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Felipe Garrido Lucero

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DE L'UNIVERSITÉ PSL
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Stable Matching Games

Soutenue par

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Al café, la música, el té con miel, y andar en bicicleta

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Jeux d'appariement stables

Le problème de l'appariement stable

Le problème d'appariement de marché un-à-un à deux côtés de Gale et Shapley [53], connu sous le nom de *problème de mariage*, consiste à trouver un appariement *stable* entre deux ensembles finis différents D et H , étant donné que chaque agent de chaque côté a un *ordre exogène (total) strict de préférence* sur les agents de l'autre côté. Formellement, chaque agent $k \in D \cup H$ est doté de $>_k$ tel que, pour tout deux agents ℓ et ℓ' du côté opposé,

$$\ell >_k \ell'$$

représente que k préfère être couplé avec ℓ plutôt qu'avec ℓ' .

Le problème du mariage se concentre sur le calcul d'un appariement μ qui associe chaque agent d'un côté à au plus un agent de l'autre côté. Le couplage μ est *stable* si aucune paire d'agents non couplés ne préfère être associée à l'autre plutôt qu'à son partenaire dans μ . Formellement, un appariement μ est *bloqué* s'il existe une paire d'agents $(d, h) \in D \times H$ telle que,

$$h >_d \mu(d) \text{ et } d >_h \mu(h)$$

où $\mu(k)$ représente le partenaire de k dans l'appariement μ . L'appariement est stable si aucune paire ne le bloque. Gale et Shapley ont utilisé un algorithme *acceptation-différée* pour prouver l'existence d'un appariement stable pour chaque instance. Leur algorithme prend l'un des côtés du marché, appelé le côté proposant, et demande à ses agents de proposer à leur option la plus préférée qui ne les a pas encore rejetés. Les agents qui reçoivent plus d'une proposition acceptent la meilleure et rejettent toutes les autres. L'algorithme se poursuit jusqu'à ce que tous les agents du côté des proposant aient été acceptés par quelqu'un. Bien que le modèle de Gale et Shapley considère deux ensembles de même taille et des préférences strictes, leur algorithme est facilement étendu à des ensembles de tailles différentes où les agents ont la possibilité de rester sans partenaire (également appelés *ordre des préférences incomplet*) et des préférences non strictes. Le calcul de l'appariement stable est exact et prend au plus $O(N^2)$ itérations avec N étant la taille de l'ensemble le plus grand.

De nombreuses propriétés de l'ensemble des appariements stables et de l'algorithme d'acceptation-différée de Gale et Shapley ont été prouvées. John Conway a prouvé une structure de lattice sur l'ensemble des appariements stables Γ (revendiquée par Gale et Shapley [53] et rapportée par Knuth [75]), c'est-à-dire que le maximum et le minimum

de deux appariements stables pour les préférences de l'un des côtés restent stables. Formellement, si μ et μ' sont deux appariements stables dans Γ , nous définissons,

$$\begin{aligned}\mu^\vee(d) &:= \max_{>d} \{\mu(d), \mu'(d)\}, \forall d \in D \\ \mu^\wedge(d) &:= \min_{>d} \{\mu(d), \mu'(d)\}, \forall d \in D\end{aligned}$$

Les appariements μ^\vee et μ^\wedge sont bien définis et appartiennent à Γ . L'appariement μ^\vee correspond au maximum entre μ et μ' pour les préférences des agents dans D . De manière analogue, μ^\wedge correspond au minimum. Une *propriété de dualité* tient puisque le maximum de deux appariements stables pour l'un des côtés correspond au minimum entre les deux mêmes appariements stables pour l'autre côté. En d'autres termes, si

$$\begin{aligned}\nu^\vee(h) &:= \max_{>h} \{\mu(h), \mu'(h)\}, \forall h \in H \\ \nu^\wedge(h) &:= \min_{>h} \{\mu(h), \mu'(h)\}, \forall h \in H\end{aligned}$$

sont l'appariement maximum et minimum pour les agents dans H , il tient,

$$\mu^\vee = \nu^\wedge \text{ et } \mu^\wedge = \nu^\vee$$

L'élément le plus élevé du lattice pour un côté, c'est-à-dire l'appariement stable le plus préféré pour lui, est l'appariement stable le moins préféré pour le côté opposé.

Dubins et Freedman [43] mais aussi Gale et Sotomayor [54] ont étudié les incitations des joueurs à mentir lorsqu'ils déclarent leurs préférences et ont prouvé que l'algorithme d'acceptation-différée de Gale et Shapley est *résistant aux comportements stratégiques* pour le côté proposant, car l'algorithme produit le meilleur appariement stable pour eux, c'est-à-dire l'élément le plus élevé du lattice.

Gusfield et Irving [60] étudient le mariage stable et ses variantes comme une riche source de problèmes et d'idées qui illustrent, à la fois, la conception et l'analyse d'algorithmes efficaces. Roth et Sotomayor [89] ont étudié la structure de l'ensemble des mariages stables et son Core. Vande Vate [100] a caractérisé l'ensemble des appariements stables dans le problème un-à-un comme des points extrêmes d'un polytope lorsque $|D| = |H|$. Rothblum [91] a étendu ce résultat au cas $|D| \neq |H|$.

Balinski et Ratier [24, 25] ont proposé une approche élégante du problème par des graphes dirigés et, comme Rothblum, ils ont caractérisé le polytope d'appariement stable dans le problème un-à-un par des inégalités linéaires, prouvant que tout point réalisable du polytope est un appariement stable et vice-versa.

Roth et Vande Vate [90] ont étudié un processus aléatoire pour trouver un appariement stable à partir d'un appariement arbitraire. Ma [79] a prouvé que l'algorithme de Roth et Vande Vate ne trouve pas tous les appariements stables. Récemment, Dworzak [45] a introduit une nouvelle classe d'algorithmes appelés *algorithmes d'acceptation-différée avec chaînes de compensation (DACC)*¹ dans laquelle les deux

¹Une classe similaire d'algorithmes a été introduite par McVitie et Wilson [80]

parties du marché peuvent faire des offres et a prouvé qu'un appariement est stable si et seulement s'il est le résultat d'un algorithme DACC. Plus précisément, les algorithmes DACC choisissent un ordre de proposition aléatoire σ sur tous les agents $D \cup H$ qui est modifié chaque fois qu'un agent est remplacé en lui permettant de proposer le suivant. En indexant les algorithmes DACC sur σ , Dworczak a prouvé qu'un appariement est stable si et seulement s'il est le résultat d'un algorithme DACC pour un certain ordre σ .

Une approche en point fixe a été faite par Fleiner [52] en formulant une généralisation matroïdal du problème du mariage stable et en étudiant la structure de lattice du modèle généralisé.

L'une des premières extensions du problème du mariage au cadre des *préférences endogènes* est le *jeu d'assignation* de Shapley et Shubik [93] dans lequel les agents d'un même couple peuvent effectuer des transferts monétaires. L'exemple principal est un *marché du logement* où les acheteurs et les vendeurs ont des utilités quasi-linéaires. Les acheteurs $d \in D$ ont des évaluations différentes $((v_{d,h})_{d \in D, h \in H})$ pour les maisons des vendeurs $h \in H$, et les vendeurs ont des coûts différents $((c_h)_{h \in H})$ sur leurs propres maisons. Chaque vendeur $h \in H$ est disposé à vendre sa maison à un certain prix $p_h \geq c_h$, et chaque acheteur $d \in D$ est intéressé à acheter la maison de h à un certain prix $p_{d,h} \leq v_{d,h}$. Une allocation est une paire (μ, p) avec μ étant un appariement entre vendeurs et acheteurs, et $p = (p_d)_{d \in D}$ étant un vecteur de transferts monétaires, où p_d est le prix offert par $d \in D$ à $\mu(d) \in H$. Les allocations dans le modèle de Shapley-Shubik sont stables s'il n'existe aucune paire non appariée (d, h) et aucun prix de transaction q de d à h tel que les deux agents se retrouvent strictement mieux lotis en échangeant. En exploitant la linéarité des fonctions de gain sur les transferts monétaires, Shapley et Shubik ont trouvé des solutions stables $\pi = (\mu, p)$ pour leur problème en utilisant la programmation linéaire où une paire primal-dual donne, respectivement, l'appariement μ et le vecteur de prix p . Remarquons la complexité polynomiale de la résolution du jeu d'affectation grâce à l'approche de programmation linéaire.

Le jeu d'affectation appartient à la classe des *jeux coopératifs à utilité transférable* car les agents d'un même couple (d, h) doivent partager leur *valeur* $v_{d,h} + c_h$ de telle sorte que personne ne préfère changer de partenaire. De plus, Shapley et Shubik ont prouvé que l'ensemble des allocations stables pour leur jeu d'affectation est exactement le *Core* du problème du marché du logement vu comme un jeu coopératif à utilité transférable.

Demange et Gale [39] ont considéré des fonctions d'utilité plus générales sur les transferts monétaires (non-quasi-linéaires), ont autorisé les transferts monétaires des deux côtés (de l'acheteur au vendeur et vice-versa), et ont prouvé que l'ensemble des allocations stables a une structure en lattice (la non-vacuité de cet ensemble a été prouvée dans [37, 87]). Demange et al. [40] ont conçu deux *mécanismes de prix ascendants* pour calculer les allocations stables du modèle d'*appariement avec des transferts* dans [39]. Pour des utilités entières, le premier algorithme converge en un nombre borné d'itérations vers une solution exacte. Pour des paiements conti-

nus, le second algorithme converge vers une allocation ε -stable en un nombre borné d'itérations. Malgré l'abondante littérature réalisée au cours des années, le problème du calcul d'allocations stables exactes pour les marchés d'appariement avec des utilités continues est toujours ouvert.

Les modèles avec transferts monétaires comme ceux présentés ci-dessus appartiennent à la classe des marchés d'appariement avec utilité transférable. Une comparaison claire entre les modèles à utilité transférable et non transférable a été faite par Echeñique et Galichon dans [47]. Galichon et al. [55] ont étudié un modèle d'appariement stable avec *utilité transférable imparfaite*, dû par exemple à la présence de taxes dans les transferts, et ont prouvé algorithmiquement l'existence de solutions stables. Dupuy et al. [44] ont étudié plus loin les effets de la taxation dans les marchés d'appariement à deux côtés, fournissant un lien continu entre les modèles d'appariement avec et sans transferts.

D'autres extensions du modèle de mariage stable au cadre des préférences endogènes considèrent *préférences aléatoires* lorsque les agents les tirent de distributions de probabilité. Gimbert et al. [57] ont étudié le nombre de *paires stables*, c'est-à-dire les paires (d, h) qui apparaissent dans un certain appariement stable, trouvant de nombreux résultats asymptotiques dépendant des distributions considérées. Les mêmes auteurs dans [59] ont étendu leur étude aux effets d'avoir des préférences aléatoires et aux manipulations possibles des agents sur la sortie des algorithmes d'appariements stables.

Marchés d'appariement un-à-plusieurs à deux côtés sont la généralisation des modèles expliqués ci-dessus au cas où les agents d'un des côtés peuvent être appariés avec plusieurs partenaires en même temps. De nombreuses applications intéressantes découlent de ces modèles. Gimbert et al. [58] ont étudié un problème de choix d'école avec information imparfaite dans lequel les étudiants ne révèlent qu'une version partielle de leurs préférences en raison du nombre limité de candidatures autorisées. Correa et al. [36] ont étudié un mécanisme centralisé pour répartir équitablement les élèves dans les écoles au Chili en donnant la priorité à l'attribution de frères et sœurs jointes. En France, des études approfondies ont été menées pour développer le mécanisme d'allocation des étudiants aux universités *Parcoursup*².

Les premiers à avoir introduit ce problème ont été, comme pour les marchés d'appariement un-à-un à deux côtés, Gale et Shapley dans leur article fondateur [53]. Ils ont prouvé que leur même algorithme d'acceptation-différée pouvait être appliqué au cas d'un-à-plusieurs. Baïou et Balinski [21, 23] ont généralisé l'approche de la théorie des graphes dans [24, 25] aux marchés d'appariement un-à-plusieurs et la caractérisation du polytope des appariements stables dans ce nouveau cadre. Comme Shapley et Shubik dans le cas un-à-un, Crawford et Knoer [37] ont étendu le modèle des marchés d'appariement un-à-plusieurs au cadre des transferts monétaires linéaires. Kelso et Crawford [71] sont allés plus loin dans cette extension en considérant tout type d'utilité transférable. Leur *modèle d'appariement des emplois* considère des tra-

²Un document français sur Parcoursup peut être trouvé [ici](#).

vailleurs et des entreprises qui s'apparient et déterminent simultanément les salaires à verser aux travailleurs. Les auteurs ont prouvé l'existence d'allocations stables pour toute situation dans laquelle les travailleurs sont des *substituts bruts* pour les entreprises: l'augmentation du salaire d'un ensemble de travailleurs ne peut jamais amener une entreprise à retirer une offre d'un travailleur dont le salaire n'a pas été augmenté.

Hatfield et Milgrom ont rédigé un article fondamental sur les marchés d'appariement un-à-plusieurs [66], le modèle d'*appariement avec des contrats*, qui étend le modèle de Kelso et Crawford en permettant aux médecins et aux hôpitaux (au lieu des travailleurs et des entreprises) de signer des *contrats* à partir d'un ensemble fini de contrats possibles X sur le marché. Les contrats sont *bilatéraux* de sorte que chaque $x \in X$ relie un médecin $x_D \in D$ avec un hôpital $x_H \in H$. Les agents sont dotés d'ordres de préférence qui définissent des *fonctions de choix* ($C_d, C_h : d \in D, h \in H$). Étant donné un ensemble de contrats possibles $X' \subseteq X$, les fonctions de choix produisent le contrat le plus préféré pour chaque médecin, et le sous-ensemble de contacts le plus préféré pour chaque hôpital. Formellement,

$$C_d(X') = \arg \max_{>_d} \{x \in X' : x_D = d\}$$

$$C_h(X') = \arg \max_{>_h} \{X'' \subseteq X' : \forall x \in X'', x_H = h, \text{ et } \forall x, x' \in X'', x_D \neq x'_D\}$$

Hatfield et Milgrom ont prouvé que l'ensemble des allocations stables est un lattice non vide et qu'un *mécanisme d'offre cumulative* atteint les extrêmes du lattice grâce au théorème du point fixe de Tarski. L'hypothèse principale qui sous-tend ce résultat est la *substituabilité* des hôpitaux.

Définition. Les éléments de X sont **substituables** pour l'hôpital h si pour tous les sous-ensembles $X' \subseteq X'' \subseteq X$, il s'avère que $X' \setminus C_h(X') \subseteq X'' \setminus C_h(X'')$

En d'autres termes, les contrats sont des substituts de h si aucun contrat rejeté ne peut être choisi en raison de l'élargissement de l'ensemble des contrats. Il a été prouvé que la substituabilité est suffisante mais pas nécessaire pour l'existence d'allocations stables dans le modèle d'appariement avec contrats et de nombreux auteurs ont travaillé pour trouver des hypothèses plus faibles [16, 63, 64, 65]. Hatfield et Milgrom ont prouvé un deuxième résultat dans leur modèle, le *théorème de l'hôpital rural*: chaque hôpital embauche le même nombre de médecins à chaque appariement stable. Les hypothèses clés de ce résultat sont la substituabilité et la *loi de la demande agrégée*.

Définition. Les préférences de l'hôpital h satisfont la **loi de la demande agrégée** si pour tout $X' \subseteq X'' \subseteq X$, $|C_h(X')| \leq |C_h(X'')|$.

En d'autres termes, l'hôpital h satisfait à la loi de la demande agrégée si, chaque fois que l'ensemble des contrats possibles s'étend, le nombre total de contrats choisis par l'hôpital h augmente faiblement.

Aygun et Sönmez [16] ont exposé que des modèles différents sont obtenus si les fonctions de choix des agents sont traitées comme des primitives ou si elles sont

induites à partir des classements de préférences dans le modèle d'appariement avec des contrats. Le modèle de Hatfield et Milgrom appartient au deuxième type, mais ils ont traité leurs fonctions de choix comme des primitives. Pour garantir véritablement l'existence d'allocations stables, une hypothèse supplémentaire, à savoir l'*insignifiance des contrats rejetés*, est également requise.

Définition. La fonction de choix de l'hôpital h satisfait à l'**insignifiance des contrats rejetés** si pour tout $X' \subseteq X$, et tout $x \in X'$, si $x \notin C_h(X' \cup \{x\})$, alors $C_h(X') = C_h(X' \cup \{x\})$.

En d'autres termes, la fonction de choix de h n'est pas pertinente pour les contrats rejetés si les contrats choisis ne changent pas en raison de la suppression des contrats rejetés de l'ensemble des contrats. La bonne nouvelle pour le travail de Hatfield et Milgrom est que l'insignifiance des contrats rejetés est satisfaite en présence de la substituabilité et de la loi de la demande agrégée.

Des extensions plus complexes, telles que le modèle de marché d'appariement plusieurs-à-plusieurs, ont également été étudiées. Un article pionnier dans l'extension du modèle de Gale-Shapley au cadre plusieurs-à-plusieurs a été réalisé par Blair [29], étudiant le cas où les travailleurs et les entreprises doivent former des partenariats (les travailleurs peuvent travailler dans plusieurs entreprises en même temps), obtenant une structure en lattice de l'ensemble des allocations stables avec salaires, étendant [71] et [75]. Baïou et Balinski [20] ont étendu des résultats connus dans les domaines un-à-un et un-à-plusieurs au domaine plusieurs-à-plusieurs. Dans [22], les mêmes auteurs sont allés plus loin dans l'extension en considérant un problème d'allocation entre fournisseurs et acquéreurs dans lequel de nombreuses copies du même bien peuvent être allouées des fournisseurs aux acquéreurs.

Un troisième problème a été introduit dans l'article fondateur de Gale et Shapley [53]: *le problème de colocataires*. Le problème des colocataires consiste en un ensemble fini pair d'agents D , chacun d'entre eux ayant un *ordre exogène strict de préférence* $>_d$ sur le reste des agents, et cherchant à s'apparier par paires. La partition de l'ensemble des agents en paires doit être *stable*, c'est-à-dire qu'aucune paire d'agents ne peut à la fois préférer abandonner ses partenaires et s'apparier ensemble. Formellement, un appariement μ (une partition de D en paires) est stable s'il n'existe pas de $(d_1, d_2) \in D \times D$ tel que $d_2 >_{d_1} \mu(d_1)$ et $d_1 >_{d_2} \mu(d_2)$.

Contrairement au problème du *mariage stable*, le problème du *colocataires* n'a pas toujours une solution stable, comme l'ont déjà observé Gale et Shapley dans leur article: Considérons quatre agents $D = \{1, 2, 3, 4\}$, chacun d'entre eux ayant les préférences strictes suivantes sur son partenaire possible,

	Préférences
1	2 > 3 > 4
2	3 > 1 > 4
3	1 > 2 > 4
4	Arbitraire

Il n'est pas possible de faire une partition stable des quatre joueurs en deux paires car l'agent apparié avec 4 trouvera toujours quelqu'un d'autre à appairer, qui préférera également changer.

Knuth a posé la question de la conception d'un algorithme permettant de calculer une répartition stable chaque fois qu'elle existe. Irving [68] a répondu par un algorithme polynomial permettant de calculer une solution si elle existe ou de signaler la non-existence d'allocations stables pour les instances sans solution. Son algorithme comporte deux phases, une première dans laquelle les agents proposent et disposent, comme dans l'algorithme d'acceptation-différée de Gale et Shapley, et une seconde dans laquelle les couples potentiels sont bannis du système jusqu'à ce qu'une solution soit trouvée ou que l'infaisabilité de l'appariement stable soit découverte (pouvant également être découverte pendant la première phase).

Tan [97] a étudié les conditions nécessaires et suffisantes pour l'existence d'appariements stables dans le problème des colocataires car il a fait valoir que l'algorithme d'Irving, bien qu'il signale la non-existence d'une solution stable, n'en illustre pas clairement la raison. Il a prouvé qu'une instance du problème des colocataires possède un appariement stable si et seulement si elle ne permet pas l'existence de *partitions stables* avec des *parties de taille impair*. Une partition stable est une partition de l'ensemble D en coalitions de taille quelconque telle qu'aucune paire d'agents de coalitions différentes ne préfère être ensemble. Les partitions stables peuvent être vues comme des partitions de D en cycles (parties) où chaque agent indique son partenaire le plus désiré.

De nombreuses extensions du problème des colocataires au cadre *utilité transférable* ont été réalisées. Comme Balinski et Ratier, Eriksson et Karlander [49] ont exploré une méthode de théorie des graphes pour étudier l'existence d'allocations stables du problème des colocataires avec utilité transférable. Comme Shapley et Shubik, Talman et Yang [96] ainsi que Chiappori et al. [34], ont utilisé l'approche de programmation linéaire pour résoudre leurs problèmes. Comme Demange et al. dans leur premier algorithme, Andersson et al. [10] ont conçu un *processus d'ajustement des prix* qui calcule, sous paiements intégraux, une allocation stable ou réfute son existence en temps fini.

Shioura [94] a fait le lien entre les colocataires à utilité transférable et le jeu d'affectation de Shapley et Shubik [93]. Plus précisément, Shioura a réduit son problème à un jeu d'affectation particulier dans un graphe biparti auxiliaire et a proposé une extension de l'algorithme d'Andersson et al. [10] pour calculer une allocation stable en cas d'existence.

A notre connaissance, la seule extension du problème de colocataires au domaine *utilité non transférable* a été réalisée par Alkan et Tuncay [9]. Dans leur modèle, chaque couple d'agents (d_1, d_2) possède une fonction de *partenariat* décroissante et continue $u_{1,2}$ qui, étant donné le gain de l'un des agents $f_1 \in \mathbb{R}$, produit le gain du partenaire $f_2 = u_{1,2}(f_1)$. Les auteurs ont prouvé que l'ensemble des allocations stables est soit vide, soit équivalent à l'ensemble des *aspirations équilibrées réalisables*. Les aspirations équilibrées sont des profils de gain $f \in \mathbb{R}^{|D|}$ pour lesquels aucun agent du

marché n'est *surdemandé*. Formellement, nous définissons l'*ensemble de demande* de l'agent d à f comme,

$$P_d(f) := \{d' \in D : f_d = u_{d,d'}(f_{d'})\}$$

Alors, f est une aspiration équilibrée si aucun agent n'appartient à plus d'un ensemble de demande. L'aspiration équilibrée existe toujours. Alkan et Tuncay ont conçu une *procédure de marché* (un mécanisme analogue à celui d'Andersson et al. pour le cadre non-transférable continu) qui, en temps polynomial, produit une aspiration équilibrée ε^3 . Chaque fois qu'une aspiration équilibrée f est réalisable par une allocation, c'est-à-dire qu'il existe un appariement μ tel que $f_d = u_{d,\mu(d)}(f_{\mu(d)})$ si d est apparié, et égal à \underline{f}_d sinon, où \underline{f}_d est le gain rationnel individuel de l'agent d (son utilité d'être célibataire), la paire (allocation, aspiration) est stable. Cependant, ce n'est pas toujours le cas et les aspirations équilibrées peuvent devoir être mises en œuvre par des semi-allocations, c'est-à-dire des appariements relaxés dans lesquels les agents peuvent être appariés avec deux partenaires et leur gain final est la moyenne de ceux reçus de chaque partenaire (probablement lié au fait d'avoir des partitions stables avec des parties impaires de taille au moins égale à trois).

Les problèmes de colocataires avec préférences ordinales, utilité transférable et utilité non-transférable peuvent être vues comme des cas particuliers de *jeux hédoniques* [32, 42] dans lesquels les agents d'un ensemble D cherchent à créer des coalitions de taille quelconque et les joueurs reçoivent une utilité dépendant des agents de leur même coalition. Les préférences peuvent être considérées de plusieurs façons. Les agents peuvent avoir *des préférences booléennes* [19], soit ils aiment ou n'aiment pas la coalition; *préférences orientées vers les amis* [72], chaque agent divise l'ensemble des autres agents en amis, ennemis et neutres, et obtient un gain de 1, -1 et 0 respectivement, pour chacun de ces agents dans sa coalition; ou *préférences orientées vers l'ennemi* où les agents classés comme ennemis donnent une utilité de $-n$ lorsqu'ils sont présents dans la coalition; entre autres. Les utilités peuvent également être de plusieurs types. Les agents peuvent avoir des utilités *additives séparable* [18], *fractionnel additive séparable* [17], ou *fonctions de paiement basées sur le rôle* [95], entre autres.

D'autres extensions dans la littérature de l'appariement stable concernent la *temporalité* du problème. Des applications pratiques comme *l'allocation de reins* [27, 103] ou le *conavettage* [8, 26, 101] ont montré que les agents (ou les articles) au sein du marché peuvent ne pas être présents en même temps car ils peuvent arriver à des moments différents. De nombreuses années ont passé jusqu'à ce que le problème du mariage stable de Gale et Shapley obtienne enfin ses premières extensions dynamiques. Damiano et Lam [38] ont défini un modèle fini à temps répété dans lequel les agents peuvent changer de partenaires entre les périodes. Tous les agents sont présents sur le marché dès le début et les utilités dépendent de celles obtenues à chaque période.

³Rappelez-vous que pour les problèmes d'appariement avec des utilités continues, trouver des solutions exactes est toujours un problème ouvert.

Caldentey et al. [31], Adan et Weiss [6], Bušić et al. [30], et Adan et al. [5], entre autres, ont travaillé dans un *modèle d'appariement stochastique* dans lequel les agents (ou articles) arrivent sur le marché dans un processus en temps discret suivant des distributions de probabilité. Les articles de chaque côté ont des types différents et un graphe biparti fixe donne les liens de compatibilité entre les articles des différents côtés.

Un problème particulier se pose dans les modèles de marché d'appariement dynamique. Dans de nombreux cas, les agents ne peuvent pas décider quand ou avec qui ils sont appariés car ils doivent suivre des politiques d'appariement prédéfinies. Par exemple, dans l'attribution de reins, les patients sont classés selon un système de points et, dès qu'un organe arrive, il est attribué au patient le mieux classé, sous réserve de compatibilité. Dans le cas du conavettage, dès qu'un usager arrive au marché, le conducteur le plus proche lui est attribué afin de minimiser le temps d'attente du client. L'une des politiques d'appariement les plus étudiées est le *premier arrivé, premier servi* (FCFS) [5, 6, 30, 31, 35, 82] dans lequel les agents qui arrivent sont appariés à l'agent compatible le plus ancien de l'autre côté. Les politiques d'appariement qui appariant les agents dès leur arrivée sont appelées *politiques gourmandes* et les modèles d'appariement stochastiques avec des politiques gourmandes peuvent être associés à *Markov processes*. Adan et Weiss [6] et Adan et al. [5] ont défini plusieurs chaînes de Markov à temps discret, trouvé des conditions d'ergodicité et dérivé leurs distributions stationnaires.

Jeux d'appariement

Dans les marchés à deux côtés réels, pour être attractif, un agent peut prendre des mesures qui ne peuvent être modélisées par des transferts monétaires. Lorsqu'une entreprise embauche un travailleur, elle peut combiner le transfert monétaire avec des avantages sociaux: assurance médicale, salle de sport, congés supplémentaires, horaires flexibles, aide à la garde d'enfants et jours de travail à distance. Le travailleur peut promettre d'être flexible, de travailler dur, d'apprendre de nouvelles technologies et de respecter le code de conduite de l'entreprise. Lorsqu'une université engage un professeur, il peut réduire ou augmenter ses tâches d'enseignement, exiger un nombre minimum de publications de premier plan, demander certaines responsabilités dans le département, etc. Le professeur peut promettre de publier dans les meilleures revues, d'être un excellent enseignant, de demander des subventions, d'accepter certaines responsabilités, d'organiser un séminaire et de superviser des doctorants. Toutes ces actions sont des décisions individuelles qui peuvent être inscrites explicitement ou implicitement dans un contrat, mais chaque agent est responsable de sa part. Chaque agent fera ce qui est nécessaire pour être accepté par l'autre partie et refusera de s'engager s'il juge la proposition de son partenaire insuffisante.

Dans cette thèse, nous présentons un nouveau modèle de marché d'appariement en supposant que les membres individuels d'un couple ou d'une coalition obtiennent

leurs gains à l'issue d'un jeu stratégique, joué simultanément avec l'appariement. Nous étudions trois modèles, généralisation des trois problèmes présentés par Gale et Shapley dans leur article fondateur [53]:

1. Jeux d'appariement un-à-un,
2. Jeux d'appariement un-à-plusieurs, et
3. Jeux d'appariement entre colocataires.

Pour chacun d'entre eux, nous présenterons le modèle, les défis et les résultats. Ensuite, nous étudierons un quatrième modèle, un marché d'appariement dynamique en temps continu dans lequel les agents arrivent et partent en suivant des distributions de probabilité.

1. *Jeux d'appariement un-à-un* considère deux ensembles finis d'agents D et H , respectivement médecins et hôpitaux, qui cherchent à s'apparier par couples. Étant donné un couple $(d, h) \in D \times H$, nous les dotons d'un jeu stratégique à deux joueurs

$$G_{d,h} = (X_d, Y_h, f_{d,h}, g_{d,h})$$

où X_d est l'ensemble action/stratégie de d , Y_h est l'ensemble action/stratégie de h , et $f_{d,h}, g_{d,h} : X_d \times Y_h \rightarrow \mathbb{R}$ sont les fonctions d'utilité de d et h , respectivement. Remarquons la dépendance du jeu à l'identité du couple: s'apparier avec un partenaire différent peut impliquer de jouer un jeu différent. Les agents peuvent préférer rester célibataires car ils seront dotés de *bénéfices rationnels individuels* $\underline{f} \in \mathbb{R}^{|D|}$ et $\underline{g} \in \mathbb{R}^{|H|}$.

Définition. Un **jeu d'appariement** est tout tuple Γ donné par,

$$\Gamma = (D, H, (G_{d,h} : d \in D, h \in H), \underline{f}, \underline{g})$$

Si un couple (d, h) est apparié, le joueur d choisit $x_d \in X_d$ et le joueur h choisit $y_h \in Y_h$, les utilités finales des agents dans l'allocation sont $f_{d,h}(x_d, y_h)$ et $g_{d,h}(x_d, y_h)$, respectivement. Une *allocation* du jeu d'appariement un-à-un est un triplet $\pi = (\mu, \vec{x}, \vec{y})$, avec μ un *appariement* (un-à-un) entre D et H (laissant éventuellement certains agents non appariés), $\vec{x} = (x_d)_{d \in D} \in \prod_{d \in D} X_d$ un *profil de stratégie* pour tous les agents dans D , et $\vec{y} = (y_h)_{h \in H} \in \prod_{h \in H} Y_h$ un *profil de stratégie* pour tous les agents dans H . Étant donné une allocation π , nous définissons le *utilité des joueurs* par,

$$f_d(\pi) := \begin{cases} f_{d,\mu(d)}(x_d, y_{\mu(d)}) & \text{si } d \text{ est apparié} \\ \underline{f}_d & \text{sinon} \end{cases}$$

$$Ng_h(\pi) := \begin{cases} g_{\mu(h),h}(x_{\mu(h)}, y_h) & \text{si } h \text{ est apparié} \\ \underline{g}_h & \text{sinon} \end{cases}$$

La première propriété que nous allons demander pour nos allocations est *la rationalité individuelle*.

Définition. Une allocation π est **individuellement rationnelle** si aucun agent ne reçoit moins que son gain individuellement rationnel.

Exemple 1. Considérons un jeu d'appariement avec un seul agent d et un agent h , tous deux ayant des IRP strictement positifs $\underline{f}_d = \underline{g}_h = \delta > 0$. Supposons que, s'ils acceptent de s'apparier, ils jouent au dilemme des prisonniers G suivant,

		Agent h	
		Coopérer	Trahir
Agent d	Coopérer	$2\delta, 2\delta$	$-\delta, 3\delta$
	Trahir	$3\delta, -\delta$	$0, 0$

Remarquez que l'appariement et le jeu de l'équilibre de Nash de G sont dominés par Pareto par le fait de rester célibataire. En d'autres termes, l'allocation dans laquelle les agents correspondent et jouent l'équilibre de Nash n'est pas individuellement rationnelle. Intuitivement, pour avoir une allocation stable dans laquelle les agents s'accordent, les joueurs doivent coopérer avec une certaine probabilité positive. \square

Exemple 2. Un problème d'appariement avec transferts linéaires, comme le jeu d'affectation de Shapley et Shubik, peut être représenté par une famille de jeux à somme constante où l'ensemble des stratégies sont $X_d = Y_h = \mathbb{R}^+$, et les fonctions de gain sont $f_{d,h}(x_d, y_h) = -x_d + y_h + a_{d,h}$ et $g_{d,h}(x_d, y_h) = x_d - y_h + b_{d,h}$, avec $a_{d,h}$ et $b_{d,h}$ représentant l'utilité d'être avec le partenaire lorsqu'il n'y a pas de transfert⁴. \square

Supposons qu'un certain processus (centralisé ou décentralisé) mène à une allocation où les agents sont appariés par paires et où chaque joueur apparié est destiné à jouer une certaine action. Nous voulons formuler les conditions nécessaires pour que cette allocation soit *durable*. Deux concepts de solutions peuvent être définis en fonction du niveau de *engagement* des joueurs avant qu'ils ne jouent leur partie. L'engagement peut être lié à la capacité des agents à signer des contrats contraignants au moment où ils sont appariés. De plus, l'engagement est une hypothèse clé dans la littérature sur l'appariement stable avec des utilités endogènes (comme l'appariement avec transferts et l'appariement avec contrats). La littérature sur l'appariement ne décrit pas le processus précis d'engagement mais définit plutôt une notion statique de stabilité et prouve, en utilisant par exemple un algorithme de *acceptation-différée*, l'existence d'une telle allocation stable. En suivant cette idée, nous étudions les résultats algorithmiques pour les deux sous-modèles.

Le premier concept de solution étudié est celui des couples appariés qui ne peuvent pas s'engager, c'est-à-dire qu'ils ne peuvent pas signer un contrat contraignant avant de jouer leur partie. Dans ce cas, pour que les joueurs ne s'écartent pas des actions prévues, ces dernières doivent constituer un *équilibre de Nash* de leur jeu.

⁴Le jeu des transferts est un jeu à somme constante car la somme des gains $f_{d,h}(x_d, y_h) + g_{d,h}(x_d, y_h) = a_{d,h} + b_{d,h}$ ne dépend pas de x_d ou y_h . Stratégiquement, ceci est équivalent à un jeu à somme nulle et c'est une instance particulière d'un jeu strictement compétitif.

Définition. Une allocation individuellement rationnelle $\pi = (\mu, \vec{x}, \vec{y})$ est,

1. **Stable à la Nash** si tous les couples appariés jouent un équilibre de Nash,
2. **Stable par paires** s'il n'existe aucune paire d'agents (d, h) et aucun équilibre de Nash (x'_d, y'_h) de $G_{d,h}$ tels que,

$$f_{d,h}(x'_d, y'_h) > f_d(\pi) \text{ et } g_{d,h}(x'_d, y'_h) > g_h(\pi)$$

La stabilité par paires est l'extension naturelle de la condition de stabilité de Gale-Shapley aux jeux d'appariement sans engagement. En utilisant un nouvel algorithme d'acceptation-différée avec concours, on prouve que chaque fois que tous les jeux $G_{d,h}$ admettent un ensemble non vide et compact d'équilibres de Nash, il existe une allocation stable par paires et stable à la Nash. De plus, une structure *semi-lattice* existe: le maximum entre deux allocations stables (par paire et à la Nash) pour les préférences de chaque partie est stable. Lorsque tous les jeux $G_{d,h}$ ont un unique paiement d'équilibre de Nash (comme dans les jeux à somme nulle ou, plus généralement, les jeux strictement compétitifs), le modèle devient un problème classique de mariage stable et ainsi, la structure complète du lattice est retrouvée. De plus, la dualité entre le maximum pour un côté et le minimum pour l'autre côté est également obtenue.

Il est important de remarquer que les jeux avec transferts monétaires, comme les travaux de Shapley-Shubik [93] et Demange-Gale [39], sont des jeux à somme constante où l'unique équilibre de Nash est $(x_d^*, y_h^*) = (0, 0)$, car le transfert nul est une stratégie strictement dominante. Pour que des transferts positifs se produisent, les joueurs doivent être capables de s'engager. Ceci est implicitement supposé dans la littérature de l'appariement avec transferts. En d'autres termes, le modèle des jeux d'appariement sans engagement, bien que naturel, ne parvient pas à capturer les modèles classiques d'appariement de la littérature. Illustrons cela.

Exemple 3. Considérons un jeu d'appariement avec seulement deux joueurs d et h , ayant tous deux une utilité fixe et positive $\delta > 0$ pour rester célibataire (leur option extérieure), de sorte que, si les agents décident de s'apparier, ils jouent un jeu à somme constante comme dans l'exemple 2, $G_{d,h} = (\mathbb{R}^+, \mathbb{R}^+, f_{d,h}, g_{d,h})$ avec des fonctions de gain données par,

$$f_{d,h}(x_d, y_h) = -x_d + y_h + 10\delta, \quad g_{d,h}(x_d, y_h) = x_d - y_h, \quad \forall x_d, y_h \geq 0$$

Le seul équilibre de Nash du jeu de transfert $G_{d,h}$ est le transfert nul $(x_d, y_h) = (0, 0)$, car tout transfert positif est une stratégie dominée. Par conséquent, une allocation dans laquelle les agents sont appariés est stable à la Nash si et seulement s'ils ne font pas de transferts. Pour un profil de transfert nul, l'agent h reçoit une utilité de 0. Cependant, il est préférable pour lui d'être célibataire car il obtient δ . Par conséquent, une allocation stable à la Nash est individuellement rationnelle si et seulement si les agents ne sont pas appariés. \square

Le deuxième concept de solution étudié correspond à celui dans lequel les joueurs **peuvent s'engager** (par exemple en signant des contrats contraignants). En faisant un abus de notation, nous appelons aussi stable par paire l'extension de la stabilité de Gale-Shapley au modèle sous engagement. En effet, les notions de stabilité des deux contextes, avec ou sans engagement, seront capturées par une notion plus générale.

Définition. Une allocation rationnelle individuelle $\pi = (\mu, \vec{x}, \vec{y})$ est **stable par paire** si aucune paire d'agents (d, h) ne peut conjointement dévier vers un certain profil de stratégie (x'_d, y'_h) dans leur jeu $G_{d,h}$ qui améliore de façon parétoïste leurs gains, c'est-à-dire,

$$f_{d,h}(x'_d, y'_h) > f_d(\pi) \text{ et } g_{d,h}(x'_d, y'_h) > g_h(\pi)$$

Remarquez que les agents ne sont plus limités à jouer dans leur ensemble d'équilibre de Nash. La définition précédente est l'extension naturelle de la stabilité de Gale-Shapley au cadre avec engagement. Un algorithme similaire à celui utilisé dans le cadre sans engagement nous permet de prouver que, pour tout cadre dans lequel tous les jeux stratégiques $G_{d,h}$ ont des ensembles de stratégies Pareto-optimales compactes et des fonctions de gain continues, il existe une allocation stable par paire. Une structure de semi-lattice existe également pour l'ensemble des allocations stables par paire: le maximum entre deux allocations stables par paire pour les préférences de chaque partie reste stable. La structure de lattice complète est retrouvée lorsque tous les jeux $G_{d,h}$ sont des jeux à somme constante (ou, plus généralement, des jeux strictement compétitifs).

Contrairement au modèle sans engagement, les jeux d'appariement avec engagement capturent avec succès les modèles d'appariement de la littérature. Par exemple, les modèles de Shapley-Shubik et de Demange-Gale sont des instances particulières des modèles de jeux d'appariement avec engagement où tous les jeux stratégiques sont strictement compétitifs. La stabilité par paires coïncide avec leurs notions de stabilité et notre algorithme d'acceptation-différée avec compétitions calcule des solutions stables pour leurs modèles. De plus, nous retrouvons la structure en lattice du modèle de Demange et Gale.

Les joueurs sont rationnels et choisissent leurs actions de manière optimale. Comme les agents peuvent s'engager, jouer de manière optimale n'implique pas nécessairement de jouer en équilibre. Dans un cadre avec engagement, les agents maximisent leurs utilités sous réserve de satisfaire les options extérieures des partenaires: les utilités potentielles qu'ils peuvent obtenir avec quelqu'un d'autre. Par exemple, une entreprise doit s'aligner sur le salaire du marché d'un travailleur afin de le garder. D'un autre point de vue, avoir une option extérieure (donnée par le salaire du marché ou par le salaire offert par une entreprise concurrente) permet au travailleur de négocier son salaire avec l'entreprise. Nous prouvons qu'une condition **équilibre de Nash contraint** doit naturellement exister.

Définition. Une allocation stable par paire $\pi = (\mu, \vec{x}, \vec{y})$ est **résistante aux négociations** si pour tout couple $(d, h) \in \mu$ et tout $(s_d, t_h) \in X_d \times Y_h$, elle tient,

1. Si $f_{d,h}(s_d, y_h) > f_d(\pi)$ alors, $(\mu, (\vec{x}_{-d}, s_d), \vec{x})$ n'est pas stable par paire,
2. Si $g_{d,h}(x_d, t_h) > g_h(\pi)$ alors, $(\mu, \vec{x}, (\vec{y}_{-h}, t_h))$ n'est pas stable par paire.

Une allocation $\pi = (\mu, \vec{x}, \vec{y})$ est *résistante aux négociations* si toute déviation profitable d'un joueur dans son jeu diminue le payment du partenaire en dessous de son option extérieure de marché. Dit autrement, en fixant y_h , x_d doit maximiser le payment du joueur d sous la contrainte de participation du joueur h , et vice-versa. Nous définissons ce type de profil de stratégie comme *équilibres de Nash sous contrainte* (voir ci-dessous pour la définition formelle).

En résumé, notre concept de solution combine une notion coopérative (la stabilité par paire de Gale-Shapley) avec une notion non coopérative (une condition d'équilibre de Nash généralisée). Un concept de solution similaire est utilisé dans les jeux de formation de réseaux: en fixant le réseau, les actions des joueurs doivent maximiser leurs gains, et pour chaque lien dans le réseau, les deux joueurs doivent être d'accord pour former ce lien (voir Jackson et Wolinsky [69] ou Bich et Morhaim [28]). Le modèle des jeux d'appariement un à un peut être considéré comme un modèle particulier de jeu en réseau où seuls les graphes bipartis sont possibles et où un lien est formé si les deux joueurs extrêmes acceptent de s'apparier.

Le concept de résistant aux négociations a fait l'objet d'un intérêt particulier dans les jeux répétés à l'infini et dans la conception de mécanismes [2, 3, 12, 51, 86, 92, 99]. Notre notion est plus étroitement liée à celle de Dewatripont [41] qui considère un cadre dans lequel les agents peuvent parvenir à un accord sur les contrats en raison de l'existence de tiers.

Nous prouvons qu'une allocation est résistante aux négociations si et seulement si tous les couples jouent des équilibres de Nash contraints pour leurs prix de réservation.

Définition. Soit (d, h) un couple dans une allocation et supposons que f_0 et g_0 représentent les gains minimums que d et h , respectivement, sont prêts à accepter afin d'être appariés avec l'autre. En d'autres termes, f_0 et g_0 sont les *prix de réservation* ou *options de sortie* des agents. Alors, leur profil stratégique (x_d, y_h) est un (f_0, g_0) -**équilibre de Nash contraint** si,

$$f_{d,h}(x_d, y_h) = \max\{f_{d,h}(s, y_h) : g_{d,h}(s, y_h) \geq g_0, s \in X_d\}$$

$$Ng_{d,h}(x_d, y_h) = \max\{g_{d,h}(x_d, t) : f_{d,h}(x_d, t) \geq f_0, t \in Y_h\}$$

Nous définissons une classe de jeux stratégiques (appelés *jeux faisables* (Définition 3.4.1)) comme les jeux qui admettent l'existence d'équilibres de Nash contraints et prouvons que: (a) lorsque tous les jeux $G_{d,h}$ sont faisables, un nouveau *processus de renégociation*, s'il converge, atteint une allocation stable par paire et résistante aux négociations, et (b) ce nouvel algorithme converge lorsque tous les jeux sont à somme constante, strictement compétitifs, potentiels ou infiniment répétés. Comme les jeux strictement compétitifs sont faisables, les résultats de Shapley-Shubik et de Demange-Gale sont récupérés et affinés.

La preuve qu'un jeu est faisable utilise les propriétés du jeu. En d'autres termes, la preuve dépend du jeu. Par exemple, pour les jeux potentiels, nous prouvons l'existence d'équilibres de Nash contraints en maximisant la fonction potentielle sur un ensemble approprié. Pour chacun des quatre jeux réalisables mentionnés, nous concevons un *oracle* pour calculer les équilibres de Nash contraints (par exemple en maximisant la fonction potentielle) qui garantissent les convergences du processus de renégociation.

Pour la plupart des jeux, les algorithmes conçus pour calculer des allocations résistantes aux négociations et stables par paires, en plus, trouvent des allocations dans lesquelles tous les couples jouent des profils stratégiques Pareto-optimaux. Pour l'algorithme d'acceptation-différée avec concours, il suffit de restreindre les profils stratégiques des agents à leurs ensembles Pareto-optimaux. Pour le processus de renégociation, il suffit de considérer des oracles qui choisissent des équilibres de Nash contraints Pareto-optimaux. Pour des jeux tels que les jeux à somme nulle, les jeux strictement compétitifs ou les jeux répétés à l'infini, notre oracle parvient à sélectionner un équilibre de Nash contraint Pareto-optimal. Cependant, ce ne sera pas la règle générale car pour de nombreux jeux, la Pareto-optimalité et la stabilité sont incompatibles, par exemple, des jeux potentiels tels que le *dilemme du prisonnier*.

Un cadre intéressant est obtenu pour les jeux d'appariement un-à-un avec engagement dans lesquels les couples jouent des jeux *non-dégénérés* infiniment répétés.

Définition. Nous disons qu'un jeu infiniment répété est **non-dégénéré** si l'ensemble des gains d'équilibre uniformes E a un intérieur non vide.

Expliquons mieux ce point. Considérons deux jeux, un *matching pennies* G_1 , et un *dilemme du prisonnier* G_2 , tous deux exprimés par les matrices de gains suivantes,

		Joueur 2	
		G_1	A
Joueur 1	A	1	-1
	B	-1	1

Table 1: Matching pennies

		Joueur 2	
		G_2	Coopérer
Joueur 1	Coopérer	2, 2	-1, 3
	Trahir	3, -1	0, 0

Table 2: Dilemme du prisonnier

L'appariement des centimes est un jeu à somme nulle avec une valeur $w = 0$ obtenue lorsque chaque agent joue A et B avec la même probabilité. Le dilemme du prisonnier est un jeu potentiel avec un gain d'équilibre de Nash $(0, 0)$ obtenu lorsque les deux joueurs trahissent. Supposons que nous répétons ces jeux une infinité de fois. Pour chacun d'entre eux, nous pouvons définir les ensembles de profils de gain *faisable* et *non puni* E_1 et E_2 , respectivement comme,

$$E_1 := \{(f, g) \in \text{co}((-1, 1), (1, -1)) : f \geq 0, g \geq 0\} \subseteq \mathbb{R}^2$$

$$E_2 := \{(f, g) \in \text{co}((2, 2), (-1, 3), (3, -1), (0, 0)) : f \geq 0, g \geq 0\} \subseteq \mathbb{R}^2$$

où $\text{co}(\cdot)$ indique l'enveloppe convexe. Figure 1 montre les ensembles E_1 et E_2 .

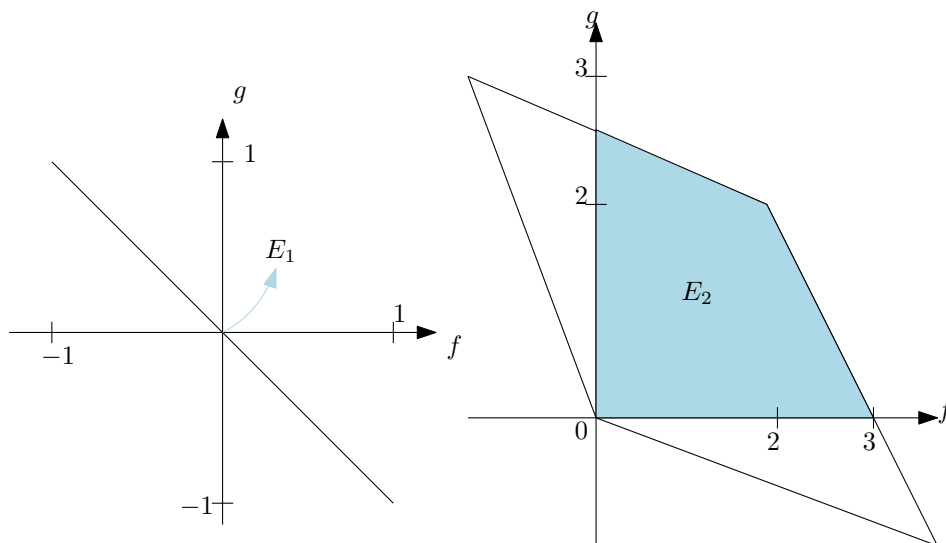


Figure 1: Ensembles de profils de gain faisable et non puni

L'ensemble E_1 correspond à un point unique dans \mathbb{R}^2 . En particulier, E_1 a un intérieur vide. Par contre, l'ensemble E_2 a un intérieur non vide comme on peut l'observer sur la figure. Par le théorème de Folk d'Aumann-Shapley [15], nous savons que les ensembles E_1 et E_2 correspondent aux ensembles des *gains d'équilibre uniformes* de chaque jeu. Par conséquent, nous obtenons que la version infiniment répétée de la correspondance des centimes est un jeu dégénéré, tandis que la version infiniment répétée du dilemme des prisonniers est non dégénérée.

L'aspect dynamique des jeux répétés permet aux couples d'oublier l'engagement initialement imposé par le modèle car les agents peuvent punir leurs partenaires pour leurs déviations d'actions. Pour être plus précis, pour les jeux non dégénérés comme la version répétée du dilemme des prisonniers ci-dessus, les gains d'équilibre de Nash contraints sont toujours (faiblement) dominés par Pareto par un gain d'équilibre uniforme. En particulier, nous pouvons toujours remplacer les équilibres de Nash contraints des couples par des équilibres uniformes sans affecter la stabilité par paire et la résistance aux négociations de l'allocation. Par conséquent, les jeux infiniment répétés (non dégénérés) satisfont la notion la plus forte de stabilité possible dans notre modèle: *Pareto-optimalité, stabilité par paire et stabilité à la Nash*.

L'exemple 3 ci-dessus a montré les limites du modèle de jeux d'appariement un à un sans engagement, à savoir qu'il ne parvient pas à capturer les modèles d'appariement classiques de la littérature comme celui de Shapley et Shubik. Ce n'est pas le cas pour un modèle avec engagement car tous les modèles discutés au début de ce chapitre sont capturés avec succès. Pour cette raison, les deux modèles suivants étudiés dans cette thèse considèrent l'engagement comme une hypothèse clé.

2. *Jeux d'appariement un-à-plusieurs avec substitués* (et engagement), la généralisation du *problème des admissions à l'université*, le deuxième problème présenté par Gale et Shapley dans [53], ainsi que la généralisation des jeux d'appariement un à

un avec engagement, considèrent que les agents d'un côté, les hôpitaux H à partir de maintenant, peuvent être appariés avec plusieurs partenaires en même temps. Formellement, étant donné un ensemble $I \subseteq D$ alloué dans $h \in H$, et étant donné des profils stratégiques $\vec{x}_I \in \prod_{d \in I} X_d$, $\vec{y}_{h,I} = (y_{h,d})_{d \in I} \in Y_h^{|I|}$, les gains des joueurs sont donnés par,

$$f_{d,h}(x_d, y_{h,d}), \forall d \in I, \text{ et } g_{I,h}(\vec{x}_I, \vec{y}_{h,I})$$

c'est-à-dire que la fonction de gain de l'hôpital h dépend de l'identité de l'ensemble des médecins I , des stratégies jouées par tous ces médecins et du profil stratégique $\vec{y}_{h,I}$ joué par h , où chaque $y_{h,d}$ représente la stratégie particulière jouée par h contre le médecin d . Les fonctions de gain des médecins sont les mêmes que dans le modèle un-à-un.

La stabilité par paires pour les jeux d'appariement un-à-un n'a pris en compte que les paires bloquantes, car seuls les couples étaient autorisés. Puisque les agents de H peuvent maintenant être appariés avec de nombreux partenaires, nous considérons la stabilité du *Core* (faible).

Définition. Une allocation $\pi = (\mu, \vec{x}, \vec{y})$ est **bloquée** par une coalition de médecins $I \subseteq D$ et un hôpital h s'il existe $(\vec{w}_I, \vec{z}_{h,I}) \in X_I \times Y_h^{|I|}$, tels que,

$$\begin{aligned} \forall d \in I, f_{d,h}(w_d, z_{h,d}) &\geq f_d(\pi), \\ \exists d \in I, f_{d,h}(w_d, z_{h,d}) &> f_d(\pi), \\ g_{I,h}(\vec{w}_I, \vec{z}_{h,I}) &> g_h(\pi), \end{aligned}$$

où $f_d(\pi)$ et $g_h(\pi)$ sont les gains de d et h , respectivement, dans π . π est **Core stable** s'il est individuellement rationnel et s'il n'est pas bloqué.

Les modèles d'appariement avec des préférences strictes (comme celui de Hatfield et Milgrom) ont la propriété que le Core et le Core faible coïncident. En raison du continuum de gains dans notre modèle, cette propriété ne se vérifie pas nécessairement. Pour obtenir l'existence d'allocations Core stables, nous dotons les hôpitaux de fonctions de gains additives séparables et de quotas. Formellement, nous disons que $h \in H$ a un *fonction de gain additive séparable* et un *quota* $q_h \in \mathbb{N}$, si,

$$g_{I,h}(\vec{x}_I, \vec{y}_h) = \begin{cases} \sum_{d \in I} g_{d,h}(x_d, y_{h,d}) & \text{si } |I| \leq q_h \\ -\infty & \text{sinon} \end{cases}$$

$$\text{où } g_{d,h} : X_d \times Y_h \rightarrow \mathbb{R}, \forall d \in I$$

Une généralisation de l'algorithme d'acceptation-différée avec concours proposé pour les jeux d'appariement de un à un prouve l'existence d'allocations stables de Core pour tout jeu d'appariement de un à plusieurs dans lequel les agents de H ont des fonctions de gain séparables additives et des quotas, et tous les agents ont des ensembles de stratégies compacts et des fonctions de gain continues. L'avantage de la séparabilité additive est que la Core stabilité et la stabilité par paires (comme

celle des jeux d'appariement un-à-un) deviennent équivalentes. En particulier, nous n'avons pas besoin d'utiliser un *mécanisme d'offre cumulative* comme celui de Hatfield et Milgrom mais seulement un algorithme d'acceptation-différée. Deux avantages importants découlent de ce changement de mécanisme: (1) les mécanismes d'offre cumulative ne donnent pas une manière constructive de calculer des allocations stables car ils ont besoin de théorèmes de point fixe pour garantir la convergence et (2) grâce à leur simplicité, la complexité des algorithmes d'acceptation-différée peut être correctement étudiée.

La séparabilité des fonctions de paiement des hôpitaux nous permet d'appliquer directement la notion de résistance aux négociation telle que définie pour les jeux d'appariement un-à-un au modèle un-à-plusieurs, affinant ainsi les modèles de Kelso-Crawford et Hatfield-Milgrom, entre autres. En effet, le même processus de renégociation peut être utilisé. En particulier, nous pouvons prouver l'existence d'allocations Core stables et résistantes aux négociations pour tout jeu d'appariement à somme nulle, strictement compétitif, potentiel ou infiniment répété.

3. *Roommates matching game* est la généralisation du troisième problème présenté par Gale et Shapley. Formellement, nous considérons un ensemble d'agents D et dotons chaque joueur $d \in D$ d'un ensemble stratégique X_d , et tout couple $(d_1, d_2) \in D \times D$ du jeu stratégique $G_{d_1, d_2} = (X_{d_1}, X_{d_2}, f_{d_1, d_2}, f_{d_2, d_1})$.

Une allocation dans ce cadre est une paire $\pi = (\mu, \vec{x})$ où μ est un *appariement un-à-un* et $\vec{x} \in \prod_{d \in D} X_d$ est un *profil de stratégie*. Notre notion de stabilité par paire capture la stabilité du problème des colocataires dans ses nombreuses variantes: préférences ordinales, utilité transférable et utilité non-transférable.

Nous nous appuyons sur les travaux d'Alkan et Tuncay sur les colocataires avec utilité non transférable pour étudier l'existence d'allocations stables par paire. Nous utilisons la *procédure de marché* conçue dans [9] pour calculer un profil de gain $\bar{f} \in \mathbb{R}^{|D|}$ qui, chaque fois qu'une allocation π peut *implémenter* celle-ci, c'est-à-dire, pour tout $d \in D$, π satisfait $f_d(\pi) = \bar{f}_d$, avec,

$$f_d(\pi) := \begin{cases} f_{d, \mu(d)}(x_d, x_{\mu(d)}) & \text{si } d \text{ est apparié} \\ \underline{f}_d & \text{sinon} \end{cases}$$

où \underline{f}_d est le gain individuel rationnel du joueur d , π est stable par paire.

Comme seuls les couples sont autorisés dans le modèle, notre notion de résistance aux négociation peut être directement appliquée, en raffinant les concepts de solution de la littérature. En particulier, notre processus de renégociation fonctionne dans n'importe quel jeu réalisable avec un oracle qui réalise la convergence de l'algorithme.

Nous terminons l'introduction du modèle des jeux d'appariement par l'observation suivante. Il existe différentes façons d'associer un jeu stratégique à un modèle de Shapley-Shubik [93], de Demange-Gale [39], ou de Hatfield-Milgrom [66], entre autres. Nos notions de Core stabilité et par paire coïncident toujours avec leur stabilité, mais chaque modélisation de jeu stratégique induit, grâce à la résistance aux négociations,

une sélection différente parmi les résultats stables. Par exemple, supposons que dans le modèle de Shapley-Shubik, il n'y a qu'un seul vendeur et un seul acheteur, le vendeur a un coût c pour la maison et l'acheteur a une valeur v pour celle-ci. Si $v < c$, il n'y a pas de possibilité d'échange. Sinon, le surplus $v - c$ est positif et tout prix p entre c et v correspond à une allocation stable par paire. Si l'interaction stratégique entre l'acheteur et le vendeur est un jeu de négociation ultimatum [1] où le vendeur est le premier proposant (resp. l'acheteur est le premier proposant), le résultat stable par paire et résistant à la renégociation choisi est $p = v$ (resp. $p = c$). D'autre part, si l'interaction stratégique est modélisée par le jeu de négociation des offres alternées de Rubinstein [84] avec des joueurs également patients, le résultat est la solution de négociation de Nash $p = (v - c)/2$. Par conséquent, différents jeux de négociation entre acheteurs et vendeurs induisent, en raison de la résistance aux négociations, un partage différent du surplus. Ceci n'est pas modélisé dans la littérature standard sur les transferts et constitue l'une des principales contributions de notre travail.

Le dernier modèle discuté dans cette thèse correspond à un *marché d'appariement dynamique en temps continu*, et c'est un travail conjoint avec Marco Scarsini. Nous considérons un marché biface de type one-to-one avec des côtés \mathcal{A} et \mathcal{B} , où les agents arrivent suivant des *processus de Poisson* avec des taux λ_A et λ_B , respectivement. Les agents peuvent naturellement quitter le marché pour cause de décès ou de fatigue à des taux respectifs de μ_A et μ_B . Les agents au sein du marché sont appariés selon une *politique d'appariement* Φ fixe, et les agents couplés quittent le marché et ne reviennent jamais. Étant donné l'état du marché à un moment donné $t > 0$, à savoir (A_t, B_t) , un agent arrivant k tire *utilité aléatoire uniforme* dans $[0, 1]$ sur tous les agents présents à ce moment-là de l'autre côté, représentant l'utilité que k obtient en s'appariant avec chacun d'eux. De même, les agents de l'autre côté tirent des utilités aléatoires uniformes dans $[0, 1]$ pour être avec k .

Les agents sont dotés de gains individuels rationnels (IRP) $\rho_k \in [0, 1], \forall k \in \mathcal{A} \cup \mathcal{B}$, c'est-à-dire leur utilité d'être célibataire, et n'acceptent jamais un appariement dont l'utilité est inférieure à ρ .

Nous étudions *algorithmes d'appariement gourmand*, c'est-à-dire les politiques d'appariement qui sont déclenchées à chaque arrivée, comme *first-come-first-served*. Étant donné une politique d'appariement gourmande fixe Φ , nous considérons la chaîne de Markov à temps continu donnée par $(Z_t = (|A_t|, |B_t|)_{t \geq 0})$ qui suit le nombre d'agents sur le marché à chaque instant $t \geq 0$. Les chaînes de Markov à temps continu peuvent être étudiées en considérant leur *chaîne de saut* $(W_n)_{n \geq 0}$ indexée uniquement sur les temps de saut, c'est-à-dire à chaque arrivée ou départ naturel d'un agent.

Les agents proposés pour être appariés avec un partenaire n'acceptent que si l'utilité obtenue est au moins leur gain rationnel individuel. Par conséquent, si a est un agent arrivant à \mathcal{A} au temps de saut n , la probabilité qu'il soit apparié est donnée par,

$$p_a := \mathbb{P}(a \text{ est apparié}) = \begin{cases} (1 - \rho_a) \cdot (1 - \prod_{b \in B_n} \rho_b) & \text{si } B_n \neq \emptyset, \\ 0 & \text{sinon.} \end{cases}$$

Les probabilités $(p_a, p_b)_{a \in \mathcal{A}, b \in \mathcal{B}}$ ne peuvent pas dépendre de l'identité des agents présents sur le marché au temps n si l'on veut obtenir une chaîne de Markov *homogène en temps*.

Définition. On dit que les agents ont des **IRPs anonymes** lorsque les probabilités $(p_a, p_b)_{a \in \mathcal{A}, b \in \mathcal{B}}$ ne dépendent pas de l'identité des joueurs mais seulement du nombre d'agents de l'autre côté du marché.

En prenant des gains rationnels individuels anonymes, la chaîne de Markov $(W_n)_n$ devient une *marche aléatoire* dans \mathbb{N}^2 . La probabilité de passer à un état supérieur est toujours inférieure à celle de passer à un état inférieur puisque la marche aléatoire ne peut pas exploser: les agents peuvent quitter le marché de deux manières, soit parce qu'ils sont partis naturellement, soit parce qu'ils ont été appariés. Les appariements se produisent à un rythme similaire à celui des arrivées (en fonction des valeurs des IRP); par conséquent, la marche aléatoire diminue ses valeurs plus souvent qu'elle ne les augmente. Ainsi, notre marche aléatoire est *non-homogène dans l'espace*.

En utilisant le critère de Foster, nous prouvons que $(W_n)_n$ est à récurrence positive (donc, en prouvant que $(Z_t)_t$ est également à récurrence positive). Ensuite, en imposant la *réversibilité*, à savoir que pour tout $(i, j) \in \mathbb{N}$ il doit être maintenu,

$$\pi(i)P(i, j) = \pi(j)P(j, i)$$

où π est une *mesure stationnaire* de $(W_n)_n$ et P sa *matrice de transition*, nous trouvons une expression récursive pour π et déduisons que la distribution stationnaire de $(Z_t)_t$ a une forme-produit.

Enfin, des expériences empiriques montrent que les politiques d'appariement averse de *acceptation-différée*, c'est-à-dire l'exécution de l'algorithme de Gale-Shapley à chaque arrivée, surpassent les autres politiques d'appariement averse comme le premier arrivé, premier servi et l'allocation aléatoire.

Nos contributions

Dans cette thèse, nous présentons le nouveau modèle de *jeux d'appariements* dans lequel un nombre fini d'agents s'apparient et reçoivent des utilités comme résultats de jeux stratégiques. Deux sous-modèles sont obtenus en fonction du niveau d'engagement des agents. Nous avons étudié plus en profondeur les jeux d'appariement avec engagement car ils sont plus difficiles mathématiquement mais ils capturent et raffinent également un grand nombre de modèles dans la littérature tels que: le *problème du mariage stable*, le *problème des admissions à l'université*, et le *problème des colocataires* de Gale et Shapley [53], le *jeu d'affectation* de Shapley et Shubik, le *problème d'appariement avec des transferts* de Demange et Gale [39], le *marché d'appariement des emplois* de Kelso et Crawford [71], l'*appariement avec contrats* de Hatfield et Milgrom [66], le *colocataires avec utilité transférable* [10], et le *colocataires avec utilité non transférable* [9]. Nos notions de Core stabilité et stabilité par paire

sont équivalentes aux notions de stabilité considérées dans chacun de ces problèmes, et notre nouvelle notion de résistance aux négociations permet de raffiner leurs solutions stables.

