Spanning Forests and phase transition

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Forêts couvrantes et transition de phase

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Résumé

L'objectif de cette thèse est d'étudier un modèle de forêts couvrantes sur un graphe muni d'un fibré vectoriel et d'une connexion unitaire. Pour les fibrés de rang 1, les configurations, combinatoires, sont des forêts couvrantes d'unicycles aléatoires tirées selon une loi déterminantale. Pour les fibrés de rang quelconque, la collection aléatoire d'arêtes est remplacée par une collection aléatoire de sous-espaces des fibres. Pour une connexion périodique unitaire, nous établissons une formule intégrale pour le noyau du processus, mettant en évidence deux phases, caractérisées par le comportement asymptotique des corrélations à grande distance.

Nous étudions par ailleurs des mesures de probabilités sur les forêts couvrantes d'unicycles sur des suites de graphes finis croissants, dépendant d'une fonction de poids sur les cycles. Nous montrons que sous certaines hypothèses sur la fonction de poids, la limite ne dépend pas des conditions au bord et peut être échantillonnée par un algorithme de marches aléatoires à boucles effacées. Sous d'autres hypothèses, nous prouvons des résultats sur la vitesse asymptotique de décroissance des corrélations avec la distance, et la taille des composantes connexes, qui s'appliquent en particulier au cas déterminantal.

Nous formulons enfin une correspondance entre les fibrés de rang 2 complexes et les fibrés de rang 1 sur le corps des quaternions, et observons, pour une connexion quaternionique unitaire et périodique, deux phases selon l'aspect commutatif ou non des quaternions, l'une correspondant au modèle usuel de forêts couvrantes uniformes et l'autre correspondant à un modèle pour lequel certains unicycles finis sont observés une infinité de fois.

Abstract

The aim of this thesis is to study a model of spanning forests on a graph endowed with a vector-bundle and a unitary connection. For rank-1 vector-bundles, the combinatorial configurations are spanning forests of random unicycles sampled according to a determinantal law. For vector-bundles of arbitrary rank, the random collection of edges is replaced by a random collection of subspaces of the fibers. For a unitary periodic connection, we establish an integral formula for the kernel of this process, revealing two phases, characterized by the asymptotic behavior of the long-distance correlations.

We also study probability measures on spanning forests of unicycles on increasing finite graph sequences, depending on a weight function on the cycles. We show that under certain assumptions on the weight function, the limit does not depend on the boundary conditions and can be sampled by a loop-erased random walk algorithm. Under other assumptions, we prove results on the asymptotic rate of decay of correlations with the distance, and the size of related components, which apply in particular to the determinantal case.

Finally, we formulate a correspondence between complex rank-2 vector-bundles and rank-1 vector-bundles on the quaternion field, and observe the existence of two phases, for a periodic unitary quaternionic connection, depending on the commutativity or non-commutativity of the quaternions weights, one corresponding to the usual model of uniform spanning forests and the other corresponding to a model for which some finite unicycles are observed infinitely many times.

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Chapter 1

Introduction

Modèles de physique statistique et forêts couvrantes aléatoires

Modèles de physique statistique et transitions de phases

La *physique statistique* est un domaine des mathématiques et de la physique théorique qui étudie le comportement à grande échelle de phénomènes aléatoires suivant des mesures de probabilités construites de façon locale. Ce domaine permet de modéliser notamment des réactions physiques au cours desquelles les interactions entre particules à échelle microscopique engendrent des phénomènes observables à échelle macroscopique. Il existe par exemple le *modèle de dimères*, ou pavage par des dominos, qui étudie la formation aléatoire de couplages parfaits qui modélise la formation de composés formés par des molécules diatomiques, les dimères.

Un autre modèle de physique statistique très étudié est le *modèle de percolation* de Fortuin-Kasteleyn (FK). Ce modèle étudie des sous-graphes aléatoires où chaque configuration possède une certaine probabilité dépendant des arêtes présentes ou absentes. Les mesures construites pour ce modèle dépendent de deux paramètres p et q. Le premier paramètre compris entre 0 et 1 favorise quand il est proche de 1 les arêtes ouvertes et le second compris strictement entre 0 et l'infini favorise quand il est supérieur à 1 les configurations avec un grand nombre de composantes connexes.

Quand le paramètre q vaut 2, le modèle est en fait équivalent au *modèle d'Ising*, un modèle de physique statistique étudiant les configurations de spins et favorisant les configurations de faible énergie, dans lesquelles les spins tendent à s'aligner. Le modèle d'Ising dépend d'un paramètre de *température* qui peut être relié au paramètre p intervenant dans le modèle de percolation FK. Quand la température est faible, le modèle est « gelé » et favorise fortement les configurations où tous les spins sont alignés tandis que lorsque la température est élevée, le modèle est plus désordonné et les configurations sont plus équiprobables. Un tel changement brutal de comportement à l'échelle macroscopique en fonction d'un paramètre régissant les interactions à échelle microscopique est appelé *transition de phase*.

Les transitions de phase apparaissent fréquemment dans l'étude de modèles de physique statistique et peuvent se caractériser de différentes manières. Dans le cas de modèles de percolation, il s'agit souvent d'une brusque *variation de la taille des composantes connexes*

en fonction du paramètre, notamment avec l'émergence d'une ou plusieurs composantes connexes infinies. Dans des modèles tels que le modèle d'Ising ou le modèle de dimères, les transitions de phase correspondent souvent à une variation brutale de la distance typique des corrélations, c'est à dire une variation de la *vitesse de décroissance des corrélations* à grande échelle, qui peut être notamment polynomiale ou exponentielle.

Les différentes phases observées peuvent également se caractériser par la prépondérance ou non des *conditions de bord* dans la construction de mesures en volume infini. En effet, les mesures en volume infini sur les modèles de physique statistique sont généralement construites en étudiant la convergence de mesures en volume fini en formant une exhaustion du graphe sur lequel le modèle est étudié. Différentes suites de mesures peuvent être construites en étudiant différentes conditions de bord sur ces exhaustions, mais lorsque les corrélations à grande distance sont suffisamment faibles, les différentes suites peuvent converger vers une unique mesure limite. Par exemple, lorsque la température dans le modèle d'Ising sur un graphe planaire est élevée, les corrélations à grande échelle sont faibles et les conditions de bord sont donc non prépondérantes. Il en résulte l'existence d'une unique *mesure de Gibbs*. Au contraire, lorsque la température est faible, les conditions au bord sont prépondérantes et entrainent l'émergence de différentes mesures en volume infini.

Une quantité caractéristique d'un modèle de physique statistique en volume infini est *l'énergie libre*. Il s'agit du taux de croissance exponentielle par unité de volume de la *fonction de partition* du modèle en volume fini, définie comme la somme des poids des configurations. Cette quantité donne de nombreuses propriétés sur un modèle et l'étude de son comportement selon les paramètres du modèle permet de comprendre également les transitions de phase du modèle.

Lorsque le paramètre q du modèle de percolation FK converge vers 0, la mesure sur les configurations favorise l'existence d'une unique composante connexe. Le modèle obtenu à la limite $q \to 0$, avec la contrainte $\frac{p}{1-p} = q^a$ avec $0 < a < 1$, est appelé *modèle de l'arbre couvrant*.

L'arbre couvrant

Définition et origines. On appelle *arbre couvrant* sur un graphe un sous-graphe connexe contenant tous les sommets et ne possèdant aucun cycle. Les arbres couvrants sont des objets bien connus de la théorie des graphes et sont très étudiés en informatique théorique pour leurs applications aux réseaux informatiques.

Le dénombrement des arbres couvrants sur un graphe remonte à Kirchhoff (1847), qui a montré des relations surprenantes entre la combinatoire des arbres couvrants et les réseaux électriques sur les graphes. Le théorème de Kirchhoff exprime le nombre d'arbres couvrants sur un graphe quelconque comme le mineur $M_{i,j}$ du *laplacien* Δ (définition 2.1) du graphe en position (i, j), avec i, j quelconques. On appelle déterminant réduit $\det{}_0$ d'un opérateur le produit de ses valeurs propres non nulles. En sommant sur tous les mineurs principaux $M_{i,i}$ de Δ pour $i \in [1, n]$, ce théorème implique que le nombre d'arbres couvrants est

$$\frac{1}{n} \det{}_0 \Delta = \frac{1}{n} \prod_i \lambda_i$$

où $\{\lambda_i\}_i$ est l'ensemble des valeurs propres non nulles de Δ.

La formule de Cayley (1889), énumérant le nombre d'arbres couvrants sur un graphe complet comme n^{n-2} où n est le nombre de sommets, en est un cas particulier.

L'étude de l'arbre couvrant aléatoire, en tant que modèle probabiliste, remonte au moins au travail de Temperley [Tem74], qui a montré que que les arbres couvrants étaient en bijection avec les pavages par des dominos. Plusieurs algorithmes ont ensuite été proposés, par Aldous et Broder ([Ald90], [Bro89]), puis par Propp et Wilson ([PW98]), pour tirer uniformément un arbre couvrant à partir de marches aléatoires simples sur le graphe. L'algorithme de Wilson [Wil96], qui permet de construire un arbre couvrant aléatoire sur un graphe fini à partir de *marches aléatoires à boucles effacées* généralise le théorème de Pemantle ([Pem91]), qui montre que l'unique chemin entre deux points dans un arbre couvrant uniforme a la même loi que la marche aléatoire à boucles effacées entre ces deux points.

Processus déterminantaux. Une conséquence du théorème de Kirchhoff est une interprétation de la probabilité de présence d'une arête lorsque le graphe est vu comme un réseau électrique. En supposant que chaque arête du graphe soit un conducteur de conductance unitaire et que deux extrémités d'une certaine arête soit connectée à une pile, la probabilité qu'une arête soit présente dans un arbre couvrant aléatoire uniforme est égale à la proportion de courant qui traverse cette arête. En considérant un graphe fini dont les arêtes sont pondérées par des poids $(c(e))_{e\in E}$ appelées conductances, une mesure de probabilité naturelle sur les arbres couvrants du graphe attribue à chaque arbre une probabilité proportionnelle au produit des conductances des arêtes:

$$\mu_c(F) = \frac{\prod_{e\in E} c(e)}{Z}$$

où $Z = \sum_F \prod_{e\in E} c(e)$ est la fonction de partition du modèle.

Le formalisme des réseaux électriques sur les graphes et l'interprétation des marches aléatoires en termes de courants électriques, principalement développés par Burton et Pemantle, leur ont permis de montrer en 1993 ([BP93]) que les arêtes d'un arbre couvrant aléatoire, distribué selon une telle mesure, forment un *processus déterminantal*, c'est à dire un processus aléatoire dont la mesure est déterminantale, au sens suivant.

Une mesure de probabilités μ sur un ensemble fini $\Omega = \{0,1\}^n$ est déterminantale s'il existe une matrice K de taille $n \times n$ appelée noyau qui vérifie la propriété suivante. Si $S \subset [1,...,n]$ alors si $x \in \Omega$ est un point aléatoire selon μ alors

$$\mathbb{P}(\forall i \in S, x_i = 1) = \det(K_{S\times S})$$

c'est-à-dire le déterminant de la matrice K où l'on a conservé uniquement les lignes et les colonnes dont les indices sont dans S.

Pour un arbre couvrant distribué selon la mesure μ_c, le noyau du processus est la matrice de transfert de courant $(Y(e_i,e_j))_{1\leq i,j\leq |E|}$ où $Y(e,e') = i^e(e')$ est le courant qui passe par e' lorsqu'on impose un courant unitaire entre les extrémités de l'arête e.

Cette propriété est l'une des propriétés fondamentales du modèle d'arbre couvrant aléatoire puisqu'elle permet des calculs explicites pour ce modèle, notamment de la fonction de partition, et constitue une des motivations principales pour son étude.

Volume infini et généralisations à d'autres modèles de forêts couvrantes

Volume infini. L'étude du modèle d'arbre couvrant, son extension en volume infini et sa généralisation à d'autres modèles s'est amplement poursuivie à la fin du 20e siècle et au début du 21e siècle. Dans [Ale95], [BLPS01], [LP16], les auteurs s'intéressent à la construction de mesures pour des modèles de forêts couvrantes sur des graphes infinis en considérant des exhaustions du graphe par des graphes finis et en étudiant la limite faible de la suite de mesures construites. Ce procédé qui est également utilisé pour la construction de mesures de Gibbs pour le modèle d'Ising fait intervenir un choix de conditions au bord dans la construction des mesures sur les graphes finis. Les auteurs présentent plusieurs choix de conditions au bord (libres, fermées et périodiques) et leurs influences sur les mesures obtenues comme mesures limites.

Forêts avec connexion. D'autres auteurs se sont ensuite intéréssés à la généralisation de ce modèle à d'autres modèles de forêts aléatoires plus généraux, notamment avec un plus grand nombre de composantes connexes, mais conservant les propriétés déterminantales et algorithmiques de l'abre couvrant, propriétés qui font l'intérêt majeur du modèle d'arbres couvrants parmi les autres modèles de physique statistique. Dans [For93], l'auteur établit des liens entre les déterminants de *laplaciens déformés par une connexion* et des sommes de poids sur des configurations d'objets combinatoires : les *forêts couvrantes d'unicycles*. Ces résultats sont de plus généralisés à d'autres types de déformation dans [KL20a]. En 2011, Kenyon a introduit dans [Ken11] un nouveau type de laplacien sur un fibré vectoriel et décrit son lien dans le cas d'un fibré de rang 1 avec les mesures déterminantales sur les forêts couvrantes d'unicycles. Pour un fibré de rang 2, il donne une formule combinatoire pour le déterminant de ce laplacien.

Graphes planaires, volume infini et diagramme de phases. Sur certains graphes *planaires*, il existe une correspondance entre les arbres couvrants et le modèle de dimères sur un graphe bipartite construit à partir du graphe d'origine. Cette correspondance, appelée *bijection de Temperley*, a motivé l'introduction d'une famille de mesures sur les forêts couvrantes enracinées sur des graphes infinis périodiques, caractérisées par leur *pente*. Cette pente correspond au changement de hauteur moyen qui caractérise les mesures de Gibbs pour le modèle de dimères, via la bijection de Temperley. L'utilisation de cette correspondance et les propriétés qui en découlent ont été étudiés dans [Sun16]. Ces mesures de Gibbs déterminées par leur pente moyenne sont également étudiées dans [Ken19] où R. Kenyon prouve qu'il s'agit de plus de mesures déterminantales et donne une expression intégrale pour leur noyau.

Des mesures sur les forêts couvrantes d'arbres enracinés peuvent également être construites en ajoutant des masses sur les sommets permettant de favoriser certains sommets comme racines des arbres de la forêt. On considère pour cela un laplacien massique qui peut être vu comme un laplacien sur un graphe modifié auquel un sommet a été ajouté et est relié à tous les autres sommets. Le modèle massique est étudié dans [Ken11] et dans [BdTR17]. Les auteurs de [BdTR17] s'intéressent notamment à un choix de masses et de conductances particulier qui rend le modèle *Z-invariant*. Cette propriété assure l'aspect local dans l'étude du modèle, en particulier des formules locales pour la fonction de Green ainsi qu'une décroissance exponentielle des corrélations. Ils obtiennent notamment une

transition de phase vers le modèle d'arbres couvrants lorsqu'un paramètre elliptique du modèle tend vers 0, caractérisée par le développement asymptotique de *l'énergie libre*.

La décroissance des corrélations ainsi que le comportement de l'énergie libre peuvent également être étudiés dans le cas des mesures à pente. Dans [KSO03] et [KO06], les auteurs s'appuient sur l'étude de la courbe spectrale du *polynôme caractéristique* du modèle de dimères pour établir des résultats sur la décroissance des corrélations et le comportement de l'énergie libre pour le modèle de dimères selon le changement de hauteur moyen des mesures. Il s'appuie sur des outils de géométrie algébrique tels que les amibes dont les propriétés sont détaillées dans [Vir02, Mik00, Yge12]. Cette technique peut être utilisée également dans le modèle de forêts couvrantes pour l'étude de la décroissance des corrélations comme on peut le voir dans [Ken19]. Les résultats peuvent également être transposés au modèle de forêts couvrantes via la bijection de Temperley comme le présente [Sun16].

Gaz arboréal. Plus récemment, les auteurs de [BCHS21], [BCH21] et de [HH23] ont étudié un modèle de forêts couvrantes non enracinées qui provient de la limite $q \to 0$ du modèle de percolation FK avec la contrainte $p = \beta q$, où le paramètre $\beta \geq 0$ est fixé. Ils étudient les limites en volume infini de ce modèle, appelé modèle du gaz arboréal, sur des réseaux et prouvent qu'il existe une valeur seuil de β à partir de laquelle, sous les mesures de Gibbs invariante par translations, il existe presque surement exactement un arbre infini.

Sujet de la thèse

Un modèle de forêts couvrantes quantiques

Le sujet de thèse, proposé par Adrien Kassel, est l'étude du modèle de **forêts couvrantes quantiques**. Cet objet d'étude est représenté dans l'illustration suivante.

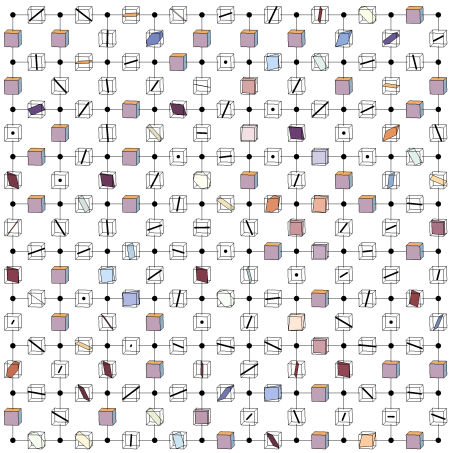

Figure 1.1 – Simulation d'une forêt couvrante quantique dans le cas de rang $N = 3$ ([KL20c])

Il s'agit d'un modèle de physique statistique intégrable avec symétrie de jauge, associé au déterminant du laplacien sur un fibré vectoriel discret au dessus d'un graphe, introduit par Adrien Kassel et Thierry Lévy dans [KL20c, KL23]. Il constitue ainsi une extension du modèle introduit dans [Ken11], en considérant des fibrés de rang quelconque.

Ce modèle est défini sur un graphe muni d'un fibré vectoriel et d'une connexion, par des opérateurs généralisant le laplacien et la fonction de Green usuelle, correspondant au cas d'un fibré de rang 1 muni d'une connexion triviale. Ce modèle généralise ainsi le modèle d'arbre couvrant usuel très étudié en particulier dans [BLPS01] et [LP16]. Le formalisme du laplacien avec une connexion, introduit en physique dans l'étude des théories de jauge sur réseaux, est détaillé dans [Ken11], qui donne une expression combinatoire du déterminant du laplacien pour des fibrés de rang 1 et 2. Les configurations combinatoires sont des forêts couvrantes d'unicycles, qui sont aléatoires selon une loi déterminantale.

Dans [KL20c], le formalisme des processus déterminantaux linéaires est introduit et fournit, pour le cas d'un graphe muni d'un fibré de rang arbitraire avec connexion unitaire dans $\mathbb{U}(N)$, une généralisation des forêts de rang 1, appelée forêt couvrante quantique. La définition de ce modèle et ses propriétés sont détaillées dans l'article en préparation [KL23]. La collection aléatoire d'arêtes de l'arbre couvrant est remplacée, dans ce modèle, par une collection aléatoire de sous-espaces des fibres, sur les arêtes du graphe. Ainsi, sur un fibré de rang N, une forêt couvrante quantique est la donnée d'un sous-espace aléatoire Q_e de $\mathbb{R}^N$ sur chaque arête e, dont la loi jointe est déterminantale. Les mesures de probabilités sur les forêts couvrantes quantiques sont déterminées par des poids matriciels sur les arêtes du graphe.

Une attention particulière est portée dans la thèse au cas des graphes périodiques avec périodicité des paramètres. Dans ce cas, on peut exhiber les phases du modèle selon l'emplacement des paramètres dans un espace qui est de dimension finie.

Les forêts couvrantes d'unicycles

Une forêt quantique peut s'interpréter de manière combinatoire en utilisant la notion de forêt couvrante d'unicycles, qui correspond au cas de rang 1. Il existe en effet N forêts couvrantes d'unicycles aléatoires dont le nombre d'occupation sur chaque arête e est la dimension $Q_e \in \{1, \dots, N\}$.

Le cas de rang 1 du modèle de forêts couvrantes quantiques correspond au modèle déterminantal de forêts couvrantes d'unicycles, associé au laplacien tordu par une connexion, introduit dans [Ken11]. Les définitions plus précises de ce modèle sont introduites dans le chapitre 2. La mesure déterminantale obtenue dans ce cas donne à chaque configuration de forêts couvrantes d'unicycles une probabilité proportionnelle au produit des poids de ses cycles où les poids dépendent de la connexion considérée.

Les forêts couvrantes aléatoires d'unicycles constituent plus généralement un modèle très riche à étudier, notamment car il est en correspondance avec des modèles de dimères, dans le cas de graphes plongés dans certaines surfaces (anneau, tore). Comme pour le modèle d'arbre couvrant qui peut être échantillonné par l'algorithme de Propp-Wilson à partir de marches aléatoires à boucles effacées, une question naturelle pour ce modèle est d'obtenir aléatoirement une forêt couvrante d'unicycles par un algorithme. Cette question a notamment été étudiée par Bouttier, Bowick, Guitter et Jeng ([BBGJ07]), qui proposent un algorithme généralisant l'algorithme de Propp-Wilson dans le cas de surface de genre

non trivial pour obtenir aléatoirement un réseau couvrant et qui utilisent cet algorithme pour l'étude du modèle de dimères sur le réseau carré à partir de la bijection de Temperley.

Dans [KK17], les auteurs s'intéressent au cas de mesures sur les forêts couvrantes d'unicycles sur un graphe quelconque, non nécessairement planaire, dépendant de poids qui ne proviennent pas nécessairement d'une connexion. Ils montrent que dans le cas où les poids sont compris entre 0 et 1, la mesure sur les forêts couvrantes d'unicyles d'un graphe fini peut être échantillonnée par un algorithme de marches aléatoires à boucles effacées où les boucles sont effacées avec probabilité égale à leurs poids. Cet algorithme dépend d'un ordre fixé sur les sommets mais la mesure ne dépend pas de l'ordre choisi. Un objectif de cette thèse est d'étudier la généralisation de cet algorithme au cas de graphes infinis et de montrer que sous certaines hypothèses de minorations des poids, la suite de mesures sur des graphes croissants converge faiblement vers une mesure en volume infini qui peut être échantillonnée par un algorithme et qui ne dépend pas de l'ordre sur les sommets choisi dans l'algorithme.

Les résultats principaux de la thèse

Les principaux résultats de cette thèse se trouvent dans les chapitres 4 à 7. Ils sont ordonnés du cadre le plus général au cadre le plus particulier (rang 1). Une fois les conditions au bord et la topologie de convergence d'une suite de mesures en volume fini vers une mesure en volume infini définies dans le chapitre 3, la première étape consiste à calculer le noyau du modèle de forêts couvrantes quantiques en rang N en volume infini avec périodicité des paramètres (chapitre 4). Ce calcul permet de mettre en évidence deux phases dans l'espace des paramètres, selon la décroissance des corrélations.

Nous nous intéressons ensuite aux forêts couvrantes d'unicycles en rang 1 et aux mesures de probabilités sur ces forêts, pondérées par les poids des cycles. Le cas particulier des poids inférieurs à 1, étudié dans le chapitre 5 constitue à la fois un outil dans l'étude des forêts couvrantes d'unicycles déterminantales et un modèle probabiliste ayant son propre intéret puisqu'il permet d'échantillonner des forêts couvrantes d'unicycles sur un graphe infini à partir d'un algorithme de marches aléatoires à boucles effacées. Nous étudions ensuite dans les chapitres 6 et 7 le comportement asymptotique des configurations distribuées selon les mesures déterminantales sur les forêts couvrantes d'unicycles associées à une connexion unitaire en rang 1, complexe ou quaternionique, selon les paramètres.

Chacun des paragraphes suivants correspond à un des chapitres 4 à 7, et est structuré de la façon suivante : nous donnons un aperçu de la problématique et des résultats existants, puis nous présentons les résultats obtenus.

Etude du noyau du modèle pour une connexion périodique

L'étude de modèles de physique statistique en volume infini sur un graphe périodique, construit comme la limite sur une exhaustion naturelle par des graphes toriques est très courante en physique statistique et permet d'exprimer le noyau du modèle en volume infini sous forme intégrale. En 1993, Burton et Pemantle ([BP93]) établissent une formule pour le noyau du modèle de forêts couvrantes uniformes comme une intégrale double d'un opérateur sur un graphe torique fini.

Pour l'étude des phases du modèle de dimères planaire, antérieure à l'étude des phases des modèles de forêts couvrantes, Kenyon, Okounkov, Sheffield ([KSO03]) considèrent un graphe doublement périodique et donnent une formule explicite pour la limite qui s'exprime comme une double intégrale faisant intervenir le *polynome caractéristique.* La position des paramètres par rapport à *l'amibe* du polynôme, déterminée par ses zéros, décrit alors les phases du modèle : liquide, gazeuse, solide. Ils donnent ensuite des formules similaires pour la famille à deux paramètres de mesures de Gibbs d'un modèle de dimères sur un graphe infini doublement périodique.

Des formules similaires sont établies pour la famille à deux paramètres de mesures de Gibbs du modèle de forêts couvrantes d'unicycles par [Ken19] et par [Sun16], appelées forêts à pentes. De telles formules sont également établies par [BdTR17] pour des forêts couvrantes d'arbres enracinées sur des graphes isoradiaux munis de masses.

Nous étudions dans le chapitre 4 des forêts couvrantes quantiques avec connexion unitaire, déterministe et périodique, sur les arêtes d'un graphe infini $\mathbb{Z}^d$-périodique. Cette connexion est alors determinée par des poids matriciels sur les arêtes du graphe et plus précisément par un choix de matrices unitaires $(M_1, ..., M_d) \in U_N(\mathbb{C})^d$ sur les demi-arêtes du graphe. Nous établissons une expression intégrale pour le noyau de la mesure déterminantale obtenue comme limite de mesures sur une exhaustion du graphe par des graphes toriques.

Théorème 1. *Le noyau de corrélation K de la mesure limite est un opérateur dont la matrice $(K_{e,e'})_{e,e'}$ s'exprime sous forme intégrale*

$$K_{e,e'} = \int_{|z_1|=1,...,|z_d|=1} \prod_j z_j^{y_j - x_j} K_{[e],[e']}(z) \frac{dz_1}{2i\pi z_1} ... \frac{dz_d}{2i\pi z_d} = \int_{z \in \mathbb{T}_d} z^{y-x} K_{[e],[e']}(z) \frac{dz}{2i\pi z}$$

où l'opérateur $K(z)$ est un endomorphisme d'un espace de dimension finie dN,

$$K(z) = \frac{1}{P(z)} d(z) t_{Com(\Delta(z))} d^*(1/z)$$

où $\Delta(z) = d^*(1/z)d(z)$, $P(z) = \det(\Delta(z))$ où les opérateurs $d(z), d^*(1/z)$ s'expriment sous forme matricielle

$$d(z) = \begin{pmatrix} I_p - z_1 M_1 \\ ... \\ I_p - z_d M_d \end{pmatrix}, \quad d^*(1/z) = \begin{pmatrix} I_p - {z_1}^{-1} M_1^{-1} & ... & I_p - {z_d}^{-1} M_d^{-1} \end{pmatrix}.$$

La preuve repose sur des décompositions périodiques des espaces de 1-formes et des techniques similaires à la transformée de Fourier discrète.

Nous étudions ensuite le comportement asymptotique des corrélations déterminées par ce noyau et mettons en évidence l'existence de deux phases, caractérisées par les décroissances exponentielles ou polynomiales des corrélations, selon l'existence ou non de zéros du polynôme caractéristique.

Nous montrons que cette dichotomie est équivalente à l'existence de vecteurs propres communs aux matrices $(M_1, ..., M_d) \in U_N(\mathbb{C})^d$ qui permettent de voir le modèle comme une superposition de sous-modèles indépendants. Dans le cas d'un fibré de rang 2, nous remarquons que ces deux phases sont caractérisées par l'aspect commutatif ou non des matrices.

Théorème 2. *On a équivalence entre les assertions suivantes :*

- *Il existe* $z = (z_1, ..., z_n) \in \mathbb{T}^d$ *tel que* $P(z) = 0$.
- *Il existe* $X \in \mathbb{C}^2$ *vecteur propre commun à toutes les matrices* M_j.

Dans le cas où $z = (z_1, ..., z_n) \in \mathbb{T}^d$ *est tel que* $P(z) = 0$*, les valeurs propres associées respectivement aux matrices* M_j *et au vecteur propre commun* X *sont* $\bar{z_j}$*. Il y a alors au plus* p *racines* $z = (z_1, ..., z_n) \in \mathbb{T}^d$ *de* P *et dans le cas* $p = 2$*, on a alors exactement deux racines* $z = (z_1, ..., z_n), \bar{z} = (\bar{z}_1, ..., \bar{z}_n) \in \mathbb{T}^d$ *et les* M_j *sont codiagonalisables et s'écrivent* $\begin{pmatrix} z_j & 0 \\ 0 & \bar{z_j} \end{pmatrix}$ *dans la base* $(X^\perp, X)$.

Mesures sur les forêts couvrantes échantillonnées par un algorithme de Wilson

La problématique naturelle d'échantillonner (de manière exacte ou parfois approchée) une mesure de probabilité par un algorithme efficace est une question essentielle dans les modèles probabilistes. Des algorithmes rapides pour obtenir un échantillon aléatoire exact d'un arbre couvrant uniforme ont été proposés par Aldous et Broder ([Ald90], [Bro89]) puis par Propp et Wilson ([PW98]). Une adaptation de l'algorithme de Propp-Wilson pour obtenir aléatoirement une forêt couvrante d'unicycles a ensuite été proposée par [BBGJ07] dans le cas d'un graphe planaire sur un anneau, puis par [KK17] pour un graphe plus général et des cycles pondérés. Une adaptation de l'algorithme de Propp-Wilson est également présentée dans [BdTR17] pour obtenir aléatoirement une forêt couvrante d'arbres enracinées sur un graphe muni de masses.

Dans le cas d'un graphe connexe fini dont les cycles sont munis de poids, une mesure de probabilité naturelle sur les forêts couvrantes d'unicycles sur ce graphe consiste à donner à une configuration une probabilité proportionnelle au produit des poids de ses cycles. Dans le cas où les poids sur les cycles orientés sont compris entre 0 et 1, les configurations aléatoires peuvent être échantillonnées par l'algorithme de marches aléatoires à boucles effacées introduit par [KK17] qui généralise l'algorithme de Wilson. Les auteurs montrent notamment que la mesure ne dépend pas de l'ordre choisi sur les sommets dans l'algorithme.

Dans le chapitre 5, nous généralisons, sous certaines hypothèses, la construction de cet algorithme aux graphes hypercubiques $\mathbb{Z}^d$, dont les arêtes correspondent aux couples de points à distance 1, dont les cycles sont munis de poids $(p(\gamma))_\gamma$ compris entre 0 et 1. Nous montrons également que la mesure μ_p, échantillonnée par cet algorithme de marches aléatoires à boucles effacées, est une mesure sur les forêts couvrantes d'unicycles du graphe infini. Nous montrons enfin le résultat suivant.

Théorème 3. *Soit* $G = \mathbb{Z}^d$ *et* p *une fonction de poids sur les cycles à valeurs dans* $[\alpha, 1]$ *avec* $\alpha > 0$*. La suite de mesures* $(\mu_{p,n})_{n\geq 1}$ *sur les forêts couvrantes d'unicycles, construites sur une exhaustion du graphe* $\mathbb{Z}^d$ *converge faiblement vers la mesure* μ_p*, quelles que soient les conditions au bord considérées. La mesure* μ_p *ne dépend pas de l'énumération des sommets choisie dans l'algorithme, et sous cette mesure, les composantes connexes du modèle sont presque surement finies.*

Dans le chapitre 5, on prouve ce résultat pour des graphes plus généraux, supposés connexes et moyennables, dont les cycles sont munis de poids $(p(\gamma))_\gamma$ compris entre 0 et 1, et tels qu'une famille de cycles vérifie une hypothèse de minoration uniforme (5.18).

Mesures déterminantales sur les forêts couvrantes d'unicycles.

Les mesures de probabilité sur les forêts d'unicycles associées au laplacien tordu par une connexion sur un fibré de rang 1 possèdent des propriétés intéressantes par leur caractère déterminantal. Les expressions combinatoires pour le déterminant du laplacien montrent que ces mesures associent également à une configuration une probabilité proportionnelle au produit de poids de cycles où les poids des cycles orientés dépendent de la connexion le long des arêtes du cycle. Néanmoins, ces poids ne sont plus à valeurs dans $[0, 1]$ et ces mesures ne peuvent donc pas être échantillonnées par l'algorithme précédent. C'est pourquoi, dans le chapitre 6, nous étudions les mesures sur les forêts couvrantes d'unicycles sur un graphe fini, associées à des poids quelconques sur les cycles.

Une question naturelle est de déterminer les changements induits dans le modèle par une variation des poids des cycles. La dépendance de la mesure d'incidence et de la fonction de partition vis-à-vis des poids est une question naturelle qui a été étudiée pour d'autres modèles de physique statistique. Des inégalités de dominations stochastiques ont notamment été établies pour les modèles de percolation.

En supposant par exemple que les fonctions de poids sur les cycles sont constantes égales à α, β, une question naturelle est d'établir une inégalité de domination stochastique de μ_α par rapport à μ_β. Néanmoins, d'après le théorème de Strassen, si deux mesures sont en situation de domination stochastique l'une par rapport à l'autre, alors il existe un couplage des deux tel que la première variable aléatoire soit incluse dans la seconde. Sur un graphe fini, deux forêts couvrantes d'unicycles qui sont incluses l'une dans l'autre sont égales puisqu'elles ont le même nombre d'arêtes (égal au nombre de sommets). Ceci impliquerait que les deux mesures μ_α et μ_β sont égales, ce qui est une contradiction. Nous montrons cependant des inégalités de comparaisons sur des événements particuliers lorsque le poids des cycles est modifié uniformément. Nous prouvons notamment qu'augmenter uniformément le poids des cycles favorise un plus grand nombre de cycles donc de composantes connexes.

Nous montrons par ailleurs que sur un graphe connexe fini, en conditionnant la mesure sur les forêts couvrantes d'unicycles par les cycles aléatoires de poids supérieurs à 1, nous obtenons une mesure échantillonnée par un algorithme de type Wilson avec des conditions au bord.

Nous appliquons ensuite ces résultats au cas des mesures déterminantales associées à des poids provenant d'une connexion complexe unitaire sur le graphe. Nous étudions notamment la limite du modèle en volume infini, qui existe par [KL23], et montrons que sous certaines hypothèses, les composantes connexes contenant un cycle sont presque surement finies. Nous montrons enfin le résultat suivant.

Théorème 4. *Si G est un graphe infini $\mathbb{Z}^d$-périodique muni d'une connexion périodique h et qu'il existe une famille finie Γ d'unicycles disjoints, telle que $\forall \gamma \in \Gamma$, $\prod_{e\in\gamma} h(e) \neq 1$ alors presque sûrement, sous la mesure μ_h associée à la connexion h, la famille Γ apparaît une infinité de fois dans la configuration aléatoire.*

Pour un cycle γ fixé, on appelle holonomie de γ la quantité $\prod_{e\in\gamma} h(e)$.

Ce résultat implique en particulier que sous l'hypothèse d'existence d'une telle famille, il existe une infinité de composantes connexes finies. Par ailleurs, sous l'hypothèse d'existence d'une telle famille, les corrélations décroissent exponentiellement vite avec la distance.

Mesures déterminantales associées à une connexion quaternionique

Le laplacien tordu par une connexion peut également être défini pour une connexion à valeurs dans le corps des quaternions et le Q-determinant de cet opérateur, comme défini par [Moo22] et [Meh04], admet également une expression combinatoire où les configurations sont des forêts couvrantes d'unicycles ([Ken11], [Kas15], [KL20a]). Cela définit naturellement une mesure de probabilités Q-determinantales sur les forêts couvrantes d'unicycles. Dans [KL20c], le changement du corps de base pour les processus déterminantaux linéaires est étudié et montre une correspondance entre les forêts couvrantes quantiques associées à un fibré de rang 2 à connexion dans $U_2(\mathbb{C})$ telle que l'holonomie de tout cycle fermé est dans $SU_2(\mathbb{C})$, et les superpositions de forêts couvrantes d'unicycles associées à une connexion à valeurs dans le corps des quaternions. Dans le chapitre 7, nous étendons cette correspondance à des graphes infinis.

Nous étudions par ailleurs le cas d'une connexion unitaire périodique à valeurs dans le corps des quaternions sur un graphe infini $\mathbb{Z}^d$-périodique. Cette connexion est alors determinée par un choix de quaternions unitaires $(h_1, ..., h_d) \in U_1(\mathbb{H})^d$ sur les demi-arêtes du graphe. Nous obtenons alors un modèle avec deux phases selon si les quaternions $(h_1, ..., h_d) \in U_1(\mathbb{H})^d$ commutent deux à deux.

Théorème 5. *Si les quaternions $(h_1, ..., h_d) \in U_1(\mathbb{H})^d$ commutent deux à deux, on retrouve le modèle usuel de forêts couvrantes uniformes étudié par [Pem91, BP93, BLPS01, LP16]. Si les quaternions unitaires $(h_1, ..., h_d) \in U_1(\mathbb{H})^d$ ne commutent pas deux à deux, on obtient un modèle limite en volume infini qui ne dépend pas des conditions de bord considérées. Pour ce modèle, tous les unicycles finis d'holonomie non triviale sont observés une infinité de fois.*

La preuve de ce théorème repose sur des résultats des chapitres 4 et 6. La preuve du dernier résultat est une adaptation du théorème 4 dans le cas d'une mesure Q-déterminantale. On remarque pour ce faire que les mesures Q-déterminantales conservent la propriété d'association négative des mesures déterminantales et qu'un Q-déterminant est nul si et seulement si la matrice est singulière au sens où elle admet une combinaison (à coefficients quaternions a priori) de ses colonnes qui vaut zéro.

Ces deux phases se distinguent également par la décroissance exponentielle ou quadratique des corrélations avec la distance.

Chapter 2

Spanning Forests on finite graphs.

In this chapter, we introduce the model which is studied in this thesis, that is to say the quantum spanning forest, defined by [KL23]. Since a quantum spanning forest can be interpreted as a superposition of quantum spanning forests of rank 1, which are easier to understand, we will define firstly the model in rank 1 before the general model of quantum spanning forests. In rank 1, quantum spanning forests, also called cycle-rooted spanning forests because of their configurations, are in some sense a generalization of spanning trees. Let us give some background about spanning trees and their generalizations before studying cycle-rooted spanning forests and quantum spanning forests.

2.1 Spanning trees and rooted spanning forests

In this section, we recall the definitions and the formalism of graphs and operators on graphs in order to define probability measures on subgraphs of a finite graph, in particular spanning forests of a finite graph. We give several examples of Boltzmann probability measures on spanning forests of a finite graph, that are probability measures which give to a configuration a probability proportional to its weight, from the easiest one (uniform spanning tree) to the ones which depend on several parameters on the graph (conductances, masses). We recall some properties of those probability measures, in particular their determinantal aspect.

2.1.1 Graphs, Cycles, Roots

We say that $G = (V, E)$ is a simple graph if V is a countable set and E is a subset of $V \times V$ such that for every $e = (x, y) \in E$, we have $-e := (y, x) \in E$. Let E_+ be a subset of E such that for every $e = (x, y) \in E$ with $x \neq y$, either $e \in E_+$ or $-e \in E_+$, but not both. We say that E_+ is the set of positively oriented edges and we denote by $E_- = \{-e : e \in E_+\}$ the set of negatively oriented edges. Then, the set of edges is $E = E_+ \cup E_-$ and this union is a disjoint one.

If a graph $G = (V, E)$ is endowed with weights $(c(e))_{e\in E} \in (\mathbb{R}_+^*)^E$ such that for every edge $e \in E_+$, $c(-e) = c(e)$ and $m(x)_{x\in V} \in \mathbb{R}_+^V$, those weights are referred to as conductances and masses, respectively.

We say that $F = (V(F), E(F))$ is a subgraph of G if F is a simple graph and if we have both inclusions $V(F) \subset V$ and $E(F) \subset E$. In particular, the set of edges of F denoted

by $E(F)$ is a subset of $E \cap (V(F) \times V(F))$. We say that F is a spanning subgraph of G if $V(F) = V$ and we say that F is the induced subgraph of G with vertex set $V(F)$ if $E(F) = E \cap (V(F) \times V(F))$.

