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Directed and Radial Random Spanning Forests in Hyperbolic Spaces

Lucas Flammant

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Pour obtenir le grade de Docteur de

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Forêts aléatoires couvrantes, dirigées et radiales, en espace hyperbolique

JURY

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Membres invités

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Remerciements

Chapter 1

Introduction

1.1 Motivations and main results

Unlike combinatorial random graphs as the Erdős–Rényi model (see [14] for their seminal paper or [6] for a very complete reference) in which all the edges occur with the same probability and independently of each other, the vertices of a geometric random graph are embedded in a metric space and the probability that an edge $\{x, y\}$ occurs, depends on geometric considerations as for example the distance between x and y . Taking into account the geometry of the ambient space, the geometric random graphs are useful to modelize many fields as statistical physics and particles systems, communication networks, population dynamics, image analysis etc. See the book of Penrose [20] for a general reference on the topic.

In 2008, Baccelli and Bordenave introduce the *Radial Spanning Tree* (RST) in the Euclidean plane to modelize communication networks [1]. The RST is a geometric random graph defined from a homogeneous Poisson Point Process (PPP) \mathcal{N} (with intensity 1) and rooted at the origin 0. It is obtained by connecting each point $z \in \mathcal{N}$ to its parent $A(z)$, defined as the closest point to z among all $z' \in \mathcal{N} \cup \{0\}$ that are closer to the origin than z (in the sense of the Euclidean distance $|\cdot|$), i.e. among $(\mathcal{N} \cup \{0\}) \cap B(0, |z|)$. A semi-infinite path of the RST is defined as a sequence of Poisson points $(z_n)_{n \geq 0} \in (\mathcal{N} \cup \{0\})^{\mathbb{N}}$ with $z_0 = 0$ and $z_n = A(z_{n+1})$ for any $n \geq 0$. A semi-infinite path $(z_n)_{n \geq 0}$ admits an asymptotic direction $\theta \in [0, 2\pi]$ if $z_n/|z_n|$ tends to $e^{i\theta}$ as n tends to infinity. The topological properties of the bi-dimensional Euclidean RST are well-understood: Baccelli and Bordenave showed that a.s. any semi-infinite path admits an asymptotic direction; a.s. every asymptotic direction is reached by at least one semi-infinite path and there exists a.s. a unique semi-infinite path in any given deterministic asymptotic direction. These results on the infinite paths are completed by Baccelli, Coupier & Tran [2].

By construction the RST is distribution invariant w.r.t. rotations but not w.r.t translations. This lack constitutes a real drawback to study some of its properties, as the vertical fluctuations of a (radial) path starting from $(n, 0)$ (with n large). To overcome this obstacle, Baccelli and Bordenave introduce an auxiliary model, called the *Directed Spanning Forest* (DSF), which approximates in distribution, locally and far from the origin the RST. The Euclidean DSF is a random forest whose vertex set is given by the homogeneous PPP \mathcal{N} . For any vector $u \in \mathbb{R}^2$, the (Euclidean) DSF with direction u is the graph obtained by connecting each point $z \in \mathcal{N}$ to the closest point to z among all points $z' \in \mathcal{N}$ that are further in the direction u , i.e. such that $\langle z' - z, u \rangle > 0$.

The RST and DSF with direction $u = -(1, 0)$ constructed on the same PPP \mathcal{N} can be locally coupled far from the origin. It is not difficult to be convinced by the fact that, with probability tending to 1 with n , any Poisson point $z \in B((n, 0), r)$ (with r not depending on n) admits the same ancestor for the RST and for the DSF. See [1]. This means that locally and far from the origin, i.e. inside the ball $B((n, 0), r)$, the radial behavior of RST paths disappears and the RST

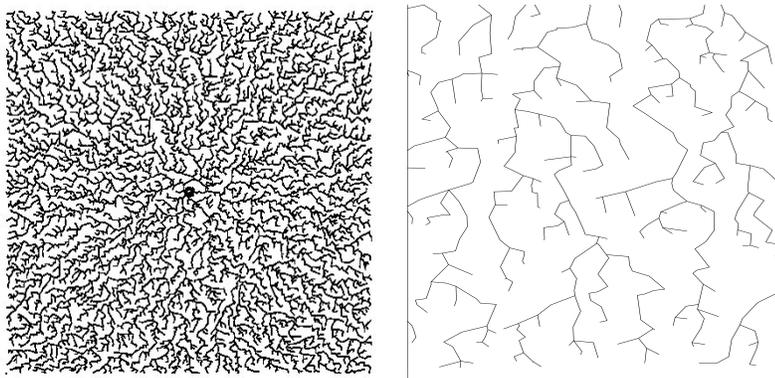


Figure 1.1: To the left, the Euclidean RST rooted at the origin and to the right the DSF with direction $u = (0, 1)$.

looks like (in distribution) to the DSF with direction $u = -(1, 0)$.

More than a tool to study the RST, the DSF is an interesting mathematical object in itself with beautiful properties. Coupier and Tran showed in 2013 that, in dimension 2, all the DSF paths coalesce (the DSF is actually a tree!) and it does not contain bi-infinite paths with probability 1 [12]. In addition, Coupier, Saha, Sarkar & Tran developed tools to split trajectories in i.i.d. blocks [11], and stated that the DSF at a diffusive scale converges in distribution to the Brownian Web. See [27] for an overview on the Brownian Web. Let us add that these new tools, extended to higher dimensions, should allow to prove that, like many other directed forests, the DSF paths coalesce in dimension $d \in \{2, 3\}$ but not in dimension $d \geq 4$, and contain no bi-infinite paths whatever the dimension. Such results have been proved for several other directed forest, e.g. see [13, 14, 26]. One main difficulty in the RST and DSF constructed on PPP lies in the fact that although their definitions are simple, complex dependencies between points are then created and are very intricate to study. In [1, 2, 12, 11], many tools specific to the Euclidean RST and DSF are developed.

Many random objects present radically different behaviours depending on whether they are considered in an Euclidean or hyperbolic setting. With the dichotomy of recurrence and transience for symmetric random walks [20], one of the most emblematic example is given by continuum percolation models. Indeed, the Poisson-Boolean model contains at most one unbounded component in \mathbb{R}^d [21] whereas it admits a non-degenerate regime with infinitely many unbounded components in the hyperbolic plane [23]. The difference is mainly explained by the fact that the hyperbolic space is non-amenable, i.e. the measure of the boundary of a large subset is not negligible w.r.t. its volume. For this reason, arguments based on comparison between volume and surface, such as the Burton & Keane argument [6], fail in hyperbolic geometry. Discovering new behaviors of the hyperbolic RST-DSF (to be defined) w.r.t. their Euclidean counterparts is the initial motivation of this thesis.

There is a growing interest for the study of random models in a hyperbolic setting. Let us cite the work of Benjamini & Schramm about the Bernoulli percolation on regular tilings and Voronoï tessellation in the hyperbolic plane [3], and the work of Calka & Tykesson about asymptotic visibility in the Poisson-Boolean model [8]. Mean characteristics of the Poisson-Voronoi tessellation have also been studied in a general Riemannian manifold by Calka, Chapron & Enriquez [6]. Recently, Hansen & Müller have studied the behavior of the critical probability $p_c(\lambda)$ (for the existence of an infinite cluster) in the percolation model on the Voronoi tessellation generated by a homogeneous PPP with intensity λ on the hyperbolic plane. On the one hand, they show in [18] that $p_c(\lambda)$ tends to $1/2$ as $\lambda \rightarrow +\infty$, confirming a conjecture of Benjamini & Schramm [3]. On the other hand, they

stated in [19] that $p_c(\lambda)$ is asymptotically equal to $\pi\lambda/3$ as $\lambda \rightarrow 0$, answering a question of [3]. Let us also mention the recent work of Linker, Mitsche, Schapira & Valesin on the contact process on random hyperbolic graphs [21].

Another motivation for this work is the following: the geometry of complex networks seems to be hyperbolic. In [5], Boguná, Papadopoulos & Krioukov put forward evidences that the hidden underlying geometry of such networks as social networks, scientific collaborator networks, computer networks is hyperbolic, and that hyperbolic geometry emerges spontaneously in the process of their formation (see Bianconi & Rahmede [4] or Fountoulakis et al. [16]). Moreover, in [5], a surprisingly good maximum likelihood fit of the Random Hyperbolic Graph model (introduced by the same authors in [20]), was shown for the embedding of the network formed by the autonomous systems of the Internet.

For any integer $d \geq 2$, the hyperbolic space \mathbb{H}^d is a d -dimensional Riemannian manifold, homogeneous, with constant negative curvature, that can be chosen equal to -1 without loss of generality. It can be represented by several models, all related by isometries: we will mainly work with the Poincaré disc model and the upper half-space. The hyperbolic space \mathbb{H}^d admits a set of ideal boundary points $\partial\mathbb{H}^d$, and $\overline{\mathbb{H}^d} := \mathbb{H}^d \cup \partial\mathbb{H}^d$ denotes the hyperbolic space endowed with its boundary. For background in hyperbolic geometry, the reader may refer to [7] or [19].

Let \mathcal{N} be a homogeneous PPP of intensity $\lambda > 0$ in \mathbb{H}^d . Recall that the intensity λ matters in the hyperbolic setting by acting on the curvature of the space, unlike in the Euclidean setting in which the intensity can be chosen equal to 1 without loss of generality. The definition of the hyperbolic RST is similar to the Euclidean case. Working with the Poincaré disc model, its vertex set is $\mathcal{N} \cup \{0\}$ and each vertex $z \in \mathcal{N}$ is connected to the closest Poisson point among those that are closer to the origin than z :

$$A(z) := \operatorname{argmin}_{z' \in (\mathcal{N} \cup \{0\}) \cap B(r)} d(z', z)$$

where $d(\cdot, \cdot)$ denotes the hyperbolic distance and $B(r)$ the corresponding ball with radius r centered at 0. The *hyperbolic Radial Spanning Tree* in \mathbb{H}^d is a tree rooted at the origin whose vertex set $V := \mathcal{N} \cup \{0\}$ and edge set $\vec{E} := \{(z, A(z)), z \in \mathcal{N}\}$. Remark that $\mathcal{N} \cup \{0\}$ does not contain isosceles triangles with probability 1: the parent $A(z)$ is then a.s. unique.

Whereas the hyperbolic RST defined above is the natural hyperbolic version of the Euclidean RST, we could define several hyperbolic versions of the Euclidean DSF because there exist several ways to follow a direction in the hyperbolic space. But one of them is more relevant since it approximates in distribution, locally and far from the origin the hyperbolic RST and we want to preserve the link between Euclidean RST and DSF for their hyperbolic counterparts. Hence, the *hyperbolic Directed Spanning Forest* is defined as follows. Working with the upper half-space, we choose to connect each Poisson point $z = (x_1, \dots, x_d) \in \mathcal{N}$ to the closest Poisson point to z (for the hyperbolic distance) among all points $z' = (x'_1, \dots, x'_d) \in \mathcal{N}$ with $x'_d > x_d$. An equivalent and more intrinsic definition of this model using horodistances is given in the sequel.

It is worth pointing out that (both Euclidean and hyperbolic) RST and DSF present some strong dependence phenomenons due to their construction rules. Let us focus on the DSF on the upper half-space. When a Poisson point z is connected to its parent $A(z)$, this automatically creates a region, precisely the upper half-space above $A(z)$ intersected with the ball $B(z, d(z, A(z)))$, which is empty of Poisson points. The parent of $A(z)$ cannot belong to this region. This means that the previous steps of a DSF path influences the next ones. Hence, the DSF presents dependence phenomenons inside a single path and between different paths.

Chapter 2 is devoted to the study of the hyperbolic DSF. After detailing its definition (with horodistances) and its first properties (no cycles, non-crossing paths and finite degree), we get a complete description of its topological properties which are radically different from the Euclidean case. Indeed, for any dimension, the hyperbolic DSF is a tree (i.e. all the DSF paths coalesce)

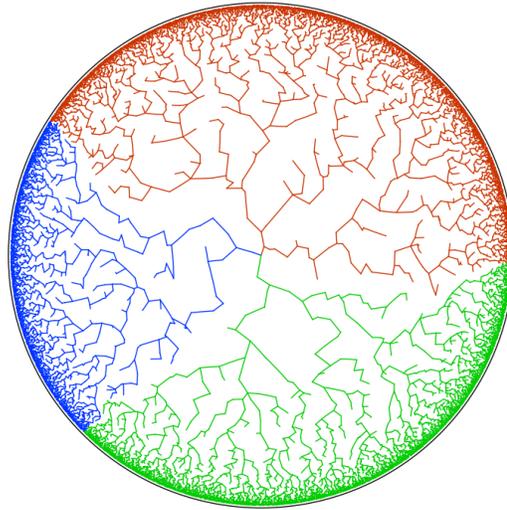


Figure 1.2: Simulation of the two-dimensional hyperbolic RST, with $\lambda = 30$, in the Poincaré disc model. The edges are represented by geodesics. The different connected components of the RST (apart from the root) are represented with different colors.

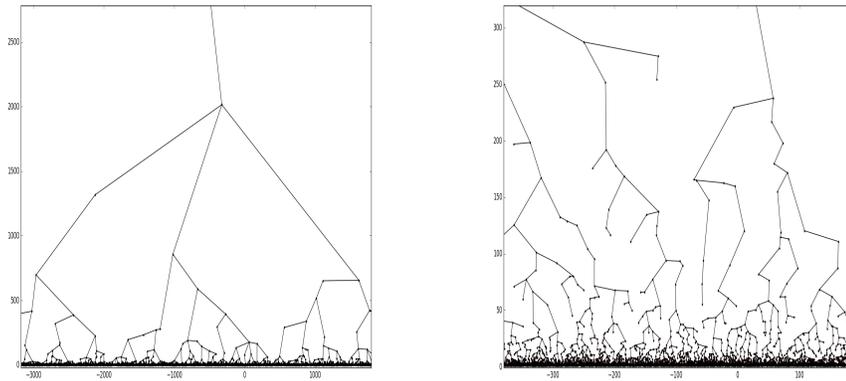


Figure 1.3: Simulation of the hyperbolic DSF in the half-space model, with $\lambda = 1$ (to the top) and $\lambda = 10$ (to the bottom). The local behaviour of the hyperbolic DSF depends on the intensity λ because the space is curved.

containing infinitely many bi-infinite branches, whose asymptotic directions are investigated.

Chapters 3 and 4 concern the hyperbolic RST and the study of its semi-infinite paths in every dimension. It is proved that a.s. every semi-infinite path admits an asymptotic direction and each asymptotic direction is reached by at least one semi-infinite path. Moreover, the RST path converging to any given (deterministic) asymptotic direction is a.s. unique. However, there exist random directions in which several RST paths converge. This set of exceptional random directions is dense in $\partial\mathbb{H}^d$ and countable in dimension $d = 2$. Finally, we show that the subset of $\partial\mathbb{H}^d$ of asymptotic directions reached by semi-infinite paths passing through a given Poisson point z has a non-empty interior. In dimension $d = 2$, this means that a.s. there is no random direction in

which more than two RST paths converge. To our knowledge, this statement which is conjectured for most of bi-dimensionnal geometric radial tree studied in the literature, has been stated for only one model, namely the last-passage percolation tree in the quadrant \mathbb{N}^2 with exponential weights, by Coupier [11].

1.2 Hyperbolic geometry

Let us recall that \mathbb{H}^{d+1} denotes the $d + 1$ -dimensional hyperbolic space, and H denotes the half-space model defined:

$$H := \{(x_1, \dots, x_d, y) \in \mathbb{R}^{d+1}, y > 0\}$$

endowed with the metric

$$ds_H^2 := \frac{dx_1^2 + \dots + dx_d^2 + dy^2}{y^2}.$$

We refer to [7] and [19] for general introduction to hyperbolic geometry.

1.2.1 The distance formula

Proposition 1.2.1 (Distance formula in the half-space model). *Let $z_1 = (x_1, y_1) \in H$ and $z_2 = (x_2, y_2) \in H$. Let $\kappa = \|x_1 - x_2\|/y_1$ and $v = y_2/y_1$. Then*

$$d(z_1, z_2) = 2 \tanh^{-1} \left(\sqrt{\frac{\kappa^2 + (v-1)^2}{\kappa^2 + (v+1)^2}} \right) = \Phi \left(\frac{\kappa^2 + (v+1)^2}{v} \right) \quad (1.2.1)$$

where $\Phi : [4, +\infty) \rightarrow \mathbb{R}_+$ is increasing and defined as

$$\Phi(t) = 2 \tanh^{-1} \left(\sqrt{1 - 4/t} \right) = \ln \left(\frac{1 + \sqrt{1 - 4/t}}{1 - \sqrt{1 - 4/t}} \right).$$

Let us recall that the height of a point $z = (x, y) \in H$ is defined as $h := \ln(y)$.

Corollary 1.2.2 (Distance between the points on the same vertical line). *Let $x \in \mathbb{R}^d$, $y_1, y_2 \in \mathbb{R}_+^*$. Consider the two points $z_1 = (x, y_1)$ and $z_2 = (x, y_2)$. Then*

$$d(z_1, z_2) = |\ln(y_2/y_1)| = |h(z_1) - h(z_2)|.$$

Corollary 1.2.3 (Distance between two points on the same horizontal hyperplane). *Let $y \in \mathbb{R}_+^*$, let $R > 0$ and $x_1, x_2 \in \mathbb{R}^d$ with $\|x_1 - x_2\| = R$. Then*

$$d(z_1, z_2) = 2 \ln \left(\frac{R}{y} \right) + o(1) \text{ when } R \rightarrow \infty.$$

where $z_i = (x_i, y)$, $i = 1, 2$. Moreover, for all $R > 0$, $d(z_1, z_2) \leq \frac{R}{y}$.

Proof of Proposition 2.2.1 and its corollaries. Step 1: It is enough to consider the case of dimension 2 ($d = 1$). Let (H^2, ds_2^2) be the two dimensional half-space endowed with the Hyperbolic metric, and (H^{d+1}, ds_{d+1}^2) be the $(d + 1)$ -dimensional half-space with Hyperbolic metric. Let $z_1 = (x_1, y_1), z_2 = (x_2, y_2) \in H^{d+1}$. Then the map

$$\begin{aligned} \varphi : H^2 &\rightarrow H^{d+1} \\ (x, y) &\mapsto \left(x_1 + x \frac{x_2 - x_1}{\|x_2 - x_1\|}, y \right) \end{aligned} \quad (1.2.2)$$

is an isometry from (H_2, ds_2^2) to $\varphi(H_2) \subset (H^{d+1}, ds_{d+1}^2)$. Recall that $\pi_x : (x, y) \mapsto x$ and $\pi_y : (x, y) \mapsto y$ denote the projections on the x -axis and on the y -axis respectively. Note that $z_1, z_2 \in \varphi(H_2)$, and

$$\pi_y \varphi^{-1}(z_1) = y_1, \quad \pi_y \varphi^{-1}(z_2) = y_2, \quad \|\pi_x \varphi^{-1}(z_2) - \pi_x \varphi^{-1}(z_1)\| = \|x_2 - x_1\|. \quad (1.2.3)$$

Thus, the distance formula (3.7.1) in H^2 immediatly implies the same formula in H^{d+1} .

Step 2: Proof in dimension 2.

Let us first apply isometry $D_{1/y_1} \circ T_{-x_1}$, that sends z_1 to $(0, e^0)$. It leads to:

$$d(z_1, z_2) = d\left((0, e^0), \left(\frac{x_2 - x_1}{y_1}, \frac{y_1}{y_2}\right)\right) = d((0, e^0), (\kappa, v)), \quad (1.2.4)$$

where κ and v has been defined in Proposition 2.2.1. \square

We wove on to show the particular cases of the distance formula, i.e. Corollaries 1.2.2 and 1.2.3.

Proof of Corollary 1.2.2. Let $x \in \mathbb{R}^d$, $y_1, y_2 \in \mathbb{R}_+^*$, and set $z_1 = (x, y_1)$, $z_2 = (x, y_2)$. Then, $\kappa = \|x - x\|/y_1 = 0$ and $v = y_2/y_1$. Thus, Proposition 2.2.1 gives,

$$d(z_1, z_2) = 2 \tanh^{-1} \left(\sqrt{\frac{(y_2/y_1 - 1)^2}{(y_2/y_1 + 1)^2}} \right) = 2 \tanh^{-1} \left(\frac{y_2 - y_1}{y_2 + y_1} \right) = \ln(y_2/y_1).$$

\square

Proof of Corollary 1.2.3. Let $y \in \mathbb{R}_+^*$, let $R > 0$ and $x_1, x_2 \in \mathbb{R}^d$ with $\|x_1 - x_2\| = R$. Then

$$d(z_1, z_2) = 2 \tanh^{-1} \left(\sqrt{\frac{(R/y)^2}{(R/y)^2 + 4}} \right) = 2 \tanh^{-1} \left(\sqrt{1 - \frac{4}{(R/y)^2 + 4}} \right).$$

When $R \rightarrow \infty$,

$$\frac{1 + \sqrt{1 - \frac{4}{(R/y)^2 + 4}}}{1 - \sqrt{1 - \frac{4}{(R/y)^2 + 4}}} \sim \left(\frac{R}{y}\right)^2 + 4 \sim \left(\frac{R}{y}\right)^2.$$

Thus

$$d(z_1, z_2) = \ln \left(\frac{1 + \sqrt{1 - \frac{4}{(R/y)^2 + 4}}}{1 - \sqrt{1 - \frac{4}{(R/y)^2 + 4}}} \right) = 2 \ln \left(\frac{R}{y} \right) + o(1) \text{ as } R \rightarrow \infty.$$

For the second part of Corollary 1.2.3, let us consider the C^1 -regular path $\gamma : [0, 1] \rightarrow H$ defined as $\gamma(t) = (x_1 + t(x_2 - x_1), y)$. Then

$$d(z_1, z_2) \leq \int_0^1 \frac{\|\gamma'(t)\|}{\pi_y(\gamma(t))} dt = \int_0^1 \frac{\|x_2 - x_1\|}{y} dt = \frac{R}{y}, \quad (1.2.5)$$

this completes the proof of Corollary 1.2.3. \square

1.2.2 Horodistance functions

Let us define the horodistance functions (see Definition 2.4 of Chapter 2). In \mathbb{H}^{d+1} , the *horodistance* formalizes the notion of "distance from a point at infinity".

Definition 1.2.4 (Horodistance functions). Let $z_0 \in \mathbb{H}^{d+1}$ be an arbitrary point, considered as the origin. Given a point at infinity $I \in \partial\mathbb{H}^{d+1}$, the *horodistance function* $\mathcal{H}_I : \mathbb{H}^{d+1} \rightarrow \mathbb{R}$ is defined as

$$\mathcal{H}_I(z) := \lim_{z' \rightarrow I} d(z, z') - d(z_0, z'). \quad (1.2.6)$$

The level sets of \mathcal{H}_I , i.e. the sets of points at the same horodistance to I , are called *horospheres* (centered at I). Horospheres in (H, ds_H^2) are represented in Figure 2.3.

Proposition 1.2.5. *In the half-space representation (H, ds_H^2) , the horodistance function \mathcal{H}_∞ is given (modulo an additive constant) by:*

$$\mathcal{H}_\infty((x, y)) = -\ln(y).$$

Proof. Without loss of generality, we can set the origin point z_0 at $(x_0, y_0) := (0, 1)$. Let $z = (x, y) \in H$. By the distance formula (Proposition 1.2.5), for any $z' = (x', y') \in H$,

$$\begin{aligned} d(z, z') - d(z_0, z') &= \ln \left[\frac{1 + \sqrt{\frac{(\|x'-x\|/y)^2 + (y'/y-1)^2}{(\|x'-x\|/y)^2 + (y'/y+1)^2}}}{1 - \sqrt{\frac{(\|x'-x\|/y)^2 + (y'/y-1)^2}{(\|x'-x\|/y)^2 + (y'/y+1)^2}}} \right] - \ln \left[\frac{1 + \sqrt{\frac{\|x'\|^2 + (y'-1)^2}{\|x'\|^2 + (y'+1)^2}}}{1 - \sqrt{\frac{\|x'\|^2 + (y'-1)^2}{\|x'\|^2 + (y'+1)^2}}} \right] \\ &= \ln \left[\frac{1 + \sqrt{\frac{\|x'-x\|^2 + (y'-y)^2}{\|x'-x\|^2 + (y'+y)^2}}}{1 + \sqrt{\frac{\|x'\|^2 + (y'-1)^2}{\|x'\|^2 + (y'+1)^2}}} \cdot \frac{1 - \sqrt{\frac{\|x'\|^2 + (y'-1)^2}{\|x'\|^2 + (y'+1)^2}}}{1 - \sqrt{\frac{\|x'-x\|^2 + (y'-y)^2}{\|x'-x\|^2 + (y'+y)^2}}} \right]. \end{aligned} \quad (1.2.7)$$

When $z' \rightarrow \infty$ (i.e. $\|x'\|^2 + \|y'\|^2 \rightarrow \infty$), $\|x' - x\|^2 + (y' \pm y)^2 \sim \|x'\|^2 + y'^2$, idem for $\|x'\|^2 + (y' \pm 1)^2$. Thus

$$\frac{1 + \sqrt{\frac{\|x'-x\|^2 + (y'-y)^2}{\|x'-x\|^2 + (y'+y)^2}}}{1 + \sqrt{\frac{\|x'\|^2 + (y'-1)^2}{\|x'\|^2 + (y'+1)^2}}} \rightarrow 1 \quad (1.2.8)$$

as $z' \rightarrow \infty$. Moreover,

$$\begin{aligned} 1 - \sqrt{\frac{\|x'\|^2 + (y'-1)^2}{\|x'\|^2 + (y'+1)^2}} &= 1 - \sqrt{1 - \frac{4y'}{\|x'\|^2 + (y'+1)^2}} \\ &\sim \frac{2y'}{\|x'-x\|^2 + (y'+1)^2} \sim \frac{2}{\|x'\|^2 + y'^2} \text{ as } z' \rightarrow \infty, \end{aligned} \quad (1.2.9)$$

and, similarly,

$$\begin{aligned} 1 - \sqrt{\frac{\|x'-x\|^2 + (y'-y)^2}{\|x'-x\|^2 + (y'+y)^2}} &= 1 - \sqrt{1 - \frac{4yy'}{\|x'-x\|^2 + (y'+y)^2}} \\ &\sim \frac{2yy'}{\|x'-x\|^2 + (y'+y)^2} \sim \frac{2y}{\|x'\|^2 + y'^2} \text{ as } z' \rightarrow \infty. \end{aligned} \quad (1.2.10)$$

Therefore, plugging (1.2.8) and (1.2.9) in (1.2.7) gives,

$$\lim_{z' \rightarrow \infty} d(z, z') - d(z_0, z') = -\ln(y),$$

this achieves the proof. \square

1.2.3 The change-of-coordinates formula

Lemma 1.2.6. *Let*

$$\mathcal{S}(d) := \frac{d\pi^{d/2}}{\Gamma(\frac{d}{2} + 1)}$$

be the $(d-1)$ -dimensional surface of the unit sphere in \mathbb{R}^d . Let $z_0 \in \mathbb{H}^{d+1}$ and let $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be measurable. Then

$$\int_{\mathbb{H}^{d+1}} f(d(z, z_0)) dz = \mathcal{S}(d) \int_{\mathbb{R}_+} f(\rho) \sinh(\rho)^d d\rho.$$

Proof. Let us consider the $(d+1)$ -dimensional open-ball model (I, ds_I^2) and we can suppose without loss of generality that $z_0 = 0$.

The first step is to compute, for $0 \leq r \leq 1$, the hyperbolic radius $\rho(r)$ of the Euclidean ball of radius r centered at 0. Consider a point $z \in I$ with $\|z\| = r$. We have

$$\rho(r) = \int_0^r \frac{4}{1-s^2} ds = 2 \tanh^{-1}(r).$$

Now, let $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be some measurable function. Recall that $\mathcal{S}(d)$ is the (Euclidean) surface of the unit sphere in \mathbb{R}^d . Then

$$\int_{\mathbb{H}^{d+1}} f(d(z, z_0)) dz = \int_I f(d(z, 0)) \frac{2^{d+1} dz}{(1-\|z\|^2)^{d+1}} = \mathcal{S}(d) \int_0^1 f(\rho(r)) r^d \frac{2^{d+1} dr}{(1-r^2)^{d+1}}$$

The change of variable $\rho = 2 \tanh^{-1}(r)$ gives:

$$\int_{\mathbb{H}^{d+1}} f(d(z, z_0)) dz = \mathcal{S}(d) \int_0^\infty f(\rho) 2^{d+1} \frac{d\rho \frac{1-r^2}{2}}{(1-r^2)^{d+1}} = \mathcal{S}(d) \int_0^\infty f(\rho) \sinh(\rho)^d d\rho,$$

which proves Lemma 2.7.2. □

1.3 The Mass Transport principle in \mathbb{R}^d

Theorem 1.3.1 (Mass Transport Principle). *Let π be some positive diagonally invariant measure on $\mathbb{R}^d \times \mathbb{R}^d$. Then for any measurable set $A \subset \mathbb{R}^d$ with nonempty interior, the following identity holds:*

$$\pi(A \times \mathbb{R}^d) = \pi(\mathbb{R}^d \times A),$$

these values can be eventually infinite.

For $s \in \mathbb{R}^d$, define $\tilde{T}_s : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}^d \times \mathbb{R}^d$ by $\tilde{T}_s(x, x') = (x + s, x' + s)$. We first show:

Lemma 1.3.2. *Let π be diagonally invariant measure on $\mathbb{R}^d \times \mathbb{R}^d$. We suppose that π is finite on compact sets. Then for all measurable set $E \subset \mathbb{R}^d \times \mathbb{R}^d$, $\pi(E) = \pi(\tilde{T}_s E)$.*

Proof of Lemma 1.3.2. Let $s \in \mathbb{R}^d$. For $R \geq 0$, let us define $K_R = [-R, R]^d$ and $\tilde{K}_R = K_R \times K_R$. Fix $R \geq 0$. Define $\Psi := \{E \in \mathcal{B}(\mathbb{R}^d \times \mathbb{R}^d), \pi(E \cap \tilde{K}_R) = \pi(\tilde{T}_s(E \cap \tilde{K}_R))\}$. Then Ψ is a monotone class since $\pi(\tilde{K}_R) < \infty$ by assumption. Define $\Psi_0 := \{A \times B, A, B \in \mathcal{B}(\mathbb{R}^d)\}$. Then Ψ_0 is a π -system and since π is diagonal invariant, for all measurable sets $A, B \subset \mathbb{R}^d$,

$$\begin{aligned} \pi((A \times B) \cap \tilde{K}_R) &= \pi((A \cap K_R) \times (B \cap K_R)) = \pi(T_s(A \cap K_R) \times T_s(B \cap K_R)) \\ &= \pi(\tilde{T}_s((A \times B) \cap \tilde{K}_R)). \end{aligned}$$

Thus $\Psi_0 \subset \Psi$. Moreover Ψ_0 generates the σ -algebra $\mathcal{B}(\mathbb{R}^d \times \mathbb{R}^d)$. Thus the Monotone Class Theorem gives that $\Psi_0 = \mathcal{B}(\mathbb{R}^d \times \mathbb{R}^d)$. That is, for all measurable set $E \subset \mathbb{R}^d \times \mathbb{R}^d$, $\pi(E \cap \tilde{K}_R) = \pi(\tilde{T}_s(E \cap \tilde{K}_R))$. Taking $R \rightarrow \infty$, we obtain $\pi(E) = \pi(\tilde{T}_s(E))$. □

Proof of Theorem 2.3.2. Let us define the following measures on \mathbb{R}^d :

$$\nu_1(A) := \pi(A \times \mathbb{R}^d), \quad \nu_2(A) := \pi(\mathbb{R}^d \times A)$$

for all measurable set $A \subset \mathbb{R}^d$. Since π is diagonally invariant, for all $s \in \mathbb{R}^d$, $\nu_1(T_s A) = \pi(T_s A \times \mathbb{R}^d) = \pi(A \times \mathbb{R}^d) = \nu_1(A)$, therefore ν_1 is translation invariant. By the same argument, ν_2 is also translation invariant.

Let us consider the case there exists some open set $A \subset \mathbb{R}^d$ such that $\nu_1(A) < \infty$. We can consider without loss of generality that A is bounded. Then ν_1 is a multiple of the Lebesgue measure. Suppose for the moment that $\nu_2(A) < \infty$. Then ν_2 is also a multiple of the Lebesgue measure. Let $B \subset \mathbb{R}^d$ be measurable with $0 < \text{Leb}(B) < \infty$, we show that $\nu_1(B) = \nu_2(B)$. Both measures $\pi(\cdot \times B)$ and $\pi(B \times \cdot)$ are σ -finite since ν_1 and ν_2 are. Thus, Fubini gives,

$$\begin{aligned} \int_{\mathbb{R}^d} \pi(T_s B \times B) ds &= \int_{s \in \mathbb{R}^d} \int_{x \in \mathbb{R}^d} \mathbf{1}_{x \in t_s B} d\pi(\{x\} \times B) ds \\ &= \int_{x \in \mathbb{R}^d} \int_{s \in \mathbb{R}^d} \mathbf{1}_{x \in T_s B} ds d\pi(\{x\} \times B) \\ &= \text{Leb}(B) \pi(\mathbb{R}^d \times B) = \nu_1(B) \end{aligned}$$

and

$$\begin{aligned} \int_{\mathbb{R}^d} \pi(B \times T_{-s} B) ds &= \int_{s \in \mathbb{R}^d} \int_{x \in \mathbb{R}^d} \mathbf{1}_{x \in t_{-s} B} d\pi(B \times \{x\}) ds \\ &= \int_{x \in \mathbb{R}^d} \int_{s \in \mathbb{R}^d} \mathbf{1}_{x \in T_{-s} B} ds d\pi(B \times \{x\}) \\ &= \text{Leb}(B) \pi(B \times \mathbb{R}^d) = \nu_2(B). \end{aligned}$$

Thus, since π is diagonal invariant and $0 < \text{Leb}(B) < \infty$,

$$\nu_1(B) = \text{Leb}(B)^{-1} \int_{\mathbb{R}^d} \pi(T_s B \times B) ds = \text{Leb}(B)^{-1} \int_{\mathbb{R}^d} \pi(B \times T_{-s} B) ds = \nu_2(B).$$

Therefore, since both ν_1 and ν_2 are multiple of the Lebesgue measure, $\nu_1 = \nu_2$. That is, for all measurable set $A \subset \mathbb{R}^d$, $\pi(A \times \mathbb{R}^d) = \pi(\mathbb{R}^d \times A)$.

This proves the theorem under the assumption $\nu_2(A) < \infty$. For $r \geq 0$, define $F_r := \{(x, x') \in \mathbb{R}^d \times \mathbb{R}^d, \|x - x'\| < r\}$ and define the measure $\pi_r(\cdot) := \pi(\cdot \cap F_r)$. Let us show that π_r is diagonally invariant. Let $A, B \subset \mathbb{R}^d$ be measurable. Since ν_1 is finite on compact sets (it is a multiple of the Lebesgue measure), so is π . Thus Lemma 1.3.2 with $E = (A \times B) \cap F_r$ gives,

$$\begin{aligned} \pi_r(A \times B) &= \pi((A \times B) \cap F_r) = \pi(\tilde{T}_s((A \times B) \cap F_r)) = \pi(\tilde{T}_s(A \times B) \cap F_r) \\ &= \pi_r(T_s A \times T_s B) \end{aligned}$$

since $F_r = T_s F_r$. Therefore π_r is diagonally invariant. Moreover, $\pi_r(A \times \mathbb{R}^d) \leq \pi(A \times \mathbb{R}^d) < \infty$. Define $A_r := \{x \in \mathbb{R}^d, d(x, A) < r\}$. Since A is bounded, so is A_r . Thus $\pi_r(\mathbb{R}^d \times A) \leq \pi(A_r \times A) \leq \pi(A_r \times \mathbb{R}^d) < \infty$. Then the previous calculations apply to π_r . By monotone convergence, for all measurable set $E \subset \mathbb{R}^d$, $\pi_r(E) \xrightarrow{r \rightarrow \infty} \pi(E)$, therefore, for all measurable set $A \subset \mathbb{R}^d$,

$$\pi(A \times \mathbb{R}^d) = \lim_{r \rightarrow \infty} \pi_r(A \times \mathbb{R}^d) = \lim_{r \rightarrow \infty} \pi_r(\mathbb{R}^d \times A) = \pi(\mathbb{R}^d \times A).$$

This achieves the proof in the case there exists some open set $A \subset \mathbb{R}^d$ such that $\pi(A \times \mathbb{R}^d) < \infty$. A symmetric argument applies if there exists some open set $A \subset \mathbb{R}^d$ such that $\pi(\mathbb{R}^d \times A) < \infty$.

Finally, the remaining case is that for all open set $A \subset \mathbb{R}^d$, $\pi(A \times \mathbb{R}^d) = \pi(\mathbb{R}^d \times A) = \infty$. The conclusion follows immediately in this case. This achieves the proof. \square

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Chapter 2

The Directed Spanning Forest in hyperbolic space

The Euclidean Directed Spanning Forest (DSF) is a random forest in \mathbb{R}^d introduced by Baccelli and Bordenave [1] and we introduce and study here the analogous tree in the hyperbolic space. The topological properties of the Euclidean DSF have been stated for $d = 2$ and conjectured for $d \geq 3$ (see further): it should be a tree for $d \in \{2, 3\}$ and a countable union of disjoint trees for $d \geq 4$. Moreover, it should not contain bi-infinite branches whatever the dimension d . In this paper, we construct the hyperbolic DSF and we give a complete description of its topological properties, which are radically different from the Euclidean case. Indeed, for any dimension, the hyperbolic DSF is a tree containing infinitely many bi-infinite branches, whose asymptotic directions are investigated.

The strategy of our proofs consists in exploiting the Mass Transport Principle, which is adapted to take advantage of the invariance by isometries. Using appropriate mass transports is the key to carry over the hyperbolic setting ideas developed in percolation and for spanning forests. This strategy provides an upper-bound for horizontal fluctuations of trajectories, which is the key point of the proofs. To obtain the latter, we exploit the representation of the forest in the hyperbolic half space.

2.1 Introduction

Many random objects present radically different behaviours depending on whether they are considered in an Euclidean or hyperbolic setting. With the dichotomy of recurrence and transience for symmetric random walks [20], one of the most emblematic example is given by continuum percolation models. Indeed, the Poisson-Boolean model contains at most one unbounded component in \mathbb{R}^d [21] whereas it admits a non-degenerate regime with infinitely many unbounded components in the hyperbolic plane [23]. The difference is mainly explained by the fact that the hyperbolic space is non-amenable, i.e. the measure of the boundary of a large subset is not negligible with respect to its volume. For this reason, arguments based on comparison between volume and surface, such as the Burton and Keane argument [6], fail in hyperbolic geometry. For background in hyperbolic geometry, the reader may refer to [7] or [19].

Hence there is a growing interest for the study of random models in a hyperbolic setting. Let us cite the work of Benjamini & Schramm about the Bernoulli percolation on regular tilings and Voronoï tessellation in the hyperbolic plane [3], and the work of Calka & Tykesson about asymptotic visibility in the Poisson-Boolean model [8]. Mean characteristics of the Poisson-Voronoi tessellation have also been studied in a general Riemannian manifold by Calka et. al. [6]. In addition, huge differences between amenable and non-amenable spaces are well known in a discrete context

[4, 18, 21].

It is in order to highlight new behaviors that we investigate the study of the hyperbolic counterpart of the Euclidean Directed Spanning Forest (DSF) defined in \mathbb{R}^d by [1]. To our knowledge, this is the first study of a spanning forest in the hyperbolic space.

Geometric random trees are well studied in the literature since it interacts with many other fields, such as communication networks, particles systems or population dynamics. We can cite the work of Norris and Turner [22] establishing some scaling limits for a model of planar aggregation. In addition, hyperbolic random graphs are well-fitted to modelize social networks [5].

The Euclidean DSF is a random forest whose introduction has been motivated by applications for communication networks. The set of vertices is given by a homogeneous Poisson Point Process (PPP) N of intensity λ in \mathbb{R}^d . For any unit vector $u \in \mathbb{R}^d$, the (Euclidean) DSF with direction u is the graph obtained by connecting each point $x \in N$ to the closest point to x among all points $x' \in N$ that are further in the direction u (i.e. such that $\langle x' - x, u \rangle > 0$).

The topological properties of the Euclidean DSF are now well-understood. Coupier and Tran showed in 2010 that, in dimension 2, it is a tree that does not contain bi-infinite branches [12]. Their proof used a Burton & Keane argument, so it cannot be carried over the hyperbolic case. In addition, Coupier, Saha, Sarkar & Tran developed tools to split trajectories in i.i.d. blocks [11], and these tools may permit to show that the Euclidean DSF is a tree in dimension 2 and 3 but not in dimension 4 and more (see [11, Remark 18, p.35]). This dichotomy and the absence of bi-infinite branches for any dimension d have been proved for similar models defined on lattices and presenting less geometrical dependencies [22, 14, 1, 13]. Indeed, compared with these models, the DSF exhibits complex geometrical dependencies : given a Poisson point $z \in N$, knowing the position of its parent $A(z)$ implies the knowledge that some region above $A(z)$, the upper part of a hyperbolic ball centered at z is empty of Poisson points, which affects the future evolution of trajectories (Figure 2.4), and thus destroys nice Markov properties available for the models on lattices mentioned above.

The hyperbolic space is a homogeneous space with constant negative curvature, that can be chosen equal to -1 without loss of generality. It can be represented by several models, all related by isometries. We will work in the $(d+1)$ -dimensional upper half-space $H := \{(x_1, \dots, x_d, y) \in \mathbb{R}^{d+1}, y > 0\}$ [7, p.69] endowed with the metric

$$ds_H^2 := \frac{dx_1^2 + \dots + dx_d^2 + dy^2}{y^2}.$$

This representation is well adapted to our problem as explained in Section 2.2. Now, let us define the hyperbolic DSF. The set of vertices is given by a homogeneous PPP N of intensity $\lambda > 0$ in (H, ds_H^2) . Given a point $x \in N$, choosing its closest vertex according to a given direction can be interpreted in different ways in the hyperbolic space. Hence several hyperbolic DSF could be considered. We choose to connect each point $z = (x_1, \dots, x_d, y) \in N$ to the closest point to z (for the hyperbolic distance) among all points $z' = (x'_1, \dots, x'_d, y') \in N$ with $y' > y$ (called the *parent* of z). An equivalent and more intrinsic definition of this model using horodistances is given in the core of the article. The main interest of this definition is the preservation of the link between the DSF and the Radial Spanning Tree (RST) existing in the Euclidean setting. The (Euclidean) RST, also defined by [1], is a random tree whose set of vertices is given by a homogeneous PPP N plus the origin 0 and defined by connecting each point $x \in N$ to the closest point to x among all points $x' \in N \cup \{0\}$ such that $\|x'\| < \|x\|$. In the Euclidean setting, the DSF approximates locally the RST in distribution far from the origin. This remains true in the hyperbolic setting for our definition of hyperbolic DSF, although the RST is not considered here. The study of the hyperbolic RST and its link with the DSF is devoted to a future work. A simulation of the hyperbolic DSF is given in Figure 2.1.

In this paper, we give a complete description of the topological properties of the hyperbolic DSF which present huge differences with the Euclidean case : whatever the dimension d , the hyperbolic

DSF is a.s. a tree (Theorem 2.1.1) and admits infinitely many bi-infinite branches (Theorem 2.1.2).

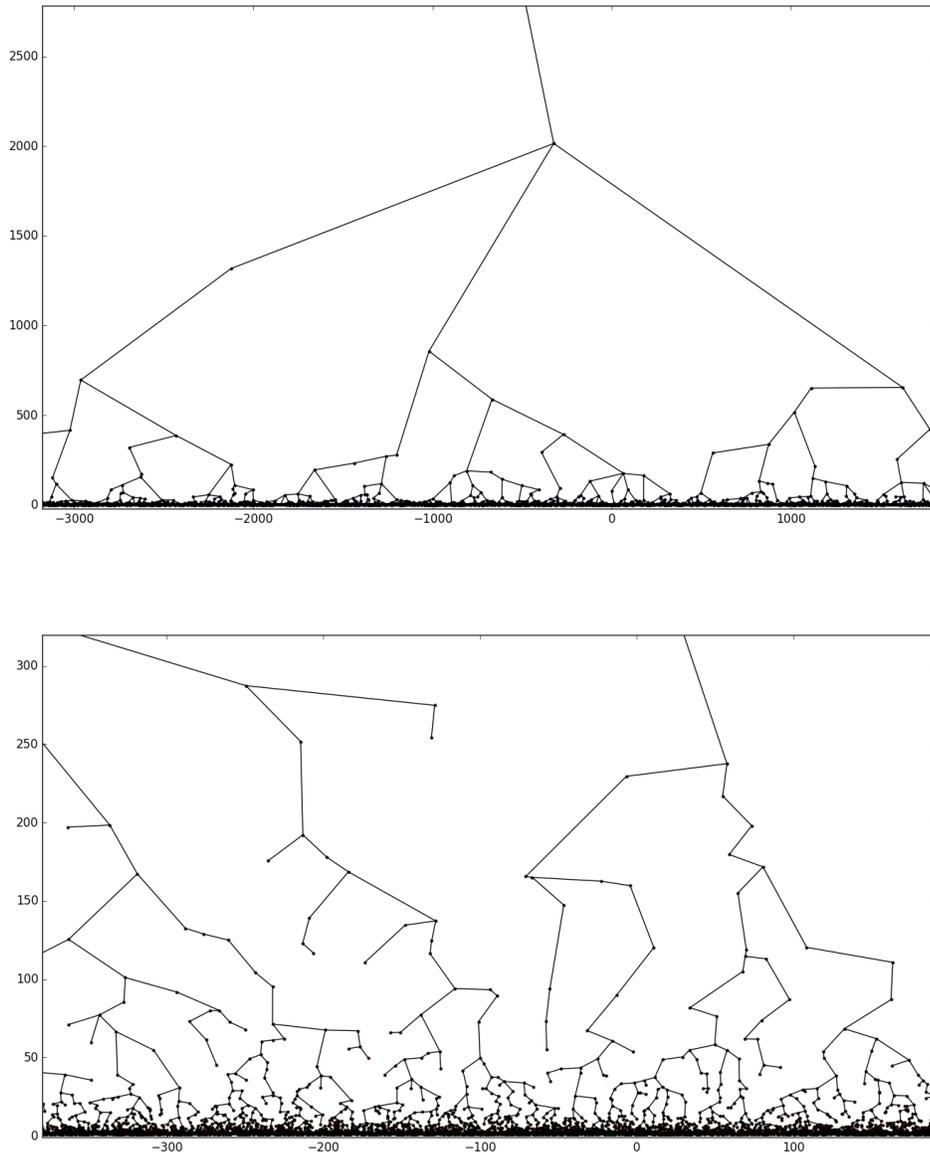


Figure 2.1: Simulation of the DSF in the half-space model, with $\lambda = 1$ (to the top) and $\lambda = 10$ (to the bottom). The local behaviour of the hyperbolic DSF depends on the intensity λ because the space is curved. For instance the average number of descendants is larger for $\lambda = 1$ than for $\lambda = 10$. But its topological properties do not depend on λ (see Theorems 2.1.1 and 2.1.2)

For the DSF, being a tree means that all branches eventually coalesce, i.e. any two points $x, y \in N$ have a common ancestor somewhere in the DSF. For any bounded measurable subset $A \in \mathbb{R}^d$, we can define its *coalescing height* τ_A as the smallest $\tau \geq 0$ such that every branches passing through $A \times \{e^0\}$ have merged below ordinate e^τ (see Definition 2.2.15). Here is our first main result:

Theorem 2.1.1. *For all $d \geq 1$ and for all intensity $\lambda > 0$, the hyperbolic DSF in dimension $d + 1$ is a.s. a tree. Moreover, if $d = 1$, for all $a > 0$, the coalescing height $\tau_{[-a,a]}$ admits exponential tail decay: for any $t > 0$,*

$$\mathbb{P} [\tau_{[-a,a]} > t] \leq 2\alpha_0 a e^{-t},$$

where the positive constant α_0 will be specified later (in Proposition 2.2.18).

The coalescence in every dimension is specific to the hyperbolic case, since in the Euclidean case, it is expected that the DSF is a tree in dimension 2 and 3 only. For $d = 1$, the coalescing height admits exponential tail decay in the hyperbolic case whereas $\mathbb{P}[\tau_{[-a,a]} > t] \in \Theta(a/\sqrt{t})$ when $t \rightarrow \infty$ in the Euclidean case (since it can be compared to the coalescing height of two Brownian motions starting from $(-a, 0)$ and $(a, 0)$ and directed to the top). The coalescence of all trajectories can be heuristically explained by the fact that two trajectories starting from the ordinate e^0 almost remain in a cone: their typical horizontal deviations at ordinate e^t are of order e^t . So, roughly speaking, they remain at the same hyperbolic horizontal distance from each other as they go up, implying that they must coalesce. This behaviour is due to the hyperbolic metric and does not occur in \mathbb{R}^d .

A bi-infinite branch can be identified as a sequence of Poisson points $(z_n)_{n \in \mathbb{Z}} \in N^{\mathbb{Z}}$ such that z_{n+1} is the parent of z_n for all $n \in \mathbb{Z}$. In the half-space representation, the boundary hyperplane $\mathbb{R}^d \times \{0\}$ are points at infinity for the hyperbolic geometry (a proper definition is given in the core of the article). We say that a bi-infinite branch converges to a point at infinity $(x, 0) \in \mathbb{R}^d \times \{0\}$ towards the past if it converges downwards to $(x, 0)$ (it is properly defined in Definition 2.2.16). Our second main result concerns bi-infinite branches and their asymptotic directions. The $(d + 1)$ -th coordinate y is seen as the time; the future is upward and the past is downward.

Theorem 2.1.2. *For all $d \geq 1$ and for all intensity $\lambda > 0$, the following assertions hold outside a set of probability zero:*

- (i) *The hyperbolic DSF admits infinitely many bi-infinite branches.*
- (ii) *Every bi-infinite branch of the hyperbolic DSF converges toward the past.*
- (iii) *For every $x \in \mathbb{R}^d$, there exists a bi-infinite branch of the hyperbolic DSF that converges to $(x, 0)$ toward the past.*
- (iv) *Such a branch is unique for almost every $x \in \mathbb{R}^d$. The set of $x \in \mathbb{R}^d$ for which there is no uniqueness is dense in \mathbb{R}^d . It is moreover countable in the bi-dimensional case (i.e. if $d = 1$).*

Moreover, for any deterministic $x \in \mathbb{R}^d$, the bi-infinite branch converging to $(x, 0)$ toward the past is unique a.s.

This result is specific to the hyperbolic case since the Euclidean DSF does not admit bi-infinite branches [12].

The existence of bi-infinite branches can be suggested by the following heuristic. In the half-space representation, because of the hyperbolic metric, the density of points decreases with the height, implying that a typical point will have a mean number of descendants larger than 1. Thus the tree of descendants of a typical point could be compared to a supercritical Galton-Watson tree and then should be infinite with positive probability. According to this heuristic, the hyperbolic DSF should admit infinitely many bi-infinite branches. On the contrary, in the Euclidean DSF, a typical point has a mean number of descendants equal to 1 (it can be seen by the Mass Transport Principle discussed later). Hence the corresponding analogy leads to a critical Galton-Watson tree which is finite a.s., which suggests that the Euclidean DSF does not admit bi-infinite branches.

The key point of the proofs is to upper-bound horizontal fluctuations of trajectories, both forward (i.e. upward) and backward (i.e. downward). Roughly speaking, we establish that a typical trajectory almost remains in a forward cone. Controlling the fluctuations of trajectories is a common technique to obtain the existence of infinite branches and to control their asymptotic directions: it is done for the RST in [1], and also by Howard & Newman in the context of first passage percolation [15].

To do it, we proceed in two steps. We first use a percolation argument to upper-bound fluctuations on a small vertical distance. Then we generalise the bound on an arbitrary vertical distance by a new technique based on the Mass Transport Principle (Theorem 2.3.2). This principle roughly says that for a given mass transport with isometries invariance properties (Definition 2.3.1), the incoming mass is equal to the outgoing mass. Most models in hyperbolic space studied in the literature are invariant by the group of all isometries, which is unimodular (i.e. the left-invariant Haar measure is also right-invariant), and the Mass Transport in the hyperbolic space [3, pp. 13-14] is well-adapted for these models. However, the hyperbolic DSF is only invariant by the group of isometries that fix a particular point at infinity, and this group is not unimodular (the invariance properties are explained in Section 2.2.3). For this reason, the Mass Transport Principle cannot be used in the same way. Instead, we introduce a slicing of H into levels $\mathbb{R}^d \times \{e^t\}$ for $t \in \mathbb{R}$, and we typically consider appropriate mass transports from $\mathbb{R}^d \times \{e^{t_1}\}$ to $\mathbb{R}^d \times \{e^{t_2}\}$ with $t_1 \leq t_2$, in order to obtain useful equalities by identifying the incoming mass and the outgoing mass.

The rest of the paper is organized as follows. In Section 2.2.1, we set some reminders on hyperbolic geometry. We also define the hyperbolic DSF in more details and we give its basic properties.

In Section 2.3, we state some technical results derived from the Mass Transport Principle in \mathbb{R}^d . These results are well fitted to take advantage of the translation invariance of the model in distribution.

In Section 2.4, we establish upper-bounds for horizontal fluctuations of forward (i.e. upward) and backward (i.e. downward) trajectories, which is the key point of the proofs. In particular, we show that a typical trajectory almost stays in a forward cone. A block control argument is used to upper-bound the fluctuations on a small vertical distance, and Mass Transport arguments are used to deduce the general bound.

In Section 2.5, we exploit the control of horizontal fluctuations to prove the coalescence in any dimensions (Theorem 2.1.1). The idea behind it is that, since two trajectories almost stay in cone, they roughly stay at the same hyperbolic horizontal distance to each other as they go up, thus they must coalesce. We also give a simpler proof of coalescence in the bi-dimensional case based on planarity.

In Section 2.6, we prove Theorem 2.1.2. We use a second moment technique to show the existence of bi-infinite branches, based on the control of forward horizontal fluctuations. We exploit the control of fluctuations backward to prove the results concerning asymptotic directions.

2.2 Definition of the hyperbolic DSF and general settings

We denote by \mathbb{N} the set of non-negative integers and by \mathbb{N}^* the set of positive integers. After recalling some facts on the hyperbolic space \mathbb{H}^{d+1} (Section 2.2.1), we consider an homogeneous PPP on \mathbb{H}^{d+1} and construct the hyperbolic DSF (Section 2.2.2).

2.2.1 The hyperbolic space and the half-space model

For $d \in \mathbb{N}^*$, the $(d+1)$ -dimensional hyperbolic space, denoted by \mathbb{H}^{d+1} , is a $(d+1)$ -dimensional Riemannian manifold, homogeneous and isotropic, and of constant negative curvature equal to -1 . The reader may refer to [7] or [19] for background on hyperbolic geometry. The space \mathbb{H}^{d+1} can be described with several isometric models and we will work in the *half-space model*, defined as the upper half-space $H := \{(x_1, \dots, x_d, y) \in \mathbb{R}^{d+1}, y > 0\}$ endowed with the metric

$$ds_H^2 := \frac{dx_1^2 + \dots + dx_d^2 + dy^2}{y^2}.$$

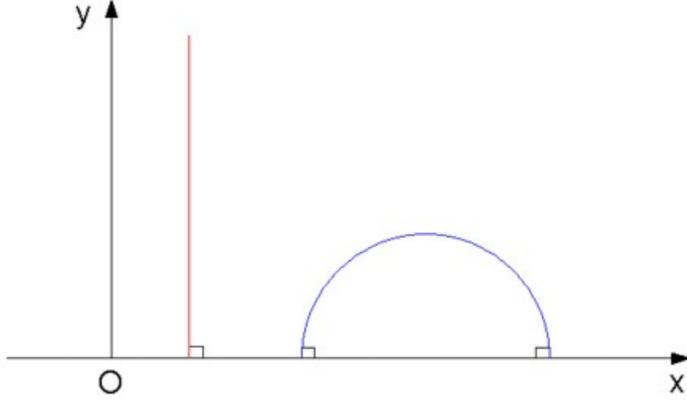


Figure 2.2: The geodesics on (H, ds_H^2) are of two types: on the one hand, the vertical straight lines $\{x = a\}$ for any $a \in \mathbb{R}^d$ (in red) and, on the other hand, the semi-circles contained in H and centered at a point of the boundary hyperplane ∂H (in blue).

The metric ds_H^2 naturally gives a volume measure μ_H on (H, ds_H^2) , given by

$$d\mu = \frac{dx_1 \dots dx_d dy}{y^{d+1}}.$$

Note that the last coordinate y plays a special role with respect to the other ones. The metric becomes smaller as we get closer to the boundary hyperplane $\partial H = \mathbb{R}^d \times \{0\}$, and this boundary is at infinite hyperbolic distance from any point of H . In the following, we will identify the point $(x_1, \dots, x_{d+1}) \in H$ with the couple $(x, y) \in \mathbb{R}^d \times \mathbb{R}_+^*$ with $x := (x_1, \dots, x_d)$ and $y := x_{d+1}$. The coordinate x is referred as the *abscissa* and y as the *ordinate*. For $z = (x, y) \in H$, we denote by $\pi_x(z) = x$ the abscissa of z and by $\pi_y(z) = y$ its ordinate. We also define its *height* as $h(z) := \ln(y)$. The height can be positive or negative depending whether $y \geq 1$ or $y \leq 1$. All along the paper, we will use the level 0 (with ordinate e^0) as a reference.

Let us set some general notation. We denote by $d(\cdot, \cdot)$ the hyperbolic distance in (H, ds_H^2) and by $\|\cdot\|$ the Euclidean norm in $\mathbb{R}^d \cup \{\infty\}$, with the convention $\|\infty\| = \infty$. For $z_1, z_2 \in H$, we denote by $[z_1, z_2]$ the geodesic between z_1 and z_2 and by $[z_1, z_2]_{eucl}$ the Euclidean segment between z_1 and z_2 . For $z \in H$ and $\rho > 0$, let $B_H(z, \rho) := \{z' \in H, d(z, z') < \rho\}$ be the hyperbolic ball centered at z of radius ρ . For $x \in \mathbb{R}^d$ and $r > 0$, let $B_{\mathbb{R}^d}(x, r) := \{x' \in \mathbb{R}^d, \|x' - x\| < r\}$ be the Euclidean ball centered at x of radius r . If there is no ambiguity, we will replace the notations $B_H(\cdot, \cdot)$ and $B_{\mathbb{R}^d}(\cdot, \cdot)$ with $B(\cdot, \cdot)$. Finally, for $z = (x, y) \in H$ and $\rho > 0$, we define the *upper semi-ball*

$$B_+(z, \rho) := B_H(z, \rho) \cap (\mathbb{R}^d \times (y, \infty)).$$

It is the part of the (hyperbolic) ball $B_H(z, \rho)$ that is above the hyperplane $\mathbb{R}^d \times \{y\}$ containing z . This hyperplane is a curved subspace of (H, ds_H^2) , so it does not split $B_H(z, \rho)$ in two isometric pieces.

We now state some useful facts about the half-space model. In (H, ds_H^2) , hyperbolic spheres are also Euclidean spheres. Moreover, the Euclidean center and the hyperbolic center belong to the same vertical line, but they do not coincide [7, Fact 1, p.86]. Hence, if $z_1, z_2, z_3 \in H$ are aligned in this order for the Euclidean metric (i.e. $z_2 \in [z_1, z_3]_{eucl}$), then $d(z_1, z_2) \leq d(z_1, z_3)$. We will use the following distance formula to do precise calculations in the half space model:

Proposition 2.2.1 (Distance formula). *Let $z_1 = (x_1, y_1) \in H$ and $z_2 = (x_2, y_2) \in H$. Let*

$\kappa = \|x_1 - x_2\|/y_1$ and $v = y_2/y_1$. Then

$$d(z_1, z_2) = 2 \tanh^{-1} \left(\sqrt{\frac{\kappa^2 + (v-1)^2}{\kappa^2 + (v+1)^2}} \right) = \Phi \left(\frac{\kappa^2 + (v+1)^2}{v} \right) \quad (2.2.1)$$

where $\Phi : [4, +\infty) \rightarrow \mathbb{R}_+$ is increasing and defined as

$$\Phi(t) = 2 \tanh^{-1} \left(\sqrt{1 - 4/t} \right) = \ln \left(\frac{1 + \sqrt{1 - 4/t}}{1 - \sqrt{1 - 4/t}} \right).$$

Remark 2.2.2. Given the ratio $v = y_2/y_1$, the distance $d(z_1, z_2)$ increases with κ . In particular, when y_1, y_2 are fixed, the distance $d(z_1, z_2)$ is minimal when $x_1 = x_2$.

