HAL
open science

Representations up to homotopy and perfect complexes over differentiable stacks

Davide Stefani

To cite this version:

Davide Stefani. Representations up to homotopy and perfect complexes over differentiable stacks. Algebraic Topology [math.AT]. Sorbonne Université, 2019. English. NNT : 2019SORUS687 . tel-04166814

HAL Id: tel-04166814

https://theses.hal.science/tel-04166814

Submitted on 20 Jul 2023

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Sorbonne Université

École doctorale de sciences mathématiques de Paris centre

Thèse de doctorat

Discipline : Mathématiques

présentée par

Davide STEFANI

Representations up to homotopy and perfect complexes over differentiable stacks

dirigée par Grégory Ginot

Soutenue le 19 Juin 2019 devant le jury composé par:

Mme Sophie CHEMLA	Sorbonne Université	examinatrice
M. Grégory GINOT	Sorbonne Université	directeur
M. Julien GRIVAUX	Sorbonne Université	examinateur
M. Camille LAURENT-GENGOUX	Université de Lorraine	examinateur
M. Hiro Lee TANAKA	Texas State University	examinteur

Au vu des rapports de:

M. Damien CALAQUE	Université de Montpellier
Mme Chengchang ZHU	Georg-August-Universitat Gottingen

Institut de mathématiques de Jussieu-Paris Rive gauche. UMR 7586.
Bote courrier 247
4 place Jussieu
75 252 Paris Cedex 05

Sorbonne Université, Campus Pierre et Marie Curie.
école doctorale de sciences mathématiques de Paris centre.
Bote courrier 290
4 place Jussieu
75 252 Paris Cedex 05

To change the world you must:
1. change yourself;
2. AND change the world.

Unknown, on a wall of Longo Maï

Acknowledgments

5 years ago, I was sitting with Henrique Burstyn on the sofas of IMPA, surrounded by the macacos of the jungle near Rio de Janeiro, questioning my choice to start a PhD. It has always been hard to me to find a sense to the research and I found my answer only in the last months: the PhD has been a wonderful opportunity to give a deeper reason to my passion for mathematics, to explore my self and the world around me, to build my personality, take consciousness of the society we are existing in and of the place I have and I would have inside it (or outside it). These pages are a unique opportunity to acknowledge all the people I came across throughout the last 4 years.

First of all I would like to thank my advisor Gregory Ginot. Since my first steps in research, he has been extremely available and kind, he spent countless hours discussing with me, shearing ideas and encouraging me. Even more important, he never judged or criticized my unusual choices and he let me the freedom to choose my own way.
I am also especially thankful to all those people who spent their time to convince me about some maths: Mauro Porta, who always found a good answer to my questions (sometimes not in the same univers that I asked it); Mathieu Anel, one of the more inspiring mathematicians I have ever met in my life; Henrique Bursztyn, who welcomed me 3 times in IMPA for over than 14 months in total; Matias Del Hoyo, Marco Robalo, Damien Calaque, Christian Ortiz, Mattia Bergomi, Moreno Andreatta. A special thanks to the referees, Damien Calaque and Chengchan Zhu, for their comments, suggestions and the time they spent to read this work.
Thanks to Matteo Pennesi, the best collaborator ever, its futuristic vision of life and music has deeply influenced my thought in the period spent together in Strasbourg. Merci aux cobureaux: Léo, Adrien, Jesua, Mathieu, Maya, tout-e-s les autres thésard-e-s du couloir 15-16.

Je pense que l'enseignement était une des expérience desquelles j'ai tiré le major profit: merci à tout-e-s mes étudiant-e-s pour l'énorme travail de déconstruction et reconstruction que j'ai du faire sur moi même pour pouvoir vous apporter ce dont vous aviez besoin. Vous avez été une des meilleurs expériences des ces derniers 4 ans.

Mon pèlerinage m'a amené à rencontrer beaucoup des personnes merveilleuses en dehors de l'académie: je tiens à coeur d'en remercier quelques unes ici pour ce qui m'ont apportés ces 4 derniers années.
Merci à tous les professeurs et guides: à Pierre, une des personnes qui plus m'a appris de ma vie. Je me rappel encore de ce jour en Octobre 2015 quand je suis venu te voir à Feeling croyant de te montrer que j'allais apprendre vite la trompette. Merci pour me faire mettre chaque semaine la barre plus haute, pour avoir toujours cru en moi, pour la dimension supérieure que t'as donné à mes actes de tous les jours et pour la constance de ton travail; à Jean-Daniel Hege, merci pour nos conversations et pour la passion que tu mets dans ce que tu fais; à Alban pour ses idées et pour m'avoir fait chanter dans ma trompette; à Dawid pour avoir réveillé mon oreille, grazie a Andy per avermi fatto ritrovare il mio io più saggio; à Philippe Aubry et Michael Alizon, pour ne jamais avoir cru en moi, et m'avoir appris tout ce que je ne cherche pas dans la musique. Grazie alla Terry per aver camminato mano nella mano, a volte senza saperlo, per tutti questi anni. Merci au musicien-ne-s rencontré-e-s à Strasbourg: à Clarissa per la sua gioia e la bellezza che ha negli occhi, à Gui Broglie pour m'avoir fait sortir le premier vrai puet en publique, à Nicolas pour les sessions de trompette, à Clement pour l'intensité des heures passés ensembles, à Marcelo pour mon premier trio et à tous les étudiants du conservatoire de Strasbourg pour m'avoir inspiré et fait prendre des grosses claques.

Obrigado a Mariana, Joana e todos os músicos da Orquestra Voadora, e de todas as fanfaras incríveis que eu encontrei nos quatros carnavais que eu fiz la no Rio. Obrigado pra ter plantado essa semente maravilhosa na minha alma.

Grazie ai bonobi per tutto quello che non posso scrivere qui: a Jared, per le mille porte che mi hai fatto aprire, per i controtempi quantistici alle 9 del mattino, per le esperienze estreme, per la saggezza infusa, per tutto quello che si vivrà ancora assieme; al mio fratello maggiore Giulio, per avermi insegnato a cucinare, a prendere il tram almeno a 45 gradi, a conoscerli e a dominarli; a Paco, per la loucura che mette in ogni atto della sua vita e di quelle degli altri; a Rembado, per avermi mostrato che i limiti dell'ingegno umano sono molto più lontani di quanto pensassi; à Fred, pour le courage de ses choix et pour avoir vu en direct son passage de physicien théorique à maçon de bal trad; a Valerio e Vito per la loro maestosa balordaggine; a Nico per la sua Zenitudine onnipresente. Grazie a tutta la famiglia dei rital per gli innumerevoli momenti prima di festa e poi di gastronomia vissuti assieme a Parigi: Ritozza, Joseba, Marzia, Mattia, Marco, Camilla, Giulia, Fabrizio, Federico, Martino...

Grazie a Inès, per 9 dei mesi più intensi della mia vita messi nero su bianco; à Margot, pour notre âme hippie dans un corps d'intellos beaugosses; a Domi, per la sua bellezza, la semplicità e l'energia che fa esplodere attorno a se; à Marion, pour

m'avoir appris le français, à Marianne, pour m'avoir appris la folie; grazie a Chiara, per i viaggi intergalattici, la connessione senza fili e il techno bloco più potente della storia; obrigado a Alice, a melhor crackuda do mundo, merci pour avoir vécu Lapa et le Brésil ensembles; Francesca, per le sue poesie, per esserci compresi nel profondo, per aver trovato un altro pezzetto di me.

Grazie a tutti i coinquilini e a tutte le persone che mi hanno ospitato nel mio errare : a Anna per essermi venuta a trovare 10 giorni e poi avermi ospitato 4 mesi (c'è da dire che ognuno di quei 10 giorni a Rio valeva un mese a Parigi), a Jerome, che a rinunciato alla proprietà privata per causa di forze maggiori (SAID!), e a Said per aver fatto da capro espiatorio al posto mio! Merci à Clara, pour ses histoires e a Fra per le chiacchierate filosofiche; grazie a Silvia, ufficialmente la migliore coloc del mondo e a Bruce D'anis pour ses BD et son art qui m'ont fait voyager. Un énorme merci aux colocs de la zonme de Montreuil, l'endroit le plus paisible d' Île de France: à Tiftif pour sa vitalité inépuisable, à Johanna pour m'avoir envoyé que de l'energie de super bonne qualité et Josephine pour avoir déjà trouvé sa place dans ce joyeux bordel. J' espère que vous allez prendre soin de ce coin de bien Être que vous avez créé au sein d'une ville aussi folle que Paris. Un grand merci à la dame des crêpes, qui a su toute seule imposer ses rythmes au va-et-vient frénétique de Jussieu.

Merci aux ami-e-s musicien-ne-s: à la fanfare invisible pour tout ce que m'a amené à voir et à comprendre de notre pauvre humanité malade, il faudra une fanfare invisible dans chaque coin du monde pour vivre ensembles l'effondrement; aux Riso Patate et Moules, à la Gab, les Laburwadima et à Los Tardos pour la musique vécue ensembles et à venir; grazie alla zietta Lisa per i nostri corsi di tromba punk; une énorme merci à la famille Kikafessa: à Charlie pour sa musique, sa présence, ses prestations de 12 heures nonstop, à Jojo parce que sans que personne l'entende, "à 4 heures du mat elle fait tout", à JB pour sa génialité indiscutable, à Teo per suonare di merda da Dio, à Matis pour me rappeler qui j'étais il y a 5 ans, à Ian pour la calme et la bonté (qu'il oublie toujours dès qu'il touche une grosse caisse), à Prune pour le jeûne qu'elle a fait malgré elle, à Jeanne pour son humour unique, à Alex, pour ce truc en commun, qu'on traites chacun à sa propre manière, a Nico e Lorenzo per aver riequilibrato la bilancia da Franco-Italiana a Italo-Francese. Je dois avouer d'être trop fier d'avoir donné l'élan initial à un aussi beau groupe, et je sais qu'on ira très loin ensembles.

Ai vecchi pilastri Sassolesi, Tax, Simon, la Sara, Frenky, la Meddi, la Dadda, la cricca di Radio Radura.

Grazie alla mia famiglia, in primis alla nonna Franca, la Nonna rosanna e la nonna Maria per la saggezza che portano e hanno portato nei loro volti, nelle loro parole e nelle loro vite. Grazie ai miei genitori, per l'amore, il supporto, la fiducia e per

l'enorme coraggio di sapersi rimettersi in discussione a 50 (e più) anni suonati di fronte a dei figli che partono così lontano. Grazie al bimbo d'oro, il modello 4.0 di gran lunga più performante di tutti i precedenti, alla Je per la sua velata tenerezza, alla Silvia per avermi reso femminista, anticapitalista, anarchico, collassologista e per aver contribuito alle mie più grandi prese di coscienza di questi ultimi anni, alla Lilli per la nostra storia insieme passata, presente e futura.

Il reste que toi, le remerciement le plus difficile, car je sais très bien de quoi te remercier, mais (comme souvent dans la vie) je ne sais pas l'exprimer avec mes mots. Du coup j'utiliserais les tiennes, toujours de loin plus sages et bien choisies: merci Lucie, pour m'avoir contaminé avec ton être, qui vit et grandit dans moi.

Abstract

This thesis is concerned with the geometry of stacks in the differential geometry context using homotopical and higher categorical techniques. These techniques becomes necessary to deal with simple stack generalizations of crucial objects such as tangent and cotangent bundles, forms on a stack, their automorphisms and more generally perfect complexes, which are one of the main object of study of this work.

In the first part of this thesis we give an overview of higher and differentiable stacks, their homotopy theory and cohomology theories. In the second part we study one representation up to homotopy of Lie groupoids and rely them with a theory of perfect complex over differentiable stacks. Among our results, we show that a representation up to homotopy on a Lie groupoid is the same as a cohesive module on its dg-algebra of smooth functions and that the correspondent dg-categories are Morita invariant. This allows us to give a definition of dg-category of perfect complexes on a differentiable stack. We moreover construct a Lie 2-groupoid of automorphisms of 2-terms complexes of vector bundles, which is a higher analogue of the classifying stack BGL_n. We conclude by giving a definition of the differentiable 2-stack of perfect complexes of amplitude $[0, 1]$ by means of a Lie 2-groupoid presenting it.

We start by an overview of the motivations behind geometric (higher) stacks, and of the homotopical notions which are crucial in a rather general context. Then we focus in more details on the context of this thesis and the results we obtained.

Key words

Lie groupoids, differentiable stacks, perfect complexes, representations up to homotopy, 2-categories, smooth nerve.

Contents

1 Stacks **25**

1.1 Categories fibered in groupoids 25
1.2 Stacks in homotopy theory 27
1.3 Higher stacks and simplicial presheaves 29
1.3.1 Jardine model structure on simplicial presheaves 29
1.3.2 Higher stacks 30
1.4 Quotient stacks 31
1.4.1 Groupoid objects 31
1.4.2 Torsors on groupoids 33
1.5 The 2-category of differentiable stacks 37
1.5.1 Lie groupoids and Stacks 38
1.5.2 The quotient stack of a Lie groupoid 41
1.5.3 Differentiable (1-)stacks 41
1.6 Some constructions for Lie groupoids 44
1.6.1 The nerve of a Lie groupoid 45
1.6.2 The Lie algebroid of a Lie groupoid 45
1.7 The 2-category of Lie groupoids 46

2 Cohomology theories for stacks **49**

2.1 Cohomology theories 49
2.1.1 Sheaf cohomology 50
2.1.2 Cohomology of coherent sheaves 51
2.1.3 Hypercohomology 51
2.1.4 Čech cohomology 51
2.1.5 De Rham cohomology 52
2.1.6 Hodge to De Rham spectral sequence 53
2.2 Cohomology of stacks 53
2.2.1 Sheaf cohomology of stacks 53
2.2.2 Čech cohomology of stacks 54
2.2.3 De Rham cohomology of stacks 55
2.3 The example of equivariant cohomology 56

2.3.1 The cotangent complex of an action Lie groupoid 56
2.4 Hodge to De Rham spectral sequence, first version 58

3 Representations up to homotopy 60
3.1 Vector bundles over stacks 60
3.2 Representations of groupoids 63
3.2.1 The general linear groupoid 65
3.3 Cohesive modules over dg-algebras 67
3.3.1 The category of graded objects 67
3.3.2 Dg-algebras, dg-modules and dg-categories 67
3.3.3 The dg-category $\mathcal{P}_{\mathcal{A}}$ 69
3.3.4 The dg-algebra of smooth functions $C^\bullet(G)$ 70
3.4 Representations up to homotopy 71
3.5 Ruth as cohesive modules over $C^\bullet(G)$ 72
3.6 The tensors D_i 75
3.7 Homotopy theory of representations up to homotopy 76
3.8 Morita invariance of $Rep^\infty(G)$ 76
3.9 Symmetric powers of representations up to homotopy 79
3.10 The tangent and cotangent complexes 80
3.11 Hodge to De Rham spectral sequence for Lie groupoids 81
3.11.1 Bott spectral sequence for differentiable stacks 82

4 The general linear 2-groupoid 83
4.1 Basics on 2-categories 83
4.2 The nerve of a 2-category 86
4.3 Defining Lie 2-groupoids 90
4.4 The general linear 2-groupoid 92
4.5 The nerve of a Lie 2-category 95
4.6 Representations as pseudo-functors 98
4.7 The stack $Perf_{[0,1]}$ 101

A Grothendieck sites, simplicial preasheaves and hypercovers 105
A.1 Simplicial sets 105
A.2 The Dold-Kan correspondence 106
A.3 Simplicial presheaves and hypercovers 106
A.4 Grothendieck sites and sheaves 108

Introduction

A homotopy theory for stacks

A modern point of view on algebraic geometry is to consider varieties and schemes as contravariant functors from the category of rings to the category of sets. More precisely, a scheme X is uniquely determined by its functor of points

$$\begin{array}{rcl} Sch^{op} & \to & Sets \\ T & \mapsto & \mathrm{Hom}(T, X) \end{array} \tag{1}$$

The local models for schemes are **spectra** of rings. Since the functor Spec : $Ring^{op} \to Sch$ is fully faithfull, we can treat the opposite category of rings as a subcategory of the category of scheme, and we call it the subcategory of affine schemes, denoted by Aff. This terminology is due to the fact that affines are the local model for general schemes. Now, the functor $\mathrm{Hom}(\cdot, X)$ of eq. (1) is uniquely determined by its value on **affine schemes**

$$\begin{tikzcd} Sch^{op} \arrow[r] & Sets \\ Aff^{op} \arrow[u, hook] \arrow[ur] & \end{tikzcd}$$

In few words, a scheme X is the same as the functor

$$\begin{array}{rcl} Aff^{op} & \to & Sets \\ R & \mapsto & \mathrm{Hom}(\mathrm{Spec}(R), X) \end{array}$$

Whenever a functor is naturally equivalent to the functor of point of an affine scheme, we will call it **representable**.

Now the question is how to go the other way round: given a functor $F : Aff^{op} \to Sets$ can we always find a scheme X such that

$$F(R) = \mathrm{Hom}(\mathrm{Spec}(R), X)?$$

To answer this question we need the notion of sheaves on categories.
It is well known that a presheaf on a topological space X is nothing more than a

contravariant functor from the category $Ouv(X)$ of open subests of X with inclusions as morphisms (i.e. $\mathrm{Hom}_{Ouv(X)}(U,V)$ is the singleton if $U \subset V$ and empty otherwise). Then a presheaf is a sheaf if and only if it satisfies a descent condition expressing that objects and morphisms can be constructed locally (see Def. 1.1.4).
The idea is to put a richer structure on the category Aff, that of a **Grothendieck site**, which is essentially the data of a topology on the category. In the algebraic case, we will need to choose an appropriate topology which is finer than the Zarisky, for example the étale, the fpqc or the fppf (cf. [61]).
Then a contravariant functor $F : Aff^{op} \to Sets$ is a presheaf for this topology, and schemes turn out to be exactly those presheaves which:

1. satisfy a **descent condition** for this topology, i.e. they are sheaves over the category Aff;
2. they admit an **algebraic (or geometric) atlas**, i.e. there are affines $\{R_i\}$, and a morphism $\coprod R_i \to F$ which is geometric, in the sense that for any affine S and morphism $S \to F$, the pullback $\coprod R_i \times_F S$ is again representable.

We put the previous ideas in the following definition:

Definition. *A scheme (or algebraic space) is a presheaf over the category $Aff = Ring^{op}$ such that:*

1. *It is a sheaf for any of the étale/fpqc/fppf topology;*
2. *it admits an algebraic atlas.*

This functorial point of view gave a new insight in the definition of stack. Stacks are geometric spaces whose local models are not affines anymore, but quotients of group actions on affines. They were introduced because of a lack of existence of moduli spaces, in particular for those moduli problem where points admit non-trivial automorphisms.

Example. Let $Diff$ be the category of differentiable manifolds, and let

$$Vect_n : Diff^{op} \to \mathcal{S}et$$

be the functor which sends a differentiable manifold M to the set of isomorphism classes of rank n vector bundles on M. A **moduli space** for $Vect_n$ is a space X such that for any manifold U, we have $Vect_n(M) = \mathrm{Hom}_{Diff}(M, X)$.

Since a morphism between manifolds is determined by its value on the neighborhood of each point, a representable functor in $Diff$ is a sheaf (we say that the topology on $Diff$ is *subcanonical*). Clearly this is not the case for $Vect_n$. In fact, for all manifolds M, if $\{U_i \to M\}$ is a trivializing family for M, then $Vect_n(U_i) = *$, the

one-point set, so for $Vect_n$ to be a sheaf would mean that every manifold admits only one isomorphism class of vector bundles of a fixed rank. The problem is that all vector bundles are locally isomorphic, but thanks to those isomorphisms one can "twist" a trivial vector bundle to obtain a non trivial one. So there is no fine moduli space for $Vect_n$ in the category of smooth manifolds $Diff$. We will see in prop. 3.1.5 that the right notion of moduli space in this case is the differentiable stack BGL_n. This kind of example was one of the main motivations for studying stacks and their geometry.

Stacks were originally defined as fibered categories satisfying some descent conditions (see def. 1.1.4), meaning that objects and morphisms can be constructed locally and glued together whenever some cocycle conditions were respected (cf. [61]). Roughly speaking, a fibered category $p : X \to C$ is a functor between categories X and C, such that the fiber $p^{-1}(U)$ over any object $U \in C$ is a groupoid. If we switch to the functorial viewpoint that we described above, we can think of a fibered category as a lax presheaf of groupoids over the category C. Then the Grothendieck construction (see thm. 1.2.1) allows to strictify those lax presheaves to actual presheaves of groupoids.

Homotopy theory provided a good framework to understand this construction, and for the development of stack theory and higher stacks. The work of Hollander (cf. [36],[23]) completed the passage to the homotopical word, by stating the descent condition in terms of homotopy theory:

Definition ([36] Def. 1.3)**.** *Let $\mathcal{C}$ be a Grothendieck site. A presheaf of groupoids F on $\mathcal{C}$, is a stack if for every cover $\{U_i \to X\}$ in $\mathcal{C}$, there is an equivalence of categories*

$$F(X) \xrightarrow{\sim} \operatorname{holim}\left(\coprod F(U_i) \rightrightarrows \coprod F(U_{ij}) \rightrightarrows \coprod F(U_{ijk}) \dots\right) \tag{2}$$

The right hand side of the equation is the homotopical limit of the simplicial diagram induced by applying F to the Čech nerve of the covering $\{U_i\}$, which is the more appropriate functorial approximation to the limit for diagrams of groupoids. It is easy to imagine that groupoids are organised in a sort of 2-category, with morphisms the functors between groupoids and 2-morphisms the natural transformations. In this world commutativity of diagrams makes no longer sense and must be replaced by commutativity up to a natural isomorphism. That's why the homotopical limit is required.

(1-)stacks are informally called *sheaves of homotopy 1-types*. The idea is that from a groupoid $\mathcal{G}$ we can construct a topological space by taking a 0-cell (a point) for each object in $\mathcal{G}$, a 1-cell (a path) for each morphism and a 2-cell for each composition. Of course this assignement is well defined only up to homotopy, and equivalent groupoids turn out to give homotopy equivalent topological spaces: indeed groupoids are a model for homotopy 1-types (cf. [28]).

The next step is to ask ourselves what is a sheaf of homotopy n-types, or more generally topological spaces (modulo homotopy equivalences). The **homotopy hypothesis** is the assertion that ∞-groupoids are equivalent to topological spaces and that n-groupoids are equivalent to homotopy n-types.

Example (Homotopy hypothesis for 1-types)**.** The (strict) 2-category of groupoids, functors, and natural transformations is equivalent to the (strict) 2-category of homotopy 1-types, continuous maps, and homotopy classes of homotopies.

Note that the homotopy hypothesis becomes a theorem or a tautology, depending on which definition of ∞ (resp. n)-groupoid one chooses.
Those questions were fruitfully adressed with the language of simplicial presheaves: they represent the best framework to study higher stacks, because they overcome the technical difficulties of the theory of n-groupoids (we will see some of them in chapter 4 for the case $n = 2$) and they bring model category in the setting of higher category theory (we can think at model categories as one of the incarnations of the abstract idea of ∞-categories, with the addictional choice of a fibrant replacement). In this framework ∞-groupoid are defined as Kan complexes, i.e. simplicial sets such that any k-horn can be filled by a k-simplex for any k. This property is usually called *k-horn filling condition*, and the ambiguity we find in the ∞-categorical word is that the filling, which represents the k-th dimensional composition, is not unique.

Example. In the case $k = 2$, this means that in the diagram below

$$\begin{tikzcd} & x_1 \arrow[dr, "f_{12}"] \arrow[d, phantom, "\Uparrow H"] & \\ x_0 \arrow[ur, "f_{01}"] \arrow[rr, "f_{02}"'] & {} & x_2 \end{tikzcd}$$

if we have two of the three morphism f_{ij}, then we can find a third morphism completing the triangle, and a homotopy $H : f_{02} \Rightarrow f_{12} \circ f_{01}$. In this example the existence of f_{02} just say that a composition exists, the existence of f_{01} and f_{12} says that we can invert morphisms. Those are the conditions defining a (lax) groupoid.

Homotopy n-types are Kan complexes K which are n-truncated, in the sense that for any $k > n$ there is an isomorphism between k-simplexes and k-horns in K (see [42] for a bible of ∞-category theory, [62] and [64] for more down to the ground definitions of n-groupoids using Kan complexes). In this case, (1-)groupoids are 1-truncated simplicial sets with horn filling conditions in any dimension.

In order to extend the definition of stacks to higher dimensions, we need to put a model structure on the category of simplicial sets and consequently on that of simplicial presheaves. In fact, the homotopy limit (2) must be computed on a

diagram of fibrant objects, and not all simplicial sets are fibrant. There is a model category structure on simplicial sets for which fibrant objects are Kan complexes; this induces a projective model structure on simplicial presheaves, where fibrations and weak equivalences are defined objectwise (see Appendix A.1 for more details). This machinery was not necessary for groupoids because they are organized in a model category were all objects are fibrant and cofibrant.
Once one is convinced that groupoids are a special case of simplicial sets, and that Kan complexes are a good model for topological spaces, it is easy to generalize the concept of *stack* (in the viewpoint of [36]) to higher dimensions:

Definition. *A simplicial presheaf $F : \mathcal{C} \to sSet$ is a stack if any hypercover $U_\bullet \to X$ induces a homotopy equivalence*

$$F'(X) \xrightarrow{\sim} \operatorname{holim}\left(\coprod F'(U_0) \rightrightarrows \coprod F'(U_1) \rightrightarrows \coprod F'(U_2) \dots\right)$$

where F' is a fibrant replacement of F.

In the definition, hypercovers must be thought as a generalization of Čech covers to higher dimensions.
Finally, one can localize the projective model structure on simplicial presheaves at hypercovers, to obtain a new model structure whose fibrant objects are exactly stacks (see Thm. 1.3.6).

Differentiable stacks and Lie groupoids

The interest in stacks is mostly due to the geometric structure one can put on it. In the previous section we never used the fact that we are working in the category of commutative rings, and all the results we exposed would be true for any Grothendieck site Aff, and simplicial presheaves on it. All the geometry of the stacks sits inside the category of affines: instead of $Ring^{op}$ one could choose $k-Mod$ (k-modules), $sk-Mod$ (simplicial k-modules), DGA (dg-algebras), $Diff$ (smooth manifolds) and so on. The geometric stacks on those categories are modeled on the geometry of affines, and on the maps that one allows for pasting them together. A stack is geometric if it admits a geometric atlas, which is a sort of "open cover by affines".
The general theory of higher geometric stacks has been extensively developped in [57] and [42]. In this work, we consider Aff to be the category of smooth manifolds with smooth morphisms, geometric stacks will be called **differentiable stacks** and we especially focus on the case of differentiable 1-stacks.
The choice of a geometric (in this case differentiable) atlas $X_0 \to \mathfrak{X}$ for a stack $\mathfrak{X}$, is equivalent to a geometric (in this case Lie) groupoid structure $X_1 \rightrightarrows X_0$, where $X_1 = X_0 \times_{\mathfrak{X}} X_0$ is, by definition of atlas, a representable sheaf (in this case a manifold).

The correspondence

$$\left\{ \begin{array}{c} \text{Differentiable stacks } \mathfrak{X} \\ + \text{ smooth atlas } X_0 \to \mathfrak{X} \end{array} \right\} \rightleftharpoons \{ \text{ Lie groupoids } X_1 \rightrightarrows X_0 \}$$

is the starting point of our work and we state it as a 2-categorical equivalence in theorem 1.7.5.

Lie groupoids are widely studied in various areas of mathematics, physics (and more), many results about stacks can give interesting counterpart in the Lie groupoids world, and conversely. In this thesis we expose some original results, jumping from one world to the other. In particular we obtain the following results:

1. We study representations up to homotopy and show that they are cohesive modules over the dg-algebra of C^∞-functions on the groupoid, which are naturally organized in a dg-category. This allows us to prove the Morita invariance of the dg-category of representations up to homotopy. As a corollary we exhibit a Hodge to De Rham spectral sequence for stacks;

2. We construct the dg(or ∞)-category of perfect complexes over a differentiable stack;

3. We study in detail the case of perfect complexes of perfect amplitude $[0, 1]$. We show that the nerve of a Lie 2-groupoid is a simplicial manifold: this allows us to define the general linear 2-groupoid $GL(V)_\bullet$ of automorphisms of a two-terms complex of vector bundles V and to show that representations up to homotopy of a Lie groupoid G on V are pseudo-functors $G \dashrightarrow GL(V)_\bullet$. We conclude with a construction of the 2-differentiable stack $Perf_{[0,1]}$.

For each of this result, we will use both techniques and results from the stacky world and from the Lie one. We present now these results in detail.

Representations up to homotopy

A Lie group G can be thought of as a smooth collection of symmetries of an abstract object. A linear representation $G \curvearrowright V$ is therefore a way to realize these symmetries on a concrete vector space V, that we will assume to be finite dimensional and real. Such a representation can be defined either as a smooth map $\rho : G \times V \to V$ satisfying $\rho^h \rho^g = \rho^{hg}$ and $\rho^1 = \mathrm{id}$, or as a Lie group morphism $G \to GL(V)$ into the general linear group. We can then study the group G by looking at its representations $G \curvearrowright V$, and this approach turns out to be very profitable.
Following the previous philosophy, a Lie groupoid $G \rightrightarrows M$ should be thought of as a smooth collection of symmetries of an abstract family parametrized by M. Lie groupoids have received much attention lately, as they provide a unifying framework

for classic geometries, and also serve as models for spaces with singularities such as orbifolds and, more generally, differentiable stacks. The infinitesimal versions of Lie groupoids are Lie algebroids, geometric objects intertwining Lie algebra bundles and (singular) foliations. Differentiation and integration set up a fruitful interaction between the two theories.

A linear representation $(G \rightrightarrows M) \curvearrowright (V \to M)$ of a Lie groupoid over a vector bundle associates to each arrow $x \xrightarrow{f} y$ a linear isomorphism $\rho^g : V^x \to V^y$ between the corresponding fibers, in a way compatible with identities and compositions. It can be presented either as a partially defined map $G \times V \to V$ or as a Lie groupoid map $G \to GL(V)$ into the general linear groupoid [21]. For example for a Lie group $G \rightrightarrows *$ those are just Lie group representations on a vector space $V \to *$ and for a manifold $M = M \rightrightarrows M$ those are just vector bundles $V \to M$.

The problem with Lie groupoid representations is that they are rather scarce, they impose strong conditions on V, and do not provide us with enough information on $G \rightrightarrows M$. This reflects in the lack of an adjoint representation, or in the limitations when establishing a Tannaka duality result for Lie groupoids (cf. [60]).

A solution for these problems was proposed by C. Arias Abad and M. Crainic, by introducing representations up to homotopy $G \curvearrowright V$ of a Lie groupoid over a graded vector bundle [4]. They can be easily defined as differentials on certain bigraded algebra of sections, or alternatively, they can be regarded as a sequence of tensors: the first one is a differential ∂ on V, the second one consists of chain maps $\rho^g : V^x \to V^y$ between the fibers, the third one $\gamma^{h,g}$ provide chain homotopies relating ρ^{hg} and $\rho^h\rho^g$, etc. Representation up to homotopy has proven to be a useful concept, for instance, when dealing with cohomology theory [4], deformations [19] and Morita equivalences [22].

However, the inconvenient is that Lie groupoids are naturally organized in a 2-category, but no homotopy theory for representations up to homotopy nor results about Morita invariance is present in the literature. This made it unclear whether a representation up to homotopy on a Lie groupoid could define any object on the corresponding quotient stack, and which kind of object.

In this work we will put a dg-categorical structure on representations up to homotopy, which allows us to show the Morita invariance and to switch to the stacky world. For this we see that representations up to homotopy are the same as cohesive modules over (curved) dg-algebras. The first advantage is that we control better their homotopy theory, the second is that instead of the classical definition from Abad and Crainic, which is modeled on Lie groupoids, our approach could be carried on for n-Lie groupoids and simplicial manifolds, so it opens some possibilities of future developpements and generalizations.

Perfect complexes over differentiable stacks

In [58], Töen writes:

> *In my opinion the true origin of derived algebraic geometry can be found in the combined works of several authors, around questions related to deformation theory of rings and schemes. On the algebraic side, André and Quillen introduced a homology theory for commutative rings, now called André-Quillen homology ([2], [49]) (...) On the algebro-geometric side, Grothendieck ([33])) and later Illusie ([38]) globalized the definition of André and Quillen and introduced the* ***cotangent complex*** *of a morphisms of schemes.*

In [3], M. Anel writes:

> *The easiest way to introduce Derived Geometry is probably the following analogy. Recall that homological algebra can be read as the enhancement of the theory of vector spaces into the theory of chain complexes, then Derived Geometry is to geometry (ordinary topological spaces, manifolds, schemes...) what chain complexes are to vector spaces.*

Both authors stress the fact that the big innovation of derived stacks is that their tangent spaces are chain complexes, and tangent spaces to classical geometric objects (manifold, affines, schemes,...) corresponds to the 0-th level of such chain complexes, just as sets can be thought as connected component of topological spaces.
Let us calculate as an example, the tangent complex to the quotient of a not-free action of a Lie groupoid.

Example. Let $G = S^1$, $X = \mathbb{R}^2$, $x \in X$ be a point, and consider the standard action of the circle on $\mathbb{R}^2$ by rotations. The infinitesimal action induces a linear map $\mathfrak{g} \to T_xX$, i.e. $\mathbb{R} \to \mathbb{R}^2$ which is injective except for $x = (0,0)$. We call $\mathfrak{g} \to T_xX$, with $\mathfrak{g}$ and T_xX in (homological) degrees 1 and 0 respectively, the tangent complex of the stack $[M/G]$ at the point x, and denote it by $\mathbb{T}_{[M/G],x}$. For $x \neq (0,0)$, the quotient M/G has the structure of a smooth manifold, and its tangent space is just

$$\mathbb{T}_{M/G,x} \simeq \mathbb{R} \simeq T_xX/\mathfrak{g} = H^0(\mathbb{T}_{[M/G],x}).$$

so the tangent space to the naive quotient is equivalent (quasi-isomorphic) to the tangent space of the quotient stack. The quotient M/G is singular at the point $x = (0,0)$ but the complex $\mathfrak{g} \to T_xX$ give a correct definition of tangent space also in this singular point, where there is no classical definition for it.

The construction of the cotangent complex can be done on affines and then globalized by a gluing procedure. Let A be a commutative k-algebra. The cotangent complex

of A is calculated as follows: we first find a simplicial resolution $A_\bullet \to A$ of A by smooth algebras, and take the totalization of the simplicial modules of Kälher differentials

$$\mathbb{L}_A := Tot(n \mapsto \Omega^1_{A_n})$$

of such simplicial algebras. One can show that $\mathbb{L}_A$ does not depend on the choice of the resolution up to quasi-isomorphism. This object controls the deformation theory of the k-algebra A.
The issue to give a well defined notion of cotangent complex for a general scheme, which is a gluing of affine schemes, i.e. commutative algebras, is that one should glue together the various cotangent spaces, which are complexes of algebras. As we pointed out above, it makes little sense to ask for isomorphisms of complexes, just as it would for isomorphisms of categories. The gluing is then performed with quasi-isomorphisms, but what does it means concretely?

