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Möbius spaces and large-scale geometry

Georg Grützner

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Möbius spaces and large-scale geometry

Espaces de Möbius et géométrie à grande échelle

Thèse de doctorat de l'université Paris-Saclay

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Thèse soutenue à Paris-Saclay, le 17 mai 2023, par

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Titre : Espaces de Möbius et géométrie à grande échelle

Mots clés : Mathématiques fondamentales, géométrie, théorie géométrique des groupes, algèbres d'opérateurs.

Résumé : Ce manuscrit développe la géométrie et l'analyse des espaces de Möbius dans les deux directions suivantes : Les espaces de Hilbert attachés aux espaces de Möbius, et la géométrie de Möbius à grande échelle.

En ce qui concerne le premier point ci-dessus, nous construisons des espaces de Sobolev H_d^α sur les $(1/2 + \alpha/Q)$ -densités associés avec une structure de Möbius \mathcal{M} de dimension Q . Nous montrons que \mathcal{M} a une constante d'Ahlfors-David uniforme. Nous utilisons cette observation pour montrer que les normes sur H_d^α sont comparables sur une grande classe de fonctions pour tous $(0 < \alpha < Q/2)$. Il s'agit d'un résultat partiel d'un programme visant à construire et étu-

dier des représentations uniformément bornées pour tous les groupes hyperboliques.

Dans la deuxième partie, nous développons la géométrie de Möbius asymptotique comme outil pour passer de la théorie discrète à la théorie continue. Nous montrons que sous de telles applications, certaines notions de dimension à grande échelle augmentent. Cela signifie, par exemple, qu'un espace $CAT(0)$ qui admet un plongement asymptotiquement Möbius dans un espace hyperbolique doit lui-même être hyperbolique. Nous construisons également un plongement asymptotiquement Möbius d'un groupe de Heisenberg de dimension infinie dans un espace de Hilbert.

Title : Möbius spaces and large scale geometry

Keywords : Pure mathematics, geometry, geometric group theory, operator algebras

Abstract : This manuscript develops the geometry and analysis of Möbius spaces in two directions : Hilbert spaces attached to Möbius spaces, and asymptotic Möbius geometry.

In Part 1, we construct Sobolev spaces H_d^α on $(1/2 + \alpha/Q)$ -densities associated with a Q -dimensional Möbius structure \mathcal{M} . We show that \mathcal{M} has a uniform Ahlfors-David constant and use this observation to show that the norms on H_d^α are comparable on a large class of functions for all $(0 < \alpha < Q/2)$. This is a partial result of a program to

construct and study uniformly bounded representations for all hyperbolic groups.

In Part 2, we study a class of large-scale Möbius maps (AM -maps) as a tool to move from the discrete to the continuous theory. We show that under such maps some large-scale notions of dimension increase. This means, for example, that a $CAT(0)$ -space which admits an AM -mapping to a hyperbolic space must itself be hyperbolic. We also construct an AM -mapping from an infinite dimensional Heisenberg group to Hilbert space.

Résumé

Une structure de Möbius \mathcal{M} sur un ensemble Z est une collection de métriques définissant les mêmes birapports. Une telle structure a une dimension de Hausdorff Q bien définie. Ce manuscrit développe la géométrie et l'analyse des espaces de Möbius dans les deux directions suivantes :

1. Les espaces de Hilbert attachés aux espaces de Möbius, et
2. la géométrie de Möbius à grande échelle.

En ce qui concerne le premier point ci-dessus, nous montrons qu'une structure de Möbius \mathcal{M} de dimension Q a une constante d'Ahlfors-David uniforme. Ainsi, de nombreux théorèmes classiques de l'analyse harmonique sur \mathbb{R}^n admettent une formulation invariante de Möbius dans le contexte de la géométrie de Möbius.

Nous utilisons cette observation pour montrer que l'opérateur de Knapp-Stein

$$(I_d^\alpha u_d)(x) = \int \frac{u_d(y)}{d(x,y)^{Q-\alpha}} d\mu_d(y), \quad (0 < \alpha < \frac{Q}{2})$$

est un opérateur continu sur l'espace L^2 à poids $(\frac{d'}{d})^\alpha$, muni d'une norme indépendante de d et d' dans \mathcal{M} . Ici, $\frac{d'}{d}(x) = \lim_{y \rightarrow x} \frac{d'(x,y)}{d(x,y)}$ est le facteur de conformité entre d et d' . Une fois ce résultat établi, nous construisons un espace de Sobolev $\mathcal{H}_d^{-\alpha}$ sur les $(\frac{1}{2} + \frac{\alpha}{Q})$ -densités. Une s -densité est une application $u : d \mapsto u_d$ de \mathcal{M} à la classe $L^0(Z)$ des fonctions mesurables sur Z , telle que $u_d = (\frac{d'}{d})^{sQ} u_{d'}$. Si on pouvait montrer que la norme sur $\mathcal{H}_d^{-\alpha}$ était à distance uniforme de la norme sur $\mathcal{H}_{d'}^{-\alpha}$ pour tout $d, d' \in \mathcal{M}$, alors pour tout $\alpha \in (0, \frac{Q}{2})$, $\mathcal{H}_d^{-\alpha}$ donnerait des représentations uniformément bornées de groupes agissant sur \mathcal{M} par des transformations de Möbius. En particulier, ces représentations interpoleraient entre la représentation quasirégulière à $\alpha = 0$ et la représentation sur les mesures à $\alpha = 1$, qui contient un cocycle propre. Nous obtenons un résultat partiel dans cette direction : nous démontrons que les normes $\|\cdot\|_d$ sur $\mathcal{H}_d^{-\alpha}$ sont comparables sur une grande classe de fonctions.

Dans la deuxième partie, nous développons la géométrie de Möbius asymptotique comme outil pour passer de la théorie discrète à la théorie continue. Une application entre espaces métriques est un plongement asymptotiquement Möbius si elle préserve presque les birapports des points qui sont à une grande distance les uns des autres. Nous montrons que sous de telles applications, certaines notions de dimension à grande échelle augmentent. Cela signifie, par exemple, qu'un espace $CAT(0)$ qui admet un plongement asymptotiquement Möbius dans un espace hyperbolique doit lui-même être hyperbolique. Nous construisons également un plongement asymptotiquement Möbius d'un groupe de Heisenberg de dimension infinie dans un espace de Hilbert.

Mots clés : Géométrie de Möbius, théorie géométrique des groupes, algèbres d'opérateurs.

Abstract

A Möbius structure \mathcal{M} on a set Z is a collection of metrics defining the same metric cross-ratios. Such a structure has a well-defined Hausdorff dimension Q . This manuscript develops the geometry and analysis of Möbius spaces in two directions:

1. Hilbert spaces attached to Möbius spaces, and
2. asymptotic Möbius geometry.

In Part 1, we show that a Möbius structure \mathcal{M} of dimension Q has a uniform Ahlfors-David constant. In summary, many classical theorems of harmonic analysis on \mathbb{R}^n admit a Möbius invariant formulation in the context of Möbius geometry.

We use this observation to show that the Knapp-Stein operator

$$(I_d^\alpha u_d)(x) = \int \frac{u_d(y)}{d(x,y)^{Q-\alpha}} d\mu_d(y), \quad (0 < \alpha < \frac{Q}{2})$$

is a continuous operator on the weighted L^2 -space $L^2((\frac{d'}{d})^\alpha d\mu_d)$, with a norm independent of d and d' in \mathcal{M} . Here $\frac{d'}{d}(x) = \lim_{y \rightarrow x} \frac{d'(x,y)}{d(x,y)}$ is the conformal factor between d and d' . From there we construct a Sobolev space $\mathcal{H}_d^{-\alpha}$ on $(\frac{1}{2} + \frac{\alpha}{Q})$ -densities. An s -density is a map $u : d \mapsto u_d$ from \mathcal{M} to the class $L^0(Z)$ of measurable functions on Z , such that $u_d = (\frac{d'}{d})^{sQ} u_{d'}$. If one could show that the norm on $\mathcal{H}_d^{-\alpha}$ is within uniform distance to the norm on $\mathcal{H}_{d'}^{-\alpha}$ for every $d, d' \in \mathcal{M}$, then for any $\alpha \in (0, \frac{Q}{2})$, $\mathcal{H}_d^{-\alpha}$ would give uniformly bounded representations of groups acting on \mathcal{M} by Möbius transformations. In particular, these representations would interpolate between the quasiregular representation at $\alpha = 0$ and the representation on measures at $\alpha = 1$, which contains a proper cocycle. We get a partial result in this direction: we prove that the norms $\|\cdot\|_d$ on $\mathcal{H}_d^{-\alpha}$ are comparable on a large class of functions.

In Part 2, we study a class of large-scale Möbius maps as a tool to move from the discrete to the continuous theory. Roughly speaking, a map between metric spaces is an asymptotic-Möbius (AM) map if it almost preserves the cross-ratio of points that are mutually far apart. We show that under such maps some large-scale notions of dimension increase. This means, for example, that a $CAT(0)$ -space which admits an AM-mapping to a hyperbolic space must itself be hyperbolic. We also construct an AM-mapping from an infinite dimensional Heisenberg group to Hilbert space.

Keywords: Möbius geometry, geometric group theory, operator algebras.

To my parents

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Chapitre 1

Introduction (Français)

Selon Buyalo et Schröder [1], une structure de Möbius \mathcal{M} sur un ensemble Z est une collection de métriques définissant les mêmes birapports. Plus précisément, on considère des quadruples (x, y, z, w) de points dans Z où aucune entrée n'apparaît trois ou quatre fois. Un tel quadruple *admissible* est associé à un *triplet de rapports croisés*

$$[x, y, z, w]_d = (d(x, y)d(z, w) : d(x, z)d(y, w) : d(x, w)d(y, z)) \in \mathbb{R}P^2. \quad (1.1)$$

Deux métriques d et d' sur Z sont dites *Möbius-équivalentes* si, pour chaque quadruple admissible, leurs triples de rapports croisés respectifs coïncident.

Une *structure de Möbius* sur un ensemble Z est une classe \mathcal{M} de métriques sur Z qui sont Möbius-équivalentes par paire.

Voici des exemples d'ensembles présentant des structures de Möbius naturelles :

1. La sphère ronde,
2. les bords des espaces symétriques de rang un,
3. les bords d'espaces de type $CAT(-1)$ [2],
4. les bords des espaces métriques fortement hyperboliques [3],
5. les bords des espaces symétriques de rang supérieur et des immeubles euclidiens [4], [5],
6. les bords de certains complexes cubiques $CAT(0)$ [5], [6].

La géométrie de Möbius présente une tension attrayante entre flexibilité et rigidité.

Les métriques Möbius-équivalentes sont conformes dans un sens très fort. Deux métriques d et d' dans une structure de Möbius sont conformes au sens où le rapport limite

$$\frac{d'}{d}(x) := \lim_{y \rightarrow x} \frac{d'(x, y)}{d(x, y)}$$

existe pour tout $x \in Z$ [3, Lemma 6]. De plus, pour tout $x \neq y \in Z$,

$$\frac{d'}{d}(x) \frac{d'}{d}(y) = \left(\frac{d'(x, y)}{d(x, y)} \right)^2.$$

On peut interpréter cette relation comme une propriété géométrique de la valeur moyenne.

Certaines propriétés flexibles de la géométrie conforme, comme l'uniformisation, persistent parfois. Cependant, la plupart de la théorie existante traite des propriétés de rigidité. Il est connu (Paulin) que les structures de Möbius aux bords des espaces à courbure négative codent la géométrie de ces espaces. Les homéomorphismes de quasi-Möbius entre les bords d'espaces métriques hyperboliques (sous des hypothèses peu restrictives), par exemple, s'étendent en des quasi-isométries [7].

Ce manuscrit développe la géométrie et l'analyse des espaces de Möbius dans deux directions

1. Les espaces de Hilbert attachés aux espaces de Möbius, et
2. la géométrie de Möbius à grande échelle.

Le deuxième point appartient à la partie rigidité de la théorie, tandis que le premier poursuit un rêve de flexibilité.

Les techniques utilisées ont été empruntées en partie 1 à l'analyse harmonique sur les groupes non-commutatifs et en partie 2 à la géométrie métrique, aux groupes de Lie nilpotents et aux espaces (de dimension infinie) $CAT(0)$.

Espaces de Möbius finis. Les bords des polyèdres euclidiens convexes peuvent être considérés comme des analogues finis des espaces de Möbius. Inspiré par le théorème de représentation conforme de Riemann¹ et le théorème d'uniformisation plus général de Poincaré et Koebe, j'ai démontré dans mon mémoire de master un théorème d'uniformisation pour les polyèdres convexes fermés dans l'espace euclidien de dimension 3 dans le sens suivant.

Theorem 1 *Chaque polyèdre convexe dans l'espace euclidien 3 a un cousin discrètement conformément équivalent inscrit dans une sphère ronde. Il est unique aux transformations de Möbius des sphères rondes près.*

Les techniques utilisées sont principalement issues de la topologie de basse dimension, en particulier la théorie de Teichmüller.

Donnons un bref résumé de ce que signifie l'équivalence conforme discrète entre polyèdres. Pour plus de détails, voir l'article [8]. Une surface triangulée consiste en une 2-variété topologique S , une subdivision finie de S en sous-ensembles (appelés faces), et des isométries des faces sur des triangles euclidiens. L'ensemble des sommets possède une collection de birapports, un pour chaque arête orientée et co-orientée. Deux surfaces triangulées sont Möbius-équivalentes s'il existe entre elles une correspondance préservant la subdivision qui préserve les birapports. Dans [9], Bobenko, Pinkall et Springborn attachent à une surface triangulée une surface hyperbolique complète d'aire finie et définissent deux surfaces triangulées comme étant discrètement conformément équivalentes si les surfaces hyperboliques correspondantes sont isométriques. L'équivalence de Möbius implique l'équivalence conforme discrète.

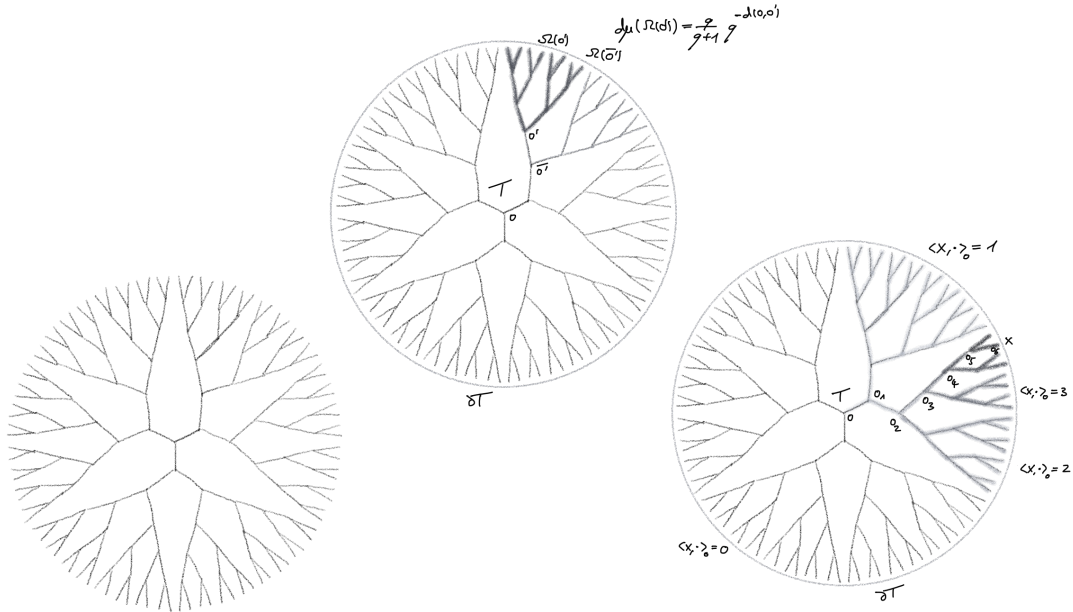
Les faces d'un polyèdre convexe dans l'espace euclidien de dimension 3 ne sont pas nécessairement des triangles. Néanmoins, il est démontré dans [8] que des triangulations de Delaunay arbitraires des faces d'un polyèdre euclidien convexe donné

1. Mentionné par Riemann dans sa dissertation : Grundlagen für allgemeine Theorie der Funktionen einer veränderlichen complexen Grösse, Göttingen, 1851

conduisent à des surfaces triangulées qui sont équivalentes du point de vue de la conformité discrète. Une triangulation de Delaunay d'une face est une triangulation dont les arêtes sont des géodésiques minimisantes et pour toute arête intérieure, la somme des deux angles des triangles euclidiens faisant face à cette arête est au plus égale à π . Par conséquent, on peut parler de polyèdres euclidiens convexes ayant des bords discrètement conformément équivalents.

Dans ce manuscrit, nous nous intéressons à la géométrie de Möbius continue. Le premier chapitre traite des applications aux questions de théorie des représentations des groupes hyperboliques. Le deuxième chapitre développe la géométrie de Möbius asymptotique comme outil pour passer de la théorie discrète à la théorie continue.

1.1 Espaces de Hilbert rattachés à des espaces de Möbius



Kazhdan a introduit la propriété (T) en 1968 comme outil pour démontrer la génération finie de réseau non uniformes dans les groupes de Lie simples de rang supérieur. Cette propriété a joué un rôle central en théorie des algèbres d'opérateurs et en théorie géométrique des groupes, voir [10].

Exemple 1 Pour un corps local \mathbb{K} , les groupes $SL_n(\mathbb{K})$, $n \geq 3$, et $Sp_{2n}(\mathbb{K})$, $n \geq 2$, ont la propriété (T). Plus généralement, tout groupe de Lie réel simple de rang au moins deux possède la propriété (T) de Kazhdan.

Exemple 2 Le groupe d'isométrie $Sp(n, 1)$, $n \geq 2$, de l'espace hyperbolique sur les quaternions, a la propriété (T).

La propriété (T) de Kazhdan se comporte d'une manière intéressante pour les groupes de Lie simples de rang un. Elle est en défaut pour deux familles, $SO(n, 1)$ et $SU(n, 1)$, et est satisfaite pour les autres, $Sp(n, 1)$ et F_4^{-20} . Des formes renforcées de la

propriété (T) ont été étudiées depuis l'article fondateur de Lafforgue. Elles ont tendance être vraies pour les groupes de Lie simples de rang supérieur et fausses pour les groupes de Lie de rang un. Mon objectif est de montrer que certaines formes renforcées de la propriété de Kazhdan sont en défaut pour de grandes classes et peut-être pour tous les groupes hyperboliques.

Ce comportement s'exprime bien en termes d'actions propres sur les espaces de Hilbert. Un groupe G de type fini a la propriété de Kazhdan si et seulement si chaque action isométrique affine de G sur un espace de Hilbert a un point fixe. Inversement, on dit qu'un groupe G de type fini a la propriété de Haagerup s'il admet une action isométrique affine sur un espace de Hilbert telle que la application d'orbite est propre (les images inverses des ensembles bornés sont bornées), voir [11]. De manière équivalente, G a la propriété de Haagerup s'il admet une représentation unitaire avec un cocycle propre.

Groupes hyp. avec propriété de Haagerup	Groupes hyp. avec propriété (T)
Groupes libres, groupes agissant proprement sur les arbres, Groupes fuchsien, $SO(n, 1)$, $SU(n, 1)$ Groupes aléatoires à densité $d < \frac{1}{6}$ [12]	$Sp(n, 1)$, $F_{4(-20)}$ Groupes aléatoires à densité $\frac{1}{3} < d < \frac{1}{2}$ [13][14]

Le type de renforcement auquel nous pensons ne concerne pas les espaces de Banach (voir [15], [16], [3], [17]), mais les actions bilipschitziennes non-isométriques sur les espaces de Hilbert. En particulier, nous voulons répondre à la question suivante posée par Y. Shalom en 2001.

Question (Shalom) : *Tout groupe hyperbolique admet-il une représentation uniformément bornée sur un espace de Hilbert avec un cocycle propre ?*

Uniformément borné signifie que les normes d'opérateur de tous les éléments du groupe sont bornées par une constante uniforme. La réponse est positive pour les groupes de Lie simples de rang un, voir [18]. Cet article montre que les représentations de la série complémentaire π_s des groupes de Lie simples de rang un admettent des structures unitaires uniformément bornées pour tous les $s \in (0, 1)$. C'est l'existence d'une structure unitaire invariante équivalente qui trouve une réponse différente selon les familles : Elle existe pour tout $s \in (0, 1)$ pour $SO(n, 1)$ et $SU(n, 1)$, et n'existe que pour un sous-intervalle n'approchant pas 0 et 1 pour $Sp(n, 1)$ et F_4^{-20} . Notons que l'invariance conforme joue un rôle crucial pour plusieurs étapes de [18].

L'étude la plus approfondie de la question de Shalom a été réalisée par A. Boyer [19] et V. Lafforgue [20]. Dans [20], Lafforgue construit une représentation avec un taux de croissance quadratique sur un espace de Hilbert avec $H^1(G, \pi) \neq 0$ pour les groupes G agissant avec des orbites infinies sur un graphe hyperbolique de Gromov de degré borné. Dans [19] le bord à l'infini ∂G d'un groupe hyperbolique est doté de la structure de Möbius découlant d'une métrique fortement hyperbolique sur G . Rappelons que le carré $Z \times Z$ d'un espace de Möbius Z porte une mesure naturelle (infinie). Boyer montre que $L^2(\partial G \times \partial G)$ est une représentation à croissance lente de G qui admet un cocycle propre.

Comme le souligne la contribution de P. Julg à [11], les représentations des séries

principale et complémentaire des groupes de Lie simples de rang un peuvent être décrites purement en termes de Möbius. Julg discute également de la dérivée Q'_1 , lorsque s tend vers 1, des formes quadratiques invariantes correspondantes. Il montre que dans les cas non-Kazhdan $SO(n, 1)$ et $SU(n, 1)$, Q'_1 est semi-définie positive avec un noyau de dimension 1, ce qui donne automatiquement un cocycle propre.

Dans le chapitre 2, nous examinons des variantes définies positives des formes quadratiques de Julg. Elles ne sont pas invariantes de Möbius, mais certaines techniques de [18] peuvent se généraliser et permettent de démontrer qu'elles sont presque invariantes de Möbius pour une grande classe de fonctions.

Plus précisément, nous travaillons dans la catégorie des espaces de Möbius (Z, \mathcal{M}) . À \mathcal{M} nous associons une classe de s -densités.

Definition 9 *Soit (Z, \mathcal{M}) un espace de Möbius de dimension de Hausdorff Q . Une densité s sur (Z, \mathcal{M}) est une correspondance $u : d \mapsto u_d$ qui affecte à chaque métrique d dans \mathcal{M} une fonction mesurable $u_d : Z \rightarrow \mathbb{C}$ telle que pour toutes les métriques d, d' dans \mathcal{M} et tous $x \in Z$, $u_d(x) = \left(\frac{d'}{d}(x)\right)^{sQ} u_{d'}(x)$.*

En supposant une régularité suffisante sur les s -densités, l'opérateur de Knapp-Stein

$$(I_d^\alpha u_d)(x) = \int \frac{u_d(y)}{d(x, y)^{Q-\alpha}} d\mu_d(y), \quad (0 < \alpha < \frac{Q}{2})$$

fait correspondre les $(\frac{1}{2} + \frac{\alpha}{Q})$ -densités aux $(\frac{1}{2} - \frac{\alpha}{Q})$ -densités, où μ_d est la mesure de Hausdorff de dimension Q associée à la métrique d , et u_d est une $(\frac{1}{2} + \frac{\alpha}{Q})$ -densité.

On est tenté de prendre une racine carrée approximative de cet opérateur et de l'appliquer deux fois. Par conséquent, notre définition de la norme $\mathcal{H}^{-\alpha}$ sur les $(\frac{1}{2} + \frac{\alpha}{Q})$ -densités sur les espaces de Möbius réguliers compacts Q -Ahlfors est la suivante

$$\|u\|_d = \|I_d^\alpha u_d\|_{L^2(d\mu_d)}. \quad (1.2)$$

Elle est positive par construction et conduit à un espace de Sobolev $\mathcal{H}_d^{-\alpha}$ de $(\frac{1}{2} + \frac{\alpha}{Q})$ -densités.

Nous voulons montrer que la norme $\|\cdot\|_d$ est presque invariante de Möbius. Nous développons une théorie de Muckenhoupt-Wheeden presque invariante de Möbius sur le bord des espaces métriques fortement hyperboliques pour prouver dans la Proposition 11 que les normes $\|\cdot\|_d$ sont comparables sur une grande classe de fonctions.

1.1.1 Résultats

Un analogue de la transformée de Cayley d'Astengo, Cowling et Di Blasio sur les espaces de Möbius donne de fortes propriétés de régularité des espaces de Möbius comme espaces métriques mesurés.

Theorem 3 *Si pour une structure de Möbius \mathcal{M} il existe $d \in \mathcal{M}$ telle que μ_d est Q -Ahlfors-David régulière, alors il existe $C \geq 1$ tel que toutes les mesures μ_d , $d \in \mathcal{M}$, sont Ahlfors-David régulières avec la même constante C .*

Nous utilisons cette observation dans le contexte des espaces métriques fortement hyperboliques X pour montrer dans le théorème 7 que les puissances des facteurs conformes $\frac{d'}{d}$ associés à la classe des métriques visuelles \mathcal{M} sur ∂X sont des poids de Muckenhoupt avec une constante A_p sur $L^p(\mu_d)$ uniformément bornée, i.e. indépendante de d et d' dans \mathcal{M} .

De là, nous prouvons la borne pondérée suivante sur l'opérateur de Knapp-Stein.

Theorem 9 *Soit X un espace fortement hyperbolique, $\{d_o\}_{o \in X}$ la structure de Möbius naturelle sur ∂X , et $0 < \alpha < \frac{Q}{q}$. Alors il existe une constante $C_\alpha > 0$ qui ne dépend que de α , telle que pour tout $d_o, d_{o'} \in \mathcal{M}$ et $f \in L^p(\mu_o)$,*

$$\left\| \left(\frac{d_{o'}}{d_o} \right)^{\frac{\alpha}{2}} \circ I_o^\alpha \circ \left(\frac{d_{o'}}{d_o} \right)^{-\frac{\alpha}{2}} f \right\|_{L^q(\mu_o)} \leq C_\alpha \|f\|_{L^p(\mu_o)}. \quad (1.3)$$

Comme résultat final nous montrons que sur un espace fortement hyperbolique les normes $\|\cdot\|_d$ sont comparables sur une grande classe de fonctions.

Theorem 11 *Soit (Z, \mathcal{M}) un espace de Möbius sphérique, Q -Ahlfors-David régulier, $0 < \alpha < \frac{Q}{2}$. Si la norme d'opérateur $\|I_o^\alpha\|_{L^2((\frac{d_{o'}}{d_o})^\alpha \mu_o)}$ est uniformément bornée, c'est-à-dire bornée par une constante indépendante de d_o et $d_{o'}$, alors il existe $C > 0$, uniformément sur \mathcal{M} , telle que toute fonction f de la famille*

$$\left\{ \left(\frac{d_{o''}}{d_o} \right)^{sQ} \right\}_{d_{o''} \in \mathcal{M}} \quad \left(s = \frac{1}{2} + \frac{\alpha}{2Q} \right)$$

satisfait à l'inégalité : $\left\| \left(\frac{d_{o'}}{d_o} \right)^{\frac{\alpha}{2}} \circ I_o^\alpha \circ \left(\frac{d_{o'}}{d_o} \right)^{-\frac{\alpha}{2}} f \right\|_{L^2(\mu_o)} \leq C \|I_o^\alpha f\|_{L^2(\mu_o)}$.

1.1.2 Conclusion et perspectives

La famille de fonctions du théorème 11 joue un rôle particulier dans le cas des arbres. Étant donné un arbre homogène T avec une origine o , soit T_n le sous-arbre de T donné par tous les sommets o' pas plus éloignés de n arêtes de o . Soit $\{d_{o'}\}_{o' \in T_n}$ la famille des distances visuelles sur ∂T associées aux sommets dans T_n . Une s -densité sur T_n , est une application $T_n \ni o' \rightarrow u_{o'}$ qui associe une fonction $u_{o'}$ sur ∂T à chaque sommet o' dans T_n . La fonction $u_{o'}$ est supposée constante sur chaque ombre d'un point du bord de T_n vu de l'origine o , et pour chaque $o, o' \in T_n$ et $x \in \partial T$, $u_o(x) = \left(\frac{d_{o'}}{d_o}(x) \right)^s u_{o'}(x)$ (nous avons normalisé les distances de telle sorte que la dimension de Hausdorff de ∂T soit de 1). Désignons cet espace de s -densités par \mathcal{D}_s^n .

Les $(\frac{1}{2} + \frac{\alpha}{2})$ -densités $o \rightarrow \left(\frac{d_{o'}}{d_o} \right)^{\frac{1}{2} + \frac{\alpha}{2}}$ avec $o' \in T_n$ forme une base orthonormale d'espace $\mathcal{D}_{\frac{1}{2} + \frac{\alpha}{2}}^n$. En effet, toute $(\frac{1}{2} + \frac{\alpha}{2})$ -densité u_o dans $\mathcal{D}_{\frac{1}{2} + \frac{\alpha}{2}}^n$ peut s'écrire comme suit

$$u_o(x) = \sum_{o' \in T} f(o') \left(\frac{d_{o'}}{d_o}(x) \right)^{\frac{1}{2} + \frac{\alpha}{2}},$$

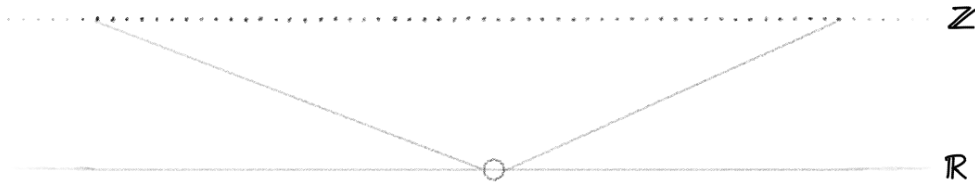
où f est une fonction (0-densité) supportée sur T_n . L'opérateur I_o^α envoie u_o à

$$I_o^\alpha(u_o)(x) = \sum_{o' \in T} f(o') \left(\frac{d_{o'}}{d_o}(x) \right)^{\frac{1}{2} - \frac{\alpha}{2}}.$$

En d'autres termes, l'opérateur I_o^α peut simplement être défini comme l'extension linéaire de l'application $(\frac{d_{o'}}{d_o}(x))^{\frac{1}{2}+\frac{\alpha}{2}} \rightarrow (\frac{d_{o'}}{d_o}(x))^{\frac{1}{2}-\frac{\alpha}{2}}$, voir [21, p. 45].

Cette construction se généralise-t-elle aux groupes hyperboliques? Cela pourrait-il aider à déduire du théorème 11 l'inégalité $\|(\frac{d_{o'}}{d_o})^{\frac{\alpha}{2}} \circ I_o^\alpha \circ (\frac{d'}{d})^{-\frac{\alpha}{2}} f\|_{L^2(\mu_o)} \leq C \|I_o^\alpha f\|_{L^2(\mu_o)}$ pour l'ensemble de l'espace vectoriel des $(\frac{1}{2} + \frac{\alpha}{2})$ -densités? Une première étape serait de prouver le théorème 11 pour la différence de deux densités dans la famille $(\frac{d_{o''}}{d_o})^{sQ}\}_{d_{o''}} \in \mathcal{M}$.

1.2 Géométrie de Möbius à grande échelle



A [finitely generated group Γ with the word metric] may appear boring and uneventful to a geometer's eye since it is discrete and the traditional local (e.g. topological and infinitesimal) machinery does not run in Γ . To regain the geometric perspective one has to change his/her position and move the observation point far away from Γ . Then the metric in Γ seen from the distance d becomes the original distance divided by d and for $d \rightarrow \infty$ the points in Γ coalesce into a connected continuous solid unity which occupies the visual horizon without any gaps or holes and fills our geometer's heart with joy. (M. Gromov, Asymptotic invariants of infinite groups, page 1)

Dans le chapitre sur la géométrie asymptotique de Möbius, une classe d'applications qui imite les applications quasi-Möbius dans le contexte de la théorie géométrique des groupes est présentée et étudiée.

Mon intérêt initial pour la géométrie de Möbius découle d'un phénomène fondamental pour la théorie des probabilités, qui a été porté à mon attention par le professeur Wendelin Werner alors qu'il étudiait l'évolution de Schramm-Löwner (SLE). En deux dimensions, le mouvement brownien a la propriété importante d'être invariant conforme (voir, par exemple, [22]). En gros, l'invariance conforme découle du fait que le mouvement brownien est la limite d'échelle de marches aléatoires simples sur différents réseaux (ce qui implique une invariance sous l'effet de l'échelle et de la rotation). Ce phénomène a des conséquences considérables pour la physique bidimensionnelle. Cependant, il semble qu'il s'agisse d'un phénomène purement bidimensionnel. Cette limitation a suscité mon intérêt pour des notions moins restrictives de la conformalité et pour les limites d'échelle d'un point de vue géométrique.

Dans [23] Pierre Pansu introduit une notion d'applications conformes à grande échelle qui imite le comportement infinitésimal des fonctions conformes. En bref, les applications conformes à grande échelle transforment des familles de boules disjointes en familles de quasi-boules faiblement superposées. Il s'agit d'une notion très flexible qui inclut, par exemple, les plongements uniformes. Cependant, cette flexibilité rend le comportement asymptotique de ces applications moins prévisible.

Nous introduisons ici une notion plus restrictive et sémantiquement plus simple qui imite le comportement de quasi-Möbius. En gros, une application entre espaces métriques est un plongement asymptotiquement Möbius si elle préserve presque le birapport des points éloignés les uns des autres. En conséquence, les conclusions des résultats présentés dans cet article sont beaucoup plus forts que ceux de [23].

Les sources d'exemples de plongement asymptotiquement Möbius sont :

1. plongements quasi-isométriques,
2. équivalences de Lipschitz sous-linéaires (c'est-à-dire des applications induisant des équivalences de Lipschitz sur des cônes asymptotiques [24]),
3. snowflaking (i.e., remplacer une métrique par une puissance de celle-ci),
4. plongements d'Assouad d'espaces métriques doublants dans \mathbb{R}^N .

Par exemple, tout groupe de Lie nilpotent ou groupe nilpotent de type fini peut être AM -plongé dans un espace euclidien de dimension suffisamment élevée [25]. De plus, nous rencontrerons des exemples de dimension infinie.

Nous avons besoin d'un critère pour déterminer si deux points d'un espace sont éloignés l'un de l'autre. Une façon d'y parvenir est de les séparer par des fonctions de croissance sous-linéaire.

Étant donné $u : \mathbb{R}_+ \rightarrow \mathbb{R}$, nous disons que les points x et y dans X sont séparés par u à partir d'une origine fixe o si $d(x, y) > u(d(x, o) + d(o, y))$. Pour simplifier, nous écrivons souvent $x, y > u$ et $|x| = d(o, x)$ pour la distance à l'origine. Nous pouvons appeler u une *jauge*, c'est-à-dire une fonction qui fixe une échelle de "vue" en fonction du "lieu". Fixer une jauge signifie décider ce qui est "proche" et "éloigné".

Definition 11 Une fonction $u : \mathbb{R}_+ \rightarrow \mathbb{R}$ est une *jauge admissible* si

1. elle est non décroissante, et
2. u croît de façon sous-linéaire, c'est-à-dire $\limsup_{r \rightarrow \infty} \frac{u(r)}{r} = 0$.

Moralement, les plongements asymptotiquement Möbius en des points très espacés sont des plongements de quasi-Möbius.

Definition 12 Une carte $f : X \rightarrow Y$ est une *carte asymptotique-Möbius* ou *AM-map* s'il existe une jauge admissible u et un homéomorphisme $\eta : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ tel que pour tout x, y, z, w dans X tel que toutes les distances δ entre $x, y, z, w > u$, alors

$$[f(x), f(y), f(z), f(w)] < \eta([x, y, z, w]).$$

Le cône *asymptotique* d'un espace métrique (X, d) , capture la géométrie à grande échelle de X . Comme le dit Gromov ci-dessus, il formalise l'idée d'instantanés de l'espace X pris par un observateur qui s'éloigne de plus en plus de X . Cette séquence d'instantanés peut se stabiliser et l'observateur a l'impression de voir un seul objet. Nous appelons cet objet le cône asymptotique de X .

1.2.1 Résultats

Theorem 12 *Soit f un plongement asymptotiquement Möbius d'un espace asymptotiquement chaîné X vers un espace Y qui préserve les suites divergentes et bornées, alors f induit une carte continue, injective et quasisymétrique g entre certains cônes asymptotiques de X et Y .*

Proposition 1 *Il existe un plongement asymptotiquement Möbius d'un groupe de Heisenberg de dimension infinie vers un espace de Hilbert.*

Notre résultat principal est que sous un plongement asymptotiquement Möbius, une notion de dimension à grande échelle augmente. Le terme pertinent dépend de la classe de groupes considérée.

En 1993, M. Gromov a introduit la notion de dimension asymptotique en tant qu'analogie à grande échelle de la dimension couvrante de Lebesgue. La dimension asymptotique d'un groupe de type fini est un invariant quasi-isométrique. Son application la plus importante remonte à Guoliang Yu, qui a montré que tout groupe de type fini avec un type d'homotopie fini et une dimension asymptotique finie satisfait la conjecture de Novikov [26].

Definition 17 *Soit X un espace métrique. Nous disons que la dimension asymptotique de X ne dépasse pas n si pour chaque recouvrement ouverte uniformément bornée \mathcal{V} de X il existe un recouvrement ouverte uniformément bornée \mathcal{U} de X de multiplicité $\leq n + 1$ telle que \mathcal{V} affine \mathcal{U} . Nous écrivons $\text{asdim } X = n$ s'il est vrai que $\text{asdim } X \leq n$ et $\text{asdim } X \not\leq n - 1$*

La dimension asymptotique d'un groupe de type fini Γ a plusieurs implications intéressantes. Par exemple, $\text{asdim } \Gamma = 0$ si et seulement si Γ est fini (Proposition 65. in [27]) et $\text{asdim } \Gamma = 1$ si et seulement si Γ est virtuellement libre (Théorème 66. in [27]).

Si G est un groupe nilpotent de type fini, alors son cône asymptotique G_ω est un groupe de Carnot [28] et la dimension asymptotique de G coïncide avec la dimension topologique de son cône asymptotique [27].

Theorem 14 *Soient G et G' des groupes nilpotents de type fini et $f : G \rightarrow G'$ un plongement asymptotiquement Möbius préservant les suites divergentes et bornées, alors $\text{asdim}(G) \leq \text{asdim}(G')$. Si G et G' sont des groupes de Lie nilpotents simplement connexes, alors $\dim(G) \leq \dim(G')$. De plus, si $\text{asdim}(G) = \text{asdim}(G')$, alors les cônes asymptotiques de G et G' sont des groupes de Lie gradués isomorphes. R Inversement, pour des groupes nilpotents dont les cônes asymptotiques sont isomorphes, il existe un plongement asymptotiquement Möbius entre eux.*

Heinrich Jung a établi au début des années 1900 que le diamètre d'un ensemble dans l'espace euclidien \mathbb{R}^n est lié au rayon de la plus petite boule englobante par une inégalité dont la constante ne dépend que de la dimension n [29]. Plus précisément, pour tout ensemble $A \subset \mathbb{R}^n$

$$r \leq d(A) \sqrt{\frac{n}{2(n+1)}}, \quad (1.4)$$

où r est le rayon de la plus petite boule englobante de A et $d(A)$ est le diamètre de A .

La *dimension géométrique* d'un espace CAT(0) X est la dimension topologique maximale de tous les sous-ensembles compacts de X [30].

Remark 6 *Il s'avère que la dimension géométrique d'un espace CAT(0) X est inférieure à n si et seulement si l'inégalité de Jung (4.1) est vraie dans X [31].*

La dimension géométrique peut être transformée en un terme de grande échelle de la manière suivante. Un espace X de CAT(0) a une dimension télescopique inférieure à n si et seulement si pour chaque $\delta > 0$ il existe un $D > 0$ tel que pour tout ensemble A dans X dont le diamètre est supérieur à D ,

$$r \leq \left(\delta + \sqrt{\frac{n}{2(n+1)}} \right) d(A). \quad (1.5)$$

Plus généralement,

Definition 18 *Un espace X dont les cônes asymptotiques sont CAT(0) a une dimension télescopique $\leq n$ si chaque cône asymptotique a une dimension géométrique $\leq n$.*

Nous adoptons la notation $\text{tele-dim}(X)$ pour la dimension télescopique de X .

