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École doctorale de sciences mathématiques de Paris centre

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présentée par

Vadim Ognov

Interacting fermions in a random medium: ground state and its entanglement entropy for a simple model

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dirigée par Frédéric Klopp

Soutenue le 26 mai 2023 devant le jury composé de :

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Résumé

Ce travail a pour but de décrire les états fondamentaux de l'opérateur de Schrödinger aléatoire associé au modèle des pièces, dans la limite thermodynamique et dans la statistique de Fermi-Dirac. Ainsi, nous considèrons les répartitions minimales, en terme d'énergie, de n électrons dans l'intervalle $[0, L]$ subdivisé en pièces par un processus de Poisson, lorsque le quotient $\frac{n}{L}$ tend vers $\rho > 0$. Ce modèle-jouet unidimensionnel réunit certains critères essentiels à une modélisation réaliste d'un ensemble de particules quantiques, à savoir une densité de particules positive ρ et un milieu aléatoire (dû aux imperfections).

Nous cherchons à saisir le comportement de ses états fondamentaux en fonction de ρ, notamment via l'énergie fondamentale par particule. Nous essayons également de quantifier l'intrication spatiale de tels états, mesurée par l'entropie d'entrelacement.

Dans le cas d'une interaction répulsive à courte portée, nous remarquons que le nombre d'électrons dans une pièce donnée admet une majoration commune à tout état fondamental. Nous optimisons parmi les groupes de pièces pouvant contenir au plus 2 particules et majorons la contribution des électrons restants pour obtenir un développement de l'énergie fondamentale par particule jusqu'à l'ordre $O(\rho^{2-\delta})$, pour tout $\delta \in (0, 1)$, dans la limite thermodynamique. Cette méthode améliore le résultat de F. Klopp et N. A. Veniaminov, sous une hypothèse plus forte cependant. Elle fournit également un facteur commun à tout état fondamental.

Par la suite, nous exprimons l'entropie d'entrelacement d'un état à répartition unique fixé, après scission en deux parties. Nous remarquons que celle-ci dépend uniquement du groupe de pièces où tombe la coupe. Nous calculons à $O(\rho^{2-\delta})$ près la moyenne de l'entropie d'entrelacement associé au facteur commun évoqué au-dessus. Nous conjecturons que, tout comme pour l'énergie fondamentale par particule, cette quantité est le terme principal de celle obtenue en considérant l'état fondamental à répartition unique tout entier.

Enfin, les résultats établis sur les systèmes de 2 particules pourraient, selon nous, s'adapter à d'autres situations.

Abstract

This work aims to describe the ground states of the random Schrödinger operator associated with the pieces model, both in the thermodynamic limit and in the Fermi-Dirac statistics. Thus, we consider the minimal distributions, in terms of energy, of n electrons in the interval $[0, L]$ subdivided into pieces by a Poisson process, when the quotient $\frac{n}{L}$ tends to $\rho > 0$. This one-dimensional toy model meets some of the essential criteria for realistic modeling of a set of quantum particles, namely a positive particle density ρ and a random medium (resulting from imperfections).

We try to understand the behavior of its ground states as a function of ρ, via the ground state energy per particle. We also try to quantify the spatial entanglement of such states, measured using entanglement entropy.

In the case of a short-range repulsive interaction, we notice that the number of electrons in a given room admits a common upper bound for any ground state. We optimize among groups of pieces that can contain at most 2 particles and majorize the remaining electrons contribution to get an expansion of the ground state energy per particle to the order $O(\rho^{2-\delta})$, for any $\delta \in (0, 1)$, in the thermodynamic limit. This method improves the results of F. Klopp and N. A. Veniaminov, but under a stronger assumption. It also provides a common factor for any ground state.

In the following, we express the entanglement entropy of a fixed single-distribution state after bipartition. We note that it depends only on the group of pieces where the split falls. We compute up to the order $O(\rho^{2-\delta})$ the mean of the entanglement entropy associated with the common factor mentioned above. We conjecture that, as for the fundamental energy per particle, this quantity is the main term of the one given by a whole single-distribution ground state.

Additionally, we believe that the results established on $2-$ particle systems may apply to other situations as well.

Introduction

Until the early 20th century, elementary particle systems were studied at the microscopic level through classical mechanics and at the macroscopic level through thermodynamics. However, quantum mechanics, which dates back to the 1920′s, turns out to be a more effective approach to describing these systems. Further, the modeling process should take into account the impurities of the background in which particles live. Such considerations led to the development of a branch of mathematical physics called the spectral theory of random Schrödinger operators. The discipline began in the 1950's with the work of the physicist P. Anderson who sought to understand the behavior of a single particle in a disordered medium [And58]. In the following, one should interpret the random background as being an inert (no time dependency), irregular structure. Two major examples of one-particle models are presented below.

The continuous Anderson model describes the propagation of a particle within a randomly perturbed crystal. The atoms of the crystal are distributed in an infinite lattice, say $\mathbb{Z}^d$. Each of them produces a repulsive potential whose intensity is modulated by a random variable. Using the paradigm of quantum mechanics, the electron is represented by a wave function ψ that satisfies the Schrödinger equation $i\partial_t\psi = h_\omega\psi$ where h_ω is a random Schrödinger operator given by

$$h_\omega = -\Delta_d + \sum_{i\in\mathbb{Z}^d} a_i(\omega)u(\cdot - i) \quad \text{on } L^2(\mathbb{R}^d) \tag{A}$$

with

- Δ_d the $d-$dimensional Laplacian operator,
- $(a_i)_{i\in\mathbb{Z}^d}$ a family of independent identically distributed nonnegative random variables with a bounded and compactly supported density g,
- u some nonnegative function whose conditions will be fixed later.

The Poisson model describes the path of a particle in an amorphous structure (such as glass). This time the atoms are randomly placed on $\mathbb{R}^d$ and each repulsive potential has the same intensity. Then, the electron satisfies the Schrödinger equation, as above, where h_ω is a random Schrödinger operator given by

$$h_\omega = -\Delta_d + \sum_{e\in X(\omega)} u(\cdot - e) \quad \text{on } L^2(\mathbb{R}^d) \tag{B}$$

where

- X is a Poisson process on $\mathbb{R}^d$ with intensity $\lambda > 0$, which means that
  - $(P1)$ for any borel set $\mathcal{B} \subset \mathbb{R}^d$ of volume $|\mathcal{B}|$, for any $k \in \mathbb{N}$, the probability of the event $\big\{\omega,\ \#(X(\omega)\cap \mathcal{B}\,) = k\big\}$ is $\frac{|\mathcal{B}|^k}{k!}e^{-\lambda|\mathcal{B}|}$,
  - $(P2)$ for any borel set $\mathcal{B}_1, \mathcal{B}_2 \subset \mathbb{R}^d$, the random variables $\#(X\cap\mathcal{B}_1)$ and $\#(X\cap\mathcal{B}_2)$ are independent.

Since the time evolution of the particle in both examples is given by $\psi(t,x) = e^{-ith_\omega}\psi(0,x)$, for any $t \in \mathbb{R}$ and $x \in \mathbb{R}^d$, we are interested in the spectral properties of the operator h_ω. Let us assume that u is a non-negative bounded measurable function on $\mathbb{R}^d$ with compact support and lower bounded in the neighborhood of the origin. Then, a phenomenon called spectral localization holds at the bottom of the spectrum for both models: there exists $\delta > 0$ such that, almost surely, on $[0,\delta]$, the operator h_ω has only a pure point spectrum with exponentially decaying eigenfunctions. Both quantum systems also exhibit dynamical localization on $[0,\delta]$, that one can interpret as the particle (which starts with an energy below δ) being trapped within a bounded region, at all times $t > 0$. For $d = 1$, there is spectral localization and dynamical localization for the entire spectrum. In the case of the continuous Anderson model, these results are stated by [AEN$^+$05] (and with weaker assumptions by [GK13]), whereas in the case of the Poisson model, they are provided by [GHK05]. The proofs are based on previous works dealing with discrete random operators (see e.g [Sto11],[Bou04]). Concerning the pre-requisites of functional analysis and spectral theory, we recommend the book by M. Reed and B. Simon [RS81].

This localization phenomenon should be compared with what happens in the case of a particle in an ideal solid. For the latter, the atoms occupy each vertex of an infinite lattice, say $\mathbb{Z}^d$, and they are perfectly identical. So, the corresponding Schrödinger operator h is given by

$$\text{(C)} \qquad h = -\Delta_d + \sum_{i\in\mathbb{Z}^d} u(x-i) \quad \text{on } L^2(\mathbb{R}^d).$$

The operator h has only an absolutely continuous spectrum [RS78]. By the RAGE theorem (e.g [Tes14]), it implies that the particle escapes any compact of $\mathbb{R}^d$ when t goes to $+\infty$. This illustrates that disorder, as modeled in the examples (A) and (B), changes the behaviour of a single particle.

A follow-up question is what happens for many indistinguishable particles in the same disordered medium. Such models should take into account a possible interaction between particles (see e.g [FA80]). In the paradigm of quantum physics, a many-body system is also represented by a wavefunction that satisfies a Schrödinger equation. For n particles and a one-particle operator h, our $n-$particle operator $H^U(n)$ has the following form

$$\text{(D)} \qquad H^U(n) = \sum_{i=1}^{n} \underbrace{\mathbf{1}\otimes\cdots\otimes\mathbf{1}}_{i-1 \text{ times}} \otimes h \otimes \underbrace{\mathbf{1}\otimes\cdots\otimes\mathbf{1}}_{n-i \text{ times}} + \sum_{i<j} U(x_i - x_j)$$

where U is a radial nonnegative potential which models the pairwise repulsive interaction. The nature of the particles, also called statistics, determines the domain of the $n-$particle operator. The states should be symmetric for bosons (such as photons), while they should be antisymmetric for fermions (such as electrons). For example, the eigenstates of a system of n non-interacting fermions are given by the Slater determinants

$$\text{(E)} \qquad \Psi_{\mathbf{a}}(x_1,\dots,x_n) = \Big(\bigwedge_{i=1}^{n}\psi_{a_i}\Big)(x_1,\dots,x_n) = \frac{1}{\sqrt{n!}}\det\Big(\big(\psi_{a_i}(x_j)\big)_{1\leqslant i,j\leqslant n}\Big)$$

where $(\psi_i)_{i\in I}$ are the eigenstates of the one-particle operator h, the $n-$tuple $\mathbf{a} = (a_1,\dots,a_n)$ belongs to I^n and any two coordinates of $\mathbf{a}$ are differents.

Using inductively the methods of the single particle case, one can retrieve the localization at the bottom of the spectrum, for a given number of particles and weak interactions. For instance, if U decays exponentially then there exists $\delta_n > 0$ such that the multi-particle continuum Anderson model $H^U_\omega(n)$ has a pure point spectrum on $[0,\delta_n]$ and it exhibits spectral and dynamical localization on $[0,\delta_n]$ (see [CdMS11][FW15]). V. Chulaevsky and Y. Suhov [CS08][CS09a][CS09b] and M. Aizenman and S. Warzel [AW09] previously addressed the issue of localization in the multi-particle discrete Anderson model.

However, in the context of condensed matter physics, infinite-sized models with a finite number of particles are not realistic. The amount of particles per unit of volume, also called the density of particles, should be positive. The common process is to define the quantum system of n interacting particles in a finite-sized region $L\Omega$, for $L > 0$ and Ω a bounded convex set in $\mathbb{R}^d$. Again, it is described by some operator

$$\text{(F)} \qquad H^U(L,n) = \sum_{i=1}^{n} \underbrace{\mathbf{1} \otimes \cdots \otimes \mathbf{1}}_{i-1 \text{ times}} \otimes h(L) \otimes \underbrace{\mathbf{1} \otimes \cdots \otimes \mathbf{1}}_{n-i \text{ times}} + \sum_{i<j} U(x_i - x_j)$$

with $h(L)$ the restriction of a one-particle operator on $L\Omega$ and U an even nonnegative potential. Then, we consider the spectral properties of $H^U(L,n)$ as the parameter L and the number of particles n satisfy the following limits

$$L \to +\infty \quad \text{and} \quad \frac{n}{L^d} \to \rho > 0.$$

This regime is called the thermodynamic limit. Veniaminov proved that, for a large class of disordered quantum systems, almost surely, the ground state energy per particle converges to a unique real number $\mathcal{E}^U(\rho)$ in the thermodynamic limit [Ven12]. Unfortunately, such general results are rare.

The above mathematical definitions of localization become obsolete for a positive density of particles. For condensed matter physicists, many-body localization refers to the remaining at long times of initial conditions within local observables of the quantum system. Such a phenomenon violates the eigenstate thermalization hypothesis of J. M. Deutsch [Deu91] and M. Srednicki [Sre94]. These theoretical postulates, as well as the transition between localized states at low energy and thermal states at high energy, are discussed by D. A. Huse and R. Nandkishore in [NH15]. In parallel, several experimental observations exhibit signatures of many-body localization in some disordered quantum systems (see e.g [SHB+15][yCHZ+16]). From a mathematical perspective, many-body localization is essentially an open topic.

A hint to discriminate the localized states from the thermal ones might be the estimation of the entanglement between given parts of the medium. Quantum mechanics violates Bell's inequality, named after J. S. Bell [BA04]. It implies that, in general, one cannot decompose a quantum system into independent subsystems. This phenomenon is called entanglement and it can be quantified through the entanglement entropy. For instance, even in the case of distinguisable particles, correlations may exist between the physical properties of several particles. Let $\mathcal{H}$ be an Hilbert space that admits the decomposition $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ and let ψ be a state of $\mathcal{H}$. The entanglement entropy $\mathcal{S}_A(\psi)$ is defined as the von Neumann entropy of η_A, meaning that

$$\text{(G)} \qquad \mathcal{S}_A(\psi) = -\mathrm{tr}\big(\eta_A \ln \eta_A\big).$$

where

- η_A is the reduced density matrix on the subspace $\mathcal{H}_A$ of the state ψ, given by

$$\eta_A = \mathrm{tr}_{\mathcal{H}_B} P_\psi = \sum_{i \in I} \big|\langle e_i, \psi \rangle\big|^2$$

  for P_ψ the orthogonal projector on ψ and any ONB $(e_i)_{i \in I}$ of the subspace $\mathcal{H}_B$.

Note that $\mathcal{S}_A(\psi) = 0$ if and only if ψ admits a factorization $\psi = \psi_A \otimes \psi_B$, with $\psi_A \in \mathcal{H}_A$ and $\psi_B \in \mathcal{H}_B$. One can define similar quantities from the bipartition of the background. For a bounded convex set $\Omega \subset \mathbb{R}^d$ and a parameter $L_\star > 0$, the entanglement entropy $\mathcal{S}_{L_\star}$, measures how far a state is from being the product of a state on the convex set $L_\star\Omega$ and a state on its complementary (with respect to the correct statistics).

Formally, it is given by the von Neumann entropy of $\eta_{L_\star}$, the reduced density matrix on $L_\star\Omega$ of the state Ψ, meaning that

$$\text{(H)} \qquad \mathcal{S}_{L_\star}(\Psi) = -\mathrm{tr}\big(\eta_{L_\star} \ln \eta_{L_\star}\big).$$

As the notion of reduced density matrix varies with the nature of the particles, one should adapt it from the above definition provided for distinguishable particles. Note that a single-particle state ψ is not entangled (i.e $\mathcal{S}_{L_\star}(\psi) = 0$) if and only if it is localized (in the common sense) in either $L_\star\Omega$ or its complementary.

Physics papers conjecture that, in the case of many-body disordered quantum systems with short-range interactions, the entanglement entropy on a convex set of the ground state would scale with the boundary area of that convex set rather than its volume (see e.g [NH15][Laf16][AL18]). Mathematical results on entanglement entropy are mostly derived for non-interacting particles, because one can express it using the infinite one-particle operator and its restrictions. For instance, A. Elgart and L. Pastur dealt with free fermions in a random medium described by the discrete Anderson model [EPS16]. For $0 < L_\star < L$ and Γ a bounded subset of $\mathbb{Z}^d$, they proved that, in the thermodynamic limit, the mean entanglement entropy on $L_\star\Gamma$ of the ground state $\Psi^0_\omega(L,n)$ of the operator $H^0_\omega(L,n)$ (see (F) for $U = 0$) may follow an area law, as

$$\text{(I)} \qquad \lim_{\substack{L\to+\infty \\ \frac{n}{L^d}\to\rho}} \mathbb{E}\Big(\mathcal{S}_{L_\star}\big(\Psi^0_\omega(L,n)\big)\Big) \underset{L_\star\to+\infty}{\sim} C_{\Gamma,\rho}\, L_\star^{d-1}.$$

The Fermi energy E_ρ of a system of non-interacting fermions is equal to the maximum amount of energy produced by a particle in the ground state for a density of particles $\rho > 0$. The asymptotic (I) holds if E_ρ belongs to the localized part of the spectrum of the one-particle random operator. From the previous discussion, it includes ρ small enough for $d \geqslant 2$ and any ρ for $d = 1$. Yet, area laws are also observed for the entanglement entropy of the ground state of some quasi-free models on the lattice $\mathbb{Z}^d$ (that generalize bosonic harmonic chains or fermionic spin chains for $d = 1$), provided that there is a gap beyond the ground state energy and the interactions are restricted to nearest neighbors [ARS15][ECP10]. On the other hand, D. Gioev and I. Klich [GK06] and R. Helling, H. Leschke, A. V. Sobolev and W. Spitzer [HLS10][LSS14] dealt with the free Fermi gas that describes non-interacting fermions that are not subjected to any external potential, so that the one-particle operator is just $h = -\Delta_d$ on $L^2(\mathbb{R}^d)$. For $0 < L_\star < L$ and Ω a bounded convex of $\mathbb{R}^d$ with a piece-wise $\mathcal{C}^1-$smooth boundary, they proved that, in the thermodynamic limit, the entanglement entropy on $L_\star\Omega$ of the ground state $\Psi^0(L,n)$ of the operator $H^0(L,n)$ satisfies a logarithmic enhanced area law, as

$$\text{(J)} \qquad \lim_{\substack{L\to+\infty \\ \frac{n}{L^d}\to\rho}} \mathcal{S}_{L_\star}\big(\Psi^0(L,n)\big) \underset{L_\star\to+\infty}{\sim} C_{\Omega,\rho}\, L_\star^{d-1} \ln L_\star.$$

Such a logarithmic correction is expected for the entanglement entropy of the ground state of some quasi-free models, provided that there is no gap above the ground state energy and/or the interactions are long-range [ECP10].

Our works focus on a specific one-dimensional disordered quantum system: the pieces model. The associated one-particle random operator h_ω has a quite simple structure. It acts as a Dirichlet-Laplacian on subintervals, the so-called pieces, whose extremities are given by a Poisson point process. Equivalently, one could consider a one-dimensional Poisson model, as in (B) for $d = 1$, whose potential u coming from each atom is a $\delta-$Dirac distribution of infinite intensity. This description was originally derived by J. M. Luttinger and H. K. Sy [LS73b]. Note that the spectrum of h_ω is pure point and it is given by

$$\sigma(h_\omega) = \Big\{\Big(\frac{j\pi}{l_\omega}\Big)^2,\ \exists \text{ pieces of length } l_\omega,\ j \in \mathbb{N}^\star\Big\}.$$

Since each eigenstate of h_ω lives on a single piece, localization holds trivially. J. M. Luttinger and H. K. Sy computed N, the integrated density of states of h_ω. The spectrum $\sigma(h_\omega)$ is almost surely a non-random set equal to the support of N. Using this complete knowledge of the one-particle operator, J. M. Luttinger and H. K. Sy [LS73a], and more recently O. Lenoble and V. Zagrebnov [LZ07], studied the pieces model in the case of non-interacting bosons and derived Bose-Einstein Condensation, which can occur at the macroscopic level when a large proportion of bosons occupies the ground state. Interacting bosons are also investigated (for e.g [KPS19],[KP21]).

Following the footsteps of F. Klopp and N. A. Veniaminov [KV20], we explore the case of fermions (or Fermi-Dirac statistics), we disturb the free system by adding a repulsive interaction between particles and we place ourselves in the thermodynamic limit. So, fixing the convex $\Omega = [0,1]$, for a parameter $L > 0$ and a number of particles $n \geqslant 1$, we define the random operator $H_\omega^U(L,n)$, as in (F), on $\bigwedge^n L^2(\mathbb{R})$ the space of $n-$antisymmetric square-integrable functions. We assume that only nearby particles may interact one to another, meaning that the support of the potential U coincides with a compact of length that is independent from the density of particles ρ. By previous discussions, one would like

($\star$) to describe how the ground state(s) of $H_\omega^U(L,n)$ behaves in the thermodynamic limit, in terms of the density of particles ρ;

($\star\star$) to confirm that the mean of the entanglement entropy on $[0, L_\star]$ of the ground state(s) of $H_\omega^U(L,n)$ does not increase with the length $L_\star$, in the thermodynamic limit, and to expand it in terms of the density of particles ρ.

By Pauli exclusion principle, the ground state for n non-interacting fermions is given by the Slater determinant (see (E)) of the n first eigenstates of the one-particle random operator h_ω. It yields that almost surely, in the thermodynamic limit, the pieces with length below $l_\rho = \pi E_\rho^{-1/2}$ are empty in the free ground state, where E_ρ is the Fermi energy of h_ω. F. Klopp and N. A. Veniaminov [KV20] use this argument to build an approximate state Ψ^{opt} for interacting fermions, under weaker assumptions on the potential U. Starting from the free ground state, they separated the most energy-contributing pairs of particles by moving one particle into an empty piece. They optimize between the energy produced by the interaction of two particles in the same piece and the energy given by the particles in the new pieces. The total contribution of other interactions is proven to be negligible. Their theorem states that, for a potential U in $L^p(\mathbb{R})$ $(p > 1)$, such that $x^3 \int_x^{+\infty} U(t)dt \to_{+\infty} 0$, the thermodynamic limit of the ground state per particle satisfies

$$\mathcal{E}^U(\rho) = \mathcal{E}^0(\rho) + \pi^2 \gamma_\star^U \rho \,|\ln \rho\,|^{-3}\big(1 + o(1)\big) \tag{K}$$

where $\gamma_\star^U > 0$ depends only on U and $o(1)$ refers to $\rho \to 0$. They also measure how close the approximate state is to any ground state using the distance between the respective $1-$ and $2-$ particle density matrices, namely $\gamma_\Psi^{(1)}$ and $\gamma_\Psi^{(2)}$, which are defined as the reduced density of the state Ψ on the one-particle space $L^2(\mathbb{R})$, and on the two-particle space $L^2(\mathbb{R}) \wedge L^2(\mathbb{R})$. In particular, when U is compactly supported, they get that, for ρ small enough,

$$\limsup_{\substack{L\to+\infty \\ \frac{n}{L}\to\rho}} \frac{1}{n} \big\|\gamma^{(1)}_{\Psi_\omega^U(L,n)} - \gamma^{(1)}_{\Psi^{\text{opt}}}\big\|_{\text{tr}} \leqslant C\rho\,|\ln\rho\,|^{-1}, \tag{La}$$

$$\limsup_{\substack{L\to+\infty \\ \frac{n}{L}\to\rho}} \frac{1}{n^2} \big\|\gamma^{(2)}_{\Psi_\omega^U(L,n)} - \gamma^{(2)}_{\Psi^{\text{opt}}}\big\|_{\text{tr}} \leqslant C\rho\,|\ln\rho\,|^{-1}. \tag{Lb}$$

where $\Psi_\omega^U(L,n)$ is a ground state of $H_\omega^U(L,n)$.

We will now explain our contribution to issues ($\star$) and ($\star\star$). For short-range interactions and a density of particles ρ small enough, there exists a minimal length $l_{\rho,U}$ such that pieces with length below $l_{\rho,U}$ are empty for any ground state. So, the random background reduces to a compilation of groups of pieces, that we call chains, such that a particle belonging to a chain cannot interact with a particle living outside this chain. This structure is, therefore, similar to the one of the non-interacting system if the chains replace the pieces. Expressing the ground state(s) turns into finding the minimizing distribution(s) of n particles among the chains. But, due to the nature of the Poisson process, large and/or overcrowed chains do not contribute much to the total energy. The ground state energy per particle is mostly, up to an error $O(\rho^{2-\delta})$, $\delta \in (0,1)$, given by isolated particle and isolated pairs of particles lying in one or two pieces. By induction, we build an approximate state Ψ^{test}. We refer to Theorem 1.2.1 for the expansion of the thermodynamic limit $\mathcal{E}^U(\rho)$. We state here the results on the 1− and 2−particle density matrices: for $\delta \in (0,1)$, for ρ small enough,

$$\text{(Ma)} \qquad \limsup_{\substack{L\to+\infty \\ \frac{n}{L}\to\rho}} \frac{1}{n}\left\|\gamma^{(1)}_{\Psi^U_\omega(L,n)} - \gamma^{(1)}_{\Psi^{\text{test}}}\right\|_{\text{tr}} \leqslant C\rho^{2-\delta},$$

$$\text{(Mb)} \qquad \limsup_{\substack{L\to+\infty \\ \frac{n}{L}\to\rho}} \frac{1}{n^2}\left\|\gamma^{(2)}_{\Psi^U_\omega(L,n)} - \gamma^{(2)}_{\Psi^{\text{test}}}\right\|_{\text{tr}} \leqslant C\rho^{2-\delta}.$$

where $\Psi^U_\omega(L,n)$ is a ground state of $H^U_\omega(L,n)$. So, the state Ψ^{test} approximates any ground state of $H^U_\omega(L,n)$ better than the state Ψ^{opt} of F. Klopp and N. A. Veniaminov. In fact, our induction gives a common factor between all ground states of $H^U_\omega(L,n)$.

It is unclear if one can improve the expansion of $\mathcal{E}^U(\rho)$ up to $O(\rho^{p-\delta})$ for any $p \geqslant 3$, with our method. Through the choice of some $p \geqslant 2$, we eliminate the chains that all together produce an energy of order at most $O(\rho^{p-\delta})$. Our strategy requires to prove the convexity with respect to the number of particles of the ground state energy for each remaining chain. This convexity is trivial for one or two particles, and unknown for three or more particles, because we do not have an a priori lower bound for the contribution of the interaction. Otherwise, our approximation of the ground state energy of $H^U_\omega(L,n)$, in the case $p=2$, relies on knowledge of the ground state energy of small 2−particle systems. For a given $p \geqslant 3$, one could consider the available systems one-by-one. The techniques that we use for the 2−particle systems may be applied for more complicated systems. But it seems tedious and we would prefer a common derivation.

Concerning the entanglement entropy, the idea is to define our quantities in the Fock spaces as in [HLS10]. This way, we remove both the issue of the indistinguishability of the particles, and the issue of the number of particles in each sub-interval. As discussed above, we know that the data of a minimizing distribution of n fermions among the chains is equivalent to a ground state of the operator $H^U_\omega(L,n)$. Such a state $\Psi^U_\omega(L,n)$ is a product of states lying in the chains, meaning that

$$\Psi^U_\omega(L,n) = \bigwedge_{I_\omega \text{ chain}} \psi^U_{I_\omega}.$$

If the parameter $L_\star$ falls in the gap between two chains then $\Psi^U_\omega(L,n)$ is not entangled. Otherwise, for a parameter $L_\star$ that falls in a chain $I_{\omega,L_\star}$, we express the entanglement entropy on $[0,L_\star]$ of $\Psi^U_\omega(L,n)$ by

$$\text{(N)} \qquad \mathcal{S}_{L_\star}\big(\Psi^U_\omega(L,n)\big) = \sum_{s=1}^{\kappa_{\omega,L_\star}} \text{tr}\,\big(K_s \ln K_s\big)$$

where

- $\kappa_{\omega,L_\star}$ is the number of particles in the chain $I_{\omega,L_\star}$,

– K_s is the operator of kernel

$$\mathcal{K}_s(x,y) = \binom{\kappa_\star}{s} \int_{[L_\star,L]^{\kappa_\star - s}} \psi^U_{I_{\omega,L_\star}}(x,z)\psi^U_{I_{\omega,L_\star}}(y,z)\,dz.$$

Again, due to the nature of the Poisson process, the larger the chains the less numerous they are. It confirms that, in the thermodynamic limit, the mean of the entanglement entropy on $[0,L_\star]$ of $\Psi^U_\omega(L,n)$, namely $\mathbb{E}\big(\mathcal{S}_{L_\star}\big(\Psi^U_\omega(L,n)\big)\big)$, admits an upper bound as function of the parameter $L_\star$. Moreover, it suggests that the main contribution to this expectation comes from small $1-$ or $2-$particle systems. Our following statements will be based on a full contribution to the mean of the large and/or overcrowed chains of order at most $O(\rho^{2-\delta})$, for $\delta \in (0,1)$, which we do not prove in this thesis. Using the approximate ground state Ψ^{test}, one may expand $\mathbb{E}\big(\mathcal{S}_{L_\star}\big(\Psi^U_\omega(L,n)\big)\big)$ in terms of the entanglement entropies given by $1-$ and $2-$particles systems up to $O(\rho^{2-\delta})$. As for the ground state energy per particle (see (K)), the leading term in the difference between this mean and the one of the free case is determined by the interaction between two particles in the same piece. One could get a formula for

$$\mathbb{E}\Big(\mathcal{S}_{L_\star}\big(\Psi^U_\omega(L,n)\big)\Big) - \mathbb{E}\Big(\mathcal{S}_{L_\star}\big(\Psi^0_\omega(L,n)\big)\Big) \tag{O}$$

up to $o\big(\rho\,|\ln\rho\,|^{-1}\big)$, in the thermodynamic limit.

Structure of the thesis

The manuscript is divided into three chapters. Chapter 1 is the main part. It covers the content of our article *Thermodynamic limit of the pieces model*, published in 2022 in Annales Henri Poincaré [Ogn22]. We give the description of the pieces model and our theorem on the thermodynamic limit of the ground state energy per particle. We add a section about the non-degeneracy of the ground state. Chapter 2 gathers recent works and conjectures on the entanglement entropy. Finally, Chapter 3 should be understood as a toolbox for the two others. We state the results on the ground state and ground state energy for two 2-particle systems.

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Chapter 1

Ground state of the pieces model

1.1 Model and first observations

1.1.1 The pieces model for Fermi-Dirac statistics

Let $X(w) = (x_k(w))_{k\in\mathbb{Z}}$ be a ***Poisson point process*** on $\mathbb{R}$ of intensity 1. Recall that the probability that a Borel set $\mathcal{B} \subset \mathbb{R}$ contain exactly k points is

$$\mathbb{P}\Big(\#\big(X(w) \cap \mathcal{B}\big) = k\Big) = \frac{|\mathcal{B}|^k}{k!} e^{-|\mathcal{B}|}$$

and for two disjoints Borel sets $\mathcal{B}_1, \mathcal{B}_2 \subset \mathbb{R}$, the events $\{X(w) \cap \mathcal{B}_1 = k_1\}$ and $\{X(w) \cap \mathcal{B}_2 = k_2\}$ are independent.

We will drop the "w" index. Without loss of generality, we assume that $\min\{k,\, x_k > 0\} = 1$ and we set $x_0 = 0$. We denote, for $L > 0$, m_L the random variable corresponding to the number of Poisson points in the box $[0, L]$,

$$m_L = \#\big(X \cap [0, L]\big). \tag{1.1.1}$$

By a large deviation principle, when L is large, with probability $1 - O(L^{-\infty})$, $m_L = L + O(L^{\frac{2}{3}})$. For $i \in [\![1, m_L]\!]$, the ***i-th piece*** is the interval $\Delta_i = [x_{i-1}, x_i]$.

For $L > 0$, we define the following one-particle random operator

$$h(L) = \bigoplus_{k=1}^{m_L} \left(-\frac{d^2}{dx^2}^{\,D}_{|\Delta_k} \right) \qquad \text{on } L^2\big([0, L]\big) \tag{1.1.2}$$

where $\frac{d^2}{dx^2}^{\,D}_{|\Delta}$ is the Dirichlet Laplacian on the interval Δ.

Now, we consider n particles in the disordered background given by $h(L)$ combined with a pairwise repulsive interaction. Using the statistic of Fermi-Dirac, the ***n-particle space*** on the box $[0, L]$ is

$$\mathfrak{H}^n(L) = \bigwedge_{i=1}^{n} L^2([0, L]). \tag{1.1.3}$$

Then, for $n \geqslant 2$, the ***pieces model*** is the random operator given by

$$H^U(L, n) = \sum_{i=1}^{n} \left(\bigotimes_{j=1}^{i-1} \mathbf{1}_{L^2([0,L])} \right) \otimes h(L) \otimes \left(\bigotimes_{j=1}^{n-i} \mathbf{1}_{L^2([0,L])} \right) + W_n \qquad \text{on } \mathfrak{H}^n(L) \tag{1.1.4}$$

where W_n is the multiplication operator

$$W_n(x_1, \ldots, x_n) = \sum_{i<j} U(x_i - x_j) \tag{1.1.5}$$

and $U : \mathbb{R} \longrightarrow \mathbb{R}$ satisfies the following assumption.

Assumption 1.1.1. *The function $U : \mathbb{R} \to \mathbb{R}$ is nonnegative, even, bounded and compactly supported. Let $s(U)$ be the support of the function U and*

$$\sigma_U = \sup_{x,y \in s(U)} |x - y|. \tag{1.1.6}$$

Under Assumption 1.1.1, the operator $H^U(L,n)$ is well-defined on $\mathcal{D}(L,n)$ given by

$$\mathcal{D}(L,n) = \mathcal{C}_0^\infty\left(\left(\bigcup_{k=1}^{m_L}]x_{k-1}, x_k[\right)^n\right) \cap \mathfrak{H}^n(L)$$

and it is nonnegative. Using perturbation theory (see e.g Chapter 6 [Tes14]), one proves that $H^U(L,n)$ is essentially self-adjoint on $\mathfrak{H}^n(L)$ and it has pure point spectrum. We tackle the issue of the non-degenerancy of the ground state of $H^U(L,n)$ in Section 1.3.

Notation 1.1.1. *For $L, n > 0$, $\Psi^U(L,n)$ designates some ground state of $H^U(L,n)$ and $E^U(L,n)$ the ground state energy of $H^U(L,n)$.*

Definition 1.1.1. *The limit $\left\{L \to +\infty,\ \frac{n}{L} \to \rho\right\}$ is called the **thermodynamic limit**. The constant ρ is the **density of particles** per unit of volume.*

In [KV20], Klopp and Veniaminov proved that, even under weaker assumptions on U, the thermodynamic limit of $n^{-1}E^U(L,n)$ exists $\mathbb{P}$-almost surely and in $L^1(\mathbb{P})$. In this chapter, we give an expansion of this limit.

1.1.2 The free operator

We denote by $H^0(L,n)$ the ***free operator*** and by $E^0(L,n)$ its ground state energy. One can give quite explicitly the thermodynamic limit of the ground state energy per particle

$$\mathcal{E}^0(\rho) := \lim_{\substack{L \to +\infty \\ \frac{n}{L} \to \rho}} \frac{E^0(L,n)}{n}. \tag{1.1.7}$$

The ground state energy $E^0(L,n)$ is exactly the sum of the n first eigenvalues of the one-particle operator $h(L)$. We know that the spectrum of $h(L)$ is

$$\sigma(h(L)) = \left\{\frac{j^2\pi^2}{|\Delta_i|^2},\ j \in \mathbb{N},\ i \in [\![1, m_L]\!]\right\}. \tag{1.1.8}$$

Since these eigenvalues only depend on the lengths of the pieces and the statistical distribution of these lengths is known, the one-particle operator $h(L)$ admits an explicit ***integrated density of states*** $N : \mathbb{R} \to \mathbb{R}_+$ (see Proposition 2.6 [KV20] or Proposition 3.2 [LZ07]). One computes that almost-surely, for $E \in \mathbb{R}$,

$$N(E) := \lim_{L \to \infty} \frac{\#\left\{\text{eigenvalues of } h(L) \text{ in } (-\infty, E]\right\}}{L} = \frac{e^{-\frac{\pi}{\sqrt{E}}}}{1 - e^{-\frac{\pi}{\sqrt{E}}}} 1_{E \geqslant 0}. \tag{1.1.9}$$

Note that N is a bijection on $\mathbb{R}_+$.

Definition 1.1.2. *The **Fermi energy** E_ρ of the one-particule operator $h(L)$ is the unique preimage of ρ under the integrated density of states N. We call **Fermi length** l_ρ the length of a piece Δ for which the ground state energy of the Dirichlet Laplacian $-\frac{d^2}{dx^2}^D_{|\Delta}$ is equal to the Fermi energy E_ρ. By formula (1.1.9),*

$$l_\rho = \frac{\pi}{\sqrt{E_\rho}} = -\log\left(\frac{\rho}{1+\rho}\right). \tag{1.1.10}$$

Then, one deduces

$$\mathcal{E}^0(\rho) = \frac{1}{\rho}\int_{-\infty}^{E_\rho} E\, dN(E). \tag{1.1.11}$$

where E_ρ is given by Definition 1.1.2. We refer to Theorem 5.14 [Ven12] for the proof.

1.1.3 The approach in term of occupations

Unlike the free operator, one cannot express the ground state energy of the pieces model with interactions by using the spectral decomposition of the one-particle operator. However, in both cases, one can talk about the number of particles in a given piece.

Definition 1.1.3. *An **occupation** is a multi-index $Q = (q_i)_{1\leqslant i\leqslant m_L}$ of norm equal to n. For a given occupation $Q = (q_i)_{1\leqslant i\leqslant m_L}$, the $Q-$**occupied space** $\mathfrak{H}_Q(L)$ is given by*

$$\mathfrak{H}_Q(L) = \bigwedge_{i=1}^{m_L}\Big(\bigwedge_{j=1}^{q_i} L^2(\Delta_i)\Big). \tag{1.1.12}$$

The n-particle space admits the decomposition

$$\mathfrak{H}^n(L) = \bigoplus_{Q \text{ occupation}} \mathfrak{H}_Q(L). \tag{1.1.13}$$

In [KV20], Klopp and Veniaminov proved that the decomposition by occupation (1.1.13) is invariant under the action of $H^U(L,n)$. Indeed, for any two occupations $Q \neq Q'$, the corresponding states $\Psi_Q, \Psi_{Q'}$ do not interact, meaning that

$$\big\langle \Psi_Q, W_n \Psi_{Q'}\big\rangle = 0.$$

Moreover, as explained in Section 1.3, the ground state energy of the restriction of the pieces model $H^U(L,n)$ to the $Q-$occupied space $\mathfrak{H}_Q(L)$ is simple.

Notation 1.1.2. *For a fixed occupation Q, the operator $H^U(L,n,Q)$ designates the restriction of $H^U(L,n)$ to the $Q-$occupied space $\mathfrak{H}_Q(L)$. We denote $\big(\Psi^U(L,n,Q), E^U(L,n,Q)\big)$ its ground state couple.*

By definition, for any occupation Q, the ground state $\Psi^U(L,n,Q)$ has exactly q_i particles in the piece Δ_i for all $i \in [\![1,m]\!]$. Furthermore, the ground state energy $E^U(L,n)$ of the operator $H^U(L,n)$ satisfies

$$E^U(L,n) = \min_{Q \text{ occupation}} E^U(L,n,Q). \tag{1.1.14}$$

In the free case ($U \equiv 0$), for any occupation Q, the ground state energy $E^0(L,n,Q)$ of the operator $H^0(L,n,Q)$ satisfies

$$E^0(L,n,Q) = \sum_{i=1}^{m_L} F^0(\Delta_i, q_i) \tag{1.1.15}$$

where we denote $F^0(\Delta,k)$ the ground state energy for k non-interacting fermionic particles in the piece Δ. Each particle lies in a Dirichlet Laplacian background in Δ. The ground state energy $F^0(\Delta,k)$ is the sum of the k first eigenvalues of the operator $-\frac{d^2}{dx^2}^{D}_{|\Delta}$. One notes that the map $k \to F^0(\Delta,k)$ is strictly convex on $\mathbb{N}$. So, by (1.1.14) and Lemma 1.4.8, the ground state energy $E^0(L,n)$ is given by the sum of the n smallest elements of the set

$$\Gamma^0 = \big\{F^0(\Delta_i,k+1) - F^0(\Delta_i,k),\, i \in [\![1,m_L]\!],\, k \in \mathbb{N}\big\}. \tag{1.1.16}$$

But one notes that $\Gamma^0 = \sigma(h(L))$ given by (1.1.8). Then the ***counting function of*** Γ^0,

$$\mathbf{N}^0(E) := \lim_{L\to+\infty} \frac{\#\Big(\Gamma^0 \cap (-\infty, E]\Big)}{L}, \tag{1.1.17}$$

is well-defined and it is equal to $N(E)$, where N is the integrated density of states of $h(L)$ (see 1.1.9). Thus, we recover the formula (1.1.11).

From now on, we restrict to finite-range interactions. We assume that the length σ_U (see Assumption 1.1.1) is independent of ρ. The following lemma is crucial for our analysis.

Lemma 1.1.1. *Let $\Psi^U(L,n)$ be a ground state of $H^U(L,n)$. For n and L large enough, with respect to the thermodynamic limit (see Definition 1.1.1) and with probability $1 - O(L^{-\infty})$, there exists a minimal length $l_{\rho,U} = -\log\left(\frac{\rho}{1+\rho}\right) - (4\sigma_U + 6)\rho$ such that*

if a piece Δ_i satisfies $|\Delta_i| < kl_{\rho,U}$, $k \in \mathbb{N}$, then, for every occupation Q,

$$\Big(P_Q\Psi^U(L,n) \neq 0\Big) \Rightarrow \Big(q_i \leqslant k-1\Big)$$

where P_Q is the orthogonal projector on $\mathfrak{H}_Q(L)$.

So, given a piece, the number of particles in this piece is bounded uniformly for any ground state. In particular, the pieces of length up to $l_{\rho,U}$ are empty for any ground state.

We will use the term ***chain*** to refer to a group of pieces of length greater than $l_{\rho,U}$ with gaps of length smaller than σ_U. Let $\mathcal{P}$ be the set of chains. Using the notations of Lemma 1.1.1, for any occupation Q such that $P_Q\Psi^U(L,n) \neq 0$, the ground state energy of $H^U(L,n,Q)$ satisfies

$$E^U(L,n,Q) = \sum_{I\in\mathcal{P}} F^U\big(I, \kappa_I(Q)\big) \tag{1.1.18}$$

where $\kappa_I(Q)$ is the number of particles in the chain I and $F^U(I,\kappa)$ is the smallest energy produced by κ particles in the chain I. Each particle lies in a Dirichlet Laplacian background for some piece of I and it is eventually submitted to the repulsive pairwise interaction U.

One should think of Equation (1.1.18) as a counterpart to Equation (1.1.15) where each chain stands for an occupied piece in the free case. If one could prove the convexity of every map $\kappa \to F^U(I,\kappa)$ then by Lemma 1.4.8, the ground state energy $E^U(L,n)$ would be given by the sum of the n smallest elements of the set $\Gamma = \{F^U(I,\kappa+1) - F^U(I,\kappa),\ I \text{ chain},\ \kappa \in \mathbb{N}\}$. Thus, we introduce, for $\kappa \geqslant 0$, the $(\kappa+1)$***-th energy level of the chain*** I given by

$$f^U(I,\kappa+1) = F^U(I,\kappa+1) - F^U(I,\kappa). \tag{1.1.19}$$

It represents the smallest amount of energy that appears if one adds a particle to a minimizing configuration of κ particles in I.

From the above discussion, one would like to use that, for every chain, $\kappa \to f^U(I,\kappa)$ is increasing. It is unclear in which cases such statement is true. Take for example three pieces in a row $I = \{\Delta_1, \Delta_2, \Delta_3\}$ of lengths $\mathbf{l}_2 = l$ and $\mathbf{l}_1 = \mathbf{l}_3 = al$ with $a \in (1,2)$. One can check that, in the free case ($U \equiv 0$), for 3 particles the best occupation is $(1,1,1)$, for 4 particles it is $(2,1,1)$ or $(1,1,2)$, and for 5 particles it is $(2,1,2)$. So

$$f^0(I,5) = f^0(I,4) = F^0(\Delta_1,2) - F^0(\Delta_1,1).$$

In this pathological case, only the difference between the interaction inputs will determine the sign of $f^U(I,5) - f^U(I,4)$. The upper bounds of the interaction of two particles (see Lemma 1.4.10) are not

efficient to conclude on this sign. Of course, such symetric cases have null probability in our model. But, for non-negligeable probability, we have a triplet of pieces of lengths l_1, l_2, l_3 such that $l_2 < l_3 \leqslant l_1 < 2l_2$ and $l_1 - l_3$ is small. For such a chain, the difference between the kinetic inputs, namely $f^0(I,5) - f^0(I,4)$, is also small. The occupation given by the free case may no longer be the best one in case of interactions. Even so, if the kinetic difference $f^0(I,5) - f^0(I,4)$ is smaller than our upper bounds of the interaction inputs, then we cannot tackle the issue of the sign of $f^U(I,5) - f^U(I,4)$.

Yet it seems relevant to search for results with the assumption of monotony for small chains and/or for few particles. More precisely, let $p \geqslant 2$ and $\mathcal{P}_p$ be the set of chains each of which carries at most p particles for any ground state, and Γ_p be the set of the p lowest energy levels of every chain that belongs to $\mathcal{P}_p$, meaning that

$$\Gamma_p = \Big\{ f^U(I,\kappa),\ I \in \mathcal{P}_p,\ \kappa \leqslant p \Big\}. \tag{1.1.20}$$

Assume that

$$\forall I \in \mathcal{P}_p, \quad \forall \kappa \leqslant p-1, \qquad f^U(I,\kappa) < f^U(I,\kappa+1). \tag{1.1.21}$$

Set $\delta \in (0,1)$. By Lemma 1.1.1 and by statistical distribution of the pieces (see Proposition 1.4.9), one proves that, for any ground state, the number of particles in ${}^c\mathcal{P}_p$, the complement of $\mathcal{P}_p$, is of order $O(n\rho^{p-\delta})$. One also controls the contribution of these particles to the ground state energy with a bound of order $O(n\rho^{p-\delta})$. Then, up to an error $O(n\rho^{p-\delta})$, the ground state energy $E^U(L,n)$ is given by the sum of the n smallest elements of Γ_p. Let $\mathbf{N}_p^U$ be the ***counting function of*** Γ_p, meaning that

$$\mathbf{N}_p^U(\lambda) := \lim_{L \to +\infty} \frac{\#\Big(\Gamma_p \cap (-\infty, \lambda]\Big)}{L}. \tag{1.1.22}$$

Using $\mathbf{N}_p^U$ as a counterpart to $\mathbf{N}^0$ (see (1.1.17)), one should get an approximation of the thermodynamic limit of the ground state energy per particle $\mathcal{E}^U(\rho)$ up to an error $O(\rho^{p-\delta})$.

1.2 Main Results

Since the interaction is repulsive, Assumption (1.1.21) is always true for $p = 2$. Following the above discussion, we study this case in depth. In the set $\mathcal{P}_2$, a chain is either a single piece with at most two particles, or a pair of pieces with at most one particle in each piece. These 2-particle systems are studied in more depth in Chapter 3. We state below the results on the first energy levels.

For two particles in a single piece, Klopp and Veniaminov proved the following result.

Proposition 1.2.1. *[KV20] Under Assumption 1.1.1, for $l > 0$, consider the operator $h^U\big([0,l],2\big)$ given by*

$$h^U\big([0,l],2\big) = \left(-\frac{d^2}{dy^2}^{\,D}_{|[0,l]} \right) \otimes \mathbf{1}_{L^2([0,l])} + \mathbf{1}_{L^2([0,l])} \otimes \left(-\frac{d^2}{dx^2}^{\,D}_{|[0,l]} \right) + U(x-y) \tag{1.2.1}$$

on $L^2\big([0,l]\big) \wedge L^2\big([0,l]\big)$.

