)^\top J_n \tau_n(M) = J_n$ for all $M \in \mathrm{SL}(2, \mathbb{C})$, see for instance McGarraghy [McG05]. Note that when n is odd, it is a symmetric matrix of signature $(k+1, k)$ if $n \equiv 1[4]$ and $(k, k+1)$ if $n \equiv 3[4]$. When n is even, J_n is antisymmetric.

We denote by ω the automorphism of $\mathrm{SL}(n, \overline{\mathbb{Q}})$ defined as $\omega(M) = (M^\top)^{-1}$ where $M^{-\top}$ is the transpose of the inverse of the matrix M. In the following proposition, we give an explicit description of the 1-cocycles τ_n-compatible with ξ. It will be used repeatedly in this chapter.

Proposition 3.1.4. *For any $n \geq 3$, a 1-cocycle $\zeta : \mathrm{Gal}(\overline{\mathbb{Q}}/F) \to \mathrm{Aut}(\mathrm{SL}(n, \overline{\mathbb{Q}}))$ is τ_n-compatible with $\xi : \sigma \mapsto P^{-1}T_\sigma^{a,b}\sigma(P)$ if and only if either*

$$\zeta : \sigma \mapsto \mathrm{Inn}(\tau_n(P^{-1}T_\sigma^{a,b}\sigma(P))) \tag{3.1}$$

or there exists a quadratic field extension $F(\sqrt{d})$ of F such that

$$\zeta : \sigma \mapsto \begin{cases} \mathrm{Inn}(\tau_n(P^{-1}T_\sigma^{a,b}\sigma(P))) & \text{if } \sigma(\sqrt{d}) = \sqrt{d} \\ \mathrm{Inn}(\tau_n(P^{-1}T_\sigma^{a,b})J_n^{-1}) \circ \ \omega \circ \mathrm{Inn}(\tau_n(\sigma(P))) & \text{if } \sigma(\sqrt{d}) = -\sqrt{d}. \end{cases} \tag{3.2}$$

Proof. Up to equivalence of cocycles, we can assume that $P = I_2$. Suppose that

$$\tau_n({}_\xi\mathrm{SL}_2(F)) < {}_\zeta\mathrm{SL}_n(F).$$

From now on, we fix $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/F)$. Since $\mathrm{Aut}(\mathrm{SL}_n(\overline{\mathbb{Q}})) = \mathrm{Inn}(\mathrm{SL}_n(\overline{\mathbb{Q}})) \rtimes \langle\omega\rangle$, see Theorem 2.8 in [PR94], $\zeta(\sigma)$ is either inner or the composition of an inner automorphism with ω.

Suppose that $\zeta(\sigma) = \mathrm{Inn}(A_\sigma)$ for some $A_\sigma \in \mathrm{PGL}(n, \overline{\mathbb{Q}})$. Then, by definition, all $M \in {}_\xi\,\mathrm{SL}_2(F)$ satisfy

$$\begin{aligned} &\tau_n(T_\sigma\sigma(M)T_\sigma^{-1}) = \tau_n(M) = A_\sigma\sigma(\tau_n(M))A_\sigma^{-1} \\ \implies &A_\sigma\tau_n(T_\sigma^{-1})\tau_n(M) = \tau_n(M)A_\sigma\tau_n(T_\sigma)^{-1} \end{aligned}$$

since $\tau_n \circ \sigma = \sigma \circ \tau_n$. Hence $A_\sigma\tau_n(T_\sigma)^{-1}$ commutes with $\tau_n({}_\xi\mathrm{SL}_2(F))$ thus with its Zariski-closure which is $\tau_n(\mathrm{SL}_2(\overline{\mathbb{Q}}))$. Schur's lemma implies that $A_\sigma = \tau_n(T_\sigma)$. It follows that if ζ is inner, it is of the form of (3.1).

Suppose that $\zeta(\sigma) = \mathrm{Inn}(A_\sigma) \circ \omega$ for some $A_\sigma \in \mathrm{PGL}(n, \overline{\mathbb{Q}})$. Then for all $M \in {}_\xi\mathrm{SL}_2(F)$

$$\begin{aligned} &\tau_n(T_\sigma\sigma(M)T_\sigma^{-1}) = \tau_n(M) = A_\sigma\omega(\sigma \circ \tau_n(M))A_\sigma^{-1} \\ \implies &\tau_n(T_\sigma\sigma(M)T_\sigma^{-1}) = \tau_n(M) = A_\sigma J_n\sigma(\tau_n(M))J_n^{-1}A_\sigma^{-1} \\ \implies &A_\sigma J_n\tau_n(T_\sigma)^{-1}\tau_n(M) = \tau_n(M)A_\sigma J_n\tau_n(T_\sigma)^{-1}. \end{aligned}$$

By the argument as above we get $A_\sigma = \tau_n(T_\sigma)J_n^{-1}$.

Since J_n is in $\mathrm{PGL}(n, F)$ and T_σ has a representative in $\mathrm{PGL}(2, F)$, for every σ, the automorphism $\zeta(\sigma)$ is fixed under the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$. Applying the 1-cocycle formula we see that

$$\zeta(\sigma\tau) = \zeta(\sigma)\zeta(\tau)$$

for all $\sigma, \tau \in \mathrm{Gal}(\overline{\mathbb{Q}}/F)$ that is, ζ is group homomorphism.

Suppose ζ is not inner. Then $\zeta^{-1}(\mathrm{Inn}(\mathrm{SL}_n(\overline{\mathbb{Q}})))$ is an index 2 subgroup of $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$. It is moreover closed and open because ζ is continuous. Thus its fixed field is a quadratic extension of F which we denote $F(\sqrt{d})$ for some $d \in F$, see Theorem 5.4.10 in [Wei06]. Finally we have shown that ζ is of the form of equation (3.2).

Reciprocally, all maps defined by (3.1) or (3.2) are 1-cocycles as shown by computation using the fact that $\tau_n(T_\sigma)$ and J_n commute. □

Finally, we compute the central simple algebra associated to inner τ_n-compatible cocycles. This is the first step towards the classification of F-forms associated to τ_n-compatible cocycles. For every 1-cocycle $\zeta : \mathrm{Gal}(\overline{\mathbb{Q}}/F) \to \mathrm{Inn}(\mathrm{SL}(n, \overline{\mathbb{Q}})) \simeq \mathrm{PGL}(n, \overline{\mathbb{Q}})$ define

$${}_\zeta\mathrm{M}_n(F) = \{M \in \mathrm{M}_n(\overline{\mathbb{Q}}) \mid \zeta(\sigma)(\sigma(M)) = M \ \forall\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/F)\}.$$

It satisfies ${}_\zeta\mathrm{M}_n(F) \otimes_F \overline{\mathbb{Q}} \simeq \mathrm{M}_n(\overline{\mathbb{Q}})$.

Proposition 3.1.5. *Let $\zeta : \mathrm{Gal}(\overline{\mathbb{Q}}/F) \to \mathrm{PGL}(n, \overline{\mathbb{Q}})$ be a 1-cocycle τ_n-compatible with ξ. Then*

$${}_\zeta\mathrm{M}_n(F) \simeq \begin{cases} \mathrm{M}_n(F) & \text{if } n \text{ is odd} \\ \mathrm{M}_{\frac{n}{2}}({}_\xi\mathrm{M}_2(F)) & \text{if } n \text{ is even.} \end{cases}$$

In particular, $\tau_n({}_\xi\mathrm{SL}_2(F))$ can be conjugated to lie inside $\mathrm{SL}_n(F)$ if and only if n is odd.

Proof. Recall that, by assumptions, ζ is inner. Consider the following short exact sequence

$$1 \to \overline{\mathbb{Q}}^* \to \mathrm{GL}_n(\overline{\mathbb{Q}}) \xrightarrow{\pi} \mathrm{PGL}_n(\overline{\mathbb{Q}}) \to 1.$$

By Proposition 1.6.4, this gives rise to a long exact sequence of sets with distinguished elements

$$\mathrm{H}^1(\mathrm{Gal}(\overline{\mathbb{Q}}/F), \mathrm{GL}_n(\overline{\mathbb{Q}})) \to \mathrm{H}^1(\mathrm{Gal}(\overline{\mathbb{Q}}/F), \mathrm{PGL}_n(\overline{\mathbb{Q}})) \xrightarrow{\delta} \mathrm{H}^2(\mathrm{Gal}(\overline{\mathbb{Q}}/F), \overline{\mathbb{Q}}^*).$$

Hilbert 90, Theorem 1.6.5, shows that $\mathrm{H}^1(\mathrm{Gal}(\overline{\mathbb{Q}}/F), \mathrm{GL}_n(\overline{\mathbb{Q}}))$ is trivial. A "twisting" argument, see Lemma 1.4 in [PR94] implies that δ is injective. Hence ${}_\zeta\mathrm{M}_n(F)$ is uniquely determined by $\delta(\zeta) \in \mathrm{H}^2(\mathrm{Gal}(\overline{\mathbb{Q}}/F), \overline{\mathbb{Q}}^*) \simeq \mathrm{Br}(\overline{\mathbb{Q}})$, see Theorem 1.6.6. We thus only need to determine the Brauer equivalence class of $\delta(\zeta)$.

For every $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/F)$ choose $M_\sigma \in \mathrm{GL}_2(\overline{\mathbb{Q}})$ such that $\pi(M_\sigma) = \xi(\sigma)$. Define $a_{\sigma,\tau} = M_\sigma \sigma(M_\tau) M_{\sigma\tau}^{-1}$ for every $\sigma, \tau \in \mathrm{Gal}(\overline{\mathbb{Q}}/F)$. Note that $\zeta(\sigma) = \tau_n(\xi(\sigma))$ for all σ. Then we have

$$\zeta(\sigma) = \tau_n(\xi(\sigma)) = \tau_n(\pi(M_\sigma)) = \pi(\tau_n(M_\sigma))$$

for every $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/F)$. Thus we see that

$$\tau_n(M_\sigma)\sigma(\tau_n(M_\tau))\tau_n(M_{\sigma\tau}^{-1}) = \tau_n(M_\sigma \sigma(M_\tau) M_{\sigma\tau}^{-1})$$

for every $\sigma, \tau \in \mathrm{Gal}(\overline{\mathbb{Q}}/F)$, so $\delta(\zeta)$ is the 2-cocycle given by $(a_{\sigma,\tau})^{n-1}$. Define

$$\begin{aligned} \psi : \mathrm{Gal}(\overline{\mathbb{Q}}/F) &\to \overline{\mathbb{Q}}^* \\ \sigma &\mapsto \det(M_\sigma) \end{aligned}$$

and note that

$$\begin{aligned} \psi(\sigma)\sigma(\psi(\tau))\psi(\sigma\tau)^{-1} &= \det(M_\sigma)\sigma(\det(M_\tau))\det(M_{\sigma\tau})^{-1} \\ &= \det(M_\sigma \sigma(M_\tau) M_{\sigma\tau}^{-1}) \\ &= \det(a_{\sigma,\tau} I_2) = (a_{\sigma,\tau})^2 \end{aligned}$$

for all $\sigma, \tau \in \mathrm{Gal}(\overline{\mathbb{Q}}/F)$. If n is odd then $(a_{\sigma,\tau})^{n-1} = \psi(\sigma)^l \sigma(\psi(\tau))^l \psi(\sigma\tau)^{-l}$ where $l = \frac{n-1}{2}$ so $(a_{\sigma,\tau})_{\sigma,\tau}$ is actually trivial in $\mathrm{H}^2(\mathrm{Gal}(\overline{\mathbb{Q}}/F), \overline{\mathbb{Q}}^*)$. If n is even then $(a_{\sigma,\tau})^{n-1} = \psi(\sigma)^l \sigma(\psi(\tau))^l \psi(\sigma\tau)^{-l} a_{\sigma,\tau}$ where $l = \frac{n-2}{2}$ so $(a_{\sigma,\tau})^{n-1}$ is equivalent to $a_{\sigma,\tau}$ in $\mathrm{H}^2(\mathrm{Gal}(\overline{\mathbb{Q}}/F), \overline{\mathbb{Q}}^*)$. Since δ is injective, ζ is trivial if n is odd and ζ is equivalent to the cocycle associated to $\mathrm{M}_{\frac{n}{2}}({}_\xi\mathrm{M}_2(F)))$ if n is even. □

3.2 The orthogonal group

The goal of this section is to classify the lattices of $\mathrm{SO}(J_n, \mathbb{R})$ that contain the image of a Fuchsian representation, i.e. Theorem A for $\mathrm{SO}(k+1, k)$. Isomorphism classes of F-forms of $\mathrm{SO}(J_n)$ are in correspondence with equivalence classes of quadratic forms over F^n that have the same discriminant as J_n, see Theorem 2.4.2. To a quadratic form B, the associated F-form is $\mathrm{SO}(B)$. Hence to determine an F-form it suffices to characterize the corresponding quadratic form.

Quadratic forms over F are classified up to equivalence by global and local invariants. The main reference for this subject is Serre's book [Ser73], that deals with the case $F = \mathbb{Q}$. See also [O'M00] §66. Let q be a non-degenerate quadratic form over F of rank n and $\mathcal{P}$ be a prime ideal of $\mathcal{O}_F$. Choose a basis of F^n such that $q(x) = \sum a_i x_i^2$. Define its *Hasse invariant in* $F_\mathcal{P}$ as

$$\mathcal{E}_\mathcal{P}(q) = \bigotimes_{i<j} (a_i, a_j)_{F_\mathcal{P}} \in \mathrm{Br}(F_\mathcal{P}).$$

To compute the Hasse invariant for quadratic forms of large rank, we use the following fact. If q and q' are two quadratic forms on F^n and F^m respectively, denote by $q \perp q'$ the induced quadratic form on $F^n \oplus F^m$. Then for all prime ideals $\mathcal{P}$,

$$\mathcal{E}_\mathcal{P}(q \perp q') = \mathcal{E}_\mathcal{P}(q) \otimes \mathcal{E}_\mathcal{P}(q') \otimes (\mathrm{disc}(q), \mathrm{disc}(q'))_{F_\mathcal{P}}.$$

The Hasse-Minkowski Theorem, Theorem 9 in Chapter IV of [Ser73], states that two quadratic forms over F are equivalent if and only if they have the same discriminant, the same signature and the same Hasse invariant in $F_{\mathcal{P}}$ for all primes $\mathcal{P}$.

Proposition 3.2.1. *Let F be a totally real number field and $a, b \in F^\times$. Let $n = 2k+1 \geq 3$. Consider the 1-cocycle*

$$\zeta : \mathrm{Gal}(\overline{\mathbb{Q}}/F) \to \mathrm{Aut}(\mathrm{SO}(J_n, \overline{\mathbb{Q}})),\ \sigma \to \tau_n(T_\sigma^{a,b}).$$

The F-form associated to ζ is $\mathrm{SO}(B)$ for $B \in \mathrm{SL}(n, F)$ a symmetric matrix such that

- *at one real place of F, B has signature equal to $(k+1, k)$ if k is even and $(k, k+1)$ if k is odd,*
- *B is positive definite at all other real places of F and*
- *B has Hasse invariant*

$$\mathcal{E}_{\mathcal{P}}(B) = \begin{cases} 1 & \text{if } k \equiv 0, 3[4] \\ (a, b)_{F_{\mathcal{P}}} \otimes (-1, -1)_{F_{\mathcal{P}}} & \text{if } k \equiv 1, 2[4] \end{cases}$$

for every prime ideal $\mathcal{P}$ of $\mathcal{O}_F$.

Let us explain how one can reconstruct the quadratic form from the 1-cocycle. The embedding $\mathrm{Aut}(\mathrm{SO}(J_n, \overline{\mathbb{Q}})) \simeq \mathrm{SO}(J_n, \overline{\mathbb{Q}}) \hookrightarrow \mathrm{GL}(n, \overline{\mathbb{Q}})$ induces a map

$$\mathrm{H}^1(\mathrm{Gal}(\overline{\mathbb{Q}}/F), \mathrm{Aut}(\mathrm{SO}(J_n, \overline{\mathbb{Q}}))) \xrightarrow{i} \mathrm{H}^1(\mathrm{Gal}(\overline{\mathbb{Q}}/F), \mathrm{GL}(n, \overline{\mathbb{Q}})).$$

The 1-cocycle $i(\zeta)$ is trivial by Hilbert Theorem 90 (Theorem 1.6.5). Thus there exists $S \in \mathrm{GL}(n, \overline{\mathbb{Q}})$ such that for all $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/F)$ we have $i(\zeta)(\sigma) = S^{-1}\sigma(S)$. The associated symmetric matrix is $S^{-\top} J_n S^{-1}$. To determine the matrix S we use the algorithm described in the proof of Hilbert Theorem 90 (Theorem 1.6.5).

Proof of Proposition 3.2.1. We will exhibit a symmetric matrix, denoted $J_n^{a,b}$, which is in the F-equivalence class associated to ζ, i.e. in the F-equivalence class of the matrix B in the Proposition.

Let $L = F(\sqrt{a}, \sqrt{b})$. Then $\zeta \in \mathrm{H}^1(\mathrm{Gal}(L/F), \mathrm{SO}(J_n, L))$. We define

$$f : L^n \to L^n,\ x \mapsto \sum_{\sigma \in \mathrm{Gal}(L/F)} \tau_n(T_\sigma^{a,b})\sigma(x).$$

If L is a degree 2 extension of F let $\sigma \in \mathrm{Gal}(L/F)$ be the non-trivial element. If L is a degree 4 extension of F, let $\sigma \in \mathrm{Gal}(L/F)$ be such that $\sigma(\sqrt{a}) = -\sqrt{a}$ and $\sigma(\sqrt{b}) = \sqrt{b}$ and let $\tau \in \mathrm{Gal}(L/F)$ be such that $\tau(\sqrt{a}) = \sqrt{a}$ and $\tau(\sqrt{b}) = -\sqrt{b}$. We denote by $(e_1, ..., e_n)$ the canonical basis of L^n.

To start, suppose $n \equiv 1[4]$. If $L = F$ then the symmetric matrix associated to ζ is J_n. We suppose that L is a degree 2 extension of F. If a or b is a square, then $(a, b)_F \simeq \mathrm{M}_2(F)$ and up to change of basis we can assume both of them is are squares. This has been treated in the previous case. Suppose that a and b are not squares. We have

$$f : L^n \to L^n,\ x \mapsto x + \begin{pmatrix} & & & 1 \\ & & \iddots & \\ & -1 & & \\ 1 & & & \end{pmatrix} \sigma(x).$$

We then define for all $1 \leq i \leq \frac{k}{2}$

$$v_{2i-1} = \frac{e_{2i-1} + e_{2k-2i+3}}{2},\ v_{2i} = \frac{e_{2i} - e_{2k-2i+2}}{2},$$

$$v_{2k-2i+2} = \frac{\sqrt{a}e_{2i} + \sqrt{a}e_{2k-2i+2}}{2},\ v_{2k-2i+3} = \frac{\sqrt{a}e_{2i-1} - \sqrt{a}e_{2k-2i+3}}{2}$$

and $v_{k+1} = \frac{e_{k+1}}{2}$. Denote

$$S^{-1} = (f(v_1), \dots, f(v_n)) = \begin{pmatrix} 1 & & & & & \sqrt{a} \\ & 1 & & & \sqrt{a} & \\ & & \dots & \dots & & \\ & & & 1 & & \\ & & \dots & \dots & & \\ & -1 & & & \sqrt{a} & \\ 1 & & & & & -\sqrt{a} \end{pmatrix}.$$

The symmetric matrix associated to ζ is $S^{-\top} J_n S^{-1}$, see (3.3), whose Hasse invariant in $F_{\mathcal{P}}$, for any prime $\mathcal{P}$, is

$$(-1,-1)_{F_{\mathcal{P}}}^{\frac{n-1}{4}} \otimes \bigotimes_{j=1,3,\dots,k-1} (a, -j(n-j))_{F_{\mathcal{P}}} \simeq (-1,-1)_{F_{\mathcal{P}}}^{\frac{n-1}{4}} \otimes (a, (-1)^{\frac{n-1}{4}})_{F_{\mathcal{P}}},$$

since $\prod_{j=1,3,\dots,k-1} j(n-j)$ is a square as can be shown by induction. Thus if $n \equiv 1[8]$, then $S^{-\top} J_n S^{-1}$ has trivial Hasse invariant in $F_{\mathcal{P}}$. If $n \equiv -3[8]$ then $S^{-\top} J_n S^{-1}$ has for Hasse invariant $(-1,-1)_{F_{\mathcal{P}}} \otimes (a,a)_{F_{\mathcal{P}}}$ for all primes $\mathcal{P}$.

We suppose that L is a degree 4 extension of F. Then $f : L^n \to L^n$

$$x \mapsto x + \begin{pmatrix} & 1 & & \\ & & \dots & \\ & & & 1 \\ -1 & & & \end{pmatrix} \tau(x) + \begin{pmatrix} & & & 1 \\ & & 1 & \\ & \dots & & \\ 1 & & & \end{pmatrix} \sigma(x) + \begin{pmatrix} & & & 1 \\ & & -1 & \\ & \dots & & \\ 1 & & & \end{pmatrix} \sigma(\tau(x)).$$

For all $1 \le i \le \frac{k}{2}$ we let

$$v_{2i-1} = \frac{\sqrt{a} e_{2i-1}}{2}, \ v_{2i} = \frac{\sqrt{b} e_{2i}}{2},$$

$$v_{2k-2i+2} = -\frac{\sqrt{ab}}{2} e_{2k-2i+2}, \ v_{2k-2i+3} = \frac{e_{2k-2i+3}}{2}$$

and $v_{k+1} = \frac{e_{k+1}}{4}$. Denote by

$$S^{-1} = (f(v_1), \dots, f(v_n)) = \begin{pmatrix} \sqrt{a} & & & & & 1 \\ & \sqrt{b} & & & \sqrt{ab} & \\ & & \dots & \dots & & \\ & & & 1 & & \\ & & \dots & \dots & & \\ & \sqrt{b} & & & -\sqrt{ab} & \\ -\sqrt{a} & & & & & 1 \end{pmatrix}.$$

The symmetric matrix associated to ζ is $S^{-\top} J_n S^{-1}$, see (3.3), whose Hasse invariant in $F_{\mathcal{P}}$, for any prime $\mathcal{P}$, is

$$(-1,-1)_{F_{\mathcal{P}}}^{\frac{n-1}{4}} \otimes \bigotimes_{j=1,3,\dots,k-1} (a, bj(n-j))_{F_{\mathcal{P}}} \simeq (-1,-1)_{F_{\mathcal{P}}}^{\frac{n-1}{4}} \otimes (a, b^{\frac{n-1}{4}})_{F_{\mathcal{P}}}.$$

Thus if $n \equiv 1[8]$, $S^{-\top} J_n S^{-1}$ has trivial Hasse invariant in $F_{\mathcal{P}}$. If $n \equiv -3[8]$, $S^{-\top} J_n S^{-1}$ has for Hasse invariant $(-1,-1)_{F_{\mathcal{P}}} \otimes (a,b)_{F_{\mathcal{P}}}$ for all primes $\mathcal{P}$.

