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THÈSE DE DOCTORAT DE

L'UNIVERSITÉ DE RENNES 1

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*Mathématiques et Sciences et Technologies
de l'Information et de la Communication*
Spécialité : *Mathématiques et leurs Interactions*

Par

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Étude schématique du schéma des arcs

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RÉSUMÉ EN FRANÇAIS

0.1 Soit V une variété définie sur un corps k que l'on va supposer par simplicité de caractéristique zéro. On définit le *schéma des arcs* $\mathcal{L}_\infty(V)$ sur V comme étant l'unique k -schéma qui, pour toute k -algèbre R , réalise l'isomorphisme suivant:

$$\mathrm{Hom}_{\mathbf{Sch}_k}(\mathrm{Spec}(R[[t]]), V) \cong \mathrm{Hom}_{\mathbf{Sch}_k}(\mathrm{Spec}(R), \mathcal{L}_\infty(V)).$$

Un point de $\mathcal{L}_\infty(V)$ est appelé un arc sur V , il est un germe formel de courbe sur V . De manière analogue, étant donné un entier naturel $m \in \mathbb{N}$ on définit le *schéma des m -jets* comme étant l'unique k -schéma qui, pour toute k -algèbre R , réalise l'isomorphisme suivant:

$$\mathrm{Hom}_{\mathbf{Sch}_k}(\mathrm{Spec}(R[[t]]/\langle t^{m+1} \rangle), V) \cong \mathrm{Hom}_{\mathbf{Sch}_k}(\mathrm{Spec}(R), \mathcal{L}_m(V)).$$

Via les isomorphismes précédents, les morphismes canoniques $R[[t]] \rightarrow R[[t]]/\langle t^{m+1} \rangle$ et $R[[t]]/\langle t^{m+1} \rangle \rightarrow R[[t]]/\langle t^{n+1} \rangle$ (pour $m \geq n \geq 0$) induisent respectivement des morphismes, dits de troncation, $\theta_{m,V}^\infty : \mathcal{L}_\infty(V) \rightarrow \mathcal{L}_m(V)$ et $\theta_{n,V}^m : \mathcal{L}_m(V) \rightarrow \mathcal{L}_n(V)$ qui vérifient la propriété de transitivité $\theta_{n,V}^m \circ \theta_{m,V}^\infty = \theta_{n,V}^\infty$. On a donc un système projectif $(\mathcal{L}_m(V), \theta_{n,V}^m)$ de k -schémas dont la limite est le schéma des arcs $\mathcal{L}_\infty(V)$.

On appelle *point base* ou *centre* d'un arc $\gamma \in \mathcal{L}_\infty(V)$ le point $\gamma(0) := \theta_{0,V}^\infty(\gamma)$ de V . Soit $\kappa(\gamma)$ le corps résiduel de l'arc γ , celui-ci correspond alors à un morphisme $\mathrm{Spec}(\kappa(\gamma)[[t]]) \rightarrow V$. L'image du point générique de $\mathrm{Spec}(\kappa(\gamma)[[t]])$ via ce morphisme est appelée le *point générique* de l'arc γ .

0.2 Dans ce mémoire nous étudions des propriétés schématiques du schéma des arcs. Dans ce sens, nous présentons des progrès dans deux directions. D'une part, nous nous intéressons à la nilpotence dans le schéma des arcs et ainsi, dans le chapitre 3, nous déterminons l'idéal de fonctions nilpotentes de $\mathcal{L}_\infty(\mathcal{C})$ qui habitent dans $\mathcal{L}_1(\mathcal{C})$, où \mathcal{C} est une courbe plane affine définie par un polynôme réduit homogène ou homogène à poids. De l'autre part, nous étudions les voisinages formels du schéma des arcs d'une variété V autour de certains points et dans le chapitre 5 nous présentons, dans le cadre des variétés toriques normales, un résultat qui compare le voisinage formel de certains points de $\mathcal{L}_\infty(V)$ associés à des valuations divisorielles sur V et celui des points rationnels suffisamment génériques dans l'adhérence des premiers. Dans la suite de ce résumé nous allons préciser les résultats obtenus dans ces deux directions.

Les chapitres 2 et 4 ont, quant à eux, une vocation à introduire le schéma des arcs et certaines de ses propriétés qui seront utiles dans la suite dans le cas du premier et, pour le deuxième, à motiver nos résultats du chapitre 5 en présentant de manière détaillée le résultat principal sur la structure locale des points rationnels du schéma des arcs : le théorème de structure de Drinfeld-Grinberg-Kazhdan.

1. SUR L'ESPACE TANGENT D'UNE COURBE DÉFINIE PAR UN POLYNÔME HOMOGENÈME À POIDS

Le lien entre la géométrie de la variété V et la structure schématique de son schéma des arcs a été peu étudié, les résultats principaux ont été obtenus par M. Mustaș et J. Sebag, voir l'Introduction (chapitre 1) pour la mise en contexte de notre étude que nous décrivons à continuation.

1.1 Soit V une k -variété affine et $\Omega_{V/k}^1$ son module de différentielles de Kähler associé. On définit l'espace tangent $T_{V/k}$ de V par $\text{Spec}(\text{Sym}(\Omega_{V/k}^1))$. La définition du schéma des 1-jets et la propriété universelle de l'algèbre symétrique impliquent que $T_{V/k}$ est isomorphe comme k -schéma à $\mathcal{L}_1(V)$ (voir le lemme 3.2.2). En particulier, les morphismes de troncation induisent via cet isomorphisme des morphismes de k -schémas $\theta_{0,V}^1 : T_{V/k} \rightarrow V$ et $\theta_{1,V}^\infty : \mathcal{L}_\infty(V) \rightarrow T_{V/k}$. Nous avons la décomposition suivante :

$$(T_{V/k})_{\text{red}} = \overline{(\theta_{0,V}^1)^{-1}(V_{\text{Reg}})} \cup (\theta_{0,V}^1)^{-1}(V_{\text{Sing}}).$$

Supposons que V est irréductible. Nous démontrons que

$$\mathcal{G}(V) := \overline{\theta_{1,V}^\infty(\mathcal{L}_\infty(V))} = \overline{(\theta_{0,V}^1)^{-1}(V_{\text{Reg}})}, \quad (1.1)$$

qui est une composante irréductible de $T_{V/k}$ que nous appelons sa *composante générale*. Nous notons par $\mathcal{N}_1(V)$ l'idéal de $\mathcal{O}(T_{V/k})$ définissant $\mathcal{G}(V)$ comme un sous-schéma fermé de $T_{V/k}$.

On considère maintenant une courbe plane \mathcal{C} définie par un polynôme $f \in B_0 := k[x_0, y_0]$. On note B_1 (respectivement B_∞) l'anneau de polynômes $B_1 := k[x_0, y_0, x_1, y_1]$ (resp. $B_\infty := k[x_i, y_i; i \in \mathbb{N}]$). La dérivation $\Delta : B_\infty \rightarrow B_\infty$ munit l'anneau B_∞ d'une structure d'anneau différentiel. On note par $[f]$ l'idéal différentiel $[f] := \langle \Delta^s(f); s \in \mathbb{N} \rangle$ et par $\{f\}$ le radical de $[f]$. Alors on peut montrer (voir les lemmes 2.3.5 et 2.3.8) que $\mathcal{L}_\infty(\mathcal{C}) \cong B_\infty/[f]$ et $\mathcal{L}_1(\mathcal{C}) \cong \text{Spec}(B_1/\langle f, \Delta(f) \rangle)$, ce qui nous permet de profiter des outils d'algèbre différentielle lorsque l'on étudie les schémas des arcs et des jets.

Par conséquence, $\mathcal{N}_1(\mathcal{C})$ s'identifie avec un idéal de $B_1/\langle f, \Delta(f) \rangle$; on va aussi noter par $\mathcal{N}_1(\mathcal{C})$ le seul idéal de l'anneau B_1 contenant $\langle f, \Delta(f) \rangle$ dont l'extension dans $B_1/\langle f, \Delta(f) \rangle$ est $\mathcal{N}_1(\mathcal{C})$. C'est cette dernière notation que nous allons privilégier dorénavant. À l'aide de (1.1) on démontre que $\mathcal{N}_1(\mathcal{C})$ coïncide avec l'idéal $\{f\} \cap B_1$ de B_1 ,

autrement dit, l'idéal des fonctions nilpotentes de $\mathcal{L}_\infty(\mathcal{C})$ qui habitent dans $\mathcal{L}_1(\mathcal{C})$. De plus, si le polynôme f est irréductible et réduit, alors $\mathcal{N}_1(\mathcal{C}) = (\langle f, \Delta(f) \rangle : \partial(f)^\infty)$ pour toute dérivée partielle non nulle $\partial(f)$ de f .

1.2 Notre étude aboutit à l'obtention d'une base de Groebner de l'idéal $\mathcal{N}_1(\mathcal{C})$ de B_1 dans les cas où f est un polynôme réduit homogène ou homogène à poids, ce qui détermine complètement l'idéal. Nous présentons ici d'abord le cas homogène.

Soit $n \in \mathbb{N}^*$ un entier positif. On considère une famille d'éléments $(\gamma_i)_{1 \leq i \leq n}$ de k^* deux à deux différents. Pour chaque entier $i \in \{1, \dots, n\}$, on pose $f_i = y_0 - \gamma_i x_0 \in B_0$ et $f = x_0^\varepsilon y_0^{\varepsilon'} \prod_{i=1}^n f_i$ avec $\varepsilon, \varepsilon' \in \{0, 1\}$. On définit, pour chaque entier $i \in \{0, \dots, n\}$, les polynômes suivants dans l'anneau B_1 :

$$\delta_i := \left(\prod_{\ell=1}^i \Delta(f_\ell) \right) \times \left(\prod_{\ell=i+1}^n f_\ell \right).$$

On considère $\mathcal{C} = \text{Spec}(B_0/\langle f \rangle)$ la courbe plane associée. Le résultat suivant est notre énoncé principal pour des polynômes homogènes; il correspond au théorème 3.5.5.

Théorème 1.3. *Soit \mathcal{C} la courbe plane réduite déterminée par le polynôme homogène f défini précédemment. La famille de polynômes de B_1*

$$\mathfrak{B} := \{y_1 x_0 - y_0 x_1, x_{h_1}^\varepsilon y_{h_2}^{\varepsilon'} \delta_i(f), i \in \{0, \dots, n\}, h_1, h_2 \in \{0, 1\}\}$$

est une base de Groebner de l'idéal $\mathcal{N}_1(\mathcal{C})$ pour l'ordre monomial $y_1 >_{\text{lex}} y_0 >_{\text{lex}} x_1 >_{\text{lex}} x_0$ dans B_1 .

1.4 Nous allons traiter maintenant le cas homogène à poids. Un polynôme $f \in B_0$ est dit *homogène à poids* de poids (w_1, w_2, w) si dans l'anneau $B_0[t]$ la formule $f(t^{w_1} x_0, t^{w_2} y_0) = t^w f(x_0, y_0)$ est vérifiée. Lorsque le corps k est algébriquement clos, si f est homogène à poids alors il existe un entier $n \geq 1$, une paire d'entiers premiers entre eux (r, s) avec $r \geq s \geq 2$ et des constantes non nulles $\lambda_1, \dots, \lambda_n \in k^*$ deux à deux différentes de telle façon que l'on peut décomposer f sous la forme $f = x_0^\varepsilon y_0^{\varepsilon'} \prod_{i=1}^n (x_0^r - \lambda_i y_0^s)$ avec $\varepsilon, \varepsilon' \in \{0, 1\}$, à k -automorphisme de B_0 près.

Pour chaque entier $i \in \{1, \dots, n\}$, on pose $\tilde{D}_{-1} := \tilde{D}_{\lambda_i, -1} := s y_1 x_0 - r y_0 x_1$ et $\tilde{D}_{\lambda_i, j_i} := \lambda_i s^{j_i} y_0^{s-j_i} y_1^{j_i} - r^{j_i} x_0^{r-j_i} x_1^{j_i}$, où $j_i \in \{0, \dots, s\}$. On pose

$$\tilde{D}_{j_1, \dots, j_n} := \tilde{D}_{\lambda_1, j_1} \cdots \tilde{D}_{\lambda_n, j_n},$$

où $j_i \in \{-1, \dots, s\}$ pour chaque $i \in \{1, \dots, n\}$. Voici notre résultat principal pour des polynômes homogènes à poids, qui correspond au théorème 3.6.6.

Théorème 1.5. *Soit \mathcal{C} la courbe plane réduite déterminée par le polynôme homogène à poids $f = x_0^\varepsilon y_0^{\varepsilon'} \prod_{i=1}^n (x_0^r - \lambda_i y_0^s)$ défini précédemment. La famille de polynômes de B_1*

$$\mathfrak{B} = \{\tilde{D}_{-1}, x_{h_1}^\varepsilon y_{h_2}^{\varepsilon'} \tilde{D}_{j_1, \dots, j_n}, j_i \in \{-1, \dots, s\}, i \in \{1, \dots, n\}, h_1, h_2 \in \{0, 1\}\}$$

est une base de Groebner de l'idéal $\mathcal{N}_1(\mathcal{C})$ pour l'ordre monomial $y_1 >_{\text{lex}} y_0 >_{\text{lex}} x_1 >_{\text{lex}} x_0$ dans B_1 .

Précisons que, même si ces deux théorèmes ne semblent donner une description complète de $\mathcal{N}_1(\mathcal{C})$ (dans les cas homogène et homogène à poids) que pour des corps algébriquement clos, il est possible d'obtenir un système de générateurs de cet idéal pour un corps arbitraire de caractéristique zéro à partir de celui obtenu dans sa clôture algébrique. Des implications de ces résultats en termes des opérateurs différentiels sont décrites dans la section 3.7.

2. UN THÉORÈME DE COMPARAISON ENTRE DES VOISINAGES FORMELS

2.1 Les voisinages formels du schéma des arcs ont été principalement étudiés pour deux classes de points. Dans le cas des points rationnels, le théorème suivant de Drinfeld, Grinberg et Kazhdan détermine la structure de leurs voisinages formels.

Théorème 2.2 (Drinfeld-Grinberg-Kazhdan). *Soit V une k -variété et $v \in V(k)$ un point rationnel de V tel que $\dim_v(V) \geq 1$. Soit $\gamma \in \mathcal{L}_\infty(V)(k)$ un arc centré dans v qui n'appartient pas à $\mathcal{L}_\infty(V_{\text{Sing}})(k)$. Alors il existe un k -schéma S de type fini, un point rationnel $s \in S(k)$ et un isomorphisme de k -schémas formels :*

$$\widehat{\mathcal{L}_\infty(V)}_\gamma \cong \widehat{S}_s \hat{\times}_k \text{Spf}(k[[T_i]_{i \in \mathbb{N}}]).$$

Le k -schéma formel \widehat{S}_s dans l'isomorphisme précédent est appelé un modèle formel fini de la paire $(\mathcal{L}_\infty(V), \gamma)$. Notons qu'il est isomorphe au spectre formel d'une k -algèbre locale essentiellement de type fini. De plus, la preuve du théorème fournit une méthode de calcul effectif d'un modèle formel.

L'autre classe d'arcs dont le voisinage formel dans le schéma des arcs a été étudié sont les points génériques des parties constructibles irréductibles du schéma des arcs, nommés *points constructibles*. Comme on expliquera dans la section 5.1, de Fernex et Docampo ont montré que ces points sont caractérisés par le fait que leur voisinage formel est noethérien.

Notre résultat principal du chapitre 5 établit une comparaison entre le voisinage formel de certains points constructibles d'une variété torique associés à des valuations divisorielles toriques et le voisinage formel des points rationnels suffisamment génériques dans l'adhérence de ces points constructibles. Nous précisons à continuation cette comparaison.

2.3 Soit V une k -variété. Un arc $\gamma \in \mathcal{L}_\infty(V)$ est dit *gras* si le morphisme associé $\text{Spec}(\kappa(\gamma)[[t]]) \rightarrow V$ est dominant. Dans ce cas, l'arc γ définit une valuation discrète ord_γ sur le corps rationnel $k(V)$ de la variété V .

Considérons maintenant une valuation divisorielle ν sur V . L'adhérence dans $\mathcal{L}_\infty(V)$ de l'ensemble des arcs gras $\gamma \in \mathcal{L}_\infty(V)$ dont la valuation associée ord_γ coïncide avec ν est appelée l'*ensemble de Nash* associé à ν et noté par \mathcal{N}_ν . C'est une partie irréductible de $\mathcal{L}_\infty(V)$ dont le point générique est un point constructible.

Le résultat de comparaison obtenu est le suivant, il correspond au théorème 5.5.19.

Théorème 2.4. *Soient k un corps de caractéristique zéro et V une k -variété affine torique normale. On considère une valuation divisorielle torique ν sur V d'ensemble de Nash associé \mathcal{N}_ν . On note η_ν le point générique de \mathcal{N}_ν et κ_ν son corps résiduel.*

Il existe un sous-ensemble ouvert non vide U_ν de \mathcal{N}_ν disjoint avec $\mathcal{L}_\infty(V_{\text{Sing}})$ tel que, pour tout arc rationnel $\alpha \in U_\nu(k)$ il existe un isomorphisme de κ_ν -schémas formels entre $\widehat{\mathcal{L}_\infty(V)_{\eta_\nu}}_{\hat{\otimes}_{\kappa_\nu} \kappa_\nu} \llbracket (T_i)_{i \in \mathbb{N}} \rrbracket$ et $\widehat{\mathcal{L}_\infty(V)_\alpha}_{\hat{\otimes}_k \kappa_\nu}$.

L'énoncé du théorème 5.5.19 est plus précis et fournit une description de $\widehat{\mathcal{L}_\infty(V)_{\eta_\nu}}$ en termes d'un modèle formel fini de $(\mathcal{L}_\infty(V), \alpha)$. On en déduit que $\widehat{\mathcal{L}_\infty(V)_{\eta_\nu}}$ est le spectre formel d'une κ_ν -algèbre locale qui est non seulement noethérienne, mais essentiellement de type fini. De plus, on a une méthode pour la calculer de manière effective, ce qui n'est pas toujours simple pour le voisinage formel d'un point constructible de $\mathcal{L}_\infty(V)$.

Remarquons que notre théorème de comparaison implique que le voisinage formel de tout point rationnel de l'ouvert $U_\nu(k)$ est constant à isomorphisme près. Ceci avait été déjà observé et démontré par Bourqui et Sebag (voir [13]) par d'autres techniques ; notre résultat relie cette invariance directement avec le point générique de l'ensemble de Nash.

INTRODUCTION

1.0.1 Let V be a variety defined over a field k , which we may assume for simplicity to be of characteristic zero. An *arc* on V can be thought of as a formal germ of curve on V , the *arc scheme* $\mathcal{L}_\infty(V)$ of V is a k -scheme that parametrizes those formal germs of curve on V . Let us be slightly more precise and define $\mathcal{L}_\infty(V)$ as the unique k -scheme realizing, for every k -algebra R , the following isomorphism:

$$\mathrm{Hom}_{\mathbf{Sch}_k}(\mathrm{Spec}(R[[t]]), V) \cong \mathrm{Hom}_{\mathbf{Sch}_k}(\mathrm{Spec}(R), \mathcal{L}_\infty(V)).$$

Intimately related with the arc scheme are the jet schemes, which can be defined in an analogous way: for any nonnegative integer $m \in \mathbb{N}$, the m -*jet scheme* $\mathcal{L}_m(V)$ of V is the unique k -scheme realizing, for every k -algebra R , the isomorphism:

$$\mathrm{Hom}_{\mathbf{Sch}_k}(\mathrm{Spec}(R[[t]]/\langle t^{m+1} \rangle), V) \cong \mathrm{Hom}_{\mathbf{Sch}_k}(\mathrm{Spec}(R), \mathcal{L}_m(V)).$$

As a first geometric interpretation, jets may be understood as higher order analogues of tangent vectors, in fact the 1-jet scheme $\mathcal{L}_1(V)$ is canonically isomorphic to the tangent space $T_{V/k}$ of V (see lemma 3.2.2) and $\mathcal{L}_0(V)$ may be identified with V itself. Jets may be seen as truncations of arcs, this comes from the existence of morphisms $\theta_{m,V}^\infty : \mathcal{L}_\infty(V) \rightarrow \mathcal{L}_m(V)$ for every $m \in \mathbb{N}$ and $\theta_{n,V}^m : \mathcal{L}_m(V) \rightarrow \mathcal{L}_n(V)$ for $n \leq m$ satisfying the transitivity property $\theta_{n,V}^m \circ \theta_{m,V}^\infty = \theta_{n,V}^\infty$. Precisely, the arc scheme can be obtained as the limit of the projective system $(\mathcal{L}_m(V), \theta_{n,V}^m)$ in the category of k -schemes,

$$\mathcal{L}_\infty(V) \cong \varprojlim_{m \in \mathbb{N}} \mathcal{L}_m(V).$$

Given an arc $\gamma \in \mathcal{L}_\infty(V)$, its *base point* or *center* $\gamma(0) := \theta_{0,V}^\infty(\gamma)$ is a point of V . The *generic point* of γ is also a point of V defined as the image of the generic point of $\mathrm{Spec}(\kappa(\gamma)[[t]])$ via the morphism $\mathrm{Spec}(\kappa(\gamma)[[t]]) \rightarrow V$ corresponding to the arc γ , where $\kappa(\gamma)$ denotes its residue field. All these notions will be formalized in sections 2.1 and 2.2.

1.0.2 Once we have presented the main objects of our study, let us tell something about their history. In 1968, John F. Nash wrote a paper relating some algebraic, geometric and topological properties of the arc scheme $\mathcal{L}_\infty(V)$ with the geometry of the singularities of the k -variety V . He realized that there is a close connection between the “families of arcs” (nowadays called *Nash components*) centered at the singular locus of V and certain data associated with resolutions of singularities (whose existence had been recently proven by H. Hironaka in 1964): the *essential exceptional divisors*. As we will see below, he formulated this connection in terms of an injective map and the question of whether this map is surjective (at least in dimension two) became known as the *Nash problem*. Although Nash’s paper remained unpublished until 1995 (see [64]), it is at the origin of the study of arcs and arc scheme *per se*. We will give later in the introduction more details about this; let us just advance that a complete answer (in every dimension) of the Nash problem was finally given between 2003 and 2013.

The study of arc scheme has been boosted from the middle of the 1990’s by the development of the theory of motivic integration. The starting point of this theory was a lecture in Orsay by M. Kontsevich in 1995, where he proved that two K -equivalent varieties have the same Hodge numbers. From V. Batyrev’s proof of the fact that two birationally equivalent complex Calabi-Yau varieties have the same Betti numbers (see [2]), crucially based on p -adic integration, Kontsevich observed that one can upgrade p -adic integration to a geometric integration theory, the *motivic integration*. It is defined as a measure theory on the arc scheme and takes values in a ring of virtual motives constructed from the Grothendieck ring of varieties (a ring obtained from the cut and paste property in the category of k -varieties). Motivic integration was then formalized and extended to singular varieties defined over a field of characteristic zero by J. Denef and F. Loeser in [29] and since then it has been developed in several directions and found many applications in diverse fields, including birational geometry and singularity theory (e.g., through the definition of new singularity invariants). This development required a deeper understanding of arc scheme which was also accomplished in those years.

However, in both the preceding approaches related with the study of the Nash problem and the motivic integration, only topological considerations and the reduced structure of arc scheme are involved. Just few things are known about the scheme structure of arc scheme; to the best of our knowledge we can find in the literature results on the local formal structure due to V. Drinfeld ([30]), M. Grinberg and D. Kazhdan ([34]), A. Reguera ([66],[67]), D. Bourqui and J. Sebag (e.g., [10], [13]), H. Mourtada and A. Reguera ([62]) and T. de Fernex and R. Docampo ([26]); and results about nilpotency on arc scheme due to M. Mustață ([63]), J. Sebag ([71], [72], [73]) and K. Kpognon and J. Sebag ([55]). In this monograph we present advances in both directions, in chapter 3 concerning nilpotency and in chapter 5 about the local structure. In the rest of the introduction, we are going to contextualize and describe them.

1.0.3 As we will explain in section 2.3, *differential algebra*, introduced by J. Ritt and E. Kolchin (see [69] and [54]) provides a description of the arc scheme of an affine k -

variety in such a way that, if an affine k -variety V is defined as a closed subscheme of an affine space by an ideal I , the corresponding arc scheme $\mathcal{L}_\infty(V)$ is the affine closed k -subscheme of an appropriate (infinite-dimensional) affine space defined by the *differential ideal* $[I]$ generated by I . This description of the arc scheme in terms of differential algebra is not only computationally advantageous, but also allows to transpose results of differential algebra into geometric properties of arc scheme. As an example, from [54, Ch. IV/§17/Proposition 10] we deduce the following result about the irreducibility of arc scheme.

Theorem 1.0.4 (Kolchin irreducibility theorem). *Let V be a k -variety. Then V is irreducible if and only if $\mathcal{L}_\infty(V)$ is irreducible.*

Unfortunately, the analogous statement for reducedness is not true, the best result that we can have is the fact that if $\mathcal{L}_\infty(V)$ is reduced, then V is reduced (see lemma 2.2.17).

Many examples of reduced varieties with non-reduced arc scheme are present in the literature, see e.g., [71]. In particular our main results in chapter 3, theorems 3.5.5 and 3.6.6, imply that this is the case of singular reduced plane curves defined by homogeneous or weighted-homogeneous polynomials.

It is then interesting to study the relation between the nilpotency in the arc scheme $\mathcal{L}_\infty(V)$ and the geometry of the base variety V . In this sense, we have the following result due to M. Mustața, which can be deduced from the main theorem in [63].

Theorem 1.0.5 (Mustața). *If V is locally a complete intersection (l.c.i) k -variety with rational singularities, then for every $m \in \mathbb{N}$ the m -jet scheme $\mathcal{L}_m(V)$ is reduced. In particular $\mathcal{L}_\infty(V)$ is reduced.*

In the case of plane curves, J. Sebag found in [71] and [72] the following characterization of the reducedness of the arc scheme.

Theorem 1.0.6 (Sebag). *Let V be a l.c.i. curve defined over k . Then V is smooth over k if and only if $\mathcal{L}_\infty(V)$ is reduced.*

Moreover, for V an integral k -variety, in [72] Sebag related the torsion elements of the module of Kähler differentials $\Omega_{V/k}^1$ of V with nilpotent functions in $\mathcal{L}_\infty(V)$, deducing the following result.

Theorem 1.0.7 (Sebag). *Let V be an integral k -variety. If the k -scheme $\mathcal{L}_\infty(V)$ is reduced, then the \mathcal{O}_V -module $\Omega_{V/k}^1$ is torsion free.*

1.0.8 The preceding results either give geometric conditions on V which assure that $\mathcal{L}_\infty(V)$ is reduced or, conversely, provide properties on V which hold when $\mathcal{L}_\infty(V)$ is reduced. However, this is not usually the case in general and in fact, even for plane curves, theorem 1.0.6 shows that the presence of singularities implies the existence of

non-trivial nilpotent elements of the arc scheme. Hence, rather than wondering how the fact of $\mathcal{L}_\infty(V)$ being reduced is related with the geometry of V , we may ask the following question: what implications does the existence of non-trivial nilpotent functions of $\mathcal{L}_\infty(V)$ have in terms of V ?

Such implications have been found by J. Sebag ([73]) and M. Gros, L. Narváez Macarro and J. Sebag ([35]) in the field of differential operators for nilpotent functions of $\mathcal{L}_\infty(V)$ which live in $\mathcal{L}_1(V)$. Let us consider the polynomial rings $B_0 := k[x_0, y_0]$ and $B_1 := k[x_0, y_0, x_1, y_1]$ and the k -derivation $\Delta : B_0 \rightarrow B_1$ such that $\Delta(x_0) = x_1$ and $\Delta(y_0) = y_1$. We consider on B_1 the degree \deg_1 such that $\deg_1(x_1) = \deg_1(y_1) = 1$ and $\deg_1(x_0) = \deg_1(y_0) = 0$. If \mathcal{C} is the affine plane curve defined by a polynomial $f \in B_0$ then its 1-arc scheme $\mathcal{L}_1(\mathcal{C})$ is isomorphic to $\text{Spec}(B_1/\langle f, \Delta(f) \rangle)$, as we will see in lemma 2.3.8. The set of nilpotent functions of $\mathcal{L}_\infty(\mathcal{C})$ which live in $\mathcal{L}_1(\mathcal{C})$ defines an ideal $\mathcal{N}_1(\mathcal{C})$ of B_1 containing $\langle f, \Delta(f) \rangle$.

The following result corresponds to [35, Corollary 4.10]. It is a generalization of [73, Corollary 1.9] where the curve was asked to be integral.

Theorem 1.0.9 (Gros-Narváez Macarro-Sebag). *Let $f \in B_0$ be a reduced polynomial and \mathcal{C} the associated affine plane curve. Then, for every homogeneous element $P \in \mathcal{N}_1(\mathcal{C})$ of degree $\deg_1(f) = d$ there exists a differential operator of the plane D such that $D(f^d) \in \langle f \rangle$ and its principal symbol can be identified with P . Conversely, for every differential operator of the plane D of order d such that $D(f^d) \in \langle f \rangle$, its principal symbol can be identified with an element of $\mathcal{N}_1(\mathcal{C})$.*

The precise statement of this result can be found in theorem 3.7.6, where the differential operators related to an element of $\mathcal{N}_1(\mathcal{C})$ are moreover identified in terms of the V -filtration of the ring of differential operators of the plane along \mathcal{C} .

Furthermore, Gros, Narváez Macarro and Sebag also obtained in [35, Theorem 5.3] the following relation between $\mathcal{N}_1(\mathcal{C})$ and Bernstein-Sato operators, which will be detailed in theorem 3.7.12.

Theorem 1.0.10 (Gros-Narváez Macarro-Sebag). *Let $f \in B_0$ be a reduced polynomial and \mathcal{C} the associated affine plane curve. Let $P(s) = \sum_i P_i s^i$ be a Bernstein-Sato operator for f (i.e., there exists a polynomial $b \in \mathbb{Q}[s]$ such that $P(f^{s+1}) = b(s)f^s$) of order $d \geq 2$. Then, each s -coefficient of its principal symbol can be identified with an element of $\mathcal{N}_1(\mathcal{C})$.*

From these two results we deduce that a complete description of the ideal $\mathcal{N}_1(\mathcal{C})$ would provide a complete description of the principal symbols of differential operators of the plane of degree d such that $D(f^d) \in \langle f \rangle$ (respectively Bernstein-Sato operators for f of order $d \geq 2$). Let us also mention that the existence of non-trivial nilpotent functions of the arc scheme of a variety has recently been used in the context of vertex algebras (see [1]).

The first approach to the computational determination of nilpotent functions of the arc scheme was made by K. Kpognon and J. Sebag in [55]. In particular, they develop

an algorithm which, given the polynomial $f \in B_0$, provides a Groebner basis of the ideal $\mathcal{N}_1(\mathcal{C})$. Unfortunately, its complexity makes it useless in a large number of examples.

1.0.11 Chapter 2 is dedicated to the theoretical study of the ideal $\mathcal{N}_1(\mathcal{C})$ for \mathcal{C} an affine plane curve associated with a reduced homogeneous or weighted-homogeneous polynomial. It is the result of a joint work with J. Sebag. In the main theorems of that chapter, we produce Groebner bases of $\mathcal{N}_1(\mathcal{C})$ in those cases. Let us begin with the homogeneous case.

Let $n \in \mathbb{N}^*$. For every integer $i \in \{1, \dots, n\}$, let $\gamma_i \in k^*$ be mutually distinct elements. For every integer $i \in \{1, \dots, n\}$, we set $f_i = y_0 - \gamma_i x_0 \in B_0$ and $f = x_0^\varepsilon y_0^{\varepsilon'} \prod_{i=1}^n f_i$ with $\varepsilon, \varepsilon' \in \{0, 1\}$. Let us consider, for every integer $i \in \{0, \dots, n\}$, the following polynomials of the ring B_1 :

$$\delta_i := \left(\prod_{\ell=1}^i \Delta(f_\ell) \right) \times \left(\prod_{\ell=i+1}^n f_\ell \right).$$

We set $\mathcal{C} = \text{Spec}(B_0/\langle f \rangle)$. The following is our main result for homogeneous polynomials, corresponding to theorem 3.5.5.

Theorem 1.0.12. *Let \mathcal{C} be the reduced affine plane curve defined by the homogeneous polynomial f above. The family*

$$\mathfrak{B} := \{y_1 x_0 - y_0 x_1, x_{h_1}^\varepsilon y_{h_2}^{\varepsilon'} \delta_i(f), i \in \{0, \dots, n\}, h_1, h_2 \in \{0, 1\}\}$$

is a Groebner basis of $\mathcal{N}_1(\mathcal{C})$ for the monomial order $y_1 >_{\text{lex}} y_0 >_{\text{lex}} x_1 >_{\text{lex}} x_0$ in B_1 .

Let us now consider the weighted homogeneous case. A polynomial $f \in B_0$ is said to be *weighted homogeneous* of weight (w_1, w_2, w) if we have the formula $f(t^{w_1} x_0, t^{w_2} y_0) = t^w f(x_0, y_0)$ in the polynomial ring $B_0[t]$. If the field k is algebraically closed, if f is weighted homogeneous then there exist an integer $n \geq 1$, a pair of coprime integers (r, s) with $r \geq s \geq 2$ and $\lambda_1, \dots, \lambda_n \in k^*$ mutually distinct such that, up to k -automorphism of B_0 , we have $f = x_0^\varepsilon y_0^{\varepsilon'} \prod_{i=1}^n (x_0^r - \lambda_i y_0^s)$ with $\varepsilon, \varepsilon' \in \{0, 1\}$.

For every integer $i \in \{1, \dots, n\}$, we set $\tilde{D}_{-1} := \tilde{D}_{\lambda_i, -1} := s y_1 x_0 - r y_0 x_1$ and $\tilde{D}_{\lambda_i, j_i} := \lambda_i s^{j_i} y_0^{s-j_i} y_1^{j_i} - r^{j_i} x_0^{r-j_i} x_1^{j_i}$, where $j_i \in \{0, \dots, s\}$. We denote

$$\tilde{D}_{j_1, \dots, j_n} := \tilde{D}_{\lambda_1, j_1} \cdots \tilde{D}_{\lambda_n, j_n},$$

where $j_i \in \{-1, \dots, s\}$ for every $i \in \{1, \dots, n\}$. We can state our main result in the weighted-homogeneous case, corresponding to theorem 3.6.6.

Theorem 1.0.13. *Let \mathcal{C} be the reduced affine plane curve defined by the weighted homogeneous polynomial f above. Then the family*

$$\mathfrak{B} = \{\tilde{D}_{-1}, x_{h_1}^\varepsilon y_{h_2}^{\varepsilon'} \tilde{D}_{j_1, \dots, j_n}, j_i \in \{-1, \dots, s\}, i \in \{1, \dots, n\}, h_1, h_2 \in \{0, 1\}\}$$

is a Groebner basis of $\mathcal{N}_1(\mathcal{C})$ for the monomial order $y_1 >_{\text{lex}} y_0 >_{\text{lex}} x_1 >_{\text{lex}} x_0$ in B_1 .

Although these theorems appear to give a complete answer for homogeneous and weighted homogeneous reduced polynomials only for algebraically closed fields, it is possible to deduce a system of generators of $\mathcal{N}_1(\mathcal{C})$ for an arbitrary field of characteristic zero from that obtained for its algebraic closure by base change. The implications of those results in terms of differential operators are described in section 3.7.

1.0.14 In order to describe and contextualize our study about the formal neighbourhoods of the arc scheme, let us go back to the Nash problem. Let V be an integral separated k -variety with singular locus V_{Sing} . A subset C of $\mathcal{L}_\infty(V)$ is said to be *fat* if there is no proper closed subvariety Z of V such that $C \subset \mathcal{L}_\infty(Z)$. Every irreducible fat subset C of $\mathcal{L}_\infty(V)$ defines a valuation ord_C of the rational field $k(V)$ of V .

We define the *Nash space* of V as $\mathcal{N}(V) = (\theta_{0,V}^\infty)^{-1}(V_{\text{Sing}})$, i.e., the set of arcs on V centered at the singular locus; it is a constructible closed subset of $\mathcal{L}_\infty(V)$. A fat irreducible component of $\mathcal{N}(V)$ is called a *Nash component*. In particular, it defines a valuation on V . All the assertions in the following theorem were proven by Nash in [64].

Theorem 1.0.15 (Nash). *Let V be an integral separated k -variety. The following assertions hold true:*

- (1) *There are only finitely many irreducible components of the Nash space $\mathcal{N}(V)$; each of them is fat and hence it is a Nash component.*
- (2) *The valuation associated with a Nash component is an essential divisorial valuation of index 1 on V .*
- (3) *The valuations associated with two different Nash components are different.*

Recall that a divisor over V is *essential* if the associated divisorial valuation ν is essential, i.e., for every resolution of singularities $\pi : Y \rightarrow V$, the center of ν on Y is the generic point of an irreducible component of $\pi^{-1}(V_{\text{Sing}})$. Let us remark that if k is a field of characteristic $p > 0$, the Nash space $\mathcal{N}(V)$ may have irreducible components contained in $\mathcal{L}_\infty(V_{\text{Sing}})$. In [50, Example 2.13] S. Ishii and J. Kollár show that this is the case for $V = \text{Spec}(k[x, y, z]/\langle x^p - y^p z \rangle)$.

The preceding theorem shows that the map associating with every Nash component C of the Nash space $\mathcal{N}(V)$ the essential divisorial valuation ord_C on V is well defined and injective; it is called the *Nash map*.

$$\left\{ \begin{array}{c} \text{Nash components} \\ \text{of } \mathcal{N}(V) \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \text{essential divisorial valuations} \\ \text{of index 1 on } V \end{array} \right\}$$

In [64], Nash observed that, on several two-dimensional examples, the Nash map was also surjective and asked whether it is the case in general. He announced that an affirmative answer seemed to be possible for surfaces, but also that the analogous in higher dimension was uncertain. This question is known as the *Nash problem*.

Problem 1.0.16 (Nash problem). *Is the Nash map above bijective?*

This question was answered in a series of articles published between 2003 and 2013, which show that Nash's intuition was indeed correct. The following theorem, corresponding to the main theorem in [33], solved the case of surfaces in 2012.

Theorem 1.0.17 (Fernández de Bobadilla-Pe Pereira). *Let k be an algebraically closed field of characteristic zero and V be an integral k -surface. Then the Nash map is bijective.*

In higher dimensions, the answer to the Nash problem is negative: S. Ishii and J. Kollár gave in 2003 counterexamples in dimension ≥ 4 (see [50, Proposition 4.5]) and T. de Fernex found in 2013 counterexamples in dimension 3 (see [24]), which completed the picture. Particularly interesting is a counterexample studied by J.M. Johnson and J. Kollár in [51, Proposition 9], which was already proposed by Nash in [64].

1.0.18 However, it remains a challenging problem to understand the image of the Nash map. There exist answers under different additional assumptions, see [48, Section 4] for a recollection of some results in this sense (other answers after the publication of this article have been given in [58, Theorem 3.3] and [25, Theorem 1.1]). In particular, for toric varieties the Nash map is bijective, as shown in [50, Theorem 3.16] for normal toric varieties and in [47, Corollary 5.12] for non-normal toric varieties.

Theorem 1.0.19 (Ishii-Kollár, Ishii). *Let k be an algebraically closed field of characteristic zero and let V be an affine toric k -variety. Then the Nash map is bijective.*

Above we have said that every fat irreducible set of $\mathcal{L}_\infty(V)$ defines a valuation on V which, in the particular case of Nash components, is an essential divisorial valuation of index 1. Conversely, given a divisorial valuation ν on V , we can construct a closed irreducible fat subset \mathcal{N}_ν of $\mathcal{L}_\infty(V)$ which defines the valuation ν . We call \mathcal{N}_ν the *Nash set* associated with ν . Then, the Nash problem consists in determining the maximal Nash sets \mathcal{N}_ν such that ν is a divisorial valuation on V . This problem may be generalized to the problem of understanding the inclusions of Nash sets $\mathcal{N}_\nu \subset \mathcal{N}_{\nu'}$ in terms of the valuations ν and ν' , see [49, Problem 3.7]. In the case of normal toric varieties and toric valuations, the relation is clear, and $\mathcal{N}_\nu \subset \mathcal{N}_{\nu'}$ if and only if, for every $f \in \mathcal{O}(V)$, we have $\nu'(f) \leq \nu(f)$. This property, and also theorem 1.0.19, illustrate the fact that (normal) toric varieties are well-behaved for questions related with this kind of problems. We profit from this in chapter 5 to obtain our result about the formal neighbourhood at points of the Nash set associated with a divisorial toric valuation.

In the general setting, the problem of understanding such inclusions of Nash sets $\mathcal{N}_\nu \subset \mathcal{N}_{\nu'}$ has been related by M. Lejeune-Jalabert in [57] to a problem of wedges lifting to a resolution of singularities (a wedge is an arc on the arc scheme $\mathcal{L}_\infty(V)$) which has been proven to be equivalent to the Nash problem by A. Reguera in [66]. This equivalence, which is at the core of the last results concerning the Nash problem

(in particular that of theorem 1.0.17), lies in the fact that the formal neighbourhood of $\mathcal{L}_\infty(V)$ at the generic point of a Nash set is noetherian.

1.0.20 The above exposition shows how the formal local structure of the arc scheme, which is our field of study, played an important role in the solution of the topological problem at the origin of the study of the arc scheme. Let us now focus on the formal neighbourhoods. As we will explain in subsection 5.1, the property of the formal neighbourhood to be noetherian holds for *constructible points* (i.e., the generic point of an irreducible constructible subset of the arc scheme; let us point out that the interest of constructible subsets of the arc scheme was already remarked in motivic integration) which are *non-degenerate*, that is, whose generic point does not belong to V_{Sing} . This is not true for arbitrary arcs. However, we have the following theorem, due to V. Drinfeld, M. Grinberg and D. Kazhdan ([30], [34]) which is crucial in the understanding of the local structure at rational points. For this reason, chapter 4 is entirely devoted to this theorem and related facts.

Theorem 1.0.21 (Drinfeld-Grinberg-Kazhdan). *Let $v \in V(k)$ be a rational point of V such that $\dim_v(V) \geq 1$. Let $\gamma \in \mathcal{L}_\infty(V)(k)$ be a non-degenerate rational point of the associated arc scheme centered at v . Then there exist an affine k -scheme S of finite type, a rational point $s \in S(k)$ and an isomorphism of formal k -schemes:*

$$\widehat{\mathcal{L}_\infty(V)}_\gamma \cong \widehat{S}_s \hat{\times}_k \text{Spf}(k[[T_i]_{i \in \mathbb{N}}]).$$

The formal k -scheme \widehat{S}_s in the above isomorphism is called a *finite formal model* of the pair $(\mathcal{L}_\infty(V), \gamma)$. An important fact is that the proof of the theorem is constructive in the sense that it provides an explicit description of a finite formal model, unlike formal neighbourhoods at constructible points which, despite of their finiteness property, are in general hard to compute explicitly.

1.0.22 Let us now explain the motivation of our study in chapter 5, which is a joint work with D. Bourqui and J. Sebag. We place ourselves in the toric setting, hence V will be a normal affine toric k -variety. In [13], Bourqui and Sebag proved the following result (see theorem 5.2.17 for a more precise statement).

Theorem 1.0.23 (Bourqui-Sebag). *Let ν be a divisorial toric valuation of V and let \mathcal{N}_ν be the corresponding Nash set. Then there exists a non-empty open subset U_ν of \mathcal{N}_ν consisting of non-degenerate arcs such that, for every $\alpha \in U_\nu(k)$, the local ring $\mathcal{O}_{\mathcal{L}_\infty(V_\sigma), \alpha}$ (and hence also $\widehat{\mathcal{O}_{\mathcal{L}_\infty(V_\sigma), \alpha}}$) is constant up to isomorphism.*

Roughly speaking, this theorem says that, for sufficiently generic non-degenerate arcs in the Nash set \mathcal{N}_ν , their formal neighbourhood is constant up to isomorphism. Moreover, they provide a finite formal model in this case. Their proof profit from the extension of the action of the torus on V to an action on $\mathcal{L}_\infty(V)$. Our innovative idea

consists in relating this constant formal neighbourhood with the formal neighbourhood at the generic point of the Nash set. We obtain the following result (see theorem 5.5.19).

Theorem 1.0.24. *Let k be a field of characteristic zero. Let V be an affine normal toric k -variety. Let ν be a toric divisorial valuation on V and \mathcal{N}_ν be the associated Nash set. Let η_ν be the generic point of \mathcal{N}_ν and κ_ν be the residue field of η_ν .*

There exists a non-empty open subset U_ν of \mathcal{N}_ν consisting of non-degenerate arcs such that for every arc $\alpha \in U_\nu(k)$ there exists an isomorphism of κ_ν -formal schemes between $\widehat{\mathcal{L}_\infty(V)}_{\eta_\nu} \hat{\otimes}_{\kappa_\nu, \kappa_\nu} \llbracket (T_i)_{i \in \mathbb{N}} \rrbracket$ and $\widehat{\mathcal{L}_\infty(V)}_\alpha \hat{\otimes}_k \kappa_\nu$.

Let us stress that theorem 5.5.19 actually gives a more precise result. In particular, it provides an explicit description of $\widehat{\mathcal{L}_\infty(V)}_{\eta_\nu}$ in terms of the formal spectrum of an essentially of finite type local κ_ν -algebra (*a priori* we only knew that such a formal neighbourhood was noetherian).

It is natural to ask whether there exist other classes of varieties for which the formal neighbourhood of a rational non-degenerate arc is generically constant on Nash sets, and, if so, whether the involved isomorphism class is encoded in some sense in the formal neighbourhood of the generic point of the Nash set in such a way that a comparison theorem similar to theorem 1.0.24 still holds. Such a result would allow to transpose some of the properties and known facts about formal neighbourhoods of rational arcs obtained from the Drinfeld-Grinberg-Kazhdan theorem to that of generic points of the Nash sets; in particular the possibility of easily computing an explicit presentation.

1.0.25 Let us briefly explain the contents of each chapter. Chapter 2 is an introductory recollection of known facts about arc scheme including its definition and existence, explicit presentations in terms of a presentation of the base variety, relation with differential algebra, some topological properties and facts about its constructible subsets and the relation with valuations. In particular, most of the notions in this introduction will be formally presented there. Chapter 3 contains our study about the nilpotent functions of $\mathcal{L}_\infty(\mathcal{C})$ living in $\mathcal{L}_1(\mathcal{C})$ for \mathcal{C} a reduced affine plane curve defined by a homogeneous or weighted homogeneous polynomial. The fourth chapter is consecrated to present the Drinfeld-Grinberg-Kazhdan theorem and some related facts. Finally, we begin chapter 5 by explaining how the non-degenerate constructible points are characterized by the noetherianity of their formal neighbourhoods and then we develop our comparison result presented in theorem 1.0.24.

1.1 CONVENTIONS AND NOTATION

Along the text, k will denote a field (of characteristic zero in chapters 3 and 5 unless otherwise stated). We denote by \mathbb{N} the set of natural numbers and we assume that $0 \in \mathbb{N}$, analogously $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$ and \mathbb{C} denote the sets of integer, rational, real and complex numbers (with the corresponding usual structures). We denote by \mathbf{Alg}_k (resp. \mathbf{Sch}_k) the category of k -algebras (resp. k -schemes) with morphisms of k -algebras (resp. morphism of k -schemes). For any category \mathcal{C} and any objects $A, B \in \mathcal{C}$, we denote by $\mathrm{Hom}_{\mathcal{C}}(A, B)$ the set of morphisms from A to B in the category.

By a ring we mean a commutative unitary ring, except in section 3.7. Let R be a ring, we denote by R^* the set of multiplicative invertible elements, by analogy $\mathbb{N}^* := \mathbb{N} \setminus \{0\}$. Let \mathfrak{i} be an ideal of R and $f \in R$, we denote by R_f the localization of R with respect to the multiplicative subset $\{f^r; r \in \mathbb{N}\}$. We denote by $\mathfrak{i} : f^\infty$ the ideal $\{g \in R : f^r g \in \mathfrak{i} \text{ for some } r \in \mathbb{N}\}$. Let R' be another ring and $\vartheta : R \rightarrow R'$ a morphism of rings. For the sake of easy reading and abusing notation, the extension ideal of \mathfrak{i} in R' via the morphism ϑ is denoted by $\vartheta(\mathfrak{i})$, or even by \mathfrak{i} if the involved morphism ϑ is clear from the context (for example if R is a subring of R').

Given a k -algebra R , we denote by $R[[t]]$ the formal power series ring in the variable t with coefficients in R , it is still a k -algebra. Let $R[X_\omega; \omega \in \Omega]$ be a polynomial ring and $(f_i)_{i \in \Lambda}$ a family of elements, we denote by $\langle f_i; i \in \Lambda \rangle$ the ideal generated by them. Let S be a R -algebra and $\{s_\omega\}_{\omega \in \Omega}$ a collection of elements in S . Let $f \in R$, then we denote by $f|_{X_\omega = s_\omega} \in S$ the image of f by the unique morphism of R -algebras $R[X_\omega] \rightarrow S$ mapping X_ω to s_ω for each $\omega \in \Omega$.

We denote by \mathbb{A}_k^n the affine n -dimensional space defined over the field k . By k -variety we mean a k -scheme of finite type. Given a k -variety V , we denote by V_{Reg} or $\mathrm{Reg}(V)$ the regular locus of V (which coincides with the locus of smooth points of the structural morphism when the field k is perfect) and by V_{Sing} or $\mathrm{Sing}(V)$ the non-regular locus with its reduced structure of closed subscheme of V . Given a k -scheme X , we denote by X_{red} the reduced closed subscheme of X . The structure sheaf of X is denoted by \mathcal{O}_X and when X is affine we simply denote $\mathcal{O}(X) := \mathcal{O}_X(X)$. If X is an affine k -scheme and $f \in \mathcal{O}(X)$ we denote by $\{f \neq 0\}$ the distinguished open subset of X where f does not vanish and by $\{f = 0\}$ the closed subset $X \setminus \{f \neq 0\}$. If \mathfrak{i} is an ideal of $\mathcal{O}(X)$ then we also denote by $\{\mathfrak{i} = 0\}$ the vanishing locus of \mathfrak{i} in X .

ARC SCHEME

The aim of this chapter is to present the central objects of our study, namely the arc scheme associated with a scheme defined over a field, as well as their most important properties and some more specific ones which are relevant for our purposes in the next chapters. In this way, this chapter is intended to be an introductory recollection of results on arc scheme which are already in the literature.

We will begin by defining, in sections 2.1 and 2.2, the functors of jets and arcs. These functors are representable (we will only give some ideas in this sense as well as precise references) and their representatives are respectively the jet and arc schemes. We complete these sections with a review of useful properties of these objects.

In section 2.3 we explain the relation between arc scheme and differential algebra and how it can be taken advantage of in order to deduce interesting properties of the arc scheme; in particular the contents in this section will be useful in the study of the nilpotency in arc scheme developed in chapter 3. In section 2.4 we will present some topological properties of arc scheme which are not only interesting by themselves, but also play an important role in the sequel. We also recall the definition of constructible subsets of a scheme and explain their characterization in arc scheme. Section 2.5 is a discussion about valuations and the arc scheme, whose relationship has been at the origin of the arc scheme through the Nash problem, as explained in the Introduction. Moreover, this link appears in chapter 5 in the case of normal toric varieties.

2.1 JET SCHEMES

We are going to follow [19, Ch. 3, §2 and §3] as the main reference for this section and the following one. There, the more general setting of relative schemes is considered and the constructions and most of the results are explicitly described and proven, otherwise

precise references are given. Here we will only consider the case of schemes defined over a field and we will omit technical proofs and arguments which go beyond the scope of this presentation, indicating where they can be found.

2.1.1 Let k be a field and X be a k -scheme. Let $m \in \mathbb{N}$. For every k -scheme Y , a Y -valued m -jet on X (or Y -valued jet of level m on X) is a morphism $Y \otimes_k k[t]/\langle t^{m+1} \rangle \rightarrow X$. We denote by $\mathcal{L}_m(X)(Y)$ the set of Y -valued m -jets on X , i.e.,

$$\mathcal{L}_m(X)(Y) := \mathrm{Hom}_{\mathbf{Sch}_k} \left(Y \otimes_k k[t]/\langle t^{m+1} \rangle, X \right).$$

The functor $\mathcal{L}_m(X)$ is the Weil restriction of $X \otimes_k k[t]/\langle t^{m+1} \rangle$ with respect to the canonical morphism of k -schemes $\mathrm{Spec}(k) \rightarrow \mathrm{Spec}(k[t]/\langle t^{m+1} \rangle)$. When $Y = \mathrm{Spec}(R)$ is affine, we also speak of a R -valued jet of level m , in this case

$$\mathcal{L}_m(X)(R) := \mathcal{L}_m(X)(\mathrm{Spec}(R)) = \mathrm{Hom}_{\mathbf{Sch}_k} \left(\mathrm{Spec}(R[t]/\langle t^{m+1} \rangle), X \right).$$

Let $g : Y' \rightarrow Y$ be a morphism of k -schemes, it induces a morphism

$$\begin{array}{ccc} \mathcal{L}_m(X)(g) : \mathcal{L}_m(X)(Y) & \longrightarrow & \mathcal{L}_m(X)(Y') \\ \gamma & \longmapsto & \gamma \circ (g \times \mathrm{Id}_{k[t]/\langle t^{m+1} \rangle}) \end{array}$$

This datum defines a contravariant functor from the category of k -schemes to that of **Sets** (which is also called a *presheaf* on the category of k -schemes), i.e.,

$$\mathcal{L}_m(X) : \mathbf{Sch}_k^{\mathrm{op}} \rightarrow \mathbf{Sets}.$$

It is called the *functor of m -jets* on X .

This construction is also functorial in X : let $f : X' \rightarrow X$ be a morphism of k -schemes, then for every k -scheme Y it induces a morphism

$$\begin{array}{ccc} \mathcal{L}_m(f)(Y) : \mathcal{L}_m(X')(Y) & \longrightarrow & \mathcal{L}_m(X)(Y) \\ \gamma & \longmapsto & f \circ \gamma, \end{array}$$

which defines a morphism of functors $\mathcal{L}_m(f) : \mathcal{L}_m(X') \rightarrow \mathcal{L}_m(X)$.

2.1.2 The following proposition, corresponding with [19, Ch. 3, Proposition 2.1.3 a)] states that the functor of jets is representable.

Proposition 2.1.3. *Let k be a field, X be a k -scheme and $m \in \mathbb{N}$. Then the functor $\mathcal{L}_m(X)$ of m -jets on X is representable.*

We also denote by $\mathcal{L}_m(X)$ the unique representative of the functor $\mathcal{L}_m(X)$ of m -jets on X . It is a k -scheme, unique (up to isomorphism of k -schemes) and we call it the *m -jet scheme* (or *scheme of jets of level m*) of X (or associated with X). In particular, if

$f : X' \rightarrow X$ is a morphism of k -schemes, the morphism of functors $\mathcal{L}_m(f) : \mathcal{L}_m(X') \rightarrow \mathcal{L}_m(X)$ induces a morphism of k -schemes between the corresponding representatives, which we also denote by $\mathcal{L}_m(f)$.

The representability of the functor of m -jets on X is deduced from the fact that it is a particular case of Weil restriction, hence general results about the representability of Weil restrictions hold. The functor $\mathcal{L}_m(X)$ is the Weil restriction of $X \otimes_k k[t]/\langle t^{m+1} \rangle$ with respect to the canonical morphism of k -schemes $\mathrm{Spec}(k) \rightarrow \mathrm{Spec}(k[t]/\langle t^{m+1} \rangle)$, the fact that it is representable is deduced from [19, Ch.3, Theorem 1.4.1]. More details about Weil restrictions can be found in [19, Ch.3, §1] and more generally in [6, Section 7.6].

Let us list some properties of the jet schemes which are proven in [19, Ch. 3, Proposition 2.1.3, Subsection 2.1.5 and Lemma 2.1.6]. They are all deduced from general properties of the Weil restrictions, all of which can be found in [19, Ch.3, §1].

Proposition 2.1.4. *Let k be a field, X be a k -scheme and $m \in \mathbb{N}$. The following properties hold true:*

- (1) *Let Y be a k -scheme and $f : Y \rightarrow X$ be a closed (resp. open) immersion, then the associated morphism of schemes $\mathcal{L}_m(f) : \mathcal{L}_m(Y) \rightarrow \mathcal{L}_m(X)$ is also a closed (resp. open) immersion.*
- (2) *For every open covering $(U_i)_{i \in I}$ of X , the family $(\mathcal{L}_m(U_i))_{i \in I}$ is an open covering of $\mathcal{L}_m(X)$.*
- (3) *Assume that the k -scheme X is affine (resp. locally of finite type, resp. locally of finite presentation, resp. quasi-compact, resp. separated). Then the k -scheme $\mathcal{L}_m(X)$ has the same property.*
- (4) *Let Y be a k -scheme and $p_1 : X \times_k Y \rightarrow X$ and $p_2 : X \times_k Y \rightarrow Y$ be respectively the first and second projections. Then the morphism of k -schemes*

$$(\mathcal{L}_m(p_1), \mathcal{L}_m(p_2)) : \mathcal{L}_m(X \times_k Y) \longrightarrow \mathcal{L}_m(X) \times_k \mathcal{L}_m(Y)$$

is an isomorphism.

2.1.5 Let $m, n \in \mathbb{N}$ be two integers such that $m \geq n$. The canonical quotient morphism $k[t]/\langle t^{m+1} \rangle \rightarrow k[t]/\langle t^{n+1} \rangle$ induces a morphism $p_n^m : \mathrm{Spec}(k[t]/\langle t^{n+1} \rangle) \rightarrow \mathrm{Spec}(k[t]/\langle t^{m+1} \rangle)$ between the corresponding affine k -schemes. Let X be a k -scheme, then for every k -scheme Y the preceding morphism induces a map

$$\begin{aligned} \theta_{n,X}^m(Y) : \mathcal{L}_m(X)(Y) &\longrightarrow \mathcal{L}_n(X)(Y) \\ \gamma &\longmapsto \gamma \circ (\mathrm{Id}_Y \times p_n^m) \end{aligned}$$

which is called the *truncation morphism from level m to n* on Y -valued jets on X . These truncation morphisms $\theta_{n,X}^m(Y)$ are (contravariantly) functorial in Y ; the associated

morphism of functors $\theta_{n,X}^m : \mathcal{L}_m(X) \rightarrow \mathcal{L}_n(X)$, which can be identified with a morphism of k -schemes via the representability, is also called the *truncation morphism* from level m to n .

Let X be a k -scheme and $\ell, m, n \in \mathbb{N}$ be integers such that $\ell \geq m \geq n$. The canonical quotient morphism $k[t]/\langle t^{\ell+1} \rangle \rightarrow k[t]/\langle t^{n+1} \rangle$ being equal to the composition $k[t]/\langle t^{\ell+1} \rangle \rightarrow k[t]/\langle t^{m+1} \rangle \rightarrow k[t]/\langle t^{n+1} \rangle$, we deduce that the truncation morphisms satisfy the transitivity property

$$\theta_{n,X}^{\ell} = \theta_{n,X}^m \circ \theta_{m,X}^{\ell}.$$

Let $f : X' \rightarrow X$ be a morphism of k -schemes and $m, n \in \mathbb{N}$ be two integers such that $m \geq n$. Then the following diagram is commutative

$$\begin{array}{ccc} \mathcal{L}_m(X') & \xrightarrow{\mathcal{L}_m(f)} & \mathcal{L}_m(X) \\ \theta_{n,X'}^m \downarrow & & \downarrow \theta_{n,X}^m \\ \mathcal{L}_n(X') & \xrightarrow{\mathcal{L}_n(f)} & \mathcal{L}_n(X) \end{array}$$

which shows that the truncation morphisms $\theta_{n,X}^m$ are functorial in X . Then the jet schemes $(\mathcal{L}_m(X))_{m \in \mathbb{N}}$ together with the truncation morphisms $\theta_{n,X}^m$ form a projective system in \mathbf{Sch}_k which is functorial in X .

Let Y be another k -scheme and $\gamma \in \mathcal{L}_m(X)(Y)$ be a Y -valued m -jet. The point $\theta_{0,X}^m(\gamma) \in X(Y)$ is called its *base point*, we also denote it by $\gamma(0)$.

A proof of the following result can be found in [19, Ch. 3, Corollary 2.2.4].

Proposition 2.1.6. *Let k be a field, X a k -variety and $m, n \in \mathbb{N}$ with $m \geq n$. The truncation morphism $\theta_{n,X}^m$ is an affine morphism of k -schemes.*

2.1.7 Let $m \in \mathbb{N}$ be an integer. The canonical morphism of k -algebras $k \rightarrow k[t]/\langle t^{m+1} \rangle$ induces a morphism of k -schemes $r_m : \text{Spec}(k[t]/\langle t^{m+1} \rangle) \rightarrow \text{Spec}(k)$. Let X and Y be two k -schemes, the preceding morphism induces a map

$$\begin{array}{ccc} s_{m,X}(Y) : X(Y) & \longrightarrow & \mathcal{L}_m(X)(Y) \\ & \longmapsto & \gamma \circ (\text{Id}_Y \times r_m) \end{array}$$

which is covariantly functorial in X and contravariantly in Y , hence it defines a morphism of k -schemes $s_{m,X} : X \rightarrow \mathcal{L}_m(X)$. Since $k \rightarrow k[t]/\langle t^{m+1} \rangle$ is a section of the canonical quotient morphism $k[t]/\langle t^{m+1} \rangle \rightarrow k$, we deduce that $s_{m,X}$ is a section of the truncation morphism $\theta_{0,X}^m$. Moreover, for every $m, n \in \mathbb{N}$ such that $m \geq n$ we have $\theta_{n,X}^m \circ s_{m,X} = s_{n,X}$. A proof of the following result can be found in [19, Ch. 3, Corollary 2.2.5], it is a consequence of [36, Corollaire 5.4.6].

Corollary 2.1.8. *Let k be a field, X a k -scheme and $m \in \mathbb{N}$. The morphism $s_{m,X}$ is a closed immersion.*

2.1.9 In particular, proposition 2.1.6 implies that, if X is an affine k -scheme and $m \in \mathbb{N}$, then $\mathcal{L}_m(X)$ is also an affine k -scheme. An explicit presentation of $\mathcal{L}_m(X)$ can be given from a presentation of X as a closed subscheme of an affine space, see subsection 2.2.9 for this construction and examples. We give here three examples.

Example 2.1.10. Let k be a field and $m \in \mathbb{N}$. Then $\mathcal{L}_m(\mathrm{Spec}(k))$ is representable by $\mathrm{Spec}(k)$ and the truncation morphisms are the identity.

Example 2.1.11. Let k be a field and X be a k -scheme. The isomorphism of k -algebras $k \rightarrow k[t]/\langle t \rangle \cong k$ induces an isomorphism of functors between $\mathcal{L}_0(X)$ and $\mathrm{Hom}_{\mathbf{Sch}_k}(\bullet, X)$. Hence the functor $\mathcal{L}_0(X)$ is representable by X , and the section $s_{0,X} : X \rightarrow \mathcal{L}_0(X)$ is an isomorphism.

Example 2.1.12. The functor of 1-jets of an affine k -variety V identifies with the tangent space of V , defined as $T_{V/k} = \mathrm{Spec}(\mathrm{Sym}(\Omega_{V/k}^1))$, where $\Omega_{V/k}^1$ is the module of Kähler differentials of V . See lemma 3.2.2 for a proof. The same arguments extend to arbitrary k -schemes.

2.2 ARC SCHEME

We will present here the construction and main properties of the arc scheme associated with a scheme defined over a field. We will mainly follow [19, Ch.3, §3].

2.2.1 Let k be a field and X be a k -scheme. As already pointed out in subsection 2.1.5, the jet schemes $(\mathcal{L}_m(X))_{m \in \mathbb{N}}$, together with the truncation morphisms $\theta_{n,X}^m$ for $m, n \in \mathbb{N}$, $m \geq n$, form a projective system in \mathbf{Sch}_k which is functorial in X . Since the truncation morphisms are affine by proposition 2.1.6, the projective limit of this system exists in the category of k -schemes (see [40, Proposition 8.2.3]). Indeed, if X is affine this limit is isomorphic to $\mathrm{Spec}(\varprojlim_{m \in \mathbb{N}} \mathcal{O}(\mathcal{L}_m(X)))$. We set

$$\mathcal{L}_\infty(X) := \varprojlim_{m \in \mathbb{N}} \mathcal{L}_m(X)$$

and we call this k -scheme the *arc scheme* of X .

2.2.2 Let $m, n \in \mathbb{N}$ be two integers with $m \geq n$. Let X be a k -scheme, then the family of canonical morphisms $(\theta_{m,X}^\infty)_{m \in \mathbb{N}}$ given by the limit $\varprojlim_{m \in \mathbb{N}} \mathcal{L}_m(X)$ provides the following commutative diagram of k -schemes.

$$\begin{array}{ccc}
\mathcal{L}_\infty(X) & \xrightarrow{\theta_{m,X}^\infty} & \mathcal{L}_m(X) \\
& \searrow \theta_{n,X}^\infty & \downarrow \theta_{n,X}^m \\
& & \mathcal{L}_n(X)
\end{array}$$

The morphism $\theta_{m,X}^\infty$ is called the *truncation morphism* of level m .

2.2.3 The following result of B. Bhatt, corresponding to [5, Theorem 4.1 and Remark 4.6], plays a fundamental role in the description of the functor of points of the arc scheme.

Theorem 2.2.4 (Bhatt). *Let R be a ring and I be an ideal such that R is I -adically complete. For every scheme X , the canonical map $X(R) \rightarrow \varprojlim_{m \in \mathbb{N}} X(R/I^m)$ is bijective.*

For the proof of this theorem, techniques of derived algebraic geometry are used. Besides the original article, a sketch of the proof can be found in [19, Ch. 3, Subsection 3.3.8].

Let R be a k -algebra. By the very definition of the arc scheme $\mathcal{L}_\infty(X)$, the set $\mathrm{Hom}_{\mathbf{Sch}_k}(\mathrm{Spec}(R), \mathcal{L}_\infty(X))$ is in bijection with $\mathrm{Hom}_{\mathbf{Sch}_k}(\mathrm{Spec}(R), \varprojlim_{m \in \mathbb{N}} \mathcal{L}_m(X))$ and, since the Hom functor preserves limits, it is also in bijection with $\varprojlim_{m \in \mathbb{N}} \mathrm{Hom}_{\mathbf{Sch}_k}(\mathrm{Spec}(R), \mathcal{L}_m(X))$.

By the very definition of the m -jet functor for $m \in \mathbb{N}$, the latter set is in bijection with the set

$$\varprojlim_{m \in \mathbb{N}} \mathrm{Hom}_{\mathbf{Sch}_k}(\mathrm{Spec}(R[t]/\langle t^{m+1} \rangle), X).$$

Then, theorem 2.2.4 applied to $R[[t]]$ implies that the set above is in bijection with $\mathrm{Hom}_{\mathbf{Sch}_k}(\mathrm{Spec}(R[[t]]), X)$. This proves that the R -points of $\mathcal{L}_\infty(X)$ are in bijective correspondence with the $R[[t]]$ -points of X , i.e.,

$$\mathcal{L}_\infty(X)(R) = \mathrm{Hom}_{\mathbf{Sch}_k}(\mathrm{Spec}(R[[t]]), X).$$

An R -arc on X is a morphism of k -schemes $\mathrm{Spec}(R[[t]]) \rightarrow X$, i.e., a $R[[t]]$ -point of X or equivalently a R -point of $\mathcal{L}_\infty(X)$. Let us recall that the set of R -arcs on X for R a k -algebra determines the arc scheme $\mathcal{L}_\infty(X)$, since it is characterized by its functor of points in the category \mathbf{Sch}_k , which is determined by its restriction to the category of affine k -schemes.

2.2.5 Let k be a field and X be a k -scheme. Let $g : R \rightarrow R'$ be a morphism of k -algebras, it induces a morphism of k -algebras $\hat{g} : R[[t]] \rightarrow R'[[t]]$ and hence a morphism

of k -schemes $\text{Spec}(\hat{g}) : \text{Spec}(R'[[t]]) \rightarrow \text{Spec}(R[[t]])$. Hence we have a morphism

$$\begin{array}{ccc} \mathcal{L}_\infty(X)(g) : \mathcal{L}_\infty(X)(R) & \longrightarrow & \mathcal{L}_\infty(X)(R') \\ \gamma & \longmapsto & \gamma \circ \text{Spec}(\hat{g}) \end{array}$$

The construction of the arc scheme is also functorial in the base k -scheme X : let $f : X' \rightarrow X$ be a morphism of k -schemes, then for every k -algebra R it induces a morphism

$$\begin{array}{ccc} \mathcal{L}_\infty(f)(R) : \mathcal{L}_\infty(X')(R) & \longrightarrow & \mathcal{L}_\infty(X)(R) \\ \gamma & \longmapsto & f \circ \gamma, \end{array}$$

which defines a morphism of k -schemes $\mathcal{L}_\infty(f) : \mathcal{L}_\infty(X') \rightarrow \mathcal{L}_\infty(X)$. Let $m \in \mathbb{N}$ be an integer. Then the following diagram is commutative

$$\begin{array}{ccc} \mathcal{L}_\infty(X') & \xrightarrow{\mathcal{L}_\infty(f)} & \mathcal{L}_\infty(X) \\ \theta_{m,X'}^\infty \downarrow & & \downarrow \theta_{m,X}^\infty \\ \mathcal{L}_m(X') & \xrightarrow{\mathcal{L}_m(f)} & \mathcal{L}_m(X) \end{array}$$

2.2.6 Let us give a concrete description of the truncation morphisms in terms of R -arcs. Let $m \in \mathbb{N}$ be an integer and R be a k -algebra. The canonical quotient morphism $R[[t]] \rightarrow R[t]/\langle t^{m+1} \rangle$ induces a morphism $p_m^\infty : \text{Spec}(R[t]/\langle t^{m+1} \rangle) \rightarrow \text{Spec}(R[[t]])$ between the corresponding affine k -schemes. Let X be a k -scheme, then for every k -algebra R the map induced by the truncation morphism $\theta_{m,X}^\infty$ is given by

$$\begin{array}{ccc} \theta_{m,X}^\infty(R) : \mathcal{L}_\infty(X)(R) & \longrightarrow & \mathcal{L}_m(X)(R) \\ \gamma & \longmapsto & \gamma \circ p_m^\infty \end{array}$$

Let R be a k -algebra and $\gamma \in \mathcal{L}_\infty(X)(R)$ be a R -arc on X . The point $\theta_{0,X}^\infty(R)(\gamma) \in \mathcal{L}_0(X)(R)$ is called its *base point* and we also denote it by $\gamma(0)$.

