The Teukolsky equation on subextremal Kerr spacetimes

Pascal Millet

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THÈSE

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L'équation de Teukolsky sur les espaces-temps de Kerr sous-extrémaux

The Teukolsky equation on subextremal Kerr spacetimes

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Résumé

Dans cette thèse, nous étudions l'équation de Teukolsky sur les espaces-temps de Kerr sous-extrémaux. Ces derniers sont des solutions explicites de l'équation d'Einstein dans le vide qui décrivent des trous noirs éternels en rotation dans un univers par ailleurs vide. L'équation de Teukolsky est issue de l'étude d'équations d'onde tensorielles sur ces espaces-temps telles que les équations de Maxwell et les équations de la gravité linéarisée. Il s'agit d'une équation aux dérivées partielles hyperbolique scalaire linéaire qui gouverne certaines composantes particulières du champ. Dans le cas de la gravité linéarisée, l'analyse de cette équation joue un role central dans l'étude de la stabilité linéaire et non linéaire des trous noirs de Kerr. Nous donnons dans un premier temps une description détaillée des notions géométriques nécessaires pour comprendre d'où vient l'équation de Teukolsky. Nous introduisons les fibrés des spineurs, les formalismes de Newman-Penrose et GHP (Geroch–Held–Penrose), les fonctions à poids spinoriels et l'opérateur de Teukolsky et nous montrons comment certaines équations d'ondes tensorielles se découplent pour certaines composantes spéciales (appelées scalaires de Teukolsky). Nous étudions ensuite le comportement en temps long des solutions de l'équation de Teukolsky avec des données initiales régulières et localisées. Quand ces dernières sont à support compact, nous prouvons que la solution admet un terme principal en temps long et nous le calculons explicitement. Si les données initiales ont seulement une décroissance polynomiale inverse, nous obtenons une borne polynomiale inverse par rapport au temps pour la solution. Une force de ces résultats est qu'ils ne sont pas limités au cas des trous noirs de Kerr à rotation lente mais sont valables dans l'intervalle sous-extrémal tout entier des paramètres du trou noir ($|a| < M$). Cette analyse est fondée sur des avancées récentes en analyse spectrale et microlocale dans le contexte de la relativité générale et requiert une description précise de la résolvante à basse énergie. Dans l'appendice de la thèse, nous fournissons des notes à propos de la théorie hyperbolique standard pour les opérateurs différentiels. En particulier nous présentons des résultats d'existence, d'unicité et d'approximation pour les solutions d'équations différentielles hyperboliques d'ordre deux sur des fibrés vectoriels. Nous fournissons également une introduction à l'utilisation des méthodes microlocales pour l'obtention d'estimées Fredholm au travers d'une application à un exemple unidimensionnel simple avec un minimum d'outils techniques.

Abstract

In this thesis, we study the Teukolsky equation on subextremal Kerr spacetimes. The latter are explicit solutions of the vacuum Einstein equations which describe eternal rotating black holes in an otherwise empty universe. The Teukolsky equation arise from the study of tensorial wave equations on these spacetimes such as the Maxwell equations and the equations of linearized gravity. It is a scalar linear hyperbolic partial differential equation which governs the behavior of some geometrically meaningful components of the field. In the case of linearized gravity, the analysis of this equation plays a central role in the study of the linear and non linear stability of Kerr black holes. We first provide a detailed description of the geometric notions necessary to understand where the Teukolsky equation comes from. We introduce the spinor bundles, the Newman-Penrose and GHP (Geroch–Held–Penrose) formalisms, the spin weighted functions and the Teukolsky operator and we show how the tensorial wave equations decouple for some special components (called the Teukolsky scalars). We then study the large time behavior of the solutions of the Teukolsky equation with regular and localized initial data. When the latter are compactly supported, we prove that the solution admits a large time leading order term and we compute it explicitly. If initial data only have an inverse polynomial decay, we obtain an inverse polynomial bound in time for the solution. A strength of these results is that they are not limited to slowly rotating Kerr black holes but are valid in the whole subextremal range of black holes parameters. This analysis is based on recent advances in microlocal and spectral analysis in the context of general relativity and requires a precise description of the resolvent operator at low energy. In the appendix of the thesis, we provide some notes about the standard theory for hyperbolic differential operators. In particular we present results of existence, uniqueness and approximation for solutions of hyperbolic differential equations of order two on vector bundles. We also provide an introduction to the use of microlocal methods to get Fredholm estimates through its application to a simple one dimensional example with minimal technical tools.

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Part I

General introduction

Chapter 1

Introduction en français

1.1 Les trous noirs de Kerr et l'équation de Teukolsky

Le sujet principal de cette thèse est l'analyse de l'équation de Teukolsky sur des espaces-temps de Kerr sous-extrémaux. Les espaces-temps de trous noirs sont des solutions explicites de l'équation d'Einstein dans le vide $\mathrm{Ric}(g) = 0$ où g est une métrique Lorentzienne et $\mathrm{Ric}(g)$ est le tenseur de Ricci. Historiquement, la première solution explicite non triviale fut découverte par Schwarzschild (voir [93]). Elle a une symétrie sphérique et décrit un trou noir immobile dans un univers asymptotiquement plat. Une famille plus générale de solutions, qui inclut les trous noir en rotation a été introduite par Kerr dans [59]. Ces solutions sont les objets centraux de cette thèse. Chaque solution est indexée par deux paramètres: M, la masse du trou noir et a, son moment angulaire par unité de masse. Les trous noirs physiques sont modélisés par une solution de Kerr sous-extrémale ($|a| < M$). La métrique de Kerr, définie sur la variété $\mathcal{M} := \mathbb{R}_t \times (r_+, +\infty)_r \times \mathbb{S}^2$ avec $r_+ = M + \sqrt{M^2 - a^2}$, a l'expression suivante en coordonnées de Boyer-Lindquist (t, r, θ, ϕ):

$$\begin{aligned} g_{M,a} =& \frac{\Delta_r - a^2 \sin^2\theta}{\rho^2}\, \mathrm{d}t^2 + \frac{4Mar\sin^2\theta}{\rho^2}\, \mathrm{d}t\, \mathrm{d}\phi - \frac{\rho^2}{\Delta_r}\, \mathrm{d}r^2 \\ & - \rho^2\, \mathrm{d}\theta^2 - \frac{\sin^2\theta}{\rho^2}((a^2+r^2)^2 - a^2\Delta_r \sin^2\theta)\, \mathrm{d}\phi^2 \\ \Delta_r :=& a^2 + r^2 - 2Mr \\ \rho^2 :=& r^2 + a^2\cos^2\theta \end{aligned}$$

Nous soulignons quelques unes de ses caractéristiques géométriques principales qui ont une importance particulière pour l'étude des ondes se propageant sur l'espace-temps de Kerr:

- Symétrie axiale (le champ de vecteurs ∂_ϕ est de Killing).
- Invariance par translations temporelles (le champ de vecteurs ∂_t est de Killing).
- La trivialité à l'infini: $g_{M,a} = \mathrm{d}t^2 - \mathrm{d}r^2 - r^2(\mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\phi^2) + O(r^{-1})$.
- L'existence de géodésiques isotropes confinées dans une région spatiale compacte à l'extérieur du trou noir (trajectoires captées).

Notons également que Δ_r s'annule quand $r = r_+$. Cependant, la singularité de la métrique en $r = r_+$ est due à un mauvais choix de coordonnées et peut être effacée en introduisant les

nouvelles coordonnées Kerr$_*$:

$$(t_*, r, \theta, \phi_*) = (t + T(r), r, \theta, \phi + A(r))$$

où

$$T(r) := \int_{3M}^{r} \frac{a^2 + r^2}{\Delta_r} \, \mathrm{d}r$$

et

$$A(r) := \int_{3M}^{r} \frac{a}{\Delta_r} \, \mathrm{d}r.$$

Dans ces coordonnées, la métrique g s'écrit:

$$\begin{aligned} g =& \frac{\Delta_r - a^2 \sin^2\theta}{\rho^2} \, \mathrm{d}t_*^2 - 2 \, \mathrm{d}t_* \, \mathrm{d}r + \frac{4Mar \sin^2\theta}{\rho^2} \, \mathrm{d}t_* \, \mathrm{d}\phi_* + 2a \sin^2\theta \, \mathrm{d}r \, \mathrm{d}\phi_* - \rho^2 \, \mathrm{d}\theta^2 \\ & - \frac{\sin^2\theta}{\rho^2}((a^2 + r^2)^2 - a^2 \Delta_r \sin^2\theta) \, \mathrm{d}\phi_*^2. \end{aligned}$$

Au vu de cette expression, il apparait que la métrique peut être étendue analytiquement à une variété plus grande. Il sera suffisant pour nous de considérer une extension de $\mathcal{M}$ définie par $\mathcal{M}_\epsilon := \mathbb{R}_{t_*} \times (r_+ - \epsilon, +\infty) \times \mathbb{S}^2_{\theta,\phi_*}$ pour un petit $\epsilon > 0$. Nous munissons $\mathcal{M}_\epsilon$ de l'extension analytique de g. L'horizon futur du trou noir est par définition $\mathfrak{H} := \mathbb{R}_{t_*} \times \{r_+\} \times \mathbb{S}^2_{\theta,\phi_*}$. Il sépare la région extérieure (correspondant à $r > r_+$) de la région intérieure $r < r_+$.

Les solutions de Kerr ont joué un role majeur dans le développement de la théorie de la relativité générale car elles sont utilisées pour modéliser le champ gravitationnel à l'extérieur d'une étoile ou d'un trou noir en rotation. Par conséquent, il est crucial de comprendre comment les champs physiques classiques (incluant les champs scalaires, les champs de Dirac, les champs de Maxwell et les ondes gravitationnelles) se comportent en présence de ce champ gravitationnel. Le cadre le plus simple est de considérer qu'ils sont suffisamment faibles pour que leur influence gravitationnelle puisse être négligée. Dans ce cadre, la géométrie sous-jacente est données par une solution de Kerr exacte. Les champs physiques sont alors décrits par des équations aux dérivées partielles hyperboliques sur la variété Lorentzienne de Kerr. Ce champ de recherche a été très actif durant ces dernières décennies. Il est connu depuis le travail de Teukolsky [101] que plusieurs des champs physiques principaux (scalaire, Maxwell et ondes gravitationnelles) sur les espaces-temps de Kerr peuvent être étudiés en utilisant l'équation de Teukolsky. Cette équation scalaire dépend d'un paramètre $s \in \frac{1}{2}\mathbb{Z}$ et capture le comportement de certaines composantes géométriquement pertinentes des différents champs (dépendant de la valeur de s: $s = 0$ correspond à une onde scalaire, $s = \pm 1$ à Maxwell et $s = \pm 2$ aux équations d'Einstein linéarisées).

Cette réduction est particulièrement utile dans le cas des équations d'Einstein linéarisées autour d'une solution de Kerr, c'est pourquoi l'équation de Teukolsky est devenue un outil central dans l'étude de la stabilité linéaire et non linéaire des trous noirs. Cette famille de problèmes a des implications théoriques profondes car la stabilité par rapport à des petites perturbations est une propriété cruciale pour que la solution soit considérée comme physiquement pertinente. En pratique, les questions de stabilités non linéaires pour les équations d'Einstein se sont révélées extrêmement ardues. Les premiers travaux sur ce sujet sont [32] pour la solution de de Sitter et [19] (voir aussi [66]) pour la solution de Minkowski. Plus récemment, des résultats de stabilité ont été obtenus pour des solutions de trous noirs: dans [50], [47] pour Kerr-de Sitter et Kerr-Newman-de Sitter, dans [60] et [23] pour Schwarzschild

(trou noir sans rotation) et dans [63, 64, 36, 62, 94] pour la solution de Kerr à rotation lente. Tous ces résultats reposent sur une description précise de la propagation des perturbations sur l'espace-temps sous-jacent au niveau linéaire. Les résultats de stabilité linéaire incluent [22, 55, 5, 57] pour Schwarzschild, [4, 42] pour Kerr à rotation lente et [35] pour les espaces-temps de Reissner-Nordström sous extrémaux. Voir aussi le récent [45] qui utilise une approche microlocale pour prouver la stabilité linéaire des trous noirs de Kerr-Newman à rotation lente et faible charge.

1.2 Contenu de la thèse

Résumé de la partie II

L'opérateur de Teukolsky tel que défini dans [101] a l'expression suivante en coordonnées de Boyer-Lindquist:

$$\begin{aligned} T_s := &\left(\frac{(a^2+r^2)^2}{\Delta_r} - a^2\sin^2\theta\right)\partial_t^2 + \frac{4Mar}{\Delta_r}\partial_t\partial_\phi + \left(\frac{a^2}{\Delta_r} - \frac{1}{\sin^2\theta}\right)\partial_\phi^2 - \Delta_r^{-s}\partial_r\Delta_r^{s+1}\partial_r \\ &- \frac{1}{\sin\theta}\partial_\theta\sin\theta\partial_\theta - 2s\left(\frac{a(r-M)}{\Delta_r} + i\frac{\cos\theta}{\sin^2\theta}\right)\partial_\phi - 2s\left(\frac{M(a^2-r^2)}{\Delta_r} - r - ia\cos\theta\right)\partial_t \\ &+ s^2\mathrm{cotan}^2\theta - s. \end{aligned} \tag{1.1}$$

Avant de nous plonger dans l'analyse de cet opérateur, il semble utile d'aborder les questions suivantes:

- Comment l'opérateur de Teukolsky émerge-t-il de l'étude d'équations tensorielles?
- Dans quel sens est-il régulier (en effet l'expression (2.1) semble avoir des singularités)?
- Que sont les formalismes de Newman-Penrose et GHP (for Geroch–Held–Penrose) couramment utilisés dans la littérature sur l'équation de Teukolsky?

Ces questions trouvent une réponse dans la partie II, qui consiste en l'article de synthèse [79]. Le principal role de cette partie est de clarifier le contexte géométrique de l'équation de Teukolsky en rassemblant des informations éparpillées dans la littérature (voir l'introduction de la partie pour les références précises) et de fournir une base autonome pour l'analyse.

Nous introduisons la définition des fibrés de spineurs qui apparaissent naturellement dans l'étude de l'équation de Teukolsky (et qui sont indispensables lorsque le paramètre s n'est pas un entier). Nous définissons également les tétrades isotropes principales de Newman-Penrose qui sont formées d'un ensemenble de quatre champs de vecteurs isotropes $(l, n, m, \overline{m})$ avec l et n réels et m complexe qui satisfont des conditions de normalisation et tels que l et n sont principaux (voir définition 1.2.1 ci-dessous).

Cette notion a un sens plus généralement pour $\mathcal{M}$ un espace-temps Ricci-plat de type D (voir la définition 1.2.2 ci-dessous). Le tenseur de Ricci étant égal à zéro pour les solution de l'équation d'Einstein dans le vide, le tenseur de Weyl est égal au tenseur de courbure défini de la façon suivante: Pour X, Y, Z, T des champs de vecteurs lisses sur $\mathcal{M}$,

$$W(X,Y,Z,T) := g(\nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z, T)$$

où ∇ désigne la connexion de Levi-Civita.

Définition 1.2.1. *Un vecteur istrope $l \in T_x\mathcal{M}$ est appelé principal si pour tous $a, b \in T_x\mathcal{M}$ tels que $g(l,a) = g(l,b) = 0$, on a $W(l,a,l,b) = 0$ où W est le tenseur de Weyl. Nous disons que le vecteur l est principal de multiplicité au moins 2 si pour tous $a, b \in T_x\mathcal{M}$ tels que $g(l,a) = 0$, $W(l,a,l,b) = 0$. Cette caractérisation des vecteurs principaux peut-être trouvé dans [86], proposition 5.5.5. Des définitions alternatives équivalentes et des propriétés additionnelles des directions principales nulles sont également fournies dans [86].*

Définition 1.2.2. *Nous définissons un espace-temps de type de Petrov D comme un espace-temps $\mathcal{M}$ tel que pour tout $x_0 \in \mathcal{M}$, il existe des champs de vecteurs isotropes indépendants (donc ne s'annulant pas) l et n dans un voisinage U de x_0 tels que:*

- *Pour tout $x \in U$, l'ensemble des vecteurs isotropes de $T_x\mathcal{M}$ est exactement $\mathbb{R}l(x) \cup \mathbb{R}n(x)$*
- *Pour tout $x \in U$, $l(x)$ et $n(x)$ sont des vecteurs principaux isotrope de multiplicité (au moins) 2. (In fact, l and n are of multiplicity exactly 2, see [86], chapter 5).*

La propriété des directions isotrope principales sur les espaces-temps de type de Petrov D Ricci-plat que nous utiliserons est la suivante:

Proposition 1.2.3. *Si nous définissons n et l comme dans la définition précédente, ils sont pregeodesiques et sans cisaillement (shear-free). Le cisaillement d'un vecteur prégéodésique l en x_0 par rapport à une famille orthonormale X, Y de $l(x_0)^\perp$ est défini par*

$$\frac{1}{2}\left(g(\nabla_Y l, Y) - g(\nabla_X l, X)\right) + \frac{i}{2}\left(g(\nabla_Y l, X) + g(\nabla_X l, Y)\right)$$

(voir la définition 5.7.1 dans [86]).

Cette tétrade, adaptée aux propriétés algébriques particulières du tenseur de Weyl d'un espace temps de type de Petrov D, est utilisée comme base pour écrire les composantes des tenseurs dans le formalisme de Newman-Penrose et peut être associée (modulo une ambiguité de signe) à base de spineurs normalisée. La tétrade la plus couramment utilisée sur un espace-temps de Kerr est la tétrade de Kinnersley définit de la façon suivante:

$$\begin{aligned} l &= \frac{r^2 + a^2}{\Delta_r}\partial_t + \partial_r + \frac{a}{\Delta_r}\partial_\phi \\ n &= \frac{r^2 + a^2}{2\rho^2}\partial_t - \frac{\Delta_r}{2\rho^2}\partial_r + \frac{a}{2\rho^2}\partial_\phi \\ m &= \frac{ia\sin\theta}{\sqrt{2}p}\partial_t + \frac{1}{\sqrt{2}p}\partial_\theta + \frac{i}{\sqrt{2}p\sin\theta}\partial_\phi \end{aligned}$$

où l'on a introduit la fonction:

$$p = r + ia\cos\theta$$

Cependant, comme nous le verrons, il n'existe pas de tétrade isotrope principale de Newman-Penrose régulière globale sur l'extérieur de l'espace-temps de Kerr et cela explique les singularités dans l'expression (2.1). Par exemple, on peut voir que le vecteur m de la tétrade de Kinnersley n'est pas lisse sur l'axe de rotation du trou noir. Les quantités qui dépendent du choix tétrade de Newman-Penrose principale isotrope sont donc mieux représentées par des fonctions sur le fibré principal des tétrades de Newman-Penrose plutôt que par des fonctions de l'espace temps. Si elle satisfont en plus une condition de compatibilité par rapport

à l'action, elles peuvent être interprétées comme les sections d'un fibré en droites complexes associé. Cette observation conduit à la définition des fonctions à poids spinoriels et des composantes à poids spinoriels des tenseurs et des spineurs. Nous caractérisons complètement le fibré des tétrades principales isotropes de Newman-Penrose sur les espaces-temps de Kerr sous-extrémaux en calculant un système complet de trivializations locales. Il se trouve que ce fibré est intimement lié au fibré de Hopf sur la sphère.

Nous prenons ensuite un peu de temps pour des rappels à propos des connections sur les fibrés principaux et nous expliquons comment la connection de Levi-Civita sur le fibré tangent donne naturellement lieu à des connections sur les différents fibrés mentionnés précédemment. La connection obtenue sur le fibré des fonctions à poids spinoriel est appelé la connection GHP et est utilisée pour définir les opérateurs GHP. Enfin, nous donnons une définition plus intrinsèqe de l'operateur de Teukolsky dans le formalisme GHP. L'intérêt de cet opérateur réside dans son lien avec les équations d'onde tensorielles. Soit F une deux-forme différentielle solution des équations de Maxwell sans source:

$$\begin{aligned} \mathrm{d}F =& 0 \\ \mathrm{div}F =& 0. \end{aligned}$$

Nous définissons les scalaires de Teukolsky par

$$\begin{aligned} \alpha^{[1]} :=& F(l, m) \\ \alpha^{[-1]} :=& M^{-\frac{2}{3}}(r - ia\cos\theta)^2 F(\overline{m}, n). \end{aligned}$$

Pour une solution $\dot{g}$ des équations d'Einstein linéarisées autour de Kerr, nous définissons le tenseur de Weyl linéarisé associé $\mathrm{d}_g W(\dot{g})$. Les scalaires de Teukolsky sont alors définis par:

$$\begin{aligned} \alpha^{[2]} :=& D_g W(\dot{g})(l, m, l, m) \\ \alpha^{[-2]} :=& M^{-\frac{4}{3}}(r - ia\cos\theta)^4 D_g W(\dot{g})(\overline{m}, n, \overline{m}, n) \end{aligned}$$

Nous montrons comment les équations de Maxwell sans source et les équations d'Einstein linéarisées se decouplent pour les scalaires de Teukolsky et font apparaitre l'opérateur de Teukolsky.

Résumé de la partie III

La partie III contient la principale contribution originale de cette thèse. Il est constitué de l'article [80], qui est une description de la décroissance en temps long (avec calcul du terme principal) pour les solutions de l'équation de Teukolsky avec des données initiales régulières et localisées. Ce résultat est valable dans tout l'intervalle sous extremal $|a| < M$ et pour tous les paramètres $s \in \frac{1}{2}\mathbb{Z}$.

Cette partie contient deux résultats principaux avec des hypothèses différentes sur la décroissance des données initiales: support compact ou décroissance polynomiale inverse modérée. Dans les deux cas, nous supposons une grande régularité Sobolev initiale et nous ne cherchons pas à optimiser cette hypothèse. Nous donnons ci-dessous des énoncés simplifiés de ces deux résultats (voir les théorèmes 8.1.1, 8.1.2, 15.0.4 et 15.0.3 pour des énoncés plus précis):

Théorème 1.2.4. *Si les données initiales ont une régularité de Sobolev suffisante et sont à support compact, les solutions du problème de Cauchy correspondant pour l'équation de Teukolsky de paramètre $s \in \mathbb{Z}$ admettent un profil (terme principal) en temps long. Ce terme principal peut être calculé explicitement.*

Théorème 1.2.5. *Si les données initiales ont une décroissance polynomiale inverse modérée, les solutions du problème de Cauchy correspondant pour l'équation de Teukolsky de paramètre $s \in \mathbb{Z}$ décroissent comme l'inverse d'un polynôme en temps. Nous obtenons une estimée valable jusqu'à l'infini isotrope.*

Le temps mentionné dans les thèorèmes précédent est mesuré par une coordonnée $\mathfrak{t}$ différente de la coordonnées de Boyer-Lindquist (voir le chapitre 9 de la partie III pour la définition précise de $\mathfrak{t}$).

Le théorème 2.2.1 atteint le taux de décroissance optimal prévu par la loi de Price (voir [89, 90, 91, 52, 37]).

Le cas $s = 0$ correspond à l'équation des ondes scalaires qui a été étudiée en profondeur y compris pour la famille sous-extrémal complète $|a| < M$. Des résultats très précis sont connus incluant le calcul du terme principal de la solution en temps long (voir [48]). Nous renvoyons le lecteur à l'introduction de la partie III pour une revue de littérature plus détaillée à propos des résultats optimaux obtenus pour ce problème.

Nous nous intéressons maintenant aux résultats existants dans le cas $s \neq 0$ pour les trous noirs de Kerr. Pour des moments angulaires petits, la décroissance intégrable de l'énergie a été prouvée pour $s = \pm 1$ par Ma [69] et pour $s = \pm 2$ par Ma [70] et indépendemment par Dafermos-Holzegel-Rodnianski [21]. Ma-Zhang [71] ont encore affiné ce résultat en obtenant la décroissance optimale (avec le calcul du terme principal) pour les spins $s = \pm 1, \pm 2$ à l'extérieur d'un trou noir de Kerr à rotation lente $|a| \ll M$ (et conditionnellement pour $|a| < M$). Dans le théorème 2.2.1, nous levons cette hypothèse de rotation lente en utilisant une méthode de preuve différente. Cette étape pour inclure l'intervalle sous-extrémal entier $|a| < M$ est importante d'un point de vue physique car on s'attend à ce que les trous noirs physiques initiallement statiques ou à rotation lente voient leur moment angulaire augmenter jusqu'à une valeur proche de M au fur et à mesure qu'ils avalent la matière et les radiations du disque d'accrétion (voir [102] pour une modélisation de ce phénomène). De plus, comme mentionné précédemment, la plupart des résultats de stabilité actuels concernent le cas de la rotation lente et des estimées valables dans l'intervalle sous-extrémal complet seront nécessaires pour étendre leur portée. Notons que parallèlement à ce travail, un résultat de flux d'énergie borné et de décroissance intégrable de l'énergie locale valable pour tout l'intervalle $|a| < M$ et pour $s \in \{0, \pm 1, \pm 2\}$ a été obtenu par Shlapentokh-Rothman-Teixeira da Costa dans [96, 95] par d'autres méthodes.

La preuve des énoncés 1.2.4 et 1.2.5 s'appuie sur de récents développements en analyse spectrale et microlocale dans le contexte de la relativité générale et s'inscrit dans la continuité de [42] et [48](voir l'introduction de la partie III pour une revue de la littérature concernant ces méthodes). Un avantage majeur de cette méthode est sa robustesse car elle sépare clairement la partie de l'analyse dépendant seulement de la structure du flot hamiltonien (cadre Fredholm) et la partie dépendant de l'expression précise de l'opérateur (analyse des modes) ou d'un opérateur modèle (l'opérateur normal effectif). Par ailleurs, elle ne repose pas sur un argument perturbatif du cas sans rotation (Schwarzschild) ce qui explique que nous ne sommes pas limités au cas du moment angulaire petit. Nous présentons maintenant la structure de la preuve.

Les théorèmes 1.2.4 and 1.2.5 sont formulés en terme de problème de Cauchy mais le problème inhomogène

$$T_s v = f$$

(où v, f ont un support borné dans le passé) est plus pratique du point de vue de l'analyse spectrale. Pour obtenir un tel problème inhomogène, une idée naturelle est de poser $v = \chi(\mathfrak{t})u$ pour $\chi \in C^\infty(\mathbb{R}_\mathfrak{t}, [0,1])$ égale à 1 dans un voisinage de $+\infty$ et à 0 dans un voisinage de $-\infty$. Avec cette définition, u et v partagent le même comportement asymptotique proche de $\mathfrak{t} = +\infty$ et f a un support compact par rapport à $\mathfrak{t}$. Dans le cas de données initiales à support compact, il est possible de choisir χ tel que f a aussi un support compact en espace. Sans l'hypothèse de support compact, nous utilisons une estimée d'énergie pour calculer le comportement de f proche de l'infini isotrope.

Ensuite, nous prenons la transformée de Fourier-Laplace par rapport à $\mathfrak{t}$ et obtenons l'équation suivante pour $\Im(\sigma)$ assez grand:

$$\hat{T}_s(\sigma)\hat{v} = \hat{f}.$$

En admettant que $\hat{T}_s$ est inversible entre des espaces bien choisis, nous obtenons pour v:

$$v(\mathfrak{t}) = \frac{1}{2\pi} \int_{\Im(\sigma)=C} e^{-i\mathfrak{t}\sigma} R(\sigma)\hat{f}(\sigma)\, \mathrm{d}\sigma$$

où $R(\sigma) = \hat{T}_s(\sigma)^{-1}$. Une estimée formelle du membre de droite en utilisant des intégrations par partie, et en supposant que $\partial_\sigma^k R(\sigma)\hat{f}(\sigma)$ est intégrable fournit:

$$|v(\mathfrak{t})| \leq e^{C\mathfrak{t}}\mathfrak{t}^{-k} \left\| \partial_\sigma^k R(\sigma)\hat{f}(\sigma) \right\|_{L^1(\mathbb{R}_\sigma)}$$

Ce calcul formel conduit à l'intuition que les meilleures estimées sont obtenues quand C est petit et qu'une estimée de décroissance polynomiale inverse correspond à $C = 0$. Les observations précédentes suggèrent les points clés suivants à aborder:

1. Prouver que $\hat{T}_s(\sigma)$ est inversible dans le demi-plan complexe supérieur entre des espaces bien choisis.

2. Prouver que $R(\sigma)$ admet une borne polynomiale quand $|\sigma| \to +\infty$ et $\Im(\sigma)$ reste dans un ensemble compact (notons que puisque nous imposons une forte régularité sur les données initiales, $\hat{f}(\sigma)$ a une forte décroissance polynomiale inverse par rapport à σ).

3. Prouver que $R(\sigma)$ est holomorphe dans le demi-plan complexe supérieur strict et continue jusqu'à l'axe réel.

4. Analyser précisément la régularité de $R(\sigma)$ sur l'axe réel. Notons que, comme montré dans [48] pour l'équation des ondes, le terme principal peut être obtenu en calculant la singularité la plus forte de $R(\sigma)f(\sigma)$ sur l'axe réel (dans notre cas, elle est située en $\sigma = 0$).

Le point 1 peut être subdivisé en trois étapes: Prouver que $\hat{T}_s(\sigma)$ est Fredholm, prouver que l'indice est zéro et prouver que le noyau est trivial. La propriété Fredholm est obtenue en collant des estimées Fredholm sur différentes régions de l'espace des phases: Près des points radiaux sur l'horizon, (en utilisant [103]), dans une petite région à l'intérieur du trou noir (en utilisant une estimée hyperbolique comme fait dans [111]) et près des points radiaux à l'infini spatial en utilisant [105, 106]. Notons que pour appliquer le résultat de [106] nous avons besoin de vérifier l'absence de noyau pour l'opérateur normal effectif $\hat{T}_s(\sigma)$ (voir la définition 13.2.5) ce qui requiert la théorie des équation hypergéométriques confluentes. Le processus de collage repose sur des estimées elliptiques et sur des estimées de propagation des singularités.

Par conséquent, nous avons besoin d'une analyse globale du flot Hamiltonien de l'opérateur qui peut être réalisée à partir du symbole principal. L'absence de noyau suit d'un résultat de stabilité modale pour l'équation de Teukolsky sur un espace-temps de Kerr sous-extrémal obtenue par Whiting dans [108] et étendu par la suite par Andersson-Ma-Paganini-Whiting in [8] (voir aussi [7] pour $\sigma = 0$). La propriété d'indice zéro suivra de la continuité de l'indice et de l'inversibilité de $\hat{T}_s(\sigma)$ pour $|\Re(\sigma)| \gg 1$.

Nous prouvons cette inversibilité ainsi que la borne polynomiale du point 2 en introduisant le paramètre semiclassique $h = \frac{1}{|\sigma|}$, l'opérateur $\hat{T}_{s,h}(z) := h^2 \hat{T}_s(h^{-1}z)$ et en prouvant une borne de la forme

$$\|u\| \lesssim h^{-2} \left\| \hat{T}_{s,h}(z) u \right\|$$

(voir Proposition 13.4.10 pour l'énoncé précis). Comme précédemment, nous collons les estimées semiclassiques obtenues sur différentes régions de l'espace des phases. Cette fois, l'analyse est gouvernée par le flot hamiltonien semiclassique dont la structure globale doit être calculée. Le flot semiclassique a une structure globale plus complexe que le flot classique et en particulier, il contient un ensemble de trajectoires captées. Une estimée globale peut être obtenue en collant une estimée près des points radiaux à l'horizon (en utilisant [103]), une estimée dans une petite région à l'intérieur du trou noir (en utilisant une version semiclassique de l'estimée hyperbolique), une estimée près des points radiaux à l'infini spatial (en utilisant [105]) et une estimée près de l'ensemble capté normalement hyperbolique (fondée sur [109] et [28]).

Le point 3 suit de l'identité de la résolvante

$$R(\sigma) - R(\sigma') = R(\sigma) \left(\hat{T}_s(\sigma') - \hat{T}_s(\sigma) \right) R(\sigma')$$

une fois que les propriétés $R(\sigma)$ (déduites de l'estimée Fredholm globale) ont été établies.

Concernant le point 4, nous obtenons une forte régularité de $R(\sigma)$ sur l'axe réel en dehors de $\sigma = 0$ en utilisant l'identité de la résolvante de façon répétée. Le nombre d'itérations est seulement limité par la régularité des données initiales (qui est supposée forte dans ce travail). Cela contraste avec la situation en zéro où le nombre d'itérations est limité par la décroissance spatiale de $\hat{f}(\sigma)$. Pour le théorème 1.2.5, la régularité que nous obtenons de cette façon et les bornes sur la résolvante proche de zéro et à l'infini suffisent à conclure en prenant la transformée de Fourier inverse. Sous les hypothèses du théorème 1.2.4, nous pouvons aller plus loin (jusqu'à $2|s| + 2$ iterations) en utilisant que $\hat{f}(\sigma)$ a plus de décroissance spatiale. Nous obtenons l'expression de la singularité principale en zéro sous la forme $\sigma^{2|s|+2} R(\sigma) w$ avec w explicite et indépendant de σ. Notons que cette étape requiert une connaissance précise du noyau et du conoyau de $\hat{T}_s(0)$ (dans des espaces plus faibles que ceux pour lesquels on a l'inversibilité) que nous calculons en utilisant la théorie des equations hypergéométriques. En adaptant une idée de [48] qui consiste à peu de chose près à effectuer la dernière itération avec $\hat{T}_s(0)$ remplacé par l'opérateur normal effectif $N_{\text{eff}}(\hat{T}_s(\sigma))$ (voir définition 13.2.5), qui gouverne la transition entre $\hat{T}_s(\sigma)$ et $\hat{T}_s(0)$ proche de l'infini spatial, nous obtenons une expression explicite de $\sigma^{2|s|+2} R(\sigma) w$ modulo des termes, qui sont négligeables en temps long.

Une fois que tous les points 1-4 sont traités, nous pouvons effectuer un argument de déformation du contour d'intégration pour obtenir:

$$v(\mathfrak{t}) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-i\sigma\mathfrak{t}} R(\sigma) \hat{f}(\sigma) \, \mathrm{d}\sigma.$$

Nous utilisons ensuite des propriétés standards de la transformée de Fourier (ainsi que des calculs de transformées de Fourier explicites) pour conclure.

Résumé de la partie IV

La partie IV contient des outils analytiques généraux utiles pour l'étude de l'équation de Teukolsky. Les deux premiers chapitres de cette partie sont dédiés à une présentation autonome de résultats d'existence, d'unicité et d'approximation pour les opérateurs différentiels hyperboliques d'ordre deux sur une variété. Le contenu de cette partie est utilisé librement dans la partie III (en particulier dans l'analyse du problème de Cauchy). Bien que les résultats obtenus soient principalement fondées sur [54] et [92], nous proposons un traitement différent de ce sujet. Le dernier chapitre est une introduction à l'utilisation de techniques microlocales pour obtenire la propriété Fredholm pour des opérateurs différentiels. Notre but n'est pas de présenter une introduction complète à cette méthode, car il y a déjà d'excellentes références pour cela (voir en particulier [104]). A la place, nous choisissons de présenter la méthode au travers de son application à un exemple 1D très simple analysé en détail et avec un minimum d'outils techniques. Cette analyse met en jeu des concepts importants tels que les estimées elliptiques, les estimées de points radiaux, la propagation des singularités (dans le contexte semi-classique). Les principes qui sous-tendent cette partie, qui est destinée aux lecteurs sans expérience dans le domaine de l'analyse microlocale, sont les suivants:

- Le caractère autonome (contrairement à la partie III);
- La simplicité des outils. Les définitions sont spécialisées au cas 1D et nous limitons au minimum l'analyse pseudodifférentielle (les multiplicateurs de Fourier sont suffisant dans ce context simple);
- La simplicité de l'exemple étudié.

Bien que l'exemple étudié dans cette partie puisse être traité autrement avec des outils purement élémentaires, nous pensons qu'il est néammoins intéressant de voir un exemple débarrassé de toute la technicité habituellement nécessaire.

Chapter 2

Introduction

2.1 Kerr black holes and the Teukolsky equation

The main topic of this thesis is the analysis of the Teukolsky equation on subextremal Kerr spacetimes. Black hole spacetimes are explicit solutions of the Einstein vacuum equation $\mathrm{Ric}(g) = 0$ where g is a Lorentzian metric and $\mathrm{Ric}(g)$ is the Ricci tensor. Historically, the first explicit non trivial solution was discovered by Schwarzschild (see [93]). It has spherical symmetry and describes a static black hole in an asymptotically flat spacetime. A more general family of solutions which includes rotating black holes was introduced by Kerr in [59]. It is the focus of this thesis. Each solution is indexed by two parameters: M, the mass of the black hole and a, the angular momentum. Physical rotating black holes are model by subextremal Kerr solution ($|a| < M$). The Kerr metric has the following expression in Boyer-Lindquist coordinates:

$$\begin{aligned} g_{M,a} =& \frac{\Delta_r - a^2 \sin^2\theta}{\rho^2}\, \mathrm{d}t^2 + \frac{4Mar\sin^2\theta}{\rho^2}\, \mathrm{d}t\, \mathrm{d}\phi - \frac{\rho^2}{\Delta_r}\, \mathrm{d}r^2 \\ & - \rho^2\, \mathrm{d}\theta^2 - \frac{\sin^2\theta}{\rho^2}((a^2+r^2)^2 - a^2\Delta_r \sin^2\theta)\, \mathrm{d}\phi^2 \\ \Delta_r :=& a^2 + r^2 - 2Mr \\ \rho^2 :=& r^2 + a^2\cos^2\theta \end{aligned}$$

We highlight some of its main geometric features which are paramount for studying waves propagating on the Kerr spacetime:

- Axial symmetry (the vector field ∂_ϕ is Killing).
- Invariance by time translations (the vector field ∂_t is Killing).
- Asymptotic flatness: $g_{M,a} = \mathrm{d}t^2 - \mathrm{d}r^2 - r^2(\mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\phi^2) + O(r^{-1})$.
- Existence of null geodesics remaining in a spatially compact subset of the exterior region for all time (trapping).

The Kerr solution played a major role in the development of general relativity theory as they are used to model the gravitational field outside of a rotating star or black hole. Therefore, it is important to understand how the classical physical fields (including scalar fields, Dirac fields, Maxwell fields and gravitational waves) behave in presence of this gravitational

field. The simplest setting is to consider that they are sufficiently weak for their gravitational influence to be negligible. In this setting, the background geometry can be set to an exact Kerr solution. The physical fields are therefore described by hyperbolic partial differential equations on the Kerr Lorentzian manifold. This area of research has been very active over the last decades. It has been known since Teukolsky's work [101] that several of the main physical fields (scalar, Maxwell and gravitational waves) on Kerr spacetimes can be studied using the Teukolsky equation. This scalar equation depends on a parameter $s \in \frac{1}{2}\mathbb{Z}$ and captures the behavior of certain geometrically meaningful components of the different fields (depending on the value of s: $s = 0$ corresponds to scalar wave, $s = \pm 1$ to Maxwell and $s = \pm 2$ to the equations of linearized gravity).

This reduction is particularly convenient in the case of the equations of linearized gravity around the Kerr solution and this is why the Teukolsky equation has become an important tool in the study of linear and non linear stability of Kerr black holes. This family of problems has deep theoretical implications since stability under small perturbations is a crucial property for the solution to be considered as physically relevant. In practice, the nonlinear stability questions for the Einstein equations have proven extremely challenging. First works on this subject are [32] for the de Sitter solution and [19] (see also more recently [66]) for the Minkowski solution. More recent stability results have been obtained for black hole solutions: in [50], [47] for Kerr-de Sitter and Kerr-Newman-de Sitter, in [60] and [23] for Schwarzschild (non rotating black holes) and in [63, 64, 36, 62, 94] for the slowly rotating Kerr solution. All these results are based on a precise description of the propagation of perturbations on the underlying spacetime at the linear level. Linear stability results include [22, 55, 5, 57] for Schwarzschild, [4, 42] for slowly rotating Kerr and [35] for subextremal Reissner-Nordström spacetimes. See also the recent [45] which makes use of microlocal methods to prove the linear stability of weakly charged and slowly rotating Kerr-Newman black holes.

2.2 Content of the thesis

Presentation of part II

The Teukolsky operator as defined in [101] has the following expression in Boyer-Lindquist coordinates:

$$\begin{aligned}&\left(\frac{(a^2+r^2)^2}{\Delta_r} - a^2\sin^2\theta\right)\partial_t^2 + \frac{4Mar}{\Delta_r}\partial_t\partial_\phi + \left(\frac{a^2}{\Delta_r} - \frac{1}{\sin^2\theta}\right)\partial_\phi^2 - \Delta_r^{-s}\partial_r\Delta_r^{s+1}\partial_r \\ &- \frac{1}{\sin\theta}\partial_\theta\sin\theta\partial_\theta - 2s\left(\frac{a(r-M)}{\Delta_r} + i\frac{\cos\theta}{\sin^2\theta}\right)\partial_\phi - 2s\left(\frac{M(a^2-r^2)}{\Delta_r} - r - ia\cos\theta\right)\partial_t \\ &+ s^2\mathrm{cotan}^2\theta - s.\end{aligned} \tag{2.1}$$

Before diving into the analysis of this operator, it seems useful to address the three following questions:

- How does the Teukolsky operator arise from tensorial equations?
- In which sense is it smooth (since expression (2.1) seems to have singularities)?
- What are the Newman-Penrose and GHP (for Geroch–Held–Penrose) formalisms commonly used in the literature on the Teukolsky equation?

These questions find an answer in part II, which consists in the review article [79]. The main role of this part is to clarify the geometric context of the Teukolsky equation by gathering material scattered in the literature (see the introduction of the part for precise references) and provides an essentially self-contained basis for the analysis.

We introduce the definition of spinor bundles which appear naturally in the study of the Teukolsky equation (and are essential when the parameter s is not an integer). We also define principal Newman-Penrose null tetrads which are a set of four null vector fields $(l, n, m, \overline{m})$ with l and n reals and m complex which satisfy normalization conditions and such that l and n are principal (see definition 4.3.1). This tetrad is used as a basis to write components of tensors in the Newman-Penrose formalism and can be associated to the choice (with a sign ambiguity) of a normalized spin frame. However, as we will see, there is no globally smooth principal Newman-Penrose null tetrad on the Kerr exterior and this fact explains the singularities in the expression (2.1). Quantities depending on the choice of the tetrad are thus better represented as functions of the principal bundle of principal Newman-Penrose null tetrads rather than functions of the spacetime. If they satisfy in addition some compatibility condition with respect to the principal action, they can be interpreted as sections of an associated complex line bundle. This observation leads to the definition of spin-weighted functions and spin-weighted components of spinors and tensors. We completely characterize the bundle of principal Newman-Penrose tetrad on the Kerr spacetimes by computing a complete system of local trivializations. It appears that this bundle is related to the Hopf bundle on the sphere. We take some time to provide some remainders about connections on principal bundles and we explain how the Levi-Civita connection on the tangent bundle naturally gives rise to connections on the different bundles previously mentioned. The connection obtained on the bundle of spin weighted functions is called the GHP connection and is used to define the GHP operators. Finally, we give a more intrinsic definition of the Teukolsky operator in the GHP formalism and we show how the source-free Maxwell equations and the equations of linearized gravity decouple on the Teukolsky scalars.

Presentation of part III

Part III contains the main original contribution of the thesis. It is made of the article [80], which is a result of decay in the large time regime (with the computation of the leading order term) for solutions of the Teukolsky equation with regular and localized initial data. This result is valid for the whole subextremal range of black hole parameters $|a| < M$ and for all $s \in \frac{1}{2}\mathbb{Z}$.

We prove two main results with different assumptions on the decay of the initial data: compact support or moderate inverse polynomial decay. In both cases, we assume a high enough initial Sobolev regularity and we do not try to optimize this assumption. We give here a very rough version of both statements (see theorems 8.1.1, 8.1.2, 15.0.4 and 15.0.3 for more precise statements):

Theorem 2.2.1. *If the initial data have enough Sobolev regularity and are compactly supported, the solution of the corresponding Cauchy problem for the Teukolsky equation of parameter $s \in \mathbb{Z}$ admits a leading order term for large time. This leading order term can be computed explicitly up to null infinity.*

Theorem 2.2.2. *If the initial data have a moderate inverse polynomial rate of decay, the solution of the corresponding Cauchy problem for the Teukolsky equation of parameter $s \in \mathbb{Z}$ decays at an inverse polynomial rate in time. We obtain an estimate valid up to null infinity.*

Theorem 2.2.1 meets the optimal decay rate provided by Price's law (see [89, 90, 91, 52, 37]).

The case $s = 0$ corresponds to the scalar wave equation which has been studied in depth including for the whole subextremal family of Kerr black holes $|a| < M$. Very precise results are known including the first term development of the solution in the large time regime (see [48]). We refer the reader to the introduction of part III for a more detailed review of literature about optimal results obtained for this problem.

We now turn to previous results in the case $s \neq 0$ on Kerr black holes. For a small angular momentum, integrated energy decay was proved for the Teukolsky equation for spin $s = \pm 1$ by Ma [69] and for spin $s = \pm 2$ by Ma [70] and independently by Dafermos-Holzegel-Rodnianski [21]. Ma-Zhang [71] further sharpened the result obtaining the optimal decay (with computation of the leading term) for spin $s = \pm 1, \pm 2$ on the exterior region of a slowly rotating Kerr black hole $|a| \ll M$ (and conditionally in the case $|a| < M$). In Theorem 2.2.1, we remove the assumption of slow rotation by using a different method of proof. This improvement to the whole subextremal range $|a| < M$ is important from a physical point of view since physical black holes which are initially static or slowly rotating are expected to increase their angular momentum close to $|a| = M$ as they swallow matter and radiations from the accretion disk (see [102] for a model of this phenomenon). Moreover, as mentioned earlier, most of the current stability results concern the slowly rotating case and estimates valid for the whole subextremal range will be needed to extend their scope. Note that in parallel to our work, a bounded energy flux and integrated local energy decay result in the full subextremal range $|a| < M$ and for $s \in \{0, \pm 1, \pm 2\}$ has been obtained by Shlapentokh-Rothman-Teixeira da Costa in [96, 95] by other methods.

The proof of statements 2.2.1 and 2.2.2 relies on recent developments in spectral and microlocal analysis in the context of general relativity and is in line with [42] and [48] (see the introduction of part III for a literature review of these methods). A major advantage of this approach is its robustness since it clearly separates the part of the analysis depending only on the structure of the Hamiltonian flow (Fredholm framework) and the part depending on the precise expression of the operator (mode analysis) or of some model operator (the effective normal operator). Moreover, it does not rely on a perturbative argument from the non rotating (Schwarzschild) case which is the reason why we are not limited to small angular momentum.

We now briefly outline the proof (see the introduction of part III for a more detailed summary). In a first step we transform the Cauchy problem to a forcing problem $T_s v = f$ (inhomogeneous equation defined on the whole spacetime) while keeping track of the behavior of the forcing term near null infinity. This is done by energy methods. We can then take the time Fourier transform of the equation and study the time Fourier transformed Teukolsky operator $\hat{T}_s(\sigma)$ which is the main focus of the part. We then compute precisely the classical and semiclassical Hamiltonian flow of the operator and perform several microlocal estimates (based on [103], [105], [106] and [28]) in different regimes:

- σ in a compact subset of $\{z \in \mathbb{C} : \Im(z) \geq 0\}$.
- σ in a neighborhood of zero.
- $\|\Re(\sigma)\| \to +\infty$ and $\Im(\sigma)$ bounded.

After gluing all these estimates together, we are left with Fredholm estimates which prove that the operator is Fredholm between suitable b-Sobolev spaces. We can then use a mode stability result ([108], [7] and [7]) to obtain the existence of the resolvent on $\{\sigma \in \mathbb{C} : \Im(\sigma) \geq 0\}$. The resolvent is moreover holomorphic on $\{\sigma \in \mathbb{C} : \Im(\sigma) > 0\}$ and continuous up to the real axis. We can then perform a contour deformation argument in the inverse Fourier formula (a priori well defined for $C \gg 1$):

$$v = \int_{\mathbb{R}+iC} \hat{T}_s(\sigma)^{-1} \hat{f} \, \mathrm{d}\sigma$$

to bring C to zero. Note that the high frequency analysis provides a bound for the resolvent in the strip $\{0 \leq \Im(\sigma) \leq C\}$ which is used to justify the contour deformation argument. The inverse Fourier representation writen on the real axis reveals the relation between the regularity of the resolvent and the time decay of the solution. We analyse precisely the regularity properties of $R(\sigma) = \hat{T}_s(\sigma)^{-1}$. The most delicate part is the regularity at zero. In case of compactly supported initial data, we manage to compute explicitly the higher singularity at zero and by inverse Fourier transformation, it becomes the principal term in the development of the solution for large time. If the initial data only have a moderate inverse polynomial decay, we are able to prove that the resolvent at zero has some amount of regularity which translates into an inverse polynomial bound for the solution.

Presentation of part IV

Part IV contains general analytic tools useful for the study of the Teukolsky operator. The first and second chapters of the part are dedicated to an essentially self contained presentation of existence, uniqueness and approximation theory for hyperbolic partial differential operators of order two on a manifold. The content of this part is freely used in Part III (in particular for the analysis of the Cauchy problem). Although the results we obtain are mainly based on [54] and [92], we propose a different treatment of the topic. The last chapter is an introduction to the use of the microlocal methods to obtain the Fredholm property of differential operators. Our goal is not to provide a complete introduction to the method, as there are already excellent references doing that in the literature (see in particular [104]). Instead, we chose to present the method through its application to a very simple one dimensional example treated in full details and with minimal technical tools. This analysis contains important concepts such as elliptic estimate, radial points estimate, propagation of singularities (in the semiclassical setting). The underlying principles of this part, which is intended for readers with no background in microlocal analysis, are the following:

- self contained nature (in contrast to part III);
- simplicity of the tools. The definitions are specialized to the 1D case and we limit the amount of pseudodifferential analysis required (Fourier multipliers are sufficient in this simple setting);
- concreteness and simplicity of the example.

Although the one dimensional example presented in this part can be treated with elementary methods, we believe that it is interesting for pedagogical purposes to see an example stripped of all technicality.

Part II

Geometric context of the Teukolsky equation

Chapter 3

Introduction

In recent years, there has been a large amount of research on the Teukolsky equation on black hole type spacetimes. For boundedness and decay of solutions on a Kerr black hole with small angular momentum, we can mention in particular [21] by Dafermos, Holzegel and Rodnianski and [71] by Ma and Zhang. In the case of the full subextremal range, [96] by Shlapenthok-Rothman and Texeira da Costa is the first paper of a series dedicated to the boundedness and decay for the Teukolsky equation. These works are motivated by the fundamental question of the stability of the Kerr spacetime as a solution of the Einstein equations. Indeed, results about the Teukolsky equation have led to multiple breakthroughs concerning linear (see [22] by Dafermos Holzegel and Rodnianski and [4] by Andersson, Bäckdahl, Blue and Ma) and non linear stability of black holes (see [61] by Klainerman and Szeftel and [23] by Dafermos, Rodnianski, Holzegel and Taylor).

However, for an analyst entering the subject, it can be difficult to collect all the information necessary to understand the geometric background of the equation. In this paper, we shall revisit the geometric background of the equation and the GHP formalism from scratch, focusing on the Kerr case.

The Teukolsky equation (introduced in [101]) is a differential equation on spin weighted functions. The geometrical framework necessary to understand these objects includes spin geometry, the Newman-Penrose formalism (see [83] and [84]) and the closely related GHP formalism (introduced in [34]). Presentations of these formalisms are given in [18] (Newman-Penrose formalism and spin geometry), [88] (Newman-Penrose formalism and spin geometry) [6](spin geometry), [3, Section 2.4] (spin geometry and GHP formalism), [1, Section 2.1] (GHP formalism), [2](GHP formalism), [43] (geometrical definition of spin weighted functions close to the one presented here and GHP formalism) and [20](interpretation of spin weighted functions as complex line bundles).

Based on the above references, this paper synthesizes in an essentially self-contained manner the minimal geometric background necessary to understand how spin weighted functions appear in the study of the Teukolsky equation and provides detailed computations in the case of the Kerr spacetime. In particular, we explain the link between an abstract definition of spin weighted functions (relying on the Newman-Penrose formalism) and a more concrete definition involving the Hopf bundle. The latter is used in [21], [96] and [100] in the context of the analysis of the Teukolsky equation on a Kerr background, but also in [38] and [30] in the context of spin weighted functions on the two dimensional sphere. For the sake of simplicity, we do not consider the interaction of the GHP formalism with the conformal structure of the spacetime (studied by Araneda in [12]).

First, we introduce the general definitions of spin weighted functions on a general Petrov D type spacetime. Second, we compute the topology of the bundles in the case of a Kerr spacetime. Lastly, we introduce the various connections and define the GHP operators. We

choose a geometrical approach using principal connections.

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3.1 Conventions and notations

In this paper, we use the sign convention $(+,-,-,-)$ for Lorentzian metric. For example, the Minkowski metric on $\mathbb{R}^4$ will be given by the matrix $\eta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$.

If we have a Cartesian product $A \times B$ we denote by pr_A the projection to A.

Let $k \in \mathbb{Z}$ and $n \in \mathbb{N}$. We denote by $[k]_n$ the image of k under the projection $\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$. More generally, we will denote by $[x]$ the image of x under the projection map when we have a quotient space.

If E is a smooth vector bundle (real or complex) over a manifold $\mathcal{M}$, we denote by E' the dual vector bundle (for every $x \in \mathcal{M}$, $(E')_x$ is the space of linear forms on E_x). We denote by $\Gamma(E)$ the set of smooth sections of E.

If ∇ is a linear connection acting on a complex vector bundle, we sometimes want to compute ∇_X where X is a section of the complexified tangent space. In this case, we simply extend the connection by $\mathbb{C}$-linearity, in other words, we define $\nabla_X := \nabla_{\Re(X)} + i\nabla_{\Im(X)}$.

Chapter 4

Definition of the bundles

Let $\mathcal{M}$ be a spacetime (4 dimensional space and time oriented Lorentzian manifold). The reader may refer to [87, Chapter 3] for the definition of a Lorentzian manifold. A time orientation is the choice of a global continuous timelike vector field and a space orientation is a continuous choice of a connected component of the set of triples (b_2, b_3, b_4) of independent spacelike vectors at each point.

Definition 4.0.1. *For $x \in \mathcal{M}$, an oriented basis of $T_x\mathcal{M}$ is a basis $(b_1, ..., b_4)$ with b_1 future oriented and timelike and (b_2, b_3, b_4) spacelike and space oriented.*

The principal motivation to introduce spin weighted functions is the following: In the case where the spacetime is of Petrov type D (see Definition 4.3.2 below), we would like to find a global Newman Penrose null tetrad containing two vectors in the principal null directions (see definition 4.3.1). This tetrad can be associated (up to a sign ambiguity) with a spin frame which is adapted to the geometry. Indeed, it ensures the vanishing of some associated spin coefficients and therefore simplifies the component expression of the Dirac operator. We can even hope to reduce tensor equations to scalar decoupled equations for some components. However, it is in general not possible to find such a tetrad (and such spin frame) globally for topological reasons (see for example the computations on Kerr in section 5 below). However, it is possible to choose a global tetrad "up to some complex factor" (and similarly for the spin frame). Rigorously, this "almost tetrad" is defined as the smooth bundle of all four vectors satisfying the Newman Penrose conditions (or the bundle of all normalized spin frame associated to it). The analog of components in this "almost tetrad" can then be defined for co-tensors (and co-spinors in the almost spin-frame). Such components can be interpreted as sections of some complex line bundle $\mathcal{B}(s, w)$. Sections of $\mathcal{B}(s, w)$ are called spin weighted functions.

4.1 Spin structure and frame of oriented orthonormal frames

We know by [33] that $\mathcal{M}$ admits a spin structure. In other words, if we denote by $\mathfrak{O}$ the $SO^+(1,3)$ principal bundle of oriented orthonormal frames on $\mathcal{M}$ (and $\pi_{\mathfrak{O}}$ is the associated projection), there exists an $SL(2,\mathbb{C})$ principal bundle $\pi_{\mathfrak{S}} : \mathfrak{S} \to \mathcal{M}$ and a double covering $p : \mathfrak{S} \to \mathfrak{O}$ such that the following diagram commutes

$$\begin{array}{ccc} \mathfrak{S} & \xrightarrow{p} & \mathfrak{O} \\ & {\scriptstyle \pi_{\mathfrak{S}}} \searrow & \swarrow {\scriptstyle \pi_{\mathfrak{O}}} \\ & \mathcal{M} & \end{array}$$

and such that for all $g \in SL(2,\mathbb{C})$ and $x \in \mathfrak{S}$, $p(x \cdot g) = p(x) \cdot \tilde{p}(g)$ where $\tilde{p} : SL(2,\mathbb{C}) \to SO^+(1,3)$ is given by the Weyl representation

$$\tilde{p} : \begin{cases} SL(2,\mathbb{C}) \to SO^+(1,3) \\ M \mapsto \left(x \in \mathbb{R}^4 \mapsto i_2^{-1} M \frac{1}{\sqrt{2}} \begin{pmatrix} x_0 + x_3 & x_1 + ix_2 \\ x_1 - ix_2 & x_0 - x_3 \end{pmatrix} M^* \right) \end{cases}$$

where

$$i_2 : \begin{cases} \mathbb{C}^4 \to \mathcal{M}_2(\mathbb{C}) \\ z \mapsto \frac{1}{\sqrt{2}} \begin{pmatrix} z_0 + z_3 & z_1 + iz_2 \\ z_1 - iz_2 & z_0 - z_3 \end{pmatrix}. \end{cases}$$

We also identify $\mathbb{C}^2 \otimes \overline{\mathbb{C}}^2$ and $\mathcal{M}_2(\mathbb{C})$ via the linear isomorphisms

$$i_1 : \begin{cases} \mathbb{C}^2 \otimes \overline{\mathbb{C}}^2 \to \mathcal{M}_2(\mathbb{C}) \\ a \otimes \overline{b} \mapsto a\overline{b}^T. \end{cases}$$

For later use, we denote by $i_0 := i_2^{-1} \circ i_1$ the identification between $\mathbb{C}^2 \otimes \overline{\mathbb{C}^2}$ and $\mathbb{C}^4$. Note that for $a \in \mathbb{C}^2$, $i_0(a \otimes \overline{a})$ is a real four dimensional vector whose components x_0, x_1, x_2, x_3 satisfy $x_0^2 - x_1^2 - x_2^2 - x_3^2 = 0$. Using the identification i_0, we have for every $g \in SL(2,\mathbb{C})$:

$$\rho(g) \otimes \overline{\rho}(g) = \mu \circ \tilde{p}(g) \tag{4.1}$$

where ρ is the canonical representation of $SL(2,\mathbb{C})$ on $\mathbb{C}^2$ and μ is the canonical representation of $SO^+(1,3)$ on $\mathbb{C}^4$ (the identification i_0 is implicit in the equality).

4.2 Spinor bundles

Vector bundle associated to a principal bundle

Given a principal bundle $\pi : \mathcal{E} \to \mathcal{B}$ and a representation ρ of the structure group G on some (real or complex) vector space V, we can form the associated vector bundle $\mathcal{F} := \mathcal{E} \times V / \sim$ where $(e,v) \sim (e',v')$ if there exists $g \in G$ such that $e \cdot g = e'$ and $\rho(g^{-1})(v) = v'$. The vector space structure is given on each fiber by $\lambda[(e,v)] = [(e,\lambda v)]$ and $[(e,v)] + [(e,v')] = [(e,v+v')]$; it does not depend on the choice of (e,v) in the equivalence class $[(e,v)]$. If $\phi : \pi^{-1}U \to U \times G$ is a local trivialization (in the sense of principal bundles) of $\mathcal{E}$, then $\phi_{\mathcal{F}} : x \in \mathcal{F} \mapsto ((\pi(x), z)$ such that $[(\phi^{-1}(\pi(x),1), z)] = x)$ is a local trivialization of $\mathcal{F}$. In these notes, $\phi_{\mathcal{F}}$ will be called the trivialization associated to ϕ. If we apply the previous construction to the bundle of oriented orthonormal frames with the canonical representation of $SO^+(1,3)$ on $\mathbb{C}^4$, we obtain the complexified tangent bundle on $\mathcal{M}$.

Remark 4.2.1. *If A and B are complex vector bundles associated to $\mathcal{E}$ for the representations ρ_A and ρ_B, then $A \otimes B$ is naturally isomorphic to the bundle associated to $\mathcal{E}$ for the action $\rho : g \mapsto \rho_A(g) \otimes \rho_B(g)$ and $\overline{A}$ is naturally isomorphic to the bundle associated to $\mathcal{E}$ for the action $\overline{\rho}_A$.*

Vector bundle associated with the spin structure

Applying the previous construction to the spin structure and the canonical representation of $SL(2,\mathbb{C})$, we get a complex vector bundle of rank 2 over $\mathcal{M}$ called the spinor bundle $\mathcal{S}$. If we choose a local trivialization Φ of the spin structure, we obtain an associated local trivialization $\Phi_{\mathcal{S}}$ of $\mathcal{S}$.

Proposition 4.2.2. *Given a local trivialization $\Phi : \pi_{\mathfrak{S}}^{-1}(U) \to U \times SL(2,\mathbb{C})$ of $\mathfrak{S}$, there exists a unique local trivialization (on the same open set) $\Psi : \pi_{\mathfrak{O}}^{-1}(U) \to U \times SO^+(1,3)$ of $\mathfrak{O}$ such that $\Psi \circ p \circ \Phi^{-1} = Id \times \tilde{p}$. We say that Φ and Ψ are compatible.*

Proof. A necessary condition on Ψ is $\Psi \circ p = (Id \times \tilde{p}) \circ \Phi$. Therefore, the uniqueness is a direct consequence of the surjectivity of $p : \pi_{\mathfrak{S}}^{-1}(U) \to \pi_{\mathfrak{O}}^{-1}(U)$. To construct Ψ, we have to show that if $p(x) = p(y)$, then $(Id \times \tilde{p}) \circ \Phi(x) = (Id \times \tilde{p}) \circ \Phi(y)$. Assume $x, y \in \mathfrak{S}$ are such that $p(x) = p(y)$. In particular $\pi_{\mathfrak{S}}(x) = \pi_{\mathfrak{S}}(y)$. By definition of a principal $SL(2,\mathbb{C})$ bundle, there exists $g \in SL(2,\mathbb{C})$ such that $y = x \cdot g$. By definition of p, $p(x) = p(x \cdot g) = p(x) \cdot \tilde{p}(g)$ and $\tilde{p}(g) = 1$ (the action of $SO^+(1,3)$ on each fiber of $\mathfrak{O}$ is free). Moreover, because Φ is a trivialization, $\Phi(x \cdot g) = (\pi_{\mathfrak{S}}(y), pr_{SL(2,\mathbb{C})}(\Phi(y))g)$. Now,

$$(Id \times \tilde{p}) \circ \Phi(x \cdot g) = (\pi_{\mathfrak{S}}(x), \tilde{p}(pr_{SL(2,\mathbb{C})}(\Phi(x)))\tilde{p}(g)) = (Id \times \tilde{p}) \circ \Phi(x).$$

So Ψ is well defined. Moreover, using that $(Id \times \tilde{p}) \circ \Phi$ is surjective and $(Id \times \tilde{p}) \circ \Phi(x) = (Id \times \tilde{p}) \circ \Phi(y)$ if and only if $p(x) = p(y)$ (the only if part comes from the fact that $(Id \times \tilde{p}) \circ \Phi(x)$ and p have exactly two preimages), Ψ is bijective. Since p is a smooth covering map and $\Psi \circ p$ is a local diffeomorphism, we obtain that Ψ is a local diffeomorphism so it is a diffeomorphism. We also have that for $g \in SO^+(1,3)$, $\Psi(x \cdot g) = (\pi_{\mathfrak{O}}(x), pr_{SO^+(1,3)}(\Psi(x))g)$. As a consequence Ψ is a local trivialization of $\mathfrak{O}$. □

Remark 4.2.3. *A local trivialization of a principal bundle can be defined by a local smooth section of the bundle (we ask that this section is the constant map equal to the neutral element of the group when written in the local trivialization). Two local sections s_1 of $\mathfrak{S}$ and s_2 of $\mathfrak{O}$ define compatible local trivializations if and only if $s_2 = p \circ s_1$. This fact could have been used to prove the proposition as well.*

Remark 4.2.4. *It is not true in general that for any local trivialization Ψ of $\mathfrak{O}$, there exists a compatible local trivialization of $\mathfrak{S}$. However it is true locally (there exist exactly 2 such compatible local trivializations corresponding to the two local lifts of the section $x \mapsto \Psi^{-1}(x, 1)$ through p).*

Remark 4.2.5. *If $\Phi \circ \Phi'^{-1}(x, g) = (x, gf(x))$ is a change of local trivialization of $\mathcal{E}$ (with $f : U \to G$ a smooth map), then $\Phi_{\mathcal{F}} \circ \Phi'^{-1}_{\mathcal{F}}(x, v) = (x, \rho(f(x))(v))$.*

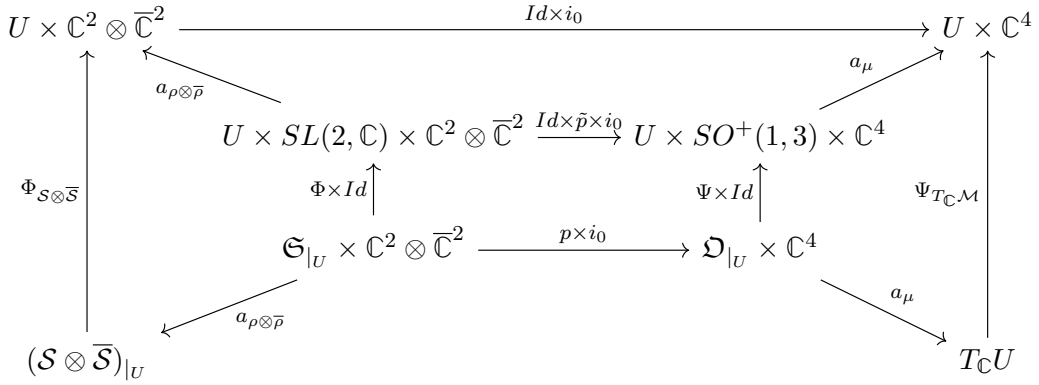
Two compatible trivializations enable to reduce locally the picture of spinors and vectors on $\mathcal{M}$ to the picture of spinors and vectors on Minkowski space. In particular, we have an analog on $\mathcal{M}$ of the previously defined identification between $\mathbb{C}^2 \otimes \overline{\mathbb{C}^2}$ and $\mathbb{C}^4$:

Proposition 4.2.6. *There exists a unique isomorphism j of complex vector bundles between $\mathcal{S} \otimes \overline{\mathcal{S}}$ and $T_{\mathbb{C}}\mathcal{M}$ such that, given any pair of compatible local trivializations (Φ, Ψ) on U, $\Psi_{\mathcal{T}_{\mathbb{C}}\mathcal{M}} \circ j \circ \Phi^{-1}_{\mathcal{S}\otimes\overline{\mathcal{S}}}$ is exactly (Id_U, i_0) where i_0 is the identification between $\mathbb{C}^2 \otimes \overline{\mathbb{C}}^2$ and $\mathbb{C}^4$.*

Proof. The uniqueness is obvious since compatible local trivializations cover $\mathcal{M}$. For the existence, we define j locally on each open set U associated to a compatible local trivialization and we have to check that all the definitions agree when they overlap. Let (Φ, Ψ) and (Φ', Ψ') be two pairs of compatible local trivializations on the same open set U. Let $g_1 : U \to SL(2,\mathbb{C})$ smooth be such that $\Phi \circ \Phi'^{-1}(x, g) = (x, gg_1(x)))$ and $g_2 : U \to SO^+(1,3)$ smooth be such that $\Psi \circ \Psi'^{-1}(x, g) = (x, gg_2(x))$. Using the compatibility (therefore $p = \Psi^{-1} \circ Id \times \tilde{p} \circ \Phi = \Psi'^{-1} \circ Id \times \tilde{p} \circ \Phi'$), we get for all $x \in U$ and all $g \in SL(2,\mathbb{C})$ $(x, \tilde{p}(g)) = (x, \tilde{p}(gg_1(x))g_2(x)^{-1})$ and we deduce $\tilde{p} \circ g_1 = g_2$. Then $\Psi_{\mathcal{T}_{\mathbb{C}}\mathcal{M}} \circ \Psi'^{-1}_{\mathcal{T}_{\mathbb{C}}\mathcal{M}} \circ (Id \times i_0) \circ \Phi'_{\mathcal{S}\otimes\overline{\mathcal{S}}} \circ \Phi^{-1}_{\mathcal{S}\otimes\overline{\mathcal{S}}}(x, v) = \big(x, \mu(\tilde{p}(g_1(x)))(i_0(\rho(g_1(x)^{-1}) \otimes \overline{\rho}(g_1(x)^{-1})(v)))\big)$ (where ρ and μ are the same as in (4.1) and we used remark 4.2.5 to compute the changes of associated trivialization).

But (4.1) states exactly that $\mu(\tilde{p}(g_1(x))) \circ i_0 \circ (\rho(g_1(x)^{-1}) \otimes \overline{\rho}(g_1(x)^{-1})) = i_0$. Therefore $\Psi_{\mathcal{T}_{\mathbb{C}}\mathcal{M}} \circ \Psi'^{-1}_{\mathcal{T}_{\mathbb{C}}\mathcal{M}} \circ (Id \times i_0) \circ \Phi'_{\mathcal{S}\otimes\overline{\mathcal{S}}} \circ \Phi^{-1}_{\mathcal{S}\otimes\overline{\mathcal{S}}}(x, v) = (Id \times i_0)(x, v)$ and the proposition is proved. □

Remark 4.2.7. *A more conceptual proof can be given using a commutative diagram. If $\pi : \mathcal{E} \to \mathcal{B}$ is a principal bundle, if V is a vector space endowed with a representation ρ of the structure group and if $\mathcal{F}$ is the corresponding associated vector bundle, we denote by $a_\rho : \mathcal{E} \times V \to \mathcal{F}$ the map sending an element on its equivalence class. Also note that if $\mathcal{E}$ is trivial, we have a canonical identification between $\mathcal{F}$ and $\mathcal{B} \times V$. Given a pair of compatible local trivializations (Φ, Ψ), we have the following commutative diagram:*



Looking at the diagram, it is clear that the map j does not depend on the choice of the pair of compatible local trivializations since $a_{\rho\otimes\overline{\rho}}$ is surjective and $j \circ a_{\rho\otimes\overline{\rho}} = a_\mu \circ (p \times i_0)$.

Remark 4.2.8. *In the following, we identify $\mathcal{S} \otimes \overline{\mathcal{S}}$ and $T_{\mathbb{C}}\mathcal{M}$ using the map j implicitly. Note that if $m = j(a \otimes \overline{b}) \in T_{\mathbb{C}}\mathcal{M}$, then $\overline{m} = j(b \otimes \overline{a})$. This comes from the analogous property on the map i_0.*

Symplectic form on spinors

Proposition 4.2.9. *There exists a unique symplectic form ϵ (section of $\mathcal{S}' \wedge \mathcal{S}'$) such that for all local trivializations Φ of $\mathfrak{S}$, we have $\epsilon_x(\Phi_{\mathcal{S}}^{-1}(x, v), \Phi_{\mathcal{S}}^{-1}(x, w)) = \det(v, w)$.*

Proof. The uniqueness is obvious. To prove the existence, we define ϵ locally and check that all the definitions agree. Let Φ and Φ' be two local trivializations of $\mathfrak{S}$ on some open set U. We denote by $g : U \to SL(2, \mathbb{C})$ the smooth map such that $\Phi \circ \Phi'^{-1}(x, h) = (x, hg(x))$. Then $\det(\rho(g(x)^{-1})(v), \rho(g(x)^{-1})(w)) = \det(g(x)^{-1}(v, w)) = \det(g(x)^{-1}) \det(v, w) = \det(v, w)$. The proposition is proved. □

Proposition 4.2.10. *We have the following equality for all $x \in \mathcal{M}$, for all $a, c \in \mathcal{S}_x$ and $\overline{b}, \overline{d} \in \overline{\mathcal{S}}_x$:*

$$g(a \otimes \overline{b}, c \otimes \overline{d}) = \epsilon(a, c)\overline{\epsilon(b, d)}$$

Proof. We fix a pair of compatible trivializations (Φ, Ψ). The equality to prove in the associated trivializations is

$$\forall a, c \in \mathbb{C}^2, \forall \overline{b}, \overline{d} \in \overline{\mathbb{C}}^2, \eta(i_0(a \otimes \overline{b}), i_0(c \otimes \overline{d})) = \det(a, c)\overline{\det(b, d)}$$

where $\eta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$ is the Minkowski metric on $\mathbb{C}^4$. We check that this equality is true using the explicit definition of i_0. □

4.3 Bundle of normalized spin frames along the principal null directions

From now on, we assume that $\mathcal{M}$ is a Petrov-type D (see the definition below) Ricci-flat spacetime. Since the Ricci tensor is zero, the Weyl tensor is equal to the curvature tensor and we choose the following convention: For X, Y, Z, T smooth vector fields on $\mathcal{M}$, $W(X,Y,Z,T) := g(\nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z, T)$ where ∇ denotes the Levi-Civita connection. Note that this convention is the same as in [88] but it is the opposite sign convention in [18].

Definition 4.3.1. *A null vector $l \in T_x\mathcal{M}$ is called principal if for all $a, b \in T_x\mathcal{M}$ such that $g(l,a) = g(l,b) = 0$, we have $W(l,a,l,b) = 0$ where W is the Weyl tensor. We say that the vector l is principal of multiplicity at least 2 if for all $a, b \in T_x\mathcal{M}$ such that $g(l,a) = 0$, $W(l,a,l,b) = 0$. This characterization of principal vectors can be found in [86], proposition 5.5.5. Alternative equivalent definitions and additional properties of principal null directions are also provided in [86].*

Definition 4.3.2. *We define a Petrov type D spacetime as a spacetime $\mathcal{M}$ such that for all $x_0 \in \mathcal{M}$, there exist linearly independent (therefore non-vanishing) null vector fields l and n in a neighborhood U of x_0 such that:*

- *For all $x \in U$, the set of principal vector fields at $T_x\mathcal{M}$ is exactly $\mathbb{R}l(x) \cup \mathbb{R}n(x)$*
- *For all $x \in U$, $l(x)$ and $n(x)$ are principal null vectors of multiplicity (at least) 2. (In fact, l and n are of multiplicity exactly 2, see [86], chapter 5).*

The property of principal null directions on Petrov-type D Ricci-flat spacetime that we need here is the following:

Proposition 4.3.3. *If we define n and l as in the previous definition, they are pregeodesic and shear-free. The shear of a pregeodesic vector field l at x_0 with respect to X, Y orthonormal family of $l(x_0)^\perp$ is defined as $\frac{1}{2}\left(g(\nabla_Y l, Y) - g(\nabla_X l, X)\right) + \frac{i}{2}\left(g(\nabla_Y l, X) + g(\nabla_X l, Y)\right)$ (see definition 5.7.1 in [86]).*

Proof. See proposition 5.9.2 in [86]. □

We define the following subset of $\mathcal{S} \times_{\mathcal{M}} \mathcal{S}$:

$$\mathcal{A} := \cup_{x\in\mathcal{M}} \{(o,\iota) \in \mathcal{S}_x \times \mathcal{S}_x : j(o \otimes \overline{o}) \text{ and } j(\iota \otimes \overline{\iota}) \text{ are independent, future oriented along null principal directions and } \epsilon(o,\iota) = 1\}$$

The bundle $\mathcal{A}$ is naturally endowed with a canonical $\mathbb{C}^*$ right action:

$$(o,\iota)\cdot z = (zo, z^{-1}\iota)$$

We also have a $\mathbb{Z}/4\mathbb{Z}$ right action given by the map (image of the generator of $\mathbb{Z}/4\mathbb{Z}$): $(o,\iota) \mapsto (i\iota, io)$

By combining the two actions (performing the action of $\mathbb{C}^*$ first) , we get a right action of $\mathbb{C}^* \rtimes_f \mathbb{Z}/4\mathbb{Z}$ (with $f : [1]_4 \mapsto (z \mapsto z^{-1})$) on $\mathcal{A}$. However, this action is not free and we can quotient by the stabilizer of any point in $\mathcal{A}$ which is the normal subgroup $H := \langle(-1, [2]_4)\rangle$ to get a free action of the group $G_{\mathcal{A}} := (\mathbb{C}^* \rtimes_f \mathbb{Z}/4\mathbb{Z})/H$. Topologically, $\mathbb{C}^* \rtimes_f \mathbb{Z}/4\mathbb{Z}$ is simply $\mathbb{C}^* \times \mathbb{Z}/4\mathbb{Z}$ which is homeomorphic to four disjoint copies of $\mathbb{C}^*$. The quotient by H identifies $\mathbb{C}^* \times \{[0]_4\}$ with $\mathbb{C}^* \times \{[2]_4\}$ and $\mathbb{C}^* \times \{[1]_4\}$ with $\mathbb{C}^* \times \{[3]_4\}$ so $G_{\mathcal{A}}$ is homeomorphic to two disjoint copies of $\mathbb{C}^*$ (more precisely the two connected components are $\mathbb{C}_0^* := \{[(z,[0]_4)], z \in \mathbb{C}^*\}$ and $\mathbb{C}_1^* := \{[(z,[1]_4)], z \in \mathbb{C}^*\}$, $\mathbb{C}_0^*$ being the connected component of the neutral element). Note that $\mathbb{C}_0^*$ is a normal subgroup of $G_{\mathcal{A}}$ isomorphic to $\mathbb{C}^*$.

Proposition 4.3.4. *The set $\mathcal{A}$ is a smooth submanifold of $\mathcal{S} \times_{\mathcal{M}} \mathcal{S}$. Moreover, $\mathcal{A}$ is a $G_{\mathcal{A}}$ principal bundle (with projection map $\pi_{\mathcal{A}} = \pi_{\mathcal{S}\times_{\mathcal{M}}\mathcal{S}|_{\mathcal{A}}}$).*

Proof. Let $x \in \mathcal{M}$. There exists an open neighborhood U of x such that there exists an oriented orthonormal tetrad (e_0, e_1, e_2, e_3) so that $e_0 + e_3$ and $e_0 - e_3$ are future oriented along the null principal directions. This tetrad gives a local trivialization Ψ of $\mathfrak{O}$. Then, even taking a smaller neighborhood, we can assume that there exists a local trivialization Φ of $\mathfrak{S}$ on U such that Φ and Ψ are compatible (see remark 4.2.4). The set $\tilde{\mathcal{A}} := \Phi_{\mathcal{S}\times_{\mathcal{M}}\mathcal{S}}(\pi_{\mathcal{A}}^{-1}(U))$ is given by $U \times A$ where $A \subset \mathbb{C}^2 \times \mathbb{C}^2$ is defined as the set of $(o, \iota) \in \mathbb{C}^2 \times \mathbb{C}^2$ such that:

$$\begin{cases} i_0(o \otimes \overline{o}) &= (\lambda, 0, 0, \lambda) \\ i_0(\iota \otimes \overline{\iota}) &= (\mu, 0, 0, -\mu) \end{cases}$$

or

$$\begin{cases} i_0(\iota \otimes \overline{\iota}) &= (\lambda, 0, 0, \lambda) \\ i_0(o \otimes \overline{o}) &= (\mu, 0, 0, -\mu) \end{cases}$$

$$\det(o, \iota) = 1$$

where λ and μ are real positive numbers. We find the following parametrization for A:

$$\alpha : \begin{cases} G_{\mathcal{A}} \to A \\ [(z, [0]_4)] \mapsto \left(\begin{pmatrix} z \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ z^{-1} \end{pmatrix} \right) \\ [(z, [1]_4)] \mapsto \left(\begin{pmatrix} 0 \\ iz^{-1} \end{pmatrix}, \begin{pmatrix} iz \\ 0 \end{pmatrix} \right) \end{cases}$$

where for $k \in \{0, 1, 2, 3\}$, $[(z, [k]_4)]$ is the class $(z, [k]_4)H$ in $G_{\mathcal{A}}$. This map is a proper injective immersion in $\mathbb{C}^2 \times \mathbb{C}^2$ and its image is A. The fact that the image is included in A can be checked directly with the definitions. The other inclusion is proved by solving the system defining A. For example, if $o = \begin{pmatrix} o_0 \\ o_1 \end{pmatrix}$ and $\lambda \in (0, +\infty)$, the condition $i_0(o \otimes \overline{o}) = (\lambda, 0, 0, \lambda)$ rewrites (by definition of i_0):

$$\left(\frac{\sqrt{2}}{2} \left(|o_0|^2 + |o_1|^2 \right), \sqrt{2}\Re(o_0\overline{o_1}), \sqrt{2}\Im(o_0\overline{o_1}), \frac{\sqrt{2}}{2} \left(|o_0|^2 - |o_1|^2 \right) \right) = (\lambda, 0, 0, \lambda)$$

This equality holds if and only if $o_1 = 0$ and $|o_0|^2 = 2\lambda$. The other cases are very similar.

This proves that A is a submanifold of $\mathbb{C}^2 \times \mathbb{C}^2$ and we deduce that $\mathcal{A}$ is a submanifold of $\mathcal{S} \times_{\mathcal{M}} \mathcal{S}$. Moreover, we check (using the definition of the action) that the maps $\Phi^{-1}_{\mathcal{S}\times_{\mathcal{M}}\mathcal{S}} \circ (Id_U \times \alpha)$ (defined around each point $x \in \mathcal{M}$) endow $\mathcal{A}$ with a structure of $G_{\mathcal{A}}$ principal bundle over $\mathcal{M}$. □

4.4 Bundle of oriented Newman-Penrose null tetrads along principal null directions

Definition 4.4.1. *Let $(n, l, m) \in T_{\mathbb{C},x}\mathcal{M}^3$ be such that $(n, l, m, \overline{m})$ is a null basis of $T_{\mathbb{C},x}\mathcal{M}$ with l and n real and future oriented. We say that (n, l, m) is oriented if $(\frac{n+l}{\sqrt{2}}, \Re(m), -\Im(m), \frac{l-n}{\sqrt{2}})$ is oriented.*

We define the following subset of $T_{\mathbb{C}}\mathcal{M}^3$:

$$\mathcal{N} := \cup_{x\in\mathcal{M}} \Big\{ (l,n,m) \in (T_{\mathbb{C},x}\mathcal{M}\setminus\{0\})^3 \,/\, g(l,l) = g(n,n) = g(m,m) = 0, l = \bar{l}, n = \overline{n},$$
$$l \text{ and } n \text{ are independent principal and future oriented}, g(l,m) = g(n,m) = 0,$$
$$g(m,\overline{m}) = -1, g(l,n) = 1, (l,n,m) \text{ is oriented in the sense of definition 4.4.1} \Big\}$$

In particular, for $(l,n,m) \in \mathcal{N}_x$, $(l,n,m,\overline{m})$ is a basis of $T_{\mathbb{C},x}\mathcal{M}$. The set $\mathcal{N}$ is endowed with a canonical $\mathbb{C}^*$ right action:

$$(l,n,m)\cdot z = \left(|z|l, |z|^{-1}n, \frac{z}{|z|}m \right)$$

We also have a $\mathbb{Z}/2\mathbb{Z}$ right action defined by the involution:

$$(l,n,m) \mapsto (n,l,\overline{m})$$

Combining these two actions (performing the action of $\mathbb{C}^*$ first), we get a right action of $\mathbb{C}^* \rtimes_g \mathbb{Z}/2\mathbb{Z}$ on $\mathcal{N}$ where $g([1]_2)(z) = z^{-1}$.

Proposition 4.4.2. *$\mathcal{N}$ is a smooth submanifold of $T_{\mathbb{C}}\mathcal{M}^3$ and is a $\mathbb{C}^* \rtimes_g \mathbb{Z}/2\mathbb{Z}$ principal bundle (for the action previously defined) with projection map $\pi_{\mathcal{N}} := (\pi_{T_{\mathbb{C}}\mathcal{M}^3})_{|\mathcal{N}}$.*

Proof. Let $x \in \mathcal{M}$, there exists an open neighborhood U of x such that there exists an oriented orthonormal tetrad (e_0,e_1,e_2,e_3) so that e_0+e_3 and e_0-e_3 are future oriented along the null principal directions. We denote by Ψ the corresponding trivialization of $\mathfrak{O}$. Then $\Psi_{T_{\mathbb{C}}\mathcal{M}^3}(\mathcal{N}) = U \times N$ where N is the subset of $(\mathbb{C}^4)^3$ of (l,n,m) such that:

$$\begin{cases} l &= \frac{\sqrt{2}}{2}\begin{pmatrix}\lambda\\0\\0\\\lambda\end{pmatrix} \\ n &= \frac{\sqrt{2}}{2}\begin{pmatrix}\lambda^{-1}\\0\\0\\-\lambda^{-1}\end{pmatrix} \\ m &= \frac{\sqrt{2}}{2}\begin{pmatrix}0\\e^{i\theta}\\-ie^{i\theta}\\0\end{pmatrix} \end{cases} \text{ or } \begin{cases} n &= \frac{\sqrt{2}}{2}\begin{pmatrix}\lambda\\0\\0\\\lambda\end{pmatrix} \\ l &= \frac{\sqrt{2}}{2}\begin{pmatrix}\lambda^{-1}\\0\\0\\-\lambda^{-1}\end{pmatrix} \\ m &= \frac{\sqrt{2}}{2}\begin{pmatrix}0\\e^{i\theta}\\ie^{i\theta}\\0\end{pmatrix} \end{cases}$$

where λ is a positive real and θ is any real.

We have the following injective proper immersion (in $(\mathbb{C}^4)^3$)

$$\beta : \begin{cases} \mathbb{C}^* \rtimes_g \mathbb{Z}/2\mathbb{Z} \to N \\ (z,[0]_2) \mapsto \left(\frac{1}{\sqrt{2}}\begin{pmatrix}|z|\\0\\0\\|z|\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}|z|^{-1}\\0\\0\\-|z|^{-1}\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}0\\\frac{z}{|z|}\\-i\frac{z}{|z|}\\0\end{pmatrix} \right) \\ (z,[1]_2) \mapsto \left(\frac{1}{\sqrt{2}}\begin{pmatrix}|z|^{-1}\\0\\0\\-|z|^{-1}\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}|z|\\0\\0\\|z|\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}0\\\frac{\overline{z}}{|z|}\\i\frac{\overline{z}}{|z|}\\0\end{pmatrix} \right) \end{cases}$$

This map has image N so N is a submanifold of $(\mathbb{C}^4)^3$ and therefore, $\mathcal{N}$ is a submanifold of $T_{\mathbb{C}}\mathcal{M}^3$. Moreover, the maps β (around each $x \in \mathcal{M}$) define the structure of a $\mathbb{C}^* \rtimes_g \mathbb{Z}/2\mathbb{Z}$ principal bundle on $\mathcal{N}$. □

We have a natural map:

$$d : \begin{cases} \mathcal{A} \to \mathcal{N} \\ (o, \iota) \mapsto (o \otimes \overline{o}, \iota \otimes \overline{\iota}, o \otimes \overline{\iota}) \end{cases}$$

Remark 4.4.3. *The fact that the map is well defined follows from proposition 4.2.10 and the following remark about orientation: If $(o, \iota) \in \mathcal{A}_x$, then if we fix a pair of compatible local trivializations (Φ, Ψ) around x, $M := (pr_{\mathbb{C}^2}(\Phi_{\mathcal{S}}(o)), pr_{\mathbb{C}^2}(\Phi_{\mathcal{S}}(\iota))) \in SL(2, \mathbb{C})$. Then by a change of compatible local trivializations (associated to the smooth maps $x \in U \mapsto M^{-1}$ and $x \in U \mapsto \tilde{p}(M^{-1})$), we can assume that $M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. It follows that $pr_{\mathbb{C}^4}\Psi_{T_{\mathbb{C}}\mathcal{M}}(o \otimes \overline{o}) = \frac{\sqrt{2}}{2}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$, $pr_{\mathbb{C}^4}\Psi_{T_{\mathbb{C}}\mathcal{M}}(\iota \otimes \overline{\iota}) = \frac{\sqrt{2}}{2}\begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}$ and $pr_{\mathbb{C}^4}\Psi_{T_{\mathbb{C}}\mathcal{M}}(o \otimes \overline{\iota}) = \frac{\sqrt{2}}{2}\begin{pmatrix} 0 \\ 1 \\ -i \\ 0 \end{pmatrix}$. Since Ψ is a trivialization of $\mathfrak{O}$ (oriented orthonormal bases), $\Psi^{-1}_{T_{\mathbb{C}}\mathcal{M}}$ sends the canonical basis of $\mathbb{R}^4$ to an oriented orthonormal basis of $T\mathcal{M}$. Therefore $(o \otimes \overline{o}, \iota \otimes \overline{\iota}, o \otimes \overline{\iota})$ is oriented in the sense of definition 4.4.1.*

Properties of d

Proposition 4.4.4. *The map d is a double covering map from $\mathcal{A}$ to $\mathcal{N}$ and for $[(z, [u]_4)] \in G_{\mathcal{A}}$ we have*

$$d(a \cdot [(z, [u]_4)]) = d(a) \cdot (z^2, [u]_2)$$

Remark 4.4.5. *The map $\begin{cases} \mathbb{C}^* \rtimes_f \mathbb{Z}/4\mathbb{Z} \to \mathbb{C}^* \rtimes_g \mathbb{Z}/2\mathbb{Z} \\ (z, [u]_4) \mapsto (z^2, [u]_2) \end{cases}$ is well defined and is a group morphism (the key point is that $f([u]_4) = g([u]_2)$). The normal subgroup H is included in the kernel so we have a group morphism $\begin{cases} G_{\mathcal{A}} \to \mathbb{C}^* \rtimes_g \mathbb{Z}/2\mathbb{Z} \\ [(z, [u]_4)] \mapsto (z^2, [u]_2) \end{cases}$. The map is surjective and its kernel is the discrete normal subgroup $\{[(1, [0]_4)], [(-1, [0]_4)]\}$. Therefore, it is a Lie group double covering map.*

Proof. Let $x \in \mathcal{M}$, we define U, Φ and Ψ as in the proof of proposition 4.3.4. Then we have $\Phi_{T_{\mathbb{C}}\mathcal{M}^3} \circ d \circ \left(\Psi^{-1}_{\mathcal{S}\times\mathcal{S}}\right)_{|_{U\times A}} = Id_U \times \hat{d}$ where

$$\hat{d} : \begin{cases} A \to N \\ \left(\begin{pmatrix} z \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ z^{-1} \end{pmatrix}\right) \mapsto \left(\frac{1}{\sqrt{2}}\begin{pmatrix} |z|^2 \\ 0 \\ 0 \\ |z|^2 \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} |z|^{-2} \\ 0 \\ 0 \\ -|z|^{-2} \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ \frac{z^2}{|z|^2} \\ -i\frac{z^2}{|z|^2} \\ 0 \end{pmatrix} \right) \\ \left(\begin{pmatrix} 0 \\ iz^{-1} \end{pmatrix}, \begin{pmatrix} iz \\ 0 \end{pmatrix}\right) \mapsto \left(\frac{1}{\sqrt{2}}\begin{pmatrix} |z|^{-2} \\ 0 \\ 0 \\ -|z|^{-2} \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} |z|^2 \\ 0 \\ 0 \\ |z|^2 \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ \frac{\overline{z}^2}{|z|^2} \\ i\frac{\overline{z}^2}{|z|^2} \\ 0 \end{pmatrix} \right) \end{cases}$$

(A and N are as defined in the proofs of propositions 4.3.4 and 4.4.2. The submanifold A is parametrized by $G_{\mathcal{A}}$ using the maps α and B is parametrized by $\mathbb{C}^* \rtimes_g \mathbb{Z}/2\mathbb{Z}$ using the map β). With this expression, we deduce the expression of the map $\beta^{-1} \circ \hat{d} \circ \alpha$:

$$\beta^{-1} \circ \hat{d} \circ \alpha : \begin{cases} G_{\mathcal{A}} \to \mathbb{C}^* \rtimes_g \mathbb{Z}/2\mathbb{Z} \\ [(z, [u]_4)] \mapsto (z^2, [u]_2) \end{cases}$$

To conclude the proof, note that $Id_U \times \left(\beta^{-1} \circ \hat{d} \circ \alpha\right)$ is the expression of the map d in local trivializations for the structure of principal bundles (the same trivializations used to define the structure in the proof of propositions 4.3.4 and 4.4.2). □

4.5 Vector bundles associated with $\mathcal{A}$ and $\mathcal{N}$ and spin weighted functions

In this section, we have to make an assumption on the topology of $\mathcal{A}$. We assume that $\mathcal{A}$ has exactly two connected components. This assumption is true for a large variety of spacetimes of interest thanks to the following proposition:

Proposition 4.5.1. *If $\mathcal{M}$ is simply connected, then $\mathcal{A}$ and $\mathcal{N}$ have exactly two connected components.*

Proof. We sketch the proof for $\mathcal{A}$, then see remark 4.5.2 to deduce the result for $\mathcal{N}$. We know that each fiber of $\mathcal{A}$ has two connected components diffeomorphic to $\mathbb{C}^*$. Then we deduce (since $\mathcal{M}$ is connected) that $\mathcal{A}$ has one or two connected components. By contradiction, assume that $\mathcal{A}$ has only one connected component. Take $x \in \mathcal{M}$, we call $\mathbb{C}_0^*$ and $\mathbb{C}_1^*$ the two connected components of $\mathcal{A}_x$. By the hypothesis, there exists a continuous path $\gamma : [0,1] \to \mathcal{A}$ such that $\gamma(0) = 1_0$ and $\gamma(1) = 1_1$. Then $\pi_{\mathcal{A}} \circ \gamma$ is a loop on $\mathcal{M}$. But $\mathcal{M}$ is simply connected. Therefore there exists a homotopy $f : [0,1] \times [0,1] \to \mathcal{M}$ between $\pi_{\mathcal{A}} \circ \gamma$ and the constant loop $t \in [0,1] \mapsto x$ such that $f_t(0) = x$ and $f_t(1) = x$ for all $t \in [0,1]$. But $\mathcal{A}$ is a fiber bundle over $\mathcal{M}$ so it has the homotopy lifting property and we can find a lift $\tilde{f}$ of f such that $\tilde{f}_0 = \gamma$. But the concatenation of $t \in [0,1] \mapsto \tilde{f}_t(0)$, $t \in [0,1] \mapsto \tilde{f}_1(t)$ and $t \in [0,1] \mapsto \tilde{f}_{1-t}(1)$ is a continuous path with values in $\mathcal{A}_x$ joining 1_0 and 1_1 which is a contradiction. □

Remark 4.5.2. *If $\mathcal{A}$ has two connected components $\mathcal{A}_0$ and $\mathcal{A}_1$, then $d(\mathcal{A}_0)$ and $d(\mathcal{A}_1)$ are disjoint. Indeed, if $d(x) = d(y)$ either $x = y$ or, by proposition 4.4.4 (and the end of remark 4.4.5), we have $y = x \cdot [(-1, [0]_4)]$ and the continuous path $t \mapsto x \cdot [(e^{it\pi}, [0]_4)]$ joins x and y. Therefore $\mathcal{N}$ also has two connected components given by $d(\mathcal{A}_0)$ and $d(\mathcal{A}_1)$ (Indeed, these two sets are connected, and there exists no continuous path from one to the other otherwise we could lift this path to a path between $\mathcal{A}_0$ and $\mathcal{A}_1$).*

Proposition 4.5.3. *We assume that $\mathcal{M}$ is connected. $\mathcal{N}$ has two connected components if and only if there exist two global smooth null future oriented vector fields l and n such that at each point $x \in \mathcal{M}$, $l(x)$ and $n(x)$ are independent and principal.*

Proof. We assume that $\mathcal{N}$ has two connected components. We choose one that we call $\mathcal{N}_0$. Then if $(l, n, m), (l', n', m') \in \mathcal{N}_0$ with $\pi_{\mathcal{N}}(l, n, m) = \pi_{\mathcal{N}}(l', n', m')$, there exists $z \in \mathbb{C}^*$ such that $(l, n, m) = (l', n', m') \cdot (z, [0]_2) = (|z|l', |z|^{-1}n', \frac{z}{|z|}m')$. In particular l and l' are positively collinear as well as n and n'. Then any convex combination of l and l' is principal null, the same for n and n' and the two are independent. This remark enables us to construct global vector fields n and l from local sections of $\mathcal{N}_0$ using a partition of unity. If we assume the existence of the global vector fields l and n, then we have the two connected components:

$\mathcal{N}_0 := \{(u,v,m) : u \text{ and } l \text{ are collinear } \text{ and } v \text{ and } n \text{ are collinear } \}$ and
$\mathcal{N}_1 := \{(u,v,m) : u \text{ and } n \text{ are collinear } \text{ and } v \text{ and } l \text{ are collinear } \}$. □

From now we assume that $\mathcal{A}$ has two connected components. We choose one component that we call $\mathcal{A}_0$ and we define $\mathcal{N}_0 := d(\mathcal{A}_0)$ (which is one of the two connected components of $\mathcal{N}$ according to remark 4.5.2). The other connected component is called $\mathcal{A}_1$ (and $\mathcal{N}_1$). This choice defines an additional notion of orientation which corresponds to an ordering of the principal null directions ($\mathcal{A}_0$ is the subset of oriented elements of $\mathcal{A}$).

The right action of the subgroup $\mathbb{C}_0^* := \{[(z,[0]_4)], z \in \mathbb{C}^*\}$ of $G_{\mathcal{A}}$ gives $\mathcal{A}_0$ the structure of a $\mathbb{C}^*$-principal bundle. Similarly, $\{(z,[0]_2), z \in \mathbb{C}^*\}$ gives $\mathcal{N}_0$ the structure of a $\mathbb{C}^*$-principal bundle. The action of $[(1,[1]_4)]$ induces a diffeomorphism between $\mathcal{A}_0$ and $\mathcal{A}_1$ (similarly, the action of $(1,[1]_2)$ induces a diffeomorphism between $\mathcal{N}_0$ and $\mathcal{N}_1$).

Remark 4.5.4. *Since $\mathbb{C}^*$ is commutative, the right action is also a left action (and we use both notations in the following).*

Let $w, s \in \frac{1}{2}\mathbb{Z}$, we have the following representation of $\mathbb{C}^*$:

$$\rho_{s,w} : \begin{cases} \mathbb{C}^* \to GL(\mathbb{C}) \\ z \mapsto (a \mapsto z^{-w-s}\overline{z}^{-w+s}a) \end{cases}$$

We define the bundle $\mathcal{B}(s,w)$ as the complex line bundle associated to $\mathcal{A}_0$ (with the right action) and the representation $\rho_{s,w}$. We have a natural identification between sections of $\mathcal{B}(s,w)$ and the set of complex valued functions f defined on $\mathcal{A}_0$ such that for all $z \in \mathbb{C}^*$

$$f(a \cdot z) = z^{w+s}\overline{z}^{w-s}f(a). \tag{4.2}$$

The identification is given by: $f \mapsto (x \mapsto [(a, f(a))]$ where a is any element of $(\mathcal{A}_0)_x)$. We call a spin weighted function with weights (s,w) any section of $\mathcal{B}(s,w)$ or equivalently (with the identification) any function on $\mathcal{A}_0$ satisfying (4.2). We denote by $W_{(s,w)}$ the set of spin weighted functions.

Remark 4.5.5. *We have the following canonical identification $\mathcal{B}(s+s', w+w') = \mathcal{B}(s,w) \otimes \mathcal{B}(s',w')$.*

Remark 4.5.6. *The number s is called the spin weight and the number w is called the boost weight in [88, Section 4.12].*

We call $\boldsymbol{o}$ (resp. $\boldsymbol{\iota}$) the first (resp. second) projection from $\mathcal{A}_0$ to $\mathcal{S}$ and we define $\boldsymbol{l} := \boldsymbol{o}\otimes\overline{\boldsymbol{o}}$, $\boldsymbol{n} := \boldsymbol{\iota}\otimes\overline{\boldsymbol{\iota}}$ and $\boldsymbol{m} = \boldsymbol{o}\otimes\overline{\boldsymbol{\iota}}$. Note that thanks to the map j the maps $\boldsymbol{l}$, $\boldsymbol{m}$ and $\boldsymbol{n}$ can be seen as $T_{\mathbb{C}}\mathcal{M}$ valued maps. Note that due to the relation $\boldsymbol{o}(u \cdot z) = z\boldsymbol{o}(u)$ (resp. $\boldsymbol{\iota}(u \cdot z) = z^{-1}\boldsymbol{\iota}(u)$) for $u \in \mathcal{A}_0$, we can identify $\boldsymbol{o}$ (resp. $\boldsymbol{\iota}$) with a smooth section of the bundle $\mathcal{B}\left(\frac{1}{2}, \frac{1}{2}\right) \otimes \mathcal{S}$ (resp. $\mathcal{B}\left(-\frac{1}{2}, -\frac{1}{2}\right) \otimes \mathcal{S}$).

The following proposition is the main reason of why we are interested in spin weighted functions.

Proposition 4.5.7 (Spin weighted components of cospinors)**.** *We denote by $\mathcal{S}_{a,b} := \otimes^a\mathcal{S}' \otimes^b \overline{\mathcal{S}}'$. There is a bijection F between the set of sections of $\mathcal{S}_{a,b}$ and the set $\prod_{I\subset[\![1,a]\!],J\subset[\![1,b]\!]} W_{(|I|-|J|+\frac{b-a}{2},|I|+|J|-\frac{a+b}{2})}$ given by $F : u \mapsto \prod_{I\subset[\![1,a]\!],J\subset[\![1,b]\!]} u_{I,J}$ where for $y \in \mathcal{A}$, $u_{I,J}(y) = u(g_1(y), \dots, g_a(y), h_1(y), \dots, h_b(y))$ with $g_i = \boldsymbol{o}$ if $i \in I$, $g_i = \boldsymbol{\iota}$ if $i \notin I$, $h_i = \overline{\boldsymbol{o}}$ if $i \in J$ and $h_i = \overline{\boldsymbol{\iota}}$ if $i \notin J$. We call the collection $F(u)$ the collection of spin weighted components of u.*

Proof. The fact that the components are actually spin weighted functions with the claimed weight is a consequence of the $\mathbb{C}$-linearity of cospinors and the fact that $o(y \cdot z) = zo(y)$ and $\iota(y \cdot z) = z^{-1}\iota(y)$. The fact that F is bijective follows from the construction of the inverse map. Indeed, if we fix $x \in \mathcal{M}$ and $y \in (\mathcal{A}_0)_x$, the spin weighted components evaluated at y give exactly the image of a basis of $\otimes^a \mathcal{S}_x \otimes^b \overline{\mathcal{S}}_x$ which correspond to the data of an element in $(\mathcal{S}_{a,b})_x$. The fact that this element does not depend on the choice of y follows from the property (4.2). □

Remark 4.5.8. *If u has some regularity as a section, its spin weighted components have the same regularity (as sections of the appropriate line bundle $\mathcal{B}(s,w)$) and vice versa. It is the major advantage of spin weighted components of a smooth tensor: they are defined globally as smooth objects while being particularly adapted to the geometry.*

Remark 4.5.9. *Thanks to the identification between $\mathcal{S} \otimes \overline{\mathcal{S}}$ and $T_{\mathbb{C}}\mathcal{M}$, we have also spin weighted components for tensor fields.*

Remark 4.5.10. *Proposition 4.5.7 can also be understood if we consider $\boldsymbol{o}$ and $\boldsymbol{\iota}$ as spin weighted spinors (elements of $\Gamma(\mathcal{B}(s,w) \otimes \mathcal{S})$). Indeed, if $u \in \mathcal{S}_{a,b}$, spin weighted components are complete contractions of $u \otimes g_1 \otimes ... \otimes g_a \otimes h_1 \otimes ... \otimes h_b$ and are therefore sections of the tensor product of factors of the form $\mathcal{B}\left(\pm\frac{1}{2}, \pm\frac{1}{2}\right)$ and $\mathcal{B}\left(\pm\frac{1}{2}, \mp\frac{1}{2}\right)$ (and by remark 4.5.5 sections of some $\mathcal{B}(s,w)$).*

Remark 4.5.11. *If (o, ι) is a local section of $\mathcal{A}_0$, it provides a local trivialization of $\mathcal{A}_0$ and therefore a local trivialization of $\mathcal{B}(s,w)$. The expression of a spin weighted component in this trivialization is obtained by replacing $\boldsymbol{o}$ by o, $\boldsymbol{\iota}$ by ι, $\boldsymbol{l}$ by l, $\boldsymbol{m}$ by m and $\boldsymbol{n}$ by n in the expression of the component. Therefore the bold font notation is handy to take local trivializations. However, $\boldsymbol{o}$ and $\boldsymbol{\iota}$ do not depend on the choice of a particular local trivialization.*

Remark 4.5.12. *This decomposition is often used after a first decomposition of the cospinor or cotensor into symmetric spinors (see [88] section 3.3 for more details about this type of decomposition). For example, because the electromagnetic tensor* F *is antisymmetric and real, it can be decomposed as:*

$$\mathrm{F} = \phi \otimes \overline{\epsilon} + \epsilon \otimes \overline{\phi} \tag{4.3}$$

(see (3.4.20) in [88] for details) where ϕ is a section of $(\mathcal{S}')^{\odot 2}$ (where $\odot$ is the symmetric product). Then, the spin weighted components of ϕ can be computed using equation (4.3)*:*

$$\begin{aligned}
\phi(\boldsymbol{o}, \boldsymbol{o}) &= \mathrm{F}(\boldsymbol{l}, \boldsymbol{m}) \\
\phi(\boldsymbol{o}, \boldsymbol{\iota}) &= \frac{1}{2}\left(\mathrm{F}(\overline{\boldsymbol{m}}, \boldsymbol{m}) + \mathrm{F}(\boldsymbol{l}, \boldsymbol{n})\right) \\
\phi(\boldsymbol{\iota}, \boldsymbol{\iota}) &= \mathrm{F}(\overline{\boldsymbol{m}}, \boldsymbol{n})
\end{aligned}$$

If we fix a local section (o, ι) of $\mathcal{A}_0$ and write the components in the associated local trivialization, we find the usual spin components of the electromagnetic tensor.

Similarly, we have the following decomposition for the Weyl tensor W (see [88, (4.6.41)]):

$$W = \Psi \otimes \overline{\epsilon} \otimes \overline{\epsilon} + \epsilon \otimes \epsilon \otimes \overline{\Psi} \tag{4.4}$$

where Ψ is a section of $(\mathcal{S}')^{\odot 4}$. We can compute the spin weighted components of Ψ from components of W:

$$\begin{aligned}
\boldsymbol{\Psi}_0 :=& \Psi(\boldsymbol{o},\boldsymbol{o},\boldsymbol{o},\boldsymbol{o}) = W(\boldsymbol{l},\boldsymbol{m},\boldsymbol{l},\boldsymbol{m})\\
\boldsymbol{\Psi}_1 :=& \Psi(\boldsymbol{o},\boldsymbol{o},\boldsymbol{o},\boldsymbol{\iota}) = W(\boldsymbol{l},\boldsymbol{m},\boldsymbol{l},\boldsymbol{n})\\
\boldsymbol{\Psi}_2 :=& \Psi(\boldsymbol{o},\boldsymbol{o},\boldsymbol{\iota},\boldsymbol{\iota}) = W(\boldsymbol{l},\boldsymbol{m},\overline{\boldsymbol{m}},\boldsymbol{n})\\
\boldsymbol{\Psi}_3 :=& \Psi(\boldsymbol{o},\boldsymbol{\iota},\boldsymbol{\iota},\boldsymbol{\iota}) = W(\boldsymbol{l},\boldsymbol{n},\overline{\boldsymbol{m}},\boldsymbol{n})\\
\boldsymbol{\Psi}_4 :=& \Psi(\boldsymbol{\iota},\boldsymbol{\iota},\boldsymbol{\iota},\boldsymbol{\iota}) = W(\overline{\boldsymbol{m}},\boldsymbol{n},\overline{\boldsymbol{m}},\boldsymbol{n})
\end{aligned}$$

Since $\boldsymbol{l}$ and $\boldsymbol{n}$ (seen as maps from $\mathcal{A}_0$ to $T_{\mathbb{C}}\mathcal{M}$) are valued in the set of principal null vectors and the spacetime is of type D, all the components vanish except $\boldsymbol{\Psi}_2$. Note that we adopted the sign convention of [88, (4.11.9)]. In [18, Chapter 1, (294)] and in [101, (1.3)], the authors add a minus sign but since the Weyl tensor also has a different sign convention, the definition is in fact the same.

Remark 4.5.13. *We can also define spin weighted components for sections of $\mathcal{S}_{a,b}\otimes\mathcal{B}(s,w)$ (using remark 4.5.5).*

4.6 Reduction of $\mathcal{A}_0$ and $\mathcal{N}_0$

To simplify the computations it is useful to find a smaller principal bundle with a representation such that the associated vector bundle is isomorphic to $\mathcal{B}(s,w)$. We can consider $\mathcal{A}_{0,r} := \mathcal{A}_0/\mathbb{R}_+^*$ (we quotient by the action of $\mathbb{R}_+^* \subset \mathbb{C}^*$). Similarly we define $\mathcal{N}_{0,r} := \mathcal{N}_0/\mathbb{R}_+^*$. We verify that the map d induces a double cover between $\mathcal{A}_{0,r}$ and $\mathcal{N}_{0,r}$ (we still call this induced map d). Moreover $\mathcal{A}_{0,r}$ and $\mathcal{N}_{0,r}$ both have the structure of a $U(1)$-principal bundle over $\mathcal{M}$.

Remark 4.6.1. *According to proposition 4.5.3, we have a global choice of independent principal real smooth null vector fields l and n. It enables us to make a global choice of representative for $\mathcal{A}_{0,r}$ and $\mathcal{N}_{0,r}$ (note that the ordering (l,n) gives a choice of connected component). We have the following identifications for the reduced bundles*

$$\begin{aligned}
\mathcal{A}_{0,r} \simeq& \{(o,\iota)\in\mathcal{S}\times\mathcal{S} : o\otimes\overline{o} = l, \iota\otimes\overline{\iota} = n\}\\
\mathcal{N}_{0,r} \simeq& \{m\in T_{\mathbb{C}}\mathcal{M} : g(m,m) = g(l,m) = g(n,m) = 0,\\
&\qquad g(m,\overline{m}) = -1, g(l,n) = 1 \text{ and } (n,l,m) \text{ is oriented}\}\,.
\end{aligned}$$

Pay attention to the fact that the identification of $\mathcal{A}_{0,r}$ depends on the particular choice of l and n but the identification of $\mathcal{N}_{0,r}$ only depends on the ordering of l and n.

Remark 4.6.2. *In both cases, the choice of l and n enables to associate each local smooth section of $\mathcal{A}_{0,r}$ (resp. $\mathcal{N}_{0,r}$) to a local smooth section of $\mathcal{A}_0$ (resp. $\mathcal{N}_0$). Therefore, when a choice of l and n has been made, we can work with $\mathcal{A}_{0,r}$ and $\mathcal{N}_{0,r}$ instead of $\mathcal{A}_0$ and $\mathcal{N}_0$.*

Chapter 5

Concrete computations in the subextremal Kerr exterior

We define the Kerr metric with mass parameter M and angular momentum per unit of mass a. We assume that $0 < a < M$ (subextremal Kerr). The Kerr metric is given by:

$$g = \left(1 - \frac{2Mr}{\rho^2}\right) \mathrm{d}t^2 + \left(4Mar\frac{\sin^2\theta}{\rho^2}\right) \mathrm{d}t\, \mathrm{d}\phi - \frac{\rho^2}{\Delta_r}\, \mathrm{d}r^2 - \rho^2\, \mathrm{d}\theta^2 - \sin^2\theta \left(r^2 + a^2 + 2Ma^2 r\frac{\sin^2\theta}{\rho^2}\right) \mathrm{d}\phi^2$$

with

$$\Delta_r := r^2 - 2Mr + a^2$$
$$\rho^2 := r^2 + a^2\cos^2\theta$$

We define $r_0 := M + \sqrt{M^2 - a^2}$ and we consider first the Kerr exterior $\mathcal{M} := \mathbb{R}_t \times (r_0, +\infty) \times \mathbb{S}^2$. We also define the $Kerr_*$ coordinates: $(t_*, r, \theta, \phi_*) = (t + T(r), r, \theta, \phi + A(r))$ with $T(r) := \int_{r_1}^r \frac{a^2+r^2}{\Delta_r}\, \mathrm{d}r$ and $A(r) = \int_{r_1}^r \frac{a}{\Delta_r}\, \mathrm{d}r$ for some arbitrary (but fixed) $r_1 \in (r_0, +\infty)$. Kerr space time is an important example of a Petrov type D spacetime (Ricci-flat). In this section we make the previous definitions explicit in this concrete case.

5.1 Complete system of trivializations

We now compute the concrete topology of the bundles in the Kerr case. We will show on the way that there is no global continuous oriented Newman Penrose tetrad (global continuous section of $\mathcal{N}_0$) nor a global continuous normalized spin frame along the null directions (global section of $\mathcal{A}_0$). Let $M > 0$ and $a < M$. We endow $\mathcal{M} := \mathbb{R}_t \times (r_0, +\infty) \times \mathbb{S}^2$ with the Kerr metric. It is a Petrov type D simply connected spacetime. We saw that in this case, $\mathcal{A}$ and $\mathcal{N}$ have two connected components (proposition 4.5.1). Then, looking at the proof of proposition 4.5.3, we see that a choice of a connected component is induced by a choice of global linearly independent smooth vector fields (l, n) which are future oriented principal null directions. Here we take (Kinnersley's tetrad):

$$l = \frac{r^2 + a^2}{\Delta_r}\partial_t + \partial_r + \frac{a}{\Delta_r}\partial_\phi \tag{5.1}$$

$$n = \frac{r^2 + a^2}{2\rho^2}\partial_t - \frac{\Delta_r}{2\rho^2}\partial_r + \frac{a}{2\rho^2}\partial_\phi \tag{5.2}$$

We use the identification in remark 4.6.1 to describe $\mathcal{A}_{0,r}$ and $\mathcal{N}_{0,r}$. We define $p = r + ia\cos\theta$. Then we can see that

$$m = \frac{ia\sin\theta}{\sqrt{2}p}\partial_t + \frac{1}{\sqrt{2}p}\partial_\theta + \frac{i}{\sqrt{2}p\sin\theta}\partial_\phi \tag{5.3}$$

is a local section of $\mathcal{N}_{0,r}$ over $\mathcal{M} \setminus \mathbb{R}_t \times (r_0, +\infty) \times \{N, S\}$ where N and S are the north and south poles of $\mathbb{S}^2$. Note that the vector field m cannot be extended to a smooth vector field on $\mathcal{M}$. However it provides a local trivialization of $\mathcal{N}_{0,r}$:

$$\Psi_m : \begin{cases} U(1) \times \mathbb{R}_t \times (r_0, +\infty) \times \left(\mathbb{S}^2 \setminus \{N, S\}\right) \to \mathcal{N}_{0,r} \\ (e^{i\rho}, x) \mapsto e^{i\rho} m(x) \end{cases}$$

Remark 5.1.1. *Note that given a local trivialization $\Psi : U(1) \times U \to \mathcal{N}_{0,r}$ on $\mathcal{N}_{0,r}$, we can define a corresponding local trivialization on $\mathcal{N}_0$ by taking*

$$\tilde{\Psi} : \begin{cases} \mathbb{C}^* \times U \to \mathcal{N}_0 \\ (z, x) \mapsto (|z| l(x), |z|^{-1} n(x), \Psi(\frac{z}{|z|}, x)) \end{cases}$$

Our next goal is to write a complete system of local trivializations on $\mathcal{N}_{0,r}$. We introduce the stereographic coordinates relative to the north pole

$$(x_N, y_N) = \left(\operatorname{cotan}\left(\frac{\theta}{2}\right)\cos\phi, \operatorname{cotan}\left(\frac{\theta}{2}\right)\sin\phi\right)$$

which are local smooth coordinates on $\mathbb{S}^2 \setminus \{N\}$. We define the following map:

$$\Psi_N : \begin{cases} U(1) \times \mathbb{R}_t \times (r_0, +\infty) \times \left(\mathbb{S}^2 \setminus \{N\}\right) \to T_{\mathbb{C}}\mathcal{M} \\ (e^{i\rho}, t, r, x_N, y_N) \mapsto \frac{e^{i\rho}}{\sqrt{2}\left(r + ia\frac{(x_N^2+y_N^2)-1}{1+x_N^2+y_N^2}\right)} \Big(\frac{2ia(x_N - iy_N)}{1+x_N^2+y_N^2}\partial_t \\ \qquad\qquad + \frac{(x_N^2+y_N^2+1)}{2}\left(-\partial_{x_N} + i\partial_{y_N}\right)\Big). \end{cases}$$

Note that Ψ_N commutes with the projection $T_{\mathbb{C}}\mathcal{M} \to \mathcal{M}$. We remark that on $U(1) \times \mathbb{R}_t \times (r_0, +\infty) \times \mathbb{S}^2 \setminus \{N, S\}$ (where $U(1)$ is identified with $U(1) \times \{\mathrm{Id}\}$) we have:

$$\Psi_m = \frac{x_N + iy_N}{\sqrt{x_N^2 + y_N^2}}\Psi_N = e^{i\phi}\Psi_N$$

where ϕ is the usual spherical coordinate on $\mathbb{S}^2$. Using that $\mathcal{N}_{0,r}$ is closed in $T_{\mathbb{C}}\mathcal{M}$ and Ψ_N is continuous, we deduce that Ψ_N has values in $\mathcal{N}_{0,r}$. Moreover, since Ψ_N is a smooth proper injective immersion as a function with values in $T_{\mathbb{C}}\mathcal{M}$, it remains such as a function with values in the submanifold $\mathcal{N}_{0,r}$. Since $U(1) \times \mathbb{R}_t \times (r_0, +\infty) \times \left(\mathbb{S}^2 \setminus \{N\}\right)$ has the same dimension as $\mathcal{N}_{0,r}$, we deduce that Ψ_N defines local coordinates on $\mathcal{N}_{0,r}$. Moreover, we check easily that it trivializes the action of $U(1)$.

We also introduce the stereographic coordinates relative to the south pole

$$(x_S, y_S) = \left(\tan\left(\frac{\theta}{2}\right)\cos\phi, \tan\left(\frac{\theta}{2}\right)\sin\phi\right)$$

which are local smooth coordinates on $\mathbb{S}^2 \setminus \{S\}$. We also define the map Ψ_S:

$$\Psi_S : \begin{cases} U(1) \times \mathbb{R}_t \times (r_0, +\infty) \times \left(\mathbb{S}^2 \setminus \{S\}\right) \to T\mathcal{M} \\ (e^{i\rho}, t, r, x_S, y_S) \mapsto \frac{e^{i\rho}}{\sqrt{2}\left(r + ia\frac{1-(x_S^2+y_S^2)}{1+x_S^2+y_S^2}\right)} \Big(\frac{2ia(x_S + iy_S)}{1+x_S^2+y_S^2}\partial_t \\ \qquad\qquad + \frac{(x_S^2+y_S^2+1)}{2}\left(\partial_{x_S} + i\partial_{y_S}\right)\Big). \end{cases}$$

On $U(1) \times \mathbb{R}_t \times (r_0, +\infty) \times \left(\mathbb{S}^2 \setminus \{N, S\}\right)$, we have:

$$\Psi_m = \frac{x_S - iy_S}{\sqrt{x_S^2 + y_S^2}} \Psi_S = e^{-i\phi} \Psi_S.$$

As previously for Ψ_N, we deduce that Ψ_S defines local coordinates on $\mathcal{N}_{0,r}$ and trivializes the action of $U(1)$.

Note that (for $\omega \in \mathbb{S}^2 \setminus \{N, S\}$ with local stereographic coordinates relative to the north pole (x_N, y_N) and relative to the south pole (x_S, y_S)),

$$\Psi_N(e^{i\rho}, t, r, x_N, y_N) = \frac{x_S - iy_S}{x_S + iy_S} \Psi_S(e^{i\rho}, t, r, x_S, y_S) = \frac{x_N - iy_N}{x_N + iy_N} \Psi_S(e^{i\rho}, t, r, x_S, y_S)$$

and we deduce

$$\Psi_S^{-1} \Psi_N(e^{i\rho}, t, r, x_N, y_N) = \left(\frac{x_N - iy_N}{x_N + iy_N} e^{i\rho}, t, r, x_N, y_N \right).$$

The complete system of local trivializations (Ψ_N, Ψ_S) enables us to show easily the following proposition:

Proposition 5.1.2. *There is no global continuous section of $\mathcal{N}_{0,r}$ (in other words, $\mathcal{N}_{0,r}$ is not the trivial bundle $U(1) \times \mathbb{R}_t \times (r_0, +\infty) \times \mathbb{S}^2$)*

Proof. We argue by contradiction. Let us assume the existence of a global continuous section f. Then we construct $f_1 := \mathrm{pr}_{U(1)} \Psi_N^{-1} \circ f_{|_{\{0\} \times \{r_0+1\} \times \mathbb{S}^2 \setminus \{N\}}}$ which is continuous ($\mathrm{pr}_{U(1)}$ being the projection onto $U(1)$). Using the stereographic coordinates relative to the north pole on $\{0\} \times \{r_0 + 1\} \times \mathbb{S}^2 \setminus \{N\}$, we can see f_1 as a function from $\mathbb{R}^2$ to $U(1)$. Using the same construction with respect to the south pole ($f_2 = \mathrm{pr}_{U(1)} \Psi_S^{-1} \circ f_{|_{\{0\} \times \{r_0+1\} \times \mathbb{S}^2 \setminus \{S\}}}$) and the identification using stereographic coordinates relative to the south pole, we obtain a continuous function f_2 from $\mathbb{R}^2$ to $U(1)$. The two constructions overlap and going through the various identifications, we get the relation $f_1(x, y) = \frac{(x+iy)^2}{x^2+y^2} f_2(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2})$ on $\mathbb{R}^2 \setminus 0$. We define $g : (0, +\infty) \times U(1) \to U(1)$ by $g(r, \omega) := f_1(r\omega)$. Since f_1 and f_2 are continuous at $(0, 0)$, g can be continuously extended by $g(0, \omega) = f_1(0)$ and $g(+\infty, \omega) = \omega^2 f_2(0)$. This extension is a homotopy between two loops with different indices hence we have a contradiction and there is no global continuous section. □

Remark 5.1.3. *A global continuous section on $\mathcal{N}_0$ composed with the projection of $\mathcal{N}_0$ onto $\mathcal{N}_{0,r}$ provides a global continuous section on $\mathcal{N}_{0,r}$. Therefore there is no global continuous section of $\mathcal{N}_0$ either. Similarly (using the map d) there is no global continuous section of $\mathcal{A}_0$ and $\mathcal{A}_{0,r}$.*

From the complete system of local trivializations on $\mathcal{N}_0$, we can deduce a complete system of local trivializations on $\mathcal{A}_0$. Indeed, the previous discussion shows that we have the sections $s_N = (l, n, e^{-i\phi} m)$ (smooth on $\mathbb{R}_t \times (r_0, +\infty) \times \mathbb{S}^2 \setminus \{N\}$) and $s_S = (l, n, e^{i\phi} m)$ (smooth on $\mathbb{R}_t \times (r_0, +\infty) \times \mathbb{S}^2 \setminus \{S\}$) of $\mathcal{N}_0$. Since $\mathbb{R}_t \times (r_0, +\infty) \times \mathbb{S}^2 \setminus \{N\}$ is simply connected and d is a double covering map, we have exactly two lifts of s_N as a local smooth section of $\mathcal{A}_0$ on $\mathbb{R}_t \times (r_0, +\infty) \times \mathbb{S}^2 \setminus \{N\}$. We fix a choice (o_N, ι_N) of such a section. Then we define $(o, \iota)_S = (e^{i\phi} o_N, e^{-i\phi} \iota_N)$ which is smooth on $\mathbb{R}_t \times (r_0, +\infty) \times \mathbb{S}^2 \setminus \{S, N\}$ and we want to prove that $(o, \iota)_S$ extends smoothly to $\mathbb{R}_t \times (r_0, +\infty) \times \mathbb{S}^2 \setminus \{S\}$. To show that, we first check that $d \circ (o, \iota)_S = s_S$ on $\mathbb{R}_t \times (r_0, +\infty) \times \mathbb{S}^2 \setminus \{S, N\}$. Let $x \in \mathbb{R}_t \times (r_0, +\infty) \times \mathbb{S}^2 \setminus \{N, S\}$; we know that there exists a unique smooth lift $\tilde{s}_S$ of s_S with value $(o, \iota)_S(x)$ at x. The set such that $\tilde{s}_S = (o, \iota)_S$ is open (we check that in an open set of trivialization containing a point x_0 such

that $\tilde{s}_S(x_0) = (o,\iota)_S(x_0))$, closed (as a subset of $\mathbb{R}_t \times (r_0,+\infty) \times \mathbb{S}^2 \setminus \{S,N\}$) by continuity of $\tilde{s}_S$ and $(o,\iota)_S$ and non empty. Therefore $(o,\iota)_S = \tilde{s}_S$. Finally, these two sections provide a complete system of trivializations of $\mathcal{A}_0$:

$$A_N^{-1} : \begin{cases} \mathbb{C}^* \times \mathbb{R}_t \times (r_0,+\infty) \times \mathbb{S}^2 \setminus \{N\} \to \mathcal{A}_0 \\ (z,x) \mapsto (zo_N(x), z^{-1}\iota_N(x)) \end{cases}$$
$$A_S^{-1} : \begin{cases} \mathbb{C}^* \times \mathbb{R}_t \times (r_0,+\infty) \times \mathbb{S}^2 \setminus \{S\} \to \mathcal{A}_0 \\ (z,x) \mapsto (zo_S(x), z^{-1}\iota_S(x)) \end{cases}$$

with change of trivializations given by the map $A_N A_S^{-1}(z,x) = (e^{i\phi}z, x)$. Note that there is no lift of the local section (l,n,m) smooth on $\mathbb{R}_t \times (r_0,+\infty) \times \mathbb{S}^2 \setminus \{N,S\}$. Indeed, the local section of $\mathcal{A}_0$ $(o_m,\iota_m) := (e^{i\frac{\phi}{2}}o_N, e^{-i\frac{\phi}{2}}\iota_N) = (e^{-i\frac{\phi}{2}}o_S, e^{i\frac{\phi}{2}}\iota_N)$ defined on $\mathbb{R}_t \times (r_0,+\infty) \times \mathbb{S}^2 \setminus \{\phi = 0\}$ is such a lift but it does not extend continuously to $\mathbb{R}_t \times (r_0,+\infty) \times \mathbb{S}^2 \setminus \{N,S\}$. We denote by A_m the local trivialization of $\mathcal{A}_0$ associated to (o_m,ι_m). It is traditionally used to write the Teukolsky operator.

Remark 5.1.4 (Bundles on an extended Kerr spacetime)**.** *We can also compute local trivializations of $\mathcal{A}_0$ and $\mathcal{N}_0$ on a larger Kerr space time. More precisely, using Kerr star coordinates (t_*, r, ω_*), we can extend the Kerr metric to a larger spacetime which is given by $\mathcal{M}_{ext} := \mathbb{R}_{t^*} \times (r_0 - \epsilon, +\infty)_r \times \mathbb{S}^2_{\omega_*}$. There are very few modifications with respect to the computations on the exterior. The main thing to note is that we cannot make the same choice of l and n as previously since they do not extend smoothly across the future horizon $\mathcal{H} = \{r = r_0\}$. Thus we renormalize them:*

$$\tilde{l} = \Delta_r l$$
$$\tilde{n} = \Delta_r^{-1} n$$

and extend them as independent future oriented principal null vector fields. However the vector field m extends smoothly to $\mathbb{R}_{t^*} \times (r_0 - \epsilon, +\infty)_r \times \left(\mathbb{S}^2_{\omega_*} \setminus \{N,S\}\right)$. Indeed we have in Kerr star coordinates:

$$m = \frac{ia\sin\theta}{\sqrt{2}p}\partial_{t_*} + \frac{1}{\sqrt{2}p}\partial_\theta + \frac{i}{\sqrt{2}p\sin\theta}\partial_{\phi_*}$$

where $p = r + ia\cos\theta = r + ia\cos\theta$ as previously. This expression defines an extension of m on $\mathcal{M}_{ext}$ such that for all $x \in \mathcal{M}_{ext}$, $(\tilde{l}(x), \tilde{n}(x), m(x)) \in \mathcal{N}_0$. As previously, we can compute explicitly a complete system of local trivializations of the bundles $\mathcal{N}_{0,r}$, $\mathcal{A}_{0,r}$, $\mathcal{N}_0$ and $\mathcal{A}_0$.

5.2 Link with the Hopf fibration

In this section, we see $\mathbb{S}^3$ as the unit quaternions group; that is to say

$$\mathbb{S}^3 := \left\{a + bi + cj + dk, (a,b,c,d) \in \mathbb{R}^4 : a^2 + b^2 + c^2 + d^2 = 1\right\} \subset \mathbb{H}.$$

For $h \in \mathbb{S}^3$, the subset of imaginary quaternions $I = \left\{bi + cj + dk : (b,c,d) \in \mathbb{R}^3\right\}$ is stable by the map $c_h : h' \mapsto hh'h^*$ (h^* is the conjugate of h in the sense of quarternions and since $h \in \mathbb{S}^3$, $h^* = h^{-1}$). The map c_h is even an orthogonal map for the usual norm on I since $(hh'h^*)(hh'h^*)^* = hh'h'^*h^* = \|h'\|^2$. Therefore, $\Psi : h \mapsto c_h$ defines a Lie group morphism from $\mathbb{S}^3$ to $O(3)$ (note that I is identified with $\mathbb{R}^3$ by sending (i,j,k) to the canonical basis). Since $\mathbb{S}^3$ is connected and $c_1 = Id$, we have that $\Psi(\mathbb{S}^3) \subset SO(3)$. Finally, we compute the kernel of Ψ. Let h be such that $c_h = Id$. Then for all $h' \in I$, we have:

$$hh'h^* = h'$$
$$hh' = h'h \text{ (since } h^*h = 1)$$

Therefore, h commutes with every element of I. But $I + Z(\mathbb{H}) = \mathbb{H}$ (where $Z(\mathbb{H}) = \mathbb{R}$ is the center of $\mathbb{H}$). Therefore, $h \in Z(\mathbb{H})$. Finally, since $h \in \mathbb{S}^3$, we find $h = 1$ or $h = -1$. Therefore, Ψ is a Lie group morphism between two connected Lie groups with finite kernel of size two. Therefore it is a double covering map. We can identify $SO(3)$ with $\mathfrak{O}_{\mathbb{S}^2}$, the bundle of oriented orthonormal frames on $\mathbb{S}^2$ (we identify a matrix in $SO(3)$ with columns C_1, C_2 and C_3 with the basis $(C_2, C_3) \in T_{C_1}\mathbb{S}^2$). We denote by $\pi_{\mathbb{S}^2} : SO(3) \to \mathbb{S}^2$ the projection when $SO(3)$ is seen as the bundle of oriented orthonormal frames on $\mathbb{S}^2$ (therefore $\pi_{\mathbb{S}^2}(M)$ is the first column of the matrix with the identification that we have chosen).

The Hopf fibration can be defined as $H = \pi_{\mathbb{S}^2} \circ \Psi : \mathbb{S}^3 \to \mathbb{S}^2$ (which is a smooth submersion). We identify $\mathbb{S}^2$ with the subset of of imaginary quaternions with unit norm. Let $u \in \mathbb{S}^2 \subset I$, by definition $H^{-1}(u) = \left\{h \in \mathbb{S}^3 : hih^* = u\right\}$. We see that we have a right smooth fiber preserving action of $U(1) := \mathbb{S}^3 \cap (\mathbb{R} + i\mathbb{R})$ on $\mathbb{S}^3$ (given by right multiplication). Moreover, this action is simply transitive on each fiber since if $h_1, h_2 \in H^{-1}(u)$, then $g := h_1^{-1}h_2$ is the only element of $\mathbb{H}$ such that $h_1 g = h_2$ and it belongs to $U(1)$ (indeed it commutes with i and has norm 1). Therefore, the Hopf fibration is a $U(1)$-principal bundle. There is a unique action of $U(1)$ on $\mathfrak{O}_{\mathbb{S}^2}$ such that for all $x \in \mathbb{S}^3$ and $g' \in U(1)$, $\Psi(x \cdot g') = \Psi(x) \cdot g'^2$ and it is defined by $c_h \cdot g = c_{hg'} = c_{-hg'}$ for any g' such that $g'^2 = g$. Writing this more explicitly, we see that for a matrix $M \in SO(3)$ with columns C_1, C_2, C_3 , $M = \Psi(h)$ for some $h \in \mathbb{S}^3$ with $\underset{(i,j,k)}{\mathrm{Mat}}\, hih^* = C_1$, $\underset{(i,j,k)}{\mathrm{Mat}}\, hjh^* = C_2$ and $\underset{(i,j,k)}{\mathrm{Mat}}\, hkh^* = C_3$. Then if $g = \cos(\rho) + i\sin(\rho) \in U(1)$ and $g' = \cos\left(\frac{\rho}{2}\right) + i\sin\left(\frac{\rho}{2}\right)$, $M \cdot g$ has columns

$$\begin{aligned}
\underset{(i,j,k)}{\mathrm{Mat}}\, hg'i(hg')^* &= C_1 \\
\underset{(i,j,k)}{\mathrm{Mat}}\, hg'j(hg')^* &= \underset{(i,j,k)}{\mathrm{Mat}} \left((\cos^2\left(\frac{\rho}{2}\right) - \sin^2\left(\frac{\rho}{2}\right))hjh^* + 2\sin\left(\frac{\rho}{2}\right)\cos\left(\frac{\rho}{2}\right) hkh^*\right) \\
&= \cos(\rho)C_2 + \sin(\rho)C_3 \\
\underset{(i,j,k)}{\mathrm{Mat}}\, hg'k(hg')^* &= -\sin(\rho)C_2 + \cos(\rho)C_3.
\end{aligned}$$

The map $Id_{\mathbb{R}_t \times (r_0, +\infty)} \times H$ enables to put the structure of a $U(1)$-principal bundle on $\mathbb{R}_t \times (r_0, +\infty) \times \mathbb{S}^3$ and similarly, we put the structure of a $U(1)$-principal bundle on $\mathbb{R}_t \times (r_0, +\infty) \times \mathfrak{O}_{\mathbb{S}^2}$.

Proposition 5.2.1. *We define the map:*

$$f : \begin{cases} \mathbb{R}_t \times (r_0, +\infty) \times \mathfrak{O}_{\mathbb{S}^2} \to T_{\mathbb{C}}\mathcal{M} \\ (t, r, (\omega, X, Y) \in \mathfrak{O}_{\mathbb{S}^2}) \mapsto -\frac{ia\langle X + iY, e_3\rangle_{\mathbb{R}^3}}{\sqrt{2}p}\partial_t + \frac{1}{\sqrt{2}p}X + \frac{i}{\sqrt{2}p}Y \in T_{(t,r,\omega)}\mathcal{M} \end{cases}$$

where elements of $\mathbb{S}^2$ (resp. $\mathfrak{O}_{\mathbb{S}^2}$) are represented by unit vectors (resp. triples of unit vectors where the first one represents the base point in $\mathbb{S}^2$) in $\mathbb{R}^3$ (and e_1, e_2, e_3 is the canonical basis of $\mathbb{R}^3$) and $p = r + i\cos(\theta)$. The notation $\langle . \rangle_{\mathbb{R}^3}$ denotes the canonical scalar product on $\mathbb{R}^3$ extended to a $\mathbb{C}$-bilinear form on $\mathbb{C}^3$ (therefore it is not hermitian).

We have $f : \mathbb{R}_t \times (r_0, +\infty) \times \mathfrak{O}_{\mathbb{S}^2} \to \mathcal{N}_{0,r}$ and it is an isomorphism of principal bundles.

Proof. Let $(t, r, \omega) \in \mathcal{M}$ and $(\omega, X, Y) \in \mathfrak{O}_{\mathbb{S}^2}$. To prove that f has values in $\mathcal{N}_{0,r}$, we have to prove that $m := -\frac{ia\langle X + iY, e_3\rangle_{\mathbb{R}^3}}{\sqrt{2}p}\partial_t + \frac{1}{\sqrt{2}p}X + \frac{i}{\sqrt{2}p}Y$ is null, orthogonal to any principal null vector and $g(m, \overline{m}) = -1$.

We can write the Kerr metric on the form:

$$\begin{aligned}
g = &\left(1 - \frac{2Mr}{\rho^2}\right) \mathrm{d}t^2 + \frac{4Mar}{\rho^2}\, \mathrm{d}t(\sin^2\theta\, \mathrm{d}\phi) - \frac{\rho^2}{\Delta_r}\, \mathrm{d}r^2 - \rho^2 g_{\mathbb{S}^2} \\
&- a^2\left(1 + \frac{2Mr}{\rho^2}\right)(\sin^2\theta\, \mathrm{d}\phi)^2
\end{aligned}$$

Moreover, if we see elements $\omega \in \mathbb{S}^2$ as units vectors $\begin{pmatrix}\omega_x\\ \omega_y\\ \omega_z\end{pmatrix}$ on $\mathbb{R}^3$, we have:

$$\sin^2\theta \, \mathrm{d}\phi = -\omega_y \, \mathrm{d}x + \omega_x \, \mathrm{d}y$$

We can therefore compute:

$$\begin{aligned} 2p^2 g(m,m) = &- a^2 \Big(\langle X + iY, e_3\rangle^2 + \langle X + iY, -\omega_y e_1 + \omega_x e_2\rangle^2 \\ &- \frac{2Mr}{\rho^2} \left(\langle X + iY, e_3\rangle - i \langle X + iY, -\omega_y e_1 + \omega_x e_2\rangle \right)^2 \Big) \end{aligned}$$

But note that $-\omega_y e_1 + \omega_x e_2 = e_3 \times \omega$, $\omega \times X = Y$ and $Y \times \omega = X$ (since ω, X, Y is an oriented orthonormal basis of $\mathbb{R}^3$). Then, by definition of the cross product on $\mathbb{R}^3$,

$$\begin{aligned} \langle X, e_3 \times \omega\rangle &= \det(X, e_3, \omega) \\ &= \det(e_3, \omega, X) \\ &= \langle \omega \times X, e_3\rangle \\ &= \langle Y, e_3\rangle \\ \langle Y, e_3 \times \omega\rangle &= - \langle X, e_3\rangle \end{aligned}$$

Therefore, $\langle X + iY, e_3 \times \omega\rangle = \langle Y - iX, e_3\rangle = -i \langle X + iY, e_3\rangle$. We deduce that $2p^2 g(m,m) = 0$.

We compute also

$$\begin{aligned} g(\sqrt{2}pm, \Delta_r l) = &\left(1 - \frac{2Mr}{\rho^2}\right) (-ia \langle X + iY, e_3\rangle) (r^2 + a^2) + \frac{2Ma^2 r}{\rho^2} \sin^2\theta (-ia \langle X + iY, e_3\rangle) \\ &+ \frac{2Mar(r^2 + a^2)}{\rho^2} \langle X + iY, e_3 \times \omega\rangle - a\rho^2 \langle X + iY, e_3 \times \omega\rangle \\ &- a^3 \left(1 + \frac{2Mr}{\rho^2}\right) \langle X + iY, e_3 \times \omega\rangle \sin^2\theta \\ = 0 \quad &(\text{using } \langle X + iY, e_3 \times \omega\rangle = -i \langle X + iY, e_3\rangle) \end{aligned}$$

Very similar computations show:

$$\begin{aligned} g(\sqrt{2}pm, \Delta_r^{-1} n) &= 0 \\ g(m, \overline{m}) &= 1 \end{aligned}$$

The previous computations show that f has values in $\mathcal{N}_{0,r}$. Moreover f is smooth and so is its inverse:

$$f^{-1} : m \in (\mathcal{N}_{0,r})_{(t,r,\omega)} \mapsto (t, r, \sqrt{2}p(\Re(\mathrm{pr}_{T_\omega \mathbb{S}^2, \partial_t} m), \Im(\mathrm{pr}_{T_\omega \mathbb{S}^2, \partial_t} m)))$$

where $\mathrm{pr}_{\mathbb{S}^2, \partial_t}$ is the linear projection onto $T_\omega \mathbb{S}^2$ parallel to ∂_t. Therefore, it is a diffeomorphism. The compatibility with projections maps is immediate. It remains to prove that f is compatible with the actions of $U(1)$. Let $e^{i\rho} \in U(1)$.

$$\begin{aligned} (\omega, X, Y) \cdot e^{i\rho} &= (\omega, \cos(\rho) X + \sin(\rho) Y, -\sin(\rho) Y + \cos(\rho) X) \\ &= (\omega, \Re(e^{i\rho}(X + iY)), \Im(e^{i\rho}(X + iY))) \end{aligned}$$

Therefore, if $(\omega, X, Y) \in \mathfrak{O}_{\mathbb{S}^2}$:

$$\begin{aligned} f(t, r, (\omega, X, Y) \cdot e^{i\rho}) &= -\frac{ia \left\langle e^{i\rho}(X + iY), e_3 \right\rangle_{\mathbb{R}^3}}{\sqrt{2}p} \partial_t + e^{i\rho}(X + iY) \\ &= e^{i\rho} f(t, r, (X, Y)) \\ &= f(t, r, (X, Y)) \cdot e^{i\rho} \end{aligned}$$

□

The double cover map $\tilde{d} := f \circ \Psi$ satisfies $\tilde{d}(x \cdot g) = \tilde{d}(x) \cdot g^2$ for all $g \in U(1)$ and $x \in \mathbb{R}_t \times (r_0, +\infty) \times \mathbb{S}^3$ and the following diagram is commutative

$$\begin{array}{ccc} \mathbb{R}_t \times (r_0, +\infty) \times \mathbb{S}^3 & \xrightarrow{\tilde{d}} & \mathcal{N}_{0,r} \\ \downarrow{\scriptstyle Id \times H} & \swarrow{\scriptstyle \pi_{\mathcal{N}_{0,r}}} & \\ \mathcal{M} & & \end{array}$$

We see that $\tilde{d} : \mathbb{R}_t \times (r_0, +\infty) \times \mathbb{S}^3 \to \mathcal{N}_{0,r}$ is very similar to $d : \mathcal{A}_{0,r} \to \mathcal{N}_{0,r}$. Indeed, the two are isomorphic as we see in the following proposition:

Proposition 5.2.2. *Given $u \in \mathbb{R}_t \times (r_0, +\infty) \times \mathbb{S}^3$ and $v \in \mathcal{A}_{0,r}$ such that $\tilde{d}(u) = d(v)$, we have a unique isomorphism of principal bundles $\mathfrak{G} : \mathbb{R}_t \times (r_0, +\infty) \times \mathbb{S}^3 \to \mathcal{A}_{0,r}$ such that the following diagram is commutative:*

$$\begin{array}{ccc} \mathbb{R}_t \times (r_0, +\infty) \times \mathbb{S}^3 & \xrightarrow{\mathfrak{G}} & \mathcal{A}_{0,r} \\ & {\scriptstyle \tilde{d}} \searrow \quad \swarrow {\scriptstyle d} & \\ & \mathcal{N}_{0,r} & \end{array}$$

and such that $\mathfrak{G}(u) = v$.

Proof. Since $\mathbb{R}_t \times (r_0, +\infty) \times \mathbb{S}^3$ is simply connected and d is a covering map, $\tilde{d}$ admits a unique lift through d to a smooth map $\mathfrak{G} : \mathbb{R}_t \times (r_0, +\infty) \times \mathbb{S}^3 \to \mathcal{A}_{0,r}$ such that $\mathfrak{G}(u) = v$. A priori $\mathfrak{G}$ is only a smooth map. It remains to show that $\mathfrak{G}$ is in fact an isomorphism of principal bundles. First note that for all $y \in \mathbb{R}_t \times (r_0, +\infty) \times \mathbb{S}^3$, we have $(Id \times H)(y) = \pi_{\mathcal{A}_{0,r}}(\mathfrak{G}(y))$ (using the commutative diagrams).

Let V be a small open subset of $\mathcal{M}$ and $y : V \to \mathbb{R}_t \times (r_0, +\infty) \times \mathbb{S}^3$ be a local section of $\mathbb{R}_t \times (r_0, +\infty) \times \mathbb{S}^3$. Then $\mathfrak{G} \circ y$ is a local section of $\mathcal{A}_{0,r}$ and $\tilde{d} \circ y$ is a local section of $\mathcal{N}_{0,r}$. Let Ψ_1 be the local trivialization of $\mathbb{R}_t \times (r_0, +\infty) \times \mathbb{S}^3$ such that $\Psi_1(x, e^{i\rho}) = e^{i\rho} \cdot y(x)$, Ψ_2 the local trivialization of $\mathcal{A}_{0,r}$ such that $\Psi_2(x, e^{i\rho}) = e^{i\rho} \cdot \mathfrak{G}(y(x))$ and Φ the local trivialization of $\mathcal{N}_{0,r}$ such that $\Phi(x, e^{i\rho}) = e^{i\rho} \cdot \tilde{d}(y(x))$. Then for $x \in V$ and $a \in U(1)$, $\Psi_2^{-1} \circ \mathfrak{G} \circ \Psi_1(x, a) = (x, \gamma(x, a))$ where γ is the unique continuous $U(1)$-valued function such that $\gamma(x, a) \cdot \mathfrak{G}(y(x)) = \mathfrak{G}(a \cdot y(x))$. In particular $\gamma(x, 1) = 1$. Moreover, $\Phi^{-1} \circ \tilde{d} \circ \Psi_1(x, a) = \Phi^{-1} \circ d \circ \Psi_2(x, a) = (x, a^2)$. For $x \in V$ and $a \in U(1)$:

$$\begin{aligned} \Phi^{-1} \circ d \circ \mathfrak{G} \circ \Psi_1(x, a) &= \Phi^{-1} \circ d \circ \Psi_2 \circ \Psi_2^{-1} \circ \mathfrak{G} \circ \Psi_1(x, a) \\ &= (x, \gamma(x, a)^2) \end{aligned}$$

And we also have

$$\begin{aligned} \Phi^{-1} \circ d \circ \mathfrak{G} \circ \Psi_1(x, a) &= \Phi^{-1} \circ \tilde{d} \circ \Psi_1(x, a) \\ &= (x, a^2) \end{aligned}$$

We deduce $\gamma(x,a)^2 = a^2$.

Then the following diagram is commutative:

$$\begin{array}{ccc} & & U(1) \\ & \overset{\gamma}{\nearrow} & \downarrow{\scriptstyle a\mapsto a^2} \\ V\times U(1) & \xrightarrow{(x,a)\mapsto a^2} & U(1) \end{array}$$

Because $a \mapsto a^2$ is a covering map, we have the uniqueness of such a continuous lift with $\gamma(x,1) = 1$. We deduce $\gamma = \mathrm{pr}_{U(1)}$. This proves that $\mathfrak{G}$ is an isomorphism of principal bundles.

□

Remark 5.2.3. *The previous proposition shows that there are exactly two choices $\mathfrak{G}_1$ and $\mathfrak{G}_2$ for the isomorphism and we have for all $a \in \mathbb{R}_t \times (r_0, +\infty) \times \mathbb{S}^3$, $\mathfrak{G}_1(a) = \mathfrak{G}_2(a \cdot (-1))$. We choose one of the two and call it $\mathfrak{G}$.*

Thanks to the previous proposition, we have now a concrete description of $\mathcal{A}_{0,r}$ and we can use it to define spin weighted functions as in [21, Section 2.2]. The concrete description avoids the reference to spin frames.

5.3 Stationarity

In this section, we introduce the notion of a trivial (vector or principal) bundle with respect to a factor in a product decomposition and we apply this notion to the bundle $\mathcal{B}(s,s)$ in the Kerr case. We consider a manifold $\mathcal{M}$ with a product decomposition $\Psi : \mathcal{M} \to \mathcal{X} \times \mathcal{Y}$ (Ψ being a fixed diffeomorphism).