We say that $\gamma = (e_1, \dots, e_n) \in E(F)^n$ is an oriented cycle of length n in F if

$$\forall i \in [1, n-1], e_i^+ = e_{i+1}^-,\ e_n^+ = e_1^- .$$

We denote by $\mathcal{C}_\rightarrow(F)$ the set of oriented cycles in F.

If $\gamma = (e_1, \dots, e_n) \in \mathcal{C}_\rightarrow(F)$ is an oriented cycle of F, we denote by γ^{-1} the cycle obtained by reversing the orientation that is to say $\gamma^{-1} = (-e_n, \dots, -e_1)$. We say that two oriented cycles γ, γ' are equivalent ($\gamma \sim \gamma'$) if $\gamma' = \gamma^{-1}$ and we define the set of non-oriented cycles by $\mathcal{C}(F) = \mathcal{C}_\rightarrow(F)/\sim$. If $[\gamma] \in \mathcal{C}(F)$, we denote by γ and γ^{-1} both oriented cycles in the equivalence class $[\gamma]$, and therefore the set of oriented cycles of F can be written as $\mathcal{C}_\rightarrow(F) = \mathcal{C}(F) \cup \mathcal{C}(F)^{-1}$.

If F has some distinguished vertices, they are called roots of F. Let $R(F)$ be the set of roots of F. If $F = (V(F), E(F))$ is a *tree*, that is to say a connected subgraph without any cycle, with a unique root, we say that F is a rooted tree. If $r \in V$ is a vertex of a finite connected graph G, there is a bijection between spanning trees of G and spanning trees of G rooted at r.

Let us recall in the following some properties about random spanning trees of a finite connected graph G.

2.1.2 Uniform spanning tree on a finite graph

If $G = (V, E)$ is a finite connected graph endowed with conductances $(c(e))_{e \in E} \in (\mathbb{R}_+^*)^E$, a spanning tree of G is a spanning subgraph $T = (V, E(T))$ which is a tree that is to say a connected graph without any cycle. There exists a natural probability measure on the set $\mathcal{T}(G)$ of spanning trees of G which gives to every spanning tree a weight proportional to the product of conductances of edges. It is defined as follows. For every $T \in \mathcal{T}(G)$,

$$\mu_c(T) = \frac{\prod_{e \in E(T) \cap E_+} c(e)}{Z_c} \tag{2.1}$$

where $Z_c = \sum_{T \in \mathcal{T}(G)} \prod_{e \in E(T) \cap E_+} c(e)$ is referred to as the partition function of the spanning tree model.

We say that f is a 0-form on G if f is a map $f : V \to \mathbb{C}$ and we say that θ is a 1-form on G if θ is an anti-symmetric map $\theta : E \to \mathbb{C}$ that is to say $\theta(-e) = -\theta(e)$ for every edge $e \in E$. Let us denote by $\Omega^0(G)$ and $\Omega^1(G)$ the spaces of 0-forms and 1-forms on G. Those spaces are endowed with inner products defined by

$$\begin{cases} \langle f, f' \rangle_{\Omega^0(G)} = \sum_{x \in V} f(x) f'(x) \\ \langle \theta, \theta' \rangle_{\Omega^1(G)} = \sum_{e \in E_+} \theta(e) \theta'(e) \end{cases}$$

Let us denote by $d : \Omega^0(G) \to \Omega^1(G)$ and $d^* : \Omega^1(G) \to \Omega^0(G)$ the operators defined for every vertex v and edge $e = (e^-, e^+)$ by

$$\begin{cases} df(e) = f(e^+) - f(e^-) \\ d^*\theta(v) = \sum_{v', (v', v) \in E} \theta((v', v)) \end{cases}$$

Those operators are adjoint for the inner products defined just above.

Definition 2.1. The Laplacian operator $\Delta_c : \Omega^0(G) \to \Omega^0(G)$ associates to a 0-form f the following 0-form:

$$\Delta_c f(v) = \sum_{v' \sim v} c(vv')(f(v) - f(v')).$$

Let C be the operator $C : \Omega^1(G) \to \Omega^1(G)$ defined by $C(\theta)(e) = c(e)\theta(e)$. Then, the Laplacian operator can be written as :

$$\Delta_c = d^* C d.$$

Kirchhoff's matrix-tree theorem [Kir] says that the number of spanning trees of a graph is given by any minor $M_{i,j}$ of the Laplacian operator $\Delta = d^*d$, with all conductances equal to 1, in position (i, j) for any i, j and it is also equal to

$$\frac{1}{n} \det{}_0 \Delta = \frac{1}{n} \prod_i \lambda_i,$$

where $\det{}_0 \Delta$ is the reduced determinant of the Laplacian operator, that is the product of its non-zero eigenvalues $\{\lambda_i\}_i$ and n is the number of vertices of G. When all conductances are equal to 1, the Laplacian operator is an Hermitian matrix since it is the product of adjoint operators.

More generally, when conductances are non trivial, the partition function Z_c is the reduced-determinant of the Laplacian operator $\Delta_c = d^*Cd$, which is also the product of adjoint operators for another inner product, as defined in [LP16].

Indeed, the space $\Omega^1(G)$ is also endowed with an inner product depending on c. If we define r by $r(e) = \frac{1}{c(e)}$ for every edge e, this inner product is defined for every θ, θ' by

$$\langle \theta, \theta' \rangle_r = \sum_{e \in E_+} r(e)\theta(e)\theta'(e).$$

This inner product is natural in terms of electric networks because if we define the energy of a 1-form as the quantity $\mathcal{E}(\theta) = \|\theta\|_r^2$, then for a current i given as the divergence of a electric potential v, we have $\mathcal{E}(\theta) = \langle i, dv \rangle_r$, which is exactly the energy in the electric sense.

For this inner product, the operators Cd and d^* are adjoint, which means that for every $f \in \Omega^0(G)$ and $\theta \in \Omega^1(G)$, we have

$$\langle \theta, Cdf \rangle_{\Omega^1(G),r} = \langle d^*\theta, f \rangle_{\Omega^0(G)}.$$

It implies the following orthogonal decomposition:

$$\Omega^1(G) = im(Cd) \oplus^{\perp_r} \ker(d^*).$$

For $e \in E$, let $\chi^e := \mathbb{1}_e - \mathbb{1}_{-e}$ denote the unit flow along e represented as a 1-form. The space $im(Cd)$, also known as the star space $\bigstar$ is spanned by the so-called stars $\sum_{e^-=x} c(e)\chi^e$, whereas the space $\ker(d^*)$, also known as the cycle space $\diamondsuit$ is spanned by the cycles $\sum_{e \in \gamma} \chi^e$ where γ is an oriented cycle.

It follows from Kirchhoff's matrix-tree theorem the following result of [Pem91].

Theorem 2.2. *[Pem91] Let us denote by $k = Cd((\Delta_c)_{\ker(d)^\perp})^{-1}d^*$ the orthogonal projection on $im(Cd)$ for the inner product $\langle , \rangle_r$. Let us consider the orthonormal basis $(\theta_e)_{e\in E_+}$ of $\Omega^1(G)$ where θ_e is defined for every $e \in E_+$ by $\theta_e = \sqrt{c(e)}\chi^e$. Let us denote by K the matrix of k in this basis. Then, under the measure μ_c, edges of a random tree T form a determinantal process whose kernel is the matrix K. For every set of edges $e_1, \dots, e_n$,*

$$\mu_c(e_1, \dots, e_n \in T) = \det(\langle K\theta_e, \theta_{e'}\rangle_r)_{e,e'\in\{e_1,\dots,e_n\}}$$

The space $\Omega^1(G)$ can also be endowed with another inner product depending on c defined for every θ, θ' by

$$\langle \theta, \theta' \rangle_c = \sum_{e\in E_+} c(e)\theta(e)\theta'(e).$$

Then, the operators d and d^*C are adjoint for this inner product and we have the following orthogonal decomposition:

$$\Omega^1(G) = im(d) \oplus^{\perp_c} \ker(d^*C).$$

If we consider the operator $k' = d((\Delta_c)_{\ker(d)^\perp})^{-1}d^*C$ of orthogonal projection on $im(d)$ for the inner product $\langle , \rangle_c$, then since k and k' are conjugate by the diagonal matrix C, they define the same determinantal process and Theorem 2.2 also holds for k' and the orthonormal basis defined as $\left(\theta_e = \frac{1}{\sqrt{c(e)}}\chi^e\right)_{e\in E}$.

This convention is often adopted in the literature, for instance in [Ken11].

Moreover, this measure is sampled by the Propp-Wilson algorithm ([Wil96, PW98]). This algorithm relies on the choice of a root r but the measure is independent of the choice of vertex r and gives to every spanning tree a weight proportional to the product of conductances of its edges. We will give more precise definitions of this algorithm in section 2.2.4.

A first possible generalization of the model of random spanning tree on a finite graph is the model of rooted spanning forest on a finite graph $G = (V, E)$ endowed with masses $m(x)_{x\in V} \in \mathbb{R}_+^V$. The model of rooted spanning forest associated to a massive Laplacian is studied in [BdTR17] on graphs, so-called isoradial graphs, and a phase transition is obtained for specific weights. We give definitions of the massive Laplacian and the model of rooted spanning forests.

2.1.3 Massive Laplacian and rooted spanning forests

Let $G = (V, E)$ be a finite graph endowed with masses $m(x)_{x\in V} \in \mathbb{R}_+^V$ and conductances $(c(e))_{e\in E} \in (\mathbb{R}_+^*)^E$.

The *massive Laplacian* operator on G is the operator $\Delta_m : \Omega^0(G) \to \Omega^0(G)$ defined for every $f \in \Omega^0(G)$, for every $v \in V$, by

$$\Delta_m f(v) = \sum_{v'\in V, v'\sim v} c_{vv'}(f(v) - f(v')) + m(v)f(v).$$

A *non-oriented rooted spanning forest* of G is a non-oriented subgraph of G which contains all vertices of the graph, all of whose connected components are rooted trees, that are graphs which contain a unique distinguished vertex.

We denote by $\mathcal{F}_R(G)$ the set of non-oriented rooted spanning forests on the graph G. A natural Boltzmann probability measure on $\mathcal{F}_R(G)$ is defined by:

$$\forall F \in \mathcal{F}_R(G), \quad \mu_{m,c}(F) = \frac{1}{Z_R(G,c,m)} \prod_{x \in R(F)} m(x) \prod_{e \in E(F) \cap E_+} c(e). \tag{2.2}$$

where $Z_R(G,c,m)$ is the partition function of the model and satisfies:

$$Z_R(G,c,m) = \sum_{F \in \mathcal{F}_R(G)} \prod_{x \in R(F)} m(x) \prod_{e \in E(F) \cap E_+} c(e) = \det(\Delta_m).$$

Just as the measure on spanning trees, this measure $\mu_{m,c}$ is determinantal. Those results rely on a correspondence between this model and a model of spanning tree without masses, adding a root r and new conductances depending on m, c. Let r be a new vertex called the root of G and add for every vertex x an edge e_x from x to r with conductances depending on c, m. The kernel is the orthogonal projection on the image by d of 0-forms on this graph which are zero on r.

2.2 Cycle-rooted spanning forests

Another possible generalization of random spanning tree on a finite graph is the model of cycle-rooted spanning forests on a finite graph $G = (V, E)$ endowed with a weight function w on cycles. When this weight function w is provided by a *unitary connection*, the measure is determinantal and its partition function is given by the determinant of an operator called the vector-bundle Laplacian ([Ken11]).

2.2.1 Vector-Bundle Laplacian and CRSF

We say that a non-oriented subgraph F of G is a *cycle-rooted spanning forest* (CRSF) if it contains all the vertices and if every connected component of F contains a unique cycle. Each connected component of F is called a *unicycle* or a *cycle-rooted tree*. We will denote by $\mathcal{U}(G)$ the set of CRSFs of G.

We say that an oriented subgraph F of G is a *oriented cycle-rooted spanning forest* (OCRSF) if it contains all the vertices and if every connected component of F contains a unique cycle, which is given an orientation. Every edge of an OCRSF is oriented towards the cycle of its connected component. We will denote by $\mathcal{U}_{\rightarrow}(G)$ the set of OCRSFs of G.

We recall the definition of the vector-bundle Laplacian associated to a unitary connection, whose determinant can be expressed as a combinatorial sum over CRSF.

Definition 2.3. ([Ken11]) Let $\mathbb{U}$ be the set of complex numbers of modulus 1. Let $h : E \to \mathbb{U}$ be a *unitary connection* of rank 1 on the graph G, in the sense that for every oriented edge e, the equality $h(-e) = h(e)^*$ holds. For every oriented cycle $\gamma = (e_1, \dots, e_p)$, the *holonomy* of γ is

$$\mathrm{hol}_h(\gamma) = \left(\prod_i h(e_i) \right).$$

Let $\Delta_h : \Omega^0(G) \to \Omega^0(G)$ be called the *vector-bundle Laplacian* associated to h and defined for every 0-form f and vertex v by

$$\Delta_h f(v) = \sum_{e\,:\,e^+=v\in E} c(e)(f(e^+) - h(e)f(e^-))$$

According to [Ken11], if for every $e \in E$, $h(e) = h_{ee^+}h_{e^-e}$, for some $h_{ee^+}, h_{e^-e} \in \mathbb{U}$, the vector-bundle Laplacian can be factorized as $\Delta_h = d_h^* C d_h$ where C is the matrix of conductances, and the operators $d_h : \Omega^0(G) \to \Omega^1(G)$ and $d_h^* : \Omega^1(G) \to \Omega^0(G)$ are defined by

$$d_h f(e) = h_{e^+e} f(e^+) - h_{e^-e} f(e^-) \tag{2.3}$$

$$d_h^* \theta(v) = \sum_{v',v'v\in E} h_{(v',v)v}\theta(v'v) \tag{2.4}$$

and are adjoint operators. The following orthogonal decompositions hold:

$$\Omega^1(G) = im(Cd_h) \oplus^{\perp_r} \ker(d_h^*) = im(d_h) \oplus^{\perp_c} \ker(d_h^* C).$$

From [For93], the following formula holds:

$$\det(\Delta_h) = \sum_{F\in\mathcal{U}(G)} \prod_{e\in E(F)\cap E_+} c(e) \prod_{[\gamma]\in\mathcal{C}(F)} (w_h(\gamma) + w_h(\gamma^{-1})),$$

where the function $w_h : \mathcal{C}_\to(G) \to \mathbb{R}_+$ is defined for every oriented cycle γ by

$$w_h(\gamma) = 1 - \mathrm{Re}(\mathrm{hol}_h(\gamma)).$$

This formula can also be written as

$$\det(\Delta_h) = \sum_{\overrightarrow{F}\in\mathcal{U}_\to(G)} \prod_{e\in E(\overrightarrow{F})} c(e) \prod_{\gamma\in\mathcal{C}_\to(F)} w_h(\gamma).$$

It defines a natural Boltzmann probability measure on $\mathcal{U}(G)$:

$$\forall F \in \mathcal{U}(G), \quad \mu_h(F) = \frac{\prod_{[\gamma]\in\mathcal{C}(F)}(w_h(\gamma) + w_h(\gamma^{-1})) \prod_{e\in E(F)\cap E_+} c(e)}{Z_h}, \tag{2.5}$$

where Z_h is called the partition function of the model

$$Z_h = \sum_{F\in\mathcal{U}(G)} \prod_{e\in E(F)\cap E_+} c(e) \prod_{[\gamma]\in\mathcal{C}(F)} (w_h(\gamma) + w_h(\gamma^{-1})) = \det(\Delta_h),$$

and a natural Boltzmann probability measure on $\mathcal{U}_\to(G)$:

$$\forall \overrightarrow{F} \in \mathcal{U}_\to(G), \quad \overrightarrow{\mu}_h(\overrightarrow{F}) = \frac{\prod_{\gamma\in\mathcal{C}_\to(\overrightarrow{F})} w_h(\gamma) \prod_{e\in E(\overrightarrow{F})} c(e)}{Z_h}, \tag{2.6}$$

where Z_h is called the partition function of the model

$$Z_h = \det(\Delta_h).$$

Those measures are equal after removal of the orientations of edges.

Like the measures on spanning trees and spanning rooted spanning forests, those measure are determinantal.

Theorem 2.4. *[Ken11] μ_h is a determinantal measure associated to the orthogonal projection on $im(Cd_h)$. For every edges $e_1, \dots, e_m \in E(G)$, if F is sampled under the measure μ_h,*

$$\mu_h(\{e_1, \dots, e_m\} \subset \mathsf{F}) = \det((K(e_i, e_j))_{1 \leq i,j \leq m})$$

where K is the matrix of the orthogonal projection on $im(Cd_h)$ for $\langle ., . \rangle_r$.

The following generalization of the model includes rooted spanning forests and cycle-rooted spanning forests, but also the usual model of random spanning tree.

2.2.2 Wired vertex-and-cycle-rooted spanning forests

Definition 2.5. Let W be a subset of vertices of G. We say that a subgraph F of G is :

- a wired or essential vertex-and-cycle-rooted spanning forest (EVCRSF) with respect to W if it contains all the vertices and every connected component of F is either a rooted tree disjoint from W, a cycle-rooted tree disjoint from W or an unrooted tree which contains a unique vertex of W, called a boundary-rooted tree.

- a wired or essential cycle-rooted spanning forest (ECRSF) with respect to W if it contains all the vertices and every connected component of F is either a cycle-rooted tree disjoint from W or an unrooted tree which contains a unique vertex of W, called a boundary-rooted tree.

- a *wired or essential oriented vertex-and-cycle-rooted spanning forest* (EOVCRSF) if it is a wired vertex-and-cycle-rooted spanning forest with respect to W and every cycle of F is given an orientation, that is to say if every connected component of F either is an oriented rooted tree or contains a unique oriented cycle or a unique vertex of W. Every edge of an EOVCRSF is oriented towards the root or the cycle or the unique point in W of its connected component.

- a *wired or essential oriented cycle-rooted spanning forest* (EOCRSF) with respect to W if every connected component of F either is a cycle-rooted tree disjoint from W which is given an orientation or is an unrooted tree which contains a unique vertex of W, called a boundary-rooted tree. Every edge of an EOCRSF is oriented towards the cycle or the unique point in W of its connected component.

We denote by $\mathcal{V}_W(G)$ and $\mathcal{U}_W(G)$ the sets of EVCRSF of G and ECRSF with respect to W, and by $\mathcal{V}_{\to W}(G)$ and $\mathcal{U}_{\to W}(G)$ the sets of EOVCRSF and EOCRSF with respect to W

We will say that F is a *vertex-and-cycle-rooted spanning forest* (VCRSF) if it contains all the vertices and if every connected component of F is either a rooted tree or contains a unique cycle, that is an EVCRSF with respect to $W = \emptyset$. We denote by $\mathcal{U}(G)$ and $\mathcal{V}(G)$ instead of $\mathcal{U}_\emptyset(G)$ and $\mathcal{V}_\emptyset(G)$ the sets of CRSF and VCRSF.

We also denote by $\mathcal{U}_\to(G)$ and $\mathcal{V}_\to(G)$ instead of $\mathcal{U}_{\to\emptyset}(G)$ and $\mathcal{V}_{\to\emptyset}(G)$ the sets of OVCRSF and OCRSF, which correspond to the case $W = \emptyset$.

Definition 2.6 (Wired measure)**.** We define a measure on $\mathcal{V}_W(G)$ called the wired measure on EVCRSF of G with boundary W, which gives to a configuration a probability proportional to the product of weights of cycles, roots and edges.

$$\mu^W_{(c,m,h)}(F) = \frac{\prod_{e\in E(F)} c(e) \prod_{x\in R(F)} m(x) \prod_{\gamma\in\mathcal{C}(F)}(w_h(\gamma) + w_h(\gamma^{-1}))}{Z^W_{(c,m,w)}} \tag{2.7}$$

When $m = 0$, $\mu^W_{(c,h)}$ has support in the set of essential cycle-rooted spanning forests $\mathcal{U}_W(G)$.

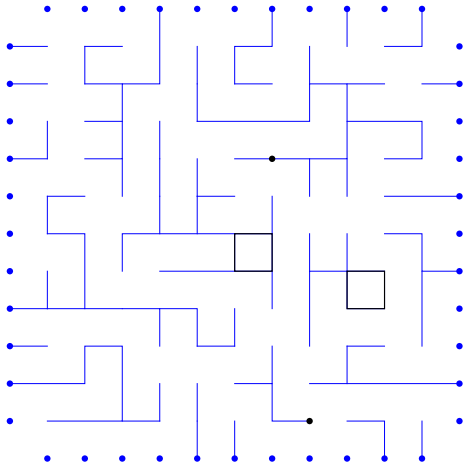

Figure 2.1 – Simulation of an EVCRSF for $c = 1, p = 0.75, m = 0.01$ and W the boundary of the square grid.

Let us emphasize that the measures defined in previous subsections correspond to the case of $W = \emptyset$, and either $h = id$, which is equivalent to $w_h = 0$, that is the measure on rooted spanning forests (2.2) or $m = 0$, that is the measure on cycle-rooted spanning forests (2.6).

When the weight functions w_h, m are both equal to 0 and $W \neq \emptyset$, this measure has support in Essential Rooted Spanning Forests (ERSF), all of whose connected component are boundary-rooted trees. In particular, when $W = \{r\}$ is a single vertex and the weight functions w_h, m are both equal to 0, this measure has support on spanning trees rooted at r. If we fix whatever vertex r, it corresponds to the measure on spanning trees as defined in Equation (2.1).

Notice that the model with non-trivial masses $(m(x))$ is equivalent to a model without masses but with a small oriented self-loop $l_x = (x, x)$ over every vertex $x \in V$ with a unitary connection defined by $h(l_x)$ and a conductance $c(l_x)$ such that

$$c(l_x)(1 - Re(\mathrm{hol}_h(l_x))) = \frac{m(x)}{2}.$$

Therefore, without lost of generality, we may assume that $m = 0$.

Definition 2.7. We define the Laplacian $\Delta_{h,W}$ with *Dirichlet boundary condition* at W as follows. For $f : V\backslash W \to \mathbb{C}$ and $v \in V\backslash W$,

$$\Delta_{h,W} f(v) = \sum_{v'\in V\backslash W, v'\sim v} c_{vv'}(f(v) - h_{v'v} f(v')).$$

This is the Laplacian Δ_h restricted to the subspace of 0-forms which are zero on the boundary W and projected back to this subspace ([Ken11, Section 8]).

Theorem 2.8. *[Ken11, Theorem 7] The measure $\mu^W_{(c,w)}$ is determinantal and*

$$\det \Delta_{h,W} = \sum_{F \in \mathcal{U}_W(G)} \prod_{\gamma \in \mathcal{C}(F)} (w_h(\gamma) + w_h(\gamma^{-1})) \prod_{e \in E(F)} c(e) = Z^W_{(c,w)}$$

The kernel of this measure is the orthogonal projection on the following subspace of 1-forms

$$\bigstar_W = Cd_h(\iota(\Omega^0(G\backslash W)))$$

where $\iota : \Omega^0(G\backslash W) \to \Omega^0(G)$ is defined by $\iota(.)(W) = 0$ and $\iota(.)(V\backslash W) = id$.

We may define other Boltzmann probability measures on cycle-rooted spanning forests associated to other weight functions w on cycles of G. When those weight functions are not provided by a unitary connection h, those measures are not determinantal measures.

2.2.3 Probability measures on oriented vertex-and-cycle-rooted spanning forests of a finite graph

In this subsection, we define probability measures on vertex-and-cycle-rooted spanning forests for a weight function on cycles which does not necessarily come from a connection h.

Let $G = (V, E)$ be a finite connected graph endowed with weights $(c(e))_{e\in E} \in (\mathbb{R}_+^*)^E$ and $m(x)_{x\in V} \in \mathbb{R}_+^V$. Let $w : \mathcal{C}_\to(G) \to \mathbb{R}_+$ be a non-negative function defined on oriented cycles of G which is symmetric under orientation reversal.

There is a natural probability measure on $\mathcal{V}_{\to W}(G)$ associated to (c, m, w), called the wired measure on essential spanning forest of G with boundary W, whose configurations have weight proportional to the product of weights of cycles, roots and edges. This measure is denoted $\mu_{(c,m,w)}$ and is defined for every EOVCRSF $F \in \mathcal{V}_{\to W}(G)$ by

$$\mu^W_{(c,m,w)}(F) = \frac{\prod_{e\in E(F)} c(e) \prod_{x\in R(F)} m(x) \prod_{\gamma\in\mathcal{C}_\to(F)} w(\gamma)}{Z_{(c,m,w)}}, \tag{2.8}$$

where $Z_{(c,m,w)}$ is called the partition function of the model

$$Z_{(c,m,w)} = \sum_{F\in\mathcal{V}_{\to W}(G)} \prod_{e\in E(F)} c(e) \prod_{x\in R(F)} m(x) \prod_{\gamma\in\mathcal{C}_\to(F)} w(\gamma).$$

Let us emphasize that for general weight functions w defined on oriented cycles, the measure is not necessarily determinantal. Nevertheless, if there exists a unitary connection h such that for all oriented cycle γ,

$$w(\gamma) = 1 - \mathrm{Re}(\mathrm{hol}_h(\gamma)),$$

then it corresponds to the determinantal measure $\mu^W_{(c,m,h)}$ as defined in (2.7), after removal of the orientations of edges.

In the remainder of this section, we will be interested in another kind of weight functions which are weight functions which take values in $[0, 1]$ and for which the corresponding measure on $\mathcal{V}_{\to W}(G)$ is sampled by an algorithm.

2.2.4 Probability measures on CRSF sampled by a random walk algorithm

Sampling a uniform spanning tree by an algorithm is a problem which has been studied since the end of the 20^{th}-century. Several algorithms have been given by Aldous and Broder ([Ald90], [Bro89]), and then by Propp and Wilson ([Wil96, PW98]). Wilson's algorithm relies on loop-erased random walks as follows. Let $x_0, x_1, \dots, x_n$ be an ordering of the vertex set V of G, and let $T_0 = \{x_0\}$. At each step i, let $(X_k^{(x_i)})_{k\geq 0}$ be a random walk on the graph G with conductances $(c(e))$ starting from x_i. Every time the random walk makes a loop, it is erased. The random walk $(X_k^{(x_i)})_{k\geq 0}$ is stopped when it reaches the set of already explored vertices denoted by $V(T_{i-1})$. At the end of the i^{th} step, let $T_i = T_{i-1} \cup L(X_k^{(x_i)})$ where $L(X_k^{(x_i)})$ is obtained from $(X_k^{(x_i)})_{k\geq 0}$ after removal of the loops. At the end, we obtain a tree T_n such that $V(T_n) = V(G)$. The random tree T_n is distributed according the measure μ_c which is defined in equation (2.1) and this measure does not depend on the ordering of the vertex set.

Just as sampling a uniform spanning tree of a finite graph, sampling a vertex-and-cycle-rooted spanning forest of a finite graph is a well studied problem and its answer relies on algorithms using loop-erased random walks inspired from the Propp-Wilson algorithm for the generation of a random spanning tree.

In [BBGJ07], authors introduce some methods to generate a random spanning web using a "cycle-popping" inspired from the Propp-Wilson algorithm.

If the weight function on cycles is $p : \mathcal{C}_{\to}(G) \to [0, 1]$, then according to [KK17], the measure $\mu_{c,p}$, corresponding to $W = \emptyset$ and $m = 0$, can be sampled by an algorithm of loop-erased random walk where we keep an oriented cycle γ, with probability $p(\gamma)$.

More precisely, let $x_1, \dots, x_n$ be an ordering of the vertex set V of G and let $\mathsf{F}_0 = \emptyset$. At each step i, let $(X_k^{(x_i)})_{k\geq 0}$ be a random walk on the graph G with conductances $(c(e))$ starting from x_i. Every time the random walk makes a loop, the oriented cycle γ is kept with probability $p(\gamma)$ or erased with probability $1 - p(\gamma)$. The random walk $(X_k^{(x_i)})_{k\geq 0}$ is stopped when it reaches the set of already explored vertices denoted by $V(\mathsf{F}_{i-1})$ or when a cycle is kept. At the end of the i^{th} step, let $\mathsf{F}_i = \mathsf{F}_{i-1} \cup L(X_k^{(x_i)})$ where $L(X_k^{(x_i)})$ is obtained from $(X_k^{(x_i)})_{n\geq 0}$ after removing all the loops except the last one if a loop is kept at the end of the i^{th} step. At the end, $V(\mathsf{F}_n) = V(G)$. Notice that the algorithm always finishes if and only if there exists at least a loop γ in G such that $p(\gamma) > 0$.

The measure $\mu_{c,p}^W$ with a wired boundary condition W can also be sampled by an algorithm. We follow the same algorithm but every time the random walk meets W, the walk stops and a new random walk starts from the next vertex in the ordering. At the beginning of the algorithm, we set $\mathsf{F}_0 = W$ instead of $\mathsf{F}_0 = \emptyset$. The algorithm always finishes if and only if there exists at least a loop γ in $G\backslash W$ such that $p(\gamma) > 0$ or if $W \neq \emptyset$.

When p is constant equal to 0, the measure $\mu_{(c,m,0)}^W$ is well defined and can also be sampled by a Wilson type algorithm if and only if $W \neq \emptyset$ or $m \neq 0$. Indeed, the model with non-trivial masses $(m(x))$ is equivalent to a model without masses but with a new boundary condition, adding a root r and new conductances on edges e_x from x to r, depending on m, c. Therefore, the measure $\mu_{(c,m,0)}^W$ can be sampled by the algorithm described just above by setting at the beginning $F_0 = W \cup \{r\}$ which is non-empty.

The measure $\mu^W_{(c,m,0)}$ has support in the set of *Essential Rooted Spanning Forests* (ERSF) with respect to W, all of whose connected components are rooted trees in $G\backslash W$ or contain a unique vertex of W. When $m = 0$, the measure $\mu^W_{(c,0,0)}$ has support in the set of *Essential Spanning Forests* (ESF) with respect to W, all of whose connected components contain a unique vertex of W.

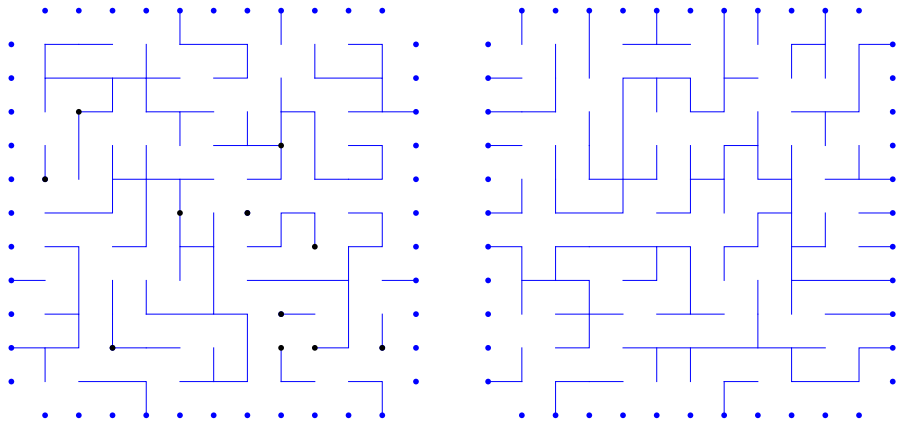

Figure 2.2 – Simulation of an ERSF for $p = 0, m = 0.01$ and of an ESF for $p = 0, m = 0$. The marked points inside are the roots of the configuration.

In [BdTR17], authors use this algorithm to sample rooted spanning forests, all of whose connected components are rooted trees.

Let us emphasize that when $W = \{r\}$ is a single vertex and the weight functions m, p are both equal to 0, then $\mu^{\{r\}}_{(c,0,0)}$ is the measure on spanning trees of G rooted at r, sampled by the usual Wilson algorithm.

More generally, when the graph G is endowed with masses $m(x)_{x\in V} \in \mathbb{R}^V_+$, and a weight function on cycles $p : \mathcal{C}_{\to}(G) \to [0, 1]$, the measure $\mu^W_{(c,m,p)}$ can be sampled by the algorithm described just above, by adding a new vertex r and conductances on edges e_x from x to r, depending on m, c and setting $F_0 = W \cup \{r\}$.

Another way to sample the probability measure $\mu^W_{(c,m,p)}$ by the Wilson type algorithm, is to add over each vertex $x \in V$, a small oriented self-loop $l_x$with weight $p(l_x)$ and conductances $(c(l_x))$ such that

$$\begin{cases} c(l_x)p(l_x) = \frac{m(x)}{2} \\ p(l_x) \leq 1. \end{cases}$$

Notice that for a unitary complex connection h, the weight function w_h takes values in $[0, 2]$. Nevertheless, if the connection h is near the identity in the sense that for every cycle γ, $\mathrm{Re}(\mathrm{hol}_h(\gamma)) \geq 0$, then the weight function w_h takes values in $[0, 1]$ and the measure $\mu_{(c,0,w_h)}$ is determinantal and sampled by the algorithm described juste above. When this is the case, if the graph is furthermore endowed with masses $m(x)_{x\in V} \in \mathbb{R}^V_+$, then choosing $h(l_x), c(l_x)$ such that

$$\begin{cases} h(l_x)c(l_x) = \frac{m(x)}{2} \\ 1 - Re(\mathrm{hol}_h(l_x)) \leq 1, \end{cases}$$

implies that $\mu_{(c,m,w_h)}$ is still sampled by the algorithm described just above and is determinantal.

In the remainder of this thesis, we will not always specify c, m which will be taken constant equal to 1,0 but it is possible to consider other conductances and masses with slight modifications.

We will denote by $\mu_w := \mu_{(1,0,w)}$ the measure which has support in the set of cycle-rooted spanning forests, all of whose connected components contain a unique cycle. There are two families of weight functions which give interesting properties for the measure μ_w:

- Weight functions w_h which are provided by a connection h which give determinantal measures and are defined by

$$\forall \gamma \in \mathcal{C}_{\rightarrow}(G), \quad w_h(\gamma) = 1 - Re(\mathrm{hol}_h(\gamma)).$$

- Weight functions p which take values in $[0, 1]$ and which give measures sampled by an algorithm described just above.

We will study properties of those measures in chapters 5 and 6 under some assumptions.

2.3 Quantum Spanning forests

The model of quantum spanning forests which is introduced in [KL20c] and studied in [KL23] is an example of determinantal linear processes which are studied in [KL20c]. We firstly recall the construction of those processes and their properties.

2.3.1 Determinantal linear processes

Determinantal linear processes are defined by a measure on the Grassmannian of a real, complex or quaternionic inner product vector space which is characterized by a self-adjoint contraction operator of the vector space. We recall the construction of those processes which is explained in [KL20c] and the particular case of quantum spanning forests on finite graph with a well.

Let E be such an inner product vector space of dimension d and $B = (e_1, \ldots, e_d)$ be an orthonormal basis of E and K the matrice in this basis of a self-adjoint contraction operator k. We define the random subspace Q of E as

$$Q = \mathrm{Vect}(e_i, i \in X),$$

where X is a finite determinantal point process (DPP) associated to the set $S = \{1, \ldots, d\}$ and $K \in M_d(\mathbb{C})$.

The random subspace Q is adapted to the basis $(e_1, \ldots, e_d)$. One can extend this definition to the case where E admits an orthogonal decomposition σ :

$$\mathsf{E} = \mathsf{E}_1 \oplus \ldots \oplus \mathsf{E}_s, \quad s \leq d.$$

If k is a self-adjoint operator with spectra included in $[0, 1]$, we get a random subspace Q adapted to the decomposition σ after concatenating uniformly sampled bases of $\mathsf{E}_1, \ldots, \mathsf{E}_s$

and considering the random subspace Q which corresponds to this basis and to the operator k. In the case where k is an orthogonal projection on a subspace H of E, we get some further properties to describe the random subspace Q.

Theorem 2.9. *[KL20c] Almost surely, the orthogonal decomposition*

$$\mathsf{E} = Q \bigoplus H^{\perp}$$

holds and the expected value of the projection $P^{Q}_{\|H^{\perp}}$ on the random subspace Q parallel to the orthogonal of H is given by the operator k:

$$\mathbb{E}(P^{Q}_{\|H^{\perp}}) = k.$$

Moreover, the incidence measure of the process has a density given by $\det k^{R}_{R}$ with respect to the Haar measure $\nu^{\mathsf{E},\sigma}$ on the set of subspaces of E adapted to the decomposition σ.

2.3.2 Quantum spanning forests on finite graphs

In this section, we recall the construction of quantum spanning forests on finite graphs, endowed with a fiber bundle and a unitary connection, from [KL20c].

Let $G = (V, E)$ be a finite graph, with $d = |E|$, endowed with N-dimensional Euclidean spaces $(F_x)_{x\in V}$ over vertices and $(F_e)_{e\in E}$ over edges, and a unitary connection h, that is to say linear isometries $h_{e,x} : F_x \to F_e$, $h_{x,e} : F_e \to F_x$ when $x \sim e$ such that for all $(x, e) \in V \times E$, the relation $h_{x,-e} = h^{-1}_{e,x}$ holds.

Definition 2.10 (0-forms and 1-forms)**.** We denote by $\Omega^0(G)$ and $\Omega^1(G)$ the spaces of bundle-valued 0-forms and 1-forms over G.

$$\Omega^0(G) = \{f : V \to (F_x)_{x\in V} \mid f(x) \in F_x \quad \forall x \in V\}.$$

$$\Omega^1(G) = \{\theta : E \to (F_e)_{e\in E} \mid \theta(e) = -\theta(-e) \in F_e \quad \forall e \in E_+\}.$$

Those spaces are endowed with inner products defined as

$$\langle \theta, \theta' \rangle_{\Omega^1(G)} = \sum_{e\in E_+} \langle \theta(e), \theta'(e) \rangle_{F_e}$$

$$\langle f, f' \rangle_{\Omega^0(G)} = \sum_{x\in V} \langle f(x), f'(x) \rangle_{F_x}.$$

Definition 2.11 (Covariant derivative and its adjoint)**.** We get a natural map

$$d_h : \Omega^0(G) \to \Omega^1(G),$$

defined by

$$d_h f(e) = h_{e^+e} f(e^+) - h_{e^-e} f(e^-).$$

Denoting by $d^{\star}_h$ its adjoint, we have for $\theta \in \Omega^1(G)$,

$$d^{\star}_h \theta(v) = \sum_{v', v'v\in E} h_{(v'v)v} \theta(v'v).$$

We define the vector-bundle Laplacian $\Delta_h : \Omega^0(G) \to \Omega^0(G)$ by $\Delta_h = d^{\star}_h \circ d_h$.

Recall from [KL20c] that we have the following orthogonal decompositions σ_1, σ_2 respectively defined as

$$\Omega^1(G) = im(d_h) \bigoplus^{\perp} \ker(d_h^\star), \qquad \Omega^1(G) = \bigoplus_{e \in E_+} L_e,$$

where $L_e = \{w \in \Omega^1(G), w(e') = 0 \; \forall \; e' \notin \{e, -e\}\}$.

Definition 2.12. [KL20c] The quantum spanning forest Q on G is the determinantal linear process associated to the orthogonal projection k on $im(d_h)$ and to the decomposition σ_2.

More precisely, let k be the orthogonal projection on $im(d_h)$, parallel to $\ker(d_h^\star)$. Let $B = (b_i)_{i \in [1, N|E|]}$ be a random orthonormal basis of $\Omega^1(G)$ obtained after concatenating uniformly sampled bases of $(L_e)_{e \in E}$. Let $Q = Vect\{b_i, i \in X\}$ be the quantum spanning forest on G associated to the connection h, where X is a determinantal point process associated to the matrix K of the operator k in basis B.

A quantum spanning forest (QSF) is a random subspace $Q = \oplus_{e \in E} Q_e$ where for every $e \in E$, the random subspace Q_e is a subspace of L_e but can be seen as a subspace of F_e since both spaces F_e and L_e are isomorphic. Indeed, every 1-form $w \in L_e$ is entirely determined by the vector $w(e) \in F_e$.

As for every DLP, the law μ of a Quantum spanning forest is characterized by its *incidence measure* Z_μ. This incidence measure is a measure on the Grassmanniann $Gr(\Omega^1(G))$ of $\Omega^1(G)$, that is the set of linear subspaces of $\Omega^1(G)$ which is linked to μ as follows:

$$\forall R \in Gr(\Omega^1(G)), Z_\mu(R) = \mu(R \subset Q),$$

where Q is a quantum spanning forest distributed according to μ.

From [KL20c] and [KL23], the incidence measure Z_μ has a *density* ρ with respect to the Haar measure ν^σ on the set of subspaces of $\Omega^1(G)$ adapted to the decomposition:

$$\Omega^1(G) = \bigoplus_{e \in E_+} L_e.$$

Furthermore, the following result gives an exact expression of the density.