Then, for large l, the ground state energy $e^U\big([0,l],2\big)$ of the operator $h^U\big([0,l],2\big)$ admits the following expansion

$$e^U\big([0,l],2\big) = \frac{5\pi^2}{l^2} + \frac{\gamma}{l^3} + o(l^{-3}) \tag{1.2.2}$$

with $\gamma > 0$ when $U \neq 0$. The first and second energy levels of the interval $[0, l]$ are given by

$$f^U([0,l],1) = \frac{\pi^2}{l^2} \quad \text{and} \quad f^U([0,l],2) = e^U([0,l],2) - \frac{\pi^2}{l^2}. \tag{1.2.3}$$

For two particles in a pair of pieces, we prove the following result.

Proposition 1.2.2. *Under Assumption 1.1.1, for $l > 0$, $d \geqslant 0$ and $a > 1$, consider the operator $h^U\big(([-al,0],[d,d+l]),(1,1)\big)$ given by*

$$\begin{aligned} h^U\Big(([-al,0],[d,d+l]),(1,1)\Big) = &\Big(-\frac{d^2}{dy^2}^{D}_{|[-al,0]}\Big) \otimes \mathbf{1}_{L^2([d,d+l])} + \mathbf{1}_{L^2([-al,0])} \otimes \Big(-\frac{d^2}{dx^2}^{D}_{|[d,d+l]}\Big) \\ &+ U(x-y) \qquad \text{on } L^2\big([-al,0]\big) \wedge L^2\big([d,d+l]\big). \end{aligned} \tag{1.2.4}$$

Then, for $d \geqslant 0$ and large $l > 0$, the ground state energy $e^U\big(([-al,0],[d,d+l]),(1,1)\big)$ of the operator $h^U\big(([-al,0],[d,d+l]),(1,1)\big)$ admits the following expansion

$$e^U\big(([-al,0],[d,d+l]),(1,1)\big) = \Big(\frac{\pi^2}{l^2} + \frac{\pi^2}{(al)^2}\Big) + \frac{\tau(d)}{a^3 l^6}\Big(1 + o(1)\Big) \tag{1.2.5}$$

with $\tau(d)$ a positive function that vanishes for $d > \sigma_U$. The first and second energy levels of the pair $\big([-al,0],[d,d+l]\big)$ are given by

$$\begin{aligned} f^U\big(([-al,0],[d,d+l]),1\big) &= \frac{\pi^2}{(al)^2} \\ \text{and} \quad f^U\big(([-al,0],[d,d+l]),2\big) &= e^U\big(([-al,0],[d,d+l]),(1,1)\big) - \frac{\pi^2}{(al)^2}. \end{aligned} \tag{1.2.6}$$

We now state our theorem. Consider, on $(0, +\infty)$, the map

$$\begin{aligned} \mathcal{J}(\lambda) = \big(1 - \sigma_U e^{-l_{\rho,U}}\big)^2 \Bigg(&\int_{\mathcal{D}_1(\lambda)} f^U([0,u],1) e^{-u}\, du + \int_{\mathcal{D}_2(\lambda)} f^U([0,u],2) e^{-u}\, du \\ &+ \int_0^{\sigma_U} \Big(\int_{\mathcal{D}_3(\lambda)} 2e^{-(u+v)} f^U\big(([-u,0],[t,v+t]),1\big)\, dudv \Big) dt \\ &+ \int_0^{\sigma_U} \Big(\int_{\mathcal{D}_4(\lambda,t)} 2e^{-(u+v)} f^U\big(([-u,0],[t,v+t]),2\big)\, dudv \Big) dt \Bigg) \end{aligned} \tag{1.2.7}$$

where $f^U(I,1)$ (resp. $f^U(I,2)$) is the first (resp. second) energy level of the chain I (see (1.2.3) and (1.2.6)),

$$\begin{aligned} \mathcal{D}_1(\lambda) &= \Big[\frac{\pi}{\sqrt{\lambda}}, 3l_{\rho,U}\Big], & \mathcal{D}_3(\lambda) &= \Big\{(x,y) \in \big[l_{\rho,U}, 2l_{\rho,U}\big]^2, y \geqslant \max\Big(x, \frac{\pi}{\sqrt{\lambda}}\Big)\Big\} \\ \mathcal{D}_2(\lambda) &= \Big[\frac{2\pi}{\sqrt{\lambda}} + \frac{\gamma}{8\pi^2}, 3l_{\rho,U}\Big], & \mathcal{D}_4(\lambda,t) &= \Big\{(x,y) \in \big[l_{\rho,U}, 2l_{\rho,U}\big]^2,\ y \geqslant x \geqslant \Big(\frac{\pi}{\sqrt{\lambda}} + \frac{\tau(t)}{2y^3}\Big)\Big\}. \end{aligned}$$

and γ (resp. $\tau(t)$) is given in Proposition 1.2.1 (resp. Proposition 1.2.2). These domains of integration come from the distribution of the first and second energy levels below λ for the chains in $\mathcal{P}_2$.

Theorem 1.2.1. *Under Assumption 1.1.1, let $l_{\rho,U} = -\log\big(\frac{\rho}{1+\rho}\big) - (4\sigma_U + 6)\rho$ be the minimal length from Lemma 1.1.1.*

For $\delta \in (0,1)$, there exists $\rho_\delta > 0$ such that for every $\rho \in (0,\rho_\delta)$ there is a Fermi energy level λ_ρ, depending only on ρ and U, such that, with probability $1-O(L^{-\infty})$, the thermodynamic limit of the ground state energy per particle satisfies

$$\mathcal{E}^U(\rho) := \lim_{\substack{L\to+\infty \\ \frac{n}{L}\to\rho}} \frac{E^U(L,n)}{n} = \frac{1}{\rho}\mathcal{J}(\lambda_\rho) + O(\rho^{2-\delta}). \tag{1.2.8}$$

where $\mathcal{J}$ is defined in (1.2.7).

Our method also provides a description of the ground state itself. Recall that, in any chain of $\mathcal{P}_2$, there is at most two particles. They are either in the same piece (Proposition 1.2.1) either in two distinct pieces (Proposition 1.2.2). From $\lambda_\rho > 0$ a Fermi energy level given by Theorem 1.2.1, we build an occupation Q^{test} such that the particles in $\mathcal{P}_2$ fill all the energy levels below λ_ρ. In other word, if one adds a particle into a chain of $\mathcal{P}_2$, then the corresponding amount of energy this addition produces would be above λ_ρ. More precisely, we set

(i) for a single piece $\Delta_i \in \mathcal{P}_2$,
$$q_i^{\text{test}} = \max\left\{q,\ f^U(\Delta_i, q) \leqslant \lambda_\rho\right\};$$

(ii) for a pair $(\Delta_j, \Delta_k) \in \mathcal{P}_2$, assuming $|\Delta_j| \leqslant |\Delta_k|$,
$$q_j^{\text{test}} = \max\Big(0,\ \max\Big\{q,\ f^U\big((\Delta_j,\Delta_k),q\big) \leqslant \lambda_\rho\Big\} - 1\Big),$$
$$q_k^{\text{test}} = \min\Big(1,\ \max\Big\{q,\ f^U\big((\Delta_j,\Delta_k),q\big) \leqslant \lambda_\rho\Big\}\Big).$$

We prove that one can complete Q^{test} on ${}^c\mathcal{P}_2$ with respect to the condition on the number of particles in a piece, given by Lemma 1.1.1. Then, set the following state

$$\Psi^{\text{test}}(L,n) = \Big(\bigwedge_{I\in\mathcal{P}_2} \psi^U\Big(I, (q_i^{\text{test}})_{i\in I}\Big)\Big) \wedge \Big(\bigwedge_{I\in{}^c\mathcal{P}_2}\bigwedge_{i\in I} \psi^0\Big(\Delta_i, q_i^{\text{test}}\Big)\Big) \tag{1.2.9}$$

where

(i) $\psi^U\big(I,(q_i)_{i\in I}\big)$ is the ground state for the interacting system with exactly q_i particles in Δ_i;

(ii) $\psi^0(\Delta,q)$ is the ground state for q non-interacting particles in Δ, given by the Slater determinant of the q first eigenfunctions of the operator $-\frac{d^2}{dx^2}^{D}_{|\Delta}$.

Remark 1.2.1. *In Subsection 1.4.2.3, we prove that the Fermi energy level λ_ρ given by Theorem 1.2.1 is greater than the Fermi energy E_ρ of the operator $h(L)$ (see Definition 1.1.2). It implies that, in the state $\Psi^{test}(L,n)$, some single pieces of length below the Fermi length l_ρ host one particle each, which used to be, in the free case, either in single pieces of length greater than $2l_\rho$, either in pairs of pieces of length greater than l_ρ.*

We compare the state $\Psi^{\text{test}}(L,n)$ to any ground state $\Psi^U(L,n)$ through the one- and two- particle density matrices, using trace norm $\|\ \|_{\text{tr}}$.

Definition 1.2.1. *For $\phi \in \mathfrak{H}^n(L)$, its* 1-***particle density matrix*** *is the operator $\gamma_\phi^{(1)}$ on $\mathfrak{H}^1(L) = L^2([0,L])$ with kernel*

$$\gamma_\phi^{(1)}(x,y) = n\int_{[0,L]^{n-1}} \phi(x,Z)\overline{\phi(y,Z)}dZ. \tag{1.2.10}$$

The 2-***particle density matrix*** *of* ϕ *is the operator* $\gamma_\phi^{(2)}$ *on* $\mathfrak{H}^2(L)$ *with kernel*

$$\gamma_\phi^{(2)}(x_1,x_2,y_1,y_2) = \frac{n(n-1)}{2}\int_{[0,L]^{n-2}} \phi(x_1,x_2,Z)\overline{\phi(y_1,y_2,Z)}dZ. \tag{1.2.11}$$

Proposition 1.2.3. *Let* $\Psi^U(L,n)$ *be a ground state of* $H^U(L,n)$*. For* $\delta \in (0,1)$*,* $\rho \in (0,\rho_\delta)$*, set the state* $\Psi^{test}(L,n)$ *according to the above construction. Then, in the thermodynamic limit, with probability* $1-O(L^{-\infty})$*, one has*

$$\frac{1}{n}\left\|\gamma^{(1)}_{\Psi^U(L,n)} - \gamma^{(1)}_{\Psi^{test}(L,n)}\right\|_{tr} \leqslant 10\rho^{2-\delta}. \tag{1.2.12}$$

We get an analogue of Proposition 1.2.3 for the 2-particle density matrix.

Proposition 1.2.4. *Let* $\Psi^U(L,n)$ *be a ground state of* $H^U(L,n)$*. For* $\delta \in (0,1)$ *and* $\rho \in (0,\rho_\delta)$*, set the state* $\Psi^{test}(L,n)$ *as above. Then, in the thermodynamic limit, with probability* $1-O(L^{-\infty})$*, one has*

$$\frac{1}{n^2}\left\|\gamma^{(2)}_{\Psi^U(L,n)} - \gamma^{(2)}_{\Psi^{test}(L,n)}\right\|_{tr} \leqslant 45\rho^{2-\delta}. \tag{1.2.13}$$

Remark 1.2.2. *The bounds given in Proposition 1.2.3 and Proposition 1.2.4 are better than the ones given in [KV20].*

1.3 Non-degeneracy

In this Section, we define the pieces model a bit differently. Indeed, it can be described as an operator-valued random variable that depends on the random vector given by the length of the pieces. More precisely, let X be a Poisson point process on $\mathbb{R}$ of intensity 1. If $\#(X \cap [0,L]) = \mathbf{m}$, we denote $\Lambda = (\mathbf{l}_k)_{1\leqslant k\leqslant \mathbf{m}}$ the random vector with values in $(\mathbb{R}_+^\star)^{\mathbf{m}}$ and $\mathbf{l}_k$ the length of the $k-$th piece. The probability law of Λ is absolutely continuous with respect to Lebesgue measure on $\mathbb{R}^{\mathbf{m}}$ [GMS83]. Let $H^U(n,\Lambda)$ be the pieces model, meaning that

$$H^U(n,\Lambda) = \sum_{i=1}^{n}\left(\bigotimes_{j=1}^{i-1}\mathbf{1}_{L^2([0,\mathbf{x_m}])}\right)\otimes h(\mathbf{x_m})\otimes\left(\bigotimes_{j=1}^{n-i}\mathbf{1}_{L^2([0,\mathbf{x_m}])}\right) + W_n \qquad \text{on } \mathfrak{H}^n(\mathbf{x_m}) \tag{1.3.1}$$

where $\mathbf{x}_m = \mathbf{l}_1 + \cdots + \mathbf{l_m}$. We denote $E^U(n,\Lambda)$ the ground state energy of $H^U(n,\Lambda)$.

1.3.1 Simplicity for a fixed occupation

We recall that the operator $H^U(n,\Lambda)$ preserves the decomposition by occupation of the $n-$particle space $\mathfrak{H}^n(\mathbf{x_m})$. For a fixed occupation Q, let $H^U(n,\Lambda,Q)$ be the restriction of $H^U(n,\Lambda)$ to the subspace $\mathfrak{H}_Q(\mathbf{x_m})$.

Proposition 1.3.1. *[KV20] For any occupation* Q*, the ground state* $\Psi^U(n,\Lambda,Q)$ *of* $H^U(n,\Lambda,Q)$ *is non-degenerate.*

Proof. We refer to Lemma 3.6 of [KV20] for the details. But it seems worthy to give the method. They prove that $H^U(n,\Lambda,Q)$ is equivalent to the Schrödinger operator $-\Delta + W_n$ with Dirichlet boundary conditions on an open connected subset of $\mathbb{R}^n$. Then, they apply a enhanced version of Perron Frobenius theorem (see XIII.43 and XIII.44 of [RS78]). □

Remark 1.3.1. *Since the 2-particle systems of Chapter 3 can be interpreted as special cases of* $H^U(n,\Lambda,Q)$*, their respective ground states are non -degenerate.*

1.3.2 Simplicity for a truncated analytic potential

Klopp and Veniaminov proved that, if U is real analytic, then, almost-surely, the ground state of the operator $H^U(L,n)$ is non-degenerate. It is unclear that the non-degenerancy of the ground state remains true for any potential U satisfying Assumption 1.1.1. Since the holomorphy is not compatible with the short-range condition, we will consider a truncated analytic function and we try to adapt the proof of Theorem 1.1 of [KV20]. So, in this subsection, we assume that

Assumption 1.3.1. *The function U satisfies for any $x \in \mathbb{R}$, $U(x) = u(x)\mathbf{1}_{[-\delta,\delta]}(x)$ for some even analytic function u.*

Lemma 1.3.1. *Assume that $\#(X \cap [0,L]) = \boldsymbol{m}$. Let $\Lambda_\star = \big((l_\star)_i\big)_{1\leqslant i\leqslant \boldsymbol{m}} \in (R_+^\star)^{\boldsymbol{m}}$ be the lengths given by an outcome of the Poisson process X. For $1 \leqslant \iota \leqslant \boldsymbol{m}$, we set*

$$\mathcal{U}_\iota = \Big\{\Lambda \in (\mathbb{R}_+^\star)^{\boldsymbol{m}},\ \boldsymbol{l}_\iota > l_{\rho,U},\ \forall i \neq \iota \quad \boldsymbol{l}_i = (l_\star)_i\Big\} \tag{1.3.2}$$

where $l_{\rho,U}$ is given by Lemma 1.1.1. Let Q be an occupation with coordinates equal to zero for the pieces of length less than $l_{\rho,U}$. We denote by $E^U(n,\Lambda,Q)$ the ground state energy of the operator $H^U(n,\Lambda,Q)$.

Then, under Assumption 1.3.1, the function $\Lambda \mapsto E^U(n,\Lambda,Q)$ admits an holomorphic extension $\widetilde{E}^U(n,\cdot\,,Q)$ on $\widetilde{\mathcal{U}}_\iota$ given by

$$\widetilde{\mathcal{U}}_\iota = \Big\{\Lambda \in \mathbb{C}^{\boldsymbol{m}},\ |\boldsymbol{l}_\iota| > l_{\rho,U},\ \forall i \neq \iota \quad \boldsymbol{l}_i = (\boldsymbol{l}_\star)_i\Big\}. \tag{1.3.3}$$

The method of the proof is to write the interaction on the basis of the eigenvalues of the free operator. We get a linear combination of integrals on domains given by Figure 3.1. Thanks to the shape of these domains and the regularity of u, each integral admit an analytic extension. It provides an analytic extension for the operator. We refer to the Appendix section for the details.

Remark 1.3.2. *By Hartogs theorem, Lemma 1.3.1 also states that $E^U(n,\Lambda,Q)$ is analytic on $\big\{\Lambda \in (\mathbb{R}_+^\star)^{\boldsymbol{m}},\ \boldsymbol{l}_i > l_{\rho,U}$ if $q_i \neq 0$, and $\boldsymbol{l}_i$ constant else$\big\}$.*

We can now give our result.

Proposition 1.3.2. *Under Assumption 1.3.1 and in the thermodynamic limit, almost surely, the ground state of the pieces model $H^U(L,n)$ is non degenerate.*

Proof. The method is based on the proof of Theorem 1.1 of [KV20].

By reducio ad absurdum, we assume that

$$\mathbb{P}\big(E^U(L,n) \text{ is of multiplicity} \geqslant 2\big) > 0 \tag{1.3.4}$$

It implies that, for some integer $m \geqslant 1$,

$$\mathbb{P}\Big(E^U(L,n) \text{ is of multiplicity} \geqslant 2 \,\Big|\, \#\big(X \cap [0,L]\big) = m\Big) > 0. \tag{1.3.5}$$

From now on, we place ourselves in the case of such an integer $\mathbf{m}$ and we use the description in terms of lengths of the pieces. For $\Lambda \in (\mathbb{R}_+^\star)^{\mathbf{m}}$, let $E^U(n,\Lambda)$ be the ground state energy of the operator $H^U(n,\Lambda)$ given by (1.3.1).

The decomposition by occupation of the $n-$particle space $\mathfrak{H}^n(\mathbf{x_m})$ gives a characterization of the degenerancy of $E^U(n,\Lambda)$.

Remark 1.3.3. *For $\Lambda \in (\mathbb{R}_+^\star)^{\boldsymbol{m}}$, the followings are equivalent:*

(i) the ground state of the operator $H^U(n,\Lambda)$ is degenerate;

(ii) there exists at least two differents occupations Q and Q' such that the corresponding ground state energies are equal to the ground state energy of $H^U(n,\Lambda)$.

Yet, Lemma 1.1.1 gives a restriction on the available occupations.

Remark 1.3.4. *If the occupation Q satisfies*

$$E^U(n,\Lambda,Q) = E^U(n,\Lambda)$$

then

$$\Big(q_i \neq 0\Big) \Rightarrow \Big(\boldsymbol{l}_i > l_{\rho,U}\Big).$$

Let Q, Q' be two differents occupations and ι be a piece such that $q_\iota \neq q'_\iota$. We consider the open $\mathcal{O}_{Q,Q'}$ given by

$$\mathcal{O}_{Q,Q'} = \Big\{\Lambda \in (\mathbb{R}_+^\star)^{\mathbf{m}},\ (q_i \neq 0) \vee (q'_i \neq 0) \quad \Rightarrow \quad (\mathbf{l}_i > l_{\rho,U})\Big\}. \tag{1.3.6}$$

We assume that there exists an open set $\mathcal{U} \subset \mathcal{O}_{Q,Q'}$ such that for any $\Lambda \in \mathcal{U}$, the evaluations in Λ of the ground state energies $E^U(n,\Lambda,Q)$ and $E^U(n,\Lambda,Q')$ are equal almost surely. By Fubini, there exists $\Lambda_\star \in \mathcal{U}$ and

$$\mathcal{U}_\iota = \Big\{\Lambda \in (\mathbb{R}_+^\star)^{\mathbf{m}},\ \mathbf{l}_\iota > l_{\rho,U},\ \forall i \neq \iota \quad \mathbf{l}_i = (\mathbf{l}_\star)_i\Big\} \tag{1.3.7}$$

such that almost surely in $\Lambda \in \mathcal{U}_\iota \cap \mathcal{U}$,

$$E^U(n,\Lambda,Q) = E^U(n,\Lambda,Q'). \tag{1.3.8}$$

Using the notations of Lemma 1.3.1, we consider

$$F : \begin{cases} \mathcal{U}_\iota & \longrightarrow & \mathbb{C} \\ \Lambda & \longmapsto & \widetilde{E}^U(n,\Lambda,Q) \end{cases} \qquad \text{and} \qquad G : \begin{cases} \mathcal{U}_\iota & \longrightarrow & \mathbb{C} \\ \Lambda & \longmapsto & \widetilde{E}^U(n,\Lambda,Q') \end{cases}. \tag{1.3.9}$$

We know that,

(i) $F(\Lambda) = G(\Lambda)$ almost surely in $\Lambda \in \mathcal{U}_\iota \cap \mathcal{U}$ by (1.3.8);

(ii) F and G are analytic on $\mathcal{U}_\iota$ by Lemma 1.3.1.

So,

$$F(\Lambda) = G(\Lambda) \qquad \forall \Lambda \in \mathcal{U}_\iota. \tag{1.3.10}$$

On the other hand, one computes

$$\frac{\partial F}{\partial \mathbf{l}_\iota}(\Lambda) \sim_{\mathbf{l}_\iota \to \infty} C\left(\frac{q_\iota}{\mathbf{l}_\iota}\right)^3 \qquad \text{and} \qquad \frac{\partial G}{\partial \mathbf{l}_\iota}(\Lambda) \sim_{\mathbf{l}_\iota \to \infty} C\left(\frac{q'_\iota}{\mathbf{l}_\iota}\right)^3 \tag{1.3.11}$$

This contradicts (1.3.10).

Thus, for any Q, Q' two differents occupations, almost surely on $\mathcal{O}_{Q,Q'}$, $E^U(n,\Lambda,Q) \neq E^U(n,\Lambda,Q')$. Combining Remark 1.3.3 and Remark 1.3.4, we get that almost surely the ground state energy $E^U(n,\Lambda)$ is of multiplicity 1. This contradicts (1.3.5).

This concludes the proof of Proposition 1.3.2.

□

1.4 Proofs

1.4.1 Occupations and ground state energies

1.4.1.1 Necessary conditions on the occupations providing ground states

We recall that the Fermi energy and the Fermi length l_ρ (see Definition 1.1.2) satisfy

$$N(E_\rho) = \rho \qquad \text{and} \qquad E_\rho = \frac{\pi^2}{l_\rho^2} \tag{1.4.1}$$

with $N(E)$ the integrated density of states of $h(L)$ (see (1.1.9)). By formula (1.1.11), in the thermodynamic limit, the ground state energy $E^0(L,n)$ is the sum of the eigenvalues of the one-particle operator $h(L)$ that are smaller than E_ρ. But the spectrum of $h(L)$ is

$$\sigma(h(L)) = \Big\{ \frac{j^2\pi^2}{|\Delta_i|^2},\, j \in \mathbb{N},\, i \in [\![1, m_L]\!] \Big\}. \tag{1.4.2}$$

Since,

$$\frac{j^2\pi^2}{|\Delta_i|^2} \leqslant E_\rho \iff j \leqslant \frac{\Delta_i}{l_\rho} \tag{1.4.3}$$

we get that no piece of a length strictly below kl_ρ can carry more than $k-1$ particles in the ground state of the free operator $H^0(L,n)$. Due to Assumption 1.1.1 of finite-range interactions, in the case of the full operator $H^U(L,n)$, we exhibit the same phenomenon for some minimal length $l_{\rho,U} < l_\rho$. The following lemma is a reformulation of Lemma 1.1.1.

Lemma 1.4.1. *Let $\Psi^U(L,n)$ be a ground state of $H^U(L,n)$. For n and L large enough, with respect to the thermodynamic limit (see Definition 1.1.1) and with probability $1 - O(L^{-\infty})$, there exists a minimal length $l_{\rho,U} = l_\rho - (4\sigma_U + 6)\rho$ such that*

if a piece Δ_i satisfies $|\Delta_i| < kl_{\rho,U}$, $k \in \mathbb{N}$, then, for every occupation Q,

$$\Big(P_Q \Psi^U(L,n) \neq 0 \Big) \Rightarrow \Big(q_i \leqslant k-1 \Big)$$

where P_Q is the orthogonal projector on $\mathfrak{H}_Q$.

Then, any ground state of $H^U(L,n)$ belongs to $\bigoplus_{Q\in\mathcal{Q}} \mathfrak{H}_Q(L)$ where $\mathfrak{H}_Q(L)$ is given in (1.1.13) and

$$\mathcal{Q} = \Big\{ (q_i) \in \mathbb{N}^m,\, \sum_{i=1}^m q_i = n \text{ and for } 1 \leqslant i \leqslant m \quad q_i \leqslant \Big\lfloor \frac{|\Delta_i|}{l_{\rho,U}} \Big\rfloor \Big\}. \tag{1.4.4}$$

This is a slight improvement of Lemma 3.25 of [KV20]. We use the same method of proof.

Proof. Let $t > 0$ be a parameter that will be set later in order to get our contradiction. For the time being, $l_{\rho,U}$ is of the form $l_\rho - t\rho$.

Assume that Δ^e is the smallest piece that does not satisfy the property of the lemma. Then, there exists $k \in \mathbb{N}$ so that $(k-1)l_{\rho,U} \leqslant |\Delta^e| < kl_{\rho,U}$ and Q^e an occupation so that Δ^e is occupied by $j = k-1+e$ particles in $P_{Q^e}\Psi^U(L,n)$ with $e \geqslant 1$. Without loss of generality, we assume that $\Psi^U(L,n) = P_{Q^e}\Psi^U(L,n)$. Our goal is to prove that one can define a state $\Phi^U(L,n)$ such that

$$\langle \Phi^U(L,n), H^U(L,n)\Phi^U(L,n)\rangle < \langle \Psi^U(L,n), H^U(L,n)\Psi^U(L,n)\rangle$$

which would contradict the statement "$\Psi^U(L,n)$ is a ground state ".

By hypothesis, there are at most $n-j+1$ pieces with some particle in the state $\Psi^U(L,n)$. The idea is to find at least $n+1$ pieces longer than $l_{\rho,U}$ that are far enough of the others pieces longer than $l_{\rho,U}$. If so, one could move the e extra particles into e of the $n+1-(n-j+1)=j$ empty such pieces without creating any interaction.

We call *interaction range* of a piece Δ the set of pieces Δ' such as the distance between Δ and Δ' is less than or equal to σ_U. Note that if $\mathbf{d}(\Delta,\Delta')>2\sigma_U$ then the intersection of the interaction range of Δ and the one of Δ' is empty. This property means that no particle can interact with some particles of both pieces Δ and Δ'. We will estimate the number of *pieces of type* $(\star)$ that satisfy the two following conditions

(i) a length between $l_{\rho,U}$ and $2l_{\rho,U}$;

(ii) no other piece of length greater than $l_{\rho,U}$ at a distance less than $2\sigma_U+1$.

For the count, we use Proposition 1.4.9 and Proposition 1.4.10. Let $\eta\in(\frac{2}{3},1)$. Since $l_\rho=-\log\left(\frac{\rho}{1+\rho}\right)$ (see (1.1.2)) and, in the thermodynamic limit, $L^{\beta-1}=o(\rho^2)$, we compute, with probability $1-O(L^{-\infty})$,

$$\begin{aligned}\#\big\{\Delta,\, l_{\rho,U}<|\Delta|<2l_{\rho,U}\big\} &= Le^{-(l_\rho-t\rho)}(1-e^{-(l_\rho-t\rho)})+O(L^\beta)\\ &= L\rho\Big(\big(1+(t-1)\rho+o(\rho)\big)\big(1-\rho+o(\rho)\big)+\,o(\rho)\Big)\\ &= L\rho\big(1+(t-1)\rho+o(\rho)\big)\end{aligned}\tag{1.4.5}$$

and, similarly,

$$\begin{aligned}\#\big\{(\Delta,\Delta'),\, |\Delta|>l_{\rho,U},\, |\Delta'|>l_{\rho,U},\, \mathbf{d}(\Delta,\Delta')\leqslant 2\sigma_U+1\big\} &= 2(2\sigma_U+1)Le^{-2(l_\rho-t\rho)}+O(L^\beta)\\ &= 2(2\sigma_U+1)L\rho\Big((\rho+o(\rho))\\ &\qquad\times\big(1+2t\rho+o(\rho))\big)+o(\rho)\Big)\\ &= 2(2\sigma_U+1)L\rho^2(1+o(\rho)).\end{aligned}\tag{1.4.6}$$

Thus, combining (1.4.5) and (1.4.6), we get that there are more than $L\rho\big(1+(t-1)\rho-2(2\sigma_U+2)\rho+o(\rho)\big)$ pieces of type $(\star)$.

From now, we choose $t=4\sigma_U+6$ such that $1+(t-1)\rho-2(2\sigma_U+2)\rho+o(\rho)=\alpha_\rho>1$. So, in the thermodynamic limit, for L and n large enough,

$$\#\{\text{ pieces of type }(\star)\}\geqslant L\rho\,\alpha_\rho\geqslant n+1.\tag{1.4.7}$$

By the pigeonhole principle, there are at least j of pieces of type $(\star)$ and for which the interaction area do not carry any particle in $\Psi^U(L,n)$. Therefore one can move the e extra particles to these slots. We get a new state $\Phi^U(L,n)$.

Before the exchange, the free energy of the piece Δ^e is

$$\begin{aligned}e^0(\Delta^e,j) &= e^0(\Delta^e,k-1)+\sum_{i=k}^{j}\frac{i^2\pi^2}{|\Delta^e|^2}\\ &= e^0(\Delta^e,k-1)+\frac{6ek^2+6e(e-1)k+(2e-1)e(e-1)}{6}\frac{\pi^2}{|\Delta^e|^2}\\ &\geqslant e^0(\Delta^e,k-1)+\frac{6ek^2+6e(e-1)k+(2e-1)e(e-1)}{6}\frac{\pi^2}{k^2l_{\rho,U}^2}\\ &\geqslant e^0(\Delta^e,k-1)+e\frac{j}{k}\frac{\pi^2}{l_{\rho,U}^2}.\end{aligned}$$

So, the e extra particles contribute to more than $e\frac{j}{k}\frac{\pi^2}{l_{\rho,U}^2}$ in $\Psi^U(L,n)$. But in $\Phi^U(L,n)$, the free energy associated to these e particles is strictly less than $e\frac{\pi^2}{l_{\rho,U}^2}$ and there is no interaction energy. So,

$$\langle \Phi^U(L,n), H^U(L,n)\Phi^U(L,n)\rangle < \langle \Psi^U(L,n), H^U(L,n)\Psi^U(L,n)\rangle$$

Thus $\Psi^U(L,n)$ can not be a ground state and this completes the proof of Lemma 1.4.1. □

1.4.1.2 Decomposition of $[0,L]$ into chains

From now on, we fix the minimal length $l_{\rho,U} = l_\rho - (4\sigma_U + 6)\rho$. According to Lemma 1.4.1, the pieces of length $l < l_{\rho,U}$ are empty for any ground state. We divide the others pieces into undecomposable groups of pieces that may interact through U. For simplicity, we identify a piece Δ_k and its index k (position). The ***length of the piece*** i is denoted by $\mathbf{l}_i$ and ***the distance between the pieces*** j and k by $\mathbf{d}_{j,k}$.

Definition 1.4.1. *The r-tuple $I = (i_1, \dots, i_r)$, with $i_1 < \dots < i_r$, is a **chain** of size r if*

(i) for every $k \in [\![1,r]\!]$, $\boldsymbol{l}_{i_k} \geqslant l_{\rho,U}$,

(ii) for every $k \in [\![1,r-1]\!]$, $\boldsymbol{d}_{i_k,i_{k+1}} \leqslant \sigma_U$,

(iii) for every $j < i_1$ such that $\boldsymbol{l}_j \geqslant l_{\rho,U}$, $\boldsymbol{d}_{j,i_1} > \sigma_U$

(iv) for every $j > i_r$ such that $\boldsymbol{l}_j \geqslant l_{\rho,U}$, $\boldsymbol{d}_{i_r,j} > \sigma_U$.

We give an example of a chain of size 3 in Figure 1.1.

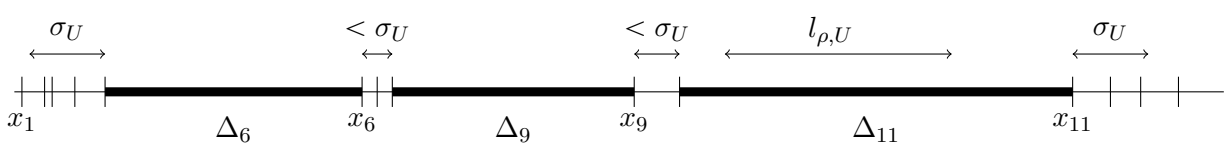

Figure 1.1: Example of a chain of size 3. *The 3-tuple (6,9,11) is a chain because (i) the pieces Δ_6, Δ_9 and Δ_{11} are longer than $l_{\rho,U}$, (ii) the distance between x_6 and x_8 and the distance between x_9 and x_{10} are shorter than σ_U, and (iii)-(iv) there is no piece longer than $l_{\rho,U}$ at distance σ_U below x_5 or above x_{11}.*

For a fixed occupation Q in the subset $\mathcal{Q}$, given by (1.4.4), let $H^U(L,n,Q)$ be the restriction of the pieces model $H^U(L,n)$ to the $Q-$occupied space $\mathfrak{H}_Q(L)$:

$$H^U(L,n,Q) = P_Q H^U(L,n) P_Q \qquad \text{on} \quad \mathfrak{H}_Q(L) = \bigwedge_{i=1}^{m}\Big(\bigwedge_{j=1}^{q_i} L^2(\Delta_i)\Big) \tag{1.4.8}$$

where P_Q is the orthogonal projector on $\mathfrak{H}_Q(L)$. As chains do not interact one with another, $H^U(L,n,Q)$ can be written as a sum of operators each of which acting on a specific chain. We list the notations and definitions for these operators.

Definition 1.4.2. *Fix I a chain. For $(q_i)_{i\in I} \in \mathbb{N}^\star$, the operator $h^U(I,(q_i)_{i\in I})$ rules the behaviour of each q_i particles in the piece Δ_i of the chain I, meaning that*

(1.4.9)

$$h^U(I,(q_i)_{i\in I}) = \sum_{\kappa=1}^{\kappa_I}\Big(\bigotimes_{j=1}^{\kappa-1}\boldsymbol{1}_{L^2([0,L])}\Big)\otimes h_I(L)\otimes\Big(\bigotimes_{j=\kappa+1}^{\kappa_I}\boldsymbol{1}_{L^2([0,L])}\Big) + W_{\kappa_I} \qquad \text{on} \quad \bigwedge_{i\in I}\Big(\bigwedge_{j=1}^{q_i} L^2(\Delta_i)\Big)$$

where

(i) $\kappa_I = \sum_{i\in I} q_i$ *is the total number of particles in* I*;*

(ii) $h_I(L)$ *is the one-particle operator defined by*

$$h_I(L) = \bigoplus_{i\in I}\left(-\frac{d^2}{dx^2}_{|\Delta_i}^{D}\right) \qquad \text{on } L^2\big([0,L]\big); \tag{1.4.10}$$

(iii) W_k *is the multiplication operator given by (1.1.5).*

Let $\psi^U(I,(q_i)_{i\in I})$ *and* $e^U(I,(q_i)_{i\in I})$ *be the ground state and the ground state energy of* $h^U(I,(q_i)_{i\in I})$*. Set* $F^U(I,0)\equiv 0$ *and for* $\kappa\in\mathbb{N}^\star$

$$F^U(I,\kappa) = \min_{\kappa_I=\kappa} e^U(I,(q_i)_{i\in I}). \tag{1.4.11}$$

For $\kappa\in\mathbb{N}^\star$*, the* κ***-th energy level of the chain*** I *is defined by*

$$f^U\big(I,\kappa\big) = F^U(I,\kappa) - F^U(I,\kappa-1). \tag{1.4.12}$$

We will now discriminate the chains by number of particles in any ground state. Fix $p\in\mathbb{N}^\star$. We denote by

$$\mathcal{P}_p = \Big\{I \text{ chain}, \sum_{i\in I}\Big\lfloor\frac{\mathbf{l}_i}{l_{\rho,U}}\Big\rfloor < (p+1)\Big\} \tag{1.4.13}$$

the set of chains that cannot carry more than p particles in any ground state of $H^U(L,n)$, and by $\mathcal{N}_p$ the set of others pieces. Then, the ground state of $H^U(L,n,Q)$ factorizes as follows

$$\Psi^U(L,n,Q) = \Psi^U_{\mathcal{P}_p}(Q)\wedge\Psi^U_{\mathcal{N}_p}(Q) \tag{1.4.14}$$

where

$$\Psi^U_{\mathcal{P}_p}(Q) = \bigwedge_{I\in\mathcal{P}_p}\psi^U(I,(q_i)_{i\in I}) \qquad \text{and} \qquad \Psi^U_{\mathcal{N}_p}(Q) = \bigwedge_{I \text{ chain}\subset\mathcal{N}_p}\psi^U(I,(q_i)_{i\in I}). \tag{1.4.15}$$

with $\psi^U(I,(q_i)_{i\in I})$ is the ground state of the operator $h^U(I,(q_i)_{i\in I})$ given by (1.4.9). Furthermore, the ground state energy of $H^U(L,n,Q)$ is written

$$E^U(L,n,Q) = E^U_{\mathcal{P}_p}(Q) + E^U_{\mathcal{N}_p}(Q) \tag{1.4.16}$$

where

$$E^U_{\mathcal{P}_p}(Q) = \sum_{I\in\mathcal{P}_p} e^U(I,(q_i)_{i\in I}) \quad \text{and} \quad E^U_{\mathcal{N}_p}(Q) = \sum_{I \text{ chain}\subset\mathcal{N}_p} e^U(I,(q_i)_{i\in I}). \tag{1.4.17}$$

with $e^U(I,(q_i)_{i\in I})$ is the ground state energy of the operator $h^U(I,(q_i)_{i\in I})$ given by (1.4.9). We study these two quantities in the next subsections.

1.4.1.3 Upper bound on the energy contribution of large and/or overcrowded chains

The following lemma gives an upper bound for the number of particles that one does not control when the occupation is known only for the chains of $\mathcal{P}_p$.

Lemma 1.4.2. *For* $p\in\mathbb{N}^\star$*, and* $\delta\in(0,1)$*, there exists* $\rho_{p,\delta}>0$ *such that, for every* $\rho\in(0,\rho_{p,\delta})$*, in the thermodynamic limit*

$$\rho^{p+\delta} \leqslant \sup_{Q\in\mathcal{Q}}\left(\frac{1}{n}\sum_{i\in\mathcal{N}_p} q_i\right) \leqslant \rho^{p-\delta}. \tag{1.4.18}$$

Proof. We recall that the statement Q belongs to the subset $\mathcal{Q}$ implies that

$$q_i \leqslant \left\lfloor \frac{\mathbf{l}_i}{l_{\rho,U}} \right\rfloor. \tag{1.4.19}$$

If $i \in \mathcal{N}_p$, we have the following options.

(i) Either $\mathbf{l}_i < l_{\rho,U}$, $q_i = 0$;

(ii) Or $\mathbf{l}_i \geqslant (p+1)l_{\rho,U}$, then, using Proposition 1.4.9 and (1.4.19), one computes

$$\sum_{i,\,\mathbf{l}_i \geqslant (p+1)l_{\rho,U}} q_i \leqslant \sum_{k=p+1}^{+\infty} kL(e^{-kl_{\rho,U}} - e^{-(k+1)l_{\rho,U}}) = (p+1)Le^{-(p+1)l_{\rho,U}}(1 + O(e^{-l_{\rho,U}}))$$

(iii) Or $i \in I$ chain of size $r \geqslant 2$ and $\sum_{j\in I} \mathbf{l}_j \geqslant (p+1)l_{\rho,U}$ and $\mathbf{l}_i < (p+1)l_{\rho,U}$; in this case by (1.4.19), $q_i \leqslant p$. For $r \leqslant p$,

$$\begin{aligned}\#\{I \text{ chain of size } r \text{ of total length } \geqslant (p+1)l_{\rho,U}\} &\leqslant \#\{r \text{ pieces of total length } \geqslant (p+1)l_{\rho,U} \\ &\qquad \text{with gaps of length } \leqslant \sigma_U\} \\ &\leqslant {\sigma_U}^{r-1} Le^{-(p+1)l_{\rho,U}}\end{aligned}$$

and

$$\begin{aligned}\#\{I \text{ chain of size } r \geqslant p+1\} &\leqslant \#\{(p+1) \text{ pieces of length } \geqslant l_{\rho,U} \text{ with gaps of length } \leqslant \sigma_U\} \\ &\leqslant {\sigma_U}^{p} Le^{-(p+1)l_{\rho,U}}.\end{aligned}$$

Then, for any $Q \in \mathcal{Q}$,

$$\frac{1}{n}\sum_{i\in\mathcal{N}_p} q_i \leqslant (p+1)\frac{L}{n}e^{-(p+1)l_{\rho,U}}(1 + O(e^{-l_{\rho,U}})) + \sum_{r=2}^{p+1} p\,{\sigma_U}^{r-1}\frac{L}{n}e^{-(p+1)l_{\rho,U}}. \tag{1.4.20}$$

As $e^{-(p+1)l_{\rho,U}} = o(\rho^{p+1-\delta})$, there exists $\rho'_{p,\delta} > 0$ such that for $\rho \in (0, \rho'_{p,\delta})$ in the thermodynamic limit

$$\frac{1}{n}\sum_{i\in\mathcal{N}_p} q_i \leqslant \rho^{p-\delta}. \tag{1.4.21}$$

This completes the proof of the right-hand side of the inequality (1.4.18).