Definition 5.3.1. *We say that a (vector or principal) bundle $p_E : E \to \mathcal{M}$ is trivial with respect to $\mathcal{X}$ in the decomposition given by Ψ if there exists an isomorphism f of (vector or principal) bundles over Ψ between E and the bundle $\mathcal{X} \times F$ (by definition it is the product of the trivial bundle $Id : \mathcal{X} \to \mathcal{X}$ and some bundle $p_F : F \to \mathcal{Y}$). In particular, we have the following commutative diagram:*

$$\begin{array}{ccc} E & \xrightarrow{f} & \mathcal{X}\times F \\ \downarrow{\scriptstyle p_E} & & \downarrow{\scriptstyle Id_{\mathcal{X}}\times p_F} \\ \mathcal{M} & \xrightarrow{\Psi} & \mathcal{X}\times\mathcal{Y} \end{array}$$

We will say that f is a semi-trivialization of the bundle.

Remark 5.3.2. *It is equivalent to say that E is (isomorphic to) the pullback of a bundle F on $\mathcal{Y}$ by the second projection (indeed, this pullback bundle is exactly the bundle $\mathcal{X} \times F$).*

Remark 5.3.3. *In the following we fix the identification between $\mathcal{M}$ and Ψ. Therefore we will assume $\mathcal{M} = \mathcal{X} \times \mathcal{Y}$.*

Remark 5.3.4. *If $f : E \to \mathcal{X} \times F$ and $f' : E \to \mathcal{X} \times F'$ are two semi-trivializations, we have $f \circ f'^{-1}(x,z) = (x, \gamma(z))$ where $\gamma : F' \to F$ is an isomorphism of vector (or principal) bundles.*

Proposition 5.3.5. *Let E be a finite rank vector bundle over $\mathcal{M} = \mathcal{X} \times \mathcal{Y}$ ($\mathcal{M}$ is paracompact since it is a smooth manifold). If $\mathcal{X}$ is contractible, then E is trivial with respect to $\mathcal{X}$.*

Proof. Let $f : [0,1] \times \mathcal{X} \times \mathcal{Y} \to \mathcal{X} \times \mathcal{Y}$ be a smooth map such that $f_0 = Id_{\mathcal{X}\times\mathcal{Y}}$ and $f_1(x,y) = (x_0, y)$ for all $(x,y) \in \mathcal{X} \times \mathcal{Y}$ (x_0 is some element of $\mathcal{X}$). We define $\mathcal{E} = f^*E$. Then, for example by proposition 1.7 in [44] (rather its direct equivalent in the smooth case, obtained by minor modifications in the proof), $\mathcal{E}_{|_{\{0\}\times\mathcal{X}\times\mathcal{Y}}}$ is isomorphic to $\mathcal{E}_{|_{\{1\}\times\mathcal{X}\times\mathcal{Y}}}$. □

Proposition 5.3.6. *Let E be a vector bundle over $\mathcal{X}\times\mathcal{Y}$. If $\mathcal{C} = (U_\alpha, \Psi_\alpha)$ is a complete system of local trivializations for E such that all the transition maps $g_{\alpha,\alpha'} \in C^\infty(U_\alpha \cap U_{\alpha'}; Gl_n(\mathbb{R}))$ are independent of the first factor (ie factorize as $g_{\alpha,\alpha'} = \tilde{g}_{\alpha,\alpha'} \circ \pi_2$ where π_2 is the second projection on $\mathcal{X} \times \mathcal{Y}$), then there exists a semi-trivialization $f_\mathcal{C} : E \to \mathcal{X} \times F_\mathcal{C}$ where $F_\mathcal{C}$ is the vector bundle on $\mathcal{Y}$ given by the transition maps $\tilde{g}_{\alpha,\alpha'}$.*

Remark 5.3.7. *We will say that the system $\mathcal{C}$ is stationary with respect to $\mathcal{X}$. As mentioned in the proof of the proposition, a stationary system of trivializations comes naturally with a semi-trivialization of E.*

Proof. We first define the vector bundle $F_\mathcal{C}$ by $\coprod_\alpha \pi_2(U_\alpha)\times\mathbb{R}^n/(\tilde{g}_{\alpha,\alpha'})$ (the family $\tilde{g}_{\alpha,\alpha'}$ satisfies the cocycle condition). We denote by i_α the natural maps from $U_\alpha \times \mathbb{R}^n$ to $F_\mathcal{C}$ (composition of the injection in the disjoint union and the projection in the quotient). These maps are continuous injective and by definition of the quotient, on $U_\alpha \cap U'_\alpha$, $i_\alpha^{-1} i_{\alpha'}(y,v) = (y, g_{\alpha,\alpha'}(y)v)$. Then we define the map $f_\mathcal{C}$ by $f_\mathcal{C}(z) = (\Psi_\alpha(z)_1, i_\alpha(\Psi_\alpha(z)_2, \Psi_\alpha(z)_3))$ if $z \in U_\alpha$ (the index 1,2 and 3 refers to the components in the product decomposition). This does not depend on the choice of α such that $z \in U_\alpha$ and $(Id_\mathcal{X} \times i_\alpha^{-1}) \circ f_\mathcal{C} \circ \Psi_\alpha^{-1}$ is the identity of $U_\alpha \times \mathbb{R}^n$. Therefore, $f_\mathcal{C}$ is an isomorphism of vector bundles. □

Proposition 5.3.8. *Conversely, if we have a semi-trivialization $f : E \to \mathcal{X} \times F$ and a complete system of local trivializations (U_α, Ψ_α) on F, we can define a complete system of local trivializations on E with transition maps independent of the first factor.*

Proof. We just take the system $(\mathcal{X} \times U_\alpha, (Id_\mathcal{X} \times \Psi_\alpha) \circ f)$. □

We now apply these notions to the study of $\mathcal{B}(s,w)$. First note that the complete system of trivializations (A_N, A_S) of $\mathcal{A}_0$ has a transition map depending only on ϕ. As a consequence, the associated sytem of trivializations of $\mathcal{B}(s,w)$ also has a transition map depending only on ϕ (see remark 4.2.5). Therefore, we are in the context of proposition 5.3.6 and $\mathcal{B}(s,w)$ is trivial with respect to the factor $\mathbb{R}_t \times (r_0, +\infty)$ (we could also have used proposition 5.3.5 but proposition 5.3.6 provides a concrete vector bundle $\mathcal{B}_{\mathbb{S}^2}(s,w)$ on $\mathbb{S}^2$ and a concrete semi-trivialization $f : \mathcal{A}_0 \to \mathbb{R}_t \times (r_0, +\infty) \times \mathcal{B}_{\mathbb{S}^2}(s,w)$ associated with the stationary system of local trivializations). The map f enables us to identify $\Gamma(\mathcal{B}(s,w))$ with $C^\infty(\mathbb{R}_t \times (r_0, +\infty), \Gamma(\mathcal{B}_{\mathbb{S}^2}(s,w)))$.

Chapter 6

Connections on the bundles and the GHP formalism

There are different methods to define the spin connection and the GHP connection, for example we could write them down explicitly in an expression involving coefficients depending only on the Levi-Civita connection and the metric (see remark 6.2.5 and proposition 6.2.10) and check that it defines a linear connection. For computational purposes, these definitions are enough but they are not insightful. The spin connection is the unique connection ∇ such that $j^*\nabla_{LC} = \nabla \otimes \nabla$ (where ∇_{LC} is the Levi-Civita connection) and $\nabla\epsilon = 0$ and these properties can be used as an alternative definition as well. However, here we choose another definition. We start with the Levi-Civita connection and move it naturally through the different bundles involved. The advantage of this method is that each step is very natural, moreover it gives a good understanding of where the GHP connection comes from. Its main drawback is that it is a little longer than the direct definitions and involves some elementary knowledge about principal connections.

Therefore, we give a brief reminder about principal connections in order to have a self contained presentation. A more detailed introduction to this topic can be found in [65, Chapter 2].

6.1 Principal connection

Definition 6.1.1. *Let $\pi_E : E \to \mathcal{M}$ be a principal bundle with structure group G (multiplicative, with neutral element denoted by 1). Let $\mathfrak{g}$ be the Lie algebra of G. A principal connection ω is a $\mathfrak{g}$-valued one-form on E such that:*

- *For all $g \in G$, $Ad_g R_g^* \omega = \omega$ where $Ad_g : \mathfrak{g} \to \mathfrak{g}$ is the adjoint representation ($Ad_g(\xi) := \frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} g \exp(t\xi) g^{-1}$) and $R_g : E \to E$ is the right action of g on E.*
- *For all $\xi \in \mathfrak{g}$ and all $e \in E$, $\omega_e(\ \mathrm{d}_1 i_e \xi) = \xi$ where we have used the map*

$$i_e : \begin{cases} G \to E \\ g \mapsto e \cdot g \end{cases}$$

Remark 6.1.2. *Differentiating the relation $\pi_E \circ i_a(g) = \pi_E(a)$, we have the inclusion $\mathrm{d}_1 i_a(\mathfrak{g}) \subset ker(\ \mathrm{d}_a \pi_E)$. Moreover, $\dim(ker(\ \mathrm{d}_a \pi_E)) = \dim(\pi_E^{-1}(\{a\})) = \dim(G) = \dim(\mathfrak{g}) = \dim\ \mathrm{d}_1 i_a(\mathfrak{g})$. Therefore $\mathrm{d}_1 i_a(\mathfrak{g}) = ker(\ \mathrm{d}_a \pi_E)$.*

Lemma 6.1.3. *For all $e \in E$, $ker(\omega_e) \oplus ker(\ \mathrm{d}_e \pi_E) = T_e E$. In particular $\mathrm{d}_e \pi_{E|ker(\omega_e)}$ is an isomorphism between $ker(\omega_e)$ and $T_{\pi_E(e)}\mathcal{M}$.*

Proof. The image of the injective map $\mathrm{d}_1 i_e$ is exactly $ker(\mathrm{d}_e\pi_E)$ (see remark 6.1.2). So the second point in the definition of ω implies that $ker(\omega_e) \cap ker(\mathrm{d}_e\pi_E) = \{0\}$ and $\dim(Ran(\omega_e)) = \dim(\mathfrak{g}) = \dim(ker(\mathrm{d}_e\pi_E))$. Moreover, $\dim(ker(\omega_e)) = \dim(E) - \dim(Ran(\omega_e))$ so we deduce that $\dim(ker(\omega_e)) + \dim(ker(\mathrm{d}_e\pi_E)) = \dim(E)$ and we have the lemma. □

From now on, for $a \in E$, we denote by $H_a := ker(\omega_a)$. The following property gives a way to construct a linear connection (on an associated vector bundle) from a principal connection.

Proposition 6.1.4. *Let $\rho : G \to GL(V)$ be a representation of G and let $\mathcal{F}$ be the vector bundle associated to this representation. Then, a smooth section s of $\mathcal{F}$ is naturally identified with a smooth function $f : E \to V$ such that $f(x \cdot g) = \rho(g^{-1})(f(x))$ and we can define (for $X \in T_x\mathcal{M}$) $(\nabla_X s)(x) := [a,\ \mathrm{d}_a f((\ \mathrm{d}_a\pi_{E|_{H_a}})^{-1}(X))]$ where a is any element of $\pi_E^{-1}(\{x\})$. With this definition, ∇ is a linear connection on $\mathcal{F}$.*

Proof. The definition does not depend on the choice of a. If we choose $a' = a \cdot g$, we have to prove that $[a,\ \mathrm{d}_a f((\ \mathrm{d}_a\pi_{E|_{H_a}})^{-1}(X))] = [a',\ \mathrm{d}_{a'} f((\ \mathrm{d}_{a'}\pi_{E|_{H_{a'}}})^{-1}(X))]$. We have

$$[a',\ \mathrm{d}_{a'} f((\ \mathrm{d}_{a'}\pi_{E|_{H_{a'}}})^{-1}(X))] = [a, \rho(g)\ \mathrm{d}_{a'} f((\ \mathrm{d}_{a'}\pi_{E|_{H_{a'}}})^{-1}(X))]$$

so it remains to prove that

$$\rho(g)\ \mathrm{d}_{a'} f((\ \mathrm{d}_{a'}\pi_{E|_{H_{a'}}})^{-1}(X)) =\ \mathrm{d}_a f((\ \mathrm{d}_a\pi_{E|_{H_a}})^{-1}(X))$$

. We have the following facts

- $\mathrm{d}_{a\cdot g}\pi_E\ \mathrm{d}_a R_g =\ \mathrm{d}_a\pi_E$ (by differentiating $\pi_E \circ R_g = \pi_E$)
- $\mathrm{d}_{a\cdot g}\pi_{E|_{H_{a\cdot g}}}(\ \mathrm{d}_a R_g)_{|_{H_a}} =\ \mathrm{d}_a\pi_{E|_{H_a}}$ using the previous point and the fact that $R_g(H_a) = H_{a\cdot g}$.
- $\mathrm{d}_{a\cdot g}\pi_{E|_{H_{a\cdot g}}}^{-1} =\ \mathrm{d}_a R_g\ \mathrm{d}_a\pi_{E|_{H_a}}^{-1}$ (using the previous point)
- $\mathrm{d}_{a\cdot g} f\ \mathrm{d}_a R_g = \rho(g^{-1})\ \mathrm{d}_a f$ (by differentiating $f(a \cdot g) = \rho(g^{-1})f(a)$)

We use that to conclude $\rho(g)\ \mathrm{d}_{a'} f((\ \mathrm{d}_{a'}\pi_{E|_{H_{a'}}})^{-1}(X)) =\ \mathrm{d}_a f((\ \mathrm{d}_a\pi_{E|_{H_a}})^{-1}(X))$. Now we have to prove that ∇ is a linear connection. We obviously have $\nabla_{\lambda X+Y} = \lambda\nabla_X + \nabla_Y$. Let h be a smooth function on $\mathcal{M}$. The section hs is associated with the function $\tilde{f} = (h \circ \pi_E)f$ and $\mathrm{d}_a\tilde{f} = f(a)\ \mathrm{d}_{\pi_E(a)}h\ \mathrm{d}_a\pi_E + h(\pi_E(a))\ \mathrm{d}_a f$. Therefore, we have

$$\mathrm{d}_a\tilde{f}((\ \mathrm{d}_a\pi_{E|_{H_a}})^{-1}(X)) =\ \mathrm{d}_{\pi_E(a)}h(X)f + h(\pi_E(a))\ \mathrm{d}_a f((\ \mathrm{d}_a\pi_{E|_{H_a}})^{-1}(X)).$$

We deduce that ∇ defines a linear connection on $\mathcal{F}$. □

A useful lemma to compute the such defined connection is the following:

Lemma 6.1.5. *We use the notation of proposition 6.1.4. Let e be a local smooth section of E around some $x_0 \in \mathcal{M}$ such that $\omega \circ\ \mathrm{d}_{x_0} e = 0$. Let $X \in T_{x_0}\mathcal{M}$ and let s be a local smooth section of $\mathcal{F}$ on some open neighborhood U of x_0 such that $s(x) = [(e(x), v(x))]$ with $v : U \to V$ smooth. Then, $\nabla_X s = [(e(x_0),\ \mathrm{d}_{x_0} v(X))]$*

Proof. The equivariant function associated to s is $f : \pi_E^{-1}(U) \to V$ such that for all $x \in U$, $f(e(x)) = v(x)$. Let $X \in T_{x_0}\mathcal{M}$. We use the definition of the connection ∇ to write

$$\nabla_X s = [(e(x_0),\ \mathrm{d}_{e(x_0)} f\left(\ \mathrm{d}_{e(x_0)}\pi_{E|_{H_{e(x_0)}}}\right)^{-1}(X))]$$

Moreover, we have $\mathrm{d}_{x_0}e(X) \in ker(\omega_{e(x_0)}) = H_{e(x_0)}$ and $\mathrm{d}_{e(x_0)}\pi_E\ \mathrm{d}_{x_0}e(X) = \mathrm{d}_{x_0}(\pi_E \circ e)(X) = X$. We deduce that $\left(\mathrm{d}_{e(x_0)}\pi_{E|H_{e(x_0)}}\right)^{-1}(X) = \mathrm{d}_{x_0}e(X)$. As a consequence

$$\begin{aligned}\nabla_X s &= [(e(x_0),\ \mathrm{d}_{e(x_0)}f\ \mathrm{d}_{x_0}e(X))] \\ &= [(e(x_0),\ \mathrm{d}_{x_0}(f\circ e)(X))] \\ &= [(e(x_0),\ \mathrm{d}_{x_0}v(X))]\end{aligned}$$

□

Remark 6.1.6. *In the case where E is the frame bundle of $\mathcal{F}$, the condition $\omega \circ \mathrm{d}_{x_0}e = 0$ amounts to saying that the derivative of the local frame vanishes at x_0. Then the previous lemma tells us that in this local frame, we can compute covariant derivatives of a section of $\mathcal{F}$ by taking usual derivatives of the coordinates.*

Remark 6.1.7. *Let E be a general principal bundle over $\mathcal{M}$ and ω a principal connection on E. Then for all $e_0 \in E$, $ker(\omega_e)$ is transverse to the fiber (and of dimension $\dim(\mathcal{M})$) by lemma 6.1.3. As a consequence, there exists a local section of E around $x_0 := \pi_E(e_0)$ such that $e(x_0) = e_0$ and $\omega_{e_0} \circ \mathrm{d}_{x_0}e_0 = 0$. Moreover, since $\dim(\mathrm{d}_{x_0}e_0(T_{x_0}\mathcal{M})) = \dim(\mathcal{M}) = \dim(ker(\omega_{e_0}))$, we have $\mathrm{d}_{x_0}e_0(T_{x_0}\mathcal{M}) = ker(\omega_{e_0})$.*

We can use lemma 6.1.5 to compute the connection in the general case.

Corollary 6.1.8. *We use the notations of proposition 6.1.4. Let e be a local smooth section of E around $x_0 \in \mathcal{M}$. Let $X \in T_{x_0}\mathcal{M}$ and s be a section of $\mathcal{F}$ on some open neighborhood U of x_0 such that $s(x) = [(e(x), v(x))]$ with $v : U \to V$ smooth. Then we have $\nabla_X s = [e(x_0),\ \mathrm{d}_{x_0}v(X) + \mathrm{d}_1\rho\omega_{e(x_0)}(\mathrm{d}_{x_0}e(X))v(x_0)]$.*

Proof. We define e' smooth section of E on a neighborhood U of x_0 such that $e'(x_0) = e(x_0)$ and $\mathrm{d}_{x_0}e'(T_{x_0}\mathcal{M}) = ker(\omega_{e(x_0)})$ (e' exists by remark 6.1.7). We define $g : U \to G$ the unique smooth map such that $e' = e \cdot g$ (in particular $g(x_0) = 1$). By the chain rule, we have:

$$\mathrm{d}_{x_0}e'(x) = \mathrm{d}_1 i_{e(x_0)}(\mathrm{d}_{x_0}g(X)) + \mathrm{d}_{x_0}e(X)$$

But by definition of $e'(x_0)$, $\omega_{e(x_0)}(\mathrm{d}_{x_0}e'(x)) = 0$. On the other hand:

$$\omega_{e(x_0)}(\mathrm{d}_{x_0}e'(x)) = \mathrm{d}_{x_0}g(X) + \omega_{e(x_0)}(\mathrm{d}_{x_0}e(X))$$

therefore $\mathrm{d}_{x_0}g(X) = -\omega_{e(x_0)}(\mathrm{d}_{x_0}e(X))$. We can now compute:

$$\begin{aligned}s(x) &= [(e(x), v(x))] = [(e(x)\cdot g(x), \rho(g(x)^{-1})v(x))] \\ \nabla_X s(x_0) &= [(e(x_0)\cdot g(x_0),\ \mathrm{d}_{x_0}(\rho(g^{-1})v)(X)] \qquad \text{(by proposition 6.1.4)} \\ &= [(e(x_0),\ \mathrm{d}_{x_0}(v)(X) - \mathrm{d}_1\rho\ \mathrm{d}_{x_0}g(X)v(x_0))] \qquad \text{(chain rule)} \\ &= [(e(x_0),\ \mathrm{d}_{x_0}(v)(X) + \mathrm{d}_1\rho\omega_{e(x_0)}(\mathrm{d}_{x_0}e(X))v(x_0))]\end{aligned}$$

□

The construction of proposition 6.1.4 behaves well with respect to the tensor product of vector bundles as we see in the following proposition:

Proposition 6.1.9. *Let E be a G principal bundle with a principal connection ω, ρ_1 be a representation of G on V_1 and ρ_2 a representation of G on V_2. Let $\mathcal{F}_1$ be the vector bundle associated to E with the representation ρ_1 and $\mathcal{F}_2$ the vector bundle associated to E with the representation ρ_2. We denote by ∇_1 (resp. ∇_2) the linear connection on $\mathcal{F}_1$ (resp. $\mathcal{F}_2$) obtained from ω thanks to proposition 6.1.4. The bundle $\mathcal{F}_1 \otimes \mathcal{F}_2$ is naturally associated to E with the representation $g \mapsto \rho_1(g) \otimes \rho_2(g)$. We denote by $\tilde{\nabla}$ the connection on $\mathcal{F}_1 \otimes \mathcal{F}_2$ obtained from ω by proposition 6.1.4. Then we have*

$$\nabla_1 \otimes \nabla_2 = \tilde{\nabla}$$

Proof. Let $x_0 \in \mathcal{M}$, let s be a local section of E on an open neighborhood U of x_0 such that $\omega \circ \mathrm{d}_{x_0} s = 0$ (it is always possible to find such a section by remark 6.1.7). Let f_1 (resp. f_2) be a smooth section of $\mathcal{F}_1$ (resp. $\mathcal{F}_2$) on U. We write $f_1 = [(s, g_1)]$ and $f_2 = [(s, g_2)]$ with $g_i : U \to V_i$ smooth. Then we have $f_1 \otimes f_2(x) = [(s(x), g_1(x) \otimes g_2(x))]$. Let $X \in T_{x_0}\mathcal{M}$. We use lemma 6.1.5 to compute

$$\begin{aligned}
\tilde{\nabla}_X (f_1 \otimes f_2)(x_0) &= [(s(x_0),\ \mathrm{d}_{x_0}(g_1 \otimes g_2)(X))] \\
&= [(s(x_0),\ \mathrm{d}_{x_0} g_1(X) \otimes g_2(x_0) + g_1(x_0) \otimes\ \mathrm{d}_{x_0} g_2(X))] \\
&= [(s(x_0),\ \mathrm{d}_{x_0} g_1(X))] \otimes [(s(x_0), g_2(x_0))] \\
&\quad + [(s(x_0), g_1(x_0))] \otimes [(s(x_0),\ \mathrm{d}_{x_0} g_2(X))] \\
&= (\nabla_1)_X f_1(x_0) \otimes f_2(x_0) + f_1(x_0) \otimes (\nabla_2)_X f_2(x_0) \qquad \text{(by lemma 6.1.5)}
\end{aligned}$$

The equality is true for pure product sections and we use linearity to conclude. □

Finally, we give a kind of reverse construction of the previous one when E is the principal bundle of frames of a vector bundle $\mathcal{F}$ of rank n (real or complex, we consider the complex case here).

Proposition 6.1.10. *We assume that we have a linear connection ∇ on $\mathcal{F}$ (complex vector bundle of rank n over a manifold $\mathcal{M}$ of positive dimension). We denote by E the $GL(n, \mathbb{C})$-principal bundle of frames. There exists a unique $\mathcal{M}_n(\mathbb{C})$-valued one-form ω on E such that for all local frames $(e_1, ..., e_n)$ around x and all $X \in T_x\mathcal{M}$,*

$$\omega_{e_1(x),...,e_n(x)}(\ \mathrm{d}_x(e_1, ..., e_n)(X)) = \underset{e_1(x),...,e_n(x)}{Mat}(\nabla_X e_1(x), ..., \nabla_X e_n(x)) \in \mathcal{M}_n(\mathbb{C})$$

where $\underset{f_1,...,f_n}{Mat}(a_1, ..., a_n)$ is the unique matrix M such that for all $i \in [\![1, n]\!]$, $a_i = \sum_{k=1}^n M_{k,i} f_k$. Moreover, ω is a principal connection on E.

Proof. For $x \in \mathcal{M}$, we denote by $\Gamma_x(E)$ the set of local smooth section of E defined on some neighborhood of x. To prove the uniqueness, it is enough to remark that for a frame $(f_1, ..., f_n)$ at x, the set

$$\{\ \mathrm{d}_x(e_1, ..., e_n)(X) : (e_1, ..., e_n) \in \Gamma_x(E) \text{ with } (e_1, ..., e_n)(x) = (f_1, ..., f_n), X \in T_x\mathcal{M}\}$$

generates $T_{(f_1,...,f_n)}E$. Indeed, by a simple construction in a local trivialization around x, we can show that

$$\{\ \mathrm{d}_x(e_1, ..., e_n)(X) : (e_1, ..., e_n) \in \Gamma_x(E) \text{ with } (e_1, ..., e_n)(x) = (f_1, ..., f_n), X \in T_x\mathcal{M}\}$$

is exactly $T_{(f_1,...,f_n)}E \setminus ker(\ \mathrm{d}_{(f_1,...,f_n)}\pi_E)$ (so we have the desired conclusion as soon as $\mathcal{M}$ has positive dimension). The existence follows from the observation that both sides of the equality are linear and if $\mathrm{d}_x(e_1, ..., e_n)(X) = \mathrm{d}_x(e'_1, ..., e'_n)(Y)$, then $(\nabla_X e_1(x), ..., \nabla_X e_n(x)) = (\nabla_Y e'_1(x), ..., \nabla_Y e'_n(x))$. Now we have to check that ω defines a principal connection on the

bundle of frames. In particular, for $(f_1, ..., f_n) \in E$ and $g \in GL(n, \mathbb{R})$, $Ad_g(R_g^* \omega)_{f_1,...,f_n} = \omega_{f_1,...,f_n}$. By the remark in the proof of uniqueness, it is enough to check it on the vectors on the form $\mathrm{d}_x(e_1, ..., e_n)(X)$ where $(e_1, ..., e_n)(x) = (f_1, ..., f_n)$. We define $(e'_1, ..., e'_n) := (e_1, ..., e_n) \cdot g$ for $g \in GL(n, \mathbb{C})$ and by definition of the right action of $GL(n, \mathbb{C})$ on E, we have $e'_i = \sum_{k=1}^n g_{k,i} e_k$. Then we have $\mathrm{d}_{(f_1,...,f_n)} R_g \ \mathrm{d}_x(e_1, ..., e_n)(X) = \mathrm{d}_x(e'_1, ..., e'_n)(X)$. We use that to compute:

$$\begin{aligned}
R_g^* \omega(\ \mathrm{d}_x(e_1, ..., e_n)(X)) &= \omega_{(f_1,...,f_n)\cdot g}(\ \mathrm{d}_x(e'_1, ..., e'_n)(X)) \\
&= \operatorname*{Mat}_{(f_1,...,f_n)\cdot g} \left(\nabla_X e'_1, ..., \nabla_X e'_n \right) \\
&= \operatorname*{Mat}_{(f_1,...,f_n)\cdot g} \left(\sum_{k=1}^n (g_{k,1} \nabla_X e_k, ..., g_{k,n} \nabla_X e_k) \right) \\
&= g^{-1} \operatorname*{Mat}_{(f_1,...,f_n)} (\nabla_X e_1, ..., \nabla_X e_n) g \\
&= (Ad_g)^{-1} \omega(\ \mathrm{d}_x(e_1, ..., e_n)(X))
\end{aligned}$$

The second property to check is that for $(f_1, ..., f_n) \in E$ and $g : (-1, 1) \to GL(n, \mathbb{C})$ smooth with $g(0) = Id$, we have $\omega(\frac{\mathrm{d}}{\mathrm{d}t}_{|t=0}(f_1, ..., f_n) \cdot g(t)) = \frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} g(t)$. To see that, we take $(x_1, ..., x_n)$ smooth coordinates around x with $(x_1, ..., x_n)(x) = 0$ and $(e_1, ..., e_n)$ a local frame around x such that $(e_1, ..., e_n)(x) = (f_1, ..., f_n)$. We define the local frame $(e'_1, ..., e'_n) := (e_1, ..., e_n) \cdot g(x_1)$. We compute (using the chain rule)

$$\mathrm{d}_x(e'_1, ..., e'_n)(\partial_{x_1}) = \frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} ((f_1, ..., f_n) \cdot g(t)) + \mathrm{d}_x(e_1, ..., e_n)(\partial_{x_1})$$

Moreover, we have

$$e'_i = \sum_{k=1}^n g_{k,i}(x_1) e_k$$

Therefore $\nabla_{\partial_{x_1}} e'_i(x) = \sum_{k=1}^n \left(\frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} g_{k,i} \right) f_k + \nabla_{\partial_{x_1}} e_k(x)$ and using the definition, we see that $\omega_{(f_1,...,f_n)}(\ \mathrm{d}_x(e'_1, ..., e'_n)(\partial_{x_1})) = \omega_{(f_1,...,f_n)}(\ \mathrm{d}_x(e_1, ..., e_n)(\partial_{x_1})) + \frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} g(t)$. Finally,we deduce:

$$\omega_{(f_1,...,f_n)} \left(\frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} ((f_1, ..., f_n) \cdot g(t)) \right) = \frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} g(t).$$

□

Proposition 6.1.11. *Let $\mathcal{F}$ be a complex vector bundle of rank n over a manifold $\mathcal{M}$ of positive dimension and let E be the $GL(n, \mathbb{C})$-principal bundle of frames. In this setting, the constructions of proposition 6.1.4 and 6.1.10 are inverse of one another.*

Proof. Let Y be a local smooth section of $\mathcal{F}$ around $x_0 \in \mathcal{M}$ and $X \in T_{x_0}\mathcal{M}$. We denote by ω the principal connection on E obtained from ∇ by going through the construction of proposition 6.1.10 and $\overline{\nabla}$ the linear connection obtained from ω by proposition 6.1.4. We prove that $\nabla_X Y = \overline{\nabla}_X Y$. To do so we choose a local basis $(e_1, ..., e_n)$ around x_0 such that $(\nabla_X e_1, ... \nabla_X e_n) = 0$ (it is always possible to construct such a basis by working in a local trivialization around x). We denote by f the function defined from a neighborhood of $\pi_E^{-1}(\{x_0\})$ to $\mathbb{C}^n$ by $f(a) = \operatorname*{Mat}_a Y(\pi_E(a))$. We define $a_0 := (e_1, ..., e_n)(x_0)$.

By definition of ω, we have $\omega_{a_0}\ \mathrm{d}_{x_0}(e_1, ..., e_n)(X) = 0$. If we write $Y = [((e_1, ..., e_n), (Y_1, ..., Y_n))]$ where Y_i are the coordinates of Y in the local basis $(e_1, ..., e_n)$,

we can use lemma 6.1.5 to deduce that $\overline{\nabla}_X Y(x_0) = [(a_0, (\ \mathrm{d}_{x_0} Y_1(X), ...,\ \mathrm{d}_{x_0} Y_n(X)))] = \sum_{k=1}^n X(Y_k)(x_0) e_k(x_0)$.

On the other hand

$$\begin{aligned}\nabla_X Y &= \sum_{k=1}^n X(Y_k)(x_0) e_k + Y_k \nabla_X e_k(x_0) \\ &= \sum_{k=1}^n X(Y_k)(x_0) e_k \text{ by definition of } (e_1, ..., e_k) \\ &= \overline{\nabla}_X Y\end{aligned}$$

We conclude that $\nabla = \overline{\nabla}$.

We also have to prove that if ω is a principal connection on E and if ∇, linear connection on $\mathcal{F}$ is obtained by proposition 6.1.4, then the principal connection $\tilde{\omega}$ on E obtained from ∇ by proposition 6.1.10 is equal to ω. Since we already know that for $x_0 \in \mathcal{M}$, $a \in E$ and $h \in \mathfrak{g}$, $\omega\ \mathrm{d}_1 i_a(h) = \tilde{\omega}\ \mathrm{d}_1 i_a(h) = h$ (by definition of a principal connection) and since $T_a E = ker(\omega_a) \oplus\ \mathrm{d}_1 i_a(\mathfrak{g})$ (see lemma 6.1.3 and remark 6.1.2), it is enough to prove that $\tilde{\omega} = 0$ on $ker(\omega_a)$. By remark 6.1.7, there exists a smooth local section $(e_1, ..., e_n)$ on an open neighborhood U around x_0 such that $(e_1, ..., e_n)(x_0) = a$ and $\mathrm{d}_{x_0}(e_1, ..., e_n)(T_{x_0}\mathcal{M}) = ker(\omega_a)$. We are reduced to proving that $\tilde{\omega}_a \circ \mathrm{d}_{x_0}(e_1, ...e_n) = 0$. By definition of $\tilde{\omega}$, this is the same as proving that for all $i \in [\![1, n]\!]$, $\nabla e_i = 0$. By definition of ∇ and lemma 6.1.5 (use the fact that $\mathrm{d}_{x_0}(e_1, ..., e_n)(T_{x_0}\mathcal{M}) = ker(\omega_a)$), it is the same as proving that if $f_i : U \to \mathbb{R}^n$ is such that $e_i = [(e_1, ..., e_n), f_i]$ then $\mathrm{d}_{x_0} f_i = 0$. This last fact is obvious since f_i are the coordinates of e_i in the local basis $(e_1, ..., e_n)$. □

Proposition 6.1.12 (pull back of a principal connection)**.** *Let A be a G_A principal bundle over $\mathcal{M}$ and B a G_B principal bundle over the same manifold $\mathcal{M}$. We assume that G_A is an embedded Lie subgroup of G_B and that we have an embedding $f : A \to B$ (with $\pi_B \circ f = \pi_A$) such that for all $g \in G_A$ and $a \in A$, $f(a \cdot g) = f(a) \cdot g$. For every principal connection ω on B such that for all $a \in A$, $ker(\omega_{f(a)}) \subset T_{f(a)} f(A)$, the one form $f^*\omega$ is a principal connection on A.*

Proof. We first have to check that $f^*\omega$ has values in $\mathfrak{g}_A$. To prove that, we remark that for all $a \in A$, $Ran(\ \mathrm{d}_a f) = ker\omega_{f(a)} \oplus\ \mathrm{d}_1 i_{f(a)} \mathfrak{g}_A$. Indeed, the right-hand side is included into the left-hand side and both sides have the same dimension ($\dim(Ran(\ \mathrm{d}_a f)) = \dim A = \dim(\mathcal{M}) + \dim(G_A)$). Moreover, $\omega(ker\omega_{f(a)} \oplus\ \mathrm{d}_1 i_{f(a)} \mathfrak{g}_B) = \mathfrak{g}_A$ so $f^*\omega$ has value in $\mathfrak{g}_A$. The two properties of principal connection for $f^*\omega$ follow directly for the corresponding one for ω. □

Remark 6.1.13. *The condition $ker(\omega_{f(a)}) \subset T_{f(a)} f(A)$ takes a particularly simple form when B is the frame bundle of a vector bundle $\mathcal{F}$ and ω comes from a linear connection ∇ on $\mathcal{F}$ (by proposition 6.1.10). Indeed, we have the following equivalence: $ker(\omega_{f(a)}) \subset T_{f(a)} f(A)$ if and only if for all $a \in A$, there exists $(e_1, ..., e_n)$ a local section of $f(A)$ around $x_0 := \pi_B(f(a))$ such that $(e_1, ..., e_n)(x_0) = a$ and for all $X \in T_{x_0} A$, $(\nabla_X e_1, ..., \nabla_X e_n) = 0$. The idea of the proof is the following: Let $a \in A$ and $x_0 := \pi_B(f(a))$. Assume $ker(\omega_{f(a)}) \subset T_{f(a)} f(A)$, then we can find a submanifold C of $f(A)$ of dimension $dim\mathcal{M}$ containing $f(a)$ with tangent space $ker(\omega_{f(a)})$ at $f(a)$. We know that $ker(\omega_{f(a)}) \cap ker(\ \mathrm{d}_{f(a)} \pi_B) = \{0\}$, as a consequence C defines a section of $f(A)$ in a small neighborhood of $f(a)$ this section is the one we are looking for. Conversely if we have a section $(e_1, ..., e_n)$ with the required properties, $\mathrm{d}_x(e_1, ..., e_n)(T_{x_0}\mathcal{M}) \subset ker(\omega_{f(a)})$ and has the same dimension, so $ker(\omega_{f(a)}) =\ \mathrm{d}_x(e_1, ..., e_n)(T_{x_0}\mathcal{M}) \subset T_{f(a)} f(A)$.*

A typical situation where the pull back appears is the following:

Proposition 6.1.14. *Let A be a G principal bundle and $\rho : G \to GL(\mathbb{C}^n)$ a representation of G on $\mathbb{C}^n$ (we can also replace $\mathbb{C}^n$ by $\mathbb{R}^n$) which is an embedding of Lie-groups. Let $\mathcal{F}$ be the vector bundle associated to A with the representation ρ. Let ω be a principal connection on A and ∇ the connection on $\mathcal{F}$ constructed by proposition 6.1.4. Let E be the principal bundle of frames on $\mathcal{F}$ and ω_E be the principal connection on E given by proposition 6.1.10. Define the embedding:*

$$f : \begin{cases} A \to E \\ a \mapsto ([a, e_1], [a, e_2], ..., [a, e_n]) \end{cases}$$

where $(e_1, ..., e_n)$ is the canonical basis of $\mathbb{C}^n$. Then $f^*\omega_E = \mathrm{d}_1\rho \circ \omega$.

Proof. First, we can check that for all $a \in A$ and $g \in G$, $f(a \cdot g) = ([a, e_1], ..., [a, e_n]) \cdot \rho(g)$. In particular for $a \in A$, $\mathrm{d}_1(f \circ i_a) = \mathrm{d}_a f \, \mathrm{d}_1 i_a = \mathrm{d}_1 i_{f(a)} \, \mathrm{d}_1\rho$. By the definition of a principal connection, we know that $\mathrm{d}_1\rho\omega_a \, \mathrm{d}_1 i_a(h) = \mathrm{d}_1\rho(h)$ and also $(\omega_E)_{f(a)} \, \mathrm{d}_1 i_{f(a)}(\, \mathrm{d}_1\rho(h)) = \mathrm{d}_1\rho(h)$. We deduce that $\mathrm{d}_1\rho\omega_a \, \mathrm{d}_1 i_a(h) = (\omega_E)_{f(a)} \, \mathrm{d}_1 i_{f(a)}(\, \mathrm{d}_1\rho(h)) = (\omega_E)_{f(a)} \, \mathrm{d}_a f \, \mathrm{d}_1 i_a(h)$. Since $T_{\pi_A(a)}A = \mathrm{d}_1 i_a(\mathfrak{g}) \oplus ker(\omega_a)$, we are now reduced to checking that $(\omega_E)_{f(a)} \, \mathrm{d}_a f(ker(\omega_a)) = 0$. We take s a local section of A in an open neighborhood U of x_0 such that $s(x_0) = a$ and $\mathrm{d}_{x_0} s(T_{x_0}\mathcal{M}) = ker(\omega_a)$. Then by lemma 6.1.5, $\nabla[s, e_i] = 0$. We deduce that for $X \in T_{x_0}\mathcal{M}$, $(\nabla_X[s, e_1], ..., \nabla_X[s, e_n]) = 0$. Finally, using the definition of ω_E we deduce that $(\omega_E)_{f(a)}(\, \mathrm{d}_{x_0} f \circ s(X)) = \underset{f(a)}{\mathrm{Mat}}(\nabla_X[s, e_1], ..., \nabla_X[s, e_n]) = 0$. Therefore $(\omega_E)_{f(a)} \circ \mathrm{d}_{x_0} f(ker(\omega_A)) = 0$ □

In some cases of interest, the condition in remark 6.1.13 is not satisfied. In these cases, the pull back is not a connection because the image is not contained in the Lie subalgebra $\mathfrak{g}_A$. The goal of the following proposition is to correct this by composing (to the left) by a projection onto $\mathfrak{g}_A$. The main downside with this construction is that we have several choices for the projection leading to different connections. However, in cases we are interested in here, there is a particularly natural choice (see the remark after the proof).

Proposition 6.1.15 (Pull Back of a connection in more complicated cases)**.** *Let A be a G_A principal bundle over $\mathcal{M}$ and B a G_B principal bundle over the same manifold $\mathcal{M}$. We assume that G_A is an embedded Lie subgroup of G_B and that we have an embedding $f : A \to B$ such that for all $g \in G_A$ and $a \in A$, $f(a \cdot g) = f(a) \cdot g$. For simplicity, we identify implicitly the Lie algebra of A and the Lie algebra of $f(A)$. Assume that we have a subspace V of $\mathfrak{g}_B$ such that:*

- $V \oplus \mathfrak{g}_A = \mathfrak{g}_B$
- $\forall g \in G_A, Ad_g(V) = V$

We denote by $q : \mathfrak{g}_B \to \mathfrak{g}_A$ the projection onto $\mathfrak{g}_A$ with kernel V. For every principal connection ω on B, we can define the $\mathfrak{g}_A$-valued one form $\tilde{\omega} := q \circ f^*\omega$. Then $\tilde{\omega}$ is a principal connection on A.

Proof. For all $g \in G_A$, we have $Ad_g(\mathfrak{g}_A) \subset \mathfrak{g}_A$ and $Ad_g(V) \subset V$ so Ad_g commutes with q. Let

$g \in G_A$ and $a \in A$,

$$\begin{aligned} Ad_g R_g^* \tilde{\omega}_a &= Ad_g \tilde{\omega}_{a\cdot g} \circ \mathrm{d}_a R_g \\ &= Ad_g \circ q(f^*\omega)_{a\cdot g} \circ \mathrm{d}_a R_g \\ &= q \circ Ad_g \omega_{f(a\cdot g)} \circ \mathrm{d}_{a\cdot g} f \circ \mathrm{d}_a R_g \\ &= q \circ Ad_g \omega_{f(a)\cdot g} \circ \mathrm{d}_{f(a)} R_g \circ \mathrm{d}_a f(a) \\ &= q \circ Ad_g (R_g^* \omega)_{f(a)} \circ \mathrm{d}_a f(a) \\ &= q \circ (f^*(Ad_g R_g^* \omega))_a \\ &= q \circ (f^*\omega)_a \\ &= \tilde{\omega}_a. \end{aligned}$$

The second property of principal connections is immediate since q is the identity on $\mathfrak{g}_A$. □

Remark 6.1.16. *In some cases, there is a natural choice for V. We recall that the Killing form of $\mathfrak{g}_B$ is by definition the symmetric bilinear form $K(x,y) = tr(ad_x ad_y)$ (where ad_x, ad_y are considered as endomorphisms of $\mathfrak{g}_B$). Note that the Killing form is invariant under every automorphism of Lie algebra of $\mathfrak{g}_B$. When K is non degenerate, we say that $\mathfrak{g}_B$ is semisimple. If moreover, $K_{|_{\mathfrak{g}_A \times \mathfrak{g}_A}}$ is also non degenerate, we can consider the very natural choice $V := \mathfrak{g}_A^\perp$ where $\mathfrak{g}_A^\perp$ is the orthogonal with respect to K (note that the two non degeneracy conditions imply that $V \oplus \mathfrak{g}_A = \mathfrak{g}_B$). The fact that $Ad_g(V) = V$ for all $g \in G_A$ is then given by the invariance of the Killing form by the Lie algebra automorphism (preserving $\mathfrak{g}_A$ since $g \in G_A$) Ad_g.*

We conclude this section by the following proposition that we do not prove (the proof follows quite easily from the definitions)

Proposition 6.1.17. *Let A be a G_A principal bundle over $\mathcal{M}$ and B be a G_B principal bundle over the same manifold $\mathcal{M}$. We assume that we have a covering Lie group morphism $\tilde{f} : G_A \to G_B$ and a smooth covering map $f : A \to B$ such that $\pi_B \circ f = \pi_A$ and for all $g \in G_A$ and $a \in A$, $f(a \cdot g) = f(a) \cdot \tilde{f}(g)$. Then, if ω is a principal connection on B, $f^*\omega$ is a principal connection on A.*

6.2 Spin connection, GHP connection and GHP operators

For deeper geometric insight on the definitions of connections and operators, see [43].

Spin connection

We denote by ∇ the Levi-Civita connection on $\mathcal{M}$. Using proposition 6.1.10, we can define a principal connection ω on the space of complex tangent frames E. We now want to apply proposition 6.1.12 (and remark 6.1.13) to the various principal bundles previously defined. We have an embedding of principal bundles $f : \mathfrak{O} \to E$. We use the following proposition and proposition 6.1.12 (and remark 6.1.13) to show that $f^*\omega$ is a principal connection on $\mathfrak{O}$. We still call it ω.

Proposition 6.2.1. *Let $y_0 \in \mathcal{M}$. For all $(f_0, ..., f_3) \in \mathfrak{O}_{y_0}$, there exists a local smooth section $(e_0, ..., e_3)$ (around y_0) of $\mathfrak{O}$ such that $(e_0, ..., e_3)(y_0) = (f_0, ..., f_3)$ and for all $X \in T_{y_0}\mathcal{M}, (\nabla_X e_0, ..., \nabla_X e_3) = 0$.*

Proof. We define $(x_0, ..., x_3)$ local normal coordinates on a neighborhood of y_0 such that $(\partial_{x_0}, ..., \partial_{x_3})(y_0) = (f_0, ..., f_3)$. Then we denote by $g_{i,j}$ the metric coefficients in these coordinates. Because coordinates are normal we have for all $X \in T_{y_0}\mathcal{M}$, for all $i \in [\![0, 3]\!]$, $\nabla_X \partial_{x_i} = 0$, $g_{i,j}(y_0) = \delta_{i,j}$ and g has vanishing first derivatives at y_0. We define $(e_0, ..., e_3)$ as the Gram-Schmidt orthonormalization of $(\partial_{x_0}, ..., \partial_{x_3})$. As a consequence there is a smooth family of upper triangular invertible matrices $S(y)$ that we can express explicitly with respect to the coefficients $g_{i,j}$ such that $S(y_0) = Id$ and $(e_1, ..., e_n)(y) = (\partial_{x_0}, ..., \partial_{x_3})(y)S(y)$. Then we have, for $X \in T_{y_0}\mathcal{M}$, $(\nabla_X e_1, ..., \nabla_X e_n) = (\nabla_X \partial_{x_0}, ..., \nabla_X \partial_{x_3}) + (\partial_{x_0}, ..., \partial_{x_3})(y)X(S)(y_0)$ where $X(S)(y_0)$ is the derivation coefficient by coefficient. Because all the derivatives of the metric coefficients $g_{i,j}$ vanish at y_0, we have $X(S)(y_0) = 0$ and $(\nabla_X e_1, ..., \nabla_X e_n) = 0$. □

We can then use proposition 6.1.17 to define a connection ω on $\mathfrak{S}$ by pulling back the principal connection on $\mathfrak{O}$. We can also define a linear connection ∇ on the spinor bundle $\mathcal{S}$ and on $\overline{\mathcal{S}}$ by using proposition 6.1.4. Concretely, note that for a real vector field X and a section $\overline{a}$ of $\overline{\mathcal{S}}$, we have the equality $\nabla_X \overline{a} = \overline{\nabla_X a}$ (it follows from the definitions). Therefore, by $\mathbb{C}$-linearity, if X is a complex vector field, $\nabla_X \overline{a} = \overline{\nabla_{\overline{X}} a}$. We used the same notation ∇ and ω for connections on different bundles (and we use the context to remove ambiguity). To show that all these definitions are consistent, it is useful to prove the following proposition:

Proposition 6.2.2. *We have $j^*\nabla = \nabla \otimes \nabla$ where the left hand side is the Levi-Civita connection on $T_{\mathbb{C}}\mathcal{M}$ and the right hand side is the connection on $\mathcal{S} \otimes \overline{\mathcal{S}}$.*

Proof. First note that $\nabla \otimes \nabla$ is the same connection as the one obtained by proposition 6.1.4 applied to the bundle $\mathcal{S} \otimes \overline{\mathcal{S}}$ seen as associated to $\mathfrak{S}$ (see proposition 6.1.9). Let $x_0 \in \mathcal{M}$. Using proposition 6.2.1, we define a local smooth section (e_0, e_1, e_2, e_3) of $\mathfrak{O}$ around x_0 such that $(\nabla e_0(x_0), ..., \nabla e_3(x_0)) = 0$. In particular, by definition of the connection on $\mathfrak{O}$, $\omega \circ \mathrm{d}_{x_0}(e_0, ..., e_3) = 0$. Let s be a local smooth section of $\mathfrak{S}$ around x_0 such that $p \circ s = (e_0, ..., e_3)$. By definition of the connection on $\mathfrak{S}$, we have $\omega \circ \mathrm{d}_{x_0} s = 0$. Moreover, the trivialization induced by s on $\mathfrak{S}$ and the one induced by $(e_0, ..., e_3)$ on $\mathfrak{O}$ are compatible. Then j written in these local trivializations is just the map $id_U \times i_0$ where $i_0 : \mathbb{C}^2 \otimes \overline{\mathbb{C}}^2 \to \mathbb{C}^4$ is an isomorphism (defined earlier). We take Z_1 a local smooth section of $\mathcal{S} \otimes \overline{\mathcal{S}}$ on a small neighborhood U of x_0, then $Z_2 = j(Z_1)$ is a local smooth section of $T_{\mathbb{C}}\mathcal{M}$ around x_0. We have $Z_1 = [(s, \tilde{Z}_1)]$ with $\tilde{Z}_1 : U \to \mathbb{C}^2 \otimes \overline{\mathbb{C}}^2$ and $Z_2 = [((e_0, ..., e_3), \tilde{Z}_2)]$ with $\tilde{Z}_2 := i_0 \circ \tilde{Z}_1$. Let $X \in T_{x_0}\mathcal{M}$. By the first remark in this proof and the definition of the connection on an associated bundle (proposition 6.1.4), we have that $(\nabla \otimes \nabla)_X Z_1 = [(s, \ \mathrm{d}_{x_0}\tilde{Z}_1(X))]$ (it uses the fact that $\omega \circ \mathrm{d}_{x_0} s = 0$ and lemma 6.1.5). We similarly have $\nabla_X Z_2 = [((e_0, ..., e_3), \ \mathrm{d}_{x_0}\tilde{Z}_2(X))] = [((e_0, ..., e_3), i_0 \ \mathrm{d}_{x_0}\tilde{Z}_1(X))]$ (it uses the fact that $\omega \circ \mathrm{d}_{x_0}(e_0, ..., e_3) = 0$, lemma 6.1.5 and the linearity of i_0). Then we conclude that $\nabla_X j \circ Z_1 = j(\nabla \otimes \nabla)_X Z_1$. Since it is true for all local smooth sections Z_1, all $X \in T_{x_0}\mathcal{M}$ and all $x_0 \in \mathcal{M}$, we have proved the proposition. □

Remark 6.2.3. *In this remark, we use the identification j implicitly. Note that ∇ is not the only connection with the property $\nabla \otimes \nabla = \nabla_{LC}$ (at least locally). Indeed, let U be an open set on which we have a smooth spin frame (o, ι) with $\epsilon(o, \iota) = 1$. Then we can define a connection ∇' on $\mathcal{S}_{|_U}$ such that for all real vector fields X on U:*

$$\nabla'_X o = (a(X) + i\mu(X))o + b(X)\iota$$
$$\nabla'_X \iota = c(X)o + (-a(X) + i\mu(X))\iota$$

where a, b and c are complex valued linear forms on U and μ is a real valued linear form on U. Moreover, we define a,b and c as follows:

$$\begin{aligned} b(X) &= -g((\nabla_{LC})_X(o\otimes\overline{o}), o\otimes\overline{\iota}) \\ c(X) &= -g((\nabla_{LC})_X(\iota\otimes\overline{\iota}), \iota\otimes\overline{o}) \\ a(X) &= \frac{1}{2}\left(g((\nabla_{LC})_X(o\otimes\overline{o}), \iota\otimes\overline{\iota}) - g((\nabla_{LC})_X(o\otimes\overline{\iota}), \iota\otimes\overline{o})\right) \end{aligned} \tag{6.1}$$

Conditions (6.1) *are necessary to have $\nabla\otimes\nabla = \nabla_{LC}$ and we now prove that they are sufficient (in the definition of ∇' we can chose freely any real linear form μ hence the lack of uniqueness). Note that $o\otimes\overline{o}$, $\iota\otimes\overline{\iota}$, $o\otimes\overline{\iota}$, $\iota\otimes\overline{o}$ is a normalized null tetrad on U. Therefore we have (using properties of the Levi-Civita connection):*

$$\begin{aligned} (\nabla_{LC})_X o\otimes\overline{o} &= 2\Re(a(X))o\otimes\overline{o} + b(X)\iota\otimes\overline{o} + \overline{b(X)}o\otimes\overline{\iota} \\ (\nabla_{LC})_X \iota\otimes\overline{\iota} &= -2\Re(a(X))\iota\otimes\overline{\iota} + c(X)o\otimes\overline{\iota} + \overline{c(X)}\iota\overline{o} \\ (\nabla_{LC})_X o\otimes\overline{\iota} &= \overline{c(X)}o\otimes\overline{o} + b(X)\iota\otimes\overline{\iota} + 2i\Im(a(X))o\otimes\overline{\iota} \\ (\nabla_{LC})_X \iota\otimes\overline{o} &= c(X)o\otimes\overline{o} + \overline{b(X)}\iota\otimes\overline{\iota} - 2i\Im(a(X))\iota\otimes\overline{o} \end{aligned}$$

To check that $\nabla'\otimes\nabla' = \nabla_{LC}$, it is enough to prove that check the equality on this tetrad. It follows from the definition of ∇'.

The previous remark shows that ∇ is not completely determined by proposition 6.2.2. However, we also have the following proposition:

Proposition 6.2.4. *We have $\nabla\epsilon = 0$*

Proof. By definition, $\nabla\epsilon = 0$ if for all $x_0 \in \mathcal{M}$, $X \in T_{x_0}\mathcal{M}$ and a,b spinor fields defined on a neighborhood U of x_0, we have:

$$X(\epsilon(a,b))(x_0) = \epsilon(\nabla_X a, b)(x_0) + \epsilon(a, \nabla_X b)(x_0).$$

We denote by $\omega_{\mathfrak{S}}$ the principal connection on $\mathfrak{S}$ previously defined (and used to define the spin connection ∇). Let s be a smooth local section of $\mathfrak{S}$ such that $(\omega_{\mathfrak{S}})_{s(x_0)}\ \mathrm{d}_{x_0}s = 0$ (which exists by remark 6.1.7). Let $\tilde{a} : U \to \mathbb{C}^2$ and $\tilde{b} : U \to \mathbb{C}^2$ smooth be such that $a(x) = [(s(x), \tilde{a}(x))]$ and $b(x) = [(s(x), \tilde{b}(x))]$. By lemma 6.1.5, we have

$$\begin{aligned} \nabla_X a(x_0) &= [(s(x_0),\ \mathrm{d}_{x_0}\tilde{a}(X))] \\ \nabla_X b(x_0) &= [(s(x_0),\ \mathrm{d}_{x_0}\tilde{b}(X))] \end{aligned}$$

Moreover, by definition of ϵ (see proposition 4.2.9), we have $\epsilon(a,b)(x) = \det(\tilde{a}(x), \tilde{b}(x))$. Therefore by bilinearity of the determinant:

$$\begin{aligned} X(\epsilon(a,b))(x_0) &= \det(\ \mathrm{d}_{x_0}\tilde{a}(X), \tilde{b}(x_0)) + \det(\tilde{a}(x_0),\ \mathrm{d}_{x_0}\tilde{b}(X)) \\ &= \epsilon(\nabla_X a, b)(x_0) + \epsilon(a, \nabla_X b)(x_0). \end{aligned}$$

□

Remark 6.2.5. *It is very useful to note that the connection ∇ on $\mathcal{S}$ is completely determined by proposition 6.2.2 and proposition 6.2.4 (in particular, we could have used these properties*

as a definition of the connection ∇ on $\mathcal{S}$). Indeed, if we fix a local basis (o,ι) of $\mathcal{S}$ such that $\epsilon(o,\iota)=1$ and X a vector field on $\mathcal{M}$, we can write:

$$\begin{aligned}\nabla_X o &= a(X)o + b(X)\iota\\ \nabla_X \iota &= c(X)o + d(X)\iota\end{aligned}$$

Using proposition 6.2.4 and $\epsilon(o,\iota)=1$, we have $a(X)=-d(X)$. By proposition 6.2.2, we also have:

$$\begin{aligned}b(X) &= -g(\nabla_X(o\otimes\overline{o}), o\otimes\overline{\iota})\\ c(X) &= -g(\nabla_X(\iota\otimes\overline{\iota}), \iota\otimes\overline{o})\\ a(X) &= \frac{1}{2}\left(g(\nabla_X(o\otimes\overline{o}), \iota\otimes\overline{\iota}) - g(\nabla_X(o\otimes\overline{\iota}), \iota\otimes\overline{o})\right)\end{aligned}$$

The coefficients of the one forms a, $-b$ and c on the null basis $(l,n,m,\overline{m}) := (o\otimes\overline{o}, \iota\otimes\overline{\iota}, o\otimes\overline{\iota}, \iota\otimes\overline{o})$ are called spin coefficients. For example, following the notation in [18] chapter 1 (286) for the spin coefficients:

$$\begin{array}{cccc}\kappa=-b(l) & \tau=-b(n) & \sigma=-b(m) & \rho=-b(\overline{m})\\ \pi=c(l) & \nu=c(n) & \mu=c(m) & \lambda=c(\overline{m})\\ \epsilon=a(l) & \gamma=a(n) & \beta=a(m) & \alpha=a(\overline{m})\end{array}$$

Remark 6.2.6. *If we require that (o,ι) is a local section of $\mathcal{A}_0$ (and as usual $(l,n,m) := d(o,\iota)$), we have that l and n are pregeodesic and therefore, by remark 6.2.5, we have:*

$$\begin{aligned}b(l) &= -g(\nabla_l l, m) = 0\\ c(n) &= -g(\nabla_n n, \overline{m}) = 0\end{aligned}$$

Moreover, since l and n are shear-free and $\Re(m), -\Im(m)$ is an orthonormal family of $l^\perp$ and $n^\perp$, we have

$$\begin{aligned}0 =& \frac{1}{2}\left(g(\nabla_{\Im(m)}l, \Im(m)) - g(\nabla_{\Re(m)}l, \Re(m))\right) - \frac{i}{2}\left(g(\nabla_{\Re(m)}l, \Im(m)) + g(\nabla_{\Im(m)}l, \Re(m))\right)\\ =& -\frac{1}{2}g(\nabla_m l, m)\end{aligned}$$

and similarly

$$\begin{aligned}0 =& -\frac{1}{2}g(\nabla_m n, m)\\ =& -\frac{1}{2}g(\nabla_{\overline{m}} n, \overline{m}) \qquad\qquad \textit{(complex conjugate)}\end{aligned}$$

Therefore we have:

$$\begin{aligned}b(m) =& 0\\ c(\overline{m}) =& 0\end{aligned}$$

In the spin-coefficient formalism, this corresponds to the vanishing of κ, λ, σ and ν.

Remark 6.2.7. *We can also compute concretely the connection ω on $\mathfrak{S}$ using proposition 6.1.14: Indeed, we have a natural embedding of $\mathfrak{S}$ in the set of frames (o,ι) of $\mathcal{S}$ given by the map f of proposition 6.1.14 (note that the representation of $SL(2,\mathbb{C})$ into $GL(\mathbb{C}^2)$ is just the natural embedding so $\mathrm{d}_1\rho$ is just the inclusion of $\mathfrak{sl}(2,\mathbb{C})$ into $M_2(\mathbb{C})$). The map f identifies $\mathfrak{S}$ with the set of frames (o,ι) such that $\epsilon(o,\iota)=1$. Therefore, proposition 6.1.14 implies that for a local smooth section s of $\mathfrak{S}$ identified with the local frame (o,ι) and $X\in T_{x_0}\mathcal{M}$,*

$$\omega(\mathrm{d}_{x_0}s(X)) = \underset{(o,\iota)(x_0)}{Mat}(\nabla_X o, \nabla_X \iota) = \begin{pmatrix} a(X) & c(X)\\ b(X) & -a(X)\end{pmatrix} \in \mathfrak{sl}(2,\mathbb{C}).$$

GHP connection

We refer the reader to subsection 4.3 for the definition of $\mathcal{A}$ and to subsection 4.4 for the definition of $\mathcal{N}$. In subsection 4.5, we defined $\mathcal{A}_0$ and $\mathcal{N}_0$ under the assumption that $\mathcal{A}$ has two connected components. We now define the connection on $\mathcal{A}_0$ and $\mathcal{N}_0$ using proposition 6.1.15.

Remark 6.2.8. *Since $\mathcal{N}_0$ can be embedded into the bundle of tangent frames (see (6.2) below), we could expect to use proposition 6.1.12 to define a connection of $\mathcal{N}_0$ on the model of what was done in subsection 6.2 to define the connection on $\mathfrak{O}$. However, this simple approach fails due to the fact that we do not have the analog of proposition 6.2.1 for $\mathcal{N}_0$ and therefore the condition of remark 6.1.13 is not satisfied in general. For example, we show that the condition does not hold in Kerr. By contradiction, assume that the condition of remark 6.1.13 holds. In particular, we deduce that near any point $x \in \mathcal{M}$, we can find a local section (l, n, m) of $\mathcal{N}_0$ defined on a neighborhood U of x such that $\nabla l(x) = \nabla n(x) = \nabla m(x) = 0$ (where ∇ is the Levi-Civita connection). Then, we can write any local smooth section (l', n', m') of $(\mathcal{N}_0)_{|_U}$ as $(|z|\, l, |z|^{-1}\, n, \frac{z}{|z|} m)$ for some smooth function $z : U \to \mathbb{C} \setminus \{0\}$. In particular, we have:*

$$\nabla_{n'} l' = \frac{n'(|z|)}{|z|} l' + |z|\, \nabla_{n'} l$$

Therefore $\nabla_{n'} l'(x) \in \mathbb{R} l'(x)$ and the spin coefficient $-g(\nabla_{n'} l', m')(x) = \tau(x) = 0$ for every $x \in \mathcal{M}$ and every smooth section (l', n', m') of $\mathcal{N}_0$. But this is a contradiction since $\tau = \frac{-ia \sin\theta}{\rho^2 \sqrt{2}}$ according to (175) in [18, Chapter 6, Section 56].

The covering map $\tilde{p} : SL(2, \mathbb{C}) \to SO^+(1, 3)$ gives a Lie algebra isomorphism $\mathrm{d}_{Id}\tilde{p} : \mathfrak{sl}(2, \mathbb{C}) \to \mathfrak{so}(1, 3)$.

We also have an embedding of principal bundles

$$f : \begin{cases} \mathcal{N}_0 \to \mathfrak{O} \\ (l, n, m) \mapsto \left(\frac{l+n}{\sqrt{2}}, \sqrt{2}\Re(m), -\sqrt{2}\Im(m), \frac{l-n}{\sqrt{2}} \right) \end{cases} \tag{6.2}$$

associated[1] to the embedding of Lie groups

$$\tilde{f} : \begin{cases} \mathbb{C}^* \to SO^+(1, 3) \\ |z| e^{i\theta} \mapsto \begin{pmatrix} \frac{1}{2}(|z| + |z|^{-1}) & 0 & 0 & \frac{1}{2}(|z| - |z|^{-1}) \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ \frac{1}{2}(|z| - |z|^{-1}) & 0 & 0 & \frac{1}{2}(|z| + |z|^{-1}) \end{pmatrix} \end{cases}$$

We have chosen f to be the unique map such that $f \circ d = p \circ \mathfrak{f}$ where $\mathfrak{f}$ is the natural embedding of $\mathcal{A}_0$ into $\mathfrak{S}$ given by $(o, \iota) = \left[\mathfrak{f}(o, \iota), \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) \right]$. Note that $\mathfrak{f}((o, \iota) \cdot z) = \mathfrak{f}(o, \iota) \cdot \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix}$, therefore, $\mathfrak{f}$ is associated with the following embedding of Lie groups:

$$\tilde{\mathfrak{f}} : \begin{cases} \mathbb{C}^* \to SL(2, \mathbb{C}) \\ z \to \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} \end{cases} .$$

[1] Meaning that for all $x \in \mathcal{N}_0$ and $g \in \mathbb{C}^*$ we have $f(x \cdot g) = f(x) \cdot \tilde{f}(g)$

The fact that $f \circ d = p \circ \mathfrak{f}$ can be checked using a pair of compatible trivializations (and the associated trivializations for $\mathcal{S}$ and $T\mathcal{M} \otimes \mathbb{C}$) in which it follows from the equality (true for every $C_1, C_2 \in \mathbb{C}^2$):

$$\tilde{p}\left(C_1, C_2\right) = \left(i_0\left(\frac{C_1 \otimes \overline{C_1} + C_2 \otimes \overline{C_2}}{\sqrt{2}}\right), i_0\left(\frac{C_1 \otimes \overline{C_2} + C_2 \otimes \overline{C_1}}{\sqrt{2}}\right), \right.$$
$$\left. -i_0\left(\frac{C_1 \otimes \overline{C_2} - C_2 \otimes \overline{C_1}}{\sqrt{2}i}\right), i_0\left(\frac{C_1 \otimes \overline{C_1} - C_2 \otimes \overline{C_2}}{\sqrt{2}}\right)\right).$$

Note that f is unique since d is surjective.

We define $H := \tilde{f}(\mathbb{C}^*)$ which is a commutative embedded Lie subgroup of $SO^+(1,3)$ and $\mathfrak{h}$ its Lie algebra (it is an abelian Lie subalgebra of $\mathfrak{so}(1,3)$). In order to apply remark 6.1.16 (and proposition 6.1.15), we want to prove that the Killing form of $\mathfrak{so}(1,3)$ is non degenerate and that its restriction to $\mathfrak{h}$ is also non degenerate.

It is easier to check this using the isomorphism of real Lie algebras $\mathrm{d}_{Id}\tilde{p}$ since we have $(\mathrm{d}_{Id}\tilde{p})^{-1}(\mathfrak{h}) = \left\{ \begin{pmatrix} z & 0 \\ 0 & -z \end{pmatrix}, z \in \mathbb{C} \right\}$ (this follows from the equality $\tilde{f}(z^2) = \tilde{p}\begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix}$). Note that we consider $\mathfrak{sl}(2,\mathbb{C})$ as a real Lie algebra here, we will denote it by $\mathfrak{sl}(2,\mathbb{C})_{\mathbb{R}}$ to emphasize this fact. The Killing form of $\mathfrak{sl}(2,\mathbb{C})_{\mathbb{R}}$ is $B(M,N) = 8\Re(\mathrm{tr}(MN))$. Indeed, if $A_{\mathbb{C}}$ is the matrix of $ad_M \circ ad_N$ in the $\mathbb{C}$-basis $E_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, E_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, E_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, the matrix of $ad_M \circ ad_N$ in the $\mathbb{R}$ basis $(E_1, E_2, E_3, iE_1, iE_2, iE_3)$ is $A_{\mathbb{R}} := \begin{pmatrix} \Re(A_{\mathbb{C}}) & -\Im(A_{\mathbb{C}}) \\ \Im(A_{\mathbb{C}}) & \Re(A_{\mathbb{C}}) \end{pmatrix}$. Therefore $\mathrm{tr}(A_{\mathbb{R}}) = 2\Re(\mathrm{tr}(A_{\mathbb{C}}))$. We use this fact to deduce the expression of the Killing form of $\mathfrak{sl}(2,\mathbb{C})_{\mathbb{R}}$ from the classical expression of the Killing form of $\mathfrak{sl}(2,\mathbb{C})_{\mathbb{C}}$ (complex Lie algebra). Let $M \in \mathfrak{sl}(2,\mathbb{C})_{\mathbb{R}}$, we assume that for all $N \in \mathfrak{sl}(2,\mathbb{C})_{\mathbb{R}}$, $B(M,N) = 0$. In particular, $B(M, \overline{M}^T) = 0$ and $B(M, -i\overline{M}^T) = 0$. Therefore $\mathrm{tr}(M\overline{M}^T) = 0$ and $M = 0$. We conclude that B is non degenerate. Now let $M = \begin{pmatrix} z & 0 \\ 0 & -z \end{pmatrix}$ (with $z \in \mathbb{C}$) be such that for all N of the form $\begin{pmatrix} z' & 0 \\ 0 & -z' \end{pmatrix}$ (with $z' \in \mathbb{C}$), $B(M,N) = 0$. Again, we have $B\left(M, \overline{M}^T\right) = 0$ and $B\left(M, -i\overline{M}^T\right) = 0$ and therefore $M = 0$. We conclude that the restriction of B to $(\mathrm{d}_{Id}\tilde{p})^{-1}(\mathfrak{h})$ is non degenerate. A direct computation shows that the orthogonal complement of $(\mathrm{d}_{Id}\tilde{p})^{-1}(\mathfrak{h})$ is $\left\{ \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}, a, b \in \mathbb{C} \right\}$. Then by remark 6.1.16 we can use proposition 6.1.15 to define a principal connection ω on $\mathcal{N}_0$. Finally we can use proposition 6.1.17 to define a principal connection ω on $\mathcal{A}_0$, using the covering map d. Note that an equivalent definition is given by applying proposition 6.1.15 for the natural embedding $\mathfrak{f} : \mathcal{A}_0 \to \mathfrak{S}$ and the orthogonal projection (with respect to the Killing form) on $\left\{ \begin{pmatrix} z & 0 \\ 0 & -z \end{pmatrix}, z \in \mathbb{C} \right\}$ (the equivalence between the two definitions follows from the equality $f \circ d = p \circ \mathfrak{f}$).

Remark 6.2.9. *Using the second definition of the connection on $\mathcal{A}_0$ and remark 6.2.7, we have that for (o, ι) a smooth section of $\mathcal{A}_0$ around $x_0 \in \mathcal{M}$ and $X \in T_{x_0}\mathcal{M}$,*

$$\omega_{(o(x_0),\iota(x_0))}(\mathrm{d}_{x_0}(o,\iota)(X)) = \pi_{\mathfrak{h}}^{\perp}\begin{pmatrix} a(X) & c(X) \\ b(X) & -a(X) \end{pmatrix}$$
$$= \begin{pmatrix} a(X) & 0 \\ 0 & -a(X) \end{pmatrix}$$

where a,b and c are the same as in remark 6.2.5 and can be computed using only $(l, n, m, \overline{m}) := d(o, \iota)$ and the Levi-Civita connection. We have denoted the Killing-orthogonal projection onto

$(\mathrm{d}_{Id}\tilde{p})^{-1}(\mathfrak{h})$ *by* $\pi_{\mathfrak{h}}^{\perp}$*. Also note that in this equality, the Lie algebra of* $\mathbb{C}^*$ *has been identified with the Lie algebra of* $\tilde{\mathfrak{f}}(\mathbb{C}^*)$ *(see proposition 6.1.15). Without the identification, we simply get* $\omega_{(o(x_0),\iota(x_0))}(\mathrm{d}_{x_0}(o,\iota)(X)) = a(X)$*.*

We can use this principal connection to define a connection Θ on $\mathcal{B}(s,w)$ by proposition 6.1.4. Now we give a way to compute concretely Θu for u a local smooth section of $\mathcal{B}(s,w)$.

Proposition 6.2.10. *Let* (o,ι) *be a local smooth section of* $\mathcal{A}_0$*. Let* u *be a local smooth section of* $\mathcal{B}(s,w)$ *on an open set* U *of* $\mathcal{M}$ *and let* $u_1 : U \to \mathbb{C}$ *be such that* $u = [(o,\iota), u_1]$*. Then for* $x_0 \in U$ *and* $X \in \mathbb{C} \otimes_{\mathbb{R}} T_{x_0}\mathcal{M}$

$$\Theta_X u = [(o,\iota)(x_0), -\left((w+s)a(X) + (w-s)\overline{a(\overline{X})}\right)u_1(x_0) + X(u_1)]$$

where $a(X) = \epsilon(\nabla_X o, \iota) = \epsilon(\nabla_X \iota, o)$ *as previously.*

Proof. By $\mathbb{C}$-linearity of both sides, it is enough to prove the result for real vectors. Let $x_0 \in U$ and $X \in T_{x_0}\mathcal{M}$ (real vector space). By corollary 6.1.8 and remark 6.2.9, we have:

$$\begin{aligned}\Theta_X u &= [(o(x_0),\iota(x_0)), X(u_1) + \mathrm{d}_1(\rho_{w,s})\omega(\mathrm{d}_{x_0}(o,\iota)(X))u_1(x_0)] \\ &= [(o(x_0),\iota(x_0)), X(u_1) + \mathrm{d}_1(\rho_{w,s})(a(X))u_1(x_0)] \\ &= [(o(x_0),\iota(x_0)), X(u_1) - ((s+w)a(X) + (w-s)\overline{a(\overline{X})})u_1(x_0)]\end{aligned}$$

□

Remark 6.2.11. *It is reassuring to check that if* u *is a smooth section of* $\mathcal{B}(s,w)$ *and* v *is a smooth section of* $\mathcal{B}(s',w')$ *and if we fix* (o,ι) *a local smooth section of* $\mathcal{A}_0$ *near* x_0 *and write* $u = [(o,\iota), u_1]$*,* $v = [(o,\iota), v_1]$*,* $u \otimes v = [(o,\iota), u_1 v_1]$*, and* $\Theta_X(u \otimes v) = [(o,\iota), w_1]$ *we have:*

$$\begin{aligned}w_1 &= \left(-(w+w'+s+s')a(X) - (w+w'-(s+s)')\overline{a(\overline{X})}\right)u_1(x_0)v_1(x_0) + X(u_1v_1) \\ &= \left(\left(-(w+s)a(X) - (w-s)\overline{a(\overline{X})}\right)u_1(x_0) + X(u_1)\right)v_1(x_0) \\ &\quad + \left(\left(-(w'+s')a(X) - (w'-s')\overline{a(\overline{X})}\right)v_1(x_0) + X(v_1)\right)u_1(x_0)\end{aligned}$$

and therefore $\Theta_X(u \otimes v) = (\Theta_X u) \otimes v + u \otimes (\Theta_X v)$*. There is a compatibility between the connection acting on different spin weighted bundles and the isomorphism* $\mathcal{B}(s,w) \otimes \mathcal{B}(s',w') = \mathcal{B}(s+s', w+w')$*.*

GHP operators

If u is a smooth section of $\mathcal{B}(s,w)$, then Θu is a smooth section of $T^*_{\mathbb{C}}\mathcal{M} \otimes \mathcal{B}(s,w)$. We can define the operators which map Θu to its spin weighted components (defined in remark 4.5.13). Equivalently these operators can be seen as contraction of the spin weighted 2-cospinor Θu with the spin weighted spinors o,ι, $\overline{o}$ and $\overline{\iota}$. These operators are called GHP operators. In this subsection, we denote by $\boldsymbol{o}$ the first projection of $\mathcal{A}_0$ and $\boldsymbol{\iota}$ the second projection. We also use the notation $\boldsymbol{l} = \boldsymbol{o} \otimes \overline{\boldsymbol{o}}$, $\boldsymbol{n} = \boldsymbol{\iota} \otimes \overline{\boldsymbol{\iota}}$ and $\boldsymbol{m} = \boldsymbol{o} \otimes \overline{\boldsymbol{\iota}}$. We define the operators

$$\text{þ} : \begin{cases}\Gamma(\mathcal{B}(s,w)) \to \Gamma(\mathcal{B}(s,w+1)) \\ u \mapsto \Theta_{\boldsymbol{l}} u\end{cases} \qquad \text{þ}' : \begin{cases}\Gamma(\mathcal{B}(s,w)) \to \Gamma(\mathcal{B}(s,w-1)) \\ u \mapsto \Theta_{\boldsymbol{n}} u\end{cases}$$

$$\text{ð} : \begin{cases}\Gamma(\mathcal{B}(s,w)) \to \Gamma(\mathcal{B}(s+1,w)) \\ u \mapsto \Theta_{\boldsymbol{m}} u\end{cases} \qquad \text{ð}' : \begin{cases}\Gamma(\mathcal{B}(s,w)) \to \Gamma(\mathcal{B}(s-1,w)) \\ u \mapsto \Theta_{\overline{\boldsymbol{m}}} u\end{cases}$$

Remark 6.2.12. *We can use proposition 6.2.10 to compute þ, þ′, ð and ð′ in a local trivialization given by a smooth local section (o, ι) of $\mathcal{A}_0$. Indeed, if $u \in \mathcal{B}(s, w)$ is written as $u = [(o, \iota), u_1]$ we have:*

$$\begin{aligned}
(þu)_1 &= \left(-(w+s)a(l) - (w-s)\overline{a(l)}\right) u_1 + l(u_1) \\
(þ'u)_1 &= \left(-(w+s)a(n) - (w-s)\overline{a(n)}\right) u_1 + n(u_1) \\
(ðu)_1 &= \left(-(w+s)a(m) - (w-s)\overline{a(\overline{m})}\right) u_1 + m(u_1) \\
(ð'u)_1 &= \left(-(w+s)a(\overline{m}) - (w-s)\overline{a(m)}\right) u_1 + \overline{m}(u_1).
\end{aligned}$$

Using spin coefficients as defined in remark 6.2.5, we have:

$$\begin{aligned}
(þu)_1 &= \left(-(w+s)\epsilon - (w-s)\overline{\epsilon}\right) u_1 + l(u_1) \\
(þ'u)_1 &= \left(-(w+s)\gamma - (w-s)\overline{\gamma}\right) u_1 + n(u_1) \\
(ðu)_1 &= \left(-(w+s)\beta - (w-s)\overline{\alpha}\right) u_1 + m(u_1) \\
(ð'u)_1 &= \left(-(w+s)\alpha - (w-s)\overline{\beta}\right) u_1 + \overline{m}(u_1).
\end{aligned}$$

Proposition 6.2.13. *There exists a unique element $\boldsymbol{b(\overline{m})}$ of $\mathcal{B}(0,1)$ such that for every local smooth section o, ι of $\mathcal{A}_0$ defined on an open set U, $\boldsymbol{b(\overline{m})}_{|_U} = [(o, \iota), b(\overline{m})]$ (where $b(\overline{m}) := -\epsilon(\nabla_{\iota\otimes\overline{o}} o, o)$).*

Proof. The only thing to prove is that the definition does not depend of the choice of the local section (o, ι). Let (o, ι) be a local smooth section of $\mathcal{A}_0$ on an open set U. Let $z : U \to \mathbb{C}^*$ be a smooth map and $(o', \iota') = (o, \iota) \cdot z = (zo, z^{-1}\iota)$ (every local smooth section on U can be written in this form). We then have:

$$\begin{aligned}
\epsilon(\nabla_{\iota'\otimes\overline{o}'} o', o') =& \epsilon(\nabla_{\overline{z}z^{-1}\iota\otimes\overline{o}}(zo), zo) \\
=& \epsilon(\overline{z}z^{-1}\overline{m}(z)o, zo) + \epsilon(\overline{z}\nabla_{\iota\otimes\overline{o}} o, zo) \\
=& z\overline{z}\epsilon(\nabla_{\iota\otimes\overline{o}} o, o) \\
=& \rho_{(0,1)}(z^{-1})\epsilon(\nabla_{\iota\otimes\overline{o}} o, o)
\end{aligned}$$

Since $[(o, \iota), -\rho_{(0,1)}(z^{-1})\epsilon(\nabla_{\iota\otimes\overline{o}} o, o)] = [(o', \iota') \cdot z^{-1}, -\epsilon(\nabla_{\iota\otimes\overline{o}} o, o)]$ by definition of $\mathcal{B}(0,1)$, we finally have:

$$[(o', \iota'), -\epsilon(\nabla_{\iota'\otimes\overline{o}'} o', o')] = [(o, \iota), -\epsilon(\nabla_{\iota\otimes\overline{o}} o, o)]$$

□

Remark 6.2.14. *Thanks to remark 4.5.5, we can see $\boldsymbol{b(\overline{m})}$ as a multiplication operator from $\mathcal{B}(s, w)$ to $\mathcal{B}(s, w+1)$.*

Remark 6.2.15. *We similarly define the spin weighted functions $\boldsymbol{b(n)} \in \mathcal{B}(1,0)$, $\boldsymbol{c(m)} \in \mathcal{B}(0,-1)$ and $\boldsymbol{c(l)} \in \mathcal{B}(-1,0)$.*

Chapter 7

Definition of the Teukolsky operator

Definition 7.0.1 (Contraction operator)**.** *Let $\phi \in (\mathcal{S}')^{\otimes n_0}$ ($n_0 \in \mathbb{N}$). We define the operator $C_{i,j}$ (with $i < j$) such that in any basis (s_0, s_1) of $\mathcal{S}$ with $\epsilon(s_0, s_1) = 1$:*

$$\begin{aligned}(C_{i,j}\phi)(s_{k_1}, ..., \hat{s_{k_i}}, ..., \hat{s_{k_j}}, ..., s_{i_{n_0}}) =& \phi(s_{k_1}, ..., s_{k_{i-1}}, s_1, s_{k_{i+1}}, ..., s_{k_{j-1}}, s_0, s_{k_{j+1}}, ..., s_{k_{n_0}}) \\ &- \phi(s_{k_1}, ..., s_{k_{i-1}}, s_0, s_{k_{i+1}}, ..., s_{k_{j-1}}, s_1, s_{k_{j+1}}, ..., s_{k_{n_0}})\end{aligned}$$

where $k_1, ...k_n \in \{0, 1\}$ and $\hat{s_{k_l}}$ means that s_{k_l} is skipped in the enumeration. This definition does not depend on the chosen basis as long as it is normalized.

Definition 7.0.2. *Let $\phi \in \Gamma((\mathcal{S}')^{\odot n_0})$. We define the operator D (Dirac operator) as $D\phi = C_{1,3}\nabla\phi \in \Gamma(\overline{\mathcal{S}}' \otimes (\mathcal{S}')^{\otimes n_0 - 1})$. Because the spinor is symmetric, we can replace 3 in the definition by any index in $\{3, ..., n_0 + 2\}$.*

Remark 7.0.3. *Note that $D\phi$ is symmetric with respect to the last $n_0 - 1$ variables but the first one has a particular status.*

Proposition 7.0.4. *We have the following relations at the level of spin weighted components:*

$$(D\phi)(\overline{\boldsymbol{o}}, \boldsymbol{o}, ..., \boldsymbol{o}) = (\eth' + \boldsymbol{c(l)})\,\phi(\boldsymbol{o}, ..., \boldsymbol{o}) - (\text{Þ} + n_0\boldsymbol{b(\overline{m})})\,\phi(\boldsymbol{\iota}, \boldsymbol{o}, ..., \boldsymbol{o}) \tag{7.1}$$

$$(D\phi)(\overline{\boldsymbol{\iota}}, \boldsymbol{o}, ..., \boldsymbol{o}) = (\text{Þ}' + \boldsymbol{c(m)})\,\phi(\boldsymbol{o}, ..., \boldsymbol{o}) - (\eth + n_0\boldsymbol{b(n)})\,\phi(\boldsymbol{\iota}, \boldsymbol{o}, ..., \boldsymbol{o}) \tag{7.2}$$

where $\boldsymbol{c(l)}$, $\boldsymbol{c(m)}$, $\boldsymbol{b(\overline{m})}$ and $\boldsymbol{b(n)}$ are seen as multiplications operators on spin-weighted functions (see remark 6.2.14 and 6.2.15)

Proof. Since both sides of the equalities are spin weighted functions of the same weight $((\frac{n_0}{2} - 1, \frac{n_0}{2})$ for the first and $(\frac{n_0}{2}, \frac{n_0}{2} - 1)$ for the second), it is enough to check the equality in a local trivialization near each point. We do it for the first equality (the second is similar). Let $x_0 \in \mathcal{M}$, let (o, ι) be a local section of $\mathcal{A}_0$. Thanks to the bold notation, it is easy to compute the components in local trivializations associated to this local section:

$$\begin{aligned}(D\phi)(\overline{o}, o, ..., o) =& (\nabla_{\iota\otimes\overline{o}}\phi)(o, ..., o) - (\nabla_{o\otimes\overline{o}}\phi)(\iota, o, ..., o) \\ =& \overline{m}(\phi(o, ..., o)) - n_0\phi(\nabla_{\overline{m}}o, o, ..., o) - l(\phi(\iota, o, ..., o)) + \phi(\nabla_l\iota, o, ..., o) \\ &+ (n_0 - 1)\phi(\iota, \nabla_l o, ..., o) \\ =& (\overline{m} - n_0 a(\overline{m}) + c(l))(\phi(o, ..., o)) \\ &- (l + n_0 b(\overline{m}) - (n_0 - 2)a(l))(\phi(\iota, o, ..., o)) \\ &+ (n_0 - 1)b(l)\phi(\iota, \iota, o, ..., o)\end{aligned}$$

By remark 6.2.6, we have $b(l) = 0$. Moreover, since $\phi(\boldsymbol{o}, ..., \boldsymbol{o}) \in \Gamma\left(\mathcal{B}\left(\frac{n_0}{2}, \frac{n_0}{2}\right)\right)$, the expression of ð' on this bundle expressed in the local trivialization induced by (o, ι) is $\overline{m} - n_0 a(\overline{m})$ by

remark 6.2.12. Similarly, the fact that $\phi(\iota, o, ..., o) \in \Gamma\left(\mathcal{B}\left(\frac{n_0}{2}-1, \frac{n_0}{2}-1\right)\right)$ gives the local expression $l-(n_0-2)a(l)$ for þ. By definition, $c(l)$ and $b(\overline{m})$ are the local expressions for $\boldsymbol{c(l)}$ and $\boldsymbol{b(\overline{m})}$ in the local trivialization induced by (o, ι). Therefore, we have the desired equality. □

Proposition 7.0.5. *We have the following relations at the level of spin weighted components:*

$$\begin{aligned} G_{\frac{n_0}{2}}\phi(\boldsymbol{o}, ..., \boldsymbol{o}) =& (þ + n_0\boldsymbol{b(\overline{m})} + \overline{\boldsymbol{b(\overline{m})}})(D\phi)(\bar{\iota}, \boldsymbol{o}, ..., \boldsymbol{o}) \\ & - (ð + \overline{\boldsymbol{c(l)}} + n_0\boldsymbol{b(n)})(D\phi)(\overline{\boldsymbol{o}}, \boldsymbol{o}, ..., \boldsymbol{o}) \end{aligned}$$

where

$$G_s = (þ + 2s\boldsymbol{b(\overline{m})} + \overline{\boldsymbol{b(\overline{m})}})(þ' + \boldsymbol{c(m)}) - (ð + \overline{\boldsymbol{c(l)}} + 2s\boldsymbol{b(n)})(ð' + \boldsymbol{c(l)})$$

is a smooth differential operator acting on the bundle $\mathcal{B}(s, s)$

Proof. We apply $(þ + n_0\boldsymbol{b(\overline{m})} + \overline{\boldsymbol{b(\overline{m})}})$ on the left of (7.2) (in proposition 7.0.4) and $(ð + \overline{\boldsymbol{c(l)}} + n_0\boldsymbol{b(n)})$ on the left of (7.1) and we take the difference. Then, we use the remarkable relation

$$(þ + n_0\boldsymbol{b(\overline{m})} + \overline{\boldsymbol{b(\overline{m})}})(ð + n_0\boldsymbol{b(n)}) - (ð + \overline{\boldsymbol{c(l)}} + n_0\boldsymbol{b(n)})(þ + n_0\boldsymbol{b(\overline{m})}) = 0 \tag{7.3}$$

as an operator acting on the bundle $\mathcal{B}\left(\frac{n_0}{2}, \frac{n_0}{2}\right)$. Therefore, the component $\phi(\boldsymbol{\iota}, \boldsymbol{o}, ..., \boldsymbol{o})$ disappears and we have the result. The last thing to prove is the relation (7.3). It is enough to prove it in a local trivialization provided by a smooth local section (o, ι) of $\mathcal{A}_0$ near each point $x_0 \in \mathcal{M}$. Therefore, we want to prove that:

$$\begin{aligned} &\left(l-(n_0-1)a(l) + \overline{a(l)} + n_0 b(\overline{m}) + \overline{b(\overline{m})}\right)(m-(n_0-2)a(m) + n_0 b(n)) \\ &-(m-(n_0-1)a(m) - \overline{a(\overline{m})} + \overline{c(l)} + n_0 b(n))(l-(n_0-2)a(l) + n_0 b(\overline{m})) = 0 \end{aligned}$$

If we use remark 6.2.5 to replace the occurrences of a,b and c by spin coefficients, we rewrite this as:

$$\begin{aligned} &(l-(n_0-1)\epsilon + \bar{\epsilon} - n_0\rho - \overline{\rho})(m-(n_0-2)\beta - n_0\tau) \\ &-(m-(n_0-1)\beta - \overline{\alpha} + \overline{\pi} - n_0\tau)(l-(n_0-2)\epsilon - n_0\rho) = 0 \end{aligned}$$

We see that this relation is a particular case of a relation between spin coefficients of a null tetrad with l and n in principal directions introduced by Teukolsky in [101] (equation (2.11)). It is the case $p = n_0 - 2$ and $q = -n_0$. □

Definition 7.0.6. *We define the Teukolsky operator $T_s := 2G_s - 4(s-1)\left(s-\frac{1}{2}\right)\boldsymbol{\Psi}_2$ (where $\boldsymbol{\Psi}_2$ was defined in remark 4.5.12). It is naturally a differential operator on $\mathcal{B}(s, s)$.*

7.1 The Teukolsky scalars for the electromagnetic fields and the linearized gravitational perturbations

In this section, we assume that we can find a smooth determination of $\boldsymbol{\Psi}_2^{\frac{2s}{3}}$ for any $s \in \frac{1}{2}\mathbb{Z}$ on the spacetime. It possible on the Kerr spacetime which is our main focus. The source-free Maxwell equations lead to the following equation on the electromagnetic spinor (definded by (4.3)):

$$D\phi = 0.$$

This equation correspond to [88, (5.1.57)] and we refer the reader to this reference for more details about the derivation. We have seen in proposition 7.0.5 that the component $\phi(\boldsymbol{o},\boldsymbol{o}) \in \mathcal{B}(1,1)$ satisfes the following equation:

$$G_1\phi(\boldsymbol{o},\boldsymbol{o}) = T_1\phi(\boldsymbol{o},\boldsymbol{o}) = 0. \tag{7.4}$$

We can perform the transformation[1] $(o,\iota) \mapsto (i\iota, io)$ which corresponds to making the other choice of connected component when defining $\mathcal{A}_0$. Since this choice was arbitrary, all the relations derived so far still hold after the transformation. As an example, from (7.4), we obtain[2]:

$$T_1'\phi(\boldsymbol{\iota},\boldsymbol{\iota}) = 0 \tag{7.5}$$

where

$$T_1' := 2(\text{þ}' + 2\boldsymbol{c(m)} + \overline{\boldsymbol{c(m)}})(\text{þ} + \boldsymbol{b}(\overline{\boldsymbol{m}})) - 2(\text{ð}' + \overline{\boldsymbol{b(n)}} + 2\boldsymbol{c(l)})(\text{ð} + \boldsymbol{b(n)})$$

is the operator obtained from T_1 after the transformation $(o,\iota) \mapsto (i\iota, io)$.

Definition 7.1.1. *We define the Teukolsky scalars for the electromagnetic field by:*

$$\begin{aligned}\alpha^{[1]} :=& \phi(\boldsymbol{o},\boldsymbol{o})\\ \alpha^{[-1]} :=& \boldsymbol{\Psi}_2^{-\frac{2}{3}}\phi(\boldsymbol{\iota},\boldsymbol{\iota}).\end{aligned}$$

Note that $\alpha^{[1]} \in \mathcal{B}(1,1)$ and $\alpha^{[-1]} \in \mathcal{B}(-1,-1)$.

The fact that $T_1\alpha^{[1]} = 0$ is exactly (7.4). Moreover, we have $T_{-1}\alpha^{[-1]} = 0$ as a consequence of (7.5) and of the following more general proposition:

Proposition 7.1.2. *Let $s \in \frac{1}{2}\mathbb{N}$. We have the following equality between operators acting on sections of $\mathcal{B}(-s,-s)$:*

$$\boldsymbol{\Psi}_2^{\frac{2s}{3}} T_{-s} \boldsymbol{\Psi}_2^{-\frac{2s}{3}} = T_s'$$

where

$$\begin{aligned}T_s' =& 2(\text{þ}' + 2s\boldsymbol{c(m)} + \overline{\boldsymbol{c(m)}})(\text{þ} + \boldsymbol{b}(\overline{\boldsymbol{m}})) - 2(\text{ð}' + \overline{\boldsymbol{b(n)}} + 2s\boldsymbol{c(l)})(\text{ð} + \boldsymbol{b(n)})\\ &- 4(s-1)\left(s - \frac{1}{2}\right)\boldsymbol{\Psi}_2\end{aligned}$$

Proof. For a Ricci-flat spacetime, the Bianchi identity implies (see [88, (4.10.9)]):

$$D\Psi = 0$$

where Ψ is the Weyl spinor already defined in (4.4). We deduce the following relations on $\boldsymbol{\Psi}_2$ (see [88, (4.12.37),(4.12.38)]):

$$\begin{aligned}\text{þ}\boldsymbol{\Psi}_2 =& -3\boldsymbol{b}(\overline{\boldsymbol{m}})\boldsymbol{\Psi}_2\\ \text{þ}'\boldsymbol{\Psi}_2 =& -3\boldsymbol{c(m)}\boldsymbol{\Psi}_2\\ \text{ð}\boldsymbol{\Psi}_2 =& -3\boldsymbol{b(n)}\boldsymbol{\Psi}_2\\ \text{ð}'\boldsymbol{\Psi}_2 =& -3\boldsymbol{c(l)}\boldsymbol{\Psi}_2.\end{aligned} \tag{7.6}$$

[1] It corresponds to the prime notation introduced in [88, (4.5.17)].

[2] We could alternatively adapt the computation presented in the proof of proposition 7.0.5

We also make use of the following relations[3] where operators are understood as acting on sections of $\mathcal{B}(-s,-s)$ (see [88, (4.12.33),(4.12.35)]):

$$\begin{aligned}\text{þ}\text{þ}' - \text{þ}'\text{þ} =& \left(\boldsymbol{c(l)} - \overline{\boldsymbol{b(n)}}\right)\text{ð} + \left(\overline{\boldsymbol{c(l)}} - \boldsymbol{b(n)}\right)\text{ð}' \\ &- 2s\left(\boldsymbol{b(n)c(l)} - \boldsymbol{\Psi}_2\right)\end{aligned} \tag{7.7}$$

$$\begin{aligned}\text{ð}\text{ð}' - \text{ð}'\text{ð} =& \left(\boldsymbol{c(m)} - \overline{\boldsymbol{c(m)}}\right)\text{þ} + \left(\overline{\boldsymbol{b(\overline{m})}} - \boldsymbol{b(\overline{m})}\right)\text{þ}' \\ &- 2s\left(\boldsymbol{b(\overline{m})c(m)} + \boldsymbol{\Psi}_2\right)\end{aligned} \tag{7.8}$$

$$\text{þ}(\boldsymbol{c(m)}) - \text{ð}(\boldsymbol{c(l)}) = -\boldsymbol{c(m)}\overline{\boldsymbol{b(\overline{m})}} + \boldsymbol{c(l)}\overline{\boldsymbol{c(l)}} + \boldsymbol{\Psi}_2 \tag{7.9}$$

$$\text{þ}'(\boldsymbol{b(\overline{m})}) - \text{ð}'(\boldsymbol{b(n)}) = -\boldsymbol{b(\overline{m})}\overline{\boldsymbol{c(m)}} + \boldsymbol{b(n)}\overline{\boldsymbol{b(n)}} + \boldsymbol{\Psi}_2 \tag{7.10}$$

From (7.6), we get:

$$\begin{aligned}\boldsymbol{\Psi}_2^{\frac{2s}{3}} T_{-s} \boldsymbol{\Psi}_2^{-\frac{2s}{3}} =& T_{-s} + 4s\boldsymbol{c(m)}\text{þ} + 4s\boldsymbol{b(\overline{m})}\text{þ}' - 4s\boldsymbol{c(l)}\text{ð} - 4s\boldsymbol{b(n)}\text{ð}' \\ &+ 4s\Big(\text{þ}(\boldsymbol{c(m)}) - \text{ð}(\boldsymbol{c(l)}) + \overline{\boldsymbol{b(\overline{m})}}\boldsymbol{c(m)} - \overline{\boldsymbol{c(l)}}\boldsymbol{c(l)} \\ &+ \boldsymbol{c(m)b(\overline{m})} - \boldsymbol{c(l)b(n)}\Big)\end{aligned}$$

Using (7.9), we can simplify the expression:

$$\begin{aligned}\boldsymbol{\Psi}_2^{\frac{2s}{3}} T_{-s} \boldsymbol{\Psi}_2^{-\frac{2s}{3}} =& T_{-s} + 4s\boldsymbol{c(m)}\text{þ} + 4s\boldsymbol{b(\overline{m})}\text{þ}' - 4s\boldsymbol{c(l)}\text{ð} - 4s\boldsymbol{b(n)}\text{ð}' \\ &+ 4s\left(\boldsymbol{\Psi_2} + \boldsymbol{c(m)b(\overline{m})} - \boldsymbol{c(l)b(n)}\right)\end{aligned}$$

Therefore, it remains to prove that:

$$\begin{aligned}T'_s - T_{-s} =& 4s\boldsymbol{c(m)}\text{þ} + 4s\boldsymbol{b(\overline{m})}\text{þ}' - 4s\boldsymbol{c(l)}\text{ð} - 4s\boldsymbol{b(n)}\text{ð}' \\ &+ 4s\left(\boldsymbol{\Psi}_2 + \boldsymbol{c(m)b(\overline{m})} - \boldsymbol{c(l)b(n)}\right)\end{aligned}$$

Using the definitions and relations (7.9) and (7.10) we have:

$$\begin{aligned}T'_s =& 2\text{þ}'\text{þ} + \left(4s\boldsymbol{c(m)} + 2\overline{\boldsymbol{c(m)}}\right)\text{þ} + 2\boldsymbol{b(\overline{m})}\text{þ}' \\ &- 2\text{ð}'\text{ð} - \left(2\overline{\boldsymbol{b(n)}} + 4s\boldsymbol{c(l)}\right)\text{ð} - 2\boldsymbol{b(n)}\text{ð}' \\ &+ 4s\left(\boldsymbol{c(m)b(\overline{m})} - \boldsymbol{c(l)b(n)}\right) - (4s^2 - 6s + 4)\boldsymbol{\Psi}_2 \\ T_{-s} =& 2\text{þ}\text{þ}' + 2\boldsymbol{c(m)}\text{þ} + \left(-4s\boldsymbol{b(\overline{m})} + 2\overline{\boldsymbol{b(\overline{m})}}\right)\text{þ}' \\ &- 2\text{ð}\text{ð}' - 2\boldsymbol{c(l)}\text{ð} - \left(2\overline{\boldsymbol{c(l)}} - 4s\boldsymbol{b(n)}\right)\text{ð}' \\ &- 4s\left(\boldsymbol{c(m)b(\overline{m})} - \boldsymbol{c(l)b(n)}\right) - (4s^2 + 6s + 4)\boldsymbol{\Psi}_2\end{aligned}$$

Therefore

$$\begin{aligned}T'_s - T_{-s} =& 2\Big(\text{þ}'\text{þ} - \text{þ}\text{þ}' + (\boldsymbol{c(l)} - \overline{\boldsymbol{b(n)}})\text{ð} + (\overline{\boldsymbol{c(l)}} - \boldsymbol{b(n)})\text{ð}'\Big) \\ &+ 2\Big(\text{ð}\text{ð}' - \text{ð}'\text{ð} + (\overline{\boldsymbol{c(m)}} - \boldsymbol{c(m)})\text{þ} + (\boldsymbol{b(\overline{m})} - \overline{\boldsymbol{b(\overline{m})}})\text{þ}'\Big) \\ &+ 4s\boldsymbol{c(m)}\text{þ} + 4s\boldsymbol{b(\overline{m})}\text{þ}' - 4s\boldsymbol{c(l)}\text{ð} - 4s\boldsymbol{b(n)}\text{ð}' \\ &+ 8s\left(\boldsymbol{c(m)b(\overline{m})} - \boldsymbol{c(l)b(n)}\right) + 12s\boldsymbol{\Psi}_2\end{aligned}$$

We conclude the proof by using (7.7) and (7.8). □

[3]The relations are obtained by writing the curvature components in the Newman-Penrose formalism.

The case of linearized gravitational perturbations is more involved. We consider a smooth family of metrics $(g_\lambda)_{\lambda\in(-1,1)}$ on $\mathcal{M}$ which are Ricci-flat and such that $g_0 = g$ is the Petrov type D metric we are interested in. Note that for $\lambda \neq 0$, we do not assume that g_λ is of Petrov type D.

Definition 7.1.3. *We define the Teukolsky scalars for the linearized gravitational perturbation* $\dot{g} = \frac{\mathrm{d}}{\mathrm{d}\lambda}_{|\lambda=0} g_\lambda$ *by*

$$\alpha^{[2]} := \mathrm{d}_g W[\dot{g}](\boldsymbol{l}, \boldsymbol{m}, \boldsymbol{l}, \boldsymbol{m})$$
$$\alpha^{[-2]} := \boldsymbol{\Psi}_2^{-\frac{4}{3}}\, \mathrm{d}_g W[\dot{g}](\overline{\boldsymbol{m}}, \boldsymbol{n}, \overline{\boldsymbol{m}}, \boldsymbol{n}).$$

where $W(g)$ *is the Weyl tensor (considered as a function of* g*) and* $\mathrm{d}_g W[\dot{g}] := \frac{\mathrm{d}}{\mathrm{d}\lambda}_{|\lambda=0} W(g_\lambda)$ *is the linearized Weyl tensor. Note that* $\alpha^{[2]} \in \mathcal{B}(2,2)$ *and* $\alpha^{[-2]} \in \mathcal{B}(-2,-2)$.

Proposition 7.1.4. *We have:*

$$T_2 \alpha^{[2]} = 0$$
$$T_{-2} \alpha^{[-2]} = 0$$

Proof. In view of proposition 7.1.2 and the fact that $\mathrm{d}_g W[\dot{g}](\overline{m}, n, \overline{m}, n)$ is obtained from $\mathrm{d}_g W[\dot{g}](l, m, l, m)$ by performing the transformation $(o, \iota) \mapsto (i\iota, io)$, it is enough to prove the first identity. We essentially reformulate the proof given in [101]. For every $\lambda \in (-1, 1)$, we fix a spin structure for the Lorentzian manifold $(\mathcal{M}, g_\lambda)$. As mentioned in the proof of proposition 7.1.2, the Bianchi identity implies for all $\lambda \in (-1, 1)$:

$$D_\lambda \Psi_\lambda = 0 \tag{7.11}$$

where D_λ is the operator D (see definition 7.0.2) defined using the spin structure of $(\mathcal{M}, g_\lambda)$ and Ψ_λ is the Weyl spinor of g_λ. At this point we are tempted to take the derivative with respect to λ at zero to get an equation involving linearized quantities. However, our definition of spinor bundles depends on g and we did not define any differential structure with respect to the metric. Therefore objects such as $\frac{\mathrm{d}}{\mathrm{d}\lambda}\Psi_\lambda$ are not defined. Although it is possible to develop a formalism for the calculus of variation with spinors (see for example [13]), we choose here to avoid this difficulty by making use of the tetrad formalism whenever a differentiation with respect to λ is needed.

Let $x \in \mathcal{M}$. Let U be a small neighborhood of x. We choose smooth families of smooth vector fields $l_\lambda, n_\lambda \in \Gamma(TU)$ and $m_\lambda, \overline{m}_\lambda \in \Gamma(T_{\mathbb{C}}U)$ such that for all $\lambda \in \mathbb{R}$,

$$g_\lambda(l_\lambda, n_\lambda) = -g(m_\lambda, \overline{m}_\lambda) = 1$$

and all the other products vanishes. We also require that (l_0, n_0, m_0) is a local section of $\mathcal{N}_0$. For each $\lambda \in (-1, 1)$, we fix a normalized smooth spin frame $(o_\lambda, \iota_\lambda)$ on U such that $o_\lambda \otimes \overline{o}_\lambda = l(\lambda)$, $o_\lambda \otimes \overline{\iota}_\lambda = m(\lambda)$ and $\iota_\lambda \otimes \overline{\iota}_\lambda = n(\lambda)$. We can use (7.11) in this tetrad to deduce that on U we have:

$$(\overline{m}_\lambda - 4a(\overline{m})_\lambda + c(l)_\lambda)(\Psi_0)_\lambda - (l_\lambda + 4b(\overline{m})_\lambda - 2a(l)_\lambda)(\Psi_1)_\lambda + 3b(l)_\lambda(\Psi_2)_\lambda = 0 \tag{7.12}$$
$$(n_\lambda - 4a(n)_\lambda + c(m)_\lambda)(\Psi_0)_\lambda - (m_\lambda + 4b(n)_\lambda - 2a(m)_\lambda)(\Psi_1)_\lambda + 3b(m)_\lambda(\Psi_2)_\lambda = 0. \tag{7.13}$$

The method of proof is exactly the same as in proposition 7.0.4. The only difference is that the coefficients $b(l)_\lambda$, $b(m)_\lambda$ are generally not zero for $\lambda \neq 0$. We can now take the derivative with respect to λ at $\lambda = 0$ in (7.12) and (7.13) (it is well defined since each object can be expressed using only the tetrad $l_\lambda, n_\lambda, m_\lambda, \overline{m}_\lambda$). We denote this derivative with a dot and we

sometimes omit the index λ when it is zero. Since $(\Psi_0)_0 = (\Psi_1)_0 = (b(m))_0 = (b(l))_0 = 0$, we obtain:

$$(\overline{m} - 4a(\overline{m}) + c(l))\dot{\Psi}_0 - (l + 4b(\overline{m}) - 2a(l))\dot{\Psi}_1 + 3\dot{b(l)}\Psi_2 = 0 \tag{7.14}$$

$$(n - 4a(n) + c(m))\dot{\Psi}_0 - (m + 4b(n) - 2a(m))\dot{\Psi}_1 + 3\dot{b(m)}\Psi_2 = 0. \tag{7.15}$$

As in the proof of proposition 7.0.5, we apply $(m - 3a(m) - \overline{a(\overline{m})} + \overline{c(l)} + 4b(n))$ to (7.14) and $(l - 3a(l) + \overline{a(l)} + 4b(\overline{m}) + \overline{b(\overline{m})})$ to (7.15) and use (7.3) to cancel the term involving $\dot{\Psi}_1$. We get:

$$G_2\dot{\Psi}_0 - 3(m - 3a(m) - \overline{a(\overline{m})} + \overline{c(l)} + 4b(n))\dot{b(l)}\Psi_2 \tag{7.16}$$

$$+3(l - 3a(l) + \overline{a(l)} + 4b(\overline{m}) + \overline{b(\overline{m})})\dot{b(m)}\Psi_2 = 0 \tag{7.17}$$

The last step is to remove the reference to $\dot{b(m)}$ and $\dot{b(l)}$ to have an equation only on $\dot{\Psi}_0$. First, we use the relations (7.6) to get:

$$m(\dot{b(l)}\Psi_2) - l(\dot{b(m)}\Psi_2) = \Psi_2\left((m - 3b(n))\dot{b(l)} - (l - 3b(\overline{m}))\dot{b(m)}\right)$$

Then we consider the following relation obtained by computing the curvature components in the tetrad formalism (it corresponds to equation (b) in [88, (4.12.32)]):

$$-\left(l_\lambda - 3a(l)_\lambda + \overline{a(l)}_\lambda + b(\overline{m})_\lambda + \overline{b(\overline{m})}_\lambda\right) b(m)_\lambda$$
$$+\left(m_\lambda - 3a(m)_\lambda - \overline{a(\overline{m})}_\lambda + \overline{c(l)}_\lambda + b(n)_\lambda\right) b(l)_\lambda = (\Psi_0)_\lambda.$$

Taking the derivative at $\lambda = 0$ and using the fact that $b(m) = b(l) = 0$ at $\lambda = 0$, we get:

$$-\left(l - 3a(l) + \overline{a(l)} + b(\overline{m}) + \overline{b(\overline{m})}\right)\dot{b(m)}$$
$$+\left(m - 3a(m) - \overline{a(\overline{m})} + \overline{c(l)} + b(n)\right)\dot{b(l)} = \dot{\Psi}_0.$$

Replacing in (7.16) and multiplying by 2 we finally obtain:

$$(2G_2 - 6\Psi_2)\dot{\Psi}_0 = T_2\dot{\Psi}_0 = 0$$

Note that since $(\Psi_0)_\lambda = W(g_\lambda)(l_\lambda, n_\lambda, l_\lambda, n_\lambda)$, we have:

$$\begin{aligned}\dot{\Psi}_0 &= \mathrm{d}_g W[\dot{g}](l,n,l,n) + 2W(g)(\dot{l},n,l,n) + 2W(g)(l,\dot{n},l,n)\\ &= \mathrm{d}_g W[\dot{g}](l,n,l,n)\end{aligned}$$

where we have used the fact that l and n are principal null vector fields to get the last line. Therefore, we have proved that the Teukolsky equation holds locally near any $x \in \mathcal{M}$ for $\alpha^{[2]}$ and therefore it holds globally. □

7.2 Formula for the Teukolsky operator on Kerr in a trivialization provided by the Kinnersley tetrad

We consider the local section (o_m, ι_m) of $\mathcal{A}_0$ defined on $U := \mathbb{R}_t \times (r_0, +\infty) \times \mathbb{S}^2 \setminus \{\phi = 0\}$. We recall that by definition $d((o_m, \iota_m)) = (l, n, m)$ where $l, n, m, \overline{m}$ is the Kinnersley tetrad defined by (5.1), (5.2) and (5.3). Note that $(-o_m, -\iota_m)$ also has the property $d((-o_m, -\iota_m)) = (l, n, m)$ (it is the only other section of $\mathcal{A}_0$ defined on U with this property). However, the expression of T_s in a local trivialization provided by a section (o, ι) only depends on $d(o, \iota)$

(this follows from the fact that it is the case for operators þ, þ', ð and ð' and for the spin weighted functions $\boldsymbol{b(\overline{m})}$, $\boldsymbol{b(n)}$ and $\boldsymbol{c(l)}$). Therefore, it is correct to speak about the formula for the Teukolsky operator in a trivialization provided by the Kinnersley tetrad without further precision. But to fix the ideas, we consider here the differential operator $(T_s)_m$ (T_s written in the local trivialization provided by A_m, it is therefore a differential operator on U). We use the expression of þ, þ', ð and ð computed in remark 6.2.12 and the definition of $\boldsymbol{b(\overline{m})}$, $\boldsymbol{b(n)}$ and $\boldsymbol{c(l)}$ to find for $s \in \frac{1}{2}\mathbb{Z}$:

$$\begin{aligned}(T_s)_m =& 2\left(l - (2s-1)a(l) + \overline{a(l)} + 2sb(\overline{m}) + \overline{b(\overline{m})}\right)(n - 2sa(n) + c(m)) \\ & - 2\left(m - (2s-1)a(m) - \overline{a(\overline{m})} + \overline{c(l)} + 2sb(n)\right)(\overline{m} - 2sa(\overline{m}) + c(l)) \\ & - 4(s-1)\left(s - \frac{1}{2}\right)\Psi_2\end{aligned}$$

where a, b, c and Ψ_2 have to be computed with respect to the tetrad $(l, n, m, \overline{m})$ (see remark 6.2.5). The computation of these coefficients is done in [18, Chapter 6, Section 56, (175), (180)] and we finally find:

$$\begin{aligned}(r^2 + a^2\cos^2(\theta))(T_s)_m = & \left(\frac{(r^2+a^2)^2}{\Delta_r} - a^2\sin^2\theta\right)\partial_t^2 + \frac{4Mar}{\Delta_r}\partial_t\partial_\phi \\ & + \left(\frac{a^2}{\Delta_r} - \frac{1}{\sin^2\theta}\right)\partial_\phi^2 - \Delta_r^{-s}\partial_r\left(\Delta_r^{s+1}\partial_r\right) \\ & - \frac{1}{\sin\theta}\partial_\theta(\sin\theta\partial_\theta) - 2s\left(\frac{a(r-M)}{\Delta_r} + \frac{i\cos\theta}{\sin^2\theta}\right)\partial_\phi \\ & - 2s\left(\frac{M(r^2-a^2)}{\Delta_r} - r - ia\cos\theta\right)\partial_t \\ & + \left(s^2\cot^2\theta - s\right).\end{aligned}$$

Part III

Optimal decay for solutions on subextremal Kerr spacetimes

Chapter 8

Introduction

The study of wave propagation on black hole spacetimes has been the subject of intense research in the last decades. An important motivation is to understand the stability properties of explicit solutions of the Einstein equations. The first works on this subject are [32] for the de Sitter solution and [19] (see also more recently [66]) for the Minkowski solution. More recently, stability results have been obtained for black hole solutions: in [50, 47] for Kerr-de Sitter and Kerr-Newman-de Sitter, in [60, 23] for Schwarzschild and in [63, 64, 36, 62, 94] for the slowly rotating Kerr solution.

All these results are based on a precise description of the propagation of perturbations on the underlying spacetime at the linear level. Linear stability results include [22, 55, 5, 57] for Schwarzschild, [4, 42] for slowly rotating Kerr and [35] for subextremal Reissner-Nordström spacetimes. See also the recent [45] which makes use of the microlocal method to prove the linear stability of weakly charged and slowly rotating Kerr-Newman black holes. As shown in some of these approaches, the Teukolsky equation introduced in [101] can be used to reduce the tensorial equations of linearized gravity to a scalar wave type equation involving gauge independent quantities. Moreover, the Teukolsky equation also encompass Maxwell equations and the scalar wave equation which are often considered as simplified model for linearized gravity. The equation depends on a spin parameter $s \in \frac{1}{2}\mathbb{Z}$ whose value changes according to the type of field being studied: $s = 0$ for scalar waves, $s = \pm\frac{1}{2}$ for sourceless neutrino fields, $s = \pm 1$ for Maxwell fields and $s = \pm 2$ for linearized gravitational perturbations.

There exists a large literature about the Teukolsky equation and the related linear wave equations. The optimal decay rate for linear perturbations on a Schwarzschild spacetime was first conjectured by Price [89, 90] and later clarified by Price-Burko [91]. The conjecture was generalized to the Kerr case in [52, 37]. The most studied case has been the wave equation $s = 0$ starting with pioneer works by Wald and Kay-Wald [107, 58] followed by many authors. The question of optimal decay on subextremal Kerr black holes is now well understood. Tataru [97] (see also [76] for a generalization) obtained the optimal local decay for a family of stationary spacetimes including the subextremal Kerr family assuming a local integrated energy estimate (this estimate holds for the *full subextremal range* of paramters as proved by Dafermos–Rodnianski–Shlapentokh-Rothman [24]). Using a different approach Donninger-Schlag-Soffer [25] obtained the optimal local decay on Schwarzschild black holes. The global optimal decay (by global we mean uniform up to null infinity) was obtained in the spherically symmetric case by Angelopoulos-Aretakis-Gajic [10, 9] where they also compute the precise asymptotic profile. The global optimal decay and leading order term are obtained by Hintz [48] for subextremal Kerr black holes using spectral methods. A similar expansion including the case of initial data with non zero Newman-Penrose charges was later obtained by Angelopoulos-Aretakis-Gajic [11] using a physical space approach. On a Reissner–Nordström spacetime, we also mention the result of Luk-Oh [68] where the authors obtained a sharp

decay for the wave equation and deduce the instability of the Cauchy horizon under scalar perturbations, highlighting the link between sharp decay of waves and the strong cosmic censorship.

For higher spin, Donninger-Schlag-Soffer [26] (later refined by Metcalfe-Tataru-Tohaneanu [77] in the case $s = \pm 1$ under an integrated energy decay assumption) obtained an explicit but sub optimal polynomial decay rate for Teukolsky solutions with spin $\pm 1, \pm 2$ on a Schwarzschild background. On a slowly rotating Kerr black hole, integrated energy decay was proved for the Teukolsky equation for spin $s = \pm 1$ by Ma [69] and for spin $s = \pm 2$ by Ma [70] and independently by Dafermos-Holzegel-Rodnianski [21]. On the Schwarzschild background, Price's law was obtained by Ma-Zhang for spin $s = \pm\frac{1}{2}, \pm 1, \pm 2$ in [72, 73]. Ma-Zhang [71] further generalized their result to the Price's law (with computation of the leading term) for spin $s = \pm 1, \pm 2$ on the exterior region of a slowly rotating Kerr black hole $|a| \ll M$ (and conditionally in the case $|a| < M$). More recently, a boundedness and decay result in the full subextremal range has been obtained by Shlapentokh-Rothman-Teixeira da Costa [96, 95]. In the current work, we prove the unconditional Price's law for the whole subextremal range of parameters ($|a| < M$) and for arbitrary spins $s \in \frac{1}{2}\mathbb{Z}$.

Our approach relies on microlocal and spectral methods (and in particular on works by Vasy [103, 105, 106], Wunsch-Zworski [109] and Dyatlov [28]). We make use of Melrose's b and scattering pseudodifferential algebras [75, 74]. A crucial point in the proof of our result is the analysis of the low energy limit of the resolvent which has been initiated in the Euclidean context by the work of Jensen-Kato [56]. Although we adopt Vasy's point of view [106] for the low energy analysis, we mention [15, 14, 16, 39, 40, 41] for a different perspective on this problem. We also highlight the recent works by Morgan and Morgan-Wunsch [81, 82] which establish a connection between the rate of convergence of a stationary Lorentzian metric towards the Minkowski metric, the regularity of the resolvent near zero energy and the pointwise decay rate of solutions of the wave equation using techniques closely related to the ones employed in the current work. For a different approach on this problem including the non-stationary case, see also the work of Looi [67]. The spectral and microlocal methods have recently led to the previously mentioned linear stability result for Einstein's equations on Kerr black holes by Häfner-Hintz-Vasy in [42] and to the sharp asymptotic description of scalar waves by Hintz [48]. A key ingredient of our proof is the mode stability result, originally obtained by Whiting in [108] and further improved by Andersson-Ma-Paganini-Whiting in [8] and Andersson-Häfner-Whiting in [7] (see also [17] for a different proof and a partial result for Kerr-de Sitter).

8.1 Main results

Let $s \in \frac{1}{2}\mathbb{Z}$. We fix a subextremal Kerr black hole spacetime with parameters $|a| < M$ (see section 9.1). Our main results concern the solution of the Cauchy problem with initial data on a spacelike hypersurface Σ_0 transversal to the future horizon and equal to the Boyer-Lindquist initial hypersurface near Boyer-Lindquist radius $r = +\infty$.

$$\begin{cases} T_s u = 0 \\ u_{|\Sigma_0} = u_0 \\ n_{\Sigma_0} u_{|\Sigma_0} = u_1 \end{cases} \tag{8.1}$$

where T_s is the Teukolsky operator (see Section 10.1) and n_{Σ_0} is the unit normal vector field to Σ_0 (with respect to the rescaled Kerr metric $\tilde{g}$, see subsection 9.1). We prove two different results depending on the assumptions on the initial data. In both cases, we do not intend to provide optimal regularity assumptions on the initial data. We denote by $\Delta^{[s]}$ the

spin weighted Laplacian. To state the results, we introduce the function $\mathfrak{t}$ whose level sets are transversal to null infinity and to the future horizon (see Subsection 9.1 for the exact definition).

Theorem 8.1.1. *Let $\alpha \in (0,1)$. Let $k \in \mathbb{N}$. There exists $N \in \mathbb{N}$ such that the following holds. For all initial data u_0, u_1 such that for all $j \leq N$, for $i \in \{0,1\}$, $r^{-\frac{1}{2}+\alpha}(r\partial_r)^{2j}u_i \in L^2(\Sigma_0)$ and $r^{-\frac{1}{2}+\alpha}(\Delta^{[s]})^j u_i \in L^2(\Sigma_0)$, we have for all $p \leq k$ and $\mathfrak{t} \geq 1$:*

$$|(\mathfrak{t}\partial_{\mathfrak{t}})^p u(\mathfrak{t}, r)| \leq C_k \mathfrak{t}^{-\alpha+} \frac{r^{s+|s|+}}{(r+\mathfrak{t})^{1+s+|s|}}$$

Theorem 8.1.2. *Let $k \in \mathbb{N}$. There exists $N \in \mathbb{N}$ and $\epsilon > 0$ such that the following holds: For all compactly supported initial data $u_0, u_1 \in H^N(\Sigma_0)$, for all $p \leq k$, all $\delta > 0$ and all $\mathfrak{t} \geq 0$ we have:*

$$\left|(\mathfrak{t}\partial_{\mathfrak{t}})^p \left(u(\mathfrak{t}, r) - \mathfrak{p}(\mathfrak{t}, r^{-1}, \omega)\right)\right| \leq C_{\delta,k} r^{-1+|s|+s+\delta} \mathfrak{t}^{-3-|s|+s-\epsilon} (t+r)^{-|s|-s}$$

where $\mathfrak{p}$ is an explicit function depending on s and on the initial data.

Remark 8.1.3. *The $\delta > 0$ in the previous theorem comes from the fact that we stated Theorem 15.0.4 in a Sobolev space with respect to r (and we lose a small power of r using the Sobolev embedding). However we could get rid of it with some extra work starting from proposition 14.2.37.*

Remark 8.1.4. *For a more precise version of these theorems and the explicit definition of $\mathfrak{p}$, see Corollary 15.0.3 and Corollary 15.0.5.*

8.2 Method of proof

Theorems 8.1.1 and 8.1.2 are stated in terms of the Cauchy problem (8.1), but the forcing problem

$$T_s v = f$$

(where v, f have bounded support in the past) is more convenient in the spectral analysis perspective. To get a forcing problem from (8.1), a natural idea is to take $v = \chi(\mathfrak{t})u$ for $\chi \in C^\infty(\mathbb{R}_{\mathfrak{t}}, [0,1])$ equal to 1 in a neighborhood of $+\infty$ and to 0 in a neighborhood of $-\infty$. With this definition, u and v share the same asymptotic behavior near $\mathfrak{t} = +\infty$ and f has compact support in $\mathfrak{t}$. In the case of initial data with compact support, it is possible to choose χ such that f also has a spatial compact support. Without the compact support hypothesis, we use energy estimates similar to what is done in [51] to compute the behavior of f near null infinity.

Then we take the Fourier-Laplace transform with respect to $\mathfrak{t}$ and obtain the following equation for $\Im(\sigma)$ large enough: $\hat{T}_s(\sigma)\hat{v} = \hat{f}$. If we are able to prove that $\hat{T}_s$ is invertible between suitable spaces, we obtain the following expression for v:

$$v(\mathfrak{t}) = \frac{1}{2\pi} \int_{\Im(\sigma)=C} e^{-i\mathfrak{t}\sigma} R(\sigma)\hat{f}(\sigma)\, \mathrm{d}\sigma$$

where $R(\sigma) = \hat{T}_s(\sigma)^{-1}$. A formal estimate of the right-hand side using integration by part, assuming $\partial_\sigma^k R(\sigma)\hat{f}(\sigma)$ is integrable provides:

$$|v(\mathfrak{t})| \leq e^{C\mathfrak{t}} \mathfrak{t}^{-k} \left\| \partial_\sigma^k R(\sigma)\hat{f}(\sigma) \right\|_{L^1(\mathbb{R}_\sigma)}$$

This formal computation leads to the intuition that better estimates are obtained when C is small and inverse polynomial estimates correspond to $C = 0$. The previous observations suggest the following key points to address:

1. Prove that $\hat{T}_s(\sigma)$ is invertible on the upper half plane between suitable spaces.

2. Prove that $R(\sigma)$ admits a polynomial bound when $|\sigma| \to +\infty$ and $\Im(\sigma)$ remains in a compact set (note that since we impose high regularity on the initial data, $\hat{f}(\sigma)$ has a high polynomial decay with respect to σ).

3. Prove that $R(\sigma)$ is holomorphic on the strictly upper half plane and continuous up to the real axis.

4. Analyze precisely the regularity of $R(\sigma)$ on the real axis. Note that as shown in [48] for the wave equation, the leading order term can be obtained by computing the highest order singularity of $R(\sigma)f(\sigma)$ on the real axis (in our case, it is localized at $\sigma = 0$).

Point 1 can be subdivided into three steps: Proving that $\hat{T}_s(\sigma)$ is Fredholm, proving that its index is zero and proving that its kernel is trivial. The Fredholm property is obtained by gluing Fredholm estimates on different regions of phase space: Near radial points on the horizon, (using [103]), in a small region inside the black hole (using a hyperbolic estimate as in [111]) and near radial points at spatial infinity using [105, 106] (see also [75]). Note that in order to apply the results of [106] we have to check the triviality of the kernel of the effective normal operator of $\hat{T}_s(\sigma)$ (see definition 13.2.5) which involves theory of the confluent hypergeometric equation. The gluing process relies on elliptic estimates and on propagation of singularities. Therefore, we need a global analysis of the classical Hamiltonian flow of the operator which can be computed from its principal symbol. Triviality of the kernel follows from the mode stability result for the Teukolsky equation on a subextremal Kerr spacetime originally obtained by Whiting in [108] and further improved by Andersson-Ma-Paganini-Whiting in [8] (see also [7] for $\sigma = 0$). The index zero property will follow from the continuity of the index and the invertibility of $\hat{T}_s(\sigma)$ for large $\Re(\sigma)$.

We prove this invertibility together with the polynomial bound of point 2 by introducing the semiclassical parameter $h = \frac{1}{|\sigma|}$, the operator $\hat{T}_{s,h}(z) := h^2\hat{T}_s(h^{-1}z)$ and proving a bound of the form $\|u\| \leq h^{-2}\left\|\hat{T}_{s,h}(z)u\right\|$ (see Proposition 13.4.10 for the precise statement). As before, we glue semiclassical estimates obtained on different regions of phase space. This time, the analysis is driven by the semiclassical Hamiltonian flow whose global structure has to be computed. The semiclassical flow has a more complicated structure than the classical flow and in particular it contains a set of trapped trajectories. It appears that a global estimates can be obtained by gluing an estimate near radial points on the horizon (using [103]), an estimate in a small region inside the black hole (using a semiclassical version of the hyperbolic estimate), an estimate near radial points at spatial infinity (using [105]) and an estimate near the normally hyperbolic trapped set (based on [109] and [28]).

Point 3 follows from the resolvent identity $R(\sigma) - R(\sigma') = R(\sigma)\left(\hat{T}_s(\sigma') - \hat{T}_s(\sigma)\right)R(\sigma')$ once the mapping properties of $R(\sigma)$ (resulting from the global Fredholm estimate) have been clarified.

Concerning point 4, we obtain the high regularity of $R(\sigma)$ on the real axis outside of $\sigma = 0$ by using repeatedly the identity of the resolvent. The number of iterations is only limited by the regularity of the initial data (which is assumed to be high in this work). This contrast with the situation at zero where the number of iterations is limited by the spatial decay of $\hat{f}(\sigma)$. For Theorem 8.1.1, the regularity we get from this analysis and the resolvent's bounds we have obtained at zero and at infinity are sufficient to conclude after taking the inverse Fourier

transform. Under the hypotheses of theorem 8.1.2, we are able to go further (up to $2|s|+2$ iterations) using that $\hat{f}(\sigma)$ has more spatial decay. We obtain the expression of the highest order singularity at zero as $\sigma^{2|s|+2}R(\sigma)w$ with w explicit independent of σ. Note that this step requires a precise knowledge of the kernel and cokernel of $\hat{T}_s(0)$ (in weaker spaces than the ones on which we have invertibility) which we compute using the theory of hypergeometric equations. Adapting an idea of [48] which roughly consists in performing a last iteration with $\hat{T}_s(0)$ replaced by the effective normal operator $N_{\text{eff}}(\hat{T}_s(\sigma))$ (see Definition 13.2.5), which governs the transition between $\hat{T}_s(\sigma)$ and $\hat{T}_s(0)$ near $x = 0$, we are then able to compute explicitly $\sigma^{2|s|+2}R(\sigma)w$ modulo terms, which are irrelevant for the late time expansion.

After all the points 1-4 have been addressed, we are ready to perform the contour deformation argument to write:

$$v(\mathfrak{t}) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-i\sigma\mathfrak{t}}R(\sigma)\hat{f}(\sigma)\,\mathrm{d}\sigma.$$

We then use standard properties of the Fourier transform (together with careful Fourier transform computations) to conclude the proof.

8.3 Outline of the paper

- In Sections 9 and 10, we introduce the notations and classical properties of pseudodifferential operators needed for the proof.

- In Section 11 we state the propositions needed to rephrase the Cauchy problem as a forcing problem while keeping track of the spatial decay of the forcing term.

- In Section 12, we compute precisely the classical and semiclassical Hamiltonian flows of the Teukolsky operator. As mentioned earlier, the structure of these flows are paramount to obtain global Fredholm and semiclassical estimates.

- In Section 13 we first get the Fredholm and semiclassical estimates on the different problematic regions of phase space. Subsection 13.1 handles the region near the horizon (with both Fredholm and semiclassical estimates), Subsection 13.2 handles the region near $r = +\infty$ (with two Fredholm estimates: one uniform for σ in a compact not containing zero and one uniform for σ in a small neighborhood of zero, and a semiclassical estimate) and Subsection 13.3 handles the region near the trapped set in the semiclassical regime. In the beginning of Subsection 13.4, we explain how a global Fredholm estimate implies the Fredholm property. We then obtain global Fredholm estimates and global semiclassical estimates. Subsection 13.5 is dedicated to the zero index property. The heart of the proof is the continuity of the index, but some care is required since spaces on which the operator is Fredholm depends on σ. The case of $\hat{T}_s(0)$ is treated separetly in Subsection 13.6 since we also need to compute its kernel and cokernel in weaker spaces to later compute the higher order singularity at $\sigma = 0$.

- Section 14 contains the analysis of the resolvent which includes its mapping properties, uniform bounds when $\sigma \to 0$ and $|\sigma| \to +\infty$ and the precise computation of the highest order singularity at $\sigma = 0$.

- In section 15, we use the previous results about the resolvent to perform the contour deformation argument and we finish the proof of the main theorems.

- Appendix A presents a quite general version of the semiclassical hyperbolic estimate that we use in the proof. In Appendix B, we obtain the absence of kernel for a class of

operators including the effective normal operator associated to the Teukolsky equation. Finally, Appendix C presents in detail the energy estimate used to translate the Cauchy problem into a forcing problem.

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Chapter 9

Geometric framework

9.1 Kerr metric

The exterior of a rotating black hole of mass M and of angular momentum a with $|a| < M$ is described by the Kerr solution to the Einstein equation [59]. We are interested in the exterior region, modeled by the smooth manifold $\mathcal{M} := \mathbb{R}_t \times (r_+, +\infty) \times \mathbb{S}^2$ where $r_+ := M + \sqrt{M^2 - a^2}$. The manifold $\mathcal{M}$ is endowed with the Kerr metric given in Boyer-Lindquist coordinates (t, r, θ, ϕ) by:

$$\begin{aligned} g =& \frac{\Delta_r - a^2 \sin^2\theta}{\rho^2} \, \mathrm{d}t^2 + \frac{4Mar\sin^2\theta}{\rho^2} \, \mathrm{d}t \, \mathrm{d}\phi - \frac{\rho^2}{\Delta_r} \, \mathrm{d}r^2 \\ & - \rho^2 \, \mathrm{d}\theta^2 - \frac{\sin^2\theta}{\rho^2}((a^2 + r^2)^2 - a^2\Delta_r \sin^2\theta) \, \mathrm{d}\phi^2 \end{aligned}$$

where

$$\begin{aligned} \rho^2 :=& r^2 + a^2\cos^2\theta \\ \Delta_r :=& r^2 - 2Mr + a^2 \end{aligned}$$

Note that Δ_r vanishes when $r = r_+$ (the other root is $r_- := M - \sqrt{M^2 - a^2}$). However, the singularity is merely a coordinate singularity. We introduce Kerr$_*$ coordinates:

$$(t_*, r, \theta, \phi_*) = (t + T(r), r, \theta, \phi + A(r))$$

where $T(r) := \int_{3M}^r \frac{a^2+r^2}{\Delta_r} \, \mathrm{d}r$ and $A(r) = \int_{3M}^r \frac{a}{\Delta_r} \, \mathrm{d}r$. In these coordinates, the metric g writes:

$$\begin{aligned} g =& \frac{\Delta_r - a^2\sin^2\theta}{\rho^2} \, \mathrm{d}t_*^2 - 2 \, \mathrm{d}t_* \, \mathrm{d}r + \frac{4Mar\sin^2\theta}{\rho^2} \, \mathrm{d}t_* \, \mathrm{d}\phi_* + 2a\sin^2\theta \, \mathrm{d}r \, \mathrm{d}\phi_* - \rho^2 \, \mathrm{d}\theta^2 \\ & - \frac{\sin^2\theta}{\rho^2}((a^2 + r^2)^2 - a^2\Delta_r \sin^2\theta) \, \mathrm{d}\phi_*^2. \end{aligned}$$

Therefore, it can be extended analytically to a larger manifold. Let $\epsilon > 0$ such that $r_- < r_+ - \epsilon$. We consider here an extension of $\mathcal{M}$ defined as $\mathcal{M}_\epsilon := \mathbb{R}_{t_*} \times (r_+ - \epsilon, +\infty) \times \mathbb{S}^2_{\theta,\phi_*}$ endowed with the analytic extension of g. The future horizon is by definition $\mathfrak{H} := \mathbb{R}_{t_*} \times \{r_+\} \times \mathbb{S}^2_{\theta,\phi_*}$.

Let ψ_1, ψ_2 be smooth, non negative and monotonic functions with $\psi_1 = 1$ on $\{r < 3M\}$, $\psi_1 = 0$ on $\{r > 4M\}$, $\psi_2 = 0$ on $\{r < 5M\}$, $\psi_2 = 1$ on $\{6M < r\}$. In our analysis, we mainly use the smooth function $\mathfrak{t} := t_* - L(r)$ where the smooth function L is defined by

$$L(r) := \begin{cases} 1 + T(r) + \int_r^{+\infty} \psi_1(r) \frac{a^2+r^2}{\Delta_r} \, \mathrm{d}r + \int_{-\infty}^r \psi_2(r) \frac{a^2+r^2}{\Delta_r} \, \mathrm{d}r \text{ for } r > r_+ \\ 1 + \int_{3M}^{+\infty} \psi_1(r) \frac{a^2+r^2}{\Delta_r} \, \mathrm{d}r \text{ for } r \leq r_+ \end{cases} . \tag{9.1}$$

Note that $\mathfrak{t} = t_* + c_1$ on $\{r < 3M\}$, $\mathfrak{t} = t - 1$ on $\{4M < r < 5M\}$ and $\mathfrak{t} = t - T(r) + c_2$ on $\{6M < r < +\infty\}$ where c_1 and c_2 are real constants which will play no role in the analysis. As a consequence of the definition, level sets of $\mathfrak{t}$ are smooth hypersurfaces transverse to the future horizon and transverse to null infinity. We also define $\varphi = \phi_* - A(r)(1 - \psi_1(r))$ (so that φ is equal to ϕ_* near the horizon and equal to ϕ far from the horizon).

We define $\Sigma_0^{\mathfrak{t}} := \mathfrak{t}^{-1}(\{0\})$ and we have that $\mathcal{M}_\epsilon$ is diffeomorphic to $\mathbb{R}_{\mathfrak{t}} \times \Sigma_0^{\mathfrak{t}}$. We introduce the coordinate $x := \frac{1}{r}$ and we define $\overline{\mathcal{M}_\epsilon}$ the smooth manifold with boundary $\mathbb{R}_{\mathfrak{t}} \times [0, \frac{1}{r_+ - \epsilon})_x \times \mathbb{S}^2_{\theta,\varphi}$ and we call the boundary $\mathscr{I}^+ := \mathbb{R}_{\mathfrak{t}} \times \{0\} \times \mathbb{S}^2_{\theta,\varphi}$ future null infinity. Note that the metric does not extend smoothly to the boundary but blows up like x^{-2}. For this reason, we will often consider the conformally rescaled metric $\tilde{g} := \rho^{-2} g$. We denote by G the metric induced by g on the cotangent bundle and by $\tilde{G}$ the metric induced by $\tilde{g}$.

9.2 Spin weighted functions

Let $s \in \frac{1}{2}\mathbb{Z}$. Let $U_N := \mathbb{S}^2 \setminus \{N\}$ and $U_S := \mathbb{S}^2 \setminus \{S\}$ where N is the north pole and S the south pole. We define $\mathcal{B}_s$, the complex line bundle over $\mathbb{S}^2$ with projection π, with local trivializations $(U_S, \mathcal{T}_S)$ and $(U_N, \mathcal{T}_N)$ and transition function from $\mathcal{T}_S$ to $\mathcal{T}_N$ given by

$$f_s^{S,N} : \begin{cases} U_N \cap U_S \to GL_1(\mathbb{C}) \\ (\theta, \phi) \mapsto e^{-2is\phi} \end{cases}$$

In other words, for every $e \in \mathcal{B}_s$ with $\pi(e) = x \in U_N \cap U_S$, if we use the notation $(x, v_N) = \mathcal{T}_N(e)$ and $(x, v_S) = \mathcal{T}_S(e)$, we have $v_N = f_s^{S,N}(x) v_S$. For conventional reason, we also introduce the (redundant) local trivialization on $U_N \cap U_S \setminus \{\phi = 0\}$, $\mathcal{T}_m$ which is such that the transition function from $\mathcal{T}_S$ to $\mathcal{T}_m$ is $f_s^{S,m}(\theta, \phi) = e^{-is\phi}$ (which has no continuous extension to $U_N \cap U_S$ if s is not an integer).

We introduce the complex line bundle $\mathcal{B}(s, s) := \pi_3^*(\mathcal{B}_s)$ over $\overline{\mathcal{M}_\epsilon}$ where π_3 is the third projection ($\pi_3 : \overline{\mathcal{M}_\epsilon} \to \mathbb{S}^2_{\theta,\varphi}$). We denote by $\tilde{\mathcal{T}}_S$ (respectively $\tilde{\mathcal{T}}_N$ and $\tilde{\mathcal{T}}_m$) the local trivialization of $\mathcal{B}(s, s)$ on $\mathbb{R}_{\mathfrak{t}} \times [0, \frac{1}{r_+ - \epsilon})_x \times U_S$ (respectively on $\mathbb{R}_{\mathfrak{t}} \times [0, \frac{1}{r_+ - \epsilon})_x \times U_N$ and $\mathbb{R}_{\mathfrak{t}} \times [0, \frac{1}{r_+ - \epsilon})_x \times U_m$) induced by $\mathcal{T}_S$ (respectively $\mathcal{T}_N$ and $\mathcal{T}_m$). The sections of $\mathcal{B}(s, s)$ are spin-weighted functions. This definition may seem a bit artificial but it will be enough for the purpose of this paper. For a more intrinsic and meaningful definition (but equivalent up to explicit isomorphism of vector bundle) and some explanation of how this bundle naturally appears in the study of tensorial equations, see [79].

If we have a vector bundle E over a manifold with boundary (or with corners) X (and E' its dual), we denote by:

- Ω_X the bundle of densities over X.
- $\Gamma(X, E)$ (or $\Gamma(E)$ for short) the space of smooth sections (smooth up to the boundary).
- $\Gamma^k(X, E)$ (or $\Gamma^k(E)$) the space of sections of regularity C^k (up to the boundary).
- $\Gamma(\overset{\circ}{X}, E)$ (or $\Gamma^\circ(E)$ for short) the space of smooth sections on the boundle $\overset{\circ}{E} := E_{|\overset{\circ}{X}}$
- $\dot{\Gamma}(X, E)$ (or $\dot{\Gamma}(E)$ for short) the space of smooth sections vanishing at infinite order at the boundary.
- $\mathcal{D}'(\overset{\circ}{X}, E)$ (or $\mathcal{D}'^\circ(E)$ for short) the space of distributional sections of $E_{|\overset{\circ}{X}}$ (the dual space of $\Gamma_c^\circ(E' \otimes \Omega_X)$ where the index c indicates that the sections have compact support in the base manifold, in this case $\overset{\circ}{X}$)

- $\dot{\mathcal{D}}'(X,E)$ (or $\dot{\mathcal{D}}'(E)$ for short) the space of supported distributional sections (the dual space of $\Gamma_c(E'\otimes\Omega_X)$)
- $\mathcal{D}'(X,E)$ (or $\mathcal{D}'(E)$ for short) the space of extendible distributional sections (the dual space of $\dot{\Gamma}_c(E'\otimes\Omega_X)$ the space of smooth compactly supported sections on X vanishing up to infinite order at ∂X). The space of extendible distributional sections can be viewed as a quotient space of distributional sections on a larger manifold. It is naturally included in the space of distributional sections on $\overset{\circ}{X}$. Note that we have a natural restriction map $\dot{\mathcal{D}}'(X)\to\mathcal{D}'(X)$ whose kernel is the space of distributions supported at the boundary.

For more details about supported and extendible distributions, see [54] (appendix B). If the boundary has several connected components, we can prescribe the extendible/ supported character of the distribution at each boundary component independently.

Note that since $\mathcal{B}(s,s)$ is a pullback of $\mathcal{B}_s$, we have a natural identification between elements of $\Gamma^\circ(\mathcal{B}(s,s))$ and $\mathbb{C}^\infty(\mathbb{R}_\mathfrak{t}\times(r_+-\epsilon,+\infty),\Gamma(\mathcal{B}_s))$ and between $\mathcal{D}'(\mathcal{B}(s,s))$ and $\mathcal{D}'(\mathbb{R}_\mathfrak{t}\times(r_+-\epsilon,+\infty),\mathcal{D}'(\mathcal{B}_s))$.

If Y,X are smooth manifolds (with or without boundary) and E is a vector bundle over X, we denote by $Y\times E$ the pullback bundle $\pi_2^*(E)$ over $Y\times X$ (where π_2 is the second projection on the product). With this notation, we have for example $\mathcal{B}(s,s)=\mathbb{R}_\mathfrak{t}\times[0,\frac{1}{r_+-\epsilon})_x\times\mathcal{B}_s$. When the context is clear, we sometimes omit the pullback in the notation, for example $\Gamma(Y\times X,E)$ will be a shortcut for $\Gamma(Y\times X,\pi_2^*(E))$.

Let Θ be the unique connection on $\mathcal{B}_s$ such that in trivialization $\mathcal{T}_m$:

$$\begin{aligned}\Theta_{\partial_\theta}=&\partial_\theta\\ \Theta_{\partial_\phi}=&\partial_\phi+is\cos\theta\end{aligned}$$

We also denote by Θ the pullback on $\mathcal{B}(s,s)$, concretely:

$$\begin{aligned}\Theta_{\partial_x}&=\partial_x\\ \Theta_{\partial_\mathfrak{t}}&=\partial_\mathfrak{t}\end{aligned}$$

This redundant notation is sometimes convenient to use Einstein's summation convention. To alleviate notations, we also sometimes omit the ∂. For example, $\Theta_\mathfrak{t}$ stands for $\Theta_{\partial_\mathfrak{t}}$.

If we have a Frechet space F, we define the Fourier transform $\mathcal{F}_\mathfrak{t}u$ of a section $u\in\mathscr{S}'(\mathbb{R},F')$ by (for all $\phi\in\mathscr{S}(\mathbb{R}_\mathfrak{t},F)$):

$$\mathcal{F}_\mathfrak{t}u(\phi)=u(\hat{\phi})$$

where $\hat{\phi}(\sigma):=\int_{\mathbb{R}_\mathfrak{t}}e^{i\mathfrak{t}\sigma}\phi(\mathfrak{t})\,\mathrm{d}\mathfrak{t}$.

Chapter 10

Analytic framework

10.1 Teukolsky operator

The Teukolsky operator (originally introduced in [101]) is a smooth second order differential operator on $\overset{\circ}{\mathcal{B}}(s,s)$. Its original expression was given in a local trivialization $\mathcal{T}_{kinner}$ defined on a dense open subset of $\mathcal{M}$ (this local trivialization is associated to the Kinnersley tetrad, in a way explained in [79]). This local trivialization does not extend to a local trivialization on a dense open subset of $\mathcal{M}_\epsilon$, this is linked to the fact that the Kinnersley tetrad degenerate at the horizon. However, the Teukolsky operator itself extends analytically to $\mathcal{M}_\epsilon$ (it can also be defined more intrinsically directly on $\mathcal{M}_\epsilon$, see [79] for details). The local trivialization $\mathcal{T}_m$ that we have introduced earlier (based on a renormalized tetrad which is smooth across the horizon) gives the following expression for T_s:

$$\begin{aligned}(T_s)_m =& -a^2\sin^2\theta\partial_{t_*}^2 - 2a\partial_{t_*}\partial_{\phi_*} - \frac{1}{\sin^2\theta}\partial_{\phi_*}^2 - \frac{1}{\sin\theta}\partial_{\theta_*}\left(\sin\theta\partial_{\theta_*}\right) - \Delta_r^{-s}\partial_{r_*}\left(\Delta_r^{s+1}\partial_{r_*}\right)\\ &- 2(a^2+r^2)\partial_{t_*}\partial_{r_*} - 2a\partial_{\phi_*}\partial_{r_*} + 4s(r-M)\partial_{r_*} - \frac{2is\cos\theta}{\sin^2\theta}\partial_{\phi_*}\\ &- 2\left((1-2s)r - ias\cos\theta\right)\partial_{t_*} + (s^2\cot^2\theta + s)\end{aligned} \tag{10.1}$$

We can check (by changing trivialization with $\mathcal{T}_S$ and $\mathcal{T}_N$) that this expression defines a unique second order smooth differential operator T_s on $\mathcal{B}(s,s)_{|\overset{\circ}{\mathcal{M}}_\epsilon}$. We can also write the operator in coordinates $(\mathfrak{t}, r, \theta, \varphi)$:

$$\begin{aligned}(T_s)_m =& a_{\mathfrak{t},\mathfrak{t}}\partial_{\mathfrak{t}}^2 + a_{\mathfrak{t},\varphi}\partial_{\mathfrak{t}}\partial_\varphi + a_{\varphi,\varphi}\partial_\varphi^2 - \frac{1}{\sin\theta}\partial_\theta\sin\theta\partial_\theta - \Delta_r^{-s}\partial_r\Delta_r^{s+1}\partial_r + a_{\mathfrak{t},r}\partial_{\mathfrak{t}}\partial_r\\ &+ a_{r,\varphi}\partial_r\partial_\varphi + 4s(r-M)\partial_r + a_\varphi\partial_\varphi + a_{\mathfrak{t}}\partial_{\mathfrak{t}} + s^2\cot^2\theta + s.\end{aligned} \tag{10.2}$$

where the coefficients $a_{\mathfrak{t},\mathfrak{t}}$, $a_{\mathfrak{t},\varphi}$, $a_{\varphi,\varphi}$, $a_{\mathfrak{t},r}$, $a_{r,\varphi}$, a_φ, $a_{\mathfrak{t}}$ are smooth, independent of $\mathfrak{t}$ and φ, with:

	$r<3M$	$4M<r<5M$	$6M<r$
$a_{\mathfrak{t},\mathfrak{t}}$	$-a^2\sin^2\theta$	$\frac{(r^2+a^2)^2}{\Delta_r} - a^2\sin^2\theta$	$-a^2\sin^2\theta$
$a_{\mathfrak{t},\varphi}$	$-2a$	$\frac{4Mar}{\Delta_r}$	$\frac{4Mar}{\Delta_r}$
$a_{\varphi,\varphi}$	$-\frac{1}{\sin^2\theta}$	$\frac{a^2}{\Delta_r} - \frac{1}{\sin^2\theta}$	$\frac{a^2}{\Delta_r} - \frac{1}{\sin^2\theta}$
$a_{\mathfrak{t},r}$	$-2(a^2+r^2)$	0	$2(a^2+r^2)$
$a_{r,\varphi}$	$-2a$	0	0
a_φ	$-\frac{2is\cos\theta}{\sin^2\theta}$	$-2s\left(\frac{a(r-M)}{\Delta_r} + i\frac{\cos\theta}{\sin^2\theta}\right)$	$-2s\left(\frac{a(r-M)}{\Delta_r} + i\frac{\cos\theta}{\sin^2\theta}\right)$
$a_{\mathfrak{t}}$	$-2\left((1-2s)r - ias\cos\theta\right)$	$-2s\left(\frac{M(r^2-a^2)}{\Delta_r} - r - ia\cos\theta\right)$	$\frac{4sMa^2-2sr(a^2+r^2)}{\Delta_r}$ $+2(s+1)r + 2ias\cos\theta$

With the previous expression, we see that

$$T_s\left(\mathscr{S}'\left(\mathbb{R}_{\mathfrak{t}},\mathcal{D}'((r_+-\epsilon,+\infty)\times\mathcal{B}_s)\right)\right)\subset\mathscr{S}'\left(\mathbb{R}_{\mathfrak{t}},\mathcal{D}'((r_+-\epsilon,+\infty)\times\mathcal{B}_s)\right).$$

We are interested in the following equation

$$T_s u = f \tag{10.3}$$

We define the differential operator $\hat{T}_s$ on $\mathbb{R}_\sigma\times(r_+-\epsilon,+\infty)_r\times\mathcal{B}_s$ as the operator such that for all $u\in\mathscr{S}'(\mathbb{R}_{\mathfrak{t}},\mathcal{D}'_{(r_+-\epsilon,+\infty)_r\times\mathcal{B}_s})$:

$$\hat{T}_s\mathcal{F}_{\mathfrak{t}}u=\mathcal{F}_{\mathfrak{t}}(T_s u).$$

If we call σ the conjugate variable of $\mathfrak{t}$, we get:

$$\begin{aligned}\left(\hat{T}_s(\sigma)\right)_m =&-a_{\mathfrak{t},\mathfrak{t}}\sigma^2-ia_{\mathfrak{t},\varphi}\sigma\partial_\varphi+a_{\varphi,\varphi}\partial_\varphi^2-\frac{1}{\sin\theta}\partial_\theta\sin\theta\partial_\theta-\Delta_r^{-s}\partial_r\Delta_r^{s+1}\partial_r-ia_{\mathfrak{t},r}\sigma\partial_r\\&+a_{r,\varphi}\partial_r\partial_\varphi+4s(r-M)\partial_r+a_\varphi\partial_\varphi-ia_{\mathfrak{t}}\sigma+s^2\cot^2\theta+s.\end{aligned}$$

Therefore, $(\hat{T}_s(\sigma))_\sigma$ is a family of smooth differential operators on $(r_+-\epsilon,+\infty)\times\mathcal{B}_s$. Note that if we call $T_s(\sigma)'$ the operator obtained with the Fourier transform with respect to $\mathfrak{t}$ replaced by the Fourier transform with respect to t, we have the relation: $\hat{T}_s(\sigma)=e^{i\sigma T(r)}T_s(\sigma)'e^{-i\sigma T(r)}$ on $r>6M$.