2.2.7 Let us now list some properties of the arc scheme. Some of them may be deduced by direct arguments and others are consequences of the very definition of the arc scheme and the analogous properties for jet schemes (see proposition 2.1.4). These results correspond to [19, Ch.3 Proposition 3.2.5, Proposition 3.2.3, Corollary 3.2.4, Corollary 3.3.7 and Lemma 3.2.2]

Proposition 2.2.8. *Let k be a field and X be a k -scheme. The following properties hold true:*

- (1) *Let Y be a k -scheme and $f : Y \rightarrow X$ be a closed (resp. open) immersion, then the associated morphism of schemes $\mathcal{L}_\infty(f) : \mathcal{L}_\infty(Y) \rightarrow \mathcal{L}_\infty(X)$ is also a closed (resp. open) immersion.*

- (2) For every open covering $(U_i)_{i \in I}$ of X , the family $(\mathcal{L}_\infty(U_i))_{i \in I}$ is an open covering of $\mathcal{L}_\infty(X)$.
- (3) Assume that the k -scheme X is quasi-compact (resp. quasi-separated, resp. separated). Then the k -scheme $\mathcal{L}_\infty(X)$ has the same property.
- (4) Let Y be a k -scheme and $p_1 : X \times_k Y \rightarrow X$ and $p_2 : X \times_k Y \rightarrow Y$ be respectively the first and second projections. Then the morphism of k -schemes

$$(\mathcal{L}_\infty(p_1), \mathcal{L}_\infty(p_2)) : \mathcal{L}_\infty(X \times_k Y) \longrightarrow \mathcal{L}_\infty(X) \times_k \mathcal{L}_\infty(Y)$$

is an isomorphism.

- (5) Let $m \in \mathbb{N}$. The truncation morphism $\theta_{m,X}^\infty : \mathcal{L}_\infty(X) \rightarrow \mathcal{L}_m(X)$ is an affine morphism of k -schemes.

2.2.9 We know by subsection 2.2.1 that the arc scheme of an affine k -scheme is an affine k -scheme itself, and the same property holds for the m -jet scheme for $m \in \mathbb{N}$ (see proposition 2.1.6). Moreover, we know that if $f : Y \rightarrow X$ is a closed immersion of k -schemes, then $\mathcal{L}_\infty(f) : \mathcal{L}_\infty(Y) \rightarrow \mathcal{L}_\infty(X)$ and $\mathcal{L}_m(f) : \mathcal{L}_m(Y) \rightarrow \mathcal{L}_m(X)$ are also closed immersions (propositions 2.1.4 and 2.2.8). We will now show an explicit presentation of the arc scheme (resp. m -jet scheme) of a closed subscheme of an affine space in terms of a presentation.

Let k be a field, E be a set and $(T_e)_{e \in E}$ be a family of indeterminates. Let $m \in \mathbb{N} \cup \{\infty\}$ and $\Lambda = \{i \in \mathbb{N} : i \leq m\}$ (in particular if $m = \infty$ then $\Lambda = \mathbb{N}$). Let $\mathbb{A}_k^E = \text{Spec}(k[T_e ; e \in E])$ be the affine k -space. We observe that, for every k -algebra R , we have the direct sum decomposition of R -modules

$$R[[t]]/\langle t^{m+1} \rangle \cong \bigoplus_{i \in \Lambda} Rt^i$$

(where we assume the left-hand term to be $R[[t]]$ for $m = \infty$). The set $\mathcal{L}_m(\mathbb{A}_k^E)(R)$ is in bijection with the set $\text{Hom}_{\mathbf{Alg}_k}(k[T_e ; e \in E], R[[t]]/\langle t^{m+1} \rangle)$ and the above decomposition provides a bijection between this set and $R^{E \times \Lambda}$. Explicitly, a R -valued m -jet γ on \mathbb{A}_k^E is the datum of a family $(\gamma_e)_{e \in E}$ of polynomials of $R[[t]]$ (or power series in $R[[t]]$ if $m = \infty$) of degree $\leq m$; hence they can be written as $\gamma_e = \sum_{i \in \Lambda} \gamma_{e,i} t^i$ and the above bijection associates with $(\gamma_e)_{e \in E}$ the family $(\gamma_{e,i})_{e \in E, i \in \Lambda} \in R^{E \times \Lambda}$. This gives an isomorphism of functors between the functor of points of $\mathcal{L}_m(\mathbb{A}_k^E)$ and the functor of points of the affine space $\mathbb{A}_k^{E \times \Lambda} = \text{Spec}(k[T_{e,i} ; e \in E, i \in \Lambda])$.

Moreover, if $n \in \mathbb{N}$ is an integer with $n < m$ then the truncation morphism $\theta_{n, \mathbb{A}_k^E}^m : \mathcal{L}_m(\mathbb{A}_k^E) \rightarrow \mathcal{L}_n(\mathbb{A}_k^E)$ corresponds with the projection $\mathbb{A}_k^{E \times \Lambda} \rightarrow \mathbb{A}_k^{(n+1)E}$ which associates with an element $(\gamma_{e,i})_{e \in E, i \in \Lambda} \in \mathbb{A}_k^{E \times \Lambda}(R)$ the element $(\gamma_{e,i})_{e \in E, 0 \leq i \leq n}$ of $\mathbb{A}_k^{(n+1)E}$ (thus we have “forgotten” the components $\gamma_{e,i}$ for $n < i \leq m$).

Let now X be a closed subscheme of the affine space \mathbb{A}_k^E defined by an ideal I of $k[T_e; e \in E]$, as already said $\mathcal{L}_m(X)$ is a closed subscheme of $\mathcal{L}_m(\mathbb{A}_k^E)$, and we can describe the corresponding ideal of $k[T_{e,i}; e \in E, i \in \Lambda]$. For every polynomial $f \in k[T_e; e \in E]$, there exists a unique family $(f_s)_{s \in \Lambda}$ of polynomials in $k[T_{e,i}; e \in E, i \in \Lambda]$ such that the following equality holds in $k[T_{e,i}; e \in E, i \in \Lambda][[t]]$:

$$f \left(\left(\sum_{i \in \Lambda} T_{e,i} t^i \right)_{e \in E} \right) = \sum_{s \in \Lambda} f_s \left((T_{e,i})_{\substack{e \in E \\ 0 \leq i \leq s}} \right) t^s \pmod{t^{m+1}} \quad (2.1)$$

Let I_m be the ideal of $k[T_{e,i}; e \in E, i \in \Lambda]$ generated by the polynomials f_s , for $f \in I$ and $s \in \Lambda$. We will check that $\mathcal{L}_m(X)$ is the closed subscheme of $\mathcal{L}_m(\mathbb{A}_k^E)$ defined by I_m .

Let R be a k -algebra and $\gamma \in \mathcal{L}_m(\mathbb{A}_k^E)(R)$ given by a family $(\gamma_e)_{e \in E}$ of polynomials of $R[[t]]$ (or power series in $R[[t]]$ if $m = \infty$) of degree $\leq m$; as seen above it corresponds with a family $(\gamma_{e,i})_{e \in E, i \in \Lambda} \in R^{E \times \Lambda}$. For every polynomial $f \in k[T_e; e \in E]$ we have the following equality in $R[[t]]$:

$$f(\gamma) = \sum_{s \in \Lambda} f_s \left((\gamma_{e,i})_{\substack{e \in E \\ 0 \leq i \leq s}} \right) t^s \pmod{t^{m+1}}. \quad (2.2)$$

We deduce that the $R[[t]]/t^{m+1}$ -point γ of \mathbb{A}_k^E belongs to $X(R[[t]]/t^{m+1})$ if and only if $f_s \left((\gamma_{e,i})_{\substack{e \in E \\ 0 \leq i \leq s}} \right) = 0$ for every $s \in \Lambda$ and every $f \in I$. Hence the functor of points of $\mathcal{L}_m(X)$ is represented by the closed subscheme of the affine space $\mathbb{A}_k^{E \times \Lambda}$ defined by the ideal generated by $(f_s)_{s \in \Lambda}$ for f in (a generating family of) the ideal I . We have proven the following proposition.

Proposition 2.2.10. *Let k be a field, E be a set and $(T_e)_{e \in E}$ be a family of indeterminates. Let $m \in \mathbb{N} \cup \{\infty\}$ and $\Lambda = \{i \in \mathbb{N} : i \leq m\}$. Let I be an ideal of the polynomial ring $k[T_e; e \in E]$ and let X be the affine k -scheme $\text{Spec}(k[T_e; e \in E]/I)$. Then the k -scheme $\mathcal{L}_m(X)$ is isomorphic to the affine k -scheme*

$$\mathcal{L}_m(X) \cong \text{Spec} \left(\frac{k[T_{e,i}; e \in E, i \in \Lambda]}{\langle f_s : f \in I, s \in \Lambda \rangle} \right),$$

where $(f_s)_{s \in \Lambda}$ are defined as in equation (2.1). Moreover, in the definition of the ideal it suffices to take f running over a generating family of I .

Example 2.2.11. Let k be a field, $n \in \mathbb{N}$ and X be the affine k -scheme (variety, in fact) $\text{Spec}(k[T_1, T_2, T_3]/\langle T_1 T_3 - T_2^{n+1} \rangle)$. The equations of the embedding of $\mathcal{L}_\infty(X)$ in the affine space $\mathbb{A}_k^{3\mathbb{N}}$ are given by the vanishing of the coefficients of the power series

$$\left(\sum_{i \in \mathbb{N}} T_{1,i} t^i \right) \left(\sum_{i \in \mathbb{N}} T_{3,i} t^i \right) - \left(\sum_{i \in \mathbb{N}} T_{2,i} t^i \right)^{n+1}$$

in the ring $k[T_{1,i}, T_{2,i}, T_{3,i}; i \in \mathbb{N}][[t]]$. This is equivalent to the following infinite system:

$$\begin{cases} T_{1,0}T_{2,0} - T_{3,0}^{n+1} & = 0 \\ T_{1,0}T_{2,1} + T_{1,1}T_{2,0} - (n+1)T_{3,0}^n T_{3,1} & = 0 \\ \dots & \dots \\ \sum_{i=0}^s T_{1,i}T_{2,s-i} - \sum_{r_1+\dots+r_{n+1}=s} T_{3,r_1} \cdots T_{3,r_{n+1}} & = 0 \\ \dots & \dots \end{cases}$$

We check that proposition 2.2.10 holds and

$$\mathcal{L}_\infty(X) \cong \text{Spec} \left(\frac{k[T_{1,i}, T_{2,i}, T_{3,i}; i \in \mathbb{N}]}{\left\langle \sum_{i=0}^s T_{1,i}T_{2,s-i} - \sum_{r_1+\dots+r_{n+1}=s} T_{3,r_1} \cdots T_{3,r_{n+1}} : s \in \mathbb{N} \right\rangle} \right).$$

2.2.12 Let k be a field and X be a k -scheme. A point γ of $\mathcal{L}_\infty(X)$ is called an *arc* of X . Let K be a field extension of the residue field $\kappa(\gamma)$ of γ , as for any scheme the point γ gives rise to a K -point of $\mathcal{L}_\infty(X)$, i.e., a K -arc $\varphi_\gamma : \text{Spec}(K[[t]]) \rightarrow X$ on X which we usually identify with γ itself. If X is an affine k -scheme, then it corresponds with a morphism of k -algebras $\gamma^* : \mathcal{O}(X) \rightarrow K[[t]]$ which we often denote also by γ in an abuse of notation.

The image of the closed point of $\text{Spec}(K[[t]])$ under φ_γ equals $\theta_{0,X}^\infty(\gamma)$, it does not depend on the choice of the field extension K . We call it the *base point* or the *center* of the arc γ and denote it by $\gamma(0)$.

The image of the generic point of $\text{Spec}(K[[t]])$ under φ_γ neither depends on the choice of K and is called the *generic point* of the arc γ ; we denote it by $\gamma(\eta)$.

Let us recall some facts from general topology. Let X be a topological space and let x, y be points of X . We say that x is a *specialization* of y and that y is a *generization* of x if x belongs to the closure of $\{y\}$ in X . We observe that a closed subset is stable under specialization and an open subset is stable under generization. Let $f : X \rightarrow X'$ be a continuous map of topological spaces and $x, y \in X$. If x is a specialization of y , then $f(x)$ is a specialization of $f(y)$. The proof of the following result can be found in [19, Ch.3, Lemma 3.4.3].

Lemma 2.2.13. *Let k be a field and X be a k -scheme. Let $\gamma \in \mathcal{L}_\infty(X)$ be an arc of X . The following assertions hold true:*

- (1) *One has $\gamma(0) \in \overline{\{\gamma(\eta)\}}$, i.e., the base point of an arc is a specialization of its generic point.*
- (2) *Let Z be a closed subscheme of X . The point γ belongs to $\mathcal{L}_\infty(Z)$ if and only if $\gamma(\eta)$ belongs to Z .*

(3) Let U be an open subscheme of X . The point γ belongs to $\mathcal{L}_\infty(U)$ if and only if $\gamma(0)$ belongs to U .

Let us stress that assertion (3) is equivalent to the fact that $\mathcal{L}_\infty(U)$ identifies with the open subscheme $(\theta_{0,X}^\infty)^{-1}(U)$ of $\mathcal{L}_\infty(X)$.

2.2.14 Let k be a field and R be a k -algebra. The canonical quotient morphism $R[[t]] \rightarrow R$ admits a section $R \rightarrow R[[t]]$ mapping $a \mapsto a + 0 \cdot t + \dots$. Let X be a k -scheme. The preceding section induces a morphism of k -schemes $s_{\infty,X} : X \rightarrow \mathcal{L}_\infty(X)$ mapping a R -point $x \in X(R)$ to a constant R -arc $s_{\infty,X}(x)$. This morphism is the limit of the family of morphisms $(s_{m,X})_{m \in \mathbb{N}}$ (see subsection 2.1.7); for every $m \in \mathbb{N}$ we have $\theta_{m,X}^\infty \circ s_{\infty,X} = s_{m,X}$ and in particular $\theta_{0,X}^\infty \circ s_{\infty,X} = \text{Id}_X$ (via the identification of $\mathcal{L}_0(X)$ and X , see example 2.1.11). The arcs belonging to $s_{\infty,X}(X)$ are called the *constant arcs* of X . A proof of the following result can be found in [19, Ch.3, Proposition 3.5.2], it is a consequence of [36, Corollaire 5.4.6].

Proposition 2.2.15. *Let k be a field and X be a k -scheme. The morphism $s_{\infty,X}$ is a closed immersion.*

2.2.16 Let us now discuss about the reduced structure of the arc scheme.

Lemma 2.2.17. *Let k be a field and X be a k -scheme. If the k -scheme $\mathcal{L}_\infty(X)$ is reduced, then the k -scheme X is reduced.*

Proof. The k -scheme $\mathcal{L}_\infty(X)$ being reduced, the morphism $\theta_{0,X}^\infty : \mathcal{L}_\infty(X) \rightarrow X$ factors through X_{red} . The factorization

$$X \xrightarrow{s_{\infty,X}} \mathcal{L}_\infty(X) \xrightarrow{\theta_{0,X}^\infty} X$$

of Id_X implies that Id_X also factorizes through X_{red} and we deduce that $X = X_{\text{red}}$. \square

The converse is not true in general (see e.g., theorems 3.5.5 and 3.6.6), hence it is interesting to study the relation between the nilpotency in arc scheme and the geometry of the base scheme. Some facts are known in this sense. The first advance in this direction was made by M. Mustață, who proved in [63] that if X is an integral locally complete intersection variety with rational singularities defined over a field of characteristic zero, then $\mathcal{L}_\infty(X)$ is reduced. J. Sebag proved in [71, Theorem 1] and [72] that if X is an integral l.c.i curve defined over a field of characteristic zero, then $\mathcal{L}_\infty(X)$ is reduced if and only if X is smooth. More generally, he proved in [72, Theorem 1.6] that if X is an integral variety defined over a field of characteristic zero such that $\mathcal{L}_\infty(X)$ is reduced, then the \mathcal{O}_X -module Ω_X^1 of Kähler differentials is torsion-free (in particular for a l.c.i integral variety, $\mathcal{L}_\infty(X)$ reduced implies that X is normal). In chapter 3 we will describe some nilpotent elements of $\mathcal{L}_\infty(X)$ for X a reduced singular plane curve over a field of characteristic zero defined by a homogeneous or weighted-homogeneous polynomial.

2.2.18 When the base scheme is a smooth variety, the arc scheme has the following property. It corresponds to [19, Ch. 3, Proposition 3.7.5].

Proposition 2.2.19. *Let k be a field and V be a smooth k -variety of pure dimension d . For every $m \in \mathbb{N} \cup \{\infty\}$ and every $n \in \mathbb{N}$ with $m \geq n$, the truncation morphism $\theta_{n,V}^m : \mathcal{L}_m(V) \rightarrow \mathcal{L}_n(V)$ is a locally trivial fibration with fiber $\mathbb{A}_k^{(m-n)d}$ (the fiber is $(\mathbb{A}_k^{\mathbb{N}})^d$ when $m = \infty$).*

In particular, when k is a perfect field and U is an affine k -subvariety of V contained in V_{Reg} , we can apply the preceding result and deduce that, $\mathcal{O}(U)$ being an integral domain, $\mathcal{O}(\mathcal{L}_m(U))$ also is for every $m \in \mathbb{N} \cup \{\infty\}$.

2.3 DIFFERENTIAL ALGEBRA AND THE ARC SCHEME

Differential algebra was introduced by Joseph Ritt and Ellis Kolchin in order to study the differential equations in an algebraic way. The main references on this subject are [54] and [69]. In this section we show how differential algebra provides a description of the arc scheme of an affine scheme which is, in particular, computationally useful. The aim is to give some properties which we will use in the study of nilpotency in chapter 3, hence we only treat the case of algebras over a field.

2.3.1 Let k be a field. Let R be a k -algebra and M a R -module. A k -derivation from R to M is a morphism of k -modules $D : R \rightarrow M$ which satisfies the Leibniz rule, i.e., for $a, b \in R$ we have $D(ab) = aD(b) + bD(a)$. We denote by $\text{Der}_k(R, M)$ the set of k -derivations from R to M , which is naturally a R -module with multiplication defined by $aD : b \mapsto a(D(b)) \in M$ for $a \in R$ and $D \in \text{Der}_k(R, M)$. A k -derivation from R to itself is also called a k -derivation on R , the R -module formed by all of them is denoted by $\text{Der}_k(R)$. The pair (R, D) consisting of a k -algebra R endowed with a k -derivation D on R is a *differential k -algebra*.

Let (R, D) be a differential k -algebra. An ideal $I \subset R$ is a *differential ideal* if and only if $D(I) \subset I$. Let I be a differential ideal of (R, D) . The differential structure of (R, D) induces canonically a differential structure on the quotient k -algebra R/I such that, for $a \in R$, the derivative of $a + I$ is $D(a) + I$. Given $S \subset R$ we denote by $[S]$ the differential ideal generated by S . It is the smallest differential ideal of (R, D) containing S ; an explicit presentation is given by $\langle D^i(s); s \in S, i \in \mathbb{N} \rangle$, i.e., it is generated as an ideal of R by the elements in S and their derivatives of higher order. The radical of $[S]$ is denoted by $\{S\}$ and it is also a differential ideal of (R, D) if k is a field of characteristic 0 (or more generally if R is a Ritt algebra, see e.g., [52, Section 4]). As usually, if S only contains the element $a \in R$, we will simply write $[a]$ and $\{a\}$ instead of $[S]$ and $\{S\}$. An ideal I of R (differential or not) is called *radical* or *reduced* if $I = \sqrt{I}$.

2.3.2 Let $n \in \mathbb{N}$. For every $m \in \mathbb{N} \cup \{\infty\}$, we denote by A_m the polynomial ring $k[T_{i,j}; i \in \{1, \dots, n\}, j \in \{0, \dots, m\}]$ with the convention $\{0, \dots, \infty\} = \mathbb{N}$. We endow the k -algebra A_∞ with the k -derivation Δ defined by $\Delta(T_{i,j}) = T_{i,j+1}$, for every integer $i \in \{1, \dots, n\}$ and every integer $j \in \mathbb{N}$. The resulting differential k -algebra (A_∞, Δ) is denoted by $k\{T_1, \dots, T_n\}$ and is called the *differential polynomial ring*. The injective morphism of k -algebras $k[T_1, \dots, T_n] \rightarrow k\{T_1, \dots, T_n\}$, defined by $T_i \mapsto T_{i,0}$, identifies the polynomial ring $k[T_1, \dots, T_n]$ with A_0 and gives rise to a structure of $k[T_1, \dots, T_n]$ -algebra on $k\{T_1, \dots, T_n\}$. In particular, by a slight abuse of notation, we will not make any difference between the rings $k[T_1, \dots, T_n]$ and A_0 .

In the k -algebra A_∞ we say that the element $T_{i,j}$ has weight j ; this weight defines a grading over A_∞ in such a way that the term $\prod_{i=1}^n T_{i,j_i}^{a_i}$ has degree $\sum_{i=1}^n a_i j_i$, where $a_i \in \mathbb{N}$ for $i \in \{1, \dots, n\}$. The homogeneous elements for this grading are called *isobaric*. We observe that the k -derivation Δ is an isobaric (i.e., homogeneous for this grading) morphism of degree 1.

Let us mention the following useful and classical statement (which is a direct consequence of [54, I/§9/Lemma 6] in the particular case of a single equation):

Lemma 2.3.3. *Let k be a field of characteristic zero. Let I be a reduced ideal of A_0 . Let $P \in A_0$. Then, the polynomial P belongs to the ideal $\{I\}$ of A_∞ if and only if it belongs to the ideal I .*

Proof. Let $P \in \{I\}$, then there exists $m \in \mathbb{N}$ such that $P^m \in [I]$. Hence we can write $P^m = \sum_{r=1}^s a_r \Delta^{e_r}(f_r)$ for $s \in \mathbb{N}$, $f_r \in I$, $a_r \in A_\infty$ and $e_r \in \mathbb{N}$ for $1 \leq r \leq s$. The elements $\Delta^{e_r}(f_r)$ are isobaric of weight e_r and we can also suppose a_r isobaric for $1 \leq r \leq s$. The element P^m being isobaric of weight 0, we deduce that $\sum_{r=1}^s a_r \Delta^{e_r}(f_r)$ also is. Hence, up to renumbering, this element equals $\sum_{r=1}^{s'} a_r f_r$ for $s' \leq s$, which belongs to I . The ideal I being reduced, we deduce that $P \in I$. The converse is clear. \square

2.3.4 We can give another description of the arc scheme of an affine scheme, slightly different from that of subsection 2.2.9, using differential algebra.

Lemma 2.3.5. *Let k be a field. Let $n, r \in \mathbb{N} \setminus \{0\}$, in the k -algebra $k[T_1, \dots, T_n]$ we consider the ideal $I := \langle f_1, \dots, f_r \rangle$. Let $X = \text{Spec}(k[T_1, \dots, T_n]/I)$. Then*

$$\mathcal{L}_\infty(X) \cong \text{Spec}(k\{T_1, \dots, T_n\}/[I]).$$

For the proof of this lemma we need some technical results. First, let us recall the generalized Leibniz rule, which in particular holds for the differential k -algebra (A_∞, Δ) .

Lemma 2.3.6. *Let (R, D) be a differential k -algebra. Let f and g be two elements in R and let $i \in \mathbb{N}$ be an integer. Then*

$$\frac{D^i(fg)}{i!} = \sum_{k=0}^i \frac{D^k(f)D^{i-k}(g)}{k!(i-k)!}.$$

Proof. Observe that the preceding equality is equivalent to

$$D^i(fg) = \sum_{k=0}^i \binom{i}{k} D^k(f)D^{i-k}(g).$$

Let us prove it by induction on $i \in \mathbb{N}$. The equality above trivially holding for $i = 0$, assume that it also holds for some $i \in \mathbb{N}$. The following equalities, based on an application of the Leibniz rule, a redefinition of the counters and well-known properties of binomial coefficients, show that the equality also holds for $i + 1$, which concludes the proof.

$$\begin{aligned} D^{i+1}(fg) &= D^i(fD(g)) + D^i(D(f)g) \\ &= \sum_{k=0}^i \binom{i}{k} D^k(f)D^{i+1-k}(g) + \sum_{k=0}^i \binom{i}{k} D^{k+1}(f)D^{i-k}(g) \\ &= \sum_{k=0}^i \binom{i}{k} D^k(f)D^{i+1-k}(g) + \sum_{k=1}^{i+1} \binom{i}{k-1} D^k(f)D^{i+1-k}(g) \\ &= \sum_{k=1}^i \left(\binom{i}{k} + \binom{i}{k-1} \right) D^k(f)D^{i+1-k}(g) + fD^{i+1}(g) + D^{i+1}(f)g \\ &= \sum_{k=0}^{i+1} \binom{i+1}{k} D^k(f)D^{i+1-k}(g). \end{aligned}$$

□

Let us now define the following map:

$$\begin{aligned} \exp(\Delta t): A_\infty &\rightarrow A_\infty[[t]] \\ f &\mapsto \sum_{i \in \mathbb{N}} \frac{\Delta^i(f)t^i}{i!} \end{aligned}$$

Lemma 2.3.7. *The map $\exp(\Delta t)$ defined above is a morphism of k -algebras.*

Proof. The k -linearity is a direct consequence of the k -linearity of the derivation Δ . Let now f and g be two polynomials in A_∞ , by definition

$$\exp(\Delta t)(fg) = \sum_{i \in \mathbb{N}} \frac{\Delta^i(fg)t^i}{i!}.$$

By the generalized Leibniz rule (lemma 2.3.6) the last term equals

$$\sum_{i \in \mathbb{N}} \sum_{k=0}^i \frac{\Delta^k(f) \Delta^{i-k}(g)}{k!(i-k)!} t^i,$$

which equals $\exp(\Delta t)(f) \exp(\Delta t)(g)$. \square

We can now prove lemma 2.3.5.

Proof of lemma 2.3.5. By proposition 2.2.10 we know that

$$\mathcal{L}_\infty(X) \cong \text{Spec} \left(\frac{A_\infty}{\langle f_{q,s} : 1 \leq q \leq r, s \in \mathbb{N} \rangle} \right).$$

Let us fix $q \in \{1, \dots, r\}$, the map $\exp(\Delta t)$ being a morphism of k -algebras we deduce that $\exp(\Delta t)(f_q) = f_q(\exp(\Delta t)(T_1), \dots, \exp(\Delta t)(T_n))$. Via the identification of A with the subring A_0 of A_∞ , it equals

$$f_q \left(\sum_{i \in \mathbb{N}} \frac{T_{1,i}}{i!} t^i, \dots, \sum_{i \in \mathbb{N}} \frac{T_{n,i}}{i!} t^i \right).$$

By definition of the family $(f_{q,s})_{\substack{1 \leq q \leq r \\ s \in \mathbb{N}}}$ of elements of A_∞ (see subsection 2.2.9, in particular equation (2.1)), the preceding element of $A_\infty[[t]]$ equals

$$\sum_{s \in \mathbb{N}} f_{q,s} \left(\left(\frac{T_{e,i}}{i!} \right)_{\substack{1 \leq e \leq n \\ 0 \leq i \leq s}} \right) t^s.$$

On the other hand, by definition

$$\exp(\Delta t)(f_q) = \sum_{s \in \mathbb{N}} \frac{\Delta^s(f_q)}{s!} t^s,$$

and identifying the coefficients of t in both series we deduce that

$$\frac{\Delta^s(f_q)}{s!} = f_{q,s} \left(\left(\frac{T_{e,i}}{i!} \right)_{\substack{1 \leq e \leq n \\ 0 \leq i \leq s}} \right). \quad (3.1)$$

Hence we have found a change of variables inducing an automorphism of A_∞ which allows to recover $\Delta^s(f_q)$ from $f_{q,s}$ (and conversely) for $1 \leq q \leq r$ and $s \in \mathbb{N}$, i.e., via this change of variables the ideal $\langle f_{q,s} : 1 \leq q \leq r, s \in \mathbb{N} \rangle$ corresponds with the ideal $[I] = \langle \Delta^s(f_q); 1 \leq q \leq r, s \in \mathbb{N} \rangle$ of A_∞ and it induces an isomorphism between the presentation of $\mathcal{L}_\infty(X)$ in proposition 2.2.10 and the k -algebra in the statement of this lemma, which concludes the proof. \square

Differential algebra also provides a presentation of the jet schemes. The arguments in the proof of lemma 2.3.5 may be immediately adapted to prove the following lemma:

Lemma 2.3.8. *Let k be a field. Let $n, r \in \mathbb{N} \setminus \{0\}$, in the k -algebra $k[T_1, \dots, T_n]$ we consider the ideal I . Let $X = \text{Spec}(k[T_1, \dots, T_n]/I)$. Then*

$$\mathcal{L}_m(X) \cong \text{Spec}(A_m / \langle \Delta^s(f); f \in I, 0 \leq s \leq m \rangle).$$

Both the presentations of $\mathcal{L}_\infty(X)$ obtained in proposition 2.2.10 and lemma 2.3.5 are useful depending on the case. The first one is combinatorially nice and is thus useful when we aim to apply combinatorial techniques as in chapter 5. On the other hand, the second description provides a differential ring and the powerful techniques from differential algebra are applicable. This is the approach followed in chapter 3.

2.3.9 As a direct consequence of lemma 2.3.5, with the same notation we deduce the following presentation of the reduced arc scheme of X :

$$\mathcal{L}_\infty(X)_{\text{red}} \cong \text{Spec}(k\{T_1, \dots, T_n\}/\{I\}).$$

Hence results in differential algebra concerning radical differential ideals have direct implications in the reduced structure of the arc scheme. Let I be a reduced ideal of the ring A_0 . We have the following classical theorem of Kolchin (see [54, Ch. IV/§17/Proposition 10]):

Theorem 2.3.10 (Kolchin irreducibility theorem). *Let k be a field of characteristic zero. Let I be a prime ideal of the ring A_0 . Then the reduced differential ideal $\{I\}$ is also prime.*

As a consequence of this theorem, if I is a reduced ideal of the ring A_0 and $I = \bigcap_{j=1}^r I_j$ is a *prime decomposition* of I (i.e., the ideal I_j is prime for every integer $j \in \{1, \dots, r\}$ and homogeneous if the ideal I is homogeneous), then for every integer $j \in \{1, \dots, r\}$ the reduced differential ideal $\{I_j\}$ is prime and

$$\{I\} = \{I_1\} \cap \dots \cap \{I_r\}. \quad (3.2)$$

In subsection 2.4.5 we will discuss the implications of this result in the description of the irreducible components of the arc scheme.

2.3.11 Density statements in arc scheme (e.g., [65, Corollary 3.7]) provide presentations of the ideal $\{I\}$ (when the ideal I is assumed to be prime) which are very useful from a computational point of view. The following general formulation can be deduced from [54, Ch. IV/S17/Proposition 10] and the more general statement in [7, Proposition 3.3] (which is valid in arbitrary characteristic); we provide a direct proof for the convenience of the reader.

Lemma 2.3.12. *Let k be a field of characteristic zero. Let I be a prime ideal of the ring A_0 . For every $H \in A_0$ such that $\text{Sing}(\text{Spec}(A_0/I)) \subset \{H = 0\}$ and $H \notin I$, we have*

$$\{I\} = ([I] : H^\infty) \quad (3.3)$$

Proof. Let $P \in ([I] : H^\infty)$. There exists an integer N such that $H^N P \in \{I\}$. From theorem 2.3.10 we deduce that $\{I\}$ is a prime ideal and by lemma 2.3.3 $H^N \notin \{I\}$, then we conclude that $P \in \{I\}$.

Let us now prove the other inclusion. First, note that the kernel of the canonical morphism $A_\infty \rightarrow (A_\infty)_H/[I](A_\infty)_H$ equals $([I] : H^\infty)$ (since $f/1 \in [I](A_\infty)_H$ is equivalent to $H^N f \in [I]$ for some $N \in \mathbb{N}$). Hence it induces an injective morphism

$$A_\infty/([I] : H^\infty) \hookrightarrow (A_\infty/[I])_H. \quad (3.4)$$

According to the notation in section 1.1, let $\{H \neq 0\} = \text{Spec}(A_0) \setminus \{H = 0\}$. From (3.4) we deduce that $\mathcal{L}_\infty(\{H \neq 0\}) \subset \{([I] : H^\infty) = 0\}$, which implies, thanks to the irreducibility of $\mathcal{L}_\infty(\text{Spec}(A_0/I))$ (by corollary 2.4.6 and the fact that I is a prime ideal), that $\{([I] : H^\infty) = 0\} = \mathcal{L}_\infty(\text{Spec}(A_0/I))_{\text{red}}$. Hence, the radical of the ideal $([I] : H^\infty)$ coincides with the ideal $\{I\}$.

Since $\text{Sing}(\text{Spec}(A_0/I)) \subset \{H = 0\}$, we deduce that $\{H \neq 0\}$ is an open subscheme of $\text{Reg}(\text{Spec}(A_0/I))$ and by proposition 2.2.8 (1) $\mathcal{L}_\infty(\{H \neq 0\})$ also is an open subscheme of $\mathcal{L}_\infty(\text{Reg}(\text{Spec}(A_0/I)))$. By proposition 2.2.19 we conclude that the localization $(A_\infty/[I])_H$ by H of the ring $A_\infty/[I]$ is a domain. Then (3.4) implies that the ideal $([I] : H^\infty)$ is prime, hence reduced, which concludes the proof. \square

Formula (3.3), in the special case of systems of algebraic equations, can be linked to corresponding formulas of Lazard and to the Rosenfeld lemma (see [54, Ch. IV/§9/Lemma 2]) in the context of differential algebra. The proof of the Kolchin irreducibility theorem (see [54, Ch. IV/§17/Proposition 10]) in particular explains how to use these results in the differential setting in order to obtain statements analogous to lemma 2.3.12 in the algebraic framework. As a direct illustration, let us stress that, for every irreducible polynomial $f \in A_0$, [54, Ch. IV/§9/Lemma 2] directly implies that

$$\{f\} = ([f] : \partial(f)^\infty).$$

for every nonzero partial derivative $\partial(f)$ of f . This formula also is a particular form of lemma 2.3.12, and the way we will apply it in chapter 3.

2.3.13 Let k' be an algebraic closure of the field k . Let I be a prime ideal of A_0 . We observe that the ideal $\{I\} \otimes_k k'$ of the ring $A_\infty \otimes_k k'$ coincides with the radical of the differential ideal generated by the ideal I in the differential ring $A'_\infty := A_\infty \otimes_k k'$. Besides, for every polynomial $P \in A'_1 := A_1 \otimes_k k'$, one can check directly from the very definition that, if $(e_i)_{i \in I}$ is a basis of the k -vector space k' , then the polynomial $P = \sum_{i \in I} P_i e_i$ (with $P_i \in A_0$ for every $i \in I$) belongs to $(\{I\} \otimes_k k') \cap A'_1$ if and only if, for every $i \in I$, we have $P_i \in \{I\}$.

2.4 TOPOLOGICAL PROPERTIES OF THE ARC SCHEME

In this section we will include some results related with the topology of the arc scheme. In particular we deduce from Kolchin theorem 2.3.10 a result about the irreducible components of the arc scheme. In a second part, we introduce the constructible subsets of the arc scheme and give some properties which will be useful in the sequel.

2.4.1 Let us first present three topological results which can be found or deduced from [50]. The proof of the following lemma is contained in the proof of [50, Lemma 2.12].

Lemma 2.4.2 (Ishii-Kollár). *Let k be a field and V be a k -variety. Then every arc $\gamma \in \mathcal{L}_\infty(V)$ is a specialization of an arc $\varphi \in \mathcal{L}_\infty(V)$ whose base point is the generic point of γ , i.e., $\varphi(0) = \gamma(\eta)$.*

The following corollary is a consequence of this lemma, see [19, Ch.3, Corollary 4.2.3].

Corollary 2.4.3. *Let k be a field and V be a k -variety. Let U be an open subscheme of V and let $Z = V \setminus U$ be its complement. Then the k -scheme $\mathcal{L}_\infty(U)$ is dense in $\mathcal{L}_\infty(V) \setminus \mathcal{L}_\infty(Z)$.*

Finally let us state the following result, corresponding to [50, Lemma 2.12]. Let us stress that, in particular, it says that the arc scheme of a reduced variety V defined over a field of characteristic zero has no irreducible component contained in the closed subscheme $\mathcal{L}_\infty(V_{\text{Sing}})$.

Lemma 2.4.4. *Let k be a field of characteristic zero and V be a reduced k -variety. Then every point $\gamma \in \mathcal{L}_\infty(V)$ whose base point belongs to V_{Sing} is a specialization of an arc $\varphi \in \mathcal{L}_\infty(V)$ whose base point belongs to V_{Sing} and whose generic point does not belong to V_{Sing} (i.e., the arc φ does not belong to $\mathcal{L}_\infty(V_{\text{Sing}})$).*

2.4.5 Let us now present the geometric counterpart of subsection 2.3.9 and in particular of the Kolchin irreducibility theorem (theorem 2.3.10). Let k be a field of characteristic zero and let I be an ideal of the polynomial ring $k[T_1, \dots, T_n]$. Let $V = \text{Spec}(k[T_1, \dots, T_n]/I)$ be the associated k -variety. Recall that, as consequence of lemma 2.3.5, we can obtain the following presentation of the reduced closed subscheme of $\mathcal{L}_\infty(V)$:

$$\mathcal{L}_\infty(V)_{\text{red}} \cong \text{Spec}(k\{T_1, \dots, T_n\}/\{I\}).$$

Then, Kolchin irreducibility theorem 2.3.10 implies the following result in terms of the arc scheme (see [19, Ch. 3/§4/Theorem 4.3.4] for an alternative proof).

Corollary 2.4.6. *Let k be a field of characteristic zero and let V be a k -variety. Let $(V_i)_{i \in I}$ be the irreducible components of V . Then $(\mathcal{L}_\infty(V_i))_{i \in I}$ are the irreducible components of $\mathcal{L}_\infty(V)$. In particular, V is irreducible if and only if $\mathcal{L}_\infty(V)$ is irreducible.*

Remark 2.4.7. The assumption of the field to be of characteristic zero in Kolchin irreducibility theorem, as well as in its version in terms of the arc scheme stated in corollary 2.4.6, is crucial. For example, let k be a field of characteristic $p > 0$ and let us consider the polynomial $f = T_1^p + T_2^p T_3 \in k[T_1, T_2, T_3]$. It is an irreducible polynomial and hence the k -variety $X = \text{Spec}(k[T_1, T_2, T_3]/\langle f \rangle)$ is irreducible. Its reduced singular locus is the closed subset $X_{\text{Sing}} = V(T_1, T_2)$, we note X_{sm} its smooth locus. Then $\mathcal{L}_\infty(X_{\text{Sing}})$ and $\mathcal{L}_\infty(X_{\text{sm}})$ are the two irreducible components of $\mathcal{L}_\infty(X)$, hence it is not irreducible although X is. A detailed proof can be found in [19, Ch. 3/§4/Remark 4.3.5].

Let us stress that, in order to study topological questions, one can always assume the base variety to be reduced, as the following lemma (corresponding to [19, Ch.3, Lemma 3.4.4]) assures. This justifies that corollary 2.4.6 is indeed a direct consequence of the Kolchin irreducibility theorem 2.3.10 although the ideal in its statement is assumed to be reduced.

Lemma 2.4.8. *Let k be a field and X be a k -scheme. The closed immersion $X_{\text{red}} \rightarrow X$ induces an isomorphism $(\mathcal{L}_\infty(X_{\text{red}}))_{\text{red}} \rightarrow (\mathcal{L}_\infty(X))_{\text{red}}$. In particular, there is a homeomorphism $\mathcal{L}_\infty(X_{\text{red}}) \rightarrow \mathcal{L}_\infty(X)$.*

2.4.9 Let us recall some definitions and results about constructible subsets of a scheme. Let X be a topological space. A subset Z of X is said to be *retrocompact* in X if, for every quasi-compact open set U of X , $Z \cap U$ is quasi-compact. If X is a scheme, a subset Z of X is retrocompact in X if and only if, for every affine open subscheme U of X , $Z \cap U$ is quasi-compact.

The set X is retrocompact in X , and every closed subset of X also is. A finite union of retrocompact subsets of X is retrocompact in X (since a finite union of quasi-compact subsets is quasi-compact). A finite intersection of open retrocompact subsets of X is a retrocompact open subset in X . If X is a noetherian topological space, then every subset of X is retrocompact in X .

Let X be a topological space. A subset C of X is said to be *constructible* in X if it belongs to the smallest set of subsets of X containing all retrocompact open subsets of X and is stable under finite intersection and complements (and hence it is also stable under finite unions). By [37, Ch. 0, Proposition 9.1.3], C is a constructible subset of X if and only if there exist finite families (U_1, \dots, U_n) and (V_1, \dots, V_n) of retrocompact open subsets of X such that $C = \bigcup_{i=1}^n (U_i \cup (X \setminus V_i))$.

In the case of a noetherian scheme, the definition of a constructible subset is simpler. Let X be a noetherian scheme. A subset C of X is constructible in X if and only if it is a finite union of locally closed subsets of X . Let us stress that this is the case of the m -jet scheme $\mathcal{L}_m(V)$ of a k -variety V , for $m \in \mathbb{N}$. Thanks to the fact that $\mathcal{L}_\infty(V)$ is the projective limit of the m -jet schemes $\varprojlim \mathcal{L}_m(V)$ with the truncation morphisms, we can also describe the constructible subsets of $\mathcal{L}_\infty(V)$ in a simpler way. Parts (a) and (b) of

the following lemma can be found under this form in [19, Ch.3, Lemma 4.5.2], which is a particular case of [40, Théorème 8.3.11]. Part (c) corresponds to [70, Lemme 4.3.9].

Lemma 2.4.10. *Let k be a field and V a k -variety. The following assertions hold true:*

- (a) *For every integer $m \in \mathbb{N}$ and every constructible subset D of $\mathcal{L}_m(V)$, the subset $(\theta_{m,V}^\infty)^{-1}(D)$ of $\mathcal{L}_\infty(V)$ is constructible; it is closed (resp. open) if D is closed (resp. open).*
- (b) *For every constructible subset C of $\mathcal{L}_\infty(V)$, there exist an integer $m \in \mathbb{N}$ and a constructible subset C_m of $\mathcal{L}_m(V)$ such that $C = (\theta_{m,V}^\infty)^{-1}(C_m)$. Moreover, if C is closed (resp. open), then C_m can be taken to be closed (resp. open).*
- (c) *For every constructible subset C of $\mathcal{L}_\infty(V)$, for every $m \in \mathbb{N}$, the subset $\theta_{m,V}^\infty(C)$ of $\mathcal{L}_m(V)$ is constructible.*

This lemma implies in particular that constructible subsets in arc scheme coincide with the so-called *cylinders* in motivic integration.

Remark 2.4.11. In [41] what we have defined to be a constructible subset of a topological space X is called a globally constructible subset of X . There, a subset C of a scheme X is defined to be constructible if every point of X is contained in an open subset U of X such that $C \cap U$ is globally constructible in U (if X is a scheme we ask U to be an affine open subscheme). Then every globally constructible subset of X is a constructible subset. In fact, for noetherian schemes the converse also holds, and this property can be extended to projective limits of noetherian schemes; see [19, Appendix, Proposition 1.3]. Hence we deduce that both notions coincide for the arc scheme of a k -variety.

2.5 VALUATIONS AND THE ARC SCHEME

2.5.1 Let k be a field, $k \subset K$ a field extension and Γ a totally ordered abelian group. We endow $\Gamma \cup \{\infty\}$ with a structure of totally ordered abelian group by fixing $d < \infty$ for every $d \in \Gamma$ and $\infty + d = \infty + \infty = \infty$. Let us recall that a *valuation* on K with values in Γ is a map $\nu : K \rightarrow \Gamma \cup \{\infty\}$ such that

- (1) $\nu(ab) = \nu(a) + \nu(b)$ for every $a, b \in K$.
- (2) $\nu(a + b) \geq \min(\nu(a), \nu(b))$ for every $a, b \in K$.
- (3) $\nu(a) = \infty$ if and only if $a = 0$.

When Γ is isomorphic to \mathbb{Z} the valuation ν is said to be a *discrete valuation*.

An integral domain R contained in K is a *valuation ring* of K if, for every $a \in K$, either $a \in R$ or $a^{-1} \in R$. Given a valuation ν on K , we denote by $R_\nu \subset K$ the associated

valuation ring, i.e., the set of elements $x \in K$ such that $\nu(x) \geq 0$. It is a local ring of maximal ideal $\{x \in K : \nu(x) > 0\}$. Conversely, given a valuation ring R of K we can construct a valuation ν of K such that $R_\nu = R$. See [77, Proposition 2.3] for this construction. Let us remark that two valuations of K can have the same ring, in this case we say that they are equivalent (and in fact they are equal up to composition with an isomorphism of ordered groups, see [77, Proposition 2.4]). Let us observe that, the unique automorphism of ordered groups in \mathbb{Z} being the identity, one can associate with a discrete valuation ring a unique valuation with values in \mathbb{Z} , which is called its *normalized valuation*.

2.5.2 Let X be an integral separated k -scheme and let ν be a valuation on the field $k(X)$. We say that ν is *centered* on X if there exists a point $x \in X$ such that the local ring $(\mathcal{O}_{X,x}, \mathfrak{m}_x)$ is contained in R_ν (as a subring, note that both can be seen as subrings of $k(X)$). If ν is centered on X then the set of points $x \in X$ such that $\nu(a) > 0$ for every $a \in \mathfrak{m}_x$ is a nonempty irreducible closed subset of X whose generic point is called the *center* of ν in X and denoted by $c_X(\nu)$.

Let X, Y be two integral separated k -schemes and $\varphi : Y \rightarrow X$ be a birational morphism of k -schemes; it induces an isomorphism $k(X) \cong k(Y)$. Let ν be a valuation on the field $k(Y)$, hence it can be seen also as a valuation on $k(X)$ via the induced isomorphism. If ν is centered on Y , then it is centered on X and $c_X(\nu) = \varphi(c_Y(\nu))$. Moreover, if φ is assumed to be proper, the valuative criterion of properness implies that, if ν is centered on X , then ν is centered on Y as well. See e.g., [77, Section 7] for more details.

2.5.3 Let k be a field and V' be an integral separated k -variety with field of functions $k(V')$. Let E be a *prime divisor* in V' , that is, an integral closed subscheme of V' of codimension 1. Assume that V' is normal (it suffices to assume that the generic point of E is normal). Then the local ring $\mathcal{O}_{V',E}$ (defined as the localization of $\mathcal{O}_{V'}$ at the generic point of E) is a one-dimensional normal noetherian local ring, and by [60, Theorem 11.2] it is a discrete valuation ring. Let us consider the order function $\text{ord}_E : k(V')^* \rightarrow \mathbb{Z}$ which associates with a nonzero element $f \in k(V')$ its order of vanishing $\text{ord}_E(f)$ at the generic point of E . Then ord_E is the extension of the normalized valuation of $\mathcal{O}_{V',E}$ to its field of fractions $k(V')$. This valuation is centered on V' and its center $c_{V'}(\text{ord}_E)$ is the generic point of E .

Now, let V be an integral separated k -variety and $\varphi : V' \rightarrow V$ be a proper birational morphism, it induces an isomorphism of fields $k(V) \cong k(V')$. Then we say that the prime divisor E in V' is a *divisor over V* . The valuation ord_E on $k(V')$ induces a valuation on $k(V)$ which is also denoted by ord_E . By subsection 2.5.2, this valuation is centered on V and its center is the generic point of $\varphi(E)$.

A valuation ν on $k(V)$ with values in \mathbb{Z} which is of the form $q\text{ord}_E$ for a triple (V', φ, E) as above and an integer $q \geq 1$ is called a *divisorial valuation* on V and we say that it is centered on V' . The integer q is called the *index* of ν and denoted by q_ν .

2.5.4 Up to the end of the section we will assume that k is a field of characteristic zero since the original references which we will cite work in this setting, although most of the results we will present are still true for perfect fields. Let V be an integral separated k -variety. Let $\gamma \in \mathcal{L}_\infty(V)$ be an arc on V with residue field $\kappa(\gamma)$, let us see it as a morphism $\gamma : \text{Spec}(\kappa(\gamma)[[t]]) \rightarrow V$. According to [47], γ is said to be *thin* if its image is contained in a strict closed subscheme of V . If γ is not thin, then it is called a *fat arc*. The following equivalent properties are characterizations of fat arcs (see [47, Proposition 2.5]):

- (i) The arc $\gamma \in \mathcal{L}_\infty(V)$ is not thin.
- (ii) The morphism $\gamma : \text{Spec}(\kappa(\gamma)[[t]]) \rightarrow V$ corresponding to γ is dominant.
- (iii) The morphism $\gamma : \text{Spec}(\kappa(\gamma)[[t]]) \rightarrow V$ maps the generic point η of $\text{Spec}(\kappa(\gamma)[[t]])$ to the generic point of V .
- (iv) The corresponding morphism of k -algebras $\gamma^* : \mathcal{O}(V) \rightarrow \kappa(\gamma)[[t]]$ is injective.
- (v) The induced morphism of local k -algebras $\gamma^* : \mathcal{O}_{V,\gamma(0)} \rightarrow \kappa(\gamma)[[t]]$ is injective.

Analogously, in [31] a subset C of $\mathcal{L}_\infty(V)$ was defined to be *thin* if there is a proper closed subvariety Z of V such that $C \subset \mathcal{L}_\infty(Z)$. An irreducible closed subset of $\mathcal{L}_\infty(V)$ which is not thin is called a *fat subset*.

Let $\gamma \in \mathcal{L}_\infty(V)$ be a fat arc. The induced morphism of local rings $\gamma^* : \mathcal{O}_{V,\gamma(0)} \rightarrow \kappa(\gamma)[[t]]$ being injective (characterization (v)), γ^* can be extended to an injective morphism of fields $\gamma^* : k(V) \rightarrow \kappa(\gamma)((t))$, where $k(V)$ is the field of functions of V . We can compose it with the t -adic valuation $\text{ord}_t : \kappa(\gamma)((t)) \rightarrow \mathbb{Z} \cup \{\infty\}$ to obtain a discrete valuation on $k(V)$:

$$\begin{aligned} \text{ord}_\gamma : k(V) &\longrightarrow \mathbb{Z} \cup \{\infty\} \\ f &\longmapsto \text{ord}_t(\gamma^*(f)). \end{aligned}$$

Observe that the morphism $\gamma : \text{Spec}(\kappa(\gamma)[[t]]) \rightarrow V$ maps the closed point of $\text{Spec}(\kappa(\gamma)[[t]])$ to the center of the valuation ord_γ , which coincides with the center of the arc, $\gamma(0)$. If C is an irreducible and fat closed subset of $\mathcal{L}_\infty(V)$ then its generic point $\gamma \in \mathcal{L}_\infty(V)$ is a fat arc, and we also denote by ord_C the valuation ord_γ on $k(V)$. Note that, for every $f \in \mathcal{O}_V$, we have $\text{ord}_C(f) = \inf \{\text{ord}_\gamma(f) : \gamma \in C \text{ is a fat arc}\}$ and indeed, for every $f \in k(V)$, we have $\text{ord}_C(f) = \inf \{\text{ord}_\gamma(f) : \gamma \in C \text{ is a fat arc}\}$.

Example 2.5.5. Let V be a smooth separated k -variety and E be a prime divisor in V . Then $(\theta_{0,V}^\infty)^{-1}(E)$ is an irreducible constructible subset of $\mathcal{L}_\infty(V)$, let γ be its generic point. Then ord_γ coincides with the divisorial valuation ord_E on $k(V)$.

Let ν be a discrete valuation on $k(V)$. In [49, Definition 2.8], S. Ishii defined the closed subset $W_V(\nu)$ as the closure of the set of fat arcs $\gamma \in \mathcal{L}_\infty(V)$ such that $\text{ord}_\gamma = \nu$. If ν is not centered on V then $W_V(\nu) = \emptyset$. Moreover, $\theta_{0,V}^\infty(W_V(\nu)) \subset \{c_V(\nu)\}$. The

following proposition corresponds to [47, Proposition 2.11] and shows that for a divisorial valuation ν the set $W_V(\nu)$ is non-empty:

Proposition 2.5.6 (Ishii). *Let k be a field of characteristic zero and V be an integral separated k -variety. For every divisorial valuation ν on V there exists a fat arc $\gamma \in \mathcal{L}_\infty(V)$ of V such that $\nu = \text{ord}_\gamma$.*

The next result, corresponding to [49, Proposition 2.9], complements proposition 2.5.6.

Proposition 2.5.7 (Ishii). *Let k be a field of characteristic zero and V and V' be integral separated k -varieties. Let $\varphi : V' \rightarrow V$ be a proper birational morphism and let ν be a divisorial valuation on V centered on V' . Then $W_V(\nu) = \overline{\mathcal{L}_\infty(\varphi)(W_{V'}(\nu))}$.*

Let us remark that in the preceding proposition the assumption on φ to be proper may be omitted, a proof in this case can be found in [19, Ch.7, Lemma 2.2.6]. Let us remark that, for ν a divisorial valuation on V , the set $W_V(\nu)$ is called in [49] a *maximal divisorial set*. It has the property to be maximal for the inclusion among all the irreducible closed fat subsets C of $\mathcal{L}_\infty(V)$ such that $\text{ord}_C = \nu$. By its relation with the Nash problem (see the Introduction) we also call it a *Nash set*, this terminology will be used in chapter 5.

2.5.8 Now we are going to present the notion of contact loci in arc scheme, first studied in [31]. They correspond with the loci of arcs on a variety defined by order of contact with a fixed subscheme. Precisely, let k be a field and V be an integral separated k -variety. Let \mathfrak{a} be an ideal sheaf of \mathcal{O}_V and $q \in \mathbb{N}$, we define the *contact loci* of \mathfrak{a} as the subsets

$$\text{Cont}^{\geq q}(V, \mathfrak{a}) = \{\gamma \in \mathcal{L}_\infty(V) : \text{ord}_\gamma(\mathfrak{a}) \geq q\}$$

and

$$\text{Cont}^q(V, \mathfrak{a}) = \{\gamma \in \mathcal{L}_\infty(V) : \text{ord}_\gamma(\mathfrak{a}) = q\}.$$

Both are constructible subsets of $\mathcal{L}_\infty(V)$ (see [31]), the subset $\text{Cont}^{\geq q}(V, \mathfrak{a})$ is closed and $\text{Cont}^q(V, \mathfrak{a})$ is locally closed in $\mathcal{L}_\infty(V)$ (and an open constructible subset of $\text{Cont}^{\geq q}(V, \mathfrak{a})$, since $\text{Cont}^q(V, \mathfrak{a}) = \text{Cont}^{\geq q}(V, \mathfrak{a}) \setminus \text{Cont}^{\geq q+1}(V, \mathfrak{a})$). If Y is the closed subscheme of V defined by \mathfrak{a} , we define $\text{Cont}^{\geq q}(V, Y)$ and $\text{Cont}^q(V, Y)$ respectively as $\text{Cont}^{\geq q}(V, \mathfrak{a})$ and $\text{Cont}^q(V, \mathfrak{a})$. We have the following relation with divisorial valuations.

Proposition 2.5.9 (Ishii). *Let k be a field of characteristic zero and V, V' be integral separated k -varieties such that V' is smooth. Let $\varphi : V' \rightarrow V$ be a proper birational morphism and E be a prime divisor in V' . Let q be an integer such that $q \geq 1$ and $\nu = \text{qord}_E$ be the divisorial valuation on V centered on V' . Then*

$$W_V(\nu) = \overline{\mathcal{L}_\infty(\varphi)(\text{Cont}^{\geq q}(V', E))}.$$

In particular, $W_V(\nu)$ is irreducible.

Sketch of the proof. This proposition corresponds to [49, Proposition 3.4] and the proof consists in showing that $W_{V'}(\nu) = \overline{\text{Cont}^{\geq q}(V', E)}$ and then applying proposition 2.5.7. \square

The following proposition is a generalization to the singular case of [31, Corollary 2.6]. It corresponds to [27, Proposition 2.12].

Proposition 2.5.10 (de Fernex-Ein-Ishii). *Let k be a field of characteristic zero and V be an integral separated affine k -variety. Let \mathfrak{a} be a non-zero ideal of \mathcal{O}_V . Let $q \in \mathbb{N}$. Then every fat irreducible component of $\text{Cont}^{\geq q}(V, \mathfrak{a})$ is a maximal divisorial set, that is, of the form $W_V(\nu)$ for ν a divisorial valuation on V .*

In fact, for a divisorial valuation ν on an affine k -variety V , an ideal \mathfrak{a} of \mathcal{O}_V and an integer $q \geq 1$, the inclusion $W_V(\nu) \subset \text{Cont}^{\geq q}(V, \mathfrak{a})$ is equivalent to $\nu(\mathfrak{a}) \geq q$.

ON THE TANGENT SPACE OF A WEIGHTED HOMOGENEOUS PLANE CURVE SINGULARITY

Let k be a field of characteristic 0. Let $\mathcal{C} = \text{Spec}(k[x, y]/\langle f \rangle)$ be a weighted homogeneous plane curve singularity with tangent space $\pi_{\mathcal{C}}: T_{\mathcal{C}/k} \rightarrow \mathcal{C}$. The Zariski closure $\mathcal{G}(\mathcal{C})$ of the set of the 1-jets on \mathcal{C} which define formal solutions (in $K[[t]]^2$ for field extensions K of k) of the equation $f = 0$ is an irreducible component of $T_{\mathcal{C}/k}$, which we call its general component, by analogy with the theory of differential equations. We denote by $\mathcal{N}_1(\mathcal{C})$ the ideal defining $\mathcal{G}(\mathcal{C})$ as a reduced closed subscheme of $T_{\mathcal{C}/k}$. In this chapter, we study from a computational point of view the ideal $\mathcal{N}_1(\mathcal{C})$.

The elements of the ideal $\mathcal{N}_1(\mathcal{C})$ define the nilpotent functions on the arc scheme $\mathcal{L}_{\infty}(\mathcal{C})$ associated with \mathcal{C} which live on $\mathcal{L}_1(\mathcal{C})$. The relation of arc and jets schemes with differential algebra presented in section 2.3 provides important tools to study these nilpotency phenomena and hence the general component of the tangent space. This is the content of section 3.2.

In section 3.3 we present some technical results of combinatorial nature which will be useful in the following sections. In section 3.4 we obtain, using differential algebra, some properties about $\mathcal{N}_1(\mathcal{C})$ in the particular case of a curve \mathcal{C} defined by a homogeneous polynomial. In section 3.5 we obtain Groebner bases of $\mathcal{N}_1(\mathcal{C})$ for a curve \mathcal{C} defined by a homogeneous polynomial and in section 3.6 we do the same thing for weighted homogeneous polynomials using the results for the homogeneous case.

Finally, elements in $\mathcal{N}_1(\mathcal{C})$ are related with differential operators, as shown in [73]. In section 3.7 we obtain applications of the results in this chapter in terms of differential operators.

Most of the contents in the present chapter have been obtained in collaboration with J. Sebag and are included in the article [61].

3.1 CONVENTIONS ON POLYNOMIALS

Let us recall from subsection 2.3.2 that, for $n \in \mathbb{N}$ and $m \in \mathbb{N} \cup \{\infty\}$, we denote by A_m the polynomial ring $k[t_{i,j}; i \in \{1, \dots, n\}, j \in \{0, \dots, m\}]$ with the convention $\{0, \dots, \infty\} = \mathbb{N}$. We denote by B_0 (resp. B_1) the polynomial ring $k[x_0, y_0]$ (resp. $k[x_0, y_0, x_1, y_1]$), which may be identified with A_0 (resp. A_1) for $n = 2$.

3.1.1 On the polynomial ring A_1 (or B_1), we will use various graded structures associated with various *degree functions*.

1. The *total degree* $\deg := \deg_{\text{tot}}$ of the polynomial ring A_1 ; for this function, the monomial $M = t_{1,0}^{a_1} \dots t_{n,0}^{a_n} t_{1,1}^{b_1} \dots t_{n,1}^{b_n}$ of A_1 is of degree $\deg_{\text{tot}}(M) = \sum_{i=1}^n (a_i + b_i)$.
2. The *partial degree* \deg_0 of the polynomial ring A_1 , where every polynomial is seen as a polynomial in the variables $t_{i,0}$ with coefficients in the ring $k[t_{i,1}; i \in \{1, \dots, n\}]$; for this function, the monomial M is of degree $\deg_0(M) = \sum_{i=1}^n a_i$.
3. The *partial degree* \deg_1 of the polynomial ring A_1 , where every polynomial is seen as a polynomial in the variables $t_{i,1}$ with coefficients in the ring $k[t_{i,0}; i \in \{1, \dots, n\}]$; for this function, the monomial M is of degree $\deg_1(M) = \sum_{i=1}^n b_i$. For this grading, a homogeneous polynomial P is said to be *1-homogeneous* of *1-degree* $\deg_1(P)$.

We say that a polynomial $P \in A_1$ is *bi-homogeneous* if the polynomial P is *simultaneously* a homogeneous polynomial for the graded structure induced by (2) and that induced by (3). Equivalently, the polynomial P is bi-homogeneous if and only if there exist two integers e, d such that one has $\deg_0(T) = e$ and $\deg_1(T) = d$ for every nonzero term T of P . The pair (d, e) is the *bi-degree* of P . Let us stress that, in particular, the polynomial P then is homogeneous for the graded structure induced by (1) (but, obviously, the converse does not hold, consider for example $P = t_{1,0} + t_{1,1}$).

3.1.2 Let k be a field of characteristic zero. Let $f \in B_0$ be a polynomial. We say that f is *weighted homogeneous* of weight (w_1, w_2, w) if we have the formula

$$f(t^{w_1}x_0, t^{w_2}y_0) = t^w f(x_0, y_0)$$

in the polynomial ring $B_0[t]$. We recall the following usual characterizations of weighted homogeneous polynomials.

Proposition 3.1.3. *Let k be a field of characteristic zero. Let $f \in B_0 \setminus k$ be a reduced polynomial. The following assertions are equivalent:*

1. The polynomial f is weighted homogeneous of weight (w_1, w_2, w) ;

2. Every monomial $x_0^i y_0^j$ of f satisfies $iw_1 + jw_2 = w$;

We assume that the field k is algebraically closed. The former assertions are equivalent to the following one:

3. There exist a k -automorphism σ of the ring B_0 , an integer $n \geq 1$, a pair of coprime integers (r, s) , with $r \geq s$, and $\lambda_1, \dots, \lambda_n \in k^*$ such that $\sigma(f) = x_0^\varepsilon y_0^{\varepsilon'} \prod_{i=1}^n (x_0^r - \lambda_i y_0^s)$ with $\varepsilon, \varepsilon' \in \{0, 1\}$.

Proof. Equivalence (1) \Leftrightarrow (2) is clear. Equivalence (2) \Leftrightarrow (3) is proved in [18, Lemmas 1,2,3]. \square

3.2 THE GENERAL COMPONENT OF THE TANGENT SPACE

As already mentioned in the introduction, the object of study of the present chapter is the general component of the tangent space of an affine plane curve. In this section, we will see that the tangent space of a k -variety is isomorphic to its 1-jet scheme and we will show how to take advantage of the relation between differential algebra and jet schemes in order to study the general component.

3.2.1 Let k be a field of characteristic zero. Let V be an affine k -variety and $\Omega_{V/k}^1$ the module of Kähler differentials of V , which is the unique $\mathcal{O}(V)$ -module representing the functor from the category of $\mathcal{O}(V)$ -modules to that of sets which sends an $\mathcal{O}(V)$ -module M to $\text{Der}_k(\mathcal{O}(V), M)$ (i.e., $\text{Hom}_{\text{Mod}_{\mathcal{O}(V)}}(\Omega_{V/k}^1, M) \cong \text{Der}_k(\mathcal{O}(V), M)$). We define the tangent space $T_{V/k}$ of V as $\text{Spec}(\text{Sym}(\Omega_{V/k}^1))$. Although we will work in the affine case, let us stress that this construction may be extended in the global case, the cotangent sheaf being the generalization of the module of Kähler differentials and the tangent sheaf being its dual.

Lemma 3.2.2. *Let k be a field and V a k -scheme. Then the tangent space $T_{V/k}$ of V is isomorphic as a k -scheme to $\mathcal{L}_1(V)$.*

Proof. We can assume that V is affine. Let C be a k -algebra such that $V = \text{Spec}(C)$. We will show that there is a bijection between the functor $\mathcal{L}_1(V)$ defined in subsection 2.1.1 and the functor of points of $\text{Spec}(\text{Sym}(\Omega_{V/k}^1))$, we can restrict to the category of affine k -schemes. Let R be a k -algebra, then it suffices to find a bijection between $\text{Hom}_{\text{Sch}_k}(\text{Spec}(R), \text{Spec}(\text{Sym}(\Omega_{V/k}^1)))$ and $\text{Hom}_{\text{Sch}_k}(\text{Spec}(R[t]/\langle t^2 \rangle), V)$ functorial in R .

On the one hand, a morphism in $\text{Hom}_{\text{Sch}_k}(\text{Spec}(R), \text{Spec}(\text{Sym}(\Omega_{V/k}^1)))$ corresponds with a morphism in $\text{Hom}_{\text{Alg}_k}(\text{Sym}(\Omega_{V/k}^1), R)$. By the universal property of symmetric algebra, it corresponds with the datum of a morphism of k -algebras $f : C \rightarrow R$ and

an element in $\text{Hom}_{\text{Mod}_C}(\Omega_{V/k}^1, R)$ which defines, by definition of $\Omega_{V/k}^1$, a derivation $D : C \rightarrow R$.

On the other hand, a morphism in $\text{Hom}_{\text{Sch}_k}(\text{Spec}(R[t]/\langle t^2 \rangle), V)$ corresponds with a morphism γ in $\text{Hom}_{\text{Alg}_k}(C, R[t]/\langle t^2 \rangle)$ which is determined, for $c \in C$, by $\gamma(c) = c_0 + c_1 t$. For $c, c' \in C$ we have $\gamma(c + c') = \gamma(c) + \gamma(c') = c_0 + c'_0 + (c_1 + c'_1)t$ and $\gamma(cc') = c_0 c'_0 + (c_0 c'_1 + c'_0 c_1)t$. If we fix $\gamma_0, \gamma_1 : C \rightarrow R$ defined by $\gamma_0(c) = c_0$ and $\gamma_1(c) = c_1$ for every $c \in C$, we deduce that γ_0 is a morphism of k -algebras and γ_1 is a k -derivation.

Hence the datum of (f, D) defines a morphism of k -algebras

$$\begin{aligned} C &\longrightarrow R[t]/\langle t^2 \rangle \\ c &\longmapsto f(b) + D(b)t \end{aligned}$$

which furnishes a bijection, functorial in R , between $\text{Hom}_{\text{Sch}_k}(\text{Spec}(R), \text{Spec}(\text{Sym}(\Omega_{V/k}^1)))$ and $\text{Hom}_{\text{Sch}_k}(\text{Spec}(R[t]/\langle t^2 \rangle), V)$. \square

From now on we identify $T_{V/k}$ and $\mathcal{L}_1(V)$ via the isomorphism in lemma 3.2.2. Then the truncation morphisms induce canonical morphisms $\theta_{0,V}^1 : T_{V/k} \rightarrow V$ and $\theta_{1,V}^\infty : \mathcal{L}_\infty(V) \rightarrow T_{V/k}$. We have the following decomposition:

$$(T_{V/k})_{\text{red}} = \overline{(\theta_{0,V}^1)^{-1}(V_{\text{Reg}})} \cup (\theta_{0,V}^1)^{-1}(V_{\text{Sing}}). \quad (2.1)$$

By lemma 2.4.4 one knows that the open subscheme $\mathcal{L}_\infty(V) \setminus \mathcal{L}_\infty(V_{\text{Sing}})$ is dense in $\mathcal{L}_\infty(V)$. If V is assumed to be irreducible, we observe that the closed subset $\overline{\theta_{1,V}^\infty(\mathcal{L}_\infty(V))}$ of $\mathcal{L}_1(V)$ is irreducible (since $\mathcal{L}_\infty(V)$ is by corollary 2.4.6) and contains the open subset $(\theta_{0,V}^1)^{-1}(V_{\text{Reg}}) = \mathcal{L}_1(V_{\text{Reg}})$ by proposition 2.2.19. On the other hand, $(\theta_{0,V}^1)^{-1}(V_{\text{Reg}})$ also is an irreducible component of $T_{V/k}$. Hence

$$\mathcal{G}(V) := \overline{\theta_{1,V}^\infty(\mathcal{L}_\infty(V))} = \overline{(\theta_{0,V}^1)^{-1}(V_{\text{Reg}})} \quad (2.2)$$

is defined to be the *general component* of $T_{V/k}$. If $K \supset k$ is a field extension, the K -points of $\mathcal{G}(V)$ hence correspond to elements of $T_{V/k}(K)$, i.e., K -valued 1-jets, which are Zariski closed to 1-jets with regular base-point.