Theorem 15 *Soit X et Y des espaces dont les cônes asymptotiques sont CAT(0). S'il existe un plongement asymptotiquement Möbius $f : X \rightarrow Y$ préservant les suites divergentes et bornées, alors la dimension télescopique augmente, c'est-à-dire $\text{tele-dim}(X) \leq \text{tele-dim}(Y)$.*

En particulier, si Y est hyperbolique, X l'est aussi.

1.2.2 Conclusion et perspectives

Un problème intéressant est le suivant. Les plongements quasi-symétriques entre groupes de Carnot sont grossièrement conformes au sens de Pansu, voir [23]. Par conséquent, ils élèvent la dimension de Hausdorff par le théorème de Pansu dans [23]. Cependant, rien n'est dit si les groupes de Carnot ont des dimensions de Hausdorff égales. Peut-on dire quelque chose dans le cas des plongements asymptotiquement Möbius ? Par exemple, dans le cas du groupe de Heisenberg vers \mathbb{R}^4 .

Chapter 2

Introduction (English)

According to Buyalo and Schröder [1], a Möbius structure \mathcal{M} on a set Z is a collection of metrics defining the same metric cross-ratios. More precisely, one considers quadruples (x, y, z, w) of points in Z where no entry occurs three or four times. Such an *admissible* quadruple is associated with a *cross-ratio triple*

$$[x, y, z, w]_d = (d(x, y)d(z, w) : d(x, z)d(y, w) : d(x, w)d(y, z)) \in \mathbb{R}P^2. \quad (2.1)$$

Two metrics d and d' on Z are called *Möbius equivalent* if for each admissible quadruple their respective cross-ratio triples coincide.

A *Möbius structure* on a set Z is a class \mathcal{M} of metrics on Z which are pairwise Möbius equivalent.

Examples of sets with natural Möbius structures are: The round sphere, boundaries of symmetric spaces of rank one, boundaries of CAT(-1) spaces [2], boundaries of strongly hyperbolic metric spaces [3], boundaries of symmetric spaces of higher rank and Euclidean buildings [4], [5], and boundaries of certain CAT(0) cube complexes [5], [6].

Möbius geometry exhibits an attractive tension between flexibility and rigidity. Möbius equivalent metrics are conformal in a very strong sense. Two metrics d and d' in a Möbius structure are conformal in sense that the limiting ratio $\frac{d'}{d}(x) := \lim_{y \rightarrow x} \frac{d'(x, y)}{d(x, y)}$ exists for all $x \in Z$ [3, Lemma 6]. Moreover, for all $x \neq y \in Z$,

$$\frac{d'}{d}(x) \frac{d'}{d}(y) = \left(\frac{d'(x, y)}{d(x, y)} \right)^2. \quad (2.2)$$

One may interpret this relation as a geometric mean-value property.

Certain flexible properties of conformal geometry, such as uniformization, sometimes persist. However, most of the existing theory deals with rigidity properties. Möbius structures on the boundaries of nonpositively curved spaces have been shown to encode the geometry of these spaces. Quasi-Möbius homeomorphisms between boundaries of hyperbolic metric spaces (satisfying mild conditions), for example, extend to quasi-isometries [7].

This manuscript develops the geometry and analysis of Möbius spaces in two directions

1. Hilbert spaces attached to Möbius spaces, and

2. asymptotic Möbius geometry.

The second item belongs to the rigidity part of the theory, while the first follows a flexibility dream.

The techniques are borrowed in part 1 from harmonic analysis on noncommutative groups and in part 2 from metric geometry, nilpotent Lie groups and (infinite dimensional) CAT(0)-spaces.

Finite Möbius spaces. The boundaries of convex Euclidean polyhedra can be regarded as finite analogues of Möbius spaces. Inspired by the Riemann mapping theorem¹ and the more general uniformization theorem of Poincaré and Koebe [32], [33], [34], I proved in my master thesis a uniformization theorem for closed convex polyhedra in Euclidean 3-space in the following sense.

Theorem 1 *Every convex polyhedron in Euclidean 3-space has a discrete-conformal equivalent cousin inscribed in a round sphere. It is unique up to Möbius transformations between round spheres.*

The techniques used are mainly from low-dimensional topology, in particular Teichmüller theory.

We give a brief summary of what *discrete conformal equivalence* means between polyhedra. For details see the paper [8]. A triangulated surface consists of a topological 2-manifold S , a finite subdivision of S into subsets (called faces), and isometries of faces onto Euclidean triangles. The vertex set has a collection of cross-ratios, one for each oriented and co-oriented edge. Two triangulated surfaces are *Möbius equivalent* if there is a subdivision-preserving mapping between them that preserves the cross-ratios. In [9], Bobenko, Pinkall, and Springborn attach to a triangulated surface a complete hyperbolic surface with cusps and define two triangulated surfaces to be discretely conformally equivalent if the corresponding hyperbolic surfaces are isometric. Möbius equivalence implies discrete conformal equivalence.

The faces of a convex polyhedron in Euclidean 3-space need not be triangles. Nevertheless, it is shown in [8] that arbitrary Delaunay triangulations of faces of a given convex Euclidean polyhedron lead to triangulated surfaces which are discrete-conformally equivalent. A *Delaunay triangulation* of a face is a triangulation whose edges are minimising geodesics and for any interior edge the sum of the two angles of the Euclidean triangles facing that edge is at most π . Therefore, one can speak of convex Euclidean polyhedra having discrete-conformal equivalent boundaries.

In this manuscript we are interested in continuous Möbius geometry. The first chapter deals with applications to questions of representation theory of hyperbolic groups. The second chapter develops asymptotic Möbius geometry as a tool to move from a discrete to a continuous theory.

1. Mentioned by Riemann in his dissertation: Grundlagen für allgemeine Theorie der Funktionen einer veränderlichen complexen Grösse, Göttingen, 1851

2.1 Hilbert spaces attached to Möbius spaces

Kazhdan introduced the property (T) in 1968 as a tool for proving finite generation of nonuniform lattices in simple Lie groups of higher rank. This property has played a central role in the theory of operator algebras and geometric group theory, see [10].

Kazhdan's property (T) behaves in an interesting way for simple Lie groups of rank one. It fails for two families, $SO(n, 1)$ and $SU(n, 1)$, and holds for the others, $Sp(n, 1)$ and F_4^{-20} . Strengthened forms of the property (T) have been studied since Lafforgue's seminal paper [20]. They tend to hold for simple Lie groups with higher rank and fail for Lie groups with rank one. My goal is to show that certain strengthened forms of the Kazhdan property fail for large classes and possibly all hyperbolic groups.

This failure is best expressed in terms of proper actions on Hilbert spaces. A finitely generated group G has the Kazhdan property if and only if every affine isometric action of G on a Hilbert space has a fixed point. Conversely, a finitely generated group G is said to have the Haagerup property if it admits an affine isometric action on a Hilbert space such that the orbit map is proper (inverse images of bounded sets are bounded), see [11]. Equivalently, G has the Haagerup property if it admits a unitary representation with a proper cocycle.

The type of reinforcement we have in mind does not concern Banach spaces (see [15], [16], [3], [17]), but nonisometric actions on Hilbert spaces. In particular, we want to address the following question posed by Y. Shalom in 2001.

Question (Shalom): *Does every hyperbolic group admit a uniformly bounded representation on a Hilbert space with a proper cocycle?*

Uniformly bounded means that the operator norms of all group elements are bounded by a uniform constant. Shalom's question is answered positively for $Sp(n, 1)$, see [35]. In [18], it is shown that the complementary series representations π_s of simple Lie groups of rank one admit uniformly bounded unitary structures for all $s \in (0, 1)$. It is the existence of an equivalent invariant unitary structure which is answered differently for different families: It exists for all $s \in (0, 1)$ for $SO(n, 1)$ and $SU(n, 1)$, and exists only for a subinterval not approaching 0 and 1 for $Sp(n, 1)$ and F_4^{-20} . Note that conformal invariance plays a crucial role for multiple steps in [18].

The most thorough study of Shalom's question was carried out by A. Boyer [19] and V. Lafforgue [20]. In [20], Lafforgue constructs a representation with quadratic growth rate on a Hilbert space with $H^1(G, \pi) \neq 0$ for groups G acting with infinite orbits on a Gromov-hyperbolic graph with bounded degree. In [19] the ideal boundary ∂G of a hyperbolic group is endowed with the Möbius structure arising from a strongly hyperbolic metric on G . Recall that the square $Z \times Z$ of a Möbius space Z carries a natural (infinite) measure. It is shown that $L^2(\partial G \times \partial G)$ is a slow-growth representation of G which admits a proper cocycle.

As pointed out in P. Julg's contribution to [11], principal and complementary series representations of simple Lie groups of rank one can be described purely in Möbius terms. Julg also discusses the derivative Q'_1 , when s tends to 1, of the corresponding invariant quadratic forms. He shows that in the non-Kazhdan cases $SO(n, 1)$ and $SU(n, 1)$, Q'_1 is positive semidefinite with a 1-dimensional kernel, which automati-

cally gives a proper cocycle.

In chapter 2 we examine positive definite variants of the quadratic forms of Julg. They are not Möbius invariant, but certain techniques in [18] generalize and help prove that they are almost Möbius invariant for a large class of functions.

Specifically, we work in the category of Möbius spaces (Z, \mathcal{M}) . With \mathcal{M} we associate a class of s -densities.

Definition 9 *Let (Z, \mathcal{M}) be a Möbius space of Hausdorff dimension Q . An s -density on (Z, \mathcal{M}) is a mapping $: d \mapsto u_d$ which assigns to each metric d in \mathcal{M} a measurable function $u_d : Z \rightarrow \mathbb{C}$ such that for all metrics $d, d' \in \mathcal{M}$ and all $x \in Z$, $u_d(x) = (\frac{d'}{d}(x))^{sQ} u_{d'}(x)$.*

Assuming sufficient regularity on s -densities, the Knapp-Stein operator

$$(I_d^{2\alpha} u_d)(x) = \int \frac{u_d(y)}{d(x, y)^{Q-2\alpha}} d\mu_d(y), \quad (0 < \alpha < \frac{Q}{2})$$

maps $(\frac{1}{2} + \frac{\alpha}{Q})$ -densities to $(\frac{1}{2} - \frac{\alpha}{Q})$ -densities, where μ_d is the Q -dimensional Hausdorff measure associated with the metric d , and u_d is an $(\frac{1}{2} + \frac{\alpha}{Q})$ -density.

One is tempted to take an approximate root of this operator and apply it twice. Therefore, our definition for the $\mathcal{H}^{-\alpha}$ -norm on $(\frac{1}{2} + \frac{\alpha}{Q})$ -densities on compact Q -Ahlfors regular Möbius spaces is

$$\|u\|_d = \|I_d^\alpha u_d\|_{L^2(d\mu_d)}. \quad (2.3)$$

It is positive by construction and leads to a Sobolev space $\mathcal{H}_d^{-\alpha}$ of $(\frac{1}{2} + \frac{\alpha}{Q})$ -densities.

We want to show that the norm $\|\cdot\|_d$ is almost Möbius invariant. We develop a nearly Möbius invariant Muckenhoupt-Wheeden theory on the boundary of strongly hyperbolic metric spaces. This allows us to prove in Theorem 11 that the norms $\|\cdot\|_d$ are comparable on a large class of functions.

2.1.1 Results

An analog of the Cayley transform of Astengo, Cowling and Di Blasio on Möbius spaces gives strong regularity properties of Möbius spaces as measure spaces.

Theorem 3 *If for a Möbius structure \mathcal{M} there exists $d \in \mathcal{M}$ such that μ_d is Q -Ahlfors-David regular, then there exists $C \geq 1$ such that every measure μ_d , $d \in \mathcal{M}$, is Ahlfors-David regular with constant C .*

We use this observation in the context of strongly hyperbolic metric spaces X to show in Theorem 7 that powers of conformal factors $\frac{d'}{d}$ associated to the class of visual metrics \mathcal{M} on ∂X are Muckenhoupt weights with uniformly bounded A_p -constant on $L^p(\mu_d)$, independent of d and d' in \mathcal{M} .

From there we prove the following weighted bound on the Knapp-Stein operator.

Theorem 9 *Let X be a strongly hyperbolic space, $\{d_o\}_{o \in X}$ be the natural Möbius structure on ∂X , and $0 < \alpha < \frac{Q}{2}$. Then there is a constant $C_\alpha > 0$ that depends only on α , such that for all $d_o, d_{o'} \in \mathcal{M}$ and $f \in L^p(\mu_o)$,*

$$\| \left(\frac{d_{o'}}{d_o} \right)^{\frac{\alpha}{2}} \circ I_o^\alpha \circ \left(\frac{d_{o'}}{d_o} \right)^{-\frac{\alpha}{2}} f \|_{L^q(\mu_o)} \leq C_\alpha \|f\|_{L^p(\mu_o)}, \quad (2.4)$$

where $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{Q}$.

As a final result we show that on strongly hyperbolic metric spaces the norms $\|\cdot\|_{d_o}$ are comparable on a large class of functions.

Theorem 11 *Let (Z, \mathcal{M}) be a spherical, Q -Ahlfors-David regular Möbius space, $0 < \alpha < \frac{Q}{2}$. If the operator norm $\|I_d^\alpha\|_{L^2((\frac{d'}{d})^\alpha \mu_d)}$ is uniformly bounded, i.e., bounded by a constant independent of d and d' , then there exists $C > 0$, uniformly over \mathcal{M} , s.t. any function f from the family*

$$\left\{ \left(\frac{d''}{d} \right)^{sQ} \right\}_{d'' \in \mathcal{M}} \quad \left(s = \frac{1}{2} + \frac{\alpha}{2Q} \right)$$

satisfies the inequality $\| \left(\frac{d'}{d} \right)^{\frac{\alpha}{2}} \circ I_d^\alpha \circ \left(\frac{d'}{d} \right)^{-\frac{\alpha}{2}} f \|_{L^2(\mu_d)} \leq C \|I_d^\alpha f\|_{L^2(\mu_d)}$.

2.1.2 Conclusion and perspectives

The family of functions in Theorem 11 plays a special role in the case of trees. Given a homogeneous tree T with origin o , let T_n be the subtree of T given by all vertices o' which are not farther than n edges from o . Let $\{d_{o'}\}_{o' \in T_n}$ be the family of visual distances on ∂T belonging to vertices in T_n . An s -density on T_n is a mapping $T_n \ni o' \rightarrow u_{o'}$ that associates with each vertex o' in T_n a function $u_{o'}$ on ∂T . The function $u_{o'}$ is assumed to be constant on each shadow of a boundary point of T_n seen from the origin o , and for each $o, o' \in T_n$ and $x \in \partial T$, $u_o(x) = \left(\frac{d_{o'}}{d_o} \right)^s u_{o'}(x)$ (we normalized the distances so that the Hausdorff dimension of ∂T is 1). Let us denote this space of s -densities by \mathcal{D}_s^n .

The $(\frac{1}{2} + \frac{\alpha}{2})$ -densities $o \rightarrow \left(\frac{d_{o'}}{d_o} \right)^{\frac{1}{2} + \frac{\alpha}{2}}$ with $o' \in T_n$ form an orthonormal basis of $\mathcal{D}_{\frac{1}{2} + \frac{\alpha}{2}}^n$. Any $(\frac{1}{2} + \frac{\alpha}{2})$ -density u_o in $\mathcal{D}_{\frac{1}{2} + \frac{\alpha}{2}}^n$ can be written as

$$u_o(x) = \sum_{o' \in T} f(o') \left(\frac{d_{o'}}{d_o} \right)^{\frac{1}{2} + \frac{\alpha}{2}},$$

where f is a function (0-density) supported on T_n . The operator I_o^α sends u_o to

$$I_o^\alpha(u_o)(x) = \sum_{o' \in T} f(o') \left(\frac{d_{o'}}{d_o} \right)^{\frac{1}{2} - \frac{\alpha}{2}}.$$

In other words, the operator $I_{d_o}^\alpha$ can simply be defined as the linear extension of the map $\left(\frac{d_{o'}}{d_o} \right)^{\frac{1}{2} + \frac{\alpha}{2}} \rightarrow \left(\frac{d_{o'}}{d_o} \right)^{\frac{1}{2} - \frac{\alpha}{2}}$, see [21, p. 45].

Can this construction be generalized to hyperbolic groups? Could this help to derive from Theorem 11 the inequality $\| \left(\frac{d_{o'}}{d_o} \right)^{\frac{\alpha}{2}} \circ I_o^\alpha \circ \left(\frac{d_{o'}}{d_o} \right)^{-\frac{\alpha}{2}} f \|_{L^2(\mu_o)} \leq C \|I_o^\alpha f\|_{L^2(\mu_o)}$ for the entire vector space of $(\frac{1}{2} + \frac{\alpha}{2})$ -densities? A first step would be to prove Theorem 11 for the difference of two densities from the family $\left\{ \left(\frac{d_{o''}}{d_o} \right)^{sQ} \right\}_{d_{o''} \in \mathcal{M}}$.

2.2 Asymptotic Möbius geometry

A [finitely generated group Γ with the word metric] may appear boring and uneventful to a geometer's eye since it is discrete and the traditional local (e.g. topological and infinitesimal) machinery does not run in Γ . To regain the geometric perspective one has to change his/her position and move the observation point far away from Γ . Then the metric in Γ seen from the distance d becomes the original distance divided by d and for $d \rightarrow \infty$ the points in Γ coalesce into a connected continuous solid unity which occupies the visual horizon without any gaps or holes and fills our geometer's heart with joy. (M. Gromov, Asymptotic invariants of infinite groups, page 1)

In the chapter on asymptotic Möbius geometry, a class of maps that mimics quasi-Möbius maps in the context of geometric group theory is presented and studied.

My initial interest in Möbius geometry stems from a phenomenon fundamental to probability theory, which was brought to my attention by Prof. Wendelin Werner while studying Schramm-Löwner-Evolution (SLE). In two dimensions, Brownian motion has the important property of being conformally invariant (see, for example, [22]). Roughly speaking, conformal invariance arises from the fact that Brownian motion is the scaling limit of simple random walks on different lattices (implying invariance under scaling and rotation). This phenomenon has far-reaching consequences for two-dimensional physics. However, it appears to be a purely two-dimensional phenomenon. This limitation has piqued my interest in less restrictive notions of conformality and in scaling limits from a geometric point of view.

In [23] Pierre Pansu introduces a notion of large-scale conformal maps that mimics the infinitesimal behavior of conformal maps. In short, large-scale conformal maps map families of disjoint balls to families of weakly overlapping quasiballs. It is a very flexible notion that includes, for example, coarse embeddings. However, this flexibility makes the asymptotic behavior of such maps less predictable.

Here we introduce a more restrictive and semantically simpler notion that mimics quasi-Möbius behavior. Roughly speaking, a map between metric spaces is an asymptotic-Möbius map if it nearly preserves the cross-ratio of points that are mutually far apart. In consequence, the results presented in this paper are much stronger than those in [23].

Sources of examples of *AM*-maps are:

1. Quasi-isometric embeddings,
2. Sublinear-bi-Lipschitz equivalences (i.e. maps inducing Lipschitz equivalences on asymptotic cones [24]),
3. Snowflaking (i.e., replacing a metric by a power of it),
4. Assouad maps from doubling metric spaces to \mathbb{R}^N .

For example, any nilpotent Lie group or finitely generated nilpotent group can be *AM*-embedded in a Euclidean space of sufficiently high dimension [25]. Moreover, we will encounter infinite dimensional examples.

We need a criterion to tell when two points in a space are far apart. One way to do this is to separate them by sublinear growing functions.

Given $u : \mathbb{R}_+ \rightarrow \mathbb{R}$, we say that the points x and y in X are separated by u from a fixed origin o if $d(x, y) > u(d(x, o) + d(o, y))$. For simplicity, we often write $x, y > u$ and $|x| = d(o, x)$ for the distance from the origin. We can call u a *gauge*, i.e., a function that sets a scale of “sight” as a function of “location”. To fix a gauge means to decide what is “near” and “far away”.

Definition 11 *A function $u : \mathbb{R}_+ \rightarrow \mathbb{R}$ is an admissible gauge if*

1. *it is non-decreasing, and*
2. *u grows sublinearly, i.e. $\limsup_{r \rightarrow \infty} \frac{u(r)}{r} = 0$.*

Morally, asymptotic-Möbius maps at widely spaced points are quasi-Möbius.

Definition 12 *A map $f : X \rightarrow Y$ is an asymptotic-Möbius map or AM-map if there exists an admissible gauge u and a homeomorphism $\eta : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that for every x, y, z, w in X such that all 6 distances between $x, y, z, w > u$, then*

$$[f(x), f(y), f(z), f(w)] < \eta([x, y, z, w]).$$

The *asymptotic cone* of a metric space (X, d) , captures the geometry on the large scale of X . As Gromov puts it above, it formalizes the idea of snapshots of the space X taken by an observer moving farther and farther away from X . This sequence of snapshots can stabilize and the observer has the impression of seeing a single object. We call this object the asymptotic cone of X .

2.2.1 Results

Theorem 12 *Let f be an AM-map from an asymptotically chained space X to a space Y that preserves diverging and bounded sequences, then f induces a continuous, injective, quasisymmetric map g between some asymptotic cones of X and Y .*

Proposition 1 *There exists an asymptotic-Möbius map from an infinite dimensional Heisenberg group to Hilbert space.*

Our main result is that under AM-mappings, a large-scale notion of dimension increases. The relevant term depends on the class of groups considered.

In 1993 M. Gromov introduced the notion of asymptotic dimension as a large scale analogue of Lebesgue’s covering dimension [36]. The asymptotic dimension of a finitely generated group is a quasi-isometric invariant. Its most prominent application goes back to Guoliang Yu, who showed that any finitely generated group with finite homotopy type and finite asymptotic dimension satisfies the Novikov conjecture [26].

Definition 17 *Let X be a metric space. We say that the asymptotic dimension of X does not exceed n if for every uniformly bounded open cover \mathcal{V} of X there exists a uniformly bounded open cover \mathcal{U} of X of multiplicity $\leq n + 1$ such that \mathcal{V} refines \mathcal{U} . We write $asdim X = n$ if it is true that $asdim X \leq n$ and $asdim X \not\leq n - 1$*

The asymptotic dimension of a finitely generated group Γ has several interesting implications. For example, $\text{asdim } \Gamma = 0$ if and only if Γ is finite (Proposition 65. in [27]) and $\text{asdim } \Gamma = 1$ if and only if Γ is virtually free (Theorem 66. in [27]).

If G is a finitely generated nilpotent group, then its asymptotic cone G_ω is a Carnot group [28] and the asymptotic dimension of G coincides with the topological dimension of its asymptotic cone [27].

Theorem 14 *Let G and G' be a finitely generated nilpotent groups and $f : G \rightarrow G'$ an AM-map preserving diverging and bounded sequences, then $\text{asdim}(G) \leq \text{asdim}(G')$. If G and G' are simply connected nilpotent Lie groups, then $\dim(G) \leq \dim(G')$. Furthermore, if $\text{asdim}(G) = \text{asdim}(G')$, then the asymptotic cones of G and G' are isomorphic graded Lie groups. Conversely, given nilpotent groups with isomorphic asymptotic cones, there exists an asymptotic-Möbius map between them.*

Heinrich Jung established in the early 1900s that the diameter of a set in Euclidean space \mathbb{R}^n is related to the radius of the smallest enclosing ball by an inequality whose constant depends only on the dimension n [29]. More precisely, for any set $A \subset \mathbb{R}^n$

$$r \leq d(A) \sqrt{\frac{n}{2(n+1)}}, \quad (2.5)$$

where r is the radius of the smallest enclosing ball of A and $d(A)$ is the diameter of A .

The *geometric dimension* of a CAT(0)-space X is the maximal topological dimension of all compact subsets in X [30].

Remark 6 *It turns out that the geometric dimension of a CAT(0)-space X is less than n if and only if Jung's inequality (4.1) is true in X [31].*

The geometric dimension can be transformed into a large-scale term in the following way. A CAT(0)-space X has telescopic dimension less than n if and only if for every $\delta > 0$ there is a $D > 0$ such that for every set A in X with diameter greater than D ,

$$r \leq \left(\delta + \sqrt{\frac{n}{2(n+1)}} \right) d(A). \quad (2.6)$$

Somewhat more general,

Definition 18 *A space X whose asymptotic cones are CAT(0) has telescopic dimension $\leq n$ if every asymptotic cone has geometric dimension $\leq n$.*

We adopt the notation $\text{tele-dim}(X)$ for the telescopic dimension of X .

Theorem 15 *Let X and Y be spaces, whose asymptotic cones are CAT(0). If there exists an AM-map $f : X \rightarrow Y$ preserving diverging and bounded sequences, then the telescopic dimension increases e.g. $\text{tele-dim}(X) \leq \text{tele-dim}(Y)$.*

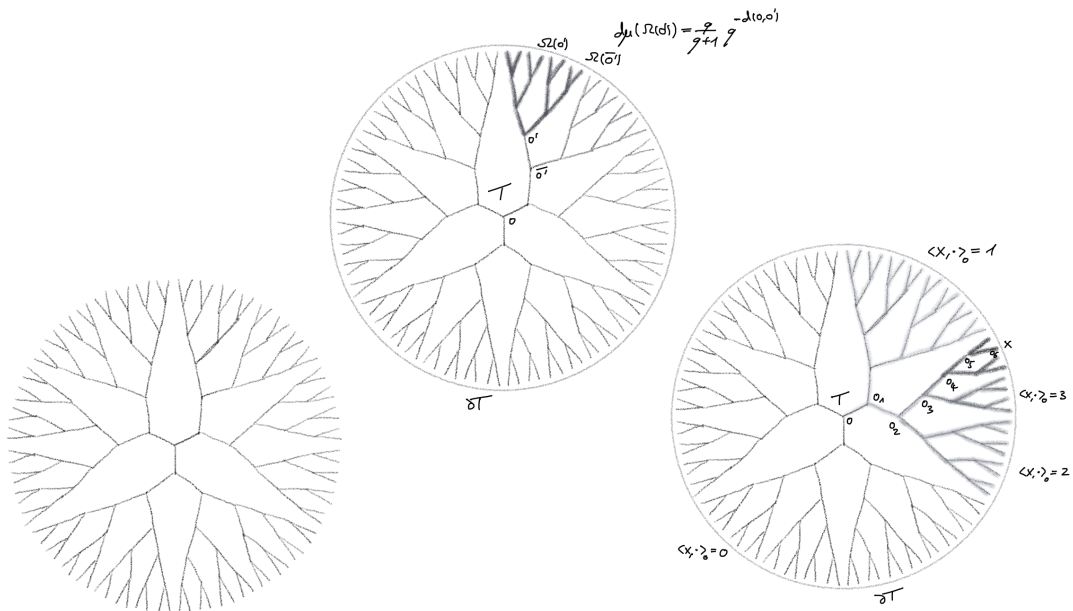
In particular, if Y is hyperbolic, so is X .

2.2.2 Conclusion and perspectives

A tantalizing problem is the following. Quasisymmetric mappings between Carnot groups are coarsely conformal in the sense of Pansu, see [23]. Therefore, they raise the Hausdorff dimension by Pansu's theorem in [23]. However, nothing is said if the Carnot groups have equal Hausdorff dimensions. Can something be said in the case of asymptotic-Möbius maps? For example, in the case of the Heisenberg group to \mathbb{R}^4 .

Chapter 3

Harmonic analysis on the boundary of hyperbolic groups



3.1 Introduction

3.1.1 Representation theory of hyperbolic groups.

Kazhdan's Property (T). A σ -compact topological group G has the Kazhdan property (T), if every affine-isometric action of G on a real Hilbert space has a fixed point. The original 1968 definition of Kazhdan, is formulated in terms of "almost" invariant vectors of representations of G on complex Hilbert spaces. The equivalence of the two formulations is known as the Delorme-Guichardet Theorem. Kazhdan introduced property (T) as a tool to prove finite generation of nonuniform lattices in simple Lie groups of higher rank. It is a representation-theoretic form of rigidity that has similarities to geometric superrigidity [37]. The property plays a central role in the theory of operator algebra and in geometric group theory. An excellent monograph on property (T) was written by Bekka, de la Harpe and Valette [10].

Example 1 For a local field \mathbb{K} , the groups $\mathrm{SL}_n(\mathbb{K})$, $n \geq 3$, and $\mathrm{Sp}_{2n}(\mathbb{K})$, $n \geq 2$, have property (T). More generally, any simple real Lie group of rank at least two has the Kazhdan property (T).

Example 2 The isometry group $\mathrm{Sp}(n, 1)$, $n \geq 2$, of the hyperbolic space over the quaternions, has property (T).

The Kazhdan property (T) shows up *geometrically* in affine isometric actions on real Hilbert spaces, *algebraically* in group cohomology with coefficients in orthogonal representations, and *analytically* as functions of conditional negative type. The connections between the three perspectives are quite straightforward. If A is an affine isometric action of a topological group G on a real Hilbert space, then the linear part π is an orthogonal representation of G , and the translation part b is a 1-cocycle with coefficients in π . b defines a class in $H^1(G, \pi)$ and $g \mapsto \|b(g)\|^2$ is a conditional negative type function on G .

In this paper we take the geometric point of view on Kazhdan's property (T).

Weak a-T-menability. It is known that hyperbolic groups can be divided into two classes: those with and those without the Kazhdan property (T).

A strong negation of the Kazhdan property (T) is the Haagerup property, also known as a-T-menability. A locally compact topological group G has the *Haagerup property* if there exists a proper affine-isometric action of G on a real Hilbert space. Morally, such a group is strongly nonrigid.

Hyp. groups with Haagerup's property	Hyp. groups with property (T)
Free groups, groups acting properly on trees, Fuchsian groups, $\mathrm{SO}(n, 1)$, $\mathrm{SU}(n, 1)$ Random groups at density $d < \frac{1}{6}$ [12]	$\mathrm{Sp}(n, 1)$, $\mathrm{F}_{4(-20)}$ Random groups at density $\frac{1}{3} < d < \frac{1}{2}$ [13][14]

Groups having property (T) do not admit unitary representations with 1-cohomology.

Higher rank Lie groups (rank ≥ 2) satisfy a certain stronger form of Kazhdan's property (T). Namely, higher rank Lie groups do not admit uniformly bounded representations with reduced cohomology.

Conversely, all rank one Lie groups admit uniformly bounded representations with reduced cohomology. This motivates the question whether all hyperbolic groups admit uniformly bounded representations with cohomology. An even stronger conjecture is often quoted and is dedicated to Y. Shalom [38].¹

Conjecture 1 (Y. Shalom) *Let G be a nonelementary hyperbolic group. There exists a uniformly bounded representation of G on a Hilbert space, for which $H^1(G, \pi) \neq 0$ and for which there exists a proper 1-cocycle.*

1. First mentioned in: Mini-Workshop: Geometrization of Kazhdan's Property (T), Mathematisches Forschungsinstitut Oberwolfach, July 7th - July 14th, 2001, Talk: Reduced cohomology of unitary representation, Y. Shalom, (Report No. 29/2001) (see question 14 in open problem session)

Shalom's conjecture is known to be true for lattices in $\mathrm{Sp}(n, 1)$ [35]. It also leads us to a conjectured classification of hyperbolic groups due to Cowling. Namely, if Shalom's conjecture is true, then we can associate to each hyperbolic group G a number $\Lambda(G)$ as follows:

$$\Lambda(G) := \inf_{\pi} \sup_{g \in G} \|\pi_g\|, \quad (3.1)$$

where the infimum is taken over all uniformly bounded representations with cohomology.

This invariant measures, in some sense, the degree to which the strong form of the Kazhdan property (T) fails in G . If there are no uniformly bounded representations with cohomology, we define $\Lambda(G)$ to be infinite.

$\Lambda(G)$ is presumably related to an invariant defined by Cowling and Haagerup [39] [40]. Cowling believes that $\Lambda(G)$ is related to the curvature of the hyperbolic group (with the word metric).

Not much is known about $\Lambda(G)$ in general. The only result in this direction is a spectral gap theorem for automorphism groups of simplicial complexes [41].

3.1.2 Fractional calculus.

At first glance, the theory of fractional calculus has little to do with the abstract representation theory and the questions therein that we addressed above. We shall see, however, that the two topics are closely related.

The classical noncompact picture.

Mon idée fondamentale est de généraliser les principes de l'analyse ordinaire, qui ne considère que les différentielles dy, d^2y, d^3y, \dots , dont les indices sont entiers positifs, et les intégrales $\int y dx, \int^2 y dx^2, \int^3 y dx^3, \dots$, que l'on [...] peut regarder comme des différentielles à indices entiers négatifs. Je propose de considérer aussi les différentielles à indices fractionnaires, et même à indices irrationnels et imaginaires. (J. Liouville, JEP, 1832)

Liouville's 1832 work, [42], marks the starting point for the study of fractional integration and differentiation. For a locally integrable function f on \mathbb{R} and a complex number α in the half-plane $\Re(\alpha) > 0$, Liouville introduces the function

$$I^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x f(t)(x-t)^{\alpha-1} dt, \quad (3.2)$$

where Γ is the gamma function and a is any base point.

The operator I^α satisfies

$$\frac{d}{dx} I^{\alpha+1} f(x) = I^\alpha f(x), \quad I^\alpha(I^\beta f) = I^{\alpha+\beta} f. \quad (3.3)$$

It is therefore useful to define *fractional differentiation* by taking enough ordinary derivatives of $I^\alpha f$.

It's worth noting that a formal treatment of the integral $I^\alpha f$ can also be found in unpublished notes by Riemann from 1847 entitled "Versuch einer allgemeinen

Auffassung der Integration und Differentiation" [43]. At that time Riemann was still a student at the University of Göttingen and most likely had no knowledge of Liouville's work. There is a nice article on Riemann's notes by Hermann Weyl from 1917 [44].

A systematic treatment of the properties of "Riemann-Liouville" integrals and derivatives for certain classes of functions ($L^p(\mathbb{R})$, Lipschitz, etc.) was given by Hardy and Littlewood in 1928 [45].

In 1949 M. Riesz published a paper which generalized the "Riemann-Liouville" integrals to higher dimensions and gave meaning to the operator I^α for negative α by means of analytic continuation [46]. In his honor the operators I^α on \mathbb{R}^n are known today as *Riesz-potential operators*.

Today these operators are studied on spaces of *homogeneous type*, i.e., on metric spaces with a doubling measure compatible with the topology (the metric must satisfy the triangle inequality only up to a constant).

The compact picture. Let \mathbb{S}^1 be the circle considered as the boundary of the open disk $\mathbb{D} = \{z : |z| < 1\}$ in the complex plane \mathbb{C} . We can endow \mathbb{S}^1 with a metric as follows. Given two points $x, y \in \mathbb{S}^1$, let $\theta(x, y) \in [0, \pi]$ be the angle at the origin 0 between x and y . Then

$$d_0(x, y) = \sin \frac{1}{2}\theta(x, y),$$

is a metric on \mathbb{S}^1 . We can forget about normalization and naively define fractional integration on \mathbb{S}^1 as

$$I_0^\alpha f(x) = \int_{\mathbb{S}^1} \frac{f(y)}{d_0(x, y)^{1-\alpha}} dy. \quad (0 < \alpha < 1)$$

Here dy is simply the ordinary Lebesgue measure by identifying \mathbb{S}^1 with $[0, 1]$. This definition turns out to be 'natural' in a very precise sense.

We endow \mathbb{D} with the Poincaré metric

$$d(z_1, z_2) = 2 \tanh^{-1} \left| \frac{z_1 - z_2}{1 - z_1 \bar{z}_2} \right|.$$

Given two points $x, y \in \mathbb{S}^1$ and one point $z \in \mathbb{D}$, then the three points x, y, z form an ideal triangle in \mathbb{D} . Let $\theta_z(x, y) \in [0, \pi]$ be the angle at z of this ideal triangle. Then again

$$d_z(x, y) = \sin \frac{1}{2}\theta_z(x, y),$$

is a metric on \mathbb{S}^1 . We call it the *visual metric* on \mathbb{S}^1 related to the point z . Let μ_z be the harmonic measure on \mathbb{S}^1 with respect to the point z , then we can define

$$I_z^\alpha f(x) = \int_{\mathbb{S}^1} \frac{f(y)}{d_z(x, y)^{1-\alpha}} d\mu_z(y). \quad (0 < \alpha < 1) \quad (3.4)$$

This definition agrees with the above when z is the origin. It turns out that I_z^α "does not depend on the choice of z " if we consider f not as a function but as $(\frac{1}{2} + \alpha)$ -density. An adhoc definition of densities on the boundary of the Poincaré disc goes as follows.

Definition 1 For $s \in \mathbb{R}$, an s -density on \mathbb{S}^1 is a map $f : z \mapsto f_z$ that associates a measurable function f_z with each point in \mathbb{D} such that for all points $z, z' \in \mathbb{D}$,

$$f_z = \left(\frac{d\mu_{z'}}{d\mu_z} \right)^s f_{z'}. \quad (3.5)$$

One can check that if f_z is a continuous $(\frac{1}{2} + \alpha)$ -density, then $z \mapsto I_z^\alpha f_z$ is a $(\frac{1}{2} - \alpha)$ -density.

We note that 1-densities are signed measures on \mathbb{S}^1 which are absolutely continuous with respect to harmonic measures. As α converges to 1, the operator I_z^α converges to an operator I^1 which takes a signed-measure μ to its total mass $\mu(\mathbb{S}^1)$.

3.1.3 Principal and complementary series representations.

s -densities and fractional integration and differentiation appear naturally in the context of induced representations. We give a brief overview of this connection in the context of Lie group theory. For a general survey see [47, Chapter VII, pp.167].

Let G be a connected semisimple Lie group with finite center. Let KAN be an Iwasawa decomposition of G . We denote M and M' as the centralizer and normalizer, respectively, of A in K .

MAN is the minimal standard parabolic subgroup of G , more compactly we write H for the subgroup MAN . For $s \in \mathbb{R}$, an s -density on the smooth manifold G/H is a tensorial object v which has only one component in coordinates and which transforms under diffeomorphisms according to the rule:

$$\phi^* v = |\det(d\phi)|^s (v \circ \phi).$$

Densities can be multiplied: The pointwise product of an s -density with an s' -density, is an $(s + s')$ -density. 1-densities coincide with smooth (absolutely continuous) measures on G/H . Formally, s -densities are smooth sections of a G -homogeneous line bundle $\mathcal{D}_s(G/H)$. The action of G on $\mathcal{D}_s(G/H)$ is given by $g \cdot v = (l_g^{-1})^* v$, where l_g is the left-translation action of G on G/H .

The finite dimensional irreducible unitary representations of MAN are all of the form $(\sigma \otimes \lambda) : man \mapsto a^\lambda \sigma(m)$, where $a \mapsto a^\lambda \in \mathbb{C}$ is a unitary character of A and σ is an irreducible unitary representation of M .

The characters $a \mapsto a^\lambda$ of A are constructed as follows. Given $\lambda \in \mathfrak{a}_\mathbb{C}^* := \text{Hom}(\mathfrak{a}, \mathbb{C})$, for $a \in A$, we write

$$a^\lambda = e^{\lambda \log a} \in \mathbb{C}.$$

The unitary characters are given by all $\lambda \in i\mathfrak{a}^*$.

The category of continuous representations of H is equivalent to the category of G -homogeneous vector bundles on G/H , by the associated vector bundle construction. The G -homogeneous vector bundle $\mathcal{D}_s(G/H)$ of s -densities, is the associated bundle of the character

$$\delta^s : a \mapsto e^{s \text{tr}(\text{ad}(\log a)|_{\mathfrak{n}})}.$$

One checks that $\delta^s(a) = |\det \text{Ad}_{\mathfrak{g}/\mathfrak{h}}(man)|^{-s}$, for all $m \in M, n \in N$, where $\text{Ad}_{\mathfrak{g}/\mathfrak{h}}$ is the adjoint-representation of H on $\mathfrak{g}/\mathfrak{h}$.