Finally, we introduce the semiclassical rescaling: $\hat{T}_{s,h}(z):=h^2\hat{T}_s(h^{-1}z)$ where $z=\frac{\sigma}{|\sigma|}$ and the small semiclassical parameter is $h=|\sigma|^{-1}$. It will be used to study the regime $\Re(\sigma)\to+\infty$ with $\Im(\sigma)$ bounded.

10.2 Sobolev spaces

We define the spatial manifolds $X:=(r_+-\epsilon,+\infty)_r\times\mathbb{S}^2$ and $\overline{X}:=[0,\frac{1}{r_+-\epsilon})_x\times\mathbb{S}^2$. We define the volume form $\mathrm{d}vol:=r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\varphi$ on X. The choice of this exact volume form is not crucial for the radial point estimates since the subprincipal symbol at a radial point does not depend on the volume form. However, we have less freedom for the computation of the subprincipal symbol at the trapped set (see Remark 13.3.2). To define the L^2 norm on $(r_+-\epsilon,+\infty)\times\mathcal{B}_s$, we also need a hermitian metric $\mathbb{m}$ on $\mathcal{B}_s$. We define it using the local trivialization $\mathcal{T}_m$. On this trivialization, for any $x\in U_m$ and $z_1,z_2\in\mathbb{C}$, $\mathbb{m}_x(z_1,z_2):=z_1\overline{z_2}$. The metric $\mathbb{m}$ on $(\mathcal{B}_s)_{|_{U_m}}$ extends uniquely to a smooth hermitian metric on $\mathcal{B}_s$. We often write $\mathbb{m}(z)$ for $\mathbb{m}(z,z)$.

The metric $\mathbb{m}$ is parallel with respect to the previously defined connection Θ.

Lemma 10.2.1. *We have* $\Theta\mathbb{m}=0$.

Proof. By continuity, it is enough to prove the equality in the local trivialization $\mathcal{T}_m$. Let $L_1,L_2\in\Gamma(\mathcal{B}_s)$, in the local trivialization, we have:

$$\begin{aligned}\Theta_{\partial_\theta}\mathbb{m}(L_1,L_2)=&\partial_\theta(\overline{L_1}L_2)-\overline{\partial_\theta(L_1)}L_2-\overline{L_1}(\partial_\theta L_2)\\=&0\\\Theta_{\partial_\phi}\mathbb{m}(L_1,L_2)=&\partial_\phi(\overline{L_1}L_2)-\overline{(\partial_\phi+is\cos\theta)L_1}L_2-\overline{L_1}(\partial_\phi+is\cos\theta)L_2\\=&is\cos\theta\overline{L_1}L_2-is\cos\theta\overline{L_1}L_2\\=&0.\end{aligned}$$

□

We can now define the following scalar product on $\Gamma_c((r_+ - \epsilon, +\infty) \times \mathcal{B}_s)$. For $u, v \in \Gamma_c((r_+ - \epsilon, +\infty)_r \times \mathcal{B}_s)$:

$$\langle u, v \rangle = \int_X \mathbb{m}_{r(x)}(u(x), v(x)) \, \mathrm{d}vol(x).$$

We define $L^2_{(r_+-\epsilon,+\infty)_r \times \mathcal{B}_s}$ as the completion of $\Gamma_c((r_+ - \epsilon, +\infty)_r \times \mathcal{B}_s)$ for the associated norm. We remark that we have the natural inclusion $L^2((r_+ - \epsilon, +\infty)_r \times \mathcal{B}_s) \subset \mathcal{D}'_{(r_+-\epsilon,+\infty)_r \times \mathcal{B}_s}$.

We now define the b-tangent bundle and the associated notations.

Definition 10.2.2. *Let $\mathcal{N}$ be a manifold with boundary of dimension n. We define ${}^bT\mathcal{N}$, the bundle of b-vectors on $\mathcal{N}$ as the bundle whose sections are smooth vector fields tangent to the boundary. In local coordinates $(y_i)_{i=0}^{n-1}$ near a point of the boundary, if y_0 is a defining function of the boundary, such vector fields write $a_0(y)y_0\partial_{y_0} + \sum_{i=1}^{n-1} a_i(y)\partial_{y_i}$ with $(a_i)_{i=0}^{n-1}$ a family of smooth functions (smooth up to the boundary). The dual bundle is denoted by ${}^bT^*\mathcal{N}$. These definitions extend to the case where $\mathcal{N}$ is a manifold with corners by defining b-vector fields as smooth vector fields tangent to each boundary face. The bundle ${}^bT\mathcal{N}$ is then used to construct other b-objects such as b-metrics which are smooth metrics on ${}^bT\mathcal{N}$ or b-volume forms which are non vanishing sections of $\Lambda^n\left({}^bT^*\mathcal{N}\right)$. For $k \in \mathbb{N}$, we denote by $Diff_b^k(E)$ the algebra of differential operators generated (as a $C^\infty(\mathcal{N})$-module) by the set $\{Id\} \cup \left\{\Theta_{X_1}...\Theta_{X_j}, j \leq k, X_i \in \Gamma({}^bT\mathcal{N})\right\}$.*

We also need to define the scattering tangent bundle.

Definition 10.2.3. *Let $\mathcal{N}$ be a manifold with boundary of dimension n. Let y_0 be a boundary defining function. We define ${}^{sc}T\mathcal{N}$, the bundle of sc-vectors on $\mathcal{N}$ as the bundle whose sections are of the form $y_0 Z$ for $Z \in \Gamma({}^bT\mathcal{N})$. In local coordinates $(y_i)_{i=0}^{n-1}$ near a point of the boundary, such vector fields write $a_0(y)y_0^2\partial_{y_0} + \sum_{i=1}^{n-1} a_i(y)y_0\partial_{y_i}$ with $(a_i)_{i=0}^{n-1}$ a family of smooth functions (smooth up to the boundary). The dual bundle is denoted by ${}^{sc}T^*\mathcal{N}$. The bundle ${}^{sc}T\mathcal{N}$ is then used to construct other sc-objects such as sc-metrics which are smooth metrics on ${}^{sc}T\mathcal{N}$ or sc-volume forms which are non vanishing sections of $\Lambda^n\left({}^{sc}T^*\mathcal{N}\right)$. For $k \in \mathbb{N}$, we denote by $Diff_{sc}^k(E)$ the algebra of differential operators generated (as a $C^\infty(\mathcal{N})$-module) by the set $\{Id\} \cup \left\{\Theta_{X_1}...\Theta_{X_j}, j \leq k, X_i \in \Gamma({}^{sc}T\mathcal{N})\right\}$ where Θ is any connection on E.*

Definition 10.2.4. *Let $\mathcal{N}$ be a compact manifold with boundary of dimension n with a boundary defining function y_0. Let E be a vector bundle over $\mathcal{N}$ with connection Θ and a metric $\mathbb{m}$. We fix a finite family of smooth vector fields $(Z_i)_{i=1}^N$ which generate ${}^bT\mathcal{N}$ as a $C^\infty(\mathcal{N})$ module and we fix a b-volume form. We define $H_b^0(E) = H_{b,h}^0(E)$ as the space $L_b^2(E)$ where the b index on L^2 indicates that integration is performed using the b-volume form. For $\tilde{r} \in \mathbb{N}$, $\tilde{r} \geq 1$, we define recursively the b-Sobolev space $H_b^{\tilde{r}}(E)$ and its semiclassical version $H_{b,h}^{\tilde{r}}(E)$ by completion of $\Gamma_c^\circ(E)$ in the norms:*

$$\|u\|_{H_b^{\tilde{r}+1}} := \|u\|_{H_b^{\tilde{r}}} + \sum_{i=1}^N \|\Theta_{Z_i} u\|_{H_b^{\tilde{r}}(E)}$$

$$\|u\|_{H_{b,h}^{\tilde{r}+1}} := \|u\|_{H_{b,h}^{\tilde{r}}} + \sum_{i=1}^N \|h\Theta_{Z_i} u\|_{H_{b,h}^{\tilde{r}}(E)}$$

We then define $H_b^{\tilde{r}}(E)$ and $H_{b,h}^{\tilde{r}}(E)$ for $\tilde{r} \in \mathbb{R}$ by interpolation and duality. For $l \in \mathbb{R}$, we then define $H_b^{\tilde{r},l} := y_0^l H_b^{\tilde{r}}$ and $H_{b,h}^{\tilde{r},l} := y_0^l H_{b,h}^{\tilde{r}}$.

Similarly, we can define the spaces $H_{sc}^{\tilde{r},l}$ and $H_{sc,h}^{\tilde{r},l}$ for $\tilde{r}, l \in \mathbb{R}$ by replacing all the b indices by sc indices in the previous definition.

Note that the previous definition makes sense even if the manifold has no boundary (then ${}^bT\mathcal{N} = {}^{sc}T\mathcal{N} = T\mathcal{N}$). We use it to define the Sobolev space $H^k(\mathcal{B}_s)$.

When the boundary has several connected components, we can specify the behavior of the sections near each component of the boundary independently. In particular, in the case of sections of $[0, \frac{1}{r_+-\epsilon}] \times \mathcal{B}_s$, we define the following hybrid spaces:

Definition 10.2.5. *We can view distributions in $\dot{\mathcal{D}}'((0, \frac{1}{r_+-\epsilon}]_x \times \mathcal{B}_s)$ as elements of $\mathcal{D}'((0, \frac{1}{r_+-\epsilon}+1)_x \times \mathcal{B}_s)$ (therefore, it makes sense to ask whether or not some $u \in \dot{\mathcal{D}}'((0, \frac{1}{r_+-\epsilon}] \times \mathcal{B}_s)$ is in $H_b^{k,l}([0, \frac{1}{r_+-\epsilon}+1]_x \times \mathcal{B}_s)$. We denote by $\dot{H}_b^{k,l} := H_b^{k,l} \cap \dot{\mathcal{D}}'((0, \frac{1}{r_+-\epsilon}]_x \times \mathcal{B}_s)$ endowed with the induced norm. We perform the same construction (with indices b replaced by sc) to get $\dot{H}_{sc}^{\tilde{r},l}$. We also get the semiclassical version of these spaces by adding an index h next to the index b (or sc) in the definition.*

Definition 10.2.6. *For distributions in $u \in \mathcal{D}'\left(\left(0, +\frac{1}{r_+-\epsilon}\right]_x \times \mathcal{B}_s\right)$, we denote by $Ext(u)$ the set of $\tilde{u} \in \mathcal{D}'((0,+\infty)_x \times \mathcal{B}_s)$ with support contained in some $x^{-1}((0,C))$ such that $\tilde{u}_{|_{(0,\frac{1}{r_+-\epsilon})}} = u$. Therefore, it makes sense to define the (possibly infinite) norm:*

$$\|u\|_{\overline{H}_b^{k,l}} := \inf_{\tilde{u} \in Ext(u)} \|\tilde{u}\|_{H_b^{k,l}([0,C]_x \times \mathcal{B}_s)}$$

We perform the same construction (with indices b replaced by sc) to get $\overline{H}_{sc}^{\tilde{r},l}$. We also get the semiclassical version of these spaces by adding an index h next to the index b in the definition.

To be consistent with the convention used in [105] and [106] (where a scattering volume form is used in the definition of b-Sobolev spaces), it is useful to introduce $\overline{H}_{(b)}^{\tilde{r},l} := \overline{H}_b^{\tilde{r},l+\frac{3}{2}}$ and $\dot{H}_{(b)}^{\tilde{r},l} = \dot{H}_b^{\tilde{r},l+\frac{3}{2}}$.

Note that the spaces $\overline{H}_{(b)}^{k,l}$ and $\dot{H}_{(b)}^{-k,-l}$ are dual to each other (using the volume form $\mathrm{d}vol$ and the metric $\mathbb{m}$ for the identifications).

For $k \in \mathbb{R}$, we define the Mellin transform of a function $f \in \dot{C}^\infty([0,+\infty), H^k(\mathcal{B}_s))$ which is Schwartz at infinity by

$$Mf(\lambda) := \int_0^{+\infty} x^{-i\lambda} f(x) \frac{\mathrm{d}x}{x}$$

By Plancherel formula, M can be extended to an isomorphism from $L_b^2([0,+\infty); H^k(\mathcal{B}_s))$ to $L^2(\mathbb{R}_\lambda, H^k(\mathcal{B}_s))$. Also note that $i\lambda Mf(\lambda) = M(x\partial_x f)(\lambda)$ and therefore, for $k \geq 0$, M extends to an isomorphism between $H_b^k([0,+\infty]_x \times \mathcal{B}_s)$ (where the boundary defining function of $x = +\infty$ is x^{-1}) and $(1+|\lambda|^2)^{-\frac{k}{2}} L^2(\mathbb{R}_\lambda, L^2(\mathbb{S}^2; \mathcal{B}_s)) \cap L^2(\mathbb{R}_\lambda, H^k(\mathbb{S}^2; \mathcal{B}_s))$.

10.3 Pseudodifferential algebras

In this section, we introduce the definitions and properties of the various Pseudodifferential algebras involved. Since we do not provide proofs for standard properties of pseudodifferential operators, we refer to [74, Chapter 4], [50, Appendix A], [54, Chapter XVIII], [104], [110] for details and proofs.

Definitions

Let E be a vector bundle over a manifold $\mathcal{N}$ (of dimension n) and F be a vector bundle over a manifold $\mathcal{N}'$ (of dimension n'). We use the notation $E \boxtimes F$ to denote

the vector bundle over $\mathcal{N} \times \mathcal{N}'$ with fiber $E_x \otimes F_y$ over $(x,y) \in \mathcal{N} \times \mathcal{N}'$. We use the notation $\Omega^s(E)$ to denote the real line bundle over $\mathcal{N}$ whose fibers are $\Omega^s(E)_x = \{u : (\Lambda^n E_x) \setminus \{0\} \to \mathbb{R}, \forall t \in \mathbb{R} \setminus \{0\}, u(t\alpha) = |t|^s u(\alpha)\}$. Note that $\Omega^1(T\mathcal{N}) = \Omega(T\mathcal{N})$ is the usual density bundle over $\mathcal{N}$.

We define the fiber radial compactification of E as the bundle obtained by adding a boundary at fiber infinity. More precisely, for $\mathfrak{m}$ a smooth positive definite metric on E, a boundary defining function of fiber infinity is $(x,\xi) \mapsto \sqrt{\mathfrak{m}_x(\xi,\xi)}^{-1}$. This definition does not depend on the choice of $\mathfrak{m}$ as two different metrics gives locally equivalent boundary defining functions. If $\mathcal{N}$ has a boundary, the definition is the same (but we require that $\mathfrak{m}$ is a smooth positive metric up to the boundary). If the boundary of $\mathcal{N}$ is not empty, the resulting bundle is a manifold with corners. We denote by $\overline{T}^*X$ the fiber radial compactification of T^*X and by S^*X fiber infinity.

Let G be an other vector bundle over $\mathcal{N}$.

Definition 10.3.1. *We define the space $S^m(E,G)$ of G-valued symbols of order m on E as the set of functions $u \in C^\infty(E,G)$ such that the following estimates hold: For all open subset U on which E is trivial, if (y,ξ) are local coordinates and K is a compact subset of U, for all $\alpha, \beta \in \mathbb{N}^n$, there exists $C > 0$ such that (uniformly on K)*

$$|\partial_y^\alpha \partial_\xi^\beta u(x,\xi)|_G \leq C \langle \xi \rangle^{m-|\beta|} \tag{10.4}$$

where $|.|_G$ is computed using a fixed (but arbitrary) metric on G. We similarly define the space of semiclassical symbols of order m $S_h^m(E,G)$ as the set of smooth h-indexed families $(u_h)_{h\in[0,1)} \in C^\infty([0,1)_h, S^m(E,G))$ such that estimate (10.4) holds uniformly with respect to $h \in [0,1)$.

In the case where $\mathcal{N}$ has a boundary with boundary defining function y_0, we also define the following space of symbols:

Definition 10.3.2. *For $l \in \mathbb{R}$, we denote by $S^{m,l}(E,G)$ the space of G-valued symbols of order (m,l) on E as the set of functions $u \in C^\infty(E|_{\overset{\circ}{\mathcal{N}}}, G)$ such that the following estimates hold: For all open subset U of $\mathcal{N}$ on which E is trivial, if (y,ξ) are local coordinates (with y_0 the boundary defining function if U intersects the boundary) and K is a compact subset of U, for all $\alpha, \beta \in \mathbb{N}^n$, there exists $C > 0$ such that (uniformly on K):*

- *If $U \cap \partial\mathcal{N} \neq \emptyset$:*
$$|\partial_y^\alpha \partial_\xi^\beta u(x,\xi)|_G \leq C \langle \xi \rangle^{m-|\beta|} y_0^{l-\alpha_0} \tag{10.5}$$
- *If $U \cap \partial\mathcal{N} = \emptyset$*
$$|\partial_y^\alpha \partial_\xi^\beta u(x,\xi)|_G \leq C \langle \xi \rangle^{m-|\beta|}. \tag{10.6}$$

In the previous estimates, $|.|_G$ is computed using a fixed (but arbitrary) metric on G. We similarly define the space of semiclassical symbols of order (m,l) $S_h^{m,l}(E,G)$ as the set of h-indexed families $(u_h)_{h\in(0,1)}$ such that for all $h \in (0,1)$, $u_h \in S^{m,l}(E,G)$ and estimates (10.5), (10.6) hold uniformly with respect to $h \in (0,1)$.

We recall the Schwartz kernel theorem applied to our setting (see theorem 4.14 in [74] and theorem 5.2.1 in [53]). In the statement of the theorem, L denotes the set of continuous linear operators from $\Gamma_c(X, \mathcal{B}_s)$ to $\mathcal{D}'(X, \mathcal{B}_s)$.

Theorem 10.3.3. *There is a one to one correspondence between L and the set of distributions $\mathcal{D}'(X \times X, \mathcal{B}_s \boxtimes (\mathcal{B}_s' \otimes \Omega(TX)))$. The correspondence is given by:*

$$S : \begin{cases} \mathcal{D}'(X \times X, \mathcal{B}_s \boxtimes (\mathcal{B}_s' \otimes \Omega(TX))) \to L \\ A \mapsto (u \in \Gamma_c(X, \mathcal{B}_s) \mapsto (v \mapsto < A, v \otimes u >)) \end{cases}$$

The distribution A is the Schwartz kernel of the operator $S(A)$.

Definition 10.3.4. *We denote by ${}^bT\overline{X}$ the vector bundle of b-vector fields on $\overline{X}$ (we denote by ${}^bT^*\overline{X}$ it dual). We denote by ${}^b\overline{T}^*\overline{X}$ the fiber radial compactification of ${}^bT^*\overline{X}$ and by ${}^bS^*\overline{X}$ fiber infinity. We define the set $\overline{L} := S\left(\mathcal{D}'\left(\overline{X}\times\overline{X}, \mathcal{B}_s \boxtimes \left(\mathcal{B}'_s \otimes \Omega({}^bT\overline{X})\right)\right)\right)$. We have the natural inclusion $\overline{L} \subset L$.*

We define $\overline{X}^2_b$ as the blow-up of the manifold with corner $\overline{X}^2$ with respect to the corner $\partial\overline{X}\times\partial\overline{X}$ with the blow-down map β. Concretely if we use primes to denote coordinates on the second factor and ω to denote coordinates on $\mathbb{S}^2$, $\overline{X}^2_b$ is obtained by the introduction of coordinates $(\rho := x+x', \tau := \frac{x-x'}{x+x'}, \omega, \omega')$ on $\overline{X}^2\backslash\{x = x' = 0\}$ and the addition of the front face which corresponds to $\{0\}_\rho\times[-1,1]_\tau\times\mathbb{S}^2_\omega\times\mathbb{S}^2_{\omega'}$ in these new coordinates. The blow-down map is then the identity map on $\overline{X}^2\backslash\{x = x' = 0\}$ and the map $(0,\tau,\omega,\omega') \mapsto ((0,\omega),(0,\omega')) \in (\partial\overline{X})^2$ on the front face (see [74, Chapter 4] for details about this construction). We denote by Δ the diagonal in $\overline{X}^2$. We define $\Delta_b := \overline{\beta^{-1}\left(\Delta \setminus \partial\overline{X}\times\partial\overline{X}\right)}$ and the front face $f = \beta^{-1}(\partial\overline{X}\times\partial\overline{X})$.

Definition 10.3.5. *If $\mathcal{E}$ is a vector bundle of order m over $\overline{X}^2_b$, we say that a distribution $A \in \mathcal{D}'(\overline{X}^2_b, \mathcal{E})$ is conormal of order m to Δ_b if it is smooth on $\overline{X}^2_b \setminus \Delta_b$ and if for every $y \in \Delta_b$, there exist:*

- *Local coordinates $(\alpha^i)_{i=0}^5$ and a local trivialization of $\mathcal{E}$ on a neighborhood U of y such that $\Delta_b \cap U = \left\{\alpha^0 = \alpha^1 = \alpha^2 = 0\right\}$. We use the notation $\alpha' := (\alpha^0,\alpha^1,\alpha^2)$ and $\alpha'' := (\alpha^3,\alpha^4,\alpha^5)$.*
- *A symbol $a \in S^m(\mathbb{R}^3\times\mathbb{R}^3, \mathbb{R}^m)$ such that $A(\alpha',\alpha'') = (2\pi)^{-3}\int_{\mathbb{R}^3} e^{i\xi\cdot\alpha'} a(\xi,\alpha'')\,\mathrm{d}\xi$ on U where the local trivialization is used implicitly to identify A with a $\mathbb{R}^m$-valued function.*

It can be checked (see for example [54, Theorem 18.2.9]) that the element a in the definition is invariantly defined in $S^m(N^*(\Delta_b), \mathcal{E}_{|\Delta_b}\otimes\Omega(N^*\Delta_b))$ modulo $S^{m-1}(N^*(\Delta_b), \mathcal{E}_{|\Delta_b}\otimes\Omega(N^*\Delta_b))$.

Definition 10.3.6. *Following [74], we define the (small) algebra of b-operators Ψ_b as the subset of $\overline{L}$ of operators whose Schwartz kernels (lifted to $\overline{X}^2_b$) are conormal to Δ_b, vanish up to infinite order at $\partial\overline{X}^2_b \setminus f$ and are properly supported in $\overline{X}^2_b$.*

For a conormal kernel A of order m (we write $A \in \Psi_b^{m,0}$), the principal symbol is therefore given by an element of

$$S^m\left(N^*\Delta_b, \left(\mathcal{B}_s \boxtimes \left(\mathcal{B}'_s \otimes \Omega({}^bT\overline{X})\right)\right)_{|\Delta_b} \otimes \Omega(N^*\Delta_b)\right)/S^{m-1}.$$

We can use the map

$$I : \begin{cases} N^*\Delta_b \to {}^bT^*\overline{X} \\ \alpha \mapsto (v \mapsto \alpha(v,0)) \end{cases}$$

with inverse

$$I^{-1} : \begin{cases} {}^bT^*\overline{X} \to N^*\Delta_b \\ \beta \mapsto ((v,v') \mapsto \beta(v) - \beta(v')) \end{cases}$$

to identify $N^*\Delta_b$ and ${}^bT^*\overline{X}$. Moreover, we have canonical isomorphisms $\Omega({}^bT\overline{X})\otimes\Omega({}^bT^*\overline{X}) = \mathbb{R}\times\overline{X}$ and $\mathcal{B}_s\otimes\mathcal{B}'_s = \mathbb{C}\times\overline{X}$. Using the identification I, the symbol of A can be seen as an element of $S^m({}^bT^*\overline{X})/S^{m-1}$ which is called the principal symbol of the operator.

Following [50, Appendix A3], we introduce the space $\Psi^{m,0}_{b,h}$ of b semiclassical operators. Note that in this case, the principal symbol is invariantly defined in $S^m_h({}^bT^*\overline{X})/hS^{m-1}_h$.

Eventually, we define $\Psi_b^{m,l}$ as the set $x^{-l}\Psi_b^{m,0}$ (and similarly $\Psi_{b,h}^{m,l} := x^{-l}\Psi_{b,h}^{m,0}$).

We now define the local model for scattering pseudodifferential operators. Let $\overline{\mathbb{R}^3_y}$ be the radial compactification of $\mathbb{R}^3_y$. We consider the manifold $\mathcal{N} := \overline{\mathbb{R}_y}^3$ and the trivial bundle $E = \mathbb{C} \times \mathcal{N}$. For $p \in S^{m,l}(\mathbb{R}^3_\xi \times \overline{\mathbb{R}^3_y}, \mathcal{L}(E,E))$, we define the operator $\mathrm{Op}(p)$ by its action on $\phi \in C_c^\infty(\mathbb{R}^3, \mathbb{C})$:

$$\mathrm{Op}(p)u(y) = (2\pi)^{-3} \int e^{i\xi\cdot(y-y')} p(y,\xi)u(y')\, \mathrm{d}y'\, \mathrm{d}\xi$$

We call $\Psi_{sc}^{m,l}(\mathbb{C} \times \mathcal{N})$ the operators obtained by this procedure. We similarly define $Op_h(p_h)$ for a symbol in $p_h \in S^{m,l}(\mathbb{R}^3_\xi \times \overline{\mathbb{R}^3_y})$ by:

$$\mathrm{Op}_h(p_h)u(y) = (2\pi h)^{-3} \int e^{ih^{-1}\xi\cdot(y-y')} p_h(y,\xi)u(y')\, \mathrm{d}y'\, \mathrm{d}\xi.$$

The corresponding operator space will be denoted by $\Psi_{sc,h}^{m,l}(\mathbb{C} \times \mathcal{N})$.

We can then use this local model to define the algebra of scattering pseudodifferential operators on $[0, \frac{1}{r_+ - \epsilon}) \times \mathcal{B}_s$ (see [104, Section 5.3.2] for more details about this construction in a more general context):

Definition 10.3.7. *We define $\Psi_{sc}^{m,l}([0, \frac{1}{r_+-\epsilon}) \times \mathcal{B}_s)$ (scattering pseudodifferential operators) as the set of continuous linear operators $A : \Gamma_c(X, \mathcal{B}_s) \to \mathcal{D}'(X, \mathcal{B}_s)$ such that:*

- *The Schwartz kernel of A is properly supported.*
- *For all χ_1, χ_2 smooth and compactly supported on some open sets of trivialization for $[0, \frac{1}{r_+-\epsilon}) \times \mathcal{B}_s$, $\chi_1 A \chi_2 \in \Psi_{sc}^{m,l}(\mathbb{C} \times \mathcal{N})$.*

The definition of $\Psi_{sc,h}^{m,l}([0, \frac{1}{r_+-\epsilon}) \times \mathcal{B}_s)$ is obtained by replacing A by a family $(A_h)_{h\in(0,1)}$ of continuous linear operators from $C_c^\infty(X, \mathcal{B}_s)$ to $\mathcal{D}'(X, \mathcal{B}_s)$ and $\Psi_{sc,h}^{m,l}(\mathbb{C} \times \mathcal{N})$ by $\Psi_{sc,h}^{m,l}(\mathbb{C} \times \mathcal{N})$ in the previous definition.

We denote by $\Psi_{b,c}^{m,l}$ the set of operator with Schwartz kernel supported on $\mathcal{M}^2_{\epsilon-\eta}$ for some $0 < \eta < \epsilon$. Standard theory of b-pseudodifferential operators (see for example [74, Section 5.9]) provides the composition rule: Let $A \in \Psi_{b,c}^{m,l}$ with principal symbol a and $B \in \Psi_{b,c}^{m',l'}$ with principal symbol b, then $AB \in \Psi_{b,c}^{m+m',l+l'}$ and has principal symbol ab. The fact that we restrict to operators with Schwartz kernel supported in $\mathcal{M}^2_{\epsilon-\eta}$ enables to reduce to the case of b-pseudodifferential operators on a compact manifold with boundary. Sometimes, we also want to take the composition of an operator $A \in \Psi_{b,c}^{m,l}$ and a differential operator $B \in x^{l'}\mathrm{Diff}_b^{m'}$. In this case we can use the locality of B to make sense of the compositions AB and BA as elements of $\Psi_{b,c}^{m+m',l+l'}$. Indeed, for any cutoff χ equal to 1 on $(r_+ - \epsilon + \eta, +\infty)_r$ and with compact support in $(r_+ - \epsilon, +\infty)_r$, $A\chi B\chi = AB$ and $\chi B\chi A = BA$ by locality of B. These compositions properties will be enough for our purpose. We have the same composition rules for semiclassical b-operators, scattering operators and semiclassical scattering operators (see for example [104]).

We say that an operator in $\Psi_b^{m,l}$ is elliptic if its principal symbol $p \in x^l S^m({}^bT^*\overline{X})/S^{m-1}$ has an inverse in other words if there exists $q \in x^{-l}S^{-m}({}^bT^*\overline{X})/S^{-m-1}$ such that $pq = qp = [1]$ where $[1]$ is the class of 1 in $S^0({}^bT^*\overline{X})/S^{-1}$. We can localize this definition near a point $(x,\xi) \in {}^bS^*\overline{X}$: we say that $P \in \Psi^{m,l}$ with principal symbol $p \in x^l S^m({}^bT^*\overline{X})/S^{m-1}$ is elliptic at (x,ξ) if there exists $q \in x^{-l}S^{-m}({}^bT^*\overline{X})/S^{m-1}$ such that $pq = qp = [g]$ with $g = 1$ on a neighborhood of (x,ξ). The subset of ${}^bS^*\overline{X}$ at which P is elliptic is denoted $\mathrm{Ell}(P)$. The

complementary subset (in ${}^bS^*\overline{X}$) is the characteristic set denoted by $\mathrm{Char}(P)$. The wavefront set of P denoted $WF(P)$ is defined negatively: $(x,\xi) \in {}^bS^*\overline{X}$ is not in the wavefront set of P if there exists $A \in \Psi_{b,c}^{0,0}$ with $(x,\xi) \in \mathrm{Ell}(A)$ such that $AP \in \Psi_{b,c}^{-\infty,l}$. All these definitions have an analog[1] in the semiclassical and scattering cases (see [104] and [110]).

In this paper, we denote the principal symbol map by $\mathfrak{s}$. In the semiclassical setting, we denote it by $\mathfrak{s}_h$.

Mapping properties

Let $m_0, l_0 \in \mathbb{R}$. Let $A \in \Psi_b^{m_0,l_0}$ (or $A \in \Psi_{sc}^{m_0,l_0}$ and $m_0 \geq 0$). Then A is bounded as an operator between the following spaces:

$$A : H_{b,c}^{m,l} \to H_{b,c}^{m-m_0,l-l_0}$$
$$A : H_{b,loc}^{m,l} \to H_{b,loc}^{m-m_0,l-l_0}$$

Where $H_{b,c}^{m,l}$ is the subspace of $\overline{H}_b^{m,l}$ whose elements have compact support in $\overline{\mathcal{M}}_\epsilon$. The space $H_{b,loc}^{m,l}$ is the subset of distributional sections u such that for all $\chi \in C_c^\infty(\overline{X})$, $\chi u \in H_{b,c}^{m,l}$. Note that operators in $\mathrm{Diff}_b^{m_0,l_0}$, $\mathrm{Diff}_{sc}^{m_0,l_0}$, $\Psi_{b,c}^{m_0,l_0}$ and $\Psi_{sc,c}^{m_0,l_0}$ are bounded from $\overline{H}_b^{m,l}$ to $\overline{H}_b^{m-m_0,l-l_0}$ and from $\dot{H}_b^{m,l}$ to $\dot{H}_b^{m-m_0,l-l_0}$. The same mapping properties holds for semiclassical pseudodifferential operators if the Sobolev spaces are replaced by their semiclassical version.

Elliptic estimates

We state the standard estimate which is a consequence of Proposition 18.1.23 in [54]:

Proposition 10.3.8. *Let $A, B \in \Psi_b^{0,0}$ be pseudodifferential operators with compactly supported Schwartz kernels in X^2 (in particular it vanishes near the boundary of $\overline{X}^2$). Let $P \in \Psi_b^{m,l}$ be a differential operator with characteristic set Σ (subset of fiber infinity). We assume that $WF(A) \cap \Sigma = \emptyset$ and $WF(A) \subset Ell(B)$. Then we have the following estimates: For every integers $N, M > 0$ and every $s \in \mathbb{R}$, there exists $C > 0$ such that:*

$$\|Au\|_{\overline{H}_{(b)}^{s+m,l}} \leq \left(\|BPu\|_{\overline{H}_{(b)}^{s,l}} + \|u\|_{\overline{H}_{(b)}^{-N,-M}} \right)$$

Remark 10.3.9. *Since the support of the Schwartz kernels of A and B are compactly supported in $X \times X$, the index l and the extendible character at $r = r_+ - \epsilon$ are irrelevant. For the same reason, the following estimates is also true:*

$$\|Au\|_{\dot{H}_{(b)}^{s+m,l}} \leq C \left(\|BPu\|_{\dot{H}_{(b)}^{s,l}} + \|u\|_{\dot{H}_{(b)}^{-N,-M}} \right)$$

We also need the semiclassical version of the b-elliptic estimate:

Proposition 10.3.10. *Let $A, B \in \Psi_{b,h}^{0,0}$ with Schwartz kernels supported inside a compact subset of $\overline{X}^2$. Let $P \in \Psi_{b,h}^{m,l}$ be a differential operator with characteristic set Σ. We assume that $WF_h(A) \cap \Sigma = \emptyset$ and $WF_h(A) \subset Ell(B)$. Then we have the following estimate: For every integers $N, M > 0$ and every $\tilde{r}, q \in \mathbb{R}$, there exists $C > 0$ such that:*

$$\|Au\|_{\overline{H}_{(b),h}^{\tilde{r}+m,q+l}} \leq C \left(\|BPu\|_{\overline{H}_{(b),h}^{\tilde{r},q}} + h \|u\|_{\overline{H}_{(b),h}^{-N,q}} \right)$$

[1] Note that $\mathrm{Ell}(A), WF(A) \subset {}^b\overline{T}^*\overline{X}$ in the semiclassical case and $\mathrm{Ell}(A), WF(A) \subset {}^{sc}S^*\overline{X} \cup ({}^{sc}T^*\overline{X})_{|_{\partial\overline{X}}}$ in the scattering case

Remark 10.3.11. *By the support condition on the Schwartz kernel, the character at $r = r_+ - \epsilon$ is irrelevant and we can replace $\overline{H}$ by $\dot{H}$.*

Remark 10.3.12. *The same estimate holds for semiclassical scattering operators. In the scattering case, we could even get an error term $h\,\|u\|_{\overline{H}^{-N,-N}_{(b)}}$ but this will not be needed.*

Propagation of singularities

We state the result in the case of differential operators. We first need to introduce the notion of Hamiltonian flow for a differential operator.

Definition 10.3.13. *Let $p \in C^\infty(T^*X, \mathbb{C})$. The Hamiltonian vector field H_p associated to p is the unique vector field on T^*X such that, for every local coordinates y on an open subset U of X:*

$$H_p = \sum_{i=1}^{n} \partial_{\xi_i} p \partial_{y_i} - \partial_{y_i} p \partial_{\xi_i}.$$

where (y, ξ) are the induced local coordinates on T^*U.

Let $P \in \mathrm{Diff}_b^{k,l}$. In particular $P \in \Psi_b^{k,l}$ and we take $x^l p \in C^\infty({}^bT^*\overline{X}, \mathbb{C})$ where p is a representative of its principal symbol and we assume that p has real values. Then, we can check that the Hamiltonian vector field $H_{x^l p}$ rescaled by the factor μ^{-k+1} extends to a vector field on ${}^b\overline{T}^*\overline{X}$ tangent to the boundary (where μ is a boundary defining function of fiber infinity). In particular, it defines a flow on fiber infinity which we call the Hamiltonian flow. It does not depend on the choice of the representative p.

In the case $P \in \mathrm{Diff}_{sc}^{k,l}$, the rescaled Hamiltonian vector field extends to a smooth vector field on ${}^{sc}\overline{T}^*\overline{X}$ tangent to the boundary and it defines a flow on fiber infinity but also on the face $\{x = 0\}$.

Finally, we define the semiclassical Hamiltonian flow for operator $P_h \in \mathrm{Diff}_{b,h}^{k,l}$ or $P_h \in \mathrm{Diff}_{sc,h}^{k,l}$. We call p_h the semiclassical principal symbol restricted to the face $h = 0$ (by a slight abuse we will call this restriction the semiclassical principal symbol in the rest of this paper). Then we have $x^l p_h \in C^\infty({}^{b/sc}\overline{T}^*X, \mathbb{C})$. We assume that p_h is real valued. Then, the Hamiltonian vector field H_{p_h} (after rescaling) extends to a vector field on ${}^{b/sc}\overline{T}^*X$ tangent to the boundaries. We call the flow of this vector field the semiclassical flow of the operator P_h.

The following proposition is a standard propagation of singularities estimate. For a proof in the scattering case (which can be adapted to treat the b case as well), see [104, Section 5.4].

Proposition 10.3.14. *Let $B_0, B_1, G \in \Psi_{b,c}^{0,0}$. Let $P \in Diff_b^{m,k}$ with real principal symbol. Assume that for every $x \in WF_b(B_1)$, there exists $t > 0$ (resp. $t < 0$) such that $e^{-tH_p}x \in Ell(B_0)$ and $(e^{-sH_p}x)_{s\in[0,t]}$ (resp. $(e^{-sH_p}x)_{s\in[t,0]}$) remains in the elliptic set of G. For every $N > 0$, there exists a constant $C > 0$ such that we have the following estimate (which holds for every u such that the right hand side is finite):*

$$\|B_1 u\|_{\overline{H}_b^{\tilde{r},l}} \leq C\left(\|GPu\|_{\overline{H}_b^{\tilde{r}-m+1,l-k}} + \|B_0 u\|_{\overline{H}_b^{\tilde{r},l}} + \|u\|_{\overline{H}_b^{\tilde{r}-N,l}}\right).$$

Remark 10.3.15. *By the assumption on the support of the Schwartz kernels of B_0, B_1 and G, we see that the behavior at the end $\{r = r_+ - \epsilon\}$ is irrelevant and we have the same estimate with $\dot{H}$ instead of $\overline{H}$.*

We also have a semiclassical version of this estimate:

Proposition 10.3.16. *Let $B_0, B_1, G \in \Psi^{0,0}_{b,h,c}$. Let $P \in Diff^{m,k}_{b,h}$ with real principal symbol. Assume that for every $x \in WF_{b,h}(B_1)$, there exists $t > 0$ such that $e^{-tH_{p_h}} x \in Ell(B_0)$ and $(e^{-sH_{p_h}} x)_{s\in[0,t]}$ remains in the elliptic set of G. For every $N > 0$, there exists a constant $C > 0$ such that we have the following estimate:*

$$\|B_1 u\|_{\overline{H}^{\tilde{r},l}_{b,h}} \leq C \left(h^{-1} \|GPu\|_{\overline{H}^{\tilde{r}-m+1,l-k}_{b,h}} + \|B_0 u\|_{\overline{H}^{\tilde{r},l}_{b,h}} + h^N \|u\|_{\overline{H}^{\tilde{r}-N,l}_{b,h}} \right).$$

Remark 10.3.17. *See also [104, Section 5.4] for a version in the scattering setting. Moreover, we stress the fact that there is also a second microlocal version of propagation of singularities estimates and elliptic estimates (including in the semiclassical regime) for an algebra $\Psi^{\tilde{r},m,l}_{b,sc}$ of operators refining the b-algebra. We refer to [105] for a detailed presentation of this algebra and its properties. We will make very little use of this second microlocal algebra here but we build on results of [105, 106] in which it plays a central role.*

Chapter 11

Cauchy problem

To state the Cauchy problem, we consider a time coordinates $t_0 = t_* + h(r) \leq$ whose level sets are transverse to the future event horizon (with $\mathrm{d}t_0$ remaining timelike up to $r_+ - 2\epsilon$) and which is equal to the usual Boyer-Lindquist coordinate when r is large. For convenience, we also require that $t_0 \geq \mathfrak{t}$ (see (C.1) for a possible concrete definition of t_0).

We consider the Cauchy problem with initial data on $\Sigma_0 := t_0^{-1}(\{0\})$. Note that the level sets of t_0 are naturally identified with X.

11.1 Cauchy Problem for smooth compactly supported initial data

For $\tilde{r} \in \mathbb{R}$, $\overline{H}^{\tilde{r}}(X, \mathcal{B}_s)$ denotes the set of restrictions to X of distributions in $H^{\tilde{r}}(\mathbb{R}_x \times \mathbb{S}^2, \mathcal{B}_s)$ (where the chosen volume form used to define the latter does not matter). These are natural spaces for the Cauchy problem with compactly supported initial data since the classical hyperbolic theory leads us to expect that $x^{-1}u$ is the restriction of of a solution on a slightly larger manifold (see the proof of proposition 11.1.1). The difference with respect to spaces $\overline{H}_b^{\tilde{r},l}(X)$ is the behavior at the boundary $\{x = 0\}$.

Proposition 11.1.1. *Let $\tilde{r} \geq 0$. Let $u_0 \in \overline{H}^{\tilde{r}+1}(\Sigma_0, \mathcal{B}_s)$, $u_1 \in \overline{H}^{\tilde{r}}(\Sigma_0, \mathcal{B}_s)$ be compactly supported. The Cauchy problem:*

$$\begin{cases} T_s u = 0 \\ u_{|\Sigma_0} = u_0 \\ \nabla^\mu t_0 \partial_\mu u_{|\Sigma_0} = u_1 \end{cases}$$

has a unique solution (in the sense of distributions) u in $H^{\tilde{r}+1}_{loc}(\mathcal{M}_\epsilon, \mathcal{B}_s)$. Moreover, for all $k \leq \tilde{r}$ (where $k \in \mathbb{N}$) we have

$$x^{-1}u \in C^0([0, +\infty)_{\mathfrak{t}}, \overline{H}^{\tilde{r}+\frac{1}{2}}(X, \mathcal{B}_s)) \cap C^k([0, +\infty)_{\mathfrak{t}}, \overline{H}^{\tilde{r}-k+\frac{1}{2}}(X, \mathcal{B}_s))$$

with the exponential bound ($C_{\tilde{r}} > 0$ is independent of T):

$$\sup_{\mathfrak{t} \in [0,T]} \left\| x^{-1}u(\mathfrak{t}) \right\|_{\overline{H}^{\tilde{r}+\frac{1}{2}}(X, \mathcal{B}_s)} \leq C_{\tilde{r}} e^{C_{\tilde{r}} T} \left(\|u_0\|_{\overline{H}^{\tilde{r}+1}(\Sigma_0, \mathcal{B}_s)} + \|u_1\|_{\overline{H}^{\tilde{r}}(\Sigma_0, \mathcal{B}_s)} \right).$$

In the statement of the Cauchy problem, ∇t_0 denotes the gradient of t_0 with respect to the metric $\tilde{g}$ (but since the support of u_1 is compact we could have chosen the gradient with respect to the Kerr metric g).

Proof. Recall that we defined $\tilde{g} := \rho^{-2} g$ and $\tilde{G}$ the associated metric on the cotangent bundle. Since the principal symbol of T_s is $\tilde{G}$ (see for example the expression in (10.1)), the metric to consider for hyperbolic theory is $\tilde{g}$ (which induces the same causal structure as g). The existence and uniqueness of u in $H^{\tilde{r}+1}_{loc}(\mathcal{M}_\epsilon, \mathcal{B}_s)$ follows from the classical hyperbolic theory on the globally hyperbolic Lorentzian manifold $\mathcal{M}_\epsilon$ (see [54, Chapter XXIII] and [92, Section 12] for an introduction to this theory). For $A \subset \mathcal{M}_\epsilon$, we denote by $J^+(A)$ the causal future of A (for more detail about this notion see [92, Section 10.2.4]). By finite speed of propagation, the support of u is contained in $J^+(K)$ where K is a compact subset of Σ_0 whose interior contains the union of the supports of u_0 and u_1. As a consequence, we can find a strictly spacelike hypersurface Σ_0' which is transverse to $\mathscr{I}^+$, which contains K and such that $J^+(K) \cap \Sigma_0' = K$. The solution u is then solution to the Cauchy problem with initial data on $\tilde{\Sigma}_0$. Note that $x^{-1} T_s x$ is smooth up to $x = 0$, and we check that we can extend it analytically on a slightly larger manifold $\tilde{\mathcal{M}}_\epsilon := \mathbb{R}_{\mathfrak{t}} \times [-\eta, \frac{1}{r_+ - \epsilon - \eta})_x \times \mathbb{S}^2$ for a small $\eta > 0$. For η sufficiently small, $\tilde{g}$ extends analytically as a Lorentzian metric on $\tilde{\mathcal{M}}_\epsilon$ and Σ_0' extends as a spacelike hypersurface $\tilde{\Sigma}_0'$ such that Σ_0' is relatively compact in $\tilde{\Sigma}_0'$. We denote by $\overline{\mathcal{M}}_\epsilon^{[0,T]} := \left(\overline{\mathcal{M}}_\epsilon \cup \left\{x = \frac{1}{r_+ - \epsilon}\right\}\right) \cap \mathfrak{t}^{-1}([0,T])$. The Cauchy problem for the operator $x^{-1} T x$ on the domain of dependence of $\tilde{\Sigma}_0'$ has a unique solution which extends $x^{-1}u$ and, by classical hyperbolic theory, the solution $x^{-1}u$ belongs to $\overline{H}^{\tilde{r}+1}(\overline{\mathcal{M}}_\epsilon^{[0,T]})$. Moreover, we can find $u^n \in x C^\infty(\overline{\mathcal{M}}_\epsilon^{[0,T]})$ such that $\lim_{n \to +\infty} x^{-1} u^n = x^{-1} u$ in $\overline{H}^{\tilde{r}+1}(\overline{\mathcal{M}}_\epsilon^{[0,T]})$ and such that the initial data (u_0^n, u_1^n) of u^n satisfies $\lim_{n \to +\infty} (u_0^n, u_1^n) = (u_0, u_1)$ in $\overline{H}^{\tilde{r}+1} \times \overline{H}^{\tilde{r}}$. Therefore, we can assume that u is smooth when performing energy estimates on $\overline{\mathcal{M}}_\epsilon^{[0,T]}$. The standard energy estimate for hyperbolic partial differential equations gives (for some constant $C > 0$ independent of T because coefficients of T_s do not depend on $\mathfrak{t}$):

$$\left\| x^{-1} u \right\|_{\overline{H}^{\tilde{r}+1}\left(\overline{\mathcal{M}}_\epsilon^{[0,T]}\right)} \leq C e^{CT} \left(\|u_0\|_{H^{\tilde{r}+1}(\Sigma_0)} + \|u_1\|_{H^{\tilde{r}}(\Sigma_0)} \right)$$

We use the trace theorem (see for example Theorem B.2.7 in [54]) to get $x^{-1}u \in C^0([0, +\infty)_{\mathfrak{t}}, \overline{H}^{\tilde{r}+\frac{1}{2}}(X, \mathcal{B}_s))$ and:

$$\sup_{\mathfrak{t} \in [0,T]} \left\| x^{-1} u(\mathfrak{t}) \right\|_{\overline{H}^{\tilde{r}+\frac{1}{2}}(X, \mathcal{B}_s)} \leq C_{\tilde{r}} e^{C_{\tilde{r}} T} \left(\|u_0\|_{\overline{H}^{\tilde{r}+1}(\Sigma_0, \mathcal{B}_s)} + \|u_1\|_{\overline{H}^{\tilde{r}}(\Sigma_0, \mathcal{B}_s)} \right)$$

Eventually, still by the trace theorem, we obtain $x^{-1}u \in C^k([0, +\infty)_{\mathfrak{t}}, \overline{H}^{\tilde{r}-k+\frac{1}{2}}(X, \mathcal{B}_s))$. □

The following proposition translates the Cauchy problem into a forcing problem and records regularity and decay properties of the forcing term.

Proposition 11.1.2. *Let $k \in \mathbb{N}$ and let $\tilde{r} > \frac{1}{2} + k$. Let u be as in Proposition 11.1.1, then there exists $v \in x C^{1+k}(\mathbb{R}_{\mathfrak{t}}, \overline{H}^{\tilde{r}-\frac{1}{2}-k}(X, \mathcal{B}_s))$ and $\mathfrak{t}_0 < \mathfrak{t}_1$ such that:*

- *$v(\mathfrak{t}) = 0$ if $\mathfrak{t} < \mathfrak{t}_0$*
- *$v(\mathfrak{t}) = u(\mathfrak{t})$ if $\mathfrak{t}_1 < \mathfrak{t}$*
- *$T_s v = f$ with $f \in C^k(\mathbb{R}_{\mathfrak{t}}, \overline{H}_b^{\tilde{r}-\frac{3}{2}-k,\infty})$, supp$(f) \subset [\mathfrak{t}_0, \mathfrak{t}_1]$.*

Proof. By finite speed of propagation, there exists a compact subset $K \subset (r_+ - \epsilon, +\infty) \times \mathbb{S}^2$ and a time interval $[\mathfrak{t}_0, \mathfrak{t}_1]$ such that $\text{supp}(u) \cap \mathfrak{t}^{-1}([\mathfrak{t}_0, \mathfrak{t}_1]) \subset [\mathfrak{t}_0, \mathfrak{t}_1] \times K$. We define $\chi \in C^\infty(\mathbb{R}, [0,1])$ equal to 0 on $(-\infty, \mathfrak{t}_0]$ and equal to 1 on $[\mathfrak{t}_1, +\infty)$ and $v = \chi u$. The fact that $v \in C^{1+k}(\mathbb{R}_{\mathfrak{t}}, \overline{H}^{\tilde{r}-\frac{1}{2}-k}(X, \mathcal{B}_s))$ follows from the same property on u. Since $T_s u = 0$, we get $T_s v = [T_s, \chi] u$ which has time support in $[\mathfrak{t}_0, \mathfrak{t}_1]$ and spatial support in K. □

Remark 11.1.3. *The function χ introduced in the proof of Proposition 11.1.2 is not the only choice. In particular, it is not necessary to choose a function of $\mathfrak{t}$. The relevant properties for $\chi \in C^\infty(\mathcal{M}_\epsilon, [0,1])$ to get $v := \chi u$ as in Proposition 11.1.2 are:*

- *$\chi = 1$ on $\{\mathfrak{t} \geq \mathfrak{t}_1\}$ for some $\mathfrak{t}_1 \in \mathbb{R}$.*
- *$\chi = 0$ on $\{\mathfrak{t} \leq \mathfrak{t}_0\} \cap supp(u)$ for some $\mathfrak{t}_0 < \mathfrak{t}_1$.*
- *$supp(\mathrm{d}\chi) \cap supp(u)$ is compact in $\mathcal{M}_\epsilon$.*

In particular, it is possible to choose a function of the form $\chi(t_0)$ with $\chi \in C^\infty(\mathbb{R}, [0,1])$ equal to 1 on $[\eta, +\infty)$ and equal to 0 on $(-\infty, 0]$ for $\eta > 0$ small enough. This freedom will be useful to highlight the dependence of the forcing term on the initial data (see Remark 15.0.6).

We are now able to take the Fourier-Laplace transform.

Corollary 11.1.4. *We use the notation of proposition 11.1.1 and assume $\tilde{r} > k + \frac{1}{2}$ for some $k \in \mathbb{N}$. We also fix $l \in \mathbb{R}$ (large decay rate). For all $\sigma \in \mathbb{C}$ such that $\Im(\sigma) > C_{\tilde{r}}$, we have the following equality between the Fourier-Laplace transforms (with respect to $\mathfrak{t}$):*

$$\hat{T}_s(\sigma)\hat{v}(\sigma) = \hat{f}(\sigma)$$

Moreover, we have that $\hat{f}$ is holomorphic on $\mathbb{C}$ with values in $\overline{H}_b^{\tilde{r}-\frac{1}{2}-k,l}$ and there exists $D > 0$ such that for all $j \in \mathbb{N}$ there exists $D_j > 0$ such that:

$$\left\| \partial_{\sigma_x}^j \hat{f}(\sigma_x + i\sigma_y) \right\|_{\overline{H}_b^{\tilde{r}-k-\frac{1}{2},l}} \leq D_j \langle \sigma_x \rangle^{-k} e^{D|\sigma_y|} \tag{11.1}$$

Proof. The fact that $\hat{u}(\sigma)$ is well defined for $\Im(\sigma) > C_{\tilde{r}}$ follows from the exponential estimate in proposition 11.1.1 and the equality follows from the definition of $\hat{T}$. The estimate on $\hat{f}$ follows from the Paley-Wiener-Schwartz theorem (see for example [53], Theorem 7.3.1) with the observation that $\partial_{\mathfrak{t}}^k(\mathfrak{t}^j f)$ is a compactly supported distribution of order zero. □

11.2 More general initial data

The goal of this section is to translate the Cauchy problem on Σ_0 with data $u_0 \in \overline{H}_b^{\tilde{r}+1,1+\alpha}$ and $u_1 \in \overline{H}_b^{\tilde{r},1+\alpha}$ into a forcing problem and to specify the properties of the Fourier transform of the forcing term.

Before stating the propositions, we need some geometric preparation. Following [51], we add another boundary on $\overline{\mathcal{M}}_\epsilon$ as follows: On $\mathcal{U} := \{\mathfrak{t} < 0\}$, we can define $\rho_0 := -\mathfrak{t}^{-1}$ and $\rho_I := -x\mathfrak{t}$ and we add the boundary: $I_0 := \{\rho_0 = 0\}$ to $\mathcal{U}$. Note that the set $\{\rho_I = 0, \rho_0 > 0\}$ corresponds to $\mathscr{I}^+ \cap \mathcal{U}$ and $\{\rho_0 = 0\}$ has been glued at the end $\mathfrak{t} = -\infty$, therefore the manifold has a corner at $\rho_0 = \rho_I = 0$. The closure of the hypersurface Σ_0 on this new manifold intersects I_0 transversally. Moreover, we consider $\mathcal{M}_\epsilon$ as included in $\mathcal{M}_{2\epsilon}$. It enables us to define extendible Sobolev estimates at $r = r_+ - \epsilon$. We call $\mathbf{M}_\epsilon := \overline{\mathcal{M}}_\epsilon \cup I_0 \cup \{r = r_+ - \epsilon\}$. Note that in this section, we do not need the boundary I_+ and its boundary defining function ρ_+ which are introduced in [51]. Indeed, it is enough for our purpose to prove a crude exponential bound with respect to $\mathfrak{t}$ which do not require a precise analysis near I_+ (a more precise estimate will then follow from the analysis of the resolvent).

Definition 11.2.1. *For $\tilde{r} \in \mathbb{N}$, we define the b-Sobolev space $\mathcal{E}^{\tilde{r}}$ of distributions which are (locally in $\boldsymbol{M}_\epsilon \cap \{t_0 \geq 0\}$): extendible (as a distribution of Sobolev order $\tilde{r}$) at $r = r_+ - \epsilon$ and at $t_0 = 0$ and of b regularity $\tilde{r}$ near $I_0 \cup \mathscr{I}^+$. Concretely, $u \in \mathcal{E}^{\tilde{r}}$ means that for any*

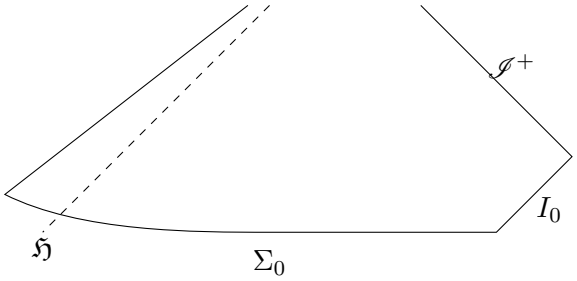

Figure 11.1: Representation of $\mathbf{M}_\epsilon \cap \{t_0 \geq 0\}$

family of smooth[1] *vector fields* $(L_i)_{i=1}^N$ *on* $\boldsymbol{M}_\epsilon \cap \{t_0 \geq 0\}$ *tangent to* $\mathscr{I}^+ \cup I_0$ *with* $N \leq \tilde{r}$, $L_1...L_N u \in L^2_{b,loc}(\boldsymbol{M}_\epsilon, \mathcal{B}_s)$.

We use the time coordinate $\tilde{\mathfrak{t}}$ on $\mathbf{M}_\epsilon \cap \{t_0 \geq 0\}$, which satisfies $c \leq \tilde{G}(\,\mathrm{d}\tilde{\mathfrak{t}},\,\mathrm{d}\tilde{\mathfrak{t}}) \leq C$ for positive constants c and C and is smooth up to $\mathscr{I}^+$.

The main proposition of this Section is obtained by adapting energy estimates from [51, Section 4.1]:

Proposition 11.2.2. *Let* $\tilde{r} \in \mathbb{N}$. *Let* $a_I < 0$ *and* $a_0 > a_I + 1$. *Let* $u_0 \in \overline{H}_b^{\tilde{r}+1,a_0}$ *and* $u_1 \in \overline{H}_b^{\tilde{r},a_0}$. *We denote by* ∇t_0 *the gradient of* t_0 *with respect to the metric* $\tilde{g}$. *The unique solution* u *to the Cauchy problem:*

$$\begin{cases} T_s u = 0 \\ u_{|\Sigma_0} = u_0 \\ \rho_0 \nabla^\mu t_0 \partial_\mu u_{|\Sigma_0} = u_1 \end{cases}$$

belongs to $\rho_0^{a_0}\rho_I^{a_I+1}\mathcal{E}^{\tilde{r}}$. *Moreover if* $k \leq \tilde{r}$ *for some* $k \in \mathbb{N}$, *it also belongs to* $C^k([-\frac{1}{2},+\infty)_{\mathfrak{t}}, \overline{H}_b^{\tilde{r}-k,a_I+1}) \cap C^0([-\frac{1}{2},+\infty)_{\mathfrak{t}}, \overline{H}_b^{\tilde{r},a_I+1})$ *and there exists* $C_{\tilde{r}} > 0$ *such that for all* $T > 0$*:*

$$\sup_{\mathfrak{t}\in[-\frac{1}{2},T]} \|u(\mathfrak{t})\|_{\overline{H}_b^{\tilde{r},a_I+1}} \leq C_{\tilde{r}} e^{C_{\tilde{r}}T} \left(\|u_0\|_{\overline{H}_b^{\tilde{r}+1,a_0}} + \|u_1\|_{\overline{H}^{\tilde{r},a_0}} \right)$$

Remark 11.2.3. *In the proof, we obtain a more precise result (namely* $u \in \rho_0^{a_0}\rho_I^{a_I+1} H^{1,\tilde{r}}_{\mathscr{I},b}$ *which is better in terms of regularity, see the proof for the definition of* $H^{1,\tilde{r}}_{\mathscr{I},b}$*). However, since we do not aim to optimize the regularity assumptions here, we allow this loss in order to stay with simpler spaces* $\mathcal{E}^{\tilde{r}}$.

We then use an idea mentioned in [51, Section 5.1] to get:

Proposition 11.2.4. *With the notations of Proposition 11.2.2, if we assume in addition that* $a_0 = 1 + \alpha$ *with* $\alpha \in (0,1)$ *and* $\tilde{r} > k + 2$ *for some* $k \in \mathbb{N} \setminus \{0\}$, *then there exists constants* $\mathfrak{t}_0 < \mathfrak{t}_1$ *and* $v \in C^k\left(\mathbb{R}_{\mathfrak{t}}, \overline{H}_b^{\tilde{r}-2-k,1-}\right)$ *such that:*

- $v(\mathfrak{t}) = 0$ *if* $\mathfrak{t} < \mathfrak{t}_0$
- $v(\mathfrak{t}) = u(\mathfrak{t})$ *if* $\mathfrak{t} > \mathfrak{t}_1$
- *For all* $\mathfrak{t} \in \mathbb{R}$, $v(\mathfrak{t}) = xv_0(\mathfrak{t}) + \overline{H}_b^{\tilde{r}-2-k,1+\alpha-}$ *where* $v_0 \in C^k(\mathbb{R}_{\mathfrak{t}}, H^{\tilde{r}-2-k}(\mathcal{B}_s))$

[1] up to the boundary of $\mathbf{M}_\epsilon$

- $T_s v = f$ *with* $f \in C^{k-1}\left(\mathbb{R}_{\mathfrak{t}}, \overline{H}^{\tilde{r}-3-k,-\frac{3}{2}+\alpha-}_{(b)}\right)$, $\mathfrak{t}(supp(f)) \subset [\mathfrak{t}_0, \mathfrak{t}_1]$

Proposition 11.2.2 and Proposition 11.2.4 allow us to reduce the study of the asymptotic behavior of the solution to the Cauchy problem to the study of the solution v of the forcing problem $T_s v = f$ provided by Proposition 11.2.4. Moreover, the properties of u, v and f stated in the propositions allow us to consider the Fourier-Laplace transform with respect to $\mathfrak{t}$:

Corollary 11.2.5. *We use the notation of Propositions 11.2.2 and 11.2.4 and assume* $\tilde{r} > k+3$ *for some* $k \in \mathbb{N}$. *For all* $\sigma \in \mathbb{C}$ *such that* $\Im(\sigma) > C_{\tilde{r}}$, *we have the following equality between the Fourier-Laplace transforms:*

$$\hat{T}_s(\sigma)\hat{v}(\sigma) = \hat{f}(\sigma)$$

Moreover, we have that $\hat{f}$ *is holomorphic on* $\mathbb{C}$ *with values in* $\overline{H}^{\tilde{r}-3-k,-\frac{3}{2}+\alpha-}_{(b)}$ *and there exists* $D > 0$ *such that for all* $j \in \mathbb{N}$ *there exists* $D_j > 0$ *such that:*

$$\left\| \partial^j_{\sigma_x} \hat{f}(\sigma_x + i\sigma_y) \right\|_{\overline{H}^{\tilde{r}-3-k,-\frac{3}{2}+\alpha-}_{(b)}} \leq D_j \left\langle \sigma_x \right\rangle^{-(k-1)} e^{D|\sigma_y|} \tag{11.2}$$

Since the detailed proofs of Proposition 11.2.2 and Proposition 11.2.4 are quite long and not essential to understand the rest of the paper, they have been added in Appendix C.

Chapter 12

Analysis of the classical and semiclassical Hamiltonian flow of $\hat{T}_s(\sigma)$

Before stating the Fredholm estimates, we analyse the structure of the classical and semiclassical Hamiltonian flows of $\hat{T}_s(\sigma)$. In particular, we identify the elliptic regions, the transport regions (i.e. the subset of the characteristic set where the Hamilton vector field does not vanish) and the radial points. In the semiclassical regime, we will see that some trajectories under the Hamiltonian flow remains in a compact region of X, a phenomenon known as trapping. Our analysis of the flow relies on [103, 27]. Note that a similar analysis for the more general Kerr-Newman spacetimes is presented in [45, Section 5].

12.1 Analysis of the classical flow

Proposition 12.1.1. *Let $\eta > 0$. The operator $\hat{T}_s(\sigma)$ is (classically) elliptic on every region of the form $\{r > 2M + \eta\} \cap \overset{\circ}{K}$ where K is a compact subset of X.*

Proof. If we denote by $\hat{T}_s(\sigma)'$ the operator obtained by Fourier transform with respect to t_* instead of $\mathfrak{t}$, we check that $\hat{T}_s(\sigma) = e^{-i\sigma L(r)}\hat{T}_s'(\sigma)e^{i\sigma L(r)}$ where L is a smooth function on X. In particular, the classical principal symbols of $\hat{T}_s(\sigma)$ and $\hat{T}_s'(\sigma)$ are the same. In coordinates (r, θ, ϕ_*), writing cotangent vectors as $\xi\, \mathrm{d}r + \zeta\, \mathrm{d}\phi_* + \eta\, \mathrm{d}\theta$, we have:

$$\begin{aligned}\mathfrak{s}(\hat{T}_s(\sigma)) =& \Delta_r \xi^2 + 2a\xi\zeta + \frac{\zeta^2}{\sin^2\theta} + \eta^2 \\ =& \left(a^2\cos^2\theta + r(r-2M)\right)\xi^2 + \left(a\sin\theta\xi + \frac{\zeta}{\sin(\theta)}\right)^2 + \eta^2\end{aligned} \tag{12.1}$$

where $\mathfrak{s}$ is the classical principal symbol map. Strictly speaking, this computation is only valid outside the rotation axis, however with a change of coordinates we can show that the operator is elliptic also in a neighborhood of the rotation axis (see (12.4) below). □

We now investigate the characteristic set and the Hamiltonian flow on it (in the region $\{r \leq 2M\}$). The Hamiltonian vector field is

$$H = (2\Delta_r\xi + 2a\zeta)\,\partial_r + \left(2a\xi + \frac{2\zeta}{\sin^2\theta}\right)\partial_{\phi_*} + 2\eta\partial_\theta - 2(r-M)\xi^2\partial_\xi + \frac{2\zeta^2\cos\theta}{\sin^3\theta}\partial_\eta \tag{12.2}$$

We now work in the fiber radial compactification $\overline{T}^*X$ of T^*X and we set $\tilde{\rho} := \frac{1}{\sqrt{\xi^2 + \frac{\zeta^2}{\sin^2\theta} + \eta^2}}$ (note that $\tilde{\rho}$ extends smoothly to the rotation axis). We set $\tilde{\xi} := \tilde{\rho}\xi$, $\tilde{\zeta} := \tilde{\rho}\frac{\zeta}{\sin\theta}$, $\tilde{\eta} := \tilde{\rho}\eta$. We

use functions $(\tilde{\rho}, (\tilde{\xi}, \tilde{\zeta}, \tilde{\eta})) \in [0, +\infty) \times \mathbb{S}^2$ to parametrize the fibers of $\overline{T}^*\mathcal{M}_\epsilon \setminus \{0\}$ away from the rotation axis. The rescaled Hamiltonian vector field (in terms of $(r, \theta, \phi_*, \tilde{\rho}, (\tilde{\xi}, \tilde{\eta}, \tilde{\zeta}))$ is

$$\begin{aligned}\tilde{H} = \tilde{\rho}H = \left(2\Delta_r\tilde{\xi} + 2a\tilde{\zeta}\sin\theta\right)\partial_r + \left(2a\tilde{\xi} + \frac{2\tilde{\zeta}}{\sin\theta}\right)\partial_{\phi_*} + 2\tilde{\eta}\partial_\theta + 2(r-M)\tilde{\xi}^3\tilde{\rho}\partial_{\tilde{\rho}} \\ - 2(r-M)\tilde{\xi}^2 Z_1 + 2\mathrm{cotan}\theta\tilde{\zeta}^2 Z_2 - 2\tilde{\eta}\tilde{\zeta}\mathrm{cotan}\theta Z_3 \end{aligned} \tag{12.3}$$

where $Z_1 := (1-\tilde{\xi}^2)\partial_{\tilde{\xi}} - \tilde{\eta}\tilde{\xi}\partial_{\tilde{\eta}} - \tilde{\zeta}\tilde{\xi}\partial_{\tilde{\zeta}}$, $Z_2 := (1-\tilde{\eta}^2)\partial_{\tilde{\eta}} - \tilde{\eta}\tilde{\zeta}\partial_{\tilde{\zeta}} - \tilde{\eta}\tilde{\xi}\partial_{\tilde{\xi}}$ and $Z_3 = (1-\tilde{\zeta}^2)\partial_{\tilde{\zeta}} - \tilde{\zeta}\tilde{\xi}\partial_{\tilde{\xi}} - \tilde{\zeta}\tilde{\eta}\partial_{\tilde{\eta}}$ are smooth vector fields on $\mathbb{S}^2$. Note that $\tilde{H}$ is smooth also on the rotation axis (see the computation in stereographic coordinates (12.5) below).

Remark 12.1.2. *Pay attention to the fact that ∂_θ in (12.3) is different from ∂_θ in (12.2) since $\tilde{\rho}$ and $\tilde{\zeta}$ depend on θ.*

Following [103] (section 6.3), we define the sets $\Lambda_+ := \{\Delta_r = 0, \eta = 0, \zeta = 0, \xi > 0\}$, $L_+ = \Lambda_+ \cap \{\tilde{\rho} = 0\}$ and $\Lambda_- := \{\Delta_r = 0, \eta = 0, \zeta = 0, \xi < 0\}$, $L_- = \Lambda_- \cap \{\tilde{\rho} = 0\}$. The vector field $\tilde{H}$ extends smoothly to fiber infinity ($\tilde{\rho} = 0$).

Definition 12.1.3. *Points of the characteristic set where the rescaled Hamitonian vector field vanishes are called radial points.*

Proposition 12.1.4. *Radial points are contained in $L_+ \cup L_-$. Moreover, $L_+ \cup L_-$ is closed for the flow of the rescaled Hamiltonian vector field.*

Proof. To cover the rotation axis, we use stereographic coordinates (of north pole) on the sphere so that: $(x_N, y_N) = \left(cotan\frac{\theta}{2}\cos\phi_*, cotan\frac{\theta}{2}\sin\phi_*\right)$. We define similarly stereographic coordinates of south pole (x_S, y_S). We know that the principal symbol is symmetric with respect to the reflection by the equatorial plan. As a consequence, it is enough to do the analysis in the stereographic coordinates of north pole. We write linear forms as $\xi\ \mathrm{d}r + \gamma\ \mathrm{d}x_N + \mu\ \mathrm{d}y_N$. We obtain:

$$\mathfrak{s}(\hat{T}_s(\sigma)) = \Delta_r\xi^2 + 2a\xi(-y_N\gamma + x_N\mu) + \frac{(1+x_N^2+y_N^2)^2}{4}(\gamma^2+\mu^2) \tag{12.4}$$

We can compute the fiber infinity defining function $\tilde{\rho}$ (and see that it extends smoothly across the rotation axis):

$$\tilde{\rho} = \frac{1}{\sqrt{\xi^2 + \frac{(1+x_N^2+y_N^2)^2}{4}(\gamma^2+\mu^2)}}$$

This expression prompts us to use the parametrization $\tilde{\rho}, (\tilde{\xi}, \tilde{\gamma}, \tilde{\mu}) \in \mathbb{S}^2$ where $\tilde{\xi} = \tilde{\rho}\xi$, $\tilde{\gamma} = \tilde{\rho}\frac{(1+x_N^2+y_N^2)\gamma}{2}$ and $\tilde{\mu} = \tilde{\rho}\frac{(1+x_N^2+y_N^2)\mu}{2}$. Note that we have the relations

$$\begin{aligned}\tilde{\zeta} &= -\frac{y_N}{\sqrt{x_N^2+y_N^2}}\tilde{\gamma} + \frac{x_N}{\sqrt{x_N^2+y_N^2}}\tilde{\mu} \\ \tilde{\eta} &= -\frac{x_N}{\sqrt{x_N^2+y_N^2}}\tilde{\gamma} - \frac{y_N}{\sqrt{x_N^2+y_N^2}}\tilde{\mu}\end{aligned}$$

We can then rewrite the rescaled Hamiltonian vector field (and see that it extends smoothly across the rotation axis):

$$\begin{aligned}\tilde{H} = \left(2\Delta_r\tilde{\xi} + 4a\frac{x_N\tilde{\mu} - y_N\tilde{\gamma}}{1+x_N^2+y_N^2}\right)\partial_r - 2a\tilde{\xi}y_N\partial_{x_N} + 2a\tilde{\xi}x_N\partial_{y_N} + (1+x_N^2+y_N^2)\left(\tilde{\gamma}\partial_{x_N} + \tilde{\mu}\partial_{y_N}\right) \\ + 2(r-M)\tilde{\xi}^3\tilde{\rho}\partial_{\tilde{\rho}} - 2(r-M)\tilde{\xi}^2 Z_1 + 2x_N Z_3 + 2y_N Z_4 + 2a\tilde{\xi}Z_5\end{aligned} \tag{12.5}$$

where Z_1 is the same as previously and can be expressed in terms of $\tilde{\xi}, \tilde{\gamma}, \tilde{\mu}$: $Z_1 = (1-\tilde{\xi}^2)\partial_{\tilde{\xi}} - \tilde{\xi}\tilde{\gamma}\partial_{\tilde{\gamma}} - \tilde{\xi}\tilde{\mu}\partial_{\tilde{\mu}}$. The vector fields Z_4 and Z_5 are smooth vector fields[1] on $\mathbb{S}^2$ and are given by:

$$\begin{aligned} Z_3 &= -\tilde{\mu}^2\partial_{\tilde{\gamma}} + \tilde{\gamma}\tilde{\mu}\partial_{\tilde{\mu}} \\ Z_4 &= \tilde{\mu}\tilde{\gamma}\partial_{\tilde{\gamma}} - \tilde{\gamma}^2\partial_{\tilde{\mu}} \\ Z_5 &= -\tilde{\mu}\partial_{\tilde{\gamma}} + \tilde{\gamma}\partial_{\tilde{\mu}} \end{aligned}$$

With the expression of $\tilde{H}$, we see that the radial points (on the domain of the coordinates (x_N, y_N, r)) are exactly solutions of the following system:

$$\begin{cases} 2\Delta_r\tilde{\xi} + 4a\dfrac{x_N\tilde{\mu} - y_N\tilde{\gamma}}{1+x_N^2+y_N^2} = 0 & (12.6) \\ -2a\tilde{\xi}y_N + (1+x_N^2+y_N^2)\tilde{\gamma} = 0 & (12.7) \\ 2a\tilde{\xi}x_N + (1+x_N^2+y_N^2)\tilde{\mu} = 0 & (12.8) \\ 2\tilde{\xi}^2(M-r)(1-\tilde{\xi}^2) = 0 & (12.9) \\ 2(r-M)\tilde{\xi}^3\tilde{\gamma} + 2\tilde{\mu}\tilde{\gamma}y_N - 2a\tilde{\mu}\tilde{\xi} - 2\tilde{\mu}^2x_N = 0 & (12.10) \\ 2(r-M)\tilde{\mu}\tilde{\xi}^3 + 2a\tilde{\gamma}\tilde{\xi} + 2\tilde{\gamma}\tilde{\mu}x_N - 2\tilde{\gamma}^2y_N = 0 & (12.11) \end{cases}$$

Assume that we have a solution $(r, x_N, y_N, \tilde{\xi}, \tilde{\gamma}, \tilde{\mu})$ of this system. Using (12.9), (and the fact that $M-r<0$ on $\mathcal{M}_\epsilon$), we deduce $\tilde{\xi} \in \{-1, 0, 1\}$. We exclude the case $\tilde{\xi} = 0$ since (12.7) and (12.8) give then $\tilde{\gamma} = \tilde{\mu} = 0$ which is impossible since $\tilde{\xi}^2 + \tilde{\gamma}^2 + \tilde{\mu}^2 = 1$. We deduce that $\tilde{\xi} \in \{-1, 1\}$ and $\tilde{\gamma} = \tilde{\mu} = 0$. Using (12.6), we deduce that $\Delta_r = 0$ and therefore $r = r_+$. Finally, equations (12.8) and (12.9) give $x_N = y_N = 0$. We deduce that the set of radial points is exactly $\left\{x_N = y_N = \Delta_r = \tilde{\mu} = \tilde{\gamma} = 0, \tilde{\xi} = \pm 1\right\} \cup \left\{x_S = y_S = \Delta_r = \tilde{\mu} = \tilde{\gamma} = 0, \tilde{\xi} = \pm 1\right\}$ (the second set is obtained by symmetry). Moreover, note that $\tilde{H}$ is tangent to $L_+ \cup L_-$ (the only non vanishing component is parallel to ∂_ϕ) and therefore, $L_+ \cup L_-$ is closed under the flow of $\tilde{H}$. □

Definition 12.1.5. *Let A be included in the characterstic set, closed under the flow of the rescaled Hamiltonian vector field and a submanifold of fiber infinity or, in the scattering setting, of the boundary face $\{x = 0\}$. Here we assume that A does not intersect the other boundary face, but we could extend the definition to the case where A is transversal to it. We say that A is a sink (resp. a source) for the Hamiltonian flow if there exists ρ_0 a non negative quadratic defining function of A within the characteristic set[2] such that:*

- *$\tilde{H}\rho_0 = -\beta_1\rho_0 - F_2 + F_3$ (resp. $\tilde{H}\rho_0 = \beta_1\rho_0 + F_2 + F_3$) with β_1, F_2, F_3 are functions defined on a neighborhood of A and β_1 is positive on A, F_3 vanishes cubically at A and $F_2 \geq 0$.*
- *There exists μ, a defining function of the boundary face containing A (fiber infinity or $\{x = 0\}$), such that $\tilde{H}\mu = -\beta_0\mu$ (resp. $\tilde{H}\mu = \beta_0\mu$) where β_0 is a function defined in a neighborhood of A with $\beta_0 > 0$ on A.*

We check that $H\tilde{\rho} = 2(r-M)\tilde{\xi}^3$. So on $\Lambda_\pm$, $H\tilde{\rho} = \pm 2(r-M)$ (with the notation of [103], it means that $\beta_0 = 2(r_+ - M)$). Moreover, the non negative homogeneous of degree zero function $\rho_0 := \tilde{\eta}^2 + \tilde{\zeta}^2$ is a quadratic defining function of $\Lambda_\pm$ inside the characteristic set of $\hat{T}_s(\sigma)$ and we have $\tilde{\rho}H\rho_0 = 4(r-M)\tilde{\xi}^3\left(\tilde{\eta}^2 + \tilde{\zeta}^2\right) = 4(r-M)\tilde{\xi}^3\rho_0$. This shows that L_+ is a source for $\hat{T}_s(\sigma)$ and L_- is a sink in the sense of [103]. Note that there is a different sign

[1] Here we represent vector fields on $\mathbb{S}^2 \subset \mathbb{R}^3_{(\tilde{\xi},\tilde{\mu},\tilde{\gamma})}$ in the canonical basis $(\partial_{\tilde{\xi}}, \partial_{\tilde{\mu}}, \partial_{\tilde{\gamma}})$

[2] The restriction of ρ_0 to the characteristic set vanishes quadratically at A and is non degenerate.

convention in [103] (the principal symbols given page 483 has a minus sign with respect to our choice).

Now we study more precisely the behavior of the bicharacteristic curves (integral curves of $\tilde{H}_{|\{\tilde{\rho}=0\}}$ included in the characteristic set of $\mathfrak{s}(\hat{T}_s(\sigma))$). Let $\lambda \mapsto f(\lambda) = (r(\lambda), \theta(\lambda), \phi(\lambda), \tilde{\xi}(\lambda), \tilde{\zeta}(\lambda), \tilde{\eta}(\lambda))$ be a bicharacteristic curve defined on the maximally extended open interval I. By definition, we have for all $\lambda \in I$

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\tilde{\xi} = 2\tilde{\xi}^2(M-r)(1-\tilde{\xi}^2) \leq 0 \tag{12.12}$$

$$\Delta_r \tilde{\xi}^2 + 2a\tilde{\xi}\tilde{\zeta}\sin\theta + \tilde{\zeta}^2 + \tilde{\eta}^2 = 0 \tag{12.13}$$

Note that if $\Delta_r > a^2$, we have

$$\begin{aligned}\Delta_r \tilde{\xi}^2 + 2a\tilde{\xi}\tilde{\zeta}\sin\theta + \tilde{\zeta}^2 + \tilde{\eta}^2 &> a^2\tilde{\xi}^2 + 2a\tilde{\xi}\tilde{\zeta}\sin\theta + \tilde{\zeta}^2 \\ &\geq a^2\tilde{\xi}^2 - 2|a||\tilde{\xi}||\tilde{\zeta}| + \tilde{\zeta}^2 \\ &\geq (|a||\tilde{\xi}| - |\tilde{\zeta}|)^2 \\ &\geq 0\end{aligned}$$

This computation ensures that the bicharacteristic curve remains in $\{r \leq 2M\}$.

Lemma 12.1.6. *We have $\tilde{\xi}\tilde{H}r < 0$ on the characteristic set when $\Delta_r < 0$.*

Proof. Since, $\tilde{H}r = 2\Delta_r\tilde{\xi} + 2a\tilde{\zeta}\sin\theta$, on the characteristic set we have:

$$\begin{aligned}0 =& \Delta_r\tilde{\xi}^2 + 2a\tilde{\xi}\tilde{\zeta}\sin\theta + \tilde{\zeta}^2 + \tilde{\eta}^2 \\ =& \tilde{\xi}\tilde{H}r - \Delta_r\tilde{\xi}^2 + \tilde{\zeta}^2 + \tilde{\eta}^2 \\ \tilde{\xi}\tilde{H}r =& \Delta_r\tilde{\xi}^2 - (\tilde{\zeta}^2 + \tilde{\eta}^2) < 0\end{aligned}$$

□

We deduce from (12.12) that $\tilde{\xi}$ is decreasing. Moreover, $\tilde{\xi}$ is bounded (since $|\tilde{\xi}| \leq 1$). Then

- Case $1 > \tilde{\xi}(0) > 0$: In this case, $\sup I = \lambda_+ < +\infty$. Otherwise, $\lim_{\lambda\to+\infty} \tilde{\xi} = 0$ (ξ has a limit because it is decreasing at least linearly when ξ is in a fixed compact subset of $(0,1)$) but this contradicts the fact that for all $\lambda \in I$, $\Delta_r\tilde{\xi}^2 + 2a\tilde{\xi}\tilde{\zeta}\sin\theta + \tilde{\zeta}^2 + \tilde{\eta}^2 = 0$. As a consequence, f leaves every compact set as $\lambda \to \lambda_+$. Using that $\left\{r \geq r_+ - \frac{\epsilon}{2}, \tilde{\rho} = 0\right\} \cap \left\{\tilde{\rho}^2 p = 0\right\}$ is compact, this means that there exists $\lambda_0 \in I$ such that for all $\lambda > \lambda_0$, $r(\lambda) < r_+ - \frac{\epsilon}{2}$. By lemma 12.1.6, r cannot reach $r_+ - \epsilon$ in the past therefore $\inf I - \infty$ and then $\lim_{\lambda\to-\infty} \tilde{\xi} = 1$ and $\lim_{\lambda\to-\infty} \tilde{\zeta} = \lim_{\lambda\to-\infty} \tilde{\eta} = 0$. Using the fact that $\Delta_r\tilde{\xi}^2 + 2a\tilde{\xi}\tilde{\zeta}\sin\theta + \tilde{\zeta}^2 + \tilde{\eta}^2 = 0$, we deduce that $\lim_{\lambda\to-\infty} r(\lambda) = r_+$. As a consequence, f tends to L_+ when $\lambda \to -\infty$.

- Case $-1 < \tilde{\xi}(0) < 0$: We show as in the first case that I is lower bounded by some $\lambda_- \in \mathbb{R}$ and there exists $\lambda_0 \in I$ such that for all $\lambda < \lambda_0$, $r(\lambda) < r_+ - \frac{\epsilon}{2}$. Moreover $\sup(I) = +\infty$ and f tends to L_- when $\lambda \to +\infty$.

- Case $|\tilde{\xi}(0)| = 1$: In this case $\tilde{\eta} = \tilde{\zeta} = 0$. Therefore, since we are on the characteristic set, $\Delta_r = 0$. We conclude that the bicharacteristic is included in L_+ or in L_-.

- Case $\tilde{\xi}(0) = 0$: This case is impossible since it would mean $\tilde{\zeta}^2 + \tilde{\eta}^2 = 1$ which contradicts $\Delta_r\tilde{\xi}^2 + 2a\tilde{\xi}\tilde{\zeta}\sin\theta + \tilde{\zeta}^2 + \tilde{\eta}^2 = 0$.

Finally, possible behavior of bicharacteristic curves are summarized by the following proposition:

Proposition 12.1.7. *Let $\lambda \mapsto f(\lambda) = (r(\lambda), \theta(\lambda), \phi(\lambda), \tilde{\xi}(\lambda), \tilde{\zeta}(\lambda), \tilde{\eta}(\lambda))$ be a bicharacteristic curve defined on the maximally extended open interval I containing 0. Then one of the following cases holds:*

1. *$I = (-\infty, \lambda_+)$ for some $\lambda_+ \in (0, +\infty)$ and there exists a small non empty interval $J := (\lambda_+ - \eta, \lambda_+)$ such that $(r \circ f)_{|_J}$ is decreasing with values in $(r_+ - \epsilon, r_+ - \frac{\epsilon}{2})$ and f tends to L_+ when $\lambda \to -\infty$. In this case $\tilde{\xi}$ is negative along the trajectory*
2. *$I = (\lambda_-, +\infty)$ for some $\lambda_- \in (-\infty, 0)$ and there exists a small non empty interval $J := (\lambda_-, \lambda_- + \eta)$ such that $(r \circ f)_{|_J}$ is increasing with values in $(r_+ - \epsilon, r_+ - \frac{\epsilon}{2})$ and f tends to L_- when $\lambda \to +\infty$.*
3. *The curve is included in L_+ (resp. in L_-). In this case $\tilde{\xi}$ is positive (resp. negative) along the trajectory.*

Proposition 12.1.8. *There exists an open neighborhood U_+ of L_+ such that if f is a maximal bicharacteristic curve defined on the open interval I containing 0 and if $f(0) \in U_+ \cap \Sigma$ (Σ is the characteristic set of p), then $(-\infty, 0) \subset I$ and f tends to L_+ as $\lambda \to -\infty$.*

Proof. We recall that by definition of ϵ, $\Delta_{r_+ - \frac{\epsilon}{2}} = a^2 + \left(r_+ - \frac{\epsilon}{2}\right)^2 - 2M\left(r_+ - \frac{\epsilon}{2}\right) < 0$. We define $0 < \alpha < 1$ such that $\Delta_{r_+ - \frac{\epsilon}{2}}(1 - \alpha^2) + 2a\alpha + \alpha^2 < 0$. As a consequence, for all $1 \geq \tilde{\xi} \geq \sqrt{1 - \alpha^2}$ and all $\tilde{\zeta}, \tilde{\eta}$ such that $\tilde{\zeta}^2 + \tilde{\eta}^2 \leq \alpha^2$ we have

$$\Delta_{r_+ - \frac{\epsilon}{2}} \tilde{\xi}^2 + 2a\tilde{\xi}\tilde{\zeta}\sin\theta + \tilde{\zeta}^2 + \tilde{\eta}^2 < 0 \tag{12.14}$$

We define $U_+ := \left\{\rho_0 < \alpha^2, \tilde{\xi} > \sqrt{1 - \alpha^2}, r > r_+ - \frac{\epsilon}{2}\right\}$. Let f and I be defined as in the statement of the proposition. Using (12.12), we have that $\tilde{\xi}$ is decreasing on I. This shows that for every $\lambda \in I \cap (-\infty, 0]$, we have $\tilde{\xi} \geq \sqrt{1 - \alpha^2}$ and therefore $\tilde{\zeta}^2 + \tilde{\eta}^2 \leq \alpha^2$. Then (12.14) ensures that for all $\lambda \in I \cap (-\infty, 0]$, $r(\lambda) > r_+ - \frac{\epsilon}{2}$. As a consequence, f remains in the compact set $\left\{r \geq r_+ - \frac{\epsilon}{2}\right\} \cap \Sigma$ on $I \cap (-\infty, 0]$ and therefore, $(-\infty, 0] \subset I$. Then using proposition 12.1.7, we get the claimed convergence. □

Remark 12.1.9. *If we take the union of the images of U_+ by the backward bicharacteristic flow, we get an open neighborhood of L_+ which is stable under the backward bicharacteristic flow with the same property as U_+.*

We can prove similarly:

Proposition 12.1.10. *There exists an open neighborhood U_- of L_- such that if f is a maximal bicharacteristic curve defined on the open interval I containing 0 and if $f(0) \in U_- \cap \Sigma$ (Σ is the characteristic set of p), then $(0, +\infty) \subset I$ and f tends to L_- as $\lambda \to +\infty$.*

12.2 Analysis of the semiclassical flow

We will use the semiclassical regime to study the behavior of the operator $\hat{T}_s(\sigma)$ when $|\sigma| \to +\infty$ and $0 \leq \Im(\sigma) \leq C$ for some fixed constant $C > 0$. Therefore, we introduce the semiclassical parameter $h = |\sigma|^{-1}$ and the rescaled operator $\hat{T}_{s,h}(z) := h^2 \hat{T}_s(h^{-1}z)$ for $z = \pm 1 + O(h)$. We denote by z_0 the element of $\{-1, 1\}$ such that $z - z_0 = O(h)$. The semiclassical principal symbol is $p_h(\xi) = -\tilde{G}(\xi - z_0\, \mathrm{d}\mathfrak{t})$ (it does not depend on the imaginary part of z). From this formula, we see that the semiclassical flow of $h^2\hat{T}_s(h^{-1}z)$ is closely related with

the geodesic flow of $\mathcal{M}_\epsilon$. Since the coordinate $\mathfrak{t}$ is not very convenient for computations, we study the effect of a change of time function (in the Fourier transform) of the form $t_2 = \mathfrak{t} + f(r)$ for a smooth function f defined on an open interval $I \subset (r_+ - \epsilon, +\infty)$. We define $U_I = I \times \mathbb{S}^2$ and $p_{2,h}(\xi) = -\tilde{G}(\xi - z_0 \, \mathrm{d}t_2)$ on $\mathcal{T}^* U_I$. In particular we have $p_{2,h}(\xi) = p_h(\xi - z_0 f'(r) \, \mathrm{d}r)$.

Lemma 12.2.1. *The map $\Psi : \xi \mapsto \xi - z_0 f'(r) \, \mathrm{d}r$ is a symplectomorphism of $T^* U_I$*

Proof. The map Ψ is smooth with smooth inverse $\omega \mapsto \omega + z_0 f'(r) \, \mathrm{d}r$. Let $y \in U_I$ and $x := (x_0, x_1, x_2)$ be smooth local coordinates around y with $x_0 = r$. The symplectic form in local coordinates is therefore: $\omega = \sum_{i=0}^{2} \mathrm{d}x_i \wedge \mathrm{d}\xi_i$ (where $\xi_x := (\xi_0, \xi_1, \xi_2)$ are the conjugated local coordinates) and $\Psi(x, \xi_0, \xi_1, \xi_2) = (x, \xi_0 - z_0 f'(r), \xi_1, \xi_2)$. Let $\xi \in T^*_y U_I$, $X = \sum_{i=0}^{3} X_{x_i} \partial_{x_i} + \sum_{i=0}^{3} X_{\xi_i} \partial_{\xi_i}$ and $Y = \sum_{i=0}^{3} Y_{x_i} \partial_{x_i} + \sum_{i=0}^{3} Y_{\xi_i} \partial_{\xi_i}$. We have:

$$\begin{aligned}
\mathrm{d}_\xi \Psi(X) &= X - X_{x_0} z_0 f'' \partial_{\xi_0} \in T_{\Psi(\xi)} T^* U_I \\
\omega(\, \mathrm{d}_\xi \Psi X, \, \mathrm{d}_\xi \Psi Y) &= \omega(X, Y) - \omega(X_{x_0} z_0 f'' \partial_{\xi_0}, Y) - \omega(X, Y_{x_0} z_0 f'' \partial_{\xi_0}) \\
&= \omega(X, Y) + Y_{x_0} X_{x_0} z_0 f'' - Y_{x_0} X_{x_0} z_0 f'' \\
&= \omega(X, Y)
\end{aligned}$$

□

Remark 12.2.2. *Note that f extends to a diffeomorphism of $\overline{T}^* U_I$ (fiber radial compactification of $T^* U_I$) preserving fiber infinity. This is consistent with the fact that the classical principal symbol is not affected by a change of time function in the Fourier transform.*

Lemma 12.2.3. *Let Ψ be a symplectomorphism between the symplectic manifolds $(\mathcal{N}_1, \omega_1)$ and $(\mathcal{N}_2, \omega_2)$. Then for any $f \in C^\infty(\mathcal{N}_2)$, we have $H_{\Psi^* f} = \Psi^* H_f$. Therefore, if γ is an integral curve of H_f, $\Psi^{-1} \circ \gamma$ is an integral curve of $H_{\Psi^* f}$.*

Proof. By definition, for $x \in \mathcal{N}_1$ and $X \in T_x \mathcal{N}_1$, we have

$$\mathrm{d}_x (f \circ \Psi)(X) = \omega_1(H_{\Psi^* f}, X)$$

But we also have:

$$\begin{aligned}
\mathrm{d}_x f \circ \Psi(X) = \; & \mathrm{d}_{\Psi(x)} f(\, \mathrm{d}_x \Psi(X)) \\
&= \omega_2(H_f(\Psi(x)), \, \mathrm{d}_x \Psi(X)) \\
&= \omega_2(\, \mathrm{d}_x \Psi(\, \mathrm{d}_x \Psi)^{-1}(H_f(\Psi(x))), \, \mathrm{d}_x \Psi(X)) \\
&= \omega_1(\Psi^* H_f, X)
\end{aligned}$$

By the non degeneracy of ω_1, we have the equality. □

Using the two previous lemmas, we see that we have an identification between the integral curves of H_{p_h} and the integral curves of $H_{p_{2,h}}$ in $T^* U_I$ (and the projections on U_I of two identified integral curves coincide). We use this identification implicitly in the remaining part of this section. In the following, we will make the three following choices in the concrete computations:

- $I = (r_+ - \epsilon, 3M)$ and f_I such that $t_* = \mathfrak{t} + f_I(r)$ (in particular, $f_I(r)$ is constant and the induced symplectomorphism is trivial)
- $I = (6M, +\infty)$, we choose the time function ${}^*t := t - T(r)$ (in particular, $f_I(r)$ is constant and therefore the induced symplectomorphism is trivial).

- $I = (r_{\min}, r_{\max})$ ($r_{\min}$ and $r_{\max}$ are constants which will be introduced in Lemmas 12.2.10 and 12.2.18), we choose f_I such that $t = \mathfrak{t} + f_I(r)$

We also need a general elementary lemma about smooth vector fields.

Lemma 12.2.4. *Let X be a smooth vector field on a manifold $\mathcal{M}$ and r be a smooth function on $\mathcal{M}$. If $\mathcal{M}$ has a boundary, we assume that X is tangent to the boundary (in order for the integral curves to be locally defined near any point of $\mathcal{M}$). Let $y \in \mathcal{M}$ be a point such that $Xr(y) = 0$ and $X^2r(y) > 0$ (resp. $X^2r(y) < 0$). Then there exists a neighborhood V of y such that for every integral curve γ of X starting at V and maximally defined on an interval J containing 0, there exists $s > 0$ in J with $-s \in J$ such that $r(\gamma(s)) > r(y)$ (resp. $r(\gamma(s)) < r(y)$), $\dot{r}(s) > 0$ (resp. $\dot{r}(s) < 0$) and $r(\gamma(-s)) > r(y)$ (resp. $r(\gamma(-s)) < r(y)$), $\dot{r}(-s) < 0$ (resp. $\dot{r}(-s) > 0$).*

Proof. We do the proof in the case $X^2r(y) > 0$. There exists a constant $\eta > 0$ and an open neighborhood U of y on which $0 < \eta < X^2r < 2\eta$. We fix $K \subset U$ a compact neighborhood of y. Since the existence time of integral curves in U is lower semi-continuous, there exists $\delta > 0$ such that every integral curve starting in K exists and remains in U on the interval $[-\delta, \delta]$. Then for an integral curve starting at K we have for every $-\delta \leq s \leq \delta$:

$$\begin{aligned} r(s) &\geq r(0) + s\dot{r}(0) + \eta s^2 \\ r(s) - r(y) &\geq (r(0) - r(y)) + s(\dot{r}(0) + \eta s) \end{aligned}$$

By shrinking K until $|\dot{r}(x)| \leq \frac{\eta\delta}{2}$ and $r(x) - r(y) > -\frac{\eta\delta^2}{4}$ for all $x \in K$, we have $r(s) - r(y) > 0$ at time δ and at time $-\delta$. Moreover, for $s \in [0, \delta]$, we have

$$\dot{r}(0) + s\eta \leq \dot{r}(s) \leq \dot{r}(0) + 2s\eta$$

while for $s \in [-\delta, 0]$, we have

$$\dot{r}(0) + 2s\eta \leq \dot{r}(s) \leq \dot{r}(0) + s\eta$$

Therefore, using $|\dot{r}(0)| \leq \frac{\eta\delta}{2}$, we get

$$\begin{aligned} \dot{r}(-\delta) &\leq -\frac{1}{2}\eta\delta < 0 \\ \dot{r}(\delta) &\geq \frac{1}{2}\eta\delta > 0. \end{aligned}$$

□

We now compute the principal symbol near the horizon (using t_* coordinates on the radial interval $I = (r_+ - \epsilon, 3M)$): We use coordinates $\xi = \xi_r \, \mathrm{d}r + \zeta \, \mathrm{d}\phi_* + \eta \, \mathrm{d}\theta$ for cotangent vectors.

$$\begin{aligned} p_h(x, \xi) =& \Delta_r \xi_r^2 + \frac{1}{\sin^2\theta}\zeta^2 + \eta^2 + 2a\zeta\xi_r - 2(a\zeta + (a^2 + r^2)\xi_r)z_0 + a^2 \sin^2\theta z_0^2 \\ =& \left(\frac{\zeta}{\sin(\theta)} - az_0 \sin\theta\right)^2 + \eta^2 + \xi_r\left(-2(a^2 + r^2)z_0 + 2a\zeta + \xi_r\Delta_r\right) \\ =& -\rho^2 G(\xi - z_0 \, \mathrm{d}t_*) \\ H_{p_h} =& 2(-(a^2 + r^2)z_0 + a\zeta + \xi_r\Delta_r)\partial_r - (2(r - M)\xi_r^2 - 4rz_0\xi_r)\partial_{\xi_r} \\ &+ \left(-2az_0 + \frac{2\zeta}{\sin^2\theta} + 2a\xi_r\right)\partial_\phi + 2\eta\partial_\theta - 2\cos\theta\left(z_0^2 a^2 \sin\theta - \frac{\zeta^2}{\sin^3\theta}\right)\partial_\eta. \end{aligned}$$

Lemma 12.2.5. *The quantities ζ, p_h and $K := \eta^2 + \left(\frac{\zeta}{\sin(\theta)} - z_0 a \sin\theta\right)^2$ are invariant under the flow. As a consequence $p_h - K = \xi_r(-2(a^2+r^2)z_0 + 2a\zeta + \xi_r\Delta_r)$ is also invariant.*

Proof. It can be shown by a direct computation using the Hamiltonian vector field but it comes from the invariance of the energy E, the norm, the angular momentum L, and the Carter constant K along geodesics. □

Lemma 12.2.6. *If $y \in S^*U_I$ (at fiber infinity) is an accumulation point of a bicharacteristic curve γ (maximally defined on some interval J). Then $\tilde{\xi}(y) = \pm 1$ and $\tilde{\eta}(y) = \tilde{\zeta}(y) = 0$. In particular we also have $\tilde{H}r(y) = 2\tilde{\xi}(y)\Delta_r = \pm 2\Delta_r$ (where $\tilde{H} := \tilde{\rho}H_{p_h}$ is the rescaled Hamiltonian vector field).*

Proof. We denote by s_n an increasing sequence in J such that $\lim_{n\to+\infty} s_n = \sup(J)$ and $\lim_{n\to+\infty} \gamma(s_n) = y$. By the conservation of $\eta^2 + \left(\frac{\zeta}{\sin\theta} - az_0\sin\theta\right)^2$, $\lim_{n\to+\infty}\tilde{\eta}(s_n) = \lim_{n\to+\infty}\tilde{\zeta}(s_n) = 0$ and therefore $\lim_{n\to+\infty}\tilde{\xi}(s_n) = \pm 1$. By continuity of $\tilde{\eta}$, $\tilde{\zeta}$ and $\tilde{\xi}$ we obtain the first part of the lemma. Then replacing in the explicit expression of the Hamiltonian flow at fiber infinity we get $\tilde{H}_{p_h} r(y) = \tilde{\xi}(y)2\Delta_r = \pm 2\Delta_r$ □

Lemma 12.2.7. *If $z_0 \in \{-1, 1\}$, then: $\left\{-(r^2 + a^2\cos^2\theta)z_0 - Mr\xi_r = 0\right\} \cap p^{-1}(\{0\}) = \emptyset$. In particular, the characteristic set has at least two connected components corresponding to both signs of $-(r^2 + a^2\cos^2\theta)z_0 - Mr\xi_r$ (sign + correspond to the future light cone and - corresponds to the past light cone). Each bicharacteristic curve remains in one of the two connected components for all time.*

Proof. We remark that the vector field $T = \partial_{t_*} - \frac{Mr}{\rho^2}\partial_r$ satisfies $g(T,T) = 1$. Therefore, if X is a non zero null vector $g(X,T) \neq 0$. Therefore, if $X^\#$ is the associated linear form, $X^\#(T) \neq 0$. In particular, we remark that for ξ such that $p_{h,z_0}(\xi) = 0$, we have $-\rho^2 G(\xi - z_0\ \mathrm{d}t_*) = 0$ therefore $\xi - z_0\ \mathrm{d}t_* = X^\#$ for some non zero null vector X. Therefore

$$-(r^2 + a^2\cos^2\theta)z_0 - Mr\xi_r = \rho^2(\xi - z_0\ \mathrm{d}t_*)(T) = \rho^2 g(X,T) \neq 0.$$

□

Lemma 12.2.8. *If $a^2\cos^2\theta + r^2 - 2Mr > 0$ (i.e outside the ergoregion), then for all ξ such that $p_{h,z_0}(\xi) = 0$, z_0 and $(r^2 + a^2\cos^2\theta)z_0 + Mr\xi_r$ have the same sign. Therefore for $z_0 = -1$, the component $(r^2 + a^2\cos^2\theta)z_0 + Mr\xi_r > 0$ of the characteristic set lies inside the ergoregion. For $z_0 = 1$ the component $(r^2 + a^2\cos^2\theta)z_0 + Mr\xi_r < 0$ of the characteristic set lies inside the ergoregion. On T^*U_I, ξ_r and $(r^2 + a^2\cos^2\theta)z_0 + Mr\xi_r$ have the same sign.*

Proof. Outside the ergoregion, T and ∂_t are both timelike future oriented. Therefore, for all null vector X, $g(T,X)$ and $g(\partial_t, X)$ have the same sign. For $p_{h,z_0}(\xi) = 0$, we have $\xi - z_0\ \mathrm{d}t_* = X^\#$ with X null and as a consequence $(\xi - z_0\ \mathrm{d}t_*)(\partial_t) = -z_0$ and $\rho^2(\xi - z_0\ \mathrm{d}t_*)(T) = -(r^2 + a^2\cos^2\theta)z_0 - Mr\xi$ have the same sign. Similarly, since ∂_r is past oriented on U_I, we have $\rho^2(\xi - z_0\ \mathrm{d}t_*)(T)$ and $(\xi - z_0\ \mathrm{d}t_*)(\partial_r) = \xi_r$ have opposite signs. However, since ∂_r is null, ξ_r can vanish (contrary to $-(r^2 + a^2\cos^2\theta)z_0 - Mr\xi_r$). □

Therefore for a fixed z_0, we can define $\Sigma_\pm := p_h^{-1}\{0\} \cap \left\{\pm((r^2 + a^2\cos^2\theta)z_0 + Mr\xi_r) > 0\right\}$. With this definition $\Sigma_{\mathrm{sgn}(-z_0)}$ lies in the ergoregion.

Remark 12.2.9. *Note that $\tilde{\rho}(r^2 + a^2\cos^2\theta)z_0 + Mr\tilde{\xi_r}$ has the same sign as $(r^2 + a^2\cos^2\theta)z_0 + Mr\xi_r$ on T^*U_I. Therefore, we can extend the definition of $\Sigma_\pm$ to include the component at fiber infinity given by $\Sigma_\pm \cap S^*U_I = \left\{p_h = 0 = \tilde{\rho}, \pm\tilde{\xi_r} > 0\right\}$. As a consequence, we have $L_\pm \subset \Sigma_\pm$.*

Lemma 12.2.10. *Let $z_0 = \pm 1$. There exists $r_+ < r_{min} < 3M$ such that: $r_+ \leq r \leq r_{min}$, $p_h = 0$ and $H_{p_h} r = 0 \Rightarrow H_{p_h}^2 r < 0$.*

For $r = r_+$ we even have that $H_{p_h} r$ does not vanish and has the sign of $\left(-(r^2 + a^2\cos^2\theta)z_0 - Mr\xi_r\right)$.

Proof. We have:

$$H_{p_h} r = 2(-(a^2 + r^2)z_0 + a\zeta + \xi_r \Delta_r).$$

First, we handle the particular case of $r = r_+$ (equivalently $\Delta_r = 0$). Using the fact that $p_h = 0$, we get $\xi_r H_{p_h} r = -\left(\frac{\zeta}{\sin\theta} - az_0\sin\theta\right)^2 - \eta^2$. If the right-hand side is equal to 0, we get $\zeta = az_0\sin^2\theta$ and therefore $H_{p_h} r = 2\left(-(a^2+r^2)z_0 + a^2 z_0\sin^2\theta\right) = -2z_0\rho^2 \neq 0$, therefore we must have $\xi_r = 0$ and $H_{p_h} r$ has the same sign as $-(r^2 + a^2\cos^2\theta)z_0 - Mr\xi_r$. If the right-hand side is negative, we have $\xi_r \neq 0$ and $-(r^2 + a^2\cos^2\theta)z_0 - Mr\xi_r$ have opposite sign (by lemma 12.2.8), therefore, $H_{p_h} r$ is non zero and has the sign of $\left(-(r^2 + a^2\cos^2\theta)z_0 - Mr\xi_r\right)$.

Now we assume $r > r_+$. If $H_{p_h} r = 0$, we have $\xi_r\left(-2(a^2+r^2)z_0 + 2a\zeta + \xi_r\Delta_r\right) = -\Delta_r\xi_r^2$. Using $p_h = 0$ (for the first line) and $H_{p_h} r = 0$ for the second line, we get:

$$\xi_r = \pm\sqrt{\frac{\eta^2 + \left(\frac{\zeta}{\sin\theta} - az_0\sin\theta\right)^2}{\Delta_r}}$$
$$\xi_r = -\frac{(-(a^2+r^2)z_0 + a\zeta)}{\Delta_r}$$

Let $\epsilon > 0$: We have 2 possibilities:

1. $|\xi_r| \geq \frac{\epsilon}{\sqrt{\Delta_r}}$

2. $|\xi_r| \geq \frac{r^2 - |a|\epsilon}{\Delta_r}$

To prove this, assume that $|\frac{\zeta}{\sin\theta} - az_0\sin\theta| \leq \epsilon$ (otherwise, we are in the first case). Then, we have

$$\begin{aligned} |a\zeta| &\leq a^2\sin^2\theta + |a\sin\theta|\epsilon \leq a^2\sin^2\theta + |a|\,\epsilon \\ |-(a^2+r^2)z_0 + a\zeta| &\geq a^2 + r^2 - |a\zeta| \\ &\geq a^2\cos^2\theta + r^2 - |a|\,\epsilon \\ &\geq r^2 - |a|\,\epsilon. \end{aligned}$$

Therefore, we are in the second case. We can use this with $\epsilon = \frac{r_+^2}{2(|a|+1)}$ and we find $|\xi_r| \geq \frac{r_+^2}{2(1+|a|)\sqrt{\Delta_r}}$ as soon as $\Delta_r \leq 1$.

Now we compute:

$$H_{p_h}^2 r = -4\xi_r(-2rz_0 + \xi_r(r-M))\Delta_r + 2(-rz_0 - \xi_r(M-r))H_{p_h}(r)$$

Therefore, when $H_{p_h} r = 0$, we have $H_{p_h}^2 r < 0$ as soon as $|\xi_r| > \frac{2r}{r-M}$ and this holds when $r > r_+$ is close to r_+ since $\frac{2r}{r-M} = o(\frac{r_+^2}{2(1+|a|)\sqrt{\Delta_r}})$ when $r \to r_+$. □

Lemma 12.2.11. *On $p_h^{-1}(\{0\})$ we have: $\xi_r H_{p_h} r \leq 0$ when $\Delta \leq 0$. If $\Delta_r < 0$ and $\xi_r \neq 0$, we have a strict inequality. If $\xi_r = 0$, $H_{p_h} r$ has the same sign as $-z_0$ (striclty) on T^*U_I.*

Proof. We have: $H_{p_h} r = 2(-(a^2+r^2)z_0 + a\zeta + \xi_r \Delta_r)$. Moreover, since $p_h = 0$ and $\Delta_r \leq 0$, we must have $\xi_r \left(-2(a^2+r^2)z_0 + 2a\zeta + \xi_r \Delta_r\right) \leq 0 \leq -\Delta_r \xi_r^2$. Therefore $\xi_r H_{p_h} r \leq 0$ with strict inequality when $\xi_r \neq 0$ and $\Delta_r < 0$. If $\xi_r = 0$, we have (on the characteristic set):

$$\left(\frac{\zeta}{\sin\theta} - az_0 \sin\theta\right)^2 + \eta^2 = 0$$

and therefore

$$\begin{aligned} \eta =& 0 \\ \zeta =& az_0 \sin^2\theta \end{aligned}$$

If we replace in the expression of $H_{p_h} r$, we get:

$$\begin{aligned} H_{p_h} r =& 2(-(a^2+r^2)z_0 + a^2 z_0 \sin^2\theta) \\ =& -2\rho^2 z_0 \end{aligned}$$

□

Lemma 12.2.12. *Let γ be a bicharacteristic curve defined on an interval J. Let $J_I = \{s \in J : r(s) \in I\}$. On each connected component of J_I, we have the alternative $\xi_r(s) = 0$ or ξ_r never vanishes.*

Proof. Assume that $\xi_r(s_0) = 0$ for some $s_0 \in J$. Then using that $p_h = 0$, we have $\eta^2 + \left(\frac{\zeta}{\sin\theta} - az_0 \sin\theta\right)^2 = 0$. Therefore $\zeta = az_0 \sin^2\theta$. This property is true for all $s \in J$ by the conservation law. Therefore for all $s \in J$ $\xi_r(-2(a^2\cos^2\theta + r^2)z_0 + \Delta_r \xi_r) = 0$. Since $\frac{2(a^2\cos^2\theta + r^2)}{\Delta_r} > \delta > 0$ for some constant δ independent of r and θ, the continuity of ξ_r implies that $\xi_r(s) = 0$ for all s in the connected component of s_0 in J_I. □

Lemma 12.2.13. *Let γ be a bicharacteristic curve defined on an interval J. We assume that there exists $s_0 \in J$ such that $r(s_0) \leq r_{min}$ and $\dot{r}(s_0) < 0$. Then we have two cases:*

- *The curve γ remains in $\{r_+ \leq r\}$ for all $s \in [s_0, \sup J)$ in which case $\sup J = +\infty$ and γ tends to L_+ or to L_-.*
- *γ reaches $\{r < r_+\}$ for some $s_1 > s_0$ and in this case, $\sup J < +\infty$ and $\lim_{s \to \sup J} r(\gamma(s)) = r_+ - \epsilon$.*

Proof. Note that by lemma 12.2.10 and lemma 12.2.11, $r \circ \gamma$ cannot have a local minimum on T^*U_I and therefore it must be strictly decreasing on $[s_0, \sup J)$. In particular, γ remains in $\{r < r_{min}\}$ for $s > s_0$ where r_{min} is defined in Lemma 12.2.10.

First assume that γ does not reach $\{r < r_+\}$. The curve γ remains in a compact set and therefore has an accumulation point $y \in \overline{T^*U_I}$ with $r(y) = r_\infty \geq r_+$ and $r_\infty = \inf\{r(\gamma(s)), s \geq s_0\}$. We denote by s_n an increasing sequence in J such that $\lim_{n \to +\infty} s_n = \sup(J)$ and $\lim_{n \to +\infty} \gamma(s_n) = y$. If y is at fiber infinity, then by lemma 12.2.6, we have $\tilde{H}_{p_h} r(y) = \pm 2\Delta_r$ and since $r_\infty \leq r(s)$ for all $s \in (s_0, \sup J)$, we conclude that $\tilde{H}_{p_h} r(y) = 0$ and therefore $r_\infty = r_+$ and $y \in L_+ \cup L_-$ (still by lemma 12.2.6). Moreover, y cannot be in T^*U_I, otherwise we have either $H_{p_h} r \neq 0$ or $H_{p_h}^2 r < 0$ by lemma 12.2.10 and both cases contradict $r_\infty = \inf_{s \in (s_0, \sup J)} r(s)$ (see lemma 12.2.4 for the second case). Therefore, the only possible accumulation points are L_+ and L_- and by lemma 12.2.7, γ cannot have both accumulation points in L_+ and in L_-. Therefore, γ tends to L_+ or to L_- at $\sup J$ (and since γ remains in a compact set $\sup J = +\infty$).

Now we assume that there exists $s_1 > s_0$ such that $r(s_1) < r_+$. By Lemma 12.2.11, $r \circ \gamma$ is then strictly decreasing on $(s_1, \sup J)$. The curve γ cannot have any accumulation point in $\{r \geq r(s_1)\}$ at $\sup J$. Indeed, at such an accumulation point y, $\tilde{H}_{p_h} r(y)$ must vanish to be consistent with the fact that r is decreasing but this never happens when $r < r_+$ (at fiber-infinity it can be ruled out since we show as previously $\tilde{H}_{p_h}(y) = \pm 2\Delta_r$ and elsewhere, it is a consequence of lemma 12.2.11). □

We have an analog of lemma 12.2.13 (with a similar proof) describing the behavior of some bicharacteritics in the past:

Lemma 12.2.14. *Let γ be a bicharacteristic curve defined on an interval J. We assume that there exists $s_0 \in J$ such that $r(s_0) \leq r_{min}$ and $\dot{r}(s_0) > 0$. Then we have two cases:*

- *The curve γ remains in $\{r_+ \leq r\}$ for all $s \in (\inf J, s_0]$ in which case $\inf J = -\infty$ and γ tends to L_+ or to L_-.*
- *γ reach $\{r < r_+\}$ for some $s_1 < s_0$ and in this case $\inf J > -\infty$ and $\lim_{s \to \inf J} r(\gamma(s)) = r_+ - \epsilon$.*

Lemma 12.2.15. *Let γ be a bicharacteristic curve with at least one point in $\{r \leq r_{min}\}$. We assume that for all $s \in J$ such that $\gamma(s) \in \{r \leq r_{min}\}$, we have $\dot{r}(s) \geq 0$. Then one of the following cases holds:*

- *Case 1: There exists $s_0 \in J$ such that $r(s_0) > r_+$. In this case, γ reaches $\{r > r_{min}\}$ in finite time s_1 and remains in this set for all $s_1 < s < \sup J$.*
- *Case 2: For all $s \in J$, $r(s) \leq r_+$. In this case, $\sup J = +\infty$ and γ tends to L_+ or to L_- when $s \to +\infty$.*

Proof. We use the notations of the lemma. Assume that there exits $s_0 \in J$ such that $r(s_0) > r_+$. Then we prove that γ reaches $\{r > r_{\min}\}$ by contradiction. Indeed, if it is not the case, there exists $y \in \overline{T^*U_I}$ with $r(y) = \sup_{s \in (s_0, \sup J)} r(\gamma(s)) > r_+$ and a sequence $s_n \in (s_0, \sup J)$ with $\lim_{n \to +\infty} s_n = \sup J$ and $\lim_{n \to +\infty} \gamma(s_n) = y$. If y is at fiber infinity, by lemma 12.2.6 $\tilde{H} r(y) = \pm 2\Delta_r \neq 0$ which contradicts $r(y) = \sup_{s \in (s_0, \sup J)} r(\gamma(s))$. If y is not at fiber infinity, for the same reason we must have $H_{p_h} r(y) = 0$ and lemma 12.2.10 implies that $H^2_{p_h} r(y) < 0$. But in this case, lemma 12.2.4 provides a point $s \in J$ with $\gamma(s) \in T^*U_I$ and $\dot{r}(s) < 0$ which contradicts the hypothesis on γ. Therefore, such an accumulation point y cannot exist and γ reaches $\{r > r_{\min}\}$ at some time $s_1 \in J$. Moreover, for all $s \in J$ with $s > s_1$, $r(s) > r_{\min}$ (otherwise we get a contradiction at $s_2 = \inf \{s \in J : s > s_1, r(s) = r_{\min}\}$ where we must have $\dot{r}(s_2) = 0$ since by hypthesis $\dot{r}(s_2) \geq 0$ and using lemma 12.2.10).

Assume that for all $s \in J$, $r(s) \leq r_+$. In particular γ remains in a compact set and $\sup J = +\infty$. Then, by compactness, there exists $y \in \overline{T^*U_I}$ with $r(y) = \sup \{r(s), s \in J\} \leq r_+$ (this implies that $\tilde{H}_{p_h} r(y) = 0$) such that y is an accumulation point for γ at $\sup J$. By lemmas 12.2.11 (for the case $r(y) < r_+$) and 12.2.10 (for the case $r(y) = r_+$), y cannot be in T^*U_I therefore y is at fiber infinity. By lemma 12.2.6, the only possibility is $y \in L_\pm$. As usual we use lemma 12.2.7 to prove that γ cannot have accumulation points both in L_+ and in L_-. We conclude that γ tends to L_+ or to L_-. □

We also have an analog of lemma 12.2.15 for past behavior:

Lemma 12.2.16. *Let γ be a bicharacteristic curve with at least one point in $\{r \leq r_{min}\}$. We assume that for all $s \in J$ such that $\gamma(s) \in \{r \leq r_{min}\}$, we have $\dot{r}(s) \leq 0$. Then:*

- *Case 1: There exists $s_0 \in J$ such that $r(s_0) > r_+$. In this case, γ reaches $\{r > r_{min}\}$ in finite time s_1 and remains in this set for all $\inf J < s < s_1$.*
- *Case 2: For all $s \in J$, $r(s) \leq r_+$. In this case, $\inf J = -\infty$ and γ tends to L_+ or to L_- when $s \to -\infty$.*

We now compute the principal symbol near infinity (using the time coordinate ${}^*t = t - T(r)$ on the radial interval $(6M, +\infty)$): We define $\xi = \xi_r \, \mathrm{d}r + \zeta \, \mathrm{d}\phi + \eta \, \mathrm{d}\theta$.

$$\begin{aligned}
p_h(\xi) =& -\rho^2 G(\xi - z_0 \, \mathrm{d}^* t) \\
=& a^2 \sin^2\theta z_0^2 - \left(\frac{a^2}{\Delta_r} - \frac{1}{\sin^2\theta}\right)\zeta^2 + \frac{4Mar}{\Delta_r}\zeta z_0 + \eta^2 + \Delta_r \xi_r^2 + 2(a^2+r^2) z_0 \xi_r \\
=& \eta^2 + \left(\frac{\zeta}{\sin\theta} + a z_0 \sin\theta\right)^2 - \frac{a^2}{\Delta_r}\zeta^2 - 2a\zeta z_0 + \Delta_r \xi_r^2 + 2(a^2+r^2) z_0 \xi_r + \frac{4Mar}{\Delta_r}\zeta z_0 \\
H_{p_h}(\xi) =& \left(2\xi_r \Delta_r + 2 z_0 (a^2+r^2)\right)\partial_r + 2\left(-a^2 z_0^2 \sin\theta + \frac{\zeta^2}{\sin^3\theta}\right)\cos\theta \partial_\eta \\
&+ 2\left(-\frac{2Maz_0\zeta}{\Delta_r} + \frac{a\zeta(M-r)(-4Mrz_0 + a\zeta)}{\Delta_r^2} + \xi_r(-2rz_0 + \xi_r(M-r))\right)\partial_{\xi_r} \\
&+ 2\eta\partial_\theta + 2\left(\frac{2Marz_0}{\Delta_r} - \zeta\left(\frac{a^2}{\Delta_r} - \frac{1}{\sin^2\theta}\right)\right)\partial_\phi
\end{aligned}$$

Remark 12.2.17. *The quantities p_h, ζ and $\eta^2 + \left(\frac{\zeta}{\sin\theta} + a z_0 \sin\theta\right)^2$ are still conserved along the bicharacteristic curves.*

Lemma 12.2.18. *There exists $r_{max} > 6M$ such that in $\{r \geq r_{max}\} \cap p_h^{-1}(\{0\})$, the following implication holds: $H_{p_h} r = 0 \Rightarrow H_{p_h}^2 r > 0$*

Proof. On the set $\{H_{p_h} r = 0\}$, we have:

$$\begin{aligned}
\xi_r =& -\frac{a^2+r^2}{\Delta_r} z_0 \\
=& -z_0 + O(r^{-1}).
\end{aligned}$$

On $\{H_{p_h} r = 0\} \cap p_h^{-1}(\{0\})$, we have:

$$\begin{aligned}
\eta^2 + \left(\frac{\zeta}{\sin\theta} - a z_0 \sin\theta\right)^2 + \left(-2az_0 - \frac{a^2 - 4Marz_0}{\Delta_r}\zeta\right)\zeta =& \Delta_r \xi_r^2 \\
=& r^2 + O(r)
\end{aligned}$$

Therefore, there exists a constant $B > 0$ and $r_{c,1} > 6M$ such that $|\zeta| \leq Br$ on this set for $r \geq r_{c,1}$. Otherwise, there would be a sequence $(r_n, \theta_n, \zeta_n, \eta_n, (\xi_r)_n)$ with $|\zeta_n| \geq n r_n$ and $\lim_{n\to\infty} r_n = +\infty$ and the left-hand side would be larger than $\frac{n^2 r_n^2}{2}$ for r large enough which is impossible. Similarly we can find $B' > 0$ and $r_{c,2} > 6M$ such that $|\eta| \leq B' r$ on this set for $r \geq r_{c,2}$. We can now compute, still on the set $\{H_{p_h} r = 0\} \cap p_h^{-1}(\{0\})$ (on this set the expression simplifies):

$$H_{p_h}^2 r = -8Marz_0\zeta + 4\frac{a\zeta(M-r)}{\Delta_r}\left(-4Mrz_0 + a\zeta\right) + 4\xi_r\left(-2rz_0 + \xi_r(M-r)\right)\Delta_r$$

Using our previous estimates, we see that the dominant term is $-4\xi_r \Delta_r r z_0$ and $H_{p_h}^2 r = 4r^3 + O(r^2)$ therefore, it is positive for r larger than some $r_c > 6M$. □

Remark 12.2.19. *Note that we are interested in the flow of the renormalized vector field* $\tilde{H}_{p_h} := r^{-1}H_{p_h}$ *which extends to a continuous vector field on* ${}^{sc}T^*(6M, +\infty] \times \mathbb{S}^2$ *(we do not need to compactify the fiber since the characteristic set of* p_h *does not intersect fiber infinity) and using smooth (up to the boundary at infinity) coordinates* $x := \frac{1}{r}$, ξ_r, $\eta_{sc} = \frac{\eta}{r}$, $\zeta_{sc} = \frac{\zeta}{r}$ *we find*

$$\begin{aligned}
\tilde{H}_{p_h} =& 2x\left(-\xi_r\left(x^2\Delta_r\right) - z_0\left(a^2x^2+1\right)\right)\partial_x \\
&+ 2\left(-\eta_{sc}\left(\xi_r\left(x^2\Delta_r\right) + z_0\left(a^2x^2+1\right)\right) + \left(-a^2x^2z_0^2\sin(\theta) + \frac{\zeta_{sc}^2}{\sin^3\theta}\right)\cos(\theta)\right)\partial_{\eta_{sc}} \\
&- 2\zeta_{sc}\left(\xi_r\left(x^2\Delta_r\right) + z_0\left(a^2x^2+1\right)\right)\partial_{\zeta_{sc}} \\
&+ \frac{2}{\left(x^2\Delta_r\right)^2}\left(-2Max^2z_0\zeta_{sc}\left(x^2\Delta_r\right) + ax^2\zeta_{sc}\left(Mx-1\right)\left(-4Mz_0 + a\zeta_{sc}\right)\right. \\
&\qquad\left. +\xi_r\left(\xi_r\left(Mx-1\right) - 2z_0\right)\left(x^2\Delta_r\right)^2\right)\partial_{\xi_r} \\
&+ 2\eta_{sc}\partial_\theta + \frac{2\left(2Max^2z_0\sin^2(\theta) + \zeta_{sc}\left(2Mx - a^2x^2\cos^2(\theta) - 1\right)\right)}{\left(x^2\Delta_r\right)\sin^2(\theta)}\partial_\phi
\end{aligned}$$

Therefore, the restriction of $\tilde{H}_{p_h}$ *to the boundary* $\{x = 0\}$ *is*

$$\begin{aligned}
2\xi_r\left(-\xi_r - 2z_0\right)\partial_{\xi_r} + \left(-2\eta_{sc}\xi_r - 2\eta_{sc}z_0 + \frac{2\zeta_{sc}^2\cos(\theta)}{\sin^3(\theta)}\right)\partial_{\eta_{sc}} + 2\zeta_{sc}\left(-\xi_r - z_0\right)\partial_{\zeta_{sc}} \\
+2\eta_{sc}\partial_\theta - \frac{2\zeta_{sc}}{\sin^2(\theta)}\partial_\phi
\end{aligned}$$

As expected, it vanishes only for $(\xi_r, \zeta_{sc}, \eta_{sc}) \in \{(0,0,0), (-2z_0, 0, 0)\}$. *We define* $\mathcal{R}_{in} := \{\xi_r = \zeta_{sc} = \eta_{sc} = x = 0\}$ *and* $\mathcal{R}_{out} := \{\xi_r = -2z_0, \zeta_{sc} = \eta_{sc} = x = 0\}$.

Lemma 12.2.20. *Let* γ *be a bicharacteristic curve of the renormalized Hamiltonian vector field maximally defined on an interval* J. *If, there exists* $s_0 \in J$ *such that* $\dot{r}(s_0) > 0$ *and* $r(s_0) \geq r_{max}$, *then* $\sup J = +\infty$ *and* $\gamma(s)$ *converges to* $\mathcal{R}_{in}$ *or to* $\mathcal{R}_{out}$ *when* $s \to +\infty$.

Proof. By the previous lemma, $\dot{r}(s) > 0$ for all $s > s_0$ (and $\gamma([s_0, \sup J)) \subset \{r_{\max} \leq r \leq +\infty\}$). Therefore $r(s)$ has a limit. First, we prove by contradiction that this limit is $+\infty$. Assume that $\lim_{s\to\sup J} r(s) = r_\infty < +\infty$. Then the curve remains in the compact set $\{r_{\max} \leq r \leq +\infty\} \cap p_h^{-1}(\{0\})$ and has therefore an accumulation point y with $r(y) = r_\infty$. However, $H_{p_h}r(y) = 0$ (otherwise, γ should cross the hypersurface $\{r = r_\infty\}$ and therefore by lemma 12.2.18, we have $H_{p_h}^2 r > 0$ but this contradicts the fact $r_\infty = \sup_{s\in[s_0,\sup J)}\gamma(s)$ by lemma 12.2.4.

Therefore, $r_\infty = +\infty$. Then by conservation of $\eta^2 + \left(\frac{\zeta}{\sin\theta} - az_0\sin\theta\right)^2$ along the flow, we have $\lim_{s\to+\infty}\eta_{sc}(s) = 0$ and $\lim_{s\to+\infty}\zeta_{sc}(s) = 0$. Finally, using that $p_h = 0$ along the flow, we get that $\lim_{s\to+\infty}\xi_r(s)(\xi_r(s) + 2z_0) = 0$ and using the continuity of $\xi_r(s)$, either $\lim_{s\to+\infty}\xi_r(s) = 0$ or $\lim_{s\to+\infty}\xi_r(s) = -2z_0$. □

The analog of lemma 12.2.20 for past behavior is

Lemma 12.2.21. *Let* γ *be a bicharacteristic curve maximally defined on an interval* J. *If, there exists* $s_0 \in J$ *such that* $\dot{r}(s_0) < 0$ *and* $r(s_0) \geq r_{max}$, *then* $\inf J = -\infty$ *and* $\gamma(s)$ *converges to* $\mathcal{R}_{in}$ *or to* $\mathcal{R}_{out}$ *when* $s \to -\infty$.

To cover the behavior of all bicharacteristic curves in T^*U_I we need the following lemma

Lemma 12.2.22. *Let γ be a bicharacteristic curve maximally defined on an interval J. We assume that for all $s \in J$ such that $r(s) \geq r_{max}$ we have $\dot{r}(s) \leq 0$. Then γ reaches $\{r < r_{max}\}$ for some $s_0 \in J$ and stays in this set for $s_0 \leq s < \sup J$.*

Proof. As in the proof of lemma 12.2.20, we can prove using lemma 12.2.18 and lemma 12.2.4 that γ cannot have an accumulation point at $\sup J$ in $\{r \geq r_{\max}\}$ (in this case, accumulation points at $r = +\infty$ are impossible since $r \circ \gamma$ is decreasing). Therefore, γ reaches $\{r < r_{\max}\}$ in finite time s_0. Moreover, if $r(s) \geq r_{\max}$ for some $s > s_0$, then we could define $s_1 = \inf\{s_0 < s < \sup J : r(s) = r_{\max}\}$. Since $\dot{r}(s_1) \leq 0$ by hypothesis and cannot be negative me must have $\dot{r}(s_1) = 0$ and by lemma 12.2.18 and lemma 12.2.4 we get a contradiction. □

Similarly we have

Lemma 12.2.23. *Let γ be a bicharacteristic curve maximally defined on an interval J. We assume that for all $s \in J$ such that $r(s) \geq r_{max}$ we have $\dot{r}(s) \geq 0$. Then γ reaches $\{r < r_{max}\}$ in finite time in the past and stays in this set.*

We now compute the principal symbol near the trapped set (using the t time coordinate on the radial interval $I = (r_{\min}, r_{\max})$): We use coordinates $\xi = \xi_r \, \mathrm{d}r + \zeta \, \mathrm{d}\phi + \eta \, \mathrm{d}\theta$ for cotangent vectors and we introduce the function $\alpha = \frac{-(r^2+a^2)z_0 + a\zeta}{\Delta_r}$.

$$\begin{aligned}
p_h(\xi) =& -\rho^2 G(\xi - z_0 \, \mathrm{d}t) \\
=& -\left(\frac{(a^2+r^2)^2}{\Delta_r} - a^2 \sin^2\theta\right) z_0^2 - \left(\frac{a^2}{\Delta_r} - \frac{1}{\sin^2\theta}\right)\zeta^2 + \left(\frac{2a(a^2+r^2)}{\Delta_r} - 2a\right)\zeta z_0 \\
&+ \eta^2 + \Delta_r \xi_r^2 \\
=& \Delta_r(\xi_r^2 - \alpha^2) + \eta^2 + \left(-az_0 \sin\theta + \frac{\zeta}{\sin\theta}\right)^2 \\
H_{p_h} =& 2\xi_r \Delta_r \partial_r + \left(\alpha(-4rz_0 - 2(r-M)\alpha) - 2\xi_r^2(r-M)\right)\partial_{\xi_r} + 2\eta\partial_\theta \\
&+ 2\cos\theta\left(\frac{\zeta^2}{\sin^3\theta} - a^2 z_0^2 \sin\theta\right)\partial_\eta + \frac{2}{\Delta_r}\left(2Maz_0 - a^2\zeta + \frac{\Delta_r}{\sin^2\theta}\zeta\right)\partial_\phi
\end{aligned}$$

Remark 12.2.24. *We still have the conserved quantities: ζ, p_h and $\eta^2 + \left(-az_0 \sin\theta + \frac{\zeta}{\sin\theta}\right)^2$.*

Note that due to Lemmas 12.2.13, 12.2.14, 12.2.20 and 12.2.21, we see that bicharacteristic curves leaving $(r_{\min}, r_{\max})$ exit any compact subset of $(r_+, +\infty) \times \mathbb{S}^2$ either in the past or in the future. Therefore, trapped geodesics remain in T^*U_I for all time. Proposition 3.2 of [27] translates[3] in our context to:

Proposition 12.2.25. *Let K_{z_0} be the union of bicharacteristic curves of $p_h(z_0)$ which are contained in T^*U_I with $I = (r_{min}, r_{max})$. Then*

$$K_{z_0} = \{G(\xi - z_0 \, \mathrm{d}t) = \xi_r = ((r-M)\alpha + 2rz_0)\Delta_r = 0\}.$$

Proposition 3.5 of [27] translates to

Proposition 12.2.26. *Let $\Gamma_\pm$ be the union of bicharacteristic curves that are contained in T^*U_I in the future (-) or in the past (+). We denote by $\hat{K}$ the projection of K_{z_0} on $\mathbb{S}^2 \times \mathbb{R}^2_{\zeta,\eta}$. Then we have*

$$\Gamma_\pm = \left\{(r, \hat{x}, \xi_r, \hat{\xi}) : (\hat{x}, \hat{\xi}) \in \hat{K}, \xi_r = \pm sgn(r - r'_{\hat{x},\hat{\xi}})\sqrt{\frac{\Phi_{\hat{x},\hat{\xi}}(r)}{\Delta_r}}\right\}$$

[3]For more details about the translation from properties of the rescaled geodesic flow and the semiclassical flow see [27] section 2.3

where

$$\Phi_{\hat{x},\hat{\xi}} = -\left(\eta^2 + \left(-az_0\sin\theta + \frac{\zeta}{\sin\theta}\right)^2\right) + \Delta_r\alpha^2$$

and $r'_{\hat{x},\hat{\xi}}$ *is the unique solution of* $\Phi_{\hat{x},\hat{\xi}}(r) = 0$.

Finally, we need the following lemma which is implicit in [27].

Lemma 12.2.27. *Let* γ *be a bicharacteristic curve included in* Γ_- *(resp.* Γ_+*). We define* $\hat{x} := (\theta,\phi)(\gamma(0))$ *and* $\hat{\xi} := (\zeta,\eta)(\gamma(0))$*. Then,* γ *is defined on an interval* J *with* $\sup(J) = +\infty$ *(resp.* $\inf(J) = -\infty$*) and*

$$\begin{aligned}\forall s \in J, (\theta,\phi,\eta,\zeta)(\gamma(s)) &\in \hat{K}\\ \xi_r(\gamma(s)) &\xrightarrow[s\to\pm\infty]{} 0\\ r(\gamma(s)) &\xrightarrow[s\to\pm\infty]{} r'_{\hat{x},\hat{\xi}}\end{aligned}$$

Therefore γ *is included in any neighborhood of* K_{z_0} *for* s *large enough (resp. for* $-s$ *large enough).*

Proof. The fact that for all $s \in J$, $(\theta,\phi,\eta,\zeta)(\gamma(s)) \in \hat{K}$ is a consequence of the fact that $\Gamma_\pm$ are stable by the flow and of the explicit expression given in proposition 12.2.26. Moreover, we see that $\Phi_{\hat{x},\hat{\xi}}$ only depends on $\hat{x}$ and $\hat{\xi}$ through $\eta^2 + \left(-az_0\sin\theta + \frac{\zeta}{\sin\theta}\right)^2$ and ζ which are constant along the flow. Therefore, for all $s \in J$, $\xi_r(\gamma(s)) = \pm\mathrm{sgn}(r - r'_{\hat{x},\hat{\xi}})\sqrt{\frac{\Phi_{\hat{x},\hat{\xi}}(r)}{\Delta_r}}$. Note that

$$\begin{aligned}\Phi_{\hat{x},\hat{\xi}}(\gamma(s)) =& \Delta_r\xi_r^2 - p_h\\ =& \Delta_r\xi_r(\gamma(s))^2\end{aligned}$$

Therefore, it is enough to prove the convergence of $r(\gamma(s))$ to $r'_{\hat{x},\hat{\xi}}$, the convergence of $\xi_r(\gamma(s))$ will follow from the fact that $\Phi_{\hat{x},\hat{\xi}}(r'_{\hat{x},\hat{\xi}}) = 0$. We do the case of $\gamma \subset \Gamma_-$, the other case is similar. To alleviate the notation, we define $r(s) := r(\gamma(s))$ and $\xi_r(s) := \xi_r(\gamma(s))$ and we denote by a dot the derivative with respect to s. For all $s \in J$, we have

$$\begin{aligned}\dot{r}(s) =& H_{p_h} r(\gamma(s))\\ =& 2\Delta_r\xi_r(s)\\ =& -\mathrm{sgn}(r - r'_{\hat{x},\hat{\xi}})\sqrt{\Delta_r\Phi_{\hat{x},\hat{\xi}}(r(s))}\end{aligned} \tag{12.15}$$

In particular $\dot{r}$ is negative when $r > r'_{\hat{x},\hat{\xi}}$ and positive when $r < r'_{\hat{x},\hat{\xi}}$. Moreover, at $r'_{\hat{x},\hat{\xi}}$ which is its only vanishing point, $\Phi_{\hat{x},\hat{\xi}}$ vanishes at order 2 (see the discussion before proposition 3.5 in [27]) therefore, the right-hand side is locally Lipschitz with respect to r. We conclude that we can solve (12.15) with initial condition $r(0)$ on an interval $\tilde{J}$ with $\sup\tilde{J} = +\infty$ and $\lim_{s\to+\infty} r(s) = r'_{\hat{x},\hat{\xi}}$. But it means that $\gamma(s)$ remains in a compact set when $s \to \sup J$ (the characterstic set of p intesected with $\left\{r'_{\hat{x},\hat{\xi}} - \epsilon \le r \le r'_{\hat{x},\hat{\xi}} + \epsilon\right\}$ is compact for ϵ small enough). Therefore $\sup J = +\infty$ and by uniqueness of the solution of (12.15), $\lim_{s\to+\infty} r(\gamma(s)) = r'_{\hat{x},\hat{\xi}}$.

Finally, the last statement of the lemma follows from the fact that

$$F := \left\{(r'_{\hat{x},\hat{\xi}}, (\theta,\phi)(\gamma(s)), 0, (\zeta,\eta)(\gamma(s))), s \in J\right\} \subset K_{z_0}$$

and $\lim_{s\to+\infty} d(\gamma(s), F) = 0$ for any distance d inducing the topology of T^*U_I. □

We can now completely describe the asymptotic behavior of bicharacteristic curves:

Proposition 12.2.28. *We define the surface $B_\epsilon = \{r = r_+ - \epsilon\} \subset {}^{sc}\overline{T}^*(r_+ - 2\epsilon, +\infty] \times \mathbb{S}^2$. In this proposition, we say that a curve γ defined on some interval J is of type (A, B) where A and B are two sets in $\{L_-, L_+, B_\epsilon, \mathcal{R}_{in}, \mathcal{R}_{out}\}$ if γ tends to A at $\inf J$ and to B at $\sup J$. Moreover, if $A = B_\epsilon$ (resp. $B = B_\epsilon$), $\inf J > -\infty$ (resp. $\sup J < +\infty$), in all the other case the corresponding bound of J is infinite.*

Let γ be a bicharacteristic curve for the renormalized Hamiltonian flow on ${}^{sc}\overline{T}^*(r_+ - \epsilon, +\infty] \times \mathbb{S}^2$ maximally defined on some interval J.

Let $z_0 = -1$.

- *If $\gamma \subset \Sigma_+$, then either $\gamma \subset L_+$ or γ is of type (L_+, B_ϵ).*
- *If $\gamma \subset \Sigma_-$, then either $\gamma \subset L_- \cup K_{z_0} \cup \mathcal{R}_{in} \cup \mathcal{R}_{out}$ or γ is of type $(\mathcal{R}_{in}, L_-)$, $(\mathcal{R}_{in}, K_{z_0})$, $(\mathcal{R}_{in}, \mathcal{R}_{out})$ or (B_ϵ, L_-), (B_ϵ, K_{z_0}), $(B_\epsilon, \mathcal{R}_{out})$, (K_{z_0}, L_-), $(K_{z_0}, \mathcal{R}_{out})$*

Let $z_0 = 1$.

- *If $\gamma \subset \Sigma_+$, then either $\gamma \subset L_+ \cup K_{z_0} \cup \mathcal{R}_{in} \cup \mathcal{R}_{out}$ or γ is of type (L_+, B_ϵ), (L_+, K_{z_0}), $(L_+, \mathcal{R}_{in})$, $(\mathcal{R}_{out}, K_{z_0})$, $(\mathcal{R}_{out}, B_\epsilon)$, $(\mathcal{R}_{out}, \mathcal{R}_{in})$,$(K_{z_0}, B_\epsilon)$, $(K_{z_0}, \mathcal{R}_{in})$.*
- *If $\gamma \subset \Sigma_-$, then either $\gamma \subset L_-$ or γ is of type (B_ϵ, L_-).*

Proof. This proposition is a combination of lemmas 12.2.13, 12.2.14, 12.2.15, 12.2.16, 12.2.27, 12.2.20, 12.2.21 and of the following observations about the Hamiltonian vector field (which rule out some cases which are a priori compatible with the lemmas):

- If $z_0 = -1$, L_+ is a source, L_- is a sink, $\mathcal{R}_{in}$ is a source, $\mathcal{R}_{out}$ is a sink.
- If $z_0 = 1$, L_+ is a source, L_- is a sink, $\mathcal{R}_{in}$ is a sink, $\mathcal{R}_{out}$ is a source.
- Bicharacteristic curves in Σ_- only cross the horizon $\{r = r_+\}$ towards the exterior of the black hole (see lemma 12.2.10)
- Bicharacteristic curves in Σ_+ only cross the horizon $\{r = r_+\}$ towards the interior of the black hole (see lemma 12.2.10)

□

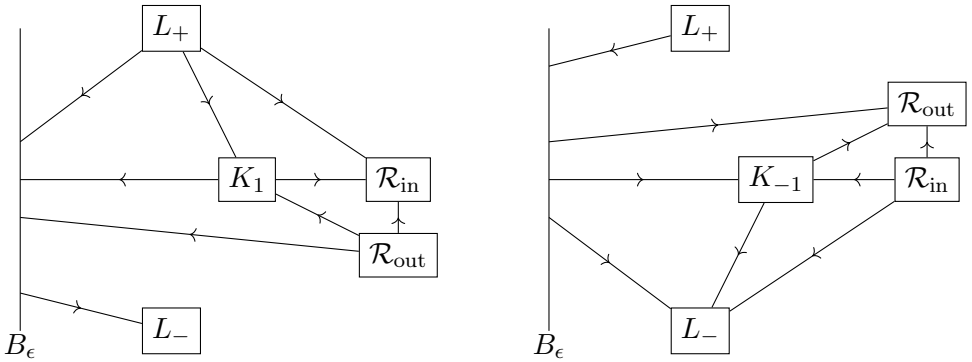

Figure 12.1: Structure of the semiclassical Hamiltonian flow (for $z_0 = 1$ on the left and $z_0 = -1$ on the right)

Finally, we state the following lemma for later use. It is a consequence of lemma 2.4 in [28] (the fact that normally hyperbolic trapping assumptions hold is proven in [27]).

Lemma 12.2.29. *For all $\epsilon > 0$ small enough, there exists a neighborhood U of K_{z_0} and ϕ_+, ϕ_- smooth functions on U such that:*

- $\{\phi_+ = 0\} = \Gamma^+ \cap U$
- $\{\phi_- = 0\} = \Gamma^- \cap U$
- *There exists $\delta > 0$ such that, $U_\delta := \{|\phi_+| < \delta, |\phi_-| < \delta, |p| < \delta\}$ is compactly contained in U.*
- $\{\phi_+, \phi_-\} > 0$ *on* U
- $H_p\phi_\pm = \mp c_\pm \phi_\pm$ *with $c_\pm$ smooth positive bounded functions on U with $c_\pm > \nu_{\min} - \epsilon$ where $\nu_{min} > 0$ is defined in Proposition 3.7 of [27] (but we will not use its exact value).*

Chapter 13

Fredholm property of $\hat{T}_s(\sigma)$

We have seen previously (see Corollary 11.1.4 and Corollary 11.2.5) that the Cauchy problem can be reduced to a forcing problem. After a Fourier transformation, we get the problem:

$$\hat{T}_s(\sigma)\hat{v}(\sigma) = \hat{f}(\sigma)$$

where $\hat{f}$ is precisely characterized. To recover properties of $\hat{v}$, we want to write it as $\hat{v}(\sigma) = \hat{T}_s(\sigma)^{-1}\hat{f}(\sigma)$. In this section, we prove that the operator $\hat{T}_s(\sigma)$ is Fredholm between appropriate spaces. This together with a mode stability result (see [108], [8] and [7]) will provide the invertibility of $\hat{T}_s(\sigma)$ between suitable spaces.