3.2.3 If V is assumed to be affine with $\mathcal{O}(V) = A_0/I$, we denote by $\mathcal{N}_1(V)$ the unique ideal of $\mathcal{O}(T_{V/k})$ such that $\mathcal{O}(\mathcal{G}(V)) = \mathcal{O}(T_{V/k})/\mathcal{N}_1(V)$. By lemmas 2.3.8 and 3.2.1, the ideal $\mathcal{N}_1(V)$ has a unique preimage in the ring A_1 which contains the ideal $\langle I, \Delta(I) \rangle$. In a slight abuse of notation, we also denote this preimage by $\mathcal{N}_1(V)$. Let us also assume V to be integral. By formula (2.2) we have

$$\mathcal{N}_1(V) := \sqrt{([I] \cap A_1) / \langle I, \Delta(I) \rangle}.$$

We check that $\sqrt{[I] \cap A_1} = \sqrt{[I]} \cap A_1 = \{I\} \cap A_1$ (since $f \in \sqrt{[I]} \cap A_1$ if and only if there exists $n \in \mathbb{N}$ such that $f^n \in [I] \cap A_1$ if and only if $f \in \sqrt{[I] \cap A_1}$). Hence by lemma 2.3.12, for every $H \in A_0$ such that $V_{\text{Sing}} \subset \{H = 0\}$ and $H \notin I$ we have

$$\mathcal{N}_1(V) = (\{I\} \cap A_1) / \langle I, \Delta(I) \rangle = (([I] : H^\infty) \cap A_1) / \langle I, \Delta(I) \rangle. \quad (2.3)$$

From this formula and lemma 2.3.5 we observe that the elements of the ideal $\mathcal{N}_1(V)$ define the nilpotent functions on the arc scheme $\mathcal{L}_\infty(V)$ which live in $\mathcal{L}_1(V)$.

Besides, we observe that $\langle I, \Delta(I) \rangle \subset (\langle I, \Delta(I) \rangle : H^\infty) \subset \mathcal{N}_1(V)$. Moreover, the kernel of the canonical morphism $A_1 \longrightarrow (A_1)_H / \langle I, \Delta(I) \rangle (A_1)_H$ equals the ideal $(\langle I, \Delta(I) \rangle : H^\infty)$ of A_1 (since $f/1 \in \langle I, \Delta(I) \rangle (A_1)_H$ is equivalent to $H^n f \in \langle I, \Delta(I) \rangle$ for some $n \in \mathbb{N}$). Hence it induces an injective morphism

$$A_1 / (\langle I, \Delta(I) \rangle : H^\infty) \hookrightarrow (A_1 / \langle I, \Delta(I) \rangle)_H. \quad (2.4)$$

Since $V_{\text{Sing}} \subset \{H = 0\}$ we deduce that $\{H \neq 0\} \subset V_{\text{Reg}}$ and hence by proposition 2.1.4 (1) $\mathcal{L}_1(\{H \neq 0\}) \cong \text{Spec}((A_1 / \langle I, \Delta(I) \rangle)_H)$ is also an open (reduced) subscheme of $\mathcal{L}_1(V_{\text{Reg}}) = (\theta_{0,V}^1)^{-1}(V_{\text{Reg}})$, thus its closure equals the general component $\mathcal{G}(V)$ by (2.2).

On the other hand, by proposition 2.2.19 we have that the localization $(A_1 / \langle I, \Delta(I) \rangle)_H$ by H of the ring $A_1 / \langle I, \Delta(I) \rangle$ is a domain. Then (2.4) implies that $A_1 / (\langle I, \Delta(I) \rangle : H^\infty)$ is also a domain and hence $(\langle I, \Delta(I) \rangle : H^\infty)$ is a prime ideal of A_1 and the irreducible closed reduced subscheme $\text{Spec}(A_1 / (\langle I, \Delta(I) \rangle : H^\infty))$ of $\mathcal{L}_1(V)$ contains $\mathcal{L}_1(\{H \neq 0\})$, which shows that $\text{Spec}(A_1 / (\langle I, \Delta(I) \rangle : H^\infty)) = \mathcal{G}(V)$ (all the involved subschemes are reduced) and

$$\mathcal{N}_1(V) = (\langle I, \Delta(I) \rangle : H^\infty). \quad (2.5)$$

3.2.4 The previous remarks, together with the consequence of theorem 2.3.10 given in equation (3.2) in subsection 2.3.9, provide the following observation:

Observation 3.2.5. *Let k be a field of characteristic zero. Let $r \in \mathbb{N}^*$. Let I be a reduced ideal of A_0 with $(I_j)_{j \in \{1, \dots, r\}}$ as prime components. We set $V = \text{Spec}(A_0/I)$ and $V_j = \text{Spec}(A_0/I_j)$ for every integer $j \in \{1, \dots, r\}$. Let $P \in A_1$. The following assertions are equivalent:*

1. *The polynomial P belongs to the ideal $\mathcal{N}_1(V)$;*
2. *The polynomial P belongs to the ideal $\bigcap_{j=1}^r (\{I_j\} \cap A_1)$;*
3. *For every integer $j \in \{1, \dots, r\}$, the polynomial P belongs to the ideal $\mathcal{N}_1(V_j) = ([I_j] : H_j^\infty) = (\langle I_j, \Delta(I_j) \rangle : H_j^\infty)$ for every H_j satisfying the assumption of lemma 2.3.12.*

In particular, if there exists an irreducible polynomial $f \in A_0$ (resp. B_0) such that $I = \langle f \rangle$, then we have $\mathcal{N}_1(V) = (\langle f, \Delta(f) \rangle : \partial(f)^\infty)$ for every nonzero partial derivative $\partial(f)$ of f . This will be the case for the curves in the following sections.

Remark 3.2.6. By analogous arguments, observation 3.2.5 can be extended for m -jet schemes of any level $m \geq 1$.

3.2.7 Let V be an affine k -variety with $\mathcal{O}(V) = A_0/I$. Since, for every generator g of I , the polynomial $\Delta(g)$ is homogeneous, with $\deg_1(\Delta(g)) = 1$, for the graded structure (3) in subsection 3.1.1 (i.e., it is isobaric of weight 1), we conclude from formula (2.3) that the ideal $\mathcal{N}_1(V)$ (in the ring A_1) is homogeneous. Besides, if the ideal I is assumed to be homogeneous (in the ring A_0), the same argument implies that the ideal $\mathcal{N}_1(V)$ (in the ring A_1) is bi-homogeneous.

3.2.8 From the expression of the ideal $\mathcal{N}_1(V)$ in (2.5) K. Kpognon and J. Sebag developed an algorithm producing a Groebner basis of it which we describe here (see also [55, Remark 4.7] and [14, Section 7]).

Let $V = \text{Spec}(A_0/I)$ be an integral variety and let $H \in A_0$ such that $V_{\text{Sing}} \subset \{H = 0\}$ and $H \notin I$. We consider the ideal $\langle I, \Delta(I), 1 - HT \rangle$ of the ring $A_1[T]$. From equation (2.5) we deduce that

$$\mathcal{N}_1(V) = \langle I, \Delta(I), 1 - HT \rangle \cap A_1.$$

Then a classical result in Elimination Theory (see [22, Ch.3, §1, Theorem 2]) assures that, if \mathcal{G} is a Groebner basis of the ideal $\langle I, \Delta(I), 1 - HT \rangle$ in $A_1[T]$ with respect to an elimination order for the variable T (for example a lexicographic order where T is the greatest variable), then $\mathcal{G} \cap A_1$ is a Groebner basis of $\mathcal{N}_1(V)$. Recall that \mathcal{G} may be constructed using the Buchberger's algorithm ([22, Ch.2, §7]).

The computations in the examples in this chapter have been made using this algorithm implemented in SageMath [76]. Let us stress that this algorithm has a complexity comparable to that of Buchberger's algorithm, so an important number of examples cannot be processed.

3.3 TECHNICAL RESULTS ON POLYNOMIALS

In this section, we establish technical results (see propositions 3.3.4 and 3.3.6) which will provide characterizations for the divisibility of elements in B_1 by the polynomial $sy_1x_0 - ry_0x_1$ for $r, s \in \mathbb{N} \setminus \{0\}$, but which are sufficiently general to be considered independently. This polynomial will play an important role in our description of the general component attached to an affine plane curve defined by the datum of a homogeneous or weighted-homogeneous polynomial, and the results in this section will be useful in the following ones. In this section we fix the lexicographic order on B_1 associated with $y_1 > y_0 > x_1 > x_0$.

3.3.1 On the set \mathbb{N}^4 , we introduce the following equivalence relation: for every pair of tuples $(a, b) \in \mathbb{N}^4 \times \mathbb{N}^4$, we say that a is equivalent to b if there exists an integer $s \in \mathbb{Z}$ such that

$$\begin{cases} b_1 = a_1 - s \\ b_2 = a_2 + s \\ b_3 = a_3 + s \\ b_4 = a_4 - s. \end{cases}$$

In this case, we write $a \sim b$.

Lemma 3.3.2. *Let $a, b \in \mathbb{N}^4$. The following assertions are equivalent:*

1. *We have $a \sim b$;*
2. *The 4-tuples a, b verify the following conditions:*

$$\begin{cases} a_1 + a_3 = b_1 + b_3 \\ a_2 + a_4 = b_2 + b_4 \\ a_1 + a_2 = b_1 + b_2 \\ a_3 + a_4 = b_3 + b_4. \end{cases}$$

Proof. We only have to prove (2) \Rightarrow (1). Let us set $s := a_1 - b_1$. We observe from equations in system (2) that

$$\begin{aligned} s &= b_2 - a_2 \\ &= b_3 - a_3 \\ &= a_4 - b_4. \end{aligned}$$

Thus, we deduce that $b_1 = a_1 - s$, $b_4 = a_4 - s$, $b_2 = a_2 + s$ and $b_3 = a_3 + s$. \square

3.3.3 Let Γ be a set of representatives of \sim in \mathbb{N}^4 . For every polynomial $P \in B_1$, there exist bi-homogeneous polynomials $P_1, \dots, P_m \in B_1$ with $P = \sum_{i=1}^m P_i$ and which satisfy the following property: for every integer $i \in \{1, \dots, m\}$, one can find a unique $\alpha_i \in \Gamma$ such that

$$P_i = \sum_{a \in \mathbb{N}^4, a \sim \alpha_i \in \Gamma} \lambda_a y_1^{a_1} y_0^{a_2} x_1^{a_3} x_0^{a_4}. \quad (3.1)$$

If we assume that the polynomial P is bi-homogeneous of bi-degree (d, e) , we observe, thanks to lemma 3.3.2, that, for every integer $i \in \{1, \dots, m\}$, there exists an integer $\ell_i \leq d + e$ such that:

$$P_i = \sum_{\substack{(a_1, a_2) \in \mathbb{N}^2 \\ a_1 + a_2 = \ell_i}} \lambda_{(a_1, a_2, d-a_1, e-a_2)} y_1^{a_1} y_0^{a_2} x_1^{d-a_1} x_0^{e-a_2}. \quad (3.2)$$

(Let us stress that, because of the assumption on P , we have $a_3 = d - a_1$ and $a_4 = e - a_2$ in formula (3.1).)

Proposition 3.3.4. *Let $P \in B_1$ be a polynomial. We set $P = \sum_{a \in \mathbb{N}^4} \lambda_a y_1^{a_1} y_0^{a_2} x_1^{a_3} x_0^{a_4}$. Let $r, s \in \mathbb{N} \setminus \{0\}$. The following assertions are equivalent:*

1. *The polynomial $sy_1x_0 - ry_0x_1$ divides the polynomial P .*
2. *We have the formula $\sum_{b \in \mathbb{N}^4, b \sim a} \lambda_b r^{b_1} s^{b_3} = 0$ for every tuple $a \in \mathbb{N}^4$.*

If we assume that the polynomial P is bi-homogeneous of bi-degree (d, e) , then the former assertions are equivalent to the following one:

- (3) *For every integer ℓ , we have the formula*

$$\sum_{\substack{(a_1, a_2) \in \mathbb{N}^2 \\ a_1 + a_2 = \ell}} \lambda_{(a_1, a_2, d - a_1, e - \ell + a_1)} r^{a_1} s^{d - a_1} = 0.$$

Proof. Assertion (3) is equivalent to assertion (2) by equation (3.2). For (1) \Rightarrow (2), we set $G = sy_1x_0 - ry_0x_1$. Let $Q \in B_1$ be a polynomial. Each term $M = \mu_m y_1^{m_1} y_0^{m_2} x_1^{m_3} x_0^{m_4}$ of Q (with $\mu_m \in k$) provides two monomials in the expression of QG , whose degrees belong to the same equivalence class by \sim , namely $s\mu_m y_1^{m_1+1} y_0^{m_2} x_1^{m_3} x_0^{m_4+1}$ and $-r\mu_m y_1^{m_1} y_0^{m_2+1} x_1^{m_3+1} x_0^{m_4}$. One checks that their sum satisfies the required property since

$$\begin{aligned} s\mu_m r^{m_1+1} s^{m_3} - r\mu_m r^{m_1} s^{m_3+1} &= \mu_m r^{m_1} s^{m_3} (rs - rs) \\ &= 0. \end{aligned} \tag{3.3}$$

Let us now prove (2) \Rightarrow (1). We may assume that

$$P = \sum_{a \in \mathbb{N}^4, a \sim \alpha \in \Gamma} \lambda_a y_1^{a_1} y_0^{a_2} x_1^{a_3} x_0^{a_4}$$

by subsection 3.3.3. By assumption, we have

$$\sum_{a \in \mathbb{N}^4, a \sim \alpha \in \Gamma} \lambda_a r^{a_1} s^{a_3} = 0.$$

We have to prove that G divides the polynomial P . Let us set

$$\text{LM}(P) = \lambda_{\tilde{a}} y_1^{\tilde{a}_1} y_0^{\tilde{a}_2} x_1^{\tilde{a}_3} x_0^{\tilde{a}_4}$$

with $\tilde{a} \sim \alpha$. Various cases occur:

◦ If $\tilde{a}_1 = 0$ then $\lambda_a = 0$ whenever $a_1 > 0$ (otherwise it would contradict the fact that the tuple \tilde{a} corresponds to $\text{LM}(P)$). But, by the definition of the relation \sim , there is no tuple $a \sim \tilde{a}$ with $a_1 = \tilde{a}_1$ different from \tilde{a} itself. Thus $P = \text{LM}(P)$ and, by assumption, we have $\lambda_{\tilde{a}} = 0$; hence, $P = 0$.

◦ We assume that $\tilde{a}_4 = 0$. By the definition of relation \sim , every tuple a equivalent to \tilde{a} must verify $a_1 \geq \tilde{a}_1$, since $a_1 = \tilde{a}_1 + a_4$. If $a_1 > \tilde{a}_1$, we deduce that $\lambda_a = 0$ because of the choice of \tilde{a} . Thus, we have $P = \text{LM}(P)$, and we conclude as formerly.

◦ We assume that $\tilde{a}_1, \tilde{a}_4 > 0$. Then the polynomial

$$P^{(1)} := P - \left(\frac{\lambda_{\tilde{a}}}{s} y_1^{\tilde{a}_1-1} y_0^{\tilde{a}_2} x_1^{\tilde{a}_3} x_0^{\tilde{a}_4-1} \right) G$$

still verifies $\sum_{a \in \mathbb{N}^4, a \sim \alpha \in \Gamma} \lambda_a r^{a_1} s^{a_3} = 0$ (by observation (3.3) applied here), and we also have $\text{LT}(P^{(1)}) < \text{LT}(P)$. Using the previous cases, we observe that this construction can be iterated. In this way, we construct $P^{(2)}$ such that $\text{LT}(P^{(2)}) < \text{LT}(P^{(1)})$ and G divides $P^{(2)} - P^{(1)}$. After a finite number t of steps (at most $\min\{\tilde{a}_1, \tilde{a}_4\}$), we will obtain $P^{(t)} = 0$, which proves the property and concludes the proof. \square

3.3.5 Let $r, s \in \mathbb{N}^*$. We introduce the morphism of B_0 -algebras

$$\tilde{e}v_1: B_1 \rightarrow B_0 \tag{3.4}$$

defined by $x_1 \mapsto sx_0$ and $y_1 \mapsto ry_0$.

Proposition 3.3.6. *Let $P \in B_1$ be a 1-homogeneous polynomial of 1-degree d . Let $r, s \in \mathbb{N} \setminus \{0\}$. The following assertions are equivalent:*

1. *The polynomial P is divisible by $sy_1x_0 - ry_0x_1$.*
2. *We have $\tilde{e}v_1(P) = 0$.*

Proof. We only have to prove (2) \Rightarrow (1). We set $P = \sum_{\substack{a \in \mathbb{N}^4 \\ a_1+a_3=d}} \lambda_a y_1^{a_1} y_0^{a_2} x_1^{a_3} x_0^{a_4}$. By assumption, we have

$$\begin{aligned} 0 &= \tilde{e}v_1(P) \\ &= \sum_{\substack{a \in \mathbb{N}^4 \\ a_1+a_3=d}} \lambda_a r^{a_1} s^{a_3} y_0^{a_1+a_2} x_0^{a_3+a_4} \end{aligned} \tag{3.5}$$

Let $(\ell, m) \in \mathbb{N}^2$. If $a_3 + a_4 = m$ and $a_1 + a_2 = \ell$, we conclude that $a_2 + a_4 = \ell - a_1 + m + a_1 - d = \ell + m - d$. Thus, the sum $P_{(\ell, m)}$ of the terms $T = \lambda_a y_1^{a_1} y_0^{a_2} x_1^{a_3} x_0^{a_4}$ of P with $a_3 + a_4 = m$ and $a_1 + a_2 = \ell$ is a bi-homogeneous polynomial of bi-degree $(d, \ell + m - d)$. Formula (3.5) implies that, for every pair of integers $(\ell, m) \in \mathbb{N}^2$, we have

$$\sum_{\substack{(a_1, a_2) \in \mathbb{N}^2 \\ a_1+a_2=\ell \\ d+a_4=m+a_1}} \lambda_{(a_1, a_2, d-a_1, \ell+m-d-a_2)} r^{a_1} s^{d-a_1} = 0.$$

We deduce from proposition 3.3.4 that each polynomial $P_{(\ell, m)}$ is divisible by $sy_1x_0 - ry_0x_1$, which concludes the proof. \square

3.4 DIFFERENTIAL PROPERTIES OF HOMOGENEOUS POLYNOMIALS

Let k be a field of characteristic zero. In this section, we establish various technical results which will be used in the next sections. We exhibit in particular an additional differential structure on the ring A_1 (see theorem 3.4.5).

Let us recall the following simple but useful relation, usually known as the Euler identity (or theorem) for homogeneous functions.

Lemma 3.4.1 (Euler identity). *Let k be a field of characteristic zero and $P \in A_0$ be a homogeneous polynomial of degree d . Then*

$$\sum_{i=1}^n \partial_{t_{i,0}}(P)t_{i,0} = dP.$$

3.4.2 Let us introduce the following k -derivations of the k -algebra A_1 . We denote by $D \in \text{Der}_k(A_1)$ (resp. $E \in \text{Der}_k(A_1)$) the derivation defined by $\sum_{i=1}^n t_{i,1}\partial_{t_{i,0}}$ (resp. $\sum_{i=1}^n t_{i,0}\partial_{t_{i,1}}$). The derivations D, E have the following first properties.

Proposition 3.4.3. *Let k be a field of characteristic zero.*

1. *For every polynomial $P \in A_0$, we have $D(P) = \Delta(P)$ and $E(P) = 0$.*
2. *For every pair $(i, j) \in \{1, \dots, n\}^2$ of integers, we have*

$$D(t_{i,1}t_{j,0} - t_{j,1}t_{i,0}) = E(t_{i,1}t_{j,0} - t_{j,1}t_{i,0}) = 0.$$

3. *For every homogeneous ideal I of the ring A_0 , we have $E(\langle I, \Delta(I) \rangle) \subset \langle I, \Delta(I) \rangle$.*
4. *Let $\text{ev}_1: A_1 \rightarrow A_0$ be the (surjective) morphism of A_0 -algebras which sends the variable $t_{i,1}$ to $t_{i,0}$ for every integer $i \in \{1, \dots, n\}$. For every reduced homogeneous ideal I of the ring A_0 , for every polynomial $P \in \mathcal{N}_1(\text{Spec}(A_0/I))$, we have $\text{ev}_1(P) \in I$.*

Proof. Assertions (1) and (2) are obvious and follow from a direct computation. Let us prove assertion (3). Let $g \in A_0$ be a nonzero homogeneous generator of I of degree $\text{deg}_0(g)$. Then, thanks to the Euler identity (lemma 3.4.1), we have

$$\begin{aligned} E(\Delta(g)) &= \sum_{i=1}^n \partial_{t_{i,0}}(g)E(t_{i,1}) \\ &= \sum_{i=1}^n \partial_{t_{i,0}}(g)t_{i,0} \\ &= \text{deg}_0(g)g. \end{aligned}$$

Let us prove assertion (4). Up to replacing the ideal I by each of its prime components, we may assume that the ideal I is prime by observation 3.2.5. Then, by subsection 3.2.3, there exist an integer N and a polynomial $H \notin I$ such that $H^N P \in \langle I, \Delta(I) \rangle$. Then, the polynomial $H^N \text{ev}_1(P)$ belongs to the ideal $\langle I, \text{ev}_1(\Delta(I)) \rangle$. We observe that, for every homogeneous polynomial $g \in A_0$, we have that $\text{ev}_1(\Delta(g))$ equals $\text{deg}_0(g)g$; hence, we deduce that $\langle I, \text{ev}_1(\Delta(I)) \rangle = I$. Thus $H^N \text{ev}_1(P)$ belongs to the prime ideal I which implies $\text{ev}_1(P) \in I$. \square

3.4.4 The following theorem can be interpreted as a formal “almost” integration of homogeneous polynomials in the ring A_1 . Precisely, we show in theorem 3.4.5 that the action of the derivation D (resp. E) on the image of E (resp. D) is near from “the” reverse action.

Theorem 3.4.5. *Let k be a field of characteristic zero. For every bi-homogeneous polynomial $P \in A_1$ of bi-degree (d, e) , with $d, e \geq 1$, there exists a positive integer α such that*

$$D(E(P)) - \alpha P \in \langle t_{i,1}t_{j,0} - t_{j,1}t_{i,0}; i, j \in \{1, \dots, n\} \rangle.$$

The same formula holds for the polynomial $E(D(P))$.

Proof. We only prove the formula for the polynomial $D(E(P))$. The proof is based on a direct computation. We have

$$\begin{aligned} D(E(P)) &= \left(\sum_{i=1}^n t_{i,1} \partial_{t_{i,0}} \right) \circ \left(\sum_{j=1}^n t_{j,0} \partial_{t_{j,1}} \right) (P) \\ &= \left(\sum_{i=1}^n t_{i,1} \partial_{t_{i,1}} (P) \right) + T \end{aligned} \quad (4.1)$$

The first parenthesis in formula (4.1) equals, by the Euler identity (lemma 3.4.1), the polynomial dP . Besides, we have

$$\begin{aligned} T &:= \sum_{i=1}^n t_{i,1} \left(\sum_{j=1}^n t_{j,0} \partial_{t_{i,0}} \partial_{t_{j,1}} (P) \right) \\ &=: \sum_{i=1}^n T_i, \end{aligned}$$

where we set, for every integer $i \in \{1, \dots, n\}$,

$$T_i := t_{i,1} \sum_{j=1}^n t_{j,0} \partial_{t_{i,0}} \partial_{t_{j,1}} (P).$$

Let us fix an integer $i \in \{1, \dots, n\}$. We write

$$\begin{aligned} T_i &= \left(t_{i,0} \sum_{j=1}^n t_{j,1} \partial_{t_{j,1}} \partial_{t_{i,0}} (P) \right) + \left(t_{i,1} \sum_{j=1, j \neq i}^n t_{j,0} \partial_{t_{i,0}} \partial_{t_{j,1}} (P) \right) \\ &\quad - \left(t_{i,0} \sum_{j=1, j \neq i}^n t_{j,1} \partial_{t_{j,1}} \partial_{t_{i,0}} (P) \right) \end{aligned} \quad (4.2)$$

For every integer $j \in \{1, \dots, n\}$, with $j \neq i$, we set

$$T_{i,j} = (t_{i,1}t_{j,0} - t_{i,0}t_{j,1}) \partial_{t_{i,0}} \partial_{t_{j,1}}(P) \in \langle t_{\alpha,1}t_{\beta,0} - t_{\beta,1}t_{\alpha,0}; \alpha, \beta \in \{1, \dots, n\} \rangle$$

and observe, thanks to the Euler identity (lemma 3.4.1), that formula (4.2) can be rewritten under the following form:

$$T_i = dt_{i,0} \partial_{t_{i,0}}(P) + \sum_{j=1, j \neq i}^n T_{i,j}. \quad (4.3)$$

Thus, formula (4.1) can be rewritten as

$$\begin{aligned} D(E(P)) &= dP + T \\ &= dP + \sum_{i=1}^n T_i \\ &= dP + d \left(\sum_{i=1}^n t_{i,0} \partial_{t_{i,0}}(P) \right) + \left(\sum_{i=1}^n \sum_{j=1, j \neq i}^n T_{i,j} \right). \end{aligned} \quad (4.4)$$

By the Euler identity applied to the last term in formula (4.4) we conclude that

$$\begin{aligned} D(E(P)) &= dP + T \\ &= dP + deP + \left(\sum_{i=1}^n \sum_{j=1, j \neq i}^n T_{i,j} \right) \\ &= d(e+1)P + \left(\sum_{i=1}^n \sum_{j=1, j \neq i}^n T_{i,j} \right) \end{aligned}$$

which concludes the proof. \square

3.4.6 From now on and up to the end of the section, we restrict ourselves to the case of affine plane curves. The polynomial $y_1x_0 - y_0x_1$ plays an important role in our study as the following lemma underlines it:

Lemma 3.4.7. *Let k be a field of characteristic zero. Let \mathcal{C} be an affine plane curve defined by the datum of a homogeneous polynomial $g \in B_0$. Then the polynomial $y_1x_0 - y_0x_1$ belongs to the ideal $\mathcal{N}_1(\mathcal{C})$.*

Proof. By observation 3.2.5, up to replacing g by each of its irreducible factors, we may assume that the polynomial g is irreducible; then, we have to prove that the polynomial $y_1x_0 - y_0x_1$ belongs to the ideal $([g] : \partial(g)^\infty)$ (for some nonzero partial derivative). Let us assume that $\partial_y(g) \neq 0$ (a symmetrical argument works if $\partial_x(g) \neq 0$). We observe that $\Delta(g) = \partial_x(g)x_1 + \partial_y(g)y_1$. We write

$$\begin{aligned} \partial_y(g)(y_1x_0 - y_0x_1) &\equiv -x_0\partial_x(g)x_1 - y_0\partial_y(g)x_1 \pmod{\Delta(g)} \\ &\equiv -\deg_0(g)x_1g \pmod{\Delta(g)}, \end{aligned}$$

which concludes the proof. \square

Example 3.4.8. Lemma 3.4.7 does not hold in higher dimension. Let us consider the hypersurface \mathcal{S} of \mathbb{A}_k^3 defined by the datum of the polynomial $f = x_0^2 + y_0^2 + z_0^2 \in k[x, y, z]$. Then $x_1y_0 - x_0y_1, x_1z_0 - x_0z_1, y_1z_0 - z_1y_0 \notin \mathcal{N}_1(\mathcal{S})$.

3.4.9 We consider the morphism of B_0 -algebras $\text{ev}_1: B_1 \rightarrow B_0$ defined by $x_1 \mapsto x_0$ and $y_1 \mapsto y_0$.

Lemma 3.4.10. *Let k be a field of characteristic zero. Let $g \in B_0$ be a reduced homogeneous polynomial with $\mathcal{C} = \text{Spec}(B_0/\langle g \rangle)$. Let $P \in B_1$ be a homogeneous polynomial (in x_1, y_1) of degree $\deg_1(P) = d$. The following assertions are equivalent:*

1. *The polynomial P belongs to $\mathcal{N}_1(\mathcal{C})$;*
2. *The polynomial g divides $\text{ev}_1(P)$ in the ring B_0 .*

Proof. By (4) in proposition 3.4.3, we only have to prove (2) \Rightarrow (1). By observation 3.2.5, we may assume that the polynomial g is irreducible up to replacing it by each of its irreducible factors. Two cases occur.

◦ Let us assume that there exists $u \in k^*$ such that $g = ux$. (By symmetrical arguments, we could prove the case $g = uy$.) In this case, we have $\mathcal{N}_1(\mathcal{C}) = \langle g, \Delta(g) \rangle = \langle x_0, x_1 \rangle$. Hence, the polynomial P belongs to $\mathcal{N}_1(\mathcal{C})$ if and only if it belongs to the kernel of the morphism of k -algebras $\text{ev}: B_1 \rightarrow k$ sending the variables x_0, x_1 to zero. Let us assume that the polynomial g divides $\text{ev}_1(P)$. Since P is 1-homogeneous, there exists $q \in k[y_0]$ such that $\text{ev}(P) = P(0, y_0, 0, y_1) = q(y_0)y_1^d$. Since $\text{ev}(\text{ev}_1(P)) = \text{ev}_1(\text{ev}(P))$, we conclude that $q = 0$. In other words, we have $P \in \mathcal{N}_1(\mathcal{C})$.

◦ Let us assume that g is not divisible by x_0 or y_0 , cases for which we have proved the property. We have the formula

$$\begin{aligned} x_0^d P(x_0, y_0, x_1, y_1) &= P(x_0, y_0, x_0 x_1, x_0 y_1) \\ &\equiv P(x_0, y_0, x_0 x_1, x_1 y_0) \pmod{y_1 x_0 - x_1 y_0} \\ &\equiv x_1^d P(x_0, y_0, x_0, y_0) \pmod{y_1 x_0 - x_1 y_0}. \end{aligned} \tag{4.5}$$

By assumption, the polynomial g divides $x_1^d P(x_0, y_0, x_0, y_0)$. By formula (4.5) and lemma 3.4.7, we conclude that $x_0^d P(x_0, y_0, x_1, y_1)$ belongs to $\mathcal{N}_1(\mathcal{C})$. By lemma 2.3.3, we conclude that $P(x_0, y_0, x_1, y_1)$ belongs to $\mathcal{N}_1(\mathcal{C})$ (which is prime). \square

Example 3.4.11. The analogue of lemma 3.4.10 does not hold in higher dimensions. Let us consider the hypersurface of \mathbb{A}_k^3 defined by the datum of the polynomial $f = x_0^2 + y_0^2 + z_0^2 \in k[x, y, z]$. Then the polynomial $x_1 y_0 - x_0 y_1$ satisfies condition (2) but does not belong to $\mathcal{N}_1(\mathcal{C})$.

Proposition 3.4.12. *Let k be a field of characteristic zero. For every affine plane curve \mathcal{C} defined by the datum of a homogeneous polynomial $g \in B_0$, the ideal $\mathcal{N}_1(\mathcal{C})$ is stable under the actions of D and E .*

In general, this assertion does not hold true. See example 3.4.13.

Proof. By observation 3.2.5, we may assume that the polynomial g is irreducible. Let $P \in \mathcal{N}_1(\mathcal{C})$. By observation 3.2.5, there exist an integer $M \in \mathbb{N}$, a nonzero partial derivative $\partial(g)$ of g and polynomials $\alpha, \beta \in B_0$ such that

$$\partial(g)^M P = \alpha g + \beta \Delta(g). \quad (4.6)$$

By proposition 3.4.3, we know that the derivation E stabilizes the ideal $\langle g, \Delta(g) \rangle$. Then, by applying E to equation (4.6), we conclude that $\partial(g)^M E(P)$ belongs to the ideal $\langle g, \Delta(g) \rangle$ which concludes the proof by observation 3.2.5. Let us prove the assertion for the derivation D . By a direct computation, we obtain $D(\Delta(g)) = D(\partial_x(g))x_1 + D(\partial_y(g))y_1$. Then we observe, thanks to the Euler identity (lemma 3.4.1), that

$$\begin{aligned} D(\Delta(g))(x_0, y_0, x_0, y_0) &= (\partial_x(\partial_x(g))x_0 + \partial_y(\partial_x(g))y_0)x_0 + (\partial_x(\partial_y(g))x_0 + \partial_y(\partial_y(g))y_0)y_0 \\ &= (\deg(g) - 1)(\partial_x(g)x_0 + \partial_y(g)y_0) \\ &= \deg(g)(\deg(g) - 1)g, \end{aligned}$$

and we conclude by lemma 3.4.10. \square

Example 3.4.13. Let us consider the hypersurface \mathcal{S} of \mathbb{A}_k^3 defined by the datum of the polynomial $f = z^3 + y^2x + y^3 \in k[x, y, z]$. One can check that $D(\Delta(f)) \notin \mathcal{N}_1(\mathcal{S})$. The polynomial $P = 3y_0x_0z_1 + 3y_0^2z_1 - y_0x_1z_0 - 2y_1x_0z_0 - 3y_0y_1z_0$ belongs to $\mathcal{N}_1(\mathcal{S}) \setminus \langle f, \Delta(f) \rangle$ but $D(P) \notin \mathcal{N}_1(\mathcal{S})$.

3.5 GENERAL COMPONENT OF AN AFFINE PLANE CURVE DEFINED BY A HOMOGENEOUS POLYNOMIAL

Let k be a field of characteristic zero. The aim of this section is to describe presentations for the ideal $\mathcal{N}_1(\mathcal{C})$ when \mathcal{C} is an affine plane curve defined by the datum of a homogeneous polynomial in B_0 .

3.5.1 We introduce the following notation. Let $m \in \mathbb{N}$. Let $g \in B_0$ be a homogeneous polynomial with $\deg_0(g) = m$ and $\mathcal{C} = \text{Spec}(B_0/\langle g \rangle)$. For every integer $i \in \mathbb{N}$, for every polynomial $g \in B_0$, we denote by $D_i(g)$ the element $D^{(i)}(g)/i$, if $i \geq 1$, and $D_0(g) = g$, which belongs to the ideal $\mathcal{N}_1(\mathcal{C})$. In particular, for every integer $i \geq m + 1$, we have $D_i(g) = 0$.

Proposition 3.5.2. *Let k be a field of characteristic zero. Let $m \geq 1$ be an integer. Let \mathcal{C} be an affine plane curve defined by the datum of a reduced homogeneous polynomial $g \in B_0$ with $\deg_0(g) = m$. The ideal $\mathcal{N}_1(\mathcal{C})$ is generated by the family $D_{-1} := y_1x_0 - y_0x_1$ and the $D_i(g)$ for every integer $i \in \{0, \dots, m\}$.*

Proof. By proposition 3.4.12, for every integer $i \in \{0, \dots, m\}$ we have $D_i(g) \in \mathcal{N}_1(\mathcal{C})$. Thanks to this observation and lemma 3.4.7, we deduce that $\langle y_1x_0 - y_0x_1 \rangle + \langle D_i(g); i \in$

$\{0, \dots, m\} \subset \mathcal{N}_1(\mathcal{C})$. Conversely, we have now to prove that $\mathcal{N}_1(\mathcal{C}) \subset \langle y_1x_0 - y_0x_1 \rangle + \langle D_i(g); i \in \{0, \dots, m\} \rangle$. We show the result by an induction on the degree d (in x_1, y_1) of the polynomials in $\mathcal{N}_1(\mathcal{C})$ considered as polynomials in the ring $B_0[x_1, y_1]$. By observation 3.2.5, we may assume that g is irreducible. If $P \in \mathcal{N}_1(\mathcal{C})$ is a polynomial with degree $d = 0$, then the polynomial g divides P by lemma 2.3.3. Let $d \geq 1$ and $P \in \mathcal{N}_1(\mathcal{C})$ with $\deg_1(P) = d$. By subsection 3.2.7, we may assume that P is bi-homogeneous. We observe that the degree of the polynomial $E(P)$ equals $d - 1$ and belongs to the ideal $\mathcal{N}_1(\mathcal{C})$ by proposition 3.4.12. By the induction hypothesis, we deduce that $E(P) \in \langle y_1x_0 - y_0x_1 \rangle + \langle D_i(g); i \in \{0, \dots, m\} \rangle$. We conclude the proof by applying the operator D to $E(P)$ thanks to theorem 3.4.5. \square

Example 3.5.3. Let k be a field of characteristic zero. Let $f = x_0y_0 \in B_0$. We observe that $\{x_0\} \cap \{y_0\} = [x_0] \cdot [y_0]$; one inclusion is clear. For the inverse one, let $P \in \{x_0\} \cap \{y_0\}$, then there exists $m \in \mathbb{N}$ such that $P^m \in [x_0] \cap [y_0]$. In this case, this is equivalent to $P \in [x_0] \cap [y_0]$: let us suppose by contradiction that $P \notin [x_0]$, then there is a term in T in the polynomial P which does not contain any variable x_i for $i \in \mathbb{N}$. Then T^m is a term in P^m which does not contain any variable x_i for $i \in \mathbb{N}$ and hence $P^m \notin [x_0]$, contradiction. The argument proving that $P \in [y_0]$ is analogous.

Thus we can write P in the form $P = \sum_{i \in \mathbb{N}} b_i x_i$ where $b_i \in k[x_i, y_j; i, j \in \mathbb{N}]$ for $i \in \mathbb{N}$. From $P \in [y_0]$ we deduce that $b_i \in [y_0]$ for $i \in \mathbb{N}$, hence $P \in [x_0] \cdot [y_0]$. Then we have the formula

$$\mathcal{N}_1(\mathcal{C}) = \{x_0\} \cap \{y_0\} \cap B_1 = ([x_0] \cdot [y_0]) \cap B_1 = \langle f, x_0y_1, x_1y_0, x_1y_1 \rangle.$$

By proposition 3.5.2, we deduce another presentation of the ideal $\mathcal{N}_1(\mathcal{C})$ given by

$$\langle f, y_1x_0 - y_0x_1, x_0y_1 + y_0x_1, x_1y_1 \rangle.$$

3.5.4 Let k be a field of characteristic zero and $t \in \mathbb{N}^*$. For every integer $i \in \{1, \dots, t\}$, let $\gamma_i \in k^*$ be mutually distinct elements. For every integer $i \in \{1, \dots, t\}$, we set $f_i = y_0 - \gamma_i x_0 \in B_0$ and $f = x_0^\varepsilon y_0^{\varepsilon'} \prod_{i=1}^t f_i$ with $\varepsilon, \varepsilon' \in \{0, 1\}$. Let us denote by J the ideal of the ring B_1 defined by

$$J := \langle f_1, \Delta(f_1) \rangle \cdot \langle f_2, \Delta(f_2) \rangle \cdots \langle f_t, \Delta(f_t) \rangle.$$

The following bi-homogeneous polynomials of the ring B_1 belong to this ideal

$$\delta_i := \left(\prod_{\ell=1}^i \Delta(f_\ell) \right) \times \left(\prod_{\ell=i+1}^t f_\ell \right) \quad (5.1)$$

for every integer $i \in \{0, \dots, t\}$. We set $\mathcal{C} = \text{Spec}(B_0/\langle f \rangle)$.

Theorem 3.5.5. *Keep the assumptions and notation of subsection 3.5.4. The family*

$$\mathfrak{B} := \{y_1x_0 - y_0x_1, x_{h_1}^\varepsilon y_{h_2}^{\varepsilon'} \delta_i(f), i \in \{0, \dots, t\}, h_1, h_2 \in \{0, 1\}\}$$

is a Groebner basis of $\mathcal{N}_1(\mathcal{C})$ for the monomial order $y_1 >_{\text{lex}} y_0 >_{\text{lex}} x_1 >_{\text{lex}} x_0$ in B_1 .

Let us stress that, if the field k is assumed to be algebraically closed, theorem 3.5.5 provides a complete answer for homogeneous polynomials. Subsection 2.3.13 explains how theorem 3.5.5 also gives an explicit Groebner basis in case the field k is not assumed to be algebraically closed.

Proof. Let us prove theorem 3.5.5. By lemma 3.4.7 and the very definition of the ideal J , we conclude that the family is contained in the ideal $\mathcal{N}_1(\mathcal{C})$. By [3, Proposition 5.38], in order to prove that the family \mathfrak{B} is a Groebner basis of the ideal $\mathcal{N}_1(\mathcal{C})$, it suffices to show that every element in $\mathcal{N}_1(\mathcal{C})$ has some term in $\langle \text{LT}(\mathfrak{B}) \rangle$. Let us denote the polynomial $\delta_{-1} := y_1x_0 - y_0x_1$. We observe that

$$\langle \text{LT}(\mathfrak{B}) \rangle = \left\langle y_1x_0, \left\{ x_{h_1}^\varepsilon y_{h_2}^{\varepsilon'} y_1^\ell y_0^{t-\ell} \right\}_{\substack{\ell \in \{0, \dots, t\} \\ h_1, h_2 \in \{0, 1\}}} \right\rangle$$

Let $P \in \mathcal{N}_1(\mathcal{C})$ that we may assume to be bi-homogeneous. We apply lemma 3.4.10 and deduce that f divides $\text{ev}_1(P)$ in the ring B_0 . Two cases occur:

- (i) Assume that $\text{ev}_1(P) \neq 0$. Then, the polynomial P has some term of the form $y_1^{a_1} y_0^{a_2} x_1^{a_3} x_0^{a_4}$ such that $\text{LT}(f) = x_0^\varepsilon y_0^{\varepsilon'} y_0^t$ divides $y_0^{a_1+a_2} x_0^{a_3+a_4}$; hence, we have $a_1 + a_2 \geq t + \varepsilon'$ and $a_3 + a_4 \geq \varepsilon$. The second inequality shows that either x_0^ε or x_1^ε divide the term $x_1^{a_3} x_0^{a_4}$.

On the other hand, for $\ell \in \{0, \dots, t\}$, the pairs $(\ell + \varepsilon', t - \ell)$, $(\ell, t - \ell + \varepsilon')$ range over all possible pairs of nonnegative integers whose sum equals $t + \varepsilon'$, and thus some monomial in $\left\{ y_{h_2}^{\varepsilon'} y_1^\ell y_0^{t-\ell} \right\}_{h_2 \in \{0, 1\}}$ divides the term $y_1^{a_1} y_0^{a_2}$. We deduce that

$y_1^{a_1} y_0^{a_2} x_1^{a_3} x_0^{a_4}$ is divisible by some monomial in $\left\{ x_{h_1}^\varepsilon y_{h_2}^{\varepsilon'} y_1^\ell y_0^{t-\ell} \right\}_{\substack{\ell \in \{0, \dots, t\} \\ h_1, h_2 \in \{0, 1\}}}$ and we have

proved the property.

- (ii) Assume that $\text{ev}_1(P) = 0$. Then, by proposition 3.3.6, the polynomial $y_1x_0 - y_0x_1$ divides P ; hence the monomial y_1x_0 divides $\text{LT}(P)$ and the property holds.

□

Remark 3.5.6. Along the whole chapter we have chosen the monomial order in B_1 to be the lexicographic one with $y_1 >_{\text{lex}} y_0 >_{\text{lex}} x_1 >_{\text{lex}} x_0$. Since we are in a homogeneous setting, the corresponding graded lexicographic order also works because it coincides with the lexicographic order. However, not every monomial order works, even if it is a lexicographic order. See example 3.5.9.

3.5.7 Let us mention the following consequence of theorem 3.5.5, which improves proposition 3.5.2.

Corollary 3.5.8. *Let k be a field of characteristic zero. Let $f \in B_0$ be a reduced homogeneous polynomial which is not divisible by neither x_0 nor y_0 . We set $\mathcal{C} = \text{Spec}(B_0/\langle f \rangle)$. The family formed by the polynomial $y_1x_0 - y_0x_1$ and the $D_i(f)$ for every integer $i \in \{0, \dots, \deg_0(f)\}$ is a Groebner basis of the ideal $\mathcal{N}_1(\mathcal{C})$ for the monomial order $y_1 >_{\text{lex}} y_0 >_{\text{lex}} x_1 >_{\text{lex}} x_0$ in B_1 .*

Proof. Let k' be an algebraic closure of k . Let us consider the differential ideal $\{f\}$ in $k\{x, y\}$ and let P be a polynomial in $\{f\} \cap B_1$. By subsection 2.3.13, the ideal $\{f\} \otimes_k k'$ equals the radical of the differential ideal generated by f in the differential ring $k'\{x, y\}$. By theorem 3.5.5, the leading term $\text{LT}(P)$ of the polynomial P is divisible (in $k'\{x, y\}$) by the leading term of some of the δ_i , for $i \in \{-1, \dots, \deg_0(f)\}$. We observe that these leading terms are the same as those of the family $\{D_i\}_{i \in \{-1, \dots, \deg_0(f)\}}$, in the notation of proposition 3.5.2. But the polynomials in this family belong to $k\{x, y\}$, which concludes the proof. \square

Example 3.5.9. Let us fix the field $k = \mathbb{Q}$. Let us consider the polynomial $f = x_0^4 + x_0^3y_0 + y_0^4$, which is irreducible in B_0 and homogeneous of degree 4. From a direct computation using the algorithm described in subsection 3.2.8 we obtain a Groebner basis of $\{f\} \cap B_1$ for the monomial order $y_1 >_{\text{lex}} y_0 >_{\text{lex}} x_1 >_{\text{lex}} x_0$:

$$\mathfrak{B} = \left\{ \begin{array}{l} y_1x_0 - y_0x_1, f, y_0^3y_1 + x_0^2x_1y_0 + x_0^3x_1, y_0^2y_1^2 + x_0x_1^2y_0 + x_0^2x_1^2, \\ y_0y_1^3 + x_1^3y_0 + x_0x_1^3, y_1^4 + x_1^3y_1 + x_1^4 \end{array} \right\}.$$

The family given in proposition 3.5.2 is the following:

$$\mathfrak{C} = \{D_{-1} := y_1x_0 - y_0x_1, D_0 := f, D_1 := 4y_0^3y_1 + x_0^3y_1 + 3x_0^2x_1y_0 + 4x_0^3x_1, \\ D_2 := 12y_0^2y_1^2 + 6x_0^2x_1y_1 + 6x_0x_1^2y_0 + 12x_0^2x_1^2, \\ D_3 := 24y_0y_1^3 + 18x_0x_1^2y_1 + 6x_1^3y_0 + 24x_0x_1^3, D_4 := 24y_1^4 + 24x_1^3y_1 + 24x_1^4\}.$$

We observe that the leading terms of the elements in \mathfrak{B} and \mathfrak{C} are the same. Hence \mathfrak{C} is also a Groebner basis of $\{f\} \cap B_1$. As an illustration of remark 3.5.6, if we consider on B_1 the monomial lexicographic order with $y_0 >_{\text{lex}} x_1 >_{\text{lex}} y_1 >_{\text{lex}} x_0$ then neither \mathfrak{C} nor \mathfrak{B} are Groebner bases of $\{f\} \cap B_1$.

3.6 GENERAL COMPONENT OF A PLANE CURVE DEFINED BY A WEIGHTED HOMOGENEOUS POLYNOMIAL

In this section we compute a system of generators of the ideal $B_1 \cap \{f\}$ when the polynomial f is weighted homogeneous. Let $(r, s) \in \mathbb{N}^2$ be a pair of coprime integers with $r > s \geq 2$. The techniques we will use are partly similar to those of the homogeneous case (see section 3.5).

3.6.1 We begin by giving an analogue to lemma 3.4.7 in the weighted homogeneous case.

Lemma 3.6.2. *Let k be a field of characteristic zero. Let $\lambda \in k^*$. Let $(r, s) \in \mathbb{N}^2$ be a pair of coprime integers with $r > s \geq 2$. Let $f = x_0^r - \lambda y_0^s \in B_0$. We set $\mathcal{C} = \text{Spec}(B_0/\langle f \rangle)$. Then the polynomial $sy_1x_0 - ry_0x_1$ belongs to the ideal $\mathcal{N}_1(\mathcal{C})$.*

Proof. The polynomial f being irreducible, we have to prove that $sy_1x_0 - ry_0x_1$ belongs to the ideal $([f] : \partial(f)^\infty)$ (for some nonzero partial derivative). Let us reason with $\partial_y(f) \neq 0$ (a symmetrical argument works for $\partial_x(f) \neq 0$). Recall that $\Delta(f) = x_1\partial_x(f) + y_1\partial_y(f)$. We write

$$\begin{aligned} \partial_y(f)(sy_1x_0 - ry_0x_1) &\equiv -sx_0\partial_x(f)x_1 - ry_0\partial_y(f)x_1 \pmod{\Delta(f)} \\ &\equiv -rsx_1f \pmod{\Delta(f)}, \end{aligned}$$

which concludes the proof. \square

3.6.3 Now we give an analogue to lemma 3.4.10. We consider the morphism of B_0 -algebras $\tilde{e}v_1 : B_1 \rightarrow B_0$ defined by $x_1 \mapsto sx_0$ and $y_1 \mapsto ry_0$.

Lemma 3.6.4. *Let k be a field of characteristic zero. Let $\lambda \in k^*$. Let $(r, s) \in \mathbb{N}^2$ be a pair of coprime integers with $r > s \geq 2$. Let $f = x_0^r - \lambda y_0^s \in B_0$. We set $\mathcal{C} = \text{Spec}(B_0/\langle f \rangle)$. Let $P \in B_1$ be a 1-homogeneous polynomial of degree $\deg_1(P) =: d$. The following assertions are equivalent:*

1. *The polynomial P belongs to $\mathcal{N}_1(\mathcal{C})$.*
2. *The polynomial $\tilde{e}v_1(P)$ belongs to the ideal $\langle f \rangle$ in the ring B_0 .*

Proof. Let $P \in \mathcal{N}_1(\mathcal{C})$. By observation 3.2.5, there exist an integer $N \in \mathbb{N}$ and a polynomial $H \notin \langle f \rangle$ in B_0 such that $H^N P \in \langle f, \Delta(f) \rangle$. Then, taking the image via $\tilde{e}v_1$, the polynomial $H^N \tilde{e}v_1(P)$ belongs to the prime ideal $\langle f, \tilde{e}v_1(\Delta(f)) \rangle = \langle f \rangle$, because $\tilde{e}v_1(\Delta(f)) = rsf$. We conclude that $\tilde{e}v_1(P)$ belongs to $\langle f \rangle$. Conversely, let $P \in B_1$ such that $\tilde{e}v_1(P) = P(x_0, y_0, sx_0, ry_0) \in \langle f \rangle \subset B_0$. We have the formula

$$\begin{aligned} s^d x_0^d P(x_0, y_0, x_1, y_1) &= P(x_0, y_0, sx_0x_1, sx_0y_1) \\ &\equiv P(x_0, y_0, sx_0x_1, rx_1y_0) \pmod{sy_1x_0 - ry_0x_1} \\ &\equiv x_1^d P(x_0, y_0, sx_0, ry_0) \pmod{sy_1x_0 - ry_0x_1}. \end{aligned} \tag{6.1}$$

By assumption, $x_1^d P(x_0, y_0, sx_0, ry_0)$ belongs to $\langle f \rangle$, then by (6.1) and lemma 3.6.2 the polynomial $s^d x_0^d P(x_0, y_0, x_1, y_1)$ belongs to $\mathcal{N}_1(\mathcal{C})$, which is a prime ideal. By lemma 2.3.3 the polynomial $P(x_0, y_0, x_1, y_1)$ belongs to $\mathcal{N}_1(\mathcal{C})$. \square

3.6.5 Let us state the main result of this section. Let $\lambda_1, \dots, \lambda_t \in k$ be nonzero elements. Let $(r, s) \in \mathbb{N}^2$ be a pair of coprime integers with $r > s \geq 2$. For every integer $i \in \{1, \dots, t\}$, we set $\tilde{D}_{-1} := \tilde{D}_{\lambda_i, -1} := sy_1x_0 - ry_0x_1$ and $\tilde{D}_{\lambda_i, j_i} := \lambda_i s^{j_i} y_0^{s-j_i} y_1^{j_i} - r^{j_i} x_0^{r-j_i} x_1^{j_i}$, where $j_i \in \{0, \dots, s\}$. For every $i \in \{1, \dots, t\}$, if $j_i \in \{-1, \dots, s\}$, we denote

$$\tilde{D}_{j_1, \dots, j_t} := \tilde{D}_{\lambda_1, j_1} \cdots \tilde{D}_{\lambda_t, j_t}. \quad (6.2)$$

Theorem 3.6.6. *Let k be a field of characteristic zero. Let $\lambda_1, \dots, \lambda_t \in k$ be nonzero and mutually distinct elements. Let $(r, s) \in \mathbb{N}^2$ be a pair of coprime integers with $r > s \geq 2$. Let $f \in B_0$ be the polynomial $f = x_0^\varepsilon y_0^{\varepsilon'} \prod_{i=1}^t (x_0^r - \lambda_i y_0^s)$ with $\varepsilon, \varepsilon' \in \{0, 1\}$. We set $\mathcal{C} = \text{Spec}(B_0/\langle f \rangle)$. Then the family*

$$\mathfrak{B} = \{\tilde{D}_{-1}, x_{h_1}^\varepsilon y_{h_2}^{\varepsilon'} \tilde{D}_{j_1, \dots, j_t}, j_i \in \{-1, \dots, s\}, i \in \{1, \dots, t\}, h_1, h_2 \in \{0, 1\}\}$$

is a Groebner basis of $\mathcal{N}_1(\mathcal{C})$ for the monomial order $y_1 >_{\text{lex}} y_0 >_{\text{lex}} x_1 >_{\text{lex}} x_0$ in B_1 .

Let us stress that, if the field k is assumed to be algebraically closed, theorem 3.6.6 provides a complete answer for weighted homogeneous polynomials by proposition 3.1.3. Subsection 2.3.13 explains how theorem 3.6.6 also gives an explicit Groebner basis in case the field k is not assumed to be algebraically closed.

Remark 3.6.7. As in the homogeneous case (see remark 3.5.6), it is not true in general that the family given in theorem 3.6.6 is a Groebner basis of $\mathcal{N}_1(\mathcal{C})$ when the monomial order on B_1 is not the lexicographic one with $y_1 >_{\text{lex}} y_0 >_{\text{lex}} x_1 >_{\text{lex}} x_0$. See example 3.6.8.

Example 3.6.8. Let us fix the field $k = \mathbb{R}$ and let $k' := \mathbb{C}$. We keep the notation in subsection 2.3.13. Let us consider the polynomial $f = x_0^7 + x_0 y_0^4$, which is weighted homogeneous of weight $(2, 3, 14)$. In $k'[x_0, y_0]$, we have $f = x_0(x_0^3 - iy_0^2)(x_0^3 + iy_0^2)$. By remark 3.6.19, which follows from the proof of theorem 3.6.6, the following family is a Groebner basis of $(\{f\} \otimes_k k') \cap B_1'$ for the monomial order $y_1 >_{\text{lex}} y_0 >_{\text{lex}} x_1 >_{\text{lex}} x_0$:

$$\mathfrak{B}' = \{2y_1x_0 - 3y_0x_1\} \cup \{(x_\ell \tilde{D}_{0,0}, x_\ell \tilde{D}_{1,0}, x_\ell \tilde{D}_{2,0}, x_\ell \tilde{D}_{2,1}, x_\ell \tilde{D}_{2,2})_{\ell \in \{0,1\}}\}.$$

For example, the polynomial $x_0 \tilde{D}_{1,0}$ equals $-2y_0^3 y_1 x_0 - 3x_0^6 x_1 + i(2y_0 y_1 x_0^4 - 3y_0^2 x_0^3 x_1)$, which has the two components $-2y_0^3 y_1 x_0 - 3x_0^6 x_1$ and $2y_0 y_1 x_0^4 - 3y_0^2 x_0^3 x_1$. In this way, from the preceding family we obtain the following one by computing the components (on the basis $\{1, i\}$) of its elements:

$$\begin{aligned} \mathfrak{C} = & \{2y_1x_0 - 3y_0x_1, f, x_1x_0^6 + x_1y_0^4, 2y_0^3y_1x_0 + 3x_0^6x_1, 2y_0y_1x_0^4 - 3y_0^2x_0^3x_1, \\ & 2y_0^3y_1x_1 + 3x_0^5x_1^2, 2y_0y_1x_0^3x_1 - 3y_0^2x_0^2x_1^2, 4y_1^2y_0^2x_0 + 9x_0^5x_1^2, 4y_1^2x_0^4 - 9y_0^2x_0^2x_1^2, \\ & 4y_1^2y_0^2x_1 + 9x_0^4x_1^3, 4y_1^2x_0^3x_1 - 9y_0^2x_0x_1^3, 8y_1^3y_0x_0 + 27x_0^4x_1^3, \\ & 12y_1^2x_0^3x_1 - 18y_0y_1x_0^2x_1^2, 8y_1^3y_0x_1 + 27x_0^3x_1^4, 12y_1^2x_0^2x_1^2 - 18y_0y_1x_0x_1^3, \\ & 16y_1^4x_0 + 81x_0^3x_1^4, 16y_1^4x_1 + 81x_0^2x_1^5\}. \end{aligned}$$

Now, from a direct computation using the algorithm described in subsection 3.2.8 we obtain a Groebner basis of $\{f\} \cap B_1$ for the monomial order $y_1 >_{\text{lex}} y_0 >_{\text{lex}} x_1 >_{\text{lex}} x_0$:

$$\mathfrak{B} = \{2y_1x_0 - 3y_0x_1, f, x_1x_0^6 + x_1y_0^4, 2y_0^3y_1x_1 + 3x_0^5x_1^2, 4y_1^2y_0^2x_1 + 9x_0^4x_1^3, \\ 8y_1^3y_0x_1 + 27x_0^3x_1^4, 16y_1^4x_1 + 81x_0^2x_1^5\}$$

We observe that $\langle \mathfrak{B} \rangle = \langle \mathfrak{C} \rangle$. However, if we consider the monomial order $x_0 >_{\text{lex}} x_1 >_{\text{lex}} y_0 >_{\text{lex}} y_1$ neither \mathfrak{C} nor \mathfrak{B} are Groebner bases of $\{f\} \cap B_1$.

3.6.9 The proof of theorem 3.6.6 is presented in subsection 3.6.18 and is based on results in subsection 3.6.12 and the present one. A key ingredient in our proof is to pass from the weighted homogeneous setting to the homogeneous one. For this, let us call $C_0 = k[u_0, v_0]$ and $C_1 = k[u_0, v_0, u_1, v_1]$. We set a pair of coprime integers $(r, s) \in \mathbb{N}^2$ with $r > s \geq 2$. We consider the morphism of k -modules $\rho : B_1 \rightarrow C_1$ given by $x_0 \mapsto u_0^s, y_0 \mapsto v_0^r, x_1 \mapsto su_0^{s-1}u_1, y_1 \mapsto rv_0^{r-1}v_1$. The following lemmas provide some properties of the morphism ρ allowing to relate both settings.

Lemma 3.6.10. *Let $P \in B_1$. Then $\rho(\tilde{\text{ev}}_1(P)) = \text{ev}_1(\rho(P))$.*

Proof. On the one hand,

$$\begin{aligned} \rho(\tilde{\text{ev}}_1(P(x_0, y_0, x_1, y_1))) &= \rho(P(x_0, y_0, sx_0, ry_0)) \\ &= P(u_0^s, v_0^r, su_0^s, rv_0^r) \end{aligned}$$

On the other hand,

$$\begin{aligned} \text{ev}_1(\rho(P(x_0, y_0, x_1, y_1))) &= \text{ev}_1(P(u_0^s, v_0^r, su_0^{s-1}u_1, rv_0^{r-1}v_1)) \\ &= P(u_0^s, v_0^r, su_0^s, rv_0^r) \end{aligned}$$

□

Lemma 3.6.11. *The morphism ρ is injective. Moreover, if $M = v_1^{a_1}v_0^{a_2}u_1^{a_3}u_0^{a_4} \in C_1$, then the following assertions are equivalent:*

1. *The monomial M belongs to the image $\text{Im}(\rho)$ of the morphism ρ .*
2. *The following conditions hold true:*

$$\left\{ \begin{array}{l} r|a_1 + a_2; \\ \frac{a_1 + a_2}{r} \geq a_1; \\ s|a_3 + a_4; \\ \frac{a_3 + a_4}{s} \geq a_3. \end{array} \right. \quad (6.3)$$

If these conditions hold, we have $\rho^{-1}(M) = \frac{1}{r^{a_1}s^{a_3}}y_1^{a_1}y_0^{\frac{a_1+a_2}{r}-a_1}x_1^{a_3}x_0^{\frac{a_3+a_4}{s}-a_3}$.

Proof. It suffices to see that a monomial $M = v_1^{a_1} v_0^{a_2} u_1^{a_3} u_0^{a_4} \in C_1$ has at most a unique preimage (which is that given in the statement). Let $\tilde{M} = \lambda_b y_1^{b_1} y_0^{b_2} x_1^{b_3} x_0^{b_4} \in B_1$, then $\rho(\tilde{M}) = \lambda_b r^{b_1} s^{b_3} v_1^{b_1} v_0^{r(b_1+b_2)-b_1} u_1^{b_3} u_0^{s(b_3+b_4)-b_3}$.

In order to have $M = \rho(\tilde{M})$, we need to find the nonnegative integer solutions of the system of equations obtained by equalling the exponents of each variable in M and $\rho(\tilde{M})$. One deduce that the solution exists if the four conditions in the statement are satisfied and in this case it is unique. After adjusting the coefficient we obtain

$$\tilde{M} = \frac{1}{r^{a_1} s^{a_3}} y_1^{a_1} y_0^{\frac{a_1+a_2}{r}-a_1} x_1^{a_3} x_0^{\frac{a_3+a_4}{s}-a_3}. \quad \square$$

3.6.12 Let $\lambda \in k^*$, we consider the weighted homogeneous polynomial $f = x_0^r - \lambda y_0^s \in B_0$. We set $g := \rho(f) = u_0^{rs} - \lambda v_0^{rs} \in C_0$; it is a homogeneous polynomial. We set $\mathcal{C} = \text{Spec}(B_0/\langle f \rangle)$ and $\mathcal{D} = \text{Spec}(C_0/\langle g \rangle)$. The following lemma relates $\mathcal{N}_1(\mathcal{C})$ and $\mathcal{N}_1(\mathcal{D})$ via the morphism ρ .

Lemma 3.6.13. *We have the equality $\rho(\mathcal{N}_1(\mathcal{C})) = \mathcal{N}_1(\mathcal{D}) \cap \text{Im}(\rho)$.*

Proof. Let $P \in \mathcal{N}_1(\mathcal{C})$, then by lemma 3.6.4 we know that $\tilde{e}v_1(P) \in \langle f \rangle$; hence we have $\rho(\tilde{e}v_1(P)) \in \rho(\langle f \rangle) \subset \langle g \rangle$. By lemma 3.6.10, it means that $ev_1(\rho(P)) \in \langle g \rangle$. Since the polynomial g is reduced and homogeneous, by lemma 3.4.10, we deduce that this condition is equivalent to $\rho(P) \in \mathcal{N}_1(\mathcal{D})$. Conversely, let $Q \in \mathcal{N}_1(\mathcal{D}) \cap \text{Im}(\rho)$. Since the morphism ρ is injective, there exists a unique $P \in B_1$ such that $\rho(P) = Q$. By lemmas 3.6.10 and 3.4.10, $\rho(\tilde{e}v_1(P)) = ev_1(\rho(P)) = ev_1(Q) \in \langle g \rangle = \langle \rho(f) \rangle$.

◦ We assume that $\rho_{|_{B_0}}(\langle f \rangle) = \langle \rho(f) \rangle \cap \text{Im}(\rho_{|_{B_0}})$ (where we see $\langle f \rangle$ and $\langle \rho(f) \rangle$ respectively as ideals in B_0 and C_0). Then, by the injectivity of the morphism ρ , we deduce that $\tilde{e}v_1(P) \in \langle f \rangle$ and conclude the proof by lemma 3.6.4.

◦ Let us prove $\rho_{|_{B_0}}(\langle f \rangle) = \langle \rho(f) \rangle \cap \text{Im}(\rho_{|_{B_0}})$. We only have to prove that $\rho_{|_{B_0}}(\langle f \rangle) \supset \langle \rho(f) \rangle \cap \text{Im}(\rho_{|_{B_0}})$. Let $R \in B_0$ such that $\rho(R) \in \langle \rho(f) \rangle$ (seen as an ideal in C_0). Then there exists a polynomial $S \in C_0$ such that $\rho(R) = S\rho(f)$. Let us show that $S \in \text{Im}(\rho)$. Each monomial $v_0^{a_2} u_0^{a_4}$ of $\rho(R)$ is, by our assumption, in the form $v_0^{a_2} u_0^{a_4} = v_0^{b_2+c_2} u_0^{b_4+c_4}$, where $v_0^{b_2} u_0^{b_4}$ (respectively $v_0^{c_2} u_0^{c_4}$) is a term of S (respectively $\rho(f)$). But $\rho(R)$ and $\rho(f)$ being in the image of ρ , by lemma 3.6.11, r divides a_2 and c_2 ; from $a_2 = b_2 + c_2$ we then deduce that r also divides b_2 . Analogously we have that $s|b_4$, then $v_0^{b_2} u_0^{b_4}$ belongs to $\text{Im}(\rho)$; hence the polynomial S also does. \square

Let $\lambda_1, \dots, \lambda_t \in k$ be nonzero elements. For every integer $i \in \{1, \dots, t\}$ we set $f_i = x_0^r - \lambda_i y_0^s$ and $\mathcal{C}_i = \text{Spec}(B_0/\langle f_i \rangle)$. We prove that lemma 3.6.13 can be extended to this setting.

Proposition 3.6.14. *Let k be a field of characteristic zero. Let $\lambda_1, \dots, \lambda_t \in k$ be nonzero and mutually distinct elements. Let $(r, s) \in \mathbb{N}^2$ be a pair of coprime integers with $r > s \geq 2$. Let $f \in B_0$ be the polynomial $f = \prod_{i=1}^t (x_0^r - \lambda_i y_0^s)$ and $g := \rho(f) \in C_0$*

its image by the morphism ρ . We set $\mathcal{C} = \text{Spec}(B_0/\langle f \rangle)$ and $\mathcal{D} = \text{Spec}(C_0/\langle g \rangle)$. Then $\rho(\mathcal{N}_1(\mathcal{C})) = \mathcal{N}_1(\mathcal{D}) \cap \text{Im}(\rho)$.

Proof. As the f_i are the irreducible factors of f , from the Kolchin irreducibility theorem (see theorem 2.3.10), we deduce the formula:

$$\mathcal{N}_1(\mathcal{C}) := \{f\} \cap B_1 = \bigcap_{i=1}^t (\{f_i\} \cap B_1) =: \bigcap_{i=1}^t \mathcal{N}_1(\mathcal{C}_i). \quad (6.4)$$

For every integer $i \in \{1, \dots, t\}$, we set $g_i := \rho(f_i) = u_0^{rs} - \lambda_i v_0^{rs} \in C_0$ and $\mathcal{D}_i = \text{Spec}(C_0/\langle g_i \rangle)$. From (6.4), the injectivity of the morphism ρ and lemma 3.6.13, we deduce the following equalities:

$$\rho(\mathcal{N}_1(\mathcal{C})) = \bigcap_{i=1}^t \rho(\mathcal{N}_1(\mathcal{C}_i)) = \bigcap_{i=1}^t (\mathcal{N}_1(\mathcal{D}_i) \cap \text{Im}(\rho)). \quad (6.5)$$

On the other hand, by subsection 2.3.13 we may assume that the field k is algebraically closed. For every integer $i \in \{1, \dots, t\}$, let $g_i^{(j)}$ ($j \in J_i$) be the irreducible factors of the polynomial g_i ; hence the decomposition $g = \prod_{i=1}^t \prod_{j \in J_i} g_i^{(j)}$ is the decomposition of g into irreducible factors. By applying the Kolchin irreducibility theorem 2.3.10, we obtain

$$\mathcal{N}_1(\mathcal{D}_i) = \{g_i\} \cap C_1 = \bigcap_{j \in J_i} \{g_i^{(j)}\} \cap C_1,$$

and

$$\mathcal{N}_1(\mathcal{D}) = \{g\} \cap C_1 = \bigcap_{i=1}^t \bigcap_{j \in J_i} \{g_i^{(j)}\} \cap C_1 = \bigcap_{i=1}^t \mathcal{N}_1(\mathcal{D}_i). \quad (6.6)$$

We conclude the proof directly from formulas (6.5) and (6.6). \square

Remark 3.6.15. Let us observe that proposition 3.6.14 and lemma 3.4.10 yield the following characterization. Let $P \in B_1$. Then, we have $P \in \mathcal{N}_1(\mathcal{C})$ if and only if $\rho(P) \in \rho(\mathcal{N}_1(\mathcal{C}))$ (because of the injectivity of the morphism ρ), if and only if $\rho(P) \in \mathcal{N}_1(\mathcal{D}) \cap \text{Im}(\rho)$, if and only if $\text{ev}_1(\rho(P)) \in \langle g \rangle = \langle g_1 \cdots g_t \rangle$.

3.6.16 We first prove the following less general version of theorem 3.6.6 which solves the case of a product of cusps.

Proposition 3.6.17. *Let k be a field of characteristic zero. Let $\lambda_1, \dots, \lambda_t \in k$ be nonzero and mutually distinct elements. Let $(r, s) \in \mathbb{N}^2$ be a pair of coprime integers with $r > s \geq 2$. Let $f \in B_0$ be the polynomial $f = \prod_{i=1}^t (x_0^r - \lambda_i y_0^s)$. We set $\mathcal{C} = \text{Spec}(B_0/\langle f \rangle)$. Then the family*

$$\mathfrak{B} = \{\tilde{D}_{-1}, \tilde{D}_{j_1, \dots, j_t} : j_i \in \{-1, \dots, s\}, i \in \{1, \dots, t\}\}$$

is a Groebner basis of $\mathcal{N}_1(\mathcal{C})$ for the monomial order $y_1 >_{\text{lex}} y_0 >_{\text{lex}} x_1 >_{\text{lex}} x_0$ in B_1 .