Concerning the left-hand side, let $Q^0 = (q_i^0)_{1\leqslant i\leqslant m_L}$ be the occupation of the ground state for the free model. By (1.4.3), we have that for $i \in [\![1, m_L]\!]$ if $\mathbf{l}_i \in [kl_\rho, (k+1)l_\rho)$ then $q_i^0 = k$. Since $l_{\rho,U} \leqslant l_\rho$, $Q^0 \in \mathcal{Q}$. So,

$$\sum_{i\in\mathcal{N}_p} q_i^0 \geqslant \sum_{i,\,\mathbf{l}_i \geqslant (p+1)l_\rho} q_i^0 = \sum_{k=p+1}^{+\infty} kL(e^{-kl_\rho} - e^{-(k+1)l_\rho}) = (p+1)Le^{-(p+1)l_\rho}(1 + O(e^{-l_\rho}))$$

As $\rho^{p+1+\delta} = o(e^{-(p+1)l_\rho})$, there exists $\rho_{p,\delta} \leqslant \rho'_{p,\delta}$ that gives the left part of the inequality (1.4.18). □

Proposition 1.4.1. *For a fixed $p \geqslant 1$, $\delta \in (0,1)$ and $Q \in \mathcal{Q}$, there exists $\rho_{p,\delta} > 0$ such that, for $\rho \in (0, \rho_{p,\delta})$, in the thermodynamic limit,*

$$E^U_{\mathcal{N}_p}(Q) \leqslant n\rho^{p-\delta} \tag{1.4.22}$$

Proof. As in Definition 1.4.2, for any chain I, we denote $\psi^U(I,(q_i)_{i\in I})$ and $e^U(I,(q_i)_{i\in I})$ the ground state and ground state energy of the operator $h^U\big(I,(q_i)_{i\in I}\big)$ given by (1.4.9). We use the notations $\psi^0\big(I,(q_i)_{i\in I}\big)$ and $e^0\big(I,(q_i)_{i\in I}\big)$ for the free case. We have

$$\big\langle \psi^U(I,(q_i)_{i\in I})\,,\, h^U\big(I,(q_i)_{i\in I}\big)\psi^U(I,(q_i)_{i\in I})\big\rangle \leqslant \big\langle \psi^0\big(I,(q_i)_{i\in I}\big)\,,\, h^U\big(I,(q_i)_{i\in I}\big)\psi^0\big(I,(q_i)_{i\in I}\big)\big\rangle$$

so

$$e^U(I,(q_i)_{i\in I}) \leqslant e^0\big(I,(q_i)_{i\in I}\big) + \big\langle \psi^0\big(I,(q_i)_{i\in I}\big), W_{\kappa_I}\psi^0\big(I,(q_i)_{i\in I}\big)\big\rangle. \tag{1.4.23}$$

Then, we compute

$$\begin{aligned} E^U_{\mathcal{N}_p}(Q) &= \sum_{I\subset\mathcal{N}_p \text{ chain}} e^U\big(I,(q_j)_{j\in I}\big) \\ &\leqslant \sum_{I\subset\mathcal{N}_p \text{ chain}} \Big(e^0\big(I,(q_j)_{j\in I}\big) + \big\langle \psi^0\big(I,(q_i)_{i\in I}\big), W_{\kappa_I}\psi^0\big(I,(q_i)_{i\in I}\big)\big\rangle\Big) \\ &\leqslant \max_{\substack{Q\in\mathcal{Q}\\ j\in\mathcal{N}_p}} \left(\frac{e^0\big([0,\mathbf{l}_j],q_j\big)}{q_j}\right) \sum_{j\in\mathcal{N}_p} q_j + \sum_{I\subset\mathcal{N}_p \text{ chain}} \big\langle \psi^0\big(I,(q_i)_{i\in I}\big), W_{\kappa_I}\psi^0\big(I,(q_i)_{i\in I}\big)\big\rangle \end{aligned} \tag{1.4.24}$$

where we use that

$$e^0\big(I,(q_j)_{j\in I}\big) = \sum_{j\in I} e^0\big([0,\mathbf{l}_j],q_j\big).$$

For any $Q\in\mathcal{Q}$ and $j\in\mathcal{N}_p$, by Lemma 1.4.1, we have $q_j l_{\rho,U}\leqslant l_j$. So,

$$e^0\big([0,\mathbf{l}_j],q_j\big) = \sum_{k=1}^{q_j} \frac{k^2\pi^2}{\mathbf{l}_j^2} \leqslant C\frac{q_j^3}{\mathbf{l}_j^2} \leqslant C\frac{q_j}{l^2_{\rho,U}} \leqslant \frac{q_j}{3} \tag{1.4.25}$$

for ρ small enough. Combining Lemma 1.4.2 and (1.4.25), there exists $\rho'_{p,\delta}>0$ so that, for $\rho\in(0,\rho'_{p,\delta})$, in the thermodynamic limit,

$$\max_{\substack{Q\in\mathcal{Q}\\ j\in\mathcal{N}_p}} \left(\frac{e^0\big([0,\mathbf{l}_j],q_j\big)}{q_j}\right) \sum_{j\in\mathcal{N}_p} q_j \leqslant \frac{n}{3}\rho^{p-\delta}. \tag{1.4.26}$$

We deal with the remaining sum in (1.4.24) using the results of Lemma 1.4.10. By skew-symmetry, we have

$$\big\langle \psi^0\big(I,(q_i)_{i\in I}\big), W_{\kappa_I}\psi^0\big(I,(q_i)_{i\in I}\big)\big\rangle = \frac{\kappa_I(\kappa_I-1)}{2}\int_{[0,L]^{\kappa_I}} U(y-x)\psi^0\big(I,(q_i)_{i\in I}\big)^2(x,y,Z)dxdydZ. \tag{1.4.27}$$

For a chain I, the ground state $\psi^0(I,(q_i)_{i\in I})$ is the Slater determinant of $\big(\phi_j^{\Delta_i}\big)_{1\leqslant j\leqslant q_i}^{i\in I}$ where, for $i\in I$, $j\in\mathbb{N}$, $\phi_j^{\Delta_i}$ is the state on $L^2(\Delta_i)$ given by

$$\phi_j^{\Delta_i}(x) = \frac{\sqrt{2}}{\sqrt{\mathbf{l}_i}}\sin\Big(\frac{\pi}{\mathbf{l}_i}j(x-x_i)\Big)\mathbf{1}_{\Delta_i}(x). \tag{1.4.28}$$

So,

$$\begin{aligned} \psi^0(I,(q_i)_{i\in I})(x_1,\dots,x_{\kappa_I}) &= \Big(\bigwedge_{i\in I}\bigwedge_{j=1}^{q_i}\phi_j^{\Delta_i}\Big)(x_1,\dots,x_{\kappa_I}) \\ &= \frac{1}{\sqrt{\kappa_I!}}\sum_{\sigma\in\mathfrak{S}_{\kappa_I}} \varepsilon(\sigma)\prod_{k=1}^{\kappa_I}\phi_{j_{\sigma(k)}}^{\Delta_{i_{\sigma(k)}}}(x_k) \end{aligned} \tag{1.4.29}$$

with

$$i_k = \inf\Big\{i \in I,\, k \leqslant \sum_{r\in I,\, r\leqslant i} q_r\Big\} \qquad \text{and} \qquad j_k = k - \sum_{r\in I,\, r\leqslant i_k-1} q_r.$$

By orthogonality of $(\phi_j^{\Delta_i})_j^i$, we know that for $\sigma,\tau \in \mathfrak{S}_{\kappa_I}$

$$\int_{[0,L]} \phi_{j_{\sigma(k)}}^{\Delta_{i_{\sigma(k)}}}(z)\phi_{j_{\tau(k)}}^{\Delta_{i_{\tau(k)}}}(z)dz = \begin{cases} 1 & \text{if } \sigma(k)=\tau(k) \\ 0 & \text{else.}\end{cases} \tag{1.4.30}$$

Then (1.4.27) becomes

$$\begin{aligned}\big\langle \psi^0\big(I,(q_i)_{i\in I}\big), W_{\kappa_I}\psi^0\big(I,(q_i)_{i\in I}\big)\big\rangle &= \frac{\kappa_I(\kappa_I-1)}{2(\kappa_I!)} \sum_{\substack{\sigma,\tau\in\mathfrak{S}_{\kappa_I}\\ \sigma(k)=\tau(k)\,\forall k\geqslant 3}} \varepsilon(\sigma)\varepsilon(\tau)\Big(\int_{[0,L]^2} U(y-x) \\ &\qquad \phi_{j_{\sigma(1)}}^{\Delta_{i_{\sigma(1)}}}(x)\phi_{j_{\tau(1)}}^{\Delta_{i_{\tau(1)}}}(x)\phi_{j_{\sigma(2)}}^{\Delta_{i_{\sigma(2)}}}(y)\phi_{j_{\tau(2)}}^{\Delta_{i_{\tau(2)}}}(y)\,dxdy\Big) \\ &= \sum_{i\in I}\sum_{1\leqslant j<k\leqslant q_i}\int_{[0,L]^2} U(y-x)\Big|\phi_j^{\Delta_i}\wedge\phi_k^{\Delta_i}\Big|^2(x,y)dxdy \\ &\quad + \sum_{h,i\in I,\, h\neq i}\sum_{j=1}^{q_h}\sum_{k=1}^{q_i}\int_{[0,L]^2} U(y-x)\Big|\phi_j^{\Delta_h}\wedge\phi_k^{\Delta_i}\Big|^2(x,y)dxdy.\end{aligned} \tag{1.4.31}$$

We apply Lemma 1.4.10,

$$\begin{aligned}\big\langle \psi^0\big(I,(q_i)_{i\in I}\big), W_{\kappa_I}\psi^0\big(I,(q_i)_{i\in I}\big)\big\rangle &\leqslant C\sum_{i\in I}\sum_{1\leqslant j<k\leqslant q_i}\frac{j^2+k^2}{\mathbf{l}_i^3} + C\sum_{h,i\in I,\, h\neq i}\sum_{j=1}^{q_h}\sum_{k=1}^{q_i}\frac{j^2k^2}{l_h^3\mathbf{l}_i^3} \\ &\leqslant C\sum_{i\in I}\frac{q_i}{l_{\rho,U}^3} + C\sum_{h,i\in I,\, h\neq i}\frac{q_hq_i}{l_{\rho,U}^6} \\ &\leqslant \frac{C}{l_{\rho,U}^3}\sum_{i\in I}q_i + \frac{C}{l_{\rho,U}^6}\Big(\sum_{i\in I}q_i\Big)^2\end{aligned} \tag{1.4.32}$$

where C depends on U and σ_U. Again by Lemma 1.4.2, there exists $\rho''_{p,\delta}>0$ so that for $\rho\in(0,\rho''_{p,\delta})$, in the thermodynamic limit

$$\frac{C}{l_{\rho,U}^3}\sum_{i\in I}q_i \leqslant \frac{n}{3}\rho^{p-\delta} \tag{1.4.33}$$

For the part with squares in 1.4.32, we adapt the proof of (1.4.18). A chain $I\subset\mathcal{N}_p$ of size $r\geqslant p+1$ of total length $l\in[kl_{\rho,U},(k+1)l_{\rho,U})$ with $k\geqslant r$ may contain at most k particles. Otherwise, the chains $I\subset\mathcal{N}$ of size $r\leqslant p$ and of total length $l\in[kl_{\rho,U},(k+1)l_{\rho,U})$ with $k\geqslant p+1$ may contain at most k particles. So,

$$\sum_{I\subset\mathcal{N}_p}\Big(\sum_{i\in I}q_i\Big)^2 \leqslant \sum_{r=p+1}^{+\infty}{\sigma_U}^{r-1}\sum_{k=r}^{\infty}k^2Le^{-kl_{\rho,U}} + \sum_{r=1}^{p}{\sigma_U}^{r-1}\sum_{k=p+1}^{+\infty}k^2Le^{-kl_{\rho,U}}.$$

We claim that, if $\sigma_Ue^{-l_{\rho,U}}<1$,

$$\exists C>0 \qquad \sum_{I\subset\mathcal{N}_p}\Big(\sum_{i\in I}q_i\Big)^2 \leqslant C\max\{1,\dots,{\sigma_U}^p\}(p+1)^2Le^{-(p+1)l_{\rho,U}}. \tag{1.4.34}$$

So, there exists $\rho'''_{p,\delta} > 0$ so that for $\rho \in (0, \rho'''_{p,\delta})$, in the thermodynamic limit

$$\frac{C}{l^6_{\rho,U}}\Big(\sum_{i\in I} q_i\Big)^2 \leqslant \frac{n}{3}\rho^{p-\delta} \tag{1.4.35}$$

Then, if ρ is small enough, combining (1.4.26), (1.4.33) and (1.4.35), the inequality (1.4.24) becomes in the thermodynamic limit

$$E^U_{\mathcal{N}_p}(Q) \leqslant n\rho^{p-\delta}. \tag{1.4.36}$$

It concludes the proof of Proposition 1.4.1. □

Remark 1.4.1. *If one replaces $\psi^U_{\mathcal{N}_p}(Q)$ by $\bigwedge_{I\subset\mathcal{N}_p}\psi^0\big(I,(q_i)_{i\in I}\big)$ then the same bound holds for the energy.*

1.4.1.4 Estimation of the energy contribution of short chains

The following proposition states that, when the number of particles in $\mathcal{P}_p$ is known, $E^U_{\mathcal{P}_p}$ is the sum of the smallest energy levels. But it requires a strong hypothesis on the monotony of the energy levels.

Assumption 1.4.1. *For a fixed $p \geqslant 1$, using the notations of Definition 1.4.2, the application*

$$f^U(I,.) : \begin{cases} [\![0,p]\!] & \longrightarrow \mathbb{R} \\ \kappa & \longmapsto f^U\big(I,\kappa\big) \end{cases} \tag{1.4.37}$$

is increasing for every chain I in $\mathcal{P}_p$.

Let Γ_p be the set whose elements are the first p energy levels for each chain I in $\mathcal{P}_p$,

$$\Gamma_p = \Big\{f^U\big(I,\kappa\big),\, I\in\mathcal{P}_p, 1\leqslant\kappa\leqslant p\Big\}. \tag{1.4.38}$$

We define a lexical order $\leqslant_p$ on Γ_p such that
(1.4.39)

$$\forall I,J\in\mathcal{P}_p,\ 1\leqslant k,l\leqslant p \qquad f^U(I,k) <_p f^U(J,l) \Longleftrightarrow \begin{cases} f^U(I,k) < f^U(J,l) \\ \text{else} \quad \text{last index of } I < \text{first index of J} \\ \text{else} \quad k<l \end{cases}$$

From now on, we denote $n_{p,Q}$ is the number of particles in $\mathcal{P}_p$ for the occupation Q.

Proposition 1.4.2. *For a fixed $p \geqslant 1$, let $\{a_k \in \Gamma_p,\, a_{k-1} <_p a_k\}$ be the ordered set given by (1.4.38) and (1.4.39). Under Assumption 1.4.1, for $r \leqslant \min(n, \#\Gamma_p)$, any occupation Q that minimizes $E^U_{\mathcal{P}_p}$ when $n_{p,Q} = r$ and $q_i \leqslant \left\lfloor \frac{l_i}{l_{\rho,U}}\right\rfloor$ for each piece Δ_i in $\mathcal{P}_p$, satisfies*

$$E^U_{\mathcal{P}_p}(Q) = \sum_{k=1}^{r} a_k. \tag{1.4.40}$$

Proof. Fix $r \leqslant \min(n, \#\Gamma_p)$. Take an occupation Q that minimizes $E^U_{\mathcal{P}_p}$ when $n_{p,Q} = r$ and $q_i \leqslant \left\lfloor \frac{l_i}{l_{\rho,U}}\right\rfloor$ for each piece Δ_i in $\mathcal{P}_p$. Then, by reductio ad absurdum, using the notations of Definition 1.4.2,

$$E^U_{\mathcal{P}_p}(Q) = \sum_{I\in\mathcal{P}_p} e^U(I,(q_i)_{i\in I}) = \sum_{I\in\mathcal{P}_p} F^U(I,\kappa_I) = \sum_{I\in\mathcal{P}_p}\sum_{j=1}^{\kappa_I} f^U(I,j)$$

with $\sum_{I\in\mathcal{P}_p}\kappa_I = r$ and $\kappa_I \leqslant \sum_{i\in I}\left\lfloor \frac{\mathbf{l}_i}{l_{\rho,U}}\right\rfloor \leqslant p$. In particular,

$$\sum_{k=1}^{r} a_k \leqslant E^U_{\mathcal{P}_p}(Q). \tag{1.4.41}$$

For the reverse inequality, we build by induction an appropriate occupation Q'. Set $Q'(0)\equiv(0,\dots,0)$. For k from 1 to r, assume that the multi-index $Q'(k-1) = \big(q'_i(k-1)\big)_{1\leqslant i\leqslant m}$ satisfies

$$E^U_{\mathcal{P}_p}\big(Q'(k-1)\big) = \sum_{s=1}^{k-1} a_s \qquad \text{and} \qquad \sum_{i=1}^{m} q'_i(k-1) = k-1.$$

We know that $a_k = f^U(I,j)$ meaning a_k is the j-th energy level of the chain I. Since $f^U(I,.)$ is increasing, we have $\{f^U(I,1),\dots,f^U(I,j-1)\} = \{a_{i_1},\dots,a_{i_{j-1}}\}$ for $1\leqslant i_1<\dots<i_{j-1}\leqslant k-1$ and for every $i>j$, $f^U(I,i)>a_k$. In particular,

$$\sum_{s=1}^{k} a_s = \sum_{s\notin\{i_1,\dots,i_{j-1},k\}} a_s + \sum_{i=1}^{j} f^U(I,i) = \sum_{s\notin\{i_1,\dots,i_{j-1},k\}} a_s + F^U(I,j).$$

We set $q'_i(k)$ for $i\in I$ so that $e^U(I,(q'_i(k))_{i\in I}) = F^U(I,j)$ and for every $i\notin I$, $q'_i(k) = q'_i(k-1)$. Then,

$$E^U_{\mathcal{P}_p}\big(Q'(k)\big) = \sum_{s=1}^{k} a_s \qquad \text{and} \qquad \sum_{i=1}^{m} q'_i(k) = k.$$

We fill the coordinates in $\mathcal{N}_p$ so that Q' is an occupation with $n_{p,Q} = r$.

It concludes the proof of Proposition 1.4.2. □

Remark 1.4.2. *We don't know yet how to prove that Assumption 1.4.1 holds when $p\geqslant 3$. The following lemma gives a partial result for chains of size 1.*

Lemma 1.4.3. *Using the notations of Definition 1.4.2, if $l < l_{\rho,U}^{\frac{3}{2}-\varepsilon}$ for $\varepsilon\in(0,\frac{1}{2})$ then for ρ small enough*

$$\forall r\in[\![1,p-1]\!] \qquad f^U\big([0,l],r+1\big) > f^U\big([0,l],r\big) \tag{1.4.42}$$

where $p = \lfloor \frac{l}{l_{\rho,U}}\rfloor$.

Proof. Assume that $l = \pi\beta^{-1}l_{\rho,U}$ and $r\geqslant 1$. Using the notations of Definition 1.4.2, let $F(r) = F^U\big([0,l],r\big)$ be the smallest energy produced by r particles in the piece $[0,l]$ and $\psi(r)$ a corresponding eigenfuntion. Also set $F^0(r)$ and $\psi^0(r)$ in the free case $U\equiv 0$. We know that

$$\psi^0(r) = \bigwedge_{i=1}^{r}\varphi_i$$

where $\varphi_i(x) = \frac{\sqrt{2}}{\sqrt{l}}\sin\big(\frac{\pi}{l}ix\big)\mathbf{1}_{[0,l]}(x)$. We compute

$$\begin{aligned}
\langle W_r\psi^0(r),\psi^0(r)\rangle &= \frac{r(r-1)}{2}\int_{[0,L]^r} U(x_1-x_2)\psi^0(r)(x_1,x_2,Z)^2dx_1dx_2dZ \text{ by skew-symmetry}\\
&= \frac{r(r-1)}{2}\frac{1}{r!}\sum_{\sigma,\sigma'\in\mathfrak{S}_r}\varepsilon(\sigma)\varepsilon(\sigma')\int U(x_1-x_2)\prod_{i=1}^{r}\varphi_{\sigma(i)}(x_i)\varphi_{\sigma'(i)}(x_i)dX\\
&= \sum_{p<q\leqslant r}\int U(x_1-x_2)\big|\varphi_p\wedge\varphi_q\big|^2(x_1,x_2)dx_1dx_2
\end{aligned}$$

by skew-symmetry and orthogonality of $(\varphi_i)_{i\geqslant 1}$. Hence, by Lemma 1.4.10,

$$\langle W_r\psi^0(r),\psi^0(r)\rangle \leqslant \sum_{p<q\leqslant r} Cl^{-3}(p^2+q^2) \leqslant Cl^{-3}r^4. \tag{1.4.43}$$

Since,

$$0 \leqslant F(r) - F^0(r) \leqslant \langle W_r\Psi^0(r),\Psi^0(r)\rangle.$$

we have

$$F(r) = F^0(r) + O(l^{-3}r^4) = \sum_{i=1}^{r}(\pi l^{-1}i)^2 + O(l^{-3}r^4) \tag{1.4.44}$$

Then,

$$F(r+1) - 2F(r) + F(r-1) = 2\pi l^{-2}r\big(1+O(l_{\rho,U}^{-2\varepsilon})\big) \tag{1.4.45}$$

as $r < l.l_{\rho,U}^{-1}$ and $l^2 l_{\rho,U}^{-3} \leqslant l_{\rho,U}^{-2\varepsilon}$. Thus one gets that for ρ small enough the r.h.s is positive. This concludes the proof of Lemma 1.4.3. □

Combining Lemma 1.4.3 and Lemma 1.4.8, we get that Assumption 1.4.1 holds when one cancels the interaction between pieces and p is less than $|\log(\rho)|$. Without restriction on the form of the interaction, the issue occurs when the growth in the free energy is less or of the order of the interaction between two pieces. More precisely, we don't know yet how to deal with the cases where the lengths of a pair of pieces $\{\Delta_i,\Delta_j\}$ satisfy

$$\exists\, k_i,k_j \in [\![1,p-1]\!] \qquad \Big|\Big(\frac{k_i}{l_i}\Big)^2 - \Big(\frac{k_j}{l_j}\Big)^2\Big| = O(l_{\rho,U}^{-6}). \tag{1.4.46}$$

Otherwise, the proof of Proposition 1.4.2 gives an approximation by induction of the restriction to $\mathcal{P}_p$ of any occupation of the ground state. To state the following corollary, we introduce a notation for the chains that produce the same element in $(\Gamma_p,\leqslant)$ where $\leqslant$ is the usual order. We set for $1\leqslant r\leqslant \#\Gamma_p$

$$\mathcal{G}_p(r) = \Big\{I\in\mathcal{P}_p,\ \exists\, 1\leqslant k\leqslant p,\ f^U(I,k) = \text{the } r\text{-th smallest element of } (\Gamma_p,\leqslant_p)\Big\}. \tag{1.4.47}$$

Corollary 1.4.1. *Under Assumption 1.4.1, there exists a sequence of occupations $\big(Q(r)\big)_{r\leqslant n}$ such that*

1. *the number of particles in the chains of $\mathcal{P}_p$ for the occupation $Q(r)$ is $n_{p,Q(r)} = r$;*
2. *the restrictions of $Q(r)_{|\mathcal{P}_p}$ and $Q(r+1)_{|\mathcal{P}_p}$ are equal except for one chain;*
3. *if Ψ^U is a ground state of H^U and Q is an occupation that satisfies $P_Q\Psi^U \neq 0$ then*

$$Q_{|\mathcal{P}_p\setminus\mathcal{G}_p(n_{p,Q})} = Q(n_{p,Q})_{|\mathcal{P}_p\setminus\mathcal{G}_p(n_{p,Q})}, \tag{1.4.48}$$

where $\mathcal{G}_p(n_{p,Q})$ is given by (1.4.47) and $Q_{|B} = (q_i)_{i\in B}$ the restriction of the multi-index to B.

The issue of the cardinal of $\mathcal{G}_p(r)$, for any $r\in[\![1,\#\Gamma_p]\!]$, looks as hard to solve as the issue of order of degeneracy of the ground state of $H^U(L,n)$. However, it seems relevant to assume that, except for some pathological Poisson point processes, one should get only few cases of equality for the energy levels of Γ_p.

Assumption 1.4.2. *For $1\leqslant r\leqslant \#\Gamma_p$, $\#\mathcal{G}_p(r)\leqslant n\rho^{p-\delta}$.*

The next proposition states that if Assumption 1.4.1 and Assumption 1.4.2 are true for some $p \geqslant 1$ then the number of particles in each piece of $\mathcal{P}_p$, except for at most $2n\rho^{p-\delta}$ chains, stays the same for any ground state.

Proposition 1.4.3. *Set $p \in \mathbb{N}^\star$, $\delta \in (0,1)$ and $\rho \in (0, \rho_\delta)$. Under Assumption 1.4.1 and Assumption 1.4.2, there exist a subset $\mathcal{F}_p$ of $\mathcal{P}_p$ and, for each piece i in $\mathcal{F}_p$, an integer $q_i^{\mathcal{F}_p}$ such that*

1. *the number of chains in $\mathcal{P}_p \backslash \mathcal{F}_p$ is less than or equal to $2n\rho^{p-\delta}$;*
2. *if Ψ^U is a ground state of H then it admits the decomposition $\Psi^U = \Phi^{U,\mathcal{F}_p} \wedge \Omega^{U,\mathcal{F}_p^c}$ with*

$$\Phi^{U,\mathcal{F}_p} = \bigwedge_{I \in \mathcal{F}_p} \psi^U\big(I, (q_i^{\mathcal{F}_p})_{i \in I}\big) \qquad \text{and} \qquad \Omega^{U,\mathcal{F}_p^c} = \sum_{Q \in \mathcal{Q}} \lambda(Q) \bigwedge_{I \notin \mathcal{F}_p} \psi^U\big(I, (q_i)_{i \in I}\big). \tag{1.4.49}$$

Proof. Let $\mathcal{F}_p$ be the set of chains I in $\mathcal{P}_p$ such that the function $r \mapsto Q(r)_{|I}$ is constant on $[\![n - 2n\rho^{p-\delta}, n]\!]$. By Corollary 1.4.1, for $r \leqslant n-1$, there is a unique chain $I \in \mathcal{P}_p$ for which $Q(r)_{\mathcal{P}_p \backslash I} = Q(r+1)_{\mathcal{P}_p \backslash I}$. So, by induction on $r \geqslant n - 2n\rho^{p-\delta}$, $\mathcal{F}_p$ is not empty and the numbers of chains in $\mathcal{P}_p \backslash \mathcal{F}_p$ is less than or equal to $2n\rho^{p-\delta}$. Then, for any piece i in $\mathcal{F}_p$, let $q_i^{\mathcal{F}_p}$ be the common value.

Let $\Psi^U(L,n)$ be a ground state of the operator $H^U(L,n)$. By Lemma 1.4.1 and Definition 1.4.2, we have the decomposition

$$\Psi^U = \sum_{Q \in \mathcal{Q}} \lambda(Q) \bigwedge_{I \text{ chain}} \psi^U(I, (q_i)_{i \in I}) \tag{1.4.50}$$

where $\psi^U(I, (q_i)_{i \in I})$ is a normalized wave function of $\mathfrak{H}^{q_I}(U_I)$ with $q_I = \sum_{i \in I} q_i$ and $U_I = \bigcup_{i \in I} \Delta_i$.

Using Lemma 1.4.2, if an occupation Q satisfies $P_Q \Psi^U \neq 0$ then the number of particles in $\mathcal{P}_p$ for Q belongs to $[\![n - n\rho^{p-\delta}, n]\!]$.

Under Assumption 1.4.2, we have

$$\bigcup_{r = n - n\rho^{p-\delta}}^{n} \mathcal{G}_p(r) \subset \mathcal{P}_p \backslash \mathcal{F}_p \tag{1.4.51}$$

because the left term, gathers all the chains that match with any r-th smallest element in $(\Gamma_p, \leqslant_p)$ for $r \in [\![n - n\rho^{p-\delta}, n]\!]$ (see (1.4.47)) while the right term gathers all the chains that match with any element of $(\Gamma_p, \leqslant_p)$ between the $(n - 2\rho^{p-\delta})$-th and the n-th ones.

Using the third point of Corollary 1.4.1, one shows that, for every chain $I \in \mathcal{F}_p$, the restriction map $(q_i)_{i \in [\![1, m_L]\!]} \mapsto (q_i)_{i \in I}$ is constant on $\{Q \in \mathcal{Q}, P_Q \Psi^U(L,n) \neq 0\}$, equal to $(q_i^{\mathcal{F}_p})_{i \in I}$. So, (1.4.50) becomes

$$\Psi^U(L,n) = \Big(\bigwedge_{I \in \mathcal{F}_p} \psi^U\big(I, (q_i^{\mathcal{F}_p})_{i \in I}\big) \Big) \wedge \Big(\sum_{Q \in \mathcal{Q}} \lambda(Q) \bigwedge_{I \notin \mathcal{F}_p} \psi^U\big(I, (q_i)_{i \in I}\big) \Big) \tag{1.4.52}$$

This concludes the proof of Proposition 1.4.3. □

1.4.2 Ground state approximation by optimization between 2−particle systems

1.4.2.1 Monotony of the energy levels in 2−particle systems

We recall that $\mathcal{P}_2$ is the set of chains each of which carries at most two particles for any ground state:

$$\mathcal{P}_2 = \Big\{ I \text{ chain}, \sum_{i \in I} \Big\lfloor \frac{\mathbf{l}_i}{l_{\rho,U}} \Big\rfloor \leqslant 2 \Big\}. \tag{1.4.53}$$

Note that a chain that belongs to $\mathcal{P}_2$ is either a unique piece or a pair of pieces. We also recall that $(\Gamma_2, \leqslant_2)$ is the ordered set of energy levels given by

$$\Gamma_2 = \Big\{ f^U\big(I,\kappa\big),\ I \in \mathcal{P}_2,\ \kappa \in \{1,2\} \Big\} \tag{1.4.54}$$

and

(1.4.55)

$$\forall I, J \in \mathcal{P}_2,\ k,l \in \{1,2\} \qquad f^U(I,k) <_2 f^U(J,l) \Longleftrightarrow \begin{cases} f^U(I,k) < f^U(J,l) \\ \text{else} \quad \text{last index of } I < \text{first index of J} \\ \text{else} \quad k < l \end{cases} .$$

Lemma 1.4.4. *With probability $1 - O(L^{-\infty})$, for ρ small enough, in the thermodynamic limit*

$$2n < \#\Gamma_2 < 2n\big(1 + (3\sigma_U + 6)\rho\big).$$

Proof. Using $e^{-l_{\rho,U}} = \rho\big(1 + (4\sigma_U + 5)\rho + o(\rho)\big)$, Proposition 1.4.9 and Proposition 1.4.10, we compute

$$\begin{aligned} \#\Gamma_2 &= 2\#\Big\{\Delta_i \in \mathcal{P}_2\Big\} + 2\#\Big\{\big(\Delta_j, \Delta_k\big) \in \mathcal{P}_2\Big\} \\ &= 2L(1 - \sigma_U e^{-l_{\rho,U}})^2\Big(\big(e^{-l_{\rho,U}} - e^{-3l_{\rho,U}}\big) + \sigma_U\big(e^{-l_{\rho,U}} - e^{-2l_{\rho,U}}\big)^2\Big) \\ &= 2L\big(1 - 2\sigma_U\rho + o(\rho)\big)\rho\big(1 + (4\sigma_U + 5)\rho + o(\rho)\big)\big(1 + \sigma_U\rho + o(\rho)\big) \\ &= 2L\rho\big(1 + (3\sigma_U + 5)\rho + o(\rho)\big) \end{aligned}$$

It concludes the proof of Lemma 1.4.4. □

We now prove that Assumption 1.4.1 holds when $p = 2$.

Lemma 1.4.5. *For $I \in \mathcal{P}_2$, $f^U(I,2) > f^U(I,1)$.*

Proof. If $I \in \mathcal{P}_2$, then we have two cases.

(i) Either $I = (i)$ is a unique piece of length $\mathbf{l}_i \in [l_{\rho,U}, 3l_{\rho,U})$. The first energy level of Δ_i is

$$f^U(\Delta_i, 1) = \frac{\pi^2}{\mathbf{l}_i^2}$$

For the second energy level of Δ_i, we use Proposition 1.2.1.

$$f^U(\Delta_i, 2) = \frac{4\pi^2}{\mathbf{l}_i^2} + \frac{\gamma}{\mathbf{l}_i^3}\Big(1 + o(1)\Big) > f^U(\Delta_i, 1). \tag{1.4.56}$$

(ii) Or $I = (j,k)$ is a pair of pieces of length $\mathbf{l}_j, \mathbf{l}_k \in [l_{\rho,U}, 2l_{\rho,U})$ separated by a gap of length $\mathbf{d}_{jk} \leqslant \sigma_U$. The first energy level of the pair $\{\Delta_j, \Delta_k\}$ is

$$f^U\Big(\{\Delta_j, \Delta_k\}, 1\Big) = \min\Big(\frac{\pi^2}{\mathbf{l}_j^2}, \frac{\pi^2}{\mathbf{l}_k^2}\Big). \tag{1.4.57}$$

Concerning the second energy level of this pair, we use Proposition 1.2.2.

$$f^U\Big(\{\Delta_j, \Delta_k\}, 2\Big) = \max\Big(\frac{\pi^2}{\mathbf{l}_j^2}, \frac{\pi^2}{\mathbf{l}_k^2}\Big) + \frac{\tau(\mathbf{d}_{jk})}{\mathbf{l}_j^3\mathbf{l}_k^3}\Big(1 + o(1)\Big) > f^U\Big(\big(\Delta_j, \Delta_k\big), 1\Big). \tag{1.4.58}$$

This completes the proof of Lemma 1.4.5. □

Combining Lemma 1.4.5 and Proposition 1.4.2, we get the following corollary.

Corollary 1.4.2. *For $r \leqslant n$, the minimum of $E^U_{\mathcal{P}_2}$ when there are exactly r particles in the chains of $\mathcal{P}_2$ is equal to the sum of the r smallest elements of Γ_2.*

1.4.2.2 Distribution of the energy levels among 2−particle systems

By Corollary 1.4.2, we need to understand the distribution of the energy levels in Γ_2. For $\lambda > 0$, we define

$$N_2^U(L,\lambda) := \frac{1}{L}\#\{\, x \in \Gamma_2,\, x \in (-\infty,\lambda]\} \quad \text{and} \quad \mathbf{N}_2^U(\lambda) := \lim_{L\to\infty} N_2^U(L,\lambda). \tag{1.4.59}$$

The application $\mathbf{N}_2^U$ is called the ***counting function of*** Γ_2. We evaluate it in the following proposition.

Proposition 1.4.4. *Define the application J by, for $\lambda > 0$,*

$$\begin{aligned} J(\lambda) := (1-\sigma_U e^{-l_{\rho,U}})^2 \Bigg(&\int_{\mathcal{D}_1(\lambda)} e^{-u}\,du + \int_{\mathcal{D}_2(\lambda)} e^{-u}\,du \\ &+ \int_0^{\sigma_U} \left(\int_{\mathcal{D}_3(\lambda)} 2e^{-(u+v)}\,dudv \right) dt + \int_0^{\sigma_U} \left(\int_{\mathcal{D}_4(\lambda,t)} 2e^{-(u+v)}\,dudv \right) dt \Bigg). \end{aligned} \tag{1.4.60}$$

where

$$\mathcal{D}_1(\lambda) = \Big[\max\Big(l_{\rho,U}, \frac{\pi}{\sqrt{\lambda}}\Big), 3l_{\rho,U}\Big], \qquad \mathcal{D}_3(\lambda) = \Big\{(x,y) \in \big[l_{\rho,U}, 2l_{\rho,U}\big]^2, y \geqslant \max\Big(x, \frac{\pi}{\sqrt{\lambda}}\Big)\Big\}$$

$$\mathcal{D}_2(\lambda) = \Big[\max\Big(2l_{\rho,U}, \frac{2\pi}{\sqrt{\lambda}} + \frac{\gamma}{8\pi^2}\Big), 3l_{\rho,U}\Big], \qquad \mathcal{D}_4(\lambda,t) = \Big\{(x,y) \in \big[l_{\rho,U}, 2l_{\rho,U}\big]^2,\, y \geqslant x \geqslant \Big(\frac{\pi}{\sqrt{\lambda}} + \frac{\tau(t)}{2y^3}\Big)\Big\}.$$

and γ (resp. $\tau(t)$) is given in Proposition 1.2.1 (resp. Proposition 1.2.2).

Then, with probability $1 - O(L^{-\infty})$, for every $\beta > 1$, the counting function $\mathbf{N}_2^U$ of Γ_2 satisfies

$$\forall \lambda > 0 \qquad \mathbf{N}_2^U(\lambda) = J(\lambda) + R_\beta \tag{1.4.61}$$

with $R_\beta = O(\rho^\beta)$.

Proof. A chain of $\mathcal{P}_2$ is either a single piece Δ_i or a pair $\{\Delta_i, \Delta_j\}$. In the first case, the energy levels of Δ_i are functions of a single parameter, the length $\mathbf{l}_i \in [l_{\rho,U}, 3l_{\rho,U}]$. When $I = \{\Delta_i, \Delta_j\}$, the energy levels of I are given by the triplet of parameters $(\mathbf{l}_i, \mathbf{l}_j, \mathbf{d}_{ij}) \in [l_{\rho,U}, 3l_{\rho,U}] \times [l_{\rho,U}, 3l_{\rho,U}] \times [0, \sigma_U]$.

Fix $\beta > 1$. We set a discretization of the above parameters with a constant step ρ^β. We get a sequence of approximated energy levels Γ_2^β. We prove that the Hausdorff distance between Γ_2 and Γ_2^β is of order $O(\rho^\beta)$. So it is sufficient to compute the counting function of Γ_2^β at order $O(\rho^\beta)$. Since the Poisson process fix the statistics of pieces, one knows how many times each approximated energy level appears in Γ_2^β. We will use the expansion of the energy levels given by Proposition 1.2.1 and Proposition 1.2.2 to replace the condition "below λ" by some conditions on the parameters.

We now give the details. For $I \in \mathcal{P}_2$, we distinguish two cases.

(i) If $I = \{\Delta_i\}$ then $\mathbf{l}_i \in [k\rho^\beta, (k+1)\rho^\beta)$ for some k and for $a \in \{1,2\}$, we approximate the a-th energy level of the piece Δ_i by

$$f_a(k) = f^U\big([0, k\rho^\beta], a\big). \tag{1.4.62}$$

The parameter k goes from $K_1 = \lfloor l_{\rho,U}\rho^{-\beta}\rfloor$ to $K_3 = \lfloor 3l_{\rho,U}\rho^{-\beta}\rfloor$. For $a \in \{1,2\}$, we define

$$p_a(k) = \#\Big\{\{\Delta_i\} \in \mathcal{P}_2,\, \mathbf{l}_i \in [k\rho^\beta, (k+1)\rho^\beta)\Big\}; \tag{1.4.63}$$

(ii) if $I = \big(\Delta_j, \Delta_k\big)$ then $\mathbf{l}_j \in [r\rho^\beta, (r+1)\rho^\beta)$, $\mathbf{l}_k \in [s\rho^\beta, (s+1)\rho^\beta)$ and $\mathbf{d}_{j,k} \in [d\rho^\beta, (d+1)\rho^\beta)$ for some r, s and d and, for $a \in \{1,2\}$, we approximate the a-th energy level of the pair (Δ_j, Δ_k) by

$$g_a(r,s,d) = f^U\Big(\big\{[-r\rho^\beta, 0], [d\rho^\beta, d\rho^\beta + s\rho^\beta]\big\}, 1\Big) \tag{1.4.64}$$

Here the parameters r, s go from K_1 to $K_2 = \lfloor 2l_{\rho,U}\rho^{-\beta}\rfloor$ and the parameter d goes from 0 to $D = \lfloor \sigma_U \rho^{-\beta}\rfloor$. For $a \in \{1,2\}$, we set

(1.4.65)
$$q_a(r,s,d) = \#\Big\{\big(\Delta_j,\Delta_k\big) \in \mathcal{P}_2,\ \mathbf{l}_j \in [r\rho^\beta,(r+1)\rho^\beta),\ \mathbf{l}_k \in [s\rho^\beta,(s+1)\rho^\beta),\ \mathbf{d}_{j,k} \in [d\rho^\beta,(d+1)\rho^\beta)\Big\}.$$

Let Γ_2^β be the set of approximated energy levels

$$\Gamma_2^\beta = \Big\{f_a(k),\ a \in \{1,2\},\ k \in [\![K_1,K_3]\!]\Big\} \cup \Big\{g_a(r,s,d),\ a \in \{1,2\}, r,s \in [\![K_1,K_2]\!],\ d \in [\![0,D]\!]\Big\}. \tag{1.4.66}$$

Lemma 1.4.6. *Recall the definition of the Hausdorff distance d_∞ on $\mathcal{P}(\mathbb{R})$. For $(A,B) \in \mathcal{P}(\mathbb{R})^2$,*

$$d_\infty(A,B) := \sup_{a\in A}\inf_{b\in B}|a-b|.$$

For $\beta > 1$, there exists $C > 0$ such that

$$d_\infty(\Gamma_2,\Gamma_2^\beta) \leqslant C\rho^\beta.$$

Proof. (of Lemma 1.4.6) By construction of Γ_2^β, from $x \in \Gamma_2$ we compute $x^\beta \in \Gamma_2^\beta$. We study the cases separately.

(i) Either $x^\beta = f_1(k)$, then x belongs to $[f_1(k+1), f_1(k)]$. Note that

$$f_1(k) - f_1(k+1) = \frac{2\pi^2}{k^3\rho^{2\beta}} + O\Big(\frac{1}{k^4\rho^{2\beta}}\Big), \tag{1.4.67}$$

(ii) Or $x^\beta = f_2(k)$, then x belongs to $[f_2(k+1), f_2(k)]$. Using (1.4.56), one computes that

$$f_2(k) - f_2(k+1) = \frac{8\pi^2}{k^3\rho^{2\beta}} + O\Big(\frac{1}{k^4\rho^{2\beta}}\Big), \tag{1.4.68}$$

(iii) Or $x^\beta = g_1(r,s,d)$. Without lost of generality, assume that $r < s$. Then x belongs to $[g_1(r,s+1,d), g_2(r,s,d)]$. Using (1.4.57), one computes that

$$g_1(r,s,d) - g_1(r,s+1,d) = \frac{2\pi^2}{s^3\rho^{2\beta}} + O\Big(\frac{1}{s^4\rho^{2\beta}}\Big), \tag{1.4.69}$$

(iv) Or $x^\beta = g_2(r,s,d)$. Without lost of generality, assume that $r < s$. Then x belongs to $[g_2(r+1,s,d), g_2(r,s,d)]$. Using (1.4.58), one computes that

$$g_2(r,s,d) - g_2(r+1,s,d) = \frac{2\pi^2}{r^3\rho^{2\beta}} + O\Big(\frac{1}{r^4\rho^{2\beta}}\Big), \tag{1.4.70}$$

So

$$\inf_{b\in\Gamma_2^\beta}|x-b| \leqslant C\frac{1}{r^3\rho^{2\beta}}$$

Since k (resp. r and s) is of order $O(l_{\rho,U}\rho^{-\beta})$, we conclude

$$\forall x \in \Gamma_2 \qquad \inf_{b\in\Gamma_2^\beta}|x-b| \leqslant C\rho^\beta.$$

□

By Lemma 1.4.4 and Lemma 1.4.6, for $\beta > 1$,

$$\frac{1}{L}\left|\#\left\{x \in \Gamma_2, x \in (-\infty, \lambda]\right\} - \#\left\{x \in \Gamma_2^\beta, x \in (-\infty, \lambda]\right\}\right| \leqslant \frac{\#\Gamma_2}{L} d_\infty(\Gamma_2, \Gamma_2^\beta) \leqslant C\rho^{\beta+1}. \tag{1.4.71}$$

As for (1.4.59), let $\mathbf{N}_{2,\beta}^U$ be the counting function of Γ_2^β

$$\forall \lambda > 0 \qquad \mathbf{N}_{2,\beta}^U(\lambda) := \lim_{L\to\infty} \frac{1}{L}\#\left\{x \in \Gamma_2^\beta, x \in (-\infty, \lambda]\right\}. \tag{1.4.72}$$

Then, for $\beta > 1$,

$$\mathbf{N}_2^U(\lambda) = \mathbf{N}_{2,\beta}^U(\lambda) + O(\rho^{\beta+1}). \tag{1.4.73}$$

We will estimate $\mathbf{N}_{2,\beta}^U$. Set $\lambda \in (\min \Gamma_2^\beta, \max \Gamma_2^\beta)$. We translate the condition "energy level smaller than λ" in term of bounds for the parameters of the discretization. For $k \in [\![K_1, K_3 - 1]\!]$,

$$f_1(k) \leqslant \lambda \qquad \Leftrightarrow \qquad k \geqslant \frac{\pi}{\sqrt{\lambda}}\rho^{-\beta} \tag{1.4.74}$$

Using the asymptotic (1.4.56), for large k, we compute that

$$f_2(k) = \frac{4\pi^2}{\left(k - \frac{\gamma}{8\pi^2\rho^\beta}\right)^2 \rho^{2\beta}} + R_k \tag{1.4.75}$$

with $R_k = o(\frac{1}{k^3\rho^{2\beta}})$.

The remainder R_k is negligible with respect to the gap between $f_2(k+1)$ and $f_2(k)$ (see (1.4.68)). It yields

$$\begin{aligned} f_2(k) \leqslant \lambda \qquad &\Leftrightarrow \qquad \frac{4\pi^2}{\left(k - \frac{\gamma}{8\pi^2\rho^\beta}\right)^2} \leqslant \lambda\rho^{2\beta} \\ &\Leftrightarrow \qquad k \geqslant \left(\frac{2\pi}{\sqrt{\lambda}} + \frac{\gamma}{8\pi^2}\right)\rho^{-\beta} \end{aligned} \tag{1.4.76}$$

For $r, s \in [\![K_1, K_2 - 1]\!]$ and $d \in [\![0, D-1]\!]$, assuming $r \leqslant s$

$$g_1(r, s, d) \leqslant \lambda \qquad \Leftrightarrow \qquad s \geqslant \max\left(r, \frac{\pi}{\sqrt{\lambda}}\rho^{-\beta}\right) \tag{1.4.77}$$

Using the asymptotic (1.4.58), for large $r < s$ and $d \in [\![0, D]\!]$, we compute

$$g_2(r, s, d) = \frac{\pi^2}{\left(r - \frac{\tau(d\rho^\beta)}{2s^3\rho^{4\beta}}\right)^2 \rho^{2\beta}} + S_{r,s,d} \tag{1.4.78}$$

with $S_{r,s,d} = o(\frac{1}{r^3\rho^{2\beta}})$.

The remainder $S_{r,s,d}$ is negligible with respect to the gap between $g_2(r+1, s, d)$ and $g_2(r, s, d)$ (see (1.4.70)). It yields, for large $r \leqslant s$,

$$\begin{aligned} g_2(r, s, d) \leqslant \lambda \qquad &\Leftrightarrow \qquad \frac{\pi^2}{\left(r - \frac{\sigma(d\rho^\beta)}{2s^3\rho^{4\beta}}\right)^2} \leqslant \lambda\rho^{2\beta} \\ &\Leftrightarrow \qquad r \geqslant \left(\frac{\pi}{\sqrt{\lambda}} + \frac{\sigma(d\rho^\beta)}{2s^3\rho^{3\beta}}\right)\rho^{-\beta} \end{aligned} \tag{1.4.79}$$

Thus, combining (1.4.74), (1.4.76), (1.4.77) and (1.4.79), for $\lambda \in (\min \Gamma_2^\beta, \max \Gamma_2^\beta)$,

$$\#\Big\{x \in \Gamma_2^\beta, x \in (-\infty, \lambda]\Big\} = \sum_{k=k_1(\lambda)}^{K_3-1} p_1(k) + \sum_{k=k_2(\lambda)}^{K_3-1} p_2(k) + \sum_{d=0}^{D-1} \sum_{(r,s)\in B(\lambda)} \varepsilon(r,s) q_1(r,s,d) + \sum_{d=0}^{D-1} \sum_{(r,s)\in C(\lambda,d)} \varepsilon(r,s) q_2(r,s,d) \tag{1.4.80}$$

where $p_a(k)$ (resp. $g_a(r,s,d)$) is given by (1.4.63) (resp. (1.4.64)) and

$$\begin{aligned}
k_1(\lambda) &:= \left\lceil \frac{\pi}{\sqrt{\lambda}} \rho^{-\beta} \right\rceil, \qquad k_2(\lambda) := \left\lceil \Big(\frac{2\pi}{\sqrt{\lambda}} + \frac{\gamma}{8\pi^2}\Big)\rho^{-\beta} \right\rceil, \\
B(\lambda) &:= \left\{(u,v) \in [\![K_1, K_2-1]\!]^2,\ v \geqslant \max\Big(u, \frac{\pi}{\sqrt{\lambda}}\rho^{-\beta}\Big)\right\}, \\
C(\lambda,d) &:= \left\{(u,v) \in [\![K_1, K_2-1]\!]^2,\ v \geqslant u \geqslant \Big(\frac{\pi}{\sqrt{\lambda}} + \frac{\tau(d\rho^\beta)}{2v^3\rho^{3\beta}}\Big)\rho^{-\beta}\right\}, \\
\varepsilon(r,s) &:= 2 \text{ if } r \neq s \text{ and } \varepsilon(r,s) := 1 \text{ otherwise.}
\end{aligned}$$

By Lemma 1.4.9, for $\eta \in (\frac{2}{3}, 1)$, with probability $1 - O(L^{-\infty})$, we have for $a \in \{1,2\}$ and for k, r, s, d

$$\begin{aligned}
p_a(k) &= L\big(1 - \sigma_U e^{-l_{\rho,U}}\big)^2 e^{-k\rho^\beta}(1 - e^{-\rho^\beta}) + r_a(k) L^\eta \\
q_a(r,s,d) &= L\big(1 - \sigma_U e^{-l_{\rho,U}}\big)^2 e^{-(r+s)\rho^\beta} \rho^\beta \big(1 - e^{-\rho^\beta}\big)^2 + s_a(r,s,d) L^\eta
\end{aligned} \tag{1.4.81}$$

with $r_a(k)$ and $s_a(r,s,d)$ bounded for every k, r, s and d.

Using dominated convergence theorem, we get

$$\begin{aligned}
\mathbf{N}_{2,\beta}^U(\lambda) &= \lim_{L\to\infty} \frac{1}{L} \#\Big\{x \in \Gamma_2^\beta, x \in (-\infty, \lambda]\Big\} \\
&= \Big(1 - \sigma_U e^{-l_{\rho,U}}\Big)^2 \Bigg(\sum_{k=k_1(\lambda)}^{K_3-1} e^{-k\rho^\beta}(1 - e^{-\rho^\beta}) + \sum_{k=k_2(\lambda)}^{K_3-1} e^{-k\rho^\beta}(1 - e^{-\rho^\beta}) \\
&\quad + \sum_{d=0}^{D-1} \sum_{(r,s)\in B(\lambda)} \varepsilon(r,s) e^{-(r+s)\rho^\beta} \rho^\beta (1 - e^{-\rho^\beta})^2 \\
&\quad + \sum_{d=0}^{D-1} \sum_{(r,s)\in C(\lambda,d)} \varepsilon(r,s) e^{-(r+s)\rho^\beta} \rho^\beta (1 - e^{-\rho^\beta})^2 \Bigg)
\end{aligned} \tag{1.4.82}$$

Let Σ_3 be the third sum in Equation (1.4.82).

$$\begin{aligned}
\Sigma_3 &:= \sum_{d=0}^{D-1} \sum_{(r,s)\in B(\lambda)} \varepsilon(r,s) e^{-(r+s)\rho^\beta} \rho^\beta (1 - e^{-\rho^\beta})^2 \\
&= \int_0^{D\rho^\beta} \Bigg(\sum_{(r,s)\in B(\lambda)} \int_{r\rho^\beta}^{(r+1)\rho^\beta} \int_{s\rho^\beta}^{(s+1)\rho^\beta} 2e^{-(u+v)}\, du dv \Bigg) dt - D\rho^\beta (1 - e^{-\rho^\beta})^2 \sum_{r=k_1(\lambda)}^{K_2-1} e^{-2r\rho^\beta}
\end{aligned} \tag{1.4.83}$$

$$= \int_0^{D\rho^\beta} \int_{\mathcal{B}_\beta(\lambda)} 2e^{-(u+v)} du dv - \frac{D}{2} \rho^{2\beta} e^{-2k_1(\lambda)\rho^\beta} \big(1 + o(1)\big) \tag{1.4.84}$$

where

$$\mathcal{B}_\beta(\lambda) = \left\{(x,y) \in \left[K_1\rho^\beta, K_2\rho^\beta\right],\, y \geqslant \max\left(x, \left\lceil \frac{\pi}{\sqrt{\lambda}}\rho^{-\beta}\right\rceil \rho^\beta\right)\right\}.$$

Set

$$\mathcal{B}(\lambda) = \left\{(x,y) \in [l_{\rho,U}, 2l_{\rho,U}]^2, y \geqslant \max\left(x, \frac{\pi}{\sqrt{\lambda}}\right)\right\}.$$

Using that, for any $x > 0$,

$$\left|x - \lceil x\rho^{-\beta}\rceil\rho^\beta\right| \leqslant \rho^\beta \qquad \text{and} \qquad \left|x - \lfloor x\rho^{-\beta}\rfloor\rho^\beta\right| \leqslant \rho^\beta$$

we get

$$\begin{aligned}
(1.4.85)\quad \left|\Sigma_3 - \int_0^{\sigma_U}\left(\int_{\mathcal{B}(\lambda)} 2e^{-(u+v)}\,dudv\right)dt\right| &\leqslant \rho^\beta\left(2e^{-2l_{\rho,U}}\int_{\mathcal{B}(\lambda)} du + 2\sigma_U\int_{\mathcal{B}(\lambda)\setminus\mathcal{B}_\beta(\lambda)} du + \sigma_U e^{-2l_{\rho,U}}\right)\\
&\leqslant \rho^\beta\left(2e^{-2l_{\rho,U}}(2l_{\rho,U})^2 + 8\sigma_U\rho^\beta + \sigma_U e^{-2l_{\rho,U}}\right)\\
&\leqslant C\rho^\beta.
\end{aligned}$$

The other terms in Equation (1.4.82) can be handled in much the same way.