13.1 Estimate near the horizon

To define the adjoint operator, we use the volume form $\mathrm{d}vol$ and the metric $\mathbb{m}$ (defined in Subsection 10.2). We now compute the subprincipal symbol, that is to say the principal symbol of $\frac{1}{2i}\left(\hat{T}_s(\sigma) - \hat{T}_s(\sigma)^*\right)$ near the horizon:

$$p_{sub} = \mathfrak{s}\left(\frac{1}{2i}\left(\hat{T}_s(\sigma) - \hat{T}_s(\sigma)^*\right)\right) = \left(-2a\Im(\sigma) + 2\frac{a}{r}\right)\zeta + 2\left((r-M)s + \frac{\Delta_r}{r} - (a^2+r^2)\Im(\sigma)\right)\xi_r$$

Proposition 13.1.1. *Let $k > m \geq \frac{1}{2} - \left(-s + \frac{(a^2+r_+^2)}{r_+-M}\Im(\sigma)\right)$ If $A, B, G \in \Psi_b^{0,0}$ with compactly supported Schwartz kernels are such that A and G are elliptic on $L_\pm$ and every forward (or backward) classical bicharacteristic curve from $WF(B)$ tends to $L_\pm$ with closure in the elliptic set of G,*

$$Au \in \overline{H}_{(b)}^{m,l} \Rightarrow \|Bu\|_{\overline{H}_{(b)}^{k,l}} \leq C\left(\left\|G\hat{T}_s(\sigma)u\right\|_{\overline{H}_{(b)}^{k-1,l}} + \|u\|_{\overline{H}_{(b)}^{-N,l}}\right)$$

Proposition 13.1.2. *Let $k < \frac{1}{2} + \left(-s + \frac{(a^2+r_+^2)}{r_+-M}\Im(\sigma)\right)$. Assume $A \in \Psi_b^{0,0}$ with compactly supported Schwartz kernel is elliptic on a neighborhood of $L_\pm$, $B \in \Psi_b^{0,0}$ and $G \in \Psi_b^{0,0}$ with compactly supported Schwartz kernels and G is elliptic on $WF(A)\cup WF(B)$. Assume also that every forward (or backward) bicharacteristic curve of $\hat{T}(\sigma)$ from a point of $WF(A)$ reaches $Ell(B)$ while remaining in $Ell(G)$. Then*

$$\|Au\|_{\dot{H}_{(b)}^{k,l}} \leq C\left(\left\|G\hat{T}_s(\sigma)^*u\right\|_{\dot{H}_{(b)}^{k-1,l}} + \|Bu\|_{\dot{H}_{(b)}^{k,l}} + \|u\|_{\dot{H}_{(b)}^{-N,l}}\right)$$

Remark 13.1.3. *These estimates are microlocalized away from $\partial\overline{X}$. As a consequence, the index l does not play any role here and could be replaced in the right hand sides by $l - N$ for any $N \in \mathbb{N}$.*

Proof. These two propositions are a mild modification of propositions 2.3 and 2.4 in [103]. There are only two differences:

1. The proposition in [103] is not stated for operators on sections of a complex vector bundles. However, in the case of a complex vector bundle of rank 1 (as it is the case here), the principal symbols of operators are scalar and the same proof can be applied (see remark 2.1 in [103]).

2. The subprincipal symbol p_{sub} of the operator $\hat{T}_s(\sigma)$ has not exactly the form required [103]. However, what is really needed in the proof is the positivity of $k-\frac{1}{2}-\frac{p_{sub}\tilde{\rho}}{\beta_0}$ in case of proposition 2.3 and the negativity of $k-\frac{1}{2}+\frac{p_{sub}\tilde{\rho}}{\beta_0}$ in case of proposition 2.4 where by definition $H\rho_{|L_\pm}=\pm\beta_0\rho$ (as mentioned previously, we have taken into account the different sign convention for the principal symbol in [103]). As a consequence, the proof works if we adapt the hypothesis on k as we did.

□

We also get the corresponding semiclassical estimates given by propositions 2.10 and 2.11 of [103] which we state for the operator $\hat{T}_{s,h}(z)=h^2\hat{T}_s(h^{-1}z)$ where $z=\frac{\sigma}{|\sigma|}$, $h=|\sigma|^{-1}$. The estimates are uniform when σ lies in the region $\{0\leq\Im(\sigma)\leq\eta,|\Re(\sigma)|\geq\eta\}$ with $\eta>0$.

Proposition 13.1.4. *Let $k>m\geq\frac{1}{2}-\left(-s+\frac{(a^2+r_+^2)}{r_+-M}\Im(h^{-1}z)\right)$. If $A,B,G\in\Psi^{0,0}_{b,h}$ with compactly supported Schwartz kernels are such that A and G are elliptic on $L_\pm$ and every forward (or backward) bicharacteristic curve from $WF_h(B)$ tends to $L_\pm$ with closure in the elliptic set of G,*

$$Au\in\overline{H}^{m,l}_{(b),h}\Rightarrow\|Bu\|_{\overline{H}^{k,l}_{(b),h}}\leq C\left(h^{-1}\left\|G\hat{T}_{s,h}(z)u\right\|_{\overline{H}^{k-1,l}_{(b),h}}+h\|u\|_{\overline{H}^{-N,l}_{(b),h}}\right)$$

Proposition 13.1.5. *Let $k<\frac{1}{2}+\left(-s+\frac{(a^2+r_+^2)}{r_+-M}\Im(h^{-1}z)\right)$. Assume $A\in\Psi^{0,0}_{b,h}$ with compactly supported Schwartz kernel is elliptic on a neighborhood of $L_\pm$, $B\in\Psi^{0,0}_{b,h}$ and $G\in\Psi^{0,0}_{b,h}$ with compactly supported Schwartz kernels and G is elliptic on $WF_h(A)\cup WF_h(B)$. Assume also that every forward (or backward) bicharacteristic curve of $\hat{T}_{s,h}(z)$ from a point of $WF_h(A)$ reaches $Ell(B)$ while remaining in $Ell(G)$. Then*

$$\|Au\|_{\dot{H}^{k,l}_{(b),h}}\leq C\left(h^{-1}\left\|G\hat{T}_{s,h}(z)^*u\right\|_{\dot{H}^{k-1,l}_{(b),h}}+\|Bu\|_{\dot{H}^{k,l}_{(b),h}}+h\|u\|_{\dot{H}^{-N,l}_{(b),h}}\right)$$

We now look at the operator $\hat{T}_s(\sigma)$ on $\left\{r<r_+-\frac{\epsilon}{3}\right\}$. To estimate the solution in this region, we use classical hyperbolic estimates (following an idea presented in [111, Section 3.2]). We need the two following propositions:

Proposition 13.1.6. *Let $u\in\dot{H}^{m,l}_{(b)}$. We denote by $\tilde{H}^s$ the space of distributions on $(-\infty,r_+-\frac{\epsilon}{3})$ which are supported in $(r_+-\epsilon,r_+-\frac{\epsilon}{3})$ and extendible at $r_+-\frac{\epsilon}{3}$ endowed with the corresponding norm. In particular we have $v:=u_{|_{r<r_+-\frac{\epsilon}{3}}}\in\tilde{H}^s$. With a slight abuse of notation, we write $\|u\|_{\tilde{H}^{s-1}}$ for $\|v\|_{\tilde{H}^{s-1}}$. For all $s\in\mathbb{R}$, we have*

$$\|u\|_{\tilde{H}^s}\leq C\left\|\hat{T}_s(\sigma)^*u\right\|_{\tilde{H}^{s-1}}$$

(with the convention that some terms may be infinite).

The next proposition is similar but with the extendible and supported ends inverted.

Proposition 13.1.7. *Let $u \in \overline{H}^{m,l}_{(b)}$ with $supp(u) \subset \left\{r \leq r_+ - \frac{\epsilon}{3}\right\}$. Then for all $s \in \mathbb{R}$, we have*

$$\|u\|_{\overline{H}^{s,l}_{(b)}} \leq C \left\|\hat{T}_s(\sigma)u\right\|_{\overline{H}^{s-1,l}_{(b)}}$$

(with the convention that some terms may be infinite).

These two propositions are consequences of the standard hyperbolic theory for second order partial differential operators. Indeed, on $\left\{r_+ - 2\epsilon < r < r_+ - \frac{\epsilon}{3}\right\}$, the operator $\hat{T}_s(\sigma)$ has (classical) principal symbol:

$$\mathfrak{s}(\hat{T}_s(\sigma)) = -\tilde{G}(\xi)$$

where $\tilde{G}$ is a Lorentzian metric. Moreover $-\tilde{G}(\mathrm{d}r) = \Delta_r$ which is uniformly negative on $\left\{r_+ - 2\epsilon < r < r_+ - \frac{\epsilon}{3}\right\}$. Therefore, $\hat{T}_s(\sigma)$ is strictly hyperbolic with respect to the level sets of r in this region. We also need a semiclassical version of the hyperbolic estimate (see Proposition A.1.7). Since it is less standard than classical hyperbolic estimate, we provide a detailed proof (in a more general setting) in the appendix, section A. This proof can easily be adapted to obtain a proof of Propositions 13.1.6 and 13.1.7.

We can now combine the previous results to get a complete microlocal estimate near the horizon (and more precisely in the full region $\{r \leq 2M\}$)

Proposition 13.1.8. *Let N and N' be integers which will be taken large. Let $k' > \frac{1}{2} + s - \frac{a^2+r_+^2}{r_+-M}\Im(\sigma)$. Let $u \in \overline{H}^{k',\infty}_{(b)}$ such that $supp(u) \subset \{r \leq 2M + D\}$ (for some $D > 0$) and $v \in \dot{H}^{-N,\infty}_{(b)}$ such that $supp(v) \subset \{r \leq 2M + D\}$. Let k be such that $k > k'$. We have*

$$\|u\|_{\overline{H}^{k,l}_{(b)}} \leq C \left(\left\|\hat{T}_s(\sigma)u\right\|_{\overline{H}^{k-1,-N'}_{(b)}} + \|u\|_{\overline{H}^{k',-N'}_{(b)}} \right)$$

$$\|v\|_{\dot{H}^{1-k,l}_{(b)}} \leq C \left(\left\|\hat{T}_s(\sigma)^* v\right\|_{\dot{H}^{-k,-N'}_{(b)}} + \|v\|_{\dot{H}^{-N,-N'}_{(b)}} \right)$$

We first need a separation lemma (following from the analysis of the dynamical structure)

Lemma 13.1.9. *There exists disjoint open subsets U_+ and U_- of $\overline{T}^*\mathcal{M}_\epsilon$ such that $\Sigma \cap \left\{r \geq r_+ - \frac{\epsilon}{2}\right\} \subset U_+ \cup U_-$, $L_+ \subset U_+$ and $L_- \subset U_-$ and*

- *For every $x \in U_+ \cap \Sigma$, the bicharacteristic curve through x tends to L_+ at $-\infty$ and is included in $\left\{r < r_+ - \frac{\epsilon}{2}\right\}$ in the future.*
- *For every $x \in U_- \cap \Sigma$, the bicharacteristic curve through x tends to L_- at $+\infty$ and is included in $\left\{r < r_+ - \frac{\epsilon}{2}\right\}$ in the past.*

Proof. In view of the dynamical analysis obtained in Proposition 12.1.7, we can take $U_1 = \left\{\tilde{\xi} > 0, \rho < 1\right\}$ and $U_2 = \left\{\tilde{\xi} < 0, \rho < 1\right\}$. □

Proof of proposition 13.1.8. These estimates are proved by combining the previous estimates in this section. We first prove the first one: Let $\Sigma_\pm := \Sigma \cap \left\{r \geq r_+ - \frac{\epsilon}{2}\right\} \cap U_\pm$ (where $U_\pm$ are defined in lemma 13.1.9) are compact subsets of $\overline{T}^*\mathcal{M}_\epsilon$. We cover them by a finite number of open sets $(U_i)_{i=1}^N$ included in $U_\pm$. Let $B_i \in \Psi_b^{0,0}$ with compactly supported Schwartz kernels

such that $WF(B_i) \subset U_i$ and $WF(I - \sum_{i=1}^N B_i) \cap \Sigma \subset \left\{ r < r_+ - \frac{\epsilon}{2} \right\}$. Let $\chi \in C^\infty(\mathbb{R})$ be such that $\chi = 1$ on $r < r_+ - \frac{\epsilon}{2}$ and $\chi = 0$ on $\left\{ r > r_+ - \frac{\epsilon}{3} \right\}$. We have:

$$\|u\|_{\overline{H}^{k,l}_{(b)}} \leq \sum_{i=1}^N \|B_i u\|_{\overline{H}^{k,l}_{(b)}} + C \|\chi u\|_{\overline{H}^{k,l}_{(b)}} + \left\| \left(I - \sum_{i=1}^N B_i \right) (1-\chi) u \right\|_{\overline{H}^{k,l}_{(b)}}$$

We have $\|B_i u\|_{\overline{H}^{k,l}_{(b)}} \leq C \left\| \hat{T}_s(\sigma) u \right\|_{\overline{H}^{k-1,-N'}_{(b)}} + \|u\|_{\overline{H}^{-N,-N'}_{(b)}}$ using proposition 13.1.1 for each B_i (here we use the regularity assumption on u). Moreover $\left\| (I - \sum_{i=1}^N B_i)(1-\chi) u \right\|_{\overline{H}^{k,l}_{(b)}} \leq \left\| \hat{T}_s(\sigma) u \right\|_{H^{k-1,l}_{(b)}} + \|u\|_{\overline{H}^{-N,-N'}_{(b)}}$ using proposition 10.3.8 (elliptic estimate) since $WF((I - \sum_{i=1}^N B_i)(1-\chi)) \cap \Sigma = \emptyset$.

Eventually, we have to bound the term $\|\chi u\|_{\overline{H}^{k,l}_{(b)}}$. Using proposition 13.1.7, we get $\|\chi u\|_{\overline{H}^{k,l}_{(b)}} \leq \left\| \hat{T}_s(\sigma) \chi u \right\|_{\overline{H}^{k-1,l}_{(b)}}$. Note that

$$\begin{aligned} \left\| \hat{T}_s(\sigma) \chi u \right\|_{\overline{H}^{k-1,l}_{(b)}} &\leq \left\| \hat{T}_s(\sigma) u \right\|_{\overline{H}^{k-1,-N'}_{(b)}} + \left\| [\hat{T}, \chi] u \right\|_{\overline{H}^{k-1,-N'}_{(b)}} \\ &\leq \left\| \hat{T}_s(\sigma) u \right\|_{\overline{H}^{k-1,-N'}_{(b)}} + C \|\tilde{\chi} u\|_{\overline{H}^{k,-N'}_{(b)}} \end{aligned}$$

where $\tilde{\chi} \in C_c^\infty$ and $\text{supp}\tilde{\chi} \subset \left\{ r_+ - \frac{\epsilon}{2} \leq r \leq r_+ - \frac{\epsilon}{3} \right\}$. Therefore

$$\begin{aligned} \|\tilde{\chi} u\|_{\overline{H}^{k,-N'}_{(b)}} &\leq \sum_{i=1}^N \|B_i \tilde{\chi} u\|_{\overline{H}^{k,-N'}_{(b)}} + \left\| \left(I - \sum_{i=1}^N B_i \right) \tilde{\chi} u \right\|_{H^{k,-N'}_{(b)}} \\ &\leq C \left\| \hat{T}_s(\sigma) u \right\|_{\overline{H}^{k-1,-N'}_{(b)}} + \|u\|_{H^{-N,-N'}_{(b)}} \end{aligned}$$

where the last line is obtained using proposition 13.1.1 for each $B_i \tilde{\chi}$ and proposition 10.3.8 for the last term.

We now prove the second estimate: We define B_i and χ in the same way but we arrange that B_0 and B_1 are such that $WF(I - B_0) \cap L_+ = \emptyset$ and $WF(I - B_1) \cap L_- = \emptyset$. We have

$$\|u\|_{\dot{H}^{1-k,l}_{(b)}} \leq \sum_{i=1}^N \|B_i u\|_{\dot{H}^{1-k,l}_{(b)}} + C \|\chi u\|_{\dot{H}^{1-k,-N'}_{(b)}} + \left\| \left(I - \sum_{i=1}^N B_i \right) (1-\chi) u \right\|_{\dot{H}^{1-k,-N'}_{(b)}}$$

Using proposition 13.1.2 for B_0 and B_1 and usual propagation of singularity for the other B_i, we get $\sum_{i=1}^N \|B_i u\|_{\dot{H}^{1-k,l}_{(b)}} \leq \left\| \hat{T}_s(\sigma)^* u \right\|_{\dot{H}^{-k,-N'}_{(b)}} + \|\chi u\|_{\dot{H}^{1-k,-N'}_{(b)}}$. As previously, using proposition 10.3.8, we can write $\left\| (I - \sum_{i=1}^N B_i)(1-\chi) u \right\|_{\dot{H}^{1-k,-N'}_{(b)}} \leq \left\| \hat{T}_s(\sigma)^* u \right\|_{\dot{H}^{-k,-N'}_{(b)}} + \|u\|_{H^{-N,-N'}_{(b)}}$. Finally, we have

$$\begin{aligned} \|\chi u\|_{\dot{H}^{1-k,-N'}_{(b)}} &\leq C \|u\|_{\tilde{H}^{1-k}} \\ &\leq C \left\| \hat{T}_s(\sigma)^* u \right\|_{\dot{H}^{-k,-N'}_{(b)}} \end{aligned}$$

where the last inequality comes from Proposition 13.1.6. □

13.2 Estimate near $x = 0$

Now we prove an estimate near the end $x = 0$. As before, we also need a semiclassical version. Contrarily to the previous estimates which remained valid up to $\sigma = 0$, it is not automatic in this case and we need to prove a uniform version of the estimate down to $\sigma = 0$. Unless otherwise indicated, computation in this section are localized on $\left\{0 \leq x \leq \frac{1}{6M}\right\}$. To be coherent with the notation in [105], we introduce the operator:

$$P(\sigma) = e^{i\frac{\sigma}{x} - 2iM\sigma\ln(x)} \frac{1}{\Delta_r} \hat{T}_s(\sigma) e^{-i\frac{\sigma}{x} + 2iM\sigma\ln(x)}.$$

Other choices are possible to fit in the framework of [105], the choice made here imposes that the coefficient of $(xD_x)^2$ is equal to 1. Therefore, the operator after conjugation (defined in [105, Section 3])

$$\hat{P}(\sigma) := e^{-i\frac{\sigma}{x}} P(\sigma) e^{i\frac{\sigma}{x}} = x^{-2iM\sigma} \frac{1}{\Delta_r} \hat{T}_s(\sigma) x^{2iM\sigma} \tag{13.1}$$

We will use this equality to translate estimates about $P(\sigma)$ into estimates about $\hat{T}_s(\sigma)$.

We have the following decomposition for $P(\sigma)$:

$$P(\sigma) = P(0) + \sigma Q - \sigma^2$$

where $P(0) \in \mathrm{Diff}^2_{sc}$ and $Q \in x\mathrm{Diff}^1_{sc}$ and explicitly (using the trivialization $\mathcal{T}_m$)

$$\begin{aligned}
P(0) =& (x^2 D_x)^2 + \frac{2i(r-M)(1-s)}{\Delta_r} x^2 D_x + \frac{1}{\Delta_r \sin\theta} D_\theta \sin\theta D_\theta + \frac{r^2 + a^2\cos^2\theta - 2Mr}{\Delta_r^2 \sin^2\theta} D_\phi^2 \\
&+ \frac{2s(\Delta_r \cos\theta + ia(M-r)\sin^2\theta)}{\Delta_r^2 \sin^2\theta} D_\phi + \frac{s^2 \mathrm{cotan}^2\theta + s}{\Delta_r} \\
Q =& \frac{4Mar}{\Delta_r^2} D_\phi - \frac{4M(2Mr - a^2)}{r\Delta_r} x^2 D_x - \frac{1}{r^2\Delta_r^2} q_0 \\
q_0 =& (4M\sigma + 2is)\, x^{-5} + x^{-4} C^\infty\left(\left[0, \frac{1}{6M}\right]_x \times \mathbb{S}^2\right)
\end{aligned}$$

Proposition 13.2.1. *Let χ_1 be a smooth cutoff compactly supported in $\left\{x < \frac{1}{6M}\right\}$ with $\chi_1 = 1$ on $\left\{x < \frac{1}{12M}\right\}$.*

Assume that $l + \frac{1}{2} - 2M\Im(\sigma) < 0$, $\tilde{r} + l + \frac{1}{2} + 2s + 2M\Im(\sigma) > 0$ and $\Im(\sigma) \geq 0$ and $\sigma \neq 0$. Then if $u \in H^{\tilde{r}',l}_{(b)}$ for some $\tilde{r}'$ such that $\tilde{r}' + l + \frac{1}{2} + 2M\Im(\sigma) > 0$, we have:

$$\|\chi_1 u\|_{H^{\tilde{r},l}_{(b)}} \leq C\left(\left\|\hat{P}(\sigma)u\right\|_{H^{\tilde{r},l+1}_{(b)}} + \|u\|_{H^{-N,l-1}_{(b)}}\right)$$

Moreover, if $v \in H^{\tilde{r}'',l''}_{(b)}$ for some $\tilde{r}'' \in \mathbb{R}$ (with no further condition on $\tilde{r}''$) and $l'' + \frac{1}{2} + 2M\Im(\sigma) > 0$, we have:

$$\|\chi_1 v\|_{H^{-\tilde{r},-(l+1)}_{(b)}} \leq C\left(\left\|\hat{P}(\sigma)^* v\right\|_{H^{-\tilde{r},-l}_{(b)}} + \|v\|_{H^{-N,-l-2}_{(b)}}\right)$$

The constants are uniform with respect to σ in a compact subset of $\{\Im(\sigma) \geq 0\} \setminus \{0\}$

Proof. First note that it is enough to prove that

$$\|\chi_1 u\|_{H^{\tilde{r},l}_{(b)}} \leq C \left(\left\| \hat{P}(\sigma)\chi_1 u \right\|_{H^{\tilde{r},l+1}_{(b)}} + \|\chi_1 u\|_{H^{-N,l-1}_{(b)}} \right) \tag{13.2}$$

and similarly for the second inequality. Indeed, we have that $[\hat{P}(\sigma), \chi_1] \in \mathrm{Diff}^1_b$ is supported away from the boundary and on the elliptic set of $\hat{P}(\sigma)$. Therefore, for $\tilde{\chi}_1$ a smooth compactly supported cutoff equal to 1 on $\mathrm{supp}(\chi_1)$ we have:

$$\begin{aligned} \left\| [\hat{P}(\sigma), \chi_1] u \right\|_{H^{\tilde{r},l+1}_{(b)}} &\leq \|\tilde{\chi}_1 u\|_{H^{\tilde{r}+1,-\infty}_{(b)}} \\ &\leq C \left\| \hat{P}(\sigma) u \right\|_{H^{\tilde{r}-1,-\infty}_{(b)}} \\ &\leq C \left\| \hat{P}(\sigma) u \right\|_{H^{\tilde{r},l+1}_{(b)}} \end{aligned}$$

We can do the same for the second inequality. Therefore, we can consider $P(\sigma)$ as an operator acting on $\left[0, \frac{1}{6M}\right)_x \times \mathcal{B}_s$.

The estimate (13.2), is exactly the kind of estimate obtained in the proof of theorem 1.1 in [105].

Note that the operator $P(\sigma)$ is not exactly of the form described in section 2 of [105] since it acts on a non trivial complex line bundle. However, as mentioned in remark 1.3 of [105], the proof of the estimate is completely parallel in this case (and since in our case the bundle is of dimension 1, it requires even less adaptation). The exact hypotheses that we use and which are sufficient to run the proof are the following (we do not intend to be as general as the adaptation of [105] to complex line bundles could be, in particular we do not consider conormal operator's coefficients and we restrict to a simpler form for $P(0)$ which is enough to treat our case): Let $X = [0, \frac{1}{6M})_x \times \partial X$ be a smooth manifold with boundary of dimension n with smooth boundary defining function x. Let $\omega_{\partial X}$ be a smooth volume form on the boundary, we denote by $\mathrm{d}vol = x^{-n-1}|\,\mathrm{d}x|\omega$. Let $p : E \to \partial X$ be a complex line bundle over ∂X and m be a smooth metric on E. We denote by $\tilde{E} = \pi_2^* E$ (the semitrivial bundle over X associated to E) and $\tilde{m} = \pi_2^* m$ the associated metric. Let $P(\sigma) = P(0) + \sigma Q - \sigma^2$ be such that

(H1)

$$P(0) = (x^2 D_x)^2 + x^2 A + (i(n-1) + a_0) x (x^2 D_x)$$

where $A \in C^\infty([0, M)_x, \mathrm{Diff}^2(E))$ and $a_0 \in C^\infty(X)$. We also require that the principal symbol of $x^{-2}P(0)$ is elliptic in the b sense. For simplicity, we imposed that the coefficients a', $a_{0,j}$ are zero with respect to the local coordinate expression (3.4) in [105], which is true in our case.

(H2)

$$Q = b_0 x (x^2 D_x) + x^2 R + b' x$$

where $R \in C^\infty([0, M)_x, \mathrm{Diff}^1(E))$ and $b_0, b' \in C^\infty(X)$. Coefficients of Q can depend smoothly on σ.

(H3) The operators x^2A and x^2R admit the following decomposition:

$$x^2A = \sum_{j=1}^{N}(A_j^*A_j + A_j^*A_j' + A_j^\dagger A_j) + A''$$
$$x^2R = \sum_{j=1}^{N}(A_j^*R_j + R_j^\dagger A_j) + R''$$

where $A_j \in xC^\infty([0,M)_x, \mathrm{Diff}^1(E)) \subset x\mathrm{Diff}^1(\tilde{E})$, $A_j', A_j^\dagger, R_j, R_j^\dagger \in xC^\infty(X)$ and $A'', R'' \in x^2C^\infty(X)$. This hypothesis is needed to enforce the conclusion of lemma 3.3 in [105] (which is automatically true when we have the precise local form given in (3.4) and (3.5), we avoid giving local expression here since it would also depend on the choice of a local trivialization of E in addition to the choice of coordinates on ∂X but it could be done).

Under the previous hypotheses, we have:

$$\begin{aligned}\hat{P}(\sigma) &:= e^{-i\frac{\sigma}{x}}P(\sigma)e^{i\frac{\sigma}{x}} \\ &= P(0) + \sigma\hat{Q} - 2\sigma\left(x^2D_x + i\frac{n-1}{2}x + x\tilde{\alpha}_+(\sigma)\right)\end{aligned}$$

with $\hat{Q} = Q - xb'$ and $\tilde{\alpha}_+(\sigma) = \frac{a_0 - b' + b_0\sigma}{2}$. Therefore the normal operator is formally identical to the one in [105] (the only difference is that we consider it as a differential operator on $\tilde{E}$). The threshold value for $\tilde{r}$ is then $-\frac{1}{2} + \Im(\alpha_+(\sigma))$ where $\alpha_+(\sigma) = \lim_{x\to 0}\tilde{\alpha}_+(\sigma)$. Similarly, the threshold value for $\tilde{r}+l$ is $-\frac{1}{2}+\Im(\alpha_-(\sigma))$ where $\alpha_-(\sigma) := \lim_{x\to 0}\tilde{\alpha}_-(\sigma)$ and $\tilde{\alpha}_-(\sigma) := \frac{a_0 + b' + b_0\sigma}{2}$.

Let (y_i) be local coordinates on ∂X. Let τ be the variable associated with x in ${}^{sc}T^*X$ and μ_i be the variable associated with y_i. Hypotheses (H1) and (H3) imply that the principal symbol of $P(0)$ in the scattering decay sense is $\tau^2 + q(y)(\mu)$ where $q(y)$ is a positive definite quadratic form (positivity comes from (H3) and definiteness from the fact that $x^{-2}P(0)$ is elliptic in the b sense). This is the correct form to run commutator estimates with the same method as in [105] (and we have the conclusion of Lemma 3.3 in [105] thanks to hypothesis (H3) to deal with the case of non real σ).

We just have to check that the hypotheses are satisfied our case.

(H1) The hypothesis is true with $a_0 = \frac{2i(Mr(1+s) - a^2 - r^2 s)}{\Delta_r} = -2is + O(x)$,

$$\begin{aligned}A =& \frac{r^2}{\Delta_r \sin\theta} D_\theta \sin\theta D_\theta + \frac{r^2\left(r^2 + a^2\cos^2\theta - 2Mr\right)}{\Delta_r^2\sin^2\theta}D_\phi^2 \\ &+ \frac{2sr^2(\Delta_r\cos\theta + ia(M-r)\sin^2\theta)}{\Delta_r^2\sin^2\theta}D_\phi + \frac{r^2\left(s^2\mathrm{cotan}^2\theta + s\right)}{\Delta_r}.\end{aligned} \tag{13.3}$$

(H2) The hypothesis is true with $b' = -\frac{1}{r\Delta_r^2}q_0 = -4M\sigma - 2is + O(x)$, $b_0 = -\frac{4M(2Mr - a^2)}{\Delta_r}$ and $R = \frac{4Mar^3}{\Delta_r^2}D_\phi$ (which is smooth as a differential operator on $\tilde{E}$).

(H3) We define the following smooth operators on $\mathcal{B}_s$ by their coordinates in local trivialization $\mathcal{T}_m$:

$$(\tilde{Z}_1)_m = -\sin\phi\partial_\theta + \cos\phi\left(-\frac{is}{\sin\theta} - \frac{\cos\theta}{\sin\theta}\partial_\phi\right)$$
$$(\tilde{Z}_2)_m = -\cos\phi\partial_\theta - \sin\phi\left(-\frac{is}{\sin\theta} - \frac{\cos\theta}{\sin\theta}\partial_\phi\right)$$
$$(\tilde{Z}_3)_m = \sqrt{1-\frac{a^2}{\Delta_r}}\partial_\phi.$$

We have

$$x^2 A = A_1^* A_1 + A_2^* A_2 + A_3^* A_3 + A_3^\dagger A_3 + A''$$
$$x^2 R = R_3^\dagger A_3$$

with

$$A_i = \frac{1}{\sqrt{\Delta_r}}\tilde{Z}_i$$
$$A_3^\dagger = \frac{2sa(M-r)}{\Delta_r\sqrt{\Delta_r\left(1-\frac{a^2}{\Delta_r}\right)}}$$
$$A_3'' = \frac{s(1-s)}{\Delta_r}$$
$$R_3^\dagger = \frac{-4iMar}{\Delta_r\sqrt{\Delta_r\left(1-\frac{a^2}{\Delta_r}\right)}}.$$

□

If we translate the previous estimate into an estimate on $\frac{1}{\Delta_r}\hat{T}_s(\sigma)$ using (13.1), we get:

Proposition 13.2.2. *Let χ_1 be a smooth cutoff compactly supported in $\left\{x < \frac{1}{6M}\right\}$ with $\chi_1 = 1$ on $\left\{x < \frac{1}{12M}\right\}$.*

Assume that $l + \frac{1}{2} < 0$, $\tilde{r} + l + \frac{1}{2} + 2s + 4M\Im(\sigma) > 0$ and $\Im(\sigma) \geq 0$ and $\sigma \neq 0$. Then if $u \in H_{(b)}^{\tilde{r}',l}$ for some $\tilde{r}'$ such that $\tilde{r}' + l + \frac{1}{2} + 2s + 4M\Im(\sigma) > 0$, we have:

$$\|\chi_1 u\|_{H_{(b)}^{\tilde{r},l}} \leq C\left(\left\|\hat{T}_s(\sigma)u\right\|_{H_{(b)}^{\tilde{r},l-1}} + \|u\|_{H_{(b)}^{-N,l-1}}\right)$$

Moreover, if $v \in H_{(b)}^{\tilde{r}'',l''}$ for some $\tilde{r}'' \in \mathbb{R}$ (with no further condition on $\tilde{r}''$) and $l'' > -\frac{1}{2}$, we have:

$$\|\chi_1 v\|_{H_{(b)}^{-\tilde{r},1-l}} \leq C\left(\left\|\hat{T}_s(\sigma)^* v\right\|_{H_{(b)}^{-\tilde{r},-l}} + \|v\|_{H_{(b)}^{-N,-l}}\right).$$

The constants are uniform with respect to σ in a compact subset of $\{\Im(\sigma) \geq 0\} \setminus \{0\}$.

We can also get the semiclassical version of this estimate (meaning uniform with respect to σ in a strip $\{0 \leq \Im(\sigma) \leq \eta, |\Re(\sigma)| > A\}$). This is done in [105], in section 5. We give the version for $\hat{T}_{s,h}(\tilde{z}) = h^2\hat{T}_s(h^{-1}\tilde{z})$ with $0 \leq \Im(\tilde{z}) \leq \eta h$ and $|\Re(\tilde{z})| \geq Ah$. Note that the non trapping assumption mentioned in [105] plays no role for the microlocal version of the estimate. Note that we use b Sobolev spaces in the statement instead of second microlocal Sobolev spaces, thus we lose some precision with respect to [105].

Proposition 13.2.3. *Let U be a neighborhood of $\mathcal{R}_{in}$ separated from fiber infinity. Let $B_0, B_1, G \in \Psi^{0,0}_{sc,h}$ with $WF(B_0) \cup WF(B_1) \cup WF(G) \subset U$, $WF(B_0) \cup WF(B_1) \subset Ell(G)$ and every bicharacteristic curve from $WF(B_0)$ reaches $Ell(B_1)$ in finite time (with the time having the same sign as $\Re(\tilde{z})$) while remaining in $Ell(G)$. If $0 \leq \Im(\tilde{z}) \leq \eta h$, $|\Re(\tilde{z})| \geq Ah$, $l + \frac{1}{2} < 0$ and $\tilde{r} \in \mathbb{R}$. For all $u \in H^{\tilde{r}',l'}_{(b),h}$ (with no conditions on $\tilde{r}'$ and l') and all $N \in \mathbb{N}$, we have a constant $C > 0$ uniform with respect to h, $\tilde{z}$ and u such that:*

$$\|B_0 u\|_{H^{\tilde{r},l}_{(b),h}} \leq C\left(h^{-1}\left\|G\hat{T}_{s,h}(\tilde{z})u\right\|_{H^{\tilde{r},l-1}_{(b),h}} + \|B_1 u\|_{H^{\tilde{r},l}_{(b),h}} + h^N \|u\|_{H^{-N,l}_{(b),h}}\right)$$

Moreover, if $u \in H^{\tilde{r}',l'}_{(b),h}$ with $l' > -\frac{1}{2}$:

$$\|B_0 u\|_{H^{-\tilde{r},1-l}_{(b),h}} \leq C\left(h^{-1}\left\|G\hat{T}_{s,h}(\tilde{z})^* u\right\|_{H^{-\tilde{r},-l}_{(b),h}} + h^N \|u\|_{H^{-N,1-l}_{(b),h}}\right)$$

We have the corresponding estimate near $\mathcal{R}_{out}$:

Proposition 13.2.4. *Let U be a neighborhood of $\mathcal{R}_{out}$ separated from fiber infinity. Let $B_0, B_1, G \in \Psi^{0}_{sc,h}$ with $WF(B_0) \cup WF(B_1) \cup WF(G) \subset U$, $WF(B_0) \cup WF(B_1) \subset Ell(G)$ and every bicharacteristic curve from $WF(B_0)$ reaches $Ell(B_1)$ in finite time (with the time having the same sign as $-\Re(\tilde{z})$) while remaining in $Ell(G)$. If $0 \leq Im(\tilde{z}) \leq \eta h$, $|\Re(\tilde{z})| \geq Ah$, $\tilde{r} + l + \frac{1}{2} + 2s > 0$ and $u \in H^{\tilde{r}',l'}_{(b)}$ with $\tilde{r}' + l' + \frac{1}{2} + 2s > 0$, then for all $N \in \mathbb{N}$, there exists $C > 0$ independent of u, h and $\tilde{z}$ such that:*

$$\|B_0 u\|_{H^{\tilde{r},l}_{(b),h}} \leq C\left(h^{-1}\left\|G\hat{T}_h(\tilde{z})u\right\|_{H^{\tilde{r},l-1}_{(b),h}} + h^N \|u\|_{H^{-N,l}_{(b),h}}\right)$$

and if $u \in H^{\tilde{r}',l'}_{(b)}$ with no condition on $\tilde{r}', l'$, we have:

$$\|B_0 u\|_{H^{-\tilde{r},1-l}_{(b),h}} \leq C\left(h^{-1}\left\|G\hat{T}^*_h(\tilde{z})u\right\|_{H^{-\tilde{r},-l}_{(b),h}} + \|B_1 u\|_{H^{-\tilde{r},1-l}_{(b),h}} + h^N \|u\|_{H^{-N,1-l}_{(b),h}}\right)$$

We also need to state an estimate which is true uniformly up to $\sigma = 0$. We begin by recalling the definition of the effective normal operator (which is compatible with Definition 2.4 in [106] although phrased in a slightly different setting).

Definition 13.2.5. *The effective normal operator of $\hat{P}(\sigma)$ denoted by $N_{\text{eff}}(\hat{P}(\sigma))$ is $\hat{P}(\sigma)$ modulo $x(x+\sigma)^2\text{Diff}^2_b([0, \frac{1}{r_+ - \epsilon})_x \times \mathcal{B}_s)$. Similarly, $N_{\text{eff}}(\hat{T}_s(\sigma))$ is $\hat{T}_s(\sigma)$ modulo $x^{-1}(x+\sigma)^2\text{Diff}^2_b([0, \frac{1}{r_+ - \epsilon})_x \times \mathcal{B}_s)$.*

Proposition 13.2.6. *Let χ_1 be a smooth cutoff compactly supported in $\left\{x < \frac{1}{6M}\right\}$ with $\chi_1 = 1$ on $\left\{x < \frac{1}{12M}\right\}$. Assume that $\alpha \in \left(l + \frac{1}{2} + s - |s|, l + \frac{3}{2} + s + |s|\right)$, $l+\tilde{r} > -\frac{1}{2}-2s$ and $l < -\frac{1}{2}$. Then, there exists $\sigma_0 > 0$ such that for all $\sigma \in \mathbb{C}$ with $\Im(\sigma) \geq 0$ and $|\sigma| \leq \sigma_0$, we have:*

$$\|(x+|\sigma|)^\alpha \chi_1 u\|_{H^{\tilde{r},l}_{(b)}} \leq C\left(\left\|(x+|\sigma|)^\alpha \hat{P}(\sigma)u\right\|_{H^{\tilde{r}-1,l+2}_{(b)}} + \|u\|_{H^{-N,l-1}_{(b)}}\right)$$

Proof. As for Proposition 13.2.1, using that $[\hat{P}(\sigma), \chi_1] \in \mathrm{Diff}^1_b$ is supported away from the boundary and on the elliptic set of $\hat{P}(\sigma)$, it is enough to prove

$$\|(x+|\sigma|)^\alpha \chi_1 u\|_{H^{\tilde{r},l}_{(b)}} \leq C\left(\left\|(x+|\sigma|)^\alpha \hat{P}(\sigma)\chi_1 u\right\|_{H^{\tilde{r}-1,l+2}_{(b)}} + \|\chi_1 u\|_{H^{-N,l-1}_{(b)}}\right). \tag{13.4}$$

The estimate is therefore localized in a neighborhood of $\{x = 0\}$ where the microlocal estimates obtained in the proof of Theorem 2.5 in [106] applies. As for the previous proposition, because we consider an operator acting on a complex line bundle, we are not exactly in the setting of [106]. However, the proof is completely parallel under the following hypotheses[1] (which are not the most general ones):

$$P(\sigma) = P(0) + \sigma Q - (1-R)\sigma^2$$

(H1) $R \in xC^\infty(X)$ and $\Im(R) \in x^2C^\infty(X)$

(H2)

$$P(0) = (x^2D_x)^2 + x^2A + (i(n-1)+\beta)x(x^2D_x) + x^2a'$$

where $A \in C^\infty([0,M)_x, \mathrm{Diff}^2(E))$ such that $\frac{x^2}{2i}(A - A^*) \in x^3C^\infty([0,M)_x, \mathrm{Diff}^1(E))$ and $a', \beta \in C^\infty(X)$. We also require that the principal symbol of $x^{-2}P(0)$ is elliptic in the b sense. For simplicity we imposed that the coefficients $a_{0,j}$ in the local coordinate expression in the proof of proposition 2.1 in [106] are zero, which is true in our case.

(H3)

$$Q = b_0 x(x^2D_x) + x^2S + \gamma x$$

where $S \in C^\infty([0,M)_x, \mathrm{Diff}^1(E))$ is principally self-adjoint and $b_0, \gamma \in C^\infty(X)$ is such that $\Im(b_0) \in xC^\infty(X)$. The coefficients of Q are supposed to be independent of σ

(H4) The operators x^2A and x^2S admit the following decomposition:

$$x^2A = \sum_{j=1}^N (A_j^*A_j + A_j^*A_j' + A_j^\dagger A_j) + A''$$
$$x^2S = \sum_{j=1}^N (A_j^*R_j + R_j^\dagger A_j) + R''$$

where $A_j \in xC^\infty([0,M)_x, \mathrm{Diff}^1(E)) \subset x\mathrm{Diff}^1(\tilde{E})$, $A_j', A_j^\dagger, R_j, R_j^\dagger \in xC^\infty(X)$ and $A'', R'' \in x^2C^\infty(X)$.

[1] We use the same notations as in the proof of proposition 13.2.1

With the previous hypotheses, we see that (2.2) and (2.3) in [106] are satisfied with $\beta_I = \Im(\beta)$ and $\beta'_I = -\Re(\beta)\frac{n-2}{2} + \Im(a')$ and $\gamma_I = \Im(\gamma)$. We have the following form for the conjugated operators:

$$\hat{P}(\sigma) = P(0) + \sigma\hat{Q} + \sigma^2\hat{R} - 2\sigma\left(x^2D_x + i\frac{n-1}{2}x + \frac{\beta-\gamma}{2}x\right)$$

where

$$\hat{Q} = Q - x\gamma$$
$$\hat{R} = R - xb_0$$

Under these hypotheses, the effective normal operator (see Definition 13.2.5) is

$$\begin{aligned} N_{\text{eff}}(\hat{P}(\sigma)) =&(x^2D_x)^2 + i(n-1)x(x^2D_x) + x^2A(0) + \beta_{|\partial_x}x^2\left(xD_x + i\frac{n-2}{2}\right) \\ &+ x^2(\beta')_{|\partial X} - 2\sigma\left(x^2D_x + i\frac{n-1}{2}x + \frac{\beta_{|\partial X} - \gamma_{|\partial X}}{2}x\right) \end{aligned}$$

where $\beta' = a' - i\beta\frac{n-2}{2}$ (therefore $\Im(\beta') = \beta'_I$). Note that $(x^2D_x)^2 + i(n-1)x(x^2D_x) + x^2A(0)$ is similar to the term Δ_{g_0} in [106].

We add two last hypotheses:

(H5) $\beta_{|\partial X} \in i\mathbb{R}$ and $\Re(\beta'_{|\partial X}) + \lambda_0 > \frac{\left(\beta_{|\partial X}\right)^2}{4} - \left(\frac{n-2}{2}\right)^2$ where λ_0 is the smallest eigenvalue of $A(0)$. This hypothesis is present in theorem 2.5 in [106] (but with $\lambda_0 = 0$).

(H6) For every λ_k eigenvalue of $A(0)$:

$$\frac{1}{2} + \frac{i\gamma_{|\partial X}}{2} + \sqrt{\left(\frac{n-2}{2}\right)^2 - \frac{\beta^2_{|\partial X}}{4} + \lambda_k + \beta'_{|\partial X}} \notin -\mathbb{N}$$
$$\frac{1}{2} - \frac{i\gamma_{|\partial X}}{2} + \sqrt{\left(\frac{n-2}{2}\right)^2 - \frac{\beta^2_{|\partial X}}{4} + \lambda_k + \beta'_{|\partial X}} \notin -\mathbb{N}$$

This hypothesis is not mentioned in [106] but it seems necessary (it is at least sufficient) to ensure that Proposition 5.4 in [106] applies to the effective normal operator with no error term. Indeed, under this hypothesis, we can prove that the effective normal operator has no kernel in the space of interest (see appendix B, note that by hypothesis (H2), $A(0)$ is an elliptic formally selfadjoint operator and by hypothesis (H5) there exists some constant $C > 0$ such that $A(0) + C$ is positive. Therefore, even replacing β' by $\beta' - C$ and taking $L := A(0) + C$, we can apply the result of appendix B).

We now check the hypothesis in our case: We write: $P(\sigma) = P(0) + \sigma Q - (1-R)\sigma^2$ with $P(0)$ already defined and

$$\begin{aligned} Q =&\frac{4Mar}{\Delta_r^2}D_\phi - \frac{4M\left(2Mr - a^2\right)}{r\Delta_r}x^2D_x - \frac{1}{r^2\Delta_r^2}q_1 \\ q_1 =&2ir^5s + r^4\left(-6iMs - 2as\cos(\theta)\right) + r^3\left(-8iM^2s + 4Mas\cos(\theta) + 2ia^2s\right) \\ &+ r^2\left(8iM^3s + 6iMa^2s + 2iMa^2 - 2a^3s\cos(\theta)\right) + r\left(-4iM^2a^2s - 4iM^2a^2\right) + 2iMa^4 \\ r^2\Delta_r^2R =& -16M^3a^2r + 4M^2a^4 - 4Mr^5 + r^4\left(4M^2 + a^2\sin^2(\theta)\right) \\ &+ r^3\left(-2Ma^2\sin^2(\theta) - 4Ma^2\right) + r^2\left(16M^4 + a^4\sin^2(\theta)\right) \end{aligned}$$

(H1) The first hypothesis is obvious with the definition of R.

(H2) We have the correct form for $P(0)$ if we define A as in (13.3) and $\beta := \frac{2i(Mr(1+s)-a^2-r^2s)}{\Delta_r} = -2is + O(x)$ and $a' = 0$. We check that

$$\frac{x^2}{2i}(A - A^*) = \frac{2sa(M-r)}{\Delta_r^2} D_\phi$$

(H3) We have the correct form for Q if we define $b_0 := -\frac{4M\left(2Mr-a^2\right)}{\Delta_r} = O(x)$, $S := \frac{4Mar^3}{\Delta_r^2} D_\phi$ (self adjoint) and $\gamma := -\frac{1}{r\Delta_r^2} q_1 = -2is + O(x)$

(H4) The decomposition of $x^2 A$ has already been checked and the decomposition of $x^2 S$ is obtained by taking $R_3^\dagger := -\frac{4iMar}{\Delta_r \sqrt{\left(1-\frac{a^2}{\Delta_r}\right)\Delta_r}}$ (and all the other terms equal to zero).

Therefore in our case, the effective normal operator is

$$\begin{aligned} N_{\text{eff}}(\hat{P}(\sigma)) =& (x^2 D_x)^2 + 2ix(x^2 D_x) \\ &+ x^2 \left(\frac{1}{\sin\theta} D_\theta \sin\theta D_\theta + \frac{1}{\sin^2\theta} D_\phi^2 + \frac{2s\cos\theta}{\sin^2\theta} D_\phi + s^2 \text{cotan}^2\theta + s \right) \\ &- 2isx^2 \left(xD_x + \frac{i}{2} \right) - sx^2 - 2\sigma \left(x^2 D_x + ix \right) \end{aligned}$$

To be coherent with the notation in [106], we study the conjugated and renormalized operator:

$$x^{\frac{-5}{2}} N(\hat{P}(0)) x^{\frac{1}{2}} = (xD_x)^2 + A(0) + \frac{1}{4} - 2isxD_x - s$$

After a Mellin transform with respect to x, we obtain:

$$\tau_b^2 - 2is\tau_b - s + A(0) + \frac{1}{4}$$

To deduce the central weight interval (the analog of (2.12) in [106], we have to compute the eigenvalues of $A(0) - s$ (which is the spin-s-weighted Laplacian). The eigenvalues of $A(0) - s$ are $(l+s)(l-s) + l$ for $l \in \mathbb{N}$ such that $l \geq |s|$. Therefore, the central interval for weights for the scattering end (the equivalent of (2.12) in [106]) is $\left(-\frac{3}{2} - s - |s|, -\frac{1}{2} - s + |s|\right)$. □

Remark 13.2.7. *For $\alpha = 0$, $l \in (-\frac{3}{2} - s - |s|, -\frac{1}{2})$, $l + \tilde{r} > -\frac{1}{2} - 2s$ and $\Im(\sigma) \geq 0$ the second resolved microlocal spaces $H^{\tilde{r},m,l}_{sc,b,res}$ used in [106] enables to state a more precise estimate:*

$$\|\chi_1 u\|_{H^{\tilde{r},\tilde{r}+l,l}_{sc,b,res}} \leq C \left(\left\| (x+|\sigma|)^{-1} \hat{P}(\sigma) u \right\|_{H^{\tilde{r}-2,l+\tilde{r}+1,l+1}_{sc,b,res}} + \|u\|_{H^{-N,l-1}_{(b)}} \right)$$

We can then use the bound (for $\epsilon \in [0,1]$): $(x+|\sigma|)^{-1} \leq |\sigma|^{\epsilon-1}(x+|\sigma|)^{-\epsilon}$ and the fact that $(x+|\sigma|)^\epsilon H^{\tilde{r}-2,l+\tilde{r}+1,l+1}_{sc,b,res} \supset H^{\tilde{r}-2,l+\tilde{r}+1,l+1+\epsilon}_{sc,b,res} \supset H^{\tilde{r}-\epsilon,l+1+\epsilon}_{(b)}$ to get:

$$\|\chi_1 u\|_{H^{\tilde{r},l}_{(b)}} \leq C \left(|\sigma|^{\epsilon-1} \left\| \hat{P}(\sigma) u \right\|_{H^{\tilde{r}-\epsilon,l+1+\epsilon}_{(b)}} + \|u\|_{H^{-N,l-1}_{(b)}} \right)$$

13.3 High frequency estimate at the trapped set

We use notations introduced in section 12.2. We recall that the operator $\hat{T}_{s,h}$ was defined at the end of Section 10.1. We begin by some preliminary computations near the trapped set (on the region U_I with $I = (r_{min}, r_{max})$).

Lemma 13.3.1. *Let $z \in \mathbb{C}$. On U_I, we have:*

$$\begin{aligned}\frac{1}{2hi}(\hat{T}_{s,h}(z) - \hat{T}^*_{s,h}(z)) =& \frac{2(\Delta_r + rs(r-M))}{r} hD_r + \frac{2a(s(M-r) + 2Mrh^{-1}\Im(z))}{\Delta_r} hD_\phi \\ &+ 2h^{-1}\Im(z)\Re(z)\left(a^2\sin^2(\theta) - \frac{(a^2+r^2)^2}{\Delta_r}\right) \\ &- 2s\Re(z)\frac{M(a^2-r^2)+r\Delta_r}{\Delta_r} + \frac{ih}{r^2}(2Mr(s+2) - a^2 - 3r^2(s+1)) \\ &+ 2\Im(z)as\cos\theta\end{aligned}$$

(where as usual, the adjoint is computed with respect to the volume form $r^2\sin\theta\ \mathrm{d}t\ \mathrm{d}r\ \mathrm{d}\phi\ \mathrm{d}\theta$). If $z = z_0$ with $z_0 \in \{-1, 1\}$, the semiclassical principal symbol of $\frac{1}{2hi}(\hat{T}_{s,h}(z) - \hat{T}^*_{s,h}(z))$ is 0 on the trapped set K_{z_0}. For $0 \leq \Im(z) \leq \eta h$ and $z = z_0 + O(h)$, the principal symbol of $\frac{1}{2hi}(\hat{T}_{s,h}(z) - \hat{T}^*_{s,h}(z))$ has the same sign as $-\Re(z)$ on the trapped set.

Remark 13.3.2. *Note that since the principal symbol of $\hat{T}_s$ is $\tilde{G}$, the most natural choice for the volume form is the one associated to the metric $\tilde{g}$ which is $\sin(\theta)\ \mathrm{d}r\ \mathrm{d}\theta\ \mathrm{d}\phi$. However, since H_{p_h} does not vanish on the trapped set, the choice of the volume form matters. The factor r^2 is harmless here since it amounts to replace $\hat{T}^*_{s,h}$ by $r^{-2}\hat{T}_{s,h}r^2$ and the difference $r^{-2}[\hat{T}_{s,h}, r^2]$ has principal symbol $r^{-2}H_{p_h}r^2 = 0$ on the trapped set.*

Proof. By a direct computation we find

$$p_{sub} := \frac{2(\Delta_r + rs(r-M))}{r}\xi - \frac{2as(r-M)}{\Delta_r}\zeta - 2z_0 s\frac{M(a^2-r^2)+r\Delta_r}{\Delta_r}$$

Then, using that $\xi = 0$ on K_{z_0}, we find:

$$p_{sub} = -\frac{2as(r-M)}{\Delta_r}\zeta - 2z_0 s\frac{M(a^2-r^2)+r\Delta_r}{\Delta_r}$$

We introduce the function $\alpha := \frac{-(r^2+a^2)z_0 + a\zeta}{\Delta_r}$. We have $a\zeta = \Delta_r\alpha + (r^2+a^2)z_0$ and, on K_{z_0}, we have $(r-M)\alpha\Delta_r = -2r\Delta_r z_0$. Therefore on K_{z_0}, we have:

$$\begin{aligned}p_{sub} &= -\frac{2s(r-M)}{\Delta_r}(\alpha\Delta_r + (r^2+a^2)z_0) - 2sz_0\frac{M(a^2-r^2)+r\Delta_r}{\Delta_r} \\ &= 4srz_0 - \frac{2sz_0(r-M)(r^2+a^2)}{\Delta_r} - 2z_0 s\frac{M(a^2-r^2)}{\Delta_r} - 2sz_0 r \\ &= 2srz_0 - \frac{2sz_0}{\Delta_r}r\Delta_r \\ &= 0.\end{aligned}$$

When $0 \leq \Im(z) \leq \eta h$ for some $\eta > 0$ and $z = z_0 + O(h)$, the previous computation shows that, on the trapped set, the only non vanishing contribution comes from the terms containing $h^{-1}\Im(z)$ and therefore:

$$p_{sub} = h^{-1}\Im(z)\left(2a^2\sin^2\theta z_0 - 2\frac{(a^2+r^2)^2}{\Delta_r}z_0 + \frac{4Mar}{\Delta_r}\zeta\right)$$

We note that this term does not depend on s and therefore, this part of the computation is not specific to the Teukolsky case (see for example the proof of (5.81) in [45]). We recall the argument below. Using that $\frac{2a(a^2+r^2)}{\Delta_r} - 2a = \frac{4Mar}{\Delta_r}$, we write:

$$\begin{aligned} A :=& 2\left(a^2\sin^2\theta z_0 - \frac{(a^2+r^2)^2}{\Delta_r}z_0\right) + \frac{4Mar}{\Delta_r}\zeta \\ =& 2(a^2+r^2)\alpha - 2(a\zeta - a^2\sin^2(\theta)z_0) \end{aligned}$$

On K_{z_0}, we have $\alpha^2\Delta_r = \eta^2 + \left(\frac{\zeta}{\sin\theta} - a\sin\theta z_0\right)^2$ and therefore:

$$\begin{aligned} (a^2+r^2)\left|\alpha\right| &\geq |a|\sqrt{\Delta_r}\sin(\theta)\left|\alpha\right| \\ &\geq \left|a\zeta - a^2\sin^2(\theta)z_0\right| \end{aligned}$$

Therefore A has the same sign as $2(a^2+r^2)\alpha$ which is equal to $-\frac{4r(a^2+r^2)}{r-M}z_0$ on K_{z_0}. □

We need the following modified version of theorem 1 in [28]:

Theorem 13.3.3. *Let $P_h \in \Psi_h^m$ be a principally scalar and principally real operator on a smooth (complex) vector bundle (with a fixed smooth hermitian inner product) over some orientable smooth manifold X (with a fixed volume form). We denote by p_h its semiclassical principal symbol, by Σ the semiclassical characteristic set $p_h^{-1}\{0\}$ and by H_b the semiclassical Hamiltonian vector field. We assume that there exists $\phi_\pm$ smooth functions defined on a bounded open set U of T^*X such that for a fixed small $\epsilon > 0$:*

1. *There exists $\delta > 0$ such that, $U_\delta := \{|\phi_+| < \delta, |\phi_-| < \delta, |p| < \delta\}$ is compactly contained in U.*
2. *$H_p\phi_\pm = \mp c_\pm\phi_\pm$ with $c_\pm$ smooth bounded positive functions on U satisfying $\inf c_\pm > \nu_{min} - \epsilon > 0$.*
3. *$\{\phi_+, \phi_-\} > 0$*

Let $V \Subset U$ be a neighborhood of $K := \{\phi_+ = \phi_- = 0\}$. We assume that $\frac{1}{2ih}\mathfrak{s}_h(P_h - P_h^*) < \frac{\nu_{\min}-\epsilon}{2}$ on K. There exist $B_0, B_1, G \in \Psi_h^0$ with:

- *$WF_h(B_0) \cup WF_h(B_1) \cup WF_h(G) \subset U$*
- *$\mathfrak{s}_h(B_0) = 1$ on V*
- *$WF_h(B_1) \cap \{\phi_+ = 0\} = \emptyset$*

such that:

$$\|B_0u\|_{L^2} \leq C\left(h^{-1}\|B_1u\|_{L^2} + h^{-2}\|GP_h(z)u\|_{L^2}\right)$$

Remark 13.3.4. *In our setting, we use Lemma 12.2.29 for the existence of $\phi_\pm$. When $z_0 = 1$, we use Theorem 13.3.3 with operator $\hat{T}_{s,h}(z)$ and when $z_0 = -1$, we use Theorem 13.3.3 with operator $-\hat{T}_{s,h}(z)$.*

Remark 13.3.5. *The principal differences with respect to [28], theorem 1 are the fact that P_h is not of the form $\tilde{P}_h - \lambda$ with $\tilde{P}_h$ selfadjoint (but the hypothesis $\frac{1}{2ih}\mathfrak{s}_h(P_h - P_h^*) < \frac{\nu_{\min}-\epsilon}{2}$ on K replaces self adjointness in the proof) and the fact that we do not use an absorbing potential (and therefore we keep the estimate microlocal). To see how the result in [28] can be adapted in this context, see [49, Theorem 4.7].*

We can now apply Theorem 13.3.3 to the operators $\hat{T}_{s,h}(z)$, $-\hat{T}_{s,h}(z)$, $\hat{T}_{s,h}(z)^*$ and $-\hat{T}_{s,h}(z)^*$ (see Lemma 12.2.29 and Lemma 13.3.1 to see that the requirements are met) to get the semiclassical estimate near the trapped set:

Proposition 13.3.6. *Let $z = z_0 + O(h)$ with $z_0 \in \{-1, 1\}$. There exists U a bounded neighborhood of the trapped set K and B_K, B_0 in $\Psi^{0,0}_{b,h}$ with B_K elliptic on a neighborhood of K, $WF(B_0) \cap \Sigma \subset \Sigma_{sgnz_0}$ with either $WF(B_0) \cap \Gamma_+ = \emptyset$ (when $z_0 = 1$ in (13.5) or $z_0 = -1$ in (13.6)) or $WF(B_0) \cap \Gamma_- = \emptyset$ (when $z_0 = -1$ in (13.5) or $z_0 = 1$ in (13.6)) such that for all $u \in \overline{H}^{\tilde{r},l}_{(b)}$ and all $v \in \dot{H}^{\tilde{r},l}_{(b)}$:*

$$\|B_K u\|_{\overline{H}^{\tilde{r},l}_{(b),h}} \leq C \left(h^{-1} \|B_0 u\|_{\overline{H}^{\tilde{r},l}_{(b),h}} + h^{-2} \left\| \hat{T}_{s,h}(z) u \right\|_{\overline{H}^{\tilde{r},l-1}_{(b),h}} + h \|u\|_{\overline{H}^{\tilde{r},l}_{(b),h}} \right) \tag{13.5}$$

$$\|B_K v\|_{\dot{H}^{\tilde{r},l}_{(b),h}} \leq C \left(h^{-1} \|B_0 v\|_{\dot{H}^{\tilde{r},l}_{(b),h}} + h^{-2} \left\| \hat{T}_{s,h}(z)^* v \right\|_{\dot{H}^{\tilde{r},l-1}_{(b),h}} + h \|v\|_{\dot{H}^{\tilde{r},l}_{(b),h}} \right) \tag{13.6}$$

13.4 Global estimates

We recall the following elementary lemma (see also [103, Section 2.6]).

Lemma 13.4.1. *Let $X_0 \subset X_1 \subset X_2$ and $Y_0 \subset Y_1 \subset Y_2$ be Banach spaces (with continuous dense inclusions). Let $P : X_1 \to Y_2$ be a bounded operator such that $P_{|X_0}$ is bounded from X_0 to Y_1. We assume that both inclusions $X_1 \subset X_2$ and $Y_0 \subset Y_1$ are compact and that there exists $C > 0$ such that for all $u \in X_1$ and all $v \in Y_1^*$:*

$$\|u\|_{X_1} \leq C \left(\|Pu\|_{Y_1} + \|u\|_{X_2} \right) \tag{13.7}$$

$$\|v\|_{Y_1^*} \leq C \left(\|P^* v\|_{X_1^*} + \|v\|_{Y_0^*} \right). \tag{13.8}$$

Note that in the estimates, the right hand side may be infinite. Under these assumptions, P is Fredholm as an operator between the Banach space $\mathfrak{X} := \{u \in X_1 : Pu \in Y_1\}$ (endowed with the norm $\|u\|^2_{\mathfrak{X}} = \|u\|^2_{X_1} + \|Pu\|^2_{Y_1}$) and Y_1.

Proof. We begin by proving that $\mathfrak{X}$ is a Banach space. Let (u_n) be a Cauchy sequence in $\mathfrak{X}$. Then by completeness of X_1, there exists $u \in X_1$ such that $\lim_{n \to +\infty} u_n = u$ for the topology of X_1. Moreover by completeness of Y_1 there exists $v \in Y_1$ such that $\lim_{n \to +\infty} Pu_n = v$ for the topology of Y_1. On the other hand, by continuity of P, we have $\lim_{n \to +\infty} Pu_n = Pu$ for the topology of Y_2. By uniqueness of the limit in Y_2, we have $Pu = v \in Y_1$ and $\lim_{n \to +\infty} u_n = u$ in $\mathfrak{X}$.

Now we prove the Fredholm property. For all $u \in Ker(P)$, we have $\|u\|_{X_1} \leq C \|u\|_{X_2}$ by estimate (C.16). By compactness of the inclusion $X_1 \subset X_2$, we deduce that $B(0,1)_{X_1} \cap Ker(P)$ is relatively compact for the topology induced by X_2. The topology induced by X_2 is the same as the topology induced by X_1 since the two norms are equivalent on $Ker(P)$. Therefore, $Ker(P)$ endowed with the norm X_1 has a relatively compact unit ball. We deduce that $\dim(Ker(P)) < +\infty$. The same argument using (C.17) and the compactness of the inclusion $Y_1^* \subset Y_0^*$ proves that $Ker(P^*) = Ran(P)^\perp$ is finite dimensional.

We now prove that $P(\mathfrak{X})$ is closed. We take $y_1, ..., y_k$ a normed basis of $Ker(P)$ and we denote by $y_1^*, ..., y_k^*$ extensions (of norm 1) of the dual basis (obtained by the Hahn-Banach theorem). Let (u_n) be a sequence in $\mathfrak{X}$ and $v \in Y_1$ such that $\lim_{n \to +\infty} Pu_n = v$ in Y_1. We have to prove that $v \in P(\mathfrak{X})$. Without loss of generality, we can assume that $y_i^*(u_n) = 0$ (replacing u_n by $u_n - \sum_{i \leq k} y_i^*(u_n) y_i$ which does not change the value of v). By contradiction assume

that u_n is unbounded in X_1. Extracting a subsequence, we can assume $\|u_n\|_{X_1} \to +\infty$. Then $\lim_{n\to+\infty} P\left(\frac{u_n}{\|u_n\|_{X_1}}\right) = 0$ in Y_1. By compactness, we can assume (after extracting a subsequence) that $\frac{u_n}{\|u_n\|_{X_1}}$ converges to $z \in X_2$ for the topology of X_2. Using inequality (C.16), we deduce that $\frac{u_n}{\|u_n\|_{X_1}}$ is Cauchy in X_1. We deduce $z \in Ker(P)$ and we have the convergence in X_1. Then by continuity, for $i = 1, ..., k$, we have $y_i^*(z) = 0$ and because $z \in Ker(P)$ we deduce $z = 0$. But it is a contradiction since $\left\|\frac{u_n}{\|u_n\|_{X_1}}\right\|_{\mathcal{X}} = 1$. So (u_n) is bounded in X_1. After extracting a subsequence, we can assume that $\lim_{n\to+\infty} u_n = u$ in X_2. Then we use (C.16) and deduce that (u_n) is Cauchy in X_1. Since we also have that (Pu_n) is Cauchy in Y_1, by completeness of $\mathfrak{X}$ and uniqueness of the limit in X_2, we have $\lim_{n\to+\infty} u_n = u$ in $\mathfrak{X}$. By uniqueness of the limit in Y_1, we deduce that $v = Pu \in P(\mathfrak{X})$. □

Definition 13.4.2. *We define the following spaces* $\mathcal{X}_\sigma^{\tilde{r},l} := \left\{u \in \overline{H}_b^{\tilde{r},l} : \hat{T}_s(\sigma)u \in \overline{H}_{(b)}^{\tilde{r},l-1}\right\}$ *endowed with the norm* $\|u\|_{\mathcal{X}_\sigma^{\tilde{r},l}} := \|u\|_{\overline{H}_{(b)}^{\tilde{r},l}} + \left\|\hat{T}_s(\sigma)u\right\|_{\overline{H}_{(b)}^{\tilde{r},l-1}}$. *We also define* $\mathcal{W}_\sigma^{\tilde{r},l} := \left\{u \in \overline{H}_{(b)}^{\tilde{r},l} : \hat{T}_s(\sigma)u \in \overline{H}_{(b)}^{\tilde{r},l}\right\}$ *endowed with the norm* $\|u\|_{\mathcal{W}_\sigma^{\tilde{r},l}} := \|u\|_{\overline{H}_{(b)}^{\tilde{r},l}} + \left\|\hat{T}_s(\sigma)u\right\|_{\overline{H}_{(b)}^{\tilde{r},l}}$.

The Fredholm estimates that we want to prove are stated in the following Theorem.

Theorem 13.4.3. *Let* $K \subset \{\sigma \in \mathbb{C} : \Im(\sigma) \geq 0\} \setminus \{0\}$ *be compact. If* $l < -\frac{1}{2}$, $\tilde{r} + l > -\frac{1}{2} - 2s$ *and* $\tilde{r} > \frac{1}{2} + s$, *then there exists* $C > 0$ *such that for all* $u \in H_{(b)}^{\tilde{r},l}$, *all* $v \in H_{(b)}^{-\tilde{r},1-l}$ *and all* $\sigma \in K$, *we have:*

$$\|u\|_{\overline{H}_{(b)}^{\tilde{r},l}} \leq C\left(\left\|\hat{T}_s(\sigma)u\right\|_{\overline{H}_{(b)}^{\tilde{r},l-1}} + \|u\|_{\overline{H}_{(b)}^{\tilde{r}-1,l-1}}\right)$$
$$\|v\|_{\dot{H}_{(b)}^{-\tilde{r},1-l}} \leq C\left(\left\|\hat{T}_s(\sigma)^*v\right\|_{\dot{H}_{(b)}^{-\tilde{r},-l}} + \|v\|_{H_{(b)}^{-\tilde{r}-1,-l}}\right)$$

Proof. The key point is that the junctions between the estimates take place in an elliptic region for $\hat{T}_s(\sigma)$ and $\hat{T}_s(\sigma)^*$, therefore, we can use the elliptic estimate to bound the commutator terms. We prove the first estimate as an example. Let $u \in \mathcal{X}_\sigma^{\tilde{r},l}$. Then we define three smooth cut-off functions χ_1, χ_2 and χ_3 such that $\chi_1 + \chi_2 + \chi_3 = 1$ and such that $\text{supp}(\chi_1) \subset (r_+ - \epsilon, 2M + D)$ (where D is defined in Proposition 13.1.8), $\chi_1 = 1$ on $(r_+ - \epsilon, 2M)$, $\text{supp}(\chi_3) \subset (6M, +\infty)$ and $\chi_3 = 1$ in a neighborhood of $+\infty$. We have:

$$\|u\|_{H_{(b)}^{\tilde{r},l}} \leq \|\chi_1 u\|_{H_{(b)}^{\tilde{r},l}} + \|\chi_2 u\|_{H_{(b)}^{\tilde{r},l}} + \|\chi_3 u\|_{H_{(b)}^{\tilde{r},l}}.$$

We use proposition 13.1.8 to bound $\|\chi_1 u\|_{H_{(b)}^{\tilde{r},l}}$, proposition 10.3.8 (elliptic estimate) to bound $\|\chi_2 u\|_{H_{(b)}^{\tilde{r},l}}$ and proposition 13.2.2 to bound $\|\chi_3 u\|_{H_{(b)}^{\tilde{r},l}}$. We get:

$$\begin{aligned}\|u\|_{H_{(b)}^{\tilde{r},l}} &\leq C\left(\left\|\hat{T}_s(\sigma)\chi_1 u\right\|_{H_{(b)}^{\tilde{r},l-1}} + \left\|\hat{T}_s(\sigma)\chi_2 u\right\|_{H_{(b)}^{\tilde{r},l-1}} + \left\|\hat{T}_s(\sigma)\chi_3 u\right\|_{H_{(b)}^{\tilde{r},l-1}} + \|u\|_{\overline{H}_{(b)}^{\tilde{r}-1,l-1}}\right) \\ &\leq C\left(\left\|\hat{T}_s(\sigma)u\right\|_{H_{(b)}^{\tilde{r},l-1}} + \sum_{i=1}^{3}\left\|[\hat{T}_s(\sigma), \chi_i]u\right\|_{H_{(b)}^{\tilde{r},l-1}} + \|u\|_{\overline{H}_{(b)}^{\tilde{r}-1,l-1}}\right)\end{aligned}$$

The terms $\left\|[\hat{T}_s(\sigma),\chi_i]u\right\|_{H_{(b)}^{\tilde{r},l-1}}$ can be estimated by an elliptic estimate (proposition 10.3.8). We prove the bound for $\hat{T}_s(\sigma)^*$ exactly in the same way. □

Remark 13.4.4. *We see in the proof that we loose some regularity away from ∂X, we could obtain more precise bounds by using second microlocal spaces (see [105]).*

Corollary 13.4.5. *Let $\sigma \in \mathbb{C}\setminus\{0\}$ with $\Im(\sigma)\geq 0$. If $l<-\frac{1}{2}$, $\tilde{r}+l>-\frac{1}{2}-2s-4M\Im(\sigma)$ and $\tilde{r}>\frac{1}{2}+s-\frac{a^2+r_+^2}{r_+-M}\Im(\sigma)$, then $\hat{T}_s(\sigma)$ is Fredholm as an operator from $\mathcal{X}_\sigma^{\tilde{r},l}$ to $\overline{H}_{(b)}^{\tilde{r},l-1}$.*

Global estimates near zero energy

Proposition 13.4.6. *Let $l<-\frac{1}{2}$, $\tilde{r}+l>-\frac{1}{2}-2s$, $\tilde{r}>\frac{1}{2}+s$ and*

$$\alpha\in\left(l+\frac{1}{2}+s-|s|\,,l+\frac{3}{2}+s+|s|\right).$$

Then, there exists $\sigma_0>0$ such that for $|\sigma|\leq\sigma_0$ and $\Im(\sigma)\geq 0$

$$\|(x+|\sigma|)^\alpha u\|_{\overline{H}_{(b)}^{\tilde{r},l}}\leq C\left(\left\|(x+|\sigma|)^\alpha\hat{T}_s(\sigma)u\right\|_{\overline{H}_{(b)}^{\tilde{r}-1,l}}+\|u\|_{\overline{H}_{(b)}^{\tilde{r}-1,l-1}}\right)$$

and the constant C is independent of σ. We also have the version coming from the second microlocalized resolved space (see Remark 13.2.7) for $l\in(-\frac{3}{2}-s-|s|\,,-\frac{1}{2})$, $l+\tilde{r}>-\frac{1}{2}-2s$, $\tilde{r}>\frac{1}{2}+s$, $\Im(\sigma)\geq 0$ and $\epsilon\in(0,1)$:

$$\|u\|_{\overline{H}_{(b)}^{\tilde{r},l}}\leq C\left(|\sigma|^{\epsilon-1}\left\|\hat{T}_s(\sigma)u\right\|_{\overline{H}_{(b)}^{\tilde{r}-\epsilon,l-1+\epsilon}}+\|u\|_{H_{(b)}^{-N,l-1}}\right)$$

Proof. The proof is very similar to the proof of proposition 13.4.3 since the gluing is made on elliptic regions. □

Lemma 13.4.7. *Let X and Y be Banach spaces. Let $P:X\to Y$ be a Fredholm operator. Let B be a Banach space which is continuously and densely included in Y, then P is a Fredholm operator from $A=P^{-1}(B)$ to B (A is endowed with the graph norm). Morever, $ind(P)\leq ind(P_{|_A})$.*

Proof. First, $ker(P)=ker(P_{|_A})$ is still finite dimensional. We have an inclusion from $B/(B\cap P(X))$ into $Y/P(X)$. Therefore, $\dim(B/(B\cap P(X)))\leq\dim(Y/P(X))<+\infty$. Moreover, $B\cap P(X)$ is closed into B since $P(X)$ is closed in Y. □

Corollary 13.4.8. *Assume:*

$$\begin{aligned}-\frac{3}{2}-s-|s|<l<-\frac{1}{2}\\ \tilde{r}+l>-\frac{1}{2}-2s\\ \tilde{r}>\frac{1}{2}+s\end{aligned}$$

Then, for every $\sigma\in\mathbb{C}\setminus\{0\}$ such that $\Im(\sigma)\geq 0$, we have that $\hat{T}_s(\sigma)$ is a Fredholm operator between $\mathcal{W}_\sigma^{\tilde{r},l}$ and $\overline{H}_{(b)}^{\tilde{r},l}$. Moreover, the index of $\hat{T}_s(\sigma)$ as an operator between $\mathcal{W}_\sigma^{\tilde{r},l}$ and $\overline{H}_{(b)}^{\tilde{r},l}$ is larger than the index as an operator between $\mathcal{X}_\sigma^{\tilde{r},l}$ and $\overline{H}_{(b)}^{\tilde{r}-1,l}$.

Proof. We use Lemma 13.4.7 with the space $B := \overline{H}^{\tilde{r},l}_{(b)}$, $Y := \overline{H}^{\tilde{r}-1,l}_{(b)}$, $X := \mathcal{X}^{\tilde{r},l}_{\sigma}$ since we know by Corollary 13.4.5 that $\hat{T}_s(\sigma)$ is Fredholm from X to Y under the hypotheses of Corollary 13.4.8. □

Remark 13.4.9. *The advantage of the previous corollary is that, once we have proved the invertibility, the last estimate in Proposition 13.4.6 translates into a uniform bound for the inverse up to $\sigma = 0$. This is not the case if $\mathcal{W}^{\tilde{r},l}_{\sigma}$ is replaced by $\mathcal{X}^{\tilde{r},l}_{\sigma}$.*

Global semiclassical estimates

Proposition 13.4.10. *Let $\eta > 0$. Let $l < -\frac{1}{2}$, $\tilde{r} + l > -\frac{1}{2} - 2s$ and $\tilde{r} > \frac{1}{2} + s$. There exits $A > 0$ such that, for all $\sigma \in \mathbb{C}$ with $0 \leq \Im(\sigma) \leq \eta$ and $|\sigma| \geq A$ and for all $u \in \overline{H}^{\tilde{r},l}_{(b),|\sigma|^{-1}}$, $v \in \dot{H}^{-\tilde{r},1-l}_{(b),|\sigma|^{-1}}$ we have:*

$$\|u\|_{\overline{H}^{\tilde{r},l}_{(b),|\sigma|^{-1}}} \leq C \left\|\hat{T}_s(\sigma)u\right\|_{\overline{H}^{\tilde{r},l-1}_{(b),|\sigma|^{-1}}}$$
$$\|v\|_{\dot{H}^{-\tilde{r},1-l}_{(b),|\sigma|^{-1}}} \leq C \left\|\hat{T}_s(\sigma)^* v\right\|_{\dot{H}^{-\tilde{r},-l}_{(b),|\sigma|^{-1}}}$$

Remark 13.4.11. *In particular, we have that $\hat{T}_s(\sigma)$ is invertible in this range of σ between $\mathcal{X}^{\tilde{r},l}_{\sigma}$ and $\overline{H}^{\tilde{r},l-1}_{(b)}$ (it is injective with closed range by the first estimate and has dense range by the third).*

Proof. We introduce the semiclassical parameter $h = |\sigma|^{-1}$, $z = h\sigma$ and the operator $\hat{T}_{s,h}(z) = h^2\hat{T}_s(h^{-1}z)$. Note that $z = z_0 + O(h)$ with $z_0 \in \{\pm 1\}$. We denote by $p_{h,z}$ the (semiclassical) principal symbol of $\hat{T}_{s,h}(z)$. To be concrete, we present the case $z_0 = 1$ for the first estimate (the other cases are similar).

The estimate is proved by using microlocal propagation of singularities and elliptic estimates in semiclassical second microlocal spaces on the model of [105, Section 5]. The difference in our case is that the non trapping hypothesis is not satisfied. We recall the structure of the proof in [105] to see where this assumption comes into place and why we can replace it by our analysis of the flow (Proposition 12.2.28) and the semiclassical estimates obtained previously (Propositions 13.1.4, 13.1.5, 13.2.3, 13.2.4, 13.3.6) and the hyperbolic semiclassical estimate[2] (Proposition A.1.7). By compactness of the semiclassical characteristic set on ${}^{sc}T^*X \cap \left\{x \leq \frac{1}{r_+ - \frac{\epsilon}{2}}\right\}$, it is enough to prove a microlocal estimate on a neighborhood of each point of the semiclassical characteristic set (near any point outside of the characteristic set, the estimate is obtained by ellipticity). Note that, with the hypotheses on $\tilde{r}$ and l, as proved in [105] and restated in Proposition 13.2.4, we have a source estimate near $\mathcal{R}_{\text{out}}$. Under the non trapping assumption, this can be used to initialize all the propagation of singularities estimates and the sink estimate at $\mathcal{R}_{\text{in}}$. In our case, by proposition 12.2.28, for every $x \in \Sigma_{p_h} \setminus \mathcal{R}_{\text{in}} \cup K_1$, there exists a neighborhood of V_x and $t \in \mathbb{R}$ such that $e^{tH_{p_h}}V_x \subset \mathcal{V}(\mathcal{R}_{\text{in}})$ or $\mathcal{V}(K_1)$ or $\mathcal{V}(L_+)$ or $\mathcal{V}(L_-)$ where $\mathcal{V}(A)$ is an arbitrary (but fixed) neighborhood of A. Therefore, a microlocal estimate of u in the neighborhood of all these sets is enough to initialize propagation of singularities, sink estimate at $\mathcal{R}_{\text{in}}$ (as proved in [105] and restated in Proposition 13.2.3) and semiclassical hyperbolic estimate on $\left\{x > \frac{1}{r_+ - \frac{\epsilon}{2}}\right\}$ (Proposition A.1.7). With

[2]Note that $\hat{T}_{s,h}(z)$ satisfies the hypotheses of proposition A.1.7 on $[r_+ - \frac{\epsilon}{2}, +\infty)$ and $\chi u \in \mathcal{H}^{\tilde{r}}_h$ if r plays the role of the time variable (and is affinely reparametrized so that $r_+ - \frac{\epsilon}{2}$ correspond to time 0 and $r_+ - \epsilon$ correspond to time T).

our assumptions on $\tilde{r}$ and l, we already have unconditional[3] estimates on a neighborhood of $\mathcal{R}_{\text{out}}$ (obtained in [105] and restated in Proposition 13.2.4), and on a neighborhood of L_+ and L_- (see Proposition 13.1.4). The only remaining estimate to get is on the neighborhood of K_1. By proposition 13.3.6, we have an estimate which require control on $WF_h(B_0)$ (where B_0 is as defined in Proposition 13.3.6 with $WF_h(B_0) \cap \Gamma_+ = \emptyset$). Since $WF_h(B_0) \cap \Gamma_+ = \emptyset$, we can get the estimate on $WF_h(B_0)$ by propagation of singularities only from the estimates on $\mathcal{R}_{\text{in}}$, L_+ and L_- which conclude the proof. □

13.5 Index zero property

Note that the spaces $\mathcal{X}_\sigma^{\tilde{r},l}$ and $\mathcal{W}_\sigma^{\tilde{r},l}$ depend on σ and this fact prevents us to use directly the stability of the index for $\hat{T}_s(\sigma)$. However, we can adapt the deformation argument in the proof of Theorem 6.1 in [42]. We begin by the following lemma which is a generalization of the stability of invertibility to the case of operators. The proof relies on an argument given in [106] (in the proof of (5.8)).

Lemma 13.5.1. *Let E, E_w, F, F_w be reflexive Banach spaces with continuous and compact inclusion $E \subset E_w$ and continuous inclusion $F \subset F_w$. and let $(P_j)_{j\in\mathbb{N}}$ be a sequence of bounded operators from E_w to F_w such that $\lim_{j\to+\infty} P_j = P_\infty$ in the operator norm topology. We also assume that we have uniform half Fredholm estimates for all $j \in \mathbb{N} \cup \{\infty\}$:*

$$\|u\|_E \leq C\left(\|P_j u\|_F + \|u\|_{E_w}\right)$$

in the strong sense that if the right-hand side is finite for some $u \in E_w$, then $u \in E$ and the inequality holds. Then, if $ker(P_\infty) \cap E = \{0\}$, there exists $N \in \mathbb{N}$ and $C' > 0$ such that for all $n \geq N$ (including $n = \infty$) and for all $u \in E_w$,

$$\|u\|_E \leq C' \|P_n u\|_F \tag{13.9}$$

in the strong sense that if the right-hand side is finite, then $u \in E$ and the inequality holds.

Remark 13.5.2. *In view of the half Fredholm estimate, $ker(P_\infty) \cap E = ker(P_\infty)$. Moreover, if $u \in E_w$ satisfies $P_n u \in F$ for some $n \in \mathbb{N} \cup \{\infty\}$, we get $u \in E$. Therefore, the strong character of the second estimate is a consequence of the strong character of the first.*

Proof. We argue by contradiction. If (13.9) is false, there exists a sequence $(j_n)_{n\in\mathbb{N}}$ in $\mathbb{N}\cup\{\infty\}$ with $\lim_{n\to+\infty} j_n = +\infty$ and $(u_n)_{n\in\mathbb{N}}$ with $\|u_n\|_E = 1$ such that $\lim_{n\to+\infty} P_{j_n} u_n = 0$ in F. There exists $v \in E$ such that $u_n \rightharpoonup v$ and by compactness of the inclusion of E into E_w, we have strong convergence in E_w. By the half Fredholm estimate, we have $\|v\|_{E_w} \geq C^{-1}$ and therefore $v \neq 0$. We now show that $P_\infty v = 0$. We estimate:

$$\begin{aligned}\|P_{j_n} u_n - P_\infty v\|_{F_w} &\leq \|(P_{j_n} - P_\infty)u_n\|_{F_w} + \|P_\infty(u_n - v)\|_{F_w} \\ &\leq \|P_{j_n} - P_\infty\|_{\mathcal{L}(E_w,F_w)} \|u_n\|_{E_w} + \|P_\infty\|_{\mathcal{L}(E_w,F_w)} \|u_n - v\|_{E_w}\end{aligned}$$

Finally, using that u_n converges to v in E_w (and in particular is bounded) and the convergence of P_{j_n} towards P_∞ in the operator norm topology, we find: $\lim_{n\to+\infty} P_{j_n} u_n = P_\infty v$ in F_w since the limit is 0 in F, we find $P_\infty v = 0$. Using the half Fredholm estimate, we find that $v \in E$ and this contradicts $ker(P_\infty) \cap E = \{0\}$. □

As a consequence of the previous lemma, we have the following proposition:

[3] In the sense which does not require an estimate in an other region of phase space.

Proposition 13.5.3. *Let E, E_w, E_s, F, F_w, F_s be reflexive Banach spaces such that $E_s \subset E \subset E_w$, $F_s \subset F \subset F_w$ are continuous and dense and $E \subset E_w$ and $F^* \subset F_s^*$ are compact. Let $(P_j)_{j\in\mathbb{N}}$ be a sequence of bounded operators in $\mathcal{L}(E_w, F_w)$ with $(P_j)_{|E_s} \in \mathcal{L}(E_s, F_s)$ such that $\lim_{j\to+\infty} P_j = P_\infty$ in both operator norm topologies (therefore we have the same convergence property for the adjoint operators). We also assume that we have uniform Fredholm estimates for all $j \in \mathbb{N}\cup\{\infty\}$:*

$$\|u\|_E \leq C\left(\|P_j u\|_F + \|u\|_{E_w}\right)$$
$$\|u\|_{F^*} \leq C\left(\left\|P_j^* u\right\|_{E^*} + \|u\|_{F_s^*}\right)$$

in the strong sense that if the right-hand side is finite, so is the left hand side and the inequality holds. Then, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $P_n : \{u \in E : P_n u \in F\} \to F$ and $P_\infty : \{u \in E : P_\infty u \in F\} \to F$ are Fredholm and have the same index.

Proof. The fact that P_n is Fredholm for any $n \in \mathbb{N}\cup\{\infty\}$ comes from lemma 13.4.1. Then we can adapt the standard Grushin problem proof of stability of the index[4] using lemma 13.5.1 on the operators and on their adjoint. □

As a consequence, we have that

Proposition 13.5.4. *Let $\sigma \in \mathbb{C}\setminus\{0\}$ with $\Im(\sigma) \geq 0$, $l < -\frac{1}{2}$, $\tilde{r} + l > -\frac{1}{2} - 2s$ and $\tilde{r} > \frac{1}{2} + s$. The index of $\hat{T}_s(\sigma)$ as a Fredholm operator from $\mathcal{X}_\sigma^{\tilde{r},l} \to \overline{H}_{(b)}^{\tilde{r},l-1}$ is zero.*

Proof. As stated in Remark 13.4.11, we know that $\hat{T}_s(\sigma)$ has an inverse for $\Re(\sigma)$ large enough. Moreover, the index is locally constant by proposition 13.5.3 (with $E := \overline{H}_{(b)}^{\tilde{r},l}$ and $F = \overline{H}_{(b)}^{\tilde{r},l-1}$, $E_w := \overline{H}_{(b)}^{\tilde{r}-1,l-1}$, $F_s := \overline{H}_{(b)}^{\tilde{r}+1,l}$, $E_s = \overline{H}_{(b)}^{\tilde{r}+3,l+1}$ and $F_w := \overline{H}_{(b)}^{\tilde{r}-3,l-2}$). □

Remark 13.5.5. *By Lemma 13.4.8, the index of $\hat{T}_s(\sigma)$ as a Fredholm operator from $\mathcal{W}_\sigma^{\tilde{r},l} \to \overline{H}_b^{\tilde{r},l-1}$ is non negative for all $\sigma \in \{\Im(\sigma) \geq 0, \sigma \neq 0\}$.*

13.6 Fredholm property for $\hat{T}_s(0)$

Near infinity, we have

$$\begin{aligned}
\hat{T}_s(0) =& \left(\frac{a^2}{\Delta_r} - \frac{1}{\sin^2\theta}\right)\partial_\phi^2 - \frac{1}{\sin\theta}\partial_\theta \sin\theta \partial_\theta - \Delta_r^{-s}\partial_r \Delta_r^{s+1}\partial_r \\
&+ 4s(r-M)\partial_r - 2s\left(\frac{a(r-M)}{\Delta_r} + i\frac{\cos\theta}{\sin^2\theta}\right)\partial_\phi + s^2\mathrm{cotan}^2\theta + s \\
\sigma_b(\hat{T}_s(0)) =& -\frac{a^2}{\Delta_r}\zeta_b^2 + |(\zeta_b, \eta_b)|_{\mathbb{S}^2}^2 + \xi_b^2 \\
N(\hat{T}_s(0)) =& -\Delta_{\mathbb{S}^2} + (xD_x)^2 + i(2s+1)xD_x - 4isxD_x + 2s\frac{\cos\theta}{\sin^2\theta}D_\phi + s^2\mathrm{cotan}^2\theta + s
\end{aligned}$$

We rewrite it:

$$N(\hat{T}_s(0)) = (xD_x)^2 + (i - 2is)xD_x + \Delta^{[s]} + s$$

where we have used the non negative spin weighted Laplacian (acting on $\mathcal{B}_s$ with spectrum $\{(l+s)(l-s)+l, l \in |s| + \mathbb{N}\}$):

$$\Delta^{[s]} := -\Delta_{\mathbb{S}^2} + \frac{2s\cos\theta}{\sin^2\theta}D_\phi + s^2\mathrm{cotan}^2\theta.$$

[4] See for example [29, Theorem C.5]

For $j \in |s| + \mathbb{N}$, we denote by Y_j the eigenspace of $\Delta^{[s]}$ associated with the eigenvalue $(j-s)(j+s)+j$. For any $\tilde{r} \in \mathbb{R}$, we denote by Π_j the operator on $H^{\tilde{r}}(\mathcal{B}_s)$ which is the identity on Y_j and which vanishes on all $Y_{j'}$ for $j' \neq j$ (it is well defined since $\oplus_{j'} Y_{j'}$ is dense in $H^{\tilde{s}}(\mathcal{B}_s)$ and it has norm smaller than 1). We denote by $\mathcal{Y}_j^{\tilde{r},l}$ the completion of $C_c^\infty((0, \frac{1}{r_+ - \epsilon}]) \otimes Y_j$ in the norm $\overline{H}_b^{\tilde{r},l}$. Note that $\mathrm{Id} \otimes \Pi_j$ (initially defined on $C_c^\infty((0, \frac{1}{r_+ - \epsilon}]) \otimes H^{\tilde{r}}(\mathcal{B}_s))$ extends by continuity to a bounded operator from $\overline{H}_{(b)}^{\tilde{r},l}$ to $\overline{H}_{(b)}^{\tilde{r},l}((0, \frac{1}{r_+ - \epsilon}], Y_j)$ which is the identity on $\mathcal{Y}_j^{\tilde{r},l}$ and zero on $\mathcal{Y}_p^{\tilde{r},l}$ for $p \neq j$. With a slight abuse of notation, we also denote this operator by Π_j. We define $\mathcal{Y}_{\leq j}^{\tilde{r},l} := \oplus_{p \leq j} \mathcal{Y}_p^{\tilde{r},l}$ and $\mathcal{Y}_{\geq j}^{\tilde{r},l} := \oplus_{p \geq j} \mathcal{Y}_p^{\tilde{r},l}$ (Hilbert sum). For $j \in \mathbb{N}$, we also introduce $\mathcal{S}_{\geq |s|+j}^{\tilde{r},l}$ as the completion of $C_c^\infty((0, \frac{1}{r_+ - \epsilon})) \otimes \left(\Gamma(\mathcal{B}_s) \cap \oplus_{\substack{|m| \geq |s|+j \\ m-s \in \mathbb{Z}}} \mathrm{Ker}(\partial_\phi - im) \right)$ in the norm of $\overline{H}_{(b)}^{\tilde{r},l}$.

Proposition 13.6.1. *Let $l \in (-\frac{3}{2} - s - |s|, -\frac{1}{2} - s + |s|)$ and $\tilde{r} > \frac{1}{2} + s$. $\hat{T}_s(0)$ is Fredholm from $\left\{ u \in H_{(b)}^{\tilde{r},l} : \hat{T}_s(0)u \in \overline{H}_{(b)}^{\tilde{r}-1,l} \right\}$ to $\overline{H}_{(b)}^{\tilde{r}-1,l}$. By lemma 13.4.7, it is also Fredholm from $\mathcal{W}_0^{\tilde{r},l}$ to $\overline{H}_{(b)}^{\tilde{r},l}$.*

Proof. We fix $\frac{1}{2} + s < \tilde{r}' < \tilde{r}$ and N a large integer such that $-N < 1 - \tilde{r}$. First note that the estimates of Proposition 13.1.8 still apply since we have not assumed $\sigma \neq 0$ in the proof. Moreover, using the principal symbol computation, we have that $\hat{T}_s(0) \in \Psi_b^{2,0}$ is elliptic in the b sense on $\left\{ x \leq \frac{1}{2M} \right\}$. Therefore, for all $u \in \overline{H}_{(b)}^{\tilde{r}',l}$ and $v \in \dot{H}_{(b)}^{-N,l}$ we get the global estimates:

$$\|u\|_{\overline{H}_{(b)}^{\tilde{r},l}} \leq C \left(\left\| \hat{T}_s(0)u \right\|_{\overline{H}_{(b)}^{\tilde{r}-1,l}} + \|u\|_{\overline{H}_{(b)}^{\tilde{r}',l}} \right)$$
$$\|v\|_{\dot{H}_{(b)}^{1-\tilde{r},-l}} \leq C \left(\left\| \hat{T}_s(0)^* v \right\|_{\overline{H}_{(b)}^{-\tilde{r},-l}} + \|v\|_{\overline{H}_{(b)}^{-N,-l}} \right)$$

To deduce a full Fredholm estimate, we improve the decay of the error term by a normal operator argument. To perform normal operator argument, we consider the slightly simpler conjugated operator:

$$\tilde{N}(\hat{T}_s(0)) := x^{-\frac{1}{2}} N(\hat{T}_s(0)) x^{\frac{1}{2}} \tag{13.10}$$
$$= (xD_x)^2 - 2isxD_x + \Delta^{[s]} + \frac{1}{4}. \tag{13.11}$$

The Mellin-transform of this operator is

$$M(\tau) := \tau^2 - 2is\tau + \Delta^{[s]} + \frac{1}{4}.$$

For $\Im(\tau) \in (-\frac{1}{2} + s - |s|, \frac{1}{2} + s + |s|)$, we have $\Re(\tau^2 - 2is\tau + \frac{1}{4}) > -|s|$. In particular, the operator $M(\tau)$ is invertible on each Y_j for $j \in |s| + \mathbb{N}$ with inverse $\left(\tau^2 - 2is\tau + (j-s)(j+s) + j + \frac{1}{4}\right)^{-1}$ which is bounded by $C\Re(\tau)^{-2}$ uniformly with respect to j when the imaginary part of τ is fixed in the interval $(-\frac{1}{2} + s - |s|, \frac{1}{2} + s + |s|)$. Let $w \in \overline{H}_{(b)}^{-N',l'}$ with support in $\{x \leq 1\}$, $l' \in (-\frac{3}{2} - s - |s|, -\frac{1}{2} - s + |s|)$ and $N' \in \mathbb{N}$. Assume that $N(\hat{T}_s(0))w \in \overline{H}_{(b)}^{\tilde{r}-1,l''}$ with $l'' \in (-\frac{3}{2} - s - |s|, -\frac{1}{2} - s + |s|)$, we have the following equality for $\Im(\tau) = -l'' + 1$:

$$\mathcal{M}x^{-\frac{1}{2}} w(\tau) = \sum_{j \in |s| + \mathbb{N}} \left(\tau^2 - 2is\tau + (j+s)(j-s) + j + \frac{1}{4} \right)^{-1} \Pi_j \mathcal{M} \left(\tilde{N}(\hat{T}_s(0)) x^{-\frac{1}{2}} w \right)$$

where the sum converges in $\langle \Re(\tau)\rangle^{-(\tilde{r}+1)} L^2(\mathbb{R}_{\Re(\tau)} \times \mathbb{S}^2)$ and is bounded by

$$\left\| C \langle \Re(\tau)\rangle^{\tilde{r}-1} \mathcal{M}\left(\tilde{N}(\hat{T}_s(0)) x^{-\frac{1}{2}} w(\tau)\right)\right\|_{L^2(\mathbb{R}_{\Re(\tau)}\times\mathbb{S}^2)}.$$

In particular, we get[5] that $w \in \overline{H}_{(b)}^{\tilde{r}+1,l''}$ and:

$$\|w\|_{\overline{H}_{(b)}^{\tilde{r}+1,l''}} \leq C \left\| N(\hat{T}_s(0)) w\right\|_{\overline{H}_{(b)}^{\tilde{r}-1,l''}}.$$

We get back to u as defined in the beginning of the proof. Let χ be a smooth cutoff with support in $\{x \leq 1\}$ equal to 1 in a neighborhood of $\{x = 0\}$. Since $N(\hat{T}_s(0)) - \hat{T}_s(0) \in x\mathrm{Diff}_b^2$ and $\hat{T}_s(0)u \in \overline{H}^{\tilde{r}-1,l}$, we deduce that $N(\hat{T}_s(0))\chi u \in \overline{H}^{\tilde{r}'-2,l}$. Therefore, we have:

$$\begin{aligned}
\|u\|_{\overline{H}_{(b)}^{\tilde{r}',l}} \leq& \|u\|_{\overline{H}_{(b)}^{\tilde{r}',l-1}} + \|\chi u\|_{\overline{H}_{(b)}^{\tilde{r}',l}} \\
\leq& \|u\|_{\overline{H}_{(b)}^{\tilde{r}',l-1}} + C\left\|N(\hat{T}_s(0))\chi u\right\|_{\overline{H}_{(b)}^{\tilde{r}'-2,l}} \\
\leq& C'\left(\|u\|_{\overline{H}_{(b)}^{\tilde{r}',l-1}} + \left\|\hat{T}_s(0)\chi u\right\|_{\overline{H}_{(b)}^{\tilde{r}'-2,l}}\right) \\
\leq& C''\left(\|u\|_{\overline{H}_{(b)}^{\tilde{r}',l-1}} + \left\|\hat{T}_s(0)\chi u\right\|_{\overline{H}_{(b)}^{\tilde{r}-1,l}}\right).
\end{aligned}$$

Using this in estimate (13.10), we get

$$\|u\|_{\overline{H}_{(b)}^{\tilde{r},l}} \leq C\left(\left\|\hat{T}_s(0)u\right\|_{\overline{H}_{(b)}^{\tilde{r}-1,l}} + \|u\|_{\overline{H}_{(b)}^{\tilde{r}',l-1}}\right).$$

We can do a similar normal operator argument for the adjoint $\hat{T}_s(0)^*$ (but this time, we find that the decay index has to belong to $(\frac{1}{2}+s-|s|\,,\frac{3}{2}+s+|s|)$) and we get an estimate of the form:

$$\|v\|_{\dot{H}_{(b)}^{1-\tilde{r},-l}} \leq C\left(\left\|\hat{T}^*(0)v\right\|_{\dot{H}_{(b)}^{-\tilde{r},-l}} + \|v\|_{\dot{H}_{(b)}^{-\tilde{r},-l-1}}\right).$$

Finally, by Lemma 13.4.1, we get the proposition. □

We now describe precisely the kernel of $\hat{T}_s(0)$ and $\hat{T}_s(0)^*$. To do so, it is easier to write $\hat{T}_s(0)$ in (r,θ,ϕ_*) coordinates:

$$\begin{aligned}
\hat{T}_s(0) &= -\Delta_r\partial_r^2 + 2((r-M)(s-1) - a\partial_{\phi_*})\partial_r + \Delta^{[s]} + s \\
r^2\hat{T}_s(0)^* r^{-2} &= -\Delta_r\partial_r^2 + 2((r-M)(-s-1) - a\partial_{\phi_*})\partial_r + \Delta^{[s]} - s
\end{aligned}$$

Where the adjoint is taken with respect to the density $r^2\sin(\theta)\,\mathrm{d}r\,\mathrm{d}\phi^*\,\mathrm{d}\theta$. For $l \in |s| + \mathbb{N}$ and $-l \leq m \leq l$ (with $m - s \in \mathbb{Z}$), we denote by $f_{l,m}$ the normalized spin weighted spherical harmonics associated to the eigenvalue $(l+s)(l-s)+l$ and such that $\partial_{\phi_*} f_{l,m} = imf_{l,m}$. These functions form a Hilbert basis of $L^2(\mathcal{B}_s)$.

Following [85, Section 9], we introduce the hypergeometric function

$$\mathbf{F}(a,b,c,z) := \sum_{k=0}^{+\infty} \frac{(a)_k (b)_k}{\Gamma(c+k)} \frac{z^k}{k!}$$

where $(a)_k := \prod_{j=0}^{k-1}(a+j)$.

[5] See Lemma 14.2.1 and 14.2.2 for more details about this step

Proposition 13.6.2. *Let $\hat{T}_s^*(0)$ denotes the adjoint of $\hat{T}_s(0)$ with respect to the volume form $r^2 \sin\theta \, \mathrm{d}r \, \mathrm{d}\phi^* \, \mathrm{d}\theta$. We have:*

$$ker(\hat{T}_s(0)) \cap \overline{H}_{(b)}^{s+\frac{1}{2},q} = \bigoplus_{\substack{|s|\leq l<-\frac{3}{2}-s-q \\ |m|\leq l}} \mathbb{C}u_{l,m}$$

$$ker(\hat{T}_s(0)^*) \cap \dot{H}_{(b)}^{\left(-s+\frac{1}{2}\right)-,-q} = \bigoplus_{\substack{|s|\leq l<\frac{1}{2}+s+q \\ |m|\leq l}} \mathbb{C}u_{l,m}^*$$

where

	If $m \neq 0$ or $s \notin \mathbb{Z}$
$u_{l,m}$	$\mathbf{F}\left(-l-s, 1+l-s, 1-s+\frac{2iam}{r_+-r_-}, \frac{r_+-r}{r_+-r_-}\right) f_{l,m}(\theta,\phi_*)$
$u_{l,m}^*$	$r^{-2}\frac{(r-r_+)_+^{-s-\frac{2iam}{r_+-r_-}}}{(r_+-r_-)^{-s-\frac{2iam}{r_+-r_-}}}\mathbf{F}\left(-l-\frac{2iam}{r_+-r_-}, 1+l-\frac{2iam}{r_+-r_-}, 1-s-\frac{2iam}{r_+-r_-}, \frac{r_+-r}{r_+-r_-}\right) f_{l,m}(\theta,\phi_*)$

	If $m = 0$ and $s \in -1-\mathbb{N}$
$u_{l,m}$	$\mathbf{F}\left(-l-s, 1+l-s, 1-s, \frac{r_+-r}{r_+-r_-}\right) f_{l,m}(\theta,\phi_*)$
$u_{l,m}^*$	$r^{-2}\mathbb{1}_{(r_+,+\infty)}\left(\frac{r-r_+}{r_+-r_-}\right)^{-s}\left(\frac{r-r_-}{r_+-r_-}\right)^{-s}\mathbf{F}\left(-l-s, 1+l-s, 1-s, \frac{r_+-r}{r_+-r_-}\right) f_{l,m}(\theta,\phi_*)$

	If $m = 0$ and $s \in \mathbb{N}$
$u_{l,m}$	$\left(\frac{r-r_+}{r_+-r_-}\right)^{s}\left(\frac{r-r_-}{r_+-r_-}\right)^{s}\mathbf{F}\left(-l+s, 1+l+s, 1+s, \frac{r_+-r}{r_+-r_-}\right) f_{l,m}(\theta,\phi_*)$
$u_{l,m}^*$	$r^{-2}\left(\frac{(l+s)!}{(l-s)!}(r_+-r_-)^{-s}\mathbb{1}_{(r_+,+\infty)}\mathbf{F}\left(-l+s, 1+l+s, 1+s, \frac{r_+-r}{r_+-r_-}\right)\right.$ $\left.+\sum_{j=0}^{s-1}\frac{(l+s-1-j)!}{(l-s+1+j)!(s-1-j)!}(r_+-r_-)^{-s+1+j}\delta_{r_+}^{(j)}\right) f_{l,m}(\theta,\phi_*)$

Proof. First, we decompose $u \in \overline{H}_{(b)}^{s+\frac{1}{2},q}$ on $\oplus_{l\geq|s|,-l\leq m\leq l}\overline{H}_{(b)}^{s+\frac{1}{2},q}([r_+-\epsilon,+\infty)) \otimes f_{l,m}$ (we denote by $u_{l,m}$ the element of $\overline{H}_{(b)}^{s+\frac{1}{2},q}([r_+-\epsilon,+\infty))$ in this decomposition). If $u \in \ker(\hat{T}_s(0))$, we have

$$\begin{aligned}\hat{T}(0)_{l,m,s}u_{l,m} :=& (-\Delta_r\partial_r^2 + 2((r-M)(s-1)-aim)\partial_r + (l-s+1)(l+s))u_{l,m} \\ =& 0.\end{aligned}$$

This is a hypergeometric equation and we can put it in canonical form (see [85] chapter 5 section 8.1) by the change of variable: $z = \frac{r-r_-}{r_+-r_-}$. With this new variable, we have:

$$\hat{T}(0)_{l,m,s} = z(1-z)\partial_z^2 + \left(1-s-\frac{2iam}{r_+-r_-}-(2-2s)z\right)\partial_z + (l-s+1)(l+s) \tag{13.12}$$

Defining $\alpha := -l-s$, $\beta := l-s+1$ and $\gamma := 1-s-\frac{2iam}{r_+-r_-}$ we get the canonical hypergeometric equation:

$$\hat{T}(0)_{l,m,s} = z(1-z)\partial_z^2 + (\gamma-(1+\alpha+\beta)z)\,\partial_z - \alpha\beta.$$

Note that we can in the same way decompose $v \in \dot{H}^{\tilde{r},-q-2}$ and if we assume that $r^{-2}v \in \mathrm{Ker}(T_s(0)^*)$, we get that for all $l \in |s|+\mathbb{N}$ and $m \in \mathbb{Z}+l$ with $|m| \leq l$:

$$\hat{T}(0)_{l,m,-s}v_{l,m} = 0$$

Therefore, we are reduced to finding for all $s \in \frac{1}{2}\mathbb{Z}$ and l, m as above the intersections $\mathrm{Ker}(\hat{T}(0)_{l,m,s}) \cap \overline{H}^{\frac{1}{2}+s,q}_{(b)}$ and $\mathrm{Ker}(\hat{T}(0)_{l,m,s}) \cap \dot{H}^{\tilde{r},-q-2}_{(b)}$. Therefore, we fix $s \in \frac{1}{2}\mathbb{Z}$, $l \in |s| + \mathbb{N}$ and $|m| \leq l$, $u_{m,l} \in \mathrm{Ker}(\hat{T}(0)_{l,m,s}) \cap \overline{H}^{\frac{1}{2}+s,q}_{(b)}$ and $v_{l,m} \in \mathrm{Ker}(\hat{T}(0)_{l,m,s}) \cap \dot{H}^{\tilde{r},-q-2}_{(b)}$.

We have three cases (because of possible integer coincidence):

- If $s \notin \mathbb{Z}$ or $m \neq 0$. In this case $1-\gamma \notin \mathbb{Z}$ and $1+\gamma-\alpha-\beta \notin \mathbb{Z}$. We have two independent solutions of the equation $\hat{T}(0)_{l,m,s}u = 0$:

$$u_1 := \mathbf{F}(\alpha, \beta, \alpha+\beta+1-\gamma, 1-z)$$

which is an analytic solution on $\mathbb{C} \setminus (-\infty, 0]$ and equal to $\frac{1}{\Gamma(\alpha+\beta+1-\gamma)}$ at $z = 1$ and

$$u_2 := (z-1)^{\gamma-\alpha-\beta}\mathbf{F}(\gamma-\beta, \gamma-\alpha, 1+\gamma-\alpha-\beta, 1-z)$$

(where we use the continuous branch of the logarithm defined on $\mathbb{C} \setminus (-\infty, 0]$ which coincides with the usual logarithm on $(0, +\infty)$) which is an analytic solution on $\mathbb{C} \setminus (-\infty, 1]$. Note that u_1 and u_2 are linearly independent. By the study of the regular singularity at infinity (by Frobenius method), we see that the indicial roots are α and β (see for example [85, Chapter 5, Section 5]). Since $\beta - \alpha \in 1 + \mathbb{N}$, we have an integer coincidence and we can find a connection formula analogous to (10.13) in [85]. It gives the following form for u_1 and u_2 when $z \in \mathbb{C} \setminus (-\infty, 1]$:

$$u_1(z) = \frac{(\beta-\alpha-1)!}{\Gamma(1-\gamma+\beta)\Gamma(\beta)} z^{-\alpha} H(z^{-1}) + c_1 z^{-\beta}\ln(z) G(z^{-1}) \tag{13.13}$$

$$u_2(z) = \frac{(\beta-\alpha-1)!}{\Gamma(\gamma-\alpha)\Gamma(1-\alpha)} z^{-\alpha} H(z^{-1}) + c_2 z^{-\beta}\ln(z) G(z^{-1}) \tag{13.14}$$

where H is holomorphic on $\mathbb{C} \setminus [1, +\infty)$ and $H(0) = 1$ and G is holomorphic on $\mathbb{C} \setminus [1, +\infty)$ and c_1, c_2 are constants. In particular since $\frac{(\beta-\alpha-1)!}{\Gamma(1-\gamma+\beta)\Gamma(\beta)} \neq 0$ we have $u_1(z) \sim \frac{(\beta-\alpha-1)!}{\Gamma(1-\gamma+\beta)\Gamma(\beta)} z^{-\alpha}$ when $z \to +\infty$. We know that $(u_{l,m})_{|_{(r_+,+\infty)}} = \lambda_1 u_1\left(\frac{r-r_-}{r_+-r_-}\right) + \lambda_2 u_2\left(\frac{r-r_-}{r_+-r_-}\right)$. In particular, $u_{l,m} - \lambda_1 u_1\left(\frac{r-r_-}{r_+-r_-}\right)$ is an extension of $\lambda_2 u_2\left(\frac{r-r_-}{r_+-r_-}\right)$ in $\overline{H}^{s+\frac{1}{2},loc}_{(b)}((r_+-\epsilon, +\infty))$. But this is impossible except if $\lambda_2 = 0$. Indeed, $u_2\left(\frac{r-r_-}{r_+-r_-}\right) = \left(\frac{r-r_+}{r_+-r_-}\right)^{\gamma-\alpha-\beta}\Gamma(1+\gamma-\alpha-\beta)^{-1} + \left(\frac{r-r_+}{r_+-r_-}\right)^{\gamma-\alpha-\beta+1} F(r-r_+)$ with F extending smoothly on $[r_+-\epsilon, +\infty)$. In particular, since $\left(\frac{r-r_++i0}{r_+-r_-}\right)^{\gamma-\alpha-\beta+1} F(r-r_+) \in \overline{H}^{(\frac{3}{2}+s)-,loc}_{(b)}$, u_2 has an extension in $\overline{H}^{\frac{1}{2}+s,loc}_{(b)}$ if and only if $\left(\frac{r-r_+}{r_+-r_-}\right)^{\gamma-\alpha-\beta}\Gamma(1+\gamma-\alpha-\beta)^{-1}$ has a such an extension which is not the case. Therefore, $(u_{l,m})_{|_{(r_+,+\infty)}} = \lambda_1 u_1$ and a similar argument shows that $(u_{l,m})_{|_{(r_+-\epsilon,r_+)}} = \lambda_1' u_1$. We deduce that there exists $a_0, ..., a_k$ such that $v_1 := (\lambda_1 - \lambda_1')\mathbb{1}_{(r_+,+\infty)}u_1 + \sum_{j=0}^k a_j \delta_{r_+}^{(j)}$ is a solution in $\overline{H}^{s+\frac{1}{2}}_{(b)}$. Using the equation, we have:

$$\begin{aligned}\hat{T}(0)_{m,l,s}v_1 =& 2(\lambda_1-\lambda_1')u_1(r_+)\left((r_+-M)s - aim\right)\delta_{r_+} \\ &+ \sum_{j=0}^{k}\Big(2a_j((r_+-M)(s+j+1) - aim)\delta_{r_+}^{(j+1)} - a_j((j+2)(j+1) \\ &\qquad +2(s-1)(j+1) - (l-s+1)(l+s))\delta_{r_+}^{(j)}\Big)\end{aligned}$$

Therefore, the coefficient in front of $\delta_{r_+}^{(k+1)}$ vanishes:

$$2a_k((r_+ - M)(s+k+1) - aim) = 0$$

Since we assumed that $m \neq 0$ or $s \notin \mathbb{Z}$, we get $a_k = 0$. By induction, we obtain $a_0 = ... = a_k = 0$. Finally, the vanishing of the coefficient of δ_{r_+} gives $\lambda_1 = \lambda_1'$. We conclude that $u_{l,m} \in \mathbb{C}u_1\left(\frac{r-r_-}{r_+-r_-}\right)$. However, in view of (13.13), we have $u_1 \in \overline{H}_{(b)}^{\frac{1}{2}+s,(-\frac{3}{2}-l-s)-}$ (and no better in term of decay). Therefore, $u_{l,m}$ can be non zero if and only if $q < -\frac{3}{2} - l - s$. We now determine the form of $v_{l,m}$. Since $v_{l,m}$ is a distribution supported on $[r_+ - \epsilon, +\infty)$ and is the solution to a non degenerate differential equation on (r_-, r_+), it has to be supported on $[r_+, +\infty)$. We also have that $(v_{l,m})_{|_{(r_+,+\infty)}} = \lambda_1 u_1 + \lambda_2 u_2$. Since $u_2\left(\frac{r-r_+}{r_+-r_-}\right) = (r-r_+)^{\gamma-\alpha-\beta}F(z)$ where F is smooth near $r - r_+$ and $\gamma - \alpha - \beta \notin \mathbb{Z}$, we have that $\tilde{u}_2 := (r-r_+)_+^{\gamma-\alpha-\beta}F(z)$ (where $x_+^{\gamma-\alpha-\beta}$ denote the unique homogeneous distribution supported in $[0,+\infty)$ whose restriction to $(0,+\infty)$ is $x^{\gamma-\alpha-\beta}$) is in $ker(\hat{T}(0)_{l,m,s}) \cap \dot{H}_{(b)}^{(\frac{1}{2}+s)-,loc}$. Therefore, $v_{l,m} - \lambda_2\tilde{u}_2 = \lambda_1 \mathbb{1}_{[0,+\infty)}u_1\left(\frac{r-r_+}{r_+-r_-}\right) + \sum_{j=0}^k c_k\delta_{r_+}^{(j)} \in ker(\hat{T}(0)_{l,m,s})$. But we have seen that in this case $\lambda_1 = a_0 = ... = a_k = 0$. Therefore,

$$v_{l,m} = \lambda_2\tilde{u}_2\left(\frac{r-r_+}{r_+-r_-}\right).$$

Using (13.13), we see that $\tilde{u}_2 \in \dot{H}_{(b)}^{(\frac{1}{2}+s)-,-\frac{3}{2}-s-l}$ and therefore λ_2 can be non zero if and only if $-q-2 < -\frac{3}{2} - s - l$.

- In the case of $m = 0$ and $s \in -\mathbb{N}$, we have $\gamma - \alpha - \beta = s \in -\mathbb{N}$. A basis of solutions for (13.12) is given by:

$$\begin{aligned}
u_1(z) :=& \mathbf{F}(\alpha,\beta,1-s,1-z)\\
u_2(z) :=& \tilde{\mathbf{H}}(\alpha,\beta,1-s,1-z)\\
:=& (-1)^s(\partial_c)_{|_{c=1-s}}\left((z-1)^{1-c}\mathbf{F}(\alpha+1-c,\beta+1-c,2-c,1-z)\right.\\
&\left. -e^{i\pi(1-c)}\prod_{j=1}^{-s}(j-\alpha)(j-\beta)\mathbf{F}(\alpha,\beta,c,1-z)\right)
\end{aligned}$$

u_1 is holomorphic on $\mathbb{C}\setminus(-\infty,0]$ and $u_1(1) = \frac{1}{(-s)!}$. u_2 is holomorphic on $\mathbb{C}\setminus(-\infty,1]$ and near $z=1$, we have:

$$u_2 = \ln(z-1)F(z-1) + (1-z)^s G(z-1) \tag{13.15}$$

with F and G holomorphic near 0, $F(0) = -\frac{(\alpha+s)_{-s}(\beta+s)_{-s}}{(-s)!}$ (where $(x)_k := \prod_{j=0}^{k-1}(x+j)$) and if $-s-1 \geq 0$, $G(0) = (-1)^{-s}(-s-1)!$. We also have that for $z \in \mathbb{C}\setminus(-\infty,1]$:

$$u_1(z) = \frac{z^{l+s}(2l)!}{l!(l-s)!}G(z^{-1}) + z^{-1-l+s}\ln(z)F(z^{-1}) \tag{13.16}$$

$$u_2(z) = z^{l+s}G_2(z^{-1}) + z^{-1-l+s}\ln(z)F(z^{-1}) \tag{13.17}$$

with F, G and G_2 holomorphic near zero with $G(0) = 1$ and Ψ denote the digamma function (see [85, Chapter 2, Section 2]).

As before, we use the fact that $u_2\left(\frac{r-r_-}{r_+-r_-}\right)$ has no extension in $\overline{H}^{\frac{1}{2}+s,loc}_{(b)}((r_+-\epsilon,+\infty))$ to conclude that there exists $\lambda \in \mathbb{C}$ such that $v_1 := u_{l,m} - \lambda u_1\left(\frac{r-r_-}{r_+-r_-}\right) = \frac{a_{-1}}{u_1(r_+)}\mathbb{1}_{(r_+-\epsilon,+\infty)}u_1 + \sum_{j=0}^{k} a_j\delta^{(j)}_{r_+}$. First note that since $v_1 = u_{l,m} - \lambda u_1 \in \overline{H}^{\frac{1}{2}+s,loc}_{(b)}$, only the a_j with $-1 \leq j < -1-s$ can be non zero therefore we can assume that $k < -1-s$ (in particular if $s = 0$, we get immediately $v_1 = 0$). Moreover we have:

$$\hat{T}(0)_{m,l,s}v_1 = \sum_{j=0}^{k}\Big(2a_j(r_+-M)(s+j+1)\delta^{(j+1)}_{r_+} \tag{13.18}$$

$$-a_j((j+2)(j+1)+2(s-1)(j+1)-(l-s+1)(l+s))\delta^{(j)}_{r_+}\Big) \tag{13.19}$$

$$+2a_{-1}(r_+-M)s\delta_{r_+} \tag{13.20}$$

and since $\hat{T}(0)_{l,m,s}v_1 = 0$ we deduce that:

$$2a_k(r_+-M)(s+k+1) = 0$$

and therefore, $a_k = 0$. By induction, we get that $v_1 = 0$ and $u_{l,m} \in \mathbb{C}u_1\left(\frac{r-r_-}{r_+-r_-}\right)$. Finally, using (13.16) we deduce that $u_{l,m}$ can be non zero if and only if $q < -\frac{3}{2} - l - s$. We now consider $v_{l,m}$. As before, $(v_{l,m})_{|_{(r_+-\epsilon,r_+)}} = 0$ and $(v_{l,m})_{|_{(r_+,+\infty)}} = \lambda_1 u_1 + \lambda_2 u_2$. We define $a_{-s-1} := 1$ and for all $-1 \leq j \leq -s-2$:

$$a_j = \frac{a_{j+1}((j+3)(j+2)+2(s-1)(j+2)-(l-s+1)(l+s))}{2(r_+-M)(s+j+1)}.$$

Note that if $s = 0$, we have only one term $a_{-1} = 1$ in the sequence and if $s < 0$, using that $((j+3)(j+2)+2(s-1)(j+2)-(l-s+1)(l+s)) = (j+2+l+s)(j+1-l+s)$ we see that a_j never vanishes for $-1 \leq j \leq -s-2$ and therefore $a_{-1} \neq 0$. We can solve explicitly the recurrence relation and we find:

$$a_j = \frac{(l-s-1-j)!}{(l+s+1+j)!(-s-1-j)!}(r_+-r_-)^{s+1+j}$$

We have (according to (13.18)) $\tilde{u}_1 := \frac{a_{-1}}{u_1(r_+)}\mathbb{1}_{(r_+,+\infty)}u_1 + \sum_{j=0}^{-s-1} a_j\delta^{(j)}_{r_+}$ is a solution in $\dot{H}^{(\frac{1}{2}+s)-,loc}_{(b)}$. As a consequence $v_1 := v_{l,m} - \frac{\lambda_1 u_1(r_+)}{a_{-1}}\tilde{u}_1$ is of the form $\lambda_2\tilde{u}_2 + \sum_{j=0}^{k} d_j\delta^{(j)}_{r_+}$ where $\tilde{u}_2$ is an extension of u_2 vanishing on $(-\infty, r_+)$. We introduce the distribution $\ln(x)_+$ which is the L^1_{loc} function equal to zero on $(-\infty, 0)$ and to $\ln(x)$ on $(0,+\infty)$. Since $u_2(z) = \ln(z-1)F(z-1) + (z-1)^s G(z-1)$ with F and G holomorphic near zero, we can choose $\tilde{u}_2$ of the form $\ln\left(\frac{r-r_+}{r_+-r_-}\right)_+ F\left(\frac{r-r_+}{r_+-r_-}\right) + \frac{(r-r_+)^s_+}{(r_+-r_-)^s}G\left(\frac{r-r_+}{r_+-r_-}\right)$ (where $(r-r_+)^s_+$ is defined for example in [53] equation (3.2.5)) if $-s \geq 1$ and of the form $\ln\left(\frac{r-r_+}{r_+-r_-}\right)_+ F\left(\frac{r-r_+}{r_+-r_-}\right) + \mathbb{1}_{(r_+,+\infty)}G\left(\frac{r-r_+}{r_+-r_-}\right)$ if $s = 0$. When $s = 0$ we can compute $\hat{T}(0)_{l,m,s}\tilde{u}_2 = -(r_+-r_-)^2F(0)\delta_{r_+}$ (using for example that in the sense of distributions $\tilde{u}_2 = \lim_{\epsilon\to 0}\mathbb{1}_{(r_++\epsilon,+\infty)}u_2$) we show as before that every coefficient $d_j = 0$, then $\lambda_2 = 0$ and finally $v_1 = 0$. If $s < 0$, we consider $(r-r_+)^{-s}\tilde{u}_2$ which is of the form $\lim_{\epsilon\to 0}\mathbb{1}_{(r_++\epsilon,+\infty)}u_3$

where u_3 is a solution of $(r-r_+)^{-s}\hat{T}(0)_{l,m,s}(r-r_+)^s u_3 = 0$. We compute

$$(r-r_+)^{-s}\hat{T}(0)_{l,m,s}(r-r_+)^s = -\Delta_r\partial_r^2 + 2((r_- - M)s + M - r)\partial_r + (l-s)(l+s) + l + s^2$$
$$\lim_{\epsilon\to 0}[(r-r_+)^{-s}\hat{T}(0)_{l,m,s}(r-r_+)^s, \mathbb{1}_{(r_++\epsilon,+\infty)}]u_3 = -s(r_+ - r_-)^{1-s}G(0)\delta_{r_+}$$

and therefore $(r-r_+)^{-s}\hat{T}(0)_{l,m,s}\tilde{u}_2 = -s(r_+-r_-)^{1-s}G(0)\delta_{r_+}$. On the other hand, we know that $\hat{T}(0)_{l,m,s}\tilde{u}_2$ is of the form $\sum_{j=0}^N c_j\delta_{r_+}^{(j)}$. We deduce that $N=-s$ and $c_{-s} = \frac{(-1)^{-s}(r_+-r_-)^{1-s}}{(-s-1)!}G(0)$. Finally, using that $G(0)\neq 0$ we show all the d_j and λ_2 have to vanish and $v_1 = 0$. Therefore, we conclude in both cases that $v_{l,m}\in\mathbb{C}\tilde{u}_1$. Using (13.16), we see that $v_{l,m}$ can be a non trivial multiple of $\tilde{u}_1$ if and only if $-q-2 < -\frac{3}{2} - l - s$.

- In the case $m=0$, $s\in 1+\mathbb{N}$: We have the two independent solutions:

$$u_1 = (z-1)^s z^s \mathbf{F}(-l+s, 1+l+s, 1+s, 1-z)$$
$$u_2 = (z-1)^s z^s \tilde{\mathbf{H}}(-l+s, 1+l+s, 1+s, 1-z)$$

Where $\tilde{\mathbf{H}}$ is the function defined in the previous case. Note that near $z=r_+$:

$$u_2\left(\frac{r-r_-}{r_+-r_-}\right) = -\frac{(-l)_s(1+l)_s}{s!}\left(\frac{r-r_+}{r_+-r_-}\right)^s \ln\left(\frac{r-r_+}{r_+-r_-}\right) + \overline{H}_{(b)}^{\frac{1}{2}+s,loc}$$

and therefore has no extension in $\overline{H}_{(b)}^{\frac{1}{2}+s,loc}$ and there exists $\lambda_1,\lambda_1'\in\mathbb{C}$ such that $(u_{l,m})_{|_{(r_+-\epsilon,r_+)}} = \lambda_1 u_1$ and $(u_{l,m})_{|_{(r_+,+\infty)}} = \lambda_1' u_1$. We prove that $\lambda_1=\lambda_1'$ and $u_{l,m}\in\mathbb{C}u_1\left(\frac{r-r_-}{r_+-r_t}\right)$ using that $u_{l,m}$ is in particular continuous by Sobolev embedding (so there is no Dirac masses at r_+), and $(\lambda_1-\lambda_1')(z-1)^s\mathbb{1}_{(r_+,+\infty)}$ is in $H^{s+\frac{1}{2},loc}$) if and only if $\lambda_1=\lambda_1'$. Since by (13.16), $u_1\left(\frac{r-r_-}{r_+-r_-}\right)\in\overline{H}_{(b)}^{\infty,(-\frac{3}{2}-l-s)-}$, we deduce that $u_{l,m}$ can be a non zero multiple of u_1 if and only if $q<-\frac{3}{2}-l-s$.

We now consider $v_{l,m}$. As before, $v_{l,m}=0$ on $(r_+-\epsilon,r_+)$ and $v_{l,m}=\lambda_1u_1+\lambda_2u_2$ on $(r_+,+\infty)$. Noting that $\tilde{u}_1 := \mathbb{1}_{(r_+,+\infty)}u_1$ is a distribution solution of the equation, we deduce that $v_1 := v_{l,m} - \lambda_1\tilde{u}_1$ is also a solution of the equation supported in $[r_+,+\infty)$. It is of the form $\lambda u_2 + \sum_{j=0}^N c_j\delta_{r_+}^{(j)}$ where $\tilde{u}_2 := \mathbb{1}_{(r_+,+\infty)}u_2$ (which is in L^1_{loc}). Using that $\tilde{u}_2 = \lim_{\epsilon\to 0}\mathbb{1}_{(r_++\epsilon,+\infty)}u_2$ and the expression of $\tilde{\mathbf{H}}(-l+s,1+l+s,1+s,1-z)$ near $z=1$, we compute:

$$\begin{aligned}
\hat{T}(0)_{l,m,s}\tilde{u}_2 =& (-1)^s s!(r_+-r_-)\delta_{r_+}\\
\hat{T}(0)_{l,m,s}\left(\lambda_2\tilde{u}_2 + \sum_{j=0}^N c_j\delta_{r_+}^{(j)}\right) =& \lambda_2(-1)^s s!(r_+-r_-)\delta_{r_+} + \sum_{j=0}^N 2c_j(r_+-M)(s+j+1)\delta_{r_+}^{(j+1)}\\
&- \sum_{j=0}^N c_j(j+2+l+s)(j+1-l+s)\delta_{r_+}^{(j)}
\end{aligned}$$

and we prove by induction that $c_N = ... = c_0 = 0$ and $\lambda_2 = 0$. As a consequence, $v_{l,m}\in\mathbb{C}\tilde{u}_1$ and it can be non zero if and only if $-q-2<-\frac{3}{2}-l-s$.

□

Chapter 14

Existence, boundedness and regularity of the resolvent

Proposition 14.0.1. *If $l < -\frac{1}{2}$, $\tilde{r} + l > -\frac{1}{2} - 2s - 4M\Im(\sigma)$ and $\tilde{r} > \frac{1}{2} + s - \frac{a^2+r_+^2}{r_+-M}\Im(\sigma)$, then the operator $\hat{T}_s(\sigma)$ is invertible from $\mathcal{X}_\sigma^{\tilde{r},l}$ to $\overline{H}^{\tilde{r},l-1}$ when $\Im(\sigma) \geq 0$ and $\sigma \neq 0$. If*

$$-\frac{3}{2} - s - |s| < l < -\frac{1}{2},$$
$$\tilde{r} + l > -\frac{1}{2} - 2s,$$
$$\tilde{r} > \frac{1}{2} + s,$$

the operator $\hat{T}_s(\sigma)$ is invertible from $\mathcal{W}_\sigma^{\tilde{r},l}$ to $\overline{H}^{\tilde{r},l}$ when $\Im(\sigma) \geq 0$.

Proof. We have already proved that the operators are Fredholm with non negative index (see Proposition 13.5.4 and Remark 13.5.5). Therefore, it remains to prove that their kernel is trivial. This is a consequence of [108] for $\Im(\sigma) > 0$, of the generalization [8] for $\Im(\sigma) = 0$, $\sigma \neq 0$. For the precise proof that the absence of mode for the radial equation imply the triviality of the kernel in our case, see the proof of case (1) in Theorem 4.5 in [7](which is based on the precise analysis of the spin-weighted spheroidal wave operator with complex aspherical parameter performed in [31]). The case $\sigma = 0$ is a consequence of the explicit computation of the kernel and cokernel of $\hat{T}_s(0)$ given in Proposition 13.6.2, see also [7]. □

As a consequence of the previous proposition, we can define the resolvent operator $R(\sigma) = \hat{T}_s(\sigma)^{-1}$ for $\Im(\sigma) \geq 0$. The following property summarizes its basic properties.

Proposition 14.0.2. *Let $c > 0$. For every $\eta \in [0,1]$ $R(\sigma)$ is a bounded operator from $\overline{H}_{(b)}^{\tilde{r},l}$ to $\overline{H}_{(b)}^{\tilde{r},l+1-\eta}$ for $\Im(\sigma) \geq 0$, $\sigma \neq 0$, $|\sigma| \leq c$ and*

$$-\frac{3}{2} - s - |s| < l + 1 - \eta < -\frac{1}{2}$$
$$\tilde{r} + l + 1 - \eta > -\frac{1}{2} - 2s$$
$$\tilde{r} > \frac{1}{2} + s.$$

Moreover, in this case, we have the following bound (uniform in $|\sigma| \leq c$ for c small enough):

$$\|R(\sigma)\|_{\mathcal{L}(\overline{H}_{(b)}^{\tilde{r},l}, \overline{H}_{(b)}^{\tilde{r},l+1-\eta})} \leq C|\sigma|^{\eta-1} \tag{14.1}$$

It is also a bounded operator from $\overline{H}^{\tilde{r},l}_{(b)}$ *to* $\overline{H}^{\tilde{r},l+1}_{(b)}$ *for* $\sigma \neq 0$ *and* $l+1 < -\frac{1}{2}$, $\tilde{r}+l+1 > -\frac{1}{2} - 2s - 4M\Im(\sigma)$ *and* $\tilde{r} > \frac{1}{2} + s - \frac{a^2+r_+^2}{r_+-M}\Im(\sigma)$ *and in this case we have the bound (uniform for* σ *in a strip* $\left\{0 \leq \Im(\sigma) \leq A, |\sigma| > \frac{1}{A}\right\}$ *for* $A > 0$*):*

$$\|R(\sigma)\|_{\mathcal{L}(\overline{H}^{\tilde{r},l}_{(b),|\sigma|^{-1}}, \overline{H}^{\tilde{r},l+1}_{(b),|\sigma|^{-1}})} \leq C \tag{14.2}$$

Proof. It is a consequence of proposition 14.0.1 and of the estimates of proposition 13.4.6 (where we can drop the error term since the kernel is trivial) and proposition 13.4.10. □

Remark 14.0.3. *Note that for* $\sigma, \sigma' \in \mathbb{C} \setminus \{0\}$ *with* $\Im(\sigma), \Im(\sigma') \geq 0$*, we have* $\mathcal{X}^{\tilde{r}+1,l}_{\sigma} \subset \mathcal{X}^{\tilde{r},l}_{\sigma'}$*. This is due to the fact that* $\hat{T}_s(\sigma) - \hat{T}_s(\sigma') \in x^{-1}Diff^1_b$*. The following identities are useful in the computations:*

(I1) If

$$\begin{aligned} l+1 &< -\frac{1}{2} \\ \tilde{r}+l+1 &> -\frac{1}{2} - 2s \\ \tilde{r} &> \frac{1}{2} + s \end{aligned}$$

then $\hat{T}_s(\sigma)R(\sigma) = Id_{\overline{H}^{\tilde{r},l}_{(b)}}$*.*

(I2) If

$$\begin{aligned} l &< -\frac{1}{2} \\ \tilde{r}+l &> -\frac{1}{2} - 2s \\ \tilde{r} &> \frac{1}{2} + s \end{aligned}$$

then $R(\sigma)\hat{T}_s(\sigma) = Id_{\mathcal{X}^{\tilde{r},l}_{\sigma}}$*.*

(I3) For any $\sigma' \neq 0$ *with* $\Im(\sigma') \geq 0$*, if*

$$\begin{aligned} l+1 &< -\frac{1}{2} \\ \tilde{r}+l &> -\frac{1}{2} - 2s \\ \tilde{r}-1 &> \frac{1}{2} + s \end{aligned}$$

then we have $R(\sigma)\hat{T}_s(\sigma)R(\sigma') = R(\sigma')$ *on* $\overline{H}^{\tilde{r},l}_{(b)}$*. Since* $R(\sigma')\overline{H}^{\tilde{r},l}_{(b)} = \mathcal{X}^{\tilde{r},l+1}_{\sigma'}$ *(using the inequalities on* $(\tilde{r}, l)$*) and since* $\mathcal{X}^{\tilde{r},l+1}_{\sigma'} \subset \mathcal{X}^{\tilde{r}-1,l+1}_{\sigma}$*, the identity is a consequence of (I2).*

Under the hypotheses:

$$\begin{aligned} l+1 &< -\frac{1}{2} \\ \tilde{r}+l &> -\frac{1}{2} - 2s \\ \tilde{r}-1 &> \frac{1}{2} + s, \end{aligned}$$

identities (I1) and (I3) enable to deduce the identity of the resolvent on $\overline{H}^{\tilde{r},l}_{(b)}$*:*

$$R(\sigma) - R(\sigma') = R(\sigma)\left(\hat{T}_s(\sigma') - \hat{T}_s(\sigma)\right)R(\sigma').$$

Similarly, under the hypotheses:

$$\begin{aligned} -\frac{3}{2} - s - |s| < & l < -\frac{1}{2} \\ \tilde{r} + l > & -\frac{1}{2} - 2s \\ \tilde{r} > & \frac{1}{2} + s \end{aligned}$$

the following resolvent identity holds on $\overline{H}^{\tilde{r},l}_{(b)}$*:*

$$R(\sigma) - R(0) = R(\sigma)\left(\hat{T}_s(0) - \hat{T}_s(\sigma)\right)R(0).$$

14.1 Regularity of the resolvent

Proposition 14.1.1. *Let* l *and* $\tilde{r}$ *be such that* $l+1 < -\frac{1}{2}$, $\tilde{r}+l > -\frac{1}{2}-2s$ *and* $\tilde{r}-1 > \frac{1}{2}+s$*. The family* $R(\sigma)$ *is holomorphic in* $\Im(\sigma) > 0$ *as a family of operators in* $\mathcal{L}(\overline{H}^{\tilde{r}+1,l}_{(b)}, \overline{H}^{\tilde{r},l+1}_{(b)})$*. Moreover, it is locally Lipschitz on* $D := \{\sigma \in \mathbb{C}, \Im(\sigma) \geq 0, \sigma \neq 0\}$ *as a family in* $\mathcal{L}(\overline{H}^{\tilde{r},l}_{(b)}, \overline{H}^{\tilde{r}-1,l+1}_{(b)})$*.*

Proof. For $\sigma, \sigma' \in D$, we have the resolvent identity on $\overline{H}^{\tilde{r},l}_{(b)}$:

$$R(\sigma) - R(\sigma') = R(\sigma)(\hat{T}_s(\sigma) - \hat{T}_s(\sigma'))R(\sigma')$$

Note that $\lim_{\sigma' \to \sigma} \frac{\hat{T}_s(\sigma) - \hat{T}_s(\sigma')}{\sigma - \sigma'} = -2\sigma a_{\mathfrak{t},\mathfrak{t}} + i a_{\mathfrak{t},\phi}\partial_\phi + i a_{\mathfrak{t},r}\partial_r + i a_{\mathfrak{t}} \in x^{-1}\mathrm{Diff}^1_b$ the limit being in the norm topology of $\mathcal{L}(\overline{H}^{\tilde{r},l+1}_{(b)}, \overline{H}^{\tilde{r}-1,l}_{(b)})$. Therefore $\hat{T}_s(\sigma)$ is holomorphic and in particular locally Lipschitz continuous in this space. Since $R(\sigma')$ is uniformly bounded in $\mathcal{L}(\overline{H}^{\tilde{r},l}_{(b)}, \overline{H}^{\tilde{r},l+1}_{(b)})$ (locally with respect to $\sigma' \in D$), we obtain the local Lipschitz continuity property for $R(\sigma)$. If we perform the same computation on $\overline{H}^{\tilde{r}+1,l}_{(b)}$, using the continuity we just proved, we have that $R(\sigma')$ is continuous in $\mathcal{L}(\overline{H}^{\tilde{r}+1,l}_{(b)}, \overline{H}^{\tilde{r},l+1}_{(b)})$. We then deduce the following limit in the norm topology of $\mathcal{L}(\overline{H}^{\tilde{r}+1,l}_{(b)}, \overline{H}^{\tilde{r}-1,l+1}_{(b)})$:

$$\lim_{\sigma' \to \sigma} \frac{R(\sigma) - R(\sigma')}{\sigma - \sigma'} = R(\sigma)\partial_\sigma \hat{T}_s(\sigma) R(\sigma)$$

and as a consequence, $R(\sigma)$ is holomorphic on the interior of D as a family in $\mathcal{L}(\overline{H}^{\tilde{r}+1,l}_{(b)}, \overline{H}^{\tilde{r}-1,l+1}_{(b)})$. Finally, using the Cauchy integral formula (see [46], theorem 3.3.11 for a proof of the formula in the case of vector valued holomorphic functions) and the continuity of $R(\sigma)$ in $\mathcal{L}(\overline{H}^{\tilde{r}+1,l}_{(b)}, \overline{H}^{\tilde{r},l+1}_{(b)})$, we deduce that the holomorphic property holds in $\mathcal{L}(\overline{H}^{\tilde{r}+1,l}_{(b)}, \overline{H}^{\tilde{r},l+1}_{(b)})$. □

Remark 14.1.2. *Note that if we assume in addition* $-\frac{3}{2} - s - |s| < l + 1$ *and use* (14.1) *(with* $\eta = 0$*), then we get* $\|R(\sigma)\|_{\mathcal{L}\left(\overline{H}^{\tilde{r},l}_{(b)}, \overline{H}^{\tilde{r}-1,l+1}_{(b)}\right)} \leq C\,|\sigma|^{-1}$ *near zero. Since for* $\eta \in [0,1]$*,* $\mathcal{L}\left(\overline{H}^{\tilde{r},l}_{(b)}, \overline{H}^{\tilde{r}-1,l+1}_{(b)}\right)$ *is continuously included in* $\mathcal{L}\left(\overline{H}^{\tilde{r},l+\eta}_{(b)}, \overline{H}^{\tilde{r}-1,l+1}_{(b)}\right)$*, we also have the continuity in this space and thanks to* (14.1) *we see that the behavior near* $\sigma = 0$ *improves when* η *increases (up to locally bounded for* $\eta = 1$*). We can get a local Hölder continuity statement by weakening a bit more the operator norm (see the proof of Proposition 14.1.3).*

The following proposition provides a global (operator norm) continuity statement up to the real axis including at $\sigma = 0$. It also provides a rough uniform bound when $|\sigma| \to +\infty$. It will be used in the contour deformation argument.

Proposition 14.1.3. *Let $-\frac{3}{2} - s - |s| < l < -\frac{1}{2}$, $\tilde{r} - 1 > \frac{1}{2} + s$ and $\tilde{r} + l - 1 > -\frac{1}{2} - 2s$. The family $R(\sigma)$ is continuous on $\{\Im(\sigma) \geq 0\}$ for the norm topology of $\mathcal{L}(\overline{H}^{\tilde{r},l}_{(b)}, \overline{H}^{\tilde{r}-1,l-}_{(b)})$. If we assume in addition that $\tilde{r} - 1 \geq 0$, then for every $C > 0$, there exists $D > 0$ such that for every σ with $0 \leq \Im(\sigma) \leq C$:*

$$\|R(\sigma)\|_{\mathcal{L}(\overline{H}^{\tilde{r},l}_{(b)}, \overline{H}^{\tilde{r}-1,l-}_{(b)})} \leq D \langle\sigma\rangle^{\tilde{r}}$$

Proof. For the continuity at 0, we use the resolvent identity:

$$R(\sigma) - R(0) = R(\sigma)\left(\hat{T}_s(0) - \hat{T}_s(\sigma)\right) R(0)$$

and under the hypotheses on l and $\tilde{r}$, there exists $C' > 0$ such that:

- $R(0)$ is well defined and bounded from $\overline{H}^{\tilde{r},l}_{(b)}$ to $\overline{H}^{\tilde{r},l}_{(b)}$ by propositions 13.6.1 and 13.6.2.
- $\left\|\hat{T}_s(0) - \hat{T}_s(\sigma)\right\|_{\mathcal{L}(\overline{H}^{\tilde{r},l}_{(b)}, \overline{H}^{\tilde{r}-1,l-1}_{(b)})} \leq C'|\sigma|$ since $\hat{T}_s(0) - \hat{T}_s(\sigma) \in x^{-1}\sigma\mathrm{Diff}^1_b$
- By (14.1), uniformly near zero we have $\|R(\sigma)\|_{\mathcal{L}(\overline{H}^{\tilde{r}-1,l-1}_{(b)}, \overline{H}^{\tilde{r}-1,l-\delta}_{(b)})} \leq C'|\sigma|^{\delta-1}$ for all $\delta > 0$ such that $l - \delta > -\frac{3}{2} - s - |s|$ and $\tilde{r} + l - 1 - \delta > -\frac{1}{2} - 2s$.

We get:

$$\|R(\sigma) - R(0)\|_{\mathcal{L}(\overline{H}^{\tilde{r},l}_{(b)}, \overline{H}^{\tilde{r}-1,l-\delta}_{(b)})} \leq C''|\sigma|^{\delta}$$

and therefore the continuity at zero.

The continuity at $\sigma \neq 0$ has already been proved in the stronger space $\mathcal{L}\left(\overline{H}^{\tilde{r},l-1}_{(b)}, \overline{H}^{\tilde{r}-1,l}_{(b)}\right)$ (see Proposition 14.1.1).

To get the estimate

$$\|R(\sigma)\|_{\mathcal{L}(\overline{H}^{\tilde{r},l}_{(b)}, \overline{H}^{\tilde{r}-1,l-}_{(b)})} \leq D \langle\sigma\rangle^{\tilde{r}}$$

we combine (14.2) and (14.1) (with $\eta = 1$) which provides uniform bounds in stronger norms than $\mathcal{L}(\overline{H}^{\tilde{r},l}_{(b)}, \overline{H}^{\tilde{r}-1,l-}_{(b)})$ (note that we compare semiclassical norms with usual norms using $\|u\|_{\overline{H}^{\tilde{r},l}_{(b)}} \langle\sigma\rangle^{-\tilde{r}} \leq \|u\|_{\overline{H}^{\tilde{r},l}_{(b),|\sigma|^{-1}}} \leq \|u\|_{\overline{H}^{\tilde{r},l}_{(b)}}$ true for $|\sigma| \geq 1$ and $\tilde{r} \geq 0$). □

The following proposition establishes higher regularity of the resolvent on the real axis.

Proposition 14.1.4. *Let $m \in \mathbb{N}$. Let l and r be such that $l+1 < -\frac{1}{2}$, $\tilde{r}+l-2m > -\frac{1}{2}-2s$ and $\tilde{r} - 2m - 1 > \frac{1}{2} + s$. We have $R \in C^m\left(\mathbb{R}_\sigma \setminus \{0\}, \mathcal{L}\left(\overline{H}^{\tilde{r},l}_{(b)}, \overline{H}^{\tilde{r}-1-2m,l+1}_{(b)}\right)\right)$ and $|\sigma|^{-m}\partial^m_\sigma R(\sigma)$ is uniformly bounded in $\mathcal{L}\left(\overline{H}^{\tilde{r},l}_{(b),|\sigma|^{-1}}, \overline{H}^{\tilde{r}-2m-1,l+1}_{(b),|\sigma|^{-1}}\right)$ for $\sigma \in \mathbb{R} \setminus (-\alpha, \alpha)$ (where $\alpha > 0$ is arbitrary).*

Moreover, if we assume in addition that $-\frac{3}{2} - s - |s| < l + 1$, there exists $\sigma_0 > 0$ such that for all $\eta \in [0, 1]$, $\sigma^{m+1-\eta}\partial^m_\sigma R(\sigma)$ is uniformly bounded in $\mathcal{L}\left(\overline{H}^{\tilde{r},l+\eta}_{(b)}, \overline{H}^{\tilde{r}-1-2m,l+1}_{(b)}\right)$ for $\sigma \in (-\sigma_0, \sigma_0) \setminus \{0\}$.

Proof. We first prove by induction that for all $m \in \mathbb{N}$, $R(\sigma)$, if $l+1 < -\frac{1}{2}$, $\tilde{r}+l-2m > -\frac{1}{2}-2s$ and $\tilde{r}-2m-1 > \frac{1}{2}+s$ then $R(\sigma) \in C^m\left(\mathbb{R}_\sigma \setminus \{0\}, \mathcal{L}\left(\overline{H}^{\tilde{r},l}_{(b)}, \overline{H}^{\tilde{r}-1-2m,l+1}_{(b)}\right)\right)$. For $m=0$, it follows from Proposition 14.1.1. Assume that the property is true for some $m \geq 0$ and assume $l+1 < -\frac{1}{2}$, $\tilde{r}+l-2(m+1) > -\frac{1}{2}-2s$ and $\tilde{r}-2(m+1)-1 > \frac{1}{2}+s$. Then using the resolvent identity, we get that $R(\sigma) \in C^1\left(\mathbb{R}_\sigma \setminus \{0\}, \mathcal{L}\left(\overline{H}^{\tilde{r},l}_{(b)}, \overline{H}^{\tilde{r}-3,l+1}_{(b)}\right)\right)$ and $\partial_\sigma R(\sigma) = R(\sigma)\partial_\sigma \hat{T}_s(\sigma) R(\sigma)$. By the induction hypothesis for all $m' \leq m$, $R(\sigma) \in C^{m'}\left(\mathbb{R}_\sigma \setminus \{0\}, \mathcal{L}\left(\overline{H}^{\tilde{r}-2(m-m')-2,l}_{(b)}, \overline{H}^{\tilde{r}-1-2(m+1),l+1}_{(b)}\right)\right)$ and $R(\sigma) \in C^{m'}\left(\mathbb{R}_\sigma \setminus \{0\}, \mathcal{L}\left(\overline{H}^{\tilde{r},l}_{(b)}, \overline{H}^{\tilde{r}-1-2m',l+1}_{(b)}\right)\right)$. We deduce that

$$\partial_\sigma R(\sigma) \in C^m\left(\mathbb{R}_\sigma \setminus \{0\}, \mathcal{L}\left(\overline{H}^{\tilde{r},l}_{(b)}, \overline{H}^{\tilde{r}-1-2(m+1),l+1}_{(b)}\right)\right)$$

and by Leibniz rule:

$$\partial_\sigma^{m+1} R(\sigma) = \sum_{i_1+i_2+i_3=m} a_{i_1,i_2,i_3} \partial_\sigma^{i_1} R(\sigma) \partial_\sigma^{i_2+1} \hat{T}_s(\sigma) \partial_\sigma^{i_3} R(\sigma) \tag{14.3}$$

where a_{i_1,i_2,i_3} are absolute combinatorial constants. To prove that $|\sigma|^{-m} \partial_\sigma^m R(\sigma)$ is bounded on $\mathbb{R}\setminus(-\alpha,\alpha)$, we use an induction. The case $m=0$ correspond to estimate (14.2) and we conclude by using the induction hypothesis in (14.3) (together with the fact that $|\sigma|^{-1}\partial_\sigma^{i_2+1}\hat{T}_s(\sigma)$ is bounded in $x^{-1}\mathrm{Diff}^1_b$ for $\sigma \in \mathbb{R}\setminus(-\alpha,\alpha)$. To prove that for all $\eta \in [0,1]$, $|\sigma|^{-m+1-\eta}\partial_\sigma^m R(\sigma)$ is bounded on $(-\sigma_0,\sigma_0)$ in the norm of $\mathcal{L}\left(\overline{H}^{\tilde{r},l+\eta}_{(b)}, \overline{H}^{\tilde{r}-1-2m,l+1}_{(b)}\right)$, we again use an induction on m. The case $m=0$ follows from estimate (14.1) and we conclude by using the induction hypothesis with $\eta \in [0,1]$ for $\partial_\sigma^{i_3} R(\sigma)$ and with $\eta = 0$ for $\partial_\sigma^{i_1} R(\sigma)$ in (14.3) (noting that $\partial_\sigma^{i_2+1}\hat{T}(\sigma)$ is uniformly bounded on $(-\sigma_0,\sigma_0)$). □

Remark 14.1.5. *Using the resolvent identity at $\sigma = 0$, we have:*

$$R(\sigma) - R(0) = R(\sigma)\left(\hat{T}_s(0) - \hat{T}_s(\sigma)\right) R(0)$$

Let $m \in \mathbb{N}$ and $l \in (-\frac{3}{2}-s-|s|, -\frac{1}{2}-s+|s|)$,$l_c \in (-\frac{3}{2}-s-|s|, -\frac{1}{2})$, $l_c \leq l \leq l_c+1$, $l_c+\tilde{r}-2m-1 > -\frac{1}{2}-2s$ and $\tilde{r}-2m-1 > \frac{1}{2}+s$. Using the fact that $\sigma^{-1}\left(\hat{T}_s(0)-\hat{T}_s(\sigma)\right)R(0)$ maps $\overline{H}^{\tilde{r},l}_{(b)}$ to $\overline{H}^{\tilde{r},l-1}_{(b)}$ (with uniform bound with respect to $\sigma \in (-\sigma_0,\sigma_0)$) and the previous proposition, we deduce that for $\sigma \in (-\sigma_0,\sigma_0)$:

$$\left\| |\sigma|^m \partial_\sigma^m (R(\sigma)-R(0)) \right\|_{\mathcal{L}(\overline{H}^{\tilde{r},l}_{(b)}, \overline{H}^{\tilde{r}-2m-1,l_c}_{(b)})} \leq C|\sigma|^{l-l_c}$$

where the constant $C > 0$ does not depend on σ.