More generally, any continuous representation (σ, \mathcal{H}) of H is associated with a G -homogeneous vector bundle \mathcal{V} . By $\Gamma(\mathcal{V})$ denote the space of continuous sections of \mathcal{V} . The induced representation of G from the representation (σ, \mathcal{H}) of H , is the following action of G on $\Gamma(\mathcal{V})$:

$$(\pi(g)s)(x) = g \cdot (s(g^{-1}x)). \quad (s \in \Gamma(\mathcal{V}), x \in G/H, g \in G)$$

This representation π is usually denoted by $\text{Ind}_H^G(\sigma)$.

The representation $\text{Ind}_H^G(\sigma)$ is said to be unitarizable if the space $\Gamma(\mathcal{V})$ admits such a pre-Hilbert structure that π extends to a unitary representation on the associated Hilbert completion. In general, unitarity of (σ, \mathcal{H}) does not imply unitarity of $\text{Ind}_H^G(\sigma)$. However, unitarity can be achieved by twisting the representation σ with half-densities.

Indeed, we can tensorize σ with $\delta^{\frac{1}{2}}$ to obtain a representation $\sigma \otimes \delta^{\frac{1}{2}}$ of H in $\mathcal{H} \otimes \mathcal{D}_{\frac{1}{2}}(\mathfrak{g}/\mathfrak{h})$. Given $a, b \in \mathcal{H}$ and $u, v \in \mathcal{D}_{\frac{1}{2}}(\mathfrak{g}/\mathfrak{h})$, we define $(a \otimes u, b \otimes v) = \langle a, b \rangle_{\mathcal{H}} u \bar{v}$, and extend it to a sesquilinear form $\mathcal{H} \otimes \mathcal{D}_{\frac{1}{2}}(\mathfrak{g}/\mathfrak{h}) \times \mathcal{H} \otimes \mathcal{D}_{\frac{1}{2}}(\mathfrak{g}/\mathfrak{h}) \rightarrow \mathcal{D}_1(\mathfrak{g}/\mathfrak{h})$. Thus, for the sections $A, B \in \Gamma(\mathcal{V} \otimes \mathcal{D}_{\frac{1}{2}}(G/H))$ their product (A, B) is in $\Gamma(\mathcal{D}_1(G/H))$, leading to a measure. We set

$$\langle A, B \rangle = \int_{G/H} (A, B).$$

The action of $\pi = \text{Ind}_H^G(\sigma \otimes \delta^{\frac{1}{2}})$ in $\Gamma(\mathcal{V} \otimes \mathcal{D}_{\frac{1}{2}}(G/H))$ is unitary, since $(\pi(g)A, \pi(g)B) = ((l_g^{-1})^*A, (l_g^{-1})^*B) = (l_g^{-1})^*(A, B)$, and the integral of the density $(l_g^{-1})^*(A, B)$ over G/H agrees with the total mass of (A, B) .

The *principal series* of unitary representations of G is parameterized by all (σ, λ) such that $\sigma \otimes \lambda$ is unitary, and is obtained by normalized induction from MAN to G . This means

$$\pi_{\sigma, \lambda} = \text{Ind}_H^G((\sigma \otimes \lambda) \otimes \delta^{\frac{1}{2}}).$$

The induction of representations on G works even if $a \mapsto a^\lambda$ is not a unitary character. Knapp and Stein call the representations $\pi_{\sigma, \lambda}$ with λ nonunitary, the *nonunitary principal series*. Roughly speaking, the *complementary series* consists of those representations in the nonunitary principal series which can be made unitary by changing the inner product.

From now on let G be of real rank 1, and $m' \in M' \setminus M$. It turns out that a necessary condition for $\pi_{\sigma, \lambda}$ to be unitarizable is that $\sigma^{m'}(m) = \sigma(m'mm'^{-1})$ is equivalent to σ , and λ has the form

$$\lambda(a) = \delta^\alpha(a),$$

with $\alpha \in \mathbb{R}$. Then

$$\pi_{\sigma, \lambda} = \text{Ind}_H^G((\sigma \otimes \delta^\alpha) \otimes \delta^{\frac{1}{2}}) = \text{Ind}_H^G(\sigma \otimes \delta^{\frac{1}{2} + \alpha}),$$

and the inner product necessarily comes from an operator $\mathcal{I}^\alpha : \mathcal{D}_{\frac{1}{2} + \alpha}(G/H) \rightarrow \mathcal{D}_{\frac{1}{2} - \alpha}(G/H)$, which maps $(\frac{1}{2} + \alpha)$ -densities to $(\frac{1}{2} - \alpha)$ -densities [48]. For $\alpha > 0$ such operators were found by Kunze and Stein in 1964 [49]. These operators amount to fractional integration operators on $K/M \cong G/H$, as does the operator (3.4) on \mathbb{S}^1 .

Knapp and Stein observed that the existence of complementary series representations (i.e., the positivity of \mathcal{I}^α) is closely related to the analytic continuation of \mathcal{I}^α to negative values of α [50].

The following result is essentially due to B. Kostant [51].

Theorem 2 (Kostant) *If $G = \mathrm{SO}(n, 1)$ or $\mathrm{SU}(n, 1)$, then \mathcal{I}^α is positive definite for any $\alpha \in]0, \frac{1}{2}[$. If $G = \mathrm{Sp}(n, 1)$, then it is positive definite if and only if $\alpha < \frac{1}{2} - \frac{1}{2(2n+1)}$.*

Thus, if $G = \mathrm{SO}(n, 1)$ or $\mathrm{SU}(n, 1)$, then the representations

$$\pi_\alpha = \mathrm{Ind}_H^G(1_H \otimes \delta^{\frac{1}{2}+\alpha}), \quad (0 < \alpha < \frac{1}{2})$$

interpolate between the quasiregular representation $\lambda_{G/H} = \mathrm{Ind}_H^G(1_H \otimes \delta^{\frac{1}{2}})$, which is weakly contained in the regular representation λ_G , and the trivial representation 1_G when α tends to $\frac{1}{2}$.

To understand the latter statement, recall the above case of \mathbb{S}^1 , that

$$I_z^\alpha f(x) = \int_{\mathbb{S}^1} \frac{f(y)}{d_z(x, y)^{1-\alpha}} d\mu_z(y). \quad (0 < \alpha < 1)$$

tends to an operator $I^1 : \mu \mapsto \mu(\mathbb{S}^1)$ on signed measures on \mathbb{S}^1 when α tends to 1 (note that we scaled α by 2). Therefore, the diagonal matrix coefficients $g \rightarrow (\pi_{1-\epsilon}(g)d\mu^{1-\epsilon}, d\mu^{1-\epsilon})$ tend to $g \rightarrow (\int_{\mathbb{S}^1} d(l_g^{-1})^*\mu)(\int_{\mathbb{S}^1} d\mu)$, where $d\mu^{1-\epsilon}$ denotes $(1-\epsilon)$ -densities. The integral of $(l_g^{-1})^*\mu$ over \mathbb{S}^1 agrees with the total mass of μ , thus the diagonal matrix coefficients converge to constant functions, showing that $\pi_\alpha \rightarrow 1_G$ in the Fell topology.

3.1.4 Conformal invariance of differential operators.

Densities reappear in connection with the conformal invariance of natural differential operators.

Natural bundles in Riemannian geometry are functors T from the category of Riemannian manifolds into the category of vector bundles belonging to representations of the general linear group. Natural operators in Riemannian geometry are natural transformations between natural bundles which associate differential operators $A_g : T(M, g) \rightarrow T'(M, g)$ with Riemannian metrics g in such a way that if ϕ is a diffeomorphism,

$$A_{\phi^*g} = \phi^*(A_g).$$

Traditionally, such a natural transformation is said to be *conformally invariant* if, in addition, there exist real numbers a and b , so that if a Riemannian metric g is conformally changed to $g' = e^{2w}g$, then

$$A_{g'} = e^{-bw} \circ A_g \circ e^{aw}.$$

This means that A can be transformed into an exactly conformally invariant operator \mathcal{A} from sections of $D_{a/Q} \otimes T$ to sections of $D_{b/Q} \otimes T'$, $Q = \text{dimension}$.

GJMS-operators On any flat conformal manifold of dimension $Q \geq 3$, for every $k \in \mathbb{Z}^{>0}$ there is a unique natural conformally invariant operator

$$\mathcal{L}^k : \mathcal{D}_{\frac{1}{2} - \frac{k}{Q}} \rightarrow \mathcal{D}_{\frac{1}{2} + \frac{k}{Q}}, \quad (3.6)$$

whose leading part is the k -th power of the Beltrami Klein operator.

For $k = 1$ this operator is the usual conformal Laplace operator, for $k = 2$ this operator is commonly known as the Paneitz operator. If Q is even, then $\mathcal{L}^{\frac{Q}{2}}$ exists and is known as the *critical GJMS-operator*. It maps a function to a multiple of the volume form [52].

Note that $\frac{1}{2} - \frac{k}{Q} + \frac{1}{2} + \frac{k}{Q} = 1$, so the quadratic form

$$v \mapsto \int v \mathcal{L}^k v \quad (3.7)$$

is conformally invariant on $(\frac{1}{2} - \frac{k}{Q})$ -densities of Sobolev class \mathcal{H}^k .

For $k = 1$, this quadratic form is positive on the Q -sphere, $Q \geq 3$, since the scalar curvature is positive.

Definition 2 *On a Q -dimensional Riemannian manifold with scalar curvature R , the conformal Laplacian is the operator acting on smooth functions u through*

$$Lu = \Delta u + \frac{Q-2}{4(Q-1)}Ru.$$

Graham and Zworski further study natural conformally invariant operators $\mathcal{L}^\alpha : \mathcal{D}_{\frac{1}{2} - \frac{\alpha}{Q}} \rightarrow \mathcal{D}_{\frac{1}{2} + \frac{\alpha}{Q}}$, $\alpha \in (0, \frac{Q}{2})$ using scattering theory on Poincaré-Einstein manifolds. They recover the above operators \mathcal{L}^k when α is an integer [53].

sub-GJMS-operators? On contact manifolds of dimension $Q = 2n + 1$, for every $k \in \{1, \dots, n + 1\}$ there exists a natural conformally invariant operator

$$\mathcal{L}^k : \mathcal{D}_{\frac{1}{2} - \frac{k}{Q+1}} \rightarrow \mathcal{D}_{\frac{1}{2} + \frac{k}{Q+1}}, \quad (3.8)$$

whose leading part is the k -th power of the sub-Laplacian [54]. For $k = 1$ this operator is the CR invariant sub-Laplacian of Jerison-Lee [55].

Roncal and Thangavelu describe natural conformally invariant operators \mathcal{L}^α on the Heisenberg group for all positive $\alpha < \frac{n+1}{2}$ [56]. This and Theorem 2.7 from [18] suggest that fractional GJMS-operators also exist in sub-Riemannian geometry.

If we believe in the existence of natural conformally invariant operators \mathcal{L}^k on sub-Riemannian manifolds whose leading part is the k -th power of the sub-Laplacian, then

$$v \mapsto \int v \mathcal{L}^k v \quad (3.9)$$

gives conformally invariant quadratic forms on all symmetric spaces of rank 1.

3.1.5 From “exact” to “almost” conformal invariance.

Natural conformally invariant operators give us conformally invariant quadratic forms. In general, they are not positive definite. To obtain a pre-Hilbert space structure on s -densities, we can replace the exact conformally invariant operator \mathcal{L}^α by the power α of the conformal Laplacian L .

Since the conformal Laplacian L is self-adjoint and positive, L^α is well-defined. Taking powers breaks the conformal invariance. Nevertheless, L^α defines for any $s \in \mathbb{R}$ an *scale invariant* operator on s -densities, which we denote by

$$\mathcal{L}_s^\alpha : \mathcal{D}_s \rightarrow \mathcal{D}_{s+\frac{2\alpha}{Q}}.$$

Scale invariant means that \mathcal{L}_s^α is unchanged when the metric is multiplied by a constant factor. We obtain a scale invariant inner product Q^α on $\frac{1}{2} - \frac{\alpha}{Q}$ -densities of Sobolev class \mathcal{H}^α . We can hope that the representation of G in $(\mathcal{D}_{\frac{1}{2}-\frac{\alpha}{Q}}(K/M), Q^\alpha)$ is uniformly bounded.

The Cayley transform trick. Let N denote the unipotent component of the Iwasawa decomposition $G = KAN$, and let M be the centralizer of A in K . N is a simply connected nilpotent Lie group acting simply transitively on the complement of a point of K/M . The orbit map $\mathcal{C} : N \rightarrow K/M$ is called the *Cayley transform*.

Since N is a group of type H , its Lie algebra \mathfrak{n} decomposes into a direct product $\mathfrak{v} \oplus \mathfrak{z}$, where $[\mathfrak{v}, \mathfrak{v}] = \mathfrak{z}$. The exponential mapping is a bijection of \mathfrak{n} onto N , and for X in \mathfrak{v} and Z in \mathfrak{z} we can denote by (X, Z) the element $\exp(X + Z)$ in N .

The Korányi norm on N is given by

$$\mathcal{N}(X, Z) = \left(\frac{|X|^4}{16} + |Z|^2 \right)^{\frac{1}{4}}.$$

Setting $d_N(n_1, n_2) = \mathcal{N}(n_1^{-1}n_2)$, gives N the structure of a Carnot-Carathéory manifold.

There exists a metric $d_{K/M}$ on K/M such that

$$d_{K/M}(\mathcal{C}((X, Z)), \mathcal{C}((X', Z'))) = \frac{d_N((X, Z), (X', Z'))}{\left((1 + \frac{|X|^2}{4})^2 + |Z|^2 \right)^{\frac{1}{4}} \left((1 + \frac{|X'|^2}{4})^2 + |Z'|^2 \right)^{\frac{1}{4}}}$$

The Cayley transform can be viewed as a means of flattening the metric $d_{K/M}$ on K/M . In the case of real hyperbolic space, it is a stereographic projection. In all cases it is conformal (Corollary 2.3 in [18]), but far from homothetic.

The Cayley transform conjugates the A -subgroup in the Iwasawa decomposition with the 1-parameter group of dilations (δ_t) of N , which are homothetic.

$$\delta_t(X, Z) = (tX, t^2Z) \quad \forall (X, Z) \in N.$$

Imagine one can prove that pulling-back densities

$$\mathcal{C}^* : (\mathcal{D}_{\frac{1}{2}-\frac{\alpha}{Q}}(K/M), Q^\alpha) \rightarrow (\mathcal{D}_{\frac{1}{2}-\frac{\alpha}{Q}}(N), Q^\alpha)$$

is invertible. Then for each element $a \in A$ there is $t \in \mathbb{R}$ such that $a = \mathcal{C} \circ \delta_t \circ \mathcal{C}^{-1}$. With respect to the inner product Q^α , \mathcal{C} is invertible and δ_t is isometric. Therefore a is invertible with norm $\leq \|\mathcal{C}\| \|\mathcal{C}^{-1}\|$ independent of a .

Let us use the Bruhat decomposition $G = KAK$. Then K acts isometrically on $(\mathcal{D}_{\frac{1}{2}-\frac{\alpha}{Q}}(K/M), Q^\alpha)$. This shows that the G action is uniformly bounded, with norm $\leq \|\mathcal{C}\|\|\mathcal{C}^{-1}\|$. Proving the boundedness of the representation of G in the pre-Hilbert space $(\mathcal{D}_{\frac{1}{2}-\frac{\alpha}{Q}}(K/M), Q^\alpha)$ amounts to estimating a single operator. This was done by Astengo, Cowling, and Di Blasio in [18].

They obtain Hilbert structures such that $\{\pi_\alpha = \text{Ind}_H^G(1_H \otimes \delta^{\frac{1}{2}+\frac{\alpha}{Q}})\}_{\alpha \in (-\frac{Q}{2}, \frac{Q}{2})}$ become uniformly bounded representations of G , which morally “interpolate” between the quasiregular representation $\lambda_{G/H} = \text{Ind}_H^G(1_H \otimes \delta^{\frac{1}{2}})$ and the trivial representation 1_G of G .

This fact is an essential step in Julg’s program to prove the Baum-Connes conjecture with coefficients for $\text{Sp}(n, 1)$ and $F_{4(-20)}$, i.e., for simple Lie groups of real rank 1 with Kazhdan property (T) [57].

Remark 1 *The bound obtained by Astengo, Cowling and Di Blasio is not uniform in α and blows up when α approaches $\frac{Q}{2}$.*

3.1.6 From Lie groups to Q -Ahlfors-David regular spaces.

General Gromov hyperbolic groups act conformally on Q -Ahlfors-David regular metric spaces. In order to construct uniform bounded representations for all hyperbolic groups that interpolate between the quasiregular representation and the trivial representation, we would like to apply the ideas of Astengo, Cowling and Di Blasio to this setting. However, we know of no analog of a conformal Laplacian in this context. Therefore, we assume its dual description instead.

Let K^α be a fundamental solution of a differential operator \mathcal{L}^α , i.e., a distribution K^α such that $\mathcal{L}^\alpha K^\alpha = \delta$. Then, for any smooth compactly supported function u , $\mathcal{L}^\alpha(K^\alpha \star u) = (\mathcal{L}^\alpha K^\alpha) \star u = \delta \star u = u$. In other words, the operator I^α , defined by

$$I^\alpha u = K^\alpha \star u : x \mapsto \int_N K^\alpha(xy^{-1})u(y) dy$$

is $(\mathcal{L}^\alpha)^{-1}$. On the Heisenberg group, the natural conformally invariant operator \mathcal{L}^α of Roncal and Thangavelu has as fundamental solution a constant multiple of $|x|^{-Q+2\alpha}$ [56]. On Iwasawa N -groups this distribution is exactly the kernel of the intertwining operators of Knapp and Stein [50].

In this work we work in the category of Möbius spaces (Z, \mathcal{M}) , i.e. sets Z together with a collection \mathcal{M} of metrics defining the same metric cross-ratios. With \mathcal{M} we associate a class of s -densities $\mathcal{D}_s(\mathcal{M})$ and the Knapp-Stein operator

$$(I_d^\alpha u_d)(x) = \int \frac{u_d(y)}{d(x, y)^{Q-\alpha}} d\mu_d(y), \quad (0 < \alpha < \frac{Q}{2})$$

where μ_d is the Q -dimensional Hausdorff measure associated with the metric d , and u_d is an s -density, where s is a function of α .

Results. We use an analog of the Cayley transform of Astengo, Cowling and Di Blasio on Möbius spaces to prove strong regularity properties of Möbius spaces as measure spaces.

Theorem 3 *If for a Möbius structure \mathcal{M} there exists $d \in \mathcal{M}$ such that μ_d is Q -Ahlfors-David regular, then there exists $C \geq 1$ such that every measure μ_d , $d \in \mathcal{M}$, is Ahlfors-David regular with constant C .*

We use this observation in the context of strongly hyperbolic metric spaces X to show in Theorem 7 that powers of conformal factors $\frac{d'}{d}$ associated to the class of visual metrics \mathcal{M} on ∂X are Muckenhoupt weights with uniformly bounded A_p -constant on $L^p(\mu_d)$, independent of d and d' in \mathcal{M} .

From there we prove the following weighted bound on the Knapp-Stein operator.

Theorem 9 *Let X be a strongly hyperbolic space, $\{d_o\}_{o \in X}$ be the natural Möbius structure on ∂X , and $0 < \alpha < \frac{Q}{q}$. Then there is a constant $C_\alpha > 0$ that depends only on α , such that for all $d_o, d_{o'} \in \mathcal{M}$ and $f \in L^p(\mu_o)$,*

$$\|(\frac{d_{o'}}{d_o})^{\frac{\alpha}{2}} \circ I_o^\alpha \circ (\frac{d_o}{d_{o'}})^{-\frac{\alpha}{2}} f\|_{L^q(\mu_o)} \leq C_\alpha \|f\|_{L^p(\mu_o)}, \tag{3.10}$$

where $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{Q}$.

The Knapp-Stein operator is self-adjoint on $L^2(d\mu_d)$ and its square has a kernel comparable to $d(x, y)^{-Q+2\alpha}$. Thus $(I_d^\alpha)^2$ is a good candidate for a Laplacian raised to the power $-\alpha$. Our guess for the $\mathcal{H}^{-\alpha}$ -norm on $(\frac{1}{2} + \frac{\alpha}{Q})$ -densities on compact Q -Ahlfors regular Möbius spaces is

$$\|u\|_d = \|I_d^\alpha u\|_{L^2(d\mu_d)}. \tag{3.11}$$

By construction it is positive and leads to a Sobolev space $\mathcal{H}_d^{-\alpha}$ of $(\frac{1}{2} + \frac{\alpha}{Q})$ -densities. We would like to say that the construction is topologically independent of the metric d . In Theorem 11, we use the previous results from this paper to prove that the norms $\|\cdot\|_d$ are comparable on a large class of functions.

3.2 Möbius spaces

3.2.1 Some basics on Möbius spaces

According to Buyalo and Schröder [1], a Möbius structure \mathcal{M} on a set Z is a collection of metrics defining the same metric cross-ratios. More precisely, one considers quadruples (x, y, z, w) of points in Z where no entry occurs three or four times. Such an *admissible* quadruple is associated with a *cross-ratio triple*

$$[x, y, z, w]_d = (d(x, y)d(z, w) : d(x, z)d(y, w) : d(x, w)d(y, z)) \in \mathbb{R}P^2. \tag{3.12}$$

Two metrics d and d' on Z are called *Möbius equivalent* if for each admissible quadruple their respective cross-ratio triples coincide.

Indeed, the theory makes it necessary to consider *extended* metrics on Z which allow the existence of a point $\bullet \in X$, such that $d(x, \bullet) = \infty$ for all $x \neq \bullet$. Such a point \bullet is always assumed to be unique and $d(\bullet, \bullet) = 0$. We call such a point an *infinitely distant* point and denote it always by \bullet . If such a point exists, we sometimes use the notation Z_\bullet for the set $Z \setminus \{\bullet\}$. If \bullet occurs once in an admissible quadruple, then the associated cross-ratio triple is $[x, y, z, \bullet]_d = (d(x, y) : d(x, z) : d(y, z))$. If it occurs twice, then $[x, y, \bullet, \bullet]_d = (0 : 1 : 1)$.

A *Möbius structure* on a set Z is a class \mathcal{M} of metrics on Z which are pairwise Möbius equivalent.

A Möbius structure \mathcal{M} defines a unique topology $\mathcal{T}_\mathcal{M}$ called the semimetric-topology, see [58]. If \mathcal{M} is induced by a metric d , then the topology \mathcal{T}_d induced by d agrees with the semimetric-topology $\mathcal{T}_\mathcal{M}$ [59, Theorem A].

Two metrics d and d' in a Möbius structure are conformal in the strong sense that the limiting ratio $\frac{d'}{d}(x) := \lim_{y \rightarrow x} \frac{d'(x, y)}{d(x, y)}$ exists for all $x \in Z \setminus \{\bullet\}$ [3, Lemma 6]. Moreover, for all $x \neq y \in Z \setminus \{\bullet\}$,

$$\frac{d'}{d}(x) \frac{d'}{d}(y) = \left(\frac{d'(x, y)}{d(x, y)} \right)^2. \quad (3.13)$$

Following Nica [3] and Sullivan [60, §4], we interpret this relation as a geometric mean-value property.

Examples of sets with natural Möbius structures are:

1. The round sphere,
2. Boundaries of symmetric spaces of rank one,
3. Boundaries of CAT(-1) spaces [2],
4. Boundaries of strongly hyperbolic metric spaces [3],
5. Boundaries of symmetric spaces of higher rank and Euclidean buildings [4], [5],
6. Boundaries of certain CAT(0)-cube complexes [5], [6].

A Möbius structure \mathcal{M} is associated with a family of Hausdorff measures $\{\mu_d\}_{d \in \mathcal{M}}$ with a uniform Hausdorff dimension Q [61].

For two metrics d, d' in a Möbius structure \mathcal{M} , the Radon-Nikodým derivative $\frac{d\mu_d}{d\mu_{d'}}$ is connected to the conformal factor $\frac{d}{d'}$ by

$$\frac{d\mu_d}{d\mu_{d'}}(x) = \frac{d}{d'}(x)^Q \quad \forall x \in Z \setminus \{\bullet\}, \quad (3.14)$$

see [3, Lemma 8].

We say that a Möbius space (Z, \mathcal{M}) or a Möbius structure \mathcal{M} is *spherical* if $\text{diam}(Z, d) \leq 1$ for all $d \in \mathcal{M}$.

3.2.2 The metric Cayley transform

The stereographic projection is a conformal bijection from the unit sphere \mathbb{S} in \mathbb{R}^{n+1} onto $\mathbb{R}^n \cup \{\infty\}$. Its generalization to Iwasawa N -groups associated to simple Lie

groups of rank 1 is known as the Cayley transformation. A pure metric analog of the Cayley transform can be given as follows.

We will use the following definition of a K -semimetric in this paper. A K -semimetric is a symmetric function $\rho(x, y)$ on Z such that there exists $K > 0$ and a (possibly extended) metric d on Z , s.t.

$$\frac{1}{K}d(x, y) \leq \rho(x, y) \leq Kd(x, y). \tag{3.15}$$

A semimetric space is a set Z together with a semimetric ρ . The semimetric ρ and the metric d induce the same topology on Z . Furthermore, one can speak of Lipschitz maps, Möbius maps or symmetric maps between semimetric spaces.

Definition 3 *Let (Z, ρ) be a semimetric space, then the Cayley transformation with respect to a point $\bullet \in Z$ is the bijection obtained by replacing ρ on Z by the semimetric*

$$\rho_{\bullet}(x, y) = \frac{\rho(x, y)}{\rho(x, \bullet)\rho(\bullet, y)} \quad \text{if } x, y \in Z \setminus \{\bullet\}, \tag{3.16}$$

$\rho_{\bullet}(x, \bullet) = \infty$ for all $x \in Z \setminus \{\bullet\}$ and $\rho_{\bullet}(\bullet, \bullet) = 0$.

If Z already has an infinitely distant point \bullet' , the convention is that $\rho_{\bullet}(x, \bullet') = \frac{1}{\rho(x, \bullet)}$.

The Cayley transform is conformal in the sense that for all admissible quadruples $(x, y, z, w) \in Z$,

$$[x, y, z, w]_{\rho} = [x, y, z, w]_{\rho_{\bullet}}.$$

A metric d is *Ptolemaic* if its Cayley transform d_{\bullet} with respect to every point $\bullet \in Z$ is again a metric. A Möbius space (Z, \mathcal{M}) for which the Cayley transform maps into itself with respect to every point \mathcal{M} is called a *Ptolemaic space*.

Proposition 2 (Buckley, Herron, Xie [62]) *The Cayley transform maps K -semimetrics to $4K^3$ -semimetrics.*

Proof: By assumption,

$$\frac{1}{K^3}d_{\bullet}(x, y) \leq \rho_{\bullet}(x, y) \leq K^3d_{\bullet}(x, y).$$

A slight modification of the semimetric d_{\bullet} yields a metric by setting

$$d(x, y) = \inf_{x=x_0, x_1, \dots, x_{k-1}, x_k=y} \sum_{j=0}^k d_{\bullet}(x_j, x_{j+1}).$$

One checks (for details see [62, Lemma 3.1], p.844) that

$$\frac{1}{4}d_{\bullet}(x, y) \leq d(x, y) \leq d_{\bullet}(x, y).$$

Thus,

$$\frac{1}{4K^3}d(x, y) \leq \rho_{\bullet}(x, y) \leq 4K^3d(x, y).$$

■

The conformal factor between the semimetric ρ_\bullet and ρ is given by

$$\frac{\rho_\bullet}{\rho}(x) = \frac{1}{\rho(x, \bullet)^2}. \quad \forall x \in Z \setminus \{\bullet\} \quad (3.17)$$

To see this, $\frac{\rho_\bullet}{\rho}(x) = \lim_{y \rightarrow x} \frac{\rho_\bullet(x, y)}{\rho(x, y)} = \lim_{y \rightarrow x} \frac{1}{\rho(x, \bullet)\rho(\bullet, y)}$, where convergence is w.r.t. the topology induced by ρ .

Both semimetrics are related by the geometric mean value property

$$\rho_\bullet(x, y)^2 = \frac{\rho_\bullet}{\rho}(x) \frac{\rho_\bullet}{\rho}(y) \rho(x, y)^2 \quad \forall x, y \in Z \setminus \{\bullet\}.$$

Let μ_ρ and μ_{ρ_\bullet} be the Q -dimensional Hausdorff measures of ρ and ρ_\bullet respectively. The definition of an s -dimensional Hausdorff measure for semimetrics is analogous to the classical one. Start with

$$\mu_{\delta, \rho_\bullet}^s(A) = \inf \left(\sum_i (\text{diam}_{\rho_\bullet} U_i)^s \mid \{U_i\} \delta\text{-cover of } A \setminus \{\bullet\} \right),$$

and define

$$\mu_{\rho_\bullet}^s(A) = \lim_{\delta \rightarrow 0} \mu_{\delta, \rho_\bullet}^s(A).$$

The Hausdorff dimension $Q = \inf(s \in \mathbb{R} \mid \mu_{\rho_\bullet}^s(Z) = 0)$ is independent of ρ and ρ_\bullet , see [61].

Proposition 3 *The Hausdorff measure μ_{ρ_\bullet} of ρ_\bullet can be expressed in terms of the Hausdorff measure μ_ρ by*

$$\mu_{\rho_\bullet}(A) = \int_A \frac{1}{\rho(z, \bullet)^{2Q}} d\mu_\rho(z), \quad (3.18)$$

for any measurable subset A of Z_\bullet .

When A is a measurable bounded set w.r.t ρ and away from \bullet , this follows from the geometric mean value property above, see [3, Lemma 8].

Proof: Fix a measurable set $A \subset Z \setminus \{\bullet\}$, that is bounded w.r.t ρ and away from \bullet , i.e. \bullet is not a limit point of A .

From the geometric mean value property, we get that

$$\left(\inf_U \frac{\rho_\bullet}{\rho} \right) \text{diam}_\rho U \leq \text{diam}_{\rho_\bullet} U \leq \left(\sup_U \frac{\rho_\bullet}{\rho} \right) \text{diam}_\rho U$$

for every $U \subset Z_\bullet$. Thus for every measurable $T \subset Z_\bullet$, we have

$$\left(\inf_T \frac{\rho_\bullet}{\rho} \right)^Q \mu_\rho(T) \leq \mu_{\rho_\bullet}(T) \leq \left(\sup_T \frac{\rho_\bullet}{\rho} \right)^Q \mu_\rho(T). \quad (3.19)$$

Assume $T \subset A$, and let $\epsilon > 0$. There exists $\eta > 0$ such that $|\frac{\rho_\bullet}{\rho}(x) - \frac{\rho_\bullet}{\rho}(y)| < \epsilon \inf_A \frac{\rho_\bullet}{\rho}$, whenever $\rho(x, y) < \eta$. Thus, if T is a measurable η -set in A , meaning that $\text{diam}(T) < \eta$, then $\sup_T \frac{\rho_\bullet}{\rho} \leq (1 + \epsilon) \inf_T \frac{\rho_\bullet}{\rho}$. This gives

$$(1 + \epsilon)^{-Q} \left(\sup_T \frac{\rho_\bullet}{\rho} \right)^Q \mu_\rho(T) \leq \int_T \left(\frac{\rho_\bullet}{\rho} \right)^Q d\mu_\rho \leq (1 + \epsilon)^Q \left(\inf_T \frac{\rho_\bullet}{\rho} \right)^Q \mu_\rho(T).$$

Together with (3.19), we get that for every measurable η -set T in A ,

$$(1 + \epsilon)^{-Q} \mu_{\rho_\bullet}(T) \leq \int_T \left(\frac{\rho_\bullet}{\rho}\right)^Q d\mu_\rho \leq (1 + \epsilon)^Q \mu_{\rho_\bullet}(T). \quad (3.20)$$

The set A may be partitioned into a finite number of measurable η -subsets. Using (3.21) for each η -piece of A , and adding up, yields

$$(1 + \epsilon)^{-Q} \mu_{\rho_\bullet}(A) \leq \int_A \left(\frac{\rho_\bullet}{\rho}\right)^Q d\mu_\rho \leq (1 + \epsilon)^Q \mu_{\rho_\bullet}(A). \quad (3.21)$$

Since ϵ is arbitrary, we conclude (3.18) for $A \subset Z \setminus \{\bullet\}$ measurable, bounded w.r.t ρ and away from \bullet . For a general measurable set S , we decompose S into a countable disjoint union of sets of type A . ■

An inverse Cayley transform from an unbounded semimetric space to a bounded one can be given as follows.

Definition 4 *Let (Z, ρ) be a semimetric space with an infinitely distant point $\bullet \in Z$. The inverse Cayley transform with respect to a center $c \in Z \setminus \{\bullet\}$ is the mapping obtained by replacing ρ by the semimetric*

$$|\rho|(x, y) = \frac{\rho(x, y)}{(1 + \rho(x, c))(1 + \rho(y, c))} \quad \text{if } x, y \in Z \setminus \{\bullet\}, \quad (3.22)$$

$$|\rho|(x, \bullet) = |\rho|(\bullet, x) = \frac{1}{(1 + \rho(x, c))} \text{ for all } x \in Z \setminus \{\bullet\}, \text{ and } |\rho|(\bullet, \bullet) = 0.$$

The inverse Cayley transform is again conformal in the sense that for all admissible quadruples $(x, y, z, w) \in Z$,

$$[x, y, z, w]_\rho = [x, y, z, w]_{|\rho|}.$$

Proposition 4 (Bonk, Kleiner [63]) *The inverse Cayley transform maps K -semimetrics to $4K^3$ -semimetrics.*

Proof: By assumption,

$$\frac{1}{K^3} \frac{d(x, y)}{(1 + d(x, c))(1 + d(y, c))} \leq \frac{\rho(x, y)}{(1 + \rho(x, c))(1 + \rho(y, c))} \leq \frac{K^3 d(x, y)}{(1 + d(x, c))(1 + d(y, c))}$$

for some metric d .

As above, a slight modification of the semimetric $|d|$ yields a metric by setting

$$\hat{d}(x, y) = \inf_{x=x_0, x_1, \dots, x_{k-1}, x_k=y} \sum_{j=0}^k |d|(x_j, x_{j+1}).$$

One checks (for details see [63] Lemma 2.2.) that

$$\frac{1}{4} |d|(x, y) \leq \hat{d}(x, y) \leq |d|(x, y).$$

So

$$\frac{1}{4K^3} \hat{d}(x, y) \leq |\rho|(x, y) \leq 4K^3 \hat{d}(x, y). \quad \blacksquare$$

Analogously to the case above, the conformal factor between the semimetric $|\rho|$ and ρ is given by

$$\frac{|\rho|}{\rho}(x) = \frac{1}{(1 + \rho(x, c))^2} \quad \forall x \in Z \setminus \{\bullet\}. \quad (3.23)$$

Both semimetrics are related by the geometric mean value property

$$|\rho|(x, y)^2 = \frac{|\rho|}{\rho}(x) \frac{|\rho|}{\rho}(y) \rho(x, y)^2 \quad \forall x, y \in Z \setminus \{\bullet\}.$$

Proposition 5 *The Q -dimensional Hausdorff measure $\mu_{|\rho|}$ of $|\rho|$ can be expressed in terms of the Hausdorff measure μ_ρ by*

$$\mu_{|\rho|}(A) = \int_{A \setminus \{\bullet\}} \frac{1}{(1 + \rho(z, c))^{2Q}} d\mu_\rho(z), \quad (3.24)$$

for any measurable subset A of Z .

The proof is analogous to the case of ρ_\bullet .

The following proposition shows that the Cayley transform of a bounded semimetric space (Z, ρ) followed by the inverse Cayley transform returns the semimetric space (Z, ρ) up to a bi-Lipschitz map. This justifies our naming convention.

Proposition 6 (Buckley, Herron, Xie, Proposition 3.5 [62]) *Let (Z, ρ) be a bounded semimetric space and let (Z, ρ_\bullet) be its Cayley transform with respect to a nonisolated point $\bullet \in Z$. The inverse Cayley transform of (Z, ρ_\bullet) with respect to any center in $Z \setminus \{\bullet\}$, is bi-Lipschitz equivalent to (Z, ρ) .*

Remark 2 *Quantitatively, if we assume that $\text{diam}(Z) = 1$, then the proof yields a bi-Lipschitz constant of $\max\{16, 4K^4\}$.*

Proof: We can assume that $\text{diam}(Z) = 1$. Choose $c \in Z \setminus \{\bullet\}$ with $\rho(c, \bullet) \geq \frac{1}{2}$.

We check that for all $x \in Z_\bullet$,

$$\frac{1}{4} \rho(x, \bullet) \leq |\rho_\bullet|(x, \bullet) \leq 2K^2 \rho(x, \bullet).$$

The upper bound is calculated as follows,

$$\begin{aligned} |\rho_\bullet|(x, \bullet) &= \frac{1}{1 + \rho_\bullet(x, c)} \\ &= \frac{\rho(x, \bullet)\rho(\bullet, c)}{\rho(x, \bullet)\rho(\bullet, c) + \rho(x, c)} \\ &\leq \frac{2\rho(x, \bullet)\rho(\bullet, c)}{\rho(x, \bullet) + 2\rho(x, c)} \leq 2K^2 \rho(x, \bullet). \end{aligned}$$

For the lower bound,

$$\begin{aligned} |\rho_\bullet|(x, \bullet) &= \frac{1}{1 + \rho_\bullet(x, c)} \\ &= \frac{\rho(x, \bullet)\rho(\bullet, c)}{\rho(x, \bullet)\rho(\bullet, c) + \rho(x, c)} \\ &\geq \frac{1}{2} \frac{\rho(x, \bullet)}{\rho(x, \bullet) + \rho(x, c)} \geq \frac{1}{4} \rho(x, \bullet). \end{aligned}$$

If $x, y \in Z_\bullet$, then

$$\begin{aligned} |\rho_\bullet|(x, y) &= \frac{\rho_\bullet(x, y)}{(1 + \rho_\bullet(c, x))(1 + \rho_\bullet(c, y))} \\ &= |\rho_\bullet|(x, \bullet) \rho_\bullet(x, y) |\rho_\bullet|(y, \bullet) \\ &= \frac{|\rho_\bullet|(x, \bullet)|\rho_\bullet|(y, \bullet)}{\rho(x, \bullet)\rho(\bullet, y)} \rho(x, y) \\ &\leq 4K^4 \rho(x, y). \end{aligned}$$

For the lower bound we get,

$$|\rho_\bullet|(x, y) \geq \frac{1}{16} \rho(x, y).$$

■

3.2.3 Uniform Ahlfors-David regularity

We have seen that associated with each Möbius structure \mathcal{M} is a family of measures $\{\mu_d\}_{d \in \mathcal{M}}$. This family of measures has regularity properties in the following sense.

Definition 5 *A measure μ on a metric space (Z, d) of Hausdorff dimension Q is said to be Ahlfors-David regular, if there exists $C > 0$ such that for every ball $B(x, r) \subset Z$, $r \in (0, \text{diam } Z)$,*

$$\frac{1}{C} r^Q \leq \mu(B(x, r)) \leq C r^Q.$$

Theorem 3 *If for a Möbius structure \mathcal{M} there exists $d \in \mathcal{M}$ such that μ_d is Q -Ahlfors-David regular, then there exists $C \geq 1$ such that every measure μ_d , $d \in \mathcal{M}$, is Ahlfors-David regular with constant C .*

This observation is crucial because it often allows us to treat Möbius spaces as uniformly Ahlfors-David regular. Much of the theory of classical harmonic analysis on Ahlfors-David regular metric spaces therefore admits a Möbius invariant analog on Möbius spaces.

The above theorem follows essentially from the fact that the Cayley transformations of two metrics d, d' of a Möbius structure \mathcal{M} are homothetic.