Proof. By applying lemma 3.6.4 to every element in \mathfrak{B} for each of the f_i , $i \in \{1, \dots, t\}$, and equality (6.4), we deduce that $\mathfrak{B} \subset \mathcal{N}_1(\mathcal{C})$. By [3, Proposition 5.38], in order to show that the family \mathfrak{B} is a Groebner basis of $\mathcal{N}_1(\mathcal{C})$ it suffices to prove that every element in $\mathcal{N}_1(\mathcal{C})$ has some term in $\langle \text{LT}(\mathfrak{B}) \rangle$. Let us compute the leading terms of the elements of \mathfrak{B} for the considered monomial order.

◦ We have $\text{LT}(sy_1x_0 - ry_0x_1) = y_1x_0$.

◦ For $\widetilde{D}_{j_1, \dots, j_t}$, let us denote $\ell := \#\{j_i : j_i \neq -1\}$. Then, for $j_i \in \{-1, \dots, s\}$, $i \in \{1, \dots, t\}$, we have

$$\begin{aligned} \widetilde{D}_{j_1, \dots, j_t} &= \widetilde{D}_{\lambda_1, j_1} \cdots \widetilde{D}_{\lambda_t, j_t} \\ &= \prod_{j_i \neq -1} (\lambda_i s^{j_i} y_0^{s-j_i} y_1^{j_i} - r^{j_i} x_0^{r-j_i} x_1^{j_i}) (sy_1x_0 - ry_0x_1)^{t-\ell} \\ &= \lambda_{i_1} \cdots \lambda_{i_\ell} s^{t-\ell+j_{i_1}+\dots+j_{i_\ell}} y_1^{t-\ell+(j_{i_1}+\dots+j_{i_\ell})} y_0^{\ell s-(j_{i_1}+\dots+j_{i_\ell})} x_0^{t-\ell} + \dots \end{aligned}$$

Hence, we deduce that $\text{LT}(\widetilde{D}_{j_1, \dots, j_t}) = y_1^{t-\ell+j_{i_1}+\dots+j_{i_\ell}} y_0^{\ell s-(j_{i_1}+\dots+j_{i_\ell})} x_0^{t-\ell}$.

We conclude that

$$\langle \text{LT}(\mathfrak{B}) \rangle = \left\langle y_1x_0, \left\{ y_1^{t-\ell+(j_{i_1}+\dots+j_{i_\ell})} y_0^{\ell s-(j_{i_1}+\dots+j_{i_\ell})} x_0^{t-\ell} \right\}_{\substack{0 \leq \ell \leq t \\ 0 \leq j_{i_1}, \dots, j_{i_\ell} \leq s}} \right\rangle.$$

Let $P \in \mathcal{N}_1(\mathcal{C})$. We aim to prove that some of its terms belongs to $\langle \text{LT}(\mathfrak{B}) \rangle$. By remark 3.6.15 we know that $\text{ev}_1(\rho(P)) \in \langle g \rangle = \langle g_1 \cdots g_t \rangle$. Two cases occur:

◦ If $\text{ev}_1(\rho(P)) = 0$, then, by lemma 3.6.10, we have $\rho(\widetilde{\text{ev}}_1(P)) = 0$. By the injectivity of the morphism ρ , we deduce that $\widetilde{\text{ev}}_1(P) = 0$. But, by proposition 3.3.6, this means that $P \in \langle sy_1x_0 - ry_0x_1 \rangle$; hence we conclude that the monomial y_1x_0 divides $\text{LT}(P)$.

◦ If $\text{ev}_1(\rho(P)) \neq 0$, then P has some term $y_1^{a_1} y_0^{a_2} x_1^{a_3} x_0^{a_4}$ such that $\text{LT}(g) = v_0^{trs}$ (we are considering the monomial order $v_1 >_{\text{lex}} v_0 >_{\text{lex}} u_1 >_{\text{lex}} u_0$ in C_1) divides $\text{ev}_1(\rho(y_1^{a_1} y_0^{a_2} x_1^{a_3} x_0^{a_4})) = \text{ev}_1(v_1^{a_1} v_0^{r(a_1+a_2)-a_1} u_1^{a_3} u_0^{s(a_3+a_4)-a_3}) = v_0^{r(a_1+a_2)} u_0^{s(a_3+a_4)}$. Thus, it implies that $trs \leq r(a_1 + a_2)$; hence, we have $ts \leq a_1 + a_2$. For $0 \leq j_1, \dots, j_t \leq s$, the pairs $(j_1 + \dots + j_t, ts - (j_1 + \dots + j_t))$ range over all possible pairs of nonnegative integers whose sum equals ts . Thus some monomial in $\{y_1^{j_1+\dots+j_t} y_0^{ts-(j_1+\dots+j_t)}\}_{0 \leq j_1, \dots, j_t \leq s}$ (which is a subset of $\text{LT}(\mathfrak{B})$, take $\ell = t$) divides the term $y_1^{a_1} y_0^{a_2}$, and hence also the term $y_1^{a_1} y_0^{a_2} x_1^{a_3} x_0^{a_4}$. \square

3.6.18 Let us prove theorem 3.6.6.

For every integer $i \in \{1, \dots, t\}$, we set $f_i = x_0^r - \lambda_i y_0^s \in B_0$, $f_{\text{cusp}} = \prod_{i=1}^t f_i \in B_0$, $g_i = \rho(f_i) = u_0^{rs} - \lambda_i v_0^{rs} \in C_0$ and $g_{\text{cusp}} = \rho(f_{\text{cusp}}) \in C_0$. The corresponding affine plane k -curves are respectively denoted by $\mathcal{C}_i = \text{Spec}(B_0/\langle f_i \rangle)$ and $\mathcal{C}_{\text{cusp}} = \text{Spec}(B_0/\langle f_{\text{cusp}} \rangle)$. We write \mathcal{C}_x (resp. \mathcal{C}_y) for the affine plane k -curve associated with the datum of x_0^ε (resp. $y_0^{\varepsilon'}$). By applying the Kolchin theorem as in the proof of proposition 3.6.14, we

deduce that $\mathcal{N}_1(\mathcal{C}) = \mathcal{N}_1(\mathcal{C}_x) \cap \mathcal{N}_1(\mathcal{C}_y) \cap \mathcal{N}_1(\mathcal{C}_{cusp})$. Then, by the injectivity of ρ and remark 3.6.15 applied to \mathcal{C}_{cusp} , we deduce that, if we take a polynomial P in $\mathcal{N}_1(\mathcal{C})$, then $\text{ev}_1(\rho(P))$ belongs to $\langle u_0^{s\varepsilon} \rangle \cap \langle v_0^{r\varepsilon'} \rangle \cap \langle g_{cusp} \rangle$. We can write $\text{ev}_1(\rho(P))$ in the form $\text{ev}_1(\rho(P)) = Qg_{cusp}$ for a polynomial $Q \in C_0$, and $u_0^{s\varepsilon}$ and $v_0^{r\varepsilon'}$ divide Qg_{cusp} . Let us recall that $g_{cusp} = \prod_{i=1}^t g_i = \prod_{i=1}^t u_0^{rs} - \lambda_i v_0^{rs} = (-1)^t \lambda_1 \cdots \lambda_t v_0^{trs} + \cdots + u_0^{trs}$. A direct calculation then shows that $u_0^{s\varepsilon} v_0^{r\varepsilon'}$ divides Q . So, the polynomial $\text{ev}_1(\rho(P))$ can be written as $Q' u_0^{s\varepsilon} v_0^{r\varepsilon'} g_{cusp}$. We conclude that the polynomial $\text{ev}_1(\rho(P))$ belongs to the ideal $\langle u_0^{s\varepsilon} v_0^{r\varepsilon'} g_{cusp} \rangle = \langle \rho(f) \rangle$. It is clear that $\mathfrak{B} \subset \mathcal{N}_1(\mathcal{C})$. By [3, Proposition 5.38], in order to show that \mathfrak{B} is a Groebner basis of $\mathcal{N}_1(\mathcal{C})$ it is sufficient to prove that every element in $\mathcal{N}_1(\mathcal{C})$ has some term in $\langle \text{LT}(\mathfrak{B}) \rangle$. From the computations in the first part of the proof of proposition 3.6.17 we deduce that

$$\langle \text{LT}(\mathfrak{B}) \rangle = \left\langle y_1 x_0, \left\{ x_{h_1}^\varepsilon y_{h_2}^{\varepsilon'} y_1^{t-\ell+j_{i_1}+\cdots+j_{i_\ell}} y_0^{\ell s-(j_{i_1}+\cdots+j_{i_\ell})} x_0^{t-\ell} \right\}_{\substack{0 \leq \ell \leq t \\ 0 \leq j_{i_1}, \dots, j_{i_\ell} \leq s \\ 0 \leq h_1, h_2 \leq 1}} \right\rangle.$$

Let P be a polynomial in $\mathcal{N}_1(\mathcal{C})$. We have already observed that $\text{ev}_1(\rho(P))$ belongs to $\langle u_0^{s\varepsilon} v_0^{r\varepsilon'} g_{cusp} \rangle$. Then, we finish the proof in an analogous way as we did in the proof of proposition 3.6.17. Two cases occur:

◦ If $\text{ev}_1(\rho(P)) = 0$, then, by lemma 3.6.10, we have $\rho(\widetilde{\text{ev}}_1(P)) = 0$. By the injectivity of the morphism ρ , we deduce that $\widetilde{\text{ev}}_1(P) = 0$. But, by proposition 3.3.6, this means that $P \in \langle sy_1 x_0 - ry_0 x_1 \rangle$; hence we conclude that the monomial $y_1 x_0$ divides $\text{LT}(P)$.

◦ If $\text{ev}_1(\rho(P)) \neq 0$ then P has some term $y_1^{a_1} y_0^{a_2} x_1^{a_3} x_0^{a_4}$ such that $\text{LT}(u_0^{s\varepsilon} v_0^{r\varepsilon'} g_{cusp}) = v_0^{trs+r\varepsilon'} u_0^{s\varepsilon}$ (we are considering the monomial order $v_1 >_{\text{lex}} v_0 >_{\text{lex}} u_1 >_{\text{lex}} u_0$ in C_1) divides $\text{ev}_1(\rho(y_1^{a_1} y_0^{a_2} x_1^{a_3} x_0^{a_4})) = \text{ev}_1(v_1^{a_1} v_0^{r(a_1+a_2)-a_1} u_1^{a_3} u_0^{s(a_3+a_4)-a_3}) = v_0^{r(a_1+a_2)} u_0^{s(a_3+a_4)}$. Hence $trs + r\varepsilon' \leq r(a_1 + a_2)$ and then $ts + \varepsilon' \leq a_1 + a_2$, and $\varepsilon s \leq s(a_3 + a_4)$ and then $\varepsilon \leq a_3 + a_4$. We deduce that $y_1^{a_1} y_0^{a_2} x_1^{a_3} x_0^{a_4}$ is divisible by some of the terms in $\left\{ x_{h_1}^\varepsilon y_{h_2}^{\varepsilon'} y_1^{j_1+\cdots+j_t} y_0^{ts-(j_1+\cdots+j_t)} \right\}_{\substack{0 \leq j_1, \dots, j_t \leq s \\ 0 \leq h_1, h_2 \leq 1}}$, which is a subset of $\text{LT}(\mathfrak{B})$ taking $\ell = t$.

Remark 3.6.19. From the last part of the proof of theorem 3.6.6 we observe that not every element in \mathfrak{B} is essential, in the sense that the final argument also holds for a subset of \mathfrak{B} and hence it is also a Groebner basis of $\mathcal{N}_1(\mathcal{C})$. Precisely, one may take only the elements of the form $x_{h_1}^\varepsilon y_{h_2}^{\varepsilon'} \widetilde{D}_{j_1, \dots, j_t}$ for $h_1, h_2 \in \{0, 1\}$, $j_i \in \{0, \dots, s\}$ and $i \in \{1, \dots, t\}$; that is, it is not necessary to consider the element $sy_1 x_0 - ry_0 x_1$ in the products. Moreover, between those elements, for every $e \in \{0, \dots, ts\}$ it suffices to consider the elements of the form $x_{h_1}^\varepsilon y_{h_2}^{\varepsilon'} \widetilde{D}_{j_1, \dots, j_t}$ for $h_1, h_2 \in \{0, 1\}$ and a unique choice of $j_1, \dots, j_t \in \{0, \dots, s\}$ such that $j_1 + \cdots + j_t = e$.

For example, the following family is a Groebner basis of $\mathcal{N}_1(\mathcal{C})$ (we keep the notation and assumptions of theorem 3.6.6):

$$\mathfrak{B} = \left\{ sy_1 x_0 - ry_0 x_1, x_{h_1}^\varepsilon y_{h_2}^{\varepsilon'} \widetilde{D}_{j_1, \dots, j_t} : h_1, h_2 \in \{0, 1\}, j_i \in \{0, \dots, s\}, i \in \{1, \dots, t\} \right\}$$

where j_i is zero unless $j_m = s$ for every integer $m < i$.

3.7 AN APPLICATION IN THE FIELD OF DIFFERENTIAL OPERATORS

For a curve \mathcal{C} defined by a reduced polynomial $f \in B_0$, there exists a relation between $\mathcal{N}_1(\mathcal{C})$ and a class of differential operators of the plane related with f . Hence the main results in sections 3.5 and 3.6 have a translation in this setting, which we will explain in the present section. First, let us introduce the basics about differential operators (see [21] for a precise reference).

3.7.1 Let k be a field of characteristic zero and R be a k -algebra. We define inductively in d the set of *differential operators* of R over k of order at most d , $\mathcal{D}_{R|k}^d$, as the following set of k -linear maps $R \rightarrow R$. For $d = 0$ we identify an element $a \in R$ with the k -linear map $a : R \rightarrow R$ corresponding with the multiplication by a , $r \mapsto ar$ for every $r \in R$. Then $\mathcal{D}_{R|k}^0 := \{a : R \rightarrow R; a \in R\}$. For $d \geq 1$ we define $\mathcal{D}_{R|k}^d$ as the set of k -linear maps $D : R \rightarrow R$ such that the bracket $[D, a] := D \circ a - a \circ D$ belongs to $\mathcal{D}_{R|k}^{d-1}$ for every $a \in R$. We call

$$\mathcal{D}_{R|k} := \bigcup_{d \in \mathbb{N}} \mathcal{D}_{R|k}^d \tag{7.1}$$

the *ring of differential operators* of R over k (with the multiplication given by the composition, it is well-defined because if $D_1 \in \mathcal{D}_{R|k}^{d_1}$ and $D_2 \in \mathcal{D}_{R|k}^{d_2}$ then $D_1 \circ D_2 \in \mathcal{D}_{R|k}^{d_1+d_2}$, see [21, Ch.3, Proposition 1.2]). We observe that this ring is not-necessarily commutative.

The ring $\mathcal{D}_{R|k}$ is filtered by the order (see (7.1)). Let us consider the corresponding *graded ring* (with the assumption $\mathcal{D}_{R|k}^{-1} = 0$)

$$\text{gr}(\mathcal{D}_{R|k}) := \bigoplus_{d \in \mathbb{N}} \mathcal{D}_{R|k}^d / \mathcal{D}_{R|k}^{d-1}. \tag{7.2}$$

For every $d \in \mathbb{N}$, the canonical projection $\sigma_d : \mathcal{D}_{R|k}^d \rightarrow \mathcal{D}_{R|k}^d / \mathcal{D}_{R|k}^{d-1}$ is called the *symbol map* of order d . If a differential operator $D \in \mathcal{D}_{R|k}$ has order $d \in \mathbb{N}$, i.e., if $D \in \mathcal{D}_{R|k}^d \setminus \mathcal{D}_{R|k}^{d-1}$, then its d -symbol $\sigma_d(D)$ is called its *principal symbol* and will be simply denoted by $\sigma(D)$.

3.7.2 Let us now consider the particular case of $R = B_0 = k[x_0, y_0]$. An element D of $\mathcal{D}_{B_0|k}$ is called a *differential operator of the plane*. By [21, Ch.3, Theorem 2.3] we know that $\mathcal{D}_{B_0|k}$ is isomorphic to the second Weyl algebra, i.e., the k -subalgebra of the algebra of k -linear maps $B_0 \rightarrow B_0$ (with the addition and the composition of maps) generated by the operators x_0, y_0 (corresponding with the multiplication maps) and $\partial_{x_0}, \partial_{y_0}$ (see [21, Ch.1, §1]). In fact, this theorem assures that if $D \in \mathcal{D}_{B_0|k}^d$, it can be written as a

formal combination with coefficients in B_0 of the form $D = \sum_{i+j \leq d} a_{i,j} \partial_{x_0}^i \partial_{y_0}^j$. If $D \in \mathcal{D}_{R|k}$ has order $d \in \mathbb{N}$, in a slight abuse of notation we will write $\sigma(D) = \sum_{i+j=d} a_{i,j} \partial_{x_0}^i \partial_{y_0}^j$.

By [21, Ch.7, Theorem 3.1] the graded ring $\text{gr}(\mathcal{D}_{B_0|k})$ is a commutative polynomial ring which can be identified with B_1 with the partial degree \deg_1 (see subsection 3.1.1). The variable x_1 corresponds with the symbol $\sigma_1(\partial_{x_0})$ and the variable y_1 with the symbol $\sigma_1(\partial_{y_0})$. This identification is the coordinate-dependent version of the intrinsic isomorphism between $\text{Sym}(\text{Der}_k(B_0))$ and $\text{gr}(\mathcal{D}_{B_0|k})$.

3.7.3 Let $f \in B_0$ be a reduced polynomial and $\mathcal{C} = \text{Spec}(B_0/\langle f \rangle)$ the associated affine plane curve. We consider the V -filtration $V_\star^\mathcal{C}$ of $\mathcal{D}_{B_0|k}$ along \mathcal{C} which is defined, for every integer $r \in \mathbb{Z}$, by

$$V_r^\mathcal{C} := \{D \in \mathcal{D}_{B_0|k} \mid \forall \ell \in \mathbb{Z} \ D(\langle f \rangle^\ell) \subset \langle f \rangle^{\ell-r}\},$$

with the convention $\langle f \rangle^s = B_0$ for $s \leq 0$. From the inductive definition of the differential operators we deduce that $\mathcal{D}_{B_0|k}^d \subset V_d^\mathcal{C}$ for every $d \in \mathbb{N}$. For the general definition and more details on the V -filtration see [17], [53] or [59]. The following result is a part of [35, Corollary 4.4], see also [73].

Lemma 3.7.4 (Gros-Narváez Macarro-Sebag). *Let k be a field of characteristic zero. Let $f \in B_0$ be a reduced polynomial and $\mathcal{C} = \text{Spec}(B_0/\langle f \rangle)$ the associated affine plane curve. Let $D \in \mathcal{D}_{B_0|k}$ be a differential operator of order $d \in \mathbb{N}$. The following assertions are equivalent:*

1. *The operator D verifies $D(f^d) \in \langle f \rangle$.*
2. *The operator D belongs to $V_{d-1}^\mathcal{C}$.*

3.7.5 Let us now state the main result relating differential operators and elements of $\mathcal{N}_1(\mathcal{C})$ for \mathcal{C} an affine plane curve defined by a reduced polynomial $f \in B_0$. We denote by $\sharp : B_1 \rightarrow B_1$ the morphism of B_0 -algebras defined by

$$P(x_0, y_0, x_1, y_1) \mapsto P^\sharp := P(x_0, y_0, -y_1, x_1).$$

Let us recall that, by subsection 3.7.2, the graded ring $\text{gr}(\mathcal{D}_{B_0|k})$ can be identified with the polynomial ring B_1 , hence in particular via this identification we can apply the morphism \sharp to the principal symbol $\sigma(D)$ of a differential operator of the plane $D \in \mathcal{D}_{B_0|k}$. The following theorem is part of [35, Corollary 4.10], combined with lemma 3.7.4, see also [73].

Theorem 3.7.6 (Gros-Narváez Macarro-Sebag). *Let k be a field of characteristic zero. Let $f \in B_0$ be a reduced polynomial and $\mathcal{C} = \text{Spec}(B_0/\langle f \rangle)$ the associated affine plane curve. Let $P \in B_1$ be a homogeneous polynomial with $\deg_1(P) = d \geq 0$. Then, the following statements are equivalent:*

- (1) The polynomial P belongs to $\mathcal{N}_1(\mathcal{C})$.
- (2) For every differential operator $D \in \mathcal{D}_{B_0|k}^d$ such that $\sigma_d(D)^\sharp = P$, we have $D(f^d) \in \langle f \rangle$.
- (3) There exist a differential operator $D \in \mathcal{D}_{B_0|k}^d$ and an integer $e \geq d$ such that $\sigma_d(D)^\sharp = P$ and $D(f^e) \in \langle f^{e-d+1} \rangle$.
- (4) For every differential operator $D \in \mathcal{D}_{B_0|k}^d$ of order d such that $\sigma_d(D)^\sharp = P$, we have $D \in V_{d-1}^{\mathcal{C}}$.
- (5) There exists a differential operator $D \in \mathcal{D}_{B_0|k}^d$ of order d such that $\sigma_d(D)^\sharp = P$ and $D \in V_{d-1}^{\mathcal{C}}$.

3.7.7 The morphism $\sharp : B_1 \rightarrow B_1$ being a B_0 -automorphism, we can denote by $\flat : B_1 \rightarrow B_1$ its inverse, which is also a B_0 -automorphism given by

$$P(x_0, y_0, x_1, y_1) \mapsto P^\flat := P(x_0, y_0, y_1, -x_1).$$

Our study of $\mathcal{N}_1(\mathcal{C})$ for \mathcal{C} an affine plane curve defined by a reduced polynomial $f \in B_0$ in sections 3.5 (for f homogeneous) and 3.6 (for f weighted homogeneous) can be combined with theorem 3.7.6 in order to deduce the following results:

Corollary 3.7.8. *Let k be a field of characteristic zero. Let $\gamma_1, \dots, \gamma_t \in k$ be nonzero and mutually distinct elements. Let $f \in B_0$ be the polynomial $f = x_0^\varepsilon y_0^{\varepsilon'} \prod_{i=1}^t (y_0 - \gamma_i x_0)$ with $\varepsilon, \varepsilon' \in \{0, 1\}$. We set $\mathcal{C} = \text{Spec}(B_0/\langle f \rangle)$. Let $D \in \mathcal{D}_{B_0|k}^d$ be a differential operator of the plane of order d such that $D(f^d) \in \langle f \rangle$ (or equivalently, such that $D \in V_{d-1}^{\mathcal{C}}$). Then, via the identification in subsection 3.7.2 between $\text{gr}(\mathcal{D}_{B_0|k})$ and B_1 , its principal symbol $\sigma_d(D)$ is a combination in B_1 of the following family of polynomials (in the notation of subsection 3.5.4):*

$$\left\{ x_1 x_0 + y_0 y_1, (x_{h_1}^\varepsilon y_{h_2}^{\varepsilon'} \delta_i(f))^\flat, i \in \{0, \dots, t\}, h_1, h_2 \in \{0, 1\} \right\}.$$

Proof. It is direct from theorem 3.7.6, theorem 3.5.5 and the fact that a Groebner basis of an ideal is in particular a system of generators. \square

Corollary 3.7.9. *Let k be a field of characteristic zero. Let $\lambda_1, \dots, \lambda_t \in k$ be nonzero and mutually distinct elements. Let $(r, s) \in \mathbb{N}^2$ be a pair of coprime integers with $r > s \geq 2$. Let $f \in B_0$ be the polynomial $f = x_0^\varepsilon y_0^{\varepsilon'} \prod_{i=1}^t (x_0^r - \lambda_i y_0^s)$ with $\varepsilon, \varepsilon' \in \{0, 1\}$. We set $\mathcal{C} = \text{Spec}(B_0/\langle f \rangle)$. Let $D \in \mathcal{D}_{B_0|k}^d$ be a differential operator of the plane of order d such that $D(f^d) \in \langle f \rangle$ (or equivalently, such that $D \in V_{d-1}^{\mathcal{C}}$). Then, via the identification in subsection 3.7.2 between $\text{gr}(\mathcal{D}_{B_0|k})$ and B_1 , its principal symbol $\sigma_d(D)$ is a combination in B_1 of the following family of polynomials (in the notation of subsection 3.6.5):*

$$\left\{ \widetilde{D}_{-1}^\flat, (x_{h_1}^\varepsilon y_{h_2}^{\varepsilon'} \widetilde{D}_{j_1, \dots, j_t})^\flat, j_i \in \{-1, \dots, s\}, i \in \{1, \dots, t\}, h_1, h_2 \in \{0, 1\} \right\}.$$

Proof. It is direct from theorem 3.7.6, theorem 3.6.6 and the fact that a Groebner basis of an ideal is in particular a system of generators. \square

3.7.10 M. Gros, L. Narváez Macarro and J. Sebag also established a relation between elements in $\mathcal{N}_1(\mathcal{C})$ for \mathcal{C} an affine plane curve defined by a reduced polynomial $f \in B_0$ and Bernstein-Sato operators for f , which gives another application of our study in sections 3.5 and 3.6. Let us first present the *Bernstein-Sato construction*, for more information see [4] or the survey [78].

Let s be a new variable, we consider the polynomial ring $\mathcal{D}_{B_0|k}[s]$. The order of a differential operator $P(s) = \sum_i P_i s^i \in \mathcal{D}_{B_0|k}[s]$ is defined as $\text{ord}(P(s)) := \max\{\text{ord}(P_i)\}$. Then the ring $\mathcal{D}_{B_0|k}[s]$ is filtered by the order and, using the identification in subsection 3.7.2 between $\text{gr}(\mathcal{D}_{B_0|k})$ and B_1 , the corresponding graded ring $\text{gr}(\mathcal{D}_{B_0|k}[s])$ can be identified with the (commutative) polynomial ring $B_1[s]$ with the unique extension of the partial degree deg_1 such that $\text{deg}_1(s) = 0$. If $\text{ord}(P(s)) = d$, then its d -symbol $\sigma_d(P(s)) := \sum_{i \in J} \sigma_d(P_i) s^i$, with $J = \{i \in \mathbb{N} : \text{ord}(P_i) = d\}$, is a 1-homogeneous polynomial of degree d in $B_1[s]$, which will be called the *principal symbol* of $P(s)$ and denoted by $\sigma(P(s))$.

Now let $f \in B_0$ be a non-constant polynomial. We denote by $B_0[f^{-1}, s]f^s$ the free module generated by the symbol f^s over the ring $B_0[f^{-1}, s]$. It is a B_0 -module and via the rule

$$\partial_{\dagger} \bullet \left(\frac{g}{f^k} f^s \right) = \partial_{\dagger} \left(\frac{g}{f^k} \right) f^s + \frac{sg}{f^{k+1}} \partial_{\dagger}(f) f^s$$

for $g \in B_0[s]$ and $\dagger \in \{x_0, y_0\}$ it becomes a left $\mathcal{D}_{B_0|k}[s]$ -module. We recall that elements in B_0 can be identified with multiplication morphisms and seen as differential operators, see subsection 3.7.1. We denote by

$$\text{ann}_{\mathcal{D}_{B_0|k}[s]}(f^s) = \left\{ P(s) \in \mathcal{D}_{B_0|k}[s] \mid P(s) \bullet f^s = 0 \right\}$$

the *parametric annihilator*. Let us recall Bernstein's theorem (see [4]):

Theorem 3.7.11 (Bernstein). *Let k be a field of characteristic zero. Let $f \in B_0$. There exist a differential operator $P(s) \in \mathcal{D}_{B_0|k}[s]$ and a non-zero polynomial $b(s) \in k[s]$ such that*

$$P(s) \bullet f^{s+1} = b(s) f^s. \tag{7.3}$$

The set of polynomials $b(s) \in k[s]$ for which an equation of the form 7.3 holds is an ideal of $k[s]$, which is non-zero by Bernstein's theorem. The polynomial ring $k[s]$ being a principal ideal domain, we call its monic generator the *Bernstein-Sato polynomial* of f . We will call a *Bernstein-Sato operator* for f any differential operator $P(s) \in \mathcal{D}_{B_0|k}[s]$ satisfying equation 7.3 for a certain $b(s) \in k[s]$. In particular, we have $P(s)f - b(s) \in \text{ann}_{\mathcal{D}_{B_0|k}[s]}(f^s)$.

The following result, corresponding to [35, Theorem 5.3], establishes the relation announced at the beginning of this subsection between Bernstein-Sato operators and the ideal $\mathcal{N}_1(\mathcal{C})$.

Theorem 3.7.12 (Gros-Narváez Macarro-Sebag). *Let k be a field of characteristic zero. Let $f \in B_0$ be a reduced polynomial and $\mathcal{C} = \text{Spec}(B_0/\langle f \rangle)$ the associated affine plane curve. Let $P(s) = \sum_i P_i s^i \in \mathcal{D}_{B_0|k}[s]$ be a differential operator of order $d \geq 0$. Then, for every integer i such that $\text{ord}(P_i) = d$, we have $\sigma_d(P_i)^\sharp \in \mathcal{N}_1(\mathcal{C})$ provided that one of the following hypotheses holds:*

- (1) *The operator $P(s)$ belongs to $\text{ann}_{\mathcal{D}_{B_0|k}[s]}(f^s)$.*
- (2) *The operator $P(s)$ is a Bernstein-Sato operator for f of order $d \geq 2$.*

3.7.13 Our study of $\mathcal{N}_1(\mathcal{C})$ for \mathcal{C} an affine plane curve defined by a reduced polynomial $f \in B_0$ in sections 3.5 (for f homogeneous) and 3.6 (for f weighted homogeneous) can be combined with theorem 3.7.12 in order to deduce the following results:

Corollary 3.7.14. *Let k be a field of characteristic zero. Let $\gamma_1, \dots, \gamma_t \in k$ be nonzero and mutually distinct elements. Let $f \in B_0$ be the polynomial $f = x_0^\varepsilon y_0^{\varepsilon'} \prod_{i=1}^t (y_0 - \gamma_i x_0)$ with $\varepsilon, \varepsilon' \in \{0, 1\}$. We set $\mathcal{C} = \text{Spec}(B_0/\langle f \rangle)$. Let $P(s) = \sum_i P_i s^i \in \mathcal{D}_{B_0|k}^d[s]$ be a Bernstein-Sato differential operator for f of order $d \geq 2$, or a differential operator in $\text{ann}_{\mathcal{D}_{B_0|k}[s]}(f^s)$ of order d . Then, via the identification in subsection 3.7.2 between $\text{gr}(\mathcal{D}_{B_0|k})$ and B_1 , for every integer i such that $\text{ord}(P_i) = d$, its principal symbol $\sigma_d(P_i)$ is a combination in B_1 of the following family of polynomials (in the notation of subsection 3.5.4):*

$$\left\{ x_1 x_0 + y_0 y_1, (x_{h_1}^\varepsilon y_{h_2}^{\varepsilon'} \delta_i(f))^\flat, i \in \{0, \dots, t\}, h_1, h_2 \in \{0, 1\} \right\}.$$

Proof. It is direct from theorem 3.7.12, theorem 3.5.5 and the fact that a Groebner basis of an ideal is in particular a system of generators. \square

Corollary 3.7.15. *Let k be a field of characteristic zero. Let $\lambda_1, \dots, \lambda_t \in k$ be nonzero and mutually distinct elements. Let $(r, s) \in \mathbb{N}^2$ be a pair of coprime integers with $r > s \geq 2$. Let $f \in B_0$ be the polynomial $f = x_0^\varepsilon y_0^{\varepsilon'} \prod_{i=1}^t (x_0^r - \lambda_i y_0^s)$ with $\varepsilon, \varepsilon' \in \{0, 1\}$. We set $\mathcal{C} = \text{Spec}(B_0/\langle f \rangle)$. Let $P(s) = \sum_i P_i s^i \in \mathcal{D}_{B_0|k}^d[s]$ be a Bernstein-Sato differential operator for f of order $d \geq 2$, or a differential operator in $\text{ann}_{\mathcal{D}_{B_0|k}[s]}(f^s)$ of order d . Then, via the identification in subsection 3.7.2 between $\text{gr}(\mathcal{D}_{B_0|k})$ and B_1 , for every integer i such that $\text{ord}(P_i) = d$, its principal symbol $\sigma_d(P_i)$ is a combination in B_1 of the following family of polynomials (in the notation of subsection 3.6.5):*

$$\left\{ \tilde{D}_{-1}^\flat, (x_{h_1}^\varepsilon y_{h_2}^{\varepsilon'} \tilde{D}_{j_1, \dots, j_t})^\flat, j_i \in \{-1, \dots, s\}, i \in \{1, \dots, t\}, h_1, h_2 \in \{0, 1\} \right\}.$$

Proof. It is direct from theorem 3.7.12, theorem 3.6.6 and the fact that a Groebner basis of an ideal is in particular a system of generators. \square

Example 3.7.16. Let us interpret corollaries 3.7.9 and 3.7.15 in the particular case of a cusp. Let $(r, s) \in \mathbb{N}^2$ be a pair of coprime integers with $r > s \geq 2$. Let $f = x_0^r - y_0^s \in B_0 = k[x_0, y_0]$ and $\mathcal{C} = \text{Spec}(B_0/\langle f \rangle)$. Let us consider the following differential operators:

$$\begin{cases} f, \\ sx_0\partial_{x_0} + ry_0\partial_{y_0}, \\ (-1)^i s^i y_0^{s-i} \partial_{x_0}^i + r^i x_0^{r-i} \partial_{y_0}^i \quad \forall i \in \{1, \dots, s\}. \end{cases} \quad (7.4)$$

Corollary 3.7.9 assures that for every differential operator $D = \sum_{i+j \leq d} a_{i,j}(x_0, y_0) \partial_{x_0}^i \partial_{y_0}^j$ on the plane, with order d , such that $D(f^d) \in \langle f \rangle$ (or equivalently, such that $D \in V_{d-1}^{\mathcal{C}}$), its principal symbol $\sigma(D) = \sum_{i+j=d} a_{i,j}(x_0, y_0) \partial_{x_0}^i \partial_{y_0}^j$ is a combination (in the Weyl algebra $\mathcal{D}_{B_0|k}$) of the differential operators in 7.4.

On the other hand, corollary 3.7.15 assures that if $P(s) = \sum_i P_i s^i$ is a Bernstein-Sato operator for f of order $d \geq 2$ or a differential operator in $\text{ann}_{\mathcal{D}_{B_0|k}[s]}(f^s)$ of order d , then each P_i of maximal order d in the expression of $P(s)$ is a combination (in the Weyl algebra $\mathcal{D}_{B_0|k}$) of the differential operators in 7.4.

In [35, Subsection 6.1] an explicit example of a Bernstein-Sato operator for $f = x_0^5 - y_0^3$ is computed; it is also shown that it satisfies corollary 3.7.15.

THE DRINFELD-GRINBERG-KAZHDAN THEOREM

The local study of schemes is a classical field in Algebraic Geometry. In the particular case of the arc scheme of an algebraic variety, questions about the formal neighbourhoods at points have been investigated, first by V. Drinfeld, M. Grinberg and D. Kazhdan for rational points and then by A. Reguera, T. de Fernex and R. Docampo for another class of schematic points called constructible points.

The present chapter is intended to review the main result in the first aforementioned direction and some related facts. Precisely, we will present a result (theorem 4.1.2) about the structure of the formal neighbourhood of the arc scheme of a variety at a rational arc. A first version under the assumption that the base field is of characteristic zero was proved by M. Grinberg and D. Kazhdan in [34]¹, this result was extended by Drinfeld for arbitrary fields (see [30, Theorem 0.1], although the preprint dates from 2002).

The aim of this chapter is to present this result, stated in theorem 4.1.2. We also give a proof in the case of affine plane curves which illustrates the arguments needed in the general setting with the advantage of requiring a simpler notation. In section 4.2 we present the finite formal models, whose existence we deduce from the Drinfeld-Grinberg-Kazhdan theorem, and some facts about them such as their (non-)uniqueness, geometric properties that they encode, invariance and effective computation.

Chapter 5, and precisely our main theorem 5.5.19, explains the interest of the present chapter, since it allows to transpose to the formal neighbourhoods of the arc scheme at some constructible points (the generic points of the Nash sets for divisorial toric valuations) some properties deduced from Drinfeld-Grinberg-Kazhdan theorem for the formal neighbourhoods at rational points of the arc scheme.

The main references in this chapter are the original articles [34] and [30] proving the Drinfeld-Grinberg-Kazhdan theorem and also [10], the survey [15] and [19, Ch.3, §5].

¹This work of Grinberg and Kazhdan was indeed motivated by a conjecture of Drinfeld in private communication, see [34, Introduction].

4.1 THE DRINFELD-GRINBERG-KAZHDAN THEOREM

The aim of this section is to present theorem 4.1.2 and discuss many facts about it, including some elements of the proof in the case of affine plane curves which essentially requires the same technical arguments as the general setting.

4.1.1 Let us recall that, for k a field, V a k -variety and $\gamma \in \mathcal{L}_\infty(V)$ an arc on V , we denote by $\widehat{\mathcal{L}_\infty(V)}_\gamma$ the formal neighbourhood of the k -scheme $\mathcal{L}_\infty(V)$ at the point γ , i.e., $\mathrm{Spf}(\widehat{\mathcal{O}_{\mathcal{L}_\infty(V), \gamma}})$.

Theorem 4.1.2 (Drinfeld-Grinberg-Kazhdan). *Let k be a field and V be a k -variety with a rational point $v \in V(k)$ such that $\dim_v(V) \geq 1$. Let $\gamma \in \mathcal{L}_\infty(V)(k)$ be a rational point of the associated arc scheme, not contained in $\mathcal{L}_\infty(V_{\mathrm{Sing}})$ and whose center $\gamma(0)$ is v . Then there exist an affine k -scheme S of finite type, a rational point $s \in S(k)$ and an isomorphism of formal k -schemes:*

$$\widehat{\mathcal{L}_\infty(V)}_\gamma \cong \widehat{S}_s \hat{\times}_k \mathrm{Spf}(k[[T_i]_{i \in \mathbb{N}}]). \quad (4.1)$$

This theorem gives a decomposition of $\widehat{\mathcal{L}_\infty(V)}_\gamma$ as a product (in the category of formal k -schemes) of a noetherian affine formal k -scheme, which we will call a *finite formal model* of the pair $(\mathcal{L}_\infty(V), \gamma)$, and an infinite-dimensional formal disk. We will see in subsection 4.2.1 that, although a finite formal model for $(\mathcal{L}_\infty(V), \gamma)$ satisfying the assumptions in the theorem is not unique, all of them are related in some sense. An important fact is that Drinfeld's proof of theorem 4.1.2 is effective in the sense that it provides a procedure to explicitly compute a presentation of a finite formal model from a presentation of V and (sufficiently many coefficients of) the arc γ .

Let $\gamma \in \mathcal{L}_\infty(V)$ an arc of V . We say that γ is a *non-degenerate arc* if its generic point $\gamma(\eta)$ does not belong to the singular locus V_{Sing} of V , i.e., if γ is not contained in V_{Sing} . We denote by $\mathcal{L}_\infty^\circ(V) := \mathcal{L}_\infty(V) \setminus \mathcal{L}_\infty(V_{\mathrm{Sing}})$ the set of non-degenerate arcs of V . In subsection 4.2.5 we will explain why the assumption in theorem 4.1.2 of the arc γ to be non-degenerate cannot be omitted.

A crucial point of the proof of theorem 4.1.2 consists in describing the formal completion $\widehat{\mathcal{L}_\infty(V)}_\gamma$ in terms of arc deformations, via its functor of points. Other important ingredients that we will recall are the Weierstrass preparation theorem for power series over a complete local ring (see subsection 4.1.9) and the reduction to the case where V is a complete intersection (see subsection 4.1.12).

4.1.3 Let k be a field and A be a topological k -algebra. We say that I is an *ideal of definition* of A if I is open and for every neighbourhood N of 0 there exists an integer $n > 0$ such that $I^n \subset N$, i.e., the sequence $(I^n)_{n \in \mathbb{N}}$ converges to 0. We say that

A is *admissible* if it is separated and complete and it possesses an ideal of definition. Let \mathbf{Adm}_k be the category of admissible k -algebras with the continuous morphisms of k -algebras (see [36, 0/§7]).

We denote by \mathbf{Lacp}_k the full subcategory of \mathbf{Adm}_k whose objects are completions of local k -algebras with residue field isomorphic to k . Let A_1, A_2 be local k -algebras in \mathbf{Lacp}_k with respective maximal ideals $\mathfrak{m}_1, \mathfrak{m}_2$ and $\varphi : A_1 \rightarrow A_2$ be a continuous morphism of k -algebras. Let $a \in \mathfrak{m}_1$, then the sequence $(a^n)_{n \in \mathbb{N}}$ converges to 0 (i.e., a is *topologically nilpotent*) and thus $\varphi(a)$ cannot be a unit because φ is continuous, hence $\varphi(a) \in \mathfrak{m}_2$. We deduce that φ is a local morphism. An object of \mathbf{Lacp}_k is a k -algebra of the form $\widehat{A} = \varprojlim A/\mathfrak{m}^n$, where A is a local k -algebra with maximal ideal \mathfrak{m} such that $k = A/\mathfrak{m}$. Then \widehat{A} is a local k -algebra with maximal ideal given by $\widehat{\mathfrak{m}} = \text{Ker}(\widehat{A} \rightarrow A/\mathfrak{m})$. It is separated and complete for the topology defined by the family of ideals $(\text{Ker}(\widehat{A} \rightarrow A/\mathfrak{m}^n))_{n \in \mathbb{N}}$.

Nevertheless, if A is not noetherian, \widehat{A} might not be complete for the $\widehat{\mathfrak{m}}$ -adic topology. An example of object in \mathbf{Lacp}_k is, for every set Λ , the k -algebra $\widehat{A} = k[[\langle T_i \rangle_{i \in \Lambda}]]$ appearing in the isomorphism of theorem 4.1.2 for $\Lambda = \mathbb{N}$. It is the completion of the polynomial k -algebra $k[\langle T_i \rangle_{i \in \Lambda}]$ with respect to its maximal ideal $\mathfrak{m} = \langle T_i; i \in \Lambda \rangle$. When the set Λ is infinite, $\widehat{\mathfrak{m}}$ does not equal $\mathfrak{m} = \widehat{\mathfrak{m}}$ and the $\widehat{\mathfrak{m}}$ -adic topology is not complete, see [43]. We denote by $\mathbf{D}_k^\Lambda := \text{Spf}(k[[\langle T_i \rangle_{i \in \Lambda}]])$ the corresponding formal scheme, which is a formal disk. For every object $A \in \mathbf{Lacp}_k$ we set $A[[\langle T_i \rangle_{i \in \Lambda}]] := A \widehat{\otimes}_k k[[\langle T_i \rangle_{i \in \Lambda}]]$, which is again an object of \mathbf{Lacp}_k .

We denote by \mathbf{Test}_k the full subcategory of \mathbf{Lacp}_k whose objects are local k -algebras with residue field isomorphic to k and nilpotent maximal ideal, we call them *test-rings*. Let us note that such a k -algebra (A, \mathfrak{m}) is canonically isomorphic to its \mathfrak{m} -adic completion, $A \cong \varprojlim A/\mathfrak{m}^n$ and hence it is an object of \mathbf{Lacp}_k (i.e., test-rings are those objects in \mathbf{Lacp}_k with nilpotent maximal ideal). We denote by $\iota : \mathbf{Test}_k \rightarrow \mathbf{Lacp}_k$ the inclusion functor. A proof of the following lemma can be found in [19, Ch. 3, Lemma 5.1.6].

Lemma 4.1.4. *Let k be a field, (R, \mathfrak{m}) a local k -algebra with residue field k and A an object in \mathbf{Test}_k . The completion morphism $R \rightarrow \widehat{R}$ induces a bijection from $\text{Hom}_{\mathbf{Lacp}_k}(\widehat{R}, A)$ to the set of local morphisms of k -algebras $R \rightarrow A$.*

4.1.5 Let us denote by $\widehat{\mathbf{Lacp}}_k$ (resp. $\widehat{\mathbf{Test}}_k$) the category of covariant functors from \mathbf{Lacp}_k (resp. \mathbf{Test}_k) to the category \mathbf{Sets} of sets. Given an object $A \in \mathbf{Lacp}_k$ we denote by h_A the functor $\text{Hom}_{\mathbf{Lacp}_k}(A, \cdot)$ which is an object of $\widehat{\mathbf{Lacp}}_k$. It is a consequence of the Yoneda lemma that, for $A, A' \in \mathbf{Lacp}_k$, the natural maps (functorial in A, A')

$$\text{Hom}_{\mathbf{Lacp}_k}(A, A') \cong \text{Hom}_{\widehat{\mathbf{Lacp}}_k}(h_A, h_{A'}) \quad (1.2)$$

are bijective. We denote by $h'_A := h_A \circ \iota$ the restriction of the functor h_A to the subcategory \mathbf{Test}_k . This gives rise to a functor

$$\begin{aligned} h' : \mathbf{Lacp}_k &\longrightarrow \widehat{\mathbf{Test}}_k \\ A &\longmapsto h'_A = \text{Hom}_{\mathbf{Lacp}_k}(A, \cdot) \end{aligned}$$

from the category \mathbf{Lacp}_k to the category of functors from \mathbf{Test}_k to \mathbf{Sets} .

Lemma 4.1.6. *Let k be a field. With the notation of this subsection, the functor h' is fully faithful.*

Idea of the proof. By equation 1.2 it is enough to see that, for every $A, A' \in \mathbf{Lacp}_k$, the natural map

$$\mathrm{Hom}_{\widehat{\mathbf{Lacp}_k}}(h_A, h_{A'}) \longrightarrow \mathrm{Hom}_{\widehat{\mathbf{Lacp}_k}}(h_A \circ \iota, h_{A'} \circ \iota)$$

is bijective. Then the proof lies on the observation that, for every object $B \in \mathbf{Lacp}_k$, if we choose a local k -algebra (R, \mathfrak{m}) with residue field k and an isomorphism $B \cong \varprojlim R/\mathfrak{m}^n$, then for every $n \in \mathbb{N}$ the k -algebra R/\mathfrak{m}^n is a test-ring. A detailed proof can be found in [19, Ch.3, Lemma 5.1.8]. \square

Lemma 4.1.6 allows to deduce the following observation, which will be a key point in the sequel.

Observation 4.1.7. *Let A, A' be two objects in \mathbf{Lacp}_k . From lemma 4.1.6 we deduce that a morphism of functors $h'_A \rightarrow h'_{A'}$ is induced by a unique morphism $A \rightarrow A'$ of admissible local k -algebras, and the first one is an isomorphism if and only if the second one is an isomorphism. Summing up, an object $A \in \mathbf{Lacp}_k$ is determined by its associated functor h'_A .*

4.1.8 Let k be a field, V a k -variety and $\gamma \in \mathcal{L}_\infty(V)(k)$ a rational point of the associated arc scheme. Let us denote by \mathfrak{m}_γ the maximal ideal of the local ring $\mathcal{O}_{\mathcal{L}_\infty(V), \gamma}$ of $\mathcal{L}_\infty(V)$ at γ . By definition, the formal neighbourhood $\widehat{\mathcal{L}_\infty(V)}_\gamma$ at γ is the formal spectrum of the \mathfrak{m}_γ -adic completion of $\mathcal{O}_{\mathcal{L}_\infty(V), \gamma}$. Then its corresponding k -algebra $\widehat{\mathcal{O}_{\mathcal{L}_\infty(V), \gamma}}$ is an object of \mathbf{Lacp}_k and hence it is determined (by observation 4.1.7) by the sets $\mathrm{Hom}_{\mathbf{Lacp}_k}(\widehat{\mathcal{O}_{\mathcal{L}_\infty(V), \gamma}}, A)$ for every test-ring A over k .

Let $(A, \mathfrak{m}_A) \in \mathbf{Test}_k$. An element $\gamma_A \in \widehat{\mathcal{L}_\infty(V)}_\gamma(A)$ is called an A -deformation of γ . Its reduction modulo \mathfrak{m}_A coincides with γ . Then the elements in $\mathrm{Hom}_{\mathbf{Lacp}_k}(\widehat{\mathcal{O}_{\mathcal{L}_\infty(V), \gamma}}, A)$ are A -deformations of γ and observation 4.1.7 implies that $\widehat{\mathcal{L}_\infty(V)}_\gamma$ is determined by the A -deformations of γ , for $A \in \mathbf{Test}_k$. Let us stress that, by lemma 4.1.4, an A -deformation γ_A corresponds with a commutative diagram

$$\begin{array}{ccc} \mathcal{O}_{\mathcal{L}_\infty(V), \gamma} & \xrightarrow{\gamma_A} & A \\ & \searrow & \downarrow \\ & & k \end{array}$$

where we have also denoted by γ_A the corresponding morphism of k -algebras, and the unnamed morphisms are the reductions modulo the respective maximal ideals. Let us

denote by $p_A : A[[t]] \rightarrow k[[t]]$ the unique local morphism of k -algebras which extends the projection $A \rightarrow A/\mathfrak{m}_A \cong k$. The diagram above also corresponds to a commutative diagram

$$\begin{array}{ccc}
 \mathcal{O}_{V,\gamma(0)} & \xrightarrow{\gamma_A} & A[[t]] \\
 & \searrow \gamma & \downarrow p_A \\
 & & k[[t]]
 \end{array} \tag{1.3}$$

or equivalently (with some abuse in the notation of the morphisms) with the diagram

$$\begin{array}{ccc}
 \text{Spec}(A[[t]]) & \xrightarrow{\gamma_A} & V \\
 p_A \uparrow & \nearrow \gamma & \\
 \text{Spec}(k[[t]]) & &
 \end{array}$$

4.1.9 An important ingredient of the proof of theorem 4.1.2 is the uniqueness in the Weierstrass division. Let (A, \mathfrak{m}) be a local ring which is complete for the \mathfrak{m} -adic topology. Let $f \in A[[t]]$, we can write $f = \sum_{m \in \mathbb{N}} f_m t^m$. Let $d \in \mathbb{N}$; we say that f is d -regular if $f_m \in \mathfrak{m}$ for every $m < d$ and f_d is a unit in A . We say that f is d -Weierstrass (or d -distinguished) if it is a monic polynomial of $A[t]$ of degree d which is d -regular. Let us denote by $\mathscr{W}_d(A)$ the set of d -Weierstrass polynomials of $A[t]$. A proof of the following theorem can be found in [56, Theorem 9.1].

Theorem 4.1.10 (Weierstrass division theorem). *Let (A, \mathfrak{m}) be a local ring which is complete for the \mathfrak{m} -adic topology. Let $f, g \in A[[t]]$ be formal power series with coefficients in A and let $d \in \mathbb{N}$. If f is d -regular, then there exists a unique pair (q, r) with $q \in A[[t]]$ and $r \in A[t]$ such that $g = qf + r$ and $\deg(r) < d$.*

Let us observe that theorem 4.1.10 implies that every $f \in A[[t]]$ which does not reduce to 0 modulo \mathfrak{m} is not a zero divisor in $A[[t]]$ (since f is d -regular for some $d \in \mathbb{N}$ and $0 = f \cdot 0$ is the unique factorization).

The following theorem provides a unique decomposition of a d -regular formal power series as a product of a d -Weierstrass polynomial and an invertible series (see [56, Theorem 9.2] for a proof).

Theorem 4.1.11 (Weierstrass preparation theorem). *Let (A, \mathfrak{m}) be a local ring which is complete for the \mathfrak{m} -adic topology. Let $f \in A[[t]]$ be a formal power series with coefficients in A and let $d \in \mathbb{N}$. If f is d -regular, then there exists a unique pair (p, u) with $p \in A[t]$ a d -Weierstrass polynomial (i.e., $p = t^d + p_{d-1}t^{d-1} + \dots + p_0$ with $p_0, \dots, p_{d-1} \in \mathfrak{m}$) and $u \in A[[t]]$ a unit.*

4.1.12 Following [10, Subsection 4.2] or [19, Ch. 3, Subsection 5.3] we know that we can use the Jacobian Criterion of smoothness to reduce the proof of theorem 4.1.2 to the case where V is a complete intersection in an affine space. Let us recall the key points of this reduction.

Let k be a field, V a k -variety and $\gamma \in \mathcal{L}_\infty(V)(k)$ a rational point of the associated arc scheme. Let U be an open subset of V containing $\gamma(0)$. Then the canonical inclusion of U in V identifies $\mathcal{L}_\infty(U)$ with an open subscheme of $\mathcal{L}_\infty(V)$ containing γ and it induces an isomorphism between $\widehat{\mathcal{L}_\infty(U)_\gamma}$ and $\widehat{\mathcal{L}_\infty(V)_\gamma}$. If we suppose U to be affine, this shows that we can reduce the proof of theorem 4.1.2 to the case where V is affine.

Let us recall the definition of the Jacobian matrix and the Jacobian criterion of smoothness. Let $r \in \mathbb{N}$ and $f_1, \dots, f_r \in k[T_1, \dots, T_n]$. The *Jacobian matrix* $J(f_1, \dots, f_r)$ is the matrix of size $r \times n$ with (i, j) -entry the partial derivative $\partial f_i / \partial T_j$ for $i \in \{1, \dots, r\}$ and $j \in \{1, \dots, n\}$. For $r \leq n$, a minor of order r of this matrix is the determinant of a square submatrix of $J(f_1, \dots, f_r)$ of size $r \times r$. This corresponds with

$$\det \left((\partial f_i / \partial T_{j_i})_{1 \leq i \leq r} \right)$$

for a choice of a family (j_1, \dots, j_r) of elements of $\{1, \dots, n\}$.

The general version of the Jacobian criterion in [6, Proposition 2.2/7] allows to obtain the following statement, from which we can deduce the subsequent particular version of the Jacobian criterion, see [19, Ch. 3, Subsection 5.3] for details.

Lemma 4.1.13. *Let k be a field and let V a the closed subscheme of \mathbb{A}_k^n defined by an ideal I of $k[T_1, \dots, T_n]$. Let x be a point of V and $r \in \mathbb{N}$. Then V is smooth over k of relative codimension r at the point x if and only if there exist $f_1, \dots, f_r \in I$, a minor g of order r of the Jacobian matrix $J(f_1, \dots, f_r)$ and an element $h \in (\langle f_1, \dots, f_r \rangle : I)$ such that $g(x)h(x) \neq 0$.*

Proposition 4.1.14 (Jacobian criterion). *Let k be a field and let V a the closed subscheme of \mathbb{A}_k^n defined by an ideal I of $k[T_1, \dots, T_n]$. The nonsmooth locus V_{Sing} of $V|k$ is the closed subscheme of V whose ideal is generated by all the products of the form gh , where, for an integer $r \in \{0, \dots, n\}$ and elements $f_1, \dots, f_r \in I$, g is a minor of order r of the Jacobian matrix $J(f_1, \dots, f_r)$ and $h \in (\langle f_1, \dots, f_r \rangle : I)$.*

The following proposition justifies the reduction to the complete intersection case for the study of the formal neighbourhoods of non-degenerate rational arcs.

Proposition 4.1.15. *Let k be a field and let V a the closed subscheme of \mathbb{A}_k^n defined by an ideal I of $k[T_1, \dots, T_n]$. Let $\gamma \in \mathcal{L}_\infty(V)(k)$ be a rational point of the associated arc scheme, not contained in $\mathcal{L}_\infty(V_{\text{Sing}})$. Then there exist an integer $r \in \{0, \dots, n\}$ and polynomials $f_1, \dots, f_r \in I$ defining a closed subscheme $W = \text{Spec}(k[T_1, \dots, T_n] / \langle f_1, \dots, f_r \rangle)$ of \mathbb{A}_k^n such that the following properties hold:*

- (a) *There exists a minor of order r of the Jacobian matrix $J(f_1, \dots, f_r)$ which does not vanish at γ .*

(b) *There exists a unique irreducible component of V containing the arc γ , and its dimension is equal to $n - r$.*

(c) *The canonical closed immersion $j : V \hookrightarrow W$ induces an isomorphism*

$$j_* : \widehat{\mathcal{L}_\infty(V)}_\gamma \xrightarrow{\sim} \widehat{\mathcal{L}_\infty(W)}_{j(\gamma)}.$$

Sketch of the proof. The existence of W and part (a) follow from the fact that γ is not contained in V_{Sing} and the Jacobian criterion (corollary 4.1.14). For part (b), we use once again the information given by the preceding application of the Jacobian criterion to deduce, by lemma 4.1.13, that X is smooth of dimension $n - r$ at the generic point of γ (the smoothness comes from $\gamma \notin \mathcal{L}_\infty(V_{\text{Sing}})$, the lemma assures that the codimension is r which coincides with the number of generators of the ideal defining W). Then there exists a unique irreducible component of V containing γ , with dimension $n - r$. For part (c), by observation 4.1.7 one has only to prove that, for every $A \in \mathbf{Test}_k$, the map $\widehat{\mathcal{L}_\infty(V)}_\gamma(A) \rightarrow \widehat{\mathcal{L}_\infty(W)}_{j(\gamma)}(A)$ induced by j is bijective. The Weierstrass division theorem 4.1.10 will be useful in this task. See [19, Ch. 3, Proposition 5.3.5] for the complete proof. \square

Let us remark that the terminology “complete intersection” is an abuse in this case. Although the arc $j(\gamma)$ is contained in an irreducible component of W of dimension $n - r$, it may happen that W has another irreducible component of dimension bigger than $n - r$ passing through the center of $j(\gamma)$. Indeed, in the next subsection we will explain that one may restrict to the irreducible component containing the arc.

4.1.16 In order to study the formal neighbourhood of a non-degenerate rational point in the arc scheme of a variety V we may assume that V is integral. Although this reduction is not necessary for the original proof of theorem 4.1.2, it remains interesting and we will justify it. We adapt here the arguments in [15, Subsection 3.5].

Lemma 4.1.17. *Let k be a field and V be a k -variety. Let $\gamma \in \mathcal{L}_\infty(V)(k)$ be a non-degenerate rational arc. Then there exists an irreducible component W of V_{red} which contains the generic point of γ such that, if we denote by $\tilde{\gamma}$ the arc induced by γ in W , then $\widehat{\mathcal{L}_\infty(V)}_\gamma$ is isomorphic to $\widehat{\mathcal{L}_\infty(W)}_{\tilde{\gamma}}$.*

Proof. The arc γ being non-degenerate, its generic point $\gamma(\eta)$ is a regular point of V and hence its local ring is a regular local ring, in particular it is a domain. The arc γ corresponds to a morphism of local k -algebras $\gamma : \mathcal{O}_{V,\gamma(0)} \rightarrow k[[t]]$ and $\mathcal{O}_{V,\gamma(\eta)}$ is isomorphic to $(\mathcal{O}_{V,\gamma(0)})_{\ker(\gamma)}$, thus it is also a domain. We deduce that there exists a unique minimal prime ideal \mathfrak{p} of $\mathcal{O}_{V,\gamma(0)}$ contained in $\ker(\gamma)$, it corresponds with a (unique) minimal prime ideal of $\mathcal{O}(V)$ contained in $\ker(\gamma)$, which defines the unique irreducible component of V_{red} which contains $\gamma(\eta)$. By observation 4.1.7, we only have to show that for every testing $(A, \mathfrak{m}_A) \in \mathbf{Test}_k$, every A -deformation γ_A of γ factorizes through this irreducible component.

Let $\gamma_A : \mathcal{O}_{V,\gamma(0)} \rightarrow A[[t]]$ be an A -deformation of γ , we have to prove that \mathfrak{p} is contained in $\ker(\gamma_A)$. By definition of an A -deformation we have $\ker(\gamma) = \gamma_A^{-1}(\mathfrak{m}_A[[t]])$ (see diagram 1.3). Let $x \in \mathfrak{p}$, since \mathfrak{p} vanishes in $(\mathcal{O}_{V,\gamma(0)})_{\ker(\gamma)}$ we deduce that there exists an element $y \notin \ker(\gamma)$ such that $xy = 0$. Hence $\gamma_A(x)\gamma_A(y) = 0$ in $A[[t]]$. But $\gamma_A(y) \notin \mathfrak{m}_A[[t]]$ and hence it is not a zero divisor in $A[[t]]$ (see subsection 4.1.9). We deduce that $\gamma_A(x) = 0$, thus $x \in \ker(\gamma_A)$.

Then, if W is the irreducible component of V_{red} defined by \mathfrak{p} and $\tilde{\gamma}$ is the arc induced by γ in W , then the A -deformations of γ in V are in bijection (functorial in A) with the A -deformations of $\tilde{\gamma}$ in W , which implies the isomorphism between the corresponding formal neighbourhoods. \square

Let us see γ as a local morphism $\gamma : \widehat{\mathcal{O}_{V,\gamma(0)}} \rightarrow k[[t]]$. A *formal branch* or *formal irreducible component* at $\gamma(0)$ which contains γ is a minimal prime ideal \mathfrak{p} of $\widehat{\mathcal{O}_{V,\gamma(0)}}$ such that $\mathfrak{p} \subset \ker(\gamma)$. In particular, if \mathfrak{p} is a formal branch at $\gamma(0)$, then γ factorizes through the quotient morphism $\widehat{\mathcal{O}_{V,\gamma(0)}} \rightarrow \widehat{\mathcal{O}_{V,\gamma(0)}}/\mathfrak{p}$. Indeed, an adaptation of the arguments in the previous proof show that every A -deformation of γ factorizes through the unique formal irreducible component at $\gamma(0)$ which contains γ . This is the content of [10, Lemma 2.4] (see also [12, Proposition 3.2]), which is stated in a slightly more general setting, for rational arcs which could be degenerate but assumed to be contained in a unique formal irreducible component (which automatically holds for non-degenerate arcs).

Lemma 4.1.18. *Let k be a field and V be an integral k -variety. Let $\gamma \in \mathcal{L}_\infty(V)(k)$ be a non-degenerate rational arc. There exists a unique minimal prime ideal \mathfrak{p} of the ring $\widehat{\mathcal{O}_{V,\gamma(0)}}$ such that the induced morphism $\gamma : \widehat{\mathcal{O}_{V,\gamma(0)}} \rightarrow k[[t]]$ of \mathbf{Lacp}_k factorizes through $\widehat{\mathcal{O}_{V,\gamma(0)}} \rightarrow \widehat{\mathcal{O}_{V,\gamma(0)}}/\mathfrak{p}$. Then, for every test-ring $(A, \mathfrak{m}_A) \in \mathbf{Test}_k$, for every A -deformation $\gamma_A \in \mathcal{L}_\infty(V)_\gamma(A)$ of γ , the induced morphism $\gamma : \widehat{\mathcal{O}_{V,\gamma(0)}} \rightarrow A[[t]]$ of \mathbf{Lacp}_k also factorizes through $\widehat{\mathcal{O}_{V,\gamma(0)}} \rightarrow \widehat{\mathcal{O}_{V,\gamma(0)}}/\mathfrak{p}$. Besides, \mathfrak{p} is the only minimal prime ideal of $\widehat{\mathcal{O}_{V,\gamma(0)}}$ with this property.*

Proof. The first assertion is a consequence of the fact that γ is non-degenerate, see [12, Proposition 3.6]. Let $(A, \mathfrak{m}_A) \in \mathbf{Test}_k$ and $\gamma_A \in \mathcal{L}_\infty(V)_\gamma(A)$; then $\ker(\gamma) = \gamma_A^{-1}(\mathfrak{m}_A[[t]])$. Let $\mathfrak{p}, \mathfrak{q}_1, \dots, \mathfrak{q}_n$ be the minimal prime ideals of the ring $\widehat{\mathcal{O}_{V,\gamma(0)}}$ and we assume that $\mathfrak{p} \subset \ker(\gamma)$, let us prove that $\mathfrak{p} \subset \ker(\gamma_A)$. Let $x \in \mathfrak{p}$. Since the ring $\widehat{\mathcal{O}_{V,\gamma(0)}}$ is reduced, we have $\mathfrak{p} \cap (\bigcap_{i=1}^n \mathfrak{q}_i) = 0$. The first assertion implies the existence of elements $y_i \in \mathfrak{q}_i$ such that $y_i \notin \ker(\gamma) = \gamma_A^{-1}(\mathfrak{m}_A[[t]])$, for $i \in \{1, \dots, n\}$. Then $xy_1 \cdots y_n = 0$ and hence $\gamma_A(x)\gamma_A(y_1) \cdots \gamma_A(y_n) = 0$. Since $\gamma_A(y_i)$ does not reduce to 0 modulo \mathfrak{m}_A , it is not a zero divisor in $A[[t]]$ (see subsection 4.1.9) and we deduce that $\gamma_A(x) = 0$, i.e., $x \in \ker(\gamma_A)$. For the last assertion, if $\mathfrak{q}_i \subset \ker(\gamma_A)$ for some $i \in \{1, \dots, n\}$, then in particular $\mathfrak{q}_i \subset \gamma_A^{-1}(\mathfrak{m}_A[[t]]) = \ker(\gamma)$ which is in contradiction with the first assertion. \square

4.1.19 Following the ideas in subsection 4.1.5 and in particular observation 4.1.7, the strategy of the proof consists in showing that there exists an affine scheme S of finite type and $s \in S(k)$ such that, for every test-ring $A \in \mathbf{Test}_k$, we have a bijection $\mathrm{Hom}_{\mathbf{Lacp}_k}(\widehat{\mathcal{O}_{\mathcal{L}_\infty(V), \gamma}}, A) \rightarrow \mathrm{Hom}_{\mathbf{Lacp}_k}(\widehat{\mathcal{O}_{S, s}}, A) \times \mathrm{Hom}_{\mathbf{Lacp}_k}(k[[T_i]_{i \in \mathbb{N}}], A)$ functorial in A . By subsection 4.1.12, we can assume that V is a complete intersection affine k -variety. By simplicity in the notation, we will present here the proof for V an affine plane curve; the arguments in the general setting are analogous.

- Let f be a polynomial in $k[x, y]$ and V be the corresponding affine k -variety $V = \mathrm{Spec}(k[x, y]/\langle f \rangle)$. We consider the non-degenerate arc $\gamma \in \mathcal{L}_\infty(V)(k)$ given (via identification with a morphism $\gamma : \mathcal{O}_V \rightarrow k[[t]]$) by $\gamma(t) = (x(t), y(t)) \in k[[t]]^2$.

The arc γ not belonging to $\mathcal{L}_\infty(V_{\mathrm{Sing}})$, by the Jacobian criterion we deduce that some partial derivative of f does not vanish at γ . We may assume that $\partial_y f(x(t), y(t)) \neq 0$ and we fix $d = \mathrm{ord}_t(\partial_y f(x(t), y(t))) \in \mathbb{N}$. We can perform the euclidean division (the Weierstrass division in theorem 4.1.10 for the complete local ring $k[[t]]$) of $x(t)$ by t^{2d} and $y(t)$ by t^d , obtaining

$$\begin{cases} x(t) = z(t)t^{2d} + \tilde{x}(t) \\ y(t) = w(t)t^d + \tilde{y}(t) \end{cases} \quad (1.4)$$

where $z(t), w(t) \in k[[t]]$, $\tilde{x}(t), \tilde{y}(t) \in k[t]$ and $\deg(\tilde{x}(t)) \leq 2d - 1$, $\deg(\tilde{y}(t)) \leq d - 1$.

- Let $(A, \mathfrak{m}_A) \in \mathbf{Test}_k$ be a test-ring. Let $x_A(t), y_A(t) \in \mathfrak{m}_A[[t]]$ in such a way that $(x(t) + x_A(t), y(t) + y_A(t)) \in \widehat{\mathcal{L}_\infty(V)_\gamma}(A)$ is an A -deformation of γ (since we clearly have $(x(t) + x_A(t), y(t) + y_A(t)) = (x(t), y(t))$ modulo \mathfrak{m}_A). Let us note that such an A -deformation of γ is determined by the pair $(x_A(t), y_A(t))$. By the Weierstrass preparation theorem 4.1.11 we deduce the existence of a unique d -Weierstrass polynomial $q_A(t) \in \mathscr{W}_d(A)$ and a unique invertible series $u_A(t) \in A[[t]]^*$ such that

$$\partial_y f(x(t) + x_A(t), y(t) + y_A(t)) = q_A(t)u_A(t). \quad (1.5)$$

The assumptions of the Weierstrass preparation theorem are satisfied because $\partial_y f(x(t) + x_A(t), y(t) + y_A(t)) \notin \mathfrak{m}_A[[t]]$ since its reduction modulo $\mathfrak{m}_A[[t]]$ equals $\partial_y f(x(t), y(t))$ which is nonzero in $k[[t]]$.

Hence, we apply the Weierstrass division theorem 4.1.10 (let us observe that $q_A(t)$ and $q_A(t)^2$ are respectively d -regular and $2d$ -regular) to deduce that we have

$$\begin{cases} x(t) + x_A(t) = (z(t) + z_A(t))q_A(t)^2 + \tilde{x}(t) + \tilde{x}_A(t) \\ y(t) + y_A(t) = (w(t) + w_A(t))q_A(t) + \tilde{y}(t) + \tilde{y}_A(t) \end{cases}$$

where $z_A(t), w_A(t) \in \mathfrak{m}_A[[t]]$, $\tilde{x}_A(t), \tilde{y}_A(t) \in \mathfrak{m}_A[t]$ with $\deg(\tilde{x}_A(t)) \leq 2d - 1$ et $\deg(\tilde{y}_A(t)) \leq d - 1$. Let us stress that $z(t), z_A(t), \tilde{x}(t), \tilde{x}_A(t), w(t), w_A(t), \tilde{y}(t)$ and $\tilde{y}_A(t)$ (and also $q_A(t)$ and $u_A(t)$) are uniquely determined.