14.2 More precise regularity near $\sigma = 0$

First we establish more precisely the mapping properties of $R(0)$ which will be used throughout the section using a normal operator argument. Then we describe precisely the singular part of $R(\sigma)f$ near zero. We use the notations Y_j, $\mathcal{Y}^{\tilde{r},l}_j$, $\mathcal{Y}^{\tilde{r},l}_{\geq j}$, $\mathcal{Y}^{\tilde{r},l}_{\leq j}$ and Π_j already introduced in Section 13.6.

Lemma 14.2.1. *Let $l \geq l_0$, $\tilde{r} \geq \tilde{r}_0$. Let $u \in \overline{H}^{\tilde{r}_0,l_0}_{(b)}$ with $u=0$ in a neighborhood of $\left\{x \geq \frac{1}{r_+}\right\}$. We assume that for all $j \in |s|+\mathbb{N}$, $u_j := \Pi_j u \in \mathcal{Y}^{\tilde{r},l}_j$ and $\sum_{p=0}^{+\infty}\int_{\mathbb{R}}(1+|\tau|^2+p^2)^{\tilde{r}}|\hat{u}_{|s|+p}(\tau-$*

$i(l+\frac{3}{2}))|^2\,\mathrm{d}\tau<+\infty$ *(where* $\hat{u}_j$ *is the Mellin transform of* u_j*). Then,* $u\in\overline{H}^{r,l}_{(b)}$ *and*

$$\|u\|_{\overline{H}^{\tilde{r},l}_{(b)}}\leq C\sum_{p=0}^{+\infty}\int_{\mathbb{R}}(1+|\tau|^2+p^2)^{\tilde{r}}\left|\hat{u}_{|s|+p}\left(\tau-i\left(l+\frac{3}{2}\right)\right)\right|^2\,\mathrm{d}\tau$$

for some constant $C>0$ *independent of* u*.*

Proof. Let u be defined as in the statement of the proposition. First note that the algebraic sum $\oplus_{j\in|s|+\mathbb{N}}\mathcal{Y}^{\tilde{r},l}_j$ is dense in $\overline{H}^{\tilde{r},l}_{(b)}$ and therefore also in $\overline{H}^{\tilde{r}_0,l_0}_{(b)}$. This density property can be obtained using first the density of $C^\infty_c((0,\frac{1}{r_+}])\otimes H^{\tilde{r}}(\mathcal{B}_s)$ in $\overline{H}^{\tilde{r},l}_{(b)}$ and then using the density of the algebraic sum $\oplus_{j\in|s|+\mathbb{N}}Y_j$ in $H^s(\mathcal{B}_s)$. This density and the orthogonality of $\mathcal{Y}^{\tilde{r}_0,l_0}_j$ and $\mathcal{Y}^{\tilde{r}_0,l_0}_j$ when $j\neq j'$ for the norm inducing scalar product $\langle v,w\rangle_{\tilde{r}_0,l_0}:=\left\langle x^{-l_0}(1-(x\partial_x)^2+\Delta^{[s]})^{\frac{\tilde{r}_0}{2}}v,x^{-l_0}(1-(x\partial_x)^2+\Delta^{[s]})^{\frac{\tilde{r}_0}{2}}w\right\rangle_{L^2_{(b)}}$ imply that the sequence $(\sum_{p=0}^N\Pi_{|s|+p}(u))_{N\in\mathbb{N}}$ converges towards u in $\overline{H}^{\tilde{r}_0,l_0}_{(b)}$. Since $\mathcal{Y}^{\tilde{r},l}_j$ and $\mathcal{Y}^{\tilde{r},l}_{j'}$ are orthogonal when $j\neq j'$ for the norm-inducing scalar product $\langle v,w\rangle_{\tilde{r},l}$, we get:

$$\begin{aligned}\left\|\sum_{p=0}^N u_{|s|+p}\right\|^2_{\overline{H}^{\tilde{r},l}_{(b)}}&=\sum_{p=0}^N\left\|u_{|s|+p}\right\|^2_{\overline{H}^{\tilde{r},l}_{(b)}}\\&=\sum_{p=0}^N\left\|(1-(x\partial_x)^2+p(p+2\,|s|)+p+|s|)^{\frac{r}{2}}u_{|s|+p}\right\|^2_{\overline{H}^{0,l}_{(b)}}\\&\leq C\sum_{p=0}^N\int_{\mathbb{R}}(1+|\tau|^2+p^2)^{\tilde{r}}|\hat{u}_{|s|+p}(\tau-i(l+\frac{3}{2}))|^2\,\mathrm{d}\tau\end{aligned}$$

where $C>0$ is some constant independent of u. We deduce that $\sum_{p=0}^{+\infty}u_{|s|+p}$ is absolutely convergent in $\overline{H}^{\tilde{r},l}_{(b)}$ and therefore, the limit (which is equal to u by uniqueness of the limit in $\overline{H}^{\tilde{r}_0,l_0}_{(b)}$) belongs to $\overline{H}^{\tilde{r},l}_{(b)}$. Moreover the computation shows that $\|u\|_{\overline{H}^{\tilde{r},l}_{(b)}}\leq C\sum_{p=0}^{+\infty}\int_{\mathbb{R}}(1+|\tau|^2+p^2)^{\tilde{r}}|\hat{u}_{|s|+p}(\tau-i(l+\frac{3}{2}))|^2\,\mathrm{d}\tau$. □

We also record the following useful lemma about the Mellin transform.

Lemma 14.2.2. *Let* $u\in\overline{H}^{\tilde{r}_0,l_0}_{(b)}$ *with* $supp(u)$ *contained in* $\{x\leq D\}$ *for some* $D>0$*. The Mellin transform* Mu *is holomorphic on* $\left\{\Im(\tau)>-(l_0+\frac{3}{2})\right\}$ *and* $\alpha\mapsto Mu(.+i\alpha)$ *is continuous from* $[-\left(l_0+\frac{3}{2}\right),+\infty)$ *to* $\left\langle 1+\tau^2+\Delta^{[s]}\right\rangle^{-\frac{\tilde{r}_0}{2}}L^2(\mathbb{R}_\tau,L^2(\mathcal{B}_s))$*. Moreover, if for* $l>l_0$*,* Mu *admits a meromorphic extension to* $\left\{\Im(\tau)>-(l+\frac{3}{2})\right\}$ *with a finite number of poles* $a_0,...a_N$ *included in some compact subset* K *of* $\left\{\Im(\tau)>-(l+\frac{3}{2})\right\}$ *and has a bound of the form:*

$$\int\left|\left(1+\tau^2+\Delta^{[s]}\right)^{\frac{\tilde{r}_0}{2}}1_{\mathbb{C}\setminus K}Mu(\tau+i\alpha)\right|^2\,\mathrm{d}\tau\leq C_0\tag{14.4}$$

for some $C_0>0$ *independent of* $\alpha\in\left(-\left(l+\frac{3}{2}\right),-\left(l_0+\frac{3}{2}\right)\right]$ *(*$1_{\mathbb{C}\setminus K}$ *being the indicator function of* $\mathbb{C}\setminus K$*). Then there exists a family of non negative integers* $(c_j)_{j=0}^N$ *and a family of complex numbers* $(b_{j,k})_{0\leq j\leq N0\leq k\leq c_j}$ *such that,* $u\in\sum_{j=0}^N\sum_{k=0}^{c_j}b_{j,k}\ln(x)^kx^{ia_j}\chi_0(x)+\overline{H}^{\tilde{r}_0,l}_{(b)}$ *where* χ_0 *is any smooth cutoff equal to* 1 *near zero and compactly supported.*

Proof. By definition, for $\Im(\tau) > -\left(l_0 + \frac{3}{2}\right)$, $Mu(\tau) = \int x^{-i\tau} u(x) \frac{\mathrm{d}x}{x}$. Since for every $\epsilon > 0$, $x^{-(l_0+\frac{3}{2})+\epsilon} u(x) \in L^1_b([0,+\infty))$, we get that Mu is holomorphic on $\left\{\Im(\tau) > -(l_0 + \frac{3}{2})\right\}$. If $\alpha, \alpha' \leq l_0$, $x^{-\alpha} u \in H^{\tilde{r}_0,0}_{(b)}$ and

$$\left\|(x^{-\alpha} - x^{-\alpha'})u\right\|_{H^{\tilde{r}_0,0}_{(b)}} \leq C \sum_{j=0}^{N'} \left\|(xD_x)^j (x^{l_0-\alpha} - x^{l_0-\alpha'})\chi_1\right\|_{L^\infty} \left\|x^{-l_0} u\right\|_{\overline{H}^{\tilde{r}_0,0}_{(b)}}$$

where $\chi_1 = 1$ on $x(\mathrm{supp}(u))$ and χ_1 is smooth and compactly supported and N' is a large integer. Using that $\lim_{\alpha \to \alpha'} \left\|(xD_x)^j (x^{l_0-\alpha} - x^{l_0-\alpha'})\chi_1\right\|_{L^\infty} = 0$ for any $j \in \mathbb{N}$, we deduce that $\alpha \mapsto x^\alpha u$ is continuous in $\dot{H}^{\tilde{r}_0,0}_{(b)}$ and therefore, by continuity of the Mellin transform from $\dot{H}^{\tilde{r}_0,0}_{(b)}$ to $\left\langle 1 + \tau^2 + \Delta^{[s]} \right\rangle^{-\frac{\tilde{r}_0}{2}} L^2(\mathbb{R}_\tau, L^2(\mathcal{B}_s))$ we conclude the first part of the proof. Now we assume that Mu admits an extension as in the statement. Then by the residue theorem, we have for all $A > 0$ and $\epsilon > 0$ small enough such that $K \subset \left\{z : -(l + \frac{3}{2}) + \epsilon \leq \Im(z) \leq -(l_0 + \frac{3}{2})\right\}$ and for all $x \in (0, +\infty)$:

$$\begin{aligned}\int_{-A}^{A} x^{i(\tau - i(l_0+\frac{3}{2}))} Mu\left(\tau - i\left(l_0 + \frac{3}{2}\right)\right) \mathrm{d}\tau = &- \sum_{j=0}^{N} 2i\pi \mathrm{Res}_{|\tau=a_j} x^{i\tau} Mu(\tau) \\ &+ \int_{-A}^{A} x^{i(\tau - i(l+\frac{3}{2})+i\epsilon)} Mu\left(\tau - i\left(l + \frac{3}{2}\right) + i\epsilon\right) \mathrm{d}\tau \\ &+ i \int_{-(l+\frac{3}{2})+\epsilon}^{-(l_0+\frac{3}{2})} x^{i(A+iy)} Mu(A + iy)\, \mathrm{d}y \\ &- i \int_{-(l+\frac{3}{2})+\epsilon}^{-(l_0+\frac{3}{2})} x^{i(-A+iy)} Mu(-A + iy)\, \mathrm{d}y\end{aligned}$$

We have the following limit in the sense of distributions:

$$\lim_{A \to +\infty} \int_{-A}^{A} x^{i(\tau - i(l_0+\frac{3}{2}))} Mu\left(\tau - i\left(l_0 + \frac{3}{2}\right)\right) \mathrm{d}\tau = u(x)$$

Moreover, (using the bound (14.4)) we get that:

$$\int_{-A}^{A} x^{i(\tau - i(l+\frac{3}{2})+i\epsilon)} Mu\left(\tau - i\left(l + \frac{3}{2}\right) + i\epsilon\right) \mathrm{d}\tau$$

has a limit $v \in x^{l+\frac{3}{2}-\epsilon} H^{\tilde{r}_0,0}_b((0,+\infty) \times \mathcal{B}_s)$ in the sense of distributions when $A \to +\infty$. We deduce that

$$f_A := i \int_{-(l+\frac{3}{2})+\epsilon}^{-(l_0+\frac{3}{2})} x^{i(A+iy)} Mu(A + iy)\, \mathrm{d}y - i \int_{-(l+\frac{3}{2})+\epsilon}^{-(l_0+\frac{3}{2})} x^{i(-A+iy)} Mu(-A + iy)\, \mathrm{d}y$$

also has a limit in the sense of distributions. We prove that this limit is zero. Let $\phi \in \Gamma_c((0,+\infty) \times \mathcal{B}_s)$. Using integration by part with the relation $(\pm A + iy)^{-K} (xD_x)^K x^{i(\pm A+iy)} = x^{i(\pm A+iy)}$, we obtain that there exists a constant $C > 0$ (independent of A) such that for a large integer K:

$$\begin{aligned}|\langle f_A, \phi \rangle| \leq C \left\|(xD_x)^K \phi\right\|_{L^1_b((0,+\infty), H^{-\tilde{r}_0}(\mathcal{B}_s))} \Bigg(&\int_{-(l+\frac{3}{2})+\epsilon}^{-(l_0+\frac{3}{2})} A^{-K} \left\|(1 + \Delta^{[s]})^{\frac{\tilde{r}_0}{2}} Mu(A + iy)\right\|^2_{L^2(\mathcal{B}_s)} \\ &+ A^{-K} \left\|(1 + \Delta^{[s]})^{\tilde{r}_0 2} Mu(A + iy)\right\|^2_{L^2(\mathcal{B}_s)}\, \mathrm{d}y\Bigg)\end{aligned}$$

Using (14.4), we get for A_0 and K large enough[1]:

$$\int_{A_0}^{+\infty}\left(\int_{-(l+\frac{3}{2})+\epsilon}^{-(l_0+\frac{3}{2})} A^{-K}\left\|(1+\Delta^{[s]})^{\frac{\tilde{r}_0}{2}}Mu(A+iy)\right\|^2_{L^2(\mathcal{B}_s)} \right.$$
$$\left. +A^{-K}\left\|(1+\Delta^{[s]})^{\frac{\tilde{r}_0}{2}}Mu(A+iy)\right\|^2_{L^2(\mathcal{B}_s)}\,\mathrm{d}y\right)\,\mathrm{d}A<+\infty.$$

We deduce that there exists a sequence A_n with $\lim_{n\to+\infty} A_n=+\infty$ such that

$$\lim_{n\to+\infty}\left(\int_{-(l+\frac{3}{2})+\epsilon}^{-(l_0+\frac{3}{2})} A_n^{-K}\left\|(1+\Delta^{[s]})^{\frac{\tilde{r}_0}{2}}Mu(A_n+iy)\right\|^2_{L^2(\mathcal{B}_s)} \right.$$
$$\left. +A_n^{-K}\left\|(1+\Delta^{[s]})^{\tilde{r}_0 2}Mu(A_n+iy)\right\|^2_{L^2(\mathcal{B}_s)}\,\mathrm{d}y\right)=0$$

Using this particular sequence, we deduce that $\lim_{A\to+\infty} f_A=0$ in the sense of distributions. Therefore writing $\sum_{k=0}^{c_j} b_{j,k}\ln(x)^k x^{ia_j}:=-2i\pi\mathrm{Res}_{|\tau=a_j}x^{i\tau}Mu(\tau)$ we get:

$$u=v+\sum_{j=0}^{N}\sum_{k=0}^{c_j} b_{j,k}\ln(x)^k x^{ia_j}$$

We use this equality with different values of ϵ and (14.4) provides for all $0<\epsilon<\epsilon_0$:

$$\|v\|_{x^{l+\frac{3}{2}-\epsilon}H_b^{\tilde{r}_0,0}}\leq C_1C_0$$

where C_1 is independent of ϵ and u. Let $\phi\in\Gamma_c((0,+\infty)\times\mathcal{B}_s)$, we have:

$$\begin{aligned}|\langle v,\phi\rangle|\leq&\|v\|_{x^{l+\frac{3}{2}-\epsilon}H_b^{\tilde{r}_0,0}}\|\phi\|_{x^{-l-\frac{3}{2}+\epsilon}H_b^{-\tilde{r}_0,0}}\\ \leq&C_1C_0\|\phi\|_{x^{-l-\frac{3}{2}+\epsilon}H_b^{-\tilde{r}_0,0}}\end{aligned}\tag{14.5}$$

But for all $N\in\mathbb{N}$, $\lim_{\epsilon\to0}\left\|\left(x^{l+\frac{3}{2}-\epsilon}-x^{l+\frac{3}{2}}\right)\phi\right\|_{H_b^{\tilde{r}_0,0}}=0$ (all b derivatives converge uniformly on the support of ϕ). Taking the limit $\epsilon\to0$ in (14.5), we get $v\in H_b^{\tilde{r}_0,l+\frac{3}{2}}$ with a norm smaller than C_1C_0. Using the support condition of u, we have:

$$u=\chi_0 v+\sum_{j=0}^{N}\sum_{k=0}^{c_j} b_{j,k}\ln(x)^k x^{ia_j}\chi_0$$

where $\|\chi_0 v\|_{\overline{H}_{(b)}^{\tilde{r}_0,l}}\leq C_2C_0$ with C_2 independent of u. □

Remark 14.2.3. *The estimate* (14.4) *can be replaced by an other similar estimate. For example, we can assume that Mu extends meromorphically to $\{\Im(\tau)>D_0\}$ (where $D_0\in\mathbb{R}$) with a finite number of poles and that we have bounds of the form:*

$$\|Mu(\tau)\|^2_{H^{\tilde{r}}(\mathcal{B}_s)}\leq C(1+\Re(\tau)^2)^{-\tilde{r}}e^{C|\Im(\tau)|}$$

for $\tau\in\{\Im(\tau)\geq D\}\setminus K$ where $D>D_0$ and $\Re(K)$ compact. We also assume that there is no pole with imaginary part equal to D. In, this case we have that $u\in\sum_{j=0}^{N}\sum_{k=0}^{c_j} b_{j,k}\ln(x)^k x^{ia_j}\chi_0+\overline{H}_{(b)}^{\tilde{r},-D-\frac{3}{2}}$.

[1]More explicitly, if $\tilde{r}_0\geq0$, we can take any $K\geq0$ and if $\tilde{r}_0<0$, we take $K\geq-\frac{\tilde{r}_0}{2}$

Lemma 14.2.4. *Let $j \in \mathbb{N}$, $-\frac{3}{2} - s - |s| < l < -\frac{1}{2} + j - s + |s|$ and $\tilde{r} + 1 > \frac{1}{2} + s$, then $\hat{T}_s(0)^{-1}(\mathcal{Y}^{\tilde{r},l}_{\geq j+|s|}) \subset \mathcal{Y}^{\tilde{r}+1,l}_{\geq j+|s|}$.*

Proof. Since $\hat{T}_s(0)$ commutes with $\Delta^{[s]}$, for $\epsilon > 0$ small enough, $\hat{T}_s(0)^{-1}(\mathcal{Y}^{\tilde{r},l}_{\geq j+|s|}) \subset \mathcal{Y}^{\tilde{r}+1,-\frac{3}{2}-s-|s|+\epsilon}_{\geq j+|s|}$. Let $u \in \hat{T}_s(0)^{-1}(\mathcal{Y}^{\tilde{r},l}_{\geq j+|s|})$. We have $\hat{T}_s(0)\chi_0 u = f^0 \in \mathcal{Y}^{\tilde{r},l}_{\geq j+|s|}$ (where χ_0 is a cutoff depending only on x and localizing near zero) and $N(\hat{T}_s(0))\chi_0 u = f^0 - x\mathrm{Diff}^2_b u =: f^1$. Applying $\Pi_{p+|s|}$ for $p \geq j$ and the Mellin transform, we get that $(\tau^2 + (i-2is)\tau + p(p+2|s|) + |s| + p + s)\widehat{\chi_0 u_{p+|s|}} = \hat{f}^1_{p+|s|}$ where $\widehat{\chi_0 u}_{p+|s|}$, is holomorphic on $\{\Im(\tau) > s + |s| - \epsilon\}$. Since $\hat{f}^0_{p+|s|}$ is holomorphic on $\left\{\Im(\tau) > -(l + \frac{3}{2})\right\}$ we deduce that $\hat{f}^1_{p+|s|}$ is holomorphic on $\left\{\Im(\tau) > \max(-(l+\frac{3}{2}), -1 + s + |s| - \epsilon)\right\}$. Since $(\tau^2 + (i-2is)\tau + p(p+2|s|) + |s| + p + s)$ has no zero on $\{-1 - j + s - |s| < \Im(\tau) < s + |s| + p\}$ we deduce that $\widehat{\chi_0 u}_{p+|s|}$ extends holomorphically to $\left\{\Im(\tau) > \max(-(l+\frac{3}{2}), -1 + s + |s| - \epsilon)\right\}$. An iteration of this argument proves that $\widehat{\chi_0 u}_{p+|s|}$ and $\hat{f}^1_{p+|s|}$ extend holomorphically to $\left\{\Im(\tau) > -(l + \frac{3}{2})\right\}$. Moreover, for any $\alpha \geq s + |s| - \epsilon$, there exists $C > 0$ independent of p such that

$$\begin{aligned}\int_{\mathbb{R}} \left|\widehat{\chi_0 u}_{p+|s|}(\tau + i\alpha)\right|^2 \left(|\tau|^2 + p^2 + 1\right)^{\tilde{r}+1} \mathrm{d}\tau &\leq C \left\|x^\alpha \chi_0 u_{p+|s|}\right\|^2_{\overline{H}^{\tilde{r}+1,0}_b} \\ &\leq C \left\|\chi_0 u_{p+|s|}\right\|^2_{\overline{H}^{\tilde{r}+1,-s-|s|+\epsilon}_b}\end{aligned}$$

and for any $\beta \geq -(l + \frac{3}{2})$,

$$\begin{aligned}\int_{\mathbb{R}} (1 + p^2 + |\tau|^2)^{\tilde{r}} \left|\hat{f}^0_{p+|s|}(\tau + i\beta)\right|^2 \mathrm{d}\tau &\leq C \left\|x^\beta f^0_{p+|s|}\right\|^2_{\overline{H}^{\tilde{r},0}_b} \\ &\leq C \left\|f^0_{p+|s|}\right\|^2_{\overline{H}^{\tilde{r},l}_{(b)}}\end{aligned}$$

Using that $\left|\tau^2 + (i-2is)\tau + p(p+2|s|) + |s| + p + s\right|^{-1} \leq C(\Re(\tau)^2 + p^2 + 1)^{-1}$ where C is uniform with respect to $p \geq j$ when $-(l+\frac{3}{2}) \leq \Im(\tau) \leq s + |s| - \epsilon$, we deduce that for any $\max(-(l+\frac{3}{2}), -1 + s + |s| - \epsilon) \leq \alpha \leq s + |s| - \epsilon$, there exists C' independent of $p \geq j$ such that:

$$\int_{\mathbb{R}} \left|\widehat{\chi_0 u}_{p+|s|}(\tau + i\alpha)\right|^2 \left(|\tau|^2 + p^2 + 1\right)^{\tilde{r}+1} \mathrm{d}\tau \leq C' \left(\left\|\chi_0 u_{p+|s|}\right\|^2_{\overline{H}^{\tilde{r}+1,-\frac{3}{2}-s-|s|+\epsilon}_{(b)}} + \left\|x^\alpha f^0_{p+|s|}\right\|^2_{\overline{H}^{\tilde{r},l}_{(b)}}\right)$$

Once again we can iterate the argument and deduce that there exists C'' independent of p such that for $\alpha = -(l + \frac{3}{2})$:

$$\int_{\mathbb{R}} \left|\widehat{\chi_0 u}_{p+|s|}(\tau + i\alpha)\right|^2 \left(|\tau|^2 + p^2 + 1\right)^{\tilde{r}+1} \mathrm{d}\tau \leq C'' \left(\left\|\chi_0 u_{p+|s|}\right\|^2_{\overline{H}^{\tilde{r}+1,-\frac{3}{2}-s-|s|+\epsilon}_{(b)}} + \left\|f^0_{p+|s|}\right\|^2_{\overline{H}^{\tilde{r},l}_{(b)}}\right)$$

Using Lemma 14.2.2, we get that for each $p \geq j$, $\chi_0 u_p \in \overline{H}^{\tilde{r}+1,l}_{(b)}$ with

$$\left\|\chi_0 u_{p+|s|}\right\|^2_{\overline{H}^{\tilde{r}+1,l}_{(b)}} \leq C'' \left(\left\|\chi_0 u_{p+|s|}\right\|^2_{\overline{H}^{\tilde{r}+1,-\frac{3}{2}-s-|s|+\epsilon}_{(b)}} + \left\|f^0_{p+|s|}\right\|^2_{\overline{H}^{\tilde{r},l}_{(b)}}\right).$$

Since the constant is uniform with respect to $p \geq j$, we deduce:

$$\begin{aligned}\sum_{p=j}^{+\infty} \int_{\mathbb{R}} \left|\widehat{\chi_0 u}_{p+|s|}\left(\tau - i\left(l + \frac{3}{2}\right)\right)\right|^2 \left(|\tau|^2 + p^2 + 1\right)^{\tilde{r}+1} \mathrm{d}\tau \leq \\ C'' \left(\left\|\chi_0 u\right\|^2_{\overline{H}^{\tilde{r}+1,-\frac{3}{2}-s-|s|+\epsilon}_{(b)}} + \left\|f^0\right\|^2_{\overline{H}^{\tilde{r},l}_{(b)}}\right)\end{aligned}$$

We conclude by Lemma 14.2.1 that $u \in \mathcal{Y}^{\tilde{r}+1,l}_{\geq j+|s|}$. □

Lemma 14.2.5. *Let $k \in \mathbb{N}$, $\alpha \in \mathbb{C}$, $\epsilon > 0$, χ a smooth cutoff localizing near zero and $c \in E \setminus \{0\}$ (where E is some Banach space) and $u = \chi(x)x^\alpha \ln(x)^k c$. Then the Mellin transform of u has a meromorphic extension to $\mathbb{C}$ with a pole of order $k+1$ at $-i\alpha$.*

Proof. We have that $(xD_x + i\alpha)^{k+1}u \in C^\infty_c((0,+\infty), E)$. Therefore, $(\tau + i\alpha)^{k+1}\hat{u}$ has a holomorphic extension to $\mathbb{C}$ which prove the lemma. □

In the remaining part of the section, $f_{l,m}$ is used with variables (θ, φ). Note that we have

$$f_{l,m}(\theta,\varphi) = e^{im(\varphi-\phi_*)} f_{l,m}(\theta,\phi^*)$$

where $\varphi - \phi_*$ is a function of r. In the following lemma, we record some mapping properties of $\hat{T}_s(0)^{-1}$. We first give an expansion of $\hat{T}_s(0)^{-1}u$ for a general $u \in \overline{H}^{\tilde{r},l}_{(b)}$ (see the statement for the precise conditions on l and $\tilde{r}$). Then, we give an expansion for u of the form $x^\alpha \ln(x)^p$ near $x = 0$. It will be useful since such terms naturally arise in the iteration in the proof of Proposition 14.2.10.

Lemma 14.2.6. *The indices $|s| + j$ denote an element of $Y_{|s|+j}$. Let $-\frac{1}{2} - s + |s| + k < l < -\frac{1}{2} - s + |s| + k + 1$ for $k \in \mathbb{N}$, $\tilde{r} + 1 > \frac{1}{2} + s$ and $u \in \overline{H}^{\tilde{r},l}_{(b)}$, then we have:*

$$\hat{T}_s(0)^{-1}u = \chi(x)x^{1-s+|s|}\Big(\sum_{j=0}^{k}\sum_{i=j}^{k} x^i v_{i,|s|+j}\Big) + \overline{H}^{\tilde{r}+1,l}_{(b)}$$

Moreover, we have for $0 \leq j \leq k$:

$$v_{j,|s|+j} = \frac{1}{1+2(|s|+j)} \sum_{|m|\leq|s|+j} \left\langle \overline{U^*_{|s|+j,m}}, u \right\rangle f_{|s|+j,m}$$

and for $0 \leq j \leq k-1$:

$$v_{j+1,|s|+j} = \sum_{|m|\leq|s|+j} \frac{\big(M(j^2 + (2|s| - s + 2)j + s(s - |s|) + 2|s| - s + 1) - iams\big)}{(1+2(|s|+j))(j+|s|+1)} \left\langle \overline{U^*_{|s|+j,m}}, u \right\rangle f_{|s|+j,m}$$

Where $U^*_{|s|+j,m}$ is the unique multiple of $u^*_{|s|+j,m} r^2 \sin(\theta)\ \mathrm{d}r\ \mathrm{d}\phi_*\ \mathrm{d}\theta$ equivalent to $r^{|s|+j-s} f_{|s|+j,m}(\theta,\varphi)\sin(\theta)\ \mathrm{d}r\ \mathrm{d}\varphi\ \mathrm{d}\theta$ when $r \to +\infty$ (and the bracket involving U^* is the usual duality bracket[2] between volume forms and functions). Let $p \in \mathbb{N}$ and χ be a smooth cutoff localizing near zero. If $\alpha = 1 - s + |s| + j - k > -s - |s| - j$ for some $k, j \in \mathbb{N}$, then for all $n \in \mathbb{N}$:

$$\hat{T}_s(0)^{-1}\chi(x)x^\alpha \ln(x)^p u_{|s|+j} = \chi(x)x^\alpha \sum_{j'=0}^{p}\sum_{i=0}^{n} x^i \ln(x)^{j'} v_{i,j',|s|+j} + \chi(x)x^\alpha \sum_{i=k}^{n} x^i \ln(x)^{p+1} w_{i,|s|+j} + \overline{H}^{\infty,\alpha+n-\frac{1}{2}-}_{(b)}$$

Moreover, if $k \geq 1$, we have $v_{0,p,|s|+j} = \frac{u_{|s|+j}}{k(1+2|s|+2j-k)}$, if $\alpha = -s + |s| + j$, $j + |s| \neq 0$ and $p = 0$, we have

$$w_{1,|s|+j} = \sum_{|m|\leq|s|+j} \frac{\langle f_{|s|+j,m}, u_{|s|+j}\rangle f_{|s|+j,m}}{2(j+|s|)(1+2j+2|s|)} \big(2M(j^2 + (2|s| - s)j + s(s - |s|)) - 2iams\big)$$

and if $\alpha = 1 - s + |s| + j$, we have $w_{0,|s|+j} = -\frac{u_{|s|+j}}{(p+1)(1+2|s|+2j)}$.

[2] For $\mathrm{d}v$ a volume form and f a function $< v, f >:= \int f\ \mathrm{d}v$.

Proof. Let $-\frac{1}{2}-s+|s|+k<l\leq -\frac{1}{2}-s+|s|+k+1$ for $k\in\mathbb{N}$ and $u\in\overline{H}^{\tilde{r},l}_{(b)}$. Let χ_0 be a smooth cutoff localizing near 0. We define $v:=\hat{T}_s(0)^{-1}u\in\overline{H}^{\tilde{r}+1,-\frac{1}{2}-s+|s|-}_{(b)}$. By definition, $\hat{T}_s(0)v=u$.

$$N(\hat{T}_s(0))\chi_0 v = u - x\mathrm{Diff}^2_b v =: f$$

Since $N(\hat{T}_s(0))$ and $\hat{T}_s(0)-N(\hat{T}_s(0))$ commutes with $\Delta^{[s]}$, we can project the equality on each eigenspace. We get for $j\in\mathbb{N}$:

$$((xD_x)^2+(i-2is)xD_x+j(j+2|s|)+|s|+j+s)\chi_0 v_{|s|+j} = f_{|s|+j}$$

We compute the Mellin transform:

$$(\tau^2+(i-2is)\tau+j(j+2|s|)+|s|+j+s)\widehat{\chi_0 v}_{|s|+j} = \hat{f}_{|s|+j} \tag{14.6}$$

Note that

$$\tau^2+(i-2is)\tau+j(j+2|s|)+|s|+j+s=(\tau-i(-1-j+s-|s|))(\tau-i(s+|s|+j)).$$

We define k as in the statement of the Proposition (i.e. the smallest integer such that $-1-(k+1)+s-|s|<-l-\frac{3}{2}$). We fix K a compact subset of $\mathbb{C}$ such that $K\cap\left\{\Im(\tau)=-l-\frac{3}{2}\right\}=\emptyset$ and K contains a neighborhood of $\mathcal{P}:=\{i(-1-j+s-|s|),0\leq j\leq k\}$ and such that if $\Im(\tau)\geq -l-\frac{3}{2}$ and $\tau\in K$, then $\tau-i\in K$. Note that for all $j\in\mathbb{N}$, $(\tau^2+(i-2is)\tau+j(j+2|s|)+|s|+j+s)^{-1}$ is meromorphic on $\left\{-1-(k+1)+s-|s|<\Im(\tau)<-\frac{1}{2}+s-|s|\right\}$ with poles of order 1 included in $\mathcal{P}$. Moreover we have a constant $C>0$ independent of j such that for every $\tau\in\left\{-l-\frac{3}{2}\leq\Im(\tau)\leq-\frac{1}{2}+s-|s|\right\}\setminus K$:

$$\left|\tau^2+(i-2is)\tau+j(j+2|s|)+|s|+j+s\right|^{-1}\leq C(\Re(\tau)^2+j^2+1)^{-1}. \tag{14.7}$$

Using (14.6) (and an induction), we have that:

- $\widehat{\chi_0 v}_{|s|+j}$ is holomorphic on $\mathcal{D}_0:=\left\{\Im(\tau)>\max(-1-j+s-|s|,-l-\frac{3}{2})\right\}$
- There exists $C>0$ independent of j such that for all $b>\max(-1-j+s-|s|,-l-\frac{3}{2})$,

$$\int_{\mathbb{R}}\left|1_K(\tau)(1+\tau^2+j^2)^{\frac{r+1}{2}}\widehat{\chi_0 v}_{|s|+j}(\tau+ib)\right|^2\,\mathrm{d}\tau\leq C\left(\left\|u_{j+|s|}\right\|^2_{\overline{H}^{\tilde{r},l}_{(b)}}+\left\|v_{j+|s|}\right\|^2_{\overline{H}^{\tilde{r}+1,-\frac{1}{2}-s+|s|-\epsilon}_{(b)}}\right).$$

We prove by induction on $m\in\mathbb{N}$ that

- $\widehat{\chi_0 v}_{|s|+j}$ has a meromorphic extension (with at most simple poles at $i(-1-j-\mathbb{N}+s-|s|)$) to $\mathcal{D}_m:=\left\{\Im(\tau)>\max(-1-j-m+s-|s|,-l-\frac{3}{2})\right\}$ for $m\in\mathbb{N}$.
- There exists $C>0$ independent of j such that for all $b>\max(-1-j-m+s-|s|,-l-\frac{3}{2})$,

$$\int_{\mathbb{R}}\left|1_{\mathbb{C}\setminus K}(\tau)(1+|\tau|^2+j^2)^{\frac{r+1}{2}}\widehat{\chi_0 v}_{|s|+j}(\tau+ib)\right|^2\,\mathrm{d}\tau\leq$$
$$C\left(\left\|u_{j+|s|}\right\|^2_{\overline{H}^{\tilde{r},l}_{(b)}}+\left\|v_{j+|s|}\right\|^2_{\overline{H}^{\tilde{r}+1,-\frac{1}{2}-s+|s|-\epsilon}_{(b)}}\right)$$

We assume the induction hypothesis for some $m \in \mathbb{N}$. We have $f_{|s|+j} = \chi_1 u_{|s|+j} - \chi_1 x\mathrm{Diff}_b^2 v_{|s|+j}$ (where χ_1 is a cutoff localizing near zero such that $\chi_1 = 1$ on the support of χ_0). We have $\chi_1 x\mathrm{Diff}_b^2 v_{|s|+j} = x\mathrm{Diff}_b^2 \chi_0 v_{|s|+j} + w$ where $w \in \chi_1 x\mathrm{Diff}_b^2(1-\chi_0)v_{|s|+j}$ has compact support with respect to x in $(0,+\infty)$ and is therefore in $\overline{H}_{(b)}^{\tilde{r}-1,\infty}$ and for any $N \in \mathbb{R}$, there exists $C_N > 0$ independent of v such that

$$\|w\|_{\overline{H}_{(b)}^{\tilde{r}-1,N}} \leq C_N \left\|v_{|s|+j}\right\|_{\overline{H}_{(b)}^{\tilde{r}+1,-\frac{1}{2}-s+|s|-\epsilon}}.$$

We deduce that $\hat{f}_{|s|+j}$ has a meromorphic extension to $\left\{\Im(\tau) > \max(-1-j-m-1+s-|s|, -l-\frac{3}{2})\right\}$ with at most simple poles at $\{i(-1-j-p+s-|s|), 1 \leq p \leq m\}$. Moreover, we have for $b > \max\left(-1-j-(m+1)+s-|s|\,, -l-\frac{3}{2}\right)$:

$$\begin{aligned}\int_{\mathbb{R}} \left|1_{\mathbb{C}\setminus K}(\tau+ib)(1+\tau^2+j^2)^{\frac{\tilde{r}-1}{2}} \hat{f}_{|s|+j}(\tau+ib)\right|^2 \mathrm{d}\tau \leq & C\left(\int_{\mathbb{R}} \left|(1+\tau^2+j^2)^{\frac{\tilde{r}}{2}}\hat{u}\right|^2 \mathrm{d}\tau\right. \\ & + \int_{\mathbb{R}} 1_{\mathbb{C}\setminus K}(\tau+ib)\left|(1+\tau^2+j^2)^{\frac{\tilde{r}+1}{2}}\widehat{\chi_0 v}_{|s|+j}(\tau+i(b+1))\right|^2 \mathrm{d}\tau \\ & \left. + \int_{\mathbb{R}} \left|(1+\tau^2+j^2)^{\frac{\tilde{r}-1}{2}}\hat{w}(\tau+ib)\right|^2 \mathrm{d}\tau\right) \\ \leq & C\left(\left\|u_{|s|+j}\right\|^2_{\overline{H}_{(b)}^{\tilde{r},l}} + \left\|v_{|s|+j}\right\|^2_{\overline{H}_{(b)}^{\tilde{r}+1,-\frac{1}{2}-s+|s|-\epsilon}}\right)\end{aligned}$$

where we used the induction hypothesis to bound the term involving $\widehat{\chi_0 v}_{|s|+j}(\tau+i(b+1))$. Using the identity $(\tau^2+(i-2is)\tau+j(j+2|s|)+|s|+j+s) = (\tau-i(-1+s-|s|-j))(\tau-i(s+|s|+j))$, we conclude that $\widehat{\chi_0 v}_{|s|+j}$ has a meromorphic extension to $\mathcal{D}_{m+1}$ with (at most) simple poles at $\{i(-1-j-p+s-|s|), 0 \leq p \leq m\}$. Using (14.7) we get

$$\int_{\mathbb{R}} \left|1_{\mathbb{C}\setminus K}(1+\tau^2+j^2)^{\frac{\tilde{r}+1}{2}}\widehat{\chi_0 v_{|s|+j}}(\tau+ib)\right|^2 \mathrm{d}\tau \leq C\int_{\mathbb{R}} \left|1_{\mathbb{C}\setminus K}(\tau+ib)(1+\tau^2+j^2)^{\frac{\tilde{r}-1}{2}}\hat{f}_{|s|+j}(\tau+ib)\right|^2 \mathrm{d}\tau.$$

We can therefore apply Lemma 14.2.2 to get :

$$v_{|s|+j} = x^{1-s+|s|+j}\sum_{m=0}^{k-j} x^m v_{m,|s|+j} + \overline{H}_{(b)}^{\tilde{r}+1,l}$$

Using this decomposition for $j \leq k$ and lemma 14.2.4 to prove that $v - \sum_{p=0}^{k} v_{|s|+p} \in \overline{H}_{(b)}^{\tilde{r}+1,l}$, we get the first claim of the lemma.

Let χ be a smooth cutoff localizing near zero. To determine exactly the form of $v_{j,|s|+j}$, we fix $0 \leq j \leq k$ and compute for every $\epsilon > 0$:

$$\begin{aligned}\left\langle \overline{U^*_{|s|+j,m}}, u\right\rangle &= \left\langle \overline{U^*_{|s|+j,m}}, \hat{T}_s(0)\hat{T}_s(0)^{-1}u\right\rangle \\ &= \left\langle \overline{U^*_{|s|+j,m}}, \hat{T}_s(0)\left(\chi\left(\frac{x}{\epsilon}\right)x^{1-s+|s|}\left(\sum_{j'=0}^{k}\sum_{i=j'}^{k} x^i v_{i,|s|+j'}\right) + \overline{H}_{(b)}^{\tilde{r}+1,l}\right)\right\rangle\end{aligned}$$

Note that $U^*_{|s|+j,m} \in \dot{H}_{(b)}^{(-s+\frac{1}{2})-,(\frac{1}{2}+s-|s|-j)-} r^2\,\mathrm{d}r\,\mathrm{d}\phi\,\mathrm{d}\theta$. In particular, it belongs to the dual space of $\overline{H}_{(b)}^{\tilde{r}-1,l}$ under the hypothesis on $\tilde{r}$ and l. Therefore, for all $g \in \overline{H}_{(b)}^{\tilde{r}+1,l}$,

$$\left\langle \overline{U^*_{|s|+j,m}}, \hat{T}_s(0)g\right\rangle = \left\langle \overline{\hat{T}_s(0)^* U^*_{|s|+j,m}}, g\right\rangle.$$

Using the fact that $\hat{T}_s(0)^* U^*_{|s|+j,m} = 0$ and the orthogonality property of $f_{|s|+j,-m}$, we get:

$$\left\langle \overline{U^*_{|s|+j,m}}, u \right\rangle = \left\langle \overline{U^*_{|s|+j,m}}, \hat{T}_s(0)\chi\left(\frac{x}{\epsilon}\right) x^{1-s+|s|+j} v_{j,|s|+j} \right\rangle$$

We use the fact that $\chi\left(\frac{x}{\epsilon}\right) x^{1-s+|s|+j} v_{j,|s|+j}$ tends to zero in $\overline{H}^{\tilde{r},(-\frac{1}{2}-s+|s|+j)-}_{(b)}$ when ϵ tends to zero, the fact that $N(\hat{T}_s(0))x^{1-s+|s|+j} v_{j,|s|+j} = 0$ and the fact that

$$U^*_{|s|+j,m} - x^{-2+s-|s|-j} f_{|s|+j,m} \sin(\theta)\, \mathrm{d}x\, \mathrm{d}\theta\, \mathrm{d}\phi \in \overline{H}^{\infty,(\frac{3}{2}+s-|s|-j)-}_{(b)} x^{-4} \sin(\theta)\, \mathrm{d}x\, \mathrm{d}\theta\, \mathrm{d}\phi$$

to obtain:

$$\left\langle \overline{U^*_{|s|+j,m}}, u \right\rangle = \lim_{\epsilon \to 0} \int_0^{+\infty} x^{-2+s-|s|-j} [N(\hat{T}_s(0), \chi\left(\frac{x}{\epsilon}\right)] x^{1-s+|s|+j}\, \mathrm{d}x \left\langle f_{|s|+j,m}, v_{j,|s|+j} \right\rangle$$

$$\begin{aligned}[N(\hat{T}_s(0)), \chi\left(\frac{x}{\epsilon}\right)] x^{1-s+|s|+j} =& -2(1-s+|s|+j)\frac{x^{2-s+|s|+j}}{\epsilon}\chi'\left(\frac{x}{\epsilon}\right) \\ & - x^{2-s+|s|+j}\partial_x\left(\frac{x}{\epsilon}\chi'\left(\frac{x}{\epsilon}\right)\right) + (1-2s)\frac{x^{2-s+|s|+j}}{\epsilon}\chi'\left(\frac{x}{\epsilon}\right)\end{aligned}$$

Using the relations

$$\int_0^{+\infty} \chi'\left(\frac{x}{\epsilon}\right)\frac{\mathrm{d}x}{\epsilon} = 1$$
$$\int_0^{+\infty} \partial_x\left(x\chi'\left(\frac{x}{\epsilon}\right)\right)\frac{\mathrm{d}x}{\epsilon} = 0,$$

we conclude that:

$$\left\langle \overline{U^*_{|s|+j,m}}, u \right\rangle = (1 + 2(|s|+j)) \left\langle f_{|s|+j,m}, v_{j,|s|+j} \right\rangle$$

Since $\left\langle f_{|s|+j,m}, v_{j,|s|+j} \right\rangle$ is the coefficient of $f_{|s|+j,m}$ in the decomposition of $v_{j,|s|+j}$, we get the claimed decomposition. In the case $0 \leq j \leq k-1$, we get (after projection on $\mathcal{Y}_{|s|+j}$):

$$\hat{T}_s(0)(v_{j,|s|+j}x^{1-s+|s|+j} + v_{j+1,|s|+j}x^{2-s+|s|+j} + \overline{H}^{\tilde{r},\min(\frac{3}{2}-s+|s|+j-,l)}) = u_{|s|+j}$$

We deduce that the coefficients of $x^{1-s+|s|+j}$ and $x^{2-s+|s|+j}$ in the expansion of $\hat{T}_s(0)(v_{j,|s|+j}x^{1-s+|s|+j} + v_{j+1,|s|+j}x^{2-s+|s|+j})$ have to vanish. The coefficient of $x^{1-s+|s|+j}$ vanish since $1-s+|s|+j$ is an indicial root of $N(\hat{T}_s(0))_{|\mathcal{Y}_{|s|+j}}$ and the vanishing or the coefficient of $x^{2-s+|s|+j}$ gives:

$$v_{j+1,|s|+j} = \sum_{|m|\leq |s|+j} \frac{\left(2M(j^2+(2|s|-s+2)j+s(s-|s|)+2|s|-s+1)-2iams\right)}{2(1+2(|s|+j))(j+|s|+1)} \left\langle \overline{U^*_{|s|+j,m}}, u \right\rangle f_{|s|+j,m}$$

We can iteratively compute the other terms.

We now prove the second claim. Let $\alpha = 1-s+|s|+j-k > -s-|s|$ for some $k \in \mathbb{N}$. Let $v = \hat{T}_s(0)^{-1} x^\alpha \ln(x)^p u_{|s|+j}$. A priori for all $\epsilon > 0$, $v \in \overline{H}^{\infty,\alpha-\frac{3}{2}-\epsilon}_{(b)}$. As previously, we have $(\tau^2 + (i-2is)\tau + j(j+2|s|) + |s| + j + s)\widehat{\chi_0 v} = \hat{f}$ with $f = x^\alpha \ln(x)^p u_{|s|+j} - x\mathrm{Diff}^2_b v$. We prove by induction on $m \in \mathbb{N}$ as previously (and using lemma 14.2.5) that we have:

- $\widehat{\chi_0 v}$ has a meromorphic extension to $\{\Im(\tau) > -\alpha - m\}$ with poles of order at most $p+1$ at $-i\alpha$, ..., $-i(\alpha+k-1)$ and poles of order at most $p+2$ at

$$-i(\alpha+k) = -i(1-s+|s|+j), ..., -i(\alpha+m-1).$$

- $\widehat{f}$ has a meromorphic extension to $\{\Im(\tau) > -\alpha - m - 1\}$ with poles of order at most $p+1$ at $-i\alpha$, ..., $-i(\alpha+k)$ and poles of order at most $p+2$ at

$$-i(\alpha+k+1) = -i(2-s+|s|), ..., -i(\alpha+m).$$

- For every $N \in \mathbb{N}$, there exists $C > 0$ such that for all $\tau \in \mathbb{C}$ with $|\Re(\tau)| \geq 1$:

$$|\hat{f}| \leq C(1+\Re(\tau)^2)^{-N} e^{C|\Im\tau|}.$$

We can then use Remark 14.2.3. At a pole θ of order at most q of $\widehat{\chi_0 v}$, the residue of $x^{i\tau}\widehat{\chi_0 v} = x^{i\theta}e^{i\ln(x)(\tau-\theta)}\widehat{\chi_0 v}$ is of the form $x^{i\theta}\sum_{j'=0}^{q-1}\frac{(i\ln(x))^{j'}}{j'!}\mathrm{Res}_{|\tau=\theta}((\tau-\theta)^{j'}\widehat{\chi_0 v})$. Therefore, we get the claimed expansion for v. To compute the exact expression of the principal term, we use the fact that

$$\hat{T}_s(0)\left(\chi(x)x^\alpha\sum_{j'=0}^{p}\sum_{i=0}^{n}x^i\ln(x)^{j'}v_{i,j',|s|+j} + \chi(x)x^\alpha\sum_{i=k}^{n}x^i\ln(x)^{p+1}w_{i,|s|+j} + \overline{H}_{(b)}^{\infty,\alpha+n-\frac{1}{2}-}\right) = \chi(x)x^\alpha\ln(x)^p u_{|s|+j}$$

together with the computation:

$$\begin{aligned}N(\hat{T}_s(0))x^\alpha\ln(x)^p v_{|s|+j} =& -(\alpha+|s|+j+s)(\alpha-|s|-j+s-1)x^\alpha\ln(x)^p v_{|s|+j} \\ &+ p(1-2s-2\alpha)x^\alpha\ln(x)^{p-1}v_{|s|+j} - p(p-1)x^\alpha\ln(x)^{p-2}v_{|s|+j}\end{aligned}$$

Remark that we could compute the other terms iteratively and in particular, in the case $p=0$ and $\alpha = -s+|s|$, we get the claimed expression for $w_{1,|s|+j}$. □

Let $C \in (0,+\infty]$. For $k \in \mathbb{N}$ we denote by $W_b^{k,\infty}((-C,C)_\sigma, H_{(b)}^{\tilde{r},l})$ the set of functions f from $(-C,C)$ to $H_{(b)}^{\tilde{r},l}$ which are k times differentiable with respect to $\sigma \in \mathbb{R}\backslash\{0\}$ and such that for all $0 \leq j \leq k$, $(\sigma\partial_\sigma)^j f$ is bounded on $I\backslash\{0\}$ where I is any bounded interval of $(-C,C)$. We can similarly define $W_b^{k,\infty}((0,C)_\sigma, H_{(b)}^{\tilde{r},l})$ (resp. $W_b^{k,\infty}((-C,0)_\sigma, H_{(b)}^{\tilde{r},l})$) for functions defined on $(0,C)$ (resp. on $(-C,0)$). We obtain from the definition that $f \in W_b^{k,\infty}(\mathbb{R}_\sigma, H_{(b)}^{\tilde{r},l})$ if and only if $f_{|(0,C)} \in W_b^{k,\infty}((-C,0)_\sigma, H_{(b)}^{\tilde{r},l})$ and $f_{|(-C,0)} \in W_b^{k,\infty}((0,C)_\sigma, H_{(b)}^{\tilde{r},l})$.

In this part, we fix $f \in C^\infty(\mathbb{R}_\sigma, H_{(b)}^{\tilde{r},l})$ with compact support. Our goal is to study the regularity of $R(\sigma)f(\sigma)$ with respect to σ in a neighborhood of zero for different values of the decay rate l ($\tilde{r}$ being considered as large).

Proposition 14.2.7. *Let $k \in \mathbb{N}$. Let $f \in C_c^\infty(\mathbb{R}_\sigma, \overline{H}_{(b)}^{\tilde{r},l})$ with $l \in (-\frac{5}{2}-s-|s|, \frac{1}{2}-s+|s|)\setminus\left\{-\frac{1}{2}-s+|s|\right\}$, $\tilde{r}+l_c-2k-1 > -\frac{1}{2}-2s$ and $\tilde{r}-2k-1 > \frac{1}{2}+s$. Let $l_c \in (-\frac{3}{2}-s-|s|, -\frac{1}{2})$ such that $l_c \leq l+1$. Then, we have that $R(\sigma)f \in C^\infty(\mathbb{R}_\sigma, \overline{H}_{(b)}^{\tilde{r},l_c}) + |\sigma|^{l-l_c}W_b^{k,\infty}(\mathbb{R}_\sigma, \overline{H}_{(b)}^{\tilde{r}-2k-1,l_c})$.*

Remark 14.2.8. *If $l = -\frac{1}{2}-s+|s|$, using the result with $l-\epsilon$ for every $\epsilon > 0$, we have that $R(\sigma)f \in C^\infty(\mathbb{R}_\sigma, \overline{H}_{(b)}^{\tilde{r},l_c}) + |\sigma|^{(l-l_c)-}W_b^{k,\infty}(\mathbb{R}_\sigma, \overline{H}_{(b)}^{\tilde{r}-2k-1,l_c})$.*

Remark 14.2.9. *Let $C \in (0, +\infty]$. With the notations of the proposition, if f is constant with respect to σ on $(0, C)$, the proof provides a more precise result. We have on $(0, C)$:*

$$R(\sigma)f = \sum_{k=0}^{N} \sigma^k v_k + |\sigma|^{l-l_c} W_b^{k,\infty}((0,C)_\sigma, \overline{H}_{(b)}^{\tilde{r}-2k-1,l_c})$$

where N denotes the integer such that $l - l_c - 1 \leq N < l - l_c$ (note that the sum is empty if $l \leq l_c$) and $v_k \in \overline{H}_{(b)}^{\tilde{r},l_c}$ with $v_0 = R(0)f$. The same equality holds if $(0, C)$ is replaced by $(-C, 0)$.

Proof. We proceed in three steps.

- *If $l \leq l_c$:* In this case, we use directly proposition 14.1.4 to conclude.
- *If $l \in (-\frac{3}{2} - s - |s|, -\frac{1}{2} - s + |s|)$ and $l > l_c$:* In this case, we use the resolvent identity to write:

  $$R(\sigma)f = R(0)f - R(\sigma)(\hat{T}_s(\sigma) - \hat{T}_s(0))R(0)f$$

  The term $R(0)f$ is in $C^\infty\left(\mathbb{R}_\sigma, \overline{H}_{(b)}^{\tilde{r}+1,l}\right)$ and $f_1 := (\hat{T}_s(\sigma) - \hat{T}_s(0))R(0)f \in \sigma C^\infty\left(\mathbb{R}_\sigma, \overline{H}_{(b)}^{\tilde{r},l-1}\right)$. We can iterate this procedure until $l - k \leq l_c$.
- *If $l \in (-\frac{1}{2} - s + |s|, \frac{1}{2} - s + |s|)$:* In this case we also write $R(\sigma)f = R(0)f + R(\sigma)(\hat{T}_s(\sigma) - \hat{T}_s(0))R(0)f$ and we use Lemma 14.2.6 to see that $R(0)f = cx^{1-s+|s|}\chi(x) + f_1$ where $c \in Y_{|s|}$ depends smoothly on σ, χ is a cutoff near zero and $f_1 \in \overline{H}_{(b)}^{\tilde{r}+1,l}$ depends smoothly on σ. Therefore, we have:

  $$R(\sigma)f = R(0)f + R(\sigma)v_1$$

  where $v_1 \in (\hat{T}_s(\sigma) - \hat{T}_s(0))C^\infty(\mathbb{R}_\sigma, Y_{|s|})x^{1-s+|s|}\chi(x) + \sigma C^\infty\left(R_\sigma, \overline{H}_{(b)}^{\tilde{r},l-1}\right)$. Note that if $-s + |s| = 0$, we have $v_1 \in \sigma C^\infty\left(R_\sigma, \overline{H}_{(b)}^{\tilde{r},l-1}\right)$ (since 1 is the indicial root of the normal operator of $\hat{T}_s(\sigma) - \hat{T}_s(0)$) and we can conclude immediately using the previous cases. If $-s + |s|$ is a positive integer, we can define $v_{k+1} = (\hat{T}_s(\sigma) - \hat{T}_s(0))R(0)v_k$ for $1 \leq k \leq -s + |s|$ and we have $R(\sigma)v_1 \in C^\infty(\mathbb{R}_\sigma, \overline{H}_{(b)}^{\tilde{r},l_c}) + (-1)^{-s+|s|-1}R(\sigma)v_{-s+|s|}$. By induction of the normal operator argument, we get that for $k \leq -s + |s|$, $v_k \in \sigma^k C^\infty(\mathbb{R}_\sigma, Y_{|s|})x^{1-k-s+|s|}\chi(x) + \sigma^k C^\infty(\mathbb{R}_\sigma, \overline{H}_{(b)}^{\tilde{r},l-k})$. Performing one last iteration, we have the indicial root cancellation which gives $v_k \in \sigma^{-s+|s|+1}C^\infty(\mathbb{R}_\sigma, \overline{H}_{(b)}^{\tilde{r},l-1+s-|s|})$ and $l - 1 + s - |s| \in (-\frac{3}{2} - s - |s|, -\frac{1}{2} - s + |s|)$, therefore we can conclude by the previous cases.

□

We now describe the behavior of $R(\sigma)f$ when f has a higher spatial decay.

Proposition 14.2.10. *Let $k \in \mathbb{N}$. Let $f \in C^\infty(\mathbb{R}_\sigma, \overline{H}_{(b)}^{\tilde{r},l})$ with $l > \frac{1}{2} - s + |s|$, $\tilde{r} - \frac{5}{2} - s - |s| - 2k > -\frac{1}{2} - 2s$, $\tilde{r} - 2k - 1 > \frac{1}{2} + s$. For any $\epsilon \in (0, 1)$ such that $l - \epsilon \geq \frac{1}{2} - s + |s|$, we have the following equality:*

$$R(\sigma)f = C^\infty(\mathbb{R}_\sigma, \overline{H}_{(b)}^{\tilde{r},-\frac{1}{2}-s+|s|-}) + \sigma^{2|s|+2}R(\sigma)v + |\sigma|^{2|s|+2+1-\epsilon}W_b^{k,\infty}(\mathbb{R}_\sigma, \overline{H}_{(b)}^{\tilde{r}-2k-1,-\frac{3}{2}-s-|s|+\epsilon-})$$

where $v = x^{-s-|s|}\chi(x)c_f + \overline{H}^{\tilde{r},-\frac{3}{2}-s-|s|+\epsilon}_{(b)}$ *for* χ *a smooth cutoff localizing near zero and* c_f *an element of* $Y_{|s|}$*. More precisely, we have:*

$$c_f = \sum_{|m|\leq|s|, m-s\in\mathbb{Z}} (-1)^{|s|-s+1}(2i)^{1+2|s|}\frac{\mathfrak{c}_m}{(2|s|)!}$$

where

$$\begin{aligned}
\mathfrak{c}_m :=& \frac{4iMs + \mathfrak{c}_m^{(2)} + 2i(1+|s|-s)\mathfrak{c}_m^{(3)} + \mathfrak{c}_m^{(4)}}{(1+2|s|)^2}\left\langle \overline{U^*_{|s|,m}}, f(0)\right\rangle f_{|s|,m}\\
\mathfrak{c}_m^{(2)} :=& 2as\beta_{|s|,m}\\
\mathfrak{c}_m^{(3)} :=& \frac{-iams + M(s(s-|s|)+2|s|-s+1)}{(|s|+1)}\\
\mathfrak{c}_m^{(4)} :=& \begin{cases} -2i(-2iams + 2Ms(s-|s|)) \text{ if } s<0\\ 0 \text{ if } s\geq 0\end{cases}\\
\beta_{|s|,m} :=& \left\langle f_{|s|,m}, \cos(\theta) f_{|s|,m}\right\rangle
\end{aligned}$$

Proof. First note that since $f(\sigma) = \sum_{i=0}^{2|s|+2}\sigma^i f_i + \sigma^{2|s|+3}C^\infty(\mathbb{R}_\sigma, \overline{H}^{\tilde{r},l}_{(b)})$ and $R(\sigma)\sigma^{2|s|+3}C^\infty(\mathbb{R}_\sigma, \overline{H}^{\tilde{r},l}_{(b)}) \subset |\sigma|^{2|s|+3}W_b^{k,\infty}(\mathbb{R}_\sigma, \overline{H}^{\tilde{r}-2k-1,-\frac{1}{2}-}_{(b)})$ (see Proposition 14.1.4), we are reduced to the case of f independent of σ. In this proof $c_{|s|}$ is an element of $Y_{|s|}$ and $c_{|s|+1}$ is an element of $Y_{|s|+1}$ but each instance (even in the same line) can be different. For this proof, we also record the expressions of $\hat{T}_s(\sigma) - \hat{T}_s(0)$ and $\hat{T}_s(0)$ near $x = 0$:

$$\begin{aligned}
\hat{T}_s(\sigma) - \hat{T}_s(0) =& a^2\sin^2\theta\sigma^2 + \frac{4Mar}{\Delta_r}\sigma D_\phi - 2(a^2+r^2)\sigma x^2 D_x\\
&- i\sigma\left(\frac{4Msa^2 - 2sr(a^2+r^2)}{\Delta_r} + 2(s+1)r + 2ias\cos\theta\right)\\
=& -2\sigma(D_x + ix^{-1}) + a^2\sin^2\theta\sigma^2 + i\sigma 4Ms + 2\sigma as\cos\theta + \sigma x\mathrm{Diff}^1_b\\
\hat{T}_s(0) =& \frac{a^2}{\Delta_r}\partial_\phi^2 + \Delta^{[s]} - \Delta_r^{-s}\partial_r\Delta_r^{s+1}\partial_r + 4s(r-M)\partial_r - 2s\frac{a(r-M)}{\Delta_r}\partial_\phi + s
\end{aligned}$$

Note that $\hat{T}_s(0)$ commutes with $\Delta^{[s]}$ but it is not the case of $\hat{T}_s(\sigma) - \hat{T}_s(0)$. We define recursively $u_1 = (\hat{T}_s(\sigma) - \hat{T}_s(0))\hat{T}_s(0)^{-1}f$ and $u_{j+1} = (\hat{T}_s(\sigma) - \hat{T}_s(0))\hat{T}_s(0)^{-1}u_j$ for $1\leq j\leq 2|s|+2$. It is not clear yet that this sequence is well defined since $\hat{T}_s(0)^{-1}$ is only defined on $\overline{H}^{\tilde{r},l}_{(b)}$ for $l > -\frac{3}{2} - s - |s|$ but we will see in the proof that u_j remains in this space when $j\leq 2|s|+1$. We prove recursively using lemma 14.2.6, the explicit form of $\hat{T}_s(\sigma) - \hat{T}_s(0)$ and the fact that $\cos\theta Y_{|s|}\subset Y_{|s|} + Y_{|s|+1}$ (see for example (2.33) in [71]) that for $j\leq |s| - s$:

$$\begin{aligned}
u_j =& \chi(x)\left(c_{|s|}\sigma^j x^{1-s+|s|-j} + \sigma^j x^{2-s+|s|-j}(c_{|s|}(j-1)\ln(x) + c_{|s|} + c_{|s|+1})\right)\\
&+ \sigma^{j+1}P(\sigma)\overline{H}^{\tilde{r},\frac{1}{2}-s+|s|-j-}_{(b)} + \sigma^j Q(\sigma)\overline{H}^{\tilde{r},\frac{1}{2}-s+|s|-j+\epsilon}_{(b)}
\end{aligned}$$

where P and Q are polynomials in σ and χ is any smooth cutoff localizing near $x = 0$. For $j = |s| - s + 1$:

$$\begin{aligned}
u_j =& \chi(x)\left(\sigma^j x^{2-s+|s|-j}((j-1)c_{|s|}\ln(x) + c_{|s|} + c_{|s|+1}) + \sigma^{j+1}P(\sigma)\overline{H}^{\tilde{r},\frac{1}{2}-s+|s|-j-}_{(b)}\right)\\
&+ \sigma^j Q(\sigma)\overline{H}^{\tilde{r},\frac{1}{2}-s+|s|-j+\epsilon}_{(b)}
\end{aligned}$$

For $|s| - s + 2 \leq j \leq 2|s| + 2$:

$$u_j = \chi(x) \left(\sigma^j x^{2-s+|s|-j} c_{|s|} + \sigma^{j+1} P(\sigma) \overline{H}^{\tilde{r}, \frac{1}{2}-s+|s|-j-}_{(b)} \right) + \sigma^j Q(\sigma) \overline{H}^{\tilde{r}, \frac{1}{2}-s+|s|-j+\epsilon}_{(b)}$$

And finally, for all $0 \leq j \leq 2|s| + 1$,

$$R(\sigma) f = \sum_{k=0}^{j} (-1)^k R(0) u_k + (-1)^{j+1} R(\sigma) u_{j+1}.$$

We then use proposition 14.1.4 to obtain

$$R(\sigma)\sigma^{3+2|s|} P(\sigma) \overline{H}^{\tilde{r}, -\frac{3}{2}-s-|s|-}_{(b)} \subset |\sigma|^{2|s|+2+1-\epsilon} W^{k,\infty}_b(\mathbb{R}_\sigma, \overline{H}^{\tilde{r}-2k-1, -\frac{3}{2}-s-|s|+\epsilon-}_{(b)}).$$

To compute precisely the $c_{|s|}$ term in the expression of $u_{2|s|+2}$, we see that we have two cases:

- Case $s \geq 0$: The sequence of terms $c_{|s|}$ in the expressions of u_j (we denote by $c^j_{|s|}$ the term appearing in u_j) can be computed from the recurrence relation (obtained using Lemma 14.2.6):

$$\begin{aligned}
c^1_{|s|} &= \sum_{|m| \leq |s|} \frac{2i(1 - s + |s|)\mathfrak{c}^{(3)}_m + 4iMs + 2as\beta_{|s|,m}}{1 + 2|s|} \left\langle \overline{U}^*_{|s|,m}, f(0) \right\rangle f_{|s|,m} \\
c^2_{|s|} &= -2i \frac{c^1_{|s|}}{1 + 2|s|} \\
c^{j+1}_{|s|} &= 2i(1 - s + |s| - j) \frac{c^j_{|s|}}{(j-1)(2 + 2|s| - j)}, \text{ for } j \geq 2
\end{aligned}$$

- Case $s < 0$: The logarithmic term in the expression of u_j for $2 \leq j \leq |s| - s + 1$ and the terms $c_{|s|}$ in the expression of u_j for $j = 2|s| + 2$ can be computed recursively (we call $c^j_{|s|}$ the coefficients appearing in these terms): First, the term $c^2_{|s|}$ is obtained by computing the logarithmic term in u_2.

$$\begin{aligned}
\alpha_2 &:= \sum_{|m| \leq |s|} \frac{2i(1 - s + |s|)\mathfrak{c}^{(3)}_m + 4iMs + 2as\beta_{|s|,m}}{1 + 2|s|} \left\langle \overline{U}^*_{|s|,m}, f(0) \right\rangle f_{|s|,m} \\
c^2_{|s|} &= 2i(-s + |s|) \left(-\frac{\alpha_2}{1 + 2|s|} \right. \\
&\qquad \left. + \sum_{|m| \leq |s|} \frac{2i}{(1 + 2|s|)^2} \left(2Ms(s - |s|) - 2iams\right) \left\langle \overline{U}^*_{|s|,m}, f(0) \right\rangle f_{|s|,m} \right)
\end{aligned}$$

  Then we have the following recursive relation (on the logarithmic terms) for $2 \leq j \leq -s + |s|$:

$$c^{j+1}_{|s|} = 2i(1 - s + |s| - j) \frac{c^j_{|s|}}{(j-1)(2 + 2|s| - j)}$$

  Then we have:

$$c^{2+2|s|}_{|s|} = \frac{2ic^{1+2|s|}_{|s|}}{(|s| - s)}$$

□

Now we want a more precise description of the term $R(\sigma)v$ of Proposition 14.2.10. We need to introduce the effective normal operator (see Definition 13.2.5). We have:

$$\begin{aligned}N_{\text{eff}}(\hat{P}(\sigma)) =&(x^2D_x)^2 + 2ix(x^2D_x)\\ &+ x^2\left(\frac{1}{\sin\theta}D_\theta \sin\theta D_\theta + \frac{1}{\sin^2\theta}D_\phi^2 + \frac{2s\cos\theta}{\sin^2\theta}D_\phi + s^2\text{cotan}^2\theta + s\right)\\ &- 2isx^2\left(xD_x + \frac{i}{2}\right) - sx^2 - 2\sigma\left(x^2D_x + ix\right)\end{aligned}$$

we deduce that

$$\begin{aligned}N_{\text{eff}}(\hat{T}_s(\sigma)) =&x^{-2}N_{\text{eff}}(\hat{P}(\sigma))\\ =&(xD_x)^2 + ixD_x + \Delta^{[s]} - 2is\left(xD_x + \frac{i}{2}\right) - 2\frac{\sigma}{x}\left(xD_x + i\right)\end{aligned}$$

For $\sigma > 0$, the change of variable $X = \frac{x}{|\sigma|}$ gives:

$$N_{\text{eff}}^+ :=(XD_X)^2 + iXD_X + \Delta^{[s]} - 2is\left(XD_X + \frac{i}{2}\right) - 2X^{-1}\left(XD_X + i\right)$$

and for $\sigma < 0$, it gives:

$$N_{\text{eff}}^- :=(XD_X)^2 + iXD_X + \Delta^{[s]} - 2is\left(XD_X + \frac{i}{2}\right) + 2X^{-1}\left(XD_X + i\right)$$

We use Definition B.0.1 (but here the variable X plays the role of x in the definition) to introduce the spaces $H_b^{\tilde{r},l,\nu}$. Using corollary B.2.3, we get that $N_{\text{eff}}^\pm$ are invertible from $\left\{u \in H_b^{\tilde{r},l,\nu} : N_{\text{eff}}^\pm(\hat{T}_s(\sigma))u \in H_b^{\tilde{r},l-1,\nu}\right\}$ to $H_b^{\tilde{r},l-1,\nu}$ where $l < -\frac{1}{2}$, $\tilde{r} + l > -\frac{1}{2} - 2s$ and $\nu \in \left(\frac{1}{2} + s - |s|, \frac{3}{2} + s + |s|\right)$.

Let B be a Banach space. For $\alpha \in \mathbb{R}$, we denote by $\mathcal{A}([0,1)_\sigma, \sigma^\alpha B)$ the space of smooth functions u from $(0,1)$ to B such that for all $k \in \mathbb{N}$, $\sup_{\sigma\in(0,1)} \left\|\sigma^{-\alpha}\left(\sigma\partial_\sigma\right)^k u\right\|_B < +\infty$.

Lemma 14.2.11. *Let χ, χ_1 be smooth cutoffs with $\chi = \chi_1 = 1$ in a neighborhood of 0 and χ_1 has compact support in $[0,1)$ while χ has compact support in $[0, \frac{1}{r_+ - \epsilon})$. Let $u \in H_b^{\infty,l,\nu}$, if $l + \nu \leq 0$, we have*

$$\chi(x)\chi_1(\sigma)u\left(\frac{x}{\sigma}\right) \in \mathcal{A}([0,1)_\sigma, \sigma^{\nu-\frac{3}{2}}\overline{H}_{(b)}^{\infty,l}).$$

If $l + \nu \geq 0$, we have for every $\mu \in (-l, \nu)$:

$$\chi(x)\chi_1(\sigma)u\left(\frac{x}{\sigma}\right) \in \mathcal{A}([0,1)_\sigma, \sigma^{\mu-\frac{3}{2}}\overline{H}_{(b)}^{\infty,-\mu}).$$

Proof. Let μ_1, μ_2 be defined as $\mu_1 = \nu$, $\mu_2 = l$ if $l + \nu \leq 0$ and by $\mu_1 = \mu \in (-l, \nu)$, $\mu_2 = -\mu$ if $l + \nu \geq 0$. The strategy is to bound a family of quantities $(B_{k,k'}(u))_{k,k'\in\mathbb{N}}$ which dominates the seminorms of $\mathcal{A}([0,1)_\sigma, \sigma^{\mu_1}\overline{H}_{(b)}^{\infty,\mu_2})$ we define:

$$B_{k,k'}(u) := \sup_{\sigma\in(0,1)} \int_0^{(r_+-\epsilon)^{-1}} \left\|\sigma^{-\nu+\frac{3}{2}}x^{-l}\left(x+\sigma\right)^{\nu+l}\left(x\partial_x\right)^{k'}\left(\sigma\partial_\sigma\right)^k \chi(x)\chi_1(\sigma)u\left(\frac{x}{\sigma}\right)\right\|^2_{H^{k'}(\mathcal{B}_s)} \frac{\mathrm{d}x}{x^4}$$

Using the inequality $(x+\sigma)^{l+\nu} \gtrsim 1$ if $(l+\nu) \leq 0$ and $(x+\sigma)^{l+\nu} \geq x^{l+\mu}\sigma^{\nu-\mu}$ if $l+\nu \geq 0$ and $-l \leq \mu \leq \nu$, we get that for all $N, N' \in \mathbb{N}$, there exists $C_{N,N'} > 0$ a constant independent of u such that:

$$\sup_{\sigma\in(0,1)} \left\| \sigma^{-\mu_1+\frac{3}{2}} (\sigma\partial_\sigma)^N \chi(x)\chi_1(\sigma) u\left(\frac{x}{\sigma}\right) \right\|_{\overline{H}^{N',\mu_2}_{(b)}} \leq C_{N,N'} \sum_{\substack{0\leq k\leq N \\ 0\leq k'\leq N'}} B_{k,k'}(u)$$

Moreover, we have (writing $X = \frac{x}{\sigma}$):

$$(x\partial_x)^k (\sigma\partial_\sigma)^{k'} \left(\chi(x)\chi_1(\sigma) u\left(\frac{x}{\sigma}\right)\right) =$$
$$\sum_{j=0}^{k}\sum_{m=0}^{k'} \binom{k}{j}\binom{k'}{m} (x\partial_x)^m \chi (\sigma\partial_\sigma)^j \chi_1 (-1)^{k'-m} (X\partial_X)^{k+k'-m-j} u(X)$$

There exists $C_{k,k'} > 0$ such that for all $j \leq k$ and $m \leq k'$:

$$\sup_{\substack{\sigma\in(0,1) \\ x\in\left(0,\frac{1}{r_+-\epsilon}\right)}} \left| \binom{k}{j}\binom{k'}{m} (x\partial_x)^m \chi (\sigma\partial_\sigma)^j \chi_1 \right| \leq C_{k,k'}$$

Making the change of variable $X = \frac{x}{\sigma}$ in the definition of $B_{k,k'}(u)$, we deduce that there exists $C'_{k,k'} > 0$ such that :

$$\begin{aligned} B_{k,k'}(u) \leq & \sum_{j=0}^{k}\sum_{m=0}^{k'} \int_0^{+\infty} C_{k,k'} \left\| \left(\frac{X}{X+1}\right)^{-l} (X+1)^\nu (X\partial_X)^{k+k'-j-m} u \right\|^2_{H^{k'}(\mathcal{B}_s)} \frac{\mathrm{d}X}{X^4} \\ \leq & C'_{k,k'} \|u\|^2_{H_b^{k+2k',l,\nu}} \end{aligned}$$

□

We define

$$N^{\pm}_{\mathrm{eff},|s|} := (XD_X)^2 + iXD_X + |s| - 2is\left(XD_X + \frac{i}{2}\right) \mp 2X^{-1}(XD_X + i)$$

since for all $|m| \leq |s|$ and $u \in H_b^{\tilde{r},l,\nu}([0,+\infty]_X)$, we have $N^{\pm}_{\mathrm{eff}} u f_{|s|,m} = (N^{\pm}_{\mathrm{eff},|s|} u) f_{|s|,m}$ we deduce that $N^{\pm}_{\mathrm{eff},|s|}$ are invertible between $\left\{u \in H_b^{\tilde{r},l,\nu}([0,+\infty]_X) : N^{\pm}_{\mathrm{eff},|s|} u \in H^{\tilde{r},l-1,\nu}([0,+\infty]_X)\right\}$ and $H_b^{\tilde{r},l-1,\nu}([0,+\infty]_X)$ where $l < -\frac{1}{2}$ and $\tilde{r} + l > -\frac{1}{2} - 2s$ and $\nu \in \left(\frac{1}{2} + s - |s|, \frac{3}{2} + s + |s|\right)$ (we can also prove it directly in the spirit of the proof of corollary B.2.3).

We define

$$\tilde{u}^{\pm} := (\pm 1)^{-s-|s|} (N^{\pm}_{\mathrm{eff},|s|})^{-1} X^{-s-|s|}.$$

Since $X^{-s-|s|} \in H_b^{\infty,(-s-|s|-\frac{3}{2})-,(s+|s|+\frac{3}{2})-}([0,+\infty]_X)$, we get $\tilde{u}^{\pm} \in H_b^{\infty,(-s-|s|-\frac{1}{2})-,(s+|s|+\frac{3}{2})-}$. Moreover, we have $\tilde{u}^{-} = (-1)^{-s-|s|}\overline{\tilde{u}^{+}}$.

Remark 14.2.12. *Using a normal operator argument on $\chi\tilde{u}^+$ and $(1-\chi)\tilde{u}^+$ (where χ is a smooth cutoff localizing near $X=0$) on the model of what is done in the proof of Proposition 14.2.6, we deduce the following asymptotic expansions:*

$$\begin{aligned}(1-\chi)\tilde{u}^+(X) =&(1-\chi)X^{-s-|s|}\left(\frac{\ln(X)}{1+2\,|s|}+b\right)+H_b^{\infty,\infty,(s+|s|+\frac{5}{2})-}\\ \chi\tilde{u}^+(X) =&\chi\left(\sum_{k=1}^{s+|s|}\frac{(-1)^{k+1}i^k(k-1)!}{2^k\,(s+|s|-k+1)}X^{k-s-|s|}+X\left(\frac{(-i)^{s+|s|+1}(s+|s|)!}{2^{s+|s|+1}}\ln(X)+b'\right)\right)\\ &+H_b^{\infty,(-\frac{3}{2})-,\infty}\end{aligned}$$

Where b, b' are complex constants (depending on s). Note that the sum in the second line is empty when $s\leq 0$.

Proposition 14.2.13. *Let $c_f\in Y_{|s|}$ be defined as in Proposition 14.2.10. Let $u^{(0)}(c_f)$ be the unique element of $Ker(\hat{T}_s(0))\cap\overline{H}_{(b)}^{\infty,-\frac{3}{2}-s-|s|-}$ such that $u^{(0)}(c_f)-x^{-s-|s|}c_f$ is of order $x^{1-s-|s|}$ at $x=0$. Let $v=x^{-s-|s|}\chi(x)c_f$ with χ a smooth cutoff localizing near $x=0$. Then for σ in a small punctured real neighborhood of zero and for any $\epsilon\in(0,1)$:*

$$\begin{aligned}R(\sigma)v =&\sigma^{-s-|s|}\chi(x)\left(\mathbb{1}_{\sigma>0}\tilde{u}^+\left(\frac{x}{\sigma}\right)+(-1)^{-s-|s|}\mathbb{1}_{\sigma<0}\overline{\tilde{u}^+}\left(-\frac{x}{\sigma}\right)\right)c_f\\ &+\left(\frac{\ln|\sigma|}{1+2|s|}-H(\sigma)2i\Im(b)\right)\left(\chi(x)x^{-s-|s|}c_f-u^{(0)}(c_f)\right)\\ &+C^\infty\left((-1,1)_\sigma,\overline{H}_{(b)}^{\infty,-\frac{1}{2}-s-|s|-\epsilon}\right)+\sigma^{\epsilon-}W_b^{\infty,\infty}(\overline{H}_{(b)}^{\infty,-\frac{1}{2}-s-|s|-\epsilon-}).\end{aligned}$$

In the previous expression, H is the Heaviside function and b is a complex constant which appears in the development of $\tilde{u}^+$ (we see that $\Im(b)=\frac{\pi}{2(2|s|+1)}$ in the proof of Lemma 14.2.27).

Remark 14.2.14. *In the proposition, c_f could be replaced by any element of $Y_{|s|}$.*

Proof. Note that for every $\sigma>0$, $\chi(x)\tilde{u}^+\left(\frac{x}{\sigma}\right)\in\overline{H}_{(b)}^{\infty,(-\frac{1}{2}-s-|s|)-}$. In particular, we have

$$R(\sigma)\hat{T}_s(\sigma)\chi(x)\tilde{u}^+\left(\frac{x}{\sigma}\right)=\chi(x)\tilde{u}^+\left(\frac{x}{\sigma}\right).$$

For $\sigma>0$, we deduce

$$R(\sigma)v=\sigma^{-s-|s|}\chi(x)\tilde{u}^+c_f+R(\sigma)(v-\sigma^{-s-|s|}\hat{T}_s(\sigma)\chi\tilde{u}^+c_f)$$

and

$$\begin{aligned}v-\sigma^{-s-|s|}\hat{T}_s(\sigma)\chi\tilde{u}^+c_f =&v-\sigma^{-s-|s|}N_{\text{eff}}^+\chi\tilde{u}^+c_f+\sigma^{-s-|s|}(N_{\text{eff}}^+-\hat{T}_s(\sigma))\chi\tilde{u}^+c_f\\ =&-\sigma^{-s-|s|}[N(\hat{T}_s(0)),\chi]\tilde{\chi}\tilde{u}^+c_f+\sigma^{-s-|s|}(N(\hat{T}_s(0))-\hat{T}_s(0))\chi\tilde{u}^+c_f\\ &+\sigma\text{Diff}_b^1\tilde{\chi}\tilde{u}^+c_f\end{aligned}$$

where $\tilde{\chi}=1$ on $\text{supp}\chi$. Note that by the mapping properties of N_{eff}^+,

$$\tilde{u}^+\in H_b^{\infty,(-\frac{1}{2}-s-|s|)-,(s+|s|+\frac{3}{2})-}([0,+\infty]_X).$$

Let χ_1 be a smooth cutoff localizing near $\sigma = 0$. By lemma 14.2.11, we have

$$\sigma^{-s-|s|}\chi_1(\sigma)\tilde{\chi}(x)\tilde{u}^+c_f \in \mathcal{A}\left([0,1)_\sigma, |\sigma|^{-1-}\overline{H}_{(b)}^{\infty,(-\frac{1}{2}-s-|s|)-}\right) \cap \mathcal{A}\left([0,1)_\sigma, |\sigma|^{0-}\overline{H}_{(b)}^{\infty,(-\frac{3}{2}-s-|s|)-}\right).$$

In particular,

$$\sigma^{-s-|s|}\chi_1(\sigma)\sigma\mathrm{Diff}_b^1\tilde{\chi}(x)\tilde{u}^+c_f \in \mathcal{A}\left([0,1)_\sigma, |\sigma|^{1-}\overline{H}_{(b)}^{\infty,(-\frac{3}{2}-s-|s|)-}\right)$$

and we can use Proposition 14.1.4 to conclude that

$$\sigma^{-s-|s|}\chi_1(\sigma)R(\sigma)\sigma\mathrm{Diff}_b^1\tilde{\chi}(x)\tilde{u}^+c_f \in |\sigma|^{(1-\eta)-}W_b^{\infty,\infty}\left(\overline{H}_{(b)}^{\infty,-\frac{3}{2}-s-|s|+\eta-}\right)$$

for $\eta \in (0,1)$. It is therefore part of the error term.

Let χ_2 be a smooth cutoff localizing near zero. For any $\epsilon \in (0,1)$, by remark 14.2.12, we have that

$$\tilde{u}^+\left(\frac{x}{\sigma}\right)c_f = \chi_2\left(\frac{|\sigma|}{x}\right)\left(\frac{x}{\sigma}\right)^{-s-|s|}\left(\frac{1}{1+2|s|}\ln\left(\frac{x}{|\sigma|}\right)+b\right)c_f + H_b^{\infty,-\frac{1}{2}-s-|s|-,s+|s|+\frac{3}{2}+\epsilon} \tag{14.8}$$

and therefore by Lemma 14.2.11 we have

$$\begin{aligned}\sigma^{-s-|s|}\chi_1(\sigma)\tilde{\chi}(x)\tilde{u}^+c_f =& \chi_1(\sigma)\tilde{\chi}(x)\chi_2\left(\frac{|\sigma|}{x}\right)x^{-s-|s|}\left(\frac{1}{1+2|s|}\ln\left(\frac{x}{|\sigma|}\right)+b\right)c_f \\ &+ \mathcal{A}\left([0,1)_\sigma, \sigma^\epsilon\overline{H}_{(b)}^{\infty,-\frac{3}{2}-s-|s|-\epsilon}\right)\end{aligned} \tag{14.9}$$

(and similarly with $\tilde{\chi}$ replaced by χ). Since $[N(\hat{T}_s(0)),\chi] \in x^\infty\mathrm{Diff}_b^1$ and $N(\hat{T}_s(0)) - \hat{T}_s(0) \in x\mathrm{Diff}_b^2$, we get

$$\begin{aligned}A :=& -\sigma^{-s-|s|}\chi_1(\sigma)[N(\hat{T}_s(0)),\chi]\tilde{\chi}\tilde{u}^+c_f + \sigma^{-s-|s|}\chi_1(\sigma)(N(\hat{T}_s(0)) - \hat{T}_s(0))\chi\tilde{u}^+c_f \\ =& -[N(\hat{T}_s(0)),\chi]\chi_1(\sigma)\tilde{\chi}(x)\chi_2\left(\frac{|\sigma|}{x}\right)x^{-s-|s|}\left(\frac{1}{1+2|s|}\ln\left(\frac{x}{|\sigma|}\right)+b\right)c_f \\ &+ (N(\hat{T}_s(0)) - \hat{T}_s(0))\chi_1(\sigma)\chi(x)\chi_2\left(\frac{|\sigma|}{x}\right)x^{-s-|s|}\left(\frac{1}{1+2|s|}\ln\left(\frac{x}{|\sigma|}\right)+b\right)c_f \\ &+ \mathcal{A}\left([0,1)_\sigma, \sigma^{\epsilon-}\overline{H}_{(b)}^{\infty,-\frac{1}{2}-s-|s|-\epsilon-}\right)\end{aligned}$$

Using Proposition 14.1.4, we get $R(\sigma)\mathcal{A}\left([0,1)_\sigma, \sigma^{\epsilon-}\overline{H}_{(b)}^{\infty,-\frac{1}{2}-s-|s|-\epsilon-}\right) \subset \sigma^{\epsilon-}W_b^{\infty,\infty}(\overline{H}_{(b)}^{\infty,-\frac{1}{2}-s-|s|-\epsilon-})$ and is therefore also an error term.

We write $\chi_2\left(\frac{\sigma}{x}\right) = 1 + \frac{\sigma}{x}\Psi\left(\frac{\sigma}{x}\right)$ with $(z\partial_z)^N\Psi(z)$ smooth and bounded for all N. We deduce:

$$\begin{aligned}\chi_1(\sigma)\chi(x)\chi_2\left(\frac{|\sigma|}{x}\right)x^{-s-|s|}\left(\frac{1}{1+2|s|}\ln\left(\frac{x}{\sigma}\right)+b\right)c_f =& \chi_1(\sigma)\chi(x)x^{-s-|s|}\left(\frac{1}{1+2|s|}\ln\left(\frac{x}{\sigma}\right)+b\right)c_f \\ &+ \sigma^{1-}\mathcal{A}([0,1)_\sigma, \overline{H}_{(b)}^{\infty,-\frac{5}{2}-s-|s|-})\end{aligned}$$

We can use Proposition 14.1.4 to prove that

$$R(\sigma)(N(\hat{T}_s(0)) - \hat{T}_s(0))\sigma^{1-}\mathcal{A}([0,1)_\sigma, \overline{H}_{(b)}^{\infty,-\frac{5}{2}-s-|s|-}) \subset \sigma^{\epsilon-}W_b^{\infty,\infty}(\overline{H}_{(b)}^{\infty,-\frac{1}{2}-s-|s|-\epsilon-})$$

and similarly

$$R(\sigma)[N(\hat{T}_s(0)),\chi]\sigma^{1-}\mathcal{A}([0,1)_\sigma,\overline{H}_{(b)}^{\infty,-\frac{5}{2}-s-|s|-}) \subset \sigma^{\epsilon-}W_b^{\infty,\infty}(\overline{H}_{(b)}^{\infty,-\frac{1}{2}-s-|s|-\epsilon-}).$$

We have on $(0,1)_\sigma$:

$$(N(\hat{T}_s(0))-\hat{T}_s(0))\chi_1(\sigma)\chi(x)x^{-s-|s|}\left(\frac{1}{1+2|s|}\ln(x)+b\right)c_f \in \overline{H}_{(b)}^{\infty,-\frac{1}{2}-s-|s|-} + C_c^\infty((0,1]_\sigma,\overline{H}_{(b)}^{\infty,-\frac{1}{2}-s-|s|-})$$

and

$$[N(\hat{T}_s(0)),\chi]\chi_1(\sigma)\tilde{\chi}(x)x^{-s-|s|}\left(\frac{1}{1+2|s|}\ln(x)+b\right) \in \overline{H}_{(b)}^{\infty,-\frac{1}{2}-s-|s|-} + C_c^\infty((0,1]_\sigma,\overline{H}_{(b)}^{\infty,-\frac{1}{2}-s-|s|-})$$

and therefore, by remark 14.2.9, we have that their image by $R(\sigma)$ is in

$$R(0)w^+ + C_c^\infty((0,1]_\sigma,\overline{H}_{(b)}^{\infty,-\frac{1}{2}-s-|s|-\epsilon-}) + \sigma^\epsilon W_b^{\infty,\infty}((0,1)_\sigma,\overline{H}_{(b)}^{\infty,-\frac{1}{2}-s-|s|-\epsilon-})$$

for all $\epsilon \in (0,1)$ where

$$\begin{aligned} w^+ :=& -[N(\hat{T}_s(0)),\chi]\tilde{\chi}x^{-s-|s|}\left(\frac{\ln(x)}{1+2|s|}+b\right)c_f \\ &+\left(N(\hat{T}_s(0))-\hat{T}_s(0)\right)\chi(x)x^{-s-|s|}\left(\frac{\ln(x)}{1+2|s|}+b\right)c_f. \end{aligned}$$

Note that $w^+ \in \overline{H}_{(b)}^{\infty,-\frac{1}{2}-s-|s|-}$.

So far, we are reduced to calculating the image by $R(\sigma)$ of

$$\begin{aligned} -\chi_1(\sigma)\frac{1}{1+2|s|}\ln(\sigma)\left(-[N(\hat{T}_s(0)),\chi]\tilde{\chi}(x)x^{-s-|s|}c_f+(N(\hat{T}_s(0))-\hat{T}_s(0))\chi(x)x^{-s-|s|}c_f\right) \\ = \chi_1(\sigma)\frac{\ln(\sigma)}{1+2|s|}\hat{T}_s(0)\chi(x)x^{-s-|s|}c_f \end{aligned}$$

where we used $\chi(x)N(\hat{T}_s(0))x^{-s-|s|}c_f = 0$ to get the right-hand side. Using that $\hat{T}_s(0)\chi(x)x^{-s-|s|}c_f \in \overline{H}_{(b)}^{\infty,-\frac{1}{2}-s-|s|-}$ ($x^{-s-|s|}c_f$ is in the kernel of the normal operator of $\hat{T}_s(0)$), we can write:

$$\chi_1(\sigma)\frac{\ln(\sigma)}{1+2|s|}R(\sigma)\hat{T}_s(0)\chi(x)x^{-s-|s|}c_f = \chi_1(\sigma)\frac{\ln(\sigma)}{1+2|s|}\tilde{v} - \chi_1(\sigma)\ln(\sigma)R(\sigma)\left(\hat{T}_s(\sigma)-\hat{T}_s(0)\right)\tilde{v}$$

where $\tilde{v} := R(0)\hat{T}_s(0)\chi(x)x^{-s-|s|}c_f$. By definition, $\tilde{v}$ is the only element in $\overline{H}_{(b)}^{\infty,-\frac{1}{2}-s-|s|-}$ such that $\hat{T}_s(0)\tilde{v} = \hat{T}_s(0)\chi(x)x^{-s-|s|}c_f$. We write $u^{(0)}(c_f)$ the element of $\mathrm{Ker}(\hat{T}_s(0))\cap\overline{H}_{(b)}^{\infty,-\frac{3}{2}-s-|s|-}$ such that $u^{(0)}(c_f) - x^{-s-|s|}c_f$ is of order $x^{1-s-|s|}$ at $x=0$. With this definition, we have $\tilde{v} = \chi(x)x^{-s-|s|}c_f - u^{(0)}(c_f)$. By proposition 14.2.7, we get that $R(\sigma)\left(\hat{T}_s(\sigma)-\hat{T}_s(0)\right)\tilde{v} \in \sigma^\epsilon W_b^{\infty,\infty}(\overline{H}^{\infty,-\frac{1}{2}-s-|s|-\epsilon-})$ and is therefore an error term. Finally, modulo error terms we get (for $\sigma>0$ small enough):

$$R(\sigma)v = \sigma^{-s-|s|}\chi(x)\tilde{u}^+\left(\frac{x}{\sigma}\right)c_f + \frac{\ln(\sigma)}{1+2|s|}(\chi(x)x^{-s-|s|}c_f - u^{(0)}(c_f)) + R(0)w^+$$

We perform the same computation to determine the form of $R(\sigma)v$ in a left neighborhood of 0 (it amounts to replacing N_{eff}^+ by N_{eff}^- and $\tilde{u}^+$ by $\tilde{u}^-$). We find that modulo error terms (for $\sigma < 0$ close to zero):

$$R(\sigma)v = (-\sigma)^{-s-|s|}\chi(x)\overline{\tilde{u}^+}\left(-\frac{x}{\sigma}\right)c_f + \frac{\ln(-\sigma)}{1+2|s|}(\chi(x)x^{-s-|s|}c_f - u^{(0)}(c_f)) + R(0)w^-$$

where

$$\begin{aligned} w^- :=& -[N(\hat{T}_s(0)),\chi]\tilde{\chi}x^{-s-|s|}\left(\frac{\ln(x)}{1+2|s|}+\overline{b}\right)c_f \\ &+\left(N(\hat{T}_s(0))-\hat{T}_s(0)\right)\chi(x)x^{-s-|s|}\left(\frac{\ln(x)}{1+2|s|}+\overline{b}\right)c_f. \end{aligned}$$

We see that $R(0)w^+\mathbb{1}_{\sigma>0} + R(0)w^-\mathbb{1}_{\sigma<0}$ has no smooth extension in a neighborhood of $\sigma = 0$ in general. We have (to get the second line we use that $N(\hat{T}_s(0))x^{-s-|s|}c_f = 0$):

$$\begin{aligned} R(0)w^+\mathbb{1}_{\sigma>0} + R(0)w^-\mathbb{1}_{\sigma<0} =& R(0)w^- + H(\sigma)R(0)(w^+ - w^-) \\ =& R(0)w^- - H(\sigma)R(0)\hat{T}_s(0)2i\Im(b)\chi(x)x^{-s-|s|}c_f \\ =& R(0)w^- - H(\sigma)2i\Im(b)\left(\chi(x)x^{-s-|s|}c_f - u^{(0)}(c_f)\right) \end{aligned}$$

The term $R(0)w^- \in \overline{H}_{(b)}^{\tilde{r},-\frac{1}{2}-s-|s|-}$ is independent of σ and can be absorbed in the error term $C^\infty\left((-1,1)_\sigma, \overline{H}_{(b)}^{\infty,-\frac{1}{2}-s-|s|-\epsilon}\right)$. □

Remark 14.2.15. *Contrary to what it seems, the principal term (i.e. $R(\sigma)v$ modulo $C^\infty\left((-1,1)_\sigma, \overline{H}_{(b)}^{\infty,-\frac{1}{2}-s-|s|-\epsilon}\right) + \sigma^{\epsilon-}W_b^{\infty,\infty}(\overline{H}_{(b)}^{\infty,-\frac{1}{2}-s-|s|-\epsilon-})$) does not depend on the choice of χ. Indeed, if χ_1 and χ_2 are two smooth cutoffs with compact support in $\left[0,\frac{1}{r_+}\right)_x$ and equal to 1 in a neighborhood of 0, we define the difference:*

$$\begin{aligned} D_{\chi_1,\chi_2} :=& \sigma^{-s-|s|}(\chi_1(x)-\chi_2(x))\left(\mathbb{1}_{\sigma>0}\tilde{u}^+\left(\frac{x}{\sigma}\right)+(-1)^{-s-|s|}\mathbb{1}_{\sigma<0}\overline{\tilde{u}^+}\left(-\frac{x}{\sigma}\right)\right)c_f \\ &+\frac{\ln|\sigma|}{1+2|s|}\left(\chi_1(x)-\chi_2(x)\right)x^{-s-|s|}c_f - H(\sigma)2i\Im(b)\left(\chi_1(x)-\chi_2(x)\right)x^{-s-|s|}c_f \end{aligned}$$

We have:

$$D_{\chi_1,\chi_2} \in C^\infty\left((-1,1)_\sigma, \overline{H}_{(b)}^{\infty,-\frac{1}{2}-s-|s|-\epsilon}\right) + \sigma^{\epsilon-}W_b^{\infty,\infty}(\overline{H}_{(b)}^{\infty,-\frac{1}{2}-s-|s|-\epsilon-})$$

This can be seen using (14.8) *combined with Lemma 14.2.11 which provides that for σ in a small interval $(0,\eta)$ with $\eta < 1$:*

$$\begin{aligned} (\chi_1(x)-\chi_2(x))\tilde{u}^+\left(\frac{x}{\sigma}\right) =& (\chi_1(x)-\chi_2(x))\left(\frac{\ln(x)-\ln(|\sigma|)}{1+2\,|s|}+b\right)\left(\frac{x}{|\sigma|}\right)^{-s-|s|}c_f \\ &+|\sigma|^{s+|s|+\epsilon}\,\mathcal{A}([0,1)_\sigma, H_{(b)}^{\infty,-\frac{1}{2}-s-|s|-}) \end{aligned}$$

and similarly for σ in an interval of the form $(-\eta,0)$.

Definition 14.2.16. *The previous proposition leads us to define*

$$\tilde{u}(\sigma,x) := \left(\mathbb{1}_{\sigma>0}\tilde{u}^+\left(\frac{x}{\sigma}\right)+(-1)^{-s-|s|}\mathbb{1}_{\sigma<0}\overline{\tilde{u}^+}\left(-\frac{x}{\sigma}\right)\right)c_f$$

Combining Proposition 14.2.10, Proposition 14.2.13 and the proof of Lemma 14.2.27 below (for the explicit value of the constant $\Im(b)$) we get

Corollary 14.2.17. *Let $k \in \mathbb{N}$. Let $f \in C^\infty(\mathbb{R}_\sigma, \overline{H}^{\tilde{r},l}_{(b)})$ with $l > \frac{1}{2} - s + |s|$, $\tilde{r} - \frac{5}{2} - s - |s| - 2k > -\frac{1}{2} - 2s$ and $\tilde{r} - 2k - 1 > \frac{1}{2} + s$. For all $\epsilon \in (0,1)$ such that $l - (1-\epsilon) \geq \frac{1}{2} - s + |s|$, we have the following equality for σ in a punctured neighborhood of zero:*

$$\begin{aligned} R(\sigma)f =& \sigma^{2-s+|s|}\chi(x)\tilde{u}(\sigma,x) + \sigma^{2+2|s|}\frac{\ln|\sigma|}{1+2|s|}(\chi(x)x^{-s-|s|}c_f - u^{(0)}(c_f)) \\ &- \sigma^{2+2|s|}H(\sigma)i\frac{\pi}{2|s|+1}\left(\chi(x)x^{-s-|s|}c_f - u^{(0)}(c_f)\right) \\ &+ |\sigma|^{2|s|+2+\epsilon-}W^{k,\infty}_b(\mathbb{R}_\sigma, \overline{H}^{\tilde{r}-2k-1,-\frac{1}{2}-s-|s|-\epsilon-}_b) + C^\infty(\mathbb{R}_\sigma, \overline{H}^{\tilde{r},-\frac{1}{2}-s-|s|-\epsilon}_{(b)}) \end{aligned}$$

for χ a smooth cutoff localizing near zero and c_f the element of $Y_{|s|}$ defined in proposition 14.2.10.

We add boundaries to $\overline{\mathcal{M}}_\epsilon$ to describe the asymptotic behavior of the the solution for large time.

Definition 14.2.18. *We first add the boundary corresponding to the boundary defining function $\mathfrak{t}^{-1}$ to $\overline{\mathcal{M}}_\epsilon$. We can then blow up the corner $\{\mathfrak{t}^{-1} = 0, x = 0\}$. By a slight abuse of notation, we still call $\overline{\mathcal{M}}_\epsilon$ the manifold with corners resulting from this procedure. Concretely, the manifold $\overline{\mathcal{M}}_\epsilon \cap \{\mathfrak{t} > 1\}$ now has three boundary faces (see Figure 14.1):*

- *K_+ with boundary defining function $(x\mathfrak{t}+1)^{-1}$.*
- *The front face of the blow up I_+ with boundary defining function $\frac{x\mathfrak{t}+1}{\mathfrak{t}}$.*
- *The face $\mathscr{I}^+$ (or rather its closure in the blow up space) with boundary defining function $\frac{x\mathfrak{t}}{x\mathfrak{t}+1}$.*

We also introduce a new function space to measure decay with respect to $\mathfrak{t}$ and x.

Definition 14.2.19. *Let $k \in \mathbb{N}$. $Z^{k,\tilde{r}}_{\alpha,\beta,\gamma}$ is the space of bounded functions u from $\mathbb{R}_\mathfrak{t}$ to $\overline{H}^{\tilde{r},\gamma}_{(b)}$ which have k bounded derivatives and such that for all $j \leq k$:*

$$\sup_{\mathfrak{t}\in\mathbb{R}} \left\| (x\mathfrak{t}+1)^\alpha \left(\frac{x\mathfrak{t}+1}{\mathfrak{t}}\right)^{-\beta} \left(\frac{x\mathfrak{t}}{x\mathfrak{t}+1}\right)^{-\gamma} (\mathfrak{t}\partial_\mathfrak{t})^j u \right\|_{\overline{H}^{\tilde{r},-\frac{3}{2}}_{(b)}} < +\infty$$

Remark 14.2.20. *Note that by Sobolev embedding, $\|u\|_{C^0} \leq \|u\|_{\overline{H}^{\frac{3}{2}+,-\frac{3}{2}+}_{(b)}}$ (note that the $-\frac{3}{2}$ offset comes from the fact that we use sc-volume form instead of a b-volume form). Therefore, if $\tilde{r} > \frac{3}{2} + m$ for some $m \in \mathbb{N}$, then for $u \in Z^{k,\tilde{r}}_{\alpha,\beta,\gamma}$, $j \leq k$ and $p \leq m$ we have the uniform bound*

$$|(\mathfrak{t}\partial_\mathfrak{t})^j (xD_x)^p u| \leq C x^{0-}(x\mathfrak{t}+1)^{-\alpha}\left(\frac{x\mathfrak{t}+1}{\mathfrak{t}}\right)^\beta \left(\frac{x\mathfrak{t}}{x\mathfrak{t}+1}\right)^\gamma$$

Remark 14.2.21. *The weights $(x\mathfrak{t}+1)^{-1}$, $\frac{x\mathfrak{t}+1}{\mathfrak{t}}$ and $\frac{x\mathfrak{t}}{x\mathfrak{t}+1}$ measure decay at K_+, I_+ and $\mathscr{I}^+$ (see Figure 14.1).*

We will need a slightly more precise version of Lemma 3.6 in [48].

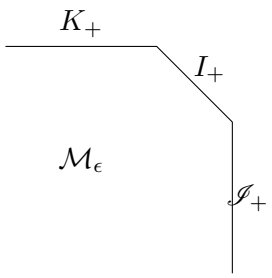

Figure 14.1: Blow up of the corner $x = 0$, $\mathfrak{t}^{-1} = 0$

Lemma 14.2.22. *Let X be a Banach space. Let $\beta > -1$ and $k > \beta + 1$, let $\hat{\Phi} \in L^1_c((-1,1)_\sigma, X) \cap |\sigma|^\beta W^{k,\infty}_b((-1,1)_\sigma \setminus \{0\}, X)$. For all $j \in \mathbb{N}$ such that $j < k - (\beta+1)$:*

$$|(\mathfrak{t}\partial_\mathfrak{t})^j \Phi(\mathfrak{t})| \leq C_j \langle \mathfrak{t} \rangle^{-1-\beta}$$

where Φ is the inverse Fourier transform of $\hat{\Phi}$.

Moreover, if $\hat{\Phi} \in C^p(\mathbb{R}_\sigma, X) \cap L^1(\mathbb{R}_\sigma, X)$ is supported on $\mathbb{R} \setminus [-\epsilon, \epsilon]$ and for $0 \leq q \leq p$, $(\sigma\partial_\sigma)^q \hat{\Phi} \in \langle \sigma \rangle^{-k+p} L^1(\mathbb{R}_\sigma, X)$ (for some $k \geq 0$) we have, for all $j \leq k$:

$$|(\partial_\mathfrak{t})^j \Phi(\mathfrak{t})| \leq C_j \langle \mathfrak{t} \rangle^{-p}$$

Proof. The proof of the first claim is formally identical to the proof of Lemma 3.6 in [48]. The difference is that we only require the lowest conormal regularity permitted by the proof ($\hat{\Phi} \in W^{k,\infty}_b((-1,1)_\sigma \setminus \{0\}, X)$ instead of $\hat{\Phi} \in W^{\infty,\infty}_b((-1,1)_\sigma \setminus \{0\}, X)$). We now prove the second claim. Since $\hat{\Phi}$ is integrable, Φ is bounded with respect to $\mathfrak{t}$. Therefore, it is enough prove the estimate for $|\mathfrak{t}| \geq 1$. We have:

$$\begin{aligned}
\Phi(\mathfrak{t}) =& \frac{1}{2\pi} \int_{\mathbb{R}\setminus[-\epsilon,\epsilon]} e^{-i\sigma\mathfrak{t}} \hat{\Phi}(\sigma) \, \mathrm{d}\sigma \\
D^j_\mathfrak{t} \Phi(\mathfrak{t}) =& \frac{1}{2\pi} \int_{\mathbb{R}\setminus[-\epsilon,\epsilon]} \sigma^j e^{-i\sigma\mathfrak{t}} \hat{\Phi}(\sigma) \, \mathrm{d}\sigma \\
=& \frac{1}{2\pi} \int_{\mathbb{R}\setminus[-\epsilon,\epsilon]} \sigma^j \hat{\Phi}(\sigma) \mathfrak{t}^{-p} (-D_\sigma)^p e^{-i\sigma\mathfrak{t}} \, \mathrm{d}\sigma \\
=& \frac{1}{2\pi} \mathfrak{t}^{-p} \int_{\mathbb{R}\setminus[-\epsilon,\epsilon]} (-1)^p e^{-i\sigma\mathfrak{t}} \sum_{l+m=p} \binom{p}{l} D^l_\sigma \sigma^j D^m_\sigma \hat{\Phi}(\sigma) \, \mathrm{d}\sigma
\end{aligned}$$

Since $D^l_\sigma \sigma^j D^m_\sigma \hat{\Phi}(\sigma)$ is integrable on $\mathbb{R} \setminus [-\epsilon, \epsilon]$ for all l, m such that $l + m = p$ we get for all $|\mathfrak{t}| \geq 1$:

$$\left\| D^j_\mathfrak{t} \hat{\Phi}(\mathfrak{t}) \right\|_X \leq C \, |\mathfrak{t}|^{-p} .$$

□

For $p, m \in \mathbb{R}$, we define the space $\mathcal{A}^{p,m}(\mathbb{R}_\sigma \setminus \{0\})$ as the set of complex valued tempered distributions f on $\mathbb{R}$ such that for all $k \in \mathbb{N}$:

$$\langle \sigma \rangle^m \left(\frac{\sigma}{\langle \sigma \rangle} \right)^{-p} (\sigma D_\sigma)^k f \in L^\infty(\mathbb{R})$$

We also define the space $S^m(\mathbb{R}_\sigma)$ (symbol of order m) as the set of complex valued smooth functions f on $\mathbb{R}$ such that for all $k \in \mathbb{N}$:

$$(D_\sigma)^k f \leq \langle \sigma \rangle^{m-k}$$

We will need the following lemma:

Lemma 14.2.23. *Let $m \leq 1$ and $p > -1$. Let $\hat{\Phi} \in \mathcal{A}^{p,m}(\mathbb{R} \setminus \{0\})$. Then we have: $\Phi \in \mathcal{A}^{m-1,p+1}(\mathbb{R} \setminus \{0\})$.*

Proof. We fix χ a smooth cutoff with compact support equal to 1 near zero. By lemma 14.2.22, we have that $\chi\hat{\Phi}$ has inverse Fourier transform in $S^{-(p+1)}(\mathbb{R})$. Using $S^{-(p+1)} + \mathcal{A}^{m-1,\infty} \subset \mathcal{A}^{m-1,p+1}$ and $(1-\chi)\hat{\Phi} \in S^{-m}$, we are reduced to proving that the Fourier transform maps S^{-m} to $\mathcal{A}^{m-1,\infty}$. This follows for example from Lemma 2.3 (and the estimate (2.7) just after) in [99, Chapter 7] for the estimate near zero and from Proposition 8.2 in [98, Chapter 3] for the estimate at infinity. □

Lemma 14.2.24. *We denote by $\mathcal{F}^{-1}$ the inverse Fourier transform. For $l \in \mathbb{R}$, we have: $\mathcal{F}^{-1}C_c^\infty(\mathbb{R}_\sigma, \overline{H}_{(b)}^{\tilde{r},l}) \subset Z^{\infty,\tilde{r}}_{\infty,\infty,l+\frac{3}{2}}$ and, for $k \geq 3 + 2|s| + \epsilon$,*

$$\mathcal{F}^{-1}|\chi(\sigma)\sigma|^{2|s|+2+\epsilon} W_b^{k,\infty}((-1,1)\setminus\{0\}, \overline{H}_{(b)}^{\tilde{r},-\frac{1}{2}-s-|s|-\epsilon}) \subset Z^{E(k-3-2|s|-\epsilon),\tilde{r}}_{3+2|s|+\epsilon,4-s+|s|,1-s-|s|-\epsilon}$$

(where $E(x)$ denotes the integer part of x and χ is smooth compactly supported in $(-1,1)$). More generally for $\beta > -1$ and $k > \beta + 1$, we have:

$$\mathcal{F}^{-1}\chi(\sigma)|\sigma|^\beta W_b^{k,\infty}((-1,1)\setminus\{0\}, \overline{H}_{(b)}^{\tilde{r},l}) \subset Z^{E(k-(\beta+1)),\tilde{r}}_{\beta+1,\beta+l+\frac{5}{2},l+\frac{3}{2}}$$

Proof. The first inclusion is a consequence of the fact that the Fourier transform maps Schwartz functions to Schwartz functions. The other inclusions are obtained by application of lemma 14.2.22. □

Definition 14.2.25. *We define:*

$$u^0_{I_+}(v) := \left(\frac{v}{v+1}\right)^{3+|s|-s} \mathcal{F}_Y^{-1}\left(Y^{2+|s|-s}\tilde{u}(Y,1)\right)(v)$$

Using the fact that $\overline{\mathcal{F}(g)} = \mathcal{F}(\breve{\bar{g}})$ (where $\breve{g}(Y) = g(-Y)$), we equivalently get:

- *If $s \in \mathbb{Z}$:*

$$u^0_{I_+}(v) := 2\left(\frac{v}{v+1}\right)^{3+|s|-s} \Re\left(\mathcal{F}_Y^{-1}\left(H(Y)Y^{2+|s|-s}\tilde{u}^+(Y^{-1})\right)(v)\right) c_f$$

- *If $s \notin \mathbb{Z}$:*

$$u^0_{I_+}(v) := 2i\left(\frac{v}{v+1}\right)^{3+|s|-s} \Im\left(\mathcal{F}_Y^{-1}\left(H(Y)Y^{2+|s|-s}\tilde{u}^+(Y^{-1})\right)(v)\right) c_f$$

The function $u^0_{I_+}$ is smooth on $(0,+\infty)$.

Remark 14.2.26. *We can compute the asymptotic expansion of $u^0_{I_+}$ at $v=0$ and $v=+\infty$. Indeed for χ a smooth compactly supported cutoff localizing near 0, using remark 14.2.12, we have:*

$$\begin{aligned}
H(Y)Y^{2+|s|-s}\chi(Y)\tilde{u}^+(Y^{-1}) =& H(Y)Y^{2+2|s|}\left(-\frac{\ln(Y)}{1+2|s|}+b\right)+\mathcal{A}^{3+2|s|-,-2-2|s|-}(\mathbb{R}\setminus\{0\})\\
H(Y)Y^{2+|s|-s}(1-\chi(Y))\tilde{u}^+(Y^{-1}) =& H(Y)\sum_{k=1}^{s+|s|}\frac{(-1)^{k+1}i^k(k-1)!}{s+|s|-k+1}Y^{2|s|+2-k}\\
&+H(Y)Y^{1+|s|-s}\left(-\frac{(-i)^{s+|s|+1}(s+|s|)!}{2^{s+|s|+1}}\ln(Y)+b'\right)\\
&+\mathcal{A}^{(1+|s|-s)-,-(|s|-s)-}(\mathbb{R}\setminus\{0\})
\end{aligned}$$

Using Lemma 14.2.23 (and the proof of Lemma 14.2.27 for the computation of $\Im(b)$), we get:

- *if $s\in\mathbb{Z}$:*

$$\begin{aligned}
u^0_{I_+}(v) =&(1-\chi(v))\frac{(-1)^{1+|s|}(2+2|s|)!}{1+2|s|}v^{-3-2|s|}c_f\\
&+\chi(v)(1+|s|-s)!\left((-1)^{1+|s|}\frac{(s+|s|)!}{2^{s+|s|+2}}+(-1)^{1+\frac{|s|-s}{2}}\frac{\Re(b')}{\pi}\right)vc_f\\
&+\mathcal{A}^{2-,(4+2|s|)-}(\mathbb{R}\setminus\{0\})c_f.
\end{aligned}$$

- *if $s\in\frac{1}{2}\mathbb{Z}$:*

$$\begin{aligned}
u^0_{I_+}(v) =&-i(1-\chi(v))\frac{(-1)^{\frac{1+2|s|}{2}}(2+2|s|)!}{1+2|s|}v^{-3-2|s|}c_f\\
&+\chi(v)i(1+|s|-s)!\left(-\frac{(-1)^{\frac{1+2|s|}{2}}(s+|s|)!}{2^{s+|s|+2}}+(-1)^{|s|-s}\frac{\Im(i^{2+|s|-s}b')}{\pi}\right)vc_f\\
&+\mathcal{A}^{2-,(4+2|s|)-}(\mathbb{R}\setminus\{0\})c_f.
\end{aligned}$$

We can now compute explicitly the value of $\Im(b)$ and the function $u^0_{I^+}$.

Lemma 14.2.27. *For $v\in(0,+\infty)$, we have:*

$$u^0_{I^+}(v)=(-i)^{2+2|s|}(2|s|)!\frac{v\left((2|s|+2)v+2(|s|-s+1)\right)}{(v+2)^{2+|s|+s}(v+1)^{3+|s|-s}}c_f.$$

Proof. Starting from the definition of $\tilde{u}^+$, we have for $X>0$:

$$(X^{-2-|s|+s}N^+_{\mathrm{eff},|s|}X^{2+|s|-s})X^{-2-|s|+s}\tilde{u}^+(X)=X^{-2-2|s|} \tag{14.10}$$

We define the operator

$$Q=-Y^2\partial_Y^2+(-2iY+2|s|+2)Y\partial_Y+2i(|s|-s+1)Y-2(|s|+1)$$

which is obtained from $X^{-2-|s|+s}N^+_{\mathrm{eff},|s|}X^{2+|s|-s}$ by the change of variable $Y=X^{-1}$. Therefore (14.10) becomes (for all $Y>0$):

$$QY^{2+|s|-s}\tilde{u}^+(Y^{-1})=Y^{2+2|s|} \tag{14.11}$$

Replacing Y by $-Y$ in (14.11), and taking the complex conjugate we obtain (for all $Y<0$):

$$(-1)^{-|s|-s}QY^{2+|s|-s}\overline{\tilde{u}^+}(-Y^{-1})=Y^{2+2|s|} \tag{14.12}$$

Combining (14.11) and (14.12), we obtain for every $Y\in\mathbb{R}\setminus\{0\}$:

$$Q\left(Y^{2+|s|-s}\tilde{u}(Y,1)\right)=Y^{2+2|s|} \tag{14.13}$$

Using Remark 14.2.12, we see that $Y\mapsto QY^{2+|s|-s}\tilde{u}(Y,1)$ understood in the sense of distributions is in $L^1_{loc}(\mathbb{R})$. Therefore it does not contain any Dirac term at $Y=0$ and equality (14.13) holds globally in the sense of distributions.

Applying the inverse Fourier transform, we get:

$$\hat{Q}\mathcal{F}_Y^{-1}\left(Y^{2+|s|-s}\tilde{u}(Y,1)\right)=i^{2+2|s|}\delta_0^{(2+2|s|)}$$

where

$$\hat{Q}=-v(v+2)\partial_v^2+2(-v(|s|+3)+s-|s|-3)\partial_v-4|s|-6.$$

Restricting to $(0,+\infty)$ we deduce:

$$\hat{Q}\mathcal{F}_Y^{-1}\left(Y^{2+|s|-s}\tilde{u}(Y,1)\right)=0.$$

The previous equation is hypergeometric. The regular singular point at infinity has indicial roots 2 and $3+2|s|$. Using the development of $u^0_{I_+}$ at infinity (see Remark 14.2.26), we deduce that, on $(0,+\infty)_v$, $\mathcal{F}_Y^{-1}\left(Y^{2+|s|-s}\tilde{u}(Y,1)\right)$ is equal to the unique (up to a constant complex factor) solution of order $v^{-3-2|s|}$ at infinity. This solution can be expressed by the mean of the general hypergeometric functions. We can also check directly that:

$$\hat{Q}\frac{(2|s|+2)v+2(|s|-s+1)}{v^{2+|s|-s}(v+2)^{2+|s|+s}}=0$$

and therefore there exists a complex constant β such that for all $v\in(0,+\infty)$:

$$\mathcal{F}_Y^{-1}\left(Y^{2+|s|-s}\tilde{u}(Y,1)\right)(v)=\beta\frac{(2|s|+2)v+2(|s|-s+1)}{v^{2+|s|-s}(v+2)^{2+|s|+s}}.$$

Moreover, using Remark 14.2.26 and Lemma 14.2.23, we have that $\mathcal{F}_Y^{-1}\left(Y^{2+|s|-s}\tilde{u}(Y,1)\right)$ belongs to $\mathcal{A}^{-2-2|s|-,3+2|s|-}$. in particular it is smooth on $(-\infty,0)$. Performing the change of variable $Z=\frac{v}{2}+1$ (to reduce to a canonical hypergeometric equation), the smoothness at $Z=0$ implies that there exists $\alpha\in\mathbb{C}$ such that for all $v\in(-\infty,0)$:

$$\mathcal{F}_Y^{-1}\left(Y^{2+|s|-s}\tilde{u}(Y,1)\right)(v)=\alpha\mathbf{F}\left(2|s|+3,2,s+|s|+3,\frac{v}{2}+1\right).$$

Since we have:

$$\begin{aligned}(1-Z)^{2+|s|-s}\mathbf{F}(2|s|+3,2,s+|s|+3,Z)=&F(s+|s|+1,s-|s|,s+|s|+3,Z)\\&=\sum_{k=0}^{|s|-s}\frac{(s+|s|+1)_k(s-|s|)_k}{(s+|s|+2+k)!}\frac{Z^k}{k!}\end{aligned}$$

We deduce that in the regime $Z\to-\infty$:

$$\mathbf{F}(2|s|+3,2,s+|s|+3,Z)\sim Z^{-2}\frac{(2|s|)!}{(|s|+s)!(2|s|+2)!}$$

The condition $\mathcal{F}_Y^{-1}\left(Y^{2+|s|-s}\tilde{u}(Y,1)\right) \in \mathcal{A}^{-2-2|s|-,3+2|s|-}$ implies $\alpha = 0$. Moreover, using Remark 14.2.26 and Lemma 14.2.23, we see that:

$$\begin{aligned}\mathcal{F}_Y^{-1}\left(Y^{2+|s|-s}\tilde{u}(Y,1)\right)(v) =& \mathcal{F}_Y^{-1}\left(Y^{2+2|s|}\left(H(Y)\left(-\frac{\ln(Y)}{1+2|s|}+b\right)+H(-Y)\left(-\frac{\ln(-Y)}{1+2|s|}+\bar{b}\right)\right)\right)\\ &+\mathcal{A}^{-3-2|s|-,4+2|s|-}\\ =&(1-\chi(v))(i\partial_v)^{2+2|s|}2\Re\mathcal{F}_Y^{-1}\left(H(Y)\left(b-\frac{\ln(Y)}{1+2|s|}\right)\right)+\mathcal{A}^{-3-2|s|-,4+2|s|-}\\ =&(1-\chi(v))(-i)^{2+2|s|}(2|s|+2)!\left(\frac{\Im(b)}{\pi}+\frac{\operatorname{sgn}(v)}{2(2|s|+1)}\right)v^{-3-2|s|}\\ &+\mathcal{A}^{-3-2|s|-,4+2|s|-}\end{aligned} \tag{14.14}$$

where χ is a smooth compactly supported cutoff equal to 1 in a neighborhood of zero. Using that $\alpha = 0$, and (near $v = -\infty$)

$$\mathcal{F}_Y^{-1}\left(Y^{2+|s|-s}\tilde{u}(Y,1)\right)(v) = (-i)^{2+2|s|}(2|s|+2)!\left(\frac{\Im(b)}{\pi}-\frac{1}{2(2|s|+1)}\right)v^{-3-2|s|}+O(v^{-4-2|s|-}),$$

we deduce $\Im(b) = \frac{\pi}{2(2|s|+1)}$. Finally, we use (14.14) at $v = +\infty$ to identify the complex factor $\beta = (-i)^{2+2|s|}(2|s|)!$. □

Lemma 14.2.28. *We have:*

$$2\chi(\mathfrak{t}^{-1})\mathcal{F}^{-1}\left(\chi(x)\chi(\sigma)\sigma^{2+|s|-s}\tilde{u}(\sigma,x)\right)(\mathfrak{t}) = \chi(x)\chi(\mathfrak{t}^{-1})\left(\frac{x\mathfrak{t}+1}{\mathfrak{t}}\right)^{3+|s|-s}u^0_{I_+}(x\mathfrak{t})+R.$$

with R smooth with bounds (for all $N, M, K \in N$):

$$\left|\partial_{\mathfrak{t}}^N(x\partial_x)^M R\right| \leq C_{N,M,K}\langle\mathfrak{t}\rangle^{-K}x^{1-s-|s|-}$$

In particular, $R \in Z^{\infty,\infty}_{\infty,\infty,1-s-|s|-}$.

Proof. Let χ be a smooth compactly supported cutoff equal to 1 near zero.

$$\begin{aligned}\chi(\mathfrak{t}^{-1})\mathcal{F}^{-1}\left(\chi(x)\chi(\sigma)\sigma^{2+|s|-s}\tilde{u}(\sigma,x)\right) =& I+I'+R\\ I :=& \chi(\mathfrak{t}^{-1})\chi(x)\mathcal{F}^{-1}\left(H(\sigma)\sigma^{2+|s|-s}\tilde{u}^+\left(\frac{x}{\sigma}\right)\right)c_f\\ I' :=& \chi(x)\chi(\mathfrak{t}^{-1})\mathcal{F}^{-1}\left((-1)^{-s-|s|}H(-\sigma)\sigma^{2+|s|-s}\overline{\tilde{u}^+\left(\frac{x}{\sigma}\right)}\right)c_f\\ R :=& -\chi(\mathfrak{t}^{-1})\mathcal{F}^{-1}\left(\chi(x)(1-\chi(\sigma))\sigma^{2+|s|-s}\tilde{u}(\sigma,x)\right)\end{aligned}$$

We see in particular that $I' = \overline{I}$ if $s \in \mathbb{Z}$ and $I' = -\overline{I}$ if $s \in \frac{1}{2}\mathbb{Z}$. Moreover, by a change of variable $Y = \frac{\sigma}{x}$ in the Fourier transform, I is exactly $\chi(\mathfrak{t}^{-1})\chi(x)x^{3+|s|-s}\mathcal{F}^{-1}\left(H(Y)Y^{2+|s|-s}\tilde{u}^+(Y)\right)(x\mathfrak{t})$. Therefore, it remains to prove the claimed bound on $\partial_{\mathfrak{t}}^N(x\partial_x)^M R$. It is enough to prove that for all $N, M, K \in \mathbb{N}$ with K large enough, there exists $C_{N,M,K} > 0$ such that for all $\mathfrak{t} > 0$ and $x > 0$:

$$\chi(x)\left|\mathcal{F}^{-1}\left(H(\sigma)(1-\chi(\sigma))\sigma^{N+2+|s|-s}(x\partial_x)^M\tilde{v}\left(\frac{x}{\sigma}\right)\right)(\mathfrak{t})\right| \leq C_{N,M,K}\mathfrak{t}^{-K}x^{1-s-|s|-}$$

Using the fact that for $0 < X < C$, we have $|(X\partial_X)^N \tilde{u}^+| \leq C_N X^{1-s-|s|-}$, we deduce that

$$\left|\chi(x)(1-\chi(\sigma))\partial_\sigma^N (x\partial_x)^M \tilde{u}^+\left(\frac{x}{\sigma}\right)\right| \leq C_N \sigma^{-N-1+s+|s|+} x^{1-s-|s|-}$$

(note that for $x \in \mathrm{supp}(\chi)$ and $\sigma \in \mathrm{supp}(1-\chi)$, we have $0 < \frac{x}{\sigma} < C$). Using these properties for $M \in \mathbb{N}$ and $K > N + 2 + |s| - s$, we deduce:

$$\begin{aligned}
A :=& \chi(x)\mathcal{F}^{-1}\left(H(\sigma)(1-\chi(\sigma))\sigma^{N+2+|s|-s}(x\partial_x)^M \tilde{u}^+\left(\frac{x}{\sigma}\right)\right)(\mathfrak{t}) \\
=& \left(\frac{1}{i\mathfrak{t}}\right)^K \chi(x)\mathcal{F}^{-1}\left(H(\sigma)\partial_\sigma^K\left((1-\chi(\sigma))\sigma^{N+2+|s|-s}(x\partial_x)^M \tilde{u}^+\left(\frac{x}{\sigma}\right)\right)\right)(\mathfrak{t}) \\
=& \left(\frac{1}{i\mathfrak{t}}\right)^K \frac{1}{2\pi}\int_0^{+\infty} e^{-i\mathfrak{t}\sigma}\chi(x)\partial_\sigma^K\left((1-\chi(\sigma))\sigma^{N+2+|s|-s}(x\partial_x)^M \tilde{u}^+\left(\frac{x}{\sigma}\right)\right)\mathrm{d}\sigma
\end{aligned}$$

and we can bound the modulus by $\mathfrak{t}^{-K} x^{1-s-|s|-} C \int_{\inf(\mathrm{supp}(1-\chi))}^{+\infty} \sigma^{N+1+2|s|-K+}\,\mathrm{d}\sigma$. □

Definition 14.2.29. *The previous proposition leads us to define* $u_{I_+} := \left(\frac{x\mathfrak{t}+1}{\mathfrak{t}}\right)^{3+|s|-s} u^0_{I_+}(x\mathfrak{t})$.

We also define $u_{K_+} := -\chi(\mathfrak{t}^{-1})\mathcal{F}^{-1}\left(\sigma^{2+2|s|}\frac{\ln|\sigma|-i\pi H(\sigma)}{2|s|+1}u^{(0)}(c_f)\right)$. *In this definition, we have used the value of the constant* $\Im(b)$ *computed in the proof of Lemma 14.2.27.*

We recall the following Fourier transform calculation:

$$u_{K_+} = -\chi(\mathfrak{t}^{-1})(-i)^{2|s|}\frac{(2+2|s|)!}{2|s|+1}\mathfrak{t}^{-3-2|s|}u^{(0)}(c_f)$$

Lemma 14.2.30. *If we define* $R := -\chi(\mathfrak{t}^{-1})\mathcal{F}^{-1}\left((1-\chi(\sigma))\sigma^{2+2|s|}\frac{\ln|\sigma|-i\pi H(\sigma)}{2|s|+1}u^{(0)}(c_f)\right)$ *We have the following bounds (for all* $N, M, K \in \mathbb{N}$*):*

$$\left|\partial_{\mathfrak{t}}^N (x\partial_x)^M R\right| \leq C_{N,M,K}\,\langle\mathfrak{t}\rangle^{-K}\,x^{-s-|s|}$$

In particular, $R \in Z^{\infty,\infty}_{\infty,\infty,-s-|s|-}$.

Proof. Note that using the definition of $u^{(0)}(c_f)$, we have that for all $M \in \mathbb{N}$ there exists $C_M > 0$ such that: $(x\partial_x)^M u^{(0)}(c_f) \leq C_M x^{-s-|s|}$. Moreover, we have that for any $p > K$, $g(\sigma) := (1-\chi(\sigma))\sigma^{2+2|s|}\frac{\ln|\sigma|-i\pi H(\sigma)}{2|s|+1} \in C^p(\mathbb{R}_\sigma, \mathbb{C})$ with support away from zero. For all $q \leq p$, there exists a constant C_p such that we have $(\sigma\partial_\sigma)^q g(\sigma) \leq C_p \sigma^{2+2|s|+}$ which is in $\sigma^{-N+p}L^1(\mathbb{R}_\sigma, \mathbb{C})$ for p large enough (for example larger than $4+N+2\,|s|$). Therefore, we can use Lemma 14.2.22 to conclude the proof. □

Definition 14.2.31. *Let* χ *be a smooth compactly supported cutoff equal to* 1 *on* $\left[-\frac{1}{2}, \frac{1}{2}\right]$ *and equal to zero on* $\mathbb{R} \setminus \left[-\frac{3}{4}, \frac{3}{4}\right]$. *For* $f \in C^\infty(\mathbb{R}_\sigma, \overline{H}^{\tilde{r},l}_{(b)})$ *with* $l > -\frac{3}{2} - s - |s|$, $\tilde{r} + l > -\frac{1}{2} - 2s$ *and* $\tilde{r} > \frac{1}{2} + s$, *we define the low energy part of the solution* $u_l := \mathcal{F}^{-1}\chi(\sigma)R(\sigma)f$ *and its high energy part* $u_h := \mathcal{F}^{-1}(1-\chi(\sigma))R(\sigma)f$.

Proposition 14.2.32. *Let* $k \in \mathbb{N}$. *Let* $f \in C^\infty(\mathbb{R}_\sigma, \overline{H}^{\tilde{r},l}_{(b)})$ *with* $l+1 < -\frac{1}{2}$, $\tilde{r}+l-2k > -\frac{1}{2}-2s$ *and* $\tilde{r} - 2k - 1 > \max(\frac{1}{2}+s, 0)$. *We assume in addition that there exists* $C > 0$ *independent of* σ *such that for all* $j \leq k$ $\left\|(\partial_\sigma)^j f(\sigma)\right\|_{\overline{H}^{\tilde{r},l}_{(b)}} \leq C\,\langle\sigma\rangle^{-1-p-}$ *with* p *large enough so that* $p - \tilde{r} + k + 1 \geq 0$. *We have that* $u_h \in Z^{p-\tilde{r}+k+1,\tilde{r}-2k-1}_{k,k+l+\frac{5}{2},l+\frac{5}{2}}$.

Proof. By proposition 14.1.4, for all $j \leq k$, $\partial_\sigma^j(1-\chi(\sigma))R(\sigma)f \in |\sigma|^{-p+j}L^1(\mathbb{R}_\sigma, \overline{H}^{\tilde{r}-2k-1,l+1}_{(b),|\sigma|^{-1}})$. In particular, since $\|u\|_{\overline{H}^{\tilde{r}-2k-1,l+1}_{(b)}} \leq |\sigma|^{\tilde{r}-2k-1}\|u\|_{\overline{H}^{\tilde{r}-2k-1,l+1}_{b,|\sigma|^{-1}}}$ uniformly when $|\sigma| \geq 1$, we have for all $j \leq k$, $\partial_\sigma^j(1-\chi(\sigma))R(\sigma)f \in |\sigma|^{-p+\tilde{r}-k-1}L^1(\mathbb{R}_\sigma, \overline{H}^{\tilde{r}-2k-1,l+1}_b)$. We conclude using lemma 14.2.22 that for all $j \leq p-\tilde{r}+k+1$:

$$\left\| \mathfrak{t}^k x^{-l-\frac{5}{2}} \partial_{\mathfrak{t}}^j u_h \right\|_{\overline{H}^{\tilde{r}-2k-1,-\frac{3}{2}}_{(b)}} \leq C_j$$

□

Proposition 14.2.33. *Let $k \in \mathbb{N}$ be such that $k \geq 4+2|s|$. Let $l > \frac{1}{2} - s + |s|$, $\tilde{r} - \frac{5}{2} - s - |s| - 2k > -\frac{1}{2} - 2s$ and $\tilde{r} - 2k - 1 > \frac{1}{2} + s$. Let $f \in C^\infty(\mathbb{R}_\sigma, \overline{H}^{\tilde{r},l}_{(b)})$ the principal term of u_l at K_+ is u_{K_+} and the principal term of u_l at I_+ is u_{I_+}. More precisely, for any $\epsilon \in (0,1)$ such that $l - (1-\epsilon) \geq \frac{1}{2} - s + |s|$, we have:*

$$\chi(\mathfrak{t}^{-1})(u_l - u_{K_+}) \in Z^{E(k-3-2|s|-\epsilon),\tilde{r}-2k-1}_{3+2|s|+\epsilon-,3-s+|s|,-s-|s|-\epsilon}$$
$$\chi(\mathfrak{t}^{-1})(u_l - u_{I_+}) \in Z^{E(k-3-2|s|-\epsilon),\tilde{r}-2k-1}_{(3+2|s|),3-s+|s|+\epsilon,-s-|s|}$$

Proof. Using Corollary 14.2.17 and Lemma 14.2.24, we see that

$$\begin{aligned}\chi(\mathfrak{t}^{-1})u_l =& \chi(\mathfrak{t}^{-1})\mathcal{F}^{-1}\left(\chi(\sigma)\sigma^{2-s+|s|}\chi(x)\tilde{u}(\sigma) + \chi(\sigma)\sigma^{2+2|s|}\frac{\ln|\sigma| - i\pi H(\sigma)}{1+2|s|}(\chi(x)x^{-s-|s|}c_f - u^{(0)}(c_f))\right) \\ &+ Z^{E(k-3-2|s|-\epsilon),\tilde{r}-2k-1}_{3+2|s|+\epsilon-,3-s+|s|+\epsilon,-s-|s|}\end{aligned}$$

Therefore, using lemma 14.2.28 and lemma 14.2.30, it is enough to prove that:

$$\chi(\mathfrak{t}^{-1})\mathcal{F}^{-1}\left(\chi(\sigma)\chi(x)\left(\sigma^{2-s+|s|}\tilde{u}(\sigma) + \sigma^{2+2|s|}\frac{\ln|\sigma| - i\pi H(\sigma)}{1+2|s|}x^{-s-|s|}c_f\right)\right) \in Z^{E(k-3-2|s|-\epsilon),\tilde{r}-2k-1}_{3+2|s|+\epsilon-,3-s+|s|,-s-|s|-\epsilon} \tag{14.15}$$

$$\chi(\mathfrak{t}^{-1})\mathcal{F}^{-1}\left(\frac{\sigma^{2+2|s|}}{1+2|s|}\left(\ln|\sigma| - i\pi H(\sigma)\right)(\chi(x)x^{-s-|s|}c_f - u^{(0)}(c_f))\right) \in Z^{E(k-3-2|s|-\epsilon),\tilde{r}-2k-1}_{(3+2|s|),3-s+|s|+\epsilon,(1-s-|s|)-} \tag{14.16}$$

We first prove the claim (14.15). Let $\tilde{\chi}$ a smooth with compact support in $(-1,1)$ and equal to 1 on $\mathrm{supp}(\chi)$. We use Remark 14.2.12 and Lemma 14.2.11 (successively on $(-\infty,0)_\sigma$ and on $(0,+\infty)_\sigma$) to obtain:

$$\begin{aligned}\chi(\sigma)\chi(x)\Big(&\sigma^{2-s+|s|}\tilde{u}(\sigma) \\ &+\sigma^{2+2|s|}\frac{\ln|\sigma| - i\pi H(\sigma)}{1+2|s|}x^{-s-|s|}c_f\Big) \in \quad \tilde{\chi}(\sigma)|\sigma|^{2+2|s|+\epsilon}W_b^{\infty,\infty}\left((-1,1)_\sigma, \overline{H}^{\infty,-\frac{3}{2}-s-|s|-\epsilon}_{(b)}\right) \\ &+\chi(x)\chi(\sigma)\sigma^{2+2|s|}\left(\frac{\ln(x)}{1+2|s|} + \overline{b}\right)c_f\end{aligned}$$

By Lemma 14.2.24, we get the claim (even with the stronger space $Z^{\infty,\infty}_{3+2|s|+\epsilon,3+|s|-s,-s-|s|-\epsilon}$).

We now show (14.16). By definition of $u^{(0)}(c_f)$, $\chi(x)x^{-s-|s|}c_f - u^{(0)}(c_f) \in \overline{H}^{\infty,(-\frac{1}{2}-s-|s|)-}_{(b)}$ and therefore (using the explicit Fourier transform of $\sigma^{2+2|s|}\frac{\ln|\sigma|-i\pi H(\sigma)}{1+2|s|}$):

$$\chi(\mathfrak{t}^{-1})\mathcal{F}^{-1}\left(\frac{\sigma^{2+2|s|}}{1+2|s|}\left(\ln|\sigma| - i\pi H(\sigma)\right)(\chi(x)x^{-s-|s|}c_f - u^{(0)}(c_f))\right) \in Z^{\infty,\infty}_{(3+2|s|),(4-s+|s|)-,(1-s-|s|)-}$$

□

We now compute the principal term at $\mathscr{I}_+$. We follow the argument in [48]. We use coordinates $v = x\mathfrak{t}$ and $\tau = \mathfrak{t}^{-1}$ which are smooth near $\mathscr{I}_+$ in the blow up space with v being a defining function of $\mathscr{I}_+$. The normal operator of T_s at $\mathscr{I}_+$ is:

$$A = -2v^{-1}\left(v\partial_v - \tau\partial_\tau\right)\left(v\partial_v - 1\right)$$

We denote by $\mathcal{N} = [0,1)_v \times [0,1)_\tau \times \mathbb{S}^2$ (the $\mathbb{S}^2$ part being the boundary at $x = 0$). We denote by $\overline{H}_b^{\tilde{r},\mu,\nu}\left((\mathcal{B}_s)_{|\mathcal{N}}\right) = v^\mu\tau^\nu\overline{H}_b^{\tilde{r},0,0}$ the usual b-Sobolev space (with b volume form) with extendible conditions at $v = 1$ and $\tau = 1$ and $\dot{H}_b^{\tilde{r},\mu,\nu}\left((\mathcal{B}_s)_{|\mathcal{N}}\right)$ the same space with supported conditions at $v = 1$ and $\tau = 1$.

Lemma 14.2.34. *Let $\tilde{r} \in \mathbb{N}$. The restriction to $\mathcal{N}$ is a continuous map from $Z^{\tilde{r},\tilde{r}}_{-\infty,\alpha,\beta}$ to $\overline{H}_b^{\tilde{r},\beta,\alpha-}$*

Proof. Let $\tilde{r} \in \mathbb{N}$. Let $u \in Z^{\tilde{r},\tilde{r}}_{-\infty,\alpha,\beta}$. As a distribution, for all $j,k \in \mathbb{N}$ with $j \leq \tilde{r}$ and $k \leq \tilde{r}$, we have $(x\partial_x)^j(\mathfrak{t}\partial_\mathfrak{t})^k u \in Z^{0,0}_{-\infty,\alpha,\beta}$. In particular for $j + k \leq \tilde{r}$, $v := (-\mathfrak{t}\partial_\mathfrak{t} + x\partial_x)^j(x\partial_x)^k u \in Z^{0,0}_{-\infty,\alpha,\beta}$. In particular, for any $\eta > 0$ we have:

$$\begin{aligned}
I :=& \int_{\mathfrak{t}\geq 1}\int_{x\leq\mathfrak{t}^{-1}} \mathfrak{t}^{2(\alpha-\beta)-1-\eta}x^{-2\beta}\left\|v(\mathfrak{t},x)\right\|^2_{L^2(\mathbb{S}^2)}x^{-1}\,\mathrm{d}x\,\mathrm{d}\mathfrak{t}\\
\leq& \max(2^{2(\alpha-\beta)+N},1)\int_{\mathfrak{t}\geq 1}\mathfrak{t}^{-1-\eta}\int_{x\leq\mathfrak{t}^{-1}}(x\mathfrak{t}+1)^{-N}\left(\frac{x\mathfrak{t}+1}{\mathfrak{t}}\right)^{-2\alpha}\left(\frac{x\mathfrak{t}}{x\mathfrak{t}+1}\right)^{-2\beta}\left\|v(\mathfrak{t},x)\right\|^2_{L^2(\mathbb{S}^2)}x^{-1}\,\mathrm{d}x\,\mathrm{d}\mathfrak{t}\\
\leq& \max(2^{2(\alpha-\beta)+N},1)\int_{\mathfrak{t}\geq 1}\mathfrak{t}^{-1-\eta}\left\|v\right\|_{Z^{0,0}_{-N,\alpha,\beta}}\,\mathrm{d}\mathfrak{t}\\
\leq& \frac{\max(2^{2(\alpha-\beta)+N},1)}{\eta}\left\|v\right\|_{Z^{0,0}_{-N,\alpha,\beta}}
\end{aligned}$$

We perform the change of variable $(\tau, v) = (\mathfrak{t}^{-1}, x\mathfrak{t})$ in the first integral and we find:

$$\int_{\tau\leq 1}\int_{v\leq 1}\tau^{\eta-2\alpha}v^{-2\beta}\left\|(\tau\partial_\tau)^j(v\partial_v)^k u(\tau,v)\right\|^2_{L^2(\mathbb{S}^2)}\frac{\mathrm{d}\tau}{\tau}\frac{\mathrm{d}v}{v} \leq \frac{\max(2^{2(\alpha-\beta)+N},1)}{\eta}\left\|v\right\|_{Z^{0,0}_{-N,\alpha,\beta}}$$

Since it is true for all j,k such that $j + k \leq \tilde{r}$, we deduce:

$$\left\|u\right\|_{\overline{H}_b^{\tilde{r},\beta,\alpha-\frac{\eta}{2}}\left((\mathcal{B}_s)_{|\mathcal{N}}\right)} \leq C\frac{\max(2^{2(\alpha-\beta)+N},1)}{\eta}\left\|v\right\|_{Z^{0,0}_{-N,\alpha,\beta}}$$

□

Proposition 14.2.35. *Let $\tilde{r} \in \mathbb{N}$. Let $f \in \overline{H}_b^{\tilde{r},\nu,\beta}\left((\mathcal{B}_s)_{|\mathcal{N}}\right)$ vanishing in a neighborhood of $\{v = 1\}$. The unique solution in $\overline{H}_b^{\tilde{r},-\infty,-\infty}\left((\mathcal{B}_s)_{|\mathcal{N}}\right)$ of the transport equation $(v\partial_v - \tau\partial_\tau)u = f$ vanishing near $\{v = 1\}$ is given by:*

$$u = -\int_0^{-\ln(v)} f(e^{-s}, v\tau e^s)\,\mathrm{d}s.$$

Moreover, for any $\alpha,\gamma \in \mathbb{R}$ with $\alpha < \gamma \leq \beta$ and $\gamma \leq \nu$, there exists $C > 0$ such that:

$$\left\|\int_0^{-\ln(v)} f(e^{-s}, v\tau e^s)\,\mathrm{d}s\right\|_{\overline{H}_b^{\tilde{r}\alpha,\beta}} \leq C\left\|f\right\|_{\overline{H}_b^{\tilde{r},\gamma,\beta}}$$

Proof. We refer the reader to the proof of Proposition C.5.1. □

Remark 14.2.36. *Let $\mu < \nu < \beta$. As a consequence of proposition 14.2.35, if $f \in \overline{H}_b^{\tilde{r},\mu,\beta-}$ has an expansion with respect to τ: $f(v,\tau) = \tau^\beta f_0(v) + g(v,\tau)$ with $f \in v^\nu C_c^\infty([0,1), H^{\tilde{r}}(\mathcal{B}_s))$ and $g \in \overline{H}_b^{\tilde{r},\mu,\beta+\epsilon}$ (for some $\epsilon > 0$), then for all $\gamma < \mu$ we have:*

$$u(v,\tau) = -\tau^\beta v^\beta \int_v^1 f_0(s) s^{-\beta-1} \, \mathrm{d}s + \overline{H}_b^{\tilde{r},\gamma,\beta+\epsilon}.$$

Note that $v^\beta \int_v^1 f_0(s) s^{-\beta-1} \, \mathrm{d}s \in v^\nu C_c^\infty([0,1), H^{\tilde{r}}(\mathcal{B}_s))$ and therefore, u admits a similar development to f.

For $\tilde{r}, l \in \mathbb{R}$, we introduce the space $\overline{H}_b^{\tilde{r},l}([0,1]_\tau)$ which is the space of b-Sobolev sections of $[0,1]_\tau \times \mathcal{B}_s$ with b-regularity $\tilde{r}$ and decay order l at the boundary $\{\tau = 0\}$ and extendible across $\{\tau = 1\}$.

Proposition 14.2.37. *Let $k \in \mathbb{N}$ such that $k \geq 8 + 4|s| + 2s$. Let $\tilde{r} - 2k - 1 > \max(1 - s + |s|, 4 + 2|s| + 2s)$ and $l > \frac{1}{2} - s + |s|$. Let $f \in C_c^{1+p}(\mathbb{R}_\mathfrak{t}, \overline{H}_{(b)}^{\tilde{r},l})$ with $p \geq \max(1, \tilde{r} - 4 - 2|s|)$ and let u be the inverse Fourier transform of $R(\sigma)\hat{f}$. Then, denoting by γ the quantity $\min(k - 4 - 2|s|, \tilde{r} - 2k - 1, p - \tilde{r} + k)$, we have:*

$$u - vu_{rad} \in \overline{H}_b^{\gamma-4-2(s+|s|),(3-s+|s|)-,2-}$$

where $u_{rad} \in \overline{H}_b^{\gamma-4-2(s+|s|),(3-s+|s|)-}([0,1]_\tau)$ and has a leading order term at $\tau = 0$ equal to $\tau^{3+|s|-s}(-i)^{2+2|s|}(2|s|)!\frac{|s|-s+1}{2^{1+|s|+s}} c_{\hat{f}}$.