Example. Let us take the example of a rank n vector bundle $E \to M$ over a smooth manifold M. If we cover M by trivial open subsets $\{U_i\}$, the data of the vector bundle E is the same as linear isomorphisms $\alpha_{i,j} : E_{|U_i \cap U_j} \to E_{|U_j \cap U_i}$ i.e. $\alpha_{i,j} : U_i \cap U_j \to GL_n$ such that

$$\alpha_{j,k} \circ \alpha_{i,j} = \alpha_{i,k} \quad \text{and} \quad \alpha_{i,i}(m) = \mathrm{id}_{\mathbb{R}^n} \ \forall m \in U_i.$$

An elegant way to reformulate what we just saw is that a vector bundle on M is the same as a Lie groupoid morphism

$$\begin{tikzcd} \coprod U_i \cap U_j \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & GL_n \arrow[d, shift left] \arrow[d, shift right] \\ \coprod U_i \arrow[r] & * \end{tikzcd} \tag{3}$$

From any such morphism we can recover a rank n vector bundle over M. Then we will deduce the well-known fact that rank n vector bundles over M are represented by the stack $BGL_n = [*/GL_n]$.
The same example can be repeated for a complex of vector bundles over M. Again, we could ask for the $\alpha_{i,j}$ to be isomorphisms of complexes over vector bundles respecting the cocycle condition. However, we saw that chain complexes have a rich homotopical structure, so it is more appropriate to ask for the $\alpha_{i,j}$ to be quasi-isomorphisms. Consequently it make no longer sense to ask for $\alpha_{j,k} \circ \alpha_{i,j} = \alpha_{i,k}$, and we would better ask for an homotopy $H_{i,j,k} : \alpha_{j,k} \circ \alpha_{i,j} \Rightarrow \alpha_{i,k}$. Again, the associativity, which was automatically satysfied, now becomes a data, and we will ask, for any i, j, k, l to have a 2-homotopy $T_{i,j,k,l}$ as follows:

$$\begin{tikzcd}
\alpha_{k,l} \circ \alpha_{j,k} \circ \alpha_{i,j,} \arrow[r, Rightarrow, "H_{j,k,l} \circ \mathrm{id}_{\alpha_{i,j}}"] \arrow[d, Rightarrow, "\mathrm{id}_{\alpha_{k,l}} \circ H_{i,j,k}"'] \arrow[dr, Rightarrow, "T_{i,j,k,l}"] & \alpha_{j,l} \circ \alpha_{i,j} \arrow[d, Rightarrow, "H_{i,j,l}"] \\
\alpha_{k,l} \circ \alpha_{i,k} \arrow[r, Rightarrow, "H_{i,k,l}"'] & \alpha_{i,l}
\end{tikzcd}$$

One can imagine that this process does not stop here, and that there will be higher homotopies (possibly trivial) respecting higher coherences for any dimension. So the more natural object constructed as a gluing of complexes of vector bundles over an open cover is not a complex of vector bundle on the manifold M.
In this framework computations are quite involved and in higher dimensions it is not possible to give explicitly all the coherences by hand: already in the case of complexes of amplitude $[0, 1]$, where all 2-homotopies (i.e. degree 2 maps of chain complexes) are zero, they are quite hard. We perform some computations for this case in chapter 4.

The problem of higher coherences was overcome with the language of model or ∞-categories. The cotangent complex can be defined as a presheaf of simplicial sets or chain complexes satisfying descent, meaning exactly that a huge list of coherences is respected, without need for explicitating it.
This approach is very profitable, but has the inconvenient that there is not a general procedure to compute the values of quasi-coherent or perfect complexes on atlases. For example, we could know that the cotangent complex $\mathbb{L}_{\mathfrak{X}}$ exists for a differentiable 1-stack $\mathfrak{X} = [X_1/X_0]$, but we do not know what kind of object it is when we pull it back on the manifold X_0: the main purpose of this thesis is to answer to this question and to study counterpart of perfect complexes in the Lie groupoids world. In chapter 3 we study this problem for differentiable 1-stacks, and show that perfect complexes correspond to representations up to homotopy on an atlas.

Let us go back to the example of the vector bundles $E \to M$ above . We saw that E is the same as the Lie groupoid morphism (4). We can interpret it as a stacks morphism $M \to [*/GL_n]$, which is (see section 1.7) a zig-zag of Lie groupoid morphisms

$$\begin{tikzcd}
M \arrow[d, shift right] \arrow[d, shift left] & U_1 \arrow[l] \arrow[r] \arrow[d, shift right] \arrow[d, shift left] & GL_n \arrow[d, shift right] \arrow[d, shift left] \\
M & U_0 \arrow[l] \arrow[r] & *
\end{tikzcd} \tag{4}$$

where the left pointing one is Morita (note that the natural Lie groupoid morphism from the Čech Lie groupoid of an open cover to the trivial Lie groupoid M is the simpler example of Morita morphism).
Then we can say that a vector bundle on M is a stack morphism from the stack M to the stack of automorphisms of rank n vector spaces, BGL_n. For the general case we should understand what the automorphisms of a chain complex of vector spaces are. We study it for two-terms vector bundles, and we call it the General Linear 2-groupoid. This is the content of a paper (cf. [20]) written in collaboration with Matias Del Hoyo which was accepted by the Pacific Journal of Mathematics.

The General Linear 2-groupoid

When $V = V_1 \oplus V_0$ is a 2-term graded vector bundle, a representation up to homotopy $G \curvearrowright V$ leads to the notion of VB-groupoid, a double structure mixing Lie groupoids and vector bundles, via a semi-direct product construction $G \ltimes V \to G$. It turns out that any VB-groupoid can be split as a semi-direct product, by choosing a horizontal lift of arrows, as proven first in [29]. This yields a 1-1 correspondence between VB-groupoids and 2-term representations up to homotopy, that can be extended to maps, and respect equivalence classes (cf. [22]). Prominent examples of VB-groupoids are the tangent and cotangent constructions. They encode the adjoint and coadjoint representations, respectively.
A VB-groupoid is an instance of a fibration of groupoids, and according to classic Grothendieck correspondence, after choosing a horizontal lift of arrows, a groupoid fibration $E \to G$ is the same as a pseudo-functor $G \dashrightarrow \{\text{Groupoids}\}$ (cf. [31]). It follows that 2-term representations up to homotopy should, in some sense, be the same as pseudo-functors. The main purpose of chapter 4 is to make this precise. We define a smooth nerve for Lie 2-categories, and prove in our first main theorem 4.5.3 that the nerve of a Lie 2-groupoid is a simplicial manifolds. Then we define a Lie 2-groupoid of automorphisms of a 2-term graded vector bundle V (in order to take care of the smooth and linear structures, we are led to fix V and restrict our attention to pseudo-functors involving the several fibers of V). In our second main theorem 4.6.3 we prove that pseudo-functors $G \dashrightarrow GL(V)_\bullet$, which are a generalization of the classification maps $G \to GL(V)_\bullet$ for actual representations, are in 1-1 correspondence with representations up to homotopy of G on V.

Finally, we see how to avoid to fix V, by constructing a general linear 2-groupoid $GL_\bullet$, whose colimit is the stack $Perf_{[0,1]}$. This allows us to prove the equivalence

$$Rep^\infty(G)_{[0,1]} \simeq \mathrm{Hom}_{St}(\mathfrak{X}, Perf_{[0,1]})$$

between representations up to homotopy on G concentrated in degrees 0 and 1 and stack morphisms from the quotient stack $\mathfrak{X}$ of G to $Perf_{[0,1]}$.

Organization of the thesis

This thesis is organized as follows:

In chapter 1 we collect some results about differentiable 1-stacks, their 2-categorical structure and homotopy theory. Those are essentially already existing/well known in the literature, but we couldn't find anywhere else organized in such a way. In sections 1.1 - 1.3 we recall the classical definition of stacks in terms of fibered category and show how to switch to the homotopical point of view. In sections 1.4 - 1.7 we define torsors over Lie groupoids and we show (cf. Thm.1.4.10 and Thm. 1.5.9) that the fibered category of smooth torsors on a Lie groupoid is a differentiable stack. Then we state the theorems relating the 2-category of Lie groupoids and that of differentiable stacks (cf. Thm. 1.7.5 and Cor. 1.7.6).

Chapter 2 is devoted to the study of cohomology theories for differentiable stacks and Lie groupoids and to see how to relate corresponding notions in those two worlds. In section 2.1 we give the definition of sheaf cohomology, hypercohomology, Čech cohomology and De Rham cohomology for manifolds, and recall how they are related to each other. In section 2.2 we extend those definitions and results to differentiable 1-stacks, essentially following [6] and [7]. In section 2.3 we study in detail the example of equivariant cohomology for an action Lie groupoid $G \times M \rightrightarrows M$. For a compact G this is equivalent to the De Rham cohomology of the stacks. We state this result in proposition 2.3.2 and we see it as a special case of the Hodge to De Rham spectral sequence (also called *Bott spectral sequence* by some authors, e.g. in [4]) of theorem 2.4.1.

In chapter 3 we study general representations up to homotopy, we show that they are equivalent to cohesive modules over the algebra of smooth functions of a Lie groupoid and we prove their Morita invariance (thm. 3.8.3), which allows us define the dg-category of perfect complex over a differentiable stack (def. 3.8.5). In section 3.1 we study in detail vector bundles over differentiable 1-stacks and on their Lie groupoid counterparts, precising some folk results: we explain what descent means in this case and we show the equivalence of the following three definition of rank n vector bundles over the differentiable stack $\mathfrak{X} = [X_0/X_1]$:

1. A sheaf on the big site $\mathfrak{X}_{big}$ such that the pullbacks on an atlas are rank n vector bundles;

2. X_1-equivariant rank n vector bundles on X_0;

3. Stack morphisms $\mathfrak{X} \to BGL_n$.

This is an interesting summary of all possible different points of view about vector bundles over stacks, which we didn't find in the literature and anticipates the ideas we will use in our main theorems. In sections 3.3 - 3.5 we define the dg-category

$\mathcal{P}_{\mathcal{A}}$ of cohesive modules over a dg-algebra $\mathcal{A}$ and we define the dg-algebra $C^\bullet(G)$ of smooth functions on (the nerve of) a Lie groupoid $G \rightrightarrows M$. Then we define representations up to homotopy of $G \rightrightarrows M$ and see that they are exactly cohesive modules over the dg-algebra $C^\bullet(G)$. In section 3.7 we use some invariance results for the dg-categories $\mathcal{P}_{\mathcal{A}}$ to show the Morita invariance of representations up to homotopy of a Lie groupoid $G \rightrightarrows M$. This is one of our main original results, it is stated in theorem 3.8.3, and extended in a more elegant and categorical language in Claim 3.8.4. As applications:

1. We define the dg-category of perfect complexes on a differentiable stack as the dg-category of representations up to homotopy on any Lie groupoid presenting it
$$\mathfrak{X} = [M/G] \quad \text{then} \quad Perf(\mathfrak{X}) := Rep^\infty(G).$$
2. We define tangent and cotangent complexes as equivalence classes of the adjoint/coadjoint representations (cf. Def. 3.10.3);
3. We state the Hodge to De Rham spectral sequence in this new framework. This establish a link between the notions of differentiable forms on a stack (i.e. sections of the symmetric powers of the cotangent complex) and De Rham forms on Lie groupoids (i.e. elements of the total complex $Tot(\Omega^p(G_q))$, see Thm. 3.11.1).

In 4 we study the case of representations up to homotopy on a two-terms graded vector bundle. We first do it in the framework of Lie groupoids, by constructing a Lie 2-groupoid of symmetries of the graded vector bundle, and then in the stacky framework, by constructing the (differentiable) 2-stack $Perf_{[0,1]}$.
Given $V = V_1 \oplus V_0 \to M$ a graded vector bundle, we construct a General Linear 2-groupoid $GL(V)_\bullet$, consisting of differentials on the fibers, quasi-isomorphims between them, and chain homotopies. It is of course a generalization to the two terms case of the General Linear groupoid of symmetries of a vector bundle. There are several non-equivalent notions of Lie 2-groupoids in the literature, some of them too strict and some other too lax for our purposes. We discuss some variants in section 4.1, and in 4.3 we introduce a notion of Lie 2-groupoid, and prove our theorem 4.4.5, asserting that $GL(V)_\bullet$ is indeed a Lie 2-groupoid. It is remarkable that even for a 2-term graded vector space V its general linear groupoid $GL(V)_\bullet$ is not a group, for it has more than one object, so groupoids arise naturally.

In the set-theoretic context there is a nerve for 2-categories that relates lax functors with simplicial maps [9, 40], which we recall in 4.2. In 4.5 we develop the smooth version of it, and our Theorem 4.5.3 shows that, even though NC is not always a simplicial manifold, it is so when the Lie 2-category C has invertible 2-arrows, in particular for a Lie 2-groupoid. This nerve construction relates our notion of Lie 2-groupoids with the simplicial approach to Lie 2-groupoids, based on the horn-filling

condition, that has received much attention lately. This can be seen as a piece of evidence supporting our definitions for Lie 2-groupoids and smooth pseudo-functors. We also compare our construction with that of [46].
Building on the previous results, that we believe are of interest in their own, we finally establish our Theorem 4.6.3, setting a 1-1 correspondence between equivalences classes of 2-term representations up to homotopy $G \curvearrowright V$ and pseudo-functors $G \dashrightarrow GL(V)_\bullet$ commuting with basic projections. It seems natural to extend this result for higher degrees, relating positively graded representations up to homotopy and maps into a general linear ∞-groupoid.
We use our results to construct a differentiable 2-stack of perfect complexes of amplitude $[0, 1]$ in section 4.7.

Chapter 1

Stacks

The classical definition of (1-)stack is given in terms of categories fibered in groupoids [61]: a stack is a category fibered in groupoid $X \to C$, where C is a Grothendieck site, and the fibration satisfy a "gluing property" with respect to the topology on C. Roughly speaking:

- we can think at C as a manifold;
- $X \to C$ being a fibration of categories means that over each open subset of C lies a **groupoid**: the fibration is then a sort of presheaf of groupoids, called **lax**;
- the gluing condition means that this presheaf is actually a sheaf, i.e. that it satisfies some nice descent conditions.

In [36] the classical definition is translated in terms of homotopy theory. This is a more natural setting, which places stacks into a larger homotopy theoretic context. In the following chapter we briefly recall the classical definition of stack and then switch to the homotopical point of view.

1.1 Categories fibered in groupoids

Let $X \xrightarrow{p} C$ be a functor. We say that an object x in X (resp. a morphism $x \to y$) is over U (resp. is over $U \to V$) if $p(x) = U$ (resp. $p(x \to y) = U \to V$). If $U' \in X$ is sent to $U = p(U') \in C$, there is a natural functor between the overcategories

$$X/U' \xrightarrow{p^*} C/U.$$

Definition 1.1.1. We say that X is a **category fibered in groupoid** or **fibered category** over C if for all such U, U':

(F1) p^* is essentially surjective;

(F2) p^* is fully faithful.

Definition 1.1.2. We denote by FibCat_C the category whose objects are categories fibered in groupoids, and whose morphisms between $X \to C$ and $Y \to C$ are functors $X \xrightarrow{F} Y$ commuting with the projection to C.

Let $X \xrightarrow{p} C$ be a category fibered in groupoids. We denote by X_U the fiber of X over U, i.e. the subcategory of X whose objects are mapped to U and whose morphisms are mapped to id_U. It follows from (F2) that this subcategory is a groupoid.

Remark 1.1.3. Let $U \xrightarrow{f} V$ be a morphism in C. It follows from (F1) and (F2) that there exists a pullback functor $f^* : X_V \to X_U$ such that for all $x \in X_U$ there is a morphism $f^*(x) \to x$ over f. Such a functor is unique up to unique natural isomorphism. Moreover, if $V \xrightarrow{f} U$ and $W \xrightarrow{g} V$ are morphisms in C, there is a unique natural isomorphism $g^* \circ f^* \simeq (f \circ g)^*$.

Definition 1.1.4. We say that a category fibered in groupoids is a **stack** if the following conditions are satisfied:

(St1) For all $U \in C$, all x, y over U and all covering families $\{U_i \xrightarrow{f_i} U\}$ the following is an equalizer diagram:

$$\mathrm{Hom}_{X_U}(x,y) \longrightarrow \prod_i \mathrm{Hom}_{X_{U_i}}(f_i^*(x), f_i^*(y)) \underset{b}{\overset{a}{\rightrightarrows}} \prod_{i,j} \mathrm{Hom}_{X_{U_{ij}}}(f_{ij}^*(x), f_{ij}^*(y))$$

where

$$\begin{tikzcd} U_{ij} = U_i \times_U U_j \arrow[r, "\pi_{ij\to i}"] \arrow[d, "\pi_{ij\to j}"'] \arrow[dr, "f_{ij}=f_{ji}"] & U_i \arrow[d, "f_i"] \\ U_j \arrow[r, "f_j"'] & U \end{tikzcd}$$

a and b send $(f_i^*(x) \xrightarrow{\phi_i} f_i^*(y))_i$ to $(\pi_{ij\to i}^*(\phi_i))_{i,j}$ and $(\pi_{ij\to j}^*(\phi_j))_{i,j}$ respectively.

(St2) For all $\{U_i \xrightarrow{f_i} U\}$ covering family, $x_i \in X_{U_i}$ and all $\pi_{ij\to i}^*(x_i) \xrightarrow{\phi_{ij}} \pi_{ij\to j}^*(x_j)$ morphism over $id_{U_{ij}}$ such that

$$\pi_{ijk\to jk}^*(\phi_{jk}) \circ \pi_{ijk\to ij}^*(\phi_{ij}) = \pi_{ijk\to ik}^*(\phi_{ik})$$

there exist a $x \in X_U$, and isomorphisms $f_i^*(x) \xrightarrow{\tau_i} x_i$ over id_{U_i} such that the

following diagrams

$$\begin{tikzcd}
 & \pi^*_{ij\to i}(x_i) \arrow[dd, "\phi_{ij}"] \\
f^*_{ij}(x) \arrow[ur, "\pi^*_{ij\to i}(\tau_i)"] \arrow[dr, "\pi^*_{ij\to j}(\tau_j)"'] & \\
 & \pi^*_{ij\to j}(x_j)
\end{tikzcd}$$

commute.

Note that in the diagram we hide the following canonical isomorphisms:

$$\pi^*_{ij\to i}(f^*_i(x)) \simeq \pi^*_{ij\to j}(f^*_j(x)) \simeq f^*_{ij}(x).$$

We call $\{x_i, \phi_{ij}\}$ a **descent datum**, and we say that it is **effective** if the existence condition in (St2) is satisfied.

Example 1.1.5. Let $X \to Ouv(T)$ be a category fibered in groupoids, such that X is a set, with the trivial category structure (only identity morphisms), and $Ouv(T)$ the category of open subsets of a topological space T with inclusion morphisms (for any two open subsets U and V, $\mathrm{Hom}_C(U, V)$ is the singleton if $U \subset V$ and empty elsewhere).
Let $\{U_i\}$ be an open cover of an open set U, $x_i \in X_{U_i}$ and $x \in X_U$. We can call the pullbacks f_i^* and f_{ij}^* "restrictions" of x to U_i and U_{ij} respectively, and see $X \to C$ as a presheaf over C, given by:

$$U \mapsto X_U$$

One can easily check that the condition (St1) is the usual descent condition for a (classical) sheaf of sets over C.

1.2 Stacks in homotopy theory

In this section we briefly recall the results contained in [36]: we translate the definition 1.1.4 to a descent condition for groupoid presheaves, and we establish a homotopy theory for stacks.
We first remark that categories fibered in groupoid are **lax presheaves of groupoids** which can be strictified by means of the Grothendieck construction (cf. Thm. 1.2.1). Then theorem 1.2.3 gives a translation of the conditions (St1) and (St2) to a descent condition for presheaves of groupoids.
We will use the notions of Grothendieck topology and presheaves on a category, which are recalled in appendix A.4.

Let $Grpd$ be the category of groupoids and functors between them (for the moment we do not care about the 2-categorical structure of $Grpd$), and $Pr(C, Grpd)$ be the category of groupoid presheaves on C, defined by

$$Pr(C, Grpd) := Fun(C^{op}, Grpd)$$

Theorem 1.2.1 (Grothendieck construction)**.** *There is an adjoint couple*

$$p : Pr(C, Grpd) \rightleftarrows \mathrm{FibCat}_C : \Gamma$$

where the unit map is an objectwise equivalence, and the counit map is a fiberwise equivalence of groupoids.

See [39] for details.
The Grothendieck construction allows us to think at fibered categories as presheaves of groupoids. The next step is to state the descent condition for stacks in an homotopical context. In particular we will need to calculate the homotopy limit 2; for this we put a model category structure on the category of groupoids:

Theorem 1.2.2 ([36], Thm. 2.1)**.** *There is a standard model structure on the category $\mathcal{G}rpd$ of (small) groupoids and functors between them such that:*

- *weak equivalences are equivalences of groupoids;*
- *fibrations are isofibrations;*
- *cofibrations are injections on objects.*

We recall now the main result of [36] which gives a definition of stack in a homotopical framework:

Theorem 1.2.3 ([36], Thm. 3.9)**.** *A category fibered in groupoids $X \to C$ is a stack if and only if for all covers $\{U_i \to U\}$ in C*

$$\Gamma(X)(U) \to \mathrm{holim}_i(\Gamma(X)(U_i)) \tag{1.1}$$

is a groupoid equivalence.

Remark 1.2.4. We can see an object of C as a groupoid presheaf on C by the Yoneda embedding:

$$\begin{array}{ccc} C & \to & Pr(C, Grpd) \\ U & \mapsto & \mathrm{Hom}(\cdot, U) \end{array}$$

where for any $V \in C$, $\hom(V, U)$ is thought as a groupoid.

We put on $Pr(C, Gprd)$ the projective model structure. Let S be the set of maps

$$S = \{\mathrm{hocolim}\, U_\bullet \to X \mid \{U_i \to X\} \text{ is a cover in } C\}$$

where U_i and X are intended as objects of $Pr(C, Gprd)$ by the Yoneda embedding.

Theorem 1.2.5 ([36] Thm 1.4)**.** *The left Bousfield localization (cf. [37]) $Pr(C, Gprd)_S$ exists and the fibrant objects in this category are exactly those which satisfy the descent condition* (1.1)*, i.e. the stacks.*

This is the model structure of 1-stacks on $Pr(C, Gprd)$, in the next section we extend these ideas to general simplicial presheaves.

1.3 Higher stacks and simplicial presheaves

We saw in the previous section that 1-stacks on C are the fibrant objects of a model structure on the category of presheaves of groupoids on C. The fibrant condition is a descent condition, which allow us to think at 1-stacks as sheaves of 1-homotopy type.
The definition of higher stacks, or sheaves of homotopy types, follows naturally in this context: we remark that model structure on presheaves of groupoids is just a truncation of a model structure on the bigger category of presheaves of simplicial sets. Higher stacks are just the fibrant objects in this larger model category.

1.3.1 Jardine model structure on simplicial presheaves

Let $sPr(C)$ be the category of simplicial presheaves on C (see appendix A.1, A.2 and A.3 for basics on simplicial sets and simplicial presheaves). We now describe the Jardine model structure on $sPr(C)$, and expose briefly the results of [23], in order to put the new definition of stack in a larger homotopy theoretic context. We need the definition of sheaves of homotopy groups:

Definition 1.3.1.

Let F be a simplicial presheaf, $X \in C$ and $s \in F(X)$.Then:

- $\pi_0 F$ is the presheaf of sets defined by $(\pi_0 F)(X) := \pi_0(F(X))$
- for $a \in F(X)_0$, $\pi_n(F, a)$ is the presheaf of groups on C/X defined by

$$\pi_n(F, a)(Y \xrightarrow{f} X) = \pi_n(F(Y), f^* a).$$

We say that a map $F \xrightarrow{\phi} G$ in $sPr(C)$ is an **isomorphism of sheaves of homotopy groups** if the induced maps $\pi_0(\phi)$ and $\pi_n(\phi, a)$ on the sheafifications are isomorphisms for all $a \in F(X)$, and all $X \in C$.

Theorem 1.3.2 (Jardine model structure)**.** *There is a proper, cofibrantly generated, simplicial model category on $sPr(C)$ such that:*

- *cofibrations are objectwise cofibrations;*

- *weak equivalences are maps which are isomorphisms on all sheaves of homotopy groups*
- *fibrations are the maps with the right lifting property with respect to the trivial cofibrations.*

We denote by $sPr(C)_J$ or $St(C)$ this model category and we will call it **the model category of stacks**. We use this terminology because the fibrant objects in this category are exactly those which satisfy an "hyperdescent" property (cf. thm 1.3.6), a generalization of the descent condition (1.1). Then the theory of 1-stacks arises as the 2-truncation of the theory of stacks:

Theorem 1.3.3 (cf. [23])**.** *The adjoint couples*

$$(S^2)^{-1}sPr(C)_J \rightleftarrows sPr(C)_J \underset{\pi_{oid}}{\overset{N}{\rightleftarrows}} Pr(C, Grpd)_S$$

induce a Quillen equivalence between $(S^2)^{-1}sPr(C)_J$ and $Pr(C, Grpd)_S$. In the equation, the first adjonction is the localization of Joyal model category at the map $S^2 \to *$.

This implies that 1-stacks on C are organized in a 2-category: the ∞-categorical structure induced by the model structure on $(S^2)^{-1}sPr(C)_J$ is 3-truncated, so that all n-morphisms are trivial for $n \geq 3$.

1.3.2 Higher stacks

Let C be a Grothendieck site and $Pr(C)$ be the category of presheaves on C. There is an interesting way to define the category of sheaves on C, which enlightens the definition of higher stacks we will give (see [59], Thm. 1.0.1). Let W_τ be the subcategory of $Pr(C)$ consisting of local isomorphisms (for the Grothendieck topology τ on $\mathcal{C}$). Let

$$Sh_\tau(C) := W_\tau^{-1}Pr(C)$$

be the localization of $Pr(C)$ at W_τ.

Theorem 1.3.4. *The natural localization morphism $s : Pr(C) \to Sh_\tau(C)$ is left exact (i.e. commutes with finite limits) and has a fully faithful right adjoint $j : Sh_\tau(C) \to Pr(C)$.*

Remark 1.3.5. In this correspondence s is equivalent to the classical sheafification functor and the essential image of j are exactly sheaves (i.e. presheaves having descent).

In this way we have defined the category of sheaves on C without even mentioning what a sheaf is: instead of considering a new class of objects, we enlarged the class of equivalences between the old ones, and we obtained a category equivalent to the "classical" category of sheaves on C. We do exactly the same thing in the more general setting of simplicial presheaves, the only difference being that the new class of local isomorphisms (hypercovers) is more complicated.
The following theorem generalizes the result of 1.2.5:

Theorem 1.3.6 ([23],Thm 6.2, Thm 7.1)**.** *Let S be the set*

$$S = \{f : Y_\bullet \to X \mid f \text{ is an hypercover and } X \in C\}.$$

Then the left Bousfield localization of the injective model structure on $sPr(C)$ at S is Quillen equivalent to the Jardine model category $sPr(C)_J$. Moreover, the fibrant objects are exactly the simplicial presheaves which are objectwise fibrant (i.e. presheaves of ∞-groupoids) and which satisfy descent for all hypercovers in S.

1.4 Quotient stacks

In this section we will study a class of stacks that arise from a group or a groupoid action. The paradigmatic example is the action of a Lie group G over a manifold M: the set of orbits $[M/G]$ will not inherit any topological/differentiable structure in general, but it is always a differentiable 1-stack.
We define the category of torsors on groupoids, and we prove in theorem 1.4.10 that it is always a stacks. This is true for a general groupoid object $X_1 \rightrightarrows X_0$ in C, and we prove in proposition 1.5.6 that on a Lie groupoid all torsors are representable, and the notion of torsor reduces to that of principal $X_1 \rightrightarrows X_0$-bundle. We can then define the quotient stack of a Lie groupoid $X_1 \rightrightarrows X_0$ as the fibered category of principal $X_1 \rightrightarrows X_0$-bundles.

1.4.1 Groupoid objects

Definition 1.4.1. A **groupoid object** in C is the data of two objects X_1 and X_0 in C, together with five morphisms:

- $s, t : X_1 \to X_0$, called respectively **source** and **target**;
- $e : X_0 \to X_1$ called **identity**;
- $m : X_1 \times_{t,X_0,s} X_1 \to X_1$, called **multiplication**;
- $i : X_1 \to X_1$, called **inverse**.

Such that

1. $s \circ \iota = t \circ \iota = id_{X_0}$, $\pi_1 \circ s = m \circ s$, $\pi_2 \circ t = m \circ t$ where $\pi_i : X_1 \times_{t,X_0,s} X_1 \to X_1$ are the projections;

2. $m(m \times id_{X_1}) = m(id_{X_1} \times m)$ (associativity of the product);

3. $m(\iota \circ s, id_{X_1}) = m(id_{X_1}, \iota \circ t) = id_{X_1}$ (unity);

4. $\cdot^{-1} \circ \cdot^{-1} = id_{X_0}$, $s \circ \cdot^{-1} = t$, $t \circ \cdot^{-1} = s$, $m(id_{X_1}, \cdot^{-1}) = \iota \circ s$ and $m(\cdot^{-1}, id_{X_1}) = \iota \circ t$ (inverse).

Those conditions can be expressed as the commutativity of some diagrams, so that the definition makes sense in any category C.
We should think at $X_1 \rightrightarrows X_0$ as a category with set of objects X_0, set of morphisms X_1 source and target s and t, composition m, identity morphisms given by e and where all morphisms are invertible, the inverse given by i. In the following we use also the notation $X_\bullet$ or X for a Lie groupoid $X_1 \rightrightarrows X_0$.

Example 1.4.2. Here are some examples of groupoid objects:

1. Any morphism $X \to Y$ in C determines a groupoid object. For this, take $X_1 = X \times_Y X$, with s and t the two projections, e the diagonal map, and i the map switching the two factors. Identifying $X_1 \times_X X_1$ with $X \times_Y X \times_Y X$, the map m is the projection onto the first and third factors.

2. For G group acting on $X \in C$, we can define the action groupoid $G \times X \rightrightarrows X$, a groupoid object in C with:

   - $X_0 =: X$ and $X_1 := G \times X$;
   - $s : G \times X \to X$ is the projection and $t : G \times X \to X$ is the action;
   - $e : X \to G \times X$ takes x to (e_G, x);
   - i takes (g, x) to $(g^{-1}, g \cdot x)$;
   - $m((g, x), (h, g \cdot x,)) = (hg, x)$

3. If $F \in Pr(C, Grpd)$, we can define a groupoid object $X_1 \rightrightarrows X_0$ in $Pr(C, Set)$: for any $W \in C$,

   - $X_0(W)$ is the set of objects in $F(W)$;
   - $X_1(W)$ is the set of morphisms in $F(W)$;
   - morphisms s, t, e, i, m are defined objectwise in the obvious way.

   It is easy to check that $X_1 \rightrightarrows X_0$ is a groupoid object in $Pr(C, Set)$. Conversely, if $X_1 \rightrightarrows X_0$ is a groupoid object in $Pr(C, Set)$, then the presheaf F which sends $W \in C$ to the groupoid whose objects are $X_0(W)$ and morphisms $X_1(W)$, with composition, inverse and identity induced respectively

by m_W, i_W and e_W, is a presheaf of groupoids. This gives an isomorphism of categories between $Pr(C, Grpd)$ and the category of groupoid objects in $Pr(C, Set)$; from now on we could use both F and $X_1 \rightrightarrows X_0$ to denote a presheaf of groupoids.

Remark 1.4.3. By the Yoneda lemma, any groupoid object $X_1 \rightrightarrows X_0$ in C can be seen as a groupoid object in $Pr(C, Set)$ (or in $Pr(C, sSet)$, the simplicial presheaves over C). This is very important, beacause the category of presheaves of sets (or of simplicial sets) over C is complete and cocomplete, i.e. all limits and colimits exist, and are simply computed pointwise. Then all limits and colimits of diagrams in C can be computed in $Pr(C, Set)$. In the following we will often state our results for presheaves over C.

Let us now look at the example 2: a priori the quotient of X by the action of G is not defined in C or does not carry all the information about G and X. For example think about:

1. an action of a topological group with non-trivial stabilizer on a topological manifold;
2. the trivial action of a topological group on a point.

In general, in the first case the quotient will not be a manifold anymore, and in the second it will always be a point, without any information about G. For this reason we are interested in defining quotients stacks, which do not loose those informations.

1.4.2 Torsors on groupoids

There is a natural notion of groupoid action that generalize that of group action:

Definition 1.4.4. Let $X_1 \rightrightarrows X_0$ be a groupoid object in $Pr(C, Set)$ (or in C), $P \in Pr(C, Set)$ (or $P \in C$) endowed with a morphism $\tau : P \to X_0$. A **left action** of $X_1 \rightrightarrows X_0$ on P is a morphisms

$$\begin{array}{ccc} X_1 \times_{t, X_0, \tau} P & \to & P \\ (f, p) & \mapsto & f \cdot p \end{array}$$

such that

$$\tau(f \cdot p) = t(f), \quad g \cdot (f \cdot p) = (g \circ f) \cdot p, \quad \mathrm{id}_{\tau(p)} \cdot p = p$$

whenever these equalities make sense. A **right action** of $X_1 \rightrightarrows X_0$ on P is a left action of $(X_1 \rightrightarrows X_0)^{op}$ (the opposite groupoid, with source and target maps exchanged) on P.

Given a groupoid object in $Pr(C, Set)$, we have an explicit description of the stackification of the associated presheaf in groupoids; by the Yoneda lemma this includes the case of a groupoid object in C.

Definition 1.4.5. Let C be a site, and $X_1 \rightrightarrows X_0$ be a groupoid object in $Pr(C, Set)$. Then $[X_0/X_1]$ is the following category:

- The objects are principal $(X_1 \rightrightarrows X_0)$-torsors, i.e. sheaves P on C equipped with:
  - An epimorphism of sheaves $P \xrightarrow{\phi} S$ for an $S \in C$;
  - A morphism $P \xrightarrow{\tau} X_0$ and a left action $X_1 \times_{t,X_0,\tau} P \to P$, such that the diagram

$$\begin{tikzcd} X_1 \times_{t,X_0,\tau} P \arrow[r] \arrow[d, "pr_1"'] & P \arrow[d, "\phi"] \\ P \arrow[r, "\phi"'] & S \end{tikzcd} \tag{1.2}$$

  commutes and the induced morphism $X_1 \times_{t,X_0,\tau} P \to P \times_S P$ is an isomorphism of sheaves. We denote such a torsor by $(P \xrightarrow{\phi} S, P \xrightarrow{\tau} X_0)$.

- Morphisms between $(P \xrightarrow{\phi} S, P \xrightarrow{\tau} X_0)$ and $(P' \xrightarrow{\phi'} S', P' \xrightarrow{\tau'} X_0)$ are commutative squares

$$\begin{tikzcd} P \arrow[r, "g"] \arrow[d, "\phi"'] & P' \arrow[d, "\phi'"] \\ S \arrow[r, "f"'] & S' \end{tikzcd} \tag{1.3}$$

  such that $\tau' \circ g = \tau$ and

$$\begin{tikzcd} X_1 \times_{t,X_0,\tau} P \arrow[r, "\mathrm{id}\times g"] \arrow[d, "\rho"'] & X_1 \times_{t,X_0,\tau'} P' \arrow[d, "\rho'"] \\ P \arrow[r, "g"'] & P' \end{tikzcd} \tag{1.4}$$

  commutes. Note that this last condition implies that the square (1.3) is a cartesian square.