Lemma 1 (Buyalo, Schroeder, Lemma 2.1 [1]) *Let (Z, \mathcal{M}) be a Möbius space and $d, d' \in \mathcal{M}$, then for any point $\bullet \in Z$ the semimetrics d_\bullet and d'_\bullet are homothetic, viz. i.e., there exists $\lambda > 0$ such that $d'_\bullet(x, y) = \lambda d_\bullet(x, y)$ for all $x, y \in Z$.*

Proof: We can assume w.l.o.g. that Z_\bullet contains two distinct points x and y . Choose λ , s.t. $d'_\bullet(x, y) = \lambda d_\bullet(x, y)$. If there exists a third point $z \in Z_\bullet$, then $[x, y, z, \bullet]_{d'_\bullet} = [x, y, z, \bullet]_{d_\bullet}$, i.e. $(d'_\bullet(x, y) : d'_\bullet(x, z) : d'_\bullet(y, z)) = (d_\bullet(x, y) : d_\bullet(x, z) : d_\bullet(y, z))$. Since $d'_\bullet(x, y) = \lambda d_\bullet(x, y)$, it follows that $d'_\bullet(x, z) = \lambda d_\bullet(x, z)$ and $d'_\bullet(y, z) = \lambda d_\bullet(y, z)$. ■

Hausdorff measures of homothetic semimetrics have strong regularity properties.

Lemma 2 *Let (Z, \mathcal{M}) be a Möbius space and $d, d' \in \mathcal{M}$, then for any point $\bullet \in Z$ the Hausdorff measure μ_{d_\bullet} is Ahlfors-David regular with a constant $C \geq 1$ if and only if, if $\mu_{d'_\bullet}$ is Ahlfors-David regular with the same constant.*

Proof: We assume that $\mu_{d'_\bullet}$ is Ahlfors-David regular with constant C .

From the conformality of the Cayley transform it follows,

$$\frac{d'_\bullet(x, z)}{d'_\bullet(x, w)} = \frac{d_\bullet(x, z)}{d_\bullet(x, w)}. \quad (x, y, w \in Z_\bullet) \quad (3.25)$$

Let $B_{d_\bullet}(x, r)$ be a ball in (Z, d_\bullet) . Without loss of generality, we can assume that $\partial B_{d_\bullet}(x, r) \neq \emptyset$. Otherwise $B_{d_\bullet}(x, r)$ is closed and there exists $0 \leq r' < r$, s.t. $\bar{B}_{d_\bullet}(x, r') = B_{d_\bullet}(x, r)$, $\partial B_{d_\bullet}(x, r') \neq \emptyset$ and we replace $B(x, r)$ with $B_{d_\bullet}(x, r')$.

Let $w \in Z$, s.t. $r = d_\bullet(x, w)$, then

$$B_{d_\bullet}(x, r) = B_{d'_\bullet}(x, d'_\bullet(x, w))$$

by the symmetry condition (3.25). So

$$\begin{aligned} \mu_{d_\bullet}(B_{d_\bullet}(x, r)) &= \int_{B_{d_\bullet}(x, r)} \left(\frac{d_\bullet(y)}{d'_\bullet(y)} \right)^Q d\mu_{d'_\bullet}(y) \\ &= \lambda^{-Q} \mu_{d'_\bullet}(B_{d_\bullet}(x, r)) \\ &= \lambda^{-Q} \mu_{d'_\bullet}(B_{d'_\bullet}(x, d'_\bullet(x, w))) \\ &\leq C \lambda^{-Q} d'_\bullet(x, w)^Q \\ &= C \left(\frac{d_\bullet(x)}{d'_\bullet(x)} \right)^Q d'_\bullet(x, w)^Q \\ &= C d_\bullet(x, w)^Q \\ &= C r^Q. \end{aligned}$$

The lower bound is obtained in an analogous way. ■

To conclude the theorem 3, it remains to show that the Cayley transform and the inverse Cayley transform preserve the Ahlfors-David regularity.

Proposition 7 (Li, Shanmugalingam, Proposition 4.1 [64]) *Let (Z, ρ) be a Q -Ahlfors-David regular semimetric space. Then the Cayley transform (Z, ρ_\bullet) of (Z, ρ) is Q -Ahlfors-David regular, with a constant depending only on the Ahlfors-David constant of (Z, ρ) , K and Q .*

Proof: W.l.o.g. we can replace ρ by its true metric approximation d . Indeed, if (Z, ρ) is Ahlfors-David regular with constant $C \geq 1$, then so is (Z, d) with constant $K^{2Q}C$.

Let C_d be the Ahlfors-David constant of measure μ_d , and let $B_{d_\bullet}(x, r)$ be any ball in (Z, d_\bullet) . We distinguish three cases, $rd(x, \bullet) \leq \frac{1}{2}$, $rd(x, \bullet) \geq 4C_d^{\frac{2}{Q}}$, and $\frac{1}{2} < rd(x, \bullet) < 4C_d^{\frac{2}{Q}}$.

Case 1 ($rd(x, \bullet) \leq \frac{1}{2}$): We are looking for two comparable balls B_d and B'_d s.t. $B_d \subset B_{d_\bullet}(x, r) \subset B'_d$.

For $y \in B_{d_\bullet}(x, r)$,

$$d(x, y) < rd(x, \bullet)d(\bullet, y). \quad (3.26)$$

Thus $d(y, \bullet) - d(\bullet, x) \leq d(x, y) < rd(x, \bullet)d(\bullet, y)$ and a rearrangement, together with $rd(x, \bullet) \leq \frac{1}{2}$, gives

$$d(y, \bullet) \leq \frac{d(\bullet, x)}{1 - rd(\bullet, x)} < 2d(\bullet, x).$$

By symmetry, we also have $d(x, \bullet) - d(\bullet, y) \leq d(x, y) < rd(x, \bullet)d(\bullet, y)$ and a rearrangement yields

$$d(y, \bullet) \geq \frac{d(\bullet, x)}{1 + rd(\bullet, x)} > \frac{2d(\bullet, x)}{3}.$$

So for $y \in B_{d_\bullet}(x, r)$

$$\frac{2d(\bullet, x)}{3} < d(y, \bullet) < 2d(x, \bullet). \quad (3.27)$$

Combining equation (3.26) and (3.27) shows that we can take

$$B'_d = B_d(x, 2rd(x, \bullet)^2).$$

Conversely, let $y \in B_d(x, \frac{2rd(x, \bullet)^2}{3})$, then

$$d(x, \bullet) - d(\bullet, y) \leq d(x, y) < \frac{2r}{3}d(x, \bullet)^2 \leq \frac{d(x, \bullet)}{3},$$

where for the last inequality we used the assumption $rd(x, \bullet) \leq \frac{1}{2}$. Thus $\frac{d(x, \bullet)}{d(y, \bullet)} \leq \frac{3}{2}$, and therefore

$$d_\bullet(x, y) = \frac{d(x, y)}{d(x, \bullet)d(\bullet, y)} < \frac{2r}{3} \frac{d(x, \bullet)}{d(y, \bullet)} < r.$$

So we can set $B_d = B_d(x, \frac{2rd(x, \bullet)^2}{3})$.

We estimate the lower bound as follows.

$$\begin{aligned} \mu_{d_\bullet}(B_{d_\bullet}(x, r)) &\geq \int_{B_d} \frac{d\mu_d(z)}{d(z, \bullet)^{2Q}} \\ &\geq \frac{\mu_d(B_d)}{2^{2Q}d(x, \bullet)^{2Q}} \\ &\geq \frac{(\frac{2rd(x, \bullet)^2}{3})^Q}{2^{2Q}C_d} \frac{1}{d(x, \bullet)^{2Q}} \\ &= \frac{r^Q}{6^Q C_d}. \end{aligned}$$

For the upper bound,

$$\begin{aligned}
\mu_{d_\bullet}(B_{d_\bullet}(x, r)) &\leq \int_{B'_d} \frac{d\mu_d(z)}{d(z, \bullet)^{2Q}} \\
&\leq \left(\frac{3}{2}\right)^{2Q} \frac{\mu_d(B'_d)}{d(x, \bullet)^{2Q}} \\
&\leq \left(\frac{3}{2}\right)^{2Q} \frac{C_d (2rd(x, \bullet))^Q}{d(x, \bullet)^{2Q}} \\
&= \left(\frac{9}{2}\right)^Q C_d r^Q.
\end{aligned}$$

Case 2: Suppose $rd(x, \bullet) \geq 4C_d^{\frac{2}{Q}}$, then in particular $Z \setminus B_d(\bullet, \frac{4C_d^{\frac{2}{Q}}}{r}) \neq \emptyset$. If $y \in Z \setminus \{B_{d_\bullet}(x, r) \cup \{\bullet\}\}$, then

$$rd(x, \bullet)d(\bullet, y) \leq d(x, y) \leq d(x, \bullet) + d(\bullet, y).$$

By rearranging and using $rd(x, \bullet) \geq 4C_d^{\frac{2}{Q}}$, we obtain

$$\begin{aligned}
d(y, \bullet) &< \frac{d(x, \bullet)}{rd(x, \bullet) - 1} \\
&\leq \frac{1}{r} \frac{4C_d^{\frac{2}{Q}}}{(4C_d^{\frac{2}{Q}} - 1)} < \frac{2}{r}.
\end{aligned}$$

Thus,

$$Z \setminus \bar{B}_d(\bullet, 2/r) \subset B_{d_\bullet}(x, r).$$

Conversely, if $y \in B_{d_\bullet}(x, r)$, then

$$d(x, \bullet) - d(\bullet, y) \leq d(x, y) \leq rd(x, \bullet)d(\bullet, y).$$

If we convert and use $rd(x, \bullet) \geq 4C_d^{\frac{2}{Q}}$, we get

$$\begin{aligned}
d(y, \bullet) &> \frac{d(x, \bullet)}{rd(x, \bullet) + 1} \\
&\geq \frac{1}{r} \frac{4C_d^{\frac{2}{Q}}}{4C_d^{\frac{2}{Q}} + 1} > \frac{2}{3r}.
\end{aligned}$$

Thus,

$$Z \setminus \bar{B}_d(\bullet, 2/r) \subset B_{d_\bullet}(x, r) \subset Z \setminus \bar{B}_d(\bullet, 2/3r).$$

We estimate the upper bound as follows,

$$\begin{aligned}
\mu_{d_\bullet}(B_{d_\bullet}(x, r)) &\leq \int_{Z \setminus \bar{B}_d(\bullet, \frac{2}{3r})} \frac{d\mu_d(z)}{d(\bullet, z)^{2Q}} \\
&\leq \sum_{i=1}^{\infty} \int_{\bar{B}_d(\bullet, \frac{2^{i+1}}{3r}) \setminus \bar{B}_d(\bullet, \frac{2^i}{3r})} \frac{d\mu_d(z)}{d(\bullet, z)^{2Q}} \\
&\leq \sum_{i=1}^{\infty} \int_{\bar{B}_d(\bullet, \frac{2^{i+1}}{3r}) \setminus \bar{B}_d(\bullet, \frac{2^i}{3r})} \frac{d\mu_d(z)}{(\frac{2^i}{3r})^{2Q}} \\
&\leq C_d \sum_{i=1}^{\infty} \frac{(\frac{2^{i+1}}{3r})^Q}{(\frac{2^i}{3r})^{2Q}} \\
&= C_d \sum_{i=0}^{\infty} 2^{-iQ} (3r)^Q \\
&= \frac{2^Q}{2^Q - 1} 3^Q C_d r^Q.
\end{aligned}$$

There is also a lower bound, as

$$\begin{aligned}
\mu_d(B_d(\bullet, 4C_d^{\frac{2}{Q}}/r) \setminus B_d(\bullet, 2/r)) &\geq \frac{4^Q C_d}{r^Q} - \frac{2^Q C_d}{r^Q} \\
&\geq \frac{2^Q C_d}{r^Q}.
\end{aligned}$$

Therefore, the lower bound can be estimated as follows

$$\begin{aligned}
\mu_{d_\bullet}(B_{d_\bullet}(x, r)) &\geq \int_{Z \setminus \bar{B}_d(\bullet, \frac{2}{r})} \frac{d\mu_d(z)}{d(\bullet, z)^{2Q}} \\
&\geq \int_{B(\bullet, \frac{4C_d^{\frac{2}{Q}}}{r}) \setminus \bar{B}_d(\bullet, \frac{2}{r})} \frac{d\mu_d(z)}{d(\bullet, z)^{2Q}} \\
&\geq \frac{2^Q C_d}{r^Q} \frac{1}{(\frac{4}{r})^{2Q} C_d^4} \\
&= \frac{r^Q}{2^{3Q} C_d^3}
\end{aligned}$$

Case 3: Suppose $\frac{1}{2} < rd(x, \bullet) < 4C_d^{\frac{2}{Q}}$. Then

$$B_{d_\bullet}(x, r/8C_d^{\frac{2}{Q}}) \subset B_{d_\bullet}(x, r) \subset B_{d_\bullet}(x, 8C_d^{\frac{2}{Q}}r).$$

The inner ball satisfies the hypothesis in case 1, so

$$\frac{r^Q}{8^Q 6^Q C_d^3} \leq \mu_{d_\bullet}(B_{d_\bullet}(x, r/8C_d^{\frac{2}{Q}})) \leq \mu_{d_\bullet}(B_{d_\bullet}(x, r)).$$

The outer ball satisfies the hypothesis of case 2, so

$$\mu_{d_\bullet}(B_{d_\bullet}(x, r)) \leq \mu_{d_\bullet}(B_{d_\bullet}(x, 8C_d^{\frac{2}{Q}}r)) \leq \frac{2^Q}{2^Q - 1} 3^Q 8^Q C_d^3 r^Q.$$

■

Proposition 8 (Li, Shanmugalingam, Proposition 3.1 [64]) *Let (Z, ρ) be an unbounded, Ahlfors-David regular semimetric space with constant $C \geq 1$. Then its inverse Cayley transform $(Z, |\rho|)$ is Ahlfors-David regular with constant at most $(2K + 1)^{2Q} 20^Q C^2$, where K is the semimetric constant of $|\rho|$.*

Proof: Let \bullet be the point at infinity in Z , i.e. $\rho(x, \bullet) = \infty$ for all $x \in Z_\bullet$. Since $(Z, |\rho|)$ is compact, it suffices to check the Ahlfors-David regularity on balls with $r \leq \frac{1}{2}$.

Let $R = \frac{1}{r} - 1$, then $\frac{1}{2r} \leq R < \frac{1}{r}$. For a sphere whose center is \bullet , we have

$$\begin{aligned} B_{|\rho|}(\bullet, r) &= \{x \in Z, \frac{1}{1 + \rho(x, c)} < r\} \cup \{\bullet\} \\ &= \{x \in Z, \rho(x, c) > \frac{1}{r} - 1\} \cup \{\bullet\} \\ &= Z \setminus \bar{B}_\rho(c, R). \end{aligned}$$

Set $\lambda = (2C_\rho^2)^{\frac{1}{Q}} > 1$, and define $B_i = B_\rho(c, \lambda^i R)$ then

$$\begin{aligned} C_\rho(\lambda^{i-1} R)^Q &= \left(\frac{\lambda^Q}{C_\rho} - C_\rho\right)(\lambda^{i-1} R)^Q \\ &= \frac{(\lambda^i R)^Q}{C_\rho} - C_\rho(\lambda^{i-1} R)^Q \\ &\leq \mu_\rho(B_i \setminus B_{i-1}), \end{aligned}$$

and

$$B_{|\rho|}(\bullet, r) = \bigcup_{i=1}^{\infty} B_i \setminus \bar{B}_{i-1}.$$

Therefore

$$\mu_{|\rho|}(B_{|\rho|}(\bullet, r)) = \sum_{i=1}^{\infty} \int_{B_i \setminus \bar{B}_{i-1}} \frac{1}{(1 + \rho(y, c))^{2Q}} d\mu_\rho(y).$$

It follows,

$$\sum_{i=1}^{\infty} \frac{\mu_\rho(B_i \setminus \bar{B}_{i-1})}{(1 + \lambda^i R)^{2Q}} \leq \mu_{|\rho|}(B_{|\rho|}(\bullet, r)) \leq \sum_{i=1}^{\infty} \frac{\mu_\rho(B_i \setminus \bar{B}_{i-1})}{(1 + \lambda^{i-1} R)^{2Q}}$$

Using $1 \leq \lambda^i R$ and the doubling property of μ_ρ ,

$$\frac{C_\rho}{4^Q \lambda^Q R^Q} \sum_{i=0}^{\infty} \lambda^{-iQ} \leq \mu_{|\rho|}(B_{|\rho|}(\bullet, r)) \leq \frac{\lambda^Q C_\rho}{R^Q} \sum_{i=0}^{\infty} \lambda^{-iQ}.$$

So

$$\frac{C_\rho}{4^Q} \frac{1}{\lambda^Q - 1} r^Q \leq \mu_{|\rho|}(B_{|\rho|}(\bullet, r)) \leq (2\lambda)^Q C_\rho \frac{\lambda^Q}{\lambda^Q - 1} r^Q.$$

A further simplification leads to,

$$\frac{1}{(4\lambda)^Q C_\rho} r^Q \leq \mu_{|\rho|}(B_{|\rho|}(\bullet, r)) \leq (4\lambda)^Q C_\rho r^Q.$$

Let $x \in Z \setminus \{\bullet\}$, we will prove this in four cases.

Case 1: Suppose $|\rho|(x, \bullet) \leq \frac{r}{2K}$, we reduce this case to the one above.

If $y \in B_{|\rho|}(x, r)$, then

$$\begin{aligned} |\rho|(\bullet, y) &\leq K(|\rho|(\bullet, x) + |\rho|(x, y)) \\ &< (K+1)r. \end{aligned}$$

So

$$B_\rho^c(x, r) \subset B_\rho^c(\bullet, (K+1)r).$$

If $y \in B_{|\rho|}(\bullet, \frac{r}{2K})$, then

$$\begin{aligned} |\rho|(x, y) &\leq K(|\rho|(x, \bullet) + |\rho|(\bullet, y)) \\ &< r. \end{aligned}$$

So

$$B_\rho^c(\bullet, \frac{r}{2K}) \subset B_\rho^c(x, r).$$

Therefore

$$\begin{aligned} \frac{1}{(2K4\lambda)^Q C_\rho} r^Q &\leq \mu_{|\rho|}(B_\rho^c(\bullet, \frac{r}{2K})) \leq \mu_{|\rho|}(B_\rho^c(x, r)) \\ &\leq \mu_{|\rho|}(B_\rho^c(\bullet, (K+1)r)) \leq ((K+1)4\lambda)^Q C_\rho r^Q. \end{aligned}$$

Case 2: Suppose $\frac{r}{2K} \leq |\rho|(x, \bullet) < 4r$.

Then as above,

$$B_{|\rho|}(x, r) \subset B_{|\rho|}(\bullet, 5Kr),$$

and therefore

$$\mu_{|\rho|}(B_{|\rho|}(x, r)) \leq (5K4\lambda)^Q C_\rho r^Q.$$

For the lower bound, note that $\frac{1}{4r} \leq 1 + \rho(x, c) \leq \frac{2K}{r}$. If $y \in B_\rho(x, \frac{1}{20r})$, then

$$1 + \rho(y, c) \geq 1 + \rho(x, c) - \rho(x, y) \geq \frac{1}{4r} - \frac{1}{20r} = \frac{1}{5r},$$

and therefore

$$|\rho|(x, y) = \frac{\rho(x, y)}{(1 + \rho(x, c))(1 + \rho(y, c))} < \frac{4r \cdot 5r}{20r} = r.$$

So

$$B_\rho(x, \frac{1}{20r}) \subset B_{|\rho|}(x, r).$$

Moreover, for $y \in B_\rho(x, \frac{1}{20r})$

$$1 + \rho(y, c) \leq 1 + \rho(x, c) + \rho(x, y) \leq \frac{2K}{r} + \frac{1}{20r} < \frac{2K+1}{r}.$$

All together,

$$\begin{aligned} \mu_{|\rho|}(B_{|\rho|}(x, r)) &\geq \int_{B_\rho(x, \frac{1}{20r})} \frac{1}{(1 + \rho(y, c))^{2Q}} d\mu_\rho(y) \\ &\geq \frac{r^{2Q}}{(2K+1)^{2Q} \mu_\rho(B_\rho(x, \frac{1}{20r}))} \\ &\geq \frac{1}{(2K+1)^{2Q} 20^Q C_\rho} r^Q \end{aligned}$$

Case 3: Suppose $|\rho|(x, \bullet) \geq 4r$ and $\rho(x, c) \leq 1$. Then for all $y \in B_{|\rho|}(x, r)$,

$$\frac{4r\rho(x, y)}{1 + \rho(y, c)} \leq \frac{\rho(x, y)}{(1 + \rho(x, c))(1 + \rho(y, c))} < r.$$

It follows that

$$4(1 + \rho(y, c) - 1 - \rho(x, c)) \leq 1 + \rho(y, c).$$

Therefore

$$1 + \rho(y, c) \leq \frac{4}{3}(1 + \rho(x, c)) \leq 3$$

since $\rho(x, c) \leq 1$ and therefore

$$\frac{1}{6}\rho(x, y) \leq |\rho|(x, y) = \frac{\rho(x, y)}{(1 + \rho(x, c))(1 + \rho(y, c))}.$$

That is, $B_{|\rho|}(x, r) \subset B_\rho(x, 6r)$ and we obtain

$$\mu_{|\rho|}(B_{|\rho|}(x, r)) \leq \int_{B_\rho(x, 6r)} \frac{1}{(1 + \rho(y, c))^{2Q}} d\mu_\rho(y) \leq \mu_\rho(B_\rho(x, 6r)) \leq C_\rho 6^Q r^Q.$$

For the lower bound, if $y \in B_\rho(x, r)$, then $|\rho|(x, y) \leq \rho(x, y) < r$ and henceforth $B_\rho(x, r) \subset B_{|\rho|}(x, r)$. So we get

$$\mu_{|\rho|}(B_{|\rho|}(x, r)) \geq \int_{B_\rho(x, r)} \frac{1}{(1 + \rho(y, c))^{2Q}} d\mu_\rho(y) \geq \frac{1}{3^{2Q}} \mu_\rho(B_\rho(x, r)) \geq \frac{1}{9^Q C_\rho} r^Q$$

Case 4: Suppose $|\rho|(x, \bullet) \geq 4r$ and $\rho(x, c) > 1$.

Define

$$D(x) = \left\{ y \in Z : \frac{4}{5}(1 + \rho(x, c)) < 1 + \rho(y, c) < \frac{4}{3}(1 + \rho(x, c)) \right\}.$$

As before, if $y \in B_{|\rho|}(x, r)$, then

$$\frac{4r\rho(x, y)}{(1 + \rho(y, c))} \leq \frac{\rho(x, y)}{(1 + \rho(x, c))(1 + \rho(y, c))} < r,$$

and

$$4\rho(x, y) < 1 + \rho(y, c).$$

The last equation together with $4(1 + \rho(y, c) - 1 - \rho(x, c)) < 1 + \rho(y, c)$ and $4(1 + \rho(x, c) - 1 - \rho(y, c)) < 1 + \rho(y, c)$ results in

$$\frac{4}{5}(1 + \rho(x, c)) < 1 + \rho(y, c) < \frac{4}{3}(1 + \rho(x, c)).$$

Also,

$$\rho(x, y) < r(1 + \rho(x, c))(1 + \rho(y, c)) < \frac{4}{3}(1 + \rho(x, c))^2 r.$$

It follows that, $B_{|\rho|}(x, r) \subset D(x) \cap B_\rho(x, \frac{4}{3}(1 + \rho(x, c))^2 r)$.

It follows,

$$\begin{aligned}
\mu_{|\rho|}(B_{|\rho|}(x, r)) &\leq \int_{D(x) \cap B_\rho(x, \frac{4}{3}(1+\rho(x, c))^2 r)} \frac{1}{(1+\rho(y, c))^{2Q}} d\mu_\rho(y) \\
&\leq \left(\frac{5}{4}\right)^{2Q} \frac{1}{(1+\rho(x, c))^{2Q}} \mu_\rho(B_\rho(x, \frac{4}{3}(1+\rho(x, c))^2 r)) \\
&\leq \left(\frac{5}{4}\right)^{2Q} \left(\frac{4}{3}\right)^Q C_\rho r^Q = \left(\frac{25}{12}\right)^Q C_\rho r^Q.
\end{aligned}$$

If $y \in B_\rho(x, \frac{4}{5}(1+\rho(x, c))^2 r)$, then

$$\begin{aligned}
\frac{4}{5}(1+\rho(x, c)) &= 1 + \rho(x, c) - \frac{1}{5}(1+\rho(x, c)) \\
&\leq 1 + \rho(x, c) - \frac{4}{5}(1+\rho(x, c))^2 r \\
&\leq 1 + \rho(x, c) - \rho(x, y) \\
&\leq 1 + \rho(y, c) \\
&\leq 1 + \rho(x, c) + \rho(x, y) \\
&< 1 + \rho(x, c) + \frac{4r}{5}(1+\rho(x, c))^2 \\
&\leq 1 + \rho(x, c) + \frac{1}{5}(1+\rho(x, c)) \\
&< \frac{4}{3}(1+\rho(x, c)).
\end{aligned}$$

Therefore $B_\rho(x, \frac{4}{5}(1+\rho(x, c))^2 r) \subset D(x)$. Moreover, knowing this inclusion,

$$\begin{aligned}
|\rho|(x, y) &= \frac{\rho(x, y)}{(1+\rho(x, c))(1+\rho(y, c))} \\
&< \frac{4}{5}(1+\rho(x, c))^2 r \frac{5}{4(1+\rho(x, c))^2} \\
&= r.
\end{aligned}$$

So $B_\rho(x, \frac{4}{5}(1+\rho(x, c))^2 r) \subset B_{|\rho|}(x, r) \subset D(x)$. It follows that

$$\begin{aligned}
\mu_{|\rho|}(B_{|\rho|}(x, r)) &\geq \int_{B_\rho(x, \frac{4}{5}(1+\rho(x, c))^2 r)} \frac{1}{(1+\rho(y, c))^{2Q}} d\mu_\rho(y) \\
&\geq \left(\frac{3}{4}\right)^{2Q} \frac{1}{(1+\rho(x, c))^{2Q}} \mu_\rho(B_\rho(x, 4(1+\rho(x, c))^2 r/5)) \\
&\geq \left(\frac{3}{4}\right)^{2Q} \left(\frac{4}{5}\right)^Q \frac{1}{C_\rho} r^Q = \left(\frac{9}{20}\right)^Q \frac{1}{C_\rho} r^Q
\end{aligned}$$

■

3.3 The boundary of hyperbolic groups

3.3.1 δ -hyperbolic spaces

In the 1980s, Mikhael Gromov introduced a notion of hyperbolicity for metric spaces that remains invariant under small changes in topology [65]. It generalizes the metric

properties of classical hyperbolic geometry and of trees. The basic definitions are remarkably simple. The *Gromov product* of two points x and y with respect to an origin o , is the nonnegative number

$$\langle x, y \rangle_o = \frac{1}{2}(d(x, o) + d(y, o) - d(x, y)). \quad (3.28)$$

In trees, $\langle x, y \rangle_o$ is the distance from o to the branch point of geodesics starting from o and passing through x and y .

A metric space (X, d) is called *Gromov hyperbolic* or δ -*hyperbolic* if for any triple of points x, y, z ,

$$\langle x, y \rangle_o \geq \min\{\langle x, z \rangle_o, \langle z, y \rangle_o\} - \delta. \quad (3.29)$$

This condition is independent of the base point in the sense that if it holds for some o and δ , it holds for all base points with 2δ .

3.3.2 Gromov hyperbolic group

A group Γ is called *Gromov hyperbolic* or *hyperbolic* for short if it acts by isometries, properly discontinuous and cocompactly on a proper, geodesic δ -hyperbolic metric space.

Examples are, among others, free groups and any group acting properly discontinuously on a locally finite tree, and the fundamental groups of closed Riemannian manifolds with strictly negative curvature.

3.3.3 Gromov boundary

Gromov hyperbolic spaces have an interesting geometry at infinity, similar to the ideal boundary of classical hyperbolic space.

The construction of the boundary goes as follows. One can say that a sequence of points x_n in X 'converges at infinity' if $\langle x_i, x_j \rangle_o \rightarrow +\infty$ as i, j goes to infinity. Two such sequences x_n, y_n are considered equivalent if $\langle x_i, y_i \rangle_o \rightarrow +\infty$ as i goes to infinity. The Gromov boundary ∂X of X is the set of equivalence classes of such sequences.

The Gromov product can be extended to the boundary by

$$\langle \eta, \xi \rangle_o = \inf \liminf_{i,j} \langle x_i, y_j \rangle_o,$$

where the infimum is taken over all representatives x_n and y_n of η and ξ , respectively. Although the choice of \liminf does not make the above definition canonical, it turns out that replacing \inf by \sup or \limsup by \liminf leads to quantities which are all boundedly close. The above definition is therefore canonical, at least at the topological level, and defines a topology for convergence at infinity on the bordification $X \cup \partial X$.

The topology of convergence at infinity on ∂X is indeed metrizable. It turns out that a metric may be found for sufficiently small ϵ that is within a constant multiple of $\exp(-\epsilon \langle \cdot, \cdot \rangle_o)$ [66, Proposition 3.21, p. 435]. We will see below that this result can be improved.

3.3.4 Roughly geodesic and strongly hyperbolic metric spaces

The classical theory of hyperbolic groups takes place in the world of geodesic metric spaces. However, the extension of the Gromov product to ∂X requires a noncanonical choice.

By smoothing the metric d , we can make the Gromov product actually extend continuously to the boundary ∂X . Moreover, the natural topology on ∂X is then induced by an explicitly given metric, the visual metric or Bourdon metric. The tradeoff is that we must move from the world of geodesic metric spaces to roughly geodesic metric spaces.

Definition 6 *A metric space (X, d) is roughly geodesic if there exists $C > 0$ such that there exists a rough geodesic between all $x, y \in X$, i.e., a mapping $\gamma : [a, b] \subset \mathbb{R} \rightarrow X$, with $\gamma(a) = x$, $\gamma(b) = y$ and such that for all $t, s \in [a, b]$, $|t - s| - C \leq d(\gamma(t), \gamma(s)) \leq |t - s| + C$.*

A rough geodesic ray is a rough geodesic $r : [0, \infty) \rightarrow X$. Two rough geodesic rays are considered equivalent if $\sup_t d(r(t), r'(t)) < \infty$. An alternative construction of the Gromov boundary of a proper and roughly geodesic space X is to consider it as the set of equivalence classes of rough geodesic rays.

The relaxation of the geodesic assumption for hyperbolic spaces to roughly geodesic was first proposed by Bonk and Schramm [67]. Hyperbolicity is a quasi-isometric invariant for roughly geodesic metric spaces and the Schwarz-Milnor lemma also holds in this context. We can therefore just as well say that a group is hyperbolic if it acts geometrically on a roughly geodesic, proper hyperbolic metric space.

Definition 7 *A metric space (X, d) is ϵ -strongly hyperbolic if for every triple of points x, y, z and an origin o in X ,*

$$\exp(-\epsilon\langle x, y \rangle_o) \leq \exp(-\epsilon\langle x, z \rangle_o) + \exp(-\epsilon\langle z, y \rangle_o).$$

Theorem 4 (Nica, Spakula, Theorem 4.2 [68]) *Let (X, d) be a ϵ -strongly hyperbolic metric space, then*

1. (X, d) is $\log(2)/\epsilon$ -hyperbolic,
2. the Gromov product extends continuously on the boundary $X \cup \partial X$,
3. and the function $d_{o,\epsilon}(\eta, \xi) = \exp(-\epsilon\langle \eta, \xi \rangle_o)$ is a metric on ∂X .

Remark 3 *In the context of roughly geodesic metric spaces, a strongly hyperbolic metric space is strongly bolic in the sense of Lafforgue [69].*

Examples of strongly hyperbolic metric spaces are $\text{CAT}(-1)$ spaces [68]. Indeed, any hyperbolic group Γ acts geometrically on a roughly geodesic and strongly hyperbolic metric space. This metric is obtained by smoothing the word metric on Γ .

Theorem 5 (Nica, Spakula, Theorem 1.3 [68]) *Let Γ be a hyperbolic group. Then the Green metric defined by a symmetric and finitely supported random walk on Γ is strongly hyperbolic.*

We give a brief summary about the Green metric [70], for details the reader can refer [71], [72]. The Green metric is a way of smoothing a word metric on a hyperbolic group Γ by a random walk $\{X_n\}$ on Γ . Its law μ must be symmetric and have a support that generates Γ . Let $F(x, y)$ be the probability that a random walk starting from x ever hits y . $F(x, y)$ is proportional to the Green function

$$G(x, y) = \sum_{n=0}^{\infty} \mathbb{P}_x[X_n = y] = \sum_{n=0}^{\infty} \mu^n(x^{-1}y),$$

where μ^n is the n -th convolution of μ , i.e., the law of X_n .

The *Green metric* between x and y in Γ is the number

$$d_G(x, y) = -\log F(x, y).$$

The Green metric is still quasi-isometric to a word metric and Γ -invariant, as well as roughly-geodesic. The first two properties follow from the nonamenability of Γ , the last one is a result of Ancona [73].

Remark 4 *For the rest of this paper, we normalize the metric on Γ so that $\epsilon = 1$.*

3.3.5 A Möbius structure on strongly hyperbolic spaces.

Strongly hyperbolic metric spaces admit a *conformal structure* on their boundary. This is in contrast to only a *quasiconformal structure* on the boundary of hyperbolic spaces.

When X is a strongly hyperbolic space, the continuity of the Gromov product allows the definition of Busemann functions

$$\beta_\xi(o, o') = 2\langle \xi, o' \rangle_o - d(o, o') = \lim_{x \rightarrow \xi} (d(x, o) - d(x, o')). \quad (o, o' \in X, \xi \in \partial X)$$

For any pair of points o, o' in a strongly hyperbolic metric space,

$$d_{o'}(\eta, \xi) = \exp\left(\frac{1}{2}(\beta_\eta(o, o') + \beta_\xi(o, o'))\right) d_o(\eta, \xi),$$

see for example [68, p.955].

This implies that for any admissible quadruple (x, y, z, w) , the cross-ratio triple $[x, y, z, w]_{d_o}$ does not depend on o , and $\{d_o\}_{o \in X}$ is a Möbius structure on ∂X .

3.3.6 Patterson-Sullivan measures

We have seen that the ideal boundary of a strongly hyperbolic metric space X is a Möbius space if it is endowed with the set of visual metrics $\mathcal{M} = \{d_o\}_{o \in X}$. The Hausdorff measure μ_o associated with the metric d_o is called the *Patterson-Sullivan measure* [2],[74],[75]. Any such measure associated with the Möbius structure \mathcal{M} has the same Hausdorff dimension Q .

On strongly hyperbolic spaces, the Radon-Nikodým derivatives $\frac{d\mu_{o'}}{d\mu_o}$ have a concrete expression in terms of Busemann functions [68],

$$\frac{d\mu_{o'}}{d\mu_o}(x) = e^{Q\beta_x(o,o')}.$$

Patterson-Sullivan measures are Ahlfors-David regular [60],[76, Proposition 6.1]. As we have seen, even more holds in the context of Möbius geometry. There is a constant $C \geq 1$ such that for any measure $\mu_o \in \{\mu_o\}_{o \in X}$, and any d_o ball $B(x, r)$,

$$\frac{1}{C} r^Q \leq \mu_o(B(x, r)) \leq C r^Q.$$

3.3.7 Sullivan's shadow lemma.

Let X be a hyperbolic group with a left-invariant hyperbolic metric that is quasi-isometric to a word metric. For every origin o and every other point $o' \in X$ there is a roughly geodesic ray γ starting from o and extending at uniform distance from o' . With respect to the Gromov product, we can assume the existence of a constant $M > 0$ such that for all $o' \in X$,

$$\sup_{x \in \partial X} \langle x, o' \rangle_o \geq d(o, o') - M. \tag{3.30}$$

The constant M can be chosen independently of the choice of an origin o .

Lemma 3 (Garncarek, Lemma 4.1 [77]) *If X is a δ -Gromov hyperbolic group, then there exists $M > 0$ such that for all o' and all o in X ,*

$$\sup_{x \in \partial X} \langle x, o' \rangle_o \geq d(o, o') - M.$$

Since for each o' in X there is a roughly geodesic ray γ starting from o and passing near o' , we can assign it a front endpoint \bullet' . More precisely, let \bullet' be a point in ∂X such that

$$\langle \bullet', o' \rangle_o \geq d(o, o') - M. \tag{3.31}$$

Proof: (of Garncarek's lemma) For any point $x \in X$ and $g \in X$, the Gromov product satisfies

$$\langle g^{-1}o, x \rangle_o + \langle go, gx \rangle_o = d(o, g^{-1}o).$$

Let $x \in \partial X$ and $\{x_i\}$ be a representative of x , then

$$d(o, g^{-1}o) = \liminf_{i \rightarrow \infty} (\langle g^{-1}o, x_i \rangle_o + \langle go, gx_i \rangle_o) \leq \langle g^{-1}o, x \rangle_o + \langle go, gx \rangle_o + 4\delta.$$

So

$$\langle go, gx \rangle_o \geq d(o, g^{-1}o) - \langle g^{-1}o, x \rangle_o - 4\delta.$$

and for two different points $y_1, y_2 \in \partial X$,

$$\begin{aligned} \sup_{x \in \partial X} \langle x, go \rangle_o &\geq \max_i \langle go, gy_i \rangle_o \\ &\geq d(o, g^{-1}o) - \min\{\langle g^{-1}o, y_1 \rangle_o, \langle g^{-1}o, y_2 \rangle_o\} - 4\delta \\ &\geq d(o, go) - \langle y_1, y_2 \rangle_o - 5\delta. \end{aligned}$$

Set $M = \inf_{y_1, y_2 \in \partial X} \langle y_1, y_2 \rangle_o + 5\delta$ ■

A ball of radius s around a point o' in X casts a shadow at infinity as seen from o . Explicitly, the shadow at infinity of a ball of radius s around o' can be defined as

$$\mathcal{O}_o^{o'}(s) = \{x \in \partial X : \langle x, o' \rangle_o \geq d(o, o') - s\} \quad (3.32)$$

The shadows at infinity are essentially balls around the endpoint of a rough geodesic ray passing almost through o' .

Lemma 4 (Sullivan's shadow lemma) *Let X be a strongly hyperbolic space. Let \bullet' be a point on the boundary for which holds*

$$\langle \bullet', o' \rangle_o \geq d(o, o') - M.$$

Then for all $s > M + \delta$

$$\bar{B}_o(\bullet', e^{-(d(o, o') - s + \delta)}) \subset \mathcal{O}_o^{o'}(s) \subset \bar{B}_o(\bullet', e^{-(d(o, o') - s - \delta)}).$$

Proof: If $x \in \mathcal{O}_o^{o'}(s)$, then

$$\begin{aligned} \langle \bullet', x \rangle_o &\geq \min\{\langle \bullet', o' \rangle_o, \langle o', x \rangle_o\} - \delta \\ &\geq \min\{d(o, o') - M, d(o, o') - s\} - \delta \\ &= d(o, o') - s - \delta. \end{aligned}$$

Therefore $d_o(\bullet', x) \leq e^{-(d(o, o') - s - \delta)}$. Conversely, if $x \in \bar{B}_o(\bullet', e^{-(d(o, o') - s + \delta)})$, then

$$\begin{aligned} \langle o', x \rangle_o &\geq \min\{\langle o', \bullet' \rangle_o, \langle \bullet', x \rangle_o\} - \delta \\ &\geq \min\{d(o, o') - M, d(o, o') - s + \delta\} - \delta \\ &= d(o, o') - s. \end{aligned}$$

The equality in the last line follows from the assumption $s > M + \delta$. ■

3.3.8 Distances associated to boundary points, the Cayley transform revisited

Let X be a strongly hyperbolic metric space and $Z = \partial X$ its ideal boundary endowed with its natural Möbius structure, i.e., a family of distances d_o indexed by points o of X :

$$d_o(x, y) = e^{-\langle x, y \rangle_o}.$$

For points $p, q \in X$,

$$\langle p, q \rangle_o = \frac{1}{2}(d(o, p) + d(o, q) - d(p, q)).$$

One then shows that $\langle p, q \rangle_o$ has a limit denoted by $\langle x, y \rangle_o$ when p tends to x and q to y .