Remark 14.2.38. *The assumptions on k, p and $\tilde{r}$ have been chosen so that $\gamma - 4 - 2(s+|s|) \geq 0$.*

Proof. First note that $\hat{f} \in C^\infty(\mathbb{R}_\sigma, \overline{H}_{(b)}^{\tilde{r},l})$ and for all $j \in \mathbb{N}$:

$$\left\| \partial_\sigma^j \hat{f}(\sigma) \right\|_{\overline{H}_{(b)}^{\tilde{r},l}} \leq C \langle \sigma \rangle^{-1-p}.$$

Combining Propositions 14.2.32 and 14.2.33, we have that $\chi(\mathfrak{t}^{-1})u - u_{I_+} \in Z^{\min(k-4-2|s|,p-\tilde{r}+k),\tilde{r}-2k-1}_{(3+2|s|)-,3-s+|s|+\epsilon,-s-|s|}$. By the explicit expression of u_{I_+} (see Lemma 14.2.27) and lemma 14.2.34, we have:

$$u \in \tau^{3+|s|+s} g_0(v) + \overline{H}_b^{\gamma,-s-|s|,3-s+|s|+\epsilon-}$$
$$g_0(v) := (-i)^{2+2|s|}(2|s|)! \frac{v((2|s|+2)v + 2(|s| - s + 1))}{(v+2)^{2+|s|+s}} c_{\hat{f}}.$$

Let χ be a smooth cutoff compactly supported in $[0,1)_v$. We have:

$$(v\partial_v - \tau\partial_\tau)(v\partial_v - 1)\chi(v)u = -\frac{v}{2}\left([T_s, \chi(v)]u + \chi(v)\hat{f} + \mathrm{Diff}_b^2 u\right)$$

We have $-\frac{v}{2}[T_s, \chi(v)]u \in \tau^{3+|s|+s} g_1(v) + \overline{H}_b^{\gamma-1,\infty,3+|s|+s+\epsilon-}$, $-\frac{v}{2}\chi(v)\hat{f} \in \overline{H}_b^{\tilde{r},3-s+|s|,\infty}$, $-\frac{v}{2}\mathrm{Diff}_b^2 u \in \tau^{3+|s|-s} g_2(v) + \overline{H}_b^{\gamma-2,1-s-|s|,3-s+|s|+\epsilon-}$ where $g_1 \in C_c^\infty((0,1)_v, \Gamma(\mathcal{B}_s))$ and $g_2 \in v^2 C_c^\infty([0,1)_v, \Gamma(\mathcal{B}_s))$ and all the terms vanishes near $\{v = 1\}$. Using proposition 14.2.35 and Remark 14.2.36 we obtain:

$$(v\partial_v - 1)\chi(v)u = \tau^{3+|s|+s} g_3(v) + \overline{H}_b^{\gamma-2,1-s-|s|-,3-s+|s|+\epsilon-}$$

where $g_3 \in v^2 C_c^\infty([0,1)_v, \Gamma(\mathcal{B}_s))$. By inverting $v\partial_v - 1$ using the Mellin transform[3] (or solving the transport equation), we get:

$$\chi(v)u \in \tau^{3+|s|+s} g_4(v) + \overline{H}_b^{\gamma-2,1-s-|s|-,3-s+|s|+\epsilon-}$$

where $g_4(v) \in v C_c^\infty([0,1)_v, \Gamma(\mathcal{B}_s))$ (we even have the explicit expression $g_4(v) = -v\int_v^1 s^{-2} g_3(s)\ \mathrm{d}s$). By uniqueness of the leading order term (as $\tau \to 0$), we deduce $g_4(v) = \chi(v) g_0(v)$. We can iterate this procedure $1 + s + |s|$ times and obtain:

$$\chi(v)u \in \tau^{3+|s|+s}\chi(v) g_0(v) + \overline{H}_b^{\gamma-2-2(|s|+s),1-,3-s+|s|+\epsilon-}$$

We deduce:

$$(v\partial_v - \tau\partial_\tau)(v\partial_v - 1)\chi(v)u \in \tau^{3+|s|+s} g_5(v) + \overline{H}_b^{\gamma-2-2(|s|+s),2-,3-s+|s|+\epsilon-}$$

where $g_5 \in v^2 C_c^\infty([0,1)_v, \Gamma(\mathcal{B}_s))$ Applying one more time Proposition 14.2.35 and Remark 14.2.36, we get:

$$(v\partial_v - 1)\chi(v)u \in \tau^{3+|s|+s} g_6(v) + R$$

with $g_6 \in v^2 C_c^\infty([0,1)_v, \Gamma(\mathcal{B}_s))$ and $R \in \overline{H}_b^{\gamma-4-2(|s|+s),2-,3-s+|s|+\epsilon-}$ (and vanishes near $v = 1$). This time, R has enough decay with respect to v so that $(i\lambda - 1)^{-1}\mathcal{M}(R)(\lambda)$ (here $\mathcal{M}(R)$ is the Mellin transformed of R with respect to v) is meromorphic on $\Im(\lambda) > -2$ with a pole of order 1 at $\lambda = -i$. As in the proof of Proposition 11.2.4, we can use a contour deformation argument to obtain that there exists $w \in \overline{H}_b^{\gamma-4-2(|s|+s),(3-s+|s|+\epsilon)-}([0,1]_v)$ such that:

$$\chi(v)u = \tau^{3+|s|-s}\chi(v) g_0(v) + vw(\tau) + \overline{H}_b^{\gamma-4-2(|s|+s),2-,3+|s|-s+\epsilon-}.$$

Since $g_0(v) = (-i)^{2+2|s|}(2\,|s|)!\frac{|s|-s+1}{2^{1+|s|+s}} c_{\hat{f}} + \tau^{3+|s|-s} v^2 C^\infty([0,1)_v, \mathcal{B}_s)$ we define

$$u_{rad} := (-i)^{2+2|s|}(2\,|s|)!\frac{|s|-s+1}{2^{1+|s|+s}} c_{\hat{f}} + w$$

and conclude the proof. □

Remark 14.2.39. *Note that by an adaptation of Theorem B.2.7. in [54], we get that for $\tilde{r} > \frac{1}{2} + j$ with $j \in \mathbb{N}$, $\dot{H}_b^{\tilde{r},\mu,\nu} \subset Z_{\infty,\nu,\mu}^{j,\tilde{r}-j-\frac{1}{2}}$ (which is a kind of converse to Lemma 14.2.34). In particular, in the setting of Proposition 14.2.37, since we have obtained in the proof:*

$$\chi(v)(u - u_{I_+}) \in \overline{H}_b^{\gamma-2-2(|s|+s),1-,3-s+|s|+\epsilon-},$$

we deduce that for $j < \gamma - 2 - 2(|s| + s) - \frac{1}{2}$, we have $\chi(\mathfrak{t}^{-1})(u - u_{I_+}) \in Z_{3+2|s|,3-s+|s|+\epsilon-,1-}^{j,\gamma-2-2(|s|+s)-\frac{1}{2}-j}$ for χ a smooth cutoff compactly supported in $[0,1)$. This is an improvement with respect to Proposition 14.2.33 since the order of decay at $\mathscr{I}^+$ is now $1-$ instead of $-s - |s|$.

[3] See for example the proof of Proposition 11.2.4 for a similar argument

Chapter 15

Contour deformation argument

We consider the forcing problem:

$$T_s u = f$$

with $f \in C_c^{1+p}(\mathbb{R}_{\mathfrak{t}}, \overline{H}^{\tilde{r},l}_{(b)})$. By the classical hyperbolic theory, there is a unique solution u of this equation which vanishes in a neighborhood of $\mathfrak{t} = -\infty$. Moreover by estimates similar to those obtained in Appendix C, there exists $C > 0$ such that $\|u\|_{\overline{H}^{\tilde{r},-N}_{(b)}} \leq Ce^{C\mathfrak{t}}$ (for some $N > 0$). In particular, we can define the Fourier-Laplace transform of u, $\hat{u}(\sigma) := \int e^{-i\sigma\mathfrak{t}} u(\mathfrak{t})\, \mathrm{d}\mathfrak{t}$ on the domain $\{\Im(\sigma) > C\}$. The equation on $\hat{u}$ is:

$$\hat{T}_s(\sigma)\hat{u}(\sigma) = \hat{f}(\sigma)$$

with $\hat{f}$ holomorphic from $\mathbb{C}$ to $\overline{H}^{\tilde{r},l}_{(b)}$). Moreover by the Paley-Wiener-Schwartz theorem (see [53, Theorem 7.3.1]), there exists $D > 0$ such that for all $k \in \mathbb{N}$ there exists $C_k > 0$ such that:

$$\left\| \partial_{\sigma_x}^k \hat{f}(\sigma_x + i\sigma_y) \right\|_{\overline{H}^{\tilde{r},l}_{(b)}} \leq C_k \langle \sigma_x \rangle^{-(1+p)} e^{D|\sigma_y|} \tag{15.1}$$

In particular, if $-\frac{3}{2} - s - |s| < l < -\frac{1}{2}$, $\tilde{r} - 2 > \frac{1}{2} + s$ and $\tilde{r} > 2 - s + |s|$, we can define $R(\sigma)\hat{f}(\sigma)$ and we have on $\Im(\sigma) = C + 1$:

$$\begin{aligned} \hat{u}(\sigma) =& R(\sigma)\hat{f}(\sigma) \\ u(\mathfrak{t}) =& \frac{1}{2\pi} \int_{\Im(\sigma)=C+1} e^{-i\sigma\mathfrak{t}} R(\sigma)\hat{f}(\sigma)\, \mathrm{d}\sigma \end{aligned}$$

Using the fact that $R(\sigma)\hat{f}(\sigma)$ is holomorphic from $\Im(\sigma) > 0$ to $\overline{H}^{\tilde{r}-1,\min\left(l+1,-\frac{1}{2}-\right)}_{(b)}$ (see proposition 14.1.1) and (14.2) in Proposition 14.0.2 and (15.1) to control the error terms (in the norm $H^{0,\min(l+1,-\frac{1}{2}-)}_{(b)}$ which is σ-uniformly smaller than $H^{\tilde{r}-1,\min\left(l+1,-\frac{1}{2}-\right)}_{(b),|\sigma|^{-1}}$ since $\tilde{r}-1 \geq 0$), we find that for every $\epsilon > 0$:

$$\int_{\Im(\sigma)=C+1} e^{-i\sigma\mathfrak{t}} R(\sigma)\hat{f}(\sigma)\, \mathrm{d}\sigma = \int_{\Im(\sigma)=\epsilon} e^{-i\sigma\mathfrak{t}} R(\sigma)\hat{f}(\sigma)\, \mathrm{d}\sigma$$

Moreover using Proposition 14.1.3, for every $\sigma \in \mathbb{R}$ we have

$$\lim_{\epsilon \to 0} \left\| R(\sigma + i\epsilon)\hat{f}(\sigma + i\epsilon) - R(\sigma) f(\sigma) \right\|_{\overline{H}^{\tilde{r}-1,l-}_{(b)}} = 0.$$

Since $\tilde{r} - 1 \geq 0$, it is also true in $\overline{H}^{0,l-}_{(b)}$ and we can combine Proposition 14.1.3, (14.2) in Proposition 14.0.2 and (15.1) to get that there exists $C_p > 0$ independent of $\epsilon \in [0, 1]$

such that: $\left\|R(\sigma+i\epsilon)\hat{f}(\sigma+i\epsilon)\right\|_{\overline{H}^{0,l-}_{(b)}} \leq C_p \langle\sigma\rangle^{-(1+p)}$. In particular if $p>0$, using Lebesgue convergence theorem (see [46] for this theorem in the case of Bochner integral, here functions $\sigma \mapsto R(\sigma+i\epsilon)f(\sigma+i\epsilon)$ is continuous in $\overline{H}^{0,l-}_{(b)}$ and therefore strongly measurable in the sense of [46]), we get:

$$u(\mathfrak{t}) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-i\sigma\mathfrak{t}} R(\sigma)\hat{f}(\sigma)\,\mathrm{d}\sigma \tag{15.2}$$

We can now state the two main theorems (and their corollaries concerning the Cauchy problem) which are based on this formula and the precise analysis of $R(\sigma)$. The first theorem concerns the forcing problem with a forcing term having moderate decay at $\mathscr{I}^+$.

Theorem 15.0.1. *Let $l=-\frac{3}{2}+\alpha$ where $\alpha\in(0,1)$. Let $k\geq \alpha+s+|s|+1$. We assume that $\tilde{r}-2k-1>1+2|s|$. Let $f\in C_c^{1+p}\left(\mathbb{R}_{\mathfrak{t}},\overline{H}^{\tilde{r},l}_{(b)}\right)$ with $p>\tilde{r}-k-1$. Let u be the unique solution of the forcing problem $\hat{T}_s u=f$ which vanishes near $\mathfrak{t}=-\infty$. There exists $C>0$ such that for all $\mathfrak{t}\in\mathbb{R}$ and for all $j\leq \min(k-\alpha-s-|s|-1, p-\tilde{r}+k+1-)$:*

$$\left\|(\mathfrak{t}\partial_{\mathfrak{t}})^j u(\mathfrak{t})\right\|_{\overline{H}^{\tilde{r}-2k-1,-\frac{3}{2}-s-|s|-}_{(b)}} \leq C\langle\mathfrak{t}\rangle^{-1-\alpha-s-|s|+}$$
$$\left\|(\mathfrak{t}\partial_{\mathfrak{t}})^j u(\mathfrak{t})\right\|_{\overline{H}^{\tilde{r}-2k-1,-\frac{1}{2}-}_{(b)}} \leq C\langle\mathfrak{t}\rangle^{-\alpha}$$

Remark 15.0.2. *Using the Sobolev embedding theorem, we have the uniform boundedness version when $\tilde{r}-2k-1>\frac{3}{2}$:*

$$\begin{aligned}\|u(\mathfrak{t},x,\omega)\| \leq& C\min(\langle\mathfrak{t}\rangle^{-1-\alpha-s-|s|+}\, x^{-s-|s|-}, \langle\mathfrak{t}\rangle^{-\alpha+}\, x^{1-})\\ \leq& 2C\frac{\langle\mathfrak{t}\rangle^{-\alpha+}\, x^{1-}}{1+(\langle\mathfrak{t}\rangle\, x)^{1+s+|s|}}\end{aligned}$$

Proof. We first use the Fourier representation formula (15.2). Then, it is a consequence of Proposition 14.2.7 (with $l_c=-\frac{3}{2}-s-|s|-$ for the first estimate and $l_c=-\frac{1}{2}-$ for the second one) and Lemma 14.2.24 (to handle the low energy part) and of Proposition 14.2.32 (to handle the high energy part). □

We now use Proposition 11.2.4 and Theorem 15.0.4 to deduce the following Corollary concerning the Cauchy problem on the hypersurface $\Sigma_0 := t_0^{-1}(\{0\})$ (see (C.1) for the precise definition of t_0). We use the notations of Subsection 11.2.

Corollary 15.0.3. *Let $\alpha\in(0,1)$. Let $k\geq\alpha+s+|s|+1$. We assume that $\tilde{r}-2k-1>1+2|s|$. Let $p>\tilde{r}-k-1$. Let $u_0\in\overline{H}_b^{\tilde{r}+6+p,1+\alpha}$ and $u_1\in\overline{H}_b^{\tilde{r}+5+p,1+\alpha}$. Let u be the solution of the Cauchy problem:*

$$\begin{cases} T_s u = 0\\ u_{|\Sigma_0} = u_0\\ \rho_0\nabla^\mu t_0\partial_\mu u_{|\Sigma_0} = u_1.\end{cases}$$

In the previous expression, $\nabla^\mu t_0$ denotes the gradient of t_0 taken with respect to the metric $\tilde{g}$. There exists $C>0$ such that for all $\mathfrak{t}\in\mathbb{R}$ and for all $j<\min(k-\alpha-s-|s|-1, p-\tilde{r}+k+1)$:

$$\left\|(\mathfrak{t}\partial_{\mathfrak{t}})^j u(\mathfrak{t})\right\|_{\overline{H}^{\tilde{r}-2k-1,-\frac{3}{2}-s-|s|-}_{(b)}} \leq C\langle\mathfrak{t}\rangle^{-1-\alpha-s-|s|+}$$
$$\left\|(\mathfrak{t}\partial_{\mathfrak{t}})^j u(\mathfrak{t})\right\|_{\overline{H}^{\tilde{r}-2k-1,-\frac{1}{2}-}_{(b)}} \leq C\langle\mathfrak{t}\rangle^{-\alpha+}$$

We go back to the forcing problem and provide a more precise result when the forcing term has higher decay:

Theorem 15.0.4. *Let $l > \frac{1}{2} - s + |s|$, $k > 8 + 4|s| + 2s + \frac{1}{2}$, $\tilde{r} - 2k - 1 > \max(\frac{3}{2} - s + |s|, \frac{9}{2} + 2|s| + 2s)$ and $p \geq \tilde{r} - 4 - 2|s|$. Let $f \in C_c^{1+p}\left(\mathbb{R}_{\mathfrak{t}}, \overline{H}^{\tilde{r},l}_{(b)}\right)$. Let u be the unique solution to the forcing problem $T_s u = f$ which vanishes near $\mathfrak{t} = -\infty$. Let χ be a smooth compactly supported cutoff equal to 1 near 0. Then, with the notations introduced in Definition 14.2.29 there $\epsilon > 0$ such that for all $j \in \mathbb{N}$ with $0 \leq j < \min(k - 8 - 2|s|, \tilde{r} - 2k - 5, p - \tilde{r} + k - 4) - 2|s| - 2s - \frac{1}{2}$:*

$$\chi(\mathfrak{t}^{-1})\left(u - \mathfrak{p}(\mathfrak{t}, x, \omega)\right) \in Z^{j, \min(k-8-2|s|, \tilde{r}-2k-5, p-\tilde{r}+k-4)-2|s|-2s-j-\frac{1}{2}}_{3+2|s|+\epsilon, 3+|s|-s+\epsilon, 1-}$$

where

$$\mathfrak{p}(\mathfrak{t}, x, \omega) := (-i)^{2+2|s|}(2|s|)!\mathfrak{t}^{-3-2|s|}\frac{(x\mathfrak{t})^{1+|s|+s}\left((2|s|+2)x\mathfrak{t} + 2(|s| - s + 1)\right)}{(x\mathfrak{t}+2)^{2+|s|+s}} u^{(0)}(c_{\hat{f}})$$

Proof. We first use the Fourier representation formula (15.2). Then, it is a consequence of Proposition 14.2.33 (note that $\mathfrak{p}$ is asymptotic to u_{K_+} near K_+ and to u_{I^+} near I^+), of Proposition 14.2.32 (to handle the high energy part) and of remark 14.2.39. □

Finally, we use Corollary 11.1.4 and Theorem 15.0.4 to deduce the following Corollary concerning the Cauchy problem on the hypersurface $\Sigma_0 := t_0^{-1}(\{0\})$ (see (C.1) for the precise definition of t_0).

Corollary 15.0.5. *Let $k > 8 + 4|s| + 2s + \frac{1}{2}$, $\tilde{r} - 2k - 1 > \max(\frac{3}{2} - s + |s|, \frac{9}{2} + 2|s| + 2s)$ and $p \geq \tilde{r} - 4 - 2|s|$. Let $u_0 \in H^{\tilde{r}+\frac{7}{2}+p}(\Sigma_0, \mathcal{B}_s)$, $u_1 \in H^{\tilde{r}+\frac{5}{2}+p}(\Sigma_0, \mathcal{B}_s)$ be compactly supported. Let u be the solution of the Cauchy problem:*

$$\begin{cases} T_s u = 0 \\ u_{|\Sigma_0} = u_0 \\ \nabla^\mu t_0 \partial_\mu u_{|\Sigma_0} = u_1. \end{cases}$$

Let χ be a smooth compactly supported cutoff equal to 1 near 0. There exists $\epsilon > 0$ such that for all $j \in \mathbb{N}$ such that $0 \leq j < \min(k - 8 - 2|s|, \tilde{r} - 2k - 5, p - \tilde{r} + k - 4) - 2|s| - 2s - \frac{1}{2}$:

$$\chi(\mathfrak{t}^{-1})\left(u - \mathfrak{p}(\mathfrak{t}, x, \omega)\right) \in Z^{j, \min(k-8-2|s|, \tilde{r}-2k-5, p-\tilde{r}+k-4)-2|s|-2s-j-\frac{1}{2}}_{3+2|s|+\epsilon, 3+|s|-s+\epsilon, 1-}$$

where

$$\mathfrak{p}(\mathfrak{t}, x, \omega) := (-i)^{2+2|s|}(2|s|)!\mathfrak{t}^{-3-2|s|}\frac{(x\mathfrak{t})^{1+|s|+s}\left((2|s|+2)x\mathfrak{t} + 2(|s| - s + 1)\right)}{(x\mathfrak{t}+2)^{2+|s|+s}} u^{(0)}(c_{\hat{f}})$$

and the function $\hat{f}$ used to compute the constant $c_{\hat{f}}$ is the Fourier transform with respect to $\mathfrak{t}$ of a function f as defined in Proposition 11.1.2.

Remark 15.0.6. *Using the freedom on the choice of the function f, we can express $c_{\hat{f}}$ in terms of the initial data. Indeed, by Remark 11.1.3, we can take a sequence of functions $\chi_n \in C^\infty(\mathbb{R})$ converging to the Heaviside function in the sense of distributions and such that $v_n = \chi_n(t_0)u$ are as in Proposition 11.1.2. We define:*

$$f_n := T_s v_n = [T_s, \chi_n(t_0)]u.$$

Since all the v_n are equal near $\mathfrak{t} = +\infty$, the constant $c_{\hat{f}_n}$ does not depend on n. Moreover, $\lim_{n\to+\infty} f_n$ exists in the sense of distribution and is equal to $f_\infty := [T_s, H(t_0)]u$ where H is the Heaviside function. Note that the function f_∞ only depends on the initial data of the Cauchy problem. Using the definition of $c_{\hat{f}_n}$, we see that $c_{\hat{f}_\infty}$ is well defined and $\lim_{n\to+\infty} c_{\hat{f}_n} = c_{\hat{f}_\infty}$. This provides a way to compute the constant $c_{\hat{f}}$ (and therefore $\mathfrak{p}$) only in terms of the initial data.

Appendix A

Hyperbolic estimate with small parameter

We begin by a general definition.

Definition A.0.1. *Let E be a (complex) vector bundle over the manifold $\mathfrak{M}$ and Θ be a real connection on E We define the gradient operator by:*

$$\mathrm{grad}_{g,\Theta} : \begin{cases} \Gamma(E) \to \Gamma(E \otimes T\mathfrak{M}) \\ u \mapsto g^{i,j}(\Theta_{\partial_j} u) \otimes \partial_i \end{cases}$$

where we used the Einstein summation convention and local coordinates on $\mathfrak{M}$ (but it does not depend on the choice).

If local coordinates are fixed on an open set U, we define the function J on U by $J(x) = \sqrt{|\det(g_{i,j}(x))|}$. We define the divergence operator by linearity using

$$\Gamma(E \otimes T\mathfrak{M}) \to \Gamma(E)$$
$$\sum_{i=1}^{n} u^i \otimes \partial_i \mapsto \sum_{i=1}^{n} J^{-1}\Theta_{\partial_i} J u^i$$

The local definition is once again independent of the choice of local coordinates. We define the operator $\Box_{g,\Theta} = \mathrm{div}_{g,\Theta}\mathrm{grad}_{g,\Theta}$.

Remark A.0.2. *Let ∇_{LC} be the Levi-Civita connection on $\mathfrak{M}$. Since we have the connection $\Theta \otimes \nabla_{LC}$ on $E \otimes F$ where F is a tensor bundle ($F = T\mathfrak{M}^{\otimes k} \otimes T^*\mathfrak{M}^{\otimes r}$ for some non negative integers k and r), we can extend the definition of* grad *and* div *naturally to these bundles.*

Let k be a smooth hermitian metric on E. We introduce the energy momentum tensor:

Definition A.0.3. *Let $u \in \Gamma^2(E)$. We call energy momentum tensor of u the tensor defined (in abstract index notation) by:*

$$T^{\delta,\gamma}(u) = \Re(k(\Theta_\mu u, \Theta_\nu u))g^{\mu,\delta}g^{\nu,\gamma} - \frac{1}{2}g^{\delta,\gamma}g^{\mu,\nu}k(\Theta_\mu u, \Theta_\nu u) + \frac{1}{2}g^{\delta,\gamma}k(u,u)$$

We remark that this tensor is symmetric and real.

Proposition A.0.4. *We have the following expression for the divergence of T:*

$$\begin{aligned} \mathrm{div}(T)^\delta =& g^{\mu,\delta}\Re(k(\Theta_\mu u, \Box_{g,\Theta} u)) + \Re(g^{\delta,\mu}g^{\gamma,\nu}k(R^{\Theta}_{\gamma,\mu}u, \Theta_\nu u)) \\ &+ \Re((\Theta_\gamma k)(\Theta_\mu u, \Theta_\nu u))(g^{\mu,\delta}g^{\nu,\gamma} - \frac{1}{2}g^{\delta,\gamma}g^{\mu,\nu}) \\ &+ \frac{1}{2}g^{\delta,\gamma}(\Theta_\gamma k)(u,u) + g^{\delta,\gamma}\Re(k(\Theta_\gamma u, u)) \end{aligned}$$

where $R^{\Theta}_{\mu,\nu}$ is the curvature tensor associated to the connection Θ.

Remark A.0.5. *We note that in the scalar case (E is a trivial bundle of rank 1), and with Θ the trivial connection and k the canonical Hermitian form on $\mathbb{C}$ we get* $\operatorname{div}(T)^\delta = g^{\mu,\delta}\Re(\partial_\mu \overline{u}\Box_g u) + g^{\delta,\gamma}\Re(\overline{u}\partial_\gamma u)$ *which is the usual result in hyperbolic estimates.*

Proof. The proposition follows from a local computation. Let $x \in \mathfrak{M}$, we compute $\operatorname{div}(T)^\delta(x)$ in normal local coordinates centered at x (we use the fact that the coordinates are normal many times in the computation):

$$\begin{aligned}\operatorname{div}(T)^\delta(x) =&\partial_\gamma T^{\delta,\gamma}\\ =&g^{\mu,\delta}g^{\nu,\gamma}\Re(\partial_\gamma k(\Theta_\mu u, \Theta_\nu u)) - \frac{1}{2}g^{\delta,\gamma}g^{\mu,\nu}\Re(\partial_\gamma k(\Theta_\mu u, \Theta_\nu u))\\ &+ \frac{1}{2}g^{\delta,\gamma}(\Theta_\gamma k)(u,u) + g^{\delta,\gamma}\Re(k(\Theta_\gamma u, u))\end{aligned}$$

Note that we used the fact that $g^{\mu,\nu}k(\Theta_\mu u, \Theta_\nu u)$ is real to introduce a real part in the second term. We have:

$$\partial_\gamma k(\Theta_\mu u, \Theta_\nu u) = (\Theta_\gamma k)(\Theta_\mu u, \Theta_\nu u) + k(\Theta_\gamma\Theta_\mu u, \Theta_\nu u) + k(\Theta_\mu u, \Theta_\gamma\Theta_\nu u)$$

therefore, we get

$$\begin{aligned}\operatorname{div}(T)^\delta(x) =&g^{\mu,\delta}\Re(k(\Theta_\mu u, g^{\nu,\gamma}\Theta_\gamma\Theta_\nu u)) + g^{\mu,\delta}g^{\nu,\gamma}\Re(k(\Theta_\gamma\Theta_\mu u - \Theta_\mu\Theta_\gamma u, \Theta_\nu u)\\ &+ \Re((\Theta_\gamma k)(\Theta_\mu u, \Theta_\nu u))(g^{\mu,\delta}g^{\nu,\gamma} - \frac{1}{2}g^{\delta,\gamma}g^{\mu,\nu})\\ &+ \frac{1}{2}g^{\delta,\gamma}(\Theta_\gamma k)(u,u) + g^{\delta,\gamma}\Re(k(\Theta_\delta u, u))\end{aligned}$$

But since the coordinates are normal around x, we get at x:

$$\begin{aligned}\Box_{g,\Theta}u(x) =&g^{\nu,\gamma}\Theta_\gamma\Theta_\nu u\\ R^{\Theta}_{\gamma,\mu}u(x) =&\Theta_\gamma\Theta_\mu u - \Theta_\mu\Theta_\gamma u\end{aligned}$$

and this conclude the proof. □

Let $\mathcal{M} = \mathbb{R}_t \times X$ with X a smooth compact manifold with boundary of dimension n. Let g be a smooth metric on $\mathcal{M}$ such that $g(\mathrm{d}t, \mathrm{d}t) = 1$. Let E be a smooth complex vector bundle of rank m over X with connection ∇ and with a smooth hermitian inner product k. Note that $C^\infty(\mathbb{R}_t, \Gamma(E))$ is naturally identified with $\Gamma(\pi_2^*(E))$ where π_2 is the second projection on the product $\mathbb{R}_t \times X$. We denote by Θ, the unique connection on $\Gamma(\pi_2^*(E))$ such that for all $u \in C^\infty(\mathbb{R}_t, \Gamma(E))$:

$$\begin{aligned}\Theta_{\partial_t}u &= \partial_t u\\ \Theta_V u &= (t \mapsto \nabla_V u(t)) \quad \text{for } V \in TX\end{aligned}$$

Definition A.0.6. *Let $\Sigma_t := \{t\} \times X$ with volume form $\operatorname{dvol}_t = \iota_{\operatorname{grad} t}\operatorname{dvol}_g$ where ι denotes the interior product and dvol_g is the volume form associated with g. We define the energy at time $t \in \mathbb{R}$ of $u \in C^2(\mathbb{R}_t, \Gamma^2(E))$ by*

$$\mathcal{E}(t)[u] = \int_{\Sigma_t} T_{\delta,\gamma}(\operatorname{grad} t)^\delta(\operatorname{grad} t)^\gamma \operatorname{dvol}_t$$

When there is no ambiguity, we use $\mathcal{E}(t)$ for $\mathcal{E}(t)[u]$.

The following lemma compares $\mathcal{E}(t)[u]$ with the H^1 norm of $u(t)$ (computed using any metric $g_{t'}$ on X). For $t' \in [T_1, T_2]$, we denote by $L^2_{t'}(E)$ the L^2 norm based on the metric $g_{t'}$ induced by g on $\Sigma_{t'}$ (which is naturally identified with X).

Lemma A.0.7. *Let $u \in C^2(\mathbb{R}_t, \Gamma^2(E))$. There exists $C > 0$ such that for all $(t', t) \in [T_1, T_2]^2$*

$$\left\|\mathrm{grad}_{g_{t'},\nabla} u(t)\right\|^2_{L^2_{t'}(E)} + \|\partial_t u(t)\|^2_{L^2_{t'}(E)} + \|u(t)\|^2_{L^2_{t'}(E)} \leq C\mathcal{E}(t).$$

In the above expression $g_{t'}$ is the metric induced on X at time T_1 and the $L^2_{t'}$ norm is taken with respect to the inner product of the bundle (for $u(t)$, this inner product is k and for $\mathrm{grad} u(t)$, it is $k \otimes g_{t'}$) and with the volume form $\mathrm{d}vol_{t'}$ (induced by the metric $g_{t'}$).

Proof. Let $x_0 \in X$, and $t_0 \in [T_1, T_2]$. We take an orthonormal basis for g at (t_0, x_0) with the first vector equal to $\mathrm{grad}\, t((t_0, x_0))$ and the other vectors $(X_i)_{i=1}^n$ in $T_{x_0}X$. There exists local coordinates $(x^\mu)_{\mu=0}^n$ around (t_0, x_0) such that $(x_i)_{i=1}^n$ are local coordinates on X on a neighborhood U of x_0 and such that $\partial_0(t_0, x_0) = \mathrm{grad}\, t$ and $\partial_i(x_0) = X_i$. To alleviate the notations, we write Θ_i for Θ_{∂_i}. We compute in these local coordinates:

$$\begin{aligned} T_{\delta,\gamma}(\mathrm{grad}\, t)^\delta(\mathrm{grad}\, t)^\gamma(x_0) =& \frac{1}{2}\Re(k(\Theta_0 u(t_0, x_0), \Theta_0 u(t_0, x_0))) + \sum_{i=0}^n \frac{1}{2}\Re(k(\Theta_i u(t_0, x_0), \Theta_i u(t_0, x_0))) \\ &+ \frac{1}{2}k(u, u) \\ \mathrm{d}vol_t(x) =& \ \mathrm{d}x^1 ... \, \mathrm{d}x^n \end{aligned}$$

Note that at (t_0, x_0):

$$\mathrm{grad}\, t = \partial_t - \ \mathrm{d}x^i(\partial_t) X_i$$

Therefore,

$$\begin{aligned} k(\partial_t u, \partial_t u) &= k(\Theta_0 u, \Theta_0 u) + 2\ \mathrm{d}x^i(\partial_t)\Re(k(\Theta_0 u, \Theta_i u)) + \ \mathrm{d}x^i(\partial_t)\ \mathrm{d}x^j(\partial_t) k(\Theta_i u, \Theta_j u) \\ &\leq (1 + C_0)k(\Theta_0 u, \Theta_0 u) + (C_0 + C_0^2 n)\sum_{i=1}^n k(\Theta_i u, \Theta_i u) \end{aligned}$$

where $C_0 := \sup_{(t,y)\in[T_1,T_2]\times S^* X_{-g_{T_1}}} \left\{ \frac{y(\partial_t)}{\sqrt{-g(y,y)}} \right\}$ where $SX_{-g_{T_1}}$ is the compact cosphere bundle associated with the Riemannian metric $-g_{T_1}$ (and since ∂_t is a smooth vector field and g is smooth, the function $(t, y) \mapsto \frac{y(\partial_t)}{\sqrt{-g_t(y,y)}}$ is smooth therefore bounded on $[T_1, T_2] \times S^* X_{-g_{T_1}}$). On the other hand we have:

$$\begin{aligned} \left\|\mathrm{grad}_{g_{t'},\nabla} u(t_0)\right\|^2_{g\otimes k}(x_0) =& \sum_{1\leq i,j\leq n} \left| g_{(t',x_0)}(\ \mathrm{d}x^i, \ \mathrm{d}x^j)\right| \Re(k(\Theta_i u, \Theta_j u)) \\ \leq& C_1 n \left(\sum_{i=1}^n \Re(k(\Theta_i u, \Theta_i u)) \right) \end{aligned}$$

Where the constant $C_1 := \sup_{(t,t',y)\in[T_1,T_2]^2\times S^* X_{-g_{T_1}}} \left\{ \frac{g_{t'}(y,y)}{g_t(y,y)} \right\}$. Moreover:

$$\mathrm{d}vol_{t'} \leq C_2\ \mathrm{d}vol_t$$

with $C_2 := \sup_{(t,t',y)\in[T_1,T_2]^2\times SOX_{-g_{T_1}}} \left\{\frac{\mathrm{d}vol_{t'}(y)}{\mathrm{d}vol_t(y)}\right\}$ where $SOX_{-g_{T_1}}$ is the bundle of oriented orthonormal bases associated with the Riemannian metric $-g_{T_1}$ which is compact. We conclude that,

$$\left\|\mathrm{grad}_{g_{t'},\nabla}u(t)\right\|^2_{L^2_{t'}(E)} + \|\partial_t u(t)\|^2_{L^2_{t'}(E)} + \|u(t)\|_{L^2_{t'}(E)} \leq C_3\mathcal{E}(t)$$

for some constant C_3 depending only on n, C_0, C_1 and C_2. □

Proposition A.0.8. *For all $u \in C^2(\mathbb{R}_t, \Gamma^2(E))$ and for all $T_1 < T_2$, there exists $C > 0$ independent of u such that for all $s \in [T_1, T_2]$:*

$$|\mathcal{E}(s) - \mathcal{E}(T_1)| \leq C\left(\int_{T_1}^s \mathcal{E}(t)\ \mathrm{d}t + \int_{T_1}^s \|\Box_{g,\Theta}u(t)\|^2_{L^2_t(E)}\ \mathrm{d}t + \left|\int_{[T_1,s]\times\partial X} i^*(\iota_{T\mathrm{grad}\, t}\ \mathrm{d}vol_g)\right|\right)$$

where the orientation on $[T_1, s]\times\partial X$ is given by an outgoing vector and $i : [T_1, T_2]\times\partial X \to [T_1, T_2]\times X$ is the inclusion map.

Proof. We use Stokes' theorem:

$$\int_{[T_1,s]\times X} \mathrm{div}_g T(\mathrm{grad}\, t)\ \mathrm{d}vol = \mathcal{E}(s) - \mathcal{E}(T_1) + \int_{[T_1,s]\times\partial X} i^*(\iota_{T\mathrm{grad}\, t}\ \mathrm{d}vol_g)$$

Decomposing $\mathrm{d}vol = \frac{\mathrm{d}t}{\sqrt{g(\mathrm{d}t,\mathrm{d}t)}}\ \mathrm{d}vol_t = \mathrm{d}t\ \mathrm{d}vol_t$, we have:

$$\int_{[T_1,s]\times X} \mathrm{div}_g T(\mathrm{grad}\, t)\ \mathrm{d}vol = \int_{T_1}^s \int_X \mathrm{div}(T)^0 + T(\nabla_{LC}\mathrm{grad}\, t)\ \mathrm{d}vol_t\ \mathrm{d}t$$

Using the the compactness of $[T_1, T_2]\times X$, we get that

$$\begin{aligned}\int_X \left|\mathrm{div}(T)^0 + T(\nabla_{LC}\mathrm{grad}\, t)\right|\ \mathrm{d}vol_t \leq& C\left(\|\Box_{g,\Theta}u\|^2_{L^2_t(E)} + \left\|\mathrm{grad}_{g_t,\nabla}u(t)\right\|^2_{L^2_t(E)} + \|\partial_t u(t)\|^2_{L^2_t(E)}\right)\\ \leq& C'(\mathcal{E}(t) + \|\Box_{g,\Theta}u\|^2_{L^2_t(E)})\end{aligned}$$

where $C, C' > 0$ are uniform with respect to $t \in [T_1, T_2]$. □

Let X_h be a family of compact manifolds with boundary included in X (the parameter h is in some arbitrary set but we will use it with $h \in (0, 1]$) and let E_h be the restriction of E to X_h. We also define the norm $\mathcal{E}_h$ as $\mathcal{E}$ but with integration on $\{t\}\times X_h$ instead of Σ_t. Let u_h be a family of smooth sections of E_h. Let $L_h : \Gamma(E) \to \Gamma(E)$ be a family of linear operator such that there exists $C > 0$ such that for all $t \in [T_1, T_2]$ and all $h \in (0, 1]$, $\|L_h u_h(t)\|^2_{L^2_t(E_h)} \leq C\mathcal{E}_h(t)$ (note that we only require this bound to be uniform for this particular family u_h).

Corollary A.0.9. *If the term $\left|\int_{[T_1,T_2]\times\partial X_h} i^*(\iota_{T\mathrm{grad}\, t}\ \mathrm{d}vol_g)\right|$ vanishes for all h (for example if X_h has no boundary or if $u_h, \Theta u_h$ vanish on $[T_1, T_2]\times\partial X_h$), we have for some $C' > 0$ independent of t and h such that:*

$$\mathcal{E}_h(t) \leq \left(\mathcal{E}_h(T_1) + \int_{T_1}^{T_2} \|(\Box_{g,\Theta} + L_h)u_h(s)\|^2_{L^2_t(E_h)}\ \mathrm{d}s\right) e^{C'(t-T_1)}$$

Proof. It follows from proposition A.0.8 applied on the manifold with boundary X_h and from Grönwall's inequality. Since they are all included in the compact manifold X and since g is smooth on $[T_1, T_2]\times X$, the constant are uniform with respect to h. □

We will use this corollary to deduce the semiclassical energy estimate.

A.1 Semiclassical hyperbolic estimate

Let $\mathcal{M} = \mathbb{R}_t \times X$ where X is a smooth compact oriented manifold of dimension n or $\mathbb{R}^n$. Let E be a smooth complex vector bundle of rank m over X with connection ∇ and with a smooth hermitian inner product k. Let g be a smooth metric on $\mathfrak{M} := \mathbb{R}_t \times X \times \mathbb{R}_\tau$ translation invariant with respect to τ and such that $\mathrm{d}t$ is timelike. We assume without loss of generality that $g(\mathrm{d}t, \mathrm{d}t) = 1$. Note that $C^\infty(\mathbb{R}_t \times \mathbb{R}_\tau, \Gamma(E))$ is naturally identified with $\Gamma(\pi_2^*(E))$ where π_2 is the second projection on the product $\mathbb{R}_t \times X \times \mathbb{R}_\tau$. We define a connection Θ on $\pi_2^*(E)$ by $C^\infty(\mathbb{R}_t \times X \times \mathbb{R}_\tau)$-linearity using the identities (for $u \in C^\infty(\mathbb{R}_t \times \mathbb{R}_\tau, \Gamma(E))$):

$$\begin{aligned}\Theta_{\partial_t} u &= \partial_t u\\ \Theta_{\partial_\tau} u &= \partial_\tau u\\ \Theta_V u &= ((t,\tau) \mapsto \nabla_V u(t,\tau)) \quad \text{for } V \in TX\end{aligned}$$

We then consider an operator P_h acting on $C^\infty(\mathbb{R}_t, \Gamma(E))$ as follows:

$$P_h u = e^{-i\frac{\tau}{h}} h^2 \Box_{g,\Theta} e^{i\frac{\tau}{h}} u + R_1(t) h\partial_t u + R_2(t) u$$

where $R_2(t)$ is a smooth family of operators in $h\Psi_h^1(X;E)$ and $R_1(t)$ is a smooth family of operators in $h\Psi_h^0(X;E)$. The first term is a priori in $C^\infty(\mathbb{R}_t \times \mathbb{R}_\tau, \Gamma(E))$ but is in fact independent of τ and we identify it with an element of $C^\infty(\mathbb{R}_t, \Gamma(E))$.

More explicitly, if we choose local coordinates $(x_1, ..., x_n)$ in X, $x_0 = t$, $x_{n+1} = \tau$, we have:

$$\begin{aligned}P_h =& h^2 \sum_{0\le\mu,\nu\le n}^{n} J^{-1}\Theta_\nu J g^{\mu,\nu}\Theta_\mu + h\sum_{\mu=0}^{n}\left(g^{\mu,n+1}\Theta_\mu + J^{-1}\Theta_\mu J g^{\mu,n+1}\right)\\ &+ g^{n+1,n+1} + R_1(t)h\partial_t + R_2(t)\end{aligned}$$

where $J := \sqrt{|\det g|}$, $R_2(t)$ is a smooth family of operators in $h\Psi_h^1(X;E)$ and $R_1(t)$ is a smooth family of operators in $h\Psi_h^0(X;E)$. Note that the principal part $A_h = h^2\sum_{0\le\mu,\nu\le n}^{n} J^{-1}\nabla_\nu J g^{\mu,\nu}\nabla_\mu + h\sum_{\mu=0}^{n}\left(g^{\mu,n+1}\nabla_\mu + J^{-1}\nabla_\mu J g^{\mu,n+1}\right) + g^{n+1,n+1}$ is invariantly defined (the expression does not depend on the chosen local coordinates and trivializations).

Lemma A.1.1. *Let $T \in (0, +\infty)$. With the previous notations, for all $s \in \mathbb{R}$, there exists a constant C_s depending on g, k, s and on the symbol norms of $R(t)$ on $[0,T]$ such that for all $\phi \in C^2([0,T];\Gamma(E))$ with $\phi(0) = \partial_t\phi(0) = 0$:*

$$\|\phi\|_{L^2([0,T],H_h^s)} + \|h\partial_t\phi\|_{L^2([0,T],H_h^{s-1})} \le C_s h^{-1}\|P_h\phi\|_{L^2([0,T],H_h^{s-1})}$$

Remark A.1.2. *If we assume that $s = 1$, it will be clear from the proof that the estimate holds for $\phi \in C^2([0,T];\Gamma^2(E))$.*

Proof. First we reduce to the case $s = 1$. We denote by g_0 the metric induced by g on $\{t = \tau = 0\}$. Let $s > 0$ and Q_s be a semiclassical pseudodifferential operator on $\Gamma(E)$ with principal symbol $(1+|\xi|_{g_0}^2)^{\frac{s}{4}} I_m$ (where I_m is the identity operator in $\mathcal{L}(E,E)$). Let $G_s := Q_s^* Q_s$ where the adjoint is taken with respect to the metric g_0 and the fiber inner product k. The principal symbol of G_s is $(|\xi|_{g_0}^2 + 1)^{\frac{s}{2}} I_m$ and therefore, G_s is semiclassically elliptic of order s and since X is compact, G_s is Fredholm between $H_h^r(E)$ and $H_h^{r-s}(E)$ for every $r \in \mathbb{R}$. Since G_s is also formally symmetric, its index is zero and since it is injective for h small enough, G_s^{-1} exists as a (uniformly with respect to h) bounded operator from $H_h^r(E)$ to $H_h^{r+s}(E)$ for every $r \in \mathbb{R}$. We show that G_s^{-1} is a semiclassical pseudodifferential operator with principal

symbol $(|\xi|_{g_0}^s+1)^{-1}I_m$. Indeed, there exists a parametrix R_s for G_s by ellipticity. Therefore, we have for any $N\in\mathbb{N}$ and $u\in\Gamma(E)$:

$$\begin{aligned}\left\|(R_s-G_s^{-1})u\right\|_{H_h^N}&\leq\left\|G_s^{-1}G_s(R_s-G_s^{-1})u\right\|_{H_h^N}\\&\leq C_r\left\|(G_sR_s-I)u\right\|_{H_h^{N-s}}\\&\leq C_rh^N\left\|u\right\|_{H_h^{-N}}\end{aligned}$$

As a consequence, $R_s-G_s^{-1}$ which was a priori only a bounded operator from H_h^{-N} to H_h^{-N+s} is in fact a (h uniformly) bounded operator from $h^{-N}H_h^{-N}$ to H^N for any $N\in\mathbb{N}$. Therefore, it is an element of $h^\infty\Psi_h^{-\infty}(E)$ and $G_s^{-1}\in R_s+h^\infty\Psi_h^{-\infty}(E)$ is a pseudodifferential operator in $\Psi_h^{-s}(E)$ with principal symbol $(|\xi|_{g_0}^2+1)^{-s}I_m$.

If the estimate of the lemma is true for $s=1$, we can then apply it to the operator $G_sP_hG_s^{-1}$ which is of the correct form and G_su.

Let $h\in(0,1]$. In order to get the inequality in the case $s=1$, we apply corollary A.0.9 to the section $\tilde{u}_h=e^{\frac{i\tau}{h}}u$ on the manifold with boundary $X\times[0,2\pi h]_\tau\subset X\times[0,2\pi]$ between times $T_1=0$ and $T_2=T$. We check that there exists $C>0$ such that for all $h\in(0,1]$ and all $t\in[0,T]$:

$$\left\|h^{-1}u(t)\right\|_{H_h^1(X)}^2+\left\|\partial_tu(t)\right\|_{L^2(X)}^2\leq C\mathcal{E}_h(t)[\tilde{u}_h]$$

Moreover, since $\tilde{u}$ is $2\pi h$ periodic with respect to τ and since $\mathrm{d}vol_g$, g and k are ∂_τ invariant, we have

$$\int_{[0,T]\times X\times\{2\pi h\}_\tau}i^*(\iota_{\mathrm{grad}\,T}\,\mathrm{d}vol_g)\,\mathrm{d}t-\int_{[0,T]\times X\times\{0\}_\tau}i^*(\iota_{\mathrm{grad}\,T}\,\mathrm{d}vol_g)\,\mathrm{d}t=0.$$

We also check that with $L_h:=h^{-1}R_1(t)\partial_t+h^{-2}R_2(t)$, we have

$$\begin{aligned}\left\|L_h\tilde{u}_h\right\|_{L^2}^2\leq&C\left(\left\|h^{-1}u\right\|_{H_h^1(E)}^2+\left\|\partial_tu\right\|_{L^2(E)}^2\right)\\\leq&C'\mathcal{E}_h(t)[\tilde{u}_h]\end{aligned}$$

with the constant C' independent of u and h. Therefore, corollary A.0.9 gives:

$$\left\|h^{-1}u(t)\right\|_{H_h^1(E)}^2+\left\|\partial_tu(t)\right\|_{L^2(E)}^2\leq C\mathcal{E}_h(t)[\tilde{u}_h]\leq Ce^{Ct}\left\|(\Box_{g,\Theta}+L_h)\tilde{u}_h(s)\right\|_{L^2([0,T],L^2(E))}^2$$

Multiplying both sides by h^2 and integrating on $[0,T]$:

$$\left\|u(t)\right\|_{L^2([0,T],H_h^1(E))}^2+\left\|h\partial_tu(t)\right\|_{L^2([0,T],L^2(E))}^2\leq C'\left(h^{-2}\left\|P_hu\right\|_{L^2([0,T],L^2(E))}^2\right)$$

where the constant C' depends on T but is independent of u and h. □

We now want to get estimates in the norm $\overline{H}_h^s$ (therefore have the same order of derivation in time and space). We need a quantitative and semiclassical version of lemma B.2.9 in [54] applied to the operator P_h. This require finer spaces with two indices of regularity. We first recall the definition of spaces $H_{(m,s)}(\mathbb{R}^n)$ and $\overline{H}_{(m,s)}(\mathbb{R}^n_+)$ following [54, Section B.2]. We begin by the definition of $H_{(m,s)}(\mathbb{R}^n)$.

Definition A.1.3. *For $m,s\in\mathbb{R}$, we denote by $H_{(m,s)}(\mathbb{R}^n)$ the set of tempered distribution u such that:*

$$\langle\xi\rangle^m\left\langle\xi'\right\rangle^s\hat{u}\in L^2(\mathbb{R}^n)$$

where $\xi = (\xi', \xi_n) \in \mathbb{R}^n$ is the dual variable of x. When $u \in H_{(m,s)}(\mathbb{R}^n)$, we define the corresponding norm:

$$\|u\|_{H_{(m,s)}(\mathbb{R}^n)} = \left\|\langle\xi\rangle^m \langle\xi'\rangle^s \hat{u}\right\|_{L^2}.$$

We get the semiclassical version of this space $H_{h,(m,s)}(\mathbb{R}^n)$ by taking the semiclassical Fourier transform

$$\mathcal{F}_h u(\xi) := (2\pi h)^{-\frac{n}{2}} \int e^{-i\frac{x\xi}{h}} u(x)\, \mathrm{d}x$$

instead of the Fourier transform in the definition (the space is the same but the norm degenerates when $h \to 0$).

We define $\mathbb{R}^n_+ := \mathbb{R}^{n-1} \times (0, +\infty)$ and we introduce the space $\overline{H}_{(m,s)}(\mathbb{R}^n_+)$:

Definition A.1.4. *Elements of $\overline{H}_{(m,s)}(\mathbb{R}^n_+)$ are distributions on R^n_+ which admits an extension in $H_{(m,s)}(\mathbb{R}^n)$. The norm is given by the infimum over all the possible extensions of the $H_{(m,s)}(\mathbb{R}^n)$ norm of the extension. Replacing $H_{(m,s)}(\mathbb{R}^n)$ by $H_{h,(m,s)}(\mathbb{R}^n)$ in the definition, we get the space $\overline{H}_{h,(m,s)}(\mathbb{R}^n_+)$.*

We generalize these spaces in the context of sections of a rank k vector bundle $[T_0, T_1]_t \times E$ where E is a rank k vector bundle over a smooth compact manifold (without boundaries).

Definition A.1.5. *We consider a covering of X by a finite family of open sets $(U_i)_{i=1}^N$ such that we have local coordinates on each U_i and the bundle E is trivial on U_i. We take a partition of unity χ_i on X subordinated to (U_i). We take $\psi_1, \psi_2 \in C^\infty(\mathbb{R}, [0,1])$ such that $\psi_1 + \psi_2 = 1$ and $\psi_1 = 0$ on $(-\infty, \frac{2T_0+T_1}{3}]_t$, $\psi_2 = 0$ on $(\frac{T_0+2T_1}{3}, +\infty)_t$. Note that modulo a translation and/or reflection in the t variable, for any distributional section u of $E_{|(T_0,T_1)\times X}$, we can identify $\chi_i \Psi_j u$ with a finite family $(v^p_{i,j})_{\substack{1\le i\le N\\ 1\le j\le 2\\ 1\le p\le k}}$ of distributions on $\mathbb{R}^{n+1}_+$ (the n first variables correspond to the local coordinates on X and the last variable correspond to t). The space $\overline{H}_{(m,r)}(E)$ (or $\overline{H}_{(m,r)}$ for short) is the space of u such that all the $v^p_{i,j}$ belongs to $\overline{H}_{(m,r)}(\mathbb{R}^{n+1}_+)$. The norm is given by:*

$$\|u\|^2_{\overline{H}_{(m,s)}} = \sum_{i,j,p} \left\|v^p_{i,j}\right\|^2_{\overline{H}_{(m,s)}(\mathbb{R}^n_+)}$$

Note that a different choice for the χ_i, the ψ_j and the local charts and trivializations induces an equivalent norm. Replacing $\overline{H}_{(m,s)}(\mathbb{R}^{n+1}_+)$ by $\overline{H}_{h,(m,s)}(\mathbb{R}^{n+1}_+)$ in the definition, we get the space $\overline{H}_{h,(m,s)}(E)$.

We use the following quantitative and semiclassical version of lemma B.2.9 in [54] applied to the operator P_h:

Lemma A.1.6. *Let $k \in \mathbb{N}$, $m, r \in \mathbb{R}$. If $u \in \overline{H}_{h,(m-k+1,r)}$ and $P_h u \in \overline{H}_{h,(m,r-k)}$, then $u \in \overline{H}_{h,(m+1,r-k)}$ and there exists a constant $C > 0$ independent of u and $h \in (0,1]$ such that:*

$$\|u\|_{\overline{H}_{h,(m+1,r-k)}} \le C\left(\|u\|_{\overline{H}_{h,(m-k+1,r)}} + \|P_h u\|_{\overline{H}_{h,(m,r-k)}}\right)$$

Proof. We prove this result by induction on k. For $k = 0$, there is nothing to prove. Assume that the result is true for some $k \in \mathbb{N}$. Then let $u \in \overline{H}_{h,(m-k,r)}$ such that $P_h u \in \overline{H}_{h,(m,r-k-1)}$. Using the expression of the operator P_h, we get:

$$\begin{aligned}\left\|h^2\partial_t^2 u\right\|_{\overline{H}_{h,(m-k-1,r-1)}} &\le \|P_h u\|_{\overline{H}_{h,(m-k-1,r-1)}} + C\|h\partial_t u\|_{\overline{H}_{h,(m-k-1,r)}} + C\|u\|_{\overline{H}_{h,(m-k-1,r+1)}}\\ &\le C\left(\|P_h u\|_{\overline{H}_{h,(m,r-k-1)}} + \|u\|_{\overline{H}_{h,(m-k,r)}}\right)\end{aligned}$$

Then, we use the semiclassical version of lemma B.2.3 (on a compact manifold with boundary) in [54] to get:

$$\begin{aligned}\|h\partial_t u\|_{\overline{H}_{h,(m-k,r-1)}} \leq & C\left(\left\|h^2\partial_t^2 u\right\|_{\overline{H}_{h,(m-k-1,r-1)}} + \|h\partial_t u\|_{\overline{H}_{h,(m-k-1,r)}}\right)\\ \leq & C\left(\|P_h u\|_{\overline{H}_{h,(m,r-k-1)}} + \|u\|_{\overline{H}_{h,(m-k,r)}}\right)\end{aligned}$$

and the same lemma applied to u gives:

$$\begin{aligned}\|u\|_{\overline{H}_{h,(m-k+1,r-1)}} \leq & C\left(\|h\partial_t u\|_{\overline{H}_{h,(m-k,r-1)}} + \|u\|_{\overline{H}_{h,(m-k,r)}}\right)\\ \leq & C\left(\|P_h u\|_{\overline{H}_{h,(m,r-k-1)}} + \|u\|_{\overline{H}_{h,(m-k,r)}}\right)\end{aligned}$$

By the induction hypothesis, we conclude that:

$$\begin{aligned}\|u\|_{\overline{H}_{h,(m+1,r-k-1)}} \leq & C\left(\|u\|_{\overline{H}_{h,(m-k+1,r-1)}} + \|P_h u\|_{\overline{H}_{h,(m,r-1-k)}}\right)\\ \leq & C\left(\|P_h u\|_{\overline{H}_{h,(m,r-k-1)}} + \|u\|_{\overline{H}_{h,(m-k,r)}}\right)\end{aligned}$$

□

Using lemma A.1.6, we can state the following proposition. For $s \in \mathbb{R}$, let $\mathcal{H}_h^s$ be the space of distributional sections of $(-\infty, T) \times E$ which can be obtained from a restriction of a distribution in $H_h^s(\mathbb{R} \times E)$ and which have support in $[0, T) \times X$.

Proposition A.1.7. *Let $s \in \mathbb{R}$. There exists $C > 0$ such that for all $u \in \cup_{N\in\mathbb{N}}\mathcal{H}_h^{-N}$:*

$$\|u\|_{\mathcal{H}_h^s} \leq Ch^{-1}\|P_h u\|_{\mathcal{H}_h^{s-1}}$$

In the strong sense that if $P_h u \in \mathcal{H}_h^{s-1}$, then $u \in \mathcal{H}_h^s$ and the inequality holds.

Proof. We prove the lemma for some fixed $h \in (0, 1]$ (but we check that the constant C does not depend on h). First, note that the combination of lemma A.1.1 and lemma A.1.6 gives the proposition for fixed $s \in \mathbb{R}$ if we assume in addition that $u \in \Gamma^N((-\infty, T] \times E)$ with support in $[0, T] \times X$ for some large integer N (depending on s). We will use this estimate on P_h and P_h^* to get the general result. We note that the formal adjoint P_h^* is of the correct form to apply the estimate (and we can put zero initial data at $t = T$ instead of $t = 0$). The dual space of $\mathcal{H}_h^s$ is $\mathcal{G}_h^{-s}$, the space of distributional sections of $(0, +\infty) \times E^*$ which can be obtained from a restriction of a distribution in $H_h^{-s}(\mathbb{R} \times E^*)$ and which have support in $(0, T] \times X$. Therefore, for $f \in \mathcal{H}^{s-1}$ and $\phi \in \Gamma([0, +\infty) \times E^*)$ with support in $[0, T] \times X$, we have:

$$\begin{aligned}|\langle\phi, f\rangle_{L^2}| \leq & \|\phi\|_{\mathcal{G}_h^{-s+1}}\|f\|_{\mathcal{H}_h^{s-1}}\\ \leq & Ch^{-1}\|P_h^*\phi\|_{\mathcal{G}_h^{-s}}\|f\|_{\mathcal{H}_h^{s-1}}\end{aligned}$$

Therefore, by Hahn-Banach, there exists $u \in \mathcal{H}_h^s$ such that $P_h u = f$ and $\|u\|_{\mathcal{H}^s} \leq Ch^{-1}\|Pu\|_{\mathcal{H}_h^{s-1}}$. By taking $f \in \Gamma((-\infty, T] \times E)$ with support in $[0, T] \times X$, the previous construction with s large enough provides u in $\Gamma^N((-\infty, T] \times E)$ with support in $[0, T] \times X$ for an arbitrary large N such that $P_h u = f$. This argument applied to P_h^* gives the density of

$$\left\{P_h^*\phi, \phi \in \Gamma^N([0, +\infty) \times E^*) \text{ with support in } [0, T] \times X\right\}$$

in $\mathcal{G}_h^s$ (for N large enough). Therefore, we have uniqueness of $u \in \mathcal{H}_h^{-M}$ such that $P_h u = f \in \mathcal{H}_h^s$ (for any M and s). Now we can prove the estimate for a general $u \in \mathcal{H}_h^{-N}$ such that

$P_h u \in \mathcal{H}_h^s$. We can take a sequence f_n of sections in $\Gamma((-\infty, T] \times E)$ with support in $[0, T] \times X$ such that $\lim_{n \to +\infty} f_n = P_h u$ in $h^{-1}\mathcal{H}_h^{s-1}$. Then, the associated u_n are in $\Gamma((-\infty, T] \times E)$ with support in $[0, T] \times X$ (we can construct u_n in Γ^N for any N and we have uniqueness). By the estimate, the sequence is Cauchy in $\mathcal{H}_h^s$ and by uniqueness of $u \in \mathcal{H}_h^{-N}$ such that $P_h u = f$, the limit is u. Therefore, $u \in \mathcal{H}_h^s$ and the estimate holds. This also proves that the set of sections $u \in \Gamma((-\infty, T] \times E)$ with support in $[0, T] \times X$ is dense for the norm $\|u\|_{\mathcal{H}_h^s} + h^{-1} \|P_h u\|_{\mathcal{H}_h^{s-1}}$. □

Appendix B

Invertibility of the effective normal operator

Let $\mathcal{N}$ be a smooth compact Riemannian manifold (without boundary). Then $X := [0,+\infty]_x \times \mathcal{N}$ is a smooth manifold with boundary and we denote by n its dimension. The boundary is given by two faces each of them is diffeomorphic to $\mathcal{N}$ and associated with the smooth boundary defining functions x and $r := x^{-1}$. Let E be a smooth complex line bundle over $\mathcal{N}$ endowed with a hermitian metric m and let $\tilde{E}$ be the bundle given by $[0,+\infty] \times E$.

Definition B.0.1. *Let $\tilde{r}, l, \nu \in \mathbb{R}$. We define the space $H_b^{\tilde{r},l,\nu}(\tilde{E})$ ($H_b^{\tilde{r},l,\nu}$ for short) as the space of sections of $\tilde{E}$ which are in the usual b-space (with a b-volume form) $H_b^{\tilde{r},l+\frac{n}{2}}$ near the end $x = 0$, in the space $H^{\tilde{r}}$ in any compact region of the interior of X and in the space $H_b^{\tilde{r},\nu-\frac{n}{2}}$ near the conic end ($x^{-1} = 0$). More concretely for $\tilde{r} \in \mathbb{N}$, if χ_1, $\chi_2 \in C^\infty([0,+\infty];[0,1])$ are such that $\chi_1 + \chi_2 = 1$, $\chi_1 = 1$ near $x = 0$, $\chi_2 = 1$ near $x^{-1} = 0$, we can define the square of the norm of $H_b^{\tilde{r},0,0}(\tilde{E})$ by the following expression:*

$$\begin{aligned}&\left\|x^{-\frac{n}{2}}(x\partial_x)^{\tilde{r}}\chi_1 u\right\|^2_{L^2_b((0,+\infty);L^2(E))} + \left\|x^{-\frac{n}{2}}\chi_1 u\right\|^2_{L^2_b((0,+\infty);H^{\tilde{r}}(E))}\\ +&\left\|x^{-\frac{n}{2}}(x\partial_x)^{\tilde{r}}\chi_2 u\right\|^2_{L^2_b((0,+\infty);L^2(E))} + \left\|x^{-\frac{n}{2}}\chi_2 u\right\|^2_{L^2_b((0,+\infty);H^{\tilde{r}}(E))}\end{aligned}$$

The spaces for $\tilde{r} \in \mathbb{R}$ are then defined by interpolation and duality. Eventually, we define $H_b^{\tilde{r},l,\nu} := \left(\frac{x}{x+1}\right)^l (x+1)^{-\nu} H_b^{\tilde{r},0,0}$.

The $\frac{n}{2}$ shift corresponds to the fact that we want the index to be consistent with the volume form $x^{-n-1}\,\mathrm{d}x\,\mathrm{d}vol_{\mathcal{N}}$ instead of a b-volume form. Let $\beta \in i\mathbb{R}$, $\gamma, \beta' \in \mathbb{C}$, $\zeta \in \mathbb{C} \setminus \{0\}$ with $\Im(\zeta) \geq 0$. We consider the following operator on $\tilde{E}$:

$$\tilde{P}(\zeta) = \left(x^2 D_x\right)^2 + i(n-1)x^3 D_x + x^2 L + \beta\left(x^3 D_x + i\frac{n-2}{2}x^2\right) \tag{B.1}$$

$$+ \beta' x^2 - 2\zeta x\left(xD_x + i\frac{n-1}{2} + \frac{\beta-\gamma}{2}\right) \tag{B.2}$$

where $\Re(\beta') > \frac{\beta^2}{4} - \left(\frac{n-2}{2}\right)^2$, L is a non negative elliptic formally selfadjoint (with respect to the metric m and volume form $\mathrm{d}vol_{\mathcal{N}}$) second order differential operator on E (in particular it is selfadjoint on $L^2(E)$ with domain $H^2(E)$). We denote by $(\lambda_k)_{k\in\mathbb{N}}$ the sequence of its eigenvalues. In this section, we prove that $\tilde{P}(\zeta)$ has an inverse when defined between suitable functional spaces. We start by proving the Fredholm property and we then prove that the kernel and cokernel are trivial.

B.1 Fredholm property

Lemma B.1.1. *Let* $\Im(\zeta) \geq 0$, $l < -\frac{1}{2} + \frac{\Im(\beta-\gamma)}{2}$, $\tilde{r} + l > -\frac{1}{2} + \frac{\Im(\beta+\gamma)}{2}$ *and*

$$\nu \in \left(1 - \frac{\Im(\beta)}{2} - \Re\sqrt{-\frac{\beta^2}{4} + \left(\frac{n-2}{2}\right)^2 + \beta'}, 1 - \frac{\Im(\beta)}{2} + \Re\sqrt{-\frac{\beta^2}{4} + \left(\frac{n-2}{2}\right)^2 + \beta'}\right).$$

The operator $\tilde{P}(\zeta)$ *is Fredholm from* $\left\{u \in H_b^{\tilde{r},l,\nu} : \tilde{P}u \in H_b^{\tilde{r},l+1,\nu-2}\right\}$ *to* $H_b^{\tilde{r},l+1,\nu-2}$.

Proof. First note that the operator $\tilde{P}(\zeta)$ is b-elliptic near $x = +\infty$ and outside any neighborhood of the end $x = 0$. Therefore by the b-elliptic estimate, for any neighborhood U of $x = 0$, if χ_1 is any cutoff equal to 1 near $x = 0$ and such that $\mathrm{supp}(\chi_1) \subset U$, we have a constant $C > 0$ and $\delta > 0$ such that for all $u \in \cup_{N\in\mathbb{N}} H_b^{-N,-N,-N}$, $v \in \cup_{N\in\mathbb{N}} H_b^{-N,-N,-N}$:

$$\|(1-\chi_1(x))u\|_{H_b^{\tilde{r},l,\nu}} \leq C\left(\left\|\tilde{P}(\zeta)u\right\|_{H_b^{\tilde{r}-2,l+1,\nu-2}} + \|(1-\chi_1(x))u\|_{H_b^{\tilde{r}-1,l-1,\nu}}\right)$$

$$\|(1-\chi_1(x))v\|_{H_b^{-\tilde{r},-l-1,-\nu+2}} \leq C\left(\left\|\tilde{P}^*(\zeta)u\right\|_{H_b^{-\tilde{r}-2,-l,-\nu}} + \|(1-\chi_1(x))v\|_{H_b^{-\tilde{r}-1,-l-2,-\nu+2}}\right)$$

in the strong sense that if the right-hand side is finite, so is the left-hand side. The normal operator of $\tilde{P}(\zeta)$ at the end $x = +\infty$ is $r^{-2}\left((rD_r)^2 - i(n-2)rD_r + L + \beta'\right)$ where $r := \frac{1}{x}$. Using a Mellin transform argument (on the model of what is done in the proof of Proposition 13.6.1) and the fact that ν is not an indicial root (by hypothesis), we get that there exists $C > 0$ and $\delta > 0$ such that:

$$\|(1-\chi_1(x))u\|_{H_b^{\tilde{r}-1,l-1,\nu}} \leq C\left(\left\|\tilde{P}(\zeta)u\right\|_{H_b^{\tilde{r}-2,l+1,\nu-2}} + \|(1-\chi_1(x))u\|_{H^{\tilde{r}-1,l-1,\nu-\delta}}\right).$$

A similar estimate also holds for the adjoint operator. Overall, we obtain:

$$\|(1-\chi_1(x))u\|_{H_b^{\tilde{r},l,\nu}} \leq C\left(\left\|\tilde{P}(\zeta)u\right\|_{H_b^{\tilde{r}-2,l+1,\nu-2}} + \|(1-\chi_1(x))u\|_{H_b^{\tilde{r}-1,l-1,\nu-\delta}}\right) \tag{B.3}$$

$$\|(1-\chi_1(x))v\|_{H_b^{-\tilde{r},-l-1,-\nu+2}} \leq C\left(\left\|\tilde{P}^*(\zeta)u\right\|_{H_b^{-\tilde{r}-2,-l,-\nu}} + \|(1-\chi_1(x))v\|_{H_b^{-\tilde{r}-1,-l-2,-\nu+2-\delta}}\right). \tag{B.4}$$

Moreover, we also have constants $C' > 0$ and $\delta > 0$ such that:

$$\|\chi_1(x)u\|_{H_b^{\tilde{r},l,\nu}} \leq C'\left(\left\|\tilde{P}(\zeta)u\right\|_{H_b^{\tilde{r},l+1,\nu-2}} + \|u\|_{H_b^{\tilde{r}-\delta,l-\delta,\nu-\delta}}\right) \tag{B.5}$$

$$\|\chi_1(x)v\|_{H_b^{-\tilde{r},-l-1,-\nu+2}} \leq C'\left(\left\|\tilde{P}^*(\zeta)v\right\|_{H_b^{-\tilde{r},-l,-\nu}} + \|v\|_{H_b^{-\tilde{r}-\delta,-l-1-\delta,-\nu+2-\delta}}\right) \tag{B.6}$$

in the strong sense that if the right-hand side is finite, so is the left-hand side. Estimates (B.5) and (B.6) follows from the fact that

$$\begin{aligned} P(\zeta) :&= e^{i\frac{\zeta}{x}}\tilde{P}(\zeta)e^{-i\frac{\zeta}{x}} \\ &= (x^2D_x)^2 + (\beta + i(n-1))\,x^3D_x + \left(L + i\beta\frac{n-2}{2} + \beta'\right)x^2 + \gamma x\zeta - \zeta^2 \end{aligned}$$

satisfies the three conditions presented in the proof of Proposition 13.2.1 with $a_0 := \beta$, $A := L + i\beta\frac{n-2}{2} + \beta'$, $\sigma := \zeta$, $Q = \gamma x$. Note that the third condition is a consequence of the

following property of L: There exists $(L_k)_{k=0}^N$ in $\mathrm{Diff}^1(E)$ and (L'_k) and L'' in $C^\infty(\mathcal{N})$ such that

$$L = \sum_{k=0}^{N} L_k^* L_k + L'_k L_k + L''$$

By using a finite partition of unity on the compact manifold $\mathcal{N}$, we can assume that L has compact support on an open set of trivialization U of E (L is not selfadjoint anymore after localization but it is still principally self adjoint). We write the operator in a local trivialization (in which m is the canonical hermitian product on $\mathbb{C}$):

$$L = \sum_{1\leq i,j\leq n} a_{i,j} D_{y_i} D_{y_j} + V$$

with $V \in \mathrm{Diff}^1(U)$ (compactly supported in U). We can assume that $a_{i,j} = a_{j,i}$ in this decomposition. Because L is principally self adjoint, $a_{i,j}$ are real valued. Let χ be smooth non negative compactly supported in U and such that $\chi = 1$ on the support of L and V. We have:

$$\begin{aligned} &L - \sum_{1\leq i,j\leq n} \left(\frac{a_{i,j}}{2} D_{y_i} + \chi D_{y_j}\right)^* \left(\frac{a_{i,j}}{2} D_{y_i} + \chi D_{y_j}\right) \\ &- \sum_{i=1}^{n} \left(\sqrt{\sum_{j=1}^{n} \frac{a_{i,j}^2}{4} + n\chi^2}\partial_{y_i}\right)^* \left(\sqrt{\sum_{j=1}^{n} \frac{a_{i,j}^2}{4} + n\chi^2}\partial_{y_i}\right) \in \mathrm{Diff}^1(U) \end{aligned}$$

Note that we use the fact that $\sqrt{\sum_{j=1}^n \frac{a_{i,j}^2}{4} + n\chi^2}$ is smooth on U. Therefore, we can take $(\frac{a_{i,j}}{2} D_{y_i} + \chi D_{y_j})$ and $\sqrt{\sum_{j=1}^n \frac{a_{i,j}^2}{4} + n\chi^2}\partial_{y_i}$ for the operators L_k. The important fact is that they generate the $C^\infty(U)$-module of smooth vector fields with support included in the support of L.

Finally, using estimates (B.3), (B.4), (B.5) and (B.6) in lemma 13.4.1, we conclude the proof. □

Remark B.1.2. *In the Fredholm estimate:*

$$\|u\|_{H_b^{\tilde{r},l,\nu}} \leq C\left(\left\|\tilde{P}(\zeta)u\right\|_{H_b^{\tilde{r},l,\nu}} + \|u\|_{H_b^{\tilde{r}-\delta,l-\delta,\nu-\delta}}\right)$$

the constant δ can be chosen equal to

$$\frac{1}{4}\min\left(r + l + \frac{1}{2} - \frac{\Im(\beta+\gamma)}{2}, \nu - 1 + \frac{\Im(\beta)}{2} + \Re\sqrt{-\frac{\beta^2}{4} + \left(\frac{n-2}{2}\right)^2 + \beta'}\right).$$

In particular, under the hypotheses of Proposition B.1.1, for $u \in Ker(\tilde{P}(\zeta))$, we can iterate the estimate to prove that $u \in H_b^{\infty,l,\nu}$.

B.2 Trivial kernel and existence of the inverse

Proposition B.2.1. *We assume that for all $k \in \mathbb{N}$,*

$$\tilde{a} := \frac{1}{2} + \frac{i\gamma}{2} + \sqrt{\left(\frac{n-2}{2}\right)^2 - \frac{\beta^2}{4} + \lambda_k + \beta'}$$

is not a non positive integer and $\Im(\zeta) \geq 0$. *If* $u \in H_b^{\tilde{r},l,\nu}(\tilde{E})$ *is in the kernel of* $\tilde{P}(\zeta)$ *with* $l < -\frac{1}{2} + \frac{\Im(\beta-\gamma)}{2}$, $\tilde{r} + l > -\frac{1}{2} + \frac{\Im(\beta+\gamma)}{2}$ *and*

$$\nu \in \left(1 - \frac{\Im(\beta)}{2} - \Re\sqrt{-\frac{\beta^2}{4} + \left(\frac{n-2}{2}\right)^2 + \beta'}, 1 - \frac{\Im(\beta)}{2} + \Re\sqrt{-\frac{\beta^2}{4} + \left(\frac{n-2}{2}\right)^2 + \beta'}\right),$$

then $u = 0$.

The following proposition will be used to prove that the kernel of the adjoint operator is trivial. We state it as a general proposition for operators of the form (B.1).

Proposition B.2.2. *We assume that for all* $k \in \mathbb{N}$, $\frac{1}{2} - \frac{i\gamma}{2} + \sqrt{\left(\frac{n-2}{2}\right)^2 - \frac{\beta^2}{4} + \lambda_k + \beta'}$ *is not a negative integer and* $\Im(\zeta) \leq 0$. *If* $u \in H_b^{\tilde{r},l,\nu}(\tilde{E})$ *is in the kernel of* $\tilde{P}(\zeta)$ *with* $l > -\frac{1}{2} + \frac{\Im(\beta-\gamma)}{2}$ *and* $\nu \in \left(1 - \frac{\Im(\beta)}{2} - \Re\sqrt{-\frac{\beta^2}{4} + \left(\frac{n-2}{2}\right)^2 + \beta'}, 1 - \frac{\Im(\beta)}{2} + \Re\sqrt{-\frac{\beta^2}{4} + \left(\frac{n-2}{2}\right)^2 + \beta'}\right)$, *then* $u = 0$.

Proof. We prove both of these propositions in the same time since the proofs only differ at the end. We first reduce the problem to a singular ordinary differential equation by diagonalizing the operator L. Finally, we use standard theory of the confluent hypergeometric equation to prove that $u = 0$.

First note that for $u \in \cup_{N\in\mathbb{N}} H_b^{-N,-N,-N}$ such that $\tilde{P}(\zeta)u = 0$, the classical elliptic regularity theory provides $u \in \Gamma((0,+\infty)_x \times E)$. We denote by $(f_i)_{i\in\mathbb{N}}$ an orthonormal family of eigenfunctions of L and by $(\lambda_i)_{i\in\mathbb{N}}$ the corresponding family of eigenvalues. We define the following operators:

$$\Pi_i : \begin{cases} \Gamma((0,+\infty)_x \times E) \to C^\infty((0,+\infty)_x) \\ u \mapsto \left(x \mapsto \int_{\mathcal{N}} m(u(x,y), f_i(y))\, \mathrm{d}vol(y)\right) \end{cases}$$

(where $\mathrm{d}vol(y)$ is the volume form on $\mathcal{N}$). The spectral theorem for compact selfadjoint operators (applied to $(L+1)^{-1}$) provides that for all $x \in (0,+\infty)$ $\lim_{n\to+\infty} \sum_{i=0}^{n} (\Pi_i u)(x) f_i = u(x)$ for the topology of $L^2(E)$.

By orthogonality of the f_i, this implies that $u(x) = 0$ if and only if $u_i(x) := \Pi_i u(x) = 0$ for all $i \in \mathbb{N}$. Moreover, using the fact that Π_i commutes with $\tilde{P}(\zeta)$ and the equality $\tilde{P}(\zeta)u = 0$, we see that $\tilde{P}_i(\zeta)u_i = 0$ for all $i \in \mathbb{N}$ where

$$\begin{aligned} \tilde{P}_i(\zeta) = &\left(x^2 D_x\right)^2 + i(n-1)x^3 D_x + x^2\lambda_i + \beta\left(x^3 D_x + i\frac{n-2}{2}x^2\right) + \beta' x^2 \\ &- 2\zeta x\left(xD_x + i\frac{n-1}{2} + \frac{\beta-\gamma}{2}\right). \end{aligned}$$

For the rest of the proof, we fix $k \in \mathbb{N}$ and we prove that $u_k = 0$. We use standard results in analysis of singular ODE. First, we can change variable $r = x^{-1}$ in the equation and we get:

$$\begin{aligned} Q_k(\zeta) =& D_r^2 - i(n-1)r^{-1}D_r + r^{-2}\lambda_k + \beta\left(-r^{-1}D_r + i\frac{n-2}{2}r^{-2}\right) + \beta' r^{-2} \\ &- 2\zeta\left(-D_r + i\frac{n-1}{2}r^{-1} + \frac{\beta-\gamma}{2}r^{-1}\right) \\ v_k(r) :=& u_k(r^{-1}) \\ \tilde{P}_k u_k =& 0 \Leftrightarrow Q_k(\zeta)v_k = 0 \end{aligned}$$

Therefore, the equation has meromorphic coefficients, has a regular singularity at $r = 0$ and a rank one singularity at $r = \infty$. The indicial equation at 0 is

$$-\alpha^2 + (2 - n + i\beta)\alpha + \left(\lambda_k + i\beta\frac{n-2}{2} + \beta'\right) = 0$$

with two roots:

$$\alpha_\pm = 1 - \frac{n}{2} + \frac{i\beta}{2} \pm \sqrt{\left(\frac{n-2}{2}\right)^2 - \frac{\beta^2}{4} + \lambda_k + \beta'}$$

(we choose the branch of the square root which extends the square root on $\mathbb{R}_+$ to $\mathbb{C}\setminus(-\infty, 0]$).

We know by Frobenius theory that the space of solutions of the equation $Q_k(\zeta)v_k = 0$ is of dimension 2 and consists of analytic functions with a holomorphic extension (at least) to $\mathbb{C} \setminus (-\infty, 0]$ where 0 is a branch point. After fixing a continuous determination of $\ln(x)$ on $\mathbb{C} \setminus (-\infty, 0]$, one of the solutions can be written as:

$$\sum_{n=0}^{+\infty} a_n r^{n+\alpha_+}$$

where the radius of convergence of the series $\sum_{n=0}^{+\infty} a_n r^n$ is infinite. The precise expression of an other independent solution as a series depends on whether or not $\alpha_+ - \alpha_- \in \mathbb{N}$ but it is equivalent to bx^{α_-} for some $b \neq 0$ (since $\alpha_+ \neq \alpha_-$). The fact that $u \in H_b^{\tilde{r},l,\nu}$ with

$$\nu \in \left(1 - \frac{\Im(\beta)}{2} - \Re\sqrt{-\frac{\beta^2}{4} + \left(\frac{n-2}{2}\right)^2 + \beta'}, 1 - \frac{\Im(\beta)}{2} + \Re\sqrt{-\frac{\beta^2}{4} + \left(\frac{n-2}{2}\right)^2 + \beta'}\right)$$

implies that u_k cannot be equivalent to $x^{-1+\frac{n}{2}+\frac{\Im(\beta)}{2}+\sqrt{-\frac{\beta^2}{4}+\left(\frac{n-2}{2}\right)^2+\beta'+\lambda_k}}$ when $x \to +\infty$. Therefore, we have $v_k = o(r^{\alpha_-})$ near $r = 0$. We deduce that $v_k = C\sum_{n=0}^{+\infty} a_n r^{n+\alpha_+}$ for some $C \in \mathbb{C}$. In particular, the function $r^{-\alpha_+}v_k$ has a holomorphic extension to $\mathbb{C}$.

Now we show that the differential equation can be written as a confluent hypergeometric equation after the transformation $\tilde{Q}_k := r^{1-\alpha_+}Q_k r^{\alpha_+}$ followed by the change of variable $z = -2i\zeta r$.

$$\begin{aligned}
\tilde{Q}_k &= -r\partial_r^2 + (1 - n - 2\alpha_+ + i\beta - 2i\zeta r)\partial_r + \zeta(-2i\alpha_+ - \beta + \gamma - i(n-1)) \\
&= 2i\zeta\left(z\partial_z^2 + (2\alpha_+ + (n-1) - i\beta - z)\partial_z + \left(-\alpha_+ + \frac{i\beta}{2} - \frac{i\gamma}{2} - \frac{n-1}{2}\right)\right) \\
&= 2i\zeta\left(z\partial_z^2 + (c - z)\partial_z - \tilde{a}\right)
\end{aligned}$$

where

$$\begin{aligned}
c &= 2\alpha_+ + n - 1 - i\beta \\
&= 1 + 2\sqrt{\left(\frac{n-2}{2}\right)^2 - \frac{\beta^2}{4} + \lambda_k + \beta'} \\
\tilde{a} &= \alpha_+ - \frac{i\beta}{2} + \frac{i\gamma}{2} + \frac{n-1}{2} \\
&= \frac{1}{2} + \frac{i\gamma}{2} + \sqrt{\left(\frac{n-2}{2}\right)^2 - \frac{\beta^2}{4} + \lambda_k + \beta'}
\end{aligned}$$

Now we assume that $\tilde{a}$ is not a non positive integer. Therefore, since $Q_k v_k = 0$, we have $\tilde{Q}_k \tilde{v}_k = 0$ where $\tilde{v}_k(z) = (r^{-\alpha_+} v_k)\left(\frac{iz}{2\zeta}\right)$. Moreover, $\tilde{v}_k$ is holomorphic on $\mathbb{C}$. Therefore, by Frobenius theory, we have $\tilde{v}_k = C\mathbf{M}(\tilde{a}, c, z)$ (where $\mathbf{M}$ is the renormalized Kummer's function, defined for example by (9.04) in [85, Chapter 7]). Note that $z \mapsto \mathbf{M}(\tilde{a}, c, z)$ is a non zero entire function except when $\tilde{a}, c \in -\mathbb{N}$ and $c \leq \tilde{a}$ in which case, it is identically zero. Using (10.09) and (10.10) in [85, Chapter 7], we see that we have the following asymptotic expansion (valid for any $N \in \mathbb{N}$) of $\mathbf{M}(\tilde{a}, c, z)$ when $|z| \to \infty$:

$$\begin{aligned}
\mathbf{M}(\tilde{a}, c, z) = & \frac{e^{-\tilde{a}i\pi} z^{-\tilde{a}}}{\Gamma(c-\tilde{a})} \sum_{k=0}^{N} (-1)^k \frac{(\tilde{a})_k (1+\tilde{a}-c)_k}{k! z^k} \\
& + \frac{e^z z^{\tilde{a}-c}}{\Gamma(\tilde{a})} \sum_{k=0}^{N} \frac{(c-\tilde{a})_k (1-\tilde{a})_k}{k! z^k} \\
& + o(z^{-N-\Re(\tilde{a})} + e^{\Re(z)} z^{-N+\tilde{a}-c}) \qquad \text{when } -\frac{\pi}{2} - \delta \leq \arg(z) \leq \frac{\pi}{2} - \delta \\
\mathbf{M}(\tilde{a}, c, z) = & \frac{e^{\tilde{a}i\pi} z^{-\tilde{a}}}{\Gamma(c-\tilde{a})} \sum_{k=0}^{N} (-1)^k \frac{(\tilde{a})_k (1+\tilde{a}-c)_k}{k! z^k} \\
& + \frac{e^z z^{\tilde{a}-c}}{\Gamma(\tilde{a})} \sum_{k=0}^{N} \frac{(c-\tilde{a})_k (1-\tilde{a})_k}{k! z^k} \\
& + o(z^{-N-\Re(\tilde{a})} + e^{\Re(z)} z^{-N+\tilde{a}-c}) \qquad \text{when } \frac{\pi}{2} - \delta \leq \arg(z) < \frac{3}{2}\pi - \delta
\end{aligned}$$

where $z^{-\tilde{a}}$ and $z^{\tilde{a}-c}$ are fixed using $\ln(z) = \ln|z| + i\mathrm{arg(z)}$ (pay attention to the fact that in the last asymptotics, arg is not the principal argument).

We now conclude the proof of proposition B.2.1. Using Remark B.1.2, we obtain that $u_k f_k \in H_b^{\infty,l,\nu}$. By the usual normal operator argument (involving the Mellin transform on the model of what is done in the proof of Proposition 13.6.1), we obtain that u_k has an asymptotic expansion into powers of x near $x = 0$ (up to an arbitrarily high order). This correspond to an asymptotic expansion for v_k into powers of $\frac{-2i\zeta}{z}$ as $|z| \to +\infty$ along some half line with $\arg(z) \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ (here we use that $\Im(\zeta) \geq 0$). If $\tilde{a}$ is not a non positive integer, this is not compatible with the expansion of $\mathbf{M}(\tilde{a}, c, z)$ except if $C = 0$. Therefore, $v_k = 0$ for all k and $u = 0$.

We now conclude the proof of proposition B.2.2. If $r \in (0, +\infty)$ and ζ in in the lower half complex plane, we have $z = -2i\zeta r$ is on some half line with argument in $[\frac{\pi}{2}, \frac{3\pi}{2}]$. Let χ be some cutoff localizing near zero. Using the expansion of $\mathbf{M}(\tilde{a}, c, z)$ and the fact that $\chi(x) x^{\alpha_+} u_k = C\mathbf{M}(\tilde{a}, c, -2i\zeta x^{-1}) \in H_b^{\tilde{r}, l+\alpha_+, \infty}$, we obtain:

- if $\arg(-2i\zeta) \in [\frac{\pi}{2}, \frac{3\pi}{2} - \delta)$, there exists $N \in \mathbb{N}$ such that:

$$\begin{aligned}
A(x) := & C\chi(x) \frac{e^{-\tilde{a}i(-\pi+\arg(-2i\zeta))} |2\zeta|^{-\tilde{a}} x^{\tilde{a}}}{\Gamma(c-\tilde{a})} \sum_{k=0}^{N} (-1)^k x^k \frac{(\tilde{a})_k (1+\tilde{a}-c)_k}{k! (-2i\zeta)^k} \\
& + C\chi(x) \frac{e^{-\frac{2i\zeta}{x} + i(\tilde{a}-c)\arg(-2i\zeta)} |2\zeta|^{\tilde{a}-c} x^{c-\tilde{a}}}{\Gamma(\tilde{a})} \sum_{k=0}^{N} x^k \frac{(c-\tilde{a})_k (1-\tilde{a})_k}{k! (-2i\zeta)^k} \in H_b^{\min(\tilde{r},0), l+\alpha_+, \infty}.
\end{aligned}$$

- if $\arg(-2i\zeta) \in [-\frac{\pi}{2} - \delta, -\frac{\pi}{2}]$, there exists $N \in \mathbb{N}$ such that:

$$A(x) := \frac{e^{-\tilde{a}i(\pi+\arg(-2i\zeta))}\,|2\zeta|^{-\tilde{a}}\,x^{\tilde{a}}}{\Gamma(c-\tilde{a})}\sum_{k=0}^{N}(-1)^k x^k \frac{(\tilde{a})_k(1+\tilde{a}-c)_k}{k!(-2i\zeta)^k}$$
$$+\frac{e^{-\frac{2i\zeta}{x}+i(\tilde{a}-c)\arg(-2i\zeta)}\,|2\zeta|^{\tilde{a}-c}\,x^{c-\tilde{a}}}{\Gamma(\tilde{a})}\sum_{k=0}^{N}x^k\frac{(c-\tilde{a})_k(1-\tilde{a})_k}{k!(-2i\zeta)^k} \in H_b^{\min(\tilde{r},0),l+\alpha_+,\infty}$$

Using that the Mellin tranform of $B(x) := e^{-\frac{2i\zeta}{x}}\chi(x)$ has a holomorphic extension to the whole complex plane[1] when $\Im(\zeta \leq 0)$ and lemma 14.2.5, we conclude that the Mellin transformed $MA(\tau)$ has a simple pole of order one at $\tau = -i\tilde{a}$ and residue of modulus $\frac{|C||2\zeta|^{\tilde{a}}}{|\Gamma(c-\tilde{a})|}$. Since on the other hand $A \in H_b^{\min(\tilde{r},0),l+\alpha_+,\infty}$, MA has to be holomorphic on $\Im(\tau) > -l - \frac{n}{2} - \alpha_+$. Since $l > -\frac{1}{2} + \frac{\Im(\beta-\gamma)}{2}$, we deduce $\frac{|C||2\zeta|^{\tilde{a}}}{|\Gamma(c-\tilde{a})|} = 0$ and (using that $c - \tilde{a}$ is not a non positive integer) $C = 0$. □

Corollary B.2.3. *Let $\zeta \in \mathbb{C} \setminus \{0\}$ and $\Im(\zeta) \geq 0$. We assume that for all $k \in \mathbb{N}$,*

$$\frac{1}{2} + i\frac{\gamma}{2} + \sqrt{\left(\frac{n-2}{2}\right)^2 - \frac{\beta^2}{4} + \lambda_k + \beta'} \notin -\mathbb{N}$$
$$\frac{1}{2} - i\frac{\overline{\gamma}}{2} + \sqrt{\left(\frac{n-2}{2}\right)^2 - \frac{\beta^2}{4} + \lambda_k + \overline{\beta'}} \notin -\mathbb{N}$$

Then $\tilde{P}(\zeta)$ is invertible from $\left\{u \in H_b^{r,l,\nu} : \tilde{P}(\zeta)u \in H_b^{r,l+1,\nu-2}\right\}$ to $H_b^{r,l+1,\nu-2}$ where $l < -\frac{1}{2} + \frac{\Im(\beta-\gamma)}{2}$ and $r + l > -\frac{1}{2} + \frac{\Im(\beta+\gamma)}{2}$ and $\nu \in \left(1 - \frac{\Im(\beta)}{2} - \Re\sqrt{-\frac{\beta^2}{4} + \left(\frac{n-2}{2}\right)^2 + \beta'}, 1 - \frac{\Im(\beta)}{2} + \Re\sqrt{-\frac{\beta^2}{4} + \left(\frac{n-2}{2}\right)^2 + \beta'}\right)$.

Proof. We already have Fredholm estimates for $\tilde{P}(\zeta)$ (see the proof of lemma B.1.1). The triviality of the kernel follows from proposition B.2.1. The adjoint operator $\tilde{P}(\zeta)^*$ has the same form as $\tilde{P}$ but with ζ replaced by $\overline{\zeta}$, β replaced by $\overline{\beta} = -\beta$, γ replaced by $\overline{\gamma}$, β' replaced by $\overline{\beta'}$. The second non integer coincidence condition therefore corresponds to the condition in proposition B.2.2. Moreover, $\tilde{P}(\zeta)^*$ goes from $H_b^{-r,-l-1,2-\nu}$ to $H_b^{r,l,\nu}$. Therefore the hypotheses of proposition B.2.2 are satisfied since:

$$-l - 1 > -\frac{1}{2} - \frac{\Im(\beta-\gamma)}{2} = -\frac{1}{2} + \frac{\Im(\overline{\beta}-\overline{\gamma})}{2}$$
$$2 - \nu \in \left(1 - \frac{\Im(\overline{\beta})}{2} - \Re\sqrt{-\frac{\overline{\beta}^2}{4} + \left(\frac{n-2}{2}\right)^2 + \overline{\beta'}}, 1 - \frac{\Im(\overline{\beta})}{2} + \Re\sqrt{-\frac{\overline{\beta}^2}{4} + \left(\frac{n-2}{2}\right)^2 + \overline{\beta'}}\right)$$

□

[1] This is a consequence of the relation $xD_xB(x) = -2\zeta x^{-1}B(x) + C_c^\infty((0,1))$

Appendix C

Proofs of Propositions 11.2.2, 11.2.4

The goal of this section is to provide detailed proofs of Propositions 11.2.2, 11.2.4 and Corollary 11.2.5. The proof of proposition 11.2.2 is based on energy estimates on the model of what is done in [51, Section 4.1]. The estimate on U_3 (defined below) follows less closely [51] but remains in the same spirit. The proof of Proposition 11.2.4 is then based on an idea presented in [51, Section 5.1].

In this section, ∇ and div denotes the Levi-Civita connection and divergence operator associated to the metric $\tilde{g}$.

For the proof it will be convenient to have a concrete defintion for t_0. Moreover, we also need a time coordinate $\tilde{\mathfrak{t}}$ whose level sets are transverse to $\mathscr{I}^+$ and $\mathfrak{H}$ and such that $\mathrm{d}\tilde{\mathfrak{t}}$ remains uniformly timelike with respect to $\tilde{G}$ up to $\mathscr{I}^+$ and up to $r_+ - 2\epsilon$. Therefore we introduce:

$$t_0 := t_* + h(r) \tag{C.1}$$

$$\tilde{\mathfrak{t}} := t_* + F(r) \tag{C.2}$$

where $h(r)$ and $F(r)$ are defined as follow. Let $\chi_0 \in C^\infty(\mathbb{R}, [0,1])$ be a smooth cutoff such that $\chi_0 = 1$ on $(-\infty, 3M]$ and $\chi_0 = 0$ near $+\infty$. Let $\chi_1 \in C^\infty(\mathbb{R}, [0,1])$ such that $\chi_1 = 0$ on $(-\infty, 5M)$ and $\chi_1 = 1$ near $+\infty$. Moreover, we can assume[1] that $\sup \mathrm{supp}(\chi_0) < \inf \mathrm{supp}(\chi_1)$ and $\chi_0 \leq \psi_1$, $\chi_1 \leq \psi_2$ (where ψ_1, ψ_2 were used to define $L(r)$ in (9.1)). We define:

$$h(r) := \begin{cases} -T(r) - \int_r^{+\infty} \chi_0(r) \left(\frac{a^2+r^2}{\Delta_r} - \frac{M^2}{a^2+r^2} \right) \mathrm{d}r \text{ if } r > r_+ \\ -\int_{3M}^{+\infty} \chi_0(r) \frac{a^2+r^2}{\Delta_r} \mathrm{d}r + \int_r^{+\infty} \chi_0(r) \frac{M^2}{a^2+r^2} \mathrm{d}r \text{ otherwise} \end{cases}$$

$$F(r) = h(r) + \int_{-\infty}^{r} \chi_1(r) \left(\frac{M^2}{\Delta_r} - \frac{(a^2+r^2)}{\Delta_r} \right) \mathrm{d}r$$

Note that $h(r)$ and $F(r)$ are smooth across the event horizon. Moreover, $\tilde{\mathfrak{t}}$ is smooth up to $\mathscr{I}^+$.

By construction, we have:

$$\mathfrak{t} + 1 \leq \tilde{\mathfrak{t}} \leq t_0 \leq t$$

An explicit computation provides:

$$\begin{aligned} \tilde{G}(\mathrm{d}t_0, \mathrm{d}t_0) =& -a^2 \sin^2\theta + \frac{(a^2+r^2)^2}{\Delta_r} - \chi_0(r)^2 \frac{(a^2+r^2)^2}{\Delta_r} \left(1 - \frac{M^2 \Delta_r}{(a^2+r^2)^2} \right)^2 \\ \geq& -a^2 \sin^2\theta + M^2 \left(2 - \frac{M^2 \Delta_r}{(a^2+r^2)^2} \right) \end{aligned}$$

[1] This assumption is used to ensure $\mathfrak{t} + 1 \leq t_0$ and be able to figure out the relative position of some particular level sets of $\mathfrak{t}$ and Σ_0 later in the proof.

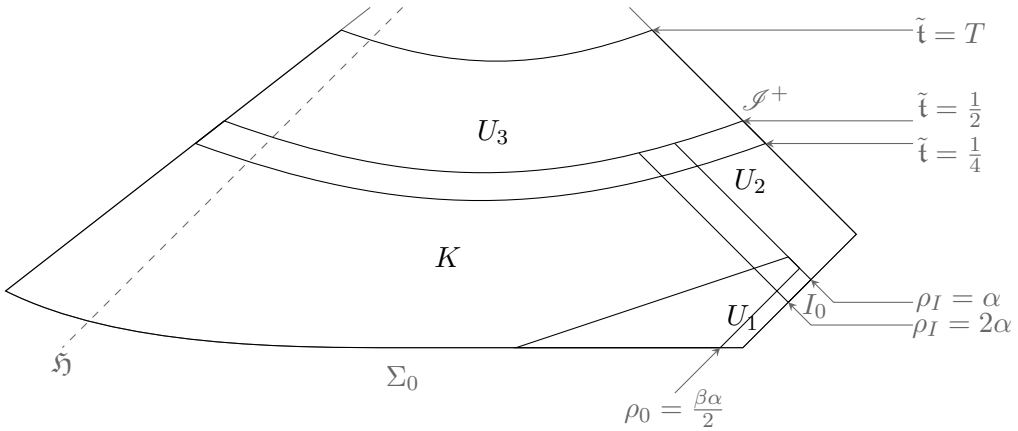

Figure C.1: Representations of the sets U_1, U_2, U_3 and K

Since on $r > r_+ - 2\epsilon$ (for ϵ sufficiently small), we have:

$$\begin{aligned} M^2 \leq r^2 &\leq a^2 + r^2 \\ \Delta_r &\leq a^2 + r^2 \end{aligned}$$

we deduce $\tilde{G}(\mathrm{d}t_0, \mathrm{d}t_0) \geq -a^2 \sin^2\theta + M^2$ and $\mathrm{d}t_0$ is timelike on $\mathcal{M}_{2\epsilon}$. Similarly, since $\mathrm{d}\tilde{\mathfrak{t}} = \mathrm{d}t_0$ for $r < \inf(\mathrm{supp}\chi_1)$ we have that $\mathrm{d}\tilde{\mathfrak{t}}$ is timelike when $r < \inf(\mathrm{supp}\chi_1)$. For $r \geq \inf(\mathrm{supp}\chi_1)$, we have:

$$\begin{aligned} \tilde{G}(\mathrm{d}\tilde{\mathfrak{t}}, \mathrm{d}\tilde{\mathfrak{t}}) =& -a^2\sin^2\theta + \frac{(a^2+r^2)^2}{\Delta_r} - \chi_1(r)^2\frac{(a^2+r^2-M^2)^2}{\Delta_r} \\ \geq& -a^2\sin^2\theta + \frac{(a^2+r^2)M^2}{\Delta_r}\left(2 - \frac{M^2}{a^2+r^2}\right) \\ \geq& M^2 - a^2 \end{aligned}$$

This proves that $\mathrm{d}\tilde{\mathfrak{t}}$ is uniformly timelike with respect to $\tilde{G}$.

Let $\alpha > 0$, $\beta > 0$ and $T > \frac{1}{2}$ be constants to be chosen later. Note that since $\mathfrak{t} + 1 \leq \tilde{\mathfrak{t}}$, we have $\{\tilde{\mathfrak{t}} \leq \frac{1}{2}\} \subset \mathcal{U}$. Our goal to prove Proposition 11.2.2 is to get estimates on $\{r \geq r_+ - \epsilon, t_0 \geq 0, \tilde{\mathfrak{t}} \leq T\}$ for all $T > \frac{1}{2}$. To achieve that, we glue four estimates together:

- An estimate on $U_1 := \{t_0 \geq 0, \rho_I \geq \alpha, \rho_0 \leq \beta\rho_I\}$
- An estimate on $U_2 := \{\tilde{\mathfrak{t}} \leq \frac{1}{2}, \rho_I \leq 2\alpha\}$
- An estimate on $U_3 := \{r_+ - \epsilon \leq r, \frac{1}{4} \leq \tilde{\mathfrak{t}} \leq T\}$
- An estimate on $K := \left\{r_+ - \epsilon \leq r, \tilde{\mathfrak{t}} \leq \frac{1}{2}, t_0 \geq 0, \rho_I \geq \alpha, \rho_0 \geq \frac{\beta\alpha}{2}\right\}$

Note that ρ_I is bounded[2] on $t_0 \geq 0$, therefore, for every $\eta > 0$, by taking β small enough, we have $\rho_0 \leq \eta$ on U_1. We denote by $\beta \mapsto \eta(\beta)$ a positive function with $\lim_{\beta\to 0}\eta(\beta) = 0$ and such that $\rho_0 \leq \eta(\beta)$ on U_1. Moreover since $\rho_0 = \frac{1}{L(r)-t_0+h(r)}$ on U, if $t_0 \geq 0$, $\frac{1}{L(r)+h(r)} \leq \rho_0$

[2]This follows from, $-x\mathfrak{t} = \frac{h(r)+L(r)-t_0}{r}$

and therefore for all $C > 0$, there exists $\eta > 0$ such that $\rho_0 \leq \eta \Rightarrow r > C$. Using this in the inequality (valid for $t_0 \geq 0$):

$$\begin{aligned}\rho_I =& \frac{h(r) + L(r) - t_0}{r}\\ \leq& \frac{h(r) + L(r)}{r}\\ \leq& 1 + o_{r\to\infty}(1),\end{aligned}$$

we deduce that for β small enough, $\rho_I \leq \frac{3}{2}$ on U_1 (in the sequel we assume that β is small enough to ensure this property). Note that K is compact included in $\mathcal{M}_{2\epsilon}$ (and can be included into a relatively compact hyperbolic region $\mathcal{R}$). Since t_0 is a global time function on $\mathcal{M}_{2\epsilon} \cap \{t_0 > -1\}$, by classical hyperbolic theory, we get:

Lemma C.0.1. *For all $s \in \mathbb{R}$, there exists $C > 0$ such that for u as in proposition 11.2.2:*

$$\|u\|_{\overline{H}^{s+1}(K)} \leq C \|u_0\|_{\overline{H}^{s+1}(\mathcal{R}\cap\Sigma_0)} + C \|u_1\|_{\overline{H}^{s}(\mathcal{R}\cap\Sigma_0)}$$

C.1 Estimate on U_1

On U_1, we have the following expression for the operator T_s:

$$T_s = \rho_I(2 - \rho_I)\partial_{\rho_I}^2 - 2\rho_0\partial_{\rho_0}\partial_{\rho_I} - \not{G} + \rho_0 \mathrm{Diff}_b^2 + \mathrm{Diff}_b^1$$

The inverse metric can be written as:

$$\tilde{G} = \rho_I(2 - \rho_I)\partial_{\rho_I}^2 - 2\rho_0\partial_{\rho_0}\partial_{\rho_I} - \not{G} + \rho_0 \mathrm{Diff}_b^2 \tag{C.3}$$

and is therefore a b-metric on U_1 (note that the only part of the boundary intersecting U_1 is I_0).

Lemma C.1.1. *If g is a b-metric (Lorentzian or Riemannian) on some manifold with boundary $\mathcal{M}$ (and we call ρ the defining function of the boundary), and if $X \in \Gamma\left({}^bT^*\mathcal{M}^{\otimes k} \otimes {}^bT\mathcal{M}^{\otimes k'}\right)$, then $\nabla X \in \Gamma({}^bT^*\mathcal{M}^{\otimes(k+1)} \otimes {}^bT\mathcal{M}^{\otimes k'})$ where ∇ is the Levi-Civita connection associated to g.*

Proof. Using the definition of the tensor product connection, we are reduced to the cases $(k, k') = (0, 1)$ and $(k, k') = (1, 0)$. Moreover, note that for all $Z \in \Gamma^b T\mathcal{M}$, $\omega \in \Gamma({}^bT^*\mathcal{M})$ and $X \in \Gamma({}^bT\mathcal{M})$, we have $\nabla_Z\omega(X) = Z(\omega(X)) - \omega(\nabla_Z X)$ with $Z(\omega(X)) \in C^\infty(\mathcal{M})$. Therefore, if for all $X \in \Gamma({}^bT\mathcal{M})$, $\nabla X \in \Gamma({}^bT^*\mathcal{M} \otimes {}^bT\mathcal{M})$, we obtain $\nabla\omega \in \Gamma({}^bT^*\mathcal{M}^{\otimes 2})$. We are thus reduced to proving the case $(k, k') = (0, 1)$. By Leibniz rule and since for $Y \in \Gamma({}^bT\mathcal{M})$ and $f \in C^\infty(\mathcal{M})$ we have $Yf \in C^\infty(\mathcal{M})$, it is enough to prove that if ρ is a local defining function of the boundary and $(y_i)_{i=1}^n$ are local coordinates on the boundary:

$$\nabla_{\rho\partial_\rho}\partial_{y_i} = a_0(\rho, y)\rho\partial_\rho + \sum_{i=1}^n a_i(\rho, y)\partial_{y_i} \tag{C.4}$$

$$\nabla_{\rho\partial_\rho}(\rho\partial_\rho) = a_0(\rho, y)\rho\partial_\rho + \sum_{i=1}^n a_i(\rho, y)\partial_{y_i} \tag{C.5}$$

$$\nabla_{\partial_{y_i}}\partial_{y_i} = a_0(\rho, y)\rho\partial_\rho + \sum_{i=1}^n a_i(\rho, y)\partial_{y_i} \tag{C.6}$$

$$\nabla_{\partial_{y_i}}(\rho\partial_\rho) = a_0(\rho, y)\rho\partial_\rho + \sum_{i=1}^n a_i(\rho, y)\partial_{y_i} \tag{C.7}$$

where coefficients $(a_\mu)_{\mu=0}^n$ are smooth up to the boundary (and can be different in each line). To prove (C.4)-(C.7), we only need to prove that $\Gamma^\beta_{\alpha,\gamma} = \rho_0^{\delta_0^\beta - \delta_0^\alpha - \delta_0^\gamma} a(\rho, y)$ with a smooth up to the boundary. Since we have

$$\Gamma^\beta_{\alpha,\gamma} = \frac{1}{2} G^{\beta,\theta} \left(\partial_\alpha g_{\theta,\gamma} + \partial_\gamma g_{\theta,\alpha} - \partial_\theta g_{\alpha,\gamma} \right)$$

we conclude the argument using the fact that $g \in {}^bT^*\mathcal{M}^{\otimes 2}$ and $G \in {}^bT\mathcal{M}^{\otimes 2}$. □

As we have no a priori information on the behavior of u near the boundary, we first prove a c-uniform estimate on $U_1^c := U_1 \cap \{t + r \leq c\}$ for $c > c_0$. We then take the limit $c \to +\infty$ and Fatou's lemma to get the estimate on U_1. Note that since U_1^c is relatively compact in $\mathcal{M}_\epsilon$, by classical hyperbolic theory we can approximate u by smooth functions u_n such that on U_1^c, $\lim_{n\to+\infty} u_n = u$ in $\overline{H}^{s+1}(U_1^c)$ and $\lim_{n\to+\infty} (u_n)_{|\Sigma_0 \cap U_1^c} = u_0$ (in $\overline{H}^{s+1}(\Sigma_0 \cap U_1^c)$) and $\lim_{n\to+\infty} (\partial_t u_n)_{|\Sigma_0 \cap U_1^c} = u_1$ (in $\overline{H}^s(\Sigma_0 \cap U_1^c)$). Therefore, we can assume for the estimates on U_1^c that u is smooth. We define the energy-momentum tensor:

$$T^{\delta,\gamma}(u) = \Re(\mathbb{m}(\Theta_\mu u, \Theta_\nu u))\tilde{G}^{\mu,\delta}\tilde{G}^{\nu,\gamma} - \frac{1}{2}\tilde{G}^{\delta,\gamma}\tilde{G}^{\mu,\nu}\mathbb{m}(\Theta_\mu u, \Theta_\nu u) + \frac{1}{2}\tilde{G}^{\delta,\gamma}\mathbb{m}(u,u).$$

This expression is an adaptation of the classical energy-momentum tensor for the scalar wave equation with the introduction of the metric $\mathbb{m}$ (due to the fact that u is valued in $\mathcal{B}_s$) and of the last term which is necessary to control the L^2 norm of u. To limit the amount of notations, we sometimes use the name of coordinates as index on tensors or vectors. For example T^{ρ_0,ρ_0} is used to denote the component $T(\mathrm{d}\rho_0, \mathrm{d}\rho_0)$. Moreover, we use the same notation for a bilinear form and for the associated quadratic form. For example $\tilde{G}(\mathrm{d}\rho_I)$ stands for $\tilde{G}(\mathrm{d}\rho_I, \mathrm{d}\rho_I)$. We define the vector fields:

$$V = \nabla^\mu \rho_I$$
$$W = \rho_0^{-2a_0} e^{A\rho_I} V$$

Note that $\tilde{G}(\mathrm{d}\rho_I) \geq \frac{\alpha}{4}$ on U_1 if β is small enough since $\alpha \leq \rho_I \leq \frac{3}{2}$ (and $\rho_0 \leq \eta(\beta)$) on U_1. We compute:

$$\mathrm{div}(T^{\mu,\nu}W_\mu) = e^{A\rho_I}\rho^{-2a_0}\left(\mathrm{div}(T)^{\rho_I} - T^{\mu,\nu}\nabla_\mu V_\nu - \frac{2a_0}{\rho_0}T^{\rho_0,\rho_I} + AT^{\rho_I,\rho_I}\right)$$

For $k, k' \in \mathbb{N}$, we denote by $\overline{\Gamma}({}^bT^*U_1^{\otimes k} \otimes {}^bTU_1^{k'})$ smooth local sections of ${}^bT^*U_1^{\otimes k} \otimes {}^bTU_1^{k'}$ defined on U_1 which have an extension as smooth sections of ${}^bT^*\mathbf{M}_\epsilon^{\otimes k} \otimes {}^bT\mathbf{M}_\epsilon^{\otimes k'}$. By lemma C.1.1, $\nabla\, \mathrm{d}\rho_I \in \overline{\Gamma}({}^bT^*U_1^{\otimes 2})$ and therefore we have:

$$\tilde{G}^{\mu,\alpha}\tilde{G}^{\nu,\beta}\nabla_\alpha V_\beta \in \overline{\Gamma}({}^bTU_1^{\otimes 2})$$
$$\tilde{G}^{\mu,\nu}\nabla_\mu V_\nu \in C^\infty(\overline{U}_1)$$

Since U_1 is relatively compact, we deduce the bound:

$$|T^{\mu,\nu}\nabla_\mu V_\nu| \leq C\left(\mathbb{m}(\partial_{\rho_I} u) + \mathbb{m}(\rho_0\partial_{\rho_0} u) + \not{G}^{\omega_i,\omega_j}\mathbb{m}(\Theta_{\omega_i} u, \Theta_{\omega_j} u) + \mathbb{m}(u)\right)$$

where (ω_0, ω_1) are unspecified local coordinates on the sphere (note that the expression $\not{G}^{\omega_i,\omega_j}\mathbb{m}(\Theta_{\omega_i} u, \Theta_{\omega_j} u)$ does not depend of the choice of these local coordinates). Similarly since $\frac{\mathrm{d}\rho_0}{\rho_0} \otimes \mathrm{d}\rho_I \in \overline{\Gamma}({}^bT^*U_1^{\otimes 2})$, we get:

$$\left|\frac{2a_0}{\rho_0}T^{\rho_0,\rho_I}\right| \leq C\left(\mathbb{m}(\partial_{\rho_I} u) + \mathbb{m}(\rho_0\partial_{\rho_0} u) + \not{G}^{\omega_i,\omega_j}\mathbb{m}(\Theta_{\omega_i} u, \Theta_{\omega_j} u) + \mathbb{m}(u)\right)$$

Finally, using the explicit expression of $\tilde{G}$ (C.3), we get:

$$\begin{aligned}
T^{\rho_I,\rho_I} =&\rho_I^2(2-\rho_I)^2\mathbb{m}(\partial_{\rho_I}u) - 2\rho_I(2-\rho_I)\Re\mathbb{m}(\partial_{\rho_I}u,\rho_0\partial_{\rho_0}u) + \mathbb{m}(\rho_0\partial_{\rho_0}u)\\
&-\frac{1}{2}\rho_I(2-\rho_I)\left(\rho_I(2-\rho_I)\mathbb{m}(\partial_{\rho_I}u) - 2\Re\mathbb{m}(\rho_0\partial_{\rho_0}u,\partial_{\rho_I}u) - \not{G}^{\omega_i,\omega_j}\mathbb{m}(\Theta_{\omega_i}u,\Theta_{\omega_j}u) - \mathbb{m}(u)\right)\\
&+R\\
=&\frac{1}{2}\rho_I^2(2-\rho_I)^2\mathbb{m}(\partial_{\rho_I}u) - \rho_I(2-\rho_I)\Re\mathbb{m}(\partial_{\rho_I}u,\rho_0\partial_{\rho_0}u) + \mathbb{m}(\rho_0\partial_{\rho_0}u)\\
&+\frac{1}{2}\rho_I(2-\rho_I)\not{G}^{\omega_i,\omega_j}\mathbb{m}(\Theta_{\omega_i}u,\Theta_{\omega_j}u) + \frac{1}{2}\rho_I(2-\rho_I)\mathbb{m}(u)\\
|R| \leq& C\rho_0\left(\mathbb{m}(\partial_{\rho_I}u) + \mathbb{m}(\rho_0\partial_{\rho_0}u) + \not{G}^{\omega_i,\omega_j}\mathbb{m}(\Theta_{\omega_i}u,\Theta_{\omega_j}u) + \mathbb{m}(u)\right)
\end{aligned}$$

Using that:

$$|\rho_I(2-\rho_I)\Re\mathbb{m}(\partial_{\rho_I}u,\rho_0\partial_{\rho_0}u)| \leq \frac{3}{4}\mathbb{m}(\rho_0\partial_{\rho_0}u) + \frac{\rho_I^2(2-\rho_I)^2}{3}\mathbb{m}(\partial_{\rho_I}u)$$

We get that for β small enough,

$$T^{\rho_I,\rho_I} \geq C\left(\mathbb{m}(\partial_{\rho_I}u) + \mathbb{m}(\rho_0\partial_{\rho_0}u) + \not{G}^{\omega_i,\omega_j}\mathbb{m}(\Theta_{\omega_i}u,\Theta_{\omega_j}u) + \mathbb{m}(u)\right)$$

on U_1 for some constant $C>0$ independent of u. Using lemma A.0.4 and the fact that the curvature $R^{\Theta}_{\mu,\nu}$ and $\Theta\mathbb{m}$ are bounded on U_1, we get:

$$|\mathrm{div}(T)^{\rho_I}| \leq C\left(\mathbb{m}(\partial_{\rho_I}u) + \mathbb{m}(\rho_0\partial_{\rho_0}u) + \not{G}^{\omega_i,\omega_j}\mathbb{m}(\Theta_{\omega_i}u,\Theta_{\omega_j}u) + \mathbb{m}(\Box_{\tilde{g},\Theta}u) + \mathbb{m}(u)\right)$$

We check that $T_s - \Box_{\tilde{g},\Theta} \in \mathrm{Diff}^1_b$ on U_1 and therefore:

$$|\mathrm{div}(T)^{\rho_I}| \leq C\left(\mathbb{m}(\partial_{\rho_I}u) + \mathbb{m}(\rho_0\partial_{\rho_0}u) + \not{G}^{\omega_i,\omega_j}\mathbb{m}(\Theta_{\omega_i}u,\Theta_{\omega_j}u) + \mathbb{m}(u) + \mathbb{m}(T_su)\right)$$

Finally, we see that if we choose A large enough, we get on U_1:

$$\mathrm{div}(T^{\mu,\nu}W_\mu) \geq e^{A\rho_I}\rho^{-2a_0}\left(AC\left(\mathbb{m}(\partial_{\rho_I}u) + \mathbb{m}(\rho_0\partial_{\rho_0}u) + \not{G}^{\omega_i,\omega_j}\mathbb{m}(\Theta_{\omega_i}u,\Theta_{\omega_j}u) + \mathbb{m}(u)\right) - C\mathbb{m}(T_su)\right)$$

where $C>0$ is a constant independent of u. Finally, we can apply the Stokes theorem on U_1^c which has boundaries included in:

- $\{t_0=0\}$, an outward normal is $-\,\mathrm{d}t_0$ which is timelike past oriented.
- $\{\rho_I=\alpha\}$, an outward normal is $-\,\mathrm{d}\rho_I$ which is timelike future oriented.
- $\{\rho_0=\beta\rho_I\}$, an outward normal is $\mathrm{d}\rho_0-\beta\,\mathrm{d}\rho_I$ which is timelike future oriented if β is small enough.
- $\{t+r=c\}$, an outward normal is $\mathrm{d}t+\mathrm{d}r$. We have $\tilde{G}(\mathrm{d}t+\mathrm{d}r) = 4Mr+O(1)$ when $r\to+\infty$ and therefore the normal is timelike future oriented on the boundary of U_1^c (provided r is large enough which is the case if β is small enough).

Since $\mathrm{d}\rho_I$ (and therefore W) is timelike past oriented, we deduce that (uniformly with respect to c in a neighborhood of $+\infty$):

$$\begin{aligned}
\int_{U_1^c} e^{A\rho_I}\rho^{-2a_0}AC\left(\mathbb{m}(\partial_{\rho_I}u) + \mathbb{m}(\rho_0\partial_{\rho_0}u) + \not{G}^{\omega_i,\omega_j}\mathbb{m}(\Theta_{\omega_i}u,\Theta_{\omega_j}u) + \mathbb{m}(u)\right)\,\mathrm{d}vol_{\tilde{g}}&\\
-C\int_{U_1^c}\rho_0^{-2a_0}e^{A\rho_I}\mathbb{m}(T_su)\,\mathrm{d}vol_{\tilde{g}} \leq \int_{U_1^c}\mathrm{div}(T^{\mu,\nu}W_\mu)\,\mathrm{d}vol_{\tilde{g}}&\\
\leq \int_{\partial U_1^c\cap\{t_0=0\}} -\rho_0^{-2a_0}e^{A\rho_I}T\left(\mathrm{d}\rho_I, \frac{\mathrm{d}t_0}{\sqrt{\tilde{G}(\mathrm{d}t_0)}}\right)\,\mathrm{d}vol_{\Sigma_0}&
\end{aligned}$$

For r large enough, we have

$$\begin{aligned}\mathrm{d}t_0 =& -\left(1+\frac{a^2+r^2}{\Delta_r\rho_I}\right)\frac{\mathrm{d}\rho_0}{\rho_0^2} - \frac{a^2+r^2}{\Delta_r}\frac{\mathrm{d}\rho_I}{\rho_0\rho_I^2}\\ G(\mathrm{d}t_0)^{-\frac{1}{2}} =& \rho_0\rho_I + \rho_0^2 C^\infty(\overline{U_1}).\end{aligned}$$

Therefore, $\frac{\mathrm{d}t_0}{\sqrt{\tilde{G}(\mathrm{d}t_0)}}$ is a uniformly timelike b-one form when restricted to $U_1\cap\{t_0=0\}$. Since $\mathrm{d}\rho_I$ is also a uniformly timelike b-one form when restricted to $U_1\cap\{t_0=0\}$ and since $\tilde{g}$ is a b-metric, we have (letting $c\to+\infty$):

$$\begin{aligned}&A\int_{U_1^c} e^{A\rho_I}\rho^{-2a_0}\left(\mathbb{m}(\partial_{\rho_I}u)+\mathbb{m}(\rho_0\partial_{\rho_0}u)+\not{G}^{\omega_i,\omega_j}\mathbb{m}(\Theta_{\omega_i}u,\Theta_{\omega_j}u)+\mathbb{m}(u)\right)\mathrm{d}vol_{\tilde{g}}\\ &\leq C\int_{U_1^c}\rho_0^{-2a_0}e^{A\rho_I}\mathbb{m}(T_su)\,\mathrm{d}vol_{\tilde{g}} + Ce^{\frac{3A}{2}}\left(\|u_0\|_{\rho_0^{a_0}H_b^1(U_1\cap\Sigma_0)}+\|u_1\|_{\rho_0^{a_0}L_b^2(U_1\cap\Sigma_0)}\right)\end{aligned}\tag{C.8}$$

with constants independent of A and u. Now we assume that $T_su=0$, in particular, in this case for all $k\geq 1$ $(\rho_0\nabla t_0)^k u$ has a trace on $\Sigma_0\cap U_1$ and we can express

$$((\rho_0\nabla t_0)^k u)_{|\Sigma_0\cap U_1} = \mathrm{Diff}_b^{k-1}(\Sigma_0\cap U_1)((\rho_0\nabla t_0)u)_{|\Sigma_0\cap U_1} + \mathrm{Diff}_b^k(\Sigma_0\cap U_1)u_{|\Sigma_0\cap U_1}.$$

and therefore, for any $Z\in\mathrm{Diff}_b^k$,

$$(Zu)_{|\Sigma_0\cap U_1} = \mathrm{Diff}_b^{k-1}(\Sigma_0\cap U_1)((\rho_0\nabla t_0)u)_{|\Sigma_0\cap U_1} + \mathrm{Diff}_b^k(\Sigma_0\cap U_1)u_{|\Sigma_0\cap U_1}.$$

We want to get the estimate in $H_b^k(U_1)$ with $k\geq 1$. We use the fact that for $Z\in\mathrm{Diff}_b^k$ defined on U_1, $[T_s,Z]\in\mathrm{Diff}_b^{k+1}(U_1)$ in what follows. Let $(Z_i)_{i=1}^N$ be operators in Diff_b^k such that $\sum_{i=1}^N\|Z_iu\|_{L_b^2(U_1)}$ is equivalent to $\|u\|_{H_b^k(U_1)}$. The estimate (C.8) applied to Z_lu gives:

$$\begin{aligned}&A\int_{U_1^c} e^{A\rho_I}\rho^{-2a_0}\mathbb{m}(\partial_{\rho_I}Z_lu)+\mathbb{m}(\rho_0\partial_{\rho_0}Z_lu)+\not{G}^{\omega_i,\omega_j}\mathbb{m}(\Theta_{\omega_i}Z_lu,\Theta_{\omega_j}Z_lu)+\mathbb{m}(Z_lu)\,\mathrm{d}vol_{\tilde{g}}\\ &\leq C\int_{U_1^c}\rho_0^{-2a_0}e^{A\rho_I}[T_s,Z_l]u\,\mathrm{d}vol_{\tilde{g}} + Ce^{\frac{3A}{2}}\left(\|u_0\|_{\rho_0^{a_0}H_b^{k+1}(U_1\cap\Sigma_0)}+\|u_1\|_{\rho_0^{a_0}H_b^k(U_1\cap\Sigma_0)}\right)\end{aligned}$$

We sum the estimates from $l=1$ to N and if we take A large enough so that the left-hand-side absorbs the term $C\int_{U_1^c}\rho_0^{-2a_0}e^{A\rho_I}[T_s,Z_l]u\,\mathrm{d}vol_{\tilde{g}}$ we get:

Lemma C.1.2. *There exists a constant $C>0$ such that for all u as in proposition 11.2.2:*

$$\|u\|_{\rho_0^{a_0}H_b^{k+1}(U_1)}\leq C\left(\|u_0\|_{\rho_0^{a_0}H_b^{k+1}(U_1\cap\Sigma_0)}+\|u_1\|_{\rho_0^{a_0}H_b^k(U_1\cap\Sigma_0)}\right)$$

in the strong sense that the left-hand side is finite whenever the right-hand side is finite and the inequality holds.

C.2 Estimate on U_2

We denote by ${}^{\mathscr{I}}TU_2$ the bundle with smooth local sections $\rho_0\partial_{\rho_0}$, $\rho_I\partial_{\rho_I}$, $\rho_I^{\frac{1}{2}}\not{\partial}$ (where $\not{\partial}$ denotes a smooth vector field on $\mathbb{S}^2$) and by ${}^{\mathscr{I}}T^*U_2$ the dual bundle. We can then define the Sobolev space $H^1_{\mathscr{I},b}$ with norm:

$$\|u\|^2_{H^1_{\mathscr{I},b}} = \int_{U_2}\mathbb{m}(\rho_0\partial_{\rho_0}u)+\mathbb{m}(\rho_I\partial_{\rho_I}u)+\rho_I\not{G}^{\omega_i,\omega_j}\mathbb{m}(\Theta_{\omega_i}u,\Theta_{\omega_j}u)\frac{\mathrm{d}\rho_0}{\rho_0}\frac{\mathrm{d}\rho_I}{\rho_I}\,\mathrm{dvol}_{\mathbb{S}^2}$$

Let $Z_1, Z_2, Z_3 \in \Gamma(T\mathbb{S}^2)$ be vector fields spanning $T\mathbb{S}^2$ (as a $C^\infty(\mathbb{S}^2)$-module). For $k \in \mathbb{N}$, we define by induction the norms:

$$\begin{aligned}
\|u\|_{H^{1,0}_{\mathscr{I},b}} &:= \|u\|_{H^1_{\mathscr{I},b}} \\
\|u\|^2_{H^{1,k+1}_{\mathscr{I},b}} &:= \|u\|^2_{H^{1,k}_{\mathscr{I},b}} + \|\rho_0\partial_{\rho_0}u\|^2_{H^{1,k}_{\mathscr{I},b}} + \|\rho_I\partial_{\rho_I}u\|^2_{H^{1,k}_{\mathscr{I},b}} + \sum_{i=1}^3 \|\Theta_{Z_i}u\|^2_{H^{1,k}_{\mathscr{I},b}}
\end{aligned}$$

Let $u \in H^{\tilde{r}+1}_{loc}(\mathcal{M}_{2\epsilon})$ with $\rho_0^{-a_0}\rho_I^{-a_I}T_s\rho_I u \in \overline{H}^{\tilde{r}}_b$. At first, we assume that $u = 0$ on $\left\{\rho_I \geq \frac{3}{2}\alpha\right\}$. We do an energy estimate with the vector field $W = \rho_0^{-2a_0}\rho_I^{-2a_I}(-(1+c_V)\rho_I\partial_{\rho_I} + \rho_0\partial_{\rho_0})$ where c_V is a constant to be chosen later.

We define the space $S^0(\mathbf{M}_\epsilon)$ as the set of functions $f \in C^\infty(\overset{\circ}{\mathbf{M}}_\epsilon)$ such that for every $Z \in \mathrm{Diff}^k_b(\mathbf{M}_\epsilon)$, Zf is bounded. Let $\otimes$ denotes the tensor product of $C^\infty(\mathbf{M}_\epsilon)$ modules. We denote by $S^0\mathrm{Diff}^k_{\mathscr{I},b} := S^0(\mathbf{M}_\epsilon) \otimes \mathrm{Diff}^k_{\mathscr{I},b}$ the set of $\mathscr{I}$-differential operators with coefficients in S^0. Similarly for a vector bundle E, we denote by $S^0\Gamma(E) = S^0(\mathbf{M}_\epsilon) \otimes_{C^\infty(\mathbf{M}_\epsilon)} \Gamma(E)$ the set of sections with coefficients in S^0. It is necessary to introduce these spaces since vector fields such as $\rho_I\partial_\phi$ are not in $\Gamma({}^{\mathscr{I}}U_2)$ since $\rho_I^{\frac{1}{2}}$ is not smooth up to the boundary. However $\rho_I\partial_\phi \in S^0\Gamma({}^{\mathscr{I}}U_2)$ which is just as suitable for our purpose.

Note that on U_2, we have:

$$\begin{aligned}
\tilde{G} =& \frac{1}{\rho_I}\left(2\rho_I^2\partial^2_{\rho_I} - 2\rho_0\rho_I\partial_{\rho_I}\partial_{\rho_0} - \rho_I\not{G} + \rho_I\Gamma({}^{\mathscr{I}}TU_2^{\otimes 2})\right) \\
\tilde{g} =& \rho_I\left(-\frac{2\,\mathrm{d}\rho_0^2}{\rho_0^2} - 2\frac{\mathrm{d}\rho_0\,\mathrm{d}\rho_I}{\rho_0\rho_I} - \frac{1}{\rho_I}\not{g} + \rho_I\Gamma\left({}^{\mathscr{I}}T^*U_2^{\otimes 2}\right)\right)
\end{aligned}$$

We consider:

$$\begin{aligned}
T_{\mu,\nu} =& \mathbb{m}(\Theta_\mu u, \Theta_\nu u) - \frac{1}{2}\tilde{g}_{\mu,\nu}\tilde{G}^{\gamma,\delta}\mathbb{m}(\Theta_\gamma u, \Theta_\delta u) + \frac{1}{2\rho_I}\tilde{g}_{\mu,\nu}\mathbb{m}(u) \\
\mathrm{div}(T_{\mu,\nu}W^\mu) =& \mathrm{div}(T)_\mu W^\mu + T_{\mu,\nu}\pi^{\mu,\nu} \\
\pi^{\mu,\nu} :=& \frac{1}{2}\left(\nabla^\mu W^\nu + \nabla^\nu W^\mu\right) \\
=& \frac{1}{2}\left(\tilde{G}^{\mu,\alpha}\partial_\alpha W^\nu + \tilde{G}^{\alpha,\nu}\partial_\alpha W^\mu - W^\alpha\partial_\alpha\tilde{G}^{\mu,\nu}\right)
\end{aligned}$$

A tedious computation provides:

$$\begin{aligned}
T_{\mu,\nu}\pi^{\mu,\nu} =& \rho_0^{-2a_0}\rho_I^{-2a_I-1}\left((2(a_I-a_0)c_V + 2a_I)\mathbb{m}(\rho_I\partial_{\rho_I}u) - 4a_I\Re\mathbb{m}(\rho_0\partial_{\rho_0}u, \rho_I\partial_{\rho_I}u) + 2a_I\mathbb{m}(\rho_0\partial_{\rho_0}u)\right. \\
&\left. + \frac{2(a_I-a_0) + 2a_Ic_V - c_V - 1}{2}\left(\rho_I\not{G}^{\omega_i,\omega_j}\mathbb{m}(\Theta_{\omega_i}u, \Theta_{\omega_j}u) + \mathbb{m}(u)\right)\right) + R \\
|R| \leq& C\rho_I^{-2a_I}\rho_0^{-2a_0}\left(\mathbb{m}(\rho_I\partial_{\rho_I}u) + \mathbb{m}(\rho_0\partial_{\rho_0}u) + \rho_I\not{G}^{\omega_i,\omega_j}\mathbb{m}(\Theta_{\omega_i}u, \Theta_{\omega_j}u) + \mathbb{m}(u)\right)
\end{aligned}$$

If we assume that $c_V > 0$, $a_I < a_0$ and $a_I < 0$, we get:

$$\begin{aligned}
&(2(a_I-a_0)c_V + 2a_I)\mathbb{m}(\rho_I\partial_{\rho_I}u) \\
&- 4a_I\Re\mathbb{m}(\rho_0\partial_{\rho_0}u, \rho_I\partial_{\rho_I}u) + 2a_I\mathbb{m}(\rho_0\partial_{\rho_0}u) \leq -c_1\mathbb{m}(\rho_I\partial_{\rho_I}u) - c_2\mathbb{m}(\rho_0\partial_{\rho_0}u) \\
&c_1 := (a_0 - a_I)c_V > 0 \\
&c_2 := -2a_I - \frac{16a_I^2}{-8a_I + 4(a_0-a_I)c_V} > 0
\end{aligned}$$

We also check that (see Proposition A.0.4 for a similar computation) that there exists $C_0 > 0$ and $C > 0$ such that on U_2 we have:

$$\begin{aligned}|\mathrm{div}(T)_\mu W^\mu| \leq & C_0 \rho_I^{-2a_I} \rho_0^{-2a_0} \left(\mathbb{m}(\rho_I \partial_{\rho_I} u) + \mathbb{m}(\rho_0 \partial_{\rho_0} u) + \rho_I \not{G}^{\omega_i,\omega_j} \mathbb{m}(\Theta_{\omega_i} u, \Theta_{\omega_j} u) \right. \\ & \left. + \mathbb{m}(u) \right) + \rho_I^{-2a_I-1} \rho_0^{-2a_0} \left(C \mathbb{m}(\rho_I \Box_{\Theta,\tilde{g}} u) + \frac{c_1}{2} \mathbb{m}(\rho_I \partial_{\rho_I}) + \frac{c_2}{2} \mathbb{m}(\rho_0 \partial_{\rho_0}) \right) \quad \text{(C.9)}\end{aligned}$$

Moreover, $\rho_I^{-1} \rho_0^{-1} T_s \rho_I \rho_0 - \Box_{\Theta,\tilde{g}} \in S^0 \mathrm{Diff}^1_{\not{g}}$ and therefore, if α is small enough we can replace $\rho_I \Box_{\Theta,g}$ by $\rho_0^{-1} T_s \rho_I \rho_0$ in (C.9) at the cost of increasing C_0. As a consequence, if we choose α small enough, we have the following inequality on U_2 for some constants $c_3 > 0$ and $C > 0$:

$$\begin{aligned}\mathrm{div}(T_{\mu,\nu} \pi^{\mu,\nu}) \leq & - c_3 \rho_0^{-2a_0} \rho_I^{-2a_I-1} \left(\mathbb{m}(\rho_I \partial_{\rho_I} u) + \mathbb{m}(\rho_0 \partial_{\rho_0} u) + \rho_I \not{G}^{\omega_i,\omega_j} \mathbb{m}(\Theta_{\omega_i} u, \Theta_{\omega_j} u) \right. \\ & \left. + \mathbb{m}(u) \right) + C \rho_I^{-2a_I-1} \rho_0^{-2a_0} \mathbb{m}(\rho_0^{-1} \rho_I^{-1} T_s \rho_I \rho_0 u) \quad \text{(C.10)}\end{aligned}$$

where constant c_3 does not depend on α (as long as it is smaller than some small threshold $\alpha_0 > 0$).

For $0 < c < \alpha$ and c' large enough, we apply the Stokes formula on $U_2^{c,c'} := U_2 \cap \{\rho_I \geq c, t + r \leq c'\}$ which is relatively compact in $\mathcal{M}_\epsilon$ (and we will take $c \to 0$ and $c' \to \infty$). The boundaries of $U_2^{c,c'}$ are included in :

- $\{\rho_I = c\}$ with exterior normal $-\mathrm{d}\rho_I$ which is timelike and future oriented for c small enough.
- $\left\{\tilde{\mathfrak{t}} = \frac{1}{2}\right\}$ with exterior normal $\mathrm{d}\tilde{\mathfrak{t}}$ which is timelike and future oriented.
- $\{t + r = c'\}$ with exterior normal $\mathrm{d}t + \mathrm{d}r$ which is timelike and future oriented when r is large (and on the boundary of U_2 we have $\tilde{\mathfrak{t}} \leq \frac{1}{2}$ and since for some constant $C_0 > 0$, $t - r - C_0 \leq \tilde{\mathfrak{t}} \leq \frac{1}{2}$, we deduce that on the boundary $t + r = c'$, $r \geq \frac{c' - C_0 - \frac{1}{2}}{2}$ is as large as we want if we chose c' large enough).
- $\{\rho_I = 2\alpha\}$ with exterior normal $\mathrm{d}\rho_I$ which is timelike and past oriented if we choose α small enough.

Since u has support in $\left\{\rho_I \leq \frac{3}{2}\alpha\right\}$, the boundary term corresponding to $\{\rho_I = 2\alpha\}$ vanishes. The other boundary terms are non negative (using that W is timelike future oriented if we choose α small enough) and we deduce:

$$-\int_{U_2^{c,c'}} \mathrm{div}(T_{\mu,\nu} W^\mu) \, \mathrm{d}vol_{\tilde{g}} \leq 0$$

Combining the previous inequality with (C.10), we get:

$$\begin{aligned}\int_{U_2^{c,c'}} \rho_0^{-2a_0} \rho_I^{-2a_I-1} \left(\mathbb{m}(\rho_I \partial_{\rho_I} u) + \mathbb{m}(\rho_0 \partial_{\rho_0} u) + \rho_I \not{G}^{\omega_i,\omega_j} \mathbb{m}(\Theta_{\omega_i} u, \Theta_{\omega_j} u) + \mathbb{m}(u) \right) \mathrm{dvol}_{\tilde{g}} & \\ \leq \frac{1}{c_3} \int_{U_2^{c,c'}} \rho_I^{-2a_I+1} \rho_0^{-2a_0} \mathbb{m}(\rho_I^{-1} \rho_0^{-1} T_s \rho_I \rho_0 u) \, \mathrm{dvol}_{\tilde{g}} \quad \text{(C.11)}\end{aligned}$$

Since the constant c_3 does not depend on c, c', we can use Fatou's lemma and prove that (C.11) holds with $U_2^{c,c'}$ replaced by U_2.

Now we introduce $Z_0 = \rho_I \partial_{\rho_I}$, $Z_1 = \rho_0 \partial_{\rho_0}$, and $Z_2, Z_3, Z_4 \in \mathrm{Diff}^1(\mathcal{B}(s))$ generating the $C^\infty(\mathbb{S}^2)$ module $\mathrm{Diff}^1(\mathcal{B}_s)$. We then have for $1 \leq i \leq 4$, $[Z_i, \rho_0^{-1} \rho_I^{-1} T_s \rho_0 \rho_I] \in \mathrm{Diff}^2_b$ and $[Z_0, \rho_0^{-1} \rho_I^{-1} T_s \rho_0 \rho_I] = -\rho_0^{-1} \rho_I^{-1} T_s \rho_0 \rho_I + \mathrm{Diff}^2_b$. We use this property together with (C.11)

(applied to $Z_i u$) to prove by induction that for all $k \in \mathbb{N}$, there exists $\alpha_k > 0$ such that for $\alpha = \alpha_k$ in the definition of U_2 and for all u with support in $\left\{\rho_I \leq \frac{3}{2}\alpha_k\right\}$:

$$\|u\|_{\rho_I^{a_I}\rho_0^{a_0} H^{1,k}_{\mathscr{I},b}(U_2)} \leq \frac{1}{c_3} \|T_s \rho_0 \rho_I u\|_{\rho_I^{a_I}\rho_0^{a_0+1} H^k_b(U_2)}$$

Note that a priori α_k has to depend on k since at each step we have to take α small enough so that the term arising from the commutator can be absorbed into the left-hand side.

Let $k \in \mathbb{N}$. Let $\chi \in C^\infty(\mathbb{R}, [0,1])$ be equal to 1 on $(-\infty, \alpha_k)$ and zero on $(\frac{3}{2}\alpha_k, +\infty)$. We do not assume anymore that u has support in $\left\{\rho_I \leq \frac{3}{2}\alpha_k\right\}$, we use the previous argument on $\chi\rho_I^{-1}\rho_0^{-1}u$ where u is as defined in proposition 11.2.2 combined with estimate on U_1 and K to control[3] the commutator term:

$$\begin{aligned}
\left\|\chi\rho_I^{-1}\rho_0^{-1}u\right\|_{\rho_I^{a_I}\rho_0^{a_0} H^{1,k}_{\mathscr{I},b}(U_2)} \leq & \frac{1}{c_3} \|T_s \chi u\|_{\rho_I^{a_I}\rho_0^{a_0+1} H^k_b(U_2)} \\
\|T_s \chi u\|_{\rho_I^{a_I}\rho_0^{a_0+1} H^k_b(U_2)} = & \|[T_s, \chi]u\|_{\rho_I^{a_I}\rho_0^{a_0+1} H^k_b(U_2)} \\
\leq & \|u\|_{\rho_0^{a_0+1}\rho_I^{a_I} H^{k+1}_b(U_2)} \\
\leq & C\left(\|u\|_{H^{k+1}(K)} + \|u\|_{\rho_0^{a_0+1} H^{k+1}_b(U_1)}\right).
\end{aligned}$$

Moreover, we have:

$$\left\|(1-\chi)\rho_I^{-1}\rho_0^{-1}u\right\|_{\rho_I^{a_I}\rho_0^{a_0} H^{1,k}_{\mathscr{I},b}(U_2)} \leq C\left(\|u\|_{\rho_0^{a_0+1} H^k_b(U_1)} + \|u\|_{H^k(K)}\right).$$

We conclude:

Lemma C.2.1. *For all $k \in \mathbb{N}$, all $a_I < 0$, $a_0 > a_I + 1$, there exists constants $C_k > 0$ and $\alpha_k > 0$ such that for u as in proposition 11.2.2 (with $\tilde{r} \geq k$):*

$$\|u\|_{\rho_I^{a_I+1}\rho_0^{a_0} H^{1,k}_{\mathscr{I},b}(U_2)} \leq C_k\left(\|u_0\|_{\rho_0^{a_0}\overline{H}^{k+1}_b(\Sigma_0)} + \|u_1\|_{\rho_0^{a_0}\overline{H}^k_b(\Sigma_0)}\right)$$

where we have taken $\alpha = \alpha_k$ in the definition of U_2.

C.3 Estimate on U_3

We define ${}^{\mathscr{I}}TU_3$ and $H^{1,k}_{\mathscr{I},b}$ which are the natural extension to the corresponding spaces defined on U_2 but since U_3 is not included in $\mathcal{U}$ and does not intersect I_0, we use vector fields $x\partial_x$, $\partial_{\mathfrak{t}}$ instead of $\rho_I\partial_{\rho_I}$ and $\rho_0\partial_{\rho_0}$. We also introduce the bundle ${}^{\beta}TU_3$ which is generated by the sections $x^{\frac{1}{2}}\partial_{\mathfrak{t}}$, $x\partial_x$ and $x^{\frac{1}{2}}\not{\partial}$ (where $\not{\partial}$ denotes a smooth vector field on $\mathbb{S}^2$). We compute:

$$\begin{aligned}
\tilde{G} =& x^{-1}\left(-2\partial_{\mathfrak{t}} x\partial_x - x\not{G} + x\Gamma\left({}^{\mathscr{I}}TU_3^{\otimes 2}\right)\right) \\
\tilde{G} =& x^{-1}\left((2M^2 - a^2\sin^2\theta)x\partial_{\mathfrak{t}}^2 - 2\partial_{\mathfrak{t}} x\partial_x - x\not{G} + xS^0\Gamma\left({}^{\beta}TU_3^{\otimes 2}\right)\right)
\end{aligned}$$

As before, we obtain the estimate on U_3 as the limit for $\eta \to 0$ of estimates on the relatively compact sets $U_{3,\eta} := U_3 \cap \left\{\frac{x}{1+\tilde{t}} \geq \eta\right\}$. We define the norm $\mathfrak{H}^1(U_3)$ by

$$\|u\|^2_{\mathfrak{H}^1(U_3)} = \int_{U_3} x\mathbb{m}(\partial_{\mathfrak{t}} u) + x^2\mathbb{m}(\partial_x u) + x\not{G}^{i,j}\mathbb{m}(\Theta_i u, \Theta_j u) + \mathbb{m}(u)\frac{\mathrm{d}x}{x}\,\mathrm{d}\tilde{\mathfrak{t}}\,\mathrm{d}^2\omega$$

[3]Note that the estimates of Lemma C.1.2 and Lemma C.0.1 do not impose any restriction on α, and we can apply them with $\alpha = \alpha_k$.

the norm $\mathfrak{H}^1(U_{3,\eta})$ with the same expression but integration on $U_{3,\eta}$. For $(A_i)_{i=1}^3$ a set of vector fields spanning the $C^\infty(\mathbb{S}^2)$ module of vector fields on $\mathbb{S}^2$, we define:

$$\|u\|^2_{\mathfrak{H}^{1,k}(U_3)} = \|u\|^2_{H_b^k(U_3)} + \sum_{i=1}^3 \left\|x^{\frac{1}{2}}\Theta_{A_i}u\right\|^2_{H_b^k(U_3)} + \left\|x^{\frac{1}{2}}\partial_{\tilde{\mathfrak{t}}}u\right\|^2_{H_b^k(U_3)} + \|x\partial_x u\|^2_{H_b^k(U_3)}$$

Let $u \in H^{\tilde{r}+1}_{loc}(\mathcal{M}_{2\epsilon})$ with $x^{-a_I}T_s xu \in \overline{H}^{\tilde{r}}_b$. At first, we assume that $u = 0$ on $\left\{\tilde{t} \leq \frac{3}{8}\right\}$. As before, for energy estimates on relatively compact sets, we can assume that u is smooth. We prove that for A large enough:

$$\left\|x^{-a_I}e^{-\frac{A\tilde{\mathfrak{t}}}{2}}\partial_{\tilde{\mathfrak{t}}}u\right\|^2_{L^2_b(U_{3,\eta})} + A\left\|x^{-a_I}e^{-\frac{A\tilde{\mathfrak{t}}}{2}}u\right\|^2_{\mathfrak{H}^1(U_3)} \leq C\left\|e^{-\frac{A\tilde{\mathfrak{t}}}{2}}x^{-a_I}T_s xu\right\|^2_{L^2_b(U_3)}.$$

Then, we prove by induction on k that for $u \in H^{k+1}_{(loc)}(\mathcal{M}_+)$ with $x^{-a_I}T_s xu \in H^k_b(U_3)$:

$$\left\|x^{-a_I}e^{-\frac{A\tilde{\mathfrak{t}}}{2}}\mathbb{m}(\partial_{\tilde{\mathfrak{t}}}u)\right\|^2_{H_b^k(U_{3,\eta})} + A\left\|x^{-a_I}e^{-\frac{A\tilde{\mathfrak{t}}}{2}}u\right\|^2_{\mathfrak{H}^{1,k}(U_T)} \leq C\left\|e^{-\frac{A\tilde{\mathfrak{t}}}{2}}x^{-a_I}T_s xu\right\|^2_{H_b^k(U_3)}.$$

We define the energy momentum tensor as follows:

$$T^{\delta,\gamma}(u) = \Re(\mathbb{m}(\Theta_\mu u, \Theta_\nu u))\tilde{G}^{\mu,\delta}\tilde{G}^{\nu,\gamma} - \frac{1}{2}\tilde{G}^{\delta,\gamma}\tilde{G}^{\mu,\nu}\mathbb{m}(\Theta_\mu u, \Theta_\nu u) + \frac{1}{2}\tilde{G}^{\delta,\gamma}x^{-1}\mathbb{m}(u)$$

We define the vector fields:

$$\begin{aligned} V :=& (-\partial_{\tilde{\mathfrak{t}}} + x\partial_x) \in \Gamma({}^{\mathscr{I}}TU_3) \\ W :=& e^{-A\tilde{\mathfrak{t}}}x^{-2a_I}V \end{aligned}$$

Then we define the energy one-form:

$$J^W_\mu := T_{\nu,\mu}W^\mu$$

The divergence is given by:

$$K^W(u) = \mathrm{div}(T)_\mu W^\mu + T_{\nu,\mu}\nabla^\nu W^\mu$$

Moreover, we have (see Proposition A.0.4 and lemma 10.2.1):

$$\begin{aligned} \mathrm{div}(T)_\mu W^\mu :=& e^{-A\tilde{\mathfrak{t}}}x^{-2a_I}\Big(\Re(\mathbb{m}(\Theta_V u, \Box_{\tilde{g},\Theta}u)) + V^\mu\Re(\tilde{G}^{\gamma,\nu}\mathbb{m}(R^\Theta_{\gamma,\mu}u, \Theta_\nu u)) \\ & -\frac{x^{-1}}{2}\mathbb{m}(u,u) + \Re(x^{-1}\mathbb{m}(\Theta_V u, u))\Big) \end{aligned}$$

Using that $R^\Theta(\partial_x, .) = R^\Theta(\partial_{\tilde{\mathfrak{t}}}, .) = 0$ (from the definition of Θ) and Cauchy-Schwarz estimates, we get for all $\epsilon > 0$, there exists $C_\epsilon > 0$ (independent of A and u) such that:

$$\begin{aligned} |\mathrm{div}(T)_\mu W^\mu| \leq& e^{-A\tilde{\mathfrak{t}}}x^{-2a_I-1}\Big(\epsilon\mathbb{m}(\partial_{\tilde{\mathfrak{t}}}u) + C_\epsilon x^2\mathbb{m}(\partial_x u) + C_\epsilon x\not{G}^{i,j}\mathbb{m}(\Theta_i u, \Theta_j u) + C_\epsilon\mathbb{m}(u) \\ & + C_\epsilon\mathbb{m}(x\Box_{\tilde{g},\Theta}u)\Big) \end{aligned}$$

where C depends only on the metric. In the sequel, we use this estimate with $\epsilon = -a_I > 0$.