Finally, if $n \equiv 1[4]$, we let $J_n^{a,b}$ be the diagonal matrix defined by

$$(J_n^{a,b})_{ii} = \begin{cases} -2a(n-i)!(i-1)! & \text{if } i \le k \text{ and } i \text{ is odd,} \\ -2b(n-i)!(i-1)! & \text{if } i \le k \text{ and } i \text{ is even,} \\ k!k! & \text{if } i = k+1, \\ 2ab(n-i)!(i-1)! & \text{if } i \ge k+2 \text{ and } i \text{ is even,} \\ 2(n-i)!(i-1)! & \text{if } i \ge k+2 \text{ and } i \text{ is odd.} \end{cases} \tag{3.3}$$

From now on, suppose $n \equiv 3[4]$**.** If $L = F$, the symmetric matrix associated to ζ is J_n. We assume that L is a degree 2 extension of F. If a or b is a square then $(a,b)_F = 1$ and up to change of basis we can assume both of them are squares. This has been treated in the previous case. Assume none of them is a square. We have

$$f : L^n \to L^n,\ x \mapsto x + \begin{pmatrix} & & & -1 & 1 \\ & & & \cdots & \\ 1 & & & & \end{pmatrix}\sigma(x).$$

We then define for all $1 \leq i \leq \frac{k+1}{2}$

$$v_{2i-1} = e_{2i-1},\ v_{2i} = e_{2i},$$

$$v_{2k-2i+2} = \sqrt{a}e_{2k-2i+2},\ v_{2k-2i+3} = -\sqrt{a}e_{2k-2i+3}$$

and $v_{k+1} = \frac{\sqrt{a}e_{k+1}}{2}$. Denote

$$S^{-1} = (f(v_1), \ldots, f(v_n)) = \begin{pmatrix} 1 & & & & & & \sqrt{a} \\ & 1 & & & & \sqrt{a} & \\ & & \cdots & & \cdots & & \\ & & & \sqrt{a} & & & \\ & & \cdots & & \cdots & & \\ & -1 & & & & \sqrt{a} & \\ 1 & & & & & & -\sqrt{a} \end{pmatrix}.$$

The symmetric matrix associated to ζ is $S^{-\top}J_nS^{-1}$, see (3.4), which has Hasse invariant

$$(-1,-1)_{F_\mathcal{P}}^{\frac{n+1}{4}} \otimes (a,-2a)_{F_\mathcal{P}} \otimes \bigotimes_{j=1,3,\ldots,k} (a,-j(n-j))_{F_\mathcal{P}} \simeq (-1,-1)_{F_\mathcal{P}}^{\frac{n+1}{4}} \otimes (a,(-1)^{\frac{n-3}{4}}a)_{F_\mathcal{P}}$$

for all primes $\mathcal{P}$ since $2\prod_{j=1,3,\ldots,k} j(n-j)$ is a square as can be shown by induction. Thus if $n \equiv -1[8]$ then $S^{-\top}J_nS^{-1}$ has trivial Hasse invariant in $F_\mathcal{P}$. If $n \equiv 3[8]$ then $S^{-\top}J_nS^{-1}$ has Hasse invariant $(-1,-1)_{F_\mathcal{P}} \otimes (a,a)_{F_\mathcal{P}}$.

We suppose that L is a degree 4 extension of F. Then $f : L^n \to L^n$

$$x \mapsto x + \begin{pmatrix} -1 & & & & \\ & 1 & & & \\ & & \cdots & & \\ & & & -1 & \end{pmatrix}\tau(x) + \begin{pmatrix} & & & -1 & -1 \\ & & & \cdots & \\ -1 & & & & \end{pmatrix}\sigma(x) + \begin{pmatrix} & & & -1 & 1 \\ & & & \cdots & \\ 1 & & & & \end{pmatrix}\sigma(\tau(x)).$$

We then define for all $1 \leq i \leq \frac{k+1}{2}$

$$v_{2i-1} = \frac{\sqrt{b}e_{2i-1}}{2},\ v_{2i} = \frac{e_{2i}}{2},$$

$$v_{2k-2i+2} = \frac{\sqrt{a}e_{2k-2i+2}}{2},\ v_{2k-2i+3} = \frac{\sqrt{ab}e_{2k-2i+3}}{2}$$

and $v_{k+1} = \frac{\sqrt{a}e_{k+1}}{4}$. Denote

$$S^{-1} = (f(v_1), \ldots, f(v_n)) = \begin{pmatrix} \sqrt{b} & & & & & & \sqrt{ab} \\ & 1 & & & & \sqrt{a} & \\ & & \cdots & & \cdots & & \\ & & & \sqrt{a} & & & \\ & & \cdots & & \cdots & & \\ & -1 & & & & \sqrt{a} & \\ -\sqrt{b} & & & & & & \sqrt{ab} \end{pmatrix}.$$

The symmetric matrix associated to ζ is $S^{-\top}J_nS^{-1}$, see (3.4), which has Hasse invariant

$$(-1,-1)_{F_\mathcal{P}}^{\frac{n+1}{4}} \otimes (a,2)_{F_\mathcal{P}} \otimes \bigotimes_{j=1,3,\ldots,k} (a,bj(n-j))_{F_\mathcal{P}} \simeq (-1,-1)_{F_\mathcal{P}}^{\frac{n+1}{4}} \otimes (a,b^{\frac{n+1}{4}})_{F_\mathcal{P}}$$

for all primes $\mathcal{P}$. Thus if $n \equiv -1[8]$, $S^{-\top} J_n S^{-1}$ has trivial Hasse invariant in $F_{\mathcal{P}}$. If $n \equiv 3[8]$, $S^{-\top} J_n S^{-1}$ has for Hasse invariant $(-1,-1)_{F_{\mathcal{P}}} \otimes (a,b)_{F_{\mathcal{P}}}$.

Finally, if $n \equiv 3[4]$, we let $J_n^{a,b}$ be the diagonal matrix defined by

$$(J_n^{a,b})_{ii} = \begin{cases} -2b(n-i)!(i-1)! & \text{if } i \leq k \text{ and } i \text{ is odd,} \\ 2(n-i)!(i-1)! & \text{if } i \leq k \text{ and } i \text{ is even,} \\ -ak!k! & \text{if } i = k+1, \\ -2a(n-i)!(i-1)! & \text{if } i \geq k+2 \text{ and } i \text{ is even,} \\ 2ab(n-i)!(i-1)! & \text{if } i \geq k+2 \text{ and } i \text{ is odd.} \end{cases} \tag{3.4}$$

□

Proposition 3.2.2. *Let Γ be an arithmetic subgroup of $\mathrm{SL}(2,\mathbb{R})$. Let F be a totally real number field and $a, b \in F^\times$ such that Γ is commensurable with the norm 1 elements of an order of $(a,b)_F$.*

Let $n = 2k+1 \geq 3$ be odd. Then $\tau_n(\Gamma)$ lies in a subgroup of $\mathrm{SO}(J_n,\mathbb{R})$ widely commensurable with $\mathrm{SO}(B,\mathcal{O}_F)$ for $B \in \mathrm{SL}(n,F)$ a symmetric matrix such that

- *at one real place of F, B has signature equal to $(k+1,k)$ if k is even and $(k,k+1)$ if k is odd,*
- *B is positive definite at all other real places of F and*
- *B has Hasse invariant*

$$\mathcal{E}_{\mathcal{P}}(B) = \begin{cases} 1 & \text{if } k \equiv 0,3[4] \\ (a,b)_{F_{\mathcal{P}}} \otimes (-1,-1)_{F_{\mathcal{P}}} & \text{if } k \equiv 1,2[4] \end{cases}$$

for every prime ideal $\mathcal{P}$ of $\mathcal{O}_F$.

Furthermore these are the only arithmetic subgroups of $\mathrm{SO}(J_n,\mathbb{R})$ that contain $\tau_n(\Gamma)$ up to wide commensurability.

Proof. Let $\xi : \mathrm{Gal}(\overline{\mathbb{Q}}/F) \to \mathrm{PGL}(2,\overline{\mathbb{Q}})$ be a 1-cocycle such that Γ is commensurable with ${}_\xi\mathrm{SL}_2(\mathcal{O}_F)$. Up to equivalence, we can assume that $\xi : \sigma \to T_\sigma^{a,b}$. Since the 1-cocycle $\zeta : \mathrm{Gal}(\overline{\mathbb{Q}}/F) \to \mathrm{Aut}(\mathrm{SO}(J_n,\overline{\mathbb{Q}}))$, $\sigma \to \tau_n(T_\sigma^{a,b})$ is τ_n-compatible with ξ, Proposition 2.2.6 guarantees that $\tau_n(\Gamma)$ lies in a lattice widely commensurable with $\mathrm{SO}(J_n^{a,b},\mathcal{O}_F)$. Here $J_n^{a,b}$ is the symmetric matrix defined in the proof of Proposition 3.2.1. Note that $\mathrm{SO}(J_n^{a,b},\mathcal{O}_F)$ is indeed a lattice of $\mathrm{SO}(J_n,\mathbb{R})$ since $J_n^{a,b}$ is positive definite over all real embeddings of F in $\mathbb{R}$ except one, see Equations (3.3) and (3.4) in the proof of Proposition 3.2.1.

Conversely, suppose that an Λ is an arithmetic subgroup of $\mathrm{SO}(J_n,\mathbb{R})$ that contains $\tau_n(\Gamma)$. Since $\mathrm{SO}(J_n,\mathbb{R})$ is simple Proposition 2.4.1 implies that Λ is widely commensurable the "integer points" of a $\mathbb{R}/L$-form of $\mathrm{SO}(J_n)$ for a number field L. Proposition 1.6.12 shows that $\mathbb{R}/L$-forms of $\mathrm{SO}(J_n)$ are $\overline{\mathbb{Q}}/L$-forms of $\mathrm{SO}(J_n)$. Hence Λ is commensurable with ${}_\eta\mathrm{SO}(J_n)(\mathcal{O}_L)$ for $\eta : \mathrm{Gal}(\overline{\mathbb{Q}}/L) \to \mathrm{SO}(J_n,\overline{\mathbb{Q}})$ a 1-cocycle. By Proposition 3.2.3 below, we can assume that $L = F$.

We show that η is τ_n-compatible with ξ. For every $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/F)$ denote by

$$\tau_n^\sigma : \mathrm{SL}_2(\overline{\mathbb{Q}}) \to \mathrm{SO}(J_n,\overline{\mathbb{Q}}),\ g \mapsto \eta(\sigma) \circ \sigma \circ \tau_n \circ (\xi(\sigma) \circ \sigma)^{-1}(g).$$

This is an algebraic morphism that coincides with τ_n on a finite index subgroup of Γ. Since any finite index subgroup of Γ is Zariski-dense in $\mathrm{SL}_2(\overline{\mathbb{Q}})$, $\tau_n = \tau_n^\sigma$. This means that $\tau_n({}_\xi\mathrm{SL}_2(F)) < {}_\eta\mathrm{SO}(J_n)(F)$. Proposition 3.1.4 concludes the proof. □

Proposition 3.2.3. *Let F be a number field and let V_F denote the set of its embeddings in $\mathbb{C}$. Pick $v \in V_F$. Let $\mathbb{H}$ be a connected F-algebraic group such that $\mathbb{H}(F_w)$ is compact for all $w \neq v$, and $G < \mathbb{H}(F_v)$ be a Zariski-closed semisimple subgroup. Let $\Gamma < G$ be a lattice such that $\Gamma < \mathbb{H}(\mathcal{O}_F)$. Then there exists an F-algebraic subgroup $\mathbb{G}$ of $\mathbb{H}$ such that*

- *$\mathbb{G}(F_v) = G$,*
- *for all $w \in V_F$ not v, $\mathbb{G}(F_w)$ is compact with the euclidean topology and*

- Γ *is commensurable with* $\mathbb{G}(\mathcal{O}_F)$.

Proof. The group $G \cap \mathbb{H}(F)$ contains Γ so is Zariski-dense in G. The proof of Proposition 5.1.8 in [Mor15] implies that there exists an F-algebraic subgroup $\mathbb{G}$ of $\mathbb{H}$ satisfying $\mathbb{G}(F_v) = G$. For all $w \in V_F$ not v, $\mathbb{G}(F_w)$ is a closed subgroup of the compact group $\mathbb{H}(F_w)$, hence is compact. Finally, since $\mathbb{G}(F_w)$ is compact for all archimedean places of F except one, $\mathbb{G}(\mathcal{O}_F)$ is a lattice in G. It contains Γ up to finite index which implies that it is commensurable to Γ. □

3.3 The exceptional group $\mathbf{G}_2$

Proposition 3.3.1. *Let Γ be an arithmetic subgroup of $\mathrm{SL}(2,\mathbb{R})$. Let F be a totally real number field and $a, b \in F^\times$ such that Γ is commensurable with the norm 1 elements of an order of $(a,b)_F$. Then $\tau_7(\Gamma)$ lies in a subgroup of $\mathbf{G}_2(\mathbb{R})$ widely commensurable with $\mathbf{G}_2^{a,b}(\mathcal{O}_F)$. Furthermore the latter is the only arithmetic subgroup of $\mathbf{G}_2(\mathbb{R})$ that contains $\tau_7(\Gamma)$ up to wide commensurability.*

Proof. Let $\xi : \mathrm{Gal}(\overline{\mathbb{Q}}/F) \to \mathrm{PGL}(2,\overline{\mathbb{Q}})$, $\sigma \mapsto T_\sigma^{a,b}$. It is a 1-cocycle and ${}_\xi\mathrm{SL}_2(\mathcal{O}_F)$ is widely commensurable with Γ. Let

$$\zeta : \mathrm{Gal}(\overline{\mathbb{Q}}/F) \to \mathbf{G}_2(\overline{\mathbb{Q}}),\ \sigma \mapsto \tau_7(T_\sigma^{a,b}).$$

It makes sense thanks to Proposition 1.5.8. It satisfies $\tau_7({}_\xi\mathrm{SL}_2(F)) < {}_\zeta\mathbf{G}_2(F)$.

We need to determine ${}_\zeta\mathbf{G}_2(F)$. Hilbert's 90, Theorem 1.6.5, shows that there exists $S \in \mathrm{GL}_7(\overline{\mathbb{Q}})$ such that for all $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/F)$ $\zeta(\sigma) = S^{-1}\sigma(S)$. As computed in the proof of Proposition 3.2.2, we can take

$$S = \begin{pmatrix} \sqrt{b} & & & & & & \sqrt{ab} \\ & 1 & & & & \sqrt{a} & \\ & & \sqrt{b} & & \sqrt{ab} & & \\ & & & \sqrt{a} & & & \\ & & -\sqrt{b} & & \sqrt{ab} & & \\ & -1 & & & & \sqrt{a} & \\ -\sqrt{b} & & & & & & \sqrt{ab} \end{pmatrix}^{-1}.$$

Hence for $M \in {}_\xi\mathrm{SL}_2(F)$, $S\tau_7(M)S^{-1} \in \mathrm{SL}(7,F)$. It appears that $\tau_7({}_\xi\mathrm{SL}_2(F))$ preserves the quadratic form J_7. Hence $S\tau_7({}_\xi\mathrm{SL}_2(F))S^{-1}$ preserves the quadratic form $S^{-\top}J_nS^{-1} = J_n^{a,b}$. Proposition 1.5.8 implies that $S\tau_7({}_\xi\mathrm{SL}_2(F))S^{-1}$ preserves $\times$ as defined in Definition 2.4.8. It follows that

$$S\tau_7({}_\xi\mathrm{SL}_2(F))S^{-1} < \mathbf{G}_2^{a,b}(F).$$

Since there is a unique 1-cocycle τ_7-compatible with ξ that has values in $\mathbf{G}_2(\overline{\mathbb{Q}})$, see Proposition 3.1.4, there is only one F-form of $\mathbf{G}_2(\overline{\mathbb{Q}})$ that contains $\tau_7({}_\xi\mathrm{SL}_2(F))$. We deduce that ${}_\zeta\mathbf{G}_2(F) \simeq \mathbf{G}_2^{a,b}(F)$. Proposition 2.2.6 shows that $\tau_7(\Gamma)$ is contained in ${}_\zeta\mathbf{G}_2(\mathcal{O}_F) \simeq \mathbf{G}_2^{a,b}(\mathcal{O}_F)$ up to finite index.

The converse statement is proven as in the proof of Proposition 3.2.2. □

3.4 The special linear group, odd dimension

Proposition 3.4.1. *Let Γ be an arithmetic subgroup of $\mathrm{SL}(2,\mathbb{R})$. Let F be a totally real number field and A a quaternion algebra over F such that Γ is commensurable with the norm 1 elements of an order of A.*

Let $n \geq 3$ be odd. Then $\tau_n(\Gamma)$ lies in a subgroup of $\mathrm{SL}(n,\mathbb{R})$ widely commensurable with the lattice $\mathrm{SU}(I_n, \sigma; \mathcal{O}_F[\sqrt{d}])$ for every $d \in \mathcal{O}_F$ which is positive at exactly the same real place of F where A splits and $\sigma \in \mathrm{Gal}(F(\sqrt{d})/F)$ non-trivial. If $F = \mathbb{Q}$, then $\tau_n(\Gamma)$ also lies in a subgroup of $\mathrm{SL}(n,\mathbb{R})$ widely commensurable with $\mathrm{SL}(n,\mathbb{Z})$.

Furthermore these are the only arithmetic subgroups of $\mathrm{SL}(n,\mathbb{R})$ that contain $\tau_n(\Gamma)$ up to commensurability.

Proof. Let $a, b \in F^\times$ such that $A \simeq (a,b)_F$. Let $\xi : \mathrm{Gal}(\overline{\mathbb{Q}}/F) \to \mathrm{PGL}(2,\overline{\mathbb{Q}})$, $\sigma \mapsto T_\sigma^{a,b}$. It is a 1-cocycle such that ${}_\xi\mathrm{SL}_2(\mathcal{O}_F)$ is widely commensurable with Γ. Let $\zeta : \mathrm{Gal}(\overline{\mathbb{Q}}/F) \to \mathrm{Aut}(\mathrm{SL}_n(\overline{\mathbb{Q}}))$ be a τ_n-compatible 1-cocycle.

If ζ is inner then Proposition 3.1.5 shows that ${}_\zeta\mathrm{SL}_n(F) \simeq \mathrm{SL}(n,F)$. However, $\mathrm{SL}(n,\mathcal{O}_F)$ is a lattice of $\mathrm{SL}(n,\mathbb{R})$ if and only if $F=\mathbb{Q}$.

Suppose from now on that ζ is not inner. Denote by $F(\sqrt{d})$ the associated quadratic extension of F. Recall from Proposition 3.1.4 that $M \in {}_\zeta\mathrm{SL}_n(F)$ if and only if

$$\tau_n(T_\sigma)\sigma(M)\tau_n(T_\sigma)^{-1} = M$$

for all $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/F)$ such that $\sigma(\sqrt{d}) = \sqrt{d}$ and

$$\tau_n(T_\sigma)J_n^{-1}\sigma(M)^{-\top}J_n\tau_n(T_\sigma)^{-1} = M$$

for all $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/F)$ such that $\sigma(\sqrt{d}) = -\sqrt{d}$. Since $\sigma \mapsto \tau_n(T_\sigma)$ is a trivial cocycle, as proven in Proposition 3.1.5, there exists $S \in \mathrm{GL}(n,\overline{\mathbb{Q}})$ such that $\tau_n(T_\sigma) = S^{-1}\sigma(S)$. The first set of equations is equivalent to

$$SMS^{-1} \in \mathrm{SL}(n, F(\sqrt{d})).$$

Secondly, for all $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/F)$ such that $\sigma(\sqrt{d}) = -\sqrt{d}$

$$\begin{aligned} &\tau_n(T_\sigma)J_n^{-1}\sigma(M)^{\top}J_n\tau_n(T_\sigma)^{-1} = M^{-1} \\ \Leftrightarrow\ &\tau_n(T_\sigma)^{-\top}\sigma(M)^{\top}\tau_n(T_\sigma)^{\top}J_nM = J_n \end{aligned}$$

since $\tau_n(T_\sigma)$ and J_n commute and since $\tau_n(T_\sigma)^{-\top} = \tau_n(T_\sigma)$,

$$\Leftrightarrow\ \sigma(SMS^{-1})^{\top}S^{-\top}J_nS^{-1}(SMS^{-1}) = S^{-\top}J_nS^{-1}.$$

As explained in the proof of Proposition 3.2.2, we can take S such that $S^{-\top}J_nS^{-1} = J_n^{a,b}$. Thus

$$S\,{}_\zeta\mathrm{SL}_n(F)S^{-1} = \mathrm{SU}(J_n^{a,b}, \sigma; F(\sqrt{d}))$$

with $\sigma \in \mathrm{Gal}(F(\sqrt{d})/F)$ non-trivial. From Proposition 2.4.4, we see that σ-Hermitian forms over $F(\sqrt{d})$ are classified up to equivalence by their rank, their discriminant and their signatures at real places of F where d is negative. Hence $J_n^{a,b}$ is equivalent to I_n. Proposition 2.2.6 shows that $\tau_n(\Gamma)$ is contained in ${}_\zeta\mathrm{SL}_n(\mathcal{O}_F)$ which is widely commensurable with $\mathrm{SU}(I_n,\sigma;\mathcal{O}_F[\sqrt{d}])$.

The converse statement is proven as in the proof of Proposition 3.2.2. □

Example 3.4.2. Consider the quaternion algebra $A = (3,3)_{\mathbb{Q}}$ with basis $\{1,i,j,k\}$ satisfying the relations of Equation (1.1). Denote by $\mathcal{O} = \mathbb{Z}[1,i,j,k]$. It is an order of A. Embedding A in $\mathrm{SL}(2,\mathbb{R})$, $\mathcal{O}^1$ is conjugated to

$$\Gamma = \left\{ \begin{pmatrix} x_0 + x_1\sqrt{3} & x_2\sqrt{3} + 3x_3 \\ x_2\sqrt{3} - 3x_3 & x_0 - x_1\sqrt{3} \end{pmatrix} \middle| \ x_i \in \mathbb{Z},\ \det = 1 \right\}.$$

Since $3x^2 + 3y^2 = 1$ has no solution $(x,y) \in \mathbb{Q}^2$, Γ is a uniform lattice of $\mathrm{SL}(2,\mathbb{R})$. On can check that it is torsion-free, so it is actually a surface group. For any $\gamma \in \Gamma$ we have

$$\tau_3(\gamma) = \begin{pmatrix} (x_0+\sqrt{3}x_1)^2 & (x_0+\sqrt{3}x_1)(\sqrt{3}x_2+3x_3) & (\sqrt{3}x_2+3x_3)^2 \\ 2(x_0+\sqrt{3}x_1)(\sqrt{3}x_2-3x_3) & x_0^2-3x_1^2+3x_2^2-9x_3^2 & 2(\sqrt{3}x_2+3x_3)(x_0-\sqrt{3}x_1) \\ (\sqrt{3}x_2-3x_3)^2 & (x_0-\sqrt{3}x_1)(\sqrt{3}x_2-3x_3) & (x_0-\sqrt{3}x_1)^2 \end{pmatrix}.$$

Hence $\tau_3(\Gamma)$ does not lie in $\mathrm{SL}(3,\mathbb{Z})$. However, letting

$$P = \begin{pmatrix} 1 & 0 & \sqrt{3} \\ 0 & \sqrt{3} & 0 \\ 1 & 0 & -\sqrt{3} \end{pmatrix},$$

we have

$$P^{-1}\tau_3(\gamma)P = \begin{pmatrix} x_0^2+3x_1^2+3x_2^2+9x_3^2 & 3x_0x_2+9x_1x_3 & 6x_0x_1-18x_2x_3 \\ 4x_0x_2-12x_1x_3 & x_0^2-3x_1^2+3x_2^2-9x_3^2 & -12x_0x_3+12x_1x_2 \\ 2x_0x_1+6x_2x_3 & 3x_0x_3+3x_1x_2 & x_0^2+3x_1^2-3x_2^2-9x_3^2 \end{pmatrix}$$

which lies in $\mathrm{SL}(3,\mathbb{Z})$.

3.5 Standard cocycle

In even dimension, the determination of the forms is more involved. This section is here to set up some technical results that will be useful in the next sections.