Example 1.4.6 (Trivial torsors)**.** Let $X_1 \rightrightarrows X_0$ be a groupoid object in $Pr(C, Set)$. For $S \in C$, $S \xrightarrow{\phi} X_0$ morphism of sheaves, there is a canonical $X_1 \rightrightarrows X_0$-torsor, well defined up to a unique isomorphism, that we call the **trivial torsor over S induced by ϕ**; it is simply $P = X_1 \times_{t,X_0,\phi} S$ endowed with the projection on S, the left action

$$\begin{array}{ccc} X_1 \times_{t,X_0,s} (X_1 \times_{t,X_0,\phi} S) & \xrightarrow{\rho} & X_1 \times_{t,X_0,\phi} S \\ (g, (f, x)) & \mapsto & (g \circ f, x) \end{array}$$

and the map

$$\begin{array}{ccc} X_1 \times_{t,X_0,\phi} S & \xrightarrow{\tau} & X_0 \\ (f,x) & \mapsto & s(f) \end{array}$$

For $S = X_0 \in C$, $\phi = id_X$, we obtain the **universal trivial** $(X_1 \rightrightarrows X_0)$**-torsor** ξ_0, which is just $t : X_1 \to X_0$ with the action induced by precomposition. Then we can think the trivial torsor induced by $\phi : S \to X_0$ as the pullback $\phi^*(\xi_0)$.

Proposition 1.4.7. *Every $(X_1 \rightrightarrows X_0)$-torsor is locally trivial.*

Proof. Let $(P \xrightarrow{\phi} S, P \xrightarrow{\tau} X)$ be such a torsor. If there is a section $S \xrightarrow{\psi} P$, then P is isomorphic to the trivial torsor $(\tau \circ \psi)^*(\xi_0)$. So we only need to find local sections for ϕ. An epimorphism from a representable sheaf always admits local sections, in fact, as $id_S \in S(S)$, there is a covering family $(U_i \xrightarrow{f_i} S)$ and sections $\sigma_i \in F$ such that $\phi(\sigma_i) = f_i$. Then we have a sheaf morphism $U_i \to P_{|U_i}$ which sends $f \in U_i(M)$ to $P(f)(\sigma_i) \in P(M)$. □

Remark 1.4.8. The trivial torsor over S induced by f admits a global section

$$\begin{array}{ccc} S & \to & X_1 \times_{t,X_0,f} S \\ s & \mapsto & (\mathrm{id}_{f(s)}, s) \end{array}$$

so trivial torsors are exactly those admitting a global section.

There is a simple description of morphisms between trivial bundles, that will be useful in the following:

Proposition 1.4.9. *Let $f, g : S \to X_0$ in C, and*

$$\mathrm{Hom}_{[X_0/X_1](S)}(f^*(\xi_0), g^*(\xi_0))$$

be the set of morphisms between the trivial bundles induced by f and g respectively, which project on the identity of S. Then there is a bijection

$$\mathrm{Hom}_{[X_0/X_1](S)}(f^*(\xi_0), g^*(\xi_0)) \to \{\phi : S \to X_1 \in C \mid s \circ \phi = f, t \circ \phi = g\}$$

Proof. By the triviality and the remark above, ψ is such a morphism iff there is a commutative diagram

$$\begin{tikzcd} X_1 \times_{t,X_0,f} S \arrow[r, "\psi"] \arrow[d, shift left] & X_1 \times_{t,X_0,g} S \arrow[d] \\ S \arrow[u, bend left, "\sigma"] \arrow[r, "id_S"'] & S \end{tikzcd}$$

where σ is the trivializing section. We obtain a morphism $\phi : S \to X_1$ which is the composition

$$S \xrightarrow{\psi\circ\sigma} X_1 \times_{t,X_0,g} S \xrightarrow{\pi} X_1.$$

The commuting diagrams

$$\begin{tikzcd} X_1 \times_{t,X_0,g} S \arrow[r] \arrow[d] & X_1 \arrow[d, "t"] \\ S \arrow[r, "g"'] & X_0 \end{tikzcd} \qquad \begin{tikzcd} X_1 \times_{t,X_0,f} S \arrow[rr, "\psi"] \arrow[dr, "\tau_f"'] & & X_1 \times_{t,X_0,g} S \arrow[dl, "\tau_g"] \\ & X_0 & \end{tikzcd}$$

assure that $s \circ \phi = f$ and $t \circ \phi = g$. Conversely, let $\phi : S \to X_1$ and $s \circ \phi = f$, $t \circ \phi = g$. Then it easy to check that the morphism

$$\begin{array}{ccc} X_1 \times_{t,X_0,f} S & \longrightarrow & X_1 \times_{t,X_0,g} S \\ (h, x) & \mapsto & (\phi(x) \circ h, x) \end{array}$$

is in $\mathrm{Hom}_{[X_0/X_1](S)}(f^*(\xi_0), g^*(\xi_0))$. This shows that the mapping in the statement is a bijection. $\square$

Theorem 1.4.10. *The category $\mathfrak{X} = [X_0/X_1]$, equipped with the forgetful functor $\mathfrak{X} \to C$ that sends $(P \xrightarrow{\phi} S, P \xrightarrow{\tau} X_0)$ to S and*

$$\begin{tikzcd} P \arrow[r, "g"] \arrow[d, "\phi"'] & P' \arrow[d, "\phi'"] \\ S \arrow[r, "f"'] & S' \end{tikzcd}$$

to $S \xrightarrow{f} S'$ is a stack over C.

Proof. First, note that $\mathfrak{X}$ is a category fibered in groupoids over C. This follow from the existence of pullbacks in the category of presheaves over C and the fact that the square (1.3) is a cartesian square. We fix a choice of pullback f^* for all $f \in C$.
Let us check the stack conditions:

- (St1) is easily verified using that if P, P' are sheaves over C, $\mathrm{Hom}_{Sh(C)}(P, P')$ is a sheaf over C;

- (St2) Let $(P_i \xrightarrow{\phi_i} U_i, P_i \xrightarrow{\tau_i} X)$,

$$\begin{tikzcd} \pi^*_{ij\to i}(P_i) \arrow[r, "\phi_{ij}"] \arrow[d, "\phi_i"'] & \pi^*_{ij\to j}(P_j) \arrow[d, "\phi_j"] \\ U_{ij} \arrow[r, "="'] & U_{ij} \end{tikzcd}$$

be a descent datum for $\mathfrak{X}$. We define the presheaf P over U as:

$$\forall V \in C \quad P(V) := \coprod_{V \to U} \operatorname{colim}_i P_i(V \times_U U_i)$$

where the projection $P \xrightarrow{\phi} U$ is clear from the definition. P is a sheaf, because the equalizer diagram is a colimit. There is a natural map $P \xrightarrow{\tau} X$ induced by the $(\tau_i)_i$: if $\sigma \in P(V)$, $\phi(\sigma) = V \to U$, let σ_i be in the preimage of σ by the map $P_i(V \times_U U_i) \to \operatorname{colim}_j P_j(V \times_U U_j)$. Then $\tau_i(\sigma_i) \in X(V \times_U U_i)$ and $\tau_i(\sigma_i \mid_{U_{ij}}) = \tau_j(\sigma_j \mid_{U_{ij}})$. Moreover $V \times_U U_i \to V$ is a covering family for V, and by assumption the topology in C is subcanonical, so there is a unique $f \in X(V)$ such that

$$\forall i \;\; V \times_U U_i \to V \xrightarrow{f} X = V \times_U U_i \xrightarrow{\tau_i(\sigma_i)} X$$

and we set $\tau(\sigma) = f$. Again, to define an action on P we use the compatibility of the actions on P_i and that the topology is subcanonical. One then checks that there is a natural isomorphism $P \mid_{U_i} \simeq P_i$ compatible with (ϕ_{ij}).

□

There are few references for a general definition of the stack $[X_0/X_1]$ in an arbitrary category C: with our definition, the category of $(X_1 \rightrightarrows X_0)$-torsors is always a stack. Then the question is whether such a torsor is representable or not in C.
It is more common in the literature to define $(X_1 \rightrightarrows X_0)$-torsors as the representable ones, i.e. what is called **smooth principal bundle** in the differentiable case, but in this case the category obtained is not always a stack. In a geometric context with some suitable properties (for example in the differentiable one), all the $(X_1 \rightrightarrows X_0)$-torsors are representable; we prove it in the next section.
In the following, we will often use the standard terminology "smooth principal bundle" instead of "torsor".

1.5 The 2-category of differentiable stacks

In this section we study differentiable 1-stacks. We first give the definition of Lie groupoid and expose some examples. Then we apply the results of section 1.4 to the special case of Lie groupoid quotients. We give the definition of differentiable stacks in terms of atlases and we show in theorem 1.5.9 that there is a 1-1 correspondence between differentiable stacks with a fixed atlas and Lie groupoids.
Then we extend it to a 2-categorical equivalence in theorem 1.7.5 and corollary 1.7.6.

1.5.1 Lie groupoids and Stacks

In this section we define Lie groupoids and give some examples of differentiable stacks. The principal references are [27] and [5].

Definition 1.5.1. A **Lie groupoid** $X_\bullet = X_1 \rightrightarrows X_0$ is a groupoid object in the category of smooth manifolds such that the source and the target maps are surjective submersions.

In summary, a Lie groupoid is the data of (see section 1.4.1):

1. two smooth manifolds X_0 and X_1 whose point are respectively the objects ad morphisms of the groupoid;
2. two surjective submersions $s, t : X_1 \to X_0$ (source and target maps);
3. a smooth embedding $\iota : X_0 \to X_1$ (identity map);
4. a surjective submersion $m : X_1 \times_{t,X_0,s} X_1 \to X_1$ (multiplication);
5. a smooth automorphism $\cdot^{-1} : X_1 \to X_1$ (inverse).

satisfying conditions 1.-4. of 1.4.1.

Example 1.5.2. Here are some examples of Lie groupoids:

- (Trivial Lie groupoid) Let M be a manifold. Then $M \rightrightarrows M$, the category with objects and morphisms the manifold M, and all structural morphisms being the identity, is a Lie groupoid, called the trivial Lie groupoid of M, denoted $[M]$ (or simply by M when the context is clear).
- (Pair groupoid) Let M be a manifold. Then $M \times M \rightrightarrows M$, the category with objects M, morphisms $M \times M$, source and target the projections, identity the diagonal $\Delta : M \to M \times M$ and inverse the map $\cdot^{-1} : M \times M \to M \times M$ which exchange the factors, is a Lie groupoid, called the **pair groupoid** of M, denoted $\mathrm{Pair}(M)$.
- Let G be a Lie group. Then we can see it as the set of automorphisms of a point: with this structure $G \rightrightarrows *$ is a Lie groupoid. Conversely, the set of morphisms of a one point Lie groupoid is a Lie group.
- Let $P \to M$ be a submersion of manifolds. Then $P \times_M P$ is a manifold, and the category $P \times_M P \rightrightarrows P$, with objects P and morphisms $P \times_M P$ inherits a natural structure of Lie groupoid. This is a special case of a smooth atlas for the manifold M seen as a differentiable stack (see section 1.5.2).

- (Action groupoid) Let G be a Lie group acting (on the left) on a manifold M. The projection on the second component and the action are two smooth submersions $G \times M \rightrightarrows M$. Since $(G \times M) \times_{t,M,s} (G \times M) = G \times G \times M$ we can define the composition and the identity by

$$\begin{array}{rcl} G \times G \times M & \to & G \times M \\ (g, h, m) & \mapsto & (gh, m) \end{array}$$

$$\begin{array}{rcl} M & \to & G \times M \\ m & \mapsto & (id, m) \end{array}$$

  It is easy to check that $G \times M \rightrightarrows M$ is a Lie groupoid.

- (Groupoid action) Let $X_1 \rightrightarrows X_0$ be a Lie groupoid and M be a manifold. A left action of $X_1 \rightrightarrows X_0$ on M is the data of a smooth map $\phi : M \to X_0$ and an action map

$$\rho : X_1 \times_{t,X_0,\phi} M \to M, \quad (f, m) \mapsto f \cdot m$$

  telle que $(g \cdot (f \cdot m)) = (g \circ f) \cdot m$.

  As before we can obtain a Lie groupoid from this action: the manifold of morphism is $X_1 \times_{t,X_0,\phi} M$ and the manifold of objects is M. Then the source and target maps are the projection and the action respectively:

$$s(f, m) = m, \quad t(f, m) = f \cdot m$$

  The other structural maps are defined easily.

- (Čech groupoids) Let M be a smooth manifold, and (U_i) be an open cover of M. The Cêch groupoid associated to the open cover (U_i) has objects the manifold of the disjoint union $\coprod_i U_i$ and morphisms the disjoint union of the intersections $\coprod_{i,j} U_i \cap U_j$.

  For any i, j, the source and target maps are, respectively, the inclusion in the first and in the second component $U_i \cap U_j \hookrightarrow U_i$ and $U_i \cap U_j \hookrightarrow U_j$.

  The identity map is $\mathrm{id} : U_i \to U_i \cap U_i$, and the composition is the inclusion

$$U_i \cap U_j \times_{U_j} U_j \cap U_k = U_i \cap U_j \cap U_k \hookrightarrow U_i \cap U_k$$

  Note that the Cêch groupoid associated to the open cover $\{M\}$ is $M \rightrightarrows M$ with all structural morphisms being the identity.

- (Pullback of groupoids) Let $a : H_\bullet \to K_\bullet$ and $b : G_\bullet \to K_\bullet$ be two groupoid morphisms. The categorical weak pullback (cf. [43]) $H_\bullet \times_{K_\bullet} G_\bullet$ is the category with:

  - Objects the triples (h, α, g) with $h \in H_0$, $g \in G_0$ and $\alpha : a(h) \to b(g)$;

– Morphisms between (h, α, g) and (h', α', g') the pairs (β, γ) with $\beta : h \to h'$, $\gamma : g \to g'$ making the following diagram commute:

$$\begin{tikzcd} a(h) \arrow[r, "\alpha"] \arrow[d, "a(\beta)"'] & b(g) \arrow[d, "b(\gamma)"] \\ a(h') \arrow[r, "\alpha'"'] & b(g') \end{tikzcd}$$

If moreover $H_\bullet$, $G_\bullet$ and $K_\bullet$ are Lie groupoids and $H_0 \to K_0$ is a submersion, then also $H_\bullet \times_{K_\bullet} G_\bullet$ is a Lie groupoid.

Example 1.5.3 (Pullback groupoid)**.** Let $G \rightrightarrows M$ be a Lie groupoid, and $\phi : P \to M$ be a surjective submersion. Since ϕ is a submersion, the fiber product $P_\phi \times_s G_t \times_\phi P$ is a manifold. It is easy to put a Lie groupoid structure on $P_\phi \times_s G_t \times_\phi P \rightrightarrows P$ with source and target the projections, and composition induced by the composition on G; we denote it by $\phi^*(G_\bullet)$. There is a natural Lie groupoid morphism $\phi^*(G_\bullet) \to G_\bullet$; we will see that this is what is called a **Morita morphism**.

Proposition 1.5.4. *Let $G_\bullet = G \rightrightarrows M$ be a Lie groupoid, $\phi : P \to M$ be a submersion,* $\mathrm{Pair}(P)$ *be the pair groupoid of P and $\phi^*(P)$ the pullback of $G \rightrightarrows M$ along ϕ. Then the groupoid pullback is equivalent to the weak pullback of groupoids*

$$\phi^*(P) \simeq \mathrm{Pair}(P) \times_{[M \times M]} G_\bullet$$

where the maps of groupoid are the diagonal morphisms

$$\begin{tikzcd} P \times P \arrow[r] \arrow[d, shift right] \arrow[d, shift left] & M \times M \arrow[d, shift right] \arrow[d, shift left] & G \arrow[l] \arrow[d, shift right] \arrow[d, shift left] \\ P \arrow[r, "\Delta"] & M \times M & M \arrow[l, "\Delta"'] \end{tikzcd}$$

Proof. Recall that the weak pullback $\mathrm{Pair}(P) \times_{[M \times M]} G_\bullet$ has objects the triples (p, α, m) with p an object in $\mathrm{Pair}(P)$, m an object in $G_\bullet$ and α a morphism in $[M \times M]$. It is easy to see that this is just the manifold P. Morphisms between p and p' are morphisms between $\phi(p)$ and $\phi(p')$. So morphisms in $\mathrm{Pair}(P) \times_{[M \times M]} G_\bullet$ are $P \times_M G \times_M P$, which are exactly the morphisms of the pullback groupoid $\phi^*(G_\bullet)$. □

Example 1.5.5 (Pullback Cêch groupoid)**.** A fundamental example which we will use in the following chapter is the pullback groupoid of an open cover: let $G \rightrightarrows M$ be a Lie groupoid and $\{U_i\}$ an open cover of M. Then there is a natural submersion $\iota : \coprod U_i \to M$, and we can consider the pullback groupoid $\iota^*(G_\bullet)$. By definition, the manifold of morphism of this groupoid is given by $\coprod_{i,j} U_i \times_s G_t \times U_j$. Here

$U_i \times_s G_t \times U_j$ is the set of morphisms in G with source in U_i and target in U_j. In general at the n-th level of the nerve, we will have

$$\coprod_{i_0,\dots,i_n} U_{i_0} \times_s G_{\ t} \times U_{i_1} \times_s G_{\ t} \times \cdots \times_t U_{i_n}$$

which we also write $U_{i_0,\dots,i_n} \times_{M^{n+1}} G^n$. Composition, source and target are defined obviously.

1.5.2 The quotient stack of a Lie groupoid

By the end of chapter 1.4.2, we pointed out that our definition of quotient stack works for any groupoid object, but that with this definition torsors are not representable in general. We now check that for a Lie groupoid $X_1 \rightrightarrows X_0$, all torsors are representable, and that they coincide with the usual notion of smooth $X_1 \rightrightarrows X_0$-principal bundles.

Proposition 1.5.6. *Let $Diff$ be the site of smooth manifolds and open covers, and $X_1 \rightrightarrows X_0$ be a Lie groupoid. Then all the $X_1 \rightrightarrows X_0$-torsors are representable i.e. they are manifold.*

Proof. By proposition 1.4.7, we know that all torsors are locally trivial. Now, a trivial torsors induced by a map $\phi : M \to X_0$ in $Diff$ is $P = X_1 \times_{t,X_0,\phi} M$. This is the pull back of the submersion t and the smooth map ϕ, so it is a manifold. Now, a sheaf over $Diff$ which is locally a manifold is globally a manifold, so we are done. □

Torsors over Lie groupoids are also called principal groupoid bundles.

1.5.3 Differentiable (1-)stacks

We know from theorem 1.3.3, that (1-)stacks are organised in a 2-category. and differentiable stacks are just the full sub 2-category of those stack which admit a smooth atlas.
By the Yoneda embedding (which is fully faithfull) we are allowed to think to a manifold as a stack over $Diff$: in the following by "manifold" we often mean its image by the Yoneda embedding.

Definition 1.5.7.

1. A stack is called **representable** if it is isomorphic to a manifold;

2. A morphism of stacks $\mathfrak{X} \to \mathfrak{Y}$ is called **representable** if for any manifold M

and morphism $M \to \mathfrak{Y}$, the homotopical fiber product

$$\begin{tikzcd} \mathfrak{X} \times_M \mathfrak{Y} \arrow[r] \arrow[d] & \mathfrak{X} \arrow[d] \\ M \arrow[r] & \mathfrak{Y} \end{tikzcd}$$

is representable;

3. a morphism of stack $\mathfrak{X} \to \mathfrak{Y}$ is a **surjective representable submersion** if it is an epimorphism of presheaves, it representable and the map $\mathfrak{X} \times_M \mathfrak{Y} \to M$ is a submersion.

Definition 1.5.8. A **differentiable stack** si a stack $\mathfrak{X}$ which admits a surjective representable submersion $X \to \mathfrak{X}$ from a manifold X. We call $X \to \mathfrak{X}$ a smooth **atlas** for $\mathfrak{X}$.

We now prove that the data of a differentiable stack with the choice of an atlas is equivalent to that of a Lie groupoid. Let $x : X_0 \to \mathfrak{X}$ be a smooth atlas for $\mathfrak{X}$. By definition, the homotopy pullback

$$\begin{tikzcd} X_0 \times_{\mathfrak{X}} X_0 \arrow[r, "t"] \arrow[d, "s"'] & X_0 \arrow[d, "x"] \\ X_0 \arrow[r, "x"] & \mathfrak{X} \end{tikzcd}$$

is representable, i.e. there exists a manifold X_1 and an isomorphims of stacks $X_1 \simeq X_0 \times_{\mathfrak{X}} X_0$. We recall that the homotopy fiber product of presheaves of groupoids is calculated objectwise as the 2-fiber product of groupoids. In this case, for any manifold U, the fiber product

$$X_0(U) \times_{\mathfrak{X}(U)} X_0(U)$$

is the set of triples (f, g, ϕ) such that $f, g \in C^\infty(U, X_0)$ and $\phi : x(f) \to x(g)$ is an isomorphism in $\mathfrak{X}(U)$ (see [27] for details).
On can easily check that the maps s, t define a groupoid structure on $X_0 \times_{\mathfrak{X}} X_0 \rightrightarrows X_0$. Moreover, by definition of an atlas, s and t are surjective submersions, so $X_0 \times_{\mathfrak{X}} X_0 \rightrightarrows X_0$ is a Lie groupoid.

Theorem 1.5.9. *Let $\mathfrak{X}$ be a differentiable stack, $x : X_0 \to \mathfrak{X}$ be a smooth atlas and $X_1 \rightrightarrows X_0$ be the correspondent Lie groupoid. Then there is a canonical isomorphism of stacks*

$$[X_0/X_1] \simeq \mathfrak{X}$$

Proof. We consider a stack morphism $\Psi : \mathfrak{X} \to [X_0/X_1]$ which, for any manifold U is defined by the groupoid morphism $\Psi_U : \mathfrak{X}(U) \to [X_0/X_1](U)$, which sends:

1. An object $u : U \to \mathfrak{X} \in \mathfrak{X}(U)$ to $U \times_{u,\mathfrak{X},x} X_0 \in [X_0/X_1]$

2. A morphisms

$$\begin{tikzcd} U \arrow[r, "u"] \arrow[d, "f"'] & \mathfrak{X} \\ U' \arrow[ur, "u'"'] & \end{tikzcd}$$

in $\mathfrak{X}(U)$ to the ($X_1 \rightrightarrows X_0$-equivariant) map

$$f \times id : U \times_{u,\mathfrak{X},x} X_0 \longrightarrow U' \times_{u',\mathfrak{X},x} X_0$$

We need to check that this is a well defined morphism of stacks, and that it is a monomorphism and an epimorphism of presheaves of groupoids.

- **Well defined**: What we need to check is that the homotopical fiber product $U \times_{u,\mathfrak{X},x} X_0$ is an $X_1 \rightrightarrows X_0$-torsor over U. We know that X_1 is isomorphic to $X_0 \times_{\mathfrak{X}} X_0$, and it is clear that $U \times_{u,\mathfrak{X},x} X_0$ is a $X_0 \times_{\mathfrak{X}} X_0 \rightrightarrows X_0$-torsor, so it is also an $X_1 \rightrightarrows X_0$-torsor. Then it is easy to check that Ψ is natural, so it is a morphism of stacks.

- **Mono**: we have to show that for any $u, u' \in \mathfrak{X}(U)$ and isomorphism $\alpha : \Psi(u) \to \Psi(u')$, there exist a unique isomorphism $\beta : u \to u'$ such that $\Psi(\beta) = \alpha$. For this we choose a covering $\{U_i\}$ of U which trivialize both $\Psi(u)$ and $\Psi(u')$. Then over each open U_i of this trivialization there are unique morphisms $\beta_i : u\mid_{U_i} \to u'\mid_{U_i}$ such that $\Psi(\beta_i) = \alpha\mid_{U_i}$. By unicity those isomorphisms concide over intersections, and since $\mathfrak{X}$ is a stack they glue to an isomorphism β such that $\Psi(\beta) = \alpha$.

- **Epi**: Let $P \to U$ be an $X_1 \rightrightarrows X_0$-torsor over U, and let $\{U_i\}$ be a trivializing open cover of U. Then there exist sections $U_i \to P$ giving maps $U_i \to P \to X_0 \to \mathfrak{X}$ such that the composition is in $\mathfrak{X}(U_i)$. So the restrictions $P_{|U_i}$ are in the image of Ψ, which means that Ψ is locally surjective, i.e. an epimorphism.

□

Example 1.5.10. Let G be a Lie group, and $BG := [*/G]$ the corresponding quotient stack. We know that for any manifold U, $[*/G](U)$ is the groupoid of smooth principal G-bundles over U.

- An atlas for BG is the map $* \to BG$ where $*$ is the point intended as a stack, which means that it is the trivial presheaf of groupoids $U \mapsto *$. Then a map $* \to BG$ is the choice, for each manifold U, of a principal G-bundle over U.

For this we have a canonical map, which send a manifold U to the trivial G-bundle $U \times G$. It is clear that this map is a surjective representable submersion $* \to BG$. Moreover, for any manifold U, we have

$$*(U) \times_{BG(U)} *(U) = \{(*, *, \phi) \mid \phi \text{ is an isomorphism in } BG(U)\} \simeq G$$

so we obtained again the Lie groupoid $G \rightrightarrows *$.

- More generally, for any manifold M, there is a canonical morphism of stacks $M \to BG$ giving an atlas of BG. In this case M is intended as the functor of points $U \to \mathrm{Hom}(U, M)$, and we just need to specify which is the image the identity morphisms id_M, which is again the trivial G-bundle $M \times G \to M$. In this case one can easily check that $M \times_{BG} M \simeq M^2 \times G$, and the Lie groupoid associated to this smooth atlas is $M^2 \times G \rightrightarrows M$.

Of course $G \rightrightarrows *$ and $M^2 \times G \rightrightarrows M$ are Morita equivalent groupoids, since they induce the same quotient stack, but they are not isomorphic.

Example 1.5.11. Let M be a manifold. The identity $id : M \to M$ is an atlas for the stack M, corresponding to the pair groupoid $\mathrm{Pair}(M) \rightrightarrows M$. Let now $\{U_i\}$ be an open cover of M. Then $\coprod_i U_i \to M$ is also an atlas for the stack M and the corresponding groupoid is the Čech groupoid of the open cover $\{U_i\}$,

$$\coprod_{i,j} U_i \cap U_j \rightrightarrows \coprod_i U_i.$$

So the Čech groupoid of any open cover of M is a smooth atlas for the stack M.

We just saw that from a differentiable stack we can obtain different Lie groupoids with different smooth atlases. Two such Lie groupoids are not always equivalent as groupoid. The correct notion of equivalence is that of **Morita equivalence**, which we introduce in the following section.

Example 1.5.12. In the case where $X_0 = *$, i.e. $X_1 = G$ is a Lie groupoid, we saw that $[*/G](U)$ are just the principal G-bundles over U.
By theorem 1.2.5 we know that stacks are organized in a 2-category, so in particular between any two stacks there exist a category of morphisms. A 2-categorical version of the Yoneda lemma works in this context: $[*/G](U) = \underline{\mathrm{Hom}}_{St}(U, [*/G])$, where $\underline{\mathrm{Hom}}_{St}$ is the category of morphisms between two stacks. This means that the principal G-bundles over U are classified by the maps of stacks $U \to [*/G]$.

1.6 Some constructions for Lie groupoids

We recall briefly some constructions related to Lie groupoid theory which we will need in the following.

1.6.1 The nerve of a Lie groupoid

We briefly recall the nerve construction for a Lie groupoid and fix some notations. Let $G \rightrightarrows M$ be a Lie groupoid. The nerve of $G \rightrightarrows M$ is the simplicial object $G_\bullet$, where G_k is the k-fold fiber product

$$G_k = \underbrace{G_1 \times_{t,M,s} \cdots \times_{t,M,s} G_1}_{k \text{ times}} = \{(g_1, g_2, \ldots, g_k) \mid t(g_{i-1}) = s(g_i)\} \tag{1.5}$$

We will often note by $G^{(k)}$ the k-fold fiber product (1.5). Since the source and the target maps are submersions, $G^{(k)}$ is a manifold. The simplicial maps are given by face maps:

$$d_i(g_1, \ldots, g_k) = \begin{cases} (g_2, \ldots, g_k) & \text{if } i = 0 \\ (g_1, \ldots, g_i g_{i+1}, \ldots g_k) & \text{if } 0 < i < k \\ (g_1, \ldots, g_{k-1}) & \text{if } i = k. \end{cases}$$

and degeneracy maps

$$s_i(g_1, \ldots g_k) = (g_1, \ldots, g_i, 1, g_{i+1}, \ldots g_k)$$

for $0 \leq i \leq k$. Note that the face maps are surjective submersions.

Definition 1.6.1. Let $X_0 \to \mathfrak{X}$ be a smooth atlas for a differentiable stack $\mathfrak{X}$. Then the nerve of the Lie groupoid $X_0 \times_{\mathfrak{X}} X_0 \rightrightarrows X_0$ defines an augmented simplicial manifold $X_\bullet \to \mathfrak{X}$ which we call a **resolution** or a **presentation** of the differentiable stack $\mathfrak{X}$. Note that the stack $\mathfrak{X}$ is the homotopy colimit of the simplicial diagram $X_\bullet$.

1.6.2 The Lie algebroid of a Lie groupoid

Lie algebroids are to Lie groupoids what Lie algebras are to Lie groups and encode their infinitesimal structure.

Definition 1.6.2. A **Lie algebroid** over a manifold M is a vector bundle $E \to M$, together with a Lie bracket on its module of sections $\Gamma(E)$, and a morphisms of vector bundles $\rho : E \to TM$ called **anchor map**, such that:

1. ρ commutes with the brackets: for all $X, Y \in \Gamma(E)$,

$$\rho[X, Y] = [\rho(X), \rho(Y)];$$

2. it satisfy the Leibniz rule: for all $X, Y \in \Gamma(E)$, $f \in C^\infty(M)$

$$[X, f \cdot Y] = \rho(X) f \cdot Y + f \cdot [X, Y]$$

Let $G \rightrightarrows M$ be a Lie groupoid with source, target and identity denoted respectively by s, t and ι. We can associate to it a vector bundle on M defined by:

$$A := \ker(ds : \iota^* TG \to TM)$$

and an anchor map $dt : A \to TM$.

Proposition 1.6.3. *With definition as above, $A \to TM$ is a Lie algebroid. We call it the* **Lie algebroid associated to** *or the* **core complex** *of the Lie groupoid $G \rightrightarrows M$.*

See [44] for details.

1.7 The 2-category of Lie groupoids

Definition 1.7.1. A morphism of Lie groupoids

$$\begin{tikzcd} X_1 \arrow[r] \arrow[d, shift right] \arrow[d, shift left] & Y_1 \arrow[d, shift right] \arrow[d, shift left] \\ X_0 \arrow[r] & Y_0 \end{tikzcd}$$

is a Morita morphism if:

1. $X_0 \to Y_0$ is a surjective submersion;
2. The following square

$$\begin{tikzcd} X_1 \arrow[r] \arrow[d] & X_0 \times X_0 \arrow[d] \\ Y_1 \arrow[r] & Y_0 \times Y_0 \end{tikzcd}$$

   is cartesian.

Example 1.7.2. Let $G \rightrightarrows M$ be a Lie groupoid, $P \to M$ a surjective submersion, and $\phi^*(G)$ the associated pullback groupoid, of example 1.5.2. Then $\phi^*(G_\bullet) \to G_\bullet$ is a Morita morphism. It is easy to see that, up to isomorphism, all Morita morphisms are constructed as pullbacks of submersions.

We need an equivalence relation, so we will define Morita equivalences to be zig-zags of Morita morphisms:

Definition 1.7.3. We say that two Lie groupoids $X_\bullet$ and $Y_\bullet$ are **Morita equivalent** if there exist a Lie groupoid $Z_\bullet$, and Morita morphisms $Z_\bullet \to X_\bullet$ and $Z_\bullet \to Y_\bullet$.

A zig-zag of Lie groupoid morphisms can be rephrased in the language of bibundles, which we briefly expose now. The two approachs are equivalent, but sometimes it is more convenient to use bibundles for computations.

Definition 1.7.4. Let $X_\bullet$ and $Y_\bullet$ be two Lie groupoids. A **smooth $(X_\bullet, Y_\bullet)$-bibundle** is a manifold M with a left $X_\bullet$-action and a right $Y_\bullet$-action which commute. A bibundle is called principal (resp. right principal, left principal) if the actions are **principal** (resp. right principal, left principal). Namely, a $(X_\bullet, Y_\bullet)$-principal bibundles is a manifold P with submersions

$$\begin{tikzcd} H_1 \arrow[d, shift left] \arrow[d, shift right] & P \arrow[dl] \arrow[dr] & G_1 \arrow[d, shift left] \arrow[d, shift right] \\ H_0 & & G_0 \end{tikzcd}$$

such that P is a principal $G_\bullet$-bundle on H_0 and a principal $H_\bullet$-bundle on G_0.

Let M and N be, respectively, a $(X_\bullet, Y_\bullet)$ and $(Y_\bullet, Z_\bullet)$-bibundles. We can define a composition bibundle as

$$N \circ M := (M \times_{Y_0} N)/Y_1$$

where the quotient is taken with respect to the diagonal action

$$y \cdot (m, n) := (m \cdot y^{-1}, y \cdot n)$$

If we ask for M and N to be right principal, then $N \circ M$ is again smooth, and it is a $(X_\bullet, Z_\bullet)$-right principal bibundle. This give a well defined composition, up to isomorphisms of bibundles. We refer to [12] for more details.

It is easy to see that Lie groupoids, Lie groupoid morphisms and right principal smooth bibundles are organized in a weak 2-category *Gpd* (or bicategory, see chapter 4.1), and weakly invertible morphisms are exactly Morita morphisms. The following theorem gives an explicit description of the 2-category of differentiable (1-) stacks.

Theorem 1.7.5 ([12], Thm. 2.18)**.** *The weak 2-category Gpd is 2-equivalent to the 2-category of differentiable stacks.*

We will not give the definition of 2-equivalence here, we just point out that Morita equivalent Lie groupoids induce isomorphic quotient stacks:

Corollary 1.7.6 ([5], Theorem 2.26)**.** *Let $X_\bullet$ and $Y_\bullet$ be two Lie groupoids, and $\mathfrak{X}$, $\mathfrak{Y}$ their quotient stacks. Then the following are equivalent:*

1. *The differentiable stacks $\mathfrak{X}$ and $\mathfrak{Y}$ are isomorphic;*
2. *The Lie groupoids $X_\bullet$ and $Y_\bullet$ are Morita equivalent;*
3. *There exist an $(X_\bullet, Y_\bullet)$-bibundle such that the two actions are principal.*

Moreover, the stack morphism between $\mathfrak{X}$ and $\mathfrak{Y}$ can be described with right principal $(X_\bullet, Y_\bullet)$-bibundles or 2-terms zig-zags of Lie groupoid morphisms, with the left pointing arrow being Morita (cf. [5], Lemmas 2.29, 2.30, 2.31). Those are often called **generalized morphisms** of Lie groupoid.