In order to replace $\Box_{\tilde{g},\Theta}$ by $x^{-1}T_s x$ in the estimate (allowing the constant C to change), we check that $\Box_{\tilde{g},\Theta} - x^{-1}T_s x \in S^0\mathrm{Diff}^1_{\mathscr{I},b}$. We compute (using trivialisation $\mathcal{T}_m$):

$$\begin{aligned}
J =& \frac{\sin\theta}{1+x^2a^2\cos^2\theta} = \sin(\theta)(1+O(x^2))\\
\Box_{\tilde{g},\Theta} - x^{-1}T_s x =& J^{-1}\Theta_\mu J\tilde{G}^{\mu,\nu}\Theta_\nu - x^{-1}T_s x\\
=& \tilde{G}^{\mu,\nu}\partial_\mu\partial_\nu + J^{-1}\partial_\mu(J\tilde{G}^{\mu,\nu})\partial_\nu + is\frac{\cos\theta}{\sin^2\theta}\partial_\phi - s^2\mathrm{cotan}^2\theta - x^{-1}T_s x\\
=& \left(s + xC^\infty\left(\left[0,\frac{1}{r_+-\epsilon}\right]_x\times\mathbb{S}^2\right)\right)\\
&+\left(2s(2M-ia\cos\theta) + xC^\infty\left(\left[0,\frac{1}{r_+-\epsilon}\right]_x\times\mathbb{S}^2\right)\right)\partial_{\tilde{\mathfrak{t}}}\\
&\left(2s + xC^\infty\left(\left[0,\frac{1}{r_+-\epsilon}\right]_x\times\mathbb{S}^2\right)\right)x\partial_x\\
&+\left(2as + xC^\infty\left(\left[0,\frac{1}{r_+-\epsilon}\right]_x\times\mathbb{S}^2\right)\right)x\partial_\phi - \frac{2a^2x^2\sin(\theta)\cos(\theta)}{a^2x^2\cos^2(\theta)+1}\partial_\theta
\end{aligned}$$

Moreover, we have:

$$\begin{aligned}
T_{\nu,\mu}\nabla^\nu W^\mu =& \frac{1}{2}\left(\tilde{G}^{\mu,\alpha}\nabla_\alpha W^\nu + \tilde{G}^{\nu,\alpha}\nabla_\alpha W^\mu\right)\Re\mathbb{m}(\Theta_\mu u,\Theta_\nu u)\\
&-\frac{1}{4}\tilde{g}_{\mu,\nu}\left(\tilde{G}^{\mu,\alpha}\nabla_\alpha W^\nu + \tilde{G}^{\nu,\alpha}\nabla_\alpha W^\mu\right)\left(\tilde{G}^{\kappa,\delta}\Re\mathbb{m}(\Theta_\kappa u,\Theta_\delta u) - x^{-1}\mathbb{m}(u)\right)
\end{aligned}$$

Using the defintion of the Levi-Civita connection, we get:

$$\pi^{\mu,\nu} := \frac{1}{2}\left(\tilde{G}^{\mu,\alpha}\nabla_\alpha W^\nu + \tilde{G}^{\alpha,\nu}\nabla_\alpha W^\mu\right) = \frac{1}{2}\left(\tilde{G}^{\mu,\alpha}\partial_\alpha W^\nu + \tilde{G}^{\alpha,\nu}\partial_\alpha W^\mu - W^\alpha\partial_\alpha\tilde{G}^{\mu,\nu}\right)$$

We get the following intermediary computation results:

$$\begin{aligned}
\tilde{G}^{x,\alpha}\partial_\alpha W^x =& Ax^{-2a_I+1}e^{-A\tilde{\mathfrak{t}}} + Ae^{-A\tilde{\mathfrak{t}}}O(x^{-2a_I+2})\\
\tilde{G}^{\tilde{\mathfrak{t}},\alpha}\partial_\alpha W^x =& -(-2a_I+1)x^{-2a_I}e^{-A\tilde{\mathfrak{t}}} + Ae^{-A\tilde{\mathfrak{t}}}O(x^{-2a_I+1})\\
\tilde{G}^{x,\alpha}\partial_\alpha W^{\tilde{\mathfrak{t}}} =& -Ax^{-2a_I}e^{-A\tilde{\mathfrak{t}}} + Ae^{-A\tilde{\mathfrak{t}}}O(x^{-2a_I+1})\\
\tilde{G}^{\tilde{\mathfrak{t}},\alpha}\partial_\alpha W^{\tilde{\mathfrak{t}}} =& (-2a_I + (2M^2-a^2\sin^2\theta)Ax)x^{-2a_I-1}e^{-A\tilde{\mathfrak{t}}} + e^{-A\tilde{\mathfrak{t}}}O(x^{-2a_I} + Ax^{-2a_I+1})
\end{aligned}$$

All the other terms of this form only contribute to the error part (i.e have a decay as a component of a section of $Ae^{-A\tilde{\mathfrak{t}}}x^{-2a_I}{}^{\mathscr{I}}TU_3\otimes{}^{\mathscr{I}}TU_3$). Note that $W^\alpha\partial_\alpha\tilde{G}^{\mu,\nu}$ belongs to $e^{-A\tilde{\mathfrak{t}}}x^{-2a_I}S^0\Gamma({}^{\mathscr{I}}TU_3\otimes{}^{\mathscr{I}}TU_3)$ and is therefore an error term. We compute:

$$\tilde{g}_{\mu,\nu}\pi^{\mu,\nu} = (A-2a_I+1)x^{-2a_I}e^{-A\tilde{\mathfrak{t}}} + Ae^{-A\tilde{\mathfrak{t}}}O(x^{-2a_I+1})$$

We conclude:

$$\begin{aligned}
T_{\nu,\mu}\nabla^\nu W^\mu =& x^{-2a_I-1}e^{-A\tilde{\mathfrak{t}}}\left(\left(-2a_I + Ax\frac{2M^2-a^2\sin^2\theta}{2}\right)\mathbb{m}(\partial_{\tilde{\mathfrak{t}}}u) + Ax^2\mathbb{m}(\partial_x u)\right.\\
&\left.+\frac{1}{2}(A-2a_I+1)\mathbb{G}^{\omega_i,\omega_j}x\mathbb{m}\left(\Theta_{\omega_i}u,\Theta_{\omega_j}u\right) + \frac{1}{2}(A-2a_I+1)\mathbb{m}(u)\right) + R(\tilde{\mathfrak{t}},x,\omega)\\
\left|R(\tilde{\mathfrak{t}},x,\omega)\right| \leq& Ax^{-2a_I}e^{-A\tilde{\mathfrak{t}}}\frac{2M^2-a^2\sin^2\theta}{4}\mathbb{m}(\partial_{\tilde{\mathfrak{t}}}u) + Cx^{-2a_I}e^{-A\tilde{\mathfrak{t}}}\mathbb{m}(\partial_{\tilde{\mathfrak{t}}}u)\\
&+Cx^{-2a_I}Ae^{-A\tilde{\mathfrak{t}}}\left(x\mathbb{m}(\partial_{\tilde{\mathfrak{t}}}u) + x^2\mathbb{m}(\partial_x u) + \mathbb{G}^{\omega_i,\omega_j}x\mathbb{m}\left(\Theta_{\omega_i}u,\Theta_{\omega_j}u\right) + \mathbb{m}(u)\right)
\end{aligned}$$

where C depends on a_I and on the bounds of the metric coefficients (and their derivatives) but (crucially) not on A (nor on T since $\tilde{G}$ is stationary). The first term is obtained when using the Cauchy-Schwarz inequality to bound (uniformly with respect to A):

$$Ax^{-2a_I}e^{-A\tilde{\mathfrak{t}}}O(1)\left|\mathbb{m}(x\partial_x u,\partial_{\tilde{\mathfrak{t}}}u)\right|\leq Ax^{-2a_I}e^{-A\tilde{\mathfrak{t}}}\frac{2M^2-a^2\sin^2\theta}{4}\mathbb{m}(\partial_{\tilde{\mathfrak{t}}}u)+CAe^{-A\tilde{\mathfrak{t}}}x^{-2a_I}x^2\mathbb{m}(\partial_x u).$$

We can find $\epsilon>0$ and $A_0>0$ (both independent of u) such that there exists a constant $C>0$ such that for all $A>A_0$ the following inequality holds on $\{x\leq\epsilon\}$:

$$\begin{aligned}K_W\geq&\frac{x^{-2a_I-1}e^{-A\tilde{\mathfrak{t}}}}{2}\left(\left(-a_I+Ax\frac{2M^2-a^2\sin^2\theta}{4}\right)\mathbb{m}(\Theta_{\tilde{\mathfrak{t}}}u)+\frac{A}{2}x^2\mathbb{m}(\Theta_x u)\right.\\&\left.+\frac{1}{4}(A-2a_I+1)\not{G}^{\omega_i,\omega_j}x\mathbb{m}\left(\Theta_{\omega_i}u,\Theta_{\omega_j}u\right)+\frac{1}{4}(A-2a_I+1)\mathbb{m}(u)-C\mathbb{m}(T_s xu)\right)\end{aligned}\tag{C.12}$$

and, even restricting ϵ, we can require that V is uniformly timelike on $\left\{\frac{\epsilon}{2}\leq x\leq\epsilon\right\}$

This is the part of the estimate near $x=0$. We can modify V on $\{x>\epsilon\}$ so that it is uniformly timelike (past oriented) on $\{x>\epsilon\}$, smooth and stationary. From now on, V denotes such a modified vector field. Note that on $\{x>\epsilon\}$, we have:

$$\left|\mathrm{div}(T)_\mu W^\mu\right|\leq Cx^{-2a_I}e^{-A\tilde{\mathfrak{t}}}\left(\mathbb{m}(\Box_{\tilde{g},\Theta}u))+\mathbb{m}(\partial_{\tilde{\mathfrak{t}}}u)+\mathbb{m}(\partial_x u)+\not{G}^{\omega_i,\omega_j}\mathbb{m}(\Theta_{\omega_i}u,\Theta_{\omega_j}u)+\mathbb{m}(u)\right)$$

for some constant C (independent of A and u).

$$\begin{aligned}T_{\nu,\mu}\nabla^\nu W^\mu=&T_{\nu,\mu}V^\mu\nabla^\nu\left(x^{-2a_I}e^{-A\tilde{\mathfrak{t}}}\right)+x^{-2a_I}e^{-A\tilde{\mathfrak{t}}}T_{\mu,\nu}\nabla^\nu V^\mu\\\left\|T_{\mu,\nu}\nabla^\nu V^\mu\right\|\leq&C\left(\mathbb{m}(\partial_{\tilde{\mathfrak{t}}}u)+\mathbb{m}(\partial_x u)+\not{G}^{\omega_i,\omega_j}\mathbb{m}(\Theta_{\omega_i}u,\Theta_{\omega_j}u)+\mathbb{m}(u)\right)\end{aligned}$$

Since V and $\frac{-2a_I\nabla x}{Ax}-\nabla\tilde{\mathfrak{t}}$ are uniformly timelike (for A large enough) on $\{x>\epsilon\}$, we get as usual in hyperbolic estimates:

$$T_{\nu,\mu}V^\mu\nabla^\nu\left(x^{-2a_I}e^{-A\tilde{\mathfrak{t}}}\right)\geq C_\epsilon Ae^{-At}\left(\mathbb{m}(\partial_{\tilde{\mathfrak{t}}}u)+\mathbb{m}(\partial_x u)+\not{G}^{\omega_i,\omega_j}\mathbb{m}(\Theta_i u,\Theta_j u)+\mathbb{m}(u)\right)$$

Therefore, if we choose A large enough, we have the following inequality on $\{x>\epsilon\}$:

$$K_W\geq Ce^{-A\tilde{\mathfrak{t}}}\left((A\mathbb{m}(\partial_{\tilde{\mathfrak{t}}}u)+A\mathbb{m}(\partial_x u)+A\not{G}^{\omega_i,\omega_j}\mathbb{m}(\Theta_{\omega_i}u,\Theta_{\omega_j}u)+A\mathbb{m}(u)-\mathbb{m}(T_s xu)\right)$$

with $C>0$ independent of u, $\tilde{\mathfrak{t}}$ and A. Combining this estimate with (C.12), we get that there exists a constant $C>0$ independent of A (as long as it is large enough) and u such that for all $(\tilde{\mathfrak{t}},x,\omega)$:

$$K_W\geq Cx^{-2a_I-1}e^{-A\tilde{\mathfrak{t}}}\left((1+Ax)\mathbb{m}(\partial_{\tilde{\mathfrak{t}}}u)+Ax^2\mathbb{m}(\partial_x u)+Ax\not{G}^{\omega_i,\omega_j}\mathbb{m}(\Theta_{\omega_i}u,\Theta_{\omega_j}u)+A\mathbb{m}(u)-\mathbb{m}(T_s xu)\right)$$

We can then apply the Stokes formula on the domain $U_{3,\eta}$ for η so small that $\tilde{G}(\mathrm{d}\frac{x}{1+\tilde{\mathfrak{t}}})=\frac{2x}{(1+\tilde{\mathfrak{t}})^3}+O\left(\frac{x^2}{(1+\tilde{\mathfrak{t}})^2}\right)>0$ on the boundary face $\left\{\frac{x}{1+\tilde{\mathfrak{t}}}=\eta,\frac{1}{4}\leq\tilde{\mathfrak{t}}\leq T\right\}$. Therefore, the outward normal is causal future oriented on each boundary face of $U_{3,\eta}$ except at the face $\left\{\tilde{\mathfrak{t}}=\frac{1}{4},\frac{x}{1+\tilde{\mathfrak{t}}}\geq\eta\right\}$ but $u=0$ near this face. Therefore, we get (for some constant $C>0$ independent of u and T):

$$\left\|x^{-a_I}e^{-\frac{A\tilde{\mathfrak{t}}}{2}}\partial_{\tilde{\mathfrak{t}}}u\right\|^2_{L^2_b(U_{3,\eta})}+A\left\|x^{-a_I}e^{-\frac{A\tilde{\mathfrak{t}}}{2}}u\right\|_{\mathfrak{H}^1(U_{3,\eta})}\leq C\left(\left\|e^{-\frac{A\tilde{\mathfrak{t}}}{2}}x^{-a_I}T_s xu\right\|_{L^2_b(U_3)}\right)$$

By the monotone convergence theorem we can take $\eta \to 0$ and this concludes the first part of the estimate.

We now prove the higher regularity statement for $k > 0$ by induction (on $U_{3,\eta}$ and then we take the limit $\eta \to 0$ as before). We assume that $u \in H^{k+1}_{loc}(\mathcal{M}_{2\epsilon})$ with $x^{-a_I} T_s x u \in H^k_{loc}(\mathcal{M}_{2\epsilon})$ and that the high regularity estimate holds at order $k-1$. As before, we also assume that $\tilde{\mathfrak{t}} \geq \frac{3}{8}$ on $\mathrm{supp}(u)$. We introduce the set of commutators $\mathcal{C} := \left\{ Z_0 := \partial_{\tilde{\mathfrak{t}}}, Z_1 := \tilde{\mathfrak{t}}\partial_{\tilde{\mathfrak{t}}} - x\partial_x, Z_2, Z_3, Z_4 \right\}$ where $Z_2, Z_3, Z_4 \in \mathrm{Diff}^1(\mathcal{B}_s)$ are generators of the $C^\infty(\mathbb{S}^2)$ module $\mathrm{Diff}^1(\mathcal{B}_s)$ modulo $\mathrm{Diff}^0(\mathcal{B}_s)$. We consider $v_i = Z_i u$ which satisfy the induction hypothesis. Therefore, we have:

$$\left\| x^{-a_I} e^{-\frac{A\tilde{\mathfrak{t}}}{2}} \partial_{\tilde{\mathfrak{t}}} v_i \right\|^2_{H^{k-1}_b(U_{3,\eta})} + A \left\| x^{-a_I} e^{-\frac{A\tilde{\mathfrak{t}}}{2}} v_i \right\|^2_{\mathfrak{H}^{1,k-1}(U_{3,\eta})} \leq C \left(\left\| e^{-\frac{A\tilde{\mathfrak{t}}}{2}} x^{-a_I} T_s x v_i \right\|^2_{H^{k-1}_b(U_{3,\eta})} \right).$$

We also have the estimate of order $k-1$ on u instead of v_i:

$$\left\| x^{-a_I} e^{-\frac{A\tilde{\mathfrak{t}}}{2}} \partial_{\tilde{\mathfrak{t}}} u \right\|^2_{H^{k-1}_b(U_{3,\eta})} + A \left\| x^{-a_I} e^{-\frac{A\tilde{\mathfrak{t}}}{2}} u \right\|^2_{\mathfrak{H}^{1,k-1}(U_{3,\eta})} \leq C \left(\left\| e^{-\frac{A\tilde{\mathfrak{t}}}{2}} x^{-a_I} T_s x u \right\|^2_{H^{k-1}_b(U_{3,\eta})} \right).$$

Note that for all $0 \leq i \leq 4$, $[x^{-1} T_s x, Z_i] \in \mathrm{Diff}^2_b$ and therefore we get:

$$\begin{aligned} \left\| x^{-a_I} e^{-\frac{A\tilde{\mathfrak{t}}}{2}} \partial_{\tilde{\mathfrak{t}}} v_i \right\|^2_{H^{k-1}_b(U_{3,\eta})} + A \left\| x^{-a_I} e^{-\frac{A\tilde{\mathfrak{t}}}{2}} v_i \right\|^2_{\mathfrak{H}^{1,k-1}(U_{3,\eta})} \leq C \Bigg(& \left\| e^{-\frac{A\tilde{\mathfrak{t}}}{2}} Z_i x^{-a_I} T_s x u \right\|^2_{H^{k-1}_b(U_{3,\eta})} \\ & + \left\| e^{-\frac{A\tilde{\mathfrak{t}}}{2}} x^{-a_I+1} u \right\|^2_{H^{k+1}_b(U_{3,\eta})} \Bigg) \end{aligned}$$

Considering the definition of $H^{1,k}_{\mathscr{I},b}(U_3)$, we deduce that $\left\| x^{-a_I+1} u \right\|_{H^{k+1}_b(U_{3,\eta})} \leq C \left\| x^{-a_I} u \right\|_{H^{1,k}_{\mathscr{I},b}(U_{3,\eta})}$ with C independent of η and u. Summing all the estimates (and the one on u) and taking A large enough, we find:

$$\left\| x^{-a_I} e^{-\frac{A\tilde{\mathfrak{t}}}{2}} \partial_{\tilde{\mathfrak{t}}} u \right\|^2_{H^k_b(U_{3,\eta})} + A \left\| x^{-a_I} e^{-\frac{A\tilde{\mathfrak{t}}}{2}} u \right\|^2_{\mathfrak{H}^{1,k}(U_{3,\eta})} \leq 2C \left(\left\| e^{-\frac{A\tilde{\mathfrak{t}}}{2}} x^{-a_I} T_s x u \right\|^2_{H^k_b(U_{3,\eta})} \right)$$

If we assume in addition that $x^{-a_I} T_s x u \in H^k_b(U_3)$, the right-hand side is independent of η and we can take the limit $\eta \to 0$.

Using the fact that $\frac{1}{2} \leq \tilde{\mathfrak{t}} \leq T$ on U_3, we get:

$$\|u\|^2_{x^{a_I} H^{1,k}_{\mathscr{I},b}(U_3)} \leq C e^{AT} \|T_s x u\|^2_{x^{a_I} H^k_b(U_3)}$$

where constants A and C do not depend on T (but could depend on k).

For u as defined in proposition 11.2.2, we apply the previous bound to $\chi(\tilde{\mathfrak{t}}) x^{-1} u$ where χ is a cutoff equal to 1 on $\left\{ \tilde{\mathfrak{t}} \geq \frac{1}{2} \right\}$ and equal to zero on $\left\{ \tilde{\mathfrak{t}} \leq \frac{3}{8} \right\}$. We can use estimates on K and U_2 (Lemma C.2.1 and Lemma C.0.1) to control[4] the term

$$\|[T_s, \chi] u\|_{x^{a_I} H^k_b(U_3)} \leq C \left(\|u\|_{\rho_0^{a_0} \rho_I^{a+1} H^{1,k}_{\mathscr{I},b}(U_2)} + \|u\|_{H^{k+1}(K)} \right)$$

Therefore we get:

Lemma C.3.1. *For all $k \in \mathbb{N}$, all $a_I < 0$ and $a_0 > a_I + 1$, there exists a constant $C > 0$ such that for all u as in proposition 11.2.2 with $\tilde{r} \geq k$ and all $T > \frac{1}{2}$:*

$$\|u\|^2_{x^{a_I+1} H^{1,k}_{\mathscr{I},b}(U_3)} \leq C e^{CT} \left(\|u_0\|^2_{\rho_0^{a_0} \overline{H}^{k+1}_b(\Sigma_0)} + \|u_1\|^2_{\rho_0^{a_0} \overline{H}^k_b(\Sigma_0)} \right)$$

[4] We use that $T_s \in \rho_I^{-1} S^0 \mathrm{Diff}^2_{\mathscr{I},b}(U_2)$ and that $\mathrm{supp}(\chi')$ is separated from I_0.

C.4 Proof of Proposition 11.2.2

Lemma C.4.1. *Let $\tilde{r} \in \mathbb{N}$. There exists $C > 0$ such that for all $v \in H^{1,\tilde{r}}_{\mathscr{I},b}$ and for all $T > 0$, $v \in C^0([-\frac{1}{2}, T]_{\mathfrak{t}}, \overline{H}^{\tilde{r}}_b)$ and:*

$$\sup_{\mathfrak{t}\in[-\frac{1}{2},T]} \|v(\mathfrak{t})\|_{\overline{H}^{\tilde{r}}_b} \leq C \|v\|_{H^{1,\tilde{r}}_{\mathscr{I},b}(\{-1\leq \mathfrak{t}\leq T+1\})}$$

Proof. First, we define two open sets of the form $(-1,+\infty)_{\mathfrak{t}} \times \left(0, \frac{1}{r_+ - \frac{3\epsilon}{2}}\right)_x \times U_i$ where U_i is diffeomorphic to a disc for $i \in \{1,2\}$ which cover $(-1,+\infty)_{\mathfrak{t}} \times \left(0, \frac{1}{r_+-\epsilon}\right)_x \times \mathbb{S}^2$. Using a subordinated partition of unity, local trivializations associated to local coordinates $(\mathfrak{t}, x, y_1, y_2))$ and after the change of variable $x = e^{-y_0}$, it is enough to prove the statement: if $v \in H^{\tilde{r}}(\mathbb{R}_{\mathfrak{t}} \times \mathbb{R}^3_y)$ and $\partial_{\mathfrak{t}} v \in H^{\tilde{r}}(\mathbb{R}_{\mathfrak{t}} \times \mathbb{R}^3_y)$ (we call $\mathcal{Y}$ the space of such functions), then $v \in C^0(\mathbb{R}_{\mathfrak{t}}, H^{\tilde{r}}(\mathbb{R}^3_y))$ and:

$$\sup_{\mathfrak{t}\in\mathbb{R}} \|v(\mathfrak{t})\|_{H^{\tilde{r}}(\mathbb{R}^3_y)} \leq C \left(\|v\|_{H^{\tilde{r}}(\mathbb{R}_{\mathfrak{t}}\times\mathbb{R}^3_y)} + \|\partial_{\mathfrak{t}} v\|_{H^{\tilde{r}}(\mathbb{R}_{\mathfrak{t}}\times\mathbb{R}^3_y)} \right)$$

for a constant $C > 0$ independent of v. We first assume that v is smooth and compactly supported. We denote by $\hat{v}$ the spatial Fourier transformed of v and by $\mathcal{F}(v)$ the total Fourier transformed of v. We have:

$$\begin{aligned} \|v(0)\|^2_{H^{\tilde{r}}(\mathbb{R}^3_y)} =& \int \langle \xi \rangle^{2\tilde{r}} \, |\hat{v}(0,\xi)|^2 \, \mathrm{d}\xi \\ \hat{v}(0,\xi) =& \frac{1}{2\pi} \int\int e^{-i\mathfrak{t}\sigma} \hat{v}(\mathfrak{t},\xi) \, \mathrm{d}\mathfrak{t} \, \mathrm{d}\sigma. \end{aligned}$$

Thus,

$$\begin{aligned} |\hat{v}(0,\xi)|^2 \leq& \frac{1}{4\pi^2} \int (1+\sigma^2)^{-1} \, \mathrm{d}\sigma \int (1+\sigma^2) \left| \int e^{-i\mathfrak{t}\sigma} \hat{v}(\mathfrak{t},\xi) \, \mathrm{d}\mathfrak{t} \right|^2 \, \mathrm{d}\sigma \\ \leq& \frac{1}{4\pi} \left(\int |\mathcal{F}(v)(\sigma,\xi)|^2 \, \mathrm{d}\sigma + \int |\mathcal{F}(\partial_t v)(\sigma,\xi)|^2 \, \mathrm{d}\sigma \right) \\ \|v(0)\|^2_{H^{\tilde{r}}(\mathbb{R}^3_y)} \leq& \frac{1}{4\pi} \left(\|v\|^2_{H^{\tilde{r}}(\mathbb{R}_{\mathfrak{t}}\times\mathbb{R}^3_y)} + \|\partial_t v\|^2_{H^{\tilde{r}}(\mathbb{R}_{\mathfrak{t}}\times\mathbb{R}^3_y)} \right). \end{aligned}$$

The previous computation proves that the trace on $\{\mathfrak{t} = 0\}$ extends continuously to a linear map from $\mathcal{Y}$ to $H^{\tilde{r}}(\mathbb{R}^3_y)$. We call this map γ_0. We define $\gamma_{\mathfrak{t}} v = \gamma_0(\tau_{\mathfrak{t}} v)$ where $\tau_\theta v$ is the pullback of v by the diffeomorphism $(t,y) \mapsto (t+\theta, y)$ ($\gamma_{\mathfrak{t}}$ is the continuous extension of the trace on $\{\mathfrak{t}\} \times R^3_y$). Note that since $\mathfrak{t} \mapsto \tau_{\mathfrak{t}} v$ is continuous and bounded from $\mathbb{R}$ to $\mathcal{Y}$, we get that $\mathfrak{t} \mapsto \gamma_{\mathfrak{t}} v$ is continuous and bounded from $\mathbb{R}$ to $H^{\tilde{r}}(\mathbb{R}^3_y)$. □

Proof of Proposition 11.2.2. In order to have weights defined on $\mathbf{M}_\epsilon$, we introduce

$$\begin{aligned} \tilde{\rho}_I :=& \chi_0(\tilde{\mathfrak{t}})\rho_I + (1 - \chi_0(\tilde{\mathfrak{t}}))x \\ \tilde{\rho}_0 :=& \chi_1(\rho_0)\rho_0 + (1 - \chi_1(\rho_0)) \end{aligned}$$

where $\chi_0(\tilde{\mathfrak{t}}) = 1$ when $\tilde{\mathfrak{t}} \leq -2$, $\chi_0(\tilde{\mathfrak{t}}) = 0$ when $\tilde{\mathfrak{t}} \geq -1$, $\chi_1(\rho_0) = 1$ when $\rho_0 \leq 1$ and $\chi(\rho_0) = 0$ on $\mathbf{M}_\epsilon \setminus \{\rho_0 \leq 2\}$. We define $U_4 := \left\{ t_0 \geq 0, \mathfrak{t} \leq T, x \leq \frac{1}{r_+ - \epsilon} \right\}$. Note that using:

$$\tilde{\mathfrak{t}} - \mathfrak{t} = F(r) + L(r)$$

we see that $\tilde{\mathfrak{t}} - \mathfrak{t}$ is bounded by some constant $C_0 > 0$. Therefore we have $U_4 \subset \tilde{U}_4$ where $\tilde{U}_4 := \left\{t_0 \geq 0, \tilde{\mathfrak{t}} \leq T + C_0, x \leq \frac{1}{r_+ - \epsilon}\right\}$. Combining Lemmas C.1.2, C.2.1 and C.3.1, we get that there exists $C > 0$ independent of T such that for all u as defined in proposition 11.2.2:

$$\|u\|^2_{\tilde{\rho}_I^{a_I+1}\tilde{\rho}_0^{a_0} H^{1,k}_{\not g,b}(U_4)} \leq Ce^{CT}\left(\|u_0\|^2_{\rho_0^{a_0}\overline{H}_b^{k+1}(\Sigma_0)} + \|u_1\|^2_{\rho_0^{a_0}\overline{H}_b^{k}(\Sigma_0)}\right)$$

By Lemma C.4.1, we deduce that for all $T > 0$, $x^{-a_I-1}u \in C^0\left([0,T]_{\mathfrak{t}}, \overline{H}_b^{\tilde{r}}\right)$ and there exists $C > 0$ such that:

$$\sup_{\mathfrak{t}\in[-\frac{1}{2},T]} \left\|x^{-a_I-1}u(t)\right\|_{\overline{H}_b^{\tilde{r}}} \leq Ce^{CT}\left(\|u_0\|^2_{\rho_0^{a_0}\overline{H}_b^{k+1}(\Sigma_0)} + \|u_1\|^2_{\rho_0^{a_0}\overline{H}_b^{k}(\Sigma_0)}\right)$$

If $\tilde{r} \geq k$ for $k \in \mathbb{N} \setminus \{0\}$, we can apply Lemma C.4.1 to $\partial_{\mathfrak{t}}^k u$ which is (locally with respect to $\mathfrak{t}$) in $H^{1,\tilde{r}-k}_{\not g,b}$. We deduce $u \in C^k(\mathbb{R}_{\mathfrak{t}}, \overline{H}_b^{\tilde{r}-k})$ □

C.5 Proof of Proposition 11.2.4

For $\tilde{r} \in \mathbb{N}$, $a_0, a_I \in \mathbb{R}$, we define the space $\mathcal{H}_b^{\tilde{r},0,0}$ as the space of sections of $\mathcal{B}(s,s)$ defined on $\mathcal{U}$ with support in $\{\rho_I \leq 1\}$ and with index of b-regularity (with respect to the boundary $\{\rho_I = 0\}\cup\{\rho_0 = 0\}$) equal to $\tilde{r}$. We then define $\mathcal{H}_b^{\tilde{r},a_I,a_0} := \rho_0^{a_0}\rho_I^{a_I}\mathcal{H}_b^{\tilde{r},0,0}$ and $\overline{\mathcal{H}}_b^{\tilde{r},a_I,a_0}$ the space of restriction to $\{\rho_0 \leq 4\}$ of elements of $\mathcal{H}_b^{\tilde{r},a_I,a_0}$. We take $\rho_0 \leq 4$ in the definition (instead for example of $\rho_0 \leq 1$) since at some point, we need to have a non empty intersection with $\mathfrak{t} \geq -\frac{1}{2}$ (which is the set where we have continuity of the solution in proposition 11.2.2).

Proposition C.5.1 (Elementary transport estimate)**.** *Let $\tilde{r} \in \mathbb{N}$ and $\beta > \alpha$. Let $\chi : \mathbb{R}_{\rho_I} \to \mathbb{R}$ be a smooth cutoff equal to 1 near 0 and vanishing for $\rho_I \geq 1$. For all $\nu \in (\alpha, \beta]$ and all $N \in \mathbb{N}$, there exists $C_{\nu,N} > 0$ such that for all $u \in \overline{\mathcal{H}}_b^{\tilde{r},-N,\beta}$:*

$$\|\chi u\|_{\overline{\mathcal{H}}_b^{\tilde{r},\alpha,\beta}} \leq C_{\nu,N}\left(\|u\|_{\overline{\mathcal{H}}_b^{\tilde{r},-N,\beta}} + \|(\rho_I\partial_{\rho_I} - \rho_0\partial_{\rho_0})u\|_{\overline{\mathcal{H}}_b^{\tilde{r},\nu,\beta}}\right)$$

in the strong sense that if the right-hand side is finite, $\chi u \in \overline{\mathcal{H}}_b^{\tilde{r},\alpha,\beta}$ and the inequality holds.

Proof. Let $f \in \mathcal{D} := \left\{\phi_{|_{(0,1)_{\rho_I}\times(0,4)_{\rho_0}\times\mathbb{S}^2}}, \phi \in \Gamma_c((0,1)_{\rho_I} \times (0,+\infty)_{\rho_0} \times \mathcal{B}_s)\right\}$. We consider the following map:

$$S : \begin{cases} \mathcal{D} \to \overline{\mathcal{H}}_b^{\tilde{r},\alpha,\beta} \\ f \mapsto -\int_0^{-\ln(\rho_I)} f(e^{-s}, \rho_I\rho_0 e^s)\,\mathrm{d}s \end{cases}$$

Note that we formally have $(\rho_I\partial_{\rho_I} - \rho_0\partial_{\rho_0})S(f) = f$. We have that $u : (v,\rho_0) \mapsto -\int_0^{-\ln(v)} f(e^{-s}, v\rho_0 e^s)\,\mathrm{d}s$ is well defined as a smooth section of $(0,1)_{\rho_I} \times (0,4)_{\rho_0} \times \mathcal{B}_s$ vanishing near $v = 1$. We have to prove that it belongs to $\overline{\mathcal{H}}_b^{\tilde{r},\alpha,\beta}$. By induction, we prove that for $k, j \in \mathbb{N}$ with $k + j \leq \tilde{r}$, there exists coefficients $a_{\mu,\nu}, b_\mu$ (independent of f) such that

$$(\rho_0\partial_{\rho_0})^k(\rho_I\partial_{\rho_I})^j u = \sum_{\mu+\nu\leq\tilde{r}} a_{\mu,\nu}(\rho_I\partial_{\rho_I})^\mu(\rho_0\partial_{\rho_0})^\nu f + \sum_{\mu=0}^{\tilde{r}} b_\mu \int_0^{-\ln(\rho_I)} ((x_2\partial_{x_2})^\mu f)(e^{-s}, \rho_I\rho_0 e^s)\,\mathrm{d}s$$

(where we have called x_2 the second variable of f). The $\overline{\mathcal{H}}_b^{0,\alpha,\beta}$-norm of the first sum in the right hand side is bounded by $C\|f\|_{\overline{\mathcal{H}}_b^{\tilde{r},\alpha,\beta}}$. Moreover we have:

$$\begin{aligned} I &:= \int_0^1\int_0^4 \rho_I^{-2\alpha}\rho_0^{-2\beta}\left|\int_0^{-\ln(\rho_I)}((x_2\partial_{x_2})^\mu f)(e^{-s},\rho_I\rho_0 e^s)\,\mathrm{d}s\right|^2 \frac{\mathrm{d}\rho_0}{\rho_0}\frac{\mathrm{d}\rho_I}{\rho_I} \\ &\leq \int_0^1\int_0^4 \rho_I^{-2\alpha}\rho_0^{-2\beta}\,|\ln(\rho_I)|\int_0^{-\ln(\rho_I)}\left|((x_2\partial_{x_2})^\mu f)(e^{-s},\rho_I\rho_0 e^s)\right|^2\,\mathrm{d}s\frac{\mathrm{d}\rho_0}{\rho_0}\frac{\mathrm{d}\rho_I}{\rho_I} \end{aligned}$$

By the change of variable $(w,z,\rho_I)=(e^{-s},\rho_I\rho_0 e^s,\rho_I)$, we find:

$$\begin{aligned} I &\leq \int_0^1\int_{\rho_I}^1\int_0^{4\frac{\rho_I}{w}} |\ln(\rho_I)|\,\rho_I^{2(\beta-\alpha)}z^{-2\beta}w^{-2\beta}\,|((x_2\partial_{x_2})^\mu f)(w,z)|^2\,\frac{\mathrm{d}z}{z}\frac{\mathrm{d}w}{w}\frac{\mathrm{d}\rho_I}{\rho_I} \\ &\leq \int_{(0,1)}|\ln(\rho_I)|\,\rho_I^{2\eta}\int_{(w,z)\in(0,1)\times(0,4)} z^{-2\beta}w^{-2\alpha-2\eta}\,|((x_2\partial_{x_2})^\mu f)(w,z)|^2\,\frac{\mathrm{d}z}{z}\frac{\mathrm{d}w}{w}\frac{\mathrm{d}\rho_I}{\rho_I} \end{aligned}$$

which holds for all $\eta\in(0,\beta-\alpha]$. We deduce that for all $\nu\in(\alpha,\beta]$ there exists $C_\nu>0$ such that:

$$I\leq C_\nu\|f\|_{\overline{\mathcal{H}}_b^{\tilde{r},\nu,\beta}}$$

Finally, we get $\|S(f)\|_{\overline{\mathcal{H}}_b^{\tilde{r},\alpha,\beta}} \leq C\|f\|_{\overline{\mathcal{H}}_b^{\tilde{r},\nu,\beta}}$. By density of $\mathcal{D}$ in $\overline{\mathcal{H}}_b^{\tilde{r},\nu,\beta}$, we get that S extends uniquely as a continuous linear map from $\overline{\mathcal{H}}_b^{\tilde{r},\nu,\beta}$ to $\overline{\mathcal{H}}_b^{\tilde{r},\alpha,\beta}$. Moreover, since $(\rho_I\partial_{\rho_I}-\rho_0\partial_{\rho_0})S(f)=f$ for $f\in\mathcal{D}$, the relation stays true in the sense of distributions for $f\in\overline{\mathcal{H}}_b^{\tilde{r},\alpha,\beta}$. Let $u\in\rho_I^{-N}\rho_0^\beta H_{b,loc}^{\tilde{r}}(\mathcal{U})$ such that $(\rho_I\partial_{\rho_I}-\rho_0\partial_{\rho_0})u\in\rho_I^\nu\rho_0^\beta H_{b,loc}^{\tilde{r}}(\mathcal{U})$. For χ as in the statement, we have $(\rho_I\partial_{\rho_I}-\rho_0\partial_{\rho_0})\chi(\rho_I)u=\chi(\rho_I)(\rho_I\partial_{\rho_I}-\rho_0\partial_{\rho_0})u+\rho_I\chi'(\rho_I)u$ and using that $(\rho_I\partial_{\rho_I}-\rho_0\partial_{\rho_0})$ has no kernel in $\overline{\mathcal{H}}_b^{\tilde{r},-\infty,-\infty}$ (since all the integral curves of $\rho_I\partial_{\rho_I}-\rho_0\partial_{\rho_0}$ starting in $\{\rho_I\leq 1,\rho_0\leq 4\}$ intersect transversally the boundary $\rho_I=1$), we deduce $S(\chi(\rho_I)(\rho_I\partial_{\rho_I}-\rho_0\partial_{\rho_0})u+\rho_I\chi'(\rho_I)u)=\chi(\rho_I)u$ on $\{\rho_I\leq 1,\rho_0\leq 4\}$. In particular:

$$\|\chi(\rho_I)u\|_{\overline{\mathcal{H}}_b^{\tilde{r},\alpha,\beta}}\leq C\left(\|\chi(\rho_I)(\rho_I\partial_{\rho_I}-\rho_0\partial_{\rho_0})u\|_{\overline{\mathcal{H}}_b^{\tilde{r},\nu,\beta}}+\left\|\rho_I\chi'(\rho_I)u\right\|_{\overline{\mathcal{H}}_b^{\tilde{r},\nu,\beta}}\right)$$

Looking at the supports, it implies:

$$\|\chi(\rho_I)u\|_{\overline{\mathcal{H}}_b^{\tilde{r},\alpha,\beta}}\leq C\left(\|(\rho_I\partial_{\rho_I}-\rho_0\partial_{\rho_0})u\|_{\overline{\mathcal{H}}_b^{\tilde{r},\nu,\beta}}+\|u\|_{\overline{\mathcal{H}}_b^{\tilde{r},-\infty,\beta}}\right)$$

□

Proof of Proposition 11.2.4. We first use Proposition 11.2.2 with $a_0-2<a_I<0$ (by hypothesis, $a_0=1+\alpha<2$). We exploit the fact that $T_s-\frac{2}{\rho_I}(\rho_I\partial_{\rho_I}-\rho_0\partial_{\rho_0})(\rho_I\partial_{\rho_I}-1)\in\mathrm{Diff}_b^2$. Since $T_s u=0$, we get:

$$(\rho_I\partial_{\rho_I}-\rho_0\partial_{\rho_0})(\rho_I\partial_{\rho_I}-1)u\in\rho_I^{a_I+2}\rho_0^{a_0}H_b^{\tilde{r}-2}(U)$$

We apply Lemma C.5.1 and get $w:=(\rho_I\partial_{\rho_I}-1)\chi(\rho_I)u\in\overline{\mathcal{H}}_b^{\tilde{r}-2,a_0-,a_0}$ where $\chi\in C^\infty([0,+\infty),[0,1])$ is a smooth cutoff equal to 1 near zero and with support in $[0,1)$. We can use the Mellin transform with respect to ρ_I which sends $\overline{\mathcal{H}}_b^{\tilde{r}-2,a_0-,a_0}$ to a space of holomorphic functions from $\{\Im(\lambda)>-a_0+\}$ to $\rho_0^{a_0}\overline{H}_b^{\tilde{r}-2}([0,4]_{\rho_0}\times\mathcal{B}_s)$ (where in this last space the sections are extendible at $\rho_0=4$ and have b-regularity $\tilde{r}-2$ at $\rho_0=0$). Therefore:

$$(i\lambda-1)\mathcal{M}(\chi u)(\lambda)=g(\lambda)$$

where g is holomorphic on $\{\Im(\lambda) > -1-\alpha+\}$.Therefore, $\mathcal{M}(\chi u)(\lambda)$ is meromorphic on the same set with only one possible simple pole at $\lambda = -i$. Moreover, by Plancherel, we have the following bound for $\epsilon > 0$:

$$\sum_{j=0}^{\tilde{r}-2} \int_{\mathbb{R}+i(\epsilon-a_0)} \Re(\lambda)^{2j} \left\| g(\lambda) \right\|^2_{\rho_0^{a_0} \overline{H}_b^{\tilde{r}-2-j}([0,4]_{\rho_0}\times\mathcal{B}_s)} \, \mathrm{d}\lambda \leq$$
$$\sum_{j=0}^{\tilde{r}-2} \int_0^1 \left\| \rho_I^{-a_0+\epsilon} (\rho_I \partial_{\rho_I})^j w(\rho_I) \right\|^2_{\rho_0^{a_0} \overline{H}_b^{\tilde{r}-2-j}([0,4]_{\rho_0}\times\mathcal{B}_s)} \frac{\mathrm{d}\rho_I}{\rho_I}$$
$$\leq C \left\| w \right\|_{\overline{\mathcal{H}}_b^{\tilde{r}-2,a_0-\epsilon,a_0}} .$$

We can also get the following bound by the Cauchy Schwarz inequality (for $\lambda \in \mathbb{R}+i(\epsilon-a_0)$):

$$\begin{aligned} \left\| g(\lambda) \right\|^2_{\rho_0^{a_0} \overline{H}_b^{\tilde{r}-2}([0,4]_{\rho_0}\times\mathcal{B}_s)} &\leq \int_0^1 \rho_I^{\epsilon} \left\| \rho_I^{-a_0+\frac{\epsilon}{2}} w(\rho_I) \right\|^2_{\rho_0^{a_0} \overline{H}_b^{\tilde{r}-2}([0,4]_{\rho_0}\times\mathcal{B}_s)} \frac{\mathrm{d}\rho_I}{\rho_I} \\ &\leq \frac{C}{\epsilon} \left\| w \right\|^2_{\overline{\mathcal{H}}_b^{\tilde{r}-2,a_0-\frac{\epsilon}{2},a_0}} \end{aligned}$$

We deduce that we can use a contour deformation argument to get (the limits are understood in the sense of distributions):

$$\begin{aligned} \chi u &= \lim_{M\to+\infty} \frac{1}{2\pi} \int_{[-M,M]+i(\epsilon-a_I)} \rho_I^{i\lambda} \frac{g(\lambda)}{i\lambda - 1} \, \mathrm{d}\lambda \\ &= \lim_{M\to+\infty} \frac{1}{2\pi} \int_{[-M,M]+i(\epsilon-1-\alpha)} \rho_I^{i\lambda} \frac{g(\lambda)}{i\lambda - 1} \, \mathrm{d}\lambda + g(-i)\rho_I \end{aligned}$$

Therefore, $\chi u = \chi(\rho_I) g(-i) \rho_I + \overline{\mathcal{H}}_b^{\tilde{r}-2,1+\alpha-\epsilon,a_0}$. We now investigate the dependency of $g(-i)$ with respect to ρ_0. Using Lemma 11.2.2, for $k \in \mathbb{N}$ with $k \leq \tilde{r}-2$ and the fact[5] that $\rho_0 = -\frac{1}{\mathfrak{t}}$, we deduce that $\rho_0 \mapsto w(\rho_0)$ belongs to $C^k([2,4]_{\rho_0}, \overline{H}_b^{\tilde{r}-2-k,1+\alpha-})$.

$$g(-i)(\rho_0) = \int_0^{+\infty} \rho_I^{-1} w(\rho_0, \rho_I) \frac{\mathrm{d}\rho_I}{\rho_I}$$

we deduce $g(-i) \in C^k\left([2,4]_{\rho_0}, H^{\tilde{r}-2-k}(\mathcal{B}_s)\right)$ and therefore $\chi u - \chi(\rho_I) g(-i)\rho_I \in C^k\left([2,4]_{\rho_0}, \overline{H}_b^{\tilde{r}-2-k,1+\alpha-}\right)$. We define $\chi_1 \in C^\infty(\mathbb{R}_{\mathfrak{t}}, [0,1])$ equal to 1 near $+\infty$ and such that $\mathrm{supp}(\chi_1') \subset [\mathfrak{t}_0, \mathfrak{t}_1] \subset (-\frac{1}{2}, -\frac{1}{4})$. Then if we define $v := \chi_1(\mathfrak{t}) u$ we get $v \in C^k(\mathbb{R}_{\mathfrak{t}}, \overline{H}_b^{\tilde{r}-2-k,1-})$. Moreover,

$$\begin{aligned} T_s v =& [T_s, \chi_1] u \\ =& \left(2a_{\mathfrak{t},\mathfrak{t}}\chi_1' \partial_{\mathfrak{t}} + a_{\mathfrak{t},\varphi}\chi_1' \partial_\varphi + a_{\mathfrak{t},r}\chi_1' \partial_r + a_{\mathfrak{t}}\chi_1' + a_{\mathfrak{t},\mathfrak{t}}\chi_1''\right) u \end{aligned}$$

Since $2a_{\mathfrak{t},\mathfrak{t}}\chi_1' \partial_{\mathfrak{t}} + a_{\mathfrak{t},\varphi}\chi_1' \partial_\varphi + a_{\mathfrak{t},r}\chi_1' \partial_r + a_{\mathfrak{t}}\chi_1' + a_{\mathfrak{t},\mathfrak{t}}\chi_1'' = 2x^{-1}\chi_1' \left(-x\partial_x + 1\right) + \mathrm{Diff}_b^1$, and $2x^{-1}\chi_1' \left(-x\partial_x + 1\right) g(-i)\rho_I = 0$ we get $T_s v \in C^{k-1}(\mathbb{R}_{\mathfrak{t}}, \overline{H}_b^{\tilde{r}-3-k,\alpha-})$ which conclude the proof of proposition 11.2.4. □

[5]The level sets of $\mathfrak{t}$ are the same than the one of ρ_0 and on such a level set ρ_I is proportional to x.

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Part IV

Appendix

Appendix A

Second order hyperbolic operator on $\mathbb{R}^{n+1}$

We begin by two lemmas in $\mathbb{R}^{n+1}$. Let g be a Lorentzian metric on $\mathbb{R}^{n+1}$ with constant coefficients outside a compact set and assume $G(\ \mathrm{d}t,\ \mathrm{d}t) > 0$. We consider a vectorial operator $P = G^{\mu,\nu}\partial_\mu\partial_\nu + R_0(t)\partial_t + R_1(t)$ with R_0 a smooth bounded family in $\Psi^0(\mathbb{R}^n;\mathbb{C}^p)$ and $R_1(t)$ a smooth bounded family in $\Psi^1(\mathbb{R}^n;\mathbb{C}^p)$. We denote by $\langle .,. \rangle$ the canonical scalar product on $\mathbb{C}^p$. We say that a function f is bounded for the C^∞-norm if it is smooth, bounded on $\mathbb{R}^{n+1}$ and for every multi-index $\alpha \in \mathbb{N}^{n+1}$, the derivative $\partial^\alpha f$ is bounded on $\mathbb{R}^{n+1}$. For example, each coefficient of the metric g is bounded for the C^∞-norm.

Lemma A.0.1 (Energy estimate on $\mathbb{R}^{n+1}$)**.** *Let $s \in \mathbb{R}$ and $T > 0$. There exists $C_{s,T} > 0$ such that for all $u \in C^2([0,T],\mathcal{S}(\mathbb{R}^n))$:*

$$\sup_{0\leq t\leq T} \|u(t)\|_{H^{s+1}(\mathbb{R}^n)} + \|\partial_t u(t)\|_{H^s(\mathbb{R}^n)} \leq C_{s,T}\left(\int_0^T \|Pu\|_{H^s(\mathbb{R}^n)}\ \mathrm{d}t + \|u(0)\|_{H^{s+1}(\mathbb{R}^n)} + \|\partial_t u(0)\|_{H^s(\mathbb{R}^n)}\right)$$

Proof. Even multiplying P by $\frac{1}{G(\ \mathrm{d}t,\ \mathrm{d}t)}$ (which is bounded in the C^∞ norm), we can assume that $G(\ \mathrm{d}t,\ \mathrm{d}t) = 1$.

We introduce the energy tensor

$$T^{\delta,\gamma}(u) = \Re(\langle\partial_\mu u, \partial_\nu u\rangle)g^{\mu,\delta}g^{\nu,\gamma} - \frac{1}{2}g^{\delta,\gamma}g^{\mu,\nu}\langle\partial_\mu u, \partial_\nu u\rangle + \frac{1}{2}g^{\delta,\gamma}\langle u, u\rangle$$

Note that

$$(\mathrm{div}T)^\delta = g^{\mu,\delta}\Re(\langle\partial_\mu u, \Box_g u\rangle) + g^{\delta,\gamma}\Re(\langle\partial_\gamma u, u\rangle)$$

We define the energy one form (here given as a vector field but we have the identification between 1-forms and vector field induced by g):

$$J := -T(u)(e^{-At}\ \mathrm{d}t, .)$$

and its divergence:

$$\begin{aligned} K :=& \mathrm{div}J \\ =& -e^{-At}(\mathrm{div}T)^0 - T^{\mu,\nu}\nabla_\mu(e^{-At}\ \mathrm{d}t)_\nu \\ =& -e^{-At}(\mathrm{div}T)^0 + AT^{0,0}e^{-At} + e^{-At}T^{\mu,\nu}\Gamma^0_{\mu,\nu} \end{aligned}$$

Were $\Gamma^{\alpha}_{\mu,0}$ are Christoffel symbols (bounded in the C^{∞} norm). There exists a constant $C>0$ independent of u and A such that

$$\begin{aligned}\left|(\mathrm{div}T)^0\right| \leq & C\left(\|\Box_g u\|^2+\sum_{\mu=0}^{n}\|\partial_\mu u\|^2+\|u\|^2\right)\\ \left|T^{\mu,\nu}\Gamma^{\alpha}_{\mu,0}g_{\alpha,\nu}\right| \leq & C\left(\sum_{\mu=0}^{n}\|\partial_\mu u\|^2\right.\\ & \left.+\|u\|^2\right)\end{aligned}$$

Since G is Lorentzian and $G^{0,0}=1$, we have that $H:=(G^{\mu,0}G^{\nu,0}-\frac{1}{2}G^{\mu,\nu})$ is positive definite Indeed, if (e_μ) is an orthonormal basis for G with $e_0=\partial_t$, $H(e_\mu,e_\nu)=\delta^0_\mu\delta^0_\nu+\frac{1}{2}\delta^\nu_\mu-\delta^0_0=\frac{1}{2}\delta^\nu_\mu$. Since G is constant outside a compact set, this definiteness is uniform with respect to x and we have a constant $c>0$ (independent of A and u) such that:

$$T^{0,0}\geq c\left(\sum_{\mu=0}^{n}\|\partial_\mu u(t)\|^2_{L^2(\mathbb{R}^n)}+\|u(t)\|^2_{L^2(\mathbb{R}^n)}\right)$$

We deduce that for A large enough:

$$K\geq e^{-At}\left(\frac{Ac}{2}\left(\sum_{\mu=0}^{n}\|\partial_\mu u(t)\|^2_{L^2(\mathbb{R}^n)}+\|u(t)\|^2\right)-C\|\Box_g u\|^2_{L^2(\mathbb{R}^n)}\right)$$

Also note (for the boundary terms when we will use the Stockes formula) that:

$$J(-\,\mathrm{d}t)=e^{-At}T^{0,0}\geq ce^{-At}\left(\sum_{\mu=0}^{n}\|\partial_\mu u(t)\|^2_{L^2(\mathbb{R}^n)}+\|u(t)\|^2_{L^2(\mathbb{R}^n)}\right)$$

The Stockes theorem on $U_T=\{0\leq t\leq T\}$ (initially, we integrate with respect to the volume form induced by the metric g but in the estimate we can come back to the canonical volume form even changing the constants) gives:

$$\begin{aligned}&ce^{-AT}\|u(T)\|^2_{H^{s+1}(\mathbb{R}^n)}+ce^{-AT}\|u(T)\|_{H^s(\mathbb{R}^n)}\\ &+\int_0^T e^{-AT}\frac{Ac}{2}\left(\|u\|^2_{H^1(\mathbb{R}^n)}+\|\partial_t u\|^2_{L^2(\mathbb{R}^n)}\right)\,\mathrm{d}t\leq C\int_0^T e^{-At}\|\Box_g u\|^2_{L^2(\mathbb{R}^n)}\,\mathrm{d}t+C\|u(0)\|^2_{H^1(\mathbb{R}^n)}\\ &\qquad\qquad+C\|\partial_t u(0)\|^2_{L^2(\mathbb{R}^n)}\end{aligned}$$

We now apply this inequality to $v:=E_s u$ where $E_s=(1+\Delta)^{\frac{s}{2}}$ where Δ is the usual laplacian on $\mathbb{R}^n$. Note that:

$$\|\Box_g v\|^2_{L^2(\mathbb{R}^n)}\leq 2\|[\Box_g,E_s]u\|^2_{L^2(\mathbb{R}^n)}+2\|E_sPu\|^2_{L^2(\mathbb{R}^n)}+2\|E_s(\Box_g-P)u\|^2_{L^2(\mathbb{R}^n)}$$

Note that $[\Box_g,E_s]\in\Psi^s(\mathbb{R}^n)\partial_t+\Psi^{s+1}(\mathbb{R}^n)$ and $E_s(\Box_g-P)\in\Psi^s(\mathbb{R}^n)\partial_t+\Psi^{s+1}(\mathbb{R}^n)$. Since $\|E_s u\|_{H^1(\mathbb{R}^n)}=\|u\|_{H^{s+1}(\mathbb{R}^n)}$ and $\|E_s\partial_t u\|_{L^2(\mathbb{R}^n)}=\|\partial_t u\|_{H^s(\mathbb{R}^n)}$, we can absorb $\|[\Box_g,E_s]u\|_{L^2(\mathbb{R}^n)}$ and $\|E_s(\Box_g-P)u\|_{L^2(\mathbb{R}^n)}$ into the left-hand side if we take A large enough. Therefore, we have:

$$\begin{aligned}&ce^{-AT}\|u(T)\|^2_{H^{s+1}(\mathbb{R}^n)}+ce^{-AT}\|u(T)\|^2_{H^s(\mathbb{R}^n)}\\ &+\int_0^T e^{-AT}\frac{Ac}{2}\left(\|u\|^2_{H^{s+1}(\mathbb{R}^n)}+\|\partial_t u\|^2_{H^s(\mathbb{R}^n)}\right)\,\mathrm{d}t\leq C\int_0^T e^{-At}\|Pu\|^2_{H^s(\mathbb{R}^n)}\,\mathrm{d}t\\ &\qquad\qquad+C\|u(0)\|^2_{H^{s+1}(\mathbb{R}^n)}+C\|\partial_t u(0)\|^2_{H^s(\mathbb{R}^n)}\end{aligned}$$

□

In order to prove uniqueness, we will need a weaker version of the previous estimate (an estimate valid for all $u \in C^2([0,T], H^s(\mathbb{R}^n))$ for all $s \in \mathbb{R}$). Even a non optimal bound is enough for uniqueness.

Lemma A.0.2. *If $\psi \in \mathcal{S}$ with $\psi(0) = 1$ and $\psi_\epsilon(x) := \psi(\epsilon x)$. Then the multiplication by ψ_ϵ is bounded in $H^s(\mathbb{R}^n)$ independently of ϵ and $(1-\psi_\epsilon)u \xrightarrow[\epsilon\to 0]{} 0$ in $H^s(\mathbb{R}^n)$.*

Proof. Let $u \in H^s(\mathbb{R}^n)$. The Fourier transform of $\psi_\epsilon u$ is $\phi_\epsilon * \hat{u}$ where $\phi_\epsilon := \epsilon^{-n}\hat{\psi}_{\epsilon^{-1}}$ is bounded in $\langle\xi\rangle^s L^1(\mathbb{R}^n)$ for all $s \in \mathbb{R}$ independently of $0 < \epsilon < 1$. As a consequence

$$\begin{aligned}
\left\| \langle\xi\rangle^{\frac{s}{2}} \phi_\epsilon * \hat{u} \right\|_{L^2}^2 &\leq \int \left(\int \langle\xi\rangle^{\frac{s}{2}} |\phi_\epsilon(\zeta)\hat{u}(\xi-\zeta)| \, \mathrm{d}\zeta \right)^2 \mathrm{d}\xi \\
&\leq \int \left(\int \langle\zeta\rangle^{|\frac{s}{2}|} \left| \phi_\epsilon(\zeta) \langle\xi-\zeta\rangle^{\frac{s}{2}} \hat{u}(\xi-\zeta) \right| \mathrm{d}\zeta \right)^2 \mathrm{d}\xi \\
&\leq \left\| \langle\zeta\rangle^{\frac{|s|}{2}} \phi_\epsilon \right\|_{L^1} \int\int \langle\xi-\zeta\rangle^s |\hat{u}(\xi-\zeta)|^2 \left| \langle\zeta\rangle^{\frac{s}{2}} \phi_\epsilon(\zeta) \right| \mathrm{d}\zeta \, \mathrm{d}\xi \quad \text{(By Hölder)} \\
&\leq \left\| \langle\zeta\rangle^{\frac{|s|}{2}} \phi_\epsilon \right\|_{L^1}^2 \|u\|_{H^s}
\end{aligned}$$

Moreover, we have $(1-\psi_\epsilon)u \xrightarrow[\epsilon\to 0]{} 0$ in $H^s(\mathbb{R}^n)$. In the Fourier side, this correspond to proving that that $\lim_{\epsilon\to 0} \langle\xi\rangle^{\frac{s}{2}} \int (\hat{u}(\xi-\zeta) - \hat{u}(\xi))\phi_\epsilon(\zeta) \, \mathrm{d}\zeta = 0$ in L^2. We begin by proving it in the case $\hat{u} \in C_c^0(\mathbb{R}^n)$ which is dense in $\langle\xi\rangle^{-\frac{s}{2}} L^2$. In this case, we first show the pointwise convergence to zero of $\langle\xi\rangle^{\frac{s}{2}} \int (\hat{u}(\xi-\zeta) - \hat{u}(\xi))\phi_\epsilon(\zeta) \, \mathrm{d}\zeta$. For $\eta > 0$, we choose $0 < C < 1$ such that $\langle\xi\rangle^{\frac{s}{2}} |\hat{u}(\xi-\zeta) - \hat{u}| \leq \eta$ for $\zeta \in B(\xi, C)$. Therefore,

$$\begin{aligned}
\left| \langle\xi\rangle^{\frac{s}{2}} \int (\hat{u}(\xi-\zeta) - \hat{u}(\xi))\phi_\epsilon(\zeta) \, \mathrm{d}\zeta \right| &\leq \eta \left\| \hat{\psi} \right\|_{L^1} + \left| \int_{\zeta > C} \langle\xi-\zeta\rangle^{\frac{s}{2}} \left(\hat{u}(\xi-\zeta) - \hat{u}(\xi)\right) \langle\zeta\rangle^{\frac{|s|}{2}} \phi_\epsilon(\zeta) \, \mathrm{d}\zeta \right| \\
&\leq \eta \left\| \hat{\psi} \right\|_{L^1} + 2 \left\| \langle\zeta\rangle^{\frac{s}{2}} \hat{u} \right\|_{L^2} \left(\int_{|\zeta|>C} \langle\zeta\rangle^{|s|} |\phi_\epsilon(\zeta)|^2 \, \mathrm{d}\zeta \right)^{\frac{1}{2}}
\end{aligned}$$

Since for all $N \in \mathbb{N}$ we have $\phi_\epsilon(\zeta) \leq C_N \epsilon^{-n} \frac{\epsilon^{2N}}{(\epsilon^2+|\zeta|^2)^N}$ We conclude that on $\zeta > C$, there exists some constant $D > 0$ independent of ϵ such that:

$$|\phi_\epsilon(\zeta)| \leq D\epsilon \langle\zeta\rangle^{-\frac{n+|s|+1}{2}}$$

and as a consequence

$$\int_{|\zeta|>C} \langle\zeta\rangle^{|s|} |\phi_\epsilon(\zeta)|^2 \, \mathrm{d}\zeta \leq D'\epsilon^2$$

with $D' < 0$. We can choose ϵ small enough so that

$$\left| \langle\xi\rangle^{\frac{s}{2}} \int (\hat{u}(\xi-\zeta) - \hat{u}(\xi))\phi_\epsilon(\zeta) \, \mathrm{d}\zeta \right| \leq 2\eta \left\| \hat{\psi} \right\|_{L^1}$$

Now we use Lebesgue theorem to prove the convergence in L^2. We prove the domination condition as follows: If ξ is such that $d(\xi, \mathrm{supp}(\hat{u})) \geq 1$:

$$\left| \int (\hat{u}(\xi-\zeta) - \hat{u}(\xi))\phi_\epsilon(\zeta) \, \mathrm{d}\zeta \right| \leq \|u\|_\infty \int_{\|\zeta\| \geq d(\xi, \mathrm{supp}(\hat{u}))} |\phi_\epsilon(\zeta)| \, \mathrm{d}\zeta$$

For $|\zeta| \geq d(\xi, \mathrm{supp}(\hat{u})) \geq 1$ and $N \in \mathbb{N}$,

$$\left\langle \frac{\zeta}{\epsilon} \right\rangle^{-N} \leq \epsilon^N d(\xi, \mathrm{supp}(\hat{u}))^{-N}$$
$$\left\langle \frac{\zeta}{\epsilon} \right\rangle^{-N} \leq (2\epsilon)^N \langle \zeta \rangle^{-N}$$

Since for all $M \in \mathbb{N}$, $|\phi_\epsilon(\zeta)| \lesssim \epsilon^{-n} \left\langle \frac{\zeta}{\epsilon} \right\rangle^{-M}$, we conclude that there exists C independent of $\epsilon < 1$ such that:

$$\left| \int (\hat{u}(\xi - \zeta) - \hat{u}(\xi)) \phi_\epsilon(\zeta) \, \mathrm{d}\zeta \right| \leq C \left\| \hat{u} \right\|_\infty d(\xi, \mathrm{supp}(\hat{u}))^{-\frac{n+1+s}{2}}$$

for ξ such that $d(\xi, \mathrm{supp}(\hat{u})) \leq 1$, we have:

$$\left| \int (\hat{u}(\xi - \zeta) - \hat{u}(\xi)) \phi_\epsilon(\zeta) \, \mathrm{d}\zeta \right| \leq 2 \left\| \hat{u} \right\|_\infty \left\| \hat{\psi} \right\|_L^1$$

In particular, combining these two bounds we get $\left| \langle \xi \rangle^{\frac{s}{2}} \int (\hat{u}(\xi - \zeta) - \hat{u}(\xi)) \phi_\epsilon(\zeta) \, \mathrm{d}\zeta \right|$ is uniformly bounded by a function in L^2 independent of ϵ. We deduce that

$$\lim_{\epsilon \to 0} \langle \xi \rangle^{\frac{s}{2}} \int (\hat{u}(\xi - \zeta) - \hat{u}(\xi)) \phi_\epsilon(\zeta) \, \mathrm{d}\zeta = 0$$

when $\hat{u} \in C_c^0(\mathbb{R}^n)$. To get the result for $\hat{u} \in \langle \xi \rangle^{-\frac{s}{2}} L^2$, we take a sequence of functions $\hat{u}_n \in C_c^0$ converging to $\hat{u}$ in $\langle \xi \rangle^{-\frac{s}{2}} L^2$. In particular, for any $\eta > 0$, there exists $n \in \mathbb{N}$ such that $\|u - u_n\|_{H^s} \leq \eta$ and

$$\|(1 - \psi_\epsilon)u\|_{H^s} \leq \|(1 - \psi_\epsilon)(u - u_n)\|_{H^s} + \|(1 - \psi_\epsilon)u_n\|$$

By the boundedeness (independently of ϵ) shown earlier the first terms is bounded by $C\eta$ and for ϵ small enough, the second term is bounded by η. Therefore,

$$\lim_{\epsilon \to 0} \|(1 - \psi_\epsilon)u\|_{H^s} = 0.$$

□

Lemma A.0.3. *Let $\psi \in \mathcal{S}(\mathbb{R}^n)$ with $\int \psi \, \mathrm{d}x = 1$. Let $s \in \mathbb{R}$. The map $\Psi_\epsilon : u \mapsto \epsilon^{-n} \psi_{\epsilon^{-1}} * u$ is bounded from H^s to H^s uniformly with respect to ϵ. Moreover, $\lim_{\epsilon \to 0} \Psi_\epsilon(u) = u$ in H^s.*

Proof. The fourier transform of $\epsilon^{-n} \psi_{\epsilon^{-1}} * u$ is $\hat{\psi}_\epsilon \hat{u}$. Therefore, the boundedness statement follows from the estimate:

$$\left\| \langle \xi \rangle^{\frac{s}{2}} \hat{\psi}_\epsilon \hat{u} \right\|_L^2 \leq \left\| \hat{\psi} \right\|_\infty \|u\|_{H^s}.$$

Moreover, since $\int \psi = 1$, we have $\hat{\psi}(0) = 1$ and therefore, $\hat{\psi}_\epsilon \hat{u}$ converge pointwise to $\hat{u}$. We have the estimate:

$$\langle \xi \rangle^{\frac{s}{2}} \hat{\psi}_\epsilon \hat{u} \leq \left\| \hat{\psi} \right\|_\infty \left(\hat{u} \langle \xi \rangle^{\frac{s}{2}} \right)$$

with $\hat{u} \langle \xi \rangle^{\frac{s}{2}} \in L^2$ independent of ϵ. Therefore, by Lebesgue theorem we get $\lim_{\epsilon \to 0} \langle \xi \rangle^{\frac{s}{2}} \hat{\psi}_\epsilon \hat{u} = \langle \xi \rangle^{\frac{s}{2}} \hat{u}$ in L^2, that is to say $\lim_{\epsilon \to 0} \Psi_\epsilon(u) = u$ in H^s. □

Lemma A.0.4 (weak energy estimate)**.** *There exists $C_{T,s} > 0$ such that for all $u \in C^2([0,T], H^s(\mathbb{R}^n))$:*

$$\sup_{[0,T]} \|u\|_{H^{s-1}(\mathbb{R}^n)} + \|\partial_t u\|_{H^{s-2}(\mathbb{R}^n)} \leq C_{T,s} \left(\int_0^T \|Pu\|_{H^{s-2}(\mathbb{R}^n)} + \|u(0)\|_{H^{s-1}(\mathbb{R}^n)} + \|\partial_t u(0)\|_{H^{s-2}(\mathbb{R}^n)} \right)$$

Proof. Let $u \in C^2([0,T], H^s(\mathbb{R}^n))$. Let $\psi \in \mathcal{S}(\mathbb{R}^n)$ such that $\psi(0) = 1$ and $\int_{\mathbb{R}^n} \psi(x)\, \mathrm{d}x = 1$. We define $\psi_\epsilon(x) = \psi(\epsilon x)$. We define

$$J_\epsilon u := \psi_\epsilon(\epsilon^{-n} \psi_{\epsilon^{-1}}) * u$$

Note that for fixed $\epsilon > 0$, J_ϵ is bounded from $H^s(\mathbb{R}^n)$ to $\mathcal{S}$ for any $s \in \mathbb{R}$. Indeed, for $\alpha, \beta \in \mathbb{N}^n$:

$$x^\alpha \partial_x^\beta J_\epsilon u(x) = \sum_{\gamma+\mu=\beta} \binom{\beta}{\gamma} x^\alpha \epsilon^{|\gamma|} (\partial_x^\gamma \psi)(\epsilon x)(\epsilon^{-n-|\mu|} (\partial_x^\mu \psi)_{\epsilon^{-1}} * u(x))$$

We can estimate

$$\begin{aligned} \left\| \epsilon^{-n} (\partial_x^\mu \psi)_{\epsilon^{-1}} * u \right\|_{L^\infty} &\leq \left\| (\epsilon\xi)^\mu \hat{\psi}(\epsilon\xi) \hat{u} \right\|_L^1 (\mathbb{R}^n) \\ &\leq \left\| \langle \xi \rangle^{-s} (\epsilon\xi)^\mu \hat{\psi}(\epsilon\xi) \right\|_{L^2} \|u\|_{H^s(\mathbb{R}^n)} \end{aligned}$$

In particular $J_\epsilon u \in C^2([0,T], \mathcal{S})$. Moreover, we have:

$$\|J_\epsilon u - u\|_{H^s} \leq \|(\psi_\epsilon - 1)u\|_{H^s} + \left\| \psi_\epsilon (u - \epsilon^{-n} \psi_{\epsilon^{-1}} * u) \right\|_H^s$$

using lemma A.0.2 and A.0.3, we get for all $t \in [0,T]$ $\lim_{\epsilon \to 0} J_\epsilon u(t) = u(t)$ in H^s. We also have for all $t \in [0,T]$ in H^s: $\lim_{\epsilon \to 0} \partial_t J_\epsilon u = \partial_t u$, $\lim_{\epsilon \to 0} \partial_t^2 J_\epsilon u = \partial_t^2 u$. We deduce that for all $t \in [0,T]$, $\lim_{\epsilon \to 0} P J_\epsilon u(t) = Pu(t)$ in H^{s-2}. Moreover, in view of the definition of P, we have that for all $v \in C^2([0,T], H^s)$, there exists a constant $C > 0$ such that for all $t \in [0,T]$, $\|Pv(t)\|_{H^{s-2}(\mathbb{R}^n)} \leq C \left(\|v(t)\|_{H^s(\mathbb{R}^n)} + \|\partial_t v(t)\|_{H^s(\mathbb{R}^n)} + \left\|\partial_t^2 v(t)\right\|_{H^s(\mathbb{R}^n)} \right)$. In particular since J_ϵ is uniformly bounded with respect to $\epsilon \in (0,1)$ there exists $D > 0$ such that for all $\epsilon \in (0,1)$,

$$\sup_{t \in [0,T]} \|P J_\epsilon u\|_{H^{s-2}(\mathbb{R}^n)} \leq D \sup_{t \in [0,T]} \left(\|u\|_{H^s(\mathbb{R}^n)} + \|\partial_t v(t)\|_{H^s(\mathbb{R}^n)} + \left\|\partial_t^2 v(t)\right\|_{H^s(\mathbb{R}^n)} \right)$$

therefore, by Lebesgue's theorem $\lim_{\epsilon \to 0} P J_\epsilon u = Pu$ is $L^1([0,T], H^{s-2}(\mathbb{R}^n))$. We apply lemma A.0.1 to $J_\epsilon u$ with $s-2$ instead of s and passing to the limit, we get:

$$\sup_{[0,T]} \|u\|_{H^{s-1}(\mathbb{R}^n)} + \|\partial_t u\|_{H^{s-2}(\mathbb{R}^n)} \leq C_{T,s} \left(\int_0^T \|Pu\|_{H^{s-2}(\mathbb{R}^n)} + \|u(0)\|_{H^{s-1}(\mathbb{R}^n)} + \|\partial_t u(0)\|_{H^{s-2}(\mathbb{R}^n)} \right)$$

□

Lemma A.0.5. *Let $s \in \mathbb{R}$ For all $f \in L^1([0,T], H^s(\mathbb{R}^n))$ and all $u_0 \in H^{s+1}(\mathbb{R}^n)$, $u_1 \in H^s(\mathbb{R}^n)$, we have a unique distribution $u \in C^0([0,T], H^{s+1}(\mathbb{R}^n)) \cap C^1([0,T], H^s(\mathbb{R}^n))$ such that $Pu = f$ and $u(0) = u_0$, $\partial_t u(0) = u_1$. Moreover, this solution satisfies the estimate of lemma A.0.1*

Proof. First we prove the uniqueness. Let $s \in \mathbb{R}$ and $u \in C^0([0,T], H^{s+1}(\mathbb{R}^n)) \cap C^1([0,T], H^s(\mathbb{R}^n))$ such that $Pu = 0$, $u(0) = 0$ and $\partial_t u(0) = 0$. Then, by theorem B.2.9 in [54] (or rather the version without the "loc"), we have for any $N \in \mathbb{N}$, $u \in \overline{H}_{(N,s+1-N)}$ in particular, for $N > \frac{5}{2}$, $u \in C^2([0,T], H^{s+1-N})$. Therefore, we can apply lemma A.0.4 and we get $u = 0$ (This argument even proves the uniqueness in any space $\overline{H}^m([0,T] \times \mathbb{R}^n)$ with $m \in \mathbb{R}$). We construct the solution using the Hahn-Banach theorem. We define for $\phi \in C_c^\infty([0,T) \times \mathbb{R}^n; \mathbb{C}^p)$:

$$Q(\phi) := \langle f, \phi \rangle_{L^2([0,T)\times\mathbb{R}^n)} - \langle u_1, \phi(0) \rangle + \langle u_0, \partial_t \phi(0) \rangle + 2 \left\langle u_0, (\partial_j (G^{0,j} \phi(0)) \right\rangle - \langle u_0, R_0(0)^* \phi(0) \rangle$$

We have:

$$\begin{aligned}
|\langle f, \phi \rangle| &\leq \|f\|_{L^1([0,T], H^s(\mathbb{R}^n))} \sup_{[0,T]} \|\phi\|_{H^{-s}(\mathbb{R}^n)} \\
|\langle u_1, \phi(0) \rangle| &\leq \|u_1\|_{H^s(\mathbb{R}^n)} \|\phi(0)\|_{H^{-s}(\mathbb{R}^n)} \\
|\langle u_0, \partial_t \phi(0) \rangle| &\leq \|u_0\|_{H^{s+1}(\mathbb{R}^n)} \|\partial_t \phi(0)\|_{H^{-s-1}(\mathbb{R}^n)} \\
\left|\left\langle u_0, 2(\partial_j G^{0,j} \phi(0)) + R_0(0)^* \phi(0) \right\rangle\right| &\lesssim \|u_0\|_{H^{s+1}(\mathbb{R}^n)} \|\phi(0)\|_{H^{-s}(\mathbb{R}^n)}
\end{aligned}$$

Therefore, we get:

$$|Q(\phi)| \lesssim \left(\|f\|_{L^1([0,T], H^s(\mathbb{R}^n)} + \|u_1\|_{H^s(\mathbb{R}^n)} + \|u_0\|_{H^{s+1}(\mathbb{R}^n)} \right) \left(\sup_{[0,T]} \|\phi\|_{H^{-s}(\mathbb{R}^n)} + \|\partial_t \phi(0)\|_{H^{-s-1}(\mathbb{R}^n)} \right)$$

Applying lemma A.0.1 to ϕ with the operator P^*, Sobolev order $-s-1$ and t replaced by $T-t$, we get:

$$|Q(\phi)| \leq C_{s,T} \left(\|f\|_{L^1([0,T], H^s(\mathbb{R}^n)} + \|u_1\|_{H^s(\mathbb{R}^n)} + \|u_0|_{H^{s+1}(\mathbb{R}^n)} \right) \left(\int_0^T \|P^* \phi(t)\|_{H^{-s-1}(\mathbb{R}^n)} \, \mathrm{d}t \right)$$

Therefore, by the Hahn-Banach theorem, there exists $u \in L^\infty([0,T], H^{s+1}(\mathbb{R}^n))$ such that for all $\phi \in C_c^\infty([0,T) \times \mathbb{R}^n; \mathbb{C}^p)$:

$$\begin{aligned}
\langle u, P^* \phi \rangle = &\langle f, \phi \rangle_{L^2([0,T)\times\mathbb{R}^n)} - \langle u_1, \phi(0) \rangle + \langle u_0, \partial_t \phi(0) \rangle \\
&+ 2 \left\langle u_0, (\partial_j (G^{0,j} \phi(0)) \right\rangle - \langle u_0, R_0(0)^* \phi(0) \rangle
\end{aligned} \tag{A.1}$$

and such that

$$\sup_{t \in [0,T]} \|u(t)\|_{H^{s+1}(\mathbb{R}^n)} \leq C_{s,T} \left(\|f\|_{L^1([0,T], H^s(\mathbb{R}^n)} + \|u_1\|_{H^s(\mathbb{R}^n)} + \|u_0\|_{H^{s+1}(\mathbb{R}^n)} \right)$$

Using (A.1) with $\phi \in C_c^\infty((0,T) \times \mathbb{R}^n)$, we get $Pu = f$. A priori, we do not know if $u \in C^0([0,T], H^{s+1}(\mathbb{R}^n)) \cap C^1([0,T], H^s(\mathbb{R}^n))$ (we only have $u \in L^\infty([0,T], H^{s+1}(\mathbb{R}^n))$) but we can use this construction together with an approximation argument to get an element in $u \in C^0([0,T], H^{s+1}(\mathbb{R}^n)) \cap C^1([0,T], H^s(\mathbb{R}^n))$. We take a sequence (f_n) of functions in $C^\infty([0,T], \mathcal{S})$ such that $\lim_{n \to +\infty} f_n = f$ in $L^1([0,T], H^s(\mathbb{R}^n))$ and (u_0^n), (u_1^n) sequences in $\mathcal{S}$ with $\lim_{n \to +\infty} u_0^n = u_0$ in H^{s+1} and $\lim_{n \to +\infty} u_1^n = u_1$ in H^s. By the previous argument for some large $N \in \mathbb{R}$ we can construct $u^n \in L^\infty([0,T], H^{N+1}(\mathbb{R}^n))$ satisfying (A.1) with f replaced by f_n, u_0 by u_0^n and u_1 by u_1^n. Since $Pu^n = f_n$ is smooth, we get (lemma B.2.9 in [54]) $u^n \in H_{(3,N-3)}$. By theorem B.2.7, $u^n \in C^2([0,T], H^{N-\frac{5}{2}})$. Therefore, if we take $N \geq s + \frac{9}{2}$, we get $u^n \in C^2([0,T], H^{s+2}(\mathbb{R}^n))$. Using (A.1) with general $\phi \in C_c^\infty((0,T) \times \mathbb{R}^n)$, we get that $u^n(0) = u_0^n$ and $\partial_t u^n(0) = u_1^n$. For $n, n' \in \mathbb{N}$ we can therefore apply estimate A.0.4 to get:

$$\begin{aligned}
\sup_{t \in [0,T]} \left\| u^n(t) - u^{n'}(t) \right\|_{H^{s+1}(\mathbb{R}^n)} + \left\| \partial_t u^n(t) - \partial_t u^{n'}(t) \right\|_{H^s(\mathbb{R}^n)} \leq &C_{T,s} \left(\int_0^T \|f_n - f_{n'}\|_{H^s(\mathbb{R}^n)} \right. \\
&\left. + \left\| u_0^n - u_0^{n'} \right\|_{H^{s+1}(\mathbb{R}^n)} + \left\| u_1^n - u_1^{n'} \right\|_{H^s(\mathbb{R}^n)} \right)
\end{aligned}$$

We deduce that (u^n) is a Cauchy sequence in the complete space $C^0([0,T], H^{s+1}(\mathbb{R}^n)) \cap C^1([0,T], H^s(\mathbb{R}^n))$. We call u the limit. We get $u(0) = u_0$ and $\partial_t u(0) = u_1$. Moreover, passing to the limit in (A.1) with $\phi \in C_c^\infty((0,T) \times \mathbb{R}^n)$, we obtain $P\phi = f$. □

Appendix B

Local existence, uniqueness and approximation

Let $(\mathcal{M}, g)$ be a smooth Lorentzian manifold which admits a global time function t. We denote by $\mathcal{M}_+ := t^{-1}((0, +\infty))$ and $\mathcal{M}_0 := t^{-1}(\{0\})$.

B.1 Causality preliminaries

Definition B.1.1. *We say that a Lorentzian manifold satisfies the weak causality condition if there is no closed causal curve. It satisfies the strong causality condition if for every point p and every neighborhood U of p, there exists a neighborhood V of p such that every causal curve segment with endpoints in V is included in U.*

Definition B.1.2. *For a point p in a Lorentzian manifold $(\mathcal{M}, g)$ and every open subset U of $\mathcal{M}$, we denote by $I^+(p, U)$ (resp. $I^-(p, U)$) the chronological future (resp. past) of p inside U. For a subset A of $\mathcal{M}$, we call $D(A)$ its domain of dependence (the set of points $p \in \mathcal{M}$ such that A intersects the causal past or the causal future of p).*

Lemma B.1.3. *The Lorentzian manifold $(\mathcal{M}, g)$ previously defined is time-oriented (by $\mathrm{d}t$) and satisfies the strong causality condition.*

Proof. First note that the weak causality condition follows from the fact that t is strictly monotonic along each causal curve. We now use a quantitative version of the previous argument to get the strong causality condition. We fix a Riemaniann metric $\tilde{g}$ on $\mathcal{M}$. Let $x \in \mathcal{M}$ and U be a neighborhood of x. By continuity of the metric at x, on a relatively compact neighborhood V of x, there exists $\epsilon > 0$ such that for all $y \in V$ and all $v \in T_yV$ causal vector $|\,\mathrm{d}t(v)| \geq \epsilon\,|v|_{\tilde{g}}$. We take $\eta > 0$ small enough so that the geodesic ball $B_{\tilde{g}}(x, \eta) \subset V$. In particular, we deduce that for any future oriented causal geodesic γ intersecting $B_{\tilde{g}}(x, \frac{\eta}{2})$ at time t_0 and being outside V at time $t_1 > t_0$, we have the length $\int_{t_0}^{t_1} \left|\dot{\gamma(s)}\right|_{\tilde{g}} \,\mathrm{d}s > \frac{\eta}{2}$ and $t(\gamma(t_1)) - t(\gamma(t_0)) > \frac{\epsilon\eta}{2}$. In particular, if we take $V' = B_{\tilde{g}}(x, \frac{\eta}{2}) \cap \left\{|t - t(x)| < \frac{\epsilon\eta}{4}\right\}$, we get that every causal curve with endpoints in V' remains in V (and therefore in U). □

Lemma B.1.4. *For all $x \in \mathcal{M}$, there exists a neighborhood U of x such that for all V convex*[1] *open set contained in U, for all $p, q \in V$ such that p is in the causal future of q, there is exactly one causal geodesic segment between p and q and this segment is included in V. In particular for all $p \in V$, $I^{\pm}(p, V) = I^{\pm}(p, \mathcal{M}) \cap V$.*

[1] In the sense of [87, Chapter 5, Definition 5.]

Proof. First, we take W a convex neighborhood of x (see proposition 7 page 130 in [87]) and U a neighborhood of x given by the strong causality condition. Therefore for all $V \subset U$ we have that every causal segment with endpoint in V remains in W. We assume in addition that V is convex. Therefore, for $p, q \in V$, let γ be a causal geodesic segment between p and q. Since $V \subset U$, we know that γ remains in W. By convexity of V, we know that there exists γ' a geodesic segment between p and q remaining in V. By convexity of W, we have uniqueness (up to reparametrization) of the geodesic segment between p and q in W and therefore $\gamma = \gamma'$ remains in V. □

Lemma B.1.5. *For every $x \in \mathcal{M}$ and every neighborhood U of x, there exists S neighborhood of x in the hypersurface $\{t = t(x)\}$ such that $D(S)$ is a relatively compact neighborhood of x included in U.*

Proof. First we fix local coordinates $(x^\mu)_{\mu=0}^n$ around x such that $x^\mu(x) = 0$, $x^0 = \frac{t-t(x)}{\sqrt{G_x(\mathrm{d}_x t, \mathrm{d}_x t)}}$ and $G^{\mu,\nu}$ is the Minkowski metric at x (it exists since we can complete $\omega_0 := \frac{\mathrm{d}_x t}{\sqrt{G_x(\mathrm{d}_x t, \mathrm{d}_x t)}}$ into an orthonormal basis (ω_μ) of $T_x^*\mathcal{M}$ and find locally functions $(x^i)_{i=1}^n$ such that $\mathrm{d}_x x^i = \omega_i$). There exists W a small Euclidean coordinate ball $B(x, \eta)$ such that for all $y \in W$ for all $u \in T_y\mathcal{M}$ for y in a neighborhood W:

$$(1-\epsilon)(u^0)^2 - (1+\epsilon)(u^i)^2 < g_{\mu,\nu}u^\mu u^\nu < (1+\epsilon)(u^0)^2 - (1-\epsilon)\sum_{i=1}^n (u^i)^2 \tag{B.1}$$

We now prove that for $S = \left\{y \in W : x^0(y) = 0, \sum_{i=1}^n (x^i)^2 < \frac{\eta^2(1-\epsilon)}{2(1+\epsilon)}\right\}$, we have $C_{\eta\sqrt{\frac{1-\epsilon}{2(1+\epsilon)}}, \sqrt{\frac{1-\epsilon}{1+\epsilon}}} \subset D(S) \subset C_{\eta\sqrt{\frac{1-\epsilon}{2(1+\epsilon)}}, \sqrt{\frac{1+\epsilon}{1-\epsilon}}}$ where $C_{\beta,\alpha}$ is the set of points such that $|x_0| < \alpha(\beta - \sqrt{\sum_{i=1}^n x_i^2})$ (in particular note that $C_{\eta\sqrt{\frac{1-\epsilon}{2(1+\epsilon)}}, \sqrt{\frac{1+\epsilon}{1-\epsilon}}} \subset B(x, \eta)$).

First inclusion: Let $y \in C_{\eta\sqrt{\frac{1-\epsilon}{2(1+\epsilon)}}, \sqrt{\frac{1-\epsilon}{1+\epsilon}}}$. There exists $0 < \beta < \eta\sqrt{\frac{1-\epsilon}{2(1+\epsilon)}}$ such that $y \in C_{\beta, \sqrt{\frac{1-\epsilon}{1+\epsilon}}}$. Without loss of generality, we assume that $x^0(y) > 0$. Let $\gamma : [0, b) \to \mathcal{M}$ be a past oriented C^1 causal curve such that $\gamma(0) = 0$ and such that γ cannot be extended to an interval of the form $[0, b')$ with $b' > b$. Since γ is causal and past oriented, we have, for all $s \in [0, b)$ such that $\gamma(s) \in B(x, \eta)$, we have:

$$(1-\epsilon)\sum_{i=1}^n (\dot\gamma^i)^2 < (1+\epsilon)(\dot\gamma^0)^2 \tag{B.2}$$

$$\dot\gamma^0 < 0 \tag{B.3}$$

We define

$$B^+ := \left\{0 \le x_0 = \sqrt{\frac{1-\epsilon}{1+\epsilon}}\left(\beta - \sqrt{\sum_{i=1}^n x_i^2}\right)\right\}$$

the upper boundary of $\overline{C_{\beta, \sqrt{\frac{1-\epsilon}{1+\epsilon}}}}$. First, assume (by contradiction) that $\gamma \cap B^+ \neq \emptyset$. Then we can take

$$s_0 = \inf\left\{s \in [0, b) : \gamma(s) \in B^+\right\}$$

and since B_+ is closed and γ is continuous, $\gamma(s_0) \in B^+$. Moreover, since $\gamma(0)^0 = y^0 < \beta$ and $\gamma(s)^0$ is decreasing, we have that $\gamma(s_0)^0 < \beta$ and in therefore $\sum_{i=1}^n (\gamma(s_0)^i)^2 \neq 0$. Since for

$s \in [0, s_0)$, $0 < \gamma(s)^0 < \sqrt{\frac{1-\epsilon}{1+\epsilon}}(\beta - \sqrt{\sum_{i=1}^n (\gamma(s)^i)^2})$, differentiating at s_0 we find that:

$$0 \leq \dot\gamma(s_0)^0 + \sqrt{\frac{1-\epsilon}{1+\epsilon}} \frac{\sum_{i=1}^n \dot\gamma(s_0)^i \gamma(s_0)^i}{\sqrt{\sum_{i=1}^n (\gamma(s_0)^i)^2}} \leq \dot\gamma(s_0)^0 + \sqrt{\frac{1-\epsilon}{1+\epsilon}} \sqrt{\sum_{i=1}^n (\dot\gamma(s_0)^i)^2}$$

Using (B.3) and (B.2), we deduce

$$\sqrt{\frac{1-\epsilon}{1+\epsilon}} \sqrt{\sum_{i=1}^n (\dot\gamma(s_0)^i)^2} < \left\| \dot\gamma(s_0)^0 \right\| \leq \sqrt{\frac{1-\epsilon}{1+\epsilon}} \sqrt{\sum_{i=1}^n (\dot\gamma(s_0)^i)^2}$$

which is a contradiction. We conclude that $\gamma([0,b)) \cap B^+ = \emptyset$. Assume by contradiction that γ remains in the compact set $\overline{C_{\beta, \sqrt{\frac{1-\epsilon}{1+\epsilon}}}} \cap \left\{x^0 \geq 0\right\}$ therefore, it has an accumulation point y_1 in this set. Since γ^0 is strictly decreasing, we must have for all $s \in [0,b)$ $\gamma^0(s) \geq x^0(y_1)$. We deduce $\int_0^b \left|\dot\gamma^0(s)\right| \, \mathrm{d}s < +\infty$ and therefore by (B.2)

$$\int_0^b \sqrt{\sum_{i=1}^n (\dot\gamma^i)^2} \, \mathrm{d}s < +\infty.$$

We conclude that $\lim_{s \to b} \gamma(s) - \gamma(0)$ exists.and in particular, y_1 is in fact the limit of γ. But this contradits the inextendibility of γ. Therefore, γ must leave $\overline{C_{\beta, \sqrt{\frac{1-\epsilon}{1+\epsilon}}} \cap \left\{x^0 \geq 0\right\}}$ and (by continuity) intersects the boundary. Since the boundary is contained in $S \cup B^+$ and $\gamma([0,b)) \cap B^+ = \emptyset$, we must have $\gamma([0,b)) \cap S \neq \emptyset$.

Second inclusion: It is enough to prove that there exists a neighborhood W of $K := \overline{C_{\eta \sqrt{\frac{1-\epsilon}{2(1+\epsilon)}}, \sqrt{\frac{1+\epsilon}{1-\epsilon}}}}$ such that for any point $y \in W \setminus K$, there exists an inextendible causal curve γ which never intersects S and with $\gamma(0) = y$. We take $W := C_{\beta, \sqrt{\frac{1+\epsilon}{1-\epsilon}}} \subset B(x, \eta)$ with $\beta > \eta \sqrt{\frac{1-\epsilon}{2(1+\epsilon)}}$. Without loss of generality, we assume that $x^0(y) \geq 0$. We consider the curve $\gamma(s) = y + \left(-s, s\sqrt{\frac{1-\epsilon}{1+\epsilon}} \frac{\tilde y}{\|\tilde y\|}\right)$ where $\tilde y = (x^i(y))_{i=1}^n$ and $\|\tilde y\|$ is the Euclidean norm of $\tilde y$ (if $\tilde y = 0$, we take any Euclidean unit vector instead of $\frac{\tilde y}{\|\tilde y\|}$). We check that $\gamma(s)$ is timelike past oriented (using (B.1)) and remains in $W \setminus \overset{\circ}{K}$ for $s \in [0, x^0(y) + \alpha]$ when $\alpha > 0$ is small enough. In particular, since $S \subset \overset{\circ}{K}$, $\gamma([0, x^0(y) + \alpha])$ does not intersect S). Then, any inextendible causal extension of γ do not intersect S since $t(\gamma(s))$ is strictly monotonic, the only intersection of γ with $\{t = t(x)\}$ is a point of $\gamma([0, x^0(y) + \alpha])$ which is not in S. □

B.2 Existence and uniqueness theory

The two main propositions are the following (for the existence, we follow [54], 23.2 but without the factorizaton method at the cost of restricting the order 2 case) but we allow P to be an operator on a (complex)vector bundle E (with connection Θ) which do not really introduce new difficulties. We take P of the form $P = \Box_{g,\Theta} + R$ with $R \in \mathrm{Diff}^1(E)$.

Proposition B.2.1. *Let $U \Subset \mathcal{M}$ (open with compact closure).*

- *Problem 1: For $s \in \mathbb{R}$, if $f \in H^s_{(loc)}(\mathcal{M})$ has support in $\overline{\mathcal{M}_+}$, there exists $u \in H^{s+1}_{(loc)}(\mathcal{M})$ with support in $\overline{\mathcal{M}_+}$ such that $Pu = f$ on U.*

- *Problem 2:* *For $s \geq 0$, if v is a vector field such that $vt = 1$ and $f \in \overline{H}^s_{(loc)}(\mathcal{M}_+)$ $u_0 \in H^{s+1}(\mathcal{M}_0)$ and $u_1 \in H^s(\mathcal{M}_0)$, there exits $u \in \overline{H}^{s+1}_{(loc)}(\mathcal{M}_+)$ such that $Pu = f$ on $\mathcal{M}_+$ and $u = u_0$ on $\mathcal{M}_0 \cap U$ and $vu = u_1$ on $\mathcal{M}_0 \cap U$.*

Remark B.2.2. *In case of problem 2, note that $u \in \overline{H}^{s+1}(\overline{\mathcal{M}_+})$ and $Pu \in \overline{H}^s$. In particular, by theoreme B.2.9 in [54], we get that $u \in \overline{H}^{loc}_{(2,s-1)}$ and therefore (by theorem B.2.7) it has a sense to consider to consider traces of u and $\partial_t u$ on $\mathcal{M}_0$.*

Proof. First step: construction of a solution to Problem 1: Let $f \in H^s_{loc}(\mathcal{M})$ with $\text{supp}(f) \subset \overline{\mathcal{M}_+}$.

We take a finite cover $(W_k)_{k=1}^N$ of $\overline{U}$ by open sets as in lemma B.2.5 below. We then take a finite cover $(V_k)_{k=1}^{N'}$ of $\overline{U}$ by open sets as in lemma B.2.3 small enough so that for every $k \in [\![1, N]\!]$ and every $k' \in [\![1, N']\!]$ either $V_{k'} \cap W_k = \emptyset$ or $V_{k'} \subset W_k$. We assume that the cover is minimal in the sense that any strict subfamily does not cover $\overline{U}$.

We then define a covering family of subsets $E_0, ..., E_M$ of the finite set $F = \{V_k, k \in \mathbb{N}\}$ and a family of times $(t^k_{\min})_{k=0}^M$ and $(t^k_{\max})_{k=0}^M$ recursively as follow:

- $t^0_{\min} = \inf t(\overline{U}) - \gamma$ where γ is small enough so that $E_0 = \left\{V_k : t^0_{min} \in t(V_k)\right\}$ is not empty.
- If we have constructed $E_0, ..., E_n$, $t^0_{\min}, ..., t^n_{\min}$ and $t^0_{\max}, ..., t^{n-1}_{\max}$ for $n \neq 0$, we define $A := \cup_{V \in \cup_{i=0}^n E_i} V$ and if $F \setminus \cup_{i=0}^n E_i \neq \emptyset$, then by minimality, $\overline{U} \setminus A \neq \emptyset$. We can therefore define $t^n_{max} := \min t\left(\overline{U} \setminus A\right)$ which exists by compactness. We take $V \in F \setminus \cup_{i=0}^n E_i$ such that $t^n_{max} \in t(V)$. Because t is an open map, there exists $\gamma > 0$ such that $t^{n+1}_{\min} := t^n_{\max} - \gamma \in t(V)$. Note that by taking γ small enough, we can ensure that $t^{n+1}_{\min} > \inf t(\overline{U})$ and, if $n \geq 1$ $t^{n+1}_{\min} > t^{n-1}_{\max}$. We finally define $E_{n+1} := \left\{V' \in F : t^{n+1}_{\min} \in t(V')\right\}$. Note that $E_{n+1} \neq \emptyset$ since $V \in E_{n+1}$. Note also that E_{n+1} is generally not disjoint from $\cup_{i=0}^n E_i$.

This recursive construction ends in a finite number of steps since $(F \setminus \cup_{i=0}^n E_i)$ is a strictly decreasing sequence of finite sets. We denote by M the stopping index (therefore $F = \cup_{i=0}^M E_i$) and we define $t^M_{\max} := \sup \overline{U} + \gamma$ with γ small enough so that $t^M_{\max} \in t(\cup V_k)$. By construction we have:

- For all $k \in [\![1, M-1]\!]$, $t^{k-1}_{\min} < \inf t^k_{\min} < t^{k-1}_{\max} < t^{k+1}_{\min} < t^k_{\max} < t^{k+1}_{\max}$.
- $[0, \sup t(\overline{U})] \subset \cup_{k=0}^M (t^k_{\min}, t^k_{\max})$
- For all $k \in [\![0, M]\!]$, $\left\{x \in U : t(x) \in (t^k_{\min}, t^k_{\max})\right\} \subset \cup_{V \in E_k} V$

Let $(\chi_k)_{k=0}^{M+1}$ be a sequence of non increasing functions in $C^\infty_c(\mathbb{R}, [0,1])$ equal to 1 near $-\infty$ and to 0 near $+\infty$ such that $\text{supp}\chi'_0 \subset (t^0_{\min}, \inf t(\overline{U}))$, for $k \in [\![1, M]\!]$, $\text{supp}\chi'_k \subset (t^k_{\min}, t^{k-1}_{\max})$ and $\text{supp}\chi'_{M+1} \subset (\sup t(\overline{U}), t^M_{\max})$. We define recursively for $k \in [\![0, M]\!]$ the distributions $u_k \in H^{s+1}_{(loc)}(\mathcal{M})$ and $g_k \in H^{s+1}_{(loc)}(t^{-1}((-\infty, t^k_{\max})))$ such that

$$t(\text{supp}(u_k)), t(\text{supp}(g_k)) \subset ((t^k_{\min}, t^k_{\max}) \cap [0, +\infty))$$

and such that

$$Pu_k = \chi_{k+1}(1 - \chi_k)f + [\chi_k, P]g_{k-1} + [P, \chi_{k+1}]g_k \tag{B.4}$$

on U (with the convention that $g_{-1} = 0$). Let $k \in [\![0, M]\!]$. We assume that g_{k-1} has been constructed (recall that $g_{-1} := 0$). For every $V \in E_k$, we choose an open set (call it $W(V)$) in the family (W_k) such that $V \subset W(V)$ and we define $v^V_k \in \overline{H}^{s+1}_{(loc)}(\mathcal{M})$ with

$t(\mathrm{supp}(u_k)) \subset (\max(t^k_{\min}, 0), +\infty)$ and $Pv_k^V = (1-\chi_k)f + [\chi_k, P]g_{k-1}$ on $W(V)$ using lemma B.2.3. We define $v_k \in \overline{H}^{s+1}_{(loc)}(\cup_{V\in E_k} V)$ by $(v_k)_{|V} = (v_k^V)_{|V}$. Note that if $V \cap V' \neq \emptyset$, we have $V \cup V' \subset W(V) \cap W(V')$ and in particular $Pv_k^V = Pv_k^{V'}$ on V and $v_k^V = v_k^{V'}$ on V since V was obtained by lemma B.2.5 (and $t^k_{\min} \in t(V) \cap t(V')$ by construction of E_k). We finally take $u_k := \psi\chi_{k+1}v_k$ and[2] $g_k := \psi v_k$ where $\psi = 1$ on $\overline{U}$ and has support in $\cup V_i$. We have the claimed equality (B.4) on U (note that the commutator term $[P, \psi]$ is harmless since it has support outside of U). Therefore, a solution to Problem 1 is $\sum_{k=0}^M u_k$ since $P(\sum_{k=0}^M u_k) = (\chi_{M+1} - \chi_0)f + [P, \chi_{M+1}]g_M = f$ on U.

Second step reduction of Problem 2 to Problem 1: We take a finite cover $(W_k)_{k=1}^N$ of $\overline{U} \cap \mathcal{M}_0$ by open subset of $\mathcal{M}$ given by lemma B.2.4 below. We take a finite cover $(V_k)_{k=1}^{N'}$ of $\overline{U} \cap \mathcal{M}_0$ by open sets as in lemma B.2.6 below small enough so that for every $k \in [\![1, N]\!]$ and every $k' \in [\![1, N']\!]$ either $V_{k'} \cap W_k = \emptyset$ or $V_{k'} \subset W_k$. We assume that the cover is minimal in the sense that any strict subfamily does not cover $\overline{U} \cap \mathcal{M}_0$. For every V, we choose $W(V)$ in the family (W_k) such that $V \subset W(V)$ and we define $w^V \in \overline{H}^{s+1}(\mathcal{M}_+)$ such that $Pw^V = f$ on $W(V)$ and $w^V = u_0$ on $W(V) \cap \mathcal{M}_0$, $vw^V = u_1$ on $W(V) \cap \mathcal{M}_0$. We then define $w \in \overline{H}^{s+1}(\mathcal{M}_+ \cap \cup_{k=1}^{N'} V_k)$ by its restriction on each V_k: $w_{|V_k\cap\mathcal{M}_+} = w^V_{|V_k\cap\mathcal{M}_+}$. Note that this is well defined sinc if $V \cap V' \neq \emptyset$, then $V' \subset W(V)$ and we have therefore $P(w^V - w^{V'}) = 0$ on $V' \cap \mathcal{M}_+$ with zero Cauchy data on $V' \cap \mathcal{M}_0$ and therefore $w^V - w^{V'} = 0$ on $V' \cap \mathcal{M}_+$ (and therefore on $V \cap V'$). Finally by compactness of $\overline{U} \cap \mathcal{M}_0$, there exists $\gamma > 0$ such that $t > 2\gamma$ on $(\overline{U} \setminus \cup_{k=1}^{N'} V_k) \cap \mathcal{M}_+$. We define $u = \psi\chi(t)w \in \overline{H}^{s+1}(\mathcal{M}_+)$ where ψ is a smooth cutoff equal to 1 in a neighborhood of $\overline{U}$ and equal to zero in a neighborhood of $\mathcal{M} \setminus (\cup_{k=1}^{N'} V_k)$ and $\chi \in C^\infty(\mathbb{R})$ is a smooth cutoff equal to one on $t < \frac{\gamma}{2}$ and equal to zero on $t > \gamma$. Note that u has the correct Cauchy data and $Pu = [P, \psi\chi]u + \psi\chi f \in H^s_{(loc)}(\mathcal{M}_+)$. We use the first step to construct $h \in H^{s+1}(\mathcal{M})$ with support in $t \geq \frac{\gamma}{2}$ and such that $Ph = (1-\chi)f - [P, \chi]f$ on U (which has support on $t \geq \frac{\gamma}{2}$). Then $u + h$ is a solution to problem 2.

□

Lemma B.2.3. *Let $s \in \mathbb{R}$. We prove that for every $x_0 \in \mathcal{M}$, there exists a neighborhood U of x_0 such that for every $f \in H^s_{loc}(\mathcal{M})$ with support in $\{t \geq t_{\min}\}$ (for some $t_{\min} \in t(U)$), there exists $u \in H^{s+1}_{loc}(\mathcal{M})$ with support in $\{t \geq t_{\min}\}$ such that $Pu = f$ on U.*

Proof. First we take V a relatively compact neighborhood of x_0 which is an open of chart for the manifold and an open of trivialization for the bundle. We choose a chart so that t is the first coordinate, and V is sent to a relatively compact open subset V of $\mathbb{R}^{n+1}$. Without loss of generality, we can assume that $t_{\min} = 0$. As usual we define, $V_0 := V \cap \{t = 0\}$. For notation simplicity, we use the same name for objects on $\mathbb{R}^{n+1}$ and objects on $\mathcal{M}$ identified by this local trivialization and coordinates chart. We get an operator $P = G^{\mu,\nu}\partial_\mu\partial_\nu + \mathrm{Diff}^1(V, \mathbb{C}^p)$ defined on V with $G^{0,0} > 0$. In order to use results of section A, we want to find a global inverse Lorentzian metric $\tilde{G}$ on $\mathbb{R}^{n+1}$ constant outside a compact which coincides with G on an open neighborhood V' of 0. To construct such a metric, we only need to take V'' so small that $G_{|V''}$ remains in a convex neighborhood of $G(x_0)$ in the space of Lorentzian matrices (seen as a subset of real values matrices) and then use a partition of unity to glue G with the constant metric equal to $G(x_0)$.

We define $\chi \in C^\infty_c(\mathbb{R}^{n+1}; [0, 1])$ with support in V' and equal to 1 in a neighborhood U of zero and we will use proposition A.0.5 with operator $\tilde{P} := \chi P + (1-\chi)\tilde{G}^{\mu,\nu}\partial_\mu\partial_\nu$ (without loss of generality, we can assume $\tilde{G}^{0,0} = 1$) and T so large that $t < T$ on V'. More precisely, if $s \geq 0$, we define $\tilde{f}$ any extension of $f_{|V}$ in $H^s(\mathbb{R}^{n+1})$ and with support in $t \geq 0$. We then perform a change of time $t' = t + 1$ so that $\tilde{f}$ is supported in $\{t' \geq 1\}$ and

[2] Note that g_k is not globally defined on $\mathcal{M}$ but it is defined on $t^{-1}((-\infty, t^k_{\max}))$

we apply lemma A.0.5 with zero initial condition which provides the solution $\tilde{u}$ (the lemma gives $\tilde{u} \in C^0([0,T], H^s(\mathbb{R}^n)) \cap C^1([0,T], H^{s+1}(\mathbb{R}^n))$ but since $Pu = f$, proposition B.2.9 in [54] gives $u \in \overline{H}^{s+1}((0,T) \times \mathbb{R}^n)$). The estimate in lemma A.0.5 then shows that $\tilde{u} = 0$ on $\{t' < 1\}$ and therefore we can extend it by zero on $\{t' \leq 0\}$. Then $\chi\tilde{u}$ is defined on $\mathcal{M}$ and has the desired property. If $s \leq 0$: We first perform the same change of coordinates as before $t' = t+1$. We show recursively that for $f \in \overline{H}_{(s,m)}(\mathbb{R}_{t'})$ with $\mathrm{supp}(f) \subset [1,+\infty) \times \mathbb{R}^n$, $s \geq -N$ with $N \in \mathbb{N}$ and $m \in \mathbb{R}$, we can find $\tilde{u} \in H_{(s+1,m)}$ with support in $[1,+\infty) \times \mathbb{R}^n$ such that $\tilde{P}\tilde{u} = f$ on $(-\infty, T+1) \times \mathbb{R}^n$). For $N = 0$ it is directly lemma A.0.5 as in the previous case. We assume that the induction hypothesis is true for some $N \in \mathbb{N}$. We take $s \geq -N-1$ and $f \in H_{(s,m)}(\mathbb{R}_{t'} \times \mathbb{R}^n)$ with $\mathrm{supp}(f) \subset [1, T+1] \times \mathbb{R}^n$. By remark after theorem B.2.4 in [54], we can find $f_0 \in H_{(s+1,m-1)}$ and $f_1 \in H_{(s+1,m)}$ both supported on $[1,+\infty) \times \mathbb{R}^n$ such that $f = f_0 + \partial_t f_1$. By the induction hypothesis, we can find $\tilde{u}_1 \in H_{(s+2,m)}$ with support in $[1,+\infty)$ such that $\tilde{P}\tilde{u}_1 = f_1$ on $(-\infty, T+1)$. We have $P\partial_t\tilde{u}_1 = \partial_t f_1 + [\tilde{P}, \partial_t]u_1$. Note that since $[\tilde{P}, \partial_t]$ is a differential operator of order two but with only derivatives with respect to t of order one. Therefore $[\tilde{P}, \partial_t]u_1 \in H_{(s+1,m-1)}$ with support in $[1,+\infty) \times \mathbb{R}^n$. Still by the induction hypothesis, we can find $\tilde{u}_0$ with $P\tilde{u}_0 = f_1 - [\tilde{P}, \partial_t]u_1$ on $(-\infty, T+1)$ and with $\tilde{u}_0 \in H^{s+2,m-1}$ with support in $[1,+\infty)$. Then it suffices to take $\tilde{u} := u_0 + \partial_t u_1$. □

Lemma B.2.4. *Let $s \geq 0$. For every $x_0 \in \mathcal{M}_0$, we can find a neighborhood U of x_0 in $\mathcal{M}$ such that for every $f \in \overline{H}^s_{(loc)}(\mathcal{M}_+)$ and every $u_0 \in H^{s+1}_{(loc)}(\mathcal{M}_0)$, $u_1 \in H^s_{(loc)}(\mathcal{M}_0)$, there exists $u \in \overline{H}^{s+1}_{(loc)}(\mathcal{M}_+)$ with $u_{|_{U_0}} = (u_0)_{|_{U_0}}$, $vu_{|_{U_0}} = (u_1)_{|_{U_0}}$ and $Pu = f$ on U (where $U_0 := U \cap \mathcal{M}_0$).*

Proof. The proof is very similar to the proof of lemma B.2.3. First we take V a relatively compact neighborhood of x_0 which is an open of chart for the manifold and an open of trivialization for the vector bundle. We choose a chart so that $V \cap \{t = 0\}$ is mapped on $t = 0$ in $\mathbb{R}^{n+1}$, x_0 is mapped to 0 and V is sent to a relatively compact open subset V of $\mathbb{R}^{n+1}$. As usual, $V_0 := V \cap \{t = 0\}$. For notation simplicity, we call by the same name objects which are identified by this local trivilalization and coordinates chart. We get an operator $P = G^{\mu,\nu}\partial_\mu\partial_\nu + \mathrm{Diff}^1(V, \mathbb{C}^p)$ defined on V with $G^{0,0} > 0$. We want to find a global inverse Lorentzian metric $\tilde{G}$ on $\mathbb{R}^{n+1}$ constant outside a compact which coincides with G on an open neighborhood V' of 0. We construct such a metric as in the proof of lemma B.2.3. If we have such a metric, we will define for $\chi \in C^\infty_c(\mathbb{R}^{n+1}; [0,1])$ with support in V' and equal to 1 in a neighborhood U of zero and we will use proposition A.0.5 with operator $\tilde{P} := \chi P + (1-\chi)\tilde{G}^{\mu,\nu}\partial_\mu\partial_\nu$ and with data equal to $\tilde{f}$ any extension of $f_{|_V}$ in $H^s(\mathbb{R}^{n+1})$, $\tilde{u}_0$ any extension of $(u_0)_{|_{V_0}}$ in $H^{s+1}(\mathbb{R}^n)$ and $\tilde{u}_1$ any extension of $(u_1)_{|_{V_0}}$ in $H^s(\mathbb{R}^n)$. If we call $\tilde{u}$ the solution (the lemma gives $\tilde{u} \in C^0([0,T], \mathbb{R}^n) \cap C^1([0,T], \mathbb{R}^n)$ but since $Pu = f$, proposition B.2.9 in [54] gives $u \in \overline{H}^{s+1}((0,T), \mathbb{R}^n)$), then $\chi\tilde{u}$ is defined on $\mathcal{M}$ and has the desired properties. □

Lemma B.2.5. *For every $x_0 \in \mathcal{M}$, x_0 has a fundamental system of open neighborhoods $\mathcal{V}(x_0)$ such that for $V \in \mathcal{V}(x_0)$, if $u \in \mathcal{D}'(V)$, $Pu = 0$ on V, and if there exists $t_{min} \in t(V)$ such that $t \geq t_{min}$ on supp(u), then $u = 0$.*

Proof. We fix U a small (relatively compact) neighborhood of trivialization and of local chart containing x_0. We choose a local chart (y_0, y') in which $(y_0, y')(x_0) = 0$ and $t(y_0, y') = t(x_0) + y_0 - |y'|^2$. We define

$$V_\epsilon = \left\{ x : |y'(x)|^2 - \epsilon^2 < y_0(x) < \epsilon^2, |y'(x)| < \sqrt{2}\epsilon \right\}$$

The metric associated with the operator is of the form $G^{\mu,\nu}$ on the local chart and by hypothesis, since $\mathrm{d}t$ is uniformly timelike on V, and since $\mathrm{d}t(0) = \mathrm{d}y_0$, we have $G^{0,0}(0) > 0$. In

particular, for ϵ small enough (depending on the sup norm of derivatives of $G^{0,0}$ on V), we have $G^{0,0} > 0$ on V_ϵ. In particular, P^* is strictly hyperbolic with respect to the level set of the function y_0. We can then apply lemma B.2.3 (with the operator P^* on the manifold V_ϵ with the time function $-y_0$) to find an open neighborhood of x_0 in which we can solve the forcing problem. We can take ϵ' so small so that $V_{\epsilon'}$ is included in this neighborhood. Therefore, for all $\phi \in C_c^\infty(V_{\epsilon'})$ we can find $f \in C^k(V_{\epsilon'})$ (for $k \in \mathbb{N}$ as large as we want) vanishing in $y_0 > \epsilon^2 - \delta$ for $\delta > 0$ such that $y_0 \leq \epsilon^2 - \delta$ on $\mathrm{supp}(\phi)$. We take $u \in \mathcal{D}'(V_{\epsilon'})$ with $Pu = 0$ on $V_{\epsilon'}$ such that there exists $t_{min} \in t(V_{\epsilon'}) = (t(x_0) - \epsilon^2, t(x_0) + \epsilon^2)$ such that $t \geq t_{min}$ on $\mathrm{supp}(u)$. Since either u or f vanishes in a neighborhood of the boundary of $V_{\epsilon'}$, we can do the following computation:

$$\begin{aligned}\langle u, \phi\rangle =& \langle u, P^* f\rangle \\ =& \langle Pu, f\rangle \\ =& 0\end{aligned}$$

Since this is true for all $\phi \in C_c^\infty(V_{\epsilon'})$, $u = 0$. Any $V_{\epsilon''}$ also works if $0 < \epsilon'' < \epsilon'$, therefore we get a fundamental system of neighborhoods with the desired property. □

Lemma B.2.6. *For every $x_0 \in \mathcal{M}_0$, x_0 has a fundamental system of open neighborhoods (in $\mathcal{M}$) $\mathcal{V}(x_0)$ such that for $V \in \mathcal{V}(x_0)$, if $u \in \overline{H}^{-N}(V^+)$ for some $N > 0$, $Pu = 0$ on V, and $u = 0$, $vu = 0$ (v is a vector field such that $vt = 1$) on V^0, then $u = 0$. As usual $V^+ := V \cap \mathcal{M}_+$ and $V^0 := V \cap \mathcal{M}_0$.*

Proof. First note that by theorem B.2.9 in [54], for V open neighborhood of x_0, for all $u \in \overline{H}^{-N}(V)$, $Pu = 0$ implies $u \in H^{loc}_{(k,-N-k)}(V)$ for all $k \in \mathbb{N}$ and by theorem B.2.7, the traces of u and vu on $V \cap \mathcal{M}_0$ are well defined and therefore, the statement makes sense. We construct V_ϵ exactly as in lemma B.2.5 and we define $V_\epsilon^+ := V_\epsilon \cap \mathcal{M}_+$ and $V_\epsilon^0 = V_\epsilon \cap \mathcal{M}_0$. As in this lemma B.2.5 if we choose $\epsilon > 0$ small enough, for any $\phi \in C_c^\infty(V_\epsilon^+)$, we can construct $f \in C^k\left(\overline{V_\epsilon^+}\right)$ (for k as large as we want) such that $f = 0$ in a neighborhood of $y_0 = \epsilon^2$ and $P^* f = \phi$. We need to prove that for every $\psi \in \overline{H}_{(2,-N-2)}(V_\epsilon^+)$,

$$\langle \psi, P^* f\rangle_{V_\epsilon^+} = \langle P\psi, f\rangle_{V_\epsilon^+} + \langle \gamma(\psi), A_1\gamma(f) + A_2\gamma(vf)\rangle_{V_\epsilon^0} + \langle \gamma(v\psi), B_1\gamma(f) + B_2\gamma(vf)\rangle_{V_\epsilon^0}$$

where γ denotes the trace operator on V_ϵ^0, $A_1 \in \mathrm{Diff}^2(V_\epsilon^0, E|_{V_\epsilon^0})$, $A_2, B_1 \in \mathrm{Diff}^1(V_\epsilon^0, E|_{V_\epsilon^0})$, $B_2 \in \mathrm{Diff}^0(V_\epsilon^0, E|_{V_\epsilon^0})$. This statement is true if $\psi \in \overline{C}_c^\infty(V_\epsilon^+)$ by integration by part (and using that $f = 0$ near $y_0 = \epsilon^2$ which gives in particular that $\gamma(f)$ and $\gamma(vf)$ are compactly supported in V_ϵ^0) and by theorem B.2.7 in [54], we can use the density of $\overline{C}_c^\infty(V_\epsilon^+)$ in $\overline{H}_{(2,-N-2)}(V_\epsilon^+)$ to get the general statement.

In particular, we can apply the equality to u and we get:

$$\langle u, \phi\rangle = 0$$

for all $\phi \in C_c^\infty(V_\epsilon^+)$. □

Lemma B.2.7. *For all $y_0 \in \mathcal{M}$, there exists an open neighborhood U of y_0 such that for every $y_1 \in U$:*

- *For every $u \in \mathcal{D}'(\mathcal{M})$ such that $\inf(t(supp(u))) \geq t(y_0)$ and $Pu = 0$ on $I^-(y_1)$ we have $u = 0$ on $I^-(x_1)$.*
- *For every $u \in \mathcal{D}'(\{t > t(y_1)\})$ such that $u_{|_{I^-(y_1)\cap\{t>t(y_0)\}}} \in \overline{H}^{-N}(I^-(y_1) \cap \{t > t(y_0)\})$ for some finite N, $Pu = 0$ on $I^-(y_1) \cap \{t > t(y_0)\}$ and the traces of u and vu on $\{t = t(y_0)\} \cap I^-(y_1)$ vanish, we have $u = 0$ on $I^-(y_1) \cap \{t > t(y_0)\}$.*

Proof. The idea is to take V convex neighborhood of y_0 given by lemma B.1.4 and $U = D(S) \subset V$ for some neighborhood S of y_0 in $\{t = t(y_0)\}$ such that $D(S)$ relatively compact neighborhood of y_0 and $\overline{D(S)} \subset V$ (exists by lemma B.1.5) We fix $y_1 \in U$, and we take normal geodesic coordinates $(x^\mu)_{\mu=0}^n$ centered at y_1 on V. We define the function $\tilde{t} = (x^0)^2 - \sum_{i=1}^n (x^i)^2$. For all $v = v^i\partial_i$, we have $\mathrm{d}\tilde{t}(v) = 2x^0v^0 - 2\sum_{i=1}^n x^iv^i$. Since coordinates are normal geodesic, Gauss's lemma gives $g_{\mu,\nu}(x)x^\mu v^\nu = x^0v^0 - \sum_{i=0}^n x^iv^i$ for all x in the image of the coordinate chart and $v \in \mathbb{R}^n$. In particular, we deduce that $\nabla\tilde{t} = x^\mu\partial_\mu$ and $g(\nabla\tilde{t}, \nabla\tilde{t}) = \tilde{t}$. We deduce that $\tilde{t}$ (defined on V) is a time coordinates on $\tilde{t}^{-1}((c, +\infty))$ for every $c > 0$. We define $Q_c := \{x^0 < 0\} \cap \{t > t(y_0)\} \cap \{\tilde{t} > c\}$ for all $c > 0$. It is clear with this definition that $Q_c \subset I^-(y_1) \cap \{t > t(y_0)\}$. Moreover, we have

$$I^-(y_1) \cap \{t > t(y_0)\} = \cup_{c>0} Q_c. \tag{B.5}$$

First note that $I^-(y_1) \cap \{t > t(y_0)\} \subset D^+(S) \subset V$. Let $y_2 \in I^-(y_1) \cap \{t > t(y_0)\}$, we have a past oriented timelike geodesic from y_1 to y_2. The geodesic is the radial one since V was given by lemma B.1.4 and $y_1, y_2 \in V$. In normal geodesic coordinates, it means that $d := (x^0(y_1) - x^0(y_2))^2 - \sum_{i=1}^n (x^i(y_1) - x^i(y_2))^2 > 0$ and $x^0(y_2) < 0$. Therefore, if we take $c < d$, we have $y_2 \in Q_c$.

Let $\phi \in C_c^\infty(I^-(y_1) \cap \{t > t(y_0)\})$. In particular, there exists $c > 0$ such that $\mathrm{supp}(\phi) \subset Q_c$. We have $\overline{Q_c}$ compact and included in $\tilde{t}^{-1}((c', +\infty)) \cap \{x^0 < 0\}$ for $c' < c$. Indeed, since $Q_c \subset D(S)$ which is relatively compact in V, we already know that $\overline{Q_c} \subset V$ is compact. Moreover, $\overline{Q_c} = \{x^0 < 0\} \cap \{t \geq t(y_0)\} \cap \{\tilde{t} \geq c\}$ which is included in $\tilde{t}^{-1}((c, +\infty)) \cap \{x^0 < 0\}$. Therefore, we can use proposition B.2.1 (Problem 1 with time coordinate $-\tilde{t}$ and Sobolev embedding) to find $f \in C^k(\tilde{t}^{-1}((c, +\infty)) \cap \{x^0 < 0\})$ for some $k \in \mathbb{N}$ (which is fixed but can be chosen as large as we want) and such that $P^*f = \phi$ and $\mathrm{supp}(f) \subset \tilde{t}^{-1}((c', +\infty)) \cap \{x^0 < 0\}$. In particular, if u is a distribution of finite order on $I^-(y_1)$ with support in $\{t \geq t(y_0)\}$ and such that $Pu = 0$ in $I^-(y_1)$, we get that $\mathrm{supp}(f) \cap \mathrm{supp}(u) \subset \overline{Q_c}$ is compact in $\tilde{t}^{-1}((c', +\infty)) \cap \{x^0 < 0\} \subset I^-(y_1)$ and therefore we have, whenever k is large enough (we use the finite order of u here):

$$\langle u, \phi\rangle = \langle u, P^*f\rangle = \langle Pu, f\rangle = 0.$$

This holds for all ϕ with support compact in $I^-(y_1) \cap \{t > t(y_0)\}$ with $c > 0$ and therefore $u = 0$ on $I^-(y_1) \cap \{t > t(y_0)\}$. Together with the support assumption on u, we get that $\mathrm{supp}(u) \cap I^-(y_1) \subset \{t = t(y_0)\}$. Using theorem B.2.9 in [54] and the fact that $Pu = 0$ on $I^-(y_1)$, we get the continuity of u with respect to t (regarded as a family in a weak Sobolev space) and deduce that $u = 0$ on $I^-(y_1)$. Note that if u is the restriction of some $\tilde{u}$ defined on $\mathcal{M}$, it has finite order since $I^-(y_1) \cap \{t > t(y_1)\} \subset D(S)$ is relatively compact in $\mathcal{M}$.

Now suppose that u is a distribution in $\overline{H}^{-N}(I^-(y_1) \cap \{t > t(y_0)\})$ for some finite N, such that $Pu = 0$ and the traces of u and vu (which exist in $H_{(loc)}^{-N-1}(\{t = t(y_0)\} \cap I^-(y_1))$ by theorem B.2.9 of [54]) on the boundary $\{t = t(y_0)\} \cap I^-(y_1)$ vanishes. For ϕ with compact support in $\tilde{t}^{-1}((c, +\infty)) \cap \{x^0 < 0\} \cap \{t > t(y_0)\}$, we can still find $f \in C^k(\tilde{t}^{-1}((c, +\infty)) \cap \{x^0 < 0\})$ for some arbitrarily large $k \in \mathbb{N}$ and such that $P^*f = \phi$ and $\mathrm{supp}(f) \subset \tilde{t}^{-1}((c', +\infty)) \cap \{x^0 < 0\}$ with $c < c'$. We have seen that $Q_{c'}$ is relatively compact in $\tilde{t}^{-1}((c, +\infty)) \cap \{x^0 < 0\}$ and therefore also in $I^-(y_1)$. In particular, $\tilde{t}^{-1}((c', +\infty)) \cap \{t = t(y_0)\} \subset \overline{Q_{c'}} \cap \{t = t(y_0)\}$ is relatively compact in $\{t = t(y_1)\} \cap I^-(y_1)$. We deduce that the trace of f on $\{t = t(y_0)\} \cap I^-(y_1)$ has compact support. We can then use an adaptation of the approximation argument in proof of lemma B.2.6 (with k large enough) to get:

$$\langle u, \phi\rangle = 0$$

□

Proposition B.2.8. *If $U \Subset \mathcal{M}$ (open with compact closure) satisfies the condition that every inextendible timelike curve for the induced metric on U intersect $\mathcal{M}_0 \cap U$ exactly once, then we have uniqueness for the restriction of u to U in proposition B.2.1. Moreover, we have a constant C independent of u (u_0, u_1 if we are in the case of Problem 2) and f (but a priori depending on U and s) such that:*

- *For problem 1:*

$$\|u\|_{\overline{H}^{s+1}(U)} \leq C \|f\|_{\overline{H}^{s}(U)}$$

- *For problem 2:*

$$\|u\|_{\overline{H}^{s+1}(U\cap\mathcal{M}_+)} \leq C \Big(\|f\|_{\overline{H}^{s}(U\cap\mathcal{M}_+)} + \|u_0\|_{\overline{H}^{s+1}(U\cap\mathcal{M}_0)} + \|u_1\|_{\overline{H}^{s}(U\cap\mathcal{M}_0)} \Big)$$

Proof. Let U be an open subset of $\mathcal{M}$ as defined in the proposition. Let u be a distribution on $\mathcal{M}$ such that $Pu = 0$ on U and $t \geq 0$ on $\mathrm{supp}(u)$. Then we prove that the set $\{s : u = 0 \text{ on } U \cap \{t < s\}\}$ is unbounded. We argue by contradiction, assuming this set is bounded (it is non empty since it contains 0). We then define s_0 its supremum. By definition, we have $t \geq s_0$ on $\mathrm{supp}(u) \cap U$. We cover the compact set $\overline{\{t = s_0\} \cap U}$ by a finite number of open sets $(V_i)_{i=1}^N$ as in lemma B.2.7. For every $x \in V_i \cap U \cap \{t \geq s_0\}$, we have $I^-(x) \cap \{t \geq 0\} \subset U$ (by the hypothesis on U) and therefore the definition of s_0 implies that $u = 0$ on $I^-(x) \cap \{t < s_0\}$ and therefore by lemma B.2.7, $u = 0$ on $I^-(x)$. We deduce that for all $i \in [\![1, N]\!]$, $u = 0$ on $U \cap V_i \cap \{t \geq s_0\}$. Since $W := \overline{U} \cap \{t \geq s_0\} \setminus \cup_{i=1}^N V_i$ is compact and $t > s_0$ on W, there exists $s' > s_0$ such that $\inf_W t \geq s'$. Therefore, we have $U \cap \{s_0 \leq t < s'\} \subset \cup_{i=1}^N V_i \cap U$ and $u = 0$ on $U \cap \{t < s'\}$. This contradicts the maximality of s_0 and therefore $u = 0$ on U.

Now, we assume that $u \in H^{-N}(\mathcal{M}_+)$ for some $N \in \mathbb{N}$ and $Pu = 0$ in $U \cap \mathcal{M}_+$, and the traces of u and vu on $\mathcal{M}_0 \cap U$ vansih. We cover the compact set $\overline{\{t = s_0\} \cap U}$ by a finite number of open sets $(V_i)_{i=1}^N$ as in lemma B.2.7. As before we get that u vanishes on $I^-(x) \cap \mathcal{M}_+$ for every $x \in U \cap V_i$. We deduce that there exists $s' > 0$ such that $u = 0$ on $\{0 < t < s'\} \cap U$. Therefore we can extend u by zero so that the extension is in $H^{-N}(\mathcal{M})$ and satisfies the hypotheses in the previous part of the proof. Therefore, $u = 0$ on U.

Now we use the open mapping theorem to prove the quantitative bounds. More precisely, P defines a linear operator between the Banach spaces

$$X^{s+1} := \Big\{ u \in \overline{H}^{s+1}(U) : Pu \in \overline{H}^{s}(U) \Big\}$$

(endowed with the graph norm) and $\overline{H}^{s}(U)$. This operator is continuous and surjective since for $f \in \overline{H}^{s}(U)$, we can take u_f the restriction to U of any solution to problem 1 (but with time function $t - \inf t(U) + 1$) with data $\tilde{f}$ ($\tilde{f}$ being any extension of f in $H^s(\mathcal{M})$ with support in $\{t \geq \inf t(U) - 1\}$). Such a solution exists by proposition B.2.1 and we have $Pu_f = f$. Moreover, if $u \in X^{s+1}$ is such that $Pu = 0$, then we can find an extension $\tilde{u}$ to $\mathcal{M}$ with support in $\{t \geq \inf t(U) - 1\}$ and by the uniqueness statement already proven, we have $\tilde{u} = 0$ on U and therefore $u = 0$. By the open mapping theorem, the inverse map is continuous and we have:

$$\|u\|_{\overline{H}^{s+1}(U)} \leq C \|Pu\|_{\overline{H}^{s}(U)}$$

Let $s \geq 0$ and let γ be the trace operator on $U \cap \mathcal{M}_0$. Then $(P, \gamma, \gamma(v.))$ defines a linear operator between the Banach spaces

$$Y^{s+1} := \left\{ u \in \overline{H}^{s+1}(U \cap \mathcal{M}_+) : Pu \in \overline{H}^s(U), \gamma \in \overline{H}^{s+1}(\mathcal{M}_0 \cap U), \gamma(vu) \in \overline{H}^s(\mathcal{M}_0 \cap U) \right\}$$

with norm:

$$\|u\|^2_{Y^{s+1}} := \|u\|^2_{\overline{H}^{s+1}(U \cap \mathcal{M}_+)} + \|\gamma(u)\|^2_{\overline{H}^{s+1}(U \cap \mathcal{M}_0)} + \|\gamma(vu)\|^2_{\overline{H}^s(U \cap \mathcal{M}_0)} + \|Pu\|^2_{\overline{H}^s(U \cap \mathcal{M}_+)}$$

and the space $\overline{H}^s(U \cap \mathcal{M}_+) \oplus \overline{H}^{s+1}(U \cap \mathcal{M}_0) \oplus \overline{H}^s(U \cap \mathcal{M}_0)$. This operator is surjective by theorem B.2.1 and injective by the uniqueness proved before. Therefore, by the open mapping theorem, the inverse is continuous and therefore we have:

$$\|u\|_{\overline{H}^{s+1}(U)} \leq C \left(\|\gamma(u)\|_{\overline{H}^{s+1}(U \cap \mathcal{M}_0)} + \|\gamma(vu)\|_{\overline{H}^s(U \cap \mathcal{M}_0)} + \|Pu\|_{\overline{H}^s(U \cap \mathcal{M}_+)} \right)$$

□

The following propositions are consequences of proposition B.2.8 and B.2.1:

Proposition B.2.9. *If $\mathcal{M}_0$ in a Cauchy hypersurface for $\mathcal{M}$, then we have a unique global solution for each problem.*

Proof. We take an increasing sequence (U_n^0) of relatively compact open subset of $\mathcal{M}_0$ such that $\cup_{n \in \mathbb{N}} U_n^0 = \mathcal{M}_0$. We call $\tilde{t}$ the global time function defined on $\mathcal{M}$ (and inducing a diffeomorphism $\mathcal{M} := \mathcal{M}_0 \times \mathbb{R}_{\tilde{t}}$, we define $U_n^{[-n,n]} := \tilde{t}^{-1}([-n,n]) \cap U_n$ (where U_n is the causal closure of U_n^0) we have $\cup_{n \in \mathbb{N}} U_n^{[-n,n]} = \mathcal{M}$. For each $n \in \mathbb{N}$ we apply Proposition B.2.1 with $U = \overset{\circ}{U}{}_n^{[-n,n]}$ to construct a solution u_n. We define u the only distribution such that $u_{|_{U_n^{[-n,n]}}} = (u_n)_{|_{U_n^{[-n,n]}}}$ (and by Proposition B.2.8, this is well defined). □

Proposition B.2.10 (Local approximation theorem)**.** *We assume that $\mathcal{M}_0$ in a Cauchy hypersurface for $\mathcal{M}$. We have a version for each problem:*

- *Let u be a global solution to Problem 1 with $f \in H^s_{(loc)}$ supported in $\overline{\mathcal{M}_+}$ (it exists by Proposition B.2.9). Then if we take (f_n) a sequence of smooth functions converging to f in $H^s_{(loc)}$ (such a sequence can be constructed using a partition of unity) and if we call u_n the solution of Problem 1 with data f_n, we have that (u_n) converges to u in $\overline{H}^{s+1}(U)$ for every relatively compact open subset $U \subset \mathcal{M}$.*
- *Let u be a global solution to Problem 2 with $f \in H^s_{(loc)}(\mathcal{M}_+)$ and initial data $u_0 \in H^{s+1}_{(loc)}(\mathcal{M}_0)$, $u_1 \in H^s_{(loc)}(\mathcal{M}_0)$ (it exists by Proposition B.2.9). Then if we take (f_n) a sequence of smooth functions converging to f in $H^s_{(loc)}$, (u_0^n) smooth functions on $\mathcal{M}_0$ converging to u_0 in $H^{s+1}_{(loc)}(\mathcal{M}_0)$ and (u_1^n) smooth functions on $\mathcal{M}_0$ converging to u_1 in $H^s_{(loc)}(\mathcal{M}_0)$ (such sequences can be constructed using partitions of unity) and if we call u_n the solution of Problem 2 with data f_n, u_0^n, u_1^n, we have that u_n converge to u in $\overline{H}^{s+1}(U)$ for every relatively compact open subset $U \subset \overline{M}_+$.*

Proof. We define V an open relatively compact subset satisfying the condition of proposition B.2.8 and containing U. It exits since $\mathcal{M}$ is globally hyperbolic. For example, we can take $V := \overset{\circ}{U}{}_n^{[-n,n]}$ for n large enough (where $U_n^{[-n,n]}$ is defined as in the proof of proposition B.2.9). Then we can use the estimate of Proposition B.2.8 on $u - u_n$ and we get the convergence in $\overline{H}^{s+1}(V)$, hence in $\overline{H}^{s+1}(U)$. □

Appendix C

Microlocal analysis: a one dimensional introduction

In this section, the goal is to present the microlocal method through its application to a simple which illustrates the different phenomena involved in the main analysis (elliptic estimates, radial points, effect of the subprincipal symbol, propagation of singularity). Our goal is not to provide a comprehensive introduction to pseudodifferential calculus, scattering and b calculus and semiclassical analysis, we refer the reader to [104], [54, Chapter XVIII], [74] and [110] for this.

Our goal is rather to study in details and in a self-contained manner an example on $\mathbb{R}$ for which we can limit the technical tools and draw pictures.

C.1 Technical preliminaries

Definition C.1.1. *Let $m \in \mathbb{R}$. We define $S^m(\mathbb{R})$ as the set of smooth functions f such that for every $k \in \mathbb{N}$ there exists $C_k > 0$ such that for all $x \in \mathbb{R}$:*

$$\left|\partial_x^k f(x)\right| \leq C_k \langle x\rangle^{m-k}.$$

For $\alpha \in \mathbb{N}$, we define the seminorms:

$$\|f\|_{S^m_\alpha} := \sum_{j=0}^{\alpha} \left\|\langle x\rangle^{j-m} \partial_x^j f\right\|_\infty$$

Definition C.1.2. *For $g \in S^m(\mathbb{R})$, we define the Fourier multiplier $g(D_x)$ by:*

$$g(D_x)u := \mathcal{F}^{-1} g(\xi)\mathcal{F}(u)$$

where $\mathcal{F}$ denotes the Fourier transform. It is well defined from $\mathcal{S}'$ to $\mathcal{S}'$ and the restriction to $\mathcal{S}$ is continuous from $\mathcal{S}$ to $\mathcal{S}$.

Definition C.1.3. *For $m, l \in \mathbb{R}$, we define the Sobolev spaces:*

$$H^{m,l} = \left\{u \in \mathcal{S}' : \langle x\rangle^l \langle D_x\rangle^m u \in L^2\right\}$$

endowed with the norm:

$$\|u\|_{H^{m,l}} = \left\|\langle x\rangle^l \langle D_x\rangle^m u\right\|_{L^2}$$

Remark C.1.4. *These Sobolev spaces correspond to scattering Sobolev spaces in the compactification of $\mathbb{R}$ given by the boundary defining function $\rho = \langle x\rangle^{-1}$.*

Remark C.1.5. *Note that $H^{m,l}$ are Banach spaces. We have $\langle x\rangle^l \langle D_x\rangle^m : H^{m,l} \to L^2$ is a surjective isometry with inverse $\langle D_x\rangle^{-l} \langle x\rangle^{-m}$. In particular, since the latter sends a Schwartz function to a Schwartz function and since $\mathcal{S}$ is dense in L^2, we deduce that $\mathcal{S}$ is dense in $H^{m,l}$. We check (using the definition) that the dual of $H^{m,l}$ is identified with $H^{-m,-l}$. Note that if $m, l \geq 0$, we have $H^{m,l} \subset L^2$. Also note that if $l' \geq l$, we have $H^{m,l'} \leq H^{m,l}$.*

Lemma C.1.6. *Let $m, l \in \mathbb{R}$. For any $m', l' \in \mathbb{R}$ such that $m' - m > 1$ and $l' - l > 3$, we have the continuous inclusion:*

$$H^{m',l'} \subset H^{m,l}.$$

Remark C.1.7. *This lemma is not optimal. We later prove the much stronger property $H^{m',l'} \subset H^{m,l}$ for all m', l' such that $m' \geq m$, $l' \geq l$ (see corollary C.1.11).*

Proof. Since $\langle D_x\rangle^{-m'} \langle x\rangle^{-l'} : L^2 \to H^{m',l'}$ and $\langle x\rangle^l \langle D_x\rangle^m : H^{m,l} \to L^2$ are isometries, it is enough to prove that $A := \langle x\rangle^l \langle D_x\rangle^{m-m'} \langle x\rangle^{-l'}$ is bounded on L^2. Let $u \in \mathcal{S}$.

$$Au(x) = \langle x\rangle^l \frac{1}{2\pi} \iint e^{i\xi(x-y)} \langle \xi\rangle^{m-m'} \langle y\rangle^{-l'} u(y) \, \mathrm{d}y \, \mathrm{d}\xi.$$

The previous integral is well defined[1] since the Fourier transform of $\langle y\rangle^{-l'} u$ is a Schwartz function of ξ . Using the relation $\langle x-y\rangle^{-2k} \langle D_\xi\rangle^{2k} e^{i\xi(x-y)} = e^{i\xi(x-y)}$ for $k \in \mathbb{N}$, we can perform an integration by part in the ξ-integral.

$$Au(x) = \langle x\rangle^l \frac{1}{2\pi} \iint e^{i\xi(x-y)} \langle x-y\rangle^{-2k} \langle D_\xi\rangle^{2k} \left(\langle \xi\rangle^{m-m'}\right) \langle y\rangle^{-l'} u(y) \, \mathrm{d}y \, \mathrm{d}\xi$$

If $l > -1$, we choose $k \in \mathbb{N}$ such that $l + \frac{1}{2} < 2k < l' - \frac{1}{2}$ (it exists since $l' - \frac{1}{2} - (l + \frac{1}{2}) > 2$). With this choice we have

$$\begin{aligned} l - 2k &< -\frac{1}{2} \\ -l' + 2k &< -\frac{1}{2} \end{aligned}$$

Using the relation $\langle x-y\rangle^{-2k} \leq \langle x\rangle^{-2k} \langle y\rangle^{2k}$ and the fact that $\langle D_\xi\rangle^{2k} \left(\langle \xi\rangle^{m-m'}\right) \lesssim \langle \xi\rangle^{m-m'}$, we deduce:

$$|Au(x)| \lesssim \langle x\rangle^{l-2k} \iint \langle \xi\rangle^{m-m'} \langle y\rangle^{2k-l'} |u(y)| \, \mathrm{d}y \, \mathrm{d}\xi$$

Using that $m - m' < -1$, we get:

$$|Au(x)| \lesssim \langle x\rangle^{l-2k} \int \langle y\rangle^{2k-l'} |u(y)| \, \mathrm{d}y$$

Using the Cauchy-Schwarz inequality and the fact that $2k - l' < -\frac{1}{2}$:

$$|Au(x)|^2 \lesssim \langle x\rangle^{2l-4k} \|u(y)\|_{L^2}^2$$

Finally, using that $2l - 4k < -1$, we get the bound $\|Au(x)\|_{L^2} \leq C \|u\|_{L^2}$ for a constant C independent of u. We conclude by density of $\mathcal{S}$ in L^2. If $l \leq -1$, we choose $-l' + \frac{1}{2} < 2k < l - \frac{1}{2}$ instead and use the relation $\langle x-y\rangle^{-2k} \leq \langle x\rangle^{2k} \langle y\rangle^{-2k}$ □

[1] (provided we first integrate with respect to y and then with respect to ξ)

Although we do not need the full machinery of pseudodifferential operators at this point, we make use of the following lemma (which has a pseudodifferential flavor):

Lemma C.1.8. *Let $f \in S^l(\mathbb{R})$ and $g \in S^m(\mathbb{R})$. For every $M \in \mathbb{N}$, there exists $N \in \mathbb{N}$ such that:*

$$g(D_x)f(x) = f(x)g(D_x) + \sum_{j=1}^{N} \frac{\partial_x^j f(x)(D_\xi^j g)(D_x)}{j!} + R_N$$

where R_N is a continuous operator on $\mathcal{S}'$ (which is continuous on $\mathcal{S}$) and bounded from $H^{-2M,-2M}$ to $H^{2M,2M}$. Moreover, the constant N can be chosen depending only on M, m and l. There exists $C > 0$ a universal constant depending only on M, l and m and there exists $\alpha, \beta \in \mathbb{N}$ (depending on M,l and m) such that

$$\|R_N\|_{\mathcal{L}(H^{-2M,-2M},H^{2M,2M})} \leq C \left\|\partial_x^{N+1} f\right\|_{S_\alpha^{l-N-1}} \left\|\partial_\xi^{N+1} g\right\|_{S_\beta^{m-N-1}}$$

Proof. Let $u \in \mathcal{S}$.

$$g(D_x)(fu)(x) = \frac{1}{2\pi} \iint e^{i\xi(x-y)} g(\xi) f(y) u(y) \, \mathrm{d}y \, \mathrm{d}\xi \tag{C.1}$$

Note that the previous integral is well defined (but the order of integration is important) since $\iint e^{i\xi(x-y)} f(y)u(y) \, \mathrm{d}y$ is a Schwartz function of the variable ξ. Let $N \in \mathbb{N}$ (it will be chosen large later). By Taylor's formula we have:

$$f(y) = f(x) + \sum_{j=1}^{N} (y-x)^j \frac{f^{(j)}(x)}{j!} + (y-x)^{N+1} \int_0^1 \frac{(1-s)^N}{N!} f^{(N+1)}(x + s(y-x)) \, \mathrm{d}s$$

Replacing in (C.1) and using integration by part in the integral with respect to ξ, we get:

$$\begin{aligned} g(D_x)(fu)(x) =& \frac{1}{2\pi} \iint e^{i\xi(x-y)} f(x) g(\xi) u(y) \, \mathrm{d}y \, \mathrm{d}\xi + \sum_{j=1}^{N} \frac{1}{2\pi} \iint e^{i\xi(x-y)} \frac{\partial_x^j f(x) D_\xi^j g(\xi)}{j!} u(y) \, \mathrm{d}y \, \mathrm{d}\xi \\ &+ \frac{1}{2\pi} \iint e^{i\xi(x-y)} R_N(x,y,\xi) u(y) \, \mathrm{d}y \, \mathrm{d}\xi \end{aligned}$$

where

$$R_N(x,y,\xi) := D_\xi^{N+1} g(\xi) \int_0^1 \frac{(1-s)^N}{N!} f^{(N+1)}(x + s(y-x)) \, \mathrm{d}s.$$

For $u \in \mathcal{S}$, we define

$$R_N u := \frac{1}{2\pi} \iint e^{i\xi(x-y)} R(x,y,\xi) u(y) \, \mathrm{d}y \, \mathrm{d}\xi.$$

We have the claimed equality on $\mathcal{S}$. Using that the other terms of the equality are continuous on $\mathcal{S}'$ and the density of $\mathcal{S}$ in $\mathcal{S}'$, we deduce that R_N has a unique continuous extension to $\mathcal{S}'$ and the equality holds on $\mathcal{S}'$. The continuity of $\mathcal{S}$ is a consequence of the equality since all the other terms are continuous from $\mathcal{S}$ to $\mathcal{S}$. The last step is to prove that $\langle x \rangle^{2M} \langle D_x \rangle^{2M} R_N \langle D_x \rangle^{2M} \langle x \rangle^{2M}$ is bounded on L^2 if N is chosen large enough. By density of

$\mathcal{S}$ in L^2 it is enough to prove the estimate for $u \in \mathcal{S}$. Note that for $k, k', k'' \in \mathbb{N}$:

$$D_x^k D_y^{k'} \left(e^{i\xi(x-y)} \langle x-y \rangle^{-2k''} \langle D_\xi \rangle^{2k''} R(x,y,\xi) \right) =$$
$$\sum_{\substack{j_1+j_2+j_3=k \\ j_1'+j_2'+j_3'=k'}} \frac{k!k'!}{j_1!j_2!j_3!j_1'!j_2'!j_3'!} (-1)^{j_1'} \xi^{j_1+j_1'} e^{i\xi(x-y)} D_x^{j_2} D_y^{j_2'} \langle x-y \rangle^{-2k''} D_x^{j_3} D_y^{j_3'} \langle D_\xi \rangle^{2k''} R_N(x,y,\xi)$$
$$D_x^{j_3} D_y^{j_3'} \langle D_\xi \rangle^{2k''} R_N(x,y,\xi) =$$
$$\langle D_\xi \rangle^{2k''} D_\xi^{N+1} g(\xi) \int_0^1 \frac{(1-s)^N}{N!} s^{j_3'} (1-s)^{j_3} f^{(N+1+j_3+j_3')}(x+s(y-x)) \, \mathrm{d}s$$
$$\left| D_x^{j_2} D_y^{j_2'} \langle x-y \rangle^{-2k''} \right| \lesssim_{k,k',k''} \langle x-y \rangle^{-2k''}$$

Moreover note that:

$$\begin{aligned} \left| f^{(N+1+j_3+j_3')}(x+s(y-x)) \right| &\lesssim \left\| \partial_x^{N+1} f \right\|_{S^{l-N-1}_{k+k'}} \langle x+s(y-x) \rangle^{l-N-1-j_3-j_3'} \\ &\lesssim \left\| \partial_x^{N+1} f \right\|_{S^{l-N-1}_{k+k'}} \langle x+s(y-x) \rangle^{l-N-1} \\ &\lesssim \left\| \partial_x^{N+1} f \right\|_{S^{l-N-1}_{k+k'}} \langle x \rangle^{l-N-1} \langle y-x \rangle^{|l-N-1|} \end{aligned}$$

Therefore, we have:

$$\begin{aligned} \left| \langle D_x \rangle^{2M} \langle D_y \rangle^{2M} e^{i\xi(x-y)} \langle x-y \rangle^{-2k''} \langle D_\xi \rangle^{2k''} R_N(x,y,\xi) \right| \lesssim_{M,k''} \\ \left\| \partial_x^{N+1} f \right\|_{S^{l-N-1}_{4M}} \left\| \partial_\xi^{N+1} g \right\|_{S^{m-N-1}_{2k''}} \langle \xi \rangle^{m-N-1+4M} \langle x \rangle^{l-N-1} \langle y-x \rangle^{|l-N-1|-2k''}. \end{aligned}$$

Using the relation

$$\langle y-x \rangle^k \leq \langle y \rangle^k \langle x \rangle^{|k|}$$

we get for $N \geq l-1$:

$$\begin{aligned} \left| \langle D_x \rangle^{2M} \langle D_y \rangle^{2M} e^{i\xi(x-y)} \langle x-y \rangle^{-2k''} \langle D_\xi \rangle^{2k''} R(x,y,\xi) \right| \lesssim_{M,k''} \\ \left\| \partial_x^{N+1} f \right\|_{S^{l-N-1}_{4M}} \left\| \partial_\xi^{N+1} g \right\|_{S^{m-N-1}_{2k''}} \langle \xi \rangle^{m-N-1+4M} \langle x \rangle^{2l-2N-2+2k''} \langle y \rangle^{N+1-l-2k''}. \end{aligned} \tag{C.2}$$

We choose $N \in \mathbb{N}$ such that $N > \max(m+4M+1, l+4M+2)$. We choose $k'' \in \mathbb{N}$ such that $N + \frac{3}{2} - l + 2M < 2k'' < 2N + \frac{3}{2} - 2l - 2M$ (exists since the interval is strictly larger than 2). With these choices, we have:

$$\begin{aligned} m - N - 1 + 4M &< -1 \\ 2l - 2N - 2 + 2k'' + 2M &< -\frac{1}{2} \\ N + 1 - l - 2k'' + 2M &< -\frac{1}{2} \end{aligned}$$

Also note that the minimal choices for N and k'' are functions of M, m and l and therefore the families of integers $\alpha := 4M$ and $\beta := 2k''$ depends only on M, m and l.