Let F be a totally real number field. Let $a, b \in F^\times$ and

$$\zeta : \mathrm{Gal}(\overline{\mathbb{Q}}/F) \to \mathrm{PGL}(2n, \overline{\mathbb{Q}}),\ \sigma \mapsto \tau_{2n}(T_\sigma^{a,b}).$$

We will show that ζ is equivalent to the following "standard" 1-cocycle.

Definition 3.5.1. The 1-cocycle defined by $\eta : \mathrm{Gal}(\overline{\mathbb{Q}}/F) \to \mathrm{PGL}(2n, \overline{\mathbb{Q}})$, $\sigma \mapsto \phi_n(T_\sigma^{a,b})$ is called the *standard 1-cocycle associated to* (a,b).

Recall that $\phi_n : \mathrm{SL}(2, \mathbb{C}) \to \mathrm{SL}(2n, \mathbb{C})$, $M \mapsto \phi_n(M)$ where $\phi_n(M)$ is the block-diagonal matrix with each block on the diagonal being equal to M.

Computations show that ${}_\eta\mathrm{M}_{2n}(F)$ are 2-by-2 block matrices with each block of the form

$$\begin{pmatrix} x_0 + \sqrt{a}x_1 & \sqrt{b}x_2 + \sqrt{ab}x_3 \\ \sqrt{b}x_2 - \sqrt{ab}x_3 & x_0 - \sqrt{a}x_1 \end{pmatrix}$$

for some $x_i \in F$.

The 1-cocycle ζ does not lift to $\mathrm{SL}(2n, \overline{\mathbb{Q}})$, preventing us to use Hilbert's 90, Theorem 1.6.5. To resolve this issue, we consider the following 1-cocycle.

Lemma 3.5.2. *The map* $\zeta\eta^{-1} : \mathrm{Gal}(\overline{\mathbb{Q}}/F) \to {}_\eta\mathrm{PGL}_n(\overline{\mathbb{Q}})$, $\sigma \mapsto \zeta(\sigma)\eta(\sigma)^{-1}$ *is a 1-cocycle.*

Note that in ${}_\eta\mathrm{PGL}_n(\overline{\mathbb{Q}})$ the action of the Galois group is twisted by η, i.e. for $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/F)$ and $M \in {}_\eta\mathrm{PGL}_n(\overline{\mathbb{Q}})$ we have $\sigma \cdot M = \eta(\sigma)\sigma(M)\eta(\sigma)^{-1}$.

Proof. For all $\sigma, \tau\, \mathrm{Gal}(\overline{\mathbb{Q}}/F)$

$$\begin{aligned} \zeta\eta^{-1}(\sigma\tau) &= \zeta(\sigma)\sigma(\zeta(\tau))\sigma(\eta(\tau))^{-1}\eta(\sigma)^{-1} \\ &= \zeta(\sigma)\eta(\sigma)^{-1}(\eta(\sigma)\sigma(\zeta(\tau)\eta(\tau)^{-1})\eta(\sigma)^{-1}) \\ &= \zeta\eta^{-1}(\sigma)\sigma \cdot \zeta\eta^{-1}(\tau). \end{aligned}$$

□

The 1-cocycle $\zeta\eta^{-1}$ lifts to a 1-cocycle $\chi : \mathrm{Gal}(\overline{\mathbb{Q}}/F) \to {}_\eta\mathrm{SL}_{2n}(\overline{\mathbb{Q}})$

$$\sigma \mapsto \begin{cases} I_{2n} & \text{if } \sigma(\sqrt{a}) = \sqrt{a} \\ \begin{pmatrix} & & & & 1 & 0 \\ & & & & 0 & 1 \\ & & \iddots & & & \\ 1 & 0 & & & & \\ 0 & 1 & & & & \end{pmatrix} & \text{if } \sigma(\sqrt{a}) = -\sqrt{a}. \end{cases}$$

Hilbert's 90, Theorem 1.6.5, tells us that there exists $P \in \mathrm{GL}(n, \overline{\mathbb{Q}})$ such that for all σ, $\zeta\eta^{-1}(\sigma) = P^{-1}\eta(\sigma)\sigma(P)\eta(\sigma)^{-1}$ and its proof gives an algorithm to determine such a matrix P. The algorithm is described in section 3.2. We give here the conclusions. If a is a square, let $P = I_{2n}$. Assume a is not a square. If n is even, let

$$P = \frac{1}{2}\begin{pmatrix} I_2 & & & & & I_2 \\ & \ddots & & & \iddots & \\ & & I_2 & I_2 & & \\ & & \frac{1}{\sqrt{a}}I_2 & -\frac{1}{\sqrt{a}}I_2 & & \\ & \iddots & & & \ddots & \\ \frac{1}{\sqrt{a}}I_2 & & & & & -\frac{1}{\sqrt{a}}I_2 \end{pmatrix}.$$

If n is odd, let

$$P = \frac{1}{2}\begin{pmatrix} I_2 & & & & & & I_2 \\ & \ddots & & & & \iddots & \\ & & I_2 & & I_2 & & \\ & & & 2I_2 & & & \\ & & \frac{1}{\sqrt{a}}I_2 & & -\frac{1}{\sqrt{a}}I_2 & & \\ & \iddots & & & & \ddots & \\ \frac{1}{\sqrt{a}}I_2 & & & & & & -\frac{1}{\sqrt{a}}I_2 \end{pmatrix}.$$

For all $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/F)$, $\zeta(\sigma) = P^{-1}\eta(\sigma)\sigma(P)$. In particular ζ and η are equivalent.

3.6 The symplectic group

Proposition 3.6.1. *Let Γ be an arithmetic subgroup of $\mathrm{SL}(2,\mathbb{R})$. Let $\mathcal{O}$ be an order of a quaternion algebra over a totally real number field such that Γ is widely commensurable with $\mathcal{O}^1$. Let $n \geq 2$. Then $\tau_{2n}(\Gamma)$ lies in a subgroup widely commensurable with $\mathrm{SU}(I_n, {}^-; \mathcal{O})$. Furthermore, this is the only lattice of $\mathrm{Sp}(2n,\mathbb{R})$ that contains $\tau_{2n}(\Gamma)$ up to commensurability.*

Proof. Let F be a totally real number field and $a, b \in F^\times$ such that $\mathcal{O}$ is an order of $(a,b)_F$. We can assume that $a, b \in \mathcal{O}_F$. Let $\xi : \mathrm{Gal}(\overline{\mathbb{Q}}/F) \to \mathrm{PGL}(2,\overline{\mathbb{Q}})$, $\sigma \mapsto T_\sigma^{a,b}$. It is a 1-cocycle such that Γ is commensurable with ${}_\xi\mathrm{SL}_2(\mathcal{O}_F)$. Let

$$\zeta : \mathrm{Gal}(\overline{\mathbb{Q}}/F) \to \mathrm{PSp}_{2n}(J_{2n}, \overline{\mathbb{Q}})$$

be the τ_{2n}-compatible 1-cocycle. We want to determine ${}_\zeta\mathrm{Sp}_{2n}(J_{2n})(F)$.

Denote η the standard 1-cocycle associated to (a,b), see Definition 3.5.1. Recall from §3.5 that $\zeta(\sigma) = P^{-1}\eta(\sigma)\sigma(P)$ for all $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/F)$. Then ${}_\zeta\mathrm{Sp}_{2n}(J_{2n})(F)$

$$\begin{aligned} &= \{M \in \mathrm{Sp}_{2n}(J_{2n})(\overline{\mathbb{Q}}) \mid \zeta(\sigma)\sigma(M)\zeta(\sigma)^{-1} = M \ \forall \sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/F)\} \\ &= \{M \in \mathrm{Sp}_{2n}(J_{2n})(\overline{\mathbb{Q}}) \mid \eta(\sigma)\sigma(PMP^{-1})\eta(\sigma)^{-1} = PMP^{-1} \ \forall \sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/F)\} \\ &= \{X \in {}_\eta\mathrm{SL}_{2n}(F) \mid X^\top P^{-\top} J_{2n} P^{-1} X = P^{-\top} J_{2n} P^{-1}\}. \end{aligned}$$

Denote by A the quaternion algebra

$$\left\{ \begin{pmatrix} x_0 + \sqrt{a}x_1 & \sqrt{b}x_2 + \sqrt{ab}x_3 \\ \sqrt{b}x_2 - \sqrt{ab}x_3 & x_0 - \sqrt{a}x_1 \end{pmatrix} \;\middle|\; x_i \in F \right\}.$$

It is isomorphic to $(a,b)_F$. Further more ${}_\eta\mathrm{SL}_{2n}(F) = \mathrm{SL}(n, A)$. Denote by ${}^-$ the conjugation of A and let

$$K = \phi_n\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right).$$

Then for all $M \in {}_\eta\mathrm{SL}_{2n}(F)$, $\overline{M}^t = (KMK^{-1})^\top$ where M^t is the matrix obtained by transposing M as an element of $\mathrm{SL}(n,A)$. We emphasize that only the position of its 2-by-2 blocks changes, those blocks are not themselves transposed. Hence

$${}_\zeta\mathrm{Sp}_{2n}(J_{2n})(F) = \{M \in {}_\eta\mathrm{SL}_{2n}(F) \mid \overline{M}^t J_{2n}^* M = J_{2n}^*\}.$$

where $J_{2n}^* = KP^{-\top}J_{2n}P^{-1}$. Computations show that J_{2n}^* is a ${}^-$-Hermitian matrix. We now show that J_{2n}^* is equivalent as a ${}^-$-Hermitian matrix to $-I_{2n}$. If n is even, let

$$N = \begin{pmatrix} I_2 & & & & & & I_2 \\ & \ddots & & & & \iddots & \\ & & I_2 & & I_2 & & \\ & & & -D & & & \\ & & D & & D & & \\ & \iddots & & & & \ddots & \\ -D & & & & & & D \end{pmatrix}$$

with

$$D = \begin{pmatrix} \frac{1}{\sqrt{a}} & 0 \\ 0 & -\frac{1}{\sqrt{a}} \end{pmatrix}.$$

If n is odd, let

$$N = \begin{pmatrix} I_2 & & & & & & I_2 \\ & \ddots & & & & \iddots & \\ & & I_2 & & I_2 & & \\ & & & 2I_2 & & & \\ & & -D & & D & & \\ & \iddots & & & & \ddots & \\ -D & & & & & & D \end{pmatrix}.$$

Then $\overline{N}^t J_{2n}^* N$ is the diagonal matrix defined by

$$(\overline{N}^t J_{2n}^* N)_{ii} = \begin{cases} -4(2n-i-1)!i! & \text{if } i \text{ is odd} \\ -4(2n-i)!(i-1)! & \text{if } i \text{ is even.} \end{cases}$$

Since Γ is an arithmetic subgroup of $\mathrm{SL}(2,\mathbb{R})$, F is totally real and A ramifies at all embeddings of F except one, which we denote by ι. As we can see in §5 of [Lew82], non-degenerate $^-$-Hermitian forms on A are classified by their signatures at all real embeddings of F except ι. Here the set of signatures is always $(0, 2n)$. Hence $-J_{2n}^*$ is equivalent as a $^-$-Hermitian form to I_{2n}. Finally

$${}_\zeta\mathrm{Sp}_{2n}(J_{2n})(F) \simeq \mathrm{SU}(I_n, {}^-; A).^{1}$$

Proposition 2.2.6 shows that $\tau_{2n}(\Gamma)$ is contained in ${}_\zeta\mathrm{Sp}_{2n}(J_{2n})(\mathcal{O}_F)$ which is widely commensurable with $\mathrm{SU}(I_n, {}^-; \mathcal{O})$.

The converse statement is proven as in the proof of Proposition 3.2.2. □

3.7 The special linear group, even dimension

Proposition 3.7.1. *Let Γ be an arithmetic subgroup of $\mathrm{SL}(2,\mathbb{R})$. Let $\mathcal{O}$ be an order of a quaternion algebra A over a totally real number field F such that Γ is widely commensurable with $\mathcal{O}^1$.*

Let $n \geq 2$. Then $\tau_{2n}(\Gamma)$ lies in a subgroup of $\mathrm{SL}(2n,\mathbb{R})$ widely commensurable with $\mathrm{SU}(I_n, {}^-\otimes\sigma; \mathcal{O}\otimes\mathcal{O}_F[\sqrt{d}])$ for every $d \in \mathcal{O}_F$ which is positive exactly at the real place where A splits and $\sigma \in \mathrm{Gal}(F(\sqrt{d})/F)$ non-trivial. If $F = \mathbb{Q}$ then $\tau_{2n}(\Gamma)$ also lies in a subgroup of $\mathrm{SL}(2n,\mathbb{R})$ widely commensurable with $\mathrm{SL}(n,\mathcal{O})$.

Furthermore, those are the only lattices of $\mathrm{SL}(2n,\mathbb{R})$ that contain $\tau_{2n}(\Gamma)$ up to commensurability.

Proof. Let $a, b \in \mathcal{O}_F$ such that $A \simeq (a,b)_F$. Let $\xi : \mathrm{Gal}(\overline{\mathbb{Q}}/F) \to \mathrm{PGL}(2,\overline{\mathbb{Q}})$, $\sigma \mapsto T_\sigma^{a,b}$. It is a 1-cocycle such that Γ is commensurable with ${}_\xi\mathrm{SL}_2(\mathcal{O}_F)$. Let $\zeta : \mathrm{Gal}(\overline{\mathbb{Q}}/F) \to \mathrm{Aut}(\mathrm{SL}_{2n}(\overline{\mathbb{Q}}))$ be a τ_{2n}-compatible 1-cocycle. We want to determine ${}_\zeta\mathrm{SL}_{2n}(F)$.

If ζ is inner, Proposition 3.1.5 shows that ${}_\zeta\mathrm{SL}_{2n}(F) \simeq \mathrm{SL}(n, A)$. However, $\mathrm{SL}(n,\mathcal{O})$ is a lattice of $\mathrm{SL}(2n,\mathbb{R})$ if and only if $F = \mathbb{Q}$.

Suppose that ζ is not inner. Denote by $F(\sqrt{d})$ the corresponding quadratic extension, see Proposition 3.1.4. We can suppose that $d \in \mathcal{O}_F$. Let $\sqrt{d}$ be a square root of d.

Recall that for any $M \in \mathrm{SL}(2n,\overline{\mathbb{Q}})$, $M \in {}_\zeta\mathrm{SL}_{2n}(F)$ if and only if

$$\tau_{2n}(T_\sigma)\sigma(M)\tau_{2n}(T_\sigma)^{-1} = M$$

for all $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/F)$ such that $\sigma(\sqrt{d}) = \sqrt{d}$ and

$$\tau_{2n}(T_\sigma)J_{2n}^{-1}\sigma(M)^{-\top}J_{2n}\tau_{2n}(T_\sigma)^{-1} = M$$

[1] Here I_n is viewed as a matrix with entries in A. Hence it is the matrix I_{2n} when we see elements of A as 2-by-2 matrices.

for all $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/F)$ such that $\sigma(\sqrt{d}) = -\sqrt{d}$. In §4.1 we proved that $\tau_{2n}(T_\sigma) = P^{-1}\eta(\sigma)\sigma(P)$. Hence $M \in {}_\zeta\mathrm{SL}_{2n}(F)$ if and only if $PMP^{-1} \in {}_\eta\mathrm{SL}_{2n}(F(\sqrt{d})) \simeq \mathrm{SL}(n, A \otimes_F F(\sqrt{d}))$ and for all $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/F)$ such that $\sigma(\sqrt{d}) = -\sqrt{d}$

$$\begin{aligned} & \sigma(M)^\top J_{2n}\sigma(P)^{-1}\eta(\sigma)^{-1}PM = J_{2n}\sigma(P)^{-1}\eta(\sigma)^{-1}P \\ \Leftrightarrow\ & \sigma(PMP^{-1})^\top\sigma(P)^{-\top}J_{2n}(\eta(\sigma)\sigma(P))^{-1}PMP^{-1} = \sigma(P)^{-\top}J_{2n}(\eta(\sigma)\sigma(P))^{-1} \\ \Leftrightarrow\ & \sigma(PMP^{-1})^\top\eta(\sigma)P^{-\top}J_{2n}P^{-1}(PMP^{-1}) = \eta(\sigma)P^{-\top}J_{2n}P^{-1} \end{aligned}$$

by applying $\sigma(\cdot)^\top$ to each side of the equation since $\sigma^2(PMP^{-1}) = PMP^{-1}$. We will show that this set of equations is equivalent to the defining equation of a group conjugated to $\mathrm{SU}(I_n, {}^-\!\otimes\sigma; A\otimes_F F(\sqrt{d}))$. Denote by

$$K = \phi_n\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right).$$

The quaternion algebra A is isomorphic to

$$\left\{ \begin{pmatrix} x_0 + \sqrt{a}x_1 & \sqrt{b}x_2 + \sqrt{ab}x_3 \\ \sqrt{b}x_2 - \sqrt{ab}x_3 & x_0 - \sqrt{a}x_1 \end{pmatrix} \;\middle|\; x_i \in F \right\}.$$

We use this isomorphism to embed A in $\mathrm{M}_2(\overline{\mathbb{Q}})$. Then for all $X \in {}_\eta\mathrm{SL}_{2n}(F(\sqrt{d}))$, $\overline{X}^t = (KXK^{-1})^\top$ where X^t is the matrix obtained by transposing X as an element of $\mathrm{SL}(n, A\otimes_F F(\sqrt{d}))$. For any $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/F)$ denote by

$$\sigma^* : {}_\eta\mathrm{SL}_{2n}(F(\sqrt{d})) \to {}_\eta\mathrm{SL}_{2n}(F(\sqrt{d}))$$

the map defined on 2-by-2 blocks by

$$\begin{pmatrix} x_0+\sqrt{a}x_1 & \sqrt{b}x_2+\sqrt{ab}x_3 \\ \sqrt{b}x_2-\sqrt{ab}x_3 & x_0-\sqrt{a}x_1 \end{pmatrix} \mapsto \begin{pmatrix} \sigma(x_0)+\sqrt{a}\sigma(x_1) & \sqrt{b}\sigma(x_2)+\sqrt{ab}\sigma(x_3) \\ \sqrt{b}\sigma(x_2)-\sqrt{ab}\sigma(x_3) & \sigma(x_0)-\sqrt{a}\sigma(x_1) \end{pmatrix}.$$

Fix $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/F)$ such that $\sigma(\sqrt{d}) = -\sqrt{d}$. Let $\tau \in \mathrm{Gal}(\overline{\mathbb{Q}}/F)$ such that $\tau(\sqrt{d}) = -\sqrt{d}$ and denote by $s = \sigma^{-1}\tau$. Then for all $X \in {}_\eta\mathrm{SL}_{2n}(F(\sqrt{d}))$

$$\begin{aligned} & \tau(X)^\top\eta(\tau)P^{-\top}J_{2n}P^{-1}X = \eta(\tau)P^{-\top}J_{2n}P^{-1} \\ \Leftrightarrow\ & \eta(\sigma s)^{-1}\sigma s(X)^\top\eta(\sigma s)P^{-\top}J_{2n}P^{-1}X = P^{-\top}J_{2n}P^{-1} \\ \Leftrightarrow\ & \eta(\sigma)^{-1}\sigma(\eta(s)^{-1}s(X)^\top\eta(s))\eta(\sigma)P^{-\top}J_{2n}P^{-1}X = P^{-\top}J_{2n}P^{-1}, \\ & \text{since } \eta(\sigma)\eta(s) = \eta(s)\eta(\sigma) \in \mathrm{PGL}(2n, \overline{\mathbb{Q}}) \\ \Leftrightarrow\ & \eta(\sigma)^{-1}\sigma(\eta(s)s(X)\eta(s)^{-1})^\top\eta(\sigma)P^{-\top}J_{2n}P^{-1}X = P^{-\top}J_{2n}P^{-1} \\ \Leftrightarrow\ & \eta(\sigma)^{-1}\sigma(X)^\top\eta(\sigma)P^{-\top}J_{2n}P^{-1}X = P^{-\top}J_{2n}P^{-1}, \\ & \text{since } s(\sqrt{d}) = \sqrt{d} \\ \Leftrightarrow\ & \sigma(\eta(\sigma)X\eta(\sigma)^{-1})^\top P^{-\top}J_{2n}P^{-1}X = P^{-\top}J_{2n}P^{-1} \\ \Leftrightarrow\ & \sigma^*(X)^\top P^{-\top}J_{2n}P^{-1}X = P^{-\top}J_{2n}P^{-1}, \\ & \text{since } \sigma(\eta(\sigma)X\eta(\sigma)^{-1}) = \sigma^*(X) \\ \Leftrightarrow\ & \sigma^*(KXK^{-1})^\top KP^{-\top}J_{2n}P^{-1}X = KP^{-\top}J_{2n}P^{-1} \\ \Leftrightarrow\ & \sigma^*(\overline{X})^t J_{2n}^* X = J_{2n}^*, \\ & \text{where } J_{2n}^* = KP^{-\top}J_{2n}P^{-1}. \end{aligned}$$

The matrix J_{2n}^* is a ${}^-\!\otimes\sigma$-Hermitian matrix. Let N be the matrix introduced in the proof of Proposition 3.6.1. Since $\sigma^*(N) = N$, $\sigma^*(\overline{N}^t)J_{2n}^*N$ is the diagonal matrix defined by

$$(\sigma^*(\overline{N})^t J_{2n}^* N)_{ii} = \begin{cases} -4(2n-i-1)!i! & \text{if } i \text{ is odd} \\ -4(2n-i)!(i-1)! & \text{if } i \text{ is even.} \end{cases}$$

We can see in §7 of [Lew82] that $^-\otimes\sigma$-Hermitian forms over $A\otimes_F F(\sqrt{d})$ are classified by their rank, their signatures at real places of F and their discriminant. Hence J_{2n}^* is equivalent to $-I_{2n}$. We conclude that

$$_\zeta\mathrm{SL}_{2n}(F)\simeq \mathrm{SU}(I_n,{}^-\otimes\sigma; A\otimes_F F(\sqrt{d})).$$

Proposition 2.2.6 shows that $\tau_{2n}(\Gamma)$ is contained in

$$_\zeta\mathrm{SL}_{2n}(\mathcal{O}_F)\simeq \mathrm{SU}(I_n,{}^-\otimes\sigma;\mathcal{O}\otimes\mathcal{O}_F[\sqrt{d}])$$

up to finite index. This is a lattice of $\mathrm{SL}(2n,\mathbb{R})$ if and only if d is positive exactly at the real place where A splits.

The converse statement is proven as in the proof of Proposition 3.2.2. □

3.8 Proof of Theorem A

Proof of Theorem A. Let G be $\mathrm{SO}(k+1,k)$, $\mathrm{Sp}(2n,\mathbb{R})$, $\mathbf{G}_2(\mathbb{R})$ or $\mathrm{SL}(n,\mathbb{R})$ with $k\geq 2$ and $n\geq 3$. Let Λ be a lattice of G that contains the image of a Fuchsian representation $\rho:\pi_1(S_g)\to G$ for some $g\geq 2$. By definition, $\rho=\tau_n\circ j$ for a suitable n where j is a discrete and faithful embedding of $\pi_1(S_g)$ into $\mathrm{SL}(2,\mathbb{R})$. Hence $j(\pi_1(S_g))$ is a lattice in $\mathrm{SL}(2,\mathbb{R})$.

By Margulis Arithmeticity Theorem, Theorem 2.2.14, Λ is arithmetic. By Proposition 2.4.1, there exists a totally real number field F such that Λ is commensurable with the $\mathcal{O}_F$-points of an F-algebraic group which is compact at all real places of F except one. Applying Proposition 3.2.3 to $\tau_n(\mathrm{GL}(2,\mathbb{R}))<G$, the same holds for $j(\pi_1(S_g))$.