Chapitre 2 conçoit un algorithme d'*acceptation-différée avec concours* (DAC) pour les marchés d'appariement un-à-un afin de calculer des allocations stables par paires sous des hypothèses classiques de la théorie des jeux. De plus, l'algorithme s'exécute en un nombre limité d'itérations de temps polynomial pour de nombreux jeux bien connus de la littérature de la théorie des jeux. Nous étendons à notre nouveau cadre les résultats classiques des marchés d'appariement un-à-un, comme la *structure de lattice* de l'ensemble des allocations stables et l'*optimalité* de l'algorithme DAC pour le côté proposant.

Parmi les contributions les plus importantes, Chapitre 3 définit une nouvelle notion de stabilité, *résistance aux négociations*: Les agents d'un même couple sont rationnels et doivent jouer leur meilleure réponse à leur partenaire. Cependant, comme les agents peuvent s'engager dans des actions, les meilleures réponses sont contraintes de garantir un certain seuil de gain aux partenaires, leur *option extérieur de marché* (gain minimum que les partenaires peuvent accepter pour être appariés). La résistance aux négociations est définie par rapport à la stabilité par paire, en raffinant tous les concepts de stabilité de la littérature que le modèle capture.

Chapitre 3 caractérise la résistance aux négociations comme une condition d'*équilibre de Nash sous contrainte* (CNE) et conçoit un processus de renégociation pour calculer les allocations stables par paire et résistantes aux négociations chaque fois que les jeux stratégiques permettent l'existence de CNE et que le processus converge. De plus, la convergence de l'algorithme est garantie pour de nombreux jeux bien connus, dont les modèles de Shapley-Shubik et de Demange-Gale. De plus, pour les mêmes jeux que l'algorithme DAC, le processus de renégociation a un nombre borné d'itérations de temps polynomial.

Chapitre 4 présente un nouveau modèle de jeux d'appariement un-à-plusieurs à deux côtés dans lequel les agents de l'un des côtés, l'ensemble H , peuvent être appariés avec de nombreux agents en même temps, capturant avec succès le modèle séminal de Hatfield et Milgrom d'*appariement avec des contrats*. [66].

L'appariement avec contrats considère un marché d'appariement un-à-plusieurs dans lequel les agents sont dotés de *fonctions de choix*. Les fonctions de choix produisent le ou les contrats les plus préférés de chaque agent, étant donné un ensemble de contrats possibles sur le marché. Les auteurs ont prouvé l'existence d'allocations stables lorsque les agents de H ont des contrats substituables.

Aygün et Sönmez [16] ont exposé que différents modèles sont obtenus si les fonctions de choix des agents sont traitées comme des primitives du modèle ou si elles sont induites à partir des classements de préférences. Bien que le modèle de Hatfield et Milgrom appartienne au deuxième type, ils ont traité leurs fonctions de choix comme des primitives. Par conséquent, pour garantir véritablement l'existence d'une allocation stable, une hypothèse supplémentaire, l'*insignifiance des contrats rejetés*, est également requise. L'une des principales contributions de Chapitre 4 est le fait

que notre modèle de jeux d'appariement un-à-plusieurs fonctionne directement avec les primitives du problème (les fonctions de gain), ce qui nous permet: (1) de mieux comprendre les raisons de l'existence d'allocations Core stables, (2) de comprendre la relation entre la stabilité par paire et Core, (3) de raffiner le modèle avec des contrats en appliquant la notion de résistance aux négociations, (4) de considérer un nombre infini de contrats sur le marché, et (5) d'étudier la complexité des algorithmes proposés.

Le point (2) est particulièrement intéressant. Chaque fois que les agents dans H ont des fonctions de gain additives séparables, nous prouvons que la Core stabilité peut être réduite à la stabilité par paire. En particulier, le *mécanisme d'offre cumulative*, comme celui utilisé par Hatfield et Milgrom, peut être remplacé par des algorithmes d'*acceptation-différée*. Par conséquent, dans le Chapitre 4, nous étendons l'algorithme DAC proposé pour les jeux d'appariement un-à-un afin d'obtenir l'existence d'allocations Core stables dans le cadre un-à-plusieurs. Sous la même condition de séparabilité additive, nous étendons la résistance aux négociations au cadre un-à-plusieurs dans Chapitre 4 en réalisant le point (3) avec la même efficacité de calcul que pour les jeux d'appariement un-à-un (point (5)).

Un troisième nouveau modèle est introduit dans Chapitre 4, le *jeu d'appariement de colocataires*, une extension du modèle de colocataires à utilité non transférable au cas où des couples d'agents obtiennent leurs utilités comme résultats d'un jeu stratégique. Comme pour l'appariement avec des contrats, notre jeu d'appariement de colocataires raffine les concepts de solution de la littérature en étudiant les allocations résistantes aux négociations.

Enfin, Chapitre 6 présente un travail conjoint avec Marco Scarsini, un modèle de marché d'appariement dynamique en temps continu où les agents peuvent arriver et quitter le marché en suivant des processus stochastiques. Nous comparons différentes *politiques d'appariement gourmandes* et étudions les conditions nécessaires pour l'existence de *distributions stationnaires* avec une forme-produit.

La thèse est structurée comme suit. Chapitre 1 introduit le modèle des jeux d'appariement sans engagement et étudie l'existence et le calcul des allocations stables par paire et stable à la Nash. Chapitre 2 introduit le modèle d'appariement avec engagement et étudie l'existence et le calcul d'allocations stables par paire. Chapitre 3 introduit résistance aux négociations, notre nouveau raffinement des notions de stabilité dans la littérature, et étudie l'existence et le calcul des allocations résistantes aux négociations. Chapitre 4 introduit deux modèles supplémentaires de jeux d'appariement: les jeux d'appariement un-à-plusieurs et les jeux d'appariement entre colocataires. Pour chacun de ces modèles, l'étude de l'existence et du calcul d'allocations Core stables et résistantes aux négociations est menée. Chapitre 5 est consacré à l'étude de la complexité de tous les algorithmes proposés dans les chapitres précédents, en prouvant leur efficacité computationnelle. Chapitre 6 introduit un modèle de marché d'appariement dynamique dans lequel les agents peuvent arriver et partir au cours du temps. Le dernier chapitre conclut la thèse, donne les lignes de recherche futures, et discute des problèmes ouverts.

General Introduction

The stable matching problem

The Gale and Shapley [53] one-to-one two-sided market matching problem, known as the *marriage problem*, consists in finding a *stable* pairing between two different finite sets D and H , given that each agent on each side has an *strict exogenous (total) preference ordering* over the agents on the other side. Formally, each agent $k \in D \cup H$ is endowed with $>_k$ such that, for any two agents ℓ and ℓ' on the opposite side, $\ell >_k \ell'$ represents that k prefers to be matched with ℓ rather than with ℓ' .

The marriage problem focuses on computing a coupling μ that associates each agent on one side to at most one agent on the other side. The coupling μ is *stable* if no uncoupled pair of agents both prefer to be paired together rather than with their partners in μ . Formally, a coupling μ is *blocked* if there exists a pair of agents $(d, h) \in D \times H$, not matched between them, such that,

$$h >_d \mu(d) \text{ and } d >_h \mu(h)$$

where $\mu(k)$ represents the partner of k in the matching μ . The matching is stable if no pair blocks it. Gale and Shapley used a *deferred-acceptance* algorithm to prove the existence of a stable matching for every instance. Their algorithm takes one of the sides of the market, called the proposer-side, and asks its agents to propose to their most preferred option that has not rejected them yet. Agents receiving more than one proposal accept the best one and reject all the others. The algorithm continues until all agents on the proposer side have been accepted by somebody. Although the model of Gale and Shapley considered two sets of the same size and strict preferences, their algorithm is easily extended to sets of different sizes where the agents have the option to remain single (also referred to as having *incomplete preference orderings*) and non-strict preferences. The computation of the stable matching is exact and takes at most $O(N^2)$ iterations with N being the size of the largest set.

Many properties of the set of stable matchings and the deferred-acceptance algorithm of Gale and Shapley have been proved. John Conway proved a lattice structure over the set of stable matchings Γ (claimed by Gale and Shapley [53] and reported by Knuth [75]), namely, the maximum and the minimum of two stable matchings for the preferences of one of the sides remain stable. Formally, if μ and μ' are two stable matchings in Γ , we define,

$$\mu^\vee(d) := \max_{>_d} \{\mu(d), \mu'(d)\}, \forall d \in D$$

$$\mu^\wedge(d) := \min_{>d} \{\mu(d), \mu'(d)\}, \forall d \in D$$

The matchings μ^\vee and μ^\wedge are well defined and belong to Γ . The matching μ^\vee corresponds to the maximum between μ and μ' for the preferences of agents in D . Analogously, μ^\wedge corresponds to the minimum. A *duality property* holds as the maximum of two stable matchings for one of the sides corresponds to the minimum between the two same stable matchings for the other side. In other words, if

$$\begin{aligned} \nu^\vee(h) &:= \max_{>h} \{\mu(h), \mu'(h)\}, \forall h \in H \\ \nu^\wedge(h) &:= \min_{>h} \{\mu(h), \mu'(h)\}, \forall h \in H \end{aligned}$$

are the maximum and minimum matching for the agents in H , it holds,

$$\mu^\vee = \nu^\wedge \text{ and } \mu^\wedge = \nu^\vee$$

In particular, the highest element in the lattice for one side, that is, the most preferred stable matching for them, is the least preferred stable matching for the opposite side.

Dubins and Freedman [43] but also Gale and Sotomayor [54] studied the incentives of the players to lie when reporting their preferences and proved that the deferred-acceptance algorithm of Gale and Shapley is *strategy-proof* for the proposer side, as the algorithm outputs the best stable matching for them, that is, the highest element in the lattice.

Gusfield and Irving [60] probe the stable marriage and its variants as a rich source of problems and ideas that illustrate both the design and analysis of efficient algorithms. Roth and Sotomayor [89] studied the structure of the set of stable matchings and its core. Vande Vate [100] characterized the set of stable matchings in the one-to-one problem as extreme points of a polytope when $|D| = |H|$. Rothblum [91] extended this result to the case $|D| \neq |H|$.

Balinski and Ratier [24, 25] proposed an elegant directed graph approach to the problem and, like Rothblum, they characterized the stable matching polytope in the one-to-one problem through linear inequalities, proving that any feasible point of the polytope is a stable matching and vice-versa.

Roth and Vande Vate [90] studied a random process to find a stable matching from some arbitrary matching. Ma [79] proved that the Roth and Vande Vate algorithm does not find all the stable matchings. Recently, Dworzak [45] introduced a new class of algorithms called *deferred acceptance with compensation chains algorithms (DACC)*⁵ in which both sides of the market can make offers and proved that a matching is stable if and only if it is the outcome of some DACC algorithm. More precisely, DACC algorithms choose a random proposing order σ over all the agents $D \cup H$ which is modified every time that an agent is replaced by allowing her to propose next. Indexing the DACC algorithms over σ , Dworzak proved that a matching is stable if and only if it is the output of a DACC algorithm for some order σ .

⁵A related class of algorithms was introduced by McVitie and Wilson [80]

Adachi [4] characterized the set of stable matchings as the fixed points of an increasing function through Tarski's fixed point theorem whenever agents have strict preferences. Another fixed-point approach was made by Fleiner [52] by formulating a matroid generalization of the stable marriage problem and studying the lattice structure of the generalized model.

One of the first extension of the marriage problem to the *endogenous preferences* setting is the *assignment game* of Shapley and Shubik [93] in which agents within the same couple can make monetary transfers. The leading example is a *housing market* where buyers and sellers have quasi-linear utilities. Buyers $d \in D$ have different evaluations $((v_{d,h})_{d \in D, h \in H})$ for the sellers' houses $h \in H$, and sellers have different costs $((c_h)_{h \in H})$ over their own houses. Each seller $h \in H$ is willing to sell her house at some price $p_h \geq c_h$, and each buyer $d \in D$ is interested in buying h 's house at a certain price $p_{d,h} \leq v_{d,h}$. An allocation is a pair (μ, p) with μ being a matching between sellers and buyers, and $p = (p_d)_{d \in D}$ being a vector of monetary transfers, where p_d is the price offered by $d \in D$ to $\mu(d) \in H$. Allocations in the Shapley-Shubik model are stable if there is no unmatched pair (d, h) and no transaction price q from d to h such that both agents end up strictly better off by trading. Exploiting the linearity of the payoff functions on the monetary transfers, Shapley and Shubik found stable solutions $\pi = (\mu, p)$ for their problem using linear programming where a pair primal-dual gives, respectively, the matching μ and the vector of prices p . Remark the polynomial complexity of solving the assignment game thanks to the linear programming approach.

The assignment game belongs to the class of *cooperative games with transferable utility* as agents within the same couple (d, h) have to split their *worth* $v_{d,h} + c_h$ in such a way nobody prefers to change their partner. Moreover, Shapley and Shubik proved that the set of stable allocations for their assignment game is exactly the *Core* of the housing market problem seen as a transferable utility cooperative game.

Demange and Gale [39] considered more general utility functions on monetary transfers (non-quasi-linear), allowed monetary transfers on both sides (from buyer to seller and vice-versa), and proved that the set of stable allocations has a lattice structure (non-emptiness of this set has been proved in [37, 87]). Demange et al. [40] designed two *ascending price mechanisms* to compute stable allocations of the *matching with transfers* model in [39]. For integer utilities, the first algorithm converges in a bounded number of iterations to an exact solution. For continuous payments, the second algorithm converges to an ε -stable allocation in a bounded number of iterations. Despite the broad literature achieved during the years, the problem of computing exact stable allocations for matching markets with continuous utilities is still open.

Models with monetary transfers as the ones above belong to the class of matching markets with transferable utility. A clear comparison between the models with transferable and non-transferable utility was made by Echenique and Galichon in [47]. Galichon et al. [55] studied a model of stable matching with *imperfect transferable utility*, due for example to the presence of taxes in the transfers, and algorithmically proved the existence of stable solutions. Dupuy et al. [44] studied further the effects of

taxation in two-sided matching markets providing a continuous link between models of matching with and without transfers.

Other extensions of the stable marriage model to the endogenous preferences setting consider *random preferences* as agents draw them from probability distributions. Gimbert et al. [57] studied the number of *stable pairs*, i.e., the pairs (d, h) that appear in some stable matching, finding many asymptotic results depending on the distributions considered. The same authors in [59] extended their study to the effects of having random preferences and the possible manipulations of the agents over the output of stable matchings algorithms.

One-to-many two-sided matching markets are the generalization of the models explained above to the case in which agents on one of the sides can be matched with many partners at the same time. Many interesting applications arise from these models. Gimbert et al. [58] studied a school choice problem with imperfect information in which students reveal only a partial version of their preferences due to a limited number of applications allowed. Correa et al. [36] studied a centralized mechanism to fairly allocate students to schools in Chile giving priority to joined siblings allocation. In France, extensive studies have been done to develop the students' allocation mechanism to universities *Parcoursup*⁶.

The first ones to introduce this problem were, as for one-to-one two-sided matching markets, Gale and Shapley in their seminal paper [53]. They proved that their same deferred-acceptance algorithm could be applied to the one-to-many setting. Baiou and Balinski [21, 23] generalized the graph-theoretic approach in [24, 25] to one-to-many matching markets and the polytope characterization of stable matchings in this new setting. Echenique and Oviedo [48] characterized the set of core stable allocations as fixed points of a map. In their model, agents are endowed with strict preferences and their characterization gives an efficient algorithm to compute stable allocations. No extra assumption is required for their characterization, but substitutability is required for the non-emptiness of the core.

As Shapley and Shubik in the one-to-one case, Crawford and Knoer [37] extended the model of one-to-many matching markets to the linear monetary transfer setting. Kelso and Crawford [71] went further in the extension by considering any kind of transferable utility. Their *job matching model* considers workers and firms that get matched and simultaneously determine salaries to be paid to the workers. The authors proved the existence of stable allocations for any setting in which workers are *gross substitutes* for the firms: increasing the salary of a set of workers can never cause a firm to withdraw an offer from a worker whose salary has not been risen.

A seminal paper in one-to-many matching markets was written by Hatfield and Milgrom [66], the *matching with contracts* model, that extends the model of Kelso and Crawford by allowing doctors and hospitals (instead of workers and firms) to sign *contracts* from a finite set of possible contracts X in the market. Contracts are *bilateral* so every $x \in X$ relates one doctor $x_D \in D$ with one hospital $x_H \in H$.

⁶A french document about Parcoursup can be found [here](#).

Agents are endowed with preference orderings that define *choice functions* ($C_d, C_h : d \in D, h \in H$). Given a set of possible contracts $X' \subseteq X$, the choice functions output the most preferred contract for each doctor, and the most preferred subset of contacts for each hospital. Formally,

$$C_d(X') = \arg \max_{>_d} \{x \in X' : x_D = d\}$$

$$C_h(X') = \arg \max_{>_h} \{X'' \subseteq X' : \forall x \in X'', x_H = h, \text{ and } \forall x, x' \in X'', x_D \neq x'_D\}$$

Hatfield and Milgrom proved that the set of stable allocations is a non-empty lattice and that a *cumulative offer mechanism* reaches the extremes of the lattice thanks to Tarski's fixed point theorem. The main assumption behind this result is *substitutability* for hospitals.

Definition. Elements of X are **substitutes** for hospital h if for all subsets $X' \subseteq X'' \subseteq X$, it holds, $X' \setminus C_h(X') \subseteq X'' \setminus C_h(X'')$

In words, contracts are substitutes for h if no rejected contract can be chosen because of the broadening of the set of contracts. Substitutability has been proved to be sufficient but not necessary for the existence of stable allocations in the matching with contracts model and many authors have worked to find weaker assumptions [16, 63, 64, 65]. Hatfield and Milgrom proved a second result in their model, the *rural hospital theorem*: every hospital hires the same number of doctors at every stable matching. The key assumptions behind this result are substitutability and the *law of aggregated demand*.

Definition. The preferences of hospital h satisfy the **law of aggregated demand** if for all $X' \subseteq X'' \subseteq X$, $|C_h(X')| \leq |C_h(X'')|$.

In words, hospital h satisfies the law of aggregated demand if every time that the set of possible contracts expands, the total number of contracts chosen by hospital h weakly rises.

Aygün and Sönmez [16] exposed that different models are obtained if agents' choice functions are treated as primitives or they are induced from preference rankings in the matching with contracts model. Hatfield and Milgrom's model belongs to the second type, however, they treated their choice functions as primitives. To truly guarantee the existence of stable allocations, an extra assumption, namely, the *irrelevance of rejected contracts*, is required as well.

Definition. The choice function of hospital h satisfies the **irrelevance of rejected contracts** if for any $X' \subseteq X$, and any $x \in X'$, if $x \notin C_h(X' \cup \{x\})$, then $C_h(X') = C_h(X' \cup \{x\})$.

In words, h 's choice function is irrelevant to rejected contracts if chosen contracts do not change due to deleting rejected contracts from the set of contracts. The good news for Hatfield and Milgrom's work is that the irrelevance of rejected contracts is satisfied when having substitutability and the law of aggregated demand.

More complex extensions such as the many-to-many matching market model have been studied as well. A pioneering article in the extension of Gale-Shapley’s model to the many-to-many setting was made by Blair [29], studying the case in which workers and firms must form partnerships (workers can work in several firms at the same time), obtaining a lattice structure of the set of stable allocations with salaries, extending [71] and [75]. Baïou and Balinski [20] extended known results in one-to-one and one-to-many to the many-to-many setting. In [22], the same authors went further in the extension by considering an allocation problem between suppliers and acquirers in which many copies of the same good can be allocated from suppliers to acquirers.

A third problem was introduced in the seminal article of Gale and Shapley [53]: *roommates*. The roommates problem, part of the non-two-sided matching market models, consists of an even finite set of agents D , each of them having an *exogenous preference ordering* $>_d$ over the rest of the agents, and seeking to match in pairs. The partition of the set of agents into pairs has to be *stable*, that is, no pair of agents, not already together, can both prefer to abandon their partners and match together. Formally, a matching μ (a partition of D in pairs) is stable if there is no $(d_1, d_2) \in D \times D$, $d_1 \neq \mu(d_2)$, such that,

$$d_2 >_{d_1} \mu(d_1) \text{ and } d_1 >_{d_2} \mu(d_2)$$

Unlike the *stable marriage* problem, the *roommates* problem does not always have a stable solution, as Gale and Shapley already observed in their article: Consider four agents $D = \{1, 2, 3, 4\}$, each of them with the following strict preferences on their possible partner,

	Preferences
1	2 > 3 > 4
2	3 > 1 > 4
3	1 > 2 > 4
4	Arbitrary

It is not possible to make a stable partition of the four players into two pairs as the agent paired with 4 will always find somebody else to get matched, who will also prefer to change.

Knuth asked the question of designing an algorithm to compute a stable allocation whenever it exists. Irving [68] answered with a polynomial algorithm to compute a solution if it exists or to report the non-existence of stable allocations for those instances without a solution. His algorithm has two phases, a first one in which agents propose and dispose, as in the deferred-acceptance algorithm of Gale and Shapley, and a second one in which potential couples are banned from the system until a solution is reached or the infeasibility of the stable matching is discovered (also possible to be discovered during the first phase).

Tan [97] studied necessary and sufficient conditions for the existence of stable matchings in the roommates problem as he argued that Irving’s algorithm, despite

reporting the non-existence of a stable solution, does not clearly illustrate the reason. He proved that an instance of the roommates problem has a stable matching if and only if it does not allow the existence of *stable partitions* with *odd parties*. A stable partition is a partition of the set D in coalitions of any size such that no pair of agents from different coalitions both prefer to be together. Stable partitions can be seen as partitions of D in cycles (parties) where each agent points to her most desired partner.

Many extensions of the roommates problem to the *transferable utility* setting have been done. As Balinski and Ratier, Eriksson and Karlander [49] explored a graph-theoretic method to study the existence of stable allocations of the roommates with transferable utility problem. As Shapley and Shubik, Talman and Yang [96] as well as Chiappori et al. [34], used the linear programming approach to solve their problems. As Demange et al. in their first algorithm, Andersson et al. [10] designed a *price adjustment process* that computes, under integral payments, a stable allocation or disproves its existence in finite time.

Shioura [94] made the connection between roommates with transferable utility and the assignment game of Shapley and Shubik [93]. More precisely, Shioura reduced his problem to a particular assignment game in an auxiliary bipartite graph and proposed an extension of the algorithm of Andersson et al. [10] to compute a stable allocation in case of existence.

Up to our knowledge, the only extension of the roommates problem to the *non-transferable utility* domain has been made by Alkan and Tuncay [9]. In their model each couple of agents (d_1, d_2) has a decreasing and continuous *partnership* function $u_{1,2}$ that, given the payoff of one of the agents $f_1 \in \mathbb{R}$, it outputs the payoff of the partner $f_2 = u_{1,2}(f_1)$. The authors proved that the set of stable allocations is either empty or equivalent to the set of *realizable balanced aspirations*. Balanced aspirations are payoff profile $f \in \mathbb{R}^{|D|}$ at the ones no agent in the market is *overdemanded*. Formally, we define the *demand set* of agent d at f as,

$$P_d(f) := \{d' \in D : f_d = u_{d,d'}(f_{d'})\}$$

Then, f is a balanced aspiration if no agent belongs to more than one demand set. Balanced aspiration always exists. Alkan and Tuncay designed a *market procedure* (an analogous mechanism to the one of Andersson et al. for the continuous non-transferable setting) that, in polynomial time, outputs an ε -balanced aspiration⁷. Whenever a balanced aspiration f is realizable by an allocation, that is, there exists a matching μ such that, $f_d = u_{d,\mu(d)}(f_{\mu(d)})$ if d is matched, and equal to \underline{f}_d otherwise, where \underline{f}_d is the individually rational payoff of agent d (her utility for being single), the pair (allocation, aspiration) is stable. However, this is not always the case and balanced aspirations may need to be implemented by semi-allocations, i.e., relaxed matchings in which agents can be matched with two partners and their final payoff are the average of those received from each partner (probably related to the fact of having stable partitions with odd-parties of size at least three).

⁷Recall that for matching problems with continuous utilities finding exact solutions is still an open problem.

Roommates with ordinal preferences, transferable utility, and non-transferable utility can be seen as particular cases of *hedonic games* [32, 42] in which agents within a set D seek to create coalitions of any size and players receive a utility depending on the agents in their same coalition. Preferences can be considered in many ways. Agents may have *Boolean preferences* [19], either they like or dislike the coalition; *friend-oriented preferences* [72], each agent partitions the set of other agents into friends, enemies, and neutral agents, and gets payoff 1, -1 , and 0 respectively, for each of those agents in her coalition; or *enemy-oriented preferences* where agents ranked as enemies give utility $-n$ when they are present in the coalition; among others. Utilities can be of many ways as well. Agents can have *additive separable* [18], *fractional additive separable* [17], or *role-based* payoff functions [95], among others.

Other extensions in the literature of stable matching concern the *temporality* of the problem. Practical applications as *kidney allocation* [27, 103] or *ride-sharing* [8, 26, 101] have showed that agents (or items) within the market may not be present at the same time as they may arrive at different moments. Many years passed until the stable marriage problem of Gale and Shapley finally got its first dynamic extensions. Damiano and Lam [38] defined a finite repeated-time model in which agents can change partners between periods. All agents are present in the market from the beginning and utilities depend on the ones obtained at each period.

Caldentey et al. [31], Adan and Weiss [6], Bušić et al. [30], and Adan et al. [5], among others, worked in a *stochastic matching model* in which agents (or items) arrive to the market in a discrete-time process following probability distributions. Items within each side have different types and a fixed bipartite graph gives the compatibility links between items from different sides.

A particular issue raises in dynamic matching market models. In many settings, agents cannot decide when or with who get matched as they have to follow predefined matching policies. For example, in kidney allocation patients are ranked according to a point system and, as soon as an organ arrives, it is assigned to the highest ranked patient subject to compatibility. In ride-sharing, as soon as a rider arrives at the market, the closest driver is assigned to her in order of minimizing the waiting time of the customer. One of the most studied matching policies is *first-come-first-served* (FCFS) [5, 6, 30, 31, 35, 82] in which arriving agents get matched to the oldest compatible agent at the other side. Matching policies that match agents as soon as they arrive are called *greedy policies* and stochastic matching models with greedy policies can be associated with *Markov processes*. Adan and Weiss [6] and Adan et al. [5] defined several discrete-time Markov chains, found conditions for ergodicity, and derived their stationary distributions.

Matching games

In real-life bilateral markets, to be attractive, an agent can take actions that cannot be modeled by monetary transfers. When a firm hires a worker, it can combine

the monetary transfer with employee perks: medical insurance, gym, extra time-off, flexible schedule, childcare assistance, and days of remote work. The worker can promise to be flexible, work hard, learn new technologies, and be respectful of the company code of conduct. When a university hires a professor, it can reduce or increase its teaching duties, require a minimum number of top publications, ask for some responsibilities in the department, etc. The professor can promise to publish in top journals, be an excellent teacher, apply for grants, accept some responsibilities, organize a seminar, and supervise Ph.D. students. All those actions are individual decisions that can be put explicitly or implicitly in a contract but each agent is responsible for her part. Each agent will do what is needed to be accepted by the other party and will refuse to engage if she judges the partner's proposition as insufficient.

In this thesis, we present a novel matching market model by supposing that individual members of a couple or coalition obtain their payoffs as the output of a strategic game, simultaneously played with the matching. We study three models, generalization of the three problems presented by Gale and Shapley in their seminal paper [53]:

1. One-to-one matching games,
2. One-to-many matching games, and,
3. Roommates matching games

For each of them, we will introduce the model, challenges, and results. Then, we will study a fourth model, a dynamic matching market in continuous time in which agents arrive and leave following probability distributions.

1. *One-to-one matching games* considers two finite sets of agents D and H , doctors and hospitals, respectively, who seek to match in couples. Given a couple $(d, h) \in D \times H$, we endow them with a strategic two-player game

$$G_{d,h} = (X_d, Y_h, f_{d,h}, g_{d,h})$$

where X_d is d 's action/strategy set, Y_h is h 's action/strategy set, and $f_{d,h}, g_{d,h} : X_d \times Y_h \rightarrow \mathbb{R}$ are the utility functions of d and h , respectively. Remark the dependence of the game on the couple's identity: matching with a different partner may imply playing a different game. Agents may prefer to remain single as they will be endowed with *individually rational payoffs* $\underline{f} \in \mathbb{R}^{|D|}$ and $\underline{g} \in \mathbb{R}^{|H|}$.

Definition. A **matching game** is any tuple Γ given by,

$$\Gamma = (D, H, (G_{d,h} : d \in D, h \in H), \underline{f}, \underline{g})$$

If a couple (d, h) gets matched, player d chooses $x_d \in X_d$ and player h chooses $y_h \in Y_h$, the agents' final utilities in the allocation are $f_{d,h}(x_d, y_h)$ and $g_{d,h}(x_d, y_h)$, respectively. An *allocation* of the one-to-one matching game is a triplet $\pi = (\mu, \vec{x}, \vec{y})$, with μ a (one-to-one) *matching* between D and H (possibly letting some agents

unmatched), $\vec{x} = (x_d)_{d \in D} \in \prod_{d \in D} X_d$ a *strategy profile* for all agents in D , and $\vec{y} = (y_h)_{h \in H} \in \prod_{h \in H} Y_h$ a *strategy profile* for all agents in H . Given an allocation π , we define the *players' utilities* by,

$$f_d(\pi) := \begin{cases} f_{d,\mu(d)}(x_d, y_{\mu(d)}) & \text{if } d \text{ is matched} \\ \underline{f}_d & \text{otherwise} \end{cases}$$

$$g_h(\pi) := \begin{cases} g_{\mu(h),h}(x_{\mu(h)}, y_h) & \text{if } h \text{ is matched} \\ \underline{g}_h & \text{otherwise} \end{cases}$$

The first property that we will ask for our allocations is *individual rationality*.

Definition. An allocation π is **individually rational** if no agent receives less than her individually rational payoff.

Example 1. Consider a matching game with only one agent d and one agent h , both having strictly positive IRPs $\underline{f}_d = \underline{g}_h = \delta > 0$. Suppose that, if they agree to match, they play the following prisoners' dilemma G ,

		Agent h	
		Cooperate	Betray
Agent d	Cooperate	$2\delta, 2\delta$	$-\delta, 3\delta$
	Betray	$3\delta, -\delta$	$0, 0$

Notice that matching and playing the Nash equilibrium of G is Pareto-dominated by remaining single. In other words, the allocation in which agents match and play the Nash equilibrium is not individually rational. Intuitively, to have a stable allocation in which agents match, players should cooperate with some positive probability. \square

Example 2. A matching problem with linear transfers, as the assignment game of Shapley and Shubik, can be represented by a family of constant-sum games where the set of strategies are $X_d = Y_h = \mathbb{R}^+$, and the payoff functions are $f_{d,h}(x_d, y_h) = -x_d + y_h + a_{d,h}$ and $g_{d,h}(x_d, y_h) = x_d - y_h + b_{d,h}$, with $a_{d,h}$ and $b_{d,h}$ representing the utility of being with the partner when there is no transfer.⁸ \square

Suppose that some (centralized or decentralized) process leads to an allocation where agents are matched in pairs and each matched player is intended to play some action. We want to formulate the necessary conditions for that allocation to be *sustainable*. Two solutions concepts can be defined depending on the players' level of *commitment* before they play their game. Commitment can be related to the capacity of the agents to sign binding contracts at the moment of getting matched. Moreover, commitment is a key assumption in the literature of stable matching with

⁸The game of transfers is a constant-sum game as the sum of payoffs $f_{d,h}(x_d, y_h) + g_{d,h}(x_d, y_h) = a_{d,h} + b_{d,h}$ does not depend on x_d or y_h . Strategically, this is equivalent to a zero-sum game and it is a particular instance of a strictly competitive game.

endogenous utilities (such as matching with transfers and matching with contracts). The literature on matching does not describe the precise commitment process but rather defines a static notion of stability and proves, using for example a *deferred-acceptance* algorithm, the existence of such a stable allocation. Following this idea, we study the algorithmic results for both submodels.

The first solution concept studied is when matched couples **cannot commit** e.g. they cannot sign a binding contract before they play their game. In this case, for the players not to deviate from the intended actions, these last must constitute a *Nash equilibrium* of their game.

Definition. An individually rational allocation $\pi = (\mu, \vec{x}, \vec{y})$ is,

1. **Nash stable** if all matched couples play a Nash equilibrium,
2. **Pairwise stable** if there is no pair of agents (d, h) and Nash equilibrium (x'_d, y'_h) of $G_{d,h}$ such that,

$$f_{d,h}(x'_d, y'_h) > f_d(\pi) \text{ and } g_{d,h}(x'_d, y'_h) > g_h(\pi)$$

Pairwise stability is the natural extension of Gale-Shapley's stability condition to the matching games without commitment setting. Using a novel deferred-acceptance with competitions algorithm, it is proved that whenever all games $G_{d,h}$ admit a non-empty and compact set of Nash equilibria, a Nash stable and pairwise stable allocation exists. In addition, a *semi-lattice structure* holds: the maximum between two Nash stable and pairwise stable allocations for each side's preferences is stable. When all games $G_{d,h}$ have a unique Nash equilibrium payoff (as in zero-sum games or, more generally, strictly competitive games), the model becomes a classical stable marriage problem and so, the full lattice structure is recovered. Moreover, the duality between the maximum for one side and the minimum for the other side is obtained as well.

It is important to remark that games with monetary transfers, such as the works of Shapley-Shubik [93] and Demange-Gale [39], are constant-sum games where the unique Nash equilibrium is $(x_d^*, y_h^*) = (0, 0)$, as the null transfer is a strictly dominant strategy. For positive transfers to occur, players must be able to commit. This is implicitly assumed in the literature of matching with transfers. In other words, the model of matching games without commitment, although natural, fails to capture the classical models of matchings from the literature. Let us illustrate this.

Example 3. Consider a matching game with only two players d and h , both having a fixed and positive utility $\delta > 0$ for remaining single (their outside option), such that, in case the agents decide to match together, they play a constant-sum game as in Example 2, $G_{d,h} = (\mathbb{R}^+, \mathbb{R}^+, f_{d,h}, g_{d,h})$ with payoff functions given by,

$$f_{d,h}(x_d, y_h) = -x_d + y_h + 10\delta, \quad g_{d,h}(x_d, y_h) = x_d - y_h, \forall x_d, y_h \geq 0$$

The only Nash equilibrium of the transfer game $G_{d,h}$ is the null transfer $(x_d, y_h) = (0, 0)$, as any positive transfer is a dominated strategy. Therefore, an allocation in

which the agents are matched is Nash stable if and only if they do not make transfers. For a null transfer profile, notice agent h gets utility 0. However, she is better off being single as she gets δ . Therefore, a Nash stable allocation is individually rational if and only if the agents are not matched. \square

The second solution concept studied corresponds to the one in which players **can commit** (e.g. by signing binding contracts). Making an abuse of notation, we call as well pairwise stable to the extension of Gale-Shapley's stability to the model under commitment. Indeed, the stability notions of both settings, with or without commitment, will be captured by a more general.

Definition. An individually rational allocation $\pi = (\mu, \vec{x}, \vec{y})$ is **pairwise stable** if no pair of agents (d, h) can jointly deviate to some strategy profile (x'_d, y'_h) in their game $G_{d,h}$ that Pareto improves their payoffs, i.e.,

$$f_{d,h}(x'_d, y'_h) > f_d(\pi) \text{ and } g_{d,h}(x'_d, y'_h) > g_h(\pi)$$

Notice that agents are not restricted to play in their Nash equilibrium set anymore. The previous definition is the natural extension of Gale-Shapley's stability to the setting with commitment. A similar algorithm to the one used in the setting without commitment allows us to prove that, for any setting in which all the strategic games $G_{d,h}$ have compact Pareto-optimal strategy sets and continuous payoff functions, there exists a pairwise stable allocation. A semi-lattice structure holds as well for the set of pairwise stable allocations: the maximum between two pairwise stable allocations for each side's preferences remains stable. The full lattice structure is recovered when all games $G_{d,h}$ are constant-sum games (or, more generally, strictly competitive games).

Unlike the model without commitment, matching games with commitment successfully capture the matching models from the literature. For example, Shapley-Shubik's and Demange-Gale's models are particular instances of matching games with commitment models where all the strategic games are strictly competitive. Pairwise stability coincides with their notions of stability and our deferred-acceptance with competitions algorithm computes stable solutions for their models. In addition, we recover the lattice structure of Demange and Gale's model.

Players are rational and choose actions optimally. As agents can commit, playing optimally does not necessarily imply playing in equilibrium. In a setting with commitment agents maximize their utilities subject to satisfy the partners' outside options: the potential utilities they can obtain with somebody else. For example, a firm has to match the market salary of a worker in order to keep her. From another point of view, having an outside option (given by the market salary or by the salary offered by a competitor firm) allows to the worker to negotiate her salary with the firm. We prove that a **constrained Nash equilibrium** condition must naturally hold.

Definition. A pairwise stable allocation $\pi = (\mu, \vec{x}, \vec{y})$ is **negotiation proof** if for any couple $(d, h) \in \mu$ and any $(s_d, t_h) \in X_d \times Y_h$, it holds,

1. If $f_{d,h}(s_d, y_h) > f_d(\pi)$ then, $(\mu, (\vec{x}_{-d}, s_d), \vec{x})$ is not pairwise stable,

2. If $g_{d,h}(x_d, t_h) > g_h(\pi)$ then, $(\mu, \vec{x}, (\vec{y}_{-h}, t_h))$ is not pairwise stable.

An allocation $\pi = (\mu, \vec{x}, \vec{y})$ is *renegotiation proof* if any profitable deviation of a player in her game decreases the partner's payoff below her market outside option. Said differently, fixing y_h, x_d must maximize the payoff of player d under the constraint of participation of player h , and vice-versa. We define this kind of strategy profile as *constrained Nash equilibria* (see below for the formal definition).

Putting all together, our solution concept combines a cooperative notion (Gale-Shapley's pairwise stability) with a non-cooperative notion (a generalized Nash equilibrium condition). A similar solution concept is used in network formation games: fixing the network, players' actions must maximize their payoffs, and for each link in the network, both players must agree to form that link (see Jackson and Wolinsky [69] or Bich and Morhaim [28]). The one-to-one matching games model can be seen as a particular network game model where only bipartite graphs are possible and a link is formed if the two extreme players agree to match.

The concept of renegotiation-proofness has received particular interest in infinitely repeated games and mechanism design [2, 3, 12, 51, 86, 92, 99]. Our notion is more closely related to Dewatripont [41] who considers a setting in which agents can achieve agreement on contracts due to the existence of third parties.

We prove that an allocation is renegotiation proof if and only if all couples play constrained Nash equilibria for their reservation prices.

Definition. Let (d, h) be a couple in an allocation and suppose that f_0 and g_0 represents the minimum payoffs that d and h , respectively, are willing to accept in order of being matched with the other one. In other words, f_0 and g_0 are the agents' *reservation prices* or *outside options*. Then, their strategy profile (x_d, y_h) is a (f_0, g_0) -**constrained Nash equilibrium** if,

$$\begin{aligned} f_{d,h}(x_d, y_h) &= \max\{f_{d,h}(s, y_h) : g_{d,h}(s, y_h) \geq g_0, s \in X_d\} \\ g_{d,h}(x_d, y_h) &= \max\{g_{d,h}(x_d, t) : f_{d,h}(x_d, t) \geq f_0, t \in Y_h\} \end{aligned}$$

We define a class of strategic games (called *feasible games* (Definition 3.4.1)) as those games that admit the existence of constrained Nash equilibria and prove that: (a) when all games $G_{d,h}$ are feasible, a novel *renegotiation process*, if it converges, reaches a pairwise-strategy-proof stable allocation and (b) this new algorithm converges when all games are constant-sum, strictly competitive, potential or infinitely repeated. As strictly competitive games are feasible, Shapley-Shubik's and Demange-Gale's results are recovered and refined.

The proof that a game is feasible uses the properties of the game. In other words, the proof is game dependent. For example, for potential games, we prove the existence of constrained Nash equilibria by maximizing the potential function over an appropriated set. For each of the four feasible games mentioned we design an *oracle* to compute constrained Nash equilibria (e.g. maximizing the potential function) which guarantee the convergences of the renegotiation process.

For most of the games, the algorithms designed to compute pairwise-renegotiation-proof allocations obtain, in addition, allocations in which all couples play Pareto-optimal strategy profiles. For the deferred-acceptance with competitions algorithm, it is enough to restrict the strategy profiles of the agents to their Pareto-optimal sets. For the renegotiation process, it is enough to consider oracles that choose Pareto-optimal constrained Nash equilibria. For games such as zero-sum, strictly competitive, or infinitely repeated games, our oracle manages to select a Pareto-optimal constrained Nash equilibria. However, this is not the general rule as for many games, e.g. potential games such as the *prisoners' dilemma*, Pareto-optimality and stability are incompatible.

An interesting setting is obtained for one-to-one matching games with commitment in which couples play *non-degenerated* infinitely repeated games.

Definition. We say that an infinitely repeated game is **non-degenerated** if the set of uniform equilibrium payoffs E has a non-empty interior.

Let us better explain this point. Consider two games, a *matching pennies* G_1 , and a *prisoners' dilemma* G_2 , both of them expressed in the following payoff matrices,

		Player 2		
		G_1	A	B
Player 1	A		1	-1
	B		-1	1

Table 3: Matching pennies

		Player 2		
		G_2	Cooperate	Betray
Player 1	Cooperate		2, 2	-1, 3
	Betray		3, -1	0, 0

Table 4: Prisoners' dilemma

Matching pennies is a zero-sum game with value $w = 0$ achieved when each agent plays A and B with equal probability. Prisoners' dilemma is a potential game with Nash equilibrium payoff $(0, 0)$ achieved when both players betray. Suppose we repeat infinitely many times these games. For each of them we can define the sets of *feasible* and *non-punished* payoff profiles E_1 and E_2 , respectively as,

$$E_1 := \{(f, g) \in \text{co}((-1, 1), (1, -1)) : f \geq 0, g \geq 0\} \subseteq \mathbb{R}^2$$

$$E_2 := \{(f, g) \in \text{co}((2, 2), (-1, 3), (3, -1), (0, 0)) : f \geq 0, g \geq 0\} \subseteq \mathbb{R}^2$$

where $\text{co}(\cdot)$ states the convex envelope. Figure 2 shows the sets E_1 and E_2 . Set E_1 corresponds to a single point in \mathbb{R}^2 . In particular, E_1 has an empty interior. On the other hand, the set E_2 has a non-empty interior as we can observe in the figure. By the Folk theorem of Aumann-Shapley [15], we know the sets E_1 and E_2 correspond to the sets of *uniform equilibrium payoffs* of each game. Therefore, we obtain that the infinitely repeated version of the matching pennies is a degenerated game, while the infinitely repeated version of the prisoners' dilemma is non-degenerated.

The dynamic aspect of repeated games allows the couples to forget the commitment initially imposed by the model as agents can punish their partners for their

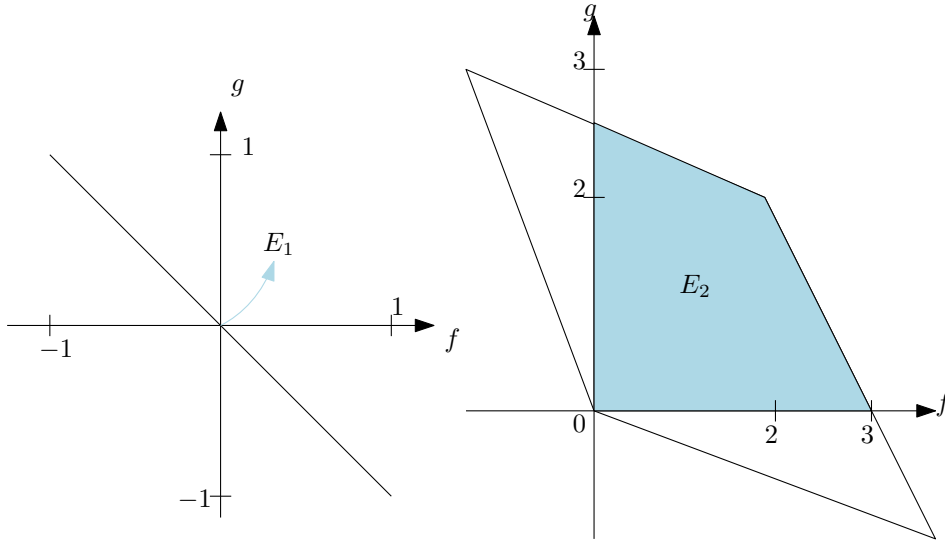


Figure 2: Feasible and non-punished payoff profiles sets

deviations in actions. To be more precise, for non-degenerated games as the repeated version of the prisoners' dilemma above, constrained Nash equilibrium payoffs are always (weakly) Pareto-dominated by a uniform equilibrium payoff. In particular, we can always replace the constrained Nash equilibria of the couples with uniform equilibria without affecting the pairwise stability and strategy proofness of the allocation. Therefore, infinitely repeated (non-degenerated) games satisfy the strongest notion of stability able in our model: *Pareto-optimality, pairwise stability, and Nash stability*.

Example 3 above showed the limits of the one-to-one matching games model without commitment, namely, it fails to capture classical matching models from the literature as the one of Shapley and Shubik. This is not the case for a model with commitment as all the models discussed at the beginning of this chapter are successfully captured. Due to this, the following two models studied in the thesis consider commitment as a key assumption.

2. *One-to-many matching games with substitutes* (and commitment), the generalization of the *college admissions problem*, the second problem presented by Gale and Shapley in [53], as well as the generalization of one-to-one matching games with commitment, consider that agents within one side, the hospitals H from now on, can be matched with many partners at the same time. Formally, given a set $I \subseteq D$ allocated in $h \in H$, and given strategy profiles $\vec{x}_I \in \prod_{d \in I} X_d$, $\vec{y}_{h,I} = (y_{h,d})_{d \in I} \in Y_h^{|I|}$, the players' payoffs are given by,

$$f_{d,h}(x_d, y_{h,d}), \forall d \in I, \text{ and } g_{I,h}(\vec{x}_I, \vec{y}_{h,I})$$

that is, the payoff function of hospital h depends on the identity of the entire set of doctors I , the strategies played by all these doctors, and the strategy profile $\vec{y}_{h,I}$ played by h , where each $y_{h,d}$ represents the particular strategy played by h against the doctor d . Doctors' payoff functions are like in the one-to-one model.

Pairwise stability for one-to-one matching games considered only blocking pairs as only couples were allowed. Since agents in H can be matched with many partners now, we consider *Core stability*.

Definition. An allocation $\pi = (\mu, \vec{x}, \vec{y})$ is **blocked** by a coalition of doctors $I \subseteq D$ and a hospital h if there exist $(\vec{w}_I, \vec{z}_{h,I}) \in X_I \times Y_h^{|I|}$, such that,

$$\begin{aligned} \forall d \in I, f_{d,h}(w_d, z_{h,d}) &> f_d(\pi), \\ g_{I,h}(\vec{w}_I, \vec{z}_{h,I}) &> g_h(\pi), \end{aligned}$$

where $f_d(\pi)$ and $g_h(\pi)$ are the payoffs of d and h , respectively, in π . π is **Core stable** if it is individually rational and it is not blocked.

To obtain the existence of Core stable allocations we endow hospitals with additive separable payoff functions and quotas. Formally, we say that $h \in H$ has an *additive separable payoff function* and *quota* $q_h \in \mathbb{N}$, if,

$$g_{I,h}(\vec{x}_I, \vec{y}_h) = \begin{cases} \sum_{d \in I} g_{d,h}(x_d, y_{h,d}) & \text{if } |I| \leq q_h \\ -\infty & \text{otherwise} \end{cases}$$

with $g_{d,h} : X_d \times Y_h \rightarrow \mathbb{R}, \forall d \in I$

A generalization of the deferred-acceptance with competitions algorithm proposed for one-to-one matching games proves the existence of Core stable allocations for any one-to-many matching game in which agents in H have additive separable payoff functions and quotas, and all agents have compact strategy sets and continuous payoff functions⁹. The advantage of additive separability is that Core stability is reduced to *pairwise* stability (as the one for one-to-one matching games). In particular, we do not need to use a *cumulative offer mechanism* as the one of Hatfield and Milgrom but only a deferred-acceptance algorithm. Two important benefits come from this change of mechanism: (1) Cumulative offer mechanisms do not give a constructive way to compute stable allocations as they need fixed point theorems to guarantee convergence and (2) thanks to their simplicity, the complexity of deferred-acceptance algorithms can be properly studied. Our characterization of core stable allocations (called *stable** in [48]) through pairwise stable allocations is in line with the work of Echenique and Oviedo [48] (called *stable* in [48]). However, our algorithm does not require fixed points theorem to obtain its convergence as we do not iterate a mapping but run a deferred-acceptance like algorithm.

The separability of hospitals' payoff functions on one payoff function per doctor allows us to directly apply the renegotiation proofness notion as defined for one-to-one matching games to the one-to-many model, refining the models of Kelso-Crawford and Hatfield-Milgrom, among others. Indeed, the same renegotiation process can be used. In particular, we can prove the existence of Core stable and renegotiation proof

⁹The discontinuity of hospitals' payoff functions due to quotas does not present issues for the deferred-acceptance with competitions algorithm.

allocations for any matching game with zero-sum, strictly competitive, potential, or infinitely repeated games.

3. *Roommates matching game* is the generalization of the third problem presented by Gale and Shapley. Formally, we consider a set of agents D and endow to each player $d \in D$ with a strategy set X_d , and to any couple $(d_1, d_2) \in D \times D$ with the strategic game $G_{d_1, d_2} = (X_{d_1}, X_{d_2}, f_{d_1, d_2}, f_{d_2, d_1})$.

An allocation in this setting is a pair $\pi = (\mu, \vec{x})$ where μ is a *one-to-one matching* and $\vec{x} \in \prod_{d \in D} X_d$ is a *strategy profile*. Our pairwise stability notion captures the stability of the roommates problem in its many variants: ordinal preferences, transferable utility, and non-transferable utility.

We leverage the work of Alkan and Tuncay in roommates with non-transferable utility to study the existence of pairwise stable allocations. We use the *market procedure* designed in [9] to compute a payoff profile $\bar{f} \in \mathbb{R}^{|D|}$ that, whenever an allocation π can *implement* it, that is, for any $d \in D$, π satisfies $f_d(\pi) = \bar{f}_d$, with,

$$f_d(\pi) := \begin{cases} f_{d, \mu(d)}(x_d, x_{\mu(d)}) & \text{if } d \text{ is matched} \\ \underline{f}_d & \text{otherwise} \end{cases}$$

where \underline{f}_d is the individually rational payoff of player d , π is pairwise stable.

As only couples are allowed in the model, our renegotiation proofness notion can be directly applied, refining the solution concepts of the literature. In particular, our renegotiation process works in any feasible game with an oracle that achieves the convergence of the algorithm.

We finish the introduction of the matching games model with the following observation. There are different ways of associating a strategic game to a Shapley-Shubik's [93], Demange-Gale's [39], or Hatfield-Milgrom's [66] model, among others. Our Core and pairwise stability notions always coincide with their stability, but each strategic game modeling induces, thanks to renegotiation proofness, a different selection among the stable outcomes. For example, suppose that in Shapley-Shubik's model, there is only one seller and one buyer, the seller has cost c for the house and the buyer has a value v for it. If $v < c$, there is no possibility of trading. Otherwise, the surplus $v - c$ is positive and any price p between c and v corresponds to a pairwise stable allocation. If the strategic interaction between the buyer and the seller is an ultimatum bargaining game [1] where the seller is the first proposer (resp. the buyer is the first proposer), the pairwise stable and renegotiation proof selected outcome is $p = v$ (resp. $p = c$). On the other hand, if the strategic interaction is modeled by the Rubinstein alternating offers bargaining game [84] with equally patient players, the outcome is the Nash bargaining solution $p = (v - c)/2$. Therefore, different bargaining games between buyers and sellers induce, due to renegotiation proofness, different sharing of the surplus. This is not modeled in the standard literature with transfers and is one of the main contributions of our work.

The last model discussed in this thesis corresponds to a *dynamic matching market in continuous time*, and it is a joint work with Marco Scarsini. We consider a one-

to-one two-sided market with sides \mathcal{A} and \mathcal{B} , where agents arrive following *Poisson processes* with rates λ_A and λ_B , respectively. Agents can naturally leave the market due to death or tiredness at rates μ_A and μ_B , respectively. Agents within the market get matched following a fixed *matching policy* Φ , and coupled agents leave the market and never return. Given the state of the market at some time $t > 0$, namely (A_t, B_t) , an arriving agent k draws *uniform random utilities* in $[0, 1]$ over all the agents present at the time on the other side, representing the utility that k gets by matching with each of them. Similarly, the agents on the other side draw uniform random utilities in $[0, 1]$ for being with k .

Agents are endowed with individually rational payoffs $\rho_k \in [0, 1], \forall k \in \mathcal{A} \cup \mathcal{B}$, i.e., their utilities for being single, and never accept a matching with utility lower than ρ .

We study *greedy matching policies*, that is, matching policies that are triggered with every arrival, as *first-come-first-served*. Given a fixed greedy matching policy Φ , we consider the continuous-time Markov chain given by $(Z_t = (|A_t|, |B_t|)_{t \geq 0})$ that follows the number of agents in the market at every moment $t \geq 0$. Continuous-time Markov chain can be studied by considering their *jump chain* $(W_n)_{n \geq 0}$ indexed only on the jump times, that is, whenever an agent arrives or naturally leaves.

Agents proposed to be matched with a partner accept only if the utility obtained is at least their individually rational payoff. Therefore, if a is an agent arriving to \mathcal{A} at the jump time n , the probability that she gets matched is given by,

$$p_a := \mathbb{P}(a \text{ gets matched}) = \begin{cases} (1 - \rho_a) \cdot (1 - \prod_{b \in B_n} \rho_b) & \text{if } B_n \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

The probabilities $(p_a, p_b)_{a \in \mathcal{A}, b \in \mathcal{B}}$ cannot depend on the identity of the agents presented on the market at time n if we want to obtain a *time-homogeneous* Markov chain.

Definition. We say that agents have **anonymous IRPs** whenever the probabilities $(p_a, p_b)_{a \in \mathcal{A}, b \in \mathcal{B}}$ do not depend on the identity of the players but only on the number of agents at the other side of the market.

Taking anonymous individually rational payoffs the Markov chain $(W_n)_n$ becomes a *random walk* in \mathbb{N}^2 . The probability of passing to a higher state is always lower than passing to a lower state since the random walk cannot explode: agents can leave the market in two ways, either because they naturally left or because they got matched. Matchings occur at a similar rate to arrivals (depending on the IRPs values) therefore, the random walk decreases its values more often than increases them. Thus, our random walk is *non-homogeneous in space*.

Using Foster's criterion we prove that $(W_n)_n$ is positive recurrence (therefore, proving that $(Z_t)_t$ is positive recurrent as well). Then, imposing *reversibility*, namely, that for any $(i, j) \in \mathbb{N}$ it must hold,

$$\pi(i)P(i, j) = \pi(j)P(j, i)$$

where π is a *stationary measure* of $(W_n)_n$ and P its *transition matrix*, we find a recursive expression for π and deduce that the stationary distribution of $(Z_t)_t$ has a product-form.

Finally, empirical experiments show that *deferred-acceptance* greedy matching policies, that is, running Gale-Shapley’s algorithm at every arrival, outperform other greedy matching policies as first-come-first-served and random allocation.

Our contributions

In this thesis, we present the novel *matching games model* in which finitely many agents match and receive utilities as the outputs of strategic games. Two submodels are obtained depending on the level of commitment of the agents. We studied more extensively matching games with commitment as they are more challenging mathematically but also capture and refine a broad number of models in the literature such as: the *stable marriage problem*, the *college admissions problem*, and the *roommates problem* of Gale and Shapley [53], the *assignment game* of Shapley and Shubik, the *matching with transfers* problem of Demange and Gale [39], the *job matching market* of Kelso and Crawford [71], the *matching with contracts* of Hatfield and Milgrom [66], the *roommates with transferable utility* problem [10], and the *roommates with non-transferable utility* problem [9]. Our notions of Core and pairwise stability are equivalent to the stability notions considered in each of these problems, and our novel renegotiation proofness notion refines their stable solutions.

Chapter 2 designs a *deferred-acceptance with competitions* (DAC) algorithm for two-sided matching markets to compute pairwise stable allocations under classical game theory assumptions. Moreover, the algorithm runs in a bounded number of polynomial-time iterations for many well-known games from the literature of game theory. We extend classical results in two-sided matching markets as the *lattice structure* of the set of stable allocations and the *optimality* of the DAC algorithm for the proposer side to our new setting.

Among the most important contributions, Chapter 3 defines a novel notion of stability, *renegotiation proofness*: Agents within the same couple are rational and must best-reply to the partner. However, as agents can commit in actions, best replies are constrained to guarantee some threshold payoff to the partners, their *outside market option* (minimum payoff that partners may accept to get matched). Renegotiation proofness is defined over pairwise stability, refining all the stability concepts from the literature that the model captures.

Chapter 3 characterizes renegotiation proofness as a *constrained Nash equilibrium* (CNE) condition and designs a renegotiation process to compute pairwise stable and renegotiation proof allocations whenever the strategic games allow the existence of CNE and the process converges. Moreover, the convergence of the algorithm is guaranteed for many well-known games, including Shapley-Shubik’s and Demange-Gale’s models. In addition, for the same games as the DAC algorithm, the renegotiation process has a bounded number of polynomial-time iterations.

Chapter 4 presents a novel two-sided one-to-many matching games model in which agents within one of the sides, namely the set H , can be matched with many agents

at the same time, successfully capturing the seminal model of Hatfield and Milgrom, *matching with contracts* [66].

Matching with contracts considers a one-to-many matching market in which agents are endowed with *choice functions*. The choice functions output the most preferred contract/contracts of each agent, given a set of possible contracts in the market. The authors proved the existence of stable allocations whenever agents in H have substitute contracts.

Aygün and Sönmez [16] exposed that different models are obtained if agents' choice functions are treated as primitives of the model or they are induced from preference rankings. Although Hatfield and Milgrom's model belongs to the second type, they treated their choice functions as primitives. Therefore, to truly guarantee the existence of stable allocation, an extra assumption, namely the *irrelevance to rejected contracts*, is required as well. One of the main contributions of Chapter 4 is the fact that our one-to-many matching games model works directly with the primitives of the problem (the payoff functions) allowing us to: (1) better understand the reasons for the existence of Core stable allocations, (2) understand the relationship between pairwise and Core stability, (3) refine the model with contracts by applying the notion of renegotiation proofness, (4) to consider infinitely many contracts in the market, and (5) to study the complexity of the proposed algorithms.

Point (2) is particularly interesting. Whenever agents in H have additive separable payoff functions, we prove that Core stability can be reduced to pairwise stability. In particular, *cumulative offer mechanism*, as the one used by Hatfield and Milgrom, can be replaced by *deferred-acceptance* algorithms. Therefore, in Chapter 4 we extend the DAC algorithm proposed for one-to-one matching games to obtain the existence of Core stable allocations in the one-to-many setting. Under the same additive separability condition, we extend renegotiation proofness to the one-to-many setting in Chapter 4 achieving point (3) with the same computational efficiency as the one for one-to-one matching games (point (5)).

A third novel model is introduced in Chapter 4, namely, the *roommates matching game*, an extension of the model of roommates with non-transferable utility to the case in which couples of agents get their utilities as the outcomes of a strategic game. As for matching with contracts, our roommates matching game refines the solution concepts from the literature by studying renegotiation proof allocations.

Finally, Chapter 6 introduces a joint work with Marco Scarsini, a dynamic matching market model in continuous-time where agents can arrive and leave the market following stochastic processes. We compare different *greedy matching policies* and study the necessary conditions for the existence of *stationary distributions* with a product-form.

The thesis is structured as follows. Chapter 1 introduces the model of matching games without commitment and studies the existence and computation of Nash-pairwise stable allocations. Chapter 2 introduces the model of matching with commitment and studies the existence and computation of pairwise stable allocations. Chapter 3 introduces renegotiation proofness, our novel refinement of the stability

notions in the literature, and studies the existence and computation of renegotiation proof allocations. Chapter 4 introduces two extra models of matching games: one-to-many matching games, and roommates matching games. For each of these models, the study of the existence and computation of Core stable and renegotiation proof allocations is conducted. Chapter 5 is dedicated to the complexity study of all the algorithms proposed in the previous chapters, proving their computational efficiency. Chapter 6 introduces a dynamic matching market model in which agents can arrive and leave during the time. The last chapter concludes the thesis, gives future research lines, and discuss some open problems.

1-to-1 Matching games without commitment: Nash stability

This chapter is devoted to studying the model of *one-to-one matching games without commitment*. One-to-one matching games, from now on, matching games, consider two finite sets of agents and match them in couples in such a way that no agent or pair of agents will have incentives to *deviate*. Matched agents receive utilities given by the outcome of a strategic game played against the partner. Therefore, deviations can occur in two ways: (1) couples of agents leaving their partners and matching together or (2) agents changing of strategy in their game.

The first kind of deviations recalls the *pairwise stability* of Gale and Shapley [53]. We will define a pairwise stability notion for our model capturing Gale-Shapley's one.

The second kind of deviations recalls us *Nash equilibria*. Agents cannot commit due to, for example, the non-existence of bilateral contracts or the incapability of the agents to punish (in utility) their partners. Thus, agents do not have any guarantee that partners will not deviate in strategies if they have profitable deviations. Therefore, for the allocation to be stable in actions, all couples will have to play Nash equilibria within their games.

We will establish the existence of allocations satisfying both notions of stability at the same time through a deferred-acceptance with competitions (DAC) algorithm and under classical game theory assumptions. Structural properties such as the lattice structure of the set of stable allocations will be then studied.

Many models can be mapped into a matching game, e.g. the **stable marriage problem** of Gale and Shapley [53], the **assignment game** of Shapley and Shubik [93], or the **matching with transfers** problem of Demange and Gale [40]. We will see how accurate is our model of matching games without commitment to capture these models.

The chapter is structured as follows. Section 1.1 presents the model of matching games and states many examples. Section 1.2 defines the notions of stability of the model. Section 1.3 studies the existence of stable allocations. Section 1.4 studies the lattice structure of the set of stable allocations. Section 1.5 shows a numerical example of the deferred-acceptance with competitions algorithm designed to compute stable allocations. Finally, Section 1.6 concludes the chapter.

1.1 Matching games model

We consider two finite sets of agents D and H that we refer to as doctors and hospitals¹. The cardinalities of D and H are denoted $|D|$ and $|H|$, respectively, and typical elements are denoted $d \in D$ and $h \in H$.

Definition 1.1.1. A **matching** μ is a mapping between D and H where each agent on one side is matched to at most one agent on the other side. If $d \in D$ and $h \in H$ are matched in μ , we will denote indistinctly $h = \mu(d)$ or $d = \mu(h)$.

When a couple $(d, h) \in D \times H$ forms, they get their payoffs as the output of a strategic game $G_{d,h} := (X_d, Y_h, f_{d,h}, g_{d,h})$, where X_d, Y_h are the strategy sets of doctor d and hospital h , respectively, and $f_{d,h}, g_{d,h} : X_d \times Y_h \rightarrow \mathbb{R}$ are their payoff functions. Denote by $X := \prod_{d \in D} X_d$ and $Y := \prod_{h \in H} Y_h$ the spaces of strategy profiles. Further assumptions (such as compactness and continuity) over the strategy sets and payoff functions will be specified later.

Definition 1.1.2. A **doctors action profile** (resp. **hospitals action profile**) is a vector $\vec{x} = (x_1, \dots, x_{|D|}) \in X$ (resp. $\vec{y} = (y_1, \dots, y_{|H|}) \in Y$). An **allocation** is a triple $\pi = (\mu, \vec{x}, \vec{y})$ in which μ is a matching, \vec{x} is a doctors action profile and \vec{y} is a hospitals action profile.

Given an allocation $\pi = (\mu, \vec{x}, \vec{y})$, the **players utilities** are defined by,

$$\begin{aligned} f_d(\pi) &:= f_{d, \mu(d)}(x_d, y_{\mu(d)}), \forall d \in D, \\ g_h(\pi) &:= g_{\mu(h), h}(x_{\mu(h)}, y_h), \forall h \in H, \end{aligned}$$

as basically, the payoff obtained in the game played against the partners.

It is natural to suppose that each agent has a utility of being single and that this utility is her/its **individually rational payoff** (IRP): she/it accepts a partner only if the payoff of their game is at least her/its IRP. Formally, each doctor $d \in D$ (resp. hospital $h \in H$) will be attributed a value $\underline{f}_d \in \mathbb{R}$ (resp. $\underline{g}_h \in \mathbb{R}$), which constitutes the utility of being single.