So, for $\lambda \in (\min\Gamma_2^\beta, \max\Gamma_2^\beta)$,

$$\begin{aligned}
(1.4.86)\qquad N_{2,\beta}^U(\lambda) = \left(1 - \sigma_U e^{-l_{\rho,U}}\right)^2 &\left(\int_{\frac{\pi}{\sqrt{\lambda}}}^{3l_{\rho,U}} e^{-u}\,du + \int_{\left(\frac{2\pi}{\sqrt{\lambda}}+\frac{\gamma}{8\pi^2}\right)}^{3l_{\rho,U}} e^{-u}\,du\right.\\
&+ \int_0^{\sigma_U}\left(\int_{\mathcal{B}(\lambda)} 2e^{-(u+v)}\,dudv\right)dt\\
&\left.+ \int_0^{\sigma_U}\left(\int_{\mathcal{C}(\lambda,t)} 2e^{-(u+v)}\,dudv\right)dt\right) + O(\rho^\beta)
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{B}(\lambda) &= \left\{(x,y) \in [l_{\rho,U}, 2l_{\rho,U}]^2, y \geqslant \max\left(x, \frac{\pi}{\sqrt{\lambda}}\right)\right\}\\
\mathcal{C}(\lambda,d) &= \left\{(x,y) \in [l_{\rho,U}, 2l_{\rho,U}]^2,\, y \geqslant x \geqslant \left(\frac{\pi}{\sqrt{\lambda}} + \frac{\tau(d)}{2y^3}\right)\right\}.
\end{aligned}$$

Combining (1.4.73) and (1.4.86), it yields

$$(1.4.87)\qquad \mathbf{N}_2^U(\lambda) = J(\lambda) + O(\rho^\beta).$$

where J is given by (1.4.60). It concludes the proof of Proposition 1.4.4. □

The following corollary states that Assumption 1.4.2 is true for $p = 2$.

Corollary 1.4.3. *Set $\delta \in (0,1)$. For every $x \in \Gamma_2$ and in the thermodynamic limit,*

$$(1.4.88)\qquad \frac{1}{n}\#\{y \in \Gamma_2,\, y = x\} = O(\rho^{2-\delta})$$

Proof. Note that each domain of integration in the RHS of (1.4.60) is smooth for $\lambda \in (0, +\infty)$. So, J is continuous on $(\min \Gamma_2, \max \Gamma_2)$. By Proposition 1.4.4, we compute for $\beta > 1$, $h > 0$ and $x \in \Gamma_2$

$$\frac{1}{n}\#\Big\{y \in \Gamma_2,\, y = x\Big\} \leqslant \frac{L}{n}\Big|\mathbf{N}_2^U(L, x+h) - \mathbf{N}_2^U(L, x-h)\Big| \tag{1.4.89}$$

$$\leqslant \frac{L}{n}\Bigg(\Big|\mathbf{N}_2^U(L, x+h) - \mathbf{N}_2^U(x+h)\Big| + \Big|\mathbf{N}_2^U(L, x-h) - \mathbf{N}_2^U(x-h)\Big| + \Big|J(x+h) - J(x-h)\Big| + O(\rho^\beta)\Bigg)$$

$$\to_{\substack{L\to+\infty \\ \frac{n}{L}\to\rho}} \frac{1}{\rho}\Big|J(x+h) - J(x-h)\Big| + O(\rho^{\beta-1}) \tag{1.4.90}$$

Taking $\beta > 2$ and $h \to 0$, we conclude the proof of Corollary 1.4.3. □

1.4.2.3 Construction of an approximated ground state

We use the counting function $\mathbf{N}_2^U$ (see (1.4.59) to build an approximate ground state for $H^U(L, n)$.

Note that, for $d \in [0, \sigma_U]$ and $\min \Gamma_2 < \lambda < \mu < \max \Gamma_2$,

$$\forall i \in \{1, 2, 3\} \quad \mathcal{D}_i(\lambda) \subsetneq \mathcal{D}_i(\mu) \qquad \text{and} \qquad \mathcal{D}_4(\lambda, d) \subsetneq \mathcal{D}_4(\mu, d). \tag{1.4.91}$$

So J is continuous and increasing on $(\min \Gamma_2, \max \Gamma_2)$. Note also that, by Lemma 1.4.4, we have, for $\lambda > \max \Gamma_2$, $\mathbf{N}_2^U(\lambda) > 2\rho$ and, for $0 < \lambda < \min \Gamma_2$, $\mathbf{N}_2^U(\lambda) = 0$.

Definition 1.4.3. *For $\beta > 2$, the $\beta-$**Fermi energy level** λ_ρ^β is the unique preimage of $\rho - R_{\beta+1}$ by the function J, with J and $R_{\beta+1}$ defined in Proposition 1.4.4. We call $\beta-$ **Fermi length** δ_ρ^β the length of a piece Δ for which the ground state energy of the Dirichlet Laplacian $-\frac{d^2}{dx^2}^D_{|\Delta}$ is equal to the $\beta-$Fermi energy level λ_ρ^β.*

By Definition 1.4.3 and Proposition 1.4.4, we have for any $\beta > 2$

$$\mathbf{N}_2^U(\lambda_\rho^\beta) = \rho \quad \text{and} \quad \delta_\rho^\beta = \frac{\pi}{\sqrt{\lambda_\rho^\beta}}. \tag{1.4.92}$$

Fix $\beta > 2$ and consider all energy levels of Γ_2 below the $\beta-$Fermi energy level λ_ρ^β and fill the chains by induction following the proof of the Proposition 1.4.2. Then, by definition, we get an occupation $Q^\beta = (q_i^\beta)_{i\in[\![1,m_L]\!]}$ for which the number of particles in $\mathcal{P}_2$ is equal to $n_{2,Q^\beta} = \min\big(n, LN_2^U(L, \lambda_\rho^\beta)\big)$ with $N_2^U(L, .)$ given by (1.4.59). For L large enough (that depends on ρ and β),

$$|N_2^U(L, \lambda_\rho^\beta) - \mathbf{N}_2^U(\lambda_\rho^\beta)| \leqslant \rho^{\beta+1}. \tag{1.4.93}$$

So, using (1.4.92), in the thermodynamic limit, the number of particles in the chains of $\mathcal{N}_2$ is less than $Cn\rho^\beta$ for some constant $C > 0$. Remembering $\beta > 2$ and the left inequality of (1.4.18), for ρ small enough, one can change $(q_i^\beta)_{i\in\mathcal{N}_2}$ so that the occupation Q^β belongs to $\mathcal{Q}$.

Using Proposition 1.4.4 and more specifically the R.H.S of (1.4.60), one can get an approximate description of Q^β in term of the pieces' lengths and δ_ρ^β. Disregarding $O(n\rho^\beta)$ particles, it means that

★ for a piece $\Delta_i \in \mathcal{P}_2$

(a) if $\mathbf{l}_i < \delta_\rho^\beta$, then $q_i^\beta = 0$

(b) if $\mathbf{l}_i \in \Big[\delta_\rho^\beta,\, 2\delta_\rho^\beta + \frac{\gamma}{8\pi^2}\Big)$ then $q_i^\beta = 1$

(c) if $\mathbf{l}_i \geqslant 2\delta_\rho^\beta + \frac{\gamma}{8\pi^2}$ then $q_i^\beta = 2$;

★ for a pair $(\Delta_j, \Delta_k) \in \mathcal{P}_2$, assume $\mathbf{l}_j \leqslant \mathbf{l}_k$

(a) if $\mathbf{l}_k < \delta_\rho^\beta$ then $q_j^\beta = q_k^\beta = 0$,

(b) if $\mathbf{l}_j \in \Big[\delta_\rho^\beta,\, \delta_\rho^\beta + \frac{\tau(\mathbf{d}_{j,k})}{2\mathbf{l}_k^3}\Big)$ then $q_j^\beta = 0$ and $q_k^\beta = 1$

(c) if $\mathbf{l}_j \geqslant \delta_\rho^\beta + \frac{\tau(\mathbf{d}_{j,k})}{2\mathbf{l}_k^3}$ then $q_j^\beta = q_k^\beta = 1$

We can compare the occupation Q^β with the occupation of the free operator Q^0. Recall that in Q^0 there are k particles in pieces of length between kl_ρ and $(k+1)l_\rho$ where l_ρ is given by (1.1.10). We compute

$$\begin{aligned}
(1.4.94)\quad \int_{\mathcal{D}_1(E_\rho)} e^{-u}\,du + \int_{\mathcal{D}_2(E_\rho)} e^{-u}\,du &= e^{-l_\rho} - e^{-3l_{\rho,U}} + e^{-2l_\rho - \frac{\gamma}{8\pi^2}} - e^{-3l_{\rho,U}} \\
&= \rho\Big(1-\rho+O(\rho^2)\Big)\Big(1+e^{-\frac{\gamma}{8\pi^2}}\rho+O(\rho^2)\Big),
\end{aligned}$$

$$\begin{aligned}
(1.4.95)\quad \int_0^{\sigma_U}\int_{\mathcal{D}_3(E_\rho)} 2e^{-(u+v)}\,dtdudv &= 2\sigma_U \int_{l_{\rho,U}}^{l_\rho}\int_{l_\rho}^{2l_{\rho,U}} e^{-(u+v)}\,dudv + 2\sigma_U \int_{2l_{\rho,U}\geqslant v\geqslant u\geqslant l_\rho} e^{-(u+v)}\,dudv \\
&= \sigma_U\rho^2\big(1+O(\rho)\big),
\end{aligned}$$

$$\begin{aligned}
(1.4.96)\quad \int_0^{\sigma_U}\int_{\mathcal{D}_4(E_\rho,t)} 2e^{-(u+v)}\,dtdudv &\leqslant 2\sigma_U \int_{2l_{\rho,U}\geqslant v\geqslant u\geqslant l_\rho} e^{-(u+v)}\,dudv \\
&= \sigma_U\rho^2\big(1+O(\rho)\big).
\end{aligned}$$

So,

$$\begin{aligned}
(1.4.97)\quad \mathbf{N}_p^U(E_\rho) &\leqslant \rho\Big(1-2\sigma_U\rho+O(\rho^2)\Big)\Big(\Big(1-\rho+O(\rho^2)\Big)\Big(1+e^{-\frac{\gamma}{8\pi^2}}\rho+O(\rho^2)\Big)+2\sigma_U\rho\Big(1+O(\rho)\Big)\Big) \\
&= \rho\Big(1+\rho\big(e^{-\frac{\gamma}{8\pi^2}}-1\big)+O(\rho^2)\Big) \\
&< \rho.
\end{aligned}$$

Thus, $E_\rho < \lambda_\rho^\beta$ meaning that $l_{\rho,U} < \delta_\rho^\beta < l_\rho$. For ρ small enough, $2l_{\rho,U} + \frac{\gamma}{8\pi^2} \geqslant 2l_\rho$ so $2\delta_\rho^\beta + \frac{\gamma}{8\pi^2} \geqslant 2l_\rho$. It means that when interactions are on, we remove one particle from pieces of length close to $2l_\rho$ but larger and put it in empty pieces of length close to l_ρ but smaller. Similarly, for pair of pieces of length close to l_ρ, one takes one particle out of the pair to fill a smaller piece that does not interact.

Hence, using (1.4.14) and (1.4.15), we define the approximated ground state

$$\Psi^\beta(L,n) = \Psi^U(L,n,Q^\beta). \tag{1.4.98}$$

Proposition 1.4.5. *Using the notations of Proposition 1.4.4, define the map $\mathcal{J}$ by*

$$\begin{aligned}
(1.4.99)\quad \mathcal{J}(\lambda) = L(1-\sigma_U e^{-l_{\rho,U}})^2\Bigg(&\int_{\mathcal{D}_1(\lambda_\rho^\beta)} f^U([0,u],1)e^{-u}\,du + \int_{\mathcal{D}_2(\lambda_\rho^\beta)} f^U([0,u],2)e^{-u}\,du \\
&+ \int_0^{\sigma_U}\int_{\mathcal{D}_3(\lambda_\rho^\beta)} 2e^{-(u+v)} f^U\big(([-u,0],[t,v+t]),1\big)\,dtdudv \\
&+ \int_0^{\sigma_U}\int_{\mathcal{D}_4(\lambda_\rho^\beta,t)} 2e^{-(u+v)} f^U\big([-u,0],[t,v+t]),2\big)\,dtdudv\Bigg).
\end{aligned}$$

For $\beta > 2$, for λ_ρ^β and $\Psi^\beta(L,n)$ defined as above, for $\delta \in (0,1)$ and $0 < \rho < \rho_\delta$ small enough, then, in the thermodynamic limit, with probability 1*,*

$$\lim_{\substack{L\to+\infty \\ \frac{n}{L}\to\rho}} \frac{\big\langle H^U(L,n)\Psi^\beta(L,n), \Psi^\beta(L,n)\big\rangle}{n} = \frac{1}{\rho}\mathcal{J}(\lambda_\rho^\beta) + O(\rho^{2-\delta}). \tag{1.4.100}$$

Proof. Fix $\beta > 2$. By construction of $\Psi^\beta(L,n)$ and using (1.4.16), we write

$$\big\langle H^U(L,n)\Psi^\beta(L,n), \Psi^\beta(L,n)\big\rangle = E^U(L,n,Q^\beta) = E^U_{\mathcal{P}_2}(Q^\beta) + E^U_{\mathcal{N}_2}(Q^\beta) \tag{1.4.101}$$

By Proposition 1.4.1, we know that, for $\delta \in (0,1)$ and $\rho \in (0, \rho_{2,\delta})$,

$$E^U_{\mathcal{N}_2}(Q^\beta) \leqslant n\rho^{2-\delta} \tag{1.4.102}$$

It gives the amount of energy produced by particles we do not control precisely. One can check that it fits with the remaining part in (1.4.100).

Otherwise, we compute $E_{\mathcal{P}_2}(Q^\beta)$ using Γ_2^β, the approximate sequence of levels of energy for the good pieces that we introduced in the proof of Proposition 1.4.4. Following the method and the notations of Proposition 1.4.4, one derives the next formula. With probability $1 - O(L^{-\infty})$ and $\eta \in (\frac{2}{3}, 1)$,

$$\begin{aligned} E_{\mathcal{P}_2}(Q^\beta) = L(1-\sigma_U e^{-l_{\rho,U}})^2 \Bigg(&\int_{\mathcal{D}_1(\lambda_\rho^\beta)} f^U([0,u],1)e^{-u}\,du + \int_{\mathcal{D}_2(\lambda_\rho^\beta)} f^U([0,u],2)e^{-u}\,du \\ &+ \int_0^{\sigma_U}\int_{\mathcal{D}_3(\lambda_\rho^\beta)} 2e^{-(u+v)} f^U\big(([-u,0],[t,v+t]),1\big)\,dtdudv \\ &+ \int_0^{\sigma_U}\int_{\mathcal{D}_4(\lambda_\rho^\beta,t)} 2e^{-(u+v)} f^U\big(([-u,0],[t,v+t]),2\big)\,dtdudv\Bigg) + O(L\rho^{\beta+1}) + O(L^\eta). \end{aligned} \tag{1.4.103}$$

Thus, in the thermodynamic limit, one derives

$$\lim_{\substack{L\to+\infty \\ \frac{n}{L}\to\rho}} \frac{\big\langle H^U(L,n)\Psi^\beta(L,n), \Psi^\beta(L,n)\big\rangle}{n} = \frac{1}{\rho}\mathcal{J}(\lambda_\rho^\beta) + O(\rho^{2-\delta}) \tag{1.4.104}$$

It concludes the proof of Proposition 1.4.5. □

Remark 1.4.3. *One could also set*

$$\Psi^\beta(L,n) = \Bigg(\bigwedge_{I\in\mathcal{P}_2} \psi^U\Big(I,(q_i^\beta)_{i\in I}\Big)\Bigg) \wedge \Bigg(\bigwedge_{I\in\mathcal{N}_2}\bigwedge_{i\in I} \psi^0\Big(\Delta_i, q_i^\beta\Big)\Bigg) \tag{1.4.105}$$

meaning that, outside of $\mathcal{P}_2$, it behaves like a free state. By Remark 1.4.1, both states (1.4.98) and (1.4.105) give, up to the order $O(\rho^{2-\delta})$, the same amount of energy per particle in the thermodynamic limit.

1.4.2.4 Comparison to the ground state energy

We compare our approximate ground state energy to the ground state energy, in the thermodynamic limit.

Proposition 1.4.6. *For $L > 0$, let $\Psi^U(L,n)$ be a ground state of $H^U(L,n)$. For $\delta \in (0,1)$ and $\beta > 3$, there exists $\rho_\delta > 0$ such that, for $\rho \in (0,\rho_\delta)$, the approximated ground state $\Psi^\beta(L,n)$, given in Subsection 1.4.2.3, satisfies in the thermodynamic limit, with probability $1 - O(L^{-\infty})$,*

$$\frac{\langle H^U(L,n)\Psi^U(L,n), \Psi^U(L,n)\rangle}{n} = \frac{\langle H^U(L,n)\Psi^\beta(L,n), \Psi^\beta(L,n)\rangle}{n} + O(\rho^{2-\delta}). \tag{1.4.106}$$

Proof. We drop the indices "L" and "n". Let Ψ^U be a ground state of H^U. Using the notations of Subsection 1.4.1.2, we have

$$\Psi^U = \sum_{Q\in\mathcal{Q}} \lambda(Q)\Psi^U_{\mathcal{P}_2}(Q) \wedge \Psi^U_{\mathcal{N}_2}(Q) \tag{1.4.107}$$

with $\lambda(Q) \in \mathbb{C}$, $\Psi^U_{\mathcal{P}_2}(Q)$ and $\Psi^U_{\mathcal{N}_2}(Q)$ given by (1.4.15). Then,

$$\begin{aligned}\langle H^U\Psi^U, \Psi^U\rangle &= \sum_{Q\in\mathcal{Q}} |\lambda(Q)|^2\Big(E^U_{\mathcal{P}_2}(Q) + E^U_{\mathcal{N}_2}(Q)\Big)\\ &\geqslant \min_{\mathcal{Q}} E^U_{\mathcal{P}_2}\end{aligned} \tag{1.4.108}$$

Fix $\beta > 3$ and $\delta \in (0,1)$. Let $\Psi^\beta = \Psi^U(Q^\beta)$ be the state given by the construction of Subsection 1.4.2.3 and $\rho_{2,\delta} > 0$ given by Proposition 1.4.1. For $\rho \in (0, \rho_{2,\delta})$, in the thermodynamic limit, we compute

$$\begin{aligned}0 \leqslant \langle H^U\Psi^\beta, \Psi^\beta\rangle - \langle H^U\Psi^U, \Psi^U\rangle &\leqslant E^U_{\mathcal{P}_2}(Q^\beta) + E^U_{\mathcal{N}_2}(Q^\beta) - \min_{\mathcal{Q}} E_{\mathcal{P}_2}\\ &\leqslant E^U_{\mathcal{P}_2}(Q^\beta) - \min_{\mathcal{Q}} E^U_{\mathcal{P}_2} + n\rho^{2-\delta}.\end{aligned} \tag{1.4.109}$$

If Q is an occupation that minimizes $E_{\mathcal{P}_2}$ on $\mathcal{Q}$ then, by Proposition 1.4.2,

$$E^U_{\mathcal{P}_2}(Q^\beta) - E^U_{\mathcal{P}_2}(Q) = \sum_{k=n_{2,Q}}^{n_{2,Q^\beta}} a_k \tag{1.4.110}$$

where $n_{2,Q}$ is the number of particles in $\mathcal{P}_2$ for the occupation Q. So,

$$(\min\Gamma_2)\frac{n_{2,Q^\beta} - n_{2,Q}}{L} \leqslant \frac{E^U_{\mathcal{P}_2}(Q^\beta) - E^U_{\mathcal{P}_2}(Q)}{L} \leqslant (\max\Gamma_2)\frac{n_{2,Q^\beta} - n_{2,Q}}{L}. \tag{1.4.111}$$

By Lemma 1.4.2,

$$0 \leqslant \frac{n_{2,Q^\beta} - n_{2,Q}}{L} \leqslant \frac{n - n_{2,Q}}{L} \leqslant \frac{n\rho^{2-\delta}}{L} \tag{1.4.112}$$

for $\rho \in (0, \rho_{2,\delta})$. Combining (1.4.111) and (1.4.112) we get

$$\lim_{\substack{L\to+\infty\\ \frac{n}{L}\to\rho}} \frac{E^U_{\mathcal{P}_2}(Q^\beta) - \min_{\mathcal{Q}} E^U_{\mathcal{P}_2}}{n} = O(\rho^{2-\delta}). \tag{1.4.113}$$

Thus, using (1.4.109) and (1.4.113), one proves that, in the thermodynamic limit,

$$\lim_{\substack{L\to+\infty\\ \frac{n}{L}\to\rho}} \frac{\langle H^U\Psi^\beta, \Psi^\beta\rangle - \langle H^U\Psi^U, \Psi^U\rangle}{n} = O(\rho^{2-\delta}) \tag{1.4.114}$$

It concludes the proof of Proposition 1.4.6. □

Combining Proposition 1.4.5 and Proposition 1.4.6, we get Theorem 1.2.1.

1.4.2.5 Distance to the 1−particle and 2−particle density matrices of ground states

We recall that for $\Psi \in \mathfrak{H}^n(L)$, we define its 1-particle density matrix $\gamma_\Psi^{(1)}$ (resp. 2-particle density matrix $\gamma_\Psi^{(2)}$) as the operator on $\mathfrak{H}^1(L)$ (resp. $\mathfrak{H}^2(L)$) given by (1.2.10) (resp. 1.2.11). The following lemma deals with the case of a vector $\Psi \in \mathfrak{H}^n(L)$ which factorizes with respect to a given partition of $[0, L]$.

Lemma 1.4.7. *[KV20] Consider $(U_i)_{1\leqslant i\leqslant r}$ a family of closed sets of $\mathbb{R}$ where $U_i \cap U_j = \varnothing$ holds for every $i \neq j$ and $|U_i|$ is finite. Set, for $(q_i)_{1\leqslant i\leqslant r} \in \mathbb{N}^r$,*

$$\Psi = \bigwedge_{i=1}^{r} \psi(i, q_i) \tag{1.4.115}$$

where $\psi(i, k)$ is a state that belongs to $\mathfrak{H}^k(U_i)$, the k-particle space on U_i. Then the 1-particle $\gamma_\Psi^{(1)}$ and the 2-particle $\gamma_\Psi^{(2)}$ admit the following decompositions

$$\gamma_\Psi^{(1)} = \sum_{i=1}^{r} \gamma_{\psi(i,q_i)}^{(1)} \tag{1.4.116}$$

and

(1.4.117)

$$\gamma_\Psi^{(2)} = \sum_{i=1}^{r} \left(\gamma_{\psi(i,q_i)}^{(2)} - \frac{1}{2}\gamma_{\psi(i,q_i)}^{(1)} \otimes \gamma_{\psi(i,q_i)}^{(1)} + \frac{1}{2}\Big(\gamma_{\psi(i,q_i)}^{(1)} \otimes \gamma_{\psi(i,q_i)}^{(1)}\Big) \circ \tau \right) + \frac{1}{2}\gamma_\Psi^{(1)} \otimes \gamma_\Psi^{(1)} - \frac{1}{2}\Big(\gamma_\Psi^{(1)} \otimes \gamma_\Psi^{(1)}\Big) \circ \tau$$

with $\tau(x_1, x_2, y_1, y_2) = (x_1, x_2, y_2, y_1)$.

We compare the 1-particle and 2-particle density matrices of our approximate ground state with those of any ground state. The following Proposition is a reformulation of Proposition 1.2.3 and Proposition 1.2.4.

Proposition 1.4.7. *Let $\Psi^U(L, n)$ be a ground state of $H^U(L, n)$. For $\delta \in (0, 1)$, $\rho \in (0, \rho_\delta)$ and $\beta > 3$, set the approximate ground state $\Psi^\beta(L, n)$ given in Subsection 1.4.2.3. Then, in the thermodynamic limit, with probability $1 - O(L^{-\infty})$, one has*

$$\frac{1}{n}\Big\|\gamma_{\Psi^U(L,n)}^{(1)} - \gamma_{\Psi^\beta(L,n)}^{(1)}\Big\|_{tr} \leqslant 10\rho^{2-\delta} \tag{1.4.118}$$

and

$$\frac{1}{n^2}\Big\|\gamma_{\Psi^U(L,n)}^{(2)} - \gamma_{\Psi^\beta(L,n)}^{(2)}\Big\|_{tr} \leqslant 45\rho^{2-\delta}. \tag{1.4.119}$$

Proof. Let $\Psi^U(L, n)$ be a ground state of $H^U(L, n)$ for large n and L. We drop the indices "L" and "n". The proof uses that both Ψ^U and Ψ^β admit a factor that fixes all but $O(n\rho^{2-\delta})$ particles. Indeed, by Lemma 1.4.5 and Corollary 1.4.3, both Assumption 1.4.1 and Assumption 1.4.2 hold for $p = 2$. So, we apply Proposition 1.4.3. There exist a subset $\mathcal{F}_2$ of $\mathcal{P}_2$ and, for each piece i in $\mathcal{F}_2$, an integer $q_i^{\mathcal{F}_2}$ such that the ground state Ψ^U admits the following factorization

$$\Psi^U = \left(\bigwedge_{I\in\mathcal{F}_2} \psi^U\big(I, (q_i^{\mathcal{F}_2})_{i\in I}\big) \right) \wedge \Omega^{U,\mathcal{F}_2^c} \tag{1.4.120}$$

where

$$\Omega^{U,\mathcal{F}_2^c} = \sum_{Q\in\mathcal{Q}} \lambda(Q) \bigwedge_{I\notin\mathcal{F}_2} \psi^U(I, (q_i)_{i\in I}) \tag{1.4.121}$$

and $\psi^U(I,(q_i)_{i\in I})$ the ground state of $h^U\big(I,(q_i)_{i\in I}\big)$ given by (1.4.9). Set

$$n^{\mathcal{F}_2} = \sum_{I\in\mathcal{F}_2}\sum_{i\in I} q_i^{\mathcal{F}_2} \tag{1.4.122}$$

the number of particles in $\mathcal{F}_2$.

Let Ψ^β be our approximated ground state given by (1.4.98). By construction, we know

$$\forall I\in\mathcal{F}_2 \quad \forall i\in I \qquad q_i^\beta = q_i^{\mathcal{F}_2}. \tag{1.4.123}$$

We denote

$$\Omega^{\beta,\mathcal{F}_2^c} = \bigwedge_{I\notin\mathcal{F}_2} \phi^U(I,(q_i^\beta)_{i\in I}) \tag{1.4.124}$$

so that

$$\Psi^\beta = \Big(\bigwedge_{I\in\mathcal{F}_2}\psi^U\big(I,(q_i^{\mathcal{F}_2})_{i\in I}\big)\Big)\wedge\Omega^{\beta,\mathcal{F}_2^c}. \tag{1.4.125}$$

We deal with the 1-particle density matrix and 2-particle density matrix separately.

(i) By Lemma 1.4.7, the 1-particle density matrix of Ψ^U satisfies

$$\gamma^{(1)}_{\Psi^U} = \sum_{I\in\mathcal{F}_2}\gamma^{(1)}_{\psi^U\left(I,(q_i^{\mathcal{F}_2})_{i\in I}\right)} + \gamma^{(1)}_{\Omega^{U,\mathcal{F}_2^c}}. \tag{1.4.126}$$

For any $\phi\in\mathfrak{H}^n(L)$, $|\phi><\phi|$ is a rank one projector and

$$\big\||\phi><\phi|\big\|_{\mathrm{tr}} = \int_{[0,L]^n}|\phi(X)|^2\,dX. \tag{1.4.127}$$

So its 1-particle density matrix $\gamma^{(1)}_\phi$ is trace class with

$$\big\|\gamma^{(1)}_\phi\big\|_{\mathrm{tr}} = \int_{[0,L]}\gamma^{(1)}_\phi(x,x)\,dx \tag{1.4.128}$$

Since $\Omega^{U,\mathcal{F}_2^c}$ is a normalized wave function of $\mathfrak{H}^{n-n^{\mathcal{F}_2}}(L)$, we compute

$$\big\|\gamma^{(1)}_{\Omega^{U,\mathcal{F}_2^c}}\big\|_{\mathrm{tr}} = n-n^{\mathcal{F}_2} \leqslant \max_{Q\in\mathcal{Q}}\sum_{i\in\mathcal{N}} q_i + 2\#\mathcal{P}_2\backslash\mathcal{F}_2 \leqslant 5n\rho^{2-\delta} \tag{1.4.129}$$

using Lemma 1.4.2 and the first point of Proposition 1.4.3 in the last inequality.

Thus,

$$\begin{aligned}\big\|\gamma^{(1)}_{\Psi^U}-\gamma^{(1)}_{\Psi^\beta}\big\|_{\mathrm{tr}} &= \big\|\gamma^{(1)}_{\Omega^{U,\mathcal{F}_2^c}}-\gamma^{(1)}_{\Omega^{\beta,\mathcal{F}_2^c}}\big\|_{\mathrm{tr}} \qquad (1.4.130)\\ &\leqslant \big\|\gamma^{(1)}_{\Omega^{U,\mathcal{F}_2^c}}\big\|_{\mathrm{tr}} + \big\|\gamma^{(1)}_{\Omega^{\beta,\mathcal{F}_2^c}}\big\|_{\mathrm{tr}}\\ &\leqslant 10n\rho^{2-\delta} \qquad (1.4.131)\end{aligned}$$

(ii) We expand the 2-particle density matrix of Ψ^U according to Lemma 1.4.7.

(1.4.132)

$$\gamma^{(2)}_{\Psi^U} = \gamma^{(2)}_{\Phi^{U,\mathcal{F}_2}} + \gamma^{(2)}_{\Omega^{U,\mathcal{F}_2^c}} + \frac{1}{2}\Big(\gamma^{(1)}_{\Phi^{U,\mathcal{F}_2}}\otimes\gamma^{(1)}_{\Omega^{U,\mathcal{F}_2^c}} + \gamma^{(1)}_{\Omega^{U,\mathcal{F}_2^c}}\otimes\gamma^{(1)}_{\Phi^{U,\mathcal{F}_2}} - \big(\gamma^{(1)}_{\Phi^{U,\mathcal{F}_2}}\otimes\gamma^{(1)}_{\Omega^{U,\mathcal{F}_2^c}}\big)\circ\tau - \big(\gamma^{(1)}_{\Omega^{U,\mathcal{F}_2^c}}\otimes\gamma^{(1)}_{\Phi^{U,\mathcal{F}_2}}\big)\circ\tau\Big).$$

For $\phi \in \mathfrak{H}^n(L)$, the corresponding 2-particle $\gamma_\phi^{(2)}$ is trace class and it satisfies

$$\left\|\gamma_\phi^{(2)}\right\|_{\mathrm{tr}} = \int_{[0,L]} \gamma_\phi^{(2)}(x_1,x_2,x_1,x_2)\,dx \tag{1.4.133}$$

Then,

$$\left\|\gamma_{\Omega^{U,\mathcal{F}_2^c}}^{(2)}\right\|_{\mathrm{tr}} = \frac{\left(n-n^{\mathcal{F}_2}\right)\left(n-n^{\mathcal{F}_2}-1\right)}{2} \leqslant \frac{25}{2}n^2\rho^{4-2\delta} \tag{1.4.134}$$

and

$$\left\|\gamma_{\Phi^{U,\mathcal{F}_2}}^{(1)} \otimes \gamma_{\Omega^{U,\mathcal{F}_2^c}}^{(1)}\right\|_{\mathrm{tr}} = \left\|\left(\gamma_{\Phi^{U,\mathcal{F}_2}}^{(1)} \otimes \gamma_{\Omega^{U,\mathcal{F}_2^c}}^{(1)}\right)\circ\tau\right\|_{\mathrm{tr}} = n^{\mathcal{F}_2}\left(n-n^{\mathcal{F}_2}\right) \leqslant 5n^2\rho^{2-\delta}. \tag{1.4.135}$$

The same inequalities hold for $\Phi^{\beta,\mathcal{F}_2}$ and $\Omega^{\beta,\mathcal{F}_2^c}$. So,

$$\left\|\gamma_{\Psi^U}^{(2)} - \gamma_{\Psi^\beta}^{(2)}\right\|_{\mathrm{tr}} \leqslant 45n^2\rho^{2-\delta}. \tag{1.4.136}$$

It concludes the proof of Proposition 1.4.7. □

1.4.3 Appendix

1.4.3.1 Convex functions and discrete optimization

Definition 1.4.4. *A function $F:\mathbb{N}\to\mathbb{R}$ is convex (resp. strictly convex) iff for every $k\geqslant 1$,*

$$F(k+1)-F(k) \geqslant F(k)-F(k-1) \qquad (\text{resp. } F(k+1)-F(k) > F(k)-F(k-1)).$$

Lemma 1.4.8. *Let $(F_i)_{1\leqslant i\leqslant p}$ be nonnegative functions defined on $\mathbb{N}$, with $F_i(0)=0$. Define*

$$F:\begin{cases}\mathbb{N}^p & \longrightarrow \mathbb{R}_+\\ (x_1,\dots,x_p) & \longmapsto \sum_{i=1}^p F_i(x_i)\end{cases} \quad \text{and} \quad G:\begin{cases}\mathbb{N} & \longrightarrow \mathbb{R}_+\\ r & \longmapsto \min_{x_1+\dots+x_p=r} F(x_1,\dots,x_p)\end{cases} \tag{1.4.137}$$

Assume that, for every i, F_i is strictly convex. Then,

1. *the function G is convex;*
2. *for $r\geqslant 1$, $G(r)$ is exactly the sum of the r smallest elements of*

$$\Gamma = \left\{F_i(k+1)-F_i(k),\ i\in[\![1,m]\!],\ k\in\mathbb{N}\right\},$$

taken with multiplicity.

Proof. 1. For $r\geqslant 1$, choose $(x_1^r,\dots,x_p^r)\in\mathbb{N}^p$ so that

$$G(r) = F(x_1^r,\dots,x_p^r).$$

We prove that one can set $(x_1^{r+1},\dots,x_p^{r+1})\in\mathbb{N}^p$ satisfying

$$\exists! j_{r+1}\in[\![1,p]\!] \qquad \left(x_{j_{r+1}}^{r+1} = x_{j_{r+1}}^r+1\right) \quad \text{and} \quad \left(\forall i\neq j_{r+1} \qquad x_i^{r+1}=x_i^r\right). \tag{1.4.138}$$

Pick $(y_1,\dots,y_p)\in\mathbb{N}^p$ with $\sum_{i=1}^p y_i = r+1$. Assume that there is $y_{i_0} > x_{i_0}^r+1$. Without loss of generality we consider $i_0=1$. Then

$$\begin{aligned}F(y_1,\dots,y_p)-F(x_1^r+1,x_2^r\dots,x_p^r) =& F(y_1-1,y_2,\dots,r+1-\sum_{i=1}^{p-1}y_i) - F(x_1^r,\dots,r-\sum_{i=1}^{p-1}x_i^r)\\ &+F_1(y_1)-F_1(y_1-1)+F_1(x_1^r)-F_1(x_1^r+1)\\ >&0\end{aligned}$$

by definition of $(x_i^r)_{1\leqslant i\leqslant p}$ and because F_1 is strictly convex from 0 to $r+1$.

So $x_i^{r+1} \leqslant x_i^r + 1$ for all i. Since $\sum_{i=1}^p x_i^{r+1} = \sum_{i=1}^p x_i^r + 1$, there is j_0 so that $x_{j_0}^{r+1} = x_{j_0}^r + 1$. Without loss of generality we can consider $j_0 = 1$. Pick $(y_1, \dots, y_p) \in \mathbb{N}^p$ with $\sum_{i=1}^p y_i = r+1$ and $y_1 = x_1^r + 1$. Then, the same calculus gives $F(y_1, \dots, y_p) \geqslant F(x_1^r + 1, x_2^r, \dots, x_p^r)$ meaning $(x_1^r + 1, x_2^r, \dots, x_p^r)$ is a minimizer of F. Thus we set $(x_i^r)_{r\geqslant 1}$ by induction and we compute

$$\begin{aligned} G(r+1) - G(r) &= F_1(x_1^r + 1) - F_1(x_1^r) \\ &> F_1(x_1^r) - F_1(x_1^r - 1) \end{aligned}$$

and for all $j \in [\![2, p]\!]$

$$G(r+1) - G(r) \geqslant F_j(x_j^r) - F_j(x_j^r - 1)$$

because

$$\sum_{i\notin\{1,j\}} F_i(x_i^r) + F_j(x_j^r) + F_1(x_1^r) \leqslant \sum_{i\notin\{1,j\}} F_i(x_i^r) + F_j(x_j^r - 1) + F_1(x_1^r + 1)$$

Hence,

$$G(r+1) - G(r) \geqslant G(r) - G(r-1).$$

2. In particular, the sequence $\big(G(r+1) - G(r)\big)_{r\geqslant 0}$ is non decreasing and it belongs to Γ. By *reductio ad absurdum*, assume that there is $a \in \Gamma \cap \big\{G(r+1) - G(r), r \geqslant 1\big\}^c$. Let r_a be such that $G(r_a) - G(r_a - 1) \leqslant a < G(r_a + 1) - G(r_a)$, and (i_a, x_a) such that $a = F_{i_a}(x_a + 1) - F_{i_a}(x_a)$. Then, $x_{i_a}^{r_a} = x_a$ and

$$F(x_1^{r_a}, \dots, x_a + 1, \dots, x_p^{r_a}) = G(r_a) + a < G(r_a + 1).$$

Contradiction.

It concludes the proof of Lemma 1.4.8. □

1.4.3.2 Statistical distribution of the pieces

We recall some results about the statistical distribution of pieces.

Proposition 1.4.8. *[KV20] With probability $1 - O(L^{-\infty})$, the largest piece has a length bounded by $\log(L)\log(\log(L))$.*

Proposition 1.4.9. *[KV20] Fix $\beta \in (\frac{2}{3}, 1)$. For L large and $a, b \in [0, \log(L)\log(\log(L))]$, with probability $1 - O(L^{-\infty})$ the number of pieces of length contained in $[a, b]$ is equal to*

$$L(e^{-a} - e^{-b}) + R_L L^\beta$$

where $|R_L|$ is bounded.

Proposition 1.4.10. *[KV20] Fix $\beta \in (\frac{2}{3}, 1)$ and $r \geqslant 2$. For L large and $(a_i)_{1\leqslant i\leqslant r}, (b_i)_{1\leqslant i\leqslant r}, (c_i)_{1\leqslant i\leqslant r-1}$ and $(d_i)_{1\leqslant i\leqslant r-1}$ some positive sequences, with probability $1 - O(L^{-\infty})$, the number of pieces such that the length of i-th piece (from left to right) is contained in $[a_i, b_i]$, the distance with the $(i+1)$-th piece is contained in $[c_i, d_i]$, is equal to*

$$L \prod_{i=1}^{r-1} (d_i - c_i) \prod_{j=1}^{r} (e^{-a_j} - e^{-b_j}) + R_L L^\beta$$

where $|R_L|$ is bounded.

The proofs of Propositions 1.4.8, 1.4.9 and 1.4.10 are in Appendix A of [KV20]. From these propositions, we derive the following lemma.

Lemma 1.4.9. *Fix $\beta \in (\frac{2}{3}, 1)$ and refer to the specific terminology in Definition 1.4.1. For L large and $a, b, c, d, f, g \in [l_{\rho,U}, \log(L)\log\log(L)]$, with probability $1 - O(L^{-\infty})$,*

1. *the number of chains of size 1 with length contained in $[a, b]$ is*

$$L(1 - \sigma_U e^{-l_{\rho,U}})^2(e^{-a} - e^{-b}) + S_L L^\beta$$

where $|S_L|$ is bounded;

2. *the number of chains of size 2 such that the length of the left piece is contained in $[a, b]$, the length of the right piece is contained in $[c, d]$ and the distance between the pieces is contained in $[f, g]$, is equal to*

$$L(1 - \sigma_U e^{-l_{\rho,U}})^2(g - f)(e^{-a} - e^{-b})(e^{-c} - e^{-d}) + S_L L^\beta$$

where $|S_L|$ is bounded.

Proof. 1. Let $\mathcal{P}_{a,b} := \{i \in [\![1, m]\!],\, \mathbf{l}_i \in [a, b]\}$. Then,

$$\{\text{chain of size 1}\} \cap \mathcal{P}_{a,b} = \mathcal{P}_{a,b} \backslash \Big(\{i \in \mathcal{P}_{a,b},\, \exists j > i,\, \mathbf{l}_j \geqslant l_{\rho,U},\, \mathbf{d}_{i,j} \leqslant \sigma_U\} \\ \cup \{i \in \mathcal{P}_{a,b},\, \exists j < i,\, \mathbf{l}_j \geqslant l_{\rho,U},\, \mathbf{d}_{j,i} \leqslant \sigma_U\} \Big)$$

We use Proposition 1.4.9, Proposition 1.4.10 and $\#(A \cup B) = \#A + \#B - \#(A \cap B)$ to conclude.

2. Let $\mathcal{R}_{a,b,c,d} := \{(i, j) \in [\![1, m]\!]^2,\, i < j,\, \mathbf{l}_i \in [a, b],\, \mathbf{l}_j \in [c, d]\}$. Then,

$$\{\text{chain of size 2}\} \cap \mathcal{R}_{a,b,c,d} = \mathcal{R}_{a,b,c,d} \backslash \Big(\{(i, j) \in \mathcal{R}_{a,b,c,d},\, \exists k > j,\, \mathbf{l}_k \geqslant l_{\rho,U},\, \mathbf{d}_{j,k} \leqslant \sigma_U\} \\ \cup \{i \in \mathcal{R}_{a,b,c,d},\, \exists k < i,\, \mathbf{l}_k \geqslant l_{\rho,U},\, \mathbf{d}_{k,i} \leqslant \sigma_U\} \Big)$$

We conclude as for (1).

1.4.3.3 Bounds for the interaction of two particles

Lemma 1.4.10. *Set, for $[a, b] \subset \mathbb{R}$ a finite interval and $i \in \mathbb{N}$,*

$$\phi_i^{[a,b]}(x) = \frac{\sqrt{2}}{\sqrt{b-a}} \sin\big(\frac{\pi}{b-a} i(x-a)\big) \mathbf{1}_{[a,b]}(x).$$

For $p, q \in \mathbb{N}$, if $l > 0$ is large enough,

$$\int U(y-x) \Big| \phi_p^{[0,l]} \wedge \phi_q^{[0,l]} \Big|^2 (x, y) dx dy \leqslant C l^{-3}(p^2 + q^2) \tag{1.4.139}$$

and if $l' > 0$ is also large enough, and $0 \leqslant d \leqslant \sigma_U$,

$$\int U(y-x) \Big| \phi_p^{[-l',0]} \wedge \phi_q^{[d,d+l]} \Big|^2 (x, y) dx dy \leqslant C l^{-3} l'^{-3} p^2 q^2 \tag{1.4.140}$$

with $C > 0$ that only depends on U.

Proof. We derive with changes of variables

$$\begin{aligned}\int U(y-x)\Big|\phi_p^{[0,l]}\wedge\phi_q^{[0,l]}\Big|^2(x,y)dxdy &= \int U(y-x)\Big(\phi_p^{[0,l]}(x)^2\phi_q^{[0,l]}(y)^2-\phi_p^{[0,l]}(x)\phi_q^{[0,l]}(y)\phi_p^{[0,l]}(y)\phi_q^{[0,l]}(x)\Big)dxdy\\
&= 4l^{-1}\int_{-l}^{l}\int_0^1 U(u)\Big(\sin^2(\pi q(ul^{-1}+v))\sin^2(\pi pv)\\
&\quad -\sin(\pi p(ul^{-1}+v))\sin(\pi q(ul^{-1}+v))\sin(\pi pv)\sin(\pi qv)\Big)dudv\\
&= 4l^{-3}\int_{-l}^{l}\int_0^1 U(u)\pi^2u^2\Big(q^2\cos^2(\pi pv)\sin^2(\pi qv)\\
&\quad -pq\cos(\pi pv)\cos(\pi qv)\sin(\pi pv)\sin(\pi qv)\Big)dudv+O(l^{-4})\\
&\leqslant 10\pi^2l^{-3}(p^2+q^2)\int_{\mathbb{R}}U(u)u^2du\end{aligned}$$

and

$$\begin{aligned}\int U(y-x)\Big|\phi_p^{[-l',0]}\wedge\phi_q^{[d,d+l]}\Big|^2(x,y)dxdy &= \int U(x-y)\phi_p^{[-l',0]}(x)^2\phi_q^{[d,d+l]}(y)^2dxdy\\
&= 4l^{-1}l'^{-1}\int_0^{l'}\int_0^{l}U(r+s+d)\sin^2(\pi prl'^{-1})\sin^2(\pi qsl^{-1})drds\\
&= 4l^{-1}l'^{-1}\int_0^{+\infty}\int_0^{u}U(u+d)\sin^2\big(\pi p(u-v)l'^{-1}\big)\sin^2\big(\pi qvl^{-1}\big)\mathrm{d}u\mathrm{d}v\\
&= 4\pi^4l^{-3}l'^{-3}p^2q^2\int_0^{+\infty}\int_0^{u}U(u+d)(u-v)^2v^2\mathrm{d}u\mathrm{d}v\\
&\quad +O(l^{-4}l'^{-3}+l^{-3}l'^{-4})\\
&\leqslant 8\pi^4l^{-3}l'^{-3}p^2q^2\int_0^{+\infty}U(u+d)u^5du.\end{aligned}$$

□

1.4.3.4 Proof of Lemma 1.3.1

For any $\Lambda\in\mathcal{U}_\iota$, there exists a unique chain I such that ι belongs to I. Since

$$E^U(n,\Lambda,Q)=\sum_{J\text{ chain}}e^U\big(J,(q_j)_{j\in J}\big)=e^U\big(I,(q_i)_{i\in I}\big)+\text{constant},\tag{1.4.141}$$

we have to give an holomorphic extension of $e^U\big(I,(q_i)_{i\in I}\big)$ on $\widetilde{\mathcal{U}}_\iota$. Set, for $\Lambda\in\mathcal{U}_\iota$,

$$S_I(\Lambda)=h^U\big(I,(q_i)_{i\in I}\big).\tag{1.4.142}$$

If one gets an holomorphic extension $\widetilde{S_I}$ of the operator-valued function S_I on $\widetilde{\mathcal{U}}_\iota$ then, by analytic perturbation theory (see II.1.8 of [Kat12]), it yields that the ground state of $\widetilde{S_I}$, namely $\widetilde{e_I}$, is an holomorphic extension of $e^U\big(I,(q_i)_{i\in I}\big)$ on $\widetilde{\mathcal{U}}_\iota$.