The proof of Proposition 2.2.1 is given in the Supplementary materials (Section 2.2.1). We now discuss some particular cases of the distance formula. For two points on a same vertical straight line, $z_1 = (x, y_1)$ and $z_2 = (x, y_2)$, their distance is $d(z_1, z_2) = |\ln(y_2/y_1)| = |h(z_1) - h(z_2)|$. This shows that the notion of height is compatible with the hyperbolic distance, this justifies the relevance of this notion. In particular, for $z = (x, e^t)$ and $\rho > 0$; consider the hyperbolic (closed) ball $\overline{B}_H(z, \rho)$. Then the *top* (i.e. the point with the highest ordinate) of $\overline{B}_H(z, \rho)$ is precisely $(x, e^{t+\rho})$, and the *bottom* (i.e. the point with the lowest ordinate) of $\overline{B}_H(z, \rho)$ is $(x, e^{t-\rho})$.

For two points on the same horizontal hyperplane, $z_1 = (x_1, y)$ and $z_2 = (x_2, y)$, denoting by $R = \|x_1 - x_2\|$ their horizontal Euclidean distance, their hyperbolic distance can be estimated when $R \rightarrow \infty$ by $d(z_1, z_2) = 2 \ln(R/y) + o(1)$. Moreover, for any $R > 0$, $d(z_1, z_2) \leq R/y$.

The hyperbolic space \mathbb{H}^{d+1} is equipped with a set of points at infinity, denoted by $\partial\mathbb{H}^{d+1}$. In the half-space model (H, ds_H^2) , the points at infinity are identified by the boundary hyperplane $\partial H = \mathbb{R}^d \times \{0\}$, plus an additional point at infinity in all directions, obtained by compactification of the closed half-space $\mathbb{R}^d \times \mathbb{R}_+$. This particular point at infinity will be denoted by ∞ .

2.2.2 Definition of the hyperbolic DSF

Poisson point processes

Let $E = \mathbb{R}^d$ or \mathbb{H}^d . For any measurable subset $A \subset E$, we denote by $|A|$ its volume (it is either $\text{Leb}(A)$ in the Euclidean case or $\mu(A)$ in the hyperbolic case). Let us denote by \mathcal{N}_S the space of locally finite subsets of E , and for $A \subset E$ measurable, let $\mathcal{N}_S(A)$ be the space of locally finite subsets of A . The spaces \mathcal{N}_S and $\mathcal{N}_S(A)$ are equipped with the σ -algebra generated by counting applications (i.e. of the form $\eta \mapsto \#(\eta \cap K)$ for any compact set K).

Definition 2.2.3 (Homogeneous Poisson point process (PPP)). For $\lambda > 0$, a point process N is called *homogeneous Poisson point process of intensity* λ if for any measurable set $A \subset E$, $\#(N \cap A)$ is distributed according to the Poisson law with parameter $\lambda|A|$.

It can be shown that there is a unique probability measure on \mathcal{N}_S satisfying this condition. Moreover, if N is a homogeneous PPP and $A_1, \dots, A_n \subset E$ are pairwise disjoint measurable subsets, then $N \cap A_1, \dots, N \cap A_n$ are mutually independent [8].

Horodistance

In \mathbb{H}^{d+1} , the *horodistance* formalizes the notion of "distance from a point at infinity".

Definition 2.2.4 (Horodistance functions). Let $z_0 \in \mathbb{H}^{d+1}$ be an arbitrary point, considered as the origin. Given a point at infinity $I \in \partial\mathbb{H}^{d+1}$, the *horodistance function* $\mathcal{H}_I : \mathbb{H}^{d+1} \rightarrow \mathbb{R}$ is defined as

$$\mathcal{H}_I(z) := \lim_{z' \rightarrow I} d(z, z') - d(z_0, z'). \quad (2.2.2)$$

The existence of the limit (2.2.2) is proved in Chapter 1, Section 1.2.2. Any change of the origin point z_0 only affects the function \mathcal{H}_I up to an additive constant. So \mathcal{H}_I is naturally defined modulo an additive constant.

The level sets of \mathcal{H}_I , i.e. the sets of points at the same horodistance to I , are called *horospheres* (centered at I). Horospheres in (H, ds_H^2) are represented in Figure 2.3.

Proposition 2.2.5. *Consider (H, ds_H^2) and recall that the boundary point ∞ has been defined in Section 2.2.1. The horodistance function \mathcal{H}_∞ is (modulo an additive constant):*

$$\mathcal{H}_\infty((x, y)) = -\ln(y).$$

We refer to the Proposition 1.2.5 of Chapter 1 for a proof.

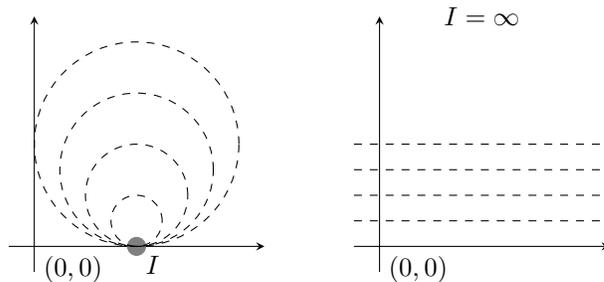


Figure 2.3: Horospheres centered at I in the half-space representation.

The DSF in hyperbolic space

We now introduce our model of DSF in \mathbb{H}^{d+1} . Fix $\lambda > 0$ and let N be a homogeneous PPP of intensity λ in \mathbb{H}^{d+1} . Consider a point at infinity $I \in \partial\mathbb{H}^{d+1}$, devoted to be the direction of the DSF or the *target point*. The choice of I is analogous to the choice of the direction vector u in the Euclidean case. In \mathbb{R}^d , each $z \in N$ is connected to its closest Poisson point among those that are "further" than z in some direction u . Similarly, in the hyperbolic case, we connect each point $z \in N$ to the closest Poisson point among those that are "further in direction I ", where being "further in direction I " is formalized by the notion of horodistance (Definition 2.2.4).

Definition 2.2.6 (Directed Spanning Forest in \mathbb{H}^{d+1}). Let $I \in \partial\mathbb{H}^{d+1}$. We call *Directed Spanning Forest (DSF)* in \mathbb{H}^{d+1} of direction I the oriented graph whose set of vertices is N and obtained by connecting each $z \in N$ to its *parent* $A(z)$ defined as

$$A(z) := \operatorname{argmin}_{z' \in N, \mathcal{H}_I(z') < \mathcal{H}_I(z)} d(z' - z).$$

A sketch of the construction is given in Figure 2.4. The choice of the direction I only affects the law of the DSF up to an isometry. Indeed, for any two points at infinity $I, I' \in \partial\mathbb{H}^{d+1}$, there exists an isometry that sends I on I' . In the following, we will work in the half-space representation (H, ds_H^2) (Section 2.2.1) and we set the direction $I = \infty$ for convenience. Indeed, the horodistance function \mathcal{H}_∞ only depends on the ordinate (Proposition 2.2.5), and $\mathcal{H}_\infty(z_1) < \mathcal{H}_\infty(z_2)$ if and only if $y_1 > y_2$. Thus, the parent of z is its closest Poisson point among those having higher ordinate than z .

Definition 2.2.6 does not specify the shape of edges, but the results announced in Theorems 2.1.1 and 2.1.2 only concern the graph structure of the hyperbolic DSF, so their veracity do not depend on the way points are connected. Here, we choose to connect each point $z \in N$ to its parent $A(z)$ by the Euclidean segment $[z, A(z)]_{\text{eucl}}$. It is more natural to represent edges with hyperbolic

geodesics, but the choice of Euclidean segments will appear more convenient for the proofs. The main reason of this choice is that we want that the y -coordinate increases along a given edge, and it is not the case using geodesics. Thus, we define the random subset $DSF \subset H$ as the union of all Euclidean segments $[z, A(z)]_{eucl}$ for $z \in N$: $DSF = \bigcup_{z \in N} [z, A(z)]_{eucl}$.

Remark 2.2.7. For $z \in N$, by definition of the parent $A(z)$, the upper semi-ball $B_+(z, d(z, A(z)))$ contains no points of N .

Convention 2.2.8. This (random) upper semi-ball $B_+(z, d(z, A(z)))$ will be more simply denoted by $B_+(z)$.

Proposition 2.2.9. *The DSF in \mathbb{H}^{d+1} is a forest a.s.*

Proof. Suppose that the hyperbolic DSF contains a cycle (z_0, \dots, z_{k-1}) . Consider the point of the cycle with the lowest ordinate. Then, by construction, both neighbors of z_i in the cycle must be parents of z_i , but z_i has only one parent, this is a contradiction. Therefore, the DSF does not contain cycles, it is a forest. \square

Proposition 2.2.10. *Almost surely, the DSF is non-crossing and has finite degree.*

Proposition 3.2.2 is show in the Appendix (Section 3.6)

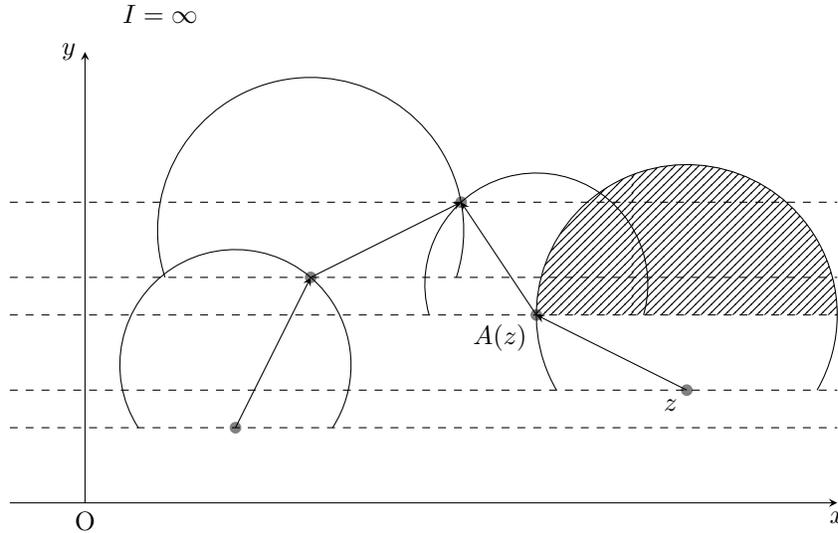


Figure 2.4: Sketch of construction of the hyperbolic DSF. This picture illustrates dependence phenomena existing in a single trajectory and between trajectories. Given a Poisson point $z \in N$, knowing the position of its parent $A(z)$ implies the knowledge that some region above $A(z)$, the upper part of a hyperbolic ball centered at z (the crosshatched area) is empty of Poisson points, which affects the future evolution of trajectories, and thus destroys nice Markov properties in the hyperbolic DSF.

General notations

If X_1, \dots, X_k are random variables, we denote by $\sigma(X_1, \dots, X_k)$ the sigma-algebra generated by X_1, \dots, X_k . If a random variable X is measurable w.r.t. $\sigma(N)$, then for $\eta \in \mathcal{N}_S$, we denote by $X(\eta)$ the value of X when $N = \eta$. Let us also denote by $T_s : x \mapsto x + s$ the translation by s in \mathbb{R}^d .

Let $z \in \text{DSF}$. Since the DSF is non-crossing [14, Section 3.1], there exists a unique $z_0 \in N$ such that $z \in [z_0, A(z_0))_{\text{eucl}}$. Then we define

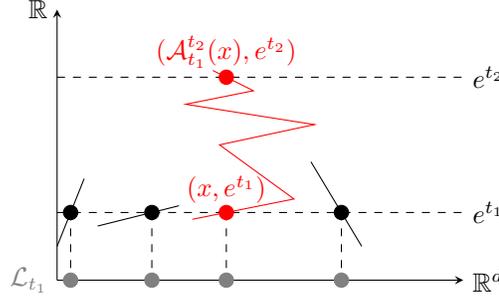
$$z_{\downarrow} := z_0, \quad z_{\uparrow} := A(z_0). \quad (2.2.3)$$

For $z \in \text{DSF}$, we define the *trajectory from z* as $[z, z_{\uparrow})_{\text{eucl}} \cup \bigcup_{n \in \mathbb{N}} [A^{(n)}(z_{\uparrow}), A^{(n+1)}(z_{\uparrow}))_{\text{eucl}}$, where $A^{(n)} := A \circ \dots \circ A$ n times.

Definition 2.2.11. For all $t \in \mathbb{R}$, we define the *level t* , denoted by \mathcal{L}_t , as the set of abscissas of points in DSF with height t :

$$\mathcal{L}_t = \{x \in \mathbb{R}^d, (x, e^t) \in \text{DSF}\}.$$

Definition 2.2.12. Let $t_1 \leq t_2$, and let $x \in \mathcal{L}_{t_1}$. The trajectory from (x, e^{t_1}) crosses the level t_2 (the hyperplane $\mathbb{R}^d \times \{e^{t_2}\}$) at most at one point. It could *a priori* never cross the level t_2 , if the y -coordinate stays indefinitely below e^{t_2} . Thus we define $\mathcal{A}_{t_1}^{t_2}(x)$ as the point $x' \in \mathbb{R}^d$ such that (x', e^{t_2}) belongs to the trajectory from (x, e^{t_1}) and we set $\mathcal{A}_{t_1}^{t_2}(x) = \infty$ if this trajectory does not cross the level t_2 . The point $\mathcal{A}_{t_1}^{t_2}(x)$ is called the *ancestor* of x .

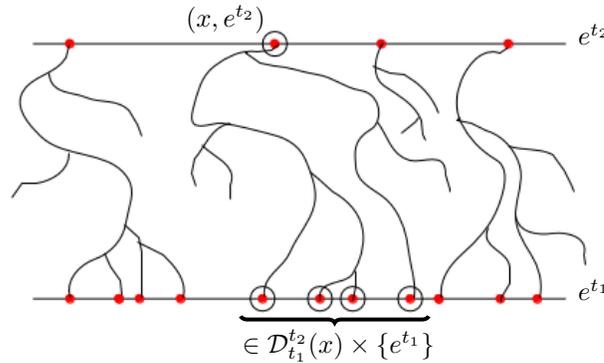


Actually, it will be shown later that the y -coordinates always goes to infinity a.s.:

Proposition 2.2.13. *Almost surely, for all $t_1 \leq t_2$, for all $x \in \mathcal{L}_{t_1}$, $\mathcal{A}_{t_1}^{t_2}(x) \neq \infty$.*

This statement is proved in Section 2.4.4.

Definition 2.2.14. Let $t_1 \leq t_2$, and let $x \in \mathcal{L}_{t_2}$. We define the sets of *descendants* of x , denoted by $\mathcal{D}_{t_1}^{t_2}(x)$, as the set of points $x' \in \mathcal{L}_{t_1}$ such that (x, e^{t_2}) belongs to the trajectory from (x', e^{t_1}) : $\mathcal{D}_{t_1}^{t_2}(x) = \{x' \in \mathcal{L}_{t_1}, \mathcal{A}_{t_1}^{t_2}(x') = x\}$.



Definition 2.2.15. Let $A \subset \mathbb{R}^d$ be measurable. We define the *coalescing height* of A , denoted by τ_A , as

$$\tau_A = \inf\{t \geq 0, \forall x, x' \in \mathcal{L}_0 \cap A \mathcal{A}_0^t(x) = \mathcal{A}_0^t(x')\} \in \mathbb{R}_+ \cup \{\infty\}.$$

It is the lowest height where all trajectories from points of $(\mathcal{L}_0 \cap A) \times \{e^0\}$ coalesce.

The following definition concerns bi-infinite branches, i.e. branches that are infinite in both directions.

Definition 2.2.16. Let $f : \mathbb{R} \rightarrow \mathbb{R}^d$. We say that f *encodes a bi-infinite branch* if for all $t_1 \leq t_2$, $f(t_2) = \mathcal{A}_{t_1}^{t_2}(f(t_1))$. In this case, the subset $\{(f(t), e^t), t \in \mathbb{R}\} \subset H$ is called a *bi-infinite branch* of the DSF.

We denote by BI the random set of functions $f : \mathbb{R} \rightarrow \mathbb{R}^d$ that encode a bi-infinite branch.

2.2.3 Preliminary properties

We will exploit invariance by isometries of the model. The family of *translations* $T_s^H : (x, y) \mapsto (x + s, y)$ for $s \in \mathbb{R}^d$ and the *dilations* $D_\lambda : (x, y) \mapsto (\lambda x, \lambda y)$ are isometries of (H, ds_H^2) [7, p.79], thus they preserve the law of N . Moreover, these isometries fix the point ∞ (isometries of (H, ds_H^2) are naturally extended to the set of points at infinity). Therefore they also preserve the horodistance function \mathcal{H}_∞ modulo an additive constant, so they preserve the graph structure of the DSF in law.

In addition, these isometries preserve Euclidean segments. Then, they preserve the law of the random subset DSF. A consequence of this translation invariance property is that, for all $t \in \mathbb{R}$, \mathcal{L}_t is a stationary point process.

It will be required to have a control of moments for the number of points of \mathcal{L}_t in a given compact set:

Proposition 2.2.17. *We have $\mathbb{E}[\#(\mathcal{L}_t \cap B_{\mathbb{R}^d}(0, R))^p] < \infty$ for all $t \in \mathbb{R}$ and $p, R \geq 0$.*

We refer to the Appendix (Section 2.7.3) for the proof.

Corollary 2.2.18. *For all $t \in \mathbb{R}$, \mathcal{L}_t has finite intensity $\alpha_0 e^{-dt}$, where α_0 is the intensity of \mathcal{L}_0 .*

Proposition 2.2.17 implies Corollary 2.2.18. By Proposition 2.2.17 with $p = 1$, \mathcal{L}_t has finite intensity for all $t \in \mathbb{R}$. Then we can define α_0 as the intensity of \mathcal{L}_0 . For $t \in \mathbb{R}$, the dilation D_{e^t} preserves the DSF in distribution, so $\mathcal{L}_t \stackrel{d}{=} e^t \mathcal{L}_0$. Then \mathcal{L}_t has finite intensity $\alpha_0 e^{-dt}$ for any $t \in \mathbb{R}$. \square

In the following, we will have to consider the law of DSF conditionally to $\{x \in \mathcal{L}_t\}$, for given $x \in \mathbb{R}^d$ and $t \in \mathbb{R}$. Thus we define the probability measure $\mathbb{P}_{x \in \mathcal{L}_t}[\cdot]$ on \mathcal{N}_S as the Palm distribution of N conditioned on $x \in \mathcal{L}_t$ (and let $\mathbb{E}_{x \in \mathcal{L}_t}[\cdot]$ its associated expectation). The definition of this probability measure follows the standard definition of Palm measures, however it should be a probability measure on all the point process (on \mathcal{N}_S) and not only on \mathcal{L}_t , that is why we need to re-define properly this probability distribution.

Proposition-definition 2.2.19 (Conditional distribution given $\{x \in \mathcal{L}_t\}$).

- (Definition) For $\Gamma \subset \mathcal{N}_S$ measurable, we define the measure μ_Γ on \mathbb{R}^d by

$$\mu_\Gamma(A) := \mathbb{E} \left[\sum_{s \in \mathcal{L}_t \cap A} \mathbf{1}_{T_{x-s} N \in \Gamma} \right] \quad (2.2.4)$$

for all measurable set $A \subset \mathbb{R}^d$. Note that μ_Γ depends on t and x .

- (Proposition) For all measurable set $\Gamma \subset \mathcal{N}_S$, the measure μ_Γ is invariant by translations and finite on compact sets.
- (Definition) Then for all measurable set $\Gamma \subset \mathcal{N}_S$, μ_Γ is a multiple of the Lebesgue measure, so we can define:

$$\mathbb{P}_{x \in \mathcal{L}_t}[\Gamma] := \frac{d\mu_\Gamma}{\alpha_0 e^{-dt} d\text{Leb}}. \quad (2.2.5)$$

- (Proposition) The map $\Gamma \mapsto \mathbb{P}_{x \in \mathcal{L}_t}[\Gamma]$ so defined is a probability measure on \mathcal{N}_S . We denote by $\mathbb{E}_{x \in \mathcal{L}_t}$ its associated expectation.

We refer to the Supplementary materials (Section 2.8.1) for the proof of Proposition-Definition 2.2.19.

Lemma 2.2.20. (*Invariance by dilations*). *Let $t, t' \in \mathbb{R}$. We have*

$$\mathbb{P}_{0 \in \mathcal{L}_t} [D_{e^{t'-t}}(N) \in \cdot] = \mathbb{P}_{0 \in \mathcal{L}_{t'}} [N \in \cdot]$$

The proof of Lemma 2.8.2 is also given in the Supplementary materials (Section 2.8.1).

2.3 The Mass Transport Principle and its consequences

In this section, we state a main ingredient of the proofs, the Mass Transport Principle and explore some consequences.

2.3.1 The Mass Transport Principle

This theorem is an adaptation of its version on the hyperbolic plane, which is due to Benjamini and Schramm [3, p.13-14].

Definition 2.3.1 (Diagonally invariant measure). Let π be some measure on $\mathbb{R}^d \times \mathbb{R}^d$ for the Borel σ -algebra. We say that π is *diagonally invariant* if for all $x \in \mathbb{R}^d$,

$$\pi(A \times B) = \pi(T_x A \times T_x B).$$

Theorem 2.3.2 (Mass Transport Principle). *Let π be some positive diagonally invariant measure on $\mathbb{R}^d \times \mathbb{R}^d$. Then for any measurable set $A \subset \mathbb{R}^d$ with nonempty interior, the following identity holds:*

$$\pi(A \times \mathbb{R}^d) = \pi(\mathbb{R}^d \times A),$$

these values can be eventually infinite.

A proof of Theorem 2.3.2 is given in the Supplementary materials (Section 1.3). The intuition behind the Mass Transport Principle can be understood as follows. The measure π describes a mass transport from \mathbb{R}^d to \mathbb{R}^d , that is, $\pi(A \times B)$ corresponds to the amount of mass transported from A to B . Then the Mass Transport Principle asserts that the outgoing mass equals the incoming mass.

In the literature, the study of percolation in hyperbolic space mostly concerns models that are invariant under any isometry of \mathbb{H}^{d+1} (see for instance, the Poisson-Boolean model studied in [23] or the Poisson-Voronoi model studied in [3]). Thus it is relevant to use the Mass Transport Principle on \mathbb{H}^{d+1} [3] to study these models. However, our model of DSF is directed, so it is only invariant under isometries that fix the target point. This group of isometries is not unimodular, so this version of the Mass Transport on \mathbb{H}^{d+1} cannot be used for the study of the DSF. Instead

of considering mass transports on \mathbb{H}^{d+1} , we typically consider mass transport from level t_1 to level t_2 (for $t_1, t_2 \in \mathbb{R}$), that is why we need the Mass Transport on \mathbb{R}^d .

We now state some consequences of the Mass Transport Principle, that play a central role in the control of horizontal fluctuations of trajectories (proofs of Theorems 2.4.4 and 2.4.5 in Section 2.4). We first define the concepts of weight function and association function (Section 2.3.2). From these objects, we construct diagonally invariant measures and obtain different equalities by identifying both sides of equality given in the Mass Transport Principle (Section 2.3.3). Proofs are given in Section 2.3.4.

2.3.2 Association functions and weight functions

Let us introduce a random variable Y independent of N , valued in some measurable space Υ . In a majority of applications, the extra random variable Y will not be necessary. However, an extra random variable will be used in the proof of (ii) in Theorem 2.4.4, because some association function using extra randomness will be considered.

Association functions

Definition 2.3.3 (Association function, general case). Let $t \in \mathbb{R}$. We call *level t -association function* or more simply *association function* a measurable function $f : \mathbb{R}^d \times \mathcal{N}_S \times \Upsilon \rightarrow \mathbb{R}^d$ such that

- f is valued in \mathcal{L}_t , more precisely

$$\forall x \in \mathbb{R}^d, f(x, N, Y) \in \mathcal{L}_t \text{ a.s.}$$

- f is *translation invariant*, in the following sense: for all $\eta \in \mathcal{N}_S$, for all $x, x' \in \mathbb{R}^d$,

$$f(x + x', T_{x'}\eta, Y) \stackrel{(d)}{=} f(x, \eta, Y) + x'.$$

We set the notation $f(x) := f(x, N, Y)$. An association function can be seen as a (translation invariant) random function from \mathbb{R}^d to \mathbb{R}^d .

For most of applications, f will not depend on Y (Y will be deterministic). This case will be referred as the *non-marked case*. In this case, the notation $f(x, \eta, Y)$ will be replaced by $f(x, \eta)$ for simplicity.

Definition 2.3.4. [Cell of a point] Let $t \in \mathbb{R}$, and let f be a level t -association function. For $x \in \mathcal{L}_t$, we define the *cell* of x as the (random) subset of \mathbb{R}^d :

$$\Lambda_f(x) := \{x' \in \mathbb{R}^d, f(x') = x\}.$$

Example 2.3.5. The most useful example to keep in mind is the following: $f(x, \eta)$ is defined as the point of $\mathcal{L}_0(\eta)$ the closest to x :

$$f(x, \eta) := \operatorname{argmin}_{x' \in \mathcal{L}_0(\eta)} \|x' - x\|.$$

Then f is a level 0-association function independent of Y (the non-marked case). Moreover, for $x \in \mathcal{L}_0$, $\Lambda_f(x)$ is the Voronoï cell of x associated to the point process \mathcal{L}_0 .

Example 2.3.6. Suppose that Y is a (homogeneous) PPP on \mathbb{R}^d independent of N . Define $f(x, \eta, \xi)$ as the point of $\mathcal{L}_0(\eta)$ the closest to x among all points $x' \in \mathcal{L}_0$ such that the ball $B(x', e^0)$ contains no points of ξ . Then f is a level 0-association function.

Another association function depending on a extra argument Y will be constructed in Section 2.4.7.

Weight functions

Definition 2.3.7 (Weight function, general case). We call *weight function* a measurable function $w : \mathbb{R}^d \times \mathcal{N}_S \times \Upsilon \rightarrow \mathbb{R}_+$ that is *translation invariant* in the following sense: for all $\eta \in \mathcal{N}_S$ and for all $x, x' \in \mathbb{R}^d$,

$$w(x, \eta, Y) \stackrel{(d)}{=} w(x + x', T_{x'}\eta, Y).$$

We set the notation

$$w(x) := w(x, N, Y).$$

A weight function can be seen as a random application from \mathbb{R}^d to \mathbb{R}_+ .

The case where w does not depend on Y will be referred as the *non-marked case*. In this case, we replace the notation $w(x, \eta, Y)$ by $w(x, \eta)$.

Example 2.3.8. Consider the function $w(x, \eta) := \mathbf{1}_{x \in \mathcal{L}_0(\eta)} \|\mathcal{A}_0^1(x)(\eta) - x\|$. It is the horizontal deviation between levels 0 and 1 of the trajectory from (x, e^0) when $x \in \mathcal{L}_0$. Then w is a weight function in the non-marked case.

Example 2.3.9. Suppose that Y is a random variable independent of N and valued in \mathbb{N}^* . We can define $w(x, \eta, n)$ as the distance ($\|\cdot\|_2$ in \mathbb{R}^d) between x and the point of \mathcal{L}_0 which is the n -th closest to x . Then w is a weight function.

Weighted association functions

Definition 2.3.10. Let $t \in \mathbb{R}$. We call *level t -weighted association function* (or more simply, *weighted association function*) a couple (f, w) , where f is an association function and w is a weight function, such that the *couple* (f, w) is translation invariant in distribution, that is, for all $\eta \in \mathcal{N}_S$ and for all $x, x' \in \mathbb{R}^d$:

$$\left(f(x + x', T_{x'}\eta, Y), w(x + x', T_{x'}\eta, Y) \right) \stackrel{(d)}{=} \left(f(x, \eta, Y) + x', w(x, \eta, Y) \right). \quad (2.3.1)$$

Note that, in the non-marked case (f and w does not depend of Y), the condition (2.3.1) is useless.

Example 2.3.11. Consider the association f introduced in Example 2.3.5: $f(x, \eta)$ is the point of $\mathcal{L}_0(\eta)$ the closest to x . Let $w(x, \eta) := \|x - f(x, \eta)\|$. Then w is a weight function and (f, w) is a weighted association function (in the non-marked case).

Example 2.3.12. Consider the association function f introduced in Example 2.3.6. Then define $w(x, \eta, \xi) := \#(\xi \cap B(0, e^0))$. Then w is a weight function, however the couple (f, w) is not a weighted association function.

2.3.3 Results derived from the Mass Transport Principle

Let us extend the Palm distribution $\mathbb{P}_{x \in \mathcal{L}_t}$ defined in Section 2.2.3 on $\sigma(N)$ to $\sigma(N, Y)$ by setting:

$$\mathbb{P}_{x \in \mathcal{L}_t}[N \in \Gamma, Y \in \Gamma'] = \mathbb{P}_{x \in \mathcal{L}_t}[N \in \Gamma] \mathbb{P}[Y \in \Gamma']$$

for all $\Gamma \in \mathcal{N}_S$ and $\Gamma' \subset \Upsilon$. This defines a probability measure on $\sigma(N, Y)$, and it also extends the notation $\mathbb{E}_{x \in \mathcal{L}_t}[X]$ to random variables X measurable w.r.t. $\sigma(N, Y)$.

Lemma 2.3.13. *Let $x \in \mathbb{R}^d$ and $t \in \mathbb{R}$. Let $w : \mathbb{R}^d \times \mathcal{N}_S \times \Upsilon \rightarrow \mathbb{R}_+$ be a weight function. Then for all measurable set $A \subset \mathbb{R}^d$,*

$$\mathbb{E} \left[\sum_{s \in \mathcal{L}_t \cap A} w(s) \right] = \alpha_0 e^{-dt} \text{Leb}(A) \mathbb{E}_{0 \in \mathcal{L}_t}[w(0)]. \quad (2.3.2)$$

Proposition 2.3.14. *Let $t_1, t_2 \in \mathbb{R}$ with $t_1 \leq t_2$. Let $w : \mathbb{R}^d \times \mathcal{N}_s \times \Upsilon \rightarrow \mathbb{R}_+$ be a weight function. Then*

$$\mathbb{E}_{0 \in \mathcal{L}_{t_1}} [w(0)] = e^{d(t_1 - t_2)} \mathbb{E}_{0 \in \mathcal{L}_{t_2}} \left[\sum_{x \in \mathcal{D}_{t_1}^{t_2}(0)} w(x) \right]. \quad (2.3.3)$$

Corollary 2.3.15 (Expected number of descendants). *Let $t_1, t_2 \in \mathbb{R}$ with $t_1 \leq t_2$. We have*

$$\mathbb{E}_{0 \in \mathcal{L}_{t_2}} [\#\mathcal{D}_{t_1}^{t_2}(0)] = e^{d(t_2 - t_1)}. \quad (2.3.4)$$

In particular, for all $x \in \mathcal{L}_{t_2}$, $\#\mathcal{D}_{t_1}^{t_2}(x) < \infty$.

Proof of Corollary 2.3.15 knowing Proposition 2.3.14. Applying Proposition 2.3.14 to $w \equiv 1$ leads to (2.3.4).

Thus, we obtain that $\mathbb{P}_{0 \in \mathcal{L}_{t_2}} [\#\mathcal{D}_{t_1}^{t_2}(0) < \infty] = 1$. Then we apply Lemma 2.3.13 with $A = \mathbb{R}^d$ to the weight function w' defined as

$$w'(x, \eta) = \mathbf{1}_{x \in \mathcal{L}_{t_2}(\eta), \#\mathcal{D}_{t_1}^{t_2}(x)(\eta) = \infty}.$$

It leads to:

$$\begin{aligned} \mathbb{E} [\#\{x \in \mathcal{L}_{t_2}, \#\mathcal{D}_{t_1}^{t_2}(x) = \infty\}] &= \mathbb{E} \left[\sum_{x \in \mathcal{L}_{t_2}} w'(x) \right] = \infty \mathbb{E}_{0 \in \mathcal{L}_{t_2}} [w'(0)] \\ &= \infty \mathbb{P}_{0 \in \mathcal{L}_{t_2}} [\#\mathcal{D}_{t_1}^{t_2}(x) = \infty] = 0. \end{aligned}$$

Thus, for all $x \in \mathcal{L}_{t_2}$, $\#\mathcal{D}_{t_1}^{t_2}(x) < \infty$. □

Proposition 2.3.16. *Let $t \in \mathbb{R}$ and let (f, w) be a level t -weighted association function. Then*

$$\mathbb{E} [w(0)] = \alpha_0 e^{-dt} \mathbb{E}_{0 \in \mathcal{L}_t} \left[\int_{\Lambda_f(0)} w(x) dx \right]. \quad (2.3.5)$$

Corollary 2.3.17 (Expected volume of a typical cell). *Let $t \in \mathbb{R}$, f be a level t -association function. Applying Proposition 2.3.16 with $w \equiv 1$ (it is easy to check that (f, w) is a weighted association function), we obtain:*

$$\mathbb{E}_{0 \in \mathcal{L}_t} [\text{Leb}(\Lambda_f(0))] = \alpha_0^{-1} e^{dt}.$$

Proposition 2.3.18. *Let $t \in \mathbb{R}$ and $p \geq 1$. Let f be a level t -association function. Then*

$$\mathbb{E}_{0 \in \mathcal{L}_t} [\text{Leb}(\Lambda_f(0))^{1+p/d}] \leq C_{p,d} e^{dt} \mathbb{E} [\|f(0)\|^p], \quad (2.3.6)$$

where $C_{p,d}$ is a positive constant that only depends on p and d .

2.3.4 Proofs

We first prove Lemma 2.3.13.

Proof of Lemma 2.3.13. We first prove the non-marked case. Let $w : \mathbb{R}^d \times \mathcal{N}_S \rightarrow \mathbb{R}_+$ be a weight function in the non-marked case.

For $\eta \in \mathcal{N}_S$, define $g(\eta) = w(x, \eta)$. By translation invariance, for all $r \in \mathbb{R}^d$, $\eta \in \mathcal{N}_S$, $w(r, \eta) = w(x, T_{x-r}\eta) = g(T_{x-r}\eta)$. In particular w is entirely determined by g .

Let $\Gamma \subset \mathcal{N}_S$ be measurable. For all measurable set $A \subset \mathbb{R}^d$,

$$\mathbb{E} \left[\sum_{s \in \mathcal{L}_t \cap A} w(s, N) \right] = \mathbb{E} \left[\sum_{s \in \mathcal{L}_t \cap A} w(x, T_{x-s}N) \right] = \mathbb{E} \left[\sum_{s \in \mathcal{L}_t \cap A} g(T_{x-s}N) \right]$$

and

$$\mathbb{E}_{x \in \mathcal{L}_t} [w(x, N)] = \mathbb{E}_{x \in \mathcal{L}_t} [g(N)].$$

Thus it suffices to prove that the identity

$$\mathbb{E} \left[\sum_{s \in \mathcal{L}_t \cap A} g(T_{x-s}N) \right] = \alpha_0 e^{-dt} \text{Leb}(A) \mathbb{E}_{x \in \mathcal{L}_t} [g(N)] \quad (2.3.7)$$

holds for all measurable functions $g : \mathcal{N}_S \rightarrow \mathbb{R}_+$ and all measurable set $A \subset \mathbb{R}^d$. Let $\Gamma \subset \mathcal{N}_S$ and $A \subset \mathbb{R}^d$ be measurable. We show (2.3.7) for $g = \mathbf{1}_\Gamma$:

$$\begin{aligned} \mathbb{E} \left[\sum_{s \in \mathcal{L}_t \cap A} g(T_{x-s}N) \right] &= \mathbb{E} \left[\sum_{s \in \mathcal{L}_t \cap A} \mathbf{1}_{T_{x-s}N \in \Gamma} \right] = \alpha_0 e^{-dt} \text{Leb}(A) \mathbb{P}_{x \in \mathcal{L}_t} [\Gamma] \text{ by (2.8.2)} \\ &= \alpha_0 e^{-dt} \text{Leb}(A) \mathbb{E}_{x \in \mathcal{L}_t} [\mathbf{1}_{N \in \Gamma}] = \mathbb{E}_{x \in \mathcal{L}_t} [g(N)], \end{aligned}$$

so (2.3.7) holds for $g = \mathbf{1}_\Gamma$. Since both sides of equality (2.3.7) are linear in g , (2.3.7) holds for all step functions. Now we pass to the limit to obtain (2.3.7) for all measurable function g . Let $g : \mathcal{N}_S \rightarrow \mathbb{R}_+$ be measurable, and consider a non-decreasing sequence $(g_n)_{n \in \mathbb{N}}$ of step functions that converges to g . By monotone convergence theorem,

$$\lim_{n \rightarrow \infty} \uparrow \sum_{s \in \mathcal{L}_t \cap A} g_n(T_{x-s}N) = \sum_{s \in \mathcal{L}_t \cap A} g(T_{x-s}N) \text{ a.s.}$$

Then by monotone convergence theorem,

$$\mathbb{E} \left[\sum_{s \in \mathcal{L}_t \cap A} g_n(T_{x-s}N) \right] \xrightarrow{n \rightarrow \infty} \mathbb{E} \left[\sum_{s \in \mathcal{L}_t \cap A} g(T_{x-s}N) \right]. \quad (2.3.8)$$

On the other hand, again by monotone convergence,

$$\mathbb{E}_{x \in \mathcal{L}_t} [g_n(N)] \xrightarrow{n \rightarrow \infty} \mathbb{E}_{x \in \mathcal{L}_t} [g(N)]. \quad (2.3.9)$$

By (2.3.8) and (2.3.9) we obtain (2.3.7) for g by passing to the limit.

We wove on to show the general case. Let us denote by \mathbb{P}_λ the distribution of N (probability measure on \mathcal{N}_S) and by \mathbb{Q} the distribution of Y (probability measure on Υ). Let $w : \mathbb{R}^d \times \mathcal{N}_S \times \Upsilon \rightarrow \mathbb{R}_+$ be a weight function in the general case. Define $\tilde{w}(x, \eta) := \int_\xi w(x, \eta, \xi) \mathbb{Q}(d\xi)$ for all $x \in \mathbb{R}^d$, $\eta \in \mathcal{N}_S$. Then \tilde{w} is a weight function in the non-marked case, so by the non-marked case applies to \tilde{w} :

$$\mathbb{E} \left[\sum_{s \in \mathcal{L}_t \cap A} \tilde{w}(s) \right] = \alpha_0 e^{-dt} \text{Leb}(A) \mathbb{E}_{x \in \mathcal{L}_t} [\tilde{w}(x)]. \quad (2.3.10)$$

For all $\eta \in \mathcal{N}_S$,

$$\sum_{s \in \mathcal{L}_t(\eta) \cap A} \tilde{w}(s, \eta) = \sum_{s \in \mathcal{L}_t(\eta) \cap A} \int_\xi w(x, \eta, \xi) \mathbb{Q}(d\xi) = \int_\xi \sum_{s \in \mathcal{L}_t(\eta) \cap A} w(s, \eta, \xi) \mathbb{Q}(d\xi).$$

Therefore

$$\begin{aligned} \mathbb{E} \left[\sum_{s \in \mathcal{L}_t \cap A} \tilde{w}(s) \right] &= \int_{\eta} \sum_{x \in \mathcal{L}_t(\eta) \cap A} \tilde{w}(s, \eta) d\mathbb{P}_{\lambda}(\eta) = \int_{\eta, \xi} \sum_{s \in \mathcal{L}_t(\eta) \cap A} w(s, \eta, \xi) \mathbb{P}_{\lambda}(d\eta) \otimes \mathbb{Q}(d\xi) \\ &= \mathbb{E} \left[\sum_{s \in \mathcal{L}_t(\eta) \cap A} w(s) \right] \end{aligned} \quad (2.3.11)$$

since N and Y are independent. On the other hand,

$$\mathbb{E}_{x \in \mathcal{L}_t} [\tilde{w}(x)] = \int_{\eta, \xi} w(x, \eta, \xi) \mathbb{P}_{x \in \mathcal{L}_t}(d\eta) \otimes \mathbb{Q}(d\xi) = \mathbb{E}_{x \in \mathcal{L}_t} [w(x)] \quad (2.3.12)$$

by definition of $\mathbb{E}_{x \in \mathcal{L}_t}$. Finally, the conclusion is obtained by combining (2.3.10), (2.3.11) and (2.3.12). \square

The proofs of Propositions 2.3.14 and 2.3.16 are based on the Mass Transport Principle.

Proof of Proposition 2.3.14. Let us define the following measure π on $\mathbb{R}^d \times \mathbb{R}^d$:

$$\pi(E) = \mathbb{E} \left[\sum_{x \in \mathcal{L}_{t_1}} \mathbf{1}_{(x, \mathcal{A}_{t_1}^{t_2}(x)) \in E} w(x) \right]$$

for all measurable set $E \subset \mathbb{R}^d \times \mathbb{R}^d$. This measure π may be interpreted as the following mass transport: from all point $x \in \mathcal{L}_{t_1}$, we transport a mass $w(x)$ to its ancestor $\mathcal{A}_{t_1}^{t_2}(x)$. The diagonally invariance of π easily follows from the translation invariance property of the model, we refer to [14, Section 5.4] for the details. By the Mass Transport Principle (Theorem 2.3.2), for any measurable set $A \subset \mathbb{R}^d$ with non-empty interior, $\pi(A \times \mathbb{R}^d) = \pi(\mathbb{R}^d \times A)$. On the one hand,

$$\pi(A \times \mathbb{R}^d) = \mathbb{E} \left[\sum_{x \in \mathcal{L}_{t_1}} \mathbf{1}_{x \in A} w(x) \right] = \mathbb{E} \left[\sum_{x \in \mathcal{L}_{t_1} \cap A} w(x) \right] = \alpha_0 e^{-dt_1} \text{Leb}(A) \mathbb{E}_{0 \in \mathcal{L}_{t_1}} [w(0)], \quad (2.3.13)$$

where Lemma 2.3.13 has been applied to w with $t = t_1$ and $x = 0$. On the other hand,

$$\begin{aligned} \pi(\mathbb{R}^d \times A) &= \mathbb{E} \left[\sum_{x \in \mathcal{L}_{t_1}} \mathbf{1}_{\mathcal{A}_{t_1}^{t_2}(x) \in A} w(x) \right] = \mathbb{E} \left[\sum_{\substack{x \in \mathcal{L}_{t_1}, \\ x' \in \mathcal{L}_{t_2} \cap A}} \mathbf{1}_{\mathcal{A}_{t_1}^{t_2}(x) = x'} w(x) \right] \\ &= \mathbb{E} \left[\sum_{x' \in \mathcal{L}_{t_2} \cap A} \sum_{x \in \mathcal{D}_{t_1}^{t_2}(x')} w(x) \right]. \end{aligned} \quad (2.3.14)$$

Consider the level t_2 -weight function

$$h(x', \eta, \xi) = \mathbf{1}_{x' \in \mathcal{L}_{t_2}} \sum_{x \in \mathcal{D}_{t_1}^{t_2}(x')(\eta)} w(x, \eta, \xi).$$

The fact that h is a weight function directly follows from translation invariance, precise computations are done in [14, Section 5.4]. Lemma 2.3.13 applied to h with $t = t_2$ and $x = 0$ gives,

$$\pi(\mathbb{R}^d \times A) \stackrel{(2.3.14)}{=} \mathbb{E} \left[\sum_{x' \in \mathcal{L}_{t_2} \cap A} \sum_{x \in \mathcal{D}_{t_1}^{t_2}(x')} w(x) \right] = \alpha_0 e^{-dt_2} \text{Leb}(A) \mathbb{E}_{0 \in \mathcal{L}_{t_2}} \left[\sum_{x \in \mathcal{D}_{t_1}^{t_2}(0)} w(x) \right]. \quad (2.3.15)$$

By combining (2.3.13), (2.3.15) and the Mass Transport Principle with some open set $A \subset \mathbb{R}^d$ verifying $\text{Leb}(A) < \infty$, we obtain (2.3.3). \square

Proof of Proposition 2.3.16. Let us consider the measure on $\mathbb{R}^d \times \mathbb{R}^d$ defined as

$$\pi(E) := \mathbb{E} \left[\int_{\mathbb{R}^d} \mathbf{1}_{(x, f(x)) \in E} w(x) dx \right]$$

for all $E \in \mathbb{R}^d \times \mathbb{R}^d$. This measure π may be interpreted as the following Mass Transport: for each point $x \in \mathbb{R}^d$, we transport a mass $w(x, f(x)) dx$ from x to $f(x)$. The measure E is diagonally invariant, thus the Mass Transport Principle applies. On the one hand:

$$\pi(A \times \mathbb{R}^d) = \mathbb{E} \left[\int_A w(x) dx \right] = \int_A \mathbb{E} [w(x)] dx = \int_A \mathbb{E} [w(0)] dx = \text{Leb}(A) \mathbb{E} [w(0)]. \quad (2.3.16)$$

where the translation invariance of (f, w) was used in the third equality. Indeed, $w(x, N, Y) \stackrel{(d)}{=} w(0, T_{-x}N, Y) \stackrel{(d)}{=} w(0, N, Y)$ so $\mathbb{E} [w(x)] = \mathbb{E} [w(0)]$ for all $x \in \mathbb{R}^d$. On the other hand,

$$\begin{aligned} \pi(\mathbb{R}^d \times A) &= \mathbb{E} \left[\int_{\mathbb{R}^d} \mathbf{1}_{f(x) \in A} w(x) dx \right] = \mathbb{E} \left[\int_{\mathbb{R}^d} \sum_{x' \in \mathcal{L}_t \cap A} \mathbf{1}_{f(x)=x'} w(x) dx \right] \\ &= \mathbb{E} \left[\sum_{x' \in \mathcal{L}_t \cap A} \int_{\mathbb{R}^d} \mathbf{1}_{f(x)=x'} w(x) dx \right] = \mathbb{E} \left[\sum_{x' \in \mathcal{L}_t \cap A} \int_{\Lambda_f(x')} w(x) dx \right]. \end{aligned} \quad (2.3.17)$$

Let $h : \mathbb{R}^d \times \mathcal{N}_S \times \Upsilon \rightarrow \mathbb{R}_+$ be defined as $h(x', \eta, \xi) = \int_{\Lambda_f(x')(\eta)} w(x, \eta, \xi) dx$. This function h is a weight function (details in [14, Section 5.4]), so, by Lemma 2.3.13 applied to h with $x = 0$,

$$\mathbb{E} \left[\sum_{x' \in \mathcal{L}_t \cap A} \int_{\Lambda_f(x')} w(x) dx \right] = \alpha_0 e^{-dt} \text{Leb}(A) \mathbb{E}_{0 \in \mathcal{L}_t} \left[\int_{\Lambda_f(0)} w(x) dx \right]. \quad (2.3.18)$$

Thus, by (2.3.17) and (2.3.18), we obtain

$$\pi(\mathbb{R}^d \times A) = \alpha_0 e^{-dt} \text{Leb}(A) \mathbb{E}_{0 \in \mathcal{L}_t} \left[\int_{\Lambda_f(0)} w(x) dx \right]. \quad (2.3.19)$$

Finally, we obtain (2.3.5) by combining (2.3.16), (2.3.19) and the Mass Transport Principle for some open set $A \subset \mathbb{R}^d$ verifying $\text{Leb}(A) < \infty$. \square

Proof of Proposition 2.3.18. Let us consider the function w defined as $w(x, \eta, \xi) = \|f(x, \eta, \xi) - x\|^p$ for $x \in \mathbb{R}^d$, $\eta \in \mathcal{N}_S$ and $\xi \in \Upsilon$. It follows from translation invariance that the couple (f, w) is a level t -weighted association function (details in [14, Section 5.4]). Proposition 2.3.16 applied to (f, w) gives,

$$\mathbb{E} [\|f(0)\|^p] = \alpha_0 e^{-dt} \mathbb{E}_{0 \in \mathcal{L}_t} \left[\int_{\Lambda_f(0)} \|f(x) - x\|^p dx \right] = \alpha_0 e^{-dt} \mathbb{E}_{0 \in \mathcal{L}_t} \left[\int_{\Lambda_f(0)} \|x\|^p dx \right] \quad (2.3.20)$$

because $f(x) = 0$ for all $x \in \Lambda_f(0)$. Suppose for the moment that the following inequality holds $\mathbb{P}_{0 \in \mathcal{L}_t}$ -almost surely:

$$\text{Leb}(\Lambda_f(0))^{1+p/d} \leq \tilde{C}_{p,d} \int_{\Lambda_f(0)} \|x\|^p dx, \quad (2.3.21)$$

where $\tilde{C}_{p,d}$ is a constant that only depends on p and d . Then

$$\mathbb{E}_{0 \in \mathcal{L}_t} [\text{Leb}(\Lambda_f(0))^{1+p/d}] \stackrel{(2.3.21)}{\leq} \tilde{C}_{p,d} \mathbb{E}_{0 \in \mathcal{L}_t} \left[\int_{\Lambda_f(0)} \|x\|^p dx \right] \stackrel{(2.3.20)}{=} \frac{\tilde{C}_{p,d} e^{dt}}{\alpha_0} \mathbb{E} [\|f(0)\|^p],$$

so (2.3.6) holds for $C_{p,d} = \tilde{C}_{p,d}/\alpha_0$. It remains to show that (2.3.21) holds $\mathbb{P}_{0 \in \mathcal{L}_t}$ -almost surely. For $r \geq 0$ we denote by $B_r := \{x \in \mathbb{R}^d, \|x\| < r\}$ the Euclidean ball of radius r centered at the origin, and we denote by $\vartheta(d) := \text{Leb}(B_1)$ the volume of the unit ball in \mathbb{R}^d . We rewrite $\|x\|^p$ as $\int_0^\infty \mathbf{1}_{r \leq \|x\|} p r^{p-1} dr$. On the event $\{0 \in \mathcal{L}_t\}$, Fubini's Theorem gives,

$$\begin{aligned} \int_{\Lambda_f(0)} \|x\|^p dx &= \int_{\mathbb{R}^d} \int_0^\infty \mathbf{1}_{x \in \Lambda_f(0)} \mathbf{1}_{r \leq \|x\|} p r^{p-1} dr dx = \int_0^\infty \int_{\mathbb{R}^d} \mathbf{1}_{x \in \Lambda_f(0) \setminus B_r} p r^{p-1} dx dr \\ &= \int_0^\infty p r^{p-1} \text{Leb}(\Lambda_f(0) \setminus B_r) dr \geq \int_0^{\left(\frac{\text{Leb}(\Lambda_f(0))}{\vartheta(d)}\right)^{1/d}} p r^{p-1} (\text{Leb}(\Lambda_f(0)) - \text{Leb}(B_r)) dr \\ &= \int_0^{\left(\frac{\text{Leb}(\Lambda_f(0))}{\vartheta(d)}\right)^{1/d}} p r^{p-1} (\text{Leb}(\Lambda_f(0)) - \vartheta(d) r^d) dr \\ &= \left[\text{Leb}(\Lambda_f(0)) r^p - \frac{\vartheta(d) p}{p+d} r^{p+d} \right]_{r=0}^{\left(\frac{\text{Leb}(\Lambda_f(0))}{\vartheta(d)}\right)^{1/d}} = \left(\frac{d}{p+d} \vartheta(d)^{-p/d} \right) \text{Leb}(\Lambda_f(0))^{1+p/d}. \end{aligned}$$

Therefore (2.3.21) holds for $\tilde{C}_{p,d} = (1 + p/d)\vartheta(d)^{p/d}$. This completes the proof. \square

2.4 Controlling fluctuations of trajectories

In order to show the main results (Theorems 2.1.1 and 2.1.2), the key point of the proofs is to upper-bound horizontal fluctuations of trajectories.

2.4.1 Cumulative Forward Deviation and Maximal Backward Deviation

We first define the Cumulative Forward Deviation (CFD) and Maximal Backward Deviation (MBD) that measure horizontal deviations, then we state the results concerning CFD and MBD.

Definition 2.4.1 (Cumulative Forward Deviation). Let $t_1 \leq t_2$. For $x \in \mathcal{L}_{t_1}$, we define the *Cumulative Forward Deviation* for x from level t_1 to level t_2 , denoted by $\text{CFD}_{t_1}^{t_2}(x)$, as

$$\text{CFD}_{t_1}^{t_2}(x) = \int_{t_1}^{t_2} \left\| \frac{\partial}{\partial s} \mathcal{A}_{t_1}^s(x) \right\| ds.$$

The quantity $\text{CFD}_{t_1}^{t_2}(x)$ can be considered as the cumulative horizontal deviations (i.e. projected on \mathbb{R}^d) of the trajectory starting from (x, e^{t_1}) between level t_1 and level t_2 . If $\mathcal{A}_{t_1}^s(x) = \infty$ for some $s \in [t_1, t_2]$, we set $\text{CFD}_{t_1}^{t_2}(x) = \infty$.

We give an equivalent definition of the quantity $\text{CFD}_{t_1}^{t_2}(x)$. We suppose that $\mathcal{A}_{t_1}^{t_2}(x) < \infty$ (else $\text{CFD}_{t_1}^{t_2}(x) = \infty$). Let us define the points $z_{start} = (x, e^{t_1})_\downarrow$ and $z_{stop} = (\mathcal{A}_{t_1}^{t_2}(x), e^{t_2})_\downarrow$. Thus z_{stop} is on the trajectory from z_{start} , let $n \in \mathbb{N}$ such that $z_{stop} = A^{(n)}(z_{start})$. Let us introduce $(x_0, e^{u_0}) = z_0 = z_{start}$, $(x_i, e^{u_i}) = z_i = A^{(i)}(z_{start})$ for $i \in \llbracket 1, n \rrbracket$. In particular, $z_n = z_{stop}$. Then:

Proposition 2.4.2 (Alternative writing of CFD).

$$\text{CFD}_{t_1}^{t_2}(x) = \begin{cases} \|\mathcal{A}_{t_1}^{t_2}(x) - x\| & \text{if } n = 0, \\ \|x_1 - x\| + \sum_{i=1}^{n-1} \|x_{i+1} - x_i\| + \|\mathcal{A}_{t_1}^{t_2}(x) - x_n\| & \text{if } n \geq 1. \end{cases}$$

Proof. If $n = 0$ then (x, e^{t_1}) and $(\mathcal{A}_{t_1}^{t_2}(x), e^{t_2})$ belong to the same edge, so the function $\frac{\partial}{\partial s} \mathcal{A}_{t_1}^s$ has constant direction. Then

$$\text{CFD}_{t_1}^{t_2}(x) = \int_{t_1}^{t_2} \left\| \frac{\partial}{\partial s} \mathcal{A}_{t_1}^s(x) \right\| ds = \left\| \int_{t_1}^{t_2} \frac{\partial}{\partial s} \mathcal{A}_{t_1}^s(x) ds \right\| = \|\mathcal{A}_{t_1}^{t_2}(x) - x\|.$$

If $n \geq 1$, then

$$\begin{aligned}
\text{CFD}_{t_1}^{t_2}(x) &= \int_{t_1}^{u_1} \left\| \frac{\partial}{\partial s} \mathcal{A}_{t_1}^s(x) \right\| ds + \sum_{i=1}^{n-1} \int_{u_i}^{u_{i+1}} \left\| \frac{\partial}{\partial s} \mathcal{A}_{t_1}^s(x) \right\| ds + \int_{u_n}^{t_2} \left\| \frac{\partial}{\partial s} \mathcal{A}_{t_1}^s(x) \right\| ds \\
&= \left\| \int_{t_1}^{u_1} \frac{\partial}{\partial s} \mathcal{A}_{t_1}^s(x) ds \right\| + \sum_{i=1}^{n-1} \left\| \int_{u_i}^{u_{i+1}} \frac{\partial}{\partial s} \mathcal{A}_{t_1}^s(x) ds \right\| + \left\| \int_{u_n}^{t_2} \frac{\partial}{\partial s} \mathcal{A}_{t_1}^s(x) ds \right\| \\
&= \|x_1 - x\| + \sum_{i=1}^{n-1} \|x_{i+1} - x_i\| + \|\mathcal{A}_{t_1}^{t_2}(x) - x_n\|.
\end{aligned}$$

For the second equality, we used the fact that, for each term of the sum, the function $s \rightarrow \frac{\partial}{\partial s} \mathcal{A}_{t_1}^s$ has constant direction in the corresponding integration interval. \square

Note that CFD upperbounds the horizontal deviations, in the following sense: for all $t_1 \leq t_2$ and $x \in \mathcal{L}_{t_1}$,

$$\|\mathcal{A}_{t_1}^{t_2}(x) - x\| = \left\| \int_{t_1}^{t_2} \frac{\partial}{\partial s} \mathcal{A}_{t_1}^s(x) ds \right\| \leq \int_{t_1}^{t_2} \left\| \frac{\partial}{\partial s} \mathcal{A}_{t_1}^s(x) \right\| ds = \text{CFD}_{t_1}^{t_2}(x).$$

Definition 2.4.3 (Maximal Backward Deviation). We define the *Maximal Backward Deviation* of x from level t_1 to level t_2 , denoted by $\text{MBD}_{t_1}^{t_2}(x)$, as

$$\text{MBD}_{t_1}^{t_2}(x) = \begin{cases} \max_{x' \in \mathcal{D}_{t_1}^{t_2}(x)} \text{CFD}_{t_1}^{t_2}(x') & \text{if } \mathcal{D}_{t_1}^{t_2}(x) \neq \emptyset \\ 0 & \text{otherwise.} \end{cases}$$

This is the maximal cumulative horizontal deviations among all trajectories between levels t_1 and t_2 and ending at (x, e^{t_2}) .

The following theorem controls the cumulative forward deviation (CFD).

Theorem 2.4.4. (*Forward fluctuations control.*) Let $p \geq 1$.

(i) Let Y be a random variable independent of N . Let X_0 be a random point of \mathcal{L}_0 (i.e. a random point of \mathbb{R}^d such that $X_0 \in \mathcal{L}_0$ a.s.), measurable w.r.t. $\sigma(N, Y)$, and such that $\mathbb{E}[\|X_0\|^p] < \infty$. Then there exists a constant $K > 0$ that only depends on p, d and λ (but not on the distribution of X_0) such that:

$$\limsup_{t \rightarrow \infty} \mathbb{E} \left[(e^{-t} \text{CFD}_0^t(X_0))^p \right] \leq K.$$

(ii) We have

$$\limsup_{t \rightarrow \infty} \mathbb{E}_{0 \in \mathcal{L}_0} \left[(e^{-t} \text{CFD}_0^t(0))^p \right] < \infty. \quad (2.4.1)$$

The intuition behind Theorem 2.4.4 is the following. Let us consider a typical trajectory. Because of the hyperbolic metric on (H, ds_H^2) , the horizontal fluctuations increase as the trajectory goes up. More precisely, the fluctuations around height h are to the order of e^h . Then the forward cumulative deviation between height 0 and height h is almost determined by the last steps of the trajectory, and it is to the order of e^h . This behaviour is typical to hyperbolic geometry. In Euclidean geometry, the fluctuations around height h are to the order of 1 for all h .

The following theorem controls the backward maximal deviation (MBD).

Theorem 2.4.5. (*Backward fluctuations control.*) For all $p \geq 1$,

$$\limsup_{h \rightarrow \infty} \mathbb{E}_{0 \in \mathcal{L}_0} [MBD_{-h}^0(0)^p] < \infty. \quad (2.4.2)$$

The intuition behind Theorem 2.4.5 is that horizontal fluctuations decrease toward the past (recall that the fluctuations around height h are of order e^h), so the sum of fluctuations between level $-\infty$ and 0 of a typical trajectory must be bounded.

The rest of this section is devoted to the proofs of Theorems 2.4.4 and 2.4.5. It will be also proved that almost surely, the y -coordinate goes to infinity along any trajectory (Proposition 2.2.13).

2.4.2 Sketch of the proofs

The proofs are organized as follows. First, we control horizontal deviations between level 0 and level δ for some small $\delta > 0$. More precisely, we prove the following proposition:

Proposition 2.4.6. *Recall that CFD has been defined in Definition 2.4.1. There exists $\delta > 0$ such that, for all $p \geq 1$,*

$$\mathbb{E}_{0 \in \mathcal{L}_0} [CFD_0^\delta(0)^p] < \infty. \quad (2.4.3)$$

In particular, for all $t \in [0, \delta]$, $\mathbb{P}_{0 \in \mathcal{L}_0}$ -a.s., $\mathcal{A}_0^t(0) \neq \infty$.

The proof of Proposition 2.4.6, based on a block control argument, is done in Section 2.4.3. In Section 2.4.4, we deduce Proposition 2.2.13 from Proposition 2.4.6.

Then, we prove (i) in Theorem 2.4.4 as follows: we propagate the control up to level δ given by Proposition 2.4.6 to obtain a control up to level t for all $t \geq 0$. It will be done by induction: from a control up to level t , we deduce a new control up to level $t + \delta$, by using Proposition 2.4.6 and mass transport arguments. The proof of (i) is done in Section 2.4.5.

In order to prove (ii), we will apply (i) to some particular X_0 measurable w.r.t $\sigma(N, Y)$, where Y is some random variable independent of N . The extra randomness that will be used in the definition of X_0 is the reason why we introduced the extra random variable Y in Section 2.3. The proof of (ii) is done in Section 2.4.7.

Finally we prove Theorem 2.4.5 in Section 2.4.8. The proof is based on (ii) in Theorem 2.4.4 and mass transport arguments.

2.4.3 Proof of Proposition 2.4.6

Step 1: a block control argument.