- Let us define the set $\mathfrak{B}(A)$ of elements $(z_A(t), \tilde{x}_A(t), \tilde{y}_A(t), q_A(t)) \in \mathfrak{m}_A[[t]] \times \mathfrak{m}_A[t]_{\leq 2d-1} \times \mathfrak{m}_A[t]_{\leq d-1} \times \mathscr{W}_d(A)$ such that

$$\begin{cases} q_A(t) \text{ divides } \partial_y f(\tilde{x}(t) + \tilde{x}_A(t), \tilde{y}(t) + \tilde{y}_A(t)) \\ q_A(t)^2 \text{ divides } f(\tilde{x}(t) + \tilde{x}_A(t), \tilde{y}(t) + \tilde{y}_A(t)) \end{cases} \quad (1.6)$$

We define a map

$$\begin{aligned} \varphi_A : \widehat{\mathcal{L}_\infty(V)}_\gamma(A) &\longrightarrow \mathfrak{B}(A) \\ (x_A(t), y_A(t)) &\longrightarrow (z_A(t), \tilde{x}_A(t), \tilde{y}_A(t), q_A(t)) \end{aligned}$$

where $z_A(t), \tilde{x}_A(t), \tilde{y}_A(t)$ and $q_A(t)$ are constructed from the A -deformation $(x_A(t), y_A(t))$ of γ as above. By construction $(z_A(t), \tilde{x}_A(t), \tilde{y}_A(t), q_A(t))$ is an element of $\mathfrak{m}_A[[t]] \times \mathfrak{m}_A[t]_{\leq 2d-1} \times \mathfrak{m}_A[t]_{\leq d-1} \times \mathcal{W}_d(A)$, we have to prove that it also satisfies conditions (1.6). For this, let us first consider the Taylor expansion of $\partial_y f(\tilde{x}(t) + \tilde{x}_A(t), \tilde{y}(t) + \tilde{y}_A(t))$ at $(x(t) + x_A(t), y(t) + y_A(t))$ (in order to make the formula shorter we omit here the explicit reference to the variable t in each element):

$$\begin{aligned} \partial_y f(\tilde{x} + \tilde{x}_A, \tilde{y} + \tilde{y}_A) &= \partial_y f(x + x_A - (z + z_A)q_A^2, y + y_A - (w + w_A)q_A) \\ &= \partial_y f(x + x_A, y + y_A) + q_A(\dots) \end{aligned}$$

By the definition of $q_A(t)$ in formula (1.5) we deduce that $q_A(t)$ divides $\partial_y f(\tilde{x}(t) + \tilde{x}_A(t), \tilde{y}(t) + \tilde{y}_A(t))$. For the second condition in formula (1.6) we consider the Taylor expansion of $f(\tilde{x}(t) + \tilde{x}_A(t), \tilde{y}(t) + \tilde{y}_A(t))$ at $(x(t) + x_A(t), y(t) + y_A(t))$:

$$\begin{aligned} f(\tilde{x} + \tilde{x}_A, \tilde{y} + \tilde{y}_A) &= f(x + x_A - (z + z_A)q_A^2, y + y_A - (w + w_A)q_A) \\ &= f(x + x_A, y + y_A) - q_A(w + w_A)\partial_y f(x + x_A, y + y_A) + q_A^2(\dots) \end{aligned}$$

The term $f(x + x_A, y + y_A)$ is zero because $(x(t) + x_A(t), y(t) + y_A(t)) \in \widehat{\mathcal{L}_\infty(V)}_\gamma(A)$ is an A -deformation of γ and by the very definition of $q_A(t)$ in formula (1.5) we deduce that $q_A(t)$ divides $\partial_y f(x(t) + x_A(t), y(t) + y_A(t))$, hence $q_A(t)^2$ divides $f(\tilde{x}(t) + \tilde{x}_A(t), \tilde{y}(t) + \tilde{y}_A(t))$. This shows that φ_A is a well-defined map.

• The next step is to prove that the map φ_A is a bijection. Let us consider an element $(z_A(t), \tilde{x}_A(t), \tilde{y}_A(t), q_A(t)) \in \mathfrak{B}(A)$. Let us recall the division in formula (1.4). We set $x_A(t), \hat{y}_A(t) \in A[[t]]$ such that

$$\begin{cases} x(t) + x_A(t) = (z(t) + z_A(t))q_A(t)^2 + \tilde{x}(t) + \tilde{x}_A(t) \\ y(t) + \hat{y}_A(t) = w(t)q_A(t) + \tilde{y}(t) + \tilde{y}_A(t) \end{cases} \quad (1.7)$$

We aim to prove that there exists a unique $w_A(t) \in \mathfrak{m}_A[[t]]$ such that

$$f(x(t) + x_A(t), y(t) + \hat{y}_A(t) + q_A(t)w_A(t)) = 0, \quad (1.8)$$

in which case $(x(t) + x_A(t), y(t) + \hat{y}_A(t) + q_A(t)w_A(t))$ would be an A -deformation of γ (the fact that $(x(t) + x_A(t), y(t) + \hat{y}_A(t) + q_A(t)w_A(t)) = (x(t), y(t))$ modulo \mathfrak{m}_A is clear by construction), i.e., an element of $\widehat{\mathcal{L}_\infty(V)}_\gamma(A)$.

Let us write the Taylor expansion of $f(x(t) + x_A(t), y(t) + \hat{y}_A(t))$ at $(\tilde{x}(t) + \tilde{x}_A(t), \tilde{y}(t) + \tilde{y}_A(t))$:

$$\begin{aligned} f(x + x_A, y + \hat{y}_A) &= f(\tilde{x} + \tilde{x}_A + (z + z_A)q_A^2, \tilde{y} + \tilde{y}_A + wq_A) \\ &= f(\tilde{x} + \tilde{x}_A, \tilde{y} + \tilde{y}_A) + q_A w \partial_y f(\tilde{x} + \tilde{x}_A, \tilde{y} + \tilde{y}_A) + q_A^2(\dots) \end{aligned}$$

From the conditions in formula (1.6) we deduce that $q_A(t)^2$ divides $f(x(t) + x_A(t), y(t) + \widehat{y}_A(t))$. Moreover, $f(x(t) + x_A(t), y(t) + \widehat{y}_A(t))$ can be written in the form $q_A(t)^2 \vartheta_A(t)$ with $\vartheta_A(t) \in \mathfrak{m}_A[[t]]$, since $f(x(t) + x_A(t), y(t) + \widehat{y}_A(t)) = 0$ modulo $\mathfrak{m}_A[[t]]$ and $q_A(t)$ is not a zero divisor in $A[[t]]$ (because $q_A(t) \notin \mathfrak{m}_A[[t]]$, see subsection 4.1.9).

On the other hand, we have the Taylor expansion of $\partial_y f(x(t) + x_A(t), y(t) + \widehat{y}_A(t))$ at $(\tilde{x}(t) + \tilde{x}_A(t), \tilde{y}(t) + \tilde{y}_A(t))$:

$$\begin{aligned} \partial_y f(x + x_A, y + \widehat{y}_A) &= \partial_y f(\tilde{x} + \tilde{x}_A + (z + z_A)q_A^2, \tilde{y} + \tilde{y}_A + wq_A) \\ &= \partial_y f(\tilde{x} + \tilde{x}_A, \tilde{y} + \tilde{y}_A) + q_A(\dots) \end{aligned}$$

and from the conditions in formula (1.6) we deduce that $q_A(t)$ divides $\partial_y f(x(t) + x_A(t), y(t) + \widehat{y}_A(t))$. Hence by the Weierstrass preparation theorem 4.1.11 there exists a unique $v_A(t) \in A[[t]]^*$ such that $\partial_y f(x(t) + x_A(t), y(t) + \widehat{y}_A(t)) = q_A(t)v_A(t)$ (by uniqueness of the factorization and the fact that $\partial_y f(x(t), y(t))$ is d -regular and $q_A(t)$ is d -Weierstrass).

Once again, let us write the Taylor expansion of $f(x(t) + x_A(t), y(t) + \widehat{y}_A(t) + q_A(t)w_A(t))$ at $(x(t) + x_A(t), y(t) + \widehat{y}_A(t))$:

$$f(x + x_A, y + \widehat{y}_A + q_A w_A) = f(x + x_A, y + \widehat{y}_A) + q_A w_A \partial_y f(x + x_A, y + \widehat{y}_A) + q_A^2 w_A^2(\dots)$$

Hence, the equation (1.8) is equivalent to the right hand side in the preceding formula being zero. Let us stress that it can be rewritten in the form

$$q_A^2 \left(\vartheta_A(t) + w_A(t)v_A(t) + w_A(t)^2 Q(w_A(t)) \right) = 0,$$

where $Q(w_A(t)) \in A[[t]]$. Since the element q_A is not a zero divisor in $A[[t]]$, this is equivalent to the existence of a unique $w_A(t) \in \mathfrak{m}_A[[t]]$ such that

$$\theta_A(t) + w_A(t) + w_A(t)^2 P(w_A(t)) = 0, \quad (1.9)$$

where $\theta_A(t) = \vartheta_A(t)v_A(t)^{-1} \in \mathfrak{m}_A[[t]]$ and $P(w_A(t)) = Q(w_A(t))v_A(t)^{-1} \in A[[t]]$.

We will use the Hensel lemma to solve the preceding equation. The ring (A, \mathfrak{m}_A) being local and complete, the ring $(A[[t]], \mathfrak{m}_A + \langle t \rangle)$ also does, and it is in particular a henselian ring. Thus the reduction of the equation $\theta_A(t) + X + X^2 P(X) = 0$ modulo $\mathfrak{m}_A + \langle t \rangle$ gives an equation $X + X^2 \overline{P}(X) = 0$ which admits the solution $X = 0$. By the Hensel lemma there exists a unique element $w_A(t) \in A[[t]]$ such that $\theta_A(t) + w_A(t) + w_A(t)^2 P(w_A(t)) = 0$.

We have to check that in fact $w_A(t) \in \mathfrak{m}_A[[t]]$. Reducing the equation $\theta_A(t) + w_A(t) + w_A(t)^2 P(w_A(t)) = 0$ modulo $\mathfrak{m}_A[[t]]$ we obtain

$$\overline{w}_A(t)(1 + \overline{w}_A(t)\overline{P}(\overline{w}_A(t))) = 0. \quad (1.10)$$

By construction $w_A(t) \in \mathfrak{m}_A + \langle t \rangle$, hence $\text{ord}_t(\overline{w}_A(t)) \geq 1$ and we deduce that $1 + \overline{w}_A(t)\overline{P}(\overline{w}_A(t))$ is invertible modulo $\mathfrak{m}_A[[t]]$. Then from equation (1.10) we deduce that $\overline{w}_A(t) = 0$ modulo $\mathfrak{m}_A[[t]]$, that is, $w_A(t) \in \mathfrak{m}_A[[t]]$.

• We have proven that φ_A is a bijection for $A \in \mathbf{Test}_k$. In addition, it follows from the proof of the Weierstrass preparation theorem that the family of bijections $(\varphi_A)_{A \in \mathbf{Test}_k}$ is functorial in A , which concludes the proof.

Remark 4.1.20. Let us define the set $\widetilde{\mathfrak{B}}(A)$ of elements $(\tilde{x}_A(t), \tilde{y}_A(t), q_A(t)) \in \mathfrak{m}_A[t]_{\leq 2d-1} \times \mathfrak{m}_A[t]_{\leq d-1} \times \mathscr{W}_d(A)$ such that

$$\begin{cases} q_A(t) \text{ divides } \partial_y f(\tilde{x}(t) + \tilde{x}_A(t), \tilde{y}(t) + \tilde{y}_A(t)) \\ q_A(t)^2 \text{ divides } f(\tilde{x}(t) + \tilde{x}_A(t), \tilde{y}(t) + \tilde{y}_A(t)) \end{cases} \quad (1.11)$$

It is clear that $\mathfrak{B}(A) = \widetilde{\mathfrak{B}}(A) \times \mathfrak{m}_A[[t]]$. In addition, we observe that $\mathfrak{m}_A[t]_{\leq 2d-1} \times \mathfrak{m}_A[t]_{\leq d-1} \times \mathscr{W}_d(A)$ is in bijection with $\mathfrak{m}_A^{2d+d+d} = \mathfrak{m}_A^{4d}$. This set is in bijection with $\mathrm{Hom}_{\mathbf{Lacp}_k}(k[[T_i]_{1 \leq i \leq 4d}], A)$. Hence we deduce that there exists an ideal $I \in k[[T_i]_{1 \leq i \leq 4d}]$ realizing the conditions (1.11) such that $\widetilde{\mathfrak{B}}(A)$ is in bijection (functorial in A) with the set $\mathrm{Hom}_{\mathbf{Lacp}_k}(k[[T_i]_{1 \leq i \leq 4d}]/I, A)$. Analogously $\mathfrak{m}_A[[t]]$ is in bijection with $\mathfrak{m}_A^{\mathbb{N}}$, which is itself in bijection with $\mathrm{Hom}_{\mathbf{Lacp}_k}(k[[T_i]_{i \in \mathbb{N}}], A)$. We deduce that $\mathfrak{B}(A)$ is in bijection with the set $\mathrm{Hom}_{\mathbf{Lacp}_k}(k[[T_i]_{1 \leq i \leq 4d}]/I \widehat{\otimes}_k k[[T_i]_{i \in \mathbb{N}}], A)$. Composing this bijection with φ_A we deduce that this set is in bijection with the set $\widehat{\mathcal{L}}_{\infty}(V)_{\gamma}(A)$ of A -deformations of γ , and this bijection is functorial in A . By observation 4.1.7 we deduce an isomorphism in \mathbf{Lacp}_k between $\widehat{\mathcal{O}}_{\mathcal{L}_{\infty}(V), \gamma}$ and $k[[T_i]_{1 \leq i \leq 4d}]/I \widehat{\otimes}_k k[[T_i]_{i \in \mathbb{N}}]$. In terms of formal k -schemes, this corresponds to an isomorphism between $\widehat{\mathcal{L}}_{\infty}(V)_{\gamma}$ and $\mathrm{Spf}(k[[T_i]_{1 \leq i \leq 4d}]/I) \widehat{\times}_k \mathrm{Spf}(k[[T_i]_{i \in \mathbb{N}}])$. Then $\mathrm{Spf}(k[[T_i]_{1 \leq i \leq 4d}]/I)$ is a finite formal model of $(\mathcal{L}_{\infty}(V), \gamma)$. Let us stress that $k[[T_i]_{1 \leq i \leq 4d}]/I$ is the completion of the localization of $k[[T_i]_{1 \leq i \leq 4d}]/I$ at the extension of the ideal $\langle T_i; 1 \leq i \leq 4d \rangle$ and hence the finite formal model can be written in the form given in the statement of theorem 4.1.2.

4.2 FINITE FORMAL MODELS AND RELATED FACTS

In this section, we will define the finite formal models of the arc scheme at a non-degenerate rational point, whose existence is guaranteed by the Drinfeld-Grinberg-Kazhdan theorem 4.1.2. Then we will discuss some facts about them such as the (non)uniqueness, existence when the assumptions of the theorem do not hold, geometric properties that they encode, invariance and effective computation.

4.2.1 Let us now interest in the noetherian affine formal k -scheme in the decomposition given by theorem 4.1.2. Let k be a field and V be a k -variety with a rational point $v \in V(k)$ such that $\dim_v(V) \geq 1$. Let $\gamma \in \mathcal{L}_{\infty}(V)(k)$ be a rational point of the associated arc scheme, not contained in $\mathcal{L}_{\infty}(V_{\mathrm{Sing}})$ and whose center $\gamma(0)$ is v . Every affine formal k -scheme \mathcal{S} which is the formal spectrum of an essentially of finite type local k -algebra such that there exists an isomorphism of formal k -schemes

$$\widehat{\mathcal{L}}_{\infty}(V)_{\gamma} \cong \mathcal{S} \widehat{\times}_k \mathrm{Spf}(k[[T_i]_{i \in \mathbb{N}}])$$

is called a *finite dimensional formal model* (or simply a *finite formal model*) of the pair $(\mathcal{L}_{\infty}(V), \gamma)$. This terminology was introduced in [16]. Hence theorem 4.1.2 may be

reformulated saying that, under the hypotheses above, the pair $(\mathcal{L}_\infty(V), \gamma)$ admits a finite formal model.

Let us observe that, for a given pair $(\mathcal{L}_\infty(V), \gamma)$, a finite formal model is not unique, even up to isomorphism: if S_s satisfies the statement of theorem 4.1.2, then $S_s \hat{\times}_k \mathbf{D}_k$ also does. However, we will be able to define a minimal formal model of $(\mathcal{L}_\infty(V), \gamma)$ which is unique up to isomorphism.

Let \mathbf{Lncp}_k be the full subcategory of \mathbf{Lacp}_k whose objects are the complete local noetherian k -algebras with residue field isomorphic to k . In particular, the section ring of a finite formal model \mathcal{S} of a pair $(\mathcal{L}_\infty(V), \gamma)$ is an element of \mathbf{Lncp}_k . A k -algebra $A \in \mathbf{Lncp}_k$ is said to be *cancellable* (in \mathbf{Lncp}_k) if there exists a k -algebra $B \in \mathbf{Lncp}_k$ such that A is isomorphic to $B[[T]]$. Let us note that for every k -algebra $A \in \mathbf{Lncp}_k$ there exist an integer $n \in \mathbb{N}$ and a non-cancellable k -algebra $A_{\min} \in \mathbf{Lncp}_k$ such that A is isomorphic to $A_{\min}[[T_1, \dots, T_n]]$. We have the following cancellation theorem, due to O. Gabber.

Theorem 4.2.2 (O. Gabber). *Let k be a field. Let $A, B \in \mathbf{Lncp}_k$ and let I, J be sets (possibly infinite). Assume that $A[[T_i]_{i \in I}]$ and $B[[U_j]_{j \in J}]$ are isomorphic in \mathbf{Lacp}_k . Then, up to exchanging A and B , there exists a finite subset $I' \subset I$ such that $A[[T_i]_{i \in I'}]$ and B are isomorphic in \mathbf{Lncp}_k . In particular, if both A and B are non-cancellable, then they are isomorphic.*

A proof of this theorem can be found in [10, Section 7]. Theorem 4.2.2 generalizes [44, Theorem 4], where the sets I and J are assumed to be finite. In [16] there is also a slightly weaker version of theorem 4.2.2.

From theorem 4.2.2 we immediately deduce that, for every k -algebra $A \in \mathbf{Lncp}_k$, the non-cancellable k -algebra $A_{\min} \in \mathbf{Lncp}_k$ is unique (up to isomorphism in \mathbf{Lncp}_k). Let us state this fact in the particular case of finite formal models.

Corollary 4.2.3. *Let k be a field and V be a k -variety with a rational point $v \in V(k)$ such that $\dim_v(V) \geq 1$. Let $\gamma \in \mathcal{L}_\infty(V)(k)$ be a rational point of the associated arc scheme, not contained in $\mathcal{L}_\infty(V_{\text{Sing}})$ and whose center $\gamma(0)$ is v . Let \mathcal{S} and \mathcal{S}' be two finite formal models of the pair $(\mathcal{L}_\infty(V), \gamma)$. Then, the k -algebras $(\mathcal{O}(\mathcal{S}))_{\min}$ and $(\mathcal{O}(\mathcal{S}'))_{\min}$ are isomorphic in \mathbf{Lncp}_k . In particular, there exist two integers $m, m' \in \mathbb{N}$ and an isomorphism (in \mathbf{Lncp}_k) $\mathcal{O}(\mathcal{S}) \hat{\otimes}_k k[[T_1, \dots, T_m]] \cong \mathcal{O}(\mathcal{S}') \hat{\otimes}_k k[[T_1, \dots, T_{m'}]]$.*

Let us recall that theorem 4.1.2 assures the existence of a finite formal model of the pair $(\mathcal{L}_\infty(V), \gamma)$ (under the corresponding assumptions). Hence corollary 4.2.3 implies that there exists a unique (up to isomorphism in \mathbf{Lncp}_k) finite formal model $\mathcal{S}_V(\gamma)$ such that $\mathcal{O}(\mathcal{S}_V(\gamma))$ is non-cancellable. We call $\mathcal{S}_V(\gamma)$ (or more precisely its isomorphism class in \mathbf{Lncp}_k) the *minimal formal model* of the pair $(\mathcal{L}_\infty(V), \gamma)$. Let us remark that the minimal formal model of $(\mathcal{L}_\infty(V), \gamma)$ is the finite formal model of minimal dimension among all the finite formal models of $(\mathcal{L}_\infty(V), \gamma)$.

Remark 4.2.4. By [15, Example 3.4 and Remark 3.5] we know that, for every k -scheme of finite type S and $s \in S(k)$ a rational point, there exist a k -variety V and a non-degenerate arc $\gamma \in \mathcal{L}_\infty(V)(k)$ such that S_s is a finite formal model of $(\mathcal{L}_\infty(V), \gamma)$. Hence, if we consider, as in [15, Remark 5.4] the non-cancellable complete noetherian local k -algebra

$$k[[T_1, \dots, T_n]] / \langle T_1^2, T_1 T_2, \dots, T_1 T_n \rangle$$

we deduce that, if we put no particular restriction on V or the non-degenerate arc $\gamma \in \mathcal{L}_\infty(V)(k)$, the dimension of a minimal formal model may be arbitrarily large.

4.2.5 The assumption in theorem 4.1.2 of the arc to be non-degenerate cannot be omitted, as shown by D. Bourqui and J. Sebag in [11]. Precisely, they found the following counterexample:

Theorem 4.2.6. *Let k be a field of characteristic zero which does not contain a root of the equation $T^2 + 1 = 0$. Let $f \in k[x, y]$ be the polynomial $x^2 + y^2$ and let \mathcal{C} be the affine plane curve defined by f . Let $\mathfrak{o} \in \mathcal{C}(k)$ be the origin of \mathbb{A}_k^2 , we also denote by \mathfrak{o} the induced constant arc in $\mathcal{L}_\infty(\mathcal{C})(k)$. Then the arc \mathfrak{o} does not satisfy the statement of theorem 4.1.2, i.e., the pair $(\mathcal{L}_\infty(\mathcal{C}), \mathfrak{o})$ does not admit a finite formal model.*

Let us observe that the decomposition given by theorem 4.1.2 implies that the nil-radical of the complete local ring $\widehat{\mathcal{O}(\mathcal{L}_\infty(V)_\gamma)}$ is nilpotent. The proof of theorem 4.2.6 consists in proving the existence of nilpotent elements of $\widehat{\mathcal{O}(\mathcal{L}_\infty(\mathcal{C})_\mathfrak{o})}$ of arbitrarily large nilpotence order. Let us stress that this example provides varieties of arbitrary dimension not satisfying the statement of theorem 4.1.2. Precisely, if V is a k -variety of arbitrary dimension and $\gamma_V \in \mathcal{L}_\infty(V)(k)$, the pair $(\mathcal{L}_\infty(V \times_k \mathcal{C}), (\gamma_V, \mathfrak{o}))$ does not admit a finite formal model.

More recently, C. Chiu and H. Hauser have completed this first answer in [20]. Precisely, [20, Theorem 1.2] implies the following theorem:

Theorem 4.2.7. *Let k be a field of characteristic zero and V be a k -variety. Let $v \in V_{\text{Sing}}(k)$, we also denote by v the induced constant arc in $\mathcal{L}_\infty(V)(k)$. Then, the pair $(\mathcal{L}_\infty(V), v)$ does not satisfy the statement of theorem 4.1.2, i.e., it does not admit a finite formal model.*

4.2.8 Let k be a field, V a k -variety and $v \in V(k)$ a rational point. It is an interesting question to study whether the minimal formal model of a pair $(\mathcal{L}_\infty(V), \gamma)$, where $\gamma \in \mathcal{L}_\infty(V)(k)$ is a non-degenerate arc with $\gamma(0) = v$, encodes some geometric information of (V, v) . In this direction we find the following result, corresponding to [12, Theorem 1.6]. Recall from lemma 4.1.18 that, if the arc γ is non-degenerate, then there is a unique formal branch at $\gamma(0)$ which contains γ .

Theorem 4.2.9 (Bourqui-Sebag). *Let k be a field, V a k -variety and $v \in V(k)$ a rational point such that $\mathcal{O}_{V,v}$ is reduced and $\dim_v(V) \geq 1$. Let $\gamma \in \mathcal{L}_\infty(V)(k)$ be a non-degenerate arc such that $\gamma(0) = v$. Then, the following conditions are equivalent:*

- (1) *The formal branch at v containing γ is smooth.*
- (2) *The formal neighbourhood $\widehat{\mathcal{L}_\infty(V)}_\gamma$ of γ in $\mathcal{L}_\infty(V)$ is isomorphic to a formal disk $\mathbf{D}_k^\mathbb{N} := \mathrm{Spf}(k[[T_i]_{i \in \mathbb{N}}])$.*
- (3) *The minimal formal model $\mathcal{S}_V(\gamma)$ of $(\mathcal{L}_\infty(V), \gamma)$ is isomorphic to $\mathrm{Spf}(k)$.*

In particular, theorem 4.2.9 says that, under those assumptions, the minimal formal model is trivial if and only if v is a smooth point of V . This fact suggests that the finite formal model of $(\mathcal{L}_\infty(V), \gamma)$ may be a measure of the singularity $(V, \gamma(0))$. We will precise this in subsection 4.2.10.

4.2.10 Let k be a field, V a k -variety and $v \in V(k)$ a rational point. Theorem 4.2.9 assures that, if v is a smooth point of V , then the formal neighbourhood of an arc centered at v is trivial and hence it does not depend on the choice of the arc. However, if v is a singular point of V , changing the involved arc while keeping the center fixed may modify the isomorphism class of the corresponding formal neighbourhood. The following example can be found in [15, Example 7.2].

Example 4.2.11. Let k be a field of characteristic different from 2 and 3. Let \mathcal{C} be the affine curve $\mathrm{Spec}(k[x, y]/\langle x^3 - y^2 \rangle)$. Let us consider the rational arcs centered at the origin $\gamma(t) = (t^2, t^3)$ and $\eta(t) = (t^4, t^6)$. One can show that $\mathcal{O}(\widehat{\mathcal{L}_\infty(\mathcal{C})}_\gamma)$ and $\mathcal{O}(\widehat{\mathcal{L}_\infty(\mathcal{C})}_\eta)$ are not isomorphic (in \mathbf{Lacp}_k).

Let us state and prove the following proposition, which corresponds to [15, Proposition 7.3].

Proposition 4.2.12. *Let k be a field and V, V' two k -varieties. Let $\gamma \in \mathcal{L}_\infty(V)(k)$, $\gamma' \in \mathcal{L}_\infty(V')(k)$ with respective centers $v = \gamma(0)$ and $v' = \gamma'(0)$. Let us suppose that there exist isomorphisms (in \mathbf{Lacp}_k) $f : \widehat{\mathcal{O}_{V,v}} \rightarrow \widehat{\mathcal{O}_{V',v'}}$ and $p : k[[t]] \rightarrow k[[t]]$ such that $p \circ \gamma = \gamma' \circ f$. Then, there exists an isomorphism of formal k -schemes $\widehat{\mathcal{L}_\infty(V)}_\gamma \cong \widehat{\mathcal{L}_\infty(V')}_{\gamma'}$.*

Proof. We may assume $\gamma = \gamma' \circ f$. By observation 4.1.7 it is enough to prove that, for every $A \in \mathbf{Test}_k$, there is a bijection between $\widehat{\mathcal{L}_\infty(V)}_\gamma(A)$ and $\widehat{\mathcal{L}_\infty(V')}_{\gamma'}(A)$. The composition with f^{-1} induces a bijection

$$\mathrm{Hom}_{\mathbf{Lacp}_k}(\widehat{\mathcal{O}_{V,v}}, A[[t]]) \xrightarrow{\circ f^{-1}} \mathrm{Hom}_{\mathbf{Lacp}_k}(\widehat{\mathcal{O}_{V',v'}}, A[[t]]).$$

By subsection 4.1.8, we can identify $\widehat{\mathcal{L}}_\infty(V)_\gamma(A)$ (resp. $\widehat{\mathcal{L}}_\infty(V')_{\gamma'}(A)$) with a subset of $\text{Hom}_{\mathbf{Lacp}_k}(\widehat{\mathcal{O}}_{V,v}, A[[t]])$ (resp. $\text{Hom}_{\mathbf{Lacp}_k}(\widehat{\mathcal{O}}_{V',v'}, A[[t]])$). Hence we have the following diagram, where the map $\widehat{\mathcal{L}}_\infty(V)_\gamma(A) \rightarrow \widehat{\mathcal{L}}_\infty(V')_{\gamma'}(A)$, obtained by restriction, is also a bijection:

$$\begin{array}{ccc} \text{Hom}_{\mathbf{Lacp}_k}(\widehat{\mathcal{O}}_{V,v}, A[[t]]) & \xrightarrow{\circ f^{-1}} & \text{Hom}_{\mathbf{Lacp}_k}(\widehat{\mathcal{O}}_{V',v'}, A[[t]]) \\ \uparrow & & \uparrow \\ \widehat{\mathcal{L}}_\infty(V)_\gamma(A) & \longrightarrow & \widehat{\mathcal{L}}_\infty(V')_{\gamma'}(A) \end{array}$$

We deduce that $\widehat{\mathcal{L}}_\infty(V)_\gamma$ and $\widehat{\mathcal{L}}_\infty(V')_{\gamma'}$ are isomorphic as formal k -schemes. \square

In example 4.2.11, the arc γ is said to be a primitive parametrization and for such arcs the formal neighbourhood is invariant. Precisely, let k be a field and \mathcal{C} be an integral k -curve, geometrically unibranch² at a rational point $c \in \mathcal{C}(k)$. The arc $\gamma \in \mathcal{L}_\infty(\mathcal{C})(k)$ is said to be a *primitive k -parametrization* at c if $\gamma(0) = c$ and for every morphism of local k -algebras $\gamma' : \mathcal{O}_{\mathcal{C},c} \rightarrow k[[t]]$ there exists a morphism of complete local k -algebras $p : k[[t]] \rightarrow k[[t]]$ such that $\gamma' = p \circ \gamma$, where γ is also seen as a morphism of local k -algebras $\gamma : \mathcal{O}_{\mathcal{C},c} \rightarrow k[[t]]$. If k is assumed to be a perfect field, the assumption that \mathcal{C} is geometrically unibranch at c guarantees the existence of primitive k -parametrizations at c . Then the following corollary is deduced from proposition 4.2.12.

Corollary 4.2.13. *Let k be a field, \mathcal{C} be a k -curve and $c \in \mathcal{C}(k)$ be a rational point. Let us assume that there exists a primitive k -parametrization $\gamma \in \mathcal{L}_\infty(\mathcal{C})(k)$ of \mathcal{C} at c . Then, the isomorphism class of the formal k -scheme $\widehat{\mathcal{L}}_\infty(\mathcal{C})_\gamma$ does not depend on the choice of the involved primitive k -parametrization. In particular, $\widehat{\mathcal{L}}_\infty(\mathcal{C})_\gamma$ and its minimal formal model $\mathcal{S}_\mathcal{C}(\gamma)$ are formal invariants of the curve singularity (\mathcal{C}, c) .*

Proof. The fact that the isomorphism class of $\widehat{\mathcal{L}}_\infty(\mathcal{C})_\gamma$ does not depend on the chosen primitive k -parametrization and is a formal invariant of (\mathcal{C}, c) is a direct consequence of proposition 4.2.12. Moreover, a primitive k -parametrization is always a non-degenerate arc and thanks to theorem 4.1.2 we deduce the existence of a finite formal model, which implies by corollary 4.2.3 that $(\mathcal{L}_\infty(\mathcal{C}), \gamma)$ admits a unique minimal formal model $\mathcal{S}_\mathcal{C}(\gamma)$. Then the formal neighbourhood $\widehat{\mathcal{L}}_\infty(\mathcal{C})_\gamma$ being a formal invariant of (\mathcal{C}, c) , we deduce that the minimal formal model also is. \square

In subsection 5.2.13 we will see that for V a normal toric variety, the formal neighbourhood of $\mathcal{L}_\infty(V)$ at a non-degenerate arc centered at the singular locus of V is also

²Recall that a scheme X is *geometrically unibranch* at a point $x \in X$ if the local ring $\mathcal{O}_{X,x}$ is geometrically unibranch. A local ring R is geometrically unibranch if R has a unique minimal prime \mathfrak{p} and the integral closure of R/\mathfrak{p} in its fraction field is a local ring whose residue field is purely inseparable over the residue field of R . See [39, 6.15.1] and [38, Ch. 0, 23.2.1] or [74, Tag 0BPZ].

invariant for every such arc belonging to a certain open subset of a Nash set corresponding to a toric divisor. The study of this invariance will be the object of chapter 5.

4.2.14 An important feature of Drinfeld's proof of theorem 4.1.2 is that it is effective in the sense that, given a presentation of a k -variety V and a suitable truncation of the arc γ (satisfying the hypotheses of the theorem), it provides a procedure to compute explicitly a presentation of a finite formal model of $(\mathcal{L}_\infty(V), \gamma)$ of the form \widehat{S}_s , for S an affine k -scheme of finite type and $s \in S(k)$ (see remark 4.1.20). In case V is an affine plane curve, D. Bourqui and J. Sebag have produced a SageMath ([76]) code of this algorithm, which can be found in [9] (unpublished).

However, this algorithm deduced from Drinfeld's proof has a high computational complexity, see [9] for a discussion about the running time for some examples. Bourqui and Sebag have developed another proof of theorem 4.1.2 valid for integral varieties defined by binomial equations, which in particular provides another algorithm for computing a finite formal model (for a given arc satisfying the assumptions of the theorem) which is computationally much more efficient. Let us describe it.

Let $n \in \mathbb{N}$ and L be a subset of $(\mathbb{N}^n \setminus \{(0, \dots, 0)\})^2$. Let V be the integral closed subvariety of \mathbb{A}_k^n defined by the ideal $\langle \mathbf{Z}^{\mathbf{a}} - \mathbf{Z}^{\mathbf{b}}; (\mathbf{a}, \mathbf{b}) \in L \rangle$ of $k[Z_1, \dots, Z_n]$, where

$$f_{\mathbf{a}, \mathbf{b}} := \mathbf{Z}^{\mathbf{a}} - \mathbf{Z}^{\mathbf{b}} = \prod_{i=1}^n Z_i^{a_i} - \prod_{i=1}^n Z_i^{b_i}.$$

By [23, Proposition 1.1.11], the ideal defining V being prime and generated by binomials, V is an affine toric variety (not necessarily normal). Let us consider a rational non-degenerate arc $\gamma \in \mathcal{L}_\infty(V)(k)$ corresponding to the datum of a family $(\gamma_i)_{1 \leq i \leq n}$ of elements of $k[[t]] \setminus \{0\}$ such that $\prod_{i=1}^n \gamma_i^{a_i} = \prod_{i=1}^n \gamma_i^{b_i}$. For $i \in \{1, \dots, n\}$ we set $c_i := \text{ord}_t(\gamma_i) \in \mathbb{N}$, $\mathbf{c} = (c_i)_{1 \leq i \leq n} \in \mathbb{N}^n$ and $d := \sum_{i=0}^n a_i c_i = \sum_{i=0}^n b_i c_i$.

Construction 4.2.15. Let $n \in \mathbb{N}$, L be a subset of $(\mathbb{N}^n \setminus \{(0, \dots, 0)\})^2$ and $\mathbf{c} = (c_i)_{1 \leq i \leq n} \in \mathbb{N}^n$. For every $(\mathbf{a}, \mathbf{b}) \in L$ we consider in $k[Z_1, \dots, Z_n]$ the element $f_{\mathbf{a}, \mathbf{b}} := \mathbf{Z}^{\mathbf{a}} - \mathbf{Z}^{\mathbf{b}}$. Let us consider the following equality in $k[t, Z_{i, s_i}; 1 \leq i \leq n, 0 \leq s_i < c_i]$:

$$f_{\mathbf{a}, \mathbf{b}} \Big|_{Z_i = t^{c_i} + \sum_{s_i=0}^{c_i-1} Z_{i, s_i} t^{s_i}} = \prod_{i=1}^n \left(t^{c_i} + \sum_{s_i=0}^{c_i-1} Z_{i, s_i} t^{s_i} \right)^{a_i} - \prod_{i=1}^n \left(t^{c_i} + \sum_{s_i=0}^{c_i-1} Z_{i, s_i} t^{s_i} \right)^{b_i} =: \sum_{s \in \mathbb{N}} f_{\mathbf{a}, \mathbf{b}, s} t^s,$$

where $f_{\mathbf{a}, \mathbf{b}, s} = 0$ for $s \geq d$. We define $W(\mathbf{c}, L)$ to be the affine closed k -subscheme of the affine space $\text{Spec}(k[Z_{i, s_i}; 1 \leq i \leq n, 0 \leq s_i < c_i])$ defined by the ideal $\langle f_{\mathbf{a}, \mathbf{b}, s}; (\mathbf{a}, \mathbf{b}) \in L, 0 \leq s < d \rangle$ and $\mathcal{W}(\mathbf{c}, L)$ the formal completion of $W(\mathbf{c}, L)$ along the origin of $\text{Spec}(k[Z_{i, s_i}; 1 \leq i \leq n, 0 \leq s_i < c_i])$.

We will show that $\mathcal{W}(\mathbf{c}, L)$ is a finite formal model of the pair $(\mathcal{L}_\infty(V), \gamma)$ (where $c_i := \text{ord}_t(\gamma_i)$) by slightly generalizing the arguments in the proof of [13, Theorem 5.2]. Let us stress that the finite formal model computed in this way is in general not minimal, but it seems to provide a finite formal model of lower dimension than the one obtained by effectively applying Drinfeld's proof, and thus closer to the minimal formal model. In [15, Subsection 3.7] the computation of a finite formal model is made for several examples using both methods and the specific one for binomial equations is much simpler in all cases.

Remark 4.2.16. Let (A, \mathbf{m}_A) be an object of \mathbf{Test}_k . Then $\text{Hom}_{\mathbf{Lacp}_k}(\mathcal{W}(\mathbf{c}, L), A)$ is in natural bijection with the set of families $\{z_{i,s_i}; i \in \{1, \dots, n\}, 0 \leq s_i < c_i\}$ of elements of \mathbf{m}_A such that for every pair $(\mathbf{a}, \mathbf{b}) \in L$ one has

$$\prod_{i=1}^n \left(t^{c_i} + \sum_{s_i=0}^{c_i-1} z_{i,s_i} t^{s_i} \right)^{a_i} - \prod_{i=1}^n \left(t^{c_i} + \sum_{s_i=0}^{c_i-1} z_{i,s_i} t^{s_i} \right)^{b_i} = 0.$$

Proposition 4.2.17. *In the preceding situation, the formal scheme $\mathcal{W}(\mathbf{c}, L)$ is a finite formal model of the pair $(\mathcal{L}_\infty(V), \gamma)$.*

Proof. By observation 4.1.7 the k -algebra $\widehat{\mathcal{O}_{\mathcal{L}_\infty(V), \gamma}}$ is determined by the functorial behaviour of the sets $\text{Hom}_{\mathbf{Lacp}_k}(\widehat{\mathcal{O}_{\mathcal{L}_\infty(V), \gamma}}, A)$ for every test-ring A over k . Let (A, \mathbf{m}_A) be an object of \mathbf{Test}_k and let us consider $\gamma_A \in \widehat{\mathcal{L}_\infty(V)}_\gamma(A)$ an A -deformation of γ , i.e., a family $(\gamma_{A,i})_{1 \leq i \leq n}$ of elements of $A[[t]]$ such that $\gamma_{A,i} = \gamma_i \pmod{\mathbf{m}_A[[t]]}$ and $\prod_{i=1}^n \gamma_{A,i}^{a_i} = \prod_{i=1}^n \gamma_{A,i}^{b_i}$.

Observe that, for $i \in \{1, \dots, n\}$, the series $\gamma_{A,i}$ is c_i -regular. Thanks to the Weierstrass preparation theorem 4.1.11 we can write $\gamma_{A,i} = p_{A,i} u_{A,i}$ with $p_{A,i}$ a c_i -Weierstrass polynomial and $u_{A,i}$ a unit in $A[[t]]$. Then we have the equality $\prod_{i=1}^n p_{A,i}^{a_i} u_{A,i}^{a_i} = \prod_{i=1}^n p_{A,i}^{b_i} u_{A,i}^{b_i}$. From the uniqueness of the Weierstrass factorization applied to $\prod_{i=1}^n p_{A,i}^{a_i} u_{A,i}^{a_i}$ (note that it is d -regular since $\prod_{i=1}^n \gamma_{A,i}^{a_i}$ is) we deduce the following equalities:

$$\prod_{i=1}^n p_{A,i}^{a_i} = \prod_{i=1}^n p_{A,i}^{b_i}, \quad (2.1)$$

$$\prod_{i=1}^n u_{A,i}^{a_i} = \prod_{i=1}^n u_{A,i}^{b_i}. \quad (2.2)$$

By remark 4.2.16 and the definition of a Weierstrass polynomial, equation (2.1) uniquely determines an element in $\text{Hom}_{\mathbf{Lacp}_k}(\mathcal{W}(\mathbf{c}, L), A)$. On the other hand, equation (2.2) defines an A -deformation of the arc $\tilde{\gamma}$ of $\mathcal{L}_\infty(V)(k)$ corresponding to the family $(\gamma_i/t^{c_i})_{1 \leq i \leq n}$ of elements of $k[[t]]$. The order of each of these series being 0, we deduce that the base point of $\tilde{\gamma}$ belongs to $\text{Spec}(k[Z_1^{\pm 1}, \dots, Z_n^{\pm 1}])$ which corresponds with the open torus of the affine space \mathbb{A}_k^n . The intersection of this torus and V is exactly the open torus

of V , which is contained in the smooth locus of V . Hence theorem 4.2.9 assures that the formal neighbourhood $\widehat{\mathcal{L}_\infty(V)}_\gamma$ is isomorphic to a formal disk $\mathbf{D}_k^\mathbb{N} := \mathrm{Spf}(k[[T_i]_{i \in \mathbb{N}}])$.

The above construction being functorial on A , we deduce that the deformation functor $A \mapsto \widehat{\mathcal{O}_{\mathcal{L}_\infty(V), \gamma}}(A)$ is isomorphic to the functor $A \mapsto \mathcal{W}(\mathbf{c}, L)(A) \widehat{\times}_k \mathrm{Spf}(k[[T_i]_{i \in \mathbb{N}}])(A)$ and hence $\mathcal{W}(\mathbf{c}, L) \widehat{\times}_k \mathrm{Spf}(k[[T_i]_{i \in \mathbb{N}}])$ is a decomposition of $\widehat{\mathcal{L}_\infty(V)}_\gamma$ as in the statement of the Drinfeld-Grinberg-Kazhdan theorem, which shows that $\mathcal{W}(\mathbf{c}, L)$ is a finite formal model of the pair $(\mathcal{L}_\infty(V), \gamma)$. \square

We stress that the formal scheme $\mathcal{W}(\mathbf{c}, L)$ only depends on the order of γ_i , for $1 \leq i \leq n$, and not on the particular chosen arc γ (see corollary 5.2.16, although it is stated in the normal toric case the same arguments hold in the general toric setting). In [13], Bourqui and Sebag show that the formal schemes in construction 4.2.15 provide a finite formal model of the pair $(\mathcal{L}_\infty(V), \gamma)$ for V a normal toric variety and γ a sufficiently generic rational arc in a Nash set (or maximal divisorial set). This is a crucial ingredient in the proof of our main theorem in chapter 5. Effective computation of the formal scheme in construction 4.2.15 on examples may be found in section 5.6.

FORMAL NEIGHBOURHOODS IN THE NASH SETS ASSOCIATED WITH TORIC VALUATIONS

We begin the chapter by completing the picture of the known results about formal neighbourhoods of the arc scheme that we started in chapter 4 for rational arcs. In section 5.1 we will introduce the constructible points of the arc scheme, which were defined by T. de Fernex and R. Docampo as the generic points of the irreducible constructible subsets of the arc space. Then, we present some results about them obtained in [26] leading to the characterization in corollary 5.1.9 of non-degenerate constructible points of the arc scheme in terms of the noetherianity of the section ring of its formal neighbourhood. In particular, the generic point of a Nash set associated with a divisorial valuation is a constructible point, and hence this finiteness result holds for its formal neighbourhood.

Let us consider the case of a normal affine toric variety V and let \mathcal{N} be the Nash set associated with a divisorial toric valuation. As we explain in subsection 5.2.13, D. Bourqui and J. Sebag showed in [13] that the formal neighbourhood of a sufficiently generic rational arc of the Nash set \mathcal{N} is constant. Note that in this case we can apply the Drinfeld-Grinberg-Kazhdan theorem to deduce that this formal neighbourhood can be decomposed as a formal product of a finite formal model and a formal disk.

The main theorem in this chapter (theorem 5.5.19) establishes a strong connection between the formal neighbourhood at such a sufficiently generic rational arc in \mathcal{N} and the formal neighbourhood at the generic point of \mathcal{N} . This connexion is made by relating a finite formal model of the first (obtained in [13]) with the latter through a direct comparison between the corresponding complete local rings. This result not only confirms the invariance already observed by Bourqui and Sebag, but also allows to transpose some properties deduced from the Drinfeld-Grinberg-Kazhdan theorem to the formal neighbourhood at the generic point of the Nash set, which is not a rational point in general. In particular, we deduce that it can be explicitly described in terms of the formal spectrum of the completion of an essentially of finite type local algebra over the residue field of the generic point.

Most of the contents in this chapter from section 5.2 are included in the preprint [8].

5.1 CONSTRUCTIBLE POINTS

In this section we will define a class of schematic points of arc scheme, recently named constructible points by T. de Fernex and R. Docampo in [26], which are characterized by the noetherianity of their formal neighbourhood. The main reference in this section is [26].

5.1.1 Let V be a scheme. A point $v \in V$ is said to be *constructible* if it is the generic point of an irreducible constructible subset of V . This definition was introduced by T. de Fernex and R. Docampo in [26]. Let us stress that the fact that a point $v \in V$ is constructible according with this definition does not mean necessarily that the subset $\{v\}$ of V is a constructible subset of V .

We have the following lemma, corresponding to [26, Lemma 10.1].

Lemma 5.1.2. *Let k be a field, V be a k -variety and $\alpha \in \mathcal{L}_\infty(V)$ a non-degenerate arc, $\alpha \notin \mathcal{L}_\infty(V_{\text{Sing}})$. Let $Y \subset V$ be the (unique) irreducible component of V such that $\alpha \in \mathcal{L}_\infty(Y)$. Then α is a constructible point of $\mathcal{L}_\infty(V)$ if and only if it is a constructible point of $\mathcal{L}_\infty(Y)$.*

5.1.3 Let k be a perfect field and V be a k -variety. Given a point $\alpha \in \mathcal{L}_\infty(V)$, we denote by $\kappa(\alpha)$ its residue field. For every $m \in \mathbb{N}$ let us denote the truncation of level m by $\alpha_m := \theta_{m,V}^\infty(\alpha) \in \mathcal{L}_m(V)$ and its residue field by $\kappa(\alpha_m)$. For convenience we will also denote α by α_∞ .

For $m \in \mathbb{N} \cup \{\infty\}$ let $\mathfrak{p}_m \in \mathcal{O}(\mathcal{L}_m(V))$ be the prime ideal defining α_m . Let us stress that $\mathfrak{p}_m/\mathfrak{p}_m^2$ is a $\kappa(\alpha_m)$ -vector space. We define the *embedding dimension* of $\mathcal{L}_m(V)$ at α_m as

$$\text{emb.dim}(\mathcal{O}_{\mathcal{L}_m(V),\alpha_m}) := \dim_{\kappa(\alpha_m)}(\mathfrak{p}_m/\mathfrak{p}_m^2).$$

We observe that, for $m' \geq m$, we have inclusions $\mathfrak{p}_m \subset \mathfrak{p}_{m'}$. Then there are natural maps $\mathfrak{p}_m/\mathfrak{p}_m^2 \rightarrow \mathfrak{p}_{m'}/\mathfrak{p}_{m'}^2$, and $\mathfrak{p}_\infty/\mathfrak{p}_\infty^2 = \varinjlim (\mathfrak{p}_m/\mathfrak{p}_m^2)$.

Let us recall [26, Corollary 9.5], which assures that for the study of the embedding dimension we can restrict to the irreducible components.

Corollary 5.1.4 (de Fernex-Docampo). *Let k be a field, V be a k -variety and $\alpha \in \mathcal{L}_\infty(V)$ a non-degenerate arc, $\alpha \notin \mathcal{L}_\infty(V_{\text{Sing}})$. Let $Y \subset V$ be the (unique) irreducible component of V such that $\alpha \in \mathcal{L}_\infty(Y)$. Then*

$$\text{emb.dim}(\mathcal{O}_{\mathcal{L}_\infty(Y),\alpha}) = \text{emb.dim}(\mathcal{O}_{\mathcal{L}_\infty(V),\alpha}).$$

The following lemma provides a useful property about local rings of finite embedding dimension. It corresponds with [26, Lemma 10.12]. Let us precise that the embedding

dimension of a k -scheme at a point can be defined analogously to the case of arc and jet schemes.

Lemma 5.1.5. *Let k be a field, V be a k -scheme and $v \in V$. Then the section ring $\widehat{\mathcal{O}_{V,v}}$ of the formal neighbourhood of V at the point v is noetherian if and only if $\text{emb.dim}(\mathcal{O}_{V,v}) < \infty$.*

Let us keep the preceding notation. For $m \in \mathbb{N}$ we define $\dim(\alpha_m) := \text{tr.deg}_k(\kappa(\alpha_m))$. Let V be a reduced and irreducible k -variety and $\alpha \in \mathcal{L}_\infty(V)$. The *jet codimension* of α in $\mathcal{L}_\infty(V)$ is defined to be

$$\text{jet.codim}(\alpha, \mathcal{L}_\infty(V)) := \lim_{m \rightarrow \infty} ((m+1) \dim(V) - \dim(\alpha_m)).$$

This limit exists (see [28, Lemma 4.13]) and is nonnegative ([26, Lemma 8.6]). The next theorem, corresponding to [26, Theorem 10.7], asserts that for a reduced and irreducible variety defined over a perfect field, the embedding dimension of its arc scheme at a point coincides with the jet codimension of this point. The proof is based on the study of the differentials of the arc scheme in the first part of [26].

Theorem 5.1.6 (de Fernex-Docampo). *Let k be a perfect field and V be an integral k -variety. Then, for every point $\alpha \in \mathcal{L}_\infty(V)$ we have*

$$\text{emb.dim}(\mathcal{O}_{\mathcal{L}_\infty(V), \alpha}) = \text{jet.codim}(\alpha, \mathcal{L}_\infty(V)).$$

5.1.7 The following theorem characterizes the non-degenerate constructible points of the arc scheme in terms of the embedding dimension. It corresponds to [26, Theorem 10.8].

Theorem 5.1.8 (de Fernex-Docampo). *Let k be a perfect field and V be a k -variety. For every $\alpha \in \mathcal{L}_\infty(V)$ the following assertions are equivalent:*

- (i) *The embedding dimension of $\mathcal{L}_\infty(V)$ at α is finite, i.e., $\text{emb.dim}(\mathcal{O}_{\mathcal{L}_\infty(V), \alpha}) < \infty$.*
- (ii) *The arc α is a constructible point of $\mathcal{L}_\infty(V)$ and it is non-degenerate, i.e., $\alpha \notin \mathcal{L}_\infty(V_{\text{Sing}})$.*

Sketch of the proof. If α is a degenerate arc, the result follows from [26, Proposition 8.7]. Otherwise, V is reduced and irreducible at the generic point $\alpha(\eta)$ and by lemma 5.1.2 and corollary 5.1.4 we can replace V by its irreducible component containing $\alpha(\eta)$ and hence assume that it is a variety. In this case the constructible points are stable points (see subsection 5.1.12) and we conclude by applying proposition 5.1.13 (stable points are characterized by their finite jet codimension) and theorem 5.1.6 (the jet codimension equals the embedding dimension). \square

From theorem 5.1.8 and lemma 5.1.5 one deduces the following result, characterizing the non-degenerate constructible points of the arc scheme in terms of the noetherianity of their formal neighbourhoods. It corresponds to [26, Corollary 10.13].

Corollary 5.1.9 (de Fernex-Docampo). *Let k be a perfect field and V be a reduced k -variety. For every $\alpha \in \mathcal{L}_\infty(V)$ the following assertions are equivalent:*

- (i) *The section ring $\widehat{\mathcal{O}_{\mathcal{L}_\infty(V), \alpha}}$ of the formal neighbourhood of $\mathcal{L}_\infty(V)$ at α is noetherian.*
- (ii) *The arc α is a constructible point of $\mathcal{L}_\infty(V)$ and it is non-degenerate, i.e., $\alpha \notin \mathcal{L}_\infty(V_{\text{Sing}})$.*

This result may be compared with theorem 4.1.2 and seen as a result about the structure of the formal neighbourhood of the arc scheme. Precisely, for a k -variety V it says that, if we consider a non-degenerate constructible point of $\mathcal{L}_\infty(V)$ instead of a non-degenerate rational point, in the decomposition of the formal neighbourhood at this point we have no longer the factor corresponding to the infinite-dimensional formal disk appearing in theorem 4.1.2, and we only find the part of finite type. However, this factor is a formal scheme of finite type defined over the residue field $\kappa(\alpha)$ which is a field extension of k which may have infinite transcendence degree over k . Our original work in the rest of the chapter consists in comparing both decompositions in the case of a normal toric variety.

5.1.10 In [26, Section 11], de Fernex and Docampo study the generic points of the maximal divisorial sets (see subsection 2.5.4 and proposition 2.5.9 for the fact that such a set is an irreducible subset of the arc scheme), which they call *maximal divisorial arcs*. In particular, they obtain the following result using theorem 5.1.8. It corresponds to [26, Corollary 11.5].

Proposition 5.1.11 (de Fernex-Docampo). *Let k be a perfect field and V be a reduced k -variety. For every divisorial valuation ν on V , the generic point γ of $W_V(\nu)$ is a constructible point of $\mathcal{L}_\infty(V)$ not belonging to $\mathcal{L}_\infty(V_{\text{Sing}})$.*

Let us stress that the fact that γ does not belong to $\mathcal{L}_\infty(V_{\text{Sing}})$ can be deduced from the fact that the subset $W_V(\nu)$ is fat.

5.1.12 Let k be a perfect field and V a k -scheme. It is a consequence of lemma 2.4.10 that the constructible subsets of $\mathcal{L}_\infty(V)$ are the so-called *weakly stable semi-algebraic subsets* of $\mathcal{L}_\infty(V)$ in [29]. Let X and Y be k -schemes of finite type and F be a k -variety (in fact it suffices to ask F to be a reduced k -scheme). Let C (resp. D) be a constructible subset of X (resp. Y). We say that a morphism $\varphi : X \rightarrow Y$ induces a *piecewise trivial fibration* $C \rightarrow D$ over D with fiber F if there exists a finite partition of $D = D_1 \sqcup \cdots \sqcup D_r$

such that, for $i \in \{1, \dots, r\}$, D_i is locally closed in Y and $C \cap \varphi^{-1}(D_i)$ is locally closed in X and isomorphic (with the reduced scheme structure) to $D_i \times F$ in such a way that φ corresponds under this isomorphism to the projection $D_i \times F \rightarrow D_i$.

Let V be an integral k -variety. According to [29], a subset $W \subset \mathcal{L}_\infty(V)$ is said to be a *stable semi-algebraic subset* (of $\mathcal{L}_\infty(V)$) if it is a constructible subset of $\mathcal{L}_\infty(V)$ (i.e., a weakly stable semi-algebraic subset) and for every $m \in \mathbb{N}$ big enough, the induced truncation morphism $\theta_{m,V}^{m+1} : \theta_{m+1,V}^\infty(\mathcal{L}_\infty(V)) \rightarrow \theta_{m,V}^\infty(\mathcal{L}_\infty(V))$ induces a piecewise trivial fibration over $\theta_{m,V}^\infty(W)$ with fiber $\mathbb{A}_k^{\dim(V)}$. In [67], for a field of characteristic zero A . Reguera defined a *stable point* of $\mathcal{L}_\infty(V)$ to be the generic point of an irreducible stable semi-algebraic subset of $\mathcal{L}_\infty(V)$. By definition, it is in particular a constructible point. As explained in [29, Subsection 2.7] a constructible subset of $\mathcal{L}_\infty(V)$ which is disjoint from $\mathcal{L}_\infty(V_{\text{Sing}})$ is a stable semi-algebraic subset by [29, Lemma 4.1], here the fact that V is a reduced k -variety is crucial. Hence a point of $\mathcal{L}_\infty(V)$ is stable if and only if it is constructible and non-degenerate. Note that in [29] they work over a field k of characteristic zero, but everything we have presented here holds for k a perfect field. In particular the proof of [29, Lemma 4.1] has been rewritten in this setting in [70] (see also [32, Proposition 4.1]). As de Fernex and Docampo explain in [26, Remark 10.4], constructible points extend the notion of stable points to nonreduced schemes. We have the following characterization of stable points which is a consequence of [29, Lemma 4.1] (see e.g., [26, Proposition 10.5]).

Proposition 5.1.13. *Let k be a perfect field and V be an integral k -variety. For every $\alpha \in \mathcal{L}_\infty(V)$ the following assertions are equivalent:*

- (i) *The jet codimension of α in $\mathcal{L}_\infty(V)$ at is finite, $\text{jet.codim}(\alpha, \mathcal{L}_\infty(V)) < \infty$.*
- (ii) *The arc α is a stable point of $\mathcal{L}_\infty(X)$.*

Let V be an integral k -variety and $\alpha \in \mathcal{L}_\infty(V)$ be a stable point. In particular, corollary 5.1.9 implies that the section ring $\widehat{\mathcal{O}_{\mathcal{L}_\infty(V), \alpha}}$ of the formal neighbourhood of $\mathcal{L}_\infty(V)$ at α is noetherian. This fact was already obtained by A. Reguera in characteristic zero. It follows from [66, Corollary 4.6] which proves that $\widehat{\mathcal{O}_{\mathcal{L}_\infty(V)_{\text{red}}, \alpha}}$ is noetherian and [67, Theorem 3.13] which proves that there is an isomorphism $\widehat{\mathcal{O}_{\mathcal{L}_\infty(V), \alpha}} \cong \widehat{\mathcal{O}_{\mathcal{L}_\infty(V)_{\text{red}}, \alpha}}$ (this last result was stated in characteristic zero but it seems that the proof can be modified to hold for perfect fields).

5.2 NORMAL TORIC VARIETIES AND THEIR ARC SCHEME

In this section we will briefly introduce the normal toric varieties and some results concerning their arc scheme which we will make use of in section 5.5. For more information about toric varieties we refer to [23] (here we will only treat some of the contents in Sections 1.1 and 1.2); concerning the arc scheme of a normal toric variety we will present

several results in [46] or [13]. Since we are studying local properties, we may restrict ourselves to the case of affine normal toric varieties.

5.2.1 Let us recall the definition of an affine toric variety. From now on and up to the end of the chapter, k is a field of characteristic zero. Let us consider $\mathbb{G}_{m,k} := \text{Spec}(k[T, T^{-1}])$, it is an algebraic group such that, for every k -algebra R , its R -points are identified with R^* , the group of units of R . We say that an algebraic group \mathcal{T} defined over k is a *split algebraic k -torus* if there exists an integer $d \in \mathbb{N}$ such that \mathcal{T} is isomorphic (as an algebraic group) to $\mathbb{G}_{m,k}^d$. We call the integer d the dimension of \mathcal{T} .

We define an *affine toric variety* to be an integral affine k -variety V containing a torus \mathcal{T} as an open subset such that the algebraic group action of the torus extends to an algebraic action $\mathcal{T} \times V \rightarrow V$. In particular, a d -dimensional torus \mathcal{T} is an affine toric variety and the affine space \mathbb{A}_k^d also is, both with open torus \mathcal{T} . In the sequel, we are will interest in normal affine toric varieties, which have a nice combinatorial description.

5.2.2 Let k be a field of characteristic zero. Let d be a positive integer and \mathcal{T} a split algebraic k -torus of dimension d . Let $N := \text{Hom}(\mathbb{G}_{m,k}, \mathcal{T})$ be the *group of cocharacters of \mathcal{T}* , which is a free \mathbb{Z} -module of rank d , i.e., a lattice isomorphic to \mathbb{Z}^d (see [42] or [45, §16]). Let $M := \text{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$ be the dual \mathbb{Z} -module of N , i.e., the *group of characters of \mathcal{T}* , which is also isomorphic to \mathbb{Z}^d . We denote by $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$ (resp. $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$) the \mathbb{R} -vector space of dimension d associated with N (resp. M). We have a \mathbb{R} -bilinear canonical map $\langle \cdot, \cdot \rangle : M_{\mathbb{R}} \times N_{\mathbb{R}} \rightarrow \mathbb{R}$ which coincides, via the identification of $N_{\mathbb{R}}$ and $M_{\mathbb{R}}$ with \mathbb{R}^d , with the usual dot product.

The points of the lattices N and M , considered as points of the associated vector spaces, are called their *integral points*. We will simply call a *cone* of $N_{\mathbb{R}}$ a strongly convex rational polyhedral cone of the vector space $N_{\mathbb{R}}$ (i.e., a convex cone generated by finitely many elements of N , which moreover does not contain any line). Let σ be a cone of $N_{\mathbb{R}}$, we consider its dual

$$\sigma^{\vee} = \{\mathbf{m} \in M_{\mathbb{R}} : \langle \mathbf{m}, \mathbf{n} \rangle \geq 0 \text{ for all } \mathbf{n} \in \sigma\}.$$

If σ is a cone of dimension d (i.e., the vector subspace spanned by σ is $N_{\mathbb{R}}$) then σ^{\vee} is also a cone of $M_{\mathbb{R}}$, otherwise it is not strongly convex (hence it may contain lines) but it is still convex and generated by finitely many elements (see [23, Propositions 1.2.4 and 1.2.12]). Moreover, $(\sigma^{\vee})^{\vee} = \sigma$. The points

$$S_{\sigma} := \sigma^{\vee} \cap M$$

form a semigroup, which we call the *semigroup associated with σ* . We have the following lemma, see e.g., [23, Proposition 1.2.17] for a proof.

Lemma 5.2.3 (Gordan Lemma). *Let N be a lattice, M be its dual \mathbb{Z} -module and $N_{\mathbb{R}}$ and $M_{\mathbb{R}}$ the associated \mathbb{R} -vector spaces. Let σ be a cone of $N_{\mathbb{R}}$. Then its associated semigroup $S_{\sigma} := \sigma^{\vee} \cap M$ is finitely generated.*

Remark 5.2.4. Regardless of the dimension of σ , by [23, Proposition 1.2.12] its strong convexity assures that σ^\vee has dimension d , i.e., the vector space spanned by σ^\vee is $M_{\mathbb{R}}$. Hence the number of generators of the semigroup S_σ is bigger or equal than d .

5.2.5 Let σ be a cone of $N_{\mathbb{R}}$. Associated with the semigroup S_σ we can consider the k -algebra

$$k[S_\sigma] := k[\chi^{\mathbf{m}}; \mathbf{m} \in S_\sigma],$$

where we may consider χ as a symbol. It is a finitely generated k -algebra by the Gordan Lemma 5.2.3. The spectrum of the k -algebra $k[S_\sigma]$ then defines a normal affine toric variety V_σ with torus \mathcal{T} (see e.g., [23, Theorem 1.2.18]). Note that every affine normal toric variety with torus \mathcal{T} is of the form V_σ for σ a cone of $N_{\mathbb{R}}$, where N is the cocharacter lattice of \mathcal{T} (see e.g., [23, Theorem 1.3.5]). For every $\mathbf{m} \in S_\sigma$, the symbol $\chi^{\mathbf{m}}$ now makes sense as the regular function on V_σ defined by \mathbf{m} . Recall that for every k -algebra R , the set $\text{Hom}_{\mathbf{Alg}_k}(k[S_\sigma], R)$ is in natural bijection with the set of semigroup morphisms $S_\sigma \rightarrow R$, where the semigroup structure on R is induced by the multiplication.

Let $\{\mathbf{m}_1, \dots, \mathbf{m}_h\}$ be the minimal set of generators of the semigroup S_σ . By remark 5.2.4 we may and shall assume in the sequel that the set $\{\mathbf{m}_1, \dots, \mathbf{m}_d\}$ is a \mathbb{Z} -basis of M . If we call z_1, \dots, z_h respectively $\chi^{\mathbf{m}_1}, \dots, \chi^{\mathbf{m}_h}$, we deduce that $\mathcal{O}(V_\sigma) := k[S_\sigma] \cong k[z_1, \dots, z_h]$.

Let us stress that, for every k -algebra R , the set $\text{Hom}_{\mathbf{Alg}_k}(\mathcal{O}(V_\sigma), R)$ is in bijection with the set of semigroup morphisms $S_\sigma \rightarrow R$, where we consider in R the multiplicative structure.

Remark 5.2.6. The closed subscheme defined by the ideal $\langle \prod_{1 \leq i \leq h} z_i \rangle$ has for support the closed set $V_\sigma \setminus \mathcal{T}$, and the same holds for the ideal $\langle \prod_{1 \leq i \leq d} z_i \rangle$. In fact, $\mathcal{O}(\mathcal{T}) = k[M] \cong k[z_1^{\pm 1}, \dots, z_d^{\pm 1}]$ (because $\{\mathbf{m}_1, \dots, \mathbf{m}_d\}$ is a \mathbb{Z} -basis of M) which is also isomorphic as a k -algebra to $k[z_1^{\pm 1}, \dots, z_h^{\pm 1}]$ (see e.g., [23, Example 1.1.13]). In particular, for every k -algebra R , the set $\text{Hom}_{\mathbf{Alg}_k}(\mathcal{O}(\mathcal{T}), R)$ is in bijection with the set of group morphisms $M \rightarrow R^*$. Moreover, \mathcal{T} is contained in the smooth locus of V_σ and hence the singular locus of V_σ is contained in the closed set $V_\sigma \setminus \mathcal{T}$.

5.2.7 An explicit description of V_σ as a closed subscheme of the affine space \mathbb{A}_k^h will be useful in the sequel. We consider the k -algebra $k[\mathbf{Z}] := k[Z_1, \dots, Z_h]$, then $\mathbb{A}_k^h = \text{Spec}(k[\mathbf{Z}])$.

Let us keep the notation in subsection 5.2.5. Let $\{\mathbf{e}_i; i \in \{1, \dots, h\}\}$ be the canonical basis of \mathbb{Z}^h . Let $\boldsymbol{\ell} = (\ell_1, \dots, \ell_h) \in \mathbb{Z}^h$, we set

$$\boldsymbol{\ell}^+ = \sum_{\ell_i \geq 0} \ell_i \mathbf{e}_i \quad \text{and} \quad \boldsymbol{\ell}^- = - \sum_{\ell_i < 0} \ell_i \mathbf{e}_i,$$

which are both elements of \mathbb{N}^h . Note that $\boldsymbol{\ell} = \boldsymbol{\ell}^+ - \boldsymbol{\ell}^-$. Given $\boldsymbol{\ell}' = (\ell'_1, \dots, \ell'_h) \in \mathbb{N}^h$, we set $\mathbf{Z}^{\boldsymbol{\ell}'} := \prod_{i=1}^h Z_i^{\ell'_i}$. Hence for $\boldsymbol{\ell} \in \mathbb{Z}^h$ we fix $F_{\boldsymbol{\ell}} := \mathbf{Z}^{\boldsymbol{\ell}^+} - \mathbf{Z}^{\boldsymbol{\ell}^-}$.

Mapping \mathbf{e}_i to $\pi(\mathbf{e}_i) := \mathbf{m}_i$ induces an exact sequence of groups

$$0 \rightarrow L \rightarrow \mathbb{Z}^h \xrightarrow{\pi} M \rightarrow 0, \quad (2.1)$$

where L is a subgroup of \mathbb{Z}^h . For $\boldsymbol{\ell} \in L$ and $\mathbf{n} \in N$, we set

$$\langle \boldsymbol{\ell}, \mathbf{n} \rangle := \left\langle \sum_{\substack{i=1 \\ \ell_i > 0}}^h \ell_i \mathbf{m}_i, \mathbf{n} \right\rangle = - \left\langle \sum_{\substack{i=1 \\ \ell_i < 0}}^h \ell_i \mathbf{m}_i, \mathbf{n} \right\rangle.$$

By [23, Proposition 1.1.9], the ideal of $k[\mathbf{Z}]$ defining V_σ is

$$\mathfrak{i}_\sigma := \langle F_\boldsymbol{\ell}; \boldsymbol{\ell} \in L \rangle. \quad (2.2)$$

The set $\{\mathbf{m}_1, \dots, \mathbf{m}_d\}$ being a \mathbb{Z} -basis of N , for $q \in \{d+1, \dots, h\}$ we can write the element \mathbf{m}_q , as a linear combination with integer coefficients (possibly negative) of $\mathbf{m}_1, \dots, \mathbf{m}_d$. Thus we have in L an element $\boldsymbol{\ell}_q = (\ell_{q,1}, \dots, \ell_{q,h})$ such that $\ell_{q,q} = 1$ and $\ell_{q,q'} = 0$ for every $q' \in \{d+1, \dots, h\} \setminus \{q\}$. The element $\boldsymbol{\ell}_q \in L$ induces an element

$$F_{\boldsymbol{\ell}_q} = Z_q \prod_{\substack{i=1 \\ \ell_{q,i} \geq 0}}^d Z_i^{\ell_{q,i}} - \prod_{\substack{i=1 \\ \ell_{q,i} < 0}}^d Z_i^{-\ell_{q,i}} \quad (2.3)$$

in the ideal \mathfrak{i}_σ . We observe that in the binomial $F_{\boldsymbol{\ell}_q}$ none of the variables Z_{d+1}, \dots, Z_h appears, excepting Z_q .