Remark 1.7.7. A smooth biprincipal $X_\bullet - Y_\bullet$ bibundle is the same as a zig-zag $X_\bullet \leftarrow Z_\bullet \rightarrow Y_\bullet$ of Morita morphisms.

Remark 1.7.8. Note that between two differentiable stacks $\mathfrak{X}$ and $\mathfrak{Y}$ there is a category of morphisms, which we denote by $\mathrm{Hom}_{St}(\mathfrak{X}, \mathfrak{Y})$. The 2-Yoneda embedding for stacks says that for $\mathfrak{X}$ a differentiable stack and M a manifold, there is a natural equivalence of groupoids

$$\mathfrak{X}(M) \simeq \mathrm{Hom}_{St}(M, \mathfrak{X})$$

Remark 1.7.9. In a large part of this thesis, we study differentiable 1-stack, which are, roughly speaking, sheaves of 1-homotopy types over the differentiable site, endowed with a smooth structure. In section 1.3 we gave the notion of higher differentiable stacks, which are sheaves of higher homotopy types. The ∞-category of these objects is obtained by a localization at hypercovers of the category of simplicial presheaves over $Diff$. In particular, a simplicial manifold can be thought as a stack by the Yoneda embedding

$$sDiff \hookrightarrow sPr(Diff)$$

By the nerve construction (cf. appendix 1.6.1), a Lie groupoid is a special case of a simplicial manifold, so we could intend it as an higher stack. From now on, we will often identify a Lie groupoid with a simplicial manifold by means of the nerve construction.

Chapter 2

Cohomology theories for stacks

In this chapter we study cohomology theories for stacks. For a (non necessarily paracompact) smooth manifolds, there are many non-equivalent cohomology theories (sheaf cohomology, De Rham cohomology, hypercohomology, Čech cohomology) which are related to one another in different, sometimes subtle ways.
We extend these definitions to differentiable stacks, we study some property and examples and we conclude by recalling in 2.4.1 the Hodge to De Rham (or Bott) spectral sequence for stacks:

Theorem (cf. [6])**.** *Let $\mathfrak{X}$ be a differentiable stack. Any cofoliation on $\mathfrak{X}$ gives rise to a E_1-spectral sequence:*

$$E_1^{m,n} = \bigoplus_{i+2k=m} H^n(\mathfrak{X}, \Lambda^i\Omega \otimes S^k\Gamma) \Rightarrow H_{DR}^{m+n}(\mathfrak{X})$$

With the approach of [6] one cannot prove that this spectral sequence is an invariant of the cofoliation, or in other words, that it does not depend on the choice of a flat atlas for the stack $\mathfrak{X}$.
In chapter 3 we state the same result for stacks, with the language of representations up to homotopy, without any choice of a cofoliation.

2.1 Cohomology theories

We first recall some notations in homological algebra:

- In the homological notation, for non-derived stacks both tangent and cotangent complexes are chain complexes, the first concentrated in non-negative degree and the second in non-positive degrees, the differential is always decreasing the degree (for the non-derived case).
- If we use cohomological notation, we only have cochain complexes and the differential increases the degree.

- We write $C_\bullet[k]$ to mean that $C_\bullet[k]_i = C_{k+i}$.

In the following we use homological notations; for example the tangent complex of a Lie groupoid is concentrated in degrees 1 and 0 and the cotangent in degrees -1 and 0.

2.1.1 Sheaf cohomology

For any sheaf $\mathcal{F}$ on a manifold M, we define the i-th **sheaf cohomology** of M with values in $\mathcal{F}$, denoted $H(M, \mathcal{F})$ as follows: let $\Gamma(M, \mathcal{F}) := \mathcal{F}(M)$. Then the global section functor

$$\begin{array}{rcl} \Gamma : AbSh(M) & \to & Ab \\ \mathcal{F} & \mapsto & \Gamma(M, \mathcal{F}) \end{array}$$

is a functor from the category of sheaves of abelian groups over M to the category of abelian groups which is left exact, i.e. for any exact sequence

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$

the following sequence is exact

$$0 \to \mathcal{F}(M) \to \mathcal{G}(M) \to \mathcal{H}(M)$$

Note that it can happen that $\mathcal{G} \to \mathcal{H}$ is surjective as map of sheaves (i.e. there are local sections) but that $\mathcal{G}(M) \to \mathcal{H}(M)$ is not surjective.

Definition 2.1.1. For any $i \geq 0$, we define the sheaf cohomology functor as the right derived functor of the global section functor Γ:

$$H^i(M, \cdot) := R^i\Gamma(M, \cdot)$$

We remind that $R^i\Gamma(M, \cdot)$ are calculated as follows: for any $\mathcal{F}$ sheaf of abelian groups over M, we take a resolution by injectives

$$\mathcal{F} \to \mathcal{I}_0 \to \mathcal{I}_1 \to \mathcal{I}_2 \to \dots$$

and we apply to it the functor Γ

$$\Gamma(M, \mathcal{I}_0) \to \Gamma(M, \mathcal{I}_1) \to \Gamma(M, \mathcal{I}_2) \to \dots$$

and we compute the cohomology groups of this cochain complex.

2.1.2 Cohomology of coherent sheaves

In the analytical and algebraic settings, coherent sheaves play a special role: the theorems A and B of Cartan say that if X is a Stein space (in the analytical case), or an affine variety (in the algebraic case), and $\mathcal{F}$ a coherent sheaf on X, then:

- $\mathcal{F}$ is spanned by its global sections, which means that there exist $\sigma_1, \ldots, \sigma_n \in \mathcal{F}(X)$ spanning all the stalks $\mathcal{F}_{X,x}$;
- $H^i(X, \mathcal{F}) = 0$ for all $i > 0$.

This imply that in the differentiable case, when we have a fiber bundle $V \to M$, which is a coherent sheaf over M, its cohomology are just the global sections

$$H^0(M, V) = \Gamma(M, V), \quad H^i(M, V) = 0$$

2.1.3 Hypercohomology

Hypercohomology extends the definition of cohomology of sheaves to the case of a chain complex of sheaves $\mathcal{F}_\bullet$ over M.
If the complex is bounded below, it is known that there exist an injective resolution $\mathcal{F}_\bullet \to \mathcal{I}_\bullet$ i.e. $\mathcal{I}_j$ is injective for all j and the map is a quasi-isomorphism. Then the hypercohomology is just defined by

$$\mathbb{H}^i(M, \mathcal{F}_\bullet) := H^i(\Gamma(\mathcal{I}_\bullet))$$

Of course sheaf cohomology is a special case of hypercohomology for a complex of sheaves concentrated in degree 0.
In the next paragraphs we will use the following well known result (cf. [32]):

Proposition 2.1.2. *There exist a spectral sequence*

$$E_1^{m,n} = H^n(M, \mathcal{F}_m) \Rightarrow \mathbb{H}^{n+m}(M, \mathcal{F}_\bullet)$$

The same result is true for complexes of sheaves on stacks, once one has extended the definitions of sheaf cohomology and hypercohomology.

2.1.4 Čech cohomology

Let $\{U_i\}$ be an open cover of the smooth manifold M. We use the standard notation $U_{i_1,\ldots,i_n} := U_{i_1} \cap \cdots \cap U_{i_n}$. The general definition of Čech cohomology of M with coefficients in $\mathcal{F}$, $\check{H}(M, \mathcal{F})$ uses a direct limit over all open covers. The **Leray theorem** says that if the covering $\{U_i\}$ is acyclic for $\mathcal{F}$, i.e. all the non zero cohomology groups vanish $H^i(U_{i_1\ldots i_n}, \mathcal{F}) = 0$, there is a simpler definition of Čech cohomology:

Definition 2.1.3. Let $\mathcal{F}$ be a sheaf on M such that the covering $\{U_i\}$ is acyclic. The simplicial diagram of abelian groups $\mathcal{F}(U_\bullet)$ becomes a cochain complex by using the alternating sum of the face maps:

$$\mathcal{F}(U)^\bullet := \bigoplus_i \mathcal{F}(U_i) \to \bigoplus_{i,j} \mathcal{F}(U_{ij}) \to \bigoplus_{i,j,k} \mathcal{F}(U_{i,j,k}) \to \dots$$

The Čech cohomology $\check{H}^\bullet(X, \mathcal{F})$ is the cohomology of the cochain complex $\mathcal{F}(U)^\bullet$.

Remark 2.1.4. For a paracompact smooth manifold, there always exists a good cover, i.e. a cover such that $U_{i_1,\dots,i_k}$ is homeomorphic to $\mathbb{R}^n$ for all $i_1, \dots, i_k$. Then the Čech cohomology can be calculated on good covers.

Proposition 2.1.5. *There is a map*

$$\check{H}^i(M, \mathcal{F}) \to H^i(M, \mathcal{F})$$

from Čech cohomology to sheaf cohomology. It is always an isomorphism for $i = 0, 1$, but it may fail to be so for higher i's. If $\mathcal{F}$ is acyclic over the $U_{i_1,\dots,i_k}$, then Cech cohomology agrees with sheaf cohomology.

Corollary 2.1.6. *Let M be a manifold which admits a good cover and $\mathcal{F}$ be a sheaf on it. Then*

$$\check{H}^i(M, \mathcal{F}) \simeq H^i(M, \mathcal{F})$$

2.1.5 De Rham cohomology

Definition/Proposition 2.1.7. The De Rham cohomology of a manifold M is equivalently:

1. the cohomology of the cochain complex

$$\Omega^0(M) \xrightarrow{d_{DR}} \Omega^1(M) \xrightarrow{d_{DR}} \Omega^2(M) \longrightarrow \dots$$

   with the usual De Rham differential d_{DR};

2. the hypercohomology $\mathbb{H}^\bullet(M, (\Omega^\bullet, d))$. where $(\Omega^\bullet, d)$ is the complex of sheaves of forms over M.

We denote it by $H^\bullet_{DR}(M)$.

Proof. The sheaves Ω^p are acyclic over manifolds, so they are injective, and the hypercohomology can be computed by $H^\bullet(\Gamma(\Omega^\bullet))$. □

Theorem 2.1.8. *Let M be a paracompact smooth manifold, and $\overline{\mathbb{R}}$ be the constant sheaf with value $\mathbb{R}$, i.e. the sheafification of the presheaf taking value $\mathbb{R}$ on any open subset of M. Then there are isomorphisms*

$$\check{H}^i(M, \overline{\mathbb{R}}) \simeq H^i_{DR}(M)$$

2.1.6 Hodge to De Rham spectral sequence

There exist a spectral sequence

$$E_1^{p,q} = H^p(M, \Omega^q) \Rightarrow H_{DR}^{p+q}(M)$$

In the spectral sequence the right hand side is the De Rham cohomology, which we have already defined. The left hand side are the Hodge cohomology groups (if we consider algebraic varieties), but in the differentiable case they are just the cohomology groups of the vector bundle $\Omega^q = \Lambda^q(T^*M)$. By Cartan's theorems

$$H^p(M, \Omega^q) = \begin{cases} 0 & \text{if } p > 0 \\ \Omega^q(M) & \text{if p=0} \end{cases}$$

so the spectral sequence is trivial in this case. This will no longer be true for stack, by a lack of good covers.

2.2 Cohomology of stacks

We now extend the previous definitions to differentiable stacks (in the following we write "stack" for "differentiable stack"). All proofs and details can be found in [6] and [7].

2.2.1 Sheaf cohomology of stacks

We want to extend the definition of sheaf cohomology to stacks. First we need to understand what is a sheaf on a stack. Let $\mathfrak{X}$ be a differentiable stack; we can associate to it a Grothendieck site $\mathfrak{X}_{big}$, called the **big site of** $\mathfrak{X}$, which is the category of morphisms from a manifold to $\mathfrak{X}$, endowed with the topology of submersions, i.e.

$$\begin{tikzcd} M_i \arrow[r] \arrow[d] & \mathfrak{X} \\ M \arrow[ur] & \end{tikzcd}$$

is a cover if and only if $\{M_i \to M\}$ is a cover. We denote this site $\mathfrak{X}_{big}$. Following the results collected in chapter 1, we can define the (model) categories of presheaves and sheaves on the stack $\mathfrak{X}$. For example a presheaf of abelian groups on the stack $\mathfrak{X}$ is just a contravariant functor

$$\mathcal{F} : \mathfrak{X}_{big}^{op} \to Ab.$$

Definition/Proposition 2.2.1. *We have:*

- *A sheaf on the stack $\mathfrak{X}$ is just a presheaf on $\mathfrak{X}$, which is a sheaf whenever restricted to any manifold;*

- *If $X_\bullet$ is the nerve of an atlas $X_0 \to \mathfrak{X}$ (see 1.6.1), then the data of a sheaf on $\mathfrak{X}$ are sheaves $\mathcal{F}_n$ on X_n, and compatible maps $\pi^*\mathcal{F}_m \to \mathcal{F}_n$ for every structure morphism $\pi : X_n \to X_m$.*

- *A sheaf $\mathcal{F}$ over $\mathfrak{X}$ is a vector bundle if $\mathcal{F}_n$ is a vector bundle for all n and the compatible maps $\pi^*\mathcal{F}_m \to \mathcal{F}_n$ are isomorphisms.*

We can now extend the notion of cohomology $H^i(\mathfrak{X}_{big}, \mathcal{F})$ for any sheaf on the big site of $\mathfrak{X}$, just by copying word by word the definition in the case of a sheaf on a manifold.

2.2.2 Čech cohomology of stacks

Let now $\mathcal{F}$ be a sheaf on a differentiable stack $\mathfrak{X}$, and let $X_\bullet$ be the resolution of a smooth atlas $X_0 \to \mathfrak{X}$. We apply $\mathcal{F}$ to the resolution and we obtain a cosimplicial abelian group $\mathcal{F}(X_\bullet)$. As usual we can made it a cochain complex just by taking the differential of alternating sums of the simplicial maps

$$\mathcal{F}(X_0) \to \mathcal{F}(X_1) \to \mathcal{F}(X_1 \times_{X_0} X_1) \to \dots$$

and then calculate the cohomology groups $H^i(\mathcal{F}(X_\bullet))$ of this cochain complex.

Proposition 2.2.2. *If $\mathcal{F}$ is acyclic on manifold, i.e. $H^i(M, \mathcal{F}) = 0 \ \forall i \neq 0$, then the cohomology groups $H^i(\mathcal{F}(X_\bullet))$ are Morita invariant, so we can set*

$$\check{H}^i(\mathfrak{X}, \mathcal{F}) := H^i(\mathcal{F}(X_\bullet))$$

for any atlas $X_0 \to \mathfrak{X}$. We call it the i-th **Čech cohomology group of $\mathfrak{X}$ with values in the sheaf $\mathcal{F}$.**

Remark 2.2.3. Note that $\mathcal{F}$ being acyclic over manifold does not mean that $H^i(\mathcal{F}(X_\bullet))$ is zero, and it is easy to find examples of stacks with non trivial cohomology groups. We are doing exactly the analogue of Čech cohomology for manifold, but in this case we do not have always a good cover for the stack $\mathfrak{X}$, so we ask the sheaf to be acyclic over manifold. In general we will need to use hypercovers if we want to make the same calculation over general sheaves (cf. [6]).

Example 2.2.4. Let $\mathfrak{X}$ be a differentiable manifold, and $\mathcal{C}^\infty$ be the sheaf of C^∞-functions on $\mathfrak{X}$, sending a manifold M to the algebra of smooth functions $C^\infty(M)$. This extends to a well defined sheaf on the stack $\mathfrak{X}$. The Čech cohomology $\check{H}^*(\mathfrak{X}, \mathcal{C}^\infty)$ is calculated by mean of a smooth atlas $G \rightrightarrows M \to \mathfrak{X}$, and is in particular the cohomology of the complex

$$C^\bullet(G) := C^\infty(M) \to C^\infty(G) \to C^\infty(G^{(2)}) \to \dots.$$

In section 3.3.2 we will see that this is a dg-algebra, called the **dg-algebra of smooth functions on the Lie groupoid** $G \rightrightarrows M$. Then the Čech cohomology of $\mathfrak{X}$ with coefficient in $\mathcal{C}^\infty$ is exactly what we will call the **differentiable cohomology** of the Lie groupoid $G \rightrightarrows M$, and the well definedness on $\mathfrak{X}$ is exactly the Morita invariance of differentiable cohomology of proposition 3.3.12.

2.2.3 De Rham cohomology of stacks

With the same notation as above, it is easy to see that we can construct a double complex

$$\begin{tikzcd}
 & & & & \\
\Omega^0(G_3) \arrow[r, "d_{DR}"] & \Omega^1(G_3) \arrow[r, "d_{DR}"] & \Omega^2(G_3) \arrow[r, "d_{DR}"] & \Omega^3(G_3) \arrow[r, "d_{DR}"] & {} \\
\Omega^0(G_2) \arrow[r, "d_{DR}"] \arrow[u, "d_{cs}"] & \Omega^1(G_2) \arrow[r, "d_{DR}"] \arrow[u, "d_{cs}"] & \Omega^2(G_2) \arrow[r, "d_{DR}"] \arrow[u, "d_{cs}"] & \Omega^3(G_2) \arrow[r, "d_{DR}"] \arrow[u, "d_{cs}"] & {} \\
\Omega^0(G_1) \arrow[r, "d_{DR}"] \arrow[u, "d_{cs}"] & \Omega^1(G_1) \arrow[r, "d_{DR}"] \arrow[u, "d_{cs}"] & \Omega^2(G_1) \arrow[r, "d_{DR}"] \arrow[u, "d_{cs}"] & \Omega^3(G_1) \arrow[r, "d_{DR}"] \arrow[u, "d_{cs}"] & {} \\
\Omega^0(G_0) \arrow[r, "d_{DR}"] \arrow[u, "d_{cs}"] & \Omega^1(G_0) \arrow[r, "d_{DR}"] \arrow[u, "d_{cs}"] & \Omega^2(G_0) \arrow[r, "d_{DR}"] \arrow[u, "d_{cs}"] & \Omega^3(G_0) \arrow[r, "d_{DR}"] \arrow[u, "d_{cs}"] & {}
\end{tikzcd}$$

where d_{DR} is the De Rham differential and d_{cs} is the cosimplicial differential given by the alternating sum of the cosimplicial maps.
We take the total complex

$$Tot(\Omega^\bullet(X_\bullet))^n := \bigoplus_{p+q=n} \Omega^q(X_p)$$

with differential $d(\omega) = d_{DR}(\omega) + (-1)^p d_{cs}(\omega)$ for any $\omega \in \omega^q(G_p)$. The sign change is introduced in order that $d^2 = 0$.

Proposition 2.2.5. *The cohomology of the complex $Tot(\Omega^\bullet(X_\bullet))$ is Morita invariant, so we can define*

$$H^i_{DR}(\mathfrak{X}) := H^i(Tot(\Omega^\bullet(X_\bullet)))$$

Remark 2.2.6. One can easily check that $\overline{\mathbb{R}} \to \Omega^1 \to \Omega^2 \to \dots$ is a resolution of the constant big sheaf $\overline{\mathbb{R}}$ on a differentiable stack $\mathfrak{X}$. So one can think the De Rham cohomology of $\mathfrak{X}$ as the sheaf cohomology with coefficients in $\overline{\mathbb{R}}$, the only difference with the "manifold case" being that the Ω^i are not acyclic over $\mathfrak{X}$ (but they are over the resolution $X_\bullet$).

2.3 The example of equivariant cohomology

In this section we study equivariant cohomology and we see how to translate the Hodge to De Rham spectral sequence in this context. Let M be a G-manifold, $\mathfrak{X}$ the quotient stack for this action, and $\mathcal{F}$ be a sheaf on $\mathfrak{X}$. By descent, global sections on $\mathfrak{X}$ are exactly G-equivariant sections on M, so we can think at equivariant cohomology as the cohomology of $\mathfrak{X}$ with coefficients in the sheaf of symmetric powers of the shifted cotangent complex $Sym^\bullet \mathbb{L}_{\mathfrak{X}}[-1]$

$$H^\bullet_G(M) \simeq H^\bullet(\mathfrak{X}, Sym^\bullet \mathbb{L}_{\mathfrak{X}}[-1])$$

With this viewpoint, proposition 2.3.2 below is a special case of theorem 2.4.1.

2.3.1 The cotangent complex of an action Lie groupoid

Let G be a Lie group. We remind that the tangent at any point of G is a Lie algebra $\mathfrak{g}$, and that the tangent bundle of G is just $TG = G \times \mathfrak{g}$ (this is actually a semidirect product).
The adjoint representation of G on $\mathfrak{g}$ is defined as follows:

- We consider the conjugation map

$$\begin{array}{rcl} \Psi : G & \to & Aut(G) \\ g & \mapsto & \Psi_g : x \mapsto gxg^{-1} \end{array}$$

- We define $Ad_g : \mathfrak{g} \to \mathfrak{g}$ to be the differential of Ψ_g at the identity. This is a Lie algebra automorphism, i.e. $Ad_g \in Aut(\mathfrak{g})$. We look at $Aut(\mathfrak{g})$ as a Lie subgroup of $Gl(\mathfrak{g})$;

- This yields the morphism

$$\begin{array}{rcl} Ad : G & \to & Aut(\mathfrak{g}) \\ g & \mapsto & Ad_g \end{array}$$

  which is a Lie group morphism, so it is a representation of G over $\mathfrak{g}$, called **the adjoint representation** of G on $\mathfrak{g}$.

- If we take the dual morphisms $Ad^*_g = \mathfrak{g}^* \to \mathfrak{g}^*$ we obtain **the coadjoint representation** of G on $\mathfrak{g}^*$.

Let now M be a G-manifold, i.e. a manifold with an action $G \times M \to M$. The tangent bundle of the action Lie groupoid

$$G \times M \rightrightarrows M$$

is $T(G \times M) = M \times \mathfrak{g}$, and we have the differential of the action map

$$M \times \mathfrak{g} \to TM$$

This is the **tangent complex** of the Lie groupoid $G \times M \rightrightarrows M$, and we consider it to be concentrated in (homological) degrees 0 and 1. By dualizing we obtain that the cotangent complex is

$$\mathbb{L}_{\mathfrak{X}} = T^*M \to M \times \mathfrak{g}^*$$

concentrated in degrees 0 and -1, so its symmetric powers are (see section 2.4)

$$Sym^n(\mathbb{L}_{\mathfrak{X}}[-1]) = \bigoplus_{2k+i=n} \Lambda^i T^*M \otimes Sym^k(\mathfrak{g}^*) = \bigoplus_{2k+i=n} \Omega^i(M) \otimes Sym^k(\mathfrak{g}^*)$$

with differentials given by

$$\alpha \otimes p(a_1, \dots, a_h) = \left((a_1, \dots, a_{h+1}) \mapsto \sum_{i=0}^{h} \iota_{\rho(a_i)} \alpha p(a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_h) \right)$$

where $\iota_{\rho(a_i)}\alpha \in \Lambda^{k-1}T^*M$ is the contraction of the k-form α on M with $\rho(a_i)$, which is a section of TM. The Lie group G acts by the adjoint representation on $Sym^\bullet(\mathfrak{g}^*)$ and by pullback on the differential forms on $\Omega^\bullet(M)$. It is easy to see that this action commutes with the differential.

Definition 2.3.1. The G-equivariant cohomology groups of M are $H^k_G(M) := h^k(\Omega^\bullet_G(M))$, where $\Omega^\bullet_G(M)$ is the **Cartan complex** of G, defined by

$$\Omega^n_G(M) := \bigoplus_{2k+i=n} (\Omega^i(M) \otimes Sym^k(\mathfrak{g}^*))^G$$

We can now state the well-known theorem relating the equivariant cohomology of a G-manifold M to the De Rham cohomology of the quotient stack $[M/G]$:

Proposition 2.3.2 ([7], Prop. 13)**.** *If G is compact, there is a natural isomorphism*

$$H^i_G(M) \to H^i_{DR}([M/G])$$

.

Proof. See [7], Lemma 12 and Proposition 13. □

We want to informally motivate why this theorem is a special case of the Hodge to De Rham spectral sequence. Let us consider again $Sym^\bullet(\mathbb{L}_{\mathfrak{X}}[-1])$: in the framework of

[6], we can consider it as a complex of sheaves on $\mathfrak{X}_{big}$, and the choice of a cofoliation allows us to decompose $Sym^n(\mathbb{L}_{\mathfrak{X}}[-1])$ as (cf. [6] chapters 3 and 4)

$$\Gamma(\mathfrak{X}, Sym^n(\mathbb{L}_{\mathfrak{X}}[-1])) = \bigoplus_{2k+i=n} \Gamma(\mathfrak{X}, T^*M \otimes Sym^k(\mathfrak{g}^*))$$

The global sections of any sheaf $\mathcal{F}$ on the stack $\mathfrak{X}$ are the following homotopy colimit:

$$\begin{aligned} \Gamma(\mathfrak{X}, \mathcal{F}) &= \operatorname{hocolim}(\Gamma(M, \mathcal{F}) \leftleftarrows \Gamma(M \times G, \mathcal{F}) \dots) \\ &= \operatorname{hocolim}(\mathcal{F}(M) \leftleftarrows \mathcal{F}(M \times G) \dots) \end{aligned}$$

if G is compact, those global sections over $\mathfrak{X}$ are the G-equivariant sections of $\mathcal{F} = Sym^n(\mathbb{L}_{\mathfrak{X}}[-1])$ on M

$$\bigoplus_{2k+i=n} \Gamma(\mathfrak{X}, T^*M \otimes Sym^k(\mathfrak{g}^*))^G = \bigoplus_{2k+i=n} (\Omega^i(M) \otimes Sym^k(\mathfrak{g}^*))^G$$

so we conclude that proposition 2.3.2 is a special instance of theorem 2.4.1.

Remark 2.3.3. The idea in the case of the action groupoid $G \times M \rightrightarrows M$ is that the data of a complex of vector bundles on $\mathfrak{X}$ (as sheaves over the big site of $\mathfrak{X}$) is just a complex of vector bundles on the manifold M with a G-action on it (where the differential is G-equivariant). Then global sections on $\mathfrak{X}$ are just the G-equivariant sections on M.
In the case of a general stack $\mathfrak{X} := [M/G]$ the data of a complex of vector bundles is more complicated, and it is given by a representation up to homotopy on the Lie groupoid $G \rightrightarrows M$. Actually, also in the case of a G-manifold the data in general will not be so nice, but for the cotangent complex we have a simple and explicit model, which is just two vector bundles over the base endowed with and action of G. In the case of a general Lie groupoid, the cotangent complex can't be described by a 2-terms complex of vector bundles with a strict action of G on it, and it is necessary to introduce the notion of **action up to homotopy** and of **representations up to homotopy**. We will do this in the next chapter.

2.4 Hodge to De Rham spectral sequence, first version

In this section we recall some result of [6], which we will re-state later in the framework of representations up to homotopy.
For a smooth manifold, the Hodge to De Rham (or Bott) spectral sequence links the Hodge cohomology groups, i.e. the cohomology groups $H^q(M, \Omega^p(M))$ of the vector bundles $\Omega^p(M) := \Lambda^i T^*M$ to the De Rham cohomology $H_{DR}(M)$ of M:

$$E_1^{p,q} = H^q(M, \Omega^p) \Rightarrow H_{DR}^{p+q}(M)$$

Theorem 2.4.1 ([6], Thm. 4.21)**.** *Let $\mathfrak{X}$ be a differentiable stack. Any cofoliation on $\mathfrak{X}$ gives rise to a E_1-spectral sequence:*

$$E_1^{p,q} = \bigoplus_{i+2k=p} H^q(\mathfrak{X}, \Lambda^i\Omega \otimes S^k\Gamma) \Rightarrow H_{DR}^{p+q}(\mathfrak{X})$$

We saw in 2.1.2 that there is a spectral sequence

$$E_1^{p,q} = H^q(\mathfrak{X}, Sym^\bullet(\mathbb{L}_\mathfrak{X}[-1])_p) \Rightarrow \mathbb{H}^{p+q}(\mathfrak{X}, Sym^\bullet(\mathbb{L}_\mathfrak{X}[-1]))$$

If the decomposition

$$\bigoplus_{i+2k=p} \Lambda^i\Omega \otimes S^k\Gamma = Sym^\bullet\mathbb{L}_\mathfrak{X}[-1]_p \tag{2.1}$$

would make sense without the choice of a cofoliation, we would have that the hypercohomology of $\mathfrak{X}$ with values in the symmetric powers of the shifted cotangent complex is isomorphic to the De Rham cohomology of $\mathfrak{X}$:

Claim 2.4.2. *Let $\mathfrak{X}$ be a differentiable stack. Then there are isomorphisms*

$$\mathbb{H}^k(\mathfrak{X}, Sym^\bullet(\mathbb{L}_X[-1])) \simeq H_{DR}^k(\mathfrak{X})$$

As we already recalled, the problem here is that the spectral sequence depends on the choice of a cofoliation (cf. [6]), and that equation (2.1) does not make sense without the machinery of cofoliations, which remains quite obscure to us. In the next chapters we try to fill this gap with the language of representations up to homotopy.

Chapter 3

Representations up to homotopy

Representations up to homotopy first appeared in [4] as a solution to the existence of an adjoint representation for a Lie groupoid $G \rightrightarrows M$. They are defined as differentials on the graded algebra of sections $C(G;E)^\bullet$, or equivalently as a sequence of tensors encoding an action "up to homotopy".

In this chapter we first define vector bundles over stack and usual representations of groupoids and explain why representations up to homotopy are generalization of those two objects. Then we point out that representations up to homotopy are exactly cohesive modules over the dg-algebra of smooth functions $C^\bullet(G)$ (cf.[10]) which are builted up from a finitely generated and projective graded module over $C^\infty(M)$, i.e. a graded vector bundle over the base manifold M. This new point of view, which we did not find in the literature, allows us to prove the Morita invariance of the dg-category Rep^∞. We apply this result to define the ∞-category of perfect complexes over a stack $\mathfrak{X}$ as the dg-category $Rep^\infty(G)$ of representations up to homotopy of any Lie groupoid $G \rightrightarrows M$ presenting $\mathfrak{X}$.

3.1 Vector bundles over stacks

We saw that stacks are generalizations of quotients of the action of a Lie group over a manifold. It is well known that the fibers of the tangent bundle of a Lie group G have a Lie algebra structure, and that if M is a G-manifold, there is a natural action of the Lie algebra $\mathfrak{g}$ over TM induced by differentiation. In the next chapters we will study the algebraic structures appearing when differentiating groupoid actions, so we will need to know what is the infinitesimal structure of a differentiable stack. For those purposes we will need the notion of vector bundle and of perfect complex over a stack. We have a good definition for the first in terms of sheaves over the big site of $\mathfrak{X}$ and for the second we will need representations up to homotopy.

In section 2.2.1, we saw that a vector bundle over a stack $\mathfrak{X}$ is a sheaf on the big site $\mathfrak{X}_{big}$ such that the pullbacks on an atlas are compatible vector bundles.
Let $\mathcal{F}$ be a vector bundle over $\mathfrak{X}$. We want to prove that the "descent datum" of a vector bundle over $\mathfrak{X}$ is an X_1-equivariant vector bundle, i.e. a vector bundle F_0 over X_0, an isomorphism of vector bundles over X_1, $\rho : s^*F_0 \to t^*F_0$ such that $\pi_2^*\rho \circ \pi_1^*\rho = m^*\rho$, where $\pi_1, \pi_2, m : X_1 \times_{X_0} X_1 \to X_1$ are respectively the projection over the first, the second component and the composition.

Proposition 3.1.1. *There is a 1-1 correspondence between vector bundles over $\mathfrak{X}$ and X_1-equivariant vector bundles over X_0.*

Proof. Let $\mathcal{F}$ be a vector bundle over $\mathfrak{X}$, and F_n be the pullback on X_n, which, by hypothesis, is a vector bundle. By definition there are linear isomorphisms

$$s^*F_0 \longrightarrow F_1 \longleftarrow t^*F_0$$

By composition we obtain an isomorphism $\rho : s^*F_0 \to t^*F_0$. Using the equalities

$$\begin{cases} s \circ \pi_1 = s \circ m \\ t \circ m = t \circ \pi_2 \\ t \circ \pi_1 = s \circ \pi_2 \end{cases}$$

we have the following diagram

$$\begin{tikzcd}
\pi_1^*s^*F_0 \arrow[r, equal] \arrow[d, "\pi_1^*\rho"] & m^*s^*F_0 \arrow[dr, "m^*\rho"] & \\
\pi_1^*t^*F_0 \arrow[dr, equal] & & m^*t^*F_0 \arrow[d, equal] \\
& \pi_2^*s^*F_0 \arrow[r, "\pi_2^*\rho"] & \pi_2^*t^*F_0
\end{tikzcd}$$

Since there are no higher homotopies, the diagram must commute on the nose, which means exactly that $\pi_2^*\rho \circ \pi_1^*\rho = m^*\rho$.
Conversely let (F_0, ψ) be a X_1-equivariant vector bundle over X_0. We should prove that there are compatible morphisms between the various pullbacks F_n on X_n. By truncation arguments we only need to prove it for $n \leq 2$. Then it is easy too see that the required conditions are necessary and sufficient for $\{F_n\}$ to be a vector bundle over $\mathfrak{X}$. $\square$

Example 3.1.2. In the case of the quotient stack $\mathfrak{X} = [M/M \times G]$ of a Lie group action on a manifold $G \curvearrowright M$, a vector bundle over $\mathfrak{X}$ is just a vector bundle over M endowed with an action of G.

A third important characterization of vector bundle over stacks is given in terms of representable morphisms of stacks. Let $\mathcal{F}$ be a vector bundle over $\mathfrak{X}$. It is easy to see that $F_1 \rightrightarrows F_0$ is a Lie groupoid (in particular a VB-groupoid over $X_1 \rightrightarrows X_0$, cf. [16]), then the quotient $\mathfrak{F} := [F_0/F_1]$ is a differentiable stack and the Lie groupoid morphism $F_\bullet \to X_\bullet$ induces a morphism of stacks $\mathfrak{F} \to \mathfrak{X}$.

Proposition 3.1.3. *Any vector bundle $\mathcal{F}$ over $\mathfrak{X}$ give rise to a morphism of stacks $\mathfrak{F} \to \mathfrak{X}$ such that:*

1. *the morphism is representable;*
2. *for every manifold U and smooth map $f : M \to \mathfrak{X}$ the pullback $U \times_{\mathfrak{X}} \mathfrak{F} \to U$ is endowed with the structure of a vector bundle;*
3. *For any $g : V \to U$ morphism of smooth manifold, the natural morphism*

$$(f \circ g)^*\mathcal{F} \to g^*(f^*\mathcal{F})$$

is a vector bundle isomorphism.