Theorem 2.13. *[KL23] On a finite graph $G = (V, E)$, if Q is a quantum spanning forest associated to a connection h then its law μ is determinantal of kernel k where k is the orthogonal projection on $im(d_h)$, in the following sense:*

- *Almost surely* $\Omega^1 = Q \bigoplus \ker(d_h^\star)$,
- $\mathbb{E}(P^Q_{\|\ker(d_h^\star)}) = k$,
- $\rho(R) = \det(k^R_R)$.

2.3.3 Trace of a QSF

Let us recall the following definitions and propositions from [KL20c].

Definition 2.14. [Holonomy of an oriented cycle] ([KL20c]) Given $\gamma_x^{\rightarrow} = (e_1, \dots, e_p)$ an oriented cycle based at $x \in V$, we call holonomy of $\gamma_x^{\rightarrow}$ the isometry $\mathrm{hol}_h(\gamma_x^{\rightarrow}) : F_x \to F_x$ defined by $\mathrm{hol}_h(\gamma_x^{\rightarrow}) = h_{e_p} \circ \ldots \circ h_{e_1}$, where $h_e = h_{ee^+} \circ h_{e^-e}$.

Definition 2.15. [Trace of a quantum spanning forest] ([KL20c]) We call *trace* of a quantum spanning forest the random variable

$$n_Q = (\dim Q_e)_{e \in E}.$$

The trace is an essential tool because it can be seen as a number of occupation when the Quantum Spanning forest is seen as a superposition of N cycle-rooted spanning forests of rank 1.

From the gauge theory introduced by Weyl and also studied by Pauli (see also [KL20c] for notations), the gauge group of the vector bundle F over $G = (V, E)$ is the Cartesian product of unitary groups

$$\mathcal{J}(F) = \{j \in \prod_{x \in V} U(F_x) \times \prod_{e \in E} U(F_e) : j_e = j_{-e} \quad \forall e \in E\}.$$

The elements of the gauge group, so-called gauge transformations act on the set of connections as follows. Let h be a connection on F. Let j be a gauge transformation. The connection $j.h$ is defined by

$$(j.h)_{e,x} = j_e \circ h_{e,x} \circ j_x^{-1}.$$

Two connections are called gauge equivalent if they are obtained from each other by applying a gauge transformation. Note that the QSF associated with a connection $j.h$ is obtained from the QSF associated with the connection h by applying a deterministic transformation. From [KL20c, KL23], the trace of a quantum spanning forest is invariant in distribution under the action of the full gauge group $\mathcal{J}(F)$.

The following Lemma, due to [KL20c] is essential because it shows that the law of the trace can be computed when random basis is fixed. We recall the proof of this lemma, since it is a good illustration of computations which link DPP and determinants of their kernel.

Lemma 2.16. *Let us consider two orthonormal bases of $\Omega^1(G)$ adapted to the decomposition in fibers $\Omega^1(G) = \oplus L_e$, where $L_e = \{w \in \Omega^1(G), w(e') = 0 \; \forall \; e' \notin \{e, -e\}\}$:*

$$\begin{cases} B = (b_1^{e_1}, \dots, b_N^{e_1}, \dots, b_1^{e_d}, \dots, b_N^{e_d}) \\ D = (d_1^{e_1}, \dots, d_N^{e_1}, \dots, d_1^{e_d}, \dots, d_N^{e_d}). \end{cases}$$

Then if K and $K' = PKP^{-1}$ are the matrices of k in those bases, and if we consider both spaces $Q = Vect(b_j, j \in X)$, $Q' = Vect(d_j, j \in Y)$ with X, Y DPP associated to K, K', then

$$(\dim(Q_e)) \overset{\mathcal{L}}{=} (\dim(Q'_e))$$

where the equality is an equality in law.

Proof. Since $(b_1^{e_i}, \dots, b_N^{e_i})$ and $(d_1^{e_i}, \dots, d_N^{e_i})$ are bases of L_{e_i} for every i, the matrix P is block-diagonal :

$$P = \begin{pmatrix} P_{e_1} & & \\ & \ddots & \\ & & P_{e_d} \end{pmatrix}$$

where P_{e_i} is the matrix between bases $(d_1^{e_i}, \dots, d_N^{e_i})$ and $(b_1^{e_i}, \dots, b_N^{e_i})$.

Let $S = [1, dN]$ be the set of indices of vectors of those bases and let us denote by $S_i = \{j \mid b_j \in \{b_1^{e_i}, \dots, b_N^{e_i}\}\}$ the set of indices of vectors of the bases of L_{e_i}.

Then, we have

$$\mathbb{E}\left[\prod_{i=1}^{d} z_i^{\dim(Q_{e_i})}\right] = \mathbb{E}\left[\prod_{i=1}^{d} z_i^{|X\cap\{j \mid b_j\in\{b_1^{e_i},\dots,b_N^{e_i}\}\}|}\right] = \mathbb{E}\left[\prod_{i=1}^{d} z_i^{|X\cap S_i|}\right] = \mathbb{E}\left[\prod_{x\in X} z_x\right]$$

where $z_x = z_i$ for every $x \in S_i$. Since P commutes with

$$Z := \begin{pmatrix} \begin{pmatrix} z_1 - 1 & & \\ & \ddots & \\ & & z_1 - 1 \end{pmatrix} & & \\ & \ddots & \\ & & \begin{pmatrix} z_d - 1 & & \\ & \ddots & \\ & & z_d - 1 \end{pmatrix} \end{pmatrix},$$

we have

$$\begin{aligned}\mathbb{E}\left[\prod_{i=1}^{d} z_i^{\dim(Q_{e_i})}\right] &= \det\left(I_d + ZK\right) = \det\left(I_d + ZPKP^{-1}\right) = \det\left(I_d + ZK'\right) \\ &= \mathbb{E}\left[\prod_{i=1}^{d} z_i^{|Y\cap S_i|}\right] = \mathbb{E}\left[\prod_{i=1}^{d} z_i^{\dim(Q'_{e_i})}\right],\end{aligned}$$

which concludes the proof. □

Let us recall the following definition, from [KL20c] which deals with direct sums of vector bundles with connections.

Definition 2.17 (Reducible connections)**.** ([KL20c]) We say that a connection h is *reducible* if there exists an integer $s \in \{2, \dots, N\}$, sub-bundles $F^{(1)}, \dots, F^{(s)}$ of F and connections $h^{(1)}, \dots, h^{(s)}$ on these sub-bundles, such that $F = F^{(1)} \oplus \dots \oplus F^{(s)}$ and the connection can be written as $h = h^{(1)} \oplus \dots \oplus h^{(s)}$. If the connection is not reducible, it is said to be *irreducible*.

The previous Lemma implies that for a reducible connection h, if we fix a basis adapted to the fibers and to the decomposition $h = h^{(1)} \oplus \dots \oplus h^{(s)}$, we obtain a matrix which is block-diagonal, and therefore, the law of the trace is the sum of traces of independent QSF adapted to $(F^{(i)}, h^{(i)}, \sigma^{(i)})$.

$$\mathbb{E}\left[\prod_i z_i^{\dim Q_{e_i}}\right] = \mathbb{E}\left[\prod_i z_i^{\dim Q_{e_i}^{h_1}}\right] \dots \mathbb{E}\left[\prod_i z_i^{\dim Q_{e_i}^{h_s}}\right],$$

where $\sigma^{(i)} : \Omega^1(\mathbb{Z}^d)^{(i)} = \oplus F_e^{(i)}$.

The following Proposition, due to [KL23] shows a stronger result on the law of a QSF for a reducible connection.

Proposition 2.18. *If τ is a refined splitting adapted to a reducible connection h, the QSF adapted to τ is a direct sum of independent QSF adapted to $(F^{(i)}, h^{(i)}, \sigma^{(i)})$. The law of n_Q is independent of the choice of τ.*

Recall that if X is DPP associated to a matrix K, $|X|$ has the same law that $\sum_{\lambda\in Sp(K)} Y_\lambda$, where Y_λ are independent Bernoulli of parameter λ for λ in the spectrum $Sp(K)$. Indeed,

$$\mathbb{E}\left[z^{|X|}\right] = \det\left(I_d + (z-1)K\right) = \prod_{\lambda\in Sp(K)} (1+(z-1)\lambda) = \mathbb{E}\left[z^{\sum Y_\lambda}\right].$$

Another consequence of Lemma 2.16 is that for a QSF Q adapted to the orthogonal projection k on $im(d_h)$, each marginal of the trace $\dim(Q_e)$ has the same law than $\sum_{\lambda\in Sp(K_{e,e})} Y_\lambda$ and therefore

$$\mathbb{E}(\dim(Q_e)) = \sum_{\lambda\in Sp(K_{e,e})} \lambda = Tr(K_{e,e}),$$

and

$$Var(\dim(Q_e)) = \sum_{\lambda\in Sp(K_{e,e})} (\lambda-\lambda^2) = Tr(K_{e,e}) - Tr(K_{e,e}^2).$$

Those equalities give the link between the mean-value of the trace of a quantum spanning forest and the trace of the kernel of this process.

2.3.4 Partition function of a QSF

The partition function is given by the determinant of the vector-bundle Laplacian operator, restricted to the orthogonal of its kernel, referred to as the reduced determinant of the Laplacian, as well as for the case of rank 1 cycle-rooted spanning forests model.

Let G be a finite graph endowed with a unitary connection h and let ν^σ be the Haar measure on the set of subspaces of $\Omega^1(G)$ adapted to the decomposition $\Omega^1(G) = \bigoplus_{e\in E(G)_+} L_e$. In the following statement, we denote by d instead of d_h the covariant derivative associated to the connection h in order to simplify the notations.

Theorem 2.19. *[KL23] We have the following expression for the reduced-determinant of the Laplacian $\Delta = d^*d$:*

$$\det{}_0(\Delta) = \det((d^*)^{|\ker(d)^\perp} d_{\ker(d)^\perp}) = \int_{Gr(E,\sigma)} \det((d^*)^{|\ker(d)^\perp}_Q d^Q_{\ker(d)^\perp}) d\nu^\sigma(Q).$$

Let μ be the law of the QSF associated to the orthogonal projection on $im(d)$. The quantity $\det((d^*)^{|\ker(d)^\perp}_Q d^Q_{\ker(d)^\perp})$ satisfies

$$\frac{\det((d^*)^{|\ker(d)^\perp}_Q d^Q_{\ker(d)^\perp})}{\det_0(\Delta)} = \frac{d\mu}{d\nu^\sigma}(Q).$$

Thus the partition function of the model is exactly $\det{}_0(\Delta)$.

We give a proof of this result since it is a good illustration of computations which link the partition function and the density of a DLP as defined in [KL20c].

Proof. Applying the Cauchy-Binet formula of [KL20c, Proposition 2.7] which is

$$\det(ba) = \int_{Gr(E,\sigma)} \det(b_Q a^Q) d\nu^\sigma(Q)$$

to the operators $b = (d^*)^{|\ker(d)^\perp}$ and $a = d_{|\ker(d)^\perp}$ gives

$$\det(\Delta^{|\ker(d)^\perp}_{|\ker(d)^\perp}) = \int_{Gr(E,\sigma)} \det((d^*)^{|\ker(d)^\perp}_Q d^Q_{|\ker(d)^\perp}) d\nu^\sigma(Q).$$

The density of the determinantal measure of the DLP associated to the orthogonal projection on im(d) is

$$\frac{d\mu}{d\nu^\sigma}(Q) = \det(k\Pi^Q + (1-k)\Pi^{Q^\perp})$$

where $k = d_{|\ker(d)^\perp} G (d^*)^{|\ker(d)^\perp}$, with G which is the inverse of the Laplacian on the invariant subspace $\ker(d)^\perp$.

Let us consider the matrix of the operator $k : \Omega^1 \to \Omega^1$ in a basis adapted to the decomposition $\Omega^1 = Q \bigoplus Q^\perp$.

$$k = \begin{pmatrix} A & B \\ B^* & D \end{pmatrix}.$$

Then, using the Schur-complement formula, we obtain

$$\begin{aligned}
\det(k\Pi^Q + (1-k)\Pi^{Q^\perp}) &= \det \begin{pmatrix} A & -B \\ B^* & I - D \end{pmatrix} \\
&= (-1)^{\dim(Q)} \det \left(\begin{pmatrix} 0 & 0 \\ 0 & I_{\dim(Q^\perp)} \end{pmatrix} - k \right) \\
&= (-1)^{\dim(Q)} \det \left(\begin{pmatrix} 0 & 0 \\ 0 & I_{\dim(Q^\perp)} \end{pmatrix} - d_{|\ker(d)^\perp} G (d^*)^{|\ker(d)^\perp} \right) \\
&= (-1)^{\dim(Q)} \frac{1}{\det(G^{-1})} \det \begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & I_{\dim(Q^\perp)} \end{pmatrix} & d_{|\ker(d)^\perp} \\ (d^*)^{|\ker(d)^\perp} & G^{-1} \end{pmatrix}.
\end{aligned}$$

Since k is an orthogonal projection of rank $p := \mathrm{rg}(d)$, we know that $\dim Q = p$ almost surely. Then $d^Q_{|\ker(d)^\perp} : \ker(d)^\perp \to Q$ and $(d^*)^{|\ker(d)^\perp}_Q : Q \to \ker(d)^\perp$ are square matrices. Then, the last determinant is equal to

$$(-1)^{\dim(Q)} \det \begin{pmatrix} d^Q_{|\ker(d)^\perp} & 0 & 0 \\ d^{Q^\perp}_{|\ker(d)^\perp} & I_{\dim(Q^\perp)} & 0 \\ \Delta & (d^*)^{|\ker(d)^\perp}_{Q^\perp} & (d^*)^{|\ker(d)^\perp}_Q \end{pmatrix}$$

and therefore,

$$\begin{aligned}
\det(k\Pi^Q + (1-k)\Pi^{Q^\perp}) &= \frac{1}{\det_0(\Delta)} \det(d^Q_{|\ker(d)^\perp}) \det(I_{\dim(Q^\perp)}) \det((d^*)^{|\ker(d)^\perp}_Q) \\
&= \frac{1}{\det_0(\Delta)} \det((d^*)^{|\ker(d)^\perp}_Q d^Q_{|\ker(d)^\perp}),
\end{aligned}$$

which concludes the proof. □

This expression of the partition function of the model as the reduced determinant of the Laplacian operator shows that the model of QSF is a good generalization of the usual model of cycle-rooted spanning forests. This theorem will allow us to obtain an integral expression for the free energy of the model of QSF (Theorem 4.17) when the size of the graph goes to infinity.

The next chapter deals with topological considerations on the model of QSF, existence of thermodynamic limits of the model when the size of the graph goes to infinity and dependence of the limit on boundary conditions.

Chapter 3

Thermodynamic limits.

In statistical physics, we try to construct measures on configurations in infinite volume, often called Gibbs measures, from limits of sequences of measures in finite volume, often called Boltzmann measures, which give to a finite configuration a probability proportional to its weight for a well-defined weight function. When configurations are defined on a underlying graph, for instance in the case of models of percolation or in the case of the Ising model, we consider a sequence of measures on configurations of growing finite subgraphs which gives at the limit an infinite volume Gibbs measure on the configurations of an infinite countable graph. To study the convergence of the sequence of measures on growing subgraphs, we need to define a topology for this convergence and to define which boundary conditions are applied on the boundaries of growing finite subgraphs.

In this chapter, we will study those questions for the model of quantum spanning forests and for the case of rank 1, so-called cycle-rooted spanning forests. We will begin this chapter with the case of cycle-rooted spanning forests since the topology is easier to understand in this case.

3.1 Rank 1 : Measures on CRSF in infinite volume and thermodynamic limits

In this section, let $G = (V, E)$ be a countably infinite connected graph. Let (G_n) be an exhaustion of G by growing finite subgraphs, that is to say an increasing sequence of subgraphs of G for the inclusion such that $G = \cup_{n\in\mathbb{N}} G_n$.

We define, in this section, sequences of measures on CRSF of a growing exhaustion of G with boundary conditions. We will see that under some assumptions, those sequences of measures converge to thermodynamic limits which are probability measures on subgraphs of the infinite graph G (called the infinite volume case).

3.1.1 Thermodynamic limits and boundary conditions

We define sequences of measures $(\mu_n)_{n\geq 1}$ on CRSF of growing subgraphs G_n of G with boundary conditions. There are two examples of boundary conditions which are often considered : free and wired boundary conditions.

Definition 3.1 (Free boundary conditions)**.** We define a measure on oriented CRSF of the graph G_n under which the configurations have a probability proportional to the product of weights of oriented cycles. After removal of orientations, it gives a measure on non oriented CRSF which is denoted by μ_n^F and called the free measure on CRSF of G_n

Definition 3.2 (Wired boundary conditions)**.** We define a measure on oriented ECRSF of the graph G_n under which the configurations, which are either connected to ∂G_n or oriented cycle-rooted trees, have a probability proportional to the product of weights of oriented cycles. After removal of orientations, it gives a measure on non oriented ECRSF of G_n which is denoted by μ_n^W and called the wired measure on ECRSF of G_n. Notice that it corresponds to the measure defined in Definition 2.6 with boundary ∂G_n.

3.1.2 Topological facts and boundary conditions

Every configuration of CRSF on G can be seen as an element of $\{0,1\}^E$. Let us recall some topological facts about $\{0,1\}^E$ (see for instance [FV17] for more details).

Since $\{0,1\}$ is compact, $\Omega = \{0,1\}^E$ is compact for the product topology and this topology is compatible with the following metric

$$d(\omega,\omega') = \sum_{e\in E} 2^{-\|e_-\|_2} 1_{\omega_e \neq \omega'_e}.$$

Therefore Ω is a compact metric space.

A function $f : \Omega \to \mathbb{R}$ is continuous for the product topology if and only if for every $\varepsilon > 0$, there exists a finite subset $\Lambda \subset E$, such that

$$\sup_{\omega,\omega' : \omega_{|\Lambda} = \omega'_{|\Lambda}} |f(\omega) - f(\omega')| \leq \varepsilon.$$

A function $f : \Omega \to \mathbb{R}$ is called local if there exists a finite set $\Lambda \subset E$ such that for every $\omega \in \Omega$, the image $f(\omega)$ is entirely determined by $\omega_{|\Lambda}$. The set of local functions is dense in the set of continuous functions $(\mathcal{C}(\Omega), ||.||_\infty)$ which is a Banach-space.

We consider $\mathcal{C}$ the smallest σ-field for which the cylinders $\mathcal{C}_{\Lambda,\varepsilon} = \{\omega \in \Omega, \omega_\Lambda = \varepsilon\}$ are measurable.

We say that a sequence of measures (μ_n) converges to the measure μ on $(\Omega, \mathcal{C})$ if and only if

$$\lim_{n\to\infty} \mathbb{E}_{\mu_n}(f) = \mathbb{E}_\mu(f),$$

for every local function f. Since the set of local functions is dense in the set of continuous functions, this topology on the set of measures on $(\Omega, \mathcal{C})$ is the same as the one induced by the notion of weak convergence.

If such a sequence of measures converges weakly towards an infinite volume measure, we have the following result on the limit measure.

Proposition 3.3. *Assume that a sequence $(\mu_n)_{n\geq 1}$ of measures on CRSF on growing subgraphs G_n of G converges weakly towards a measure μ, and let F be distributed according to μ. Then μ-almost surely, every finite connected component of F contains exactly one non-oriented cycle γ which has a non-trivial weight, in the sense that $w(\gamma) + w(\gamma^{-1}) \neq 0$.*

Proof. Let $x \in V$ and let T be a finite connected subgraph of G which contains x and which satisfies one of the following properties:

- T has strictly more than one cycle.
- T contains a non-oriented cycle $[\gamma]$ such that $w(\gamma) + w(\gamma^{-1}) = 0$, where γ and γ^{-1} are both orientations of $[\gamma]$.
- T has no cycle.

Let $cc(x)$ be the connected component of x in F. Notice that the event $\{cc(x) = T\}$ is an event with finite support since its support is included in the set of edges which have at least one extremity in T.

Let m be large enough such that $T \subset G_{m-1}$. Then, the event

$$\{cc(x) = T\} = \{cc(x) \cap G_m = T\}$$

has support in G_m. For every $n \geq m$, μ_n is supported on CRSF on G_n, all of whose cycles satisfy $w(\gamma) + w(\gamma^{-1}) > 0$. Therefore, if F_n is distributed according to the measure μ_n,

$$\mu_n(cc(x) \cap G_m = T) = \mu_n(cc(x)_{\mathsf{F}_n} = T) = 0.$$

We have the convergence $\mu_n \to \mu$ on configurations with finite support. Then,

$$\mu_n(cc(x) \cap G_m = T) \to \mu(cc(x) \cap G_m = T).$$

Finally, we obtain $\mu(cc(x) = T) = 0$. Since G is countable, almost surely, every finite connected component of F has exactly one cycle and its cycle has non-trivial weight. □

3.2 Measures on QSF in infinite volume and thermodynamic limits

In this section, we study sequences of probability measures on quantum spanning forests of growing finite graphs. To study the convergence of those measures, we give some properties on the weak convergence topology on the space of configurations of quantum spanning forests. We show that, as well as for the case of rank 1, the weak convergence is equivalent to a notion of convergence for local functions. Then, considering kernels associated with determinantal measures on quantum spanning forests, we show some stochastic inequalities between different boundary conditions. All results of this section can be applied in particular for determinantal measures on cycle-rooted spanning forests in the case of a rank 1 connection.

3.2.1 Topological facts for QSF in rank N

Every configuration of QSF on G can be seen as an element of $\Omega = Gr(\mathbb{C}^N)^E$. Here $Gr(\mathbb{C}^N)$ denotes the Grassmannian of linear subspaces of $\mathbb{C}^N$, that is the space which contains every linear subspace of $\mathbb{C}^N$. It is a compact metric space equipped with a distance d.

Therefore $\Omega = Gr(\mathbb{C}^N)^E$ is compact for the product topology and this topology is compatible with the following metric

$$d(Q, Q') = \sum_{e \in E} 2^{-\|e_-\|_2} d(Q_e, Q'_e).$$

Therefore Ω is a compact metric space.

Notice that for every $\delta > 0$, there exists a finite set Λ large enough such that,

$$\sup_{Q,Q' : Q_{|\Lambda} = Q'_{|\Lambda}} d(Q, Q') \leq \delta. \tag{3.1}$$

Since Ω is a compact set, a function $f : \Omega \to \mathbb{R}$ is continuous for the product topology if and only if it is uniformly continuous. Then, a function $f : \Omega \to \mathbb{R}$ is continuous if and only if for every $\varepsilon > 0$, there exists a constant $\delta > 0$ such that for every $Q, Q' \in \Omega$,

$$(d(Q, Q') \leq \delta) \Rightarrow (|f(Q) - f(Q')| \leq \varepsilon).$$

A function $f : \Omega \to \mathbb{R}$ is called local if there exists a finite set $\Lambda \subset E$ such that $f(Q)$ is entirely determined by $Q_{|\Lambda}$. The set of local functions is dense in the set of continuous functions $(\mathcal{C}(\Omega), ||.||_\infty)$ which is a Banach-space. Indeed, let $f : \Omega \to \mathbb{R}$ be a continuous function and Λ_n a sequence of growing finite subsets of E such that $E = \cup \Lambda_n$. Then define $f_n : Q \to f(Q_{\Lambda_n})$ where $(Q_{\Lambda_n})_e = Q_e 1_{e \in \Lambda_n}$. If $\varepsilon > 0$, there exists $\delta > 0$ of uniform continuity for f and from equation (3.1), for n large enough, we have for every $Q \in \Omega$, the inequality $d(Q_{\Lambda_n}, Q) \leq \delta$ which implies $|f_n(Q) - f(Q)| \leq \varepsilon$ and then f_n converges uniformly towards f.

We consider $\mathcal{C}$ the smallest σ-field which makes the cylinders $\mathcal{C}_{\Lambda,\varepsilon} = \{Q \in \Omega, Q_\Lambda = \varepsilon\}$ measurable. We say that a sequence of measures (μ_n) converges to the measure μ on $(\Omega, \mathcal{C})$ if and only if

$$\lim_{n \to \infty} \mathbb{E}_{\mu_n}(f) = \mathbb{E}_\mu(f)$$

for every local function f. Since the set of local functions is dense in the set of continuous functions, this topology on the set of measures on $(\Omega, \mathcal{C})$ is the same as the one induced by the notion of weak convergence.

The weak-convergence of a sequence of measures μ_n on QSF of an exhaustion of G by finite graphs (G_n) is equivalent to the weak-convergence of the sequence of associated incidence measures. Let us recall from Subsection 2.3.2 that the incidence measure associated with μ_n has a density ρ_n with respect to the Haar measure ν_n on the Grassmannian $Gr(\mathbb{C}^N)^{E(G_n)}$.

Assume that for every n, the measure μ_n is determinantal of kernel k_n. Then if $\Lambda \subset E$ is a finite set of edges and $R = \oplus_{e \in \Lambda} R_e$, then if for every $e \in \Lambda$, we fix a basis $(b_i^e)_{i \in \dim R_e}$, then the density of the incidence measure evaluated in R is given by

$$\rho_n(R) = \det\left((k_n)_R^R\right) = \det\left((k_n)_{(b_i^e)}^{(b_i^e)}\right).$$

Therefore, the weak-convergence of the sequence (μ_n) towards a determinantal measure μ of kernel k whose associated incidence measure has a density ρ with respect to

the Haar measure ν on $Gr(\mathbb{C}^N)^E$ is equivalent to the convergence $\rho_n(R) \to \rho(R)$ for every finite set $\Lambda \subset E$ and every subspace $R = \oplus_{e\in\Lambda} R_e$, which is equivalent to the following statement: for every finite family of edges $(e_1, \dots, e_k)$, for every family of linearly independent vectors of the corresponding fibers $((b_i^{e_j})_i)_{j\in[1,k]}$,

$$\det\left((k_n)^{(b_i^{e_j})}_{(b_i^{e_j})}\right) \to \det\left(k^{(b_i^{e_j})}_{(b_i^{e_j})}\right).$$

Therefore, the weak-convergence of the sequence (μ_n) towards a determinantal measure μ of kernel k is equivalent to the convergence of the matrix of $(k_n)^{e_1,\dots,e_k}_{e_1,\dots,e_k}$ towards the matrix of $k^{e_1,\dots,e_k}_{e_1,\dots,e_k}$ in any family of orthonormal basis of $(F_{e_i})_i$ for any finite family of edges $(e_1, \dots, e_k)$.

Notice that if we change the basis of a fiber to another orthonormal basis, the matrix which is obtained is the conjugate of the previous matrix by an orthogonal matrix and therefore, the previous convergence does not depend on the choice of bases of fibers.

Therefore, the weak-convergence of a sequence of measures (μ_n) towards a determinantal measure μ of kernel k is equivalent to the convergence of $\langle k_n\theta_e^p, \theta_{e'}^l\rangle$ towards $\langle k\theta_e^p, \theta_{e'}^l\rangle$ for every $e \neq e' \in E$, where $\theta_e^p \in \Omega^1(G)$ satisfies $\theta_e^p(e') = 0$ and $\theta_e^p(e)$ is the p-th vector of the canonical orthonormal basis of $\mathbb{C}^N$.

3.2.2 Kernels of the determinantal measures in infinite volume

Since in the case of quantum spanning forests, the convergence of sequences of determinantal measures on quantum spanning forests of growing finite graphs relies on the convergence of their kernels, we define measures in finite volume with boundary conditions from their kernels and study their convergence.

Recall from [KL23] that on an infinite graph the splitting $\Omega^1(G) = im(d_h) \oplus \ker(d_h^*)$ need not hold, and therefore, we cannot define a natural quantum spanning forest in infinite volume from the projection on $im(d_h)$ parallel to $\ker(d_h^*)$.

Let us borrow some more notations from [KL23].

Definition 3.4. Let $\Omega^0_{\ell^2}(F) = \{f \in \Omega^0(F) : \|f\|_{\Omega^0} < \infty\}$ be the Hilbert space of integrable 0-forms and $\Omega^1_{\ell^2}(F) = \{w \in \Omega^1(F) : \|w\|_{\Omega^1} < \infty\}$ be the Hilbert spaces of integrable 1-forms. Let $\bigstar_{\ell^2}$ be the closure in $\Omega^1_{\ell^2}(F)$ of the image by d_h of the space of elements of $\Omega^0(F)$ with finite support and $\diamondsuit_{\ell^2}$ be the closure in $\Omega^1_{\ell^2}(F)$ of the space of elements of $\ker d_h^*$ with finite support.

From [KL23], we have the orthogonal decomposition

$$\Omega^1_{\ell^2}(F) = \bigstar_{\ell^2} \oplus d_h(\mathcal{H}) \oplus \diamondsuit_{\ell^2} \tag{3.2}$$

where $\mathcal{H} = \{f \in \Omega^0 : \Delta_h f = 0 \text{ and } d_h f \in \Omega^1_{\ell^2}(F)\}$ is the space of harmonic 0-forms.

Free and wired conditions on growing subgraphs G_n are defined as follows. Let us consider for every n, the operators ι_n and ε_n which are defined by:

$$\iota_n : \Omega^0(G_n) \to \Omega^0(G_{n+1}) \qquad \varepsilon_n : \Omega^1(G_n) \to \Omega^1(G_{n+1})$$
$$f \mapsto \begin{cases} f(x) \text{ if } x \in G_n \\ 0 \text{ else} \end{cases} \qquad \theta \mapsto \begin{cases} \theta(e) \text{ if } e \in G_n \\ 0 \text{ else} \end{cases}$$

For every n, let us define

$$\bigstar_n^W := d_n(\iota_{n-1}(\Omega^0(G_{n-1})))$$

the space of exact 1-forms for wired conditions and

$$\diamondsuit_n := \varepsilon_{n-1} \ker(d_{n-1}^*)$$

the space of cycles with free conditions.

Definition 3.5 (Free and wired QSF)**.** For every n, we denote by (μ_n^F), (μ_n^W) determinantal measures on QSF of G_n whose kernels are orthogonal projections on $\diamondsuit_n^\perp$ and $\bigstar_n^W$. Those measures are called free and wired measures on QSF of G_n.

For a vector-bundle of rank 1, those measures coincide with free and wired measures as defined in subsection 3.1.1. Indeed, the free and the wired measure on cycle-rooted spanning forests as defined in subsection 3.1.1 are respectively the measure on cycle-rooted spanning forests of G_n and the measure on essential cycle-rooted spanning forests with respect to $W = \partial G_n$, as defined in Definition 2.6. Those measures are determinantal and their kernels are respectively the orthogonal projections on $\diamondsuit_n^\perp$ and $\bigstar_n^W$.

One can think of the wired boundary condition as gluing together every vertices of the boundary. The free boundary conditions corresponds to the case where edges outgoing of the graph G_n are removed and none of the vertices are glued together.

From [KL23], the sequence of measures (μ_n^F) (respectively (μ_n^W)) on QSF on the graphs (G_n) are determinantal measures whose kernels are orthogonal projections on decreasing (respectively increasing) subspaces of $\Omega^1_{\ell^2}(F)$. This sequence converges in law towards a measure on spanning subgraphs of G denoted by μ^F (respectively μ^W). The measures μ^F and μ^W are determinantal and their kernels are the orthogonal projections on the spaces $\bigstar_{\ell^2} = \overline{\cup_n \bigstar_n^W}^{\ell^2}$ and $\diamondsuit_{\ell^2} = \overline{\cup_n \diamondsuit_n}^{\ell^2}$. From Equation (3.2), the free and the wired measures μ^W and μ^F are equal if and only if $d_h(\mathcal{H}) = \{0\}$. We will see later (Lemma 6.18) that under Assumption 6.17 on the connection h, this equality is satisfied but it doesn't seem to be a necessary condition.

Let us emphasize that those statements hold in particular for determinantal measures on CRSF associated with a connection on a vector bundle of rank 1.

3.3 Periodic boundary conditions for periodic graphs

In this section, we consider a graph G embedded in $\mathbb{R}^d$ which is $\mathbb{Z}^d$-periodic, in the sense that there exists a lattice in $\mathbb{R}^d$ such that the graph G is invariant under the action of the lattice. We can consider for instance the graph $G = \mathbb{Z}^d$ where the set of edges is the set of couples of points at distance 1. We call fundamental domain the basis of the lattice under which the graph is invariant.

We will consider an exhaustion (G_n) of growing finite subgraphs of G which will be seen as graphs on growing tori where the left-right boundaries and top-bottom boundaries are glued together. We will prove in this section that those boundary conditions are stuck between free and wired boundary conditions, which is consistent with the intuition.

In order to make clear definitions, we will give some examples for $G = \mathbb{Z}^d$ for some fixed $d \geq 2$ but definitions hold in the case where the graph has a larger fundamental domain.

3.3.1 Graphs on tori and periodic boundary measures

We define graphs on tori and measures on QSF of finite graphs seen as graphs on tori.

Definition 3.6. Let $(G_n)_{n\geq 1}$ be an increasing sequence of connected induced subgraphs of G with respective vertex set V_n and let ∂G_n be the set of vertices which are connected by an edge to the complement of G_n in G. Assume that the exhaustion is constructed such that it is invariant by the action of the lattice, that is to say that the top-bottom boundaries and left-right boundaries can be glued. Let $\tilde{G}_n = G_n/\sim$ be the graph constructed from G_n where $\sim$ is the equivalence relation defined by gluing boundaries.

To illustrate this definition, assume that $G = \mathbb{Z}^d$ and $m \geq 1$ is fixed. Let $(G_n)_{n\geq 1}$ be the increasing sequence of connected induced subgraphs of $\mathbb{Z}^d$ with respective vertex set $V_n = [-nm, nm]^d$ and let

$$\partial G_n = \{v \in G_n : \ ||v||_\infty = nm\}$$

In this case, we have $\tilde{G}_n = G_n/\sim$ where $\sim$ is the equivalence relation :

$$\begin{cases} v \sim v' \ \Leftrightarrow \exists j \in [1,d], \quad v' = v \pm 2nmb_j \\ e \sim e' \ \Leftrightarrow \exists j \in [1,d], \quad e' = e \pm 2nmb_j \end{cases}$$

where (b_j) is the canonical basis of $\mathbb{R}^d$.

In the following, we will consider a sequence (μ_n^P) of measures on QSF on graphs G_n seen as graphs on the torus.

Definition 3.7 (Graph on the torus)**.** We denote by $\tilde{e}$ the equivalence class of an edge e. Let $\Omega^1(\tilde{G}_n)$ be the space of 1-forms on the graph $\tilde{G}_n$ and $\Omega^0(\tilde{G}_n)$ be the space of 0-forms on the graph $\tilde{G}_n$. Those spaces are equipped with the inner products

$$\langle \tilde{\theta}_1, \tilde{\theta}_2 \rangle_{\Omega^1(\tilde{G}_n)} = \sum_{e \in \tilde{G}_n} \tilde{\theta}_1(e)\tilde{\theta}_2(e)$$

$$\langle \tilde{f}_1, \tilde{f}_2 \rangle_{\Omega^0(\tilde{G}_n)} = \sum_{x \in \tilde{G}_n} \tilde{f}_1(x)\tilde{f}_2(x)$$

and the operators $\tilde{d}_n : \Omega^0(\tilde{G}_n) \to \Omega^1(\tilde{G}_n)$ and $\tilde{d}_n^* : \Omega^1(\tilde{G}_n) \to \Omega^0(\tilde{G}_n)$ as defined in Subsection 2.3.2 are adjoints for those inner products.

Definition 3.8 (QSF on $\tilde{G}_n$.)**.** Let $\bigstar_{\text{per}}^n := \tilde{d}_n(\Omega^0(\tilde{G}_n))$ be the subspace of $\Omega^1(\tilde{G}_n)$ which contains the periodic exact forms. For the inner product on $\Omega^1(\tilde{G}_n)$, the orthogonal decomposition holds :

$$\Omega^1(\tilde{G}_n) = \tilde{d}_n(\Omega^0(\tilde{G}_n)) \oplus \ker(\tilde{d}_n^*).$$

Let $\tilde{K}_n$ be the orthogonal projection on $\bigstar_{\text{per}}^n := \tilde{d}_n(\Omega^0(\tilde{G}_n))$. We denote by $\tilde{\mu}_n^P$ the measure on QSF on $\tilde{G}_n$ which is determinantal with kernel $\tilde{K}_n$.

Definition 3.9. Let $r_n : \{\text{Sub-bundles of } \tilde{G}_n\} \to \{\text{Sub-bundles of } G_n\}$ defined by

$$r_n(\tilde{Q}) = \oplus_{e \in G_n} \tilde{Q}_{\tilde{e}}$$

Let us define the measure μ_n^P on sub-bundles of G_n which is called the periodic boundary measure as the pushforward measure $\mu_n^P = r_{n*}\tilde{\mu}_n^P$.

In the case of a determinantal measure on CRSF associated with a vector-bundle of rank 1, the corresponding map r_n which is considered is the following map :

$$\begin{aligned} r_n : \{\text{Subgraphs of } \tilde{G}_n\} &\to \{\text{Subgraphs of } G_n\} \\ \tilde{F} &\mapsto \{e \in G_n | \tilde{e} \in \tilde{F}\} \end{aligned}$$

Then, the measure μ_n^P on subgraphs of G_n which is called the periodic boundary measure is defined as the pushforward measure $\mu_n^P = r_{n*}\tilde{\mu}_n^P$.

By definition of the pushforward measure, if $e_1, \dots, e_k \in E(G_n)$,

$$\mu_n^P(e_1, \dots, e_k \in \mathsf{F}) = \tilde{\mu}_n^P(\tilde{e}_1, \dots, \tilde{e}_k \in r_n^{-1}(\mathsf{F})) = \det(((\tilde{K}_n)_{\tilde{e}_i, \tilde{e}_j})_{1 \le i,j \le k})$$

Notice that under the measure μ_n^P, configurations which are obtained are not necessarily cycle-rooted forests when they are seen as configurations on the graph G_n. Once configurations are seen as configurations on the graph $\tilde{G}_n$ on the torus, they are CRSF and have probabilities proportional to the product of weights of cycles.

In the following, we consider for every n the measure μ_n^P on sub-bundles of G_n for a vector bundle of rank N and we will try to compare this measure with free and wired measures which have been introduced in Subsection 3.2.2.

Recall that measures (μ_n^W) and (μ_n^F) are determinantal and their kernels are orthogonal projections on $\bigstar_n^W$ and $\diamondsuit_n^\perp$, where $\bigstar_n^W$ and $\diamondsuit_n^\perp$ are subspaces of $\Omega^1(G)$. Therefore, we will establish a correspondence between the space $\Omega^1(\tilde{G}_n)$ and a subspace of $\Omega^1(G)$, such that $\tilde{K}_n$ is the conjugate by an isometry of an orthogonal projection on a subspace of $\Omega^1(G)$.

3.3.2 Correspondence with subspaces of $\Omega^1(G)$.

Let $\Omega^1_{\text{per}}(G_n)$ be the subspace of $\Omega^1(G)$ whose functions have support in $E(G_n)$ and satisfy for every $e_1, e_2 \in E(\partial G_n)$, such that the equality $\tilde{e}_1 = \tilde{e}_2$ holds in $\tilde{G}_n$, $\theta(e_1) = \theta(e_2)$.

For every n, let us define a map φ_n between $\Omega^1(\tilde{G}_n)$ and $\Omega^1_{\text{per}}(G_n)$ as follows. For every $\theta \in \Omega^1(\tilde{G}_n)$, we define for every edge e,

$$\varphi_n(\theta)(e) = \begin{cases} \theta(\tilde{e}) & \text{if } e \in G_n \backslash \partial G_n \\ \frac{\theta(\tilde{e})}{\sqrt{n(e)}} & \text{if } e \in \partial G_n \end{cases}$$

where $n(e)$ is defined by $n(e) = |\{e' \in \partial G_n | \tilde{e'} = \tilde{e}\}|$.

Lemma 3.10. *φ_n is an isometry. Therefore, for every n, $\varphi_n \circ \tilde{K}_n \circ \varphi_n^{-1}$ is the orthogonal projection on $\varphi_n(\bigstar^n_{per})$ for the inner product on $\Omega^1(G)$.*

Proof. Let $\tilde{\theta}_1, \tilde{\theta}_2 \in \Omega^1(\tilde{G}_n)$.

$$\begin{aligned}\langle \varphi_n(\tilde{\theta}_1), \varphi_n(\tilde{\theta}_2)\rangle_{\Omega^1(G_n)} &= \sum_{e\in E(G_n)} \langle \varphi_n(\tilde{\theta}_1)(e), \varphi_n(\tilde{\theta}_2)(e)\rangle_{F_e} \\ &= \sum_{e\in E(G_n\setminus\partial G_n)} \langle \tilde{\theta}_1(\tilde{e})\tilde{\theta}_2(\tilde{e})\rangle_{F_e} + \sum_{e\in E(\partial G_n)} \frac{1}{n(e)} \langle \tilde{\theta}_1(\tilde{e}), \tilde{\theta}_2(\tilde{e})\rangle_{F_e} \\ &= \sum_{e\in E(G_n\setminus\partial G_n)} \langle \tilde{\theta}_1(\tilde{e}), \tilde{\theta}_2(\tilde{e})\rangle_{F_e} + \sum_{\tilde{e}\in E(\partial \tilde{G}_n)} \langle \tilde{\theta}_1(\tilde{e}), \tilde{\theta}_2(\tilde{e})\rangle_{F_e} \\ &= \langle \tilde{\theta}_1, \tilde{\theta}_2\rangle_{\Omega^1(\tilde{G}_n)}\end{aligned}$$

Since $\tilde{K}_n$ is the orthogonal projection on $\bigstar^n_{\text{per}}$ for the inner product on $\Omega^1(\tilde{G}_n)$, the conjugate $\varphi_n \circ \tilde{K}_n \circ \varphi_n^{-1}$ is the orthogonal projection on $\varphi_n(\bigstar^n_{\text{per}})$ for the inner product on $\Omega^1(G)$.