We pave \mathbb{R}^d with cubes of edge length R , where $R > 0$ is sufficiently large and will be chosen later. For $a = (a_1, \dots, a_d) \in \mathbb{Z}^d$, let us define the cube $K_a := \prod_{1 \leq i \leq d} [R(a_i - 1/2), R(a_i + 1/2))$. Let us also define the bottom and top cells $\Psi_a^b := K_a \times [e^0, e^\delta)$ and $\Psi_a^t := K_a \times [e^\delta, e]$, where $0 < \delta \leq 1/2$ is sufficiently small and will be chosen later.

For $a \in \mathbb{Z}^d$, we say that K_a is *good* if Ψ_a^b contains no points of N , and Ψ_a^t contains at least one point of N , i.e. we define the event

$$\text{Good}(a) := \{N \cap \Psi_a^b = \emptyset\} \cap \{N \cap \Psi_a^t \neq \emptyset\}.$$

Note that the event $\text{Good}(a)$ only depends on $N \cap (\Psi_a^b \cup \Psi_a^t)$ and the cells $(\Psi_a^b \cup \Psi_a^t)$ are disjoint, so the events $\text{Good}(a)$ are mutually independent. Moreover they have the same probability by invariance by horizontal translations.

For $m \in \mathbb{N}$, we say that K_a is *m-very good*, if K_a is good and if all cubes at distance at most m of the K_a are good:

$$\text{VeryGood}_m(a) := \bigcap_{\substack{a'=(a'_1, \dots, a'_d) \in \mathbb{Z}^d, \\ \|a'-a\|_\infty \leq m}} \text{Good}(a').$$

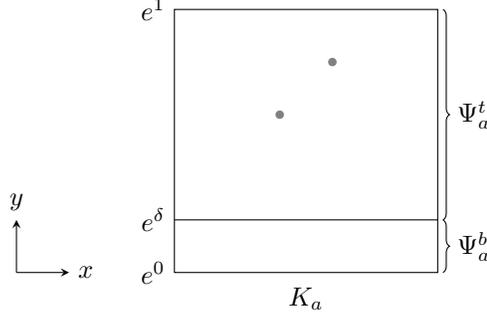


Figure 2.5: A good cube

We can consider the random field $V_m : \mathbb{Z}^d \rightarrow \{0, 1\}$ defined as $V_m(a) = \mathbf{1}_{\text{VeryGood}_m(a)}$ for all $a \in \mathbb{Z}^d$. We denote by $\Upsilon_m(X)$ the connected component of the subgraph induced by $\{a \in \mathbb{Z}^d, V_m(a) = 0\}$ containing the origin if $V_m(0) = 0$ (otherwise we set $\Upsilon_m = \emptyset$). This is the connected component of (indices of) non *m-very good* cubes containing the origin. Let $\tilde{\Upsilon}_m = \bigcup_{a \in \Upsilon_m} K_a$. We also define $\rho_m := \sup\{\|a\|, a \in \Upsilon_m\} \in \mathbb{N} \cup \{\infty\}$ the radius of Υ_m , with the convention $\sup \emptyset = 0$. Note that those quantities depend on δ and R .

In order to prove Proposition 2.4.6, we will prove that, for m large enough, any trajectory from a 0-level point in $K_0 \times \{e^0\}$ crosses the level δ at most "just after" it exits $\tilde{\Upsilon}_m \times [e^0, \infty)$, and that we can choose R, δ such that Υ_m is small (i.e. its radius admits exponential moment).

More precisely, we will use the two following lemmas. The first lemma asserts that, when a trajectory (projected on the x -axis hyperplane) crosses a *m-very good* cube for m large enough, then it crosses the level δ not far from this cube.

Lemma 2.4.7. *There exists $m \in \mathbb{N}$ depending only on d such that, almost surely, for all $R \geq 1$ and for all $\delta \in (0, 1/2]$, the following happens: for all *m-very good* cube K_a , for all $0 \leq t \leq t' \leq \delta$ and for all $x \in \mathcal{L}_t \cap K_a$,*

$$\|\mathcal{A}_t^{t'}(x) - Ra\| \leq mR. \quad (2.4.4)$$

The next lemma asserts that, R and δ can be chosen such that the radius ρ_m of the "bad" component admits exponential moments.

Lemma 2.4.8. *For all $m \in \mathbb{N}$, there exists $R \geq 1$ and $\delta \in (0, 1/2]$ such that for all $a \geq 1$,*

$$\mathbb{P}[\rho_m > a] \leq e^{-Ca}. \quad (2.4.5)$$

Lemma 2.4.7 will be proved in Step 4. In order to show that the radius admits exponential moment, we will use a theorem due to Liggett, Schonmann and Stacey [18, Theorem 0.0, p.75] to show that the field $(V_m(a))_{a \in \mathbb{Z}^d}$ is dominated from below by a product random field with density ρ that can arbitrarily close to 1 as $\mathbb{P}[\text{VeryGood}_m(0)]$ is close to 1. Lemma 2.4.8 will be proved in Step 5.

Choose m that satisfies Lemma 2.4.7. Then choose $R, \delta > 0$ that satisfies Lemma 2.4.8 for the value of m previously chosen. We will prove in Steps 2 and 3 that Proposition 2.4.6 holds for the chosen value of δ , assuming Lemmas 2.4.7 and 2.4.8.

Step 2: we show that almost surely, for all $x \in \mathcal{L}_0 \cap K_0$, and for all $t' \in [0, \delta]$,

$$\|\mathcal{A}_0^{t'}(x) - x\| \leq R(\sqrt{d} + m + \rho_m + 1). \quad (2.4.6)$$

The intuition behind this inequality is the following. If $x \in \mathcal{L}_0 \cap K_0$, when the trajectory from (x, e^0) exits the "bad" component Υ_m , which has radius ρ_m , it crosses a m -very good cube. Then Lemma 2.4.7 asserts that the trajectory should exit the strip $\mathbb{R}^d \times [e^0, e^\delta]$ at most at distance mR from the center of this cube.

The conclusion is immediate if $\rho_m = \infty$, so we suppose that $\rho_m < \infty$ in the following. Let $x \in \mathcal{L}_0 \cap K_0$ and $t' \in [0, \delta]$. If K_0 is very good, then by Lemma 2.4.7 applied with $t = 0$, $\|\mathcal{A}_0^{t'}(x)\| \leq mR$, so $\|\mathcal{A}_0^{t'}(x) - x\| \leq mR + \|x\| \leq R(m + \sqrt{d}/2)$, so we are done.

Suppose that K_0 is not m -very-good. Then $\Upsilon_m \neq \emptyset$. Let us define the outer-boundary of Υ_m and $\tilde{\Upsilon}_m$ by $\partial_{out} \Upsilon_m = \{a \in \mathbb{Z}^d, a \notin \Upsilon_m \text{ and } \exists a' \in \Upsilon_m, \|a - a'\| = 1\}$.

By definition $\partial_{out} \Upsilon_m$ is made of (indices of) very good cubes. Moreover, for all $a \in \partial_{out} \Upsilon_m$, $\|a\| \leq \rho_m + 1$. Since $\rho_m < \infty$, $\tilde{\Upsilon}_m$ is bounded so there are a finite number of points of N in $\tilde{\Upsilon}_m \times [e^0, e^{t'}]$. Then the trajectory starting from (x, e^0) should exit $\tilde{\Upsilon}_m \times [e^0, e^{t'}]$, so by continuity it should cross $\partial(\tilde{\Upsilon}_m \times [e^0, e^{t'}])$. Consider the first time (i.e. the lowest level) when the trajectory crosses $\partial(\tilde{\Upsilon}_m \times [e^0, e^{t'}])$, i.e.

$$t_{min} = \min\{t > 0, (\mathcal{A}_0^t(x), e^t) \in \partial(\tilde{\Upsilon}_m \times [e^0, e^{t'}])\}.$$

The time t_{min} is well-defined since $\partial(\tilde{\Upsilon}_m \times [e^0, e^{t'}])$ is closed. If $t_{min} = t'$ (Case 1 in Figure 2.6), then for all $t \in [0, t']$, $\mathcal{A}_0^t(x) \in \tilde{\Upsilon}_m$, so $\|\mathcal{A}_0^t(x) - x\| \leq \rho_m R + \|x\| \leq R(\rho_m + \sqrt{d}/2) + R\sqrt{d}/2 = R(\rho_m + \sqrt{d})$, so we are done. Otherwise (Case 2 in Figure 2.6), $t_{min} < t'$. In this case, $\mathcal{A}_0^{t_{min}}(x) \in \partial\tilde{\Upsilon}_m$ so $\mathcal{A}_0^{t_{min}}(x) \in K_a$ for some $a \in \partial_{out} \Upsilon_m$. Since $a \in \partial\Upsilon_m$, K_a is a very good cube, therefore by Lemma 2.4.7, $\|\mathcal{A}_0^{t'}(x) - Ra\| \leq \|\mathcal{A}_0^{t'}(\mathcal{A}_0^{t_{min}}(x)) - Ra\| \leq mR$. Then $\|\mathcal{A}_0^{t'}(x) - x\| \leq \|Ra\| + \|Ra - \mathcal{A}_0^{t'}(x)\| + \|x\| \leq R(\rho_m + 1) + mR + R\sqrt{d}/2 = R(\sqrt{d} + m + \rho_m + 1)$, this completes the proof of (2.4.6).

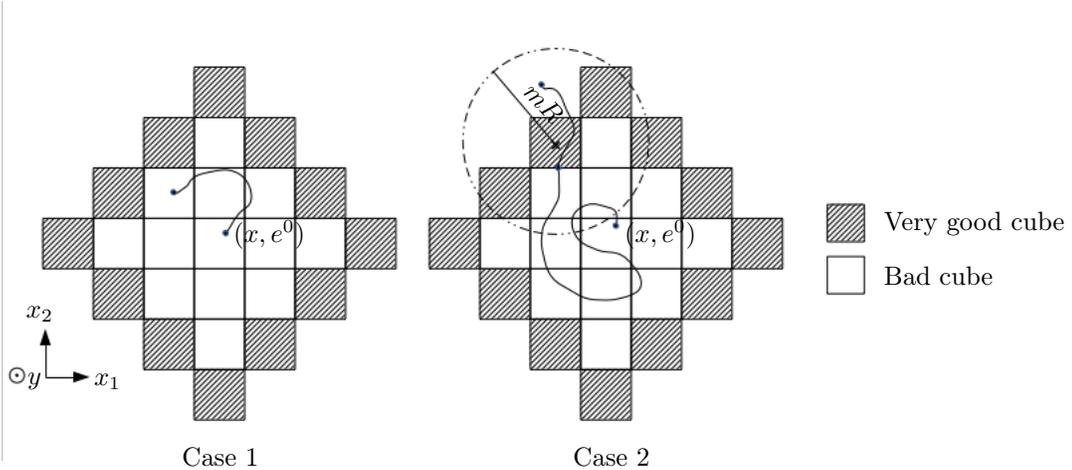


Figure 2.6: Representation of the trajectory from (x, e^0) below level t'

Step 3: end of proof of Proposition 2.4.6. Let $x \in \mathcal{L}_0 \cap K_0$. By Inequality (2.4.6) proved in Step 2, the trajectory starting from (x, e^0) is entirely contained in the cylinder $\mathcal{C} := B(0, R(\sqrt{d} + m + \rho_m + 1)) \times [e^0, e^\delta]$ before exiting the strip $\mathbb{R}^d \times [e^0, e^\delta]$. Then this portion of trajectory is made

of Euclidean segments whose horizontal deviations are upperbounded by $2R(\sqrt{d} + m + \rho_m + 1)$. Moreover, the number of segments is (roughly) upperbounded by $1 + \#(N \cap \mathcal{C})$. Then

$$\text{CFD}_0^\delta(x) \leq 2R(\sqrt{d} + m + \rho_m + 1)(1 + \#(N \cap \mathcal{C})). \quad (2.4.7)$$

By construction, ρ_m admits exponential moments, and $\#(N \cap \mathcal{C})$ admits exponential moments, therefore, $2R(\sqrt{d} + m + \rho_m + 1)(1 + \#(N \cap \mathcal{C})) \in L^p$ for all $p \geq 1$.

Now, let $p \geq 1$. Lemma 2.3.13 applied to the weight function $g(x, \eta) = \text{CFD}_0^\delta(x)(\eta)^p$, with $A = [-1/2, 1/2]^d$, gives:

$$\begin{aligned} \alpha_0 \mathbb{E}_{0 \in \mathcal{L}_0} \left[\text{CFD}_0^\delta(x)^p \right] &= \mathbb{E} \left[\sum_{x \in [-1/2, 1/2]^d} \text{CFD}_0^\delta(x)^p \right] \\ &\stackrel{(2.4.7)}{\leq} \mathbb{E} \left[\#(\mathcal{L}_0 \cap [-1/2, 1/2]^d) \left[2R(\sqrt{d} + m + \rho_m + 1)(1 + \#(N \cap \mathcal{C})) \right]^p \right]. \end{aligned} \quad (2.4.8)$$

By Proposition 2.2.17, $\#(\mathcal{L}_0 \cap [-1/2, 1/2]^d) \in L^2$. Moreover, $2R(\sqrt{d} + m + \rho_m + 1)(1 + \#(N \cap \mathcal{C})) \in L^{2p}$ by the previous discussion. Thus Cauchy-Schwartz gives,

$$\mathbb{E} \left[\#(\mathcal{L}_0 \cap [-1/2, 1/2]^d) \left[2R(\sqrt{d} + m + \rho_m + 1)(1 + \#(N \cap \mathcal{C})) \right]^p \right] < \infty, \quad (2.4.9)$$

so, combining (2.4.8) and (2.4.9), we obtain $\mathbb{E}_{0 \in \mathcal{L}_0} \left[\text{CFD}_0^\delta(0)^p \right] < \infty$, this proves Proposition 2.4.6.

Step 4: proof of Lemma 2.4.7. Consider $m \in \mathbb{N}$ large enough that will be chosen later. Let $R \geq 1$ and $\delta \in [0, 1/2]$. Let $a \in \mathbb{Z}^d$ and suppose that K_a is an m -very good cube. Let $0 \leq t \leq t' \leq \delta$ and $x \in \mathcal{L}_t \cap K_a$, define $z_0 = (x, e^t)$ and let $z = z_{0\downarrow} \in N$ (recall that the notation $z_{0\downarrow}$ has been defined by 2.2.3). Let $B = B_{\mathbb{R}^d}(Ra, Rm)$. By definition of a m -very good cube, none of the $K_{a'}$ when $\|a' - a\|_\infty \leq m$ contains points of N and since B is included in the union of $K_{a'}$ for $\|a' - a\|_\infty \leq m$, $N \cap (B \times [e^0, e^t]) = \emptyset$. Thus $A(z) \notin B \times [e^0, e^t]$.

Suppose $m \geq R\sqrt{d}/2$. Then $K_a \subset B$, so $z_0 = (x, e^0) \in K_a \times \{1\} \subset B \times [e^0, e^t]$. So $[z_0, A(z)]_{\text{eucl}}$ must cross either $B \times \{e^t\}$, or $\partial B \times [e^0, e^t]$. In the first case, $\mathcal{A}_t^t(x) \in B$, so we are done. Now we eliminate the case $[z_0, A(z)]_{\text{eucl}} \cap (\partial B \times [e^0, e^t]) \neq \emptyset$.

The following arguments are illustrated in Figure 2.7. Suppose by contradiction that $[z_0, A(z)]_{\text{eucl}}$ crosses $\partial B \times [e^0, e^t]$, and denote by z_1 the intersection point. Since $z_1 \in [z_0, A(z)]_{\text{eucl}}$, $d(z_0, z_1) \leq d(z_0, A(z))$ (recall that $d(\cdot, \cdot)$ denotes the hyperbolic distance). Since K_a is a good cube, by definition of a good cube there exists some $z_2 \in N \cap \Psi_a^t$. Since $h(z_2) \geq \delta \geq h(z_0) > h(z)$, $d(z, A(z)) < d(z, z_2)$ by definition of the parent. So $d(z, z_1) \leq d(z, A(z)) < d(z, z_2)$. Consider $\Xi := \{z' \in H, d(z', z_1) \leq d(z', z_2)\}$. Ξ is the region of H containing z_1 delimited by the mediator totally geodesic hyperplane of $[z_1, z_2]$. Since $y(z_1) \leq e^\delta \leq y(z_2)$, Ξ is either a semi (Euclidean) ball centered at a point of $\mathbb{R}^d \times \{0\}$ if $y(z_1) < y(z_2)$, or a half-space of H delimited by a vertical hyperplane if $y(z_1) = y(z_2)$. In both cases, Ξ is convex (for the Euclidean metric). Therefore, since $z \in \Xi$ by the previous discussion and $z_1 \in \Xi$ by definition, by convexity $z_0 \in \Xi$, so

$$d(z_0, z_1) \leq d(z_0, z_2). \quad (2.4.10)$$

Recall that $z_0 = (x, e^t)$, and we set $z_1 = (x_1, y_1)$, $z_2 = (x_2, y_2)$. So $\|x_1 - Ra\| = mR$, $y_1 \in [e^t, e^t]$, $x_2 \in K_a$ so $\|x_2 - Ra\| \leq R\sqrt{d}/2$ and $y_2 \in [e^t, e^0]$. We use the distance formula (Proposition 2.2.1) to compare $d(z_0, z_1)$ and $d(z_0, z_2)$. We obtain

$$d(z_0, z_1) = \Phi \left(\frac{(\|x_1 - x\|/e^t)^2 + (y_1/e^t + 1)^2}{y_1/e^t} \right).$$

Since $e^t \leq y_1 \leq e^{t'}$ and $t \geq 0$, $1 \leq y_1/e^t \leq e^{t'}$. Since $x \in K_a$, $\|x - Ra\| \leq R\sqrt{d}/2$ so $\|x_1 - x\| \geq \|x_1 - Ra\| - \|Ra - x\| \geq R(m - \sqrt{d}/2)$. Therefore, since Φ is increasing

$$\begin{aligned} d(z_0, z_1) &\geq \Phi \left(\frac{\left(\frac{R(m - \sqrt{d}/2)}{e^{t'}} \right)^2 + 4}{e^{t'}} \right) \geq \Phi \left(R^2(m - \sqrt{d}/2)^2 e^{-3t'} + 4e^{-t'} \right) \\ &\geq \Phi \left(R^2(m - \sqrt{d}/2)^2 e^{-3/2} + 4e^{-1/2} \right), \end{aligned} \quad (2.4.11)$$

since $t' \leq \delta \leq 1/2$. Moreover

$$d(z_0, z_2) = \Phi \left(\frac{(\|x_2 - x\|/e^t)^2 + (y_2/e^t + 1)^2}{y_2/e^t} \right).$$

Since $1 \leq e^t \leq y_2 \leq e$, $1 \leq y_2/e^t \leq e$. Since $x, x_2 \in K_a$, $\|x_2 - x\| \leq R\sqrt{d}$. Therefore, since Φ is increasing,

$$d(z_0, z_1) \leq \Phi \left(R^2 d e^{-2t} + (e + 1)^2 \right) \leq \Phi \left(R^2 d + (e + 1)^2 \right). \quad (2.4.12)$$

Comparing the bounds in (2.4.11) and (2.4.12), a sufficient condition for $d(z_0, z_1) > d(z_0, z_2)$ is that:

$$\begin{aligned} &\Phi \left(R^2(m - \sqrt{d}/2)^2 e^{-3/2} + 4e^{-1/2} \right) > \Phi \left(R^2 d + (e + 1)^2 \right) \\ \iff &R^2(m - \sqrt{d}/2)^2 e^{-3/2} + 4e^{-1/2} > R^2 d + (e + 1)^2 \\ \iff &R^2 > \frac{(e+1)^2 - 4e^{-1/2}}{(m - \sqrt{d}/2)^2 e^{-3/2} - d}. \end{aligned} \quad (2.4.13)$$

As $m \rightarrow \infty$, $\frac{(e+1)^2 - 4e^{-1/2}}{(m - \sqrt{d}/2)^2 e^{-3/2} - d} \rightarrow 0$, so we can choose m large enough such that $\frac{(e+1)^2 - 4e^{-1/2}}{(m - \sqrt{d}/2)^2 e^{-3/2} - d} < 1$. For this value of m , and since $R \geq 1$, we have $d(z_0, z_1) > d(z_0, z_2)$, which contradicts (2.4.10). This proves Lemma 2.4.7.

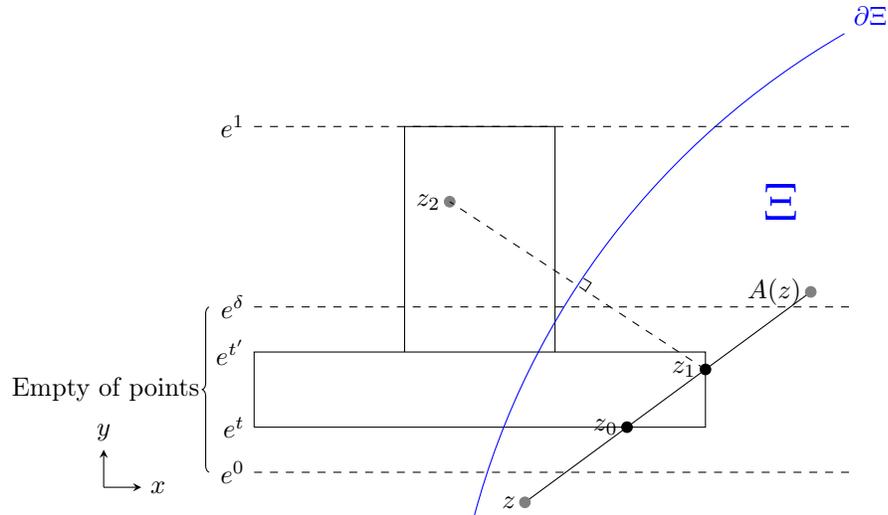


Figure 2.7: Case we want to rule out: the situation implies $d(z_0, z_1) \leq d(z_0, z_2)$, whereas computations lead to $d(z_0, z_1) > d(z_0, z_2)$.

Step 5: proof of Lemma 2.4.8. Let $m \in \mathbb{N}$. By translation invariance, $\mathbb{P}[\text{Good}(a)] = \mathbb{P}[\text{Good}(0)]$ for all $a \in \mathbb{Z}^d$. Since the events $\text{Good}(a)$ are mutually independent, for all $a \in \mathbb{Z}^d$, $\mathbb{P}[\text{VeryGood}_m(a)] = \mathbb{P}[\text{Good}(0)]^{(2m+1)^d}$. By definition, for $a \in \mathbb{Z}^d$, the event $\text{VeryGood}_m(a)$ only depends on the events $\text{Good}(a')$ with $\|a' - a\|_\infty \leq m$. In particular, the events $\text{VeryGood}_m(a)$ are not mutually independent. However, the dependencies are only local. Let $a, a' \in \mathbb{Z}^d$ such that $\|a - a'\|_\infty > 2m$. For all $a'' \in \mathbb{Z}^d$, we can't have both $\|a'' - a\|_\infty \leq m$ and $\|a'' - a'\|_\infty \leq m$. Therefore, $\text{VeryGood}_m(a)$ is independent of the family of events $(\text{VeryGood}_m(a'))_{a' \in \mathbb{Z}^d, \|a' - a\|_\infty > 2m}$. So the field $(V_m(a))_{a \in \mathbb{Z}^d}$ is $2m$ -dependant.

Thus Theorem 0.0 of [18] tells us that there exists a non-decreasing function $\chi : [0, 1] \rightarrow [0, 1]$ verifying $\lim_{t \rightarrow 1} \chi(t) = 1$ (and independent of the parameters R, δ) such that, if $(Y_a)_{a \in \mathbb{Z}^d}$ is a product random field of intensity $\chi(\mathbb{P}[V_m(0) = 1])$, then $(V_m(a))_{a \in \mathbb{Z}^d} \succeq_{st} (Y_a)_{a \in \mathbb{Z}^d}$ for the product order on $\{0, 1\}^{\mathbb{Z}^d}$.

It is well known that there exists some $\tilde{p}_c > 0$ such that for all $p < \tilde{p}_c$, in the product random field $(Y_a)_{a \in \mathbb{Z}^d}$ of density p , the radius of the cluster containing the origin admits exponential moments, see for instance Chap 6. of [16]. Choose such $\tilde{p}_c > 0$. Pick $p'_c < 1$ such that $\chi(p) > 1 - \tilde{p}_c$ for all $p > p'_c$ (it is possible since $\lim_{p \rightarrow 1} \chi(p) = 1$), and set $p_c^* = p_c'^{(2m+1)^{-d}} < 1$. It is shown in the next paragraph that $\mathbb{P}[\text{Good}(0)] > p_c^*$ for judiciously chosen R, δ . Then $\mathbb{P}[\text{VeryGood}_m(a)] = \mathbb{P}[\text{Good}(0)]^{(2m+1)^d} > p_c^*$. Therefore $\chi(\mathbb{P}[\text{VeryGood}_m(a)]) > 1 - \tilde{p}_c$ by our choice of p'_c . So the field $(Y_a)_{a \in \mathbb{Z}^d}$ is a product random field with density larger than $1 - \tilde{p}_c$. By our choice of \tilde{p}_c , it implies that the radius of the component of $\{a \in \mathbb{Z}^d, Y_a = 0\}$ containing the origin admits exponential moments, which implies that ρ_m admits exponential moments by stochastic domination.

It remains to show that we can choose $R \geq 1$ and $\delta \in (0, 1/2]$ such that $\mathbb{P}[\text{Good}(0)] > p_c^*$. Since Ψ_0^b and Ψ_0^t are disjoint, by independence

$$\mathbb{P}[\text{Good}(0)] = \mathbb{P}[N \cap \Psi_a^b = \emptyset] \mathbb{P}[N \cap \Psi_a^t \neq \emptyset] = \exp(-\lambda\mu(\Psi_a^b)) (1 - \exp(-\lambda\mu(\Psi_a^t))).$$

We have

$$\mu(\Psi_a^b) = \int_{K_0} \int_1^{e^\delta} \frac{1}{y^d} dy dx = R^d \frac{1 - e^{-(d+1)\delta}}{d+1}$$

and

$$\mu(\Psi_a^t) = \int_{K_0} \int_{e^\delta}^e \frac{1}{y^d} dy dx = R^d \frac{e^{-(d+1)\delta} - e^{-(d+1)}}{d+1}.$$

Let $\kappa := \frac{e^{-(d+1)/2} - e^{-(d+1)}}{d+1}$. If $\delta < 1/2$, then

$$\mu(\Psi_a^t) \geq R^d \frac{e^{-(d+1)/2} - e^{-(d+1)}}{d+1} = \kappa R^d.$$

Since $1 - \exp(-\lambda\kappa R^d) \rightarrow 1$ when $R \rightarrow \infty$, we can pick R large enough such that $1 - \exp(-\lambda\kappa R^d) > \sqrt{p_c^*}$.

Now R is chosen, and at R fixed, $\mu(\Psi_a^b) \rightarrow 0$ when $\delta \rightarrow 0$, so $\exp(-\lambda\mu(\Psi_a^b)) \rightarrow 1$ when $\delta \rightarrow 0$. We pick δ small enough (and also smaller than $1/2$) such that $\exp(-\lambda\mu(\Psi_a^b)) > \sqrt{p_c^*}$. For this choice of (R, δ) ,

$$\begin{aligned} \mathbb{P}[\text{Good}(0)] &= \exp(-\lambda\mu(\Psi_a^b)) (1 - \exp(-\lambda\mu(\Psi_a^t))) \\ &\geq \exp(-\lambda\kappa R^d) (1 - \exp(-\lambda\mu(\Psi_a^t))) > p_c^*, \end{aligned}$$

this proves Lemma 2.4.8 and completes the proof of Proposition 2.4.6.

2.4.4 Proof of Proposition 2.2.13

In the following, we consider $\delta > 0$ such that Proposition 2.4.6 holds. Let us deduce Proposition 2.2.13 from Proposition 2.4.6.

Proof of Proposition 2.2.13. By Proposition 2.4.6, $\mathbb{P}_{0 \in \mathcal{L}_0} [\mathcal{A}_0^\delta(0) = \infty] = 0$. Recall that $\mathcal{A}_0^\delta(x)(\eta)$ is the value of $\mathcal{A}_0^\delta(x)$ when $N = \eta$. Define the weight function (in the non-marked case) $w(x, \eta) = \mathbf{1}_{x \in \mathcal{L}_0(\eta)}$ and $\mathcal{A}_0^\delta(x)(\eta) = \infty$ for $x \in \mathbb{R}^d$, $\eta \in \mathcal{N}_S$. Lemma 2.3.13 applied with $A = \mathbb{R}^d$ gives,

$$\mathbb{E}[\#\{x \in \mathcal{L}_0, \mathcal{A}_0^\delta(x) = \infty\}] = \mathbb{E} \left[\sum_{x \in \mathcal{L}_0} w(x) \right] = \infty \mathbb{E}_{0 \in \mathcal{L}_0} [w(0)] = \infty \mathbb{P}_{0 \in \mathcal{L}_0} [\mathcal{A}_0^\delta(x) = \infty] = 0.$$

Thus a.s., for all $x \in \mathcal{L}_0$, $\mathcal{A}_0^\delta(x) \neq \infty$.

The dilation invariance property of the model implies that, for all $h \in \mathbb{R}$, a.s., for all $x \in \mathcal{L}_h$, $\mathcal{A}_h^{h+\delta}(x) \neq \infty$. We define

$$H_0 := \sup\{h \geq 0, \forall x \in \mathcal{L}_0, \mathcal{A}_0^h(x) \neq \infty\} \in [0, \infty]$$

Note that $H_0 \geq \delta$ by the previous discussion. Suppose that $\mathbb{P}[H_0 < \infty] > 0$. Then there exists some (deterministic) $h_0 \geq 0$ such that $\mathbb{P}[h_0 < H_0 < h_0 + \delta] > 0$. On this event, there exists some $x \in \mathcal{L}_0$ such that $\mathcal{A}_0^{h_0}(x) \neq \infty$ but $\mathcal{A}_{h_0}^\delta(\mathcal{A}_0^{h_0}(x)) = \mathcal{A}_0^{h_0+\delta}(x) = \infty$. Therefore $x' := \mathcal{A}_0^{h_0}(x)$ satisfies $\mathcal{A}_{h_0}^\delta(x') = \infty$. So $\mathbb{P}[\exists x \in \mathcal{L}_{h_0}, \mathcal{A}_{h_0}^\delta(x) = \infty] > 0$, which contradicts the previous discussion. So $H_0 = \infty$ a.s., i.e. a.s. for all $h \geq 0$ and for all $x \in \mathcal{L}_0$, $\mathcal{A}_0^h(x) \neq \infty$.

By dilations invariance, the same result is true for each level $t \in \mathbb{R}$: for all $t \in \mathbb{R}$, a.s., for all $x \in \mathcal{L}_t$ and for all $h \geq 0$, $\mathcal{A}_t^{t+h}(x) \neq \infty$. Therefore, almost surely, $\forall t \in \mathbb{Q}, \forall h \in \mathbb{R}_+, \mathcal{A}_t^{t+h}(x) \neq \infty$. Every trajectory crosses a rational level $t \in \mathbb{Q}$, since it is the case for every non-horizontal Euclidean segments. Thus we can replace \mathbb{Q} by \mathbb{R} in the above conclusion. This completes the proof. \square

2.4.5 Proof of (i) in Theorem 2.4.4

Let $p \geq 1$. Recall that $\delta > 0$ is chosen according to Proposition 2.4.6.

The strategy of the proof consists in iterating the control of horizontal deviation up to level δ given by Proposition 2.4.6 to obtain a control up to level t for all $t > 0$. It will be shown that, for all $t \geq 0$,

$$\mathbb{E} \left[\left(\frac{\|X_0\| + \text{CFD}_0^{t+\delta}(X_0)}{e^{t+\delta}} \right)^{p-1/p} \right] \leq \varphi \left(\mathbb{E} \left[\left(\frac{\|X_0\| + \text{CFD}_0^t(X_0)}{e^t} \right)^p \right]^{1/p} \right), \quad (2.4.14)$$

where

$$\varphi(s) = e^{-\delta} s + C_0 s^{\frac{d}{p+d}}, \quad (2.4.15)$$

where $C_0 > 0$ is a constant that only depends of p, d, λ .

The key point is that the function φ defined in (2.4.15) admits a fixed point. As it will be shown later, the factor $e^{-\delta}$ in the first term of the r.h.s. of (2.4.15) comes from the dilation invariance. Because of the metric of (H, ds^2) , the horizontal fluctuations of the lowest part of a trajectory are compressed by rescaling so they have negligible impact on the total cumulated deviations. This is specific to hyperbolic geometry; in Euclidean geometry, the same argument leads to a roughly non-optimal upper-bound of horizontal deviations.

Step 1: we prove (i) assuming (2.4.14).

By assumption, $\mathbb{E} \left[\left(\frac{\|X_0\| + \text{CFD}_0^0(X_0)}{e^0} \right)^p \right]^{1/p} = \mathbb{E} [\|X_0\|^p]^{1/p} < \infty$, so by iterating (2.4.14), since φ is non-decreasing, we get for all $n \in \mathbb{N}$,

$$\mathbb{E} \left[\left(\frac{\|X_0\| + \text{CFD}_0^{n\delta}(X_0)}{e^{n\delta}} \right)^p \right]^{1/p} \leq \varphi^n \left(\mathbb{E} [\|X_0\|^p]^{1/p} \right), \quad (2.4.16)$$

where $\varphi^n = \varphi \circ \dots \circ \varphi$ n times. Let $t \geq 0$ and $n = \lceil t/\delta \rceil$ (thus $\delta(n-1) < t \leq \delta n$). Then, using the fact that $t \mapsto \text{CFD}_0^t(X_0)$ is non-decreasing,

$$\begin{aligned} \mathbb{E} \left[\left(e^{-t} \text{CFD}_0^t(X_0) \right)^p \right]^{1/p} &\leq \mathbb{E} \left[\left(\frac{\|X_0\| + \text{CFD}_0^t(X_0)}{e^t} \right)^p \right]^{1/p} = e^{n\delta - t} \mathbb{E} \left[\left(\frac{\|X_0\| + \text{CFD}_0^t(X_0)}{e^{n\delta}} \right)^p \right]^{1/p} \\ &\leq e^\delta \mathbb{E} \left[\left(\frac{\|X_0\| + \text{CFD}_0^{n\delta}(X_0)}{e^{n\delta}} \right)^p \right]^{1/p} \stackrel{(2.4.16)}{\leq} e^\delta \varphi^n \left(\mathbb{E} [\|X_0\|^p]^{1/p} \right). \end{aligned} \quad (2.4.17)$$

The function φ is continuous, non-decreasing and admits a unique positive fixed point $s_0 = \left(\frac{C_0}{1-e^{-\delta}} \right)^{1+d/p}$ such that

$$\begin{cases} \varphi(s) > s & \text{if } 0 < s < s_0 \\ \varphi(s) = s & \text{if } s \in \{0, s_0\} \\ \varphi(s) < s & \text{if } s > s_0. \end{cases}$$

Therefore, since $\mathbb{E} [\|X_0\|^p]^{1/p} \in (0, \infty)$, $\varphi^n \left(\mathbb{E} [\|X_0\|^p]^{1/p} \right) \rightarrow s_0$ when $n \rightarrow \infty$. Combining this with (2.4.17), we obtain that (i) holds for $K = e^\delta s_0$.

Step 2: we show that (2.4.14) holds for all $t \geq 0$.

Let $t \geq 0$. By Minkowski inequality,

$$\begin{aligned} &\mathbb{E} \left[\left(\|X_0\| + \text{CFD}_0^{t+\delta}(X_0) \right)^p \right]^{1/p} \\ &\leq \mathbb{E} \left[\left(\|X_0\| + \text{CFD}_0^t(X_0) \right)^p \right]^{1/p} + \mathbb{E} \left[\left(\text{CFD}_0^{t+\delta}(X_0) - \text{CFD}_0^t(X_0) \right)^p \right]^{1/p}, \end{aligned}$$

so, multiplying both sides by $e^{-t-\delta}$, we obtain

$$\begin{aligned} &\mathbb{E} \left[\left(\frac{\|X_0\| + \text{CFD}_0^{t+\delta}(X_0)}{e^{t+\delta}} \right)^p \right]^{1/p} \leq e^{-\delta} \mathbb{E} \left[\left(\frac{\|X_0\| + \text{CFD}_0^t(X_0)}{e^t} \right)^p \right]^{1/p} \\ &+ e^{-t-\delta} \mathbb{E} \left[\left(\text{CFD}_0^{t+\delta}(X_0) - \text{CFD}_0^t(X_0) \right)^p \right]^{1/p}. \end{aligned} \quad (2.4.18)$$

The first term in the r.h.s. of (2.4.18) corresponds to $e^{-\delta}s$, with $s = \mathbb{E} \left[\left(e^{-t} (\|X_0\| + \text{CFD}_0^t(X_0)) \right)^p \right]^{1/p}$. The factor $e^{-\delta}$ comes from the rescaling and is crucial for the existence of the fixed point.

It remains to upperbound the second term by $C_0 s^{\frac{d}{p+d}}$. We use Proposition 2.3.16 to rewrite the quantity $\mathbb{E} \left[\left(\text{CFD}_0^{t+\delta}(X_0) - \text{CFD}_0^t(X_0) \right)^p \right]$. Let us introduce the level t -weighted association function (f_t, w_t) defined as

$$\begin{cases} f_t(x, \eta, \xi) = \mathcal{A}_0^t(X_0(T_{-x}\eta, \xi) + x)(\eta) \in \mathcal{L}_t(\eta) \\ w_t(x, \eta, \xi) = \text{CFD}_t^{t+\delta}(f_t(x, \eta, \xi))^p. \end{cases}$$

for all $x \in \mathbb{R}^d$, $\eta \in \mathcal{N}_S$ and $\xi \in \Upsilon$. Checking that (f_t, w_t) is well-defined and is a level t -weighted association function is done in [14, Section 6.6]. Proposition 2.3.16 applied to (f_t, w_t) gives,

$$\mathbb{E}[w_t(0)] = \alpha_0 e^{-dt} \mathbb{E}_{0 \in \mathcal{L}_t} \left[\int_{\Lambda_{f_t}(0)} w_t(x) dx \right]. \quad (2.4.19)$$

We have

$$\mathbb{E}[w_t(0)] = \mathbb{E} \left[\text{CFD}_t^{t+\delta} (\mathcal{A}_0^t(X_0))^p \right] = \mathbb{E} \left[\left(\text{CFD}_0^{t+\delta}(X_0) - \text{CFD}_0^t(X_0) \right)^p \right]$$

and, since for all $x \in \Lambda_{f_t}(0)$, $f_t(x) = 0$,

$$\mathbb{E}_{0 \in \mathcal{L}_t} \left[\int_{\Lambda_{f_t}(0)} w_t(x) dx \right] = \mathbb{E}_{0 \in \mathcal{L}_t} \left[\int_{\Lambda_t(0)} \text{CFD}_t^{t+\delta}(0)^p dx \right] = \mathbb{E}_{0 \in \mathcal{L}_t} \left[V_t(0) \text{CFD}_t^{t+\delta}(0)^p \right]$$

where $V_t(0) = \text{Leb}(\Lambda_{f_t}(0))$. Then (2.4.19) can be rewritten as:

$$\mathbb{E} \left[\left(\text{CFD}_0^{t+\delta}(X_0) - \text{CFD}_0^t(X_0) \right)^p \right] = \alpha_0 e^{-dt} \mathbb{E}_{0 \in \mathcal{L}_t} \left[V_t(0) \text{CFD}_t^{t+\delta}(0)^p \right]. \quad (2.4.20)$$

By Proposition 2.3.18 applied to f_t ,

$$\mathbb{E}_{0 \in \mathcal{L}_t} \left[V_t(0)^{1+p/d} \right] \leq C_{p,d} e^{dt} \mathbb{E} \left[\|\mathcal{A}_0^t(X_0)\|^p \right] \leq C_{p,d} e^{dt} \mathbb{E} \left[(\|X_0\| + \text{CFD}_0^t(X_0))^p \right] \quad (2.4.21)$$

since $\|\mathcal{A}_0^t(X_0)\| \leq \|X_0\| + \text{CFD}_0^t(X_0)$. Thus Hölder inequality gives,

$$\begin{aligned} \mathbb{E}_{0 \in \mathcal{L}_t} \left[V_t(0) \text{CFD}_t^{t+\delta}(0)^p \right] &\leq \mathbb{E}_{0 \in \mathcal{L}_t} \left[V_t(0)^{1+p/d} \right]^{\frac{d}{p+d}} \mathbb{E}_{0 \in \mathcal{L}_t} \left[\text{CFD}_t^{t+\delta}(0)^{p+d} \right]^{\frac{p}{p+d}} \\ &\stackrel{(2.4.21)}{\leq} C_{p,d}^{\frac{d}{p+d}} e^{\frac{d^2 t}{p+d}} \mathbb{E} \left[(\|X_0\| + \text{CFD}_0^t(X_0))^p \right]^{\frac{d}{p+d}} \mathbb{E}_{0 \in \mathcal{L}_t} \left[\text{CFD}_t^{t+\delta}(0)^{p+d} \right]^{\frac{p}{p+d}}. \end{aligned} \quad (2.4.22)$$

Since

$$\text{CFD}_t^{t+\delta}(0)(N) = e^t \text{CFD}_0^\delta(0)(D_{e^{-t}}N),$$

by dilation invariance (Lemma 2.8.2),

$$\mathbb{E}_{0 \in \mathcal{L}_t} \left[\text{CFD}_t^{t+\delta}(0)^{p+d} \right] = e^{t(p+d)} \mathbb{E}_{0 \in \mathcal{L}_0} \left[\text{CFD}_0^\delta(0)^{p+d} \right].$$

Then (2.4.22) can be rewritten as

$$\begin{aligned} &\mathbb{E}_{0 \in \mathcal{L}_t} \left[V_t(0) \text{CFD}_t^{t+\delta}(0)^p \right] \\ &\leq C_{p,d}^{\frac{d}{p+d}} e^{t p + \frac{d^2 t}{p+d}} \mathbb{E} \left[(\|X_0\| + \text{CFD}_0^t(X_0))^p \right]^{\frac{d}{p+d}} \mathbb{E}_{0 \in \mathcal{L}_0} \left[\text{CFD}_0^\delta(0)^{p+d} \right]^{\frac{p}{p+d}}. \end{aligned} \quad (2.4.23)$$

Then

$$\begin{aligned} &\mathbb{E} \left[\left(\text{CFD}_0^{t+\delta}(X_0) - \text{CFD}_0^t(X_0) \right)^p \right] \stackrel{(2.4.20)}{=} \alpha_0 e^{-dt} \mathbb{E}_{0 \in \mathcal{L}_t} \left[V_t(0) \text{CFD}_t^{t+\delta}(0)^p \right] \\ &\stackrel{(2.4.23)}{\leq} \alpha_0 C_{p,d}^{\frac{d}{p+d}} e^{t(p-d) + \frac{d^2 t}{p+d}} \mathbb{E} \left[(\|X_0\| + \text{CFD}_0^t(X_0))^p \right]^{\frac{d}{p+d}} \mathbb{E}_{0 \in \mathcal{L}_0} \left[\text{CFD}_0^\delta(0)^{p+d} \right]^{\frac{p}{p+d}} \\ &\leq \tilde{C}_0 e^{\frac{p^2 t}{p+d}} \mathbb{E} \left[(\|X_0\| + \text{CFD}_0^t(X_0))^p \right]^{\frac{d}{p+d}} \end{aligned} \quad (2.4.24)$$

where $\tilde{C}_0 = \alpha_0 C_{p,d}^{\frac{d}{p+d}} \mathbb{E}_{0 \in \mathcal{L}_0} \left[\text{CFD}_0^\delta(0)^{p+d} \right]^{\frac{p}{p+d}} < \infty$ by Proposition 2.4.6. We rewrite (2.4.24) as

$$\begin{aligned} & e^{-t-\delta} \mathbb{E} \left[\left(\text{CFD}_0^{t+\delta}(X_0) - \text{CFD}_0^t(X_0) \right)^p \right]^{1/p} \\ & \leq \tilde{C}_0^{1/p} e^{-\delta - \frac{t\delta}{p+d}} \mathbb{E} \left[\left(\|X_0\| + \text{CFD}_0^t(X_0) \right)^p \right]^{\frac{d}{p(p+d)}} \\ & = \tilde{C}_0^{1/p} e^{-\delta} \mathbb{E} \left[\left(\frac{\|X_0\| + \text{CFD}_0^t(X_0)}{e^t} \right)^p \right]^{\frac{d}{p(p+d)}}. \end{aligned} \quad (2.4.25)$$

Combining (2.4.18) and (2.4.25) we obtain that (2.4.14) holds with $C_0 = \tilde{C}_0^{1/p} e^{-\delta}$. This completes the proof.

2.4.6 A geometrical lemma

We now prove the following lemma:

Lemma 2.4.9. *For all $p \geq 1$, we have*

$$\mathbb{E} \left[\left(\min_{x \in \mathcal{L}_0} \|x\| \right)^p \right] < \infty.$$

This will be used in the proof of (ii) in Theorem 2.4.4, and several times in the following.

Proof. We will in fact prove that $\min_{x \in \mathcal{L}_0} \|x\|$ admits exponential moments. Choose $A > 0$ large enough such that, for $x_1, x_2 \in \mathbb{R}^d$, if $\|x_1 - x_2\| \geq A$ then $d((x_1, e^0), (x_2, e^0)) \geq 6$. For $n \in \mathbb{N}$, define

$$p_n := (Ane_1, 0), \quad B_n^1 := B_H(p_n, 1), \quad B_n^3 := B_H(p_n, 3).$$

Let us also define

$$B_n^{1-} = B_n^1 \cap (\mathbb{R}^d \times (0, e^0)), \quad B_n^{1+} = B_n^1 \cap (\mathbb{R}^d \times [e^0, \infty)).$$

For $n \in \mathbb{N}$, we now define the event E_n meaning that there is exactly one point of N in B_n^{1-} , exactly one point of N in B_n^{1+} and no more points in B_n^3 :

$$E_n := \{ \#(N \cap B_n^{1-}) = \#(N \cap B_n^{1+}) = 1 \text{ and } \#(N \cap B_n^3) = 0 \}.$$

The event E_n only depend on the process N inside the ball B_n^3 , and the balls $(B_n^3)_{n \in \mathbb{N}}$ are pairwise disjoint by our choice of A , so the events (E_n) are mutually independent. Moreover they all have the same probability $p > 0$. It is shown in the next paragraph that, on E_n , $\min_{x \in \mathcal{L}_0} \|x\| \leq An + 3$. Consider $n_{\min} = \min\{n \in \mathbb{N}, E_n \text{ occurs}\}$. The random variable n_{\min} is distributed according to a geometric distribution so it admits exponential moments. Since $\min_{x \in \mathcal{L}_0} \|x\| \leq An_{\min} + 3$, it implies Lemma 2.4.9.

It remains to show that E_n implies $\min_{x \in \mathcal{L}_0} \|x\| \leq An + 3$. Fix $\eta \in E_n$ and consider z_- (resp. z_+) the unique point in $\eta \cap B_n^{1-}$ (resp. $\eta \cap B_n^{1+}$). For any $z \in B^+(z_1)(\eta)$,

$$(z, p_n) \leq d(z, z_1) + d(z_1, p_n) \leq d(z_1, z_2) + d(z_1, p_n) \leq 2d(z_1, p_n) + d(p_n, z_2) \leq 3,$$

so $B^+(z_1)(\eta) \subset B_n^3$. Since $\eta \cap B_n^3$ contains no more points than z_1 and z_2 , this implies that $B^+(z_1)(\eta) = \emptyset$, so $[z_1, z_2]_{\text{eucl}} \in \text{DSF}(\eta)$. Consider the intersection point $z = (x_0, e^0)$ of $[z_1, z_2]_{\text{eucl}}$ and the hyperplane $\mathbb{R}^d \times \{1\}$. Then $z \in B_n^3$ so, by the discussion below Proposition 2.2.1, $\|x - Ane_1\| \leq d(z, p_n) \leq 3$. Thus $\|x\| \leq An + 3$, and $x \in \mathcal{L}_0(\eta)$, so $\min_{x \in \mathcal{L}_0(\eta)} \|x\| \leq An + 3$. This completes the proof of Lemma 2.4.9. \square

2.4.7 Proof of (ii) in Theorem 2.4.4

Let $p \geq 1$. Let us consider some level 0-association function f verifying $\mathbb{E} [\|f(0)\|^{2p}] < \infty$ and $\mathbb{E}_{0 \in \mathcal{L}_0} [\text{Leb}(\Lambda_f(0))^{-1}] < \infty$. The construction of such a f is done later, we assume its existence for the moment.

We need to upper-bound the quantity $\mathbb{E}_{0 \in \mathcal{L}_0} \left[(e^{-t} \text{CFD}_0^t(0))^p \right]$, but, as it is shown below, part (i) of Theorem 2.4.4 and mass transport arguments allow us to control the quantity $\mathbb{E}_{0 \in \mathcal{L}_0} \left[\text{Leb}(\Lambda_f(0)) (e^{-t} \text{CFD}_0^t(0))^{p'} \right]$ for any p' .

Thus we apply Cauchy-Schwartz inequality to make the desired quantity appear:

$$\begin{aligned} & \mathbb{E}_{0 \in \mathcal{L}_0} \left[\left(\frac{\text{CFD}_0^t(0)}{e^t} \right)^p \right] \\ & \leq \mathbb{E}_{0 \in \mathcal{L}_0} \left[\text{Leb}(\Lambda_f(0)) \left(\frac{\text{CFD}_0^t(0)}{e^t} \right)^{2p} \right]^{1/2} \mathbb{E}_{0 \in \mathcal{L}_0} [\text{Leb}(\Lambda_f(0))^{-1}]^{1/2}. \end{aligned} \quad (2.4.26)$$

We now upper-bound the first factor in the r.h.s. of (2.4.26). By (i) applied to $X_0 := f(0)$, since $\mathbb{E} [\|f(0)\|^{2p}] < \infty$,

$$\limsup_{t \rightarrow \infty} \mathbb{E} \left[\left(\frac{\text{CFD}_0^t(f(0))}{e^t} \right)^{2p} \right] < \infty. \quad (2.4.27)$$

Let

$$w(x, \eta, \xi) = \left(\frac{\text{CFD}_0^t(f(x, \eta, \xi))(\eta)}{e^t} \right)^{2p}.$$

Then (f, w) is a level 0-weighted association function, so by Proposition 2.3.16,

$$\begin{aligned} \mathbb{E} \left[\left(\frac{\text{CFD}_0^t(f(0))}{e^t} \right)^{2p} \right] &= \alpha_0 \mathbb{E}_{0 \in \mathcal{L}_0} \left[\int_{\Lambda_f(0)} \left(\frac{\text{CFD}_0^t(f(x))}{e^t} \right)^{2p} dx \right] \\ &= \alpha_0 \mathbb{E}_{0 \in \mathcal{L}_0} \left[\text{Leb}(\Lambda_f(0)) \left(\frac{\text{CFD}_0^t(0)}{e^t} \right)^{2p} \right]. \end{aligned} \quad (2.4.28)$$

Therefore by (2.4.27) and (2.4.28),

$$\limsup_{t \rightarrow \infty} \mathbb{E}_{0 \in \mathcal{L}_0} \left[\text{Leb}(\Lambda_f(0)) \left(\frac{\text{CFD}_0^t(0)}{e^t} \right)^{2p} \right] < \infty. \quad (2.4.29)$$

Finally, combining (2.4.26), (2.4.29) and the assumption $\mathbb{E}_{0 \in \mathcal{L}_0} [\text{Leb}(\Lambda_f(0))^{-1}] < \infty$, we obtain (2.4.1), so (ii) is proved.

It remains to show that there exists a level 0-association function f such that $\mathbb{E} [\|f(0)\|^{2p}] < \infty$ and $\mathbb{E}_{0 \in \mathcal{L}_0} [\text{Leb}(\Lambda_f(0))^{-1}] < \infty$. Let $\Upsilon = \mathbb{R}^d / \mathbb{Z}^d$ be the d -dimensional torus, and let Y be a random variable independent of N and uniformly distributed on $\mathbb{R}^d / \mathbb{Z}^d$.

Now, fix $\eta \in \mathcal{N}_S$ and $\xi \in \mathbb{R}^d / \mathbb{Z}^d$. Let us construct $f(x, \eta, \xi)$ for all $x \in \mathbb{R}^d$ as follows. We pave \mathbb{R}^d by cubes of size 1 such that (any representative for) ξ is a node of the grid. More precisely, let $u = (u_1, \dots, u_d) \in \mathbb{R}^d$ be a representative for ξ , and define

$$\mathcal{K}(\xi) = \left\{ \prod_{i=1}^d [u_i + a_i, u_i + a_i + 1), a = (a_1, \dots, a_d) \in \mathbb{Z}^d \right\}.$$

Clearly, this definition does not depend of the choice of the representative u , so $\mathcal{K}(\xi)$ is well-defined. We construct $f(\cdot, \eta, \xi)$ separately on each cube $K \in \mathcal{K}(\xi)$. Let $K \in \mathcal{K}(\xi)$ and let $b = (b_1, \dots, b_d) \in \mathbb{R}^d$ be the vertex of K with the lowest coordinates (that is, such that $K = \prod_{i=1}^d [b_i, b_i + 1)$). Let $n(K) := \#(\mathcal{L}_0(\eta) \cap K)$ be the number of 0-level points in K . If $n(K) = 0$, then, for all $x \in K$, we set $f(x, \eta, \xi)$ to be the point of $\mathcal{L}_0(\eta)$ the closest to x (in case of equality pick, say, the smallest for the lexicographical order). Now suppose $n(K) \geq 1$. We divide K into $n(K)$ equal slices: for $1 \leq j \leq n(K)$, we set

$$\mathcal{S}_j(K) = \prod_{i=1}^{d-1} [b_i, b_i + 1) \times \left[b_d + \frac{j-1}{n(K)}, b_d + \frac{j}{n(K)} \right).$$

Let $x_1, \dots, x_{n(K)}$ be the $n(K)$ points of $\mathcal{L}_0(\eta) \cap K$ (in, say, the lexicographical order). For $1 \leq j \leq n(K)$, we send the slice $\mathcal{S}_j(K)$ on x_j , i.e. for all $x \in \mathcal{S}_j(K)$, we set $f(x, \eta, \xi) = x_j$.

We now show that f is a level 0-association function. First, for all $x \in \mathbb{R}^d$, $\eta \in \mathcal{N}_S$, $\xi \in \Upsilon$, $f(x) \in \mathcal{L}_0$ by construction. We now check the translation invariance. Let $x, x' \in \mathbb{R}^d$ and $\eta \in \mathcal{N}_S$. By construction, for all $u \in \mathbb{R}^d$, $f(x + x', T_x \eta, \overline{u + x'}) = f(x, \eta, \overline{u}) + x'$, where \overline{x} denotes the class of x in $\mathbb{R}^d / \mathbb{Z}^d$. Then, since $Y \stackrel{(d)}{=} Y + \overline{u}$, $f(x + x', T_x \eta, Y) = f(x, \eta, Y) + x'$, so f is an association function.

We move on to show that $\mathbb{E} [\|f(0)\|^{2p}] < \infty$. By construction $f(0)$ is either the point of \mathcal{L}_0 the closest to 0, or a point of the cube K_0 containing the origin. Thus, almost surely, $\|f(0)\| \leq \min_{x \in \mathcal{L}_0} \|x\| \vee \sqrt{d}$. By Lemma 2.4.9, $\mathbb{E} [(\min_{x \in \mathcal{L}_0} \|x\|)^{2p}] < \infty$ therefore $\mathbb{E} [\|f(0)\|^{2p}] < \infty$.

We finally show that $\mathbb{E}_{0 \in \mathcal{L}_0} [\text{Leb}(\Lambda_f(0))^{-1}] < \infty$. For $x \in \mathcal{L}_t$, we note $K(x)$ the (random) cube of $\mathcal{K}(Y)$ containing x . By construction, almost surely, $\Lambda_f(0)$ contains at least a slice of volume $1/n(K(0))$ (plus eventually additional points contained in empty cubes), therefore

$$\text{Leb}(\Lambda_f(0)) \geq \frac{1}{n(K(0))} \implies \mathbb{E}_{0 \in \mathcal{L}_0} [\text{Leb}(\Lambda_f(0))^{-1}] \leq \mathbb{E}_{0 \in \mathcal{L}_0} [n(K(0))]. \quad (2.4.30)$$

Let us introduce the weight function $w(x, \eta, \xi) := \mathbf{1}_{x \in \mathcal{L}_0} n(K(0))(\eta, \xi)$. Applying Lemma 2.3.13 to w with $A = [-1/2, 1/2]^d$ leads to

$$\mathbb{E} \left[\sum_{x \in \mathcal{L}_0 \cap [-1/2, 1/2]^d} n(K(x)) \right] = \alpha_0 \mathbb{E}_{0 \in \mathcal{L}_0} [n(K(0))]. \quad (2.4.31)$$

Since a.s. $K(x) \subset [-3/2, 3/2]^d$ for all $x \in [-1/2, 1/2]^d$,

$$\begin{aligned} \mathbb{E} \left[\sum_{x \in \mathcal{L}_0 \cap [-1/2, 1/2]^d} n(K(x)) \right] &\leq \mathbb{E} [\#(\mathcal{L}_0 \cap [-1/2, 1/2]^d) \#(\mathcal{L}_0 \cap [-3/2, 3/2]^d)] \\ &\leq \mathbb{E} [\#(\mathcal{L}_0 \cap [-3/2, 3/2]^d)^2] \\ &< \infty \text{ by Proposition 2.2.17.} \end{aligned} \quad (2.4.32)$$

Combining (2.4.30), (2.4.31) and (2.4.32), we obtain that $\mathbb{E}_{0 \in \mathcal{L}_0} [\text{Leb}(\Lambda_f(0))^{-1}] < \infty$. This completes the proof.

2.4.8 Proof of Theorem 2.4.5

In order to prove Theorem 2.4.5, we will use that, for all $a, t \geq 0$ and $p \geq 1$:

$$\mathbb{E}_{0 \in \mathcal{L}_0} [\text{MBD}_{-t-a}^0(0)^p] \leq 2^{p-1} e^{a(d-p)} (\mathbb{E}_{0 \in \mathcal{L}_0} [\text{MBD}_{-t}^0(0)^p] + \mathbb{E}_{0 \in \mathcal{L}_0} [\text{CFD}_0^a(0)^p]). \quad (2.4.33)$$

Step 1: we prove Theorem 2.4.5 assuming (2.4.33). We can suppose $p > d$ since the result for $p > d$ immediately implies the result for all $p \geq 1$. Then $2^{p-1}e^{a(d-p)} \rightarrow 0$ when $a \rightarrow \infty$, so we can choose $a_0 > 0$ such that $2^{p-1}e^{a_0(d-p)} < 1$. For $n \in \mathbb{N}$, we define:

$$S_n := \sup_{\substack{t \in [na_0, \\ (n+1)a_0]} \mathbb{E}_{0 \in \mathcal{L}_0} [\text{MBD}_{-t}^0(0)^p].$$

Then we need to prove that $\limsup_{n \rightarrow \infty} S_n < \infty$. Let $s \in [0, a_0]$. Applying (2.4.33) with $t = 0$ and $a = s$ leads to

$$\mathbb{E} [\text{MBD}_{-s}^0(0)^p] \leq 2^{p-1}e^{s(d-p)} \mathbb{E}_{0 \in \mathcal{L}_0} [\text{CFD}_0^s(0)^p]$$

since $\mathbb{E}_{0 \in \mathcal{L}_0} [\text{MBD}_0^0(0)^p] = 0$. Using the fact that $\mathbb{P}_{0 \in \mathcal{L}_0}$ -a.s., the function $s \mapsto \text{CFD}_0^s(0)$ is non-decreasing and that $e^{s(d-p)} \leq 1$, we obtain

$$\forall s \in [0, a_0], \mathbb{E} [\text{MBD}_{-s}^0(0)^p] \leq 2^{p-1} \mathbb{E}_{0 \in \mathcal{L}_0} [\text{CFD}_0^{a_0}(0)^p] < \infty \quad (2.4.34)$$

since $\mathbb{E}_{0 \in \mathcal{L}_0} [\text{CFD}_0^{a_0}(0)^p] < \infty$ by (ii) in Theorem 2.4.4. Therefore $S_0 < \infty$.