Example 3.1.4. Let M be a manifold, and $\coprod_{I\times I} U_{i,j} \rightrightarrows \coprod_I U_i$ be the Čech groupoid of a trivializing cover $\{U_i\}$ of M. A rank n vector bundle over M can be described by the gluing morphisms between the trivializations over the various U_i. This is the data, for each $(i,j) \in I \times I$, of a morphism $\alpha_{i,j} : U_{i,j} \to GL_n$ satisfying a cocycle condition:

$$\forall i,j,k \quad \alpha_{i,i} = \mathrm{id}, \quad \alpha_{i,j} \circ \alpha_{j,i} = \mathrm{id}, \quad \alpha_{i,j} \circ \alpha_{j,k} = \alpha_{i,k}$$

If we think at GL_n as a Lie groupoid over the point $GL_n \rightrightarrows *$, it is easy to see that those cocycle conditions are exactly the Lie groupoid morphisms from $\coprod_{I\times I} U_{i,j} \rightrightarrows \coprod_I U_i$ to $GL_n \rightrightarrows *$.

Proposition 3.1.5. *The stack $BGL_n = [*/GL_n]$ represents vector bundles of rank n, i.e. for any manifold M, $BGL_n(M)$ is the groupoid of rank n vector bundles over M.*

Proof. By the Yoneda lemma (see 1.7.8),

$$BGL_n(M) = \mathrm{Hom}_{St}(M, BGL_n).$$

Moreover the morphisms between two differentiable stack are just Morita morphisms between some Lie groupoids presenting them.
Now, the manifold M is itself a trivial Lie groupoid $M \rightrightarrows M$ with all structural morphisms being the identity, and $GL_n \rightrightarrows *$ is a Lie groupoid presenting BGL_n. Clearly it is not true that morphisms from $M \rightrightarrows M$ to $GL_n \rightrightarrows *$ encode all vector bundles over M, but considering Morita morphisms from $M \rightrightarrows M$ to $GL_n \rightrightarrows *$ means that we can choose a covering of M and taking morphisms from the Čech groupoid of the cover to $GL_n \rightrightarrows *$, which gives exactly vector bundles over M, as we just saw in example 3.1.4. □

Then the category of rank n vector bundles over any differentiable stack $\mathfrak{X}$ can be defined as morphisms from $\mathfrak{X}$ to BGL_n:

Proposition 3.1.6. *The category of rank n vector bundles over a stack $\mathfrak{X}$ is equivalent to the category of morphisms* $\mathrm{Hom}_{St}(\mathfrak{X}, BGL_n)$.

Corollary 3.1.7. *Let $G \rightrightarrows M$ be a Lie groupoid presenting $\mathfrak{X}$. Then the category of rank n vector bundles over $\mathfrak{X}$ is equivalent to the category of Morita morphisms from $G \rightrightarrows M$ to $GL_n \rightrightarrows *$.*

3.2 Representations of groupoids

In the previuos section, we saw what a vector bundle over a differentiable stack is, and we have been able to translate it into a descent data on the Lie groupoid of an atlas. We ask now what should be a complex of vector bundles over a differentiable stack. In the algebraic context this corresponds to the category of perfect complexes. Those are defined essentially by descent: first, for any affine scheme R one defines an ∞-category of quasi-coherent complexes $QC(R)$, whose homotopy category $Ho(QC(R))$ is the usual derived category $D_{QCoh}(R)$ of perfect complexes on R. Then one shows that the functor

$$\begin{array}{rcl} QC : Aff & \to & \infty - Cat \\ R & \mapsto & QC(R) \end{array}$$

can be extended to an ∞-functor on the whole category of stacks

$$\begin{tikzcd} Aff \arrow[r] \arrow[d, hook] & \infty - Cat \\ St(Aff) \arrow[ur, dotted] & \end{tikzcd}$$

So for any stack $X \in St(Aff)$ there is an ∞-category of quasi-coherent complexes over it. The ∞-category of perfect complexes on X is roughly speaking the full ∞-subcategory of $QC(X)$ whose objects are locally quasi-isomorphic to a complex of vector bundles.
For our purposes we do not need to understand the whole ∞-category of perfect complexes on a differentiable stack (for this we would need to show the descent in the differentiable context), but we will try to understand the simpler case of a differentiable 1-stack, i.e. the quotient of a Lie groupoid.

Let $\mathfrak{X} = [X_0/X_1]$ be such a differentiable stack. A perfect complex on $\mathfrak{X}$ turns out to be a complex of vector bundles on X_0 endowed with a representation up to homotopy of the groupoid $X_1 \rightrightarrows X_0$. This is a natural generalization of the fact that the descent datum of a vector bundle on $\mathfrak{X}$ is a X_1-equivariant vector bundle over X_0.

Example 3.2.1. Let us now describe this idea in detail for the simpler case of $BG = [*/G]$, and the resolution $X_n = G^n$ with the usual simplicial structure.
We saw the descent datum of a vector bundles $\mathcal{F}$ on $\mathfrak{X}$ are vector bundles $\mathcal{F}_n$ over X_n with some compatibilities, given by the structure morphisms. Let $V^\bullet$ the pullback (or restriction) of $\mathcal{F}^\bullet$ to $X_0 = *$. In this case $V^\bullet$ is just a complex of vector spaces. As we saw in the proof of proposition 3.1.1, the compatibilities of $\mathcal{F}_0^\bullet$ and $\mathcal{F}_1^\bullet$ reduce to an isomorphism $s^*\mathcal{F}_0^\bullet \to t^*\mathcal{F}_0^\bullet$.
However, we know that for complexes the condition of being isomorphic is too strict, and quasi-isomorphims behave better in this context. So we only ask for a morphism of complexes of vector bundles which is a quasi-isomorphism on each fiber. In our example the source and target maps are equal and send G to the point $*$, so $s^*V^\bullet = t^*V^\bullet = V^\bullet \times G$ is a complex of trivial vector bundles over G. Then a quasi-isomorphism $\rho : V^\bullet \times G \to V^\bullet \times G$ is just, for any $g \in G$, a quasi-isomorphism $\rho_g : V^\bullet \to V^\bullet$.
In the case of a vector bundle, we saw in proposition 3.1.1 that the second level of coherence gives the equation: $\pi_2^*\rho \circ \pi_1^*\rho = m^*\rho$. In our example we have the following diagram of morphism of complexes

$$\begin{tikzcd} V^\bullet \times G^2 \arrow[r, "\pi_1^*\rho"] \arrow[rr, bend right, "m^*\rho"'] & V^\bullet \times G^2 \arrow[r, "\pi_2^*\rho"] & V^\bullet \times G^2 \end{tikzcd}$$

i.e., for any $g, h \in G$

$$\begin{tikzcd} V^\bullet \arrow[r, "\rho_g"] \arrow[rr, bend right, "\rho_{gh}"'] & V^\bullet \arrow[r, "\rho_h"] & V^\bullet \end{tikzcd}$$

Once again, it is not natural to ask for an equality of morphisms of chain complexes, and we ask for a chain homotopy

$$\gamma : \pi_2^*\rho \circ \pi_1^*\rho \Rightarrow m^*\rho$$

i.e. for any $g, h \in G$ a chain homotopy

$$\gamma_{g,h} : \rho_h \circ \rho_g \Rightarrow \rho_{gh}$$

One can imagine the following steps: for any $g, h, l \in G$ there is a tetrahedron whose faces are the chain homotopies

$$\begin{tikzcd} \rho_g\rho_h\rho_l \arrow[r, Rightarrow, "\gamma_{g,h}\circ id_l"] & \rho_{gh}\rho_l \arrow[r, Rightarrow, "\gamma_{gh,l}"] & \rho_{ghl} \\ \rho_g\rho_h\rho_l \arrow[r, Rightarrow, "id_g\circ\gamma_{h,l}"] & \rho_g\rho_{hl} \arrow[r, Rightarrow, "\gamma_{g,hl}"] & \rho_{ghl} \end{tikzcd}$$

and instead of asking for those two chain homotopies to be equal, we will ask for a 2-homotopy between them and so on...

We will see in the next section that representations up to homotopy precisely encode this huge structure of higher homotopies and coherences, and we will study in detail the case of a two terms chain complex in chapter 4.

3.2.1 The general linear groupoid

Given $G \rightrightarrows M$ a Lie groupoid and $E \to M$ a vector bundle, a **representation** $G \curvearrowright E$ can be defined as a map $\rho : G \times_M E \to E$, $\rho(x \xrightarrow{g} y, e) = \rho_g(e)$, such that

- $\rho_g : E_x \to E_y$ is linear,
- $\rho_{\mathrm{id}} = \mathrm{id}$
- $\rho_h \rho_g = \rho_{hg}$.

A **pseudo-representation** is a sort of non-associative action, it is defined analogously but just requiring (i) and (ii). We also call it a **quasi-action** of G on E.

Example 3.2.2. If $G \rightrightarrows *$ is a Lie group, viewed as a Lie groupoid with a single object, then its representations are the usual representations of Lie groups. If $M \rightrightarrows M$ is a manifold, viewed as a Lie groupoid with only identity arrows, then its representations are the vector bundles over M. More generally, if $G \times M \rightrightarrows M$ is the groupoid arising from a Lie group action $G \curvearrowright M$, then a representation $(G \times M) \curvearrowright E$ is the same as an equivariant vector bundle.

Example 3.2.3. Given M a manifold, a representation $\mathrm{Pair}(M) \curvearrowright E$ of its pair groupoid is the same as a trivialization of E. Given a surjective submersion $q : M \to N$, a representation $M \times_N M \curvearrowright E$ of the submersion groupoid (cf. [21]) is the same as an isomorphism $E \cong q^*E'$ with a pullback vector bundle. This can be further generalized to a foliation $F \subset TM$, which yields a holonomy groupoid $\mathrm{Hol}(F) \rightrightarrows M$, whose representations are the same as foliated bundles.

Example 3.2.4. Let $\mathbb{R}P^2$ denote the real projective plane, and let $E \to \mathbb{R}P^2$ be its tautological line bundle. Since it is not trivial there cannot be a representation of the pair groupoid $\mathrm{Pair}(\mathbb{R}P^2) \curvearrowright E$. Still, we can define a pseudo-representation $\mathrm{Pair}(\mathbb{R}P^2) \curvearrowright E$, by defining for instance $\rho_{(\ell',\ell)}(v)$ as the orthogonal projection of $v \in \ell$ over ℓ'.

Let the **General Linear Groupoid** (see eg. [21]) be defined as

$$GL(E) := \{(x, y, \phi) \mid x, y \in M,\ \phi : E_x \to E_y\}$$

It is easy to check that with the obvious source and target maps and the natural smooth structure there is a Lie groupoid structure on $GL(E) \rightrightarrows M$.
By means of the exponential law, a Lie groupoid representation can be described as a Lie groupoid morphism into the General Linear Groupoid

$$\rho^{\#} : (G \rightrightarrows M) \to (GL(E) \rightrightarrows M) \qquad \rho^{\#}(g) = \rho_g$$

In the case of a pseudo-representation we still have a smooth map $G \to GL(E)$ between the arrow spaces, compatible with source and target but that may fail to preserve the multiplication. This viewpoint allow ones to treat representations as Lie groupoid morphisms

$$\begin{tikzcd} G \arrow[r] \arrow[d, shift right] \arrow[d, shift left] & GL(E) \arrow[d, shift right] \arrow[d, shift left] \\ M \arrow[r, "id"] & M \end{tikzcd}$$

and it is specially useful when dealing with differentiation and integration.

Remark 3.2.5. As we just saw, if we simply ask for the square(s) in the diagram above to commute, we obtain a **pseudo-representation**, which has no reason to commute with the composition. However a morphism of simplicial manifolds between the nerves $G^\bullet$ and $GL(E)^\bullet$ gives always a Lie groupoid morphism on the truncation. Consider the following diagram:

$$\begin{tikzcd} G^{(3)} \arrow[r] \arrow[d, shift right] \arrow[d] \arrow[d, shift left] \arrow[d, shift left=2] & GL(E)^{(3)} \arrow[d, shift right] \arrow[d] \arrow[d, shift left] \arrow[d, shift left=2] \\ G^{(2)} \arrow[r] \arrow[d, shift right] \arrow[d] \arrow[d, shift left] & GL(E)^{(2)} \arrow[d, shift right] \arrow[d] \arrow[d, shift left] \\ G \arrow[r] \arrow[d, shift right] \arrow[d, shift left] & GL(E) \arrow[d, shift right] \arrow[d, shift left] \\ M \arrow[r, "id"'] & M \end{tikzcd} \tag{3.1}$$

Then:

- The commutativity of the lower squares, as already said, gives a quasi-action of G over E.
- The commutativity of the central squares means that this quasi-action commutes with composition.
- The commutativity of the top squares is automatic, and is given by the fact that the composition is associative for any Lie groupoid.

In the next chapter we will construct the general linear 2-groupoid, which is not a Lie groupoid anymore, and in that case the commutativity of the top squares will be an additional data, i.e. an homotopy between different ways of composing 3 composable morphisms.

3.3 Cohesive modules over dg-algebras

In this section we recall the definitions of dg-algebra, dg-module, cohesive module and of the dg-category $\mathcal{P}_A$ of cohesive modules over a dg-algebra $\mathcal{A}$.

3.3.1 The category of graded objects

Let C be a semi-simple Grothendieck category (cf. [41]) endowed with a monoidal structure (for example the category of $\mathbb{K}$-vector spaces). We can form the monoidal category of graded objects in C, denoted $\mathcal{G}r(C)$. Objects are sequences $(M^p)_{p\in\mathbb{Z}}$, morphisms are the graded vector spaces

$$\mathrm{Hom}_{\mathcal{G}r(C)}(M,L)^r := \prod_p \mathrm{Hom}_C(M^p, L^{p+r}), \quad \text{for } r \in \mathbb{Z}$$

The tensor product of two graded objects is given by

$$(M \otimes L)^n := \bigoplus_{p+q=n} M^p \otimes L^q$$

The tensor product of two morphisms $f : M \to M'$ and $g : L \to L'$ of degrees r and s respectively is a morphism of degree $r+s$ such that the n-th component is given by

$$(-1)^{ps} f^p \otimes g^q : M^p \otimes L^q \to M'^{p+r} \otimes L'^{q+s}, \quad p+q=n \tag{3.2}$$

3.3.2 Dg-algebras, dg-modules and dg-categories

Definition 3.3.1. A differential graded (or dg-) algebra is a graded object $A^\bullet$, with a degree 1 map $d : A^\bullet \to A^\bullet$, and a degree 0 map $m : A^\bullet \otimes A^\bullet \to A^\bullet$ such that:

1. $(A^\bullet, m)$ is a graded algebra;
2. $d \circ d = 0$, i.e. d is a differential for $A^\bullet$;
3. It satisfies the "graded Leibniz rule"

$$d(m(a \otimes b)) = m(da \otimes b) + (-1)^{\deg a} m(a \otimes db).$$

where $\deg(a)$ is the degree of a.

Example 3.3.2 (The dg-algebra of a Lie algebroid)**.** Let $\rho : A \to TM$ be a Lie algebroid on a smooth manifold M. By definition $\Gamma(A)$ is endowed with a Lie algebra bracket $[\cdot,\cdot]$ such that for any $f \in C^\infty(M)$, $x, y \in \Gamma(A)$ we have

$$[x, fy] = f[x,y] + (\rho(x)f)y$$

The **dg-algebra of A-differential forms** is the graded vector bundle $\Omega^\bullet(A) := \Gamma(\Lambda^\bullet A^*)$ endowed with the usual exterior product $\wedge$ and the differential $d : \Omega^\bullet(A) \to \Omega^{\bullet+1}(A)$ defined by

$$\begin{aligned} d\eta(x_1, \dots, x_k) &:= \sum_i (-1)^{i+1} \rho(x_i)(\eta(x_1, \dots, \hat{x_i}, \dots x_k)) \\ &+ \sum_{i<j} (-1)^{i+j} \eta([x_i, x_j], \dots, \hat{x_i}, \dots, \hat{x_j} \dots, x_k) \end{aligned}$$

Definition 3.3.3. A right dg-module $M^\bullet$ over a dg-algebra $(A^\bullet, m_A, d_A)$ is a graded object endowed with a differential d_M and a morphism of graded objects

$$\cdot : M^\bullet \otimes A^\bullet \to M^\bullet$$

which is a graded right action of $A^\bullet$ on $M^\bullet$ and which respects the graded Leibniz rule

$$d_M(x \cdot a) = d_M(x) \cdot a + (-1)^{\deg a} x \cdot d_A(a).$$

Definition 3.3.4. A **differential graded category** (or dg-category) is a category enriched over complexes of vector spaces.

Explicitly, a dg-category is a category $\mathcal{C}$ such that for any two objects x and y, the morphisms between them form a complex of vector spaces $(\mathcal{C}(x, y)^\bullet, d)$. Moreover, for any three objects x, y and z the composition is a morphisms of complexes

$$\mathcal{C}(x, y)^\bullet \otimes \mathcal{C}(y, z)^\bullet \to \mathcal{C}(x, z)^\bullet$$

which satisfies some properties of associativity and unit.

Definition 3.3.5. A **dg-functor** $F : \mathcal{C} \to \mathcal{D}$ between dg-categories is a functor such that the maps

$$\mathcal{C}(x, y)^\bullet \to \mathcal{D}(Fx, Fy)^\bullet$$

are maps of complexes. It is called a **quasi-equivalence** if the maps above are all quasi-isomorphisms and the induced map between the homotopy categories

$$\operatorname{Ho} F : \operatorname{Ho} \mathcal{C} \to \operatorname{Ho} \mathcal{D}$$

is an equivalence of categories.

The purpose of this chapter is to see that representations up to homotopy are special kinds of dg-modules over the dg-algebra $\mathcal{C}^\bullet(G)$, called **cohesive modules**, and to study some properties of their dg-category.

3.3.3 The dg-category $\mathcal{P}_{\mathcal{A}}$

Let $\mathcal{A} = ((A^\bullet, d), m)$ be a dg-algebra and $M^\bullet$ a graded right module over the k-algebra A^0.

Definition 3.3.6. A $\mathbb{Z}$-connection D is a k-linear map

$$D : M^\bullet \otimes_{A^0} A^\bullet \to M^\bullet \otimes_{A^0} A^\bullet$$

of total degree one, satisfying the graded Leibiniz rule.

We denote by $\mathcal{P}^\bullet_{\mathcal{A}}(\mathcal{M}, \mathcal{N})$ the complex of morphisms of $\mathcal{A}$-dg-modules between $\mathcal{M}$ and $\mathcal{N}$.

Remark 3.3.7. By definition, a $\mathbb{Z}$-graded connection D is determined by its value on $M^\bullet$, so we also note by D the restriction

$$D : M^\bullet \to M^\bullet \otimes_{A^0} A^\bullet.$$

For each $k \geq 0$ we have a component

$$D_k : M^\bullet \to M^{\bullet - k+1} \otimes_{A^0} A^k$$

so we obtain the decomposition $D = D_0 + D_1 + D_2 + \dots$, where D_1 is a connection in the usual sense and D_i is A^0-linear for all $i \neq 1$.

Definition 3.3.8 ([10], def. 2.3.2)**.** For a dg-algebra $\mathcal{A} = ((A^\bullet, d), m)$ we define the dg-category $\mathcal{P}_{\mathcal{A}}$:

1. An object $\mathcal{M} = (M^\bullet, D_M)$ in $\mathcal{P}_{\mathcal{A}}$, which we call a **cohesive module** is a bounded graded right module $M^\bullet$ over $\mathcal{A}$, which is finitely generated and projective, together with a $\mathbb{Z}$-connection

   $$D_{\mathcal{M}} : M^\bullet \otimes_{A^0} A^\bullet \to M^\bullet \otimes_{A^0} A^\bullet$$

   such that $D^2_{\mathcal{M}} = 0$.

2. The degree k morphisms $\mathrm{Hom}_{\mathcal{P}_{\mathcal{A}}}(\mathcal{M}, \mathcal{N})^k$ between two cohesive modules $\mathcal{M} = (M^\bullet, D_M)$ and $\mathcal{N} = (N^\bullet, D_N)$ are the morphisms

   $$\phi : M^\bullet \otimes_{A^0} A^\bullet \to N^\bullet \otimes_{A^0} A^\bullet$$

   of degree k such that $\forall a \in A^\bullet, \quad \phi(ea) = \phi(e)a$.

The differential on $\mathrm{Hom}_{\mathcal{P}_{\mathcal{A}}}(\mathcal{M}, \mathcal{N})^\bullet$ is defined in the standard way

$$\delta(\phi)(m) = D_{\mathcal{N}}(\phi(m)) - (-1)^{\deg \phi} \phi(D_{\mathcal{M}}(m))$$

and satisfy $\delta^2 = 0$. Again, since ϕ is determined by its restriction to $M^\bullet$, we have $\phi = \phi_0 + \phi_1 + \phi_2 + \dots$ with

$$\phi_j : M^\bullet \to N^{\bullet + k - j} \otimes_{A^0} A^j$$

Proposition 3.3.9 (cf. [10] Prop 2.3.3)**.** *For $\mathcal{A} = ((A^\bullet, d), m)$ a dg-algebra, $\mathcal{P}_{\mathcal{A}}$ is a dg-category.*

3.3.4 The dg-algebra of smooth functions $C^\bullet(G)$

Let $G \rightrightarrows M$ be a Lie groupoid and $G^\bullet$ be its nerve. We apply the $\mathcal{C}^\infty$-functor to obtain the cosimplicial vector space $\mathcal{C}^\infty(G^\bullet)$. We denote by:

- d_i^* the coface maps
$$d_i^* : \mathcal{C}^\infty(G^{(n)}) \to \mathcal{C}^\infty(G^{(n+1)})$$
induced by $d_i : G^{(n+1)} \to G^{(n)}$;
- s_i^* the codegeneracy maps
$$s_i^* : \mathcal{C}^\infty(G^{(n)}) \to \mathcal{C}^\infty(G^{(n+1)})$$
induced by $s_i : G^{(n)} \to G^{(n+1)}$ (see section 1.6.1).

We can think at $\mathcal{C}^\infty(G^\bullet)$ as a graded abelian group by forgetting the cosimplicial structure. We endow it with a graded multiplication

$$(f \cdot h)(g_1, \dots g_{k+p}) = (-1)^{kp} f(g_1, \dots, g_k) h(g_{k+1}, \dots, g_{k+p})$$

for $f \in C^k(G)$, $h \in C^p(G)$, which makes it a graded algebra. The cosimplicial differential $d_{cs} := \sum (-1)^i d_i^*$ makes $(C^\bullet(G), \cdot, d_{cs})$ a dg-algebra, which we will note simply $C^\bullet(G)$. We call it the **dg-algebra of smooth functions on** G.

We can alternatively consider the normalized dg-algebra $\check{C}^\bullet(G)$, which appears in the Dold-Kan correspondence. The Dold-Kan correspondence (see Appendix A.2) establish an equivalence between the category of cosimplicial abelian groups (resp. cosimplicial rings) and the category of positively graded cochain complexes (resp. dg-rings). The equivalence sends $\mathcal{C}^\infty(G^\bullet)$ to $\check{C}^\bullet(G)$, defined by

$$\check{C}^n(G) := \mathcal{C}^\infty(G^{(n)}) / \sum_{i=1}^{n} \partial_i \mathcal{C}^\infty(G^{(n-1)}) \simeq \bigcap_{i=0}^{n-1} \ker(\mu_i)$$

with differential the remaining 0-th face map d_0^*. Again, $(\check{C}^\bullet(G), \cdot, \partial_0)$ is a dg-algebra. Those two dg-algebras are equivalent:

Theorem 3.3.10 ([28], chapter II)**.** *The inclusion $\check{C}^\bullet(G) \hookrightarrow C^\bullet(G)$ is a quasi-isomorphism.*

Definition 3.3.11. We define the **differentiable cohomology** of a Lie groupoid $G \rightrightarrows M$ as the cohomology of the complex $C^\bullet(G)$.

Proposition 3.3.12 (Morita invariance of differentiable cohomology)**.** *A Morita equivalence between groupoids $G \rightrightarrows M$ and $H \rightrightarrows N$ induce a quasi-isomorphism between the dg-algebras $C^\bullet(G)$ and $C^\bullet(H)$, so the differentiable cohomology is a Morita invariant for Lie groupoids.*

Note that the invariance for differentiable cohomology is specific to the differentiable setting, because it is based on the existence of partitions of unity and Mayer-Vietoris arguments (see for example [18] lemma 1, p.7).

Proof. Let $\mathfrak{X} = [M/G] = [N/H]$ be the quotient stack. By definition, the usual cohomology of the dg-algebras $C^\bullet(G)$ and $C^\bullet(H)$ calculate the Čech cohomology $\check{H}^*(\mathfrak{X}; \mathcal{C}^\infty)$ which, by proposition 2.2.2, is well defined. We conclude that $C^\bullet(G)$ and $C^\bullet(H)$ have the same cohomology, i.e. they are quasi-isomorphic dg-algebras. □

3.4 Representations up to homotopy

In this section we briefly recall the classical definitions of representations up to homotopy of [4]. Then we show that they are exactly cohesive modules over the dg-algebra $C^\bullet(G)$. We define the dg-category of representations up to homotopy on a Lie groupoid $G \rightrightarrows M$ as $\mathcal{P}_{C^\bullet(G)}$ and we prove that they are Morita invariant.

In 3.2 we defined a representation of a groupoid in terms of the general linear groupoid of a vector bundle. We now give an algebraic definition of a representation, which is easier to generalize to the case of representations up to homotopy. We saw in example 3.3.4 that $(\mathcal{C}^\bullet(G), \cdot, d_{cs})$ is a dg-algebra. Given a vector bundle E over M we form the graded vector space $C^\bullet(G; E)$, whose degree k part is

$$C^k(G; E) := \Gamma(G^{(k)}, t^*E)$$

with $t(g_1, \ldots, g_k) = t(g_k)$, which is naturally a (right) $C^\bullet(G)$-graded module: given $\eta \in C^k(G, E)$, $f \in C^{k'}(G)$, the product $\eta \cdot f \in C^{k+k'}(G; E)$ is defined by

$$(\eta \cdot f)(g_1, \ldots, g_{k+k'}) := (-1)^{kk'} \eta(g_1, \ldots, g_k) f(g_{k+1}, \ldots, g_{k+k'})$$

Then a quasi-action λ of G on E induces a degree one operator

$$D_\lambda : C^\bullet(G; E) \to C^{\bullet+1}(G; E)$$

$$D_\lambda(\eta)(g_1, \ldots, g_{k+1}) = (-1)^k \{\lambda_{g_1} \eta(g_2, \ldots, g_{k+1}) + \sum_{i=1}^{k+1} (-1)^i \eta(d_k(g_1, \ldots, g_{k+1}))\}$$

The normalized subspace $\check{C}^\bullet(G; E)$ of $C^\bullet(G; E)$ consists of those η such that $s_i^*(\eta) = 0$ for all i.

Lemma 3.4.1 ([4], Lemma 2.6)**.** The construction $\lambda \to D_\lambda$ induces a 1-1 correspondence between quasi-actions λ of G on E and degree 1 operators D_λ on the graded $C^\bullet(G)$-module $C^\bullet(G; E)$ satisfying the graded Leibniz rule.
Moreover:

1. λ is unital if and only if D_λ preserves the normalized subspace $\check{C}^\bullet(G;E)$.
2. λ is an action if and only if it is unital and $D_\lambda^2 = 0$ (i.e. iff $(C^\bullet(G;E), D_\lambda)$ is a right dg-module over $C^\bullet(G)$).

The lemma says that actions of G on E correspond to dg-module structures on $C^\bullet(G;E)$. We do exactly the same thing on graded vector bundles.
Let $E = \bigoplus_{k_1 \leq i \leq k_2} E_i$ be a bounded graded vector bundle over M. The space of E-valued cochains on G will be considered with total grading

$$C(G;E)^n := \bigoplus_{k+l=n} C^k(G;E^l)$$

We say that $\eta \in C^k(G;E^l)$ has bidegree (k,l). Clearly $C(G;E)^\bullet$ is a right graded $C^\bullet(G)$-module, and the additional graduation is preserved by the action of $C^\bullet(G)$.

Definition 3.4.2 ([4] Definition 3.1)**.** A representation up to homotopy of $G \rightrightarrows M$ on a graded vector bundle E over M is a linear operator, called the **structure operator** which raises the total degree by one

$$D : C(G;E)^\bullet \to C(G;E)^{\bullet+1}$$

satisfying $D^2 = 0$ and the Leibniz identity

$$D(\eta \cdot f) = D(\eta) \cdot f + (-1)^k \eta \cdot d(f)$$

for any $\eta \in C(G;E)^k$ and $f \in C^\bullet(G)$.

3.5 Ruth as cohesive modules over $C^\bullet(G)$

In this section we show that representations up to homotopy of a Lie groupoid $G \rightrightarrows M$ are exactly cohesive modules over the dg-algebra $C^\bullet(G)$ (Thm. 3.5.4).

The global sections $\Gamma(E)$ of a vector bundle $E \to M$, are a finitely generated and projective $\mathcal{C}^\infty(M)$-module. We use the following proposition to give a useful interpretation of the dg-module $C(G;E)^\bullet$

Proposition 3.5.1. *Let $E \to M$ and $F \to N$ be finite dimensional vector bundles over M and N respectively, $\phi : N \to M$ and*

$$\begin{tikzcd} F \arrow[r, "\psi"] \arrow[d] & E \arrow[d] \\ N \arrow[r, "\phi"] & M \end{tikzcd}$$

a morphism of vector bundles which is an isomorphism on fibers. Then there is a canonical isomorphism of $\mathcal{C}^\infty(N)$-modules

$$\Gamma(E) \otimes_{\mathcal{C}^\infty(M)} \mathcal{C}^\infty(N) \to \Gamma(F)$$

Proof. We give the construction of the map, and we refer to [30] for the complete proof:

- We put a $\mathcal{C}^\infty(M)$-module structure on $\mathcal{C}^\infty(N)$ with multiplication

$$\begin{array}{rcl} \mathcal{C}^\infty(M) \otimes \mathcal{C}^\infty(N) & \to & \mathcal{C}^\infty(N) \\ f \otimes g & \mapsto & (f \circ \phi) \cdot g \end{array}$$

- The tensor product of $\mathcal{C}^\infty(M)$-modules $\Gamma(E) \otimes_{\mathcal{C}^\infty(M)} \mathcal{C}^\infty(N)$ has a natural structure of right $\mathcal{C}^\infty(N)$-module with multiplication $(\sigma \otimes g) \cdot g' := \sigma \otimes gg'$ for all $\sigma \in \Gamma(E)$, $g, g' \in \mathcal{C}^\infty(N)$;

- There is a natural map of $\mathcal{C}^\infty(N)$-modules

$$\begin{array}{rcl} \Gamma(E) \otimes_{\mathcal{C}^\infty(M)} \mathcal{C}^\infty(N) & \to & \Gamma(F) \\ g \otimes \sigma & \mapsto & \tilde{\sigma} \end{array}$$

  where $\tilde{\sigma}(n) = \psi^{-1}(\sigma(\phi(n)))$.

It is easy to see that this map is an isomorphism. □

Corollary 3.5.2. *The graded module $C(G;E)^\bullet$ (resp. $\check{C}(G;E)^\bullet$) is isomorphic to the graded module $\Gamma(E) \otimes_{\mathcal{C}^\infty(m)} C^\bullet(G)$ (resp. $\Gamma(E) \otimes_{\mathcal{C}^\infty(m)} \check{C}^\bullet(G)$).*

Proof. By proposition 3.5.1 there are natural isomorphisms

$$\Gamma(E) \otimes_{\mathcal{C}^\infty(M)} \mathcal{C}^\infty(G^{(k)}) \to \Gamma(G^{(k)}, t^*E) = C(G;E)^k$$

It is easy to see that this isomorphism can be restricted to normalized subspaces

$$\Gamma(E) \otimes_{\mathcal{C}^\infty(M)} \check{C}^k(G) \to \check{C}(G;E)^k.$$

□

Now, one can pass from vector bundles over M to finitely generated and projective modules over $\mathcal{C}^\infty(M)$ by means of the global section funtor Γ. The smooth Serre-Swan theorem states that this is actually an equivalence for connected manifolds:

Theorem 3.5.3 (Smooth Serre-Swan theorem, cf. [47])**.** *Let M be a connected smooth manifold, $FinVect(M)$ the category of finite dimensional vector bundles over M and $FinMod(M)$ be the category of finitely generated and projective modules over the algebra $C^\infty(M)$. Then the global section functor $\Gamma : FinVect(M) \to FinMod(M)$ sending a vector bundle E to the $C^\infty(M)$-module of sections $\Gamma(E)$ is an equivalence of categories.*

We can finally characterize representations up to homotopy of a Lie groupoid $G \rightrightarrows M$ as cohesive modules over the dg-algebra $C^\bullet(G)$.

Theorem 3.5.4. *Representations up to homotopy of a Lie groupoid $G \rightrightarrows M$ are exactly cohesive modules over the dg-algebra $C^\bullet(G)$.*

Proof. Using theorem 3.5.3 and corollary 3.5.2 we know that there is an equivalence between the category of graded modules on $\mathcal{C}^\infty(M)$ which are finitely generated and projective and that of graded $C^\bullet(G)$-modules of the form $C(G;E)^\bullet$ for some vector bundle E on M. By this correspondence, $\mathbb{Z}$-graded connections on $C^\bullet(G)$-modules correspond exactly to representation up to homotopy structures on $C(G;E)^\bullet$. $\square$

We can now give a clear definition of the dg-category of representations up to homotopy on a given Lie groupoid $G \rightrightarrows M$:

Definition 3.5.5. The dg-category of representations up to homotopy on a Lie groupoid $G \rightrightarrows M$ is $\mathcal{R}ep^\infty(G) := \mathcal{P}_{C^\bullet(G)}$.

Note that definition 3.3.8 gives an explicit description of k-morphisms between representations up to homotopy.
From now on, we denote by $\mathcal{M} = (M^\bullet, D)$ a cohesive module over $\mathcal{C}^\infty(M)$ and by $\mathcal{E} = (E^\bullet, D)$ a representation up to homotopy in the usual sense.

We would like to have a simple statement saying that the assignement $G \mapsto Rep^\infty(G)$, sending a Lie groupoid to the dg-category of representations up to homotopy on it, is well defined on stacks, i.e. it "pass to the quotient by Morita morphisms". If we could prove that there is an ∞-functor from dg-algebras to dg-categories

$$\begin{array}{rcl} dgA & \to & dgCat \\ A & \mapsto & \mathcal{P}_A \end{array}$$

the proposition 3.3.12 would immediately lead the result. Unfortunately such a ∞-functor does not exist: one can find quasi-isomorphic dg-algebras with non-equivalent associated dg-categories of cohesive modules (cf. [11]).

However, we will see that the statement is true for algebras of smooth functions of Morita equivalent Lie groupoids. Cohesive modules on the dg-algebra $C^\bullet(G)$ of smooth functions of a Lie groupoid $G \rightrightarrows M$ are, roughly speaking, vector bundles over M, endowed with an action of $C^\bullet(G)$ on the sections. The idea now is that Morita morphisms are very simple in the differentiable setting: they can just be described by surjective submersions $P \to M$ on the base manifolds. A vector bundle on M can be pulled back to a vector bundle on P and consequently the action of the dg-algebra, giving a dg-functor $\mathcal{P}_{C^\bullet(G)} \to \mathcal{P}_{C^\bullet(P)}$. The point is that this functor has an explicit quasi-inverse $\mathcal{P}_{C^\bullet(P)} \to \mathcal{P}_{C^\bullet(G)}$ thanks to the existence of local sections for $P \to M$ and a partition of the unity on M.
We will study in detail this equivalence in section 3.8.