Let $\Gamma=j(\pi_1(S))$. There exists a quaternion algebra A over F such that Γ is commensurable with the norm 1 elements of an order of A. We classified the lattices of G that contain $\tau_n(\Gamma)$ in Proposition 3.2.2 for $G=\mathrm{SO}(k+1,k)$, Proposition 3.3.1 for $G=\mathbf{G}_2(\mathbb{R})$, Proposition 3.4.1 for $G=\mathrm{SL}(2k+1,\mathbb{R})$, Proposition 3.6.1 for $G=\mathrm{Sp}(2n,\mathbb{R})$ and Proposition 3.7.1 for $G=\mathrm{SL}(2n,\mathbb{R})$.

To deduce Theorem A, we proceed as above for all arithmetic lattices Γ of $\mathrm{SL}(2,\mathbb{R})$ which are cocompact. We thus consider all quaternion algebras A that ramifies at all real embeddings of F but one, for all totally real number fields F, except $\mathrm{M}_2(\mathbb{Q})$ when $F=\mathbb{Q}$. Indeed, Γ is non-cocompact only in the case $F=\mathbb{Q}$ and $A\simeq\mathrm{M}_2(\mathbb{Q})$. The classification of Table 1 follows from the classification of lattices of $\mathbf{G}_2(\mathbb{R})$ in Proposition 2.4.11 and the classification of lattices $\mathrm{Sp}(2n,\mathbb{R})$ in Proposition 2.4.7. The case where $G=\mathrm{SO}(k+1,k)$ with $k\equiv 1,2[4]$ is more delicate.

Suppose that $k\equiv 1,2[4]$. The classification of lattices of $\mathrm{SO}(k+1,k)$ is stated in Proposition 2.4.6. Let $\mathrm{SO}(B,\mathcal{O}_F)$ be such a lattice and consider $\sigma:F\hookrightarrow\mathbb{R}$ to be the embedding under which B is not positive definite. Denote by S the set of finite places $\mathcal{P}$ of F such that

$$\mathcal{E}_\mathcal{P}(B)\otimes(-1,-1)_\mathcal{P}\neq 1.$$

Let V_F denote the set of infinite places of F. We now show that $S\cup V_F\setminus\{\sigma\}$ has even cardinality. By Hilbert's Reciprocity Law (Theorem 0.9.10 in [MR03]), $\mathcal{E}_v(B)\otimes(-1,-1)_v\neq 1$ at an even number of places v of F. The number of infinite places $v\in V_F$ where $\mathcal{E}_v(B)\otimes(-1,-1)_v\neq 1$ is $|V_F|-1$ since

$$\mathcal{E}_v(B)=\begin{cases}1 & \text{if } v\neq\sigma\\ -1 & \text{if } v=\sigma.\end{cases}$$

Hence $|S|\equiv|V_F|-1$ [2] which implies that $S\cup V_F\setminus\{\sigma\}$ has even cardinality. We deduce from Theorem 1.2.12 that there exists a unique, up to isomorphism, quaternion algebra A over F that ramifies exactly at the places $S\cup V_F\setminus\{\sigma\}$. Then $A\simeq\mathrm{M}_2(\mathbb{Q})$ if and only if $S\cup V_F\setminus\{\sigma\}$ is empty. This case corresponds to the quadratic form q_0. □

Chapter 4

Zariski-dense Hitchin representations in lattices

The goal of this chapter is to prove Theorem B. Recall that a Hitchin representation is a representation that is the continuous deformation of a Fuchsian representation is the character variety.

Theorem B. *Let Λ be a lattice of G listed in Table 1. Suppose that Λ is not widely commensurable with $\mathrm{SL}(2k+1,\mathbb{Z})$ or $\mathrm{SL}(n,\mathcal{O})$ for any order $\mathcal{O}$. Then there exists $g \geq 2$ such that Λ contains the image of infinitely many $\mathrm{MCG}(S_g)$-orbits of Zariski-dense Hitchin representations of $\pi_1(S_g)$.*

Given an arithmetic lattice Γ of $\mathrm{SL}(2,\mathbb{R})$, we classified in Chapter 3 the lattices that contain $\tau_n(\Gamma)$ up to wide commensurability. When Γ is a surface group, τ_n is a Fuchsian representation of Γ. In this chapter, we *"bend"* these representations. This technique requires to find a simple closed curve on the surface S_g which has a big centralizer in the lattice. When the lattice is uniform, this is done in Lemma 4.2.1 by a topological argument which relies on the fact that the image of a Fuchsian element only consists of regular elements. For non-uniform lattices, however, we use non-abelian Galois cohomology to compute explicitly the centralizer of a curve. For the lattices $\mathrm{SL}(2k+1,\mathbb{Z})$ and $\mathrm{SL}(n,\mathcal{O})$, the centralizer does not contain enough elements to perform a single bending.

We then present the Strong Approximation Theorem, a theorem that will help us prove that bending procedure provides infinitely many mapping class group orbits of Hitchin representations. We gathered in Section 4.5 some results about matrix groups over finite fields for which we did not find a reference in the litterature.

4.1 Bending Fuchsian representations

Let Γ be a cocompact arithmetic subgroup of $\mathrm{SL}(2,\mathbb{R})$. Let G be $\mathrm{SL}(n,\mathbb{R})$, $\mathrm{Sp}(2n,\mathbb{R})$, $\mathrm{SO}(k+1,k)$ or G_2. Propositions 3.2.2, 3.3.1, 3.4.1, 3.6.1 and 3.7.1 tell us that there exists an arithmetic subgroup Λ of G such that $\tau_n(\Gamma) < \Lambda$. By Selberg's Lemma (Theorem 2.1.5) there exists a finite index subgroup of Γ which is torsion free and hence a surface group. Thus τ_n induces a representation of a surface group into an arithmetic subgroup of G which, by definition, is a Fuchsian representation.

The image of this representation is not Zariski-dense in G since it lies in $\tau_n(\mathrm{GL}(2,\mathbb{R}))$. We will deform it so that it becomes Zariski-dense. The technique used is called *bending*, as introduced by Johnson and Millson [JM86]. The Zariski-closure of Hitchin representations have been classified by Guichard, see Theorem 0.8.

Let S be a closed orientable surface of genus at least 2 and γ be a simple closed curve on S. We say that γ is *separating* if the complement of its image has two connected components, otherwise it is *non-separating*. Let $j : \pi_1(S) \to \mathrm{PSL}(2,\mathbb{R})$ be a discrete and faithful representation. Let $\rho = \tau_n \circ j$. Choose $B \in \mathrm{PSL}(n,\mathbb{R})$ which commutes with $\rho([\gamma])$.

Suppose γ is separating and denote by C and D the two connected components of the complement of its image. Van Kampen's theorem tells us that the fundamental group of S is an amalgamated product:

$$\pi_1(S) = \pi_1(C) *_{[\gamma]} \pi_1(D)$$

Denote by

$$\rho^1 : \pi_1(C) \to \mathrm{PSL}(n,\mathbb{R})\ g \mapsto \rho(g)$$
$$\rho^2 : \pi_1(D) \to \mathrm{PSL}(n,\mathbb{R})\ g \mapsto B\rho(g)B^{-1}$$

Together they induce a new representation

$$\rho_B : \pi_1(S) \to \mathrm{PSL}(n,\mathbb{R}).$$

Suppose γ is non-separating and denote by C the complement of its image. Then the fundamental group of S is an HNN-extension of the fundamental group of C. More precisely let T be a tubular neighborhood of the image of γ in S. The curve γ separates T into two connected components which we denote by T_1 and T_2. Denote by $i : T_1 \hookrightarrow C$ the inclusion. Pick $p_1 \in T_1$, $p_2 \in T_2$ and $\eta : [0;1] \to S$ a path from p_1 to p_2 that does not intersect γ. Let

$$\phi : \pi_1(T_2,p_2) \xrightarrow{\sim} \pi_1(T_1,p_1),\ [\gamma] \mapsto [\eta^{-1} \circ \gamma \circ \eta].$$

Suppose $\pi_1(C,p_1) = \langle g_1,...,g_k\rangle$. Then

$$\pi_1(S) = \langle g_1,...,g_k,s \mid i_*(\phi(g)) = s^{-1}i_*(g)s\ \forall g \in \pi_1(T_1,p_1)\rangle.$$

Denote by

$$\rho^1 : \pi_1(C,p_1) \to \mathrm{PSL}(n,\mathbb{R}),\ g \mapsto \rho(g)$$
$$\rho^2 : \langle s\rangle \to \mathrm{PSL}(n,\mathbb{R}),\ s^k \mapsto (B\rho(s))^k$$

Together they induce a representation

$$\rho_B : \pi_1(S) \to \mathrm{PSL}(n,\mathbb{R})$$

since elements of $i_*(\pi_1(T_1,p_1))$ are actually powers of γ.

We call ρ_B the *bending* of ρ using B.

Lemma 4.1.1. *The group $\rho_B(\pi_1(S))$ lies in a conjugate of $\tau_n(\mathrm{SL}(2,\mathbb{R}))$ if and only if $B \in \tau_n(\mathrm{GL}(2,\mathbb{R}))$.*

Proof. Suppose $\rho_B(\pi_1(S))$ lies in $P\tau_n(\mathrm{PSL}(2,\mathbb{R}))P^{-1}$ for some $P \in \mathrm{PSL}(n,\mathbb{R})$. The group $\rho_B(\pi_1(C))$ is Zariski-dense in $\tau_n(\mathrm{PSL}(2,\mathbb{R}))$, since $j(\pi_1(C))$ is Zariski-dense in $\mathrm{PSL}(2,\mathbb{R})$. Hence the equality

$$\tau_n(\mathrm{PSL}(2,\mathbb{R})) = P\tau_n(\mathrm{PSL}(2,\mathbb{R}))P^{-1}.$$

Thus $\mathrm{Inn}(P)$ induces an automorphism of $\tau_n(\mathrm{PSL}(2,\mathbb{R}))$. We deduce that there exists $X \in \mathrm{PGL}(2,\mathbb{R})$ such that $\tau_n(X)P^{-1}$ commutes with all elements of $\tau_n(\mathrm{PSL}(2,\mathbb{R}))$ which is an absolutely irreducible subgroup of $\mathrm{PSL}(n,\mathbb{R})$. Schur's lemma implies that $P = \tau_n(X)$. Consequently $\rho_B(\pi_1(S))$ lies in $\tau_n(\mathrm{PSL}(2,\mathbb{R}))$.

Suppose now γ is separating. Then $\rho_B(\pi_1(D)) = B\rho(\pi_1(D))B^{-1}$ lies in $B\tau_n(\mathrm{PSL}(2,\mathbb{R}))B^{-1}$ which is thus equal to $\tau_n(\mathrm{PSL}(2,\mathbb{R}))$. By the above argument we get that $B \in \tau_n(\mathrm{PGL}(2,\mathbb{R}))$. Suppose γ is non-separating, then $B\rho(s) \in \tau_n(\mathrm{PSL}(2,\mathbb{R}))$ so $B \in \tau_n(\mathrm{PSL}(2,\mathbb{R}))$. □

Lemma 4.1.2. *The group ρ_B preserves a bilinear form represented by a matrix $J \in \mathrm{PGL}(n,\mathbb{R})$ if and only if $J_n = B^\top J_n B$ in $\mathrm{PGL}(n,\mathbb{R})$ and in that case $J = J_n$.*

Proof. Suppose ρ_B preserves J. Then for all $s \in \pi_1(C)$

$$J^{-1}J_n = \rho(s)^{-1}J^{-1}J_n\rho(s)$$

so $J^{-1}J_n$ commutes with $\rho(\pi_1(C))$, and so with $\tau_n(\mathrm{PSL}(2,\mathbb{R}))$. The latter is absolutely irreducible, so Schur's lemma implies that J is a scalar multiple of J_n. Finally ρ_B preserves J_n.

Suppose γ is separating. For all $s \in \pi_1(D)$ we have

$$\begin{aligned}(B\rho(s)B^{-1})^\top J_n(B\rho(s)B^{-1}) &= J_n \\ \implies \rho(s)^\top B^\top J_n B\rho(s) &= B^\top J_n B\end{aligned}$$

so $\rho(\pi_1(D))$ preserves J_n and $B^\top J_n B$ which thus must be equal in $\mathrm{PGL}(n,\mathbb{R})$ by the same argument as above. Suppose γ is non-separating. Then

$$(B\rho(s))^\top J_n(B\rho(s)) = J_n \implies B^\top J_n B = J_n.$$

□

Lemma 4.1.3. *The group $\rho_B(\pi_1(S))$ lies in a conjugate of $\mathbf{G}_2(\mathbb{R})$ if and only if $B \in \mathbf{G}_2(\mathbb{R})$.*

Proof. Suppose $\rho_B(\pi_1(S)) \subset P\mathbf{G}_2(\mathbb{R})P^{-1}$ for some $P \in \mathrm{PSL}(n,\mathbb{R})$. Then $\rho(\pi_1(C))$ lies in $P\mathbf{G}_2(\mathbb{R})P^{-1}$ and so does $\tau_7(\mathrm{PSL}(2,\mathbb{R}))$. Thus

$$P^{-1}\tau_7(\mathrm{PSL}(2,\mathbb{R}))P$$

is a Lie subgroup of $\mathbf{G}_2(\mathbb{R})$. Denote by $\mathfrak{h} = \mathrm{d}_{I_7}\tau_7(\mathfrak{sl}_2(\mathbb{R}))$ and $\mathfrak{g}_2$ the Lie algebra of $\mathbf{G}_2(\mathbb{R})$. The algebra $\mathfrak{h}$ is a principal $\mathfrak{sl}_2$-subalgebra of $\mathfrak{g}_2$ in the sense of Kostant §5 [Kos59] since it acts irreducibly[1] on $\mathbb{R}^7$. There exists $g \in \mathbf{G}_2(\mathbb{R})$ such that

$$P^{-1}\mathfrak{h}P = g\mathfrak{h}g^{-1}$$

since all autormorphisms of $\mathfrak{g}_2$ are inner, as can be seen in Gündogan [Gü10]. We can deduce that $X \mapsto PgX(Pg)^{-1}$ induces an inner autormorphism of $\mathfrak{h}$. There exists $M \in \mathrm{PSL}(2,\mathbb{R})$ such that for all $X \in \mathfrak{h}$ we have $gPX(gP)^{-1} = \tau_7(M)X\tau_7(M)^{-1}$ which implies that $\tau_7(M)^{-1}gP$ commutes with all elements of $\mathfrak{h}$. We conclude that $g^{-1}\tau_7(M) = P$ since $\mathfrak{h}$ acts absolutely irreducibly on $\mathbb{R}^7$ and $P \in \mathbf{G}_2(\mathbb{R})$. We have shown that $\rho_B(\pi_1(S)) \subset \mathbf{G}_2(\mathbb{R})$. Suppose γ is separating. Then $B\rho(\pi_1(D))B^{-1} \subset \mathbf{G}_2(\mathbb{R})$ and its Zariski closure, which is $B\tau_7(\mathrm{GL}(2,\mathbb{R}))B^{-1}$, is also in $\mathbf{G}_2(\mathbb{R})$. By the same argument as above, we show that $B \in \mathbf{G}_2(\mathbb{R})$. If γ is non-separating, then $B\rho(s) \in \mathbf{G}_2(\mathbb{R})$ so $B \in \mathbf{G}_2(\mathbb{R})$. □

4.2 A simple closed curve and its centralizer in Λ

We saw in the last section how to bend a Fuchsian representation ρ by a matrix B such that ρ_B is Zariski-dense. However, if ρ has image in a lattice Λ, it is not clear that there exists B such that ρ_B is Zariski-dense and has image in Λ. One thus needs to find a simple closed curve on the surface which has a big enough centralizer in Λ. This is the object of this section.

The image of a Hitchin representation in G contains only *regular* elements, i.e. elements whose centralizer has dimension $\mathrm{rank}_{\mathbb{R}}(G)$, see Labourie [Lab06]. If Λ is uniform, the centralizer of regular elements in Λ is always of rank as big as possible.

Lemma 4.2.1. *Let G be a semisimple Lie group of non-compact type and let Λ be a uniform lattice of G. Let $\gamma \in \Lambda$ be regular. Then $\mathrm{Comm}_\Lambda(\gamma) := \{g \in \Lambda \mid g\gamma = \gamma g\}$ is an abelian group of rank equal to $\mathrm{rank}_{\mathbb{R}}G$.*

Remark 4.2.2. The proof of this lemma follows an argument of Hamenstädt that appears in her talk [Ham17]. See also Prasad and Raghunathan [PR72] for similar results.

Proof. Let X be the symmetric space of G. It has nonpositive curvature. Since Λ is cocompact, there exists $r > 0$ such that for all $p \in \mathrm{X}$, the covering map $\pi : \mathrm{X} \to \mathrm{X}/\Lambda$ is a diffeomorphism between the balls $\mathrm{B}(p,r)$ and $\mathrm{B}(\pi(p),r)$.

[1] One can show that a $\mathfrak{sl}_2$-subalgebra that acts irreducibly on $\mathbb{R}^7$ is principal by using Theorem 5.3 in [Kos59] and using the Hasse diagram of the representation of $\mathbf{G}_2(\mathbb{R})$ for the fundamental weight associated to the short root, see for example Figure 1 in [Sam20].

Let $\mathfrak{a}$ be the Cartan subalgebra of the Lie algebra of G such that $\gamma \in \exp(\mathfrak{a})$. It is unique since γ is regular. Then $\mathfrak{a}$ embeds in X as a maximal flat. It turns out that

$$\{g \in \Lambda \mid \mathrm{Ad}(g)\mathfrak{a} = \mathfrak{a}\} = \mathrm{Comm}_\Lambda(\gamma).$$

For simplicity, we suppose that Λ is torsion-free. We can do so by Selber's Lemma (Theorem 2.1.5). Consider $M := \mathfrak{a}/\mathrm{Comm}_\Lambda(\gamma)$. It is a flat manifold and as such isometric to $\mathbb{R}^l \times (\mathbb{S}^1)^s$ where $l + s = \dim(\mathfrak{a}) = \mathrm{rank}_{\mathbb{R}}(G)$. In fact s is the rank of $\mathrm{Comm}_\Lambda(\gamma)$. Thus we want to show that $l = 0$.

Suppose that $l \geq 1$. Thus there is a copy of $\mathbb{R}$ in M. Pick a geodesic representative of γ in M. It is a circle. Denote by γ_n the translate of γ by length n along a fixed copy of $\mathbb{R}$. We now see M as immersed in X/Λ. Since X/Λ is compact Arzelà–Ascoli's Theorem tells us that the sequence of geodesics γ_n converges uniformly in X/Λ. Hence there exist $n < m$ such that the Hausdorff distance between γ_n and γ_m is less that $\frac{r}{2}$.

Pick a lift of γ_n in $\mathfrak{a}$ and denote it $\widetilde{\gamma_n}$. Consider a lift of γ_m in X which is at distance less than $\frac{r}{2}$ from $\widetilde{\gamma_n}$ at some point and denote it $\widetilde{\gamma_m}$. The curve $\widetilde{\gamma_m}$ have to stay in a tubular neighborhood of $\widetilde{\gamma_n}$ of radius $\frac{r}{2}$. The Flat Strip Theorem (Proposition 5.1. in [EO73]) implies that $\widetilde{\gamma_n}$ and $\widetilde{\gamma_m}$ lie in a maximal flat of X. Since $\widetilde{\gamma_n}$ is regular, it has to be $\mathfrak{a}$.

Since the $(m-n)$-translate of γ_n is γ_m, there exists an element $g \in \Lambda$ that sends the $(m-n)$-translate of $\widetilde{\gamma_n}$ to $\widetilde{\gamma_m}$. Because $\widetilde{\gamma_m}$ lies in $\mathfrak{a}$, g preserves $\mathfrak{a}$ and $g \in \mathrm{Comm}_\Lambda(\gamma)$. On the other hand, translates of γ in M are supposed to be at distance at least 1 from each other. This is a contradiction. □

When Λ is non-uniform previous the lemma does not hold. However, using Galois cohomology, we can describe the commutator of a particular simple closed curve. This could also be used for uniform lattices and allows one to avoid Lemma 4.2.1.

Let A be a quaternion division algebra over a totally real number field F that splits over exactly one real place of F. Up to isomorphism, it is of the form $(a,b)_F$ with $a, b \in \mathcal{O}_F$ not squares. Denote by $\{1, i, j, ij\}$ a basis of A such that $i^2 = a$, $j^2 = b$ and $ij = -ji$. We embed A in $\mathrm{M}_2(\overline{\mathbb{Q}})$ using the map defined by

$$1 \mapsto I_2,\ i \mapsto \begin{pmatrix} \sqrt{a} & 0 \\ 0 & -\sqrt{a} \end{pmatrix},\ j \mapsto \begin{pmatrix} 0 & \sqrt{b} \\ \sqrt{b} & 0 \end{pmatrix},\ ij \mapsto \begin{pmatrix} 0 & \sqrt{ab} \\ -\sqrt{ab} & 0 \end{pmatrix}. \tag{4.1}$$

Thus $\mathrm{SL}(1, A)$ corresponds to the obtained matrices which have determinant 1.

Let $\mathcal{O}$ be the order $\mathcal{O}_F[1, i, j, k]$. Then

$$\mathrm{SL}(1, \mathcal{O}) \simeq \left\{ \begin{pmatrix} x_0 + \sqrt{a}x_1 & \sqrt{b}x_2 + \sqrt{ab}x_3 \\ \sqrt{b}x_2 - \sqrt{ab}x_3 & x_0 - \sqrt{a}x_1 \end{pmatrix} \;\middle|\; x_i \in \mathcal{O}_F,\ \det = 1 \right\}.$$

Diagonal elements of this group are in bijective correspondence with the solutions of the Pell equation $x_0^2 - ax_1^2 = 1$ and are thus infinite in number by Dirichlet Unit's Theorem (see Theorem 0.4.2. in [MR03]).

By Selberg's Lemma (Theorem 2.1.5), $\mathrm{SL}(1, \mathcal{O})$ contains a finite index subgroup Γ which is torsion free. This group has to be a surface group since $\mathrm{SL}(1, \mathcal{O})$ is cocompact. Hence $\Gamma \simeq \pi_1(S_g)$ for a surface S_g of genus $g \geq 2$.