Let $n \in \mathbb{N}$. Then

$$\begin{aligned} S_{n+1} &= \sup_{\substack{t \in [na_0, \\ (n+1)a_0]} \mathbb{E} [\text{MBD}_{-t-a_0}^0(0)^p] \\ (2.4.33) \quad &\leq \sup_{\substack{t \in [na_0, \\ (n+1)a_0]} 2^{p-1}e^{a_0(d-p)} (\mathbb{E}_{0 \in \mathcal{L}_0} [\text{MBD}_{-t}^0(0)^p] + \mathbb{E}_{0 \in \mathcal{L}_0} [\text{CFD}_0^{a_0}(0)^p]) \\ &= 2^{p-1}e^{a_0(d-p)} \left(\sup_{\substack{t \in [na_0, \\ (n+1)a_0]} \mathbb{E}_{0 \in \mathcal{L}_0} [\text{MBD}_{-t}^0(0)^p] + \mathbb{E}_{0 \in \mathcal{L}_0} [\text{CFD}_0^{a_0}(0)^p] \right) \\ &= \varphi(S_n) \end{aligned} \quad (2.4.35)$$

where $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is defined as

$$\varphi(t) = 2^{p-1}e^{a_0(d-p)} (t + \mathbb{E}_{0 \in \mathcal{L}_0} [\text{CFD}_0^{a_0}(0)^p]).$$

The function φ is well-defined since $\mathbb{E}_{0 \in \mathcal{L}_0} [\text{CFD}_0^{a_0}(0)^p] < \infty$. By iterating (2.4.35), since φ is non-decreasing, we get $S_n \leq \varphi^n(S_0)$, where $\varphi^n = \varphi \circ \dots \circ \varphi$ n times. Since $2e^{a_0(d-p)} < 1$, φ is a contraction linear mapping, it admits a finite fixed point t_0 and $\varphi^n(S_0) \rightarrow t_0$. Therefore

$$\limsup_{t \rightarrow \infty} \mathbb{E}_{0 \in \mathcal{L}_0} [\text{MBD}_{-t}^0(0)^p] = \limsup_{n \rightarrow \infty} S_n \leq \limsup_{n \rightarrow \infty} \varphi^n(S_0) = t_0 < \infty.$$

This proves Theorem 2.4.5.

Step 2: we show (2.4.33).

Let $a, t \geq 0$ and $p \geq 1$. For $x \in \mathcal{L}_0$, we have

$$\begin{aligned} \text{MBD}_{-t-a}^0(x) &= \max_{x'' \in \mathcal{D}_{-t-a}^0(x)} \text{CFD}_{-t-a}^0(x'') \\ &= \max_{x'' \in \mathcal{D}_{-t-a}^0(x)} (\text{CFD}_{-t-a}^{-a}(x'') + \text{CFD}_{-a}^0(\mathcal{A}_{-t-a}^{-a}(x''))) \\ &= \max_{x' \in \mathcal{D}_{-a}^0(x)} \max_{x'' \in \mathcal{D}_{-t-a}^{-a}(x')} (\text{CFD}_{-t-a}^{-a}(x'') + \text{CFD}_{-a}^0(x')) \\ &= \max_{x' \in \mathcal{D}_{-a}^0(x)} (\text{MBD}_{-t-a}^{-a}(x') + \text{CFD}_{-a}^0(x')). \end{aligned}$$

Therefore

$$\begin{aligned}
& \mathbb{E}_{0 \in \mathcal{L}_0} [\text{MBD}_{-t-a}^0(0)^p] & (2.4.36) \\
&= \mathbb{E}_{0 \in \mathcal{L}_0} \left[\max_{x \in \mathcal{D}_{-a}^0(0)} (\text{MBD}_{-t-a}^{-a}(x) + \text{CFD}_{-a}^0(x))^p \right] \\
&\leq \mathbb{E}_{0 \in \mathcal{L}_0} \left[\sum_{x \in \mathcal{D}_{-a}^0(0)} (\text{MBD}_{-t-a}^{-a}(x) + \text{CFD}_{-a}^0(x))^p \right] \\
&\leq \mathbb{E}_{0 \in \mathcal{L}_0} \left[\sum_{x \in \mathcal{D}_{-a}^0(0)} 2^{p-1} (\text{MBD}_{-t-a}^{-a}(x)^p + \text{CFD}_{-a}^0(x)^p) \right] \\
&= 2^{p-1} \left(\mathbb{E}_{0 \in \mathcal{L}_0} \left[\sum_{x \in \mathcal{D}_{-a}^0(0)} \text{MBD}_{-t-a}^{-a}(x)^p \right] + \mathbb{E}_{0 \in \mathcal{L}_0} \left[\sum_{x \in \mathcal{D}_{-a}^0(0)} \text{CFD}_{-a}^0(x)^p \right] \right),
\end{aligned}$$

where Jensen was used in the second inequality. Now, we use the Mass Transport Principle to rewrite the quantities $\mathbb{E}_{0 \in \mathcal{L}_0} \left[\sum_{x \in \mathcal{D}_{-a}^0(0)} \text{MBD}_{-t-a}^{-a}(x)^p \right]$ and $\mathbb{E}_{0 \in \mathcal{L}_0} \left[\sum_{x \in \mathcal{D}_{-a}^0(0)} \text{CFD}_{-a}^0(x)^p \right]$. Let us introduce the two weight functions w_1 and w_2 defined as

$$w_1(x, \eta) = \mathbf{1}_{x \in \mathcal{L}_{-a}(\eta)} \text{MBD}_{-t-a}^{-a}(x)^p(\eta), \quad w_2(x, \eta) = \mathbf{1}_{x \in \mathcal{L}_{-a}(\eta)} \text{CFD}_{-a}^0(x)^p(\eta).$$

Applying Proposition 2.3.14 to w_1 with $t_1 = -a, t_2 = 0$ leads to:

$$\mathbb{E}_{0 \in \mathcal{L}_0} \left[\sum_{x \in \mathcal{D}_{-a}^0(0)} \text{MBD}_{-t-a}^{-a}(x)^p \right] = e^{ad} \mathbb{E}_{0 \in \mathcal{L}_{-a}} [\text{MBD}_{-t-a}^{-a}(0)^p]. \quad (2.4.37)$$

For all $\eta \in \mathcal{N}_S$ such that $0 \in \mathcal{L}_{-a}[\eta]$, we have

$$\text{MBD}_{-t-a}^{-a}(0)(\eta) = e^{-ap} \text{MBD}_{-t}^0(0)(D_{e^a} \eta),$$

so by scale invariance (Lemma 2.8.2 applied with $t = -a, t' = 0$),

$$\mathbb{E}_{0 \in \mathcal{L}_{-a}} [\text{MBD}_{-t-a}^{-a}(0)^p] = e^{-ap} \mathbb{E}_{0 \in \mathcal{L}_0} [\text{MBD}_{-t}^0(0)^p]. \quad (2.4.38)$$

Combining (2.4.37) and (2.4.38), we obtain:

$$\mathbb{E}_{0 \in \mathcal{L}_0} \left[\sum_{x \in \mathcal{D}_{-a}^0(0)} \text{MBD}_{-t-a}^{-a}(x)^p \right] = e^{a(d-p)} \mathbb{E}_{0 \in \mathcal{L}_0} [\text{MBD}_{-t}^0(0)^p]. \quad (2.4.39)$$

The same calculations with w_2 lead to

$$\mathbb{E}_{0 \in \mathcal{L}_0} \left[\sum_{x \in \mathcal{D}_{-a}^0(0)} \text{CFD}_{-a}^0(x)^p \right] = e^{a(d-p)} \mathbb{E}_{0 \in \mathcal{L}_0} [\text{CFD}_0^a(0)^p]. \quad (2.4.40)$$

Finally, we rewrite (2.4.36) using (2.4.39) and (2.4.40), we obtain (2.4.33). It completes the proof.

2.5 Proof of coalescence

In this section we prove Theorem 2.1.1.

2.5.1 A short proof in dimension 1 + 1

We first prove Theorem 2.1.1 in the bi-dimensional case (i.e. $d = 1$). It is based on planarity, so it only works for $d = 1$. A general (but more complex) proof of coalescence in all dimensions is given after. A useful consequence of planarity we shall need is the following:

Lemma 2.5.1. *Suppose $d = 1$. Let $t \geq 0$ and $x \in \mathcal{L}_t$. If $x_1, x_2, x_3 \in \mathcal{L}_0$ are such that $x_1 < x_2 < x_3$, and if $x_1, x_3 \in \mathcal{D}_0^t(x)$, then $x_2 \in \mathcal{D}_0^t(x)$.*

Proof of Lemma 2.5.1. Let $x_1, x_2, x_3 \in \mathcal{L}_0$ such that $x_1 < x_2 < x_3$, and $x_1, x_3 \in \mathcal{D}_0^t(x)$. Since the DSF is non-crossing [14, Section 3.1], the trajectory from (x_2, e^0) cannot cross the trajectories from (x_1, e^0) and (x_3, e^0) . The point (x, e^t) belongs to both trajectories from (x_1, e^0) and (x_3, e^0) , so it also belongs to the trajectory from (x_2, e^0) , so $x_2 \in \mathcal{D}_0^t(x)$. \square

For $t \geq 0$, we define B_t as the set of points of level t that have descendants at level 0; for $x \in B_t$, we define $M_t(x)$ as the left-most descendant of x at level 0:

$$B_t := \{x \in \mathcal{L}_t, \mathcal{D}_0^t(x) \neq \emptyset\}, \quad M_t(x) = \inf \mathcal{D}_0^t(x) \in \{-\infty\} \cup \mathbb{R}.$$

We now prove:

Lemma 2.5.2. *Almost surely, for all $x \in B_t$, $M_t(x) = \min \mathcal{D}_0^t(x)$ (or equivalently $M_t(x) > -\infty$), that is, each point $x \in B_t$ admits a left-most descendant at level 0.*

Proof of lemma 2.5.2. Since $\mathcal{D}_0^t(x)$ is locally finite and non empty for all $x \in B_t$, it suffices to show that $\inf \mathcal{D}_0^t(x) > -\infty$ for all $x \in B_t$ a.s. Consider the event

$$A := \{\exists x \in B_t, \inf \mathcal{D}_0^t(x) = -\infty\}.$$

On A , we show that there is a unique $x \in B_t$ such that $\inf \mathcal{D}_0^t(x) = -\infty$. Indeed, suppose that $\inf \mathcal{D}_0^t(x') = -\infty$ for some $x' \in B_t$. Let $\hat{x} \in \mathcal{D}_0^t(x)$. Pick $\hat{x}' \in \mathcal{D}_0^t(x')$ such that $\hat{x}' < \hat{x}$ (such a \hat{x}' exists since $\inf \mathcal{D}_0^t(x) = -\infty$). Then pick $\hat{x}'' \in \mathcal{D}_0^t(x)$ such that $\hat{x}'' < \hat{x}'$. By Lemma 2.5.1, $\hat{x}' \in \mathcal{D}_0^t(x)$ which implies $x = x'$.

Suppose that $\mathbb{P}[A] > 0$. Then, conditionally to A , we can define X as the (random) unique $x \in B_t$ such that $\mathcal{D}_0^t(x) = -\infty$. Since the event A is invariant by translations, the distribution of N conditioned by A is also invariant by translations. Therefore the law of X must be invariant by translations, but there's no probability distribution on \mathbb{R} invariant by translations. This is a contradiction, therefore $\mathbb{P}[A] = 0$. \square

We call *level t -separating points* the points $M_t(x)$ for $x \in B_t$. We denote by $S_t := \{M_t(x), x \in B_t\}$ the set of level t -separating points. Let us prove:

Lemma 2.5.3. *If $S_t \cap [-a, a] = \emptyset$, then $\tau_{[-a, a]} \leq t$.*

Proof. Suppose that $S_t \cap [-a, a] = \emptyset$. Let $x, x' \in [-a, a]$ with $x < x'$, and suppose that $\mathcal{A}_0^t(x) \neq \mathcal{A}_0^t(x')$. Consider $\hat{x} = M_t(\mathcal{A}_0^t(x'))$. Suppose that $\hat{x} \leq x$. Thus $\hat{x} \leq x < x'$, and by construction \hat{x} and x' have the same ancestor at level t . Then by Lemma 2.5.1, $x \in \mathcal{D}_0^t(\mathcal{A}_0^t(x'))$, which contradicts $x \neq x'$. Therefore $\hat{x} > x$. Moreover $\hat{x} \leq x'$ by construction, so $x < \hat{x} < x'$, therefore $\hat{x} \in [-a, a]$. But $\hat{x} \in S_t$, this contradicts $S_t \cap [-a, a] = \emptyset$. Thus for all $x, x' \in [-a, a]$, $\mathcal{A}_0^t(x) = \mathcal{A}_0^t(x')$, which implies $\tau_{[-a, a]} \leq t$. \square

We show that the level t -separating points are rare when t is large. We apply the Mass Transport Principle (Theorem 2.3.2) on \mathbb{R} with the following mass transport: from each point with descendants at level 0 we transport a unit mass to its left-most descendant.

The following measure π expresses this mass transport:

$$\pi(A \times B) = \mathbb{E} \left[\sum_{x \in B_t \cap A} \mathbf{1}_{M_t(x) \in B} \right].$$

The measure π is diagonally invariant because horizontal translations preserve the DSF's distribution. Then, by the Mass Transport Principle, $\pi(A \times \mathbb{R}) = \pi(\mathbb{R} \times A)$ for all $A \subset \mathbb{R}$ with non-empty interior. On the one hand,

$$\pi(A \times \mathbb{R}) = \mathbb{E}[\#(B_t \cap A)] \leq \mathbb{E}[\#(\mathcal{L}_t \cap A)] = \alpha_0 e^{-t} \text{Leb}(A) \quad (2.5.1)$$

and, on the other hand,

$$\pi(\mathbb{R} \times A) = \mathbb{E} \left[\sum_{x \in B_t} \mathbf{1}_{M_t(x) \in A} \right] = \mathbb{E} \left[\sum_{\substack{x \in B_t, \\ x' \in A}} \mathbf{1}_{x' = M_t(x)} \right] = \mathbb{E}[\#(S_t \cap A)]. \quad (2.5.2)$$

Therefore, combining (2.5.1) and (2.5.2) with $A = [-a, a]$, we obtain $\mathbb{E}[\#(S_t \cap [-a, a])] \leq 2a\alpha_0 e^{-t}$. Hence

$$\mathbb{P}[S_t \cap [-a, a] \neq \emptyset] \leq \mathbb{E}[\#(S_t \cap [-a, a])] \leq 2a\alpha_0 e^{-t}.$$

By Lemma 2.5.3, it implies that $\mathbb{P}[\tau_{[-a, a]} > t] \leq 2a\alpha_0 e^{-t}$, which proves the second statement of Theorem 2.1.1. The fact that the DSF is almost surely a tree immediately follows. Indeed,

$$\{\text{The DSF is a tree}\} = \{\forall a \in \mathbb{N}, \exists t \in \mathbb{N}, \tau_{[-a, a]} \leq t\} = \bigcap_{a \in \mathbb{N}} \downarrow \bigcup_{t \in \mathbb{N}} \uparrow \{\tau_{[-a, a]} \leq t\}.$$

Therefore

$$\mathbb{P}[\text{The DSF is a tree}] = \lim_{a \in \mathbb{N}} \downarrow \lim_{t \in \mathbb{N}} \uparrow \mathbb{P}[\tau_{[-a, a]} \leq t].$$

Since for all $a > 0$, $\mathbb{P}[\tau_{[-a, a]} \leq t] \geq 1 - 2a\alpha_0 e^{-t} \xrightarrow{t \rightarrow \infty} 1$, we obtain

$$\mathbb{P}[\text{The DSF is a tree}] = \lim_{a \in \mathbb{N}} \downarrow 1 = 1,$$

which proves Theorem 2.1.1 for $d = 1$.

2.5.2 General case: ideas of the proof

We move on to show the coalescence for all dimensions d . Let us consider two trajectories starting from level 0. The choice of those trajectories will be discussed later. We want to prove that those two trajectories coalesce.

The intuition behind the coalescence can be understood as follows. We can deduce from Theorem 2.4.4 that the two trajectories stay almost in a cone. That is, for A large enough and for all height h large enough, their projection on \mathbb{R}^d at height h are contained in $\mathbb{B}_{\mathbb{R}^d}(0, Ae^h)$ with high probability. That is, they stay close to each other as they go up. Then, at each height, they have a positive probability to coalesce, so they must coalesce.

This is true because the metric of H becomes larger as the ordinate increases, so this behaviour is specific to the Hyperbolic geometry. In Euclidean space, the two trajectories move away from each other as they go up, so the same argument cannot be used. Indeed, we expect that the DSF in \mathbb{R}^d with $d \geq 4$ does not coalesce.

The idea of the proof is the following. We suppose by contradiction that the two trajectories do not coalesce with positive probability. We consider some height h large enough such that, with high probability, the process N below height h almost determines if the two trajectories coalesce or not. Then, on the event of non-coalescence and apart from an event of small probability, the probability of coalescence conditionally to the process N below height h is close to 0.

On the other hand, with high probability, for some large fixed $A > 0$, both trajectories are contained in the cylinder $B_{\mathbb{R}^d}(0, Ae^h) \times (0, e^h)$ (that is, the two trajectories are not too far from each other). Thus we show that we can modify the process above height h in a way that forces the two trajectories to coalesce, and we show that the set of configurations above height h that forces coalescence has probability bounded below independently of h . This contradicts the fact that the probability of coalescence knowing the process N below height h can be close to 0 with macroscopic probability.

Some technical difficulties are due to the geometry of the model and the fact that a modification of the point process above height h can affect trajectories below height h .

2.5.3 Introduction and notations

Let $d \geq 1$. The following notations are illustrated in Figure 2.8. Let $p_1, p_2 \in \mathbb{Q}^d \times (\mathbb{Q} \cap (0, e^0))$ two points of H below level 0 with rational coordinates. We define Z_1 (resp. Z_2) as the (random) point of $N \cap (\mathbb{R}^d \times (0, e^0))$ the closest to p_1 (resp. p_2):

$$Z_i = \underset{\substack{(x,y) \in N, \\ y < 1}}{\operatorname{argmin}} d(p_i, Z_i).$$

for $i = 1, 2$. We will prove that the trajectories from Z_1 and Z_2 coalesce almost surely, i.e. that a.s. there exists $n \geq 0$ such that $A^n(Z_1) = A^n(Z_2)$, where $A^n = A \circ \dots \circ A$ n times. If this is proved, then the result will be true almost surely for all $p_1, p_2 \in \mathbb{Q}^d \times (\mathbb{Q} \cap (0, e^0))$ simultaneously, which implies that the whole DSF coalesces a.s.

For $t \geq 0$, define $k_i(t)$ as the unique non-negative integer such that $[A^{k_i(t)}Z_i, A^{k_i(t)+1}Z_i]_{eucl}$ crosses the level t . It is well defined a.s. because each trajectory starting below the level t crosses the level t a.s.

Let $A, h, M, \delta, \varepsilon > 0$ be five parameters that will be chosen later. We define

$$\begin{aligned} K(M, h, \delta) &:= \{(x, y) \in H, y \geq e^{h-\delta} \text{ and } d(z, (0, e^h)) < M\} \\ &= B_H((0, e^h), M) \cap (\mathbb{R}^d \times (e^{h-\delta}, \infty)). \end{aligned}$$

Note that $K(M, h, \delta) = D_{e^h}K(M, 0, \delta)$ so $K(M, h, \delta)$ and $K(M, 0, \delta)$ are isometric.

Let $\mathcal{F}_{in}(M, h, \delta)$ (resp. $\mathcal{F}_{out}(M, h, \delta)$) be the σ -algebra on \mathcal{N}_S generated by the process N inside $K(M, h, \delta)$ (resp. outside $K(M, h, \delta)$).

Finally, define

$$\text{Slice}(A, h, \delta) := B_{\mathbb{R}^d}(0, Ae^h) \times (e^{h-\delta}, e^h).$$

Notice that $\text{Slice}(A, h, \delta) = D_{e^h}\text{Slice}(A, 0, \delta)$, so $\text{Slice}(A, h, \delta)$ and $\text{Slice}(A, 0, \delta)$ are isometric.

We now introduce the following events. The event CO means that trajectories from Z_1 and Z_2 coalesce:

$$\text{CO} := \{\exists n \geq 0, A^n(Z_1) = A^n(Z_2)\}.$$

The event $\text{Cyl}(A, h)$ means that below the level h , trajectories from Z_1 and Z_2 are entirely contained in the cylinder $B_{\mathbb{R}^d}(0, Ae^h) \times (0, e^h)$:

$$\begin{aligned} \text{Cyl}(A, h) &:= \{\text{For } i = 1, 2, \text{ for all } 0 \leq n \leq k_i(h), \\ &A^n(Z_i) \in B_{\mathbb{R}^d}(0, Ae^h) \times (0, e^h)\}. \end{aligned}$$

Let us also define

$$\text{EmptySlice}(A, h, \delta) := \{N \cap \text{Slice}(A, h, \delta) = \emptyset\},$$

$$\text{Approx}(M, \varepsilon, \delta, h) := \{|\mathbb{P}[\text{CO} | \mathcal{F}_{out}(M, h, \delta)] - \mathbf{1}_{\text{CO}}| < \varepsilon\}.$$

2.5.4 Heart of proof of coalescence

The proof of coalescence is based on the three following lemmas.

Lemma 2.5.4. *We have*

$$\lim_{A \rightarrow \infty} \liminf_{h \rightarrow \infty} \mathbb{P}[\text{Cyl}(A, h)] = 1.$$

Lemma 2.5.5. *Let $M, \varepsilon, \delta > 0$. We have*

$$\lim_{h \rightarrow \infty} \mathbb{P}[\text{Approx}(M, \varepsilon, \delta, h)] = 1.$$

Lemma 2.5.6. *Let $A, \delta > 0$. There exist $M, \varepsilon > 0$, such that, for all $h > \delta$:*

$$\text{Cyl}(A, h) \cap \text{EmptySlice}(A, h, \delta) \subset \{\mathbb{P}[\text{CO} | \mathcal{F}_{\text{out}}(M, h, \delta)] > \varepsilon\}. \quad (2.5.3)$$

Lemmas 2.5.4, 2.5.5 and 2.5.6 are proved in Sections 2.5.5, 2.5.6 and 2.5.7 respectively. We assume them for the moment and we prove that trajectories from Z_1 and Z_2 coalesce almost surely, i.e. $\mathbb{P}[\text{CO}] = 1$. Let us suppose by contradiction that $\mathbb{P}[\text{CO}] < 1$. We choose the parameters $A, h, M, \delta, \varepsilon$ as follows:

- We first choose A large enough such that $\liminf_{h \rightarrow \infty} \mathbb{P}[\text{Cyl}(A, h)] > 1 - \mathbb{P}[\text{CO}^c]/3$, it is possible by Lemma 2.5.4. Then we pick h_0 large enough such that for all $h \geq h_0$, $\mathbb{P}[\text{Cyl}(A, h)] > 1 - \mathbb{P}[\text{CO}^c]/3$.
- Then we choose $\delta > 0$ small enough such that, for all $h \geq 0$, $\mathbb{P}[\text{EmptySlice}(A, h, \delta)] > 1 - \mathbb{P}[\text{CO}^c]/3$. It is possible since $\mathbb{P}[\text{EmptySlice}(A, h, \delta)]$ does not depend on h by dilatation invariance and, for all $h > 0$ (recall that μ is the Hyperbolic volume):

$$\lim_{\delta \rightarrow 0} \mu(\text{Slice}(A, h, \delta)) = \mu \left(\bigcap_{\delta \downarrow 0} \downarrow \text{Slice}(A, h, \delta) \right) = \mu(B_{\mathbb{R}^d}(0, Ae^h) \times \{e^h\}) = 0,$$

so

$$\mathbb{P}[\text{EmptySlice}(A, h, \delta)] = \exp(-\lambda \mu(\text{Slice}(A, h, \delta))) \xrightarrow{\delta \rightarrow 0} 1.$$

- Then we choose $M, \varepsilon > 0$ such that, for all $h > \delta$, inclusion (2.5.3) holds. It is possible by Lemma 2.5.6.
- We finally choose h large enough (and larger than h_0 and δ) such that $\mathbb{P}[\text{Approx}(M, \varepsilon, \delta, h)] > 1 - \mathbb{P}[\text{CO}^c]/3$. It is possible by Lemma 2.5.5.

Define

$$E := \text{Cyl}(A, h) \cap \text{EmptySlice}(A, h, \delta) \cap \text{Approx}(M, \varepsilon, \delta, h) \cap \text{CO}^c.$$

With this choice of parameters, all the events $\text{Cyl}(A, h)$, $\text{EmptySlice}(A, h, \delta)$ and $\text{Approx}(M, \varepsilon, \delta, h)$ have probability larger than $1 - \mathbb{P}[\text{CO}^c]/3$. Therefore

$$\mathbb{P}[\text{Cyl}(A, h)^c \cup \text{EmptySlice}(A, h, \delta)^c \cup \text{Approx}(M, \varepsilon, \delta, h)^c \cup \text{CO}] < 3 \frac{\mathbb{P}[\text{CO}^c]}{3} + \mathbb{P}[\text{CO}] = 1.$$

Then $\mathbb{P}[E] > 0$. On E , since both $\text{Approx}(M, \varepsilon, \delta, h)$ and CO^c occur, $\mathbb{P}[\text{CO} | \mathcal{F}_{\text{out}}(M, h, \delta)] < \varepsilon$; on the other hand, by (2.5.3), on E , $\mathbb{P}[\text{CO} | \mathcal{F}_{\text{out}}(M, h, \delta)] > \varepsilon$. This is a contradiction, therefore the assumption $\mathbb{P}[\text{CO}] < 1$ is wrong, so $\mathbb{P}[\text{CO}] = 1$. This proves Theorem 2.1.1.

2.5.5 Proof of Lemma 2.5.4

The proof is based on Theorem 2.4.4 and Markov inequality. For $i \in \{1, 2\}$, define

$$C_i(A, h) := \{\text{For all } 0 \leq n \leq k_i(h), A^n(Z_i) \in B_{\mathbb{R}^d}(0, Ae^h) \times (0, e^h)\}.$$

Then $\text{Cyl}(A, h) = C_1(A, h) \cap C_2(A, h)$, therefore it suffices to prove that, for $i = 1, 2$, $\lim_{A \rightarrow \infty} \liminf_{h \rightarrow \infty} \mathbb{P}[C_i(A, h)] = 1$.

Let $i \in \{1, 2\}$. Let us define X_i to be the (random) point of \mathcal{L}_0 such that (X_i, e^0) belongs to the trajectory from Z_i . Now, we write $C_i(A, h)$ as the intersection of two events $C_i^-(A, h)$ and $C_i^+(A, h)$, that means respectively that the trajectory is contained in the cylinder below (resp. above) the level 0. We define

$$C_i^-(A, h) := \{\text{For all } 0 \leq n \leq k_i(0), A^n(Z_i) \in B_{\mathbb{R}^d}(0, Ae^h) \times (0, e^0)\}$$

and

$$C_i^+(A, h) := \left\{ \sup_{0 \leq t \leq h} \|\mathcal{A}_0^t(X_i)\| \leq Ae^h \right\}.$$

Thus $C_i(A, h) = C_i^-(A, h) \cap C_i^+(A, h)$. Clearly, for all $A > 0$, $\lim_{h \rightarrow \infty} \mathbb{P}[C_i^-] = 1$ because the trajectory from Z_i goes above the level 0. It remains to show that $\lim_{A \rightarrow \infty} \liminf_{h \rightarrow \infty} \mathbb{P}[C_i^+(A, h)] = 1$.

We would like to apply Theorem 2.4.4 to X_i with, say $p = 1$, but this would demand showing $\mathbb{E}[\|X_i\|] < \infty$. A workaround is done by using the following trick. Let $B \geq 0$. Define

$$X_i^B = \begin{cases} X_i & \text{if } \|X_i\| \leq B \\ \operatorname{argmin}_{x \in \mathcal{L}_0} \|x\| & \text{otherwise.} \end{cases}$$

Using Proposition 2.4.9 with $p = 1$, we get that $\mathbb{E}[\|X_i^B\|] < \infty$. Thus Theorem 2.4.4 applied to X_i^B with $p = 1$ gives,

$$\limsup_{h \rightarrow \infty} \mathbb{E} \left[\frac{\sup_{0 \leq t \leq h} \|\mathcal{A}_0^t(X_i^B)\|}{e^h} \right] \leq \limsup_{h \rightarrow \infty} \mathbb{E} \left[\frac{\|X_i^B\| + \text{CFD}_0^h(X_i^B)}{e^h} \right] < \infty. \quad (2.5.4)$$

By Markov inequality,

$$\begin{aligned} \limsup_{h \rightarrow \infty} \mathbb{P}[\sup_{0 \leq t \leq h} \|\mathcal{A}_0^t(X_i^B)\| > Ae^h] &\leq \limsup_{h \rightarrow \infty} \frac{\mathbb{E}[\sup_{0 \leq t \leq h} \|\mathcal{A}_0^t(X_i^B)\|]}{Ae^h} \\ &= A^{-1} \limsup_{h \rightarrow \infty} \mathbb{E} \left[e^{-h} \sup_{0 \leq t \leq h} \|\mathcal{A}_0^t(X_i^B)\| \right] \\ &\rightarrow 0 \text{ when } A \rightarrow \infty, \end{aligned} \quad (2.5.5)$$

by (2.5.4).

We now need to replace X_i^B by X_i in (2.5.5). It will be done by taking $B \rightarrow \infty$. For $A, B, h > 0$,

$$\left\{ \sup_{0 \leq t \leq h} \|\mathcal{A}_0^t(X_i^B)\| \leq Ae^h \text{ and } X_i^B = X_i \right\} \subset \{C_i^+(A, h)\},$$

so

$$\mathbb{P}[C_i^+(A, h)] \geq 1 - \mathbb{P}[\sup_{0 \leq t \leq h} \|\mathcal{A}_0^t(X_i^B)\| > Ae^h] - \mathbb{P}[X_i \neq X_i^B].$$

Thus, for all $B \geq 0$,

$$\begin{aligned} \lim_{A \rightarrow \infty} \liminf_{h \rightarrow \infty} \mathbb{P}[C_i^+(A, h)] &\geq 1 - \lim_{A \rightarrow \infty} \limsup_{h \rightarrow \infty} \mathbb{P}[\sup_{0 \leq t \leq h} \|\mathcal{A}_0^t(X_i^B)\| > Ae^h] - \mathbb{P}[X_i^B \neq X_i] \\ &\stackrel{(2.5.5)}{\geq} 1 - \mathbb{P}[X_i^B \neq X_i]. \end{aligned} \quad (2.5.6)$$

Since $\mathbb{P}[X_i^B \neq X_i] \leq \mathbb{P}[X_i > B] \xrightarrow{B \rightarrow \infty} 0$, we obtain the wanted result by taking $B \rightarrow \infty$ in (2.5.6).

This proves Lemma 2.5.4.

2.5.6 Proof of Lemma 2.5.5

Let $M, \varepsilon, \delta > 0$. For $h \geq 0$, we denote by \mathcal{F}_{h-} the σ -algebra generated by the process N on $\mathbb{R}^d \times (0, e^{h-\delta})$. Since $\mathbb{R}^d \times (0, e^{h-\delta}) \subset K(M, h, \delta)^c$, $\mathcal{F}_{h-} \subset \mathcal{F}_{out}(M, h, \delta)$. Since $\bigcup_{h \uparrow \infty} \mathcal{F}_{h-} = \sigma(N)$, the martingale convergence theorem gives,

$$\lim_{h \rightarrow \infty} \mathbb{E}[\mathbf{1}_{CO} | \mathcal{F}_{h-}] = \mathbf{1}_{CO} \text{ a.s.} \quad (2.5.7)$$

We define

$$E_1 := \left\{ \left| \mathbb{E}[\mathbf{1}_{CO} | \mathcal{F}_{h-}] - \mathbf{1}_{CO} \right| \geq \frac{\varepsilon}{2} \right\}, \quad E_2 := \left\{ \left| \mathbb{E}[\mathbf{1}_{CO} | \mathcal{F}_{out}(M, h, \delta)] - \mathbb{E}[\mathbf{1}_{CO} | \mathcal{F}_{h-}] \right| \geq \frac{\varepsilon}{2} \right\}.$$

By (2.5.7), $\lim_{h \rightarrow \infty} \mathbb{P}[E_1] = 0$.

Suppose for the moment that $\lim_{h \rightarrow \infty} \mathbb{P}[E_2] = 0$. Then

$$\lim_{h \rightarrow \infty} \mathbb{P}[E_1^c \cap E_2^c] = 1 - \lim_{h \rightarrow \infty} \mathbb{P}[E_1 \cup E_2] = 1.$$

Moreover, on $E_1^c \cap E_2^c$, by triangular inequality,

$$\left| \mathbb{E}[\mathbf{1}_{CO} | \mathcal{F}_{out}(M, h, \delta)] - \mathbf{1}_{CO} \right| < \varepsilon.$$

Therefore, $\lim_{h \rightarrow \infty} [\text{Approx}(M, \varepsilon, \delta, h)] = 1$, which proves Lemma 2.5.5.

It remains to show that $\lim_{h \rightarrow \infty} \mathbb{P}[E_2] = 0$. Let us define

$$X(h) := \mathbb{E}[\mathbf{1}_{CO} | \mathcal{F}_{out}(M, h, \delta)].$$

Since $\mathcal{F}_{h-} \subset \mathcal{F}_{out}$, we have $\mathbb{E}[\mathbf{1}_{CO} | \mathcal{F}_{h-}] = \mathbb{E}[X(h) | \mathcal{F}_{h-}]$, so E_2 can be rewritten as $\{|X(h) - \mathbb{E}[X(h) | \mathcal{F}_{h-}]| \geq \varepsilon/2\}$. By Markov inequality,

$$\mathbb{P} \left[\left| X(h) - \mathbb{E}[X(h) | \mathcal{F}_{h-}] \right| \geq \frac{\varepsilon}{2} \middle| \mathcal{F}_{h-} \right] \leq \frac{2\mathbb{E} \left[\left| X(h) - \mathbb{E}[X(h) | \mathcal{F}_{h-}] \right| \middle| \mathcal{F}_{h-} \right]}{\varepsilon} \text{ a.s.}$$

So, on the one hand, by triangular inequality and because $X(h) \geq 0$ a.s.,

$$\mathbb{P}[E_2 | \mathcal{F}_{h-}] \leq \frac{2\mathbb{E} \left[X(h) + \mathbb{E}[X(h) | \mathcal{F}_{h-}] \middle| \mathcal{F}_{h-} \right]}{\varepsilon} = \frac{4\mathbb{E}[X(h) | \mathcal{F}_{h-}]}{\varepsilon} \text{ a.s.}$$

On the other hand, again by triangular inequality and because $1 - X(h) \geq 0$ a.s.,

$$\mathbb{P}[E_2 | \mathcal{F}_{h-}] \leq \frac{2\mathbb{E} \left[(1 - X(h)) + (1 - \mathbb{E}[X(h) | \mathcal{F}_{h-}]) \middle| \mathcal{F}_{h-} \right]}{\varepsilon} = \frac{4\mathbb{E}[1 - X(h) | \mathcal{F}_{h-}]}{\varepsilon} \text{ a.s.}$$

Thus

$$\mathbb{P} \left[E_2 \middle| \mathcal{F}_{h-} \right] \leq \frac{4(\mathbb{E}[X(h) | \mathcal{F}_{h-}] \wedge (1 - \mathbb{E}[X(h) | \mathcal{F}_{h-}]))}{\varepsilon} \text{ a.s.}$$

Since $\mathbb{E}[X(h) | \mathcal{F}_{h-}] = \mathbb{E}[\mathbf{1}_{CO} | \mathcal{F}_{h-}] \rightarrow \mathbf{1}_{CO}$ when $h \rightarrow \infty$,

$$\frac{4(\mathbb{E}[X(h) | \mathcal{F}_{h-}] \wedge (1 - \mathbb{E}[X(h) | \mathcal{F}_{h-}]))}{\varepsilon} \leq \frac{4|\mathbb{E}[X(h) | \mathcal{F}_{h-}] - \mathbf{1}_{CO}|}{\varepsilon} \xrightarrow{h \rightarrow \infty} 0 \text{ a.s.}$$

Therefore $\lim_{h \rightarrow \infty} \mathbb{P}[E_2 | \mathcal{F}_{h-}] = 0$ a.s., so dominated convergence theorem gives that $\lim_{h \rightarrow \infty} \mathbb{P}[E_2] = 0$. This completes the proof of Lemma 2.5.5.

2.5.7 Proof of Lemma 2.5.6

Let us introduce the following notation. For $\eta \in \mathcal{N}_S$, we define

$$\eta_{in} := \eta \cap K(M, h, \delta), \quad \eta_{out} = \eta \cap K(M, h, \delta)^c.$$

In particular, $N_{in} = N \cap K(M, h, \delta)$ and $N_{out} = N \cap K(M, h, \delta)^c$, are two independent PPPs of intensity λ on $K(M, h, \delta)$ and $K(M, h, \delta)^c$ respectively.

Let $A, \delta > 0$ and consider $\eta \in \text{Cyl}(A, h) \cap \text{EmptySlice}(A, h, \delta)$. The idea of the proof is to build an event $\text{FC}(\eta)$ on $\mathcal{N}_S(K(M, h, \delta))$, of probability bounded below by some $\varepsilon > 0$ independent of η , such that, when we replace the process η inside the box $K(M, h, \delta)$ by some $\eta'_{in} \in \text{FC}(\eta)$, then we force trajectories from Z_1 and Z_2 to coalesce.

For $i = 1, 2$, consider the point $(x_i, y_i) = A^{k_i(h)}(Z_i)(\eta)$ (i.e. the highest point of N in the trajectory below the level h). Notice that $\|x_i\| \leq Ae^h$ since $\eta \in \text{Cyl}(A, h)$. We define three balls as follows:

$$B_i^{down} = B_H \left((x_i, e^{h-\delta/2}), \frac{\delta}{2} \right) \text{ for } i = 1, 2, \quad B^{up} = B_H \left((0, e^{h+\delta/2}), \frac{\delta}{2} \right).$$

We now make the choice of M . For $h \geq 0$, define

$$\Xi(h) := \left\{ z \in H, d \left(z, B_{\mathbb{R}^d}(0, Ae^h) \times \{e^{h-\delta/2}\} \right) < \delta/2 \right\}.$$

So $\Xi(h)$ is a compact set and $\Xi(h) = D_{e^h}\Xi(0)$, therefore $\Xi(h)$ and $\Xi(0)$ are isometric. By construction, since $\|x_1\|, \|x_2\| \leq Ae^h$, $B_1^{down}, B_2^{down} \subset \Xi(h)$. Let us pick M large enough such that, for all $z^{down} \in \Xi(0)$ and for all $z^{up} \in B_H((0, e^{\delta/2}), \delta/2)$,

$$B_H(z^{down}, d(z^{down}, z^{up})) \subset B_H((0, e^0), M).$$

Since

$$\Xi(h) = D_{e^h}\Xi(0), \quad B^{up} = D_{e^h}B_H((0, e^0), \delta/2),$$

this choice of M guarantees that, for all $h \geq 0$, for all $z^{down} \in \Xi(h)$ and $z^{up} \in B^{up}$,

$$B_H(z^{down}, d(z^{down}, z^{up})) \subset B_H((0, e^h), M). \quad (2.5.8)$$

We define $\text{FC}(\eta)$ as the event on $\mathcal{N}_S(K(M, h, \delta))$ where there is exactly one point in B_1^{down} , exactly one point in B_2^{down} , exactly one point in B^{up} and no other point in $K(M, h, \delta)$:

$$\begin{aligned} \text{FC}(\eta) &:= \left\{ \eta'_{in} \in \mathcal{N}_S(K(M, h, \delta)), \right. \\ &\quad \#(\eta'_{in} \cap B_1^{down}) = 1 \text{ and } \#(\eta'_{in} \cap B_2^{down}) = 1 \text{ and } \#(\eta'_{in} \cap B^{up}) = 1 \\ &\quad \left. \text{and } \eta'_{in} \setminus (B_1^{down} \cup B_2^{down} \cup B^{up}) = \emptyset \right\}. \end{aligned}$$

This defines an event $\text{FC}(\eta)$ on $\mathcal{N}_S(K(M, h, \delta))$ for any $\eta \in \text{Cyl}(A, h) \cap \text{EmptySlice}(A, h, \delta)$.

We will use the two following claims.

Claim 2.5.7. *For any $\eta \in \text{Cyl}(A, h) \cap \text{EmptySlice}(A, h, \delta)$, the event $\text{FC}(\eta)$ forces coalescence, that is: for all $\eta'_{in} \in \text{FC}(\eta)$, $\eta' := \eta_{out} \cup \eta'_{in} \in \text{CO}$.*

Claim 2.5.8. *There exists $\varepsilon > 0$ independent of h such that $\mathbb{P}[N_{in} \in \text{FC}(\eta)] > \varepsilon$ for all $\eta \in \text{Cyl}(A, h) \cap \text{EmptySlice}(A, h, \delta)$.*

Suppose for the moment Claims 2.5.7 and 2.5.8. Choose ε as in Claim 2.5.8. Then, since, \mathcal{F}_{in} and \mathcal{F}_{out} are independent, for any $\eta \in \text{Cyl}(A, h) \cap \text{EmptySlice}(A, h, \delta)$,

$$\mathbb{P}[\text{CO} | N_{out} = \eta_{out}] = \mathbb{P}[\eta_{out} \cup N_{in} \in \text{CO}] \geq \mathbb{P}[N_{in} \in \text{FC}(\eta)] > \varepsilon,$$

so Lemma 2.5.6 is proved.

We now prove Claim 2.5.7. Let $\eta'_{in} \in FC(\eta)$ and consider $\eta' := \eta_{out} \cup \eta'_{in}$. Define z_1^{up} (resp. z_2^{up}) as the unique point of B_1^{up} (resp. B_2^{up}) and z^{up} as the unique point of B^{up} .

If we change the point process N inside the box $K(M, h, \delta)$, we potentially change trajectories from Z_1 and Z_2 below the level $h - \delta$, so care is required. However, we will see that this is not a real problem. Since Z_1, Z_2 are measurable w.r.t the process N below level 0 (i.e. $N \cap \mathbb{R}^d \times (0, e^0)$), and since $K(M, h, \delta) \in \mathbb{R}^d \times (e^{h-\delta}, \infty)$, changing the point process in $K(M, h, \delta)$ does not affect the positions of Z_1 and Z_2 . That is, $Z_1(\eta) = Z_1(\eta')$, $Z_2(\eta) = Z_2(\eta')$.

For $i = 1, 2$, we show that, for the realisation η' , the trajectory from Z_i contains z^{up} (which proves that trajectories from Z_1 and Z_2 coalesce).

By our assumption on M (2.5.8) and since $z_i^{down} \in B_i^{down} \subset \Xi(h)$,

$$B_H(z_i^{down}, d(z_i^{down}, z^{up})) \subset B_H((0, e^h), M).$$

Thus, since z^{up} is higher than z_i^{down} and the only points in $K(M, h, \delta)$ are $z_1^{down}, z_2^{down}, z^{up}$, the parent of z_1^{down} is necessarily z_2^{down} or z^{up} , and the parent of z_2^{down} is necessarily z_1^{down} or z^{up} . In all cases, z^{up} is on both trajectories from z_1^{down} and z_2^{down} .

Now let us define

$$\kappa_i := \max\{l \in \llbracket 0, k_i(h) \llbracket, A^l(Z_i)(\eta) = A^l(Z_i)(\eta')\} \cup \{k_i(h)\},$$

and

$$z_i^{sep} := A^{\kappa_i}(Z_i).$$

We show that the new parent of z_i^{sep} is one of the three points z_1^{down}, z_2^{down} or z^{up} (that is, $A(z_i^{sep})(\eta') \in \{z_1^{down}, z_2^{down}, z^{up}\}$). This implies that the trajectory from Z_i contains z^{up} for the realisation η' .

Suppose $\kappa_i < k_i(h)$ (that is, the change inside $K(M, h, \delta)$ affects the trajectory from Z_i before the highest point below the level h , it is Case 1 in Figure 2.8). Set

$$(x_i^{sep+}, y_i^{sep+}) = A(z_i^{sep})(\eta).$$

By assumption $y_i^{sep+} < e^h$. Since $\eta \in \text{Cyl}(A, h, \delta)$, $\|x_i^{sep+}\| \leq Ae^h$. Thus, since $\eta \in \text{EmptySlice}(A, h, \delta)$, $y_i^{sep+} < e^{h-\delta}$. Therefore $A(z_i^{sep})(\eta) \in \eta'$, so the new parent of z_i^{sep} is (strictly) closer to z_i^{sep} than its previous parent; in particular the new parent cannot be in η . The only possibility is that the new parent of z_i^{sep} is one of the three points z_1^{down}, z_2^{down} or z^{up} (i.e. $A(z_i^{sep})(\eta') \in \{z_1^{down}, z_2^{down}, z^{up}\}$).

Consider now the remaining case (Case 2 in Figure 2.8), $\kappa_i = k_i(h)$ (that is, the trajectory is unchanged up to $A^{k_i(h)}(Z_i)$). In this case, $z_i^{sep} = A^{k_i(h)}(Z_i)$. Since $\eta \in \text{EmptySlice}(A, h, \delta)$, and since $\|x_i\| \leq Ae^h$ (recall that (x_i, y_i) are coordinates of $A^{k_i(h)}(Z_i)$), $y_i \leq e^{h-\delta}$. Therefore, z_i^{down} is higher than z_i^{sep} . Then it suffices to show that z_i^{sep} is closer to z_i^{down} than to its previous parent, i.e. $d(z_i^{sep}, z_i^{down}) < d(z_i^{sep}, A(z_i^{sep})(\eta))$. Indeed, it implies that the new parent of z_i^{sep} cannot be in η , so it is necessarily z_1^{down}, z_2^{down} or z^{up} . It is the case if

$$B_i^{down} \subset B^{sep} := B_H(z_i^{sep}, d(z_i^{sep}, A(z_i^{sep})(\eta))).$$

The inclusion follows from the construction of B_i^{down} . Indeed, by construction, the center of B_i^{down} and z_i^{sep} have the same abscissa, so B_i^{down} and B^{sep} are centered at the same abscissa. Since the height of $A(z_i^{sep})(\eta)$ is larger than h , the top (i.e. highest point) of the ball B^{sep} has height larger than h . Moreover, its bottom (i.e. lowest point) has height smaller than $h - \delta$ since z_i^{sep} has height smaller than $h - \delta$. On the other hand, the top of B_i^{down} has height e^h and its bottom has height $e^{h-\delta}$. It follows that $B_i^{down} \subset B^{sep}$. This proves Claim 2.5.7.

We finally prove Claim 2.5.8. The event $FC(\eta)$ can be realised in two different ways: (i) there is exactly one point of N_{in} in B^{up} , exactly one point in $B_1^{down} \cap B_2^{down}$ and no other points in $K(M, h, \delta)$, or (ii) there is exactly one point in B^{up} , exactly one point in $B_1^{down} \setminus B_2^{down}$, exactly one point in $B_2^{down} \setminus B_1^{down}$, and no other points in $K(M, h, \delta)$. It is not hard to see that, whatever $\eta \in \text{Cyl}(A, h) \cap \text{EmptySlice}(A, h, \delta)$, one of the two possibilities occurs with uniformly lower-bounded probability, depending whether $B_1^{down} \cap B_2^{down}$ is large or not. The details are given in [14, Section 7.7].

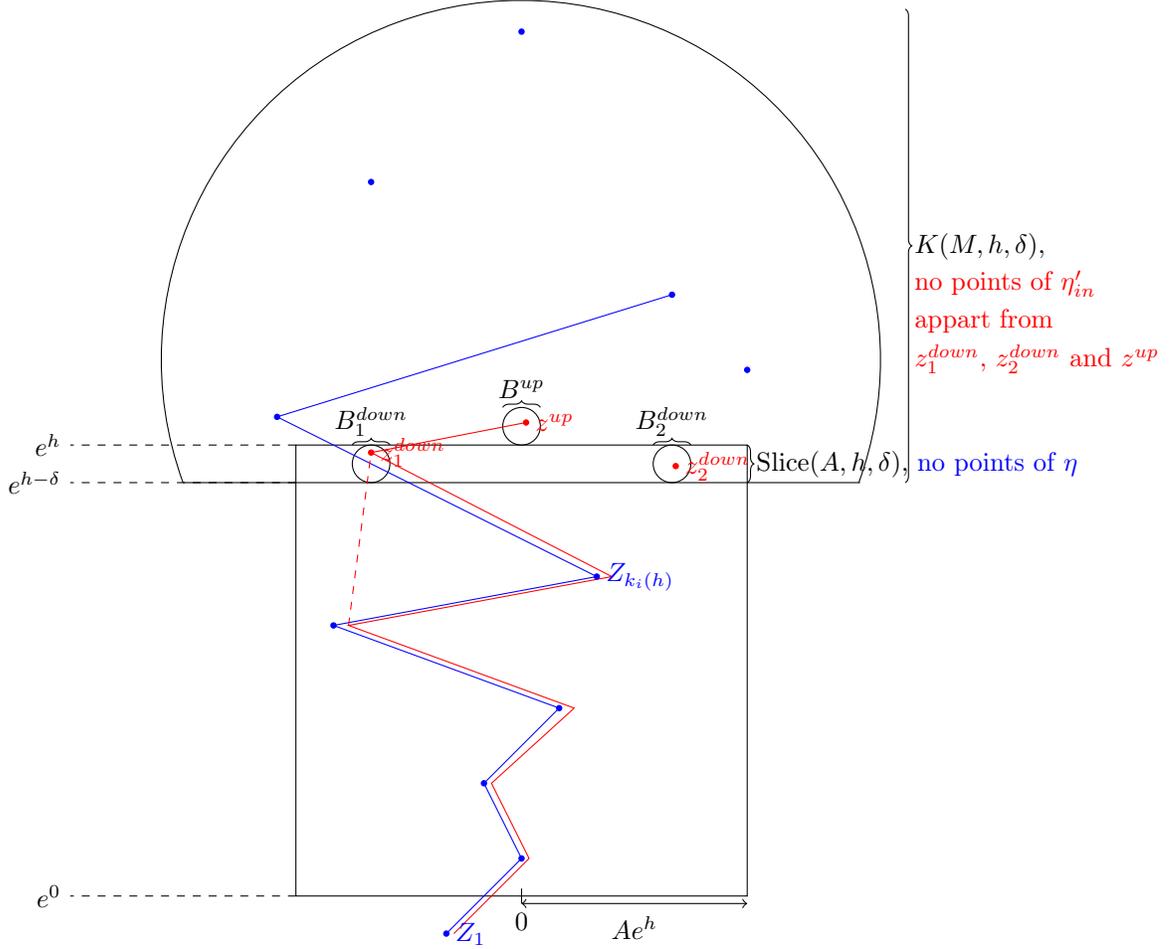


Figure 2.8: If we replace the point process η inside $K(M, h, \delta)$ by some $\eta'_{in} \in \text{FC}(\eta)$, then, for $i = 1, 2$, the new trajectory (in red) from Z_i is forced to go through z_{up} . Either it follows the previous trajectory (in blue) up to $Z_{k_i(h)}$ which is connected to z_1^{down} , z_2^{down} or z_{up} (Case 2, represented in full line), or it takes a short cut (Case 1, represented in dashed line).

2.6 The bi-infinite branches and their asymptotic directions

In this section we prove Theorem 2.1.2.

2.6.1 Notations and sketch of the proof

Let us first introduce some notations. Recall that BI is the set of functions $f : \mathbb{R} \rightarrow \mathbb{R}^d$ that encode a bi-infinite branch (see Definition 2.2.16).

Let us denote by \mathcal{I} the set of (abscissas of) points at infinity in $\mathbb{R}^d \times \{0\}$ that are the limit of at least one infinite branch in the direction of the past:

$$\mathcal{I} := \{x \in \mathbb{R}^d, \exists f \in \text{BI}, \lim_{t \rightarrow -\infty} f(t) = x\}.$$

Definition 2.6.1. Let $t \in \mathbb{R}$ and $x \in \mathcal{L}_t$. We call the *cell* of x , denoted by $\Psi_t(x)$, the set of abscissas x' of points at infinity in $\mathbb{R}^d \times \{0\}$ such that there exists a infinite branch in the direction

of past starting from (x, e^t) that converges to $(x', 0)$:

$$\Psi_t(x) := \{x' \in \mathbb{R}^d, \exists f \in \text{BI}, f(t) = x \text{ and } \lim_{t \rightarrow -\infty} f(t) = x'\}. \quad (2.6.1)$$

Thus $\mathcal{I} = \bigcup_{x \in \mathcal{L}_0} \Psi_t(x)$. In Step 1, we show that every infinite branch in the direction of the past converges to a point at infinity (point (ii) in Theorem 2.1.2), which is a direct consequence of Theorem 2.4.5. Then we show in Step 2 that the DSF is *straight* with probability 1 (Proposition 3.2.7). Recall that the Maximal Backward Deviation (MBD) has been defined in Definition 2.4.3.

Proposition 2.6.2. *The following occurs with probability 1:*

$$\forall x \in \mathcal{L}_0, \lim_{\substack{n \rightarrow \infty \\ n \in \mathbb{N}}} \max_{x' \in \mathcal{D}_{-n}^0(x)} \liminf_{t \rightarrow \infty} \text{MBD}_{-t}^{-n}(x') = 0. \quad (2.6.2)$$

The property (2.6.2) is called the *straightness* property. For simplicity, $\liminf_{t \rightarrow \infty} \text{MBD}_{-t}^{-n}(x')$ will be denoted by $\text{MBD}_{-\infty}^{-n}(x')$.

The rest proof is organized as follows. In Step 3, we show that the cells $\Psi_t(x)$ are closed and we check measurably conditions on $\Psi_t(x)$ and \mathcal{I} . In Step 4, we use a second moment technique to show that there exists infinitely many bi-infinite branches (the point (i)). It will follow that \mathcal{I} is dense in \mathbb{R}^d .

Then we show that \mathcal{I} is a closed subset of \mathbb{R}^d (Step 5). To do this, it is sufficient to show that the family of cells $\{\Psi_0(x), x \in \mathcal{L}_0\}$ is locally finite, that is, every ball $B \subset \mathbb{R}^d$ intersects finitely many cells. Thus it follows that \mathcal{I} is a dense closed subset of \mathbb{R}^d , therefore $\mathcal{I} = \mathbb{R}^d$. It proves (iii).

In Step 6, we prove (iv). The uniqueness follows from coalescence.

2.6.2 Step 1: proof of (ii)

By Theorem 2.4.5 applied with $p = 1$ and Fatou Lemma:

$$\mathbb{E}_{0 \in \mathcal{L}_0} [\text{MBD}_{-\infty}^0(x)] \leq \liminf_{h \rightarrow \infty} \mathbb{E}_{0 \in \mathcal{L}_0} [\text{MBD}_{-h}^0(0)] \leq \limsup_{h \rightarrow \infty} \mathbb{E}_{0 \in \mathcal{L}_0} [\text{MBD}_{-h}^0(0)] < \infty. \quad (2.6.3)$$

Then, almost surely, for all $x \in \mathcal{L}_0$, $\text{MBD}_{-\infty}^0(x) < \infty$. For any $f \in \text{BI}$, and any $h \geq 0$,

$$\int_{-h}^0 \|f'(t)\| dt \leq \text{MBD}_{-h}^0(f(0)),$$

thus

$$\int_{-\infty}^0 \|f'(t)\| dt \leq \liminf_{h \rightarrow \infty} \text{MBD}_{-h}^0(f(0)) = \text{MBD}_{-\infty}^0(f(0)) < \infty \text{ by (2.6.3),}$$

so $\lim_{t \rightarrow -\infty} f(t)$ exists. Then any bi-infinite branch admits an asymptotic direction toward the past.

2.6.3 Step 2: proof of straightness

The proof of straightness is based on Theorem 2.4.5. It is equivalent to prove the following statement:

$$\mathbb{P}_{0 \in \mathcal{L}_0} \left[\lim_{\substack{n \rightarrow \infty \\ n \in \mathbb{N}}} \max_{x \in \mathcal{D}_{-n}^0(0)} \text{MBD}_{-\infty}^{-n}(x) \right] = 1.$$

Let $n \in \mathbb{N}$. Consider the weight function

$$w(x, \eta) := \mathbf{1}_{x \in \mathcal{L}_{-n}} \text{MBD}_{-\infty}^{-n}(x)^{2d}.$$

Proposition 2.3.14 applied to w with $t_1 = -n$ and $t_2 = 0$ gives,

$$\mathbb{E}_{0 \in \mathcal{L}_{-n}} [\text{MBD}_{-\infty}^{-n}(0)^{2d}] = e^{-dn} \mathbb{E}_{0 \in \mathcal{L}_0} \left[\sum_{x \in \mathcal{D}_{-n}^0(0)} \text{MBD}_{-\infty}^{-n}(x)^{2d} \right].$$

Thus

$$\begin{aligned} \mathbb{E}_{0 \in \mathcal{L}_0} \left[\max_{x \in \mathcal{D}_{-n}^0(0)} \text{MBD}_{-\infty}^{-n}(x)^{2p} \right] &\leq \mathbb{E}_{0 \in \mathcal{L}_0} \left[\sum_{x \in \mathcal{D}_{-n}^0(0)} \text{MBD}_{-\infty}^{-n}(x)^{2d} \right] \\ &= e^{dn} \mathbb{E}_{0 \in \mathcal{L}_{-n}} [\text{MBD}_{-\infty}^{-n}(0)^{2d}] = e^{-dn} \mathbb{E}_{0 \in \mathcal{L}_0} [\text{MBD}_{-\infty}^0(0)^{2d}]. \end{aligned} \quad (2.6.4)$$

where invariance by dilation (Lemma 2.8.2) was used in the last equality (with $t = -n$ and $t' = 0$).

By Theorem 2.4.5 applied to any $p \geq 1$ and Fatou Lemma,

$$\mathbb{E}_{0 \in \mathcal{L}_0} [\text{MBD}_{-\infty}^0(0)^p] = \mathbb{E}_{0 \in \mathcal{L}_0} \left[\liminf_{t \rightarrow \infty} \text{MBD}_{-t}^0(0)^p \right] \leq \liminf_{t \rightarrow \infty} \mathbb{E}_{0 \in \mathcal{L}_0} [\text{MBD}_{-t}^0(0)^p] < \infty. \quad (2.6.5)$$

Thus, by taking $p = 2d$,

$$\mathbb{E}_{0 \in \mathcal{L}_0} \left[\sum_{n \in \mathbb{N}} \max_{x \in \mathcal{D}_{-n}^0(0)} \text{MBD}_{-\infty}^{-n}(x)^{2d} \right] \stackrel{(2.6.4)}{\leq} \sum_{n \in \mathbb{N}} e^{-dn} \mathbb{E}_{0 \in \mathcal{L}_0} [\text{MBD}_{-\infty}^0(0)^{2d}] \stackrel{(2.6.5)}{<} \infty.$$

It follows that $\sum_{n \in \mathbb{N}} \max_{x \in \mathcal{D}_{-n}^0(0)} \text{MBD}_{-\infty}^{-n}(x)^{2d} < \infty$ $\mathbb{P}_{0 \in \mathcal{L}_0}$ -a.s., this implies Proposition 3.2.7.

2.6.4 Step 3: the cells $\Psi_t(x)$ are closed and measurability conditions

Definition 2.6.3. A point $(x, e^t) \in \text{DSF}$ is said to be *connected to infinity* if for all $t' \leq t$, $\mathcal{D}_{t'}^t(x) \neq \emptyset$. We denote by $\text{DSF}^\infty \subset \text{DSF}$ the set of points that are connected to infinity.

For $t \in \mathbb{R}$, $x_0 \in \mathcal{L}_t$, we define the random subset of descendants of (x_0, e^t) :

$$D_t(x_0) := \{(x, e^{t'}) \in \text{DSF}^\infty, x \in \mathcal{D}_{t'}^t(x_0)\} \subset \text{DSF}^\infty.$$

The facts that $\Psi_t(x_0)$ is closed for all $t \in \mathbb{R}$ and $x_0 \in \mathcal{L}_t$ a.s. will be deduced from the following lemma:

Lemma 2.6.4. *The following occurs outside a set of probability zero: for all $t \in \mathbb{R}$, $x_0 \in \mathcal{L}_t$ and $x \in \mathbb{R}^d$, $x \in \Psi_t(x_0)$ if and only if $(x, 0) \in \overline{D_t(x_0)}$ (where $\bar{\cdot}$ denotes the closure operator).*

Lemma 2.6.4 implies that, outside a set of probability zero, for all $t \in \mathbb{R}$ and $x_0 \in \mathcal{L}_t$, $\Psi_t(x_0) = \overline{D_t(x_0)} \cap (\mathbb{R}^d \times \{0\})$ is closed in \mathbb{R}^d . It can also be deduced from Lemma 2.6.4 that the maps

$$\begin{aligned} \Phi_t : \mathbb{R}^d \times \mathbb{R}^d \times \mathcal{N}_S &\rightarrow \mathbb{R} \\ (x, x_0, \eta) &\mapsto \mathbf{1}_{x_0 \in \mathcal{L}_t(\eta)} \mathbf{1}_{x \in \Psi_t(x_0)(\eta)}, \end{aligned}$$

for any $t \in \mathbb{R}$, and

$$\begin{aligned} \Phi' : \mathbb{R}^d \times \mathcal{N}_S &\rightarrow \mathbb{R} \\ (x, \eta) &\mapsto \mathbf{1}_{x \in \mathcal{I}(\eta)}, \end{aligned}$$

are measurable if \mathcal{N}_S is equipped with the completed σ -algebra (for the probability measure \mathbb{P}), which will be required in the following steps. The details concerning these measurability conditions are given in [14, Section 8.4].