Proposition 9 *Let $x, y \in \partial X$. As $p \in X$ tends to a point $z \in \partial X \setminus \{x, y\}$,*

$$e^{d(o, p)} d_p(x, y)$$

tends to a number denoted by $d_{z,o}(x, y)$, given by

$$d_{z,o}(x, y) = \frac{d_o(x, y)}{d_o(z, x)d_o(z, y)}. \quad (3.33)$$

As o is changed, $d_{z,o}$ is multiplied by a positive constant,

$$d_{z,o'} = e^{\beta_z(o',o)} d_{z,o}.$$

Proof: Indeed, let $q, q' \in X$ be close to $x, y \in \partial X$.

$$\begin{aligned} \langle q, q' \rangle_p - \langle q, q' \rangle_o - d(o, p) &= \frac{1}{2}(d(p, q) + d(p, q') - d(q, q')) \\ &\quad - \frac{1}{2}(d(o, q) + d(o, q') - d(q, q')) - d(o, p) \\ &= \frac{1}{2}(d(p, q) - d(o, q) - d(o, p)) \\ &\quad + \frac{1}{2}(d(p, q') - d(o, q') - d(o, p)) \\ &= -\langle p, q \rangle_o - \langle p, q' \rangle_o, \end{aligned}$$

which tends to $-\langle z, x \rangle_o - \langle z, y \rangle_o$ as $p \rightarrow z, q \rightarrow x, q' \rightarrow y$. So

$$e^{-d(o,p)} \frac{d_o(x, y)}{d_p(x, y)}$$

tends to $d_o(z, x)d_o(z, y)$ when p tends to z . This gives the formula

$$d_{z,o}(x, y) = \frac{d_o(x, y)}{d_o(z, x)d_o(z, y)}.$$

Furthermore,

$$\begin{aligned} \frac{d_{z,o'}(x, y)}{d_{z,o}(x, y)} &= \lim_{p \rightarrow z} \frac{e^{d(o',p)} d_p(x, y)}{e^{d(o,p)} d_p(x, y)} \\ &= \lim_{p \rightarrow z} e^{d(o',p) - d(o,p)} \\ &= e^{\beta_z(o',o)}. \end{aligned}$$

■

Although the semimetric $d_{z,o}$ is undefined at z , it belongs to the Möbius class in an extended sense: by the convention that $d_{z,o}(z, x) = +\infty$ for all $x \neq z$, the semimetric defines the same cross ratios.

The metric associated with an ideal boundary point is defined only up to a multiplicative constant. The same is true for the Hausdorff Q -measure.

Corollary 1 *The rescaled Hausdorff Q -measure $e^{Q d(o,p)} \mu_p$ converges setwise on $\partial X \setminus \{z\}$ when $p \in X$ tends to $z \in \partial X$, to the Hausdorff Q -measure $\mu_{z,o}$ belonging to $d_{z,o}$,*

i.e. for every measurable set $A \in \partial X \setminus \{z\}$, $\lim_{p \rightarrow z} e^{Qd(o,p)} \mu_p(A) = \mu_{z,o}(A)$. This measure satisfies, for $x \neq z$,

$$\frac{d\mu_{z,o}}{d\mu_o}(x) = (d_o(z,x))^{-2Q}.$$

As o is changed, $\mu_{z,o}$ is multiplied by a positive constant,

$$\mu_{z,o'} = e^{Q\beta_z(o',o)} \mu_{z,o}.$$

Proof: Indeed, let A be an open set such that z is not a limitpoint of A . W.l.o.g. suppose the points p are far enough from o such that $A \subset \partial X \setminus \mathcal{O}_o^p(M + 2\delta)$ for all points p in the sequence, and the endpoints \bar{p} of rays starting from o and passing within uniform distance from p are all ϵ close to z . Then

$$e^{d(o,p)} \frac{d_p}{d_o}(x) = e^{2\langle x,p \rangle_o}$$

is dominated by

$$\left(\frac{e^{2\delta}}{e^{-\langle x,z \rangle_o} - \epsilon} \right)^2,$$

which is integrable on B . Thus, by dominated convergence,

$$\lim_{p \rightarrow z} e^{Qd(o,p)} \mu_p(A) = \int_A d_o(z,x)^{-2Q} d\mu_o(x). \quad (3.34)$$

If z is a limitpoint of A , then clearly the right-hand-side in (3.34) is infinite. For the left-hand-side, there exists a ball $B \subset A$ with limitpoint z , and $e^{Qd(o,p)} \mu_p(B) \geq e^{Qd(o,p)} \frac{d_p(x,z)^Q}{C}$ for x the center of B . This lower bound tends to infinity when p tends to z . Therefore, (3.34) makes sense in this case as well. The arguments above can be extended to the Borel-sigma algebra on $\partial X \setminus \{z\}$, since μ_d are Radon-measures. \blacksquare

Example 3 Let X be a symmetric space of rank 1. Then $d_{z,o}$ is a left-translation invariant dilation-homogeneous distance on the unipotent radical N of $\text{Isom}(X)$. If X has constant curvature, $d_{z,o}$ is a Euclidean distance.

Indeed, $d_{z,o}$ is preserved by the stabilizer of z and the horosphere passing through o in $\text{Isom}(X)$ containing N , and N acts simply transitively on $\partial X \setminus \{z\}$. The dilations of N are realized by the 1-parameter subgroup A of $\text{Isom}(X)$, whose elements each multiply $d_{z,o}$ by a constant. When X has constant curvature, the stabilizer of z in $\text{Isom}(X)$ is two-point transitive on $\partial X \setminus \{z\}$ such that $d_{z,o}$ is a function of Euclidean distance. By dilation homogeneity, this function is homogeneous of degree 1, so $d_{z,o}$ is a constant multiple of the Euclidean distance.

3.4 Harmonic analysis on the boundary of hyperbolic groups

3.4.1 Introduction

A central tool of classical harmonic analysis on \mathbb{R}^n are maximal functions. Maximal functions allow to obtain quantitative results on the average of functions.

The best known of all maximal functions is the Hardy-Littlewood maximal function.

Definition 8 *The Hardy-Littlewood maximal function of f , is*

$$M(f)(x) = \sup_{B \ni x} \frac{1}{|B|} \int_B |f(y)| dy \quad (3.35)$$

This operator behaves well on L^p -spaces.

Theorem 6 (see for example [78]) *The Hardy-Littlewood maximal operator maps $L^1(\mathbb{R}^n)$ to $L^{1,\infty}(\mathbb{R}^n)$ with constant at most 3^n and also $L^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ for $1 < p < \infty$ with constant at most $3^{n/p} p(p-1)^{-1}$.*

Weighted inequalities The study of weighted inequalities for maximal functions appeared as a natural extension of the theory. A main question is the characterization of all measures μ on \mathbb{R}^n for which there exists $A > 0$, s.t.

$$\int_{\mathbb{R}^n} M(f)(x)^p d\mu(x) \leq A \int_{\mathbb{R}^n} |f(x)|^p d\mu(x), \quad (3.36)$$

for any p , $1 < p < \infty$ and $f \in L^p(d\mu)$.

The complete solution was given 50 years ago by Muckenhoupt [79], who set up a class of functions A_p such that (3.36) and other similar inequalities, hold exactly when $d\mu(x) = w(x)^p dx$ with $w \in A_p$.

Weighted inequalities for singular integrals are another source of examples where Muckenhoupt's A_p condition plays a central role. If T is a singular integral operator of convolution type with some regularity assumptions for the kernel, then there exists $A > 0$, such that for all $f \in L^p(d\mu)$,

$$\int_{\mathbb{R}^n} |Tf(x)|^p d\mu(x) \leq A \int_{\mathbb{R}^n} |f(x)|^p d\mu(x),$$

if and only if $d\mu(x) = w(x)^p dx$, with $w \in A_p$.

3.4.2 Muckenhoupt weights

It turns out that the class of measures μ consists exactly of those measures which are absolutely continuous with respect to the Lebesgue measure $d\mu(x) = w(x)^p dx$, and that w satisfies the A_p inequality

$$\llbracket w \rrbracket_p = \sup_B \left(\frac{1}{|B|} \int_B w^p dx \right)^{\frac{1}{p}} \left(\frac{1}{|B|} \int_B \left(\frac{1}{w} \right)^{p'} dx \right)^{\frac{1}{p'}} < \infty, \quad (3.37)$$

where p' denotes the harmonic conjugate of p .

Remark 5 *In the literature, one often finds a slightly different but equivalent definition of the class A_p . It amounts to setting $v = w^p$ and defining A_p as the class of functions v such that $(\llbracket v^{\frac{1}{p}} \rrbracket_p)^p$ is finite.*

Example 4 The function $|x|^\alpha$ belongs to A_p on \mathbb{R}^n , if and only if $-n < \alpha p < n(p-1)$.

Below we list some simple but useful observations on general properties of Muckenhoupt weights.

Duality. Muckenhoupt weights admit an obvious "duality". If $w \in A_p$, then $w^{-1} \in A_{p'}$ with the same Muckenhoupt constant.

Alternative definition. There is an alternative definition of the class A_p which is more closely related in notation to Hardy-Littlewood's maximal function. For any locally integrable function f , we denote by f_B the average of f over a ball B , i.e., $f_B = \frac{1}{|B|} \int_B f(x) dx$. We further denote by $w^p(B)$ the number $\int_B w(x)^p dx$. Then w belongs to the class A_p if and only if the p -th power of f_B is controlled by the mean of f^p with respect to the measure $w(x)^p dx$. Specifically, this means,

$$(f_B)^p \leq \frac{c}{w^p(B)} \int_B f^p w^p dx \quad (3.38)$$

holds for all nonnegative functions f and all balls B . The minimal constant c for which this inequality is true coincides with $(A_p(w))^p$.

Suppose $w \in A_p$.

$$f_B = \frac{1}{|B|} \int_B f(x) dx = \frac{1}{|B|} \int_B f w w^{-1} dx.$$

We apply Hölder's inequality to the right-hand side with exponent p .

$$f_B \leq \frac{1}{|B|} \left(\int_B f^p w^p dx \right)^{\frac{1}{p}} \left(\int_B w^{-p'} dx \right)^{\frac{1}{p'}}.$$

Together with Muckenhoupt's inequality (3.37) establishes the above inequality (3.38), with minimal constant $c \leq (A_p(w))^{\frac{1}{p}}$. Conversely, if (3.38) holds for w , we take f to be $(w + \epsilon)^{-p'}$. Then (3.38) together with $(1-p')p = -p'$ implies that

$$\frac{1}{|B|} \left(\int_B (w(x) + \epsilon)^{-p'} dx \right)^{p-1} w^p(B) \leq c.$$

Taking the p -th root and passing to the limit $\epsilon \rightarrow 0$, we obtain Muckenhoupt's inequality (3.37).

Doubling property. Any absolutely continuous measure with respect to Lebesgue's measure, $d\mu(x) = w(x)^p dx$, is doubling if $w \in A_p$. A measure μ is doubling if there is a constant $D > 0$, s.t.

$$\mu(2B) \leq D\mu(B),$$

where $2B$ denotes the ball B with twice its radius. If $w \in A_p$, then the fact that the measure w^p is doubling, follows directly from (3.38) if we set $B = 2B'$ and $f = \chi_{B'}$. The doubling constant is explicitly given by $c2^{np}$.

A_∞ -property. We say that a weight is of class A_∞ , if for any γ , $0 < \gamma < 1$, there exists a δ , $0 < \delta < 1$, so that for every ball B and for all subsets $F \subset B$,

$$|F| \geq \gamma|B| \implies w(F) \geq \delta w(B). \quad (3.39)$$

Notice that if $w \in A_p$, then w^p is of class A_∞ . Indeed, use the inequality (3.38) above with $f = \chi_F$. Then $(|F|/|B|)^p \leq c(w^p(F)/w^p(B))$, which gives inequality (3.39) with $\delta = \gamma^p/c$.

By passing to complements i.e. setting F to $B \setminus F$ above. We get the equivalent definition, that $w \in A_\infty$ if, for all γ , $0 < \gamma < 1$, there exists δ , $0 < \delta < 1$, so that for all balls B and all subsets $E \subset B$,

$$|E| \leq \gamma|B| \implies w(E) \leq \delta w(B). \quad (3.40)$$

3.4.3 Generalized Muckenhoupt weights.

Muckenhoupt and Wheeden [80] generalized the class A_p to the operator

$$I^\alpha(f)(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy, \quad (3.41)$$

for $0 < \alpha < n$. It is often called a fractional integration operator on \mathbb{R}^n . For $\alpha = 0$ it is a Calderon-Zygmund singular integral.

Let $1 < p < \frac{n}{\alpha}$, and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, then

$$\left(\int_{\mathbb{R}^n} |I^\alpha(f)(x)|^q w(x)^q dx \right)^{\frac{1}{q}} \leq A \left(\int_{\mathbb{R}^n} |f(x)|^p w(x)^p dx \right)^{\frac{1}{p}},$$

for a nonnegative function w if and only if,

$$\llbracket w \rrbracket_{q,p} = \sup_B \left(\frac{1}{|B|} \int_B w^q dx \right)^{\frac{1}{q}} \left(\frac{1}{|B|} \int_B \left(\frac{1}{w} \right)^{p'} dx \right)^{\frac{1}{p'}} < \infty.$$

We say that the weight w belongs to the class $A_{q,p}$.

3.4.4 Muckenhoupt weights associated with Möbius structures

In this section, X will be a strongly hyperbolic metric space and its ideal boundary ∂X carries the natural Möbius structure $\{d_o\}_{o \in X}$ of visual distances. We will show that powers of the conformal factors

$$\frac{d_{o'}}{d_o}(x) = e^{\beta_x(o,o')},$$

are Muckenhoupt weights with respect to μ_o , with uniformly bounded $A_{p,q}$ constant.

Indeed, we can admit o and or o' as ideal points. This gives a Muckenhoupt theory on Möbius structures which is invariant under the Cayley transform and therefore treats the compact and noncompact picture as equivalent to each other.

The definitions and notations introduced above for \mathbb{R}^n can be readily generalized to this situation by replacing the Lebesgue measure dx by the Patterson-Sullivan measure μ_o . In fact, μ_o is Ahlfors-David regular and thus doubling. The remarks made above about the properties of A_p -weights on \mathbb{R}^n generalize easily to this setting as well and will not be repeated here.

Theorem 7 *Let ∂X be the boundary of a strongly hyperbolic metric space, equipped with the Möbius structure $\{d_o\}_{o \in X}$ of visual metrics. Let $1 < p \leq q < \infty$ and $\alpha < \frac{Q}{q}$. There exists a constant $C > 0$, s.t. for all $o, o' \in X$,*

$$\left[\left(\frac{d_{o'}}{d_o} \right)^{\frac{\alpha}{2}} \right]_{p,q} = \sup_B \left(\frac{1}{\mu_o(B)} \int_B \left(\frac{d_{o'}}{d_o} \right)^{\frac{\alpha q}{2}} d\mu_o \right)^{\frac{1}{q}} \left(\frac{1}{\mu_o(B)} \int_B \left(\frac{d_{o'}}{d_o} \right)^{-\frac{\alpha p'}{2}} d\mu_o \right)^{\frac{1}{p'}} \leq C < \infty.$$

Clearly, for any pair of points o, o' and $1 < p \leq q < \infty$, the weight $\left(\frac{d_{o'}}{d_o} \right)^{\frac{\alpha}{2}}$ is a Muckenhoupt weight of class $A_{p,q}$ since the conformal factors $\frac{d_{o'}}{d_o}$ are bounded above and away from 0. The significance comes from the fact that the bound on the $A_{p,q}$ constant is independent of o and o' .

We denote by $A(\mathcal{M})_{p,q}^\alpha$ or simply $A(\mathcal{M})$ or A the minimal $C > 0$, such that the above inequality is true for all $o, o' \in X$.

The proof shows that the constant $A(\mathcal{M})_{p,q}^\alpha$ cannot be finite for $\alpha = \frac{Q}{q}$. This will become relevant again when we discuss bounded representations and the limit when α tends to $\frac{Q}{q}$.

Theorem 7 states in particular that for any $\alpha < \frac{Q}{q}$, $\left(\frac{d_{o'}}{d_o} \right)^{\frac{\alpha}{2}}$ is of class A_p for any $1 < p \leq q$. Thus by the duality property of the Muckenhoupt weights, $\left(\frac{d_{o'}}{d_o} \right)^{-\frac{\alpha}{2}}$ is of class $A_{p'}$. The following corollaries follow from the doubling property of the Muckenhoupt weights.

Corollary 2 *For all $d_o \in \mathcal{M}$, $\alpha < \frac{Q}{q}$, the measure*

$$U \mapsto \int_U \left(\frac{d_{o'}}{d_o} \right)^{\frac{\alpha q}{2}} d\mu_o$$

is doubling with a doubling constant $(DA)^q$, where D is the doubling constant of the measures belonging to \mathcal{M} .

Corollary 3 *For all $d_o \in \mathcal{M}$, $\alpha < \frac{Q}{2p}$, the measure*

$$U \mapsto \int_U \left(\frac{d_{o'}}{d_o} \right)^{-\frac{\alpha p'}{2}} d\mu_o$$

is doubling with a doubling constant $(DA)^{p'}$, where D is the doubling constant of the measures belonging to \mathcal{M} .

Proof: (of the theorem 7) Let B be a ball in ∂X with center z and radius r , and o and o' two points in X .

Let \bullet be a front endpoint in Z of a geodesic starting from o and passing within bounded distance from o' . We denote by $|x|$ the distance of a point x in ∂X to \bullet , with respect to the metric d_o .

We denote by $B^s = B(\infty, e^\delta e^{-(d(o,o')-s)})$, and by $r(s)$ the radius of B^s . Sullivan's shadow lemma states that the shadow $\mathcal{O}_o'(s)$ is contained in B^s if $s \geq M + 2\delta$.

We will examine 3 cases separately. Pick an arbitrary large positive integer $n \geq 2$,

1. Case 1: $|z| \geq (1 + \frac{1}{n})r$,
2. Case 2: $|z| \leq (1 - \frac{1}{n})r$,
3. Case 3: $(1 - \frac{1}{n})r \leq |z| \leq (1 + \frac{1}{n})r$.

Case 1: For technical reasons, we distinguish two subcases:

1. Case 1A: $|z| \geq (1 + \frac{1}{n})r$ and $B \cap \mathcal{O}_o'(M + 2\delta) = \emptyset$,
2. Case 1B: $|z| \geq (1 + \frac{1}{n})r$ and $B \cap \mathcal{O}_o'(M + 2\delta) \neq \emptyset$.

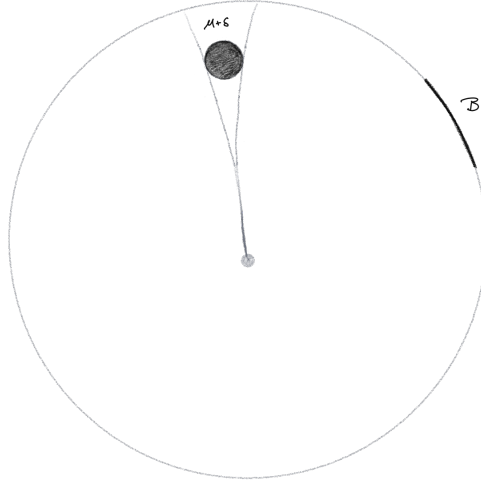


Figure 3.1 – Geometric situation in case 1A

In case 1A, we have the nice advantage of being able to replace $e^{-\langle x, o' \rangle_o}$ with the true distance $|x|_o$. Indeed, for all x in B ,

$$e^{-2\delta\alpha q} |x|^{-\alpha q} \leq e^{\alpha q \langle x, o' \rangle_o} \leq e^{2\delta\alpha q} |x|^{-\alpha q},$$

and

$$e^{-2\delta\alpha p'} |x|^{\alpha p'} \leq e^{-\alpha p' \langle x, o' \rangle_o} \leq e^{2\delta\alpha p'} |x|^{\alpha p'}.$$

This follows from the following. $B \cap \mathcal{O}_o'(M + 2\delta) = \emptyset$ is equivalent to

$$\forall x \in B, \quad \langle x, o' \rangle_o < d(o, o') - M - 2\delta.$$

So

$$\begin{aligned} \langle x, \bullet \rangle_o &\geq \min\{\langle x, o' \rangle_o, \langle o', \bullet \rangle_o\} - 2\delta \\ &\geq \min\{\langle x, o' \rangle_o, d(o, o') - M\} - 2\delta \\ &\geq \langle x, o' \rangle_o - 2\delta. \end{aligned}$$

The converse follows from the fact that B does not intersect the ball $B(\bullet, e^{-d(o,o')-M})$ either. So

$$\forall x \in B, \quad \langle x, \bullet \rangle_o \leq d(o, o') - M.$$

And therefore

$$\begin{aligned} \langle x, o' \rangle_o &\geq \min\{\langle x, \bullet \rangle_o, \langle \bullet, o' \rangle_o\} - 2\delta \\ &\geq \min\{\langle x, \bullet \rangle_o, d(o, o') - M\} - 2\delta \\ &\geq \langle x, \bullet \rangle_o - 2\delta. \end{aligned}$$

This proves the comparison claim.

This allows us to use the triangle inequality to obtain upper bounds. Since $|z| \geq (1 + \frac{1}{n})r$,

$$\forall x \in B: \quad |x| \geq |z| - r \geq |z| - \frac{n}{n+1}|z| = \frac{|z|}{n+1},$$

and

$$\forall x \in B, \quad |x| \leq |z| + r \leq |z| + \frac{n}{1+n}|z| = \frac{2n+1}{n+1}|z|.$$

Thus,

$$\begin{aligned} \int_B \left(\frac{d_{o'}}{d_o}(x)\right)^{\frac{\alpha q}{2}} d\mu_o &= e^{-\frac{\alpha q}{2}d(o,o')} \int_B e^{\alpha q \langle x, o' \rangle_o} d\mu_o \\ &\leq e^{-\frac{\alpha q}{2}d(o,o')} e^{2\delta \alpha q} \int_B |x|^{-\alpha q} d\mu_o \\ &\leq e^{-\frac{\alpha q}{2}d(o,o')} e^{2\delta \alpha q} (n+1)^{\alpha q} |z|^{-\alpha q} \mu_o(B), \end{aligned}$$

and

$$\begin{aligned} \int_B \left(\frac{d_{o'}}{d_o}(x)\right)^{-\frac{\alpha p'}{2}} d\mu_o &= e^{\frac{\alpha p'}{2}d(o,o')} \int_B e^{-\alpha p' \langle x, o' \rangle_o} d\mu_o \\ &\leq e^{\frac{\alpha p'}{2}d(o,o')} e^{2\delta \alpha p'} \int_{2B} |x|^{\alpha p'} d\mu_o \\ &\leq e^{\frac{\alpha p'}{2}d(o,o')} e^{2\delta \alpha p'} \frac{(2n+1)^{\alpha p'}}{(n+1)^{\alpha p'}} |z|^{\alpha p'} \mu_o(B). \end{aligned}$$

This implies,

$$\left(\int_B \left(\frac{d_{o'}}{d_o}\right)^{\frac{\alpha q}{2}} d\mu_o\right)^{\frac{1}{q}} \left(\int_B \left(\frac{d_{o'}}{d_o}\right)^{-\frac{\alpha p'}{2}} d\mu_o\right)^{\frac{1}{p'}} \leq e^{4\delta \alpha} (2n+1)^\alpha \mu_o(B)^{\frac{1}{q} + \frac{1}{p'}},$$

which proves the case 1A.

In case 1B, $B \cap \mathcal{O}'_o(M+2\delta) \neq \emptyset$. The shadow $\mathcal{O}'_o(M+2\delta)$ is contained in the ball $B^{M+2\delta}$, with $r_{(M+2\delta)} = e^{-(d(o,o')-M-3\delta)}$. That B intersects the shadow $\mathcal{O}'_o(M+2\delta)$ implies that the radius r is comparable to $r_{(M+2\delta)}$. More precisely, since $|z| \geq (1 + \frac{1}{n})r$,

$$r \leq n(|z| - r) \leq n r_{(M+2\delta)}.$$

This implies that,

$$B \subset B(\infty, r_{(M+2\delta)} + 2r) \subset B(\infty, (1+2n)r_{(M+2\delta)}) \subset \mathcal{O}'_o(M + \log(1+2n) + 4\delta).$$

From this follows,

$$\forall x \in B : \quad \langle x, o' \rangle_{o'} \geq d(o, o') - (M + \log(1 + 2n) + 4\delta).$$

We use this inequality for the upper bound of the p' measure.

$$\begin{aligned} \int_B \left(\frac{d_{o'}}{d_o}\right)^{-\frac{\alpha p'}{2}} d\mu_o &= e^{\frac{\alpha p'}{2}d(o,o')} \int_B e^{-\alpha p' \langle x, o' \rangle_{o'}} d\mu_o \\ &\leq e^{-\frac{\alpha p'}{2}d(o,o')} e^{\alpha p'(M+\log(1+2n)+4\delta)} \mu_o(B) \end{aligned}$$

The corresponding upper bound for the q -measure is straightforward.

$$\int_B \left(\frac{d_{o'}}{d_o}\right)^{\frac{\alpha q}{2}} d\mu_o \leq e^{\frac{\alpha q}{2}d(o,o')} \mu_o(B).$$

This implies,

$$\left(\int_B \left(\frac{d_{o'}}{d_o}\right)^{\frac{\alpha q}{2}} d\mu_o\right)^{\frac{1}{q}} \left(\int_B \left(\frac{d_{o'}}{d_o}\right)^{-\frac{\alpha p'}{2}} d\mu_o\right)^{\frac{1}{p'}} \leq e^{\alpha(M+4\delta)} (1 + 2n)^\alpha \mu_o(B)^{\frac{1}{q} + \frac{1}{p'}},$$

which completely proves the case 1.

Case 2: Let S_+ be the set of all $s \in \mathbb{R}^+$, such that $B \subset \mathcal{O}_o^{o'}(s)$. Let s_+^* be the infimum of S_+ , and

$$s_+ := d(o, o') - \delta + \log(|z| + r).$$

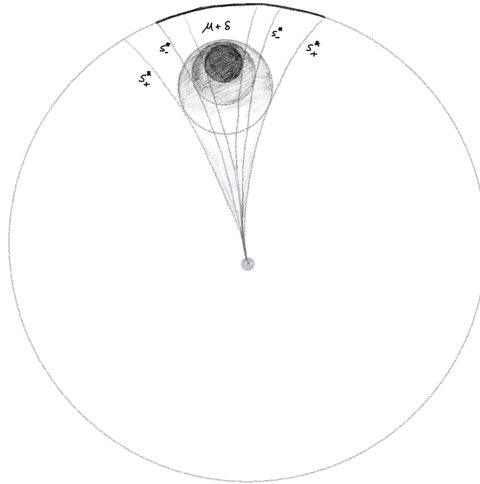


Figure 3.2 – Geometric situation in case 2A

Let S_- be the set of all $s \in \mathbb{R}^+$, such that $\mathcal{O}_o^{o'}(s) \subset B$. Note that if $s \in S_-$, then for all $s' \leq s$, $s' \in S_-$. Let s_-^* be the supremum of S_- .

Recall that we denoted by B^s the ball $B(\infty, e^\delta e^{-(d(o,o')-s)})$ and by $r(s)$ its radius.

For technical reasons, we again distinguish two subcases:

1. Case 2A: $|z| \leq (1 - \frac{1}{n})r$ and $\mathcal{O}_o^{o'}(M + 2\delta) \subset B$,
2. Case 2B: $|z| \leq (1 - \frac{1}{n})r$ and $\mathcal{O}_o^{o'}(M + 2\delta) \not\subset B$.

In case 2A, $s_+ > M + 2\delta$ and Sullivan's shadow lemma implies $s_+ \in S_+$. Thus,

$$M + 2\delta \leq s_+^* \leq s_+.$$

For the upper bound of $\int_B \left(\frac{d_{o'}}{d_o}(x)\right)^{\frac{\alpha q}{2}} d\mu_o$, we first note that

$$\int_B \left(\frac{d_{o'}}{d_o}(x)\right)^{\frac{\alpha q}{2}} d\mu_o \leq \int_{\mathcal{O}_{o'}(s_+)} \left(\frac{d_{o'}}{d_o}(x)\right)^{\frac{\alpha q}{2}} d\mu_o.$$

The right-hand side has an explicit decomposition by integration by parts.

$$\begin{aligned} \int_{\mathcal{O}_{o'}(s_+)} \left(\frac{d_{o'}}{d_o}(x)\right)^{\frac{\alpha q}{2}} d\mu_o &= e^{\frac{\alpha q}{2}d(o,o')} \int_0^\infty \alpha q e^{-\alpha q s} \mu_o(\mathcal{O}_{o'}(s) \cap \mathcal{O}_{o'}(s_+)) ds \\ &= e^{\frac{\alpha q}{2}d(o,o')} \int_0^{M+2\delta} \alpha q e^{-\alpha q s} \mu_o(\mathcal{O}_{o'}(s)) ds \\ &\quad + e^{\frac{\alpha q}{2}d(o,o')} \int_{M+2\delta}^{s_+} \alpha q e^{-\alpha q s} \mu_o(\mathcal{O}_{o'}(s)) ds \\ &\quad + e^{\frac{\alpha q}{2}d(o,o')} e^{-\alpha q s_+} \mu_o(\mathcal{O}_{o'}(s_+)) \end{aligned}$$

Further calculations yield,

$$\begin{aligned} \int_{\mathcal{O}_{o'}(s_+)} \left(\frac{d_{o'}}{d_o}(x)\right)^{\frac{\alpha q}{2}} d\mu_o &\leq e^{\frac{\alpha q}{2}d(o,o')} \int_0^{M+2\delta} \alpha q e^{-\alpha q s} \mu_o(B^{M+2\delta}) ds \\ &\quad + e^{\frac{\alpha q}{2}d(o,o')} \int_{M+2\delta}^{s_+} \alpha q e^{-\alpha q s} \mu_o(B^s) ds \\ &\quad + e^{\frac{\alpha q}{2}d(o,o')} e^{-\alpha q s_+} \mu_o(B^{s_+}) \\ &\leq e^{\frac{\alpha q}{2}d(o,o')} \mu_o(B^{M+2\delta}) \\ &\quad + C e^{\delta Q} e^{-Qd(o,o')} e^{\frac{\alpha q}{2}d(o,o')} \int_{M+2\delta}^{s_+} \alpha q e^{(Q-\alpha q)s} ds \\ &\quad + e^{\frac{\alpha q}{2}d(o,o')} e^{-\alpha q s_+} \mu_o(B^{s_+}) \\ &\leq e^{\frac{\alpha q}{2}d(o,o')} \mu_o(B^{M+2\delta}) \\ &\quad + C e^{\delta Q} e^{-Qd(o,o')} e^{\frac{\alpha q}{2}d(o,o')} \left(\frac{\alpha q}{Q-\alpha q}\right) e^{(Q-\alpha q)s_+} \\ &\quad + e^{\frac{\alpha q}{2}d(o,o')} e^{-\alpha q s_+} \mu_o(B^{s_+}) \\ &= e^{-\frac{\alpha q}{2}d(o,o')} e^{\alpha q \delta} e^{-\alpha q(M+2\delta)} r_{(M+2\delta)}^{-\alpha q} \mu_o(B^{M+2\delta}) \\ &\quad + C e^{-\frac{\alpha q}{2}d(o,o')} e^{\alpha q \delta} \left(\frac{\alpha q}{Q-\alpha q}\right) r_{(s_+)}^{Q-\alpha q} \\ &\quad + e^{-\frac{\alpha q}{2}d(o,o')} e^{\alpha q \delta} r_{(s_+)}^{-\alpha q} \mu_o(B^{s_+}) \\ &\leq C e^{-\frac{\alpha q}{2}d(o,o')} e^{\alpha q \delta} r_{(M+2\delta)}^{Q-\alpha q} \\ &\quad + C e^{-\frac{\alpha q}{2}d(o,o')} e^{\alpha q \delta} \left(\frac{\alpha q}{Q-\alpha q}\right) r_{(s_+)}^{Q-\alpha q} \\ &\quad + C e^{-\frac{\alpha q}{2}d(o,o')} e^{\alpha q \delta} r_{(s_+)}^{Q-\alpha q} \\ &\leq C e^{-\frac{\alpha q}{2}d(o,o')} e^{\alpha q \delta} \left(\frac{2Q-\alpha q}{Q-\alpha q}\right) r_{(s_+)}^{Q-\alpha q} \end{aligned}$$

Thus,

$$\int_B \left(\frac{d_{o'}}{d_o}\right)^{\frac{\alpha q}{2}} d\mu_o \leq C e^{-\frac{\alpha q}{2}d(o,o')} e^{\alpha q \delta} \left(\frac{2Q - \alpha q}{Q - \alpha q}\right) (|z| + r)^{Q - \alpha q},$$

and using the assumption $|z| \leq (1 - \frac{1}{n})r$,

$$\int_B \left(\frac{d_{o'}}{d_o}\right)^{\frac{\alpha q}{2}} d\mu_o \leq C e^{-\frac{\alpha q}{2}d(o,o')} e^{\alpha q \delta} \left(\frac{2Q - \alpha q}{Q - \alpha q}\right) \left(\frac{2n + 1}{n}\right)^{Q - \alpha q} r^{Q - \alpha q}.$$

The upper bound for the p' measure is much simpler.

$$\begin{aligned} \int_B \left(\frac{d_{o'}}{d_o}\right)^{-\frac{\alpha p'}{2}} d\mu_o &= e^{-\frac{\alpha p'}{2}d(o,o')} \int_{-\infty}^{d(o,o')} \alpha p' e^{\alpha p' s} \mu_o(B \setminus \mathcal{O}_o'(s)) ds \\ &= e^{-\frac{\alpha p'}{2}d(o,o')} \int_{-\infty}^{s^*} \alpha p' e^{\alpha p' s} \mu_o(B \setminus \mathcal{O}_o'(s)) ds \\ &\leq e^{-\frac{\alpha p'}{2}d(o,o')} e^{\alpha p' s^*} \mu_o(B) \\ &= e^{\frac{\alpha p'}{2}d(o,o')} e^{-\alpha p' \delta} (|z| + r)^{\alpha p'} \mu_o(B) \\ &\leq e^{\frac{\alpha p'}{2}d(o,o')} e^{-\alpha p' \delta} \left(\frac{3n - 1}{n}\right)^{\alpha p'} r^{\alpha p'} \mu_o(B). \end{aligned}$$

For the last inequality we used the assumption $|z| \leq (1 - \frac{1}{n})r$.

Both of the above bounds imply,

$$\begin{aligned} \left(\int_B \left(\frac{d_{o'}}{d_o}\right)^{\frac{\alpha q}{2}} d\mu_o\right)^{\frac{1}{q}} \left(\int_B \left(\frac{d_{o'}}{d_o}\right)^{-\frac{\alpha p'}{2}} d\mu_o\right)^{\frac{1}{p'}} \\ \leq C^{\frac{1}{q} + \frac{1}{p'}} \left(\frac{2Q - \alpha q}{Q - \alpha q}\right)^{\frac{1}{q}} \left(\frac{2n + 1}{n}\right)^{\frac{Q}{q} - \alpha} \left(\frac{3n - 1}{n}\right)^{\alpha} r^{Q(\frac{1}{q} + \frac{1}{p'})} \\ \leq C^{2(\frac{1}{q} + \frac{1}{p'})} \left(\frac{2Q - \alpha q}{Q - \alpha q}\right)^{\frac{1}{q}} \left(\frac{2n + 1}{n}\right)^{\frac{Q}{q} - \alpha} \left(\frac{3n - 1}{n}\right)^{\alpha} \mu_o(B)^{\frac{1}{q} + \frac{1}{p'}}, \end{aligned}$$

which proves the case 2A.

Case 2B: We recall that the shadow $\mathcal{O}_o'(M + 2\delta)$ is contained in the ball $B^{M+2\delta}$, with $r_{(M+2\delta)} = e^{-(d(o,o') - M - 3\delta)}$. This, together with the assumption $|z| \leq (1 - \frac{1}{n})r$, implies that

$$\frac{r}{n} \leq r - |z| \leq r_{(M+2\delta)}.$$

Thus,

$$B \subset B(\infty, |z| + r) \subset B(\infty, (2n - 1)r_{(M+2\delta)}) \subset \mathcal{O}_o'(M + \log(2n - 1) + 4\delta),$$

where the last inclusion again follows from the shadow lemma. This implies

$$\forall x \in B : \langle x, o' \rangle_{o'} \geq d(o, o') - (M + \log(2n - 1) + 4\delta).$$

This inequality provides the following upper bound for the p' measure.

$$\begin{aligned} \int_B \left(\frac{d_{o'}}{d_o}\right)^{-\frac{\alpha p'}{2}} d\mu_o &= e^{\frac{\alpha p'}{2}d(o,o')} \int_B e^{-\alpha p' \langle x, o' \rangle_{o'}} d\mu_o \\ &\leq e^{-\frac{\alpha p'}{2}d(o,o')} e^{\alpha p' (M + 4\delta)} (2n - 1)^{\alpha p'} \mu_o(B). \end{aligned}$$

The upper bound of the q -measure is easy to determine.

$$\int_B \left(\frac{d_{o'}}{d_o} (x) \right)^{\frac{\alpha q}{2}} d\mu_o \leq e^{\frac{\alpha q}{2} d(o, o')} \mu_o(B).$$

Both bounds together imply,

$$\left(\int_B \left(\frac{d_{o'}}{d_o} \right)^{\frac{\alpha q}{2}} d\mu_o \right)^{\frac{1}{q}} \left(\int_B \left(\frac{d_{o'}}{d_o} \right)^{-\frac{\alpha p'}{2}} d\mu_o \right)^{\frac{1}{p'}} \leq e^{\alpha(M+4\delta)} (2n-1)^\alpha \mu_o(B)^{\frac{1}{q} + \frac{1}{p'}},$$

which completely proves case 2.

Case 3: As above, let S_+ be the set of all $s \in \mathbb{R}^+$, such that $B \subset \mathcal{O}_o'(s)$. Let s_+^* be the infimum of S_+ .

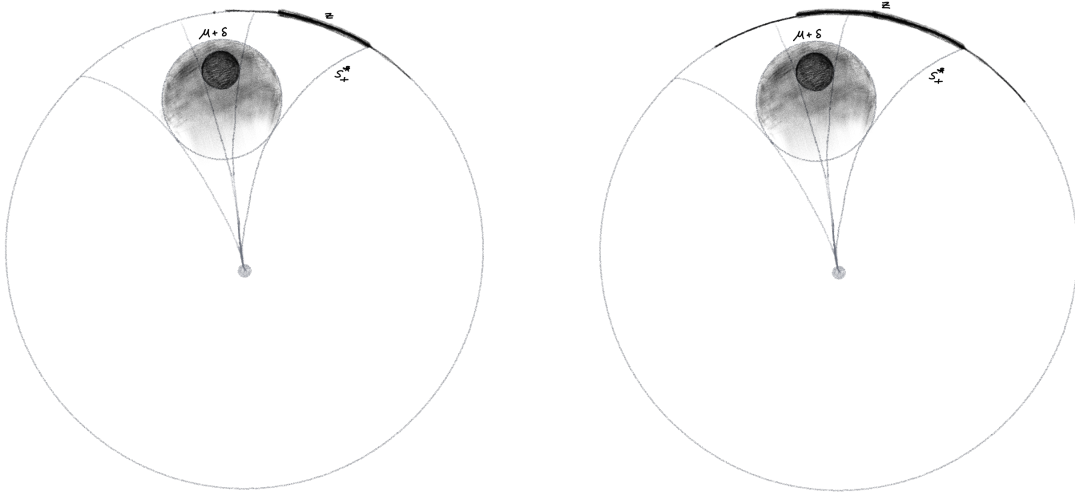


Figure 3.3 – Geometric situation in case 3A

Again, we distinguish two subcases:

1. Case 3A: $(1 - \frac{1}{n})r \leq |z| \leq (1 + \frac{1}{n})r$ and $s_+^* \geq M + 2\delta$,
2. Case 3B: $(1 - \frac{1}{n})r \leq |z| \leq (1 + \frac{1}{n})r$ and $s_+^* < M + 2\delta$.

In case 3A we can apply the shadow lemma to

$$s_+ := d(o, o') - \delta + \log(|z| + r),$$

to conclude that $s_+ \in S_+$. Thus,

$$\forall x \in B : \langle x, o' \rangle_o \geq d(o, o') - s_+ = -\log(|z| + r) + \delta \geq -\log\left(\frac{2n+1}{n}r\right) + \delta,$$

where in the last inequality we used the assumption $|z| \leq (1 + \frac{1}{n})r$.