Lemma 4.2.3. *There exists a simple closed curve on S_g whose representative in Γ is diagonal for the canonical basis.*

Proof. We will show that the generator of the diagonal subgroup of $\mathrm{SL}(1, \mathcal{O})$ represents a simple closed curve. First of all, there exists diagonal elements in Γ because it is a finite index subgroup of $\mathrm{SL}(1, \mathcal{O})$. Diagonal elements in Γ form a cyclic group. Let D be a generator and denote by γ the closed curve it represents. Then γ is simple if and only if its lifts to $\mathbb{H}^2$ are disjoints from each other. We identify

$$\partial\mathbb{H}^2 = \mathbb{P}(\mathbb{R}^2) \xrightarrow{\sim} \hat{\mathbb{R}},\ [x : y] \mapsto \begin{cases} \frac{x}{y} & \text{if } y \neq 0 \\ \infty & \text{otherwise.} \end{cases}$$

The curve γ has a lift α which is the geodesic joining 0 to ∞ since D is diagonal. To check if γ is simple we only need to check that the images of this lift under Γ are either α or do not intersect α. Let

$$g = \begin{pmatrix} x_0 + \sqrt{a}x_1 & \sqrt{b}x_2 + \sqrt{ab}x_3 \\ \sqrt{b}x_2 - \sqrt{ab}x_3 & x_0 - \sqrt{a}x_1 \end{pmatrix}$$

be an element of Γ which is not diagonal. The image of α under g is the geodesic joining $g(0)$ to $g(\infty)$. Note that neither of those can be 0 or ∞ since Γ is a discrete subgroup. We need to prove that the product $g(0)g(\infty)$ is positive. Computation gives

$$\begin{aligned} g(0)g(\infty) =& \frac{(x_0 + \sqrt{a}x_1)(\sqrt{b}x_2 + \sqrt{ab}x_3)}{(\sqrt{b}x_2 - \sqrt{ab}x_3)(x_0 - \sqrt{a}x_1)} < 0 \\ &\Leftrightarrow (x_0^2 - ax_1^2)(bx_2^2 - abx_3^2) < 0 \\ &\Leftrightarrow (x_0^2 - ax_1^2)(x_0^2 - ax_1^2 - 1) < 0 \ \text{ because } \det(g) = 1, \\ &\Leftrightarrow 0 < x_0^2 - ax_1^2 < 1. \end{aligned}$$

Since $x_0^2 - ax_1^2 \in \mathcal{O}_F$, its Galois norm, i.e. the product of its embeddings into $\mathbb{C}$, lies in $\mathbb{Z}$. So we cannot have $0 < \sigma(x_0^2 - ax_1^2) < 1$ for all embeddings σ. However, for all embeddings σ except one, $\sigma(a) < 0$ and $\sigma(b) < 0$. For these embeddings $0 < \sigma(x_0^2 - ax_1^2) = \sigma(1 + bx_2^2 - abx_3^2) < 1$. □

Let F be a number field. Denote by $\xi : \mathrm{Gal}(\overline{\mathbb{Q}}/F) \to \mathrm{PGL}(2, \overline{\mathbb{Q}})$, $\sigma \mapsto T_\sigma^{a,b}$. Let $\zeta : \mathrm{Gal}(\overline{\mathbb{Q}}/F) \to \mathrm{Aut}(\mathrm{SL}_n(\overline{\mathbb{Q}}))$ be 1-cocycle τ_n-compatible with ξ. We compute the centralizer of a proximal diagonal element in ${}_\zeta\mathrm{SL}_n(F)$. Let

$$\begin{pmatrix} w_1 & & \\ & \ddots & \\ & & w_n \end{pmatrix}, \ \prod_i w_i = 1 \tag{4.2}$$

be a diagonal matrix.

Lemma 4.2.4. *If ζ is inner, the diagonal elements of ${}_\zeta\mathrm{SL}_n(F)$ are of the form (4.2) with $w_i \in F(\sqrt{a})$ and $\sigma(w_i) = w_{n-i+1}$ for all i with $\sigma \in \mathrm{Gal}(F(\sqrt{a})/F)$ non-trivial.*

If ζ is not inner let $F(\sqrt{d})$ be the associated extension of F. Then:

- *if $\sqrt{a} \in F(\sqrt{d})$ then the diagonal elements of ${}_\zeta\mathrm{SL}_n(F)$ are of the form (4.2) with $w_i \in F(\sqrt{a})$ and $w_i\sigma(w_i) = 1$ for all i with $\sigma \in \mathrm{Gal}(F(\sqrt{a})/F)$ non-trivial;*
- *if $\sqrt{a} \notin F(\sqrt{d})$ then the diagonal elements of ${}_\zeta\mathrm{SL}_n(F)$ are of the form (4.2) with $w_i \in F(\sqrt{a}, \sqrt{d})$, $\sigma(w_i) = w_{n-i+1}$ for all i with $\sigma \in \mathrm{Gal}(F(\sqrt{a}, \sqrt{d})/F(\sqrt{d}))$ non-trivial and $w_i\tau(w_{n-i+1}) = 1$ for all i with $\tau \in \mathrm{Gal}(F(\sqrt{a}, \sqrt{d})/F(\sqrt{a}))$ non-trivial.*

Proof. Since ζ is τ_n-compatible with ξ, Proposition 3.1.4 describes ζ explicitly.

First suppose ζ is inner. We are looking for diagonal matrices $D \in \mathrm{SL}(n, \overline{\mathbb{Q}})$ such that $\zeta(\theta) \circ \theta(D) = D$ for all $\theta \in \mathrm{Gal}(\overline{\mathbb{Q}}/F)$, i.e.

$$\tau_n(T_\theta)\theta(D)\tau_n(T_\theta)^{-1} = D.$$

If θ fixes $\sqrt{a}$ the previous equation tells us that D has coefficients in $F(\sqrt{a})$. If θ does not fix $\sqrt{a}$ then the equation tells us that $\theta(w_{n-i+1}) = w_i$ for all i. Hence the result.

Suppose now that ζ is not inner. We are looking for diagonal matrices $D \in \mathrm{SL}(n, \overline{\mathbb{Q}})$ such that $\zeta(\theta) \circ \theta(D) = D$ for all $\theta \in \mathrm{Gal}(\overline{\mathbb{Q}}/F)$, i.e.

$$\tau_n(T_\theta)\theta(D)\tau_n(T_\theta)^{-1} = D$$

whenever θ fixes $\sqrt{d}$ and

$$\tau_n(T_\theta)J_n^{-1}\theta(D)^{-1}J_n\tau_n(T_\theta)^{-1} = D$$

whenever θ doesn't fix $\sqrt{d}$.

Suppose that $\sqrt{a} \in F(\sqrt{d})$. The first equation tells us that D has coefficients in $F(\sqrt{a})$ and the second that $w_i\theta(w_i) = 1$ for $\theta \in \mathrm{Gal}(F(\sqrt{a})/F)$ non-trivial.

We assume that $\sqrt{a} \notin F(\sqrt{d})$. If θ fixes $\sqrt{d}$ then we conclude as in the previous case that D has coefficients in $F(\sqrt{a}, \sqrt{d})$. If θ doesn't fix $\sqrt{d}$ and doesn't fix $\sqrt{a}$ then the equation we get is $\theta(w_{n-i+1})^{-1} = w_i$ for all i. The case where θ fixes $\sqrt{a}$ but not $\sqrt{d}$ can be deduced from the previous equation. Hence the result. □

Remark 4.2.5. One can show that for every closed curve $\gamma \in \Gamma$, there exists a basis $\{1, i, j, k\}$ of A satisfying Equation (1.1) such that $\gamma \in F(i)$, see Theorem 2.9.8 in [MR03]. Hence the rank of the centralizer of a curve in Λ is independent of the curve and can be read of Lemma 4.2.4.

4.3 Strong Approximation Theorem

In this section we recall the Strong Approximation Theorem and set up some technical lemmas so that we can apply the theorem to the representations we will build. It will be needed in the last part to prove that our constructions provide infinitely many mapping class group orbits of representations.

Recall that if R is a ring and $a \in \mathrm{R} \setminus \{0\}$ is a non-nilpotent element, then we denote by R_a the localization of R at the multiplicative set $\{a^n | n \in \mathbb{N}\}$.

Theorem 2.5.5 (Strong Approximation, Weisfeiler [Wei84])**.** *Let $\mathbb{G}$ be a connected $\overline{\mathbb{Q}}$-algebraic group which is almost simple and simply connected. Let Γ be a finitely generated Zariski-dense subgroup of $\mathbb{G}(\overline{\mathbb{Q}})$. Denote by*

$$\mathrm{R} = \mathbb{Z}[\mathrm{Tr}(\mathrm{Ad}(\Gamma))].$$

Then there exists $a \in \mathrm{R}$, a finite index subgroup $\Gamma' < \Gamma$ and a structure of a group scheme $\mathbb{G}_0$ over R_a that become isomorphic to $\mathbb{G}$ over $\overline{\mathbb{Q}}$ such that $\Gamma' < \mathbb{G}_0(\mathrm{R}_a)$ and Γ' is dense in

$$\varprojlim_{|\mathrm{R}_a/I|<\infty} \mathbb{G}_0\big(\mathrm{R}_a/I\big),$$

where the projective limit is over ideals of R_a.

Theorem 2.5.5 will be used to distinguish mapping class groups orbits of representations. The problem is that the ring R depends on Γ. We thus need to show that, in our setting, this dependence can be removed. This is done in Proposition 4.3.4.

Definition 4.3.1. Let F be a number field. An *order* of F is a subring $\mathrm{R} \subset \mathcal{O}_F$ which is also a $\mathbb{Z}$-module of rank $[F : \mathbb{Q}]$.

Lemma 4.3.2. *Let F be a totally real number field and $\mathbb{G}$ be a connected semisimple F-algebraic group which is compact over all real places of F except one. Let $\Gamma < \mathbb{G}(\mathcal{O}_F)$ be a finitely generated subgroup which is Zariski-dense in $\mathbb{G}$. Then $\mathbb{Z}[\mathrm{Tr}(\mathrm{Ad}(\Gamma))]$ is an order of F.*

Proof. Denote by $K = \mathbb{Q}(\mathrm{Tr}(\mathrm{Ad}(\Gamma))) \subset F$. We want to show that K equals F. By Theorem 1 in Vinberg [Vin71], there is a $\mathbb{C}$-basis β of $\mathfrak{g}$, the Lie algebra of $\mathbb{G}(\mathbb{C})$, such that for all $\gamma \in \Gamma$

$$\mathrm{Mat}_\beta(\mathrm{Ad}(\gamma)) \in \mathrm{GL}(n, K) \text{ with } n = \dim_{\mathbb{C}} \mathfrak{g}.$$

Since Γ is Zariski-dense in $\mathbb{G}$, $\mathrm{Ad}(\Gamma)$ is Zariski-dense in $\mathrm{Ad}(\mathbb{G})$. Hence $\mathrm{Ad}(\mathbb{G})$ is defined over K, see Theorem 14.4 in Chapter AG of Borel [Bor91]. Denote by σ the real place of F such that $\mathbb{G}(F_\sigma)$ is non-compact. Suppose $K \neq F$. There is a real place ι of F different from σ such that $\sigma_{|K} = \iota_{|K}$. Since $\mathbb{G}$ is defined over K, $\mathbb{G}(F_\sigma)$ is isomorphic to $\mathbb{G}(F_\iota)$ while over $\mathbb{R}$ one is compact and the other is not. This is a contradiction. Hence K equals F.

Now $\mathbb{Z}[\mathrm{Tr}(\mathrm{Ad}(\Gamma))]$ is a finitely generated torsion free $\mathbb{Z}$-module, so it is free. Since

$$\mathbb{Z}[\mathrm{Tr}(\mathrm{Ad}(\Gamma))] \otimes_{\mathbb{Z}} \mathbb{Q} \simeq \mathbb{Q}(\mathrm{Tr}(\mathrm{Ad}(\Gamma))),$$

$\mathbb{Z}[\mathrm{Tr}(\mathrm{Ad}(\Gamma))] = F$ is of rank $[F : \mathbb{Q}]$. Thus it is an order of F. □

Lemma 4.3.3. *Let $\mathrm{R} \subset \mathcal{O}_F$ be an order of a number field F. Then for all but finitely many prime ideals $\mathcal{P} \subset \mathcal{O}_F$*

$$\mathrm{R}/\mathrm{R}\cap\mathcal{P} = \mathcal{O}_F/\mathcal{P}. \tag{4.3}$$

Proof. Suppose not. Since R is an order of F, the index of R in $\mathcal{O}_F$ is finite. Denote it by n. Let $\mathcal{P} \subset \mathcal{O}_F$ be a prime ideal that does not satisfy (4.3) such that its residue field has characteristic $p > n$. It exists since for every prime number, there is only finitely many prime ideals in $\mathcal{O}_F$ that contain it.

One has

$$\mathrm{R}/\mathrm{R}\cap\mathcal{P} \subset \mathcal{O}_F/\mathcal{P}.$$

As a subring of a finite field, $\mathrm{R}/\mathrm{R}\cap\mathcal{P}$ is a field. Recall that

$$\mathrm{R}\mathcal{P}/\mathcal{P} \simeq \mathrm{R}/\mathrm{R}\cap\mathcal{P}.$$

Hence

$$\left(\mathcal{O}_F/\mathcal{P}\right) / \left(\mathrm{R}/\mathrm{R}\cap\mathcal{P}\right) \simeq \left(\mathcal{O}_F/\mathcal{P}\right) / \left(\mathrm{R}\mathcal{P}/\mathcal{P}\right) \simeq \mathcal{O}_F/\mathrm{R}\mathcal{P}.$$

The latter must contain at least p elements, so $\mathcal{O}_F/\mathrm{R}$ contains at least p elements. This is a contradiction. □

The following proposition is an adaptation of the Strong Approximation Theorem, Theorem 2.5.5, to our context.

Proposition 4.3.4. *Let F be a totally real number field and $\mathbb{G}$ be a connected almost simple and simply connected F-algebraic group which is compact over all real places of F except one. Let $\Gamma < \mathbb{G}(F)$ be a finitely generated Zariski-dense subgroup. There exists a finite index subgroup $\Gamma' < \Gamma$, $a \in \mathcal{O}_F$ and a group scheme structure $\mathbb{G}_0$ over $(\mathcal{O}_F)_a$ on $\mathbb{G}$ such that $\Gamma' < \mathbb{G}_0((\mathcal{O}_F)_a)$ and for all prime ideals $\mathcal{P}$ of $\mathcal{O}_F$ but finitely many, Γ' surjects onto*

$$\mathbb{G}_0\left(\mathcal{O}_F/\mathcal{P}\right).$$

Remark 4.3.5. The above makes sense since for all prime ideals $\mathcal{P}$ of $\mathcal{O}_F$ that do not contain a,

$$(\mathcal{O}_F)_a/\mathcal{P}_a \simeq \mathcal{O}_F/\mathcal{P}.$$

Proof. Let $\mathrm{R} = \mathbb{Z}[\mathrm{Tr}(\mathrm{Ad}(\Gamma))]$. By the Strong Approximation Theorem, Theorem 2.5.5, there exists $a \in \mathrm{R}$, a finite index subgroup $\Gamma' < \Gamma$ and a group scheme structure $\mathbb{G}_0$ over R_a on $\mathbb{G}$ such that $\Gamma' < \mathbb{G}_0(\mathrm{R}_a)$ and Γ' is dense in

$$\varprojlim_{|\mathrm{R}_a/I|<\infty} \mathbb{G}_0\big(\mathrm{R}_a/I\big).$$

Since $\mathrm{R} \subset \mathcal{O}_F$ we can also consider $\mathbb{G}_0$ as a group scheme over $(\mathcal{O}_F)_a$. Let $\mathcal{P}$ be a prime ideal in $\mathcal{O}_F$ that does not contain a and such that

$$\mathrm{R}/\mathrm{R}\cap\mathcal{P} = \mathcal{O}_F/\mathcal{P}.$$

Since R is an order of F (see Lemma 4.3.2), Lemma 4.3.3 implies that there is only finitely many prime ideals that do not satisfy both of those assumptions. As explained in Remark 2.5.6, Γ' surjects onto

$$\mathbb{G}_0\big(\mathrm{R}_a/(\mathrm{R}\cap\mathcal{P})_a\big) \simeq \mathbb{G}_0\big(\mathrm{R}/(\mathrm{R}\cap\mathcal{P})\big) \simeq \mathbb{G}_0\big(\mathcal{O}_F/\mathcal{P}\big).$$

□

4.4 Proof of Theorem B

Recall that we would like to prove that every lattice Λ in G as in Table 1, which is not widely commensurable with $\mathrm{SL}(2k+1,\mathbb{Z})$ or $\mathrm{SL}(n,\mathcal{O})$ for any order $\mathcal{O}$, contains infinitely many mapping class group orbits of Zariski-dense Hitchin representations, for some fixed genus. The case $\mathrm{SL}(2k+1,\mathbb{Z})$ and $\mathrm{SL}(n,\mathcal{O})$ are excluded since the computations of Lemma 4.2.4 show that the centralizer of the given curve lie in $\mathrm{SO}(k+1,k)$ or $\mathrm{Sp}(2n,\mathbb{R})$ depending on the parity of n. This prevents us from bending the representation in a Zariski-dense one with image in those lattices. Note that in [LT20], Long and Thistlethwaite settled the case $\mathrm{SL}(2k+1,\mathbb{Z})$ using two consecutive bendings.

Proof of Theorem B for $G=\mathrm{Sp}(2n,\mathbb{R})$. Let $n\geq 2$. Let $\Lambda<\mathrm{Sp}(2n,\mathbb{R})$ be a lattice. By Proposition 2.4.7 there exists a quaternion division algebra $A\simeq(a,b)_F$ over a totally real number field F and an order $\mathcal{O}$ of A such that Λ is widely commensurable with $\mathrm{SU}(I_n,{}^-;\mathcal{O})$. Proposition 3.6.1 shows that $\tau_{2n}(\mathcal{O}^1)$ lies in a lattice widely commensurable with Λ.

Let S be the quotient $\mathbb{H}^2/\mathcal{O}^1$. Up to finite cover, it is a closed surface of genus at least 2. We begin by exhibiting one Zariski dense representation of $\pi_1(S)$ in Λ.

Denote by ρ the Fuchsian representation of $\pi_1(S)$ induced by τ_{2n}. We embed $\mathcal{O}^1$ in $\mathrm{SL}(2,\mathbb{R})$ using the map defined in Equation (4.1). Let γ be the simple closed curve on S given by Lemma 4.2.3. By Lemma 4.2.4, there exists $B\in\Lambda$ which commutes with $\rho(\gamma)$, has only positive eigenvalues and such that $B\notin\tau_{2n}(\mathrm{GL}(2,\mathbb{R}))$. For instance, one can take

$$B=\mathrm{Diag}(w^2,\dots,w^2,\sigma(w)^2,\dots,\sigma(w)^2)$$

where $w\in\mathcal{O}_F[\sqrt{a}]^\times$ is an infinite order element and $\sigma\in\mathrm{Gal}(F(\sqrt{a})/F)$ is non-trivial. The bent representation ρ_B of ρ by B along γ has Zariski-dense image in $\mathrm{Sp}(2n,\mathbb{R})$, see Theorem 0.8 and Lemma 4.1.1.

We now exhibit infinitely many Mapping Class Group orbits of representations of $\pi_1(S)$ in Λ. The sequence of representations ρ_{B^k} are Zariski-dense for all $k\geq 1$. We will show that this sequence give rise to infinitely many $\mathrm{MCG}(S)$-orbits of Hitchin representations. Let $k\geq 1$ and denote by Γ_k the image of ρ_{B^k}. Since F-forms of Sp_{2n} are simply connected, see §2.1.13 in [PR94], the Strong Approximation Theorem, specifically Proposition 4.3.4, implies that a finite index subgroup $\Gamma_k'<\Gamma_k$ surjects onto $\mathrm{Sp}(2n,\mathcal{O}_F/\mathcal{P})$ for all but finitely many prime ideals $\mathcal{P}$ of $\mathcal{O}_F$. Hence

$$\mathrm{Tr}(\Gamma_k')\equiv\mathcal{O}_F \mod \mathcal{P}$$

for all but finitely many $\mathcal{P}$ since $\mathrm{Tr}(\mathrm{Sp}(2n,\mathbb{F}_q))=\mathbb{F}_q$ for all q a power of an odd prime. A proof of the latter is provided in the Appendix, Lemma 4.5.1.

Suppose that $\{\rho_{B^k},k\geq 1\}$ is contained in a finite number of orbits under $\mathrm{MCG}(S)$. Then for all but finitely many prime ideals $\mathcal{P}$, the reduction of $\mathrm{Tr}(\Gamma_k')$ modulo $\mathcal{P}$ is surjective for all $k\geq 1$. Note that there exists $\mathrm{P}\in\mathbb{Z}[\mathrm{X}]$ of degree at least 2, such that $\mathrm{Tr}(\tau_{2n}(M))=\mathrm{P}(\mathrm{Tr}(M))$ for all M. There exists infinitely many prime ideals $\mathcal{P}$ such that

$$\mathcal{O}_F/\mathcal{P}\to\mathcal{O}_F/\mathcal{P},\ x\mapsto\mathrm{P}(x)$$

is not a surjection. Indeed, if $q\equiv 1 \mod m$ with $m\geq 2$, there are no polynomial of degree m that induces a bijection of $\mathbb{F}_q$, see Corollary 1.8 in [Sha12]. Pick a prime ideal $\mathcal{P}$ such that the above map is not a surjection and such that the reduction $\mathrm{Tr}(\Gamma_k')$ modulo $\mathcal{P}$ is surjective for all $k\geq 1$. Let $k\geq 1$ such that B^k is trivial modulo $\mathcal{P}$. Then

$$\mathrm{Tr}(\Gamma_k')=\mathrm{Tr}(\rho(\pi_1(S)))=\mathrm{P}(\mathrm{Tr}(\mathcal{O}^1)) \mod \mathcal{P}$$

which is not $\mathcal{O}_F/\mathcal{P}$. This is a contradiction. □

Proof of Corollary D. The corollary follows from Theorem B where we proved that all lattices of $\mathrm{Sp}(4,\mathbb{R})$ not widely commensurable with $\mathrm{Sp}(4,\mathbb{Z})$ contain Zariski-dense Hitchin representations. The case $\mathrm{Sp}(4,\mathbb{Z})$ is settled in [LT18]. □

Proof of Theorem B for $G = \mathrm{SL}(2k+1,\mathbb{R})$. Let $n = 2k+1 \geq 3$ be odd. Let F be a totally real number field and $d \in \mathcal{O}_F$ which is positive at exactly one real place of F. Let $A \simeq (a,b)_F$ be any division algebra over F and $\mathcal{O}$ be an order in A. Proposition 3.4.1 implies that $\tau_n(\mathcal{O}^1)$ lies in a subgroup Λ of $\mathrm{SL}(n,\mathbb{R})$ which widely commensurable with $\mathrm{SU}(I_n,\sigma;\mathcal{O}_F[\sqrt{d}])$ where $\sigma \in \mathrm{Gal}(F(\sqrt{d})/F)$ is non-trivial.

The quotient $S := \mathbb{H}^2/\mathcal{O}^1$ is a closed surface of genus at least 2 up to finite cover. Let ρ be the Fuchsian representation of $\pi_1(S)$ induced by τ_n. We embed $\mathcal{O}^1$ in $\mathrm{SL}(2,\mathbb{R})$ using the map defined in Equation (4.1). Let γ be the simple closed curve on S given by Lemma 4.2.3. By Lemma 4.2.4, there exists $B \in \Lambda$ which commutes with $\rho(\gamma)$, has only positive eigenvalues and such that $B \notin \mathrm{SO}(J_n,\mathbb{R})$. If $\sqrt{a} \in F(\sqrt{d})$ one can take

$$B = \mathrm{Diag}(w^4, 1, \dots, 1, w^{-2}, w^{-2})$$

where $w \in \mathcal{O}_F[\sqrt{a}]^\times$ is an infinite order element. Suppose that $\sqrt{a} \notin F(\sqrt{d})$. Denote by $\sigma \in \mathrm{Gal}(F(\sqrt{a},\sqrt{d})/F)$ the element that satisfies $\sigma(\sqrt{a}) = -\sqrt{a}$ and $\sigma(\sqrt{d}) = \sqrt{d}$. Denote by $\tau \in \mathrm{Gal}(F(\sqrt{a},\sqrt{d})/F)$ the element that satisfies $\tau(\sqrt{a}) = \sqrt{a}$ and $\tau(\sqrt{d}) = \sqrt{d}$. Let $\theta \in \mathcal{O}_F[\sqrt{d}]^\times$ be an infinite order element. Then $\theta\sigma(\theta)\tau(\theta)\sigma(\tau(\theta)) = \pm 1$ and $\sigma(\theta) = \theta$ so $\theta^2\tau(\theta)^2 = 1$ and $\theta^2\sigma(\theta)^2 = \theta^4 \neq 1$. Hence one can take the matrix

$$B = \mathrm{Diag}(\theta^2, \theta^{-2}, 1, \dots, 1, \theta^{-2}, \theta^2).$$

The bent representation ρ_B of ρ by B along γ has Zariski-dense image in $\mathrm{SL}(n,\mathbb{R})$, see Lemma 4.1.2.