3.6 The tensors D_i

We can now give an important geometric interpretation of the tensors

$$D_k : M^\bullet \to M^{\bullet-k+1} \otimes_{\mathcal{C}^\infty(M)} C^k(G)$$

of remark 3.3.7.

Proposition 3.6.1. *Let $N^\bullet = \Gamma(E^\bullet)$ be the graded module of sections of a graded vector bundle $E\bullet \to M$. Let $\underline{\mathrm{End}}^\bullet(E^\bullet) \to M$ be the graded vector space of linear endomorphisms of $E^\bullet$. Then there are isomorphisms*

$$\mathrm{Hom}_{\mathcal{C}^\infty(M)}(N^\bullet, N^{\bullet-k+1} \otimes_{\mathcal{C}^\infty(M)} C^k(G)) \simeq C^k(G; \underline{\mathrm{End}}^{1-k}(E^\bullet))$$

for all $k \geq 0$.

Proof. It is well known that for modules A and B on the algebra R

$$\mathrm{Hom}(A, B) \simeq A^* \otimes_R B$$

We apply this to the left hand side to obtain

$$\begin{array}{rcl} \mathrm{Hom}_{\mathcal{C}^\infty(M)}(N^\bullet, N^{\bullet-k+1} \otimes_{\mathcal{C}^\infty(M)} C^k(G)) & \simeq & (N^\bullet)^* \otimes_{\mathcal{C}^\infty(M)} (N^{\bullet-k+1} \otimes_{\mathcal{C}^\infty(M)} C^k(G)) \\ & \simeq & \mathrm{Hom}_{\mathcal{C}^\infty(M)}(N^\bullet, N^{\bullet-k+1}) \otimes_{\mathcal{C}^\infty(M)} C^k(G) \\ \text{fully faithfullness in Serre-Swan} & \simeq & \underline{\mathrm{End}}^{1-k}(E^\bullet) \otimes_{\mathcal{C}^\infty(M)} C^k(G) \\ \text{prop. 3.5.1} & \simeq & C^k(G; \underline{\mathrm{End}}^{1-k}(E^\bullet)) \end{array}$$

□

Corollary 3.6.2 ([4], Proposition 3.2)**.** *There is a bijective correspondence between representations up to homotopy of G on the graded vector bundle $E^\bullet$ and sequences $\{D_k\}_{k\geq 0}$ of elements $D_k \in C^k(G; \underline{\mathrm{End}}^{1-k}(E^\bullet))$ which, for all $k \geq 0$, satisfy:*

$$\sum_{j=1}^{k-1} (-1)^j D_{k-1}(g_1, \ldots, g_j g_{j+1}, \ldots, g_k) = \sum_{j=0}^{k} (-1)^j D_j(g_1, \ldots, g_j) \circ D_{k-j}(g_{j+1}, \ldots, g_k) \tag{3.3}$$

The same idea idea can be applied to morphisms $\phi : \mathcal{E} \to \mathcal{F}$, which turn out to be sequences $\{\phi_k\}_{k\geq 0}$ of elements $\phi_k \in C^k(G; \underline{\mathrm{Hom}}^{-k}(E^\bullet, F^\bullet))$ satisfying some coherences.

Remark 3.6.3. In the theorem:

1. $D_0 : E^\bullet \to E^{\bullet+1}$ is a differential ∂ on the graded vector bundle $E^\bullet$;
2. $D_1 : E^\bullet \to E^\bullet$ is our quasi-action ρ_g of G on $E^\bullet$, and equation (3.3) says that it is a chain map $\rho\partial = \partial\rho$;

3. $D_2 : E^\bullet \to E^{\bullet-1}$ is an homotopy $\gamma_{g,h}$, and equation (3.3) says that

$$\rho_g \circ \rho_h - \rho_{gh} = \partial \circ \gamma_{g,h} + \gamma_{g,h}\partial$$

We will see this point of view more in detail in chapter 3.

Example 3.6.4 (Vector bundles over stacks)**.** We saw in section 3.1 that a vector bundle on a stack $\mathfrak{X}$ is the same as an X_1-equivariant vector bundle on X_0. In the framework of representations up to homotopy we can intepret a graded vector bundle $\mathfrak{E}^\bullet$ on $\mathfrak{X}$ as a representation up to homotopy on $(E^\bullet, D)$ of $X_1 \rightrightarrows X_0$ such that $D = D_1$, i.e. there is no differential nor higher homotopies, and the action of X_1 on $E^\bullet$ is just D_1.

3.7 Homotopy theory of representations up to homotopy

In this section we define the cohomology of representations up to homotopy and we prove the Morita invariance of the dg-category of representations up to homotopy $\mathcal{R}ep^\infty$. We conclude by giving a definition of the dg-category of perfect complexes over a differentiable 1-stack $\mathfrak{X}$.

Definition 3.7.1. The differentiable cohomology of a representation up to homotopy $\mathcal{M} = (M^\bullet, D)$ is the cohomology of the complex $(M^\bullet, D)$.

Note that if $G \rightrightarrows M$ is a Lie groupoid, and $E \to M$ is the trivial line bundle on M, then $C(G; E)^\bullet$ is isomorphic to $C^\bullet(G)$ as $C^\bullet(G)$-dg-modules. Then the differentiable cohomology of E is just the differentiable cohomology of the Lie groupoid $G \rightrightarrows M$.

Definition 3.7.2. A degree 0 morphism ϕ in $\mathcal{R}ep^\infty(G)$ is called **homotopy equivalence** if it induces an isomorphism in the homotopy category $Ho(\mathcal{R}ep^\infty(G))$.

Proposition 3.7.3 (cf. [10], Prop.2.5.2)**.** *A closed morphism $\phi \in \mathcal{R}ep_0^\infty(\mathcal{E}, \mathcal{F})$ is an homotopy equivalence if and only if $\phi_0 : E^\bullet \to F^\bullet$ (see definition 3.3.8) is a quasi-isomorphism of complexes of vector bundles.*

Corollary 3.7.4. *Let $\phi : \mathcal{E} \to \mathcal{F}$ be a morphism of representations up to homotopy such that $\phi_0 : E^\bullet \to F^\bullet$ is a quasi -isomorphism of vector bundles. Then $\mathcal{E}$ and $\mathcal{F}$ have isomorphic differentiable cohomology.*

3.8 Morita invariance of $Rep^\infty(G)$

In this section we prove that Morita equivalent groupoids have quasi-equivalent dg-categories of representations up to homotopy. For this we take a Morita morphism

of Lie groupoids $H \to G$ and we show that it induces naturally two dg-functors $Rep^\infty(H) \leftrightarrows Rep^\infty(G)$, such that the compositions are equivalences of dg-categories.

First, recall from example 1.7.2 that a Morita morphism of Lie groupoid with target $G \rightrightarrows M$ is just the data of a submersion $\phi : P \to M$, and let $\tilde{P}$ be the pullback groupoid $\tilde{P}_\bullet := \phi^*(G_\bullet)$. Let $P_M := P \times_M P \rightrightarrows P$ be the Lie groupoid associated to the submersion $P \to M$ (see example 1.5.2).

Lemma 3.8.1. There is a quasi-isomorphim of dg-algebras

$$C^\bullet(\tilde{P}) \to C^\bullet(P_M) \otimes_{C^\infty(M)} C^\bullet(G)$$

where the differentials are respectively $d_{\tilde{P}}$ and $d_{P_M} \otimes 1 + 1 \otimes d_G$.

Proof. We just remark that by proposition 3.3.12 the Lie groupoid morphisms $\tilde{P} \to G$ and $P_M \to [M]$ induce quasi-isomorphisms of dg-algebras $C^\bullet(G) \to C^\bullet(\tilde{P})$ and $C^\bullet(P_M) \to C^\bullet([M])$, and by construction $C^\bullet([M]) \simeq C^\infty(M)$. □

This allows us to construct two dg-functors associated to this Morita morphism:

1. A representation up to homotopy of G is a cohesive module on $C^\bullet(G)$, i.e. a graded vector bundle $E \to M$ and a degree 1 differential d_E on $C^\bullet(G) \otimes_{C^\infty(M)} \Gamma(E)$ satisfying the Leibniz rule. We send it to

$$C^\bullet(P_M) \otimes_{C^\infty(M)} C^\bullet(G) \otimes_{C^\infty(M)} \Gamma(E)$$

which, by lemma 3.8.1 is quasi-isomorphic to $C^\bullet(\tilde{P}) \otimes_{C^\infty(M)} \Gamma(E)$, with differential $d_{P_M} \otimes 1 + 1 \otimes d_E$. This is clearly a representation up to homotopy of $\tilde{P}$. We denote by $\phi^* : Rep^\infty(G) \to Rep^\infty(\tilde{P})$ the corresponding dg-functor.

2. A representation up to homotopy on $\tilde{P}$ is a graded vector bundle $F \to P$ and a degree 1 differential d_F on satisfying the Leibniz rule. By lemma 3.8.1 we have a quasi isomorphisms

$$C^\bullet(\tilde{P}) \otimes_{C^\infty(P)} \Gamma(F) \simeq C^\bullet(G) \otimes_{C^\infty(M)} \big(C^\bullet(P_M) \otimes_{C^\infty(P)} \Gamma(F)\big) \tag{3.4}$$

Since $C^\bullet(P_M)$ is quasi-isomorphic to $C^\infty(M)$, (3.4) is quasi-isomorphic, as a $C^\bullet(G)$-dg-module, to $C^\bullet(G) \otimes_{C^\infty(P)} \Gamma(F)$, with differential d_F. This last is a cohesive module on $C^\bullet(G)$. We denote by ϕ_* the corresponding dg-functor

$$\begin{array}{rcl} \phi_* : Rep^\infty(\tilde{P}) & \to & Rep^\infty(G) \\ C^\bullet(\tilde{P}) \otimes_{C^\infty(P)} \Gamma(F) & \mapsto & C^\bullet(G) \otimes_{C^\infty(P)} \Gamma(F). \end{array}$$

Proposition 3.8.2. *The dg-functors ϕ^* and ϕ_* are quasi-inverse to one another, and preserve differentiable cohomology.*

Proof. We need to prove that

$$\phi_* \circ \phi^* \simeq \mathrm{id}_{Rep^\infty(G)} \text{ and } \phi^* \circ \phi_* \simeq \mathrm{id}_{Rep^\infty(\tilde{P})}.$$

Let $(E, d_E) \in Rep^\infty(G)$. Then $\phi_* \circ \phi^*(E, d_E)$ is the graded $C^\bullet(G)$-module

$$C^\bullet(G) \otimes_{C^\infty(M)} \big(C^\bullet(P_M) \otimes_{C^\infty(M)} \Gamma(E)\big)$$

endowed with the differential $d_{P_M} \otimes 1 + 1 \otimes d_E$. Since $(C^\bullet(P_M), d_{P_M})$ is quasi-isomorphic to $C^\infty(M)$, then $\phi_* \circ \phi^*(E, d_E)$ is quasi-isomorphic to $(C^\bullet(G) \otimes_{C^\infty(M)} \Gamma(E), d_E) = (E, d_E)$. We conclude that $\phi_* \circ \phi^*$ is equivalent to $\mathrm{id}_{Rep^\infty(G)}$.

For the other composition, let (F, d_F) be a $C^\bullet(\tilde{P})$-dg-module. Then $\phi^* \circ \phi_*(F, d_F)$ is the $C^\bullet(\tilde{P})$-dg-module

$$C^\bullet(P_M) \otimes_{C^\infty(M)} C^\bullet(P_M) \otimes_{C^\infty(M)} C^\bullet(G) \otimes_{C^\infty(M)} \Gamma(F)$$

endowed with the differential $d_{P_M} \otimes 1 + 1 \otimes d_F$. Since $C^\bullet(P_M)$ is quasi-isomorphic to $C^\infty(M)$, the multiplication

$$m : C^\bullet(P_M) \otimes_{C^\infty(M)} C^\bullet(P_M) \to C^\bullet(P_M)$$

is a quasi-isomorphism of dg-algebras and induces a quasi-isomorphism of $C^\bullet(\tilde{P})$-dg-modules

$$\phi^* \circ \phi_*(F, d_F) \to (F, d_F)$$

(the Leibniz rule ensures that m commutes with the differentials), so $\phi^* \circ \phi_*$ is equivalent to $\mathrm{id}_{Rep^\infty(\tilde{P})}$. We conclude that the dg-functors

$$\phi^* : Rep^\infty(G) \leftrightarrows Rep^\infty(\tilde{P}) : \phi_*$$

induce an equivalence of dg-categories.

Finally, we remark that ϕ_* obviously preserves differentiables cohomology. By the first part of the theorem this implies that ϕ^* also does, and we are done. □

The previous results immediately lead to the Morita invariance for representations up to homotopy:

Theorem 3.8.3. *Let $G \rightrightarrows M$ and $H \rightrightarrows N$ be Morita equivalent Lie groupoids. Then the categories $\mathcal{R}ep^\infty(G)$ and $\mathcal{R}ep^\infty(H)$ are quasi-equivalent (see def.3.3.5).*

We claim that theorem 3.8.3 can be refined into a more complete and elegant way to state this descent result, which we will not prove here:

Claim 3.8.4. *Let*

$$Perf : Diff^{op} \to DG - Cat$$

be the (∞-)functor from the (classical) category of manifolds to the ∞-category of dg-categories, which sends a manifold M to the dg-category of representations up to homotopy of M, i.e. graded vector bundles over M. Then $Perf$ has descent and the extended functor

$$\mathrm{Perf} : St(Diff)^{op} \to DG - Cat$$

sends a differentiable stack $\mathfrak{X}$ to the dg-category $\mathcal{R}ep^{\infty}(G)$, where $G \rightrightarrows M$ is any Lie groupoid presenting $\mathfrak{X}$.

We conclude by giving a definition of the dg-category of perfect complexes over a differentiable stack and their cohomology:

Definition 3.8.5. The dg-category of perfect complexes over a differentiable stack $\mathfrak{X}$ is

$$Perf(\mathfrak{X}) := \mathcal{R}ep^{\infty}(G)$$

for any Lie groupoid $G \rightrightarrows M$ presenting $\mathfrak{X}$.

Definition 3.8.6. The cohomology of $\mathfrak{X}$ with values in a perfect complex $\mathfrak{F}^{\bullet}$ is

$$H^{\bullet}(\mathfrak{X}, \mathfrak{F}^{\bullet}) := H^{\bullet}_{diff}(G; E)$$

for any Lie groupoid presenting $\mathfrak{X}$ and representation up to homotopy presenting $\mathfrak{F}^{\bullet}$.

Thanks to theorem 3.8.4 this is a well definition of $Perf(\mathfrak{X})$ and $H^{\bullet}(\mathfrak{X}, \mathfrak{F}^{\bullet})$.

Example 3.8.7 (Vector bundles over stacks again)**.** Following the example 3.6.4, we can define graded vector bundles over stacks as those perfect complexes which admit a presentation $(E^{\bullet}, D)$ such that $D = D_1$.

3.9 Symmetric powers of representations up to homotopy

In [1] is shown the existence and unicity of a tensor product on the homotopy categories of representations up to homotopy. This yields a definition of tensor product for complexes of vector bundles over stacks by Morita invariance.
Here we just remind the principal properties of this functor, and we show that it descends to stacks. We call a morphism of representations up to homotopy $\zeta : E \to F$ **strict** if it is just a morphism between the complexes underlying E and F.

Proposition 3.9.1 ([4], Def. 3.25, [1])**.** *Let $G \rightrightarrows M$ be a Lie groupoid, and $q \in \mathbb{N}$. There exist a unique ∞-functor*

$$\mathrm{S}^q : Rep^\infty(G) \to Rep^\infty(G)$$

such that:

1. *For any $E \in Rep^\infty(G)$, the complex underlying $\mathrm{S}^q(E)$ is the symmetric power of the differential graded complex underlying E;*
2. *For any $E \in Rep^\infty(G)$, the quasi-action of $\mathrm{S}^q(E)$ is the diagonal quasi-action on $Sym^q(E)$ induced by the quasi-action underlying E;*
3. *S^q preserves strict morphisms and quasi-isomorphisms;*
4. *for any $\Phi : E \to F \in Rep^\infty(G)$, the morphism of complexes underlying $\mathrm{S}^q\,\Phi$ is q-th symmetric power of the morphism of complexes underlying Φ.*

Corollary 3.9.2. *The q-th symmetric power is well defined on perfect complexes.*

Proof. Two representations up to homotopy represent the same perfect complex if and only if they are quasi-isomorphic. Since S^q preserves quasi-isomorphisms, we conclude. □

Definition 3.9.3. Let $\mathfrak{F}$ be a perfect complex over a differentiable stack $\mathfrak{X}$. Let E be a representation up to homotopy presenting $\mathfrak{F}$. The q-th symmetric power of $\mathfrak{F}$, denoted $\mathrm{S}^q\,\mathfrak{F}$, is the equivalence class of $\mathrm{S}^q\,E$.

3.10 The tangent and cotangent complexes

For any Lie groupoid, the choice of a connection induces a representation up to homotopy on the Lie algebroid $A \to TM$, called the **adjoint representation**. This is the generalization of the adjoint representation of a Lie group to the framework of Lie groupoids, and it is not a classical representation anymore. So representations up to homotopy arise naurally as a generalization of Lie group representations.

We resume here some results of [4] about the adjoint/coadjoint representations up to homotopy.

Proposition 3.10.1. *Let $G \to M$ be a Lie groupoid with Lie algebroid $A \xrightarrow{\rho} TM$, with A in degree 0, TM in degree 1 and ρ the anchor map.*

1. *The choice of a connection σ on G induces a representation up to homotopy Ad_σ called the adjoint representation up to homotopy on G.*

2. *If σ and σ' are two connections on G, there is a canonical isomorphism $Ad_\sigma \to Ad_{\sigma'}$.*

We note by Ad_G the isomorphism class of $Ad_\sigma \in Rep^\infty(G)$. By duality we define the coadjoint representation Ad^*_G.

Proposition 3.10.2. *Let G and H be Morita equivalent Lie groupoids. Then they have quasi-isomorphic adjoint and coadjoint representations up to homotopy.*

Proof. One can find a proof of this fact in [22], using that:

- under the Grothendieck construction the adjoint/coadjoint representation up to homotopy is sent to the tangent/cotangent VB-groupoid and quasi-isomorphisms of representations up to homotopy are sent to VB-Morita equivalences;
- the tangent/cotangent groupoids of Morita equivalent groupoids are VB-Morita equivalent.

□

We can now define the tangent/cotangent complex of vector bundles over a differentiable stack $\mathfrak{X}$:

Definition 3.10.3. The tangent (resp. cotangent) complex $T_\mathfrak{X}$ (resp. $L_\mathfrak{X}$) of a differentiable stack $\mathfrak{X}$ is the equivalence class of Ad_G (of Ad^*_G) for any Lie groupoid G presenting $\mathfrak{X}$.

3.11 Hodge to De Rham spectral sequence for Lie groupoids

In this section we recall the Hodge to De Rham spectral sequence of [4] relating the definition of differentiable form on a stack $\mathfrak{X}$ to the double complex of forms $\Omega^p(G_q)$ on the nerve of a Lie groupoid G presenting $\mathfrak{X}$; we use our definition of perfect complexes to obtain the same result on stack (cor. 3.11.2).

Theorem 3.11.1 ([4], Thm. 4.1)**.** *Let $G \rightrightarrows M$ be a Lie groupoid. Then for the cohomology of the complex*

$$\Omega^q(M) \xrightarrow{d_h} \Omega^q(G) \xrightarrow{d_h} \Omega^q(G^{(2)}) \xrightarrow{d_h} \dots$$

one has

$$H^p_{d_h}(\Omega^q(G^\bullet)) \simeq H^{p-q}(G; \mathrm{S}^q\, Ad^*_G)$$

Corollary 3.11.2. *Let $\mathfrak{X}$ be a differentiable stack and $G \rightrightarrows M$ a Lie groupoid presenting it. Then*

$$H^p(\mathfrak{X}_{big}; \Omega^q) \simeq H^{p-q}(\mathfrak{X}; \mathrm{S}^q\, L_\mathfrak{X}[-1])$$

Proof. Since $G^\bullet \to \mathfrak{X}$ is a resolution of $\mathfrak{X}$, we have

$$H^p(\mathfrak{X}_{big}; \Omega^q) = H^p(\Omega^q(G^\bullet))$$

and by definition of cohomology of perfect complexes

$$H^{p-q}(\mathfrak{X}; \mathrm{S}^q\, L_{\mathfrak{X}}[-1]) = H^{p-q}(G; \mathrm{S}^q\, Ad^*_G)$$

□

3.11.1 Bott spectral sequence for differentiable stacks

We saw in theorem 2.4.1 that the choice of a cofoliation on a differentiable stack $\mathfrak{X}$ induces a E_1-spectral sequence

$$\bigoplus_{i+2k=p} H^q(\mathfrak{X}; \Lambda^i\Omega \otimes S^k\gamma) \Rightarrow H^{p+q}_{DR}(\mathfrak{X}) \tag{3.5}$$

but with techniques of [6] it was not possible to prove it independently of such a choice. In [4], the same spectral sequence was shown for Lie groupoids:

Theorem 3.11.3 ([4], Thm. 4.3)**.** *Let G be a Lie groupoid. There is a spectral sequence converging to the De Rham cohomology of G*

$$E_1^{p,q} = H^{p-q}_{diff}(G; \mathrm{S}^q(Ad^*)) \Rightarrow H^{p+q}_{DR}(G) \tag{3.6}$$

In light of the results of this chapter, namely corollary 3.9.2, and section 3.10, the proof of thm. 4.3 in [4] can be adapted to the framework of stacks:

Corollary 3.11.4. *Let $\mathfrak{X}$ be a differentiable stack. There is a spectral sequence converging to the De Rham cohomology of $\mathfrak{X}$*

$$E_1^{p,q} = H^{p-q}_{diff}(\mathfrak{X}; \mathrm{S}^q\, L_{\mathfrak{X}}[-1]) \Rightarrow H^{p+q}_{DR}(\mathfrak{X}) \tag{3.7}$$

We think that it is possible to obtain a decomposition similar to equation (3.5) of [6], but we could not be able to prove it here.

Chapter 4

The general linear 2-groupoid

In this chapter we expose the content of [20], a paper issued from a collaboration with Matias Del Hoyo, already accepted by the Pacific Journal of Mathematics. In the final section (4.7) we added a new part, aimed at building the stack $Perf_{[0,1]}$, which was not part of that collaboration.

4.1 Basics on 2-categories

We review here definitions and basic facts on set-theoretic 2-categories that are fundamental for the rest of the chapter. We give a definition of 2-groupoid, compare it with others in the literature, and discuss the notion of lax functors. We refer to [13, 40, 43] for further details. The material here is preparatory, to set notations and conventions and to serve as a quick reference. We often use the notation gf for the composition $g \circ f$.

A **2-category** C is a category enriched over the category of small categories. It has three levels of structure: objects, arrows between objects, and arrows between arrows or **2-cells**, whose collections we denote by C_0, C_1, C_2 respectively. We use letters $x, y, \dots$ for objects, $f, g, \dots$ for arrows, and $\alpha, \beta, \dots$ for 2-cells.

$$\begin{tikzcd} x \arrow[r, bend left, "g"] \arrow[r, bend right, "f"'] \arrow[r, phantom, "\Uparrow\alpha"] & y \end{tikzcd}$$

The arrows and 2-cells between two fixed objects x, y form a category $C(x, y)$, whose composition we denote by $\bullet$. For each triple x, y, z there is a composition functor $C(x, y) \times C(y, z) \xrightarrow{\circ} C(x, z)$ and a unit $\mathrm{id}_x \in C(x, x)$ satisfying the axioms encoded

in the following commutative diagrams:

$$\begin{tikzcd}
& C(x,y)\times C(y,z)\times C(z,w) \arrow[dl, "\mathrm{id}\times\circ"'] \arrow[dr, "\circ\times\mathrm{id}"] & \\
C(x,y)\times C(y,w) \arrow[dr, "\circ"'] & & C(x,z)\times C(z,w) \arrow[dl, "\circ"] \\
& C(x,w) &
\end{tikzcd}
\qquad
\begin{tikzcd}
& C(x,y) \arrow[dl, "\mathrm{id}\times\mathrm{id}_y"'] \arrow[dr, "\mathrm{id}_x\times\mathrm{id}"] \arrow[dd, "\mathrm{id}"] & \\
C(x,y)\times C(y,y) \arrow[dr, "\circ"'] & & C(x,x)\times C(x,y) \arrow[dl, "\circ"] \\
& C(x,y) &
\end{tikzcd}$$

Example 4.1.1. The paradigmatic example of a 2-category is that of small categories, functors and natural transformations. Another basic example is that of spaces, continuous maps and (homotopy classes of) homotopies.

We are interested in 2-groupoids. For us, a **2-groupoid** G is a 2-category such that (i) it is small, in the sense that G_0 is a set, (ii) every 2-cell is invertible, and (iii) every arrow $x \xrightarrow{f} y$ is invertible up to homotopy, namely there exists $y \xrightarrow{g} x$ and 2-cells $g \circ f \cong \mathrm{id}_x$ and $f \circ g \cong \mathrm{id}_y$. Some references demand the arrows to be invertible on the nose. We call such groupoids **strict**. Let us remark that our fudamental example, that of the general linear 2-groupoid, is not strict.

Example 4.1.2. A topological space X yields a 2-groupoid $\pi_2(X)$ whose objects are the points of X, arrows are the continuous paths $I \to X$, and 2-cells are (homotopy classes of) path homotopies. Composition is given by juxtaposition, moving through each path at double speed. A non-constant path is only invertible up to homotopy, hence $\pi_2(X)$ is not strict.

A simple characterization of (small) 2-categories and strict 2-groupoids is by using *double structures*, namely diagrams of compatible structures as below, where compatible means that the horizontal structural maps are functorial with respect to the vertical structures.

$$\begin{tikzcd}
G_2 \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift right] \arrow[d, shift left] & G_0 \arrow[d, shift right] \arrow[d, shift left] \\
G_1 \arrow[r, shift left] \arrow[r, shift right] & G_0
\end{tikzcd}$$

However, our notion of 2-groupoid does not benefit much out of this perspective. The following lemma, that is automatic for strict groupoids but works in general, will be useful later.

Lemma 4.1.3. If G is a 2-groupoid and $x \xrightarrow{f} y$ is an arrow in G, then the left multiplication functor $L_f : G(y,z) \to G(x,z)$ is an equivalence of categories for any z. The same holds for right multiplication.

Proof. A 2-cell $\alpha : f \Rightarrow g$ defines a natural isomorphism $L_f \Rightarrow L_g$, since the 2-cells are invertible. Then, given an arbitrary f, and picking g a quasi-inverse, we have $\mathrm{id}_{G(x,x)} = L_{\mathrm{id}_x} \cong L_g L_f$ and analogously $\mathrm{id}_{G(y,y)} = L_{\mathrm{id}_y} \cong L_f L_g$. $\square$

A **functor** $\phi : C \to D$ between 2-categories consists of functions $\phi_i : C_i \to D_i$ preserving all the structure *on the nose*. This notion is sometimes too rigid because it involves many identities between functors. A useful variant is that of a (normal) **lax functor** $\phi : C \dashrightarrow D$, which consists of three maps $\phi_i : C_i \to D_i$ preserving source, target, units and the composition $\bullet$, but only preserving $\circ$ up to a given natural transformation. More precisely, it is also given a map

$$\phi_{1,1} : C_1 \times_{C_0} C_1 \to D_2 \qquad \phi_{1,1}(g, f) : \phi_1(g \circ f) \Rightarrow \phi_1(g) \circ \phi_1(f)$$

ruling the failure of associativity of $\circ$ and satisfying the coherence axioms below:

i) $\phi_{1,1}(\mathrm{id}, f) = \mathrm{id} = \phi_{1,1}(f, \mathrm{id})$ (normality)

ii) $(\phi_2(b) \circ \phi_2(a)) \bullet \phi_{1,1}(g, f) = \phi_{1,1}(g', f') \bullet \phi_2(b \circ a)$

iii) $(\phi_{1,1}(h, g) \circ \phi_1(f)) \bullet \phi_{1,1}(hg, f) = (\phi_1(h) \circ \phi_{1,1}(g, f)) \bullet \phi_{1,1}(h, gf)$

When the structure 2-cells $\phi_{1,1}(g, f)$ are invertibles the lax functor is called a **pseudo-functor**. These notions are very interesting even when C is a usual category, viewed as a 2-category with only identity 2-cells. To ease the notation we will often write ϕ instead of ϕ_i, etc.

Example 4.1.4. Given $\pi : G \to H$ an epimorphism of groups, a set-theoretic section $\sigma : H \to G$, $\sigma(1_G) = 1_H$, leads to a pseudo-functor $\phi : H \dashrightarrow \{\text{Groups}\}$, where G is viewed as a 2-groupoid with one object and only identity 2-cells, and $\{\text{Groups}\}$ is the 2-category of groups, morphisms, and inner automorphisms as 2-cells. Here $\phi(*) = K$ is the kernel of π, $\phi(h)$ is given by conjugation by $\sigma(h)$, and $\phi(h', h)$ is the conjugation by $\sigma(h')\sigma(h)\sigma(h'h)^{-1}$. The lax functor is an actual functor if and only if σ is a morphism.

We also need to deal with morphisms between lax functors (cf. [13]). Given $\phi, \psi : C \dashrightarrow D$ lax functors between 2-categories, a **lax transformation** $H : \phi \Rightarrow \psi$ associates to each $x \in C_0$ an arrow $H_x : \phi(x) \to \psi(x)$ and to each arrow $f : x \to y$ a 2-cell $H_f : \psi(f)H_x \Rightarrow H_y\phi(f)$ satisfying

i) $H_{\mathrm{id}_x} = \mathrm{id}_{H_x}$ (normality)

ii) $(H_y\phi(\alpha)) \bullet H_f = H_g \bullet (\psi(\alpha)H_x)$, and

iii) for each pair of composable arrows $x \xrightarrow{f} y \xrightarrow{g} z$ there is a commutative prism with vertical faces H_g, H_f, H_{gf} and horizontal faces given by the structural

2-cells of ϕ, ψ.

$$\begin{tikzcd}
\phi(x) \arrow[rr] \arrow[dr] \arrow[dd] & & \phi(z) \arrow[dd] \\
& \phi(y) \arrow[ur] \arrow[dd] & \\
\psi(x) \arrow[rr] \arrow[dr] & & \psi(z) \\
& \psi(y) \arrow[ur] &
\end{tikzcd}$$

Such an H is a **lax equivalence** if the H_x are invertible up to a 2-cell and the H_f are invertible.

Remark 4.1.5. Example 4.1.4 can be easily extended to those special epimorphisms defining fibered categories (see section 1.1). The outcome is the *Grothendieck correspondence* between equivalence classes of fibred categories $E \to C$ and pseudo-functors $C \dashrightarrow \{\text{Categories}\}$. This is the first and most important example of lax functors. The main goal of this part can be considered to be a smooth linear variant of this correspondence.

4.2 The nerve of a 2-category

We already defined the nerve for Lie groupoids. We discuss here the nerve for 2-categories and 2-groupoids. We explain its behavior with respect to lax functors, and we use it to relate 2-groupoids with the weak approach to higher categories based on the horn filling condition. Some references for this are [15, 9, 35, 40].
For simplicial sets we will adopt the notation of appendix A.1.

Recall (cf. appendix A.1) that given $\mathcal{C}$ a category, and given $\phi : \Delta \to \mathcal{C}$ a covariant functor, that should be thought of as a model for simplices in $\mathcal{C}$, we can define a **singular functor** $\phi^* : \mathcal{C} \to \{\text{Simplicial sets}\}$ that associates to each object $X \in \mathcal{C}$ a simplicial set by the formula $(\phi^* X)_n = \hom_{\mathcal{C}}(\phi([n]), X)$.
We are concerned with the nerve construction for 2-categories, namely the singular functor defined when $\mathcal{C}$ is the category of 2-categories and lax functors, and $\phi([n]) = [n]$ is viewed as a 2-category with only identity 2-cells. Thus, if C is a 2-category, then its **nerve** NC has as n-simplices the lax functors $u : [n] \dashrightarrow C$, and its simplicial operators are given by pre-composition. Note that $NC_0 = C_0$ and $NC_1 = C_1$ consist of the objects and arrows of C, respectively, and NC_2 consists of triangles that are commutative up to a given 2-cell:

$$\begin{tikzcd}
& y \arrow[dr, "g"] \arrow[d, phantom, "\Uparrow\alpha"] & \\
x \arrow[ur, "f"] \arrow[rr, "h"'] & & z
\end{tikzcd}$$

To describe the higher simplices, note that a lax functor $u : [n] \dashrightarrow C$ can be thought of as a labelling of an abstract n-simplex, where u_i are objects at the vertices, $u_{j,i}$ are arrows at its edges, and $u_{k,j,i}$ are 2-cells corresponding to each triangle. For each tetrahedron on the simplex the following equation among 2-cells must hold:

$$\begin{tikzcd}
 & u_{l,i} \arrow[dl, Rightarrow, "u_{l,k,i}"'] \arrow[dr, Rightarrow, "u_{l,j,i}"] & \\
u_{l,k}u_{k,i} \arrow[dr, Rightarrow, "u_{l,k}u_{k,j,i}"'] & & u_{l,j}u_{j,i} \arrow[dl, Rightarrow, "u_{l,k,j}u_{j,i}"] \\
 & u_{l,k}u_{k,j}u_{j,i} &
\end{tikzcd}$$

The above data completely determines the nerve NC in the sense that it is *3-coskeletal*, namely $NC_k = \{\partial\Delta^k \to NC\}$ for $k > 3$.
A fundamental feature of the classic nerve for 1-categories is that it defines a fully faithful functor, it embeds the category of (small) categories into that of simplicial sets. Extending this, there is the following proposition for the nerve of 2-categories, which also provides information about the 2-cells. Here, by a **simplicial homotopy** we mean a simplicial map $X \times \Delta^1 \to Y$.