So the problem reduces to the study of the operator-valued function S_I given by (1.4.142), or more precisely the operator-valued function R_I where, for $t<0$ and $\Lambda\in\mathcal{U}_\iota$,

$$R_I:\begin{cases}\mathbb{R}_-\times\mathcal{U}_\iota & \longrightarrow \mathcal{B}\big(\mathfrak{H}^{\kappa_I}(\mathbf{x_m})\big)\\ (t,\Lambda) & \longmapsto \big(t-S_I(\Lambda)\big)^{-1}\end{cases}.\tag{1.4.143}$$

We recall that
(1.4.144)
$$h^U(I,(q_i)_{i\in I}) = \sum_{\kappa=1}^{\kappa_I}\Big(\bigotimes_{j=1}^{\kappa_I-1}\mathbf{1}_{L^2([0,\mathbf{x_m}])}\Big)\otimes h_I\otimes\Big(\bigotimes_{j=\kappa+1}^{\kappa_I}\mathbf{1}_{L^2([0,\mathbf{x_m}])}\Big)+W_{\kappa_I}\qquad\text{on}\qquad\bigwedge_{i\in I}\Big(\bigwedge_{j=1}^{q_i}L^2(\Delta_i)\Big)$$

where

(i) $\kappa_I=\sum_{i\in I}q_i$ is the total number of particles in I;

(ii) h_I is the one-particle operator defined by

$$h_I=\bigoplus_{i\in I}\Big(-\frac{d^2}{dx^2}^{\,D}_{|\Delta_i}\Big)\qquad\text{on } L^2\big([0,\mathbf{x_m}]\big); \tag{1.4.145}$$

(iii) W_k is the multiplication operator given by

$$W_k(x_1,\dots,x_k)=\sum_{i<j}U(x_j-x_i). \tag{1.4.146}$$

So, for any $t<0$ and $\Lambda\in\mathcal{U}_\iota$,

$$R_I(t,\Lambda)=R_I^0(t,\Lambda)\big(1-W_{\kappa_I}R_I^0(t,\Lambda)\big)^{-1} \tag{1.4.147}$$

with $R_I^0(\cdot,\Lambda)$ the resolvant of the free operator $h^0\big(I,(q_i)_{i\in I}\big)$.

We denote

$$\mathfrak{N}^q=\big\{(n_1,\dots,n_q)\in\mathbb{N}^q,\,0<n_1<\dots<n_q\big\} \tag{1.4.148}$$

The eigenstates of the free operator $h^0\big(I,(q_i)_{i\in I}\big)$ are given by

$$\psi_\nu=\bigotimes_{i\in I}\bigwedge_{j=1}^{q_i}\phi^{\Delta_i}_{\nu^i_j} \tag{1.4.149}$$

where

$$\nu=(\nu^i_j)^{i\in I}_{1\leqslant j\leqslant q_i}\in\prod_{i\in I}\mathfrak{N}^{q_i}\qquad\text{and}\qquad\phi^{\Delta_i}_k(x)=\frac{\sqrt{2}}{\sqrt{\mathbf{l}_i}}\sin\Big(\pi k\frac{x-\mathbf{x}_i}{\mathbf{l}_i}\Big)\mathbf{1}_{\Delta_i}(x). \tag{1.4.150}$$

Remember that $(\phi^{\Delta_i}_k)_{k\geqslant0}$ is an orthonormal basis of $L^2(\Delta_i)$ and, for $i\in I$,

$$\bigwedge_{j=1}^{q_i}\phi^{\Delta_i}_{\nu^i_j}(x_1,\dots,x_{q_i})=\frac{1}{\sqrt{q_i!}}\sum_{\sigma\in\mathfrak{S}_{q_i}}\varepsilon(\sigma)\prod_{k=1}^{q_i}\phi^{\Delta_i}_{\nu^i_{\sigma(k)}}(x_k). \tag{1.4.151}$$

First, we prove that for $t<0$, $R_I^0(t,\cdot)$ admits an holomorphic extension on $\widetilde{\mathcal{U}}_\iota$. We compute $\langle R_I^0(t,\,\Lambda)\psi_\nu,\psi_\mu\rangle$ for any $\mu,\nu\in\prod_{i\in I}\mathfrak{N}^{q_i}$ and $\Lambda\in\mathcal{U}_\iota$.

$$\langle R_I^0(t,\,\Lambda)\psi_\nu,\psi_\mu\rangle=\frac{1}{t-\sum_{i\in I}\sum_{j=1}^{q_i}\Big(\frac{\nu^i_j\pi}{l_i}\Big)^2}\,\delta_{\nu,\mu}=\frac{1}{t-\text{constant}-\big(\frac{\pi}{l_\iota}\big)^2\Big(\sum_{j=1}^{q_\iota}(\nu^\iota_j)^2\Big)}\,\delta_{\nu,\mu}. \tag{1.4.152}$$

So, $\langle R_I^0(t,\,\cdot)\psi_\nu,\psi_\mu\rangle$ has an holomorphic extension, namely $\widetilde{R_I^0(t,\,\cdot)}(\psi_\nu,\psi_\mu)$, on $\widetilde{\mathcal{U}}_\iota$. We also note that $\sup_{\nu,\mu\in\prod_{i\in I}\mathfrak{N}^{q_i}}\big|\widetilde{R_I^0(t,\,\cdot)}(\psi_\nu,\psi_\mu)\big|$ is uniformly bounded on $\widetilde{\mathcal{U}}_\star$. Then, there is a unique operator valued application

$$\widetilde{R_I^0(t,\,\cdot)}:\widetilde{\mathcal{U}}_\star\longrightarrow\mathcal{B}\big(\mathfrak{H}^{\kappa_I}(\mathbf{x_m})\big)$$

such that $\langle \widetilde{R_I^0(t,\cdot)}\psi_\nu, \psi_\mu\rangle = \widetilde{R_I^0(t,\cdot)}(\psi_\nu, \psi_\mu)$ for any $\nu, \mu \in \prod_{i\in I} \mathfrak{N}^{q_i}$. We deduce from above discussion that $\widetilde{R_I^0(t,\cdot)}$ is holomorphic on $\mathcal{U}_\star$.

On the other hand, for any $\mu, \nu \in \prod_{i\in I} \mathfrak{N}^{q_i}$, we compute $\big\langle W_{\kappa_I}\psi_\nu, \psi_\mu\big\rangle$. One notices that

$$\#\big\{(i,j), \nu_j^i \neq \mu_j^i\big\} > 2 \quad \Rightarrow \quad \big\langle W_{\kappa_I}\psi_\nu, \psi_\mu\big\rangle = 0.$$

We enumerate the other cases. Assume that $I = (i_1, \dots, i_r)$.

(i) If $\nu = \mu$ then

$$\begin{aligned}
\big\langle W_{\kappa_I}\psi_\nu, \psi_\nu\big\rangle &= \sum_{k=1}^{r} \frac{q_{i_k}(q_{i_k}-1)}{2} \int_{\Delta_{i_k}^{q_{i_k}}} U\big(x_1 - x_2\big) \Big| \bigwedge_{j=1}^{q_{i_k}} \phi_{\nu_j^{i_k}}^{\Delta_{i_k}}(X) \Big|^2 dX \\
&\quad + \sum_{k=1}^{r-1} q_{i_k} q_{i_{k+1}} \int_{\Delta_{i_k}^{q_{i_k}} \times \Delta_{i_{k+1}}^{q_{i_{k+1}}}} U\big(y_1 - x_1\big) \\
&\qquad\qquad \Big| \bigwedge_{j=1}^{q_{i_k}} \phi_{\nu_j^{i_k}}^{\Delta_{i_k}}(X) \bigwedge_{j=1}^{q_{i_{k+1}}} \phi_{\nu_j^{i_{k+1}}}^{\Delta_{i_{k+1}}}(Y) \Big|^2 dX dY \\
&= \sum_{k=1}^{r} \sum_{1\leqslant a<b\leqslant q_{i_k}} \int_{\Delta_{i_k}^2} U\big(x_1 - x_2\big) \Big| \phi_{\nu_a^{i_k}}^{\Delta_{i_k}} \wedge \phi_{\nu_b^{i_k}}^{\Delta_{i_k}} \Big|^2 (x_1, x_2)\, dx_1 dx_2 \\
&\quad + \sum_{k=1}^{r} \sum_{a=1}^{q_{i_k}} \sum_{b=1}^{q_{i_{k+1}}} \int_{\Delta_{i_k} \times \Delta_{i_{k+1}}} U\big(y_1 - x_1\big) \Big| \phi_{\nu_a^{i_k}}^{\Delta_{i_k}}(x_1) \phi_{\nu_b^{i_{k+1}}}^{\Delta_{i_{k+1}}}(y_1) \Big|^2 dx_1 dy_1 \\
&= \sum_{k=1}^{r} \sum_{1\leqslant a<b\leqslant q_{i_k}} \int_{[0,1]^2} U\big(\mathbf{l}_{i_k}(x-y)\big) \Big| \phi_{\nu_a^{i_k}} \wedge \phi_{\nu_b^{i_k}} \Big|^2 (x,y)\, dx dy \\
&\quad + \sum_{k=1}^{r} \sum_{a=1}^{q_{i_k}} \sum_{b=1}^{q_{i_{k+1}}} \int_{[0,1]^2} U\big(\mathbf{l}_{i_{k+1}} y + \mathbf{l}_{i_k} x + \mathbf{d}_{i_k, i_{k+1}}\big) \Big| \phi_{\nu_a^{i_k}}(x) \phi_{\nu_b^{i_{k+1}}}(y) \Big|^2 dx dy
\end{aligned} \tag{1.4.153}$$

with some changes of variables for the last equality and the notation $\phi_k(x) = \sqrt{2}\sin(\pi k x)\mathbf{1}_{[0,1]}(x)$.

(ii) Else there is a unique couple $(k_\star, a_\star)$ such that $\nu_{a_\star}^{i_{k_\star}} \neq \mu_{a_\star}^{i_{k_\star}}$ then

(1.4.154)

$$\begin{aligned}
\big\langle W_{\kappa_I}\psi_\nu, \psi_\nu\big\rangle &= \frac{q_{i_{k_\star}}(q_{i_{k_\star}}-1)}{2}\int_{\Delta_{i_{k_\star}}^{q_{i_{k_\star}}}} U\big(x_1-x_2\big)\bigwedge_{j=1}^{q_{i_{k_\star}}}\phi_{\nu_j^{i_{k_\star}}}^{\Delta_{i_{k_\star}}}(X)\bigwedge_{j=1}^{q_{i_{k_\star}}}\phi_{\mu_j^{i_{k_\star}}}^{\Delta_{i_{k_\star}}}(X)\,dX \\
&\quad + \sum_{k\in\{k_\star-1,\,k_\star\}} q_{i_k}q_{i_{k+1}}\int_{\Delta_{i_k}^{q_{i_k}}\times\Delta_{i_{k+1}}^{q_{i_{k+1}}}} U\big(y_1-x_1\big) \\
&\qquad \bigwedge_{j=1}^{q_{i_k}}\phi_{\nu_j^{i_k}}^{\Delta_{i_k}}(X)\bigwedge_{j=1}^{q_{i_{k+1}}}\phi_{\nu_j^{i_{k+1}}}^{\Delta_{i_{k+1}}}(Y)\bigwedge_{j=1}^{q_{i_k}}\phi_{\mu_j^{i_k}}^{\Delta_{i_k}}(X)\bigwedge_{j=1}^{q_{i_{k+1}}}\phi_{\mu_j^{i_{k+1}}}^{\Delta_{i_{k+1}}}(Y)\,dXdY \\
&= \sum_{b=1}^{q_{i_{k_\star}}}\int_{\Delta_{i_{k_\star}}^2} U\big(x-y\big)\Big|\phi_{\nu_{a_\star}^{i_{k_\star}}}^{\Delta_{i_{k_\star}}}\wedge\phi_{\nu_b^{i_{k_\star}}}^{\Delta_{i_{k_\star}}}\Big|(x,y)\Big|\phi_{\mu_{a_\star}^{i_{k_\star}}}^{\Delta_{i_{k_\star}}}\wedge\phi_{\mu_b^{i_{k_\star}}}^{\Delta_{i_{k_\star}}}\Big|(x,y)\,dxdy \\
&\quad + \sum_{b=1}^{q_{i_{k_\star-1}}}\int_{\Delta_{i_{k_\star-1}}\times\Delta_{i_{k_\star}}} U\big(y_1-x_1\big)\phi_{\nu_b^{i_{k_\star-1}}}^{\Delta_{i_{k_\star-1}}}(x_1)\phi_{\nu_{a_\star}^{i_{k_\star}}}^{\Delta_{i_{k_\star}}}(y_1)\phi_{\mu_b^{i_{k_\star-1}}}^{\Delta_{i_{k_\star-1}}}(x_1)\phi_{\mu_{a_\star}^{i_{k_\star}}}^{\Delta_{i_{k_\star}}}(y_1)\,dx_1dy_1 \\
&\quad + \sum_{b=1}^{q_{i_{k_\star}}}\int_{\Delta_{i_{k_\star-1}}\times\Delta_{i_{k_\star}}} U\big(y_1-x_1\big)\phi_{\nu_{a_\star}^{i_{k_\star}}}^{\Delta_{i_{k_\star}}}(x_1)\phi_{\nu_b^{i_{k_\star+1}}}^{\Delta_{i_{k_\star+1}}}(y_1)\phi_{\mu_{a_\star}^{i_{k_\star}}}^{\Delta_{i_{k_\star}}}(x_1)\phi_{\mu_b^{i_{k_\star+1}}}^{\Delta_{i_{k_\star+1}}}(y_1)\,dx_1dy_1 \\
&= \sum_{b=1}^{q_{i_{k_\star}}}\int_{[0,1]^2} U\big(\mathbf{l}_{i_{k_\star}}(x-y)\big)\Big|\phi_{\nu_{a_\star}^{i_{k_\star}}}\wedge\phi_{\nu_b^{i_{k_\star}}}\Big|(x,y)\Big|\phi_{\mu_{a_\star}^{i_{k_\star}}}\wedge\phi_{\mu_b^{i_{k_\star}}}\Big|(x,y)\,dxdy \\
&\quad + \sum_{b=1}^{q_{i_{k_\star-1}}}\int_{[0,1]^2} U\big(\mathbf{l}_{i_{k_\star}}y+\mathbf{l}_{i_{k_\star-1}}x+\mathbf{d}_{i_{k_\star-1},i_{k_\star}}\big)\phi_{\nu_b^{i_{k_\star-1}}}(x)\phi_{\nu_{a_\star}^{i_{k_\star}}}(y)\phi_{\mu_b^{i_{k_\star-1}}}(x)\phi_{\mu_{a_\star}^{i_{k_\star}}}(y)\,dxdy \\
&\quad + \sum_{b=1}^{q_{i_{k_\star}}}\int_{[0,1]^2} U\big(\mathbf{l}_{i_{k_\star+1}}y+\mathbf{l}_{i_{k_\star}}x+\mathbf{d}_{i_{k_\star},i_{k_\star+1}}\big)\phi_{\nu_{a_\star}^{i_{k_\star}}}(x)\phi_{\nu_b^{i_{k_\star+1}}}(y)\phi_{\mu_{a_\star}^{i_{k_\star}}}(x)\phi_{\mu_b^{i_{k_\star+1}}}(y)\,dxdy
\end{aligned}$$

with the same changes of variables.

(iii) Else there is a unique pair of couple, $(j_\star, a_\star)$ and $(k_\star, b_\star)$, such that $\nu_{a_\star}^{i_{j_\star}} \neq \mu_{a_\star}^{i_{j_\star}}$ and $\nu_{b_\star}^{i_{k_\star}} \neq \mu_{b_\star}^{i_{k_\star}}$.

(iii-i) If $j_\star = k_\star$ then

(1.4.155)

$$\begin{aligned}
\big\langle W_{\kappa_I}\psi_\nu, \psi_\nu\big\rangle &= \frac{q_{i_{k_\star}}(q_{i_{k_\star}}-1)}{2}\int_{\Delta_{i_{k_\star}}^{q_{i_{k_\star}}}} U\big(x_1-x_2\big)\bigwedge_{j=1}^{q_{i_{k_\star}}}\phi_{\nu_j^{i_{k_\star}}}^{\Delta_{i_{k_\star}}}(X)\bigwedge_{j=1}^{q_{i_{k_\star}}}\phi_{\mu_j^{i_{k_\star}}}^{\Delta_{i_{k_\star}}}(X)\,dX \\
&\quad + \sum_{k\in\{k_\star-1,\,k_\star\}} q_{i_k}q_{i_{k+1}}\int_{\Delta_{i_k}^{q_{i_k}}\times\Delta_{i_{k+1}}^{q_{i_{k+1}}}} U\big(y_1-x_1\big) \\
&\qquad \bigwedge_{j=1}^{q_{i_k}}\phi_{\nu_j^{i_k}}^{\Delta_{i_k}}(X)\bigwedge_{j=1}^{q_{i_{k+1}}}\phi_{\nu_j^{i_{k+1}}}^{\Delta_{i_{k+1}}}(Y)\bigwedge_{j=1}^{q_{i_k}}\phi_{\mu_j^{i_k}}^{\Delta_{i_k}}(X)\bigwedge_{j=1}^{q_{i_{k+1}}}\phi_{\mu_j^{i_{k+1}}}^{\Delta_{i_{k+1}}}(Y)\,dXdY \\
&= \int_{\Delta_{i_{k_\star}}^2} U\big(x_1-x_2\big)\Big|\phi_{\nu_{a_\star}^{i_{k_\star}}}^{\Delta_{i_{k_\star}}}\wedge\phi_{\nu_{b_\star}^{i_{k_\star}}}^{\Delta_{i_{k_\star}}}\Big|(x_1,x_2)\Big|\phi_{\mu_{a_\star}^{i_{k_\star}}}^{\Delta_{i_{k_\star}}}\wedge\phi_{\mu_{b_\star}^{i_{k_\star}}}^{\Delta_{i_{k_\star}}}\Big|(x_1,x_2)\,dx_1dx_2+0 \\
&= \int_{[0,1]^2} U\big((\mathbf{l}_{i_{k_\star}}(x-y)\big)\Big|\phi_{\nu_{a_\star}^{i_{k_\star}}}\wedge\phi_{\nu_{b_\star}^{i_{k_\star}}}\Big|(x,y)\Big|\phi_{\mu_{a_\star}^{i_{k_\star}}}\wedge\phi_{\mu_{b_\star}^{i_{k_\star}}}\Big|(x,y)\,dxdy
\end{aligned}$$

(iii-ii) Else $k_\star = j_\star + 1$ then

(1.4.156)

$$\begin{aligned}\big\langle W_{\kappa_I}\psi_\nu, \psi_\nu\big\rangle &= 0 + q_{i_{j_\star}} q_{i_{k_\star}} \int_{\Delta_{i_{j_\star}}^{q_{i_{j_\star}}} \times \Delta_{i_{k_\star}}^{q_{i_{k_\star}}}} U\big(y_1 - x_1\big) \\ &\qquad \bigwedge_{c=1}^{q_{i_{j_\star}}} \phi_{\nu_c^{i_{j_\star}}}^{\Delta_{i_{j_\star}}}(X) \bigwedge_{c=1}^{q_{i_{k_\star}}} \phi_{\nu_c^{i_{k_\star}}}^{\Delta_{i_{k_\star}}}(Y) \bigwedge_{c=1}^{q_{i_{j_\star}}} \phi_{\mu_c^{i_{j_\star}}}^{\Delta_{i_{j_\star}}}(X) \bigwedge_{c=1}^{q_{i_{k_\star}}} \phi_{\mu_c^{i_{k_\star}}}^{\Delta_{i_{k_\star}}}(Y)\, dXdY \\ &= \int_{\Delta_{i_{j_\star}} \times \Delta_{i_{k_\star}}} U\big(y_1 - x_1\big) \phi_{\nu_{a_\star}^{i_{j_\star}}}^{\Delta_{i_{j_\star}}}(x_1) \phi_{\nu_{b_\star}^{i_{k_\star}}}^{\Delta_{i_{k_\star}}}(y_1) \phi_{\mu_{a_\star}^{i_{j_\star}}}^{\Delta_{i_{j_\star}}}(x_1) \phi_{\mu_{b_\star}^{i_{k_\star}}}^{\Delta_{i_{k_\star}}}(y_1)\, dx_1 dy_1 \\ &= \int_{[0,1]^2} U\big(\mathbf{l}_{i_{k_\star}} y + \mathbf{l}_{i_{j_\star}} x + \mathbf{d}_{i_{j_\star}, i_{k_\star}}\big) \phi_{\nu_{a_\star}^{i_{j_\star}}}(x) \phi_{\nu_{b_\star}^{i_{k_\star}}}(y) \phi_{\mu_{a_\star}^{i_{j_\star}}}(x) \phi_{\mu_{b_\star}^{i_{k_\star}}}(y)\, dxdy\end{aligned}$$

(iii-iii) Else $|j_\star - k_\star| \geqslant 2$ then

$$\big\langle W_{\kappa_I}\psi_\nu, \psi_\nu\big\rangle = 0. \tag{1.4.157}$$

Thus, for any $\mu, \nu \in \prod_{i\in I} \mathfrak{N}^{q_i}$, the application

$$T_I(\psi_\nu, \psi_\mu) : \Lambda \longmapsto \big\langle W_{\kappa_I}\psi_\nu, \psi_\nu\big\rangle \tag{1.4.158}$$

is a linear combination of the maps

$$\gamma_{a,b,c,d} : l \longmapsto \int_{[0,1]^2} U\big(l(x-y)\big)\phi_a(x)\phi_b(y)\phi_c(x)\phi_d(y)dxdy \qquad \text{for } a,b,c,d \in \mathbb{N} \tag{1.4.159}$$

and

(1.4.160)

$$\delta_{a,b,c,d,h} : (l, l') \longmapsto \int_{[0,1]^2} U\big(ly + l'x + h\big)\phi_a(x)\phi_b(x)\phi_c(y)\phi_d(y)dxdy \qquad \text{for } a,b,c,d \in \mathbb{N} \text{ and } 0 \leqslant h < \sigma_U.$$

Set $a,b,c,d \in \mathbb{N}$. For $l > 0$, using $U(x) = u(x)\mathbf{1}_{[-\sigma_U,\sigma_U]}$, we have

$$\begin{aligned}\gamma_{a,b,c,d}(l) &= \int_{[0,\frac{\sigma_U}{l}]^2} u\big(l(y-x)\big)\phi_a(x)\phi_b(x)\phi_c(y)\phi_d(y)dxdy \\ &\quad + \int_{\frac{\sigma_U}{l}}^1 \phi_a(x)\phi_b(x)\Big(\int_{x-\frac{\sigma_U}{l}}^x u\big(l(y-x)\big)\phi_c(y)\phi_d(y)dy\Big)dx \\ &\quad + \int_{\frac{\sigma_U}{l}}^1 \phi_c(y)\phi_d(y)\Big(\int_{y-\frac{\sigma_U}{l}}^y u\big(l(y-x)\big)\phi_a(x)\phi_b(x)dx\Big)dy. \\ &= \left(\frac{\sigma_U}{l}\right)^2 \int_{[0,1]^2} u(\sigma^U(t-s))\phi_a(\frac{\sigma_U s}{l})\phi_b(\frac{\sigma_U s}{l})\phi_c(\frac{\sigma_U}{l}t)\phi_d(\frac{\sigma_U t}{l})dsdt \\ &\quad + \frac{(1-\sigma_U)\sigma_U}{l^2}\int_{[0,1]^2} u(\sigma_U(1-t))\phi_a\Big(s + (1-s)\frac{\sigma_U}{l}\Big)\phi_b\Big(s + (1-s)\frac{\sigma_U}{l}\Big) \\ &\qquad\qquad \phi_c\Big(s + (t-s)\frac{\sigma_U}{l}\Big)\phi_d\Big(s + (t-s)\frac{\sigma_U}{l}\Big)\, dsdt \\ &= \left(\frac{\sigma_U}{l}\right)^2 \int_{[0,1]^2} u(\sigma^U(t-s))\phi_a(\frac{\sigma_U s}{l})\phi_b(\frac{\sigma_U s}{l})\phi_c(\frac{\sigma_U}{l}t)\phi_d(\frac{\sigma_U t}{l})dsdt \\ &\quad + \frac{(1-\sigma_U)\sigma_U}{l^2}\int_{[0,1]^2} u(\sigma_U(1-t))\phi_c\Big(s + (1-s)\frac{\sigma_U}{l}\Big)\phi_d\Big(s + (1-s)\frac{\sigma_U}{l}\Big) \\ &\qquad\qquad \phi_a\Big(s + (t-s)\frac{\sigma_U}{l}\Big)\phi_b\Big(s + (t-s)\frac{\sigma_U}{l}\Big)\, dsdt\end{aligned} \tag{1.4.161}$$

Since $(\phi_k)_{k\geqslant 1}$ and u are holomorphic, the above computation yields that $\gamma_{a,b,c,d}$ admits a holomorphic extension $\widetilde{\gamma}_{a,b,c,d}$ on $\{|l| > l_{\rho,U}\}$.

Similarly, we get, for $0 \leqslant h < \sigma_U$ and $l, l' > 0$,

$$\begin{aligned}(1.4.162)\quad \delta_{a,b,c,d,h}(l,l') &= \int_0^{\frac{\sigma_U-h}{l'}} \phi_a(x)\phi_b(x)\Big(\int_0^{\frac{-l'x+\sigma_U-h}{l}} u\big(ly+l'x+h\big)\phi_c(y)\phi_d(y)\,dy\Big)\,dx \\ &= \frac{\sigma_U-h}{l'}\int_{[0,1]^2} \frac{(\sigma_U-h)(1-s)}{l} u\big((\sigma_U-h)s+h\big)\phi_a\Big(\frac{(\sigma_U-h)s}{l'}\Big)\phi_b\Big(\frac{(\sigma_U-h)s}{l'}\Big) \\ &\qquad \phi_c\Big(\frac{(\sigma_U-h)t(1-s)}{l}\Big)\phi_d\Big(\frac{(\sigma_U-h)t(1-s)}{l}\Big)\,dsdt\end{aligned}$$

We expand $\delta_{a,b,c,d,h}$ on $\{(l,l') \in \mathbb{C}^2,\ |l|, |l'| > l_{\rho,U}\}$, using the expression (1.4.162). We denote $\widetilde{\delta}_{a,b,c,d,h}$ this extension. It is holomorphic on both variables separately.

By substituting $\widetilde{\gamma}_{a,b,c,d}$ (resp. $\widetilde{\delta}_{a,b,c,d,h}$) instead of $\widetilde{\gamma}_{a,b,c,d}$ (resp. $\widetilde{\delta}_{a,b,c,d,h}$), one proves that $T_I(\psi_\nu,\psi_\mu)$ has an holomorphic extension $\widetilde{T_I}(\psi_\nu,\psi_\mu)$ on $\mathcal{U}_\star$ given by (**??**). From the points (i), (ii) and (iii), and using the expressions (1.4.161) and (1.4.162), we know that $\sup_{\nu,\mu\in\prod_{i\in I}\mathfrak{N}^{q_i}} \big|\widetilde{T_I}(\psi_\nu,\psi_\mu)\big|$ is uniformly bounded on $\widetilde{\mathcal{U}}_\iota$. Then, there is a unique operator valued application

$$\widetilde{T_I} : \widetilde{\mathcal{U}}_\iota \longrightarrow \mathcal{B}\big(\mathfrak{H}^{\kappa_I}(\mathbf{x_m})\big)$$

such that $\langle \widetilde{T_I}\psi_\nu, \psi_\mu\rangle = \widetilde{T_I}(\psi_\nu,\psi_\mu)$ for any $\nu,\mu \in \prod_{i\in I}\mathfrak{N}^{q_i}$. We deduce from above discussion that $\widetilde{T_I}$ is holomorphic on $\widetilde{\mathcal{U}}_\iota$.

Then, we combine the previous results with (1.4.147) to set an extension for R_I. For t large enough, $\|\widetilde{T_I}\widetilde{R_I^0(-t,\,\cdot)}\| < 1$, the serie of general term $\Big(\widetilde{T_I}\widetilde{R_I^0(-t,\,\cdot)}\Big)^n$ is normally convergent. So, $\Big(1-\widetilde{T_I}\widetilde{R_I^0(-t,\,\cdot)}\Big)^{-1}$ is holomorphic on $\widetilde{\mathcal{U}}_\iota$. Thus, for t large enough, we set on $\widetilde{\mathcal{U}}_\iota$ the holomorphic operator-valued $\widetilde{R^I(-t,\cdot)}$ given by

$$(1.4.163)\qquad \widetilde{R^I(-t,\Lambda)} = \widetilde{R^0(-t,\Lambda)}\Big(1-\widetilde{T_I}\widetilde{R_I^0(-t,\,\cdot)}\Big)^{-1} \qquad \forall \Lambda \in \widetilde{\mathcal{U}}_\iota$$

Note that if $\mathbf{l}_\iota$ is real, we recover that $\widetilde{R_I(t,\,\Lambda)}$ is the resolvant of $h^U\big(I,(q_i)_{i\in I}\big)$. So, we get an holomorphic extension of S_I on $\widetilde{\mathcal{U}}_\iota$.

It concludes the proof of Lemma 1.3.1.

Chapter 2

Entanglement entropy of the pieces model

2.1 Bipartite setting

For $0 < L_\star < L$, the one-particle space $\mathfrak{H} = L^2([0, L])$ admits the following spatial decomposition

$$\mathfrak{H} = L^2\big([0, L_\star]\big) \oplus L^2\big([L_\star, L]\big). \tag{2.1.1}$$

We denote $\mathfrak{H}_\star = L^2\big([0, L_\star]\big)$. Our definition of the entanglement entropy with respect to $\mathfrak{H}_\star$ will follow the steps of Helling, Leschke and Spitzer [HLS10]. For $n \geqslant 2$, the spatial decomposition of the n-particule space $\mathfrak{H}^n = \bigwedge^n \mathfrak{H}$ does not give a product, as the number of particles in $[0, L_\star]$ is a variable. We bypass this issue using the Fock space description.

Definition 2.1.1. *Let $\mathfrak{H}$ be a one-particle Hilbert space. The corresponding* ***fermionic Fock space*** *$\Gamma_{a.s}(\mathfrak{H})$ is the following graded algebra*

$$\Gamma_{a.s}(\mathfrak{H}) = \mathbb{C} \bigoplus_{N=1}^{\infty} \mathfrak{H}^N.$$

where $\mathfrak{H}^N = \bigwedge^N \mathfrak{H}$.

Proposition 2.1.1. *[AF01] Let $\mathfrak{H} = \mathfrak{K}_1 \oplus \mathfrak{K}_2$ be a bipartite decomposition in closed Hilbert spaces. The application ι, given by*

$$\iota(e_{i_1} \wedge \cdots \wedge e_{i_k} \wedge f_{j_1} \wedge \cdots \wedge f_{j_l}) = e_{i_1} \wedge \cdots \wedge e_{i_k} \otimes f_{j_1} \wedge \cdots \wedge f_{j_l} \tag{2.1.2}$$

for $(e_1, \dots)$ (resp. $(f_1, \dots)$) an orthonormal basis of $\mathfrak{K}_1$ (resp. $\mathfrak{K}_2$), provides an isomorphim ι between $\Gamma_{a.s}(\mathfrak{H})$ and $\Gamma_{a.s}(\mathfrak{K}_1) \otimes \Gamma_{a.s}(\mathfrak{K}_2)$.

Then, we state

Definition 2.1.2. *Let $\mathfrak{H} = \mathfrak{K}_1 \oplus \mathfrak{K}_2$ be a bipartite decomposition in closed Hilbert spaces. Let η be a pure state in $\mathfrak{H}$, meaning that η is a positive trace-class operator on $\Gamma_{a.s}(\mathfrak{H})$ with $tr\, \eta = 1$ and $tr\, \eta^2 = 1$.*

The ***reduced density matrix of*** *η* ***in*** *$\mathfrak{K}_1$, denoted η_1, is the unique bounded operator on $\Gamma_{a.s}(\mathfrak{K}_1)$ such that*

$$\forall O \in \mathcal{B}\big(\Gamma_{a.s}(\mathfrak{K}_1)\big) \qquad tr_{\mathcal{B}(\Gamma_{a.s}(\mathfrak{H}))}\big(\eta \iota^{-1}\big(O \otimes \mathbf{1}_{\Gamma_{a.s}(\mathfrak{K}_2)}\big)\iota\big) = tr_{\mathcal{B}(\Gamma_{a.s}(\mathfrak{K}_1))}(\eta_1 O). \tag{2.1.3}$$

where ι is the isomorphism of Proposition 2.1.1.

The ***entanglement entropy of*** *η* ***with respect to*** *the subspace $\mathfrak{K}_1$, denoted $\mathcal{S}_{\mathfrak{K}_1}(\eta)$, is given by the von Neumann entropy of the reduced density matrix η_1, meaning that*

$$\mathcal{S}_{\mathfrak{K}_1}(\eta) = -tr(\eta_1 \ln(\eta_1)). \tag{2.1.4}$$

From now on, we consider a ground state $\Psi^U(L,n)$ of the pieces model $H^U(L,n)$ given by (1.1.4) that belongs to a unique $Q-$occupied space $\mathfrak{H}_Q(L)$. We set

$$\Psi^U = \cdots \oplus 0 \oplus \Psi^U(L,n) \oplus 0 \oplus \dots \qquad \text{in } \Gamma_{a.s}(\mathfrak{H}) \tag{2.1.5}$$

and

$$\eta^U = \langle\, .\,, \Psi^U \rangle\, \Psi^U \qquad \text{on } \Gamma_{a.s}(\mathfrak{H}). \tag{2.1.6}$$

The rank-one operator η^U is a pure state in $\mathfrak{H}$.

Notation 2.1.1. *$\eta^U_\star$ designates the reduced density matrix of η^U in $\mathfrak{H}_\star$ and $\mathcal{S}_\star(\eta^U)$ is the entanglement entropy of η^U with respect to $\mathfrak{H}_\star$.*

2.2 Results and perspectives

Recall that, by Chapter 1, a ground state $\Psi^U(L,n) \in \mathfrak{H}_Q(L)$ admits the following decomposition

$$\Psi^U(L,n) = \bigwedge_{I\in\mathcal{P}} \psi^U\big(I, (q_i)_{i\in I}\big). \tag{2.2.1}$$

where $\mathcal{P}$ designates the set of the chains.

We claim that the entanglement entropy associated to a given realization of the point process depends only on the chain in which the bipartition falls. More precisely, the entanglement entropy of η^U with respect to $\mathfrak{H}_\star$ is given by

$$\mathcal{S}_\star(\eta^U) = -\sum_{k=1}^{\kappa_\star} \operatorname{tr}\big(K_s \ln K_s\big) \tag{2.2.2}$$

with

(*i*) $\kappa_\star$ is the number of particles in the chain $I_\star$ that contains $L_\star$,

(*ii*) K_s is the operator of kernel

$$\mathcal{K}_s(x,y) = \binom{\kappa_\star}{s} \int_{[L_\star,L]^{\kappa_\star - s}} \psi^U_{I_\star}(x,z)\psi^U_{I_\star}(y,z)\, dz.$$

We would like to use this formula to express the expected value of the entanglement entropy $\mathcal{S}_\star(\eta^U)$ in the thermodynamic limit. Our intuition is that, as for the ground state energy, if one substract the expected value of the free case, the first correction would be given by the chains of $\mathcal{P}_2$. By Proposition 1.4.3, for $\delta \in (0,1)$ and $\rho \in (0,\rho_\delta)$, a ground state $\Psi(L,n)$ factorizes as $\Psi^U = \Phi^{U,\mathcal{F}_2} \wedge \Omega^{U,\mathcal{F}_2^c}$ with $\mathcal{F}_2$ a subset of $\mathcal{P}_2$,

$$\Phi^{U,\mathcal{F}_2} = \bigwedge_{I\in\mathcal{F}_2} \psi^U\big(I, (q_i^{\mathcal{F}_2})_{i\in I}\big) \tag{2.2.3}$$

and the number of particles in $\mathcal{F}_2^c$ is of order $O(n\rho^{2-\delta})$. We set

$$\widetilde{\eta^U} = \big\langle\, .\,, \Phi^{U,\mathcal{F}_2} \big\rangle\, \Phi^{U,\mathcal{F}_2}. \tag{2.2.4}$$

We estimate the entanglement entropy for the $2-$particle systems. For two particles in a single piece, we prove the following result.

Proposition 2.2.1. *For $l > l_\star > 0$, set $t_\star = l_\star/l$. Let $s^U_{l_\star}\big([0,l],2\big)$ be the entanglement entropy corresponding to the ground state of $h^U\big([0,l],2\big)$ and with respect to $L^2\big([0,l_\star]\big)$. Then, there exists $\delta \in (0,1)$ such that the entanglement entropy of the system with two particles in $[0,l]$ satisfies*

$$\begin{aligned} s^U_{l_\star}\big([0,l],2\big) = s^0_{l_\star}\big([0,l],2\big) &- 2\Big(\int_{[0,t_\star]^2} \psi_0\psi^l_\perp\Big)\Big(1+\ln\big(\|\psi_0 \mathbf{1}_{[0,t_\star]^2}\|^2\big)\Big) \\ &- 2\langle\psi^l_\perp,\xi_+\rangle\big(1+\ln\mu_+\big) - 2\langle\psi^l_\perp,\xi_-\rangle\big(1+\ln\mu_-\big) \\ &- 2\Big(\int_{[t_\star,1]^2} \psi_0\psi^l_\perp\Big)\Big(1+\ln\big(\|\psi_0 \mathbf{1}_{[t_\star,1]^2}\|^2\big)\Big) + o\big(l^{-(1+\delta)}\big) \end{aligned} \tag{2.2.5}$$

where

(i) $s^0_{l_\star}\big([0,l],2\big)$ designates the entanglement entropy associated to the non-interacting ground state;

(ii) $\psi_0 = \phi_2 \wedge \phi_1$ and $\psi^l_\perp$ are defined in Proposition 3.1.2;

(iii) μ_+ and μ_- are the eigenvalues of $\widetilde{K_0}$ with kernel

$$\forall x,y \in [0,t_\star] \qquad \widetilde{\mathcal{K}_0}(x,y) = 2\int_{t_\star}^1 \psi_0(x,z)\psi_0(y,z)\,dz;$$

(iv) $\xi_\pm$ is given by

$$\xi_\pm = \langle u_\pm,\phi_2\rangle\big(u_\pm\otimes\phi_1\big) - \langle u_\pm,\phi_1\rangle\big(u_\pm\otimes\phi_2\big) \tag{2.2.6}$$

with (u_+,u_-) the eigenstates of $\widetilde{K_0}$.

By Corollary 3.3.1, the first non-trivial terms in formula (2.2.5) are bounded from above by $O\big(l^{-1}\big)$. We think that these terms are of order $O\big(l^{-1}\big)$. In the case of a pair of interacting particles in two distinct pieces, we get that the expansion of the entanglement entropy does not have term of such order.

Proposition 2.2.2. *For $l > 0$, $d \geqslant 0$ and $a \geqslant 1$, we consider $l_\star \in (-al, d+l)$. Let $s^U_{l_\star}\big(([-al,0],[d,d+l]),2\big)$ be the entanglement entropy corresponding to the ground state of $h^U\big(([-al,0],[d,d+l]),(1,1)\big)$ and with respect to $L^2\big([-al,l_\star]\big)$.*

Then, the entanglement entropy of the system with one particle in each interval of the pair $([-al,0],[d,d+l])$ satisfies

$$s^U_{l_\star}\big(([-al,0],[d,d+l]),2\big) = s^0_{l_\star}\big(([-al,0],[d,d+l]),2\big) + O\big(l^{-3}\big)$$

where $s^0_{l_\star}\big(([-al,0],[d,d+l]),2\big)$ denote the entanglement entropy associated to the non-interacting ground state.

Using the notations of Proposition 2.2.1, Proposition 2.2.2 and, as for $\mathcal{J}$ (see (1.2.7)), we consider for $\lambda \in (0,+\infty)$

$$\begin{aligned} \mathcal{L}(\lambda) = \big(1-\sigma_U e^{-l_{\rho,U}}\big)^2 \Bigg(&\int_{\mathcal{D}_1(\lambda)}\int_0^u s^U_v([0,u],1)e^{-u}\,dudv + \int_{\mathcal{D}_2(\lambda)}\int_0^u s^U_v([0,u],2)e^{-u}\,dudv \\ &+ \int_0^{\sigma_U}\Big(\int_{\mathcal{D}_3(\lambda)}\int_{-u}^{v+t} 2e^{-(u+v)}s^U_w\big(([-u,0],[t,v+t]),1\big)\,dudvdw\Big)dt \\ &+ \int_0^{\sigma_U}\Big(\int_{\mathcal{D}_4(\lambda,t)}\int_{-u}^{v+t} 2e^{-(u+v)}s^U_w\big(([-u,0],[t,v+t]),2\big)\,dudvdw\Big)dt\Bigg). \end{aligned} \tag{2.2.7}$$

Applying a construction similar to Theorem 1.2.1, we get

Theorem 2.2.1. *In the thermodynamic limit, the expected value of the entanglement entropy $\mathcal{S}_\star\big(\widetilde{\eta^U}\big)$, see (2.2.4), satifies for any $\delta \in (0,1)$*

$$\mathbb{E}\big[\mathcal{S}_\star\big(\widetilde{\eta^U}\big)\big] = \mathcal{L}(\lambda_\rho) + o(\rho^{2-\delta}) \tag{2.2.8}$$

where λ_ρ is given by Theorem 1.2.1 and $\mathcal{L}$ is defined in (2.2.7).

Following the above discussion, we conjecture that in the thermodynamic limit, for $\delta \in (0,1)$,

$$\mathbb{E}\big[\mathcal{S}_\star(\eta^U)\big] = \mathbb{E}\big[\mathcal{S}_\star\big(\widetilde{\eta^U}\big)\big] + o\big(\rho^{2-\delta}\big). \tag{2.2.9}$$

We already know that

- if $L_\star$ belongs to a piece that is not in the convex hull of any chain then $\mathcal{S}_\star(\eta^U) = 0$;
- if $L_\star$ belongs to the convex hull of a given chain then, with probability $O(\rho^{5/2})$, $L_\star$ is in the convex hull of a chain in $\mathcal{P}\backslash\mathcal{P}_2$; in fact,

  Lemma 2.2.1. *For $r \geqslant 1$, let $(a_i)_{i\leqslant r}, (b_i)_{i\leqslant r}, (\alpha_i)_{i\leqslant r}, (\beta_i)_{i\leqslant r}$ be positive sequences. Under conditions of comptatibility, the probability that $L_\star$ belongs to a chain of size 1 such that for every $i \leqslant r$*

  - *the distance to the left end of the $i-$th piece is contained in $[a_i, a_t + \alpha_i]$,*
  - *the distance to the right end of the $i-$th piece is contained in $[b_i, b_i + \beta_i]$,*

  is equal to

$$\big(1 - \sigma_U e^{-l_{\rho,U}}\big)^2 \Big(\prod_{i\leqslant r} \alpha_i\beta_i\Big) \exp\Big(-\sum_{i\leqslant r}\big(b_i + \beta_i - (a_i + \alpha_i)\big)\Big).$$

So, we are looking for a uniform (in the variable ρ) bound for the entanglement entropy corresponding to the ground state of a $k-$particle subsystem. In fact, an exponential majoration in the variable k would be sufficient.

Finally, one could also use the expansions given in Proposition 2.2.1 and Proposition 2.2.2 to compute the thermodynamic limit of

$$\mathbb{E}\big[\mathcal{S}_\star(\eta^U) - \mathcal{S}_\star(\eta^0)\big] \tag{2.2.10}$$

up to an error $o\big(\rho|\ln\rho|^{-(1+\varepsilon)}\big)$ for some $\varepsilon \in (0,1)$.

2.3 Proofs

2.3.1 Expression of the reduced density matrix

In this section, we fix a realization of the Poisson point process. Let $\Psi^U(L,n) \in \mathfrak{H}_Q(L)$ be a ground state of $H^U(L,n)$ so that

$$\Psi^U(L,n) = \bigwedge_{I\in\mathcal{P}} \psi^U\big(I, (q_i)_{i\in I}\big). \tag{2.3.1}$$

where $\mathcal{P}$ designates the set of the chains. We assume that the cut $L_\star$ falls into the convex hull $\mathcal{C}\big(I_\star\big)$ of the chain $I_\star$. We denote

(i) $\mathcal{P}_{\text{int}}$ the set of chains strictly included to $[0, L_\star)$, and $\Psi^U_{\mathcal{P}_{\text{int}}} = \bigwedge_{I\in\mathcal{P}_{\text{int}}} \psi^U\big(I, (q_i)_{i\in I}\big)$;

(ii) $\mathcal{P}_{\text{ext}}$ the set of chains strictly included to $(L_\star, L]$, and $\Psi^U_{\mathcal{P}_{\text{ext}}} = \bigwedge_{I\in\mathcal{P}_{\text{ext}}} \psi^U\big(I, (q_i)_{i\in I}\big)$;

(iii) $\psi^U_{I_\star} = \psi^U\big(I_\star, (q_i)_{i\in I_\star}\big)$;

(iv) $\kappa_\star = \sum_{i\in I_\star} q_i$ the number of particles in the chain $I_\star$.

Proposition 2.3.1. *Using the above notations, we set, for $0 \leqslant s \leqslant \kappa_\star$, the kernel $\mathcal{K}_s$ such that*

$$\forall\, x, y \in \Big(\mathcal{C}(I_\star) \cap [0, L_\star]\Big)^s \qquad \mathcal{K}_s(x,y) = \binom{\kappa_\star}{s} \int_{[L_\star, L]^{\kappa_\star - s}} \psi^U_{I_\star}(x,z)\overline{\psi^U_{I_\star}(y,z)}\, dz. \tag{2.3.2}$$

We write K_s the corresponding self-adjoint operator on $\bigwedge^s L^2\big(\mathcal{C}(I_\star) \cap [0, L_\star]\big)$.

Then, the reduced density matrix $\eta^U_\star$ is the operator on $\Gamma_{a.s}\big(\mathfrak{H}_\star\big)$ defined by

$$\eta^U_\star = \iota^{-1}\Bigg(\cdots \oplus 0 \oplus \bigoplus_{s=0}^{\kappa_\star} \Big(\big\langle\, .\,, \psi^U_{\mathcal{P}_{int}}\big\rangle \psi^U_{\mathcal{P}_{int}} \otimes K_s\Big) \oplus 0 \oplus \dots\Bigg)\iota \tag{2.3.3}$$

where ι is the isomorphism of Proposition 2.1.1 with respect to the spatial decomposition

$$\mathfrak{H}_\star = L^2\big([0, L_\star]\backslash\mathcal{C}(I_\star)\big) \oplus L^2\big(\mathcal{C}(I_\star) \cap [0, L_\star]\big).$$

Proof. For $r, s, u, v \geqslant 0$, let

(i) $(\xi_{r,k})_{k\in\mathbb{N}}$ be an orthonormal basis of $\bigwedge^r L^2\big([0, L_\star]\backslash\mathcal{C}(I_\star)\big)$ with $\xi_{\kappa_{\rm int},0} = \Psi^U_{\mathcal{P}_{\rm int}}$;

(ii) $(e_{s,k})_{k\in\mathbb{N}}$ be an orthonormal basis of $\bigwedge^s L^2\big(\mathcal{C}(I_\star) \cap [0, L_\star]\big)$;

(iii) $(f_{u,k})_{k\in\mathbb{N}}$ be an orthonormal basis of $\bigwedge^u L^2\big(\mathcal{C}(I_\star) \cap [L_\star, L]\big)$;

(iv) $(\varsigma_{v,k})_{k\in\mathbb{N}}$ be an orthonormal basis of $\bigwedge^v L^2\big([0, L]\backslash([0, L_\star] \cup \mathcal{C}(I_\star))\big)$ with $\varsigma_{\kappa_{\rm ext},0} = \Psi^U_{\mathcal{P}_{\rm ext}}$.

Set $\Phi_{\substack{r,s,u,v\\ i,j,k,l}} = \cdots \oplus 0 \oplus \xi_{r,i} \wedge e_{s,j} \wedge f_{u,k} \wedge \varsigma_{v,l} \oplus 0 \oplus \dots$ for $(r, s, u, v, i, j, k, l) \in \mathbb{N}^8$. Then $\Big(\Phi_{\substack{r,s,u,v\\ i,j,k,l}}\Big)$ is a orthonormal basis of $\Gamma_{a.s}(\mathfrak{H})$.