5.2.8 Let us now consider the arc scheme $\mathcal{L}_\infty(V_\sigma)$ of the toric variety V_σ . We denote by \mathbf{Z}_∞ the set of variables $\{Z_{i,s} : i \in \{1, \dots, h\}, s \in \mathbb{N}\}$. By lemma 2.3.5 we deduce that

$$\mathcal{L}_\infty(V_\sigma) \cong \text{Spec}(k[\mathbf{Z}_\infty]/[\mathfrak{i}_\sigma]),$$

where $[\mathfrak{i}_\sigma]$ is the differential ideal generated by \mathfrak{i}_σ . For every $\boldsymbol{\ell} \in L$, the elements $F_{\boldsymbol{\ell},s} \in k[\mathbf{Z}_\infty]$ may be characterized by the following equality in $k[\mathbf{Z}_\infty][[t]]$:

$$F_\boldsymbol{\ell} \Big|_{Z_i = \sum_{s \in \mathbb{N}} Z_{i,s} t^s} = \sum_{s \in \mathbb{N}} F_{\boldsymbol{\ell},s} t^s. \quad (2.4)$$

Hence we will make an abuse justified by proposition 2.2.10 and denote also by $[\mathfrak{i}_\sigma]$ the ideal of $k[\mathbf{Z}_\infty]$ generated by the elements $\{F_{\boldsymbol{\ell},s} : \boldsymbol{\ell} \in L, s \in \mathbb{N}\}$. In another abuse of notation, sometimes we will also denote by $k[z_{i,s}; i \in \{1, \dots, h\}, s \in \mathbb{N}]$ the k -algebra $\mathcal{O}(\mathcal{L}_\infty(V_\sigma))$.

Let $\mathcal{L}_\infty^\circ(V_\sigma)$ be the open set of $\mathcal{L}_\infty(V_\sigma)$ defined by

$$\mathcal{L}_\infty^\circ(V_\sigma) := \mathcal{L}_\infty(V_\sigma) \setminus \mathcal{L}_\infty(V_\sigma \setminus \mathcal{T}).$$

Let us remark that, by lemma 2.2.13, an arc $\alpha \in \mathcal{L}_\infty(V_\sigma)$ belongs to $\mathcal{L}_\infty^\circ(V_\sigma)$ if and only if its generic point $\alpha(\eta)$ belongs to \mathcal{T} . By remark 5.2.6, we deduce that α is

non-degenerate. Remark 5.2.6 also assures that $\alpha \in \mathcal{L}_\infty^\circ(V_\sigma)$ if and only if for every $\mathbf{m} \in S_\sigma$, $\alpha^*(\chi^{\mathbf{m}}) \neq 0$ and that this is also equivalent to having, for every $1 \leq i \leq d$, $\alpha^*(z_i) \in \kappa(\alpha)[[t]] \setminus \{0\}$. Therefore one has

$$\mathcal{L}_\infty^\circ(V_\sigma) = \bigcap_{i=1}^d \bigcup_{s \in \mathbb{N}} \{z_{i,s} \neq 0\}.$$

5.2.9 Let \mathbf{n} be a point of $\sigma \cap N$. For $f \in \mathcal{O}(V_\sigma)$, we set

$$\text{ord}_{\mathbf{n}}(f) = \langle f, \mathbf{n} \rangle := \inf_{\chi^{\mathbf{m}} \in f} \langle \mathbf{m}, \mathbf{n} \rangle$$

where $\chi^{\mathbf{m}} \in f$ means that the coefficient of $\chi^{\mathbf{m}}$ in the decomposition of f is not zero. It extends to a valuation on $k(V_\sigma)$ which we denote also by $\text{ord}_{\mathbf{n}}$ and which is a divisorial valuation on V_σ , as explained in [46, Definition 5.2]. We call such a valuation a *divisorial toric valuation* on V_σ . One sees that $\mathbf{n} \mapsto \text{ord}_{\mathbf{n}}$ is a bijection between $\sigma \cap N$ and the set of divisorial toric valuations on V_σ . From now on we shall identify the later set with $\sigma \cap N$.

Let $\alpha \in \mathcal{L}_\infty^\circ(V_\sigma)$. By subsection 5.2.5, the morphism of k -algebras $\alpha^* : \mathcal{O}(V_\sigma) \rightarrow \kappa(\alpha)[[t]]$ corresponds with a morphism of semigroups which, abusing notation, we also denote by $\alpha^* : S_\sigma \rightarrow \kappa(\alpha)[[t]]$. As seen in subsection 5.2.8, for every $\mathbf{m} \in S_\sigma$, $\alpha^*(\mathbf{m}) := \alpha^*(\chi^{\mathbf{m}}) \neq 0$. Then the composition

$$\text{ord}(\alpha) := \text{ord}_t \circ \alpha^* : S_\sigma \rightarrow \mathbb{N}$$

is a well-defined morphism of semigroups. It extends uniquely to a group morphism $\mathbf{n}_\alpha : M \rightarrow \mathbb{Z}$ which is nonnegative on S_σ . The lattice N being the dual of M and $(\sigma^\vee)^\vee = \sigma$, we deduce that \mathbf{n}_α can be identified with the unique element of $\sigma \cap N$ satisfying that, for every $\mathbf{m} \in S_\sigma$, $\text{ord}(\alpha)(\mathbf{m}) = \langle \mathbf{m}, \mathbf{n}_\alpha \rangle$.

Remark 5.2.10. Observe that, the arc $\alpha \in \mathcal{L}_\infty^\circ(V_\sigma)$ not being fat in general (except when its generic point is the generic point of \mathcal{T}), we cannot define its associated valuation ord_α of $k(V_\sigma)$ (see subsection 2.5.4). When this valuation is defined, it does not coincide in general with the divisorial toric valuation $\text{ord}_{\mathbf{n}_\alpha}$, although $\text{ord}_\alpha(\chi^{\mathbf{m}}) = \text{ord}_{\mathbf{n}_\alpha}(\chi^{\mathbf{m}})$ for every $\mathbf{m} \in S_\sigma$.

For every $\mathbf{n} \in \sigma \cap N$, we set

$$\mathcal{L}_\infty^\circ(V_\sigma)_{\mathbf{n}} := \{\alpha \in \mathcal{L}_\infty^\circ(V_\sigma); \mathbf{n}_\alpha = \mathbf{n}\}$$

$$\text{and } \mathcal{L}_\infty^\circ(V_\sigma)_{\geq \mathbf{n}} := \{\alpha \in \mathcal{L}_\infty^\circ(V_\sigma); \mathbf{n}_\alpha \in \mathbf{n} + \sigma\}.$$

Let us observe that, if $\mathbf{n}' \in \mathbf{n} + \sigma$, there exists $\mathbf{n}'' \in \sigma$ such that $\mathbf{n}' = \mathbf{n} + \mathbf{n}''$. Let $\mathbf{m} \in S_\sigma$, then $\langle \mathbf{m}, \mathbf{n}' \rangle = \langle \mathbf{m}, \mathbf{n} + \mathbf{n}'' \rangle = \langle \mathbf{m}, \mathbf{n} \rangle + \langle \mathbf{m}, \mathbf{n}'' \rangle$ which implies $\langle \mathbf{m}, \mathbf{n}' \rangle \geq \langle \mathbf{m}, \mathbf{n} \rangle$ since $\langle \mathbf{m}, \mathbf{n}'' \rangle \geq 0$. In particular, $\alpha \in \mathcal{L}_\infty^\circ(V_\sigma)_{\geq \mathbf{n}}$ if and only if $\text{ord}(\alpha)(\mathbf{m}) \geq \langle \mathbf{m}, \mathbf{n} \rangle$ for every $\mathbf{m} \in S_\sigma$.

The following lemma will be useful for describing the generic points of the Nash sets (or maximal divisorial sets) associated with divisorial toric valuations.

Lemma 5.2.11. *Let us keep the notation in this section. Let $\mathbf{n} \in \sigma \cap N$.*

(i) *One has*

$$\mathcal{L}_\infty^\circ(V_\sigma)_{\geq \mathbf{n}} = \mathcal{L}_\infty^\circ(V_\sigma) \cap \bigcap_{i=1}^h \bigcap_{s=0}^{\langle \mathbf{m}_i, \mathbf{n} \rangle - 1} \{z_{i,s} = 0\}.$$

(ii) *One has*

$$\mathcal{L}_\infty^\circ(V_\sigma)_\mathbf{n} = \mathcal{L}_\infty^\circ(V_\sigma) \cap \left(\bigcap_{i=1}^h \bigcap_{s=0}^{\langle \mathbf{m}_i, \mathbf{n} \rangle - 1} \{z_{i,s} = 0\} \right) \cap \bigcap_{i=1}^d \{z_{i, \langle \mathbf{m}_i, \mathbf{n} \rangle} \neq 0\}.$$

(iii) *The closure of $\mathcal{L}_\infty^\circ(V_\sigma)_\mathbf{n}$ coincides with the Nash set $\mathcal{N}_\mathbf{n} = \mathcal{N}_{\text{ord}_\mathbf{n}} = W_{V_\sigma}(\text{ord}_\mathbf{n})$ (see subsection 2.5.4) associated with the divisorial toric valuation \mathbf{n} .*

(iv) *One has*

$$\mathcal{L}_\infty^\circ(V_\sigma)_\mathbf{n} = \mathcal{N}_\mathbf{n} \cap \mathcal{L}_\infty^\circ(V_\sigma) \cap \bigcap_{i=1}^d \{z_{i, \langle \mathbf{m}_i, \mathbf{n} \rangle} \neq 0\}.$$

In particular $\mathcal{L}_\infty^\circ(V_\sigma)_\mathbf{n}$ is a non-empty open subset of $\mathcal{N}_\mathbf{n}$.

Proof. Let us prove assertion (i). By definition, $\alpha \in \mathcal{L}_\infty^\circ(V_\sigma)_{\geq \mathbf{n}}$ if and only if for every $\mathbf{m} \in S_\sigma$ one has $\text{ord}(\alpha)(\mathbf{m}) \geq \langle \mathbf{m}, \mathbf{n} \rangle \in \mathbb{N}$ which, by lemma 5.2.12 applied to the semigroup morphism $\text{ord}(\alpha)$, is equivalent to say that for every $1 \leq i \leq h$ one has $\text{ord}(\alpha)(\mathbf{m}_i) = \text{ord}(\alpha)(z_i) \geq \langle \mathbf{m}_i, \mathbf{n} \rangle$ and thus assertion (i) holds true.

Let $\alpha \in \mathcal{L}_\infty^\circ(V_\sigma)_{\geq \mathbf{n}}$. Note that $\mathbf{n}_\alpha = \mathbf{n}$ if and only if $\text{ord}(\alpha)(\mathbf{m}) = \langle \mathbf{m}, \mathbf{n} \rangle$ for every $\mathbf{m} \in S_\sigma$. Applying lemma 5.2.12 to the semigroup morphism $\text{ord}(\alpha)$ we deduce that this is equivalent to $\text{ord}(\alpha)(\mathbf{m}_i) = \langle \mathbf{m}_i, \mathbf{n} \rangle$ for $1 \leq i \leq d$, which proves assertion (ii).

For a proof of (iii), see [49, Example 2.10]. Let us note that the inclusion of $\mathcal{N}_\mathbf{n}$ in the closure of $\mathcal{L}_\infty^\circ(V_\sigma)_\mathbf{n}$ is deduced from the fact that, if $\eta_\mathbf{n}$ is the generic point of $\mathcal{N}_\mathbf{n} = W_{V_\sigma}(\text{ord}_\mathbf{n})$ (which is irreducible by proposition 2.5.9), then $\text{ord}_{\eta_\mathbf{n}}(\chi^\mathbf{m}) = \langle \mathbf{m}, \mathbf{n} \rangle$ for every $\mathbf{m} \in S_\sigma$, which implies that $\eta_\mathbf{n}$ belongs to $\mathcal{L}_\infty^\circ(V_\sigma)_\mathbf{n}$ by definition.

Assertion (iv) is a straightforward topological consequence of (ii) and (iii). Note also that the element of $\mathcal{L}_\infty^\circ(V_\sigma)(k)$ corresponding to the semigroup morphism $S_\sigma \rightarrow k[[t]]$, $\mathbf{m} \mapsto t^{\langle \mathbf{m}, \mathbf{n} \rangle}$ gives rise to an arc lying in $\mathcal{L}_\infty^\circ(V_\sigma)_\mathbf{n}$, which is therefore nonempty. \square

Lemma 5.2.12. *Let us keep the preceding notation in this section, in particular let N be a lattice isomorphic to \mathbb{Z}^d and M be its dual lattice, let σ be a cone of $N_\mathbb{R}$ and $S_\sigma = \sigma^\vee \cap M$ be its associated semigroup. Let $\{\mathbf{m}_1, \dots, \mathbf{m}_h\}$ be the minimal set of generators of S_σ in such a way that $\{\mathbf{m}_1, \dots, \mathbf{m}_d\}$ is a \mathbb{Z} -basis of M . Let $f : S_\sigma \rightarrow \mathbb{N}$ be a semigroup morphism and let $\mathbf{n} \in \sigma \cap N$. Then the following assertions are equivalent:*

(a) $f(\mathbf{m}) \geq \langle \mathbf{m}, \mathbf{n} \rangle$ for every $\mathbf{m} \in S_\sigma$.

(b) $f(\mathbf{m}_i) \geq \langle \mathbf{m}_i, \mathbf{n} \rangle$ for every $1 \leq i \leq h$.

Moreover, the following assertions are equivalent:

$$(a') f(\mathbf{m}) = \langle \mathbf{m}, \mathbf{n} \rangle \text{ for every } \mathbf{m} \in S_\sigma.$$

$$(b') f(\mathbf{m}_i) = \langle \mathbf{m}_i, \mathbf{n} \rangle \text{ for every } 1 \leq i \leq h.$$

$$(c') f(\mathbf{m}_i) = \langle \mathbf{m}_i, \mathbf{n} \rangle \text{ for every } 1 \leq i \leq d.$$

Proof. For the first part, it is clear that (a) implies (b). For the converse, let $\mathbf{m} \in S_\sigma$, then it can be written as $\mathbf{m} = \sum_{i=1}^h \ell_i \mathbf{m}_i$, where $\ell_i \in \mathbb{N}$ for $1 \leq i \leq h$. Then $f(\mathbf{m}) = f\left(\sum_{i=1}^h \ell_i \mathbf{m}_i\right)$ and f being a morphism of semigroups, this equals $\sum_{i=1}^h f(\ell_i \mathbf{m}_i)$. By (b), it is greater or equal to $\sum_{i=1}^h \ell_i \langle \mathbf{m}_i, \mathbf{n} \rangle = \langle \mathbf{m}, \mathbf{n} \rangle$, the last equality holding by bilinearity.

For the second part, (a') implies (b') and (b') implies (c') are clearly true. For (c') implies (a'), let $\mathbf{m} \in S_\sigma$, then it can be written as $\mathbf{m} = \sum_{i=1}^d \ell_i \mathbf{m}_i$, where $\ell_i \in \mathbb{Z}$ for $1 \leq i \leq d$. Hence

$$\mathbf{m} + \sum_{\ell_i < 0} -\ell_i \mathbf{m}_i = \sum_{\ell_i \geq 0} \ell_i \mathbf{m}_i, \quad (2.5)$$

and we observe that both terms are elements of S_σ .

On the one hand, the image by f of the left-hand term of (2.5) equals $f(\mathbf{m}) + \sum_{\ell_i < 0} -\ell_i f(\mathbf{m}_i)$ which is equal to $f(\mathbf{m}) + \sum_{\ell_i < 0} -\ell_i \langle \mathbf{m}_i, \mathbf{n} \rangle$ by (c').

On the other hand, the image by f of the right-hand term of (2.5) equals $\sum_{\ell_i \geq 0} \ell_i f(\mathbf{m}_i)$, which is equal to $\sum_{\ell_i \geq 0} \ell_i \langle \mathbf{m}_i, \mathbf{n} \rangle$ by (c').

Thus we have the equality $f(\mathbf{m}) + \sum_{\ell_i < 0} -\ell_i \langle \mathbf{m}_i, \mathbf{n} \rangle = \sum_{\ell_i \geq 0} \ell_i \langle \mathbf{m}_i, \mathbf{n} \rangle$, which is equivalent to $f(\mathbf{m}) = \sum_{i=1}^d \ell_i \langle \mathbf{m}_i, \mathbf{n} \rangle$ and by bilinearity we have $f(\mathbf{m}) = \langle \mathbf{m}, \mathbf{n} \rangle$. Moreover, $f(\mathbf{m}) = \langle \mathbf{m}, \mathbf{n} \rangle \in \mathbb{N}$ since $\mathbf{n} \in \sigma \cap N$ and $\mathbf{m} \in \sigma^\vee \cap M$. \square

5.2.13 Let us explain here some of the known results up to now on our subject of study in the next sections. At the origin of those results there is the following proposition of S. Ishii, see [46, Proposition 2.6].

Proposition 5.2.14 (Ishii). *Let k be a field, V be a k -variety and G be an algebraic group of finite type defined over k such that V admits an action of G . Then, for $m \in \mathbb{N} \cup \{\infty\}$, the k -scheme $\mathcal{L}_m(V)$ admits a canonical action of $\mathcal{L}_m(G)$ induced from the action of G on V .*

Let us keep the notation of this section. Then, applying proposition 5.2.14 in the case of toric varieties S . Ishii obtains the following result, corresponding to [46, Theorem 4.1 (ii)].

Proposition 5.2.15 (Ishii). *Let us keep the notation in this section. Let $\alpha_1, \alpha_2 \in \mathcal{L}_\infty^\circ(V_\sigma)(k)$. Then the semigroup morphisms valuations $\text{ord}(\alpha_1)$ and $\text{ord}(\alpha_2)$ are equal if and only if there exists $\gamma \in \mathcal{L}_\infty(\mathcal{T})(k)$ such that $\alpha_1 = \gamma \cdot \alpha_2$.*

From proposition 5.2.15 and the fact that, for every $\gamma \in \mathcal{L}_\infty(\mathcal{T})(k)$, the morphism $\alpha \mapsto \gamma \cdot \alpha$ is an automorphism of $\mathcal{L}_\infty(V_\sigma)$, D. Bourqui and J. Sebag have deduced the following result, the first one about the invariance of the formal neighbourhoods of the arc scheme. It corresponds to [13, Corollary 3.3].

Corollary 5.2.16 (Bourqui-Sebag). *Let us keep the notation in this section. Let $\alpha_1, \alpha_2 \in \mathcal{L}_\infty^\circ(V_\sigma)(k)$ such that $\text{ord}(\alpha_1) = \text{ord}(\alpha_2)$. Then the local rings $\mathcal{O}_{\mathcal{L}_\infty(V_\sigma), \alpha_1}$ and $\mathcal{O}_{\mathcal{L}_\infty(V_\sigma), \alpha_2}$ are isomorphic (as local k -algebras), and hence their respective completions $\widehat{\mathcal{O}_{\mathcal{L}_\infty(V_\sigma), \alpha_1}}$ and $\widehat{\mathcal{O}_{\mathcal{L}_\infty(V_\sigma), \alpha_2}}$ are also isomorphic (as complete local k -algebras).*

That is, the formal neighbourhood of $\mathcal{L}_\infty(V_\sigma)$ at a point $\alpha \in \mathcal{L}_\infty^\circ(V_\sigma)(k)$ only depends on the associated semigroup morphism, and not on the choice of the arc. This fact, together with lemma 5.2.11 part (iv), implies the following result, corresponding to [13, Theorem 4.2].

Theorem 5.2.17 (Bourqui-Sebag). *Let us keep the notation in this section. Let $\mathbf{n} \in \sigma \cap N$ and let $\mathcal{N}_\mathbf{n}$ be the corresponding Nash set. Then $\mathcal{L}_\infty^\circ(V_\sigma)_\mathbf{n}$ is an open subset of $\mathcal{N}_\mathbf{n} \cap \mathcal{L}_\infty^\circ(V_\sigma)$ such that, for every $\alpha \in \mathcal{L}_\infty^\circ(V_\sigma)_\mathbf{n}(k)$, the local ring $\mathcal{O}_{\mathcal{L}_\infty(V_\sigma), \alpha}$ (and hence also $\widehat{\mathcal{O}_{\mathcal{L}_\infty(V_\sigma), \alpha}}$) is constant.*

That is, the formal neighbourhood of $\mathcal{L}_\infty(V_\sigma)$ is constant for rational points in an open subset of the Nash set associated with a toric divisorial valuation. Moreover, in the same article ([13, Theorem 5.2]) they show that construction 4.2.15 provides an explicit finite formal model of $\mathcal{L}_\infty(V_\sigma)_\alpha$ for $\alpha \in \mathcal{L}_\infty^\circ(V_\sigma)_\mathbf{n}(k)$, which is a stronger result that we will need in section 5.5 (see theorem 5.5.15).

On the other hand, by proposition 5.1.11 we know that, given $\mathbf{n} \in \sigma \cap N$, the generic point $\eta_\mathbf{n}$ of the associated Nash set $\mathcal{N}_\mathbf{n}$ is a non-degenerate constructible point and by corollary 5.1.9 we know that $\widehat{\mathcal{O}_{\mathcal{L}_\infty(V_\sigma), \eta_\mathbf{n}}}$ is noetherian. In the rest of the chapter we will relate this local k -algebra with $\widehat{\mathcal{O}_{\mathcal{L}_\infty(V_\sigma), \alpha}}$ for $\alpha \in \mathcal{L}_\infty^\circ(V_\sigma)_\mathbf{n}(k)$. In particular, our result also implies the invariance obtained in theorem 5.2.17.

5.3 TECHNICAL MACHINERY FOR COMPUTING THE FORMAL NEIGHBOURHOOD AT THE GENERIC POINT OF THE NASH SET

In this section we develop the technical results which we will use in section 5.5 to obtain a convenient presentation of the formal neighbourhood of the generic point of the Nash set associated with a divisorial toric valuation. The main result of this section is theorem 5.3.7, whose hypotheses are formulated in a somewhat abstract form. In section 5.5 we will check that these hypotheses hold in the toric setting.

It should be noted that the problem of computing sensible presentations of the formal neighbourhood of generic points of the Nash sets was considered more generally by A. Reguera in [67, 68] and A. Reguera and H. Mourtada in [62]. The approach used here, based on a direct application of a version of the Hensel lemma for an infinite set of variables, is somewhat different.

5.3.1 We first state a version of the Hensel lemma for an arbitrary set of variables, whose proof is basically the same as in the case of a finite set of variables.

Proposition 5.3.2. *Let $(\mathcal{A}, \mathfrak{M}_{\mathcal{A}})$ be a complete local ring with residue field κ . Let I be a set and $\mathbf{Y} = \{Y_i\}_{i \in I}$ be a collection of indeterminates. Let J be a set and $\{F_j; j \in J\}$ be a collection of elements in $\mathcal{A}[\mathbf{Y}]$. For $\mathbf{y} \in \mathcal{A}^I$, we denote by $\mathbf{J}_{\mathbf{y}}$ the \mathcal{A} -linear application $\mathcal{A}^I \rightarrow \mathcal{A}^J$ induced by the jacobian matrix $[\partial_{Y_i} F_j]_{\mathbf{Y}=\mathbf{y}}$, and by $\mathbf{F}|_{\mathbf{Y}=\mathbf{y}} \in \mathcal{A}^J$ the J -tuple $(F_j|_{\mathbf{Y}=\mathbf{y}})_{j \in J}$.*

We assume that there exists $\mathbf{y}^{(0)} \in \mathcal{A}^I$ such that:

1. One has $\mathbf{F}|_{\mathbf{Y}=\mathbf{y}^{(0)}} = 0 \pmod{\mathfrak{M}_{\mathcal{A}}}$.
2. The κ -linear application $\kappa^I \rightarrow \kappa^J$ deduced from $\mathbf{J}_{\mathbf{y}^{(0)}}$ by reduction modulo $\mathfrak{M}_{\mathcal{A}}$ is invertible.

Then there exists a unique element $\mathbf{y} = (\mathcal{Y}_i) \in \mathcal{A}^I$ such that:

1. One has $\mathbf{F}|_{\mathbf{Y}=\mathbf{y}} = 0$.
2. For every $i \in I$, one has $\mathcal{Y}_i = y_i^{(0)} \pmod{\mathfrak{M}_{\mathcal{A}}}$.

Proof. We begin with two remarks.

First, note that though in this context the jacobian matrix may have an infinite number of rows and columns, each row has only a finite number of nonzero entries, thus $\mathbf{J}_{\mathbf{y}}$ is well defined for any \mathbf{y} in \mathcal{A}^I . Also, by assumption, there exists an \mathcal{A} -linear application

$$\mathbf{K}_{\mathbf{y}^{(0)}} : \mathcal{A}^J \rightarrow \mathcal{A}^I$$

such that $\mathbf{K}_{\mathbf{y}^{(0)}}\mathbf{J}_{\mathbf{y}^{(0)}} = \text{Id}_{\mathcal{A}^I} \pmod{\mathfrak{M}_{\mathcal{A}}}$ and $\mathbf{J}_{\mathbf{y}^{(0)}}\mathbf{K}_{\mathbf{y}^{(0)}} = \text{Id}_{\mathcal{A}^J} \pmod{\mathfrak{M}_{\mathcal{A}}}$.

Second, note that the Taylor formula assures that, for $\mathbf{y} \in \mathcal{A}^I$, there exists a family $\{\mathbf{H}_{i_1, i_2} : i_1, i_2 \in I\}$ of elements of $\mathcal{A}[\mathbf{Y}]^J$, depending on \mathbf{y} and the F_j 's, such that for every $j \in J$, $H_{i_1, i_2, j} = 0$ for all but finitely many (i_1, i_2) and for every $\mathbf{z} \in \mathcal{A}^I$ one has

$$\mathbf{F}|_{\mathbf{Y}=\mathbf{y}+\mathbf{z}} = \mathbf{F}|_{\mathbf{Y}=\mathbf{y}} + \mathbf{J}_{\mathbf{y}}(\mathbf{z}) + \sum_{i_1, i_2 \in I} z_{i_1} z_{i_2} \mathbf{H}_{i_1, i_2}|_{\mathbf{Y}=\mathbf{y}+\mathbf{z}}. \quad (3.1)$$

Note that here and elsewhere the notation we use is a condensed form for writing a possibly infinite number of relations, each of them being easily verified.

We show by induction that for every $e \geq 0$, there exists a family $\mathbf{y}^{(e)} = (y_i^{(e)}; i \in I)$ of elements of \mathcal{A} , unique modulo $\mathfrak{M}_{\mathcal{A}}^{e+1}$, such that $\mathbf{y}^{(e)} = \mathbf{y}^{(0)} \pmod{\mathfrak{M}_{\mathcal{A}}}$ and $\mathbf{F}|_{\mathbf{Y}=\mathbf{y}^{(e)}} = 0 \pmod{\mathfrak{M}_{\mathcal{A}}^{e+1}}$. The case $e = 0$ is given by our assumptions.

Now take $e \in \mathbb{N}$ and assume that our induction statement holds for e . Consider the equation

$$\mathbf{F}|_{\mathbf{Y}=\mathbf{y}^{(e)}+\mathbf{z}} = 0 \pmod{\mathfrak{M}_{\mathcal{A}}^{e+2}} \quad (3.2)$$

with unknown $\mathbf{z} = (z_i) \in \mathcal{A}^I$ such that $\mathbf{z} = 0 \pmod{\mathfrak{M}_{\mathcal{A}}^{e+1}}$, this being assured by the uniqueness of $\mathbf{y}^{(e)}$ in the induction assumption. Since $\mathbf{y}^{(e)} = \mathbf{y}^{(0)} \pmod{\mathfrak{M}_{\mathcal{A}}}$, the jacobian matrices $[\partial_{Y_i} F_j]|_{\mathbf{Y}=\mathbf{y}^{(0)}}$ and $[\partial_{Y_i} F_j]|_{\mathbf{Y}=\mathbf{y}^{(e)}}$ are equal modulo $\mathfrak{M}_{\mathcal{A}}$. Since $\mathbf{z} = 0 \pmod{\mathfrak{M}_{\mathcal{A}}^{e+1}}$, one thus has

$$\mathbf{J}_{\mathbf{y}^{(e)}}(\mathbf{z}) = \mathbf{J}_{\mathbf{y}^{(0)}}(\mathbf{z}) \pmod{\mathfrak{M}_{\mathcal{A}}^{e+2}}.$$

Thus by (3.1) and using again $\mathbf{z} = 0 \pmod{\mathfrak{M}_{\mathcal{A}}^{e+1}}$, equation (3.2) is equivalent to

$$\mathbf{J}_{\mathbf{y}^{(0)}}(\mathbf{z}) = -\mathbf{F}|_{\mathbf{Y}=\mathbf{y}^{(e)}} \pmod{\mathfrak{M}_{\mathcal{A}}^{e+2}}. \quad (3.3)$$

By assumption, $\mathbf{F}|_{\mathbf{Y}=\mathbf{y}^{(e)}} = 0 \pmod{\mathfrak{M}_{\mathcal{A}}^{e+1}}$. Thus by the first remark above, applying the linear map $\mathbf{K}_{\mathbf{y}^{(0)}}$ to (3.3) we obtain

$$\mathbf{z} = -\mathbf{K}_{\mathbf{y}^{(0)}}(\mathbf{F}|_{\mathbf{Y}=\mathbf{y}^{(e)}}) \pmod{\mathfrak{M}_{\mathcal{A}}^{e+2}}. \quad (3.4)$$

Using $\mathbf{z} = 0 \pmod{\mathfrak{M}_{\mathcal{A}}^{e+1}}$ one more time, by the first remark above and applying the linear map $\mathbf{J}_{\mathbf{y}^{(0)}}$ to (3.4) we deduce that equation (3.3) is indeed equivalent to (3.4). Since $\mathbf{F}|_{\mathbf{Y}=\mathbf{y}^{(e)}} = 0 \pmod{\mathfrak{M}_{\mathcal{A}}^{e+1}}$, equation (3.4) provides a solution \mathbf{z} such that $\mathbf{z} = 0 \pmod{\mathfrak{M}_{\mathcal{A}}^{e+1}}$.

In order to show the uniqueness of the solution modulo $\mathfrak{M}_{\mathcal{A}}^{e+2}$, note that if $\mathbf{w} \in \mathcal{A}^I$ is such that $\mathbf{w} = 0 \pmod{\mathfrak{M}_{\mathcal{A}}^{e+1}}$, one has by (3.1)

$$\mathbf{F}|_{\mathbf{Y}=\mathbf{y}^{(e)}+\mathbf{w}} = \mathbf{F}|_{\mathbf{Y}=\mathbf{y}^{(e)}} + \mathbf{J}_{\mathbf{y}^{(e)}}(\mathbf{w}) \pmod{\mathfrak{M}_{\mathcal{A}}^{e+2}},$$

thus

$$\mathbf{K}_{\mathbf{y}^{(0)}}(\mathbf{F}|_{\mathbf{Y}=\mathbf{y}^{(e)}+\mathbf{w}}) = \mathbf{K}_{\mathbf{y}^{(0)}}(\mathbf{F}|_{\mathbf{Y}=\mathbf{y}^{(e)}}) + \mathbf{K}_{\mathbf{y}^{(0)}}(\mathbf{J}_{\mathbf{y}^{(e)}}(\mathbf{w})) \pmod{\mathfrak{M}_{\mathcal{A}}^{e+2}}$$

and finally

$$\mathbf{K}_{\mathbf{y}^{(0)}}(\mathbf{F}|_{\mathbf{Y}=\mathbf{y}^{(e)}+\mathbf{w}}) = \mathbf{K}_{\mathbf{y}^{(0)}}(\mathbf{F}|_{\mathbf{Y}=\mathbf{y}^{(e)}}) + \mathbf{w} \pmod{\mathfrak{M}_{\mathcal{A}}^{e+2}}.$$

□

5.3.3 We consider the following general setting and notation for the rest of this section. Let A be a k -algebra which is a domain. Let Ω be a finite set, I be a set, $\mathbf{X} = \{X_\omega\}_{\omega \in \Omega}$ and $\mathbf{Y} = \{Y_i; i \in I\}$ be collections of indeterminates. Set

$$A[\mathbf{X}] := A[\{X_\omega\}_{\omega \in \Omega}] \quad \text{and} \quad A[\mathbf{X}, \mathbf{Y}] := A[\{X_\omega\}_{\omega \in \Omega}, \{Y_i; i \in I\}].$$

We denote by $\langle \mathbf{X} \rangle$ the prime ideal $\langle X_\omega; \omega \in \Omega \rangle$ of $A[\mathbf{X}]$. In accordance with the conventions given in section 1.1, for any $A[\mathbf{X}]$ -algebra B , we often still denote by $\langle \mathbf{X} \rangle$ the extension of the ideal $\langle \mathbf{X} \rangle$ to B .

5.3.4 The following lemma will be useful in the proof of theorem 5.3.7.

Lemma 5.3.5. *Assume that we are in the setting described in subsection 5.3.3. Let \mathfrak{h} be an ideal of $A[\mathbf{X}, \mathbf{Y}]$ such that:*

$$(i) \quad \text{One has } \langle \mathbf{X} \rangle + \mathfrak{h} = \langle \mathbf{X}, \mathbf{Y} \rangle.$$

Assume moreover that there exists an $A[\mathbf{X}]$ -algebra morphism $\widehat{\varepsilon}: A[\mathbf{X}, \mathbf{Y}] \rightarrow \text{Frac}(A)[[\mathbf{X}]]$ such that:

$$(ii) \quad \text{For every } i \in I \text{ one has } \widehat{\varepsilon}(Y_i) = Y_i \pmod{\mathfrak{h}} \text{ in the ring } \text{Frac}(A)[[\mathbf{X}]][\mathbf{Y}].$$

$$(iii) \quad \text{For every } i \in I \text{ one has } \widehat{\varepsilon}(Y_i) \in \langle \mathbf{X} \rangle.$$

Then, the $\langle \mathbf{X} \rangle$ -adic completion of the localization $(A[\mathbf{X}, \mathbf{Y}]/\mathfrak{h})_{\langle \mathbf{X} \rangle}$ is isomorphic to $\text{Frac}(A)[[\mathbf{X}]]/\widehat{\varepsilon}(\mathfrak{h})$.

Remark 5.3.6. Assume that the hypotheses of the lemma hold. Let \mathfrak{g} be any ideal of $A[\mathbf{X}, \mathbf{Y}]$ containing \mathfrak{h} such that $\langle \mathbf{X} \rangle + \mathfrak{h} = \langle \mathbf{X} \rangle + \mathfrak{g}$. Then \mathfrak{g} also satisfies the hypotheses of the lemma, with the same morphism $\widehat{\varepsilon}$. Whenever the ideals $\widehat{\varepsilon}(\mathfrak{h})$ and $\widehat{\varepsilon}(\mathfrak{g})$ coincide, the lemma shows that the $\langle \mathbf{X} \rangle$ -adic completions of $(A[\mathbf{X}, \mathbf{Y}]/\mathfrak{h})_{\langle \mathbf{X} \rangle}$ and $(A[\mathbf{X}, \mathbf{Y}]/\mathfrak{g})_{\langle \mathbf{X} \rangle}$ are isomorphic. However, the condition $\langle \mathbf{X} \rangle + \mathfrak{h} = \langle \mathbf{X} \rangle + \mathfrak{g}$ is not sufficient to assure that $\widehat{\varepsilon}(\mathfrak{h}) = \widehat{\varepsilon}(\mathfrak{g})$: take $A[\mathbf{X}, \mathbf{Y}] = k[x, y]$, $\mathbf{X} = \{x\}$, $\mathbf{Y} = \{y\}$, the ideals $\mathfrak{h} := \langle y - x \rangle$ and $\mathfrak{g} := \langle x, y \rangle$ and the morphism of $k[x]$ -algebras $\widehat{\varepsilon}: k[x, y] \rightarrow k[[x]]$ with $\widehat{\varepsilon}(y) = x$. Then $\widehat{\varepsilon}(\mathfrak{h}) = 0$ and $\widehat{\varepsilon}(\mathfrak{g}) = \langle x \rangle$.

Proof. Note that (iii) shows that $\widehat{\varepsilon}(\mathfrak{h})$ is contained in $\langle \mathbf{X} \rangle$, thus $\text{Frac}(A)[[\mathbf{X}]]/\widehat{\varepsilon}(\mathfrak{h})$ is a complete noetherian local ring with maximal ideal $\langle \mathbf{X} \rangle$. Moreover (i) and the fact that A is a domain show that $\langle \mathbf{X} \rangle$ is indeed a prime ideal of $A[\mathbf{X}, \mathbf{Y}]/\mathfrak{h}$.

Let $e \geq 1$. Let π_e be the composition of $\widehat{\varepsilon}$ with the quotient morphism

$$\text{Frac}(A)[[\mathbf{X}]] \rightarrow \text{Frac}(A)[[\mathbf{X}]]/\widehat{\varepsilon}(\mathfrak{h}) \rightarrow \text{Frac}(A)[\mathbf{X}]/(\widehat{\varepsilon}(\mathfrak{h}) + \langle \mathbf{X} \rangle^e).$$

Thanks to (iii), any element of $A[\mathbf{X}, \mathbf{Y}]$ whose constant term is not zero is sent by $\widehat{\varepsilon}$ to an invertible element of $\text{Frac}(A)[[\mathbf{X}]]$. Thus π_e induces a morphism

$$A[\mathbf{X}, \mathbf{Y}]_{\langle \mathbf{X}, \mathbf{Y} \rangle} \rightarrow \text{Frac}(A)[\mathbf{X}]/(\widehat{\varepsilon}(\mathfrak{h}) + \langle \mathbf{X} \rangle^e)$$

which in turn induces a morphism

$$\tilde{\pi}_e: A[\mathbf{X}, \mathbf{Y}]_{\langle \mathbf{X}, \mathbf{Y} \rangle} / (\mathfrak{h} + \langle \mathbf{X} \rangle^e) \rightarrow \text{Frac}(A)[\mathbf{X}] / (\widehat{\varepsilon}(\mathfrak{h}) + \langle \mathbf{X} \rangle^e).$$

Note that since $\mathfrak{h} + \langle \mathbf{X} \rangle = \langle \mathbf{X}, \mathbf{Y} \rangle$, one has $\mathfrak{h} + \langle \mathbf{X} \rangle^e = \mathfrak{h} + \langle \mathbf{X}, \mathbf{Y} \rangle^e$. Thus in order to obtain the claimed isomorphism, it suffices to show that $\tilde{\pi}_e$ is an isomorphism for any $e \geq 1$. Since the natural inclusion $A[\mathbf{X}, \mathbf{Y}]_{\langle \mathbf{X}, \mathbf{Y} \rangle} \subset \text{Frac}(A)[\mathbf{X}, \mathbf{Y}]_{\langle \mathbf{X}, \mathbf{Y} \rangle}$ is an isomorphism, surjectivity is clear.

Let us show injectivity. This amounts to show that if $P \in \text{Frac}(A)[\mathbf{X}, \mathbf{Y}]$ lies in $\text{Ker}(\pi_e)$, then $P \in \mathfrak{h} + \langle \mathbf{X} \rangle^e$. By assumption (ii), for any $P \in A[\mathbf{X}, \mathbf{Y}]$, one has $\widehat{\varepsilon}(P) = P \pmod{\mathfrak{h}}$ in the ring $\text{Frac}(A)[[\mathbf{X}]][[\mathbf{Y}]]$. In particular in the ring $\text{Frac}(A)[\mathbf{X}, \mathbf{Y}]$ one has

$$\widehat{\varepsilon}(P) + \langle \mathbf{X} \rangle^e = P + \mathfrak{h} + \langle \mathbf{X} \rangle^e \quad \text{and} \quad \widehat{\varepsilon}(\mathfrak{h}) + \langle \mathbf{X} \rangle^e \subset \mathfrak{h} + \langle \mathbf{X} \rangle^e.$$

Now if $P \in \text{Frac}(A)[\mathbf{X}, \mathbf{Y}]$ lies in $\text{Ker}(\pi_e)$, then one has $\widehat{\varepsilon}(P) + \langle \mathbf{X} \rangle^e \subset \widehat{\varepsilon}(\mathfrak{h}) + \langle \mathbf{X} \rangle^e$. Therefore, by the above properties, one has $P + \mathfrak{h} + \langle \mathbf{X} \rangle^e \subset \mathfrak{h} + \langle \mathbf{X} \rangle^e$. Thus $P \in \mathfrak{h} + \langle \mathbf{X} \rangle^e$. That concludes the proof. \square

Now we can state and prove the main result of the section.

Theorem 5.3.7. *Assume that we are in the setting described in subsection 5.3.3; we assume moreover that the set I is of the shape $\Gamma \times \mathbb{N}$ where Γ is a finite set.*

Let \mathfrak{h} be an ideal of $A[\mathbf{X}, \mathbf{Y}]$ such that:

(A) *The ideal \mathfrak{h} contains a collection of elements $\{H_{\gamma,s}, \gamma \in \Gamma, s \in \mathbb{N}\}$ of the form $H_{\gamma,s} = Y_{\gamma,s}U_{\gamma,s} + E_{\gamma,s}$ such that for every $\gamma \in \Gamma$ and every $s \in \mathbb{N}$:*

(A1) *$U_{\gamma,s}$ is a unit in A .*

(A2) *There exists a family $(E_{\gamma,s,r}) \in A[\mathbf{X}]^{\text{NU}\{-1\}}$ such that $E_{\gamma,s,r} \in \langle \mathbf{X} \rangle$ for $r \geq s$, $E_{\gamma,s,r} = 0$ for all but a finite number of r , and one has*

$$E_{\gamma,s} = E_{\gamma,s,-1} + \sum_{r \in \mathbb{N}} E_{\gamma,s,r} \cdot Y_{\gamma,r}.$$

(B) *Let $(y_{\gamma,s}) \in A^{\Gamma \times \mathbb{N}}$ be the unique family of elements of A such that for every $\gamma \in \Gamma$ and $s \in \mathbb{N}$, one has $H_{\gamma,s}|_{Y_{\gamma,r}=y_{\gamma,r}} = 0 \pmod{\langle \mathbf{X} \rangle}$; then the ideal $\langle \mathbf{X} \rangle + \mathfrak{h}$ is contained in the ideal $\langle \mathbf{X} \rangle + \langle Y_{\gamma,s} - y_{\gamma,s}; (\gamma, s) \in \Gamma \times \mathbb{N} \rangle$.*

Then there exists an $A[\mathbf{X}]$ -algebra morphism $\widehat{\varepsilon}: A[\mathbf{X}, \mathbf{Y}] \rightarrow \text{Frac}(A)[[\mathbf{X}]]$ such that:

(i) *For every $(\gamma, s) \in \Gamma \times \mathbb{N}$ one has $\widehat{\varepsilon}(H_{\gamma,s}) = 0$.*

(ii) *For every $(\gamma, s) \in \Gamma \times \mathbb{N}$, one has $\widehat{\varepsilon}(Y_{\gamma,s}) = Y_{\gamma,s} \pmod{\mathfrak{h}}$ in the ring $\text{Frac}(A)[[\mathbf{X}]][[\mathbf{Y}]]$.*

(iii) *For every ideal \mathfrak{g} containing \mathfrak{h} and such that $\langle \mathbf{X} \rangle + \mathfrak{h} = \langle \mathbf{X} \rangle + \mathfrak{g}$, the $\langle \mathbf{X} \rangle$ -adic completion of the localization $(A[\mathbf{X}, \mathbf{Y}]/\mathfrak{g})_{\langle \mathbf{X} \rangle}$ is $\langle \mathbf{X} \rangle$ -adically isomorphic to $\text{Frac}(A)[[\mathbf{X}]]/\widehat{\varepsilon}(\mathfrak{g})$.*

Assume moreover that:

(C) For every $\gamma \in \Gamma$, one has $E_{\gamma,0,-1} \in A[\mathbf{X}] \setminus \langle \mathbf{X} \rangle$.

Then one has in addition:

(iv) For every $\gamma \in \Gamma$, $\hat{\varepsilon}(Y_{\gamma,0})$ is a unit in $\text{Frac}(A)[[\mathbf{X}]]$.

Proof. First note that for each $\gamma \in \Gamma$, the reduction of the $H_{\gamma,s}$'s modulo $\langle \mathbf{X} \rangle$ gives a triangular and invertible A -linear system in the $Y_{\gamma,s}$'s. Thus the existence and uniqueness of $(y_{\gamma,s})$ in assumption (B) is a straightforward consequence of assumption (A). In fact, up to dividing $H_{\gamma,s}$ by $U_{\gamma,s}$ and modifying the $E_{\gamma,s,r}$'s, one may assume that for every γ, s, r one has $E_{\gamma,s,r} \in \langle \mathbf{X} \rangle$ and that for every γ, s one has

$$H_{\gamma,s} = Y_{\gamma,s} - y_{\gamma,s} + \sum_{r=0}^{s-1} \alpha_{\gamma,r}(Y_{\gamma,r} - y_{\gamma,r}) + E_{\gamma,s,-1} + \sum_{r \in \mathbb{N}} E_{\gamma,s,r}(Y_{\gamma,r} - y_{\gamma,r}), \quad (3.5)$$

where the $\alpha_{\gamma,r}$'s are elements of A .

Let us now apply proposition 5.3.2 with $\mathcal{A} = \text{Frac}(A)[[\mathbf{X}]]$ and $\{F_j; j \in J\} = \{H_{\gamma,s}; (\gamma, s) \in \Gamma \times \mathbb{N}\}$, this shows the existence of a family $\{\mathcal{Y}_{\gamma,s}; \gamma \in \Gamma, s \in \mathbb{N}\}$ of elements of $\text{Frac}(A)[[\mathbf{X}]]$ such that for every $(\gamma, s) \in \Gamma \times \mathbb{N}$ one has $\mathcal{Y}_{\gamma,s} = y_{\gamma,s} \pmod{\langle \mathbf{X} \rangle}$ and $H_{\gamma,s}|_{Y_{\gamma,r}=\mathcal{Y}_{\gamma,r}} = 0$. Thus mapping $Y_{\gamma,s}$ to $\mathcal{Y}_{\gamma,s}$ defines an $A[\mathbf{X}]$ -algebra morphism $\hat{\varepsilon}: A[\mathbf{X}, \mathbf{Y}] \rightarrow \text{Frac}(A)[[\mathbf{X}]]$ such that (i) holds.

For every $\gamma \in \Gamma$, (3.5) and a straightforward induction on s shows that for every s one has $Y_{\gamma,s} - y_{\gamma,s} \in \langle \mathbf{X} \rangle + \mathfrak{h}$. By assumption (B), one then has $\langle \mathbf{X} \rangle + \mathfrak{h} = \langle \mathbf{X} \rangle + \langle Y_{\gamma,s} - y_{\gamma,s}; (\gamma, s) \in \Gamma \times \mathbb{N} \rangle$. Thus $\text{Frac}(A)[[\mathbf{X}]][\mathbf{Y}]/\mathfrak{h}$ is a noetherian local ring with maximal ideal $\langle \mathbf{X} \rangle$.

On the other hand, (3.5) shows that for every $(\gamma, s) \in \Gamma \times \mathbb{N}$, since $H_{\gamma,s} \in \mathfrak{h}$ and $\hat{\varepsilon}(H_{\gamma,s}) = 0$, one has in the ring $\text{Frac}(A)[[\mathbf{X}]][\mathbf{Y}]$ the following relation, obtained by applying $\hat{\varepsilon}$ to (3.5) and subtracting (3.5):

$$\mathcal{Y}_{\gamma,s} - Y_{\gamma,s} = - \sum_{r=0}^{s-1} \alpha_{\gamma,r}(\mathcal{Y}_{\gamma,r} - Y_{\gamma,r}) - \sum_{r \geq 0} E_{\gamma,s,r}(\mathcal{Y}_{\gamma,r} - Y_{\gamma,r}) \pmod{\mathfrak{h}}.$$

Thus by a straightforward induction one gets that $\mathcal{Y}_{\gamma,s} - Y_{\gamma,s} \in \langle \mathbf{X} \rangle^e + \mathfrak{h}$ for every γ, s and $e \geq 1$ (note that we have assumed that $E_{\gamma,s,r} \in \langle \mathbf{X} \rangle$), and finally by the Krull intersection theorem $\mathcal{Y}_{\gamma,s} - Y_{\gamma,s} \in \mathfrak{h}$ for every γ, s . Thus (ii) holds.

Recalling that $\mathcal{Y}_{\gamma,s} - y_{\gamma,s} \in \langle \mathbf{X} \rangle$ by construction of $\mathcal{Y}_{\gamma,s}$, (iii) then follows from an application of lemma 5.3.5 (replacing $Y_{\gamma,s}$ with $Y_{\gamma,s} - y_{\gamma,s}$) and remark 5.3.6.

Assumption (C) is equivalent to the property $y_{\gamma,0} \in A \setminus \{0\}$. Then $y_{\gamma,0}$ is a unit in $\text{Frac}(A)[[\mathbf{X}]]$, and since $\mathcal{Y}_{\gamma,0} = y_{\gamma,0} \pmod{\langle \mathbf{X} \rangle}$, we deduce that $\mathcal{Y}_{\gamma,0} = \hat{\varepsilon}(Y_{\gamma,0})$ also is a unit, and (iv) holds. \square

Remark 5.3.8. In the statement of the theorem, if one assumes that (A) holds and that moreover \mathfrak{h} is generated by the $H_{\gamma,s}$'s and some elements of $\langle \mathbf{X} \rangle$, then (B) automatically holds. Indeed, the above proof shows that, without changing the ideal generated by the $H_{\gamma,s}$'s, one may assume that for every γ, s one has $H_{\gamma,s} = Y_{\gamma,s} - y_{\gamma,s} \pmod{\langle \mathbf{X} \rangle}$.

5.3.9 In the preceding notation, the proof of theorem 5.3.7 is based on the existence of a family $\{\mathcal{Y}_{\gamma,s}; \gamma \in \Gamma, s \in \mathbb{N}\}$ of elements of $\text{Frac}(A)[[\mathbf{X}]]$ such that for every $(\gamma, s) \in \Gamma \times \mathbb{N}$ one has $\mathcal{Y}_{\gamma,s} = y_{\gamma,s} \pmod{\langle \mathbf{X} \rangle}$ and $H_{\gamma,s}|_{Y_{\gamma,r}=\mathcal{Y}_{\gamma,r}} = 0$. The existence of this family is assured by proposition 5.3.2 and hence in order to explicitly compute the ring $\text{Frac}(A)[[\mathbf{X}]]/\widehat{\varepsilon}(\mathfrak{h})$ in part (iii) of theorem 5.3.7 on particular examples we have to effectively apply proposition 5.3.2. Let us show the recursive construction of the truncations modulo $\langle \mathbf{X} \rangle^e$ for $e \in \mathbb{N}$ of such a family $\{\mathcal{Y}_{\gamma,s}; \gamma \in \Gamma, s \in \mathbb{N}\}$, checking that the desired conditions hold.

Construction 5.3.10. *Assume that we are in the setting described in theorem 5.3.7, in particular its assumptions hold. Then, for every $(\gamma, s) \in \Gamma \times \mathbb{N}$, we can construct a family $\{y_{\gamma,s}^{(e)}\}_{e \geq 1}$ of elements of $A[\mathbf{X}] \subset A[\mathbf{X}, \mathbf{Y}]$ (and hence also belonging to $\text{Frac}(A)[\mathbf{X}]$) such that:*

- (i) For every $e \in \mathbb{N} \setminus \{0\}$ we have $Y_{\gamma,s} = y_{\gamma,s}^{(e)} \pmod{\langle \mathbf{X} \rangle^e + \mathfrak{h}}$ in the ring $A[\mathbf{X}, \mathbf{Y}]$.
- (ii) For every $e_1, e_2 \in \mathbb{N}$ such that $e_1 \geq e_2 \geq 1$ we have $y_{\gamma,s}^{(e_1)} = y_{\gamma,s}^{(e_2)} \pmod{\langle \mathbf{X} \rangle^{e_2}}$ in the ring $A[\mathbf{X}]$.
- (iii) For every $e \in \mathbb{N} \setminus \{0\}$ we have $H_{\gamma,s}|_{Y_{\gamma,r}=y_{\gamma,r}^{(e)}} = 0 \pmod{\langle \mathbf{X} \rangle^e}$ in the ring $A[\mathbf{X}, \mathbf{Y}]$.

We will prove parts (i) and (ii) by a double induction on the nonnegative integers s and e . For this, we will state and prove two lemmas that perform each step of the respective inductions. First we will need to fix some notation. Let $e \in \mathbb{N} \setminus \{0\}$, the quotient morphism $A[\mathbf{X}, \mathbf{Y}] \rightarrow A[\mathbf{X}, \mathbf{Y}]/\langle \mathbf{X} \rangle^e$ admits a canonical section. We denote by ς_e their composition

$$\varsigma_e : A[\mathbf{X}, \mathbf{Y}] \rightarrow A[\mathbf{X}, \mathbf{Y}]/\langle \mathbf{X} \rangle^e \rightarrow A[\mathbf{X}, \mathbf{Y}].$$

Let us stress that the image of $A[\mathbf{X}]$ by ς_e is contained in $A[\mathbf{X}]$. For every $(\gamma, s) \in \Gamma \times \mathbb{N}$ we set $y_{\gamma,s}^{(0)} := Y_{\gamma,s}$. Let us recall from assumption (A) in theorem 5.3.7 that $H_{\gamma,s}$ is of the form $H_{\gamma,s} = Y_{\gamma,s}U_{\gamma,s} + E_{\gamma,s}$, with $U_{\gamma,s}$ a unit in A and $E_{\gamma,s} = E_{\gamma,s,-1} + \sum_{r \in \mathbb{N}} E_{\gamma,s,r} \cdot Y_{\gamma,r}$ as in assumption (A2), i.e., $E_{\gamma,s,r} \in A[\mathbf{X}]$ for $r \in \mathbb{N} \cup \{-1\}$, $E_{\gamma,s,r} \in \langle \mathbf{X} \rangle$ for $r \geq s$ and $E_{\gamma,s,r} = 0$ for all but a finite number of r . For $\gamma \in \Gamma$ and $e, s \in \mathbb{N}$ with $e \geq 1$, we define recursively on e and s the morphisms of $A[\mathbf{X}]$ -algebras

$$\varepsilon_s^e : \begin{array}{ccc} A[\mathbf{X}, \mathbf{Y}] & \longrightarrow & A[\mathbf{X}, \mathbf{Y}] \\ Y_{\gamma,r} & \longmapsto & \begin{cases} y_{\gamma,r}^{(e)} & \text{for } r < s \\ y_{\gamma,r}^{(e-1)} & \text{for } r \geq s \end{cases} \end{array},$$

and the elements of $A[\mathbf{X}]$

$$y_{\gamma,s}^{(e)} := -\zeta_e(\varepsilon_s^e(E_{\gamma,s}))U_{\gamma,s}^{-1}. \quad (3.6)$$

Lemma 5.3.11. *Keep the previous notation. We fix $(\gamma, s) \in \Gamma \times \mathbb{N}$ and $e \in \mathbb{N} \setminus \{0\}$. We suppose that:*

- (a) *For every $r \in \mathbb{N}$ with $r < s$, there exists an element $y_{\gamma,r}^{(e)}$ in $A[\mathbf{X}]$ such that $Y_{\gamma,r} = y_{\gamma,r}^{(e)} \pmod{\langle \mathbf{X} \rangle^e + \mathfrak{h}}$ in the ring $A[\mathbf{X}, \mathbf{Y}]$.*
- (b) *If $e \geq 2$ then for every $r \in \mathbb{N}$ there exists an element $y_{\gamma,r}^{(e-1)}$ in $A[\mathbf{X}]$ such that $Y_{\gamma,r} = y_{\gamma,r}^{(e-1)} \pmod{\langle \mathbf{X} \rangle^{e-1} + \mathfrak{h}}$ in the ring $A[\mathbf{X}, \mathbf{Y}]$ (note that for $e = 1$ the equality still holds but $y_{\gamma,r}^{(0)}$ does not belong to $A[\mathbf{X}]$).*

Then:

- (i) *The element $y_{\gamma,s}^{(e)} := -\zeta_e(\varepsilon_s^e(E_{\gamma,s}))U_{\gamma,s}^{-1}$ belongs to $A[\mathbf{X}]$.*
- (ii) *We have $H_{\gamma,s} = \zeta_e(Y_{\gamma,s}U_{\gamma,s} + \varepsilon_s^e(E_{\gamma,s})) \pmod{\langle \mathbf{X} \rangle^e + \mathfrak{h}}$ in the ring $A[\mathbf{X}, \mathbf{Y}]$.*
- (iii) *We have $Y_{\gamma,s} = y_{\gamma,s}^{(e)} \pmod{\langle \mathbf{X} \rangle^e + \mathfrak{h}}$ in the ring $A[\mathbf{X}, \mathbf{Y}]$.*

This lemma performs each induction step in the proof of part (i) in construction 5.3.10. Precisely, it assures that the elements $y_{\gamma,s}^{(e)} := -\zeta_e(\varepsilon_s^e(E_{\gamma,s}))U_{\gamma,s}^{-1}$ in $A[\mathbf{X}]$ are well-defined and may be used in the subsequent steps of the induction. Let us stress that the assumptions of this lemma imply that, for $e > 1$, the image of the morphism ε_s^e is contained in the subring $A[\mathbf{X}]$ of $A[\mathbf{X}, \mathbf{Y}]$.

Proof of lemma 5.3.11. Let us prove part (i). For $e > 1$, the elements $y_{\gamma,r}^{(e)}$ (resp. $y_{\gamma,r}^{(e-1)}$) belonging to $A[\mathbf{X}]$ for every $r < s$ (resp. for every $r \in \mathbb{N}$) by assumption, we deduce that the image of the morphism ε_s^e is contained in $A[\mathbf{X}]$. Hence $\varepsilon_s^e(E_{\gamma,s}) \in A[\mathbf{X}]$ and also $\zeta_e(\varepsilon_s^e(E_{\gamma,s})) \in A[\mathbf{X}]$. We consider now the case $e = 1$. For $r < s$ we have by assumption $y_{\gamma,r}^{(1)} \in A[\mathbf{X}]$, hence $\varsigma_1(\varepsilon_s^1(E_{\gamma,s,r}Y_{\gamma,r})) = \varsigma_1(E_{\gamma,s,r}y_{\gamma,r}^{(1)}) \in A[\mathbf{X}]$. For $r \geq s$, by assumption (A1) in theorem 5.3.7 we have $E_{\gamma,s,r} \in \langle \mathbf{X} \rangle$, hence $\varsigma_1(\varepsilon_s^1(E_{\gamma,s,r}Y_{\gamma,r})) = \varsigma_1(E_{\gamma,s,r}Y_{\gamma,r}) = 0$. And obviously $\varsigma_1(\varepsilon_s^1(E_{\gamma,s,-1})) = \varsigma_1(E_{\gamma,s,-1}) \in A[\mathbf{X}]$. Then we also have $\varsigma_1(\varepsilon_s^1(E_{\gamma,s})) \in A[\mathbf{X}]$. Finally, for every $e \geq 1$, the element $U_{\gamma,s}$ being a unit in A , we deduce that $y_{\gamma,s}^{(e)} := -\zeta_e(\varepsilon_s^e(E_{\gamma,s}))U_{\gamma,s}^{-1}$ belongs to $A[\mathbf{X}]$.

Let us now prove part (ii). We consider $H_{\gamma,s} = Y_{\gamma,s}U_{\gamma,s} + E_{\gamma,s}$. Then, by assumption (A) in theorem 5.3.7 and the very definition of ζ_e , we deduce that $\zeta_e(Y_{\gamma,s}U_{\gamma,s} + \varepsilon_s^e(E_{\gamma,s})) = Y_{\gamma,s}U_{\gamma,s} + \zeta_e(\varepsilon_s^e(E_{\gamma,s}))$, since no term in $Y_{\gamma,s}U_{\gamma,s}$ belongs to $\langle \mathbf{X} \rangle$. We only have to prove that $E_{\gamma,s} = \zeta_e(\varepsilon_s^e(E_{\gamma,s})) \pmod{\langle \mathbf{X} \rangle^e + \mathfrak{h}}$ in the ring $A[\mathbf{X}, \mathbf{Y}]$. Let us see that we have in fact $E_{\gamma,s} = \varepsilon_s^e(E_{\gamma,s}) \pmod{\langle \mathbf{X} \rangle^e + \mathfrak{h}}$ (recall that $\zeta_e(\varepsilon_s^e(E_{\gamma,s})) = \varepsilon_s^e(E_{\gamma,s}) \pmod{\langle \mathbf{X} \rangle^e}$) by the very definition of ζ_e . By linearity of the morphism ε_s^e , this is equivalent to see that $E_{\gamma,s,-1} + \sum_{r \in \mathbb{N}} E_{\gamma,s,r} \cdot Y_{\gamma,r} = E_{\gamma,s,-1} + \sum_{r \in \mathbb{N}} E_{\gamma,s,r} \cdot \varepsilon_s^e(Y_{\gamma,r}) \pmod{\langle \mathbf{X} \rangle^e + \mathfrak{h}}$. We have three cases:

- Obviously $E_{\gamma,s,-1} = E_{\gamma,s,-1} \pmod{\langle \mathbf{X} \rangle^e + \mathfrak{h}}$.
- For $r < s$ we have $E_{\gamma,s,r} \cdot \varepsilon_s^e(Y_{\gamma,r}) = E_{\gamma,s,r} \cdot y_{\gamma,r}^{(e)}$ and from assumption (a) we deduce that $E_{\gamma,s,r} \cdot Y_{\gamma,r} = E_{\gamma,s,r} \cdot y_{\gamma,r}^{(e)} \pmod{\langle \mathbf{X} \rangle^e + \mathfrak{h}}$.
- For $r \geq s$ we have $E_{\gamma,s,r} \cdot \varepsilon_s^e(Y_{\gamma,r}) = E_{\gamma,s,r} \cdot y_{\gamma,r}^{(e-1)}$. If $e = 1$ then by the very definition $E_{\gamma,s,r} \cdot y_{\gamma,r}^{(0)} = E_{\gamma,s,r} \cdot Y_{\gamma,r}$. Let us suppose $e > 1$. By assumption (b) we have $Y_{\gamma,r} = y_{\gamma,r}^{(e-1)} \pmod{\langle \mathbf{X} \rangle^{e-1} + \mathfrak{h}}$. The element $E_{\gamma,s,r}$ belonging to $\langle \mathbf{X} \rangle$, we conclude that $E_{\gamma,s,r} \cdot Y_{\gamma,r} = E_{\gamma,s,r} \cdot y_{\gamma,r}^{(e-1)} \pmod{\langle \mathbf{X} \rangle^e + \mathfrak{h}}$.

Let us prove part (iii). The element $H_{\gamma,s}$ belonging to the ideal \mathfrak{h} , from part (ii) we deduce that $\varsigma_e(Y_{\gamma,s}U_{\gamma,s} + \varepsilon_s^e(E_{\gamma,s})) = 0 \pmod{\langle \mathbf{X} \rangle^e + \mathfrak{h}}$ and hence $Y_{\gamma,s}U_{\gamma,s} + \varsigma_e(\varepsilon_s^e(E_{\gamma,s})) = 0 \pmod{\langle \mathbf{X} \rangle^e + \mathfrak{h}}$ (since no term in $Y_{\gamma,s}U_{\gamma,s}$ belongs to $\langle \mathbf{X} \rangle$). The element $U_{\gamma,s}$ is still invertible modulo $\langle \mathbf{X} \rangle^e + \mathfrak{h}$, thus the preceding equality is equivalent to $Y_{\gamma,s} = -\varsigma_e(\varepsilon_s^e(E_{\gamma,s}))U_{\gamma,s}^{-1} \pmod{\langle \mathbf{X} \rangle^e + \mathfrak{h}}$ and the property holds by definition of $y_{\gamma,s}^{(e)}$. \square

The following lemma performs each induction step in the proof of part (i) in construction 5.3.10.

Lemma 5.3.12. *Keep the previous notation. We fix $\gamma \in \Gamma$ and the integers $s \in \mathbb{N}$ and $e_1, e_2, r \in \mathbb{N} \setminus \{0\}$ with $e_2 \leq e_1$. Let us suppose that the family of elements $\{y_{\gamma,r}^{(e)}\}_{\substack{e \geq 1 \\ r \in \mathbb{N}}}$ in $A[\mathbf{X}]$ defined above satisfies the following assumptions:*

- (a) For every $r \in \mathbb{N}$ with $r < s$, we have $\varsigma_{e_2}(y_{\gamma,r}^{(e_1)}) = y_{\gamma,r}^{(e_2)}$.
- (b) For every $r \in \mathbb{N}$, if $e_2 \geq 2$ we have $\varsigma_{e_2-1}(y_{\gamma,r}^{(e_1-1)}) = y_{\gamma,r}^{(e_2-1)}$.

Then $\varsigma_{e_2}(y_{\gamma,s}^{(e_1)}) = y_{\gamma,s}^{(e_2)}$.

Proof. By definition of $y_{\gamma,s}^{(e_1)}$, we have $\varsigma_{e_2}(y_{\gamma,s}^{(e_1)}) = \varsigma_{e_2}(-\varsigma_{e_1}(\varepsilon_s^{e_1}(E_{\gamma,s}))U_{\gamma,s}^{-1})$, which equals $-\varsigma_{e_2}(\varepsilon_s^{e_1}(E_{\gamma,s}))U_{\gamma,s}^{-1}$ (since $\varsigma_{e_2} \circ \varsigma_{e_1} = \varsigma_{e_2}$ for $e_2 \leq e_1$). On the other hand, by definition $y_{\gamma,s}^{(e_2)} := -\varsigma_{e_2}(\varepsilon_s^{e_2}(E_{\gamma,s}))U_{\gamma,s}^{-1}$. Then we only have to prove that $\varsigma_{e_2}(\varepsilon_s^{e_1}(E_{\gamma,s})) = \varsigma_{e_2}(\varepsilon_s^{e_2}(E_{\gamma,s}))$. By linearity, the definition of the morphisms $\varepsilon_s^{e_1}$ and $\varepsilon_s^{e_2}$ and the form of $E_{\gamma,s}$ according to assumption (A2) in theorem 5.3.7, this is equivalent to see that

$$\varsigma_{e_2}(E_{\gamma,s,-1}) + \sum_{r \in \mathbb{N}} \varsigma_{e_2}(E_{\gamma,s,r} \cdot \varepsilon_s^{e_1}(Y_{\gamma,r})) = \varsigma_{e_2}(E_{\gamma,s,-1}) + \sum_{r \in \mathbb{N}} \varsigma_{e_2}(E_{\gamma,s,r} \cdot \varepsilon_s^{e_2}(Y_{\gamma,r})).$$

We have three cases:

- Obviously $\varsigma_{e_2}(E_{\gamma,s,-1}) = \varsigma_{e_2}(E_{\gamma,s,-1})$.
- For $r < s$ we have $E_{\gamma,s,r} \cdot \varepsilon_s^{e_1}(Y_{\gamma,r}) = E_{\gamma,s,r} \cdot y_{\gamma,r}^{(e_1)}$ and $E_{\gamma,s,r} \cdot \varepsilon_s^{e_2}(Y_{\gamma,r}) = E_{\gamma,s,r} \cdot y_{\gamma,r}^{(e_2)}$. Then $\varsigma_{e_2}(E_{\gamma,s,r} \cdot \varepsilon_s^{e_1}(Y_{\gamma,r})) = \varsigma_{e_2}(E_{\gamma,s,r} \cdot \varsigma_{e_2}(y_{\gamma,r}^{(e_1)}))$, which equals $\varsigma_{e_2}(E_{\gamma,s,r} \cdot y_{\gamma,r}^{(e_2)}) = \varsigma_{e_2}(E_{\gamma,s,r} \cdot \varepsilon_s^{e_2}(Y_{\gamma,r}))$ by assumption (a).

- For $r \geq s$ we have $E_{\gamma,s,r} \cdot \varepsilon_s^{e_1}(Y_{\gamma,r}) = E_{\gamma,s,r} \cdot y_{\gamma,r}^{(e_1-1)}$ and $E_{\gamma,s,r} \cdot \varepsilon_s^{e_2}(Y_{\gamma,r}) = E_{\gamma,s,r} \cdot y_{\gamma,r}^{(e_2-1)}$. Moreover, the element $E_{\gamma,s,r}$ belongs to $\langle \mathbf{X} \rangle$. If $e_2 = 1$ then $\varsigma_1(E_{\gamma,s,r} \cdot \varepsilon_s^{e_1}(Y_{\gamma,r})) = 0$ and also $\varsigma_1(E_{\gamma,s,r} \cdot \varepsilon_s^{e_2}(Y_{\gamma,r})) = 0$. Let us suppose $e_2 > 1$, then $\varsigma_{e_2}(E_{\gamma,s,r} \cdot \varepsilon_s^{e_1}(Y_{\gamma,r})) = \varsigma_{e_2}(E_{\gamma,s,r} \cdot \varsigma_{e_2-1}(y_{\gamma,r}^{(e_1-1)}))$, which equals $\varsigma_{e_2}(E_{\gamma,s,r} \cdot y_{\gamma,r}^{(e_2-1)}) = \varsigma_{e_2}(E_{\gamma,s,r} \cdot \varepsilon_s^{e_2}(Y_{\gamma,r}))$ by assumption (b).

Then the equality above holds, which concludes the proof. \square

Now we can prove that the family $\{y_{\gamma,s}^{(e)}\}_{e \geq 1}$ defined recursively as in (3.6) is well defined and satisfies conditions (i), (ii) and (iii) in construction 5.3.10. Parts (i) and (ii) will be proven by double induction on e and s and in each step we will apply respectively lemmas 5.3.11 and 5.3.12, hence we will check that their assumptions hold.

Proof of construction 5.3.10.

- Let us prove part (i). Set $\gamma \in \Gamma$. We will reason by double induction on $e \geq 1$ and $s \in \mathbb{N}$. Recall that, by assumption, $y_{\gamma,s}^{(0)} := Y_{\gamma,s}$ for $s \in \mathbb{N}$.

Let us fix $e = 1$ and $s = 0$. By definition, the morphism ε_0^1 is the identity on $A[\mathbf{X}, \mathbf{Y}]$. Assumptions (a) and (b) in lemma 5.3.11 are trivially satisfied and hence by part (i) $y_{\gamma,0}^{(1)} := -\varsigma_1(\varepsilon_0^1(E_{\gamma,0}))U_{\gamma,0}^{-1}$ belongs to $A[\mathbf{X}]$. By part (iii) we have $Y_{\gamma,0} = y_{\gamma,0}^{(1)} \pmod{\langle \mathbf{X} \rangle + \mathfrak{h}}$ in the ring $A[\mathbf{X}, \mathbf{Y}]$. Let us stress that in fact $y_{\gamma,0}^{(1)} := -\varsigma_1(E_{\gamma,0,-1})U_{\gamma,0}^{-1}$, hence if $E_{\gamma,0,-1} \in A[\mathbf{X}] \setminus \langle \mathbf{X} \rangle$ as in assumption (C) in theorem 5.3.7 we deduce that $y_{\gamma,0}^{(1)}$ belongs to $A \setminus \{0\}$ and then is a unit in $\text{Frac}(A)[\mathbf{X}]$.