We extend the agent sets D and H by adding to each of them the so-called **empty players** d_0, h_0 who, in our future algorithms, will be respect the following rules: (1) empty players have empty strategy sets and null payoff functions, (2) they can be matched with as many agents as needed, but never between them, and (3) any player matched with an empty player receives her/its IRP as payoff. We denote $D_0 := D \cup \{d_0\}$ and $H_0 := H \cup \{h_0\}$.

Definition 1.1.3. A tuple $\Gamma = (D_0, H_0, \{G_{d,h} : (d, h) \in D \times H\}, \underline{f}, \underline{g})$ will be called a **matching game**.

¹Unlike Gale and Shapley who considered men and women, we prefer to use the notation of doctors and hospitals of Hatfield and Milgrom [66] to be consistent with the more general model presented in Chapter 4.

To illustrate our model, we consider the following leading examples.

Example 1.1.4. Consider a matching game with only one agent d and one agent h , both having strictly positive IRPs $\underline{f}_d = \underline{g}_h = \delta > 0$. Suppose that, if they agree to match, they play the following prisoners' dilemma G ,

		Agent h	
		Cooperate	Betray
Agent d	Cooperate	$2\delta, 2\delta$	$-\delta, 3\delta$
	Betray	$3\delta, -\delta$	$0, 0$

Notice that matching and playing the Nash equilibrium of G is Pareto-dominated by remaining single. Intuitively, to have a stable allocation in which agents match, players should cooperate with some positive probability. \square

Example 1.1.5. Consider a matching game with only one agent d and one agent h , both having strictly positive IRPs $\underline{f}_d = \underline{g}_h = \delta > 0$. Suppose that, if they agree to match, they play the following game G :

		Agent h	
		A	B
Agent d	A	$4\delta, \delta/2$	$0, 0$
	B	$0, 0$	$\delta/2, 4\delta$

With the same intuition as in Example 1.1.4, notice that none of the pure Nash equilibria of G should be stable. Indeed, in both pure N.E. there is a player that is worst off than being single, and therefore, she should not prefer to be matched. \square

Example 1.1.6. Consider a matching game with only one agent d and one agent h , both having strictly positive IRPs $\underline{f}_d = \underline{g}_h = \delta > 0$. Suppose that, if they agree to match, they play a constant-sum game

$$G = (\mathbb{R}_+, \mathbb{R}_+, f_{d,h}, g_{d,h}), \text{ such that for any } x, y \geq 0,$$

$$f_{d,h}(x, y) = 10\delta - x + y$$

$$g_{d,h}(x, y) = x - y$$

Game G corresponds to a transfer game in which each player increases her payoff thanks to the transfer of the partner, and decreases it due to her transfer. Since positive transfers are always strictly dominated by the null transfer, the only Nash equilibrium of G is $(x, y) = (0, 0)$. However, h is better off being unmatched than matching and receiving a null transfer. Thus, matching and playing the Nash equilibrium of the game is unstable.

Notice that being single is Pareto-dominated by agent d offering a money transfer $x \in [\delta, 9\delta]$ to agent h and h accepting to be matched. \square

Example 1.1.7. Consider a market with a set of n buyers $N = \{1, \dots, n\}$, and one seller h of an indivisible good. Buyers in N have strictly positive values for the good, $(v_d)_{d \in N} \subseteq \mathbb{R}_+$. The seller has a reservation price c , strictly positive as well. Let us set the IRPs of the agents equal to 0. Since the good is indivisible, only one buyer can be matched with the seller. If a couple buyer-seller (d, h) is created and the monetary transfers (x_d, y_h) ² is agreed, the item is sold from h to d at the price $p = x_d - y_h$. In particular, the utilities of the players are $f_d(x_d, y_h) = v_d - p = v_d - x_d + y_h$, $g_h(x_d, y_h) = p - c = x_d - y_h - c$, and $f_{d'} = 0$ for any other buyer $d' \in N \setminus \{d\}$. If $c > \max\{v_i : i \in N\}$, notice no buyer is willing to pay for the good as they are better off being unmatched. \square

1.2 Nash and pairwise stability

The first condition we will ask to any allocation is *individual rationality*.

Definition 1.2.1. An allocation $\pi = (\mu, \vec{x}, \vec{y})$ is **individually rational** if for any $d \in D$ and any $h \in H$, $f_d(\pi) \geq \underline{f}_d$ and $g_h(\pi) \geq \underline{g}_h$.

Suppose d and h agree to match and intend to play, respectively, the actions x_d and y_h . If no specific reason forces them to respect that agreement (no binding contracts, no possibility of future punishment in repeated interaction, etc) then, for (x_d, y_h) to be stable, it must constitute a Nash equilibrium of $G_{d,h}$.

Definition 1.2.2. An allocation $\pi = (\mu, \vec{x}, \vec{y})$ is **Nash stable** if for any matched couple $(d, h) \in \mu$, $(x_d, y_h) \in \text{N.E}(G_{d,h})$, i.e., (x_d, y_h) is a Nash equilibrium of $G_{d,h}$.

As players can remain single or match a better partner, a pairwise stability condition à la Gale-Shapley must also be satisfied for an allocation to be stable.

Definition 1.2.3. An individually rational allocation $\pi = (\mu, \vec{x}, \vec{y})$ is **pairwise stable** if there is no $(d, h) \in D \times H$ and no Nash equilibrium $(x'_d, y'_h) \in \text{N.E}(G_{d,h})$ such that $f_{d,h}(x'_d, y'_h) > f_d(\pi)$ and $g_{d,h}(x'_d, y'_h) > g_h(\pi)$.

Pairwise stability asks for the non-existence of (Nash)-*blocking pairs*, i.e., there is no pair $(d, h) \in D \times H$, that can be paired and play a Nash equilibrium in their game strictly improving their payoffs in π . A Nash stable and pairwise stable allocation is called Nash-pairwise stable.

1.3 Existence and computation of Nash-pairwise stable allocations

The next theorem proves, under the usual Nash equilibrium existence conditions, that a Nash-pairwise stable allocation exists and can be computed algorithmically.

² $x_d \geq 0$ means buyer pays to the seller, $y_h \geq 0$ means the seller pays to the buyer.

Theorem 1.3.1. *If for any couple (d, h) the set of Nash equilibria of the game $G_{d,h}$ is non-empty and compact,³ and the payoff functions are continuous, then the set of Nash-pairwise stable allocations is also non-empty and compact.*

Instead of directly proving Theorem 1.3.1, we prove a more general one as it will be useful for the model with commitment studied in Chapter 2 as well. Suppose that each potential couple (d, h) is restricted to choose their contracts from a set $C_{d,h} \subseteq X_d \times Y_h$.

Definition 1.3.2. *An individually rational allocation will be **pairwise stable with respect to the family** $\mathcal{C} := (C_{d,h} : d \in D, h \in H)$, if no pair (d, h) can match together, play a strategy profile in their set $C_{d,h}$, and strictly increase their payoffs.*

Theorem 1.3.3. *Suppose that all the sets in the family \mathcal{C} are non-empty and compact, and the agents' payoff functions are continuous. Then, there exists a pairwise stable allocation with respect to the family \mathcal{C} .*

Theorem 1.3.1 is a corollary of Theorem 1.3.3 when considering $C_{d,h} = N.E(G_{d,h})$, $\forall (d, h) \in D \times H$. Indeed, pairwise stability is guaranteed by Theorem 1.3.3, while Nash stability holds as all couples are restricted to play only Nash equilibria. Theorem 1.3.3 is proved in two steps:

1. We design a deferred-acceptance with competitions (DAC) algorithm to compute an ε -approximation of a pairwise stable allocation (Definition 1.3.8).
2. As the sets $C_{d,h}$ are compact and the payoff functions are continuous, accumulation points as $\varepsilon \rightarrow 0$ exist and any of them will be a pairwise stable allocation.

The pseudo-code of the algorithm used in step 1 (Algorithm 1.1) is similar to the first of the two algorithms proposed by Demange, Gale, and Sotomayor [40]. Our deferred-acceptance with competitions algorithm takes one of the sides (the doctors for the rest of the chapter) and asks its unmatched agents to propose, one by one, a contract from the family \mathcal{C} to their most preferred option. The proposal is computed such that the proposed agent is always better off by accepting it. If the proposed agent is already matched a competition between the two agents is triggered. The winner remains and the loser proposes again. Therefore, each iteration of Algorithm 1.1 has two phases: a **proposal** and a **competition**. For a numerical example, check Section 1.5.

Let us explain the two phases that compose an iteration of the DAC algorithm.

Proposal phase. Let $d \in D'$ be the proposer. Given the current allocation π (initially empty) that generates a hospitals' payoff vector $g(\pi) = (g_h(\pi))_{h \in H}$, d computes his optimal proposal as,

$$(h, x, y) \in \arg \max \{f_{d,h}(x, y) : g_{d,h}(x, y) \geq g_h(\pi) + \varepsilon, h \in H_0, (x, y) \in C_{d,h}\} \quad (1.1)$$

³Whenever a game $G_{d,h}$ has convex and compact strategy sets, and utility functions are own-quasi concave and continuous, or discontinuous but better-reply-secure [88], the set of Nash equilibria is non-empty and compact.

Algorithm 1.1: Deferred-acceptance with competitions algorithm

```

1 Input:  $\Gamma = (D_0, H_0, (G_{d,h} : (d, h) \in D \times H), \underline{f}, \underline{g})$  a matching game,  $\varepsilon > 0$ 
2 Set  $D' \leftarrow D$  as the set of single doctors, and  $\underline{g}_h(\pi) \leftarrow \underline{g}_h, \forall h \in H$ 
3 while  $D' \neq \emptyset$  do
4   | Let  $d \in D'$ . Compute his optimal proposal
   |  $(h, x, y) \in \arg \max \{f_{d,h}(x, y) : g_{d,h}(x, y) \geq g_h(\pi) + \varepsilon, h \in H_0, (x, y) \in C_{d,h}\}$ 
5   | if  $h$  is single then
6   |   |  $d$  is automatically accepted
7   | end
8   | else
9   |   |  $d$  and  $\mu(h)$  compete for  $h$  as in a second-price auction. The winner
   |   | passes to be the new partner of  $h$  and the loser is included in  $D'$ 
10  | end
11 end

```

The solution of Problem (1.1) consists in h , the most preferred hospital of doctor d , and $(x, y) \in C_{d,h}$, the strategy profile that d proposes to h to play. As an abuse of notation, we may call optimal proposal only to (x, y) , omitting the proposed hospital h . Problem (1.1) is always feasible as d can always propose to h_0 . If h is single, d is automatically accepted and the algorithm picks a new proposer in D' .

Competition phase. If the proposed agent h is matched, namely with an agent d' , a competition between d and d' starts. In the stable marriage problem, the competition is the simple comparison between the places that d and d' occupy in h 's ranking. In our case, as agents have strategies, a competition is analogous to a *second-price auction*. Let β_d be the reservation price of d , solution to the following problem,

$$\beta_d := \max \{f_{d,h'}(x, y) : g_{d,h'}(x, y) \geq g_{h'}(\pi) + \varepsilon, h' \in H_0 \setminus \{h\}, (x, y) \in C_{d,h'}\} \quad (1.2)$$

Analogously, we compute $\beta_{d'}$. Reservation prices are the highest payoff that d and d' can get by matching with somebody else. In other words, these values represent the lowest payoffs that each agent is willing to accept to be with h . d 's bid λ_d (and analogously for d') is computed by,

$$\lambda_d := \max \{g_{d,h}(x, y) : f_{d,h}(x, y) \geq \beta_d, (x, y) \in C_{d,h}\} \quad (1.3)$$

The winner is the doctor with the highest bid. Finally the winner, namely d , pays the second highest bid. Formally, d solves,

$$\max \{f_{d,h}(x, y) : g_{d,h}(x, y) \geq \lambda_{d'}, (x, y) \in C_{d,h}\} \quad (1.4)$$

The loser is included in D' and a new proposer is chosen.

Remark 1.3.4. A defeated doctor cannot propose right away to the same hospital as she is unable to increase the hospital's payoff in ε . This is crucial for the convergence of the DAC algorithm.

Remark 1.3.5. The output of the DAC algorithm (Algorithm 1.1) corresponds to an ε -approximation. This is in line with the matching literature with transfer [40, 66, 71]. The problem of computing a 0-stable allocation remains open, in our case as well as in the literature with transfers.

We focus in proving that our DAC algorithm (Algorithm 1.1) ends in finite time and its output corresponds to an ε -pairwise stable allocation (Definition 1.3.8).

Theorem 1.3.6. *The DAC algorithm ends in finite time.*

Proof. Since the strategy sets are compact and the payoff functions are continuous, they are bounded. By construction, hospitals' payoffs strictly increase with every proposal. Thus, the algorithm ends in a finite number of iterations. \square

Remark 1.3.7. Due to the monotonicity of hospitals' payoffs, once a doctor matches with h_0 , she leaves the market and remains single forever.

Forcing the doctors to increase hospitals' payoffs by (at least) ε with every proposal guarantees the finiteness of the algorithm. However, we lose accuracy as the algorithm outputs an ε -pairwise stable allocation.

Definition 1.3.8. Let $\pi = (\mu, \vec{x}, \vec{y})$ be an allocation and $\varepsilon > 0$ fixed. A pair $(d, h) \in D \times H$ is an ε -**blocking pair** if there exists a strategy profile $(x'_d, y'_h) \in C_{d,h}$ such that

$$f_{d,h}(x'_d, y'_h) > f_d(\pi) + \varepsilon \text{ and } g_{d,h}(x'_d, y'_h) > g_h(\pi) + \varepsilon$$

An allocation is ε -**pairwise stable** with respect to the family \mathcal{C} if it is ε -individually rational (no agent gets ε less than her/its IRP) and does not have any ε -blocking pair.

To prove the correctness of the DAC algorithm (Algorithm 1.1) we need some technical results.

Lemma 1.3.9. *Let d be a doctor proposing to a hospital h , currently having a payoff g_h . Let λ_d be d 's bid. Then, it always holds that $\lambda_d \geq g_h + \varepsilon$.*

Proof. Since d proposed to h , there exists a strategy profile (x, y) such that the triple (h, x, y) is solution to Problem (1.1). Therefore, the triple $(x, y, g_h + \varepsilon)$ is a feasible solution of Problem (1.3). Thus, $\lambda_d \geq g_h + \varepsilon$. \square

Lemma 1.3.10. *Let (x, y) be the solution of Problem (1.4). Then, $g_{d,h}(x, y)$ is always upper bounded by λ_d , d 's bid during the competition.*

1.3. Existence and computation of Nash-pairwise stable allocations

Proof. Let d (proposer) and d' (current partner) be two doctors competing for h and suppose, without loss of generality, that d wins. Let (x^*, y^*) be d 's optimal proposal and $(\lambda_d, \hat{x}, \hat{y})$ be the solution of Problem (1.3) for player d . Then, the pair (\hat{x}, \hat{y}) is a feasible solution of d 's Problem (1.4), as $g_{d,h}(\hat{x}, \hat{y}) = \lambda_d > \lambda_{d'}$. Consider any strategy profile (x', y') such that $g_{d,h}(x', y') > g_{d,h}(\hat{x}, \hat{y})$. If (x', y') satisfies $f_{d,h}(x', y') \geq \beta_d$, we obtain a contradiction as $(\lambda_d, \hat{x}, \hat{y})$ is solution of Problem (1.3) for player d . Therefore, the solution (x, y) of Problem (1.4) satisfies $g_{d,h}(x, y) \leq g_{d,h}(\hat{x}, \hat{y}) = \lambda_d$. \square

We are ready to prove the correctness of the DAC algorithm (Algorithm 1.1).

Theorem 1.3.11. *The allocation π , output of the DAC algorithm, is ε -pairwise stable with respect to the family \mathcal{C} .*

Proof. Let $\pi := (\mu, \vec{x}, \vec{y})$ be the output of Algorithm 1.1 and suppose it is not ε -pairwise stable. Let (d, h) be an ε -blocking pair of π , and suppose, without loss of generality, that $h \neq \mu(d)$. Let T be the last iteration at which d proposed. In particular, at time T , doctor d proposed to $\mu(d)$ and not to h , and for any posterior proposal to $\mu(d)$, d won the competition. Since d won all the posterior competitions, in particular, by Lemmas 1.3.9 and 1.3.10,

$$f_d(\pi) \geq \max\{f_{d,h'}(\vec{x}, \vec{y}) : g_{d,h'}(\vec{x}, \vec{y}) \geq g_{h'}(\pi) + \varepsilon, h' \in H_0, (\vec{x}, \vec{y}) \in C_{d,h'}\}$$

as $f_d(\pi)$ cannot be lower than any of the reservation prices computed by d during each of her competitions. Since (d, h) is an ε -blocking pair, there exists $(\vec{x}, \vec{y}) \in C_{d,h}$ such that $f_{d,h}(\vec{x}, \vec{y}) > f_d(\pi) + \varepsilon$ and $g_{d,h}(\vec{x}, \vec{y}) > g_h(\pi) + \varepsilon$. Then

$$f_d(\pi) < \max\{f_{d,h'}(\vec{x}, \vec{y}) : g_{d,h'}(\vec{x}, \vec{y}) \geq g_{h'}(\pi) + \varepsilon, h' \in H_0, (\vec{x}, \vec{y}) \in C_{d,h'}\}$$

which is a contradiction. \square

From the existence of ε -pairwise stable allocations, we are finally able to prove the existence of 0-pairwise stable allocations (Theorem 1.3.3) passing through the compactness of the Nash equilibrium sets, continuity of payoff functions, and the finiteness of players.

Proof. **0-Pairwise stable allocations existence (Theorem 1.3.3).** Consider $\varepsilon > 0$. Let $\pi_\varepsilon := (\mu_\varepsilon, \vec{x}_\varepsilon, \vec{y}_\varepsilon)$ be the output of the DAC algorithm (Algorithm 1.1). Thus, π_ε is an ε -pairwise stable allocation with respect to the family \mathcal{C} (Theorem 1.3.11). Consider a sequence of these profiles $(\pi_\varepsilon)_\varepsilon$ with ε going to 0, and a subsequence $(\pi_{\varepsilon_k})_k$ such that $(\vec{x}_{\varepsilon_k}, \vec{y}_{\varepsilon_k})_k$ converges to a fixed strategy profile (\vec{x}, \vec{y}) , which exists as the sets $(C_{d,h} : (d, h) \in D \times H)$ are compact sets.

Since there is a finite number of possible matchings, consider a subsubsequence $(\pi_{\varepsilon_{k_l}})_l$ such that $\mu_{\varepsilon_{k_l}} = \mu, \forall l \in \mathbb{N}$, with μ a fixed matching. As $(\vec{x}_{k_l}, \vec{y}_{k_l}) \rightarrow (\vec{x}, \vec{y})$ when $l \rightarrow \infty$, the sequence π_{k_l} converges to $\pi := (\mu, \vec{x}, \vec{y})$, with μ a matching and (\vec{x}, \vec{y}) a strategy profile. Moreover, as ε_{k_l} goes to 0, as for each l the allocation π_{k_l} is ε_{k_l} -pairwise stable, as the payoff functions are continuous, and as the definition of pairwise stability only includes inequalities, π is pairwise stable. \square

1.4 Lattice structure

In Gale-Shapley's model, whenever each player's preferences are strict, the set of pairwise stable matching is a **lattice**⁴. In our model, a semi-lattice structure is satisfied whenever **Nash equilibrium payoffs are different across games**, i.e., if w and w' are two Nash equilibrium payoffs of any player with two different partners, then $w \neq w'$. This is generically satisfied, i.e., when all games $G_{d,h}$ are *generic independent bimatrix games*.

Theorem 1.4.1. *Let Γ be a matching game where all Nash equilibrium payoffs are different across games. Then, the set of Nash-pairwise stable allocations Π is a semi-lattice. Moreover, whenever all games have a unique Nash equilibrium payoff, the set Π is a full lattice.*

Proof. Let $\pi = (\mu, \vec{x}, \vec{y}), \pi' = (\mu', \vec{x}', \vec{y}') \in \Pi$ be two Nash-pairwise stable allocations. Consider the new triple $\pi^\vee := (\mu^\vee, \vec{x}^\vee, \vec{y}^\vee)$ defined by

$$\mu_d^\vee = \arg \max \left\{ f_{d,\mu(d)}(x_d, y_{\mu(d)}) ; f_{d,\mu'(d)}(x'_d, y'_{\mu'(d)}) \right\}, \forall d \in D,$$

with the corresponding Nash equilibrium $(x_d^\vee, y_{\mu^\vee(d)}^\vee)$. As Nash equilibrium payoffs are different, partner and Nash equilibrium are uniquely defined when considering the maximum, so μ^\vee is well defined. Even more, it is Nash stable as all couples play a Nash equilibrium of their game. Regarding pairwise stability, suppose that an agent $k \in D \cup H$ gets less than her/its IRP in π^\vee . Then, the respective matching could not be individually rational either, which is a contradiction. Let $(d, h) \notin \mu^\vee$ be a (Nash) blocking pair of π^\vee , i.e., there exists a Nash equilibrium $(\bar{x}, \bar{y}) \in N.E(G_{d,h})$ such that $f_{d,h}(\bar{x}, \bar{y}) > f_d(\pi^\vee)$ and $g_{d,h}(\bar{x}, \bar{y}) > g_h(\pi^\vee)$. From the first inequality, it holds that $f_{d,h}(\bar{x}, \bar{y}) > f_d(\pi)$, therefore $g_{d,h}(\bar{x}, \bar{y}) \leq g_h(\pi)$, as π is pairwise stable. Remark that $g_h(\pi^\vee)$ is either equal to $g_h(\pi)$ or to $g_h(\pi')$. Since $g_{d,h}(\bar{x}, \bar{y}) > g_h(\pi^\vee)$, it must be that $g_h(\pi^\vee) = g_h(\pi')$. Then, as π' is pairwise stable, $f_{d,h}(\bar{x}, \bar{y}) \leq f_d(\pi')$, which contradicts that $f_{d,h}(\bar{x}, \bar{y}) > \max\{f_d(\pi), f_d(\pi')\}$.

Finally, suppose that all games have a unique Nash equilibrium. In this case, finding a Nash-pairwise stable allocation is equivalent to finding a stable allocation of Gale-Shapley's problem using the Nash equilibrium payoffs to determine the preferences of the agents, hence a full lattice holds. \square

Observe that constant-sum, and more generally strictly competitive games, whenever they have a Nash equilibrium, the equilibrium payoff is unique.

As in Gale-Shapley's model, our DAC algorithm in which doctors propose and hospitals dispose converges to the best Nash-pairwise stable matching for doctors. We state this result without proof as it will be proved later in a more general case.

Theorem 1.4.2. *If all Nash equilibrium payoffs are different across games and ε is small enough, the output of Algorithm 1.1, when $C_{d,h} = N.E(G_{d,h}), \forall (d, h) \in D \times H$, is the highest element (with respect to the proposer side) of the semi-lattice.*

⁴John Conway was the first to prove this result, which was reported by Knuth in [75]

Proof. It follows from Theorem 2.5.3 as the identification property (Definition 2.3.1) holds when Nash equilibrium payoffs are all different. \square

Exchanging the roles of doctors and hospitals when computing the maximum between two Nash-pairwise stable allocations, also outputs a Nash-pairwise stable allocation. Even more, when all games have a unique Nash equilibrium, the minimum between two elements in Π can be computed and the result, $(\mu^\wedge, \vec{x}^\wedge, \vec{y}^\wedge)$ for doctors and $(\nu^\wedge, \vec{x}^\wedge, \vec{y}^\wedge)$ for hospitals, also belongs to Π . In addition, as in Gale-Shapley's model, the duality property $(\mu^\vee, \vec{x}^\vee, \vec{y}^\vee) = (\nu^\wedge, \vec{x}^\wedge, \vec{y}^\wedge)$ and $(\mu^\wedge, \vec{x}^\wedge, \vec{y}^\wedge) = (\nu^\vee, \vec{x}^\vee, \vec{y}^\vee)$ holds.

We finish the model without commitment by examining the Nash-pairwise stable allocation of our examples.

Example 1.1.4. The only Nash equilibrium of the prisoners' dilemma is to play (B, B) . Then, as both players are better off being single than playing the equilibrium, the only Nash-pairwise stable allocation is the one in which players do not match. \square

Example 1.1.5. The coordination game has three Nash equilibria: two pure ones (A, A) , (B, B) , and one mixed $((8/9, 1/9), (1/9, 8/9))$, with $(\delta/9, \delta/9)$ as payoff profile. As for the three Nash equilibria at least one of the agents is worst off than being single, once again, the only Nash-pairwise stable allocation is the one in which players remain single. \square

Example 1.1.6. The only Nash equilibrium of the constant-sum game is $x = y = 0$, as any positive transfer $x > 0$ (resp. $y > 0$) is a strictly dominated strategy for d (resp. for h). Thus, for any Nash stable allocation π in which the players are matched, their payoffs are $f_d(\pi) = 10\delta$ and $g_h(\pi) = 0$. As this allocation is not individually rational for h , the only Nash-pairwise stable allocation is the one in which agents do not match. \square

Example 1.1.7. Similarly to the previous example, the only Nash equilibrium is to pay 0 for the good. However, as the seller has a null IRP, it is not individually rational for her to sell the good without receiving a positive payment. Therefore, the only Nash-pairwise stable allocation is the one in which nobody buys the good.

The Nash-pairwise stable allocations found in these examples are not Pareto-optimal. In Example 1.1.4 players can match, cooperate, and end up both better off. In Example 1.1.5 agents can match, play each strategy with equal probability, and end up both better off. In Example 1.1.6 d can propose $x \geq \delta$ to h and both agents end up better off. In Example 1.1.7, the buyer with the higher valuation can pay any price between her valuation and the one of the seller, and both, buyer and seller, end up better off. These predictions are possible only if both agents believe that the other one will honor her promise, which is assumed in the matching with transfer and matching with contract literature. The following chapters study the model where players can commit (for example by signing binding contracts).

1.5 A numerical example

Consider a matching game with three doctors $D = \{d_1, d_2, d_3\}$ and three hospitals $H = \{h_1, h_2, h_3\}$, with the following preferences,

$$A = \begin{array}{|c|c|c|} \hline 83 & 85 & 99 \\ \hline 74 & 13 & 15 \\ \hline 58 & 49 & 54 \\ \hline \end{array} \quad B = \begin{array}{|c|c|c|} \hline 69 & 6 & 28 \\ \hline 88 & 2 & 70 \\ \hline 72 & 18 & 9 \\ \hline \end{array}$$

$A(d, h)$ (resp. $B(d, h)$) represents the fixed utility that $d \in D$ (resp. $h \in H$) receives if she (resp. it) matches with h (resp. d). Suppose that all agents have null IRPs, i.e., $(\underline{f}_d, \underline{g}_h) = (0, 0), \forall (d, h) \in D \times H$. Suppose, in addition, that if a couple (d, h) is created, as in Example 1.1.6, they play a constant-sum game. Formally, if $x_d, y_h \geq 0$ are d and h respective monetary transfers, their utilities are,

$$\begin{aligned} f_{d,h}(x_d, y_h) &= A(d, h) - x_d + y_h \\ g_{d,h}(x_d, y_h) &= B(d, h) + x_d - y_h \end{aligned}$$

As already discussed, Nash stable allocations for matching with transfer problems are those in which no agent makes a positive transfer. Due to this, computing a Nash-pairwise stable allocation is equivalent to computing a Gale-Shapley's pairwise stable allocations, where ordinal preferences are deduced from the matrices A and B . The DAC algorithm (Algorithm 1.1) becomes equivalent to Gale-Shapley's deferred-acceptance algorithm in which doctors propose and competitions are reduced to preference comparisons. We summarize the DAC algorithm in Algorithm 1.2.

Algorithm 1.2: Deferred-acceptance with competitions algorithm

```

1 Set  $D' \leftarrow \{d_1, d_2, d_3\}$  as the set of unmatched doctors and  $g_h(\pi) = 0, \forall h \in H$ 
2 while  $D' \neq \emptyset$  do
3   | Let  $d \in D'$  and  $h \in H$  such that
      |
      |  $A(d, h) = \arg \max\{A(d, \cdot) : B(d, h) > g_h(\pi)\}$ 
4   | if  $h$  is single then
5   |   |  $d$  is automatically accepted
6   | end
7   | else
8   |   |  $h$  chooses
      |   |  $\arg \max\{A(d, h); A(\mu(h), h)\}$ 
      |   | and the loser is included in  $D'$ 
9   | end
10 end

```

Let us run Algorithm 1.2 iteration by iteration.

Iter 1. Suppose d_1 is the first proposer. Looking at the first row of matrix A , d_1 prefers to propose to h_2 , who accepts her.

Iter 2. Suppose d_3 is the second proposer. From $A(d_3, \cdot)$, d_3 proposes to h_1 , who accepts her.

Iter 3. Finally, d_2 proposes as she is the only remaining doctor. From the payoff matrix, d_2 proposes to h_1 and has to compete against d_3 . From the first column of B , the winner is d_2 . d_3 becomes single and proposes again.

Iter 4. d_3 computes her optimal proposal. As $B(d_3, h_1) \leq g_{h_1}(\pi)$, d_3 proposes to her second best option, h_3 . As h_3 is already matched with d_1 , they compete. h_3 prefers to d_1 so d_3 is rejected.

Iter 5. Finally, d_3 proposes to h_2 as this is her only feasible proposal. d_3 is accepted and the algorithm ends.

The output is $\mu = ((d_1, h_3), (d_2, h_1), (d_3, h_2))$. We see that d_1 and d_2 are matched with their best possible options, so they will never prefer to change to another partner. Regarding d_3 , although she would prefer to change to another partner, neither h_1 nor h_3 would accept her, as each of them prefers their actual partners. We conclude that $\pi = (\mu, \vec{0}, \vec{0})$ is a Nash-pairwise stable allocation.

1.6 Conclusions

In this chapter, we have studied the model of one-to-one matching games without commitment. Due to the inability of the agents to commit in actions, Nash stability (Definition 1.2.2) rises as a natural desired property for allocations. In addition, extending the work of Gale and Shapley, we have defined pairwise stable allocations (Definition 1.2.3) as those allocations if which no pair of agents can abandon their partners, matching together, playing a Nash equilibrium of their game, and end up both strictly better off. We have designed a deferred-acceptance with competitions (DAC) algorithm (Algorithm 1.1) to compute ε -Nash-pairwise stable allocations under classical game theory assumptions. Even more, we are able to prove the existence of 0-Nash-pairwise stable allocations under the same assumptions (Theorem 1.3.1). Following the results of Conway and Knuth [75], we have proved that the set of Nash-pairwise stable allocations is a lattice.

Regardless the positive results obtained for the matching games without commitment model, Examples 1.1.4 to 1.1.7 show its inability to capture classical stable matching models from the literature due to the incapacity of the agents to commit in actions. Commitment is a classical assumption in models such as matching with transfers [39, 93] and matching with contracts [66]. Therefore, the rest of this thesis and, in particular, Chapters 2 and 3, will be devoted to study the model of *one-to-one matching games under commitment*.

1-to-1 Matching games with commitment: Pairwise stability

Chapter 1 studied the model of matching games without commitment, proved the existence of stable allocations for this setting, and studied the lattice structure of their set. However, many examples showed how the model fails to capture classical models of stable matching from the literature. Motivated by this, and being in line with the literature on matching with transfers [39, 93] and matching with contracts [66] in which agents commit, this chapter is devoted to studying the model of *one-to-one matching games with commitment*.

As the model in Chapter 1, one-to-one matching games with commitment consider two finite sets of agents and match them in couples. Potential couples are endowed with strategic games and agents' utilities within an allocation are the outputs of the corresponding game. Agents can deviate in two ways, either changing of partner or changing strategy.

Making an abuse of notation, the extension of Gale-Shapley's *stability* to the setting with commitment will be called *pairwise stability*, exactly as in the setting without commitment. Pairwise stability will deal with agents abandoning their partners and matching together. To prove their existence we will use the same deferred-acceptance with competitions (DAC) algorithm presented in the previous chapter. Properties such as the lattice structure or the optimality of the DAC algorithm will be proved.

Unlike pairwise stability without commitment, pairwise stability under commitment will capture the stability notions in the literature of stable matching and therefore, it will recover the solutions of the classical models as the **stable marriage problem** of Gale and Shapley [53], the **Assignment game** of Shapley and Shubik [93], or the **matching with transfers** problem of Demange and Gale [40].

Agents' deviations on actions will require a novel theoretic framework, which is delegated to Chapter 3.

This chapter is structured as follows. Section 2.1 presents the model of matching games with commitment. Section 2.2 introduces the notion of pairwise stability and algorithmically proves the existence of pairwise stable allocations. Properties such as the lattice structure of the set of pairwise stable allocations or the proposer-optimality of the DAC algorithm are studied in Sections 2.3 to 2.5. Section 2.6 formally shows how to map the assignment problem of Shapley-Shubik [93] and the matching with transfers of Demange-Gale [39] to matching games under commitment. Section 2.7 shows a numerical example of the DAC algorithm designed to compute pairwise stable allocations. Finally, Section 2.8 concludes the chapter.

2.1 Matching games model

We briefly recall the model of matching games presented in Chapter 1. We consider two finite sets of agents D and H , called **doctors** and **hospitals**, respectively. Each couple $(d, h) \in D \times H$ is endowed with a **strategic game** $G_{d,h} = (X_d, Y_h, f_{d,h}, g_{d,h})$. Agents have **individually rational payoffs** (IRPs) $(\underline{f}, \underline{g})$ representing the personal valuation for being single. We extend the sets D and H to D_0 and H_0 , respectively, by adding to each of them an *empty player* such that any agent matched with one of these empty players gets her IRP as payoff.

A **matching game** is any tuple

$$\Gamma = (D_0, H_0, \{G_{d,h} : (d, h) \in D \times H\}, \underline{f}, \underline{g})$$

and **allocations** in Γ are any triplet $\pi = (\mu, \vec{x}, \vec{y})$ where μ is a matching, $\vec{x} \in \prod_{d \in D} X_d$ is a doctors' strategy profile, and $\vec{y} \in \prod_{h \in H} Y_h$ is a hospitals' strategy profile.

Given an allocation π , agents' payoffs are given by,

$$f_d(\pi) = \begin{cases} f_{d,\mu(d)}(x_d, y_{\mu(d)}) & \text{if } d \text{ is matched} \\ \underline{f}_d & \text{otherwise} \end{cases}$$

$$g_h(\pi) = \begin{cases} g_{\mu(h),h}(x_{\mu(h)}, y_h) & \text{if } h \text{ is matched} \\ \underline{g}_h & \text{otherwise} \end{cases}$$

where $\mu(k)$ states the partner of k in μ , for any $k \in D \cup H$.

2.2 Pairwise stability and its existence

Suppose that partners within a couple can commit to playing a specific action profile before playing their game. This allows them to enlarge their set of feasible contracts well beyond their set of Nash equilibria. This leads to the following stability notion.

Definition 2.2.1. An individually rational allocation $\pi = (\mu, \vec{x}, \vec{y})$ is **pairwise stable** if there are no blocking pairs, i.e., there is no $(d, h) \in D \times H$ and $(x'_d, y'_h) \in X_d \times Y_h$ such that

$$f_{d,h}(x'_d, y'_h) > f_d(\pi) \text{ and } g_{d,h}(x'_d, y'_h) > g_h(\pi)$$

Notice that a matched couple can block a coalition in Definition 2.2.1. In particular, in a pairwise stable allocation couples cannot play Pareto-dominated strategy profiles of their game. In models with strict preferences or monetary transfers this is not an issue as only agents not already matched between them can be blocking pairs. In our model, as payoffs are continuous, we need to make the distinction. This will become important when mixing pairwise stability and renegotiation proofness later on the thesis.

Pairwise stability in the setting with commitment corresponds to pairwise stability with respect to the family $\mathcal{C} = (C_{d,h} = X_d \times Y_h, \forall d \in D, \forall h \in H)$ (Definition 1.3.2).

Compared to the pairwise stability without commitment (Definition 1.2.3) defined in Section 1.2, Definition 2.2.1 allows the players to choose any feasible strategy profile without being restricted to a Nash equilibrium (or to a non-dominated strategy). Let us see the impact of such a change in our leading examples.

Example 1.1.4. Recall the prisoners' dilemma matching game in which two agents d and h with positive IRPs δ can match and play,

		Agent h	
		Cooperate	Betray
Agent d	Cooperate	$2\delta, 2\delta$	$-\delta, 3\delta$
	Betray	$3\delta, -\delta$	$0, 0$

Matching and playing the Nash equilibrium of the game is not individually rational. However, notice that not matching is not pairwise stable as matching and cooperating blocks it. Suppose that players get matched and play a mixed strategy $(x, y) = ((p, 1 - p), (q, 1 - q))$. Their expected payoffs are,

$$f_{d,h}(x, y) = 2\delta pq + 3\delta(1 - p)q - \delta p(1 - q) = \delta(3q - p)$$

$$g_{d,h}(x, y) = 2\delta pq + 3\delta p(1 - q) - \delta(1 - p)q = \delta(3p - q)$$

Imposing individual rationality we get the condition,

$$q \in \left[\frac{1+p}{3}, 3p-1 \right], \text{ such that } p, q \in [0, 1]$$

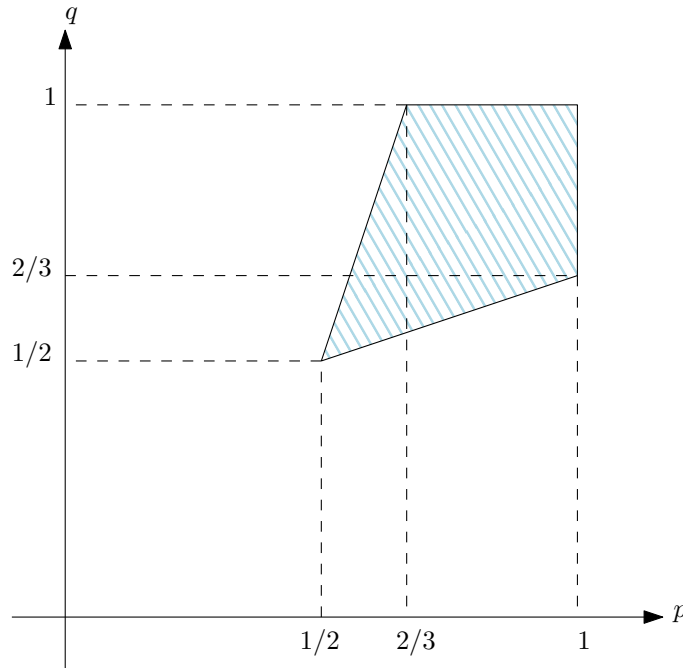


Figure 2.1: Pairwise stable allocations Example 1.1.4

2.2. Pairwise stability and its existence

Figure 2.1 shows the possible values that p and q can take such that matching together and playing the strategy profile $((p, 1 - p), (q, 1 - q))$ is pairwise stable (represented by the light blue region). Notice that we obtain a continuum of solutions. \square

Example 1.1.5. Recall the coordination matching game in which two agents d and h with positive IRPs δ can match and play,

		Agent h	
		A	B
Agent d	A	$4\delta, \delta/2$	$0, 0$
	B	$0, 0$	$\delta/2, 4\delta$

Matching and playing a pure Nash equilibrium, or remaining unmatched, are both individually irrational. Taking mixed strategies $(x, y) = ((p, 1 - p), (1 - q, q))$ (notice the order of the mixed strategies), the expected payoffs of the agents are,

$$f_{d,h}(x, y) = 4\delta p(1 - q) + \frac{\delta}{2}(1 - p)q = 4\delta p + \frac{\delta q}{2} - \frac{9\delta}{2}pq$$

$$g_{d,h}(x, y) = 4\delta(1 - p)q + \frac{\delta}{2}p(1 - q) = 4\delta q + \frac{\delta p}{2} - \frac{9\delta}{2}pq$$

Imposing individual rationality we obtain,

$$q \in \left[\frac{2 - p}{8 - 9p}, \frac{8p - 2}{9p - 1} \right], \text{ such that } p, q \in [0, 1]$$

Once again, we obtain a continuum of solutions. \square

Example 1.1.6. Recall the transfer matching game in which two agents d and h with positive IRPs δ can match and play,

$$G = (\mathbb{R}_+, \mathbb{R}_+, f, g), \text{ such that for any } x, y \geq 0,$$

$$f_{d,h}(x, y) = 10\delta - x + y$$

$$g_{d,h}(x, y) = x - y$$

As we already intuited, not matching, or matching and playing null transfers are not stable. Imposing individual rationality we find that any allocation in which agents match and the transfer profile satisfies $x - y \in [\delta, 9\delta]$, is pairwise stable. This is exactly the prediction of Shapley-Shubik's and Demange-Gale's models. \square

Example 1.1.7. Recall the auction example in which a seller sells an indivisible good to a set of n buyers. Buyers have valuations for the good and let us assume that they satisfy $v_1 > v_2 \geq \dots \geq v_n$. The seller has valuation c for the good as well and everybody has null IRP. If a couple buyer-seller (d, h) is created and the monetary

transfers (x_d, y_h) is agreed, the item is sold from h to d at the price $p = x_d - y_h$. In particular, the utilities of the players are,

$$\begin{aligned} f_{d,h}(x_d, y_h) &= v_d - p = v_d - x_d + y_h, \\ g_{d,h}(x_d, y_h) &= p - c = x_d - y_h - c, \end{aligned}$$

Finally, let us assumed $v_1 \geq c$ (otherwise no buyer has incentive to buy the item). An allocation in this problem is pairwise stable if and only if the first buyer and the seller decide to trade at some price $p_1 \in [\max(c, v_2), v_1]$. Notice that no other buyer is willing to pay a price higher than p_1 as this is not individually rational. Due to this, the seller has no incentive in abandoning buyer 1 and matching with somebody else. As, in addition, the allocation is individually rational for buyer 1 and the seller, the allocation is, indeed, pairwise stable.

As for the no commitment setting, the general deferred-acceptance with competitions algorithm (Algorithm 1.1) can be used to compute ε -pairwise stable allocations for the model with commitment. Algorithm 2.1 recalls the DAC algorithm. For a numerical example, check Section 2.7.

Algorithm 2.1: Deferred-acceptance with competitions algorithm

```

1 Input:  $\Gamma = (D_0, H_0, (G_{d,h} : (d, h) \in D \times H), \underline{f}, \underline{g})$  a matching game,  $\varepsilon > 0$ 
2 Set  $D' \leftarrow D$  as the set of single doctors, and  $g_h(\pi) \leftarrow \underline{g}_h, \forall h \in H$ 
3 while  $D' \neq \emptyset$  do
4   | Let  $d \in D'$ . Compute her optimal proposal
   |  $(h, x, y) \in \arg \max \{ f_{d,h}(x, y) : g_{d,h}(x, y) \geq g_h(\pi) + \varepsilon, h \in H_0, (x, y) \in C_{d,h} \}$ 
5   | if  $h$  is single then
6   |   |  $d$  is automatically accepted
7   | end
8   | else
9   |   |  $d$  and  $\mu(h)$  compete for  $h$  as in a second-price auction. The winner
   |   | passes to be the new partner of  $h$  and the loser is included in  $D'$ 
10  | end
11 end

```

We recall as well the two results that allow us to conclude the existence of pairwise stable allocations.

Theorem 2.2.2. *The DAC algorithm ends in finite time.*

Theorem 2.2.3. *The allocation π , output of the DAC algorithm, is ε -pairwise stable with respect to the family \mathcal{C} .*

The existence of pairwise stable allocations in the model with commitment is obtained by taking $C_{d,h} = PO(G_{d,h})$, where $PO(G_{d,h})$ is the set of Pareto-optimal strategy sets of game $G_{d,h}$, for any $d \in D$ and $h \in H$.

Theorem 2.2.4. *For any matching game with non-empty compact Pareto-optimal strategy sets and continuous payoff function there always exists a pairwise stable allocation (Definition 2.2.1).*

2.3 Semi-lattice structure

As for the stable allocations of the model without commitment, a partial lattice structure is also satisfied by the set of pairwise stable allocations under commitment. However, to be able to define a proper order between allocations, an *identification property* is needed.

Definition 2.3.1. Let $\pi = (\mu, \vec{x}, \vec{y})$ and $\pi' = (\mu', \vec{x}', \vec{y}')$ be two pairwise stable allocations (Definition 2.2.1). We say that π and π' satisfy the **identification property** if whenever a doctor/hospital has the same utility in both matchings, she/it has the same partner in both. That is,

$$\forall d \in D, \text{ if } f_{d,\mu(d)}(x_d, y_{\mu(d)}) = f_{d,\mu'(d)}(x'_d, y'_{\mu'(d)}) \implies \mu(d) = \mu'(d),$$

and analogously for $h \in H$.

The identification property is satisfied by any matching model with strict preferences. In our model this is not necessarily true as we have continuum payoff functions. However, it can be satisfied “generically” speaking, in the sense that discretizing the strategy spaces and perturbing the payoff functions, Definition 2.3.1 holds for any pair of allocations and, therefore, for any pair of pairwise stable allocations. In particular, it is satisfied by the model without binding contracts when all games have different Nash equilibrium payoffs.

The identification property is also satisfied by Demange-Gale’s [39] model with transfers and it is crucial in their lattice structure proof. Indeed, given two allocations π^1, π^2 , consider the sets of agents that prefer matching 1, matching 2, or are indifferent, respectively by,

$$\begin{aligned} P^1 &:= \{d \in D : f_d(\pi^1) > f_d(\pi^2)\}, & Q^1 &:= \{h \in H : g_h(\pi^1) > g_h(\pi^2)\} \\ P^2 &:= \{d \in D : f_d(\pi^2) > f_d(\pi^1)\}, & Q^2 &:= \{h \in H : g_h(\pi^2) > g_h(\pi^1)\} \\ P^0 &:= \{d \in D : f_d(\pi^1) = f_d(\pi^2)\}, & Q^0 &:= \{h \in H : g_h(\pi^1) = g_h(\pi^2)\} \end{aligned}$$

Demange and Gale proved that, in their setting, agents in P^1 are assigned with agents in Q^2 , and those in P^2 with the ones in Q^1 . This property holds in their model as every time an agent increases its payoff, its partner’s payoff decreases (see Sections 2.4 and 2.6 below).

Consider the new triple $\pi^\vee := (\mu^\vee, \bar{x}^\vee, \bar{y}^\vee)$ defined by

$$\mu_d^\vee = \arg \max \left\{ f_{d,\mu(d)}(x_d, y_{\mu(d)}) ; f_{d,\mu'(d)}(x'_d, y'_{\mu'(d)}) \right\}, \forall d \in D,$$

with the corresponding strategy profile $(x_d^\vee, y_{\mu^\vee(d)}^\vee)$ (which is uniquely defined when a doctor is not matched to the same hospital and can be chosen arbitrarily otherwise).

Theorem 2.3.2. *Under the identification property, $(\mu^\vee, \bar{x}^\vee, \bar{y}^\vee)$ is an pairwise stable allocation.*

Proof. The proof has two steps. We first prove that μ^\vee is a well defined matching and then, that the triple $(\mu^\vee, \bar{x}^\vee, \bar{y}^\vee)$ is pairwise stable. For the first part, let $d, k \in D$ be two doctors such that $\mu_d^\vee = \mu_k^\vee = h$, with $h \neq h_0$. Suppose $h = \mu(d) = \mu'(k)$. It follows that

$$\arg \max \left\{ f_{d,h}(x_d, y_h) ; f_{d,\mu'(d)}(x'_d, y'_{\mu'(d)}) \right\} = \arg \max \left\{ f_{k,\mu(k)}(x_k, y_{\mu(k)}) ; f_{k,h}(x'_k, y'_h) \right\}$$

as both are equal to h . By the identification property, as since $h \neq \mu(k)$, it holds,

$$f_{k,\mu(k)}(x_k, y_{\mu(k)}) < f_{k,h}(x'_k, y'_h)$$

Since π' is pairwise stable, it must hold that $g_{k,h}(x'_k, y'_h) > g_{d,h}(x_d, y_h)$, otherwise (d, h) would block π' using (x_d, y_h) (observe that $g_{k,h}(x'_k, y'_h) \neq g_{d,h}(x_d, y_h)$ by the identification property). Consequently, as π is pairwise stable, it must hold that

$$f_{k,h}(x'_k, y'_h) < f_{k,\mu(k)}(x_k, y_{\mu(k)}),$$

otherwise (k, h) blocks π using (x'_k, y'_h) . This contradicts that

$$h = \arg \max \{ f_{k,\mu(k)}(x_k, y_{\mu(k)}); f_{k,h}(x'_k, y'_h) \}$$

Thus μ^\vee is a well-defined matching.

Regarding pairwise stability, suppose (d, h) is a blocking pair for $(\mu^\vee, \bar{x}^\vee, \bar{y}^\vee)$, using some strategy profile (s, t) . In particular, it holds

$$f_{d,h}(s, t) > \max \{ f_{d,\mu(d)}(x_d, y_{\mu(d)}); f_{d,\mu'(d)}(x'_d, y'_{\mu'(d)}) \} \quad (2.1)$$

Suppose $h = h_0$. If $\mu(d) \neq h_0$ (resp. if $\mu'(d) \neq h_0$) then μ (resp. μ') is not individually rational, which is a contradiction. Otherwise, $\mu(d) = \mu'(d) = h$, which is also a contradiction with Equation (2.1). Hence $h \neq h_0$. Let d' be the partner of h in π^\vee . Then, (d', h) are together in one of the matchings μ or μ' , and (d, h) can block it using the strategy profile (s, t) : a contradiction. Thus $(\mu^\vee, \bar{x}^\vee, \bar{y}^\vee)$ is pairwise stable. \square

If μ^\wedge denotes the min operation matching in which each doctor is coupled with her worst hospital between π and π' , the output is not necessarily a pairwise stable matching. In that sense, we only have a *semi-lattice structure*. The semi-lattice structure remains valid if we exchange the roles of doctors and hospitals: if ν^\vee gives

to each hospital its best doctor between μ and μ' , the output is an pairwise stable allocation. Finally, note that the classical equality between the max for doctors/min for hospitals operations in the Gale-Shapley model does not hold in our model ($\mu^\wedge \neq \nu^\vee$). Quite the opposite can be true. For example, if all strategic games are common-interest games, the max for doctors coincides with the max for hospitals ($\nu^\vee = \mu^\vee$).

A full lattice structure can be recovered by taking extra assumptions on the strategy games. The following section presents the formal results.

2.4 Lattice structure & Strictly competitive games

The set of pairwise stable allocations Π results to be a semi-lattice whenever the identification property holds. This structure can be extended to a full lattice whenever players play constant-sum games or, more generally, strictly competitive games, i.e., games in which whenever a player's payoff increases (resp. decreases), the partner's payoff decreases (resp. increases).

Aumann [13], who introduced monotone games, considered the class \mathcal{S} of strictly competitive games in which payoff functions are obtained via monotone transformations of a zero-sum game. Although he conjectured that \mathcal{S} covers all strictly competitive games, the proof is only known for finite games [7] (see Section 5.3). Along this section, we will use Aumann's conjecture.

Definition 2.4.1. A two-person game $G_{d,h} = (X_d, Y_h, f_{d,h}, g_{d,h})$ is a **strictly competitive game**, that we denote $G_{d,h} \in \mathcal{S}$, if there exists a zero-sum game $G'_{d,h} = (X_d, Y_h, u_{d,h})$ and two monotone increasing¹ functions ϕ and φ such that,

$$f_{d,h}(\cdot, \cdot) = \phi(u_{d,h}(\cdot, \cdot)) \text{ and } g_{d,h}(\cdot, \cdot) = \varphi(u_{d,h}(\cdot, \cdot))$$

In other words, whenever a game is strictly competitive, we can always transform the game into a zero-sum game. This will allow us to reduce the study of strictly competitive games to zero-sum games in the following sections.

Remark 2.4.2. The identification property is obtained for free when games are strictly competitive games.

Theorem 2.4.3. *Let Γ be a matching game in which all games are strictly competitive games. Then, the set Π of pairwise stable allocations is a full lattice.*

Proof. From Theorem 2.3.2, Π is a semi-lattice as the maximum of two pairwise stable allocations is well defined. To finish the proof, it is enough with checking that the minimum between two allocations is well defined as well. Let $\pi, \pi' \in \Pi$ be two pairwise stable allocations and define $\pi^\wedge := (\mu^\wedge, \vec{x}^\wedge, \vec{y}^\wedge)$ as,

$$\mu_d^\wedge := \arg \min \{ f_{d,\mu(d)}(x_d, y_{\mu(d)}); f_{d,\mu'(d)}(x'_d, y'_{\mu'(d)}) \}, \forall d \in D$$

¹We consider increasing functions in order to apply these results in the following section. However, the same results hold when considering monotone decreasing functions.

with the respective strategy profile. Given $d \in D$, notice that (we omit the strategy profiles to simplify the notation),

$$\begin{aligned}
 \mu_d^\wedge &= \arg \min \{ \varphi^{-1}(u_{d,\mu(d)}(\cdot, \cdot)); \varphi^{-1}(u_{d,\mu'(d)}(\cdot, \cdot)) \} \\
 &= \varphi^{-1}(\arg \min \{ u_{d,\mu(d)}(\cdot, \cdot); (u_{d,\mu'(d)}(\cdot, \cdot)) \}) \\
 &= \varphi^{-1}(\arg \min \{ -\phi(g_{d,\mu(d)}(\cdot, \cdot)); -\phi(g_{d,\mu'(d)}(\cdot, \cdot)) \}), \\
 &= \varphi^{-1}(\arg \max \{ \phi(g_{d,\mu(d)}(\cdot, \cdot)); \phi(g_{d,\mu'(d)}(\cdot, \cdot)) \}), \\
 &= \varphi^{-1} \circ \phi(\arg \max \{ g_{d,\mu(d)}(\cdot, \cdot); g_{d,\mu'(d)}(\cdot, \cdot) \}), \\
 &= \arg \max \{ g_{d,\mu(d)}(\cdot, \cdot); g_{d,\mu'(d)}(\cdot, \cdot) \} \\
 &= \nu_d^\vee
 \end{aligned}$$

As ν^\vee is always well defined (under the identification property), μ^\wedge is a proper matching. Moreover, π^\vee remains pairwise stable, recovering the duality of ν^\vee and μ^\wedge . \square

2.5 DAC algorithm's optimality

The set of pairwise stable allocations Π may be an infinite semi-lattice depending on the number of stable payoffs. For small enough values of ε , the identification property handles this issue. Consider ε^* defined by,

$$\varepsilon^* = \min_{d \in D} \left\{ |f_{d,\mu(d)}(x_d, y_{\mu(d)}) - f_{d,\mu'(d)}(x'_d, y'_{\mu'(d)})| : (\mu, \vec{x}, \vec{y}), (\mu', \vec{x}', \vec{y}') \in \Pi \right\} \quad (2.2)$$

Then, fixing $\varepsilon \in (0, \varepsilon^*)$, if the identification property holds for any two ε -pairwise stable allocations, Π can be endowed with the partial order $>_D$,

$$\forall \pi, \pi' \in \Pi, \pi \succ_D \pi' \iff f_d(\pi) \geq f_d(\pi'), \forall d \in D \text{ and } \exists d \in D, f_d(\pi) \geq f_d(\pi') + \varepsilon$$

Whenever the identification property holds and $\varepsilon \in (0, \varepsilon^*)$, all maximal elements of $(\Pi, >_D)$ correspond to a unique allocation, denoted π_D . Although strategy profiles between two maximal elements may differ, the difference between each doctor's utilities is not greater than ε . Hence, considering equivalent all allocations where doctors' utilities do not differ by more than ε , the best ε -pairwise stable allocation for doctors is well defined and it is (ε -)unique. As in Gale-Shapley, we aim to prove that the deferred-acceptance with competitions algorithm (Algorithm 2.1) outputs a maximal element π_D of $(\Pi, >_D)$.

To prove this property, given a doctor $d \in D$, we define her **best stable hospital** $h(d) \in H_0$ as,

$$(h(d), x_d, y_h) \in \arg \max \{ f_{d,h}(x'_d, y'_h) : ((d, h), x'_d, y'_h) \in \pi, \text{ for some } \pi \in \Pi \} \quad (2.3)$$

From the uniqueness of the maximal matching in Π , we obtain the following lemma.

Lemma 2.5.1. *Under the identification property and for $\varepsilon \in (0, \varepsilon^*)$, the following two properties hold:*

2.5. DAC algorithm's optimality

1. For any two different doctors $d, d' \in D$, $h(d) \neq h(d')$.
2. Given an allocation $\pi = (\mu, \vec{x}, \vec{y})$, if $f_d(\pi) > f_d(\pi_D) + \varepsilon$ for some doctor d , then π is not ε -pairwise stable.

Lemma 2.5.2. *Let $\varepsilon \in (0, \varepsilon^*)$ and suppose the identification property holds in Π . Let t be an iteration of the DAC algorithm (Algorithm 2.1) and $d \in D$ be the current proposer doctor. Let $h(d) \in H$ be d 's best stable hospital. Then, there always exists a strategy profile $(x, y) \in X_d \times Y_{h(d)}$ such that $g_{d,h(d)}(x, y) \geq g_{h(d)}^t + \varepsilon$, with $g_{h(d)}^t$ being the payoff of $h(d)$ at time t .*

Proof. Suppose there exist $t \geq 1$ and $d \in D$, such that,

$$g_{d,h(d)}(x, y) < g_{h(d)}^t + \varepsilon, \forall (x, y) \in X_d \times Y_{h(d)}$$

In particular, $h(d)$ must be matched with someone as its current payoff is higher than its IRP. Let $k = \mu(h(d))$, that is, $h(d)$'s partner at time t . Since k is matched with $h(d)$, she is not matched with her best stable hospital $h(k)$. Let $r < t$ be the iteration when k proposed to $h(d)$. She did it either because she achieved a higher payoff than with $h(k)$, or because $h(k)$ was not a feasible solution to the optimal proposal problem (Problem 1.1). If the first case holds, by the identification property and as $\varepsilon < \varepsilon^*$, there exists a contract $(x', y') \in X_k \times Y_{h(d)}$ such that

$$f_{k,h(d)}(x', y') > f_{k,h(k)}(\pi_D) + \varepsilon$$

Thus, any proposition made to $h(d)$ between iterations r and t was won by k , as $h(d)$ continued being her best option (in particular better than $h(k)$), and it increased the payoff of $h(d)$. However, if (x'', y'') is the strategy profile played by k and $h(d)$ at iteration t , k gets a payoff ε higher than with $h(k)$ by matching with $h(d)$ and playing (x'', y'') , and for any $(x, y) \in X_d \times Y_{h(d)}$,

$$g_{d,h(d)}(x, y) < g_{k,h(d)}(x'', y'') + \varepsilon$$

Therefore, $(k, h(d))$ is an ε -blocking pair of π_D , a contradiction. Suppose that $h(k)$ was not a feasible solution for k at time r , as $h(k)$ was matched with another doctor and getting a payoff too high. Inductively, build a finite sequence of doctors $\{d_1, \dots, d_\ell\}$ (chronologically ordered) such that none of them was able to propose her best stable hospital. Consider the first of them, d_1 , and let k be the partner of $h(d_1)$ when d_1 proposed. Then, k and $h(d_1)$ form a blocking pair of π_D , a contradiction. \square

As a final result on the semi-lattice structure, we prove that our deferred-acceptance with competitions algorithm (Algorithm 2.1), when $C_{d,h} = PO(G_{d,h}), \forall (d, h) \in D \times H$, converges to the “best” ε -pairwise stable matching for doctors, for ε small enough.

Theorem 2.5.3. *If the identification property holds and ε is small enough, the output of the DAC algorithm, when $C_{d,h} = PO(G_{d,h}), \forall (d, h) \in D \times H$, is doctors ε -optimal.*

Proof. Let $d \in D$ be any doctor. From Lemma 2.5.2, along all iterations of the DAC algorithm, d can always propose to $h(d)$. Therefore, d 's final payoff is bounded from below by the one she can get with $h(d)$. Since this is also the maximum payoff that doctors can get in an ε -pairwise stable allocation, we conclude the proof. \square

Let us end this section by observing that arguments similar to [43] allow us to show that, under the identification property and for small enough values of ε , the deferred-acceptance with competitions algorithm is strategy-proof on the doctors' side: if doctors revealed their utility functions to a designer who runs the algorithm, doctors' ε -dominant strategy is to submit their true utility functions. In other words, doctors cannot increase their payoff by more than ε by misreporting their utility functions. Taking $\varepsilon \rightarrow 0$, we obtain the existence of a strategy-proof pairwise-stable allocation for doctors (and so, exchanging the roles, also for hospitals).

2.6 Shapley-Shubik and Demange-Gale models

Particular cases of strictly competitive matching games are the Shapley-Shubik assignment game [93] and the Demange-Gale matching with transfers model [39]. This section is dedicated to formally establish the connection between their models and ours. The Assignment game consists of a housing market with buyers and sellers. Each seller has a house to sell and each buyer is interested in buying a house. A solution to this problem is a pair (μ, \vec{p}) , with μ a matching between sellers and buyers, and \vec{p} a vector of positive monetary transfers from buyers to sellers. Each seller $h \in H$, has a cost of her house c_h , and each buyer $d \in D$, has a valuation $v_{d,h}$ for h 's house. If seller h sells her house to d at price $p_{d,h} \geq 0$, their payoffs are,

$$f_{d,h}(p_{d,h}) = v_{d,h} - p_{d,h}, \quad g_{d,h}(p_{d,h}) = p_{d,h} - c_h$$

respectively for buyer and seller. Demange and Gale generalized the problem by considering that whenever two agents d and h are paired, their payoffs are given by some strictly increasing and continuous payoff functions $\phi_{d,h}(t)$ for d , and $\psi_{d,h}(-t)$ for h , with $t \in \mathbb{R}$ being the net transfer from d to h ($t \geq 0$ means that d pays t to h and $t \leq 0$ means that h pays $-t$ to d).

This model can be mapped into a matching game in which all couples play strictly competitive games. Formally, given $(d, h) \in D \times H$, consider the strategic game $G_{d,h} = (X_d, Y_h, f_{d,h}, g_{d,h})$, with $X_d = Y_h = \mathbb{R}_+$, and,

$$f_{d,h}(x_d, y_h) = \phi_{d,h}(y_h - x_d), \quad g_{d,h}(x_d, y_h) = \psi_{d,h}(x_d - y_h)$$

As $G_{d,h}$ belongs to \mathcal{S} , we can apply the results on Section 2.4 and recover the existence of pairwise stable allocation and their lattice structure.

It is important to remark that, although the strategy sets in the Demange-Gale matching game are not compact, as transfers are naturally bounded by players' valuation or by the individually rational payoffs, the problem can easily be compactified, satisfying the assumptions for the existence of pairwise stable allocations (Theorem 2.2.4).

Remark 2.6.1. The unique Nash equilibrium of the strictly competitive games is $x^* = y^* = 0$: no monetary transfers in the Demange-Gale's problem (or zero prices in the Shapley-Shubik housing market). Hence, Nash stability (Definition 1.2.2) is incompatible with the solutions of the matching with transfer models. As will be seen in the next section, this is not the case with our new notion of *renegotiation proofness*.

2.7 A numerical example

Consider a matching game with three doctors $D = \{d_1, d_2, d_3\}$ and three hospitals $H = \{h_1, h_2, h_3\}$, with the following preferences,

$$A = \begin{array}{|c|c|c|} \hline 83 & 85 & 99 \\ \hline 74 & 13 & 15 \\ \hline 58 & 49 & 54 \\ \hline \end{array} \quad B = \begin{array}{|c|c|c|} \hline 69 & 6 & 28 \\ \hline 88 & 2 & 70 \\ \hline 72 & 18 & 9 \\ \hline \end{array}$$

$A(d, h)$ (resp. $B(d, h)$) represents the fixed utility that $d \in D$ (resp. $h \in H$) receives if she (resp. it) matches with h (resp. d). Suppose that all agents have null IRPs, i.e., $(\underline{f}_d, \underline{g}_h) = (0, 0), \forall (d, h) \in D \times H$. Suppose, in addition, that if a couple (d, h) is created, as in the transfer matching game example (Example 1.1.6), they play a constant-sum game. Formally, if $x_d, y_h \geq 0$ are d and h respective monetary transfers, their utilities are,

$$\begin{aligned} f_{d,h}(x_d, y_h) &= A(d, h) - x_d + y_h \\ g_{d,h}(x_d, y_h) &= B(d, h) + x_d - y_h \end{aligned}$$

Unlike Nash stability, as it will be seen now, the pairwise stability concept under commitment will output an allocation in which (some) agents will do make a positive transfer. We recall the DAC algorithm for this example.