We use this inequality to obtain the upper bound for the p' measure.

$$\begin{aligned} \int_B \left(\frac{d_{o'}}{d_o} (x) \right)^{-\frac{\alpha p'}{2}} d\mu_o &= e^{\frac{\alpha p'}{2} d(o, o')} \int_B e^{-\alpha p' \langle x, o' \rangle_o} d\mu_o \\ &\leq e^{\frac{\alpha p'}{2} d(o, o')} \left(\frac{2n+1}{n} \right)^{\alpha p'} e^{-\alpha p' \delta} r^{\alpha p'} \mu_o(B). \end{aligned}$$

For the upper bound of $\int_B \left(\frac{d_{o'}}{d_o}\right)^{\frac{\alpha q}{2}} d\mu_o$, we constrain above again by the shadow $\mathcal{O}_o'(s_+)$.

$$\int_B \left(\frac{d_{o'}}{d_o}\right)^{\frac{\alpha q}{2}} d\mu_o \leq \int_{\mathcal{O}_o'(s_+)} \left(\frac{d_{o'}}{d_o}\right)^{\frac{\alpha q}{2}} d\mu_o.$$

The right-hand side is bounded as in case 2A. Thus,

$$\int_B \left(\frac{d_{o'}}{d_o}\right)^{\frac{\alpha q}{2}} d\mu_o \leq C e^{-\frac{\alpha q}{2}d(o,o')} e^{\alpha q \delta} \left(\frac{2Q - \alpha q}{Q - \alpha q}\right) (|z| + r)^{Q - \alpha q},$$

and using the assumption $|z| \leq (1 + \frac{1}{n})r$,

$$\int_B \left(\frac{d_{o'}}{d_o}\right)^{\frac{\alpha q}{2}} d\mu_o \leq C e^{-\frac{\alpha q}{2}d(o,o')} e^{\alpha q \delta} \left(\frac{2Q - \alpha q}{Q - \alpha q}\right) \left(\frac{2n + 1}{n}\right)^{Q - \alpha q} r^{Q - \alpha q}.$$

Both bounds together imply,

$$\begin{aligned} & \left(\int_B \left(\frac{d_{o'}}{d_o}\right)^{\frac{\alpha q}{2}} d\mu_o\right)^{\frac{1}{q}} \left(\int_B \left(\frac{d_{o'}}{d_o}\right)^{-\frac{\alpha p'}{2}} d\mu_o\right)^{\frac{1}{p'}} \\ & \leq C^{2(\frac{1}{q} + \frac{1}{p'})} \left(\frac{2Q - \alpha q}{Q - \alpha q}\right)^{\frac{1}{q}} \left(\frac{2n + 1}{n}\right)^{\frac{Q}{q}} \mu_o(B)^{\frac{1}{q} + \frac{1}{p'}}, \end{aligned}$$

which proves the case 3A.

Case 3B: In this case $B \subset \mathcal{O}_o'(M + 2\delta)$. It follows that,

$$\forall x \in B, \quad \langle x, o' \rangle_{o'} \geq d(o, o') - (M + 2\delta).$$

This inequality yields the following upper bound for the p' measure.

$$\begin{aligned} \int_B \left(\frac{d_{o'}}{d_o}\right)^{-\frac{\alpha p'}{2}} d\mu_o &= e^{\frac{\alpha p'}{2}d(o,o')} \int_{2B} e^{-\alpha p' \langle x, o' \rangle_{o'}} d\mu_o \\ &\leq e^{-\frac{\alpha p'}{2}d(o,o')} e^{\alpha p'(M+2\delta)} \mu_o(B). \end{aligned}$$

The corresponding upper bound for the q -measure is simple.

$$\int_B \left(\frac{d_{o'}}{d_o}\right)^{\frac{\alpha q}{2}} d\mu_o \leq e^{\frac{\alpha q}{2}d(o,o')} \mu_o(B).$$

Both bounds together imply,

$$\left(\int_B \left(\frac{d_{o'}}{d_o}\right)^{\frac{\alpha q}{2}} d\mu_o\right)^{\frac{1}{q}} \left(\int_B \left(\frac{d_{o'}}{d_o}\right)^{-\frac{\alpha p'}{2}} d\mu_o\right)^{\frac{1}{p'}} \leq e^{\alpha(M+2\delta)} \mu_o(B)^{\frac{1}{q} + \frac{1}{p'}}.$$

This completely proves case 3 and ends the proof. ■

Theorem 7 is formulated with the compact picture in mind. However, the theorem can easily be extended to treat the compact and the noncompact picture.

Theorem 8 *There exists a constant $C > 0$, s.t. for all $o, o' \in X \cup \partial X$, and $1 < p \leq q < \infty$, $\alpha < \frac{Q}{q}$,*

$$\left\| \left(\frac{d_{o'}}{d_o}\right)^{\frac{\alpha}{2}} \right\|_{p,q} = \sup_B \left(\frac{1}{\mu_o(B)} \int_B \left(\frac{d_{o'}}{d_o}\right)^{\frac{\alpha q}{2}} d\mu_o \right)^{\frac{1}{q}} \left(\frac{1}{\mu_o(B)} \int_B \left(\frac{d_{o'}}{d_o}\right)^{-\frac{\alpha p'}{2}} d\mu_o \right)^{\frac{1}{p'}} \leq C < \infty,$$

where the supremum is taken over balls in $\partial X \setminus \{o\}$.

Proof: Let $o, o' \in X$. By multiplying both integrands by $e^{\frac{\alpha}{2}d(o,o')}$ and $e^{-\frac{\alpha}{2}d(o,o')}$, respectively,

$$\left(\frac{1}{\mu_o(B)} \int_B \left(\frac{d_{o'}}{d_o} \right)^{\frac{\alpha q}{2}} d\mu_o \right)^{\frac{1}{q}} \left(\frac{1}{\mu_o(B)} \int_B \left(\frac{d_{o'}}{d_o} \right)^{-\frac{\alpha p'}{2}} d\mu_o \right)^{\frac{1}{p'}}$$

is equal to

$$\left(\frac{1}{\mu_o(B)} \int_B e^{\alpha q \langle x, o' \rangle_o} d\mu_o(x) \right)^{\frac{1}{q}} \left(\frac{1}{\mu_o(B)} \int_B e^{-\alpha p' \langle x, o' \rangle_o} d\mu_o(x) \right)^{\frac{1}{p'}}.$$

We are tempted to take o' to the boundary. Since X is strongly hyperbolic, the integrand converges pointwise as $o' \rightarrow z \in \partial X$. Careful treatment requires the case where $x \in \mathcal{O}_o^{o'}(M+2\delta)$. For the sake of readability, we set $\mathcal{O}_o^{o'} = \mathcal{O}_o^{o'}(M+2\delta)$. Each of these integrals decomposes, for the first as

$$\int_B e^{\alpha q \langle x, o' \rangle_o} d\mu_o = \int_B \chi_{B \setminus \mathcal{O}_o^{o'}}(x) e^{\alpha q \langle x, o' \rangle_o} d\mu_o(x) + \int_{B \cap \mathcal{O}_o^{o'}} e^{\alpha q \langle x, o' \rangle_o} d\mu_o(x).$$

The function $\chi_{B \setminus \mathcal{O}_o^{o'}}(x) e^{\alpha q \langle x, o' \rangle_o}$ is dominated by the integrable function $e^{2\delta \alpha q} d_o(x, z)^{-\alpha q}$ and converges pointwise on $B \setminus \{z\}$ to $e^{\alpha q \langle x, z \rangle_o}$. The second integral is bounded by

$$e^{\alpha q d(o, o')} \mu_o(\mathcal{O}_o^{o'}) \leq e^{Q(M+3\delta)} e^{(\alpha q - Q)d(o, o')}.$$

Thus, the integral $\int_{B \cap \mathcal{O}_o^{o'}} e^{\alpha q \langle x, o' \rangle_o} d\mu_o(x)$ vanishes as $o' \rightarrow z \in \partial X$. By dominant convergence

$$\lim_{o' \rightarrow z} \int_B e^{\alpha q \langle x, o' \rangle_o} d\mu_o = \int_B e^{\alpha q \langle x, z \rangle_o} d\mu_o(x) = \int_B \left(\frac{d_{z,o}}{d_o} \right)^{\frac{\alpha q}{2}} d\mu_o.$$

An equivalent argument holds for the p' -part. This implies,

$$\left\| \left(\frac{d_{z,o}}{d_o} \right)^{\frac{\alpha}{2}} \right\|_{p,q} = \sup_B \left(\frac{1}{\mu_o(B)} \int_B \left(\frac{d_{z,o}}{d_o} \right)^{\frac{\alpha q}{2}} d\mu_o \right)^{\frac{1}{q}} \left(\frac{1}{\mu_o(B)} \int_B \left(\frac{d_{z,o}}{d_o} \right)^{-\frac{\alpha p'}{2}} d\mu_o \right)^{\frac{1}{p'}} \leq C.$$

Since $d_{z,o'} = e^{\beta z(o, o')} d_{z,o}$, the expression does not depend on the choice of o in $d_{z,o}$.

The transition from compact to noncompact picture, works in a similar way. Let $z \in \partial X$, and let B be a ball which does not contain z as a limitpoint. Let p be a sequence of points that is $p \rightarrow z$. We assume that all points are close enough to z , i.e., $B \subset \partial X \setminus \mathcal{O}_o^p(M+2\delta)$ for all points p . Further, suppose that all endpoints \bar{p} of rays starting from o and extending at uniform distance from p are ϵ close to z . Then,

$$e^{\langle x, p \rangle_o}$$

is dominated by

$$\frac{e^{2\delta}}{e^{-\langle x, z \rangle_o} - \epsilon},$$

which is integrable on B . By multiplying both integrands by $e^{\frac{\alpha}{2}d(o,p)}$ and $e^{-\frac{\alpha}{2}d(o,p)}$, respectively,

$$\left(\frac{1}{\mu_p(B)} \int_B \left(\frac{d_o}{d_p} \right)^{\frac{\alpha q}{2}} d\mu_p \right)^{\frac{1}{q}} \left(\frac{1}{\mu_p(B)} \int_B \left(\frac{d_o}{d_p} \right)^{-\frac{\alpha p'}{2}} d\mu_p \right)^{\frac{1}{p'}}$$

is equal to

$$\left(\frac{1}{\mu_p(B)} \int_B e^{-\alpha q \langle x, p \rangle_o} d\mu_p(x) \right)^{\frac{1}{q}} \left(\frac{1}{\mu_p(B)} \int_B e^{\alpha p' \langle x, p \rangle_o} d\mu_p(x) \right)^{\frac{1}{p'}}.$$

We can express for example the first integral with respect to o as

$$\frac{1}{e^{Qd(o,p)} \mu_p(B)} \int_B e^{(Q-\alpha q) \langle x, p \rangle_o} d\mu_o(x).$$

The integrand is dominated, as we discussed above. By dominate convergence

$$\begin{aligned} \lim_{p \rightarrow z} \frac{1}{e^{Qd(o,p)} \mu_p(B)} \int_B e^{(Q-\alpha q) \langle x, p \rangle_o} d\mu_o(x) &= \frac{1}{\mu_{z,o}(B)} \int_B e^{(Q-\alpha q) \langle x, z \rangle_o} d\mu_o(x) \\ &= \frac{1}{\mu_{z,o}(B)} \int_B \left(\frac{d_o}{d_{z,o}} \right)^{\frac{\alpha q}{2}} d\mu_{z,o} \end{aligned}$$

An equivalent argument holds for the p' integral. ■

3.4.5 Some classical theorems revisited.

The theory developed above allows us to revisit some classical theorems of harmonic analysis [81],[79],[82] and present them in a Möbius invariant way on the boundary of a strongly hyperbolic space. All statements make use of lemmas of Calderon-Zygmund type on spaces of homogeneous type. However, in order to maintain uniformity, attention must be paid to the constants. One may for example study uniform regularity properties of the Hardy-Littlewood maximal function

$$M_o(f)(x) = \sup_{B \ni x} \frac{1}{\mu_o(B)} \int_B |f(y)| d\mu_o,$$

or of the singular integral

$$T_o(f)(x) = \int_Z \frac{f(y)}{d_o(x,y)^Q} d\mu_o(y)$$

In this paper we will study uniform regularity properties of the fractional integration operator

$$I_o^\alpha(f)(x) = \int_Z \frac{f(y)}{d_o(x,y)^{Q-\alpha}} d\mu_o(y).$$

3.4.6 Calderon-Zygmund type lemmas on Möbius spaces.

In this section we will revisit some classical lemmas of the Calderon-Zygmund type. The goal is to highlight the Möbius invariance of the constants involved.

In a Möbius space (Z, \mathcal{M}) , a ball in (Z, \mathcal{M}) is a metric ball for some $d \in \mathcal{M}$. A metric ball for us is always of finite radius, we speak of infinite balls if we mean otherwise. A family of balls in (Z, \mathcal{M}) is a family of metric balls for some $d \in \mathcal{M}$.

In this section (Z, \mathcal{M}) is assumed to be uniformly Ahlfors-David regular. We denote the Ahlfors-David constant by C .

Lemma 5 *Let $\{B_\iota\}_{\iota \in \mathcal{I}}$ be a family of balls in (Z, \mathcal{M}) contained in a fixed ball B . Then there is a countable subset $\{B_i\}_{i \in I}$, $I \subset \mathcal{I}$, such that*

1. $B_i \cap B_j = \emptyset$ if $i \neq j$,
2. Every B_ι is contained in some $5B_i$,
3. For $d \in \mathcal{M}$ such that $\{B_\iota\}_{\iota \in \mathcal{I}}$ is a family of balls in (Z, d) , $\mu_d(\cup_{\iota \in \mathcal{I}} B_\iota) \leq C_Q \sum_{i \in I} \mu_d(B_i)$.

and where $C_Q = 5^Q C^2$ is a constant depending only on the Ahlfors-David constant C and the dimension Q of \mathcal{M} .

Proof: We use Vitali's selection procedure. Since $\{B_\iota\}$ is a family of balls in (Z, \mathcal{M}) , by assumption there exists $d \in \mathcal{M}$ s.t. $\{B_\iota\}$ and B is a family of balls in (Z, d) .

Choose B_1 so that the radius $r(B_1)$ (w.r.t d) of B_1 is at least $\frac{1}{2} \sup\{r(B_\iota) : \iota \in \mathcal{I}\}$. If B_1, \dots, B_k have been chosen, choose B_{k+1} disjoint from B_1, \dots, B_k such that $r(B_{k+1}) \geq \frac{1}{2} \sup\{r(B_\iota) : B_\iota \text{ is disjoint from } B_1, \dots, B_k\}$.

Let $\{B_i\}_{i \in I}$ be the obtained collection of disjoint d -balls. We will show that for any B_ι there exists a B_i s.t. $B_\iota \cap B_i \neq \emptyset$ and the radius of B_i is at least half of the radius of B_ι .

Ahlfors-David regularity implies that $\lim_{i \rightarrow \infty} r(B_i) = 0$ since

$$\sum_{i \in I} \frac{r(B_i)^Q}{C} \leq \sum_{i \in I} \mu_d(B_i) \leq \mu_d(B).$$

Take the first k with $r(B_{k+1}) < \frac{1}{2} r(B_\iota)$. By construction B_ι must intersect one of the balls B_1, \dots, B_k , say B_i . Since $i \leq k$, $r(B_i) \geq \frac{1}{2} r(B_\iota)$.

If $B_i = B(x_i, r_i)$, then $B_\iota \subset B(x_i, 5r_i)$ and property 2 holds. Furthermore,

$$\begin{aligned} \mu_d(\cup_{\iota \in \mathcal{I}} B_\iota) &\leq \mu_d(\cup_{i \in I} B(x_i, 5r_i)) \\ &\leq \sum_{i \in I} \mu_d(B(x_i, 5r_i)) \\ &\leq 5^Q C^2 \sum_{i \in I} \mu_d(B_i), \end{aligned}$$

which gives property 3 and proves the lemma. ■

The classical theory on the bounds of fractional integration in Euclidean space is based on the decomposition of Calderon and Zygmund by dyadic cubes. We adapt this idea to our setting as follows.

For each k in \mathbb{Z} , let $\{\hat{B}_j^k\}_j$ be a sequence of d -balls of radius 3^{k-1} , maximal with respect to the property that $\hat{B}_i^k \cap \hat{B}_j^k = \emptyset$ for $i \neq j$. Set $B_j^k = 3\hat{B}_j^k$, we call $\{B_j^k\}_j$ *dyadic d -balls of order k* in Z . We call $\{B_j^k\}_j$ *dyadic balls of order k* in (Z, \mathcal{M}) if $\{B_j^k\}_j$ are dyadic d -balls of order k for some $d \in \mathcal{M}$. A *dyadic d -ball* in Z is a dyadic

d -ball of some order $k \in \mathbb{Z}$. A *family of dyadic balls* in (Z, \mathcal{M}) is a family of dyadic d -balls for some $d \in \mathcal{M}$.

Lemma 6 *If $\{B_j^k\}_j$ is a family of dyadic balls of order k in (Z, \mathcal{M}) , then*

1. $\frac{1}{3}B_i^k \cap \frac{1}{3}B_j^k = \emptyset$ for $i \neq j$.
2. $\sum_j \chi_{B_j^k} \leq M$, where M is a constant that depends only on the Ahlfors-David constant C and dimension Q of \mathcal{M} (and not on k).
3. Let $d \in \mathcal{M}$ be such that $\{B_j^k\}_j$ are dyadic d -balls, then every d -ball of radius 3^{k-1} is contained in at least one of the d -balls B_j^k .

Proof: Let $d \in \mathcal{M}$ be such that $\{B_j^k\}_j$ is a family of dyadic d -balls. Suppose $x \in B_j^k$, $1 \leq k \leq N$, then $B_j^k \subset B(x, 2 \times 3^k)$ for $1 \leq j \leq N$. According to the Ahlfors-David regularity $\mu_d(B(x, 2 \times 3^k)) \leq C(2 \times 3^k)^Q \leq 6^Q C^2 \mu_d(\hat{B}_j^k)$ and therefore $N \mu_d(B(x, 2 \times 3^k)) \leq 6^Q C^2 \sum_{j=1}^N \mu_d(\hat{B}_j^k) = 6^Q C^2 \mu_d(\cup_{j=1}^N \hat{B}_j^k) \leq 6^Q C^2 \mu_d(B(x, 2 \times 3^k))$. So $N \leq 6^Q C^2$ and we can set $M = 6^Q C^2$.

For the third property, we choose $x \in Z$, the d -ball $B(x, 3^{k-1})$ intersects at least one of the \hat{B}_j^k by the maximality property of $\{\hat{B}_j^k\}_j$. But then $B(x, 3^{k-1})$ is contained in B_j^k since for each $w \in B(x, 3^{k-1})$ and $z \in B(x, 3^{k-1}) \cap \hat{B}_j^k$,

$$\begin{aligned} d(x_j^k, w) &\leq d(x_j^k, z) + d(z, x) + d(x, w) \\ &< 3^{k-1} + 3^{k-1} + 3^{k-1} \\ &= 3^k. \end{aligned}$$

■

Lemma 7 *Let $\{B_l\}_{l \in \mathcal{I}}$ be a family of dyadic balls in (Z, \mathcal{M}) . If $\{B_j\}_{j \in \mathcal{J}}$ is a collection of maximal balls with respect to the inclusion in $\{B_l\}_{l \in \mathcal{I}}$, then the balls $\{\frac{1}{3}B_j\}_{j \in \mathcal{J}}$ are pairwise disjoint.*

Proof: Note that due to the construction $\frac{1}{3}B_i^k \cap \frac{1}{3}B_j^k = \emptyset$ for dyadic balls of the same degree when $i \neq j$. Suppose $z \in \frac{1}{3}B_i^k \cap \frac{1}{3}B_j^l$ with $l < k$. Then $B_j^l \subset B_i^k$, because if $y \in B_j^l$,

$$\begin{aligned} d(x_i^k, y) &\leq d(x_i^k, z) + d(z, x_j^l) + d(x_j^l, y) \\ &< 3^{k-1} + 3^{l-1} + 3^l \\ &\leq 3^k. \end{aligned}$$

From this and from the maximality of $\{B_j\}_{j \in \mathcal{J}}$ follows the lemma. ■

3.4.7 Fractional integration on the boundary of strongly hyperbolic spaces.

Theorem 9 *Let X be a strongly hyperbolic space, $\{d_o\}_{o \in X}$ be the natural Möbius structure on ∂X , and $0 < \alpha < \frac{Q}{q}$. Then there is a constant $C_\alpha > 0$ that depends only on α , such that for all $d_o, d_{o'} \in \mathcal{M}$ and $f \in L^p(\mu_o)$,*

$$\|(\frac{d_{o'}}{d_o})^{\frac{\alpha}{2}} \circ I_o^\alpha \circ (\frac{d_{o'}}{d_o})^{-\frac{\alpha}{2}} f\|_{L^q(\mu_o)} \leq C_\alpha \|f\|_{L^p(\mu_o)}, \quad (3.42)$$

where $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{Q}$.

Proof: (Of Theorem 9) We may decompose f into its positive and negative part, writing $f = f_+ - f_-$. Clearly $\|f_{+/-}\|_{L^p(\mu_o)} \leq \|f\|_{L^p(\mu_o)}$. Since $\left(\frac{d_{o'}}{d_o}\right)^{\frac{\alpha}{2}} \circ I_o^\alpha \circ \left(\frac{d_{o'}}{d_o}\right)^{-\frac{\alpha}{2}}$ preserves positivity, applying Minkowski's inequality to $\left\| \left(\frac{d_{o'}}{d_o}\right)^{\frac{\alpha}{2}} \circ I_o^\alpha \circ \left(\frac{d_{o'}}{d_o}\right)^{-\frac{\alpha}{2}} f_+ - \left(\frac{d_{o'}}{d_o}\right)^{\frac{\alpha}{2}} \circ I_o^\alpha \circ \left(\frac{d_{o'}}{d_o}\right)^{-\frac{\alpha}{2}} f_- \right\|_{L^q(\mu_o)}$ reveals that we may restrict to the case of f being positive.

Showing

$$\left\| \left(\frac{d_{o'}}{d_o}\right)^{\frac{\alpha}{2}} \circ I_o^\alpha \circ \left(\frac{d_{o'}}{d_o}\right)^{-\frac{\alpha}{2}} f \right\|_{L^q(\mu_o)} \leq C_\alpha \|f\|_{L^p(\mu_o)},$$

is equivalent to

$$\left(\int_{\partial X} [I_o^\alpha(f)(x)]^q \left(\frac{d_{o'}}{d_o}\right)^{\frac{\alpha q}{2}} d\mu_o(x) \right)^{\frac{1}{q}} \leq C_\alpha \left(\int_{\partial X} f(x)^p \left(\frac{d_{o'}}{d_o}\right)^{\frac{\alpha p}{2}} d\mu_o(x) \right)^{\frac{1}{p}}.$$

This is further equivalent to

$$\left(\int_{\partial X} [I_o^\alpha \circ \left(\frac{d_{o'}}{d_o}\right)^{-\frac{\alpha p'}{2}}(f)(x)]^q \left(\frac{d_{o'}}{d_o}\right)^{\frac{\alpha q}{2}} d\mu_o(x) \right)^{\frac{1}{q}} \leq C_\alpha \left(\int_{\partial X} f(x)^p \left(\frac{d_{o'}}{d_o}\right)^{\frac{\alpha(1-p')p}{2}} d\mu_o(x) \right)^{\frac{1}{p}}.$$

By duality of $L^q\left(\left(\frac{d_{o'}}{d_o}\right)^{\frac{\alpha q}{2}} d\mu_o\right)$ with $L^{q'}\left(\left(\frac{d_{o'}}{d_o}\right)^{\frac{\alpha q}{2}} d\mu_o\right)$, by the functional $\iota(h) : g \mapsto \int hg \left(\frac{d_{o'}}{d_o}\right)^{\frac{\alpha q}{2}} d\mu_o$, we need to show that

$$\begin{aligned} & \int_{\partial X} I_o^\alpha \circ \left(\frac{d_{o'}}{d_o}\right)^{-\frac{\alpha p'}{2}}(f)(x) g(x) \left(\frac{d_{o'}}{d_o}\right)^{\frac{\alpha q}{2}} d\mu_o(x) \\ & \leq C_\alpha \left(\int_{\partial X} f(x)^p \left(\frac{d_{o'}}{d_o}\right)^{-\frac{\alpha p'}{2}} d\mu_o(x) \right)^{\frac{1}{p}} \left(\int_{\partial X} g(x)^{q'} \left(\frac{d_{o'}}{d_o}\right)^{\frac{\alpha q}{2}} d\mu_o(x) \right)^{\frac{1}{q'}}. \end{aligned}$$

for all $f, g \geq 0$ bounded.

We bound the operator I_o^α by its dyadic approximation

$$I_o^{\alpha, \text{dy.}} f(x) := \sum_{B: x \in B \text{ dyadic}} \mu_o(B)^{\frac{\alpha}{Q}-1} \int_B f d\mu_o.$$

In fact,

$$I_o^\alpha f(x) = \int_{\partial X} \frac{f(y)}{d_o(x, y)^{Q-\alpha}} d\mu_o(y) \leq C_\alpha \sum_{B: x \in B \text{ dyadic}} \mu_o(B)^{\frac{\alpha}{Q}-1} \int_B f d\mu_o, \quad (3.43)$$

where $C_\alpha = 9^{(Q-\alpha)} C^{1-\frac{\alpha}{Q}}$ with C the Ahlfors-David constant and Q the dimension of \mathcal{M} . This follows from the fact that if $3^{k-1} \leq d_o(x, y) \leq 3^k$, then both x and y are in $B(x, 3^k)$, which by Lemma 6.3 is contained in some B_j^{k+1} . So $\mu_o(B_j^{k+1})^{1-\frac{\alpha}{Q}} \leq C^{1-\frac{\alpha}{Q}} 3^{(Q-\alpha)(k+1)} \leq 9^{(Q-\alpha)} C^{1-\frac{\alpha}{Q}} d_o(x, y)^{Q-\alpha}$ and thus

$$d_o(x, y)^{\alpha-Q} \leq C_\alpha \mu_o(B_j^{k+1})^{\frac{\alpha}{Q}-1} \chi_{B_j^{k+1}}(x) \chi_{B_j^{k+1}}(y) \leq C_\alpha \sum_{B \text{ dyadic}} \mu_o(B)^{\frac{\alpha}{Q}-1} \chi_B(x) \chi_B(y).$$

The bound (3.43) follows by multiplying both sides by $f(y)$ and integrating with respect to y . Again multiplying both sides by $g(x)\left(\frac{d_{o'}}{d_o}(x)\right)^{\frac{\alpha q}{2}}$ and integrating with respect to x yields

$$\begin{aligned} & \int_{\partial X} I_o^\alpha \circ \left(\frac{d_{o'}}{d_o}\right)^{-\frac{\alpha p'}{2}} (f)(x) g(x) \left(\frac{d_{o'}}{d_o}\right)^{\frac{\alpha q}{2}} d\mu_o \\ & \leq C_\alpha \sum_{B \text{ dyadic}} \mu_o(B)^{\frac{\alpha}{Q}-1} \left(\int_B f(x) \left(\frac{d_{o'}}{d_o}\right)^{-\frac{\alpha p'}{2}} d\mu_o \right) \left(\int_B g(x) \left(\frac{d_{o'}}{d_o}\right)^{\frac{\alpha q}{2}} d\mu_o \right). \end{aligned} \quad (3.44)$$

By Theorem 7 we have

$$\left(\int_B \left(\frac{d_{o'}}{d_o}\right)^{\frac{\alpha q}{2}} d\mu_o(x) \right)^{\frac{1}{q}} \left(\int_B \left(\frac{d_{o'}}{d_o}\right)^{-\frac{\alpha p'}{2}} d\mu_o(x) \right)^{\frac{1}{p'}} \leq C_\alpha \mu_o(B)^{1-\frac{\alpha}{Q}}.$$

Define the measures

$$M_q(B) := \int_B \left(\frac{d_{o'}}{d_o}\right)^{\frac{\alpha q}{2}} d\mu_o,$$

and

$$M_{p'}(B) := \int_B \left(\frac{d_{o'}}{d_o}\right)^{-\frac{\alpha p'}{2}} d\mu_o.$$

The sum on the right side of inequality (3.44), can be further bounded as follows.

$$\begin{aligned} & \sum_{B \text{ dyadic}} \mu_o(B)^{\frac{\alpha}{Q}-1} \left(\int_B f \left(\frac{d_{o'}}{d_o}\right)^{-\frac{\alpha p'}{2}} d\mu_o \right) \left(\int_B g \left(\frac{d_{o'}}{d_o}\right)^{\frac{\alpha q}{2}} d\mu_o \right) \\ & \leq C_\alpha \sum_{B \text{ dyadic}} M_{p'}(B)^{-\frac{1}{p'}} M_q(B)^{-\frac{1}{q}} \left(\int_B f \left(\frac{d_{o'}}{d_o}\right)^{-\frac{\alpha p'}{2}} d\mu_o \right) \left(\int_B g \left(\frac{d_{o'}}{d_o}\right)^{\frac{\alpha q}{2}} d\mu_o \right) \\ & = C_\alpha \sum_{B \text{ dyadic}} M_{p'}(B)^{\frac{1}{p}} \left(\frac{1}{M_{p'}(B)} \int_B f \left(\frac{d_{o'}}{d_o}\right)^{-\frac{\alpha p'}{2}} d\mu_o \right) \\ & \quad M_q(B)^{\frac{1}{q'}} \left(\frac{1}{M_q(B)} \int_B g \left(\frac{d_{o'}}{d_o}\right)^{\frac{\alpha q}{2}} d\mu_o \right) \\ & \leq C_\alpha \left(\sum_{B \text{ dyadic}} M_{p'}(B)^{\frac{r}{p}} \left(\frac{1}{M_{p'}(B)} \int_B f dM_{p'} \right)^r \right)^{\frac{1}{r}} \\ & \quad \left(\sum_{B \text{ dyadic}} M_q(B)^{\frac{r'}{q'}} \left(\frac{1}{M_q(B)} \int_B g dM_q \right)^{r'} \right)^{\frac{1}{r'}} \end{aligned}$$

where for the last inequality we have used Hölder's inequality with $p < r < q$ (recall that $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{Q}$, $\alpha > 0$). It remains only to show that

$$\left(\sum_{B \text{ dyadic}} M_{p'}(B)^{\frac{r}{p}} \left(\frac{1}{M_{p'}(B)} \int_B f dM_{p'} \right)^r \right)^{\frac{1}{r}} \leq C_p \left(\int_{\partial X} f^p dM_{p'} \right)^{\frac{1}{p}}$$

and

$$\left(\sum_{B \text{ dyadic}} M_q(B)^{\frac{r'}{q'}} \left(\frac{1}{M_q(B)} \int_B g dM_q \right)^{r'} \right)^{\frac{1}{r'}} \leq C_{q'} \left(\int_{\partial X} g^{q'} dM_q \right)^{\frac{1}{q'}}$$

for some C_p and $C_{q'}$ depending only on p , q' and the Ahlfors-David constant C .

This follows from Lemma 8 and Lemma 9 below with $\beta = r/p > 1$ for $M_{p'}$ and $\beta = r'/q' > 1$ for M_q . \blacksquare

Lemma 8 *Let $\beta > 1$ and Γ a collection of dyadic balls in ∂X , then for some $K \geq 1$, that depends only on the Möbius structure \mathcal{M} and β ,*

$$\sum_{B \in \Gamma, B \subset B'} M_{p'}(B)^\beta \leq K M_{p'}(B')^\beta, \quad \forall B' \in \Gamma. \quad (3.45)$$

Proof: Indeed, let the radius $r(B')$ of B' be equal to 3^{k-1} for some $k \in \mathbb{Z}$. Then by Lemma 6.3, B' is contained in some B_j^k . Analogously, every B is contained in some $B_i^l \subset B_j^k$. Thus,

$$\sum_{B \in \Gamma, B \subset B'} M_{p'}(B)^\beta \leq \sum_{l=k}^{-\infty} \sum_{i: B_i^l \subset B_j^k} M_{p'}(B_i^l)^{\beta-1} M_{p'}(B_i^l). \quad (3.46)$$

By Theorem 7, powers of conformal factors are in some A_p , therefore they are in A_∞ , see (3.39). By uniform Ahlfors-David regularity, there exists a $\delta \in (0, 1)$ and a $\gamma \in (0, 1)$, such that for every pair $B_i^l, B_{i'}^l$, $\mu_o(\delta B_i^l) \leq \gamma \mu_o(B_{i'}^l)$. If we assume that $\delta B_i^l \subset B_{i'}^l$, then by the A_∞ -condition (3.40), there exists $\epsilon < 1$, such that $M_{p'}(\delta B_i^l) \leq \epsilon M_{p'}(B_{i'}^l)$. Finally, pick the maximal $n \in \mathbb{Z}$, such that $3^n \leq \delta$, then also $M_{p'}(3^n B_i^l) \leq \epsilon M_{p'}(B_{i'}^l)$

Notice that n and ϵ , only depend on the uniform Ahlfors-David-constant C , that is an invariant of \mathcal{M} .

Suppose that $l - k \leq n$, we write B_i^k for the dyadic ball of radius 3^k and origin as B_i^l . Then

$$\begin{aligned} M_{p'}(B_i^l) &= M_{p'}(3^{l-k} B_i^k) \\ &= M_{p'}(3^{n \lfloor \frac{l-k}{n} \rfloor} B_i^k) \\ &\leq M_{p'}(3^{n \lfloor \frac{l-k}{n} \rfloor} B_i^k) \\ &\leq \epsilon^{\lfloor \frac{l-k}{n} \rfloor} M_{p'}(B_i^k). \end{aligned} \quad (3.47)$$

Notice that $\lfloor \frac{l-k}{n} \rfloor$ is a positive integer that tends to infinity when l tends to negative infinity. Since $B_i^k \subset B_j^{k+1}$, we get

$$M_{p'}(B_i^l) \leq \epsilon^{\lfloor \frac{l-k}{n} \rfloor} M_{p'}(B_j^{k+1}).$$

Now

$$\begin{aligned}
\sum_{l=k}^{-\infty} \sum_{i: B_i^l \subset B_j^k} M_{p'}(B_i^l)^{\beta-1} M_{p'}(B_i^l) &\leq \sum_{l=k}^{k+n} \sum_{i: B_i^l \subset B_j^k} M_{p'}(B_i^l)^{\beta-1} M_{p'}(B_i^l) \\
&\quad + \sum_{l < k+n}^{-\infty} \sum_{i: B_i^l \subset B_j^k} M_{p'}(B_i^l)^{\beta-1} M_{p'}(B_i^l) \\
&\leq \sum_{l=k}^{k+n} M_{p'}(B_j^{k+1})^{\beta-1} \sum_{i: B_i^l \subset B_j^k} M_{p'}(B_i^l) \\
&\quad + \sum_{l < k+n}^{-\infty} \epsilon^{(\beta-1)\lfloor \frac{l-k}{n} \rfloor} M_{p'}(B_j^{k+1})^{\beta-1} \sum_{i: B_i^l \subset B_j^k} M_{p'}(B_i^l) \\
&\leq n M M_{p'}(B_j^{k+1})^\beta + \left(\sum_{l < k+n}^{-\infty} \epsilon^{(\beta-1)\lfloor \frac{l-k}{n} \rfloor} \right) M M_{p'}(B_j^{k+1})^\beta,
\end{aligned}$$

where in the last inequality we have used Lemma 6.2. Lastly, using the doubling property of $M_{p'}$ we replace B_j^{k+1} by B' . Since $\beta > 1$ and $\epsilon < 1$, we obtain the inequality (3.45) with a K depending only on \mathcal{M} and β as required. \blacksquare

Lemma 9 *Let M be a doubling measure on a metric space Z with doubling constant $C_M \geq 1$, $\beta \geq 1$ and Γ a collection of dyadic balls in X s.t.*

$$\sum_{B \in \Gamma, B \subset B'} M(B)^\beta \leq K M(B')^\beta, \quad \forall B' \in \Gamma,$$

for some $K \geq 1$. Then

$$\left(\sum_{B \in \Gamma} M(B)^\beta \left(\frac{1}{M(B)} \int_B f dM \right)^{p\beta} \right)^{\frac{1}{p\beta}} \leq C_p \left(\int_Z f^p dM \right)^{\frac{1}{p}},$$

for all $f \geq 0$, $1 < p < \infty$ and where C_p depends only on p , β , K and the doubling constant C_M .

Proof: We show that the map $f \mapsto \left(\frac{1}{M(B)} \int_B f dM \right)_{B \in \Gamma}$ takes $L^\infty(Z, dM)$ to $l^\infty(\Gamma, M(B)^\beta)$ and $L^1(Z, dM)$ to $l^{\beta, \infty}(\Gamma, M(B)^\beta)$. The lemma then follows by applying Marcinkiewicz's interpolation theorem [83].

The (∞, ∞) -boundedness is clear with constant 1.

We show that the mapping from $L^1(Z, dM)$ to $l^{\beta, \infty}(\Gamma, M(B)^\beta)$ is bounded. Assume w.l.o.g. that f is bounded with compact support and fix $\lambda > 0$. Let $\{Q_j\}_{j \in J}$ be the maximal dyadic balls B in Γ such that $\frac{1}{M(B)} \int_B |f| dM > \lambda$. Then

$$\begin{aligned}
\sum_{B \in \Gamma: \frac{1}{M(B)} \int_B |f| dM > \lambda} M(B)^\beta &\leq \sum_{j \in J} \sum_{B \subset Q_j} M(B)^\beta \\
&\leq K \sum_{j \in J} M(Q_j)^\beta \\
&\leq K C_M^{2\beta} \left(\sum_{j \in J} M\left(\frac{1}{3}Q_j\right) \right)^\beta
\end{aligned}$$

where in the second inequality we used the summation assumption over Γ and in the last inequality we used the doubling property of M and the fact that $\beta \geq 1$.

By Lemma 5, there exists a pairwise disjoint subcollection $\{Q_i\}_{i \in I}$ of dyadic balls $\{Q_j\}_{j \in J}$ with the property that every Q_j , $j \in J$, is contained in a $5Q_i$, $i \in I$. Note also that the collection $\{\frac{1}{3}Q_j\}$ is pairwise disjoint by lemma 7. Therefore

$$\begin{aligned} \sum_{j \in J} M\left(\frac{1}{3}Q_j\right) &\leq \sum_{i \in I} \sum_{j \in J: Q_j \subset 5Q_i} M\left(\frac{1}{3}Q_j\right) \\ &\leq \sum_{i \in I} M(5Q_i) \\ &\leq C_M^3 \sum_{i \in I} M(Q_i) \\ &\leq \frac{C_M^3}{\lambda} \sum_{i \in I} \int_{Q_i} |f| dM \\ &\leq \frac{C_M^3}{\lambda} \int_X |f| dM. \end{aligned}$$

Combining the above, it can be seen that

$$\sum_{B \in \Gamma: \frac{1}{M(B)} \int_B f dM > \lambda} M(B)^\beta \leq \left(K^{\frac{1}{\beta}} C_M^5 \frac{\int_X |f| dM}{\lambda} \right)^\beta,$$

which says that $f \mapsto \left(\frac{1}{M(B)} \int_B f dM\right)_{B \in \Gamma}$ is of weak $(1, \beta)$ type.

Thus Marcinkiewicz's interpolation theorem says, that for each $\theta \in (0, 1)$, $f \mapsto \left(\frac{1}{M(B)} \int_B f dM\right)_{B \in \Gamma}$ is of type (p_θ, q_θ) with $p_\theta = \frac{1}{\theta}$ and $q_\theta = \frac{\beta}{\theta}$. Take $\theta = \frac{1}{p}$, which gives the result. ■

3.5 A quadratic form on Möbius spaces

3.5.1 Densities on Möbius spaces

Definition 9 Let (Z, \mathcal{M}) be a Möbius space of Hausdorff dimension Q . An s -density on (Z, \mathcal{M}) is a mapping $d \mapsto u_d$ which assigns to each metric d in \mathcal{M} a measurable function $u_d: Z \rightarrow \mathbb{C}$ such that for all metrics $d, d' \in \mathcal{M}$ and all $x \in Z$, $u_d(x) = \left(\frac{d'}{d}(x)\right)^{sQ} u_{d'}(x)$.