Indeed, if $f \in \Omega^1(G)$, $\varphi_n \circ \tilde{K}_n \circ \varphi_n^{-1}(f) \in \varphi_n(\bigstar^n_{\text{per}})$ and if $\theta \in \varphi_n(\bigstar^n_{\text{per}})$, $\theta = \varphi_n(\tilde{\theta})$ with $\tilde{\theta} \in \bigstar^n_{\text{per}}$, then,

$$\langle f - \varphi_n \circ \tilde{K}_n \circ \varphi_n^{-1}(f), \theta\rangle = \langle \varphi_n^{-1}(f) - \tilde{K}_n \circ \varphi_n^{-1}(f), \tilde{\theta}\rangle = 0$$

since $\varphi_n^{-1}(f) - \tilde{K}_n \circ \varphi_n^{-1}(f) \in (\bigstar^n_{\text{per}})^\perp$ for the inner product on $\Omega^1(\tilde{G}_n)$, and then,

$$f - \varphi_n \circ \tilde{K}_n \circ \varphi_n^{-1}(f) \in (\varphi_n(\bigstar^n_{\text{per}}))^\perp.$$

which concludes the proof. □

In the same way, we can define a map ψ_n for every n between $\Omega^0(\tilde{G}_n)$ and $\Omega^0_{\text{per}}(G_n)$ where $\Omega^0_{\text{per}}(G_n)$ is the subspace of $\Omega^0(G)$ whose functions have support in $V(G_n)$ and such that for every $v_1, v_2 \in V(\partial G_n)$, such that $\tilde{v}_1 = \tilde{v}_2 \in \tilde{G}_n$, $f(v_1) = f(v_2)$.

For every $f \in \Omega^0(\tilde{G}_n)$, we define for every vertex v,

$$\psi_n(f)(v) = \begin{cases} f(\tilde{v}) & \text{if } v \in G_n\setminus\partial G_n \\ \frac{f(\tilde{v})}{\sqrt{n(v)}} & \text{if } v \in \partial G_n \end{cases}$$

where $n(v)$ is defined by $n(v) = |\{v' \in \partial G_n | \tilde{v'} = \tilde{v}\}|$. This map is an isometry from $\Omega^0(\tilde{G}_n)$ to $\Omega^0_{\text{per}}(G_n)$.

3.3.3 Comparison with free and wired measures

In this part, we will establish some stochastic inequalities between the measure with periodic boundary conditions and measures with free and wired conditions.

Lemma 3.11. *As subspaces of $\Omega^1(G_n)$, we have*

$$\bigstar^W_n := d_n(\iota_{n-1}(\Omega^0(G_{n-1}))) \subset \varphi_n(\bigstar^{per}_n).$$

Proof. Since ψ_n is a bijective isometry, we have

$$\iota_{n-1}\Omega^0(G_{n-1}) \subset \Omega^0_{\text{per}}(G_n) = \psi_n(\Omega^0(\tilde{G}_n)).$$

Since ψ_n^{-1} is the identity on $\iota_{n-1}\Omega^0(G_{n-1})$ and φ_n is the identity on $\Omega^1_{|\partial G_n=0}(\tilde{G}_n)$ and since $\iota_{n-1}\Omega^0(G_{n-1}) \subset \psi_n(\Omega^0(\tilde{G}_n))$, we get the following equality on $\iota_{n-1}\Omega^0(G_{n-1})$,

$$\varphi_n \circ \tilde{d}_n \circ \psi_n^{-1} = d_n.$$

As subspaces of $\Omega^1(G_n)$, we have

$$\bigstar_n^W := d_n(\iota_{n-1}(\Omega^0(G_{n-1}))) \subset \varphi_n \circ \tilde{d}_n \circ \psi_n^{-1}(\iota_{n-1}(\Omega^0(G_{n-1}))) \subset \varphi_n(\tilde{d}_n(\Omega^0(\tilde{G}_n))),$$

which is precisely the inclusion we wanted to prove. □

Denote by $\diamondsuit_n^{\text{per}}$ the orthogonal of $\bigstar_{\text{per}}^n$ for the inner product on $\Omega^1(\tilde{G}_n)$. Since φ_n is an isometry, the orthogonal of $\varphi_n(\bigstar_{\text{per}}^n)$ for the inner product on $\Omega^1(G_n)$ which is the restriction of the inner product on $\Omega^1(G)$ is $\varphi_n(\diamondsuit_n^{\text{per}})$.

Since $\tilde{d}_n$ and $\tilde{d}_n^*$ are adjoints for the inner product on $\Omega^1(\tilde{G}_n)$, we have

$$\diamondsuit_n^{\text{per}} = \ker \tilde{d}_n^* = \ker \tilde{K}_n.$$

Lemma 3.12. *As subspaces of $\Omega^1(G_n)$, we have*

$$\diamondsuit_n = \varepsilon_{n-1}(\ker(d_{n-1}^*)) \subset \varphi_n(\diamondsuit_n^{per}).$$

Proof. Let $\theta \in \diamondsuit_n$. First of all, notice that $\theta \in \Omega^1_{\text{per}}(G_n)$ since $\theta \in im(\varepsilon_{n-1})$. Since $\theta \in \diamondsuit_n$, we have $\theta = \varepsilon_{n-1}\theta_0$ with $\theta_0 \in \ker(d_{n-1}^*)$. Then, if $x \in G_{n-1}$, we have

$$d_n^*(\theta)(x) = d_{n-1}^*(\theta_0)(x) = 0,$$

by definition of θ_0. If $x \notin G_{n-1}$,

$$d_n^*(\theta)(x) = 0$$

since $\varepsilon_n\theta(e) = 0 \quad \forall e \in G_n \backslash G_{n-1}$.

Moreover, $\varphi_n^{-1}(\theta) \in \Omega^1(\tilde{G}_n)$ and for every e, such that $e^{\pm} \in \partial G_n$,

$$\varphi_n^{-1}(\theta)(\tilde{e}) = 0.$$

Then, we have

$$\tilde{d}_n^*(\varphi_n^{-1}(\theta)) = d_n^*(\theta) = 0.$$

Therefore, $\theta \in \varphi_n(\diamondsuit_n^{\text{per}})$. □

Those inclusion of sets imply the following stochastic inequalities on the measures in finite volume with boundary conditions.

Proposition 3.13. *For every finite subset $\Lambda \in E(G)$, we have for n large enough, the following inequalities which hold for every $R = \oplus_{e\in\Lambda} R_e$,*

$$\rho_n^W(R) \leq \rho_n^P(R) \leq \rho_n^F(R).$$

where ρ_n^W, ρ_n^P and ρ_n^F are the respective densities of the incidence measures associated with the measures μ_n^W, μ_n^P and μ_n^F, as defined in Subsection 2.3.2.

Proof. Let $\Lambda \in E(G)$ be a finite set of edges. Let n be large enough such that $\Lambda \subset G_{n-1}$.

Let θ be a 1-form with compact support in Λ. Let us denote by $\tilde{\theta}_n$, the 1-form in $\Omega^1(\tilde{G}_n)$ such that $\varphi_n^{-1}(\theta) = \tilde{\theta}_n$. For every $e \in G_{n-1}$,

$$\varphi_n \circ \tilde{K}_n \circ \varphi_n^{-1}(\theta)(e) = \tilde{K}_n \tilde{\theta}_n(\tilde{e}).$$

In particular, if $e_1, \dots, e_k \in \Lambda$, and if $l \in [1, N]$, denoting by $\theta^l_{e_i} = (1_{e_i^+} - 1_{e_i^-})b_l$, where $(b_l)_{l\in[1,N]}$ is the canonical basis of $\mathbb{C}^N$, we have :

$$\det\left(\left(\tilde{K}_n \tilde{\theta}^l_{e_i}(\tilde{e}_j)\right)_{i,j\in[1,k],l\in[1,N]}\right) = \det\left(\left(\varphi_n \circ \tilde{K}_n \circ \varphi_n^{-1}(\theta^l_{e_i})(e_j)\right)_{i,j\in[1,k],l\in[1,N]}\right).$$

Since $\varphi_n \circ \tilde{K}_n \circ \varphi_n^{-1}$ is the orthogonal projection on $\varphi_n(\bigstar_n^{\mathrm{per}})$ parallel to $\varphi_n(\diamondsuit_n^{\mathrm{per}})$ for the inner product on $\Omega^1(G)$, Lemma 3.11 and Lemma 3.12 show that

$$\begin{aligned}\det\left(\left(K_n^W \theta^l_{e_i}(e_j)\right)_{i,j\in[1,k],l\in[1,N]}\right) &\leq \det\left(\left(\tilde{K}_n \tilde{\theta}^l_{e_i}(\tilde{e}_j)\right)_{i,j\in[1,k],l\in[1,N]}\right)\\ &\leq \det\left(\left(K_n^F \theta^l_{e_i}(e_j)\right)_{i,j\in[1,k],l\in[1,N]}\right).\end{aligned}$$

From subsection 3.2.1, we deduce that for every $R = \oplus_{e\in\Lambda} R_e$,

$$\rho_n^W(R) \leq \rho_n^P(R) \leq \rho_n^F(R),$$

which is the desired statement. □

For the case of a connection on a rank 1 vector-bundle, this statement can be written in an easier way: for every $e_1, \dots, e_k \in E(G)$, then for n large enough,

$$\mu_n^W(e_1, \dots, e_k \in F) \leq \mu_n^P(e_1, \dots, e_k \in F) \leq \mu_n^F(e_1, \dots, e_k \in F).$$

We immediately deduce the following result.

Corollary 3.14. *If (μ_n^W) and (μ_n^F) converge weakly to the same limit μ, then (μ_n^P) also converges weakly to this limit μ.*

Recall that the sequences of measures (μ_n^W), (μ_n^F) on QSF on the sequence of growing subgraphs (G_n) with wired or free boundary conditions converge towards thermodynamic limits μ^F and μ^W, which are determinantal measures whose kernels are the orthogonal projections on spaces $\diamondsuit_{\ell^2}^{\perp}$ and $\bigstar_{\ell^2}$.

We will see later (Lemma 6.18) that under an assumption on the connection (6.17), the spaces $\diamondsuit_{\ell^2}^{\perp}$ and $\bigstar_{\ell^2}$ are equal and the measures μ^W and μ^F are equal. A consequence of Corollary 3.14 is that under this assumption, the sequences of measures (μ_n^P), (μ_n^W) and (μ_n^F) converge weakly to the same limit μ.

Chapter 4

Periodic thermodynamic limits.

In this chapter, we study a particular case of the model of the so-called quantum spanning forests in infinite volume, as defined in [KL20c] and section 2.3. We construct limit measures on quantum spanning forests on a $\mathbb{Z}^d$-periodic graph by taking the limit of measures on an exhaustion by finite graphs on growing tori of the form $\mathbb{Z}^d/(n\mathbb{Z})^d$, with n tending to infinity.

In [BP93], authors study the limit of measures on uniform spanning trees on an exhaustion of $\mathbb{Z}^d$ by finite graphs on growing tori of the form $\mathbb{Z}^d/(n\mathbb{Z})^d$, with n tending to infinity and prove the following transfer current formula for the kernel:

$$K_{e,e'} = \int_{|z|=1,|w|=1} K_{[e],[e']}(z,w) z^{x_e - x_{e'}} w^{y_e - y_{e'}} \frac{dz}{2i\pi z} \frac{dw}{2i\pi w}. \tag{4.1}$$

where $K(z,w)$ is a matrix indexed by the edges of the fundamental domain G_1, called the transfer current operator and $e = [e] + (x_e, y_e)$ where $[e]$ denotes the image of e in the fundamental domain G_1.

We prove in this chapter some similar formulas for the kernel of a quantum spanning forest associated to a unitary periodic connection on a vector-bundle of rank N.

When the fundamental domain of the graph is of size 1 and the connection is periodic, the model depends on a deterministic choice of unitary matrices $(M_1, \dots, M_d) \in U_N(\mathbb{C})^d$ on the half-edges of $\mathbb{Z}^d$, which defines a so-called unitary connection of rank N on the complex trivial bundle of dimension N over the graph $\mathbb{Z}^d$. We identify two phases in this model, according to the decay of correlations with the distance. These two phases are characterized by the existence or not of a common eigenvector to the matrices $(M_1, \dots, M_d) \in U_N(\mathbb{C})^d$. The existence of such a vector makes the connection reducible and the model becomes a superposition of independent models of inferior rank.

4.1 Quantum spanning forests on growing tori

In this section, we consider a graph G embedded in $\mathbb{R}^d$ which is $\mathbb{Z}^d$-periodic endowed with a connection h which is $\mathbb{Z}^d$-periodic, in the sense that there exists a lattice in $\mathbb{R}^d$ such that the graph G and the connection h are invariant under the action of the lattice.

Let us consider an exhaustion of G by a sequence of finite subgraphs (G_n) such that for every n, the graph G_n contains n^d copies of the fundamental domain G_1. We denote

by $(\tilde{G}_n)$ the subgraphs which are obtained by gluing opposite boundaries on a torus, and by $(\tilde{K}_{h,n})$ the kernels of the measures (μ_n^P) associated to the connection h and periodic boundary conditions.

We will prove the following theorem for quantum spanning forest, which was proved by [Ken11] for CRSF for a non necessarily unitary complex-valued connection of rank 1 on planar graphs.

Theorem 4.1. *When $n \to \infty$, we have for every $e, e' \in E$,*

$$(\tilde{K}_{h,n})_{e,e'} \to \int_{|z_1|=1,\dots,|z_d|=1} \tilde{K}_{h[e],[e']}(z_1,\dots,z_d) z_1^{x_1-y_1} \dots z_d^{x_d-y_d} \frac{dz_1}{2i\pi z_1} \dots \frac{dz_d}{2i\pi z_d}$$

where $\tilde{K}_{h[e],[e']}(z_1,\dots,z_d)$ is the orthogonal projection on $d_{h(z_1,\dots,z_d)}(\Omega^0(\tilde{G}_1))$ for a new connection $h(z_1,\dots,z_d)$ which is obtained from the connection h by multiplying the connection on edges between two copies of the fundamental domain in direction j by z_j.

From section 3.3 and subsection 3.2.1, this theorem will imply that the sequence of measures (μ_n^P) on QSF of $\tilde{G}_n$, associated to the sequence of kernels $(\tilde{K}_{h,n})$ converges weakly, when $n \to \infty$, towards a determinantal measure on QSF of G, whose kernel is given by the integral expression of Theorem 4.1. Finally, from Corollary 3.14, in the case where the free measure and the wired measure coincide in infinite volume, the kernel of the limit measure will be given by this integral expression.

4.1.1 Fourier decomposition of 1-forms

We introduce a periodic decomposition of the spaces $\Omega^0(\tilde{G}_n)$, $\Omega^1(\tilde{G}_n)$ in subsets which are invariant under the action of the Laplacian and the orthogonal projection on $im(d_{h,n})$.

We denote by $[x]$ or $[e]$ the unique vertex or edge translated from x or e which is in G_1. For every j, we also denote by t_j the vector such that for every x, $x + t_j$ is the translated of x in direction j.

Definition 4.2. For $(z_1,\dots,z_d) \in U_n^d$, let $E_{z_1,\dots,z_d} \subset \Omega^0(\tilde{G}_n)$ and $F_{z_1,\dots,z_d} \subset \Omega^1(\tilde{G}_n)$ be the spaces of 0-forms and 1-forms on $\tilde{G}_n$, respectively, which satisfy the following periodicity condition, for every $x \in G_n$ and every $e \in E(G_n)$,

$$\begin{cases} f(x+t_j) = z_j f(x) & \forall\, f \in E_{z_1,\dots,z_d} \\ \theta(e+t_j) = z_j \theta(e) & \forall\, \theta \in F_{z_1,\dots,z_d} \end{cases}$$

Lemma 4.3. *We have the following decomposition*

$$\Omega^0(\tilde{G}_n) = \bigoplus_{(z_1,\dots,z_d)\in U_n^d} E_{z_1,\dots,z_d}$$

$$\Omega^1(\tilde{G}_n) = \bigoplus_{(z_1,\dots,z_d)\in U_n^d} F_{z_1,\dots,z_d}$$

Moreover for $\tilde{d}_{h,n}$ and $\tilde{d}^\star_{h,n}$ which are defined as in section 2.2, we have the following inclusions:

$$\begin{cases} \tilde{d}_{h,n}(E_{z_1,\dots,z_d}) \subset F_{z_1,\dots,z_d}. \\ \tilde{d}^\star_{h,n}(F_{z_1,\dots,z_d}) \subset E_{z_1,\dots,z_d}. \end{cases}$$

Proof. Let $(\lambda_{z_1,\dots,z_d})_{z_1,\dots,z_d\in U_n} \in \mathbb{C}^{n^d}$ and $(f_{z_1,\dots,z_d})_{z_1,\dots,z_d\in U_n} \in \Omega^0(\tilde{G}_n)^{n^d}$ be such that for every $z_1,\dots,z_d \in \mathbb{U}_n$, $f_{z_1,\dots,z_d} \in E_{z_1,\dots,z_d}$ and we have

$$\sum_{z_1,\dots,z_d\in U_n} \lambda_{z_1,\dots,z_d} f_{z_1,\dots,z_d} = 0.$$

Let $x \in G_1$. Then $\forall\, (\alpha_1,\dots\alpha_d) \in [0,n]^d$,

$$\sum_{z_1,\dots,z_d\in U_n} \lambda_{z_1,\dots,z_d} f_{z_1,\dots,z_d}(x) z_1^{\alpha_1}\dots z_d^{\alpha_d} = 0.$$

Let $(\alpha_1,\dots\alpha_{d-1}) \in [0,n]^{d-1}$ and $\alpha_d \in [0,n]$,

$$\sum_{z_d\in U_n} \left(\sum_{z_1,\dots,z_{d-1}\in U_n} \lambda_{z_1,\dots,z_d} f_{z_1,\dots,z_d}(x) z_1^{\alpha_1}\dots z_{d-1}^{\alpha_{d-1}} \right) z_d^{\alpha_d} = 0.$$

Define $\chi = e^{2i\pi/n}$. Then $U_n = \{\chi^l,\ l \in [0,n-1]\}$. For all $\alpha_d \in [0,n]$,

$$\sum_{l\in[0,n-1]} \left(\sum_{z_1,\dots,z_{d-1}\in U_n} \lambda_{z_1,\dots,\chi^l} f_{z_1,\dots,\chi^l}(x) z_1^{\alpha_1}\dots z_{d-1}^{\alpha_{d-1}} \right) (\chi^{\alpha_d})^l = 0.$$

We get a polynomial of degree $n-1$ which takes n roots thus which is equal to the nul-polynomial. Thus, for all $(\alpha_1,\dots\alpha_{d-1}) \in [0,n]^{d-1}$, and $z_d \in U_n$,

$$\sum_{z_1,\dots,z_{d-1}\in U_n} \lambda_{z_1,\dots,z_d} f_{z_1,\dots,z_d}(x) z_1^{\alpha_1}\dots z_{d-1}^{\alpha_{d-1}} = 0.$$

By recurrence, we get for all $z_1,\dots,z_d \in U_n$,

$$\lambda_{z_1,\dots,z_d} f_{z_1,\dots,z_d}(x) = 0.$$

Since it is true for every $x \in G_1$, if $(z_1,\dots,z_d) \in U_n^d$ is such that $f_{z_1,\dots,z_d} \neq 0$, evaluating in $x \in G_1$ such that $f_{z_1,\dots,z_d}(x) \neq 0$, we have

$$\lambda_{z_1,\dots,z_d} = 0.$$

The equality of spaces comes from the equality of dimension since the spaces of 0-forms and of 1-forms on $\tilde{G}_n$ are of dimension $Nn^d|G_1|$ and $Nn^d|E(G_1)|$.

Indeed, 1-forms of $F_{z_1,\dots,z_d}$ are uniquely determined by images of edges of G_1 and 0-forms of $E_{z_1,\dots,z_d}$ are uniquely determined by images of vertices of G_1, thus $F_{z_1,\dots,z_d}$ is of dimension $N|E(G_1)|$ and $E_{z_1,\dots,z_d}$ is of dimension $N|G_1|$.

Let $f \in E_{z_1,\dots,z_d}$.

$$\begin{aligned}\tilde{d}_{h,n} f(e+t_j) &= h_{ee^-} f(e^- + t_j) - h_{ee^+} f(e^+ + t_j) = (h_{ee^-} f(e^-) - h_{ee^+} f(e^+)) z_j \\ &= \tilde{d}_{h,n} f(e) z_j.\end{aligned}$$

The proof is the same for the other inclusion. □

For $z = (z_1,\dots,z_d)$, we denote by $\tilde{d}_{h,n}(z) : E_{z_1^{-1},\dots,z_d^{-1}} \to F_{z_1^{-1},\dots,z_d^{-1}}$ the compression of $\tilde{d}_{h,n}$ and by $\tilde{d}^\star_{h,n}(1/z) : F_{z_1^{-1},\dots,z_d^{-1}} \to E_{z_1^{-1},\dots,z_d^{-1}}$ the compression of $\tilde{d}^\star_{h,n}$. The operators $\tilde{d}_{h,n}(z)$ and $\tilde{d}^\star_{h,n}(1/z)$ are adjoint as compressions of $\tilde{d}_{h,n}, \tilde{d}^*_{h,n}$.

In the next subsection, we explain why those notations are well-chosen in the case of a fundamental domain of size 1 (Lemma 4.4).

Case of a fundamental domain of size 1. We consider in this section quantum spanning forests on $G = \mathbb{Z}^d$ and the exhaustion by finite graphs $G_m = [-m, m]^d$ on growing tori, gluing opposite boundaries. We fix $m \in \mathbb{N}$ and we define $n = 2m$. Then, the graph G_m seen on a torus by gluing opposite boundaries is isomorphic to the finite graph $\tilde{G}_n := G/(n\mathbb{Z})^d$.

In the following, we denote by $(b_j)_{1\leq j\leq d}$ the canonical basis of $\mathbb{C}^d$, which is defined by the following formula $b_j = (0, \dots, 0, 1, 0, \dots, 0) \in \mathbb{C}^d$ and $(b'_k)_{1\leq k\leq N}$ the canonical basis of $\mathbb{C}^N$, that is to say $b'_k = (0, \dots, 0, 1, 0, \dots, 0) \in \mathbb{C}^N$.

Assume that the graph $\tilde{G}_n = G/(n\mathbb{Z})^d$ is endowed with a fiber bundle of rank N and a connection h which is periodic in the sense that there exists $(M_1, \dots, M_d) \in U_N(\mathbb{C})^d$ a set of deterministic matrices such that if we consider

$$E_+ = \{xy | \exists j \in [1, d], y - x = b_j\}$$

a set of oriented edges of E, then, if $e \in E_+$, $j \in [1, d]$ such that $e = (x, y)$ with $y - x = b_j$, we have in the canonical basis of $\mathbb{C}^N$,

$$h_{xe} = M_j, \quad h_{ey} = I_N.$$

$$h_{ex} = M_j^{-1}, \quad h_{ye} = I_N.$$

We denote by $e_{j,+}$ the edges of E_+ whose end is $0_{\mathbb{Z}^d}$, that is to say the edges $(-b_j, 0)$. Since the fundamental domain is of size 1, then the 0-forms are entirely determined by the image of 0 and the 1-forms are entirely determined by the images of $(e_{j,+})$. Notice that the family of 1-forms $(1_{e_{j,+}} - 1_{-e_{j,+}})b'_k$ for $j \in [1, d], k \in [1, N]$ is a orthonormal basis of $\Omega^1(G_1)$.

Lemma 4.4. *We still denote by*

$$\begin{cases} \tilde{d}_{h,n}(z) : E_{z_1^{-1},\dots,z_d^{-1}} \to F_{z_1^{-1},\dots,z_d^{-1}} \\ \tilde{d}^\star_{h,n}(1/z) : F_{z_1^{-1},\dots,z_d^{-1}} \to E_{z_1^{-1},\dots,z_d^{-1}} \end{cases}$$

compressions of $\tilde{d}_{h,n}$ and $\tilde{d}^\star_{h,n}$. Then, we can write in bases $(1_0 b'_k)_k$ and $((1_{e_{j,+}} - 1_{-e_{j,+}})b'_k)_{j,k}$,

$$\tilde{d}_{h,n}(z) = \begin{pmatrix} I_N - z_1 M_1 \\ \dots \\ I_N - z_d M_d \end{pmatrix}$$

and

$$\tilde{d}^*_{h,n}(1/z) = \begin{pmatrix} I_N - {z_1}^{-1} M_1^{-1} & \dots & I_N - {z_d}^{-1} M_d^{-1} \end{pmatrix}.$$

Recall that $\tilde{d}_{h,n}(z)$ and $\tilde{d}^\star_{h,n}(1/z)$ are adjoint as compressions of adjoint operators. Their expressions in bases show that their matrices are transposed conjugate of each other.

Proof. If $e \in E_+$, and $j \in [1, d]$ is such that $e = xy$ with $y - x = b_j$,

$$\tilde{d}_{h,n} f(e) = f(e^+) - M_j f(e^-).$$

In particular, if $f \in E_{(z_1,\dots,z_d)}$,

$$\tilde{d}_{h,n} f(e) = (I_N - z_j^{-1} M_j) f(e^+).$$

For all $j \in [1,d]$, there are two edges $e_{j,+}$ et $e_{j,-}$ whose final extremity is 0, one in E_+, the other in E_-. Since $-e_{j,-} = e_{j,+} + b_j$, we have

$$\tilde{d}^*_{h,n}\theta(x) = \sum_{j\in[1,d]} (h_{e_{j,+},x}\theta(e_{j,+}) + h_{e_{j,-},x}\theta(e_{j,-})) = \sum_{j\in[1,d]} (\theta(e_{j,+}) - M_j^{-1}\theta(-e_{j,-}))$$

In particular, if $\theta \in F_{z_1,\dots,z_d}$, $\tilde{d}^*_{h,n}\theta(0) = \sum_{j\in[1,d]}(I_N - z_j M_j^{-1})\theta(e_{j,+})$.

Finally, if $f \in E_{z_1,\dots,z_d}$, $f = \Big(f(0)\Big)$,

$$\tilde{d}_{h,n}f = \begin{pmatrix} (\tilde{d}_{h,n}f)(e_{1,+}) \\ \dots \\ (\tilde{d}_{h,n}f)(e_{d,+}) \end{pmatrix} = \begin{pmatrix} f(0) - z_1^{-1}M_1 f(0) \\ \dots \\ f(0) - z_d^{-1}M_d f(0) \end{pmatrix}$$

On the same way, if $\theta \in F_{z_1,\dots,z_d}$, $\theta = \begin{pmatrix} \theta(e_{1,+}) \\ \dots \\ \theta(e_{d,+}) \end{pmatrix}$,

$$\tilde{d}^*_{h,n}\theta(0) = \sum_{j\in[1,d]} (I_N - z_j M_j^{-1})\theta(e_{j,+}),$$

which is exactly what we wanted to prove. □

This Lemma justifies the notations $\tilde{d}_{h,n}(z)$ and $\tilde{d}^*_{h,n}(1/z)$. Let us emphasize that those operators have the same matrices as the operators $\tilde{d}_{h',1}$ and $\tilde{d}^*_{h',1}$ on $\Omega^1(\tilde{G}_1)$ where the connection h has been replaced by a connection h' defined by $(z_1M_1, \dots, z_dM_d)$.

In the following subsection, we generalize this result to the case of a larger fundamental domain.

On a larger fundamental domain. In the remainder of this section, assume that the fundamental domain can be of size larger than 1. The following result hold for any periodic connection h which is determined by its values between fibers over the fundamental domain. Notice that h is not given anymore by unitary matrices $(M_1, \dots, M_d)$.

We notice that as well as in the case of a fundamental domain of size 1, 0-forms of $E_{z_1^{-1},\dots,z_d^{-1}}$ are determined by their values on G_1 and $(z_1^{-1}, \dots, z_d^{-1})$-periodic 1-forms of $F_{z_1^{-1},\dots,z_d^{-1}}$ are determined by their values on edges of $\tilde{G}_1$ and their values on $\tilde{G}_n$ are given by multiplying the value on a vertex or an edge in the fundamental domain by z_j^{-1} for each translation of the graph in direction j.

Recall that we denote by $[x]$ or $[e]$ the unique vertex or edge translated from x or e which is in G_1. For every j, we also denote by t_j the vector such that for every x, $x + t_j$ is the translated of x in direction j.

A slight modification of the proof of Lemma 4.4 implies the following lemma.

Lemma 4.5. *Identifying $E_{z_1^{-1},\dots,z_d^{-1}}$ with $\Omega^0(\tilde{G}_1)$ and $F_{z_1^{-1},\dots,z_d^{-1}}$ with $\Omega^1(\tilde{G}_1)$, the operators $\tilde{d}_{h,n}(z)$ and $\tilde{d}^*_{h,n}(1/z)$ have the same action as the operators*

$$\begin{cases} \tilde{d}_{h(z_1,\dots,z_d)} : \Omega^0(\tilde{G}_1) \to \Omega^1(\tilde{G}_1) \\ \tilde{d}^*_{h(z_1,\dots,z_d)} : \Omega^1(\tilde{G}_1) \to \Omega^0(\tilde{G}_1) \end{cases}$$

for a new connection $h(z_1, \dots, z_d)$ on $\tilde{G}_1$ which is obtained from the connection h multiplying the connection by z_j on half-edges $([x], [e]) \in \tilde{G}_1$ for oriented edges $e = (x, y) \in G_1$ such that,

$$\begin{cases} [y] = y, \\ [x] = x + t_j. \end{cases}$$

In other words, we multiply the connection by z_j on half-edges $([x], [e]) \in \tilde{G}_1$ for oriented edges in G_1 which "enter in the graph G_1" in direction j.

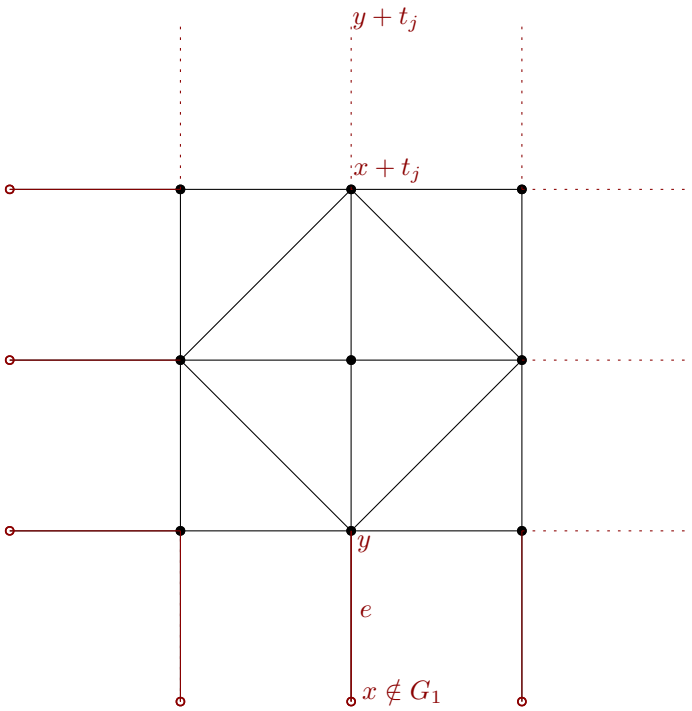

Figure 4.1 – G_1 contains all solid edges and all full vertices. Once the boundaries are glued on the torus, red vertices are glued to corresponding black vertices and the connection is multiplied by z_j on red edges.

This lemma means that if for every $f \in E_{z_1^{-1},\dots,z_d^{-1}}$, we denote by $[f]$ the corresponding 0-form in $\Omega^0(\tilde{G}_1)$ and for every $\theta \in F_{z_1^{-1},\dots,z_d^{-1}}$, we denote by $[\theta]$ the corresponding 1-form in $\Omega^1(\tilde{G}_1)$, then we have for every $f \in E_{z_1^{-1},\dots,z_d^{-1}}, \theta \in F_{z_1^{-1},\dots,z_d^{-1}}$,

$$\begin{cases} [\tilde{d}_{h,n} f] = \tilde{d}_{h(z_1,\dots,z_d)}[f], \\ [\tilde{d}^*_{h,n} \theta] = \tilde{d}^*_{h(z_1,\dots,z_d)}[\theta]. \end{cases}$$

Proof. Let e be such an edge and $f \in E_{z_1^{-1},\dots,z_d^{-1}}$. Then,

$$\tilde{d}_{h,n} f(e) = h_{y,e} f(y) - h_{x,e} f(x) = h_{y,e} f([y]) - h_{x,e} f([x] - t_j) = h_{y,e} f([y]) - z_j h_{x,e} f([x]).$$

Let $x \in G_1$ such that x is on the boundary of G_1. Assume that there exists an oriented edge $e = (x, y) \notin G_1$, with $y = [y] + t_j$. Then, $e = [e] + t_j$. Then, if $\theta \in F_{z_1^{-1},\dots,z_d^{-1}}$,

$$\begin{aligned}\tilde{d}^*_{h,n}\theta(x) &= \left(\sum_{e' \neq -e : e'^+ = x} h_{e',x}\theta(e')\right) + h_{-e,x}\theta(-e) \\ &= \left(\sum_{e' \neq -e : e'^+ = x} h_{e',x}\theta(e')\right) + z_j^{-1}h_{x,e}^{-1}\theta([-e]) \\ &= \left(\sum_{e' \neq -e : e'^+ = x} h_{e',x}\theta(e')\right) + (z_j h_{x,e})^{-1}\theta([-e]).\end{aligned}$$

If another edge $e' \neq -e$ is such that $e'^+ = x$ and $e' \in \tilde{G}_1 \setminus G_1$, replacing e' by $[e']$ also multiplies the connection by z_j depending on the direction of edge e'. It finishes the proof. □

4.1.2 Diagonalisation of the Laplacian operator

The Laplacian operator plays a fundamental role in the model because its reduced determinant gives the partition function of the model. We recall the definition of the Laplacian, twisted by a connection h:

$$\begin{aligned}\tilde{\Delta}_{h,n} : \Omega^0(\tilde{G}_n) &\to \Omega^0(\tilde{G}_n) \\ f &\mapsto \tilde{d}^\star_{h,n}\tilde{d}_{h,n}f.\end{aligned}$$

The following lemma shows that the Laplacian on G_n for a periodic connection can be block diagonalized and therefore, its reduced determinant can be computed as a product of determinants of Laplacians compressed on smaller spaces of periodic forms. As for the operators $\tilde{d}_{h,n}$ and $\tilde{d}^*_{h,n}$, the action of the Laplacian on those spaces is in correspondence with the action of Laplacian operators on the fundamental domain on the torus $\tilde{G}_1$ associated to a modified connection.

Lemma 4.6. *Each space $E_{z_1^{-1},\dots,z_d^{-1}}$ is invariant under the action of the Laplacian $\tilde{\Delta}_{h,n}$ and if $f \in E_{z_1^{-1},\dots,z_d^{-1}}$, we have*

$$\tilde{\Delta}_{h,n}f = \tilde{\Delta}_{h,n}(z)f,$$

where $\tilde{\Delta}_{h,n}(z) = \tilde{d}^\star_{h,n}(1/z)\tilde{d}_{h,n}(z)$ is the compression of the Laplacian on the invariant subspace $E_{z_1^{-1},\dots,z_d^{-1}}$. Moreover, we have the following decompositions:

$$\begin{cases}\ker(\tilde{d}_{h,n}) = \bigoplus \ker(\tilde{d}_{h,n}) \cap E_{z_1,\dots,z_d}, \\ \ker(\tilde{d}_{h,n})^\perp = \bigoplus \ker(\tilde{d}_{h,n})^\perp \cap E_{z_1,\dots,z_d}.\end{cases}$$

Then, the Laplacian compressed on the orthogonal of its kernel can be block-diagonalized and its reduced determinant is given by

$$\det{}_0(\tilde{\Delta}_{h,n}) = \prod_{\xi_1^n=1,\dots,\xi_d^n=1} \det{}_0(\tilde{\Delta}_{h,n}(\xi_1,\dots,\xi_d)).$$

Proof. The first statement is an immediate consequence of Lemma 4.3.

Let $f = \sum_{z_1,\dots,z_d} f_{z_1,\dots,z_d}$ with $f_{z_1,\dots,z_d} \in E_{z_1,\dots,z_d}$ be a 0-form of $\Omega^0(\tilde{G}_n)$ and assume that $f \in \ker(\tilde{d}_{h,n})$. Then we have the equality $0 = \tilde{d}_{h,n}f = \sum_{z_1,\dots,z_d} \tilde{d}_{h,n}(f_{z_1,\dots,z_d})$ with $\tilde{d}_{h,n}(f_{z_1,\dots,z_d}) \in F_{z_1,\dots,z_d}$. Since the sum

$$\Omega^1(\tilde{G}_n) = \bigoplus F_{z_1,\dots,z_d}$$

is a direct one, we have for all $(z_1,\dots,z_d)$, $\tilde{d}_{h,n}(f_{z_1,\dots,z_d}) = 0$. Thus for all $(z_1,\dots,z_d)$, we deduce that $f_{z_1,\dots,z_d} \in \ker(\tilde{d}_{h,n}) \cap E_{z_1,\dots,z_d}$. Therefore, we deduce the following decomposition

$$\ker(\tilde{d}_{h,n}) = \bigoplus \ker(d_{h,n}) \cap E_{z_1,\dots,z_d}.$$

Since the graph $\tilde{G}_n$ is finite, we have $\ker(\tilde{d}_{h,n})^\perp = im(\tilde{d}^*_{h,n})$. Therefore,

$$\ker(\tilde{d}_{h,n})^\perp = im(\tilde{d}^*_{h,n}) = \tilde{d}^*_{h,n}(\Omega^1) = \tilde{d}^*_{h,n}(\bigoplus F_{z_1,\dots,z_d}).$$

For all $(z_1,\dots,z_d)$, we know that $\tilde{d}^*_{h,n}(F_{z_1,\dots,z_d}) \subset im(\tilde{d}^*_{h,n}) \cap E_{z_1,\dots,z_d}$. Since the sum

$$\Omega^0(\tilde{G}_n) = \bigoplus E_{z_1,\dots,z_d}$$

is a direct one, the sum of the spaces $im(\tilde{d}^*_{h,n}) \cap E_{z_1,\dots,z_d}$ is also a direct sum and we have

$$\begin{aligned}\ker(\tilde{d}_{h,n})^\perp &= \tilde{d}^*_{h,n}(\bigoplus F_{z_1,\dots,z_d}) \\ &\subset \bigoplus im(\tilde{d}^*_{h,n}) \cap E_{z_1,\dots,z_d} \\ &= \bigoplus \ker(\tilde{d}_{h,n})^\perp \cap E_{z_1,\dots,z_d}\end{aligned}$$

and the other inclusion holds since $\forall (z_1,\dots,z_d)$, $\ker(\tilde{d}_{h,n})^\perp \cap E_{z_1,\dots,z_d} \subset \ker(\tilde{d}_{h,n})^\perp$. Both decompositions prove that the Laplacian restricted to the orthogonal of its kernel can be block-diagonalized. □

From Lemma 4.5, identifying $E_{z_1^{-1},\dots,z_d^{-1}}$ with $\Omega^0(\tilde{G}_1)$, the compression $\tilde{\Delta}_{h,n}(z_1,\dots,z_d)$ of the operator $\tilde{\Delta}_{h,n}$ on $E_{z_1^{-1},\dots,z_d^{-1}}$ has the same action as the Laplacian operator

$$\tilde{\Delta}_{h(z_1,\dots,z_d)} : \Omega^0(\tilde{G}_1) \to \Omega^0(\tilde{G}_1),$$

in the sense that if for every $f \in E_{z_1^{-1},\dots,z_d^{-1}}$, we denote by $[f]$ the corresponding 0-form in $\Omega^0(\tilde{G}_1)$, then we have for every $f \in E_{z_1^{-1},\dots,z_d^{-1}}$,

$$[\tilde{\Delta}_{h,n}f] = \tilde{\Delta}_{h(z_1,\dots,z_d)}[f].$$

Let $\tilde{\mathcal{G}}_{h,n}$ be the inverse of the Laplacian on the orthogonal complement to $\ker(\tilde{d}_{h,n})$, also called the *Green function* of the model. From Lemma 4.6, each space $E_{z_1^{-1},\dots,z_d^{-1}}$ is invariant under the action of $\tilde{\mathcal{G}}_{h,n}$ and the compression $\tilde{\mathcal{G}}_{h,n}(z_1,\dots,z_d)$ of the operator $\tilde{\mathcal{G}}_{h,n}$ on $E_{z_1^{-1},\dots,z_d^{-1}}$, has the same action than the Green function :

$$\tilde{\mathcal{G}}_{h(z_1,\dots,z_d)} : \Omega^0(\tilde{G}_1) \to \Omega^0(\tilde{G}_1).$$

In the following, we compute the kernel of the quantum spanning forest on $\tilde{G}_n$.