Let $\varepsilon = 1$. The initial payoff profiles are,

$$f^0 = (0, 0, 0), \quad g^0 = (0, 0, 0)$$

Let us consider the same order that for Section 1.5:

Iter 1. Suppose d_1 is the first proposer. She solves

$$\begin{aligned} &\max\{A(d_1, h) - x_{d_1} + y_h : B(d_1, h) + x_{d_1} - y_h \geq g_h^0(\pi) + \varepsilon, h \in H\} \\ \iff &\max\{A(d_1, h) + B(d_1, h) - (g_h^0(\pi) + \varepsilon) : h \in H\} \\ \iff &\max\{83 + 69 - 1, 85 + 6 - 1, 99 + 28 - 1\} = 151 \end{aligned}$$

The optimal proposal for doctor d_1 is $(h_1, 0, 68)$ ², i.e. d_1 proposes to h_1 and takes the highest possible profit from their transfer game by offering to h_1 exactly its IRP plus

²Any transfer profile (x_d, y_h) satisfying $-x_d + y_h = 68$ is also an optimal proposal. Taking, in particular, an optimal proposal in which one of the agents makes a null transfer will be useful for the next section.

Algorithm 2.2: Deferred-acceptance with competitions algorithm

```

1 Set  $D' \leftarrow \{d_1, d_2, d_3\}$  as the set of single doctors, and  $g_h(\pi) \leftarrow 0, \forall h \in H$ 
2 while  $D' \neq \emptyset$  do
3   Let  $d \in D'$ . Compute her optimal proposal  $(h, x_d, y_h)$  solving,
      
$$\begin{aligned} & \max A(d, h') - x_d + y_{h'} \\ & \text{s.t. } B(d, h') + x_d - y_{h'} \geq g_{h'}(\pi) + \varepsilon, \\ & h' \in \{h_0, h_1, h_2, h_3\}, x_d, y_{h'} \geq 0 \end{aligned}$$

4   if  $h$  is single then
5     |  $d$  is automatically accepted
6   end
7   else
8     |  $d$  and  $\mu(h)$  compete for  $h$ : Each of them computes,
      
$$\max\{B(d, h) + x_d - y_h : A(d, h) - x_d + y_h \geq \beta_d\}$$

      with  $\beta_d$  being  $d$ 's reservation price. The doctor with the highest bid
      wins and pays the second price. The loser is included in  $D'$ 
9   end
10 end
    
```

ε . Since h_1 is single, d_1 is accepted. The payoff profiles at the end of iteration 1 are,

$$f^1 = (151, 0, 0), \quad g^1 = (1, 0, 0)$$

Iter 2. Suppose d_3 proposes next. Similarly, she solves the problem,

$$\begin{aligned} & \max\{A(d_3, h) - x_{d_3} + y_h : B(d_3, h) + x_{d_3} - y_h \geq g_h^1(\pi) + \varepsilon, h \in H\} \\ \iff & \max\{A(d_3, h) + B(d_3, h) - (g_h^1(\pi) + \varepsilon) : h \in H\} \\ \iff & \max\{58 + 72 - (1 + 1), 49 + 18 - 1, 54 + 9 - 1\} = 128 \end{aligned}$$

The optimal proposal for doctor d_3 is $(h_1, 0, 70)$. Since h_1 is already matched, d_1 and d_3 compete. We compute their reservation prices β . Recall that reservation prices correspond to the highest payoffs that each doctor can obtain with somebody else different from h_1 (the most preferred hospital of the doctors). Formally,

$$\begin{aligned} \beta_1 &= \max\{A(d_1, h) - x_{d_1} + y_h : B(d_1, h) + x_{d_1} - y_h \geq g_h^1(\pi) + \varepsilon, h \in \{h_2, h_3\}\} \\ &= \max\{85 + 6 - 1, 99 + 28 - 1\} = 126 \end{aligned}$$

Analogously, $\beta_3 = 66$. We compute the bids.

$$\lambda_1 = \max\{B(d_1, h_1) + x_{d_1} - y_{h_1} : A(d_1, h_1) - x_{d_1} + y_{h_1} \geq 126\}$$

2.7. A numerical example

$$\begin{aligned} \iff \lambda_1 &= \max\{69 + x_{d_1} - y_{h_1} : 83 - x_{d_1} + y_{h_1} \geq 126\} \\ \iff \lambda_1 &= \max\{69 + x_{i_1} - y_{j_1} : -43 \geq x_{d_1} - y_{h_1}\} = 26 \end{aligned}$$

Similarly for d_3 ,

$$\lambda_3 = \max\{72 + x_{d_3} - y_{h_1} : 58 - x_{d_3} + y_{h_1} \geq 66\} = 64$$

Since d_3 's bid is the highest, she wins the competition. Finally, she pays the second highest price,

$$\begin{aligned} &\max\{A(d_3, h_1) - x_{d_3} + y_{h_1} : B(d_3, h_1) + x_{d_3} - y_{h_1} \geq \lambda_1\} \\ \iff &\max\{58 - x_{d_3} + y_{h_1} : 46 \geq -x_{d_3} + y_{h_1}\} = 58 + 46 = 104 \end{aligned}$$

Therefore, d_3 decreases her bid until getting a payoff of 104. In other words, the final transfer profile between d_3 and h_1 is $(x_{d_3}, y_{h_1}) = (0, 46)$. The payoff profiles at the end of iteration 2 are,

$$f^2 = (0, 0, 104), \quad g^2 = (26, 0, 0)$$

Iter 3. Suppose d_2 proposes next. Her optimal proposal is,

$$\begin{aligned} &\max\{A(d_2, h) - x_{d_2} + y_h : B(d_2, h) + x_{d_2} - y_h \geq g_h^2(\pi) + \varepsilon, h \in H\} \\ \iff &\max\{A(d_2, h) + B(d_2, h) - (g_h^2(\pi) + \varepsilon) : h \in H\} \\ \iff &\max\{74 + 88 - (26 + 1), 13 + 2 - 1, 15 + 70 - 1\} = 135 \end{aligned}$$

Thus, the optimal proposal of doctor d_2 is $(h_1, 0, 61)$, and has to compete against d_3 . The reservation prices of the doctors are $\beta_2 = 84$ and $\beta_3 = 66$. The bids are given by,

$$\begin{aligned} \lambda_2 &= \max\{B(d_2, h_1) + x_{d_2} - y_{h_1} : A(d_2, h_1) - x_{d_2} + y_{h_1} \geq \beta_2\} = 78 \\ \lambda_3 &= \max\{B(d_3, h_1) + x_{d_3} - y_{h_1} : A(d_3, h_1) - x_{d_3} + y_{h_1} \geq \beta_{d_3}\} = 64 \end{aligned}$$

Since λ_2 is the highest bid, d_2 is the winner of the competition. She decreases her offer to match the one of d_3 ,

$$\max\{A(d_2, h_1) - x_{d_2} + y_{h_1} : B(d_2, h_1) + x_{d_2} - y_{h_1} \geq \lambda_3\} = 98$$

Therefore, the final transfer profile between d_2 and h_1 is $(x_{d_2}, y_{h_1}) = (0, 24)$. The payoff profiles at the end of iteration 3 are,

$$f^3 = (0, 98, 0), \quad g^3 = (64, 0, 0)$$

Iter 4. Suppose d_1 proposes next. She solves,

$$\begin{aligned} &\max\{A(d_1, h) - x_{d_1} + y_h : B(d_1, h) + x_{d_1} - y_h \geq g_h^3(\pi) + \varepsilon, h \in H\} \\ \iff &\max\{A(d_1, h) + B(d_1, h) - (g_h^3(\pi) + \varepsilon) : h \in H\} \end{aligned}$$

$$\iff \max\{83 + 69 - (64 + 1), 85 + 6 - 1, 99 + 28 - 1\} = 126$$

Notice how h_1 is not the most preferred option for d_1 anymore, as h_1 's payoff is too high for her. Instead, she prefers to propose to h_3 the transfer profile $(0, 27)$. As j_3 is unmatched, d_1 is accepted. The payoff profiles at the end of iteration 4 are,

$$f^4 = (126, 98, 0), \quad g^4 = (64, 0, 1)$$

Iter 5. Finally, d_3 proposes again as she is the only single doctor. Her optimal proposal is,

$$\begin{aligned} & \max\{A(d_3, h) - x_{d_3} + y_h : B(d_3, h) + x_{d_3} - y_h \geq g_h^4(\pi) + \varepsilon, h \in H\} \\ \iff & \max\{A(d_3, h) + B(d_3, h) - (g_h^4(\pi) + \varepsilon) : h \in H\} \\ \iff & \max\{58 + 72 - (64 + 1), 49 + 18 - 1, 54 + 9 - (1 + 1)\} = 66 \end{aligned}$$

Notice that d_3 does not want to propose to h_1 anymore, due to having lost the competition against d_2 . d_3 offers to h_2 the transfer profile $(0, 17)$. Since h_2 is single, d_3 is automatically accepted and the algorithm stops. The final payoff profiles are,

$$f^5 = (126, 98, 66), \quad g^5 = (64, 1, 1)$$

The deferred-acceptance with competitions algorithm outputs $\pi = (\mu, \vec{x}, \vec{y})$ where

$$\mu = ((d_1, h_3), (d_2, h_1), (d_3, h_2))$$

and $(\vec{x}, \vec{y}) = ((0, 0, 0), (24, 17, 27))$. The final payoffs of the players are,

$$(f(\pi), g(\pi)) = ((126, 98, 66), (64, 1, 1))$$

To check the ε -pairwise stability of π , we compute the reservation prices of one of the sides and check that none of them is ε higher than the players' payoff. Formally, for $d \in D$, we compute β_d by,

$$\max\{A(d, h) - x_d + y_h : B(d, h) + x_d - y_h \geq g_h(\pi) + \varepsilon, h \in H \setminus \{\mu(d)\}, x_d, y_h \geq 0\}$$

and check if for every $d \in D$, $\beta_d \leq f_d(\pi) + \varepsilon$, that is,

$$\max\{A(d, h) + B(d, h) - (g_h(\pi) + \varepsilon), h \in H \setminus \{\mu(d)\}\} \leq f_d(\pi) + \varepsilon$$

We obtain that $\beta = (\beta_{d_1}, \beta_{d_2}, \beta_{d_3}) = (89, 83, 65)$. Since doctors' payoffs in π are strictly higher than their outside options, none of them has the incentive to change partners. We conclude that π is ε -pairwise stable, for $\varepsilon = 1$.

Remark 2.7.1. As the payoff matrices have integer values, the computed solution is indeed a 0-pairwise stable allocation.

2.8 Conclusions

The one-to-one matching games model with commitment is studied in this chapter and the notion of *pairwise stability* (Definition 2.2.1) is extended to this setting, capturing the solution concepts of Gale-Shapley [53], Shapley-Shubik [93], and Demange-Gale [39], among others.

Using the deferred-acceptance with competitions algorithm (Algorithm 2.1) designed in Chapter 1 the existence of ε -pairwise stable allocations is established under classical game theory assumptions. Even more, as for Nash stable allocations, we are able to prove the existence of 0-pairwise stable allocations under the same assumptions (Theorem 2.2.4).

A semi-lattice structure (Section 2.3) of the set of pairwise stable allocations has been established for any matching game in which an identification property (Definition 2.3.1) holds. Even more, we were able to obtain a full lattice structure for zero-sum and strictly competitive matching games (Section 2.4), in particular recovering the results of Demange and Gale.

Pairwise stability studies the deviations in which pairs of agents prefer to match together abandoning their assigned partner. However, no stability in terms of strategies has been covered so far. The following chapter is devoted to working on this issue, defining the novel concept of *renegotiation proofness*, refining the literature solutions.

1-to-1 Matching games with commitment: Renegotiation proofness

Chapter 1 showed that restricting agents' deviations on actions by making them play Nash equilibria is too demanding and we fail to capture the classical models of stable matching. As agents can commit in actions we will be able to relax Nash stability to the novel notion of *renegotiation proofness* in which agents will play constrained Nash equilibria, i.e., strategy profiles in which both players within a couple best reply to the partner subject to a *participation constraint*. Renegotiation proofness will refine pairwise stability (studied in Chapter 2). Thus, it will refine the solution concepts of the literature as well.

Constrained Nash equilibria will result to be the solutions to quasi-variational inequalities. Therefore, their existence cannot be always guaranteed. Due to this, a new class of games, *feasible games*, is considered. We prove that many well-known games from the literature on game theory are feasible. The proof of feasibility is game dependent as it uses the properties of each of them.

We design a *renegotiation process* which, for any feasible matching game instance and starting from any pairwise stable allocation, outputs a pairwise-renegotiation-proof stable allocation if it converges. Finally, we prove the convergence of the algorithm for many feasible games by designing game dependent oracles.

Renegotiation proofness of an allocation and Pareto-optimality in each game will not always be compatible. For static games as constant-sum or strictly competitive games, however, we will design oracles that will choose Pareto-optimal constrained Nash equilibria for each couple.

Interesting properties will rise for matching games in which couples play infinitely repeated games. Thanks to the ability of the agents to punish their partners in case of deviation in actions, formal commitment between agents will not be required anymore and the equivalence between renegotiation proofness and allocations in which agents play only uniform equilibria (Nash stability (Definition 1.2.2)) will be obtained. In other words, the model with commitment and without commitment will coincide. Even more, we will show how to compute allocations in which agents play Pareto-optimal uniform equilibria, obtaining the most powerful notion of stability in our model: Pareto-optimality, pairwise stability, and Nash stability.

The chapter is structured as follows. Section 3.1 introduces the notion of *renegotiation proofness* and applies it to the examples considered in the previous chapters. Section 3.2 characterizes the pairwise stable and renegotiation proof allocations for any matching game in which all couples play the same game. Section 3.3 introduces constrained Nash equilibria and characterizes them as the solutions of a quasi-variational

equation. Section 3.4 introduces the novel class of *feasible games* and proves that many well-known games are feasible. Section 3.5 designs the *renegotiation process* algorithm, shows that any output of the algorithm is pairwise stable and renegotiation-proof, and proves its convergence for many feasible games. Section 3.6 shows the extra properties satisfied by the allocations of matching games in which agents play infinitely repeated games. Section 3.7 shows a numerical example of the renegotiation process. Finally, Section 3.8 concludes the chapter.

3.1 Renegotiation proofness

We start by introducing the following useful notation: Given a strategy profile $\vec{x} \in X_D$, and $s_d \in X_d$ a particular strategy of some fixed doctor $d \in D$, we write (\vec{x}_{-d}, s_d) to the strategy profile obtained when replacing x_d , the strategy of doctor d in \vec{x} , by s_d .

Definition 3.1.1. A **pairwise stable** allocation $\pi = (\mu, \vec{x}, \vec{y})$ is **renegotiation proof** if for any couple $(d, h) \in \mu$ and any $(s_d, t_h) \in X_d \times Y_h$, it holds,

1. If $f_{d,h}(s_d, y_h) > f_d(\pi)$ then, $(\mu, (\vec{x}_{-d}, s_d), \vec{y})$ is not pairwise stable,
2. If $g_{d,h}(x_d, t_h) > g_h(\pi)$ then, $(\mu, \vec{x}, (\vec{y}_{-h}, t_h))$ is not pairwise stable.

Condition 1 says that no matched doctor d can profitably deviate in actions without breaking the pairwise stability of the allocation or, in other words, without creating a blocking pair or violating the IRP of some agent. Condition 2 is the dual property for hospitals.

In terms of contract theory, an allocation will be renegotiation proof if any profitable deviation of an agent from the agreed strategy profile is rejected by the partner as she will prefer her outside option or changing of partner.

Renegotiation proofness can be written as a generalized Nash equilibrium condition subject to a participation constraint: players must choose contracts that maximize their payoff under the constraint that the partners still agree to sign the contract (Proposition 3.5.2).

An interesting family of strategic games in which the output of the deferred-acceptance with competitions algorithm is not only pairwise stable but also renegotiation proof is the class of **common interest games**, in which $f_{d,h} = g_{d,h}$ for any couple $(d, h) \in D \times H$, as every time a doctor maximizes her payoff, she also does it for her partner hospital. In general, however, the constructed pairwise stable allocation will not be renegotiation proof. We will see in the following sections how, under some assumptions on the family of strategic games, one can, from any pairwise stable allocation, construct a pairwise stable and renegotiation proof one.

Let us see the impact on renegotiation proofness in our leading examples.

Example 1.1.4. Recall the prisoners' dilemma matching game example with two players, both with positive IRP δ , and payoff matrix,

		Agent h	
		Cooperate	Betray
Agent d	Cooperate	$2\delta, 2\delta$	$-\delta, 3\delta$
	Betray	$3\delta, -\delta$	$0, 0$

We have found that any allocation in which agents match and play a mixed strategy $(x, y) = ((p, 1 - p), (q, 1 - q))$ satisfying,

$$q \in \left[\frac{1+p}{3}, 3p-1 \right], \text{ such that } p, q \in [0, 1] \tag{3.1}$$

is pairwise stable. An allocation is renegotiation proof if any profitable deviation in actions of a player breaks its pairwise stability. Recall that the expected payoffs of the agents are $f_d(x, y) = \delta(3q - p)$ and $g_h(x, y) = \delta(3p - q)$.

Since there are no other couples, the only way to break the pairwise stability of the allocation is to decrease the payoff of the partner below δ by deviating. Imposing the agents' payoffs equal to δ , we obtain the strategy profile $(x^*, y^*) = ((1/2, 1/2), (1/2, 1/2))$ (the bottom-left vertex of the light blue region in Figure 2.1). Let us prove that matching and playing (x^*, y^*) is pairwise stable and renegotiation proof. First of all, since (x^*, y^*) satisfies Equation (3.1) the allocation is pairwise stable. Take $\varepsilon > 0$ and suppose that player d deviates to $x' = (1/2 - \varepsilon, 1/2 + \varepsilon)$. Then,

$$f_d(x', y^*) = \delta \left(3 \cdot \frac{1}{2} - \frac{1}{2} + \varepsilon \right) = f_d(x^*, y^*) + \delta\varepsilon > f_d(x^*, y^*),$$

so d has a profitable deviation. In general, agents profitably deviate if they decrease the probability of cooperating. It follows that,

$$g_h(x', y^*) = \delta \left(3 \cdot \left(\frac{1}{2} - \varepsilon \right) - \frac{1}{2} \right) = g_h(x^*, y^*) - 3\delta\varepsilon = \delta(1 - 3\varepsilon) < \delta$$

Thus, any profitable deviation of player d from (x^*, y^*) decreases the payoff of player h below δ violating its individual rationality. By symmetry, the same holds for any profitable deviation of player h from (x^*, y^*) . Therefore, matching together and playing (x^*, y^*) is pairwise stable and renegotiation proof. \square

Remark 3.1.2. It is easy to check that matching and playing $((1/2, 1/2), (1/2, 1/2))$ in the previous example is the only pairwise stable and renegotiation proof allocation. In particular, from the continuum of pairwise stable allocations, there is only one that is also renegotiation proof.

Example 1.1.5. Recall the coordination matching game example with two players, both with positive IRPs δ , and payoff matrix,

		Agent h	
		A	B
Agent d	A	$4\delta, \delta/2$	$0, 0$
	B	$0, 0$	$\delta/2, 4\delta$

3.1. Renegotiation proofness

We have found that any allocation in which agents match and play a strategy profile $(x, y) = ((p, 1 - p), (1 - q, q))$ satisfying,

$$q \in \left[\frac{2 - p}{8 - 9p}, \frac{8p - 2}{9p - 1} \right], \text{ such that } p, q \in [0, 1]$$

is pairwise stable. As for the previous example, setting the agents' payoff equal to their IRPs, we obtain two possible solutions $((2/3, 1/3), (1/3, 2/3))$ and $((1/3, 2/3), (2/3, 1/3))$. It is easy to check that any profitable deviation of an agent from these strategy profiles reduces the partner's payoff below δ , breaking the pairwise stability of the allocation. Even more, these are the only strategy profiles with this property. Therefore, we obtain (only) two pairwise stable and renegotiation proof allocations. \square

Example 1.1.6. Recall the transfer matching game with two agents, both with positive IRPs δ , who play the constant-sum game,

$$\begin{aligned} G &= (\mathbb{R}_+, \mathbb{R}_+, f, g), \text{ such that for any } x, y \geq 0, \\ f(x, y) &= 10\delta - x + y \\ g(x, y) &= x - y \end{aligned}$$

Pairwise stability implies that players match and make a transfer profile (x, y) such that $\delta \leq x - y \leq 9\delta$. If $x - y > \delta$, decreasing slightly x increases d 's payoff without violating h 's individual rationality. Thus, renegotiation proofness implies that $x - y = \delta$. If $x - y = \delta$ and $y > 0$, decreasing slightly y increases h 's payoff without inciting d to leave the couple. Thus, an allocation is pairwise stable and renegotiation proof if and only if d and h agree to match, $x = \delta$ and $y = 0$. We recover the competitive price equilibrium of Shapley and Shubik. \square

Example 1.1.7. Recall the auction example in which a seller sells an indivisible good to a set of n buyers. Buyers have valuations $v_1 > v_2 \geq \dots \geq v_n$ for the good, the seller has valuation c for the good, and everybody has null IRP. If a couple buyer-seller (d, h) is created and the monetary transfers (x_d, y_h) is agreed, the item is sold from h to d at the price $p = x_d - y_h$. In particular, the utilities of the players are,

$$\begin{aligned} f_d(x_d, y_h) &= v_d - p = v_d - x_d + y_h, \\ g_h(x_d, y_h) &= p - c = x_d - y_h - c, \end{aligned}$$

Finally, recall we have assumed $v_1 \geq c$. We have found that pairwise stability implies that the good is sold to buyer 1 for a price $p_1 \in [\max(c, v_2), v_1]$. In this continuum, the unique renegotiation proof allocation is $p_1 = \max(c, v_2)$ (we recover the outcome of the *2nd price auction*). Indeed, if $p_1 > \max(c, v_2)$, decreasing slightly p_1 increases buyer 1's payoff without breaking the pairwise stability as the seller still prefers to be matched with her. If $p_1 = \max(c, v_2)$ and $c > v_2$, decreasing slightly p_1 turns seller's payoff negative making her to prefer to become unmatched. Similarly, if $v_2 > c$, decreasing

slightly p_1 creates a blocking pair as buyer 2 and the seller could match together and increase strictly their payoffs by trading the good at some price between p_1 and v_2 .

As another example, Section 3.2 discusses the particular case of **symmetric matching games**, in which all couples play the same game, i.e when strategy sets and payoff functions are couple independent.

3.2 Symmetric matching games

Along this section and to keep things simple, suppose that D and H have the same size and that all agents have very low individually rational payoffs, so they prefer to be matched rather than being single. The main conclusion is: in a large class of situations, pairwise stable and renegotiation proof allocations exist and are those where all couples get the same Pareto-optimal payoff.

Formally, suppose that $f_{d,h} = f$, $g_{d,h} = g$, $X_d = S$, $Y_h = T$, for any couple $(d, h) \in D \times H$, meaning that the game played in every couple is independent of the identities of the players. Denote by G the two-player game with action sets S and T and payoff functions f and g . Suppose in addition that S and T are **convex** sets and that f and g are **continuous** and **own-payoff q-concave** (Definition 3.2.1).

Definition 3.2.1. A function f is **q-concave** if for any (s, s', u) , such that $f(s) \geq u$ and $f(s') > u$, then, $f(s'') > u$, for any $s'' \in (s, s')$.

Observe that if $s \mapsto f(s, t)$ is concave, linear, or strictly quasi-concave then, it is q-concave. Thus, we cover the class of finite games in mixed strategies (§4.7.2 [77]). Also, q-concavity implies quasi-concavity.

Definition 3.2.2. A game G satisfies the **Pareto transfer property** if for any pair of Pareto-optimal payoffs (f_1, g_1) , (f_2, g_2) such that $f_1 < f_2$ and $g_2 < g_1$, there exists a Pareto-optimal strategy profile (s', t') satisfying,

$$f_1 < f(s', t') < f_2 \text{ and } g_2 < g(s', t') < g_1$$

Expressed differently, a game satisfying the Pareto transfer property has a Pareto frontier that allows to continuously transfer utility from one player to the other, which is the case in many games. The following theorem characterizes the pairwise stable and renegotiation proof allocations for symmetric matching games.

Theorem 3.2.3. *Let (s, t) be Pareto-optimal in G . Let μ be any matching and suppose that any matched couple (d, h) in μ plays $(x_d, y_h) = (s, t)$. Then, (μ, \vec{s}, \vec{t}) is pairwise stable and renegotiation proof. Conversely, if $\pi = (\mu, \vec{x}, \vec{y})$ is pairwise stable and renegotiation proof, then any matched couple, except at most one, plays Pareto-optimally in its game. If moreover, the game satisfies the Pareto transfer property, then all couples playing Pareto-optimally share the same payoff.*

3.3. Constrained Nash equilibria

Proof. Let (s, t) be Pareto-optimal in G and suppose that any matched couple (d, h) in μ plays $(x_d, y_h) = (s, t)$. It is not possible to find a blocking pair since any deviation from (s, t) reduces the payoff of one of the agents. Therefore, the allocation is pairwise stable. Regarding renegotiation proofness, consider an arbitrary matched couple (d, h) and suppose that player d has a profitable deviation s' in G . In particular, h 's payoff strictly decreases when d deviates. Because of continuity,

$$\exists \alpha \in (0, 1) : g(s', t) < g(\alpha s + (1 - \alpha)s', t) < g(s, t)$$

Because of q -concavity,

$$f(\alpha s + (1 - \alpha)s', t) > f(s, t)$$

Consequently, any d' not matched with h increases her payoff, as well as h 's payoff, by forming a couple and playing $(\alpha s + (1 - \alpha)s', t)$. Thus, the deviation of player d creates the blocking pair (d', h) , breaking the pairwise stability.

Conversely, consider two matched couples (d_1, h_1) and (d_2, h_2) with payoffs (f_1, g_1) and (f_2, g_2) . Suppose that $f_1 \leq f_2$. If (f_2, g_2) is Pareto-dominated, d_1 can replace d_2 by proposing to h_2 a Pareto-optimal improvement, so the pair (d_1, h_2) blocks the allocation. A contradiction, thus (f_2, g_2) is Pareto optimal. Therefore, all couples, except perhaps one, play Pareto-optimally in their game. Suppose now the game satisfies the Pareto transfer property and that couples 1 and 2 are playing Pareto optimally but their payoffs differ. Without loss of generality, suppose that

$$f_1 = f(x_1, y_1) < f(x_2, y_2) = f_2$$

Then, $g_1 = g(x_1, y_1) > g(x_2, y_2) = g_2$. By the Pareto transfer property, there exists a Pareto-optimal strategy profile (s', t') such that

$$f_2 > f(s', t') > f_1 \text{ and } g_1 > g(s', t') > g_2$$

Thus, (d_1, h_2) is a blocking pair of π , a contradiction. Consequently, all couples share the same payoff. \square

Theorem 3.2.3 shows that pairwise stability and renegotiation proofness have the feature to induce cooperation (e.g. Pareto optimality) but also some uniformity in behaviors (couples share the same payoffs), in the case of symmetric matching games.

3.3 Constrained Nash equilibria

In this section the concept of constrained Nash equilibrium (CNE) is introduced and expressed through a quasi-variational inequality formulation. The renegotiation proofness characterization by constrained Nash equilibria is delegated to Section 3.5.

Constrained Nash equilibria are defined for any two-player game and do not depend on the whole matching game Γ considered. Because of this, during Sections 3.3 and 3.4 we will only consider one strategic game $G = (X, Y, f, g)$, where X, Y are compact strategy sets and f, g are continuous payoff functions. We endow each player with an *outside option*, $f_0, g_0 \in \mathbb{R}$, respectively.

Remark 3.3.1. Outside options will play a key role in the characterization of renegotiation proofness: When agents have outside options equal to their reservation payoffs, an allocation will be renegotiation proof if and only if all couples of players play a constrained Nash equilibrium. Further assumptions and how to compute these values are studied in Section 3.5.

Definition 3.3.2. A strategy profile $(x, y) \in X \times Y$ is (f_0, g_0) -feasible if $f(x, y) \geq f_0$ and $g(x, y) \geq g_0$.

In words, a strategy profile is feasible for the players, for a given pair of outside options, if they can achieve at least these payoffs when playing that strategy profile.

Definition 3.3.3. A (f_0, g_0) -feasible strategy profile (x', y') is a (f_0, g_0) -constrained Nash equilibrium (CNE) if it satisfies:

$$\begin{aligned} f(x', y') &= \max\{f(x, y') : g(x, y') \geq g_0, x \in X\} \\ g(x', y') &= \max\{g(x', y) : f(x', y) \geq f_0, y \in Y\} \end{aligned} \quad (3.2)$$

We denote the set of (f_0, g_0) -constrained Nash equilibria as $\text{CNE}(f_0, g_0)$.

A strategy profile satisfies Equations (3.2) if any player's profitable deviation decreases the partner's payoff below her outside option. Equations (3.2) can be written as a quasi-variational inequality (QVI) [50, 61, 83] with point-to-set mappings that may fail to be lower semi-continuous. Due to this, the existence of constrained Nash equilibria cannot be always guaranteed. The rest of this section is devoted to showing this characterization, while the following section will deal with the existence of CNE.

Given a pair of outside options (f_0, g_0) , consider the point-to-set mappings,

$$K_1(y') := \{x \in X : g(x, y') \geq g_0\}, \quad K_2(x') := \{y \in Y : f(x', y) \geq f_0\} \quad (3.3)$$

A strategy profile (x', y') is a (f_0, g_0) -constrained Nash equilibrium if it solves,

$$\begin{aligned} f(x', y') &= \max_{s \in K_1(y')} f(s, y'), \\ g(x', y') &= \max_{t \in K_2(x')} g(x', t) \end{aligned} \quad (3.4)$$

Remark 3.3.4. Note that, for a given strategy profile (x', y') , the sets (3.3) are convex if the strategy sets X, Y are convex as well and the payoff functions are own-concave.

Equations (3.4) can be easily rewritten as a quasi-variational inequality [61]. We state this result without proof.

Proposition 3.3.5. *Let X, Y be closed convex subsets of \mathbb{R}^n and \mathbb{R}^m , respectively. In addition, suppose that the payoff functions f, g are concave and differentiable on the player's own strategy. Then, a strategy profile (x', y') satisfies Equation (3.4) if and only if it satisfies,*

$$\begin{aligned} \nabla_x f(x', y')^T \cdot (x' - s) &\geq 0, \forall s \in K_1(y') \\ \nabla_y g(x', y')^T \cdot (y' - t) &\geq 0, \forall t \in K_2(x') \end{aligned} \quad (3.5)$$

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Setting $K(x', y') = K_1(y') \times K_2(x')$, $L = [\nabla_x f, \nabla_y g]$, a strategy profile (x', y') satisfies Equations (3.4) if and only if it is solution to the QVI(K, L).

Proposition 3.3.5 states that any solution to the QVI(K, L) will satisfy Equations (3.2), however, it is not direct that the solution will be a feasible strategy profile.

Proposition 3.3.6. *Consider a strategy profile (x', y') , solution of QVI(K, L) as defined in Proposition 3.3.5. Then, (x', y') is a (f_0, g_0) -feasible strategy profile.*

Proof. Let (x', y') be a solution of QVI(K, L). It holds, $f(x', y') \geq f_0$, as $y' \in K_2(x')$, and $g(x', y') \geq g_0$, as $x' \in K_1(y')$. Therefore, (x', y') is (f_0, g_0) -feasible. \square

From Propositions 3.3.5 and 3.3.6, the constrained Nash equilibrium problem is a generalized Nash equilibrium problem in which the point-to-set mappings K_1 and K_2 give the feasible deviations of each player.

Considering convex and compact strategy sets X, Y , and continuously differentiable concave payoff functions f, g (standard assumptions on game theory), most of the conditions on the existence of QVI solutions theorems are satisfied [50, 61, 83]. Even more, being X, Y convex sets, f, g continuous and own-payoff concave functions, the applications K_1 and K_2 are closed and convex. By choosing outside options not too restrictive, we can easily obtain the non-emptiness of the point-to-set mappings. The upper semicontinuity is a consequence of the payoff functions' continuity and the fact that K_1 and K_2 , for some given strategy profile, are polytopes. Indeed, consider $(y_n) \subseteq Y$, $x \in X$ and $x_n \in K_1(y_n), \forall n \geq 0$. Consider in addition that $y_n \rightarrow y$ and $x_n \rightarrow x$. In particular, for any $n \geq 0$, $g(x_n, y_n) \geq g_0$, as $x_n \in K_1(y_n)$. Taking $n \rightarrow \infty$, $g_0 \leq g(x_n, y_n) \rightarrow g(x, y)$, as g is a continuous function. Thus, $x \in K_1(y)$. The only missing assumption for existence is the lower semicontinuity of the point-to-set mappings. This cannot be guaranteed.

3.4 Feasible games

Constrained Nash equilibria are the key to obtaining renegotiation proof allocations. Since CNE are not guaranteed to exist as the previous section explained, we will consider the following class of games.

Definition 3.4.1. A two-person game G is **feasible** if for any pair of outside options $(f_0, g_0) \in \mathbb{R}^2$, which admits at least one (f_0, g_0) -feasible strategy profile, there exists a (f_0, g_0) -CNE.

Feasibility is a necessary condition for the existence of pairwise stable and renegotiation proof allocations. Let us illustrate it with an example.

Example 1.1.6. Recall the transfer matching game example in which two agents d and h with positive IRPs δ can match and play,

$$G = (\mathbb{R}_+, \mathbb{R}_+, f, g), \text{ such that for any } x, y \geq 0,$$

$$f_{d,h}(x, y) = 10\delta - x + y$$

$$g_{d,h}(x, y) = x - y$$

Taking $(f_0, g_0) = (\delta, \delta)$, any transfer profile (x, y) satisfying $x - y \in (\delta, 9\delta)$ is (f_0, g_0) -feasible. The (f_0, g_0) -CNE of this game corresponds to the renegotiation proof allocation found before: $x = \delta$ and $y = 0$. \square

Feasible two-player games are strongly related to the solution set of the quasi-variational inequality formulation (Equations (3.5)). As we know that the non-emptiness of the solution set cannot be guaranteed, we conjecture that not all two-player games can be feasible. The following example¹ proves this conjecture.

Example 3.4.2. Let G be the following finite game played in mixed strategies.

	L	M	R
T	2,1	-10,-10	3,0
M	3,0	2,1	-10,-10
B	-10,-10	3,0	2,1

Game G has only one Nash equilibrium, completely mixed, with payoffs $-5/3$ and -3 for players 1 and 2, respectively. Take null outside options, i.e., $f_0 = g_0 = 0$. The set of (f_0, g_0) -feasible strategy profiles is non-empty as, for example, (T, L) , (M, L) , (M, M) , (B, M) , (T, R) , and (B, R) belong to it. However, there is no (f_0, g_0) -constrained Nash equilibrium. Let us prove this formally. Let $Z_{1,2}$ be the set of feasible contracts for players 1 and 2 and

$$(x, y) = ((x_1, x_2, x_3), (y_1, y_2, y_3))$$

be a mixed strategy profile that achieves some payoff (f', g') , in which each coordinate corresponds to playing Top, Medium, and Bottom, respectively for player 1, and Left, Medium, and Right, respectively for player 2.

If any of the players plays a pure strategy, the other player can improve her payoff by a unilateral deviation, still respecting the outside option of the first player. For example, consider that player 1 plays $x = (1, 0, 0)$, i.e., she plays Top. Then, player 2 can deviate to play Left with probability 1, increasing her payoff and still giving to player 1 a positive payoff, that is, respecting her outside option. In the same way, if player 2 plays Left, player 1 can deviate to play Medium. Thus, no pure strategy can be a constrained equilibrium.

The same holds if any of the players play a mixed strategy without full support. Consider that player 1 plays $(x_1, x_2, 0)$. Then, player 2 can deviate and play Left with probability 1 if x_1 is large enough, or a mixed strategy mixing only Left and Medium if x_2 is large enough. In any of the two cases, players converge to play pure strategies, which we already saw cannot be a constrained Nash equilibrium.

¹We want to thank Eilon Solan for having suggested this example.

3.4. Feasible games

Consider that both players play mixed strategies with full support. The Nash equilibrium of the game not belonging to $Z_{1,2}$, it cannot be the case that players play $(1/3, 1/3, 1/3)$. Without loss of generality, assume that $x_1 > 1/3 \geq x_2$. The expected payoff of player 2 is given by,

$$g' = g(x, y) = y_1(12x_1 + 21x_2 - 11) + y_2(-9x_1 + 12x_2 - 1) + (1 - x_1 - 11x_2)$$

It holds that x_1, x_2, y_1, y_2 are strictly positive and $x_1 + x_2 < 1$, $y_1 + y_2 < 1$, since players have full support. Then, $-9x_1 + 12x_2 - 1 < 0$, so player 2 can deviate and increase her own payoff by decreasing y_2 . The expected payoff of player 1 is,

$$f' = f(x, y) = y_1(11x_1 + 25x_2 - 12) + y_2(-14x_1 + 11x_2 + 1) + (x_1 - 12x_2 + 2)$$

It holds $-14x_1 + 11x_2 + 1 < 0$, so player 1 increases her payoff if y_2 decreases as well. Therefore, there exists a profitable deviation for player 2 that still guarantees to player 1 her outside option. Intuitively, since player 1 is more likely to play Top, it makes sense that both players improve their payoff if player 2 decreases the probability of playing Medium, so they avoid getting -10 . We conclude that (f', g') is not the payoff of a constrained equilibrium payoff, and therefore, the game G is not feasible. \square

Once stated that not all two-player games are feasible, we dedicate the rest of this section to prove the following theorem, showing the richness of the class of feasible games.

Theorem 3.4.3. *The class of feasible games includes,*

1. *Constant-sum games with a value*
2. *Strictly competitive games with an equilibrium*
3. *Potential games*
4. *Infinitely repeated games*

The proof that a game is feasible relies on the characteristic of the game. In other words, the proof is game-dependent. Therefore, we prove Theorem 3.4.3 in several subsections. In addition, we briefly recall each of the games mentioned in the theorem.

Infinitely repeated games, besides being feasible, will satisfy many interesting properties such as Nash stability (Definition 1.2.2) and Pareto-optimality in each game. The formal study is done in Section 3.4.4.

Remark 3.4.4. Extensive form games with perfect information result to be feasible as well. The proof is delegated to Appendix A.1 since we will not study these games during the rest of the thesis.

3.4.1 Zero-sum games with a value are feasible

A two-person game $G = (X, Y, f, g)$ is a zero-sum game if players' payoff functions satisfy $f(\cdot, \cdot) = -g(\cdot, \cdot) =: u(\cdot, \cdot)$.

Definition 3.4.5. Given (f_0, g_0) player's outside options, with $f_0 \leq g_0$, a strategy profile $(x, y) \in X \times Y$ is **feasible** if and only if it satisfies

$$f_0 \leq u(x, y) \leq g_0$$

A feasible payoff profile (x', y') is a (f_0, g_0) -**CNE** if for any $(x, y) \in X \times Y$, it holds that, if $u(x, y) > u(x', y')$ then, $u(x, y) > g_0$ and, if $u(x', y) < u(x', y')$ then, $u(x', y) < f_0$.

Proof. **Zero-sum games with a value are feasible.** Let $G = (X, Y, u)$ be a zero-sum game, with X, Y compact convex subsets of topological vector spaces and u separately continuous. Suppose the game G has a value w and by continuity of u and compactness of X and Y , players have optimal strategies (x^*, y^*) . Let (x', y') be a feasible contract $(f_0 \leq u(x', y') \leq g_0)$. The analysis is split into three cases.

Case 1. $f_0 \leq w \leq g_0$. The optimal contract (x^*, y^*) is feasible. Since (x^*, y^*) is a Nash equilibrium, it is a (f_0, g_0) -constrained Nash equilibrium.

Case 2. $w < f_0 < g_0$. Consider the set $A(f_0) := \{x \in X : \exists y \in Y, u(x, y) \geq f_0\}$. Since (x', y') is a feasible contract, $A(f_0)$ is non-empty. Consider the optimization problem

$$\sup [\inf \{u(x, y) : u(x, y) \geq f_0, y \in Y\} : x \in A(f_0)] \quad (\text{P})$$

For a given $x_0 \in A(f_0)$, the set $\{y \in Y : g(x_0, y) \geq f_0\}$ is bounded and so, there exists an infimum $y_0(x_0)$. Thus, as the set $A(f_0)$ is also bounded, there exists a supremum x_0 . Let $(x_0, y_0(x_0))$ be the pair supremum-infimum solution of (P). It holds that $u(x_0, y_0(x_0)) \geq f_0$ by construction. Suppose that $u(x_0, y_0(x_0)) > f_0$. Since $w < f_0$, it holds $w < f_0 < u(x_0, y_0(x_0))$. Considering the optimal contract (x^*, y^*) , it holds $u(x_0, y^*) \leq u(x^*, y^*) = w < f_0 < u(x_0, y_0(x_0))$. By continuity of the function $u(x_0, \cdot)$, there exists $\lambda \in (0, 1)$ such that $u(x_0, y_\lambda) = f_0$, with $y_\lambda = \lambda y^* + (1 - \lambda)y_0(x_0) \in Y$. This contradicts the fact that $(x_0, y_0(x_0))$ is the solution to (P). Thus, $u(x_0, y_0(x_0)) = f_0$. If this strategy profile is a constrained Nash equilibrium, the study of the second case is done. If not, consider $y_t \in Y$ as the convex combination between $y_0(x_0)$ and y^* with t computed by,

$$t := \sup \{\tau \in [0, 1] : y_\tau := (1 - \tau)y_0(x_0) + \tau y^* \text{ and } \exists x_\tau \in X, u(x_\tau, y_\tau) = f_0\} \quad (3.6)$$

t exists as for $\tau = 0$, there exists x_0 such that $u(x_0, y_0(x_0)) = f_0$. In addition, $y_t \neq y^*$, since the contract (x^*, y^*) is a saddle point, $u(x^*, y^*) = w < f_0$ and any deviation of player 1 decreases the payoff. Notice that any profitable deviation of player 2 decreases the payoff below f_0 , as $u(x_t, y_t) = f_0$. Suppose there exists $\hat{x} \in X$ such that $f_0 = u(x_t, y_t) < u(\hat{x}, y_t) \leq g_0$. As a summary, it holds: $u(x^*, y^*) = w < f_0 = u(x_t, y_t) < u(\hat{x}, y_t)$ with y_t in the interval $(y_0(x_0), y^*)$. Once again, by the continuity of u and the convexity of $X \times Y$, there exists an element z in the interval (y_t, y^*) and

3.4. Feasible games

some $x_z \in X$, such that $u(x_z, z) = f_0$, contradicting the definition of t in (3.6). Thus, (x_t, y_t) is a constrained Nash equilibrium.

Case 3. $f_0 < g_0 < w$. Analogous to case 2. □

Zero-sum games with a value are not only feasible but for any (f_0, g_0) -CNE (x, y) , it is satisfied,

$$u(x, y) = \text{median}(f_0, w, g_0)$$

This equation will be particularly useful later. Let us illustrate the previous proof with the following example.

Example 3.4.6. Consider the *matching pennies* game (Table 3.1) played in mixed strategies,

		Player 2	
		A	B
Player 1	A	1	-1
	B	-1	1

Table 3.1: Matching pennies

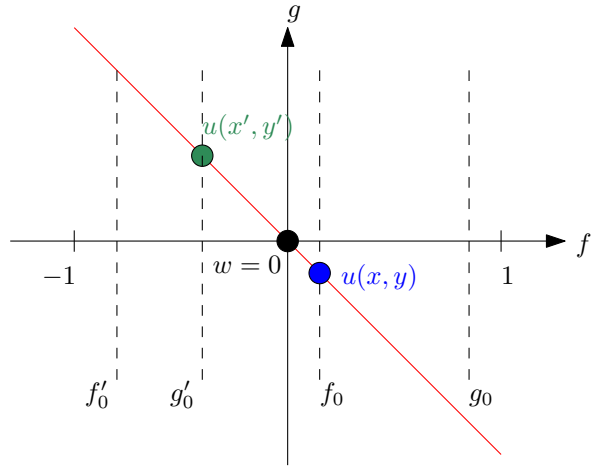


Figure 3.1: Feasible payoffs region

Matching pennies is a zero-sum game of value $w = 0$ and optimal mixed strategies $(x^*, y^*) = (1/2, 1/2)$. Figure 3.1 represents the payoff function u (in red), the value w (black dot), two pair of outside options (f_0, g_0) and (f'_0, g'_0) for the players, and two strategy profile payoffs $u(x, y)$ (blue dot) and $u(x', y')$ (green dot). It holds

$$w < f_0 < g_0 \text{ and } f'_0 < g'_0 < w$$

being, respectively, the second and third cases studied during the previous proof. We can observe that $(x, y) \in \text{CNE}(f_0, g_0)$ and $(x', y') \in \text{CNE}(f'_0, g'_0)$. Indeed, if $w < f_0 < g_0$ holds, any profitable deviation of player 2 from (x, y) makes the payoff of the game lower than f_0 , so player 1 receives less than her outside option. Analogously, in the case $f'_0 < g'_0 < w$, if player 1 can deviate from (x', y') increasing her payoff, player 2 receives less than g'_0 . □

Remark 3.4.7. From the nature of zero-sum games, we obtain that the constrained Nash equilibrium selection is always **Pareto-optimal**.

3.4.2 Strictly competitive games with an equilibrium are feasible

Definition 3.4.8. A two-player game $G = (X, Y, f, g)$ is **strictly competitive** if for any strategy profile $(x, y) \in X \times Y$, it holds,

$$\begin{aligned} \forall x' \in X : f(x', y) > f(x, y), g(x', y) < g(x, y), \text{ and,} \\ \forall y' \in Y : g(x, y') > g(x, y), f(x, y') < f(x, y) \end{aligned}$$

Recall the class \mathcal{S} of strictly competitive games à la Aumann, that is, all strictly competitive games obtained as monotone transformations of zero-sum games. To prove Theorem 3.4.3 we will use Aumann's conjecture that \mathcal{S} covers the entire class of competitive games, although we will restrict the analysis to finite strictly competitive games (the most general case for the one Aumann's conjecture has been proved so far) at the moment of studying the complexity of our algorithms (Section 5.3).

Consider a strictly competitive game $G = (X, Y, f, g)$, with X, Y compact sets and f, g continuous payoff functions. Let φ, ϕ be increasing functions such that the game $G' = (X, Y, u)$ is a zero-sum game where $f = \varphi \circ u$ and $g = \phi \circ u$. Nash equilibria of G and G' coincide, and Nash equilibrium payoffs are the image through the increasing functions from one game to another. In particular, if w is the value of G' , then $(\varphi^{-1}(w), \phi^{-1}(w))$ is a Nash equilibrium payoff of G . Let (f_0, g_0) be outside options of the players in G , and let (x^*, y^*) be a (f_0, g_0) -constrained Nash equilibrium. Consider the corresponding outside options in G' given by $f'_0 := \varphi(f_0)$, $g'_0 := -\phi(g_0)$. Indeed, f'_0 and g'_0 are outside options for the players in their zero-sum game as, for any $(x, y) \in X \times Y$ such that $f_0 \leq f(x, y)$ and $g_0 \leq g(x, y)$, it holds

$$f'_0 \leq \varphi(f_0) \leq \varphi(f(x, y)) = -\phi(g(x, y)) \leq g'_0$$

Also, (x^*, y^*) is (f'_0, g'_0) -feasible in G' , and it is direct that (x^*, y^*) is a (f'_0, g'_0) -constrained Nash equilibrium, as increasing functions preserve inequalities. From the proof of zero-sum games' feasibility, we conclude the following theorem.

Theorem 3.4.9. *Let $G = (X, Y, f, g)$ be a strictly competitive game, $G' = (X, Y, u)$ a zero-sum game, and φ, ϕ increasing functions such that the game $f = \varphi \circ u$, $g = \phi \circ u$. Suppose that G' has a value w (if and only if G has a Nash equilibrium). Then, given f_0, g_0 outside options in G , which admit a feasible contract, there always exists a (f_0, g_0) -CNE (x^*, y^*) of G . In addition, it holds*

$$\begin{aligned} f(x^*, y^*) &= \text{median}\{f_0, \varphi^{-1}(-\phi(g_0)), \varphi^{-1}(w)\} \\ g(x^*, y^*) &= \text{median}\{\phi^{-1}(-\varphi(f_0)), g_0, \phi^{-1}(-w)\} \end{aligned}$$

Remark 3.4.10. As for zero-sum games, the selection of constrained Nash equilibria is always **Pareto-optimal** for strictly competitive games.

3.4.3 Potential games are feasible

A two-person game $G = (X, Y, f, g)$ is a potential game if there exists a(n exact) potential function $\phi : X \times Y \rightarrow \mathbb{R}$ such that, $\forall x, x' \in X, y, y' \in Y$ it satisfies,

$$\begin{aligned}\phi(x', y') - \phi(x, y') &= f(x', y') - f(x, y), \text{ and} \\ \phi(x', y') - \phi(x', y) &= g(x', y') - g(x', y)\end{aligned}$$

Proof. Potential games are feasible. Let $G = (X, Y, f, g)$ be a potential game with potential function ϕ . Let (f_0, g_0) be outside options and Z_0 , be the set of all (f_0, g_0) -feasible contracts. Suppose $Z_0 \neq \emptyset$. We aim to prove that Z_0 includes at least one (f_0, g_0) -CNE. Consider

$$(x', y') \in \arg \max \{ \phi(x, y) : (x, y) \in Z_0 \}$$

Remark (x', y') always exists as Z_0 is a non-empty compact set and ϕ is continuous. It holds that (x', y') is (f_0, g_0) -feasible. Consider $x \in X$ such that $f(x, y') > f(x', y')$ and $g(x, y') \geq g_0$. In particular, $f(x, y') > f(x', y') \geq f_0$, so $(x, y') \in Z_0$. Moreover, $\phi(x, y') > \phi(x', y')$ as $f(x, y') > f(x', y')$. This contradicts that (x', y') belongs to the argmax of ϕ in Z_0 . The same holds for player 2. Thus, (x', y') is a (f_0, g_0) -CNE. \square

Unlike the previous feasible games studied, Pareto-optimality and constrained Nash equilibria are not always compatible for potential games. Consider, for example, the following prisoners' dilemma,

		Player 2	
		Cooperate	Betray
Player 1	Cooperate	2, 2	-1, 3
	Betray	3, -1	0, 0

Given outside options equal to 0 for both agents, the oracle designed in the previous proof selects the Nash equilibrium of the game, which is Pareto-dominated by cooperating. Pairwise stability (Definition 2.2.1) considers that all couples play Pareto-optimally as matched agents can also be blocking pairs. We recall this issue is not present in models with strict preferences or monetary transfers.

For matching games where all games are potential games we will be able to compute “weak” pairwise stable and renegotiation proof allocations where the only possible blocking pairs for the allocation will be matched couples not playing Pareto-optimally.

3.4.4 Infinitely repeated games are feasible

Consider a two-person finite game in mixed strategies, $G = (X, Y, f, g)$, called the **stage game**, that is played in discrete time $k = \{1, \dots, K, \dots\}$ after observing the past

history of plays $h_k = ((x_1, y_1), \dots, (x_{k-1}, y_{k-1}))$. Given $K \in \mathbb{N}$, consider the K -stages game G^K defined by the payoff functions

$$f(K, \sigma_1, \sigma_2) := \frac{1}{K} \mathbb{E}_\sigma \left[\sum_{k=1}^K f(x_k, y_k) \right], \quad g(K, \sigma_1, \sigma_2) := \frac{1}{K} \mathbb{E}_\sigma \left[\sum_{k=1}^K g(x_k, y_k) \right]$$

where $\sigma_1 : \cup_{k=1}^\infty (X \times Y) \rightarrow X$ and $\sigma_2 : \cup_{k=1}^\infty (X \times Y) \rightarrow Y$ are the players' *behavioral strategies*. We define the **uniform game** G_∞ as the game obtained by taking $K \rightarrow \infty$ in G^K .

To state the definition of constrained Nash equilibrium for uniform games, we need some preliminary concepts.

Definition 3.4.11. Consider the set of **feasible payoffs**

$$co(f, g) := co\{(f(x, y), g(x, y)) \in \mathbb{R}^2 : (x, y) \in X \times Y\}$$

in which co stands for the convex envelope.

Definition 3.4.12. Given the **punishment level** of players 1 and 2,

$$\alpha := \min_{y \in Y} \max_{x \in X} f(x, y) \text{ and } \beta := \min_{x \in X} \max_{y \in Y} g(x, y) \quad (3.7)$$

We define the set of **uniform equilibrium payoffs** (Folk theorem of Aumann-Shapley [15]) as

$$E = \{(\bar{f}, \bar{g}) \in co(f, g) : \bar{f} \geq \alpha, \bar{g} \geq \beta\}$$

Definition 3.4.13. Consider $f_0, g_0 \in \mathbb{R}$ outside options for player 1 and player 2, respectively. The set of **acceptable payoffs** is defined as

$$E(f_0, g_0) := \{(\bar{f}, \bar{g}) \in co(u, v) : \bar{f} \geq f_0 \text{ and } \bar{g} \geq g_0\}$$

We are ready to define the constrained Nash equilibria of a uniform game.

Definition 3.4.14. A strategy profile $\sigma = (\sigma_1, \sigma_2)$ is called a **constrained uniform equilibrium** of G_∞ if:

1. $\forall \varepsilon > 0$, σ is a (f_0, g_0) - ε -constrained equilibrium of any long enough finitely repeated game, that is, $\exists K_0, \forall K \geq K_0, \forall (\tau_1, \tau_2)$:
 - (a) If $f(K, \tau_1, \sigma_1) > f(K, \sigma) + \varepsilon$ then $g(K, \tau_1, \sigma_2) < g_0$,
 - (b) If $g(K, \sigma_1, \tau_2) > g(K, \sigma) + \varepsilon$ then $f(K, \sigma_1, \tau_2) < f_0$, and
2. $[(f(K, \sigma), g(K, \sigma))]_K$ has a limit $[f(\sigma), g(\sigma)]$ in \mathbb{R}^2 as K goes to infinity, with $f(\sigma) \geq f_0, g(\sigma) \geq g_0$.

The set of constrained uniform equilibrium payoffs is denoted as $E^\infty(f_0, g_0)$.

3.4. Feasible games

A uniform game will be feasible if every time that the set of acceptable payoffs is non-empty, the set of constrained uniform equilibrium payoffs is non-empty as well.

Definition 3.4.15. G_∞ is **feasible** if whenever $E(f_0, g_0)$ is non-empty, $E^\infty(f_0, g_0)$ is non-empty as well.

By the Folk theorem of Aumann-Shapley [15], the following proposition holds.

Proposition 3.4.16. Any payoff in $E \cap E(f_0, g_0)$ can be achieved by a uniform equilibrium and so by a constrained uniform equilibrium.

We are ready to prove the feasibility of infinitely repeated games (Theorem 3.4.3).

Proof. Infinitely repeated games are feasible. Suppose $E(f_0, g_0)$ is non-empty. We aim to show that $E^\infty(f_0, g_0)$ is non-empty as well. Recall the punishment levels α and β of the players (Definition 3.4.12). The analysis is split into four cases.

Case 1. $g_0 \geq \beta$ and $f_0 \geq \alpha$. It holds that $E(f_0, g_0) \subseteq E$. Then, by Proposition 3.4.16, $E^\infty(f_0, g_0) = E(f_0, g_0)$. Since $E(f_0, g_0)$ is non-empty, $E^\infty(f_0, g_0)$ is non-empty as well.

Case 2. $g_0 < \beta$ and $f_0 < \alpha$. It holds that $E \subset E(f_0, g_0)$. Thus, $E^\infty(f_0, g_0)$ contains E (by Proposition 3.4.16) and so, it is non-empty.

Case 3. $g_0 < \beta$ and $f_0 \geq \alpha$. If $F := E(f_0, g_0) \cap E$ is non-empty, by Proposition 3.4.16, all elements on F belong to $E^\infty(f_0, g_0)$. Otherwise, consider (f', g') defined by

$$g' := \max\{\bar{g} : \exists \bar{f} \text{ s.t. } (\bar{f}, \bar{g}) \in E(f_0, g_0)\}, \quad f' \in \{\bar{f} : (\bar{f}, g') \in E(f_0, g_0)\}$$

As $E(f_0, g_0)$ is a non-empty closed set, (f', g') indeed exists and it belongs to $E(f_0, g_0)$. Consider the strategy profile σ' in which the players follow a pure plan which yields the payoff (f', g') . If player 1 deviates, player 2 punishes her at the level α , and if player 2 deviates, player 1 ignores the deviation and continues to follow the pure plan.

Player 1 cannot gain more than ε by deviating. Indeed, if she does, player 2 punishes her by reducing his payoff to α . Since $(f', g') \in E(f_0, g_0)$, it holds that $f' \geq f_0 \geq \alpha$ and so, this deviation is not profitable. For player 2, suppose there exists $K \in \mathbb{N}$ and $\varepsilon > 0$ such that she can obtain a payoff $g'' > g' + \varepsilon$ by deviating at stage K . Let f'' be the average payoff of player 1 at stage K after the deviation of player 2. Since (f'', g'') is an average payoff of the K -stages game, it is feasible. It cannot hold that $f'' \geq f_0$, since it would contradict the definition of g' , as the payoff (f'', g'') would be acceptable. Thus, $f'' < f_0$. We conclude that σ' is a constrained equilibrium and then, $(f', g') \in E^\infty(f_0, g_0)$.

Case 4. $g_0 \geq \beta$ and $f_0 < \alpha$. Analogously to case 3. □

Remark 3.4.17. The constrained Nash equilibrium choice can always be done **Pareto-optimally**. Moreover, except for some ill cases, the existence of feasible uniform equilibria holds. We study this in Section 3.6.

3.5 From pairwise stable to renegotiation proof

Constrained Nash equilibria capture renegotiation proofness when considering the appropriated outside options (the players' reservation payoffs). Let $\Gamma = (D, H, (G_{d,h} : d \in D, h \in H, f, g))$ be a matching game.

Definition 3.5.1. Given $\pi = (\mu, \vec{x}, \vec{y})$ an allocation and $(d, h) \in \mu$ an arbitrary matched couple, we define their **reservation payoffs** by,

$$\begin{aligned} f_d^\pi &:= \max\{f_{d,h'}(s, t) : h' \in H_0 \setminus \{h\}, g_{d,h'}(s, t) > g_{h'}(\pi), (s, t) \in X_d \times Y_{h'}\}, \\ g_h^\pi &:= \max\{g_{d',h}(s, t) : d' \in D_0 \setminus \{d\}, f_{d',h}(s, t) > f_{d'}(\pi), (s, t) \in X_{d'} \times Y_h\}, \end{aligned} \quad (3.8)$$

Reservation payoffs are the best payoffs that d and h can get outside of their couple by matching with a partner who may accept them.

We are finally ready to characterize the renegotiation proof allocations through constrained Nash equilibria.

Proposition 3.5.2. *A pairwise stable allocation $\pi = (\mu, \vec{x}, \vec{y})$ is renegotiation proof if and only if for any $(d, h) \in \mu$, (x_d, y_h) is a (f_d^π, g_h^π) -constrained Nash equilibria, where f_d^π and g_h^π are the agents' reservation payoffs.*

Proof. Suppose that all couples play constrained Nash equilibria. Let $(d, h) \in \mu$ be an arbitrary matched couple and (x_d, y_h) be their (f_d^π, g_h^π) -CNE. Suppose there exists $s \in X_d$ such that $f_{d,h}(s, y_h) > f_{d,h}(x_d, y_h)$. In particular,

$$f_{d,h}(s, y_h) > \max\{f_{d,h}(\ell, y_h) : g_{d,h}(\ell, y_h) \geq g_h^\pi, \ell \in X_d\}$$

Thus, $g_{d,h}(s, y_h) < g_h^\pi$. Let d' be the player that attains the maximum in g_h^π . Then, (d', h) is a blocking pair of the pairwise stability of π . For h the proof is analogous.

Conversely, suppose π is renegotiation proof. Let $(d, h) \in \mu$ be an arbitrary couple and (x_d, y_h) be their strategy profile. Then, for any $s \in X_d$ such that $f_{d,h}(s, y_h) > f_{d,h}(x_d, y_h)$, it holds that $g_{d,h}(s, y_h) < g_h^\pi$. Thus,

$$f_{d,h}(x_d, y_h) \geq \max\{f_{d,h}(\ell, y_h) : g_{d,h}(\ell, y_h) \geq g_h^\pi, \ell \in X_d\}$$

For player h the proof is analogous. □

Proposition 3.5.2 gives the first insight into the design of an algorithm to compute renegotiation proof allocations: It is enough to modify the strategy profiles of each couple by constrained Nash equilibria under outside options equal to their reservation payoffs. A second insight will come from the following result.

Proposition 3.5.3. *Let $\pi = (\mu, \vec{x}, \vec{y})$ be an allocation. Then, π is pairwise stable if and only if $f_d^\pi \leq f_d(\pi)$ and $g_h^\pi \leq g_h(\pi)$, for any $(d, h) \in D \times H$, where f_d^π and g_h^π are agents' reservation payoffs (Equations (3.8)).*

3.5. From pairwise stable to renegotiation proof

Proof. Suppose that π is pairwise stable and let $d \in D$ be a doctor such that $f_d^\pi > f_d(\pi)$. Thus, there exists $h \in H_0 \setminus \{\mu(d)\}$ and $(s, t) \in X_d \times Y_h$ such that

$$g_{d,h}(s, t) > g_h(\pi) \text{ and } f_d^\pi = f_{d,h}(s, t)$$

It is clear that (d, h) is a blocking pair of π , so we obtain a contradiction. The same conclusion holds if for any $h \in H$, $g_h^\pi > g_h(\pi)$.

Conversely, suppose that for any $(d, h) \in D \times H$, $f_d^\pi \leq f_d(\pi)$ and $g_h^\pi \leq g_h(\pi)$. Let $(d, h) \in D \times H$ be a blocking pair of π . Then, there exists $(s, t) \in X_d \times Y_h$ such that

$$f_{d,h}(s, t) > f_d(\pi) \text{ and } g_{d,h}(s, t) > g_h(\pi)$$

In particular, notice that $f_d^\pi \geq f_{d,h}(s, t)$ and $g_h^\pi \geq g_{d,h}(s, t)$, as each of player can offer to the other one more than their current payoffs. We obtain a contradiction. \square

If π is a pairwise stable allocation, Proposition 3.5.3 implies that the reservation payoffs of all couples are never greater than their current payoffs. Therefore, if $(d, h) \in \mu$, (x_d, y_h) is always (f_d^π, g_h^π) -feasible and, if their game is feasible, there always exists a (f_d^π, g_h^π) -constrained Nash equilibrium (\hat{x}_d, \hat{y}_h) . Moreover, replacing the current strategy profile with the CNE will not create blocking pairs. Thus, the pairwise stability is always preserved. With this in mind, a renegotiation process is designed. It will output a pairwise stable and renegotiation proof allocation for any pairwise stable allocation used as an input. Intuitively, it will replace one by one the strategy profiles of the couples by a CNE, using at each iteration the reservation payoffs (Equation (3.8)) as outside options.

If at any iteration a couple replaces (x_d, y_h) by a (f_d^π, g_h^π) -feasible Nash equilibrium, they will keep playing it during all posterior iterations. If couples cannot replace their strategy profile with a Nash equilibrium, the choice of a constrained Nash equilibrium is made by an **oracle**. Our *renegotiation process* is summarized in Algorithm 3.1. For a numerical example, check Section 3.7.

The convergence of Algorithm 3.1 does not directly hold as the reservation payoffs change at each iteration. Indeed, replacing a strategy profile with a constrained Nash equilibrium may decrease the payoff of an agent. Therefore, her reservation payoff may also change and the constrained Nash equilibrium may not be an equilibrium anymore. Nevertheless, if after changing all strategy profiles of π , the reservation payoffs remain invariant, the current allocation is indeed renegotiation proof and the algorithm stops.

The proof of the convergence of the renegotiation process (Algorithm 3.1) is game-dependent as the choice of the oracle is different for each class of games. First, we state the proof of its correctness. Then, we state the proof that the renegotiation process converges for each of the classes of games.