Since $\Psi^U = \cdots \oplus 0 \oplus \xi_{\kappa_{\rm int},0} \wedge \psi^U_{I_\star} \wedge \varsigma_{\kappa_{\rm ext},0} \oplus 0 \oplus \dots$, we compute for $O \in \mathcal{B}(\Gamma_{a.s}(\mathfrak{H}_\star))$

$$\begin{aligned}\mathrm{tr}_{\mathcal{B}(\Gamma_{a.s}(\mathfrak{H}))}\big(\eta^U \iota^{-1}\big(O \otimes \mathbf{1}\big)\iota\big) &= \sum_{r,s,u,v} \sum_{i,j,k,l} \Big\langle \Phi_{\substack{r,s,u,v\\ i,j,k,l}}, \Psi^U_L \Big\rangle \Big\langle \Psi^U_L, \iota^{-1}\big(O \otimes \mathbf{1}\big)\iota \Phi_{\substack{r,s,u,v\\ i,j,k,l}} \Big\rangle \\ &= \sum_{s+u=\kappa_\star} \sum_{j,k} \Big\langle \Phi_{\substack{\kappa_{\rm int},s,u,\kappa_{\rm ext}\\ 0,j,k,0}}, \Psi^U_L \Big\rangle \Big\langle \Psi^U_L, \iota^{-1}\big(O \otimes \mathbf{1}\big)\iota\, \Phi_{\substack{\kappa_{\rm int},s,u,\kappa_{\rm ext}\\ 0,j,k,0}} \Big\rangle\end{aligned} \tag{2.3.4}$$

Using that

$$\iota^{-1}\big(O \otimes \mathbf{1}\big)\iota\, \Phi_{\substack{\kappa_{\rm int},s,u,\kappa_{\rm ext}\\ 0,j,k,0}} = \big(O(\cdots \oplus 0 \oplus \xi_{\kappa_{\rm int},0} \wedge e_{s,j} \oplus 0 \oplus \dots)\big) \wedge \big(\cdots \oplus 0 \oplus f_{u,k} \wedge \varsigma_{\kappa_{\rm ext},0} \oplus 0 \oplus \dots\big) \tag{2.3.5}$$

and

$$\Big\langle \Phi_{\substack{\kappa_{\rm int},s,u,\kappa_{\rm ext}\\ 0,j,k,0}}, \Psi^U \Big\rangle = \big\langle e_{s,j} \wedge f_{u,k}, \psi^U_{I_\star} \big\rangle \tag{2.3.6}$$

we get

$$\mathrm{tr}_{\mathcal{B}(\Gamma_{a.s}(\mathfrak{H}))}\big(\eta^U \iota^{-1}\big(O \otimes \mathbf{1}\big)\iota\big) = \sum_{s+u=\kappa_\star} \sum_{i,j,k} \big\langle e_{s,i} \wedge f_{u,k}, \psi^U_{I_\star} \big\rangle \big\langle e_{s,j} \wedge f_{u,k}, \psi^U_{I_\star} \big\rangle \big\langle \Phi_{\substack{\kappa_{\rm int},s\\ 0,i}}, O\Phi_{\substack{\kappa_{\rm int},s\\ 0,j}} \big\rangle \tag{2.3.7}$$

with the notation $\Phi^{r,s}_{i,j} = \cdots \oplus 0 \oplus \xi_{r,i} \wedge e_{s,j} \oplus 0 \oplus \dots$. We note that for $0 \leqslant s \leqslant \kappa_\star$

(2.3.8)

$$\begin{aligned}\mathcal{K}_s(x,y) &= \binom{\kappa_\star}{s} \int_{[L_\star,L]^{\kappa_\star - s}} \psi^U_{I_\star}(x,z)\psi^U_{I_\star}(y,z)\,dz \\ &= \sum_k \left(\sqrt{\binom{\kappa_\star}{s}} \int_{[L_\star,L]^{\kappa_\star-s}} \psi^U_{I_\star}(x,u) f_{\kappa_\star - s,k}(u)\,du\right)\left(\sqrt{\binom{\kappa_\star}{s}} \int_{[L_\star,L]^{\kappa_\star-s}} \psi^U_{I_\star}(y,v) f_{\kappa_\star-s,k}(v)\,dv\right)\end{aligned}$$

and for any j,k

(2.3.9)

$$\begin{aligned}\left\langle e_{s,j} \wedge f_{\kappa_\star - s,k}\,, \psi^U_{I_\star}\right\rangle &= \int_{[0,L]^{\kappa_\star}} \left(e_{s,j} \wedge f_{\kappa_\star-s,k}\right)(x)\psi^U_{I_\star}(x)\,dx \\ &= \frac{1}{\sqrt{\binom{\kappa_\star}{s}}} \sum_{\substack{A \subset [\![1,\kappa_\star]\!] \\ |A|=s}} \varepsilon(A) \int_{[0,L]^{\kappa_\star}} e_{s,j}(x_A) f_{\kappa_\star-s,k}(x_{A^c})\psi^U_{I_\star}(x)\,dx \\ &\qquad \text{where } x_A = (x_i)_{i\in A} \text{ and } \varepsilon(A) \text{ is the signature of the partition } A \cup A^c \\ &\qquad \text{(see for example Appendix C of [KV20]} \\ &= \frac{1}{\sqrt{\binom{\kappa_\star}{s}}} \sum_{\substack{A \subset [\![1,\kappa_\star]\!] \\ |A|=s}} \int_{[0,L]^{\kappa_\star}} e_{s,j}(x_A) f_{\kappa_\star-s,k}(x_{A^c})\psi^U_{I_\star}(x_A,x_{A^c})\,dx_A dx_{A^c} \\ &\qquad \text{by skew-symmetry of } \psi^U_{I_\star} \\ &= \sqrt{\binom{\kappa_\star}{s}} \int_{[0,L]^{\kappa_\star}} e_{s,j}(y) f_{\kappa_\star-s,k}(z)\psi^U_{I_\star}(y,z)\,dydz\end{aligned}$$

so

$$\begin{aligned}\langle K_s e_{s,i}\,, e_{s,j}\rangle &= \int_{[0,L_\star]^s}\int_{[0,L_\star]^s} \mathcal{K}_s(x,y) e_{s,i}(y) e_{s,j}(x)\,dxdy \\ &= \sum_k \langle \psi^U_{I_\star}\,, e_{s,i} \wedge f_{\kappa_\star-s,k}\rangle\langle \psi^U_{I_\star}\,, e_{s,j} \wedge f_{\kappa_\star-s,k}\rangle.\end{aligned} \tag{2.3.10}$$

Thus,

$$\mathrm{tr}_{\mathcal{B}(\Gamma_{a.s}(\mathfrak{H}))}\left(\eta^U \iota^{-1}\left(O \otimes \mathbf{1}\right)\iota\right) = \sum_{s=0}^{\kappa_\star}\sum_{i,j} \left\langle K_s e_{s,i}\,, e_{s,j}\right\rangle \left\langle \Phi^{\kappa_{\mathrm{int}},s}_{0,i}\,, O\Phi^{\kappa_{\mathrm{int}},s}_{0,j}\right\rangle \tag{2.3.11}$$

On the other hand, we define the self-adjoint operator $\theta_\star$ on $\Gamma_{a.s}\left(\mathfrak{H}_\star\right)$ by

$$\theta_\star = \iota^{-1}\left(\cdots \oplus 0 \oplus \bigoplus_{s=0}^{\kappa_\star} \left(\langle\,.\,, \psi^U_{\mathcal{P}_{\mathrm{int}}}\rangle \psi^U_{\mathcal{P}_{\mathrm{int}}} \otimes K_s\right) \oplus 0 \oplus \dots\right)\iota \tag{2.3.12}$$

that is

$$\theta_\star \Phi^{r,s}_{i,j} = \begin{cases} \cdots \oplus 0 \oplus \xi_{\kappa_{\mathrm{int}},0} \wedge (K_s e_{s,j}) \oplus 0 \oplus \dots & \text{if } r = \kappa_{\mathrm{int}}, 0 \leqslant s \leqslant \kappa_\star \text{ and } i = 0 \\ 0 & \text{else}\end{cases} \tag{2.3.13}$$

Then, for $O \in \mathcal{B}(\Gamma_{a.s}(\mathfrak{H}_\star))$,

$$\mathrm{tr}_{\mathcal{B}(\Gamma_{a.s}(\mathfrak{H}_\star))}\big(\theta_\star O\big) = \sum_{r,s}\sum_{i,j}\big\langle \theta_\star \Phi_{\substack{r,s\\i,j}}\,, O\Phi_{\substack{r,s\\i,j}}\big\rangle \tag{2.3.14}$$

$$= \sum_{s=0}^{\kappa_\star}\sum_{j,k}\big\langle K_s e_{s,j}\,, e_{s,k}\big\rangle\big\langle \Phi_{\substack{\kappa_{\mathrm{int}},s\\0,k}}\,, O\Phi_{\substack{\kappa_{\mathrm{int}},s\\0,j}}\big\rangle$$

$$= \mathrm{tr}_{\mathcal{B}(\Gamma_{a.s}(\mathfrak{H}))}\big(\eta^U \iota^{-1}\big(O\otimes \mathbf{1}\big)\iota\big) \tag{2.3.15}$$

using (2.3.11). We conclude that $\eta^U_\star = \theta_\star$ by Definition 2.1.2. □

Then, the entanglement entropy of η^U with respect to $\mathfrak{H}_\star$ satisfies

$$\mathcal{S}_\star(\eta^U) = -\sum_{k=1}^{\kappa_\star}\mathrm{tr}\big(K_s \ln K_s\big) \tag{2.3.16}$$

with K_s given by the kernel (2.3.2). We note that it only depends on the chain $I_\star$ that contains $L_\star$, and, more precisely, the ground state of the associated subsystem.

In the next subsection, we develop the case of the chains containing 2 particles.

2.3.2 Entanglement entropy for 2−particle systems

For each 2−particle system, we adapt the bipartite setting of Section 2.1. Using the results of Chapter 3, we prove the first order expansions of the associated entanglement entropies.

2.3.2.1 Case of a single piece

Proof. (of Proposition 2.2.1) Set $l > l_\star > 0$. Applying Definition 2.1.2 and Proposition 2.3.1, the entanglement entropy $s^U_{l_\star}\big([0,l],2\big)$ is given by

$$s^U_{l_\star}\big([0,l],2\big) = \sum_{s=0}^{2}\mathrm{tr}\big(K_s \ln K_s\big) \tag{2.3.17}$$

where the operator K_s admits the following kernel

$$\forall\, x,y\in[0,l_\star]^s \qquad \mathcal{K}_s(x,y) = \binom{2}{s}\int_{[l_\star,l]^{2-s}} \psi^U([0,l],2)(x,z)\psi^U([0,l],2)(y,z)\,dz \tag{2.3.18}$$

with $\psi^U([0,l],2)$ the ground state of the 2-particle system in $[0,l]$ (see Definition 1.4.2).

Set $t = \frac{l_\star}{l} > 0$. We use the notations of Section 3.1, Chapter 3. Using the isometry O^l defined in (3.1.2), we note that the eigenvalues of K_s are those of the operator F_s acting on $L^2([0,t]^s)$ with kernel

$$\forall x,y\in[0,t]^s \qquad \mathcal{F}_s(x,y) = \binom{2}{s}\int_{[t,1]^{2-s}} \psi^l_0(x,z)\psi^l_0(y,z)\,dz. \tag{2.3.19}$$

where ψ^l_0 is the ground state of the 2-particle system h^l. We recall that

$$h^l = -\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} + U^l \qquad \text{on } L^2\big([0,1]\big)\wedge L^2\big([0,1]\big) \tag{2.3.20}$$

and the decomposition

$$\psi^l_0 = \frac{\psi_0 + \psi^l_\perp}{\sqrt{1+\|\psi^l_\perp\|^2}}. \tag{2.3.21}$$

- First, we consider the scalar F_0 that satisifies

$$F_0 = c_l\Big(\|\psi_0 \mathbf{1}_{[t,1]^2}\|^2 + 2\int_{[t,1]^2} \psi_0\psi_\perp^l + \|\psi_\perp^l \mathbf{1}_{[t,1]^2}\|^2\Big) \tag{2.3.22}$$

with $c_l = \big(1 + \|\psi_\perp^l\|^2\big)^{-1}$. By Proposition 3.1.2,

$$\|\psi_\perp^l\|^2 \leqslant \frac{C}{l^2} \quad \text{and} \quad \Big|\int_{[t,1]^2} \psi_0\psi_\perp^l\Big|^2 \leqslant \frac{C}{l^2} \tag{2.3.23}$$

So,

$$-\operatorname{tr}\big(K_0 \ln K_0\big) = -\|\psi_0 \mathbf{1}_{[t,1]^2}\|^2 \ln\big(\|\psi_0 \mathbf{1}_{[t,1]^2}\|^2\big) - 2\Big(\int_{[t,1]^2} \psi_0\psi_\perp^l\Big)\Big(1 + \ln\big(\|\psi_0 \mathbf{1}_{[t,1]^2}\|^2\big)\Big) + O\Big(\frac{1}{l^2}\Big). \tag{2.3.24}$$

The term $\int_{[t,1]^2} \psi_0\psi_\perp^l$ is at most of order $O\big(\frac{1}{l}\big)$. We do not have proof of the corresponding lower bound.

- Simirlarly, we consider the rank-one operator F_2 that satisfies

$$F_2 = c_l\big(\psi_0 + \psi_\perp^l\big)\mathbf{1}_{[0,t]^2} \otimes \big(\psi_0 + \psi_\perp^l\big)\mathbf{1}_{[0,t]^2}. \tag{2.3.25}$$

Since $\|\psi_\perp^l\|^2 \leqslant \frac{C}{l^2}$, we get

$$-\operatorname{tr}\big(K_2 \ln K_2\big) = -\|\psi_0 \mathbf{1}_{[0,t]^2}\|^2 \ln\big(\|\psi_0 \mathbf{1}_{[0,t]^2}\|^2\big) - 2\Big(\int_{[0,t]^2} \psi_0\psi_\perp^l\Big)\Big(1 + \ln\big(\|\psi_0 \mathbf{1}_{[0,t]^2}\|^2\big)\Big) + O\Big(\frac{1}{l^2}\Big). \tag{2.3.26}$$

- Finally, we set, for $x, y \in (0,t)$,

$$\begin{aligned}
\mathcal{I}_0(x,y) &= 2\int_t^1 \psi_0(x,z)\psi_0(y,z)\,dz \\
\mathcal{I}_\times(x,y) &= 2\int_t^1 \big(\psi_0(x,z)\psi_\perp^l(y,z) + \psi_\perp^l(x,z)\psi_0(y,z)\big)\,dz \\
\mathcal{I}_\perp(x,y) &= 2\int_t^1 \psi_\perp^l(x,z)\psi_\perp^l(y,z)\,dz
\end{aligned}$$

and the corresponding operator $I_0, I_\times$ and $I_\perp$ such that

$$F_1 = c_l\big(I_0 + I_\times + I_\perp\big) \tag{2.3.27}$$

We prove that $I_0 + I_\times$ is a operator of rank at most 4. We show that $-\operatorname{tr}\big(F_1 \ln F_1\big)$ is given by the two highest eigenvalues of $I_0 + I_\times$ up to a controlled error. We state that, to prove (2.2.5), it is sufficient to get that $-\operatorname{tr}\big(I_\perp \ln I_\perp\big)$ is $O(l^{-(1+\delta)})$.

For $k \geqslant 1$, let $\phi_k = \sqrt{2}\sin(\pi k\,.\,)$ be the $k-$th eigenstate of $-\frac{d^2}{dx^2}^{D}_{\|[0,1]}$. We have $\psi_0 = \phi_2 \wedge \phi_1$. We compute for any $(x,y) \in [0,t]^2$,

$$\mathcal{I}_0(x,y) = \|\tau_2\|^2 \upsilon_1(x)\upsilon_1(y) + \|\tau_1\|^2 \upsilon_2(x)\upsilon_2(y) - 2\langle\tau_1,\tau_2\rangle \upsilon_1(x)\upsilon_2(y) \tag{2.3.28}$$

and

$$\mathcal{I}_\times(x,y) = \upsilon_2(x)\omega_1(y) - \upsilon_1(x)\omega_2(y) + \upsilon_2(y)\omega_1(x) - \upsilon_1(y)\omega_2(x) \tag{2.3.29}$$

where we have set

$$\upsilon_k = \phi_k \mathbf{1}_{[0,t]}, \quad \tau_k = \phi_k \mathbf{1}_{[t,1]} \quad \text{and} \quad \omega^k = \int_0^t \psi_\perp^l(x,z)\phi_k(z)\,dz. \tag{2.3.30}$$

We estimate the eigenvalues of $I_0 + I_\times$. Let u_+, u_- be the eigenstates of $\mathcal{I}_0$ with respective eigenvalues $\mu_+ > \mu_- > 0$. If $\upsilon_i = a_+^i u_+ + a_-^i u_-$ then

$$\begin{aligned} I_0 + I_\times &= \mu_+ u_+ \otimes u_+ + \mu_- u_- \otimes u_- + \upsilon_2 \otimes \omega_1 - \upsilon_1 \otimes \omega_2 + \omega_1 \otimes \upsilon_2 - \omega_2 \otimes \upsilon_1 \\ &= \Big(\sqrt{\mu_+} u_+ + \frac{1}{\sqrt{\mu_+}}\big(a_+^2\omega_1 - a_+^1\omega_2\big)\Big) \otimes \Big(\sqrt{\mu_+} u_+ + \frac{1}{\sqrt{\mu_+}}\big(a_+^2\omega_1 - a_+^1\omega_2\big)\Big) \\ &\quad + \Big(\sqrt{\mu_-} u_- + \frac{1}{\sqrt{\mu_-}}\big(a_-^2\omega_1 - a_-^1\omega_2\big)\Big) \otimes \Big(\sqrt{\mu_-} u_- + \frac{1}{\sqrt{\mu_-}}\big(a_-^2\omega_1 - a_-^1\omega_2\big)\Big) \\ &\quad - \frac{1}{\sqrt{\mu_+}}\big(a_+^2\omega_1 - a_+^1\omega_2\big) \otimes \frac{1}{\sqrt{\mu_+}}\big(a_+^2\omega_1 - a_+^1\omega_2\big) \\ &\quad - \frac{1}{\sqrt{\mu_-}}\big(a_-^2\omega_1 - a_-^1\omega_2\big) \otimes \frac{1}{\sqrt{\mu_-}}\big(a_-^2\omega_1 - a_-^1\omega_2\big) \end{aligned} \tag{2.3.31}$$

We note that, using Proposition 3.1.2, for $k \in \{1,2\}$

$$\|\omega_k\|^2 = \int_0^t \Big| \int_t^1 \psi_\perp^l(x,z)\phi_k(z)\,dz\Big|^2 dz \leqslant \|\phi_k\|^2 \|\psi_\perp^l\|^2 \leqslant \frac{C}{l^2}.$$

So,

$$I_0 + I_\times = v_+ \otimes v_+ + v_- \otimes v_- + R \tag{2.3.32}$$

where we have set

$$v_\pm = \sqrt{\mu_\pm} u_\pm + \frac{1}{\sqrt{\mu_\pm}}\big(a_\pm^2\omega_1 - a_\pm^1\omega_2\big) \tag{2.3.33}$$

and where R is a negative operator of norm $O(l^{-2})$ and of rank at most 2. The eigenvalues of $v_+ \otimes v_+ + v_- \otimes v_-$ satisfies, for $\delta \in (0,1)$,

$$\begin{aligned} \lambda_\pm &= \frac{\|v_+\|^2 + \|v_-\|^2 \pm \sqrt{\big(\|v_+\|^2 - \|v_-\|^2\big)^2 + 4\big|\langle v_+, v_-\rangle\big|^2}}{2} \\ &= \mu_\pm + 2\big\langle u_\pm, a_\pm^2\omega_1 - a_\pm^1\omega_2\big\rangle + o\Big(\frac{1}{l^{1+\delta}}\Big). \end{aligned} \tag{2.3.34}$$

Since, for $k \in \{1,2\}$, one has

$$\langle u_\pm, \omega_k\rangle = \int_0^t \int_t^1 \psi_\perp^l(x,z) u_\pm(x)\phi_k(z)\,dz = \langle \psi_\perp^l, u_\pm \otimes \phi_k\rangle, \tag{2.3.35}$$

and $a_\pm^k = \langle u_\pm, \phi_k\rangle$, for $\xi_\pm$ defined by (2.2.6) one has

$$\lambda_\pm = \mu_\pm + 2\langle \psi_\perp^l, \xi_\pm\rangle + o\Big(\frac{1}{l^{1+\delta}}\Big). \tag{2.3.36}$$

Let $(e_i)_{i>0}$ be an ONB of $L^2([0,t])$ and $(e_i)_{i<0}$ be an ONB of $L^2([t,1])$. We decompose $\psi_\perp^l$ as

$$\psi_\perp^l = \sum_{i<j} a_{i,j}(e_i \wedge e_j). \tag{2.3.37}$$

Then, we compute for any $x, y \in [0,t]$

$$\begin{aligned}\mathcal{I}_\perp(x,y) &= 2\sum_{i<j,r<s} a_{i,j}a_{k,l}\int_t^1 (e_i\wedge e_j)(x,z)(e_r\wedge e_s)(y,z)dz \\ &= \sum_{j,s>0}\sum_{i<0} a_{i,j}a_{i,s}e_j(x)e_s(y) \\ &= \sum_{i<0}\chi_i(x)\chi_i(y)\end{aligned} \tag{2.3.38}$$

where we set

$$\chi_i = \sum_{j>0} a_{i,j}e_j. \tag{2.3.39}$$

So, $I_\perp$ is the sum of the rank-one operator $\big(\chi_i\otimes\chi_i\big)_{i<0}$. We shall to use the following lemma.

Lemma 2.3.1. *Let $(u_n)_{n\geqslant 1}$ be a sequence in a Hilbert space $\mathcal{H}$ and let $\Sigma = \sum_{n\geqslant 1}(u_n\otimes u_n)$. Assume that there exist $\sigma > 1$, $\alpha > 1$ and $C > 0$ such that for any $n \geqslant 1$,*

$$\|u_n\|^2 \leqslant \frac{C}{l^\sigma n^\alpha}. \tag{2.3.40}$$

Then, for any $\beta \in \big(2(1+\alpha)^{-1}, 1\big)$, there exists $C > 0$ such that

$$tr\big(\Sigma^\beta\big) \leqslant \frac{C}{l^{\sigma\beta}} \tag{2.3.41}$$

Proof. (of Lemma 2.3.1) Let μ_k^n be the $k-$th eigenvalue of $\Sigma_n = \sum_{j=1}^n u_j\otimes u_j$ the $n-$th partial sum of Σ. We set $\mu_k^n = 0$ for $k > n$. There exists a sequence (α_k^n) such that

$$0 \leqslant \mu_k^{n+1} - \mu_k^n = \alpha_k^{n+1}\|u_{n+1}\|^2 \qquad \text{and} \qquad \sum_{k=1}^{n+1}\alpha_k^{n+1} = 1 \tag{2.3.42}$$

because $\Sigma_{n+1} \geqslant \Sigma_n$ and

$$\|u_{n+1}\|^2 = \mathrm{tr}\big(\Sigma_{n+1} - \Sigma_n\big) = \sum_{k=1}^{n+1}\big(\mu_k^{n+1} - \mu_k^n\big) = \Big(\sum_{k=1}^{n+1}\alpha_k^{n+1}\Big)\|u_{n+1}\|^2.$$

This is an application of a Lidskii result (see Theorem 1.20 of [Sim05])

Set $\beta \in \big(2(1+\alpha)^{-1}, 1\big)$. We consider for $N \geqslant 1$

$$\begin{aligned}\mathrm{tr}\Big(\big(\Sigma_{N+1}\big)^\beta\Big) &= \sum_{k=1}^{N+1}\big(\mu_k^{N+1}\big)^\beta \\ &= \sum_{n=1}^{N+1}\Big(\sum_{k=1}^{n+1}\big(\mu_k^{n+1}\big)^\beta - \sum_{k=1}^{n}\big(\mu_k^n\big)^\beta\Big) + \mu_1^1 \\ &= \sum_{n=1}^{N+1}\sum_{k=1}^{n+1}\Big(\big(\mu_k^{n+1}\big)^\beta - \big(\mu_k^n\big)^\beta\Big) + \mu_1^1 \\ &= \sum_{n=1}^{N+1}\sum_{k=1}^{n+1}\Big(\big(\mu_k^n + \alpha_k^{n+1}\|u_n\|^2\big)^\beta - \big(\mu_k^n\big)^\beta\Big) + \mu_1^1.\end{aligned}$$

Since $x \to x^\beta$ is concave on $(0,+\infty)$ (so subadditive), we get

$$\begin{aligned}\operatorname{tr}\Big(\big(\Sigma_{N+1}\big)^\beta\Big) &\leqslant \sum_{n=1}^{N+1}\sum_{k=1}^{n+1}\Big(\alpha_k^{n+1}\|u_{n+1}\|^2\Big)^\beta + \mu_1^1\\ &\leqslant \sum_{n=1}^{N+1}\|u_{n+1}\|^{2\beta}(n+1)^{1-\beta}\Big(\sum_{k=1}^{n+1}\alpha_k^{n+1}\Big)^\beta + \mu_1^1 \quad \text{by Hölder}\\ &\leqslant \sum_{n=1}^{N+1}\|u_{n+1}\|^{2\beta}(n+1)^{1-\beta} + \mu_1^1\end{aligned}$$

using the property (2.3.42).
The assumption (2.3.40) implies, for any $N \geqslant 1$,

$$\operatorname{tr}\Big(\big(\Sigma_{N+1}\big)^\beta\Big) \leqslant \frac{C}{l^{\sigma\beta}}\sum_{k=1}^{N+1}(n+1)^{1-(1+\alpha)\beta} + \frac{C}{l^\sigma}. \tag{2.3.43}$$

Using $\beta \in \big(2(1+\alpha)^{-1},1\big)$, we conclude that there exist $\sigma > 1$ and $C > 0$ such that for any $N \geqslant 1$

$$\operatorname{tr}\Big(\big(\Sigma_{N+1}\big)^\beta\Big) \leqslant \frac{C}{l^{\sigma\beta}}. \tag{2.3.44}$$

Thus, by monotone convergence, we state that there exists $C > 0$ such that

$$\operatorname{tr}\big(\Sigma^\beta\big) \leqslant \frac{C}{l^{\sigma\beta}}. \tag{2.3.45}$$

□

For any $i < 0$,

$$\begin{aligned}\|\chi_i\|^2 &= \sum_{j>0}\big|\langle\psi_\perp^l, e_i\wedge e_j\rangle\big|^2\\ &= \int_0^t\Big|\int_t^1\psi_\perp^l(x,y)e_i(x)\,dx\Big|^2\,dy.\end{aligned} \tag{2.3.46}$$

We set $e_i(x) = \frac{1}{\sqrt{1-t}}\sin\Big(|i|\pi\frac{x-t}{1-t}\Big)$. Since ψ_0 and ψ_0^l are in $H_0^1\big([0,1]^2\big)$ so does $\psi_\perp^l$. Then, by integration by parts,

$$\frac{1}{\sqrt{1-t}}\int_t^1\psi_\perp^l(x,y)\sin\Big(|i|\pi\frac{x-t}{1-t}\Big)\,dx = -\frac{\sqrt{1-t}}{|i|\pi}\int_t^1\frac{\partial\psi_\perp^l}{\partial x}(x,y)\Big(1-\cos\Big(|i|\pi\frac{x-t}{1-t}\Big)\Big)\,dx \tag{2.3.47}$$

By Cauchy-Schwarz,

$$\begin{aligned}\Big|\int_t^1\psi_\perp^l(x,y)e_i(x)\,dx\Big|^2 &\leqslant \frac{1-t}{i^2\pi^2}\Big(\int_t^1\Big|\frac{\partial\psi_\perp^l}{\partial x}(x,y)\Big|^2\,dx\Big)\Big(\int_t^1\Big|1-\cos\Big(|i|\pi\frac{x-t}{1-t}\Big)\Big|^2\,dx\Big)\\ &\leqslant \frac{4}{i^2\pi^2}\int_0^1\Big|\frac{\partial\psi_\perp^l}{\partial x}(x,y)\Big|^2\,dx\end{aligned}$$

So, for any $i < 0$,

$$\|\chi_i\|^2 \leqslant \frac{4}{i^2\pi^2}\Big\|\frac{\partial\psi_\perp^l}{\partial x}\Big\|^2 \leqslant \frac{C}{i^2}\Big\|\sqrt{h_\perp^0 - e_0^l}\,\psi_\perp^l\Big\|^2 \leqslant \frac{C}{i^2}\times\frac{1}{l} \tag{2.3.48}$$

where the last inequality is obtained from (3.3.16) and (3.3.22). On the other hand, we have by Proposition 3.1.2

$$\|\chi_i\|^2 \leqslant \|\psi_\perp^l\|^2 \leqslant \frac{C}{l^2}. \tag{2.3.49}$$

Thus, for any $i < 0$ and $\theta \in (0,1)$,

$$\|\chi_i\|^2 \leqslant \frac{C}{l^{2-\theta} i^{2\theta}}. \tag{2.3.50}$$

With $\theta \in (\frac{1}{2}, 1)$ and $\beta \in \big(2(1+2\theta)^{-1}, 1\big)$, we derive from Lemma 2.3.1 that there exists $C > 0$ such that

$$-\operatorname{tr}\big(I_\perp^\beta\big) \leqslant \frac{C}{l^{(2-\theta)\beta}}. \tag{2.3.51}$$

Let $(\lambda_k)_{k\geqslant 1}$ be the decreasing sequence of the eigenvalues of the Hilbert-Schmidt operator $v_+ \otimes v_+ + v_- \otimes v_- + I_\perp$. Since $0 \leqslant I_\perp \leqslant \|\psi_\perp^l\|^2 \leqslant Cl^{-2}$, one gets the following bounds:

$$\lambda_+ \leqslant \lambda_1 \leqslant \lambda_+ + \frac{C}{l^2} \quad \text{and} \quad \lambda_- \leqslant \lambda_2 \leqslant \lambda_- + \frac{C}{l^2}. \tag{2.3.52}$$

Otherwise, we shall apply twice a lemma resulting from the Courant-Fischer-Weyl min-max principle.

Lemma 2.3.2. *[Kir08] Let K be a compact selfadjoint operator on an Hilbert space $\mathcal{H}$ with eigenvalues $\mu_1 \geqslant \mu_2 \geqslant \dots$, $u \in \mathcal{H}$ and $\alpha \in \mathbb{R}$. Then $K + \alpha\langle \cdot, u\rangle u$ has eigenvalues $(\lambda_k)_{k\in\mathbb{N}}$ with $\lambda_k \leqslant \mu_{k-1}$ for any $k \geqslant 2$.*

Since $x \to x^\beta$ is increasing for any $\beta \in (0,1)$, we derive from Lemma 2.3.2 that

$$-\sum_{k\geqslant 3} \lambda_k^\beta \leqslant -\operatorname{tr}\big(I_\perp^\beta\big). \tag{2.3.53}$$

Finally, we consider $(\sigma_k)_{k\leqslant 1}$ the decreasing sequence of the eigenvalues of the Hilbert-Schmidt operator F_1, defined in (2.3.19). Using (2.3.27) and (2.3.32), we have

$$F_1 = c_l\Big(v_+ \otimes v_+ + v_- \otimes v_- + I_\perp + R\Big). \tag{2.3.54}$$

with $-Cl^{-2} \leqslant R \leqslant 0$. So, for any $k \geqslant 1$,

$$\max\big(0, c_l(\lambda_k - Cl^{-2})\big) \leqslant \sigma_k \leqslant c_l \lambda_k. \tag{2.3.55}$$

Combining it with (2.3.53), we get that for $\beta \in (0,1)$

$$-\sum_{k\geqslant 3} \sigma_k^\beta \leqslant -c_l^\beta \operatorname{tr}\big(I_\perp^\beta\big) \tag{2.3.56}$$

So, using (2.3.51), there exists $\delta \in (0,1)$ such that

$$-\sum_{k\geqslant 3} \sigma_k \ln \sigma_k = o\big(l^{-(1+\delta)}\big) \tag{2.3.57}$$

On the other hand, from (2.3.55), (2.3.52) and (2.3.36), we get

$$\sigma_1 = \mu_+ + 2\langle \psi_\perp^l, \xi_+\rangle + o\Big(\frac{1}{l^{1+\delta}}\Big) \tag{2.3.58}$$

so

$$-\sigma_1 \ln \sigma_1 = \mu_+ \ln(\mu_+) + \langle \psi_\perp^l, \xi_+ \rangle \big(1 + \ln \mu_+\big) + o\big(\frac{1}{l^{1+\delta}}\big) \tag{2.3.59}$$

and similar formulas holds for σ_2 replacing $+$ by $-$.

Thus, there exists $\delta \in (0,1)$ such that

$$-\operatorname{tr}\big(K_1 \ln K_1\big) = -\sum_{*\in\{+,-\}} \Big(\mu_* \ln(\mu_*) + \langle \psi_\perp^l, \xi_* \rangle \big(1 + \ln \mu_*\big)\Big) + o\big(\frac{1}{l^{1+\delta}}\big) \tag{2.3.60}$$

with $\xi_\pm$ given by (2.2.6).

Combining (2.3.24), (2.3.26) and (2.3.60), we get that if $L_\star$ is at distance greater than $cl_{\rho,U}$ then, with interactions, the entanglement entropy admits the following expansion:

$$\begin{aligned} s_{l_\star}^U\big([0,l],2\big) = s_{l_\star}^0\big([0,l],2\big) &- 2\Big(\int_{[0,t_\star]^2} \psi_0 \psi_\perp^l\Big)\Big(1 + \ln\big(\|\psi_0 \mathbf{1}_{[0,t_\star]^2}\|^2\big)\Big) \\ &- 2\langle \psi_\perp^l, \xi_+ \rangle \big(1 + \ln \mu_+\big) - 2\langle \psi_\perp^l, \xi_- \rangle \big(1 + \ln \mu_-\big) \\ &- 2\Big(\int_{[t_\star,1]^2} \psi_0 \psi_\perp^l\Big)\Big(1 + \ln\big(\|\psi_0 \mathbf{1}_{[t_\star,1]^2}\|^2\big)\Big) + o\big(l_{\rho,U}^{-(1+\delta)}\big) \end{aligned} \tag{2.3.61}$$

It concludes the proof of Proposition 2.2.1. □

2.3.2.2 Case of two pieces

Proof. (of Proposition 2.2.2) Set $l > 0$, $d \geqslant 0$, $a > 1$ and $l_\star \in [-al, d+l]$. Applying Definition 2.1.2 and Proposition 2.3.1, the entanglement entropy $s_{l_\star}^U\big(([-al,0],[d,d+l]),2\big)$ is given by

$$s_{l_\star}^U\big(([-al,0],[d,d+l]),2\big) = \sum_{s=0}^{2} \operatorname{tr}\big(K_s \ln K_s\big) \tag{2.3.62}$$

where the operator K_s admits the following kernel

$$\begin{aligned} \forall\, x,y \in [-al, l_\star]^s \qquad \mathcal{K}_s(x,y) = \binom{2}{s} \int_{[l_\star, l+d]^{2-s}} &\psi^U\big(([-al,0],[d,d+l]),(1,1)\big)(x,z) \times \\ &\psi^U\big(([-al,0],[d,d+l]),(1,1)\big)(y,z)\, dz \end{aligned} \tag{2.3.63}$$

with $\psi^U\big(([-al,0],[d,d+l]),(1,1)\big)$ the ground state of the 2-particle system in the pair $([-al,0],[d,d+l])$ (see Definition 1.4.2).

Figure 2.1 shows the three possible cases for the cut $l_\star$.

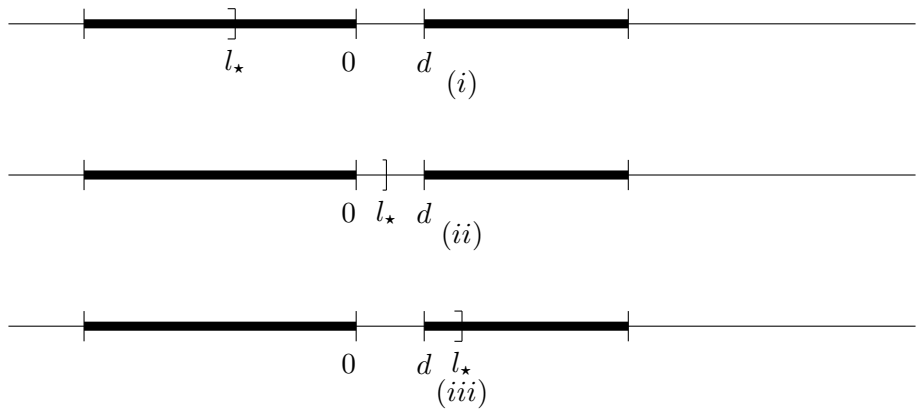

Figure 2.1: Cases of cut $l_\star$ in a pair of pieces.

We deal with these cases separately.

(i) First, since the ground state $\psi^U\big(([-al,0],[d,d+l]),(1,1)\big)(x,y)$ is equal to zero if both x and y are in the same interval we have

$$K_2 \equiv 0. \tag{2.3.64}$$

Set $t = \frac{al-l_\star}{al} > 0$. We will use the notations of Section 3.2, Chapter 3. Using the isometry $O^{l,a}$ given by (3.2.2), the eigenvalues of K_s, $s \in \{0,1\}$, are equal to those of the operator F_s on $L^2([t,1]^s)$ defined by kernel

$$\forall x,y \in [t,1]^s \qquad \mathcal{F}_s(x,y) = \int_{[0,t]^{1-s}\times[0,1]} \psi_0^{l,a}(x,z)\psi_0^{l,a}(y,z)\,dz. \tag{2.3.65}$$

where $\psi_0^{l,a}$ is the ground state of $h^{l,a}$. We recall that

$$h^{l,a} = -\frac{1}{a^2}\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} + U^{l,a} \tag{2.3.66}$$

and the decomposition

$$\psi_0^{l,a} = \frac{\psi_0^a + \psi_\perp^{l,a}}{\sqrt{1+\|\psi_\perp^{l,a}\|^2}}. \tag{2.3.67}$$

- For $t \in [0,1]$, the scalar $\mathcal{F}_0$ is given by

$$\mathcal{F}_0 = c_{l,a}\Big(\big\|\psi_0^a \mathbf{1}_{[0,t]\times[0,1]}\big\|^2 + 2\int_0^t\int_0^1 \psi_0^a(x,y)\psi_\perp^{l,a}(x,y)\,dxdy + \big\|\psi_\perp^{l,a}\mathbf{1}_{[0,t]\times[0,1]}\big\|^2\Big) \tag{2.3.68}$$

with $c_{l,a} = \big(1+\|\psi_\perp^{l,a}\|^2\big)^{-1}$. By Proposition 3.2.2, we know that

$$\big\|\psi_\perp^{l,a}\mathbf{1}_{[0,t]\times[0,1]}\big\|^2 \leqslant \frac{C}{a^3 l^6}$$

and

$$\Big|\int_0^t\int_0^1 \psi_0^a(x,y)\psi_\perp^{l,a}(x,y)\,dxdy\Big| \leqslant \|\psi_0^a\|\|\psi_\perp^{l,a}\| \leqslant \frac{C}{a^{3/2}l^3}$$

So,

$$-\operatorname{tr}(K_0 \ln K_0) = -\|\phi_1 \mathbf{1}_{[0,t]}\|^2 \ln\Big(\|\phi_1 \mathbf{1}_{[0,t]}\|^2\Big) + O\Big(\frac{1}{l^3}\Big). \tag{2.3.69}$$

- On the other hand, we denote

$$\psi_0^a = \phi_1 \otimes \phi_1 \qquad \text{and} \qquad \psi_\perp^{l,a} = \sum_{i,j} a_{i,j}(f_i \otimes \phi_j) \tag{2.3.70}$$

in the basis $(f_i \otimes \phi_j)_{i\in\mathbb{Z}, j>0}$ with $(f_i)_{i>0}$ an ONB of $L^2([0,t])$, $(f_i)_{i<0}$ an ONB of $L^2([t,1])$ and, for $j>0$, $\phi_j = \sin(\pi j \cdot)$. Then, we compute for any $x, y \in [t,1]$,

$$\begin{aligned}\mathcal{F}_1(x,y) &= c_{l,a} \int_0^1 \big(\psi_0^a + \psi_\perp^{l,a}\big)(x,z)\big(\psi_0^a + \psi_\perp^{l,a}\big)(y,z)\, dz \\ &= c_{l,a}\Big(\phi_1(x)\phi_1(y) + \phi_1(x)\gamma_1(y) + \gamma_1(x)\phi_1(y) + \sum_{j>0} \gamma_j(x)\gamma_j(y)\Big)\end{aligned}$$

with

$$\gamma_j = \sum_{i<0} a_{i,j} f_i. \tag{2.3.71}$$

Thus,

$$F_1 = c_{l,a}\Big(\upsilon \otimes \upsilon + \widetilde{F_1}\Big) \tag{2.3.72}$$

where $\upsilon = \phi_1 \mathbf{1}_{[t,1]} + \gamma_1$ and $\widetilde{F_1}$ is the sum of the rank-one operators $\big(\gamma_j \otimes \gamma_j\big)_{j\geqslant 2}$. The unique eigenvalue of the operator $\upsilon \otimes \upsilon$ is $\mu = \|\upsilon + \gamma_1\|^2$. By Proposition 3.2.2,

$$|\langle \phi_1, \gamma_1 \rangle| \leqslant \|\gamma_1\| \leqslant \|\psi_\perp^{l,a}\| \leqslant \frac{C}{a^{3/2} l^3}$$

So,

$$\mu = \|\phi_1 \mathbf{1}_{[t,1]}\|^2 + O\big(\frac{1}{l^3}\big) \tag{2.3.73}$$

We will prove that for $\delta \in (0,1)$,

$$-\operatorname{tr}(F_1 \ln F_1) = \|\phi_1 \mathbf{1}_{[t,1]}\|^2 \ln\big(\|\phi_1 \mathbf{1}_{[t,1]}\|^2\big) + O\big(\frac{1}{l^3}\big). \tag{2.3.74}$$

We state that it is sufficient to get that $-\operatorname{tr}\big(\widetilde{F_1} \ln \widetilde{F_1}\big)$ is $O(l^{-3})$. Let $(\lambda_k)_{k\geqslant 1}$ be the decreasing sequence of eigenvalues of F_1. Since $0 \leqslant \widetilde{F_1} \leqslant \|\psi_\perp^{l,a}\|^2$, we have, by Proposition 3.2.2,

$$c_{l,a}\mu \leqslant \lambda_1 \leqslant c_{l,a}\big(\mu + \|\psi_\perp^{l,a}\|^2\big) \tag{2.3.75}$$

so

$$\lambda_1 = \|\phi_1 \mathbf{1}_{[t,1]}\|^2 + O\big(l^{-3}\big). \tag{2.3.76}$$

On the other hand, we apply Lemma 2.3.2 to F_1. So, for $\beta \in (0,1)$,

$$\sum_{k\geqslant 2} \lambda_k^\beta \leqslant c_{l,a}^\beta \mathrm{tr}\big(\widetilde{F_1}^\beta\big) \tag{2.3.77}$$

It remains to prove for some $\beta \in (0,1)$

$$\mathrm{tr}\big(\widetilde{F_1}^\beta\big) = O(l^{-3}). \tag{2.3.78}$$

We shall apply Lemma 2.3.1 and, thus, we estimate $\|\gamma_j\|^2$ for $j > 0$.

$$\begin{aligned} \|\gamma_j\|^2 &= \sum_{j>0} \big|\big\langle \psi_\perp^{l,a}, f_i \otimes \phi_j \big\rangle\big|^2 \\ &= \int_t^1 \Big| \int_0^1 \psi_\perp^{l,a}(x,y)\phi_j(y)\,dy \Big|^2 dx. \end{aligned} \tag{2.3.79}$$

Since ψ_0^a and $\psi_0^{l,a}$ are in $H_0^1\big([0,1]^2\big)$ so does $\psi_\perp^{l,a}$. Then, by integration by parts,

$$\int_0^1 \psi_\perp^{l,a}(x,y)\sin(\pi j y)\,dy = -\frac{1}{\pi i}\int_0^1 \frac{\partial \psi_\perp^{l,a}}{\partial y}(x,y)\Big(1-\cos(\pi j y)\Big)\,dy \tag{2.3.80}$$

So, by Cauchy-Schwarz, for any $j > 0$,

$$\|\gamma_j\|^2 \leqslant \frac{4}{j^2\pi^2}\Big\|\frac{\partial \psi_\perp^{l,a}}{\partial y}\Big\|^2 \leqslant \frac{C}{j^2}\Big\|\sqrt{h_\perp^{0,a} - e_0^{l,a}}\,\psi_\perp^{l,a}\Big\|^2 \leqslant \frac{C}{j^2} \times \frac{1}{a^3 l^4} \tag{2.3.81}$$

where the last inequality is obtained from (3.3.16) and (3.3.86). We conclude to (2.3.78) using Lemma 2.3.1. Thus,

$$-\mathrm{tr}(K_1 \ln K_1) = \|\phi_1 \mathbf{1}_{[t,1]}\|^2 \ln\big(\|\phi_1 \mathbf{1}_{[t,1]}\|^2\big) + O\big(l^{-3}\big). \tag{2.3.82}$$

(ii) Each interval contains exactly one particle so

$$K_0 \equiv 0 \quad \text{and} \quad K_2 \equiv 0. \tag{2.3.83}$$

In fact, the case (ii), meaning with a cut in the gap is similar to the case (i) if the cut is exactly at 0. We conclude that

$$-\mathrm{tr}(K_1 \ln K_1) = O\big(l^{-3}\big). \tag{2.3.84}$$

(iii) We note that by definition of $\psi^U\big(([-al,0],[d,d+l]),(1,1)\big)$,

$$K_0 \equiv 0. \tag{2.3.85}$$

Set $t = \frac{l_\star - d}{l} > 0$. Again, using the isometry $O^{l,a}$ given by 3.2.2, the eigenvalues of K_s, $s \in \{1,2\}$, are eaqual to those of F_s with respective kernel

$$\forall x,y \in [0,1] \qquad \mathcal{F}_1(x,y) = \int_t^1 \psi_0^{l,a}(x,z)\psi_0^{l,a}(y,z)\,dz \tag{2.3.86}$$

$$\forall x_1,y_1 \in [0,1], \quad \forall x_2,y_2 \in [0,t] \qquad \mathcal{F}_2(x_1,x_2,y_1,y_2) = \psi_0^{l,a}(x_1,x_2)\psi_0^{l,a}(y_1,y_2) \tag{2.3.87}$$

where $\psi_0^{l,a}$ is the ground state of $h^{l,a}$ given by (2.3.66).