By integration by part with respect to y we have

$$\begin{aligned} v(x) :=& \langle x\rangle^{2M} \langle D_x\rangle^{2M} R_N \langle D_x\rangle^{2M} \langle x\rangle^{2M} u \\ =& \frac{\langle x\rangle^{2M}}{2\pi} \iint \langle D_x\rangle^{2M} \left(e^{i\xi(x-y)} R_N(x,y,\xi)\right) \langle D_y\rangle^{2M} \langle y\rangle^{2M} u(y)\, \mathrm{d}y\, \mathrm{d}\xi \\ =& \frac{\langle x\rangle^{2M}}{2\pi} \iint \langle D_x\rangle^{2M} \langle D_y\rangle^{2M} \left(e^{i\xi(x-y)} R_N(x,y,\xi)\right) \langle y\rangle^{2M} u(y)\, \mathrm{d}y\, \mathrm{d}\xi \end{aligned}$$

Using the relation $\langle x-y\rangle^{-2k''} \langle D_\xi\rangle^{2k''} e^{i\xi(x-y)} = e^{i\xi(x-y)}$ and integrating by part[2] with respect to ξ, we find:

$$v(x) = \langle x\rangle^{2M} \iint \langle D_x\rangle^{2M} \langle D_y\rangle^{2M} \left(e^{i\xi(x-y)} \langle x-y\rangle^{-2k''} \langle D_\xi\rangle^{2k''} R_N(x,y,\xi)\right) \langle y\rangle^{2M} u(y)\, \mathrm{d}y\, \mathrm{d}\xi$$

We now compute the L^2 norm of v. Note that by (C.2) and the fact that $m-N-1+4M < -1$:

$$\begin{aligned} |v(x)| \lesssim_{M,m,l}& \\ \left\|\partial_x^{N+1} f\right\|_{S^{l-N-1}_{\alpha_M}} \left\|\partial_\xi^{N+1} g\right\|_{S^{m-N-1}_{\beta_M}} & \langle x\rangle^{2l-2N-2+k''+2M} \iint \langle \xi\rangle^{m-N-1+4M} \langle y\rangle^{N+1-l-k''+2M} |u(y)|\, \mathrm{d}y\, \mathrm{d}\xi \\ \lesssim_{M,l,m} \left\|\partial_x^{N+1} f\right\|_{S^{l-N-1}_{\alpha_M}} \left\|\partial_\xi^{N+1} g\right\|_{S^{m-N-1}_{\beta_M}} & \langle x\rangle^{2l-2N-2+k''+2M} \int \langle y\rangle^{N+1-l-k''+2M} |u(y)|\, \mathrm{d}y \end{aligned}$$

Using the Cauchy-Schwarz inequality and the fact that $2N+2-2l-2k''+4M < -1$:

$$|v(x)|^2 \lesssim_{M,l,m} \left\|\partial_x^{N+1} f\right\|^2_{S^{l-N-1}_{\alpha}} \left\|\partial_\xi^{N+1} g\right\|^2_{S^{m-N-1}_{\beta}} \langle x\rangle^{4l-4N-4+2k''+4M} \|u\|^2_{L^2}$$

Finally, using that $4l-4N-4+2k''+4M < -1$, we deduce that there exists $C>0$ depending only on M, l and m such that:

$$\|v\|_{L^2} \leq C \left\|\partial_x^{N+1} f\right\|_{S^{l-N-1}_{\alpha}} \left\|\partial_\xi^{N+1} g\right\|_{S^{m-N-1}_{\beta}} \|u\|_{L^2}\,.$$

□

Proposition C.1.9. *Let $m,l,r,q \in \mathbb{R}$. Let $f \in S^l(\mathbb{R})$ and $g \in S^m(\mathbb{R})$. The operator $g(D_x)$ is bounded from $H^{r,q}$ to $H^{r-m,q}$. The operator multiplication by f is bounded from $H^{r,q}$ to $H^{r,q-l}$. Moreover, there exists a constant $C>0$ and an integer α, both depending only on m,r and q such that:*

$$\|g(D_x)\|_{\mathcal{L}(H^{r,q},H^{r-m,q})} \leq C \|g\|_{S^m_\alpha}\,.$$

Similarly, there exists a constant $C'>0$ and an integer β, both depending only on l,r and q such that:

$$\|f\|_{\mathcal{L}(H^{r,q},H^{r,q-l})} \leq C \|f\|_{S^l_\beta}\,.$$

[2]Since the double integral is not absolutely convergent some care is required. We use the fact that the ξ dependence of $R_N(x,y,\xi)$ can be factored out and we perform the integration by part in the second integral.

Proof. Note that the first claim reduces to proving that $\langle x\rangle^q \langle D_x\rangle^{m-r} g(D_x) \langle D_x\rangle^{-m} \langle x\rangle^{-q}$ are bounded on L^2. We choose $M \in \mathbb{N}$ such that $2M > \max(3, q+3)$. With this choice we have the continuous inclusions $H^{2M,2M} \subset H^{0,q}$ and $L^2 \subset H^{-2M,-2M}$ by lemma C.1.6. Therefore $\langle x\rangle^q$ is bounded from $H^{2M,2M}$ to L^2. We use lemma C.1.8 with this M and we get $N \in \mathbb{N}$ such that:

$$\langle D_x\rangle^{r-m} g(D_x) \langle D_x\rangle^{-m} \langle x\rangle^{-q} = \sum_{j=0}^{N} f_j(x) g_j(D_x) + R_N$$

where $f_j \in S^{-q}(\mathbb{R})$, $g_j \in S^0(\mathbb{R})$ and R_N is bounded from $H^{-2M,-2M}$ to $H^{2M,2M}$. We deduce that

$$\langle x\rangle^q \langle D_x\rangle^{m-r} g(D_x) \langle D_x\rangle^{-m} \langle x\rangle^{-q} = \sum_{j=0}^{N} \langle x\rangle^q f_j(x) g_j(D_x) + \langle x\rangle^q R_N$$

Since $\langle x\rangle^q f_j(x)$ and $g_j(\xi)$ are bounded for the uniform norm, the operators $\langle x\rangle^q f_j(x) g_j(D_x)$ are bounded on L^2 and by our choice of M, $\langle x\rangle^q R_N$ is bounded on L^2.

We now prove the second claim. We have to prove that $\langle x\rangle^{q-l} \langle D_x\rangle^m f(x) \langle D_x\rangle^{-m} \langle x\rangle^{-q}$ is bounded on L^2. We use lemma C.1.8 with N a large integer to be chosen later and we get:

$$\langle x\rangle^{q-l} \langle D_x\rangle^m f(x) \langle D_x\rangle^{-m} \langle x\rangle^{-q} = \sum_{j=1}^{N} f_j(x) g_j(D_x) \langle D_x\rangle^{-m} \langle x\rangle^{-q} + R_N \langle D_x\rangle^{-m} \langle x\rangle^{-q}$$

where $f_j \in S^q$ and $g_j \in S^m(\mathbb{R})$. Since $\langle D_x\rangle^{-m} \langle x\rangle^{-q}$ is bounded from L^2 to $H^{m,q}$ if we take N large enough so that R_N is bounded from $H^{-2M,-2M}$ to $H^{2M,2M}$ with $-2M < \min(m-1, q-3, -3)$, we get that the last term is bounded on L^2. We apply lemma C.1.8 again in each term of the sum:

$$f_j(x) g_j(D_x) \langle D_x\rangle^{-m} \langle x\rangle^{-q} = \sum_{k=1}^{N} f_{j,k}(x) g_{j,k}(D_x) + f_j R_{N_j}$$

where $f_{j,k}, g_{j,k} \in S^0(\mathbb{R})$. Since f_j is bounded from $H^{0,q}$ to L^2, if we take N_j large enough, we get that R_{N_j} is bounded from $H^{-2M,-2M}$ to $H^{2M,2M}$ with $2M > \max(3, q+3)$ and then $f_j R_{N_j}$ is bounded on L^2. Since all the terms $f_{j,k}(x) g_{j,k}(D_x)$ are bounded on L^2, we conclude. To obtain the quantitative bounds, we go through the argument and we use the explicit form of the f_j, g_j, $f_{j,k}$, $g_{j,k}$ and the quantitative bound for the error term provided by lemma C.1.8 (with remark that the choice of M in each application of lemma C.1.8 can be done as a function of m, r, q or l, r, q). □

Remark C.1.10. *Using the proposition with $\langle x\rangle^{-l} \langle D_x\rangle^{-m}$ and $\langle D_x\rangle^m \langle x\rangle^l$ we obtain that $\langle x\rangle^{-l} \langle D_x\rangle^{-m}$ is an isomorphism between L^2 and $H^{m,l}$. Therefore the choice of the order of the operators in the definition of $H^{m,l}$ does not matter (up to equivalence of norm).*

Corollary C.1.11. *Let $m, l, m', l' \in \mathbb{R}$ be such that $m \leq m'$ and $l \leq l'$. We have the continuous inclusion: $H^{m',l'} \subset H^{m,l}$.*

Proof. Since $\langle D_x\rangle^{-m'} \langle x\rangle^{-l'}$ is an isometry from L^2 to $H^{m',l'}$, the corollary is a consequence of the boundedness of $\langle x\rangle^l \langle D_x\rangle^{m-m'} \langle x\rangle^{-l'}$ on L^2. But since $\langle x\rangle^{-l'} \in S^{-l}(\mathbb{R})$, $\langle x\rangle^l \in S^l(\mathbb{R})$ and $\langle \xi\rangle^{m-m'} \in S^0(\mathbb{R})$ it is a consequence of proposition C.1.9. □

We are now able to prove the following improvement of lemma C.1.8:

Proposition C.1.12. *Let $f \in S^l(\mathbb{R})$ and $g \in S^m(\mathbb{R})$. For every $N \in \mathbb{N}$ we have:*

$$g(D_x)f(x) = f(x)g(D_x) + \sum_{j=1}^{N} \frac{\partial_x^j f(x)(D_\xi^j g)(D_x)}{j!} + R_N$$

where R_N is a continuous operator on $\mathcal{S}'$ (which is continuous on $\mathcal{S}$) and for every $r, q \in \mathbb{R}$, R_N is bounded from $H^{r,q}$ to $H^{r-m+N+1,q-l+N+1}$.

Proof. Let $r, q \in \mathbb{R}$, we have to prove that $g(D_x)f(x) - f(x)g(D_x) - \sum_{j=1}^{N} \frac{\partial_x^j f(x)(D_\xi^j g)(D_x)}{j!}$ is bounded from $H^{r,q}$ to $H^{r-m+N+1,q-l+N+1}$. We choose M such that $H^{r,q} \subset H^{-2M,-2M}$ and $H^{2M,2M} \subset H^{r-m+N+1,q-l+N+1}$. By lemma C.1.8, there exists N' (and without loss of generality, we can assume $N' \geq N$) such that:

$$g(D_x)f(x) - f(x)g(D_x) - \sum_{j=1}^{N} \frac{\partial_x^j f(x)(D_\xi^j g)(D_x)}{j!} = \sum_{j=N+1}^{N'} \frac{\partial_x^j f(x)(D_\xi^j g)(D_x)}{j!} + R_{N'}$$

with $R_{N'}$ bounded from $H^{r,q}$ to $H^{r-m+N+1,q-l+N+1}$. Since for $N+1 \leq j \leq N'$ we have $\partial_x^j f \in S^{l-N-1}$ and $D_\xi^j g \in S^{m-N-1}$, we conclude using proposition C.1.9. □

The quantitative bound in proposition C.1.9 can be used to obtain the following families of approximations:

Proposition C.1.13. *We fix $m > 0$, $l > 0$, $r, q \in \mathbb{R}$. Let $\chi \in C_c^\infty(\mathbb{R})$ with $\chi = 1$ in a neighborhood of zero. The families of multiplication operators $\langle \epsilon x \rangle^{-l}$ and $\chi(\epsilon x)$ are uniformly bounded (with respect to $\epsilon \in (0,1)$) in $\mathcal{L}(H^{r,q}, H^{r,q})$ and for every $\delta > 0$, they converge to the identity in $\mathcal{L}(H^{r,q}, H^{r,q-\delta})$. Similarly, the families $\langle \epsilon D_x \rangle^{-m}$ and $\chi(\epsilon D_x)$ are bounded in $\mathcal{L}(H^{r,q}, H^{r,q})$ and for every $\delta > 0$, they converge to the identity in $\mathcal{L}(H^{r,q}, H^{r-\delta,q})$.*

Proof. In view of proposition C.1.9, it is the consequence of the observation that $f_\epsilon : x \mapsto \langle \epsilon x \rangle^{-l}$ and $g_\epsilon : x \mapsto \chi(\epsilon x)$ are bounded for any seminorm S_k^0, $k \in \mathbb{N}$ and for every $k \in \mathbb{N}$ and $\delta > 0$, $\lim_{\epsilon \to 0} \|f_\epsilon - 1\|_{S_k^\delta} = \lim_{\epsilon \to 0} \|g_\epsilon - 1\|_{S_k^\delta} = 0$. □

Proposition C.1.14. *Let $m, l, m', l' \in \mathbb{R}$ be such that $m' > m$ and $l' > l$. The inclusion $H^{m',l'} \subset H^{m,l}$ is compact.*

Proof. Using the isomorphism $\langle x \rangle^{l'} \langle D_x \rangle^{m'}$, we can reduce to the case $m' = l' = 0$. By proposition C.1.13, for $m < 0$ and $l < 0$, the identity operator in $\mathcal{L}(L^2, H^{m,l})$ is the limits when $\epsilon \to 0$ of $\chi(\epsilon x)\chi(\epsilon D_x)$. Since the set of compact operators is closed for the operator norm topology, it is enough to prove that $K_\epsilon := \chi(\epsilon x)\chi(\epsilon D_x)$ is compact (as an operator from L^2 to $H^{m,l}$) for $\epsilon > 0$. Let (u_n) be a bounded sequence in L^2. For all $n \in \mathbb{N}$, $K_\epsilon u_n$ has support in $X := \epsilon^{-1}\text{supp}\chi$ which is compact. We now show that the family $(K_\epsilon u_n)$ is equicontinuous and bounded for the uniform norm. We have:

$$\begin{aligned}
\|K_\epsilon u_n\|_\infty &\leq \|\chi\|_\infty \|\chi(\epsilon D_x) u_n\|_\infty \\
&\leq \|\chi\|_\infty (2\pi)^{-1} \|\chi(\epsilon\xi)\hat{u}_n\|_{L^1} \\
&\leq \frac{1}{\sqrt{2\epsilon\pi}} \|\chi\|_\infty \|\chi\|_{L^2} \|u_n\|_{L^2}
\end{aligned}$$

Therefore, the sequence $(K_\epsilon u_n)_{n\in\mathbb{N}}$ is bounded for the uniform norm. Let $x, y \in \mathbb{R}$,

$$\begin{aligned}
|K_\epsilon u_n(x) - K_\epsilon u_n(y)| \leq & \frac{1}{\sqrt{2\epsilon\pi}} |\chi(\epsilon x) - \chi(\epsilon y)| \, \|\chi\|_{L^2} \, \|u_n\|_{L^2} \\
& + \frac{\|\chi\|_\infty}{2\pi} \left| \int \left(e^{i\xi x} - e^{i\xi y} \right) \chi(\epsilon\xi) \hat{u}_n(\xi) \, \mathrm{d}\xi \right| \\
\leq & \sqrt{\frac{\epsilon}{2\pi}} \, \|\partial_x \chi\|_\infty \, |x-y| \, \|\chi\|_{L^2} \, \|u_n\|_{L^2} \\
& + |x-y| \, \frac{\|\chi\|_\infty}{2\pi} \int |\xi \chi(\epsilon\xi) \hat{u}_n(\xi)| \, \mathrm{d}\xi \\
\leq & \, |x-y| \left(\sqrt{\frac{\epsilon}{2\pi}} \, \|\chi\|_{L^2} \, \|\partial_x \chi\|_\infty + \frac{1}{\sqrt{2\epsilon^3\pi}} \, \|\chi\|_\infty \, \|\xi\chi(\xi)\|_{L^2} \right) \|u_n\|_{L^2}
\end{aligned}$$

The sequence is uniformly Lipschitz. We can use the Arzela-Ascoli theorem to extract a subsequence converging uniformly to a continuous function u with support in $\epsilon^{-1}\mathrm{supp}\chi$. Since the sequence is uniformly bounded, this uniform convergence implies the convergence in L^2 by Lebesgue theorem. Since the topology of $H^{m,l}$ is weaker, the sequence converges in $H^{m,l}$. We deduce that the operator K_ϵ is compact. □

We introduce the following hybrid spaces (which have a different decay at $-\infty$ and at $+\infty$).

Definition C.1.15. *We fix two monotonic smooth non negative cutoffs χ_+ and χ_- such that $\chi_- = 1$ on a neighborhood of $-\infty$, $\chi_+ = 1$ on a neighborhood of $+\infty$ and $\chi_- + \chi_+ = 1$. For $m, l, l' \in \mathbb{R}$, we introduce the space $H^{m,l,l'}$ as the space of distributions $u \in \mathcal{S}'$ such that $\chi_- u \in H^{m,l}$ and $\chi_+ u \in H^{m,l'}$. For $u \in H^{m,l,l'}$, we define the norm:*

$$\|u\|^2_{H^{m,l,l'}} = \|\chi_- u\|^2_{H^{m,l}} + \|\chi_+ u\|^2_{H^{m,l'}} \, .$$

With this norm, $H^{m,l,l'}$ is a Banach space. We also define $S^{l,l'}(\mathbb{R})$ the set of functions $f \in C^\infty(\mathbb{R})$ such that $\chi_- f \in S^l(\mathbb{R})$ and $\chi_+ f \in S^{l'}(\mathbb{R})$.

Remark C.1.16. *Note that for $l < l_0$, $l' < l'_0$ and $m < m_0$, the inclusion $H^{m_0,l_0,l'_0} \subset H^{m,l,l'}$ is compact. Indeed, if we have a bounded sequence (u_n) in H^{m_0,l_0,l'_0} then $(\chi_- u_n)$ is bounded in H^{m_0,l_0} and $(\chi_+ u_n)$ is bounded in H^{m_0,l'_0} and and by proposition C.1.14, we can extract a subsequence such that $\chi_- u_n$ converges in $H^{m,l}$ and $\chi_+ u_n$ converges in $H^{m,l'}$ therefore the subsequence converges in $H^{m,l,l'}$.*

We have the following natural extension of proposition C.1.9.

Proposition C.1.17. *Let $m, l, l' \in \mathbb{R}$. Let $m_0 \in \mathbb{R}$. Let $g \in S^{m_0}(\mathbb{R})$. The family of operators $g(D_x)$ is bounded from $H^{m,l,l'}$ to $H^{m-m_0,l,l'}$. Moreover, let $f \in S^{l_0,l'_0}(\mathbb{R})$. The operator multiplication by f is bounded from $H^{m,l,l'}$ to $H^{m,l-l_0,l'-l'_0}$ (in particular if $f \in S^{l_0}(\mathbb{R})$ the operator multiplication by f is bounded from $H^{m,l,l'}$ to $H^{m,l-l_0,l'-l_0}$).*

Proof. We have $\chi_-(x) g(D_x) = g(D_x)\chi_-(x) + [\chi_-(x), g(D_x)]$ where $g(D_x)\chi_-(x)$ is bounded from $H^{m,l}$ to $H^{m-m_0,l}$ and $[\chi_-(x), g(D_x)]$ is bounded from $H^{m,l-N}$ to $H^{m-m_0,l+N}$ for every $N \in \mathbb{N}$. A similar observation for $\chi_+(x) g(D_x)$ leads to the following bound:

$$\begin{aligned}
\|\chi_- g(D_x) u\|^2_{H^{m,l}} + \|\chi_- g(D_x) u\|^2_{H^{m,l'}} & \lesssim \|\chi_- u\|^2_{H^{m-m_0,l}} + \|\chi_+ u\|^2_{H^{m-m_0,l}} + \|u\|_{H^{m-m_0,\min(l,l')}} \\
& \lesssim \|u\|^2_{H^{m-m_0,l,l'}} \, .
\end{aligned}$$

The second claim is obtained by writing $\chi_- f = \tilde{\chi}_- f \chi_-$ with $\tilde{\chi}_- = 1$ on $\mathrm{supp}(\chi_-)$ and $\tilde{\chi}_- = 0$ on a neighborhood of $+\infty$ so that $\tilde{\chi}_- f \in S^{l_0}(\mathbb{R})$ (and similarly for $\chi_+ f$). □

C.2 Analysis of $D_x + V(x) + z$

One of the simplest possible example to study is the operator $P(z) := D_x + V(x) + z$ where $z \in \mathbb{R}$ and V is a complex valued potential $V \in S^{-1}(\mathbb{R})$. Moreover, we assume that V has a homogeneous leading order term at $x = +\infty$ and $x = -\infty$. In other words, if we fix a smooth cutoff χ_0 (rep. χ_1) localizing near $x = -\infty$ (resp. $x = +\infty$), there exist complex constants $v_{-\infty}$ and $v_{+\infty}$ in $\mathbb{C}$ such that $V(x) - \chi_0(x)v_{-\infty}\left|x\right|^{-1} - \chi_1(x)v_{+\infty}\left|x\right|^{-1} \in S^{-2}$ ($S^{-1-\delta}$ for $\delta > 0$ would be enough but we want to keep things simple here). Our first goal is to find spaces $\mathfrak{X}$ and Y such that $P(z)$ is Fredholm from $\mathfrak{X}$ to Y. This can be achieved by elementary methods (see subsection C.3) but our goal is to illustrate the microlocal approach.

Analysis of the scattering classical flow

We consider the compactification $\overline{\mathbb{R}}$ of $\mathbb{R}$ obtained by adding the points $\{-\infty\}$ and $\{+\infty\}$. We endow $\overline{\mathbb{R}}$ with a structure of manifold with boundaries using the boundary defining functions $\rho_+ = \frac{1}{x}$ (near $x = +\infty$) and $\rho_- = -\frac{1}{x}$ (near $x = -\infty$). The scattering cotangent bundle ${}^{sc}T^*\overline{\mathbb{R}}$ is the bundle which coincides with the cotangent bundle on the interior of $\overline{\mathbb{R}}$ but, locally near each boundary face, smooth sections are of the form $a(\rho_\pm)\frac{\mathrm{d}\rho_\pm}{\rho_\pm^2}$ for a a smooth function (while smooth sections of $T^*\overline{\mathbb{R}}$ are of the form $a(\rho_\pm)\ \mathrm{d}\rho_\pm$). Note that $\mathrm{d}x$ uniquely extends to a global non vanishing smooth section of ${}^{sc}T^*\overline{\mathbb{R}}$. Therefore, it provides a basis of the fiber at each point and we introduce the corresponding coordinate of the fiber ξ_{sc}. The fiber radial compactification of ${}^{sc}T^*\overline{\mathbb{R}}$ denoted by ${}^{sc}\overline{T}^*\overline{\mathbb{R}}$ is obtained by adding the points $\xi_{sc} = +\infty$ and $\xi_{sc} = -\infty$ to each fiber. The boundary defining function of fiber infinity is $\left|\xi_{sc}\right|^{-1}$. The manifold ${}^{sc}\overline{T}^*\overline{\mathbb{R}}$ is a manifold with corners (see figure C.1). The scattering principal symbol of a differential operator is given by the leading order term[3] at each boundary face of the function $p(x, \xi)$ obtained by replacing each derivative ∂_x in the expression of the operator by $i\xi_{sc}$. We construct the Hamiltonian vector field $H_p := \partial_{\xi_{sc}}\partial_x - \partial_x p\partial_{\xi_{sc}}$ which can be rescaled by $\rho_\pm^{l-1}\left|\xi_{sc}\right|^{-m+1}$ where l is the polynomial order of growth of p near the faces $x = \pm\infty$ and m is the order of p at the faces $\xi_{sc} = \pm\infty$.

The scattering principal symbol of P at the boundary faces $x = \pm\infty$ is

$$p_{sc}(x, \xi_{sc}) = \xi_{sc} + z$$

and $p_{sc}(x, \xi_{sc}) = \xi_{sc}$ at the boundary faces $\xi_{sc} = \pm\infty$. In particular, we see that the order at the boundary is zero and the order at fiber infinity is one. The rescaled principal symbol at fiber infinity is ± 1 which is invertible and the principal symbol both at $x = -\infty$ and at $x = +\infty$ is $\xi_{sc} + z$ which is invertible when $z \in \mathbb{C} \setminus \mathbb{R}$. It has characteristic set $\{\xi_{sc} = -z\}$ when $z \in \mathbb{R}$. These points are radial points[4] since the rescaled Hamiltonian vector field is $H_p = \mp\rho_\pm\partial_{\rho_\pm}$ near $x = -\infty$ and $x = +\infty$.

Therefore, we will get a Fredholm estimate by combining an elliptic estimate and two radial point estimates. We fix $z \in \mathbb{R}$.

We begin by the following elliptic estimate:

Proposition C.2.1. *Let χ_1 be a smooth compactly supported cutoff such that $\chi_1 = 1$ in a neighborhood of $-z$. For all $m, l, l' \in \mathbb{R}$ and all $N \in \mathbb{N}$, there exists $C > 0$ such that for all $u \in \cup_{j\in\mathbb{N}} H^{-j,-j}$:*

$$\left\|u\right\|_{H^{m,l,l'}} \leq C\left(\left\|Pu\right\|_{H^{m-1,l,l'}} + \left\|\chi_1(D_x)u\right\|_{H^{m,l,l'}} + \left\|u\right\|_{H^{m-N,l-N,l'-N}}\right) \tag{C.3}$$

[3] If it exists.

[4] Points of the characteristic set where the rescaled Hamiltonian vector field vanishes.

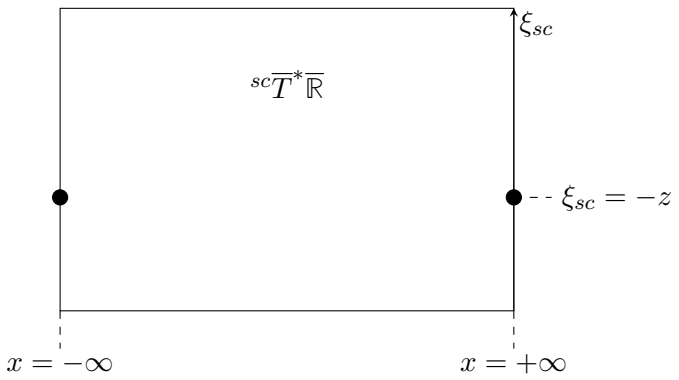

Figure C.1: Representation of the fiber radial compactification of the scattering cotangent bundle of $\overline{\mathbb{R}}$.

in the strong sense that if the right-hand side is finite, then $u \in H^{m,l,l'}$ and the inequality holds. Moreover, if χ_2 is any smooth compactly supported cutoff, for all $N \in \mathbb{N}$, there exists a constant $C > 0$ such that:

$$\|\chi_2(x)u\|_{H^{m,l,l'}} \leq C\left(\|Pu\|_{H^{m-1,l,l'}} + \|u\|_{H^{m-N,l-N,l'-N}}\right) \tag{C.4}$$

Proof. First, note that by induction, it is enough to prove the estimate for $N = 1$. We define the function $f : \xi \mapsto (1 - \chi_1(\xi))(\xi + z)^{-1}$ which belongs to $S^{-1}(\mathbb{R})$. Therefore, using the relation

$$f(D_x)P = 1 - \chi_1(D_x) + f(D_x)V(x)$$

and proposition C.1.17, we get the first estimate. To prove the second one, we use the relation:

$$\begin{aligned}\chi_2(x) =& f(D_x)P\chi_2(x) + \chi_1(D_x)\chi_2(x) + f(D_x)V(x)\chi_2(x)\\ =& f(D_x)\chi_2(x)P + \frac{1}{i}f(D_x)\chi_2'(x) + \chi_1(D_x)\chi_2(x) + f(D_x)V(x)\chi_2(x).\end{aligned}$$

We consider a smooth compactlys supported cutoff $\tilde{\chi}_2$ equal to 1 on the support of χ_2 (and therefore we have $\tilde{\chi}_2(x)\chi_2'(x) = \chi_2'(x)$ and $\chi_2(x)\tilde{\chi}_2(x) = \chi_2(x)$). We can then use proposition C.1.17 to get:

$$\|\chi_2(x)u\|_{H^{m,l,l'}} \leq C\left(\|Pu\|_{H^{m-1,l,l'}} + \|\tilde{\chi}_2(x)u\|_{H^{m-1,l-1,l'-1}}\right)$$

We can then iterate the procedure to get a bound of $\|\tilde{\chi}_2(x)u\|_{H^{m-1,l-1,l'-1}}$. After N iteration, we obtain the claimed estimate. □

Corollary C.2.2. *Let χ_1 be a smooth compactly supported cutoff such that $\chi_1 = 1$ in a neighborhood of $-z$. Let χ_3 and χ_4 be smooth monotonic cutoffs such that $\chi_3 = 1$ in a neighborhood of $-\infty$, $\chi_3 = 0$ in a neighborhood of $+\infty$, $\chi_4 = 0$ in a neighborhood of $-\infty$ and $\chi_4 = 1$ in a neighborhood of $+\infty$. For all $m, l, l' \in \mathbb{R}$ and all $N \in \mathbb{N}$, there exists a constant $C > 0$ such that for all $u \in \cup_{j\in\mathbb{N}} H^{-j,-j}$ we have:*

$$\|u\|_{H^{m,l,l'}} \leq C\left(\|Pu\|_{H^{m-1,l,l'}} + \|\chi_3(x)\chi_1(D_x)u\|_{H^{m,l,l'}} + \|\chi_4(x)\chi_1(D_x)u\|_{H^{m,l,l'}} + \|u\|_{H^{m-N,l-N,l'-N}}\right)$$

in the strong sense that if the right-hand side is finite, so is the left-hand side and the inequality holds.

Proof. Once again, by induction it is enough to prove the estimate for $N = 1$. We bound the error term in estimate (C.3):

$$\begin{aligned}\|\chi_1(D_x)u\|_{H^{m,l,l'}} \lesssim \|\chi_3(x)\chi_1(D_x)u\|_{H^{m,l,l'}} + \|\chi_4(x)\chi_1(D_x)u\|_{H^{m,l,l'}} \\ + \|(1-\chi_3-\chi_4)\chi_1(D_x)u\|_{H^{m,l,l'}}\end{aligned}$$

By estimate (C.4), we have:

$$\begin{aligned}\|(1-\chi_3-\chi_4)\chi_1(D_x)u\|_{H^{m,l,l'}} &\lesssim \|P\chi_1(D_x)u\|_{H^{m-1,l,l'}} + \|\chi_1(D_x)u\|_{H^{m-N,l-N,l'-N}} \\ &\lesssim \|\chi_1(D_x)Pu\|_{H^{m-1,l,l'}} + \|[V(x),\chi(D_x)]u\|_{H^{m-1,l,l'}} + \|u\|_{H^{m-N,l-N,l'-N}}\end{aligned}$$

Since $\chi_1(D_x)$ is bounded on $H^{m-1,l,l'}$ and $[V(x),\chi(D_x)]$ is bounded from $H^{m-1,l-1,l'-1}$ to $H^{m-1,l,l'}$, we get:

$$\|u\|_{H^{m,l,l'}} \lesssim \|Pu\|_{H^{m-1,l,l'}} + \|\chi_3(x)\chi_1(D_x)u\|_{H^{m,l,l'}} + \|\chi_4(x)\chi_1(D_x)u\|_{H^{m,l,l'}} + \|u\|_{H^{m-1,l-1,l'-1}}$$

□

We now prove the radial point estimates which are based on a commutator argument. we begin by the estimate near the radial point at $x = -\infty$.

Proposition C.2.3. *Let χ_1 be a smooth monotonic compactly supported cutoff equal to 1 in a neighborhood of $-\infty$ and equal to zero on a neighborhood of 0. Let χ_2 be a smooth compactly supported cutoff equal to 1 in a neighborhood of $-z$. Let $\tilde{\chi}_1$ with the same properties as χ_1 and equal to 1 on supp(χ_1). Let $N \in \mathbb{N}$. Let $\alpha_1 > \Im(v_{-\infty})$ and $\max(\Im(v_{-\infty}), \alpha_1 - \frac{1}{2}) < \alpha_1' < \alpha_1$ and $-\Im(v_{-\infty}) < \alpha_2$. There exists a constant $C > 0$ such that for all $u \in \cup_{j\in\mathbb{N}} H^{-j,-j}$ with $\tilde{\chi}_1(x)u \in H^{m-1,-\frac{1}{2}+\alpha_1'}$ and all $v \in \cup_{j\in\mathbb{N}} H^{-j,-j}$ with $\tilde{\chi}_1(x)v \in H^{m-1,-\alpha_2-1}$:*

$$\|\chi_1(x)\chi_2(D_x)u\|_{H^{m,-\frac{1}{2}+\alpha_1}} \leq C\left(\|\chi_1(x)Pu\|_{H^{m-1,\frac{1}{2}+\alpha_1}} + \|u\|_{H^{-N,-\frac{1}{2}+\alpha_1',-N}}\right) \tag{C.5}$$

$$\|\chi_1(x)\chi_2(D_x)v\|_{H^{m,-\frac{1}{2}-\alpha_2}} \leq C\left(\|\chi_1(x)Pv\|_{H^{m-1,\frac{1}{2}-\alpha_2}} + \|v\|_{H^{-N,-1-\alpha_2,-N}}\right) \tag{C.6}$$

in the strong sense that if the right-hand side is finite, so is the left-hand side and the inequality holds.

Remark C.2.4. *The fact that the threshold (here $\Im(v_{-\infty})$) is given by the subprincipal symbol of the operator at the radial point is a general fact in radial point estimates.*

Proof. In radial point estimates (see [104, Section 5.4.7]), the general principle is that we obtain an estimate by using a commutator argument with a cutoff version of the weight (either the fiber infinity defining function or the boundary defining function depending on where the radial set is located). In our case, it suggests to introduce the following commutator operator: $A := \chi_2(D_x)\chi_1(x)^2 |x|^{\pm 2\alpha} \chi_2(D_x)$. Moreover, the commutator argument requires additional decay and we introduce the family $A_\epsilon := \chi_2(D_x)\chi_1(x)^2 |x|^{2\alpha} \langle \epsilon x\rangle^{-\eta} \chi_2(D_x)$. The amount of extra decay η depends on whether we prove (C.5) (source point estimate) or (C.6) (sink point estimate).

Source estimate: First note that since

$$\chi_1(x)\chi_2(D_x) - \chi_1(x)\chi_2(D_x)\tilde{\chi}(x) = \chi_1(x)[\tilde{\chi}_1(x), \chi_2(D_x)]$$

is bounded from $H^{-j,-j}$ to $H^{j,j}$ for all $j \in \mathbb{N}$ (lemma C.1.9) and since $\chi_1(x)P\tilde{\chi}_1(x) = \chi_1(x)P$, it is enough to prove the estimate with u replaced by $\tilde{\chi}_1(x)u$. Therefore, without loss of generality, we assume that $u \in \cap_{j\in\mathbb{N}} H^{m-1,-\frac{1}{2}+\alpha_1',j}$ and $Pu \in \cap_{j\in\mathbb{N}} H^{m-1,\frac{1}{2}+\alpha_1,j}$. We take $A = \chi_2(D_x)\chi_1(x)^2 |x|^{2\alpha_1} \chi_2(D_x)$ and $\eta = 2(\alpha_1 - \alpha_1')$. Therefore the operator A_ϵ is bounded from

$H^{m-1,-\frac{1}{2}+\alpha_1'}$ to $H^{-m+1,-\frac{1}{2}-\alpha_1'}$ and the following expressions make sense for $u \in H^{m-1,-\frac{1}{2}+\alpha_1'}$ such that $Pu \in H^{m-1,\frac{1}{2}+\alpha_1}$:

$$\langle A_\epsilon u, Pu\rangle - \langle Pu, A_\epsilon u\rangle$$
$$\langle (PA_\epsilon - A_\epsilon P + (P^* - P)A_\epsilon)u, u\rangle$$

However, the equality between the two expressions is not immediate since the term $\langle PA_\epsilon u, u\rangle$ is not well defined a priori (proposition C.1.9 only gives $PA_\epsilon u \in H^{-m+1,-\frac{1}{2}-\alpha_1'}$ which has not enough decay). Following [104, Section 5.4.7], we prove the equality by an additional regularization argument. Namely for all $\mu \in (0,1)$, we have the equality:

$$\left\langle A_\epsilon \langle \mu x\rangle^{-1} u, Pu\right\rangle - \left\langle P\langle \mu x\rangle^{-1} u, A_\epsilon u\right\rangle = \left\langle (PA_\epsilon - A_\epsilon P + (P^* - P)A_\epsilon)\langle \mu x\rangle^{-1} u, u\right\rangle \quad \text{(C.7)}$$

By proposition C.1.13, we have that $\langle \mu x\rangle^{-1}$ is bounded uniformly with respect to $\mu \in (0,1)$ in

$$\mathcal{L}(H^{m-1,-\frac{1}{2}+\alpha_1'}, H^{m-1,-\frac{1}{2}+\alpha_1'}).$$

Therefore, by the Banach-Alaoglu theorem (and the Eberlein-Smulian theorem), we can extract a weakly convergent subsequence in $H^{m-1,-\frac{1}{2}+\alpha_1'}$, $\lim_{n\to+\infty} \langle \mu_n x\rangle^{-1} u = u_\infty$. On the other hand (still by proposition C.1.13), we have $\lim_{\mu\to 0} \langle \mu x\rangle^{-1} u = u$ in $H^{m-1,-\frac{1}{2}+\alpha_1'-\delta}$ for $\delta > 0$ and by uniqueness of the limit in the sense of distributions, we get $u_\infty = u$. The weak convergence in $H^{m-1,-\frac{1}{2}+\alpha_1'}$ implies that:

$$\lim_{n\to+\infty} \left\langle A_\epsilon \langle \mu_n x\rangle^{-1} u, Pu\right\rangle = \langle A_\epsilon u, Pu\rangle$$
$$\lim_{n\to+\infty} \left\langle (PA_\epsilon - A_\epsilon P + (P^* - P)A_\epsilon)\langle \mu_n x\rangle^{-1} u, u\right\rangle = \langle (PA_\epsilon - A_\epsilon P)u, u\rangle$$

Finally,

$$P\langle \mu_n x\rangle^{-1} u = \langle \mu_n x\rangle^{-1} Pu + i\mu_n^2 x \langle \mu_n x\rangle^{-3} u.$$

By the quantitative bound in proposition C.1.9, we get $\lim_{n\to+\infty} i\mu_n^2 x\langle \mu_n x\rangle^{-3} u = 0$ in $H^{m-1,\frac{1}{2}+\alpha_1'}$. As before, we can extract a subsequence of (μ_n) (that we still call (μ_n)) such that $\langle \mu_n x\rangle^{-1} Pu$ converges weakly towards Pu in $H^{m-1,\frac{1}{2}+\alpha_1}$ and (and therefore also in $H^{m-1,\frac{1}{2}+\alpha_1'}$ since $\alpha_1' < \alpha_1$). We conclude:

$$\lim_{n\to+\infty} \left\langle P\langle \mu_n x\rangle^{-1} u, A_\epsilon u\right\rangle = \langle Pu, A_\epsilon u\rangle$$

We have therefore proved that:

$$\langle A_\epsilon u, Pu\rangle - \langle Pu, A_\epsilon u\rangle = \langle (PA_\epsilon - A_\epsilon P + (P^* - P)A_\epsilon)u, u\rangle \quad \text{(C.8)}$$

We compute the right-hand side:

$$\begin{aligned}
\frac{PA_\epsilon - A_\epsilon P + (P^* - P)A_\epsilon}{-i} =& \chi_2(D_x)\chi_1(x)|x|^{2\alpha_1-1}\langle \epsilon x\rangle^{2\alpha_1'-2\alpha_1}\left(2\chi_1'(x)|x|\right.\\
&\left.-2\chi_1(x)\left(\alpha_1 - (\alpha_1-\alpha_1')\frac{(\epsilon x)^2}{1+(\epsilon x)^2} - \Im(v_{-\infty})\right)\right)\chi_2(D_x)\\
&+ \chi_2(D_x)\chi_1(x)^2 2\Im(V(x) - v_{-\infty}|x|^{-1})|x|^{2\alpha_1}\langle x\rangle^{2\alpha_1'-2\alpha_1}\chi_2(D_x)\\
&+ i[V(x) - 2i\Im(V(x)), \chi_2(D_x)]\chi_1(x)^2|x|^{2\alpha_1}\langle x\rangle^{2\alpha_1'-2\alpha_1}\chi_2(D_x)\\
&+ i\chi_2(D_x)\chi_1(x)^2|x|^{2\alpha_1}\langle x\rangle^{2\alpha_1'-2\alpha_1}[\chi_2(D_x), V(x)]
\end{aligned} \quad \text{(C.9)}$$

Note that

$$\chi_1(x)\left(2\chi_1'(x)\left|x\right|-2\chi_1(x)\left(\alpha_1-(\alpha_1-\alpha_1')\frac{(\epsilon x)^2}{1+(\epsilon x)^2}-\Im(v_{-\infty})\right)\right)\geq 2\chi_1(x)^2(\alpha_1'-\Im(v_{-\infty}))$$

Let

$$\begin{aligned}G:=&-\chi_2(D_x)\chi_1(x)^2 2\Im(V(x)-v_{-\infty}\left|x\right|^{-1})\left|x\right|^{2\alpha_1}\left\langle\epsilon x\right\rangle^{2\alpha_1'-2\alpha_1}\chi_2(D_x)\\&+i[V(x)-2i\Im(V(x)),\chi_2(D_x)]\chi_1(x)^2\left|x\right|^{2\alpha_1}\left\langle\epsilon x\right\rangle^{2\alpha_1'-2\alpha_1}\chi_2(D_x)\\&+i\chi_2(D_x)\chi_1(x)^2\left|x\right|^{2\alpha_1}\left\langle\epsilon x\right\rangle^{2\alpha_1'-2\alpha_1}[\chi_2(D_x),V(x)]\end{aligned}$$

By lemma C.1.17 and lemma C.1.12, $\left\langle D_x\right\rangle^N\left(\chi_-\left\langle x\right\rangle^{-\alpha_1+1}+\chi_+\left\langle x\right\rangle^N\right)G$ is bounded from $H^{-N,-1+\alpha_1,-N}$ to L^2 (uniformly with respect to $\epsilon\in(0,1)$). We deduce that there exists $C>0$ independent of $\epsilon\in(0,1)$ such that:

$$\begin{aligned}-i\left\langle i(PA_\epsilon-A_\epsilon P+(P^*-P)A_\epsilon)u,u\right\rangle\geq&2(\alpha_1'-\Im(v_{-\infty}))\left\|\left|x\right|^{\alpha_1-\frac{1}{2}}\left\langle\epsilon x\right\rangle^{\alpha_1'-\alpha_1}\chi_1(x)\chi_2(D_x)u\right\|_{L^2}^2\\&-C\left\|u\right\|_{H^{-N,-1+\alpha_1,-N}}^2\end{aligned}$$

The previous equation is a lower bound for the right-hand side of (C.8). We get an upper bound for the left-hand side by the Cauchy-Schwarz inequality:

$$\begin{aligned}\left|\left\langle Pu,A_\epsilon u\right\rangle\right|\leq&\frac{1}{2(\alpha_1'-\Im(v_{-\infty}))}\left\|\left\langle\epsilon x\right\rangle^{\alpha_1'-\alpha_1}\left|x\right|^{\alpha_1+\frac{1}{2}}\chi_1(x)\chi_2(D_x)Pu\right\|_{L^2}^2\\&+\frac{\alpha_1'-\Im(v_{-\infty})}{2}\left\|\left|x\right|^{\alpha_1-\frac{1}{2}}\left\langle\epsilon x\right\rangle^{\alpha_1'-\alpha_1}\chi_1(x)\chi_2(D_x)u\right\|_{L^2}^2\end{aligned}$$

We deduce:

$$\begin{aligned}\left\|\left|x\right|^{\alpha_1-\frac{1}{2}}\left\langle\epsilon x\right\rangle^{\alpha_1'-\alpha_1}\chi_1(x)\chi_2(D_x)u\right\|_{L^2}^2\leq&\frac{1}{(\alpha_1'-\Im(v_{-\infty}))^2}\left\|\left|x\right|^{\alpha_1+\frac{1}{2}}\chi_1(x)\chi_2(D_x)Pu\right\|_{L^2}^2\\&+\frac{C}{\alpha_1'-\Im(v_{-\infty})}\left\|u\right\|_{H^{-N,-1+\alpha_1,-N}}^2\end{aligned}$$

We deduce that the left-hand side is bounded uniformly with respect to $\epsilon\in(0,1)$. By the Banach-Alaoglu theorem (and the Eberlein-Smulian theorem), we extract a weakly convergent sequence in L^2. Since the sequence is norm convergent (proposition C.1.13) in $H^{0,-\delta}$ for $\delta>0$ and using the uniqueness of the limit in the sense of distributions, we have that the weak limit is in fact $\left|x\right|^{\alpha_1-\frac{1}{2}}\chi_1(x)\chi_2(D_x)u$. Therefore $\left|x\right|^{\alpha_1-\frac{1}{2}}\chi_1(x)\chi_2(D_x)u\in L^2$ and by weak lower semicontinuity of the norm we obtain:

$$\begin{aligned}\left\|\left|x\right|^{\alpha_1-\frac{1}{2}}\chi_1(x)\chi_2(D_x)u\right\|_{L^2}^2\leq&\frac{1}{(\alpha_1'-\Im(v_{-\infty}))^2}\left\|\left|x\right|^{\alpha_1+\frac{1}{2}}\chi_1(x)\chi_2(D_x)Pu\right\|_{L^2}^2\\&+\frac{C}{\alpha_1'-\Im(v_{-\infty})}\left\|u\right\|_{H^{-N,-1+\alpha_1,-N}}^2\end{aligned}$$

We write

$$\left|x\right|^{\alpha_1+\frac{1}{2}}\chi_1(x)\chi_2(D_x)=\left|x\right|^{\alpha_1+\frac{1}{2}}\tilde{\chi}_1(x)\chi_2(D_x)\chi_1(x)+\left|x\right|^{\alpha_1+\frac{1}{2}}\tilde{\chi}_1(x)[\chi_1(x),\chi_2(D_x)].$$

We use that $|x|^{\alpha_1+\frac{1}{2}}\,\tilde{\chi}_1(x)\chi_2(D_x)$ is bounded from $H^{m-1,\frac{1}{2}+\alpha_1}$ and $|x|^{\alpha_1+\frac{1}{2}}\,\tilde{\chi}_1(x)[\chi_1(x),\chi_2(D_x)]$ is bounded from $H^{-j,-j}$ to $H^{j,j}$ for all $j\in\mathbb{N}$ by proposition C.1.9 to get:

$$\left\||x|^{\alpha_1+\frac{1}{2}}\,\chi_1(x)\chi_2(D_x)Pu\right\|_{L^2}^2 \lesssim \|\chi_1(x)Pu\|_{H^{m-1,\frac{1}{2}+\alpha_1}} + \|u\|_{H^{-N,-1+\alpha_1,-N}}$$

Furthermore, using that on the support of χ_1 we have

$$|x|^{\alpha_1-\frac{1}{2}} \gtrsim \langle x\rangle^{\alpha_1-\frac{1}{2}}$$

we get:

$$\left\||x|^{\alpha_1-\frac{1}{2}}\,\chi_1(x)\chi_2(D_x)u\right\|_{L^2}^2 \geq \|\chi_1(x)\chi_2(D_x)u\|_{H^{0,-\frac{1}{2}+\alpha_1}}^2$$

For the last step, we take $\tilde{\chi}_2$ a smooth compactly cutoff equal to 1 on the support of χ_2.

$$\begin{aligned}\chi_1(x)\chi_2(D_x)u =&\chi_1(x)\tilde{\chi}_2(D_x)\chi_2(D_x)u\\ =&\tilde{\chi}_2(D_x)\chi_1(x)\chi_2(D_x)u + [\chi_1(x),\tilde{\chi}_2(D_x)]\chi_2(D_x)u\end{aligned}$$

By proposition C.1.9, $[\chi_1(x),\tilde{\chi}_2(D_x)]\chi_2(D_x)$ is bounded from $H^{-N,-1+\alpha_1,-N}$ to $H^{m,-\frac{1}{2}+\alpha_1}$, $\langle D_x\rangle^m\,\tilde{\chi}_2(D_x)$ is bounded on $H^{0,-\frac{3}{2}+\alpha_1}$ and

$$\begin{aligned}\|\chi_1(x)\chi_2(D_x)u\|_{H^{m,-\frac{1}{2}+\alpha_1}} \lesssim& \|\langle D_x\rangle^m\,\tilde{\chi}_2(D_x)\chi_1(x)\chi_2(D_x)u\|_{H^{0,-\frac{1}{2}+\alpha_1}} + \|u\|_{H^{-N,-1+\alpha_1,-N}}\\ \lesssim& \|\chi_1(x)\chi_2(D_x)u\|_{H^{0,-\frac{1}{2}+\alpha_1}} + \|u\|_{H^{-N,-1+\alpha_1,-N}}\end{aligned}$$

We deduce:

$$\|\chi_1(x)\chi_2(D_x)u\|_{H^{m,-\frac{1}{2}+\alpha_1}} \lesssim \|\chi_1(x)\chi_2(D_x)Pu\|_{H^{m-1,\frac{1}{2}+\alpha_1}} + \|u\|_{H^{-N,-1+\alpha_1,-N}}\,.$$

Sink estimate: We now prove the second estimate. As before, it is enough to prove the estimate for $\tilde{\chi}_1(x)v$, and therefore without loss of generality we can assume $v\in\cap_{j\in\mathbb{N}}H^{m-1,-1+\alpha_2,j}$. We take the commutator

$$A := \chi_2(D_x)\chi_1(x)^2\,|x|^{-2\alpha_2}\,\chi_2(D_x)$$

and $\eta=2$. As before we have:

$$\langle A_\epsilon v, Pv\rangle - \langle Pv, A_\epsilon v\rangle = \langle (PA_\epsilon - A_\epsilon P + (P^*-P)A_\epsilon)v, v\rangle$$

This time it is immediate without the need of a second regularization since we have taken η large enough (the fact that we can allow a larger power of $\langle\epsilon x\rangle^{-1}$ is due to the corresponding term in the commutator having the correct sign unlike in the source estimate). We compute the commutator:

$$\begin{aligned}i(PA_\epsilon - A_\epsilon P + (P^*-P)A_\epsilon) =&\chi_2(D_x)\chi_1(x)\,|x|^{-2\alpha_2-1}\,\langle\epsilon x\rangle^{-2}\left(2\chi_1'(x)\,|x|\right.\\ &\left.+2\chi_1(x)\left(\alpha_2+\frac{(\epsilon x)^2}{1+(x\epsilon)^2}+\Im(v_{-\infty})\right)\right)\chi_2(D_x)\\ &+\chi_2(D_x)\chi_1(x)^2 2\Im(V(x)-|x|^{-1}\,v_{-\infty})\,\langle\epsilon x\rangle^{-2}\,|x|^{-2\alpha_2}\,\chi_2(D_x)\\ &+i[V(x)-2i\Im(V(x)),\chi_2(D_x)]\chi_1(x)^2\,\langle\epsilon x\rangle^{-2}\,|x|^{-2\alpha_2}\,\chi_2(D_x)\\ &+i\chi_1(x)^2\,\langle\epsilon x\rangle^{-2}\,|x|^{-2\alpha_2}\,[\chi_2(D_x),V(x)]\end{aligned}$$

This time, the terms $2\chi_1'(x)\left|x\right|$ and $2\chi_1(x)\left(\alpha_2+\frac{(\epsilon x)^2}{1+(x\epsilon)^2}+\Im(v_{-\infty})\right)$ have opposite signs. As in the source estimate, defining

$$\begin{aligned}G :=&\chi_2(D_x)\chi_1(x)^2 2\Im(V(x)-\left|x\right|^{-1}v_{-\infty})\left\langle \epsilon x\right\rangle^{-2}\left|x\right|^{-2\alpha_2}\chi_2(D_x)\\&+i[V(x)-2i\Im(V(x)),\chi_2(D_x)]\chi_1(x)^2\left\langle \epsilon x\right\rangle^{-2}\left|x\right|^{-2\alpha_2}\chi_2(D_x)\\&+i\chi_1(x)^2\left\langle \epsilon x\right\rangle^{-2}\left|x\right|^{-2\alpha_2}[\chi_2(D_x),V(x)]\end{aligned}$$

we have $\left\langle D_x\right\rangle^N\left(\chi_-\left\langle x\right\rangle^{\alpha_2+1}+\chi_+\left\langle x\right\rangle^N\right)G$ is bounded from $H^{-N,-1-\alpha_2,-N}$ to L^2 (uniformly with respect to $\epsilon\in(0,1)$) by lemma C.1.17 and lemma C.1.12. We deduce that there exists $C>0$ such that:

$$\begin{aligned}\langle i(PA_\epsilon-A_\epsilon P+(P^*-P)A_\epsilon)v,v\rangle\gtrsim&\left\|\left|x\right|^{-\frac{1}{2}-\alpha_2}\left\langle \epsilon x\right\rangle^{-1}\chi_1(x)\chi_2(D_x)v\right\|_{L^2}^2\\&-\left\|\left|\chi_1'(x)x\right|^{\frac{1}{2}}\left\langle \epsilon x\right\rangle^{-1}\chi_1(x)\chi_2(D_x)v\right\|_{L^2}^2-C\left\|u\right\|_{H^{-N,-1-\alpha_2,-N}}^2\end{aligned}$$

We bound $\langle Pv,A_\epsilon v\rangle$ exactly as in the first case and (using also the uniform boundedness of $\left\langle \epsilon x\right\rangle^{-2}$) we get uniformly with respect to $\epsilon\in(0,1)$:

$$\begin{aligned}\left\|\left|x\right|^{-\frac{1}{2}-\alpha_2}\left\langle \epsilon x\right\rangle^{-1}\chi_1(x)\chi_2(D_x)v\right\|_{L^2}^2\lesssim&\left\|\chi_1(x)Pv\right\|_{H^{m-1,\frac{1}{2}-\alpha_2}}^2+\left\|\left|\chi_1'(x)x\right|^{\frac{1}{2}}\chi_1(x)\chi_2(D_x)v\right\|_{L^2}^2\\&+\left\|v\right\|_{H^{-N,-1-\alpha_2,-N}}^2\end{aligned}$$

Using again a weak convergence argument, we deduce that $\left|x\right|^{-\frac{1}{2}-\alpha}\chi_1(x)\chi_2(D_x)v\in L^2$ and

$$\begin{aligned}\left\|\left|x\right|^{-\frac{1}{2}-\alpha}\chi_1(x)\chi_2(D_x)v\right\|_{L^2}^2\lesssim&\left\|Pv\right\|_{H^{m-1,\frac{1}{2}-\alpha}}^2+\left\|\left|\chi_1'(x)x\right|^{\frac{1}{2}}\chi_1(x)\chi_2(D_x)v\right\|_{L^2}^2\\&+\left\|v\right\|_{H^{-N,-1-\alpha_2,-N}}^2\end{aligned}$$

Since $\left\|\left|\chi_1'(x)x\right|^{\frac{1}{2}}\chi_1(x)\chi_2(D_x)v\right\|_{L^2}^2\lesssim\left\|v\right\|_{H^{-N,-1-\alpha_2,-N}}^2$, we can finish the argument as in the source estimate. □

A completely parallel proof enables to get the estimate near the radial point at $x=+\infty$.

Proposition C.2.5. *Let χ_1 be a smooth monotonic compactly supported cutoff equal to 1 in a neighborhood of $+\infty$ and equal to zero on a neighborhood of 0. Let χ_2 be a smooth compactly supported cutoff equal to 1 in a neighborhood of $-z$. Let $\tilde{\chi}_1$ be defined as χ_1 and equal to 1 on supp(χ_1). Let $N\in\mathbb{N}$. Let $\beta_1>-\Im(v_{+\infty})$ and $\max(-\Im(v_{+\infty}),\beta_1-\frac{1}{2})<\beta_1'<\beta_1$ and $\Im(v_{+\infty})<\beta_2$. There exists a constant $C>0$ such that for all $u\in\cup_{j\in\mathbb{N}}H^{-j,-j}$ such that $\tilde{\chi}_1(x)u\in H^{m-1,-\frac{1}{2}+\beta_1'}$ and all $v\in\cup_{j\in\mathbb{N}}H^{-j,-j}$ such that $\tilde{\chi}_1(x)v\in H^{m-1,-\beta_2-1}$:*

$$\left\|\chi_1(x)\chi_2(D_x)u\right\|_{H^{m,-\frac{1}{2}+\beta_1}}\leq C\left(\left\|\chi_1(x)Pu\right\|_{H^{m-1,\frac{1}{2}+\beta_1}}+\left\|u\right\|_{H^{-N,-N,-\frac{1}{2}+\beta_1'}}\right)\tag{C.10}$$

$$\left\|\chi_1(x)\chi_2(D_x)v\right\|_{H^{m,-\frac{1}{2}-\beta_2}}\leq C\left(\left\|\chi_1(x)Pv\right\|_{H^{m-1,\frac{1}{2}-\beta_2}}+\left\|v\right\|_{H^{-N,-N,-1-\beta_2}}\right)\tag{C.11}$$

in the strong sense that if the right-hand side is finite, so is the left-hand side and the inequality holds.

We have several choices to obtain Fredholm estimates. Indeed, since the radial points are isolated, we can choose freely to use a source or sink estimate at any given radial point. Note that the situation would have been different if we had characteristic curves linking radial points (see the semiclassical analysis below).

We can now state all the Fredholm estimates following from our analysis:

Proposition C.2.6. *Let $m \in \mathbb{R}$, $\Im(v_{-\infty}) < \alpha_1$, $-\Im(v_{-\infty}) < \alpha_2$, $\max(\Im(v_{-\infty}), \alpha_1 - \frac{1}{2}) < \alpha_1' < \alpha_1$, $-\Im(v_{+\infty}) < \beta_1$, $\Im(v_{+\infty}) < \beta_2$, $\max(-\Im(v_{+\infty}), \beta_1 - \frac{1}{2}) < \beta_1' < \beta_1$. There exists a constant $C > 0$ such that for all $u \in \cup_{N\in\mathbb{N}} H^{-N,-N}$, we have (in the strong sense that whenever the right-hand side is finite, so is the left-hand side and the inequality holds):*

$$\|u\|_{H^{m,-\frac{1}{2}+\alpha_1,-\frac{1}{2}+\beta_1}} \leq C\left(\|Pu\|_{H^{m-1,\frac{1}{2}+\alpha_1,\frac{1}{2}+\beta_1}} + \|u\|_{H^{m-1,-\frac{1}{2}+\alpha_1',-\frac{1}{2}+\beta_1'}}\right) \tag{C.12}$$

$$\|u\|_{H^{m,-\frac{1}{2}-\alpha_2,-\frac{1}{2}-\beta_2}} \leq C\left(\|Pu\|_{H^{m-1,\frac{1}{2}-\alpha_2,\frac{1}{2}-\beta_2}} + \|u\|_{H^{m-1,-1-\alpha_2,-1-\beta_2}}\right) \tag{C.13}$$

$$\|u\|_{H^{m,-\frac{1}{2}+\alpha_1,-\frac{1}{2}-\beta_2}} \leq C\left(\|Pu\|_{H^{m-1,\frac{1}{2}+\alpha_1,\frac{1}{2}-\beta_2}} + \|u\|_{H^{m-1,-\frac{1}{2}+\alpha_1',-1-\beta_2}}\right) \tag{C.14}$$

$$\|u\|_{H^{m,-\frac{1}{2}-\alpha_2,-\frac{1}{2}+\beta_1}} \leq C\left(\|Pu\|_{H^{m-1,\frac{1}{2}-\alpha_2,\frac{1}{2}+\beta_1}} + \|u\|_{H^{m-1,-1-\alpha_2,-\frac{1}{2}+\beta_1'}}\right) \tag{C.15}$$

Proof. All the estimates are proved similarly: we use the elliptic estimate of corollary C.2.2 and then the radial point estimate (propositions C.2.3 and C.2.5) to bound the error terms $\chi_3(x)\chi_1(D_x)u$ and $\chi_3(x)\chi_1(D_x)u$ using either the source or the sink estimate. For (C.12) we use two sources estimates, for (C.13) we use two sink estimates, for (C.14) we use a source estimate near $-\infty$ and a sink estimate near $+\infty$ and for (C.15) we use the source estimate near $+\infty$ and the sink estimate near $-\infty$.

We give some details in the case of (C.14). Let $u \in H^{m-1,-\frac{1}{2}+\alpha_1',-1-\beta_2}$ such that $Pu \in H^{m-1,\frac{1}{2}+\alpha_1,\frac{1}{2}-\beta_2}$ (otherwise the right-hand side is infinite). Let χ_1 be a smooth compactly supported cutoff such that $\chi_1 = 1$ in a neighborhood of $-z$. Let χ_3 and χ_4 be smooth monotonic cutoffs such that $\chi_3 = 1$ in a neighborhood of $-\infty$, $\chi_3 = 0$ on $\mathrm{supp}(\chi_+)$, $\chi_4 = 0$ on $\mathrm{supp}(\chi_-)$ and $\chi_4 = 1$ in a neighborhood of $+\infty$.

Since $\chi_- u \in H^{m-1,-\frac{1}{2}+\alpha_1'}$, and $\chi_- Pu \in H^{m-1,\frac{1}{2}+\alpha_1}$ we can use the source estimate (C.5) (with N large enough) to get $\chi_3 u \in H^{m,-\frac{1}{2}+\alpha_1}$ and:

$$\|\chi_3(x)\chi_1(D_x)u\|_{H^{m,-\frac{1}{2}+\alpha_1}} \lesssim \|\chi_3(x)Pu\|_{H^{m-1,\frac{1}{2}+\alpha_1}} + \|u\|_{H^{m-1,-\frac{1}{2}+\alpha_1',-1-\alpha_2}}$$

Similarly, we can use (C.11) to get:

$$\|\chi_4(x)\chi_1(D_x)u\|_{H^{m,-\frac{1}{2}-\beta_2}} \lesssim \|\chi_4(x)Pu\|_{H^{m-1,\frac{1}{2}-\beta_2}} + \|u\|_{H^{m-1,-\frac{1}{2}+\alpha_1',-1-\alpha_2}}$$

We use corollary C.2.2 (with N large enough):

$$\begin{aligned}\|u\|_{H^{m,-\frac{1}{2}+\alpha_1,-\frac{1}{2}-\beta_2}} \lesssim & \|Pu\|_{H^{m-1,-\frac{1}{2}+\alpha_1,-\frac{1}{2}-\beta_2}} + \|\chi_3(x)\chi_1(D_x)u\|_{H^{m,-\frac{1}{2}+\alpha_1,-\frac{1}{2}-\beta_2}} \\ & + \|\chi_4(x)\chi_1(D_x)u\|_{H^{m,-\frac{1}{2}+\alpha_1,-\frac{1}{2}-\beta_2}} + \|u\|_{H^{m-1,-\frac{1}{2}+\alpha_1',-1-\beta_2}}\end{aligned}$$

Combining all these estimates, we get the claim. □

Estimates of proposition C.2.6 are called Fredholm estimates because of the following general result.

Lemma C.2.7. *Let $X_0 \subset X_1 \subset X_2$ and $Y_0 \subset Y_1 \subset Y_2$ be Banach spaces (with continuous dense inclusions). Let $P : X_1 \to Y_2$ be a bounded operator such that $P_{|X_0}$ is bounded from X_0 to Y_1. We assume that both inclusions $X_1 \subset X_2$ and $Y_0 \subset Y_1$ are compact and that there exists $C > 0$ such that for all $u \in X_1$ and all $v \in Y_1^*$:*

$$\|u\|_{X_1} \leq C\left(\|Pu\|_{Y_1} + \|u\|_{X_2}\right) \tag{C.16}$$

$$\|v\|_{Y_1^*} \leq C\left(\|P^*v\|_{X_1^*} + \|v\|_{Y_0^*}\right). \tag{C.17}$$

Note that in the previous estimate, the right hand side may be infinite. Under these assumptions, P is Fredholm as an operator between the Banach space $\mathfrak{X} := \{u \in X_1 : Pu \in Y_1\}$ (endowed with the norm $\|u\|_{\mathfrak{X}}^2 = \|u\|_{X_1}^2 + \|Pu\|_{Y_1}^2$) and Y_1.

Proof. We begin by proving that $\mathfrak{X}$ is a Banach space. Let (u_n) be a Cauchy sequence in $\mathfrak{X}$. Then by completeness of X_1, there exists $u \in X_1$ such that $\lim_{n\to+\infty} u_n = u$ for the topology of X_1. Moreover by completeness of Y_1 there exists $v \in Y_1$ such that $\lim_{n\to+\infty} Pu_n = v$ for the topology of Y_1. On the other hand, by continuity of P, we have $\lim_{n\to+\infty} Pu_n = Pu$ for the topology of Y_2. By uniqueness of the limit in Y_2, we have $Pu = v \in Y_1$ and $\lim_{n\to+\infty} u_n = u$ in $\mathfrak{X}$.

Now we prove the Fredholm property. For all $u \in Ker(P)$, we have $\|u\|_{X_1} \leq C\|u\|_{X_2}$ by estimate (C.16). By compactness of the inclusion $X_1 \subset X_2$, we deduce that $B(0,1)_{X_1} \cap Ker(P)$ is relatively compact for the topology induced by X_2. The topology induced by X_2 is the same as the topology induced by X_1 since the two norms are equivalent on $Ker(P)$. Therefore, $Ker(P)$ endowed with the norm X_1 has a relatively compact unit ball. We deduce that $\dim(Ker(P)) < +\infty$. The same argument using (C.17) and the compactness of the inclusion $Y_1^* \subset Y_0^*$ proves that $Ker(P^*) = Ran(P)^\perp$ is finite dimensional.

We now prove that $P(\mathfrak{X})$ is closed. We take $y_1, ..., y_k$ a normed basis of $Ker(P)$ and we denote by $y_1^*, ..., y_k^*$ extensions (of norm 1) of the dual basis (obtained by the Hahn-Banach theorem). Let (u_n) be a sequence in $\mathfrak{X}$ and $v \in Y_1$ such that $\lim_{n\to+\infty} Pu_n = v$ in Y_1. We have to prove that $v \in P(\mathfrak{X})$. Without loss of generality, we can assume that $y_i^*(u_n) = 0$ (replacing u_n by $u_n - \sum_{i\leq k} y_i^*(u_n)y_i$ which does not change the value of v). By contradiction assume that u_n is unbounded in X_1. Extracting a subsequence, we can assume $\|u_n\|_{X_1} \to +\infty$. Then $\lim_{n\to+\infty} P\left(\frac{u_n}{\|u_n\|_{X_1}}\right) = 0$ in Y_1. By compactness, we can assume (after extracting a subsequence) that $\frac{u_n}{\|u_n\|_{X_1}}$ converges to $z \in X_2$ for the topology of X_2. Using inequality (C.16), we deduce that $\frac{u_n}{\|u_n\|_{X_1}}$ is Cauchy in X_1. We deduce $z \in Ker(P)$ and we have the convergence in X_1. Then by continuity, for $i = 1, ..., k$, we have $y_i^*(z) = 0$ and because $z \in Ker(P)$ we deduce $z = 0$. But it is a contradiction since $\left\|\frac{u_n}{\|u_n\|_{X_1}}\right\|_{\mathcal{X}} = 1$. So (u_n) is bounded in X_1. After extracting a subsequence, we can assume that $\lim_{n\to+\infty} u_n = u$ in X_2. Then we use (C.16) and deduce that (u_n) is Cauchy in X_1. Since we also have that (Pu_n) is Cauchy in Y_1, by completeness of $\mathfrak{X}$ and uniqueness of the limit in X_2, we have $\lim_{n\to+\infty} u_n = u$ in $\mathfrak{X}$. By uniqueness of the limit in Y_1, we deduce that $v = Pu \in P(\mathfrak{X})$. □

We deduce the following Fredholm statement:

Proposition C.2.8. *Let Y be a Banach space of distributions. Let $m, l, l' \in \mathbb{R}$. We define $\mathfrak{X}^{m,l,l'}(Y) := \left\{u \in H^{m,l,l'} : Pu \in Y\right\}$ (endowed with the graph norm). Let $\alpha_1 > \Im(v_{-\infty})$,$\alpha_2 > -\Im(v_{-\infty})$, $\beta_1 > -\Im(v_{+\infty})$ and $\beta_2 > \Im(v_{+\infty})$. The following operators are Fredholm:*

- $P : \mathfrak{X}^{m,-\frac{1}{2}+\alpha_1,-\frac{1}{2}+\beta_1}(H^{m-1,\frac{1}{2}+\alpha_1,\frac{1}{2}+\beta_1}) \to H^{m-1,\frac{1}{2}+\alpha_1,\frac{1}{2}+\beta_1}$
- $P : \mathfrak{X}^{m,-\frac{1}{2}-\alpha_2,-\frac{1}{2}-\beta_2}(H^{m-1,\frac{1}{2}-\alpha_2,\frac{1}{2}-\beta_2}) \to H^{m-1,\frac{1}{2}-\alpha_2,\frac{1}{2}-\beta_2}$
- $P : \mathfrak{X}^{m,-\frac{1}{2}+\alpha_1,-\frac{1}{2}-\beta_2}(H^{m-1,\frac{1}{2}+\alpha_1,\frac{1}{2}-\beta_2}) \to H^{m-1,\frac{1}{2}+\alpha_1,\frac{1}{2}-\beta_2}$
- $P : \mathfrak{X}^{m,-\frac{1}{2}-\alpha_2,-\frac{1}{2}+\beta_1}(H^{m-1,\frac{1}{2}-\alpha_2,\frac{1}{2}+\beta_1}) \to H^{m-1,\frac{1}{2}-\alpha_2,\frac{1}{2}+\beta_1}$

In the case of this particularly simple operator $D_x + V(x) + z$, proposition C.2.8 can also be recovered by elementary methods as we partially show in the following section (in the case $m = 1$ for simplicity).

C.3 Fredholm property by an elementary method

It is instructive to perform a completely explicit analysis in this simple case. Let $z \in \mathbb{R}$. Let $F(x) = \int_0^x V(s)\ \mathrm{d}s$. In particular, we have $F(x) = v_{+\infty} \ln(x) + O(1)$ when $x \to +\infty$ and $F(x) = -v_{-\infty} \ln|x| + O(1)$ when $x \to -\infty$. For the operator $P = D_x + V(x) + z$ and $f \in L^1_{loc}(\mathbb{R})$, standard ordinary differential equation theory enables to find all the solutions in $\mathcal{S}'$ of $(D_x + V(x) + z)u = f$. They are of the form:

$$u_C(x) = i \int_0^x e^{i(F(s)-F(x)+z(s-x))} f(s)\ \mathrm{d}s + Ce^{-i(F(x)+zx)}$$

for $C \in \mathbb{C}$. Let $\alpha_1 > \Im(v_{-\infty})$ and $\beta_1 > -\Im(v_{+\infty})$ We prove that

$$P : \mathfrak{X}^{1,-\frac{1}{2}+\alpha_1,-\frac{1}{2}+\beta_1}(H^{0,\frac{1}{2}+\alpha_1,\frac{1}{2}+\beta_1}) \to H^{0,\frac{1}{2}+\alpha_1,\frac{1}{2}+\beta_1}$$

is Fredholm of index -1 (the other cases are similar). By the development of F, there exists $C > 0$ such that for $x \geq 1$:

$$\begin{aligned} C^{-1}x^{\Im(v_{+\infty})} \leq \left|e^{-i(F(x)+zx)}\right| &\leq Cx^{\Im(v_{+\infty})} \\ \left|D_x e^{-i(F(x)+zx)}\right| &\leq Cx^{\Im(v_{+\infty})} \end{aligned}$$

and similarly, for $x \leq 1$:

$$\begin{aligned} C^{-1}\left|x\right|^{-\Im(v_{-\infty})} \leq \left|e^{i(F(x)+zx)}\right| &\leq C\left|x\right|^{-\Im(v_{-\infty})} \\ \left|D_x e^{i(F(x)+zx)}\right| &\leq C\left|x\right|^{-\Im(v_{-\infty})} \end{aligned}$$

We deduce that $Ker(P) \cap H^{1,-\frac{1}{2}+\alpha_1,-\frac{1}{2}+\beta_1} = \{0\}$ and $Ker(P^*) \cap H^{0,-\frac{1}{2}-\alpha_1,-\frac{1}{2}-\beta_1} = \mathbb{C}e^{-i(\overline{F(x)}+zx)}$. Therefore, it remains to prove that if $f \in H^{0,\frac{1}{2}+\alpha_1,\frac{1}{2}+\beta_1}$ and

$$\int_{-\infty}^{+\infty} e^{i(F(s)+zs)} f(s)\ \mathrm{d}s = 0,$$

then for some value of the constant C, $u_C \in H^{1,-\frac{1}{2}+\alpha_1,-\frac{1}{2}+\alpha_2}$. We take

$$C = \int_{-\infty}^{0} e^{i(F(s)+zs)} f(s)\ \mathrm{d}s.$$

We have

$$u_C(x) = \int_{-\infty}^{x} e^{i(F(s)-F(x)+z(s-x)} f(s)\ \mathrm{d}s = \int_{x}^{+\infty} e^{i(F(s)-F(x)+z(s-x)} f(s)\ \mathrm{d}s$$

(where the second equality comes from the hypothesis on f).

$$\begin{aligned} |\chi_-(x)u(x)|^2 \lesssim \chi_-(x)^2 \left\langle x\right\rangle^{-2\Im(v_{-\infty})} \int_{-\infty}^{x} \left\langle s\right\rangle^{-1-\alpha_1+\Im(v_{-\infty})}\ \mathrm{d}s \int_{-\infty}^{x} \left\langle s\right\rangle^{1+\alpha_1+\Im(v_{-\infty})} |f(s)|^2\ \mathrm{d}s \\ \lesssim \left\langle x\right\rangle^{-\alpha_1-\Im(v_{-\infty})} \int_{-\infty}^{x} \left\langle s\right\rangle^{1+\alpha_1+\Im(v_{-\infty})} |f(s)|^2\ \mathrm{d}s \end{aligned}$$

Similarly:

$$\begin{aligned} |\chi_+(x)u(x)|^2 \lesssim \chi_+(x)^2 \left\langle x\right\rangle^{2\Im(v_{+\infty})} \int_{x}^{+\infty} \left\langle s\right\rangle^{-1-\beta_1-\Im(v_{+\infty})}\ \mathrm{d}s \int_{x}^{+\infty} \left\langle s\right\rangle^{1+\beta_1-\Im(v_{+\infty})} |f(s)|^2\ \mathrm{d}s \\ \lesssim \left\langle x\right\rangle^{-\beta_1+\Im(v_{+\infty})} \int_{x}^{+\infty} \left\langle s\right\rangle^{1+\beta_1-\Im(v_{+\infty})} |f(s)|^2\ \mathrm{d}s \end{aligned}$$

We deduce (introducing $M := \sup \operatorname{supp}(\chi_-)$):

$$\begin{aligned}\int_{\mathbb{R}} \langle x\rangle^{-1+2\alpha_1} |\chi_-(x)u(x)|^2 \, \mathrm{d}x &\lesssim \int_{\mathbb{R}} \chi_-(x)^2 \langle x\rangle^{-1+\alpha_1-\Im(v_{-\infty})} \int_{-\infty}^{x} \langle s\rangle^{1+\alpha_1+\Im(v_{-\infty})} |f(s)|^2 \, \mathrm{d}s \, \mathrm{d}x \\ &\lesssim \int_{-\infty}^{M} \int_{s}^{M} \chi_-(x)^2 \langle x\rangle^{-1+\alpha_1-\Im(v_{-\infty})} \, \mathrm{d}x \langle s\rangle^{1+\alpha_1+\Im(v_{-\infty})} |f(s)|^2 \, \mathrm{d}s \\ &\lesssim \int_{-\infty}^{M} \langle s\rangle^{1+2\alpha_1} \|f(s)\|^2 \, \mathrm{d}s \\ &\lesssim \|f\|^2_{H^{0,\frac{1}{2}+\alpha_1,\frac{1}{2}+\beta_1}}\end{aligned}$$

Similarly,

$$\int_{\mathbb{R}} \langle x\rangle^{-1+2\beta_1} |\chi_+(x)u(x)|^2 \, \mathrm{d}x \lesssim \|f\|^2_{H^{0,\frac{1}{2}+\alpha_1,\frac{1}{2}+\beta_1}}$$

A similar computation provides:

$$\left\| \langle x\rangle^{-1+2\alpha_1} D_x\chi_- u \right\|^2_{L^2} \lesssim \|f\|^2_{H^{0,\frac{1}{2}+\alpha_1,\frac{1}{2}+\beta_1}}$$

and

$$\left\| \langle x\rangle^{-1+2\beta_1} D_x\chi_+ u \right\|^2_{L^2} \lesssim \|f\|^2_{H^{0,\frac{1}{2}+\alpha_1,\frac{1}{2}+\beta_1}} .$$

This concludes the proof.

C.4 Semiclassical analysis

Often, we want to know when the operator is invertible and we want uniform bounds for the inverse when $z \to +\infty$. In this section, we introduce the semiclassical tools which will be used to answer this question. We introduce the semiclassical parameter $h \in (0,1)$.

As before, it will be enough for our simple example to use combinations of Fourier multipliers and multiplication operators. Note that semiclassical Fourier multipliers will be of the form $g(hD_x)$ for $g \in S^m$.

Definition C.4.1. *We introduce the semiclassical family of Sobolev spaces. For $m, l \in \mathbb{R}$, we define $H_h^{m,l}$ as $H^{m,l}$ endowed with the norm:*

$$\|u\|_{H_h^{m,l}} := \left\| \langle x\rangle^l \langle hD_x\rangle^m u \right\|_{L^2}$$

Remark C.4.2. *Note that for all $h > 0$, the norm of $H_h^{m,l}$ is equivalent to the norm of $H^{m,l}$. However, this definition become interesting when we study the limit $h \to 0$. For example, for a family $(u_h)_{h\in(0,1)}$ of elements in $H^{m,l}$ we are interested by quantities such as: $\sup_{h\in(0,1)} \|u_h\|_{H_h^{m,l}}$. If $m \geq 0$, the explicit bound in proposition C.1.9 provides in particular that $\sup_{h\in(0,1)} \|u_h\|_{H_h^{m,l}} \leq \sup_{h\in(0,1)} \|u_h\|_{H^{m,l}}$ but the converse does not hold (see the family $u_h = \chi(h^{-1}x)$ with $\chi \in C_c^\infty(\mathbb{R})$, $l = 0$, $m = 1$).*

Proposition C.4.3. *Let $l, m \in \mathbb{R}$. If $m \geq 0$, we have the following relations between norms (uniformly with respect to h):*

$$h^m \|.\|_{H^{m,l}} \lesssim_{l,m} \|.\|_{H_h^{m,l}} \lesssim_{l,m} \|.\|_{H^{m,l}} .$$

If $m \leq 0$, we have (uniformly with respect to h):

$$h^{-m} \|.\|_{H_h^{m,l}} \lesssim_{l,m} \|.\|_{H^{m,l}} \lesssim_{l,m} \|.\|_{H_h^{m,l}}$$

Proof. In both cases it is a consequence of the fact that the families of operators:

$$\langle x\rangle^l \langle D_x\rangle^{-|m|} \langle hD_x\rangle^{|m|} \langle x\rangle^{-l}$$

and

$$\langle x\rangle^l h^{|m|} \langle D_x\rangle^{|m|} \langle hD_x\rangle^{-|m|} \langle x\rangle^{-l}$$

are h-uniformly bounded on L^2 which follows from the quantitative bound in proposition C.1.9 together with the fact that $\langle\xi\rangle^{-|m|} \langle h\xi\rangle^{|m|}$ and $h^{|m|} \langle\xi\rangle^{|m|} \langle h\xi\rangle^{-|m|}$ have h-uniformly bounded seminorms in $S^0(\mathbb{R})$. □

By the explicit bound in proposition C.1.9 we have:

Proposition C.4.4. *Let $m_0 \in \mathbb{R}$. Let $g \in S^{m_0}(\mathbb{R})$. For every $m, l \in \mathbb{R}$, the family of operators $g(hD_x)$ is h-uniformly bounded from $H_h^{m,l}$ to $H_h^{m-m_0,l}$.*

Proof. We have to prove that $\langle x\rangle^l g(hD_x) \langle hD_x\rangle^{-m_0} \langle x\rangle^{-l}$ is h-uniformly bounded on L^2. It is a consequence of the explicit bound in proposition C.1.9 (noting that $\langle h\xi\rangle^{-m_0} g(h\xi)$ has all its $S^0(\mathbb{R})$-seminorms h-uniformly bounded). □

Remark C.4.5. *For $l' \geq l$ and $m \in \mathbb{R}$ it follows from the definition that: $\|.\|_{H_h^{m,l}} \leq \|.\|_{H_h^{m,l'}}$. An application of proposition C.4.4 proves that $\langle hD_x\rangle^{m-m'}$ is h-uniformly bounded on $H^{m',l'}$ for $m' \geq m$ and any $l' \in \mathbb{R}$. Therefore we deduce that for $l' \geq l$ and $m' \geq m$, we have h-uniformly:*

$$\|.\|_{H_h^{m,l}} \lesssim_{m-m',l} \|.\|_{H_h^{m',l'}}$$

We also have a semiclassical version of proposition C.1.8:

Proposition C.4.6. *Let $g \in S^m(\mathbb{R})$ and $f \in S^l(\mathbb{R})$. For all $M \in \mathbb{N}$, there exists $N \in \mathbb{N}$ such that*

$$g(hD_x)f(x) = f(x)g(hD_x) + \sum_{j=1}^{N} h^j \frac{\partial_x^j f(x)(D_\xi^j g)(hD_x)}{j!} + h^{N+1} R_{N,h}$$

where $R_{N,h}$ is a continuous operator on $\mathcal{S}'$ (which is continuous on $\mathcal{S}$) and $hR_{N,h}$ is h-uniformly bounded from $H_h^{-2M,-2M}$ to $H_h^{2M,2M}$.

Proof. By the proof of proposition C.1.8, we have for $N \in \mathbb{N}$:

$$g(hD_x)f(x) - f(x)g(hD_x) - \sum_{j=1}^{N} h^j \frac{\partial_x^j f(x)(D_\xi^j g)(hD_x)}{j!} = h^{N+1} R_{N,h}$$

where for $u \in \mathcal{S}$:

$$\begin{aligned} R_{N,h}u =& \frac{1}{2\pi} \iint e^{i\xi(x-y)} R_{N,h}(x,y,\xi) u(y) \, \mathrm{d}y \, \mathrm{d}\xi \\ R_{N,h}(x,y,\xi) :=& (D_\xi^{N+1} g)(h\xi) \int_0^1 \frac{(1-s)^N}{N!} f^{(N+1)}(x + s(y-x)) \, \mathrm{d}s \\ =& R_N(x,y,h\xi) \end{aligned}$$

where $R_N(x,y,\xi)$ was already analyzed in the proof of proposition C.1.8. A computation very similar to the one performed there provides the following h-uniform bound (for $k'' \in \mathbb{N}$ and $N \geq l-1$):

$$\left|\langle hD_x\rangle^{2M}\langle hD_y\rangle^{2M} e^{i\xi(x-y)}\langle x-y\rangle^{-2k''}\langle D_\xi\rangle^{2k''} R_N(x,y,h\xi)\right| \lesssim_{M,k''} \left\|\partial_x^{N+1} f\right\|_{S^{l-N-1}_{4M}} \left\|\partial_\xi^{N+1} g\right\|_{S^{m-N-1}_{2k''}} \langle h\xi\rangle^{m-N-1+4M}\langle x\rangle^{2l-2N-2+2k''}\langle y\rangle^{N+1-l-2k''} \tag{C.18}$$

We choose N and k'' exactly as in the proof of proposition C.1.8. We define:

$$\begin{aligned} v(x) &:= \langle x\rangle^{2M}\langle hD_x\rangle^{2M} R_N \langle hD_x\rangle^{2M}\langle x\rangle^{2M} u \\ &= \langle x\rangle^{2M} \iint \langle hD_x\rangle^{2M}\langle hD_y\rangle^{2M}\left(e^{i\xi(x-y)}\langle x-y\rangle^{-2k''}\langle D_\xi\rangle^{2k''} R_N(x,y,h\xi)\right)\langle y\rangle^{2M} u(y)\,\mathrm{d}y\,\mathrm{d}\xi \end{aligned}$$

where the second line is obtained by integration by part exactly as in the proof of proposition C.1.8. Using the bound (C.18) (for simplicity, we do not keep track of the dependence with respect to the seminorms of f and g since we do not use it) and the fact that $m-N-1+4M < -1$:

$$\begin{aligned} |v(x)| &\lesssim \iint \langle h\xi\rangle^{m-N-1+4M}\langle x\rangle^{2l-2N-2+2k''}\langle y\rangle^{N+1-l-2k''}|u(y)|\,\mathrm{d}\xi\,\mathrm{d}y \\ &\lesssim h^{-1}\langle x\rangle^{2l-2N-2+2k''}\int \langle y\rangle^{N+1-l-2k''}|u(y)|\,\mathrm{d}y \end{aligned}$$

The Cauchy-Schwarz inequality and the fact that $2l-2N-2+2k'' < -\frac{1}{2}$, $N+1-l-2k'' < -\frac{1}{2}$ provides (h-uniformly):

$$\|v\|_{L^2}^2 \lesssim h^{-1}\|u\|_{L^2}$$

We deduce that $hR_{N,h}$ is h-uniformly bounded from $H_h^{-2M,-2M}$ to $H_h^{2M,2M}$. This concludes the proof. □

We can use the previous proposition to obtain the h-uniform boundedness of multiplication operators:

Proposition C.4.7. *Let $l,m \in \mathbb{R}$. Let $f \in S^{l_0}(\mathbb{R})$. The operator multiplication by f is h-uniformly bounded from $H_h^{m,l}$ to $H_h^{m,l-l_0}$.*

Proof. We have to prove that the operator $\langle x\rangle^{l-l_0}\langle hD_x\rangle^m f(x)\langle hD_x\rangle^{-m}\langle x\rangle^{-l}$ is h-uniformly bounded on L^2. We use proposition C.4.6 with N and M large enough to write:

$$\langle hD_x\rangle^m f(x) = \sum_{j=0}^{N} h^j f_j(x) g_j(hD_x) + h^{N+1} R_N$$

with $f_j \in S^l(\mathbb{R})$, $g_j \in S^{-m}(\mathbb{R})$ and hR_N is h-uniformly bounded from $H_h^{m,l}$ to $H_h^{0,-l+l_0}$ (note that by remark C.4.5, it is the case as soon as $-2M \leq m$, $-2M \leq l$, $0 \leq 2M$, $-l+l_0 \leq 2M$). We deduce that $\langle x\rangle^{l-l_0} h^{N+1} R_N \langle hD_x\rangle^{-m}\langle x\rangle^{-l}$ is h-uniformly bounded on L^2. Since by the quantitative bound in C.1.9 $g_j(hD_x)\langle hD_x\rangle^m$ is h-uniformly bounded on $H^{0,l}$ and $\langle x\rangle^{l-l_0} f_j$ is bounded from $H^{0,l}$ to L^2, this concludes the proof. □

The following improvement of proposition C.4.6 is often convenient:

Proposition C.4.8. *Let $g \in S^m(\mathbb{R})$ and $f \in S^l(\mathbb{R})$. For $N \in \mathbb{N}$, we have:*

$$g(hD_x)f(x) = f(x)g(hD_x) + \sum_{j=1}^{N} h^j \frac{\partial_x^j f(x)(D_\xi^j g)(hD_x)}{j!} + h^{N+1} R_{N,h}$$

where $R_{N,h}$ is a continuous operator on $\mathcal{S}'$ (which is continuous on $\mathcal{S}$) and for all $r, q \in \mathbb{R}$, $R_{N,h}$ is h-uniformly bounded from $H_h^{r,q}$ to $H_h^{r-m+N+1, r-l+N+1}$.

Proof. On the model of the proof of proposition C.1.12, we use proposition C.4.6 for N' large enough and we use propositions C.4.4 and C.4.7 to bound the explicit terms with index larger than $N+1$ in the expansion. □

Similarly to the previous section, we introduce the semiclassical hybrid spaces (with different decay index at $x = -\infty$ and at $x = +\infty$)

Definition C.4.9. *Let $m, l, l' \in \mathbb{R}$. Let χ_+ and χ_- be smooth non negative cutoffs such that $\chi_+ = 1$ in a neighborhood of $+\infty$ and $\chi_- = 1$ in a neighborhood of $-\infty$ and such that $\chi_- + \chi_+ = 1$. The space $H_h^{m,l,l'}$ is the vector space $H^{m,l,l'}$ endowed with the norm*

$$\|u\|^2_{H_h^{m,l,l'}} = \|\chi_- u\|^2_{H_h^{m,l}} + \|\chi_+ u\|^2_{H_h^{m,l'}}$$

We have the following natural extension of propositions C.4.4 and C.4.7.

Proposition C.4.10. *Let $m, l, l' \in \mathbb{R}$ Let $m_0 \in \mathbb{R}$. Let $g \in S^{m_0}(\mathbb{R})$. The family of operators $g(hD_x)$ is h-uniformly bounded from $H_h^{m,l,l'}$ to $H_h^{m-m_0,l,l'}$. Let $f \in S^{l_0,l_0'}(\mathbb{R})$, the operator multiplication by f is h-uniformly bounded from $H_h^{m,l,l'}$ to $H_h^{m,l-l_0,l'-l_0'}$.*

Proof. We have $\chi_-(x)g(hD_x) = g(hD_x)\chi_-(x) + [\chi_-(x), g(hD_x)]$ where $g(hD_x)\chi_-(x)$ is h-uniformly bounded from $H_h^{m,l}$ to $H_h^{m,l}$ and $[\chi_-(x), g(hD_x)]$ is bounded from $H_h^{m,l-N}$ to $H_h^{m,l+N}$ for every $N \in \mathbb{N}$. A similar observation for $\chi_+(x)g(hD_x)$ leads to the following bound:

$$\begin{aligned}\|\chi_- g(hD_x)u\|^2_{H_h^{m,l}} + \|\chi_+ g(hD_x)u\|^2_{H_h^{m,l'}} &\lesssim \|\chi_- u\|^2_{H_h^{m,l}} + \|\chi_+ u\|^2_{H_h^{m,l}} + \|u\|_{H_h^{m,\min(l,l')}} \\ &\lesssim \|u\|_{H_h^{m,l,l'}}.\end{aligned}$$

The proof for f follows from C.4.7. □

C.5 Analysis of the operator $P_h = hD_x + hV(x) + 1$

We consider the semiclassical family of operators $P_h = hD_x + hV(x) + 1$. The semiclassical principal symbol (obtained by replacing $h\partial_x$ by $i\xi_{sc}$ and keeping only the principal order with respect to h) is:

$$p_h(x, \xi) = \xi_{sc} + 1$$

and the characteristic set (which is now a subset of ${}^{sc}\overline{T}^*\overline{\mathbb{R}}$ in contrast to the classical case where it was a subset of $\partial\left({}^{sc}\overline{T}^*\overline{\mathbb{R}}\right)$) is the closure of $\{\xi_{sc} = -1\}$ in ${}^{sc}\overline{T}^*\overline{\mathbb{R}}$. The semiclassical Hamiltonian vector field is:

$$H_{p_h} = 2\partial_x$$

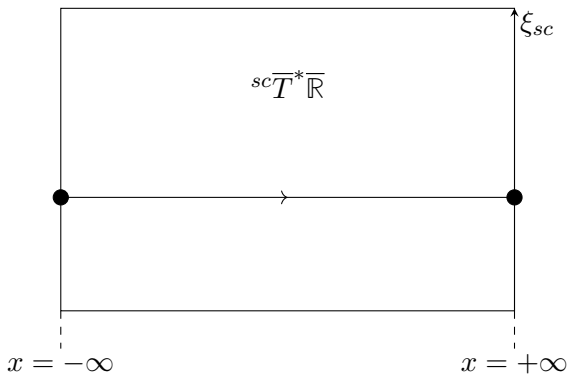

Figure C.2: Semiclassical Hamiltonian flow on the characteristic set of P_h.

Near $x = -\infty$, using the boundary defining function $\rho_- = -x^{-1}$, we have $H_{p_h} = \rho_-^2 \partial_{\rho_-}$ (and the rescaled Hamiltonian vector field near $x = -\infty$ is $\rho_-^{-1} H_{p_h} = \rho_- \partial_{\rho_-}$ and the point $\{\xi_{sc} = -1, \rho_- = 0\}$ is a radial point. Similarly, the point $\{\xi_{sc} = -1, \rho_+ = 0\}$ is a radial point. This time both radial points are linked by a characteristic curve. Therefore, we have to use a source radial point estimate at one end and a sink radial point estimate at the other end and propagate control from the source to the sink by propagation of singularities. As before we begin by the elliptic estimate to control the operator outside of the characteristic set:

Proposition C.5.1. *Let χ be a smooth compactly supported cutoff which is equal to 1 in the neighborhood of -1. Let $m, l, l' \in \mathbb{R}$. For all $N \in \mathbb{N}$, there exists a constant $C >$ (independent of h) such that for all $u \in \cup_{j\in\mathbb{N}} H^{-j,-j}$:*

$$\|u\|_{H_h^{m,l,l'}} \leq C \left(\|P_h u\|_{H_h^{m-1,l,l'}} + \|\chi(hD_x)u\|_{H_h^{m,l,l'}} + h^N \|u\|_{H_h^{m-N,l-N,l'-N}} \right)$$

as usual with the convention that some terms may be infinite.

Proof. As in the classical case, by induction, it is enough to prove it for $N = 1$. We have the relation:

$$(hD_x - 1)^{-1}(1 - \chi(hD_x))P_h = 1 - \chi(hD_x) + h(hD_x - 1)^{-1}(1 - \chi(hD_x))V(x).$$

By proposition C.4.10 we obtain that $(hD_x - 1)^{-1}(1 - \chi(hD_x))$ is h-uniformly bounded from $H_h^{m-1,l,l'}$ to $H_h^{m,l,l'}$ and $(hD_x - 1)^{-1}(1 - \chi(hD_x))V(x)$ is h-uniformly bounded from $H_h^{m-1,l-1,l'-1}$ to $H_h^{m,l,l'}$. □

We state the semiclassical version of the radial point estimates. We begin by the estimate near the radial point at $x = -\infty$:

Proposition C.5.2. *Let χ_1 be a smooth monotonic compactly supported cutoff equal to 1 in a neighborhood of $-\infty$ and equal to zero on a neighborhood of 0. Let χ_2 be a smooth compactly supported cutoff equal to 1 in a neighborhood of -1. Let χ_3 be a non negative smooth compactly supported cutoff such that $\chi_3 = 1$ on $supp(\chi_1')$. Let $\tilde{\chi}_1$ be defined as χ_1 and equal to 1 on $supp(\chi_1)$. Let $N \in \mathbb{N}$. Let $\alpha_1 > \Im(v_{-\infty})$, $\max(\Im(v_{-\infty}), \alpha_1 - \frac{1}{2}) < \alpha_1' < \alpha_1$ and $\alpha_2 > -\Im(v_{-\infty})$. We assume that the support of χ_1 is sufficiently close to $-\infty$ so that $\alpha_1' > \Im(V(x)\,|x|)$ and $\alpha_2 > -\Im(V(x)\,|x|)$ on $supp(\chi_1)$. There exists a constant $C > 0$ (independent of h) such that for all $u \in \cup_{j\in\mathbb{N}} H^{-j,-j}$ such that $\tilde{\chi}_1(x)u \in H_h^{m-1,-\frac{1}{2}+\alpha_1'}$ and all*

$v \in \cup_{j\in\mathbb{N}} H_h^{-j,-j}$ *such that* $\tilde{\chi}_1(x)v \in H_h^{m-1,-1-\alpha_2}$*:*

$$\|\chi_1(x)\chi_2(hD_x)u\|_{H_h^{m,-\frac{1}{2}+\alpha_1}} \leq C\left(h^{-1}\|\chi_1(x)P_hu\|_{H_h^{m-1,\frac{1}{2}+\alpha_1}} + h^{\frac{1}{2}}\|u\|_{H_h^{-N,-\frac{1}{2}+\alpha_1',-N}}\right) \quad \text{(C.19)}$$

$$\begin{aligned}\|\chi_1(x)\chi_2(hD_x)v\|_{H_h^{m,-\frac{1}{2}-\alpha_2}} \leq C\Big(&h^{-1}\|\chi_1(x)P_hv\|_{H_h^{m-1,\frac{1}{2}-\alpha_2}} + \|\chi_3(x)\chi_2(hD_x)v\|_{H_h^{m,-\frac{1}{2}-\alpha_2}} \\ &+h^{\frac{1}{2}}\|v\|_{H_h^{-N,-1-\alpha_2,-N}}\Big)\end{aligned} \quad \text{(C.20)}$$

in the strong sense that if the right-hand side is finite, so is the left-hand side and the inequality holds.

Proof. The proof is very similar to the proof of proposition C.2.3 and we therefore provide less details.

Source estimate (C.19)**:** As in the proof of proposition C.2.3, we can assume without loss of generality that $u \in \cap_{j\in\mathbb{N}} H^{m-1,-\frac{1}{2}+\alpha_1',j}$. We perform a commutator argument with

$$\begin{aligned}A =&\chi_2(hD_x)\chi_1(x)^2\,|x|^{2\alpha_1}\,\chi_2(hD_x)\\ A_\epsilon =&\chi_2(hD_x)\chi_1(x)^2\,|x|^{2\alpha_1}\,\langle\epsilon x\rangle^{2(\alpha_1'-\alpha_1)}\,\chi_2(hD_x).\end{aligned}$$

Exactly as in the proof of proposition C.2.3, we obtain:

$$\langle A_\epsilon u, P_hu\rangle - \langle P_hu, A_\epsilon u\rangle = \langle (P_hA_\epsilon - A_\epsilon P_h + (P_h^* - P_h)A_\epsilon)u, u\rangle \quad \text{(C.21)}$$

The commutator computation is also very similary but we have an extra factor h.

$$\begin{aligned}\frac{P_hA_\epsilon - A_\epsilon P_h + (P_h^* - P_h)A_\epsilon}{-i} =&h\chi_2(hD_x)\chi_1(x)\,|x|^{2\alpha_1-1}\,\langle\epsilon x\rangle^{2\alpha_1'-2\alpha_1}\,\Big(2\chi_1'(x)\,|x|\\ &-2\chi_1(x)\left(\alpha_1 - (\alpha_1-\alpha_1')\frac{(\epsilon x)^2}{1+(\epsilon x)^2} - \Im(V(x)\,|x|)\right)\Big)\,\chi_2(hD_x)\\ &+ ih[V(x) - 2i\Im(V(x)), \chi_2(hD_x)]\chi_1(x)^2\,|x|^{2\alpha_1}\,\langle\epsilon x\rangle^{2\alpha_1'-2\alpha_1}\,\chi_2(hD_x)\\ &+ ih\chi_2(hD_x)\chi_1(x)^2\,|x|^{2\alpha_1}\,\langle\epsilon x\rangle^{2\alpha_1'-2\alpha_1}\,[\chi_2(hD_x), V(x)]\end{aligned}$$

We denote by $\eta := \inf_{x\in\mathrm{supp}\chi_1} \alpha_1' - \frac{\Im(V(x))}{|x|} > 0$ We deduce that there exists $C > 0$ independent of h and ϵ such that:

$$\begin{aligned}\langle -i(P_hA_\epsilon - A_\epsilon P_h + (P_h^* - P_h)A_\epsilon)u, u\rangle \geq&2h\eta\left\||x|^{\alpha_1-\frac{1}{2}}\,\langle\epsilon x\rangle^{\alpha_1'-\alpha_1}\,\chi_1(x)\chi_2(hD_x)u\right\|_{L^2}^2\\ &- Ch^2\|u\|^2_{H_h^{-N,-1+\alpha_1,-N}}.\end{aligned}$$

On the other hand we have:

$$\begin{aligned}|\langle P_hu, A_\epsilon u\rangle| \leq&\frac{1}{2h\eta}\left\|\langle\epsilon x\rangle^{\alpha_1'-\alpha_1}\,|x|^{\alpha_1+\frac{1}{2}}\,\chi_1(x)\chi_2(hD_x)Pu\right\|_{L^2}^2\\ &+\frac{h\eta}{2}\left\||x|^{\alpha_1-\frac{1}{2}}\,\langle\epsilon x\rangle^{\alpha_1'-\alpha_1}\,\chi_1(x)\chi_2(hD_x)u\right\|_{L^2}^2.\end{aligned}$$

We deduce (after a weak convergence argument identical to the one in proof of proposition C.2.3):

$$\left\||x|^{\alpha_1-\frac{1}{2}}\,\chi_1(x)\chi_2(hD_x)u\right\|_{L^2}^2 \leq \frac{1}{h^2\eta^2}\left\||x|^{\alpha_1+\frac{1}{2}}\,\chi_1(x)\chi_2(hD_x)Pu\right\|_{L^2}^2 + \frac{C}{\eta}h\|u\|^2_{H_h^{-N,-1+\alpha_1,-N}}.$$

Exactly as in the proof of proposition C.2.3 but using propositions C.4.8 and C.4.10 instead of propositions C.1.12 and C.1.9, we obtain (uniformly with respect to h):

$$\|\chi_1(x)\chi_2(hD_x)u\|_{H_h^{m,-\frac{1}{2}+\alpha_1}} \lesssim h^{-1}\|\chi_1(x)P_hu\|_{H_h^{m-1,\frac{1}{2}+\alpha_1}} + h^{\frac{1}{2}}\|u\|_{H_h^{-N,-\frac{1}{2}+\alpha_1',-N}}$$

Sink estimate (C.20)**:** As in the proof of proposition C.2.3, we can assume without loss of generality that $v \in H^{m-1,-1-\alpha_2}$. We make a commutator estimate with

$$A_\epsilon := \chi_2(hD_x)\chi_1(x)^2 |x|^{-2\alpha_2} \langle \epsilon x\rangle^{-2} \chi_2(hD_x).$$

We have:

$$\langle A_\epsilon u, P_h u\rangle - \langle P_h u, A_\epsilon u\rangle = \langle (P_h A_\epsilon - A_\epsilon P_h + (P_h^* - P_h)A_\epsilon)u, u\rangle . \tag{C.22}$$

The commutator is:

$$\begin{aligned} i(P_hA_\epsilon - A_\epsilon P_h + (P_h^* - P_h)A_\epsilon) =& h\chi_2(hD_x)\chi_1(x)|x|^{-2\alpha_2-1}\langle\epsilon x\rangle^{-2}\left(2\chi_1'(x)|x|\right.\\ &\left.+2\chi_1(x)\left(\alpha_2 + \frac{(\epsilon x)^2}{1+(x\epsilon)^2} + \Im(V(x)|x|)\right)\right)\chi_2(hD_x)\\ &+ hi[V(x) - 2i\Im(V(x)), \chi_2(hD_x)]\chi_1(x)^2|x|^{-2\alpha_2}\langle\epsilon x\rangle^{-2}\chi_2(hD_x)\\ &+ h\chi_2(hD_x)\chi_1(x)^2|x|^{-2\alpha_2}\langle\epsilon x\rangle^{-2}[\chi_2(hD_x), V(x)] \end{aligned}$$

As in the proof of proposition C.2.3, we have terms of opposite signs in the commutator. We get (h, ϵ-uniformly):

$$\begin{aligned} \langle i(P_hA_\epsilon - A_\epsilon P_h + (P_h^* - P_h)A_\epsilon)v, v\rangle \gtrsim& h\left\||x|^{-\frac{1}{2}-\alpha_2}\langle\epsilon x\rangle^{-1}\chi_1(x)\chi_2(hD_x)v\right\|_{L^2}^2\\ &- h\left\||\chi_1'(x)x|^{\frac{1}{2}}\langle\epsilon x\rangle^{-1}\chi_1(x)\chi_2(hD_x)v\right\|_{L^2}^2 - h^2\|v\|^2_{H^{-N,-1-\alpha_2,-N}}. \end{aligned}$$

The difference with respect to proposition C.2.3 is that we cannot bound

$$\left\||\chi_1'(x)x|^{\frac{1}{2}}\langle\epsilon x\rangle^{-1}\chi_1(x)\chi_2(hD_x)v\right\|_{L^2}^2$$

using an elliptic estimate since P_h is not semiclassically elliptic on $\mathrm{supp}(\chi_1') \times \{-1\}_{\xi_{sc}}$. We bound $|\langle P_hu, A_\epsilon u\rangle|$ exactly as in the source case and we get h-uniformly (after a weak convergence argument):

$$\left\||x|^{-\frac{1}{2}-\alpha_2}\chi_1(x)\chi_2(hD_x)v\right\|_{L^2}^2 \lesssim h^{-2}\|P_hv\|^2_{H^{m-1,\frac{1}{2}-\alpha_2}} + \|\chi_3(x)\chi_2(hD_x)v\|_{L^2}^2 + h\|v\|^2_{H^{-N,-1-\alpha_2,-N}}$$

Using propositions C.4.8 and C.4.4, we get (uniformly with respect to h):

$$\begin{aligned} \|\chi_1(x)\chi_2(hD_x)v\|_{H_h^{m,-\frac{1}{2}-\alpha_2}} \leq& C\left(h^{-1}\|\chi_1(x)P_hv\|_{H_h^{m-1,\frac{1}{2}-\alpha_2}} + \|\chi_3(x)\chi_2(hD_x)v\|_{H_h^{m,-\frac{1}{2}-\alpha_2}}\right.\\ &\left.+h^{\frac{1}{2}}\|v\|_{H_h^{-N,-1-\alpha_2,-N}}\right) \end{aligned}$$

□

We have the analog proposition for the radial point near $x = +\infty$:

Proposition C.5.3. *Let χ_1 be a smooth monotonic compactly supported cutoff equal to 1 in a neighborhood of $+\infty$ and equal to zero on a neighborhood of 0. Let χ_2 be a smooth compactly supported cutoff equal to 1 in a neighborhood of -1. Let χ_3 be a non negative smooth compactly supported cutoff such that $\chi_3 = 1$ on $supp(\chi_1')$. Let $\tilde{\chi}_1$ be defined as χ_1 and equal to 1 on $supp(\chi_1)$. Let $N \in \mathbb{N}$. Let $\beta_1 > -\Im(v_{+\infty})$, $\max(-\Im(v_{+\infty}), \beta_1 - \frac{1}{2}) < \beta_1' < \beta_1$ and $\beta_2 > \Im(v_{+\infty})$. We assume that χ_1 is supported sufficiently close to $+\infty$ so that $\beta_1' > -\Im(V(x)\,|x|)$ and $\beta_2 > \Im(V(x)\,|x|)$ on $supp(\chi_1)$. There exists a constant $C > 0$ (independent of h) such that for all $u \in \cup_{j\in\mathbb{N}} H^{-j,-j}$ such that $\tilde{\chi}_1(x)u \in H_h^{m-1,-\frac{1}{2}+\beta_1'}$ and all $v \in \cup_{j\in\mathbb{N}} H_h^{-j,-j}$ such that $\tilde{\chi}_1(x)v \in H_h^{m-1,-1-\beta_2}$:*

$$\|\chi_1(x)\chi_2(hD_x)u\|_{H_h^{m,-\frac{1}{2}+\beta_1}} \leq C\left(h^{-1}\|\chi_1(x)P_hu\|_{H_h^{m-1,\frac{1}{2}+\beta_1}} + h^{\frac{1}{2}}\|u\|_{H_h^{-N,-N,-\frac{1}{2}+\beta_1'}}\right) \quad \text{(C.23)}$$

$$\begin{aligned}\|\chi_1(x)\chi_2(hD_x)v\|_{H_h^{m,-\frac{1}{2}-\beta_2}} \leq & C\left(h^{-1}\|\chi_1(x)P_hv\|_{H_h^{m-1,\frac{1}{2}-\beta_2}} + \|\chi_3(x)\chi_2(hD_x)v\|_{H_h^{m,-\frac{1}{2}-\beta_2}}\right.\\ & \left. + h^{\frac{1}{2}}\|v\|_{H_h^{-N,-N,-1-\beta_2}}\right)\end{aligned} \quad \text{(C.24)}$$

in the strong sense that if the right-hand side is finite, so is the left-hand side and the inequality holds.

The last estimate we need is a propagation estimate along the semiclassical Hamiltonian flow.

Proposition C.5.4. *Let χ_2 be a smooth compactly supported cutoff equal to 1 in a neighborhood of -1. Let χ_1, χ_3, χ_4 be non negative smooth compactly supported cutoffs such that $\chi_1 = 1$ on $supp(\chi_3) \cup supp(\chi_4)$ and for every $x \in supp(\chi_4)$ there exists $t \in (0, +\infty)$ such that $e^{-tH_p}(x,-1) \in \{(x,-1) : \chi_3(x) = 1\}$ and for all $s \in (0,t)$, $e^{sH_p}(x,-1) \subset \{(x,-1) : \chi_1(x) = 1\}$. For all $m, l \in \mathbb{R}$ and $N \in \mathbb{N}$, there exists $C > 0$ independent of h such that for all $u \in H^{-N,-N}$:*

$$\|\chi_4(x)\chi_2(hD_x)u\|_{H_h^{m,l}} \leq C\left(h^{-1}\|\chi_1(x)P_hu\|_{H_h^{m-1,l}} + \|\chi_3(x)\chi_2(hD_x)u\|_{H_h^{m,l}} + h^{\frac{1}{2}}\|u\|_{H_h^{-N,-N}}\right)$$

in the strong sense that if the right-hand side is finite, so is the left-hand side and the inequality holds.

Remark C.5.5. *We can easily adapt the proof to get the estimate if we assume that for every $x \in supp(\chi_4)$ there exists $t \in (-\infty, 0)$ (instead of $(0, +\infty)$) such that $e^{-tH_p}(x,-1) \in \{(x,-1) : \chi_3(x) = 1\}$ and for all $s \in (t,0)$, $e^{sH_p}(x,-1) \subset \{(x,-1) : \chi_1(x) = 1\}$. In other words, we can propagate the estimate forward and backward along the Hamiltonian flow of P_h.*

Proof. The proof of this proposition is based on a commutator estimate with a function exponentially decreasing along the Hamiltonian flow except on the support of χ_3. By the hypothesis (using that $e^{-tH_p}(x,-1) = (x-2t,-1)$), we have $\inf\{x \in \mathbb{R} : \chi_3(x) = 1\} < \inf \operatorname{supp}(\chi_4)$. We define a smooth compactly supported function χ_5 (see figure C.3) such that $\operatorname{supp}(\chi_5) \subset \left\{\chi_1 \geq \frac{1}{2}\right\}$, $\chi_5(x) = 1$ on $\operatorname{supp}(\chi_4)$ and $\chi_5'(x) = -b(x) + e(x)$ with $b, e \geq 0$ and $\operatorname{supp}(e) \subset \left\{\chi_3 \geq \frac{1}{2}\right\}$ (for example, we can arrange that b is supported in a arbitrarily small neighborhood of $\inf\{\chi_3 = 1\}$). For $M > 0$ a large constant to be chosen later, we define the commutator $A = \chi_2(hD_x)e^{-Mx}\chi_5(x)^2\chi_2(hD_x)$. We have:

$$\langle P_hu, Au\rangle - \langle Au, P_hu\rangle = \langle (AP_h - P_hA + (P_h - P_h^*)A)u, u\rangle\,.$$

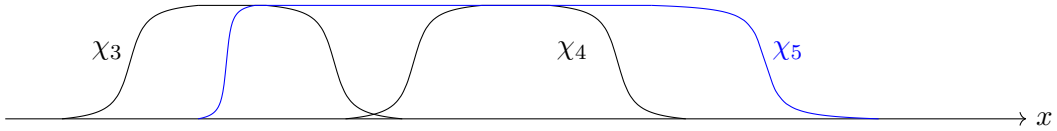

Figure C.3: Representation of the relative positions of χ_3, χ_4 and χ_5.

Moreover:

$$\begin{aligned} i(AP_h - P_hA + (P_h - P_h^*)A) =& h\chi_2(hD_x)e^{-Mx}\chi_5(x)\left((M - 2\Im(V(x))\chi_5(x)\right. \\ &\left. +2b(x) - 2e(x)\right)\chi_2(hD_x) \\ &+ h[V(x) + 2i\Im(V(x)), \chi_2(hD_x)]e^{-Mx}\chi_5(x)^2\chi_2(hD_x) \\ &+ h\chi_2(hD_x)e^{-Mx}\chi_5(x)^2[\chi_2(hD_x), V(x)] \end{aligned}$$

We deduce that for $M > 1 + 2\sup_{x\in\mathrm{supp}(\chi_5)}|V(x)|$, we have uniformly with respect to h:

$$\begin{aligned} \langle i(AP_h - P_hA)u, u\rangle \gtrsim& h\left\|e^{-\frac{Mx}{2}}\chi_5(x)\chi_2(hD_x)u\right\|_{L^2}^2 - h\left\|\chi_3(x)\chi_2(hD_x)u\right\|_{L^2}^2 \\ &- h^2\left\|u\right\|_{H_h^{-N,-N}}^2 \end{aligned}$$

On the other hand we have for $\eta > 0$:

$$|\langle P_hu, Au\rangle| \leq \frac{1}{2h\eta}\left\|e^{-\frac{Mx}{2}}\chi_1(x)\chi_2(hD_x)P_hu\right\|_{L^2}^2 + \frac{\eta h}{2}\left\|e^{-\frac{Mx}{2}}\chi_5(x)\chi_2(hD_x)u\right\|_{L^2}^2$$

We deduce that for η small enough we have (h-uniformly),

$$\begin{aligned} \|\chi_4(x)\chi_2(hD_x)u\|_{L^2}^2 \lesssim& \left\|e^{-\frac{Mx}{2}}\chi_5(x)\chi_2(hD_x)u\right\|_{L^2}^2 \\ \lesssim& h^{-2}\left\|e^{-\frac{Mx}{2}}\chi_1(x)\chi_2(hD_x)P_hu\right\|_{L^2}^2 + \|\chi_3(x)\chi_2(hD_x)u\|_{L^2}^2 + h\left\|u\right\|_{H_h^{-N,-N}}^2 \end{aligned}$$

Finally using propositions C.4.8 and C.4.4, we get the existence of a constant $C > 0$ (independent of h and u) such that:

$$\|\chi_4(x)\chi_2(hD_x)u\|_{H_h^{m,l}} \leq C\left(h^{-1}\left\|\chi_1(x)P_hu\right\|_{H_h^{m-1,l}} + \left\|\chi_3(x)\chi_2(hD_x)u\right\|_{H_h^{m,l}} + h^{\frac{1}{2}}\left\|u\right\|_{H_h^{-N,-N}}\right)$$

□

Combining all the semiclassical estimates, we get:

Proposition C.5.6. *Let $m \in \mathbb{R}$. Let $\alpha_1 > \Im(v_{-\infty})$,$\alpha_2 > -\Im(v_{-\infty})$, $\max(-\frac{1}{2}+\alpha_1, \Im(v_{-\infty})) < \alpha_1' < \alpha_1$, $\beta_1 > -\Im(v_{+\infty})$, $\beta_2 > \Im(v_{+\infty})$ and $\max(-\frac{1}{2}+\beta_1, -\Im(v_{+\infty})) < \beta_1' < \beta_1$. There exists $h_0 \in (0,1)$ and $C > 0$ independent of $h < h_0$ such that for all $u \in H_h^{m-1,-\frac{1}{2}+\alpha_1',-1-\beta_2}$ such that $P_hu \in H_h^{m-1,\frac{1}{2}+\alpha_1,\frac{1}{2}-\beta_2}$ and all $v \in H_h^{m-1,-1-\alpha_2,-\frac{1}{2}+\beta_1'}$ such that $P_hv \in H_h^{m-1,\frac{1}{2}-\alpha_2,\frac{1}{2}+\beta_1}$.*

$$\begin{aligned} \|u\|_{H_h^{m,-\frac{1}{2}+\alpha_1,-\frac{1}{2}-\beta_2}} \leq& Ch^{-1}\|P_hu\|_{H_h^{m-1,\frac{1}{2}+\alpha_1,\frac{1}{2}-\beta_2}} \\ \|v\|_{H_h^{m,-\frac{1}{2}-\alpha_2,-\frac{1}{2}+\beta_1}} \leq& Ch^{-1}\|P_hv\|_{H_h^{m-1,\frac{1}{2}-\alpha_2,\frac{1}{2}+\beta_1}} \end{aligned}$$

Proof. The first estimate is obtained by a source estimate near $x = -\infty$ and a sink estimate near $x = +\infty$. The second one is obtained similarly but with a source estimate near $x = +\infty$ and a sink estimate near $x = -\infty$. We prove in details the first one. Let $u \in H_h^{m-1,-\frac{1}{2}+\alpha_1',-1-\alpha_2}$ such that $P_h u \in H_h^{m-1,\frac{1}{2}+\alpha_1,\frac{1}{2}-\alpha_2}$. Let χ_0 be a smooth monotonic compactly supported cutoff equal to 1 in a neighborhood of $-\infty$ and supported on a small neighborhood of $-\infty$. Let χ_1 be a smooth monotonic compactly supported cutoff equal to 1 in a neighborhood of $+\infty$ and supported in a small neighborhood of $+\infty$. Let χ_2 be a smooth compactly supported cutoff equal to 1 in a neighborhood of -1. Let χ_3 be a non negative smooth compactly supported cutoff such that $\chi_3 = 1$ on $\mathrm{supp}(\chi_1')$. By propositions C.5.2 and C.5.3, we obtain $\chi_2(hD_x)u \in H^{m,-\frac{1}{2}+\alpha_1,-\frac{1}{2}-\alpha_2}$ and (uniformly with respect to h):

$$\begin{aligned}
\|\chi_0(x)\chi_2(hD_x)u\|_{H_h^{m,-\frac{1}{2}+\alpha_1}} \lesssim & h^{-1}\|\chi_0(x)P_h u\|_{H_h^{m-1,\frac{1}{2}+\alpha_1}} + h^{\frac{1}{2}}\|u\|_{H_h^{m-1,-\frac{1}{2}+\alpha_1',-1-\beta_2}} \\
\|\chi_1(x)\chi_2(hD_x)u\|_{H_h^{m,-\frac{1}{2}-\beta_2}} \lesssim & h^{-1}\|\chi_1(x)P_h u\|_{H_h^{m-1,\frac{1}{2}-\beta_2}} + \|\chi_3(x)\chi_2(hD_x)u\|_{H_h^{m,-\frac{1}{2}-\alpha_2}} \\
& + h^{\frac{1}{2}}\|u\|_{H_h^{m-1,-\frac{1}{2}-\alpha_1',-1-\beta_2}}
\end{aligned}$$

Let χ_5 be a smooth non negative compactly supported cutoff such that $\chi_5 = 1$ on $\mathrm{supp}(\chi_3) \cup \mathrm{supp}(1-\chi_0-\chi_1)$ and χ_6 be a smooth compactly supported cutoff equal to 1 on some open interval and such that $\mathrm{supp}\chi_6 \subset \{\chi_0 = 1\}$ and $\sup \mathrm{supp}(\chi_6) < \inf \mathrm{supp}(\chi_5)$. Let $\tilde{\chi}_5$ be a smooth compactly supported cutoff such that $\tilde{\chi}_5 = 1$ on $[\inf \mathrm{supp}\chi_6, \sup \mathrm{supp}\chi_5]$. Note that for all $x \in \mathrm{supp}(\chi_5)$, there exists $t \in (0,+\infty)$ such that $e^{-tH_p}(x,-1) \in \{(x,-1) : \chi_6(x) = 1\}$ and for all $s \in [0,t]$, $e^{-sH_p}(x,-1) \in \{(x,-1) : \tilde{\chi}_5(x) = 1\}$. We can therefore apply proposition C.5.4 and we obtain (h-uniformly):

$$\begin{aligned}
\|\chi_5(x)\chi_2(hD_x)u\|_{H_h^{m,-\frac{1}{2}+\alpha_1}} &\lesssim \|\tilde{\chi}_5(x)P_h u\|_{H_h^{m-1,\frac{1}{2}+\alpha_1}} + \|\chi_6(x)\chi_2(hD_x)u\|_{H_h^{m,-\frac{1}{2}+\alpha_1}} \\
&\quad + h^{\frac{1}{2}}\|u\|_{H_h^{m-1,-\frac{1}{2}-\alpha_1',-1-\alpha_2}} \\
&\lesssim \|P_h u\|_{H_h^{m-1,\frac{1}{2}+\alpha_1,\frac{1}{2}-\beta_2}} + \|\chi_0(x)\chi_2(hD_x)u\|_{H_h^{m,-\frac{1}{2}+\alpha_1}} \\
&\quad + h^{\frac{1}{2}}\|u\|_{H_h^{m-1,-\frac{1}{2}-\alpha_1',-1-\alpha_2}}
\end{aligned}$$

Since we have (see proposition C.4.7) h uniformly:

$$\begin{aligned}
\|(1-\chi_0(x)-\chi_1(x))\chi_2(hD_x)u\|_{H_h^{m,-\frac{1}{2}+\alpha_1}} &\lesssim \|\chi_5(x)\chi_2(hD_x)u\|_{H_h^{m,-\frac{1}{2}+\alpha_1}} \\
\|\chi_3(x)\chi_2(hD_x)u\|_{H_h^{m,-\frac{1}{2}-\alpha_2}} &\lesssim \|\chi_5(x)\chi_2(hD_x)u\|_{H_h^{m,-\frac{1}{2}+\alpha_1}}
\end{aligned}$$

we deduce:

$$\|\chi_2(hD_x)u\|_{H_h^{m,-\frac{1}{2}+\alpha_1,-\frac{1}{2}-\beta_2}} \lesssim h^{-1}\|P_h u\|_{H^{m-1,\frac{1}{2}+\alpha_1,\frac{1}{2}-\beta_2}} + h^{\frac{1}{2}}\|u\|_{H_h^{m-1,-\frac{1}{2}+\alpha_1',-1-\beta_2}}$$

Therefore, we can use the elliptic estimate of proposition C.5.1 to get $u \in H^{m,-\frac{1}{2}+\alpha_1,-\frac{1}{2}-\alpha_2}$ and uniformly with respect to h:

$$\|u\|_{H_h^{m,-\frac{1}{2}+\alpha_1,-\frac{1}{2}-\beta_2}} \lesssim h^{-1}\|P_h u\|_{H^{m-1,\frac{1}{2}+\alpha_1,\frac{1}{2}-\beta_2}} + h^{\frac{1}{2}}\|u\|_{H_h^{m-1,-\frac{1}{2}+\alpha_1',-1-\beta_2}}$$

For h small enough, we can absorb the term $h\|u\|_{H_h^{m-1,-\frac{1}{2}+\alpha_1',-1-\beta_2}}$ in the left-hand side. We conclude that for h small enough we have (h-uniformly):

$$\|u\|_{H_h^{m,-\frac{1}{2}+\alpha_1,-\frac{1}{2}-\beta_2}} \lesssim h^{-1}\|P_h u\|_{H_h^{m-1,\frac{1}{2}+\alpha_1,\frac{1}{2}-\beta_2}}.$$

□

The main outcome from proposition C.5.6 is that for all $\alpha_1, \alpha_2, \beta_1$ and β_2 as in the proposition, the following operators are invertible[5] if h is small enough:

- $P_h : \mathfrak{X}^{m,-\frac{1}{2}+\alpha_1,-\frac{1}{2}-\beta_2}(H^{m-1,\frac{1}{2}+\alpha_1,\frac{1}{2}-\beta_2}) \to H^{m-1,\frac{1}{2}+\alpha_1,\frac{1}{2}-\beta_2}$
- $P_h : \mathfrak{X}^{m,-\frac{1}{2}-\alpha_2,-\frac{1}{2}+\beta_1}(H^{m-1,\frac{1}{2}-\alpha_2,\frac{1}{2}+\beta_1}) \to H^{m-1,\frac{1}{2}-\alpha_2,\frac{1}{2}+\beta_1}$

Taking $h = \frac{1}{z}$ (for $z > 1$) we deduce that $D_x + V(x) + z$ is invertible between these spaces for z large enough. We can also deduce a (crude) bound for the norm of the inverse uniform with respect to $z > z_0$ using proposition C.4.3. For example, if $m = 1$, we get that there exists $z_0 > 0$ and a constant $C > 0$ such that for all $z > z_0$ we have:

$$\left\|(D_x + V(x) + z)^{-1}\right\|_{\mathcal{L}\left(H^{0,\frac{1}{2}+\alpha_1,\frac{1}{2}-\beta_2},H^{1,-\frac{1}{2}+\alpha_1,-\frac{1}{2}-\beta_2}\right)} \leq Cz$$
$$\left\|(D_x + V(x) + z)^{-1}\right\|_{\mathcal{L}\left(H^{0,\frac{1}{2}+\alpha_1,\frac{1}{2}-\beta_2},H^{0,-\frac{1}{2}+\alpha_1,-\frac{1}{2}-\beta_2}\right)} \leq C$$

[5]Semiclassical estimates provide $Ker(P_h) = Ker(P_h^*) = \{0\}$ and the closed range property of P_h.

Bibliography

[1] Steffen Aksteiner and Lars Andersson. Linearized gravity and gauge conditions. *Classical and Quantum Gravity*, 28(6):065001, 2011.

[2] Steffen Aksteiner, Lars Andersson, and Thomas Bäckdahl. New identities for linearized gravity on the Kerr spacetime. *Physical Review D*, 99(4):044043, 2019.

[3] Lars Andersson, Thomas Bäckdahl, and Pieter Blue. Geometry of black hole spacetimes. In Thierry Daudé, Dietrich Häfner, and Jean-Philippe Nicolas, editors, *Asymptotic analysis in general relativity*, chapter 2, pages 9–79. Cambridge University Press, 2018.

[4] Lars Andersson, Thomas Bäckdahl, Pieter Blue, and Siyuan Ma. Stability for linearized gravity on the kerr spacetime. *arXiv:1903.03859*, 2019.

[5] Lars Andersson, Pieter Blue, and Jinhua Wang. Morawetz estimate for linearized gravity in Schwarzschild. *Ann. Henri Poincaré*, 21(3):761–813, 2020.

[6] Lars Andersson, Thomas Bäckdahl, and Pieter Blue. Spin geometry and conservation laws in the Kerr spacetime. *Surveys in Differential Geometry*, 20(1):183–226, 2015.

[7] Lars Andersson, Dietrich Häfner, and Bernard F Whiting. Mode analysis for the linearized Einstein equations on the Kerr metric: the large $\mathfrak{a}$ case. *to appear in J. Eur. Math. Soc.*, 2022.

[8] Lars Andersson, Siyuan Ma, Claudio Paganini, and Bernard F. Whiting. Mode stability on the real axis. *J. Math. Phys.*, 58(7):072501, 19, 2017.

[9] Yannis Angelopoulos, Stefanos Aretakis, and Dejan Gajic. Late-time asymptotics for the wave equation on spherically symmetric, stationary spacetimes. *Adv. Math.*, 323:529–621, 2018.

[10] Yannis Angelopoulos, Stefanos Aretakis, and Dejan Gajic. A vector field approach to almost-sharp decay for the wave equation on spherically symmetric, stationary spacetimes. *Ann. PDE*, 4(2):120, 2018. Id/No 15.

[11] Yannis Angelopoulos, Stefanos Aretakis, and Dejan Gajic. Late-time tails and mode coupling of linear waves on Kerr spacetimes. *Advances in Mathematics*, 417:108939, 2023.

[12] Bernardo Araneda. Conformal invariance, complex structures and the Teukolsky connection. *Classical and Quantum Gravity*, 35(17):175001, 2018.

[13] Thomas Bäckdahl and Juan A. Valiente Kroon. A formalism for the calculus of variations with spinors. *Journal of Mathematical Physics*, 57(2):022502, 18, 2016.

[14] Jean-François Bony and Dietrich Häfner. Low frequency resolvent estimates for long range perturbations of the Euclidean Laplacian. *Math. Res. Lett.*, 17(2):301–306, 2010.

[15] Jean-François Bony and Dietrich Häfner. The semilinear wave equation on asymptotically Euclidean manifolds. *Commun. Partial Differ. Equations*, 35(1):23–67, 2010.

[16] Jean-François Bony and Dietrich Häfner. Local energy decay for several evolution equations on asymptotically Euclidean manifolds. *Ann. Sci. Éc. Norm. Supér. (4)*, 45(2):311–335, 2012.

[17] Marc Casals and Rita Teixeira da Costa. Hidden spectral symmetries and mode stability of subextremal Kerr (-ds) black holes. *arXiv:2105.13329*, 2021.

[18] Subrahmanyan Chandrasekhar. *The mathematical theory of black holes*. Oxford University Press, 1998.

[19] Demetrios Christodoulou and Sergiu Klainerman. *The global nonlinear stability of the Minkowski space*, volume 41 of *Princeton Math. Ser.* Princeton, NJ: Princeton University Press, 1993.

[20] WD Curtis and David E Lerner. Complex line bundles in relativity. *Journal of Mathematical Physics*, 19(4):874–877, 1978.

[21] Mihalis Dafermos, Gustav Holzegel, and Igor Rodnianski. Boundedness and decay for the Teukolsky equation on Kerr spacetimes. I: The case $|a| \ll M$. *Ann. PDE*, 5(1):118, 2019. Id/No 2.

[22] Mihalis Dafermos, Gustav Holzegel, and Igor Rodnianski. The linear stability of the Schwarzschild solution to gravitational perturbations. *Acta Math.*, 222(1):1–214, 2019.

[23] Mihalis Dafermos, Gustav Holzegel, Igor Rodnianski, and Martin Taylor. The non-linear stability of the schwarzschild family of black holes. *arXiv:2104.08222*, 2021.

[24] Mihalis Dafermos, Igor Rodnianski, and Yakov Shlapentokh-Rothman. Decay for solutions of the wave equation on Kerr exterior spacetimes. III: The full subextremal case $|a| < M$. *Ann. Math. (2)*, 183(3):787–913, 2016.

[25] Roland Donninger, Wilhelm Schlag, and Avy Soffer. A proof of Price's Law on Schwarzschild black hole manifolds for all angular momenta. *Adv. Math.*, 226(1):484–540, 2011.

[26] Roland Donninger, Wilhelm Schlag, and Avy Soffer. On pointwise decay of linear waves on a Schwarzschild black hole background. *Commun. Math. Phys.*, 309(1):51–86, 2012.

[27] Semyon Dyatlov. Asymptotics of linear waves and resonances with applications to black holes. *Commun. Math. Phys.*, 335(3):1445–1485, 2015.

[28] Semyon Dyatlov. Spectral gaps for normally hyperbolic trapping. *Ann. Inst. Fourier*, 66(1):55–82, 2016.

[29] Semyon Dyatlov and Maciej Zworski. *Mathematical theory of scattering resonances*, volume 200. American Mathematical Soc., 2019.

[30] Michael Eastwood and K. Paul Tod. Edth - a differential operator on the sphere. *Math. Proc. Camb. Philos. Soc.*, 92:317–330, 1982.

[31] Felix Finster and J. Smoller. A spectral representation for spin-weighted spheroidal wave operators with complex aspherical parameter. *Methods and Applications of Analysis*, 23:35–118, 03 2016.

[32] Helmut Friedrich. On the existence of n-geodesically complete or future complete solutions of Einstein's field equations with smooth asymptotic structure. *Commun. Math. Phys.*, 107:587–609, 1986.

[33] Robert Geroch. Spinor structure of space-times in general relativity. ii. *Journal of Mathematical Physics*, 11(1):343–348, 1970.

[34] Robert Geroch, Alan Held, and Roger Penrose. A space-time calculus based on pairs of null directions. *Journal of Mathematical Physics*, 14(7):874–881, 1973.

[35] Elena Giorgi. The linear stability of Reissner-Nordström spacetime: the full subextremal range $|Q| < M$. *Commun. Math. Phys.*, 380(3):1313–1360, 2020.

[36] Elena Giorgi, Sergiu Klainerman, and Jérémie Szeftel. A general formalism for the stability of Kerr. *arXiv:2002.02740*, 2020.

[37] Reinaldo J. Gleiser, Richard H. Price, and Jorge Pullin. Late-time tails in the Kerr spacetime. *Classical Quantum Gravity*, 25(7):6, 2008. Id/No 072001.

[38] Joshua N Goldberg, Alan J MacFarlane, Ezra T Newman, Fritz Rohrlich, and EC George Sudarshan. Spin-s spherical harmonics and ð. *Journal of Mathematical Physics*, 8(11):2155–2161, 1967.

[39] Colin Guillarmou and Andrew Hassell. Resolvent at low energy and Riesz transform for Schrödinger operators on asymptotically conic manifolds. I. *Math. Ann.*, 341(4):859–896, 2008.

[40] Colin Guillarmou and Andrew Hassell. Resolvent at low energy and Riesz transform for Schrödinger operators on asymptotically conic manifolds. II. *Ann. Inst. Fourier*, 59(4):1553–1610, 2009.

[41] Colin Guillarmou, Andrew Hassell, and Adam Sikora. Resolvent at low energy. III: The spectral measure. *Trans. Am. Math. Soc.*, 365(11):6103–6148, 2013.

[42] Dietrich Häfner, Peter Hintz, and András Vasy. Linear stability of slowly rotating Kerr black holes. *Invent. Math.*, 223(3):1227–1406, 2021.

[43] G Harnett. The ghp connection: a metric connection with torsion determined by a pair of null directions. *Classical and Quantum Gravity*, 7(10):1681, 1990.

[44] Allen Hatcher. Vector bundles and K-theory. *http://www. math. cornell. edu/˜ hatcher*, 2003.

[45] Lili He. The linear stability of weakly charged and slowly rotating Kerr-Newman family of charged black holes. *arXiv:2301.08557*, 2023.

[46] Einar Hille and Ralph Saul Phillips. *Functional analysis and semi-groups*, volume 31. American Mathematical Soc., 1996.

[47] Peter Hintz. Non-linear stability of the Kerr-Newman-de Sitter family of charged black holes. *Ann. PDE*, 4(1):131, 2018. Id/No 11.

[48] Peter Hintz. A sharp version of Price's law for wave decay on asymptotically flat spacetimes. *Commun. Math. Phys.*, 389(1):491–542, 2022.

[49] Peter Hintz and András Vasy. Global analysis of quasilinear wave equations on asymptotically kerr-de sitter spaces. *International Mathematics Research Notices*, 2016(17):5355–5426, 2016.

[50] Peter Hintz and András Vasy. The global non-linear stability of the Kerr-de Sitter family of black holes. *Acta Math.*, 220(1):1–206, 2018.

[51] Peter Hintz and András Vasy. Stability of Minkowski space and polyhomogeneity of the metric. *Ann. PDE*, 6(1):146, 2020. Id/No 2.

[52] Shahar Hod. Mode-coupling in rotating gravitational collapse of a scalar field. *Physical Review D*, 61(2):024033, 1999.

[53] Lars Hörmander. *The analysis of linear partial differential operators I: Distribution theory and Fourier analysis*. Springer, 2007.

[54] Lars Hörmander. *The analysis of linear partial differential operators III: Pseudo-differential operators*. Springer Science & Business Media, 2007.

[55] Pei-Ken Hung, Jordan Keller, and Mu-Tao Wang. Linear stability of Schwarzschild spacetime: decay of metric coefficients. *J. Differ. Geom.*, 116(3):481–541, 2020.

[56] Arne Jensen and Tosio Kato. Spectral properties of Schrödinger operators and time-decay of the wave functions. *Duke Math. J.*, 46:583–611, 1979.

[57] Thomas William Johnson. The linear stability of the Schwarzschild solution to gravitational perturbations in the generalised wave gauge. *Ann. PDE*, 5(2):92, 2019. Id/No 13.

[58] Bernard S. Kay and Robert M. Wald. Linear stability of Schwarzschild under perturbations which are non- vanishing on the bifurcation 2-sphere. *Classical Quantum Gravity*, 4:893–898, 1987.

[59] Roy P Kerr. Gravitational field of a spinning mass as an example of algebraically special metrics. *Phys. Rev. Lett.*, 11(5):237, 1963.

[60] Sergiu Klainerman and Jérémie Szeftel. *Global nonlinear stability of Schwarzschild spacetime under polarized perturbations*, volume 210 of *Ann. Math. Stud.* Princeton, NJ: Princeton University Press, 2020.

[61] Sergiu Klainerman and Jérémie Szeftel. *Global Nonlinear Stability of Schwarzschild Spacetime under Polarized Perturbations:(AMS-210)*, volume 395. Princeton University Press, 2020.

[62] Sergiu Klainerman and Jeremie Szeftel. Kerr stability for small angular momentum. *arXiv:2104.11857*, 2021.

[63] Sergiu Klainerman and Jérémie Szeftel. Construction of GCM spheres in perturbations of Kerr. *Ann. PDE*, 8(2):153, 2022. Id/No 17.

[64] Sergiu Klainerman and Jérémie Szeftel. Effective results on uniformization and intrinsic GCM spheres in perturbations of Kerr. *Ann. PDE*, 8(2):89, 2022. Id/No 18.

[65] Shoshichi Kobayashi and Katsumi Nomizu. *Foundations of differential geometry*, volume 1. New York, London, 1963.

[66] Hans Lindblad and Igor Rodnianski. The global stability of Minkowski space-time in harmonic gauge. *Ann. Math. (2)*, 171(3):1401–1477, 2010.

[67] Shi-Zhuo Looi. Pointwise decay for the wave equation on nonstationary spacetimes. *Journal of Mathematical Analysis and Applications*, page 126939, 2022.

[68] Jonathan Luk and Sung-Jin Oh. Proof of linear instability of the Reissner-Nordström Cauchy horizon under scalar perturbations. *Duke Math. J.*, 166(3):437–493, 2017.

[69] Siyuan Ma. Uniform energy bound and Morawetz estimate for extreme components of spin fields in the exterior of a slowly rotating Kerr black hole. I: Maxwell field. *Ann. Henri Poincaré*, 21(3):815–863, 2020.

[70] Siyuan Ma. Uniform energy bound and Morawetz estimate for extreme components of spin fields in the exterior of a slowly rotating Kerr black hole. II: linearized gravity. *Commun. Math. Phys.*, 377(3):2489–2551, 2020.

[71] Siyuan Ma and Lin Zhang. Sharp decay for Teukolsky equation in Kerr spacetimes. *arXiv:2111.04489*, 2021.

[72] Siyuan Ma and Lin Zhang. Price's law for spin fields on a Schwarzschild background. *Ann. PDE*, 8(2):100, 2022. Id/No 25.

[73] Siyuan Ma and Lin Zhang. Sharp decay estimates for massless Dirac fields on a Schwarzschild background. *J. Funct. Anal.*, 282(6):112, 2022. Id/No 109375.

[74] Richard Melrose. *The Atiyah-Patodi-singer index theorem*. CRC Press, 1993.

[75] Richard B. Melrose. Spectral and scattering theory for the Laplacian on asymptotically Euclidean spaces. In *Spectral and scattering theory. Proceedings of the 30th Taniguchi international workshop, held at Sanda, Hyogo, Japan*, pages 85–130. Marcel Dekker, 1994.

[76] Jason Metcalfe, Daniel Tataru, and Mihai Tohaneanu. Price's law on nonstationary space-times. *Adv. Math.*, 230(3):995–1028, 2012.

[77] Jason Metcalfe, Daniel Tataru, and Mihai Tohaneanu. Pointwise decay for the Maxwell field on black hole space-times. *Adv. Math.*, 316:53–93, 2017.

[78] Aaron Meurer, Christopher P. Smith, Mateusz Paprocki, Ondřej Čertík, Sergey B. Kirpichev, Matthew Rocklin, AMiT Kumar, Sergiu Ivanov, Jason K. Moore, Sartaj Singh, Thilina Rathnayake, Sean Vig, Brian E. Granger, Richard P. Muller, Francesco Bonazzi, Harsh Gupta, Shivam Vats, Fredrik Johansson, Fabian Pedregosa, Matthew J. Curry, Andy R. Terrel, Štěpán Roučka, Ashutosh Saboo, Isuru Fernando, Sumith Kulal, Robert Cimrman, and Anthony Scopatz. Sympy: symbolic computing in Python. *PeerJ Computer Science*, 3:e103, January 2017.

[79] Pascal Millet. Geometric background for the Teukolsky equation revisited. *arXiv:2111.03347*, 2021.

[80] Pascal Millet. Optimal decay for solutions of the Teukolsky equation on the Kerr metric for the full subextremal range $|a| < m$. *arXiv:2302.06946*, 2023.

[81] Katrina Morgan. The effect of metric behavior at spatial infinity on pointwise wave decay in the asymptotically flat stationary setting. *arXiv:2006.11324*, 2020.

[82] Katrina Morgan and Jared Wunsch. Generalized price's law on fractional-order asymptotically flat stationary spacetimes. *arXiv:2105.02305*, 2021.

[83] Ezra Newman and Roger Penrose. An approach to gravitational radiation by a method of spin coefficients. *Journal of Mathematical Physics*, 3(3):566–578, 1962.

[84] Ezra Newman and Roger Penrose. Errata: an approach to gravitational radiation by a method of spin coefficients. *Journal of Mathematical Physics*, 4(7):998–998, 1963.

[85] Frank Olver. *Asymptotics and special functions*. CRC Press, 1997.

[86] Barrett O'Neill. *The geometry of Kerr black holes*. Courier Corporation, 2014.

[87] Barret O'Neil. *Semi-Riemannian Geometry*. Academic Press, New York, 1983.

[88] Roger Penrose and Wolfgang Rindler. *Spinors and space-time: Volume 1, Two-spinor calculus and relativistic fields*, volume 1. Cambridge University Press, 1984.

[89] Richard H Price. Nonspherical perturbations of relativistic gravitational collapse. I. Scalar and gravitational perturbations. *Physical Review D*, 5(10):2419, 1972.

[90] Richard H Price. Nonspherical perturbations of relativistic gravitational collapse. II. Integer-spin, zero-rest-mass fields. *Physical Review D*, 5(10):2439, 1972.

[91] Richard H Price and Lior M Burko. Late time tails from momentarily stationary, compact initial data in Schwarzschild spacetimes. *Physical Review D*, 70(8):084039, 2004.

[92] Hans Ringström. *The Cauchy problem in general relativity*, volume 6. European Mathematical Society, 2009.

[93] Karl Schwarzschild. Über das gravitationsfeld einer kugel aus inkompressibler flüssigkeit nach der einsteinschen theorie. *Sitzungsberichte der königlich preußischen Akademie der Wissenschaften zu Berlin*, pages 424–434, 1916.

[94] Dawei Shen. Construction of GCM hypersurfaces in perturbations of Kerr. *arXiv:2205.12336*, 2022.

[95] Yakov Shlapentokh-Rothman and Rita Teixeira da Costa. Boundedness and decay for the Teukolsky equation on Kerr in the full subextremal range $|a| < m$: physical space analysis. *arXiv:2302.08916*, 2023.

[96] Yakov Shlapentokh-Rothman and Rita Teixeira da Costa. Boundedness and decay for the Teukolsky equation on Kerr in the full subextremal range $|a| < m$: frequency space analysis. *arXiv:2007.07211*, 2020.

[97] Daniel Tataru. Local decay of waves on asymptotically flat stationary space-times. *Am. J. Math.*, 135(2):361–401, 2013.

[98] Michael Taylor. *Partial differential equations I: Basic theory*, volume 115. Springer Science & Business Media, 2011.

[99] Michael Taylor. *Partial differential equations II: Qualitative studies of linear equations*, volume 116. Springer Science & Business Media, 2013.

[100] Rita Teixeira da Costa. Mode stability for the Teukolsky equation on extremal and subextremal Kerr spacetimes. *Communications in Mathematical Physics*, 378(1):705–781, 2020.

[101] Saul A Teukolsky. Perturbations of a rotating black hole. I. Fundamental equations for gravitational, electromagnetic, and neutrino-field perturbations. *The Astrophysical Journal*, 185:635–647, 1973.

[102] Kip S Thorne. Disk-accretion onto a black hole. ii. evolution of the hole. *Astrophys. J.*, 191:507–520, 1974.

[103] András Vasy. Microlocal analysis of asymptotically hyperbolic and Kerr-de Sitter spaces (with an appendix by Semyon Dyatlov). *Invent. Math.*, 194(2):381–513, 2013.

[104] András Vasy. A minicourse on microlocal analysis for wave propagation. In Thierry Daudé, Dietrich Häfner, and Jean-Philippe Nicolas, editors, *Asymptotic analysis in general relativity*, volume 443, pages 219–374. Cambridge University Press Cambridge, 2018.

[105] András Vasy. Limiting absorption principle on Riemannian scattering (asymptotically conic) spaces, a Lagrangian approach. *Commun. Partial Differ. Equations*, 46(5):780–822, 2021.

[106] András Vasy. Resolvent near zero energy on Riemannian scattering (asymptotically conic) spaces, a Lagrangian approach. *Commun. Partial Differ. Equations*, 46(5):823–863, 2021.

[107] Robert M Wald. Note on the stability of the schwarzschild metric. *Journal of Mathematical Physics*, 20(6):1056–1058, 1979.

[108] Bernard F. Whiting. Mode stability of the Kerr black hole. *J. Math. Phys.*, 30(6):1301–1305, 1989.

[109] Jared Wunsch and Maciej Zworski. Resolvent estimates for normally hyperbolic trapped sets. *Ann. Henri Poincaré*, 12(7):1349–1385, 2011.

[110] Maciej Zworski. *Semiclassical analysis*, volume 138. American Mathematical Society, 2012.

[111] Maciej Zworski. Resonances for asymptotically hyperbolic manifolds: Vasy's method revisited. *J. Spectr. Theory*, 6(4):1087–1114, 2016.