4.1.3 Kernel of the process and characteristic polynomial

The kernel of the quantum spanning forest on $\tilde{G}_n$ associated to the connection h is the operator of orthogonal projection on $im(\tilde{d}_{h,n})$. This operator may be written in the form:

$$\begin{aligned}\tilde{K}_{h,n} : \Omega^1(\tilde{G}_n) &\to \Omega^1(\tilde{G}_n)\\ \theta &\mapsto \tilde{d}_{h,n}\tilde{\mathcal{G}}_{h,n}\tilde{d}^{\star}_{h,n}\theta\end{aligned}$$

Lemma 4.3 and Lemma 4.6 imply that each space $F_{z_1^{-1},\dots,z_d^{-1}}$ is invariant under the action of $\tilde{K}_{h,n}$ and if $\theta \in F_{z_1^{-1},\dots,z_d^{-1}}$, then

$$\tilde{K}_{h,n}\theta = \tilde{K}_{h,n}(z)\theta$$

where for $z = (z_1, \dots, z_d)$, $\tilde{K}_{h,n}(z) = \tilde{d}_{h,n}(z)\tilde{\mathcal{G}}_{h,n}(z)\tilde{d}^{\star}_{h,n}(1/z)$ is the compression of the operator $\tilde{K}_{h,n}$ on $F_{\xi_1^{-1},\dots,\xi_d^{-1}}$.

From Lemma 4.5, identifying $F_{\xi_1^{-1},\dots,\xi_d^{-1}}$ with $\Omega^1(\tilde{G}_1)$, the compression $\tilde{K}_{h,n}(z)$ has the same action as the operator of orthogonal projection on $\tilde{d}_{h(\xi_1,\dots,\xi_d)}(\Omega^1(\tilde{G}_1))$:

$$\tilde{K}_{h(\xi_1,\dots,\xi_d)} : \Omega^1(\tilde{G}_1) \to \Omega^1(\tilde{G}_1).$$

Definition 4.7. We call *characteristic polynomial* of the model, associated to a connection h and denote by $P_h(z_1, \dots, z_d)$ the determinant of the Laplacian operator

$$\tilde{\Delta}_{h(z_1,\dots,z_d)} : \Omega^0(\tilde{G}_1) \to \Omega^0(\tilde{G}_1)$$

associated to the connection $h(z_1, \dots, z_d)$.

Notice that $P_h(z_1, \dots, z_d)$ is indeed a Laurent polynomial in $z_1, \dots, z_d$ because of the multi-linearity of the determinant and the fact that the new connection $h(z_1, \dots, z_d)$ is obtained multiplying the connection on some edges by the variables $(z_j, z_j^{-1})_j$.

In the case of a fundamental domain of size 1 and a connection given by matrices $(M_1, \dots, M_d)$ as defined in section 4.1.1, the expression of the characteristic polynomial of this model is given by

$$P_h(z_1, \dots, z_d) = \det\left(2dI_N - \sum_{i=1}^{d}(z_iM_i + (z_iM_i)^{-1})\right). \tag{4.2}$$

Recall from Lemma 4.6 that the reduced determinant of the Laplacian associated to h on the graph $\tilde{G}_n$ is given by

$$\det{}_0(\tilde{\Delta}_{h,n}) = \prod_{z_1^n=1,\dots,z_d^n=1} \det{}_0\tilde{\Delta}_{h(z_1,\dots,z_d)}. \tag{4.3}$$

When $(z_1, \dots, z_d)$ is such that $P_h(z_1, \dots, z_d) \neq 0$, then by definition of the reduced determinant,

$$\det{}_0\tilde{\Delta}_{h(z_1,\dots,z_d)} = P_h(z_1, \dots, z_d).$$

Expression of the kernel of the process in a basis In this part, we define an orthonormal basis of $\Omega^1(\tilde{G}_n)$ which can be easily decomposed according to the orthogonal decomposition

$$\Omega^1 = \bigoplus_{z \in U_n^d} F_z.$$

We will then explicit the action of $\tilde{K}_n$ on this basis.

Definition 4.8. Let $\chi = e^{2i\pi/n}$. Let e be an edge of $\tilde{G}_n$. We can write $e = [e] + (x_1, \dots, x_d)$ where $[e] \in G_1$. We define the functions $(\theta_e^i)_{1 \le i \le N}$ as follows. If $e' = [e'] + (y_1, \dots, y_d)$, define

$$\theta_e^i(e') = \frac{1}{n^d} \sum_{0 \le l_1, \dots, l_d \le n-1} \prod_k (\chi^{l_k})^{-x_k} (\chi^{l_k})^{y_k} \theta_{[e]}^i([e'])$$

where $\theta_{[e]}^i$ is defined in $\Omega^1(G_1)$ by

$$\theta_{[e]}^i([e']) = \begin{cases} b_i & \text{if } e' = [e] \\ -b_i & \text{if } [e'] = -[e] \\ 0 & \text{else} \end{cases}$$

where $(b_i)_i = ((0, \dots 0, 1, 0, \dots, 0)^t)_i$ is the canonical basis of $\mathbb{C}^N$.

Definition 4.9. For every $(\xi_1, \dots, \xi_d) \in \mathbb{T}^d$, we denote by

$$\left((\tilde{K}_h)_{[e],[e']}^{i,j}(\xi_1, \dots, \xi_d) \right)_{i,j \in [1,N]^2, [e],[e'] \in E(G_1)}$$

the matrix of the operator $\tilde{K}_{h(\xi_1,\dots,\xi_d)} : \Omega^1(\tilde{G}_1) \to \Omega^1(\tilde{G}_1)$.

Recall that when $(\xi_1, \dots, \xi_d) \in U_n^d$, the operator $\tilde{K}_{h(\xi_1,\dots,\xi_d)}$ has the same action as the compression of $(\tilde{K}_{h,n})$ on the space of $(\xi_1^{-1}, \dots, \xi_d^{-1})$-periodic 1-forms, of dimension $N|E(G_1)|$, which has been identified to $\Omega^1(\tilde{G}_1)$.

For edges $[e], [e']$ of the fundamental domain in E_+,

$$(\tilde{K}_h)_{[e],[e']}^{i,j}(\xi_1, \dots, \xi_d) = \langle (\tilde{K}_{h(\xi_1,\dots,\xi_d)}) \theta_{[e']}^i, \theta_{[e]}^j \rangle.$$

The following theorem gives the expression of the kernel in a basis of $\Omega^1(\tilde{G}_n)$ which uses the notations defined above.

Theorem 4.10. *The family of functions $(\theta_e^i)_{1 \le i \le N, e \in \tilde{G}_n \cap E_+}$ is an orthonormal basis of $\Omega^1(\tilde{G}_n)$ and the matrix of the operator $\tilde{K}_{h,n} : \Omega^1(\tilde{G}_n) \to \Omega^1(\tilde{G}_n)$ can be written in this basis*

$$\begin{aligned} (\tilde{K}_{h,n})_{e,e'}^{i,j} &= \langle (\tilde{K}_{h,n}) \theta_{e'}^i, \theta_e^j \rangle \\ &= \frac{1}{n^d} \sum_{{\xi_1}^n = 1, \dots, {\xi_d}^n = 1} \prod_{k=1}^d (\xi_k)^{x_k - y_k} (\tilde{K}_h)_{[e],[e']}^{i,j}(\xi_1, \dots, \xi_d). \end{aligned}$$

Proof.

- For $e \in \tilde{G}_n$, $1 \leq i \leq N$, the 1-form $\theta_e^i \in \Omega^1(\tilde{G}_n)$ satisfies

$$\theta_e^i(e') = \begin{cases} b_i & \text{if } e' = e, \\ -b_i & \text{if } e' = -e, \\ 0 & \text{else.} \end{cases}$$

Indeed, if for some k, $x_k \neq y_k$, $\theta_e^i(e')$ can be factorized by $\sum_{l\in[0,n-1]}(\chi^l)^{y_k-x_k} = 0$. Thus $\langle \theta_e^i, \theta_{e'}^j \rangle = 1_{i=j}(1_{e=e'} - 1_{e=-e'})$, which proves that the family is an orthonormal basis.

- For $0 \leq l_1, \ldots, l_d \leq n-1$, $1 \leq i \leq N$ and $[e] \in G_1$ the 1-form in $\Omega^1(\tilde{G}_n)$

$$\psi^i_{e,l_1,\ldots,l_d} : [e'] + (y_1, \ldots, y_d) \in \tilde{G}_n \mapsto \prod_{k=1}^{d} (\chi^{l_k})^{y_k} \theta^i_{[e]}([e'])$$

is in $F_{\chi^{l_1},\ldots,\chi^{l_d}}$ and is identified to $\theta^i_{[e]}$ in $\Omega^1(\tilde{G}_1)$. Then,

$$\begin{aligned}(\tilde{K}_{h,n})(\psi^i_{e,l_1,\ldots,l_d})([e'] + (y_1, \ldots, y_d)) &= \prod_{k=1}^{d} (\chi^{l_k})^{y_k} (\tilde{K}_{h,n})(\psi^i_{e,l_1,\ldots,l_d})([e']) \\ &= \prod_{k=1}^{d} (\chi^{l_k})^{y_k} \left(\tilde{K}_{h(\chi^{-l_1},\ldots,\chi^{-l_d})}\right)(\theta^i_{[e]})([e']).\end{aligned}$$

Moreover, we have

$$\theta_e^i(e') = \frac{1}{n^d} \sum_{0\leq l_1,\ldots,l_d\leq n-1} \quad \prod_{k=1}^{d} (\chi^{l_k})^{-x_k} \psi^i_{e,l_1,\ldots,l_d}(e'),$$

and this decomposition is the unique decomposition of θ_e^i in $\Omega^1(\tilde{G}_n) = \bigoplus_{z\in U_n^d} F_z$.

- If $e = [e] + (x_1, \ldots, x_d) \in \tilde{G}_n$, $e' = [e'] + (y_1, \ldots, y_d) \in \tilde{G}_n$, we have

$$\begin{aligned}(\tilde{K}_{h,n})\theta_e^i(e') &= \frac{1}{n^d} \sum_{0\leq l_1,\ldots,l_d\leq n-1} \quad \prod_{k=1}^{d} (\chi^{l_k})^{-x_k} (\tilde{K}_{h,n})(\psi^i_{e,l_1,\ldots,l_d})(e') \\ &= \frac{1}{n^d} \sum_{0\leq l_1,\ldots,l_d\leq n-1} \quad \prod_{k=1}^{d} (\chi^{-l_k})^{x_k-y_k} \left(\tilde{K}_{h(\chi^{-l_1},\ldots,\chi^{-l_d})}\right)(\theta^i_{[e]})([e']) \\ &= \frac{1}{n^d} \sum_{{\xi_1}^n=1,\ldots,{\xi_d}^n=1} \quad \prod_{k=1}^{d} (\xi_k)^{x_k-y_k} (\tilde{K}_{h(\xi_1,\ldots,\xi_d)})\theta^i_{[e]}([e']).\end{aligned}$$

Considering for $1 \leq j \leq N$ the coordinate maps e_j^* associated to the canonical basis of $\mathbb{C}^N$ finishes the proof. □

4.2 Study of the roots of the characteristic polynomial in the unit torus

In this section, we will study the roots of the characteristic polynomial in the unit torus, in order to establish the convergence of the sequence of kernels of the measures on quantum spanning forests on growing tori of size n when n goes to infinity. This result will imply the convergence of the sequence of measures towards a determinantal measure whose kernel is the limit of the sequence of kernels.

The following Lemma due to [KL23] will be useful to obtain an equivalent condition to the existence of roots of the characteristic polynomial in the unit torus.

Lemma 4.11. *[KL23] If a section f is a horizontal section for a connection h, which means that $f \in \ker d_h \backslash \{0\}$, then for any vertex x and any loop based in x, denoted by $\gamma \in L_x$, we have $f(x) \in \ker(id_{F_x} - hol_h(\gamma))$. Furthermore, this condition is a sufficient one, which means that for every vertex x,*

$$ev_x(\ker d_h) = \cap_{\gamma \in L_x(G)} \ker(id_{F_x} - hol_h(\gamma)).$$

4.2.1 Roots of the characteristic polynomial on the unit torus

Fundamental domain of size 1. Assume that the fundamental domain is of size 1 and the connection is given by matrices $(M_1, \dots, M_d)$ as defined in section 4.1.1. We will prove that in this case, the existence of roots of the characteristic polynomial is equivalent to the existence of common eigenvectors to the matrices $(M_1, \dots, M_d)$.

Recall that the expression of the characteristic polynomial of this model is given by

$$P_h(z_1, \dots, z_d) = \det\left(2dI_N - \sum_{i=1}^{d}(z_i M_i + (z_i M_i)^{-1})\right),$$

and that the Laplacian operator on $\Omega^0(\tilde{G}_1)$ associated to the connection $h(z_1, \dots, z_d)$ is

$$\tilde{\Delta}_1(z_1, \dots, z_d) = d^\star(z_1^{-1}, \dots, z_d^{-1})d(z_1, \dots, z_d) = \sum_{i=1}^{d}(I_p - z_i^{-1}M_i^{-1})(I_p - z_i M_i).$$

Theorem 4.12. *For a square lattice with a fundamental domain of size 1, the following properties are equivalent:*

- *There exists $z = (z_1, \dots, z_d) \in \mathbb{T}^d$ such that $P(z) = 0$.*
- *There exists $X \in \mathbb{C}^N$ a common eigenvector to the matrices M_j.*

In the case where $z = (z_1, \dots, z_d) \in \mathbb{T}^d$ is such that $P(z) = 0$, the eigenvalues associated to the matrices M_j and to the common eigenvector X are $\bar{z_j}$.

There are at most N distinct roots $z = (z_1, \dots, z_d) \in \mathbb{T}^d$ of P.

In the case $N = 2$, if we assume that the matrices are in $SU_2(\mathbb{C})$ and that the connection is singular, then we get two roots $z = (z_1, \dots, z_d), \bar{z} = (\bar{z}_1, \dots, \bar{z}_d) \in \mathbb{T}^d$ and the matrices M_j are co-diagonalizable and can be written as $\begin{pmatrix} z_j & 0 \\ 0 & \bar{z_j} \end{pmatrix}$ in the basis $(X^\perp, X)$.

Proof. We give two proofs of the equivalence, the first one is based on the geometry of the fundamental domain and the other one is based on explicit calculations of the characteristic polynomial.

1. Let $z = (z_1, \dots, z_n) \in \mathbb{T}^d$ such that $P(z) = 0$. Then $\tilde{\Delta}(z)$ is non injective. In particular, there exists $f \in \ker(\tilde{d}(z)) \backslash \{0\}$ because $\tilde{d}(z)$ et $\tilde{d}^*(z)$ are adjoints. From Lemma 4.11, a section in the kernel of $\tilde{d}(z)$, so-called horizontal section, satisfies for all x the following relation depending on the holonomy of whatever loop based in x:

$$\mathrm{ev}_x(\ker\ \tilde{d}(z)) = \cap_{\gamma \in L_x(G_1)} \ker(id_{F_x} - \mathrm{hol}_{z,G_1}(\gamma)).$$

   In particular, considering the loops γ based in $(0, \dots, 0)$ on the fundamental domain, seen as a graph $\tilde{G}_1$ on the torus, of holonomy $z_j M_j$ for $j \in [1, d]$, we get

$$f(0) \in \ker(id_p - z_j M_j) \quad \forall j \in [1, d].$$

   All the matrices M_j have a common eigenvector $X = f(0)$ associated respectively to the eigenvalues $\bar{z_j}$ of modulus 1.

$$M_j X = \bar{z_j} X \quad \forall j \in [1, \dots, d].$$

2. For the other proof, let us write explicitly the characteristic polynomial :

$$\begin{aligned} P(z_1, \dots, z_d) &= \det(2dI_N - \sum_j (z_j M_j + \bar{z_j} M_j^{-1})) \\ &= \det(H) = \prod_{i=1}^{N} m_i(z) \end{aligned}$$

   where $H := (\sum_j 2I_N - (z_j M_j + \bar{z_j} M_j^{-1}))$ and $(m_i(z))$ are the eigenvalues of H which are in $\mathbb{R}_+$ since H is Hermitian and semidefinite. Indeed,

$$\begin{aligned} \langle HX, X\rangle &= \langle \sum_j 2dI_N - (z_j M_j + \bar{z_j} M_j^{-1})X, X\rangle \\ &= \sum_j 2d\langle X, X\rangle - z_j\langle M_j X, X\rangle - \bar{z_j}\langle M_j^{-1} X, X\rangle \\ &= \sum_j 2d(\langle X, X\rangle - Re(z_j\langle M_j X, X\rangle)), \end{aligned}$$

   where $|Re(z_j\langle M_j X, X\rangle)| \leq |\langle z_j M_j X, X\rangle| \leq \langle X, X\rangle$ since M_j is in $U_N(\mathbb{C})$. This inequality is an equality if and only if $X = 0$ or X is a common eigenvector to the matrices M_j associated to the eigenvalues $\bar{z}_j$. Thus, $\langle HX, X\rangle = 0$ if and only if we have $X = 0$ or the matrices M_j have a common eigenvector associated to the eigenvalues $\bar{z}_j$. It proves that P vanishes on $\mathbb{T}^d$ if and only if H is not definite if and only if the matrices M_j have a common eigenvector.

To end the proof, let us recall that the matrices M_j are unitary. In particular,

$$M_j^* X = z_j X$$

If $Y \in X^\perp \backslash \{0\}$, then, $\langle M_j Y, X\rangle = \langle Y, M_j^* X\rangle = \langle Y, z_j X\rangle = 0$.

If $N = 2$, then $\dim X^\perp = 1$, thus $M_j Y \in X^\perp = Vect(Y)$. Then Y is a common eigenvector to the matrices M_j. Finally, if the matrices M_j have determinant 1, Y is an eigenvector associated to the z_j and the matrices $M_1, \dots, M_d$ can be written in the basis (Y, X) of $\mathbb{C}^2$ as $\begin{pmatrix} z_j & 0 \\ 0 & \bar{z_j} \end{pmatrix}$. □

Larger fundamental domain and other lattices. Let G be a $\mathbb{Z}^d$-periodic graph with a fundamental domain G_1, endowed with a periodic connection h defined by its values on G_1.

Let $x \in G_1$. Let $\mathcal{C}_x(\tilde{G_1})$ be the set of closed cycles of $\tilde{G_1}$ rooted at x.

Theorem 4.13. *The following properties are equivalent:*

- *There exists $X \in \mathbb{C}^N$ a common eigenvector to all holonomies $hol_h(\tilde{\gamma})$ for $\tilde{\gamma} \in \mathcal{C}_x(\tilde{G_1})$ associated with eigenvalue 1 for contractible cycles.*
- *There exists $z = (z_1, \dots, z_n) \in \mathbb{T}^d$ such that $P(z) = 0$.*

In the case where $z = (z_1, \dots, z_n) \in \mathbb{T}^d$ is such that $P(z) = 0$, there exists a common eigenvector X to the holonomies $hol_h(\tilde{\gamma}_j(x))$ associated with the eigenvalues $\bar{z_1}, \dots, \bar{z_N}$. In particular, there are at most N distinct roots $z = (z_1, \dots, z_n) \in \mathbb{T}^d$ of P.

Proof. Assume that the characteristic polynomial P vanishes on $\mathbb{T}^d$ and let $\underline{z} \in \mathbb{T}^d$ be a root of the characteristic polynomial. Then, there exists a horizontal section $f \in \ker(\tilde{d}(z)) \backslash \{0\}$ because $\tilde{d}(z)$ et $\tilde{d}^*(z)$ are adjoints. From Lemma 4.11, it implies the existence of $X \in \mathbb{C}^N$ such that

$$X \in \cap_{\gamma \in L_x(G)} \ker(id_{F_x} - \mathrm{hol}_{h(z_1,\dots,z_d),\tilde{G}_1}(\gamma)).$$

This vector X is a common eigenvector to all holonomies $hol_h(\tilde{\gamma})$ for $\tilde{\gamma} \in \mathcal{C}_x(\tilde{G_1})$ associated with eigenvalue 1 for contractible cycles and eigenvalue $\bar{z}_1^{i_1} \dots \bar{z}_d^{i_d}$ for a cycle of homotopy class $(i_1, \dots, i_d)$. If there exists such an eigenvector X, the 0-form defined by $f(x) = X$ and for every y, if γ is a path from x to y, $f(y) = hol_h(\gamma)X$, is in $\ker(d_h) \cap E_{z_1,\dots,z_d}$, that is $\ker(d_h(\bar{z}_1, \dots, \bar{z}_d))$, thus, the characteristic polynomial P vanishes at $(\bar{z}_1, \dots, \bar{z}_d)$. □

For some lattices such as the triangular lattice, this result gives explicit equivalent conditions to the existence of roots of the characteristic polynomial in the unit torus in terms of common eigenvectors of the matrices which define the connection.

Corollary 4.14. *For a triangular lattice with a fundamental domain of size 1, endowed with a periodic connection defined by unitary matrices M_1, M_2, M_3, the following properties are equivalent:*

- *There exists $z = (z, w) \in \mathbb{T}^2$ such that $P(z, w) = 0$.*
- *There exists $X \in \mathbb{C}^N$ a common eigenvector to the matrices M_1, M_2, M_3, associated respectively to eigenvalues $\bar{z}, \bar{w}, \bar{z}\bar{w}$, such that*

$$M_1 M_3 X = M_2 X.$$

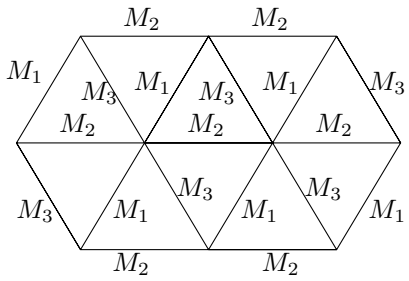

Figure 4.2 – Triangular planar lattice endowed with a periodic connection h.

Proof. There exists $(z, w) \in \mathbb{T}^2$ such that $P(z, w) = 0$ if and only if there exists a horizontal section $f \in \ker(\tilde{d}(z))\backslash\{0\}$ because $\tilde{d}(z)$ et $\tilde{d}^*(z)$ are adjoints. From Lemma 4.11, it is equivalent to the existence of $X \in \mathbb{C}^N$ such that

$$X \in \cap_{\gamma \in L_v(G)} \ker(id_{F_v} - \mathrm{hol}_{h(z,w),\tilde{G}_1}(\gamma)).$$

From Lemma 4.5, for any $z = (z, w) \in \mathbb{T}^2$, the connection $h(z, w)$ on the graph $\tilde{G}_1$ on the torus is as follows:

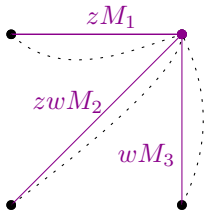

Figure 4.3 – The fundamental domain on the torus $\tilde{G}_1$ endowed with the connection $h(z, w)$.

Therefore, for any $z = (z, w) \in \mathbb{T}^2$, we have $P(z, w) = 0$ if and only if there exists X in $\cap_{\gamma \in L_v(G)} \ker(I_N - \mathrm{hol}_{h(z,w),\tilde{G}_1}(\gamma))$ that is X such that

$$zM_1X = zwM_2X = wM_3X = X,$$

which concludes the proof. □

Those results imply in particular that the characteristic polynomial has at most N distinct roots in the unit torus. The finite number of roots is a key tool to show the convergence of the kernels sequence.

4.2.2 Thermodynamic limit of the model

In this subsection, we study the convergence of the kernel $\tilde{K}_{h,n}$ when $n \to \infty$.

Lemma 4.15. *When $n \to \infty$, we have for every $e, e' \in E$ and every $1 \leq i, j \leq N$,*

$$(\tilde{K}_{h,n})^{i,j}_{e,e'} \to \int_{|z_1|=1,\ldots,|z_d|=1} (\tilde{K}_h)^{i,j}_{[e],[e']}(z_1, \ldots, z_d) z_1^{x_1-y_1} \ldots z_d^{x_d-y_d} \frac{dz_1}{2i\pi z_1} \ldots \frac{dz_d}{2i\pi z_d}.$$

Proof. Let us write explicitly the n-roots of unity as $\xi_l = \exp(\frac{2il\pi}{n})$ for $1 \leq l \leq n$. Then

$$\begin{aligned}(\tilde{K}_{h,n})^{i,j}_{e,e'} &= \frac{1}{n^d} \sum_{{\xi_1}^n=1,\dots,{\xi_d}^n=1} \prod_k (\xi_k)^{x_k-y_k} (\tilde{K}_h)^{i,j}_{[e],[e']}(\xi_1,\dots,\xi_d) \\ &= \frac{1}{n^d} \sum_{1\leq l_1,\dots,l_d\leq n} \prod_k (e^{\frac{2il_k\pi}{n}})^{x_k-y_k} (\tilde{K}_h)^{i,j}_{[e],[e']}(e^{\frac{2il_1\pi}{n}},\dots,e^{\frac{2il_d\pi}{n}})\end{aligned}$$

The map $(z_1,\dots,z_d) \mapsto (\tilde{K}_h)(z_1,\dots,z_d)$ remains bounded on the unit torus since it is the matrix of an orthogonal projection and has at most a finite number of discontinuity points given by the roots in the unit torus of the characteristic polynomial. Therefore, the Riemannian sum converges towards the following limit

$$(\tilde{K}_{h,n})^{i,j}_{e,e'} \to \int_{|z_1|=1,\dots,|z_d|=1} \prod_k z_k^{x_k-y_k} (\tilde{K}_h)^{i,j}_{[e],[e']}(z_1,\dots,z_d) \frac{dz_1}{2i\pi z_1} \frac{dz_d}{2i\pi z_d},$$

which is the desired limit. □

Let us define the operator K on $\Omega^1(\mathbb{Z}^d)$ by

$$\langle K\theta_{e'}, \theta_e\rangle = \int_{|z_1|=1,\dots,|z_d|=1} (\tilde{K}_h)_{[e],[e']}(z_1,\dots,z_d) z_1^{x_1-y_1} \dots z_d^{x_d-y_d} \frac{dz_1}{2i\pi z_1} \dots \frac{dz_d}{2i\pi z_d}.$$

Lemma 4.15 and topological considerations of the subsection 3.2.1 imply the following statement about the sequence of measures (μ_n^P) as defined in section 3.3. Notice that this result is proved in [Ken11] for non-unitary complex periodic connections of rank 1.

Theorem 4.16. *The sequence of measures (μ_n^P) converges weakly towards a determinantal measure μ^P on the Grassmannian of the vector bundle of rank N over G which is associated to the operator K. This measure defines a quantum spanning forest on G whose kernel has the following integral expression :*

$$K_{e,e'} = \int_{z\in\mathbb{T}^d} z^{x-y} (\tilde{K}_h)_{[e],[e']}(z) \frac{dz}{2i\pi z}. \tag{4.4}$$

Recall that this theorem relies on the following property : If $\Lambda \subset E$ is a finite subset and $R = \oplus_{e\in\Lambda} R_e$ with R_e a subspace of F_e for every $e \in \Lambda$, then for m large enough,

$$\rho_m^P(R) = \det((\tilde{K}_m)_{(b_i^e)})_{e\in\Lambda,(b_i^e)_{i\leq \dim R_e}}) \to \det((K)_{(b_i^e)})_{e\in\Lambda,(b_i^e)_{i\leq \dim R_e}}).$$

where for every $e \in \Lambda$, the family $(b_i^e)_{i\leq \dim R_e}$ is a basis of R_e and ρ_m^P is the density of the incidence measure $R \mapsto \mu_m^P(R \subset Q)$, as defined in Subsection 2.3.2.

In particular, in the case where the free and the wired measures coincide in infinite volume, we deduce from Corollary 3.14 that the free and the wired measure coincide with μ^P which means that those measures are determinantal of kernel K defined by the integral expression (4.4).

Recall from Equation (3.2) that the free and the wired measures coincide in infinite volume if and only if $d_h(\mathcal{H}) = \{0\}$ where $\mathcal{H} = \{f \in \Omega^0 : \Delta_h f = 0 \text{ and } d_h f \in \Omega^1_{\ell^2}(F)\}$ is the space of harmonic 0-forms. We will see later (Lemma 6.18) that under Assumption 6.17 on the connection h, this equality is satisfied but it does not seem to be a necessary condition to get the equality between the free and the wired measure. Indeed, this assumption is not satisfied by the trivial connection, for which the free and the wired measures on uniform spanning forests are equal (see [LP16]).

4.2.3 Free-energy of the model

By analogy with other statistical physics models, we define the free energy of the infinite volume measure obtained as the limit of a sequence of measures μ_n on an exhaustion of the infinite graph by growing finite subgraphs as the following quantity:

$$F = -\lim_{n\to\infty} \frac{1}{n^d} \log(|F_n|),$$

where F_n is the partition function of the measure μ_n.

Theorem 4.17. *The free energy of the limit measure μ^P is given by the following integral expression :*

$$F = -\int_{z\in\mathbb{T}^d} \log(|P_h(z)|),$$

that is the Mahler measure of the polynomial P_h, as studied in [BZ20, GM21, BGMP22].

For a fundamental domain of size 1, the free energy is given by

$$F = -\int_{(z_1,\dots,z_d)\in\mathbb{T}^d} \log\left(|\det\left(2dI_N - \sum_{i=1}^{d}(z_iM_i + (z_iM_i)^{-1})\right)|\right).$$

Proof. From [KL23], as recalled in (section 2.3, Theorem 2.19), the partition function of the model is the reduced determinant $\det{}_0(\tilde{\Delta}_{h,n})$ of the Laplacian operator associated to h on the graph $\tilde{G}_n$. From the diagonal decomposition of the Laplacian operator restricted to the orthogonal of the kernel of the covariant derivative (Equation (4.3)), this determinant is given by $\det{}_0(\tilde{\Delta}_{h,n}) = \prod_{z_1^n=1,\dots,z_d^n=1} \det{}_0\tilde{\Delta}_{h(z_1,\dots,z_d)}$.

By definition of the free energy, we have

$$\begin{aligned}
F &= -\lim_{n\to\infty} \frac{1}{n^d} \log(|\det{}_0(\tilde{\Delta}_{h,n})|) \\
&= -\lim_{n\to\infty} \frac{1}{n^d} \log\left(\prod_{z_1^n=1,\dots,z_d^n=1} |\det{}_0\tilde{\Delta}_{h(z_1,\dots,z_d)}|\right) \\
&= -\lim_{n\to\infty} \frac{1}{n^d} \sum_{z_1^n=1,\dots,z_d^n=1} \log(|\det{}_0\tilde{\Delta}_{h(z_1,\dots,z_d)}|) \\
&= -\lim_{n\to\infty} \frac{1}{n^d} \sum_{1\le l_1,\dots,l_d\le n} \log\left(|\det{}_0\tilde{\Delta}_{h\left(e^{\frac{2il_1\pi}{n}},\dots,e^{\frac{2il_d\pi}{n}}\right)}|\right).
\end{aligned}$$

The function $\underline{z} \mapsto \det{}_0\tilde{\Delta}_{h(z_1,\dots,z_d)}$ does not vanish on $\mathbb{T}^d = \{|z_1| = 1,\dots,|z_d| = 1\}$ and has at most a finite number of discontinuity points, corresponding to the roots of the characteristic polynomial on the unit torus. On a neighborhood of each discontinuity point, the map $(z_1,\dots,z_d) \in \mathbb{T}^d \mapsto \det{}_0\tilde{\Delta}_{h(z_1,\dots,z_d)}$ coincides with $P_h(z_1,\dots,z_d)$ since it does not vanish, and thus is a polynomial function. Therefore, the function obtained by applying the logarithm, that is to say $(z_1,\dots,z_d) \mapsto \log\left(|\det{}_0\tilde{\Delta}_{h\left(e^{\frac{2il_1\pi}{n}},\dots,e^{\frac{2il_d\pi}{n}}\right)}|\right)$ is integrable. In particular, we obtain the convergence of the Riemannian sum towards

$$-\int_{|z_1|=1,\dots,|z_d|=1} \log(|\det{}_0\tilde{\Delta}_{h(z_1,\dots,z_d)}|)\frac{dz_1}{2i\pi z_1}\frac{dz_d}{2i\pi z_d}.$$

For almost every $(z_1, \dots, z_d) \in \mathbb{T}^d$, the characteristic polynomial does not vanish, then the quantity $\det {}_0\tilde{\Delta}_{h(z_1,\dots,z_d)}$ coincides with $P_h(z_1, \dots, z_d)$. Therefore, we can replace the first quantity by the second one in the integral, which concludes the proof of the first statement. From Equation (4.2), for a fundamental domain of size 1, the characteristic polynomial is given by

$$P_h(z_1, \dots, z_d) = \det\left(2dI_N - \sum_{i=1}^{d}(z_i M_i + (z_i M_i)^{-1})\right),$$

which concludes the proof of the second statement. □

In the following, we will consider the determinantal measure in infinite volume whose kernel is given by the integral expression and study the asymptotic behavior under this measure, that is the *edge-to-edge correlations* decay.

4.3 Edge-to-edge correlations decay

In this section, we still denote by G an infinite $\mathbb{Z}^d$-periodic graph endowed with a connection h which is $\mathbb{Z}^d$-periodic. We assume that the free and the wired measure on QSF of G coincide and we denote by μ this measure which is determinantal of kernel K defined by the integral expression :

$$K_{e,e'} = \int_{z\in\mathbb{T}^d} z^{x-y}(K_h)_{[e],[e']}(z)\frac{dz}{2i\pi z}$$

where we recall that

$$(K_h)(z) = d_h(z)\tilde{\Delta}_h(z)^{-1}d_h^*(z) = \frac{1}{P_h(z)}d_h(z)t_{\mathrm{Com}(\tilde{\Delta}_h(z))}d_h^*(z) \tag{4.5}$$

where the polynomial P_h is the characteristic polynomial associated to the connection h.

We will show in this section that there are different phases in the model depending on the edge-to-edge correlations' decay and that those phases depend on the existence of zeros of the characteristic polynomial in the unit torus.

Edge-to-edge correlations do not depend on the bases

Definition 4.18. We call edge-to-edge correlations the quantities $Cov(n_e, n_{e'})_{e,e'}$ where we recall that for every $e \in E$, the random variable n_e is the dimension of the random subspace Q_e of the fiber F_e.

If we fix a basis $(b_i^e)_{1\le i\le N}$ of F_e for every $e \in E$, the law of the couple $(n_e, n_{e'})$ is the law of $(\sum_i 1_{b_i\in Q_e}, \sum_i 1_{b_i\in Q_{e'}})$.

By bi-linearity of the covariance, we have

$$Cov(n_e, n_{e'}) = Cov(\sum_i 1_{b_i\in Q_e}, \sum_j 1_{b_j\in Q_{e'}})) = \sum_{i,j} Cov(1_{b_i\in Q_e}, 1_{b_j\in Q_{e'}}).$$

By definition of the kernel of the process, we have the following formula for the covariances

$$\begin{aligned} Cov(1_{b_i\in Q_e}, 1_{b_j\in Q_{e'}}) &= \mathbb{P}(b_i \in Q_e, b_j \in Q_{e'}) - \mathbb{P}(b_i \in Q_e)\mathbb{P}(b_j \in Q_{e'}) \\ &\propto \det \begin{pmatrix} K_{b_i^e,b_i^e} & K_{b_j^{e'},b_i^e} \\ K_{b_i^e,b_j^{e'}} & K_{b_j^{e'},b_j^{e'}} \end{pmatrix} - K_{b_i^e,b_i^e}K_{b_j^{e'},b_j^{e'}} \\ &\propto -K_{b_j^{e'},b_i^e}K_{b_i^e,b_j^{e'}} \\ &\propto -K^{j,i}_{e',e}K^{j,i}_{e,e'}. \end{aligned}$$

Recall that when we change the bases of fibers, we replace K by PKP^{-1} where P is a block-diagonal unitary matrix. Therefore, the decay of edge-to-edge correlations does not depend on the choice of the bases of fibers.

We will see in the following that the edge-to-edge correlations' decay depends on the existence of roots of the characteristic polynomial in the unit torus, which depends on the values of the unitary connection.

4.3.1 Regular and singular connections

In the following, we will consider the following definition of a regular connection.

Definition 4.19 (Regular connection)**.** If $\forall \underline{z} \in \mathbb{T}^d, P(\underline{z}) \neq 0$, we say that the connection is regular. Otherwise, we say that the connection is singular.

Recall that in [KL23], the following definition of a regular connection is given.

Definition 4.20. [KL23] A connection is said to be singular if $\ker(d_h) \neq \{0\}$. Otherwise, we say that the connection is regular.

We will prove in the following that both definitions are equivalent.

Proposition 4.21. *For a square lattice with a fundamental domain of size 1, both definitions of a regular connection are equivalent.*

Proof. Assume that $\ker(d_h) \neq \{0\}$. Consider $f \in \ker(d_h)\backslash\{0\}$ and denote

$$F = vect(f(v), v \in \mathbb{Z}^d).$$

We have $F \neq \{0\}$ and if $v \in \mathbb{Z}^d, \forall j \in [1, d]$, $f(v + t_j) = M_j f(v)$. Then, $\forall i, j \in [1, d]$,

$$f(v + t_j + t_i) = M_i M_j f(v) = M_j M_i f(v).$$

Therefore, for every $x \in F$, $\forall i, j \in [1, d]$, $M_i M_j x = M_j M_i x$. Then, F is invariant under M_i since $M_i f(v) = f(v+t_i)$. F is an invariant nonzero subspace on which $M_1, \dots, M_d$ commute, then they are co-diagonalizable on this subspace and have a common eigenvector.

Assume that there exists $\underline{z} \in \mathbb{T}^d$ such that $P(\underline{z}) = 0$, thus there exists X a common eigenvector for the eigenvalues $z_1, \dots, z_d$. Define f by $f(0) = X$ and for every $(n_1, \dots, n_d) \in \mathbb{Z}^d$, define $f(n_1, \dots, n_d) = z_1^{n_1} \dots z_d^{n_d} X$. Notice that

$$d_h f(e) = z_1^{n_1} \dots z_d^{n_d} X - M_j z_1^{n_1} \dots z_j^{n_j - 1} \dots z_d^{n_d} X = z_1^{n_1} \dots z_d^{n_d} X - z_1^{n_1} \dots z_d^{n_d} X = 0.$$

Thus $f \in \ker(d_h)$. □

Slight modifications of the proof of Proposition 4.21 imply the following theorem in the case of larger fundamental domain and other lattices.

Let G be a $\mathbb{Z}^d$-periodic graph with a fundamental domain G_1, endowed with a periodic connection h defined by its values on G_1. Let $x \in G_1$. Let V_x be the set of vertices translated from x. Note that when the fundamental domain is of size 1, V_x is G. Let $x+t_1, \dots, x+t_d$ be the set of vertices of G obtained by translating x by the action of $\mathbb{Z}^d$.

Let $\mathcal{C}_x(\tilde{G}_1)$ be the set of closed cycles of $\tilde{G}_1$ rooted at x. Let $(\gamma_1(x), \dots, \gamma_d(x))$ be a family of shortest paths in G, where for every i, $\gamma_i(x)$ joins x to $x+t_i$ and if $\tilde{\gamma}_i(x)$ is the image of $\gamma_i(x)$ in $\mathcal{C}_x(\tilde{G}_1)$, the homotopy class of $\tilde{\gamma}_i(x)$ is exactly 1 in direction i.

Note that the family $(\tilde{\gamma}_1(x), \dots, \tilde{\gamma}_d(x))$ forms a basis of the first homology group of the torus of dimension d. More generally, if $\tilde{\gamma} \in \mathcal{C}_x(\tilde{G}_1)$ is a cycle of non trivial homotopy class, and if γ is a path in G lifting $\tilde{\gamma}$, let $x+t_\gamma \in V_x$ be the translated vertex from x which is the end of the path γ.