- We denote

$$\psi_0^{l,a} = \frac{\psi_0^a + \psi_\perp^{l,a}}{\sqrt{1+\|\psi_\perp^{l,a}\|^2}} \qquad \text{with} \qquad \psi_0^a = \phi_1 \otimes \big(\sum_{j\in\mathbb{Z}} b_j g_j\big) \qquad \text{and} \qquad \psi_\perp^{l,a} = \sum_{i,j} a_{i,j}(\phi_i \otimes g_j) \tag{2.3.88}$$

in the basis $(\phi_i \otimes g_j)_{i>0, j\in\mathbb{Z}}$ with $(g_i)_{i>0}$ an ONB of $L^2([0,t])$, $(g_i)_{i<0}$ an ONB of $L^2([t,1])$ and, for $i>0$, $\phi_i = \sin(\pi i \cdot)$. Then, for any $x, y \in [0,1]$,

$$\begin{aligned}\mathcal{F}_1(x,y) &= c_{l,a}\Big(\|\phi_1 \mathbf{1}_{[t,1]}\|^2 \phi_1(x)\phi_1(y) + \sum_{i>0}\sum_{j<0} b_j a_{i,j}\big(\phi_1(x) f_i(y) + f_i(x)\phi_1(y)\big) \\ &\quad + \sum_{i,k>0}\sum_{j<0} a_{i,j}a_{k,j} f_i(x) f_k(y)\Big) \\ &= c_{l,a}\Big(\alpha\phi_1(x)\phi_1(y) + \phi_1(x)\xi(y) + \xi(x)\phi_1(y) + \sum \gamma_j(x)\gamma_j(y)\Big)\end{aligned}$$

where we have set

$$\alpha = \|\phi_1 \mathbf{1}_{[t,1]}\|^2, \qquad \gamma_j = \sum_{i>0} a_{i,j} f_i \quad \text{and} \quad \xi = \sum_{j<0} b_j \gamma_j. \tag{2.3.89}$$

We define

$$u = \sqrt{\alpha}\phi_1 + \frac{1}{\sqrt{\alpha}}\xi, \qquad v = \frac{1}{\sqrt{\alpha}}\xi \quad \text{and} \quad \widetilde{\mathcal{F}_1} = \sum_{j<0} \gamma_j \otimes \gamma_j \tag{2.3.90}$$

so that

$$F_1 = c_{l,a}\Big(u\otimes u - v \otimes v + \widetilde{\mathcal{F}_1}\Big). \tag{2.3.91}$$

By Proposition 3.2.2, we have

$$\|\xi\|^2 \leqslant \big(\sum_{j<0} \langle \phi_1, g_j\rangle\langle \psi_\perp^{l,a}, f_i \otimes g_j\rangle\big)^2 \leqslant \|\phi_1 \mathbf{1}_{[t,1]}\|^2 \|\psi_\perp^{l,a}\|^2 \leqslant \frac{C}{a^3 l^6}. \tag{2.3.92}$$

So, the eigenvalues of $u\otimes u$ and $v \otimes v$, respectively μ_+ and μ_-, satisfy

$$\mu_+ = \|\phi_1 \mathbf{1}_{[t,1]}\|^2 + O(l^{-3}) \quad \text{and} \quad \mu_- = O(l^{-6}). \tag{2.3.93}$$

We will prove that for $\delta \in (0,1)$,

$$-\operatorname{tr}(F_1 \ln F_1) = -\|\phi_1 \mathbf{1}_{[t,1]}\|^2 \ln\big(\|\phi_1 \mathbf{1}_{[t,1]}\|^2\big) + O\big(l^{-3}\big). \tag{2.3.94}$$

Again, we state that it is sufficient to get that $-\operatorname{tr}\widetilde{\mathcal{F}_1}\ln\widetilde{\mathcal{F}_1}$ is $O\big(l^{-3}\big)$. We denote by $\widetilde{G_1}$ the Hilbert-Schmit operator $u\otimes u + \widetilde{F_1}$. If $(\lambda_k)_{k\geqslant 1}$ are the eigenvalues of $\widetilde{G_1}$, with $\lambda_1 \geqslant \lambda_2 \geqslant \dots$, then, since $0 \leqslant \widetilde{F_1} \leqslant \|\psi_\perp^{l,a}\|^2$,

$$\mu_+ \leqslant \lambda_1 \leqslant \mu_+ + \|\psi_\perp^{l,a}\|^2, \tag{2.3.95}$$

and, by Lemma 2.3.2, for $\beta \in (0,1)$,

$$\sum_{k\geqslant 2} \lambda_k^\beta \leqslant \operatorname{tr}\Big(\widetilde{F_1}^\beta\Big). \tag{2.3.96}$$

Let $(\sigma_k)_{k\geqslant 1}$ be the decreasing sequence of eigenvalues of F_1. Since $0 \leqslant F_1 = c_{l,a}(G_1 - v\otimes v)$, we have

$$\max\big(0, c_{l,a}(\lambda_k - \mu_-)\big) \leqslant \sigma_k \leqslant c_{l,a}\lambda_k. \tag{2.3.97}$$

Using (2.3.93), (2.3.95) and Proposition 3.2.2, we get

$$\sigma_1 = \|\phi_1 \mathbf{1}_{[t,1]}\|^2 + O(l^{-3}). \tag{2.3.98}$$

On the other hand, combining (2.3.97) and (2.3.96), it gives

$$\sum_{k\geqslant 2} \sigma_k^\beta \leqslant c_{l,a}\,\mathrm{tr}\Big(\widetilde{F_1}^\beta\Big). \tag{2.3.99}$$

It only remains to prove that for some $\beta \in (0,1)$

$$-\,\mathrm{tr}\widetilde{\mathcal{F}_1}\ln\widetilde{\mathcal{F}_1} = O\big(l^{-3}\big). \tag{2.3.100}$$

We shall use Lemma 2.3.1. We set, for $j<0$, $g_j(x) = \frac{1}{\sqrt{1-t}}\sin\Big(|j|\pi\frac{x-t}{1-t}\Big)$. By integration by parts,

$$\int_t^1 \psi_\perp^{l,a}(x,y) g_j(y)\,dy = -\frac{\sqrt{1-t}}{|j|\pi}\int_t^1 \frac{\partial\psi_\perp^l}{\partial y}(x,y)\Big(1-\cos\Big(|j|\pi\frac{y-t}{1-t}\Big)\Big)\,dx. \tag{2.3.101}$$

So,

$$\begin{aligned}
\|\gamma_j\|^2 &= \sum_{i>0}\big|\langle\psi_\perp^{l,a}, \phi_i\otimes g_j\big|^2 \\
&= \int_0^1\Big|\int_t^1 \psi^{l,a} g_j(y)\,dy\Big|^2\,dx \\
&\leqslant \frac{4}{j^2\pi^2}\Big\|\frac{\partial\psi_\perp^{l,a}}{\partial y}\Big\|^2 && \text{by above observation and Cauchy-Schwarz} \\
&\leqslant \frac{C}{j^2}\Big\|\sqrt{h_\perp^{0,a} - e_0^{l,a}}\,\psi_\perp^{l,a}\Big\|^2 \\
&\leqslant \frac{C}{j^2}\times\frac{1}{a^3 l^4} && \text{using (3.3.16) and (3.3.86).}
\end{aligned} \tag{2.3.102}$$

We conclude to (2.3.100) using Lemma 2.3.1. Thus,

$$-\,\mathrm{tr}(K_1\ln K_1) = -\|\phi_1\mathbf{1}_{[t,1]}\|^2\ln\big(\|\phi_1\mathbf{1}_{[t,1]}\|^2\big) + o\big(l^{-3}\big). \tag{2.3.103}$$

- On the other hand, the operator $\mathcal{F}_2$ is rank-one with positive eigenvalue λ given by

$$\lambda = c_{l,a}\Big(\|\psi_0^a\mathbf{1}_{[0,1]\times[0,t]}\|^2 + 2\int_0^1\int_0^t \psi_0^a(x,y)\psi_\perp^{l,a}(x,y)\,dxdy + \|\psi_\perp^{l,a}\mathbf{1}_{[0,1]\times[0,t]}\|^2\Big) \tag{2.3.104}$$

We know, by Proposition 3.2.2, that

$$\|\psi_\perp^{l,a}\|^2 \leqslant \frac{C}{a^3 l^6}$$

and

$$\Big|\int_0^1\int_0^t \psi_0^a(x,y)\psi_\perp^{l,a}(x,y)\,dxdy\Big| \leqslant \|\psi_0^a\|\|\psi_\perp^{l,a}\| \leqslant \frac{C}{a^{3/2}l^3}.$$

So,

$$-\,\mathrm{tr}(K_2\ln K_2) = -\|\phi_1\mathbf{1}_{[0,t]}\|^2\ln\Big(\|\phi_1\mathbf{1}_{[0,t]}\|^2\Big) + O\big(l^{-3}\big). \tag{2.3.105}$$

As a conclusion, we prove that if the cut $L_\star$ lies in a pair of pieces then up to an error $O(l^{-3})$, the entanglement entropy behaves similarly to the free case:

$$s^U_{l_\star}\big(([-al,0],[d,d+l]),2\big) = s^0_{l_\star}\big(([-al,0],[d,d+l]),2\big) + O\big(l^{-3}\big). \tag{2.3.106}$$

It concludes the proof of Proposition 2.2.2. □

Chapter 3

2−particle systems

3.1 Interaction in a single piece

Let $U : \mathbb{R} \to \mathbb{R}$ be nonnegative, even, bounded and compactly supported function, and let σ_U be the diameter of its support.

We consider two fermions in an interval $[0, l]$ that repel one another because of the potential U, that is, on $L^2\big([0,l]\big) \wedge L^2\big([0,l]\big)$, the following operator

$$h^U\big([0,l],2\big) = \left(-\frac{d^2}{dy^2}^{\,D}_{|[0,l]} \right) \otimes \mathbf{1}_{L^2([0,l])} + \mathbf{1}_{L^2([0,l])} \otimes \left(-\frac{d^2}{dx^2}^{\,D}_{|[0,l]} \right) + U(x-y) \tag{3.1.1}$$

with Dirichlet boundary conditions.

We set

$$O^l : \begin{cases} L^2\big([0,1]^2\big) & \longrightarrow L^2\big([0,l]^2\big) \\ \phi & \longmapsto l^{-1}\, \phi \circ (\theta^l)^{-1} \end{cases} \qquad \text{where} \qquad \theta^l(x,y) = (lx, ly). \tag{3.1.2}$$

Then, the operator $h^U([0,l],2)$ is unitarily equivalent, by action of O^l, to the operator $l^{-2}h^l$ where

$$h^l = -\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} + U^l \qquad \text{on } L^2\big([0,1]\big) \wedge L^2\big([0,1]\big) \tag{3.1.3}$$

with Dirichlet boundary conditions and $U^l(x,y) = l^2 U\big(l(x-y)\big)$. The operator h^l is essentially self-adjoint and its ground state is non-degenerate (see Proposition 1.3.1). We denote (e_0^l, ψ_0^l) the ground state couple of the operator h^l.

One checks that the eigenvalues of the free operator h^0 are

$$e_{p,q} = \pi^2(p^2+q^2) \qquad (e_0 := e_{1,1}) \tag{3.1.4}$$

for $p > q \geqslant 1$, with the corresponding eigenfunctions

$$\psi_{p,q}(x,y) = \sqrt{2}\,\big(\sin(\pi p x)\sin(\pi q y) - \sin(\pi p y)\sin(\pi q x)\big) \qquad (\psi_0 := \psi_{2,1}). \tag{3.1.5}$$

Klopp and Veniaminov give the expansion of the ground state energy of h^l.

Proposition 3.1.1. *[KV20] For $l > 0$, the ground state energy e_0^l of the operator h^l, given by (3.1.3), admits the following expansion*

$$e_0^l = 5\pi^2 + \frac{\gamma}{l} + o(l) \tag{3.1.6}$$

with $\gamma > 0$ when $U \neq 0$.

We use their results to study the ground state itself.

Proposition 3.1.2. *Let (e_0^l, ψ_0^l) be the ground state couple of h^l given by (3.1.3). Decompose the ground state ψ_0^l on Span$(\psi_0) \oplus$ Span$(\psi_0)^\perp$,*

$$\psi_0^l = \frac{1}{\sqrt{1 + \|\psi_\perp^l\|^2}} \Big(\psi_0 + \psi_\perp^l\Big) \tag{3.1.7}$$

Then,

$$\|\psi_\perp^l\|^2 = \frac{C}{l^2} + o\Big(\frac{1}{l^2}\Big). \tag{3.1.8}$$

We also get an L^2-expansion of the ground state (see Corollary 3.3.1).

3.2 Interaction between two pieces

Let $U : \mathbb{R} \to \mathbb{R}$ be nonnegative, even, bounded and compactly supported function, and let σ_U be the diameter of its support.

This time, we consider two fermions lying in separate pieces $[-al, 0]$ and $[d, d+l]$. The corresponding operator is given by

$$\begin{aligned} h^U\Big(([-al,0],[d,d+l]),(1,1)\Big) = &\Big(-\frac{d^2}{dy^2}^{D}_{|[-al,0]}\Big) \otimes \mathbf{1}_{L^2([d,d+l])} + \mathbf{1}_{L^2([-al,0])} \otimes \Big(-\frac{d^2}{dx^2}^{D}_{|[d,d+l]}\Big) \\ &+ U(x-y) \qquad \text{on } L^2\big([-al,0]\big) \wedge L^2\big([d,d+l]\big). \end{aligned} \tag{3.2.1}$$

for $l > 0$, $d \geqslant 0$ and $a > 1$, and with Dirichlet boundary conditions.

We set

$$O^{l,a} : \begin{cases} L^2\big([0,1]^2\big) & \longrightarrow L^2\big([-al,0] \times [d,d+l]\big) \\ \phi & \longmapsto a^{-1/2} l^{-1}\, \phi \circ (\theta^{l,a})^{-1} \end{cases} \qquad \text{where} \qquad \theta^{l,a}(x,y) = (-alx, ly + d). \tag{3.2.2}$$

Then, the operator $h^U\Big(([-al,0],[d,d+l]),(1,1)\Big)$ is unitarily equivalent, by action of $O^{l,a}$, to the operator $l^{-2}h^{l,a}$ where

$$h^{l,a} = -\frac{1}{a^2}\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} + U^{l,a} \tag{3.2.3}$$

with Dirichlet boundary conditions and $U^{l,a}(x,y) = l^2 U(alx + ly + d)$. Similarly, the operator $h^{l,a}$ is essentially self-adjoint and its ground state is non-degenerate (see Proposition 1.3.1). We denote $(e_0^{l,a}, \psi_0^{l,a})$ the ground state couple of the operator $h^{l,a}$.

One checks that the eigenvalues of the free operator $h^{0,a}$ are

$$e_{p,q}^a = \pi^2(a^{-2}p^2 + q^2) \qquad (e_0^a := e_{1,1}^a) \tag{3.2.4}$$

for $p, q \geqslant 1$, with the corresponding eigenfunctions

$$\psi_{p,q}^a(x,y) = 2\sin(\pi p x)\sin(\pi q y) \qquad (\psi_0^a := \psi_{1,1}^a). \tag{3.2.5}$$

As for the case of a single piece, we give an expansion of the ground state energy.

Proposition 3.2.1. *For $d \geqslant 0$, $a \geqslant 1$ and large $l > 0$, the ground state energy $e_0^{l,a}$ of the operator $h_0^{l,a}$ admits the following expansion*

$$e_0^{l,a} = \pi^2\Big(1 + \frac{1}{a^2}\Big) + \frac{\tau(d)}{a^3 l^4}\Big(1 + o(1)\Big) \tag{3.2.6}$$

with $\tau(d)$ a positive function that vanishes for $d > \sigma_U$.

We also study the ground state itself.

Proposition 3.2.2. *Let $(e_0^{l,a}, \psi_0^{l,a})$ be the ground state couple of $h^{l,a}$ given by (3.2.3). Decompose the ground state $\psi_0^{l,a}$ on $Span(\psi_0^a) \oplus Span(\psi_0^a)^\perp$,*

$$\psi_0^{l,a} = \frac{1}{\sqrt{1 + \|\psi_\perp^{l,a}\|^2}}\Big(\psi_0^a + \psi_\perp^{l,a}\Big) \tag{3.2.7}$$

Then,

$$\|\psi_\perp^{l,a}\|^2 = \frac{C}{a^3 l^6} + o\Big(\frac{1}{l^6}\Big). \tag{3.2.8}$$

3.3 Proofs

In this section, the index $(*)$ is empty or equal to (a). It refers to either the 2-particle system in single piece or the 2-particle system in two pieces.

3.3.1 Firsts observations

To begin with, one remarks that for both cases

$$0 < e_0^{(*)} \leqslant e_0^{l,(*)} \leqslant e_0^{(*)} + \big\langle U^{l,(*)}\psi_0^{(*)}, \psi_0^{(*)}\big\rangle. \tag{3.3.1}$$

Since U is compactly supported (see Assumption **??**), the interaction $U^{l,(*)}$ is restricted to the grey areas given by Figure 3.1. As $l \to \infty$, the area on the left square (i) converges to the diagonal with speed $O(l^{-1})$, but the area on the right square (ii) goes to $(0,0)$ with speed $O(l^{-2})$.

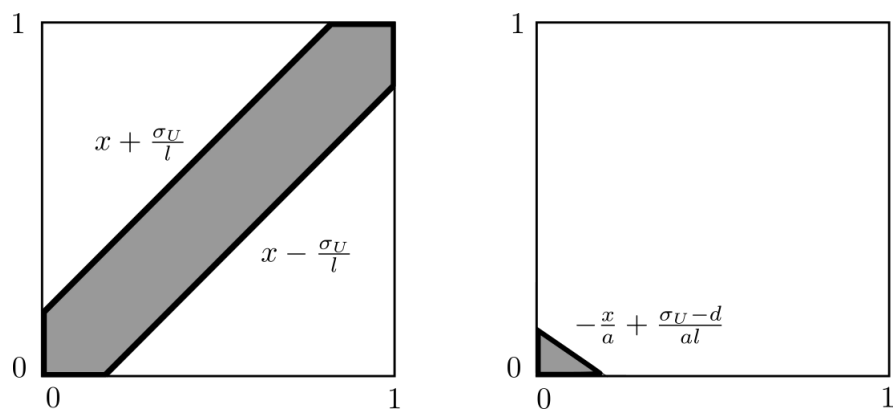

Figure 3.1: Locations of the interaction after the scaling to $[0,1]^2$. (i) *on the left the case of two particles in a single piece of length l;* (ii) *on the right the case of two particles in two distinct pieces of lengths l, al with $a \geqslant 1$ and at distance $d \leqslant \sigma_U$.*

Then, one sets for each case a partial isometry $\Gamma_{(*)}^l$ that emphasizes the grey area behaviour. More precisely, we define

(i) for a single piece,

$$\Gamma^l : \begin{cases} L^2\big([0,1]^2\big) & \longrightarrow L^2(\mathbb{R}\times[0,1]) \\ f & \longmapsto \frac{1}{\sqrt{l}}\mathbf{1}_{\Omega^l}(f\circ\gamma^{-1}) \end{cases} \tag{3.3.2}$$

where $\Omega^l = \Big\{(u,y)\in\mathbb{R}\times[0,1],\ y+\frac{u}{l}\in(0,1)\Big\}$ and $\gamma : \begin{cases} [0,1]^2 & \longrightarrow \mathbb{R}\times[0,1] \\ (x,y) & \longmapsto \big(l(x-y),y\big) \end{cases}$;

(ii) for two pieces

$$\Gamma^{l,a} : \begin{cases} L^2\big([0,1]^2\big) & \longrightarrow L^2(\mathbb{R}^2_{+\star}) \\ f & \longmapsto \frac{1}{l\sqrt{a}}\mathbf{1}_{\Omega^{l,a}}(f\circ(\gamma^a)^{-1}) \end{cases} \tag{3.3.3}$$

where $\Omega^{l,a} = \Big\{(u,v)\in\mathbb{R}^2_{+\star},\ \frac{u}{l}\in(0,1),\ \frac{v}{al}\in(0,1)\Big\}$ and $\gamma^a : \begin{cases} [0,1]^2 & \longrightarrow \mathbb{R}^2_{+\star} \\ (x,y) & \longmapsto (lx,lay) \end{cases}$.

One can use these partial isometries to get an upper bound for the perturbation of the ground state energy due to the interaction. For $l>0$

$$\begin{aligned} \big\langle U^l\psi_0,\psi_0\big\rangle &= \big\langle \Gamma^l U^l\psi_0,\Gamma^l\psi_0\big\rangle \\ &= 2l\int_{\mathbb{R}\times[0,1]} U(u)\Big(\sin\big(\pi(y+ul^{-1})\big)\sin\big(2\pi y\big)-\sin\big(\pi y\big)\sin\big(2\pi(y+ul^{-1})\big)\Big)^2\,dudy \\ &= 2l^{-1}\int_{\mathbb{R}\times[0,1]} U(u)\pi^2u^2\Big(4\cos^2(2\pi y)\sin^2(\pi y)+\cos^2(\pi y)\sin^2(2\pi y) \\ &\quad -4\cos(2\pi y)\cos(\pi y)\sin(2\pi y)\sin(\pi y)\Big)dudy + O(l^{-2}) \quad \text{using Taylor expansions} \\ &\leqslant Cl^{-1} \quad \text{since } U \text{ is compactly supported} \end{aligned} \tag{3.3.4, 3.3.5}$$

whereas $l>0$, $d>0$ and $a\geqslant 1$

$$\begin{aligned} \big\langle U^{l,a}\psi_0^a,\psi_0^a\big\rangle &= \big\langle \Gamma^{l,a}U^{l,a}\psi_0^a,\Gamma^{l,a}\psi_0^a\big\rangle \\ &= 4a^{-1}\int_{\mathbb{R}^2_{+\star}} U(u+v+d)\sin^2\big(\pi ul^{-1}\big)\sin^2\big(\pi v(al)^{-1}\big)\,dudv \\ &= 4a^{-3}l^{-4}\pi^2\int_{\mathbb{R}^2_{+\star}} U(u+v+d)u^2v^2dudv + O(a^{-2}l^{-5}) \quad \text{using Taylor expansions} \\ &\leqslant Ca^{-3}l^{-4} \quad \text{since } U \text{ is compactly supported.} \end{aligned} \tag{3.3.6}$$

We recover orders in $l>0$ which are similar to those given in Proposition 3.1.1 and Proposition 3.2.1. We would like to change these inequalities into approximations.

3.3.2 Perturbation theory for the ground state and ground state energy

Following from above discussion, we apply the decomposition $\mathrm{Span}(\psi_0^{(*)})\oplus\mathrm{Span}(\psi_0^{(*)})^\perp$ to each system. Using $\Pi_0^{(*)} = |\psi_0^{(*)}\rangle\langle\psi_0^{(*)}|$ the orthogonal projection on $\mathrm{Span}(\psi_0^{(*)})$, $\Pi_\perp^{(*)}$ the orthogonal projection on $\mathrm{Span}(\psi_0^{(*)})^\perp$ and $h_\perp^{l,(*)} = h^{l,(*)}_{|\mathrm{Span}(\psi_0^a)^\perp}$, the operator becames, for $e>0$,

$$h^{l,(*)} - e = \begin{pmatrix} e_0^{(*)} + \Pi_0^{(*)}U^{l,(*)}\Pi_0^{(*)} - e & \Pi_0^{(*)}U^{l,(*)}\Pi_\perp^{(*)} \\ \Pi_\perp^{(*)}U^{l,(*)}\Pi_0 & \Pi_\perp^{(*)}(h_\perp^{l,(*)}-e)\Pi_\perp^{(*)} \end{pmatrix} \tag{3.3.7}$$

Since $h_\perp^{l,(*)} - e_0^{l,(*)}$ is invertible, the operator $h_\perp^{l,(*)} - e_0^{l,(*)}$ admits a Schur complement $C^{l,(*)}$ given by

$$C^{l,(*)} = e_0^{(*)} + \Pi_0^{(*)} U^{l,(*)} \Pi_0^{(*)} - e_0^{l,(*)} - \Pi_0^{(*)} U^{l,(*)} \Pi_\perp^{(*)} \left(h_\perp^{l,(*)} - e_0^{l,(*)} \right)^{-1} \Pi_\perp^{(*)} U^{l,(*)} \Pi_0^{(*)}. \tag{3.3.8}$$

So, $e_0^{l,(*)}$ is the ground state energy iff $C^{l,(*)} = 0$, meaning that

$$e_0^{l,(*)} = e_0^{(*)} + \left\langle \psi_0^{(*)}, U^{l,(*)} - U^{l,(*)} \Pi_\perp^{(*)} \left(h_\perp^{l,(*)} - e_0^{l,(*)} \right)^{-1} \Pi_\perp^{(*)} U^{l,(*)} \psi_0^{(*)} \right\rangle \tag{3.3.9}$$

The proofs of Proposition 3.1.1 and Proposition 3.2.1 focus on the estimation of this second term.

On the other hand, using (3.3.7), we have

$$\psi_\perp^{l,(*)} = -\Pi_\perp^{(*)} \left(h_\perp^{l,(*)} - e_0^{l,(*)} \right)^{-1} \Pi_\perp^{(*)} U^{l,(*)} \psi_0^{(*)} \tag{3.3.10}$$

The proofs of Proposition 3.1.2 and Proposition 3.2.2 focus on the estimation of the norm of the right-hand side.

We derive the resolvant

$$R_\perp^{l,(*)}(z) = \Pi_\perp^{(*)} \left(h_\perp^{l,(*)} - e_0^{l,(*)} \right)^{-1} \Pi_\perp^{(*)} \tag{3.3.11}$$

that appears in both (3.3.9) and (3.3.10), using Krein's formula,

$$R_\perp^{l,(*)}(z) = \sqrt{R_\perp^{0,(*)}(z)} \left(1 + \sqrt{R_\perp^{0,(*)}(z)} U^{l,(*)} \sqrt{R_\perp^{0,(*)}(z)} \right)^{-1} \sqrt{R_\perp^{0,(*)}(z)}. \tag{3.3.12}$$

Applying the following algebraic tool

$$a(1 + ab^2a)^{-1}ab = a^2b(1 + ba^2b)^{-1} \tag{3.3.13}$$

for $a = \sqrt{R_\perp^{0,(*)}(z)}$, $b = \sqrt{U^{l,(*)}}$ and the notation

$$T^{l,(*)} = \sqrt{U^{l,(*)}} \sqrt{R_\perp^{0,(*)}((e_0^{l,(*)})}, \tag{3.3.14}$$

the equation (3.3.9) becomes

$$\begin{aligned} e_0^{l,(*)} &= e_0^{(*)} + \left\langle \sqrt{U^{l,(*)}} \psi_0^{(*)}, \left(I - T^{l,(*)} \left(T^{l,(*)} \right)^\star \right) \left(1 + T^{l,(*)} \left(T^{l,(*)} \right)^\star \right)^{-1} \right) \sqrt{U^{l,(*)}} \psi_0^{(*)} \right\rangle \\ &= e_0^{(*)} + \left\langle \sqrt{U^{l,(*)}} \psi_0^{(*)}, \left(1 + T^{l,(*)} \left(T^{l,(*)} \right)^\star \right)^{-1} \sqrt{U^{l,(*)}} \psi_0^{(*)} \right\rangle \end{aligned} \tag{3.3.15}$$

and the equation (3.3.10) becomes

$$\psi_\perp^{l,(*)} = -R_\perp^{0,(*)}(e_0^{l,(*)}) \sqrt{U^{l,(*)}} \left(1 + T^{l,(*)} \left(T^{l,(*)} \right)^\star \right)^{-1} \sqrt{U^{l,(*)}} \psi_0^{(*)}. \tag{3.3.16}$$

The following step of both proofs manages to understand the behaviour in $l \to \infty$ of the L^2-function $\sqrt{U^{l,(*)}} \psi_0^{(*)}$ and the bounded operator $\left(1 + T^{l,(*)} \left(T^{l,(*)} \right)^\star \right)^{-1}$. We rescale these quantities through the partial isometry $\Gamma^{l,(*)}$, given by 3.3.2 or 3.3.3, to amplify the impact of the interaction. Yet the methods to compute the rescaled datas differ depending on the cases. We develop the specificities below.

3.3.2.1 Case of a single piece

Proof. (of Proposition 3.1.1, see [KV20])

We refer to Subsection 6.1 of [KV20] for the technical details of this proof. Following from the discussion of the previous section, Klopp and Veniaminov proved that

(i) the sequence of functions $\Big(\sqrt{l}\Gamma^l\sqrt{U^l}\psi^0_{(i)}\Big)_{l>0}$ converges to $\phi\otimes\chi_0$ on $L^2\big(\mathbb{R}\times[0,1]\big)$, where

$$\forall u\in\mathbb{R}, y\in[0,1],\qquad \phi(u)=u\sqrt{U(u)}\qquad\text{and}\qquad \chi_0(y)=\pi\sqrt{2}\Big(\sin(3\pi y)-\sin(\pi y)\Big) \tag{3.3.17}$$

(ii) the sequence of operators $\Big(\Gamma^l\Big(1+T^l\Big(T^l\Big)^\star\Big)^{-1}\Big(\Gamma^l\Big)^\star\Big)_{l>0}$ converges strongly to the bounded operator $\big(1+K\big)^{-1}\otimes I$ on $L^2(\mathbb{R}\times[0,1])$, where K is the kernel operator on $L^2(\mathbb{R})$ given by

$$\forall u,u'\in\mathbb{R}\qquad K(u,v)=\frac{1}{2}\sqrt{U(u)}\Big(|u+u'|-|u-u'|\Big)\sqrt{U(u')} \tag{3.3.18}$$

Thus, they deduce from (3.3.15) that

$$\begin{aligned} e_0^l &= e_0+\Big\langle\sqrt{U^l}\psi_0,\Big(1+T^l\Big(T^l\Big)^\star\Big)^{-1}\sqrt{U^l}\psi_0\Big\rangle\\ &= e_0+\frac{1}{l}\Big\langle\phi\otimes\chi_0,\Big(\big(1+K\big)^{-1}\otimes I\Big)\phi\otimes\chi_0\Big\rangle+o\Big(\frac{1}{l}\Big)\\ &= 5\pi^2+\frac{1}{l}\|\chi_0\|^2\Big\langle\phi,\big(1+K\big)^{-1}\phi\Big\rangle+o\Big(\frac{1}{l}\Big). \end{aligned} \tag{3.3.19}$$

They set $\gamma=\|\chi_0\|^2\Big\langle\phi,\big(1+K\big)^{-1}\phi\Big\rangle$. It concludes the proof of Proposition 3.1.1. □

We now give the details for the result on the ground state itself, which is inspired by the techniques of [KV20].

Proof. (of Proposition 3.1.2)

From the discussion of the previous section, we apply (3.3.16).

$$\Big\|\psi^l_\perp\Big\|^2=\Big\|R^0_\perp\big(e^l_0\big)\sqrt{U^l}\Big(1+T^l\Big(T^l\Big)^\star\Big)^{-1}\sqrt{U^l}\psi_0\Big\|^2 \tag{3.3.20}$$

where we denote $T^l=\sqrt{U^l}\sqrt{R^0_\perp(e^l_0)}$ and $R^0_\perp(z)$ is the resolvant of the restriction of the free operator to $\mathrm{Span}(\psi_0)^\perp$. Using the basis of the eigenfunctions $(\psi_{p,q})_{p>q\geqslant 1}$, we expand

$$\begin{aligned} \Big\|\psi^l_\perp\Big\|^2 &= \sum_{\substack{p>q\\(p,q)\neq(2,1)}}\Big|\Big\langle\psi_{p,q},R^0_\perp(e^l_0)\sqrt{U^l}\Big(1+T^l\Big(T^l\Big)^\star\Big)^{-1}\sqrt{U^l}\psi_0\Big\rangle\Big|^2\\ &= \sum_{\substack{p>q\\(p,q)\neq(2,1)}}\frac{1}{e_{p,q}-e^l_0}\Big|\Big\langle\psi_{p,q},(T^l)^\star\big(1+T^l(T^l)^\star\big)^{-1}\sqrt{U^l}\psi_0\Big\rangle\Big|^2. \end{aligned} \tag{3.3.21}$$

With the notations (3.3.17) and (3.3.18) (or Subsection 6.1 of [KV20]), one can notice that

$$\begin{aligned} \Big\|T^{l\star}\big(1+T^lT^{l\star}\big)^{-1}\sqrt{U^l}\psi_0\Big\|^2 &= \Big\langle\sqrt{U^l}\psi_0,\big(1+T^lT^{l\star}\big)^{-1}T^lT^{l\star}\big(1+T^lT^{l\star}\big)^{-1}\sqrt{U^l}\psi_0\Big\rangle\\ &= \frac{1}{l}\big\langle\phi\otimes\chi_0,\big[K(1+K)^{-2}\otimes I\big]\,\phi\otimes\chi_0\big\rangle+o(l^{-1})\\ &\leqslant\frac{C}{l}. \end{aligned} \tag{3.3.22}$$

Since the left-hand term of (3.3.22) is an upper bound of the right-hand side of (3.3.21), this last inequality implies that

$$\|\psi_\perp^l\|^2 \leqslant Cl^{-1}. \tag{3.3.23}$$

Yet we get an upper bound that is not of the expected order (see Proposition 3.1.2). The factor l^{-1} that is missing will come form the deformation of the basis $(\psi_{p,q})_{p>q}$ through the partial isometry given Γ^l given by (3.3.2).

We define for $p > q \geqslant 1$

$$\kappa_{p,q}^l = \left\langle \Gamma^l\sqrt{U^l}\psi_{p,q}, \left(1+\Gamma^l T^l T^{l\star}\Gamma^{l\star}\right)^{-1}\Gamma^l\sqrt{U^l}\psi_0\right\rangle. \tag{3.3.24}$$

so that Equation (3.3.21) becomes

$$\left\|\psi_\perp^l\right\|^2 = \sum_{\substack{p>q\geqslant 1\\(p,q)\neq(2,1)}} \frac{\left|\kappa_{p,q}^l\right|^2}{(e_{p,q}-e_0^l)^2}. \tag{3.3.25}$$

First, we prove that the sequence $\left(l\kappa_{p,q}^l\right)_{l>0}$ admits a limit. For $p > q \geqslant 1$, we define the functions

$$\chi_{p,q} : \begin{cases} [0,1] & \to \quad \mathbb{R} \\ x & \mapsto \frac{\pi}{\sqrt{2}}\Big((p+q)\sin\big(\pi(p-q)x\big) - (p-q)\sin\big(\pi(p+q)x\big)\Big) \end{cases}. \tag{3.3.26}$$

Remark that $\chi_0 = \chi_{2,1}$.

Lemma 3.3.1. *For any $p \geqslant q \geqslant 1$, the sequence $\left(l\kappa_{p,q}^l\right)_{l>0}$ converges to*

$$\left\langle \phi\otimes\chi_{p,q}\,, \left[\left(1+K\right)^{-1}\otimes I\right]\phi\otimes\chi_0\right\rangle = \left\langle \phi\,, \left(1+K\right)^{-1}\phi\right\rangle\langle\chi_{p,q},\chi_0\rangle. \tag{3.3.27}$$

with the notations (3.3.24), (3.3.17), (3.3.26) and (3.3.18).

Proof. (of Lemma 3.3.1) For $(u,y)\in\Omega^l$, we use the formulas for sine and cosine to write

$$\left(\Gamma^l\sqrt{U^l}\psi_{p,q}\right)(u,y) = \sqrt{l}\sqrt{U(u)}\,\psi_{p,q}\left(y+\frac{u}{l},y\right) = f_{p,q}^{l,+}(u,y) + f_{p,q}^{l,-}(u,y) + g_{p,q}^{l,+}(u,y) + g_{p,q}^{l,-}(u,y) \tag{3.3.28}$$

where

$$\begin{aligned}
f_{p,q}^{l,+}(u,y) &= \sqrt{2l}\sqrt{U(u)}\cos\left(\pi\frac{p-q}{2}\frac{u}{l}\right)\sin\left(\pi\frac{p+q}{2}\frac{u}{l}\right)\sin\Big(\pi(p-q)y\Big),\\
f_{p,q}^{l,-}(u,y) &= -\sqrt{2l}\sqrt{U(u)}\cos\left(\pi\frac{p+q}{2}\frac{u}{l}\right)\sin\left(\pi\frac{p-q}{2}\frac{u}{l}\right)\sin\Big(\pi(p+q)y\Big),\\
g_{p,q}^{l,+}(u,y) &= \sqrt{2l}\sqrt{U(u)}\sin\left(\pi\frac{p+q}{2}\frac{u}{l}\right)\sin\left(\pi\frac{p-q}{2}\frac{u}{l}\right)\cos\Big(\pi(p+q)y\Big),\\
g_{p,q}^{l,-}(u,y) &= -\sqrt{2l}\sqrt{U(u)}\sin\left(\pi\frac{p+q}{2}\frac{u}{l}\right)\sin\left(\pi\frac{p-q}{2}\frac{u}{l}\right)\cos\Big(\pi(p-q)y\Big).
\end{aligned}$$

From this decomposition and some Taylor expansions, one gets that

$$\sqrt{l}\left(\Gamma^l\sqrt{U^l}\psi_{p,q}\right)(u,y) = \phi(u)\chi_{p,q}(y)\mathbf{1}_{\Omega^l}(u,y) + \frac{1}{l}u^2\sqrt{U(u)}\varepsilon_{p,q}(u,y)\mathbf{1}_{\Omega^l}(u,y) \tag{3.3.29}$$

where ϕ is given by (3.3.17), $\chi_{p,q}$ by (3.3.26) and $\varepsilon_{p,q}$ is continuous and bounded on $\mathbb{R}\times[0,1]$. So, the sequence $\left(\sqrt{l}\Gamma^l\sqrt{U^l}\psi_{p,q}\right)_{l>0}$ converges to $\phi\otimes\chi_{p,q}$ on $L^2(\mathbb{R}\times[0,1])$.

On the other hand, we recall that the sequence $\Big(\big(1+\Gamma^l T^l T^{l\star}\Gamma^{l\star}\big)^{-1}\Big)_{l>0}$ is uniformly bounded by 1 and it converges to $\big(1+K\big)^{-1}\otimes I$ strongly (see subsection 6.1 of [KV20]).

Thus, the sequence $\big(l\kappa^l_{p,q}\big)_{l>0}$ converges to

$$\Big\langle \phi\otimes\chi_{p,q}\,,\big[\big(1+K\big)^{-1}\otimes I\big]\phi\otimes\chi_0\Big\rangle = \Big\langle \phi\,,\big(1+K\big)^{-1}\phi\Big\rangle\langle\chi_{p,q},\chi_0\rangle. \tag{3.3.30}$$

It concludes the proof of Lemma 3.3.1. □

The next lemma shows that one can approximate the quantity

$$l^2\left\|\psi^l_\perp\right\|^2 = \sum_{\substack{p>q\geqslant 1\\(p,q)\neq(2,1)}} \frac{\left|l\kappa^l_{p,q}\right|^2}{(e_{p,q}-e^l_0)^2}. \tag{3.3.31}$$

by a finite sum.

Lemma 3.3.2. *For any $\varepsilon>0$, there exists $r_\varepsilon\geqslant 1$ such that*

$$C^l_\varepsilon := \sum_{\substack{p>q\geqslant r_\varepsilon\\(p,q)\neq(2,1)}} \frac{\left|l\kappa^l_{p,q}\right|^2}{(e_{p,q}-e^l_0)^2} \tag{3.3.32}$$

is uniformly bounded by ε.

Proof. (of Lemma 3.3.2) First we transform (3.3.31) by applying the change of variables ($j=q, k=p-q$), decomposing of the function $\Gamma^l\sqrt{U^l}\psi_{p,q}$ into (3.3.28) from the proof of Lemma 3.3.1, and setting $s^l=\big(1+\Gamma^l T^l T^{l\star}\Gamma^{l\star}\big)^{-1}\Gamma^l\sqrt{U^l}\psi_{2,1}$, so that

$$l^2\left\|R^l_\perp(e^l_0)U^l\Pi_0\right\|^2 = \sum_{\substack{a,b\in\{f,g\}\\ \bullet,\circ\in\{+,-\}}}\sum_{j,k\geqslant 1}\frac{l\big\langle a^{l,\bullet}_{j+k,j},s^l\big\rangle\overline{l\big\langle b^{l,\circ}_{j+k,k},s^l\big\rangle}}{\Big(\frac{\pi^2}{2}\big((2j+k)^2+k^2\big)-e^l_0\Big)^2}. \tag{3.3.33}$$

We deal separately with each of the right-hand terms of (3.3.33). However we will develop the general idea during the estimation of the first term (that is $a=f$, $b=g$, $\bullet=+$, $\circ=-$). Let $\varepsilon>0$ and $i_\varepsilon\geqslant 1$ such that

$$\sum_{i\geqslant i_\varepsilon} i^{-\frac{3}{2}}\leqslant\varepsilon. \tag{3.3.34}$$

(i) In this part, we consider the quadric term in

$$l\big\langle f^{l,+}_{j+k,j},s^l\big\rangle = l\int_{\Omega^l}\sqrt{2l}\sqrt{U(u)}\cos\Big(\pi\frac{k}{2}\frac{u}{l}\Big)\sin\Big(\pi\frac{2j+k}{2}\frac{u}{l}\Big)\sin\Big(\pi ky\Big)s^l(u,y)\,dudy. \tag{3.3.35}$$

Here, the idea is to interpret the integral in the second variable as $k-th$ coefficient of the L^2-function $h^l(u,.)$ in the basis $(\sqrt{2}\sin(i\pi.))_{i\in\mathbb{N}}$. So, we write the following bound.

$$\big|l\big\langle f^{l,+}_{j+k,j},s^l\big\rangle\big|\leqslant\pi\sqrt{l}\min\Big(\frac{2j+k}{2},l\Big)\int_{\mathbb{R}}|u|\sqrt{U(u)}\,\big|\alpha^l_k(u)\big|\,du \tag{3.3.36}$$

where

$$\alpha^l_k(u) := \sqrt{2}\int_0^1\sin\big(\pi ky\big)s^l(u,y)dy. \tag{3.3.37}$$

But the sequence $(\sqrt{l}s^{\,l})_{l>0}$ is uniformly bounded in $L^2(\mathbb{R}\times[0,1])$. By Subsection 6.1 of [KV20],

$$\begin{aligned}\|s^{\,l}\|^2 &= \left\langle \sqrt{U^l}\psi_0, \left(1+T^lT^{l\star}\right)^{-2}\sqrt{U^l}\psi_0\right\rangle\\ &= \frac{1}{l}\left\langle \phi\otimes\chi_0, \left[(1+K)^{-2}\otimes I\right]\phi\otimes\chi_0\right\rangle + o(l^{-1})\\ &\leqslant \frac{C}{l}.\end{aligned}$$

So,

$$\int_{\mathbb{R}}\sum_{k\in\mathbb{N}}\left|\alpha_k^l(u)\right|^2du = \int_{\mathbb{R}}\int_0^1\left|s^{\,l}(u,y)\right|^2dydu \leqslant \frac{C}{l}. \tag{3.3.38}$$

Combining (3.3.36) and (3.3.38), we compute

(3.3.39)

$$\begin{aligned}\sum_{j,k\geqslant i_\varepsilon}\frac{\left|l\left\langle f^{l,+}_{j+k,j}, s^{\,l}\right\rangle\right|^2}{\left(\frac{\pi^2}{2}\left((2j+k)^2+k^2\right)-e_0^l\right)^2} &\leqslant l\sum_{j,k\geqslant i_\varepsilon}\frac{\min\left(\frac{2j+k}{2},\,l\right)^2}{\left(\frac{\pi^2}{2}\left((2j+k)^2+k^2\right)-e_0^l\right)^2}\left(\int_{\mathbb{R}}|u|\sqrt{U(u)}\left|\alpha_k^l(u)\right|du\right)^2\\ &\leqslant Cl\sum_{j,k\geqslant i_\varepsilon}\frac{1}{\pi^4(2j)^2}\left(\int_{\mathbb{R}}|u|\sqrt{U(u)}\left|\alpha_k^l(u)\right|du\right)^2\\ &\leqslant C\varepsilon\end{aligned}$$

where the last line uses the Cauchy-Schwarz inequality and the property (3.3.34).