Proof of Lemma 2.6.4. Let $t \in \mathbb{R}$ and $x_0 \in \mathcal{L}_t$. It is clear that $x \in \Psi_t(x_0)$ implies that $(x, 0) \in \overline{D_t(x_0)}$. Let us suppose $(x, 0) \in \overline{D_t(x_0)}$ and we show that $x \in \Psi_t(x_0)$. Let us construct a sequence $(x'_n) \in (\mathbb{R}^d)^\mathbb{N}$ inductively such that, at each step $n \in \mathbb{N}$, $x'_n \in \mathcal{L}_{t-n}$, $x'_{n+1} \in \mathcal{D}_{t-n-1}^{t-n}(x'_n)$ and $(x, 0) \in \overline{D_{t-n}(x'_n)}$.

We set $x'_0 = x_0$. Let $n \in \mathbb{N}$ and suppose that x'_n has been constructed. Since

$$D_{t-n}(x'_n) = (\mathbb{R}^d \times (0, e^{t-n-1}]) \cap \left(\bigcup_{x'' \in \mathcal{D}_{t-n-1}^{t-n}(x'_n)} D_{t-n-1}(x'') \right)$$

and $\#\mathcal{D}_{t-n-1}^{t-n}(x) < \infty$ with probability 1 (Corollary 2.3.15), it follows that

$$\overline{D_{t-n}(x'_n)} \cap (\mathbb{R}^d \times \{0\}) = \bigcup_{x'' \in \mathcal{D}_{t-n-1}^{t-n}(x'_n)} (\overline{D_{t-n-1}(x'')} \cap (\mathbb{R}^d \times \{0\})). \quad (2.6.6)$$

Since $(x, 0) \in \overline{D_{t-n}(x'_n)}$, by (2.6.6) it is possible to choose $x'_{n+1} \in \mathcal{D}_{t-n-1}^{t-n}(x'_n)$ such that $(x, 0) \in \overline{D_{t-n-1}(x'_{n+1})}$.

This construction defines a sequence $(x'_n)_{n \in \mathbb{N}}$ such that, for all $n \in \mathbb{N}$, $(x'_n, 0) \in \overline{D_{t-n}(x'_n)}$. The sequence of points (x'_n, e^{t-n}) naturally defines an infinite branch toward the past and it remains to show that this branch converges to $(x, 0)$ toward the past. By Step 1, this branch converges to some point at infinity, thus it suffices to show that $x'_n \rightarrow x$ as $n \rightarrow \infty$.

Let us assume for the moment that

$$\forall n \in \mathbb{N}, \|x'_n - x\| \leq \text{MBD}_{-\infty}^{t-n}(x'_n). \quad (2.6.7)$$

Then, by the straightness property (Proposition 3.2.7),

$$\|x'_n - x\| \leq \text{MBD}_{-\infty}^{t-n}(x'_n) \leq \max_{x'' \in \mathcal{D}_{t-n}^{t-n}(x_0)} \text{MBD}_{\infty}^{t-n}(x'') \rightarrow 0 \text{ as } n \rightarrow \infty.$$

It remains to prove (2.6.7). Let $n \in \mathbb{N}$. Let $\varepsilon > 0$. Since $x \in \overline{D_{t-n}(x'_n)}$, there exists $t_2 \leq t - n$ and $x_2 \in \mathcal{D}_{t_2}^{t-n}(x'_n)$ such that $(x_2, t_2) \in \text{DSF}^\infty$ and $\|x_2 - x\| < \varepsilon$. Let $t_3 \leq t_2$. Since $(x_2, t_2) \in \text{DSF}^\infty$, there exists some $x_3 \in \mathcal{D}_{t_3}^{t_2}(x_2)$. Thus,

$$\begin{aligned} \|x'_n - x\| &\leq \|x'_n - x_2\| + \|x_2 - x\| \leq \text{CFD}_{t_2}^{t-n}(x_2) + \varepsilon \\ &\leq \text{CFD}_{t_3}^{t-n}(x_3) + \varepsilon \leq \text{MBD}_{t_3}^{t-n}(x'_n) + \varepsilon. \end{aligned}$$

Thus $\|x'_n - x\| \leq \text{MBD}_{t_3}^{t-n}(x'_n) + \varepsilon$ for any t_3 small enough, then

$$\|x'_n - x\| \leq \liminf_{t' \rightarrow \infty} \text{MBD}_{t'}^{t-n} + \varepsilon = \text{MBD}_{-\infty}^{t-n}(x'_n) + \varepsilon.$$

Since this is true for any $\varepsilon > 0$, we obtain (2.6.7), this completes the proof of Lemma 2.6.4. \square

2.6.5 Step 4: \mathcal{I} is nonempty and dense in \mathbb{R}^d

The main part of the proof consists in proving that $\mathcal{I} \neq \emptyset$ (i.e. the point (i) in Theorem 2.1.2). The density will follow easily. The proof is based on a second moment method, Theorem 2.4.4 and Lemma 2.3.18. For $t \geq 0$, let us define the level t -association function f_t as follows:

$$f_t(x, \eta) := \mathcal{A}_0^t \left(\underset{x' \in \mathcal{L}_0(\eta)}{\text{argmin}} \|x' - x\| \right) (\eta),$$

for any $x \in \mathbb{R}^d$ and $\eta \in \mathcal{N}_S$. That is, we consider the point x' of $\mathcal{L}_0(\eta)$ the closest to x and we follow the trajectory from x' up to the level t . We apply a second moment method on $V_t(0) := \text{Leb}(\Lambda_{f_t}(0))$ (recall that $\Lambda_{f_t}(\cdot)$ is defined in Definition 2.3.4). First, Corollary 2.3.17 gives that $\mathbb{E}_{0 \in \mathcal{L}_t}[V_t(0)] = \alpha_0^{-1} e^{dt}$, where α_0 has been defined in Proposition 2.2.18. We now apply Theorem 2.4.4 to upper-bound $\mathbb{E}_{0 \in \mathcal{L}_t}[V_t(0)^2]$.

Let $X_0 := \arg\min_{x \in \mathcal{L}_0} \|x\|$ be the point of \mathcal{L}_0 the closest to 0. By Proposition 2.4.9, $\mathbb{E}[\|X_0\|^p] < \infty$. Then (i) in Theorem 2.4.4 applied to X_0 with $p = d$ gives,

$$\limsup_{t \rightarrow \infty} e^{-dt} \mathbb{E}[\|\mathcal{A}_0^t(X_0)\|^d] \leq \limsup_{t \rightarrow \infty} \mathbb{E} \left[\left(\frac{\|X_0\| + \text{CFD}_0^t(X_0)}{e^t} \right)^d \right] < \infty. \quad (2.6.8)$$

Applying Proposition 2.3.18 to f_t with $p = d$, we obtain

$$\mathbb{E}_{0 \in \mathcal{L}_t}[V_t(0)^2] \leq C_{d,d} e^{dt} \mathbb{E}[\|f_t(0)\|^d] = C_{d,d} e^{dt} \mathbb{E}[\|\mathcal{A}_0^t(X_0)\|^d]. \quad (2.6.9)$$

Then

$$\limsup_{t \rightarrow \infty} e^{-2dt} \mathbb{E}[V_t(0)^2] \stackrel{(2.6.9)}{\leq} \limsup_{t \rightarrow \infty} C_{d,d} e^{-dt} \mathbb{E}[\|\mathcal{A}_0^t(0)\|^d] \stackrel{(2.6.8)}{<} \infty. \quad (2.6.10)$$

By scale invariance (Lemma 2.8.2) and Cauchy-Schwartz,

$$\mathbb{P}_{0 \in \mathcal{L}_0}[\mathcal{D}_{-t}^0(0) \neq \emptyset] = \mathbb{P}_{0 \in \mathcal{L}_t}[\mathcal{D}_0^t(0) \neq \emptyset] = \mathbb{P}_{0 \in \mathcal{L}_t}[V_t(0) > 0] \geq \frac{\mathbb{E}_{0 \in \mathcal{L}_0}[V_t(0)]^2}{\mathbb{E}_{0 \in \mathcal{L}_0}[V_t(0)^2]}.$$

Thus

$$\begin{aligned} \mathbb{P}_{0 \in \mathcal{L}_0}[\forall t \geq 0, \mathcal{D}_{-t}^0(0) \neq \emptyset] &= \lim_{t \rightarrow \infty} \mathbb{P}_{0 \in \mathcal{L}_0}[\mathcal{D}_{-t}^0(0) \neq \emptyset] \geq \liminf_{t \rightarrow \infty} \frac{e^{-2dt} \mathbb{E}_{0 \in \mathcal{L}_0}[V_t(0)]^2}{e^{-2dt} \mathbb{E}_{0 \in \mathcal{L}_0}[V_t(0)^2]} \\ &= \frac{\alpha_0^{-2}}{\limsup_{t \rightarrow \infty} e^{-2dt} \mathbb{E}_{0 \in \mathcal{L}_0}[V_t(0)^2]} > 0 \text{ by (2.6.10)}. \end{aligned}$$

Note that, if the DSF has finite degree (which happens with probability 1), then a point (x, e^0) with $x \in \mathcal{L}_0$ belongs to an infinite branch if and only if for all $t \geq 0$, $\mathcal{D}_{-t}^0(x) \neq \emptyset$. Thus

$$\mathbb{P}_{0 \in \mathcal{L}_0}[0 \text{ belongs to a bi-infinite branch}] > 0.$$

It easily follows that there exists a bi-infinite branch with positive probability, and bi-infinite branches converge in the direction of the past by Step 1, so $\mathbb{P}[\mathcal{I} \neq \emptyset] > 0$. Since the event $\mathcal{I} \neq \emptyset$ is translation invariant, it implies $\mathbb{P}[\mathcal{I} \neq \emptyset] = 1$ by ergodicity.

We move on to show that \mathcal{I} is dense in \mathbb{R}^d almost surely. Let $x \in \mathbb{R}^d$ and $r > 0$. By translation invariance, $\mathcal{I} \stackrel{d}{=} T_x \mathcal{I}$. Moreover, by dilation invariance, for all $r' > 0$, $\mathcal{I} \stackrel{d}{=} D_{r/r'} \mathcal{I}$. Thus, since $\mathcal{I} \neq \emptyset$ a.s.,

$$\begin{aligned} \mathbb{P}[\mathcal{I} \cap B(x, r) \neq \emptyset] &= \mathbb{P}[T_x \mathcal{I} \cap B(x, r) \neq \emptyset] = \mathbb{P}[\mathcal{I} \cap B(0, r) \neq \emptyset] \\ &= \mathbb{P}[D_{r/r'} \mathcal{I} \cap B(0, r) \neq \emptyset] = \mathbb{P}[\mathcal{I} \cap B(0, r') \neq \emptyset] \rightarrow 1 \text{ as } r' \rightarrow \infty, \end{aligned}$$

thus $\mathbb{P}[\mathcal{I} \cap B(x, r) \neq \emptyset] = 1$. Since \mathbb{R}^d admits a countable basis, it follows that \mathcal{I} is dense in \mathbb{R}^d almost surely.

2.6.6 Step 5: \mathcal{I} is closed in \mathbb{R}^d

Since $\Psi_0(x)$ is closed in \mathbb{R}^d for all $x \in \mathcal{L}_0$ by Step 3, it is sufficient to show that the family of cells $(\Psi_0(x))_{x \in \mathcal{L}_0}$ is locally finite almost surely, i.e. for any ball $B \subset \mathbb{R}^d$ of radius 1, $B \cap \Psi_0(x) \neq \emptyset$ for

finitely many $x \in \mathcal{L}_0$. Let $B \subset \mathbb{R}^d$ be some ball of radius 1, it will be shown that B intersects finitely many cells a.s. The conclusion will immediately follow since \mathbb{R}^d admits a countable basis. By translation invariance it is enough to consider $B(0, e^0)$.

For $t \in \mathbb{R}$ and $x \in \mathcal{L}_t$, we define the radius of the cell $\Psi_t(x)$, denoted by $\text{Rad}_t(x)$, as

$$\text{Rad}_t(x) := \sup_{x' \in \Psi_t(x)} \|x' - x\|.$$

with the convention $\sup \emptyset = 0$, where $\Psi_t(x)$ is defined in (2.6.1). We now show that $\text{Rad}_0(x) \leq \text{MBD}_{-\infty}^0(x)$. Let $x' \in \Psi_0(x)$. There exists $f \in \text{BI}$ such that $f(0) = x$ and $\lim_{t \rightarrow -\infty} f(t) = x'$. Thus

$$\|x' - x\| = \left\| \int_{-\infty}^0 f'(t) dt \right\| \leq \int_{-\infty}^0 \|f'(t)\| dt \leq \liminf_{t \rightarrow -\infty} \text{MBD}_t^0(x) = \text{MBD}_{-\infty}^0(x).$$

Since this is true for each $x' \in \Psi_0(x)$, $\text{Rad}_0(x) \leq \text{MBD}_{-\infty}^0(x)$. Thus, by (2.6.5), for any $p \geq 1$, $\mathbb{E}_{0 \in \mathcal{L}_0}[\text{Rad}_0(0)^p] < \infty$. For $x \in \mathcal{L}_0$, we now define the *augmented cell* of x by the set of points x' that are at distance at most 1 from $\Psi_0(x)$:

$$\Psi'_0(x) := \{x' \in \mathbb{R}^d, \exists x'' \in \Psi_0(x), \|x'' - x'\| < 1\}.$$

Note that $\Psi_0(x) \cap B(0, e^0) \neq \emptyset$ if and only if $0 \in \Psi'_0(x)$, this is the reason why $\Psi'_0(x)$ has been introduced. Thus what we want to show is that $0 \in \Psi'_0(x)$ for finitely many $x \in \mathcal{L}_0$. It is done by the Mass Transport Principle. From each $x \in \mathcal{L}_0$, we transport a unit mass from x to each unit volume of $\Psi'_0(x)$. It corresponds to the measure π defined as

$$\pi(E) := \mathbb{E} \left[\sum_{x \in \mathcal{L}_0} \int_{\Psi'_0(x)} \mathbf{1}_{(x, x') \in E} dx' \right].$$

for all $E \subset \mathbb{R}^d \times \mathbb{R}^d$. Let A be a nonempty open subset of \mathbb{R}^d . On the one hand,

$$\pi(A \times \mathbb{R}^d) = \mathbb{E} \left[\sum_{x \in \mathcal{L}_0 \cap A} \text{Leb}(\Psi'_0(x)) \right] = \alpha_0 \text{Leb}(A) \mathbb{E}_{0 \in \mathcal{L}_0}[\text{Leb}(\Psi'_0(0))],$$

where Lemma 2.3.13 is used in the second equality. On the other hand,

$$\begin{aligned} \pi(\mathbb{R}^d \times A) &= \mathbb{E} \left[\sum_{x \in \mathcal{L}_0} \int_{\Psi'_0(x)} \mathbf{1}_{x' \in A} dx' \right] = \mathbb{E} \left[\sum_{x \in \mathcal{L}_0} \int_A \mathbf{1}_{x' \in \Psi_0(x)} dx' \right] \\ &= \mathbb{E} \left[\int_A \#\{x \in \mathcal{L}_0, x' \in \Psi'_0(x)\} dx' \right] = \int_A \mathbb{E}[\#\{x \in \mathcal{L}_0, x' \in \Psi'_0(x)\}] dx' \\ &= \int_A \mathbb{E}[\#\{x \in \mathcal{L}_0, 0 \in \Psi'_0(x)\}] dx' = \text{Leb}(A) \mathbb{E}[\#\{x \in \mathcal{L}_0, 0 \in \Psi'_0(x)\}]. \end{aligned}$$

Fubini was used in second, third and fourth equality and translation invariance was used in the fifth equality. Thus, since π is diagonally invariant, the Mass Transport Principle gives,

$$\mathbb{E}[\#\{x \in \mathcal{L}_0, 0 \in \Psi'_0(x)\}] = \alpha_0 \mathbb{E}_{0 \in \mathcal{L}_0}[\text{Leb}(\Psi'_0(0))].$$

Denoting by $\vartheta(d)$ the volume of the unit ball in \mathbb{R}^d , we have

$$\mathbb{E}_{0 \in \mathcal{L}_0}[\text{Leb}(\Psi'_0(0))] \leq \mathbb{E}_{0 \in \mathcal{L}_0}[\vartheta(d)(\text{Rad}_0(0) + 1)^d] \leq 2^{d-1} \vartheta(d) (\mathbb{E}_{0 \in \mathcal{L}_0}[\text{Rad}_0(0)^d] + 1) < \infty,$$

it follows that $\mathbb{E}[\#\{x \in \mathcal{L}_0, 0 \in \Psi'_0(x)\}] < \infty$, this proves that the family $\{\Psi_0(x), x \in \mathcal{L}_0\}$ is locally finite almost surely.

Therefore, \mathcal{I} is dense and closed in \mathbb{R}^d , thus $\mathcal{I} = \mathbb{R}^d$. This proves (iii) in Theorem 2.1.2.

2.6.7 Step 6 proof of (iv)

Let us call \mathcal{I}' the set of (abscissas of) points in $\mathbb{R}^d \times \{0\}$ which are the limit in the direction of past of at least two bi-infinite branches:

$$\mathcal{I}' := \{x \in \mathbb{R}^d, \exists f_1, f_2 \in \text{BI}, f_1 \neq f_2 \text{ and } \lim_{t \rightarrow -\infty} f_1(t) = \lim_{t \rightarrow -\infty} f_2(t) = x\}.$$

The proof that $\Phi' : (x, \eta) \mapsto \mathbf{1}_{x \in \mathcal{I}'(\eta)}$ is measurable, done in Step 3, can be easily adapted to show that $(x, \eta) \mapsto \mathbf{1}_{x \in \mathcal{I}'(\eta)}$ is also measurable. By translation invariance and Fubini,

$$\mathbb{E}[\text{Leb}(\mathcal{I}')] = \mathbb{E} \left[\int_{\mathbb{R}^d} \mathbf{1}_{x \in \mathcal{I}'} dx \right] = \int_{\mathbb{R}^d} \mathbb{P}[x \in \mathcal{I}'] dx = \int_{\mathbb{R}^d} \mathbb{P}[0 \in \mathcal{I}'] dx = \infty \mathbb{P}[0 \in \mathcal{I}'].$$

Thus, in order to show that $\text{Leb}(\mathcal{I}') = 0$ a.s., we will prove that $\mathbb{P}[0 \in \mathcal{I}'] = 0$.

Consider the set of bi-infinite branches that converges to $(0, 0)$ in the direction of the past. For $t \in \mathbb{R}$, let $\mathcal{P}(t)$ be the set of t -level points through which these branches pass:

$$\mathcal{P}(t) := \{f(t), f \in \text{BI and } \lim_{t \rightarrow -\infty} f(t) = 0\} = \{x \in \mathcal{L}_t, 0 \in \Psi_t(x)\}.$$

We define the *coalescing time* of 0, denoted by τ_0 , as the first time t for which all branches converging to $(0, 0)$ in the direction of the past have coalesced:

$$\tau_0 := \inf\{t \in \mathbb{R}, \#\mathcal{P}(t) = 1\} \in \mathbb{R} \cup \{-\infty, +\infty\}.$$

Let us show that $\tau_0 < +\infty$ a.s. It has been shown in Step 3 that the family of cells $\{\Psi_0(x), x \in \mathcal{L}_0\}$ is locally finite, so $\#\mathcal{P}(0) < \infty$ a.s. (and it is also true that $\#\mathcal{P}(t) < \infty$ for all t). By coalescence (Theorem 2.1.1), there exists a.s. some $t \geq 0$ such that a.s. (and it is also true that $\#\mathcal{P}(t) < \infty$ for all t). By coalescence (Theorem 2.1.1) trajectories starting from the points $\{(x, e^0), x \in \mathcal{P}(0)\}$ coalesce before time t . For such a t , $\#\mathcal{P}(t) = 1$, therefore $\tau_0 < \infty$ a.s.

By dilation invariance, for all $t \in \mathbb{R}$, $\tau_0 \stackrel{d}{=} \tau_0 + t$, therefore the only possibility is that $\tau_0 = -\infty$ a.s. This implies that $\#\mathcal{P}(t) = 1$ for all $t \in \mathbb{R}$ a.s., so there exists a unique $f \in \text{BI}$ such that $\lim_{t \rightarrow -\infty} f(t) = 0$. This shows that $\mathbb{P}[0 \in \mathcal{I}'] = 0$ so \mathcal{I}' has measure zero almost surely.

We move on to show that \mathcal{I}' is dense in \mathbb{R}^d . We first show that $\mathcal{I}' \neq \emptyset$ a.s. Let us suppose that $\mathcal{I}' = \emptyset$ with positive probability. On the event $\{\mathcal{I}' = \emptyset\}$, the cells $\{\Psi_0(x), x \in \mathcal{L}_0\}$ are pairwise disjoint. So for all $x \in \mathcal{L}_0$,

$$\Psi_0(x)^c = \bigcup_{\substack{x' \in \mathcal{L}_0 \\ x' \neq x}} \Psi_0(x').$$

Since the cells $\Psi_0(x)$ are closed in \mathbb{R}^d (Step 3) and the family $\{\Psi_0(x) x \in \mathcal{L}_0\}$ is locally finite (Step 5), both $\Psi_0(x)$ and $\Psi_0(x)^c$ must be closed in \mathbb{R}^d . By connectivity, this implies that $\Psi_0(x)$ is \emptyset or \mathbb{R}^d and there is unique $x \in \mathbb{R}^d$ such that $\Psi_0(x) = \mathbb{R}^d$. Then, conditioning to the event $\{\mathcal{I}' = \emptyset\}$, the law of the unique random $X \in \mathcal{L}_0$ such that $\Psi_0(X) = \mathbb{R}^d$ must be translation invariant, which is impossible. Therefore $\mathbb{P}[\mathcal{I}' = \emptyset] = 0$.

We now show that \mathcal{I}' is dense in \mathbb{R}^d by the same argument that have been use to show that \mathcal{I} is dense. For any $x \in \mathbb{R}^d$ and $0 < \varepsilon < R < \infty$, by translation and dilation invariance,

$$\mathbb{P}[\mathcal{I}' \cap B(x, \varepsilon)] = \mathbb{P}[\mathcal{I}' \cap B(0, R)] \rightarrow 1 \text{ as } R \rightarrow \infty,$$

so $\mathbb{P}[\mathcal{I}' \cap B(x, \varepsilon)] = 1$. Since \mathbb{R}^d admits a countable basis, we can conclude that \mathcal{I}' is dense in \mathbb{R}^d almost surely.

The last point is to show that \mathcal{I}' is countable in the bi-dimensional case ($d = 1$). Note that, for $x \in \mathbb{R}^d$, $x \in \mathcal{I}'$ if and only if there exists some level $t \in \mathbb{R}$ and two points $x_1, x_2 \in \mathcal{L}_t$ with

$x_1 \neq x_2$ such that $x \in \Psi_t(x_1) \cap \Psi_t(x_2)$. Moreover the level t can be chosen rational without loss of generality. Thus it suffices to show that, for a given level $t \in \mathbb{Q}$, $\cup_{x_1, x_2 \in \mathcal{L}_t, x_1 \neq x_2} (\Psi_t(x_1) \cap \Psi_t(x_2))$ is countable. Let us consider the set $L_t^\infty := \{x \in \mathcal{L}_t, \Psi_t(x) \neq \emptyset\}$. Since it is a discrete subset of \mathbb{R} , let us index its elements by \mathbb{Z} in the ascending order: $\mathcal{L}_t^\infty = \{x_n, n \in \mathbb{Z}\}$. It has been shown that, for $n \in \mathbb{Z}$, $\Psi_t(x_n) \subset \mathbb{R}$ is closed (Step 3) and bounded (Step 5); moreover it has to be connected by planarity. Thus $\Psi_t(x_n)$ is a segment (eventually reduced to a single point); let us write $\Psi_t(x_n) = [a_n, b_n]$ for all $n \in \mathbb{Z}$. Again by planarity, $b_n \leq a_{n+1}$ for all $n \in \mathbb{N}$ (else a trajectory from $(b_n, 0)$ should cross a trajectory from $(a_{n+1}, 0)$). Moreover, since the segments $[a_n, b_n]$ cover \mathbb{R} , $b_n \leq a_{n+1}$ so $a_n = b_{n+1}$ for all $n \in \mathbb{Z}$. Finally, the set of points in \mathbb{R} belonging to two different cells $[a_n, b_n]$ are exactly the set of extremities $\{a_n, n \in \mathbb{Z}\}$, so it is countable. This completes the proof.

We can wonder what are the possible numbers of bi-infinite branches sharing a same asymptotic direction toward the past. This question is unsolved, but we can give the following conjecture:

Conjecture 2.6.5. *Almost surely, the maximal number of bi-infinite branches sharing a same asymptotic direction toward the past is $d + 1$. That is,*

$$\max_{x \in \mathbb{R}^d} \#\{f \in BI, \lim_{t \rightarrow -\infty} f(t) = x\} = d + 1.$$

The intuition behind this conjecture can be explained as follows. Let us consider the family of cells $\{\Psi_t(x), x \in \mathcal{L}_t\}$ for a given level $t \in \mathbb{R}$. They cover \mathbb{R}^d and they do not overlap except for boundaries. A boundary point shared by k cells corresponds to an asymptotic direction with k branches that have not coalesced at level t . It is reasonable to expect that it exists $d + 1$ cells sharing a same boundary point, but that it does not exist $d + 2$ cells overlapping at a same point. If this is true for every level $t \in \mathbb{R}$, it implies the existence of $d + 1$ branches sharing a same asymptotic direction but the non-existence of $d + 2$ such branches.

2.7 Appendix : first properties of the hyperbolic DSF

In this section, we show Proposition 3.2.2.

2.7.1 The edges never cross

Let us show that the DSF is non-crossing a.s. We first run out the case $d \geq 2$. Almost surely, N does not contain four coplanary points, so two edges never cross.

In the following, we suppose $d = 1$. Recall that $\pi_y : (x, y) \mapsto y$ is the projection on the y -coordinate. Let $z_1, z_2 \in \mathcal{N}_S$ and suppose that $[z_1, A(z_1)]_{eucl} \cap [z_2, A(z_2)]_{eucl} \neq \emptyset$. We denote by P_{eucl} the intersection point of $[z_1, A(z_1)]_{eucl}$ and $[z_2, A(z_2)]_{eucl}$. Let us suppose that there are no two points z_1, z_2 with $\pi_y(z_1) = \pi_y(z_2)$ (this happens with probability 0). We will prove the following:

Claim 2.7.1. *The geodesics $[z_1, A(z_1)]$ and $[z_2, A(z_2)]$ meet at one point P_{hyp} .*

We suppose Claim 2.7.1 for the moment. We have $\pi_y(A(z_1)) > \pi_y(P_{eucl}) > \pi_y(z_2)$, thus by definition of the parent, $d(z_2, A(z_2)) < d(z_2, A(z_1))$. Then

$$\begin{aligned} d(z_2, P_{hyp}) + d(P_{hyp}, A(z_2)) &= d(z_2, A(z_2)) < d(z_2, A(z_1)) \\ &\leq d(z_2, P_{hyp}) + d(P_{hyp}, A(z_1)), \end{aligned} \tag{2.7.1}$$

so $d(P_{hyp}, A(z_2)) < d(P_{hyp}, A(z_1))$. On the other hand, interchanging z_1 and z_2 in the previous calculation leads to $d(P_{hyp}, A(z_1)) < d(P_{hyp}, A(z_2))$. This is a contradiction. Therefore $[z_1, A(z_1)]_{eucl} \cap [z_2, A(z_2)]_{eucl} = \emptyset$.

It remains to show Claim 2.7.1. For $i = 1, 2$, consider the simple closed curve supported on $[z_i, A(z_i)] \cup [z_i, A(z_i)]_{eucl}$. Let us denote by R_i the region of H inside this closed curve. We now show that R_i contains no point of N . Both $[z_i, A(z_i)]_{eucl}$ and $[z_i, A(z_i)]$ are contained in $B_H(z_i, d(z_i, A(z_i)))$ since $B_H(z_i, d(z_i, A(z_i)))$ is a Euclidean ball so it is convex for both Hyperbolic and Euclidean metrics. Moreover, $\pi_y(A(z_i)) > \pi_y(z_i)$, so both $[z_i, A(z_i)]$ and $[z_i, A(z_i)]_{eucl}$ are contained in the upper-half plane $\mathbb{R}^d \times (\pi_y(z_i), \infty)$. Thus, both $[z_i, A(z_i)]$ and $[z_i, A(z_i)]_{eucl}$ are contained in $B^+(z_i)$. By simple connexity, $R_i \subset B^+(z_i)$. Thus, since $N \cap B^+(z_i) = \emptyset$, R_i contains no points of N .

By assumption $[z_1, A(z_1)]_{eucl}$ crosses $[z_2, A(z_2)]_{eucl}$ exactly once, and none of the extremities z_1 and $A(z_1)$ belong to R_2 . Thus $[z_1, A(z_1)]_{eucl}$ should cross $[z_2, A(z_2)]$ exactly once. Now, consider $[z_2, A(z_2)]$. None of the extremities z_2 and $A(z_2)$ belong to R_1 , so by the same argument, $[z_2, A(z_2)]$ crosses $[z_1, A(z_1)]$ exactly once. This proves Claim 2.7.1 and achieves the proof of Proposition ??.

2.7.2 The DSF has finite degree

We move on to show that the DSF is locally finite a.s. Fix the origin $z_0 := (0, e^0)$. Consider $N' = N \cup \{z_0\}$ and consider the DSF on N' . Since N is a Poisson Point Process, N' has same law as the Palm version of N conditioned that $z_0 \in N$. The origin z_0 has one parent almost surely, so it has to be shown that z_0 has finitely many sons almost surely. We apply Campbell formula [8]. Consider the function

$$\begin{aligned} F : \mathcal{N}_S \times \mathbb{H}^d &\rightarrow \mathbb{R}_+ \\ (\eta, z) &\mapsto \mathbf{1}_{B_+(z, d(z, 0)) \cap N = \emptyset} \end{aligned} \quad (2.7.2)$$

For $z \in N$, if z is a son of z_0 then $B_+(z, d(z, z_0)) = \emptyset$ so $F(N \setminus \{z\}, z) = 1$. Therefore,

$$\begin{aligned} \mathbb{E} \left[\#\{z \in N, (z, z_0) \in \vec{E}\} \right] &\leq \mathbb{E} \left[\sum_{z \in N} F(N \setminus \{z\}, z) \right] \\ &= \int_{\mathbb{H}^d} \mathbb{E} [F(N, z)] dz \\ &= \int_{\mathbb{H}^d} \exp(-\lambda \mu(B^+(z, d(z, z_0)))) dz, \\ &= \int_{\mathbb{H}^d} \exp(-\lambda \mu(B^+(0, d(z, z_0)))) dz, \end{aligned} \quad (2.7.3)$$

where Campbell formula was used in the first equality. The last inequality holds since, for all $\rho > 0$ $B^+(z, \rho)$ have same volume as $B^+(z_0, \rho)$ by isometry invariance. We now rewrite the integral above using the following coordinates transformation formula:

Lemma 2.7.2. *Let $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$. Then*

$$\int_{\mathbb{H}^d} f(d(z, z_0)) dz = \int_{\mathbb{R}_+} s(\rho) f(\rho) d\rho \quad (2.7.4)$$

$s : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is some function. This function verify $s(\rho) \sim \beta e^{d\rho}$ when $\rho \rightarrow \infty$ for some constant $\beta > 0$.

The proof of lemma 2.7.2 is given in Chapter 1 (see Proposition 2.7.2). This formula applied to $f(\rho) = \mu(B^+(z_0, \rho))$ and (2.7.3) lead to:

$$\mathbb{E} \left[\#\{z \in N, (z, z_0) \in \vec{E}\} \right] \leq \int_{\mathbb{R}_+} s(\rho) e^{-\lambda \mu(B^+(z_0, \rho))} d\rho. \quad (2.7.5)$$

In order to show that the right-hand side is finite, we need to lower-bound $\mu(B^+(z_0, \rho))$. Suppose for the moment that, for all ρ large enough,

$$\mu(B^+(z_0, \rho)) \geq e^{d\rho/3}, \quad (2.7.6)$$

Then

$$\begin{aligned} \mathbb{E} \left[\#\{z \in N, (z, z_0) \in \vec{E}\} \right] &\leq \int_{\mathbb{R}_+} s(\rho) e^{-\lambda\mu(B^+(z_0, \rho))} d\rho \\ &\leq \int_{\mathbb{R}_+} s(\rho) \exp(-\lambda e^{d\rho/3}) d\rho \\ &< \infty \end{aligned} \quad (2.7.7)$$

since $s(\rho) \sim \beta \exp(d\rho) \ll \exp(\lambda e^{d\rho/3})$. Thus 0 has a finite number of sons almost surely, this shows that the DSF is locally finite almost surely.

It remains to show (2.7.6). Let $\rho > 0$. Consider the cylinder

$$\mathcal{C}_\rho := B_{\mathbb{R}^d}(0, e^{\frac{2}{5}\rho}) \times [1, e^\rho - e^{-\rho}]. \quad (2.7.8)$$

The claim is that, when ρ is large enough, $\mathcal{C}_\rho \subset B^+(0, \rho)$. Indeed, by the discussion below Corollary 1.2.2, it follows that the Euclidean center of $B_H(z_0, \rho)$ is $(e^\rho + e^{-\rho})/2$, thus by reflectional symmetry with respect to the hyperplane $\mathbb{R}^d \times \{(e^\rho + e^{-\rho})/2\}$, it suffices to show that $B_{\mathbb{R}^d}(0, e^{\frac{2}{5}\rho}) \subset B_H(0, \rho)$ for ρ large enough. It follows from Corollary 1.2.3 that, for r large enough, $d((0, e^0), (x, e^0)) \leq 5/2 \ln(r)$ for all $x \in B_{\mathbb{R}^d}(0, r)$, thus, for ρ large enough $B_{\mathbb{R}^d}(0, e^{\frac{2}{5}\rho}) \subset B_H(z_0, \rho)$ and the claim is proved.

Finally, we can easily compute the volume of \mathcal{C}_ρ :

$$\mu(\mathcal{C}_\rho) = \eta e^{\frac{2}{5}d\rho} \int_1^{e^\rho - e^{-\rho}} \frac{dz}{z^{d+1}} \sim \frac{\eta}{d} e^{\frac{2}{5}d\rho} \text{ when } \rho \rightarrow \infty. \quad (2.7.9)$$

where η denotes the volume of the (Euclidean) unit ball in \mathbb{R}^d . Thus, for ρ large enough,

$$\mu(B^+(z_0, \rho)) \geq \mu(\mathcal{C}_\rho) \geq \frac{\eta}{2d} e^{\frac{2}{5}d\rho} \geq e^{d\rho/3}, \quad (2.7.10)$$

this achieves the proof.

2.7.3 Controlling the number of points at a given level

We finally prove Proposition 2.2.17. By the dilation invariance property of model, it is enough to show it for $t = 0$. Let $R > 0$. We will in fact prove that $(\#\mathcal{L}_0 \cap B(0, R))^p$ admits exponential moments. Let $n \in \mathbb{N}$, and $L > 0$ depending on n that will be chosen later. Let us partition $\#\mathcal{L}_0 \cap B(0, R)$ in two sets:

$$\begin{aligned} E_{\leq L} &:= \{x \in \#\mathcal{L}_0 \cap B(0, R), d((x, e^0)_\downarrow, (0, e^0)) \leq L\}, \\ E_{> L} &:= \{x \in \#\mathcal{L}_0 \cap B(0, R), d((x, e^0)_\downarrow, (0, e^0)) > L\}. \end{aligned}$$

We have

$$\mathbb{P}[\#\mathcal{L}_0 \cap B(0, R) \geq n] \leq \mathbb{P}[E_{\leq L} \geq n] + \mathbb{P}[E_{> L} \neq \emptyset]. \quad (2.7.11)$$

Then we will upperbound the two terms of this sum.

Step 1: we upperbound $\mathbb{P}[E_{\leq L} \geq n]$.

Clearly, $\#E_{\leq L} \leq \#(N \cap B_H((0, e^0), L))$. Let us denote by \mathcal{V} the Hyperbolic volume of $\#(N \cap B_H((0, e^0), L))$. We use the following lemma to estimate \mathcal{V} :

Lemma 2.7.3 (Volume of a Hyperbolic ball). *For any $z_0 \in \mathbb{H}^{d+1}$,*

$$\mu(B_{(z_0, \rho)}) \sim \frac{\mathcal{S}(d)}{d2^d} e^{d\rho} \text{ when } \rho \rightarrow \infty.$$

This Lemma follows from Lemma 2.7.2 applied to $f = \mathbf{1}_{[0, \rho]}$ and easy computations. Then, when $L \rightarrow \infty$, large enough, $\mathcal{V} \sim \mathcal{S}(d)/(d2^d)e^{dL} = O(e^{dL})$. So $\#N \cap B_H((0, e^0), L)$ is distributed according to a Poisson law of parameter $\lambda\mathcal{V} \leq Ce^{dL}$ for some constant C large enough. We use the following Chernoff bound [25]:

Lemma 2.7.4 (Chernoff bound for a Poisson distribution). *If X is distributed according to a Poisson law of parameter $\alpha > 0$, then, for $n \geq \alpha$,*

$$\mathbb{P}[X \geq n] \leq \frac{e^{-\alpha}(e\alpha)^n}{n^n}.$$

See [25] for a proof. Applying this bound to $\#N \cap B_H((0, e^0), L)$ leads to:

$$\mathbb{P}[\#E_{\leq L} \geq n] \leq \mathbb{P}[\#(N \cap B_H((0, e^0), L)) \geq n] \leq \frac{\exp(-Ce^{dL})(Ce^{dL+1})^n}{n^n} \quad (2.7.12)$$

if $n \geq Ce^{dL}$ and for L large enough.

Step 2: we upperbound $\mathbb{P}[E_{>L} \neq \emptyset]$.

For $x \in E_{>L}$, by triangular inequality and Corollary ??,

$$\begin{aligned} d((x, e^0)_\downarrow, (x, e^0)_\uparrow) &\geq d((x, e^0), (x, e^0)_\downarrow) \\ &\geq d((x, e^0)_\downarrow, (0, e^0)) - d((x, e^0), (0, e^0)) \end{aligned}$$

The second part of Corollary 1.2.3 gives that $d((x, e^0), (0, e^0)) \leq \|x\| \leq R$. Then $d((x, e^0)_\downarrow, (x, e^0)_\uparrow) \geq d((x, e^0)_\downarrow, (0, e^0)) - R$. Thus $N \cap B^+((x, e^0)_\downarrow, d((x, e^0)_\downarrow, (0, e^0)) - R) = \emptyset$. Let us define the set

$$E'_{>L} := \{z \in N, d(z, (0, e^0)) > L \text{ and } N \cap B^+(z, d(z, (0, e^0)) - R) = \emptyset\}.$$

Then $\#E_{>L} \leq \#E'_{>L}$. Consider the function $f : \mathcal{N}_S \times H \rightarrow \mathbb{R}_+$ defined as

$$f(z, \eta) := \mathbf{1}_{d(z, (0, e^0)) > L} \mathbf{1}_{B^+(z, d(z, (0, e^0)) - R) = \emptyset}.$$

By Campbell formula [8] and Fubini,

$$\mathbb{E}[\#E'_{>L}] = \mathbb{E} \left[\sum_{z \in N} f(z, N \setminus \{z\}) \right] = \mathbb{E} \left[\int_H f(z, N) dz \right] = \int_H \mathbb{P}[z \in E'] dz. \quad (2.7.13)$$

For $z \in H$ such that $d(z, (0, e^0)) \leq L$, $\mathbb{P}[z \in E'_{>L}] = 0$. If $d(z, (0, e^0)) > L$, then

$$\mathbb{P}[z \in E'_{>L}] = \exp(-\lambda\mu(B^+(z, d(z, (0, e^0)) - R))) \stackrel{(2.7.6)}{\leq} \exp\left(-\lambda e^{d(d(z, (0, e^0)) - R)/3}\right). \quad (2.7.14)$$

Thus, using the change of coordinates formula (Lemma 2.7.2),

$$\begin{aligned} \mathbb{E} [\#E'_{>L}] &\stackrel{(2.7.13),(2.7.14)}{\leq} \int_H \exp\left(-\lambda e^{d(d(z,(0,e^0))-R)/3}\right) dz \\ &= \mathcal{S}(d) \int_L^\infty \sinh(\rho)^d \exp\left(-\lambda e^{d(\rho-R)/3}\right) d\rho \\ &\leq \mathcal{S}(d) \int_L^\infty \exp\left(d\rho - \lambda e^{d(\rho-R)/3}\right) d\rho \text{ for } L \text{ large enough.} \end{aligned}$$

For L large enough, since $d\rho - \lambda e^{d(\rho-R)/3} \leq -e^{\rho/4}$,

$$\mathbb{E} [\#E'_{>L}] \leq \int_L^\infty e^{-e^{\rho/4}} d\rho \leq \int_L^\infty e^{\rho/4} e^{-e^{\rho/4}} = \left[-4e^{-e^{\rho/4}}\right]_L^\infty = 4e^{-e^{L/4}}. \quad (2.7.15)$$

Step 3: conclusion. We now combine upperbounds obtained in Step 1 and Step 2. Let us take

$$L = \frac{1}{d} \ln\left(\frac{n}{2C}\right),$$

then $Ce^{dL} = n/2$. Consider n large enough such that upperbounds (2.7.12) and (2.7.15) are satisfied. Then

$$\begin{aligned} \mathbb{P}[\#\mathcal{L}_0 \cap B(0, R) \geq n] &\stackrel{(2.7.11)}{\leq} \mathbb{P}[E_{\leq L} \geq n] + \mathbb{P}[E_{>L} \neq \emptyset] \\ &\stackrel{(2.7.12),(2.7.15)}{\leq} \frac{\exp(-n/2)(en/2)^n}{n^n} + 4 \exp\left(-e^{1/(4d) \ln(n/(2C))}\right) \\ &= \left(\frac{e^{1/2}}{n}\right)^n + 4 \exp\left(-\left(\frac{n}{2C}\right)^{\frac{1}{4d}}\right) \\ &\leq e^{-n^{1/(5d)}} \text{ for } n \text{ large enough.} \end{aligned}$$

Therefore $\#\mathcal{L}_0 \cap B(0, R) \in L^p$ for all $p \geq 1$, this achieves the proof of Proposition 2.2.17.

2.8 Appendix: Palm measure, conditioning the DSF on $x \in \mathcal{L}_t$

In this section, we define the Palm distribution of the DSF conditioned that $x \in \mathcal{L}_t$ for given $x \in \mathbb{R}^d$ and $t \in \mathbb{R}$. We refer to [?, Definition 2.6] for the definition of the hyperbolic DSF [?, Definition 2.11] for the definition of the point process \mathcal{L}_t .

2.8.1 Definition of $\mathbb{P}_{x \in \mathcal{L}_t}$

Let us fix $x \in \mathbb{R}^d, t \in \mathbb{R}$. We define the probability measure on \mathcal{N}_S corresponding to N conditioned by the event $\{x \in \mathcal{L}_t\}$. We follow the classic definition of Palm measures [8]. Recall that, for $s \in \mathbb{R}^d, T_s$ denotes the translation by s .

Proposition-definition 2.8.1 (Conditional distribution given $\{x \in \mathcal{L}_t\}$).

- (Definition) For $\Gamma \subset \mathcal{N}_S$ measurable, we define the measure μ_Γ on \mathbb{R}^d by

$$\mu_\Gamma(A) := \mathbb{E} \left[\sum_{s \in \mathcal{L}_t \cap A} \mathbf{1}_{T_{x-s}N \in \Gamma} \right] \quad (2.8.1)$$

for all measurable set $A \subset \mathbb{R}^d$. Note that μ_Γ depends on t and x .

- (Proposition) For all measurable set $\Gamma \subset \mathcal{N}_S$, the measure μ_Γ is invariant by translations and finite on compact sets.
- (Definition) Then for all measurable set $\Gamma \subset \mathcal{N}_S$, μ_Γ is a multiple of the Lebesgue measure, so we can define:

$$\mathbb{P}_{x \in \mathcal{L}_t}[\Gamma] := \frac{d\mu_\Gamma}{\alpha_0 e^{-dt} d\text{Leb}}, \quad (2.8.2)$$

where α_0 is the intensity of \mathcal{L}_0 .

- (Proposition) The map $\Gamma \mapsto \mathbb{P}_{x \in \mathcal{L}_t}[\Gamma]$ so defined is a probability measure on \mathcal{N}_S . We denote by $\mathbb{E}_{x \in \mathcal{L}_t}$ its associated expectation.

Thus we defined a probability measure on \mathcal{N}_S . In the following, if an event E (resp. a random variable X) is measurable w.r.t. $\sigma(N)$, we denote by $\mathbb{P}_{x \in \mathcal{L}_t}[E]$ (resp. $\mathbb{E}_{x \in \mathcal{L}_t}[X]$) the probability of E (resp. the expectation of X) when N is distributed according to the probability measure $\mathbb{P}_{x \in \mathcal{L}_t}$.

Proof. We begin with the second point. Let $\Gamma \subset \mathcal{N}_S$ be measurable. For all measurable set $A \subset \mathbb{R}^d$ and $r \in \mathbb{R}^d$:

$$\begin{aligned} \mu_\Gamma(T_r A) &= \mathbb{E} \left[\sum_{s \in \mathcal{L}_t \cap T_r A} \mathbf{1}_{T_{x-s} N \in \Gamma} \right] = \mathbb{E} \left[\sum_{s \in (T_{-r} \mathcal{L}_t) \cap A} \mathbf{1}_{T_{x-s} (T_{-r} N) \in \Gamma} \right] \\ &= \mathbb{E} \left[\sum_{s \in \mathcal{L}_t \cap A} \mathbf{1}_{T_{x-s} N \in \Gamma} \right] = \mu_\Gamma(A). \end{aligned}$$

We used the fact that the translation T_{-r} preserves the DSF distribution in the fourth equality. Then μ_Γ is invariant by translations. Let $K \subset \mathbb{R}^d$ be some compact set. Then $\mu_\Gamma(K) \leq \mathbb{E}[\#(\mathcal{L}_t \cap K)] = e^{-dt} \alpha_0 \text{Leb}(K) < \infty$ because \mathcal{L}_t has finite intensity $e^{-dt} \alpha_0$. This proves the second point.

For the fourth point, we have, by taking $A = [0, 1]^d$ in the definition of μ_Γ ,

$$\mathbb{P}_{x \in \mathcal{L}_t}[\Gamma] = \alpha_0^{-1} e^{dt} \mathbb{E} \left[\sum_{s \in \mathcal{L}_t \cap [0, 1]^d} \mathbf{1}_{T_{x-s} N \in \Gamma} \right].$$

We show that $\mathbb{P}_{x \in \mathcal{L}_t}$ is a positive measure. First, $\mathbb{P}_{x \in \mathcal{L}_t}[\Gamma] > 0$ for all measurable set $\Gamma \subset \mathcal{N}_S$ and $\mathbb{P}_{x \in \mathcal{L}_t}[\emptyset] = 0$. The σ -additivity follows from the monotone convergence theorem.

Moreover

$$\mathbb{P}_{x \in \mathcal{L}_t}[\mathcal{N}_S] = \alpha_0^{-1} e^{dt} \mathbb{E}[\#\mathcal{L}_t \cap [0, 1]^d] = 1,$$

so $\mathbb{P}_{x \in \mathcal{L}_t}$ is a probability measure on \mathcal{N}_S . □

Lemma 2.8.2. (*Invariance by dilations*). *Let $t, t' \in \mathbb{R}$. We have*

$$\mathbb{P}_{0 \in \mathcal{L}_t}[D_{e^{t'-t}}(N) \in \cdot] = \mathbb{P}_{0 \in \mathcal{L}_{t'}}[N \in \cdot]$$

Proof. For all measurable set $\Gamma \subset \mathcal{N}_S$ and $A \subset \mathbb{R}^d$, we consider

$$\mu_\Gamma(A) = \mathbb{E} \left[\sum_{s \in \mathcal{L}_t \cap A} \mathbf{1}_{T_{-s} N \in \Gamma} \right], \quad \mu'_\Gamma(A) = \mathbb{E} \left[\sum_{s \in \mathcal{L}_{t'} \cap A} \mathbf{1}_{T_{-s} N \in \Gamma} \right].$$

Let $A \subset \mathbb{R}^d$ such that $\text{Leb}(A) > 0$. By definition

$$\begin{aligned} \mathbb{P}_{0 \in \mathcal{L}_t} [D_{e^{t'-t}}(N) \in \Gamma] &= \mathbb{P}_{0 \in \mathcal{L}_t} [N \in D_{e^{t'-t}}(\Gamma)] = \frac{\mu_{D_{e^{t'-t}}\Gamma}(A)}{\alpha_0 e^{-dt} \text{Leb}(A)} \\ &= \frac{\mathbb{E} \left[\sum_{s \in \mathcal{L}_t \cap A} \mathbf{1}_{T_{-s}N \in D_{e^{t'-t}}\Gamma} \right]}{\alpha_0 e^{-dt} \text{Leb}(A)}. \end{aligned} \quad (2.8.3)$$

We have

$$\begin{aligned} \mathbb{E} \left[\sum_{s \in \mathcal{L}_t \cap A} \mathbf{1}_{T_{-s}N \in D_{e^{t'-t}}\Gamma} \right] &= \mathbb{E} \left[\sum_{s \in \mathcal{L}_t \cap A} \mathbf{1}_{D_{e^{t'-t}} \circ T_{-s}N \in \Gamma} \right] \\ &= \mathbb{E} \left[\sum_{s \in e^{t'-t}(\mathcal{L}_t \cap A)} \mathbf{1}_{D_{e^{t'-t}} \circ T_{-e^{t'-t}^{-1}s}N \in \Gamma} \right]. \end{aligned} \quad (2.8.4)$$

Since $D_{e^{t'-t}} \circ T_{-e^{t'-t}^{-1}s} = T_{-s} \circ D_{e^{t'-t}}$ and $N \stackrel{(d)}{=} D_{e^{t'-t}}N$,

$$\begin{aligned} \mathbb{E} \left[\sum_{s \in e^{t'-t}(\mathcal{L}_t \cap A)} \mathbf{1}_{D_{e^{t'-t}} \circ T_{-e^{t'-t}^{-1}s}N \in \Gamma} \right] &= \mathbb{E} \left[\sum_{s \in \mathcal{L}_{t'} [D_{e^{t'-t}}N] \cap e^{t'-t}A} \mathbf{1}_{T_{-s} \circ D_{e^{t'-t}}N \in \Gamma} \right] \\ &= \mathbb{E} \left[\sum_{s \in \mathcal{L}_{t'} [N] \cap e^{t'-t}A} \mathbf{1}_{T_{-s}N \in \Gamma} \right]. \end{aligned} \quad (2.8.5)$$

Combining (2.8.3), (2.8.4) and (2.8.5), we obtain

$$\begin{aligned} \mathbb{P}_{0 \in \mathcal{L}_t} [D_{e^{t'-t}}(N) \in \Gamma] &= \frac{\mathbb{E} \left[\sum_{s \in \mathcal{L}_{t'} \cap e^{t'-t}A} \mathbf{1}_{T_{-s}N \in \Gamma} \right]}{\alpha_0 e^{-dt} \text{Leb}(A)} = \frac{\mu'_{\Gamma}(e^{t'-t}A)}{\alpha_0 e^{-dt'} \text{Leb}(e^{t'-t}A)} \\ &= \mathbb{P}_{0 \in \mathcal{L}_{t'}} [N \in \Gamma], \end{aligned}$$

so Lemma 2.8.2 is proved. \square

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Chapter 3

The Radial Spanning Tree in hyperbolic space

In this chapter, we define and analyze an extension to the d -dimensional hyperbolic space of the Radial Spanning Tree (RST) introduced by Baccelli and Bordenave in the two-dimensional Euclidean space [1]. In particular, we will focus on the description of the infinite branches of the tree. The properties of the two-dimensional Euclidean RST are extended to the hyperbolic case in every dimension: almost surely, every infinite branch admits an asymptotic direction and each asymptotic direction is reached by at least one infinite branch. Moreover, the branch converging to any deterministic asymptotic direction is unique almost surely. To obtain results for any dimension, a completely new approach is considered here. Our strategy mainly relies on the two following ingredients. First, the hyperbolic metric allows us to obtain fine control of the branches' fluctuations in the hyperbolic DSF without using planarity arguments. Then, we couple the hyperbolic RST with the hyperbolic DSF introduced and studied in Chapter 2.

3.1 Introduction

Geometric random trees are well studied in the literature since they interact with many other fields, such as communication networks, particles systems or population dynamics. Several works have established scaling limits for two-dimensional radial trees [10, 9] and translation invariant forests [11, 22, 14]. In addition, random spanning trees appear in the context of first passage percolation [15]. A complete introduction to geometric random graphs is given in Penrose [20].

The Radial Spanning Tree (RST) is a random tree whose introduction in the two-dimensional Euclidean space has been motivated by applications for communication networks [1]. The construction of this tree is the same on the plane \mathbb{R}^2 or on the hyperbolic space \mathbb{H}^d (presented below). The set of vertices is given by a homogeneous Poisson Point Process (PPP) \mathcal{N} of intensity λ . The RST rooted at the origin 0 is the graph obtained by connecting each point $z \in \mathcal{N}$ to its parent $A(z)$, defined as the closest point to z among all points $z' \in \mathcal{N} \cup \{0\}$ that are closer to the origin than z . This defines a random tree rooted at the origin with a radial structure. An infinite backward path is defined as a sequence of Poisson points $(z_n)_{n \geq 0} \in (\mathcal{N} \cup \{0\})^{\mathbb{N}}$ with $z_0 = 0$ and $z_n = A(z_{n+1})$ for any $n \geq 0$. Given an infinite path, we will say that the forward direction is towards 0 and the backward direction is towards infinity.

The topological properties of the bi-dimensional Euclidean RST are well-understood. Baccelli and Bordenave showed that almost surely, any infinite backward path admits an asymptotic direction; moreover, a.s., every asymptotic direction is reached by at least one infinite backward path and there exists a.s. a unique infinite path in any given deterministic asymptotic direction [1]. These results on the infinite paths are completed by Baccelli, Coupier & Tran [2].

For any integer $d \geq 2$, the hyperbolic space \mathbb{H}^d is a d -dimensional Riemannian manifold with constant negative curvature, that can be chosen equal to -1 without loss of generality. It admits a set of ideal boundary points $\partial\mathbb{H}^d$, and $\overline{\mathbb{H}^d} := \mathbb{H}^d \cup \partial\mathbb{H}^d$ denotes the hyperbolic space endowed with its boundary. It is a non-amenable space, i.e. the measure of the boundary of a large subset is not negligible with respect to its volume. The hyperbolic space is defined in more details in [7] and [19].

There is a growing interest for the study of random models in a hyperbolic setting. Benjamini and Schramm establish percolation results on regular tilings and Voronoï tessellation in the hyperbolic plane [3]. Mean characteristics of the Poisson-Voronoï tessellation have also been considered in a general Riemannian manifold by Calka et al. [6]. This interest is explained by at least two reasons. First, hyperbolic random graphs are well-fitted to model social networks [5]. In addition, strong differences have been noticed for properties of random models depending whether they are considered in an Euclidean or hyperbolic setting. Indeed, some hyperbolic random graphs admits a non-degenerate regime with infinitely many unbounded components in the hyperbolic space [23, 16], which is generally not the case in the Euclidean space. In addition, behaviours of non-amenable spaces are well studied in a discrete context [4, 18, 21].

Thus it is natural to consider and study the hyperbolic RST, which we define in the same way as the Euclidean RST. A simulation of the two-dimensional hyperbolic RST is given in Figure 3.1. In this paper, we extend the results of Baccelli and his coauthors to hyperbolic geometry **in every dimension**. Here is our main result:

Theorem 3.1.1. *For any dimension $d \geq 1$ and any intensity λ , the following happens:*

- (i) *almost surely, any infinite backward path $(z_n)_{n \in \mathbb{N}}$ admits an asymptotic direction, i.e. there exists $z_\infty \in \partial\mathbb{H}^{d+1}$ such that $\lim_{n \rightarrow \infty} z_n = z_\infty$ (in the sense of the topology of $\overline{\mathbb{H}^{d+1}}$);*
- (ii) *almost surely, for any $I \in \partial\mathbb{H}^{d+1}$, there exists an infinite backward path (z_n) with asymptotic direction I (i.e. such that $\lim_{n \rightarrow \infty} z_n = I$);*
- (iii) *for any deterministic boundary point $I \in \partial\mathbb{H}^{d+1}$, the path with asymptotic direction I is almost surely unique;*
- (iv) *the set of boundary points with two infinite backward paths is dense in $\partial\mathbb{H}^{d+1}$;*
- (v) *this set is moreover countable in the bi-dimensional case (i.e. $d = 1$).*

Establishing the results announced in Theorem 3.1.1 in every dimension constitutes the main originality of this paper. For the two reasons explained further, the proofs of Baccelli and Bordenave in the 2D-Euclidean setting [1] cannot be generalised to higher dimensions.

In both contexts \mathbb{R}^2 and \mathbb{H}^{d+1} , for any $d \geq 1$, the proofs of (i), (ii), (iv) and (v) of Theorem 3.1.1 follow the strategy of Howard and Newman [15], which is to show that the tree is *straight*, that is, the descendants subtree of a vertex far from the origin is included in a thin cone. To prove that the 2D-Euclidean RST is straight, Baccelli and Bordenave used a translation invariant model derived from the RST: the Directed Spanning Forest (DSF), which constitutes a local approximation of the RST far from the origin [1]. They exploit the theory of Markov chains to bound from above fluctuations of trajectories in the DSF and then, they deduce the straightness of the RST via planarity. This strategy cannot be generalised to higher dimensions. However, in \mathbb{H}^d , we manage to control the angular deviations of branches in the RST without resorting to an auxiliary model, which required planarity in the Euclidean setting. The hyperbolic metric guarantees that angular deviations decay exponentially fast with the distance to the origin, which is strong enough to show straightness.

In addition, in the Euclidean context, the uniqueness part (point (iii) in Theorem 3.1.1) is only proved in dimension 2 since it strongly uses planarity [15, 1], and the strategy of proof cannot be generalised to higher dimensions. To prove (iii) in \mathbb{H}^d , our strategy consists in exploiting the link

existing between the hyperbolic RST and another random graph, the hyperbolic DSF, defined and studied in Flammant 2019 [12], which is the hyperbolic counterpart of the Euclidean DSF used by Baccelli and Bordenave. Roughly speaking, the hyperbolic DSF can be defined as the limit of the hyperbolic RST when the origin point tends to an ideal boundary point. Similarly to the Euclidean setting, it constitutes a local approximation of the RST far from the origin. The proof of (iii) exploits the coalescence of the hyperbolic DSF (i.e. it is almost surely a tree) [12, Theorem 1.1], which is a non-trivial fact obtained by exploiting the mass-transport principle, and a local coupling between the two models.

After defining the hyperbolic RST and giving its basic properties, we define two quantities that encode angular fluctuations along trajectories, the Cumulative angular Forward Deviations (CFD) and the Maximal Backward Deviations (MBD). We then establish upper-bounds of these quantities: first, we upper-bound the Maximal Backward Deviations in a thin annulus of width $\delta > 0$ (Proposition 3.2.5) and then we deduce a global control of MBD in the whole space (Proposition 3.2.6), that roughly says that angular deviations decay exponentially fast with the distance to the origin. From this upper-bound, we deduce that the RST is straight in the sense of Howard & Newman (Proposition 3.2.7). The points (i), (ii), (iv) and (v) in Theorem 3.1.1 can be deduced from straightness and the upper-bound of MBD given by Proposition 3.2.6. The point (iii) (the uniqueness part) is done by exploiting a local coupling existing between the RST and the DSF far from the origin.

The rest of paper is organized as follows. In Section 3.2, we set some reminders of hyperbolic geometry and we define the hyperbolic RST. Then, we give its basic properties and a road-map of the proofs. We also announce the upper-bounds of angular deviations (Propositions 3.2.5 and 3.2.6) and the straightness property (Proposition 3.2.7). The proof of Theorem 3.1.1 is done in Section 3.3. Proposition 3.2.5 is proved in Section 3.4 and the proofs of Propositions 3.2.6 and 3.2.7 are done in Section 3.5.

3.2 Definitions, notations and basic properties

We denote by \mathbb{N} the set of non-negative integers and by \mathbb{N}^* the set of positive integers. In the rest of the paper, c (resp. C) will be some small (resp. large) constant whose value can change from a line to another.

3.2.1 The hyperbolic space

We refer to [7] or [19] for a complete introduction to hyperbolic geometry. For $d \in \mathbb{N}^*$, the $(d+1)$ -dimensional hyperbolic space, denoted by \mathbb{H}^{d+1} , is a $(d+1)$ -dimensional Riemannian manifold of constant negative curvature -1 that can be defined by several isometric models. One of them is the open-ball model consisting in the unit open ball

$$I = \{(x_1, \dots, x_{d+1}) \in \mathbb{R}^{d+1}, x_1^2 + \dots + x_{d+1}^2 < 1\} \quad (3.2.1)$$

endowed with the following metric:

$$ds_I^2 := 4 \frac{dx_1^2 + \dots + dx_{d+1}^2}{1 - x_1^2 - \dots - x_{d+1}^2}. \quad (3.2.2)$$

This model is rotation invariant. The metric becomes smaller as we get closer to the boundary unit sphere ∂I , and this boundary is at infinite distance from the center 0.