Let us fix $s > 0$, assume that, for every $0 \leq r < s$, the element $y_{\gamma,r}^{(1)}$ in $A[\mathbf{X}]$ satisfies $Y_{\gamma,r} = y_{\gamma,r}^{(1)} \pmod{\langle \mathbf{X} \rangle + \mathfrak{h}}$ in the ring $A[\mathbf{X}, \mathbf{Y}]$. Then assumption (a) in lemma 5.3.11 holds and also (trivially) assumption (b). Hence by part (i) in that lemma $y_{\gamma,s}^{(1)} := -\varsigma_1(\varepsilon_s^1(E_{\gamma,s}))U_{\gamma,s}^{-1}$ belongs to $A[\mathbf{X}]$ and by part (iii) we have $Y_{\gamma,s} = y_{\gamma,s}^{(1)} \pmod{\langle \mathbf{X} \rangle + \mathfrak{h}}$ in the ring $A[\mathbf{X}, \mathbf{Y}]$. Let us stress that, for every $s \in \mathbb{N}$, $y_{\gamma,s}^{(1)}$ coincides with the element $y_{\gamma,s}$ in the statement of theorem 5.3.7.

Let us fix $e > 1$, assume that for every $e', r \in \mathbb{N}$ with $1 \leq e' < e$ there exists an element $y_{\gamma,r}^{(e')}$ in $A[\mathbf{X}]$ such that $Y_{\gamma,r} = y_{\gamma,r}^{(e')} \pmod{\langle \mathbf{X} \rangle^{e'} + \mathfrak{h}}$ in the ring $A[\mathbf{X}, \mathbf{Y}]$ (we will refer to this assumption as the outer induction assumption).

Let us fix $s = 0$, then assumption (a) in lemma 5.3.11 trivially holds and assumption (b) holds by the outer induction assumption. Hence by part (i) in that lemma $y_{\gamma,0}^{(e)} := -\varsigma_e(\varepsilon_0^e(E_{\gamma,0}))U_{\gamma,0}^{-1}$ belongs to $A[\mathbf{X}]$ and by part (iii) we have $Y_{\gamma,0} = y_{\gamma,0}^{(e)} \pmod{\langle \mathbf{X} \rangle^e + \mathfrak{h}}$ in the ring $A[\mathbf{X}, \mathbf{Y}]$.

Let us fix $s > 0$, assume that, for every $0 \leq r < s$, the element $y_{\gamma,r}^{(e)}$ in $A[\mathbf{X}]$ satisfies $Y_{\gamma,r} = y_{\gamma,r}^{(e)} \pmod{\langle \mathbf{X} \rangle^e + \mathfrak{h}}$ in the ring $A[\mathbf{X}, \mathbf{Y}]$ (we will refer to this assumption as the inner induction assumption). Then assumption (a) in lemma 5.3.11 holds by the inner induction assumption and assumption (b) also holds by the outer one. Hence by part (i) in that lemma $y_{\gamma,s}^{(e)} := -\varsigma_e(\varepsilon_s^e(E_{\gamma,s}))U_{\gamma,s}^{-1}$ belongs to $A[\mathbf{X}]$ and by part (iii) we have $Y_{\gamma,s} = y_{\gamma,s}^{(e)} \pmod{\langle \mathbf{X} \rangle^e + \mathfrak{h}}$ in the ring $A[\mathbf{X}, \mathbf{Y}]$.

• Let us prove part (ii). Set $\gamma \in \Gamma$. Let $e_1, e_2 \in \mathbb{N}$ with $1 \leq e_2 \leq e_1$; in order to prove that $y_{\gamma,s}^{(e_1)} = y_{\gamma,s}^{(e_2)} \pmod{\langle \mathbf{X} \rangle^{e_2}}$ in the ring $A[\mathbf{X}]$, by the very definition of the morphism ς_{e_2} it suffices to prove that $\varsigma_{e_2}(y_{\gamma,s}^{(e_1)}) = y_{\gamma,s}^{(e_2)}$. The property clearly holds for $e_1 = e_2$, for the case $e_1 > e_2$ we will reason by double induction on $e_1 \geq 2$ and $s \in \mathbb{N}$.

Let us fix $e_1 = 2$, then $e_2 = 1$. We also fix $s = 0$, then assumptions (a) and (b) in lemma 5.3.12 trivially hold, hence $\varsigma_1(y_{\gamma,0}^{(2)}) = y_{\gamma,0}^{(1)}$.

Fix now $s > 0$ and assume that for every $r \in \mathbb{N}$ with $r < s$ we have $\varsigma_1(y_{\gamma,r}^{(2)}) = y_{\gamma,r}^{(1)}$. Then assumption (a) in lemma 5.3.12 holds and (trivially) also assumption (b). Thus $\varsigma_1(y_{\gamma,s}^{(2)}) = y_{\gamma,s}^{(1)}$.

Let us fix $e_1 > 2$, assume that for every $e'_1, e'_2 \in \mathbb{N}$ with $1 \leq e'_2 < e'_1 < e_1$ and every $r \in \mathbb{N}$ we have $\varsigma_{e'_2}(y_{\gamma,r}^{(e'_1)}) = y_{\gamma,r}^{(e'_2)}$ (we will refer to this assumption as the outer induction assumption). Let $e_2 \in \mathbb{N}$ such that $1 \leq e_2 < e_1$.

Let us fix $s = 0$, then assumption (a) in lemma 5.3.12 trivially holds and assumption (b) holds by the outer induction assumption. Hence $\varsigma_{e_2}(y_{\gamma,0}^{(e_1)}) = y_{\gamma,0}^{(e_2)}$.

Let us fix $s > 0$, assume that for every $r \in \mathbb{N}$ with $r < s$ we have $\varsigma_{e_2}(y_{\gamma,r}^{(e_1)}) = y_{\gamma,r}^{(e_2)}$ (we will refer to this assumption as the inner induction assumption). Then assumption (a) in lemma 5.3.12 holds by the inner induction assumption and assumption (b) holds by the outer one (and trivially if $e_2 = 1$). We deduce that $\varsigma_{e_2}(y_{\gamma,s}^{(e_1)}) = y_{\gamma,s}^{(e_2)}$.

• Let us prove part (iii). We set $\gamma \in \Gamma$. For the sake of simplicity, we define for every $e \in \mathbb{N} \setminus \{0\}$ the morphism of $A[\mathbf{X}]$ -algebras $\varepsilon^e : A[\mathbf{X}, \mathbf{Y}] \rightarrow A[\mathbf{X}]$ given by $\varepsilon^e(Y_{\gamma,r}) = y_{\gamma,r}^{(e)}$ for every $r \in \mathbb{N}$, it is well-defined by part (i). Let us fix $e \in \mathbb{N} \setminus \{0\}$ and $s \in \mathbb{N}$. Then the property $H_{\gamma,s}|_{Y_{\gamma,r}=y_{\gamma,r}^{(e)}} = 0 \pmod{\langle \mathbf{X} \rangle^e}$ in the ring $A[\mathbf{X}, \mathbf{Y}]$ can be rewritten as $\varepsilon^e(H_{\gamma,s}) = 0 \pmod{\langle \mathbf{X} \rangle^e}$ in the ring $A[\mathbf{X}]$, which is equivalent to $\varsigma_e(\varepsilon^e(H_{\gamma,s})) = 0$ by definition of the morphism ς_e .

By the very definition of $y_{\gamma,r}^{(e)}$, we have $y_{\gamma,s}^{(e)}U_{\gamma,s} + \varsigma_e(\varepsilon_s^e(E_{\gamma,s})) = 0$ in $A[\mathbf{X}]$. On the other hand, $\varsigma_e(\varepsilon^e(H_{\gamma,s})) = y_{\gamma,s}^{(e)}U_{\gamma,s} + \varsigma_e(\varepsilon^e(E_{\gamma,s}))$, since $\varsigma_e(y_{\gamma,s}^{(e)}) = y_{\gamma,s}^{(e)}$ and $U_{\gamma,s} \in A$. Then it suffices to prove that $\varsigma_e(\varepsilon^e(E_{\gamma,s})) = \varsigma_e(\varepsilon_s^e(E_{\gamma,s}))$. By linearity, the definition of the morphisms ε^e and ε_s^e and the form of $E_{\gamma,s}$ according to assumption (A2) in theorem 5.3.7, this is equivalent to see that

$$\varsigma_e(E_{\gamma,s,-1}) + \sum_{r \in \mathbb{N}} \varsigma_e(E_{\gamma,s,r} \cdot \varepsilon^e(Y_{\gamma,r})) = \varsigma_e(E_{\gamma,s,-1}) + \sum_{r \in \mathbb{N}} \varsigma_e(E_{\gamma,s,r} \cdot \varepsilon_s^e(Y_{\gamma,r})).$$

We have three cases:

- Obviously $\varsigma_e(E_{\gamma,s,-1}) = \varsigma_e(E_{\gamma,s,-1})$.
- For $r < s$ we have $E_{\gamma,s,r} \cdot \varepsilon_s^e(Y_{\gamma,r}) = E_{\gamma,s,r} \cdot y_{\gamma,r}^{(e)}$ and also $E_{\gamma,s,r} \cdot \varepsilon^e(Y_{\gamma,r}) = E_{\gamma,s,r} \cdot y_{\gamma,r}^{(e)}$, hence $\varsigma_e(E_{\gamma,s,r} \cdot \varepsilon^e(Y_{\gamma,r})) = \varsigma_e(E_{\gamma,s,r} \cdot \varepsilon_s^e(Y_{\gamma,r}))$.
- For $r \geq s$ we have $E_{\gamma,s,r} \cdot \varepsilon_s^e(Y_{\gamma,r}) = E_{\gamma,s,r} \cdot y_{\gamma,r}^{(e-1)}$ and $E_{\gamma,s,r} \cdot \varepsilon^e(Y_{\gamma,r}) = E_{\gamma,s,r} \cdot y_{\gamma,r}^{(e)}$. Moreover, the element $E_{\gamma,s,r}$ belongs to $\langle \mathbf{X} \rangle$. If $e = 1$ then $\varsigma_1(E_{\gamma,s,r} \cdot \varepsilon^1(Y_{\gamma,r})) = 0$

and also $\varsigma_1(E_{\gamma,s,r} \cdot \varepsilon_s^1(Y_{\gamma,r})) = 0$. Let us suppose $e > 1$, then $\varsigma_e(E_{\gamma,s,r} \cdot \varepsilon^e(Y_{\gamma,r})) = \varsigma_e(E_{\gamma,s,r} \cdot \varsigma_{e-1}(y_{\gamma,r}^{(e)}))$, which equals $\varsigma_e(E_{\gamma,s,r} \cdot y_{\gamma,r}^{(e-1)})$ by part (ii) and hence $\varsigma_e(E_{\gamma,s,r} \cdot \varepsilon_s^e(Y_{\gamma,r}))$.

Then the equality above holds, which concludes the proof. \square

5.4 TECHNICAL MACHINERY FOR THE COMPARISON THEOREM

In this section we will obtain the crucial technical result (theorem 5.4.6) allowing to establish our comparison theorem in section 5.5. As for theorem 5.3.7, the hypotheses are formulated in a somewhat abstract form, and in section 5.5 we will verify that these hypotheses hold in the toric setting.

5.4.1 Let k be a field of characteristic zero. If K is a field extension of k , we denote by \mathbf{LcCpl}_K the category of complete local k -algebras with residue field k -isomorphic to K . We begin with an elementary yet useful lemma.

Lemma 5.4.2. *Let K be a field, \mathcal{A} be an object of \mathbf{LcCpl}_K and \mathfrak{a} and \mathfrak{b} two ideals of \mathcal{A} such that for every object \mathcal{B} of \mathbf{LcCpl}_K one has the inclusion*

$$\{\varphi \in \text{Hom}_{\mathbf{LcCpl}_K}(\mathcal{A}, \mathcal{B}) : \mathfrak{b} \subset \text{Ker}(\varphi)\} \subset \{\varphi \in \text{Hom}_{\mathbf{LcCpl}_K}(\mathcal{A}, \mathcal{B}) : \mathfrak{a} \subset \text{Ker}(\varphi)\}.$$

Then one has the inclusion $\mathfrak{a} \subset \mathfrak{b}$.

Proof. We apply the assumption with φ the quotient morphism $\mathcal{A} \rightarrow \mathcal{A}/\mathfrak{b}$. \square

Notation 5.4.3. Let Δ be a finite set and \mathbf{Y} be the set of indeterminates $\{Y_\delta; \delta \in \Delta\}$. Let R be a ring. Let $\mathcal{Y}(t) := \{\mathcal{Y}_\delta(t) : \delta \in \Delta\}$ be a family of elements in the power series ring $R[[t]]$. Let $P \in R[\mathbf{Y}]$. Then we define the family $\{P_{s,\mathcal{Y}(t)} : s \in \mathbb{N}\}$ of elements of R by the following equality in $R[[t]]$:

$$P|_{Y_\delta = \mathcal{Y}_\delta(t)} = \sum_{s \in \mathbb{N}} P_{s,\mathcal{Y}(t)} t^s. \quad (4.1)$$

Remark 5.4.4. Let us keep the same notation as before. Let S be another ring, $\varphi: R \rightarrow S$ is a ring morphism. We also denote by φ the induced morphisms $R[\mathbf{Y}] \rightarrow S[\mathbf{Y}]$ and $R[[t]] \rightarrow S[[t]]$ obtained by applying φ coefficientwise. Then for every $s \in \mathbb{N}$ one has $\varphi(P_{s,\mathcal{Y}(t)}) = \varphi(P)_{s,\varphi(\mathcal{Y}(t))}$.

5.4.5 Now we can state and prove the main result of the section.

Theorem 5.4.6. *Let K be a field extension of k , Δ be a finite set and \mathbf{Y} be the set of indeterminates $\{Y_\delta; \delta \in \Delta\}$. Let $(d_\delta) \in \mathbb{N}^\Delta$ be a family of nonnegative integers. Let \mathbf{X} be the set of variables $\{X_{\delta,j}; \delta \in \Delta, 0 \leq j < d_\delta\}$. We denote by $\langle \mathbf{X} \rangle$ the maximal ideal of the power series ring $K[[\mathbf{X}]]$.*

Let Ω be a (possibly infinite) set, and let $\{P_\omega\}_{\omega \in \Omega}$ be a family of elements in the polynomial ring $K[\mathbf{Y}]$ such that, for every $\omega \in \Omega$:

(I) One may write $P_\omega = \prod_{\delta \in \Delta} Y_\delta^{u_{\omega,\delta}^+} - \prod_{\delta \in \Delta} Y_\delta^{u_{\omega,\delta}^-}$, where $u_{\omega,\delta}^+, u_{\omega,\delta}^- \in \mathbb{N}$.

(II) One has $P_\omega|_{Y_\delta=t^{d_\delta}} = 0$ in $K[t]$, in other words

$$\sum_{\delta \in \Delta} d_\delta u_{\omega,\delta}^+ = \sum_{\delta \in \Delta} d_\delta u_{\omega,\delta}^- =: c_\omega.$$

Let $\{x_{\delta,j} : \delta \in \Delta, j \geq d_\delta\}$ be a family of elements in $K[[\mathbf{X}]]$. For $\delta \in \Delta$, set

$$\mathcal{Y}_\delta(t) := \sum_{j=0}^{d_\delta-1} X_{\delta,j} t^j + \sum_{j \geq d_\delta} x_{\delta,j} t^j \in K[[\mathbf{X}]][[t]]$$

and

$$\widetilde{\mathcal{Y}}_\delta(t) := \sum_{j=0}^{d_\delta-1} X_{\delta,j} t^j + t^{d_\delta} \in K[[\mathbf{X}]][[t]]$$

We assume:

- (a) for every $\delta \in \Delta$, x_{δ,d_δ} is a unit;
- (b) for every $\omega \in \Omega$ and every $s \geq c_\omega$, one has $P_{\omega,s,\mathcal{Y}(t)} = 0$.

We consider the following ideals of $K[[\mathbf{X}]]$: $\mathfrak{a} := \langle \{P_{\omega,s,\mathcal{Y}(t)} : \omega \in \Omega, s \in \mathbb{N}\} \rangle$ and $\mathfrak{b} := \langle \{P_{\omega,s,\widetilde{\mathcal{Y}}(t)} : \omega \in \Omega, s \in \mathbb{N}\} \rangle$.

Then $K[[\mathbf{X}]]/\mathfrak{a}$ and $K[[\mathbf{X}]]/\mathfrak{b}$ are isomorphic objects of \mathbf{LcCpl}_K .

Proof. By assumption (a), for every $\delta \in \Delta$, the series $\mathcal{Y}_\delta(t)$ is a d_δ -regular element of $K[[\mathbf{X}]][[t]]$. Thus by the Weierstrass preparation theorem (see theorem 4.1.11), there exists a family $\{\mathfrak{X}_{\delta,j} : \delta \in \Delta, 0 \leq j < d_\delta\}$ of elements of the maximal ideal $\langle \mathbf{X} \rangle$ of $K[[\mathbf{X}]]$ and a family $\{U_{\delta,r} : \delta \in \Delta, r \in \mathbb{N}\}$ of elements of $K[[\mathbf{X}]]$ with $U_{\delta,0}$ an unit, such that, setting

$$W_\delta(t) := t^{d_\delta} + \sum_{j=0}^{d_\delta-1} \mathfrak{X}_{\delta,j} t^j \quad \text{and} \quad U_\delta(t) := \sum_{r \in \mathbb{N}} U_{\delta,r} t^r$$

one has

$$\mathcal{Y}_\delta(t) = W_\delta(t)U_\delta(t). \tag{4.2}$$

Identifying the t -coefficients in the latter equation yields the following relations in $K[[\mathbf{X}]]$:

$$X_{\delta,j} = \mathfrak{X}_{\delta,j}U_{\delta,0} + \sum_{r=0}^{j-1} \mathfrak{X}_{\delta,r}U_{\delta,j-r}, \quad 0 \leq j < d_\delta.$$

Since $U_{\delta,0}$ is a unit, we deduce that the element of $\text{Hom}_{\mathbf{LcCpl}_K}(K[[\mathbf{X}]], K[[\mathbf{X}]])$ sending $X_{\delta,j}$ to $\mathfrak{X}_{\delta,j}$ for $\delta \in \Delta$ and $0 \leq j < d_\delta$ is an isomorphism.

Setting

$$\mathbf{c} := \left\langle \left\{ P_{\omega,s,\{W_\delta(t)\}} : \omega \in \Omega, s \in \mathbb{N} \right\} \right\rangle,$$

the above isomorphism shows that $K[[\mathbf{X}]][[t]]/\mathbf{b}$ and $K[[\mathbf{X}]][[t]]/\mathbf{c}$ are isomorphic objects in \mathbf{LcCpl}_K . To conclude the proof, we show that $\mathbf{a} = \mathbf{c}$ using lemma 5.4.2.

Let $(\mathcal{B}, \mathfrak{m}_{\mathcal{B}})$ be an object in \mathbf{LcCpl}_K , and let φ be an element of $\text{Hom}_{\mathbf{LcCpl}_K}(K[[\mathbf{X}]], \mathcal{B})$. We still denote by φ the induced morphism $K[[\mathbf{X}]][[t]] \rightarrow \mathcal{B}[[t]]$ obtained by applying φ coefficientwise.

Let us assume that for every $\omega \in \Omega$ one has $P_\omega|_{Y_\delta=\varphi(\mathcal{Y}_\delta(t))} = 0$. One has to show that for every $\omega \in \Omega$ one has $P_\omega|_{Y_\delta=\varphi(W_\delta(t))} = 0$.

From our assumption and hypothesis (I) we deduce the following equality in $\mathcal{B}[[t]]$:

$$\prod_{\delta \in \Delta} \varphi(\mathcal{Y}_\delta(t))^{u_{\omega,\delta}^+} = \prod_{\delta \in \Delta} \varphi(\mathcal{Y}_\delta(t))^{u_{\omega,\delta}^-},$$

which can be rewritten, using equation 4.2, as

$$\prod_{\delta \in \Delta} \varphi(W_\delta(t))^{u_{\omega,\delta}^+} \prod_{\delta \in \Delta} \varphi(U_\delta(t))^{u_{\omega,\delta}^+} = \prod_{\delta \in \Delta} \varphi(W_\delta(t))^{u_{\omega,\delta}^-} \prod_{\delta \in \Delta} \varphi(U_\delta(t))^{u_{\omega,\delta}^-}. \quad (4.3)$$

Note that for every $\delta \in \Delta$, $\varphi(W_\delta(t))$ is a Weierstrass polynomial of degree d_δ in $\mathcal{B}[[t]]$ and $\varphi(U_\delta(t))$ is a unit in $\mathcal{B}[[t]]$, since $\varphi(U_{\delta,0})$ is.

By uniqueness of the Weierstrass factorization in $\mathcal{B}[[t]]$, one gets the equality

$$\prod_{\delta \in \Delta} \varphi(W_\delta(t))^{u_{\omega,\delta}^+} = \prod_{\delta \in \Delta} \varphi(W_\delta(t))^{u_{\omega,\delta}^-} \quad (4.4)$$

which means exactly that $P_\omega|_{Y_\delta=\varphi(W_\delta(t))} = 0$.

Conversely, assume that for every $\omega \in \Omega$ one has $P_\omega|_{Y_\delta=\varphi(W_\delta(t))} = 0$, in other words, that (4.4) holds, and let us show that for every $\omega \in \Omega$ one has $P_\omega|_{Y_\delta=\varphi(\mathcal{Y}_\delta(t))} = 0$. Let $\widetilde{W}_\omega(t) \in \mathcal{B}[[t]]$ be the common value of both members of (4.4). Note that $\widetilde{W}_\omega(t)$ is a Weierstrass polynomial of degree c_ω . On the other hand, one has

$$P_\omega|_{Y_\delta=\varphi(\mathcal{Y}_\delta(t))} = \widetilde{W}_\omega(t) \left(\prod_{\delta \in \Delta} \varphi(U_\delta(t))^{u_{\omega,\delta}^+} - \prod_{\delta \in \Delta} \varphi(U_\delta(t))^{u_{\omega,\delta}^-} \right).$$

By assumption (b), $P_\omega|_{Y_\delta=\varphi(\mathcal{Y}_\delta(t))}$ is an element of the polynomial ring $\mathcal{B}[t]$ with degree less than c_ω . By the uniqueness of the Weierstrass division by $\widetilde{W}_\omega(t)$ in $\mathcal{B}[[t]]$ one concludes that $P_\omega|_{Y_\delta=\varphi(\mathcal{Y}_\delta(t))} = 0$. \square

5.5 A COMPARISON THEOREM BETWEEN FORMAL NEIGHBOURHOODS

In this section we will make use of the results in sections 5.3 and 5.4 in order to obtain the main comparison theorem as an application of the results in those sections to the toric setting. It should be noted that our results provide basically two approaches for computing effectively the formal neighbourhood of the generic point of the Nash set associated with a divisorial toric valuation. The first one is based on an effective implementation of the Hensel lemma crucially used in section 5.3 (see subsection 5.3.9). The second one takes advantage of the comparison theorem to use exactly the same techniques as in the case of rational arcs described in [13]. The latter seems to be much more efficient in practice. See section 5.6 below for explicit examples of computation.

5.5.1 We retain the notation introduced in section 5.2. In particular, V_σ is the affine toric k -variety of dimension d associated with a cone σ and presented as $k[\mathbf{Z}]/\mathfrak{i}_\sigma$, where $\mathbf{Z} = \{Z_i : i \in \{1, \dots, h\}\}$ and \mathfrak{i}_σ is generated by the binomial elements $\{F_\ell = \mathbf{Z}^{\ell^+} - \mathbf{Z}^{\ell^-} ; \ell \in L\}$, L being a subgroup of \mathbb{Z}^h (see subsection 5.2.7). Moreover, denoting by \mathbf{Z}_∞ the set of variables $\{Z_{i,s} : i \in \{1, \dots, h\}, s \in \mathbb{N}\}$, the arc scheme $\mathcal{L}_\infty(V_\sigma)$ associated with the affine toric variety V_σ may be identified with the affine scheme $\text{Spec}(k[\mathbf{Z}_\infty]/[\mathfrak{i}_\sigma])$; the ideal $[\mathfrak{i}_\sigma]$ is generated by the elements $\{F_{\ell,s} ; \ell \in L, s \in \mathbb{N}\}$ (see subsection 5.2.8).

The following proposition gives an explicit description of the generic point of the Nash set associated with a divisorial toric valuation (see subsection 5.2.9).

Proposition 5.5.2. *Let $\mathbf{n} \in \sigma \cap N$ be a divisorial toric valuation, $\mathcal{N}_\mathbf{n}$ be the associated Nash set and $\eta_\mathbf{n}$ be the generic point of $\mathcal{N}_\mathbf{n}$. Let $\mathfrak{a}_\mathbf{n}$ be the ideal $\langle \{Z_{i,s_i} : 1 \leq i \leq h, 0 \leq s_i < \langle \mathbf{m}_i, \mathbf{n} \rangle\} \rangle$ of $k[\mathbf{Z}_\infty]$. Let $G_\mathbf{n} := \prod_{i=1}^d Z_{i, \langle \mathbf{m}_i, \mathbf{n} \rangle}$ and $g_\mathbf{n}$ be the image of $G_\mathbf{n}$ in $\mathcal{O}(\mathcal{L}_\infty(V_\sigma))$. Then:*

- (i) *The prime ideal of $\mathcal{O}(\mathcal{L}_\infty(V_\sigma))$ corresponding with $\eta_\mathbf{n}$ is the radical of the image of $\mathfrak{a}_\mathbf{n}$ in $\mathcal{O}(\mathcal{L}_\infty(V_\sigma))$.*
- (ii) *The point $\eta_\mathbf{n}$ belongs to the distinguished open subset $\{g_\mathbf{n} \neq 0\}$ of $\mathcal{L}_\infty(V_\sigma)$. The prime ideal of the localization $\mathcal{O}(\mathcal{L}_\infty(V_\sigma))_{g_\mathbf{n}}$ corresponding with $\eta_\mathbf{n}$ is the extension of $\mathfrak{a}_\mathbf{n}$ to $\mathcal{O}(\mathcal{L}_\infty(V_\sigma))_{g_\mathbf{n}}$.*

Proof. Assertion (i) follows from lemma 5.2.11: from part (iv) we know that $\mathcal{L}_\infty^\circ(V_\sigma)_\mathbf{n}$ is an open subset of $\mathcal{N}_\mathbf{n}$, hence their generic points coincide. By part (ii) it is the intersection of the open subset $\mathcal{L}_\infty^\circ(V_\sigma) \cap \bigcap_{i=1}^d \{z_{i, \langle \mathbf{m}_i, \mathbf{n} \rangle} \neq 0\}$ of $\mathcal{L}_\infty(V)$ (recall that $\mathcal{L}_\infty^\circ(V_\sigma)$ is open) with the closed subset $\left(\bigcap_{i=1}^h \bigcap_{s=0}^{\langle \mathbf{m}_i, \mathbf{n} \rangle - 1} \{z_{i,s} = 0\}\right)$, which is the closed subset of $\mathcal{L}_\infty(V)$ defined by the ideal $\langle \{z_{i,s_i} : 1 \leq i \leq h, 0 \leq s_i < \langle \mathbf{m}_i, \mathbf{n} \rangle\} \rangle$. Then the generic point of $\mathcal{L}_\infty^\circ(V_\sigma)_\mathbf{n}$ is the radical of this ideal and we finish by observing

that $\langle \{z_{i,s_i} : 1 \leq i \leq h, 0 \leq s_i < \langle \mathbf{m}_i, \mathbf{n} \rangle\} \rangle$ is the extension in $\mathcal{O}(\mathcal{L}_\infty(V_\sigma))$ of the ideal \mathfrak{a}_n . Let us observe that the fact that the open subset $\bigcap_{i=1}^d \{z_{i,\langle \mathbf{m}_i, \mathbf{n} \rangle} \neq 0\}$ equals $\{\prod_{i=1}^d z_{i,\langle \mathbf{m}_i, \mathbf{n} \rangle} \neq 0\}$ allows to deduce that the generic point belongs to the distinguished open subset $\{g_n \neq 0\}$ of $\mathcal{L}_\infty(V_\sigma)$.

Let us now prove assertion (ii). By (i), it is enough to show that the k -algebra $R := k[\mathbf{Z}_\infty]_{G_n}/[\mathfrak{i}_\sigma] + \mathfrak{a}_n$ is a domain. Let us show that its functor of points is isomorphic to the functor of points of the k -algebra $\mathcal{O}(\mathcal{L}_\infty(\mathcal{T}))$, the latter being a domain since \mathcal{T} is a smooth irreducible variety (see proposition 2.2.19).

Let A be a k -algebra. By the very definition of R the set $\text{Hom}_{\mathbf{Alg}_k}(R, A)$ is in natural bijection with the set of k -algebra morphisms $\varphi' : \mathcal{O}(V_\sigma) \rightarrow A[[t]]$ such that $\text{ord}_t(\varphi'(\chi^{m_i})) \geq \langle \mathbf{m}_i, \mathbf{n} \rangle$ for $1 \leq i \leq h$ and $\text{ord}_t(\varphi'(\chi^{m_i})) = \langle \mathbf{m}_i, \mathbf{n} \rangle$ for $1 \leq i \leq d$. By subsection 5.2.5 this set is in bijection with the set of semigroup morphisms $\varphi : S_\sigma \rightarrow A[[t]]$ (here $A[[t]]$ has its multiplicative structure) such that $\text{ord}_t(\varphi(\mathbf{m}_i)) \geq \langle \mathbf{m}_i, \mathbf{n} \rangle$ for $1 \leq i \leq h$ and $\text{ord}_t(\varphi(\mathbf{m}_i)) = \langle \mathbf{m}_i, \mathbf{n} \rangle$ for $1 \leq i \leq d$. Hence we can apply lemma 5.2.12 to the semigroup morphism $\text{ord}_t \circ \varphi : S_\sigma \rightarrow \mathbb{N}$ and deduce that $\text{ord}_t(\varphi(\mathbf{m})) = \langle \mathbf{m}, \mathbf{n} \rangle$ for $\mathbf{m} \in S_\sigma$.

Then φ uniquely corresponds with a semigroup morphism $\psi : S_\sigma \rightarrow (A[[t]])^*$ which extends uniquely to a group morphism $\bar{\psi} : M \rightarrow (A[[t]])^*$. By remark 5.2.6, such a group morphism corresponds uniquely with an element of $\text{Hom}_{\mathbf{Alg}_k}(\mathcal{O}(\mathcal{L}_\infty(\mathcal{T})), A)$, which concludes the proof. \square

5.5.3 Recall from subsection 5.2.7 the existence of a family $\{F_{\ell_q} ; d+1 \leq q \leq h\}$ of elements of the ideal \mathfrak{i}_σ ; let us denote by $\mathfrak{j} := \langle F_{\ell_q} ; d+1 \leq q \leq h \rangle$ the ideal of $k[\mathbf{Z}]$ generated by them. We have the following technical lemmas.

Lemma 5.5.4. *Let \mathfrak{i} be an ideal of $k[\mathbf{Z}]$. Let $d \leq h$ and $F \in k[\mathbf{Z}]$ such that F lies in the quotient ideal $\mathfrak{i} : (\prod_{i=1}^d Z_i)^\infty$. Let $(c_i) \in \mathbb{N}^d$ and \mathfrak{a} be the ideal $\langle Z_{i,s} : 1 \leq i \leq h, 0 \leq s \leq c_i \rangle$ of $k[\mathbf{Z}_\infty]$. Let $G = \prod_{i=1}^d Z_{i,c_i}$. Then in the localization $k[\mathbf{Z}_\infty]_G$, the ideal $[\langle F \rangle]$ is contained in the ideal $[\mathfrak{i}] + \mathfrak{a}$.*

Proof. For simplicity, let us denote by $\text{HS} : k[\mathbf{Z}] \rightarrow k[\mathbf{Z}_\infty][[t]]$ the morphism of k -algebras given by $\text{HS}(F) := F|_{Z_i = \sum_{s \in \mathbb{N}} Z_{i,s} t^s} = \sum_{s \in \mathbb{N}} F_s t^s$. By proposition 2.2.10 and lemma 2.3.5 we may assume that the t -coefficients of $\text{HS}(F)$ generate the ideal $[\langle F \rangle]$.

Let $H \in \mathfrak{i}$ and $e \in \mathbb{N}$ such that $(\prod_{i=1}^d Z_i)^e F = H$. Applying HS and using the very definition of \mathfrak{a} , one obtains the relation

$$\prod_{i=1}^d (t^{c_i} [Z_{i,c_i} + t(\dots)])^e \text{HS}(F) = \text{HS}(H) \pmod{\mathfrak{a}[[t]]}.$$

Thus, setting $K := \sum_{i=1}^d e c_i$, for $s < K$ one has $H_s \in \mathfrak{a}[[t]]$ and one may write

$$\prod_{i=1}^d [Z_{i,c_i} + t(\dots)]^e \text{HS}(F) = \sum_{s \geq 0} H_{s+K} t^s \pmod{\mathfrak{a}[[t]]}.$$

By the definition of G , the series $\prod_{i=1}^d [Z_{i,c_i} + t(\dots)]^e$ is invertible in $k[\mathbf{Z}_\infty]_G[[t]]$. That concludes the proof. \square

Lemma 5.5.5. *Let us keep the notation of subsection 5.2.7, in particular σ is a cone of $N_{\mathbb{R}}$, $\{\mathbf{m}_1, \dots, \mathbf{m}_h\}$ is a system of generators of the semigroup S_σ such that its subset $\{\mathbf{m}_1, \dots, \mathbf{m}_d\}$ forms a \mathbb{Z} -basis of M and $\mathfrak{i}_\sigma := \langle F_\ell; \ell \in L \rangle$ is the ideal of $k[\mathbf{Z}]$ defining V_σ . Let $\mathfrak{j} := \langle F_{\ell_q} : d+1 \leq q \leq h \rangle$.*

(i) *Set $G_d := \prod_{i=1}^d Z_i$. For every $\ell \in L$, F_ℓ lies in the quotient ideal $\mathfrak{j} : G_d^\infty$. In other words, the ideal \mathfrak{i}_σ vanishes in $k[\mathbf{Z}]_{G_d}/\mathfrak{j}$.*

(ii) *Let $\mathbf{n} \in \sigma \cap N$ and $\mathfrak{a}_\mathbf{n}$ be the ideal $\langle \{Z_{i,s_i} : 1 \leq i \leq h, 0 \leq s_i < \langle \mathbf{m}_i, \mathbf{n} \rangle\} \rangle$ of $k[\mathbf{Z}_\infty]$. Let $G_\mathbf{n} := \prod_{i=1}^d Z_{i, \langle \mathbf{m}_i, \mathbf{n} \rangle}$. Then the ideals $[\mathfrak{i}_\sigma] + \mathfrak{a}_\mathbf{n}$ and $[\mathfrak{j}] + \mathfrak{a}_\mathbf{n}$ coincide in the localization $k[\mathbf{Z}_\infty]_{G_\mathbf{n}}$.*

Proof. Set $G_h := \prod_{i=1}^h Z_i$. Since $\{\ell_q : d+1 \leq q \leq h\}$ spans the lattice L , [75, Lemma 12.2] shows that \mathfrak{i}_σ vanishes in $k[\mathbf{Z}]_{G_h}/\mathfrak{j}$. But (2.3) shows that the natural morphism $k[\mathbf{Z}]_{G_d} \rightarrow k[\mathbf{Z}]_{G_h}$ induces an isomorphism $k[\mathbf{Z}]_{G_d}/\mathfrak{j} \cong k[\mathbf{Z}]_{G_h}/\mathfrak{j}$. This shows (i).

By (i) and lemma 5.5.4, in the localization $k[\mathbf{Z}_\infty]_{G_\mathbf{n}}$, the ideal $[\mathfrak{i}_\sigma]$ is contained in $[\mathfrak{j}] + \mathfrak{a}_\mathbf{n}$. Since the inclusion $[\mathfrak{j}] \subset [\mathfrak{i}_\sigma]$ holds by definition, one deduces that (ii) also holds. \square

The ideal \mathfrak{j} of the ring $k[\mathbf{Z}]$ defines an affine k -scheme $W := \text{Spec}(k[\mathbf{Z}]/\mathfrak{j})$ which contains V_σ as a closed subscheme. Then $\mathcal{L}_\infty(W)$ may be identified with $\text{Spec}(k[\mathbf{Z}_\infty]/[\mathfrak{j}])$ with $[\mathfrak{j}] = \langle F_{\ell_q, s}; d+1 \leq q \leq h, s \in \mathbb{N} \rangle$. The closed immersion $V_\sigma \rightarrow W$ induces a closed immersion $\mathcal{L}_\infty(V_\sigma) \rightarrow \mathcal{L}_\infty(W)$ between the corresponding arc schemes (see proposition 2.2.8). For $\mathbf{n} \in \sigma \cap N$, let $\eta'_\mathbf{n}$ be the image of $\eta_\mathbf{n}$ by this closed immersion. We shall reduce the computation of the formal neighbourhood of $\mathcal{L}_\infty(V_\sigma)$ at $\eta_\mathbf{n}$ to that of the formal neighbourhood of $\mathcal{L}_\infty(W)$ at $\eta'_\mathbf{n}$. We will say that we are in the toric setting in the former situation and (abusing terminology) in the complete intersection setting in the latter.

Lemma 5.5.6. *Let $\mathbf{n} \in \sigma \cap N$ be a divisorial toric valuation, $\mathcal{N}_\mathbf{n}$ be the associated Nash set and $\eta_\mathbf{n}$ be the generic point of $\mathcal{N}_\mathbf{n}$. Let $\eta'_\mathbf{n}$ be the image of $\eta_\mathbf{n}$ by the closed immersion $\mathcal{L}_\infty(V_\sigma) \rightarrow \mathcal{L}_\infty(W)$. Let $\mathfrak{a}_\mathbf{n}$ be the ideal $\langle \{Z_{i,s_i} : 1 \leq i \leq h, 0 \leq s_i < \langle \mathbf{m}_i, \mathbf{n} \rangle\} \rangle$ of $k[\mathbf{Z}_\infty]$. Let $G_\mathbf{n} := \prod_{i=1}^d Z_{i, \langle \mathbf{m}_i, \mathbf{n} \rangle}$ and $g'_\mathbf{n}$ be the image of $G_\mathbf{n}$ in $\mathcal{O}(\mathcal{L}_\infty(W))$.*

Then the point $\eta'_\mathbf{n}$ belongs to the distinguished open set $\{g'_\mathbf{n} \neq 0\}$ of $\mathcal{L}_\infty(W)$, and the prime ideal of $\mathcal{O}(\mathcal{L}_\infty(W))_{g'_\mathbf{n}}$ corresponding with $\eta'_\mathbf{n}$ is the extension of $\mathfrak{a}_\mathbf{n}$ to $\mathcal{O}(\mathcal{L}_\infty(W))_{g'_\mathbf{n}}$.

Proof. Let $\mathfrak{p}_\mathbf{n}$ (resp. $\mathfrak{p}'_\mathbf{n}$) be the prime ideal of $k[\mathbf{Z}_\infty]/[\mathfrak{i}_\sigma]$ (resp. $k[\mathbf{Z}_\infty]/[\mathfrak{j}]$) corresponding to $\eta_\mathbf{n}$ (resp. $\eta'_\mathbf{n}$). Then $\mathfrak{p}'_\mathbf{n}$ is the preimage of $\mathfrak{p}_\mathbf{n}$ via the quotient morphism $k[\mathbf{Z}_\infty]/[\mathfrak{j}] \rightarrow$

$k[\mathbf{Z}_\infty]/[\mathfrak{i}_\sigma]$. We have the following commutative diagram, where all the morphisms are the canonical quotient ones:

$$\begin{array}{ccc} k[\mathbf{Z}_\infty] & \longrightarrow & k[\mathbf{Z}_\infty]/[\mathfrak{i}_\sigma] \\ \downarrow & \nearrow & \\ k[\mathbf{Z}_\infty]/[\mathfrak{j}] & & \end{array}$$

This diagram induces via localization the following commutative diagram:

$$\begin{array}{ccc} k[\mathbf{Z}_\infty]_{G_n} & \longrightarrow & (k[\mathbf{Z}_\infty]/[\mathfrak{i}_\sigma])_{g_n} \\ \downarrow & \nearrow & \\ (k[\mathbf{Z}_\infty]/[\mathfrak{j}])_{g'_n} & & \end{array}$$

We will also denote by \mathfrak{a}_n (resp. \mathfrak{p}_n) the extension of this ideal in $k[\mathbf{Z}_\infty]_{G_n}$ (resp. $(k[\mathbf{Z}_\infty]/[\mathfrak{i}_\sigma])_{g_n}$). By proposition 5.5.2, \mathfrak{p}_n is the extension of \mathfrak{a}_n in the ring $(k[\mathbf{Z}_\infty]/[\mathfrak{i}_\sigma])_{g_n}$ and it is still a prime ideal. Note that $g'_n \notin \mathfrak{p}'_n$, otherwise by the commutativity of the first diagram we would have $g_n \in \mathfrak{p}_n$ which is not the case. The extension of \mathfrak{p}'_n in $(k[\mathbf{Z}_\infty]/[\mathfrak{j}])_{g'_n}$ is the preimage of the prime ideal \mathfrak{p}_n via the morphism induced by the quotient one in the localizations. We conclude by the commutativity of the second diagram and the fact that $[\mathfrak{i}_\sigma] + \mathfrak{a}_n$ and $[\mathfrak{j}] + \mathfrak{a}_n$ coincide in $k[\mathbf{Z}_\infty]_{G_n}$ by lemma 5.5.5 (ii). \square

Notation 5.5.7. For $q \in \{1, \dots, h\}$ and $\mathbf{r} \in \mathbb{N}^q$ we denote by $\mathbf{Z}_{\leq q, \leq r_i}$ the set of variables $\{Z_{i, s_i}; 1 \leq i \leq q, 0 \leq s_i \leq r_i\}$. If $q = h$ we write $\mathbf{Z}_{\bullet, \leq r_i}$ instead of $\mathbf{Z}_{\leq h, \leq r_i}$. We define similarly $\mathbf{Z}_{\leq q, \geq r_i}$, $\mathbf{Z}_{\geq q, \leq r_i}$ and so on.

5.5.8 The following lemma shows that we can apply theorem 5.3.7 in the complete intersection setting.

Lemma 5.5.9. *Let $\mathbf{n} \in \sigma \cap N$ be a divisorial toric valuation.*

Let $G_n := \prod_{i=1}^d Z_{i, \langle \mathbf{m}_i, \mathbf{n} \rangle}$ and A be the k -algebra $k[\mathbf{Z}_{\leq d, \geq \langle \mathbf{m}_i, \mathbf{n} \rangle}]_{G_n}$.

Let Ω be the finite set $\{(i, s_i); i \in \{1, \dots, h\}, 0 \leq s_i < \langle \mathbf{m}_i, \mathbf{n} \rangle\}$. For $\omega \in \Omega$, set $X_\omega := Z_\omega$. Set $\Gamma = \{d+1, \dots, h\}$. For $q \in \Gamma$ and $s \in \mathbb{N}$, set $Y_{q,s} := Z_{q, \langle \mathbf{n}, \mathbf{m}_q \rangle + s}$.

Let \mathfrak{h} be the extension of the ideal $[\mathfrak{j}]$ in $k[\mathbf{Z}_\infty]_{G_n}$. For $s \in \mathbb{N}$ and $q \in \Gamma$, set $H_{q,s} := F_{\ell_q, \langle \mathbf{n}, \ell_q \rangle + s}$ (recall from subsection 5.2.7 the definition of $\langle \ell_q, \mathbf{n} \rangle$). Then with this notation the hypotheses in theorem 5.3.7 hold true. Moreover, if \mathfrak{g} is the extension of the ideal $[\mathfrak{i}_\sigma]$ in $k[\mathbf{Z}_\infty]_{G_n}$, then $\langle \mathbf{X} \rangle + \mathfrak{h} = \langle \mathbf{X} \rangle + \mathfrak{g}$.

Proof. Note that with the notation of the statement, one has in particular $A[\mathbf{X}, \mathbf{Y}] = k[\mathbf{Z}_\infty]_{G_n}$ and the ideal $\langle \mathbf{X} \rangle$ corresponds to $\mathfrak{a}_n = \langle \mathbf{Z}_{\bullet, \langle \mathbf{m}_i, \mathbf{n} \rangle} \rangle$.

Let us show that assumption (A) in theorem 5.3.7 holds. Pick up $q \in \{d+1, \dots, h\}$. Set

$$\Lambda_q^+ := \{i \in \{1, \dots, d\}; \ell_{q,i} > 0\}, \quad \Lambda_q^- := \{i \in \{1, \dots, d\}; \ell_{q,i} < 0\}$$

$$\Theta_{q,s}^+ := \{(r_q, (r_{i,k}; i \in \Lambda_q^+, 1 \leq k \leq \ell_{q,i})); r_q, r_{i,k} \in \mathbb{N}, r_q + \sum_{i \in \Lambda_q^+} \sum_{k=1}^{\ell_{q,i}} r_{i,k} = s\}$$

$$\text{and } \Theta_{q,s}^- := \{(r_{i,k}; i \in \Lambda_q^-, 1 \leq k \leq -\ell_{q,i}); r_{i,k} \in \mathbb{N}, \sum_{i \in \Lambda_q^-} \sum_{k=1}^{-\ell_{q,i}} r_{i,k} = s\}.$$

Then by (2.3) and (2.4), the polynomial $F_{\ell_{q,s}}$ has the following form:

$$F_{\ell_{q,s}} = \sum_{(r_q, r_{i,k}) \in \Theta_{q,s}^+} Z_{q,r_q} \prod_{i \in \Lambda_q^+} \prod_{k=1}^{\ell_{q,i}} Z_{i,r_{i,k}} - \sum_{(r_{i,k}) \in \Theta_{q,s}^-} \prod_{i \in \Lambda_q^-} \prod_{k=1}^{-\ell_{q,i}} Z_{i,r_{i,k}}. \quad (5.1)$$

Note that setting $r_q = \langle \mathbf{m}_q, \mathbf{n} \rangle + s$ and $r_{i,k} = \langle \mathbf{m}_i, \mathbf{n} \rangle$ for $1 \leq k \leq \ell_{q,i}$ defines an element of $\Theta_{q, \langle \ell_q, \mathbf{n} \rangle + s}^+$. Set

$$U_q := \prod_{i \in \Lambda_q^+} \prod_{k=1}^{\ell_{q,i}} Z_{i, \langle \mathbf{m}_i, \mathbf{n} \rangle}.$$

By the definition of $G_{\mathbf{n}}$, U_q is an invertible element of $k[\mathbf{Z}_{\leq d, \geq \langle \mathbf{m}_i, \mathbf{n} \rangle}]_{G_{\mathbf{n}}}$.

Set

$$E_{q,s,-1} := \sum_{\substack{(r_q, r_{i,k}) \in \Theta_{q, \langle \ell_q, \mathbf{n} \rangle + s}^+ \\ r_q < \langle \mathbf{m}_q, \mathbf{n} \rangle}} Z_{q,r_q} \prod_{i \in \Lambda_q^+} \prod_{k=1}^{\ell_{q,i}} Z_{i,r_{i,k}} - \sum_{(r_{i,k}) \in \Theta_{q, \langle \ell_q, \mathbf{n} \rangle + s}^-} \prod_{i \in \Lambda_q^-} \prod_{k=1}^{-\ell_{q,i}} Z_{i,r_{i,k}}.$$

For $r \in \mathbb{N}$, set $\delta_{r,s} = 1$ if $r = s$ and 0 otherwise, and

$$E_{q,s,r} =: -\delta_{s,r} U_q + \sum_{\substack{(r_{i,k}; i \in \Lambda_q^+, 1 \leq k \leq \ell_{q,i}); r_{i,k} \in \mathbb{N} \\ (\langle \mathbf{m}_q, \mathbf{n} \rangle + r, (r_{i,k})) \in \Theta_{q, \langle \ell_q, \mathbf{n} \rangle + s}^+}} \prod_{i \in \Lambda_q^+} \prod_{k=1}^{\ell_{q,i}} Z_{i,r_{i,k}}.$$

Thus by (5.1), one has

$$F_{\ell_{q, \langle \ell_q, \mathbf{n} \rangle + s}} = U_q Z_{q, \langle \mathbf{m}_q, \mathbf{n} \rangle + s} + E_{q,s,-1} + \sum_{r \in \mathbb{N}} E_{q,s,r} Z_{q, \langle \mathbf{m}_q, \mathbf{n} \rangle + r}.$$

Since $\Lambda_q^\pm \subset \{1, \dots, d\}$, it is clear that for $r \in \mathbb{N}$ one has $E_{q,s,r} \in k[\mathbf{Z}_{\leq d, \bullet}]$ and $E_{q,s,-1} \in k[\mathbf{Z}_{\leq d, \bullet} \cup \mathbf{Z}_{> d, < \langle \mathbf{m}_i, \mathbf{n} \rangle}]$.

Thus (A1) is satisfied, and in order to show that (A2) also holds, it remains to prove that for any $r \geq s$, each monomial of $E_{q,s,r}$ contains a variable Z_{i,r_i} with $i \in \{1, \dots, d\}$

and $r_i < \langle \mathbf{m}_i, \mathbf{n} \rangle$. Take $(r_{i,k}; i \in \Lambda_q^+, 1 \leq k \leq \ell_{q,i})$ a family of nonnegative integers such that $(\langle \mathbf{m}_q, \mathbf{n} \rangle + r, (r_{i,k})) \in \Theta_{q, \langle \ell_q, \mathbf{n} \rangle + s}^+$, that is

$$\langle \mathbf{m}_q, \mathbf{n} \rangle + r + \sum_{i \in \Lambda_q^+} \sum_{k=1}^{\ell_{q,i}} r_{i,k} = \langle \ell_q, \mathbf{n} \rangle + s.$$

We have to show that either at least one of the $r_{i,k}$'s is $< \langle \mathbf{m}_i, \mathbf{n} \rangle$ or $r = s$ and $r_{i,k} = \langle \mathbf{m}_i, \mathbf{n} \rangle$ for every i, k (the latter case corresponds to the monomial $U_q Z_{q, \langle \mathbf{m}_q, \mathbf{n} \rangle + s}$). Assume $r_{i,k} \geq \langle \mathbf{m}_i, \mathbf{n} \rangle$ for every i, k . Then

$$\langle \ell_q, \mathbf{n} \rangle + s = \langle \mathbf{m}_q, \mathbf{n} \rangle + r + \sum_{i \in \Lambda_q^+} \sum_{k=1}^{\ell_{q,i}} r_{i,k} \geq r + \left\langle \mathbf{m}_q + \sum_{i \in \Lambda_q^+} \ell_{k,i} \mathbf{m}_i, \mathbf{n} \right\rangle = r + \langle \ell_q, \mathbf{n} \rangle.$$

If $r > s$ this is a contradiction. If $r = s$, the first minoration must be an equality, which imposes $r_{i,k} = \langle \mathbf{m}_i, \mathbf{n} \rangle$ for every i, k .

Let us prove that (C) holds. We have to show that

$$E_{q,0,-1} := \sum_{\substack{(r_q, r_{i,k}) \in \Theta_{q, \langle \ell_q, \mathbf{n} \rangle}^+ \\ r_q < \langle \mathbf{m}_q, \mathbf{n} \rangle}} Z_{q, r_q} \prod_{i \in \Lambda_q^+} \prod_{k=1}^{\ell_{q,i}} Z_{i, r_{i,k}} - \sum_{(r_{i,k}) \in \Theta_{q, \langle \ell_q, \mathbf{n} \rangle}^-} \prod_{i \in \Lambda_q^-} \prod_{k=1}^{-\ell_{q,i}} Z_{i, r_{i,k}}$$

does not belong to the ideal $\langle \mathbf{Z}_{\bullet, \langle \mathbf{m}_i, \mathbf{n} \rangle} \rangle$. All the terms in the left hand sum lie in the ideal, and arguing similarly as above, one sees that the only monomial in $E_{q,0,-1}$ not belonging to the above ideal corresponds to $r_{i,k} = \langle \mathbf{m}_i, \mathbf{n} \rangle$ in the right hand sum. Thus one has $E_{q,0,-1} = \prod_{i \in \Lambda_q^-} Z_{i, \langle \mathbf{m}_i, \mathbf{n} \rangle}^{-\ell_{q,i}} \pmod{\langle \mathbf{Z}_{\bullet, \langle \mathbf{m}_i, \mathbf{n} \rangle} \rangle}$ which allows to conclude.

Let us show that (B) holds. Since \mathfrak{h} is the extension of the ideal $[\mathfrak{j}]$ in $k[\mathbf{Z}_\infty]_{G_n}$, it is generated by the union of the families $\{H_{q,s}; q \in \Gamma, s \in \mathbb{N}\}$ and $\{F_{\ell_q, s}; q \in \Gamma, s \in \mathbb{N}, s < \langle \ell_q, \mathbf{n} \rangle\}$.

Arguing similarly as above, one sees using (5.1) that in case $s < \langle \ell_q, \mathbf{n} \rangle$ every monomial of $F_{\ell_q, s}$ must contain a variable $Z_{i,r}$ with $r < \langle \mathbf{m}_i, \mathbf{n} \rangle$. Thus \mathfrak{h} is generated by some elements of $\langle \mathbf{X} \rangle$ and the $H_{\gamma, s}$'s. By remark 5.3.8, assumption (B) holds in this case.

Finally, by part (ii) in lemma 5.5.5 the last assertion holds true. □

5.5.10 Combining the preceding results and also theorem 5.3.7, we obtain the following corollary which provides a presentation of the formal neighbourhoods we are interested in in both the toric and the complete intersection settings. In particular, it assures that these two formal neighbourhoods are isomorphic.

Corollary 5.5.11. *Let \mathbf{n} be a divisorial toric valuation of $\sigma \cap N$. There exists a $k[\mathbf{Z}_{\leq d, \geq \langle \mathbf{m}_i, \mathbf{n} \rangle}, \mathbf{Z}_{\bullet, \langle \mathbf{m}_i, \mathbf{n} \rangle}]$ -algebra morphism $\hat{\varepsilon}: k[\mathbf{Z}_\infty] \rightarrow k(\mathbf{Z}_{\leq d, \geq \langle \mathbf{m}_i, \mathbf{n} \rangle})[[\mathbf{Z}_{\bullet, \langle \mathbf{m}_i, \mathbf{n} \rangle}]]$ such that :*

(i) The section ring of the formal neighbourhood of $\mathcal{L}_\infty(V_\sigma)$ at η_n (resp. of $\mathcal{L}_\infty(W)$ at η'_n) are both isomorphic to the complete noetherian local ring

$$k(\mathbf{Z}_{\leq d, \geq \langle m_i, n \rangle}) \llbracket \mathbf{Z}_{\bullet, < \langle m_i, n \rangle} \rrbracket / \widehat{\varepsilon}(\langle \mathbf{j} \rangle).$$

(ii) For every $i \in \{1, \dots, h\}$, $\widehat{\varepsilon}(Z_{i, \langle m_i, n \rangle})$ is invertible.

(iii) For every $q \in \{d+1, \dots, h\}$ and $s \in \mathbb{N}$, we have $\widehat{\varepsilon}(F_{\ell_q, \langle \ell_q, n \rangle + s}) = 0$.

Proof. Lemma 5.5.9 assures that, with the identifications in its statement we can apply theorem 5.3.7. By lemma 5.5.6, this shows part (i) in the complete intersection setting, i.e., the section ring of the formal neighbourhood of $\mathcal{L}_\infty(W)$ at η'_n is isomorphic to

$$k(\mathbf{Z}_{\leq d, \geq \langle m_i, n \rangle}) \llbracket \mathbf{Z}_{\bullet, < \langle m_i, n \rangle} \rrbracket / \widehat{\varepsilon}(\langle \mathbf{j} \rangle),$$

where the morphism $\widehat{\varepsilon} : k[\mathbf{Z}_\infty] \rightarrow k(\mathbf{Z}_{\leq d, \geq \langle m_i, n \rangle}) \llbracket \mathbf{Z}_{\bullet, < \langle m_i, n \rangle} \rrbracket$ is determined by theorem 5.3.7. Parts (ii) and (iii) are also given by theorem 5.3.7.

It remains to prove part (i) in the toric case. By proposition 5.5.2, the application of theorem 5.3.7 to the extension \mathfrak{g} of the ideal $\langle \mathbf{i}_\sigma \rangle$ in $k[\mathbf{Z}_\infty]_{G_n}$ (which can be done by lemma 5.5.9) allows to deduce that the section ring of the formal neighbourhood of $\mathcal{L}_\infty(V_\sigma)$ at η_n is isomorphic to

$$k(\mathbf{Z}_{\leq d, \geq \langle m_i, n \rangle}) \llbracket \mathbf{Z}_{\bullet, < \langle m_i, n \rangle} \rrbracket / \widehat{\varepsilon}(\langle \mathbf{i}_\sigma \rangle),$$

with the same morphism $\widehat{\varepsilon}$ as in the complete intersection case. Then we only have to prove that $\widehat{\varepsilon}(\langle \mathbf{i}_\sigma \rangle)$ and $\widehat{\varepsilon}(\langle \mathbf{j} \rangle)$ coincide as ideals of $k(\mathbf{Z}_{\leq d, \geq \langle m_i, n \rangle}) \llbracket \mathbf{Z}_{\bullet, < \langle m_i, n \rangle} \rrbracket$.

It is clear that $\widehat{\varepsilon}(\langle \mathbf{j} \rangle) \subset \widehat{\varepsilon}(\langle \mathbf{i}_\sigma \rangle)$. For the converse inclusion, let $F \in \mathbf{i}_\sigma$. By lemma 5.5.5 part (i), there exists an integer $e \in \mathbb{N}$ such that $\left(\prod_{i=1}^d Z_i \right)^e F \in \mathbf{j}$. Recall the morphism of k -algebras $\text{HS} : k[\mathbf{Z}] \rightarrow k[\mathbf{Z}_\infty] \llbracket t \rrbracket$ from the proof of lemma 5.5.4. Then we have $\text{HS} \left(\left(\prod_{i=1}^d Z_i \right)^e \right) \text{HS}(F) \in \langle \mathbf{j} \rangle \llbracket t \rrbracket$. Recall now that $\widehat{\varepsilon}$ is a $k[\mathbf{Z}_{\leq d, \geq \langle m_i, n \rangle}, \mathbf{Z}_{\bullet, < \langle m_i, n \rangle}]$ -algebra morphism, hence we deduce that

$$\text{HS} \left(\left(\prod_{i=1}^d Z_i \right)^e \right) \widehat{\varepsilon}(\text{HS}(F)) \in \widehat{\varepsilon}(\langle \mathbf{j} \rangle) \llbracket t \rrbracket. \quad (5.2)$$

Moreover, the element $Z_{i, \langle m_i, n \rangle}$ being invertible in $k(\mathbf{Z}_{\leq d, \geq \langle m_i, n \rangle}) \llbracket \mathbf{Z}_{\bullet, < \langle m_i, n \rangle} \rrbracket$ for every $i \in \{1, \dots, d\}$, we deduce that $\text{HS} \left(\left(\prod_{i=1}^d Z_i \right)^e \right)$ does not reduce to 0 modulo the maximal ideal $\langle \mathbf{Z}_{\bullet, < \langle m_i, n \rangle} \rangle$.

Note that the ideal $\widehat{\varepsilon}(\langle \mathbf{j} \rangle)$ is contained in the maximal ideal $\langle \mathbf{Z}_{\bullet, < \langle m_i, n \rangle} \rangle$ (otherwise the section ring of the formal neighbourhood of $\mathcal{L}_\infty(W)$ at η'_n would be the zero ring by part (i) applied to the complete intersection setting, which is not true). In the canonical

quotient ring $k(\mathbf{Z}_{\leq d, \geq \langle \mathbf{m}_i, \mathbf{n} \rangle})[[\mathbf{Z}_{\bullet, \langle \mathbf{m}_i, \mathbf{n} \rangle}]]/\widehat{\varepsilon}(\mathfrak{j})$, which is also a complete local ring, the image of $\text{HS} \left(\left(\prod_{i=1}^d Z_i \right)^e \right)$ still does not reduce to 0 modulo the maximal ideal and hence it is not a zero divisor by the Weierstrass division theorem (see subsection 4.1.9). From (5.2) we deduce that in $k(\mathbf{Z}_{\leq d, \geq \langle \mathbf{m}_i, \mathbf{n} \rangle})[[\mathbf{Z}_{\bullet, \langle \mathbf{m}_i, \mathbf{n} \rangle}]][[t]]/\widehat{\varepsilon}(\mathfrak{j})$ we have the equality

$$\text{HS} \left(\left(\prod_{i=1}^d Z_i \right)^e \right) \widehat{\varepsilon}(\text{HS}(F)) = 0.$$

Thus $\widehat{\varepsilon}(\text{HS}(F))$ belongs to $\widehat{\varepsilon}(\mathfrak{j})[[t]]$ for every $F \in \mathfrak{i}_\sigma$, which shows that $\widehat{\varepsilon}(\mathfrak{i}_\sigma) \subset \widehat{\varepsilon}(\mathfrak{j})$. \square

Remark 5.5.12. Let us point out that lemma 5.5.9 also assures that we can perform the effective construction in subsection 5.3.9 of the truncations of $\widehat{\varepsilon}(Z_{i,j})$ modulo the powers of the ideal $\langle \mathbf{Z}_{\bullet, \langle \mathbf{m}_i, \mathbf{n} \rangle} \rangle$ for $d + 1 \leq i \leq h$ and $j \geq \langle \mathbf{m}_i, \mathbf{n} \rangle$, which determine the morphism $\widehat{\varepsilon}$ (see construction 5.3.10). This provides a way for explicitly computing the formal neighbourhood $\widehat{\mathcal{L}_\infty(V_\sigma)_{\eta_n}}$ as a limit. In section 5.6 we will show this on several examples.

5.5.13 Let us recall the definition of some objects in [13, Subsection 5.1], adapted to our notation. We denote by $\tilde{\varepsilon}: k[\mathbf{Z}_\infty] \rightarrow k[[\mathbf{Z}_{\bullet, \langle \mathbf{m}_i, \mathbf{n} \rangle}]]$ the unique k -algebra morphism mapping, for every $i \in \{1, \dots, h\}$, $Z_{i,s}$ to $Z_{i,s}$ for $s < \langle \mathbf{m}_i, \mathbf{n} \rangle$, $Z_{i, \langle \mathbf{m}_i, \mathbf{n} \rangle}$ to 1 and $Z_{i,s}$ to 0 for $s > \langle \mathbf{m}_i, \mathbf{n} \rangle$.

For $L' \subseteq L$, let $W(\mathbf{n}, L')$ be the affine closed k -subscheme of the affine space $\text{Spec}(k[\mathbf{Z}_{\bullet, \langle \mathbf{m}_i, \mathbf{n} \rangle}])$ defined by the ideal $\langle \tilde{\varepsilon}(F_{\ell,s}); \ell \in L', s \in \mathbb{N} \rangle$ and $\mathcal{W}(\mathbf{n}, L')$ the formal completion of $W(\mathbf{n}, L')$ along the origin of $\text{Spec}(k[\mathbf{Z}_{\bullet, \langle \mathbf{m}_i, \mathbf{n} \rangle}])$. Let us remark that, setting $c_i := \langle \mathbf{m}_i, \mathbf{n} \rangle$ for $i \in \{1, \dots, h\}$, $\mathbf{c} := (c_i)_{1 \leq i \leq h}$ and identifying an element $\ell \in L$ with the pair (ℓ_+, ℓ_-) , then $\mathcal{W}(\mathbf{n}, L')$ coincides with the formal k -scheme $\mathcal{W}(\mathbf{c}, L')$ in construction 4.2.15.

Remark 5.5.14. Let $(\mathcal{A}, \mathfrak{M}_{\mathcal{A}})$ be an object of \mathbf{LcCpl}_k . Then $\text{Hom}_{\mathbf{LcCpl}_k}(\mathcal{W}(\mathbf{n}, L'), \mathcal{A})$ is in natural bijection with the set of families $\{z_{i,s}; i \in \{1, \dots, h\}, 0 \leq s < \langle \mathbf{m}_i, \mathbf{n} \rangle\}$ of elements of $\mathfrak{M}_{\mathcal{A}}$ such that for every element $\ell \in L'$ one has

$$F_\ell|_{Z_i = \sum_{s=0}^{\langle \mathbf{m}_i, \mathbf{n} \rangle - 1} z_{i,s} t^s + t^{\langle \mathbf{m}_i, \mathbf{n} \rangle}} = 0.$$

Let us recall [13, Theorem 5.2], adapted to our notation.

Theorem 5.5.15 (Bourqui-Sebag). *Let k be a field of characteristic zero. Let $\mathbf{n} \in N \cap \sigma$ be a divisorial toric valuation and $\alpha \in \mathcal{L}_\infty(V_\sigma)_\mathbf{n}^\circ(k)$. For an appropriate choice of $L' \subseteq L$ such that $\{\ell_q; d + 1 \leq q \leq h\} \subseteq L'$, the formal k -scheme $\mathcal{W}(\mathbf{n}, L')$ is a finite formal model of the pair $(\mathcal{L}_\infty(V_\sigma), \alpha)$, that is, the formal neighbourhood of $\mathcal{L}_\infty(V_\sigma)$ at α is isomorphic to $\mathcal{W}(\mathbf{n}, L') \widehat{\otimes}_k k[[\mathfrak{T}_i]_{i \in \mathbb{N}}]$.*

The following lemma shows that, for the computation of formal neighbourhoods of k -rational arcs on $\mathcal{L}_\infty(V_\sigma)$, one may also reduce to the complete intersection setting.

Lemma 5.5.16. *Let $\mathbf{n} \in \sigma \cap N$ be a divisorial toric valuation and L' be a subset of L such that $\{\ell_q; d+1 \leq q \leq h\} \subseteq L'$. Then $\mathcal{W}(\mathbf{n}, L')$ is isomorphic, as a formal k -scheme, to $\mathcal{W}(\mathbf{n}, \{\ell_q; d+1 \leq q \leq h\})$.*

Thanks to this lemma, for any $L' \subseteq L$ such that $\{\ell_q; d+1 \leq q \leq h\} \subseteq L'$ and any $\mathbf{n} \in \sigma \cap N$, we may denote $\mathcal{W}(\mathbf{n}, L')$ by $\mathcal{W}(\mathbf{n})$.

Proof. By remark 5.5.14, there is, for every object $(\mathcal{A}, \mathfrak{M}_{\mathcal{A}})$ of \mathbf{LcCpl}_k , a natural inclusion $\mathrm{Hom}_{\mathbf{LcCpl}_k}(\mathcal{W}(\mathbf{n}, L'), \mathcal{A}) \subset \mathrm{Hom}_{\mathbf{LcCpl}_k}(\mathcal{W}(\mathbf{n}, \{\ell_q; d+1 \leq q \leq h\}), \mathcal{A})$. To conclude, it suffices to show that this is an equality. Let $\{z_{i,s}; i \in \{1, \dots, h\}, 0 \leq s < \langle \mathbf{m}_i, \mathbf{n} \rangle\}$ be a family of elements of $\mathfrak{M}_{\mathcal{A}}$ such that, setting

$$z_i(t) := \sum_{s=0}^{\langle \mathbf{m}_i, \mathbf{n} \rangle - 1} z_{i,s} t^s + t^{\langle \mathbf{m}_i, \mathbf{n} \rangle},$$

one has, for every $d+1 \leq q \leq h$, $F_{\ell_q}|_{Z_i=z_i(t)} = 0$. Let $\ell \in L'$. By lemma 5.5.5 there exists a positive integer e such that $(\prod_{i=1}^d Z_i)^e F_{\ell} \in \langle F_{\ell_q}; d+1 \leq q \leq h \rangle$. Thus

$$\left(\prod_{i=1}^d z_i(t) \right)^e F_{\ell}|_{Z_i=z_i(t)} = 0.$$

Since $z_i(t)$ is a Weierstrass polynomial in $\mathcal{A}[[t]]$, it is not a zero divisor (see subsection 4.1.9). Thus one infers that $F_{\ell}|_{Z_i=z_i(t)} = 0$, which concludes the proof. \square

The following proposition performs the aimed comparison in the complete intersection setting.

Proposition 5.5.17. *Let $\mathbf{n} \in \sigma \cap N$ be a divisorial toric valuation, we consider the field $K := k(\mathbf{Z}_{\leq d, \geq \langle \mathbf{m}_i, \mathbf{n} \rangle})$. Then the residue field of $\eta'_{\mathbf{n}}$ is isomorphic to K and the formal neighbourhood of $\mathcal{L}_{\infty}(W)$ at the point $\eta'_{\mathbf{n}}$ is isomorphic, as a formal K -scheme, to $K \widehat{\otimes}_k \mathcal{W}(\mathbf{n}, \{\ell_q; d+1 \leq q \leq h\})$.*

Proof. We still denote by $\tilde{\varepsilon}$ the composition of the morphism defined in subsection 5.5.13 with the natural inclusion morphism $k[[\mathbf{Z}_{\bullet, \langle \mathbf{m}_i, \mathbf{n} \rangle}]] \rightarrow K[[\mathbf{Z}_{\bullet, \langle \mathbf{m}_i, \mathbf{n} \rangle}]]$.