Theorem 4.22. *The following properties are equivalent:*

- *The connection h is singular in the sense that* $\ker d_h \neq \{0\}$.
- *There exists $\underline{z} = (z_1, \dots, z_n) \in \mathbb{T}^d$ such that $P(\underline{z}) = 0$.*

Proof. Assume that the connection is singular in the sense that $\ker d_h \neq \{0\}$ and let f be a non-zero horizontal section on G. Let $x \in G_1$. Let $F(x) = Vect(\{f(v), v \in V_x\})$. For every $\tilde{\gamma} \in \mathcal{C}_x(\tilde{G}_1)$, for every $v \in V_x$, $hol_h(\tilde{\gamma})f(v) = f(v+t_\gamma) \in F(x)$. Therefore, the space $F(x)$ is invariant under every $\tilde{\gamma} \in \mathcal{C}_x(\tilde{G}_1)$. Furthermore, if $\tilde{\gamma}_1, \tilde{\gamma}_2 \in \mathcal{C}_x(\tilde{G}_1)$, then if $v \in V(x)$,

$$hol_h(\tilde{\gamma}_1)hol_h(\tilde{\gamma}_2)f(v) = f(v+t_{\gamma_1}+t_{\gamma_2}) = f(v+t_{\gamma_2}+t_{\gamma_1}) = hol_h(\tilde{\gamma}_2)hol_h(\tilde{\gamma}_1)f(v).$$

Then, there exists $X \in F(x)$ a common eigenvector to every $hol_h(\tilde{\gamma})$ for $\tilde{\gamma} \in \mathcal{C}_x(\tilde{G}_1)$. In particular, the vector $X \in \mathbb{C}^N$ is a common eigenvector to $hol_h(\tilde{\gamma}_1(x)), \dots, hol_h(\tilde{\gamma}_d(x))$ associated with unitary complex eigenvalues $z_1, \dots, z_d$. If $\tilde{\gamma} \in \mathcal{C}_x(\tilde{G}_1)$ is a contractible cycle on the torus, then for every $v \in V_x$, $hol_h(\tilde{\gamma})f(v) = f(v)$ since $f \in \ker d_h$. Then, $hol_h(\tilde{\gamma})$ is the identity on $F(x)$ and $hol_h(\tilde{\gamma})X = X$.

Then, for such an eigenvector X, the 0-form defined by $f(x) = X$ and for every y, if γ is a path from x to y, $f(y) = hol_h(\gamma)X$, is in $\ker(d_h) \cap E_{z_1,\dots,z_d}$, that is $\ker(d_h(\bar{z}_1, \dots, \bar{z}_d))$ and thus, the characteristic polynomial P vanishes at $(\bar{z}_1, \dots, \bar{z}_d)$.

Assume now that the characteristic polynomial P vanishes on $\mathbb{T}^d$ and let $\underline{z} \in \mathbb{T}^d$ be a root of the characteristic polynomial and $f \in \ker d_h(\underline{z})$ be a horizontal section for the connection $h(\underline{z})$ on $\tilde{G}_1$.

Defining f on G as in the proof of Proposition 4.21 by $(z_1, \dots, z_d)$-periodicity, that is for every $y \in G_1$ and $(n_1, \dots, n_d) \in \mathbb{Z}^d$,

$$f(y + (n_1, \dots, n_d)) = z_1^{n_1} \dots z_d^{n_d} f(y),$$

to obtain a horizontal section of G for the connection h concludes the proof. □

4.3.2 Exponential decay of correlations for a regular connection

In this section, we assume that the connection is regular, that this to say that $\underline{z} \mapsto P(\underline{z})$ does not vanish on $\mathbb{T}^d$. Let us recall from Equation (4.5) that for two edges $e, e' \in G$, we have

$$K_{e,e'} = \int_{\underline{z}\in\mathbb{T}^d} \underline{z}^{y-x} \frac{1}{P(\underline{z})} (d(\underline{z}) t_{\mathrm{Com}(\tilde{\Delta}(\underline{z}))} d^*(1/\underline{z}))_{[e],[e']} \frac{d\underline{z}}{2i\pi\underline{z}}. \tag{4.6}$$

Theorem 4.23. *If we choose to isolate a variable w and write $\tilde{z} = (z_1, \dots, \hat{w}, \dots, z_d) \in \mathbb{T}_{d-1}$, then we have $c > 0, \beta < 1$ such that for all $y \in \mathbb{Z}$ and $\tilde{\alpha} \in \mathbb{Z}^{d-1}$,*

$$C_y := \int_{\underline{z}\in\mathbb{T}^d} w^y \tilde{z}^{\tilde{\alpha}} \frac{1}{P(\underline{z})} \frac{d\underline{z}}{2i\pi\underline{z}} \leq c \exp(-\beta|y|).$$

Proof. Since $\underline{z} \mapsto P(\underline{z})$ is continuous on the compact set $\mathbb{T}_d$ and with strictly positive values, we know that $P(\underline{z}) \geq m$ with $m > 0$. By Fubini Lebesgue we can integrate the integrand in whatever order. Then we have

$$\begin{aligned} C_y &= \int_{\underline{z}\in\mathbb{T}^d} w^y \tilde{z}^{\tilde{\alpha}} \frac{1}{P(\underline{z})} \frac{d\underline{z}}{2i\pi\underline{z}} \\ &= \int_{\tilde{z}\in\mathbb{T}^{d-1}} \tilde{z}^{\tilde{\alpha}} \left(\int_{w\in\mathbb{T}} w^y \frac{1}{P(\tilde{z}, w)} \frac{dw}{2i\pi w} \right) \frac{d\tilde{z}}{2i\pi\tilde{z}} := \int_{\tilde{z}\in\mathbb{T}^{d-1}} \tilde{z}^{\tilde{\alpha}} I(\tilde{z}) \frac{d\tilde{z}}{2i\pi\tilde{z}}, \end{aligned}$$

where

$$I(\tilde{z}) = \left(\int_{w\in\mathbb{T}} w^y \frac{1}{P(\tilde{z}, w)} \frac{dw}{2i\pi w} \right) = \left(\int_{\theta\in[0,2\pi]} e^{i\theta y} \frac{1}{P(\tilde{z}, e^{i\theta y})} \frac{d\theta}{2\pi} \right).$$

Let $\tilde{z} \in \mathbb{T}^{d-1}$. $P(\tilde{z}, .)$ is a polynomial with one variable which does not vanish on $\mathbb{T}$ and which takes a finite number of roots. Then, there exists $r(\tilde{z}) < 1$ such that $P(\tilde{z}, .)$ does not have any root in

$$A_{r(\tilde{z})} = \{w \in \mathbb{C}, r(\tilde{z}) \leq |w| \leq \frac{1}{r(\tilde{z})}\}.$$

Since $\tilde{z} \in \mathbb{T}^{d-1} \mapsto (\lambda_1(\tilde{z}), \dots, \lambda_n(\tilde{z}))$ where $(\lambda_1(\tilde{z}), \dots, \lambda_n(\tilde{z}))$ is the set of roots of $P(\tilde{z}, .)$ is continuous, the application $\tilde{z} \mapsto r(\tilde{z})$ is continuous on a compact set, thus there exists $0 < r < 1$ such that $r(\tilde{z}) \leq r$ for all $\tilde{z}$ and $P(\tilde{z}, .)$ does not have any root in A_r for all $\tilde{z}$.

For all $\tilde{z}$, the application $w \mapsto w^y \frac{1}{P(\tilde{z},w)} \frac{1}{2i\pi w}$ is holomorphic on A_r as a rational fraction whose denominator does not vanish on A_r.

Up to replacing r by $r < r' < 1$, we also know that $w \mapsto w^y \frac{1}{P(\tilde{z},w)} \frac{1}{2i\pi w}$ is holomorphic on a neighborhood of the compact domain A_r and by Cauchy's theorem,

$$I(\tilde{z}) = \int_{|w|=1} w^y \frac{1}{P(\tilde{z}, w)} \frac{dw}{2i\pi w} = \int_{|w|=r} w^y \frac{1}{P(\tilde{z}, w)} \frac{dw}{2i\pi w} = \int_{|w|=\frac{1}{r}} w^y \frac{1}{P(\tilde{z}, w)} \frac{dw}{2i\pi w}.$$

Then

$$|I(\tilde{z})| \leq \min \left(\left| \int_{|w|=r} w^y \frac{1}{P(\tilde{z}, w)} \frac{dw}{2i\pi w} \right|, \left| \int_{|w|=\frac{1}{r}} w^y \frac{1}{P(\tilde{z}, w)} \frac{dw}{2i\pi w} \right| \right).$$

Therefore

$$|I(\tilde{z})| \leq \min\left(r^y \frac{1}{\min_{|w|=r}|P(\tilde{z},w)|}, \frac{1}{r^y}\frac{1}{\min_{|w|=1/r}|P(\tilde{z},w)|}\right),$$

and

$$|I(\tilde{z})| \leq c(\tilde{z}) r^{|y|},$$

where $c(\tilde{z}) = \frac{1}{\min_{|w|=r,|w|=1/r}|P(\tilde{z},w)|}$ where the minimum is strictly positive because $P(\tilde{z},.)$ is continuous with strictly positive values on the compact set $\{|w| = r, |w| = 1/r\}$. Moreover, $\tilde{z} \mapsto c(\tilde{z})$ is continuous on a compact set, thus it reaches its bounds, thus there exists $c > 0$ such that

$$|I(\tilde{z})| \leq c r^{|y|}.$$

Finally, we obtain

$$|C_y| \leq \int_{\tilde{z}\in\mathbb{T}^{d-1}} |\tilde{z}|^{\tilde{\alpha}} |I(\tilde{z})| \frac{d\tilde{z}}{2i\pi\tilde{z}} \leq c r^{|y|} = c\exp(-\ln(1/r)|y|),$$

which concludes the proof. □

This theorem implies the following result on the rate of decay of edge-to-edge correlations for a regular connection.

Corollary 4.24. *If the function $\underline{z} \mapsto P(\underline{z})$ does not vanish on $\mathbb{T}^d$, then, for every couple of edges $(e, e') \in E(\mathbb{Z}^d)^2$, the edge-to-edge correlation $Cov(n_e, n_{e'})$ decreases at exponential rate with the distance $|e - e'| \to \infty$.*

Proof. From Equation (4.6), there exists a polynomial Q such that

$$\tilde{K}_{[e],[e']}(z_1,\ldots,z_d) = \frac{(Q(\underline{z}))_{[e],[e']}}{P(\underline{z})}.$$

Then, Theorem 4.23 implies that the quantity $|K_{e,e'}|$ decreases at an exponential rate when the distance $|e - e'| \to \infty$. From the beginning of Section 4.3, it implies the exponential decay of $Cov(n_e, n_{e'})$. □

Let us emphasize that Corollary 4.24 holds for periodic connections on fundamental domains of any size or for other lattices such as the triangular lattice.

Note that in each direction, the exponential rate of decay is larger than the logarithm of the inverse of the largest radius such that the characteristic polynomial does not vanish in the annulus centered around the unit circle. In particular, the further the roots of P are from the unit torus, the faster the edge-to-edge correlations decrease.

In the following, we will prove that when the connection is singular, then the model is in some way equivalent to a superposition of spanning trees and a model for which the connection is regular. When this is the case, the rate of decay of the edge-to-edge correlations in some directions is not faster than polynomial with the distance.

4.3.3 Singular and reducible connections

Recall the definition 2.17 of a reducible connection as defined in [KL20c] and [KL23].

Definition 4.25 (Reducible connections)**.** [KL23] We say that the connection is reducible if there exists sub-bundles $F^{(1)}, F^{(2)}$ of $F = (F_e)_{e\in\mathbb{Z}^d}$ and connections $h^{(1)}, h^{(2)}$ on these sub-bundles such that $F = F^{(1)} \oplus F^{(2)}, h = h^{(1)} \oplus h^{(2)}$. Then the splitting $\Omega^1(\mathbb{Z}^d) = \oplus F_e$ can be refined in

$$\Omega^1(\mathbb{Z}^d) = F_e^{(1)} \oplus F_e^{(2)}.$$

Periodic unitary connection of rank 1 on $\mathbb{Z}^d$. Notice that if h is a unitary periodic complex connection of rank 1 on a vector-bundle $L^{(i)}$ of rank 1 over the square lattice $\mathbb{Z}^d$, then h is given by some unitary complex numbers $(u_1, \dots, u_d) \in \mathbb{C}^d$, in the sense that for every $j \in [1, d]$,

$$h(x, x + t_j) = u_j$$

where (t_i) is the set of translation-vectors, that is here the canonical basis of $\mathbb{R}^d$.

Let us recall that the case of rank 1 with a trivial connection $h_0 = id$ gives a measure μ_{id} which is the uniform spanning forests measure as studied in [Pem91, BP93, BLPS01, LP16].

In [Ken19] and [Sun16], the model of determinantal cycle-rooted spanning forests associated with a connection is studied for non unitary complex periodic connections on planar periodic graphs. It is established in particular that for a periodic unitary complex connection h, the measure μ_h coincides with μ_{id} and the model has the same law as the uniform spanning forest. Temperley's bijection (see [KPW00]) between spanning forests and dimers and the quadratic decay for the dimer model (see [KSO03]) imply that edge-to-edge correlations have quadratic decay.

The equality between μ_h and μ_{id} for a periodic unitary complex connection h as above relies on the fact that the connection h is gauge equivalent to the trivial connection. This equality can be established with a change of variable in the integral expression of the kernel of the model and therefore we obtain the following extension of the result of [Ken19] and [Sun16] for non planar graphs. Furthermore, we justify that the edge-to-edge correlations decrease polynomially in that case.

Theorem 4.26. *(see [Ken19, Sun16] for planar graphs) If h is a unitary periodic connection of rank 1 as above, the measure μ_h coincides with μ_{id} and the model has the same law as the uniform spanning forest. Edge-to-edge correlations have polynomial decay of order $\frac{1}{|e-e'|^{2d}}$.*

In particular, from [Pem91], under this measure, almost surely, there are only infinite connected components.

Proof. From Equation (4.4) and Equation (4.1), note that the kernel K^h of the model in this case is not the same as the kernel K^{id} of the uniform spanning forest but is the conjugate of the kernel of the uniform spanning tree model by a diagonal unitary matrix as follows.

For every e, e' edges of $\mathbb{Z}^d$,

$$\begin{aligned}K^h_{e,e'} &= \int_{|z_1|=1,\dots,|z_d|=1} \tilde{K}^h_{[e],[e']}(z_1,\dots,z_d) z_1^{x_1-y_1}\dots z_d^{x_d-y_d}\frac{dz_1}{2i\pi z_1}\dots\frac{dz_d}{2i\pi z_d}\\ &= \int_{|z_1|=1,\dots,|z_d|=1} \tilde{K}^{id}_{[e],[e']}(z_1u_1,\dots,z_du_d) z_1^{x_1-y_1}\dots z_d^{x_d-y_d}\frac{dz_1}{2i\pi z_1}\dots\frac{dz_d}{2i\pi z_d}\\ &= (u_1^{x_1}\dots u_d^{x_d})^{-1}K^{id}_{e,e'}u_1^{y_1}\dots u_d^{y_d} = (UK^{id}U^*)_{e,e'}\end{aligned}$$

where $U = diag(u(e))_{e\in E(\mathbb{Z})}$ is a diagonal unitary matrix, whose entries are

$$u([e]+(x_1,\dots,x_d)) = \prod_{i=1}^d u_i^{-x_i}.$$

Then the generating polynomial of the measure μ_h is given by

$$\begin{aligned}g_{\mu_h}(z_1,\dots,z_n) &= \det(I+(diag(z_1,\dots,z_n)-I)K^h_{e_1,\dots,e_n})\\ &= \det(I+(diag(z_1,\dots,z_n)-I)(UK^{id}U^*)_{e_1,\dots,e_n})\\ &= \det(U(I+(diag(z_1,\dots,z_n)-I)K^{id})U^*)_{e_1,\dots,e_n}\\ &= \det(I+(diag(z_1,\dots,z_n)-I)K^{id})_{e_1,\dots,e_n}\\ &= g_{\mu_{id}}(z_1,\dots,z_n).\end{aligned}$$

Then the generating polynomial of the measure μ_h is the same as the generating polynomial of the uniform spanning forest and μ_h coincides with μ_{id}.

It is well known that in the case of the uniform spanning forest, the kernel is the transfer current matrix and that the transfer current $T(e,e')$ is related to the Green's function as follows:

$$T(e,e') = G(e^-,e'^-) - G(e^+,e'^-) - G(e^-,e'^+) + G(e^+,e'^+).$$

From [LL10, Uch98, MS22], for a symmetric random walk on $\mathbb{Z}^d$, there exists a constant a depending on d such that as $\|x-y\|$ goes to infinity in $\mathbb{Z}^d$,

$$G(x,y) = \frac{a}{\|x-y\|^{d-2}} + O\left(\frac{1}{\|x-y\|^d}\right) \tag{4.7}$$

Writing $e=(x,x+e_i), e'=(y,y+e_j)$, when the quantity $|e-e'|$ goes to infinity,

$$\begin{aligned}T(e,e') &= G(x,y) - G(x,y+e_j) - G(x+e_i,y) + G(x+e_i,y+e_j)\\ &= \frac{a}{\|x-y\|^{d-2}} - \frac{a}{\|x-y-e_j\|^{d-2}} - \frac{a}{\|x+e_i-y\|^{d-2}} + \frac{a}{\|x+e_i-y-e_j\|^{d-2}}\\ &+ O\left(\frac{1}{\|x-y\|^d}\right)\end{aligned}$$

Then, if $x-y=(z_1,\dots,z_d)$,

$$\begin{cases}\|x-y\|^2 = \sum z_k^2\\ \|x+e_i-y\|^2 = \sum_{k\neq i} z_k^2 + (z_i+1)^2 = \|x-y\|^2 + 1 + 2z_i\\ \|x-y-e_j\|^2 = \sum_{k\neq j} z_k^2 + (z_j-1)^2 = \|x-y\|^2 + 1 - 2z_j\\ \|x+e_i-y-e_j\|^2 = \sum_{k\neq i,j} z_k^2 + (z_i+1)^2 + (z_j-1)^2 = \|x-y\|^2 + 2z_i - 2z_j + 2.\end{cases}$$

Therefore, we have

$$\begin{cases} \|x+e_i-y\|^{d-2} = \left(\|x-y\|^2\left(1+\frac{1+2z_i}{\|x-y\|^2}\right)\right)^{(d-2)/2} \\ \|x-y-e_j\|^{d-2} = \left(\|x-y\|^2\left(1+\frac{1-2z_j}{\|x-y\|^2}\right)\right)^{(d-2)/2} \\ \|x+e_i-y-e_j\|^{d-2} = \left(\|x-y\|^2\left(1+\frac{2-2z_j+2z_i}{\|x-y\|^2}\right)\right)^{(d-2)/2}, \end{cases}$$

which implies when $x-y=(z_1,\dots,z_d)$ that

$$\begin{aligned} &T(e,e') \\ &= \frac{a\left(1-\left(1+\frac{2z_i+1}{\sum z_k^2}\right)^{-\frac{d-2}{2}}-\left(1+\frac{1-2z_j}{\sum z_k^2}\right)^{-\frac{d-2}{2}}+\left(1+\frac{2-2z_j+2z_i}{\sum z_k^2}\right)^{-\frac{d-2}{2}}\right)}{\|x-y\|^{d-2}} \\ &+ O\left(\frac{1}{\|x-y\|^d}\right) \\ &= \frac{\left(a\frac{d-2}{2}(1+2z_i+1-2z_j-2+2z_j-2z_i)\right)}{\|x-y\|^d} + O\left(\frac{1}{\|x-y\|^d}\right) \\ &= O\left(\frac{1}{\|x-y\|^d}\right). \end{aligned}$$

Thus, coefficients of the transfer current matrix $T(e,e')$ have polynomial decay of order $O\left(\frac{1}{|e-e'|^d}\right)$. The polynomial decay cannot be uniformly faster than $\frac{1}{|e-e'|^d}$. Indeed, if we consider $\beta>1$, and x_0,x,y_0,y such that $|x_0-y_0|\sim n$ and $|x-x_0|\sim|y-y_0|\sim n^\beta$, we have when $(x_0,\dots,x)$ and $(y_0,\dots,y)$ are shortest paths from x_0 to x and from y_0 to y,

$$G_{x,y}-G_{x_0,y_0}=\sum_i\sum_j T(x_ix_{i+1},y_jy_{j+1})$$

If coefficients of the transfer current matrix $T(e,e')$ have polynomial decay of order α then

$$|G_{x,y}-G_{x_0,y_0}|\le n^{-(\alpha-2\beta)}.$$

But we also know from Equation (4.7), that $|G_{x,y}-G_{x_0,y_0}|\sim n^{-(d-2)}$. Then we must have $\alpha-2\beta<d-2$. Therefore, $\alpha<d-2+2\beta<d$. To conclude the proof, the polynomial decay of correlations of order $\frac{1}{|e-e'|^{2d}}$ relies on the expression of correlations in terms of coefficients of the transfer current matrix recalled at the beginning of Section 4.3. □

Let us mention that [KW14, Theorem 1] establishes that for $d\ge3$, on a simply connected finite domain $D\subset\mathbb{Z}^d$, approximated in a "good" way by embedded connected weighted graphs, the transfer current matrix converges, up to a local weight factor, to the differential of Green's function on D. It implies that correlations of the spanning tree model have polynomial decay of order $\frac{1}{n^{2d}}$.

Singular connections are reducible. Assume in the following that the graph is the square lattice $G=\mathbb{Z}^d$ for some fixed $d\ge2$ with a fundamental domain of size 1 is endowed with a connection which is given by some unitary matrices $M_1,\dots,M_d\in U_N(\mathbb{C})$ with $N>1$.

Theorem 4.27. *If a connection is given by $M_1, \dots, M_d \in U_N(\mathbb{C})$ with $N > 1$ is singular, then it is reducible. The pair (F, h) is isomorphic to*

$$(\oplus_{1\leq i\leq p} L^{(i)} \oplus B_{N-p} \quad , \quad \oplus_{1\leq i\leq p} h^{(i)} \oplus h_{N-p}),$$

where p is at least the number of distinct roots of the characteristic polynomial in the unit torus, h_{N-p} is a regular connection and for every $1 \leq i \leq p$, $L^{(i)}$ is a vector-bundle of rank 1 spanned by the i^{th} common eigenvector and $h^{(i)}$ is a periodic connection of rank 1 given by $(u_1^i, \dots, u_d^i) \in \mathbb{C}^d$, the-$i^{th}$ eigenvalues of the matrices $M_1, \dots, M_d$.

Proof. From Theorem 4.12, for every root $\underline{z} = (z_1, \dots, z_d)$ of the characteristic polynomial P, there exists a unit vector X which is a common eigenvector to the matrices M_j.

We construct an orthogonal family $X_1, ..., X_p$ of common eigenvectors of matrices (M_j) by recursion over N, such that the matrices (M_j) restricted to $Vect(X_1, \dots, X_p)^\perp$ do not have any common eigenvector.

If X_1 is a common eigenvector of matrices (M_j) obtained for the root $\underline{z}$, we can consider an orthonormal basis $(e_2, \dots, e_N)$ of $X_1^\perp$ and this subspace is invariant under the action of matrices (M_j). Then, considering the model with basis $(X_1, e_2, \dots, e_N)$ does not change the characteristic polynomial P since it only conjugates the Laplacian operator. Then, P is the product of the determinant P_{N-1} of the Laplacian on the sub-bundle of rank $N-1$ and basis $(e_2, \dots, e_N)$ and the determinant P_1 of the Laplacian on the sub-bundle of rank 1 and basis (X_1). Both determinants P_{N-1} and P_1 are Laurent polynomials with d variables and by construction the only root of P_1 in the unit torus is $\underline{z}$. In particular, if $|r(P)|$ is the number of distinct roots of P in the unit torus and $|r(P_{N-1})|$ is the number of distinct roots of P_{N-1} in the unit torus, we get

$$|r(P_{N-1})| \geq |r(P)| - 1.$$

By assumption of recursion on P_{N-1}, there exists an orthogonal family $X_2, ..., X_p$ of $X_1^\perp$ which is a family of eigenvectors of the $(M_j)_{X_1^\perp}$, with $p - 1 \geq |r(P_{N-1})|$, such that the matrices $(M_j)_{X_1^\perp}$ restricted to $Vect(X_2, \dots, X_p)^\perp$ do not have any common eigenvector. Therefore, we obtain an orthogonal family $(X_1, ..., X_p)$ of common eigenvectors of matrices (M_j), with $p \geq |r(P_{N-1})| + 1 \geq |r(P)|$, such that the matrices (M_j) restricted to $Vect(X_1, \dots, X_p)^\perp$ do not have any common eigenvector.

Then, if we complete the family $X_1, ..., X_p$ into a basis of $\mathbb{C}^N$ over fibers, the pair (F, h) is isomorphic to $(\oplus_{1\leq i\leq p} L^{(i)} \oplus B_{N-p} \quad , \quad \oplus_{1\leq i\leq p} h^{(i)} \oplus h_{N-p})$ where h is the connection given by matrices (M_j) restricted to $Vect(X_1, \dots, X_p)^\perp$ whose characteristic polynomial does not vanish on $\mathbb{T}^d$. Therefore, h_{N-p} is a regular connection. □

Polynomial decay of correlations for singular connections. The following corollary is a consequence of Theorem 4.27 and of Theorem 4.26 which is used as a tool to compute the decay of edge-to-edge correlations for a reducible connection.

Corollary 4.28. *If a connection is given by $M_1, \dots, M_d \in U_N(\mathbb{C})$ with $N > 1$ is singular, then for some sequences edges $(e_k)_k, (e'_k)_k$ such that $|e_k - e'_k| \to_{k\to\infty} \infty$, we have*

$$Cov(n_{e_k}, n_{e'_k}) \to 0$$

at polynomial rate of order $\frac{1}{|e_k - e'_k|^{2d}}$ for $(e_k)_k, (e'_k)_k$ well-chosen.

Proof. Assume that the connection is singular. Thus, from Theorem 4.27, the connection is reducible and there exists an orthogonal family $(X_1, \dots, X_p)$ of $\mathbb{C}^N$ of eigenvectors of the matrices M_j. Then, if we complete this family in an orthonormal basis of $\mathbb{C}^N$, for this choice of basis for each fiber of the bundle, the kernel of the process is the matrix $(K^{i,j}_{e,e'})_{e,e'\in E, i,j\in[1,N]}$ where the sub-matrix obtained for every $i, j \in [1, p]$ is a block-diagonal matrix, all of whose blocks $(K^{i,i}_{e,e'})_{e,e'\in E}$ are the kernels associated to the rank 1 vector bundle $L^{(i)}$ and the connection $h^{(i)}$ of rank 1 given by $(u^i_1, \dots, u^i_d) \in \mathbb{C}^d$.

From Theorem 4.26, the quantity $|K^{i,i}_{e,e'}|$ decreases at a polynomial rate of order $\frac{1}{|e-e'|^{2d}}$ when $|e - e'|$ goes to infinity for some edges e, e'. Indeed, when $i \in [1, p]$,

$$-|K^{i,i}_{e,e'}|^2 = Cov(1_{b_i\in Q_e}, 1_{b_i\in Q_{e'}}) = Cov(1_{e\in T_i}, 1_{e'\in T_i})$$

where T_i has the same law as a uniform spanning forest, as in [BP93, BLPS01].

Therefore, $|Cov(n_e, n_{e'})|$ is a sum of terms with a positive sign, some of which decrease at a polynomial rate of order $\frac{1}{|e-e'|^{2d}}$ and other terms decrease at an exponential rate. Therefore,

$$|Cov(n_e, n_{e'})| = \sum_{i,j} |K^{i,j}_{e,e'}|^2 \geq |K^{1,1}_{e,e'}|^2 = |Cov(1_{e\in T_1}, 1_{e'\in T_1})|$$

decreases at a polynomial rate of order $\frac{1}{|e-e'|^{2d}}$ for e, e' well chosen. □

Null-correlations for diagonally-translated edges. In this subsection, we are interested in the correlations between edges which are translated on the diagonal for a specific class of connections on the square lattice $\mathbb{Z}^d$.

Theorem 4.29. *Let $M \in U_N(\mathbb{C})$. For all $(M^{\varepsilon_1}, \dots, M^{\varepsilon_d})$ with $(\varepsilon_1, \dots, \varepsilon_d) \in \{\pm 1\}^d$ which define the kernel K as above, for all couple of edges $e, e' = e + (\varepsilon_1 x, \dots, \varepsilon_d x)$ with $x \in \mathbb{Z}\backslash\{0\}$, we have*

$$K_{e,e'} = 0_{M_N(\mathbb{C})}.$$

Proof. Assume firstly that $M_1 = \dots = M_d =: M \in U_N(\mathbb{C})$. Let $e, e' \in E$ edges such that $e' = e + (x, \dots, x)$ with $x \in \mathbb{Z}\backslash\{0\}$. Then $[e], [e']$ are equal in G_1 and can be written as e_k with $k \in [1, d]$. Then,

$$\begin{aligned} K_{e,e'} &= \int_{|z_1|=1,\dots,|z_d|=1} K_{[e],[e']}(z_1, \dots, z_d) z_1^x \dots z_d^x \frac{dz_1}{2i\pi z_1} \dots \frac{dz_d}{2i\pi z_d} \\ &= \frac{1}{(2i\pi)^d} \int_{\underline{z}\in\mathbb{T}^d} K_{[e],[e']}(\underline{z}) \underline{z}^{\underline{x}} \frac{dz}{\underline{z}}. \end{aligned}$$

And if $\sigma \in S_d$ and $\underline{\sigma(\underline{z})} =: (z_{\sigma(1)}, \dots, z_{\sigma(d)})$, then since $\underline{x} = (x, \dots, x)$, we have

$$\frac{\underline{z}^{\underline{x}}}{\underline{z}} = \frac{\underline{\sigma(\underline{z})}^{\underline{x}}}{\underline{\sigma(\underline{z})}} \quad \forall \underline{z} \in \mathbb{T}^d.$$

Since $M_1 = \dots = M_d = M$,

$$K_{e_k,e_k}(\underline{z}) = K_{e_{\sigma(k)},e_{\sigma(k)}}(\underline{\sigma(\underline{z})}) \quad \forall \underline{z} \in \mathbb{T}^d.$$

Then, with the change of variable $\underline{z} \mapsto \underline{\sigma(z)}$,

$$K_{e,e'} = \frac{1}{(2i\pi)^d} \int_{\underline{z}\in\mathbb{T}^d} \left(\frac{1}{d!} \sum_{\sigma\in S_d} K_{e_{\sigma(k)},e_{\sigma(k)}}(\underline{\sigma(z)}) \right) \underline{z}^{\underline{x}} \frac{d\underline{z}}{\underline{z}}$$
$$= \frac{1}{d!} \frac{1}{(2i\pi)^d} \int_{\underline{z}\in\mathbb{T}^d} \left(\sum_{\sigma\in S_d} K_{e_{\sigma(k)},e_{\sigma(k)}}(\underline{z}) \right) \underline{z}^{\underline{x}} \frac{d\underline{z}}{\underline{z}}.$$

Since $\sum_{\sigma\in S_d} K_{e_{\sigma(k)},e_{\sigma(k)}}(\underline{z})$

$$= \sum_{\sigma\in S_d} (d(\underline{z})\tilde{\Delta}(\underline{z})^{-1} d^*(1/\underline{z}))_{\sigma(k),\sigma(k)}$$
$$= \sum_{\sigma\in S_d} (I_N - z_{\sigma(k)}M)(\sum_m 2I_N - z_m M - (z_m M)^{-1})^{-1}(I_N - z_{\sigma(k)}^{-1}M^{-1})$$
$$= \sum_{1\le i\le d} \sum_{\{\sigma\in S_d, \sigma(k)=i\}} (I_N - z_i M)(I_N - z_i^{-1}M^{-1})(\sum_m 2I_N - z_m M - (z_m M)^{-1})^{-1}$$
$$= (d-1)! \sum_{1\le i\le d} (I_N - z_i M)(I_N - z_i^{-1}M^{-1})(\sum_m 2I_N - z_m M - (z_m M)^{-1})^{-1}$$
$$= (d-1)!\tilde{\Delta}(\underline{z})\tilde{\Delta}(\underline{z})^{-1} = (d-1)!I_N.$$

we have

$$K_{e,e'} = \frac{(d-1)!}{d!} \frac{1}{(2i\pi)^d} \int_{\underline{z}\in\mathbb{T}^d} (I_N)\, \underline{z}^{\underline{x}} \frac{d\underline{z}}{\underline{z}} = 0_{M_N(\mathbb{C})}.$$

We consider now the model with $(M_1, \dots, M_i^{-1}, \dots, M_d)$. If $(z_1, \dots, z_d) \in \mathbb{T}_d$,

$$K_{[e],[e']}^{(M_1,\dots,M_i^{-1},\dots,M_d)}(z_1, \dots, \bar{z}_i, \dots, z_d) = K_{[e],[e']}^{(M_1,\dots,M_i,\dots,M_d)}(z_1, \dots, z_i, \dots, z_d)$$

Then, with the change of variable $z_i \mapsto \bar{z}_i$,

$$K_{e,e'}$$
$$= \int_{|z_1|=1,\dots,|z_d|=1} K_{[e],[e']}^{(M_1,\dots,M_i^{-1},\dots,M_d)}(z_1, \dots, \bar{z}_i, \dots, z_d) z_1^x \dots \bar{z}_i^{-x} \dots z_d^x \prod_j \frac{dz_j}{2i\pi z_j}$$
$$= \int_{|z_1|=1,\dots,|z_d|=1} K_{[e],[e']}^{(M_1,\dots,M_i,\dots,M_d)}(z_1, \dots, z_i, \dots, z_d) z_1^x \dots z_i^x \dots z_d^x \frac{dz_1}{2i\pi z_1} \dots \frac{dz_d}{2i\pi z_d}$$
$$= 0_{M_N(\mathbb{C})}$$

which concludes the proof. □

This theorem extends the result [LP16, Theorem 4.8] which holds for the uniform spanning forest obtained as the limit of uniform spanning trees.

It follows from this theorem that the edge-to-edge correlations do not decrease at a polynomial rate in all directions.

Larger fundamental domains and other lattices For larger fundamental domains, the following result gives a link between the number of roots of the characteristic polynomial on the unit torus and the dimension of the kernel of the covariant derivative.

Theorem 4.30. *Each distinct root of the characteristic polynomial gives an eigenvector which defines a periodic horizontal 0-form in* $\ker d_h$*. Then, the dimension of* $\ker(d_h)$ *is larger than the number of distinct roots of the characteristic polynomial* P *in the unit torus and smaller than the number of roots of the characteristic polynomial* P *in the unit torus counted with multiplicity.*

Proof. On the one hand, if $(f_1, \dots, f_k)$ is an orthogonal basis of $\ker d_h$, then the eigenvectors $X_1, \dots, X_k$ to the holonomies $hol_h(\tilde{\gamma}_1(x)), \dots, hol_h(\tilde{\gamma}_d(x))$, obtained from the construction in the proof of Theorem 4.22 are such that for every i, $X_i \in Vect(\{f_i(v), v \in V_x\})$, and thus $X_1, \dots, X_k$ are orthogonal vectors and the characteristic polynomial has at least k roots counted with multiplicity.

On the other hand, if P has $|r(P)|$ distinct roots, as in the proof of Theorem 4.27, we construct by recursion an orthogonal family of common eigenvectors $(X_1, \dots, X_p)$ to the holonomies of cycles which form a basis of the fundamental group, with $p \geq |r(P)|$. Those eigenvectors are associated with eigenvalues which are given by roots of P.

It gives a family of p orthogonal periodic sections $(f_1, \dots, f_p)$ (in the sense of Definition 4.2), whose periodicity is given by the associated eigenvalues, which are in $\ker d_h$. Therefore, $\ker d_h$ is of dimension larger than p and then larger than $|r(P)|$. □

Theorem 4.30 implies the following result, which relies on a statement of [KL23]. We give the proof for completeness.

Corollary 4.31. *If a periodic connection is singular, then it is reducible. The pair* (F, h) *is isomorphic to*

$$(\oplus_{1\leq i\leq p} L^{(i)} \oplus B_{N-p}, \quad \oplus_{1\leq i\leq p} h^{(i)} \oplus h_{N-p}),$$

where p *is the dimension of* $\ker d_h$*,* h_{N-p} *is a regular connection and for every* $1 \leq i \leq p$*,* $L^{(i)}$ *is a line bundle endowed with a trivial connection* $h^{(i)}$*.*

Proof. Let $(f_1, \dots, f_p)$ be an orthogonal basis of $\ker d_h$. For each $i \in \{1, \dots, p\}$, define, for each vertex v and for each edge e,

$$L^{(i)}_v = \mathbb{C} f_i(v), \quad L^{(i)}_e = \mathbb{C} h_{e,e^-} f_i(e^-)$$

Finally, let B^{N-p} be the orthogonal in the vector bundle F of the sum $\oplus L^{(i)}$. Therefore, the pair (F, h) is isomorphic to

$$(\oplus_{1\leq i\leq p} L^{(i)} \oplus B_{N-p}, \quad \oplus_{1\leq i\leq p} h^{(i)} \oplus h_{N-p}),$$

and by construction, the restriction h_{N-p} is a regular connection of rank $N-p$ on B_{N-p} and for every $1 \leq i \leq p$, $L^{(i)}$ is a vector-bundle of rank 1 and $h^{(i)}$ is the trivial connection. □

Note that from [Uch98], the asymptotic development of the Green function holds more generally on other $\mathbb{Z}^d$-periodic lattices (see also [KW14] for a comparison of the transfer current kernel with the second derivative of the Green function on $\mathbb{R}^d$). In particular, when a connection is singular and thus reducible, the edge-to-edge correlations also decrease at polynomial rate.

Chapter 5

Wilson algorithm measures on spanning forests.

In this chapter, we study probability measures on cycle-rooted spanning forests associated with a weight function on cycles which takes values in $[0,1]$, as defined in Section 2.2.

Let $G = (V, E)$ be a finite connected graph endowed with a weight function on cycles $p : \mathcal{C}_{\to}(G) \to [0,1]$, which is symmetric under orientation reversal.

If $W \subset V$, recall that we say that F is a wired (or essential) oriented cycle-rooted spanning forest with respect to W if every connected component of F is either an oriented cycle-rooted tree disjoint from W or an unrooted tree which contains a unique vertex of W, seen as a tree rooted at that vertex. We denote by $\mathcal{U}_{\to W}(F)$ the set of essential oriented cycle-rooted spanning forests with respect to W. There exists a natural Boltzmann probability measure on $\mathcal{U}_{\to W}(F)$, denoted by $\mu^W_{c,p}$ which is defined for every $F \in \mathcal{U}_{\to W}(F)$ by

$$\mu^W_{c,p}(F) = \frac{\prod_{e\in F} c(e) \prod_{\gamma\in\mathcal{C}_{\to}(F)} p(\gamma)}{Z^W_{c,p}}$$

where $Z^W_{c,p} = \sum_{F\in\mathcal{U}_{\to W}(F)} \prod_{e\in E} c(e) \prod_{\gamma\in\mathcal{C}_{\to}(F)} p(\gamma)$ is the partition function.

The weight function p does not necessarily come from a connection h and therefore, the measure $\mu^W_{c,p}$ is not necessarily determinantal. As recalled in subsection 2.2.4, the measure $\mu^W_{c,p}$ can be sampled by an algorithm of loop-erased random walk where we keep an oriented cycle γ with probability $p(\gamma)$.

- Let $x_1, \dots, x_n$ be an ordering of the vertex set V of G and let $\mathsf{F}_0 = W$.
- At each step i, let $(X^{(x_i)}_n)_{n\geq 0}$ be a random walk on the graph G with conductances $(c(e))$ starting from x_i. Every time the random walk makes a loop, the oriented cycle γ is kept with probability $p(\gamma)$ or erased with probability $1 - p(\gamma)$.
- The random walk $(X^{(x_i)}_n)_{n\geq 0}$ is stopped when it meets W or reaches the set of already explored vertices denoted by $V(\mathsf{F}_{i-1})$ or when a cycle is kept. At the end of the i^{th} step, let $\mathsf{F}_i = \mathsf{F}_{i-1} \cup L(X^{(x_i)}_n)$ where $L(X^{(x_i)}_n)$ is obtained from $(X^{(x_i)}_n)_{n\geq 0}$ after removing all the loops except the last one if a loop is kept at the end of the i^{th} step.

- At the end, $V(\mathsf{F}_n) = V(G_n)$. Notice that the algorithm always finishes if and only if there exists at least a loop γ in G such that $p(\gamma) > 0$ or if $W \neq \emptyset$.