We now show that the sequence of representations ρ_{B^l} give rise to infinitely many $\mathrm{MCG}(S)$-orbits. Let $l \geq 1$ and denote by Γ_l the image of ρ_{B^l}. Since F-forms of SL_n are simply connected, the Strong Approximation Theorem (Theorem 2.5.5 and Proposition 4.3.4) implies that for all but finitely many prime ideals $\mathcal{P}$ of $\mathcal{O}_F$, a finite index subgroup $\Gamma_l' < \Gamma_l$ surjects onto

$$\begin{cases} \mathrm{SL}_n\left(\mathcal{O}_F/\mathcal{P}\right) & \text{if } d \text{ is a square in } \mathcal{O}_F/\mathcal{P} \\ \mathrm{SU}\left(I_n,\sigma_0;\mathcal{O}_F/\mathcal{P}[\sqrt{d}]\right) & \text{if } d \text{ is not a square in } \mathcal{O}_F/\mathcal{P} \end{cases}$$

for

$$\sigma_0 \in \mathrm{Gal}\left(\mathcal{O}_F/\mathcal{P}(\sqrt{d})/\mathcal{O}_F/\mathcal{P}\right)$$

non-trivial. Lemmas 4.5.1 and 4.5.2 in the Appendix show that

$$\mathrm{Tr}(\Gamma_l') = \begin{cases} \mathcal{O}_F/\mathcal{P} \mod \mathcal{P} & \text{if } d \text{ is a square in } \mathcal{O}_F/\mathcal{P} \\ \mathcal{O}_F/\mathcal{P}[\sqrt{d}] \mod \mathcal{P} & \text{if } d \text{ is not a square in } \mathcal{O}_F/\mathcal{P} \end{cases}$$

for all but finitely many prime ideals $\mathcal{P}$.

Suppose that $\{\rho_{B^l}, l \geq 1\}$ is contained in a finite number of orbits under $\mathrm{MCG}(S)$. Then for all but finitely many prime ideals $\mathcal{P}$, the reduction of $\mathrm{Tr}(\Gamma_l')$ is surjective modulo $\mathcal{P}$ for all $l \geq 1$. As before, there exists $\mathrm{P} \in \mathbb{Z}[\mathrm{X}]$ such that $\mathrm{Tr}(\tau_n(M)) = \mathrm{P}(\mathrm{Tr}(M))$ for all M. By our assumption, there is a prime ideal $\mathcal{P}$ such that the reduction of $\mathrm{Tr}(\Gamma_l')$ is surjective for all $l \geq 1$ and the map

$$\mathcal{O}_F/\mathcal{P} \to \mathcal{O}_F/\mathcal{P},\ x \mapsto \mathrm{P}(x)$$

is not a surjection (see Corollary 1.8 in [Sha12]). Let $l \geq 1$ such that B^l is trivial modulo $\mathcal{P}$. Then

$$\mathrm{Tr}(\Gamma_l') = \mathrm{Tr}(\rho(\pi_1(S))) = \mathrm{P}(\mathrm{Tr}(\mathcal{O}^1)) \mod \mathcal{P}.$$

This is a contradiction. □

Proof of Corollary C. The corollary follows from Theorem B for $G = \mathrm{SL}(p,\mathbb{R})$ together with [LT20] and Proposition 2.4.3. □

Proof of Theorem B for $G = \mathrm{SL}(2n,\mathbb{R})$. Let $n \geq 2$. Let F be a totally real number field. Let A be a quaternion algebra over F which splits at exactly one real place σ among the real places of F and $\mathcal{O}$ be an order of A. Let $d \in \mathcal{O}_F$ which is positive exactly at σ. Proposition 3.7.1 shows that $\tau_{2n}(\mathcal{O}^1)$ lies in a lattice Λ of $\mathrm{SL}(2n,\mathbb{R})$ widely commensurable with $\mathrm{SU}(I_n,{}^-\otimes\sigma;\mathcal{O}\otimes\mathcal{O}_F[\sqrt{d}])$. The proof now follows exactly the argument of the proof of Theorem B for $G = \mathrm{SL}(2k+1,\mathbb{R})$. □

Proof of Theorem B for $G = \mathbf{G}_2(\mathbb{R})$. Let Λ be a lattice in $\mathbf{G}_2(\mathbb{R})$. By Proposition 2.4.11, there exists a totally real number field F and $a, b \in \mathcal{O}_F$ such that for all embeddings $\sigma : F \to \mathbb{R}$ except one, $\sigma(a)$ and $\sigma(b)$ are negative and Λ is widely commensurable with $\mathbf{G}_2^{a,b}(\mathcal{O}_F)$. Let $A = (a, b)_F$ if $F = \mathbb{Q}$ and A be any quaternion division algebra over $\mathbb{Q}$ otherwise. Let $\mathcal{O}$ be an order of A. Proposition 3.3.1 shows that $\tau_7(\mathcal{O}^1)$ lies in Λ up to wide commensurability.

The proof follows the argument of Theorem B for $G = \mathrm{Sp}(2n, \mathbb{R})$, using Lemma 4.5.4 in the Appendix. □

Since F-forms of SO_n are not simply connected, we cannot apply the Strong Approximation Theorem as stated in Proposition 4.3.4. We will instead use Theorem 5.1 in Nori [Nor87]. Before doing so, we need to prove the following lemma.

Lemma 4.4.1. *Let Γ be a group and $G_1, \dots, G_n$ be centerless connected simple Lie groups. For all $1 \leq i \leq n$, let $\rho_i : \Gamma \to G_i$ be a Zariski-dense representation. Suppose that there does not exist $i \neq j$ and a continuous isomorphism $\phi : G_i \to G_j$ satisfying $\phi \circ \rho_i = \rho_j$. Then*

$$\rho_1 \times \cdots \times \rho_n : \Gamma \to G_1 \times \cdots \times G_n$$

is Zariski-dense.

Proof. We will show the lemma by induction on $n \geq 1$. The case $n = 1$ is clear. Let $n \geq 2$ and assume the result has been shown for $n-1$. Denote by H the Zariski-closure of the image of $\rho \coloneqq \rho_1 \times \cdots \times \rho_n$. We want to show that H is equal to $G \coloneqq G_1 \times \cdots \times G_n$.

Let $\pi_i : G \to G_i$ be the projection onto the i-th coordinate. By Lemma 3.1 in [DK02], $\pi_i(H) = G_i$. Let $p_i : G \to G/G_i$ be the projection that forgets about the i-th coordinate. Using the induction hypothesis and Lemma 3.1 in [DK02], $p_i(H) = G/G_i$.

Denote by

$$H_i = H \cap (\{e\} \times \ldots \times \{e\} \times G_i \times \{e\} \times \ldots \times \{e\}) \overset{\pi_i}{\hookrightarrow} G_i.$$

Let $h \in H_i$ and $g \in G_i$. There exists $k \in H$ such that $\pi_i(k) = g$. Hence

$$g\pi_i(h)g^{-1} = \pi_i(khk^{-1}) \in \pi_i(H_i)$$

which proves that $\pi_i(H_i)$ is normal in G_i.

Suppose that for some $1 \leq i \leq n$, $\pi_i(H_i) = G_i$. Denote by

$$K_i = H \cap (G_1 \times \cdots \times G_{i-1} \times \{e\} \times G_{i+1} \times \cdots \times G_n) \overset{p_i}{\hookrightarrow} G/G_i.$$

Pick $g \in G/G_i$. There exists $h \in H$ such that $p_i(h) = g$. Denote by h_i its i-th coordinate. There exists $k \in H_i$ such that $\pi_i(k) = h_i$. Hence $hk^{-1} \in K_i$ and is send to g under p_i. This shows that $K_i \hookrightarrow G/G_i$ is an isomorphism. Finally $H = G$.

Otherwise, suppose that $\pi_i(H_i)$ is trivial for all i. Pick $g \in G/G_n$. There exists a unique $\phi(g) \in G_n$ such that $(g, \phi(g)) \in H$. Indeed if (g, h) and (g, h') are in H then $(e, h^{-1}h') \in \pi_n(H_n)$ which is trivial. If follows from

$$(g, \phi(g))(g', \phi(g')) = (gg', \phi(g)\phi(g'))$$

that ϕ is a group homomorphism. Since $\pi_n(H) = G_n$, ϕ is surjective.

For each $1 \leq i \leq n-1$, the restriction of ϕ to G_i is either trivial or an isomorphism. Indeed the image of G_i under ϕ is a normal subgroup of G_n. Hence there exists $1 \leq i \leq n-1$ such that $\phi_{|G_i} : G_i \to G_n$ is an isomorphism. Since its graph is closed, it is continuous. The same holds also for its inverse. Finally $\phi_{|G_i} \circ \rho_i = \rho_j$. Contradiction. □

Proof of Theorem B for $G = \mathrm{SO}(k+1, k)$. Let $n = 2k + 1 \geq 5$.

We first deal with the case $k \equiv 1, 2[4]$. Let Λ be a lattice in $\mathrm{SO}(k+1, k)$ not widely commensurable with $\Lambda_{\mathbb{Q}}$. Proposition 2.4.6 shows that it is widely commensurable to $\mathrm{SO}(Q, \mathcal{O}_F)$ for F a totally real number field, $Q \in \mathrm{SL}(n, F)$ a symmetric matrix which is positive definite at all real places of F except one such real place σ for which Q has signature equal to $(k+1, k)$ if k is even and $(k, k+1)$ if k is odd.

We want to apply Proposition 3.2.2 to an order of a suitable quaternion algebra over F. Denote by S the set of finite places $\mathcal{P}$ of F such that

$$\mathcal{E}_{\mathcal{P}}(Q) \otimes (-1,-1)_{\mathcal{P}} \neq 1.$$

We show that $S \cup V_F \setminus \{\sigma\}$ has even cardinality. By Hilbert's Reciprocity Law, see Theorem 0.9.10 in [MR03], $\mathcal{E}_v(Q) \otimes (-1,-1)_v \neq 1$ at an even number of places v of F. The number of infinite places $v \in V_F$ where $\mathcal{E}_v(Q) \otimes (-1,-1)_v \neq 1$ is $|V_F| - 1$ since

$$\mathcal{E}_v(Q) = \begin{cases} 1 & \text{if } v \neq \sigma \\ -1 & \text{if } v = \sigma. \end{cases}$$

Hence $|S| \equiv |V_F| - 1$ [2] which implies that $S \cup V_F \setminus \{\sigma\}$ has even cardinality.

Theorem 7.3.6 in [MR03] implies that there exists a quaternion algebra A over F that does not split exactly at $S \cup V_F \setminus \{\sigma\}$. Let $\mathcal{O}$ be an order of A. Proposition 3.2.2 tells us that $\tau_n(\mathcal{O}^1)$ lies in a subgroup of $\mathrm{SO}(J_n, \mathbb{R})$ which, up to wide commensurability, we can assume to be $\mathrm{SO}(Q, \mathcal{O}_F)$.

Let $S = \mathbb{H}^2/\mathcal{O}^1$. Up to finite cover, we can assume that S is a closed surface of genus at least 2. We embed $\mathcal{O}^1$ in $\mathrm{SL}(2,\mathbb{R})$ using the map defined in Equation (4.1). Let γ be the simple closed curve on S given by Lemma 4.2.3. The map τ_n induces a Fuchsian representation $\rho : \pi_1(S) \to \mathrm{SO}(J_n, \mathbb{R})$. Lemma 4.2.4 implies that there exists $B \in \mathrm{SO}(Q, \mathcal{O}_F)$ which commutes with $\rho(\gamma)$, has only positive eigenvalues but which is not in $\tau_n(\mathrm{GL}(2,\mathbb{R}))$. For instance, one can take

$$B = \mathrm{Diag}(w^2, \sigma(w)^2, 1, \dots, 1, w^2, \sigma(w)^2)$$

where $w \in \mathcal{O}_F[\sqrt{a}]^\times$ is an infinite order element and $\sigma \in \mathrm{Gal}(F(\sqrt{a})/F)$ is non-trivial. The bent representation ρ_B of ρ by B along the curve γ has Zariski-dense image in $\mathrm{SO}(J_n, \mathbb{R})$ as Lemma 4.1.1 shows. Furthermore the sequence of representations ρ_{B^l} are Zariski-dense for all $l \geq 1$ and give rise to infinitely many $\mathrm{MCG}(S)$-orbits, as we will show now. Fix $l \geq 1$ and let $\Gamma_l = \rho_{B^l}(\pi_1(S))$.

For every prime ideal $\mathcal{P}$ of $\mathcal{O}_F$, denote by

$$\pi_{\mathcal{P}} : \mathrm{SO}(Q, \mathcal{O}_F) \to \mathrm{SO}(Q, \mathcal{O}_F/\mathcal{P})$$

the reduction modulo $\mathcal{P}$. Let $\Omega(Q, \mathcal{O}_F/\mathcal{P})$ be the commutator subgroup of $\mathrm{SO}(Q, \mathcal{O}_F/\mathcal{P})$. We first prove that $\pi_{\mathcal{P}}(\Gamma_l)$ contains $\Omega(Q, \mathcal{O}_F/\mathcal{P})$ for every prime ideal $\mathcal{P}$ except a finite number of them. Up to multiplying Q be a constant, we can assume that it has coefficients in $\mathcal{O}_F$. Hence $\mathrm{SO}(Q)$ is an $\mathcal{O}_F$-group scheme. Denote it by $\mathbb{G}$ to simplify the exposition. Now $\Gamma_l < \mathbb{G}(\mathcal{O}_F) \simeq \mathrm{Res}_{\mathcal{O}_F/\mathbb{Z}}\mathbb{G}(\mathbb{Z})$ which is a subgroup of $\mathrm{GL}_{nd}(\mathbb{Z})$ where d is the degree of F. We have

$$\begin{aligned} \Gamma_l &\hookrightarrow \mathrm{Res}_{\mathcal{O}_F/\mathbb{Z}}\mathbb{G}(\mathbb{R}) \simeq \mathrm{SO}(\sigma_1(Q), \mathbb{R}) \times \cdots \times \mathrm{SO}(\sigma_d(Q), \mathbb{R}) \\ \gamma &\mapsto (\sigma_1(\gamma), \dots, \sigma_d(\gamma)) \end{aligned}$$

where the σ_i are the embeddings of F in $\mathbb{R}$, see §2.1.2 in [PR94]. Since Γ_l is Zariski-dense in $\mathrm{SO}(Q, \mathcal{O}_F)$, it is Zariski-dense in $\mathrm{SO}(\sigma_i(Q), \mathbb{R})$ for every embedding $\sigma_i : F \hookrightarrow \mathbb{R}$. Suppose that there exists $i \neq j$ such that $\sigma_i \circ \rho_{B^l}$ is conjugated to $\sigma_j \circ \rho_{B^l}$. Then for all $x \in \mathrm{Tr}(\Gamma_l)$, $\sigma_i(x) = \sigma_j(x)$. We showed in Lemma 4.3.2 that $F = \mathbb{Q}(\mathrm{Tr}(\mathrm{Ad}(\Gamma_l))) \subset \mathbb{Q}(\mathrm{Tr}(\Gamma_l))$. Hence σ_i and σ_j agree on F, which is not possible. By Lemma 4.4.1, Γ_l is Zariski-dense in $\mathrm{Res}_{\mathcal{O}_F/\mathbb{Z}}\mathbb{G}$.[2]

Let $m \geq 1$ and p be a prime. For any subgroup $A < \mathrm{GL}_m(\mathbb{F}_p)$, we denote by A^+ the subgroup of A generated by the elements of A of order p. It is a normal subgroup. Let

$$r_p : \mathrm{GL}_m(\mathbb{Z}) \to \mathrm{GL}_m(\mathbb{F}_p)$$

be the reduction map. Theorem 2.5.7 shows that for all but finitely many primes p

$$\mathrm{Res}_{\mathcal{O}_F/\mathbb{Z}}\mathbb{G}(\mathbb{F}_p)^+ < r_p(\Gamma_l) < \mathrm{Res}_{\mathcal{O}_F/\mathbb{Z}}\mathbb{G}(\mathbb{F}_p).$$

[2] Note that Γ_l lies in the connected component of $\mathrm{SO}(k+1, k)$.

Recall that for any prime p

$$\mathrm{Res}_{\mathcal{O}_F/\mathbb{Z}}\mathbb{G}(\mathbb{F}_p) \simeq \mathbb{G}(\mathbb{F}_p \otimes_{\mathbb{Z}} \mathcal{O}_F) \simeq \mathbb{G}(\mathcal{O}_F/p\mathcal{O}_F) \simeq \prod_{\mathcal{P}|p} \mathbb{G}(\mathcal{O}_F/\mathcal{P})$$

where $\mathcal{P}$ are prime ideals in $\mathcal{O}_F$. For almost all prime ideals $\mathcal{P}$

$$\mathbb{G}(\mathcal{O}_F/\mathcal{P}) \simeq \mathrm{SO}(I_n, \mathcal{O}_F/\mathcal{P})$$

since for almost all prime ideals $\mathcal{P}$, Q is non-degenerate over $\mathcal{O}_F/\mathcal{P}$ and is thus equivalent to I_n or to λI_n where λ is not a square in $\mathcal{O}_F/\mathcal{P}$. Now

$$\mathrm{Res}_{\mathcal{O}_F/\mathbb{Z}}\mathbb{G}(\mathbb{F}_p)^+ \simeq \prod_{\mathcal{P}|p} \mathrm{SO}(I_n, \mathcal{O}_F/\mathcal{P})^+ = \prod_{\mathcal{P}|p} \Omega(I_n, \mathcal{O}_F/\mathcal{P})$$

since $\Omega(I_n, \mathcal{O}_F/\mathcal{P})$ is a normal and simple index 2 subgroup of $\mathrm{SO}(I_n, \mathcal{O}_F/\mathcal{P})$, see Chapter 3 §5 in [Suz82]. Thus

$$r_p(\Gamma_l) \simeq \prod_{\mathcal{P}|p} \pi_{\mathcal{P}}(\Gamma_l) \implies \Omega(I_n, \mathcal{O}_F/\mathcal{P}) < \pi_{\mathcal{P}}(\Gamma_l)$$

for almost all prime ideals $\mathcal{P}$. Finally, using Lemma 4.5.3 in the Appendix, we conclude that the reduction of $\mathrm{Tr}(\Gamma_l)$ modulo $\mathcal{P}$ is surjective for almost all prime ideals $\mathcal{P}$. The proof ends the same way as the proof of Theorem B for $G = \mathrm{Sp}(2n, \mathbb{R})$.

We now deal with the case $k \equiv 0, 3[4]$. Let F be a totally real number field. We want to show that $\Lambda_F = \mathrm{SO}(q, \mathcal{O}_F)$, see the §2.3 for the definition, contains infinitely many mapping class group orbits of Zariski-dense Hitchin representations. Let A to be any quaternion algebra over F that splits exactly at the real place where q is not positive definite. Let $\mathcal{O}$ be an order in A and denote $S = \mathbb{H}^2/\mathcal{O}^1$. From now on, the proof goes as in the case $k \equiv 1, 2[4]$. □

4.5 Appendix: traces over finite fields

In the proof of Theorem B, mapping class group orbits of Hitchin representations are distinguished by their trace. Using the Strong Approximation Theorem, Theorem 2.5.5, we infer properties on traces of representations from the trace of matrix groups over finite fields. This section is dedicated to establish easy lemmas which we did not find in the literature. One can avoid these lemmas by proving the analogous of Lemma 5.4.1, that will appear in the last chapter.

Let q be a power of an odd prime.

Lemma 4.5.1. *For any $n \geq 1$, $\mathrm{Tr}(\mathrm{Sp}(2n, \mathbb{F}_q)) = \mathbb{F}_q$.*

Proof. For $n = 1$, the result is proven finding explicit matrices. For $n \geq 2$, it follows from a block embedding of $\mathrm{Sp}(2, \mathbb{F}_q)$ in $\mathrm{Sp}(2n, \mathbb{F}_q)$. □

Let $n \geq 2$. Denote by $\mathrm{SU}(I_n, \mathbb{F}_q) = \{M \in \mathrm{SL}(n, \mathbb{F}_{q^2}) | \overline{\sigma}(M)^\top M = I_n\}$ with $\overline{\sigma} \in \mathrm{Gal}(\mathbb{F}_{q^2}/\mathbb{F}_q)$ non-trivial.

Lemma 4.5.2. *For any $n \geq 3$, $\mathrm{Tr}(\mathrm{SU}(I_n, \mathbb{F}_q)) = \mathbb{F}_{q^2}$.*

Proof. We can embed $\mathrm{SU}(I_3, \mathbb{F}_q)$ in $\mathrm{SU}(I_n, \mathbb{F}_q)$ for any $n \geq 3$ using

$$M \mapsto \begin{pmatrix} M & \\ & I_{n-2} \end{pmatrix}.$$

Hence it suffices to prove the result for $n = 3$.

The map

$$\mathrm{N} : \mathbb{F}_{q^2} \to \mathbb{F}_q,\ x \mapsto x\overline{\sigma}(x)$$

is surjective. Indeed, any generator of the cyclic group $\mathbb{F}_{q^2}^\times$ is mapped under N to a generator of $\mathbb{F}_q^\times$. Let $a \in \mathbb{F}_{q^2}$. There exists $b \in \mathbb{F}_{q^2}$ such that $\mathrm{N}(b) = -a - \overline{\sigma}(a)$. Let

$$M = \begin{pmatrix} a & b & 1 \\ \overline{\sigma}(b) & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Then $\det(M) = 1$ and $\overline{\sigma}(M)^\top JM = J$ where

$$J = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Thus a conjugate of M lies in $\mathrm{SU}(I_3, \mathbb{F}_p)$. Since $\mathrm{Tr}(M) = a - 1$, $\mathrm{Tr}(\mathrm{SU}(I_3, \mathbb{F}_p)) = \mathbb{F}_{p^2}$. □

Let $n \geq 1$. We denote by $\Omega(I_n, \mathbb{F}_q)$ the commutator subgroup of $\mathrm{SO}(I_n, \mathbb{F}_q)$. It is an index 2 subgroup, see [Suz82] paragraph 5 chapter 3.

Lemma 4.5.3. *For any $n \geq 4$, $\mathrm{Tr}(\Omega(I_n, \mathbb{F}_q)) = \mathbb{F}_q$.*

Proof. Suppose that $n = 4$. Pick $a \in \mathbb{F}_q^\times$. Let

$$M = \begin{pmatrix} 0 & 0 & -a & a \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ a^{-1} & 0 & 0 & 0 \end{pmatrix}.$$

We can check that $\det(M) = 1$ and that $M^\top JM = J$ where

$$J = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Since $\det(J) = 1$, J is congruent to I_4. Hence a conjugate of M^2 lies in $\Omega(I_4, \mathbb{F}_q)$. Since $\mathrm{Tr}(M^2) = -2a + 4$ and $\mathrm{Tr}(I_4) = 4$, $\mathrm{Tr}(\Omega(I_4, \mathbb{F}_q)) = \mathbb{F}_q$.

For $n \geq 4$, the proof follows from an embedding of $\Omega(I_4, \mathbb{F}_q)$ in $\Omega(I_n, \mathbb{F}_q)$. □

Let $\mathrm{G}_2(\mathbb{F}_q)$ be the finite group of Lie type associated to the Dynkin diagram G_2 embedded in $\mathrm{SL}_7(\mathbb{F}_q)$ via its 7-dimensional irreducible representation.