By corollary 5.5.11 and the very definition of $\mathcal{W}(\mathbf{n}, \{\ell_q; d+1 \leq q \leq h\})$, it is enough to show that the quotients of $K[[\mathbf{Z}_{\bullet, \langle \mathbf{m}_i, \mathbf{n} \rangle}]]$ by the ideals $\langle \widehat{\varepsilon}([j]) \rangle = \langle \widehat{\varepsilon}(F_{\ell_q, s}); d+1 \leq q \leq h, s \in \mathbb{N} \rangle$ on the one hand and $\langle \tilde{\varepsilon}([j]) \rangle = \langle \tilde{\varepsilon}(F_{\ell_q, s}); d+1 \leq q \leq h, s \in \mathbb{N} \rangle$ on the other hand, are isomorphic.

For this, we aim to apply theorem 5.4.6. Set $\Delta := \{1, \dots, h\}$. For $i \in \Delta$, set $d_i := \langle \mathbf{m}_i, \mathbf{n} \rangle$; for $0 \leq s < \langle \mathbf{m}_i, \mathbf{n} \rangle$ set $X_{i,s} := Z_{i,s}$ and for $s \geq \langle \mathbf{m}_i, \mathbf{n} \rangle$ set $x_{i,s} := \widehat{\varepsilon}(Z_{i,s})$. For $i \in \Delta$, set

$$\mathcal{Y}_i(t) := \sum_{s \in \mathbb{N}} \widehat{\varepsilon}(Z_{i,s}) t^s = \sum_{s=0}^{\langle \mathbf{m}_i, \mathbf{n} \rangle - 1} X_{i,s} t^s + \sum_{s \geq \langle \mathbf{m}_i, \mathbf{n} \rangle} x_{i,s} t^s$$

$$\text{and } \tilde{\mathcal{Y}}_i(t) = \sum_{s=0}^{\langle \mathbf{m}_i, \mathbf{n} \rangle - 1} X_{i,s} t^s + t^{\langle \mathbf{m}_i, \mathbf{n} \rangle}.$$

For every $P \in k[\mathbf{Z}]$, we then have (see notation 5.4.3 and remark 5.4.4) $P_{s, \mathbf{y}(t)} = \widehat{\varepsilon}(P_s)$ and $P_{s, \tilde{\mathbf{y}}(t)} = \widetilde{\varepsilon}(P_s)$. Set $\Omega := \{d+1, \dots, h\}$ and for $q \in \Omega$ set $P_q := F_{\ell_q}$, then $u_{q,i}^+ = (\ell_q^+)_i$ and $u_{q,i}^- = (\ell_q^-)_i$.

Assumption (a) is a consequence of corollary 5.5.11. With our identifications, the nonzero integer c_q defined in the statement of theorem 5.4.6 is

$$c_q = \sum_{i=1}^h (\ell_q^+)_i \langle \mathbf{m}_i, \mathbf{n} \rangle = \left\langle \sum_{\substack{i=1 \\ \ell_{q,i} \geq 0}}^h \ell_{q,i} \mathbf{m}_i, \mathbf{n} \right\rangle = \langle \ell_q, \mathbf{n} \rangle.$$

Then still by corollary 5.5.11, for every $s \in \mathbb{N}$ we have $\widehat{\varepsilon}(F_{\ell_q, \langle \ell_q, \mathbf{n} \rangle + s}) = 0$, that is $F_{\ell_q, \langle \ell_q, \mathbf{n} \rangle + s, \mathbf{y}(t)} = 0$ and assumption (b) holds. That concludes the proof. \square

5.5.18 Now we can state the main theorem of the chapter. It illustrates the striking fact that not only the isomorphism class of the formal neighbourhood of a generic k -rational arc of the Nash set associated with a divisorial toric valuation is constant (as observed in [13], see subsection 5.2.13) but moreover the involved isomorphism class is encoded in some sense in the formal neighbourhood of the generic point of the Nash set.

Theorem 5.5.19. *Let k be a field of characteristic zero. Let σ be a cone in $N_{\mathbb{R}}$ and V_{σ} be the associated affine normal toric k -variety. Let $\mathbf{n} \in \sigma \cap N$ be a divisorial toric valuation. Let $\eta_{\mathbf{n}}$ be the generic point of the Nash set $\mathcal{N}_{\mathbf{n}}$ and $\kappa(\eta_{\mathbf{n}})$ its residue field. Let $\mathcal{W}(\mathbf{n})$ be the noetherian formal k -scheme defined in subsection 5.5.13.*

Then there exists a nonempty open set $U_{\mathbf{n}}$ of the Nash set $\mathcal{N}_{\mathbf{n}}$ such that:

- (i) *The formal neighbourhood of $\mathcal{L}_{\infty}(V_{\sigma})$ at $\eta_{\mathbf{n}}$ is isomorphic, as a formal $\kappa(\eta_{\mathbf{n}})$ -scheme, to $\kappa(\eta_{\mathbf{n}}) \widehat{\otimes}_k \mathcal{W}(\mathbf{n})$. In particular it is isomorphic to the formal spectrum of the completion of an essentially of finite type local $\kappa(\eta_{\mathbf{n}})$ -algebra.*
- (ii) *For any arc $\alpha \in U_{\mathbf{n}}(k)$, the formal neighbourhood of $\mathcal{L}_{\infty}(V_{\sigma})$ at α is isomorphic, as a formal k -scheme, to $\mathcal{W}(\mathbf{n}) \widehat{\otimes}_k k[[T_i]_{i \in \mathbb{N}}]$.*

Proof. We take $U_{\mathbf{n}} := \mathcal{L}_{\infty}(V_{\sigma})_{\mathbf{n}}^{\circ}$. For part (i), proposition 5.5.17 assures that the formal neighbourhood of $\mathcal{L}_{\infty}(W)$ at the point $\eta'_{\mathbf{n}}$ is isomorphic, as a formal $\kappa(\eta'_{\mathbf{n}})$ -scheme, to $\kappa(\eta'_{\mathbf{n}}) \widehat{\otimes}_k \mathcal{W}(\mathbf{n})$. Corollary 5.5.11(i) implies that the residue fields $\kappa(\eta_{\mathbf{n}})$ and $\kappa(\eta'_{\mathbf{n}})$ are isomorphic and the formal neighbourhood of $\mathcal{L}_{\infty}(V_{\sigma})$ at the point $\eta_{\mathbf{n}}$ is isomorphic, as a formal $\kappa(\eta_{\mathbf{n}})$ -scheme, to the formal neighbourhood of $\mathcal{L}_{\infty}(W)$ at $\eta'_{\mathbf{n}}$ and hence to $\kappa(\eta_{\mathbf{n}}) \widehat{\otimes}_k \mathcal{W}(\mathbf{n})$.

Part (ii) is a direct consequence of theorem 5.5.15, which assures that there exists an appropriate choice of $L' \subseteq L$ such that $\{\ell_q; d+1 \leq q \leq h\} \subseteq L'$ such that the

formal neighbourhood of $\mathcal{L}_\infty(V_\sigma)$ at α is isomorphic to $\mathcal{W}(\mathbf{n}, L') \hat{\otimes}_k k[[\langle T_i \rangle_{i \in \mathbb{N}}]]$. Lemma 5.5.16 shows that $\mathcal{W}(\mathbf{n}, L')$ is isomorphic, as a formal k -scheme, to $\mathcal{W}(\mathbf{n}, \{\ell_q; d+1 \leq q \leq h\}) =: \mathcal{W}(\mathbf{n})$, which concludes the proof. \square

5.6 EXAMPLES

Let us consider an explicit example of computation of the formal neighbourhood of the generic point of the Nash set associated with a toric valuation.

5.6.1 Let $N = M = \mathbb{Z}^2$, σ be the cone of \mathbb{R}^2 generated by $(1, 0)$ and $(1, 2)$ and V_σ be the associated affine toric variety. The semigroup S_σ is minimally generated by $\mathbf{m}_1 = (0, 1)$, $\mathbf{m}_2 = (1, 0)$ and $\mathbf{m}_3 = (2, -1)$. We observe that \mathbf{m}_1 and \mathbf{m}_2 form a \mathbb{Z} -basis of M and the relation $\mathbf{m}_1 + \mathbf{m}_3 = 2\mathbf{m}_2$ (corresponding to $\ell = (1, -2, 1)$) generates all the nontrivial relations between elements of S_σ . Thus, setting $F := Z_1 Z_3 - Z_2^2$, the ideal of V_σ in $k[Z_1, Z_2, Z_3]$ is the ideal generated by F ; we observe that V_σ is a complete intersection. The ideal of $\mathcal{L}_\infty(V_\sigma)$ in the ring $k[\mathbf{Z}_\infty] = k[Z_{1,s}, Z_{2,s}, Z_{3,s}; s \in \mathbb{N}]$ is generated by $\{F_s; s \in \mathbb{N}\}$ where $F_s = \sum_{r=0}^s (Z_{1,s-r} Z_{3,r} - Z_{2,s-r} Z_{2,r})$.

We now consider the divisorial toric valuation ord_n of V_σ corresponding to $\mathbf{n} = (1, 1) \in \sigma \cap N$, then $\langle \mathbf{m}_i, \mathbf{n} \rangle = 1$ for $i = 1, 2, 3$ and $\langle \ell, \mathbf{n} \rangle = 2$. The prime ideal of $\mathcal{O}(V_\sigma)$ corresponding to the generic point η_n of the Nash set associated with ord_n is the radical of the image of the ideal $\langle Z_{1,0}, Z_{2,0}, Z_{3,0} \rangle$ of $k[\mathbf{Z}_\infty]$ (see proposition 5.5.2). The residue field of η_n is isomorphic to $K := k(Z_{1,s}, Z_{2,s}; s \geq 1)$.

According to lemma 5.5.9, in the notation of theorem 5.3.7 the set Γ is a singleton and for $s \in \mathbb{N}$ we fix $Y_s := Z_{3,s+1}$ and $H_s := F_{s+2}$. For $s \geq 2$, the proof of lemma 5.5.9 shows that we can write F_s under the desired form $F_s = Z_{3,s-1} U_{s-2} + E_{s-2,-1} + \sum_{r \in \mathbb{N}} E_{s-2,r} Z_{3,r}$ with the identifications $U := U_{s-2} := Z_{1,1}$, $E_{s-2,-1} := -\sum_{r=0}^s Z_{2,s-r} Z_{2,r} + Z_{1,s} Z_{3,0}$ and $E_{s-2,r} := Z_{1,s-r}$ if $1 \leq r < s-1$ or $r = s$ and $E_{s-2,r} := 0$ otherwise. We also denote $E_{s-2} = E_{s-2,-1} + \sum_{r \in \mathbb{N}} E_{s-2,r} Z_{3,r}$.

Denote by $\{z_{3,s}; s \geq 1\}$ the unique family of elements of K such that for every $s \geq 2$, one has

$$\sum_{r=1}^{s-1} (Z_{1,s-r} \cdot z_{3,r} - Z_{2,s-r} Z_{2,r}) = 0.$$

Note that the latter is a triangular invertible K -linear system in the $z_{3,s}$'s.

Now let $\{\mathcal{Z}_{3,s}; s \geq 1\}$ be the unique family of elements of $K[[Z_{1,0}, Z_{2,0}, Z_{3,0}]]$ such that

1. for every $s \geq 1$, one has $\mathcal{Z}_{3,s} = z_{3,s} \pmod{\langle Z_{1,0}, Z_{2,0}, Z_{3,0} \rangle}$;
2. for every $s \geq 2$, one has

$$Z_{1,s} Z_{3,0} - Z_{2,0} Z_{2,s} + \sum_{r=1}^s (Z_{1,s-r} \cdot \mathcal{Z}_{3,r} - Z_{2,s-r} Z_{2,r}) = 0.$$

Explicit truncations of the series $\mathcal{Z}_{3,s}$ may be obtained by applying effectively the Hensel lemma, as in subsection 5.3.9. In this first example, let us compute the first few truncations of the elements $\mathcal{Z}_{3,s}$ for $s \geq 1$ in order to illustrate the recursive construction in 5.3.10.

We keep the notation in subsection 5.3.9. Although we have already given the expression of E_{s-2} for $s \geq 2$, it will be useful to display the following elements:

$$\begin{aligned} E_0 &= Z_{1,0}Z_{3,2} + Z_{1,2}Z_{3,0} - 2Z_{2,0}Z_{2,2} - Z_{2,1}^2 \\ E_1 &= Z_{1,0}Z_{3,3} + Z_{1,2}Z_{3,1} + Z_{1,3}Z_{3,0} - 2Z_{2,1}Z_{2,2} - 2Z_{2,0}Z_{2,3} \\ E_2 &= Z_{1,0}Z_{3,4} + Z_{1,2}Z_{3,2} + Z_{1,3}Z_{3,1} + Z_{1,4}Z_{3,0} - Z_{2,2}^2 - 2Z_{2,1}Z_{2,3} - 2Z_{2,0}Z_{2,4} \end{aligned}$$

We define $z_{3,1}^{(1)} := -\varsigma_1(\varepsilon_0^1(E_0))U^{-1}$. In this case the morphism ε_0^1 is the identity on $K[\mathbf{Z}_\infty]$ and we obtain

$$z_{3,1}^{(1)} = -\varsigma_1(Z_{1,0}Z_{3,2} + Z_{1,2}Z_{3,0} - 2Z_{2,0}Z_{2,2} - Z_{2,1}^2)Z_{1,1}^{-1} = \frac{Z_{2,1}^2}{Z_{1,1}}.$$

Now $z_{3,2}^{(1)} := -\varsigma_1(\varepsilon_1^1(E_1))U^{-1}$. By definition, $\varepsilon_1^1(Z_{3,1}) = z_{3,1}^{(1)}$ and ε_1^1 is the identity over any other variable of $K[\mathbf{Z}_\infty]$. Hence

$$\begin{aligned} z_{3,2}^{(1)} &= -\varsigma_1 \left(Z_{1,0}Z_{3,3} + \frac{Z_{1,2}Z_{2,1}^2}{Z_{1,1}} + Z_{1,3}Z_{3,0} - 2Z_{2,1}Z_{2,2} - 2Z_{2,0}Z_{2,3} \right) Z_{1,1}^{-1} \\ &= -\frac{Z_{1,2}Z_{2,1}^2}{Z_{1,1}^2} + \frac{2Z_{2,1}Z_{2,2}}{Z_{1,1}}. \end{aligned}$$

In the same way $z_{3,3}^{(1)} := -\varsigma_1(\varepsilon_2^1(E_2))U^{-1}$, and the morphism ε_2^1 is defined by $\varepsilon_2^1(Z_{3,1}) = z_{3,1}^{(1)}$, $\varepsilon_2^1(Z_{3,2}) = z_{3,2}^{(1)}$ and is the identity over any other variable of $K[\mathbf{Z}_\infty]$. Hence

$$\begin{aligned} z_{3,3}^{(1)} &= -\varsigma_1 \left(Z_{1,0}Z_{3,4} + Z_{1,2} \left(-\frac{Z_{1,2}Z_{2,1}^2}{Z_{1,1}^2} + \frac{2Z_{2,1}Z_{2,2}}{Z_{1,1}} \right) + \frac{Z_{1,3}Z_{2,1}^2}{Z_{1,1}} + Z_{1,4}Z_{3,0} - Z_{2,2}^2 \right. \\ &\quad \left. - 2Z_{2,1}Z_{2,3} - 2Z_{2,0}Z_{2,4} \right) Z_{1,1}^{-1} \\ &= \frac{Z_{1,2}^2Z_{2,1}^2}{Z_{1,1}^3} - \frac{2Z_{1,2}Z_{2,1}Z_{2,2}}{Z_{1,1}^2} - \frac{Z_{1,3}Z_{2,1}^2}{Z_{1,1}^2} + \frac{Z_{2,2}^2}{Z_{1,1}} + \frac{2Z_{2,1}Z_{2,3}}{Z_{1,1}}. \end{aligned}$$

Analogously we can compute $z_{3,s}^{(1)}$ for $s > 3$ and we observe that $z_{3,s} = z_{3,s}^{(1)}$ for $s \geq 1$. Let us now construct the truncations modulo $\langle Z_{1,0}, Z_{2,0}, Z_{3,0} \rangle^2$. We define $z_{3,1}^{(2)} := -\varsigma_2(\varepsilon_0^2(E_0))U^{-1}$. The morphism ε_0^2 is defined by $\varepsilon_0^2(Z_{3,r}) = z_{3,r}^{(1)}$ for $r \geq 1$

and is the identity over any other variable of $K[\mathbf{Z}_\infty]$. In particular its image lies in $K[Z_{1,0}, Z_{2,0}, Z_{3,0}]$. Thus

$$\begin{aligned} z_{3,1}^{(2)} &= -\varsigma_2 \left(Z_{1,0} \left(-\frac{Z_{1,2}Z_{2,1}^2}{Z_{1,1}^2} + \frac{2Z_{2,1}Z_{2,2}}{Z_{1,1}} \right) + Z_{1,2}Z_{3,0} - 2Z_{2,0}Z_{2,2} - Z_{2,1}^2 \right) Z_{1,1}^{-1} \\ &= \frac{Z_{2,1}^2}{Z_{1,1}} + \frac{Z_{1,0}Z_{1,2}Z_{2,1}^2}{Z_{1,1}^3} - \frac{2Z_{1,0}Z_{2,1}Z_{2,2}}{Z_{1,1}^2} - \frac{Z_{1,2}Z_{3,0}}{Z_{1,1}} + \frac{2Z_{2,0}Z_{2,2}}{Z_{1,1}}. \end{aligned}$$

Note that $z_{3,1}^{(2)} = z_{3,1}^{(1)} \pmod{\langle Z_{1,0}, Z_{2,0}, Z_{3,0} \rangle}$.

Finally $z_{3,2}^{(2)} := -\varsigma_2(\varepsilon_1^2(E_1))U^{-1}$. The morphism ε_1^2 is defined by $\varepsilon_1^2(Z_{3,1}) = z_{3,1}^{(2)}$, $\varepsilon_1^2(Z_{3,r}) = z_{3,r}^{(1)}$ for $r \geq 2$ and is the identity over any other variable of $K[\mathbf{Z}_\infty]$. Hence

$$\begin{aligned} z_{3,2}^{(2)} &= -\varsigma_2 \left(Z_{1,0} \left(\frac{Z_{1,2}^2Z_{2,1}^2}{Z_{1,1}^3} - \frac{2Z_{1,2}Z_{2,1}Z_{2,2}}{Z_{1,1}^2} - \frac{Z_{1,3}Z_{2,1}^2}{Z_{1,1}^2} + \frac{Z_{2,2}^2}{Z_{1,1}} + \frac{2Z_{2,1}Z_{2,3}}{Z_{1,1}} \right) \right. \\ &\quad \left. + Z_{1,2} \left(\frac{Z_{1,0}Z_{1,2}Z_{2,1}^2}{Z_{1,1}^3} - \frac{2Z_{1,0}Z_{2,1}Z_{2,2}}{Z_{1,1}^2} - \frac{Z_{1,2}Z_{3,0}}{Z_{1,1}} + \frac{2Z_{2,0}Z_{2,2}}{Z_{1,1}} + \frac{Z_{2,1}^2}{Z_{1,1}} \right) \right. \\ &\quad \left. + Z_{1,3}Z_{3,0} - 2Z_{2,1}Z_{2,2} - 2Z_{2,0}Z_{2,3} \right) Z_{1,1}^{-1} \\ &= -\frac{Z_{1,2}Z_{2,1}^2}{Z_{1,1}^2} + \frac{2Z_{2,1}Z_{2,2}}{Z_{1,1}} - \frac{2Z_{1,0}Z_{1,2}^2Z_{2,1}^2}{Z_{1,1}^4} + \frac{4Z_{1,0}Z_{1,2}Z_{2,1}Z_{2,2}}{Z_{1,1}^3} + \frac{Z_{1,0}Z_{1,3}Z_{2,1}^2}{Z_{1,1}^3} \\ &\quad - \frac{Z_{1,0}Z_{2,2}^2}{Z_{1,1}^2} - \frac{2Z_{1,0}Z_{2,1}Z_{2,3}}{Z_{1,1}^2} + \frac{Z_{1,2}^2Z_{3,0}}{Z_{1,1}^2} - \frac{2Z_{1,2}Z_{2,0}Z_{2,2}}{Z_{1,1}^2} - \frac{Z_{1,3}Z_{3,0}}{Z_{1,1}} + \frac{2Z_{2,0}Z_{2,3}}{Z_{1,1}}. \end{aligned}$$

And we also note that $z_{3,2}^{(2)} = z_{3,2}^{(1)} \pmod{\langle Z_{1,0}, Z_{2,0}, Z_{3,0} \rangle}$. In order to compute $z_{3,3}^{(2)}$ we have to obtain first $z_{3,4}^{(1)}$. Observe that, since we have computed $z_{3,1}^{(2)}$ and $z_{3,2}^{(2)}$, we can already obtain $z_{3,1}^{(3)} := -\varsigma_3(\varepsilon_0^3(E_0))U^{-1} = -\varsigma_3(Z_{1,0}z_{3,2}^{(2)} + Z_{1,2}Z_{3,0} - 2Z_{2,0}Z_{2,2} - Z_{2,1}^2)Z_{1,1}^{-1}$.

Hence one has

$$\begin{aligned} z_{3,1} &= \frac{Z_{2,1}^2}{Z_{1,1}} + \frac{Z_{1,0}Z_{1,2}Z_{2,1}^2}{Z_{1,1}^3} - \frac{2Z_{1,0}Z_{2,1}Z_{2,2}}{Z_{1,1}^2} - \frac{Z_{1,2}Z_{3,0}}{Z_{1,1}} + \frac{2Z_{2,0}Z_{2,2}}{Z_{1,1}} \pmod{\langle Z_{1,0}, Z_{2,0}, Z_{3,0} \rangle^2}, \\ z_{3,2} &= -\frac{Z_{1,2}Z_{2,1}^2}{Z_{1,1}^2} + \frac{2Z_{2,1}Z_{2,2}}{Z_{1,1}} - \frac{2Z_{1,0}Z_{1,2}^2Z_{2,1}^2}{Z_{1,1}^4} + \frac{4Z_{1,0}Z_{1,2}Z_{2,1}Z_{2,2}}{Z_{1,1}^3} + \frac{Z_{1,0}Z_{1,3}Z_{2,1}^2}{Z_{1,1}^3} \\ &\quad - \frac{Z_{1,0}Z_{2,2}^2}{Z_{1,1}^2} - \frac{2Z_{1,0}Z_{2,1}Z_{2,3}}{Z_{1,1}^2} + \frac{Z_{1,2}^2Z_{3,0}}{Z_{1,1}^2} - \frac{2Z_{1,2}Z_{2,0}Z_{2,2}}{Z_{1,1}^2} - \frac{Z_{1,3}Z_{3,0}}{Z_{1,1}} + \frac{2Z_{2,0}Z_{2,3}}{Z_{1,1}} \\ &\quad \pmod{\langle Z_{1,0}, Z_{2,0}, Z_{3,0} \rangle^2} \end{aligned}$$

and

$$z_{3,3} = \frac{Z_{1,2}^2Z_{2,1}^2}{Z_{1,1}^3} - \frac{2Z_{1,2}Z_{2,1}Z_{2,2}}{Z_{1,1}^2} - \frac{Z_{1,3}Z_{2,1}^2}{Z_{1,1}^2} + \frac{Z_{2,2}^2}{Z_{1,1}} + \frac{2Z_{2,1}Z_{2,3}}{Z_{1,1}} \pmod{\langle Z_{1,0}, Z_{2,0}, Z_{3,0} \rangle}.$$

We observe that explicit computations quickly become cumbersome.

Then the formal neighbourhood of $\eta_{\mathbf{n}}$ in $\mathcal{L}_{\infty}(V_{\sigma})$ is isomorphic to the formal spectrum of

$$K[[Z_{1,0}, Z_{2,0}, Z_{3,0}]]/\langle Z_{1,0}Z_{3,0} - Z_{2,0}^2, Z_{1,1}Z_{3,0} + Z_{1,0}Z_{3,1} - 2Z_{2,0}Z_{2,1} \rangle.$$

Note that it is not clear that the latter is the completion of an essentially of finite type local K -algebra.

Using our comparison theorem, the computation of the formal neighbourhood of $\eta_{\mathbf{n}}$ in $\mathcal{L}_{\infty}(V_{\sigma})$ may also be done in the following much more straightforward way. First we use construction 4.2.15 to compute the formal scheme $\mathcal{W}(\mathbf{n})$ defined in 5.5.13, which coincides with $\mathcal{W}((1, 1, 1), \{(1, 0, 1), (0, 2, 0)\})$ in the notation of construction 4.2.15. We have the following equality in $k[Z_{1,0}, Z_{2,0}, Z_{3,0}, t]$:

$$F|_{Z_j=t+Z_{j,0}} = (t + Z_{1,0})(t + Z_{3,0}) - (t + Z_{2,0})^2 = (Z_{1,0} + Z_{3,0} - 2Z_{2,0})t + Z_{1,0}Z_{3,0} - Z_{2,0}^2.$$

We deduce that

$$\mathcal{W}(\mathbf{n}) = \text{Spf} \left(\frac{k[[Z_{1,0}, Z_{2,0}, Z_{3,0}]]}{\langle Z_{1,0}Z_{3,0} - Z_{2,0}^2, Z_{1,0} + Z_{3,0} - 2Z_{2,0} \rangle} \right)$$

and then the formal neighbourhood of $\eta_{\mathbf{n}}$ in $\mathcal{L}_{\infty}(V_{\sigma})$ is isomorphic to

$$\text{Spf} \left(\frac{K[[Z_{1,0}, Z_{2,0}, Z_{3,0}]]}{\langle Z_{1,0}Z_{3,0} - Z_{2,0}^2, Z_{1,0} + Z_{3,0} - 2Z_{2,0} \rangle} \right).$$

In addition, it is not difficult to see that $\mathcal{W}(\mathbf{n})$ is isomorphic to $\text{Spf}(k[[Z_{1,0}, Z_{2,0}]]/\langle Z_{1,0}^2 \rangle)$.

5.6.2 Let us consider the following more general setting than in example 5.6.1 Let $N = M = \mathbb{Z}^2$, σ be the cone of \mathbb{R}^2 generated by $(1, 0)$ and $(1, d)$ for $d \geq 1$ and V_{σ} be the associated affine toric variety. The semigroup S_{σ} is minimally generated by $\mathbf{m}_1 = (0, 1)$, $\mathbf{m}_2 = (1, 0)$ and $\mathbf{m}_3 = (d, -1)$. We observe that \mathbf{m}_1 and \mathbf{m}_2 form a \mathbb{Z} -basis of M and the relation $\mathbf{m}_1 + \mathbf{m}_3 = d\mathbf{m}_2$ (corresponding to $\ell = (1, -d, 1)$) generates all the nontrivial relations between elements of S_{σ} . Thus, setting $F := Z_1Z_3 - Z_2^d$, the ideal of V_{σ} in $k[Z_1, Z_2, Z_3]$ is the ideal generated by F ; we observe that V_{σ} is a complete intersection. The ideal of $\mathcal{L}_{\infty}(V_{\sigma})$ in the ring $k[\mathbf{Z}_{\infty}] = k[Z_{1,s}, Z_{2,s}, Z_{3,s}; s \in \mathbb{N}]$ is generated by $\{F_s; s \in \mathbb{N}\}$ where

$$F_s = \sum_{r=0}^s Z_{1,s-r}Z_{3,r} - \sum_{r_1+\dots+r_d=s} Z_{2,r_1} \cdots Z_{2,r_d}.$$

We now consider the divisorial toric valuation $\text{ord}_{\mathbf{n}}$ of V_{σ} corresponding to $\mathbf{n} = (a, b) \in \sigma \cap N$; let us observe that $0 \leq b \leq da$. then $\langle \mathbf{m}_1, \mathbf{n} \rangle = b$, $\langle \mathbf{m}_2, \mathbf{n} \rangle = a$, $\langle \mathbf{m}_3, \mathbf{n} \rangle = da - b$ and $\langle \ell, \mathbf{n} \rangle = da$. The prime ideal of $\mathcal{O}(V_{\sigma})$ corresponding to the generic point $\eta_{\mathbf{n}}$ of the Nash set associated with $\text{ord}_{\mathbf{n}}$ is the radical of the image of the

ideal $\langle Z_{1,s_1}, Z_{2,s_2}, Z_{3,s_3}; 0 \leq s_1 < b, 0 \leq s_2 < a, 0 \leq s_3 < da - b \rangle$ of $k[\mathbf{Z}_\infty]$ (see proposition 5.5.2). The residue field of η_n is isomorphic to $K := k(Z_{1,s_1}, Z_{2,s_2}; s_1 \geq b, s_2 \geq a)$.

Corollary 5.5.11 assures the existence of a family of elements $\{\mathcal{Z}_{3,s_3}; s_3 \geq da - b\}$ of $K[[Z_{1,s_1}, Z_{2,s_2}, Z_{3,s_3}; 0 \leq s_1 < b, 0 \leq s_2 < a, 0 \leq s_3 < da - b]]$ such that the formal neighbourhood of η_n in $\mathcal{L}_\infty(V_\sigma)$ is isomorphic to the formal spectrum of

$$\frac{K[[Z_{1,s_1}, Z_{2,s_2}, Z_{3,s_3}; 0 \leq s_1 < b, 0 \leq s_2 < a, 0 \leq s_3 < da - b]]}{\left\langle \sum_{r=0}^{\min(da-b-1,s)} Z_{1,s-r} Z_{3,r} + \sum_{r=\min(da-b-1,s)}^s Z_{1,s-r} \mathcal{Z}_{3,r} - \sum_{r_1+\dots+r_d=s} Z_{2,r_1} \cdots Z_{2,r_d}; 0 \leq s < da \right\rangle}.$$

Note that the truncations of the elements in the family $\{\mathcal{Z}_{3,s_3}; s_3 \geq da - b\}$ modulo $\langle Z_{1,s_1}, Z_{2,s_2}, Z_{3,s_3}; 0 \leq s_1 < b, 0 \leq s_2 < a, 0 \leq s_3 < da - b \rangle^e$ for $e \geq 1$ may be explicitly computed using subsection 5.3.9, as we did in example 5.6.1 for $d = 2$ and $a = b = 1$.

Using our comparison theorem 5.5.19 we deduce that the computation of the formal neighbourhood of η_n in $\mathcal{L}_\infty(V_\sigma)$ may also be done in the following way. First we compute the formal scheme $\mathcal{W}(\mathbf{n})$ defined in 5.5.13, which coincides with

$$\mathcal{W}((b, a, da - b), \{((1, 0, 1), (0, d, 0))\})$$

in the notation of construction 4.2.15. We have the following equality in the polynomial ring $k[Z_{1,s_1}, Z_{2,s_2}, Z_{3,s_3}, t; 0 \leq s_1 < b, 0 \leq s_2 < a, 0 \leq s_3 < da - b]$:

$$\begin{aligned} F \Big|_{Z_j=t^{\langle m_j, \mathbf{n} \rangle} + \sum_{s_j=0}^{\langle m_j, \mathbf{n} \rangle - 1} Z_{j,s_j} t^{s_j}} &= \left(t^b + \sum_{s_1=0}^{b-1} Z_{1,s_1} t^{s_1} \right) \left(t^{da-b} + \sum_{s_3=0}^{da-b-1} Z_{3,s_3} t^{s_3} \right) - \left(t^a + \sum_{s_2=0}^{a-1} Z_{2,s_2} t^{s_2} \right)^d \\ &=: \sum_{s=0}^{da-1} \tilde{F}_s t^s, \end{aligned}$$

where $\tilde{F}_s \in k[Z_{1,s_1}, Z_{2,s_2}, Z_{3,s_3}; 0 \leq s_1 < b, 0 \leq s_2 < a, 0 \leq s_3 < da - b]$ for every $s < da$.

We deduce that

$$\mathcal{W}(\mathbf{n}) = \text{Spf} \left(\frac{k[[Z_{1,s_1}, Z_{2,s_2}, Z_{3,s_3}; 0 \leq s_1 < b, 0 \leq s_2 < a, 0 \leq s_3 < da - b]]}{\langle \tilde{F}_s; 0 \leq s < da \rangle} \right)$$

and then the formal neighbourhood of η_n in $\mathcal{L}_\infty(V_\sigma)$ is isomorphic to

$$\text{Spf} \left(\frac{K[[Z_{1,s_1}, Z_{2,s_2}, Z_{3,s_3}; 0 \leq s_1 < b, 0 \leq s_2 < a, 0 \leq s_3 < da - b]]}{\langle \tilde{F}_s; 0 \leq s < da \rangle} \right).$$

5.6.3 Let us present an example which illustrates the passage to the complete intersection setting. Let $N = M = \mathbb{Z}^2$, σ be the cone of \mathbb{R}^2 generated by $(1, 0)$ and $(2, 3)$ and V_σ be the associated affine toric variety. The dual cone σ^\vee is generated by $(0, 1)$

and $(3, -2)$ and the semigroup S_σ is minimally generated by $\mathbf{m}_1 = (1, 0)$, $\mathbf{m}_2 = (0, 1)$, $\mathbf{m}_3 = (2, -1)$ and $\mathbf{m}_4 = (3, -2)$. We observe that \mathbf{m}_1 and \mathbf{m}_2 form a \mathbb{Z} -basis of M . We find the relations $\mathbf{m}_2 + \mathbf{m}_3 = 2\mathbf{m}_1$, $2\mathbf{m}_2 + \mathbf{m}_4 = 3\mathbf{m}_1$, $\mathbf{m}_1 + \mathbf{m}_4 = 2\mathbf{m}_3$ and $\mathbf{m}_2 + \mathbf{m}_4 = \mathbf{m}_1 + \mathbf{m}_3$ between generators of S_σ ; observe that the first two relations correspond to the elements $\mathbf{l}_3 = (-2, 1, 1, 0)$ and $\mathbf{l}_4 = (-3, 2, 0, 1)$ of L in the notation of subsection 5.2.7. The preceding relations give respectively rise to the polynomials $F_3 := Z_2Z_3 - Z_1^2$, $F_4 = Z_2^2Z_4 - Z_1^3$, $G := Z_1Z_4 - Z_3^2$ and $Z_2Z_4 - Z_1Z_3$ of $k[Z_1, Z_2, Z_3, Z_4]$. Observe that, by construction of \mathbf{l}_3 (resp. \mathbf{l}_4), the variable Z_4 (resp. Z_3) does not appear in F_3 (resp. F_4). The ideal of V_σ in $k[Z_1, Z_2, Z_3, Z_4]$ is $\langle F_3, F_4, G, H \rangle = \langle F_3, G, H \rangle$, hence

$$V_\sigma = \text{Spec}(k[Z_1, Z_2, Z_3, Z_4]/\langle F_3, G, H \rangle).$$

Since $\dim(V_\sigma) = \dim(\sigma)$, the surface V_σ is not a complete intersection. According with subsection 5.5.3 we denote by W the affine variety defined by the ideal $\langle F_3, F_4 \rangle$ of $k[Z_1, Z_2, Z_3, Z_4]$, it clearly contains V_σ as a closed subscheme.

The ideal of $\mathcal{L}_\infty(V_\sigma)$ (resp. $\mathcal{L}_\infty(W)$) in the ring $k[\mathbf{Z}_\infty] = k[Z_{1,s}, Z_{2,s}, Z_{3,s}, Z_{4,s}; s \in \mathbb{N}]$ is $\langle F_{3,s}, G_s, H_s; s \in \mathbb{N} \rangle$ (resp. $\langle F_{3,s}, F_{4,s}; s \in \mathbb{N} \rangle$), where, for $s \in \mathbb{N}$,

$$\begin{aligned} F_{3,s} &= \sum_{r=0}^s (Z_{2,s-r}Z_{3,r} - Z_{1,s-r}Z_{1,r}), \\ F_{4,s} &= \sum_{r_1+r_2+r_3=s} (Z_{2,r_1}Z_{2,r_2}Z_{4,r_3} - Z_{1,r_1}Z_{1,r_2}Z_{1,r_3}), \\ G_s &= \sum_{r=0}^s (Z_{1,s-r}Z_{4,r} - Z_{3,s-r}Z_{3,r}), \\ H_s &= \sum_{r=0}^s (Z_{2,s-r}Z_{4,r} - Z_{1,s-r}Z_{3,r}). \end{aligned}$$

We now consider the divisorial toric valuation ord_n of V_σ corresponding to $\mathbf{n} = (1, 1) \in \sigma \cap N$, then $\langle \mathbf{m}_i, \mathbf{n} \rangle = 1$ for $i = 1, 2, 3, 4$, $\langle \mathbf{l}_3, \mathbf{n} \rangle = 2$ and $\langle \mathbf{l}_4, \mathbf{n} \rangle = 3$. The prime ideal of $\mathcal{O}(V_\sigma)$ corresponding to the generic point η_n of the Nash set associated with ord_n is the radical of the image of the ideal $\langle Z_{1,0}, Z_{2,0}, Z_{3,0}, Z_{4,0} \rangle$ of $k[\mathbf{Z}_\infty]$ (see proposition 5.5.2). Let η'_n be the image of η_n via the closed immersion $\mathcal{L}_\infty(V_\sigma) \rightarrow \mathcal{L}_\infty(W)$, by lemma 5.5.6 the prime ideal of $\mathcal{O}(W)$ corresponding to η'_n is also the radical of the image of the ideal $\langle Z_{1,0}, Z_{2,0}, Z_{3,0}, Z_{4,0} \rangle$ of $k[\mathbf{Z}_\infty]$. The residue field of η_n (and η'_n) is isomorphic to $K := k(Z_{1,s}, Z_{2,s}; s \geq 1)$.

Corollary 5.5.11 assures the existence of a family of elements $\{\mathcal{Z}_{3,s}\mathcal{Z}_{4,s}; s \geq 1\}$ of $K[[Z_{1,0}, Z_{2,0}, Z_{3,0}, Z_{4,0}]]$ such that the formal neighbourhoods of η'_n in $\mathcal{L}_\infty(W)$ and of η_n in $\mathcal{L}_\infty(V_\sigma)$ are isomorphic to the formal spectrum of

$$\frac{K[[Z_{1,0}, Z_{2,0}, Z_{3,0}, Z_{4,0}]]}{\left\langle \begin{aligned} &Z_{3,0}Z_{2,0} - Z_{1,0}^2, Z_{3,0}Z_{2,1} + \mathcal{Z}_{3,1}Z_{2,0} - 2Z_{1,0}Z_{1,1}, \\ &Z_{4,0}Z_{2,0}^2 - Z_{1,0}^3, 2Z_{4,0}Z_{2,0}Z_{2,1} + \mathcal{Z}_{4,1}Z_{2,0}^2 - 3Z_{1,0}^2Z_{1,1}, \\ &2Z_{4,0}Z_{2,0}Z_{2,2} + 2Z_{4,0}Z_{2,1}^2 + 2\mathcal{Z}_{4,1}Z_{2,0}Z_{2,1} + \mathcal{Z}_{4,2}Z_{2,0}^2 - 3Z_{1,0}^2Z_{1,2} - 3Z_{1,0}Z_{1,1}^2 \end{aligned} \right\rangle}.$$

The construction in subsection 5.3.9 shows how to explicitly compute the truncations modulo $\langle Z_{1,0}, Z_{2,0}, Z_{3,0}, Z_{4,0} \rangle^e$ for $e \geq 1$ of the elements of the family $\{\mathcal{Z}_{3,s}, \mathcal{Z}_{4,s}; s \geq 1\}$. Let us compute some of them. In the notation of theorem 5.3.7 and subsection 5.3.9 we have $\Gamma = \{3, 4\}$ and for $s \in \mathbb{N}$ we have $Y_{\gamma,s} := Z_{\gamma,s+1}$ for $\gamma \in \{3, 4\}$, $H_{3,s} := F_{3,s+2}$, $H_{4,s} := F_{4,s+3}$, $U_{3,s} := U_3 := Z_{2,1}$ and $U_{4,s} := U_4 := Z_{2,1}^2$. Then $F_{3,s+2} = Z_{3,s+1}U_{3,s} + E_{3,s}$ and $F_{4,s+3} = Z_{4,s+1}U_{4,s} + E_{4,s}$ for $s \in \mathbb{N}$, it will be useful to display some of the $E_{\gamma,s}$:

$$E_{3,0} = Z_{3,0}Z_{2,2} + Z_{3,2}Z_{2,0} - 2Z_{1,0}Z_{1,2} - Z_{1,1}^2,$$

$$E_{3,1} = Z_{3,0}Z_{2,3} + Z_{3,1}Z_{2,2} + Z_{3,3}Z_{2,0} - 2Z_{1,0}Z_{1,3} - 2Z_{1,1}Z_{1,2},$$

$$E_{4,0} = 2Z_{4,0}Z_{2,0}Z_{2,3} + 2Z_{4,0}Z_{2,1}Z_{2,2} + 2Z_{4,1}Z_{2,0}Z_{2,2} + 2Z_{4,2}Z_{2,0}Z_{2,1} + Z_{4,3}Z_{2,0}^2 - 3Z_{1,0}^2Z_{1,3} - 6Z_{1,0}Z_{1,1}Z_{1,2} - Z_{1,1}^3,$$

$$E_{4,1} = 2Z_{4,0}Z_{2,0}Z_{2,4} + 2Z_{4,0}Z_{2,1}Z_{2,3} + 2Z_{4,1}Z_{2,0}Z_{2,3} + Z_{4,0}Z_{2,2}^2 + 2Z_{4,1}Z_{2,1}Z_{2,2} + 2Z_{4,2}Z_{2,0}Z_{2,2} + 2Z_{4,3}Z_{2,0}Z_{2,1} + Z_{4,4}Z_{2,0}^2 - 3Z_{1,0}^2Z_{1,4} - 6Z_{1,0}Z_{1,1}Z_{1,3} - 3Z_{1,0}Z_{1,2}^2 - 3Z_{1,1}^2Z_{1,2},$$

$$E_{4,2} = 2Z_{4,0}Z_{2,0}Z_{2,5} + 2Z_{4,0}Z_{2,1}Z_{2,4} + 2Z_{4,1}Z_{2,0}Z_{2,4} + 2Z_{4,0}Z_{2,2}Z_{2,3} + 2Z_{4,1}Z_{2,1}Z_{2,3} + 2Z_{4,2}Z_{2,0}Z_{2,3} + Z_{4,1}Z_{2,2}^2 + 2Z_{4,2}Z_{2,1}Z_{2,2} + 2Z_{4,3}Z_{2,0}Z_{2,2} + 2Z_{4,4}Z_{2,0}Z_{2,1} + Z_{4,5}Z_{2,0}^2 - 3Z_{1,0}^2Z_{1,5} - 6Z_{1,0}Z_{1,1}Z_{1,4} - 6Z_{1,0}Z_{1,2}Z_{1,3} - 3Z_{1,1}^2Z_{1,3} - 3Z_{1,1}Z_{1,2}^2.$$

We observe that the form of F_3 is analogous to that of the only equation in example 5.6.1, hence the same procedure gives

$$\begin{aligned} \mathcal{Z}_{3,1} &= \frac{Z_{1,1}^2}{Z_{2,1}} + \frac{2Z_{1,0}Z_{1,2}}{Z_{2,1}} + \frac{Z_{2,0}Z_{2,2}Z_{1,1}^2}{Z_{2,1}^3} - \frac{2Z_{2,0}Z_{1,1}Z_{1,2}}{Z_{2,1}^2} - \frac{Z_{3,0}Z_{2,2}}{Z_{2,1}} \pmod{\langle Z_{1,0}, Z_{2,0}, Z_{3,0}, Z_{4,0} \rangle^2} \\ \mathcal{Z}_{3,2} &= -\frac{Z_{2,2}Z_{1,1}^2}{Z_{2,1}^2} + \frac{2Z_{1,1}Z_{1,2}}{Z_{2,1}} \pmod{\langle Z_{1,0}, Z_{2,0}, Z_{3,0}, Z_{4,0} \rangle} \end{aligned}$$

Let us now compute some truncations of $\mathcal{Z}_{4,1}$. Observe that, since the elements $Z_{4,2}$ and $Z_{4,3}$ appear in $E_{4,0}$, in order to compute $\mathcal{Z}_{4,1} = z_{4,1}^{(2)} \pmod{\langle Z_{1,0}, Z_{2,0}, Z_{3,0}, Z_{4,0} \rangle^2}$ we need $z_{4,2}^{(1)}$ and $z_{4,3}^{(1)}$.

We define $z_{4,1}^{(1)} := -\varsigma_1(\varepsilon_0^1(E_{4,0}))U_4^{-1}$. The morphism ε_0^1 is the identity on $K[\mathbf{Z}_\infty]$ so

$$z_{4,1}^{(1)} = -\varsigma_1(E_{4,0})Z_{2,1}^{-2} = \frac{Z_{1,1}^3}{Z_{2,1}^2}.$$

Now $z_{4,2}^{(1)} := -\varsigma_1(\varepsilon_1^1(E_{4,1}))U_4^{-1}$. By definition, $\varepsilon_1^1(\mathcal{Z}_{3,1}) = z_{3,1}^{(1)}$, $\varepsilon_1^1(\mathcal{Z}_{4,1}) = z_{4,1}^{(1)}$ and ε_1^1 is the identity over any other variable of $K[\mathbf{Z}_\infty]$. Hence

$$z_{4,2}^{(1)} = -\varsigma_1(\varepsilon_1^1(E_{4,1}))Z_{2,1}^{-2} = \frac{3Z_{1,1}^2Z_{1,2}}{Z_{2,1}^2} - \frac{2Z_{2,2}Z_{1,1}^3}{Z_{2,1}^3}.$$

In the same way, $z_{4,3}^{(1)} := -\varsigma_1(\varepsilon_1^2(E_{4,2}))U_4^{-1}$ and in this case $\varepsilon_1^2(Z_{\gamma,1}) = z_{\gamma,1}^{(1)}$, $\varepsilon_1^2(Z_{\gamma,2}) = z_{\gamma,2}^{(1)}$ for $\gamma \in \{3, 4\}$ and ε_1^2 is the identity over any other variable of $K[\mathbf{Z}_\infty]$. Hence

$$z_{4,3}^{(1)} = -\varsigma_1(\varepsilon_1^2(E_{4,2}))Z_{2,1}^{-2} = \frac{3Z_{1,1}^2Z_{1,3}}{Z_{2,1}^2} + \frac{3Z_{1,1}Z_{1,2}^2}{Z_{2,1}^2} - \frac{6Z_{2,2}Z_{1,1}^2Z_{1,2}}{Z_{2,1}^3} - \frac{2Z_{2,3}Z_{1,1}^3}{Z_{2,1}^3} + \frac{3Z_{2,2}^2Z_{1,1}^3}{Z_{2,1}^4}.$$

We can compute in an analogous way $z_{4,s}^{(1)}$ for $s > 3$. Let us compute $z_{4,1}^{(2)} := -\varsigma_2(\varepsilon_0^2(E_{4,0}))U_4^{-1}$. In this case, $\varepsilon_0^2(Z_{\gamma,s}) = z_{\gamma,s}^{(1)}$ for $\gamma \in \{3, 4\}$ and $s \geq 1$ and ε_0^2 is the identity over any other variable of $K[\mathbf{Z}_\infty]$. Thus

$$z_{4,1}^{(2)} = -\varsigma_2(\varepsilon_0^2(E_{4,0}))Z_{2,1}^{-2} = \frac{Z_{1,1}^3}{Z_{2,1}^2} - \frac{6Z_{2,0}Z_{1,1}^2Z_{1,2}}{Z_{2,1}^3} + \frac{6Z_{1,0}Z_{1,1}Z_{1,2}}{Z_{2,1}^2} + \frac{2Z_{2,0}Z_{2,2}Z_{1,1}^3}{Z_{2,1}^4} - \frac{2Z_{4,0}Z_{2,2}}{Z_{2,1}}.$$

We observe that $z_{4,1}^{(2)} = z_{4,1}^{(1)} \pmod{\langle Z_{1,0}, Z_{2,0}, Z_{3,0}, Z_{4,0} \rangle}$. We compute the successive truncations $z_{4,s}^{(e)}$ for $s, e \geq 1$ in an analogous way. Hence we have

$$\begin{aligned} \mathcal{Z}_{4,1} &= \frac{Z_{1,1}^3}{Z_{2,1}^2} - \frac{6Z_{2,0}Z_{1,1}^2Z_{1,2}}{Z_{2,1}^3} + \frac{6Z_{1,0}Z_{1,1}Z_{1,2}}{Z_{2,1}^2} + \frac{2Z_{2,0}Z_{2,2}Z_{1,1}^3}{Z_{2,1}^4} - \frac{2Z_{4,0}Z_{2,2}}{Z_{2,1}} \\ &\pmod{\langle Z_{1,0}, Z_{2,0}, Z_{3,0}, Z_{4,0} \rangle^2} \end{aligned}$$

$$\mathcal{Z}_{4,2} = \frac{3Z_{1,1}^2Z_{1,2}}{Z_{2,1}^2} - \frac{2Z_{2,2}Z_{1,1}^3}{Z_{2,1}^3} \pmod{\langle Z_{1,0}, Z_{2,0}, Z_{3,0}, Z_{4,0} \rangle}$$

$$\begin{aligned} \mathcal{Z}_{4,2} &= \frac{3Z_{1,1}^2Z_{1,3}}{Z_{2,1}^2} + \frac{3Z_{1,1}Z_{1,2}^2}{Z_{2,1}^2} - \frac{6Z_{2,2}Z_{1,1}^2Z_{1,2}}{Z_{2,1}^3} - \frac{2Z_{2,3}Z_{1,1}^3}{Z_{2,1}^3} + \frac{3Z_{2,2}^2Z_{1,1}^3}{Z_{2,1}^4} \\ &\pmod{\langle Z_{1,0}, Z_{2,0}, Z_{3,0}, Z_{4,0} \rangle}. \end{aligned}$$

Let us stress that, according with subsection 5.3.9, the computation of $z_{3,s}^{(e)}$ and $z_{4,s}^{(e)}$ for $s, e \geq 1$ should be done simultaneously so that the morphism ε_r^e for $r \in \mathbb{N}$ and $e \geq 1$ is well defined at each step. However, since the variables $Z_{3,r}$ (resp. $Z_{4,r}$) do not appear in $F_{4,s}$ (resp. $F_{3,s}$) for $r, s \in \mathbb{N}$ because of the definition of F_4 (resp. F_3), the effective construction of $z_{3,s}^{(e)}$ and $z_{4,s}^{(e)}$ may be done separately by defining ε_r^e to be the identity over the non-concerned variables in each case.

Using our comparison theorem 5.5.19, the computation of the formal neighbourhood of $\eta_{\mathbf{n}}$ in $\mathcal{L}_\infty(V_\sigma)$ may also be done in the following much more straightforward way. First we compute the formal scheme $\mathcal{W}(\mathbf{n})$ defined in subsection 5.5.13, which coincides with

$$\mathcal{W}((1, 1, 1, 1), \{((0, 1, 1, 0), (2, 0, 0, 0)), ((0, 2, 0, 1), (3, 0, 0, 0))\})$$

in the notation of construction 4.2.15. We have the following equality in the polynomial ring $k[Z_{1,0}, Z_{2,0}, Z_{3,0}, Z_{4,0}, t]$:

$$F_3|_{Z_j=t+Z_{j,0}} = (t + Z_{2,0})(t + Z_{3,0}) - (t + Z_{1,0})^2 = (Z_{2,0} + Z_{3,0} - 2Z_{1,0})t + Z_{2,0}Z_{3,0} - Z_{1,0}^2.$$

$$\begin{aligned} F_4|_{Z_j=t+Z_{j,0}} &= (t + Z_{2,0})^2(t + Z_{4,0}) - (t + Z_{1,0})^3 \\ &= (2Z_{2,0} + Z_{4,0} - 3Z_{1,0})t^2 + (Z_{2,0}^2 + 2Z_{2,0}Z_{4,0} - 3Z_{1,0}^2)t + Z_{2,0}^2Z_{4,0} - Z_{1,0}^3. \end{aligned}$$

We deduce that

$$\mathcal{W}(\mathbf{n}) = \text{Spf} \left(\frac{k[[Z_{1,0}, Z_{2,0}, Z_{3,0}, Z_{4,0}]]}{\left\langle \begin{array}{l} Z_{2,0}Z_{3,0} - Z_{1,0}^2, Z_{2,0} + Z_{3,0} - 2Z_{1,0} \\ Z_{2,0}^2Z_{4,0} - Z_{1,0}^3, Z_{2,0}^2 + 2Z_{2,0}Z_{4,0} - 3Z_{1,0}^2, 2Z_{2,0} + Z_{4,0} - 3Z_{1,0} \end{array} \right\rangle} \right)$$

and hence the formal neighbourhood of $\eta_{\mathbf{n}}$ in $\mathcal{L}_{\infty}(V_{\sigma})$ (and also that of $\eta'_{\mathbf{n}}$ in $\mathcal{L}_{\infty}(W)$) is isomorphic to

$$\text{Spf} \left(\frac{K[[Z_{1,0}, Z_{2,0}, Z_{3,0}, Z_{4,0}]]}{\left\langle \begin{array}{l} Z_{2,0}Z_{3,0} - Z_{1,0}^2, Z_{2,0} + Z_{3,0} - 2Z_{1,0} \\ Z_{2,0}^2Z_{4,0} - Z_{1,0}^3, Z_{2,0}^2 + 2Z_{2,0}Z_{4,0} - 3Z_{1,0}^2, 2Z_{2,0} + Z_{4,0} - 3Z_{1,0} \end{array} \right\rangle} \right).$$

BIBLIOGRAPHY

- [1] Tomoyuki Arakawa and Andrew R. Linshaw, *Singular support of a vertex algebra and the arc space of its associated scheme*, Representations and nilpotent orbits of Lie algebraic systems, Progr. Math., vol. 330, Birkhäuser/Springer, Cham, 2019, pp. 1–17. MR 3971726
- [2] Victor V. Batyrev, *Birational Calabi-Yau n -folds have equal Betti numbers*, New trends in algebraic geometry (Warwick, 1996), London Math. Soc. Lecture Note Ser., vol. 264, Cambridge Univ. Press, Cambridge, 1999, pp. 1–11. MR 1714818
- [3] Thomas Becker and Volker Weispfenning, *Gröbner bases*, Graduate Texts in Mathematics, vol. 141, Springer-Verlag, New York, 1993, A computational approach to commutative algebra.
- [4] I. N. Bernšteĭn, *Analytic continuation of generalized functions with respect to a parameter*, Funkcional. Anal. i Priložen. **6** (1972), no. 4, 26–40. MR 0320735
- [5] Bhargav Bhatt, *Algebraization and Tannaka duality*, Camb. J. Math. **4** (2016), no. 4, 403–461. MR 3572635
- [6] Siegfried Bosch, Werner Lütkebohmert, and Michel Raynaud, *Néron models*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 21, Springer-Verlag, Berlin, 1990. MR 1045822
- [7] David Bourqui and Mercedes Haiech, *On the nilpotent functions at a non-degenerate arc*, Manuscripta Math. (2020), in press.
- [8] David Bourqui, Mario Morán Cañón, and Julien Sebag, *On the behaviour of formal neighborhoods in the Nash sets associated with toric valuations: a comparison theorem*, preprint.
- [9] David Bourqui and Julien Sebag, *A SAGE implementation of Drinfeld’s arguments, and some variations*, available at <https://perso.univ-rennes1.fr/david.bourqui/sageDGK.pdf>.
- [10] ———, *The Drinfeld-Grinberg-Kazhdan theorem for formal schemes and singularity theory*, Confluentes Math. **9** (2017), no. 1, 29–64.
- [11] ———, *The Drinfeld-Grinberg-Kazhdan theorem is false for singular arcs*, J. Inst. Math. Jussieu **16** (2017), no. 4, 879–885. MR 3680347
- [12] ———, *Smooth arcs on algebraic varieties*, J. Singul. **16** (2017), 130–140. MR 3670512

- [13] ———, *Finite formal model of toric singularities*, J. Math. Soc. Japan **71** (2019), no. 3, 805–829. MR 3984243
- [14] ———, *Arc schemes of affine algebraic plane curves and torsion Kähler differential forms*, Arc Schemes and Singularities, World Scientific (Europe), 2020, pp. 99–111.
- [15] ———, *The local structure of arc schemes*, Arc Schemes and Singularities, World Scientific (Europe), 2020, pp. 69–97.
- [16] A. Bouthier, B. C. Ngô, and Y. Sakellaridis, *On the formal arc space of a reductive monoid*, Amer. J. Math. **138** (2016), no. 1, 81–108. MR 3462881
- [17] Nero Budur, *On the V -filtration of \mathcal{D} -modules*, Geometric methods in algebra and number theory, Progr. Math., vol. 235, Birkhäuser Boston, Boston, MA, 2005, pp. 59–70. MR 2159377
- [18] Leonardo Meireles Câmara, *On the classification of quasi-homogeneous curves*, preprint, <https://arxiv.org/pdf/1009.1664.pdf>.
- [19] Antoine Chambert-Loir, Johannes Nicaise, and Julien Sebag, *Motivic integration*, Progress in Mathematics, vol. 325, Birkhäuser/Springer, New York, 2018. MR 3838446
- [20] Christopher Chiu and Herwig Hauser, *On the formal neighborhood of degenerate arcs*, preprint.
- [21] S. C. Coutinho, *A primer of algebraic D -modules*, London Mathematical Society Student Texts, vol. 33, Cambridge University Press, Cambridge, 1995. MR 1356713
- [22] David Cox, John Little, and Donal O’Shea, *Ideals, varieties, and algorithms*, Undergraduate Texts in Mathematics, Springer-Verlag, New York, 1992, An introduction to computational algebraic geometry and commutative algebra.
- [23] David A. Cox, John B. Little, and Henry K. Schenck, *Toric varieties*, Graduate Studies in Mathematics, vol. 124, American Mathematical Society, Providence, RI, 2011. MR 2810322
- [24] Tommaso de Fernex, *Three-dimensional counter-examples to the Nash problem*, Compos. Math. **149** (2013), no. 9, 1519–1534. MR 3109732
- [25] Tommaso de Fernex and Roi Docampo, *Terminal valuations and the Nash problem*, Invent. Math. **203** (2016), no. 1, 303–331. MR 3437873
- [26] ———, *Differentials on the arc space*, Duke Math. J. **169** (2020), no. 2, 353–396. MR 4057146
- [27] Tommaso de Fernex, Lawrence Ein, and Shihoko Ishii, *Divisorial valuations via arcs*, Publ. Res. Inst. Math. Sci. **44** (2008), no. 2, 425–448. MR 2426354
- [28] Tommaso de Fernex and Mircea Mustața, *The volume of a set of arcs on a variety*, Rev. Roumaine Math. Pures Appl. **60** (2015), no. 3, 375–401. MR 3436273

- [29] Jan Denef and François Loeser, *Germes of arcs on singular algebraic varieties and motivic integration*, *Invent. Math.* **135** (1999), no. 1, 201–232. MR 1664700
- [30] Vladimir Drinfeld, *The Grinberg–Kazhdan formal arc theorem and the Newton groupoids*, *Arc Schemes and Singularities*, World Scientific (Europe), 2020, pp. 37–56.
- [31] Lawrence Ein, Robert Lazarsfeld, and Mircea Mustață, *Contact loci in arc spaces*, *Compos. Math.* **140** (2004), no. 5, 1229–1244. MR 2081163
- [32] Lawrence Ein and Mircea Mustață, *Jet schemes and singularities*, *Algebraic geometry—Seattle 2005. Part 2*, *Proc. Sympos. Pure Math.*, vol. 80, Amer. Math. Soc., Providence, RI, 2009, pp. 505–546. MR 2483946
- [33] Javier Fernández de Bobadilla and María Pe Pereira, *The Nash problem for surfaces*, *Ann. of Math. (2)* **176** (2012), no. 3, 2003–2029. MR 2979864
- [34] M. Grinberg and D. Kazhdan, *Versal deformations of formal arcs*, *Geom. Funct. Anal.* **10** (2000), no. 3, 543–555.
- [35] Michel Gros, Luis Narváez Macarro, and Julien Sebag, *Arc scheme and Bernstein operators*, *Arc Schemes and Singularities*, World Scientific (Europe), 2020, pp. 279–295.
- [36] A. Grothendieck, *Éléments de géométrie algébrique. I. Le langage des schémas*, *Inst. Hautes Études Sci. Publ. Math.* (1960), no. 4, 228. MR 217083
- [37] ———, *Éléments de géométrie algébrique. III. Étude cohomologique des faisceaux cohérents. I*, *Inst. Hautes Études Sci. Publ. Math.* (1961), no. 11, 167. MR 217085
- [38] ———, *Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. I*, *Inst. Hautes Études Sci. Publ. Math.* (1964), no. 20, 259. MR 173675
- [39] ———, *Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. II*, *Inst. Hautes Études Sci. Publ. Math.* (1965), no. 24, 231. MR 199181
- [40] ———, *Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. III*, *Inst. Hautes Études Sci. Publ. Math.* (1966), no. 28, 255. MR 217086
- [41] A. Grothendieck and J. A. Dieudonné, *Éléments de géométrie algébrique. I*, *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*, vol. 166, Springer-Verlag, Berlin, 1971. MR 3075000
- [42] Alexander Grothendieck, *Généralités sur les groupes algébriques affines. groupes algébriques affines commutatifs*, *Séminaire Claude Chevalley* **1** (1956–1958) (fr), exposé 4.
- [43] Mercedes Haiech, *Non-complete completions*, *Arc Schemes and Singularities*, World Scientific (Europe), 2020, pp. 57–68.

- [44] Eloise Hamann, *On power-invariance*, Pacific J. Math. **61** (1975), no. 1, 153–159. MR 407009
- [45] James E. Humphreys, *Linear algebraic groups*, Springer-Verlag, New York-Heidelberg, 1975, Graduate Texts in Mathematics, No. 21. MR 0396773
- [46] Shihoko Ishii, *The arc space of a toric variety*, J. Algebra **278** (2004), no. 2, 666–683. MR 2071659
- [47] ———, *Arcs, valuations and the Nash map*, J. Reine Angew. Math. **588** (2005), 71–92. MR 2196729
- [48] ———, *Jet schemes, arc spaces and the Nash problem*, C. R. Math. Acad. Sci. Soc. R. Can. **29** (2007), no. 1, 1–21. MR 2354631
- [49] ———, *Maximal divisorial sets in arc spaces*, Algebraic geometry in East Asia—Hanoi 2005, Adv. Stud. Pure Math., vol. 50, Math. Soc. Japan, Tokyo, 2008, pp. 237–249. MR 2409559
- [50] Shihoko Ishii and János Kollár, *The Nash problem on arc families of singularities*, Duke Math. J. **120** (2003), no. 3, 601–620.
- [51] Jennifer M. Johnson and János Kollár, *Arc spaces of cA -type singularities*, J. Singul. **7** (2013), 238–252. MR 3094648
- [52] Irving Kaplansky, *An introduction to differential algebra*, second ed., Hermann, Paris, 1976, Actualités Scientifiques et Industrielles, No. 1251, Publications de l’Institut de Mathématique de l’Université de Nancago, No. V. MR 0460303
- [53] M. Kashiwara, *Vanishing cycle sheaves and holonomic systems of differential equations*, Algebraic geometry (Tokyo/Kyoto, 1982), Lecture Notes in Math., vol. 1016, Springer, Berlin, 1983, pp. 134–142. MR 726425
- [54] E. R. Kolchin, *Differential algebra and algebraic groups*, Academic Press, New York-London, 1973, Pure and Applied Mathematics, Vol. 54.
- [55] Kodjo Kpogon and Julien Sebag, *Nilpotency in arc scheme of plane curves*, Comm. Algebra **45** (2017), no. 5, 2195–2221.
- [56] Serge Lang, *Algebra*, third ed., Graduate Texts in Mathematics, vol. 211, Springer-Verlag, New York, 2002. MR 1878556
- [57] M. Lejeune-Jalabert, *Arcs analytiques et résolution minimale des singularités des surfaces quasi homogènes*, Séminaire sur les Singularités des Surfaces, Springer Berlin Heidelberg, 1980, pp. 303–336.
- [58] Monique Lejeune-Jalabert and Ana J. Reguera, *Exceptional divisors that are not uniruled belong to the image of the Nash map*, J. Inst. Math. Jussieu **11** (2012), no. 2, 273–287. MR 2905305
- [59] B. Malgrange, *Polynômes de Bernstein-Sato et cohomologie évanescence*, Analysis and topology on singular spaces, II, III (Luminy, 1981), Astérisque, vol. 101, Soc. Math. France, Paris, 1983, pp. 243–267. MR 737934

- [60] Hideyuki Matsumura, *Commutative ring theory*, second ed., Cambridge Studies in Advanced Mathematics, vol. 8, Cambridge University Press, Cambridge, 1989, Translated from the Japanese by M. Reid. MR 1011461
- [61] Mario Morán Cañón and Julien Sebag, *On the tangent space of a weighted homogeneous plane curve singularity*, J. Korean Math. Soc. **57** (2020), no. 1, 145–169. MR 4052713
- [62] Hussein Mourtada and Ana J. Reguera, *Mather discrepancy as an embedding dimension in the space of arcs*, Publ. Res. Inst. Math. Sci. **54** (2018), no. 1, 105–139. MR 3749346
- [63] Mircea Mustață, *Jet schemes of locally complete intersection canonical singularities*, Invent. Math. **145** (2001), no. 3, 397–424, With an appendix by David Eisenbud and Edward Frenkel.
- [64] John F. Nash, Jr., *Arc structure of singularities*, Duke Math. J. **81** (1995), no. 1, 31–38 (1996), A celebration of John F. Nash, Jr. MR 1381967
- [65] Johannes Nicaise and Julien Sebag, *Greenberg approximation and the geometry of arc spaces*, Comm. Algebra **38** (2010), no. 11, 4077–4096.
- [66] Ana J. Reguera, *A curve selection lemma in spaces of arcs and the image of the Nash map*, Compos. Math. **142** (2006), no. 1, 119–130. MR 2197405
- [67] ———, *Towards the singular locus of the space of arcs*, Amer. J. Math. **131** (2009), no. 2, 313–350.
- [68] ———, *Coordinates at stable points of the space of arcs*, J. Algebra **494** (2018), 40–76. MR 3723170
- [69] Joseph Fels Ritt, *Differential Algebra*, American Mathematical Society Colloquium Publications, Vol. XXXIII, American Mathematical Society, New York, N. Y., 1950. MR 0035763
- [70] Julien Sebag, *Intégration motivique sur les schémas formels*, Bull. Soc. Math. France **132** (2004), no. 1, 1–54.
- [71] ———, *Arcs schemes, derivations and Lipman’s theorem*, J. Algebra **347** (2011), 173–183.
- [72] ———, *A remark on Berger’s conjecture, Kolchin’s theorem, and arc schemes*, Arch. Math. (Basel) **108** (2017), no. 2, 145–150.
- [73] ———, *On logarithmic differential operators and equations in the plane*, Illinois J. Math. **62** (2018), no. 1-4, 215–224. MR 3922414
- [74] The Stacks Project Authors, *Stacks Project*, <https://stacks.math.columbia.edu>, 2020.
- [75] Bernd Sturmfels, *Gröbner bases and convex polytopes*, University Lecture Series, vol. 8, American Mathematical Society, Providence, RI, 1996. MR 1363949
- [76] The Sage Developers, *Sagemath, the Sage Mathematics Software System (Version 8.9)*, 2019, <https://www.sagemath.org>.

- [77] Michel Vaquié, *Valuations, Resolution of singularities* (Obergurgl, 1997), Progr. Math., vol. 181, Birkhäuser, Basel, 2000, pp. 539–590. MR 1748635
- [78] Uli Walther, *Survey on the D -module f^s* , Commutative algebra and noncommutative algebraic geometry. Vol. I, Math. Sci. Res. Inst. Publ., vol. 67, Cambridge Univ. Press, New York, 2015, With an appendix by Anton Leykin, pp. 391–430. MR 3525478

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Titre : Étude schématique du schéma des arcs

Mots clés : Schéma des arcs, singularités, courbes planes, espace tangent, voisinages formels, variétés toriques, bases de Gröbner.

Résumé : Le schéma des arcs associé à une variété algébrique définie sur un corps paramètre les germes formels de courbes que l'on peut tracer sur la variété considérée. Nous étudions certaines propriétés schématiques locales du schéma des arcs d'une variété. Étant donnée une courbe affine plane singulière définie par un polynôme réduit homogène ou homogène à poids, nous calculons, principalement par des arguments d'algèbre différentielle, des présentations de l'idéal définissant l'adhérence du lieu lisse de l'espace tangent qui est toujours une composante irréductible de cet espace.

En particulier, nous obtenons une base de Gröbner de cet idéal, ce qui nous permet de décrire les fonctions de l'espace tangent de la variété qui sont nilpotentes dans le schéma des arcs. Par ailleurs, nous étudions le voisinage formel dans le schéma des arcs d'une variété torique normale de certains arcs appartenant à l'ensemble de Nash associé à une valuation divisorielle torique. Nous établissons un théorème de comparaison, dans le schéma des arcs, entre le voisinage formel du point générique de l'ensemble de Nash et celui d'un arc rationnel suffisamment général dans ce même ensemble de Nash.

Title: Study of the scheme structure of arc scheme

Keywords: Arc scheme, singularity theory, plane curves, tangent space, formal neighbourhoods, toric varieties, Gröbner bases.

Abstract: The arc scheme associated with an algebraic variety defined over a field parameterizes the formal germs of curves lying on the considered variety. We study some local schematic properties of the arc scheme of a variety. Given an affine plane curve singularity defined by a reduced homogeneous or weighted homogeneous polynomial, we compute, mainly using arguments from differential algebra, presentations of the ideal defining the Zariski closure of the smooth locus of the tangent space, which is always an irreducible component of this space.

In particular, we obtain a Groebner basis of such ideal, which gives a complete description of the functions of the tangent space of the variety which are nilpotent in the arc scheme. On the other hand, we study the formal neighbourhood in the arc scheme of a normal toric variety of certain arcs belonging to the Nash set associated with a divisorial toric valuation. We establish a comparison theorem, in the arc scheme, between the formal neighbourhood of the generic point of the Nash set and that of a sufficiently generic rational arc in the same Nash set.