Proposition 4.2.1 (cf. [9])**.** *The nerve $C \mapsto NC$ defines a fully faithful functor from the category of (small) 2-categories and (normal) lax functors to the category of simplicial sets. Moreover, if $\phi, \psi : C \dashrightarrow D$ are lax functors and every 2-cell in D is invertible, then there is a lax transformation $H : \phi \Rightarrow \psi$ if and only if there is a simplicial homotopy $\tilde{H} : N\phi \cong N\psi$.*

Sketch of proof. Given a simplicial map $\tilde{\phi} : NC \to ND$, we can define a lax functor $\phi : C \dashrightarrow D$ such that $N\phi = \tilde{\phi}$ by setting $\phi_0 = \tilde{\phi}_0$, $\phi_1 = \tilde{\phi}_1$, and defining ϕ_2 and $\phi_{1,1}$ as restrictions of $\tilde{\phi}_2$ to the following type of triangles. The simplicial identities on $\tilde{\phi}$ imply the axioms of lax functor on ϕ, and that $N\phi = \tilde{\phi}$, proving the first assertion.

$$\begin{tikzcd}
 & y \arrow[dr, "\mathrm{id}_y"] \arrow[d, phantom, "\Uparrow\alpha"] & \\
x \arrow[ur, "f"] \arrow[rr, "f"'] & & y
\end{tikzcd}
\qquad
\begin{tikzcd}
 & y \arrow[dr, "g"] \arrow[d, phantom, "\Uparrow\mathrm{id}_{gf}"] & \\
x \arrow[ur, "f"] \arrow[rr, "gf"'] & & z
\end{tikzcd}$$

Regarding the second one, given $\phi, \psi : C \dashrightarrow D$ lax functors, while a lax transformation $H : \phi \cong \psi$ associates to an arrow $x \xrightarrow{f} y$ a 2-cell filling a commutative square, a simplicial homotopy $\tilde{H} : N\phi \cong N\psi$ should provide a triangulation of that square:

$$\begin{tikzcd}
\phi(x) \arrow[r, "\phi(f)"] \arrow[d, "H_y"'] \arrow[dr, phantom, "\Downarrow H_f"] & \phi(y) \arrow[d, "H_x"] \\
\psi(x) \arrow[r, "\psi(f)"'] & \psi(y)
\end{tikzcd}
\qquad
\begin{tikzcd}
\phi(x) \arrow[r, "\phi(f)"] \arrow[d, "\tilde{H}_x"'] \arrow[dr, "\Uparrow\tilde{H}_{f,0}", "\Downarrow\tilde{H}_{f,1}"'] & \phi(y) \arrow[d, "\tilde{H}_y"] \\
\psi(x) \arrow[r, "\psi(f)"'] & \psi(y)
\end{tikzcd}$$

where $\tilde{H}_{f,0}$ and $\tilde{H}_{f,1}$ are short for $\tilde{H}(s_1(f), s_0(\mathrm{id}_{[1]}))$ and $\tilde{H}(s_0(f), s_1(\mathrm{id}_{[1]}))$. The lax transformation H induces a simplicial homotopy $\tilde{H}$ by setting $\tilde{H}_{f,0} = \mathrm{id}$ and $\tilde{H}_{f,1} = H_f$. Conversely, if every 2-cell on D is invertible, we can define an H out of $\tilde{H}$ by setting $H_f = \tilde{H}_{f,1} \bullet (\tilde{H}_{f,0})^{-1}$. □

Another fundamental feature of the classic nerve is the following characterization of its image: a simplicial set is the nerve of a category if and only if every inner horn $(0 < k < n)$ admits a filling, and this filling is unique for $n > 1$. Similarly, it is the nerve of a groupoid if and only if the same holds for every horn, inner or not.

$$\begin{tikzcd} \Lambda^n_k \arrow[r, "\forall"] \arrow[d] & X \\ \Delta^n \arrow[ur, dashed, "\exists(!)"'] & \end{tikzcd}$$

This motivates an approach to higher category theory that has received much attention lately. A simplicial set X is then a **weak m-category** if every inner horn in X admits a filling, and the filling is unique for $n > m$, and X is a **weak m-groupoid** if the same holds for every horn, inner or not. The missing face of the horn, provided by the filling, should be though of as a *composition*, defined up to homotopy, of the remaining faces. Next proposition relates 2-groupoids with weak 2-groupoids via the nerve functor. Similar results are discussed in [24].

Proposition 4.2.2. *Given C a 2-category, NC is a weak 2-category if and only if every 2-cell of C is invertible, and NC is a weak 2-groupoid if and only if C is a 2-groupoid.*

Proof. Since NC is 3-coskeletal, every (n, k)-horn has a unique filling for $n \geq 5$. For $n = 2$ the horizontal composition of arrows provide inner horn-fillings, and the fillings of the outer horns correspond to the existence of quasi-inverses. So let us study the cases $n = 3, 4$.
For $n = 3$, given a 2-cell $\alpha : f \Rightarrow g : x \to y$, we can build a $(3, 1)$-horn with faces as below,

$$\begin{tikzcd} & y & \\ x \arrow[ur, "g"] \arrow[r, "g"] \arrow[rr, "f"'] & y \arrow[u, "\mathrm{id}"] \arrow[ur, "\mathrm{id}"'] \arrow[d, phantom, "\alpha\Uparrow"] & y \arrow[ul, "\mathrm{id}"'] \arrow[ul, phantom, "\mathrm{id}\Rightarrow"] \arrow[ul, phantom, "\mathrm{id}\Rightarrow"] \\ & & \end{tikzcd}$$

and the remaining face of a filling will give a right inverse $\beta : g \Rightarrow f$ to α, showing that inner-horn filling implies that every 2-cell is invertible. Conversely, a horn gives

three 2-cells, that correspond to three sides on a square as below:

$$\begin{tikzcd}
u_{3,0} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & u_{3,1}u_{1,0} \arrow[d, Rightarrow] \\
u_{3,2}u_{2,0} \arrow[r, Rightarrow] & u_{3,2}u_{2,1}u_{1,0}
\end{tikzcd}$$

In an inner horn, either the 2-cell on the top or in the left is missing, but since every 2-cell is invertible, we can fill the square by taking the obvious composition. In an outer horn, either the 2-cell on the bottom or on the right is missing, and assuming C is a 2-groupoid, we can get the missing face by factoring the triple composition by either $u_{3,2}$ or $u_{1,0}$ as it follows from 4.1.3.
For $n = 4$, the 2-skeleton of a 4-simplex u gives the edges of a cube as below:

$$\begin{tikzcd}
& u_{4,0} \arrow[dl, Rightarrow] \arrow[d, Rightarrow] \arrow[dr, Rightarrow] & \\
u_{4,1}u_{1,0} \arrow[d, Rightarrow] \arrow[dr, Rightarrow] & u_{4,2}u_{2,0} \arrow[dl, Rightarrow] \arrow[dr, Rightarrow] & u_{4,3}u_{3,0} \arrow[d, Rightarrow] \arrow[dl, Rightarrow] \\
u_{4,2}u_{2,1}u_{1,0} \arrow[dr, Rightarrow] & u_{4,3}u_{3,1}u_{1,0} \arrow[d, Rightarrow] & u_{4,3}u_{3,2}u_{2,0} \arrow[dl, Rightarrow] \\
& u_{4,3}u_{3,2}u_{2,1}u_{1,0} &
\end{tikzcd}$$

Each face of the 4-simplex corresponds to the commutativity of the corresponding face of the cube. The bottom face commutes because of the compatibility between horizontal and vertical composition. Since every 2-cell is invertible, five commuting faces on the cube imply that the other is commutative as well, thus every horn admits a unique filling, concluding the proof. □

Remark 4.2.3. Other ways to associate a simplicial set to a 2-category C are by regarding it as a double category with a trivial side, applying twice the classic nerve, and reducing the resulting bisimplicial set by using the *diagonal* d or the *total functor* T, also known as bar or codiagonal:

$$\text{2-categories} \xrightarrow{N^2} \text{bisimplicial sets} \overset{d,T}{\rightrightarrows} \text{simplicial sets}$$

It is shown in [15] that TN^2C and dN^2C are equivalent to NC from a homotopy viewpoint. We remark here that, when C is a strict 2-groupoid there is actually an isomorphism $TN^2C \cong NC$, which is completely determined by the following formula for 2-cells.

$$\begin{tikzcd}
x \arrow[r, "h"] \arrow[d, equal] \arrow[dr, phantom, "\Uparrow\alpha"] & y \arrow[r, "g"] \arrow[d, equal] & z \\
x \arrow[r, "f"'] \arrow[d, equal] & y & \\
x & &
\end{tikzcd}
\quad \mapsto \quad
\begin{tikzcd}
& y \arrow[dr, "f"] \arrow[d, phantom, "\Uparrow g\alpha^{-1}"] & \\
x \arrow[ur, "g"] \arrow[rr, "gh"'] & & z
\end{tikzcd}$$

4.3 Defining Lie 2-groupoids

We discuss here the smooth versions of 2-categories and 2-groupoids we are going to work with, provide some examples, and discuss other uses for those terms in the literature.

A **Lie 2-category** C is, roughly speaking, a 2-category internal to the category of smooth manifolds. It consists of a (small) 2-category as defined before, on which (i) the sets of objects C_0, arrows C_1 and 2-cells C_2 are equipped with manifold structures; (ii) the source and target maps $s, t : C_i \to C_{i-1}$ of 2-cells and arrows are surjective submersions, and (iii) the units $u : C_{i-1} \to C_i$ and the multiplications $\circ$ and $\bullet$ are smooth. Functors $\phi : C \to D$ between Lie 2-categories are easy to define, as 2-functors for which the three maps $\phi_i : C_i \to D_i$ are smooth.

Example 4.3.1. Let $(\mathbb{R}, \cdot)$ be the multiplicative monoid of real numbers, viewed as a Lie 2-category with a single object, space of arrows $\mathbb{R}$, and both horizontal and vertical composition equal to the multiplication. This is a Lie 2-category on which not every 2-cell is invertible.

Let G be a Lie 2-category that, from the set-theoretic viewpoint, is also a 2-groupoid, as defined in the previous sections. In order to define when G is a Lie 2-groupoid we have to make sense of smooth inversions. For 2-cells this is clear, because there is an inversion map $i : G_2 \to G_2$, and we can require it to be smooth. For arrows this is less clear: since inversion is only defined up to homotopy, there is not an inversion map in general. Note that, since source and target $G_2 \to G_1$ are surjective submersions, the sets of 2-horns $N_{2,i}G = \hom(\Lambda^2_i, NG)$ define manifolds:

$$N_{2,0}G = \left\{ \begin{tikzcd} & y \\ x \arrow[ur, "f"] \arrow[r, "h"'] & z \end{tikzcd} \right\} \qquad N_{2,1}G = \left\{ \begin{tikzcd} & y \arrow[dr, "g"] & \\ x \arrow[ur, "f"] & & z \end{tikzcd} \right\} \qquad N_{2,2}G = \left\{ \begin{tikzcd} & y \arrow[dr, "g"] & \\ x \arrow[rr, "h"'] & & z \end{tikzcd} \right\}$$

We will discuss a smooth structure on the whole nerve NG in the following sections. For now, we just endow N_2G with a manifold structure using the following fibered product.

$$\begin{tikzcd} N_2G \arrow[r] \arrow[d] & N_{2,1}G \arrow[d, "m"] \\ G_2 \arrow[r, "t"'] & G_1 \end{tikzcd}$$

We define G to be a **Lie 2-groupoid** if, besides being a Lie 2-category and a 2-groupoid, (i) the inversion of 2-cells $i : G_2 \to G_2$ is smooth, and (ii) the following restriction maps are surjective submersions:

$$d_{2,0} : N_2G \to N_{2,0}G \qquad d_{2,2} : N_2G \to N_{2,2}G.$$

We say that the Lie 2-groupoid is **strict** if it is set-theoretic strict and the inversion of arrows $i : G_1 \to G_1$ is smooth. The smooth structure on N_2G also allow us to make sense of lax functors in the smooth setting. We define a **smooth lax functor** betwen Lie 2-categories $\phi : C \dashrightarrow D$ as a lax functor such that ϕ_0, ϕ_1 and the map $(\phi_2, \phi_{1,1}) : N_2C \to N_2D$ is smooth.

Example 4.3.2. Given K an abelian Lie group, we can see it as the 2-cells of a Lie 2-category with one object and one arrow, and where both multiplications $\bullet$ and $\circ$ agree with that of K. The resulting 2-category $K \rightrightarrows * \rightrightarrows *$ is in fact a Lie 2-groupoid. A similar thing can be done with a bundle of abelian Lie groups $G \rightrightarrows M$, such as a torus bundle. This *delooping* construction stays within the finite dimensional setting and plays a key role for instance in the theory of *gerbes*.

We recall briefly the **Dold-Kan correspondence**, from appendix A.2. When $\mathcal{C}$ is an abelian category, eg. that of vector spaces, then a simplicial object $X : \Delta^\circ \to \mathcal{C}$ gives rise to a chain complex (X'_n, ∂) by defining $X'_n = \cap_{i>0} \ker(d_i : X_n \to X_{n-1})$ and $\partial = d_0$. It turns out that this construction yields an equivalence of categories between simplicial objects and positively graded chain complexes. The horn-filling condition translates into the abelian setting, in such a way that categories and groupoids both correspond to 2-term complexes, and linear natural transformations correspond to chain homotopies.

Example 4.3.3. By a **linear 2-category** we mean a Lie 2-category V on which the V_i are (real finite dimensional) vector spaces and the structure maps are linear. They are examples of Lie 2-groupoids. Viewing them as double linear categories, and applying Dold-Kan correspondence both horizontally and vertically, we encode such a V into a 3-term complex as below.

$$\begin{tikzcd} V'_2 \arrow[r] \arrow[d] & 0 \arrow[d] \\ V'_1 \arrow[r] & V_0 \end{tikzcd}$$

Remark 4.3.4. We warn the reader about different uses for the word Lie 2-groupoid within the literature other than the one we have introduced, which suits our fundamental example. In [46],[51] and other references, they use the word to refer to what we called strict Lie 2-groupoid, they demand a fonctorial choice of the inverse of arrows to exists, so our notion is more general. In [64], [25] and other references Lie 2-groupoids are defined as smooth versions of weak 2-groupoids, they do not require the existence of a well-defined composition. We will see later that a smooth version of the nerve functor for Lie 2-categories allow us to regard our Lie 2-groupoids as examples of them.

4.4 The general linear 2-groupoid

Here we show our first main theorem, asserting that the symmetries of a (2-term) graded vector space or bundle can be endowed with the structure of a Lie 2-groupoid, which we call the **general linear 2-groupoid**. This construction extends the general linear groupoid of a vector bundle without a grading (see eg. [21]).

Throughout this section, let $V = V_1 \oplus V_0 \to M$ be a graded vector bundle over a smooth manifold M. We will first describe the set-theoretic structure of its general linear 2-groupoid $GL(V)_\bullet$ and then take care of the smoothness. From the set-theoretic viewpoint we have:

i) An object $\partial^x \in GL(V)_0$ is a differential $\partial^x : V_1^x \to V_0^x$ on the fiber $V^x = V_0^x \oplus V_1^x$;

ii) An arrow $\alpha : \partial^x \to \partial^y \in GL(V)_1$ is a couple of linear maps $\alpha_1 : V_1^x \to V_1^y$, $\alpha_0 : V_0^x \to V_0^y$, defining a quasi-isomorphism between V^x and V^y;

$$\begin{tikzcd} V_1^x \arrow[r, "\alpha_1"] \arrow[d, "\partial^x"'] & V_1^y \arrow[d, "\partial^y"] \\ V_0^x \arrow[r, "\alpha_0"'] & V_0^y \end{tikzcd}$$

iii) A 2-cell $R : \alpha \to \alpha' : \partial^x \to \partial^y$ on $GL(V)_2$ is a chain homotopy, given by a linear map $R : V_0^x \to V_1^y$ such that $R\partial^x = \alpha_1 - \alpha_1'$ and $\partial^y R = \alpha_0 - \alpha_0'$.

$$\begin{tikzcd} V_1^x \arrow[r, "\alpha_1"] \arrow[d, "\partial^x"'] & V_1^y \arrow[d, "\partial^y"] \\ V_0^x \arrow[r, "\alpha_0"'] \arrow[ur, "R"] & V_0^y \end{tikzcd}$$

The multiplication $\circ$ in $GL(V)_\bullet$ is the composition of maps, and the multiplication $\bullet$ is the composition of chain homotopies, which is just the sum of the corresponding maps R. Every 2-cell is invertible, and every arrow is invertible up to a 2-cell. Thus we have a well-defined 2-groupoid $GL(V)_\bullet$. Via Dold-Kan we can embed it into the 2-category of linear categories.

Remark 4.4.1. Even when $M = *$ our construction $GL(V)_\bullet$ yields a 2-groupoid and not what one might call a 2-group, for there are many objects and not just one. Fixing an object ∂ on $GL(V)_\bullet$, its isotropy 2-groupoid $GL(V)_\partial$ can be compared with the construction studied in [50].

Next we show that $GL(V)_\bullet$ inherits a smooth structure from certain vector bundles. To ease the notation, given $A, B \to M$ vector bundles, we write $[A, B] \to M$ for the inner-hom vector bundle. Then we have that:

i) $GL(V)_0$ identifies with the total space of $[V_1, V_0] \to M$;

ii) $GL(V)_1$ is a subspace of $E = [\pi_1^* V_1, \pi_1^* V_0] \oplus [\pi_2^* V_1, \pi_2^* V_0] \oplus [\pi_1^* V_1, \pi_2^* V_1] \oplus [\pi_1^* V_0, \pi_2^* V_0]$, a vector bundle over $M \times M$, where $\pi_i : M \times M \to M$ are the obvious projections;

iii) $GL(V)_2$ is the set-theoretic fiber product $GL(V)_1 \times_{M \times M} [\pi_1^* V_0, \pi_2^* V_1]$.

The issue here is to show that $GL(V)_1 \subset E$ is a submanifold. Then $GL(V)_2$ will identify with a fibered product along a submersion, in fact with a pullback vector bundle. This issue is rather subtle and will require a careful analysis. The first step in our argument is to provide a simple system of equations describing $GL(V)_1 \subset E$.

Lemma 4.4.2. We can write $GL(V)_1 = F \cap U_1 \cap U_0$ where

$$\begin{aligned} F &= \{(\partial^x, \partial^y, \alpha_0, \alpha_1) \in E : \alpha_0 \partial^x = \partial^y \alpha_1\} \\ U_1 &= \{(\partial^x, \partial^y, \alpha_0, \alpha_1) \in E : \ker(\partial^x) \cap \ker(\alpha_1) = 0\} \\ U_0 &= \{(\partial^x, \partial^y, \alpha_0, \alpha_1) \in E : \mathrm{im}(\partial^y) + \mathrm{im}(\alpha_0) = V_0^y\} \end{aligned}$$

Proof. An element $(\partial^x, \partial^y, \alpha_0, \alpha_1)$ belongs to F if and only if the corresponding square of vector space maps commute, it belongs to U_1 if and only if the morphism between the fibers is injective in degree 1 homology, and belongs to U_0 if and only if it is surjective in degree 0 homology. Since both fibers V^x, V^y, as 2-term complexes, have the same Euler characteristic $\dim V_0 - \dim V_1$, then so do their homologies, and therefore the two inequalities $\dim H_1(V^x) \leq \dim H_1(V^y)$ and $\dim H_0(V^x) \geq \dim H_0(V^y)$ imply that α is in fact a quasi-isomorphism. □

The subset F can be seen as the preimage of the zero section of the following map between the total space of vector bundles over $M \times M$, where $E' = [\pi_1^* V_1, \pi_2^* V_0]$.

$$\phi : E \to E' \qquad \phi(\partial^x, \partial^y, \rho_1, \alpha_0) = \alpha_0 \partial^x - \partial^y \alpha_1$$

This map is quadratic and its rank is not constant in general, as next example shows.

Example 4.4.3. Let $M = *$ and $V_0 = V_1 = \mathbb{R}$. Then $GL(V)_0 \cong \mathbb{R}$, $E \cong \mathbb{R}^4$ and F identifies with $\{(x, y, z, w) \in \mathbb{R}^4 : xy - zw = 0\}$, that is not a submanifold of $\mathbb{R}^4$. This examples shows that if we define the general linear 2-category $gl(V)$ as we have defined $GL(V)_\bullet$, but without imposing the quasi-isomorphism axiom, then $gl(V)$ cannot be made a Lie 2-category in a reasonable way.

Next we show that the map ϕ above has maximal rank over the opens U_i, and since the zero section $0_{M \times M} \subset E'$ is closed embedded, the same holds for $GL(V)_1$.

Proposition 4.4.4. *The map $\phi : E \to E'$ has maximal rank over the opens U_i.*

Proof. Let $p = (\partial^x, \partial^y, \alpha_1, \alpha_0) \in U_1$ and let $q = \phi(p) = \alpha_0\partial^x - \partial^y\alpha_1$. To show that $d\phi_p : T_pE \to T_qE'$ is surjective we argue by realizing vectors as 1-jets of curves. Given $\gamma(t) \in E'$, $\gamma(0) = q$, we want to lift the curve γ to a curve on E through p. By using local trivializations of V we can assume $x(t) = x$ and $y(t) = y$ are fixed. Let us suppose that $p \in U_1$, the other case is analogous. Since $\ker \partial^x \cap \ker \alpha_1 = 0$, the linear map $(\partial^x, \alpha_1) : V_1^x \to V_0^x \oplus V_1^y$ is a monomorphism, and therefore it admits a linear retraction $(\tilde{\partial}^x, \tilde{\alpha}_1) : V_0^x \oplus V_1^y \to V_1^x$. Then the curve $\tilde{\gamma}(t) = (\partial^x, \gamma(t)\tilde{\alpha}_1, \alpha_1, \gamma(t)\tilde{\partial}^x) \in E$ is a lift as required. □

Theorem 4.4.5. *Given $V = V_1 \oplus V_0$ a graded vector bundle, its general linear 2-groupoid $GL(V)_\bullet$ inherits a natural structure of a Lie 2-groupoid.*

Proof. As we have already discussed, $GL(V)_0$ identifies $[V_1, V_0]$, $GL(V)_1 \subset E$ with the preimage of a closed embedded submanifold along a maximal rank map, and $GL(V)_2$ is a fiber product along a submersion. It is straightforward to check that with these definitions the structure maps of $GL(V)_\bullet$ are smooth, including the inversion of 2-cells. It only remains to show that the restriction maps below are surjective submersions:

$$d_{2,0} : N_2G \to N_{2,0}G \qquad d_{2,2} : N_2G \to N_{2,2}G$$

Let us show it for $d_{2,0}$, the other case is analogous. We argue again by lifting curves. We start with $\alpha(t) : \partial^{x(t)} \to \partial^{y(t)}$ and $\gamma(t) : \partial^{x(t)} \to \partial^{z(t)} \in GL(V)_1$, defining a curve on $N_{(2,0)}G$, and in order to lift it to N_2G, we want to define $\beta(t) : \partial^{y(t)} \to \partial^{z(t)}$ and $R(t) : \gamma(t) \Rightarrow \beta(t)\alpha(t)$. Working locally we can again assume $x = x(t), y = y(t), z = z(t)$ are fixed. The monomorphism $(\alpha_1(t), \partial^x(t)) : V_1^x \to V_1^y \oplus V_0^y$ admits a retraction $\tilde{\alpha}_1(t), \tilde{\partial}^x(t)$, and by basic argument on linear algebra, we can take it smooth on t. Then the following short exact sequence splits smoothly,

$$0 \to V_1^x \xrightarrow{(\alpha_1(t), \partial^x(t))} V_1^y \oplus V_0^y \xrightarrow{(\partial^y(t), \alpha_0(t))} V_0^y \to 0$$

and we gain a section $(\tilde{\partial}^y(t), \tilde{\alpha}_0(t))$. We can define $\beta_i(t) = \gamma_i(t)\tilde{\alpha}_i(t)$ and $R(t) = \gamma_1(t)\tilde{\partial}^x$. □

Remark 4.4.6. Let us denote by $GL'(V)_\bullet \subset GL(V)_\bullet$ the open Lie 2-groupoid with the same objects, arrows the invertible chain maps, and 2-cells the chain homotopies. This is a strict Lie 2-groupoid, somehow simpler than our version, and both agree around the units, thus both should behave in the same way with respect to *differentiation*, even though this process is not yet clear. See [50] for a related discussion. But regarding our purposes, this simpler construction $GL(V)'_\bullet$ is not satisfactory, there are representations up to homotopy of Lie groupoids that cannot be invertible. An example is the adjoint representation of the pair groupoid of the sphere $\mathrm{Pair}(S^2)$, or of any other non-parallelizable manifold. We will come back to this later.

4.5 The nerve of a Lie 2-category

We deal here with the problem of endowing the nerve NC of a Lie 2-category C with a reasonable smooth structure. We show with a simple example that for general C this may not be possible. Our second main theorem shows that if every 2-cell is invertible then NC is indeed a simplicial manifold, and this happens for instance if C is a Lie 2-groupoid.

Given C a Lie 2-category, we define its **ambient** simplicial manifold AC for the nerve NC, roughly speaking, by considering arbitrary collections $\{u_{k,j,i}\}$ of 2-cells and disregarding any compatibility. More precisely, we define AC by

$$A_nC = \prod_{[2]\xrightarrow{a}[n]} C_2 \qquad u \in A_nC,\ b : [n] \to [m] \ \Rightarrow\ b^*(u)_a = u_{b\circ a} \in A_mC$$

This way AC is a well-defined simplicial manifold, and every face map is a surjective submersion, for it is just the projection onto some of the coordinates. There is a **canonical inclusion** $\phi : NC \to AC$ defined by the formula $\phi(u)_a = (u \circ a)_{2,1,0}$, where $u \in N_nC$, $u : [n] \dashrightarrow C$, and $a : [2] \to [n]$. In other words, $\phi(u)$ keeps track of the 2-cells corresponding to each triangle, and by means of the identities, the arrows on the edges and the objects on the vertices. Since every simplex in NC is determined by its 2-skeleton, the map ϕ is injective. We are concern with the question of whether $\phi(N_nC) \subset A_nC$ is a submanifold, which is not the case in general.

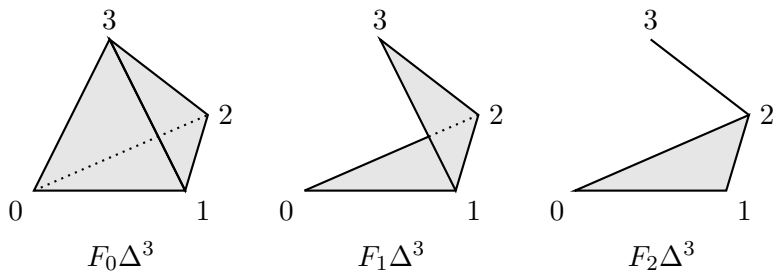
Example 4.5.1. Let $(\mathbb{R}, \cdot)$ be the multiplicative monoid viewed as a Lie 2-category as described in Example 4.3.1. Then $N_0C = \{*\}$, $N_1C = \{\mathrm{id}_*\}$, and $N_2C = \mathbb{R}$, but $N_3C \subset A_3C$ is not a submanifold. Disregarding the degenerate coordinates, we can identify N_3C with tuples $(x, y, z, w) \in \mathbb{R}^4$ such that $xy = zw$, the equation corresponding to the commutativity of the tetrahedron.

For C a 1-category, a simplex $u \in N_nC$ is the same as a chain of n composable arrows, so we can write N_nC as an iterated fiber product, and use this to define a smooth structure on it. Next we develop a similar combinatorial description for simplices $u \in N_nC$, where C is a 2-category whose 2-cells are invertible.
We see Δ^{n-1} inside Δ^n by using the face d_n, and define a decreasing filtration

$$\Delta^n = F_0\Delta^n \supset F_1\Delta^n \supset \cdots \supset F_{n-1}\Delta^n \supset \Delta^{n-1}$$

by setting $F_k\Delta^n = \{a : [m] \to [n] \mid a(m) < n \text{ or } a(0) \geq k\}$, namely $F_k\Delta^n$ is the union of Δ^{n-1} with the last face of dimension k. As an example, we depict the

filtration for $n = 3$:



Define $N_n^k C = \{F_k\Delta^n \to NC\}$. Note that $N_n^0 C = N_n C$, that we have projections $N_n^k C \to N_n^{k+1} C$, and that $N_n^{n-1} C = N_{n-1} \times_{C_0} C_1$ is the set-theoretic fiber product over $u \mapsto u_n$ and s.

Proposition 4.5.2. *If every 2-cell of C is invertible then there are set-theoretic fiber products:*

$$\begin{tikzcd} N_n^{k-1} C \arrow[r] \arrow[d] & C_2 \arrow[d, "t"] \\ N_n^k C \arrow[r, "\phi_n^k"'] & C_1 \end{tikzcd} \qquad \phi_n^k(u) = u_{n,k} \circ u_{k,k-1}$$

Proof. The inclusion $F_{k+1}\Delta^n \to F_k\Delta^n$ has all the vertices on its image, all the edges except for (n, k), and all the triangles except for (n, l, k), with $k < l < n$. Thus, given $u : F_k\Delta^n \to NC$, if we know its restriction u' to $F_{k+1}\Delta^n$ and the 2-cell α corresponding to the triangle $(n, k+1, k)$, then we have all the vertices, we recover the edge (n, k) as the source of α, and we recover the 2-cells corresponding to the triangles (n, l, k) inductively on $l - k$ by means of the equation:

$$u_{n,l,k} = (u_{n,l} \circ u_{l,k+1,k})^{-1} \bullet (u_{n,l,k+1} \circ u_{k+1,k}) \bullet u_{n,k+1,k}$$

This shows that the map $N_n^k C \to N_n^{k+1} C \times_{C_1} C_2$ is injective.

To see that it is also surjective, we need to check that, given $u' : F_{k+1}\Delta^n \to NC$ and given $\alpha : u'_{n,k} \Rightarrow u'_{n,k+1} u'_{k+1,k}$, the above equations can be used to define a simplicial map $u : F_k\Delta^n \to NC$. This translates into showing that for every tetrahedron (l, k, j, i) the above equation holds. The only tetrahedrons that deserve an explanation are those of the type (n, l', l, k) with $k < l < l' < n$. Moreover, if $l = k + 1$ then the equation holds by the construction of u. So let us assume that

$k+1<l$. The 4-simplex corresponding to $(n,l',l,k+1,k)$ yields a cube as below:

$$\begin{tikzcd}
 & u_{n,k} \arrow[dl, Rightarrow] \arrow[d, Rightarrow] \arrow[dr, Rightarrow] & \\
u_{n,k+1}u_{k+1,k} \arrow[d, Rightarrow] \arrow[dr, Rightarrow] & u_{n,l}u_{l,k} \arrow[dl, Rightarrow] \arrow[dr, Rightarrow] & u_{n,l'}u_{l',k} \arrow[d, Rightarrow] \arrow[dl, Rightarrow] \\
u_{n,l}u_{l,k+1}u_{k+1,k} \arrow[dr, Rightarrow] & u_{n,l'}u_{l',k+1}u_{k+1,k} \arrow[d, Rightarrow] & u_{n,l'}u_{l',l}u_{l,k} \arrow[dl, Rightarrow] \\
 & u_{n,l'}u_{l',l}u_{l,k+1}u_{k+1,k} &
\end{tikzcd}$$

We want to see that the back right face commutes. But we know that: the back left face commutes by definition of $u_{n,l,k}$; the upper face commutes by definition of $u_{n,l',k}$; the left front face commutes for it factors through $u_{k+1,k}$; the right front face commutes for it factors through $u_{n,l'}$; and the bottom face commutes for $\circ$ and $\bullet$ are mutually distributibe. Hence the result. $\square$

We can now prove our second main theorem.

Theorem 4.5.3. *Given C a Lie 2-category, if its 2-arrows are (smoothly) invertible, then the nerve NC is naturally a simplicial manifold.*

Proof. We endow each N_nC with a smooth structure inductively. For $n=0,1$ we do it by means of the obvious identifications $N_0C=C_0$ and $N_1C=C_1$. For larger n we use the filtration and fiber products on previous proposition, noting that one of the maps is always a surjective submersion, and using the standard transversality criterion. Hence N_nC is a closed embedded submanifold of the product

$$N_nC \subset N_{n-1}C \times \prod_{(i+1,i)} C_1 \times \prod_{(n,i+1,i)} C_2$$

We will prove that, for these smooth structures, the canonical inclusion $\phi : N_nC \to A_nC$ into the ambient is a closed embedding. This implies that (i) the smooth structures that we have defined on N_nC do not depend on the particular filtration we have used, and that (ii) the simplicial maps on NC are smooth and NC is a simplicial manifold.
For each triple (k,j,i), we have to show that the composition $\phi_{k,j,i} = \pi_{k,j,i}\phi : N_nC \to A_nC \to C_2$ is smooth. By projecting on the first coordinate of the above product, and using an inductive argument, we solve the case $n>k$. By projecting on the other coordinates we solve the cases $(n,i+1,i)$. It remains to study the other projections $\phi_{n,j,i}$. But such a projection can be written as an expression involving the other coordinates and the multiplications $\circ$ and $\bullet$, that are smooth. A similar argument applies also to the degenerate coordinates. $\square$

It follows from our theorem that the nerve of a Lie 2-groupoid is a simplicial manifold, and that a smooth pseudo-functors $\phi : G \dashrightarrow G'$ is the same as a simplicial smooth map $\phi : NG \to NG'$. Next we present a less immediate corollary.

Corollary 4.5.4. *With the above hypothesis, the face maps $d_i : N_nC \to N_{n-1}C$ are surjective submersions.*

Proof. This is more a corollary of the proof rather than of the statement. When $i = n$ it follows by factoring d_n through the filtration, for each projection $N_n^kC \to N_n^{k+1}C$ is the base-change of a surjective submersion, as well as $N_n^{n-1}C \to N_{n-1}C$. When $i \neq n$ we can argue similarly, but now using a different filtration of Δ^n, by complexes containing the face $d_i(\Delta^{n-1})$. □

We finish this section by developing a smooth version of 4.2.2, setting a bridge between our theory and that of weak Lie 2-categories and weak Lie 2-groupoids, as defined in [35, 64]. A simplicial manifold X is a **weak Lie m-category** or a **weak Lie m-groupoid** if the corresponding restrictions maps $X_n \to X_{n,k}$ are surjective submersions, for some reasonable smooth structure on the space of (n, k)-horns. The space of horns $X_{n,k}$ can be expressed as an equalizer

$$\prod_{i\neq k} X_{n-1} \rightrightarrows \prod_{i,j\neq k} X_{n-1},$$

which may not exists in the category of manifolds. In general this is argued by an inductive argument. In our case, when $X = NC$ is the nerve of a Lie 2-category with invertible 2-arrows, it follows from our construction that $X_n \to \prod_{i\neq k} X_{n-1}$ is a closed embedded submanifold for $n > 3$ and for $n = 3, k = 2$. The case $n = 3, k = 1$ follows by using a symmetric filtration on the simplex. Therefore, since X_n is also a set-theoretic equalizer, we conclude that $X_n \cong X_{n,k}$ is a diffeomorphism in these cases. The case $n = 2$ is easy, and therefore we can conclude:

Proposition 4.5.5. *Let C be a Lie 2-category on which every 2-arrow is invertible. Then NC is a weak Lie 2-category. Moreover, NC is a weak Lie 2-groupoid if and only if C is a Lie 2-groupoid.*

Remark 4.5.6. The main theorem on [46] shows that if G is a strict Lie 2-groupoid then TN^2G is a weak Lie 2-groupoid. Thus, in light of the isomorphism described in 4.2.3, our theorem can be regarded as an extension of that to non-strict Lie 2-groupoid. This is crucial for us, for our fundamental example $GL(V)_\bullet$ is not strict.

4.6 Representations as pseudo-functors

In this section we study representations up to homotopy of chapter 3 in the particular case of 2-term vector bundles $V = V_1 \oplus V_0$, and present our main theorem, stating

a 1-1 correspondence between representations $G \curvearrowright V$ and pseudo-functors $G \dashrightarrow GL(V)_\bullet$.