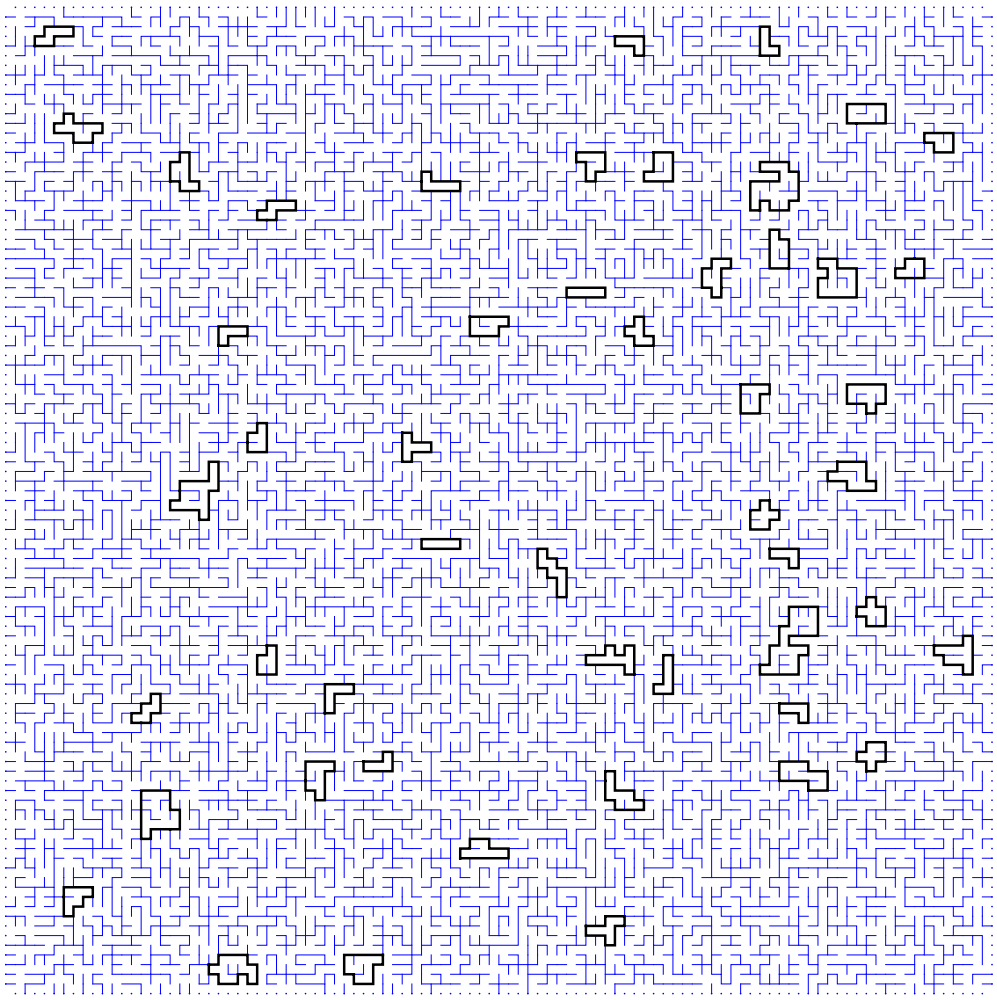

Figure 5.1 – Simulation of an ECRSF for $p : \gamma \mapsto 0.75$ and $c = 1$.

When the size of the graph is large, we observe for positive constant values of p, with high probability, that the size of the cycles remains small and that the points which are far away from the boundary are not connected to the boundary.

In the next section, we will introduce a tool, so called p-loop erased random walks, which will be useful to extend the algorithm described just above for finite graphs to infinite graphs in order to define an infinite volume measure on cycle-rooted spanning forests of an infinite graph.

In the remainder of this chapter, we will assume that the conductances are equal to 1 in order to simplify the notations but it is possible to consider other conductances by replacing every simple random walk on G by random walks with jump probabilities proportional to conductances.

5.1 p-Loop erased random walks and rooting times

In the following, we will consider a countably infinite connected graph $G = (V, E)$, with bounded vertex degree, exhausted by an increasing sequence $(G_n)_{n\geq 1}$ of connected induced subgraphs of G, with respective vertex set V_n. We denote by ∂G_n the subset of V_n of vertices which are connected by an edge to the complement of G_n in G. For every $v \in V$, we denote by $\mathbb{P}_v$ the law of a simple random walk on G starting from v.

We assume that G is endowed with a weight function $p : \mathcal{C}_{\rightarrow}(G) \rightarrow [0, 1]$ on cycles, which is symmetric under orientation reversal, and we assume that the following assumption is satisfied for the exhaustion (G_n) of the graph G and the weight function p.

Assumption 5.1. *There exist $\alpha > 0$ and $\beta > 0$, such that for every $n \in \mathbb{N}^*$, for every vertex $v \in \partial G_n$, for every random walk $(X_k)_{k\geq 0}$ on G, starting from v, there exists a loop γ_v in $G_{n+1}\backslash(G_n \cup \partial G_{n+1})$ which satisfies $p(\gamma_v) \geq \alpha$ and $\mathbb{P}_v((X_1, \ldots, X_{|\gamma_v|}) = \gamma_v) > \beta$.*

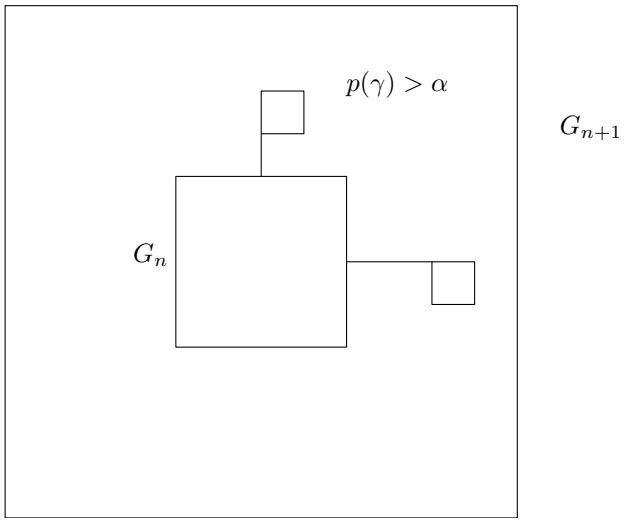

Figure 5.2 – Assumption 5.1

5.1.1 Hitting times

In the following we denote by v_0 a vertex of G_1.

Definition 5.2. If C is a subset of the vertex set V, we define for a random walk (X_k) the hitting time of C, that is to say

$$T_C := \min\{k \geq 0 | X_k \in C\}.$$

Notice that in this definition, T_C can be equal to 0 if the random walk starts from a vertex of C.

Definition 5.3. Let (X_k) be a simple random walk starting from v_0. Let (T_n) be the sequence of random hitting times of ∂G_n for the random walk (X_k), that is to say

$$T_n := T_{\partial G_n} = \min\{k \geq 0 | X_k \in \partial G_n\}.$$

Lemma 5.4. *The hitting-time T_n is finite almost-surely for every $n \in \mathbb{N}^*$. Furthermore,*

$$\lim_{k\to\infty} \mathbb{P}_{v_0}(T_n \geq k) = \mathbb{P}_{v_0}(T_n = \infty) = 0.$$

Proof. Let $n \in \mathbb{N}^*$. Almost surely, T_n is finite because almost surely if $k \geq 1$, there exists a time such that the random walk makes k consecutive steps in the same direction. Therefore, the random walk exits every finite ball in finite time almost surely. Since the events $(T_n \geq k)$ are decreasing in k (for a fixed n) with respect to inclusion, the monotone convergence theorem implies

$$\lim_{k\to\infty} \mathbb{P}_{v_0}(T_n \geq k) = \mathbb{P}_{v_0}\left(\bigcap_{k\geq 1} \{T_n \geq k\}\right) = \mathbb{P}_{v_0}(T_n = \infty),$$

which concludes the proof. □

5.1.2 Rooting time

Let (X_n) be a simple random walk on G starting from v_0 and let (Y_n) be a sequence of independent random variables of uniform law on $[0, 1]$, which are independent of the random variables X_n.

We want to define a p-loop-erased random walk such that, if at time n, the random walk (X_n) closes a loop γ_n, the loop is kept if $Y_n \leq p(\gamma_n)$ and erased else.

Given (X_n, Y_n), we construct a sequence of random walks $((Z_n^k)_n)_k$ as follows. We define recursively $(Z_n^k)_{n\geq 1}$ for $k \in \mathbb{N}^*$. Let $(Z_n^1) = (X_n)$ and given $(Z_n^k)_n$, let us consider the first time n_k such that Z_n^k closes a loop that is to say

$$n_k = \min\{j > n_{k-1} \in \mathbb{N}^* | Z_j^k \in \{Z_0^k, \dots, Z_{j-1}^k\}\}.$$

Then, let n_k' be the time of the beginning of the loop, that is to say

$$n_k' = \min\{j \in \mathbb{N}, Z_j^k = Z_{n_k}^k\}.$$

Therefore, the loop which is closed at time n_k is the loop $\gamma_{n_k} := (Z_{n_k'}^k, \dots, Z_{n_k}^k)$ Finally, if $Y_{n_k} \geq p(\gamma_{n_k})$, then define for every $n \in \mathbb{N}$,

$$Z_n^{k+1} = \begin{cases} Z_{n_k}^k & \text{if } n_k' \leq n \leq n_k, \\ Z_n^k & \text{else.} \end{cases}$$

The random walk $(Z_n^{k+1})_{n\geq 0}$ is obtained from $(Z_n^k)_{n\geq 0}$ by erasing the loop γ_{n_k}.

Otherwise, if $Y_{n_k} \leq p(\gamma_{n_k})$, for every $m \geq k + 1$ and $n \in \mathbb{N}$, let

$$Z_n^m = \begin{cases} Z_n^k & \text{if } n \leq n_k, \\ Z_{n_k}^k & \text{else.} \end{cases}$$

The random walk $(Z_n^m)_{n\geq 0}$ is obtained from $(Z_n^k)_{n\geq 0}$ by stopping the random walk at time n_k.

Definition 5.5. If (X_n) is a simple random walk on G starting from v_0 and (Y_n) is a sequence of independent random variables of uniform law on $[0,1]$, which are independent of the X_n, we say that $(n_k)_{n\geq 1}$ is the sequence of random times where the random walk (X_n) closes a loop γ_{n_k}. Let T_r be called the *random rooting time* for (X_n, Y_n) that is to say the first time where a loop is kept:

$$T_r := \min\{n_k | Y_{n_k} \leq p(\gamma_{n_k})\}.$$

where $\min \emptyset = +\infty$. If k is such that $T_r = n_k$, then let $(Z_n)_{n\leq T_r} = (Z_n^k)_{n\leq T_r}$ be called the *p-loop-erased random walk* obtained from (X_n, Y_n).

Let us emphasize that if T_r is finite, then there exists a k such that $T_r = n_k$ and then the p-loop erased random walk $(Z_n)_{n\leq T_r}$ is well defined and is obtained from $(X_n)_{n\leq T_r}$, erasing every loop excepted the last one. Here, the loop-erased random walk is indexed on the same time set than the random walk (X_n). Nevertheless, Z_n does not depend only on $(X_k)_{k\leq n}$.

For instance, if there exists exactly two disjoint loops $(X_{n_1'}, \dots, X_{n_1})$, $(X_{n_2'}, \dots, X_{n_2})$ with $n_2' \geq n_1$ and if $Y_{n_1} > p(X_{n_1'}, \dots, X_{n_1}), Y_{n_2} \leq p(X_{n_2'}, \dots, X_{n_2})$, then $T_r = n_2$ and

$$Z_n = \begin{cases} X_{n_1} & \text{if } n_1' \leq n \leq n_1, \\ X_n & \text{if } n_1 \leq n \leq n_2. \end{cases}$$

In particular, for $n_1' \leq n \leq n_1$, Z_n depends on $(X_k)_{k\leq n_1}$.

5.1.3 The rooting time is almost surely finite.

We will show in this subsection that the rooting time T_r is a stopping time and that almost surely, it is finite.

Definition 5.6. Let $(\mathcal{F}_n)_n$ be the filtration adapted to the process $((X_n, Y_n))_n$, that is defined by

$$\mathcal{F}_n = \sigma(X_0, \dots, X_n, Y_0, \dots, Y_n),$$

which is the smallest sigma-field which makes the $(X_i)_{0\leq i\leq n}, (Y_i)_{1\leq i\leq n}$ measurable.

Lemma 5.7. *For every $m \in \mathbb{N}^*$, the hitting time of ∂G_m, denoted by T_m, is a stopping time with respect to the filtration $(\mathcal{F}_n)_n$. The rooting time T_r is also a stopping time with respect to the filtration $(\mathcal{F}_n)$. Moreover, for every $m \in \mathbb{N}$, if we consider the σ-field adapted to the stopping time T_m, defined by*

$$\mathcal{F}_{T_m} = \{A \in \mathcal{F} : \forall k \geq 0, \{T_m \leq k\} \cap A \in \mathcal{F}_k\},$$

then, the event $\{T_m < T_r\}$ is in $\mathcal{F}_{T_m}$.

Proof. Let $m \in \mathbb{N}$. The events $(\{T_m \geq k\} = \{X_1, \dots, X_k \in G_m \backslash \partial G_m\})_k$ are measurable with respect to $(\mathcal{F}_k)_k$ and therefore T_m is a stopping time.

The proof for T_r relies on the fact that the event $\{T_r \leq k\}$ only depends on the random walk $(X_n, Y_n)_{n\leq k}$ and therefore, for every k, the event $\{T_r \geq k\}$ is in $\mathcal{F}_k$.

Let $k \in \mathbb{N}$. Let us define a family of functions $(F_i)_{1\leq i\leq k}$ such that

$$\max_{1\leq i\leq k} F_i(X_0,\dots,X_k,Y_0,\dots,Y_k) = 1$$

if and only if the event $\{T_r \leq k\}$ is satisfied. Let $(\tilde{Z}^1_n)_{n\leq k} = (X_n)_{n\leq k}$ and $\tilde{n}_0 = 0$. Then, we define recursively a sequence $((\tilde{Z}^i_n)_{n\leq k})_{1\leq i\leq k}$ and a sequence $(\tilde{n}_i)_{i\leq k}$ as follows.

Define $\tilde{n}_i : (X_0,\dots,X_k) \mapsto \min\{\tilde{n}_{i-1} < j \leq k | \tilde{Z}^i_j \in \{\tilde{Z}^i_0,\dots,\tilde{Z}^i_{j-1}\}\}$ with the convention $\min\emptyset = +\infty$.

If $\tilde{n}_i(X_0,\dots,X_k) = +\infty$, define $F_i(X_0,\dots,X_k,Y_0,\dots,Y_k) = 0$ and $\tilde{Z}^{i+1} = \tilde{Z}^i$.

Else, let $\tilde{n}'_i$ be the first time $j < \tilde{n}_i$ such that $\tilde{Z}^i_{n_i} = \tilde{Z}^i_{\tilde{n}'_i}$ and if $Y_{\tilde{n}_i} \leq p(\gamma_{\tilde{n}_i})$, let

$$\begin{cases} F_i(X_0,\dots,X_k,Y_0,\dots,Y_k) = 1_{Y_{\tilde{n}_i}\leq p(\gamma_{\tilde{n}_i})} \\ \tilde{Z}^{i+1} = \tilde{Z}^i, \end{cases}$$

and otherwise, let for $n \leq k$,

$$\tilde{Z}^{i+1}_n = \begin{cases} \tilde{Z}^i_{\tilde{n}_i} & \text{if } \tilde{n}'_i \leq n \leq \tilde{n}_i \\ \tilde{Z}^i_n & \text{else.} \end{cases}$$

Observe that the sequence $(n_i)_i$ is increasing and while i is such that $n_i \leq k$, we have $\tilde{n}_i = n_i$ and $\tilde{Z}^i = Z^i$. For i such that $n_i \geq k$, $\tilde{n}_i = +\infty$ and $F_i(X_0,\dots,Y_k) = 0$. In particular, $\max_{1\leq i\leq k} F_i(X_0,\dots,X_k,Y_0,\dots,Y_k) = 1$ if and only if there exists i such that $n_i \leq k$ and such that $Y_{n_i} \leq p(\gamma_{n_i})$, which is exactly the event $\{T_r \leq k\}$.

The event $\{T_m < T_r\}$ is in $\mathcal{F}_{T_m}$ because if $k \geq 0$,

$$\{T_m \leq k\} \cap \{T_m < T_r\} = \cup_{1\leq i\leq k} (\{T_m = i\} \cap \{T_r > i\}) \in \mathcal{F}_k.$$

which concludes the proof. □

Let us emphasize that T_r is a stopping time for the filtration $(\mathcal{F}_n)$ even if (Z_n) is not adapted to the filtration $(\mathcal{F}_n)$. Indeed, Z_n depends on (X_k) for $k \geq n$. Lemma 5.7 is a useful tool to show that the rooting time is almost surely finite for a simple random walk starting from v_0.

Lemma 5.8. *Under Assumption 5.1, the rooting time T_r for a simple random walk (X_n) starting from v_0 and (Y_n) as defined in Definition 5.5 is finite almost surely and $\mathbb{P}_{v_0}(T_r > T_m)$ decays exponentially fast to 0 with m. More precisely, there exists $\delta \in]0,1[$ such that*

$$\mathbb{P}_{v_0}(T_m < T_r) \leq \delta^m.$$

Proof. Let $m \in \mathbb{N}^*$ be fixed. The process $((X_k,Y_k))_{k\geq 0}$ satisfies the strong Markov property. Therefore, conditional on the event $\{T_m < \infty\}$ which is almost sure by Lemma 5.4, for every $k \geq 0$, the pair of random variables (X_{T_m+k}, Y_{T_m+k}) is independent of $\mathcal{F}_{T_m}$ given (X_{T_m}, Y_{T_m}).

From Assumption 5.1, there exists a loop $\gamma_{X_{T_m}}$ which lies inside $G_{m+1}\backslash(G_m \cup \partial G_{m+1})$ with weight larger than α and such that the probability that a random walk $(X_{T_m+k})_k$ makes this loop $\gamma_{X_{T_m}}$ is greater than β.

Let us denote by $A_{X_{T_m}}$ the event that the random walk $(X_{T_m+k})_k$ makes this loop $\gamma_{X_{T_m}}$, and let us denote by $B_{X_{T_m}}$ the event $\{Y_{T_m+|\gamma_{X_{T_m}}|} \leq p(\gamma_{X_{T_m}})\}$. Conditional on X_{T_m}, the events $A_{X_{T_m}}$ and $B_{X_{T_m}}$ are independent and have probabilities $\mathbb{P}_{X_{T_m}}(\gamma_{X_{T_m}}) \geq \beta$ and $p(\gamma_{X_{T_m}}) \geq \alpha$.

The event $A_{X_{T_m}} \cap \{Y_{T_m+|\gamma_{X_{T_m}}|} \leq p(\gamma_{X_{T_m}})\}$ for the random walk $(X_{T_m+k}, Y_{T_m+k})_{k\geq 0}$ starting from $(X_{T_m}, Y_{T_m}) \in \partial G_m$ has a probability greater than $\alpha\beta$.

Conditional on (X_{T_m}, Y_{T_m}), it is independent of $\mathcal{F}_{T_m}$, therefore from Lemma 5.7, it is independent of the event $\{T_m < T_r\}$.

Conditional on $T_m < T_r$, if the event $A_{X_{T_m}} \cap \{Y_{T_m+|\gamma_{X_{T_m}}|} \leq p(\gamma_{X_{T_m}})\}$ is satisfied, then the event $\{T_{m+1} > T_r\}$ is satisfied. Indeed, we show that on this event, the random walk keeps a loop before reaching ∂G_{m+1} and therefore $T_{m+1} > T_r$.

Let i be the largest integer such that $n_i \leq T_m$. Then, by construction of $(Z_n^{i+1})_n$, assuming $T_r > T_m \geq n_i$, (Z_n^{i+1}) coincides with (X_n) after time n_i and therefore after time T_m. For $n \leq T_m$, $Z_n^{i+1} \in \{X_0, \dots, X_{T_m}\}$ by construction and therefore $Z_n^{i+1} \in G_m$.

If the event $A_{X_{T_m}} \cap \{Y_{T_m+|\gamma_{X_{T_m}}|} \leq p(\gamma_{X_{T_m}})\}$ is satisfied, then, for n between $T_m + 1$ and $T_m + |\gamma_{X_{T_m}}|$, we have $Z_n^{i+1} \in G_{m+1} \backslash (G_m \cup \partial G_{m+1})$, and therefore, for such a n,

$$Z_n^{i+1} \notin (Z_0^{i+1}, \dots, Z_{T_m}^{i+1}).$$

Since we have $n_{i+1} \geq T_m$ by assumption on i, we have necessarily $n_{i+1} = T_m + |\gamma_{X_{T_m}}|$. Since the event $\{Y_{T_m+|\gamma_{X_{T_m}}|} \leq p(\gamma_{X_{T_m}})\}$ is satisfied by assumption and $T_r > n_i$, we have $T_r = n_{i+1} = T_m + |\gamma_{X_{T_m}}|$, and since $A_{X_{T_m}}$ is satisfied, for $T_m + 1 \leq n \leq T_m + |\gamma_{X_{T_m}}|$, we have $X_n \in G_{m+1} \backslash (G_m \cup \partial G_{m+1})$ and therefore $T_{m+1} > T_m + |\gamma_{X_{T_m}}| = T_r$.

Therefore, denoting by $\delta := 1 - \alpha\beta < 1$,

$$\begin{aligned}
\mathbb{P}_{v_0}(T_{m+1} < T_r \mid T_m < T_r) &\leq 1 - \mathbb{P}_{(X_{T_m}, Y_{T_m})}(A_{X_{T_m}} \cap \{Y_{T_m+|\gamma_{X_{T_m}}|} \leq \alpha\} \mid T_m < T_r) \\
&= 1 - \mathbb{P}_{(X_{T_m}, Y_{T_m})}(A_{X_{T_m}} \cap \{Y_{T_m+|\gamma_{X_{T_m}}|} \leq \alpha\}) \\
&= 1 - \mathbb{P}_{X_{T_m}}(A_{X_{T_m}})\mathbb{P}(Y_{T_m+|\gamma_{X_{T_m}}|} \leq \alpha) \\
&\leq 1 - \alpha\beta = \delta.
\end{aligned}$$

This inequality holds for every $m \in \mathbb{N}^*$ and δ does not depend on m. Then, writing

$$\mathbb{P}_{v_0}(T_{m+1} < T_r) = \mathbb{P}_{v_0}(T_{m+1} < T_r \mid T_m < T_r)\mathbb{P}_{v_0}(T_m < T_r),$$

we obtain by induction on m the exponential decay of the following probability:

$$\mathbb{P}_{v_0}(T_m < T_r) \leq \delta^m.$$

Let $\varepsilon > 0$. For fixed m large enough, $\delta^m \leq \varepsilon$.

For k large enough, we have $\mathbb{P}_{v_0}(T_m \geq k) \leq \varepsilon$. Then for k large enough, we have

$$\begin{aligned}
\mathbb{P}_{v_0}(T_r \geq k) &= \mathbb{P}_{v_0}(\{T_r \geq k\} \cap \{T_m \geq k\}) + \mathbb{P}_{v_0}(\{T_r \geq k\} \cap \{T_m \leq k-1\}) \\
&\leq \mathbb{P}_{v_0}(T_m \geq k) + \mathbb{P}_{v_0}(T_r > T_m) \leq 2\varepsilon.
\end{aligned}$$

Therefore we have shown that for every $\varepsilon > 0$, for k large enough, $\mathbb{P}_{v_0}(T_r \geq k) < 2\varepsilon$, which means that

$$\lim_{k\to\infty} \mathbb{P}_{v_0}(T_r \geq k) = 0.$$

Therefore, from the monotone convergence theorem,

$$\mathbb{P}_{v_0}(T_r = \infty) = \mathbb{P}_{v_0}(\bigcap_{k\geq 1}\{T_r \geq k\}) = \lim_{k\to\infty} \mathbb{P}_{v_0}(T_r \geq k) = 0.$$

This concludes the proof. □

The proof of Lemma 5.8 can be adapted to show that the rooting time is almost surely finite for a random walk starting from another vertex of G, even if this vertex is not in G_1, as follows.

Lemma 5.9. *Let $(X_n^{(x)})$ be a random walk starting from $x \in G$ and let (Y_n) be the process defined in Definition 5.5. Under Assumption (5.1), the rooting time T_r for $(X_n^{(x)})$ is finite almost surely and $\mathbb{P}_x(T_r > T_n)$ decays exponentially fast to 0 with n.*

Proof. Notice that x is not anymore in G_1 and therefore the bound $\mathbb{P}_x(T_n \leq T_r) \leq \delta^n$ does not hold.

Nevertheless, if m_x is such that $x \in G_{m_x}$, then for $n \geq m_x$, the proof of Lemma 5.8 shows that for the loop-erased random walk starting from x,

$$\mathbb{P}_x(T_{n+1} < T_r | T_n < T_r) \leq \delta,$$

and therefore, for $n \geq m_x$,

$$\mathbb{P}_x(T_n < T_r) \leq \delta^{n-m_x}.$$

Therefore $\mathbb{P}_x(T_n \leq T_r)$ tends to 0 exponentially fast with n and an argument similar to the one given in the proof of Lemma 5.8 gives that T_r is finite almost surely. □

Lemma 5.9 shows that if we start a simple random walk on G from a vertex v, almost surely T_r is finite. It implies that almost surely the sequence $((Z_n^k)_{n\geq 0})_{k\geq 0}$ is constant eventually and its limit $(Z_n)_{n\geq 0}$ is well defined with $(Z_n)_{n\geq T_r}$ constant.

5.1.4 p-loop-erased random walk with a boundary condition

Let us briefly recall our current notations. We still assume that (X_n) is a simple random walk on G starting from any vertex v, (Y_n) is a sequence of independent random variables of uniform law in $[0, 1]$, which are independent of the X_n and $W \subset V$ is a deterministic set of vertices.

We define in this subsection a p-loop-erased random walk obtained from $(X_n, Y_n)_{n\geq 0}$ with a boundary condition W.

Definition 5.10. Let T_W be the hitting time of W, and let T_r be the rooting time of the process $(X_n, Y_n)_{n\geq 0}$. Let $T_f = \min(T_r, T_W)$ be called the *ending time* of $(X_n, Y_n)_{n\geq 0}$ with boundary condition W.

Given $(X_n, Y_n)_{n\leq T_W}$, we construct a p-loop erased random walk $(Z_n^W)_n$ with boundary conditions W as follows. We define recursively n_k^W and $(Z_n^{k,W})_{n\geq 0}$ for $k \in \mathbb{N}^*$. Let $(Z_n^{1,W})_{n\leq T_W} = (X_n)_{n\leq T_W}$ and $n_0^W = 0$.

Then, we define recursively a sequence $((Z_n^{i,W})_{n\leq T_W})_{i\geq 1}$ and a sequence $(n_i)_{i\leq k}$ as follows.

Let $n_i^W : (X_0, \dots, X_k) \mapsto \min\{n_{i-1}^W < j \leq T_W | Z_j^{i,W} \in \{Z_0^{i,W}, \dots, Z_{j-1}^{i,W}\}\}$ be the i-th loop-closing time before reaching W, where $\min \emptyset = \infty$.

If $n_i^W(X_0, \dots, X_k) = \infty$, let us define $Z^{i+1,W} = Z^{i,W}$.

Else, let $n_i'^W$ be the first time $j < n_i^W$ such that $Z_{n_i^W}^{i,W} = Z_{n_i'^W}^{i,W}$ and if $Y_{n_i^W} \leq p(\gamma_{n_i^W})$, let for every $m \geq i+1$,

$$Z_n^{m,W} = \begin{cases} Z_n^{i,W} & \text{if } n \leq n_i^W, \\ Z_{n_i^W}^{i,W} & \text{else,} \end{cases}$$

and otherwise, for $n \leq T_W$, let

$$Z_n^{i+1,W} = \begin{cases} Z_{n_i^W}^{i,W} & \text{if } n_i'^W \leq n \leq n_i^W, \\ Z_n^{i,W} & \text{else.} \end{cases}$$

Notice that $Z_n^{i+1,W}$ is obtained from $Z_n^{i,W}$ by erasing the first loop which ends before T_W. While i is small enough such that $n_i \leq T_W$, , we have $n_i^W = n_i$ and $Z^{i+1,W} = Z^{i+1}$.

Proposition 5.11. *Almost surely, T_f is finite and $((Z_n^{i,W})_{n\leq T_f})_{i\geq 1}$ is constant eventually.*

We define the p-loop erased random walk with boundary conditions W as

$$(Z_n^W)_{n\leq T_f} = \lim_{i\to\infty} (Z_n^{i,W})_{n\leq T_f}.$$

- *If $T_f = T_W$, $(Z_n^W)_{n\leq T_f} = (Z_n^{i_f})_{n\leq T_W}$ where $i_f = \min\{i | n_i > T_W\}$.*
- *If $T_f = T_r$, $(Z_n^W)_{n\leq T_f} = (Z_n)_{n\leq T_f}$ where (Z_n) denotes the p-loop-erased random walk without any boundary condition.*

Proof. Recall from Lemma 5.9 that T_r is finite almost surely. Since $T_f \leq T_r$, the ending time T_f is almost surely finite. Assume that $T_r < \infty$. Recall that the sequence (n_i) is strictly increasing.

If $T_W < T_r$, then T_W is finite and there exists i such that $n_i > T_W$ and then we have $n_i^W = \infty$ and $Z^{m,W} = Z^{i,W}$ for $m \geq i$. Let $i_f = \min\{i | n_i > T_W\}$ be the first index such that it happens. Then $n_{i_f-1} \leq T_W$ and $(Z_n^{i_f,W})_{n\leq T_W} = (Z_n^{i_f})_{n\leq T_W}$. Then, for $m \geq i_f$, $n_m > T_W$ and then $n_m^W = \infty$. Then, for every $m \geq i_f$,

$$(Z_n^{m,W})_{n\leq T_f} = (Z_n^{i_f,W})_{n\leq T_f} = (Z_n^{i_f})_{n\leq T_f}.$$

Else, there exists i such that $T_r = n_i \leq T_W$. Then, $n_i^W = n_i$ and $Y_{n_i^W} \leq p(\gamma_{n_i^W})$ and for $m \geq i$, $(Z_n^{m,W})_{n\leq T_r} = (Z_n^{i,W})_{n\leq T_r} = (Z_n^i)_{n\leq T_r}$. Since $n_i = T_r$, then we have $(Z_n^i)_{n\leq T_r} = (Z_n)_{n\leq T_r}$ and therefore, $(Z_n^W)_{n\leq T_f} = (Z_n^W)_{n\leq T_f}$ where (Z_n) denotes the p-loop erased random walk without any boundary condition. □

Notice that a p-loop-erased random walk with boundary condition W is obtained from $(X_n, Y_n)_{n\leq \min(T_r,T_W)}$ erasing every loop except the last one if $T_r < T_W$. Let us emphasize that when $T_r > T_W$, the p-loop erased random walk with boundary conditions $(Z_n^W)_{n\leq T_W}$ is not equal to the p-loop erased random walk $(Z_n)_{n\leq T_W}$ stopped at T_W.

In the next section, we will construct a probability measure on CRSF of an infinite graph from an algorithm based on the sampling of p loop-erased random walks.

5.2 Measures on CRSF in infinite volume and thermodynamic limits

In this section, we still consider a countably infinite connected graph G, an exhaustion $(G_n)_{n\geq 1}$ and a weight function $p \in [0,1]$ on cycles which satisfies Assumption 5.1.

We will construct in this section a measure defined by a Wilson type algorithm on infinite graphs associated with an ordering of the vertex set. Then, we will show that for whatever ordering of the vertex set, the measure in infinite volume is the weak limit of sequences of measures on cycle-rooted spanning forests of growing finite graphs associated with the weight function p and free or wired boundary conditions. We will finally conclude that the measure in infinite volume does not depend on the ordering of the vertex set.

5.2.1 Sampling algorithm for a fixed ordering on an infinite graph

We construct a probability measure on cycle-rooted spanning forests of G which is sampled by an algorithm of Wilson type with respect to an ordering of the infinite vertex set V. This measure will depend on the weight function p.

We may also construct such a probability measure on vertex-and-cycle-rooted spanning forests depending on masses and conductances with slight modifications but we assume here that $m = 0$ and $c = 1$ in order to simplify notations.

Definition 5.12. If $B = \{e_1, \dots, e_n\}$ is a finite subset of size n of E, and if we are given $\varepsilon_1, \dots, \varepsilon_n \in \{0,1\}^n$, let us define the corresponding cylinder as

$$C_{\varepsilon_1,\dots,\varepsilon_n} := \{(w_i)_{i\in|E|} \in \{0,1\}^{|E|} | w_1 = \varepsilon_1, \dots, w_n = \varepsilon_n\}.$$

Let $\mathcal{C} := \sigma((\mathcal{C}_{\varepsilon_1,\dots,\varepsilon_n})_{\varepsilon_1,\dots,\varepsilon_n})$ be the smallest σ-field which makes the cylinders measurable.

Let φ be an ordering of the vertex set V of G, in the sense of a bijection $\varphi : \mathbb{N} \to V$. Let $(v_i)_{i\geq 1}$ be the sequence of vertices of G with ordering φ, that is $(v_i)_i = (\varphi(i))_i$.

We will construct a measure on CRSF of G by means of a family of p-loop-erased random walks with boundary conditions which are defined recursively. This family will be obtained deterministically from a family of independent simple random walks and a family of independent Bernoulli laws, following results of Section 5.1.

Definition 5.13. Let $((X_n^{(x)})_{n\geq 1})_{x\in G}, ((Y_n^{(x)})_{n\geq 1})_{x\in G}$ be independent random variables such that for all $x \in G$, $(X_n^{(x)})_n$ is a simple random walk on G starting from x and $(Y_n^{(x)})_n$ is a sequence of independent random variables with uniform law on $[0,1]$.

For a fixed $x \in V$, consider the sequence $\left((X_n^{(x)}, Y_n^{(x)})\right)_{n\geq 1}$ and denote by T_r^x the rooting time of the p-LERW that is to say the first time n such that $(X_n^{(x)})_n$ closes a loop γ_n such that the inequality $p(\gamma_n) \geq Y_n^{(x)}$ holds.

Under Assumption 5.1 on p, Lemma 5.9 shows that $T_r^{v_1}$ is almost surely finite and therefore the first step of the algorithm finishes, whatever the first vertex in the ordering is.

Definition 5.14. For a fixed random data $((X_n^{(x)})_{n\geq 1})_{x\in V}, ((Y_n^{(x)})_{n\geq 1})_{x\in V}$ as above, we construct the subgraphs (F_i) recursively. Let $\mathsf{F}_0 = \emptyset$. Let $i \in \mathbb{N}^*$ and assume that F_{i-1} is constructed. Denote by $T_f^{\underline{v}_i}$ the ending time of $((X_n^{(v_i)})_{n\geq 1}, (Y_n^{(v_i)})_{n\geq 1})$ with boundary condition $V(\mathsf{F}_{i-1})$, that is

$$T_f^{\underline{v}_i} = \min(T_r^{v_i}, T_{V(\mathsf{F}_{i-1})}),$$

where $\underline{v}_i = (v_1, \dots, v_i)$ refers to all previous vertices with respect to the ordering.

Let $\mathsf{F}_i = \mathsf{F}_{i-1} \cup L(v_i)$ be the component obtained at the end of step i, where $L(v_i)$ is the p-LERW with boundary condition $V(\mathsf{F}_{i-1})$ obtained from $\left((X_n^{(v_i)}, Y_n^{(v_i)})\right)_{n\geq 1}$ until $T_f^{\underline{v}_i}$.

Each step i of the algorithm finishes either if the random walk reaches a connected component created during a previous step or if the random walk is rooted to a loop. Notice that $T_f^{\underline{v}_i}$ is the time where the i^{th}-step of the algorithm with ordering φ finishes. Recall that under Assumption 5.1 on p, Proposition 5.11 implies that $T_f^{\underline{v}_i}$ is finite almost surely.

Lemma 5.15. *For every bijection $\varphi : \mathbb{N} \to V$, there exists a measure μ_φ on $(\mathcal{U}(G), \mathcal{C})$ which is sampled by the previous algorithm with ordering φ. The measure on finite cylinders corresponds to finite random configurations which are sampled in a finite time.*

Proof. Sample a sequence $((X_n^{(x)})_{n\geq 1})_{x\in V}, ((Y_n^{(x)})_{n\geq 1})_{x\in V}$ like in Definition 5.13.

From Definition 5.14, we obtain a configuration of CRSF on G by $\mathsf{F} = \cup_{i\geq 1}\mathsf{F}_i$. The configuration F is well defined since it is a deterministic function of

$$((X_n^{(x)})_{n\geq 1})_{x\in V}, ((Y_n^{(x)})_{n\geq 1})_{x\in V}.$$

Let μ_φ be the law of F associated to a random choice of $((X_n^{(x)})_{n\geq 1})_{x\in V}, ((Y_n^{(x)})_{n\geq 1})_{x\in V}$, that is to say the push-forward by the algorithm of the measure which gives

$$((X_n^{(x)})_{n\geq 1})_{x\in V}, ((Y_n^{(x)})_{n\geq 1})_{x\in V}.$$

Then, μ_φ is a measure on $(\mathcal{U}(G), \mathcal{C})$. Let B be a finite subset of size n of E, with edges $e_1, \dots, e_n$ and let $\varepsilon_1, \dots, \varepsilon_n \in \{0, 1\}^n$. Let K be the finite set of vertices containing all the extremities of edges of B and vertices which are preceding those vertices for the order φ. Let us consider the previous algorithm for vertices $v_1, \dots, v_{|K|}$ for the ordering φ. Almost surely, the algorithm to construct $\mathsf{F}_{|K|}$ ends in finite time. The constructed graph $\mathsf{F}_{|K|}$ is a random subgraph of G which is spanning for K and therefore it is spanning for B. Then $\mu_\varphi(C_{\varepsilon_1,\dots,\varepsilon_n})$ is the probability that the random configuration $F_{|B}$ obtained from the previous construction satisfies

$$\mathsf{F}_{|B} \in C_{\varepsilon_1,\dots,\varepsilon_n}.$$

The measure μ_φ restricted to $(2^B, \mathcal{C})$ is the law of the random configuration $\mathsf{F}_{|B}$, which is sampled in a finite time. □

In the following, we will show that the measure in infinite volume constructed from an enumeration of the vertex set V does not depend on the choice of the enumeration of the vertices. The proof of this statement will rely on a comparison between the measure μ_φ for an ordering φ and a measure on CRSF on a large finite subgraph of G. We will see that, under some assumptions, the thermodynamic limit of sequences of measures on CRSF of growing subgraphs coincides with the measure sampled by the previous algorithm and does not depend on the ordering of the infinite vertex set.

5.2.2 Thermodynamic limits of the Wilson measures

Assume that Assumption 5.1 on the existence of a lower bound $\alpha > 0$ on the weight of a family of loops holds. We will show in this section that the measure defined by the Wilson type algorithm on infinite graphs is the weak limit of sequences of measures on cycle-rooted spanning forests of growing finite graphs associated to the weight function p and free or wired boundary conditions.

Let $n \in \mathbb{N}$ and φ_n be an ordering of the vertex set V_n of the graph G_n. We consider the measures μ_n^F and μ_n^W as defined in Subsection 3.1.1. Those measures are sampled by the algorithm introduced in Section 2.2.4 and recalled at the beginning of the chapter, with respective boundary conditions $\emptyset$ and ∂G_n. According to [KK17], the measures μ_n^F and μ_n^W do not depend on the ordering of the vertices of $G_n \backslash \partial G_n$. Therefore, the measures μ_n^F and μ_n^W are also sampled by the algorithm described just above (Definition 5.14) on the finite graph G_n with the setting $F_0 = \partial G_n$ at the beginning.

Theorem 5.16. *Let φ be an ordering of $V(G)$ in the sense of a bijection $\varphi : \mathbb{N} \to V(G)$. Let (G_n) be an increasing exhaustion of G, and let $(\mu_n^F), (\mu_n^W)$ be the corresponding sequences of probability measures on CRSF of G_n with free and wired boundary conditions, respectively. The sequences of probability measures (μ_n^F) and (μ_n^W) converge weakly to the measure μ_φ.*

Proof. We consider an event $B \in 2^E$ which depends on only finitely many edges, and we consider K_0 the set of vertices incident to the edges on which B depends. Let K be the union of K_0 and the set of vertices that precede some vertex in K_0 in the ordering φ of the vertices. Let n be large enough such that $K \subset G_n$.

Let us construct a coupling $(\mathsf{F}, \tilde{\mathsf{F}_n}^F, \tilde{\mathsf{F}_n}^W)$ of random configurations obtained from the same random data $((X_n^{(x)})_{n\geq 1})_{x\in G}, (Y_n^{(x)})_{n\geq 1, x\in G}$, such that the law of F is μ_φ, the law of $\tilde{\mathsf{F}_n}^F$ is μ_n^F and the law of $\tilde{\mathsf{F}_n}^W$ is μ_n^W and such that the three configurations coincide with high probability on B.