For simplicity, we denote s -densities compactly by u or $u_d(d\mu_d)^s$, where μ_d is the Hausdorff Q -dimensional measure on Z . The space of s -densities $\mathcal{D}_s(\mathcal{M})$ on Z , forms a vector space under pointwise addition.

Densities can be multiplied. The pointwise product of an s -density with an s' -density is an $(s + s')$ -density. 1-densities may be identified with signed measures that are absolutely continuous with respect to a Hausdorff measure of Q dimension, see example 6. $\frac{1}{2}$ -densities u such that $\int u^2 < \infty$ have a natural Möbius invariant

inner product, multiply them and integrate the resulting measure. This works only for $s = \frac{1}{2}$. To generate invariant inner products for other values of s , we need more structure.

Example 5 *A measurable function f on Z is a 0-density.*

Example 6 *A signed measure μ on Z that is absolutely continuous with respect to a Hausdorff measure of Q dimension defines a 1-density. In fact, for every metric d' in the class there is a measurable function $u_{d'}$ such that $d\mu = u_{d'} d\mu_{d'}$. Since*

$$\frac{d\mu_{d'}}{d\mu_d} = \left(\frac{d'}{d}\right),$$

$$u_d d\mu_d = d\mu = u_{d'} d\mu_{d'} = u_{d'} \left(\frac{d'}{d}\right)^Q d\mu_d,$$

thus for almost any $x \in Z$,

$$u_d(x) = \left(\frac{d'}{d}(x)\right)^Q u_{d'}(x).$$

Conversely, for every 1-density, $|u_d| d\mu_d$ defines a measure that is absolutely continuous with respect to the Hausdorff Q -dimensional measure μ_d . This motivates to consider s -densities as fractional measures.

Example 7 *The distance $(x, y) \mapsto d(x, y)$ as a 2 point function is a $(-\frac{1}{2Q}, -\frac{1}{2Q})$ bi-density, i.e. a $(-\frac{1}{2Q})$ -density in each component. Indeed, going from one distance d' to another d in the Möbius class,*

$$\frac{d(x, y)}{d'(x, y)} = \left(\frac{d'}{d}(x)\right)^{-1/2} \left(\frac{d'}{d}(y)\right)^{-1/2}.$$

Example 8 *On the ideal boundary of strongly hyperbolic spaces, we consider the s -densities associated with the Möbius structure $\{d_o\}_{o \in X}$ as maps $o \mapsto u_o$. These maps transform under a change of origin as*

$$u_o(x) = e^{sQ\beta_x(o, o')} u_{o'}(x).$$

3.5.2 Densities viewed from an ideal origin

On the boundary of a strongly hyperbolic space X we can extend the definition of densities $o \mapsto u_o$ to the boundary ∂X . Recall that the sequence of measures $\{e^{Qd(o, p)} \mu_p\}$ converges to a measure $\mu_{z, o}$ if $p \rightarrow z \in \partial X$.

Proposition 10 *Let u be an s -density on ∂X , and let $x \in \partial X$. Since $p \in X$ tends to a point $z \in \partial X \setminus \{x\}$,*

$$e^{-Qs d(o, p)} u_p(x)$$

tends to a number denoted by $u_{z, o}(x)$ and given by

$$u_{z, o}(x) = u_o(x) d_o(z, x)^{2Qs}. \quad (3.48)$$

As o is changed, $u_{z, o}$ is multiplied by a positive constant,

$$u_{z, o'} = e^{Qs \beta_z(o', o)} u_{z, o}.$$

Proof: Indeed, by definition,

$$\frac{u_p(x)}{u_o(x)} = e^{sQ\beta_x(p,o)}.$$

Thus

$$\begin{aligned} e^{-Qs d(o,p)} \frac{u_p(x)}{u_o(x)} &= e^{-sQ(d(o,p) - \beta_x(p,o))} \\ &= e^{-2Qs \langle x, p \rangle_o}, \end{aligned}$$

where we used $\beta_x(o, p) = 2\langle x, p \rangle_o - d(o, p)$. The right-hand-side converges pointwise to $d_o(z, x)^{2Qs}$ as $p \rightarrow z$.

Furthermore,

$$\begin{aligned} \frac{u_{z,o'}(x)}{u_{z,o}(x)} &= \lim_{p \rightarrow z} \frac{e^{-Qs d(o',p)} u_p(x)}{e^{-Qs d(o,p)} u_p(x)} \\ &= \lim_{p \rightarrow z} e^{-Qs(d(o',p) - d(o,p))} \\ &= e^{-Qs \beta_z(o',o)}. \end{aligned}$$

■

For $z \neq x$, the number $d_o(z, x)^{-2Qs}$ is indeed the sQ power of $\frac{d_{z,o}}{d_o}(x)$. To see this, recall that for $x, y \in \partial X \setminus \{z\}$, $d_{z,o}(x, y) = \frac{d_o(x,y)}{d_o(x,z)d_o(z,y)}$. In particular, the notation $u = u_{z,o}(d\mu_{z,o})^s$ is valid.

3.5.3 Fractional integration revisited

Let (Z, \mathcal{M}) be a Möbius space of dimension Q . The Riesz-Markov-Kakutani theorem states that the dual space of continuous compactly supported functions can be identified with the space of signed measures (with minor regularity assumptions). This establishes a duality between the spaces of 0-densities and 1-densities.

The Riesz-Markov-Kakutani duality is transferable to s -densities. Given a $(1 - s)$ -density v , then

$$u \mapsto \int uv, \quad (u \in \mathcal{D}_s(\mathcal{M}))$$

is an element of \mathcal{D}_s^* . Morally, \mathcal{D}_{1-s} and \mathcal{D}_s are dual to each other. The space $\mathcal{D}_{\frac{1}{2}}$ is dual to itself, it carries the inner product $(u, v) = \int_Z uv$.

P. Julg [11] noted that the operator

$$\mathcal{I}^s(u)(x) = \int_Z \frac{u_d(y)}{d(x, y)^{2Q(1-s)}} \mu_d(y), \quad \left(\frac{1}{2} < s < 1\right)$$

maps s -densities to $(1 - s)$ -densities. Indeed,

$$\int_Z \frac{u_d(y)}{d(x, y)^{2Q(1-s)}} d\mu_d(y) = \left(\frac{d'}{d}(x)\right)^{(1-s)Q} \int_Z \frac{u_{d'}(y)}{d'(x, y)^{2Q(1-s)}} d\mu_{d'}(y).$$

$\mathcal{I}^s(u)$ thus transforms as $(1 - s)$ -density.

Replacing s by $\frac{1}{2} + \frac{\alpha}{Q}$, we get the fractional integration operator

$$I_d^{2\alpha}(f)(x) = \int_Z \frac{f(y)}{d(x, y)^{Q-2\alpha}} d\mu_d(y) \quad (0 < \alpha < \frac{Q}{2})$$

defined on functions.

The operator \mathcal{I}^s yields a Möbius invariant quadratic form,

$$Q^s(u) = \int_Z u \mathcal{I}^s(u). \quad (u \in \mathcal{D}_s)$$

On the boundary of the real hyperbolic plane, the operator I_d^s naturally appears in the theory of s -Poisson transformations [84]. We will illustrate this fact with the instructive example of trees.

3.5.4 The case of regular trees

On a tree T of degree $q+1$, the harmonic measure of a vertex o is the unique (obvious) probability measure μ_o on $Z = \partial T$ which is invariant under all automorphisms fixing o . Since it coincides with the Patterson-Sullivan measure, the Radon-Nikodým derivatives are exponentials of Busemann functions,

$$\frac{d\mu_{o'}}{d\mu_o}(\zeta) := P(o, o', \zeta) = q^{\beta_\zeta(o, o')} \tag{3.49}$$

for all $\zeta \in Z$ and vertices $x, y \in T$. Here $\beta_\zeta(o, o')$ tends to $+\infty$ when o' tends to ζ .

In this section we allow s to be any complex number. Fix an origin $o \in T$. Consider the s -Poisson transform defined for functions $u : Z \rightarrow \mathbb{C}$ by

$$\mathcal{P}^s(u)(o') = \int_Z P(o, o', \zeta)^{1-s} u(\zeta) d\mu_o(\zeta). \tag{3.50}$$

For $s = 0$, the change of variable formula (3.49) shows that $\mathcal{P}^0(u)$ does not depend on the choice of o , \mathcal{P}^0 is the usual Poisson transform yielding harmonic functions on T . This invariance is preserved for $s \neq 0$, provided one considers functions on Z as s -densities.

Densities on the ideal boundary of a regular tree We give a definition related to the natural Möbius structure of visual metrics associated with vertices in T .

Definition 10 *An s -density on Z is a map $v : o \mapsto u_o$ that associates a measurable function $u_o : Z \rightarrow \mathbb{C}$ with each vertex of T such that for all vertices $o, o' \in T$ and $\zeta \in Z$,*

$$u_o(\zeta) = P(o, o', \zeta)^s u_{o'}(\zeta). \tag{3.51}$$

The image of \mathcal{P}^s consists of eigenfunctions of the Laplacian on T .

Theorem 10 ([21] page 37) *For $s \neq k\pi i / \ln q$, let u be an s -density on Z , and let $f = \mathcal{P}^s(u)$. Then f does not depend on the choice of the origin o , and*

$$\Delta f = \rho(s)f, \quad \text{where } \rho(s) = 1 - \frac{q^s + q^{1-s}}{1 + q}. \tag{3.52}$$

Conversely, any $\rho(s)$ -eigenfunction of the Laplacian on T is the image by \mathcal{P}^s of a unique finitely additive measure (viewed as an s -density) on Z .

Since $\rho(1-s) = \rho(s)$, the operator

$$\mathcal{I}^s = (\mathcal{P}^s)^{-1} \circ \mathcal{P}^{1-s}$$

is well-defined. It maps $(1-s)$ -densities to s -densities, i.e. the quadratic form

$$Q^s : v \mapsto \int_Z v \mathcal{I}^{1-s}(v)$$

is well-defined on \mathcal{D}_s . If $\Re(s) = \frac{1}{2}$, the Hermitian form is

$$H^s : v \mapsto \int_Z \bar{v} \mathcal{I}^{1-s}(v)$$

is well-defined.

Lemma 10 ([21] page 45) *1. For all $s \in \mathbb{C}$ for which $\Re(s) = \frac{1}{2}$ holds, H^s is positive definite.*

2. Q^s is real iff $\Im(s) \in \frac{\pi}{\log q} \mathbb{Z}$. In these cases, Q^s is positive definite iff $0 < \Re(s) < 1$. Otherwise, it has exactly 1 positive direction.

The proof amounts to an explicit computation of eigenvectors and eigenvalues of \mathcal{I}^s (acting on functions, i.e., without considering densities). The spectrum of \mathcal{I}^{1-s} is the set

$$\text{sp}(\mathcal{I}^{1-s}) = \{1\} \cup \frac{q^{-s} - q^s}{q^{1-s} - q^{s-1}} q^{(2s-1)\mathbb{N}^*}.$$

Both cases yield families of unitary representations of $\text{Aut}(T)$. The first family is called *principal series*, the second *complementary series*.

Proposition 11 *Be $s \in \mathbb{C}$ with $\frac{1}{2} < \Re(s) < 1$. Fix a vertex $o \in T$. Then, for $u \in \mathcal{D}_s$,*

$$\begin{aligned} Q^s(u) &= \frac{1 - q^{-2}}{1 - q^{-2s}} \int \int_{Z \times Z} q^{2(1-s)\langle x, y \rangle_o} u_o(x) u_o(y) d\mu_o(x) d\mu_o(y) \\ &= \frac{1 - q^{-2}}{1 - q^{-2s}} \int \int_{Z \times Z} d_o(x, y)^{-2(1-s) \log q} u_o(x) u_o(y) d\mu_o(x) d\mu_o(y). \end{aligned} \tag{3.53}$$

Except for one normalization constant, this quadratic form agrees exactly with the previously defined one.

Proof: (of proposition 11) It relies on Proposition 1.3 on page 40 of [21], which describes the inverse of the s -Poisson transform. Here is a formula for the finitely additive measure m whose s -Poisson transform is the eigenfunction f . Fix the origin o . For the edge e , let (o'', o') be the endpoints of e , where o'' lies between o and o' .

Let \mathcal{O}_o' be the set of endpoints of the subtree intersected by e and facing o , i.e., the shadow of o' as seen from o . Note that $\mu_o(\mathcal{O}_o') = \frac{q}{q+1} q^{-d(o,o')}$. Then

$$m(\mathcal{O}_o') = \frac{1}{q^{(1-s)} - q^{-(1-s)}} q^{-(1-s)d(o,o'')} (f(o') - q^{-(1-s)} f(o'')).$$

Suppose $f = \mathcal{P}^{1-s}(u)$, where u is a $(1-s)$ -density realized at the origin o by the function u_o , i.e.

$$f(o') = \int_Z P(o, o', y)^s u_o(y) d\mu_o(y).$$

If o' is close enough to a point $x \in Z$, $x \neq y$, then o'' lies between y and o' along a geodesic, so $\beta_y(o', o'') = -1$, $\beta_y(o, o'') = \beta_y(o, o') - 1$, $P(o, o'', y) = q^{-1} P(o, o', y)$,

$$P(o, o', y)^s - q^{-(1-s)} P(o, o'', y)^s = (1 - q^{-1}) P(o, o', y)^s, \quad (y \notin \mathcal{O}_o')$$

On the other hand, if $y \in \mathcal{O}_o'$, then $P(o, o'', y) = q P(o, o', y)$,

$$P(o, o', y)^s - q^{-(1-s)} P(o, o'', y)^s = (1 - q^{2s-1}) P(o, o', y)^s, \quad (y \in \mathcal{O}_o').$$

This gives,

$$\begin{aligned} f(o') - q^{-(1-s)} f(o'') &= \int_Z (P(o, o', y)^s - q^{-(1-s)} P(o, o'', y)^s) u_o(y) d\mu_o(y) \\ &= \int_{\mathcal{O}_o'} (P(o, o', y)^s - q^{-(1-s)} P(o, o'', y)^s) u_o(y) d\mu_o(y) \\ &\quad + \int_{Z \setminus \mathcal{O}_o'} (P(o, o', y)^s - q^{-(1-s)} P(o, o'', y)^s) u_o(y) d\mu_o(y) \\ &= (1 - q^{2s-1}) \int_{\mathcal{O}_o'} P(o, o', y)^s u_o(y) d\mu_o(y) \\ &\quad + (1 - q^{-1}) \int_{Z \setminus \mathcal{O}_o'} P(o, o', y)^s u_o(y) d\mu_o(y) \end{aligned}$$

Therefore

$$\begin{aligned} m(\mathcal{O}_o') &= \frac{1 - q^{2s-1}}{q^{(1-s)} - q^{-(1-s)}} \int_{\mathcal{O}_o'} q^{-(1-s)d(o,o'')} P(o, o', y)^s u_o(y) d\mu_o(y) \\ &\quad + \frac{1 - q^{-1}}{q^{(1-s)} - q^{-(1-s)}} \int_{Z \setminus \mathcal{O}_o'} q^{-(1-s)d(o,o'')} P(o, o', y)^s u_o(y) d\mu_o(y) \\ &= \frac{q^{(1-s)}(1 - q^{2s-1})}{q^{(1-s)} - q^{-(1-s)}} \int_{\mathcal{O}_o'} q^{-(1-s)d(o,o')} P(o, o', y)^s u_o(y) d\mu_o(y) \\ &\quad + \frac{q^{(1-s)}(1 - q^{-1})}{q^{(1-s)} - q^{-(1-s)}} \int_{Z \setminus \mathcal{O}_o'} q^{-(1-s)d(o,o')} P(o, o', y)^s u_o(y) d\mu_o(y) \end{aligned}$$

Note that

$$q^{d(o,o')} P(o, o', y) = q^{d(o,o') + \beta_y(o,o')}.$$

This gives,

$$\begin{aligned} m(\mathcal{O}_o') &= \frac{q^{(1-s)}(1 - q^{2s-1})}{q^{(1-s)} - q^{-(1-s)}} \int_{\mathcal{O}_o'} q^{-d(o,o')} q^{s(d(o,o') + \beta_y(o,o'))} u_o(y) d\mu_o(y) \\ &\quad + \frac{q^{(1-s)}(1 - q^{-1})}{q^{(1-s)} - q^{-(1-s)}} \int_{Z \setminus \mathcal{O}_o'} q^{-(1-s)d(o,o')} P(o, o', y)^s u_o(y) d\mu_o(y) \end{aligned}$$

Since o' tends to $x \neq y$, $q^{d(o,o')}P(o,o',y) = q^{d(o,o')+\beta_y(o,o')}$ is stationary. In fact, it remains constant once $d(o,o') \geq \langle x,y \rangle_o$, hence its limit is $q^{2\langle x,y \rangle_o}$. Note also that the first integral becomes negligible only when $\Re(s) < \frac{1}{2}$. We remark that this s is not the same as the one in the hypothesis, but rather needs to be replaced by $1-s$ later on. So if \mathcal{O}'_o is small enough, and $\Re(s) < \frac{1}{2}$,

$$\begin{aligned} m(\mathcal{O}'_o) &\sim \frac{1-q^{-1}}{1-q^{-2(1-s)}} \int_Z q^{-d(o,o')} q^{2s\langle x,y \rangle_o} u_o(y) d\mu_o(y) \\ &\sim \frac{1-q^{-1}}{1-q^{-2(1-s)}} \frac{1+q}{q} \left(\int_Z q^{2s\langle x,y \rangle_o} u_o(y) d\mu_o(y) \right) \mu_o(\mathcal{O}'_o). \end{aligned}$$

In other words, m has density

$$\frac{1-q^{-2}}{1-q^{-2(1-s)}} \int_Z q^{2s\langle x,y \rangle_o} u_o(y) d\mu_o(y)$$

with respect to μ_o .

Replace now s with $1-s$. By definition, $Q^s(u)$ amounts to the integration of u_o with respect to this measure, so

$$Q^s(u) = \frac{1-q^{-2}}{1-q^{-2s}} \int_{Z \times Z} q^{2(1-s)\langle x,y \rangle_o} u_o(y) u_o(x) d\mu_o(y) d\mu_o(x).$$

■

3.6 Uniformly bounded representations

Given a Möbius space (Z, \mathcal{M}) we have seen that the quadratic form

$$Q^s(u) = \int_{Z \times Z} d(x,y)^{-2Q(1-s)} u_d(x) u_d(y) \mu_d(x) \mu_d(y), \quad \left(\frac{1}{2} < s < 1\right)$$

on s -densities, is exactly Möbius invariant. In other words, Q^s does not depend on a choice of $d \in \mathcal{M}$. A thorny question remains about the positivity of Q^s .

In the world of simple Lie groups of real rank 1, Q^s occurs as an invariant inner product of complementary series representations. The necessary and sufficient conditions for the positivity of Q^s were given by B. Kostant [51]. For (Z, \mathcal{M}) as a Möbius space on the boundary of $SO(n, 1)$ and $SU(n, 1)$, Q^s is positive definite on the whole strip $s \in (\frac{1}{2}, 1)$. This is in contrast to the case of $Sp(n, 1)$, for which Q^s is positive definite only if $s \in (\frac{1}{2}, 1 - \frac{1}{2n+1})$. The existence of a point s^* for which Q^s is not positive definite for all $s \in [s^*, 1)$ is typical for groups with Kazhdan property T such as $Sp(n, 1)$ or $F_{4(-20)}$.²

In the limit $s \rightarrow 1$, the quadratic form Q^s degenerates to $Q^1(u) = \left(\int_Z u\right)^2$ on measures. However, the derivative of Q^s with respect to $s = 1$ again yields an invariant quadratic form

$$H(u) = \left. \frac{dQ^s(u)}{ds} \right|_{s=1} = 2Q \int_{Z \times Z} \log d(x,y) u_d(x) u_d(y) \mu_d(x) \mu_d(y).$$

2. $F_{4(-20)}$ is the rank one real form of the simple complex Lie group of type F_4 .

P. Julg [11] gives a short proof of the fact that H is positive definite in the case of $\mathrm{SO}(n, 1)$ and $\mathrm{SU}(n, 1)$. This also follows from the positivity of Q^s on the entire strip $s \in (\frac{1}{2}, 1)$. The affine space

$$\mathcal{A} = \{u \in D^1 : H(u) < \infty, \int_Z u = 1\},$$

is Möbius invariant. The actions of $\mathrm{SO}(n, 1)$ and $\mathrm{SU}(n, 1)$ on (\mathcal{A}, H) are isometric and proper, confirming the well-known fact that these groups have the Haagerup property.

The above discussion shows that for a strongly hyperbolic space X the positivity of Q^s on ∂X fails in general. Overcoming this restriction is the topic of the next section.

3.6.1 A Hilbert space attached to Möbius spaces

In section 3.5.3 we saw the fractional integration operator

$$I_d^{2\alpha}(u) = \int_Z d(x, y)^{-Q+2\alpha} u_d(y) d\mu_d(y), \quad (0 < \alpha < \frac{Q}{2}) \quad (3.54)$$

on $(\frac{1}{2} + \frac{\alpha}{Q})$ -densities, which is exactly Möbius invariant. We may view

$$I_d^\alpha(u) = \int_Z d(x, y)^{-Q+\alpha} u_d(y) d\mu_d(y), \quad (0 < \alpha < \frac{Q}{2})$$

as an approximate root of (3.54). It is self-adjoint on $L^2(d\mu_d)$ and its square has kernel

$$K(x, y) = \int_Z d(x, z)^{-Q+\alpha} d(z, y)^{-Q+\alpha} d\mu_d(z).$$

This kernel is comparable to $d(x, y)^{-Q+2\alpha}$ and $(I_d^\alpha)^2$ is indeed a good candidate for a Laplacian raised to the power $-\alpha$. Therefore, our guess for the $\mathcal{H}^{-\alpha}$ -norm on $(\frac{1}{2} + \frac{\alpha}{Q})$ -densities on compact Q -Ahlfors regular Möbius spaces is

$$\|u\|_d = \|I_d^\alpha u_d\|_{L^2(d\mu_d)}.$$

It is positive by construction and leads to a Sobolev space $\mathcal{H}_d^{-\alpha}$ of $(\frac{1}{2} + \frac{\alpha}{Q})$ -densities.

Uniform boundedness Apriori, the construction of $\mathcal{H}_d^{-\alpha}$ depends on the choice of the metric d within the Möbius structure. We can ask whether it is canonical, i.e. for fixed $0 < \alpha < \frac{Q}{2}$, does there exist a $C > 0$, s.t.

$$\frac{1}{C} \|u\|_d \leq \|u\|_{d'} \leq C \|u\|_d, \quad (3.55)$$

for all $d, d' \in \mathcal{M}$.

If G is a group acting on Z , then G acts naturally on $\mathcal{D}_s(\mathcal{M})$ by $g \cdot u = g \cdot u_d (dg_* \mu_d)^s$, where $g \cdot u_d$ is left-translation on functions and $g_* \mu_d$ denotes the pushforward measure of μ_d by g . If G acts on Z through Möbius automorphisms, then $\|g \cdot u\|_d = \|u\|_{g^* d}$.

Here g^*d denotes the pullback of d by g . In fact, $\frac{dg_*\mu_d}{d\mu_d}(x) = \left(\frac{(g^{-1})^*d}{d}(x)\right)^Q = \left(\frac{d}{g^*d}(g^{-1}x)\right)^Q$. For better readability we set $\frac{1}{2} + \frac{\alpha}{Q} = s$.

$$\begin{aligned}
\|g \cdot u\|_d^2 &= \|I_d^\alpha(g \cdot u_d (dg_*\mu_d)^s)\|_{L^2(\mu_d)}^2 \\
&= \int_Z \left(\int_Z \frac{u_d(g^{-1}y) \left(\frac{(g^{-1})^*d}{d}(y)\right)^{sQ}}{d(x,y)^{Q-\alpha}} d\mu_d(y) \right)^2 d\mu_d(x) \\
&= \int_Z \left(\int_Z \frac{u_d(g^{-1}y) \left(\frac{d}{g^*d}(g^{-1}y)\right)^{sQ}}{g^*d(g^{-1}x, g^{-1}y)^{Q-\alpha}} d\mu_d(y) \right)^2 d\mu_d(x) \\
&= \int_Z \left(\int_Z \frac{u_d(y) \left(\frac{d}{g^*d}(y)\right)^{sQ}}{g^*d(x,y)^{Q-\alpha}} d(g^{-1})_*\mu_d(y) \right)^2 d(g^{-1})_*\mu_d(x) \\
&= \int_Z \left(\int_Z \frac{u_{g^*d}(y)}{g^*d(x,y)^{Q-\alpha}} d\mu_{g^*d}(y) \right)^2 d\mu_{g^*d}(x) \\
&= \|u\|_{g^*d}^2
\end{aligned}$$

If G maps \mathcal{M} into itself, then (3.55) implies,

$$\frac{1}{C}\|u\|_d \leq \|g \cdot u\|_d \leq C\|u\|_d. \quad (\forall g \in G) \quad (3.56)$$

In other words, the action of G on $\mathcal{H}^{-\alpha}$ is uniformly bounded.

Lemma 11 *For fixed $0 < \alpha < \frac{Q}{2}$, there exists a $C > 0$, s.t.*

$$\frac{1}{C}\|u\|_d \leq \|u\|_{d'} \leq C\|u\|_d,$$

for all $d, d' \in \mathcal{M}$ if and only if for all $d, d' \in \mathcal{M}$,

$$\left\| \left(\frac{d'}{d}\right)^{\frac{\alpha}{2}} \circ I_d^\alpha \circ \left(\frac{d'}{d}\right)^{-\frac{\alpha}{2}} u_d \right\|_{L^2(\mu_d)} \leq C \|I_d^\alpha u_d\|_{L^2(\mu_d)}. \quad (3.57)$$

Proof: In fact, due to the symmetry of the problem, it is only necessary to show the upper bound in (3.55) for all $d, d' \in \mathcal{M}$. Rewriting the norm $\|u\|_{d'}$ in terms of a fixed background measure μ_d , we get

$$\begin{aligned}
\|u\|_{d'}^2 &= \|I_{d'}^\alpha(u_{d'} (d\mu_{d'})^{\frac{1}{2} + \frac{\alpha}{Q}})\|_{L^2(\mu_{d'})}^2 \\
&= \int_Z \left(\int_Z \frac{u_{d'}(y)}{d'(x,y)^{Q-\alpha}} d\mu_{d'}(y) \right)^2 d\mu_{d'}(x) \\
&= \int_Z \left(\int_Z \frac{\left(\frac{d}{d'}(y)\right)^{\frac{Q}{2} + \alpha} u_d(y)}{\left(\frac{d'}{d}(x)\frac{d'}{d}(y)\right)^{\frac{Q}{2} - \frac{\alpha}{2}} d(x,y)^{Q-\alpha}} d\mu_{d'}(y) \right)^2 d\mu_{d'}(x) \\
&= \int_Z \left(\left(\frac{d}{d'}(x)\right)^{\frac{Q}{2} - \frac{\alpha}{2}} \int_Z \frac{\left(\frac{d}{d'}(y)\right)^{Q + \frac{\alpha}{2}} u_d(y)}{d(x,y)^{Q-\alpha}} d\mu_{d'}(y) \right)^2 d\mu_{d'}(x) \\
&= \int_Z \left(\left(\frac{d}{d'}(x)\right)^{-\frac{\alpha}{2}} \int_Z \frac{\left(\frac{d}{d'}(y)\right)^{\frac{\alpha}{2}} u_d(y)}{d(x,y)^{Q-\alpha}} d\mu_d(y) \right)^2 d\mu_d(x) \\
&= \left\| \left(\frac{d'}{d}\right)^{\frac{\alpha}{2}} \circ I_d^\alpha \circ \left(\frac{d'}{d}\right)^{-\frac{\alpha}{2}} u_d \right\|_{L^2(\mu_d)}^2.
\end{aligned}$$

■

Directly showing the inequality $\|(\frac{d'}{d})^{\frac{\alpha}{2}} \circ I_d^\alpha \circ (\frac{d'}{d})^{-\frac{\alpha}{2}} u_d\|_{L^2(\mu_d)} \leq C \|I_d^\alpha u_d\|_{L^2(\mu_d)}$, proves to be difficult. However, the weaker statement $\|(\frac{d'}{d})^{\frac{\alpha}{2}} \circ I_d^\alpha \circ (\frac{d'}{d})^{-\frac{\alpha}{2}} u_d\|_{L^2(\mu_d)} \leq C \|u_d\|_{L^2(\mu_d)}$, follows from Theorem 9, when \mathcal{M} is a spherical Möbius structure.

Indeed, by Theorem 9, $\|(\frac{d'}{d})^{\frac{\alpha}{2}} \circ I_d^\alpha \circ (\frac{d'}{d})^{-\frac{\alpha}{2}} u_d\|_{L^q(\mu_d)} \leq C \|u_d\|_{L^p(\mu_d)}$, with $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{Q}$. Thus $1 < p \leq q$ we may set $q = 2$, and then by Hölder's inequality

$$\|(\frac{d'}{d})^{\frac{\alpha}{2}} \circ I_d^\alpha \circ (\frac{d'}{d})^{-\frac{\alpha}{2}} f\|_{L^2(\mu_d)} \leq C \|f\|_{L^p(\mu_d)} \leq C \|f\|_{L^2(\mu_d)}. \quad (3.58)$$

This means that the operator norm $\|I_d^\alpha\|_{L^2((\frac{d'}{d})^\alpha \mu_d)}$ of I_d^α is uniformly bounded by a constant C . We will see that for spherical Möbius structures the weaker statement (3.58) implies the stronger one (3.57) for a family of special functions.

Powers of conformal factors. In a Möbius space (Z, \mathcal{M}) , the conformal factors raised to the power of sQ are s -densities on Z . It is clear that for any metric $d' \in \mathcal{M}$, the function

$$d \mapsto \left(\frac{d'}{d}\right)^{sQ}$$

transforms as an s -density. Morally, $(\frac{d'}{d})^{sQ}$ is the s -density, that is, the constant function 1 on the metric d' . We recall that I_d^α generates an operator on s -densities for a given s as a function of α . It turns out that $s = \frac{1}{2} + \frac{\alpha}{2Q}$. This fact implies strong regularity properties on spherical Möbius spaces.

Theorem 11 *Let (Z, \mathcal{M}) be a spherical, Q -Ahlfors-David regular Möbius space, $0 < \alpha < \frac{Q}{2}$. If the operator norm $\|I_d^\alpha\|_{L^2((\frac{d'}{d})^\alpha \mu_d)}$ is uniformly bounded, i.e., bounded by a constant independent of d and d' , then there exists $C > 0$, uniformly over \mathcal{M} , s.t. any function f from the family*

$$\left\{ \left(\frac{d''}{d}\right)^{sQ} \right\}_{d'' \in \mathcal{M}} \quad \left(s = \frac{1}{2} + \frac{\alpha}{2Q} \right)$$

satisfies the inequality $\|(\frac{d'}{d})^{\frac{\alpha}{2}} \circ I_d^\alpha \circ (\frac{d'}{d})^{-\frac{\alpha}{2}} f\|_{L^2(\mu_d)} \leq C \|I_d^\alpha f\|_{L^2(\mu_d)}$.

This theorem holds in particular for the natural Möbius structure $\{d_o\}_{o \in X}$ on a strongly hyperbolic space X .

We set $s = \frac{1}{2} + \frac{\alpha}{2Q}$. We denote the dual of s by $s' = \frac{1}{2} - \frac{\alpha}{2Q}$. The conformal invariance of I_d^α on s -densities, manifests itself explicitly as follows.

$$\begin{aligned} I_d^\alpha \left(\left(\frac{d'}{d}\right)^{sQ} \right)(x) &= \int_Z \frac{(\frac{d'}{d}(y))^{sQ}}{d(x, y)^{Q-\alpha}} d\mu_d(y) \\ &= \int_Z \frac{(\frac{d'}{d}(y))^{sQ}}{(\frac{d}{d'}(x) \frac{d}{d'}(y))^{s'Q} d'(x, y)^{Q-\alpha}} d\mu_d(y) \\ &= \left(\frac{d'}{d}(x)\right)^{s'Q} \int_Z \frac{(\frac{d'}{d}(y))^Q}{d'(x, y)^{Q-\alpha}} d\mu_d(y) \\ &= \left(\frac{d'}{d}(x)\right)^{s'Q} \int_Z \frac{1}{d'(x, y)^{Q-\alpha}} d\mu_{d'}(y). \end{aligned}$$

If \mathcal{M} is a spherical Möbius structure, then $\int_Z \frac{1}{d'(x,y)^{Q-\alpha}} d\mu_{d'}(y)$ is finite.

Lemma 12 *If \mathcal{M} is a spherical, Q -Ahlfors-David regular Möbius structure, then there exists $C > 0$ s.t. $\forall d \in \mathcal{M}$,*

$$1 + \frac{Q - \alpha}{\alpha} \frac{1}{C} \leq \int_Z \frac{1}{d(x,y)^{Q-\alpha}} d\mu_d(y) \leq 1 + \frac{Q - \alpha}{\alpha} C.$$

Proof: We use integration by parts to integrate over balls.

$$\begin{aligned} \int_Z \frac{1}{d(x,y)^{Q-\alpha}} d\mu_d(y) &= \int_Z (Q - \alpha) \int_0^\infty t^{-Q+\alpha-1} \chi_{\{t \geq d(x,y)\}} dt d\mu_d(y) \\ &= (Q - \alpha) \int_0^\infty t^{-Q+\alpha-1} \mu_d(B_d(x,t)) dt \\ &= 1 + (Q - \alpha) \int_1^\infty t^{-Q+\alpha-1} \mu_d(B_d(x,t)) dt. \end{aligned}$$

The last line takes advantage of the fact that \mathcal{M} is spherical. The uniform Ahlfors-David regularity of \mathcal{M} , gives a $C \geq 1$ such that $\frac{t^Q}{C} \leq \mu_d(B_d(x,t)) \leq Ct^Q$. Using this above and integrating out, we prove the lemma. \blacksquare

We are now in a position to prove the proposition 11.

Proof: (of Theorem 11) Recall that we set $s = \frac{1}{2} + \frac{\alpha}{2Q}$ and $s' = \frac{1}{2} - \frac{\alpha}{2Q}$. For any three metrics $d, d', d'' \in \mathcal{M}$ we have,

$$\begin{aligned} \left(\frac{d'}{d}\right)^{\frac{\alpha}{2}} \circ I_d^\alpha \circ \left(\frac{d'}{d}\right)^{-\frac{\alpha}{2}} \left(\left(\frac{d''}{d}\right)^{sQ}\right)(x) &= \left(\frac{d'}{d}(x)\right)^{\frac{\alpha}{2}} \int_Z \frac{\left(\frac{d'}{d}(y)\right)^{-\frac{\alpha}{2}} \left(\frac{d''}{d}(y)\right)^{sQ}}{d(x,y)^{Q-\alpha}} d\mu_d(y) \\ &= \left(\frac{d'}{d}(x)\right)^{\frac{\alpha}{2}} \int_Z \frac{\left(\frac{d'}{d}(y)\right)^{-\frac{\alpha}{2}} \left(\frac{d''}{d}(y)\right)^{sQ}}{\left(\frac{d}{d''}(x) \frac{d}{d''}(y)\right)^{s'Q} d''(x,y)^{Q-\alpha}} d\mu_d(y) \\ &= \left(\frac{d'}{d}(x)\right)^{\frac{\alpha}{2}} \left(\frac{d''}{d}(x)\right)^{s'Q} \int_Z \frac{\left(\frac{d'}{d}(y)\right)^{-\frac{\alpha}{2}} \left(\frac{d''}{d}(y)\right)^Q}{d''(x,y)^{Q-\alpha}} d\mu_d(y) \\ &= \left(\frac{d'}{d}(x)\right)^{\frac{\alpha}{2}} \left(\frac{d''}{d}(x)\right)^{s'Q} \int_Z \frac{\left(\frac{d'}{d}(y)\right)^{-\frac{\alpha}{2}}}{d''(x,y)^{Q-\alpha}} d\mu_{d''}(y) \\ &= \left(\frac{d''}{d}(x)\right)^{\frac{Q}{2}} \left(\frac{d'}{d}(x)\right)^{\frac{\alpha}{2}} \left(\frac{d}{d''}(x)\right)^{\frac{\alpha}{2}} I_{d''}^\alpha \left(\left(\frac{d'}{d}\right)^{-\frac{\alpha}{2}}\right)(x) \\ &= \left(\frac{d''}{d}(x)\right)^{\frac{Q}{2}} \left(\frac{d'}{d''}(x)\right)^{\frac{\alpha}{2}} I_{d''}^\alpha \left(\left(\frac{d'}{d}\right)^{-\frac{\alpha}{2}}\right)(x). \end{aligned}$$

Thus,

$$\left\| \left(\frac{d'}{d}\right)^{\frac{\alpha}{2}} \circ I_d^\alpha \circ \left(\frac{d'}{d}\right)^{-\frac{\alpha}{2}} \left(\left(\frac{d''}{d}\right)^{sQ}\right) \right\|_{L^2(d\mu_d)} = \left\| I_{d''}^\alpha \left(\left(\frac{d'}{d}\right)^{-\frac{\alpha}{2}}\right) \right\|_{L^2\left(\left(\frac{d'}{d''}\right)^\alpha \mu_{d''}\right)}$$

By assumption there exists $C > 0$ s.t

$$\left\| I_{d''}^\alpha \left(\left(\frac{d'}{d}\right)^{-\frac{\alpha}{2}}\right) \right\|_{L^2\left(\left(\frac{d'}{d''}\right)^\alpha \mu_{d''}\right)} \leq C \left\| \left(\frac{d'}{d}\right)^{-\frac{\alpha}{2}} \right\|_{L^2\left(\left(\frac{d'}{d''}\right)^\alpha \mu_{d''}\right)}.$$

But $\|(\frac{d'}{d})^{-\frac{\alpha}{2}}\|_{L^2((\frac{d'}{d''})^\alpha \mu_{d''})} = \|(\frac{d''}{d})^{s'Q}\|_{L^2(\mu_d)}$, and lemma 12 implies for a spherical Möbius structure,

$$\|(\frac{d''}{d})^{s'Q}\|_{L^2(\mu_d)} \sim \|I_d^\alpha((\frac{d''}{d})^{sQ})\|_{L^2(\mu_d)}.$$

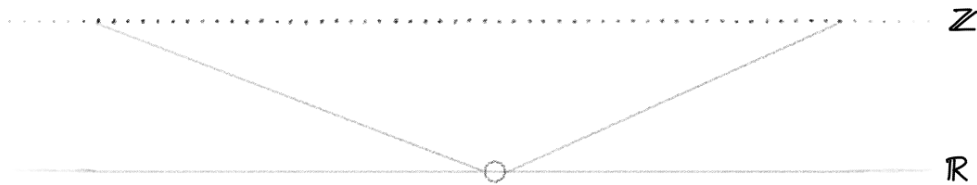
Thus there exists $C' > 0$ independently of d, d', d'' , s.t.

$$\|(\frac{d'}{d})^{\frac{\alpha}{2}} \circ I_d^\alpha \circ (\frac{d'}{d})^{-\frac{\alpha}{2}}((\frac{d''}{d})^{sQ})\|_{L^2(d\mu_d)} \leq C' \|I_d^\alpha((\frac{d''}{d})^{sQ})\|_{L^2(\mu_d)}.$$

■

Chapter 4

Asymptotic-Möbius maps



4.1 Introduction

In 1999, Oded Schramm investigated the scaling limits of loop-erased random walks (LERW) [85]. Schramm observed that the conformal invariance of the limit process γ in the upper half plane \mathbb{H} , implies its encoding in corresponding conformal mappings $f_t : \mathbb{H} \setminus \gamma \rightarrow \mathbb{H}$ satisfying Loewner's differential equation

$$\frac{\partial f_t}{\partial t}(z) = \frac{\partial f_t}{\partial z}(z) \frac{2}{\zeta(t) - z}.$$

$\zeta(t)$ is the driving function defined on the ideal boundary \mathbb{R} of \mathbb{H} . If $\zeta(t) = \sqrt{\kappa}B_t$, where B_t is Brownian motion on \mathbb{R} , then one speaks of γ as the Schramm-Loewner-Evolution (SLE) of speed κ . This observation revolutionized two-dimensional statistical physics (see, e.g., a review of SLE by G. F. Lawler [86]).