Theorem 3.5.4 (The renegotiation process is correct). *If the renegotiation process (Algorithm 3.1) converges, its output is pairwise stable and renegotiation proof.*

Algorithm 3.1: Renegotiation process

```

input :  $\pi = (\mu, \vec{x}, \vec{y})$  pairwise stable allocation
1  $t \leftarrow 1, \pi(t) \leftarrow \pi$ 
2 while True do
3   for  $(d, h) \in \mu$  do
4     Compute the reservation payoffs  $f_d^{\pi(t)}$  and  $g_h^{\pi(t)}$  (Equation (3.8))
5     Choose  $(x_d^*, y_h^*) \in \text{CNE}(f_d^{\pi(t)}, g_h^{\pi(t)})$  and set  $(x_d^{t+1}, y_h^{t+1}) \leftarrow (x_d^*, y_h^*)$ 
6   end
7   if  $\forall (d, h) \in \mu, (x_d^{t+1}, y_h^{t+1}) = (x_d^t, y_{h,d}^t)$  then
8     Output  $\pi(t)$ 
9   end
10   $t \leftarrow t + 1$ 
11 end
    
```

Proof. Let $\pi = (\mu, \vec{x}, \vec{y})$ be the input of Algorithm 3.1. By construction, whenever the algorithm converges, the output is renegotiation proof. Concerning pairwise stability, we aim to prove that if π_t , the allocation before iteration t , is pairwise stable then, π_{t+1} is pairwise stable as well. Let $(d, \mu(d))$ be a couple that changes of strategy profile at iteration t . Let $(x_d, y_{\mu(d)})$ be their strategy profile at iteration t and $(\hat{x}_d, \hat{y}_{\mu(d)})$ at time $t + 1$. Suppose there exists (i, j) a blocking pair of π_{t+1} . If $i \neq d$ (and analogously if $j \neq \mu(d)$) then $f_{i, \mu(i)}(\pi_{t+1}) = f_{i, \mu(i)}(\pi_t)$. Thus, it cannot hold that both $i \neq d$ and $j \neq \mu(d)$, otherwise the pair (i, j) would also block π_t . Without loss of generality, suppose that $i = d$. In particular, $j \neq \mu(d)$ because i and j are not a couple. It holds,

$$g_{d,j}(s, r) > g_{\mu(j),j}(\pi_{t+1}) = g_{\mu(j),j}(\pi_t)$$

where $(s, r) \in X_d \times Y_j$ is the strategy profile used by (b, j) to block π . Then, if f_d^π is d 's reservation payoff at iteration t (computed by Equation (3.8)), it holds

$$f_d^\pi \geq f_{d,j}(s, r) = f_{i,j}(s, r) > f_{i, \mu(i)}(\pi_{t+1}) = f_{d, \mu(d)}(\hat{x}_d, \hat{y}_{\mu(d)})$$

This contradicts the fact that $(\hat{x}_d, \hat{y}_{\mu(d)})$ is (f_d^π, g_h^π) -feasible. \square

The finiteness of the algorithm for zero-sum games needs a preliminary result.

Lemma 3.5.5. *Let Γ be a matching game where all strategic games are zero-sum games with a value. Let $\pi = (\mu, \vec{x}, \vec{y})$ be a pairwise stable allocation and $(d, h) \in \mu$ be a matched couple. Let $w_{d,h}$ be the value of their game. Consider the sequence of reservation payoffs of (d, h) denoted by $(f_d^{\pi(t)}, g_h^{\pi(t)})_t$, with t being the iterations of Algorithm 3.1. If there exists t^* such that $w_{d,h} \leq f_d^{\pi(t^*)}$ (resp. $w_{d,h} \geq g_h^{\pi(t^*)}$), then the subsequence $(f_d^{\pi(t)})_{t \geq t^*}$ (resp. $(g_h^{\pi(t)})_{t \geq t^*}$) is non increasing (resp. non decreasing).*

Proof. Suppose there exists an iteration t in which $w_{d,h} \leq f_d^{\pi(t)} \leq g_h^{\pi(t)}$, so couple (d, h) switches its payoff to $f_d^{\pi(t)}$ (recall that constrained Nash equilibrium payoffs

are equal to the median between the value of the game and the players' reservation payoffs). Let (\hat{x}_d, \hat{y}_h) be the constrained Nash equilibrium played by (d, h) at iteration t . Since (\hat{x}_d, \hat{y}_h) must be $(f_d^{\pi(t+1)}, g_h^{\pi(t+1)})$ -feasible, in particular it holds $f_d^{\pi(t+1)} \leq f_{d,h}(\hat{x}_d, \hat{y}_h) = f_d^{\pi(t)}$. Therefore, the sequence of reservation payoffs starting from t is non-increasing. \square

Theorem 3.5.6 (Convergence renegotiation process zero-sum games). *For any oracle, the renegotiation process converges for zero-sum games with a value.*

Proof. At the beginning of Algorithm 3.1, each couple (d, h) belongs to one of the following cases: $f_d^\pi \leq w_{d,h} \leq g_h^\pi$, $w_{d,h} \leq f_d^\pi \leq g_h^\pi$, or $f_d^\pi \leq g_h^\pi \leq w_{d,h}$. In the first case, the couple plays a Nash equilibrium and never changes it afterward. In the second case, as f_d^π is non increasing for d (Lemma 3.5.5) and bounded from below by $w_{d,h}$, her sequence of reservation payoffs converges. Analogously, the sequence of reservation payoffs for h converges on the third case. Therefore, the renegotiation process converges. \square

For strictly competitive games the convergence of the renegotiation process is a corollary of the one for zero-sum games. We state it without proof.

Theorem 3.5.7 (Convergence renegotiation process strictly competitive games). *For any oracle, the renegotiation process converges for strictly competitive games in \mathcal{S} with an equilibrium.*

Proof. Corollary of Theorem 3.5.6. \square

Shapley-Shubik's and Demange-Gale's models can be mapped into a matching game in which all strategic games $G_{d,h}$ are included in the class \mathcal{S} of strictly competitive games (see Section 2.6). Our results, therefore, apply directly to their works proving the existence of allocations that are not only pairwise stable but also renegotiation proof. The refinement induced by renegotiation proofness crucially depends on the choice of the strategic games $G_{d,h}$. For example, if we model the game between a buyer and a seller as an ultimatum game [1] where the buyer is the first proposer, she gets all the surplus, while when the first proposer is the seller, she is the one who gets all the surplus. However, if the game is an alternative offer bargaining game [84], the surplus is shared equally.

Theorem 3.5.8 (Convergence renegotiation process potential games). *There exists an oracle for potential games such that the renegotiation process converges.*

Proof. Consider a couple $(d, h) \in \mu$ and $(\hat{x}_d^t, \hat{y}_h^t)_t$ their sequence of constrained Nash equilibria along the iterations. Since $(\hat{x}_d^{t-1}, \hat{y}_h^{t-1})$ is always feasible for the following iteration (Proposition 3.5.3), the sequence $\phi_{d,h}(\hat{x}_d^t, \hat{y}_h^t)_t$ is non-decreasing over t . Then, as the potential functions are continuous and the strategy sets are compact, the sequences $(\phi_{d,h}(\hat{x}_d^t, \hat{y}_h^t))_t$ are converging for any couple (d, h) . Thus, the renegotiation process converges. \square

Theorem 3.5.9 (Convergence renegotiation process infinitely repeated games). *There exists an oracle for infinitely repeated games such that the renegotiation process converges.*

Proof. Let π be a pairwise stable allocation, t an iteration, and $(d, h) \in \mu$ a couple. Let $(f_d^{\pi(t)}, g_h^{\pi(t)})$ be their reservation payoffs at iteration t , and consider $F_t := E \cap E(f_d^{\pi(t)}, g_h^{\pi(t)})$. If F_t is non-empty, there exists a $(f_d^{\pi(t)}, g_h^{\pi(t)})$ -feasible uniform equilibrium for (d, h) , so they keep playing it forever. If $F_t = \emptyset$, without loss of generality, assume that $f_d^{\pi(t)} \geq \alpha$ and $g_h^{\pi(t)} < \beta$, where α and β are the punishment levels of d and h , respectively. Consider the oracle used in the proof of the feasibility of infinitely repeated games (Theorem 3.4.3). Let (f^t, g^t) be the $(f_d^{\pi(t)}, g_h^{\pi(t)})$ -constrained Nash equilibrium payoff chosen at iteration t by the oracle, so

$$g^t := \max\{\bar{g} \in \mathbb{R} : \exists \bar{f} \in \mathbb{R} \text{ such that } (\bar{f}, \bar{g}) \in E(f_d^{\pi(t)}, g_h^{\pi(t)})\}$$

If $g^t \geq \beta$, $(f^t, g^t) \in E$ and then, F_t is non-empty, a contradiction. Thus, $g^t < \beta$. Let $(f_d^{\pi(t+1)}, g_h^{\pi(t+1)})$ be the couple's reservation payoffs at the following iteration, and set again $F_{t+1} = E \cap E(f_d^{\pi(t+1)}, g_h^{\pi(t+1)})$. If F_{t+1} is non-empty, they play a uniform equilibrium. Otherwise, since $g_h^{\pi(t+1)} \leq g^t < \beta$ (Proposition 3.5.3), in particular it holds that $f_d^{\pi(t+1)} \geq \alpha$ and $g_h^{\pi(t+1)} < \beta$. Let (f^{t+1}, g^{t+1}) be the new constrained Nash equilibrium payoff found by the oracle. Since pairwise stability implies that $(f^t, g^t) \in E(f_d^{\pi(t+1)}, g_h^{\pi(t+1)})$ (Proposition 3.5.3), $g^{t+1} \geq g^t$. In addition, as F_{t+1} is empty, $g^{t+1} < \beta$. Thus, h 's sequence of constrained Nash equilibrium payoffs $(g^t)_t$ is non-decreasing and bounded from above by β . Therefore, the sequence converges to a fixed payoff, and then, the renegotiation process converges as well. \square

3.6 Pareto-optimality, pairwise stability, and Nash stability.

Consider a couple who plays a two-player infinitely repeated game. For simplicity, we omit the identity of the agents. Given a pair of reservation payoffs (f^0, g^0) , we recall the oracle designed for the feasibility of infinitely repeated games proof. Let $F := E(f_0, g_0) \cap E$, where $E(f_0, g_0)$ is the set of acceptable payoffs and E is the set of uniform equilibrium payoffs. For any case in which F is non-empty, the oracle picks a (f_0, g_0) -feasible uniform equilibrium as constrained Nash equilibrium. Otherwise, the oracle computes a (f_0, g_0) -CNE. It is interesting to study in which cases F is empty.

Consider a *matching pennies* game played in mixed strategies (Table 3.2, also studied in Example 3.4.6) repeated infinitely many times. The infinitely repeated version of the matching pennies game has 0 as the only uniform equilibrium payoff. Consider reservation payoffs (f_0, g_0) such that $w < f_0 \leq g_0$, as in Figure 3.2. It follows that $E(f_0, g_0)$ is the continuum of values in the blue line. It holds $F = \{0\} \cap E(f_0, g_0) = \emptyset$. The only constrained uniform equilibrium payoff corresponds to the limit point between the red and blue line, that is, when player 1 gets exactly f_0 as payoff.

		Player 2	
		A	B
Player 1	A	1	-1
	B	-1	1

Table 3.2: Matching pennies

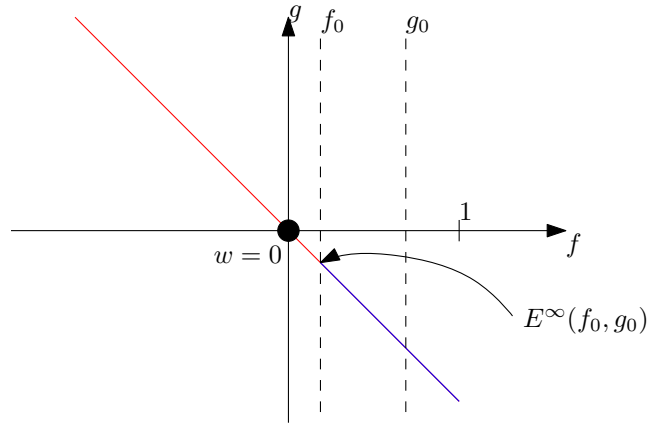


Figure 3.2: Matching pennies

Any constant-sum game used as stage game for an infinitely repeated game will present the same behavior as matching pennies. However, for most of the rest of the stage games that we may consider, the intersection set F will be non-empty. Let us illustrate this with another example. Let G be a prisoners' dilemma (Table 3.3) played in mixed strategies and repeated infinitely many times.

		Player 2	
		Cooperate	Betray
Player 1	Cooperate	2, 2	-1, 3
	Betray	3, -1	0, 0

Table 3.3: Prisoner's dilemma

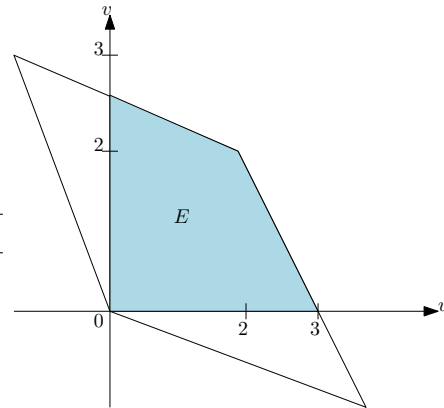
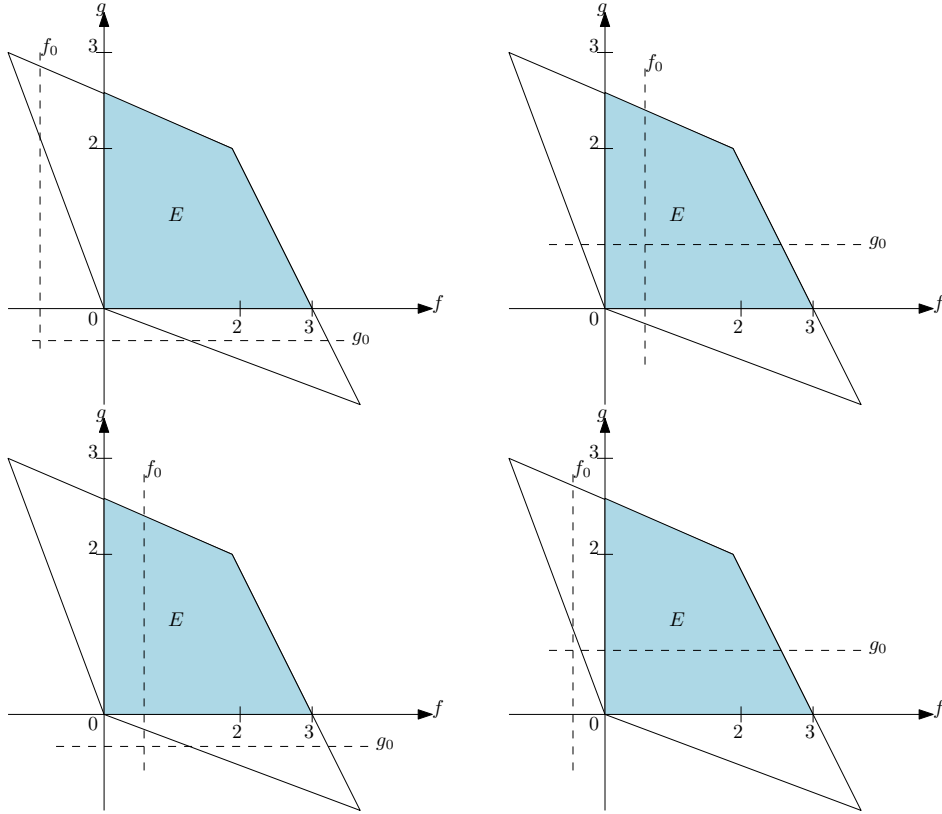


Figure 3.3: Uniform equilibrium payoffs

The punishment levels of the agents being equal to $(0, 0)$ (the unique Nash equilibrium payoff), the set of uniform equilibrium payoffs (Figure 3.3) is,

$$E = \{(\bar{f}, \bar{g}) \in co(\{(2, 2), (-1, 3), (3, -1), (0, 0)\}) : \bar{f} \geq 0 \text{ and } \bar{g} \geq 0\}$$

Given a pair of reservation payoffs (f_0, g_0) such that there exists at least one feasible contract, the following figures show the four possible cases studied in the proof of the feasibility of infinitely repeated games.



Thanks to the non-emptiness of the interior of E , given that for (f_0, g_0) there exists at least one feasible contract, the intersection between E and $E(f_0, g_0)$ gives a non-empty set. Therefore, we can always find a (f_0, g_0) -feasible uniform equilibrium of the repeated game in which no agent has incentives to deviate. This is not the case for infinitely repeated games with constant-sum games as stage games. To an infinitely repeated game that has a set of uniform equilibrium payoffs with a non-empty interior we call it **non-degenerated**.

Consider next a matching game $\Gamma = (D_0, H_0, \{G_{d,h} : (d, h) \in D \times H\}, \underline{f}, \underline{g})$. We recall the notion of Nash stability given in Chapter 1.

Definition 1.2.2. An allocation $\pi = (\mu, \vec{x}, \vec{y})$ is **Nash stable** if for any matched couple $(d, h) \in \mu$, $(x_d, y_h) \in \text{N.E}(G_{d,h})$, i.e., (x_d, y_h) is a Nash equilibrium of $G_{d,h}$.

The following theorem proves the equivalence between the models with and without commitment when Γ is an infinitely repeated matching game.

Theorem 3.6.1. *Let Γ be a matching game in which each strategic game $G_{d,h}$, for $(d, h) \in D \times H$, is a non-degenerated infinitely repeated game. Let $\pi = (\mu, \vec{\sigma}_D, \vec{\sigma}_H)$ be a pairwise stable allocation, with $\vec{\sigma}_D$ and $\vec{\sigma}_H$ profiles of behavioral strategies. Then, there always exists a pairwise stable (Definition 2.2.1) and Nash stable (Definition 1.2.2) allocation $\pi' = (\mu, \vec{\tau}_D, \vec{\tau}_H)$ that weakly Pareto-dominates π . In addition, an allocation is renegotiation proof allocation if and only if it is Nash stable.*

3.6. Pareto-optimality, pairwise stability, and Nash stability.

Remark that π and π' in the previous theorem have the same matching μ .

Proof. That renegotiation proof allocations and Nash stable allocations are equivalent comes from the previous discussion. As the games are non-degenerated, for each couple there always exists a feasible uniform equilibrium. Therefore, the allocation is renegotiation proof if and only if the oracle can always choose a uniform equilibrium as constrained Nash equilibrium.

Regarding the first point, consider a couple $(d, h) \in \mu$ and their behavioral strategy profile (σ_d, σ_h) in π . The payoff profile $(f_{d,h}(\sigma_d, \sigma_h), g_{d,h}(\sigma_d, \sigma_h)) \in \mathbb{R}^2$ belongs to the convex envelope of the stage game payoffs. Since the game is non-degenerated, there exists a payoff profile $(f_d^*, g_h^*) \geq (f_{d,h}(\sigma_d, \sigma_h), g_{d,h}(\sigma_d, \sigma_h))$ that belongs to the set of uniform equilibrium payoffs. Therefore, there exists a uniform equilibrium (τ_d, τ_h) satisfying,

$$f_{d,h}(\tau_d, \tau_h) = f_d^* \geq f_{d,h}(\sigma_d, \sigma_h) = f_d(\pi) \text{ and } g_{d,h}(\tau_d, \tau_h) = g_h^* \geq g_{d,h}(\sigma_d, \sigma_h) = g_h(\pi)$$

We conclude the proof. \square

We have remarked that the oracle designed to compute constrained Nash equilibria in infinitely repeated games selects a Pareto-optimal outcome (Remark 3.4.17). Consequently, the pairwise stable and renegotiation proof allocations in matching games with non-degenerated infinitely repeated keep the property that all couples play Pareto-optimally within their games. In this setting, we achieve the strongest stability criterion in our model: *Pareto-optimality, pairwise stability, and Nash stability*.

In the same setting, commitment loses its importance as mutually beneficial contract can be reached via a uniform Nash equilibria (since, as observed above, renegotiation proofness is always possible).

For matching games with degenerated infinitely repeated games (for example, when couples play infinitely repeated transfer games) commitment can still be removed as deviations from a prescribed strategy path can be punished afterward. This is not possible for static games.

In real-life applications (marriages, civil unions, professors and universities, doctors and hospitals, workers and firms, etc) interactions are made in the long term and *contracts* specify the actions to be made by each agent at each stage of the relationship. Deviations in actions at some stage can be punished in two ways: partners can punish using the game actions or they can just break the couple and match with another agent. From this point of view, the pairwise stable and renegotiation proof allocations that we have defined constitute stationary (on the matching) stable solutions. This is the first insight into obtaining a *dynamic model of matching games* in which, at every stage, a matching game is played, i.e., at every stage all agents decide which partner to have and which strategies to play against that partner.

Pareto-optimality, pairwise stability, and Nash stability are solutions achievable only in matching games with non-degenerated infinitely repeated games. In general,

for any matching game in which the sets of Pareto-optimal payoffs of each strategic game can be obtained as Nash equilibrium payoffs and the vector of individually rational payoffs is Pareto dominated, the same arguments given for infinitely repeated games can be applied. The following simultaneous Nash bargaining game example is a static game in which the strongest stability notion can be obtained as well.

Example 3.6.2. Consider a matching game with only one agent d and one agent h , both with null individually rational payoffs. If agents agree to match, suppose they play a *Nash simultaneous bargaining game*: Let I be a pie of size 1. Independently and simultaneously, both of them propose a split of the pie $(x, 1 - x)$ and $(1 - y, y)$. If the proposals coincide, the split is implemented and the players get the proposed utilities. Otherwise, I is wasted and both agents get 0. The individually rational payoffs are Pareto-dominated by any split in which both agents get at least something. Moreover, all strategy profiles in which the agents agree in the split are Nash equilibria and are Pareto-optimal. As for non-degenerate infinitely repeated games, any pairwise stable and renegotiation proof allocation achieves Pareto-optimality, pairwise stability, and Nash stability. \square

3.7 A numerical example

Consider a matching game with three doctors $D = \{d_1, d_2, d_3\}$ and three hospitals $H = \{h_1, h_2, h_3\}$, with the following preferences,

$$A = \begin{array}{|c|c|c|} \hline 83 & 85 & 99 \\ \hline 74 & 13 & 15 \\ \hline 58 & 49 & 54 \\ \hline \end{array} \quad B = \begin{array}{|c|c|c|} \hline 69 & 6 & 28 \\ \hline 88 & 2 & 70 \\ \hline 72 & 18 & 9 \\ \hline \end{array}$$

$A(d, h)$ (resp. $B(d, h)$) represents the fixed utility that $d \in D$ (resp. $h \in H$) receives if she (resp. it) matches with h (resp. d). Suppose that all agents have null IRPs, i.e., $(\underline{f}_d, \underline{g}_h) = (0, 0), \forall (d, h) \in D \times H$. Suppose, in addition, that if a couple (d, h) is created, as in the transfer matching game example (Example 1.1.6), they play a constant-sum game. Formally, if $x_d, y_h \geq 0$ are d and h respective monetary transfers, their utilities are,

$$\begin{aligned} f_{d,h}(x_d, y_h) &= A(d, h) - x_d + y_h \\ g_{d,h}(x_d, y_h) &= B(d, h) + x_d - y_h \end{aligned}$$

The pairwise stable allocation π found by the deferred-acceptance with competitions algorithm (Algorithm 2.2) in the previous chapter (Section 2.7) is known to be the most preferred stable allocation by the proposer side. However, there is a continuum of strategy profiles that achieve the same payoffs that π . Indeed, for any $\delta \in [0, \min\{y_{h_1}, y_{h_2}, y_{h_3}\}]$, the shifted allocation $\pi_\delta = (\mu, \vec{x} + \delta, \vec{y} - \delta)$ gives the same payoffs to the agents and, therefore, it remains pairwise stable. From this continuum of solutions, there is only one that is renegotiation proof.

3.7. A numerical example

Let us start this section by studying the constrained Nash equilibria. Recall that $(x_d, y_h) \in \mathbb{R}_+^2$ is a (f_d^π, g_h^π) -CNE if and only if,

$$\begin{aligned} f_{d,h}(x_d, y_h) &= \max\{f_{d,h}(s, y_h) : g_{d,h}(s, y_h) \geq g_h^\pi, s \geq 0\} \\ &= \max\{A(d, h) - s + y_h : B(d, h) + s - y_h \geq g_h^\pi, s \geq 0\} \\ &\iff x_d = [g_h^\pi + y_h - B(d, h)]_+, \text{ and} \\ g_{d,h}(x_d, y_h) &= \max\{g_{d,h}(x_d, t) : f_{d,h}(x_d, t) \geq f_d^\pi, t \geq 0\} \\ &= \max\{B(d, h) + x_d - t : A(d, h) - x_d + t \geq f_d^\pi, t \geq 0\} \\ &\iff y_h = [f_d^\pi + x_d - A(d, h)]_+ \end{aligned}$$

where $[\cdot]_+ = \max\{\cdot, 0\}$. We state the pseudo-code of the renegotiation process for this problem in Algorithm 3.2.

Algorithm 3.2: Renegotiation process

```

1 Input:  $\pi$  a pairwise stable allocation
2 repeat
3   for  $(d, h) \in \mu$  do
4     Compute  $(f_d^\pi, g_h^\pi)$  and set,
            $x_d = [g_h^\pi + y_h - B(d, h)]_+$  and  $y_h = [f_d^\pi + x_d - A(d, h)]_+$ 
5   end
6 until Convergence;

```

Let us run the renegotiation process iteration by iteration.

Iter 1. Let (d_1, h_3) be the first couple. Since d_1 is already playing a dominating strategy, only h_3 can have a profitable deviation. Then, as $x_{d_1} = 0$, h_3 's new transfer is given by,

$$y_{h_3} = [f_{d_1}^\pi - A(d_1, h_3)]_+$$

We compute the reservation payoff of d_1 ,

$$\begin{aligned} f_{d_1}^\pi &= \max\{f_{d_1,h}(s, t) : g_{d_1,h}(s, t) > g_h(\pi), h \in \{h_0, h_1, h_2\}, s, t \geq 0\} \\ &= \max\{A(d_1, h) - s + t : B(d_1, h) + s - t > g_h(\pi), h \in \{h_0, h_1, h_2\}, s, t \geq 0\} \\ &= \max\{A(d_1, h) + B(d_1, h) - g_h(\pi), h \in \{h_0, h_1, h_2\}\} \\ &= \max\{0; 83 + 69 - 64; 85 + 6 - 1\} = 90 \end{aligned}$$

Therefore, $y_{h_3} = [90 - 99]_+ = 0$ and the payoffs of the couple become,

$$(f_{d_1}(\pi), g_{h_3}(\pi)) = (99, 28)$$

Notice how the couple passes to play their Nash equilibrium, that is, none of the agents makes a positive transfer. This is not a coincidence. Recall that d_1 and h_3

play a strictly competitive game. Therefore, we have an explicit formula for the constrained Nash equilibrium payoffs given the reservation payoffs of the players and the Nash equilibrium payoff. In particular, whenever the Nash equilibrium of the game is feasible, the couple plays it. Let us check this by computing h_3 's reservation payoff.

$$\begin{aligned} g_{d_3}^\pi &= \max\{g_{d,h_3}(s,t) : f_{d,h_3}(s,t) > f_d(\pi), d \in \{d_0, d_2, d_3\}, s, t \geq 0\} \\ &= \max\{B(d, h_3) + s - t : A(d, h_3) - s + t > f_d(\pi), d \in \{d_0, d_2, d_3\}, s, t \geq 0\} \\ &= \max\{A(d, h_3) + B(d, h_3) - f_d(\pi), d \in \{d_0, d_2, d_3\}\} \\ &= \max\{0; 15 + 70 - 98; 54 + 9 - 66\} = 0 \end{aligned}$$

Thus, the Nash equilibrium being feasible for the agents, the oracle picks it as constrained Nash equilibrium. Remark that, as the couple plays a Nash equilibrium, they will not change of strategy profile again.

Iter 2. Consider the second couple (d_2, h_1) . Again, only h_1 can deviate as d_2 already plays a dominating strategy. Since d_1 and h_3 modified their payoffs in the previous iteration, the reservation payoffs of d_2 and h_1 maybe have changed. Due to this, we need to recompute them. Formally,

$$\begin{aligned} f_{d_2}^\pi &= \max\{\underline{f}_{d_2}; A(d_2, h_2) + B(d_2, h_2) - g_{h_2}(\pi); A(d_2, h_3) + B(d_2, h_3) - g_{h_3}(\pi)\} \\ &= \max\{0; 13 + 2 - 1; 15 + 70 - 28\} = 57 \\ g_{h_1}^\pi &= \max\{\underline{g}_{h_1}; A(d_1, h_1) + B(d_1, h_1) - f_{d_1}(\pi); A(d_3, h_1) + B(d_3, h_1) - f_{d_3}(\pi)\} \\ &= \max\{0; 83 + 69 - 99; 58 + 72 - 66\} = 64 \end{aligned}$$

Once again, notice that the Nash equilibrium (no transfers) between d_2 and h_1 is feasible, as they achieve the payoff profile $(74, 88)$. Therefore, the couple passes to make null transfers.

Iter 3. Consider the last couple (d_3, h_2) and compute their reservation payoffs,

$$\begin{aligned} f_{d_3}^\pi &= \max\{\underline{f}_{d_3}; A(d_3, h_1) + B(d_3, h_1) - g_{h_1}(\pi); A(d_3, h_3) + B(d_3, h_3) - g_{h_3}(\pi)\} \\ &= \max\{0; 58 + 72 - 88; 54 + 9 - 28\} = 42 \\ g_{h_2}^\pi &= \max\{\underline{g}_{h_2}; A(d_1, h_2) + B(d_1, h_2) - f_{d_1}(\pi); A(d_2, h_2) + B(d_2, h_2) - f_{d_2}(\pi)\} \\ &= \max\{0; 85 + 6 - 99; 13 + 2 - 74\} = -8 \end{aligned}$$

The Nash equilibrium of the third couple being feasible, they get as payoffs, $(f_{d_3}(\pi), g_{h_2}(\pi)) = (49, 18)$. As all couples switched to play the Nash equilibrium of their game, the algorithm stops.

Remark 3.7.1. Starting from the best pairwise stable allocation for doctors and applying the renegotiation process (Algorithm 3.2) we have found the best solution for doctors in the model without transfers (i.e. the Gale-Shapley original model). This result does not always hold. As seen in the transfer matching game example (Example 1.1.6), the unique pairwise stable and renegotiation proof allocation is the one in which the first player offers a positive transfer δ to the second player.

3.8 Conclusions

We have introduced *renegotiation proofness* (Definition 3.1.1), a novel concept in the literature of stable matching which refines the pairwise stability concept studied in Chapter 2. We have characterized the renegotiation proof allocations as those allocations in which all agents play *constrained Nash equilibria* (Definition 3.3.3, Proposition 3.5.2), that is, best-reply strategy profiles subject to participation constraints, and show them to be solutions to quasi-variational inequalities.

We have introduced the class of *feasible game* (Definition 3.4.1) as all the strategic two-player games in which a constrained Nash equilibrium exists for any pair of reservation payoffs that admits feasible strategies (Definition 3.3.2). The class of feasible games is been proved to be rich enough to include many well-known games in the literature on game theory (Theorem 3.4.3).

Leveraging the renegotiation proofness characterization through constrained Nash equilibria we have designed a renegotiation process that, starting from any pairwise stable allocation, outputs a pairwise stable and renegotiation proof one for any feasible matching game whenever the process converges (Theorem 3.5.4). Moreover, we have established the convergences for many feasible games.

For static games such as zero-sum games or strictly competitive games, the constrained Nash equilibrium selection is always Pareto-optimal, so pairwise stable and renegotiation proof allocations are locally Pareto-optimal as well. This does not hold for every static feasible game as potential games have shown. Interesting results are obtained when considering matching games in which couples play infinitely repeated games. The ability to punish the partners in case of deviations dismisses the need for commitment in the couple making equivalent the models with and without commitment as any pairwise stable and renegotiation proof allocation is Nash stable as well. Moreover, the constrained Nash equilibrium selection for infinitely repeated games can always be Pareto-optimal, obtaining the strongest notion of stability in our model: Pareto-optimally, pairwise stability, and Nash stability.

Two research lines are followed from the work on Chapters 2 and 3. Chapter 4 extends the model of one-to-one matching games under commitment to one-to-many matching games under commitment where hospitals can be matched to many doctors at the time. Chapter 5 conducts the complexity study of the algorithms presented in these chapters.

1-to-Many Matching games with commitment

The previous chapters introduced and studied the model of one-to-one matching games in which agents from two sets are paired and simultaneously play strategic games. Two models were deduced depending on the level of commitment of the agents. For a model without commitment (Chapter 1), Nash stability naturally arises. Although the non-emptiness and the lattice structure of the set of Nash-pairwise stable allocations can be guaranteed, we have observed the incapacity of this model to capture the classic models in the literature of stable matching. In exchange, these models are successfully captured by the matching games with commitment model. Similar results have been established for the model with commitment such as the non-emptiness of the set of pairwise stable allocations under classical game theoretical assumptions, the lattice structure of the set of pairwise stable allocations, and the proposer-optimality of the deferred-acceptance with competitions (DAC) algorithm designed to compute pairwise stable allocations (Chapter 2).

A second and novel notion of stability was then considered for the model with commitment, *renegotiation proofness*, which refines the set of pairwise stable allocations. We have proved the non-emptiness of the set of pairwise stable and renegotiation proof allocations for any matching feasible game in which the renegotiation process converges (Chapter 3).

The present chapter is devoted to extending the work done in the previous two chapters by studying the mathematical model of one-to-many matching games under commitment in which hospitals are allowed to be matched with many doctors at the same time.

One-to-many matching games present new challenges for the existence of stable allocations and the design of algorithms to compute them. However, in addition to the models already captured by one-to-one matching games, one-to-many matching games get a much broader class of models in stable matching markets, being among the most important, the *matching job market* model of Kelso and Crawford [71], the *matching with contracts* model of Hatfield and Milgrom [66], *hedonic games* [42], the *roommates problem* of Gale and Shapley [53, 68, 75], the *roommates problem with transferable utility* [10, 34, 49, 74, 96], and the *roommates problem with non-transferable utility* of Alkan and Tuncay [9].

This chapter is structured as follows. Section 4.1 presents the model of one-to-many matching games and its notion of Core stability. Two submodels of one-to-many matching games in which we can study the existence of Core stable allocations are

then considered in Sections 4.2 and 4.3, respectively. Section 4.4 extends renegotiation proofness to the same two submodels and uses the renegotiation process of Chapter 3 to prove their existence. Finally, Section 4.5 concludes the chapter.

4.1 One-to-many matching games

In this section, we present the mathematical model of one-to-many matching games, the generalization of the one-to-one matching games model studied in the previous chapters, give many examples of models that can be mapped into a one-to-many matching game, and introduce the notion of Core stability.

4.1.1 Mathematical model

Consider two finite sets of agents D and H that we refer to as **doctors** and **hospitals**, respectively. Simultaneously to get matched, agents may play actions from a given set of actions and receive some utilities depending on the matching and the action profiles chosen. Formally, every doctor $d \in D$ and every hospital $h \in H$ is endowed with a **set of strategies** X_d and Y_h , respectively, subsets of a topological space. Given $I \subseteq D$ we denote $X_I := \prod_{d \in I} X_d$ and $Y_h^{|I|}$ to h 's strategy power set. For the sake of simplicity, we will omit the parenthesis $|\cdot|$ and write Y_h^I to refer to the power set.

Doctors choose only one strategy from their strategy sets (e.g. the number of weekly hours they want to work), while hospitals pick as many strategies as assigned doctors have (e.g. the salary of each doctor). The payoff of a hospital depends on the strategies played by all its doctors as well as the ones played by the hospital itself. Doctors' payoffs, in exchange, depend on their strategy, the particular strategy that their hospital plays against them, and the strategies of all the other doctors allocated in the same hospital (from now on, their colleagues). Formally, given a set of doctors I allocated in a hospital h , the agents' payoff functions are,

$$\begin{aligned} \forall d \in I, f_{d,I,h} : X_d \times X_I \times Y_h &\rightarrow \mathbb{R}, \\ g_{I,h} : X_I \times Y_h^I &\rightarrow \mathbb{R} \end{aligned} \tag{4.1}$$

Notice the redundancy on d 's payoff function as d belongs to I . To avoid the over-charged but more accurate notation $f_{d,I \setminus \{d\},h}$, we prefer to make an abuse of notation.

Players are endowed with **individually rational payoffs** (IRPs). Formally, for every doctor $d \in D$ (resp. hospital $h \in H$), we consider a value $\underline{f}_d \in \mathbb{R}$ (resp. $\underline{g}_h \in \mathbb{R}$), representing the utility for being unmatched.

Definition 4.1.1. A (one-to-many) **matching game** is given by

$$\Gamma = \left(D, H, (X_d)_{d \in D}, (Y_h)_{h \in H}, (\underline{f}_d)_{d \in D}, (\underline{g}_h)_{h \in H} \right)$$

and payoff functions given as in (4.1).

We extend the notion of allocation given in the previous chapters to the one-to-many setting. In this new setting, a matching μ corresponds to any mapping between D and H . We extend the sets $D_0 := D \cup \{d_0\}$ and $H_0 := H \cup \{h_0\}$ by adding the empty players d_0 and h_0 , such that any agent matched with one of them gets her IRP as payoff.

Definition 4.1.2. An **allocation** corresponds to any triplet $\pi = (\mu, \vec{x}, \vec{y})$ in which,

$$\begin{aligned} \mu &: D \rightarrow H \text{ is a **matching**,} \\ \vec{x} &\in X_D \text{ is a **doctors' strategy profile**} \\ \vec{y} &:= (\vec{y}_h)_{h \in H} \text{ is **hospital h's strategy profile**, where each } \vec{y}_h \in Y_h^{\mu(h)} \end{aligned}$$

For a coalition $(I, h) \in \mu$, where $I \subseteq D$ and $h \in H$, we write indistinctly $I = \mu(h)$ and $\mu(d) = h, \forall d \in I$. For a doctor $d \in D$ allocated in hospital h , we use $\mu(h)$ to denote d 's colleagues¹.

Given an allocation $\pi = (\mu, \vec{x}, \vec{y})$, $h \in H$ and $d \in \mu(h)$, their utilities at π correspond to,

$$\begin{aligned} f_d(\pi) &:= f_{d, \mu(h), h}(x_d, \vec{x}_{\mu(h)}, y_{d, h}) = f_{d, \mu(h), h}(x_d, (x_{d'})_{d' \in \mu(h)}, y_{d, h}), \\ g_h(\pi) &:= g_{\mu(h), h}(\vec{x}_{\mu(h)}, \vec{y}_{\mu(h), h}) = g_{\mu(h), h}((x_{d'})_{d' \in \mu(h)}, (y_{d', h})_{d' \in \mu(h)}) \end{aligned}$$

Doctors' utility depends on the identity of their hospital, the identity of their colleagues, their own strategy, the strategies of their colleagues, and the particular strategy played by the hospital against them. Hospitals' utility depends on the identity of their doctors, the strategies of these doctors, and the strategies played by the hospitals against each of their doctors. Let us illustrate the model with the following examples.

Example 4.1.3. Consider a set of $2n$ doctors $D = \{1, \dots, 2n\}$, each of them with a ranking $r_d \in \mathbb{N}$, and suppose that $r_1 > r_2 > \dots > r_{2n}$ (the higher the ranking, the better the doctor). Consider a set of two hospitals $H = \{h_1, h_2\}$, and that each of them has a prestige $p_h \in \mathbb{N}$, with $p_1 > p_2$ (the higher the prestige, the better the hospital). Take all strategy sets empty. Given a matching μ , $h \in H$ and $d \in \mu(h)$, agents' payoffs are given by,

$$f_d(\mu) := p_h + \sum_{d' \in \mu(h)} r_{d'} \text{ and } g_h(\mu) := \begin{cases} \sum_{d' \in \mu(h)} r_{d'} & \text{if } |\mu(h)| \leq q_h \\ -\infty & \text{otherwise} \end{cases}$$

where $q_1, q_2 \in \mathbb{N}$ are fixed and known values. In words, doctors' payoffs correspond to the aggregated ranking of their colleagues and their own ranking (the higher, the better) plus the prestige of their hospital. For hospitals, their utility is given by the aggregated ranking of their doctors up to a capacity q_h . \square

¹Notice again the abuse of notation as we consider d inside of her colleagues set.

Example 4.1.4. Consider a set D of n doctors, a set H of m hospitals, with $m > \lceil n/2 \rceil$, suppose that all strategy sets are empty (as in the previous example), and all agents have null IRPs. Given $I \subseteq D$, $d \in D$, and $h \in H$, suppose the payoff functions are given by,

$$f_{d,I,h} = \begin{cases} v_{d,d'} & \text{if } I = \{d'\} \\ -M & \text{if } |I| > 1 \end{cases} \quad \text{and } g_{I,h} \equiv 0$$

where $(v_{d,d'}, d, d' \in D)$ are fixed positive real values and $M \gg 1$. In words, doctors receive a positive utility if they have only one colleague and a negative utility otherwise. In addition, hospitals have null payoff functions independent of the doctors assigned to them, and doctors' utilities do not depend on the hospital they are assigned to. Since there are more hospitals than half of the doctors, doctors in groups of more than two agents can always opt by matching in couples in order to end up better off. The resulting matching game corresponds to a classical *roommates problem with exogenous preferences*. \square

Example 4.1.5. Consider a set of three doctors $D = \{1, 2, 3\}$ and two hospitals $H = \{a, b\}$, all agents with empty strategy sets. The following table shows the utility of each doctor for belonging to each possible subset of doctors, independent of the assigned hospital, (doctors get utility only in the coalitions they appear in)

		Subset of doctors						
		{1}	{2}	{3}	{1,2}	{1,3}	{2,3}	{1,2,3}
Doctors	1	0	-	-	1	-2	-	-1
	2	-	0	-	1	-	-2	-1
	3	-	-	0	-	1	2	3

Endow hospitals with null payoff functions. Remark that agents 1 and 2 prefer to be together rather than being single, and none of them wants to be with 3. The resulting matching game corresponds to an *hedonic game*. \square

4.1.2 Core stability

This section is devoted to extend the pairwise stability notion studied in Chapter 2 to the model of one-to-many matching games. Renegotiation proofness, studied in Chapter 3, is delegated to Section 4.4. As pairwise stability for one-to-one matching games, *Core stability* will generalize the stability notions from the literature on one-to-many matching markets, in particular, the one of Hatfield and Milgrom [66].

Definition 4.1.6. An allocation $\pi = (\mu, \vec{x}, \vec{y})$ is **blocked** by a coalition (I, h) , with $I \subseteq D$ and $h \in H$, if there exist $(\vec{w}_I, \vec{z}_{h,I}) \in X_I \times Y_h^I$, such that,

$$\forall d \in I, f_{d,I,h}(w_d, \vec{w}_I, z_{h,d}) > f_d(\pi),$$

$$g_{I,h}(\vec{w}_I, \vec{z}_{h,I}) > g_h(\pi),$$

π is **core stable** if it is **individually rational** (no agent gets less than her IRP) and it is not blocked.

An allocation is core stable if (1) agents' IRPs are satisfied, so nobody prefers to abandon their partners and become single, and (2) we cannot find a set of doctors and a hospital who prefer to abandon their assigned partners and matching together as they end up better off. The coalition (I, h) in the previous definition is called a **blocking coalition**.

Core stability captures *stability* in the matching with contracts and roommates setting, *pairwise stability* in the stable marriage problem, and *core stability* in the assignment game, matching with transfers, and hedonic games settings. In addition, it extends pairwise stability for one-to-one matching games to the one-to-many setting. Core notions from economy are also captured by our stability notion [67] when fixing the matching μ .

Let us compute the core stable allocations of our examples.

Example 4.1.3. Recall the example with $2n$ doctors ordered by ranking and two hospitals ordered by prestige. Suppose that hospitals have capacities $q_1 = q_2 = n$. Let π be an individually rational allocation such that a doctor $d \in \{n + 1, \dots, 2n\}$ is matched with h_1 . As the capacities of both hospitals are equal to n and agents have low IRPs, there must be a doctor $d' \in \{1, \dots, n\}$ matched with h_2 . Notice that the coalition (I, h_1) , with $I = \{1, \dots, n\}$, blocks π as all doctors in I increase strictly their payoffs if they match with h_1 , as well as h_1 . Thus, for π to be core stable, it must hold that the best n doctors are assigned to h_1 , and the n worst to h_2 . Therefore, we obtain that the only core stable allocation for this matching game is the one that *segregates* the doctors by their rankings. \square

Example 4.1.4. Recall the roommates example in which a set of n doctors seek to get matched in couples. A matching μ is core stable if there is no pair of doctors $(d_1, d_2) \in D \times D$, such that,

$$v_{d_1, d_2} > f_{d_1}(\mu) \text{ and } v_{d_2, d_1} \geq f_{d_2}(\mu)$$

In particular, as there are more hospitals than half of the doctors, any pair of doctors can form a couple in case of being matched with a bigger group. A matching μ is core stable if and only if it is stable for the roommates problem². \square

Example 4.1.5. Recall the hedonic game example in which three doctors seek to get matched in coalitions and get payoffs as shown in Table 4.1. A partition μ of the set of doctors is core stable if there is no set of doctors I , not matched between them, such that matching together, all of them end up weakly better off and at least

²Assume, without loss of generality, that $v_{d, d'} \neq v_{d, d''}$, for any d, d', d'' in D .

		Subsets of doctors						
		{1}	{2}	{3}	{1,2}	{1,3}	{2,3}	{1,2,3}
Doctors	1	0	-	-	1	-2	-	-1
	2	-	0	-	1	-	-2	-1
	3	-	-	0	-	1	2	3

Table 4.1: Hedonic game

one of them ends up strictly better off. Therefore, the core stable allocations are the partitions $\mu_1 = (\{1, 2, a\}, \{3, b\})$ and $\mu_2 = (\{1, 2, b\}, \{3, a\})$, equivalent in terms of utility, which correspond to the *core stable partition* of the hedonic game when dropping the hospitals from the allocation. \square

Core stability has been extensively studied in the literature on stable matching (in the context of each model). In *one-to-one two-sided markets*, Gale and Shapley [53] proved that a pairwise stable allocation always exists for the stable marriage problem (a result that still holds with sets of different sizes and non-strict preferences). Shapley and Shubik [93] did the same for the case of linear monetary transfers through linear programming. In Chapter 1 we proved the existence of Nash-pairwise stable allocations for any matching game without commitment with compact Nash equilibrium sets and continuous payoff functions, while in Chapter 2 we did the same for pairwise stable allocations in matching games with commitment. For *one-to-many matching markets* Crawford and Knoer [37] proved the existence of stable allocations in the case of matching with linear monetary transfers. Kelso and Crawford [71] defined a job market in which workers and firms get matched and define wages to be paid to the workers. They proved the existence of stable allocations under gross-substitute conditions. Hatfield and Milgrom [66] extended this result to any finite set of contracts also under a substitutability assumption³.

In non-two-sided markets in which a partition of the set of agents has to be found, the existence of stable allocations becomes trickier. Gale and Shapley gave a small instance in which the roommates problem fails to have a stable allocation (Example 4.1.7) while Knuth [75] proved that roommates problems can have multiple ones. Let us show Gale-Shapley's example.

Example 4.1.7. Consider four players $\{1, 2, 3, 4\}$, each of them with the strict (ordinal) preferences on their possible partner given in Table 4.2. It is not possible to make a stable partition of the four players into two pairs as the agent paired with player 4 will always find somebody else to get matched, who will also prefer to change. \square

Knuth asked the question of designing an algorithm to compute a stable allocation whenever it exists. Irving [68] answered with an efficient algorithm to compute a solution if it exists or to report the non-existence of stable allocations for those

³Aygün and Sönmez [16] remarked that the irrelevance of rejected contracts is also needed.

	Preferences
Players	1 2 > 3 > 4
	2 3 > 1 > 4
	3 1 > 2 > 4
	4 Arbitrary

Table 4.2: Roommates problem

instances without a solution. Tan [97] found necessary and sufficient conditions for the existence of stable matchings in the roommates problem: the non-existence of *stable partitions* with *odd parties*.

Roommates with transferable utility has received attention lately [10, 34, 49, 74, 96]. Andersson et al. [10] designed a *price adjustment process* that computes, under integral payments, a stable allocation or disproves its existence in finite time. Shioura [94] made the connection between roommates with transferable utility and the assignment game of Shapley and Shubik [93]. More precisely, Shioura reduced his problem to a particular assignment game in an auxiliary bipartite graph and proposed an extension of the algorithm of Andersson et al. [10] to compute a stable allocation in case of existence. Up to our knowledge, the only extension of the roommates problem to the *non-transferable utility* domain has been made by Alkan and Tuncay [9].

For more complex problems such as *hedonic games* in which any size of coalitions are allowed, the existence of core allocations becomes a much harder problem.

In this chapter we aim to design efficient algorithms to find core stable and renegotiation proof allocations, if they exist, or to report the non-existence of these allocations when the problem does not allow them. In order to have any hope of designing tractable algorithms, we will focus on two particular models: (*one-to-many*) *additive separable matching games* and *roommates matching games*, being both submodels of one-to-many matching games.

4.2 1st submodel: (1-to-many) Additive separable matching games

The additive separable matching games submodel rises as a generalization to Hatfield and Milgrom's work [66]. Therefore, we begin this section by recalling their model and results.

4.2.1 Matching with contracts model

Matching with contracts, defined by Hatfield and Milgrom (H&M) [66], considers two finite sets D and H of doctors and hospitals, respectively, and a finite set of contracts X . Contracts are *bilateral*, so each $x \in X$ is related to only one doctor $x_D \in D$ and one hospital $x_H \in H$. Agents have *choice functions* $(C_d, C_h, \forall d \in D, \forall h \in H)$, such

that, given a set of contracts $X' \subseteq X$, $C_d(X')$ outputs the most preferred contract of d among those in X' , and $C_h(X')$ outputs the most preferred subset of contracts of h among those in X' .

Due to the way that H&M treated the hospitals' choice functions, these must satisfy two assumptions for the existence of stable allocations: *substitutability* (Definition 4.2.1 or page 918 [66]) and *irrelevance of rejected contracts* (IRC) (Definition 4.2.1, or page 6 [16]).

Definition 4.2.1. Contracts are **substitutes** for hospital h if for any $X' \subseteq X$ and any pair of different contracts $x, x' \in X$, if $x \notin C_h(X')$, then $x \notin C_h(X' \cup \{x'\})$. The choice function of hospital h satisfies the **irrelevance of rejected contracts** (IRC) if for any $Y \subseteq X$, and any $z \notin Y$, if $z \notin C_h(Y \cup \{z\})$ then $C_h(Y) = C_h(Y \cup z)$.

A set of contracts is stable if no agent prefers to abandon her contract and no coalition of doctors can get matched with a hospital weakly improving everybody's allocation with at least one doctor and the hospital ending up strictly better off.

Definition 4.2.2. A set of contracts $X' \subseteq X$ is **stable** if, $\bigcup_{d \in D} C_d(X') = \bigcup_{h \in H} C_h(X') = X'$ and there is no $h \in H$ and $X'' \neq C_h(X')$, such that, $X'' = C_h(X' \cup X'') \subseteq \bigcup_{d \in D} C_d(X' \cup X'')$.

Using fixed-point techniques from lattice theory that allowed to guarantee the convergence of a *cumulative offer mechanism* (COM), H&M proved that the set of stable allocations is a non-empty lattice. We claim that an easier existence proof can be conducted under substitutability, as a deferred-acceptance kind of algorithm is enough to compute a stable allocation. Our claim lies in the following result, unobserved in [66].

Proposition 4.2.3. *Let $X' \subseteq X$ be an allocation and suppose that contracts are substitutes for hospitals and the IRC property is satisfied. X' is stable if and only if it is individually rational and there is no doctor d , hospital h , and contract $x \in X \setminus X'$, with $x_D = d, x_H = h$, such that $x = C_d(X' \cup \{x\})$ and $x \in C_h(X' \cup \{x\})$.*

Remark that the second property of Proposition 4.2.3 corresponds to **pairwise stability**, adapted to the matching with contracts setting.

Proof. Suppose X' is stable. Let d be a doctor and $x, x' \in X$ be two contracts related to doctor d ($x_D = x'_D = d$), being x' her contract in the allocation ($x' \in X'$) and x any other contract ($x \in X \setminus X'$), such that d prefers x to x' ($x = C_d(X' \cup \{x\})$). Let $h = x_H$ be the hospital related to contract x . Set $X'' = (X' \setminus \{x'\}) \cup \{x\}$ as the allocation obtained when replacing x' by x and consider $X''|_h := \{y \in X'' : y_H = h\}$. Notice that $X''|_h = X'|_h \cup \{x\}$. It follows that $X''|_h \subseteq \bigcup_{d \in D} C_d(X' \cup X''|_h)$ as the only change is the new contract x , the one is preferred by d over x' . Since X' is stable, it follows that,

$$X''|_h \neq C_h(X' \cup X''|_h) \iff [X'|_h \cup \{x\}] \neq C_h(X' \cup \{x\}) \iff x \notin C_h(X' \cup \{x\})$$

Conversely, suppose X' is individually rational and there is no doctor d , hospital h , and contract $x \in X \setminus X'$, with $x_D = d, x_H = h$, such that $x = C_d(X' \cup \{x\})$ and $x \in C_h(X' \cup \{x\})$. Suppose X' is not stable, so there exist h a hospital and a set of contracts $X'' \neq C_h(X')$ such that, $X'' = C_h(X' \cup X'') \subseteq \bigcup_{d \in D} C_d(X' \cup X'')$. Let $x \in X'' \setminus C_h(X')$ and d the related doctor to x ($x_D = d$). Since $x \in C_h(X' \cup X'')$, we have that x belongs to $C_h(Z)$ for any $Z \subseteq X' \cup X''$, such that $x \in Z$ (substitutability + IRC). In particular, $x \in C_h(X' \cup \{x\})$. Since $x = C_d(X' \cup \{x\})$ as doctors choose only one contract among the proposed ones, we obtain a contradiction. \square

Pairwise stable allocations can be computed by a deferred-acceptance-like algorithm, such as Algorithm 4.1. In the pseudo-code, we have extended the set of contracts X to $X_0 = X \cup \{\emptyset\}$ where \emptyset represents being unmatched. In addition, given a set of contracts $Y \subseteq X$, and k an agent, we denote by $Y|_k$ the subset of contracts in Y that are related to k .

Algorithm 4.1: Deferred-acceptance with contracts algorithm

- 1 **Input:** (D, H, X_0) a matching with contracts instance.
 - 2 Set $D' \leftarrow D$ as the set of unmatched doctors.
 - 3 Set $Y, Z \leftarrow \emptyset$ as the sets of accepted and rejected contracts, respectively.
 - 4 **while** $D' \neq \emptyset$ **do**
 - 5 Let $d \in D'$ and $x \in \arg \max C_d(X_0 \setminus Z)$.
 - 6 If $x = \emptyset$, update $D' \leftarrow D' \setminus \{d\}$ and start again. Otherwise, let $h = x_H$ be the concerned hospital.
 - 7 Let $W = C_h(Y \cup \{x\})$ and $\overline{W} = (Y|_h \cup \{x\}) \setminus W$. Update $Y, Z \leftarrow (Y \setminus \overline{W}) \cup W, Z \cup \overline{W}$.
 - 8 Update D' as it corresponds.
 - 9 **end**
 - 10 Output Y
-

Algorithm 4.1 takes an unmatched doctor and asks her to propose her most preferred contract among all those that have not been already rejected. The concerned hospital then chooses its most preferred subset of contracts among all the proposed ones. The rejected contracts are stored, the set of unmatched doctors is updated accordingly, and a new iteration starts. The algorithm keeps running until all doctors have been allocated to a hospital or the non-allocated ones prefer to remain unmatched.

Algorithm 4.1 always converges as the set of contracts is finite and the set of rejected contracts is increasing. Moreover, the output Y is always pairwise stable as for any doctor d and any contract $x \in X \setminus Y$, with $x = C_d(Y \cup \{x\})$, necessarily x was rejected along the algorithm. Therefore, $x \notin C_h(Y \cup \{x\})$, where $h = x_H$. From Proposition 4.2.3, we conclude that Algorithm 4.1 outputs a stable allocation whenever hospitals have substitute contracts and satisfy the irrelevance of rejected

contracts property. Observe that even if one of the conditions fails to be satisfied, a pairwise stable allocation always exists for any instance of the H&M model.

4.2.2 Additive separable matching games

Let Γ be a matching game (Definition 4.1.1) and relax the doctors' payoff functions dependence on colleagues, that is, redefine the payoff functions as,

$$f_{d,h} : X_d \times Y_h \rightarrow \mathbb{R}, \forall d \in I, \text{ and } g_{I,h} : X_I \times Y_h^I \rightarrow \mathbb{R} \quad (4.2)$$

where $I \subseteq D$ and $h \in H$. The relaxation from the general matching game model to the additive separable matching game sub model corresponds to Kelso and Crawford's [71] main assumption.

We will endow hospitals with **quotas**, representing the maximum number of doctors they can receive. Formally, we say hospital $h \in H$ has quota $q_h \in \mathbb{N}$, if for any $I \subseteq D$ and strategy profiles $\vec{x}_I \in X_I, \vec{y}_h \in Y_h^I$, their payoff function becomes:

$$\tilde{g}_{I,h}(\vec{x}_I, \vec{y}_h) = \begin{cases} g_{I,h}(\vec{x}_I, \vec{y}_h) & \text{if } |I| \leq q_h \\ -\infty & \text{otherwise} \end{cases}$$

As individual rationality of the hospitals will exclude allocations that don't respect the quotas, by abuse notation, **we denote \tilde{g} also by g** .

Remark 4.2.4. A model without quotas can be easily obtained by taking $\vec{q} \equiv |D|$ for all hospitals. Similarly, taking $\vec{q} \equiv 1$ we recover the one-to-one matching models in the literature [53, 93, 40].

Next, we introduce the notion of additive separability.

Definition 4.2.5. Let h be a hospital. We say that h 's payoff function is **additive separable** if for every doctor $d \in D$ there exists a function $g_{d,h} : X_d \times Y_h \rightarrow \mathbb{R}$, such that

$$g_h(\pi) = \begin{cases} \sum_{d \in \mu(h)} g_{d,h}(x_d, y_{h,d}) & \text{if } |\mu(h)| \leq q_h \\ -\infty & \text{otherwise} \end{cases}$$

for $\pi = (\mu, \vec{x}, \vec{y})$ any allocation.

Notice that in additive separable matching games, every tuple

$$G_{d,h} := (X_d, Y_h, f_{d,h}, g_{d,h})$$

defines a two-player game. In particular, we properly capture the model of one-to-one matching games under commitment studied in Chapters 2 and 3. The next example shows how a matching with linear transfer problem can be mapped to an additive separable matching game (more precisely, poly-matrix constant-sum games).

Example 4.2.6. A set of H buyers bid for the items offered by D sellers. Buyers are allowed to buy as many items as they prefer while sellers have only one indivisible item to sell. Buyers $h \in H$ and sellers $d \in D$ have non-negative valuations for the items $u_{h,d}$ and v_d , respectively. If a buyer h buys the items of a set of sellers $J \subseteq D$, paying $\vec{p}_h := (p_{h,d})_{d \in J}$, agents' payoffs are,

$$f_{d,h}(p_{h,d}) = p_{h,d} - v_d, \forall d \in J, \text{ and } g_{J,h}(\vec{p}_h) = \sum_{d \in J} (u_{h,d} - p_{h,d}).$$

Remark that only buyers are strategic in this example while sellers only care about the highest bidder. \square

4.2.3 Link between matching games and the H&M model

Now that we have some examples at hand, we can formally show the links between our model to that of H&M and anticipate the advantages of working with our model instead. Interestingly, there are many mappings to associate a H&M model to ours, while the converse is unique.

Consider a matching with contracts model (D, H, X) . Endow all agents with strategy sets: for $d \in D$ and $h \in H$, we let $X_d = \{x \in X : x|_D = d\}$ and $Y_h = \{x \in X : x|_H = h\}$. Under an extra assumption on the preferences in the H&M model (such as the strong axiom of revealed preferences), there exists a utility representation [33] for each agent over X : u_d for a doctor d and v_h for a hospital h , that we can w.l.o.g. assume positive. Given $I \subseteq D$, $h \in H$, and strategy profiles $(\vec{w}_I, \vec{z}_{h,I}) \in X_I \times Y_h^I$, define the payoff function of a doctor to be,

$$f_{d,h}(w_d, z_{d,h}) = \begin{cases} u_d(w_d) & \text{if } w_d = z_{d,h} \\ -1 & \text{otherwise} \end{cases}$$

The mapping means that if both parts agree on a contract that is available to them, they get the (positive) utility of that contract; otherwise, they get -1. We do similarly for the hospitals.

Conversely, given a matching game in which all agents are endowed with strategy sets (X_d, Y_h) and payoff functions, consider the set of contracts $Z := \bigcup_{(d,h) \in D \times H} (X_d \times Y_h)$ and associate the agents' IRPs to an empty contract \emptyset , included in Z . Then, the matching game can be uniquely mapped into a matching with contracts, where the choice functions are defined, $\forall Z' \subseteq Z$ as,

$$\begin{aligned} z = C_d(Z') &\iff f_{d,h}(z) = \max\{f_{d,h}(z') : z' \in Z', h = z_H\} \\ Z'' = C_h(Z') &\iff g_{I,h}(Z'') = \max\{g_{I,h}(Z''') : Z''' \subseteq Z', I = \{z_D \in D, z \in Z'''\}\} \end{aligned}$$

Hence, computing the choice functions of hospitals may not be so easy, even in the separable additive case (they are the Argmax of the utility functions), and requires an exponential number of resources as we need to specify them for each coalition

of doctors. In addition, choice functions are black boxes and do not allow us to understand the incentives behind agents' choices of contracts.

Working with strategies and utilities solves these issues. In addition, (1) the *ir-relevance to rejected contracts* assumption is automatically satisfied (thanks to the strong axiom of weak preferences) (2) we can work with infinitely many contracts (finiteness is crucial in H&M), (3) we can refine the set of stable solutions using renegotiation-proofness, (4) we can identify interesting matching problems where computing renegotiation-proof stable allocations is polynomial.

4.2.4 Core stability: 1st submodel

The core is a classical solution concept from cooperative game theory and has been widely studied [14, 56]. The non-emptiness of the core cannot be always guaranteed without extra assumptions on the game. We aim to establish sufficient conditions for the set of core stable allocations in our model to be non-empty.

Strict preferences over a finite set of contracts is a common assumption [62, 63, 64, 65, 66] in matching markets as it makes equivalent the concepts of **core** (Definition 4.1.6) and **weak-core** (all agents within a coalition must end up weakly better off and at least one of them strictly better off to block an allocation). This equivalence does not necessarily hold in our model due to the continuum of payoffs. Thus, we focus in studying core stable allocations by proving that any pairwise stable allocation is core stable under additive separability (Proposition 4.2.8).

Remark that, generically, the equivalence holds as we can discretize the set of strategy profiles and perturb the players' payoffs to get strict preferences. Moreover, the DA algorithm (Algorithm 4.1) can be applied to construct core stable allocations in matching games by (1) mapping the model to a matching with contracts model, (2) discretizing with mesh ε our continuum set of strategy profiles, (3) perturbing the payoffs up to ε to avoid ties in the preferences, (4) applying Algorithm 4.1 to find a pairwise stable allocation, and finally (5) tending ε to zero. Clearly, this is too costly. Additive separability will allow us to reduce the computational complexity of this task by extending our deferred-acceptance with competition (DAC) algorithm from the one-to-one matching games model. Then, the same assumption will allow us to apply our renegotiation process to compute core stable and renegotiation proof allocations.

The approach will be to reduce core stability to pairwise stability. Let Γ be a matching game and suppose that all hospitals have additive separable payoff functions and quotas (Definition 4.2.5).

Definition 4.2.7. An individually rational allocation $\pi = (\mu, \vec{x}, \vec{y})$ is **blocked by a pair** (d, h) , if there exists $(w_d, z_{h,d}) \in X_d \times Y_h$, such that $f_{d,h}(w_d, z_{h,d}) > f_d(\pi)$ and

$$g_{d,h}(w_d, z_{h,d}) > \begin{cases} g_{d,h}(x_d, y_{h,d}) & \text{if } \mu(d) = h, \\ \min_{d' \in \mu(h)} g_{d',h}(x_{d'}, y_{h,d'}) & \text{if } |\mu(h)| = q_h, \mu(d) \neq h, \\ 0 & \text{if } |\mu(h)| < q_h, \mu(d) \neq h, \end{cases}$$

π is **pairwise stable** if it is not blocked by any pair.

When $d = \mu(h)$, blocking means that the strategy profile used by (d, h) in π is not Pareto-optimal in their game, so they can jointly deviate to a strictly better outcome. Notice that for unit hospitals quotas $q \equiv 1$, we recover a one-to-one model where pairwise stability and core stability coincide.

Example 4.2.6. Recall the multi-item auction example. For simplicity, consider two buyers $H = \{\alpha, \beta\}$, four sellers $D = \{a, b, c, d\}$, all sellers having the same valuation $v = 1$ for their items, and all agents with a null IRP. In addition, suppose that buyers α and β have valuations $u_\alpha = (10, 10, 2, 2)$ and $u_\beta = (2, 2, 10, 10)$, respectively, for the sellers' items. The core stable allocations corresponds to any $\pi = (\mu, (p_\alpha, p_\beta))$ with $\mu = ((\alpha, a), (\alpha, b), (\beta, c), (\beta, d))$ and the buyers' strategy profiles $p_\alpha = (x_1, x_2, 0, 0)$ and $p_\beta = (0, 0, x_3, x_4)$, with $2 \leq x_i \leq 10$ for $i \in \{1, 2, 3, 4\}$, meaning, for example, that α pays x_1 monetary units to a and x_2 to b . We obtain a continuum of core stable allocations. Remark that the two extremes $p_\alpha \equiv p_\beta \equiv 2$ and $p_\alpha \equiv p_\beta \equiv 10$ correspond to the 2nd price auction outcome and 1st price auction outcome, respectively. \square

We prove, first of all, that any pairwise stable allocation is core stable.