The volume measure on (I, ds_I^2) , denoted by Vol_I , is given by

$$d\text{Vol}_I = 2^{d+1} \frac{dx_1 \dots dx_{d+1}}{1 - x_1^2 - \dots - x_{d+1}^2}. \quad (3.2.3)$$

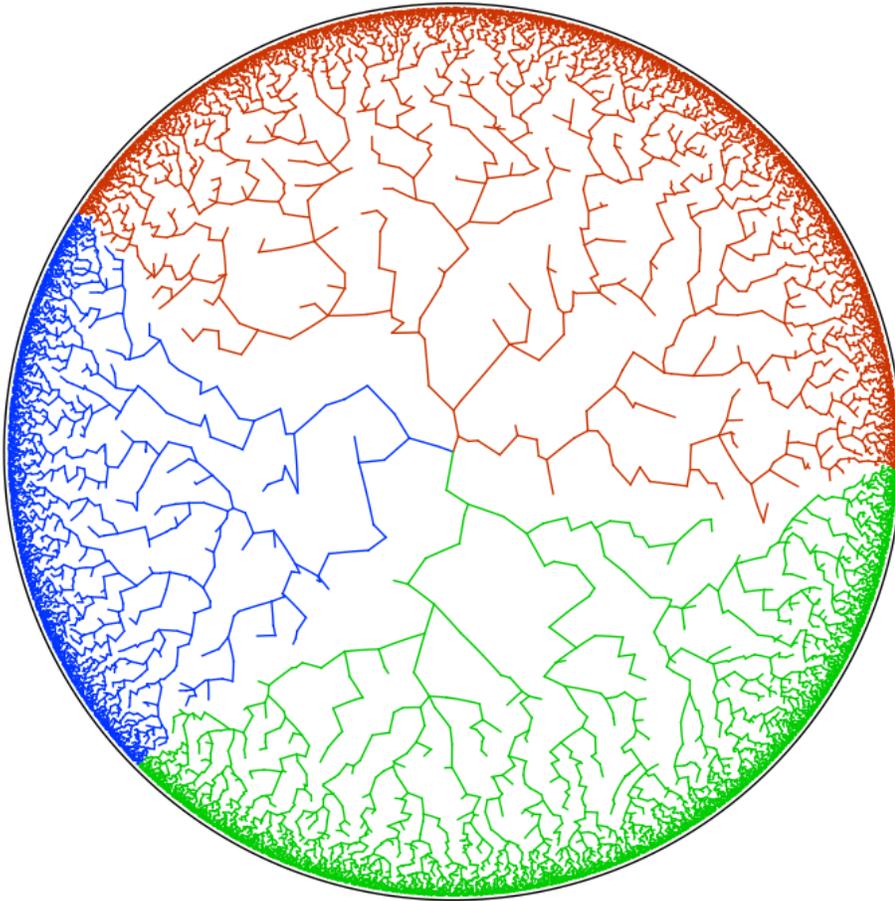


Figure 3.1: Simulation of the two-dimensional hyperbolic RST, with $\lambda = 30$, in the Poincaré disc model. The edges are represented by geodesics. The different connected components of the RST (apart from the root) are represented with different colors.

In this model (I, ds_I^2) , the geodesics are of two types: the diameters of I and the arcs that are perpendicular to the boundary unit sphere ∂I . We refer to discussion [7, P.80] for a proof. Moreover, this model is conformal, which means that the hyperbolic angle between two geodesics corresponds to their Euclidean angle in the open-ball representation.

An important fact about hyperbolic geometry is that all points and all directions play the same role. More precisely, \mathbb{H}^{d+1} is homogeneous and isotropic. It means that the group of isometries of \mathbb{H}^{d+1} acts transitively on the unit tangent bundle of \mathbb{H}^{d+1} : given two points $x, y \in \mathbb{H}^{d+1}$ and two unit tangent vectors $u \in T_x \mathbb{H}^{d+1}$, $v \in T_y \mathbb{H}^{d+1}$, there exists an isometry g of \mathbb{H}^{d+1} such that $g(x) = y$ and that pushes forward u on v . The notations T_x , T_y and the vocabulary relating to Riemannian geometry are defined in [17]. We refer to [19], Proposition 1.2.1 p.5, for a proof.

The hyperbolic space \mathbb{H}^{d+1} is naturally equipped with a set of points at infinity, and the most natural way to identify these points is to use the open-ball model. In (I, ds_I^2) , the set of points at infinity is identified by the boundary unit sphere ∂I . We denote by $\partial \mathbb{H}^{d+1}$ the boundary set (represented by the boundary unit sphere in (I, ds_I^2)) and by $\overline{\mathbb{H}^{d+1}} := \mathbb{H}^{d+1} \cup \partial \mathbb{H}^{d+1}$ the hyperbolic space \mathbb{H}^{d+1} plus the set of points at infinity, with the topology given by the closed ball. A point

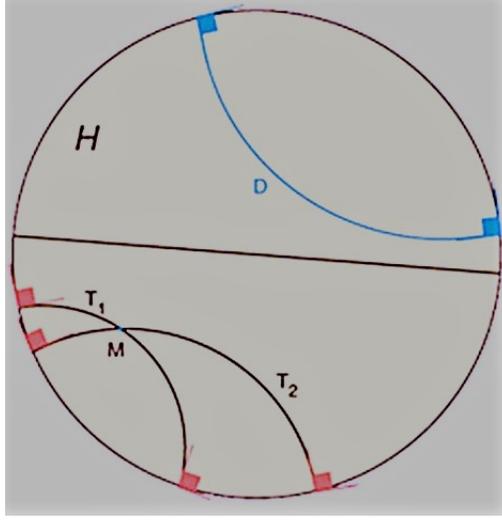


Figure 3.2: Geodesics in the open ball model

$z_\infty \in \partial\mathbb{H}^{d+1}$ is called *ideal point* or *point at infinity*.

We denote by $d(\cdot, \cdot)$ the hyperbolic distance in \mathbb{H}^{d+1} , and by $\|\cdot\|$ the Euclidean norm in \mathbb{R}^d , with the convention $\|\infty\| = \infty$. Let us denote by Vol the volume measure on \mathbb{H}^{d+1} . For $z_1, z_2 \in \overline{\mathbb{H}^{d+1}}$, let us denote by $[z_1, z_2]$ the geodesic between z_1 and z_2 . Moreover, we set the notations:

$$[z_1, z_2] := [z_1, z_2] \setminus \{z_2\}, \quad]z_1, z_2] := [z_1, z_2] \setminus \{z_1\}, \quad]z_1, z_2[\setminus (\{z_2\} \cup \{z_1\}).$$

Let us denote by $[z_1, z_2]$ (resp. $(z_1, z_2]$) the semi-geodesic passing through z_2 (resp. z_1) and ending at z_1 (resp. z_2). For $z_1, z_2, z_3 \in \overline{\mathbb{H}^{d+1}}$, $\widehat{z_1 z_2 z_3}$ is the measure of the corresponding (non-oriented) hyperbolic angle. For any subset $B \subset \mathbb{H}^{d+1}$, \overline{B} denotes the closure of B in $\overline{\mathbb{H}^{d+1}}$. For any point $z \in \mathbb{H}^{d+1}$ and $\theta > 0$, $\text{Cone}(z, \theta) := \{z' \in \mathbb{H}^{d+1}, \widehat{z_0 z z'} \leq \theta\}$ is defined as the cone of apex 0 and aperture θ (if $\theta \geq \pi$ then $\text{Cone}(z, \theta)$ is the whole space \mathbb{H}^{d+1}). In addition, for $r > 0$ and $z \in \mathbb{H}^{d+1}$, we define

$$B_{S(r)}(z, \theta) := \text{Cone}(z, \theta) \cap S(r). \quad (3.2.4)$$

Let $0 \in \mathbb{H}^{d+1}$ be some arbitrary origin point (it can be thought as the center of the ball in the open-ball representation), which will play the role of the root of the RST. For $z \in \mathbb{H}^{d+1}$ and $r > 0$, we denote by $B(z, r) := \{z' \in H, d(z, z') < r\}$ (resp. $S(z, r) := \{z' \in H, d(z, z') = r\}$) the hyperbolic ball (resp. *sphere*) centered at z of radius r , and we set $B(r) := B(0, r)$ (resp. $S(r) := s(0, r)$). For $x \in \mathbb{R}^d$ and $r > 0$, let us also denote by $B_{\mathbb{R}^d}(x, r) := \{x' \in \mathbb{R}^d, \|x' - x\| < r\}$ the Euclidean ball centered at x of radius r .

Let us denote by \mathbb{S}^d the unit Euclidean sphere in \mathbb{R}^{d+1} and by ν its d -dimensional volume measure. Since the RST is a rooted graph, a convenient way to represent points in \mathbb{H}^{d+1} is to use polar coordinates. Recall that 0 is the origin point. For any point $z \in \mathbb{H}^{d+1}$, we denote by $z = (r; u)$ its polar coordinates w.r.t. 0: r is its distance to 0 and $u \in UT_0\mathbb{H}^{d+1} \simeq \mathbb{S}^d$ is its direction ($UT_0\mathbb{H}^{d+1}$ is the unitary tangent space of 0 in \mathbb{H}^{d+1}). In polar coordinates, the volume measure Vol is given by

$$d\text{Vol}(r; u) = \sinh(r)^d dr d\nu(u). \quad (3.2.5)$$

A direct consequence of this is that the volume of a ball of radius r is given by:

$$\text{Vol}(B(r)) = \int_0^r \sinh(r)^d dr d\nu(\mathbb{S}^d) = \mathcal{S}_d \sinh(r)^d \simeq e^{dr} \text{ when } r \rightarrow \infty, \quad (3.2.6)$$

where \mathcal{S}^d is the d -dimensional volume of \mathbb{S}^d .

The hyperbolic law of cosines [24, p.13] is a well adapted tool to compute distances using polar coordinates. Given $z_1 = (r_1; u_1)$, $z_2 = (r_2, u_2) \in \mathbb{H}^{d+1}$, the hyperbolic law of cosines gives,

$$\cosh d(z_1, z_2) = \cosh(r_1) \cosh(r_2) - \langle u_1, u_2 \rangle \sinh(r_1) \sinh(r_2). \quad (3.2.7)$$

3.2.2 The hyperbolic RST

In the rest of the paper, the dimension d and the intensity $\lambda > 0$ are fixed. Let \mathcal{N} be a homogeneous PPP of intensity λ in \mathbb{H}^{d+1} . The definition of the hyperbolic RST is similar to the Euclidean case. The set of vertices is $\mathcal{N} \cup \{0\}$. Each vertex $z \in \mathcal{N}$ is connected to the closest Poisson point among those that are closer to the origin than z :

Definition 3.2.1 (Radial Spanning Tree in \mathbb{H}^{d+1}). For any $z = (r; u) \in \mathcal{N}$, the *parent* of z is defined as

$$A(z) := \operatorname{argmin}_{z' \in \mathcal{N} \cap B(r)} d(z', z).$$

We call *Radial Spanning Tree (RST)* in \mathbb{H}^{d+1} rooted at 0 the oriented graph (V, \vec{E}) where

$$V := \mathcal{N} \cup \{0\}, \quad \vec{E} := \{(z, A(z)), z \in \mathcal{N}\}.$$

It is possible to assume that $\mathcal{N} \cup \{0\}$ does not contain isosceles triangles, since this event has probability 1. Thus the ancestor $A(z)$ is well-defined.

For $z \in \mathcal{N} \cup \{0\}$ and $k \in \mathbb{N}$, let us define $A^{(k)}(z) = A \circ \dots \circ A$ k times and $A^{(-k)}(z) = \{z' \in \mathcal{N}, A^{(k)}(z') = z\}$ (in particular $A^{(-1)}(z)$ is the set of daughters of z). For $z \in \mathcal{N}$ and $r \geq 0$, let us define

$$B^+(z, r) := B(z, r) \cap B(0, d(0, z)) \text{ and } B^+(z) := B^+(z, d(z, A(z))).$$

By definition of the parent, $B^+(z) \cap \mathcal{N} = \emptyset$ for all $z \in \mathcal{N}$.

Definition 3.2.1 does not specify the shape of edges, but the results announced in Theorem 3.1.1 only concern the graph structure of the hyperbolic RST, so their veracity does not depend on the geometry of edges. It is more natural to represent edges with hyperbolic geodesics, but we do another choice which will appear more convenient for the proofs. Given $z_1 = (r_1; u_1)$, $z_2 = (r_2; u_2) \in \mathbb{H}^{d+1}$ such that $0 \notin [z_1, z_2]$, we define a path $[z_1, z_2]^*$, in an isotropic way, verifying the two following conditions:

- i) the distance to the origin 0 is monotonous along the path $[z_1, z_2]^*$,
- ii) the distance to z_1 is also monotonous along this path.

It will be necessary for the proofs that the shape of edges satisfy conditions (i) and (ii), and the geodesic $[z_1, z_2]$ does not verify condition (i) in general. Since $0 \notin [z_1, z_2]$, u_1 and u_2 are not antipodal, thus one can consider the unique geodesic path $\gamma_{u_1, u_2} : [0, 1] \rightarrow UT_0\mathbb{H}^{d+1}$ on the sphere with constant speed connecting u_1 to u_2 ($UT_0\mathbb{H}^{d+1}$ is the unitary tangent space of 0 in \mathbb{H}^{d+1}). Hence we define the path $[z_1, z_2]^*$ as

$$\begin{aligned} [0, 1] &\rightarrow \mathbb{H}^{d+1} \\ t &\mapsto ((1-t)r_1 + tr_2; \gamma_{u_1, u_2}(\phi_{r_1, r_2, \widehat{u_1, u_2}}(t))), \end{aligned} \quad (3.2.8)$$

where $\phi_{r_1, r_2, \widehat{u_1, u_2}} : [0, 1] \rightarrow [0, 1]$ is defined as:

$$\phi_{r_1, r_2, \widehat{u_1, u_2}}(t) := \frac{1}{\widehat{u_1, u_2}} \arccos \left(\frac{(1-t) \sinh(r_1) + t \cos(\widehat{u_1, u_2}) \sinh(r_2)}{\sinh((1-t)r_1 + tr_2)} \right).$$

This function $\phi_{r_1, r_2, \widehat{u_1, u_2}}$ is built to ensure that the distance to the origin z_1 is monotonous along the path $[z_1, z_2]^*$. Indeed, by the hyperbolic law of cosines (3.2.7),

$$\begin{aligned} & \cosh d(z_1, ((1-t)r_1 + tr_2; \gamma_{u_1, u_2}(\phi(t))) \\ &= \cosh(r_1) \cosh((1-t)r_1 + tr_2) - \cos(\phi(t)(\widehat{u_1, u_2})) \sinh(r_1) \sinh((1-t)r_1 + tr_2) \\ &= t [\cosh(r_1) \cosh(r_2) - \cos(\widehat{u_1, u_2}) \sinh(r_1) \sinh(r_2)] \end{aligned}$$

is monotonous in t .

We define $[z_1, z_2]^* := [z_1, z_2]^* \setminus \{z_2\}$ and $]z_1, z_2]^* := [z_1, z_2]^* \setminus \{z_1\}$. It is possible to assume that \mathcal{N} does not contain two points z_1, z_2 such that $0 \in [z_1, z_2]$ since this event has probability 1. Let us now define the random set RST by connecting each point $z \in \mathcal{N}$ to $A(z)$ by the path $[z, A(z)]^*$:

$$\text{RST} := \bigsqcup_{z \in \mathcal{N}} [z, A(z)]^*.$$

It may exist some points z belonging to several paths $[z_1, A(z_1)]^*, \dots, [z_k, A(z_k)]^*$; in that case, z is counted with multiplicity k in RST. Formally, $\text{RST} = \bigcup_{z \in \mathcal{N}} [z, A(z)]^* \times \{z\} \subset \mathbb{H}^{d+1} \times \mathbb{H}^{d+1}$, i.e. an element $z = (z', z'') \in \text{RST}$ is a couple where $z' \in \mathbb{H}^{d+1}$ is a point of the RST and z'' is the root of an edge containing z' . For $z = (z', z'') \in \text{RST}$, we define

$$z_{\downarrow} = z'', \quad z_{\uparrow} = A(z'').$$

In the following, we will commit an abuse of notations by considering that $\text{RST} \subset \mathbb{H}^{d+1}$ and identifying an element $z = (z', z'') \in \text{RST}$ to the corresponding point $z' \in \mathbb{H}^{d+1}$. Given $z \in \text{RST}$, let $n := \min\{k \geq 0, A^{(k)}(z_{\uparrow}) = 0\}$ be the number of steps required to reach the origin from z_{\uparrow} ; we define the *trajectory* from z as

$$\pi(z) := [z, z_{\uparrow}]^* \cup \bigcup_{k=0}^{n-1} [A^{(k)}(z_{\uparrow}), A^{(k+1)}(z_{\uparrow})]^*.$$

For $r > 0$, we define the *level* r as

$$\mathcal{L}_r := \text{RST} \cap S(r).$$

For $0 < r \leq r'$ and for $z' \in \mathcal{L}_{r'}$, the *ancestor at level* r of z' , denoted by $\mathcal{A}_r^{r'}(z')$ is the intersection point of $\pi(z')$ and $S(r)$. For $0 < r \leq r'$ and for $z \in \mathcal{L}_r$, the *set of descendents at level* r' is defined as $\mathcal{D}_r^{r'}(z) := \{z' \in \mathcal{L}_{r'}, z \in \pi(z')\}$ (we extend the notation for $z \notin \mathcal{L}_r$ by setting $\mathcal{D}_r^{r'}(z) := \emptyset$). For $z = (r; u) \in \text{RST}$, the *descendents subtree* of z is defined as $\mathcal{D}(z) := \bigcup_{r' \geq r} \mathcal{D}_r^{r'}(z)$. In addition, we call *infinite backward path* a sequence $(z_i)_{i \in \mathbb{N}} \in (\mathbb{H}^{d+1})^{\mathbb{N}}$ such that $z_0 = 0$ and $z_i = A(z_{i+1})$ for all $i \geq 0$.

Let us end this section with basic properties about RST proved in Appendix 3.6.

Proposition 3.2.2. *The RST is a tree and it has finite degree a.s. Moreover, in the bi-dimensional case ($d = 1$), the representation of the RST obtained by connecting each vertex $z \in \mathcal{N}$ to its parent $A(z)$ by the geodesic $[z, A(z)]$ (instead of $[z, A(z)]^*$) is planar, i.e. there is no two points $z_1, z_2 \in \mathcal{N}$ such that $[z_1, A(z_1)] \cap [z_2, A(z_2)] \neq \emptyset$.*

3.2.3 Sketch of proofs

In order to prove our main result (Theorem 3.1.1), the key point is to upper-bound angular deviations of trajectories. We first introduce two quantities, the *Cumulative Forward Angular Deviations (CFD)* and *Maximal Backward Deviations (MBD)* to quantify those fluctuations.

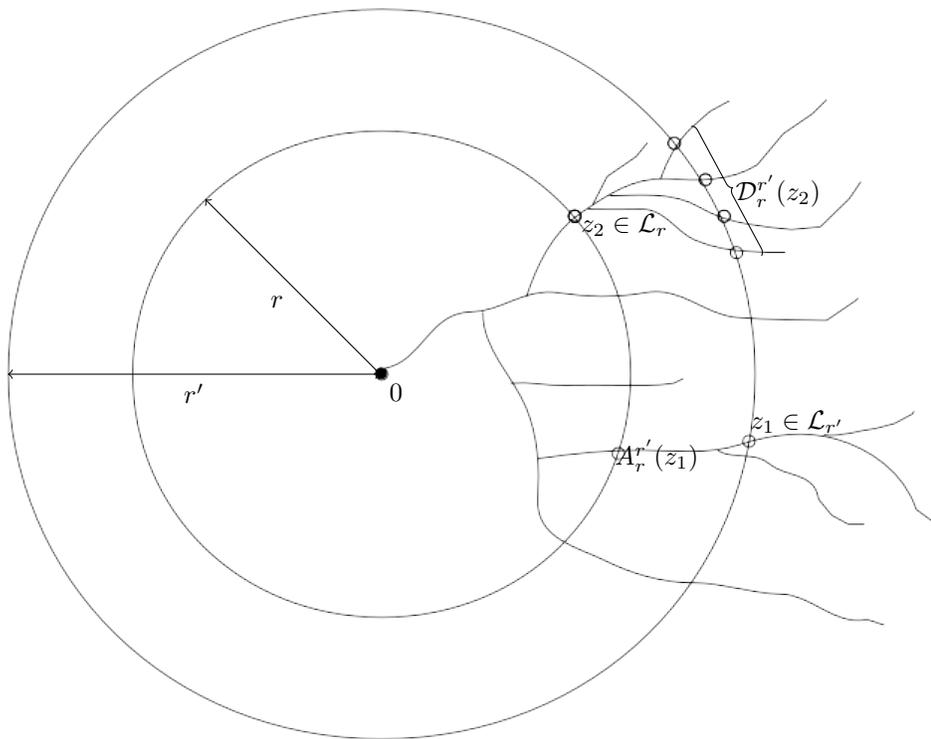


Figure 3.3: Representation of levels r and r' , the ancestor $A_r^{r'}(\cdot)$ and the set of descendants $\mathcal{D}_r^{r'}(\cdot)$

Definition 3.2.3 (Cumulative Forward angular Deviations). Let $0 < r \leq r'$ and $z' \in S(r')$. If $z' \notin \text{RST}$, we set $\text{CFD}_r^{r'}(z') = 0$ by convention and we now suppose that $z' \in \text{RST}$. Let $z := \mathcal{A}_r^{r'}(z')$. We define the *Cumulative Forward angular Deviations* of z' between levels r and r' as

$$\text{CFD}_r^{r'}(z') := \begin{cases} z'_\downarrow 0z & \text{if } z_\downarrow = z'_\downarrow, \\ z'_\downarrow 0z'_\uparrow + \sum_{k=0}^{n-1} A^{(k)}(z'_\uparrow) 0A^{(k+1)}(z'_\uparrow) + z'_\downarrow 0z & \text{else,} \end{cases}$$

where n is the unique non negative integer such that $A^{(n)}(z'_\uparrow) = z_\downarrow$.

Definition 3.2.4 (Maximal Backward angular Deviations). Let $0 < r \leq r'$ and $z \in S(r)$. We define the *Maximal Backward angular Deviations* between levels r and r' as

$$\text{MBD}_r^{r'}(z) := \begin{cases} 0 & \text{if } z \notin \text{RST}, \\ \sup_{r'' \in [r, r']} \max_{z'' \in \mathcal{D}_r^{r''}(z)} \text{CFD}_r^{r''}(z'') & \text{if } z \in \text{RST}. \end{cases}$$

We extend the definition to $r' = \infty$ by setting:

$$\text{MBD}_r^\infty(z) := \lim_{r' \rightarrow \infty} \text{MBD}_r^{r'}(z),$$

the limit exists since $r' \mapsto \text{MBD}_r^{r'}(z)$ is non-decreasing.

These quantities will be upper-bounded in two steps. First, a percolation argument is used to control angular deviations in any annulus of width $\delta > 0$ for some small $\delta > 0$ (Proposition 3.2.5) and then we deduce a global control of angular deviations (Proposition 3.2.6). Recall that $B_{S(r)}(\cdot, \cdot)$ is defined in (3.2.4).

Proposition 3.2.5. *There exists $\delta > 0$ such that, for any $p \geq 1$, there exists $C = C(d, p) > 0$ such that for any $r > 0$, $\theta \geq 0$ and any direction $u \in \mathbb{S}^d$,*

$$\mathbb{E} \left[\sum_{z \in B_{S(r)}(u, \theta) \cap \text{RST}} \left(\text{MBD}_r^{r+\delta}(z) \right)^p \right] \leq C \theta^d e^{r(d-p)}, \quad (3.2.9)$$

Proposition 3.2.6. *For any p large enough, there exists some constant $C_{\#} > 0$ such that, for any $0 < r_0 < \infty$, $A > 0$ and any direction $u \in \mathbb{S}^1$,*

$$\mathbb{E} \left[\sum_{z \in B_{S(r_0)}(u, Ae^{-r_0}) \cap \text{RST}} \left(\text{MBD}_{r_0}^\infty(z) \right)^p \right] \leq C_{\#} A^d e^{-r_0 p}.$$

These controls of angular deviations will be first used to show that the RST is *straight* (Proposition 3.2.7). The straightness property is the key to show (i), (ii) and (iv) in Theorem 3.1.1.

Proposition 3.2.7 (straightness property). *Almost surely, the following happens. For any $\varepsilon > 0$, there exists some $R_0 > 0$, such that, for any radius $r_0 \geq R_0$, for any $z \in \text{RST}$ with $d(0, z) \geq r_0$, the descendants subtree $\mathcal{D}(z)$ is contained in a cone of apex 0 and aperture $e^{-(1-\varepsilon)r_0}$, i.e. for any $z', z'' \in \mathcal{D}(z)$, $\widehat{z'0z''} \leq e^{-(1-\varepsilon)r_0}$.*

The proof of (iii) in Theorem 3.1.1 exploits the controls of angular deviations (Proposition 3.2.6) and the link existing between the RST and the hyperbolic Directed Spanning Forest introduced in [12]: the DSF approximates locally the RST far from the origin. The unicity of the infinite backward path with some given deterministic asymptotic direction has been shown for the DSF [5], and the local coupling existing between the two models permits to show that this property remains true for the RST.

3.3 Proof of Theorem 3.1.1

Here we assume that Propositions 3.2.6 and 3.2.7 are proved and we show that it implies Theorem 3.1.1.

3.3.1 The existence part: proof of (i),(ii),(iv) and (v)

It will be shown in this section that any infinite backward path admits an asymptotic direction and that any ideal boundary point is the asymptotic direction of an infinite backward path (point (i) and (ii) in Theorem 3.1.1). The strategy consists, in exploiting the straightness property (Proposition 3.2.7):

Let (z_n) be an infinite backward path, we prove that (z_n) admits an asymptotic direction. For $n \geq 0$, let us decompose z_n in polar coordinates: $z_n = (r_n; u_n)$. Proposition 3.2.7 immediately implies that the sequence $(u_n)_{n \geq 0}$ is a Cauchy sequence in $UT_0\mathbb{H}^{d+1} \simeq \mathbb{S}^d$, thus it converges, so (z_n) converges to some boundary point $z_\infty \in \partial\mathbb{H}^{d+1}$.

Let $\Psi = \{\lim_{n \rightarrow \infty} z_n, (z_n) \text{ is an infinite backward path}\} \subset \partial\mathbb{H}^{d+1}$ be the set of asymptotic directions reached by at least one infinite backward path. In order to prove that $\Psi = \partial\mathbb{H}^{d+1}$, we proceed in two steps: we first show that Ψ is dense in $\partial\mathbb{H}^{d+1}$, then we show that Ψ is closed in $\partial\mathbb{H}^{d+1}$.

Since the RST is an infinite tree with finite degree a.s. (Proposition 3.2.2), there exists an infinite backward path from 0 and the corresponding infinite backward path converges to an ideal boundary point by the previous paragraph, thus $\Psi \neq \emptyset$ almost surely.

We denote by $\text{Stab}(0)$ the set of isometries that fix 0, in particular it contains rotations centered at 0. Let B be an open subset of $\partial\mathbb{H}^{d+1}$. Since $\partial\mathbb{H}^{d+1} \simeq \mathbb{S}^d$ is compact, there exists finitely many isometries $\gamma_1, \dots, \gamma_k \in \text{Stab}(0)$ such that $\bigcup_{i=1, \dots, k} \gamma_i B = \partial\mathbb{H}^{d+1}$. The random set RST is invariant in distribution by $\text{Stab}(0)$, so the events $\{\Psi \cap \gamma_i B \neq \emptyset\}$ all have the same probability. Since $\Psi \neq \emptyset$ almost surely, $\mathbb{P}\left[\bigcup_{i=1, \dots, k} \{\Psi \cap \gamma_i B \neq \emptyset\}\right] = 1$ therefore $\mathbb{P}(\Psi \cap B \neq \emptyset) > 0$. In addition, for any neighborhood $\Phi \subset \overline{\mathbb{H}^{d+1}}$ of B , the event $\{\Psi \cap B \neq \emptyset\}$ is entirely determined by $\mathcal{N} \cap \Phi$, therefore it has probability 0 or 1. Thus $\Psi \cap B \neq \emptyset$ almost surely. Since the topology on $\partial\mathbb{H}^{d+1}$ admits a countable basis, Ψ is almost surely dense in $\partial\mathbb{H}^{d+1}$.

It remains to show that Ψ is a closed subset of $\partial\mathbb{H}^{d+1}$. Let $I \in \overline{\Psi}$ (recall that $\overline{\Psi}$ is the closure of Ψ in $\overline{\mathbb{H}^{d+1}}$). We construct by induction a sequence $(z_n)_{n \geq 1} \in \mathcal{N}^{\mathbb{N}^*}$ is an infinite backward path such that, for any $i \in \mathbb{N}$, $I \in \overline{\mathcal{D}(z_i)}$. Suppose $0, \dots, z_{i-1}$ already defined such that $z_j = A(z_{j+1})$ for $0 \leq j \leq i-2$ and $I \in \overline{\mathcal{D}(z_{i-1})}$. Since the vertex z_{i-1} has finitely many daughters, there exists some $z \in A^{(-1)}(z_{i-1})$ such that $I \in \overline{\mathcal{D}(z)}$. Thus we define z_i as such a z .

We now use straightness to show that the infinite backward path (z_n) constructed above converges to I (and thus $I \in \Psi$). This infinite backward path converges to some $I' \in \partial\mathbb{H}^{d+1}$ by (i). Let $\varepsilon > 0$, by Proposition 3.2.7 there exists some $i \geq 0$ such that $\mathcal{D}(z_i)$ (and thus $\overline{\mathcal{D}(z_i)}$) is contained in a cone of apex 0 and aperture at most ε . Since both I and I' belong to $\overline{\mathcal{D}(z_i)}$, $\widehat{IOI'} \leq \varepsilon$. Thus $I = I'$, which achieves the proof of the existence part.

Proof of (iv) and (v) Let us denote by $\Psi' \subset \partial\mathbb{H}^{d+1}$ the set of asymptotic directions with two infinite backward paths. To show (iv), we first show that, a.s. $\Psi' \neq \emptyset$. For $z \in \text{RST}$, let us define $\Psi_z \subset \partial\mathbb{H}^{d+1}$ as the set of asymptotic directions of infinite backward paths from z . By the same argument as in Step 2, Ψ_z is a closed subset of $\partial\mathbb{H}^{d+1}$. By (ii), a.s., there exists at least two infinite backward paths, so there exists a.s. some level $r_0 > 0$ with two points connected to infinity. Thus $\{\Psi_z, z \in \mathcal{L}(r_0)\}$ is a covering of $\partial\mathbb{H}^{d+1}$ by closed subsets, where at least two of them are nonempty. Since $\partial\mathbb{H}^{d+1}$ is connected, it implies that there exists $z_1, z_2 \in \mathcal{L}_{r_0}$ such that $\Psi_{z_1} \cap \Psi_{z_2} \neq \emptyset$. Thus $\Psi' \neq \emptyset$ a.s.

We use the same argument as in Step 2 to deduce that Ψ' is dense. Let B be an open subset of $\partial\mathbb{H}^{d+1}$. Since $\partial\mathbb{H}^{d+1} \simeq \mathbb{S}^d$ is compact, there exists finitely many isometries $\gamma_1, \dots, \gamma_k \in \text{Stab}(0)$ such that $\bigcup_{i=1, \dots, k} \gamma_i B = \partial\mathbb{H}^{d+1}$. The random set RST is invariant in distribution by $\text{Stab}(0)$, so the events $\Gamma_i := \{\Psi' \cap \gamma_i B \neq \emptyset\}$ all have the same probability. Since $\Psi \neq \emptyset$ almost surely, $\mathbb{P}\left[\bigcup_{i=1, \dots, k} \Gamma_i\right] = 1$ therefore $\mathbb{P}(\Gamma_i) > 0$. In addition, for any neighborhood $\Phi \subset \overline{\mathbb{H}^{d+1}}$ of B , the event Γ_i is entirely determined by $\mathcal{N} \cap \Phi$, therefore it has probability 0 or 1. Thus $\Psi' \cap B \neq \emptyset$ almost surely. Since the topology on $\partial\mathbb{H}^{d+1}$ admits a countable basis, Ψ' is almost surely dense in $\partial\mathbb{H}^{d+1}$.

The proof of (v) is done by exploiting the planarity in the bi-dimensional case (Proposition 3.2.2). Let us associate to any $z_\infty \in \Psi'$ a couple of vertices $P(z_\infty) = (z_1, z_2) \in \mathcal{N}^2$ with $z_1 \neq z_2$ such that $z_\infty \in \Psi_{z_1} \cap \Psi_{z_2}$. By planarity, such an application P must be injective. Indeed, if $z_\infty \neq z'_\infty$ are such that $P(z_\infty) = P(z'_\infty) = (z_1, z_2)$, then there exists four distinct backward infinite paths joining z_∞ to z_1 , z_∞ to z_2 , z'_∞ to z_1 and z'_∞ to z_2 . This implies that two paths among them intersect each other, even if the representation of edges are replaced by geodesics, which contradicts planarity. Therefore Ψ' is a.s. countable in the case $d = 1$.

3.3.2 The uniqueness part: proof of (iii)

The strategy is to exploit the link between the hyperbolic RST and the hyperbolic DSF. Let us consider the $(d+1)$ -dimensional half-space model (H, ds^2) :

$$H = \{(x_1, \dots, x_d, y) \in \mathbb{R}^{d+1}, y > 0\}, \quad ds^2 = \frac{dx_1^2 + \dots + dx_d^2 + dy^2}{y^2}.$$

In the following, we will identify the point $(x_1, \dots, x_{d+1}) \in H$ with the couple $(x, y) \in \mathbb{R}^d \times \mathbb{R}_+^*$ with

$$x := (x_1, \dots, x_d), \quad y := x_{d+1}. \quad (3.3.1)$$

The coordinate x is referred as the *abscissa* and y as the *ordinate*. Let us remind that, in the half-space representation, the boundary set $\partial\mathbb{H}^{d+1}$ is identified as the boundary hyper-plane $\mathbb{R}^d \times \{0\}$, plus an additional point at infinity denoted by ∞ , obtained by compactifying the closed half-space $\mathbb{R}^d \times \mathbb{R}_+$. Let us define I_∞ as the boundary point represented by $(0, 0)$ in H .

Let \mathcal{N} be a PPP inside (H, ds^2) , and, for any $h \geq 0$, let us define RST(h) as the Radial Spanning Tree of \mathcal{N} with origin $O(h) := (0, e^h)$, and let us define DSF as the Directed Spanning Forest of \mathcal{N} with direction ∞ (defined in [5]). Given $z \in \mathcal{N}$ let $A_{\text{DSF}}(z)$ be the parent of z in DSF, and let $A_{\text{RST}(h)}(z)$ be the parent of z in RST(h). We will also consider, for any given $h \geq 0$, the direction toward I_∞ defined as $u := (0, \dots, -1) \in UT_{O(h)}\mathbb{H}^{d+1} \simeq \mathbb{S}^d$. The proof is based on the two following propositions. The next one asserts that the RST(h) and the DSF coincide in a given compact set when h is large enough.

Proposition 3.3.1 (Coupling between RST and DSF). *Let $K \subset \mathbb{H}^{d+1}$ be some compact set. Then*

$$\lim_{h \rightarrow \infty} \mathbb{P}[\forall z \in \mathcal{N} \cap K, A_{\text{RST}(h)}(z) = A_{\text{DSF}}(z)] = 1.$$

For $A, a, h \geq 0$, let us define:

$$\begin{aligned} \text{Vois}(A, h) &:= \text{Cone}_{O(h)}(I_\infty, Ae^{-h}) \setminus B(O(h), h), \\ \text{Vois}'(A, a, h) &:= (B(O(h), h+a) \cap \text{Cone}_{O(h)}(I_\infty, Ae^{-h})) \setminus B(O(h), h), \\ \text{Vois}''(A, a, h) &:= \text{Cone}_{O(h)}(I_\infty, Ae^{-h-a}) \setminus B(O(h), h+a), \end{aligned}$$

where $\text{Cone}_{z_0}(z, \theta)$ denotes the cone with apex z_0 , direction z and aperture θ . Let us also define:

$$\begin{aligned} \text{Cyl}(A) &:= B_{\mathbb{R}^d}(0, A) \times \left(0, \frac{3}{2}\right] \subset H, \\ \text{Cyl}'(A, a) &:= B_{\mathbb{R}^d}(0, A) \times \left[\frac{1}{2}e^{-a}, \frac{3}{2}\right] \subset H, \\ \text{Cyl}''(A, a) &:= B_{\mathbb{R}^d}(0, Ae^{-a}) \times \left(0, \frac{3}{2}e^{-a}\right] \subset H. \end{aligned}$$

The sets $\text{Vois}(A, h)$, $\text{Vois}'(A, a, h)$, $\text{Vois}''(A, a, h)$ and $\text{Cyl}(A)$, $\text{Cyl}'(A, a)$, $\text{Cyl}''(A, a)$ are represented in Figure 3.4. We will use the following geometrical fact:

Lemma 3.3.2. *For any $A, a \geq 0$, h can be chosen large enough such that*

$$\text{Vois}(A, h) \subset \text{Cyl}(A), \quad \text{Vois}'(A, a, h) \subset \text{Cyl}'(A, a), \quad \text{and} \quad \text{Vois}''(A, a, h) \subset \text{Cyl}''(A, a). \quad (3.3.2)$$

For $A, h \geq 0$, let us define the event $E(A, h)$ saying that every infinite backward path converging to I_∞ in $\text{RST}(h)$ restricted to the annulus $\mathbb{H}^{d+1} \setminus S(O(h), h)$ are contained in $\text{Vois}(A, h)$:

$$E(A, h) := \{\forall z \in \text{RST}(h) \cap (\mathbb{H}^{d+1} \setminus S(O(h), h)), I_\infty \in \overline{\mathcal{D}_{\text{RST}(h)}(z)} \implies z \in \text{Vois}(A, h)\},$$

where $\mathcal{D}_{\text{RST}(h)}(z)$ denotes the descendants subtree of z in $\text{RST}(h)$. Let us also define, for $A, h, a > 0$, the event $E'(A, h, a)$ saying that every infinite backward path converging to I_∞ in $\text{RST}(h)$ restricted to the annulus $\mathbb{H}^{d+1} \setminus S(O(h), h+a)$ are contained in $\text{Vois}''(A, a, h)$:

$$E'(A, a, h) := \{\forall z \in \text{RST}(h) \cap (\mathbb{H}^{d+1} \setminus S(O(h), h+a)), I_\infty \in \overline{\mathcal{D}_{\text{RST}(h)}(z)} \implies z \in \text{Vois}''(A, a, h)\}.$$

The following proposition asserts that, uniformly in h , the events $E(A, h)$ and $E(A, a, h)$ occur with high probability when A is large.

Proposition 3.3.3. *We have*

$$\lim_{A \rightarrow \infty} \liminf_{h \rightarrow \infty} \mathbb{P}[E(A, h)] = 1, \quad \lim_{A \rightarrow \infty} \liminf_{h \rightarrow \infty} \mathbb{P}[E'(A, a, h)] = 1 \text{ for any } a \geq 0.$$

Let us assume Propositions 3.3.1, 3.3.3 and Lemma 3.3.2 for the moment and let us prove part (iii) of Theorem 3.1.1. For $h \geq 0$, let us define the event

$$U(h) := \{\text{there is a unique infinite backward path converging to } I_\infty \text{ in } \text{RST}(h)\}$$

By isometries invariance, $\mathbb{P}[U(h)]$ is independent of h , and let us suppose for contradiction that $q := \mathbb{P}[U(h)^c] > 0$. For $A, a, h \geq 0$, let us define the event

$$\text{CO}(A, a, h) := \{\forall z \in \mathcal{N} \cap \text{Cyl}'(A, a), A_{\text{RST}(h)}(z) = A_{\text{DSF}}(z)\}$$

By Proposition 3.3.3, A can be chosen such that

$$\liminf_{h \rightarrow \infty} \mathbb{P}[E(A, h)] > 1 - q/4, \quad \mathbb{P}[E'(A, a, h)] > 1 - q/4.$$

Then, by Proposition 3.3.1 applied to the compact set $K := \text{Cyl}'(A, a)$ and Lemma 3.3.2, h can be chosen large enough such that inclusions (3.3.2) hold and such that

$$\mathbb{P}[\text{CO}(A, a, h)] > 1 - q/4, \quad \mathbb{P}[E(A, h)] \geq 1 - q/4 \text{ and } \mathbb{P}[E'(A, a, h)] \geq 1 - q/4.$$

Let us define the event $Z(a)$ as

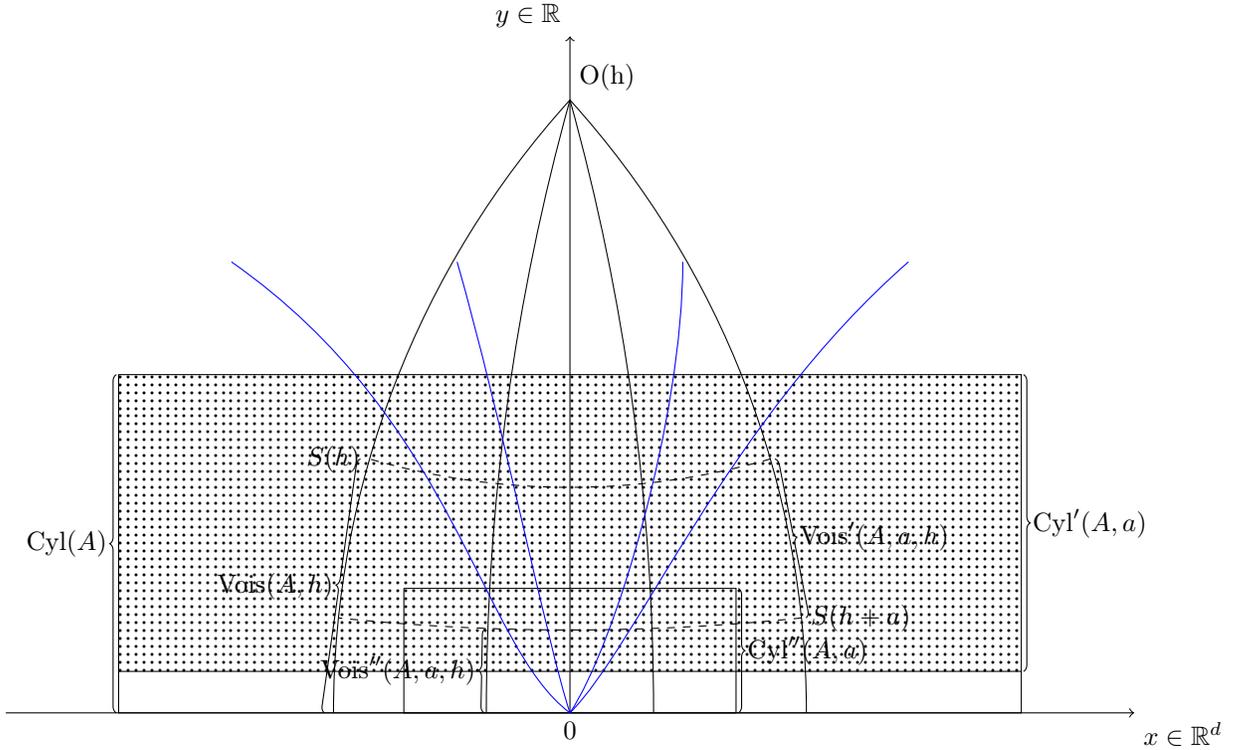
$$Z(a) := U^c \cap E(A, h) \cap E'(A, a, h) \cap \text{CO}(A, a, h),$$

and define

$$Z := \bigcap_{a_0 > 0} \bigcup_{a \geq a_0} Z(a).$$

For the choices of q, h done before, $\mathbb{P}[Z(a)] \geq q/4$ and so $\mathbb{P}[Z] \geq q/4 > 0$. On the event $Z(a)$, and because inclusion (3.3.2) holds, there exists two infinite backward paths in $\text{RST}(h)$ whose restrictions to $\mathbb{R}^d \times (0, 3/2]$ are contained in $\text{Cyl}(A, a)$ and converging to I_∞ , and intersecting $\text{Cyl}''(A, a)$. These two infinite backward paths coincide with those of DSF inside $\text{Cyl}'(A, a)$. Thus, in DSF, there exists two infinite backward paths contained in $\text{Cyl}(A, a)$ and intersecting $\text{Cyl}''(A, a)$. Therefore, since this is true for all $a > 0$, on the event Z , it is possible to construct two infinite backward paths converging to I_∞ in DSF using the fact that DSF is locally finite (it is true by [5, Proposition 9]). However, by [5, Theorem 3], there almost surely a unique infinite backward path converging to I_∞ in DSF. This leads to a contradiction, which achieves the proof.

Figure 3.4: Representation of the sets $\text{Vois}(A, h)$, $\text{Vois}'(A, a, h)$, $\text{Vois}''(A, a, h)$ and $\text{Cyl}(A)$, $\text{Cyl}'(A, a)$, $\text{Cyl}''(A, a)$. The backward paths of $\text{RST}(h)$ converging to 0 (in blue) are all contained in $\text{Vois}''(A, a, h)$ up to level $h + a$ and contained in $\text{Vois}(A, a, h)$ up to level h . In the dashed area ($\text{Cyl}'(A, a)$), the DSF and $\text{RST}(h)$ coincide.



Proof of Proposition 3.3.1. Let us define, for $z = (x, y) \in \mathcal{N}$ and $h \geq 0$:

$$B_{\text{DSF}}^+(z) := B(z, d(z, A(z))) \cap (\mathbb{R}^d \times (y, \infty)), \quad B_{\text{RST}(h)}^+(z) := B(z, d(z, A(z))) \cap (y, \infty).$$

Let $K \subset \mathbb{H}^{d+1}$ be some compact set. For any given $z \in \mathcal{N} \cap K$, $h \geq 0$, $A_{\text{DSF}}(z) = A_{\text{RST}(h)}(z)$ if and only if $\mathcal{N} \cap (B_{\text{DSF}}^+(z) \Delta B_{\text{RST}(h)}^+(z)) = \emptyset$. For any $z \in \mathbb{H}^{d+1}$, $\text{Vol}(B_{\text{DSF}}^+(z) \Delta B_{\text{RST}(h)}^+(z)) \rightarrow 0$ as

$h \rightarrow \infty$ (recall that Vol is the hyperbolic volume). Campbell formula [8] gives,

$$\begin{aligned} & \mathbb{E} \left[\#\{z' \in \mathcal{N} \cap B(z, d(z, A(z))), \mathcal{N} \cap (B_{\text{DSF}}^+(z) \Delta B_{\text{RST}(h)}^+(z')) \neq \emptyset\} \right] \\ &= \lambda \int_{B(z, d(z, A(z)))} \mathbb{P} \left[\mathcal{N} \cap (B_{\text{DSF}}^+(z) \Delta B_{\text{RST}(h)}^+(z')) \neq \emptyset \right] d\text{Vol}(z) \\ &= \lambda \int_{B(z, d(z, A(z)))} 1 - \exp(-\lambda \text{Vol}(B_{\text{DSF}}^+(z) \Delta B_{\text{RST}(h)}^+(z))) d\text{Vol}(z) \rightarrow 0 \text{ as } h \rightarrow \infty \end{aligned}$$

by dominated convergence. Proposition 3.3.1 follows. \square

Proof of Proposition 3.3.3. \square

For $n \in \mathbb{N}$ and $h \geq 0$, let us define the event

$$F_n(h) := \{\exists z \in B_{S(h)}(u, 2^{n+1}e^{-h}), \text{MBD}_h^\infty(z) > 2^n e^{-h}\}.$$

We now show that for $n \in \mathbb{N}$ and $h \geq 0$, $E(2^n, h)^c \subset \bigcup_{m \geq n} F_m(h)$. If $E(2^n, h)^c$ does not occur, then there exists some $z \in S(h) \setminus B_{S(h)}(u, 2^n e^{-h})$ such that $I_\infty \in \overline{\mathcal{D}_{\text{RST}(h)}(z)}$, so, for the value of $m \in \mathbb{N}_{\geq n}$ such that $2^m \leq z\widehat{0I_\infty} < 2^{m+1}$, $z \in B_{S(h)}(u, 2^{m+1}e^{-h})$ and $\text{MBD}_h^\infty(z) > 2^n e^{-h} \geq z\widehat{0I_\infty} \geq 2^m$, thus $F_m(h)$ occurs.

Therefore

$$\mathbb{P}[E(2^n, h)] \geq 1 - \sum_{m \geq n} \mathbb{P}[F_m(h)].$$

We now upper-bound $\mathbb{P}[F_n(h)]$. On $F_n(h)$, the following occurs:

$$\sum_{z \in B_{S(h)}(u, 2^{n+1}e^{-h}) \cap \text{RST}} \text{MBD}_h^\infty(z)^p > 2^{np} e^{-ph}, \quad (3.3.3)$$

thus, by Markov inequality,

$$\mathbb{P}[F_n(h)] \leq 2^{-np} e^{ph} \mathbb{E} \left[\sum_{z \in B_{S(h)}(u, 2^{n+1}e^{-h}) \cap \text{RST}} \text{MBD}_h^\infty(z)^p \right] \stackrel{\text{Prop.3.2.6}}{\leq} C_{\text{fl}} 2^{n(d-p)}$$

for some $C_{\text{fl}} > 0$ depending only on p . Combining this with (3.3.3) for some $p > d$ leads to

$$\mathbb{P}[E(2^n, h)] \geq 1 - C_{\text{fl}} \frac{2^{m(d-p)}}{1 - 2^{d-p}},$$

this proves the first part of Proposition 3.3.3. The second part be deduced from the first part by applying the dilation $(x, y) \mapsto (e^a x, e^a y)$ (which is an isometry of (H, ds^2)).

Proof of Lemma 3.3.2. Let $A, a, h \geq 0$, and let $z = (x, y) \in \text{Vois}(A, h)$. Considering the totally geodesic plane containing I_∞, z and $O(h)$ (represented by a half-plane in H), it is possible to suppose $d = 1$ without loss of generality. We apply the distance and angle formulas in (H, ds^2) (Propositions 3.7.1 and 3.7.2).

Let $z = (x, y) \in \text{Vois}(A, h)$. On the one hand, $z\widehat{O(h)I_\infty} \leq Ae^{-h}$, so, taking h large enough such that $Ae^{-h} < \pi/2$,

$$\arctan \left| \frac{2xe^h}{e^{2h} - x^2 - y^2} \right| \leq Ae^{-h},$$

thus, for h large enough,

$$|x|e^{-h} \leq \arctan |2xe^{-h}| \leq \arctan \left| \frac{2xe^h}{e^{2h} - x^2 - y^2} \right| \leq Ae^{-h},$$

so $|x| \leq A$.

On the other hand, $d(O(h), z) \geq h$, so

$$\begin{aligned} & 2 \tanh^{-1} \left(\sqrt{1 - \frac{4ye^h}{A^2 + (y + e^h)^2}} \right) \\ & \stackrel{|x| \leq A}{\geq} 2 \tanh^{-1} \left(\sqrt{1 - \frac{4ye^{-h}}{(xe^{-h})^2 + (ye^{-h} + 1)^2}} \right) = d(O(h), z) \geq h, \end{aligned}$$

so

$$e^h \leq \frac{1 + \sqrt{1 - \frac{4ye^h}{A^2 + (y + e^h)^2}}}{1 - \sqrt{1 - \frac{4ye^h}{A^2 + (y + e^h)^2}}} = \frac{2}{1 - \sqrt{1 - \frac{4ye^h}{A^2 + (y + e^h)^2}}} - 1 = \frac{e^h}{y} + o(e^h) \text{ when } h \rightarrow \infty,$$

for h large enough this implies $y \leq 3/2$. The two other inclusions are shown by similar computations. \square

3.4 Proof of Proposition 3.2.5

We use a bloc control argument similar to [5, Section 6.3]. Let $\delta > 0$ small and $A > 0$ large that will be chosen later. For $r_0 \geq 2$ and $z \in S(r_0)$, let us define

$$\begin{aligned} \Psi_1(r_0, z) &:= \text{Cone}(z, 3Ae^{-r_0}) \cap (B(r_0 + \delta) \setminus B(r_0)), \\ \Psi_2(r_0, z) &:= \text{Cone}(z, Ae^{-r_0}) \cap (B(r_0) \setminus B(r_0 - 1)). \end{aligned} \quad (3.4.1)$$

A point $z \in S(r_0)$ is said to be *good* if the following event $G(r_0, z)$ occurs:

$$G(r_0, z) := \{\mathcal{N} \cap \Psi_1(r_0, z) = \emptyset \text{ and } \mathcal{N} \cap \Psi_2(r_0, z) \neq \emptyset\}. \quad (3.4.2)$$

A good point is represented in Figure 3.5.

Let us define the random subsets $\hat{\chi}(r_0) \subset S(r_0)$ and $\chi(r_0) \subset \mathbb{H}^{d+1} \setminus \{0\}$ as

$$\hat{\chi}(r_0) := \bigcup_{\substack{z \in S(r_0), \\ G(r_0, z) \text{ occurs}}} \text{Cone}(z, Ae^{-r_0}), \quad \chi(r_0) := \bigcup_{z \in \hat{\chi}(r_0)}]0, z) \subset \mathbb{H}^{d+1} \setminus \{0\}. \quad (3.4.3)$$

The region $\chi(r_0)$ is the *controlled region*, where the cumulative forward deviations in the annulus $S(r_0 + \delta) \setminus S(r_0)$ will be upper-bounded. This control of fluctuations are given by the following lemma:

Lemma 3.4.1. *There exists some deterministic constant $C_{geom} > 0$ such that for any $r_0 \geq 2$, $s \in [r_0, r_0 + \delta]$, and $z \in \mathcal{L}_s \cap \chi(r_0)$, $CFD_{r_0}^s(z) \leq C_{geom}e^{-r_0}$.*

Lemma 3.4.1 will be proved in Section 3.4.4.

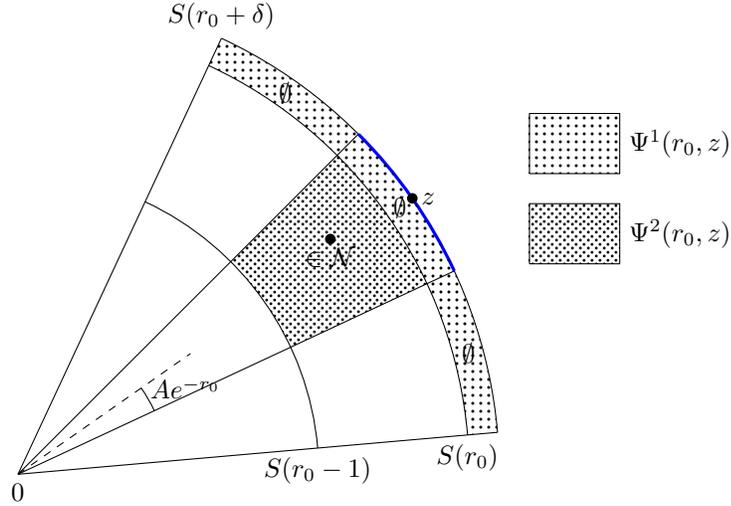


Figure 3.5: The point z is a *good point*; the fluctuations of trajectories crossing $B_{S(r_0)}(z, Ae^{-r_0})$ (in blue) are well controlled.

For any $r_0 \geq 2$, we cover the sphere $S(r_0)$ by balls of angular radius e^{-r_0} such that the number of balls overlapping at a given point never exceeds some constant K . To proceed, we use the following lemma:

Lemma 3.4.2. *There exists $K = K(d, p) \in \mathbb{N}^*$ such that, for any $r_0 \geq 2$, there exist a non-negative integer $N(r_0) \geq 0$ and a family of points $z_1, \dots, z_{N(r_0)} \in S(r_0)$ such that:*

- $\bigcup_{1 \leq i \leq N(r_0)} B_{S(r_0)}(z_i, e^{-r_0}) = S(r_0)$,
- $\forall z \in S(r_0), \#\{1 \leq i \leq N(r_0), z \in B_{S(r_0)}(z_i, e^{-r_0})\} \leq K$.

Moreover, there exists $C_{ball} = C_{ball}(K, d) > 0$ such that, for any $r_0 \geq 2$, $z \in S(r_0)$ and $A \geq 1$, the number of balls intersecting $B_{S(r_0)}(z, Ae^{-r_0})$ is upper-bounded by $C_{ball}A^d$:

$$\begin{aligned} \exists C_{ball} > 0, \forall r_0 \geq 2, \forall z \in S(r_0), \forall A \geq 1, \\ \#\{1 \leq i \leq N(r_0), B_{S(r_0)}(z_i, e^{-r_0}) \cap B_{S(r_0)}(z, Ae^{-r_0}) \neq \emptyset\} \leq C_{ball}A^d. \end{aligned}$$

We refer to Section 3.4.4 for the proof of Lemma 3.4.2.

For $1 \leq i \leq N(r_0)$, z_i is said to be *inhibited* if the ball $B_{S(r_0)}(z_i, e^{-r_0})$ intersects $S(r_0) \setminus \hat{\chi}$, and the corresponding event is denoted by $\text{Inhib}(i)$. Let $\Psi(r_0) \subset \llbracket 1, N(r_0) \rrbracket$ be the union of all inhibited balls:

$$\Psi(r_0) := \bigcup_{\substack{1 \leq i \leq N(r_0), \\ B_{S(r_0)}(z_i, e^{-r_0}) \cap (S(r_0) \setminus \hat{\chi}) \neq \emptyset}} B_{S(r_0)}(z_i, e^{-r_0}).$$

The region Ψ is the *augmented uncontrolled region*, that contains $(\mathbb{H}^{d+1} \setminus \{0\}) \setminus \chi$. For $z \in S(r_0)$, let $\hat{\text{Cl}}(z)$ be the cluster of z in $\Psi(r_0)$ and let us also define (recall that $]0, z)$ is the semi-geodesic starting at 0 and containing z , without 0):

$$\text{Cl}(z) = \bigcup_{z \in \hat{\text{Cl}}(z)}]0, z) \subset \mathbb{H}^{d+1} \setminus \{0\}.$$

We define the *angular radius* of $\text{Cl}(z)$ as $\text{Rad}(z) := \sup_{z' \in \text{Cl}(z)} \widehat{z'0z}$.

The next lemma asserts that the connected components of the augmented uncontrolled region Ψ are small (the radius admits exponential tail decay):

Lemma 3.4.3. *There exists $\delta > 0$ small enough, $A > 0$ large enough and some constant $c_{dec} > 0$ such that, for any B large enough, $r_0 \geq 2$ and $z \in S(r_0)$,*

$$\mathbb{P}[e^{r_0} \text{Rad}(z) > B] \leq e^{-c_{dec} B}.$$

Lemma 3.4.3 is proved in Section 3.4.4.

In addition, we need a control of the number of points in a given region of a sphere $S(r_0)$, which is given by the next lemma:

Lemma 3.4.4. *For any $p \geq 1$, there exists a constant $C = C(d, p) > 0$ such that, for any $r_0 \geq 0$ and any direction $z \in B_{S(r_0)}$,*

$$\mathbb{E}[\#(\mathcal{L}_{r_0} \cap B_{S(r_0)}(z, e^{-r_0}))^p] \leq C.$$

We refer to Section 3.4.4 for the proof of Lemma 3.4.4. Let us choose $A, \delta > 0$ as in Lemma 3.4.3 and C_{geom} as in Lemma 3.4.1.