There are generalizations for dimensions higher than 2 [87]. However, the strength of conformal techniques in higher dimensions is severely limited. This is particularly clear in Liouville's 1850 theorem.¹ While all simply connected domains in \mathbb{R}^2 are conformally equivalent by the Riemann mapping theorem, any smooth conformal mapping on a domain of \mathbb{R}^n , with $n > 2$, is a composition of translations, similarities, orthogonal transformations and inversions.

1. Monge 1850, pp. 609 - 616, a note contributed by Liouville as editor (Note VI: Extension au cas des trois dimensions de la question du tracé géographique.)

Suitable classes of functions on metric spaces that capture conformal behavior are the classes of quasisymmetric (QS) maps, quasi-Möbius (QM) maps, and quasiconformal (QC) maps. Quasisymmetric maps transform balls into "quasiballs." The definition is based on the fact that conformal maps transform infinitesimal balls into infinitesimal ellipsoids. Quasi-Möbius maps preserve almost the cross ratio of points, and quasiconformal maps are an infinitesimal version of quasisymmetric maps. We always have $QS \Rightarrow QM \Rightarrow QC$ [88].

We are therefore interested in classes of functions that converge under scaling limits to the classes QS, QM or QC. In the eyes of a geometric group theorist, we are looking for classes of maps between finitely generated groups Γ , with the word metric, which remain invariant under small changes in topology and which become conformal in the above sense when viewed from a distance. Specifically, we require that the class of maps is invariant under composition by quasi-isometries.

This chapter presents and examines a class of maps that mimics quasi-Möbius maps in the context of geometric group theory.

In [23] Pierre Pansu introduces a notion of large-scale conformal maps that mimics the infinitesimal behavior of conformal maps. In short, large-scale conformal maps map families of disjoint balls onto families of weakly overlapping quasiballs. It is a very flexible notion that includes, for example, coarse embeddings. However, this flexibility makes the asymptotic behavior of such maps less predictable.

Instead, we introduce a more restrictive and semantically simpler notion that mimics quasi-Möbius behavior. Roughly speaking, a map between metric spaces is an asymptotic-Möbius map if it nearly preserves the cross ratio of points that are a large distance apart. As a result, the results presented in this paper have stronger conclusions than those in [23].

We postpone the precise definition of asymptotic-Möbius (AM) maps until later. Sources of examples of AM-maps are:

1. Quasi-isometric embeddings,
2. Sublinear-bi-Lipschitz equivalences (i.e., maps inducing Lipschitz equivalences on asymptotic cones [24]),
3. Snowflaking (i.e., replacing a metric by a power of it),
4. Assouad maps from doubling metric spaces to \mathbb{R}^N .

For example, any nilpotent Lie group or finitely generated group can be AM-embedded in a Euclidean space of sufficiently high dimension [25]. Moreover, we will encounter infinite dimensional examples.

4.1.1 Results

Our main result is that under AM-mappings a large-scale term of dimension increases. The relevant term depends on the class of the groups considered.

Theorem 14 *Let G and G' be a finitely generated nilpotent groups and $f : G \rightarrow G'$ an AM-map preserving diverging and bounded sequences, then $\text{asdim}(G) \leq \text{asdim}(G')$. If G and G' are simply connected nilpotent Lie groups, then $\dim(G) \leq \dim(G')$. Furthermore, if $\text{asdim}(G) = \text{asdim}(G')$, then the asymptotic cones of G and G' are*

isomorphic graded Lie groups. Conversely, given nilpotent groups with isomorphic asymptotic cones, there exists an asymptotic-Möbius map between them.

In the world of CAT(0)-spaces, the analogous theorem takes the following form.

Theorem 15 *Let X and Y be spaces, whose asymptotic cones are CAT(0). If there exists an AM-map $f : X \rightarrow Y$ preserving diverging and bounded sequences, then the telescopic dimension increases e.g. $\text{tele-dim}(X) \leq \text{tele-dim}(Y)$.*

4.2 Asymptotic-Möbius maps

4.2.1 Quasi-Möbius maps

Quasi-Möbius maps were introduced by Jussi Väisälä, among others, as a means of studying quasisymmetric maps and quasiconformal maps. Unlike quasisymmetric maps, quasi-Möbius maps do not have a point fixed at infinity.

Let X be a metric space, and $x, y, z, w \in X$ a quadruple of distinct points. Their *cross-ratio* is

$$[x, y, z, w] = \frac{d(x, z)d(y, w)}{d(x, w)d(y, z)}.$$

If X is unbounded, then the cross ratio extends to the one-point compactification of X [88]. A map $f : X \rightarrow Y$ is a *quasi-Möbius embedding* if there is a homeomorphism $\eta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that for all quadruples of distinct points $x, y, z, w \in X$,

$$[f(x), f(y), f(z), f(w)] \leq \eta([x, y, z, w]).$$

If f is a homeomorphism, then f is quasi-Möbius iff its inverse is quasi-Möbius.

Examples of quasi-Möbius maps are

1. the stereographic projection $\mathbb{R}^n \rightarrow S^n$,
2. the Cayley transformations (the complex, quaternionic and octonionic analogues of the stereographic projection),
3. the inversions $x \mapsto \frac{x}{|x|^2}$ in Banach spaces.

4.2.2 Asymptotic-Möbius maps

We need a criterion to tell when two points in a space are far apart. One way to do this is to separate them by sublinear growing functions. This idea was used by Y. Cornuier to define sublinear-bi-Lipschitz equivalences [24].

Given $u : \mathbb{R}_+ \rightarrow \mathbb{R}$, we say that the points x and y in X are separated by u from a fixed origin o if $d(x, y) > u(d(x, o) + d(o, y))$. For simplicity, we often write $x, y > u$ and $|x| = d(o, x)$ for the distance from the origin. We can call u a *gauge*, i.e., a function that sets a scale of "sight" as a function of "location". To fix a gauge means to decide what is "near" and "far away".

Definition 11 A function $u : \mathbb{R}_+ \rightarrow \mathbb{R}$ is an admissible gauge if

1. it is non-decreasing, and
2. u grows sublinearly, i.e. $\limsup_{r \rightarrow \infty} \frac{u(r)}{r} = 0$.

Morally, asymptotic-Möbius maps at widely spaced points are quasi-Möbius.

Definition 12 A map $f : X \rightarrow Y$ is an asymptotic-Möbius map or AM-map if there exists an admissible gauge u and a homeomorphism $\eta : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that for every x, y, z, w in X such that all 6 distances between $x, y, z, w > u$, then

$$[f(x), f(y), f(z), f(w)] < \eta([x, y, z, w]).$$

At some point we will need a coarse version of path connectivity.

Definition 13 A metric space X is asymptotically chained, if for every origin $o \in X$ there exists an admissible gauge v such that for all $x, y \in X$ there exists a chain $x_1 = x, \dots, x_{k+1} = y$ satisfying

$$\max_{i \in 1 \dots k} \{d(x_i, x_{i+1})\} < v(|x| + |y|).$$

Our most important technical step will be the following theorem, the proof of which will occupy section 4.2.4 after some introductory definitions.

Theorem 12 Let f be an AM-map from an asymptotically chained space X to a space Y that preserves diverging and bounded sequences, then f induces a continuous, injective, quasisymmetric map g between some asymptotic cones of X and Y .

4.2.3 Asymptotic-cones

The *asymptotic cone* of a metric space (X, d) , captures the geometry on the large scale of X . Roughly speaking, it formalizes the idea of snapshots of the space X taken by an observer moving farther and farther away from X . This sequence of snapshots can stabilize and the observer has the impression of seeing a single object. We call this object the asymptotic cone of X .

If R -balls $B(o, R)$ with metric $R^{-1}d$ are uniformly pre-compact, then the asymptotic cones can be constructed concretely by pointed Gromov-Hausdorff convergence [89], [90] In general, however, we have to resort to constructions via ultrafilters.

Definition 14 A (nonprincipal) ultrafilter ω over \mathbb{N} is a set of subsets of \mathbb{N} satisfying the following conditions:

1. If $A, B \in \omega$ then $A \cap B \in \omega$.
2. If $A \in \omega, A \subset B \subset \mathbb{N}$, then $B \in \omega$.
3. For every $A \subset \mathbb{N}$, either $A \in \omega$ or $\mathbb{N} \setminus A \in \omega$.
4. No finite subset of \mathbb{N} is in ω .

Equivalently, ω is a finitely additive probability measure on \mathbb{N} such that every subset has measure either 0 or 1 and every finite subset has measure 0.

If a statement $A(n)$ holds for all $n \in B$, where B belongs to an ultrafilter ω , then $A(n)$ is said to hold ω -almost surely.

Definition 15 *Let ω be a (nonprincipal) ultrafilter over \mathbb{N} . An ω -limit of a sequence of points $\{x_n\}$ in a topological space X is a point x in X such that for any neighborhood U of x the relation $x_n \in U$ holds ω -almost surely.*

If X is a Hausdorff space, then the ω -limit of a sequence is unique. We denote this point by $\lim_{\omega} x_n$.

Definition 16 *The ultrapower of a set X with respect to an ultrafilter ω , denoted by X^{ω} , consists of equivalence classes of sequences $\{x_n\}$, $x_n \in X$, where two sequences $\{x_n\}$ and $\{y_n\}$ are identical if and only if $x_n = y_n$ ω -almost surely.*

We adopt the notation $\{x_n\}_{\omega}$ for the equivalence class of sequences.

Let (X, d) be a metric space, ω an ultrafilter over \mathbb{N} and o an origin in X . Furthermore, let a sequence of numbers $\{\lambda_n\}$ be given, with $\lim_{\omega} \lambda_n = \infty$. Then there is a pseudo-metric on the ultrapower X^{ω} by setting the distance between $x = \{x_n\}_{\omega}$ and $y = \{y_n\}_{\omega}$ to

$$d_{\omega}(x, y) = \lim_{\omega} \frac{d(x_n, y_n)}{\lambda_n}.$$

Let $X_{\omega}(\lambda_n)$ be the set of equivalence classes of elements $x = \{x_n\}_{\omega}$ in X^{ω} satisfying $\lim_{\omega} \frac{d(x_n, o)}{\lambda_n} < \infty$ and two elements $x = \{x_n\}_{\omega}$ and $y = \{y_n\}_{\omega}$ of X^{ω} are identical if and only if $d_{\omega}(x, y) = 0$. The metric space $(X_{\omega}(\lambda_n), d_{\omega})$ is called an *asymptotic cone* of (X, d) .

For points in the asymptotic cone $(X_{\omega}(\lambda_n), d_{\omega})$ of (X, d) we use the notation x_{ω} .

Lemma 13 *If $x_{\omega} \neq y_{\omega}$, then for every admissible gauge u and all sequences $\{x_n\}$, $\{y_n\}$ representing x_{ω} and y_{ω} , respectively,*

$$d(x_n, y_n) > u(|x_n| + |y_n|), \quad \omega - a.s.$$

Proof: If there is an admissible function u and sequences $\{x_n\}$, $\{y_n\}$ such that

$$d(x_n, y_n) \leq u(|x_n| + |y_n|), \quad \omega - a.s.$$

then $\omega - a.s.$,

$$\frac{d(x_n, y_n)}{\lambda_n} \leq \frac{u(|x_n| + |y_n|)}{\lambda_n}.$$

So

$$\lim_{\omega} \frac{d(x_n, y_n)}{\lambda_n} = 0.$$

■

Lemma 14 *Let X be an asymptotically chained metric space. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in X representing the same point x_ω different from the origin o_ω . Then, for any admissible gauge u , either*

$$x_n, y_n > u, \quad \omega - a.s.$$

or there exists a sequence $\{w_n\}$ representing the same point s.t

$$x_n, w_n > u \quad \omega - a.s.$$

and

$$y_n, w_n > u \quad \omega - a.s.$$

Proof: Suppose $d(x_n, y_n) \leq u$ $\omega - a.s.$ and assuming $|x_n|, |y_n| \rightarrow +\infty$.

Since X is asymptotically chained, there is an admissible gauge v and a chain $x_1^n = o, \dots, x_{k+1}^n = y_n$ such that

$$\max_{i \in 1, \dots, k} \{d(x_i^n, x_{i+1}^n)\} < v(|y_n|).$$

Assuming n is large enough so that $2u(4|y_n|) < |y_n|$, then at least one point $x_{i_n}^n$ in the chain satisfies

$$2u(4|y_n|) < d(y_n, x_{i_n}^n) \leq 2u(4|y_n|) + v(|y_n|).$$

Define $w_n = x_{i_n}^n$.

For n large enough,

$$\begin{aligned} 2d(x_n, y_n) &\leq 2u(|x_n| + |y_n|) \\ &\leq |x_n| + |y_n| \\ &\leq 2|y_n| + d(x_n, y_n). \end{aligned}$$

It follows,

$$d(x_n, y_n) \leq u(4|y_n|).$$

and therefore

$$u(4|y_n|) < d(x_n, w_n) \leq 3u(4|y_n|) + v(|y_n|).$$

Choose N s.t. $\forall n \geq N$

$$3u(4|y_n|) + v(|y_n|) \leq 2|y_n|.$$

Then

$$\begin{aligned} u(|x_n| + |w_n|) &\leq u(|x_n| + |y_n| + d(y_n, w_n)) \\ &\leq u(2|y_n| + d(x_n, y_n) + 2u(4|y_n|) + v(|y_n|)) \\ &\leq u(2|y_n| + 3u(4|y_n|) + v(|y_n|)) \\ &\leq u(4|y_n|) \\ &< d(x_n, w_n). \end{aligned}$$

This means $x_n, w_n > u$ and

$$\frac{d(x_n, w_n)}{\lambda_n} \leq \frac{3u(4|y_n|) + v(|y_n|)}{\lambda_n} \rightarrow 0.$$

Related,

$$\begin{aligned}
u(|y_n| + |w_n|) &\leq u(2|y_n| + d(y_n, w_n)) \\
&\leq u(2|y_n| + 2u(4|y_n|) + v(|y_n|)) \\
&\leq u(4|y_n|) \\
&< d(y_n, w_n).
\end{aligned}$$

■

4.2.4 Proof of Theorem 12

Proof: The case when X is bounded is trivial, so we can assume that X is unbounded. Let u be an admissible gauge as given by the definition of f . Fix an origin $o \in X$. Let B_n be the ball of radius n around o . For each n , choose a point z_n in B_n such that

$$d(f(o), f(z_n))$$

is 'maximal'. More precisely, let $\epsilon > 0$ and choose z_n such that

$$d(f(o), f(z_n)) > \sup_{x \in B_n} d(f(o), f(x)) - \epsilon.$$

The supremum on the right hand side exists. Otherwise, one could construct a bounded sequence in B_n mapped by f to an unbounded sequence. This contradicts the assumptions.

Moreover, the four-point condition

$$[f(x), f(w), f(y), f(z)] \leq \eta([x, w, y, z]),$$

reduces to the three-point condition

$$\frac{d(f(x), f(y))}{d(f(x), f(z))} \leq \eta\left(\frac{d(x, y)}{d(x, z)}\right),$$

by bringing w to a point at ∞ .

In what follows, we often abbreviate the notation $f(x)$ to x' .

Define

$$\lambda_n = d(o, z_n)$$

and

$$\lambda'_n = d(o', z'_n).$$

By construction and assumption over f , the sequence λ'_n is divergent. Consequently, the sequence λ_n must also be divergent by the assumption over f .

(1) We show that if $\frac{d(o, x_n)}{\lambda_n}$ is bounded, then $\frac{d(o', f(x_n))}{\lambda'_n}$ is also bounded.

If $x_\omega = o_\omega$, then $d(o, x_n) \leq \lambda_n$ for n large, therefore $d(o', x'_n) \leq \lambda'_n$.

If $x_\omega, z_\omega, o_\omega$ are all distinct, then by lemma 13 $\{x_n\}, \{z_n\}, \{o\}$ are separated by u ω -a.s., and the Möbius- condition yields

$$\frac{d(o', x'_n)}{\lambda'_n} = \frac{d(o', x'_n)}{d(o', z'_n)} < \eta\left(\frac{d(o, x_n)}{d(o, z_n)}\right) = \eta\left(\frac{d(o, x_n)}{\lambda_n}\right) \quad \omega - a.s.$$

Thus $\frac{d(o', x'_n)}{\lambda'_n}$ is bounded.

If $x_\omega = z_\omega$, then by lemma 14 there are $\{w_n\}$ representing the same point, s.t. $\{w_n\}, \{x_n\}$ are u separated and $\{w_n\}, \{z_n\}$ are u separated. In particular, $\{w_n\}, \{x_n\}, \{o\}$ and $\{w_n\}, \{z_n\}, \{o\}$ are u separated. Again using the Möbius condition,

$$\begin{aligned} \frac{d(o', x'_n)}{\lambda'_n} &= \frac{d(o', x'_n)}{d(o', z'_n)} \\ &= \frac{d(o', x'_n)}{d(o', w'_n)} \frac{d(o', w'_n)}{d(o', z'_n)} \\ &< \eta\left(\frac{d(o, x_n)}{d(o, w_n)}\right) \eta\left(\frac{d(o, w_n)}{d(o, z_n)}\right) \\ &= \eta\left(\frac{d(o, x_n)}{\lambda_n} \frac{\lambda_n}{d(o, w_n)}\right) \eta\left(\frac{d(o, w_n)}{\lambda_n}\right) \\ &\rightarrow \eta(1)^2 \end{aligned}$$

Thus $\frac{d(o', x'_n)}{\lambda'_n}$ is bounded.

(2) The mapping $\{x_n\} \rightarrow \{f(x_n)\}$ is well-defined from $X_\omega(\lambda_n)$ to $Y_\omega(\lambda'_n)$.

If $\{x_n\}$ and $\{y_n\}$ are two sequences representing the same point different from the origin o_ω , then by Lemma 14 and Lemma 13 either $\{x_n\}, \{y_n\}, \{o\}$ are separated by u , or there exists a sequence $\{w_n\}$ representing the same point, s.t. $\{x_n\}, \{w_n\}, \{o\}$ and $\{y_n\}, \{w_n\}, \{o\}$ are separated by u .

In the first case,

$$\begin{aligned} \frac{d(x'_n, y'_n)}{\lambda'_n} &= \frac{d(o', x'_n)}{\lambda'_n} \frac{d(x'_n, y'_n)}{d(x'_n, o')} \\ &\leq \frac{d(o', x'_n)}{\lambda'_n} \eta\left(\frac{d(x_n, y_n)}{d(x_n, o)}\right) \\ &= \frac{d(o', x'_n)}{\lambda'_n} \eta\left(\frac{d(x_n, y_n)}{\lambda_n} \frac{\lambda_n}{d(x_n, o)}\right) \\ &\rightarrow 0 \end{aligned}$$

In the second case,

$$\begin{aligned} \frac{d(x'_n, y'_n)}{\lambda'_n} &\leq \frac{d(x'_n, w'_n)}{\lambda'_n} + \frac{d(w'_n, y'_n)}{\lambda'_n} \\ &\leq \frac{d(o', x'_n)}{\lambda'_n} \frac{d(x'_n, w'_n)}{d(x'_n, o')} + \frac{d(o', y'_n)}{\lambda'_n} \frac{d(y'_n, w'_n)}{d(y'_n, o')} \\ &\leq \frac{d(o', x'_n)}{\lambda'_n} \eta\left(\frac{d(x_n, w_n)}{d(x_n, o)}\right) + \frac{d(o', y'_n)}{\lambda'_n} \eta\left(\frac{d(y_n, w_n)}{d(y_n, o)}\right) \\ &\leq \frac{d(o', x'_n)}{\lambda'_n} \eta\left(\frac{d(x_n, w_n)}{\lambda_n} \frac{\lambda_n}{d(x_n, o)}\right) + \frac{d(o', y'_n)}{\lambda'_n} \eta\left(\frac{d(y_n, w_n)}{\lambda_n} \frac{\lambda_n}{d(y_n, o)}\right) \\ &\rightarrow 0 \end{aligned}$$

Therefore, in both cases $x'_\omega = y'_\omega$.

Let $\{x_n\}$ be a sequence representing the origin o_ω . If $\{x_n\}$ is a bounded sequence, then under the assumption over f $\{x'_n\}$ is also bounded and thus $x'_\omega = o'_\omega$.

If $d(o, x_n)$ diverges, then $\{x_n\}$ and $\{o\}$ are u -separated. We can choose $y_\omega \neq o_\omega$, then by lemma 13, $\{x_n\}, \{y_n\}, \{o\}$ are u -separated. Thus,

$$\begin{aligned} \frac{d(o', x'_n)}{\lambda'_n} &= \frac{d(o', y'_n)}{\lambda'_n} \frac{d(o', x'_n)}{d(o', y'_n)} \\ &\leq \frac{d(o', y'_n)}{\lambda'_n} \eta\left(\frac{d(o, x_n)}{d(o, y_n)}\right) \\ &= \frac{d(o', y'_n)}{\lambda'_n} \eta\left(\frac{d(o, x_n)}{\lambda_n} \frac{\lambda_n}{d(o, y_n)}\right) \\ &\rightarrow 0. \end{aligned}$$

Therefore, $o'_\omega = x'_\omega$.

(3) The map $g : \{x_n\} \rightarrow \{f(x_n)\}$ is quasisymmetric and in particular continuous and injective.

Let $\{w_n\}, \{x_n\}, \{y_n\}$ be sequences representing three distinct points w_ω, x_ω and y_ω in $X_\omega(d_n)$. By lemma 13, $\{w_n\}, \{x_n\}, \{y_n\}$ are pairwise u separated. So

$$\frac{d(x'_n, y'_n)}{d(x'_n, w'_n)} < \eta\left(\frac{d(x_n, y_n)}{d(x_n, w_n)}\right),$$

equivalently

$$\frac{d(x'_n, y'_n)}{\lambda'_n} \frac{\lambda'_n}{d(x'_n, w'_n)} < \eta\left(\frac{d(x_n, y_n)}{\lambda_n} \frac{\lambda_n}{d(x_n, w_n)}\right).$$

Taking the ω limit, we get

$$\frac{d_\omega(x'_\omega, y'_\omega)}{d_\omega(x'_\omega, w'_\omega)} \leq \eta\left(\frac{d_\omega(x_\omega, y_\omega)}{d_\omega(x_\omega, w_\omega)}\right).$$

So, if $g : \{x_n\} \mapsto \{f(x_n)\}$ is not constant, then it is continuous and injective. Note that $\{o'\}$ and $\{f(z_n)\}$ are not the same points, so g is indeed not constant. ■

4.3 Examples

4.3.1 Sublinear-Lipschitz equivalences

In [24], Y. Cornulier introduces *sublinear-Lipschitz maps*. A map $f : X \rightarrow Y$ between metric spaces is a sublinear-Lipschitz map if there is an admissible gauge $u : \mathbb{R}_+ \rightarrow \mathbb{R}$ such that

$$d(f(x), f(y)) \leq Cd(x, y) + C'u(|x| + |y|), \quad \forall x, y \in X,$$

for some constants $C, C' > 0$.

Two sublinear-Lipschitz maps f, f' are equivalent if there is an admissible gauge v and a constant $C'' > 0$ such that

$$d(f(x), f'(x)) \leq C''v(|x|)$$

for all $x \in X$.

Sublinear-Lipschitz maps between metric spaces form a category. Taking asymptotic cones, we obtain a functor from the sublinear-Lipschitz category to the Lipschitz category. The sublinear-Lipschitz category is in a sense the maximal category with such a property.

The isomorphisms in the sublinear-Lipschitz category are called sublinear-Lipschitz equivalences or SBE maps.

Proposition 12 *Every SBE-mapping $f : X \rightarrow Y$ is an AM-mapping with linear η .*

Proof: If f is SBE, then f is bi-Lipschitz except at scales below an admissible gauge v .

Indeed, an SBE map satisfies

$$c'd(x, y) - C'u(|x| + |y|) \leq d(f(x), f(y)) \leq cd(x, y) + Cu(|x| + |y|),$$

for some gauge u .

Fix the gauge $v = 2\frac{C'}{C}u$, and let $x, y \in X$ s.t. $x, y > v$. Then

$$\begin{aligned} d(f(x), f(y)) &\leq cd(x, y) + Cu(|x| + |y|) \\ &\leq \left(c + \frac{C'}{2C}\right)d(x, y), \end{aligned}$$

and

$$\begin{aligned} d(f(x), f(y)) &\geq c'd(x, y) - C'u(|x| + |y|) \\ &\geq \frac{c'}{2}d(x, y) \end{aligned}$$

So there exists $D > 0$, s.t. for all $x, y > v$,

$$\frac{1}{D}d(x, y) \leq d(f(x), f(y)) \leq Dd(x, y).$$

In particular, f is an AM-mapping with linear η . ■

4.3.2 Assouad-type maps

How can one recognize when a metric space is bi-Lipschitz equivalent to an Euclidean space? (S. Semmes in [91], 1996)

Simple as it sounds, this question is not obvious. If a metric space admits a bi-Lipschitz embedding in \mathbb{R}^n , then it is clearly doubling. The converse is not true, however; the 3-dimensional Heisenberg group with the Carnot-Carathéodory metric is doubling, but does not admit a bi-Lipschitz embedding in \mathbb{R}^n for any n .

Doubling in the context of a metric space (X, d) means, that there exists a constant $D > 0$ such that for any $x \in X$ and $r > 0$, the ball $B(x, r) = \{y \in X : d(x, y) < r\}$ can be covered by at most D balls of radius $\frac{r}{2}$.

Assouad's embedding theorem [25] [92] states that any snowflake $X^\alpha = (X, d^\alpha)$, $0 < \alpha < 1$, of a doubling metric space admits a bi-Lipschitz embedding in a Euclidean space.

It is clear that the Assouad embedding of a doubling metric space into Euclidean space is an asymptotic-Möbius map. In what follows, we construct an example of an Assouad mapping from an infinite dimensional Heisenberg group into a Hilbert space. This construction follows straightforwardly from a construction of Lee and Naor for the finite dimensional case [93].

Let H be an infinite dimensional complex Hilbert space. H carries the symplectic form $\Omega(a, b) = \text{Im}(\langle a, b \rangle)$.

The *infinite-dimensional Heisenberg group* \mathcal{H}_Ω , is the set of tuples (a, t) with $a \in H$, $t \in \mathbb{R}$ and the group law

$$(a, t)(a', t') = (a + a', t + t' + 2\Omega(a, a')).$$

Let G be a group with identity element e . A group seminorm on G is a function $G \rightarrow [0, \infty)$ satisfying $N(g^{-1}) = N(g)$ for all $g \in G$, $N(gh) \leq N(g) + N(h)$ for all $g, h \in G$ and $N(e) = 0$. Moreover, if $N(g) = 0$ if and only if $g = e$, then N is a group norm on G .

The function

$$N(a, t) = \sqrt{\sqrt{\|a\|^4 + t^2} + \|a\|^2},$$

is a group norm on \mathcal{H}_Ω . Namely, if N_1 and N_2 are group seminorms, then $\sqrt{N_1^2 + N_2^2}$ is a group seminorm. In this case, $N_1 = (\|a\|^4 + t^2)^{\frac{1}{4}}$ is the Koranyi norm. Given N ,

$$d_N((a, t), (a', t')) = N((a, t)(a', t')^{-1})$$

is a right invariant metric on \mathcal{H}_Ω .

Let G be a group, then a Hermitian kernel on G is a complex-valued function $K : G \times G \rightarrow \mathbb{C}$ satisfying $K(g, h) = \overline{K(h, g)}$ for all $g, h \in G$.

A Hermitian kernel on G is positive definite if

$$\sum_{i,j}^n K(g_i, g_j) c_i \overline{c_j} \geq 0$$

for all $g_1, \dots, g_n \in G$ and for all complex numbers $c_1, \dots, c_n \in \mathbb{C}$, with equality if and only if the c_i vanish. A Hermitian kernel on G is conditionally negative definite if

$$\sum_{i,j}^n K(g_i, g_j) c_i \overline{c_j} \leq 0$$

for all $g_1, \dots, g_n \in G$ and for all complex numbers $c_1, \dots, c_n \in \mathbb{C}$ satisfying $\sum_i^n c_i = 0$.

Theorem 13 (Schönberg [94]) *Let G be a group, $K : G \times G \rightarrow \mathbb{R}$ a real-valued kernel on G satisfying $K(g, g) = 0 \forall g \in G$. Then K is conditionally negative definite if and only if there exists a Hilbert space H and a function $T : G \rightarrow H$ such that $\forall g, h \in G : K(g, h) = \|T(g) - T(h)\|^2$.*

Thus, if d_N is conditionally negative definite, then there is a Hilbert space H and an isometry $T : (\mathcal{H}_\Omega, \sqrt{d_N}) \rightarrow H$. In particular, the mapping $T : (\mathcal{H}_\Omega, d_N) \rightarrow H$ is an AM mapping.

A complex-valued function F on a group G satisfying $F(g^{-1}) = \overline{F(g)}$ for all $g \in G$ induces a Hermitian kernel on G by defining $K_F(g, h) := F(gh^{-1})$. The function F is called positive definite if K_F is a positive definite Hermitian kernel on G .

Lemma 15 *For any $\lambda \in \mathbb{R}$, the function*

$$\Phi_\lambda(a, t) = e^{-|\lambda|\|a\|^2 + i\lambda t}$$

is positive definite on \mathcal{H}_Ω .

Proof: Indeed, the function Φ_λ satisfies the condition $\Phi_\lambda((a, t)^{-1}) = \overline{\Phi_\lambda((a, t))}$ and induces the kernel

$$K((a, s), (b, t)) := \exp(-|\lambda|\|a - b\|^2 + i\lambda(s - t - 2\Omega(a, b))).$$

The kernel can be rewritten as a product of three exponentials

$$\exp(-|\lambda|(\|a\|^2 + \|b\|^2)) \exp(i\lambda(s - t)) \exp(2|\lambda|(\operatorname{Re}(\langle a, b \rangle) - i \operatorname{sign}(\lambda) \Omega(a, b))).$$

The product of positive definite kernels is positive definite. The first two factors are positive definite because for all $c_1, \dots, c_n \in \mathbb{C}$ the matrix $(c_i \overline{c_j})_{i,j}$ is positive semidefinite. In particular, also the matrix $(\exp(i\lambda(s_i - t_j)) c_i \overline{c_j})_{i,j}$. The exponential of any positive definite kernel is again a positive definite kernel ([95] Proposition 8.2.). Thus the kernel K is positive definite if

$$\operatorname{Re}(\langle a, b \rangle) - i \operatorname{sign}(\lambda) \Omega(a, b)$$

is positive definite. This is $\langle a, b \rangle$ if $\lambda < 0$ and $\overline{\langle a, b \rangle}$ if $\lambda \geq 0$ and thus clearly positive definite. \blacksquare

Proposition 13 $d_N : \mathcal{H}_\Omega \times \mathcal{H}_\Omega \rightarrow H$ *is conditionally negative definite.*

Proof: The existence of $\frac{1}{2}$ -stable distributions implies that for all $\epsilon > 0$ there exists a non-negative integrable function $\varphi_\epsilon : \mathbb{R} \rightarrow [0, \infty)$ s.t. its Fourier transform $\hat{\varphi}_\epsilon(t) = e^{-\epsilon\sqrt{|t|}}$. Note that $\frac{1}{2\pi}\hat{\hat{\varphi}}_\epsilon(x) = \varphi_\epsilon(x)$.

By the above lemma,

$$F_\epsilon(a, t) = \int_{\mathbb{R}} e^{-|\lambda|\|a\|^2 + i\lambda t} \varphi_\epsilon(\lambda) d\lambda$$

is positive definite on \mathcal{H}_Ω . Let

$$h_k(x) = \frac{k}{\pi} \frac{1}{k^2 + x^2}$$

denote the Cauchy distribution with scale parameter $k > 0$. We write around

$$\begin{aligned} F_\epsilon(a, t) &= \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\lambda t} \widehat{h}_{\|a\|^2}(\lambda) \widehat{\varphi}_\epsilon(\lambda) d\lambda \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\lambda t} \widehat{h_{\|a\|^2} * \widehat{\varphi}_\epsilon}(\lambda) d\lambda \\ &= (h_{\|a\|^2} * \widehat{\varphi}_\epsilon)(t). \end{aligned}$$

Since F_ϵ is positive definite, $\frac{1-F_\epsilon}{\epsilon}$ is conditionally negative definite and

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \frac{1 - F_\epsilon(a, t)}{\epsilon} &= \lim_{\epsilon \rightarrow 0} \left[h_{\|a\|^2} * \frac{1 - \widehat{\varphi}_\epsilon}{\epsilon} \right] \\ &= \lim_{\epsilon \rightarrow 0} \int_{\mathbb{R}} \frac{1 - e^{-\epsilon\sqrt{|x|}}}{\epsilon} h_{\|a\|^2}(t - x) dx \\ &= \frac{\|a\|^2}{\pi} \int_{\mathbb{R}} \frac{\sqrt{|x|}}{\|a\|^4 + (t - x)^2} dx. \end{aligned}$$

We now show that for all $r, t \in \mathbb{R}$

$$r^2 \int_{\mathbb{R}} \frac{\sqrt{|x|}}{r^4 + (t - x)^2} dx = \pi \sqrt{\sqrt{r^4 + t^2} + r^2}.$$

By changing variables $x = r^2 y$ and $s = t/r^2$, the left hand side can be written as

$$\int_0^\infty \left(\frac{1}{(1 + (s - y)^2)} + \frac{1}{(1 + (s + y)^2)} \right) \sqrt{y} dy$$

This integral is equal to

$$\lim_{r \rightarrow 0} \lim_{R \rightarrow \infty} \frac{1}{2} \int_{C_{r,R}} \left(\frac{1}{1 + (s - z)^2} + \frac{1}{1 + (s + z)^2} \right) \sqrt{z} dz,$$

where $C_{r,R}$ is the keyhole contour with a branch cut along the positive real axis. The integrand has simple poles at $i \pm s$ and $-i \pm s$. By $\text{Res}\left(\frac{\sqrt{z}}{1+(s-z)^2}, i+s\right) = \frac{\sqrt{i+s}}{2i}$, $\text{Res}\left(\frac{\sqrt{z}}{1+(s-z)^2}, -i+s\right) = \frac{\sqrt{-i+s}}{-2i}$, $\text{Res}\left(\frac{\sqrt{z}}{1+(s+z)^2}, i-s\right) = \frac{\sqrt{i-s}}{2i}$ and $\text{Res}\left(\frac{\sqrt{z}}{1+(s+z)^2}, -i-s\right) = \frac{\sqrt{-i-s}}{-2i}$. Now adding all the residuals and multiplying by $2\pi i$, we obtain the above equation as

$$\frac{\pi}{2} (\sqrt{i+s} - \sqrt{-i+s} + \sqrt{i-s} - \sqrt{-i-s}).$$

This further simplifies to

$$\begin{aligned} \pi \text{Re}(\sqrt{i+s} + \sqrt{i-s}) &= \pi (\text{Re}(\sqrt{i+s}) + \text{Im}(\sqrt{i+s})) \\ &= \pi \left(\sqrt{\frac{\sqrt{1+s^2} + s}{2}} + \sqrt{\frac{\sqrt{1+s^2} - s}{2}} \right) \\ &= \pi \sqrt{\sqrt{1+s^2} + 1} \end{aligned}$$

So we have seen that the N -norm is a limit of conditionally negative definite functions on \mathcal{H}_Ω , so it is conditionally negative definite. \blacksquare

4.4 Applications to dimension theory

4.4.1 Dimension theory of finitely-generated groups

In 1993 M. Gromov introduced the notion of asymptotic dimension as a large scale analogue of Lebesgue's covering dimension [36]. The asymptotic dimension of a finitely generated group is a quasi-isometric invariant. Its most prominent application goes back to Guoliang Yu, who showed that any finitely generated group with finite homotopy type and finite asymptotic dimension satisfies the Novikov conjecture [26].

Definition 17 *Let X be a metric space. We say that the asymptotic dimension of X does not exceed n if for every uniformly bounded open cover \mathcal{V} of X there exists a uniformly bounded open cover \mathcal{U} of X of multiplicity $\leq n + 1$ such that \mathcal{V} refines \mathcal{U} . We write $\text{asdim}X = n$ if it is true that $\text{asdim}X \leq n$ and $\text{asdim}X \not\leq n - 1$*

The asymptotic dimension of a finitely generated group Γ has several interesting implications. For example, $\text{asdim} \Gamma = 0$ if and only if Γ is finite (Proposition 65. in [27]) and $\text{asdim} \Gamma = 1$ if and only if Γ is virtually free (Theorem 66. in [27]).

If G is a finitely generated nilpotent group, then its asymptotic cone G_ω is a Carnot group [28] and the asymptotic dimension of G coincides with the topological dimension of its asymptotic cone [27].

Theorem 14 *Let G and G' be a finitely generated nilpotent groups and $f : G \rightarrow G'$ an AM-map preserving diverging and bounded sequences, then $\text{asdim}(G) \leq \text{asdim}(G')$. If G and G' are simply connected nilpotent Lie groups, then $\dim(G) \leq \dim(G')$. Furthermore, if $\text{asdim}(G) = \text{asdim}(G')$, then the asymptotic cones of G and G' are isomorphic graded Lie groups. Conversely, given nilpotent groups with isomorphic asymptotic cones, there exists an asymptotic-Möbius map between them.*

Proof: The mapping f induces a quasisymmetric mapping g between the asymptotic cones G_ω and G'_ω of G and G' , respectively.

In particular, g is an injective mapping from G_ω to G'_ω and hence the topological dimensions increase. Since G and G' are nilpotent, $\text{asdim}(G) = \text{top-dim}(G_\omega) \leq \text{top-dim}(G'_\omega) = \text{asdim}(G')$.

The case for nilpotent Lie groups follows from the fact that for nilpotent Lie groups $\text{top-dim}(G) = \text{top-dim}(G_\omega)$.

If $\text{asdim}(G) = \text{asdim}(G')$, the invariance of domain theorem implies that g is a homeomorphism. Since both g and g^{-1} are quasisymmetric, they are differentiable by Pansu's theorem a.e., and the differential of g is a graded isomorphism of groups at almost every point [96].

The converse follows from Cornulier [97, Proposition 2.9] saying that the asymptotic cones of G and G' are bi-Lipschitz equivalent if and only if there exists an SBE map between G and G' . In particular, there exists an asymptotic Möbius map between them. ■

4.4.2 Dimension theory of CAT(0)-spaces

Heinrich Jung established in the early 1900s that the diameter of a set in Euclidean space \mathbb{R}^n is related to the radius of the smallest enclosing ball by an inequality whose constant depends only on the dimension n [29]. More precisely, for any set $A \subset \mathbb{R}^n$

$$r \leq d(A) \sqrt{\frac{n}{2(n+1)}}, \quad (4.1)$$

where r is the radius of the smallest enclosing ball of A and $d(A)$ is the diameter of A .

The *geometric dimension* of a CAT(0)-space X is the maximal topological dimension of all compact subsets in X [30].

Remark 6 *It turns out that the geometric dimension of a CAT(0)-space X is less than n if and only if Jung's inequality (4.1) is true in X [31].*

The geometric dimension can be transformed into a large-scale term in the following way. A CAT(0)-space X has telescopic dimension less than n if and only if for every $\delta > 0$ there is a $D > 0$ such that for every set A in X with diameter greater than D ,

$$r \leq \left(\delta + \sqrt{\frac{n}{2(n+1)}} \right) d(A). \quad (4.2)$$

Somewhat more general,

Definition 18 *A space X whose asymptotic cones are CAT(0) has telescopic dimension $\leq n$ if every asymptotic cone has geometric dimension $\leq n$.*

We adopt the notation $\text{tele-dim}(X)$ for the telescopic dimension of X .

Theorem 15 *Let X and Y be spaces, whose asymptotic cones are CAT(0). If there exists an AM-map $f : X \rightarrow Y$ preserving diverging and bounded sequences, then the telescopic dimension increases e.g. $\text{tele-dim}(X) \leq \text{tele-dim}(Y)$.*

The theorem follows straight from theorem 12, remark 6 and the fact that the topological dimension increases under injective continuous maps.

It follows that there can be no AM-mapping from Euclidean buildings of finite rank r and hence telescopic dimension r to a Euclidean building of lower rank.

If there is a AM-mapping from a CAT(0)-space to a hyperbolic metric space, then X is also a hyperbolic metric space.

Appendix A

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