Proposition 4.2.8. *Let Γ be an additive separable matching game Then, any pairwise stable allocation $\pi = (\mu, \vec{x}, \vec{y})$ is core stable.*

Proof. Suppose π is pairwise stable but not core stable. Let (I, h) be a blocking coalition, that is, there exist $(\vec{w}_I, \vec{z}_{h,I}) \in X_I \times Y_h^I$, such that,

$$\begin{aligned} \forall d \in I, f_{d,h}(w_d, z_{h,d}) &> f_d(\pi), \\ g_{I,h}(\vec{w}_I, \vec{z}_{h,I}) &> g_h(\pi) \end{aligned}$$

Since h has an additive separable payoff function, it follows,

$$\sum_{d \in I} g_{d,h}(w_d, z_{h,d}) > \sum_{d \in \mu(h)} g_{d,h}(x_d, y_{h,d})$$

In particular, there must exist $d \in I$ and $d' \in \mu(h)$ such that

$$g_{d,h}(w_d, z_{h,d}) > g_{d',h}(x_{d'}, y_{h,d'})$$

Since doctor d increases strictly her payoff with the deviation, (d, h) is a blocking pair of π , which is a contradiction. \square

Proposition 4.2.8 states that in order to prove the non-emptiness of the set of core stable allocations, it is enough with finding a pairwise stable allocation. The rest of the section is devoted to prove the following theorem.

Our characterization of core stable allocations (called stable* in [48]) through pairwise stable allocations is in line with the work of Echenique and Oviedo [48] (called stable in [48]). However, our algorithm does not require fixed points theorem to obtain its convergence as we do not iterate a mapping but run a deferred-acceptance like algorithm.

Theorem 4.2.9. *Suppose that the matching game is additive separable with quota, agents' strategy sets are compact, payoff functions are continuous, and that for each (d, h) the set of Pareto-optimal strategy profiles in the game between d and h is closed⁴. Then, there exists a pairwise stable allocation.*

The **sketch of proof** of Theorem 4.2.9 is the following.

1. Fixing $\varepsilon > 0$, we extend the deferred-acceptance with competitions (DAC) algorithm to compute an ε -pairwise stable allocation in the one-to-many setting.
2. As the Pareto-optimal strategy profile sets are compact and the payoff functions are continuous, taking $\varepsilon \rightarrow 0$ gives the existence of a 0-pairwise stable allocation.

The DAC algorithm (Algorithm 4.2) asks the unmatched doctors to propose, one by one, a strategy profile to their most preferred hospital. The proposal is computed such that a hospital is always better off by accepting it. As long as hospitals have not reached their quota, we keep the empty doctor matched with them so proposer doctors can always join the hospital if they desire it. Therefore, our DAC fills the hospitals until one agent has to replace somebody to be able to enter. In case the proposal includes replacing a real doctor (i.e. different from the empty one), a competition (analogous to a *second price auction*) between the two doctors starts. The winner stays at the hospital while the loser becomes unmatched. Unless a hospital has reached its quota, we always suppose that d_0 is allocated in the hospital and can be replaced.

Algorithm 4.2: Deferred-acceptance with Competitions algorithm

```

1 Input:  $\Gamma$  a matching game,  $\varepsilon > 0$ ,
2 Set  $D' \leftarrow D$  as the set of unmatched doctors and match each hospital to  $d_0$ 
3 while  $D' \neq \emptyset$  do
4   | Let  $d \in D'$  and  $(h, d', x_d, y_{h,d})$  be the solution of Problem (4.3).
5   | if  $|\mu(h)| < q_h$  then
6   |   |  $d$  is accepted
7   | end
8   | else
9   |   |  $d$  and  $d'$  compete for  $h$  as in a second-price auction. The winner
10  |   | stays at  $h$ , goes out of  $D'$ , and the loser is included in  $D'$ 
11 end
    
```

Two phases compose an iteration of the DAC algorithm. In the **proposal phase** an unmatched doctor d , given the current hospitals' payoff profile $g(\pi) = (g_h(\pi))_{h \in H}$, solves the following **optimal proposal problem**,

$$\max_{\substack{h \in H_0 \\ (w, z) \in X_d \times Y_h}} \left[f_{d,h}(w, z) : g_{d,h}(w, z) \geq \min_{d' \in \mu(h)} g_{d',h}(x_{d'}, y_{h,d'}) + \varepsilon \right] \quad (4.3)$$

⁴Although rare (non generic), there are games with non-closed Pareto-optimal strategy profiles sets. Appendix A.2 includes an example.

The solution to Problem (4.3) consists in h , d 's most preferred hospital, d' the doctor that d desires to replace (possibly d_0), and $(x_d, y_{h,d})$ the strategy profile that d proposes to play to h . The solution $(h, d', x_d, y_{h,d})$ is called the **optimal proposal** of doctor d . Making an abuse of notation, we may call optimal proposal as well to $(h, x_d, y_{h,d})$ or even to $(x_d, y_{h,d})$. Ties are broken by favoring hospitals. When the optimal proposal includes $d' = d_0$, the proposer joins the hospital without replacing d_0 (except when a hospital reaches its quota).

If the optimal proposal includes a doctor $d' \neq d_0$, the **competition phase** starts and both doctors play a *second-price auction*. We define the **reservation payoff** of doctor d , β_d , (and analogously the one of d') as the optimal value of the problem,

$$\max_{\substack{h' \in H_0 \setminus \{h\} \\ (w,z) \in X_d \times Y_{h'}}} \left[f_{d,h'}(w, z) : g_{d,h'}(w, z) \geq \min_{d' \in \mu(h')} g_{d',h'}(x_{d'}, y_{h',d'}) + \varepsilon \right] \quad (4.4)$$

In the case of several solutions, ties are broken by favoring hospitals. Given β_d the reservation payoff of d , her bid λ_d (and analogously the one of d') is computed by,

$$\max [g_{d,h}(w, z) : f_{d,h}(w, z) \geq \beta_d, w \in X_d, z \in Y_h] \quad (4.5)$$

The winner is the doctor with the highest bid (in case of a tie the winner is the current partner), who reduces her proposal to match the one of the loser. Formally, if d wins, she solves,

$$\max [f_{d,h}(w, z) : g_{d,h}(w, z) \geq \lambda_{d'}, w \in X_d, z \in Y_h] \quad (4.6)$$

Ties are broken by choosing the proposal that maximizes h 's utility. The loser is included in D' and a new proposer is chosen.

Theorem 4.2.10. *The DAC algorithm ends in finite time.*

Proof. Since the strategy sets are compact and the payoff functions are continuous, they are bounded. By construction, hospitals' payoffs strictly increase with every proposal. Thus, the algorithm ends in a finite number of iterations. \square

Forcing the doctors to increase hospitals' payoffs by at least ε with every proposal guarantees the finiteness of the DAC algorithm. However, we lose accuracy as the algorithm outputs an ε -pairwise stable allocation (Definition 4.2.12).

Definition 4.2.11. An allocation $\pi = (\mu, \vec{x}, \vec{y})$ is **ε -individually rational** if for any $d \in D$ and $h \in H$, $f_d(\pi) \geq \underline{f}_d + \varepsilon$ and $g_h(\pi) \geq \underline{g}_h + \varepsilon$.

Definition 4.2.12. An ε -individually rational allocation $\pi = (\mu, \vec{x}, \vec{y})$ is **ε -blocked by a pair** (d, h) , if there exists $(w_d, z_{h,d}) \in X_d \times Y_h$, such that $f_{d,h}(w_d, z_{h,d}) > f_d(\pi) + \varepsilon$ and

$$g_{d,h}(w_d, z_{h,d}) > \begin{cases} g_{d,h}(x_d, y_{h,d}) + \varepsilon & \text{if } \mu(d) = h, \\ \min_{d' \in \mu(h)} g_{d',h}(x_{d'}, y_{h,d'}) + \varepsilon & \text{if } |\mu(h)| = q_h, \mu(d) \neq h, \\ \varepsilon & \text{if } |\mu(h)| < q_h, \mu(d) \neq h, \end{cases}$$

π is **ε -pairwise stable** if it is not ε -blocked by any pair.

4.3. 2nd submodel: Roommates problem

To prove the correctness of the DAC algorithm we need one last result.

Lemma 4.2.13. *The optimal value of Problem (4.6) is strictly higher than the winner's reservation payoff.*

Proof. Let d (proposer) and d' (matched doctor) be two doctors competing for a place in hospital h and suppose d is the winner. The result holds as the optimal solution of Problem (4.6) is always a feasible solution of Problem (4.5) as $\lambda_d \geq \lambda_{d'}$. \square

Theorem 4.2.14. *Let $\varepsilon > 0$ be fixed. Then, the DAC algorithm outputs an ε -pairwise stable allocation.*

Proof. Let $\pi = (\mu, \vec{x}, \vec{y})$ be the output of Algorithm 4.2. By construction, π is ε -individually rational. Suppose π is not ε -pairwise stable, so there exists (d, h) and $(w_d, z_{h,d}) \in X_d \times Y_h$, such that,

$$f_{d,h}(w_d, z_{h,d}) > f_d(\pi) + \varepsilon, \text{ and } g_{d,h}(w_d, z_{h,d}) > g_{d',h}(x_{d'}, y_{h,d'}) + \varepsilon$$

for some $d' \in \mu(h)$. Let T be the last time that d proposed. Since hospitals' payoffs are increasing, in particular, $(h, w_d, z_{h,d})$ was a feasible solution of d' 's Problem (4.3) at time T . Therefore, the only way that $(h, w_d, z_{h,d})$ was not the optimal solution is because d achieved a payoff strictly higher than $f_{d,h}(w_d, z_{h,d})$. During any posterior iteration, d was always able to propose to h and replace somebody. In particular, for the last competition of d , $\beta_d \geq f_{d,h}(w_d, z_{h,d})$, thus by Lemma 4.2.13, $f_d(\pi) \geq \beta_d \geq f_{d,h}(w_d, z_{h,d})$, which contradicts that d belongs to the blocking pair. \square

The existence of 0-pairwise stable allocations is a consequence of Theorem 4.2.14, passing through the compactness of the Pareto-optimal strategy sets, the continuity of payoff functions, and the finiteness of players. Therefore, from Proposition 4.2.8, we conclude the existence of a core stable allocation.

4.3 2nd submodel: Roommates problem

The roommates problem, defined by Gale and Shapley [53], consists of a set D of agents, each of them having strict preferences for the rest of the agents in D , and seeking to match in couples. Many authors [10, 34, 49, 74, 96] extended the model to the *transferable utility* case and studied the existence of stable allocations. Up to our knowledge, Alkan and Tuncay [9] is the only article to work roommates with *non-transferable utility*. Their model will be the base for our extension. We start by explaining more in detail the model in [9] keeping our language of doctors.

Let D be a finite set of *doctors* and $\underline{f} = (f_d)_{d \in D} \in \mathbb{R}^{|D|}$ be their *individually rational payoff* profile. For every potential couple $(d_1, d_2) \in D$, we consider a *partnership function* $u_{1,2} : \mathbb{R} \rightarrow \mathbb{R}$ such that $u_{1,2}(f_{d_2})$ is the utility that agent d_1 achieves when her partner d_2 achieves f_{d_2} . In particular, it holds $u_{1,2} = u_{2,1}^{-1}$. The partnership functions are assumed to be *continuous*, *decreasing*, and $u_{1,2}(f_{d_2}) < \infty$, for any $d_1, d_2 \in D$.

Unlike our approach in the previous chapters where we focused on the players' strategies, Alkan and Tuncay focused on payoff profiles.

Definition 4.3.1. A **payoff profile** is a vector $f = (f_d)_{d \in D} \in \mathbb{R}^{|D|}$. A payoff profile is **blocked** by a couple (d_1, d_2) if there exists $(f'_{d_1}, f'_{d_2}) \in \mathbb{R}^2$ such that $f'_{d_1} > f_{d_1}$, $f'_{d_2} > f_{d_2}$, and $f'_{d_1} = u_{1,2}(f'_{d_2})$. An **individually rational** payoff profile, i.e., $f_d \geq \underline{f}_d$ for any doctor d , is **stable** if it cannot be blocked.

Given a payoff profile, we focus next on finding a matching that can implement it.

Definition 4.3.2. A **matching** μ is a partition of D in pairs and singletons. A payoff profile f is **realizable** by a matching μ if $f_{d_1} = u_{1,2}(f_{d_2})$, for any $(d_1, d_2) \in \mu$, and the pair (μ, f) is called an **allocation**. Finally, a **stable allocation** is any allocation in which the payoff profile is stable.

Alkan and Tuncay characterized the stable allocations as any allocation (μ, f) satisfying,

$$f_d = \max \left\{ \underline{f}_d, \max_{d' \neq d} u_{d,d'}(f_{d'}) \right\}, \forall d \in D \quad (4.7)$$

A payoff profile satisfying Equation (4.7) is called an **aspiration**. Therefore, to solve the roommates with non-transferable utility problem the goal is to find an aspiration realizable by some matching.

To study when an aspiration is realizable, the authors worked with demand sets. Given a payoff profile f and $d \in D$, we define d 's **demand set** P_d at f as,

$$P_d(f) := \{d' \in D \setminus \{d\} : f_d = u_{d,d'}(f_{d'})\}$$

that is, the set of all agents with who d can achieve the payoff f_d . Notice $d_2 \in P_{d_1}(f)$ if and only if $d_1 \in P_{d_2}(f)$.

We define a **submarket** at f as any pair of disjoint sets of doctors $(B, S) \subseteq D \times D$ such that,

1. For any $d \in B$, $f_d > \underline{f}_d$
2. The demand set of every B -player is in S
3. There exists a matching μ such that for every $d \in S$, $\mu(d) \in B \cap P_d(f)$, or d is unmatched.

Notice that a submarket (B, S) is not asked to be a partition of D , as there may be doctors that do not belong to B nor S . By (3), it always holds $|S| \leq |B|$. A market in which $|S| = |B|$ is called a **balanced market**. An aspiration f that generates a balanced market is called a **balanced aspiration**. The following result states the existence of aspirations with balanced markets (Theorem 1, page 10 [9]).

Theorem 4.3.3. *The set of balanced aspirations is non-empty.*

Moreover, Alkan and Tuncay designed a market procedure to compute balanced aspiration that converges in a polynomial number of iterations. A second theorem links balanced aspirations and stable allocations (Theorem 2, page 11 [9]).

Theorem 4.3.4. *The set of stable allocations is either empty or equal to the set of balanced aspirations.*

To be precise, Alkan and Tuncay proved that whenever the set of balanced aspirations does not coincide with the set of stable allocations, it coincides with the set of *semistable allocations*. Semistable allocations are the relaxation of stable allocations in which agents can be matched with two partners at the time and their final payoff is the average payoff obtained with her two partners. Therefore, the set of stable allocations is non-empty if and only if it coincides with the set of balanced aspirations if and only if the set of semistable allocations is empty.

The market procedure in [9] designed to compute balanced aspirations starts from any aspiration, generates a piecewise linear path of aspirations, and stops in a bounded number of steps at a balanced aspiration. At every iteration, given the current aspiration, the mechanism computes the demand sets of every agent. In case there exists a balanced submarket, the mechanism stops. Otherwise, the mechanism identifies a submarket (B, S) with $|S| < |B|$ and alters the aspiration continuously along a suitable *direction*. The direction is reset when the submarket changes.

Remark the analogy between the market procedure just explained and the *increasing price mechanism* (IPM) of Andersson et al. [10]. The IPM computes the demand sets of the agents given a payoff profile, computes an over-demanded set (Definition 4.3.7), and increases in one unit the utility of its agents. In the market procedure of Alkan and Tuncay, S is over-demanded by B .

The suitable *direction* used to alter the aspiration is computed by a second mechanism, called the *direction procedure*. Starting from a submarket (B, S) , with $|S| < |B|$, the procedure computes $\lambda > 0$ such that increasing the payoffs of the agents in S by λ , decreasing the payoff of the agents in B by λ , and letting unchanged the payoffs of the agents outside the submarket, the shifted submarket and the original one remain the same. The following result (Lemma 7, page 29 [9]) states the correctness of the direction procedure.

Lemma 4.3.5. *If (B, S) is a bipartite submarket at an aspiration f , then there exists a direction \vec{e} with,*

$$\begin{aligned} e_d &< 0, \forall d \in B, \\ e_d &> 0, \forall d \in S, \\ e_d &= 0, \forall d \notin B \cup S, \end{aligned}$$

such that (B, S) is a bipartite submarket at $f + \lambda\vec{e}$, for all sufficiently small $\lambda > 0$.

Finally, the following result (Theorem 4, page 22 [9]) states the correctness and finiteness of the market procedure.

Theorem 4.3.6. *The market procedure reaches a balanced aspiration in a bounded number of steps.*

The remaining question is whether a balanced aspiration can be implemented.

Definition 4.3.7. *Given an aspiration f and a set of doctors I , we define $\mathcal{O}(I) := \{d \in D : P_d(f) \subseteq I\}$ as the set of agents that demand I . I is **overdemanded** if $|\mathcal{O}(I)| > |I|$.*

Notice that at a balanced aspiration f no set of doctors is over-demanded. Therefore, considering the undirected graph $\text{Gr} = (D, E)$, with D the set of doctors and $(d_1, d_2) \in E$ if and only if $d_2 \in P_{d_1}(f)$, Gr can always be decomposed as a disjoint union of cycles. Then, the aspiration f is implementable by a matching whenever there exists a decomposition of Gr including only even-cycles, as for odd-cycles at least one agent will need to be matched with two partners. We remark the connection with Tan [97] works in the existence of stable matchings for the roommates problem with endogenous preferences.

We finish this section by extending the model of roommates with non-transferable utility to the setting of matching games. Consider a matching game (Definition 4.1.1),

$$\Gamma = \left(D, H, (X_d)_{d \in D}, (f_{d,d'})_{d,d' \in D}, (\underline{f}_d)_{d \in D} \right)$$

where we have taken null hospitals' payoff functions, empty hospitals' strategy sets, hospitals' quotas $\vec{q} \equiv 2$, and we have relaxed the doctors' payoff functions dependence on the allocated hospital (as done in Example 4.1.4). Allocations become pairs $\pi = (\mu, \vec{x})$ where μ is a partition of the set D into pairs and singletons and $\vec{x} \in X_D$ is a doctors' strategy profile. Given an allocation π , doctors' payoffs are given by,

$$f_d(\pi) = \begin{cases} f_{d,\mu(d)}(x_d, x_{\mu(d)}), & \text{if } d \text{ is matched} \\ \underline{f}_d & \text{otherwise} \end{cases}$$

In case of no confusion, we will omit the set H from the matching game.

4.3.1 Core stability: 2nd submodel

Consider a roommates matching game as described above. We leverage the work of Alkan and Tuncay to compute core stable allocations of our roommates matching games. Remark that core stable allocations and pairwise stable allocations are trivially equivalent in the roommates setting (with and without strict preferences).

Definition 4.3.8. Let $\pi = (\mu, \vec{x})$ be an individually rational allocation. We say that π is **pairwise stable** if, there is no pair $(d, d') \in D \times D$, and strategy profile $(x_d, x_{d'}) \in X_d \times X_{d'}$, such that,

$$f_d(x_d, x_{d'}) > f_d(\pi) \text{ and } f_{d'}(x_{d'}, x_d) > f_{d'}(\pi)$$

Since an already matched couple can block a matching in case they do not play weakly-Pareto optimally, we obtain the monotony of the partnership functions.

From the existence of balanced aspirations (Theorem 4.3.6) in [9] we can conclude the following result.

Theorem 4.3.9. *Given Γ a roommates matching game with continuous payoff functions, let $\bar{f} \in \mathbb{R}^{|D|}$ be the balanced aspiration computed by the market procedure of Alkan and Tuncay. Then, the set of pairwise stable allocations is non-empty if and only if \bar{f} can be implemented by an allocation (μ, \bar{x}) .*

4.4 Renegotiation proofness 1-to-many matching games

This section is devoted to extending *renegotiation proofness* to the two submodels considered in the previous section. We recall the definition of renegotiation proofness given in Chapter 3 for the one-to-one model.

Definition 4.4.1. A pairwise stable allocation $\pi = (\mu, \bar{x}, \bar{y})$ is **renegotiation proof** if for any couple $(d, h) \in \mu$ and any $(s_d, t_h) \in X_d \times Y_h$, it holds,

1. If $f_{d,h}(s_d, y_h) > f_d(\pi)$ then, $(\mu, (\bar{x}_{-d}, s_d), \bar{x})$ ⁵ is not pairwise stable,
2. If $g_{d,h}(x_d, t_h) > g_h(\pi)$ then, $(\mu, \bar{x}, (\bar{y}_{-h}, t_h))$ is not pairwise stable.

Definition 4.4.1 can be applied without issues to the roommates submodel as agents are matched in couples and to the additive separability matching games submodel.

Being able to apply the same renegotiation proofness definition for one-to-one matching games to our two submodels will allow us to use the *renegotiation process* from Chapter 3 to compute pairwise stable and renegotiation proof allocations, adapted to each submodel.

4.4.1 1st submodel: Additive separable matching games

Consider an additive separable matching game

$$\Gamma = \left(D_0, H_0, (X_d)_{d \in D}, (Y_h)_{h \in H}, (f_{d,h}, g_{d,h})_{d \in D, h \in H}, \underline{f}, \underline{g} \right)$$

such that, given an allocation $\pi = (\mu, \bar{x}, \bar{y})$, agents' payoffs are given by,

$$\begin{aligned} \forall d \in D, f_d(\pi) &= f_{d, \mu(d)}(x_d, y_{\mu(d), d}) \\ \forall h \in H, g_h(\pi) &= \begin{cases} \sum_{d \in \mu(h)} g_{d,h}(x_d, y_{h,d}) & \text{if } |\mu(h)| \leq q_h \\ -\infty & \text{otherwise} \end{cases} \end{aligned}$$

We adapt the renegotiation process used for one-to-one matching games to compute pairwise stable and renegotiation proof allocations of the additive separable matching games submodel. Let us see first the notions of *reservation payoff* and *constrained Nash equilibrium* in this setting.

⁵Given a strategy profile $\bar{x} \in \prod_{d \in D} X_d$, (\bar{x}_{-d}, s_d) denotes the strategy profile obtained when player d replaces her strategy $x_d \in \bar{x}$ by $s_d \in X_d$.

Definition 4.4.2. Given $\pi = (\mu, \vec{x}, \vec{y})$ an allocation, $(I, h) \in \mu$ an arbitrary matched coalition, and $d \in I$, we define f_d^π , the **reservation payoff of d** , as the optimal value of the problem,

$$\begin{aligned} & \max f_{d,h'}(s, t) \\ & \text{s.t. } g_{d,h'}(s, t) > \min_{d' \in \mu(h)} g_{d',h'}(x_{d'}, y_{h',d'}) \\ & h' \in H_0 \setminus \{h\}, (s, t) \in X_d \times Y_{h'} \end{aligned}$$

that is, the highest payoff that d can achieve by matching with another hospital. Similarly, we define h 's **reservation payoff g_h^π** as the optimal value of the problem,

$$\begin{aligned} & \max g_{d',h}(s, t) \\ & \text{s.t. } f_{d',h}(s, t) > f_{d',\mu(d')}(x_{d'}, y_{\mu(d'),d'}) \\ & d' \in D_0 \setminus \{\mu(h)\}, (s, t) \in X_{d'} \times Y_h \end{aligned}$$

that is, the highest payoff that h can achieve by replacing one of its doctors.

It may be intuitive to think that hospitals should have a reservation payoff for each of their doctors. The intuition is correct. However, as reservation payoffs depend on the agents outside of the coalition, the hospital has exactly the same reservation payoff for each of its doctors. In particular, no doctor should decrease her contribution to her hospital payoff below g^π as the hospital will have incentives to replace her.

As for one-to-one matching games, constrained Nash equilibria are strategy profiles in which each agent is best-replying to the partner subject to offering her/it at least the reservation payoff.

Definition 4.4.3. Let $\pi = (\mu, \vec{x}, \vec{y})$ be an allocation and $(f_d^\pi, g_h^\pi : (d, h) \in D \times H)$ be the agents' reservation payoffs. Let $(I, h) \in \mu$ and $d \in I$. The strategy profile $(x_d, y_{h,d})$ is (f_d^π, g_h^π) -**feasible** if $f_{d,h}(x_d, y_{h,d}) \geq f_d^\pi$ and $g_{d,h}(x_d, y_{h,d}) \geq g_h^\pi$. We say $(x_d, y_{h,d})$ is a (f_d^π, g_h^π) -**constrained Nash equilibrium** if it is (f_d^π, g_h^π) -feasible and,

$$\begin{aligned} f_{d,h}(x_d, y_{h,d}) &= \max\{f_{d,h}(s, y_{h,d}) : g_{d,h}(s, y_{h,d}) \geq g_h^\pi, s \in X_d\} \\ g_{d,h}(x_d, y_{h,d}) &= \max\{g_{d,h}(x_d, t) : f_{d,h}(x_d, t) \geq f_d^\pi, t \in Y_h\} \end{aligned}$$

We denote $\text{CNE}(f_d^\pi, g_h^\pi)$ to the set of (f_d^π, g_h^π) -constrained Nash equilibria.

The following result was proved for one-to-one matching games (Proposition 3.5.2). As it can be straightforwardly applied to the current setting, we state it without proof.

Proposition 4.4.4. *A pairwise stable allocation $\pi = (\mu, \vec{x}, \vec{y})$ is renegotiation proof if and only if for any $(I, h) \in \mu$ and $d \in I$, the strategy profile $(x_d, y_{h,d})$ is a (f_d^π, g_h^π) -constrained Nash equilibria, where f_d^π and g_h^π are the agents' reservation payoffs.*

Algorithm 4.3: Renegotiation process

```

input :  $\pi = (\mu, \vec{x}, \vec{y})$  pairwise stable allocation
1  $t \leftarrow 1, \pi(t) \leftarrow \pi$ 
2 while True do
3   for  $(d, h) \in \mu$  do
4     | Compute the reservation payoffs  $f_d^{\pi(t)}$  and  $g_h^{\pi(t)}$  (Definition 4.4.2)
5     | Choose  $(x_d^*, y_{h,d}^*) \in \text{CNE}(f_d^{\pi(t)}, g_h^{\pi(t)})$  and set  $(x_d^{t+1}, y_{h,d}^{t+1}) \leftarrow (x_d^*, y_{h,d}^*)$ 
6   end
7   if  $\forall (d, h) \in \mu, (x_d^{t+1}, y_{h,d}^{t+1}) = (x_d^t, y_{h,d}^t)$  then
8     | Output  $\pi(t)$ 
9   end
10   $t \leftarrow t + 1$ 
11 end

```

Algorithm 4.3 states the *renegotiation process*, adapted to the additive separable matching games submodel.

From the correctness of the renegotiation process for one-to-one matching games (Theorem 3.5.4) and the finiteness in zero-sum games (Theorem 3.5.6), strictly competitive games (Theorem 3.5.7), potential games (Theorem 3.5.8), and infinitely repeated games (Theorem 3.5.9) we obtain the following result.

Theorem 4.4.5. *Whenever Algorithm 4.3 converges, its output is a pairwise stable and renegotiation proof allocation of the additive separable matching games submodel. Moreover, the algorithm converges for zero-sum games with a value, strictly competitive games with an equilibrium, potential games, and infinitely repeated games.*

4.4.2 2nd submodel: Roommates problem

Consider a roommates matching game

$$\Gamma = (D_0, H, (X_d)_{d \in D}, (f_{d,d'})_{d,d' \in D}, \underline{f})$$

The renegotiation proofness can be, without issues, adapted to our roommates submodel. We formally explain the reservation payoff computation and the definition of constrained Nash equilibria to then, announce the adapted version of the renegotiation process.

Definition 4.4.6. Given $\pi = (\mu, \vec{x})$ an allocation and $d \in D$ a doctor. We define f_d^π , the **reservation payoff** of agent d , by

$$f_d^\pi := \max\{f_{d,d'}(s, t) : d' \in D_0 \setminus \{\mu(d)\}, f_{d',d}(t, s) > f_{d'}(\pi), (s, t) \in X_d \times X_{d'}\} \quad (4.8)$$

Given a couple $(d_1, d_2) \in \mu$ and their reservation payoffs $f_{d_1}^\pi$ and $f_{d_2}^\pi$, their strategy profile (x_{d_1}, x_{d_2}) is $(f_{d_1}^\pi, f_{d_2}^\pi)$ -feasible if,

$$f_{d_1, d_2}(x_{d_1}, x_{d_2}) \geq f_{d_1}^\pi \text{ and } f_{d_2, d_1}(x_{d_2}, x_{d_1}) \geq f_{d_2}^\pi$$

(x_{d_1}, x_{d_2}) is a $(f_{d_1}^\pi, f_{d_2}^\pi)$ -constrained Nash equilibrium if it is $(f_{d_1}^\pi, f_{d_2}^\pi)$ -feasible and,

$$\begin{aligned} f_{d_1, d_2}(x_{d_1}, x_{d_2}) &= \max\{f_{d_1, d_2}(s, x_{d_2}) : f_{d_2, d_1}(x_{d_2}, s) \geq f_{d_2}^\pi, s \in X_{d_1}\} \\ f_{d_2, d_1}(x_{d_2}, x_{d_1}) &= \max\{f_{d_2, d_1}(t, x_{d_1}) : f_{d_1, d_2}(x_{d_1}, t) \geq f_{d_1}^\pi, t \in X_{d_2}\} \end{aligned}$$

We denote $\text{CNE}(f_{d_1}^\pi, f_{d_2}^\pi)$ to the set of $(f_{d_1}^\pi, f_{d_2}^\pi)$ -constrained Nash equilibria.

We are ready to announce the renegotiation process adapted to the roommates submodel.

Algorithm 4.4: Renegotiation process

```

input :  $\pi = (\mu, \vec{x})$  pairwise stable allocation
1  $t \leftarrow 1, \pi(t) \leftarrow \pi$ 
2 while True do
3   for  $(d_1, d_2) \in \mu$  do
4     | Compute the reservation payoffs  $f_{d_1}^{\pi(t)}$  and  $f_{d_2}^{\pi(t)}$  (Equation (4.8))
5     | Choose  $(x_{d_1}^*, x_{d_2}^*) \in \text{CNE}(f_{d_1}^{\pi(t)}, f_{d_2}^{\pi(t)})$  and set  $(x_{d_1}^{t+1}, x_{d_2}^{t+1}) \leftarrow (x_{d_1}^*, x_{d_2}^*)$ 
6   end
7   if  $\forall (d_1, d_2) \in \mu, (x_{d_1}^{t+1}, x_{d_2}^{t+1}) = (x_{d_1}^t, x_{d_2}^t)$  then
8     | Output  $\pi(t)$ 
9   end
10   $t \leftarrow t + 1$ 
11 end

```

From the correctness and finiteness of the renegotiation process for one-to-one matching games (Theorem 3.5.4), we obtain the following result.

Theorem 4.4.7. *Whenever Algorithm 4.4 converges, its output is a pairwise stable and renegotiation proof allocation of the roommates submodel. Moreover, the algorithm converges for zero-sum games with a value, strictly competitive games with an equilibrium, potential games, and infinitely repeated games.*

4.5 Conclusions

In this chapter we have extended the model of one-to-one matching games under commitment (Chapters 2 and 3) to the one-to-many setting in which hospitals can be matched with many doctors at the time. Many models in one-to-many matching markets can be captured by one-to-many matching games as the *matching job market*

model of Kelso and Crawford [71], the *matching with contracts* model of Hatfield and Milgrom [66], and *hedonic games* [42]. Moreover, one-to-many matching games also capture non-two-sided matching market problems as the *roommates problem* of Gale and Shapley [53, 68, 75], the *roommates problem with transferable utility* [10, 34, 49, 74, 96], the *roommates problem with non-transferable utility* [9].

We have studied the existence of core stable and renegotiation proof allocations for two submodels of one-to-many matching games: *additive separable matching games* (Section 4.2) and *roommates* (Section 4.3). For the additive separable matching games submodel, we have designed a deferred-acceptance with competitions algorithm (Algorithm 4.2) (extension of the one used for one-to-one matching games) that computes a core stable allocation whenever all strategy sets are compact, payoff functions are continuous and, in addition, the sets of Pareto-optimal strategy profiles are closed.

For the roommates submodel, we have leveraged the work of Alkan and Tuncay. Using their market procedure we are able to compute payoff profiles that, whenever an allocation π can implement them (Definition 4.3.2), π results to be *core stable*. Moreover, the market procedure output \bar{f} is realizable, i.e., we can ensure the existence of core stable allocations, every time that the demand graph at \bar{f} can be decomposed in the disjoint union of even-cycles.

Regarding renegotiation proofness, we have managed to extend the results in Chapter 3 for one-to-one matching games to the two submodel: additive separable matching games and roommates matching games. For the first model the extension is a consequence of the additive separability. For the second model, as agents keep matching in couples, the extension of renegotiation proofness is straightforward. For both submodels we have studied how to compute the agents' *reservation payoffs*, the *constrained Nash equilibria*, and we have adapted the *renegotiation process*. As for one-to-one matching games, we can compute renegotiation proof allocations whenever players play zero-sum games with a value, strictly competitive games with an equilibrium, potential games, and infinitely repeated games (Theorems 4.4.5 and 4.4.7).

The complexity study of the algorithms presented here is conducted in the following chapter. We will show that for many well-known matching games our algorithms are efficient as they have a bounded number of polynomial-time iterations.

A possible future research line is the mix between the two one-to-many matching games submodels, roommates and additive separable matching games, studied in this chapter. The mixed setting would define a model in which couples of doctors are assigned to hospitals, and agents' utilities within the same triplet depend on the strategies and identities of the partners. Interesting applications may rise from this mix, for example, the siblings' schools allocation problem of Correa et al. [36].

Complexity study

The deferred-acceptance algorithm of Gale and Shapley [53] is guaranteed to converge in at most $O(N^2)$ iterations, where N is the size of the biggest market side. Shapley and Shubik [93] also achieved a polynomial complexity when computing their stable allocations thanks to the linearity of the payoff functions over the payments, which allowed them to solve their problem by linear programming. In the roommates problem, Irving [68] designed a polynomial algorithm to compute a stable matching of the problem, in case of existence, or to report the non-existence of stable allocations. The first of the two algorithms designed by Demange et al. [40], an increasing price mechanism with integer payoffs, is also guaranteed to converge in polynomial time to an exact stable allocation. The same holds for Andersson et al. who adapted this last algorithm to roommates with (integer) transferable utility [10]

As we have mentioned in previous chapters, computing exact stable solutions for matching problems with non-integer utilities is still an open problem. Therefore, authors have designed approximation schemes for their problems. The second algorithm of Demange et al. [40] converges in a bounded number of iterations T to an ε -stable solution, with $T \propto \frac{1}{\varepsilon}$. The same holds for the continuous market procedure of Alkan and Tuncay [9] in the roommates with non-transferable utility setting.

In this chapter, we extend the literature results proving that our stable allocation computation algorithms for matching games have a bounded number of polynomial-time iterations whenever players play finite zero-sum games in mixed strategies, finite strictly competitive games in mixed strategies, or infinitely repeated games with finite stage games in mixed strategies. We will focus on the two submodels of one-to-many matching games studied in Chapter 4: (1) additive separable (with quotas) matching games and (2) roommates. First of all, we formalize the notion of bi-matrix (or finite) game in mixed strategies.

Definition 5.0.1. A two-player game $G = (X, Y, f, g)$ is called a **bi-matrix game in mixed strategies** if there exist S, T finite strategy sets such that,

$$X := \Delta(S) = \left\{ x \in [0, 1]^{|S|} : \sum_{s \in S} x(s) = 1 \right\}$$

$$Y := \Delta(T) = \left\{ y \in [0, 1]^{|T|} : \sum_{t \in T} y(t) = 1 \right\}$$

correspond to the simplex of S and T , respectively, and the payoff functions are,

$$f(x, y) := xAy = \sum_{s \in S} \sum_{t \in T} A(s, t)x(s)y(t)$$

$$g(x, y) := xBy = \sum_{s \in S} \sum_{t \in T} B(s, t)x(s)y(t)$$

where $x \in X, y \in Y$, and $A, B \in \mathbb{R}^{|S| \times |T|}$ are payoff matrices. With this in mind, we can define a bi-matrix matching game as any matching game in which all two-player games are bi-matrix games in mixed strategies.

1. An **additive separable bi-matrix matching game** is any additive separable matching game

$$\Gamma = (D_0, H_0, (X_d)_{d \in D}, (Y_h)_{h \in H}, (f_{d,h}, g_{d,h})_{d \in D, h \in H}, (\underline{f}_d)_{d \in D}, (\underline{g}_h)_{h \in H})$$

where each two-player game $G_{d,h} := (X_d, Y_h, f_{d,h}, g_{d,h})$ is a finite game in mixed strategies, i.e., there exist finite pure strategy sets S_d, T_h such that $X_d = \Delta(S_d)$, $Y_h = \Delta(T_h)$ and,

$$\begin{aligned} f_{d,h}(x_d, y_{h,d}) &= x_d A_{d,h} y_{h,d} \\ g_{d,h}(x_d, y_{h,d}) &= x_d B_{d,h} y_{h,d} \end{aligned}$$

with $A_{d,h}, B_{d,h} \in \mathbb{R}^{|S_d| \times |T_h|}$ payoff matrices for all $d \in D$ and $h \in H$.

2. A **roommates bi-matrix matching game** is any roommate matching game

$$\Gamma = (D_0, (X_d)_{d \in D}, (f_{d,d'})_{d, d' \in D}, (\underline{f}_d)_{d \in D})$$

where each two-player game $G_{d,d'} := (X_d, X_{d'}, f_{d,d'}, f_{d',d})$ is a finite game in mixed strategies, i.e., there exist finite pure strategy sets $S_d, S_{d'}$ such that $X_d = \Delta(S_d)$, $X_{d'} = \Delta(S_{d'})$ and,

$$\begin{aligned} f_{d,d'}(x_d, x_{d'}) &:= x_d A_{d,d'} x_{d'} \\ f_{d',d}(x_{d'}, x_d) &:= x_{d'} A_{d',d} x_d \end{aligned}$$

with $A_{d,d'}, A_{d',d} \in \mathbb{R}^{|S_d| \times |S_{d'}|}$ payoff matrices.

For examples of bi-matrix matching games, check Example 1.1.4, Example 1.1.5, and Examples 4.1.4 and 4.1.5. The rest of the chapter is structured as it follows. Section 5.1 recalls the algorithms in Chapter 4 and explains their complexity issues related to the presence of quadratic constrained quadratic programming problems. Section 5.2 makes the formal complexity study for matching games in which couples play zero-sum matching games. Section 5.3 extends the results to the case in which couples play strictly competitive games. Section 5.4 makes the formal complexity study for matching games in which couples play infinitely repeated matching games. Section 5.5 concludes the chapter.

5.1 Algorithms to compute stable allocations in matching games

In this section, we recall the algorithms presented in Chapter 4 to compute stable allocations in additive separable matching games and roommates matching games.

5.1.1 Deferred-acceptance with competitions algorithm

Consider an additive separable bi-matrix matching game

$$\Gamma = (D_0, H_0, G_{d,h} = (X_d, Y_h, f_{d,h}, g_{d,h})_{d \in D, h \in H}, (\underline{f}_d)_{d \in D}, (\underline{g}_h)_{h \in H})$$

Algorithm 5.1 states the deferred-acceptance with competitions (DAC) algorithm adapted to this model.

Algorithm 5.1: DAC algorithm: Additive separable case

```

1 Input:  $\Gamma$  a matching game,  $\varepsilon > 0$ ,
2 Set  $D' \leftarrow D$  as the set of unmatched doctors
3 while  $D' \neq \emptyset$  do
4   | Let  $d \in D'$  and  $(h, d', x_d, y_{h,d})$  be a solution to,
      |
      | 
$$\begin{aligned} & \max w A_{d,h} z \\ & \text{s.t. } w B_{d,h} z \geq \min_{d' \in \mu(h)} x_{d'} B_{d',h} y_{h,d'} + \varepsilon \\ & h \in H_0, (w, z) \in X_d \times Y_h \end{aligned} \tag{5.1}$$

      |
5   | if  $|\mu(h)| < q_h$  then
6   |   |  $d$  is accepted
7   | end
8   | else
9   |   |  $d$  and  $d'$  compete for  $h$  as in a second-price auction. The winner
10  |   | stays at  $h$ , goes out of  $D'$ , and the loser is included in  $D'$ 
11  | end
12 end

```

As all the games are finite games played in mixed strategies, all agents have compact strategy sets and continuous payoff functions. Therefore, the DAC algorithm is guaranteed to converge to an ε -pairwise stable allocation (Theorem 4.2.14). Moreover, the convergence is done in a finite number of iterations as we prove now.

Theorem 5.1.1. *The deferred-acceptance with competitions algorithm converges in a bounded number $T \propto \frac{1}{\varepsilon}$ of iterations.*

Proof. For every hospital, $h \in H$, consider the value,

$$G_h := \max\{B_{d,h}(s, t) - \underline{g}_h : d \in D_0, s \in S_d, t \in T_h\}$$

and let $G_{\max} := \max_{h \in H} G_h$ be the maximum of them. By construction, Algorithm 5.1 increases hospitals' payoffs at each iteration by at least ε . Therefore, the number of iterations is bounded by $T := \frac{1}{\varepsilon} G_{\max}$. \square

Notice that G_{\max} does not depend on the number of players nor the number of pure strategies per player but only on the values of the payoff matrices. Therefore, taking bounded payoff matrices, T only depends on the relaxation rate ε .

We aim to study next under which assumptions the iterations of the DAC algorithm have polynomial complexity. Let us recall the two phases that form an iteration of our DAC algorithm.

Proposal phase. Let $d \in D'$ be the proposer. Given the current allocation π (initially empty) that generates a hospitals' payoff vector $g(\pi) = (g_h(\pi))_{h \in H}$, d computes his optimal proposal (h, d', x, y) solving,

$$\max \left\{ xA_{d,hy} : xB_{d,hy} \geq \min_{d' \in \mu(h)} xB_{d',hy} + \varepsilon, h \in H_0, (x, y) \in X_d \times Y_h \right\} \quad (1.1)$$

Competition phase. If the optimal proposal includes a doctor $d' \neq d$, a second-price auction competition between d and d' starts. Let β_d be the reservation payoff of d , solution to the following problem,

$$\max \left\{ xA_{d,h'y} : xB_{d,h'y} \geq \min_{d' \in \mu(h')} xB_{d',h'y} + \varepsilon, h' \in H_0 \setminus \{h\}, (x, y) \in X_d \times Y_{h'} \right\} \quad (1.2)$$

Analogously, we compute $\beta_{d'}$. Then, d 's bid (and analogously for d') is computed by,

$$\lambda_d := \max \left\{ xB_{d,hy} : xA_{d,hy} \geq \beta_d, (x, y) \in X_d \times Y_h \right\} \quad (1.3)$$

The winner is the doctor with the highest bid. Finally the winner, namely d , pays the second highest bid. Formally, d solves,

$$\max \left\{ xA_{d,hy} : xB_{d,hy} \geq \lambda_{d'}, (x, y) \in X_d \times Y_h \right\} \quad (1.4)$$

Remark that all the optimization problems solved during an iteration of Algorithm 5.1 have a quadratically constrained quadratic programming (QCQP) structure¹ [11, 46, 78]. Particular complexity issues will arise when solving this kind of optimization problems (check Section 5.1.4).

5.1.2 Market procedure

The market procedure of Alkan and Tuncay [9] studied in Section 4.3 for roommates with non-transferable utility computes a balanced aspiration $f \in \mathbb{R}^{|D|}$. f corresponds to the payoff profile that players must have at any pairwise stable allocation. As their existence theorem (Theorem 4.3.4) states, a pairwise stable allocation will exist if and only the output of the market procedure is realizable by a proper allocation (and not a semiallocation allowing half-partnerships).

In this chapter we will deal with the following complexity issue of the pairwise stable allocations computation for the roommates submodel: Given a couple of doctors $(d_1, d_2) \in D \times D$ and strategy profiles $(f_1, f_2) \in \mathbb{R}^2$ such that,

$$f_{d_1} = u_{1,2}(f_{d_2})$$

¹Problem 1.1 can be decomposed in $|H|$ QCQP sub-problems.

we aim to compute $(x_{d_1}, x_{d_2}) \in X_{d_1} \times X_{d_2}$ such that,

$$f_{d_1}(x_{d_1}, x_{d_2}) = f_1 \text{ and } f_{d_2}(x_{d_2}, x_{d_1}) = f_2$$

Being able to find the strategy profile that achieves the payoffs (f_1, f_2) , for any payoff profile, will allow us to solve two challenges of this model: (1) the computation of the *demand sets* of the players at every iteration of the market procedure, and (2) given a balanced aspiration f , the computation of the strategy profile $\vec{x} \in X_D$ such that (μ, \vec{x}) implements f , whenever this can be done.

Consider a bi-matrix game $G = (X, Y, A, B)$ with $X = \Delta(S)$ and $Y = \Delta(T)$ being simplex, and A, B payoff matrices. Given $(u, v) \in \mathbb{R}^2$, we aim to find $x \in X, y \in Y$, such that,

$$\begin{aligned} xAy = u &\iff \sum_{s \in S} \sum_{t \in T} A(s, t)x_s y_t = u \\ xBy = v &\iff \sum_{s \in S} \sum_{t \in T} B(s, t)x_s y_t = v \end{aligned}$$

The previous system of quadratic equations can be seen as a QCQP with a constant objective function. Therefore, being able to solve QCQPs in polynomial time would allow us to compute the strategy profile of a couple that achieves a given payoff profile. We will come back to this problem later.

5.1.3 Renegotiation process

Chapter 3 presented a *renegotiation process* that, starting from any pairwise stable allocation, outputs a pairwise stable and renegotiation process allocation whenever players play feasible games and the algorithm converges. Moreover, it established the convergence of the algorithm for zero-sum, strictly competitive, potential, and infinitely repeated matching games. Although the convergence is guaranteed, to obtain an upper bound for the number of iterations, an ε -version of the renegotiation process needs to be considered.

We present the notion of ε -renegotiation proofness, characterize it by ε -constrained Nash equilibria, and define the class of ε -feasible games. Then, considering the ε -version of the renegotiation process, we will prove its finiteness with an upper bound $T \propto \frac{1}{\varepsilon}$. All the results presented in Chapter 3 can be generalized to their ε -version. For the sake of completeness, we present the formal proofs.

Along this section we will work with the *additive separable matching games* model, however, all the results are straightforwardly applicable to roommates matching games.

Definition 5.1.2. An ε -**pairwise stable** allocation $\pi = (\mu, \vec{x}, \vec{y})$ is ε -**renegotiation proof** if for any pair $(I, h) \in \mu$, any $d \in I$, and any $(s_d, t_h) \in X_d \times Y_h$, it holds,

1. If $f_{d,h}(s_d, y_h) > f_{d,h}(x_d, y_{h,d}) + \varepsilon$ then, $(\mu, (\vec{x}_{-d}, s_d), \vec{y})^2$ is not ε -pairwise stable,

²Given a strategy profile $\vec{x} \in \prod_{d \in D} X_d$, (\vec{x}_{-d}, s_d) denotes the strategy profile obtained when player d replaces her strategy $x_d \in \vec{x}$ by $s_d \in X_d$.

2. If $g_{d,h}(x_d, t_{d,h}) > g_{d,h}(x_d, y_{h,d}) + \varepsilon$ then, $(\mu, \vec{x}, (\vec{y}_{-(d,h)}, t_{d,h}))$ is not ε -pairwise stable.

Renegotiation proof allocations can be characterized as all those allocations in which all agents play constrained Nash equilibria (Proposition 3.5.2). An analogous version holds for ε -renegotiation proofness.

Definition 5.1.3. Let $\pi = (\mu, \vec{x}, \vec{y})$ be an allocation, $(I, h) \in \mu$ be a matched pair, and $d \in I$ a fixed doctor. We define the ε -**reservation payoffs** of d and h , $f_d^\pi(\varepsilon)$, $g_h^\pi(\varepsilon)$, respectively, as the optimal values of the problems,

$$\begin{aligned} & \max_{\substack{k \in H_0 \setminus \{h\} \\ (s,t) \in X_d \times Y_k}} \left\{ f_{d,k}(s, t) : g_{d,k}(s, t) > \min_{d' \in \mu(k)} g_{d',k}(x_{d'}, y_{h,d'}) + \varepsilon \right\} \\ & \max_{\substack{k \in D_0 \setminus \{I\} \\ (s,t) \in X_k \times Y_h}} \left\{ g_{k,h}(s, t) : f_{k,h}(s, t) > f_{k,\mu(k)}(x_k, y_{\mu(k),k}) + \varepsilon \right\} \end{aligned} \quad (5.2)$$

We recall a previous discussion about hospitals' reservation payoffs: It may be intuitive to think that hospitals should have a reservation payoff for each of their doctors. The intuition is correct. However, as reservation payoffs depend on the agents outside of the *couple*, the hospital has exactly the same reservation payoff for each of its doctors. In particular, no doctor d should decrease her contribution to the hospital h 's payoff below $g_h^\pi(\varepsilon)$, otherwise, the hospital will have incentives to replace d . We give next, the definition of ε -constrained Nash equilibria.

Definition 5.1.4. Given an allocation $\pi = (\mu, \vec{x}, \vec{y})$, a pair $(I, h) \in \mu$, a doctor $d \in I$, and their reservation payoffs $(f_d^\pi(\varepsilon), g_h^\pi(\varepsilon))$, a strategy profile $(x'_d, y'_{h,d}) \in X_d \times Y_h$ is

1. ε -**feasible** if $f_{d,h}(x'_d, y'_{h,d}) + \varepsilon \geq f_d^\pi(\varepsilon)$ and $g_{d,h}(x'_d, y'_{h,d}) + \varepsilon \geq g_h^\pi(\varepsilon)$,
2. an ε - $(f_d^\pi(\varepsilon), g_h^\pi(\varepsilon))$ -**constrained Nash equilibrium (CNE)** if it is ε -feasible and it satisfies,

$$\begin{aligned} f_{d,h}(x'_d, y'_{h,d}) + \varepsilon & \geq \max\{f_{d,h}(s, y'_{h,d}) : g_{d,h}(s, y'_{h,d}) + \varepsilon \geq g_h^\pi(\varepsilon), s \in X_d\}, \\ g_{d,h}(x'_d, y'_{h,d}) + \varepsilon & \geq \max\{g_{d,h}(x'_d, t) : f_{d,h}(x'_d, t) + \varepsilon \geq f_d^\pi(\varepsilon), t \in Y_h\} \end{aligned}$$

We denote the set of ε - $(f_d^\pi(\varepsilon), g_h^\pi(\varepsilon))$ -CNE by ε -CNE $(f_d^\pi(\varepsilon), g_h^\pi(\varepsilon))$.

We extend the characterization of renegotiation proof allocations through constrained Nash equilibria (Proposition 3.5.2) to the ε -case.

Theorem 5.1.5. *An ε -pairwise stable allocation $\pi = (\mu, \vec{x}, \vec{y})$ is ε -renegotiation proof if and only if for any pair $(I, h) \in \mu$ and $d \in I$, $(x_d, y_{h,d})$ is an ε - $(f_d^\pi(\varepsilon), g_h^\pi(\varepsilon))$ -constrained Nash equilibria, where $(f_d^\pi(\varepsilon), g_h^\pi(\varepsilon))$ are the agents' reservation payoffs (Equation (5.2)).*

Proof. Suppose that all couples play constrained Nash equilibria. Let $(d, h) \in \mu$ be a couple and $(x_d, y_{h,d})$ be their ε - $(f_d^\pi(\varepsilon), g_h^\pi(\varepsilon))$ -CNE. Suppose there exists $x'_d \in X_d$ such that,

$$f_{d,h}(x'_d, y_{h,d}) > f_{d,h}(x_d, y_{h,d}) + \varepsilon$$

It follows,

$$f_{d,h}(x'_d, y_{h,d}) > \max\{f_{d,h}(s, y_{h,d}) : g_{d,h}(s, y_{h,d}) + \varepsilon \geq g_h^\pi(\varepsilon), s \in X_d\}$$

Thus, $f_{d,h}(x'_d, y_{h,d}) + \varepsilon < g_h^\pi(\varepsilon)$. Let d' be the player that attains the maximum in $g_h^\pi(\varepsilon)$. Then, (d', h) is an ε -blocking pair of π . For player h the proof is analogous.

Conversely, suppose π is ε -renegotiation proof. Let $(d, h) \in \mu$ be a couple and $(x_d, y_{h,d})$ be their strategy profile. For any $x'_d \in X_d$ such that

$$f_{d,h}(x'_d, y_{h,d}) > f_{d,h}(x_d, y_{h,d}) + \varepsilon$$

it holds, $g_{d,h}(x'_d, y_{h,d}) + \varepsilon < g_h^\pi(\varepsilon)$. Thus,

$$f_{d,h}(x_d, y_{h,d}) + \varepsilon \geq \max\{f_{d,h}(s, y_{h,d}) : g_{d,h}(s, y_{h,d}) + \varepsilon \geq g_h^\pi(\varepsilon), s \in X_d\}$$

For player h the proof is analogous. □

ε -Constrained Nash equilibria are not guaranteed to exist in every bi-matrix game. Due to this, we extend the class of feasible games.

Definition 5.1.6. A two-person game is called ε -feasible if for any pair of reservation payoffs which admits at least one ε -feasible strategy profile, there exists an ε -constrained Nash equilibrium for the same pair of reservation payoffs.

The class of 0-feasible games contains all zero-sum games with a value, strictly competitive games with an equilibrium, potential games, and infinitely repeated games (Theorem 3.4.3). We will present the formal proof that zero-sum games with a value, strictly competitive games with an equilibrium, and infinitely repeated games are ε -feasible as well. Although we leave potential games out of the proof, we conjecture they belong to the class of ε -feasible games too.

Theorem 5.1.7. *The class of ε -feasible games includes zero-sum games with a value, strictly competitive games with an equilibrium, and infinitely repeated games.*

As for 0-feasible games, the proof of Theorem 5.1.7 is game dependent, and therefore, it has to be made for each class of games. Thus, we give the formal proofs with the complexity study. Algorithm 5.2 shows the pseudo-code of the ε -renegotiation process.

Theorem 5.1.8. *If Algorithm 5.2 converges, its output is an ε -pairwise stable and ε -renegotiation proof allocation.*

Proof. By construction, the output of Algorithm 5.2 is ε -renegotiation proof (Theorem 5.1.5). Regarding ε -pairwise stability, we prove that π always remains ε -pairwise stable at every iteration T . For $T = 0$ it holds as the input of Algorithm 5.2 is ε -pairwise stable. Suppose that for some $T > 0$, $\pi(T)$ is ε -pairwise stable but there

Algorithm 5.2: Renegotiation process

```

input :  $\pi = (\mu, \vec{x}, \vec{y})$   $\varepsilon$ -pairwise stable allocation
1  $t \leftarrow 1, \pi(t) \leftarrow \pi$ 
2 while True do
3   for  $(d, h) \in \mu$  do
4     | Compute the reservation payoffs  $f_d^{\pi(t)}$  and  $g_h^{\pi(t)}$  (Equation (5.2))
5     | Choose  $(x_d^*, y_{h,d}^*) \in \varepsilon\text{-CNE}(f_d^{\pi(t)}, g_h^{\pi(t)})$  and set  $(x_d^{t+1}, y_{h,d}^{t+1}) \leftarrow (x_d^*, y_{h,d}^*)$ 
6   end
7   if  $\forall (d, h) \in \mu, (x_d^{t+1}, y_{h,d}^{t+1}) = (x_d^t, y_{h,d}^t)$  then
8     | Output  $\pi(t)$ 
9   end
10   $t \leftarrow t + 1$ 
11 end

```

exists an ε -blocking pair (d, h) of $\pi(T + 1)$. Then, there exists $(x^*, y^*) \in X_d \times Y_h$ such that

$$f_{d,h}(x^*, y^*) > f_d(\pi(T + 1)) + \varepsilon \text{ and } g_{d,h}(x^*, y^*) > \min_{k \in \mu^{T+1}(h)} g_{k,h'}(x_k^{T+1}, y_{h,k}^{T+1}) + \varepsilon$$

Necessarily, d or h changed of strategy profile at T , otherwise (d, h) would also block $\pi(T)$. Without loss of generality, suppose d did. It follows,

$$f_{d,h}(x^*, y^*) > f_d(\pi(T + 1)) + \varepsilon = f_{d,\mu(h)}(x', y') + \varepsilon \geq f_d^{\pi(T)}(\varepsilon) \geq f_{d,h}(x^*, y^*)$$

where $f_d^{\pi(T)}(\varepsilon)$ is d 's reservation payoffs at time T , $(x', y') \in \varepsilon\text{-CNE}(f_d^{\pi(T)}(\varepsilon), g_{\mu(h)}^{\pi(T)}(\varepsilon))$ is the CNE chosen by $(d, \mu(d))$ at time T , and the last inequality comes from Equation (5.2). We obtain a contradiction. \square

Section 3.5 proved the convergence of the renegotiation process (Algorithm 3.1) for different feasible games. We extend these proofs to the ε -version of the algorithm (Algorithm 5.2) during its complexity analysis. The next section discusses the issues of having QCQP problems in our algorithms.

5.1.4 Quadratically constrained quadratic programs

The main issue in the complexity study of our algorithms is the presence of quadratically constrained quadratic programming (QCQP) problems [11, 46, 78]. As we have already remarked, the optimization problems solved during an iteration of the deferred-acceptance with competitions algorithm, the demand sets of the market procedure of Alkan and Tuncay, the allocation computation given a balanced aspiration, the computation of the reservation payoffs during the renegotiation process, or even

the constrained Nash equilibria computation, all of them have the following structure,

$$\begin{aligned} \max \quad & xAy \\ \text{s.t.} \quad & xBy \geq c \\ & x \in X, y \in Y \end{aligned} \tag{5.3}$$

where A, B are real-valued matrices, $c \in \mathbb{R}$, and X, Y are simplex. For negative semi-definite matrices A and B , Problem (5.3) corresponds to a convex problem and can be solved in polynomial time. However, in its most general case, Problem (5.3) is NP-hard. Luckily, for zero-sum games, strictly competitive games, and infinitely repeated games we will manage to reduce these problems to a polynomial number of linear programs.

Linear programming is one of the most useful tools to prove the polynomial complexity of given problems. The first polynomial algorithms for linear programming problems were published by Khachiyan [73] and Karmarkar [70]. For our analysis, we will refer to the complexity result of Vaidya [98].

Theorem 5.1.9 (Vaidya'89). *Let P be a linear program with m constraints, n variables, and such its data takes L bits to be encoded. Then, in the worst case, P can be solved in $O((n + m)^{1.5}nL)$ elementary operations.*

We split the complexity analysis into three sections, one per each kind of the matching game mentioned: zero-sum, strictly competitive, and infinitely repeated. The three algorithms recalled in the previous sections are studied in each setting.

5.2 Zero-sum matching games

Consider a matching game Γ in which all strategic games are finite zero-sum matrix games in mixed strategies, from now on, a zero-sum matching game. Notice Γ can be either a additive separable matching game or a roommates matching game. When needed, we will specify the submodel considered.

We study the complexity of the three algorithms recalled in the previous section: *deferred-acceptance with competitions* (Algorithm 5.1), *market procedure*, and *renegotiation process* (Algorithm 5.2), when the matching game Γ is a zero-sum matching game (with the corresponding submodel). The following subsections will split the analysis for each algorithm. All the presented results will use the following main theorem.

Theorem 5.2.1. *Let $G = (X, Y, A, B)$ be a finite zero-sum game in mixed strategies, where $X = \Delta(S)$, $Y = \Delta(T)$ are simplexes with S, T pure strategy sets, and A, B are payoff matrices. Given a vector c , the QCQP Problem (5.3),*

$$\begin{aligned} \max \quad & xAy \\ \text{s.t.} \quad & xBy \geq c \\ & x \in X, y \in Y \end{aligned}$$

5.2. Zero-sum matching games

can be solved in $O(|S| \cdot |T|)$ comparisons.

To prove Theorem 5.2.1 we need a preliminary result. Notice, first of all, that since G is a zero-sum game, the QCQP Problem (5.3) can be rewritten as

$$\begin{aligned} \max \quad & xAy \\ \text{s.t.} \quad & xAy \leq c \\ & x \in X, y \in Y \end{aligned} \tag{5.4}$$

Therefore, solving the previous optimization problem is equivalent to finding a strategy profile (x, y) such that $xAy = \min\{c, \max A\}$ ³. Without loss of generality it can be always considered $\min A \leq c \leq \max A$ since replacing c by $\min\{c, \max A\}$ does not change at all Problem (5.4) and for $c < \min A$ the problem is infeasible.

Lemma 5.2.2. *Given a matrix payoff A and $c \in \mathbb{R}$, with $\min A \leq c \leq \max A$, there always exists $(x, y) \in X \times Y$, such that $xAy = c$, with x or y being a pure strategy.*

Proof. Let $s \in S$ be a pure strategy for player 1 in G , such that there exist $t, t' \in T$, with $A(s, t) \leq c \leq A(s, t')$. Then, there exists $\lambda \in [0, 1]$ such that $\lambda A(s, t) + (1 - \lambda)A(s, t') = c$. Even more, λ is explicitly given by

$$\lambda = \frac{c - A(s, t)}{A(s, t') - A(s, t)} \tag{5.5}$$

Suppose that such a pure strategy s does not exist, so for any $s \in S$, either $A(s, t) \leq c, \forall t \in T$, or $A(s, t) \geq c, \forall t \in T$. Let $t \in T$ be any pure strategy of player 2. Then, since $\min A \leq c \leq \max A$, there exists $s, s' \in S$ such that $A(s, t) \leq c \leq A(s', t)$. Thus, considering λ given by,

$$\lambda = \frac{c - A(s, t)}{A(s', t) - A(s, t)} \tag{5.6}$$

it holds that $\lambda A(s, t) + (1 - \lambda)A(s', t) = c$. □

We are ready to prove the complexity of solving the QCQP problem for a zero-sum game (Theorem 5.2.1).

Proof of Theorem 5.2.1. The complexity of solving the QCQP Problem (5.4) corresponds to the one of finding the pure strategies used in the convex combination of Lemma 5.2.2's proof and then computing the corresponding λ . Let

$$S^+ := \{s \in S : \exists t \in T, A(s, t) \geq c\} \text{ and } S^- := \{s \in S : \exists t \in T, A(s, t) \leq c\}$$

These sets are computed in $|S| \cdot |T|$ comparisons, as in the worst case we have to check all coefficients in A . As $\min A \leq c \leq \max A$, both sets are non-empty. If $S^+ \cap S^- \neq \emptyset$,

³We introduce the notation $\max A := \max_{s,t} A(s, t)$ and $\min A := \min_{s,t} A(s, t)$

there exist $s \in S$ and $t, t' \in T$ such that $A(s, t) \leq c \leq A(s, t')$, so Equation (5.5) gives the solution desired. Otherwise, there exists $t \in T$ and $s, s' \in S$ such that $A(s, t) \leq c \leq A(s', t)$, and Equation (5.6) gives the solution desired. Computing the intersection of S^+ and S^- has complexity $O(|S|)$. In either case (the intersection is empty or not), finding the pure strategies needed for the convex combination takes at most $|T|$ comparisons. Finally, computing λ requires a constant number of operations for the sizes of the strategy sets. Adding all up, we obtain the complexity result stated. \square

5.2.1 Deferred-acceptance with competitions algorithm

Suppose Γ is a zero-sum additive separable matching game. We aim to prove the following result.

Theorem 5.2.3 (Complexity). *Let $d \in D$ be a proposer doctor. Let h be the proposed hospital and d' be the doctor that d wants to replace. If d is the winner of the competition, the entire iteration of the DAC algorithm (Algorithm 5.1) has complexity,*

$$O\left(\left[|H| \cdot |D| + (|S_d| + |S_{d'}|) \cdot \sum_{h' \in H} |T_{h'}|\right] L\right)$$

where L represents the number of bits required to encode all the data.

The proof of Theorem 5.2.3 is split in several results, each of them being a corollary of the complexity result for the general QCQP problem (Theorem 5.2.1).

Corollary 5.2.4. *d 's optimal proposal can be computed in*

$$O\left(|H| \cdot |D| + |S_d| \cdot \sum_{h' \in H} |T_{h'}|\right)$$

comparisons.