We studied in chapter 3 representations up to homotopy of a Lie groupoid G over a graded vector bundle $V = \oplus V_i$. We defined it as a degree 1 differential D on a space of sections $\Gamma(NG, V)$ of V over the nerve of G inducing a graded module structure. By decomposing $D = \oplus D_i$ into bi-homogeneous components, we can reinterpret D as a pseudo-representation over a complex (V, ∂) with homotopies controlling its associativity. We recall here the 2-term case, the simplest new case, using an homological convention.

Proposition 4.6.1 (cf. [22, 29])**.** *If $V = V_1 \oplus V_0$, then a representation up to homotopy $G \curvearrowright V$ is the same as a tuple $(\partial, \rho_1, \rho_0, \gamma)$, where $\partial : V_1 \to V_0$ is a linear map, $\rho_i : G \curvearrowright V_i$ are pseudo-representations commuting with ∂, and*

$$\gamma : (x \xrightarrow{g} y \xrightarrow{h} z) \mapsto (\gamma_{h,g} : \rho_{hg} \Rightarrow \rho_h \rho_g)$$

is a curvature tensor satisfying

$$\rho_1^{g_3} \circ \gamma^{g_2,g_1} - \gamma^{g_3 g_2, g_1} + \gamma^{g_3, g_2 g_1} - \gamma^{g_3,g_2} \circ \rho_0^{g_1} = 0.$$

A morphism $\phi : V \to W$ is the same as a triple (ϕ_1, ϕ_0, μ) where $\phi_i : V_i \to W_i$ are vector bundle chain maps and $\mu : (x \xrightarrow{g} y) \mapsto (\mu_g : V_0^x \to W_1^y)$ is a tensor satisfying $\rho\phi - \phi\rho = \mu\partial + \partial\mu$ and

$$\phi_1 \gamma^{h,g} + \mu_g \rho_2^h + \rho_1^g \mu_h - \mu^{gh} - \gamma^{h,g} \phi_0 = 0.$$

The **point-wise homology** of a 2-term representation $G \curvearrowright V$ consists of $H_1^x(V) = \ker \partial^x$ and $H_0^x(V) = \operatorname{coker} \partial^x$. If the rank of ∂ is constant then $H_1(V)$ and $H_0(V)$ are vector bundles and there is an induced representation over them. A representation up to homotopy V whose point-wise homology vanishes is called **acyclic**. A morphisms $\phi : V \to W$ of 2-term representations up to homotopy inducing isomorphims on the point-wise homology is called an **equivalence**.

Example 4.6.2. Given $\rho : \mathrm{Pair}(P^2) \curvearrowright E$ the pseudo-representation discussed in 3.2.4, we can define an acyclic representation up to homotopy $\mathrm{Pair}(P^2) \curvearrowright E \oplus E$ by setting $\partial = \mathrm{id}$, $\rho_1 = \rho_0 = \rho$ and $\gamma = \rho - \rho\rho$. The same can be done for any pseudo-representation.

We are now ready to present our main theorem. Given a Lie groupoid $G \rightrightarrows M$ we have a canonical projection $\pi_G : G \to P(M)$ that just remembers the source and target of an arrow. Given a 2-term vector bundle $V \to M$, we have a canonical projection $\pi_V : GL(V)_\bullet \to P(M)$ that only remembers the base-points on the vector bundle. Then we have

Theorem 4.6.3. *Given $G \rightrightarrows M$ a Lie groupoid and $V = V_1 \oplus V_0 \to M$ a graded vector bundle, there is a 1-1 correspondence between equivalence classes of representations up to homotopy $\rho : G \curvearrowright V$ and of pseudo-functors $\phi : G \dashrightarrow GL(V)_\bullet$ satisfying $\pi_V \phi = \pi_G$.*

Proof. This is a rather direct consequence of the constructions and results collected during our work. In light of our construction of the general linear groupoid (cf. 4.4.5) and in light of our definition of smooth pseudofunctors, supported on the simplicial characterization (cf. 4.5.3), a pseudo-functor $\phi : G \dashrightarrow GL(V)_\bullet$ is a simplicial map $\phi : NG \to NGL(V)_\bullet$, the degree 0 component ϕ_0 is the same as a differential ∂ on V, the degree 1 component ϕ_1 gives a pseudo-representation ρ on V compatible with ∂, and the degree 2 component ϕ_2 yields a curvature tensor $(x \xrightarrow{g} y \xrightarrow{h} z) \mapsto (\gamma_{h,g} : \rho_{hg} \Rightarrow \rho_h \rho_g)$, defining a 2-term representation up to homotopy, as characterized in proposition 4.6.1. Analogously, a smooth lax equivalence $H : \phi \Rightarrow \psi$ between two pseudo-functors consists of smooth maps $H : M \to GL(V)_1$, $H : G \to GL(V)_2$, which correspond to the components ϕ_0, μ of a morphism of representations described in 4.6.1, the component ϕ_1 being determined by ϕ, ψ and ϕ_0. It is straightforward to check that this way we get a 1-1 correspondence between equivalence classes of smooth pseudo-functors and representations up to homotopy. $\square$

If we combine our main theorem with the main theorem of [29] then we get what we might call a smooth linear variant of Grothendieck correspondence (cf. 4.1.5).

$$\left\{\begin{matrix}\text{VB-groupoids}\\ \Gamma \to G\end{matrix}\right\} \quad \rightleftharpoons \quad \left\{\begin{matrix}\text{2-term RUTH}\\ G \curvearrowright V_1 \oplus V_0\end{matrix}\right\} \quad \rightleftharpoons \quad \left\{\begin{matrix}\text{pseudo-functors}\\ G \dashrightarrow GL(V)_\bullet\end{matrix}\right\}$$

Corollary 4.6.4. *Given $G \rightrightarrows M$ a Lie groupoid, there is a 1-1 correspondence between equivalence classes of VB-groupoids $\Gamma \to G$ and of pseudo-functors $\phi : G \dashrightarrow GL(V)_\bullet$ satisfying $\pi_V \phi = \pi_G$.*

We close this section by outlining two different problems related to our results, the first related to the infinitesimal picture and the second with higher versions of our results.

Remark 4.6.5. In [45], an infinitesimal analog to our main theorem was announced. It is commonly accepted that weak higher Lie groupoids and higher Lie algebroids are related by a theory of differentiation and integration, though the details of such a theory are yet to be understood. Within this context, we expect that the differentiation of our general linear 2-groupoid is the object $gl(V)$ introduced there, and that the differentiation and integration of maps will provide an alternative approach to the integration of 2-term representations up to homotopy, other that that of [16].

Remark 4.6.6. It is natural to expect our results to remain valid on higher degrees. The construction of the general linear groupoid seems plausible to be generalized for more general graded vector bundles. The understanding of pseudo-functors within this context seems to be less clear, though a complete immersion into the simplicial approach would solve this issue. Expectations here should be curbed, for even disregarding the smooth and linear structures, such a higher analog for Grothendieck correspondence is still unknown.

4.7 The stack $Perf_{[0,1]}$

In chapter 3 we saw how to relate perfect complexes on a differentiable stack $\mathfrak{X}$ to representations up to homotopy on a given presentation $G \rightrightarrows M$ of $\mathfrak{X}$. Then in chapter 4, we constructed a Lie 2-groupoid $GL(V)_\bullet$ of automorphisms of a 2-terms graded vector bundle $V = V_0 \oplus V_1 \to M$, and we showed that representations up to homotopy of a Lie groupoid $G \rightrightarrows M$ on V are exactly morphisms of Lie 2-groupoids (which we called "pseudo-functors") $G \dashrightarrow GL(V)_\bullet$. In this last section we will use these two results to construct a stack $Perf_{[0,1]}$ of perfect complexes of amplitude $[0,1]$, where amplitude $[0,1]$ means that we can find a Lie groupoid presenting $\mathfrak{X}$ such that the pullback on it is a representation up to homotopy concentrated in degrees 0 and 1.
For this, we construct a weak Lie 2-groupoid $GL_\bullet$ such that:

1. $GL_\bullet$ does not depend on the Lie groupoid presenting $\mathfrak{X}$ nor on a choice of vector bundles;

2. The 2-category of morphisms from $G \rightrightarrows M$ to $GL_\bullet$ is equivalent to the full sub-dg-category of $Rep^\infty(G)$ of those representations which are concentrated in degrees 0 and 1;

3. The quotient 2-stack $Perf_{[0,1]}$ of $GL_\bullet$ is 2-geometric and $\mathrm{Hom}(\mathfrak{X}, Perf_{[0,1]})$ is the (2-)category of perfect complexes of amplitude $[0,1]$ over $\mathfrak{X}$.

In this section we will denote in the same way a Lie groupoid and its simplicial nerve.

Definition 4.7.1. The **general linear weak 2-groupoid** $GL_\bullet$ is

$$GL_\bullet = \bigoplus_{(n,m)\in\mathbb{N}\times\mathbb{N}} GL(\mathbb{R}^n_0 \oplus \mathbb{R}^m_1)$$

where $\mathbb{R}^n_0$ and $\mathbb{R}^m_1$ are respectively the degree 0 and 1 terms of a two terms graded vector bundle over the point $*$.

Here we prefer to think at $GL_\bullet$ as a simplicial manifold, so the direct sum should be intended as a direct sum of simplicial manifolds.

Remark 4.7.2. Let $G \rightrightarrows \mathbb{R}^k$ be a Lie groupoid with trivial base. Then a two term graded vector bundle over $\mathbb{R}^k$ is trivial and is determined, up to isomorphism, by the data of ranks n and m, of the degree 0 and 1 terms. By the results of section 4.6, we deduce that pseudo-functors from $G \rightrightarrows \mathbb{R}^k$ to $GL_\bullet$ are in 1-1 correspondence with **all** representations up to homotopy of the Lie groupoid $G \rightrightarrows \mathbb{R}^k$.

We recall from section 1.7, that morphisms of differentiable stacks from $\mathfrak{X} = [M/G]$ to $\mathfrak{Y} = [H/N]$ correspond to zig-zags

$$X_\bullet \tilde{\leftarrow} Z_\bullet \to Y_\bullet$$

the left pointing morphism being a Morita morphism. We are concerned with generalized morphisms with source a Lie groupoid and target a 2-Lie groupoid. In this special case, generalized morphisms are defined as above.

Definition 4.7.3. We define $Perf_{[0,1]}$ as the homotopy colimit of the simplicial manifold $GL_\bullet$, in the ∞-category $St(Diff)$.

Claim 4.7.4. *Let $Rep^\infty(G)_{[0,1]}$ be the full sub-dg-category of $Rep^\infty(G)$ of those representations on graded vector bundles concentrated in degrees 0 and 1, and let $\mathfrak{X} = [M/G]$. Then there is an equivalence*

$$Rep^\infty(G)_{[0,1]} \simeq \mathrm{Hom}_{St}(\mathfrak{X}, Perf_{[0,1]})$$

We will not show the ∞-categorical equivalence here, we only sketch the 1-1 correspondence between the objects on the homotopy categories. By theorem 4.6.3, there is a 1-1 correspondence between representations up to homotopy of $G \rightrightarrows M$ on $V = V_0 \oplus V_1$ and morphisms of simplicial manifolds $G_\bullet \to NGL(V)_\bullet$ (or pseudo functors $G_\bullet \dashrightarrow GL(V)_\bullet$). Moreover the perfect complexes of amplitude $[0,1]$ on a differentiable stack $\mathfrak{X}$ are

$$Perf_{[0,1]}(\mathfrak{X}) = \bigcup_{V=V_0\oplus V_1} \left\{ \begin{matrix} \text{pseudo-functors} \\ G \dashrightarrow GL(V)_\bullet \end{matrix} \right\}$$

where V varies over all possibles 2-terms graded vector bundles over M.
The idea now is that the choice of V_0 and V_1 depends on the choice of a cocycle on an open cover $\{U_i\}$ of M, so instead of choosing for each element of the union a graded vector bundle, we could just refine the base manifold M, by taking the pullback Čech groupoid $(\coprod U_i)_\bullet$ of a trivializing cover. Note that a vector bundle on $\coprod U_i$ is always trivial, so morphisms $(\coprod U_i)_\bullet \to GL_\bullet$ (ou pseudo functors $(\coprod U_i)_\bullet \dashrightarrow GL_\bullet$) are already all the representations up to homotopy of $(\coprod U_i)_\bullet$, as we saw in remark 4.7.2. Then we should just show that this includes all possible choices of vector bundles V on M. This is a sort of generalization of proposition 3.1.5 to the two terms case.

Sketch of proof. Let $G_\bullet \to GL(V)_\bullet$ be a representation up to homotopy of $G \rightrightarrows M$ on V. The Hom-stack $\mathrm{Hom}_{St(Diff)}(\mathfrak{X}, Perf_{[0,1]})$ has objects the zig-zags

$$G_\bullet \tilde{\leftarrow} H_\bullet \to GL_\bullet$$

with $H_\bullet \to G_\bullet$ Morita morphism (see [26], lemma 2.18). We now find a Morita morphism $H_\bullet \to G_\bullet$ and a morphism $H_\bullet \to GL_\bullet$ giving the same representation up to homotopy.

Let $\{U_i \simeq \mathbb{R}^k\}$ be a trivializing open cover of M, and $(\coprod U_i)_\bullet$ be the associated pullback groupoid (see 1.5.2). Let $\alpha^0_{ij} : U_{ij} \to GL_{k_0}$ and $\alpha^1_{ij} : U_{ij} \to GL_{k_1}$ be the cocycles associated to the vector bundles V_k for $k = 0, 1$, of ranks k_0 and k_1 respectively, with respect to the open cover $\{U_i\}$.
We remind from theorem 4.6.1 that the data of a morphisms $G \to GL(V_0 \oplus V_1)$ is:

1. for each $x \in M$, a differential $\partial_x : V^0_x \to V^1_x$;
2. for each $f : x \to y$ in G, a quasi-isomorphism $\rho_f : \partial_x \to \partial_y$;
3. for each couple of composable arrows $x \xrightarrow{f} y \xrightarrow{g} z$, an homotopy $\gamma_{h,g} : \rho_{hg} \Rightarrow \rho_h \rho_g$.

Satisfying some compatibilities. Now, the data of a morphism $(\coprod U_i)_\bullet \to GL(\mathbb{R}^{K_0} \oplus \mathbb{R}^{k_1}$ is:

1. for each $i, x \in U_i$, a differential $\partial_{i,x} : \mathbb{R}^{k_0} \to \mathbb{R}^{k_1}$;
2. for each $i, j, x \in U_i,\ y \in U_j, f : x \to y$ in G, a quasi-isomorphism

$$\begin{tikzcd} \mathbb{R}^{k_0} \arrow[r, "\rho^0_{f,ij}"] \arrow[d, "\partial_{i,x}"'] & \mathbb{R}^{K_0} \arrow[d, "\partial_{j,y}"] \\ \mathbb{R}^{K_1} \arrow[r, "\rho^1_{f,ij}"'] & \mathbb{R}^{K_1} \end{tikzcd}$$

3. for each $i, j, k,\ x \in U_i,\ y \in U_j,\ z \in U_k$ and couple of composable arrows $x \xrightarrow{f} y \xrightarrow{g} z$, an homotopy $\gamma_{h,g,ijk} : \rho_{hg,ik} \Rightarrow \rho_{h,ij}\rho_{g,jk}$.

By construction of the pullback Čech groupoid (cf. 1.5.5), morphisms in $(\coprod U_i)_\bullet$ are

$$\bigoplus_{ij} U_i \times_M G \times_M U_j$$

We remark that $U_i \cap U_j$ is the subset of $U_i \times_M G \times_M U_j$ of identity functions, and that for $l = 0, 1$ we can define

$$\alpha^l_{ij} : U_{ij} \to GL_{k_l} \quad \text{by } \alpha^l_{ij}(x) := \rho^l_{\mathrm{id}_x, ij}$$

then $\{\alpha^l_{ij}\}_{ij}$ is a cocyle up to homotopy for the open cover $\{U_i\}$. If we add the condition that the homotopies $\gamma_{\mathrm{id}_x,\mathrm{id}_x,ijk}$ are trivial for all x, then we obtain that $\{\alpha^l_{ij}\}_{ij}$ is a honest strict cocycle for $l = 0, 1$, defining vector bundles E_0 and E_1 on M, and that we can glue the various $\rho_{f,ij}$ and $\gamma_{h,g,ijk}$ to obtain a representation up to homotopy of $G \rightrightarrows M$ on $E_0 \oplus E_1$. This mean that the representations up to homotopy $(\coprod U_i)_\bullet \to GL_\bullet$ include all representations up to homotopy $G_\bullet \to GL(V)_\bullet$ for all two terms graded vector bundle $V = V_0 \oplus V_1$ on M, as the subset of those representations which are a strict cocycle when restricted to the subgroupoid $\coprod_{ij} U_{ij} \rightrightarrows \coprod U_i$. $\square$

Appendix A

Sites, simplicial preasheaves and hypercovers

A.1 Simplicial sets

As usual, let $[n] = \{n, n-1, \cdots, 1, 0\}$ denotes the ordinal of $n+1$ element, and Δ the category of finite ordinals and order preserving maps, spanned by the elementary maps

$$d^i : [n-1] \to [n] \quad d^i(k) = \begin{cases} k & k < i \\ k+1 & k \geq i \end{cases}$$

$$s^j : [n+1] \to [n] \quad s^j(k) = \begin{cases} k & k \leq j \\ k-1 & k > j \end{cases}$$

which satisfy the so-called simplicial identities. Then a **simplicial set** is a contravariant functor $X : \Delta^\circ \to \{\text{Sets}\}$. It can be described as a sequence of sets $X_n = X([n])$ and a collection of *face* $d_i = X(d^i)$ and *degeneracy* $s_j = X(s^j)$ operators satisfying the (dual) simplicial identities. Maps of simplicial sets are natural transformations, or equivalently, sequences of maps compatible with the faces and degeneracies. Simplicial objects on a category $\mathcal{C}$ are defined analogously.

Example A.1.1. A simple but fundamental example is the n**-simplex** Δ^n. From the functorial viewpoint, it is the one represented by the ordinal $[n]$. Thinking of Δ^n as a graded set with further structure, it is freely generated by an element of type $[n]$, namely $\mathrm{id}_{[n]}$. By Yoneda Lemma, a map $\Delta^n \to X$ is the same as an element in X_n. The **border** $\partial\Delta^n \subset \Delta^n$ is spanned by all the faces of the generator, and the **horn** $\Lambda^n_k \subset \Delta^n$ by all the faces but the k-th.

Given $\mathcal{C}$ a category, and given $\phi : \Delta \to \mathcal{C}$ a covariant functor, that should be thought of as a model for simplices in $\mathcal{C}$, we can define a **singular functor** ϕ^* :

$\mathcal{C} \to \{\text{Simplicial sets}\}$ that associates to each object $X \in \mathcal{C}$ a simplicial set by the formula $(\phi^* X)_n = \hom_{\mathcal{C}}(\phi([n]), X)$. In other words, $\phi^* X$ is the restriction of the contravariant functor represented by X to Δ via ϕ.

Example A.1.2. When $\mathcal{C}$ is the category of topological spaces and $\phi([n])$ is the topological n-simplex, then $\phi^* X = SX$ is the **singular simplicial set** associated to X, used to define its homology. When $\mathcal{C}$ is the category of (small) categories and $\phi([n]) = [n]$, where we see an ordinal as a category by setting an arrow $i \to j$ if $i \leq j$, then $\phi^* C = NC$ is the **nerve** of the category, whose n-simplices are chains of n composable arrows, and faces and degeneracies are given by droping an extremal arrow, composing two consecutive ones, or inserting an identity.

A.2 The Dold-Kan correspondence

The Dold-Kan correspondence is an equivalence between the category $Ch^{\geq 0}$ of non-negatively graded cochain complexes of abelian groups and the category Ab^{Δ} of cosimplicial abelian groups.
This equivalence is defined by a pair of inverse functor

$$N : Ab^{\Delta} \leftrightarrow Ch^{\geq 0} : K$$

For a cosimplicial abelian ring $C^{\bullet}$, the cochain complex $N(C)^{\bullet}$, called the **normalized Moore complex** of $C^{\bullet}$ is defined by

$$N(C)^n := C^n / \sum_{i=1}^{n} \partial_i C^{n-1} \simeq \bigcap_{i=0}^{n-1} \ker(d_i : C^n \to C^{n-1})$$

For a non-negatively graded cochain complex (V, d), the cosimplicial abelian group $K(V)^{\bullet}$ is defined by

$$K(V)^n := \bigoplus_{i=1}^{n} \begin{pmatrix} n \\ i \end{pmatrix} V^i \simeq V^i \otimes \Lambda^i \mathbb{Z}^n$$

If in addiction V is a dg-ring, we can put a product on $K(V)$ making it a cosimplicial ring.

Proposition A.2.1 (cf. [17])**.** *The pair of functors (N, K) establish an equivalence of categories. Moreover, the restriction of K to dg-rings establish an equivalence between the category of positively graded dg-rings and cosimplicial rings.*

A.3 Simplicial presheaves and hypercovers

We recall now some results from [23].

Definition A.3.1. A **simplicial presheaf** on a category C is a functor from C to the category of simplicial sets $C^{op} \to sSet$.

Definition A.3.2. Let $X \in sPr(C)$ and S be a simplicial set. We denote by $\mathrm{Hom}(S, X)$ the limit

$$\mathrm{Hom}(S, X) := \lim_{\Delta^k \to S} X_k$$

Definition A.3.3. Let $f : X \to Y$ be a morphism in $sPr(C)$. Then the **matching object** $M_k(f)$ is the limit

$$\hom(\partial\Delta^k, X) \times_{\hom(\partial\Delta^k, Y)} Y_k$$

and we denote by $\mu_k(f)$ the natural map from X_k to $M_k(f)$. When Y is the final object we write $M_k(X)$ instead of $M_k(f)$.

Definition A.3.4. Let $f : E \to B$ be a morphism of simplicial presheaves. We say that f is a **generalized cover** if for all $X \in C$ and morphisms $X \to B$, there is a cover $\{U_i \to X\}$ such that for all i, $U_i \to B$ lifts through f.

Definition A.3.5. A morphism of simplicial presheaves $f : U_\bullet \to V_\bullet$ is an **hypercover** if and only if each U_k is a coproduct of representables and the maps $\mu_k(f) : U_k \to M_k(f)$ are generalized covers.
We say that $f : U_\bullet \to V_\bullet$ is an n-hypercover if moreover $\mu_k(f) : U_k \to M_k(f)$ is an isomorphism for all $k \geq n$.

We can convince ourselves that hypercovers are exactly the higher analogue of the local isomorphisms between presheaves of sets by analyzing the hypercovers between n-truncated object, for low n.

Example A.3.6 (Hypercovers between presheaves of sets). The 0-truncated objects in $sPr(C)$ are exactly the presheaves of sets, i.e. the constant simplicial objects in $Pr(C)$. Let us see what would be the weak equivalences between two such objects in the localization of $Pr(C)$ by hypercovers: we recall that in the Bousfield localization $sPr(C)_S$ weak equivalences are exactly the S-local weak equivalences where S is the set of hypercovers. Those are exactly the cofibrations $G_1 \to G_2$ such that for all fibrant S-local objects F,

$$\mathrm{Hom}(G_2, F) \to \mathrm{Hom}(G_1, F)$$

is a trivial Kan fibration. From [36] we deduce that:

- S-local objects are just $F \in Pr(C)$ such that for all generalized covers $U \to X$, $F(X) \to F(U)$ is a bijection, i.e. sheaves on C;

- S-local equivalences are morphism $G_1 \to G_2$ such that for all sheaves F,

$$\mathrm{Hom}(G_2, F) \to \mathrm{Hom}(G_1, F) \tag{A.1}$$

  is an isomorphism.

This is exactly what we did in the introduction of section 1.3.2. In fact, two presheaves are locally weak equivalent (i.e. they have isomorphic sheafification) if and only if the map (A.1) is an isomorphism for all sheaves F.

Example A.3.7. 0-hypercovers are isomorphisms of simplicial manifolds. 1-hypercovers between the nerve of two Lie groupoids are just the nerve of Morita morphisms.

A.4 Grothendieck sites and sheaves

A sheaf (of sets) over a topological space X can be seen as a contravariant functor from the category $\mathcal{O}uv(X)$ of open subsets of X to $Sets$, which satysfies a descent condition, meaning that sections are local. The notion of Grothendieck topology allows us to define sheaves over more general category than $\mathcal{O}uv(X)$.

Definition A.4.1. Let C be a category and c an object in C, and $f : d \to c$ be a morphism in C.

- A **sieve** S_c for c is a subfunctor of $\mathrm{Hom}(-, c)$.
- The pullback of S_c is defined by the following fiber product

$$f^* S_c := S_c \times_{\mathrm{Hom}(-,c)} \mathrm{Hom}(-, d)$$

- A **Grothendieck topology** τ on C is the data, for each object $c \in C$, of a collection $\tau(c)$ of distinguished sieves on c which satisfy the following axioms:
  1. (Base change) The pullback of a covering sieve is a covering sieve;
  2. (Locality) Let S be a covering sieve for c, and T be a sieve on c. If for any object $d \in C$ and any arrow $f \in S(d)$, f^*T is a covering sieve for d, then T is a covering sieve.
  3. (Identity) $\mathrm{Hom}(-, X)$ is a covering sieve for any $X \in C$.

In order to understand this definition it suffices to look at the example of $\mathcal{O}uv(X)$. The objects are just open subsets of X, and $\mathrm{Hom}(U, V) = \{\emptyset\}$ (the singleton) if $U \subset V$, and $\emptyset$ otherwise.
Then a sieve on U is just a collection of open subsets of U. One can verify that if one choses open covers as covering sieves, this gives exactly the notion of topology on the topological space X.

Bibliography

[1] Abad C.A. , Crainic M. , Dherin B.: Tensor products of representations up to homotopy. Journal of Homotopy and Related Structures, vol. 6(2), 2011, pp.239288.

[2] André M.: Homologie des algèbres commutatives. Actes du Congrès International des Mathématiciens (Nice, 1970), Tome 1, pp. 301-308. Gauthier-Villars, Paris, 1971.

[3] Anel M. : The Geometry of Ambiguity, An introduction to the ideas of Derived Geometry, this book.

[4] Arias Abad C. , Crainic M.; Representations up to homotopy and Bott's spectral sequence for Lie groupoids; Adv. Math. 248 (2013), 416–452.

[5] Behrend K., Xu P.: Differentiable stacks and gerbes; J. Symplectic Geom., Volume 9, Number 3 (2011), 285-341.

[6] Behrend K.: On the De Rham cohomology of Differential and Algebraic Stacks; Advances in Mathematics Volume 198, Issue 2, 20 December 2005, Pages 583-622

[7] Behrend K.: cohomology of stacks (lectures).
https://www.math.ubc.ca/ behrend/CohSta-1.pdf

[8] Benzeghly B.: Étude explicite de quelques champs géométriques (PhD Thesis). https://tel.archives-ouvertes.fr/tel-00868795

[9] Blanco V. , Bullejos M. , Faro E.; A full and faithful nerve for 2-categories; Applied categorical structures 13 (3), 223-233.

[10] Block J. : Duality and Equivalence of Module Categories in Noncommutative Geometry Ii: Mukai Duality for Holomorphic Noncommutative Tori (2005) .

[11] Block J. , Daenzer C.: Mukai duality for gerbes with connection. Journal für die reine und angewandte Mathematik (Crelles Journal), Volume 2010, Issue 639, Pages 131171, ISSN (Online) 1435-5345, ISSN (Print)

[12] Blohmann, C. (2008). Stacky Lie groups. International Mathematics Research Notices, 2008.

[13] Borceux F. ; Handbook of Categorical Algebra 1: Basic Category Theory; Cambridge University Press, 2008.

[14] Borisov, D., Noel, J. (2011). Simplicial approach to derived differential manifolds. arXiv preprint arXiv:1112.0033.

[15] Bullejos M. , Cegarra A. ; On the geometry of 2-categories and their classifying spaces; K-Theory 29 (2003), 211229.

[16] Bursztyn H. , Cabrera A. , Del Hoyo M.; Vector bundles over Lie groupoids and Lie algebroids; Adv. Math. 290 (2016), 163207.

[17] Castiglioni, J. L., Cortinas, G. (2004). Cosimplicial versus DG-rings: a version of the DoldKan correspondence. Journal of Pure and Applied Algebra, 191(1-2), 119-142.

[18] Crainic, M. (2003). Differentiable and algebroid cohomology, van Est isomorphisms, and characteristic classes. Commentarii Mathematici Helvetici, 78(4), 681-721

[19] Crainic, M., Mestre, J. N., Struchiner, I. (2015). Deformations of Lie groupoids. arXiv preprint arXiv:1510.02530.

[20] del Hoyo, M., Stefani, D. (2017). The General Linear 2-Groupoid. arXiv preprint arXiv:1706.07152.

[21] del Hoyo M.; Lie groupoids and their orbispaces; Port. Math. 70 (2013), 161–209.

[22] del Hoyo, M., Ortiz, C. (2016). Morita equivalences of vector bundles. arXiv preprint arXiv:1612.09289.

[23] Dugger, D., Hollander, S., Isaksen, D. C. (2004, January). Hypercovers and simplicial presheaves. In Mathematical Proceedings of the Cambridge Philosophical Society (Vol. 136, No. 1, pp. 9-51). Cambridge University Press.

[24] J. Duskin; Simplicial matrices and the nerves of weak n-categories I; CT2000 Conference (Como) Theory Appl. Categ., 9 (2001/02), pp. 198308.

[25] Getzler, E. (2009). Lie theory for nilpotent-algebras. Annals of mathematics, 271-301.

[26] Ginot, G., Stiénon, M. (2008). G-gerbes, principal 2-group bundles and characteristic classes. arXiv preprint arXiv:0801.1238.

[27] Ginot, G. Introduction to differentiable Stacks (and gerbes, moduli spaces...). https://webusers.imj-prg.fr/ gregory.ginot/papers/DiffStacksIGG2013.pdf

[28] Goerss, P. G., Jardine, J. F. (2009). Simplicial homotopy theory. Springer Science and Business Media.

[29] A. Gracia-Saz, R. Mehta: VB-groupoids and representation theory of Lie groupoids; Preprint arxiv:1007.3658v4.

[30] Greub, W., Halperin, S., Vanstone, R. (1972). Connections, Curvature, and Cohomology V1: De Rham cohomology of manifolds and vector bundles. Academic press.

[31] Grothendieck A. : Revêtements étales et groupe fondamental (SGA 1); Lecture notes in mathematics 224 (1971), Springer-Verlag.

[32] Grothendieck, A. (1957). Sur quelques points d'algebre homologique. Tohoku Mathematical Journal, Second Series, 9(2), 119-183.

[33] Grothendieck A. : Catégories cofibrées additives et complexe cotangent relatif. Lecture Notes in Mathematics, No. 79 Springer-Verlag, Berlin-New York 1968 ii+167 pp.

[34] Hall, J., Neeman, A., Rydh, D. (2018). One positive and two negative results for derived categories of algebraic stacks. Journal of the Institute of Mathematics of Jussieu, 1-25.

[35] Henriques, A. (2008). Integrating-algebras. Compositio Mathematica, 144(4), 1017-1045.

[36] Hollander, S. (2008). A homotopy theory for stacks. Israel Journal of Mathematics, 163(1), 93-124.

[37] Hovey, M. (2007). Model categories (No. 63). American Mathematical Soc.

[38] Illusie L. : Complexe cotangent et déformations. I. Lecture Notes in Mathematics, Vol. 239. Springer-Verlag, Berlin-New York, 1971. xv+355 pp.

[39] Johnstone, P. T. (2002). Sketches of an elephant: A topos theory compendium (Vol. 2). Oxford University Press.

[40] Lack S.: A 2-categories companion; Towards higher categories. Springer New York (2010) 105-191.

[41] Lefèvre-Hasewaga K.: Sur les A_∞-catégories (PhD thesis). https://arxiv.org/abs/math/0310337

[42] Lurie, J. (2009). Higher Topos Theory (AM-170). Princeton University Press.

[43] Mac Lane S.: Categories for the working mathematician (2nd ed.); Graduate Text in Math. 5. Springer (1998).

[44] Marle, C. M. (2008). Calculus on Lie algebroids, Lie groupoids and Poisson manifolds. arXiv preprint arXiv:0806.0919.

[45] Mehta R.: Representing representations up to homotopy; Slides from a talk at Berkeley (2014); https://math.berkeley.edu/ libland/gone-fishing-2014/mehta.pdf

[46] Mehta R. , Tang X.: From double Lie groupoids to local Lie groupoids; Bulletin of the Brazilian Mathematical Society 42 (2011), 651-681.

[47] Nestruev J.: Smooth manifolds and observables. Graduate texts in mathematics, 220, Springer-Verlag, ISBN 0-387-95543-7

[48] Pantev T., Ton, B., Vaquié, M., Vezzosi, G. (2013). Shifted symplectic structures. Publications mathématiques de l'IHÉS, 117(1), 271-328. ISO 690

[49] Quillen D.: On the (co-) homology of commutative rings. 1970 Applications of Categorical Algebra (Proc. Sympos. Pure Math., Vol. XVII, New York, 1968) pp. 65-87. Amer. Math. Soc., Providence, R.I.

[50] Sheng Y., Zhu C.: Integration of semidirect product Lie 2-algebras; Int. J. Geom. Methods Mod. Phys. Vol. 9, No. 5 (2012) 1250043.

[51] Sheng, Y., Zhu, C.: Higher extensions of Lie algebroids. Communications in Contemporary Mathematics, 19(03), 1650034.

[52] Hirschowitz, A., Simpson, C. Descente pour les n-champs, preprint available at. arXiv preprint math.AG/9807049.

[53] Spivak, D. I. (2010). Derived smooth manifolds. Duke Mathematical Journal, 153(1), 55-128.

[54] Stefani D. , Mémoire de M2.

[55] Swan, R. G. (1962). Vector bundles and projective modules. Transactions of the American Mathematical Society, 105(2), 264-277.

[56] Toën, B., Vaquié, M. (2007, May). Moduli of objects in dg-categories. In Annales scientifiques de lEcole normale supérieure (Vol. 40, No. 3, pp. 387-444). No longer published by Elsevier.

[57] Toën, B., Vezzosi, G. (2008). Homotopical Algebraic Geometry II: Geometric Stacks and Applications: Geometric Stacks and Applications (Vol. 2). American Mathematical Soc..

[58] Toën, B. (2014). Derived algebraic geometry. arXiv preprint arXiv:1401.1044.

[59] Toën, B., Vezzosi, G. (2005). Homotopical algebraic geometry I: Topos theory. Advances in mathematics, 193(2), 257-372.

[60] Trentinaglia G. : Tannaka duality for proper Lie groupoids; J. Pure Appl. Algebra 214 (2010) no. 6 pp. 750768.

[61] Vistoli, A. (2005). Grothendieck topologies, fibered categories and descent theory. Mathematical Surveys and Monographs, 123, 1.

[62] Wolfson J.: descent for n-bundles. Advances in Mathematics, vol. 288 (2016), pp. 527-575.

[63] Xu, P. (2004). Momentum maps and Morita equivalence. Journal of Differential Geometry, 67(2), 289-333.

[64] Zhu C.: n-Groupoids and stacky groupoids; International Mathematics Research Notices 21 (2009) 4087-4141. arXiv:math/0609420 [math.DG].