3.4.1 Step 1: a deterministic upper-bound of $\text{MBD}_{r_0}^{r_0+\delta}(\cdot)$

For $z \in S(r_0)$, let us define

$$M(z) := \#\mathcal{N} \cap \text{Cl}(z) \cap (S(r_0 + \delta) \setminus S(r_0)). \quad (3.4.4)$$

This step is devoted to the proof of the following upper-bound: almost surely, for any $z \in S(r_0)$,

$$\text{MBD}_{r_0}^{r_0+\delta}(z)^p \leq 2^{p-1} (2^p \text{Rad}(z)^p (M(z) + 1)^p + C_{\text{geom}}^p e^{-pr_0}). \quad (3.4.5)$$

The quantity $\text{MBD}_{r_0}^{r_0+\delta}(z)$ takes into account backward paths from z that ends (in the backward direction) before level $r_0 + \delta$ and backward paths reaching level $r_0 + \delta$. For $z \in S(r_0)$, we define $\text{Stop}(z)$ as the set of ending points of backward paths from z stopping before level $r_0 + \delta$ (recall that $\mathcal{D}(z) := \bigcup_{r \geq r_0} \mathcal{D}_r^r(z)$):

$$\text{Stop}(z) := \{z' = (r'; u') \in \mathcal{N} \cap \mathcal{D}(z), r_0 \leq r' < r_0 + \delta, A^{(-1)}(z') = \emptyset\}.$$

For any $z' \in \mathcal{D}_{r_0}^{r_0+\delta}(z) \cup \text{Stop}(z)$, one of the two following cases occur. Either the branch from z' to z stays inside $\text{Cl}(z)$, or it crosses $\chi(r_0)$. Let us define \mathcal{C} (resp. \mathcal{C}') as the set of couples (z, z') such that the branch from z' to z crosses $\chi(r_0)$ (resp. does not cross $\chi(r_0)$):

$$\begin{aligned} \mathcal{C} &:= \{(z, z'), z \in S(r_0), z' \in \mathcal{D}_{r_0}^{r_0+\delta}(z) \cup \text{Stop}(z), \exists s \in [r_0, r_0 + \delta], \mathcal{A}_s^{r_0+\delta}(z') \in \chi\}, \\ \mathcal{C}' &:= \{(z, z'), z \in S(r_0), z' \in \mathcal{D}_{r_0}^{r_0+\delta}(z) \cup \text{Stop}(z), \forall s \in [r_0, r_0 + \delta], \mathcal{A}_s^{r_0+\delta}(z') \notin \chi\}. \end{aligned}$$

Moreover, for any $(z, z') \in \mathcal{C}$, we define $\text{hit}(z, z')$ as the highest level where the branch from z' to z hits $\chi(r_0)$:

$$\text{hit}(z, z') := \max\{s \in [r_0, r_0 + \delta], \mathcal{A}_s^{r_0+\delta}(z') \in \chi\},$$

where the max is over a set which is finite a.s.

Let $1 \leq i \leq N(r_0)$. Let $z \in B_{S(r_0)}(z_i, e^{-r_0})$, $z' = (r'; u') \in \mathcal{D}_{r_0}^{r_0+\delta}(z) \cup \text{Stop}(z)$, we now upper-bound $\text{CFD}_{r_0}^{r'}(z')$. Let us first consider the case where $(z, z') \in \mathcal{C}'$ (the branch between z and z'

does not cross $\chi(r_0)$). Then this branch stays inside $\text{Cl}(z)$. Thus it crosses at most $M(z)$ points of \mathcal{N} , therefore

$$\text{CFD}_{r_0}^{r'}(z') \leq 2\text{Rad}(z)(M(z) + 1).$$

In the other case, $(z, z') \in \mathcal{C}$, let $z'' = (r''; u'') := \text{hit}(z, z')$. Let $p \geq 1$. Then, by Jensen inequality,

$$\text{CFD}_{r_0}^{r'}(z')^p \leq \left(\text{CFD}_{r_0}^{r''}(z'') + \text{CFD}_{r''}^{r_0+\delta}(z') \right)^p \leq 2^{p-1} \left(\text{CFD}_{r_0}^{r''}(z'')^p + \text{CFD}_{r''}^{r_0+\delta}(z')^p \right)$$

By the same argument as in the previous case,

$$\text{CFD}_{r_0}^{r''}(z'') \leq 2\text{Rad}(z)(M(z) + 1),$$

and, by Lemma 3.4.1, $\text{CFD}_{r''}^{r_0+\delta}(z') \leq C_{\text{geom}} e^{-r_0}$, since, by definition of z'' , the part of trajectory between z'' and z is included in $\text{Cl}(z)$. Thus,

$$\text{CFD}_{r_0}^{r'}(z')^p \leq 2^{p-1} \left(2^p \text{Rad}(z)^p (M(z) + 1)^p + C_{\text{geom}}^p e^{-pr_0} \right). \quad (3.4.6)$$

The upper-bound (3.4.6) holds whatever (z, z') belongs to \mathcal{C} or \mathcal{C}' . It follows that (3.4.5) holds for any $z \in S(r_0)$.

3.4.2 Step 2: a control of the tail decay of $M(z)$

Recall that, for $z \in S(r_0)$, $M(z)$ is defined in (3.4.4). In this step, it is shown that, for any $z \in S(r_0)$:

$$\mathbb{E} [M(z)^{4p}] \leq C \quad (3.4.7)$$

for some $C = C(p) > 0$.

The quantity $M(z)$ is the number of Poisson points inside a random part of the thin annulus $S(r_0 + \delta) \setminus S(r_0)$, whose diameter admits exponential tail decay (Lemma 3.4.3).

Let $z \in S(r_0)$. For given $R \geq 0$, let us define

$$\text{Reg}(R) := \text{Cone}(z_i, Re^{-r_0}) \cap (S(r_0 + \delta) \setminus S(r_0)).$$

For any $R, m \geq 0$,

$$\{M(z) > m\} \subset \{\text{Rad}(z) > Re^{-r_0}\} \cup \{\#(\mathcal{N} \cap \text{Reg}(R)) > m\}$$

thus

$$\mathbb{P}[M(z) > m] \leq \mathbb{P}[\text{Rad}(z) > Re^{-r_0}] + \mathbb{P}[\#(\mathcal{N} \cap \text{Reg}(R)) > m]. \quad (3.4.8)$$

By Lemma 3.4.3, $\mathbb{P}[\text{Rad}(z) > Re^{-r_0}] \leq e^{-c_{\text{dec}} R}$. The random variable $\#(\mathcal{N} \cap \text{Reg}(R))$ is distributed according to the Poisson law with parameter $\lambda \text{Vol}(\text{Reg}(R))$. Recall that ν the d -dimensional volume measure on \mathbb{S}^d . Denoting by u the direction of z ,

$$\lambda \nu(\text{Reg}(R)) = \lambda \nu(\{u', \langle u, u' \rangle < Re^{-r_0}\}) \text{Vol}((S(r_0 + \delta) \setminus S(r_0))) \leq CR^d. \quad (3.4.9)$$

for some constant $C > 0$ independent of r_0, R , since $\text{Vol}(S(r_0 + \delta)) = O(e^{dr_0})$ by (3.2.6). Thus $\#(\mathcal{N} \cap \text{Reg}(R)) \leq_{st} \mathcal{P}(CR^d)$. Thus Chernoff bound for the Poisson distribution [25] leads to:

$$\mathbb{P}[\#(\mathcal{N} \cap \text{Reg}(R)) \geq m] \leq \frac{e^{-CR^d} (CeR^d)^m}{m^m} \quad (3.4.10)$$

for any $m \geq CR^d$. Let us chose $R = (m/(2eC))^{1/d}$ (thus $m = 2eCR^d$). It leads to:

$$\mathbb{P}[\#\mathcal{N} \cap \text{Reg}(R) \geq m] \leq \left(\frac{e^{-1/(2e)}}{2}\right)^m \leq \left(\frac{1}{2}\right)^m. \quad (3.4.11)$$

Finally, we combine (3.4.8), Lemma 3.4.3 and (3.4.11) to obtain:

$$\begin{aligned} \mathbb{P}[\#\mathcal{N} \cap \text{Cl}(z) \cap (S(r_0 + \delta) \setminus S(r_0)) > m] &\leq e^{-c_{\text{dec}}R} + \left(\frac{1}{2}\right)^m \\ &= \exp\left(-c_{\text{dec}}(m/2eC)^{1/d}\right) + \left(\frac{1}{2}\right)^m \leq \exp(-cm^{1/d}) \end{aligned} \quad (3.4.12)$$

for some $C > 0$. Therefore:

$$\begin{aligned} &\mathbb{E}\left[\#\mathcal{N} \cap \text{Cl}(z) \cap (S(r_0 + \delta) \setminus S(r_0))^{4p}\right] \\ &= \int_0^\infty \mathbb{P}\left[\#\mathcal{N} \cap \text{Cl}(z) \cap (S(r_0 + \delta) \setminus S(r_0)) > m^{1/(4p)}\right] dm \\ &\stackrel{(3.4.12)}{\leq} \int_0^\infty \exp(-cm^{1/(4dp)}) dm < \infty, \end{aligned}$$

which proves (3.4.7).

3.4.3 Step 3: conclusion

By (3.4.5), for any $1 \leq i \leq N(r_0)$,

$$\begin{aligned} \sum_{z \in B_{S(r_0)}(z_i, e^{-r_0}) \cap \text{RST}} \text{MBD}_{r_0}^{r_0+\delta}(z)^p &\leq 2^{p-1} \#\{\text{RST} \cap B_{S(r_0)}(z_i, e^{-r_0})\} \\ &\quad (2^p \text{Rad}(z)^p (M(z) + 1)^p + C_{\text{geom}}^p e^{-pr_0}). \end{aligned}$$

It follows by Cauchy-Schwartz and Minkowski that

$$\begin{aligned} &\mathbb{E}\left[\sum_{z \in B_{S(r_0)}(z_i, e^{-r_0}) \cap \text{RST}} \text{MBD}_{r_0}^{r_0+\delta}(z)^p\right] \\ &\leq C \mathbb{E}\left[\#\{\text{RST} \cap B_{S(r_0)}(z_i, e^{-r_0})\}^2\right]^{1/2} \mathbb{E}\left[(\text{Rad}(z_i)^p M(z)^p + e^{-pr_0})^2\right]^{1/2} \\ &\leq C \mathbb{E}\left[\#\{\text{RST} \cap B_{S(r_0)}(z_i, e^{-r_0})\}^2\right]^{1/2} \left(\mathbb{E}[\text{Rad}(z_i)^{2p} M(z)^{2p}]^{1/2} + e^{-pr_0}\right) \\ &\leq C \mathbb{E}\left[\#\{\text{RST} \cap B_{S(r_0)}(z_i, e^{-r_0})\}^2\right]^{1/2} \\ &\quad \times \left(\mathbb{E}[\text{Rad}(z_i)^{4p}]^{1/4} \mathbb{E}[M(z)^{4p}]^{1/4} + e^{-pr_0}\right) \end{aligned} \quad (3.4.13)$$

for some $C = C(p) > 0$. By Lemma 3.4.4 applied to $p = 2$,

$$\mathbb{E}\left[\#\{\text{RST} \cap B_{S(r_0)}(z_i, e^{-r_0})\}^2\right] \leq C, \quad (3.4.14)$$

for C independent of r_0, z_i . By Lemma 3.4.3,

$$\begin{aligned} \mathbb{E}[\text{Rad}(z_i)^{4p}] &= e^{-4pr_0} \int_0^\infty \mathbb{P}[e^{4pr_0} \text{Rad}(z_i)^{4p} > B] dB \\ &= e^{-pr_0} \int_0^\infty \mathbb{P}[e^{4r_0} \text{Rad}(z_i) > B^{\frac{1}{4p}}] dB \\ &\leq e^{-4pr_0} \int_0^\infty e^{-c_{\text{dec}} B^{\frac{1}{4p}}} dB < \infty. \end{aligned} \quad (3.4.15)$$

Then, by combining (3.4.13), (3.4.14), (3.4.15) and (3.4.7),

$$\mathbb{E} \left[\sum_{z \in B_{S(r_0)}(z_i, e^{-r_0}) \cap \text{RST}} \text{MBD}_{r_0}^{r_0 + \delta}(z)^p \right] \leq C e^{-pr_0}, \quad (3.4.16)$$

for some $C = C(p) > 0$.

The final step is to sum over all i such that $B_{S(r_0)}(z_i, e^{-r_0})$ intersects $B_{S(r_0)}(u, \theta)$ for any given $u \in \mathbb{S}^d$ and $\theta > 0$. Let $\theta > 0$. By Lemma 3.4.2 it can be assumed that $B_{S(r_0)}(z_i, e^{-r_0})$ intersects $B_{S(r_0)}(z_i, \theta)$ for at most $C_{\text{ball}} e^{dr_0} \theta^d$ values of $i \in \llbracket 1, N(r_0) \rrbracket$. Therefore

$$\begin{aligned} \mathbb{E} \left[\sum_{z \in B_{S(r_0)}(u, \theta) \cap \text{RST}} \text{MBD}_{r_0}^{r_0 + \delta}(z)^p \right] &= \sum_{\substack{1 \leq i \leq N(r_0), \\ B_{S(r_0)}(z_i, e^{-r_0}) \\ \cap B_{S(r_0)}(u, \theta) \neq \emptyset}} \mathbb{E} \left[\sum_{z \in B_{S(r_0)}(z_i, e^{-r_0}) \cap \text{RST}} \text{MBD}_{r_0}^{r_0 + \delta}(z)^p \right] \\ &\stackrel{(3.4.16)}{\leq} C e^{-pr_0} \#\{1 \leq i \leq N(r_0), B_{S(r_0)}(z_i, e^{-r_0}) \cap B_{S(r_0)}(u, \theta) \neq \emptyset\} \\ &\leq C e^{(d-p)r_0} \theta^d, \end{aligned}$$

which achieves the proof of Proposition 3.2.5.

3.4.4 Proof of Lemmas 3.4.1, 3.4.2, 3.4.3, 3.4.4

Proof of Lemma 3.4.1. Let $r_0 \leq s \leq r_0 + \delta$ and $z_1 \in \chi(r_0) \cap \mathcal{L}(s)$. Since $z_1 \in \chi(r_0)$, $G(z_1)$ occurs. Let $z_0 = (r_0; u_0) := z_{1\downarrow}$, $z_2 = (r_2, u_2) := \mathcal{A}_s^{r_0}(z_1)$, $z_4 = (r_3, u_3) := z_{1\uparrow}$ and $z_3 \in [z_0, z_4]^*$ is such that $\widehat{z_1 0 z_4} = 3Ae^{-r_0}$. Since $G(z_1)$ occurs, there exists some $z' = (r'; u') \in \Psi_2(r_0, z_1)$. Let us suppose that $\widehat{z_1 0 z_2} \geq 3Ae^{-r_0}$. It will be shown that

$$d(z_0, z') \leq d(z_0, z_3). \quad (3.4.17)$$

It implies by construction of $[z_0, z_4]^*$ that $d(z_0, z') \leq d(z_0, z_4)$, which contradicts the fact that $z_4 = A(z_0)$. It follows that $\widehat{z_1 z' z_2} \leq 3Ae^{-r_0}$, and since $\Psi_1(r_0, z_1) \cap \mathcal{N} = \emptyset$, it implies that $\text{CFD}_s^{r_0}(z_1) \leq 3Ae^{-r_0}$, so Lemma 3.4.1 holds with $C_{\text{geom}} = 3A$.

We move on to show (3.4.17). By the hyperbolic law of cosines (3.2.7),

$$\begin{aligned} \cosh d(z_0, z') &= \cosh(r_0) \cosh(r') - \cos(\widehat{z_0 0 z'}) \sinh(r_0) \sinh(r') \\ &= 1 + (1 - \cos(\widehat{z_0 0 z'})) \sinh(r_0) \sinh(r') \\ &\leq 1 + \left(1 - \cos\left(\widehat{z_0 0 z_1} + Ae^{-r_0}\right)\right) \sinh(r_0) \sinh(r') \\ &\leq 1 + \left(1 - \cos\left(\widehat{z_0 0 z_1} + Ae^{-r_0}\right)\right) \sinh(r_0)^2 \end{aligned}$$

and

$$\begin{aligned} \cosh d(z_0, z_3) &= \cosh(r_0) \cosh(r_3) - \cos(\widehat{z_0 0 z_3}) \sinh(r_0) \sinh(r_3) \\ &= 1 + (1 - \cos(\widehat{z_0 0 z_3})) \sinh(r_0) \sinh(r_3) \\ &= 1 + \left(1 - \cos\left(\widehat{z_0 0 z_1} + \widehat{z_1 0 z_3}\right)\right) \sinh(r_0) \sinh(r_3) \\ &\geq 1 + \left(1 - \cos\left(\widehat{z_0 0 z_1} + 3Ae^{-r_0}\right)\right) \sinh(r_0) \sinh(r_3) \\ &\geq 1 + \left(1 - \cos\left(\widehat{z_0 0 z_1} + Ae^{-r_0}\right)\right) \sinh(r_0)^2. \end{aligned}$$

This proves (3.4.17), which achieves the proof of Lemma 3.4.1. \square

Proof of Lemma 3.4.2. Proving the first part of Lemma 3.4.2 is equivalent to show that there exists some $K \in \mathbb{N}$ such that, for any $\varepsilon > 0$, the Euclidean unit sphere \mathbb{S}^d can be covered by balls of radius ε such that the number of balls overlapping some given point $x \in \mathbb{S}^d$ is bounded by K , which is a standard fact.

We move on to show the second part, i.e. the existence of $C_{\text{ball}} > 0$ such that, for any $r_0 \geq 2$, $z \in S(r_0)$ and $A \geq 1$, the number of balls intersecting $B_{S(r_0)}(z, Ae^{-r_0})$ is upper-bounded by $C_{\text{ball}}A^d$. Let $u \in \mathbb{S}^d$ be the direction of z and let $A \geq 1$. For $i \in \llbracket 1, N(r_0) \rrbracket$, the ball $B_{S(r_0)}(z_i, e^{-r_0})$ intersects $B_{S(r_0)}(z, Ae^{-r_0})$ if and only if $\widehat{z_i 0 z} \leq (A+1)e^{-r_0}$. Thus

$$\left(\bigcup_{\substack{1 \leq i \leq N(r_0), \\ B_{S(r_0)}(z_i, e^{-r_0}) \cap \\ B_{S(r_0)}(z, Ae^{-r_0}) \neq \emptyset}} B_{S(r_0)}(z_i, e^{-r_0}) \right) \subset B_{S(r_0)}(z, (A+2)e^{-r_0}).$$

Recall that ν denotes the d -dimensional volume measure on \mathbb{S}^d . There exists $C > 0$ such that, for any $r_0 \geq 2$,

$$\nu\{u, \widehat{uu_0} \leq e^{-r_0}\} \geq Ce^{-r_0 d}.$$

Moreover,

$$\nu\{u, \widehat{uu_0} \leq (A+2)e^{-r_0}\} \leq ((A+2)e^{-r_0})^d,$$

thus the number of balls intersecting $B_{S(r_0)}(z, Ae^{-r_0})$ is upper-bounded by:

$$K \frac{\nu\{u, \widehat{uu_0} \leq (A+2)e^{-r_0}\}}{\nu\{u, \widehat{uu_0} \leq e^{-r_0}\}} \leq \frac{K((A+2)e^{-r_0})^d}{ce^{-r_0 d}} \stackrel{A \geq 1}{\leq} \frac{3^d K}{c} A^d,$$

the conclusion follows. \square

Proof of Lemma 3.4.3. Recall that, for $1 \leq i \leq N(r_0)$, $B_{S(r_0)}(z_i, e^{-r_0})$ is said to be inhibited if $B_{S(r_0)}(z_i, e^{-r_0}) \cap (S(r_0) \setminus \chi) \neq \emptyset$. Let us first estimate the probability that a given z_i is inhibited.

Let $1 \leq i \leq N(r_0)$. Let us consider the following events:

$$\begin{aligned} E(i) &:= \{\mathcal{N} \cap \text{Cone}(z_i, (3A+1)e^{-r_0}) \cap (B(r_0+\delta) \setminus B(r_0)) = \emptyset\}, \\ E'(i) &:= \{\mathcal{N} \cap \text{Cone}(z_i, (A-1)e^{-r_0}) \cap (B(r_0) \setminus B(r_0-1)) \neq \emptyset\}. \end{aligned}$$

We now show that $E(i) \cap E'(i) \subset \text{Inhib}(i)^c$. Let $z \in B_{S(r_0)}(z_i, e^{-r_0})$. By triangular inequality,

$$\begin{aligned} \Psi_1(r_0, z) &\subset \text{Cone}(z_i, (3A+1)e^{-r_0}) \cap (B(r_0+\delta) \setminus B(r_0)) \quad \text{and} \\ \text{Cone}(z_i, (A-1)e^{-r_0}) \cap (B(r_0+\delta) \setminus B(r_0)) &\subset \Psi_2(r_0, z). \end{aligned}$$

Therefore, on the event $E(i) \cap E'(i)$, z is good (i.e. $G(r_0, z)$ occurs). Thus, $\text{Inhib}(i)^c$ occurs, which shows that $E(i) \cap E'(i) \subset \text{Inhib}(i)^c$. It follows that $\mathbb{P}[\text{Inhib}(i)] \leq \mathbb{P}[E(i)^c] + \mathbb{P}[E'(i)^c]$. Since

$$\begin{aligned} &\text{Vol}(\text{Cone}(z_i, (3A+1)e^{-r_0}) \cap (B(r_0+\delta) \setminus B(r_0))) \\ &= \int_{r_0}^{r_0+\delta} \nu\{u', \langle u, u' \rangle \leq (3A+1)e^{-r_0}\} \sinh(r)^d dr \leq C\delta A^d \end{aligned}$$

for some $C > 0$ independent of A, r_0, δ . Thus

$$\mathbb{P}[E(i)^c] \leq 1 - e^{-\lambda c \delta A^d}$$

An analogous computation for $E'(i)$ leads to:

$$\mathbb{P}[E'(i)^c] \leq e^{-\lambda c A^d}$$

for some $c > 0$ independent of A, r_0, δ .

We move on to show that A and δ can be chosen such that $\mathbb{P}[\text{Inhib}(i)] \leq 2C_{\text{ball}}^{-1}(6A+4)^{-d}/3$ for any $1 \leq i \leq N(r_0)$. Indeed, let us first chose A such that $e^{-\lambda c A^d} C_{\text{ball}}(6A+4)^d \leq 1/3$. Then let us chose δ such that $1 - e^{-\lambda c \delta A^d} \leq C_{\text{ball}}^{-1}(6A+4)^{-d}/3$. Hence

$$\mathbb{P}[\text{Inhib}(z_i)] \leq \mathbb{P}[E(i)^c] + \mathbb{P}[E'(i)^c] \leq 1 - e^{-\lambda c \delta A^d} + e^{-\lambda c A^d} \leq \frac{2}{3} C_{\text{ball}}^{-1}(6A+4)^{-d}.$$

We finally show that, for this choice of A, δ , there exists $c_{\text{dec}} > 0$ such that for any B large enough, $r_0 \geq 2$ and $z \in S(r_0)$,

$$\mathbb{P}[e^{r_0} \text{Rad}(z) > B] \leq e^{-c_{\text{dec}} B}.$$

Fix $r_0 \geq 2$ and $z \in S(r_0)$. For given k , let us denote by $P(k)$ the set of sequences z_{i_0}, \dots, z_{i_k} among the $\{z_i, 1 \leq i \leq N(r_0)\}$ such that:

1. $d(z_{i_0}, z) \leq e^{-r_0}$,
2. for any $0 \leq j \leq k-1$, $d(z_{i_j}, z_{i_{j+1}}) \leq (6A+4)e^{-r_0}$,
3. for any $0 \leq j, j' \leq k$, $d(z_{i_j}, z_{i_{j'}}) \geq (6A+2)e^{-r_0}$.

Let us also denote by $\hat{P}(k) \subset P(k)$ the set of sequences z_{i_0}, \dots, z_{i_k} verifying 1., 2., 3. and such that:

4. z_{i_0}, \dots, z_{i_k} are inhibited.

It can be noticed that $\{\text{Rad}(z) \geq k(6A+4)\} \subset \{\hat{P}(k) \neq \emptyset\}$ for any $k \in \mathbb{N}$, thus it is enough to upper-bound $\{\hat{P}(k) \neq \emptyset\}$ for any $k \in \mathbb{N}$. Let $(z_{i_0}, \dots, z_{i_k}) \in P(k)$. For $0 \leq j \leq k$, the event $\text{Inhib}(z_{i_j})$ only depends on the Poisson process \mathcal{N} inside $\text{Cone}(z_{i_j}, (3A+1)e^{-r_0})$, therefore, by 3. the events $\text{Inhib}(z_{i_j})$ are mutually independent. Thus

$$\mathbb{P}[(z_{i_0}, \dots, z_{i_k}) \in \hat{P}(k)] = \mathbb{P}[\text{Inhib}(z_{i_0})]^{k+1} \leq \left(\frac{2}{3}\right)^{k+1} C_{\text{ball}}^{-(k+1)} (6A+4)^{-d(k+1)}.$$

By Lemma 3.4.2, for any $1 \leq i \leq N(r_0)$, the number of balls intersecting $B_{S(r_0)}(z, \theta)$ is upper-bounded by $C_{\text{ball}} e^{dr_0} \theta^d$. Thus

$$\#P(k) \leq C_{\text{ball}}^{k+1} (6A+4)^{d(k+1)}.$$

It follows that

$$\begin{aligned} \mathbb{P}[\text{Rad}(z) \geq k(6A+4)] &\leq \mathbb{P}[\hat{P}(k) \neq \emptyset] \leq \mathbb{E}[\#\hat{P}(k)] \\ &\leq C_{\text{ball}}^{k+1} \left(\frac{2}{3}\right)^{k+1} (6A+4)^{d(k+1)} C_{\text{ball}}^{-(k+1)} (6A+4)^{-d(k+1)} \leq \left(\frac{2}{3}\right)^{k+1}. \end{aligned}$$

Lemma 3.4.3 follows. \square

Proof of Lemma 3.4.4. Let $r_0, M > 0$ and $z \in S(r_0)$. Let $h \geq 0$ that will be fixed later. We divide the set $L = \{z' \in \mathcal{N}, [z', A(z')]^* \in B_{S(r_0)}(z, e^{-r_0})\}$ into two subsets $L_{\leq h}$ and $L_{> h}$ according to the length of $[z', A(z')]$:

$$L_{\leq h} := \{z' \in L, d(z', z) \leq h\}, \quad L_{> h} := \{z' \in L, d(z', z) > h\}. \quad (3.4.18)$$

Thus $L = L_{\leq h} \cup L_{> h}$, and

$$\mathbb{P}[\#L \geq M] \leq \mathbb{P}[\#L_{\leq h} \geq M] + \mathbb{P}[L_{> h} \neq \emptyset]. \quad (3.4.19)$$

We first upper-bound $\mathbb{P}[\#L_{\leq h} \geq M]$. Since $L_{\leq h} \subset B(z, h)$,

$$\mathbb{P}[\#L_{\leq h} \geq M] \leq \mathbb{P}[\#(\mathcal{N} \cap B(z, h)) \geq M].$$

By (3.2.6), $\text{Vol}(B(z, h)) \leq Ce^{dh}$, for some $C > 0$ independent of r_0 . So the random variable $\#(\mathcal{N} \cap B(z, h))$ is stochastically dominated by a Poisson law with parameter $C\lambda e^{dh}$, thus, by the Chernoff bound for the Poisson distribution [25],

$$\mathbb{P}[\#L_{\leq h} \geq M] \leq \frac{e^{-C\lambda e^{dh}} (C\lambda e^{dh})^M}{M^M}. \quad (3.4.20)$$

The second step is to upper-bound $\mathbb{P}[\#L_{>h} \neq \emptyset]$. Recall that, for $r \geq 0$,

$$B^+(z, r) := B(z, r) \cap B(0, d(0, z)).$$

For any $z' \in L_{>h}$, by triangular inequality, denoting by z^* the meeting point of $[z', A(z')]^*$ and $S(r_0)$, $d(z', A(z')) \geq d(z', z^*) \geq d(z', z) - d(z^*, z)$. The hyperbolic law of cosines (3.2.7) gives,

$$\begin{aligned} d(z^*, z) &= \text{arcosh}(\cosh(r_0)^2 - \cos(\widehat{z^*0z}) \sinh(r_0)^2) \\ &\leq \text{arcosh}(\cosh(r_0)^2 - \cos(e^{-r_0}) \sinh(r_0)^2) \\ &= \text{arcosh}(1 + (1 - \cos(e^{-r_0})) \sinh(r_0)^2) \\ &\leq C_{\text{dis}} \end{aligned}$$

for some $C_{\text{dis}} > 0$ independent of r_0 . Thus $d(z', A(z')) \geq d(z', z) - C_{\text{dis}}$, so $B^+(z, (d(z', z) - C_{\text{dis}})_+)$ is empty of points. Therefore, by Campbell formula [8],

$$\begin{aligned} \mathbb{P}[L_{>h} \neq \emptyset] &\leq \mathbb{E}[\#L_{>h}] \\ &= \lambda \int_{B(r_0) \setminus B(h)} \mathbb{P}[z' \in L_{>h}] dz' \\ &\leq \lambda \int_{B(r_0) \setminus B(h)} \mathbb{P}[B^+(z, (d(z', z) - C_{\text{dis}})_+) \cap \mathcal{N} = \emptyset] dz' \\ &= \lambda \int_{B(r_0) \setminus B(h)} \mathbb{P}[\exp(-\lambda \text{Vol}(B^+(z, (d(z', z) - C_{\text{dis}})_+)))] dz' \\ &\stackrel{(3.6.1)}{\leq} \lambda \int_{B(r_0) \setminus B(h)} \mathbb{P}[\exp(-\lambda ce^{-d(d(z', z) - C_{\text{dis}})/2})] dz' \\ &\stackrel{(3.2.5)}{=} \lambda \mathcal{S}(d) \int_h^{r_0} \exp(-\lambda ce^{-rd/2}) \sinh(r)^d dr, \end{aligned}$$

where $\mathcal{S}(d)$ denotes the surface area of the Euclidean unit ball \mathbb{S}^d .

When $r \rightarrow \infty$,

$$\exp(-\lambda ce^{-rd/2}) \sinh(r)^d = o\left(e^{-rd/3} e^{-rd/3}\right)$$

thus

$$\lambda \mathcal{S}(d) \int_h^\infty \exp(-\lambda ce^{-rd/2}) \sinh(r)^d dr = o\left(e^{-e^{-rd/3}}\right),$$

thus, for h large enough, and for any $r_0 \geq 0$,

$$\mathbb{P}[L_{>h} \neq \emptyset] \leq e^{-e^{-hd/3}}. \quad (3.4.21)$$

Finally, combining (3.4.19), (3.4.20) and (3.4.21) with $h = -\ln(M/(2C\lambda))/d$ leads to Lemma 3.4.4. \square

3.5 Proof of Propositions 3.2.6 and 3.2.7

We first prove Proposition 3.2.6.

Step 1: Let us fix $p > 3d/2$. For any $r_0 > 0$, $n \in \mathbb{N}$, let us define

$$S_n(A, r_0) := \sum_{z \in B_{S(r_0)}(u, Ae^{-r_0}) \cap \text{RST}} \text{MBD}_{r_0}^{r_0+n\delta}(z)^{2p}.$$

The strategy of the proof is to construct a family of non-negative random variables $(Y_n^M(A, r_0))_{r_0, A, M \geq 0, n \in \mathbb{N}}$ and $(Y^M(A, r_0))_{r_0, A, M \geq 0}$ such that

- (1) almost surely, $Y_n^M(A, r_0) \uparrow Y^M(A, r_0)$ when $n \rightarrow \infty$ for any $M, A, r_0 \geq 0$;
- (2) $\sup_{A, r_0} \mathbb{P}[Y^M(A, r_0) \geq M] = O(M^{-2/3})$ when $M \rightarrow \infty$;
- (3) the following implication holds almost surely:

$$S_n(A, r_0) \leq (M \wedge Y_n^M(A, r_0))A^d e^{-2r_0 p} \implies S_{n+1}(A, r_0) \leq Y_{n+1}^M(A, r_0)A^d e^{-2r_0 p}$$

Let us suppose for the moment that such random variables $Y_n^M(A, r_0)$ and $Y^M(A, r_0)$ exist. Let $A, r_0 \geq 0$ and $M \geq 0$. On the event $\{Y^M(A, r_0) \leq M\}$, it can be shown by induction that $S_n(A, r_0) \leq MA^d e^{-2r_0 p}$ for any $n \geq 0$. Indeed, $S_0 = 0$, and if $S_n(A, r_0) \leq Y_n^M(A, r_0)A^d e^{-2r_0 p}$, then, using that

$$M \wedge Y_n^M(A, r_0) \stackrel{(1)}{\leq} M \wedge Y^M(A, r_0) \leq Y^M(A, r_0) \text{ since we are on the event } \{Y^M(A, r_0) \leq M\}.$$

So $S_{n+1}(A, r_0) \leq Y_{n+1}^M(A, r_0)A^d e^{-2r_0 p}$ by (3), which achieves the induction.

Thus, for any $A, r_0, M \geq 0$,

$$\mathbb{P}[S_n(A, r_0) \geq MA^d e^{-2r_0 p}] \leq \mathbb{P}[Y^M(A, r_0) \geq M] \leq CM^{-2/3} \text{ by (2)}. \quad (3.5.1)$$

for M large enough and some constant $C > 0$ independent of A, r_0, M . It follows that

$$\begin{aligned} C' &:= \sup_{A, r_0} \mathbb{E} \left[S_n(A, r_0)^{1/2} A^{-d/2} e^{r_0 p} \right] = \sup_{A, r_0} \int_0^\infty \mathbb{P}[S_n(A, r_0)^{1/2} A^{-d/2} e^{r_0 p} \geq M] dM \\ &= \sup_{A, r_0} \int_0^\infty \mathbb{P} [S_n(A, r_0) \geq M^2 A^d e^{-2r_0 p}] \stackrel{(3.5.1)}{\leq} \int_0^\infty CM^{-4/3} dM < \infty. \end{aligned} \quad (3.5.2)$$

Let $K := \#(\mathcal{N} \cap B_{S(r_0)}(u, Ae^{-r_0}))$. Let us apply Cauchy-Schwartz with the inner product defined by $\langle X, Y \rangle = \mathbb{E}[\sum_i X_i Y_i]$,

$$\begin{aligned} &\mathbb{E} \left[\sum_{z \in B_{S(r_0)}(u, Ae^{-r_0}) \cap \text{RST}} \left(\text{MBD}_{r_0}^{r_0+n\delta}(z) \right)^p \right] \\ &\leq \mathbb{E} \left[\sum_{z \in B_{S(r_0)}(u, Ae^{-r_0}) \cap \text{RST}} \left(\text{MBD}_{r_0}^{r_0+n\delta}(z) \right)^{2p} \right]^{1/2} \mathbb{E} [\#B_{S(r_0)}(u, Ae^{-r_0})]^{1/2} \\ &= \mathbb{E} [S_n(A, r_0)^{1/2}] \mathbb{E} [\#B_{S(r_0)}(u, Ae^{-r_0})]^{1/2} \\ &\stackrel{(3.5.2)}{\leq} C' A^{d/2} e^{-r_0 p} \mathbb{E} [\#B_{S(r_0)}(u, Ae^{-r_0}) \cap \text{RST}]^{1/2}. \end{aligned} \quad (3.5.3)$$

Let us show that $\mathbb{E} [\#B_{S(r_0)}(u, Ae^{-r_0}) \cap \text{RST}] \leq CA^d$ for some $C > 0$ independent of A, r_0 . We use the covering of $S(r_0)$ by balls of radius e^{-r_0} introduced by Lemma 3.4.2 in Section 3.4. For any $1 \leq i \leq N(r_0)$, by Proposition 3.4.4 applied with $p = 1$, $\mathbb{E}[\#\text{RST} \cap B_{S(r_0)}(z_i, 1)] \leq C$ for C independent of r_0, z_i . By Lemma 3.4.2, the number of balls intersecting $B_{S(r_0)}(z_i, 1)$ is bound by $C_{\text{ball}}A^d$. It follows that $\mathbb{E} [\#B_{S(r_0)}(u, Ae^{-r_0}) \cap \text{RST}] \leq CA^d$.

Thus, by (3.5.3),

$$\mathbb{E} \left[\sum_{z \in B_{S(r_0)}(u, Ae^{-r_0}) \cap \text{RST}} \left(\text{MBD}_{r_0}^{r_0+n\delta}(z) \right)^p \right] \leq Ce^{-r_0p}. \quad (3.5.4)$$

Since $r \mapsto \text{MBD}_{r_0}^r(z)$ is non-decreasing for any $z \in S(r)$,

$$\sum_{z \in B_{S(r_0)}(u, Ae^{-r_0}) \cap \text{RST}} \left(\text{MBD}_{r_0}^\infty(z) \right)^p = \lim_{n \rightarrow \infty} \uparrow \sum_{z \in B_{S(r_0)}(u, Ae^{-r_0}) \cap \text{RST}} \left(\text{MBD}_{r_0}^{r_0+n\delta}(z) \right)^p. \quad (3.5.5)$$

Proposition 3.2.6 follows by (3.5.4) and by monotone convergence theorem.

Step 2: we build the random variables $Y_n^M(A, r_0)$ and $Y^M(A, r_0)$.

Let $A, r_0 \leq 0$, let $n \in \mathbb{N}$. The strategy is to upper-bound S_{n+1} in function of S_n . Fix $z \in B_{S(r_0)}(u, Ae^{-r_0})$. The quantity $\text{MBD}_{r_0}^{r_0+n\delta}$ takes into account finite backwards paths that stop before level $r_0 + n\delta$ and those (potentially infinite) that continue after level $r_0 + n\delta$. Let us define the random set $\text{Stop}(z)$ as the set of ending points (in the backward direction) of finite paths from z stopping before level $r_0 + n\delta$:

$$\text{Stop}(z) := \{z' = (r'; u') \in \mathcal{N} \cap \mathcal{D}(z), r_0 \leq r' \leq r_0 + n\delta, A^{-1}(z') = \emptyset\} \subset \mathcal{N}.$$

By definition of $\text{MBD}_{r_0}^{r_0+n\delta}(z)$ (resp. $\text{MBD}_{r_0}^{r_0+(n+1)\delta}(z)$) (Definition 3.2.4),

$$\text{MBD}_{r_0}^{r_0+n\delta}(z) = \max_{z'=(r';u') \in \text{Stop}(z)} \text{CFD}_{r_0}^{r'}(z') \vee \max_{z' \in \mathcal{D}_{r_0}^{r_0+n\delta}(z)} \text{CFD}_{r_0}^{r_0+n\delta}(z') \quad (3.5.6)$$

and

$$\begin{aligned} & \text{MBD}_{r_0}^{r_0+(n+1)\delta}(z) \\ &= \max_{z'=(r';u') \in \text{Stop}(z)} \text{CFD}_{r_0}^{r'}(z') \vee \max_{z' \in \mathcal{D}_{r_0}^{r_0+n\delta}(z)} \left(\text{CFD}_{r_0}^{r_0+n\delta}(z') + \text{MBD}_{r_0+n\delta}^{r_0+(n+1)\delta}(z') \right). \end{aligned} \quad (3.5.7)$$

For any $p \geq 1$, $a, b \geq 0$ and $t \in [0, 1]$, Jensen inequality gives,

$$(a+b)^p = \left(t \frac{a}{t} + (1-t) \frac{b}{1-t} \right)^p \leq t \left(\frac{a}{t} \right)^p + (1-t) \left(\frac{b}{1-t} \right)^p = t^{1-p} a^p + (1-t)^{1-p} b^p. \quad (3.5.8)$$

Applying (3.5.8) with $t = 1/n^2$ leads to:

$$\begin{aligned}
& \text{MBD}_{r_0}^{r_0+(n+1)\delta}(z)^{2p} \\
& \stackrel{(3.5.7)}{=} \max_{z'=(r';u') \in \text{Stop}(z)} \text{CFD}_{r_0}^{r'}(z')^{2p} \vee \max_{z' \in \mathcal{D}_{r_0}^{r_0+n\delta}(z)} \left(\text{CFD}_{r_0}^{r_0+n\delta}(z') + \text{MBD}_{r_0+n\delta}^{r_0+(n+1)\delta}(z') \right)^{2p} \\
& \leq \max_{z'=(r';u') \in \text{Stop}(z)} \text{CFD}_{r_0}^{r'}(z')^{2p} \vee \\
& \quad \max_{z' \in \mathcal{D}_{r_0}^{r_0+n\delta}(z)} \left[\left(1 - \frac{1}{n^2}\right)^{1-2p} \left(\text{CFD}_{r_0}^{r_0+n\delta}(z')^{2p} \right) + n^{4p-2} \left(\text{MBD}_{r_0+n\delta}^{r_0+(n+1)\delta}(z')^{2p} \right) \right] \\
& \leq \left(1 - \frac{1}{n^2}\right)^{1-2p} \left[\max_{z'=(r';u') \in \text{Stop}(z)} \left(\text{CFD}_{r_0}^{r'}(z')^{2p} \right) \vee \max_{z' \in \mathcal{D}_{r_0}^{r_0+n\delta}(z)} \left(\text{CFD}_{r_0}^{r_0+n\delta}(z')^{2p} \right) \right] \\
& \quad + n^{4p-2} \max_{z' \in \mathcal{D}_{r_0}^{r_0+n\delta}(z)} \left[\text{MBD}_{r_0+n\delta}^{r_0+(n+1)\delta}(z')^{2p} \right] \\
& \stackrel{(3.5.6)}{=} \left(1 - \frac{1}{n^2}\right)^{1-2p} \text{MBD}_{r_0}^{r_0+n\delta}(z)^{2p} + n^{4p-2} \max_{z' \in \mathcal{D}_{r_0}^{r_0+n\delta}(z)} \left[\text{MBD}_{r_0+n\delta}^{r_0+(n+1)\delta}(z')^{2p} \right]. \quad (3.5.9)
\end{aligned}$$

Summing (3.5.9) over all $z \in B_{S(r_0)}(u, Ae^{-r_0})$ leads to:

$$S_{n+1} \leq \left(1 - \frac{1}{n^2}\right)^{1-2p} S_n + n^{4p-2} \left(\sum_{\substack{z \in B_{S(r_0)}(u, Ae^{-r_0}) \\ z' \in \mathcal{D}_{r_0}^{r_0+n\delta}(z)}} \text{MBD}_{r_0+n\delta}^{r_0+(n+1)\delta}(z')^{2p} \right). \quad (3.5.10)$$

Let us place on the event $\{S_n \leq MA^d e^{-2r_0 p}\}$. Then, for any $z \in B_{S(r_0)}(u, Ae^{-r_0})$,

$$\text{MBD}_{r_0}^{r_0+n\delta}(z)^{2p} \leq S_n \leq MA^d e^{-2r_0 p},$$

so, for any $z' \in \mathcal{D}_{r_0}^{r_0+n\delta}(z)$, $z'0z \leq M^{1/2p} A^{d/(2p)} e^{-r_0}$, so

$$\begin{aligned}
\widehat{z'0I_\infty} & \leq \widehat{z'0z} + \widehat{z0I_\infty} \\
& \leq M^{\frac{d}{2p}} A^{\frac{d}{2p}} e^{-r_0} + Ae^{-r_0} \text{ since } z \in B_{S(r_0)}(u, Ae^{-r_0}) \\
& \leq Ae^{-r_0} \left(M^{\frac{d}{2p}} + 1 \right) \text{ since } \frac{d}{2p} \leq 1.
\end{aligned}$$

Therefore, for any $z \in B_{S(r_0)}(u, Ae^{-r_0})$,

$$\mathcal{D}_{r_0}^{r_0+n\delta}(z) \subset B_{S(r_0+n\delta)} \left(u, Ae^{-r_0} \left(M^{\frac{d}{2p}} + 1 \right) \right). \quad (3.5.11)$$

Let us define

$$Z_n(A, r_0) := \sum_{z \in B_{S(r_0+(n+1)\delta)}(u, Ae^{-r_0} (M^{1/(2p)} + 1)) \cap \text{RST}} \text{MBD}_{r_0+n\delta}^{r_0+(n+1)\delta}(z')^{2p}. \quad (3.5.12)$$

By (3.5.11),

$$\sum_{\substack{z \in B_{S(r_0)}(u, Ae^{-r_0}) \\ z' \in \mathcal{D}_{r_0}^{r_0+n\delta}(z)}} \text{MBD}_{r_0+n\delta}^{r_0+(n+1)\delta}(z')^{2p} \leq Z_n(A, r_0), \quad (3.5.13)$$

thus, combining (3.5.10) and (3.5.13), on the event $\{S_n \leq MA^d e^{-2r_0 p}\}$,

$$S_{n+1} \leq \left(1 - \frac{1}{n^2}\right)^{1-2p} S_n + n^{4p-2} Z_n. \quad (3.5.14)$$

This upper-bound of S_{n+1} suggests the following definition of the random variables $Y_n^M(A, r_0)$. We set $Y_0^M(A, r_0) := 0$, and for any $n \geq 0$,

$$Y_{n+1}^M(A, r_0) := \left(1 - \frac{1}{n^2}\right)^{1-2p} Y_n^M(A, r_0) + n^{4p-2} A^{-d} e^{2r_0 p} Z_n(A, r_0).$$

Let us also define $Y^M(A, r_0) := \lim_{n \rightarrow \infty} \uparrow Y_n^m(A, r_0)$ (this is well-defined since $n \rightarrow Y_n^m(A, r_0)$ is non-decreasing). We first show that the random variables $Y_n^M(A, r_0)$ verify (3): for $n \in \mathbb{N}$, on the event $\{S_n \leq (M \wedge Y_n^M(A, r_0)) A^d e^{-2r_0 p}\}$, by (3.5.14),

$$\begin{aligned} S_{n+1} &\leq \left(1 - \frac{1}{n^2}\right)^{1-2p} S_n + n^{4p-2} Z_n(A, r_0) \\ &\leq \left(1 - \frac{1}{n^2}\right)^{1-2p} Y_n^M(A, r_0) A^d e^{-2r_0 p} + n^{4p-2} Z_n(A, r_0) \\ &= A^d e^{-2r_0 p} \left[\left(1 - \frac{1}{n^2}\right)^{1-2p} Y_n^M(A, r_0) + n^{4p-2} A^{-d} e^{2r_0 p} Z_n(A, r_0) \right] \\ &\leq A^d e^{-2r_0 p} Y_{n+1}^M(A, r_0). \end{aligned}$$

Thus the random variables $Y_n^M(A, r_0)$ verify (3). We move on to show that $(Y_n^M(A, r_0))_{n, M, A, r_0}$ and $(Y^M(A, r_0))_{M, A, r_0}$ also verify (2). To proceed, we upper-bound $\mathbb{E}[Y_n^M(A, r_0)]$ by induction on n .

For any M, A, r_0, n , Proposition 3.2.5 applied for $\theta = Ae^{-r_0} (M^{1/(2p)} + 1)$ gives,

$$\begin{aligned} \mathbb{E}[Z_n(A, r_0)] &\leq C \left(Ae^{-r_0} \left(M^{\frac{1}{2p}} + 1 \right) \right)^d e^{(d-2p)(r_0+n\delta)} \\ &= CA^d \left(M^{\frac{1}{2p}} + 1 \right)^d e^{-2pr_0+n(d-2p)\delta}. \end{aligned} \quad (3.5.15)$$

Let us define, for any $n \in \mathbb{N}$,

$$p(n) := \left(1 - \frac{1}{n^2}\right)^{1-2p}, \quad q(n) := n^{4p-2} e^{n(d-2p)\delta} \quad (3.5.16)$$

and

$$P(n) := \prod_{k=0}^{n-1} p(k), \quad Q(n) := \sum_{k=0}^{n-1} q(k). \quad (3.5.17)$$

with the convention $P(0) = 1$ and $Q(0) = 0$. It can be noticed that

$$\lim_{n \rightarrow \infty} P(n) < \infty, \quad \lim_{n \rightarrow \infty} Q(n) < \infty \text{ since } d - 2p < 0. \quad (3.5.18)$$

Let us show by induction on n that $\mathbb{E}[Y_n^M(A, r_0)] \leq C(M^{1/(2p)} + 1)^d P(n)Q(n)$ for any $n \in \mathbb{N}$. The

assertion is clear for $n = 0$ and, for $n \geq 0$,

$$\begin{aligned}
\mathbb{E}[Y_{n+1}^M(A, r_0)] &= \left(1 - \frac{1}{n^2}\right)^{1-2p} \mathbb{E}[Y_n^M(A, r_0)] + n^{4p-2} A^{-d} e^{2r_0 p} \mathbb{E}[Z_n(A, r_0)] \\
&= p(n) \mathbb{E}[Y_n^M(A, r_0)] + A^{-d} e^{2r_0 p - n(d-2p)\delta} q(n) \mathbb{E}[Z_n(A, r_0)] \\
&\stackrel{(3.5.15)}{\leq} p(n) \mathbb{E}[Y_n^M(A, r_0)] + C \left(M^{\frac{1}{2p}} + 1\right)^d q(n) \\
&\leq C \left(M^{\frac{1}{2p}} + 1\right)^d [p(n)P(n)Q(n) + q(n)] \text{ by induction hypothesis} \\
&= C \left(M^{\frac{1}{2p}} + 1\right)^d [P(n+1)Q(n) + q(n)] \\
&\leq C \left(M^{\frac{1}{2p}} + 1\right)^d P(n+1)Q(n+1) \text{ since } P(n+1) \leq 1, \tag{3.5.19}
\end{aligned}$$

which achieves the induction. Thus, there exists some constant $C > 0$ such that, for any $M, A, r_0 \geq 0$, for any $n \in \mathbb{N}$, $\mathbb{E}[Y_n^M(A, r_0)] \leq C (M^{1/(2p)} + 1)^d$. By monotone convergence,

$$\mathbb{E}[Y^M(A, r_0)] \leq C \left(M^{1/(2p)} + 1\right)^d.$$

Thus, for any $M, A, r_0 \geq 0$, Markov inequality gives,

$$\mathbb{P}[Y^M(A, r_0) \geq M] \leq \frac{C (M^{1/(2p)} + 1)^d}{M} = O(M^{-2/3})$$

since $2p > 3d$. Thus the family of random variables $Y^M(A, r_0)$ verifies (2). This achieves the proof.

Proof of Proposition 3.2.7. This is a direct consequence of Proposition 3.2.6. Let $\varepsilon > 0$ and let us choose p such that $d/p < \varepsilon$. Applying Proposition 3.2.6 with $A = \pi e^{r_0}$ gives that, for any $r_0 \geq 0$,

$$\mathbb{E} \left[\sum_{z \in S(r_0) \cap \text{RST}} (\text{MBD}_{r_0}^\infty(z))^p \right] \leq C e^{r_0(d-p)}. \tag{3.5.20}$$

Thus:

$$\mathbb{E} \left[\sum_{n \in \mathbb{N}} \left(\max_{z \in \mathcal{L}_n} \text{MBD}_n^\infty(z) e^{(1-\varepsilon)n} \right)^p \right] \leq \sum_{n \in \mathbb{N}} C e^{n((d-p)+(1-\varepsilon)p)} < \infty$$

since $\varepsilon < d/p$. Therefore, a.s.,

$$\lim_{n \rightarrow \infty} e^{(1-\varepsilon)n} \max_{z \in \mathcal{L}_n} \text{MBD}_n^\infty(z) \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Moreover, $r_0 \mapsto \max_{z \in \mathcal{L}_{r_0}} \text{MBD}_{r_0}^\infty$ is non-increasing, so for any $n \leq r_0 < n+1$,

$$e^{(1-\varepsilon)r_0} \max_{z \in \mathcal{L}_{r_0}} \text{MBD}_{r_0}^\infty(z) \leq \max_{z \in \mathcal{L}_n} \text{MBD}_n^\infty(z) e^{(1-\varepsilon)r_0} \leq e^{1-\varepsilon} \max_{z \in \mathcal{L}_n} \text{MBD}_n^\infty(z) e^{(1-\varepsilon)n}, \tag{3.5.21}$$

thus

$$\lim_{r_0 \rightarrow \infty} \max_{z \in \mathcal{L}_{r_0}} \text{MBD}_{r_0}^\infty(z) e^{(1-\varepsilon)r_0} \rightarrow 0 \text{ as } r_0 \rightarrow \infty.$$

Define R_0 such that, for any $r_0 \geq R_0$, $e^{(1-\varepsilon)r_0} \max_{z \in \mathcal{L}_{r_0}} \text{MBD}_{r_0}^\infty(z) \leq 1/2$. For any $r_0 \geq R_0$, $z \in \mathcal{L}_{r_0}$, $z_1, z_2 \in \mathcal{D}(z)$, defining $r_1 := d(0, z_1)$ and $r_2 := d(0, z_2)$,

$$\begin{aligned} \widehat{z_1 0 z_2} &\leq \widehat{z_1 0 z} + \widehat{z 0 z_2} \leq \text{CFD}_{r_0}^{r_1}(z_1) + \text{CFD}_{r_0}^{r_2}(z_2) \leq \text{MBD}_{r_0}^{r_1}(z) + \text{MBD}_{r_0}^{r_2}(z) \\ &\leq 2\text{MBD}_{r_0}^\infty(z) \leq 2 \max_{z' \in \mathcal{L}_{r_0}} \text{MBD}_{r_0}^\infty(z') \leq e^{-(1-\varepsilon)r_0}. \end{aligned} \quad (3.5.22)$$

This achieves the proof of Proposition 3.2.7.

3.6 Appendix A: proof of Proposition 3.2.2

We first show that the RST is a tree. If the RST contains some loop z_0, \dots, z_n , then the furthest vertex to the origin in the loop, say z_i , must have two parents, which contradicts the definition of the RST. Moreover, for some given vertex $z \in \mathcal{N}$, the sequence $(d(A^{(k)}(z), 0))_k$ is decreasing. In addition, since $\mathcal{N} \cap B(r)$ is finite for any $r \geq 0$, there is no infinite decreasing sequence $(d(A^{(k)}(z), 0))_k$. Thus $A^{(k)}(z) = 0$ for some finite $k \geq 0$. Therefore, the RST is a connected graph, so it is a tree.

We move on to show that the RST is locally finite. Let $z_0 = (r_0; u_0) \in \mathcal{N} \cup \{0\}$. Let us assume for the moment that, for any $z = (r; u) \in \mathbb{H}^{d+1}$ and $r' > 0$,

$$\text{Vol}(B^+(z, r')) \geq ce^{d(r' \wedge r)/2}. \quad (3.6.1)$$

for some c independent of z, r . For $z_0 = (r_0; u_0)$, $z = (r; u) \in \mathbb{H}^d$, let us define

$$a(z, z_0) = \mathbf{1}_{r > r_0} \mathbf{1}_{B^+(z, d(z, z_0)) \cap \mathcal{N} = \emptyset}.$$

Thus, for any $z_0 \in \mathcal{N}$, $z_0 = A(z)$ if and only if $a(z, z_0) = 1$. By Campbell formula [8],

$$\begin{aligned} \mathbb{E} \left[\#\{z_0 \in \mathcal{N}, \#A^{(-1)}(z_0) = \infty\} \right] &= \mathbb{E} \left[\sum_{z_0 \in \mathcal{N}} \mathbf{1}_{\sum_{z \in \mathcal{N}} a(z, z_0) = \infty} \right] + \mathbb{P} \left[\sum_{z \in \mathcal{N}} a(z, 0) = \infty \right] \\ &= \mathbb{P} \left[\sum_{z \in \mathcal{N}} a(z, 0) = \infty \right] + \lambda \int_{\mathbb{H}^{d+1}} \mathbb{P} \left[\sum_{z \in \mathcal{N}} a(z, z_0) = \infty \right] dz_0, \end{aligned}$$

thus it suffices to show that $\mathbb{P} \left[\sum_{z \in \mathcal{N}} a(z, z_0) = \infty \right] = 0$ for any $z_0 \in \mathbb{H}^{d+1}$. Let $z_0 = (r_0; u_0) \in \mathbb{H}^{d+1}$. Note that, if $d(z, z_0) \geq r_0$, then $0 \in B^+(z, d(z, z_0))$ so $a(z, z_0) = 0$. Thus, Campbell formula gives,

$$\begin{aligned} \mathbb{E} \left[\sum_{z \in \mathcal{N}} a(z, z_0) \right] &= \lambda \int_{\mathbb{H}^{d+1}} \mathbb{E} [a(z, z_0)] dz \leq \lambda \int_{\mathbb{H}^{d+1}} \mathbf{1}_{d(z, z_0) < r_0} \mathbb{P}[B^+(z, d(z, z_0)) \cap \mathcal{N} = \emptyset] dz \\ &= \lambda \int_{\mathbb{H}^{d+1}} \mathbf{1}_{d(z, z_0) < r_0} \mathbb{P} \left[\exp(-\lambda \text{Vol}(B^+(z, d(z, z_0)))) \right] dz \\ &\stackrel{(3.6.1)}{\leq} \lambda \int_{\mathbb{H}^{d+1}} \mathbb{P} \left[\exp(-\lambda ce^{-d/2} d(z, z_0)) \right] dz \\ &\stackrel{(3.2.5)}{=} \lambda \mathcal{S}(d) \int_0^\infty \exp(-\lambda ce^{-r^d/2}) \sinh(r)^d dr, \\ &< \infty, \end{aligned}$$

where $\mathcal{S}(d)$ denotes the surface area of the Euclidean unit ball \mathbb{S}^d . Thus $\mathbb{P} \left[\sum_{z \in \mathcal{N}} a(z, z_0) = \infty \right] = 0$.

It remains to show (3.6.1). Let $\rho \geq 0$ and $0 \leq r \leq r_0$. Let $z' \in S(r')$. Using the hyperbolic law of cosines in the triangle $0zz'$ (3.2.7),

$$\begin{aligned} z' \in B^+(z, \rho) &\iff d(z, z') \leq \rho \\ &\iff \cosh(\rho) \geq \cosh(r_0) \cosh(r) - \cos(\widehat{z0z'}) \sinh(r_0) \sinh(r) \\ &\iff \cos(\widehat{z0z'}) \geq \frac{\cosh(r_0) \cosh(r) - \cosh(\rho)}{\sinh(r_0) \sinh(r)}. \end{aligned}$$

A study of the function $r \mapsto \frac{\cosh(r_0) \cosh(r) - \cosh(\rho)}{\sinh(r_0) \sinh(r)}$ shows that, if $\rho \geq 1$, then this quantity is non-decreasing when $r_0 - 1 \leq r \leq r_0$. Thus, for $\rho \geq 1$ and $1 \leq r_0 - 1 \leq r \leq r_0$,

$$\frac{\cosh(r_0) \cosh(r) - \cosh(\rho)}{\sinh(r_0) \sinh(r)} \leq \frac{\cosh(r_0)^2 - \cosh(\rho)}{\sinh(r_0)^2} = 1 - \frac{\cosh(\rho) - 1}{\sinh(r_0)^2} \leq 1 - Ce^{\rho-2r_0}$$

for some $C > 0$ independent of r_0, r, ρ . Thus, there exists $C > 0$ independent of r_0, r, ρ such that, if $\rho \geq 1$ and $1 \leq r_0 - 1 \leq r \leq r_0$, if $\widehat{z0z'} \leq Ce^{\rho/2-r_0}$ then $z' \in B^+(z, \rho)$. Therefore, by (3.2.5), if $\rho \geq 1$ then (recall that ν is the d -dimensional volume measure on \mathbb{S}^d and u is the direction of z),

$$\begin{aligned} \text{Vol}(B^+(z, \rho)) &\geq \int_{r_0-1}^{r_0} \nu\{u', \langle u', u \rangle \leq ce^{\rho/2-r_0}\} \sinh(r)^d dr \\ &\geq c \int_{r_0-1}^{r_0} \left[e^{d\rho/2-dr_0} \wedge 1 \right] \sinh(r)^d dr \geq ce^{d(\rho \wedge r_0)/2} \end{aligned}$$

Therefore, the RST has finite degree a.s.

It remains to show that the geodesics $[z, A(z)]$ for $z \in \mathcal{N}$ do not cross a.s. in the bi-dimensional case ($d = 1$). Let us suppose that there are no two points z_1, z_2 with $d(0, z_1) = d(0, z_2)$ (this happens with probability 0). Let $z_1 = (r_1; u_1), z_2 = (r_2; u_2) \in \mathcal{N}$ and let us set $A(z_1) := (r'_1; u'_1), A(z_2) := (r'_2; u'_2)$. Suppose that $[z_1, A(z_1)]$ and $[z_2, A(z_2)]$ meet at some point $P_{hyp} := (r_{hyp}; u_{hyp})$. We have $r'_1 < r_{hyp} < r_2$, thus by definition of the parent, $d(z_2, A(z_2)) < d(z_2, A(z_1))$. Then

$$\begin{aligned} d(z_2, P_{hyp}) + d(P_{hyp}, A(z_2)) &= d(z_2, A(z_2)) < d(z_2, A(z_1)) \\ &\leq d(z_2, P_{hyp}) + d(P_{hyp}, A(z_1)), \end{aligned} \quad (3.6.2)$$

so $d(P_{hyp}, A(z_2)) < d(P_{hyp}, A(z_1))$. On the other hand, interchanging z_1 and z_2 in the previous calculation leads to $d(P_{hyp}, A(z_1)) < d(P_{hyp}, A(z_2))$. This is a contradiction. Therefore $[z_1, A(z_1)] \cap [z_2, A(z_2)] = \emptyset$. This achieves the proof of Proposition 3.2.2.

3.7 Appendix B: computing distances and angles in the half-space model

Proposition 3.7.1 (Distance formula). *Let $z_1 = (x_1, y_1) \in H$ and $z_2 = (x_2, y_2) \in H$. Let $\kappa = \|x_1 - x_2\|/y_1$ and $v = y_2/y_1$. Then*

$$d(z_1, z_2) = 2 \tanh^{-1} \left(\sqrt{\frac{\kappa^2 + (v-1)^2}{\kappa^2 + (v+1)^2}} \right) = 2 \tanh^{-1} \left(\sqrt{1 - \frac{4v}{\kappa^2 + (v+1)^2}} \right). \quad (3.7.1)$$

We refer to [5, Proposition 5] for a proof.