Proof. d 's optimal proposal is computed by solving,

$$\begin{aligned} & \max x A_{d,h'} y \\ & s.t \ x A_{d,h'} y \leq \max_{d' \in \mu(h')} x_{d'} A_{d',h'} y_{h',d'} - \varepsilon \\ & \quad h' \in H_0, x \in X_d, y \in Y_{h'} \end{aligned} \tag{5.7}$$

Problem (5.7) can be solved by dividing it in $|H|$ sub-problems (one per hospital) and taking the best of the $|H|$ solutions. Once computed the right-hand side of each subproblem, they get the structure of the general QCQP Problem 5.4 so they need a polynomial number of comparisons to be solved (Theorem 5.2.1). Computing the right-hand side for each of them takes $|D|$ comparisons in the worst case. The complexity stated comes from putting it all together. \square

Remark 5.2.5. d 's reservation payoff when competing for h can be computed by solving Problem (5.7) leaving h out of the feasible region. Therefore, its complexity is bounded by the one in Corollary 5.2.4.

Corollary 5.2.6. *The computation of the reservation payoff β_d of doctor d plus her bid λ_d during a competition takes*

$$O\left(|H| \cdot |D| + |S_d| \cdot \sum_{h' \in H} |T_{h'}|\right)$$

comparisons.

Proof. d 's bid is computed by,

$$\begin{aligned} \min \quad & xA_{d,hy} \\ \text{s.t.} \quad & xA_{d,hy} \geq \beta_d \\ & x \in X_d, y \in Y_h \end{aligned} \tag{5.8}$$

and takes $O(|S_d| \cdot |T_h|)$ comparisons (Theorem 5.2.1). Adding this to the complexity of computing β_d , we obtain the stated result. \square

Finally, we study the optimization problem solved by the winner.

Corollary 5.2.7. *The final strategy profile played by the winner of a competition can be computed in $O(|S_d| \cdot |T_h|)$ comparisons.*

Proof. Let $\lambda_{d'}$ be the bid of d' . d solves,

$$\begin{aligned} \max \quad & xA_{d,hy} \\ \text{s.t.} \quad & xA_{d,hy} \leq \lambda_{d'} \\ & x \in X_d, y \in Y_h \end{aligned} \tag{5.9}$$

Problem (5.9) has the same structure of Problem (5.4). Therefore, we can solve it in $O(|S_d| \cdot |T_h|)$ comparisons. \square

The complexity of an entire iteration of the DAC algorithm (Theorem 5.2.3) is obtained by adding up the complexity results given in Corollaries 5.2.4, 5.2.6 and 5.2.7. We omit its formal proof.

Remark 5.2.8. If there are at most N players in each side and at most k pure strategies per player, Theorem 5.2.3 proves that each iteration of the DAC algorithm (Algorithm 5.1) takes $O((N^2 + k^2)L)$ number of elementary operations in being solved, hence is polynomial. As the number of iterations does not depend on the size of the problem but only on ε , we conclude that computing an ε -pairwise stable allocation for a one-to-many zero-sum matching game is a polynomial problem

5.2.2 Market procedure

Let Γ be a zero-sum roommates matching game. Computing the demand sets of the doctors for a zero-sum roommates matching game becomes particularly easy. Consider $f \in \mathbb{R}^{|D|}$ and two doctors $(d_1, d_2) \in D \times D$. Notice that,

$$\begin{aligned} d_2 \in P_{d_1}(f) &\iff f_{d_1} = u_{1,2}(f_{d_2}) \\ &\iff f_{d_1} = -f_{d_2} \\ &\iff f_{d_1} + f_{d_2} = 0 \end{aligned}$$

In other words, to compute the demand set of a given doctor d , it is enough with checking whether the sum of the payoff of both agents is equal to zero. We directly state the following result.

Theorem 5.2.9. *Computing the demand sets of all doctors during an iteration of the market procedure, given the current payoff profile $f \in \mathbb{R}^{|D|}$, has complexity $O(|D|^2L)$, where L is the number of bits required to encode all the data.*

The previous theorem exploits that to compute the doctors' demand sets, it is not needed to compute the strategy profiles but only to compare the agents' payoffs.

Theorem 5.2.10. *Let $f \in \mathbb{R}^{|D|}$ be the output of the market procedure and suppose there exists a matching μ such that for any doctor $d \in D$, $f_d + f_{\mu(d)} = 0$ if d is matched and $f_d = \underline{f}_d$ if d is unmatched. Then, computing the strategy profile $\vec{x} \in X_D$ such that,*

$$f_{d,\mu(d)}(x_d, x_{\mu(d)}) = f_d, \forall d \text{ matched}$$

has complexity

$$O\left(\sum_{(d_1, d_2) \in \mu} |S_{d_1}| \cdot |S_{d_2}| L\right)$$

where L is the number of bits required to encode all the data.

Proof. Given a couple $(d_1, d_2) \in \mu$, we aim to find $(x_{d_1}, x_{d_2}) \in X_{d_1} \times X_{d_2}$ such that,

$$\begin{aligned} f_{d_1, d_2}(x_{d_1}, x_{d_2}) &= f_{d_1} \\ f_{d_2, d_1}(x_{d_2}, x_{d_1}) &= f_{d_2} \end{aligned}$$

Since the strategic game of d_1 and d_2 is a zero-sum game, it is enough with computing (x_{d_1}, x_{d_2}) such that,

$$\begin{aligned} f_{d_1, d_2}(x_{d_1}, x_{d_2}) &= f_{d_1} \\ \iff x_{d_1} A_{d_1, d_2} x_{d_2} &= f_{d_1} \\ \iff \sum_{s \in S_1} \sum_{s' \in S_2} A_{d_1, d_2}(s, s') x_{d_1}(s) x_{d_2}(s') &= f_{d_1} \end{aligned}$$

which corresponds to a quadratic equation with $|S_1| + |S_2|$ variables. Notice this problem has the exact form $xAy = c$ studied in Lemma 5.2.2 that, we have already discussed, can be solved in $O(|S_1| \cdot |S_2|)$ comparisons. We conclude the proof by considering the sum over all couples in μ plus the number of bits required to encode the data. \square

Checking whether the output of the market procedure is realizable by an allocation (and not a semiallocation) can be easily done in the case of zero-sum matching games. Given $f \in \mathbb{R}^{|D|}$ the output, divide the set of doctors D as,

$$\begin{aligned} D_+ &= \{d \in D : f_d > 0\} \cup \{d_0\} \\ D_- &= \{d \in D : f_d < 0\} \cup \{d_0\} \\ D_\sim &= \{d \in D : f_d = 0\} \end{aligned}$$

Then, find a correspondence μ between D_+ and D_- such that for any pair of matched agents, the sum of their payoffs in f is zero, or the agents are matched with the empty doctor in case their payoff in f was equal to their IRP. Notice the correspondence can be computed in $|D|^2$ comparisons by checking all possible combinations. Regarding D_\sim , divide the set into two equal parts and match the agents between them, or the empty doctor in case their IRPs are equal to zero and D_\sim has an odd number of doctors. In case this procedure outputs a proper matching μ , the matching game has an pairwise stable allocation. Otherwise, the set of pairwise stable allocations is necessarily empty as the outcome of the market procedure cannot be realizable by an allocation.

Remark 5.2.11. If there are at most N doctors and k pure strategies per doctor, computing all the demand sets during an iteration of the market procedure has complexity $O(N^2L)$. Similarly, in case the outcome of the market procedure can be implemented by an allocation, finding the strategy profiles of the agents within the allocation has complexity $O(Nk^2L)$.

5.2.3 Renegotiation process

We focus now on the computation of renegotiation proof allocations. Since the same algorithm can be used for both models, additive separable matching games and room-mates matching games, we will only work on the first case. All results can be directly applied to the second model.

Suppose Γ is an additive separable matching game in which each strategic game $G_{d,h} = (X_d, Y_h, A_{d,h})$ is a finite zero-sum game in mixed strategies with value $w_{d,h}$, where $A_{d,h}$ is the payoff matrix. We aim to prove the following result.

Theorem 5.2.12 (CNE Complexity). *Let (d, h) be a couple and $G_{d,h} = (X_d, Y_h, A_{d,h})$ be their bi-matrix zero-sum game with value $w_{d,h}$. Let (f_d, g_h) be a pair of reservation payoffs. Then,*

1. $G_{d,h}$ is ε -feasible,
2. For any $(x', y') \in \varepsilon\text{-CNE}(f_d, g_h)$, it holds

$$x' A_{d,h} y' = \text{median}\{f_d - 2\varepsilon, w_{d,h}, g_h + 2\varepsilon\}$$

3. Computing an $\varepsilon\text{-CNE}(x', y')$ has complexity

$$O\left(\max\{|S_d|, |T_h|\}^{2.5} \cdot \min\{|S_d|, |T_h|\} \cdot L_{d,h}\right),$$

where S_d, T_h are the pure strategy sets of the players and $L_{d,h}$ is the number of bits required to encode the matrix $A_{d,h}$.

We will make use of the following lemma.

Lemma 5.2.13. *Let $s_1, s_2 \in S_d$ be two pure strategies for player d , (x^*, y^*) be the optimal strategies of the players, and $(x, y) \in X_d \times Y_h$ be a strategy profile such that x only has s_1, s_2 in its support. Consider $\tau \in (0, 1)$ and define $y_\tau := (1 - \tau)y + \tau y^*$. Suppose that $x A_{d,h} y_\tau = f_d$ but $s_1 A_{d,h} y_\tau \neq f_d \neq s_2 A_{d,h} y_\tau$. Finally, suppose that $w_{d,h} < f_d$. Then, there always exists $\tau' \in (\tau, 1)$, and a pure strategy $s \in S_d$ such that $s A_{d,h} y_{\tau'} = f_d$.*

Proof. It holds,

$$x A_{d,h} y_\tau = x_{s_1} \cdot s_1 A_{d,h} y_\tau + x_{s_2} \cdot s_2 A_{d,h} y_\tau = f_d$$

with $x_{s_1} + x_{s_2} = 1$, $x_{s_1}, x_{s_2} \in [0, 1]$. Since $s_1 A_{d,h} y_\tau$ and $s_2 A_{d,h} y_\tau$ are both different from f_d , we can suppose (without loss of generality) that $s_1 A_{d,h} y_\tau > f_d$ and $s_2 A_{d,h} y_\tau < f_d$. Then, as $x^* A_{d,h} y^* = w_{d,h} < f_d$ and (x^*, y^*) is a saddle point, $s_1 A_{d,h} y^* \leq w_{d,h} < f_d$. As $y_{\{\tau=1\}} = y^*$, by continuity, there exists $\tau' \in (\tau, 1)$ such that, $s_1 A_{d,h} y_{\{\tau=1\}} < f_d = s_1 A_{d,h} y_{\tau'}$. \square

Lemma 5.2.13 can be easily extended to mixed strategies of any finite support.

Proof of Theorem 5.2.12. Let (x^*, y^*) be the optimal strategies of the players, i.e., the strategy profile that achieves the value of the game $x^* A_{d,h} y^* = w_{d,h}$. We split the proof into three cases.

1. Suppose that $f_d - 2\varepsilon \leq w_{d,h} \leq g_h + 2\varepsilon$. In particular, the value of the game is ε -feasible for both agents. Since it is also a saddle point so agents do not have profitable deviations, (x^*, y^*) is an $\varepsilon\text{-}(f_d, g_h)\text{-CNE}$. From Von Neumann's theorem, we know that $(x^*, y^*, w_{d,h})$ can be obtained from the solutions of the pair primal-dual problems,

$$\begin{array}{ll} (P) & \min \langle c, x \rangle \\ & x A_{d,h} \geq b \\ & x \geq 0 \end{array} \qquad \begin{array}{ll} (D) & \max \langle b, y \rangle \\ & A_{d,h} y \leq c \\ & y \geq 0 \end{array}$$

5.2. Zero-sum matching games

where the variables satisfy $x \in X_d$, $y \in Y_h$, and the vectors c, d are given and are equal to 1 in every coordinate. If (x', y') is the primal-dual solution and z is their optimal value, the optimal strategies of player d and h are given by $(x^*, y^*) = (x'/z, y'/z)$, and they achieve the value of the game $w_{d,h}$. From Vaidya's linear programming complexity result (Theorem 5.1.9), the number of elementary operations needed to solve the primal-dual problem and computing (x^*, y^*) is

$$O\left((|S_d| + |T_h|)^{1.5} \max\{|S_d|, |T_h|\} L_{d,h}\right)$$

2. Suppose that $w_{d,h} < f_d - 2\varepsilon \leq g_h + 2\varepsilon$. Let (x_0, y_0) be an ε -feasible strategy profile. Consider the set

$$\Lambda(f_d) := \{x \in X_d : \exists y \in Y_h, xA_{d,h}y + 2\varepsilon \geq f_d\}$$

Notice $\Lambda(f_d)$ is non-empty as (x_0, y_0) belongs to it. Consider the problem,

$$\sup [\inf \{xA_{d,h}y : xA_{d,h}y + 2\varepsilon \geq f_d, y \in Y_h\} : x \in \Lambda(f_d)] \quad (5.10)$$

Since the set $\{xA_{d,h}y + 2\varepsilon \geq f_d, y \in Y_h\}$, for a given x , is bounded, as well as the set $\Lambda(f_d)$, there exists a solution (x, y) of Problem (5.10). Moreover, computing (x, y) has complexity $O(|T_h| \cdot |S_d|^{2.5}L)$ as Problem (5.10) is equivalent to solve $|T_h|$ linear programming problems, each of them with $|S_d|$ variables and 1 constraint, and then considering the highest value between them.

By construction, $xA_{d,h}y + 2\varepsilon \geq f_d$. Suppose $xA_{d,h}y + 2\varepsilon > f_d$. It follows,

$$xA_{d,h}y > f_d - 2\varepsilon > w_{d,h} = x^*A_{d,h}y^* \geq xA_{d,h}y^*$$

where the last inequality holds as (x^*, y^*) is a saddle point. Then, there exists $y' \in (y, y^*)$ such that $xA_{d,h}y' = f_d - 2\varepsilon$. This contradicts that (x, y) is solution of Problem (5.10). If (x, y) is an ε - (f_d, g_h) -CNE, the proof is over. Otherwise, consider the problem,

$$t := \sup\{\tau \in [0, 1] : y_\tau := (1 - \tau)y + \tau y^* \text{ and } \exists x_\tau \in X_d, x_\tau A_{d,h}y_\tau = f_d - 2\varepsilon\} \quad (5.11)$$

t exists as for $\tau = 0$, $xA_{d,h}y = f_d - 2\varepsilon$. In addition, $y_t \neq y^*$ as $x^*A_{d,h}y^* < f_d - 2\varepsilon$ and (x^*, y^*) is a saddle point. From Lemma 5.2.13, if $xA_{d,h}y_\tau = f_d$ for some value $\tau \in (0, 1)$, then there always exists a pure strategy $s \in S_d$ and $\tau \leq \tau' < 1$ such that $sA_{d,h}y_{\tau'} = f_d$. Thus, solving Problem (5.11) is equivalent to solve each of the next linear problems,

$$t_s := \sup\{\tau \in [0, 1] : y_\tau := (1 - \tau)y + \tau y^* \text{ and } sA_{d,h}y_\tau = f_d - 2\varepsilon\}, \forall s \in S_d,$$

and then, considering $t := \max_{s \in S_d} t_s$. Each t_s can be computed in constant time over $|S_d|$ and $|T_h|$, as the linear programming problem associated has only one variable and one constraint. Finally, computing the maximum of all t_s takes $|S_d|$ comparisons. We

claim that (x_t, y_t) is an ε - (f_d, g_h) -CNE. Let $x' \in X_d$ such that $x' A_{d,h} y_t \leq g_h + \varepsilon$. We aim to prove that $x' A_{d,h} y_t \leq x_t A_{d,h} y_t + \varepsilon$. Suppose $x' A_{d,h} y_t > x_t A_{d,h} y_t + \varepsilon$. It holds,

$$x' A_{d,h} y^* \leq w_{d,h} = x^* A_{d,h} y^* < f_d - 2\varepsilon = x_t A_{d,h} y_t < x_t A_{d,h} y_t + \varepsilon < x' A_{d,h} y_t$$

Then, there exists $z \in X_d$ and $y' \in (y_t, y^*)$ such that $z A_{d,h} y' = f_d - 2\varepsilon$, contradicting that t is solution of Problem (5.11).

Regarding player h , let $y' \in Y_h$ such that $x_t A_{d,h} y' + \varepsilon \geq f_d$. We aim to prove that $x_t A_{d,h} y' \geq x_t A_{d,h} y_t - \varepsilon$, which follows from,

$$x_t A_{d,h} y' \geq f_d - \varepsilon = f_d - 2\varepsilon + \varepsilon = x_t A_{d,h} y_t + \varepsilon > x_t A_{d,h} y_t - \varepsilon$$

We conclude that $(x_t, y_t) \in \varepsilon$ -CNE (f_d, g_h) .

3. Suppose that $f_d - 2\varepsilon \leq g_h + 2\varepsilon < w_{d,h}$. Analogously⁴ to case 2, there exists an ε - (f_d, g_h) -CNE (x, y) satisfying $x A_{d,h} y = g_h + 2\varepsilon$.

Finally, the complexity given at the theorem's state is obtained when taking the maximum complexity between the three cases. \square

As a corollary of Theorem 5.2.12 we obtain the following result.

Corollary 5.2.14. *Given an allocation $\pi = (\mu, \vec{x}, \vec{y})$, computing all games' values is a polynomial problem and its complexity is bounded by*

$$O\left(\sum_{(d,h) \in \mu} (|S_d| + |T_h|)^{1.5} \max\{|S_d|, |T_h|\} L_{d,h}\right)$$

Proof. Let $(I, h) \in \mu$ be a matched pair, $d \in I$ a doctor, and $G_{d,h} = (X_d, Y_h, A_{d,h})$ be a zero-sum game. The proof of Theorem 5.2.12 in its first case proves that computing $w_{d,h}$ takes at most $O((|S_d| + |T_h|)^{1.5} \max\{|S_d|, |T_h|\} L_{d,h})$ elementary operations, where S_d, T_h are the players' strategy sets and $L_{d,h}$ is the number of bits required to encode the matrix $A_{d,h}$. Summing up all the couples, we obtain the stated complexity. \square

The complexity of one iteration of the renegotiation process corresponds to the complexity of computing the reservation payoffs and a constrained Nash equilibrium for each couple. As we can have at most $|D|$ couples, the complexity of an entire iteration of the renegotiation process (Algorithm 5.2) is bounded by,

$$O\left(\sum_{d \in D} \left[|H| \cdot |D| + |S_d| \cdot \sum_{h \in H} |T_h| + \max\{|S_d|, |T_{\mu(d)}|\}^{2.5} \cdot \min\{|S_d|, |T_{\mu(d)}|\} \right] \cdot L\right)$$

where L is the number of bits required to encode all the problem data.

⁴An analogous version of Lemma 5.2.13 has to be proved as well. As the proof follows exactly the same arguments, we do not present this result.

5.2. Zero-sum matching games

Remark 5.2.15. Considering N agents per side and k pure strategies per agent, the complexity of an entire iteration of the renegotiation process (Algorithm 5.2) is bounded by,

$$O\left(N^4 k^{3.5} L\right)$$

Hence, it is polynomial.

The renegotiation process in its original version is known to converge for zero-sum matching games. However, no bound could be given to the number of iterations. For the ε -version, in exchange, we are able to guarantee a bound $T \propto \frac{1}{\varepsilon}$, with T not depending on the problem size.

Theorem 5.2.16 (Convergence). *Let Γ be a bi-matrix zero-sum matching game such that each game $G_{d,h}$ has a value $w_{d,h}$. Let $\pi = (\mu, \vec{x}, \vec{y})$ be an ε -pairwise stable allocation, input of the ε -renegotiation process (Algorithm 5.2), the one defines a profile of ε -reservation payoffs $(f_d^\pi(\varepsilon), g_h^\pi(\varepsilon))_{d \in I, (I,h) \in \mu}$. Then, the number of iterations of Algorithm 5.2 is bounded by*

$$\frac{1}{\varepsilon} \max_{d \in I, (I,h) \in \mu} \{f_d^\pi(\varepsilon) - w_{d,h}, w_{d,h} - g_h^\pi(\varepsilon)\}$$

To prove Theorem 5.2.16 we will make use of the following lemma.

Lemma 5.2.17. *Let Γ be a matching game as in Theorem 5.2.16. Let $\pi = (\mu, \vec{x}, \vec{y})$ be an ε -pairwise stable allocation, (I, h) be a matched pair, and $d \in I$. Consider the sequence of reservation payoffs of (d, h) denoted by $(f_d^{\pi(t)}(\varepsilon), g_h^{\pi(t)}(\varepsilon))_t$, with t being the iterations of the renegotiation process (Algorithm 5.2). If there exists t^* such that $w_{d,h} \leq f_d^{\pi(t^*)}(\varepsilon) - 2\varepsilon$ (resp. $w_{d,h} \geq g_h^{\pi(t^*)}(\varepsilon) + 2\varepsilon$), then the subsequence $(f_d^{\pi(t)}(\varepsilon))_{t \geq t^*}$ (resp. $(g_h^{\pi(t)}(\varepsilon))_{t \geq t^*}$) decreases (resp. increases) at least ε at each step.*

Proof. Suppose there exists an iteration t in which $w_{d,h} \leq f_d^{\pi(t)}(\varepsilon) - 2\varepsilon \leq g_h^{\pi(t)}(\varepsilon) + 2\varepsilon$, so couple (d, h) switches its payoff to $f_d^{\pi(t)}(\varepsilon) - 2\varepsilon$ (Theorem 5.2.12). Let (\hat{x}_d, \hat{y}_h) be the ε - $(f_d^{\pi(t)}(\varepsilon), g_h^{\pi(t)}(\varepsilon))$ -CNE played by (d, h) at iteration t . Since (\hat{x}_d, \hat{y}_h) must be ε - $(f_d^{\pi(t+1)}(\varepsilon), g_h^{\pi(t+1)}(\varepsilon))$ -feasible⁵, in particular, it holds $f_d^{\pi(t+1)}(\varepsilon) \leq \hat{x}_d A_{d,h} \hat{y}_h + \varepsilon = f_d^{\pi(t)}(\varepsilon) - \varepsilon$. Therefore, the sequence of reservation payoffs starting from t decreases at least in ε at each step. \square

Finally, we prove the convergence of the ε -renegotiation process in a $T \propto \frac{1}{\varepsilon}$ number of iterations.

Proof of Theorem 5.2.16. At the beginning of the renegotiation process (Algorithm 5.2), all couples (d, h) belong to one (not necessarily the same) of the following cases: $f_d^\pi(\varepsilon) - 2\varepsilon \leq w_{d,h} \leq g_h^\pi(\varepsilon) + 2\varepsilon$, $w_{d,h} \leq f_d^\pi(\varepsilon) - 2\varepsilon \leq g_h^\pi(\varepsilon) + 2\varepsilon$ or $f_d^\pi(\varepsilon) - 2\varepsilon \leq g_h^\pi(\varepsilon) + 2\varepsilon \leq w_{d,h}$. In the first case, the couple plays their Nash equilibrium and

⁵ ε -version of Proposition 3.5.3, seen in Theorem 5.1.8's proof.

never changes it afterward. In the second case, as $f_d^\pi(\varepsilon)$ is strictly decreasing for d (Lemma 5.2.17) and bounded from below by $w_{d,h}$, the sequence of reservation payoffs converges in at most $\frac{1}{\varepsilon}(f_d^\pi(\varepsilon) - w_{d,h})$ iterations. Analogously, the sequence of reservation payoffs for h converges on the third case in finite time. Therefore, Algorithm 5.2 converges it at most $\frac{1}{\varepsilon} \max_{(d,h) \in \mu} \{f_d^\pi(\varepsilon) - w_{d,h}, w_{d,h} - g_h^\pi(\varepsilon)\}$ iterations. \square

Let $T := \max\{\max A_{d,h} - \min A_{d,h} : (d, h) \in D \times H\}$. The following table summarizes the complexity results found for zero-sum matching games.

Algorithms	Complexity/It	Max N ^o It
DAC	$O((N^2 + k^2)L)$	T/ε
Market procedure:	-	$p(N)$
Demand sets	$O(N^2L)$	-
Implementation	$O(Nk^2L)$	-
Renegotiation process	$O(N^4k^{3.5}L)$	T/ε

Table 5.1: Complexity zero-sum matching games: N players per side, k strategies per player, L bits to encode the data, and $p(N)$ a polynomial on N .

5.3 Strictly competitive matching games

The class \mathcal{S} of strictly competitive games, initially defined by Aumann [13], was fully characterized by Adler, Daskalakis, and Papadimitriou [7] in the bi-matrix case.

Definition 5.3.1. A bimatrix game $G = (S, T, A, -B)$, with S, T finite pure strategy sets and $A, -B$ payoff matrices, is called a **strictly competitive game** if for any $x, x' \in \Delta(S), y, y' \in \Delta(T)$, $xAy - x'Ay'$ and $xBy - x'By'$ have always the same sign.

Definition 5.3.2. Given two matrices $A, B \in \mathbb{R}^{m \times n}$, we say that B is an **affine variant** of A if for some $\lambda > 0$ and unrestricted $\mu \in \mathbb{R}$, $B = \lambda A + \mu U$, where U is $m \times n$ all-ones matrix.

Adler et al. proved the following result.

Theorem 5.3.3. *If for all $x, x' \in X$ and $y, y' \in Y$, $xAy - x'Ay'$ and $xBy - x'By'$ have the same sign, then B is an affine variant of A . Even more, the affine transformation is given by,*

$$A = \frac{a^{max} - a^{min}}{b^{max} - b^{min}} [B - b^{min}U] + a^{min}U, \text{ with } \begin{cases} a^{max} := \max A, & a^{min} := \min A \\ b^{max} := \max B, & b^{min} := \min B \end{cases}$$

If $a^{max} = a^{min}$, then it also holds that $b^{max} = b^{min}$ (and vice-versa), in which case clearly A and B are affine variants.

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Theorem 5.3.3 allows us to extend all the results obtained in the previous section for zero-sum matching games to strictly competitive matching games. First of all, we prove that computing the affine transformations is a polynomial problem.

Theorem 5.3.4. *Let Γ be a matching game in which all strategic games $G_{d,h} = (S_d, T_h, A_{d,h}, -B_{d,h})$ are bi-matrix strictly competitive games. Let Γ' be the affine transformation of Γ in which all couples play zero-sum games. Then, computing Γ' has complexity*

$$O\left(|D| + |H| + \sum_{d \in D} \sum_{h \in H} |S_d| \cdot |T_h|\right)$$

Proof. In order to obtain Γ' , besides computing all zero-sum games, we also need to compute all the new individually rational payoffs. Let $(d, h) \in D \times H$ be a potential couple that plays a strictly competitive game $(S_d, T_h, A_{d,h}, -B_{d,h})$. The complexity of computing their affine transformation to a zero-sum game $(S_d, T_h, B_{d,h}, -B_{d,h})$ is $O(|S_d| \cdot |T_h|)$, as we need to compute $a_{d,h}^{max}$, $a_{d,h}^{min}$, $b_{d,h}^{max}$, and $b_{d,h}^{min}$. Regarding the individually rational payoffs $(\underline{f}_d, \underline{g}_h)$, set $\alpha_{d,h} := \frac{a_{d,h}^{max} - a_{d,h}^{min}}{b_{d,h}^{max} - b_{d,h}^{min}}$. We take $\alpha_{d,h}$ so it is always lower or equal to 1 (at least one of the two ways of taking the affine transformation guarantees this). Given $(x, y) \in X_d \times Y_h$ a strategy profile, notice that,

$$xA_{d,h}y \geq \underline{f}_d \iff xB_{d,h}y \geq \frac{\underline{f}_d - (a_{d,h}^{min} - b_{d,h}^{min}\alpha_{d,h})}{\alpha_{d,h}}, \quad (5.12)$$

$$x(-B_{d,h})y \geq \underline{g}_h \iff xB_{d,h}y \leq -\underline{g}_h \quad (5.13)$$

where we have used that $xUy = 1$. Unlike a “standard” matching game in which each player has a unique IRP that works for all possible partners, in the transformed game Γ' doctors will have one IRP per hospital, given by Equation (5.12). Formally, let

$$\underline{f}'_{d,h} := \frac{\underline{f}_d - (a_{d,h}^{min} - b_{d,h}^{min}\alpha_{d,h})}{\alpha_{d,h}}, \forall d \in D, h \in H$$

Then, a doctor d accepts to be matched with hospital h if and only her payoff is greater or equal than $\underline{f}'_{d,h}$. Regarding hospitals, it is enough considering $\underline{g}'_h := -\underline{g}_h$. Computing each coefficient takes constant time on the size of the agent sets and strategy sets. Thus, the complexity of transforming the IRPs is $O(|D| + |H|)$ plus some factor indicating the number of required bits. \square

5.3.1 Deferred-acceptance with competitions algorithm

The analysis of the DAC algorithm (Algorithm 5.1) complexity is not affected by the fact that doctors may have personalized IRPs for hospitals. Thus, from the complexity results of zero-sum games (Theorems 5.2.3 and 5.3.4) we conclude the following.

Corollary 5.3.5. *Computing ε -pairwise stable allocations in bi-matrix strictly competitive matching games is a polynomial problem as the DAC algorithm has a bounded number of iterations, each of them with complexity $O((N^2 + k^2)L)$, where N bounds the number of players in the biggest side, k bounds the number of pure strategies per player and L is the number of bits required to encode all the data.*

5.3.2 Market procedure

Let Γ be a roommates matching game such that for each couple $(d, d') \in D \times D$, their game $G_{d,d'}$ belongs to the class of bi-matrix strictly competitive games in \mathcal{S} .

The complexity results obtained for zero-sum games in the computation of the demand sets (Theorem 5.2.9) and the mechanism to implement the output of the market procedure (Theorem 5.2.10) can be extended to Γ thanks to the affine transformation result (Theorem 5.3.3). We conclude directly the following result.

Corollary 5.3.6. *Computing the demand sets of all agents during an iteration of the market procedure has complexity $O(N^2L)$ where N bounds the number of doctors and L is the number of bits required to encode all the data. In addition, the complexity of computing an allocation π that implements the output f of the market procedure (if it exists) is $O(Nk^2L)$, where k bounds the number of pure strategies per player.*

5.3.3 Renegotiation process

As in the zero-sum case, we start with the complexity of computing a constrained Nash equilibrium. Let $G_{d,h} = (X_d, Y_h, A_{d,h}, -B_{d,h})$ be a bi-matrix strictly competitive game in mixed strategies and (f_d, g_h) be a pair of reservation payoffs. Let (x, y) be an ε - (f_d, g_h) -feasible strategy profile, that is,

$$xA_{d,h}y + \varepsilon \geq f_d \text{ and } x(-B)_{d,h}y + \varepsilon \geq g_h \iff xA_{d,h}y + \varepsilon \geq f_d \text{ and } xBy \leq -g_h - \varepsilon$$

It follows,

$$\begin{aligned} xA_{d,h}y + \varepsilon \geq f_d &\iff x(\alpha_{d,h}[B_{d,h} - b_{d,h}^{\min}U] + a_{d,h}^{\min}U)y + \varepsilon \geq f_d \\ &\iff \alpha_{d,h}xB_{d,h}y + (a_{d,h}^{\min} - b_{d,h}^{\min}\alpha_{d,h})xUy + \varepsilon \geq f_d \\ &\iff \alpha_{d,h}xB_{d,h}y + (a_{d,h}^{\min} - b_{d,h}^{\min}\alpha_{d,h}) + \varepsilon \geq f_d \\ &\iff xB_{d,h}y + \varepsilon \geq \frac{f_d - (a_{d,h}^{\min} - b_{d,h}^{\min}\alpha_{d,h})}{\alpha_{d,h}} - \varepsilon \cdot \frac{1 - \alpha_{d,h}}{\alpha_{d,h}} \end{aligned}$$

Recall we have taken $\alpha_{d,h} \in [0, 1]$. Thus, in the zero-sum game $G'_{d,h} = (X_d, Y_h, B_{d,h})$, considering the pair (f'_d, g'_h) of reservation payoffs given by,

$$f'_d := \frac{f_d - (a_{d,h}^{\min} - b_{d,h}^{\min}\alpha_{d,h})}{\alpha_{d,h}} - \varepsilon \cdot \frac{1 - \alpha_{d,h}}{\alpha_{d,h}} \text{ and } g'_h := -g_h, \quad (5.14)$$

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the sets of feasible strategy profiles, as well as the sets of CNE of $G_{d,h}$ and $G'_{d,h}$, coincide. Therefore, to compute an ε - (f_d, g_h) -constrained Nash equilibrium of the strictly competitive game, we can use the following scheme:

1. Compute the transformation from $A_{d,h}$ to $B_{d,h}$ and define the zero-sum game $G'_{d,h}$.
2. Consider the new reservation payoffs (f'_d, g'_h) as in Equation (5.14).
3. Compute an ε - (f'_d, g'_h) -CNE for the zero-sum game, namely (x', y') .

Proposition 5.3.7. *The scheme above computes an ε - (f'_d, g'_h) -CNE of $G_{d,h}$.*

Proof. Let (x', y') be an ε - (f'_d, g'_h) -CNE of the zero-sum game $G'_{d,h}$. It holds,

1. $g'_h + \varepsilon \geq x' B_{d,h} y' \geq f'_d - \varepsilon$
2. For any $x \in X_d$ such that $x B_{d,h} y' \leq g_h + \varepsilon$, $(x - x') B_{d,h} y' \leq \varepsilon$
3. For any $y \in Y_h$ such that $x' B_{d,h} y + \varepsilon \geq f_d$, $x' B_{d,h} (y' - y) \leq \varepsilon$

From (1) we obtain that $x'(-B_{d,h})y' \geq -g'_h - \varepsilon = g_h - \varepsilon$, and $x' B_{d,h} y' \geq f'_d - \varepsilon$, which implies that $x' A_{d,h} y' \geq f_d - \varepsilon$, so (x', y') is (f_d, g_h) -feasible in the game $G_{d,h}$. Let $x \in X_d$ such that $x(-B_{d,h})y' + \varepsilon \geq g_h$. Then, $x B_{d,h} y' \leq g'_d - \varepsilon$. From (2), $(x - x') B_{d,h} y' + \varepsilon$. Noticing that $\alpha_{d,h}(x - x') B_{d,h} y' = (x - x') A_{d,h} y'$, we obtain that $(x - x') A_{d,h} y' \leq \alpha_{d,h} \varepsilon \leq \varepsilon$, as $\alpha_{d,h}$ was taken lower of equal than 1. Analogously, suppose there is $y \in Y_h$ such that $x' A_{d,h} y + \varepsilon \geq f_d$. Then, $x' B_{d,h} y + \varepsilon \geq f'_d$. From (3), $x'(-B_{d,h})(y - y') \leq \varepsilon$. Therefore, (x', y') is an ε -CNE of $G_{d,h}$. \square

From Proposition 5.3.7 and the complexity of computing a constrained Nash equilibrium of a zero-sum game (Theorem 5.2.12), we obtain the following result.

Corollary 5.3.8. *Let $G_{d,h} = (S_d, T_h, A_{d,h}, -B_{d,h})$ be a bi-matrix strictly competitive game and (f_d, g_h) be a pair of reservation payoffs. The complexity of computing an ε - (f_d, g_h) -constrained Nash equilibrium is*

$$O\left(\max\{|S_d|, |T_h|\}^{2.5} \cdot \min\{|S_d|, |T_h|\} \cdot L_{d,h}\right)$$

with $L_{d,h}$ the number of bits required to encode the payoff matrices.

Finally, from the bounded number of iterations of the renegotiation process for zero-sum games (Theorem 5.2.16) we deduce the following.

Corollary 5.3.9. *The ε -renegotiation process (5.2) ends in a finite number of iterations $T \propto \frac{1}{\varepsilon}$ in bi-matrix strictly competitive games.*

Let $T := \max\{\max A_{d,h} - \min A_{d,h} : (d, h) \in D \times H\}$. The following table summarizes the complexity results found.

Algorithms	Complexity/It	Max N ^o It
DAC	$O((N^2 + k^2)L)$	T/ε
Market procedure:	-	$p(N)$
Demand sets	$O(N^2L)$	-
Implementation	$O(Nk^2L)$	-
Renegotiation process	$O(N^4k^{3.5}L)$	T/ε
Affine Transformation	$O(N^2k^2)$	-

Table 5.2: Complexity strictly competitive matching games: N players per side, k strategies per player, L bits to encode the data, and $p(N)$ a polynomial on N

5.4 Infinitely repeated matching games

For each potential pair $(d, h) \in D \times H$, let $G_{d,h} = (X_d, Y_h, A_{d,h}, B_{d,h})$ be a finite bi-matrix game in mixed strategies, with $X_d = \Delta(S_d), Y_h = \Delta(T_h)$, where all matrices have only rational entries. Given $K \in \mathbb{N}$, consider the K -stages game $G_{d,h}^K$ defined by the payoff functions,

$$f_{d,h}(K, \sigma_d, \sigma_h) = \frac{1}{K} \mathbb{E}_\sigma \left[\sum_{k=1}^K A_{d,h}(s_k, t_k) \right], \quad g_{d,h}(K, \sigma_d, \sigma_h) = \frac{1}{K} \mathbb{E}_\sigma \left[\sum_{k=1}^K B_{d,h}(s_k, t_k) \right],$$

where $\sigma_d : \cup_{k=1}^\infty (S_d \times T_h) \rightarrow X_d$ is a behavioral strategy for player d and $\sigma_h : \cup_{k=1}^\infty (S_d \times T_h) \rightarrow Y_h$ is a behavioral strategy for player h . We define the uniform game $G_{d,h}^\infty$ as the limit of $G_{d,h}^K$ when K goes to infinity.

Definition 5.4.1. A matching game Γ is a **bi-matrix infinitely repeated matching game** if every strategic game is a uniform game as explained above.

To study the complexity of computing pairwise-renegotiation proof allocations in infinitely repeated games, we start by studying the complexity of solving the general QCQP Problem (5.3). We state the proof for a pair (doctor, hospital) although it can be straightforwardly applied to roommates.

Proposition 5.4.2. Let $(d, h) \in D \times H$ be a pair, $G_{d,h} = (X_d, Y_h, A_{d,h}, B_{d,h})$ their finite stage game in mixed strategies and $c \in \mathbb{R}$, such that $c \leq \max B_{d,h}$. The complexity of solving the QCQP Problem (5.3) in $G_{d,h}^\infty$ is $O((|S_d| \cdot |T_h|)^{2.5} L_{d,h})$, where $L_{d,h}$ is the number of bits required to encode the stage game.

To prove Proposition 5.4.2 we will use the following result.

Lemma 5.4.3. Let $(d, h) \in D \times H$ be a pair and let $(\bar{f}, \bar{g}) \in \mathbb{R}^2$ be a payoff vector in the set of feasible payoffs,

$$\text{co}(A_{d,h}, B_{d,h}) := \{(A_{d,h}(s, t)B_{d,h}(s, t)) \in \mathbb{R}^2 : s \in S_d, t \in T_h\}^6$$

⁶ $\text{co}(A)$ refers to the convex envelope of the set A .

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Then, there exists a pure strategy profile σ of $G_{d,h}^\infty$ that achieves (\bar{f}, \bar{g}) . In addition, the number of elementary operations used to compute σ is bounded by $O((|S_d| \cdot |T_h|)^{2.5} L_{d,h})$, where $L_{d,h}$ is the number of bits required to encode the matrices $A_{d,h}$ and $B_{d,h}$.

Proof. Consider the following system with $|S_d| \cdot |T_h|$ variables and three linear equations,

$$\begin{aligned} \sum_{s,t} A_{d,h}(s,t)\lambda_{s,t} &= \bar{f}, \\ \sum_{s,t} B_{d,h}(s,t)\lambda_{s,t} &= \bar{g}, \quad \lambda \in \Delta(S_d \times T_h) \end{aligned} \tag{5.15}$$

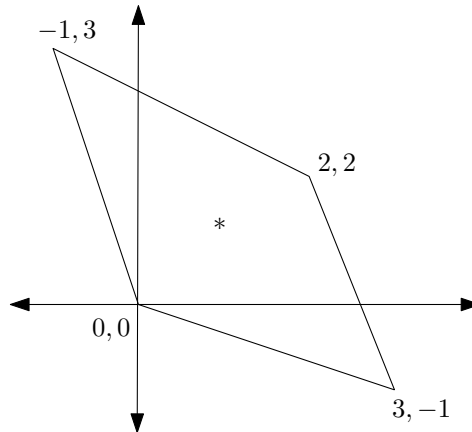
System (5.15) can be solved in $O((|S_d| \cdot |T_h|)^{2.5} L_{d,h})$ elementary operations. Since matrices $A_{d,h}$ and $B_{d,h}$ have rational entries, the solution has the form $(\lambda_{s,t})_{s,t} = (\frac{p_{s,t}}{q_{s,t}})_{s,t}$ with each $p_{s,t}, q_{s,t} \in \mathbb{N}$. Let $N_\lambda = \text{lcm}(q_{s,t} : (s,t) \in S_d \times T_h)$ be the least common multiple of all denominators. The number of elementary operations to compute N_λ is bounded by $O((|S_d| \cdot |T_h|)^2)$. Enlarge each fraction of the solution so all denominators are equal to N_λ , i.e. $\lambda = (\frac{p'_{s,t}}{N_\lambda})_{s,t}$. Suppose that $S_d = \{s_1, s_2, \dots, s_d\}$ and $T_h = \{t_1, t_2, \dots, t_h\}$. Let σ be the strategy profile in which players play (s_1, t_1) the first p'_{s_1, t_1} -stages, then (s_1, t_2) the next p'_{s_1, t_2} -stages, then (s_1, t_3) the next p'_{s_1, t_3} -stages and so on until playing (s_d, t_h) during p'_{s_d, t_h} -stages, and then they repeat all infinitely. By construction, $(f_{d,h}(\sigma), g_{d,h}(\sigma)) = (\bar{f}, \bar{g})$. \square

Let us illustrate the previous result with an example.

Example 5.4.4. Consider the following prisoners' dilemma G played infinitely many times by a couple (d, h) .

		Agent h	
		Cooperate	Betray
Agent d	Cooperate	2, 2	-1, 3
	Betray	3, -1	0, 0

The following figure shows the convex envelope of the pure payoff profiles.



Consider $(\bar{f}, \bar{g}) = (1, 1) \in \text{co}(A_{d,h}, B_{d,h})$, represented in the figure by the star. Notice that $(1, 1)$ can be obtained as the convex combination of $\frac{1}{4}(0, 0) + \frac{1}{4}(3, -1) + \frac{1}{4}(-1, 3) + \frac{1}{4}(2, 2)$. Therefore, (d, h) can obtain $(1, 1)$ in their infinitely repeated game by playing (B, B) the first four rounds, (C, B) the second four rounds, (B, C) the third four rounds, (C, C) the fourth four rounds, and cycling like this infinitely many times. As every 16 rounds the couple obtains $(1, 1)$, in the limit, their average payoff converges to (\bar{f}, \bar{g}) . \square

Finally, we prove the complexity result of solving the QCQP problem (Proposition 5.4.2).

Proof of Proposition 5.4.2. Consider the following optimization problem,

$$\begin{aligned} \max_{\lambda \in \Delta(S_d \times T_h)} \quad & \sum_{s \in S_d} \sum_{t \in T_h} A_{d,h}(s, t) \lambda_{s,t} \\ \text{s.t.} \quad & \sum_{s \in S_d} \sum_{t \in T_h} B_{d,h}(s, t) \lambda_{s,t} \geq c \end{aligned} \quad (5.16)$$

Problem (5.16) is a linear programming problem with $|S_d| \cdot |T_h|$ variables and two constraints and its optimal value (\bar{f}, \bar{g}) coincides with the optimal value of the QCQP Problem (5.3). Therefore, any strategy profile σ that achieves (\bar{f}, \bar{g}) , is a solution of the QCQP Problem (5.3). The stated complexity is obtained from solving Problem (5.16) and applying Lemma 5.4.3 to compute σ . \square

5.4.1 Deferred-acceptance with competitions algorithm

The polynomial complexity of solving the QCQP general problem (Proposition 5.4.2) allows us to prove the main result of this section.

Theorem 5.4.5 (Complexity). *Let $d \in D$ be the proposer doctor. Let h be the proposed hospital and d' be the doctor that d wants to replace. If d is the winner of the competition, the entire iteration of the DAC algorithm (Algorithm 5.1) has complexity,*

$$O \left(|H| \cdot |D| + |S_d|^{2.5} \sum_{h' \in H} |T_{h'}|^{2.5} L_{d,h'} + |S_{d'}|^{2.5} \sum_{h' \in W} |T_{h'}|^{2.5} L_{d',h'} \right)$$

where $L_{i,j}$ is the number of bits required to encode the payoff matrices of (i, j) .

Proof. The optimal proposal problem is split into $|H|$ problems. Each subproblem needs $|D|$ comparisons to compute the right-hand side and then, they have the complexity stated in Proposition 5.4.2. Thus, the optimal proposal computation has complexity,

$$O \left(|H| \cdot |D| + \sum_{h' \in H} (|S_d| \cdot |T_{h'}|)^{2.5} L_{d,h'} \right)$$

Computing the reservation payoff and the bid of each competitor has exactly the same complexity as the optimal proposal computation, considering the respective set of strategies. Finally, the problem solved by the winner has complexity $O((|S_d| \cdot |T_h|)^{2.5} \cdot L_{d,h})$. Summing up, we obtain the complexity stated in the theorem. \square

Remark 5.4.6. If there are at most N players in each side and at most k pure strategies per player, Theorem 5.4.5 proves that each iteration of the DAC algorithm (Algorithm 5.1) takes $O(N^2 + Nk^5L)$ number of elementary operations in being solved, hence it is polynomial. As the number of iterations is bounded by $Y \propto \frac{1}{\varepsilon}$ (Theorem 5.1.1), we conclude that computing an ε -pairwise stable allocation for a infinitely repeated matching game is a polynomial problem.

5.4.2 Market procedure

Let Γ be a roommates matching game such that for each couple $(d, d') \in D \times D$, their game $G_{d,d'}$ is an infinitely repeated game with a bi-matrix stage game. Consider a payoff profile $f \in \mathbb{R}^{|D|}$ and a fixed doctor $d \in D$. Notice that,

$$\begin{aligned} d' \in P_d(f) &\iff f_d = u_{d,d'}(f_{d'}) \\ &\iff (f_d, f_{d'}) \in \text{co}(A_{d,d'}, A_{d',d}) \\ &\iff \exists \lambda = (\lambda_{s,s'})_{s \in S_d, s' \in S_{d'}} \subseteq [0, 1], \sum_{s,s'} \lambda_{s,s'} = 1 : \\ &\quad f_d = \sum_{s,s'} A_{d,d'}(s, s') \lambda_{s,s'} \text{ and } f_{d'} = \sum_{s,s'} A_{d',d}(s, s') \lambda_{s,s'} \end{aligned}$$

Therefore, to determine if a doctor belongs to the demand set of d , it is enough with computing the coefficients of their convex combination which, we know, has a polynomial complexity (Lemma 5.4.3). We can conclude the following result.

Theorem 5.4.7. *The complexity of computing the demand sets of all doctors during an iteration of the market procedure has complexity*

$$O\left(\sum_{(d,d') \in D \times D} |S_d| \cdot |S_{d'}| L_{d,d'}\right)$$

where $L_{d,d'}$ is the number of bits required to encode the payoff matrices of the pair (d, d') .

Remark the previous result can be refined as we only need to check once every couple, since belonging to the demand set is a symmetric property. Therefore, once determined that $d' \in P_d(f)$, we directly obtain that $d \in P_{d'}(f)$.

Regarding the implementation of the market procedure's output, once having the good pairs of doctors given by a matching μ , it is enough with solving the linear system of equations from Lemma 5.4.3 for each of the couples. In particular, we can conclude the following result.

Theorem 5.4.8. Let $f \in \mathbb{R}^{|D|}$ be the output of the market procedure and suppose there exists a matching μ such that for any doctor $d \in D$, $(f_d, f_{\mu(d)}) \in co(A_{d,\mu(d)}, A_{\mu(d),d})$, if d is matched, and $f_d = \underline{f}_d$, if d is unmatched. Then, computing the strategy profile $\vec{x} \in X_D$ such that,

$$f_{d,\mu(d)}(x_d, x_{\mu(d)}) = f_d, \forall d \text{ matched}$$

has complexity

$$O\left(\sum_{(d,d') \in \mu} |S_d| \cdot |S_{d'}| L_{d,d'}\right)$$

where $L_{d,d'}$ is the number of bits required to encode the data of the stage game $G_{d,d'}$.

Remark 5.4.9. If there are at most N doctors and k pure strategies per doctor, computing all the demand sets during an iteration of the market procedure has complexity $O(N^2 k^2 L)$. Similarly, in case the outcome of the market procedure can be implemented by an allocation, finding the strategy profiles of the agents within the allocation has complexity $O(Nk^2 L)$.

5.4.3 Renegotiation process

We recall some of the definitions of infinitely repeated games given in Chapter 3. Let $co(A_{d,h}, B_{d,h})$ be the set of *feasible payoffs* (Definition 3.4.11), and $\alpha_{d,h}, \beta_{d,h}$ be, respectively, the *punishment levels*⁷ (Definition 3.4.12) of players $d \in D$ and $h \in H$. We define the set of *uniform equilibrium payoffs* by,

$$E_{d,h} := \{(\bar{f}, \bar{g}) \in co(A_{d,h}, B_{d,h}) : \bar{f} \geq \alpha_{d,h}, \bar{g} \geq \beta_h\}$$

From the Folk theorem of Aumann-Shapley [15], we know that $E_{d,h}$ is exactly the set of uniform equilibrium payoff of $G_{d,h}^\infty$.

Definition 5.4.10. Let $\pi = (\mu, \vec{x}, \vec{y})$ be an allocation. For every pair of reservation payoffs $(f_d^\pi(\varepsilon), g_h^\pi(\varepsilon))$ and $\varepsilon > 0$, we define the ε -**acceptable payoffs set** as

$$E_{d,h}(f_d^\pi(\varepsilon), g_h^\pi(\varepsilon)) := co(A_{d,h}, B_{d,h}) \cap \{(\bar{f}, \bar{g}) \in \mathbb{R}^2 : \bar{f} + \varepsilon \geq f_d^\pi(\varepsilon), \bar{g} + \varepsilon \geq g_h^\pi(\varepsilon)\}$$

Finally, we define ε -constrained Nash equilibria for uniform games.

Definition 5.4.11. A strategy profile $\sigma = (\sigma_d, \sigma_h)$ is an ε - $(f_d^\pi(\varepsilon), g_h^\pi(\varepsilon))$ -constrained Nash equilibrium of $G_{d,h}^\infty$ if,

1. $\forall \bar{\varepsilon} > \varepsilon, \exists K_0, \forall K \geq K_0, \forall (\tau_d, \tau_h),$
 - (a) if $f_{d,h}(K, \tau_d, \sigma_h) > f_{d,h}(K, \sigma) + \bar{\varepsilon}$ then, $g_{d,h}(K, \tau_d, \sigma_h) + \varepsilon < g_h^\pi(\varepsilon),$
 - (b) if $g_{d,h}(K, \sigma_d, \tau_h) > g_{d,h}(K, \sigma) + \bar{\varepsilon}$ then, $f_{d,h}(K, \sigma_d, \tau_h) + \varepsilon < f_d^\pi(\varepsilon)$

⁷The punishment levels depend on the partner's identity.

5.4. Infinitely repeated matching games

2. $(f_{d,h}(K, \sigma), g_{d,h}(K, \sigma)) \xrightarrow{K \rightarrow \infty} (f_{d,h}(\sigma), g_{d,h}(\sigma)) \in \mathbb{R}^2$ with $f_{d,h}(\sigma) + \varepsilon \geq f_d^\pi(\varepsilon)$, and $g_{d,h}(\sigma) + \varepsilon \geq g_h^\pi(\varepsilon)$

The set of ε - $(f_d^\pi(\varepsilon), g_h^\pi(\varepsilon))$ -CNE payoffs is denoted $E_{d,h}^\infty(f_d^\pi(\varepsilon), g_h^\pi(\varepsilon))$.

We begin the complexity analysis by studying the computation of ε -CNE.

Theorem 5.4.12 (CNE Complexity). *Let $G_{d,h}^\infty$ be an infinitely repeated game as defined above. Given any players' reservation payoffs $(f_d^\pi(\varepsilon), g_h^\pi(\varepsilon)) \in \mathbb{R}^2$ such that $E_{d,h}(f_d^\pi(\varepsilon), g_h^\pi(\varepsilon))$ is non-empty, the complexity of computing an ε - $(f_d^\pi(\varepsilon), g_h^\pi(\varepsilon))$ -CNE is at most, $O(|S_d| \cdot |T_h|)^{2.5} L_{d,h}$, where $L_{d,h}$ is the number of bits required to encode the data of the stage game $G_{d,h}$.*

We split the proof of Theorem 5.4.12 in the following three lemmas. First, from the Folk theorem of Aumann-Shapley [15], the following holds.

Lemma 5.4.13. *It holds in $E_{d,h} \cap E_{d,h}(f_d^\pi(\varepsilon), g_h^\pi(\varepsilon)) \subseteq E_{d,h}^\infty(f_d^\pi(\varepsilon), g_h^\pi(\varepsilon))$.*

Whenever the intersection in Lemma 5.4.13 is non-empty, there exists a uniform equilibrium payoff profile (\bar{f}, \bar{g}) that belongs to $E_{d,h}^\infty(f_d^\pi(\varepsilon), g_h^\pi(\varepsilon))$. Combined with Lemma 5.4.3 that states the complexity of finding a strategy profile that achieves a given payoff profile, we obtain a uniform equilibrium that achieves (\bar{f}, \bar{g}) with the complexity stated in Theorem 5.4.12. The following lemma provides sufficient conditions for that intersection to be non-empty.

Lemma 5.4.14. *Let $(f_d^\pi(\varepsilon), g_h^\pi(\varepsilon))$ be a pair of reservation payoffs such that the set $E_{d,h}(f_d^\pi(\varepsilon), g_h^\pi(\varepsilon))$ is non-empty. Then, $E_{d,h} \cap E_{d,h}(f_d^\pi(\varepsilon), g_h^\pi(\varepsilon))$ is non-empty if either $f_d^\pi(\varepsilon) - \varepsilon \geq \alpha_{d,h}$ and $g_h^\pi(\varepsilon) - \varepsilon \geq \beta_{d,h}$, or $f_d^\pi(\varepsilon) - \varepsilon < \alpha_{d,h}$ and $g_h^\pi(\varepsilon) - \varepsilon < \beta_{d,h}$.*

Proof. In the first case, $E_{d,h}(f_d^\pi(\varepsilon), g_h^\pi(\varepsilon)) \subseteq E_{d,h}$, thus the intersection between them is equal to $E_{d,h}(f_d^\pi(\varepsilon), g_h^\pi(\varepsilon))$, which is non-empty. In the second case, $E_{d,h} \subseteq E_{d,h}(f_d^\pi(\varepsilon), g_h^\pi(\varepsilon))$ and therefore, the intersection is non-empty. \square

This yields the two following missing cases.

Lemma 5.4.15. *Let $(f_d^\pi(\varepsilon), g_h^\pi(\varepsilon))$ be a pair of reservation payoffs such that the set $E_{d,h}(f_d^\pi(\varepsilon), g_h^\pi(\varepsilon))$ is non-empty. Then, computing an ε -CNE has complexity $O(|S_d| \cdot |T_h|)^{2.5} L_{d,h}$ either if $f_d^\pi(\varepsilon) - \varepsilon \geq \alpha_{d,h}$ and $g_h^\pi(\varepsilon) - \varepsilon < \beta_{d,h}$, or $f_d^\pi(\varepsilon) - \varepsilon < \alpha_{d,h}$ and $g_h^\pi(\varepsilon) - \varepsilon \geq \beta_{d,h}$.*

Proof. Suppose the first case, $f_d^\pi(\varepsilon) - \varepsilon \geq \alpha_{d,h}$ and $g_h^\pi(\varepsilon) - \varepsilon < \beta_{d,h}$. Let $F := E_{d,h}(f_d^\pi(\varepsilon), g_h^\pi(\varepsilon)) \cap E_{d,h}$. If F is non-empty, the result holds from Lemma 5.4.13. Suppose F is empty and consider the payoff profile $(\bar{f}, \bar{g}) \in \text{co}(A_{d,h}, B_{d,h})$ given by

$$\bar{g} = \max\{g \in \text{co}(A_{d,h}, B_{d,h}) : \exists f \in \mathbb{R}, (f, g) \in E_{d,h}(f_d^\pi(\varepsilon), g_h^\pi(\varepsilon))\}$$

Computing (\bar{f}, \bar{g}) can be done in $O(|S_d| \cdot |T_h|)^{2.5} L_{d,h}$ elementary operations by solving the system of linear equations with $(\lambda_{s,t})_{s \in S_d, t \in T_h}$ variables (Problem (5.16)) exchanging the roles of the matrices. Shift the payoff profile to $(\bar{f}, \bar{g} + \varepsilon)$, assuming that

increasing \bar{g} by ε does not take the payoff out of the convex envelope (if it does it, h has reached its highest possible payoff, so it does not have any profitable deviation). Let σ be a strategy in $G_{d,h}^\infty$ that achieves $(\bar{f}, \bar{g} + \varepsilon)$, computable in $O((|S_d| \cdot |T_h|)^{2.5} L_{d,h})$ (Lemma 5.4.3). Consider next σ' the strategy profile in which d and h play following σ at every stage, such that if d deviates, h punishes her decreasing her payoff to $\alpha_{d,h}$, and if h deviates, d ignores it and keeps playing according to σ . We claim that σ' is an ε - $(f_d^\pi(\varepsilon), g_h^\pi(\varepsilon))$ -CNE. Indeed, it is feasible as their limit payoff profile is $(\bar{f}, \bar{g} + \varepsilon)$. In addition, remark that s does not have profitable deviations as h punishes her and

$$\bar{f} \geq f_d^\pi(\varepsilon) - \varepsilon \geq \alpha_{d,h}$$

Finally, let $K \in \mathbb{N}$ and $\bar{\varepsilon} > \varepsilon$ such that h can deviate at time K and get $g' \geq (\bar{g} + \varepsilon) + \bar{\varepsilon}$. Let f' be the payoff of d until the stage K . Notice that $(f', g') \in \text{co}(A_{d,h}, B_{d,h})$ since (f', g') is an average payoff profile of the K -stage game. Suppose that $f' \geq f_d^\pi(\varepsilon) - \varepsilon$, so $(f', g') \in E_{d,h}(f_d^\pi(\varepsilon), g_h^\pi(\varepsilon))$. Then,

$$\bar{f} \geq f' \geq \bar{f} + \varepsilon + \bar{\varepsilon} > \bar{f}$$

which is a contradiction. Therefore, $f' < f_d^\pi(\varepsilon) - \varepsilon$. Thus, σ' is an ε - $(f_d^\pi(\varepsilon), g_h^\pi(\varepsilon))$ -CNE. For the second case in which $f_d^\pi(\varepsilon) - \varepsilon < \alpha_{d,h}$ and $g_h^\pi(\varepsilon) - \varepsilon \geq \beta_{d,h}$, the argument is analogous. \square

As all the possible cases are covered by Lemmas 5.4.14 and 5.4.15, we conclude the proof of Theorem 5.4.12 regarding the complexity of computing constrained Nash equilibria. Making a similar computation to the one for zero-sum matching games, we can bound the complexity of an entire iteration of the ε -renegotiation process (Algorithm 5.2) by,

$$O\left(\sum_{d \in D} \left[|H| \cdot |D| + \sum_{h \in H} (|S_d| \cdot |T_h|)^{2.5} + |S_d| \cdot |T_{\mu(d)}|^{2.5} \right] \cdot L\right)$$

where the first two terms come from the reservation payoffs computation, the last one from the constrained Nash equilibria computation, and L is the number of bits required to encode the entire data.

Remark 5.4.16. Considering N agents per side and k pure strategies per player, the complexity of an iteration of the ε -renegotiation process (Algorithm 5.2) can be bounded by $O(N^3 k^5 L)$.

Finally, we study the convergence of the algorithm for infinitely repeated games.

Theorem 5.4.17 (Convergence). *Let $\pi = (\mu, \sigma_D, \sigma_H)$ be an ε -pairwise stable allocation. Let $(f_d^\pi(\varepsilon), g_h^\pi(\varepsilon))_{(d,h) \in \mu}$ be the ε -reservation payoffs generated by π . Then, there exists an oracle for computing ε -CNE such that, starting from π , the ε -renegotiation process (Algorithm 5.2) converges in at most*

$$\frac{1}{\varepsilon} \left(\max_{(d,h) \in \mu} \{ \max\{ \alpha_{d,h} - f_d^\pi(\varepsilon), \beta_{d,h} - g_h^\pi(\varepsilon) \} \} \right)$$

iterations, where $\alpha_{d,h}, \beta_{d,h}$ are the punishment levels of (d, h) .

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Proof. Let $(d, h) \in \mu$ be a couple and $(f_d^\pi(\varepsilon), g_h^\pi(\varepsilon))$ be their reservation payoffs at the beginning of Algorithm 5.2. Notice that one of the following four cases must hold:

1. $f_d^\pi(\varepsilon) - \varepsilon \leq \alpha_{d,h}$ and $g_h^\pi(\varepsilon) - \varepsilon \leq \beta_{d,h}$,
2. $f_d^\pi(\varepsilon) - \varepsilon \geq \alpha_{d,h}$ and $g_h^\pi(\varepsilon) - \varepsilon \geq \beta_{d,h}$,
3. $f_d^\pi(\varepsilon) - \varepsilon \geq \alpha_{d,h}$ and $g_h^\pi(\varepsilon) - \varepsilon < \beta_{d,h}$,
4. $f_d^\pi(\varepsilon) - \varepsilon < \alpha_{d,h}$ and $g_h^\pi(\varepsilon) - \varepsilon \geq \beta_{d,h}$

Let $F_{d,h} := E_{d,h} \cap E_{d,h}(f_d^\pi(\varepsilon), g_h^\pi(\varepsilon))$ and suppose it is non-empty. Then, there exists a feasible uniform equilibrium for (d, h) , so the couple changes only once of strategy profile and never again. Suppose $F_{d,h}$ is empty. Necessarily it must hold case (3) or (4). Suppose $f_d^\pi(\varepsilon) - \varepsilon \geq \alpha_{d,h}$ and $g_h^\pi(\varepsilon) - \varepsilon < \beta_{d,h}$ and consider the oracle given in the proof of Lemma 5.4.15. Then, the couple passes to gain $(\bar{f}, \bar{g} + \varepsilon)$, where

$$\begin{aligned}\bar{g} &= \max\{g : \exists f, (f, g) \in E_{d,h}(f_d^\pi(\varepsilon), g_h^\pi(\varepsilon))\} \\ \bar{f} &\in \{f : (f, \bar{g}) \in E_{d,h}(f_d^\pi(\varepsilon), g_h^\pi(\varepsilon))\}\end{aligned}$$

Let $(f_d^{\pi(1)}(\varepsilon), g_h^{\pi(1)}(\varepsilon))$ be the couple's reservation payoffs at the next iteration and consider again $F_{d,h} := E_{d,h} \cap E_{d,h}(f_d^{\pi(1)}(\varepsilon), g_h^{\pi(1)}(\varepsilon))$. If $F_{d,h}$ is non-empty, the couple passes to play a feasible uniform equilibrium. Otherwise, the oracle computes a new payoff profile (\bar{f}', \bar{g}') such that

$$\begin{aligned}\bar{g}' &= \max\{g : \exists f, (f, g) \in E_{d,h}(f_d^{\pi(1)}(\varepsilon), g_h^{\pi(1)}(\varepsilon))\} \\ \bar{f}' &\in \{f : (f, \bar{g}') \in E_{d,h}(f_d^{\pi(1)}(\varepsilon), g_h^{\pi(1)}(\varepsilon))\}\end{aligned}$$

Since $\pi(1)$ is ε -pairwise stable, it holds $f'_d \leq \bar{f} + \varepsilon$, $g'_h \leq (\bar{g} + \varepsilon) + \varepsilon$. Therefore, $(\bar{f}, \bar{g} + \varepsilon) \in E_{d,h}(f_d^{\pi(1)}(\varepsilon), g_h^{\pi(1)}(\varepsilon))$ and then, $\bar{g}' \geq \bar{g} + \varepsilon$. We conclude that at each iteration, either the couple changes to play a feasible uniform equilibrium, or player h increases its payoff in at least ε . Since its payoff is bounded by its punishment level, the sequence converges in $T \propto \frac{1}{\varepsilon}$ iterations. If case (4) holds, the conclusion is the same: at each iteration, either the couple plays a feasible uniform equilibrium or player d increases by at least ε her payoff. Again, we obtain a $T \propto \frac{1}{\varepsilon}$ bound for the number of iterations. Thus, we obtain the number of iterations given in the statement of the theorem by considering the worst possible case. \square

Remark 5.4.18. Adding Theorem 5.4.17 to Remark 5.4.16, we can conclude that computing an ε -renegotiation proof allocation for an infinitely repeated matching game is a polynomial problem.