Lemma 4.5.4. $\mathrm{Tr}(\mathrm{G}_2(\mathbb{F}_q)) = \mathbb{F}_q$.

Proof. The group $\mathrm{G}_2(\mathbb{F}_q)$ is the automorphism group of the unique octonion algebra $\mathbb{O}$ over $\mathbb{F}_q$, see §1.10 of [SV00]. We can describe it as follows. As an $\mathbb{F}_q$-vector space $\mathbb{O} = \mathrm{M}_2(\mathbb{F}_q) \oplus \mathrm{M}_2(\mathbb{F}_q)$. Let $^-$ denote the conjugation of the quaternion algebra $\mathrm{M}_2(\mathbb{F}_q)$. For any $A \in \mathrm{M}_2(\mathbb{F}_q)$, $\overline{A}$ is the transpose of the cofactor matrix of A. The multiplication is defined by

$$(A_1, B_1) \cdot (A_2, B_2) = (A_1A_2 - \overline{B_2}B_1, B_2A_1 + B_1\overline{A_2}),$$

see §1.5 in [SV00]. We endow $\mathbb{O}$ with the quadratic form

$$(A, B) \mapsto \det(A) + \det(B).$$

An automorphism of $\mathbb{O}$ induces an element of $\mathrm{G}_2(\mathbb{F}_q)$ by its restriction to the orthogonal of $\langle (I_2, 0) \rangle$.

Let $a \in \mathbb{F}_q$ and consider the following automorphism

$$\varphi_a : \mathbb{O} \to \mathbb{O},\ (A, B) \to (A, XB) \text{ where } X = \begin{pmatrix} a & 1 \\ -1 & 0 \end{pmatrix}.$$

It is an automorphism of $\mathbb{O}$ and thus defines an element of $\mathrm{G}_2(\mathbb{F}_q)$. Picking a basis of $\mathbb{O}$, computations show that it has trace $a + 3$. Hence any element of $\mathbb{F}_q$ is the trace of an element of $\mathrm{G}_2(\mathbb{F}_q)$. □

Part III

Maximal representations

Chapter 5

Maximal representations in lattices

In this chapter, we first prove Proposition E. Recall that a maximal diagonal representation is a representation of the form $\phi_n \circ \rho$ for $\rho : \pi_1(S_g) \to \mathrm{PSL}(2,\mathbb{R})$ discrete and faithful.

Proposition E. *Every lattice of* $\mathrm{Sp}(2n,\mathbb{R})$*, not widely commensurable with* $\mathrm{Sp}(2n,\mathbb{Z})$ *when* n *is odd, contains the image of a maximal diagonal representation of some genus. Conversely, when* n *is odd,* $\mathrm{Sp}(2n,\mathbb{Z})$ *does not contain the image of a maximal diagonal representations of any genus.*

To do so, we classify ϕ_n*-compatible* cocycles and determine the F-forms of Sp_{2n} associated to such cocycles. We then establish Theorem F, which is the goal of this chapter.

Theorem F. *Let* $n \geq 2$ *and* Λ *be a lattice of* $\mathrm{Sp}(2n,\mathbb{R})$*, not widely commensurable with* $\mathrm{Sp}(2n,\mathbb{Z})$ *when* n *is odd. Then there exists* $g \geq 2$ *such that* Λ *contains infinitely many* $\mathrm{MCG}(S_g)$*-orbits of Zariski-dense maximal representations of* $\pi_1(S_g)$*. Furthermore these representations are continuous deformations of maximal diagonal representations.*

For this, we show that the commutator of a simple closed curve in the lattice can be described as a lattice itself. We can then apply a bending deformation. We appeal to the classification of Zariski-closures of maximal representations by Hamlet and Pozzetti [HP14] to prove that the bend representations are Zariski-dense. Finally, using the Strong Approximation Theorem, we show that this constructions provides infinitely many mapping class group orbits of representations. Contrary to the proof of Theorem B, the traces of the bend representation is no longer sufficient to distinguish mapping class group orbits.

5.1 Compatible cocycles

Let $n \geq 2$. Define

$$K = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and $K_n = \phi_n(K)$. The latter induces a symplectic form that is preserved by the image of ϕ_n. From now on, for any field $k \subset \mathbb{C}$, we denote by $\mathrm{Sp}(2n,k)$ the group $\mathrm{Sp}(K_n,k) = \{M \in \mathrm{SL}(2n,k) \mid M^\top K_n M = K_n\}$.

The *Kronecker product* of two matrices $A = (a_{ij}) \in \mathrm{M}_n(\mathbb{C})$ and $B \in \mathrm{M}_m(\mathbb{C})$ is the matrix $A \otimes B \in \mathrm{M}_{mn}(\mathbb{C})$ defined by

$$A \otimes B = \begin{pmatrix} a_{11}B & \dots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{n1}B & \dots & a_{nn}B \end{pmatrix}.$$

It satisfies $(A \otimes B)(C \otimes D) = AC \otimes BD$ and $\det(A \otimes B) = \det(A)^m \det(B)^n$. The map

$$\begin{aligned} \mathrm{O}(I_n,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C}) &\to \mathrm{Sp}(2n,\mathbb{C}) \\ (M,A) &\mapsto M \otimes A. \end{aligned}$$

is a Lie group homomorphism. Note that $\phi_n(A) = I_n \otimes A$.

Lemma 5.1.1. *The centralizer of* $\mathrm{Im}(\phi_n)$ *in* $\mathrm{Sp}(2n,\mathbb{C})$ *is* $\mathrm{O}(I_n,\mathbb{C})\otimes I_2$.

Proof. It is clear that $\mathrm{O}(I_n,\mathbb{C})\otimes I_2$ commutes with $\mathrm{Im}(\phi_n)$. Reciprocally, if a matrix X commutes with $\mathrm{Im}(\phi_n)$ then each of its 2-by-2 blocks have to be scalar multiple of I_2, i.e. X has to be of the form $M\otimes I_2$. Then $X\in\mathrm{Sp}(K_n,\mathbb{C})$ if and only if $M\in\mathrm{O}(I_n,\mathbb{C})$. □

Let F be a totally real number field and $\xi:\mathrm{Gal}(\overline{\mathbb{Q}}/F)\to\mathrm{PGL}(2,\overline{\mathbb{Q}})$ be a 1-cocycle. We say that a 1-cocycle $\zeta:\mathrm{Gal}(\overline{\mathbb{Q}}/F)\to\mathrm{PSp}(K_n,\overline{\mathbb{Q}})$ is *ϕ_n-compatible* with ξ if for all $\sigma\in\mathrm{Gal}(\overline{\mathbb{Q}}/F)$ and all $A\in\mathrm{SL}(2,\overline{\mathbb{Q}})$

$$\phi_n({}_\xi\mathrm{SL}_2(F))<{}_\zeta\mathrm{Sp}(K_n)(F)$$

For instance $\phi_n(\xi):\mathrm{Gal}(\overline{\mathbb{Q}}/F)\to\mathrm{PSp}(K_n,\overline{\mathbb{Q}})$ is a 1-cocycle ϕ_n-compatible with ξ.

Lemma 5.1.2. *A 1-cocycle* $\zeta:\mathrm{Gal}(\overline{\mathbb{Q}}/F)\to\mathrm{PSp}(K_n,\overline{\mathbb{Q}})$ *is ϕ_n-compatible with ξ if and only if* $\zeta=\eta\otimes\xi$ *where* $\eta:\mathrm{Gal}(\overline{\mathbb{Q}}/F)\to\mathrm{PO}(I_n,\overline{\mathbb{Q}})$ *is a 1-cocycle.*

Proof. Let $\zeta:\mathrm{Gal}(\overline{\mathbb{Q}}/F)\to\mathrm{PSp}(2n,\overline{\mathbb{Q}})$ be a 1-cocycle ϕ_n-compatible with ξ. Fix $\sigma\in\mathrm{Gal}(\overline{\mathbb{Q}}/F)$. For all $A\in{}_\xi\mathrm{SL}_2(\overline{\mathbb{Q}})$

$$\zeta(\sigma)\sigma(\phi_n(A))\zeta(\sigma)^{-1}=\phi_n(\xi(\sigma)\sigma(A)\xi(\sigma)^{-1})$$

which implies that $\phi_n(\xi(\sigma))^{-1}\zeta(\sigma)$ commutes with $\phi_n({}_\xi\mathrm{SL}_2(\overline{\mathbb{Q}}))$. The latter is Zariski-dense in $\mathrm{Im}(\phi_n)$. Hence for all $\sigma\in\mathrm{Gal}(\overline{\mathbb{Q}}/F)$

$$\zeta(\sigma)=\phi(\xi(\sigma))(\eta(\sigma)\otimes I_2)=\eta(\sigma)\otimes\xi(\sigma)$$

for some $\eta(\sigma)\in\mathrm{PO}(I_n,\overline{\mathbb{Q}})$. Since ζ is a 1-cocycle, $\eta:\mathrm{Gal}(\overline{\mathbb{Q}}/F)\to\mathrm{PO}(I_n,\overline{\mathbb{Q}})$ is also a 1-cocycle. Conversely for any such 1-cocycle η, $\eta\otimes\xi$ is ϕ-compatible with ξ. □

5.2 Forms of $\mathbf{Sp}(2n,\mathbb{R})$

The goal of this section is to compute the forms associated to compatible cocycles. We prove here Proposition E.

Consider the short exact sequence of groups

$$1\to\mu_2\to\mathrm{Sp}(2n,\overline{\mathbb{Q}})\to\mathrm{PSp}(2n,\overline{\mathbb{Q}})\to 1$$

where μ_2 is the group $\{\pm I_2\}$. It induces a long exact sequence of sets with distinguished elements

$$1\to\mathrm{H}^1(\mathrm{Gal}(\overline{\mathbb{Q}}/F),\mathrm{PSp}(2n,\overline{\mathbb{Q}}))\xrightarrow{\delta_{2n}}\mathrm{H}^2(\mathrm{Gal}(\overline{\mathbb{Q}}/F),\mu_2),$$

see §1.3.2 and Proposition 2.7 in [PR94]. Since PSp_{2n} is simply connected, Theorem 6.20 in [PR94] implies that δ_{2n} is surjective.

Since μ_2 is abelian, $\mathrm{H}^2(\mathrm{Gal}(\overline{\mathbb{Q}}/F),\mu_2)$ is a group. Under the isomorphism described in Theorem 1.6.6, it is isomorphic to the 2-torsion of the Brauer group, denoted by ${}_2\mathrm{Br}(F)$. Since F is a number field, every element of ${}_2\mathrm{Br}(F)$ is Brauer equivalent to a quaternion algebra (Theorem 20 in Chapter X of [Alb61]).

Consider the short exact sequence of groups

$$1\to\mu_2\to\mathrm{O}(I_n,\overline{\mathbb{Q}})\xrightarrow{\pi}\mathrm{PO}(I_n,\overline{\mathbb{Q}})\to 1. \tag{5.1}$$

It induces a connecting map

$$\mathrm{H}^1(\mathrm{Gal}(\overline{\mathbb{Q}}/F),\mathrm{PO}(I_n,\overline{\mathbb{Q}}))\xrightarrow{\partial_n}\mathrm{H}^2(\mathrm{Gal}(\overline{\mathbb{Q}}/F),\mu_2). \tag{5.2}$$

Contrary to δ_{2n}, ∂_n is not necessarily surjective since $\mathrm{O}(I_n,\overline{\mathbb{Q}})$ is not a simply connected algebraic group.

Lemma 5.2.1. *If n is even, ∂_n is surjective. If n is odd, ∂_n is trivial.*

Before proving the lemma, we introduce a notation. Let $a,b\in F^\times$ and pick square roots of a and b in $\overline{\mathbb{Q}}$. Recall

$$T^{a,b}:\mathrm{Gal}(\overline{\mathbb{Q}}/F)\to \mathrm{PGL}(2,\overline{\mathbb{Q}})$$
$$\sigma\mapsto\begin{cases} I_2 & \text{if } \sigma(\sqrt{a})=\sqrt{a} \text{ and } \sigma(\sqrt{b})=\sqrt{b}\\ \begin{pmatrix}1&0\\0&-1\end{pmatrix} & \text{if } \sigma(\sqrt{a})=\sqrt{a} \text{ and } \sigma(\sqrt{b})=-\sqrt{b}\\ \begin{pmatrix}0&1\\1&0\end{pmatrix} & \text{if } \sigma(\sqrt{a})=-\sqrt{a} \text{ and } \sigma(\sqrt{b})=\sqrt{b}\\ \begin{pmatrix}0&1\\-1&0\end{pmatrix} & \text{if } \sigma(\sqrt{a})=-\sqrt{a} \text{ and } \sigma(\sqrt{b})=-\sqrt{b}. \end{cases}$$

It is a 1-cocycle. Explicit computations show that

$${}_{T^{a,b}}\mathrm{M}_2(\overline{\mathbb{Q}})=\left\{\begin{pmatrix}x_0+\sqrt{a}x_1 & \sqrt{b}x_2+\sqrt{ab}x_3\\ \sqrt{b}x_2-\sqrt{ab}x_3 & x_0-\sqrt{a}x_1\end{pmatrix}\ \middle|\ x_i\in F\right\}\simeq (a,b)_F.$$

Proof of Lemma 5.2.1. Suppose that n is even. Let $a,b\in F^\times$. Let

$$\chi_n=\phi_{\frac{n}{2}}(T^{a,b}):\mathrm{Gal}(\overline{\mathbb{Q}}/F)\to \mathrm{PO}(I_n,\overline{\mathbb{Q}}).$$

For all $\sigma\in\mathrm{Gal}(\overline{\mathbb{Q}}/F)$ pick a lift M_σ of $\chi_n(\sigma)\in \mathrm{O}(I_n,\overline{\mathbb{Q}})$. By construction of the connecting map, for all $\sigma,\tau\in\mathrm{Gal}(\overline{\mathbb{Q}}/F)$, $\partial_n(\chi_n)_{\sigma\tau}=M_\sigma\sigma(M_\tau)M_{\sigma\tau}^{-1}$. Then for all σ,τ

$$\partial_n(\chi_n)_{\sigma\tau}=\partial_2(\chi_2)_{\sigma\tau}.$$

Explicit computations show that, under the isomorphism $\mathrm{H}^2(\mathrm{Gal}(\overline{\mathbb{Q}}/F),\mu_2)\simeq {}_2\mathrm{Br}(F)$, $\partial_2(\chi_2)$ corresponds to $(a,b)_F$. Since this is true for every $a,b\in F^\times$, ∂_2 and hence ∂_n is surjective.

Suppose that n is odd. Let $\eta:\mathrm{Gal}(\overline{\mathbb{Q}}/F)\to\mathrm{PO}(I_n,\overline{\mathbb{Q}})$ be a 1-cocycle. For all $\sigma\in\mathrm{Gal}(\overline{\mathbb{Q}}/F)$ pick $M_\sigma\in\mathrm{O}(I_n,\overline{\mathbb{Q}})$ such that $\pi(M_\sigma)=\eta(\sigma)$, where π is the map in (5.1). By construction of ∂_n, for all $\sigma,\tau\in\mathrm{Gal}(\overline{\mathbb{Q}}/F)$

$$\partial_n(\eta)_{\sigma\tau}I_n=M_\sigma\sigma(M_\tau)M_{\sigma\tau}^{-1}\in\mu_2.$$

Define $f:\mathrm{Gal}(\overline{\mathbb{Q}}/F)\to\overline{\mathbb{Q}}$, $\sigma\mapsto\det(M_\sigma)$. Then for all $\sigma,\tau\in\mathrm{Gal}(\overline{\mathbb{Q}}/F)$

$$\partial_n(\eta)_{\sigma\tau}=f(\sigma)\sigma(f(\tau))f(\sigma\tau)^{-1}$$

which implies that $\partial_n(\eta)$ is trivial. □

Lemma 5.2.2. *Let $\eta:\mathrm{Gal}(\overline{\mathbb{Q}}/F)\to\mathrm{PO}(I_n,\overline{\mathbb{Q}})$ and $\xi:\mathrm{Gal}(\overline{\mathbb{Q}}/F)\to\mathrm{PGL}(2,\overline{\mathbb{Q}})$ be two 1-cocycles. Then*

$$\delta_{2n}(\eta\otimes\xi)=\delta_2(\xi)\partial_n(\eta).$$

Proof. For each $\sigma\in\mathrm{Gal}(\overline{\mathbb{Q}}/F)$ pick $M_\sigma\in\mathrm{O}(I_n,\overline{\mathbb{Q}})$ and $A_\sigma\in\mathrm{SL}(2,\overline{\mathbb{Q}})$ such that the projectivization of M_σ and A_σ are $\eta(\sigma)$ and $\xi(\sigma)$ respectively. Then $M_\sigma\otimes A_\sigma$ has for projectivization $\eta(\sigma)\otimes\xi(\sigma)$. For all $\sigma,\tau\in\mathrm{Gal}(\overline{\mathbb{Q}}/F)$

$$\begin{aligned}\delta_{2n}(\eta\otimes\xi)_{\sigma\tau}I_{2n}&=(M_\sigma\otimes A_\sigma)\sigma(M_\tau\otimes A_\tau)(M_{\sigma\tau}\otimes A_{\sigma\tau})^{-1}\\&=(M_\sigma\sigma(M_\tau)M_{\sigma\tau}^{-1})\otimes(A_\sigma\sigma(A_\tau)A_{\sigma\tau}^{-1})\\&=\partial_n(\eta)_{\sigma\tau}I_n\otimes\delta_2(\xi)_{\sigma\tau}I_2\end{aligned}$$

□

We can now prove Proposition G.

Proposition G. *Let Γ be an arithmetic lattice in $\mathrm{SL}(2,\mathbb{R})$ which is commensurable with the elements of norm 1 in an order $\mathcal{O}_\Gamma$ of a quaternion algebra over a totally real number field F.*

If n is even, $\phi_n(\Gamma) < \mathrm{SU}(I_n, {}^-; \mathcal{O})$, up to wide commensurability, for any order $\mathcal{O}$ in any quaternion algebra over F that splits at exactly one place of F. Furthermore, up to wide commensurability, these are the only lattices of $\mathrm{Sp}(2n,\mathbb{R})$ that contain $\phi_n(\Gamma)$.

If n is odd, $\phi_n(\Gamma) < \mathrm{SU}(I_n, {}^-; \mathcal{O}_\Gamma)$, up to wide commensurability. Furthermore, up to wide commensurability, this is the only lattice of $\mathrm{Sp}(2n,\mathbb{R})$ that contains $\phi_n(\Gamma)$.

Proof of Proposition G. The quaternion algebra of which $\mathcal{O}_\Gamma$ is an order is of the form $(a,b)_F$ for some $a, b \in F^\times$. The 1-cocycle $I_n \otimes T^{a,b} : \mathrm{Gal}(\overline{\mathbb{Q}}/F) \to \mathrm{PSp}(2n,\overline{\mathbb{Q}})$ is ϕ_n-compatible with $T^{a,b}$. Hence $\phi_n({}_{T^{a,b}}\mathrm{SL}_2(F)) < {}_{I_n\otimes T^{a,b}}\mathrm{Sp}_{2n}(F)$. By Proposition 2.2.6 $\phi_n(\Gamma) < {}_{I_n\otimes T^{a,b}}\mathrm{Sp}_{2n}(\mathcal{O}_F)$ up to wide commensurability. For any $M \in \mathrm{SL}(2n,\overline{\mathbb{Q}})$

$$\begin{aligned}
&M \in {}_{I_n\otimes T^{a,b}}\mathrm{Sp}_{2n}(F)\\
\Leftrightarrow &(I_n \otimes T^{a,b}_\sigma)\sigma(M)(I_n \otimes T^{a,b}_\sigma)^{-1} = M\\
\Leftrightarrow &\begin{cases} M \in \mathrm{M}_n({}_{T^{a,b}}\mathrm{M}_2(F)) \text{ and} \\ M^\top K_n M = K_n \end{cases}\\
\Leftrightarrow &\begin{cases} M \in \mathrm{M}_n({}_{T^{a,b}}\mathrm{M}_2(F)) \text{ and} \\ \overline{M}^t M = I_{2n} \end{cases}
\end{aligned}$$

since $\overline{M}^t = (K_n M K_n^{-1})^\top$, where ${}^-$ is the conjugation on ${}_{T^{a,b}}\mathrm{M}_2(F)$ and M^t is the transposition of M viewed as an n-by-n matrix. It follows that ${}_{I_n\otimes T^{a,b}}\mathrm{Sp}_{2n}(\mathcal{O}_F)$ is widely commensurable with $\mathrm{SU}(I_n, {}^-; \mathcal{O}_\Gamma)$. This proves one of the two implications for n odd.

Suppose that n is even. Let $(c,d)_F$ be a quaternion algebra over F. Let $(a_{st}) \in \mathrm{H}^2(\mathrm{Gal}(\overline{\mathbb{Q}}/F), \mu_2)$ be a 2-cocycle that corresponds to the quaternion algebra $(c,d)_F$. By Lemma 5.2.1 there exists a 1-cocycle $\eta : \mathrm{Gal}(\overline{\mathbb{Q}}/F) \to \mathrm{PO}(I_n, \overline{\mathbb{Q}})$ such that $\partial_n(\eta) = \delta_2(T^{c,d})^{-1}(a_{st})$. By Lemma 5.2.2 $\delta_{2n}(\eta \otimes T^{c,d}) = (a_{st})$. By Lemma 5.1.2, $\eta \otimes T^{c,d}$ is ϕ_n-compatible with $T^{a,b}$. By Proposition 2.2.6, $\phi(\Gamma) < {}_{\eta\otimes T^{c,d}}\mathrm{Sp}_{2n}(\mathcal{O}_F)$ up to finite index. We now determine ${}_{\eta\otimes T^{c,d}}\mathrm{Sp}_{2n}(\mathcal{O}_F)$.

Consider the map

$$\begin{aligned}
(\eta \otimes \xi)(I_n \otimes T^{c,d})^{-1} : \mathrm{Gal}(\overline{\mathbb{Q}}/F) &\to {}_{I_n\otimes T^{c,d}}\mathrm{PSp}_{2n}(\overline{\mathbb{Q}})\\
\sigma &\mapsto (\eta(\sigma) \otimes \xi(\sigma))(I_n \otimes T^{c,d}_\sigma)^{-1}.
\end{aligned}$$

where we endow ${}_{I_n\otimes T^{c,d}}\mathrm{PSp}_{2n}(\overline{\mathbb{Q}})$ with the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$ defined by

$$\sigma \cdot M = (I_n \otimes T^{c,d}_\sigma)\sigma(M)(I_n \otimes T^{c,d}_\sigma)^{-1}.$$

It is a 1-cocycle with respect to this action. Moreover, it lifts to a 1-cocycle in ${}_{I_n\otimes T^{c,d}}\mathrm{Sp}_{2n}(\overline{\mathbb{Q}})$ and thus is trivial, see Proposition 2.7 in [PR94]. Hence there exists $S \in \mathrm{Sp}(2n,\overline{\mathbb{Q}})$ such that for all $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/F)$

$$\eta(\sigma) \otimes T^{c,d}_\sigma = S^{-1}(I_n \otimes T^{c,d}_\sigma)\sigma(S).$$

It follows that $M \in {}_{\eta\otimes T^{c,d}}\mathrm{Sp}_{2n}(F)$ if and only if

$$\begin{aligned}
&\begin{cases} SMS^{-1} \in \mathrm{M}_n({}_{T^{c,d}}\mathrm{M}_2(F)) \text{ and} \\ (SMS^{-1})^\top S^{-\top} K_n S^{-1}(SMS^{-1}) = S^{-\top} K_n S^{-1} \end{cases}\\
\Leftrightarrow\ &SMS^{-1} \in {}_{I_n\otimes T^{c,d}}\mathrm{Sp}_{2n}(F)
\end{aligned}$$

since $S^\top K_n S = K_n$. Finally ${}_{\eta\otimes T^{c,d}}\mathrm{Sp}_{2n}(\mathcal{O}_F)$ is widely commensurable with $\mathrm{SU}(I_n, {}^-; \mathcal{O})$ for $\mathcal{O}$ an order in $(c,d)_F$. Note that the latter is a lattice of $\mathrm{Sp}(2n,\mathbb{R})$ if and only if $(c,d)_F$ splits at exactly one real place of F.