(ii) The case of the quadric term in

$$\left|l\left\langle f^{l,-}_{j+k,j}, s^{\,l}\right\rangle\right| \leqslant \sqrt{l}\min\left(\frac{k}{2},\,l\right)\int_{\mathbb{R}}|u|\sqrt{U(u)}\left|\alpha_{2j+k}^l(u)\right|du \tag{3.3.40}$$

is quite similar to (i). Using (3.3.40) and (3.3.38), we compute

(3.3.41)

$$\begin{aligned}\sum_{j,k\geqslant i_\varepsilon}\frac{\left|l\left\langle f^{l,-}_{j+k,j}, s^{\,l}\right\rangle\right|^2}{\left(\frac{\pi^2}{2}\left((2j+k)^2+k^2\right)-e_0^l\right)^2} &\leqslant l\sum_{j,k\geqslant i_\varepsilon}\frac{\min\left(\frac{k}{2},\,l\right)^2}{\left(\frac{\pi^2}{2}\left((2j+k)^2+k^2\right)-e_0^l\right)^2}\left(\int_{\mathbb{R}}|u|\sqrt{U(u)}\left|\alpha_{2j+k}^l(u)\right|du\right)^2\\ &\leqslant Cl\sum_{j,k\geqslant i_\varepsilon}\frac{1}{\pi^4k^2}\left(\int_{\mathbb{R}}|u|\sqrt{U(u)}\left|\alpha_{2j+k}^l(u)\right|du\right)^2\\ &\leqslant C\varepsilon.\end{aligned}$$

(iii) For the quadric term in

$$l\left\langle g^{l,+}_{j+k,j}, s^{\,l}\right\rangle = l\int_{\Omega^l}\sqrt{2l}\sqrt{U(u)}\sin\left(\pi\frac{2j+k}{2}\frac{u}{l}\right)\sin\left(\pi\frac{k}{2}\frac{u}{l}\right)\cos\Big(\pi(2j+k)y\Big)s^l(u,y)\,dudy \tag{3.3.42}$$

we remark that the integral in the second coordinate is also the $2j+k-th$ coefficient of the L^2- function in the basis $\left(\sqrt{2}\cos(i\pi.)\right)_{i\in\mathbb{N}}$. In the same way of (3.3.38), if we denote

$$\beta_k^l(u) = \sqrt{2}\int_0^1\cos(\pi ky)s^{\,l}(u,y)dy \tag{3.3.43}$$

then

$$\left|l\left\langle g^{l,+}_{j+k,j}, s^{\,l}\right\rangle\right| \leqslant \frac{1}{\sqrt{l}} \min\left(\frac{2j+k}{2},\, l\right) \min\left(\frac{k}{2},\, l\right) \int_{\mathbb{R}} |u|\sqrt{U(u)}\, \left|\beta^l_{2j+k}(u)\right| du \tag{3.3.44}$$

and

$$\sum_{k\in\mathbb{N}} \left|\beta^l_k(u)\right|^2 = \int_0^1 \left|s^{\,l}(u,y)\right|^2 dy \leqslant \frac{C}{l}. \tag{3.3.45}$$

So, applying (3.3.44), Cauchy-Schwarz inequality, (3.3.45) and

$$\min\left(\frac{k}{2},\, l\right)^2 \leqslant \sqrt{2j+k}\, l^{\frac{3}{2}},$$

we derive

(3.3.46)

$$\begin{aligned}
\sum_{j,k\geqslant i_\varepsilon} \frac{\left|l\left\langle g^{l,+}_{j+k,j}, s^{\,l}\right\rangle\right|^2}{\left(\frac{\pi^2}{2}\left((2j+k)^2+k^2\right)-e^l_0\right)^2} &\leqslant \frac{1}{l} \sum_{j,k\geqslant i_\varepsilon} \frac{\min\left(\frac{2j+k}{2},\, l\right)^2 \min\left(\frac{k}{2},\, l\right)^2}{\left(\frac{\pi^2}{2}\left((2j+k)^2+k^2\right)-e^l_0\right)^2} \left(\int_{\mathbb{R}} |u|\sqrt{U(u)}\, \left|\beta^l_k(u)\right| du\right)^2 \\
&\leqslant C\frac{1}{l} \sum_{j,k\geqslant i_\varepsilon} \frac{l^{\frac{3}{2}}}{(2j)^{\frac{3}{2}}} \left(\int_{\mathbb{R}} |u|\sqrt{U(u)}\, \left|\beta^l_k(u)\right| du\right)^2 \\
&\leqslant C\varepsilon\frac{1}{\sqrt{l}}.
\end{aligned}$$

Similarly, we get

$$\sum_{j,k\geqslant i_\varepsilon} \frac{\left|l\left\langle g^{l,-}_{j+k,j}, s^{\,l}\right\rangle\right|^2}{\left(\frac{\pi^2}{2}\left((2j+k)^2+k^2\right)-e^l_0\right)^2} \leqslant C\varepsilon\frac{1}{\sqrt{l}}. \tag{3.3.47}$$

(iv) Compiling (3.3.36), (3.3.40) and (3.3.38), we get

(3.3.48)

$$\begin{aligned}
\sum_{j,k\geqslant i_\varepsilon} \frac{\left|l\left\langle f^{l,+}_{j+k,j}, s^{\,l}\right\rangle l\left\langle f^{l,-}_{j+k,j}, s^{\,l}\right\rangle\right|}{\left(\frac{\pi^2}{2}\left((2j+k)^2+k^2\right)-e^l_0\right)^2} &\leqslant l \sum_{j,k\geqslant i_\varepsilon} \frac{\min\left(\frac{2j+k}{2},\, l\right) \min\left(\frac{k}{2},\, l\right)}{\left(\frac{\pi^2}{2}\left((2j+k)^2+k^2\right)-e^l_0\right)^2} \left(\int_{\mathbb{R}} |u|\sqrt{U(u)}\, \left|\alpha^l_k(u)\right| du\right) \\
&\qquad \times \left(\int_{\mathbb{R}} |u|\sqrt{U(u)}\, \left|\alpha^l_{2j+k}(u)\right| du\right) \\
&\leqslant Cl \sum_{j,k\geqslant j_\varepsilon} \frac{1}{(2j+k)^2} \left(\int_{\mathbb{R}} |u|\sqrt{U(u)}\, \left|\alpha^l_k(u)\right| du\right) \\
&\qquad \times \left(\int_{\mathbb{R}} |u|\sqrt{U(u)}\, \left|\alpha^l_{2j+k}(u)\right| du\right) \\
&\leqslant C\varepsilon.
\end{aligned}$$

(v) Using (3.3.36), (3.3.44), (3.3.38) and (3.3.45), we get

(3.3.49)
$$\begin{aligned}\sum_{j,k\geqslant i_\varepsilon}\frac{\left|l\left\langle f^{l,+}_{j+k,j},s^l\right\rangle l\left\langle g^{l,+}_{j+k,j},s^l\right\rangle\right|}{\left(\frac{\pi^2}{2}\left((2j+k)^2+k^2\right)-e^l_0\right)^2}&\leqslant\sum_{j,k\geqslant i_\varepsilon}\frac{\min\left(\frac{2j+k}{2},\,l\right)^2\min\left(\frac{k}{2},\,l\right)}{\left(\frac{\pi^2}{2}\left((2j+k)^2+k^2\right)-e^l_0\right)^2}\Big(\int_{\mathbb{R}}|u|\sqrt{U(u)}\left|\alpha^l_k(u)\right|du\Big)\\&\qquad\times\Big(\int_{\mathbb{R}}|u|\sqrt{U(u)}\left|\beta^l_{2j+k}(u)\right|du\Big)\\&\leqslant C\sqrt{l}\sum_{j,k\geqslant j_\varepsilon}\frac{1}{(2j+k)^{\frac{3}{2}}}\Big(\int_{\mathbb{R}}|u|\sqrt{U(u)}\left|\alpha^l_k(u)\right|du\Big)\\&\qquad\times\Big(\int_{\mathbb{R}}|u|\sqrt{U(u)}\left|\beta^l_{2j+k}(u)\right|du\Big)\\&\leqslant C\varepsilon\frac{1}{\sqrt{l}}.\end{aligned}$$

One may check that the same inquality holds for any $\bullet,\circ\in\{+,-\}$:

$$\sum_{j,k\geqslant i_\varepsilon}\frac{\left|l\left\langle f^{l,\bullet}_{j+k,j},s^l\right\rangle l\left\langle g^{l,\circ}_{j+k,j},s^l\right\rangle\right|}{\left(\frac{\pi^2}{2}\left((2j+k)^2+k^2\right)-e^l_0\right)^2}\leqslant C\varepsilon\frac{1}{\sqrt{l}}. \tag{3.3.50}$$

(vi) Finally, as for (3.3.46), we use (3.3.45) and

$$\min\left(\frac{k}{2},\,l\right)^2\leqslant\sqrt{2j+k}\,l^{\frac{3}{2}},$$

to compute

(3.3.51)
$$\begin{aligned}\sum_{j,k\geqslant i_\varepsilon}\frac{\left|l\left\langle g^{l,+}_{j+k,j},s^l\right\rangle l\left\langle g^{l,-}_{j+k,j},s^l\right\rangle\right|}{\left(\frac{\pi^2}{2}\left((2j+k)^2+k^2\right)-e^l_0\right)^2}&\leqslant\frac{1}{l}\sum_{j,k\geqslant i_\varepsilon}\frac{\min\left(\frac{2j+k}{2},\,l\right)^2\min\left(\frac{k}{2},\,l\right)^2}{\left(\frac{\pi^2}{2}\left((2j+k)^2+k^2\right)-e^l_0\right)^2}\Big(\int_{\mathbb{R}}|u|\sqrt{U(u)}\left|\beta^l_k(u)\right|du\Big)\\&\qquad\times\Big(\int_{\mathbb{R}}|u|\sqrt{U(u)}\left|\beta^l_{2j+k}(u)\right|du\Big)\\&\leqslant C\sqrt{l}\sum_{j,k\geqslant i_\varepsilon}\frac{1}{(2j)^{\frac{3}{2}}}\Big(\int_{\mathbb{R}}|u|\sqrt{U(u)}\left|\beta^l_k(u)\right|du\Big)\\&\qquad\times\Big(\int_{\mathbb{R}}|u|\sqrt{U(u)}\left|\beta^l_{2j+k}(u)\right|du\Big)\\&\leqslant C\varepsilon\frac{1}{\sqrt{l}}.\end{aligned}$$

Combining (3.3.39), (3.3.41), (3.3.46), (3.3.47), (3.3.48), (3.3.50) and (3.3.51), we get that for any $\varepsilon>0$ there exists $i_\varepsilon\geqslant 1$ such that

$$\sum_{\substack{a,\,b\in\{f,g\}\\ \bullet,\,\circ\in\{+,-\}}}\sum_{j,k\geqslant i_\varepsilon}\frac{l\left\langle a^{l,\bullet}_{j+k,j},s^l\right\rangle\overline{l\left\langle b^{l,\circ}_{j+k,k},s^l\right\rangle}}{\left(\frac{\pi^2}{2}\left((2j+k)^2+k^2\right)-e^l_0\right)^2}. \tag{3.3.52}$$

is uniformly bounded by ε, meaning that the left-hand side of (3.3.33) can be approximated by a finite sum. It concludes the proof of Lemma 3.3.2. □

For $\varepsilon > 0$, let $r_\varepsilon \geqslant 1$ be the integer given by Lemma 3.3.2. Using Lemma 3.3.1 and $|e_0^l - e_{2,1}| \leqslant \frac{C}{l}$, we have

$$l^2 \sum_{\substack{1\leqslant q<p\leqslant r_\varepsilon \\ (p,q)\neq(2,1)}} \frac{\left|\kappa_{p,q}^l\right|^2}{(e_{p,q}-e_0^l)^2} = \Big|\Big\langle \phi\,,\big(1+K\big)^{-1}\phi\Big\rangle\Big|^2 \sum_{\substack{1\leqslant q<p\leqslant r_\varepsilon \\ (p,q)\neq(2,1)}} \frac{\left|\langle \chi_{p,q},\chi_{2,1}\rangle\right|^2}{(e_{p,q}-e_{2,1})^2} + D_\varepsilon^l \tag{3.3.53}$$

with $D_\varepsilon^l \to 0$ as $l \to \infty$ (for $\varepsilon > 0$ fixed). We recall that

$$\chi_{p,q}(x) = \pi\Big((p+q)\sin\big(\pi(p-q)x\big) - (p-q)\sin\big(\pi(p+q)x\big)\Big).$$

Remark that

$$\langle \chi_{p,q},\chi_{2,1}\rangle = \begin{cases} \frac{3\pi^2(2q+1)}{4} & \text{if } p-q=1 \\ -\frac{\pi^2(2q+3)}{4} & \text{if } p-q=3 \\ 0 & \text{else.} \end{cases} \tag{3.3.54}$$

Then,

$$\begin{aligned} l^2 \sum_{\substack{p>q\geqslant 1 \\ (p,q)\neq(2,1)}} \frac{\left|\kappa_{p,q}^l\right|^2}{(e_{p,q}-e_0^l)^2} = \frac{1}{16}\Big|\Big\langle \phi\,,\big(1+K\big)^{-1}\phi\Big\rangle\Big|^2 \Bigg(\sum_{2\leqslant q<r_\varepsilon} \frac{9(2q+1)^2}{\Big((q+1)^2+q^2-5)\Big)^2} \\ + \sum_{1\leqslant q<r_\varepsilon} \frac{(2q+3)^2}{\Big((q+3)^2+q^2-5)\Big)^2}\Bigg) + C_\varepsilon^l + D_\varepsilon^l \end{aligned} \tag{3.3.55}$$

with $|C_\varepsilon^l| \leqslant \varepsilon$ uniformly in l by Lemma 3.3.2. Since $\varepsilon > 0$ can be as small as wanted and by convergence of the sums of the right-hand side of (3.3.55), we conclude that

(3.3.56)

$$\|\psi_\perp^l\|^2 = \frac{1}{l^2}\frac{\Big|\Big\langle \phi\,,\big(1+K\big)^{-1}\phi\Big\rangle\Big|^2}{16}\Bigg(\sum_{q\geqslant 2}\frac{9(2q+1)^2}{\Big((q+1)^2+q^2-5)\Big)^2} + \sum_{q\geqslant 1}\frac{(2q+3)^2}{\Big((q+3)^2+q^2-5)\Big)^2}\Bigg) + o\Big(\frac{1}{l^2}\Big).$$

It concludes the proof of Proposition 3.1.2. □

In fact, the proof of Proposition 3.1.2 also provides a result on the expansion of the ground state ψ_0^l of h^l. We define,

$$\Theta = \frac{\langle \phi\,,\big(1+K\big)^{-1}\phi\rangle}{4}\Bigg(\sum_{q\geqslant 2}\frac{3(2q+1)}{2q^2+2q-5}\,\psi_{q+1,q} - \sum_{q\geqslant 1}\frac{2q+3}{2q^2+6q+4}\,\psi_{q+3,q}\Bigg). \tag{3.3.57}$$

where

(i) ϕ is the L^2-function given by $\phi(u) = u\sqrt{U(u)}$ for $u \in \mathbb{R}$;

(ii) K is the operator on $L^2(\mathbb{R})$ given by the kernel

$$K(u,v) = \frac{1}{2}\Big(\sqrt{U(u)}\big(|u+v|-|u-v|\big)\sqrt{U(v)}\Big);$$

(iii) $(\psi_{p,q})_{p>q}$ are the eigenfunctions of h^0 (see 3.1.5).

Since $(\psi_{p,q})_{p>q}$ is an ONB of $L^2\big([0,1]^2\big)$ and the coefficients in the right hand side of **??** are square summable, Θ is well defined.

Corollary 3.3.1. *Using the notations of Proposition 3.1.2, we decompose the ground state ψ_0^l of h^l on Span$(\psi_0) \oplus$ Span$(\psi_0)^\perp$ as*

$$\psi_0^l = \frac{1}{\sqrt{1+\|\psi_\perp^l\|^2}}\Big(\psi_0 + \psi_\perp^l\Big). \tag{3.3.58}$$

The orthogonal component $\psi_\perp^l$ admits the following L^2-expansion

$$\psi_\perp^l = \frac{1}{l}\Theta\Big(1+o(1)\Big) \tag{3.3.59}$$

with Θ given by (3.3.57).

Proof. Using the notations of the proof of Proposition 3.1.2 and from (3.3.16), we have

$$\psi_\perp^l = -\sum_{\substack{p>q\geqslant 1\\(p,q)\neq(2,1)}} \frac{\kappa_{p,q}^l}{e_{p,q}-e_0^l}\,\psi_{p,q}. \tag{3.3.60}$$

By Lemma 3.3.2, for any $\varepsilon > 0$, there exists $r_\varepsilon \geqslant 1$ so that the vector Y_ε, defined as

$$Y_\varepsilon := \sum_{p>q\geqslant r_\epsilon} \frac{\kappa_{p,q}^l}{e_{p,q}-e_0^l}\,\psi_{p,q}, \tag{3.3.61}$$

admits an L^2-norm that is bounded uniformly by ε. So, by Lemma 3.3.1,

$$\psi_\perp^l = \frac{1}{l}\Big(\Theta_{r_\varepsilon} + Y_\varepsilon + Z_\varepsilon^l\Big) \tag{3.3.62}$$

where

$$\Theta_r = -\frac{\langle \phi\,, \big(1+K\big)^{-1}\phi\rangle}{4}\left(\sum_{q=2}^{r}\frac{3(2q+1)}{2q^2+2q-5}\,\psi_{q+1,q} - \sum_{q=1}^{r}\frac{2q+3}{2q^2+6q+4}\,\psi_{q+3,q}\right), \tag{3.3.63}$$

$$\Theta_r \to_{r\to+\infty} \Theta \quad \text{and} \quad \big\|Z_\varepsilon^l\big\|^2 \to_{l\to+\infty} 0 \tag{3.3.64}$$

Thus,

$$\big\|l\psi_\perp^l - \Theta\big\|^2 \to_{l\to+\infty} 0. \tag{3.3.65}$$

It concludes the proof of Corollary 3.3.1. □

3.3.2.2 Case of two pieces

Proof. (of Proposition 3.2.1) From the discussion of Section 3.3, one would like to express the ground state energy $e_0^{l,a}$ using (3.3.9),

$$e_0^{l,a} = e_0^a + \Big\langle \psi_0^a, U^{l,a} - U^{l,a}R_\perp^{0,a}\big(e_0^{l,a}\big)U^{l,a}\psi_0^a\Big\rangle \tag{3.3.66}$$

where $z \to R_\perp^{l,a}(z)$ is the resolvant of the restriction of the operator $h^{l,a}$ to Span$(\psi_0^a)^\perp$.

We already got the upper bound

$$e_0^{l,a} \leqslant e_0^a + \frac{C}{l^4} \tag{3.3.67}$$

(see 3.3.6). Now, we prove that one can replace $R_\perp^{l,a}(e_0^{l,a})$ with $R_\perp^{l,a}(e_0^a)$ in (3.3.66) for some negligible cost. Set $\delta e = e_0^{l,a} - e_0^a$ and remark that, as $\|R_\perp^{l,a}(e_0^a)\| \leqslant (e_{1,2}^a - e_0^a)^{-1}$,

$$R_\perp^l(e_0^{l,a}) - R_\perp^l(e_0^a) = \sum_{n\geqslant 1} R_\perp^l(e_0^a)^{n+1}(\delta e)^n = O(\delta e) \leqslant Cl^{-4}. \tag{3.3.68}$$

So,

$$\begin{aligned}\left|\langle \psi_0, U^{l,a}\big(R_\perp^{l,a}(e_0^{l,a}) - R_\perp^{l,a}(e_0^a)\big)U^{l,a}\psi_0^a\rangle\right| &\leqslant \left\|\sqrt{U^{l,a}}\Pi_0^a\right\|^2 \left\|\sqrt{U^{l,a}}\big(R_\perp^{l,a}(e_0^{l,a}) - R_\perp^{l,a}(e_0^a)\big)\sqrt{U^{l,a}}\right\| \\ &\leqslant Cl^{-6}.\end{aligned}$$

using (3.3.6), (3.3.68) and $\|U^l\| \leqslant l^2\|U\|_\infty$. It implies that

$$e_0^{l,a} = e_0^a + \left\langle \psi_0^a, U^{l,a} - U^{l,a}R_\perp^{0,a}\big(e_0^a\big)U^{l,a}\psi_0^a\right\rangle + O\Big(l^{-6}\Big) \tag{3.3.69}$$

Thus, we transformed the implicit equation (3.3.66) to an explicit equation plus a reminder which is negligible compared to the expected order $O(l^{-4})$.

Then, as in Section 3.3 (see (3.3.15), we derive

$$e_0^{l,a} = e_0^a + \left\langle \sqrt{U^{l,a}}\psi_0^a, \left(1 + T^{l,a}\Big(T^{l,a}\Big)^\star\right)^{-1}\sqrt{U^{l,a}}\psi_0^a\right\rangle + O(l^{-6}) \tag{3.3.70}$$

with

$$T^{l,a} = \sqrt{U}^{l,a}\sqrt{R_\perp^{l,a}(e_0^a)}. \tag{3.3.71}$$

Similarly to the proof of Proposition 3.1.1, we estimate the limit $l \to \infty$ of the function $\Gamma^{l,a}\sqrt{U^{l,a}}\psi_0^a$ and the bounded operator $\Gamma^{l,a}\left(1 + T^{l,a}\Big(T^{l,a}\Big)^\star\right)^{-1}\Big(\Gamma^{l,a}\Big)^\star$, where $\Gamma^{l,a}$ is the partial isometry given by (3.3.3).

First, one computes

$$\begin{aligned} l^2\Gamma^{l,a}\sqrt{U^{l,a}}\psi_0^a(u,v) &= \frac{2l^2}{\sqrt{a}}\sqrt{U(u+v+d)}\sin\left(\pi\frac{u}{l}\right)\sin\left(\pi\frac{v}{al}\right)\mathbf{1}_{\Omega^{l,a}}(u,v) \\ &= \frac{2\pi^2}{\sqrt{a}}\sqrt{U(u+v+d)}\Big(\frac{uv}{a} + \frac{uv}{l}g_{l,a}(u,v)\Big)\mathbf{1}_{\Omega^{l,a}}(u,v) \end{aligned} \tag{3.3.72}$$

where $g_{l,a}$ is a bounded continuous function. So, by dominated convergence theorem, the sequence $\left(a^{3/2}l^2\Gamma^{l,a}\sqrt{U^{l,a}}\psi_0^a\right)_{l>0}$ admits the following limit in $L^2(\mathbb{R}_{+\star}^2)$ when $l \to +\infty$:

$$\varphi(u,v) = 2\pi^2\sqrt{U(u+v+d)}uv. \tag{3.3.73}$$

Otherwise, we use the eigencouples $\Big((e_{p,q}^a, \psi_{p,q}^a)\Big)_{p,q\geqslant 1}$ (see (3.2.4) and (3.2.5)) to compute the kernel of $\Gamma^{l,a}T^{l,a}\Big(\Gamma^{l,a}T^{l,a}\Big)^\star$. If $f \in \mathcal{C}_c^\infty(\mathbb{R}_{+\star}^2)$ then, for l large enough, $f \in \mathcal{C}_c^\infty(\Omega^{l,a})$ and

(3.3.74)

$$\Gamma^{l,a}T^{l,a}\Big(\Gamma^{l,a}T^{l,a}\Big)^\star f(u,v) = \frac{4}{al^2}\sum_{(p,q)\neq(1,1)}\frac{\mathbf{1}_{\Omega^l}(u,v)}{\pi^2(\frac{p^2}{l^2} + \frac{q^2}{(al)^2}) - \frac{e_0^a}{l^2}}\sqrt{U(u+v+d)}\sin\left(\pi u\frac{p}{l}\right)\sin\left(\pi v\frac{q}{al}\right)G_f\left(\frac{p}{l}, \frac{q}{al}\right)$$

where $G_f(\xi,\eta) = \int_{\mathbb{R}_{+\star}^2}\sqrt{U(u'+v'+d)}f(u',v')\sin(\pi u'\xi)\sin(\pi v'\eta)\,du'dv'$. By Riemann's summation, the limit for $l \to +\infty$ of (3.3.74) is

$$L(u,v) = \frac{4}{\pi^2}\mathbf{1}_{\mathbb{R}_{+\star}^2}(u,v)\sqrt{U(u+v+d)}\int_{\mathbb{R}_{+\star}^2}\frac{1}{x^2+y^2}\sin(\pi ux)\sin(\pi vy)G_f(x,y)\,dxdy. \tag{3.3.75}$$

Using $g(s,t) = \sqrt{U(s+t+d)}f(s,t)$ and its Fourier transform $\mathcal{F}_g(\xi,\eta) = \int_{\mathbb{R}^2_{+\star}} g(s,t)e^{is\xi+it\eta}\,dsdt$, one computes

$$G_f(x,y) = \frac{1}{4}\Big(- \mathcal{F}_g(x,y) + \mathcal{F}_g(x,-y) - \mathcal{F}_g(-x,-y) + \mathcal{F}_g(-x,y)\Big).$$

Then, (3.3.75) becomes

$$L(u,v) = -\frac{1}{\pi^2}\mathbf{1}_{\mathbb{R}^2_{+\star}}(u,v)\sqrt{U(u+v+d)}\iint\limits_{\mathbb{R}^2}\frac{1}{x^2+y^2}\sin(\pi ux)\sin(\pi vy)\mathcal{F}_g(x,y)\,dxdy. \tag{3.3.76}$$

Lemma 3.3.3. *Define S on $\mathcal{C}_c^\infty(\mathbb{R}^2_{+\star})$ such that $Sf = L$, given by (3.3.76). Then, the operator S is well-defined and is extended to a bounded operator on $L^2(\mathbb{R}^2_{+\star})$.*

Proof. (of Lemma 3.3.3) We first prove that, for $(u,v) \in \mathbb{R}^2_{+\star}$, $L(u,v)$, given by (3.3.76), is well-defined. We consider the singularities separately.

1. For $(\alpha,\beta) \in \mathbb{R}^2$, we have

$$\frac{1}{x^2+y^2}\sin(\alpha x)\sin(\beta y)\mathcal{F}_g(x,y) \sim_{(0,0)} \frac{xy}{x^2+y^2}\alpha\beta\mathcal{F}_g(0,0)$$

   It gives the integrability in $(0,0)$.

2. By the Paley Wiener theorem, as $f \in \mathcal{C}_c^\infty(\mathbb{R}^2_{+\star})$, $\mathcal{F}_g$ is an entire function and $|\mathcal{F}_g(x,y)| \leqslant \frac{C_j}{(1+|y|)^j}$ for $j \geqslant 1$. Then,

$$\left|\frac{1}{x^2+y^2}\sin(\alpha x)\sin(\beta y)\mathcal{F}_g(x,y)\right| \leqslant \frac{C_j}{(x^2+y^2)(1+|y|)^j}.$$

   It gives the integrability at $\pm\infty$.

So, S is well-defined on $\mathcal{C}_c^\infty(\mathbb{R}^2_{+\star})$

Take $h \in \mathcal{C}_c^\infty(\mathbb{R}^2)$. We compute

$$\begin{aligned}\int_{\mathbb{R}^2_+} U(u+v+d)\Big|\int_{\mathbb{R}^2}\frac{\sin(xu)\sin(yv)}{x^2+y^2}h(x,y)dxdy\Big|^2 dudv &\leqslant \|h\|_{L^2}^2\int U(u+v+d)\frac{\sin^2(xu)\sin^2(yv)}{(x^2+y^2)^2}\\ &\leqslant \|h\|_{L^2}^2\Bigg(\int U(u+v+d)\int_{[1,\infty)^2}\frac{1}{(x^2+y^2)^2}\\ &\quad + \int U(u+v+d)u^2v^2\int_{[0,1]^2}\frac{x^2y^2}{(x^2+y^2)^2}\Bigg)\\ &\leqslant C\|h\|_{L^2}^2.\end{aligned}$$

Since the Fourier transform is unitary and U is bounded, we get that S admits an extension on $L^2(\mathbb{R}^2_{+\star})$.

It concludes the proof of Lemma 3.3.3. □

Thus, by Lemma 3.3.3, the sequence $\left(\Gamma^{l,a}T^{l,a}\left(\Gamma^{l,a}T^{l,a}\right)^\star\right)_{l>0}$ converges strongly to some operator S. So does $\left(1+\Gamma^{l,a}T^{l,a}\left(\Gamma^{l,a}T^{l,a}\right)^\star\right)^{-1}$ to $\left(1+S\right)^{-1}$. The limit only depends on U and d.

For any positive self-adjoint operator A on a Hilbert space $\mathcal{H}$, we know $\|(1+A)^{-1}\|_{\mathcal{B}(\mathcal{H})} \leqslant 1$. Then, combining it with (3.3.73) and (3.3.76), for l large, it yields

$$\left\langle a^{3/2}l^2\Gamma^{l,a}\sqrt{U^{l,a}}\psi_0^a, \left(1+\Gamma^{l,a}T^{l,a}\left(\Gamma^{l,a}T^{l,a}\right)^\star\right)^{-1}a^{3/2}l^2\Gamma^{l,a}\sqrt{U^{l,a}}\psi_0^a\right\rangle = \left\langle\varphi, \left(1+S\right)^{-1}\varphi\right\rangle + o(1). \tag{3.3.77}$$

Thus, we deduce from (3.3.70) that

$$e_0^{l,a} = e_0^a + \Big\langle \sqrt{U^{l,a}}\psi_0^a, \Big(1 + T^{l,a}\Big(T^{l,a}\Big)^\star\Big)^{-1}\sqrt{U^{l,a}}\psi_0^a\Big\rangle + O(l^{-6}) \tag{3.3.78}$$
$$= 2\pi^2 + \frac{1}{a^3 l^4}\Big\langle \varphi, \Big(1+S\Big)^{-1}\varphi\Big\rangle + o\Big(\frac{1}{l^4}\Big).$$

We set $\tau(d) = \Big\langle \varphi, \Big(1+S\Big)^{-1}\varphi\Big\rangle_{L^2(\mathbb{R}^2_{+\star})}$. It concludes the proof of Proposition 3.2.1.

□

Proof. (of Proposition 3.2.2) We apply (3.3.10) from Section 3.3.

$$\Big\|\psi_\perp^{l,a}\Big\| = \Big\|R_\perp^{l,a}(e_0^{l,a})U^{l,a}\psi_0^a\Big\| \tag{3.3.79}$$

where $z \to R_\perp^{l,a}(z)$ is the resolvant of the restriction of the operator $h^{l,a}$ to $\mathrm{Span}(\psi_0^a)^\perp$ As for the proof of Proposition 3.2.1, we prove that one can replace $R_\perp^{l,a}(e_0^{l,a})$ with $R_\perp^{l,a}(e_0^a)$ for some negligible cost. Since $\|R_\perp^{l,a}(e_0)\| \leqslant (e_{1,2}^a - e_{1,1}^a)^{-1}$ we have

$$R_\perp^{l,a}(e_0^{l,a}) - R_\perp^{l,a}(e_0^a) = \sum_{n\geqslant 1} R_\perp^{l,a}(e_0^a)^{n+1}(e_0^{l,a} - e_0)^n = O(e_0^{l,a} - e_0^a) \leqslant Ca^{-3}l^{-4}. \tag{3.3.80}$$

So, using (3.3.6),

$$\Big\|\big(R_\perp^{l,a}(e_0^{l,a}) - R_\perp^{l,a}(e_0^a)\big)U^{l,a}\Pi_0^a\Big\| \leqslant Cl^{-4}|\langle\psi_0^a, U^{l,a}\psi_0^a\rangle| \leqslant Ca^{-6}l^{-8}. \tag{3.3.81}$$

It yields that

$$\Big\|\psi_\perp^{l,a}\Big\| = \Big\|R_\perp^{l,a}(e_0^a)U^{l,a}\psi_0^a\Big\| + O(a^{-6}l^{-8}) \tag{3.3.82}$$

Thus, we transformed the implicit formula (3.3.79) to an explicit formula plus a reminder which is negligible compared to the expected order $O(a^{-2}l^{-3})$.

Then, as for (3.3.16), we derive

$$\Big\|\psi_\perp^{l,a}\Big\|^2 = \Big\|R_\perp^{0,a)}(e_0^a)\sqrt{U^{l,a}}\Big(1 + T^{l,a}\Big(T^{l,a}\Big)^\star\Big)^{-1}\sqrt{U^{l,a}}\psi_0^a\Big\|^2 + o(a^{-4}l^{-6}) \tag{3.3.83}$$

where

$$T^{l,a} = \sqrt{U^{l,a}}\sqrt{R_\perp^{0,a}(e_0^a)}. \tag{3.3.84}$$

One computes, with the basis of the eigenvalues $\Big(\psi_{p,q}^a\Big)_{p,q\geqslant 1}$,

$$\|\psi_\perp^{l,a}\|^2 = \sum_{(p,q)\neq(1,1)}\Big|\Big\langle\psi_{p,q}^a, R_\perp^{0,a)}(e_0^a)\sqrt{U^{l,a}}\Big(1 + T^{l,a}\Big(T^{l,a}\Big)^\star\Big)^{-1}\sqrt{U^{l,a}}\psi_0^a\Big\rangle\Big|^2 \tag{3.3.85}$$
$$= \sum_{(p,q)\neq(1,1)}\frac{1}{e_{p,q}^a - e_0^a}\Big|\Big\langle\psi_{p,q}^a, T^{l,a\star}\big(1 + T^{l,a}T^{l,a\star}\big)^{-1}\sqrt{U^{l,a}}\psi_0^a\Big\rangle\Big|^2.$$

Using the notation(3.3.73) and Lemma 3.3.3, one can notice that

(3.3.86)
$$\begin{aligned}\left\|T^{l,a\star}\big(1+T^{l,a}T^{l,a\star}\big)^{-1}\sqrt{U^{l,a}}\psi_0^a\right\|^2 &= \left\langle\sqrt{U^{l,a}}\psi_0^a,\big(1+T^{l,a}T^{l,a\star}\big)^{-1}T^{l,a}T^{l,a\star}\big(1+T^{l,a}T^{l,a\star}\big)^{-1}\sqrt{U^{l,a}}\psi_0^a\right\rangle\\ &= \frac{1}{a^3l^4}\langle\varphi,S(1+S)^{-2}\varphi\rangle+o(l^{-4})\\ &\leqslant \frac{C}{a^3l^4}.\end{aligned}$$

Since the left-hand term of (3.3.86) is an upper bound of the right-hand side of (3.3.85), this last inequality implies that

$$\|\psi_\perp^{l,a}\|^2\leqslant Ca^{-3}l^{-4} \tag{3.3.87}$$

Yet we get an upper bound that is not of the expected order (see Proposition 3.1.2). The factor $a^{-1}l^{-2}$ that is missing will come form the deformation of the basis $(\psi_{p,q}^a)_{p,q}$ through the partial isometry given $\Gamma^{l,a}$ given by (3.3.3).

For $p,q\geqslant 1$, set

$$\kappa_{p,q}^{l,a}=\big\langle\Gamma^{l,a}\sqrt{U^{l,a}}\psi_{p,q}^a,(1+\Gamma^{l,a}T^{l,a}T^{l,a\star}\Gamma^{l,a\star})^{-1}\Gamma^{l,a}\sqrt{U^{l,a}}\psi_{1,1}^a\big\rangle \tag{3.3.88}$$

so that Equation (3.3.85) becomes

$$\|\psi_\perp^{l,a}\|^2=\sum_{(p,q)\neq(1,1)}\frac{\big|\kappa_{p,q}^{l,a}\big|^2}{(e_{p,q}^a-e_0^a)^2}. \tag{3.3.89}$$

From the proof of Proposition 3.2.1, we know that the sequence $\left(\left(1+\Gamma^{l,a}T^{l,a}\Big(\Gamma^{l,a}T^{l,a}\Big)^\star\right)^{-1}\right)_{l>0}$ converges strongly to $\Big(1+S\Big)^{-1}$ for some bounded operator S on $L^2(\mathbb{R}_{+\star}^2)$ (see Lemma 3.3.3). We also have that the sequence $\big(a^{\frac32}l^2\Gamma^{l,a}\sqrt{U^{l,a}}\psi_{1,1}^a\big)_{l>0}$ converges in $L^2(\mathbb{R}_{+\star}^2)$ to the function

$$\varphi:(u,v)\mapsto 2\pi^2\sqrt{U(u+v+d)}uv. \tag{3.3.90}$$

Thus,

$$a^{\frac32}l^2\left(1+\Gamma^{l,a}T^{l,a}\Big(\Gamma^{l,a}T^{l,a}\Big)^\star\right)^{-1}\Gamma^{l,a}\sqrt{U^{l,a}}\psi_{1,1}^a\longrightarrow_{l\to\infty}(1+S)^{-1}\varphi. \tag{3.3.91}$$

Let $g^{l,a}$ be the bounded continuous function that satisfies

$$2\sin(\frac{\pi x}{l})\sin(\frac{\pi y}{al})=2\pi^2xy\Big(\frac{1}{al^2}+\frac{xy}{l^4}g^{l,a}(x,y)\Big). \tag{3.3.92}$$

We also set

$$M^{l,a}(x,y)=\int_{\mathbb{R}_{+\star}^2}\left(1+\Gamma^{l,a}T^{l,a}\Big(\Gamma^{l,a}T^{l,a}\Big)^\star\right)^{-1}(x,y,z,t)zt\,dzdt \tag{3.3.93}$$

$$\text{and}\qquad R^{l,a}(x,y)=\int_{\mathbb{R}_{+\star}^2}\left(1+\Gamma^{l,a}T^{l,a}\Big(\Gamma^{l,a}T^{l,a}\Big)^\star\right)^{-1}(x,y,z,t)z^2t^2g^{l,a}(z,t)\,dzdt.$$

Remark that the functions $M^{l,a}$ and $R^{l,a}$ are uniformly bounded on $L^2(\mathbb{R}^2_{+\star})$. Given that,

$$\forall (p,q) \qquad \Gamma^{l,a}\sqrt{U^{l,a}}\psi^a_{p,q}(x,y) = \frac{2}{\sqrt{a}}\sqrt{U(x+y+d)}\sin(\frac{\pi p x}{l})\sin(\frac{\pi q y}{al}), \tag{3.3.94}$$

we expand

$$\begin{aligned}\left|\kappa^{l,a}_{p,q}\right|^2 =& \frac{1}{a^4 l^4}\Big(\int_{\mathbb{R}^2_{+\star}} \sqrt{U(x+y+d)}\sin(\frac{\pi p x}{l})\sin(\frac{\pi q y}{al})M^{l,a}(x,y)\,dxdy\Big)^2 \\ &+ \frac{2}{a^{5/2}l^6}\Big(\int_{\mathbb{R}^2_{+\star}} \sqrt{U(x+y+d)}\sin(\frac{\pi p x}{l})\sin(\frac{\pi q y}{al})M^{l,a}(x,y)\,dxdy\Big)\times \\ &\Big(\int_{\mathbb{R}^2_{+\star}} \sqrt{U(x+y+d)}\sin(\frac{\pi p x}{l})\sin(\frac{\pi q y}{al})R^{l,a}(x,y)\,dxdy\Big) \\ &+ \frac{1}{al^8}\Big(\int_{\mathbb{R}^2_{+\star}} \sqrt{U(x+y+d)}\sin(\frac{\pi p x}{l})\sin(\frac{\pi q y}{al})R^{l,a}(x,y)\,dxdy\Big)^2.\end{aligned} \tag{3.3.95}$$

Then, we consider the following kernel on $L^2(\mathbb{R}^2_{+\star}) \times L^2(\mathbb{R}^2_{+\star})$

$$Z^{l,a}(x,y,\xi,\sigma) = \frac{l^2}{a}\sum_{(p,q)\neq(1,1)} \frac{\sqrt{U(x+y+d)}\sin(\frac{\pi p x}{l})\sin(\frac{\pi q y}{al})\sin(\frac{\pi p \xi}{l})\sin(\frac{\pi q \sigma}{al})\sqrt{U(\xi+\sigma+d)}}{\left(\pi^2(p^2+a^{-2}q^2)-e^a_0\right)^2}. \tag{3.3.96}$$

Indeed, using (3.3.95), Equation (3.3.89) becomes

$$\left\|R^{l,a}_{\perp}(e^a_0)U^{l,a}\psi^a_0\right\|^2 = \frac{1}{a^3 l^6}\big\langle M^{l,a}, Z^{l,a}M^{l,a}\big\rangle + \frac{2}{a^{\frac{3}{2}}l^8}\big\langle M^{l,a}, Z^{l,a}R^{l,a}\big\rangle + \frac{1}{l^{10}}\big\langle R^{l,a}, Z^{l,a}R^{l,a}\big\rangle. \tag{3.3.97}$$

We define

$$Z(x,y,\xi,\sigma) = \int_{\mathbb{R}^2_{+\star}} \frac{\sqrt{U(x+y+d)}\sin(\pi r x)\sin(\pi s y)\sin(\pi r \xi)\sin(\pi s \sigma)\sqrt{U(\xi+\sigma+d)}}{\pi^4(r^2+s^2)^2}\,drds. \tag{3.3.98}$$

Let f, g be two functions of $L^2(\mathbb{R}^2_{+\star})$. Applying Cauchy-Schwartz, we compute

$$\Big|\Big\langle f, \big(Z^{l,a}-Z^a\big)g\Big\rangle\Big| \leqslant \|f\|_{L^2}\|g\|_{L^2}\Big(\int \left|Z^{l,a}-Z\right|^2\Big)^{\frac{1}{2}} \tag{3.3.99}$$

Because of our assumption on U, the domain of integration of the last integral is some compact $\mathcal{C}$ that does not depend on l. But by Riemann's summations, for any $(x,y,\xi,\sigma) \in \mathcal{C}$, the limit of the sequence $\big(Z^{l,a}(x,y,\xi,\sigma)\big)_{l>0}$ is exactly $Z(x,y,\xi,\sigma)$. So,

$$\|Z^{l,a}-Z\|_{L^2(\mathbb{R}^2_{+\star})\to L^2(\mathbb{R}^2_{+\star})} \longrightarrow_{l\to\infty} 0 \tag{3.3.100}$$

We deal separately with the three terms of right-hand side of (3.3.97).

(i) For the first scalar product $\big\langle M^{l,a}, Z^{l,a}M^{l,a}\big\rangle$, as $(M^{l,a})_{l>0}$ is uniformly bounded on $L^2(\mathbb{R}^2_{+\star})$ and it converges to $(1+S)^{-1}\varphi$, we get

$$\big\langle M^{l,a}, Z^{l,a}M^{l,a}\big\rangle \longrightarrow_{l\to\infty} \big\langle (1+S)^{-1}\varphi, Z(1+S)^{-1}\varphi\big\rangle \tag{3.3.101}$$

(ii) For the second scalar product, since $(M^{l,a})_{l>0}$ and $(R^{l,a})_{l>0}$ are uniformly bounded on $L^2(\mathbb{R}^2_{+\star})$, we have that

$$\big\langle M^{l,a}, Z^{l,a}R^{l,a}\big\rangle \leqslant C \tag{3.3.102}$$

for some constant $C > 0$.

(iii) Similarly one gets that

$$\left\langle R^{l,a}, Z^{l,a} R^{l,a} \right\rangle \leqslant C \tag{3.3.103}$$

for some constant $C > 0$.

Combining (3.3.97), (3.3.101), (3.3.102) and (3.3.103), we conclude that

$$l^6 \|\psi_\perp^{l,a}\|^2 = \frac{1}{a^3} \langle \varphi, (1+S)^{-1} Z (1+S)^{-1} \varphi \rangle + o(1) \tag{3.3.104}$$

where Z is given by (3.3.98). Thus,

(3.3.105)

$$\|\psi_\perp^{l,a}\|^2 = \frac{1}{a^3 l^6} \int_{\mathbb{R}^2_{+\star}} \Big(\int_{\mathbb{R}^2_{+\star}} \frac{\sqrt{U(x+y+d)} \sin(\pi r x) \sin(\pi s y)}{r^2 + s^2} \big((1+S)^{-1} \varphi \big)(x,y)\, dx dy \Big)^2 dr ds + o\Big(\frac{1}{l^6}\Big)$$

It concludes the proof of Proposition 3.2.2. □

Remark 3.3.1. *Unlike the case of a single piece (see Corollary 3.3.1), the proof of Proposition 3.2.2 does not provide an approximation of the ground state when $l \to \infty$. With $\kappa_{p,q}^{l,a}$ given by (3.3.88), we express*

$$\psi_\perp^{l,a} = - \sum_{(p,q) \neq (1,1)} \frac{\kappa_{p,q}^{l,a}}{e_{p,q}^a - e_0^a} \psi_{p,q}^a.$$

But, for any $(p,q) \neq (1,1)$, the sequence $\big(a^2 l^3 \kappa_{p,q}^{l,a}\big)_{l>0}$ converges to 0.

Bibliography

[AEN+05] Michael Aizenman, Alexander Elgart, Serguei Naboko, Jeffrey H. Schenker, and Gunter Stolz. Moment analysis for localization in random Schrödinger operators. *Inventiones mathematicae*, 163(2):343–413, oct 2005.

[AF01] Robert Alicki and Mark Fannes. *Quantum dynamical systems*. Oxford University Press, jul 2001.

[AL18] Fabien Alet and Nicolas Laflorencie. Many-body localization: An introduction and selected topics. *C R Phys*, 19(6):498–525, sep-oct 2018.

[And58] Philip W. Anderson. Absence of diffusion in certain random lattices. *Physical Review*, 109(5):1492–1505, mar 1958.

[ARS15] Houssam Abdul-Rahman and Günter Stolz. A uniform area law for the entanglement of eigenstates in the disordered XY chain. *Journal of Mathematical Physics*, 56(12):121901, dec 2015.

[AW09] Michael Aizenman and Simone Warzel. Localization bounds for multiparticle systems. *Communications in Mathematical Physics*, 290(3):903–934, apr 2009.

[BA04] John Stewart Bell and Alain Aspect. *Speakable and Unspeakable in Quantum Mechanics*. Cambridge University Press, jun 2004.

[Bou04] Jean Bourgain. On localization for lattice Schrödinger operators involving Bernoulli variables. In *Lecture Notes in Mathematics*, pages 77–99. Springer Berlin Heidelberg, 2004.

[CdMS11] Victor Chulaevsky, Anne Boutet de Monvel, and Yuri Suhov. Dynamical localization for a multi-particle model with an alloy-type external random potential. *Nonlinearity*, 24(5):1451–1472, mar 2011.

[CS08] Victor Chulaevsky and Yuri Suhov. Wegner bounds for a two-particle tight binding model. *Communications in Mathematical Physics*, 283(2):479–489, may 2008.

[CS09a] Victor Chulaevsky and Yuri Suhov. Eigenfunctions in a two-particle Anderson tight binding model. *Communications in Mathematical Physics*, 289(2):701–723, jan 2009.

[CS09b] Victor Chulaevsky and Yuri Suhov. Multi-particle Anderson localisation: Induction on the number of particles. *Mathematical Physics, Analysis and Geometry*, 12(2):117–139, jan 2009.

[Deu91] Joshua M. Deutsch. Quantum statistical mechanics in a closed system. *Physical Review A*, 43(4):2046–2049, feb 1991.

[ECP10] Jens Eisert, Marcus Cramer, and Martin B. Plenio. Area laws for the entanglement entropy. *Reviews of Modern Physics*, 82(1):277–306, feb 2010.

[EPS16] Alexander Elgart, Leonid Pastur, and Mariya Shcherbina. Large block properties of the entanglement entropy of free disordered fermions. *Journal of Statistical Physics*, 166(3-4):1092–1127, nov 2016.

[FA80] L. Fleishman and Philip W. Anderson. Interactions and the anderson transition. *Physical Review B*, 21(6):2366–2377, mar 1980.

[FW15] Michael Fauser and Simone Warzel. Multiparticle localization for disordered systems on continuous space via the fractional moment method. *Reviews in Mathematical Physics*, 27(04):1550010, may 2015.

[GHK05] François Germinet, Peter Hislop, and Abel Klein. On localization for the Schrödinger operator with a Poisson random potential. *Comptes Rendus Mathematique*, 341(8):525–528, oct 2005.

[GK06] Dimitri Gioev and Israel Klich. Entanglement entropy of fermions in any dimension and the Widom conjecture. *Physical Review Letters*, 96(10):100503, mar 2006.

[GK13] François Germinet and Abel Klein. A comprehensive proof of localization for continuous Anderson models with singular random potentials. *Journal of the European Mathematical Society*, 15(1):53–143, 2013.

[GMS83] L. N. Grenkova, Stanislas A. Molchanov, and Yuriy N. Sudarev. On the basic states of one-dimensional disordered structures. *Communications in Mathematical Physics*, 90:101–123, 1983.

[HLS10] Robert Helling, Hajo Leschke, and Wolfgang Spitzer. A special case of a conjecture by Widom with implications to fermionic entanglement entropy. *International Mathematics Research Notices*, jun 2010.

[Kat12] Tosio Kato. *Perturbation Theory for Linear Operators*. Springer Berlin Heidelberg, December 2012.

[Kir08] Werner Kirsch. An invitation to random Schrödinger operators. *Panoramas et Synthèses*, 25:1–119, 2008.

[KP21] Joachim Kerner and Maximilian Pechmann. On the effect of repulsive pair interactions on Bose–Einstein condensation in the Luttinger–Sy model. *Proceedings of the American Mathematical Society*, 149(8):3499–3513, may 2021.

[KPS19] Joachim Kerner, Maximilian Pechmann, and Wolfgang Spitzer. Bose-Einstein condensation in the Luttinger–Sy model with contact interaction. *Annales Henri Poincaré*, 20(6):2101–2134, feb 2019.

[KV20] Frédéric Klopp and Nikolaj A. Veniaminov. Interacting electrons in a random medium: a simple one-dimensional model. *Frontiers in analysis and probability*, pages 91–242, nov 2020.

[Laf16] Nicolas Laflorencie. Quantum entanglement in condensed matter systems. *Physics Reports*, 646:1–59, aug 2016.

[LS73a] Joaquin M. Luttinger and H. K. Sy. Bose-Einstein condensation in a one-dimensional model with random impurities. *Physical Review A*, 7(2):712–720, feb 1973.

[LS73b] Joaquin M. Luttinger and H. K. Sy. Low-lying energy spectrum of a one-dimensional disordered system. *Physical Review A*, 7(2):701–712, feb 1973.

[LSS14] Hajo Leschke, Alexander V. Sobolev, and Wolfgang Spitzer. Scaling of Rényi entanglement entropies of the free Fermi-gas ground state: A rigorous proof. *Physical Review Letters*, 112(16):160403, apr 2014.

[LZ07] Olivier Lenoble and Valentin A. Zagrebnov. Bose–Einstein condensation in the Luttinger–Sy model. *Markov Processes And Related Fields*, 13(2):441–468, 2007.

[NH15] Rahul Nandkishore and David A. Huse. Many-body localization and thermalization in quantum statistical mechanics. *Annual Review of Condensed Matter Physics*, 6(1):15–38, mar 2015.

[Ogn22] Vadim Ognov. Thermodynamic limit of the pieces' model. *Annales Henri Poincaré*, sep 2022.

[RS78] Michael Reed and Barry Simon. *Methods of Modern Mathematical Physics, IV: Analysis of Operators*. Elsevier Science & Techn., may 1978.

[RS81] Michael Reed and Barry Simon. *Methods of Modern Mathematical Physics, I: Functional Analysis, Second Edition*. Elsevier Science Publishing Co Inc, feb 1981.

[SHB+15] Michael Schreiber, Sean S. Hodgman, Pranjal Bordia, Henrik P. Lüschen, Mark H. Fischer, Ronen Vosk, Ehud Altman, Ulrich Schneider, and Immanuel Bloch. Observation of many-body localization of interacting fermions in a quasirandom optical lattice. *Science*, 349(6250):842–845, aug 2015.

[Sim05] Barry Simon. *Trace Ideals and Their Applications, Second Edition*. American Mathematical Society, 2005.

[Sre94] Mark Srednicki. Chaos and quantum thermalization. *Physical Review E*, 50(2):888–901, aug 1994.

[Sto11] Günter Stolz. An introduction to the mathematics of Anderson localization. *Entropy and the quantum II. Contemp. Math*, 552:71–108, 2011.

[Tes14] Gerald Teschl. *Mathematical methods in quantum mechanics : with applications to Schrödinger operators*. American Mathematical Society, 2014.

[Ven12] Nikolaj A. Veniaminov. The Existence of the thermodynamic limit for the system of interacting quantum particles in random media. *Annales Henri Poincaré*, 14(1):63–94, may 2012.

[yCHZ+16] Jae yoon Choi, Sebastian Hild, Johannes Zeiher, Peter Schauß, Antonio Rubio-Abadal, Tarik Yefsah, Vedika Khemani, David A. Huse, Immanuel Bloch, and Christian Gross. Exploring the many-body localization transition in two dimensions. *Science*, 352(6293):1547–1552, jun 2016.