The following table summarizes the complexity results found.

Algorithms	Complexity/It	N ^o It	Constants
DAC	$O(N^3k^5L)$	C_1/ε	$C_1 \leq \max_{d,h}(\max A_{d,h} - \min A_{d,h})$
Market procedure:	-	$p(N)$	-
Demand sets	$O(N^2kL)$	-	-
Implementation	$O(Nk^2L)$	-	-
Renegotiation process	$O(N^2k^5L)$	C_2/ε	$C_2 \leq \max_{d,h} \max\{D_d, D_h\}$ $D_d := \max A_{d,h} - \min A_{d,h}$ $D_h := \max B_{d,h} - \min B_{d,h}$

Table 5.3: Complexity infinitely repeated games: N players per side, k strategies per player, L bits to encode the data, $p(N)$ polynomial on N , and $A_{d,h}, B_{d,h}$ payoff matrices of the stage games of $(d, h) \in D \times H$.

5.5 Conclusions

In this chapter we have provided the complexity study of the algorithms designed in Chapter 4 for three classes of bi-matrix matching games: zero-sum, strictly competitive, and infinitely repeated. For our additive separable matching games submodel, we have proved that our algorithms converge to an ε -pairwise-renegotiation proof allocation in the three classes of matching games in a bounded number of iterations, with a bound only depending on ε . Each iteration of the algorithms having a polynomial complexity, our deferred-acceptance with competitions algorithm and renegotiation process are efficient algorithms.

For our roommates submodel we have used the work of Alkan and Tuncay [9] to establish the existence of pairwise stable allocations whenever their market procedure's output can be implemented by an allocation. We have proved that for zero-sum games, strictly competitive games, and infinitely repeated games, the computations of the demand sets and the allocation that implements the output of the procedure (if it exists), are polynomial problems over the number of doctors and their number of pure strategies. Moreover, we have given a procedure to determine the existence of an allocation implementing the output of the market procedure for zero-sum and strictly competitive matching games.

Alkan and Tuncay's proofs for the correctness and complexity of the market procedure and direction procedure have been given only in the quasi-linear case. Nevertheless, they claim their procedures achieve the same results in the general non-quasi-linear setting. A deeper analysis of these results is in the list of incoming works.

The ε -renegotiation process works as well for the roommates submodel. Moreover, its complexity results hold for roommates matching games with couples playing zero-sum, strictly competitive, and infinitely repeated games. Therefore, using the complexity results claimed by Alkan and Tuncay together with our results, we conclude that the mechanism designed to compute pairwise stable and renegotiation proof allocations for the roommates submodel is efficient as well.

Dynamic matching games

The last chapter of this thesis is devoted to studying a dynamic model of one-to-one matching markets. This is a joint work with Marco Scarsini. Although it is not directly related to the models seen in the previous chapters, this is the first attempt to extend the matching games model to a dynamic setting in which initially not all agents are present on the market but arrive with the time. In addition, we adopt a different approach by developing a stochastic process model.

The problem of stable matchings in two-sided markets in its static version was firstly presented by Gale and Shapley in their seminal paper [53]. Many applications to their model have been established as matching schools and students, doctors and hospitals, firms and workers, or patients and donors, among others. A solution to Gale-Shapley's problem is a matching from which no pair of agents prefer to abandon their assigned partners and matching together. Such a pair, if it exists, is known as a *blocking pair*. A matching that does not admit any blocking pair is called *pairwise stable* (Definition 6.1.4).

Many years passed until dynamic versions of the stable matching problem appeared. Damiano and Lam [38] defined a finite repeated-time model in which agents can change partners from one period to another. All agents are present in the market from the beginning and their final utilities depend on the ones obtained during each period.

Closer to the work in this chapter, Caldentey et al. [31], Adan and Weiss [6], Bušić et al. [30], and Adan et al. [5], among others, gave a different dynamic extension to the stable matching problem, known as the *stochastic matching model*. In this model, agents (or items) arrive on the market in a discrete-time process. Items within each side have different types and a fixed bipartite graph gives the compatibility links between items from different sides. In addition, at every period exactly one item arrives on each side of the market.

We study a stochastic matching model in which agents arrive at the market following *Poisson processes*. Unlike the literature, we consider that agents can naturally leave the market (e.g. due to the death of the patient, the task deadline, or the tiredness of the agent). Agents have cardinal preferences (utilities) over the agents on the other side of the market and utility thresholds (*individually rational payoffs* (IRPs)) representing agents' ability to reject possible partners proposed by the system. Matched agents leave the market while unmatched agents remain on it.

Unlike Damiano-Lam's model, in a stochastic matching model agents cannot decide when or with who get matched as they follow a predefined matching policy. One of the most studied matching policies is *first-come-first-served* (FCFS)

[5, 6, 30, 31, 35, 82] in which arriving agents get matched to the oldest compatible agent at the other side. Matching policies in which agents get matched as soon as they arrive (subject to availability/compatibility constraints) are called *greedy matching policies* or simple *greedy policies*. Besides FCFS, in this chapter, we consider two extra greedy matching policies: *Random allocation*, in which arrival agents are allocated randomly among the already present agents on the market (subject to individual rationality), and a *deferred-acceptance* allocation in which every arrival triggers a Gale-Shapley's algorithm run.

Stochastic matching models with greedy policies can be associated with Markov processes. Adan and Weiss [6] found conditions for ergodicity and derived the stationary distribution of a discrete-time Markov chain, resulting to be of product form. Adan et al. [5] defined other possible Markov chains for the same model, proved their ergodicity, and derived their stationary distributions with a product form as well. Finally, they computed the matching rates between the different types of agents.

We derive a continuous-time Markov chain (CTMC) from our stochastic model given by the number of agents on the market whenever agents have *anonymous IRPs* (Definition 6.2.3). Passing through the jump chain of the CTMC, we prove the positive recurrence of the Markov chain (Theorem 6.2.8). Imposing reversibility we obtain a recurrence expression for its stationary distribution (Theorem 6.2.9) which implies that our stationary distribution also has a product form. We study further the extra assumptions imposed due to the reversibility and show that everything is reduced to agents' IRPs (Theorem 6.2.11). Finally, we empirically measure the performance of the three matching policies in terms of social welfare.

The model studied in this chapter differentiates from the ones in the literature as (1) our dynamic occurs in continuous time with an infinite horizon, (2) agents can naturally leave the market, and (3) our Markov chain follows the number of agents in the market while others usually follow the whole sets of agents (including, for example, the agents' identity). Although not studied in this chapter, another difference will rise from the agents' ability to *influence* the possible matchings by lying in their IRPs. This model will be an extension of ours in future works.

The chapter is structured as follows. Section 6.1 states the stochastic matching model and explains the different greedy matching policies considered in this chapter. Section 6.2 studies the continuous-time Markov chain obtained when considering the number of agents per side, shows its positive recurrence, derives its stationary distribution, and deduces conditions on agents' IRPs from imposing reversibility. Section 6.3 makes empirical comparisons of the social welfare of the three greedy policies. Section 6.4 concludes the chapter.

6.1 The model

To avoid confusion with the matching game models studied before, this chapter has its own terminology and notation. We consider a two-sided market with sides \mathcal{A} and

\mathcal{B} in which agents arrive according to a Poisson process and get matched according to a fixed *matching policy* Φ . Matched agents leave the market while unmatched agents remain on it. *A-agents* (those arriving to \mathcal{A}) arrive at rate $\lambda_A > 0$. Analogously, *B-agents* arrive at rate $\lambda_B > 0$. Only matchings between *A-agents* and *B-agents* are allowed. Finally, unmatched agents leave the market at rate $\mu_A > 0$ and $\mu_B > 0$, respectively. This is a continuous-time model with an infinite horizon.

For every time $t \geq 0$, the market state is denoted by (A_t, B_t) , where A_t is the set of *A-agents* at time t and B_t is the set of *B-agents* at time t . The market starts initially empty so $(A_0, B_0) = (\emptyset, \emptyset)$. The full market will be denoted $(\mathcal{A}, \mathcal{B})$, and a realization of it will be denoted $(A_t, B_t)_{t \geq 0}$.

In the absence of a matching policy, standard results from the theory of birth-and-death processes can be used to compute the expected number of agents in the market. The number of agents in the market is a stable birth and death process¹. Moreover, taking $N_A(t) := |A_t|$ (and analogously $N_B(t)$),

$$\begin{aligned} \mathbb{E}[N_A(t) \mid N_A(0) = 0] &= \frac{\lambda_A}{\mu_A} (1 - e^{-\mu_A t}), \quad \forall t \geq 0, \\ \mathbb{E}[N_B(t) \mid N_B(0) = 0] &= \frac{\lambda_B}{\mu_B} (1 - e^{-\mu_B t}), \quad \forall t \geq 0. \end{aligned} \tag{6.1}$$

Figure 6.1 shows two examples of 300 average market realizations each with no matching policy. We can observe how, on average, the number of agents per side follows Equation (6.1).

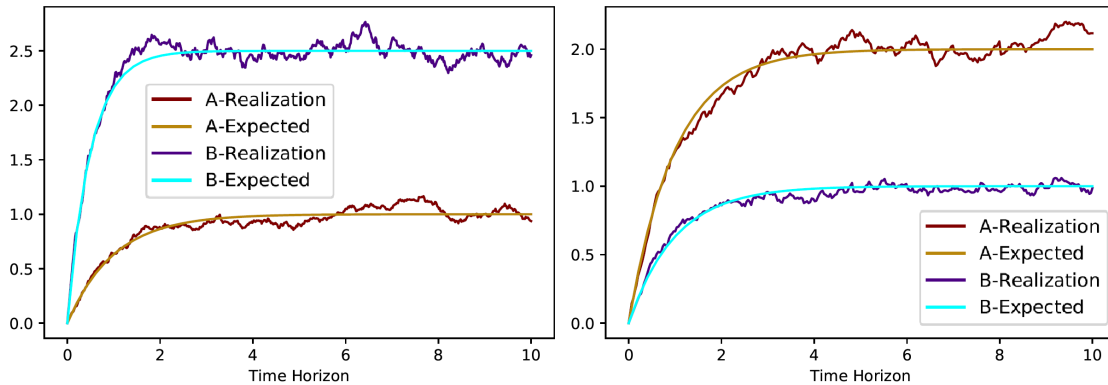


Figure 6.1: Average number of agents in the market - 300 samples. *Left*: $\lambda_B = 5$, $\mu_B = 2$, $\lambda_A = \mu_A = 1$, *Right*: $\lambda_A = 2$, $\mu_A = \lambda_B = \mu_B = 1$

To model matching, we assume that each agent, on each side, has a random cardinal preference for each agent on the other side. Formally, $\forall t \geq 0, \forall a \in A_t, \forall b \in$

¹We say that a birth and death process is stable if the queue length defines a positive recurrent Markov chain. For a detailed description of birth and death process, as well as $M/M/\infty$ queues, we refer the reader to Kulkarni [76] § 7.3.

B_t , we define,

$$\begin{aligned} X_{a,b} &\sim U[0, 1] \text{ is agent } a\text{'s utility when matched with agent } b, \\ Y_{b,a} &\sim U[0, 1] \text{ is agent } b\text{'s utility when matched with agent } a, \end{aligned}$$

where $U[0, 1]$ is the uniform distribution on $[0, 1]$. These utilities do not change over time. Agents may prefer to remain single rather than matching with some of the agents on the other side. For every $k \in \mathcal{A} \cup \mathcal{B}$, the symbol $\rho_k \in [0, 1]$ will denote agent k 's utility for remaining single, i.e., k 's **individually rational payoff**.

Definition 6.1.1. Matching a pair of agents $(a, b) \in \mathcal{A} \times \mathcal{B}$ is **individually rational** if and only if,

$$X_{a,b} \geq \rho_a \text{ and } Y_{b,a} \geq \rho_b. \quad (6.2)$$

If Equation (6.2) holds, we say that b is acceptable for a and a is acceptable for b .

Individually rational payoffs represent the agents' ability to reject partners proposed by the matching policy or central planner. To capture this ability, we will focus on computing individually rational allocations.

Remark 6.1.2. Along this chapter, individually rational payoffs will be fixed and public (for the central planner). A natural extension to our model will be the strategic setting in which agents announce their IRPs to the system when arriving and where the goal is the design of strategyproof matching policies, i.e., the design of matching policies that incentivize the agents to truthfully announce their values.

In a one-to-one static market, a matching is a correspondence between the two sides of agents such that no agent can have more than one partner. For dynamic markets, a matching is a mapping that gives, for every time t , a static matching of the market (A_t, B_t) . We state this formally.

Definition 6.1.3. Let $(A_t, B_t)_t$ be a realization of the market $(\mathcal{A}, \mathcal{B})$. A **dynamic matching** $\gamma = (\gamma_t)_{t \geq 0}$ is a continuous-time sequence of static matchings, i.e., for every $t \geq 0$, γ_t is a static matching of (A_t, B_t) .

In addition to individual rationality, we are interested in matchings that are stable à la Gale and Shapley. However, this stability notion needs also to be extended to the dynamic case. First of all, we state Gale and Shapley's definition in our context.

Definition 6.1.4. Let (A, B) be a static market. A static matching γ between A and B is blocked by a pair of agents $(a, b) \in A \times B$ if $(a, b) \notin \gamma$ and $X_{a,b} > X_{a,\gamma(a)}$, $Y_{b,a} > Y_{b,\gamma(b)}$, where $\gamma(k)$ represents the partner of agent k in the matching γ . A matching is **pairwise stable** if it is not blocked by any pair of agents.

Remark 6.1.5. Definition 6.1.4 corresponds to external stability in the matching games model.

Definition 6.1.6. Let $\gamma = (\gamma_t)_{t \geq 0}$ be a dynamic matching. We say that γ is **stable** if for every $t \geq 0$, the matching γ_t is pairwise stable for the static market (A_t, B_t) .

Gale and Shapley proved that for any static market with strict ordinal preferences and sides of the same size, there always exists a pairwise stable matching. Even more, such matching can be computed in polynomial time thanks to their deferred acceptance algorithm. Their result is straightforwardly generalized to the case with different side sizes and generic cardinal preferences. We can use Gale and Shapley's results to prove the following one.

Theorem 6.1.7. *For any realization of the two-sided market $(\mathcal{A}, \mathcal{B})$, there always exists a stable matching.*

Proof. Consider a dynamic two-sided market $(\mathcal{A}, \mathcal{B})$. Let ℓ be the arrival time of an agent to any of the sides. Fix the static market (A_ℓ, B_ℓ) and run Gale-Shapley's algorithm. Let γ_ℓ be its output. It holds that γ_ℓ is pairwise stable for (A_ℓ, B_ℓ) . For any $t \in \mathbb{R}_+$, define $\ell(t)$ to be the largest arrival time (on either side) that is not larger than t . Then, construct the sequence γ by defining $\gamma_t := \gamma_{\ell(t)}$. Thus, γ is a stable matching of $(\mathcal{A}, \mathcal{B})$. \square

The proof of Theorem 6.1.7 constructs a stable matching by running Gale and Shapley's algorithm at every arrival time and then, extending the static pairwise stable matchings to the rest of the times. Since matched agents get out of the market, no matching can be blocked by future arriving agents.

6.1.1 Greedy matching policies

Running a matching policy at every arrival time defines a *greedy (matching) policy*, since no foresight is considered. Depending on the matching policy run, a different greedy policy is obtained. Greedy policies are observed in settings in which agents do not have the ability to wait to be matched due to, for example, shortly expected lifetimes, as in kidney allocation [27, 103]. In this chapter we will study three of these policies:

1. *First-come-first-served* (FCFS): It corresponds to the most used policy in kidney allocation due to its high fairness level. Kidneys are allocated to patients that have been longer on the market (subject to individual rationality).
2. *Fully-random* (FR): The arrival agents are allocated randomly among the agents that may accept them (individual rationality).
3. *Deferred-acceptance* (DA): Agents are matched at every arrival through a deferred-acceptance algorithm as the one used to prove Theorem 6.1.7.

Although kidney allocation cannot be mapped into our model as organs do not have utilities, the matching policies known in this context can be used as benchmarks to test our deferred-acceptance policy.

Dynamic markets with greedy policies define an interesting stochastic process when considering the number of agents on each side of the market. The following section will examine this process.

6.2 Greedy policy and its stochastic process

We consider a dynamic two-sided market as explained in the previous section together with a greedy matching policy. Given fixed values of $\lambda_A, \lambda_B, \mu_A, \mu_B > 0$, the sequences $(|A_t|, |B_t| : t \geq 0)$, corresponding to the number of A -agents and B -agents on the market at every moment, define a continuous-time Markov chain

$$(Z_t)_t = \{Z_t = (|A_t|, |B_t|), t \geq 0\} \in \mathbb{N} \times \mathbb{N}$$

Let $(W_n)_n = \{W_n := (|A_{\tau_n}|, |B_{\tau_n}|) : n \in \mathbb{N}\}$ be its associated jump chain, where τ_n indicates the time of the n -th natural event (arrival or departure of an agent). For simplicity, we will omit the cardinality parenthesis $|\cdot|$ when referring to these processes in the future. Similarly, we will write (A_n, B_n) for the discrete-time Markov chain instead of (A_{τ_n}, B_{τ_n}) .

To study $(Z_t)_t$ we will focus on its jump chain $(W_n)_n$. Four possible events can happen for every $n \in \mathbb{N}$: An A -agent arrives, a B -agent arrives, an A -agent naturally leaves, and a B -agent naturally leaves. Arrivals increase the number of agents, departures decrease them. However, arrivals trigger the matching policy, which may match agents and therefore, decrease the number of agents in the market. Thus, the number of agents in the market increases only if the matching policy produces no new matchings. The following lemma details the possible outcomes of an arrival.

Lemma 6.2.1. *Suppose that an agent k arrives at the market and a matching policy is run. The possible outputs are:*

1. *One and only one matching is created, which necessarily includes agent k . In this case, the total number of agents in the market decreases by 1.*
2. *No new matching is created. In this case, the total number of agents in the market increases by 1.*

Proof. Suppose, without loss of generality, that the arriving agent k is an A -agent. If the B -side is empty, no new matching can be created and the number of agents in the market increases by 1 (there is one new A -agent). Suppose the B -side is non-empty and that a matching $(a, b) \in \mathcal{A} \times \mathcal{B}$ is created. If $a \neq k$, a and b were already present on the market and therefore, we get a contradiction: they could have been matched when the last of them arrived. Thus, necessarily, $a = k$. In particular, at most one couple is created every time that we run the matching policy, and this couple includes the arriving agent. Since k and b got matched, they leave the market, and the number of B -agents decreases by 1. Finally, if k does not want to get matched with any of the agents in the B -side, then the matching policy does not produce any new matching, and the number of A -agents increases by 1. \square

From Lemma 6.2.1 we observe that $(W_n)_n$ only increases or decreases by one unit in one of its coordinates at every time. In particular, $(W_n)_n$ is a *random walk* in $\mathbb{N} \times \mathbb{N}$. We aim at computing the transition matrix of the jump chain. We will make use of the following proposition.

Proposition 6.2.2. *Let a be an A -agent arriving at the market at time n . Then,*

$$\mathbb{P}(a \text{ gets matched}) = \begin{cases} (1 - \rho_a) \cdot (1 - \prod_{b \in B_n} \rho_b) & \text{if } B_n \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

A similar result holds for B -agents.

Proof. If the B -side is empty when a arrives, the probability of getting matched is zero. Suppose that B_n is non-empty. For a to get matched, there must exist at least one B -agent b such that a and b are mutually acceptable. It holds that

$$\begin{aligned} \mathbb{P}(\exists b \in B_n \text{ s.t. } a \text{ is acceptable for } b) &= 1 - \mathbb{P}(a \text{ is not acceptable for anybody}) \\ &= 1 - \prod_{b \in B_n} \rho_b, \end{aligned}$$

where we have used the fact that utilities are drawn from a uniform distribution. Finally,

$$\begin{aligned} \mathbb{P}(a \text{ gets matched}) &= \mathbb{P}(\exists b \in B_n \text{ s.t. } a \text{ and } b \text{ are mutually acceptable}) \\ &= (1 - \rho_a) \cdot \mathbb{P}(\exists b \in B_n \text{ s.t. } a \text{ is acceptable for } b). \end{aligned}$$

From this, we obtain the desired probability. \square

Remark that the term $\prod_{b \in B_n} \rho_b$ depends on the players' identity (through their IRPs). To obtain a time-homogeneous Markov chain, we add the following assumption to our model.

Definition 6.2.3. We say that agents have **anonymous IRPs** whenever the probability of getting matched when arriving at the market (and the matching policy is run) does not depend on the identity of the players but only on the number of agents at the other side of the market.

From now on, we consider anonymous IRPs. Notice that different greedy policies can still have different matching probabilities even under anonymous IRPs.

To be able to simultaneously treat the cases of B_n non-empty and B_n empty, we use the convention that $\prod_{j \in J} \rho_j = 1$ if $J = \emptyset$ and all ρ_j are in $[0, 1]$. From Proposition 6.2.2, we obtain the transition matrix of the jump chain.

Theorem 6.2.4. *Let P be the transition matrix of the Markov chain $(W_n)_n$. Let $p(j)$ denote the probability that an arriving A -agent gets matched when the B -side has j*

6.2. Greedy policy and its stochastic process

agents, and $q(i)$ the probability that an arriving B -agent gets matched when the A -side has i agents (see Proposition 6.2.2). Then, for any $i, j, k, \ell \in \mathbb{N}$,

$$P((i, j), (k, \ell)) = \frac{1}{(\lambda_A + \lambda_B + i\mu_A + j\mu_B)} \cdot \begin{cases} \lambda_A \cdot (1 - p(j)) & \text{if } k = i + 1, \ell = j \\ \lambda_B \cdot q(i) + i \cdot \mu_A & \text{if } k = i - 1, \ell = j \\ \lambda_B \cdot (1 - q(i)) & \text{if } k = i, \ell = j + 1 \\ \lambda_A \cdot p(j) + j \cdot \mu_B & \text{if } k = i, \ell = j - 1 \\ 0 & \text{otherwise} \end{cases}$$

Proof. Given the current state $W_n = (i, j)$, there are four possible events: an A -agent arrives (with rate λ_A) (and, with probability $p(j)$ gets matched and with probability $(1 - p(j))$ does not get matched), a B -agent arrives (with rate λ_B) (and, with probability $q(i)$ gets matched and with probability $(1 - q(i))$ does not get matched), an A -agent naturally leaves (with rate $i \cdot \mu_A$), and a B -agent naturally leaves (with rate $j \cdot \mu_B$). Notice that

$$\mathbb{P}(\text{an } A\text{-agent arrives}) = \frac{\lambda_A}{\lambda_A + \lambda_B + i\mu_A + j\mu_B}$$

as this is the probability that the minimum of four independent exponential random variables with parameters $\lambda_A, \lambda_B, i\mu_A, j\mu_B$, respectively, is the one with parameter λ_A . It follows that

$$\begin{aligned} P((i, j), (i + 1, j)) &= \mathbb{P}(\text{an } A\text{-agent arrives and does not get matched}) \\ &= \mathbb{P}(\text{an } A\text{-agent arrives})\mathbb{P}(\text{this agent does not get matched}) \\ &= \frac{\lambda_A}{\lambda_A + \lambda_B + i\mu_A + j\mu_B} \cdot (1 - p(j)) \end{aligned}$$

The other cases can be obtained similarly. \square

Remark 6.2.5. The transition probabilities depend on the current state. In particular, the probability that the market size increases is decreasing with respect to the number of agents. The random walk $(W_n)_n$ is naturally pushed back to the origin due to the matching policy and the $M/M/\infty$ queue behavior. Our Markov chain $(W_n)_n$ is therefore, a *non-homogeneous (in space) random walk* [81].

From Theorem 6.2.4 we can easily identify the Q -matrix of our continuous-time Markov chain.

Theorem 6.2.6. *Let Q be the Q -matrix of the continuous-time Markov chain $(Z_t)_t$. Taking p and q as in Theorem 6.2.4, it holds that for any $(i, j) \in \mathbb{N}^2$,*

$$\begin{aligned} Q((i, j), (i + 1, j)) &= \lambda_A \cdot (1 - p(j)) \\ Q((i, j), (i - 1, j)) &= \lambda_B \cdot q(i) + i \cdot \mu_A \end{aligned}$$

$$\begin{aligned}
 Q((i, j), (i, j + 1)) &= \lambda_B \cdot (1 - q(i)) \\
 Q((i, j), (i, j - 1)) &= \lambda_A \cdot p(j) + j \cdot \mu_B \\
 Q((i, j), (i, j)) &= -(\lambda_A + \lambda_B + i \cdot \mu_A + j \cdot \mu_B)
 \end{aligned}$$

and $Q((i, j), (k, \ell)) = 0$ otherwise.

6.2.1 Positive recurrence

We first compute the stationary distribution of the jump chain $(W_n)_n$, and use the Q -matrix to compute the stationary distribution of the continuous-time Markov chain $(Z_t)_t$. We use Foster's criterion [Theorem 2.6.4 [81]] to prove that $(W_n)_n$ is positive recurrent.

Lemma 6.2.7 (Foster's criterion). *An irreducible Markov chain $(W_n)_n$ on a countable state space Σ is positive recurrent if and only if there exist a positive function $f : \Sigma \rightarrow \mathbb{R}_+$, a finite non-empty set $E \subseteq \Sigma$, and $\varepsilon > 0$, such that*

1. $\mathbb{E}[f(W_{n+1}) - f(W_n) \mid W_n = i] \leq -\varepsilon, \forall i \in \Sigma \setminus E,$
2. $\mathbb{E}[f(W_{n+1}) \mid W_n = i] < \infty, \forall i \in E.$

Theorem 6.2.8. *The jump chain $(W_n)_n$ with transition matrix P given by Theorem 6.2.4 is positive recurrent.*

Proof. Consider the function $f(x, y) = x^2 + y^2$. Let $(i, j) \in \mathbb{N} \times \mathbb{N}$ and define

$$\Lambda(i, j) := \lambda_A + \lambda_B + i\mu_A + j\mu_B. \quad (6.3)$$

Then,

$$\begin{aligned}
 \mathbb{E}(f(W_{n+1}) - f(W_n) \mid W_n = (i, j)) &= \frac{1}{\Lambda(i, j)} \cdot [\lambda_A + \lambda_B - 2i^2\mu_A - 2j^2\mu_B \\
 &+ i(2\lambda_A(1 - p(j)) - 2\lambda_Bq(i) + \mu_A) + j(2\lambda_B(1 - q(i)) - 2\lambda_Ap(j) + \mu_B)]
 \end{aligned}$$

$\Lambda(i, j)$ being always positive, we only need to care about the expression within the squared parenthesis. Notice it can be split in the sum of two concave parabolas,

$$\begin{aligned}
 \phi(i) &:= -2i^2\mu_A + i(2\lambda_A(1 - p(j)) - 2\lambda_Bq(i) + \mu_A) + \lambda_A \\
 \psi(j) &:= -2j^2\mu_B + j(2\lambda_B(1 - q(i)) - 2\lambda_Ap(j) + \mu_B) + \lambda_B
 \end{aligned}$$

Setting $c := 2\lambda_A(1 - p(j)) - 2\lambda_Bq(i) + \mu_A$, the zeros of ϕ are,

$$i_{1,2} = \frac{-c \pm \sqrt{c^2 + 8\mu_A\lambda_A}}{4\lambda_A\mu_A}$$

In particular, one of them is positive and the other one is negative. Let i_1 be the positive zero. Then, there exists $\bar{i} > i_1$ such that for any $i > \bar{i}$, $\phi(i) \leq -1$. Similarly,

we can find $\bar{j} \in \mathbb{N}$ such that for any $j > \bar{j}$, $\psi(j) \leq -1$. Considering $E = \{0, 1, \dots, \bar{i}\} \times \{0, 1, \dots, \bar{j}\}$, we conclude that the first condition of Foster's criterion holds².

We now check the second condition. Let $i, j \in E$. Then,

$$\begin{aligned} \mathbb{E}(f(W_{n+1}) \mid W_n = (i, j)) &\leq \frac{\bar{j}^2 \bar{i} \mu_A + \bar{i}^2 \bar{j} \mu_B}{\lambda_A + \lambda_B} \\ &+ \frac{\bar{i}^3 \mu_A + \bar{i}^2 (\lambda_A + \lambda_B + 2\mu_A) + \bar{i} (2\lambda_A + 2\lambda_B + \mu_A) + \lambda_A}{\lambda_A + \lambda_B} \\ &+ \frac{\bar{j}^3 \mu_B + \bar{j}^2 (\lambda_A + \lambda_B + 2\mu_B) + \bar{j} (2\lambda_B + 2\lambda_A + \mu_B) + \lambda_B}{\lambda_A + \lambda_B} \end{aligned}$$

where we have bounded all probabilities by 1 and took all signs to be positive. We obtain that

$$\mathbb{E}(f(W_{n+1}) \mid W_n = (i, j)) < \infty, \quad \forall (i, j) \in E.$$

Thanks to Foster's criterion we obtain that $(W_n)_n$ is positive recurrent. \square

6.2.2 Reversibility

A useful technique to compute the stationary distribution of a Markov chain is to use its *reversibility*, i.e., to check if the reversed Markov chain and the original one are in balance. Reversibility is a strong assumption and not all positive recurrent Markov chains satisfy it (for a detailed explanation of reversibility, we refer the reader to Pardoux [85], § 2.7 and § 7.7). In particular, our jump chain $(W_n)_n$ will not always be reversible as we will need extra assumptions on agents' IRPs (Theorem 6.2.11).

Before showing the reversibility assumption on agents' IRPs, we use the *detailed balanced equations* (Equation (6.6)) to deduce a stationary measure of the jump chain and to show that the stationary distribution of $(W_n)_n$ has a product form as well.

Theorem 6.2.9. *Let π be a stationary measure of $(W_n)_n$. Then, π satisfies the recursion,*

$$\pi(i+1, j) = \pi(i, j) \cdot \frac{\Lambda(i+1, j) \cdot \lambda_A (1 - p(j))}{\Lambda(i, j) \cdot (\lambda_B q(i+1) + (i+1)\mu_A)}, \quad \forall i, j \in \mathbb{N} \quad (6.4)$$

$$\pi(i, j+1) = \pi(i, j) \cdot \frac{\Lambda(i, j+1) \cdot \lambda_B (1 - q(i))}{\Lambda(i, j) \cdot (\lambda_A p(j+1) + (j+1)\mu_B)}, \quad \forall i, j \in \mathbb{N} \quad (6.5)$$

where $\Lambda(i, j)$ is given by Equation (6.3). In particular, π has a product form.

Proof. A Markov chain is said to be reversible if and only if the following detailed balanced equations hold:

$$\pi(i)P(i, j) = \pi(j)P(j, i), \quad \forall i, j \in \mathbb{N}. \quad (6.6)$$

²The fact that c depends on i and j is not a problem as we can bound $p(j)$ and $q(i)$ by 0 and then find the highest \bar{i} that satisfies the first condition of Foster's criterion.

Recalling that $P((i, j), (k, \ell)) \neq 0$ only for $(k, \ell) \in \{i \pm 1, j \pm 1\}$, Equation (6.6) becomes,

$$\pi(i+1, j)P((i+1, j), (i, j)) = \pi(i, j)P((i, j), (i+1, j)), \quad (6.7)$$

$$\pi(i, j+1)P((i, j+1), (i, j)) = \pi(i, j)P((i, j), (i, j+1)). \quad (6.8)$$

Equation (6.7) holds if and only if

$$\pi(i+1, j) \cdot \frac{\lambda_B q(i+1) + (i+1)\mu_A}{\Lambda(i+1, j)} = \pi(i, j) \cdot \frac{\lambda_A(1-p(j))}{\Lambda(i, j)}.$$

From where we obtain Equation (6.4). Equation (6.5) is derived analogously. \square

The stationary distribution of the continuous-time Markov chain can be easily obtained as a consequence of Theorem 6.2.9.

Corollary 6.2.10. *The continuous-time Markov chain is positive recurrent and one invariant measure of Π is given by,*

$$\Pi(i, j) = \pi(i, j)/q(i, j), \forall i, j \in \mathbb{N} \times \mathbb{N}$$

where $q(i, j) = -Q((i, j), (i, j))$. In particular, normalizing Π we obtain the invariant distribution of the continuous-time Markov chain.

Finally, we show the reversibility conditions for our Markov chain.

Theorem 6.2.11. *The jump chain $(W_n)_n$ is reversible if and only if the probabilities p and q satisfy,*

$$(1-p(j)) \cdot (1-q(i+1)) = (1-q(i)) \cdot (1-p(j+1)), \forall i, j \in \mathbb{N} \quad (6.9)$$

Proof. Let $i, j \in \mathbb{N}$. Using Equations (6.4) and (6.5) separately on $\pi(i+1, j+1)$, we get

$$\begin{aligned} \pi(i+1, j+1) &\stackrel{(6.4)}{=} \pi(i, j+1) \cdot \frac{\Lambda(i+1, j+1) \cdot \lambda_A(1-p(j+1))}{\Lambda(i, j+1) \cdot (\lambda_B q(i+1) + (i+1)\mu_A)} \\ \pi(i+1, j+1) &\stackrel{(6.5)}{=} \pi(i+1, j) \cdot \frac{\Lambda(i+1, j+1) \cdot \lambda_B(1-q(i+1))}{\Lambda(i+1, j) \cdot (\lambda_A p(j+1) + (j+1)\mu_B)} \end{aligned}$$

We obtain

$$\frac{\pi(i, j+1) \cdot \lambda_A(1-p(j+1))}{\Lambda(i, j+1) \cdot (\lambda_B q(i+1) + (i+1)\mu_A)} = \frac{\pi(i+1, j) \cdot \lambda_B(1-q(i+1))}{\Lambda(i+1, j) \cdot (\lambda_A p(j+1) + (j+1)\mu_B)}$$

Applying Equation (6.4) to $\pi(i+1, j)$, the right-hand side becomes

$$\pi(i, j) \cdot \frac{\Lambda(i+1, j) \cdot \lambda_A(1-p(j))}{\Lambda(i, j) \cdot (\lambda_B q(i+1) + (i+1)\mu_A)} \cdot \frac{\lambda_B(1-q(i+1))}{\Lambda(i+1, j) \cdot (\lambda_A p(j+1) + (j+1)\mu_B)}$$

Similarly, the left-hand side becomes

$$\pi(i, j) \cdot \frac{\Lambda(i, j+1) \cdot \lambda_B(1-q(i))}{\Lambda(i, j) \cdot (\lambda_A p(j+1) + (j+1)\mu_B)} \cdot \frac{\lambda_A(1-p(j+1))}{\Lambda(i, j+1) \cdot (\lambda_B q(i+1) + (i+1)\mu_A)}$$

Equalizing the two sides, we obtain Equation (6.9). Therefore, satisfying the detailed balanced equations is equivalent to satisfying Equation (6.9). \square

The three greedy policies considered in this chapter result to be reversible for uniform IRPs, that is, whenever all agents have the same value ρ . Figure 6.2 shows three stationary distributions for the continuous-time Markov chain obtained with a deferred-acceptance matching policy, for fixed values $\lambda_A = \lambda_B = 10$, $\mu_A = \mu_B = 1$, and variable values of ρ . On the left there is the *full-rejection case* i.e., $\rho_k = 1, \forall k \in \mathcal{A} \times \mathcal{B}$, on the middle we have taken $\rho_k = 0.66$, and on the right, $\rho_k = 0.33$, for all agents in the market. We observe how the stationary distribution moves to the origin as ρ decreases, showing, on average, a market with fewer agents on it.

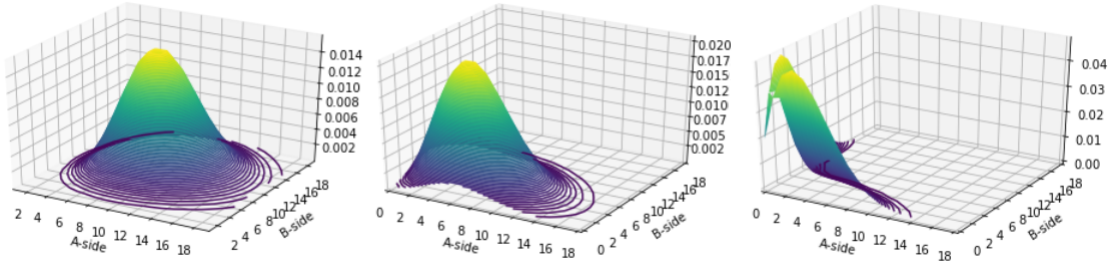


Figure 6.2: Stationary distribution CTMC. *Left:* $\rho = 1$, *Middle:* $\rho = 0.66$, *Right:* $\rho = 0.33$

In the *full-acceptance case*, i.e., when $\rho_k = 0$ for every agent k , at every time t at least one of the two sides (not always necessarily the same one) is empty. The stationary distribution can be projected in one dimension over the entire integer set, where positive values correspond to a non-empty A -side, and negative values correspond to a non-empty B -side. Figure 6.3 shows the stationary distribution of the jump chain and the continuous-time Markov chain, when choosing $\lambda_A = 10$, $\mu_A = \mu_B = 1$, and increasing λ_B from 1 to 10.

We can observe that for low values of λ_B the stationary distribution concentrates on the positive side as it is more probable to see A -agents in the market, and goes left for higher values of λ_B . In particular, it gets concentrated around 0 when both sides have equal values of λ , as in the full-acceptance case, on average, we do not see people in the market for equal rates of arrival and departure.

We end this section by giving an interpretation to Equation (6.9), represented in Figure 6.4. The left-hand side $(1-p(j))(1-q(i+1))$ corresponds to the probability that, given that the market state is (i, j) , an A -agent arrives, does not get matched, and then, a B -agent arrives, and does not get matched, either; therefore, the market

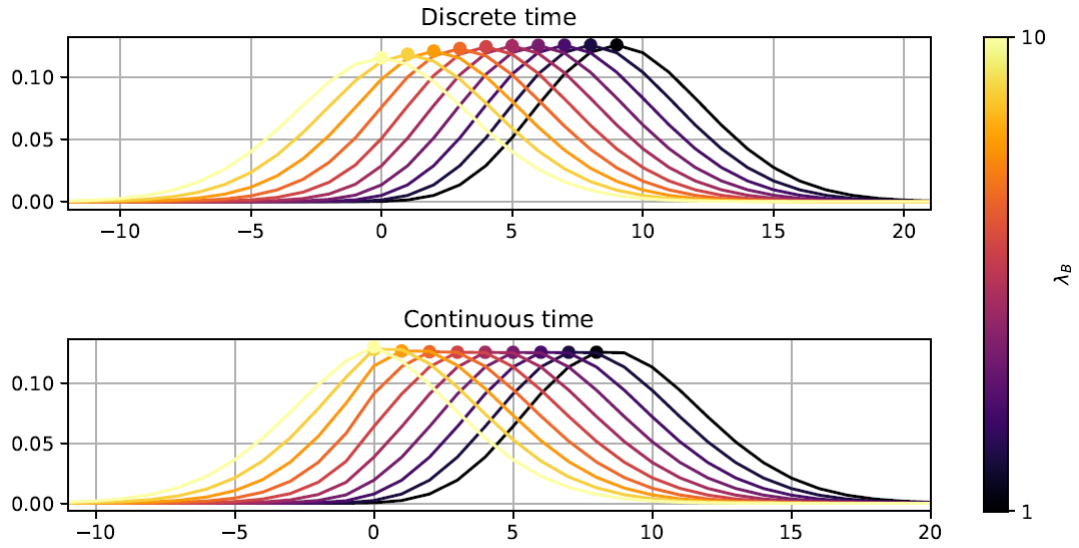


Figure 6.3: Stationary distribution variable $\lambda_B \in \{1, \dots, 10\}$, $\lambda_A = 10$, $\mu_A = \mu_B = 1$

passes to the state $(i + 1, j + 1)$. The right-hand-side is the second way that the market can pass from (i, j) to $(i + 1, j + 1)$. Namely, first a B -agent arrives and does not get matched (which holds with probability $1 - q(i)$), and then an A -agent arrives and does not get matched, either (which holds with probability $1 - p(j + 1)$). For the Markov chain to be reversible, we need these two transition probabilities to be equal.

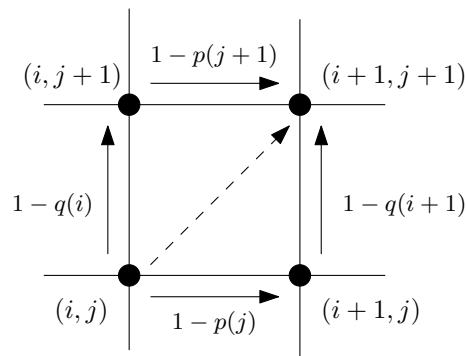


Figure 6.4: Possible ways for the market to pass from (i, j) to $(i + 1, j + 1)$

Going further, imposing reversibility to our Markov chain forces to get equal transition probabilities from $(0, 0)$ to $(i, j) \in \mathbb{N} \times \mathbb{N}$, by any trajectory of the kind $(k, \ell) \rightarrow (k + 1, \ell)$ or $(k, \ell) \rightarrow (k, \ell + 1)$ at each time.

6.3 Social Welfare

Our three greedy matching policies define a similar continuous-time Markov chain from the point of view of the number of agents on the market. However, the matched couples differ and therefore, each policy achieves a different level of social welfare.

Definition 6.3.1. Let $\gamma = (\gamma_n)_n$ be a sequence such that, for each $n \geq 0$, γ_n is a static matching of (A_n, B_n) . The **social welfare** of γ is defined as $SW(\gamma) = \sum_{n \geq 0} [\sum_{a \in A_n} u_a + \sum_{b \in B_n} v_b]$, where

$$u_a = \begin{cases} X_{a,\gamma(a)} & \text{if } a \text{ is matched in } \gamma, \\ \rho_a & \text{otherwise,} \end{cases} \quad v_b = \begin{cases} Y_{b,\gamma(b)} & \text{if } b \text{ is matched in } \gamma, \\ \rho_b & \text{otherwise.} \end{cases}$$

It is interesting to remark that, unlike the static model of stable marriage in which the proposer side is always better off with respect to the proposed side, as utilities are drawn from a uniform distribution, this phenomenon does not appear in the dynamic case. Figure 6.5 shows the average (per player and after 500 samples) welfare achieved by each side with a DA matching policy with A -side proposing. On the left, there is a symmetric market with equal arrival rates and equal departure rates, and variable (the same for all agents) $\rho \in [0, 1]$, while on the right there is a market in which A -agents arrive in a higher rate and stay longer in the market than B -agents. In symmetric markets, both sides achieve similar welfare, independent of the side that proposes. In asymmetric markets, it is the thin side that achieves higher average welfare as the DA matching policy benefits the agents that have more available partners with who to get matched when arriving at the market.

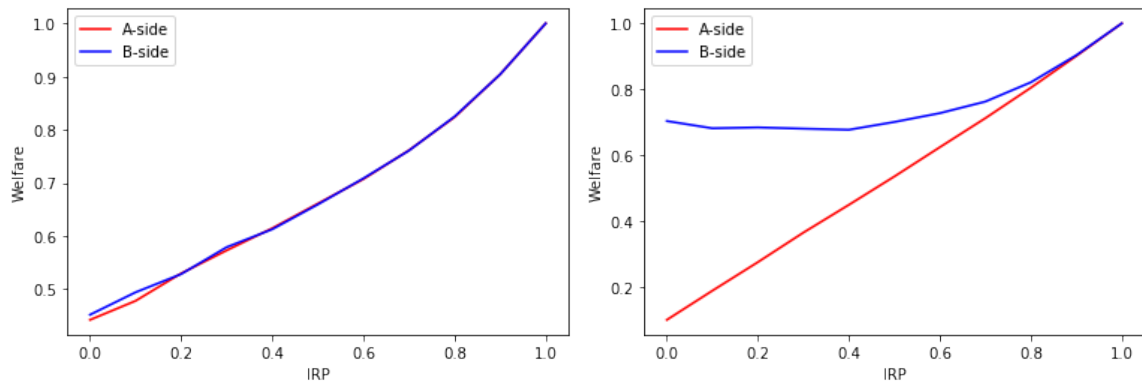


Figure 6.5: Welfare comparison DA matching policy with A -side as proposers. *Left*: Symmetric market, *Right*: Unbalanced market with more A -agents

As agents IRPs go to 1, the average social welfare goes to 1 as well as the rate of unmatched agents over the total number of agents converges also to 1. In terms of total social welfare, it is not difficult to see that a DA greedy policy outperforms the random and FCFS policies, as DA is the only one that maximizes the utility of the

agents at the moment of the matching. Figure 6.6 shows social welfare comparisons between the three matching policies for variable rate values. As Figure 6.5, on the left we have a symmetric market while on the right an unbalanced market, in both cases taking low values of $\rho \in \{0, 0.1, 0.2\}$, as the social welfare curves quickly converge to similar values for higher IRPs.

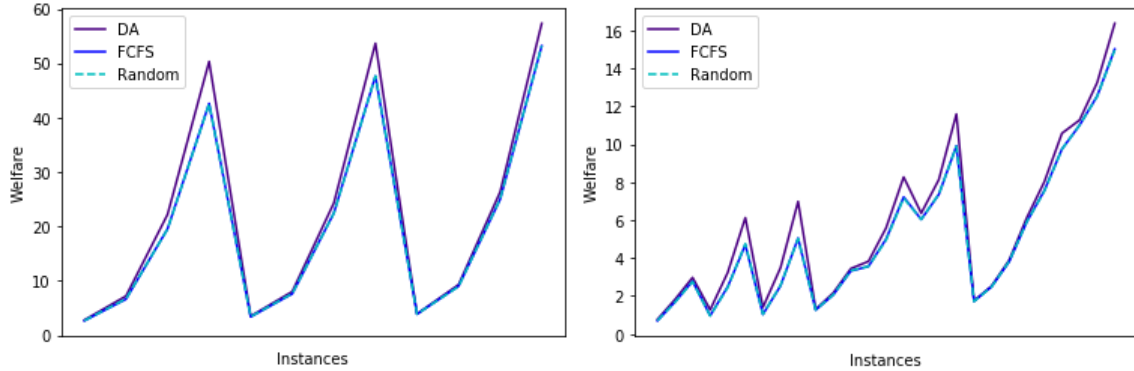


Figure 6.6: Social welfare comparison. *Left*: Symmetric market, *Right*: Unbalanced market with more A -agents

6.4 Conclusions

In this chapter, we have introduced a novel model of dynamic matching markets in which agents arrive and leave following stochastic processes. Agents within the market get matched by three possible greedy matching policies: *first-come-first-served*, *fully-random*, and *deferred-acceptance*. We have studied the evolution of the matching market under *anonymity* (Definition 6.2.3) with respect to the number of agents per side by using continuous-time Markov chains (CTMC). Our CTMC resulting to be positive recurrent (Theorem 6.2.8), we have established conditions on agents' individually rational payoffs to obtain a product-form stationary distribution (Theorem 6.2.9) from imposing reversibility (Theorem 6.2.11). Finally, we have empirically compared the matching policies in terms of social welfare and found that a deferred-acceptance kind of policy outperforms the other two.

This model is the base to study strategic behavior in matching markets where agents can affect the assignments by misreporting their individually rational payoffs at the moment of entering the market. Allowing the agents to strategically behave raises new challenges for the design of matching policies. As a future research line, we plan to extend the model in this chapter to the design of strategyproof matching policies and compare their performance against the state of the art in terms of social welfare, computational efficiency, and/or fairness.

In the mid-term, we plan to extend the model to a matching game setting in which agents do not only announce their individually rational payoffs at the moment of entering the market but also get their utilities as the outcomes of a game played

6.4. Conclusions

against the partner. The extension of internal stability and its relationship with the strategic announcements when arriving at the market may raise new and useful solutions concepts for real-life applications.

Conclusions

Sixty years ago, David Gale and Lloyd S. Shapley wrote one of the most influential articles in the literature of mathematics, economics, and computer sciences: *College admission and the stability of marriage*. In their paper, the two authors introduced the problem of allocating students to colleges, the *college admission problem*, and pairing men and women, the *stable marriage problem*. In less than twenty pages the authors presented their models, defined their solution concepts, algorithmically proved the existence of stable outcomes for any instance, and discussed/proved several important properties such as the structure of the set of solutions or the optimality of their algorithm. The article does not present any equation and still, no one could question its mathematical preciseness. The impact of the article was immediate as it allowed to prove that the *resident allocation system* in the USA was favoring hospitals over doctors.

The literature on stable matching has been developed in all possible directions, passing by different kinds of allocations as one-to-one, one-to-many, or many-to-many, different kinds of markets as two-sided, one-sided, or non-two-sided, different kinds of preferences and utilities, different kinds of stable solutions, different kinds of temporalities, etc. In addition, each model uses its own methods to prove the existence of the solutions sought.

Computer scientists have gotten interested in designing efficient algorithms to compute stable allocations motivated by the broad number of applications of the stable matching model. The work done in this thesis had as objective to centralize many of the historical models of matching markets, refine their solution concepts, and propose efficient algorithms to solve them. It considers a mix between matching market models and game theory models. Game theory studies the strategic behavior of the agents making it the perfect approach to study the strategic aspect of the agents inside of a matching market.

We have presented four models capturing different branches of matching markets:

1. *One-to-one two-sided* matching games,
2. *One-to-many two-sided* matching games,
3. *One-to-one non-two-sided* matching games and,
4. *Dynamic one-to-one two-sided* matching market.

The first three models endow the agents within the market with strategy sets and payoff functions. Matchings consist, respectively, of a one-to-one mapping between two sets D and H , a one-to-many mapping between two sets D and H , and a partition

into pairs and singletons of a set D . Allocations are then formed by considering one strategy per player, determining utilities for all the agents within the market depending on the identity of the partners in the matching, their own strategies, and the strategies of the partners. Naturally, two stability notions rise for these models: *core stability*, the generalization of Gale-Shapley's stability to the setting in which agents have strategies and deals with pairs or coalitions of agents abandoning their partners and preferring to match together; and *renegotiation proofness*, that deals with the deviations of the agents in actions within each strategic game.

Algorithmically we have proved the existence of core stable allocations for the first two submodels. We have designed an efficient *deferred-acceptance with competitions* algorithm, generalization of the one of Gale and Shapley, and proved its correctness and finiteness under classical game theory assumptions. In particular, for the second model, we have showed that simpler algorithms than the one proposed in the literature can be used to compute the desired stable solutions. Leveraging literature results, we have studied the existence of core stable allocations for the third model as well.

Considering a model with rational agents that play strategies shows its full potential when considering the second notion of stability, *renegotiation proofness*, which refines the stability notions captured from the literature. We have proved that whenever agents can compromise with their partners in the chosen strategies, a constrained equilibrium condition rises: Agents cooperate enough to guarantee some minimum threshold utility to the partners searching to maximize their own utilities.

We have defined a solid mathematical framework for renegotiation proof allocations by characterizing them as those allocations in which all matched pairs play *constrained Nash equilibria*, that is, solutions to an appropriated quasi-variational inequality. Moreover, we have defined a new class of games, *feasible games*, to study the existence of games admitting constrained Nash equilibria, and proved that many well-known games are feasible. We have designed an efficient algorithm to compute renegotiation proof allocations for the three first mentioned models. Finally, as a further refinement of the literature solutions, we have shown how to obtain core stable and renegotiation proof allocations in which all couples play Pareto-optimally.

Many future research lines start from the work done in this thesis for the three first models. A mix between our second and third model, that is, a model in which couples of doctors are assigned to hospitals and all agents are endowed with strategy sets, may find interesting applications in real life as the allocation of siblings to schools.

A further understating of the set of feasible games is in the list of future works. The proof that each studied game is feasible depends on the characteristics of the game, making it a proof game dependent. It is in our interest to find a global theory able to characterize all feasible games. Similarly, we are interested in finding a global theory able to explain the convergence of our *renegotiation process*.

The general computation of constrained Nash equilibria is also an interesting line to follow. Results from the quasi-variational inequalities domain may expose nice and

interesting properties of them. Another idea³ is to study constrained Nash equilibria as solutions to *complementarity problems* [102].

Regarding the non-two-sided model of roommates matching games, the results of Alkan and Tuncay need a deeper understanding as they have only treated the quasi-linear case (although they claimed their results hold for the non-quasi-linear case). The extension of the roommates problem to the non-transferable utility setting has not received much interest from the stable matching community. Following this research line may produce fruitful results, both from a theoretical and practical point of view. We aim to further develop this.

Before this thesis, the challenge of computing exact solutions for stable matchings with continuous utilities was an open problem. Sadly, the problem will remain open for now. Moreover, the efficient design of algorithms to compute exact solutions is (ironically) a hard problem and needs further research.

Our fourth model⁴ studied a dynamic one-to-one two-sided matching market in which agents from two sets arrive and leave the market following stochastic processes. This model is the first attempt to extend matching games to the dynamic setting. We have compared a *deferred-acceptance* kind of matching policy against two literature benchmarks, *first-come-first-served*, usually used because of its high fairness level, and *fully random*. We have studied stochastic processes generated by the markets and obtained many interesting properties as the existence of stationary distributions with product form.

Two research lines are considered for this model. The first one is to endow the agents with strategy sets and define a framework mixing dynamic versions of core stability and renegotiation proofness. The work done in static matching games where agents play infinitely repeated games may be a useful tool.

The second line is to allow the agents to announce their *individually rational payoffs* when entering the market and to study the *strategy-proof* design of matching policies to avoid strategic behaviors from the agents. Techniques from market design may be useful for this challenge. Moreover, we could leverage literature results to obtain strategy-proof and fair matching policies.

³We thank Alfred Galichon for having suggested this idea.

⁴Joint work with Marco Scarsini

Complementary results

A.1 Extensive form games with perfect information are feasible

Consider a finite set of players I and a set P of nodes of a tree. For each $p \in P$, we define its set of successor nodes in the tree by $S(p)$. For each $i \in I$ we consider a set P^i of nodes of the tree where i has to play.

Definition A.1.1. A **strategy** σ^i for player i is an application on P^i which associates to each position $p \in P^i$ a successor node in $S(p)$.

The set of terminal nodes or results is denoted by R , and the payoff of player i is given by $u^i(r)$ with $r \in R$.

Consider outside options $u_0^i \in \mathbb{R}$ for all players $i \in I$. For player i we consider the set of terminal nodes R_0^i such that her payoff is at least her outside option.

Definition A.1.2. We call the game G **admissible** if there exists at least one strategy σ such that $F[\sigma]$, the final node reached by σ , belongs to $\bigcap_{i \in I} R_0^i$, so players can obtain at least their outside options.

It is straightforward to prove (by induction on the length) that the game is admissible if and only if $\bigcap_{i \in I} R_0^i$ is non-empty.

Definition A.1.3. Let G be an admissible game and let $\sigma = (\sigma^i)_{i \in I}$ be a strategy profile such that $F[\sigma] \in \bigcap_{i \in I} R_0^i$. σ is a **constrained equilibrium** if for any $i \in I$ and for any strategy τ^i of player i , if $u^i(F[\tau^i, \sigma^{-i}]) > u^i(F[\sigma])$ then $F[\tau^i, \sigma^{-i}] \notin \bigcap_{i \in I} R_0^i$, that is, every time a player has a profitable deviation from σ , there is another player for whom the terminal node is not feasible.

A strategy profile σ naturally induces a strategy in the subgames $G[p]$, where p is a node of the tree and $G[p]$ is the game starting from p . As with subgame-perfect equilibrium, we can define the notion of constrained subgame-perfect equilibrium.

Definition A.1.4. A strategy profile σ is a **constrained subgame-perfect equilibrium** if for each position p such that $G[p]$ is admissible, the continuation strategy $\sigma[p]$ induced by σ is a constrained equilibrium of $G[p]$.

Definition A.1.5. A finite perfect information game G is called **feasible** if any time that there exists a strategy profile σ such that $F[\sigma] \in \bigcap_{i \in I} R_0^i$, then there exists τ a constrained subgame-perfect equilibrium such that $F[\tau] \in \bigcap_{i \in I} R_0^i$.

We are ready to prove the feasibility of extensive form games (Theorem 3.4.3).

Proof. **Extensive form games are feasible.** We present two proofs, one by forward induction and another one by backward induction.

Forward induction. Let i be the player who plays at the first node r , the root of the tree. For $n = 1$, note that G corresponds to a game where only i plays, and then the outputs are announced. Player i will choose the action that maximizes her payoff subject to choosing an ending node in $\bigcap_{i \in I} R_0^i$, obtaining a strategy σ in which nobody can deviate without violating the constraint. Therefore, σ is a constrained equilibrium that is subgame perfect because the game has no subgame.

Suppose that any perfect information game of length n is feasible, and let G be a game of length $n + 1$. Suppose G is admissible, so there exists a path from the root to a terminal node that gives all players a payoff greater than or equal to their outside options. We aim to prove that G has a constrained subgame-perfect equilibrium.

Let $S(r)$ be the set of all possible nodes that i can choose when she plays for the first time, that is, $S(r)$ is the set of successors of the root of the tree. Since G is admissible, there is at least one node $p \in S(r)$ such that $G[p]$ is also admissible. Let $S' \subseteq S(r)$ be the set of all nodes p in $S(r)$ such that $G[p]$ is admissible. By induction, all subgames $G[p]$ with $p \in S'$ are feasible, so for each of them there exists σ'_p a constrained subgame-perfect equilibrium.

Consider the strategies $\sigma_p := (p, \sigma'_p)$ in which player i chooses a node $p \in S'$ in her first turn and then, in the subgame $G[p]$, players follow the constrained equilibrium σ'_p . From all strategies σ_p with $p \in S'$, consider the one that maximizes i 's payoff, called σ_{p^*} . We claim that σ_{p^*} is a constrained subgame-perfect equilibrium of game G . Indeed, for p^* the induced strategy $\sigma_{p^*}[p^*]$ coincides with σ'_{p^*} so it corresponds to a constrained subgame-perfect equilibrium. Moreover, since i chose the best node $p \in S'$ for her, she has no incentive to deviate to any other node in S' . Therefore, the strategy σ_{p^*} is a constrained subgame-perfect equilibrium, since choosing any other node in $S(r) \setminus S'$ yields a non-admissible subgame.

Backward induction. Consider the longest path from the root r to some terminal node t in R . Note that there is a unique path from r to t since the tree is an acyclic graph. Let i_t be the last player to play in this path and consider the set R_{i_t} of all the terminal nodes that i_t can choose. The set is non-empty since $t \in R_{i_t}$. Next, consider the set of all nodes in R_{i_t} that belong to $\bigcap_{i \in I} R_0^i$. If the intersection $\bigcap_{i \in I} R_0^i \cap R_{i_t}$ is non-empty, player i_t can choose a final node that gives all players at least their outside option. If this is the case, i_t picks the one that maximizes her payoff. On the other hand, if the intersection is empty, anyway player i_t picks a node maximizing her own payoff. Note that, in the first case player i_t has no incentive to deviate to any other node in $\bigcap_{i \in I} R_0^i$, so if she has a profitable deviation, there will be a player who receives less than her outside option. In the second case, player i_t is playing in a non-admissible game.

Once i_t has chosen her node, we erase all the nodes of R_{i_t} from the tree and replace the node in which i_t had to play by the terminal node in which all players receive the

payoff related to the choice of i_t . Once this is done, we come back to the first step of finding the longest path from the root to some terminal node. We iterate until reducing the tree to just the root r .

Since each time a player has to decide the final node to reach in the tree, either she chooses a constrained equilibrium or a non-admissible game, the final result of this procedure leads to either finding a constrained subgame-perfect equilibrium of the game G , or proving that G is a non-admissible game, so we conclude that G is feasible. \square

A.2 A non-closed set of Pareto-optimal payoffs

Consider a two-player game with a payoff sets as in Figure A.1, consisting in the points A, B, C , and D , and the lines connecting them.

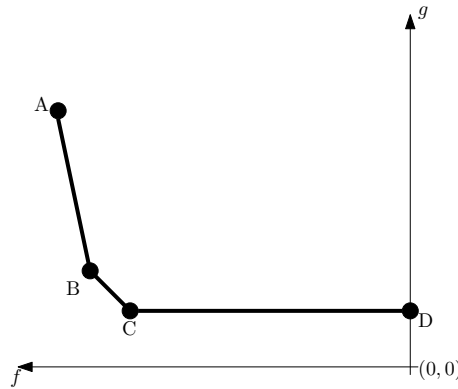


Figure A.1: A non-closed set of Pareto optimal strategies

As the point C and the whole line connecting C and D is Pareto-dominated by D , we obtain a non-closed and non-connected set of Pareto-optimal payoffs.

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RÉSUMÉ

Les marchés d'appariement stables ont été l'un des domaines d'étude les plus importants en mathématiques, en économie et en informatique au cours des soixante dernières années. Cette thèse vise à contribuer à la littérature en définissant un nouveau cadre capable d'englober et d'affiner de nombreux travaux séminaux tels que Gale et Shapley, Shapley et Shubik, Demange et Gale, Crawford et Knoer, Kelso et Crawford, Hatfield et Milgrom, Andersson et al., et d'autres.

Les jeux d'appariement ajoutent une nouvelle dimension au problème de l'appariement stable en permettant aux agents d'une coalition de jouer à des jeux stratégiques et de recevoir des paiements comme résultats. Deux modèles sont obtenus en fonction du niveau d'engagement des agents. Bien qu'un modèle sans engagement puisse être le plus intuitif, il ne parvient pas à capturer les modèles classiques mentionnés ci-dessus. En revanche, un modèle avec engagement permet de le faire avec succès.

Le modèle avec engagement est naturellement confronté à deux notions de stabilité : *stabilité par paire*, la généralisation des notions de stabilité classiques dans la littérature, et *résistance aux renégociations*, une nouvelle notion de stabilité concernant les déviations des actions des agents dans chaque jeu, qui affine les solutions stables de tous les modèles capturés.

Des algorithmes efficaces sont conçus pour calculer les allocations stables par paire et résistantes aux renégociations sous des hypothèses classiques de la théorie des jeux. Premièrement, nous proposons une généralisation de l'algorithme d'*acceptation-différée* de Gale et Shapley pour calculer des allocations stables par paire. Ensuite, un nouvel *processus de renégociation* calcule une allocation résistante aux renégociations lorsque les jeux stratégiques joués satisfont une condition de *faisabilité* et que l'algorithme converge.

La *faisabilité* des jeux est une propriété nouvelle. Nous la caractérisons en utilisant les *équilibres de Nash sous contrainte*, c'est-à-dire les meilleures stratégies de réponse soumises à des *contraintes de participation*, et nous prouvons que de nombreux jeux bien connus de la littérature de la théorie des jeux satisfont la faisabilité.

La dernière partie de cette thèse porte sur un marché d'appariement dynamique biface dans lequel les agents arrivent et partent en suivant des processus stochastiques. Différentes politiques d'appariement et les processus stochastiques générés par le nombre d'agents sur le marché sont étudiés et les conditions d'existence d'une distribution stationnaire avec une forme de produit sont trouvées.

MOTS CLÉS

Matching, Engagement, Stabilité, Équilibre de Nash sous contrainte, Temps polynomial, Jeux d'appariement

ABSTRACT

Stable matching markets have been one of the most important studied domains in mathematics, economy, and computer sciences, during the last sixty years. This thesis aims to contribute to the literature by defining a novel framework able to encompass and refine many seminal works such as Gale and Shapley, Shapley and Shubik, Demange and Gale, Crawford and Knoer, Kelso and Crawford, Hatfield and Milgrom, Andersson et al., and others.

Matching games add a new dimension to the problem of stable matching by allowing the agents within a coalition to play strategic games and receive payments as outputs. Two models are obtained depending on the level of commitment of the agents. Although a model without commitment may be the most intuitive one, it fails to capture the classical models mentioned above. In exchange, a model with commitment achieves the capture successfully.

The model with commitment naturally faces two stability notions: *pairwise stability*, the generalization of the classical stability notions in the literature, and *renegotiation proofness*, a novel stability notion concerning the agents' deviations in actions within each game, which refine the stable solutions of all the captured models.

Efficient algorithms are designed to compute pairwise stable and renegotiation proof allocations under classical game theory assumptions. First, we propose a generalization of the classical *deferred-acceptance* algorithm of Gale and Shapley to compute pairwise stable allocations. Second, a novel *renegotiation process* computes a renegotiation proof allocation whenever the strategic games played satisfy a *feasibility* condition and the algorithm converges.

Feasibility in games is a novel property. We characterize it using *constrained Nash equilibria*, that is, best-reply strategies subject to *participation constraints*, and prove that many well-known games from the literature of game theory are feasible.

The last part of this thesis works on a dynamic two-sided matching market in which agents arrive and leave following stochastic processes. Different matching policies and the stochastic processes generated by the number of agents in the market are studied and conditions for the existence of stationary distribution with product form are found.

KEYWORDS

Matching, Commitment, Stability, Constrained Nash equilibrium, Polynomial-time, Matching game