Conversely, suppose that an Λ is an arithmetic subgroup of $\mathrm{Sp}(2n,\mathbb{R})$ that contains $\phi_n(\Gamma)$. Since $\mathrm{Sp}(2n,\mathbb{R})$ is simple, Λ is widely commensurable with ${}_\zeta\mathrm{Sp}_{2n}(\mathcal{O}_L)$ for L a number field and $\zeta : \mathrm{Gal}(\overline{\mathbb{Q}}/L) \to$

$\mathrm{PSp}_{2n}(\overline{\mathbb{Q}})$ a 1-cocycle. By Proposition 3.2.3 we can assume that $L = F$. We show that ζ is ϕ_n-compatible with $T^{a,b}$. For every $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/F)$ denote by

$$\phi_n^\sigma : \mathrm{SL}_2(\overline{\mathbb{Q}}) \to \mathrm{Sp}(2n, \overline{\mathbb{Q}}),\ g \mapsto \zeta(\sigma) \circ \sigma \circ \phi_n \circ \sigma^{-1}((T_\sigma^{a,b})^{-1} g T_\sigma^{a,b}).$$

This is an algebraic morphism that coincides with ϕ_n on a finite-index subgroup of Γ. Since any finite-index subgroup of Γ is Zariski-dense in $\mathrm{SL}_2(\overline{\mathbb{Q}})$, $\phi_n = \phi_n^\sigma$. This means that $\tau_n({}_{T^{a,b}}\mathrm{SL}_2(F)) < {}_\eta\mathrm{Sp}_{2n}(F)$. Lemma 5.1.2 concludes the proof. □

It implies Proposition E together with Proposition 2.4.7 and Lemma 3.2.3.

5.3 Commutator of a diagonal element

The representations that we will build in Section 5.5 will be bendings of a maximal diagonal representations ρ along a simple closed curve γ. This requires to find such a γ with big enough centralizer in Λ. In this section, we show that for suitable Λ, diagonal elements have a big centralizer in Λ.

Let $\lambda > 1$ and $D = \mathrm{Diag}(\lambda, \lambda^{-1}) \in \mathrm{SL}(2, \mathbb{R})$. The commutator of $\phi_n(D)$ in $\mathrm{M}_{2n}(\mathbb{C})$ is the set of matrices of the form

$$\begin{pmatrix} a_{11} & 0 & a_{12} & 0 & \dots & a_{1n} & 0 \\ 0 & b_{11} & 0 & b_{12} & \dots & 0 & b_{1n} \\ a_{21} & 0 & a_{22} & 0 & \dots & a_{2n} & 0 \\ 0 & b_{21} & 0 & b_{22} & \dots & 0 & b_{2n} \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ a_{n1} & 0 & a_{n2} & 0 & \dots & a_{nn} & 0 \\ 0 & b_{n1} & 0 & b_{n2} & \dots & 0 & b_{nn} \end{pmatrix} \tag{5.3}$$

where $(a_{ij}), (b_{ij}) \in \mathrm{M}_n(\mathbb{C})$.

Lemma 5.3.1. *The commutator of $\phi_n(D)$ in $\mathrm{Sp}(K_n, \mathbb{C})$ is the group of matrices of the form (5.3) where $(a_{ij}) \in \mathrm{GL}(n, \mathbb{C})$ and $(b_{ij}) = (a_{ij})^{-\top}$.*

Proof. Let $M \in \mathrm{Sp}(K_n, \mathbb{C})$ that commutes with $\phi_n(D)$. Consider P the matrix defined on the canonical basis of $\mathbb{C}^{2n}$ by $Pe_k = e_{2k-1}$ and $Pe_{n+k} = e_{2k}$ for all $1 \leq k \leq n$. Then

$$P^{-1}MP = \begin{pmatrix} M_1 & \\ & M_2 \end{pmatrix}$$

with $M_1, M_2 \in \mathrm{M}_n(\mathbb{C})$. The matrix M preserves K_n, i.e. $M^\top K_n M = K_n$, if and only if $P^{-1}MP$ preserves $P^\top K_n P$. Since

$$P^\top K_n P = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix},$$

$M_2 = M_1^{-\top}$. □

Let $(a, b)_F$ be a quaternion algebra that splits at exactly one real place of F. Let $\eta : \mathrm{Gal}(\overline{\mathbb{Q}}/F) \to \mathrm{PO}(I_n, \overline{\mathbb{Q}})$ be a 1-cocycle.

Lemma 5.3.2. *For all $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/F)$, the commutator of $\phi_n(D)$ in $\mathrm{Sp}(2n, \overline{\mathbb{Q}})$ is stable under conjugation by $\eta(\sigma) \otimes T_\sigma^{a,b}$.*

Proof. Fix $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/F)$. The subgroup $\mathrm{GL}(n, \overline{\mathbb{Q}})$ is stable by $\eta(\sigma) \otimes I_2$ since the latter is an element of $\mathrm{PGL}(n, \overline{\mathbb{Q}})$, seen in $\mathrm{PSp}(2n, \overline{\mathbb{Q}})$. Moreover, $\mathrm{GL}(n, \overline{\mathbb{Q}})$ is stable by $I_n \otimes T_\sigma^{a,b}$ as shown by direct computations. Hence $\mathrm{GL}(n, \overline{\mathbb{Q}})$ is stable by their product. □

Consider the F-algebraic group $\mathbb{H}$ that associated to any F-algebra R the subgroup of $\mathrm{Sp}(2n, \mathrm{R})$ that consists of matrices of the form (5.3) with $\det((a_{ij})_{ij}) = 1$. It is isomorphic to the F-algebraic group SL_n. The main result of this section is the following lemma.

Lemma 5.3.3. *Let σ_0 be the only real place of F over which $(a,b)_F$ splits. Then $\mathbb{H}(\mathbb{R}^{\sigma_0}) \cap_{\eta \otimes T^{a,b}} \mathrm{Sp}_{2n}(\mathcal{O}_F)$ is a lattice in $\mathbb{H}(\mathbb{R}^{\sigma_0})$.*

Proof. For all $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/F)$, $\eta(\sigma) \otimes T^{a,b}_\sigma$ induces an automorphism of $\mathbb{H}(\overline{\mathbb{Q}})$. Hence $\eta \otimes T^{a,b}$ defines a 1-cocycle in $\mathrm{Aut}(\mathbb{H}(\overline{\mathbb{Q}}))$ and thus correspond to an F-form of $\mathbb{H}$. Denote the latter by $\mathbb{G}$. Since $\mathbb{G}$ is semisimple, Borel-Harish-Chandra Theorem [BHC62] implies that $\mathbb{G}(\mathcal{O}_F)$ is a lattice of

$$\prod_{\sigma : F \hookrightarrow \mathbb{R}} \mathbb{G}(\mathbb{R}^\sigma).$$

For all $\sigma \neq \sigma_0$, $\mathbb{G}(\mathbb{R}^\sigma)$ is a closed subgroup of ${}_{\eta \otimes T^{a,b}}\mathrm{Sp}_{2n}(\mathbb{R}^\sigma)$ which is a compact Lie group. Lemma 2.2.8 implies that $\mathbb{G}(\mathcal{O}_F)$ is a lattice of $\mathbb{G}(\mathbb{R}^{\sigma_0}) = \mathbb{H}(\mathbb{R}^{\sigma_0})$. □

5.4 Reduction to finite fields

To prove that the construction given in the next section gives rise to infinitely many mapping class group orbits of representations, we again use the Strong Approximation Theorem, Theorem 2.5.5. However, the argument used in the proof of Theorem B do not suffice to distinguish mapping class group orbits in this setting. Let Γ be an arithmetic subgroup of $\mathrm{SL}(2,\mathbb{R})$. The key point of the proof of Theorem B is that $\mathrm{Tr}(\tau_n(\Gamma))$ is not surjective modulo $\mathcal{P}$ for infinitely many prime ideals $\mathcal{P}$ of $\mathcal{O}_F$. However, $\mathrm{Tr}(\phi_n(\Gamma))$ is surjective modulo $\mathcal{P}$ for almost all $\mathcal{P}$. This is why we need a finer control on the reduction of $\phi_n(\Gamma)$ modulo prime ideals.

Let $\mathcal{O}$ be an order of a quaternion algebra A over a number field F such that A splits at exactly one real place of F. Let $\Gamma < \mathrm{SU}(I_n, {}^-; \mathcal{O})$ be a Zariski-dense subgroup. Consider the F-algebraic group defined as a functor from F-algebras to groups by

$$\mathrm{R} \mapsto \mathrm{SU}(I_n, {}^-; A \otimes_F \mathrm{R}).$$

It is a connected almost simple and simply connected algebraic group. Hence the Strong Approximation Theorem, Theorem 2.5.5, implies that there exists $a \in \mathcal{O}_F$ such that Γ surjects $\mathrm{Sp}(2n, \mathcal{O}_F/\mathcal{P})$ for any prime ideal $\mathcal{P}$ not containing a.

Let $\mathcal{P}$ be a prime ideal not containing a. Denote by $\mathbb{F}_q$ its residue field and by $\pi : \mathrm{SU}(I_n, {}^-; \mathcal{O}) \to \mathrm{Sp}(2n, \mathbb{F}_q)$ the reduction map. It is not true that two Zariski-dense subgroups Γ_1 and Γ_2 of $\mathrm{SU}(I_n, {}^-; \mathcal{O})$ that are conjugate under $\mathrm{Sp}(2n,\mathbb{R})$ satisfy $\pi(\Gamma_1) \simeq \pi(\Gamma_2)$. For instance, one can conjugate the principal congruence subgroup of level 2 of $\mathrm{SL}(2,\mathbb{Z})$ into a subgroup which is no longer trivial modulo 2. Nevertheless, we have the following.

Lemma 5.4.1. *Let $\Gamma < \mathrm{SU}(I_n, {}^-; \mathcal{O})$ be a Zariski-dense subgroup and let $g \in \mathrm{GL}(2n,\mathbb{C})$ such that $g\Gamma g^{-1} < \mathrm{SU}(I_n, {}^-; \mathcal{O})$. If $\pi(\Gamma) = \mathrm{Sp}(2n,\mathbb{F}_q)$, then $\pi(g\Gamma g^{-1}) = \mathrm{Sp}(2n,\mathbb{F}_q)$.*

Proof. We first show that there is a scalar multiple of g that lies in $\mathrm{GL}(n,A)$. Consider the following F-algebra

$$F\Gamma = \{\sum a_i \gamma_i \mid \gamma_i \in \Gamma\} \subset \mathrm{M}_n(A)$$

where the sums are finite. Proposition 2.2 in [Bas80] shows that $F\Gamma$ is a central simple algebra of dimension $4n^2$. Hence $F\Gamma = \mathrm{M}_n(A)$. It follows that the conjugation by g preserves $\mathrm{M}_n(A)$. Since all automorphisms of $\mathrm{M}_n(A)$ are inner, see Corollary 2.9.9 in [MR03], we can assume that $g \in \mathrm{GL}(n,A)$.

Denote by $F_\mathcal{P}$ the completion of F at $\mathcal{P}$ for the valuation $\nu_\mathcal{P}$ and $\mathrm{R}_\mathcal{P}$ its ring of integers. We can view Γ and g in $\mathrm{GL}(n, A \otimes_F F_\mathcal{P})$. Up to scaling, we can assume that $g \in \mathrm{M}_n(\mathcal{O} \otimes_{\mathcal{O}_F} \mathrm{R}_\mathcal{P})$ and that g is non-trivial modulo $\mathcal{P}$. Let Nrd denote the reduced norm of $\mathrm{GL}(n, A \otimes_F F_\mathcal{P})$. We claim that $\nu_\mathcal{P}(\mathrm{Nrd}(g)) = 0$.

There exists $h \in \mathrm{M}_n(\mathcal{O} \otimes_{\mathcal{O}_F} \mathrm{R}_\mathcal{P})$ such that $gh = \mathrm{Nrd}(g)$. This follows from the fact that there exists a polynomial P with coefficients in $\mathrm{R}_\mathcal{P}$ with constant term $\mathrm{Nrd}(g)$ such that $\mathrm{P}(g) = 0$. Suppose that $\nu_\mathcal{P}(\mathrm{Nrd}(g)) > 1$. Let $k \in \mathbb{N}$ such that h is trivial modulo $\mathcal{P}^k$ but not modulo $\mathcal{P}^{k+1}$. Reducing $g\Gamma h \subset \mathrm{Nrd}(g)\mathrm{M}_n(\mathcal{O} \otimes_{\mathcal{O}_F} \mathrm{R}_\mathcal{P})$ modulo $\mathcal{P}$ we have

$$\overline{g}\,\mathrm{Sp}(2n,\mathbb{F}_q)\overline{h} = 0.$$

Since $\overline{g} \neq 0$, $\overline{h} = 0$. Hence $k \geq 1$.

From $\mathrm{Nrd}(g)^{2n-2}gh = \mathrm{Nrd}(h)$, we see that viewed in a splitting field of $A \otimes_F F_{\mathcal{P}}$, $\mathrm{Nrd}(g)^{2n-2}g$ is the cofactor matrix of h. If h is trivial modulo $\mathcal{P}^k$, its cofactor matrix is trivial modulo $\mathcal{P}^{(2n-1)k}$. Hence $\nu_{\mathcal{P}}(\mathrm{Nrd}(g)) > k$. Let ω be a uniformizer of $\mathrm{R}_{\mathcal{P}}$. Reducing the equation

$$g\Gamma\frac{h}{\omega^k} \subset \frac{\mathrm{Nrd}(g)}{\omega^k}\mathrm{M}_n(\mathcal{O} \otimes_{\mathcal{O}_F} \mathrm{R}_{\mathcal{P}})$$

modulo $\mathcal{P}$, we have that $\frac{h}{\omega^k}$ is trivial. This is a contraction.

Hence $\nu_{\mathcal{P}}(\mathrm{Nrd}(g)) = 0$, i.e. the reduction of g modulo $\mathcal{P}$ is invertible. This implies that $\pi(g\Gamma g^{-1}) = g\pi(\Gamma)g^{-1}$ which concludes the proof. □

5.5 Proof of Theorem F

Write the canonical decomposition

$$\mathbb{R}^{2n} = \mathrm{V}_1 \oplus \ldots \oplus \mathrm{V}_n$$

where $\mathrm{V}_i = \mathbb{R}^2$. Denote by Δ the subgroup of $\mathrm{Sp}(2n, \mathbb{R})$ that preserves this decomposition. Explicitly

$$\Delta = \left\{ \begin{pmatrix} A_1 & & \\ & \ddots & \\ & & A_n \end{pmatrix} \middle| \; A_i \in \mathrm{SL}(2, \mathbb{R}) \right\}.$$

Lemma 5.5.1. *Let H be the Zariski closure of a maximal representation in $\mathrm{Sp}(2n, \mathbb{R})$. Suppose that $\Delta < H$ and that H acts irreducibly on $\mathbb{R}^{2n}$. Then $H = \mathrm{Sp}(2n, \mathbb{R})$.*

Proof. Corollary 4 in [BIW09] states that H is reductive and its associated symmetric space is Hermitian. Denote by $\mathfrak{h}$ the Lie algebra of H that we see embedded in $\mathfrak{sp}(2n, \mathbb{R})$. Since $\Delta < H$, the centralizer of H in $\mathrm{Sp}(2n, \mathbb{R})$ has to be discrete. Hence $\mathfrak{h}$ has no center and so is semi-simple.

Write $\mathfrak{h} = \mathfrak{k} \oplus \mathfrak{h}_0$ where $\mathfrak{k}$ is a maximal compact ideal in $\mathfrak{h}$. Consider a $\mathfrak{sl}_2(\mathbb{R})$-subalgebra of $\mathfrak{h}$. We denote by π the composition of the inclusion with the projection to $\mathfrak{k}$

$$\mathfrak{sl}_2(\mathbb{R}) \hookrightarrow \mathfrak{k} \oplus \mathfrak{h}_0 \to \mathfrak{k}.$$

The kernel of π cannot be trivial since a compact algebra does not contain $\mathfrak{sl}_2(\mathbb{R})$. Since $\mathfrak{sl}_2(\mathbb{R})$ is simple, π has to be trivial. Hence $\mathfrak{sl}_2(\mathbb{R})$ is a subalgebra of $\mathfrak{h}_0$. The Lie algebra of Δ is then also a subalgebra of $\mathfrak{h}_0$. This implies that $\mathfrak{k}$ centralizes the Lie algebra of Δ, so $\mathfrak{k}$ is trivial.

Corollary 4 in [BIW09] shows that the inclusion $\mathfrak{h} \hookrightarrow \mathfrak{sp}(2n, \mathbb{R})$ is tight, i.e. the norm of the Kähler form on the symmetric space of $\mathrm{Sp}(2n, \mathbb{R})$ and the norm of its pullback are equal. Tight embeddings are classified by Hamlet and Pozzetti in [HP14] §5.2. In particular, Table 6 and its interpretation show that if $\mathfrak{h}$ were non-simple then it would act reducibly on $\mathbb{R}^{2n}$. Knowing that H contains Δ, the classification implies that $H = \mathrm{Sp}(2n, \mathbb{R})$. □

We can now prove Theorem F.

Proof of Theorem F. Let $n \geq 2$. Let Λ be a lattice in $\mathrm{Sp}(2n, \mathbb{R})$ not widely commensurable with $\mathrm{Sp}(4k + 2, \mathbb{Z})$ for any k. By Proposition 2.4.7 and Proposition G, there exists a totally real number field F and a quaternion division algebra A over F such that A splits over exactly one real place ι of F and an order $\mathcal{O}$ of A satisfying

$$\phi(\mathcal{O}^1) < \Lambda.$$

Since A is a division algebra, $\mathcal{O}^1$ embeds as a cocompact lattice in $\mathrm{SL}(2, \mathbb{R})$. Let $S = \mathbb{H}^2/\mathcal{O}^1$. Up to finite cover, it is a closed surface.

Up to isomorphism, we can write $A = (a, b)_F$ with $a, b \in \mathcal{O}_F$ such that a and b are positive at ι. Hence, up to finite index, we can suppose that A^1 is embedded in $\mathrm{SL}(2, \mathbb{R})$ with image in

$$\left\{ \begin{pmatrix} x_0 + \sqrt{a}x_1 & \sqrt{b}x_2 + \sqrt{ab}x_3 \\ \sqrt{b}x_2 - \sqrt{ab}x_3 & x_0 - \sqrt{a}x_1 \end{pmatrix} \mid x_i \in F, \; \det = 1 \right\}.$$

Lemma 4.2.3 show that there exists a simple closed curve γ on S with image in $\mathrm{SL}(2,\mathbb{R})$ which is diagonal. Note that $A^1 = {}_{T^{a,b}}\mathrm{SL}_2(F)$ and thus $\Lambda = {}_{\eta\otimes T^{a,b}}\mathrm{Sp}_{2n}(\mathcal{O}_F)$.

Let $g \geq 2$ and $j : \pi_1(S) \to \mathcal{O}^1$ be a injective morphism. There exists automorphism φ_i of $\pi_1(S)$, $1 \leq i \leq n$, that fixes γ such such that the representations $j_i := j \circ \varphi_i$ are pairwise non $\mathrm{PGL}(2,\mathbb{R})$-conjugate. These automorphism can be induced by Dehn twists along a simple closed curve disjoint from γ. Define

$$\rho : \pi_1(S) \xrightarrow{\prod j_i} \mathrm{SL}(2,\mathbb{R})^n \hookrightarrow \mathrm{Sp}(K_n,\mathbb{R})$$

where the last map is the diagonal embedding. The image of ρ is in Λ and has Zariski-closure $\mathrm{SL}(2,\mathbb{R})^n$ by Lemma 4.4.1.

Let $\mathrm{SL}(n,\mathbb{R})$ be the subgroup of $\mathrm{Sp}(2n,\mathbb{R})$ that consists of matrices of the form (5.3) with $\det((a_{ij})_{ij}) = 1$. By Lemma 5.3.3, the centralizer of $\rho(\gamma)$ in $\Lambda \cap \mathrm{SL}(n,\mathbb{R})$ is a lattice in $\mathrm{SL}(n,\mathbb{R})$. Hence by Borel Density Theorem, Theorem 2.1.8, there exists $B \in \Lambda \cap \mathrm{SL}(n,\mathbb{R})$ such that if $I \subset \{1,\dots,n\}$ satisfies

$$B(\oplus_{i\in I}\mathrm{V}_i) = \oplus_{j\in J}\mathrm{V}_j$$

for some $J \subset \{1,\dots,n\}$ then $I = \varnothing$ or $I = \{1,\dots,n\}$. We will deform the representation ρ by B using a bending deformation. See Section 4.1 for the definition of the bend representation ρ_B. Since B can be continuously deformed into I_n within $\mathrm{SL}(n,\mathbb{R})$, ρ_B can be continuously deformed into ρ.

We show that ρ_B acts irreducibly on $\mathbb{R}^{2n}$. Let $\mathrm{V} \subset \mathbb{R}^{2n}$ be an invariant subset of ρ_B. Since the Zariski-closure of ρ_B contains Δ, V has to be invariant under Δ. Hence there exists $I \subset \{1,\dots,n\}$ such that $\mathrm{V} = \oplus_{i\in I}\mathrm{V}_i$. If γ is separating, $B\mathrm{V}$ has to be invariant by Δ. If γ is non-separating, V has to be invariant by B. In any case, we conclude that V is either $\{0\}$ or $\mathbb{R}^{2n}$. Hence ρ_B acts irreducibly on $\mathbb{R}^{2n}$. Its Zariski-closure contains Δ so Lemma 5.5.1 implies that ρ_B is Zariski-dense in $\mathrm{Sp}(2n,\mathbb{R})$.

We claim that the sequence of representations $(\rho_{B^k})_{k\geq 1}$ give rise to infinitely many $\mathrm{MCG}(S)$-orbits of representations. Suppose not. Denote by Γ_k the image of ρ_{B^k}. There exists $k_1,\dots,k_l$ such that the image every Γ_k is conjugated to the image of Γ_{k_i} for some $1 \leq i \leq l$. By the Strong Approximation Theorem, see Theorem 4.3.4, there exists a finite set Ω of prime ideals of $\mathcal{O}_F$ such that for every $1 \leq i \leq l$ and every prime ideal $\mathcal{P} \notin \Omega$, the reduction of Γ_{k_i} modulo $\mathcal{P}$ is $\mathrm{Sp}(2n,\mathcal{O}_F/\mathcal{P})$. By Lemma 5.4.1, the reduction of every Γ_k modulo $\mathcal{P} \notin \Omega$ is surjective. Pick $\mathcal{P} \notin \Omega$ and k such that B^k is trivial modulo $\mathcal{P}$. Then the reduction of ρ_{B^k} modulo $\mathcal{P}$ is equal to the reduction of ρ and thus is not surjective. This is a contradiction. □

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