Proposition 3.7.2 (Angle formula). *Let $z = (x, y) \in H$ and let $h \geq 0$. Recall that $O(h) = (0, e^h)$ and $I_\infty = (0, 0)$. If $y < e^h$, then*

$$I_\infty \widehat{O(h)z} = \arctan \left| \frac{2xe^h}{e^{2h} - x^2 - y^2} \right|$$

Proof of Proposition 3.7.2. The proof is done by considering the isometry that sends the half-space model on the Poincaré disc model [7]. The Poincaré disc model is defined as:

$$I = \{(x, y) \in \mathbb{R}^2, x^2 + y^2 < 1\}$$

endowed with the following metric:

$$ds_I^2 := 4 \frac{dx^2 + dy^2}{1 - x^2 - y^2}. \quad (3.7.2)$$

The application $\phi : H \rightarrow I$ defined as:

$$(x, y) \mapsto \frac{1}{x^2 + (y+1)^2} (x^2 + y^2 - 1, -2x)$$

is an isometry sending $(0, 1)$ on $(0, 0)$ [7]. Let us compose it with the dilation of factor e^{-h} to build an isometry from H to I sending $O(h)$ on $(0, 0)$. We obtain the application ϕ' defined as

$$\begin{aligned} \phi'(x, y) &= \frac{1}{(e^{-h}x)^2 + (e^{-h}y + 1)^2} ((e^{-h}x)^2 + (e^{-h}y)^2 - 1, -2e^{-h}x) \\ &= \frac{1}{x^2 + (y + e^h)^2} (x^2 + y^2 - e^{2h}, -2xe^h) \end{aligned}$$

The Poincaré disc model is conform (i.e. the hyperbolic angles correspond to angles in the disc model), and the geodesics containing the origin $(0, 0)$ are represented by straight lines, thus for any two points $z_1, z_2 \in I$, the hyperbolic angle $\widehat{z_1 0 z_2}$ coincides with the Euclidean one. For any $z = (x, y) \in H$, $\widehat{z O(h) I_\infty} = \phi'(z) \widehat{0 \phi'(I_\infty)}$, where the second angle is taken in the disc model. Since $\phi'(I_\infty) = (-1, 0)$, if $y < e^h$ then $\widehat{z O(h) I_\infty} < \frac{\pi}{2}$, thus

$$\widehat{z O(h) I_\infty} = \arctan \left| \frac{2xe^h}{x^2 + y^2 - e^{2h}} \right|.$$

□

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Chapter 4

Thick track at infinity

The last chapter is devoted to another refinement of the result by Howard and Newman [15]. When the tree is *straight*, their theorem says that each boundary point $I \in \partial\mathbb{H}^d$ is the asymptotic direction of a semi-infinite branch of the considered tree and that the set of boundary points that are asymptotic directions for at least two semi-infinite branches is a dense set (and countable in dimension $d = 2$). It is then natural to investigate the existence of boundary points targeted by more than two semi-infinite paths. For the RST, this last question remains open in the Euclidean space, but in the present chapter, we resolve it in the hyperbolic setting. We prove in Theorem 4.1.1 that any infinite subtree of the RST generates a thick track on $\partial\mathbb{H}^d$. In the bi-dimensional case, this implies there is no (random) asymptotic direction $I \in \partial\mathbb{H}^2$ with more than two semi-infinite branches (Corollary 4.1.2) which completes the description of semi-infinite branches and their asymptotic directions of the hyperbolic RST started in Chapter 3.

Recall that the tree is *straight* in the sense that the descendants subtree of a vertex far from the origin is included in a thin cone. In the Euclidean setting, the straightness of the RST is proved by Baccelli and Bordenave [1] using that the DSF was a translation invariant model that constitutes a local approximation of the RST far from the origin. They exploit the theory of Markov chains to bound from above fluctuations of trajectories in the DSF and then, they deduce the straightness of the RST via planarity. Their strategy hence cannot be generalised to higher dimensions. In Chapter 3, the straightness of the hyperbolic RST is obtained in any dimension by estimates of the angular deviations of branches. These estimates can be carried without resorting to an auxiliary model and hence allows to overcome planarity arguments. The hyperbolic metric guarantees that angular deviations decay exponentially fast with the distance to the origin, which is strong enough to show straightness.

4.1 Main result

We consider in this chapter the Radial Spanning Tree (RST) on the hyperbolic space \mathbb{H}^d defined in Chapter 3. We will work here with the unit open ball model (see Section 3.2.1 of Chapter 3, page 79). The boundary of \mathbb{H}^d is here identified with the unit sphere. In the sequel, For $z \in \mathbb{H}^d$ and $\rho \geq 0$, $B(z, \rho)$ denotes the hyperbolic ball centered at z of radius ρ . Also, we denote by $S(z, \rho)$ the sphere of radius ρ and center z . For $\theta \geq 0$, $\text{Cone}(z; \theta)$ is the cone of axis z and aperture θ :

$$\text{Cone}(z, \theta) = \{z' \in \mathbb{H}^d, \widehat{z0z'} \leq \theta\}.$$

We set $B_{S(r)}(z, \theta) = \text{Cone}(z; \theta) \cap S(r)$.

The RST is constructed on a Poisson point process with intensity λ of the hyperbolic space \mathbb{H}^d , denoted by \mathcal{N} . For a point $z \in \mathcal{N}$, let us denote by

$$\mathcal{D}(z) := \bigcup_{n \geq 0} A^{(-n)}(z) = \{z' \in \mathcal{N}, \exists n \geq 0, A^{(n)}(z') = z\}$$

the set of Poisson points in the descending subtree of z . See Section 3.2.2 of Chapter 3 for notations previously introduced. Let $\mathcal{D}^\infty(z)$ be the set of boundary points reached by descendants of z , i.e. the set of boundary points $I \in \partial\mathbb{H}^d$ which are the asymptotic directions of semi-infinite paths of descendants of z . In this chapter, we prove that any Poisson point z having an infinite number of descendants generates a thick track at infinity, i.e. its set of boundary points $\mathcal{D}^\infty(z)$ has a non-empty interior.

Theorem 4.1.1. *A.s. for any vertex $z \in \mathcal{N}$, either z admits a finite number of descendants or $\text{Int}(\mathcal{D}^\infty(z)) \neq \emptyset$.*

Since the hyperbolic RST satisfies the finite degree property (see [6, Proposition 2.2]), the subtree rooted at a given vertex z having infinitely many descendants, contains a semi-infinite branch which admits an asymptotic direction by [6, Theorem 1.1 (i)]. In other words, $\mathcal{D}^\infty(z) \neq \emptyset$. What Theorem 4.1.1 really means is

$$\text{a.s. for any } z \in \mathcal{N} \quad \mathcal{D}^\infty(z) \neq \emptyset \Rightarrow \text{Int}(\mathcal{D}^\infty(z)) \neq \emptyset .$$

In the bi-dimensional case, Theorem 4.1.1 completes the description of semi-infinite branches and their asymptotic directions of the hyperbolic RST started in [6], based on the strategy of Howard and Newman [7] and on the fact that the hyperbolic RST is *straight* (see [6, Prop. 2.7]). The first three items are given by [6, Theorem 1.1].

Corollary 4.1.2. *The following properties concern the hyperbolic RST in \mathbb{H}^2 .*

- (o) *A.s. any semi-infinite branch admits an asymptotic direction and, for any (random) $I \in \partial\mathbb{H}^2$, the hyperbolic RST contains a semi-infinite branch with asymptotic direction I .*
- (i) *For any (deterministic) $I \in \partial\mathbb{H}^2$, the hyperbolic RST a.s. contains a unique semi-infinite branch with asymptotic direction I .*
- (ii) *A.s. the set of $I \in \partial\mathbb{H}^2$ such that the hyperbolic RST contains at least two semi-infinite branches with asymptotic direction I is dense and countable in $\partial\mathbb{H}^2$.*
- (iii) *A.s. there is no (random) $I \in \partial\mathbb{H}^2$ with more than two semi-infinite branches.*

Item (o) says that any asymptotic direction $I \in \partial\mathbb{H}^2$ is targeted by at least one semi-infinite branch and exactly one if I is deterministic (Item (i)). There exist by Item (ii) asymptotic directions which are targeted by at least two semi-infinite branches but these directions are random and few (only countable in dimension 2). Finally, Theorem 4.1.1 specifies there is no random asymptotic direction targeted by more than two semi-infinite branches.

Proof of Corollary 4.1.2 (iii). Assume the hyperbolic RST contains three different semi-infinite branches having the same asymptotic direction $I \in \partial\mathbb{H}^2$, say γ , γ' and γ'' . Let us denote by $r_0 > 0$ a radius such that outside the ball $B(0, r_0)$ the three paths are disjoint (they share no vertices). Since the hyperbolic RST satisfies the non-crossing path property, we can also assume by planarity that γ' is trapped between the two other paths in the following sense. For any radius $r > r_0$ large enough let (r, θ_r) (resp. (r, θ'_r) , (r, θ''_r)) be the intersection point between the semi-infinite branch γ (resp. γ' , γ'') and the circle with radius r and centered at the origin. Then, modulo 2π , $\theta_r < \theta'_r < \theta''_r$ and $\theta''_r - \theta_r \rightarrow 0$ as $r \rightarrow \infty$. Henceforth, any vertex z' of $\gamma' \setminus (\gamma \cup \gamma'')$ admits infinitely many descendants but by planarity $\text{Int}(\mathcal{D}^\infty(z')) = \emptyset$. This occurs with null probability thanks to Theorem 4.1.1. \square

In the rest of this section, we now prove Theorem 4.1.1. Let us introduce some notations. Let $r_0 > 0$ and consider the closest Poisson point to $(r_0; 0)$ (with polar coordinates) that we denote:

$$z_0 = \operatorname{argmin}_{z \in \mathcal{N}} d(z, (r_0; 0)).$$

Let us define by $E(r_0)$ the event on which z_0 generates an infinite subtree $\mathcal{D}(z_0)$ of the hyperbolic RST with a set of boundary points $\mathcal{D}^\infty(z_0)$ of empty interior:

$$E(r_0) = \{ \mathcal{D}^\infty(z_0) \neq \emptyset \text{ and } \operatorname{Int}(\mathcal{D}^\infty(z_0)) = \emptyset \}. \quad (4.1.1)$$

For $\delta > 0$, let us consider the set $\mathcal{G}(\delta)$ of descendants z of z_0 whose direct ancestors $A(z)$ are not too close to z :

$$\mathcal{G}(\delta) = \{ z \in \mathcal{D}(z_0), d(z, A(z)) \geq \delta \}$$

and the set of corresponding radii:

$$\mathcal{H}(\delta) = \{ r > 0, \exists u \in [0, 2\pi), (r; u) \in \mathcal{G}(\delta) \}.$$

Lemma 4.1.3 establishes that, on the event $E(r_0)$, the descending subtree $\mathcal{D}(z_0)$ a.s. admits infinitely many elements in $\mathcal{G}(\delta)$ for $\delta > 0$ small enough.

Lemma 4.1.3. *For any $\delta > 0$ small enough, the event $E(r_0)$ is included in $\{\mathcal{H}(\delta) \text{ is unbounded}\}$, up to some negligible event.*

For any radius $r > 0$, let $\mathcal{N}_{B(r)} = \mathcal{N} \cap B(r)$ be the PPP \mathcal{N} restricted to the ball $B(r)$. Lemma 4.1.4 says that the r.v. $\mathbb{P}(E(r_0) | \mathcal{N}_{B(r)})$ is a.s. bounded away from 1 uniformly on $r \in \mathcal{H}(\delta)$.

Lemma 4.1.4. *For any $\delta > 0$ small enough, there exists $\varepsilon > 0$ such that a.s.*

$$\forall r \in \mathcal{H}(\delta), \mathbb{P}(E(r_0) | \mathcal{N}_{B(r)}) \leq 1 - \varepsilon. \quad (4.1.2)$$

Combining Lemmas 4.1.3 and 4.1.4, we will now prove that $\mathbb{P}(E(r_0)) = 0$ from which Theorem 4.1.1 follows. Proofs of Lemmas 4.1.3 and 4.1.4 are respectively postponed to Sections 4.2 and 4.3.

Proof of Theorem 4.1.1. Choose $\delta > 0$ small enough and pick $\varepsilon = \varepsilon(\delta) > 0$ as in Lemma 4.1.4. On the event $E(r_0)$, the martingale convergence theorem gives that:

$$\lim_{r \rightarrow \infty} \mathbb{P}(E(r_0) | \mathcal{N}_{B(r)}) = \mathbf{1}_{E(r_0)} = 1. \quad (4.1.3)$$

Besides, on the event $E(r_0)$, the set $\mathcal{H}(\delta)$ is a.s. unbounded and, provided $r \in \mathcal{H}(\delta)$, the conditional probability $\mathbb{P}(E(r_0) | \mathcal{N}_{B(r)})$ is smaller than $1 - \varepsilon$. This implies:

$$\liminf_{r \rightarrow \infty} \mathbb{P}(E(r_0) | \mathcal{N}_{B(r)}) < 1 - \varepsilon, \quad (4.1.4)$$

on the event $E(r_0)$. Both statements (4.1.3) and (4.1.4) are compatible only if $\mathbb{P}(E(r_0)) = 0$.

For any angle u , let $E(r_0; u)$ be the event on which the closest Poisson point z to $(r_0; u)$ generates an infinite subtree whose set of boundary points $\mathcal{D}^\infty(z)$ has an empty interior. With this notation, $E(r_0; 0) = E(r_0)$. By isotropy of the model and what precedes, $\mathbb{P}(E(r_0; u)) = \mathbb{P}(E(r_0)) = 0$. Hence, the union of the events $E(r_0; u)$ with rational radius r_0 and rational angle u has null probability. Since any Poisson point is the closest one to some rational element $(r_0; u)$ of \mathbb{H}^d , we can conclude that a.s. any vertex $z \in \mathcal{N}$ satisfies $\mathcal{D}^\infty(z) = \emptyset$ (which is equivalent to the finiteness of $\mathcal{D}(z)$) or $\operatorname{Int}(\mathcal{D}^\infty(z)) \neq \emptyset$. \square

4.2 Proof of Lemma 4.1.3

Let us consider $d \geq 1$ and $0 < \delta < 1$. We denote by \mathbb{S}^d the unit sphere of \mathbb{H}^d . The proof of Lemma 4.1.3 is based on a percolation argument. To set it up, we need a covering of the hyperbolic space \mathbb{H}^d , far from the origin, with a uniform control of overlappings. The next result is a rewriting of Lemma 4.2 of [6].

Lemma 4.2.1. *For any real number $r \geq 2$, there exists a family $(z_{r,m})_{1 \leq m \leq I(r)}$ of the (centered) sphere $S(r)$ with radius r such that:*

$$\bigcup_{1 \leq m \leq I(r)} B_{S(r)}(z_{r,m}, e^{-r}) = S(r) ,$$

where $B_{S(r)}(z_{r,m}, e^{-r}) = B(z_{r,m}, e^{-r}) \cap S(r)$.

Moreover there exists $K = K(d) > 0$ such that for any $r \geq 2$, $z \in S(r)$ and $A \geq 1$,

$$\text{Card}\{1 \leq m \leq I(r) : B_{S(r)}(z_{r,m}, e^{-r}) \cap B_{S(r)}(z, Ae^{-r}) \neq \emptyset\} \leq KA^d . \quad (4.2.1)$$

We consider the following covering. For an integer r larger than 2 and $m \in \{1, \dots, I(r)\}$, the block $B_{r,m}$ is defined as the part of the annulus $B(0, r+1) \setminus B(0, r)$ based on $B_{S(r)}(z_{r,m}, e^{-r})$:

$$B_{r,m} := \text{Cone}(z_{r,m}, e^{-r}) \cap (B(0, r+1) \setminus B(0, r)) . \quad (4.2.2)$$

Let us note that in dimension $d = 2$, one can replace this covering by a partition (see Remark 4.2.3 below).

Two blocks $B_{r,m}$ and $B_{r',m'}$ are said *adjacent*, denoted by $B_{r,m} \sim B_{r',m'}$, if they are at distance from each other less than δ . Lemma 4.2.2 asserts that the volumes of $B_{r,m}$'s are uniformly bounded and the graph generated by the adjacency relation admits a bounded degree.

Lemma 4.2.2. *There exist two positive constants C_1 and C_2 only depending on d such that for all integer radius $r \geq 2$ and $m \in \{1, \dots, I(r)\}$,*

$$\text{Vol}(B_{r,m}) \leq C_1 \quad (4.2.3)$$

and

$$\text{Card}\{(r', m') : B_{r',m'} \sim B_{r,m}\} \leq C_2 . \quad (4.2.4)$$

Proof. Let us first prove (4.2.3). It suffices to write $\text{Vol}(B_{r,m})$ as

$$\int_r^{r+1} \sinh^d(u) du \times \text{Vol}_{S^d}(\text{Cone}((1;0), e^{-r}) \cap S^d)$$

where S^d is the unit sphere in \mathbb{R}^d and Vol_{S^d} is the Haar measure on S^d . Thus

$$\text{Vol}(B_{r,m}) = \int_r^{r+1} \sinh^d(u) du \times \text{Vol}_{S^d}(\text{Cone}((1;0), e^{-r}) \cap S^d) \leq C \int_r^{r+1} e^{du} du e^{-rd} \leq C$$

for some $C > 0$.

For the second inequality, two blocks $B_{r',m'}$ and $B_{r,m}$ are adjacent whenever $B_{r',m'}$ overlaps $\{z \in H ; d(z, B_{r,m}) \leq \delta\}$. Since $\delta < 1$ this forces r' to be in $\{r-1, r, r+1\}$ and $B_{r',m'}$ to overlap $\text{Cone}(z_{r,m}, (1+\delta)e^{-r})$. Now, Lemma 4.2.1 allows us to bound the number of such blocks. Using (4.2.1) on the sphere $S(r')$, with $z = z_{r,m}$ and $A = (1+\delta)e^{r'-r} \leq 2e$, the constant $C_2 := 3K(2e)^d$ suits. \square

Remark 4.2.3. For the dimension $d = 2$, we partition \mathbb{H}^2 into rings $B(0, (n+1) \log 2) \setminus B(0, n \log 2)$. The n th ring is paved by isometric blocs $B_{n,m}$ for $m \in \{0, \dots, 2^n - 1\}$:

$$B_{n,m} := \left\{ (r; \theta) \in \mathbb{H} \mid r \in [n \log 2, (n+1) \log 2], \theta \in \left[m \frac{2\pi}{2^n}, (m+1) \frac{2\pi}{2^n} \right] \right\}.$$

Two blocks $B_{n,m}$ and $B_{n',m'}$ are considered adjacent (which is denoted by $B_{n,m} \sim B_{n',m'}$) if $B_{n,m} \cap B_{n',m'} \neq \emptyset$ (i.e. the two blocs share an edge or a corner). This relation of adjacency defines a graph with degree bounded by 8.

Given $\delta < 1$, a block $B_{r,m}$ is said δ -bad if it contains a Poisson point z such that $d(z, A(z)) < \delta$. Let us first prove that:

$$\lim_{\delta \rightarrow 0} \sup_{r,m} \mathbb{P}(B_{r,m} \text{ is } \delta\text{-bad}) = 0. \quad (4.2.5)$$

To do it, we use the Campbell formula combined with (4.2.3). For any (r, m) ,

$$\begin{aligned} \mathbb{P}(B_{r,m} \text{ is } \delta\text{-bad}) &\leq \mathbb{E}[\#\{z \in B_{r,m} \cap \mathcal{N}, d(z, A(z)) < \delta\}] \\ &\leq \mathbb{E}[\#\{z \in B_{n,m} \cap \mathcal{N}, B(z, \delta) \cap \mathcal{N} \neq \emptyset\}] \\ &= \int_{B_{r,m}} \lambda \mathbb{P}(B(z, \delta) \cap \mathcal{N} \neq \emptyset) dz \\ &= \lambda \text{Vol}(B_{r,m}) \left(1 - e^{-\lambda \text{Vol}(B(z, \delta))}\right) \end{aligned}$$

which tends to 0 as $\delta \rightarrow 0$ uniformly on (r, m) thanks to (4.2.3).

Thus, let us prove that we can choose $\delta > 0$ small enough so that the set of bad blocks

$$\Psi := \left\{ B_{r,m} : r \in \mathbb{N} \setminus \{0, 1\}, 1 \leq m \leq I(r) \text{ which are } \delta\text{-bad} \right\}$$

is a.s. subcritical w.r.t. the adjacency relation. This can be done through an adaptation of the Peierls argument. Given (r, m) , let $\mathcal{C}_{n,m}(k)$ be the set of paths with length k starting from the block $B_{n,m}$. Inequality (4.2.4) leads to

$$\text{Card}(\mathcal{C}_{n,m}(k)) \leq (C_2)^k.$$

A path $\pi = (B_1, \dots, B_k)$ in $\mathcal{C}_{n,m}(k)$ is said δ -bad if all the blocks B_1, \dots, B_k it contains are δ -bad. Then, the probability for $B_{n,m}$ to belong to an infinite connected component of δ -bad blocks is upperbounded by

$$\sum_{\pi \in \mathcal{C}_{n,m}(k)} \mathbb{P}(\pi \text{ is } \delta\text{-bad}),$$

for any k . Consider a set of pairwise non-adjacent blocks B_{i_1}, \dots, B_{i_l} in Ψ . The adjacency relation has been defined so that the events $\{B_{i_j} \text{ is } \delta\text{-bad}\}$'s are mutually independent. Moreover, given $\pi = (B_1, \dots, B_k)$ in $\mathcal{C}_{n,m}(k)$, it is possible to find some indices $\{i_1, \dots, i_l\} \subset \{1, \dots, k\}$, with $l = l(\pi) \geq k/C_2$, such that the blocks B_{i_1}, \dots, B_{i_l} are pairwise non-adjacent. Henceforth,

$$\mathbb{P}(\pi \text{ is } \delta\text{-bad}) \leq \mathbb{P}(B_{i_1}, \dots, B_{i_l} \text{ are } \delta\text{-bad}) \leq \eta^{k/C_2},$$

where $\eta := \sup_{r,m} \mathbb{P}(B_{r,m} \text{ is } \delta\text{-bad})$. Choosing δ small enough so that $C_2 \eta^{1/C_2}$ is strictly smaller than 1, we then obtain that a.s. the block $B_{r,m}$ cannot belong to an unbounded connected component of δ -bad blocks. Consequently, for this choice of δ , the set Ψ is a.s. subcritical.

To conclude the proof of Lemma 4.1.3, let us pick $0 < \delta < 1$ as above and assume that $\mathcal{D}^\infty(z_0)$ is non empty, i.e. the descending subtree of z_0 admits (at least) one infinite backward path $(z_n)_{n \geq 0}$. Since the Poisson point process \mathcal{N} is a.s. locally finite, the backward path $(z_n)_{n \geq 0}$ cannot be stuck, from some index, inside a bounded connected component of δ -bad blocks. So it eventually comes out of each connected component of δ -bad blocks— which are a.s. all bounded. Two cases must

be distinguished. Either $(z_n)_{n \geq 0}$ visits infinitely many δ -good blocks where of course a block is said *good* if it is not bad. Hence, an infinite number of the z_n 's satisfy $d(z_n, A(z_n)) \geq \delta$. Their distances to the origin, which are also a.s. infinitely many, all belong to $\mathcal{H}(\delta)$. Or, the backward path $(z_n)_{n \geq 0}$ jumps infinitely many times from a bad connected component to another one. But, by construction, two different (bad) connected components are at distance from each other at least δ . As a consequence, in both cases, the set $\mathcal{H}(\delta)$ is unbounded.

4.3 Proof of Lemma 4.1.4

For $r_0 > 0$, recall that z_0 is the closest Poisson point to $(r_0; 0)$ and the event $E(r_0)$ is defined by $\mathcal{D}^\infty(z_0) \neq \emptyset$ and $\text{Int}(\mathcal{D}^\infty(z_0)) = \emptyset$. Let $\delta > 0$ small enough as in Lemma 4.1.3. So, for a.e. configuration $\eta \in E(r_0)$, there exists an unbounded sequence of radii (depending on η) in $\mathcal{H}(\delta)$. Let $r > r_0$ be one of them. In the sequel we will work conditionally to $\mathcal{N}_{B(r)} = \eta_{B(r)}$. Let $z_1 = (r; u) \in \mathcal{G}(\delta)$ (in particular $z_1 \in \mathcal{N}$). By isotropy of the model and conditionally to $\mathcal{N}_{B(r)} = \eta_{B(r)}$, we can assume from now on that $u = 0$.

Let h, δ' be positive parameters to be specified later. We set $z_2 = (r + h; 0) \in \mathbb{H}^d$ (aligned with z_1). Below, we will ask that the ball $B(z_2, \delta')$ contains a (single) Poisson point z whose set of descendants $\mathcal{D}(z)$ generates a thick track at infinity (plus additional conditions, see the event F_0 below). This is made possible thanks to Lemma 4.3.1 which expresses that a positive proportion of Poisson points in a small ball satisfies $\text{Int}(\mathcal{D}^\infty(\cdot)) \neq \emptyset$.

Lemma 4.3.1. *There exist $A > 0$ (large) and $c_0 > 0$ such that, for any $\delta' > 0$ and any $h_0 > 0$ large enough, for any $r > 0$, there exists $h \in [h_0, h_0 + A]$ such that:*

$$\mathbb{E} \left[\#\{z \in \mathcal{N} \cap B(z_2, \delta'), \text{Int}(\mathcal{D}^\infty(z)) \neq \emptyset\} \right] \geq c_0 \text{Vol}(B(\delta')) . \quad (4.3.1)$$

Note also that among the previous parameters, only $h = h(r)$ may depend on r .

Besides, it will be convenient that w.h.p. the set $\mathcal{D}(z)$ is not sensitive to the configuration $\eta_{B(r)}$. This property is encoded by the event $\text{Deter}(r, h, \delta')$ and Lemma 4.3.2. Let us set

$$\text{Deter}(r, h, \delta') = \{\eta' : \forall z \in \mathcal{N} \cap B(z_2, \delta'), \forall \eta'', \mathcal{D}(z)[\eta''_{B(r)} \cup \eta'_{B(r)^c}] = \mathcal{D}(z)[\eta']\} .$$

The event $\text{Deter}(r, h, \delta')$ has a probability tending to 1 as $h \rightarrow \infty$ uniformly on r :

Lemma 4.3.2. *For any $\delta' > 0$,*

$$\lim_{h \rightarrow \infty} \sup_{r > 0} \mathbb{P}(\text{Deter}(r, h, \delta')) = 1 .$$

Let us now introduce the event $F = F(r, h, \delta')$ specifying that the ball $B(z_2, \delta')$ contains a unique Poisson point z whose set of descendants $\mathcal{D}(z)$ generates a thick track at infinity and does not depend on what happens in $B(r)$:

$$F = \left\{ \exists z \in \mathcal{N} \cap B(z_2, \delta') \text{ s.t. } \begin{array}{l} \mathcal{N} \cap B(z_2, \delta') = \{z\}, \text{Int}(\mathcal{D}^\infty(z)) \neq \emptyset, \\ \text{Deter}(r, h, \delta') \text{ and } \mathcal{D}(z) \subset B(r + h + \delta')^c \end{array} \right\} . \quad (4.3.2)$$

Lemmas 4.3.1 and 4.3.2 together allow to bound from below the probability of F :

Lemma 4.3.3. *Let A, c_0 as in Lemma 4.3.1. For any $\delta' > 0$ small enough, there exists $\varepsilon = \varepsilon(\delta', c_0) > 0$ such that for any $h_0 > 0$ large enough, for any $r > 0$, there exists $h \in [h_0, h_0 + A]$ such that:*

$$\mathbb{P}(F(r, h, \delta')) \geq \varepsilon . \quad (4.3.3)$$

Note also that among the previous parameters, only $h = h(r)$ may depend on r .

Proof of Lemma 4.3.3. Let A, c_0 as in Lemma 4.3.1. Lemma 4.4.1 in Section 4.4 states that for any $\delta' > 0$ small enough, the following holds uniformly on $r, h > 0$:

$$\begin{aligned} \max \left\{ \mathbb{P}(\exists z \in B(z_2, \delta'), \mathcal{D}(z) \not\subset B(r+h+\delta')^c), \mathbb{E}[\#\mathcal{N} \cap B(z_2, \delta') \mathbf{1}_{\#\mathcal{N} \cap B(z_2, \delta') \geq 2}] \right\} &\leq \frac{c_0}{10} \text{Vol}(B(\delta')) . \\ \max \left\{ \mathbb{P}(\mathcal{D}(z) \not\subset B(r+h+\delta')^c), \mathbb{E}[\#\mathcal{N} \cap B(z_2, \delta') \mathbf{1}_{\#\mathcal{N} \cap B(z_2, \delta') \geq 2}] \right\} &\leq \frac{c_0}{10} \text{Vol}(B(\delta')) . \end{aligned}$$

Given such $\delta' > 0$, we thus choose h_0 large enough according to Lemmas 4.3.1 and 4.3.2 such that for any $r > 0$, there exists $h = h(r) \in [h_0, h_0 + A]$ such that (4.3.1) and

$$\mathbb{P}(\text{Deter}(r, h, \delta')) \geq 1 - \frac{c_0}{10} \text{Vol}(B(\delta')) .$$

Remark that h_0 is uniform on r , only $h = h(r)$ may depend on r . Henceforth, we can write

$$\begin{aligned} \mathbb{P}(\exists! z \in \mathcal{N} \cap B(z_2, \delta'), \text{Int}(\mathcal{D}^\infty(z)) \neq \emptyset) &\geq \mathbb{E}[\#\{z \in \mathcal{N} \cap B(z_2, \delta') : \text{Int}(\mathcal{D}^\infty(z)) \neq \emptyset\}] \\ &\quad - \mathbb{E}[\#\mathcal{N} \cap B(z_2, \delta') \mathbf{1}_{\#\mathcal{N} \cap B(z_2, \delta') \geq 2}] \\ &\geq \frac{9c_0}{10} \text{Vol}(B(\delta')) \end{aligned}$$

and thus

$$\begin{aligned} \mathbb{P}(F(r, h, \delta')) &\geq \mathbb{P}(\exists! z \in \mathcal{N} \cap B(z_2, \delta'), \text{Int}(\mathcal{D}^\infty(z)) \neq \emptyset) - \frac{2c_0}{10} \text{Vol}(B(\delta')) \\ &\geq \frac{c_0}{2} \text{Vol}(B(\delta')) . \end{aligned}$$

So $\varepsilon = \frac{c_0}{2} \text{Vol}(B(\delta'))$ is suitable. \square

For $M > 0$, let us consider the subset $U = U(r, h, \delta', M)$ of \mathbb{H}^d defined as

$$U = \left(B(r+h+\delta') \setminus (B(r) \cup B(z_2, \delta')) \right) \cap \text{Cone}(0, Me^{-r}) . \quad (4.3.4)$$

Given a configuration $\eta' \in F$, if we remove all its points inside the set U then, for the resulting configuration η'_{U^c} , the ancestor of z_- the only Poisson point in $B(z_2, \delta')$ whose existence is ensured by the event F is z_1 which itself is a descendant of z_0 . This leads to $\text{Int}(\mathcal{D}^\infty(z_0)) \neq \emptyset$, i.e. $\eta'_{U^c} \in E(r_0)^c$. This construction requires that $r \in \mathcal{H}(\delta)$ (there is no Poisson points too close to z_1).

Lemma 4.3.4. *For any h and $M > h + 1$, for any $r \in \mathcal{H}(\delta)$ and larger than r_0 , for any $\delta' < \delta/2$, the following statement holds. For each configuration η' such that $\eta'_{B(r)} = \eta_{B(r)}$ and $\eta' \in F(r, h, \delta')$ then $\eta'_{U^c} \in E(r_0)^c$.*

We are now ready to prove Lemma 4.1.4.

Proof of Lemma 4.1.4. Parameters δ' , h and r are chosen according to Lemma 4.3.3. Because it requires $\text{Deter}(r, h, \delta')$, the event $F(r, h, \delta')$ does not depend on $\mathcal{N}_{B(r)}$. So, by Lemma 4.3.3,

$$\mathbb{P}(F(r, h, \delta') | \mathcal{N}_{B(r)} = \eta_{B(r)}) = \mathbb{P}(F(r, h, \delta')) \geq \varepsilon .$$

The simplifying assumption $u = 0$ made at the beginning of the section is justified here: conditionally to $\mathcal{N}_{B(r)}$, the angle u (of z_1 and z_2) becomes deterministic and, by isotropy of the model, the conditional probability of F does not depend on the angle of z_2 .

Thus, for $r \in \mathcal{H}(\delta)$, $r > r_0$, and $0 < \delta' < \delta/2$, we can apply Lemma 4.3.4:

$$\begin{aligned} \varepsilon \leq \mathbb{P}((\eta'_{U^c}) \in E(r_0)^c | \mathcal{N}_{B(r)} = \eta_{B(r)}) &\leq \mathbb{P}((\emptyset, \eta'_{U^c}) \in E(r_0)^c | \mathcal{N}_{B(r)} = \eta_{B(r)}) \\ &= \mathbb{P}(\eta'_U = \emptyset, \eta' \in E(r_0)^c | \mathcal{N}_{B(r)} = \eta_{B(r)}) \\ &\leq \mathbb{P}(\eta' \in E(r_0)^c | \mathcal{N}_{B(r)} = \eta_{B(r)}) . \end{aligned}$$

In conclusion $\mathbb{P}(E(r_0) | \mathcal{N}_{B(r)} = \eta_{B(r)})$ is smaller than $1 - \varepsilon$ for any $r \in \mathcal{H}(\delta)$ and $r > r_0$. \square

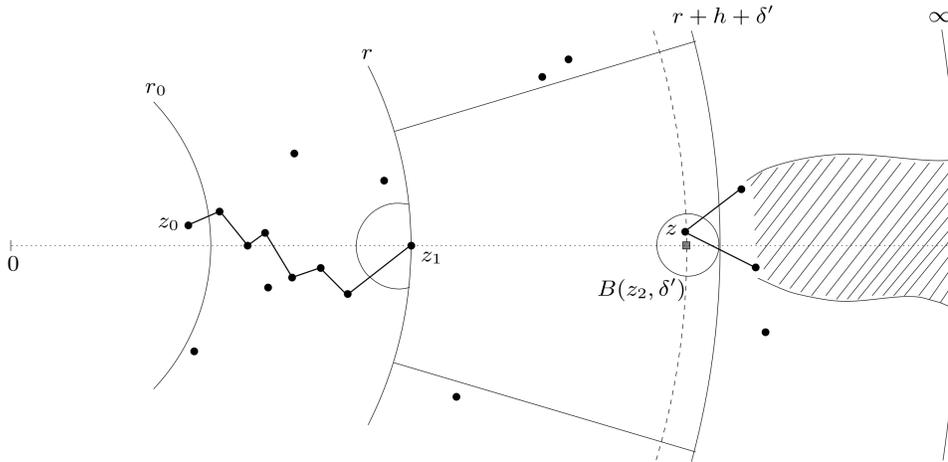


Figure 4.1: This picture represents the geometric construction elaborated in Section 4.3. Black circles are Poisson points. The ball $B(z_2, \delta')$ whose (deterministic) center z_2 is marked by a grey square, contains only one Poisson point, namely z . Its set of descendants $\mathcal{D}(z)$ is represented by the hatched region: on the picture, it satisfies $\text{Int}(\mathcal{D}^\infty(z)) \neq \emptyset$ (the bold black curve). When the set U is emptying of Poisson points then $A(z) = z_1$ which will contradict $E(r_0)$.

This section ends with the proof of Lemma 4.3.4.

Proof of Lemma 4.3.4. Consider a configuration η' which is equal to η inside the ball $B(r)$ and which belongs to the event $F(r, h, \delta')$. Let us first assume that, for the configuration η'_{U^c} , the Poisson point z (whose existence is given by F) is connected to z_1 , i.e.

$$A(z)[\eta'_{U^c}] = z_1. \quad (4.3.5)$$

Let us prove that $\eta'_{U^c} \in E(r_0)^c$. Since $\eta' \in F(r, h, \delta')$, the set of descendants $\mathcal{D}(z)[\eta']$ of z is included in $B(r + h + \delta')^c$. So, removing Poisson points of η'_{U^c} does not modify edges $(z', A(z'))$ of the RST, for $A(z') \in \mathcal{D}(z)$, and may only add new descendants. Hence, $\mathcal{D}(z)[\eta']$ is included in $\mathcal{D}(z)[\eta'_{U^c}]$ which leads to $\text{Int}(\mathcal{D}^\infty(z))[\eta'_{U^c}] \neq \emptyset$ since $\text{Int}(\mathcal{D}^\infty(z))[\eta'] \neq \emptyset$. Finally, using $A(z)[\eta'_{U^c}] = z_1$, we get $\text{Int}(\mathcal{D}^\infty(z_1))[\eta'_{U^c}] \neq \emptyset$ and the same holds for z_0 since z_1 is a descendant of z_0 . In conclusion, under the assumption (4.3.5), we have proved that $\eta'_{U^c} \in E(r_0)^c$.

It then remains to show (4.3.5). Let R be the hyperbolic distance between z and z_1 . It is sufficient to prove that $B^+(z, R)$ is included in $U \cup B^+(z_1, \delta)$ since, after removing Poisson points in U , z_1 would become the closest Poisson point to z . Here we use that $r \in \mathcal{H}(\delta)$, i.e. the ball $B^+(z_1, \delta)$ contains no other Poisson points except z_1 . Let us first remark that

$$R = d(z, z_1) \leq d(z, z_2) + d(z_2, z_1) \leq \delta' + h \leq h + 1$$

with $\delta' < 1$. So, taking $M \geq h + 1$ gives $B^+(z, R) \subset \text{Cone}(0, Me^{-r})$. Hence, it remains to prove that the set $V = B^+(z, R) \cap B(r)$ is included in $B^+(z_1, \delta)$. To do it, let us introduce v_1, v_2 and v_3 as the intersection points between the geodesic $(0z)$ and resp. the sphere with radius $r + h$ (and center 0), the sphere with radius r (and center 0) and the half-sphere of $B^+(z, R)$. Let us denote by w the symmetric of z_1 w.r.t. the geodesic $(0z)$. With this notations, V is a spherical cap whose basis is an (Euclidean) ball with (Euclidean) diameter $[w, z_1]$ and top point v_3 . We have $d(v_2, z_1) \leq d(v_1, z_2) \leq \delta'$. Thus, by symmetry, $d(w, z_1) \leq d(w, v_2) + d(v_2, z_1) = 2d(v_2, z_1) \leq 2\delta'$. Since v_3, v_2 and z are on the same geodesic, we can write:

$$d(v_3, v_2) = d(v_3, z) - d(z, v_2) \leq R - h$$

since v_3 belongs to $B^+(z, R)$. With $R \leq h + \delta'$, we get $d(v_3, z_1) \leq d(v_3, v_2) + d(v_2, z_1) \leq 2\delta'$. Whenever $\delta' < \delta/2$, the three points w , z_1 and v_3 are inside $B^+(z_1, \delta)$ which forces the spherical cap V to be included in $B^+(z_1, \delta)$ too. \square

4.4 Proofs of additional lemmas

Lemma 4.3.1 does not occur in the Euclidean case, since, in the Euclidean case, the probability that a typical point at level r belongs to an infinite branch tends to 0 as $r \rightarrow \infty$.

Proof of Lemma 4.3.1. Step 1. Let $r > 0$. By Cauchy-Schwarz inequality,

$$\mathbb{E}[\#\{z \in \mathcal{L}_r, \text{Vol}(\mathcal{D}^\infty(z)) > 0\}] \geq \frac{\mathbb{E}[\sum_{z \in \mathcal{L}_r} \text{Vol}(\mathcal{D}^\infty(z))]^2}{\mathbb{E}[\sum_{z \in \mathcal{L}_r} \text{Vol}(\mathcal{D}^\infty(z))^2]}. \quad (4.4.1)$$

By [6, Theorem 1.1], $\partial\mathbb{H}^{d+1} = \bigcup_{z \in \mathcal{L}_{r_0}} \mathcal{D}_{r_0}^\infty(z)$, so

$$\mathbb{E}\left[\sum_{z \in \mathcal{L}_{r_0}} \text{Vol}(\mathcal{D}_{r_0}^\infty(z))\right] \geq c \quad (4.4.2)$$

for some $c > 0$ independent of r_0 . We now upper-bound the denominator in (4.4.1). For any $z \in \mathcal{L}_{r_0}$, let us upper-bound $\text{Vol}(\mathcal{D}_{r_0}^\infty(z))$ using $\text{MBD}_{r_0}^\infty(z)$:

$$\text{Vol}(\mathcal{D}^\infty(z)) \leq C \text{MBD}_{r_0}^\infty(z)^d \quad (4.4.3)$$

for some $C > 0$ independent of r_0 . Thus

$$\sum_{z \in \mathcal{L}_{r_0}} \text{Vol}(\mathcal{D}_{r_0}^\infty(z))^2 \leq C \sum_{z \in \mathcal{L}_{r_0}} \text{MBD}_{r_0}^\infty(z)^{2d}. \quad (4.4.4)$$

Let us now apply the upper-bound of angular deviations: [6, Proposition 2.6] applied with $A = \pi e^{r_0}$ and $p = 2d$ gives,

$$\mathbb{E}\left[\sum_{z \in S_{r_0}} \text{MBD}_{r_0}^\infty(z)^{2d}\right] \leq C e^{-dr_0} \quad (4.4.5)$$

for some $C > 0$ independent of r_0 . For any $z \in \mathcal{L}_{r_0}$,

Finally, combining (4.4.1), (4.4.2), (4.4.4) and (4.4.5) gives,

$$\frac{\mathbb{E}[\#\{z \in \mathcal{L}_{r_0}, \text{Vol}(\mathcal{D}^\infty(z)) > 0\}]}{\mathbb{E}[\#\mathcal{L}_{r_0}]} \geq \frac{ce^{dr_0}}{\mathbb{E}[\#\mathcal{L}_{r_0}]} \geq c \quad (4.4.6)$$

for $c > 0$ independent of r_0 , where inequality [6, Lemma 4.4] has been used in the last inequality.

Step 2. Conclusion. Let $c > 0$ satisfying (4.4.6). We chose $A > 0$ large enough such that, uniformly in $r > 0$ for r large enough. If $\delta > 0$ is small enough,

$$\frac{\mathbb{E}[\#\{z \in \mathcal{L}_r, z^\dagger \notin B(r+A-\delta) \setminus B(r+\delta) \text{ and } \text{Vol}(\mathcal{D}^\infty(z)) > 0\}]}{\mathbb{E}[\#\mathcal{L}_r]} \leq c/2. \quad (4.4.7)$$

We have

$$\mathbb{E}[\#\mathcal{L}_r] \sim C e^{dr}, \quad \mathbb{E}[\#\mathcal{N} \cap B(r+A) \setminus B(r)] = \lambda \text{Vol}(B(r+A) \setminus B(r)) \sim e^{dr}. \quad (4.4.8)$$

for some $C > 0$. So, for $r > 0$ large enough,

$$\frac{\mathbb{E}[\#\{z \in S(r+A) \setminus B(r), \text{Vol}(\mathcal{D}^\infty(z)) > 0\}]}{\mathbb{E}[\#\mathcal{N} \cap B(r+A) \setminus B(r)]} \geq c' \frac{\mathbb{E}[\#\{z \in \mathcal{L}_{r_0}, z^\dagger \in B(r-A) \text{ and } \text{Vol}(\mathcal{D}^\infty(z)) > 0\}]}{\mathbb{E}[\#\mathcal{L}_{r_0}]} \geq c' c / (4.4.9)$$

with $c' > 0$ uniform in r . By Campbell formula,

$$\mathbb{E}[\#\{z \in S(r+A) \setminus B(r), \text{Vol}(\mathcal{D}^\infty(z)) > 0\}] = \lambda \int_{S(r+A) \setminus B(r)} \mathbb{P}[\text{Vol}(\mathcal{D}^\infty(z)) > 0 | z \in \mathcal{N}] dz \quad (4.4.10)$$

Let $\delta > 0$. By Campbell formula and Fubini,

$$\begin{aligned} & \int_{S(r+A) \setminus B(r)} \mathbb{E}[\#\{z' \in B(z, \delta), \text{Vol}(\mathcal{D}^\infty(z)) > 0\}] dz \\ &= \lambda \int_{S(r+A) \setminus B(r)} \int_{B(z, \delta)} \mathbb{P}[\text{Vol}(\mathcal{D}(z')) > 0 | z' \in \mathcal{N}] dz' dz \\ &= \lambda \int_{z' \in B(r+A+\delta) \setminus B(r-\delta)} \text{Vol}(B(z', \delta) \cap B(r+A) \setminus B(r)) \mathbb{P}[\text{Vol}(\mathcal{D}(z')) > 0 | z' \in \mathcal{N}] dz' \\ &\geq \lambda \text{Vol}(B(0, \delta)) \int_{z' \in B(r+A-\delta) \setminus B(r+\delta)} \mathbb{P}[\text{Vol}(\mathcal{D}(z')) > 0 | z' \in \mathcal{N}] dz' \\ &\geq \text{Vol}(B(0, \delta)) \mathbb{E}[\#\{z \in S(r+A-\delta) \setminus B(r+\delta), \text{Vol}(\mathcal{D}^\infty(z)) > 0\}] \geq c \text{Vol}(B(r+A-\delta) \setminus B(r+\delta)). \end{aligned}$$

for some $c > 0$ uniform in r, δ . Thus there exists $z \in B(r+A-\delta) \setminus B(r+\delta)$ such that $\mathbb{E}[\#\{z' \in B(z, \delta), \text{Vol}(\mathcal{D}^\infty(z)) > 0\}] \geq c \text{Vol}(0, \delta)$. By rotation invariance, there exists some radius $r' \in [r+\delta, r+A-\delta]$ such that for any $z \in S(r')$, $\mathbb{E}[\#\{z' \in B(z, \delta), \text{Vol}(\mathcal{D}^\infty(z)) > 0\}]$ which concludes the proof of Lemma 4.3.1. \square

Proof of Lemma 4.3.2. Let $A > 0$ that will be chosen later. Let $r, h > 0$ and $z \in S(r+h)$. Let us denote by χ the union of descending subtrees of every Poisson point in $B(z, \delta')$:

$$\chi := \bigcup_{z' \in \mathcal{N} \cap B(z, \delta')} D(z').$$

We have

$$\text{Deter}(r, h, z) \subset \{\exists z' = (r', u') \in \chi, d(z', (A(z'))) \geq r' - r + h\}.$$

Thus

$$\text{Deter}(r, h, z) \subset \{\exists z' = (r', u') \in \text{Cone}(u', Ae^{-r-h}), d(z', (A(z'))) \geq r' - r + h\} \cup \{\chi \not\subset \text{Cone}(u', Ae^{-r-h})\}$$

It will be shown that

$$\{\exists z' = (r', u') \in \mathcal{N} \cap \text{Cone}(u', Ae^{-r-h}), d(z', (A(z'))) \geq r' - r + h\} \leq CA^d \int_{r+h}^{\infty} e^{-d(r-r'+h)} e^{-\lambda c e^{(r'-r+h)/3}} d\lambda \quad (4.4.11)$$

and

$$\mathbb{P}[\{\chi \not\subset \text{Cone}(u, Ae^{-r-h})\}] \leq \frac{C}{A}. \quad (4.4.12)$$

where $C > 0$ is some constant independent of r, h, z . Let us prove (4.4.11). The conclusion follows from (4.4.11) and (4.4.12): for any $A > 0$,

$$\lim_{h \rightarrow \infty} \sup_{r > 0} \mathbb{P}[\text{Deter}(r, h, \cdot)] \leq \frac{C}{A}.$$

Since this is true for any $A > 0$, the conclusion follows. By Campbell formula,

$$\begin{aligned}
& \mathbb{P} [\exists z' = (r', u') \in \text{Cone}(u', Ae^{-r-h}), d(z', (A(z'))) \geq r' - r + h] \\
& \leq \mathbb{E} [\#\{z' = (r', u') \in \text{Cone}(u, Ae^{-r-h}), d(z', (A(z'))) \geq r' - r + h\}] \\
& = \lambda \int_{\text{Cone}(u, Ae^{-r-h})} \mathbb{P} [B^+(z', d(0, z') - r + h) dz' \cap \mathcal{N} = \emptyset] \\
& = \lambda \int_{r' \geq r+h} \int_{u' \in \mathbb{S}^d, \widehat{u'u} \leq Ae^{-r-h}} \mathbb{P} [B^+(z', r' - r + h) dz' \cap \mathcal{N} = \emptyset] \sinh(r')^d du' dr' \text{ by [6, (2.5)]} \\
& \leq C \int_{r+h}^{\infty} A^d e^{-d(r-r'+h)} e^{-\lambda \text{Vol}(B^+(z', r'-r+h))} dr' \\
& \leq C \int_{r+h}^{\infty} A^d e^{-d(r-r'+h)} e^{-\lambda ce^{(r'-r+h)/3}} dr'.
\end{aligned}$$

We now prove (4.4.12). We use the control of angular deviations established in [6, Proposition 2.6], with $p = 2d$:

$$\mathbb{E} \left[\sum_{z'' \in \mathcal{L}_{r+h-\delta}} \text{CFD}_{r+h-\delta}^{\infty}(z'')^{2d} \right] \leq C e^{-d(h+r-\delta)}.$$

where we recall that $\text{CFD}_r^{\infty}(z)$ is the cumulative forward angular deviations defined in [6, Definitions 2.3] or Definition 3.2.3 page 85 of Chapter 3.

In addition, for the maximal backward angular deviations $\text{MBD}_r^{\infty}(z)$ defined in [6, Definitions 2.4] (see also Definition 3.2.4 page 85 of Chapter 3),

$$\begin{aligned}
& \sum_{z'=(r';u') \in (\mathcal{N} \cap S(r+h) \setminus S(r+h-\delta))} \text{MBD}_{r'}^{\infty}(z')^d \tag{4.4.13} \\
& \leq \sum_{z'=(r';u') \in (\mathcal{N} \cap S(r+h) \setminus S(r+h-\delta))} \text{MBD}_{r'}^{\infty}(\mathcal{A}_{r+h-\delta}^{r'}(z'))^d \\
& \leq \sum_{z'' \in \mathcal{L}_{r+h-\delta}} \#\{z' \in \mathcal{N} \cap (S(r+h) \setminus S(r+h-\delta)) \cap D(z'')\} \text{MBD}_{r'}^{\infty}(z')^d,
\end{aligned}$$

thus Cauchy-Schwarz gives,

$$\begin{aligned}
& \mathbb{E} \left[\sum_{z'' \in \mathcal{L}_{r+h-\delta}} \text{MBD}_{r'}^{\infty}(z')^d \right] \tag{4.4.14} \\
& \leq \mathbb{E} \left[\sum_{z'=(r';u') \in (\mathcal{N} \cap S(r+h) \setminus S(r+h-\delta))} \#\{z' \in \mathcal{N} \cap (S(r+h) \setminus S(r+h-\delta)) \cap D(z'')\}^2 \right]^{1/2} \\
& \times \mathbb{E} \left[\sum_{z'=(r';u') \in (\mathcal{N} \cap S(r+h) \setminus S(r+h-\delta))} \text{MBD}_{r'}^{\infty}(\mathcal{A}_{r+h-\delta}^{r'}(z'))^{2d} \right]^{1/2}.
\end{aligned}$$

Thus, combining (4.4.13) and (4.4.14) and [6, Proposition 2.6]

$$\mathbb{E} \left[\sum_{z'' \in \mathcal{L}_{r+h-\delta}} \text{MBD}_{r'}^{\infty}(z')^d \right] \leq C e^{-2d(r+h)}. \tag{4.4.15}$$

□

Lemma 4.4.1. *For any $\delta' > 0$ small enough, the following holds uniformly on $r, h > 0$:*

$$\max \left\{ \mathbb{P}(\exists z \in B(z_2, \delta'), \mathcal{D}(z) \not\subset B(r+h+\delta')^c), \mathbb{E}[\#\mathcal{N} \cap B(z_2, \delta') \mathbf{1}_{\#\mathcal{N} \cap B(z_2, \delta') \geq 2}] \right\} \leq \frac{c_0}{10} \text{Vol}(B(\delta')).$$

Proof. We have, for δ' small enough:

$$\mathbb{E}[\#\mathcal{N} \cap B(z_2, \delta') \mathbf{1}_{\#\mathcal{N} \cap B(z_2, \delta') \geq 2}] \leq \frac{c_0}{10} \text{Vol}(B(\delta')).$$

Let $z \in B(z_2, \delta')$. By Campbell formula,

$$\begin{aligned} & \mathbb{E}[\#\{z' \in \mathcal{N}, A(z') = z \text{ and } z' \notin B(r+h+\delta)\}] \\ &= \lambda \int_{B(r+h+\delta) \setminus B(r+h)} \mathbb{P}[A(z') = z | z' \in \mathcal{N}] dz' \\ &= \lambda \int_{B(r+h+\delta) \setminus B(r+h)} \mathbb{P}[B^+(z', d(z, z')) \cap \mathcal{N} = \emptyset] dz' \\ &= \lambda \int_{B(r+h+\delta) \setminus B(r+h)} \exp(-\lambda \text{Vol}(B^+(z', d(z, z')))) dz' \end{aligned}$$

It has been established in [6, Proposition 3.17] thus $\text{Vol}(B^+(z', \rho)) \geq Ce^{-\rho/3}$ for some $C > 0$ when ρ is large enough. Thus

$$\begin{aligned} & \mathbb{E}[\#\{z' \in \mathcal{N}, A(z') = z \text{ and } z' \notin B(r+h+\delta)\}] \\ &= \lambda \int_{B(r+h+\delta) \setminus B(r+h)} \exp(-\lambda \text{Vol}(B^+(z', d(z, z')))) dz' \\ &\leq \lambda \int_{B(r+h+\delta) \setminus B(r+h)} \exp(-C\lambda/3d(z, z')) dz' \\ &\leq C \int_{\rho \in \mathbb{R}_+} ((\text{Vol}(B(r+h+\delta) \setminus B(r+h)) \cap B(z, \rho)) \exp(-C\lambda\rho/3)) d\rho \text{ by Fubini(4.4.16)} \\ &\leq C\delta. \end{aligned} \tag{4.4.17}$$

Then, again by Campbell formula,

$$\begin{aligned} & \mathbb{P}(\exists z \in B(z_2, \delta'), \mathcal{D}(z) \not\subset B(r+h+\delta')^c) \\ &\leq \mathbb{E}[\#\mathcal{N} \cap B(z_2, \delta'), \exists z' \in B(r+h+\delta') \setminus B(r+h), A(z') = z] \\ &= \lambda \int_{z \in B(z_2, \delta')} \mathbb{P}[\exists z' \in \mathcal{N}, A(z') = z \text{ and } z' \notin B(r+h+\delta)] dz \\ &\leq \lambda \int_{z \in B(z_2, \delta')} \mathbb{E}[\#\{z' \in \mathcal{N}, A(z') = z \text{ and } z' \notin B(r+h+\delta)\}] dz \\ &\leq C\delta^{d+1} \text{ by (4.4.16)} \\ &= o(\text{Vol}(z_2, \delta')) \text{ when } \delta' \rightarrow 0. \end{aligned}$$

This concludes the proof of Lemma 4.4.1. \square

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English Summary

In this thesis, two models of geometric spanning random trees in hyperbolic geometry are studied: the hyperbolic DSF (*Directed Spanning Forest*) and the hyperbolic RST (*Radial Spanning Tree*). These are random graphs constructed on Poisson point processes in the hyperbolic space \mathbb{H}^d of dimension $d \geq 2$ and which extend the definition of the DSF and the RST proposed in the Euclidean setting by Baccelli and Bordenave (2007). The study and properties of these objects are, in general, fundamentally different between the two geometries.

The DSF in \mathbb{R}^d is defined as follows. Any point x of the Poisson process is linked to its ancestor, defined as the closest point of the Poisson process among all those lying in the half-space bounded by the hyperplane passing through x and of chosen normal direction. To define the hyperbolic DSF in \mathbb{H}^d , we give ourselves a point at infinity \mathcal{I} (belonging to the boundary $\partial\mathbb{H}^d$). Each point x is connected to the nearest point of the Poisson process among all those which are closer to \mathcal{I} than x (with respect to horodistances). The RST is a radial tree which can be defined in both \mathbb{R}^d and \mathbb{H}^d , relative to an origin O . There are several possible definitions of the hyperbolic RST in \mathbb{H}^d , but one is naturally associated with the hyperbolic DSF and this is the one we will study.

After an introduction in Chapter 1, recalling the motivations and presenting the main tools used in this thesis, the hyperbolic DSF is studied in Chapter 2. We describe precisely its properties, which differ from the Euclidean case. In particular, whatever the dimension, the hyperbolic DSF is a tree (all trajectories coalesce) that contains an infinite number of bi-infinite branches, of which we study the asymptotic directions. Chapters 3 and 4 deal with the hyperbolic RST and the study of its semi-infinite branches, in dimensions greater than or equal to $d = 2$. Almost certainly, each semi-infinite branch has an asymptotic direction and each angle is the asymptotic direction of at least one semi-infinite branch. The semi-infinite trajectory of the RST that converges to a given deterministic direction is almost certainly unique. However, the set of (random) directions that are the boundary directions of several semi-infinite RST trajectories is dense in $\partial\mathbb{H}^d$ and countable in dimension $d = 2$. We show that the subset of $\partial\mathbb{H}^d$ which consists of the asymptotic directions reached by the semi-infinite trajectories of the RST passing through a Poisson point z , has non-empty interior. In dimension $d = 2$, this means that there is almost certainly no random direction that is the limit of three semi-infinite branches of the RST. This result, often conjectured for radial geometric tree models, has only been proved to our knowledge in one previous case (for a last-pass percolation model).

Key words: continuum percolation, hyperbolic space, stochastic geometry, random geometric tree, Directed Spanning Forest, Mass Transport Principle, Poisson point processes.

Résumé

Dans cette thèse, deux modèles d'arbres aléatoires géométriques couvrants en géométrie hyperbolique sont étudiés : la DSF (*Directed Spanning Forest*) et le RST (*Radial Spanning Tree*). Il s'agit de graphes aléatoires construits à partir de processus ponctuels de Poisson dans l'espace hyperbolique \mathbb{H}^d de dimension $d \geq 2$ et qui étendent la définition de la DSF et du RST proposée en Euclidien par Baccelli et Bordenave (2007). L'étude et les propriétés de ces objets sont, de manière générale, fondamentalement différentes entre les deux géométries.

La DSF dans \mathbb{R}^d est définie de la façon suivante. Tout point x du processus de Poisson est relié à son ancêtre, défini comme étant le point du processus de Poisson le plus proche de x parmi tous ceux qui se trouvent dans le demi-espace délimité par l'hyperplan passant par x et de normale une direction choisie. Pour définir la DSF dans \mathbb{H}^d , on se donne un point à l'infini \mathcal{I} (appartenant au bord $\partial\mathbb{H}^d$). Chaque point x est relié au point du processus de Poisson le plus proche parmi tous ceux qui sont plus proches de \mathcal{I} que x (au sens de l'horodistance). Le RST est un arbre radial qui peut quant à lui être défini à la fois dans \mathbb{R}^d et \mathbb{H}^d , par rapport à un point origine O . Il existe plusieurs définitions possible du RST dans \mathbb{H}^d , mais l'une est naturellement associée à DSF hyperbolique et c'est celle que nous étudierons.

Après une introduction au Chapitre 1, rappelant les motivations et présentant les principaux outils utilisés dans cette thèse, la DSF hyperbolique est étudiée au Chapitre 2. Nous décrivons précisément ses propriétés, qui diffèrent du cas Euclidien. En particulier, quelle que soit la dimension, la DSF hyperbolique est un arbre (toutes les trajectoires coalescent) qui contient une infinité de branches bi-infinies, dont nous étudions les directions asymptotiques. Les Chapitres 3 et 4 concernent le RST hyperbolique et l'étude de ses branches semi-infinies, en dimensions supérieures ou égales à $d = 2$. Presque sûrement, chaque branche semi-infinie admet une direction asymptotique et chaque angle est la direction asymptotique d'au moins une branche semi-infinie. La trajectoire semi-infinie du RST qui converge vers une direction déterministe donnée est unique presque sûrement. Cependant, l'ensemble des directions (aléatoires) qui sont les directions limites de plusieurs trajectoires semi-infinies du RST est dense dans $\partial\mathbb{H}^d$ et dénombrable en dimension $d = 2$. Nous montrons que le sous-ensemble de $\partial\mathbb{H}^d$ qui est constitué des directions asymptotiques atteintes par les trajectoires semi-infinies du RST passant par un point de Poisson z , est d'intérieur non vide. En dimension $d = 2$, cela signifie que presque sûrement, il n'existe pas de direction aléatoire qui soit limite de trois branches semi-infinies du RST. Ce résultat, souvent conjecturé pour les modèles d'arbres géométriques radiaux, n'avait été démontré à notre connaissance que dans un seul cas (pour un modèle de percolation de dernier passage).

Mots clés : percolation, géométrie hyperbolique, géométrie stochastique, arbre aléatoire, forêt couvrante dirigée, forêt couvrante radiale, branches semi-infinies, transport de masse, processus ponctuel de Poisson.