We denote by $\tilde{\varphi}_n$ the ordering $\varphi_{|K}$ completed in an ordering on G_n.

We follow the algorithm for every vertex of K following the ordering $\varphi_{|K}$. If at one step of the algorithm, the random walk $(X_n^{(x)})$ starting from a vertex $x \in G_n$ reaches ∂G_n, the configuration F is obtained following the random walk in the infinite graph $\mathbb{Z}^d$ until the step of the algorithm finishes and $\tilde{\mathsf{F}_n}^F, \tilde{\mathsf{F}_n}^W$ is obtained following the random walk with boundary conditions. More precisely, $\tilde{\mathsf{F}_n}^W$ is obtained from the p-loop erased random

walks with boundary conditions ∂G_n and $\tilde{\mathsf{F}_n}^F$ is obtained following the random walk on the graph G_n, until the end of the step, that is the ending time of the process $(X_n^{(x)}, Y_n^{(x)})_n$.

Once every vertex of K has been explored, we complete the configuration F following the ordering φ and we complete the configurations $\tilde{\mathsf{F}_n}^F$, $\tilde{\mathsf{F}_n}^W$ on G_n following the ordering $\tilde{\varphi}_n$, with boundary conditions.

Let us denote by $E(K)$ the set of edges whose extremal vertices are in K. The configurations F, $\tilde{\mathsf{F}_n}^F$ and $\tilde{\mathsf{F}_n}^W$ obtained from the algorithm are respectively subgraphs of $\mathbb{Z}^d$ and G_n. The three configurations are spanning subgraphs of G_n and in particular spanning subgraphs of $(K, E(K))$.

$\mathsf{F}_{E(K)}$ is the restriction to $(K, E(K))$ of a random configuration following the law μ_φ. Since G_n is finite, $\tilde{\mathsf{F}}_{nE(K)}^{W,F}$ is the restriction to $(K, E(K))$ of a random configuration following the law $\mu_n^{W,F}$ and this law does not depend on $\tilde{\varphi}_n$ (see [KK17]).

Since B depends only on edges whose endpoints are in K_0, we know that

$$|\mu_\varphi(B) - \mu_n^{F,W}(B)| \leq \mathbb{P}(\mathsf{F}_{E(K)} \neq \tilde{\mathsf{F}}_{nE(K)}).$$

Following the previous algorithm, while each step starting from a vertex of K finishes before the random walk reaches ∂G_n, both configurations $\mathsf{F}_{E(K)}, \tilde{\mathsf{F}}_{nE(K)}$ which are obtained are equal. Therefore, from the union bound,

$$\begin{aligned}\mathbb{P}(\mathsf{F}_{E(K)} \neq \tilde{\mathsf{F}}_{nE(K)}) &\leq \mathbb{P}\Big(\bigcup_{i\in[|K|]} \{T_n^{v_i} \leq T_f^{\underline{v}_i}\}\Big) \leq \sum_{i\in[|K|]} \mathbb{P}(T_n^{v_i} \leq T_f^{\underline{v}_i}) \\ &\leq \sum_{i\in[|K|]} \mathbb{P}(T_n^{v_i} \leq T_r^{v_i}) \leq |K| \max_{i\in[|K|]} \mathbb{P}(T_n^{v_i} \leq T_r^{v_i}).\end{aligned}$$

From Lemma 5.9, we obtain when $n \to \infty$,

$$|\mu_\varphi(B) - \mu_n^{W,F}(B)| \leq |K| \max_{i\in[|K|]} \mathbb{P}(T_n^{v_i} \leq T_r^{v_i}) \to 0,$$

which implies the weak convergence of $(\mu_n^{W,F})$ towards μ_φ. □

5.2.3 Independence on the ordering

Recall that for every n, the measure μ_n^W is sampled by an algorithm and does not depend on the ordering of vertices of G_n chosen in the algorithm. Combined with Theorem 5.16, this independence implies the following result.

Theorem 5.17. *Let φ be an ordering of the vertices, as defined as a bijection $\varphi : \mathbb{N} \to V$. Let p be a weight function satisfying Assumption 5.1. Let μ_φ be the measure on the cycle-rooted spanning forests of G associated with the algorithm of loop-erased random walk with weights $p(\gamma)$. The measure μ_φ does not depend on φ.*

Proof. Let φ, τ be two orderings, with $(v_i) = (\varphi(i))$, $(w_i) = (\tau(i))$ let K_1 and K_2 be respectively the sets of vertices that precede some vertex in K_0 in the ordering φ (resp. τ) and let n large enough such that $K_1 \cup K_2 \subset G_n$. Then, from Theorem 5.16,

$$|\mu_\varphi(B) - \mu_\tau(B)| \leq |K_1| \max_{i\in[|K_1|]} \mathbb{P}_{v_i}(T_n \leq T_r) + |K_2| \max_{i\in[|K_2|]} \mathbb{P}_{w_i}(T_n \leq T_r) \to 0.$$

This shows that both distributions in infinite volume coincide on cylinders and therefore the measure in infinite volume does not depend on the ordering of the vertices. □

For a weight function p satisfying Assumption 5.1, we will denote by μ_p the corresponding probability measure on CRSFs of G, which does not depend on the ordering of the vertices. The measure μ_p is also the limit of free and wired sequences of measures on cycle-rooted spanning forests of growing subgraphs and therefore, the limit does not depend on the boundary conditions.

5.3 Study of the configurations sampled under the Wilson measure

In this section, we will study the asymptotic behavior of configurations and the rate of decay of correlations with the distance for the measure μ_p which is sampled by an algorithm of p-loop erased random walks in infinite volume and which will be called the *Wilson measure*. We will study this measure under the following assumption.

Assumption 5.18. *There exists $\alpha > 0, \beta > 0, M, M' > 0, C > 0, d \in \mathbb{N}$, a family of oriented cycles $C \subset \mathcal{C}_\to(G)$ and for every $x \in G$, an increasing sequence (B_n^x) of subgraphs of G, exhausting G and containing x such that :*

- *For every $\gamma \in C$, $\alpha \leq p(\gamma) \leq 1$.*
- *For every $v \in \partial B_n^x$, there exists a loop $\gamma_v \in C \cap (B_{n+1}^x \backslash (B_n \cup \partial B_{n+1}^x))$ such that the probability for a random walk starting from v of making this loop γ_v is greater than β.*
- *For every x, for every $n \in \mathbb{N}$, $M'n \leq d(x, \partial B_n^x) \leq Mn$, and $|\partial B_n^x| \leq Cn^d$.*

Assumption 5.18 implies Assumption 5.1 and is satisfied in particular if the graph and the weight function w on cycles are invariant under translations and if Assumption 5.1 is satisfied for an exhaustion $(G_n)_n$ such that $d(0, \partial G_n) \sim Mn$ when n tends to infinity, where $M > 0$.

In this section, we will only study properties of the configurations sampled under the Wilson measure in infinite volume.

5.3.1 Every connected component is finite for the Wilson measure

Definition 5.19. For every vertex x and every subset $A \subset G$, we denote by $\{x \leftrightarrow A\}$ the event $\{A \cap C_x \neq \emptyset\}$ where C_x is the connected component of x in the random configuration sampled under μ. In particular, $\{x \leftrightarrow y\}$ means that x and y are in the same connected component.

We will denote by $T_{m,x}^x$ the hitting-time of ∂B_m^x for the random walk $(X_n^{(x)})$. Recall that T_m^x is the hitting-time of ∂G_m for the random walk $(X_n^{(x)})$.

Lemma 5.20. *Under Assumption 5.18, there exists $\delta > 0$ such that the following inequality holds for every m, for every $x \in G$*

$$\mathbb{P}_x(\{T_r^x \geq T_{m,x}^x\}) \leq \delta^m.$$

Proof. Let $x \in G$. Under Assumption 5.18, Assumption 5.1 is satisfied for the vertex x, and therefore, if we denote by $T_{m,x}^x$ the hitting time of ∂B_m^x for a p-loop erased random walk starting from x, and T_r^x its rooting time, Lemma 5.8 gives the existence of a $0 < \delta < 1$ such that the following inequality holds for every m,

$$\mathbb{P}_x(\{T_r^x \geq T_{m,x}^x\}) \leq \delta^m,$$

where $\delta = 1 - \alpha\beta$ for parameters α, β of 5.18. In particular, δ does not depend on x, which concludes the proof. □

Lemma 5.21. *Let $\delta > 0$ as in Lemma 5.20 and M as in Assumption 5.18. Let x, y be two vertices of G and denote by $d(x, y)$ the distance between x and y that is to say the length of the shortest path from x to y. Then, if $n \leq \frac{d(x,y)}{2M}$,*

$$\mu_p(x \leftrightarrow y) \leq 2\delta^n.$$

Proof. According to Theorem5.17, the measure μ_p does not depend on the ordering of the vertices. We may choose an ordering φ in which x and y are the first two vertices.

Since $n \leq \frac{d(x,y)}{2M}$, $d(x, \partial B_n^x) \leq Mn \leq \frac{d(x,y)}{2}$ and $d(y, \partial B_n^y) \leq Mn \leq \frac{d(x,y)}{2}$. If x and y are in the same connected component in a configuration obtained from this algorithm, we know that either for the p-loop erased random walk starting from x or for the one starting from y, we have $\{T_r^x \geq T_{f\varphi}^x \geq T_{n,x}^x\}$ or $\{T_r^y \geq T_{f\varphi}^y \geq T_{n,x}^y\}$. Indeed, if $T_{f\varphi}^x \leq T_{n,x}^x$ and $T_{f\varphi}^y \leq T_{n,y}^x$, then the p-loop erased random walk starting from x and from y cannot intersect, and form two disjoint connected component in the configuration. Therefore, from the union bound,

$$\mu_p(x \leftrightarrow y) \leq \mathbb{P}_x(\{T_r^x \geq T_{n,x}^x\}) + \mathbb{P}_y(\{T_r^y \geq T_{n,y}^y\}) \leq 2\delta^n,$$

which concludes the proof. □

Theorem 5.22. *μ_p-almost surely, for every $x \in V$ of G, the connected component of x is finite.*

Proof. Let $x \in V$. For every $n \in \mathbb{N}$, for every $y \in \partial B_n^x$, $d(x, y) \geq M'n$. Let $n' = \lfloor \frac{M'n}{2M} \rfloor$. Then $n' \leq \frac{d(x,y)}{2M}$, and therefore, from Lemma 5.21,

$$\mu_p(x \leftrightarrow y) \leq 2\delta^{n'}.$$

Then, from the union bound, the following upper bound on the probability that the connected component of x contains vertices of the boundary of B_n^x holds for every $n \in \mathbb{N}$, with $\delta' = \delta^{\frac{M'}{2M}}$,

$$\mathbb{P}(x \leftrightarrow \partial B_n^x) \leq \sum_{y \in \partial B_n^x} \mathbb{P}(x \sim y) \leq 2|\partial B_n^x| \delta^{\lfloor \frac{M'n}{2M} \rfloor} \leq 2Cn^d \delta'^n.$$

Then, from the monotone convergence theorem, we have

$$\mathbb{P}(x \leftrightarrow \infty) = \mathbb{P}(\cap_n \{x \leftrightarrow \partial B_n^x\}) = \lim \mathbb{P}(x \leftrightarrow \partial B_n^x) = 0.$$

Since G is countable, we know that μ-almost surely, for every $x \in G$, the connected component of x is finite. □

5.3.2 Exponential decay of correlations for the Wilson measure

We still assume that weights are in $[0, 1]$ and satisfy Assumption 5.18. Let $m \in \mathbb{N}$ and let $e_1 = (x_1, y_1)$ and $e_2 = (x_2, y_2)$ be such that $d(\{x_1, y_1\}, \{x_2, y_2\}) \geq m$.

If F is a CRSF following the law μ_p it can be sampled from the algorithm described in section 5.2.1 and from Theorem 5.17, it does not depend on the chosen ordering of vertices, therefore we may assume that the first four vertices of the ordering φ are x_1, y_1, x_2, y_2.

Let us consider in the following, four independent couples of sequences of random variables $(X_n^{x_1}, Y_n^{x_1})$, $(X_n^{y_1}, Y_n^{y_1})$, $(X_n^{x_2}, Y_n^{x_2})$, $(X_n^{y_2}, Y_n^{y_2})$, as defined in section 5.3.1.

For $i \in \{1, 2\}$, let us denote by A_i the event that both p-loop-erased random walks obtained from $(X_n^{x_i}, Y_n^{x_i})_n$, $(X_n^{y_i}, Y_n^{y_i})_n$ starting from x_i, y_i are rooted before leaving the subgraphs $B_{m/2}^{x_i}, B_{m/2}^{y_i}$, that is to say that

$$A_i = \{T_r^{x_i} < T_{x_i, m/2}^{x_i}\} \cap \{T_r^{y_i} < T_{y_i, m/2}^{y_i}\}.$$

Lemma 5.23. *Conditional on $A_1 \cap A_2$, the events $\{e_1 \in \mathsf{F}\}$ and $\{e_2 \in \mathsf{F}\}$ are independent.*

Proof. Let F_4 be the subgraph obtained after the first four runs of the algorithm.

Notice that, once F_4 has been sampled, during every subsequent run of the algorithm, the p-loop-erased random walk stops if it reaches x_1, y_1, x_2, y_2 because F_4 contains x_1, y_1, x_2, y_2. Therefore, for F the configuration obtained following the algorithm in infinite volume, we have the following equality of events for $i \in [1, 2]$,

$$\{e_i \in \mathsf{F}\} = \{e_i \in \mathsf{F}_4\}.$$

Let $(Z_n^{x_1})_{n \leq T_r^{x_1}}$ be the p-loop-erased random walk obtained from $(X_n^{x_1}, Y_n^{x_1})$ and let

$$W_1 = V((Z_n^{x_1})_{n \leq T_r^{x_1}})$$

be the set of vertices explored by this p-loop-erased random walk. Let $(Z_n^{y_1})_{n \leq \min(T_r^{y_1}, T_{W_1}^{y_1})}$ be the p-loop erased random walk starting from y_1 with boundary condition W_1. Let $\mathfrak{F}_1$ be the subgraph given by $(Z_n^{x_1})_{n \leq T_r^{x_1}}, (Z_n^{y_1})_{n \leq \min(T_r^{y_1}, T_{W_1}^{y_1})}$.

Let $(Z_n^{x_2})_{n \leq T_r^{x_2}}$ be the p-loop-erased random walk obtained from $(X_n^{x_2}, Y_n^{x_2})$ and let

$$W_2 = V((Z_n^{x_2})_{n \leq T_r^{x_2}})$$

be the set of vertices explored by this p-loop erased random walk. Let $(Z_n^{y_2})_{n \leq \min(T_r^{y_2}, T_{W_2}^{y_2})}$ be the p-loop-erased random walk starting from y_1 with boundary condition W_2. Let $\mathfrak{F}_2$ be the subgraph given by $(Z_n^{x_2})_{n \leq T_r^{x_2}}, (Z_n^{y_2})_{n \leq \min(T_r^{y_2}, T_{W_2}^{y_2})}$.

Let us emphasize that the p-loop erased random walk corresponding to the third and the fourth runs of the algorithm has boundary conditions $V(\mathfrak{F}_1)$, corresponding to the configuration created during the first two runs of the algorithm. Therefore, in general, $\mathfrak{F}_1$ and $\mathfrak{F}_2$ are not disjoint and their union is not the component created after four runs of the algorithm.

If A_1 is satisfied, $\mathfrak{F}_1$ is contained in $B^{x_1}_{m/2} \cup B^{y_1}_{m/2}$ and if A_2 is satisfied,

$$\begin{cases} T^{x_2}_r < T^{x_2}_{x_2,m/2} < T_{V(\mathfrak{F}_1)}, \\ T^{y_2}_r < T^{y_2}_{y_2,m/2} < T_{V(\mathfrak{F}_1)}, \end{cases}$$

and therefore, the third and the fourth runs finish before the p-loop erased random walks reach $V(\mathfrak{F}_1)$, that is to say that the p-loop erased random walks with boundary condition $V(\mathfrak{F}_1)$ coincides with the p-loop erased random walks without this boundary condition (see Proposition 5.11). Therefore, if A_1 and A_2 are satisfied, $\mathfrak{F}_1$ and $\mathfrak{F}_2$ are disjoint connected components and their union is exactly the component created after four runs of the algorithm.

In particular, if $A_1 \cap A_2$ is satisfied, for $i \in [1,2]$, $\{e_i \in \mathsf{F}\}$ is satisfied if and only if $\{e_i \in \mathfrak{F}_i\}$ is satisfied.

We show that conditional on $A_1 \cap A_2$, the random configurations $\mathfrak{F}_1$ and $\mathfrak{F}_2$ are independent. Recall that $(Z^{x_1}_n), (Z^{y_1}_n)$ and $(Z^{x_2}_n), (Z^{y_2}_n)$ are independent and $\mathfrak{F}_1, A_1$ only depends on $(Z^{x_1}_n), (Z^{y_1}_n)$ and $\mathfrak{F}_2, A_2$ only depends on $(Z^{x_2}_n), (Z^{y_2}_n)$. Therefore, if F_1, F_2 are some fixed configurations,

$$\mathbb{P}(\mathfrak{F}_1 = F_1, A_1, \mathfrak{F}_2 = F_2, A_2) = \mathbb{P}(\mathfrak{F}_1 = F_1, A_1)\mathbb{P}(\mathfrak{F}_2 = F_2, A_2).$$

Therefore, using independence of $\{\mathfrak{F}_i = F_i\} \cap A_i$ and A_j for $i \neq j$, we obtain

$$\begin{aligned} \mathbb{P}(\mathfrak{F}_1 = F_1, \mathfrak{F}_2 = F_2 | A_1, A_2) &= \frac{\mathbb{P}(\mathfrak{F}_1 = F_1, A_1)\mathbb{P}(\mathfrak{F}_2 = F_2, A_2)}{\mathbb{P}(A_1 \cap A_2)} \\ &= \mathbb{P}(\mathfrak{F}_1 = F_1 | A_1)\mathbb{P}(\mathfrak{F}_2 = F_2 | A_2) \\ &= \mathbb{P}(\mathfrak{F}_1 = F_1 | A_1 \cap A_2)\mathbb{P}(\mathfrak{F}_2 = F_2 | A_1 \cap A_2). \end{aligned}$$

Therefore, conditional on $A_1 \cap A_2$, the random variables $\mathfrak{F}_1$ and $\mathfrak{F}_2$ are still independent.

Therefore, conditional on $A_1 \cap A_2$, $\{e_1 \in \mathsf{F}\} = \{e_1 \in \mathfrak{F}_1\}$ and $\{e_2 \in \mathsf{F}\} = \{e_2 \in \mathfrak{F}_2\}$ are independent. □

As a consequence, we obtain the following decay of correlations

Theorem 5.24. *There exists a parameter $\iota < 1$ such that for every m large enough,*

$$\mu_p(e_2 \in \mathsf{F})\mu_p(e_1 \in \mathsf{F}) - \iota^m \leq \mu_p(\{e_2 \in \mathsf{F}\} \cap \{e_1 \in \mathsf{F}\}) \leq \mu_p(e_2 \in \mathsf{F})\mu_p(e_1 \in \mathsf{F}) + \iota^m.$$

Proof. The event $\{e_2 \in \mathsf{F}\} \cap \{e_1 \in \mathsf{F}\}$ can be decomposed as the following disjoint union:

$$(\{e_2 \in \mathsf{F}\} \cap \{e_1 \in \mathsf{F}\} \cap A_1 \cap A_2) \cup \left(\{e_1 \in \mathsf{F}\} \cap \{e_2 \in \mathsf{F}\} \cap (A_1^{\complement} \cup A_2^{\complement})\right).$$

Since $\{e_1 \in \mathsf{F}\} \cap \{e_2 \in \mathsf{F}\} \cap (A_1^{\complement} \cup A_2^{\complement})$ is included in $A_1^{\complement} \cup A_2^{\complement}$, it has probability less than $\mu_p(A_1^{\complement} \cup A_2^{\complement})$.

From Lemma 5.8, there exists some $\delta < 1$ such that from the union bound, we get

$$\mu_p(A_1^{\complement} \cup A_2^{\complement}) \leq 4\delta^{m/2}.$$

For the other term, we use the independence of $\{e_1 \in \mathsf{F}\}$ and $\{e_2 \in \mathsf{F}\}$ conditional on $A_1 \cap A_2$ proved in Lemma 5.23, which implies

$$\begin{aligned}\mu_p(\{e_2 \in \mathsf{F}\} \cap \{e_1 \in \mathsf{F}\} \cap A_1 \cap A_2) &= \mu_p(\{e_1 \in \mathsf{F}\} \cap \{e_2 \in \mathsf{F}\} | A_1 \cap A_2)\mu_p(A_1 \cap A_2) \\ &= \frac{\mu_p(\{e_1 \in \mathsf{F}\} \cap A_1 \cap A_2)\mu_p(\{e_2 \in \mathsf{F}\} \cap A_1 \cap A_2)}{\mu_p(A_1 \cap A_2)} \\ &\leq \frac{\mu_p(\{e_1 \in \mathsf{F}\})\mu_p(\{e_2 \in \mathsf{F}\})}{\mu_p(A_1 \cap A_2)}.\end{aligned}$$

Using again the lower bound on $\mu_p(A_1 \cap A_2)$ which comes from Lemma 5.8, we have

$$\mu_p(A_1 \cap A_2) \geq 1 - 4\delta^{m/2}.$$

Let $\eta < 1$ be such that for m large enough,

$$4\delta^{m/2} < \eta^m.$$

Therefore, we have the following upper bound on $\frac{1}{\mu_p(A_1 \cap A_2)}$,

$$\frac{1}{\mu_p(A_1 \cap A_2)} \leq \frac{1}{1-\eta^m} = \sum_{k \geq 0} \eta^{mk} = 1 + \sum_{k \geq 1} \eta^{mk} \leq 1 + \sum_{k \geq m} \eta^k \leq 1 + \frac{\eta^m}{1-\eta}.$$

Therefore we get

$$\mu_p(\{e_2 \in \mathsf{F}\} \cap \{e_1 \in \mathsf{F}\} \cap A_1 \cap A_2) \leq \mu_p(e_1 \in \mathsf{F})\mu_p(e_2 \in \mathsf{F})(1 + \frac{\eta^m}{1-\eta}).$$

For the other inequality, notice that $\frac{\mu_p(\{e_1 \in \mathsf{F}\} \cap A_1 \cap A_2)\mu_p(\{e_2 \in \mathsf{F}\} \cap A_1 \cap A_2)}{\mu_p(A_1 \cap A_2)}$ is larger than

$$\begin{aligned}&(\mu_p(\{e_1 \in \mathsf{F}\}\mu_p(\{e_2 \in \mathsf{F}\}) - 2\mu_p(A_1^{\complement} \cup A_2^{\complement}) \\ &\geq (\mu_p(\{e_1 \in \mathsf{F}\}\mu_p(\{e_2 \in \mathsf{F}\}) - 2\eta^m.\end{aligned}$$

Therefore,

$$\mu_p(\{e_2 \in \mathsf{F}\} \cap \{e_1 \in \mathsf{F}\} \cap A_1 \cap A_2) \geq (\mu_p(\{e_1 \in \mathsf{F}\}\mu_p(\{e_2 \in \mathsf{F}\}) - 2\eta^m,$$

and since the event $\{e_2 \in \mathsf{F}\} \cap \{e_1 \in \mathsf{F}\}$ contains the event$\{e_2 \in \mathsf{F}\} \cap \{e_1 \in \mathsf{F}\} \cap A_1 \cap A_2$, we get

$$\begin{aligned}\mu_p(\{e_2 \in \mathsf{F}\} \cap \{e_1 \in \mathsf{F}\}) &\geq \mu_p(\{e_2 \in \mathsf{F}\} \cap \{e_1 \in \mathsf{F}\} \cap A_1 \cap A_2) \\ &\geq (\mu_p(\{e_1 \in \mathsf{F}\}\mu_p(\{e_2 \in \mathsf{F}\}) - 2\eta^m.\end{aligned}$$

Finally, considering $\iota < 1$ such that for m large enough, both inequalities $2\eta^m < \iota^m$ and $\eta^m \frac{1}{1-\eta} + 4\delta^{m/2} < \iota^m$ hold, concludes the proof. □

Chapter 6

Measures on CRSF associated to weights larger than 1 and determinantal measures.

In this chapter, we study probability measures on cycle-rooted spanning forests of an infinite graph $G = (V, E)$ depending on a weight function w on cycles of the graph. We saw in chapters 2 and 5 that there are two families of probability measures on cycle-rooted spanning forests of a finite graph, one which is determinantal and one which is sampled by a random walks algorithm. Considering an exhaustion of G by a sequence of growing finite subgraphs gives rise to infinite volume measures, obtained as weak limit of probability measures sequences. In the case of determinantal measures associated to a unitary connection, the weak convergence of the sequence of measures is given by the convergence of the sequence of kernels (see chapters 3 and 4), whereas in the case of measures sampled by an extension of the Propp-Wilson algorithm, the weak convergence of the sequence of measures is given by bounds on the rooting time (see chapter 5). Both families generalize Uniform Spanning Forests and are limits of probability measures on increasing sequences of finite subgraphs.

In this chapter, we will be interested in the comparison between Boltzmann probability measures on cycle-rooted spanning forests of a finite graph depending on a weight function w. Then, we will use properties on the measures studied in chapter 5 as a tool to understand properties of determinantal measures on cycle-rooted spanning forests of an infinite graph associated to a unitary complex connection.

6.1 Comparison between families of measures.

6.1.1 Comparison between measures when w is uniformly increased

Let $G = (V, E)$ be a finite graph endowed with a non-negative function $w : \mathcal{C}_{\rightarrow}(G) \rightarrow \mathbb{R}_+$ defined on oriented cycles of G which is symmetric under orientation reversal.

For a subgraph F of G, we will denote by $\mathcal{C}_+(F)$ and call *positive cycles* the set of non-oriented cycles of F of weight strictly larger than 1 and denote by $\mathcal{C}_-(F)$ and call *negative cycles* the set of non-oriented cycles of G of weight less than or equal to 1.

Assume that there exists $C > 1$ such that $w = Cp$ with $p : \mathcal{C}_{\to}(F) \to [0,1]$ a non-negative function defined on oriented cycles of G which is symmetric under orientation reversal. We denote by μ_w and μ_p the measures on $\mathcal{U}(G)$ defined as follows :

$$\mu_w(F) = \frac{\prod_{\gamma \in \mathcal{C}(F)} 2w(\gamma)}{Z_w}, \qquad \mu_p(F) = \frac{\prod_{\gamma \in \mathcal{C}(F)} 2p(\gamma)}{Z_p},$$

where Z_w, Z_p are the partition functions of the model as defined in section 2.2.

Theorem 6.1. *Let W be the set of wired vertices and let us denote by F a random cycle-rooted spanning forest on G sampled under the measure μ_w^W or μ_p^W. For every event A, the following equality links both measures:*

$$\mu_w^W(A) = \mu_p^W(A) \frac{\mathbb{E}_p^W(C^{|\mathcal{C}(\mathsf{F})|}|A)}{\mathbb{E}_p^W(C^{|\mathcal{C}(\mathsf{F})|})}.$$

If A is an event such that the inequality $\mathbb{E}_p^W(C^{|\mathcal{C}(\mathsf{F})|}|A) \geq \mathbb{E}_p^W(C^{|\mathcal{C}(\mathsf{F})|})$ holds, then this event is more likely to happen for the measure μ_w^W:

$$\mu_w^W(A) \geq \mu_p^W(A).$$

This theorem says that if an event "raises" the number of cycles, in the sense that, conditional on this event, the mean-value of $C^{|\mathcal{C}(\mathsf{F})|}$ is larger, then it is more likely for the measure μ_w^W than for the measure μ_p^W. In the same way, an event which "reduces" the number of cycles is more likely for the measure μ_p^W than for the measure μ_w^W.

Proof. Let $F \in \mathcal{U}_W(G)$.

$$\mu_w^W(F) = \frac{1}{Z_w^W} C^{|\mathcal{C}(F)|} \prod_{\gamma \in \mathcal{C}(F)} 2p(\gamma) = \frac{Z_p^W}{Z_w^W} C^{|\mathcal{C}(F)|} \mu_p^W(F).$$

In particular, if A is an event,

$$\mathbb{E}_w^W(1_A) = \frac{Z_p^W}{Z_w^W} \sum_{F \in \mathcal{U}_W(G)} 1_A(F) C^{|\mathcal{C}(F)|} \mu_p^W(F) = \frac{Z_p^W}{Z_w^W} \mathbb{E}_p^W(1_A C^{|\mathcal{C}(\mathsf{F})|}).$$

Then, for $A = \Omega$, it gives $\frac{Z_p^W}{Z_w^W} = \frac{1}{\mathbb{E}_p^W(C^{|\mathcal{C}(\mathsf{F})|})}$, which concludes the proof. □

A consequence of Theorem 6.1 is the following theorem where we apply the result to a family of events which "raises" the number of cycles.

Theorem 6.2. *Let W be the set of wired vertices and let us denote by F a random vertex-and-cycle-rooted spanning forest on G sampled under the measure μ_w^W or μ_p^W. The following inequality holds for every $k \geq 1$,*

$$\mu_w^W(|\mathcal{C}(\mathsf{F})| \geq k) \geq \mu_p^W(|\mathcal{C}(\mathsf{F})| \geq k).$$

Proof. Let us consider the events $(A_k)_{k\geq 1}$ defined by

$$A_k = \{|\mathcal{C}(\mathsf{F})| \geq k\}.$$

Let $k \geq 1$. By definition of A_k, the following inequality holds,

$$\mathbb{E}_p^W(C^{|\mathcal{C}(\mathsf{F})|}|A_k) \geq \mathbb{E}_p^W(C^k|A_k) = C^k \geq \mathbb{E}_p^W(C^{|\mathcal{C}(\mathsf{F})|}|A_k^{\complement}).$$

Since we can write the following decomposition:

$$\mathbb{E}_p^W(C^{|\mathcal{C}(\mathsf{F})|}) = \mathbb{E}_p^W(C^{|\mathcal{C}(\mathsf{F})|}|A_k)\mathbb{P}_p^W(A_k) + \mathbb{E}_p^W(C^{|\mathcal{C}(\mathsf{F})|}|A_k^{\complement})(1-\mathbb{P}_p^W(A_k)),$$

the previous inequality gives

$$\begin{aligned}\mathbb{E}_p^W(C^{|\mathcal{C}(\mathsf{F})|}|A_k) &= \mathbb{E}_p^W(C^{|\mathcal{C}(\mathsf{F})|}|A_k)\mathbb{P}_p^W(A_k) + \mathbb{E}_p^W(C^{|\mathcal{C}(\mathsf{F})|}|A_k)(1-\mathbb{P}_p^W(A_k)) \\ &\geq \mathbb{E}_p^W(C^{|\mathcal{C}(\mathsf{F})|}).\end{aligned}$$

Therefore, from Theorem 6.1, $\mu_w^W(A_k) \geq \mu_p^W(A_k)$, which concludes the proof. □

Remark 6.3. *Let w_h be a weight function which comes from a unitary connection h. Since w_h takes values in $[0,2]$, if p_h is defined by $p_h(\gamma) = \frac{w_h(\gamma)}{2}$, then p_h takes values in $[0,1]$. Then Theorem 6.2 applied to $C = 2$ implies that a spanning forest distributed according the determinantal measure μ_h is more likely to have a lot of cycles than a spanning forest sampled by the algorithm of loop-erased random walks associated to p_h.*

Assume that p is defined by $p = \min(1, w)$. This weight function is studied in [FB22] and since it takes values in $[0,1]$, it corresponds to a measure which can be sampled by a cycle-popping algorithm and which is linked to the measure associated to w by the following formula.

Lemma 6.4. *Let W be the set of wired vertices and let us denote by F a random cycle-rooted spanning forest on G sampled under the measure μ_w^W or μ_p^W. For every event A, the following equality links both measures.*

$$\mu_w^W(A) = \mu_p^W(A)\frac{\mathbb{E}_p^W(\prod_{\gamma\in\mathcal{C}_+(\mathsf{F})} 2w(\gamma)|A)}{\mathbb{E}_p^W(\prod_{\gamma\in\mathcal{C}_+(\mathsf{F})} 2w(\gamma))}.$$

Proof. Let $F \in \mathcal{U}_W(G)$. Then $\mu_w^W(F) = \frac{Z_p^W}{Z_w^W}\prod_{\gamma\in\mathcal{C}_+(F)} 2w(\gamma)\mu_p^W(F)$. In particular, if A is an event,

$$\mathbb{E}_w^W(1_A) = \frac{Z_p^W}{Z_w^W}\sum_{F\in\mathcal{U}_W(G)} 1_A(F)\prod_{\gamma\in\mathcal{C}_+(F)} 2w(\gamma)\mu_p^W(F) = \frac{Z_p^W}{Z_w^W}\mathbb{E}_p^W(1_A\prod_{\gamma\in\mathcal{C}_+(\mathsf{F})} 2w(\gamma)).$$

Then, for $A = \Omega$, it gives $\frac{Z_p^W}{Z_w^W} = \frac{1}{\mathbb{E}_p^W(\prod_{\gamma\in\mathcal{C}_+(\mathsf{F})} 2w(\gamma))}$, which concludes the proof. □

Remark 6.5. *Since the following inequality holds,*

$$\mathbb{E}_p^W\left(\prod_{\gamma\in\mathcal{C}_+(\mathsf{F})} 2w(\gamma) \mid C_+(\mathsf{F}) \neq \emptyset\right) \geq 2 \geq \mathbb{E}_p^W\left(\prod_{\gamma\in\mathcal{C}_+(\mathsf{F})} 2w(\gamma) \mid C_+(\mathsf{F}) = \emptyset\right) = 1,$$

the previous theorem implies

$$\mu_w^W(C_+(\mathsf{F}) \neq \emptyset) \geq \mu_p^W(C_+(\mathsf{F}) \neq \emptyset).$$

6.1.2 Algorithm conditional on cycles with weights larger than 1.

Let $G = (V, E)$ be a finite graph endowed with a non-negative function $w : \mathcal{C}_{\rightarrow}(G) \rightarrow \mathbb{R}_+$ defined on oriented cycles of G which is symmetric under orientation reversal.

Definition 6.6. Let $W \subset G$ be a set of vertices of G (which can be empty). Let w_- be a weight function defined from the weight function w as follows:

$$\begin{cases} w_{-|\mathcal{C}_-(G\backslash W)} = w_{|\mathcal{C}_-(G\backslash W)}, \\ w_{-|\mathcal{C}_+(G\backslash W)} = 0. \end{cases}$$

Theorem 6.7. *Let C be a subset of $\mathcal{C}_+(G\backslash W)$ and let A be the set of vertices which are extremities of edges in C. Let F be an ECRSF with respect to W sampled according μ^W. Conditional on $\mathcal{C}_+(\mathsf{F}) = C$, the random forest $\mathsf{F}\backslash C$ has the same law than a ECRSF with respect to $A \cup W$ with weight function w_-.*

Proof. Let $F_0 \in \mathcal{U}_W(G)$.

$$\mu^W(\mathsf{F} = F_0 | \mathcal{C}_+(F) = C) = \frac{\mu(\mathsf{F} = F_0 \cap \mathcal{C}_+(\mathsf{F}) = C)}{\mu(\mathcal{C}_+(\mathsf{F}) = C)}.$$

Notice that this quantity is null if $C_+(F_0) \neq C$. Then, if $C_+(F_0) = C$ and $F_0 \in \mathcal{U}_W(G)$, every connected component of F_0 either contains a unique cycle in $C_-(G)$ or a unique cycle in C or is connected to a unique point in W. Therefore, every connected component of $F_0\backslash C$ either contains a unique cycle in $C_-(G)$ or is connected to a unique point in $W \cup A$ which means that $F_0\backslash C \in \mathcal{U}_{W\cup A}(G)$.

Then, the measure $\mu^W(.|C_+(F) = C)$ has support in

$$\mathcal{U}_W^C(G) = \{F \in \mathcal{U}_W(G) | C_+(F) = C\} = \{F \in \mathcal{U}_W(G) | F = C \cup F_-, F_- \in \mathcal{U}_{W\cup A}(G)\},$$

and if $F_0 \in \mathcal{U}_W^C(G)$,

$$\begin{aligned} \mu^W(\mathsf{F} = F_0 | \mathcal{C}_+(F) = C) &= \frac{\prod_{\gamma\in C} w(\gamma) \prod_{\gamma\in\mathcal{C}_-(F_0)} w(\gamma)}{\sum_{F\in\mathcal{U}_W(G)|\mathcal{C}_+(F)=C} \prod_{\gamma\in C} w(\gamma) \prod_{\gamma\in\mathcal{C}_-(F)} w(\gamma)} \\ &= \frac{\prod_{\gamma\in\mathcal{C}_-(F_0)} w(\gamma)}{\sum_{F\in\mathcal{U}_W^C(G)} \prod_{\gamma\in\mathcal{C}_-(F)} w(\gamma)}. \end{aligned}$$

Writing every $F \in \mathcal{U}_W^C(G)$ on a unique way as $C \cup F_-$ with $F_- \in \mathcal{U}_{W\cup A}(G)$,

$$\begin{aligned} \mu^W(\mathsf{F} = F_0 | \mathcal{C}_+(\mathsf{F}) = C) &= \frac{\prod_{\gamma\in\mathcal{C}_-(F_{0-})} w(\gamma)}{\sum_{F_-\in\mathcal{U}_{W\cup A}(G)} \prod_{\gamma\in\mathcal{C}_-(F_-)} w(\gamma)} \\ &= \mu^{W\cup A}_{w_{\mathcal{C}_-(G\backslash W)}}(F_{0-}) = \mu^{W\cup A}_{w_{\mathcal{C}_-(G\backslash W)}}(F_0\backslash C). \end{aligned}$$

Finally,

$$\mu^W(\mathsf{F}\backslash C = .|\mathcal{C}_+(\mathsf{F}) = C) = \mu^{W\cup A}_{w_{\mathcal{C}_-(G\backslash W)}}(.),$$

which concludes the proof. □

Since $w_{\mathcal{C}_-(G\setminus W)}$ takes values in $[0,1]$ by definition of $C_-(G\setminus W)$, the measure $\mu^{W\cup A}_{w_{\mathcal{C}_-(G\setminus W)}}$ can be sampled by the wired Wilson algorithm (Subsection 2.2.4) with boundary conditions $A\cup W$.

Therefore, under the measure μ^W, conditional on $\mathcal{C}_+(F)$, a ECRSF with respect to W can be sampled from an algorithm which samples a ECRSF with respect to W and extremities of edges in $\mathcal{C}_+(F)$.

6.2 Infinite volume measures

In this section, we consider a countably infinite connected graph $G=(V,E)$ with bounded vertex degree, and an exhaustion (G_n) of G.

Recall from section 3.1.1 that for a weight function w on cycles, we can define sequences denoted by (μ^W_n), (μ^F_n) of free and wired measures on the CRSF on G_n. When the weight function w is provided by a complex unitary connection, those sequences converge and under some assumptions, they converge towards the same limit.

We study in this section properties of the configurations sampled under the free or wired measures which correspond to a unitary connection h, in the case where the connection is such that Assumption 5.18 holds for the function w_{h_-} obtained from the weight function w_h as defined in Definition 6.6.

6.2.1 Comparison between the determinantal measure and a Wilson measure

Let h be a unitary complex connection such that Assumption 5.18 holds for the function w_{h_-} obtained from the weight function w_h. Let $\mu^W_{h,n}$ be the wired measure on ECRSF of G_n with wired boundary conditions on ∂G_n, corresponding to the connection h.

Let us denote by p_h the weight function which is defined by $p_h(\gamma)=\frac{w_h(\gamma)}{2}$. Let us denote by $\mu_{p,n}$ the measure on ECRSF with boundary conditions ∂G_n and weight function p_h. Since the weight function p_h satisfies Assumption 5.18 of the previous chapter, the measure in infinite volume μ_p associated to p_h is well defined and μ_p is the weak-limit of the sequence $(\mu_{p,n})_n$. From Theorem 5.22, μ_p almost surely, every connected component is finite and therefore almost surely there are infinitely many connected components.

We deduce from Theorem 6.2 and from this result the following result.

Proposition 6.8. *For every integer $n\geq 1$, let F_n be a configuration sampled according to the measure $\mu^W_{h,n}$, and let $\mathcal{C}(\mathsf{F}_n)$ be the set of cycles of F_n. Then, for every $k\geq 1$, when $n\to\infty$,*

$$\mu^W_{h,n}(|\mathcal{C}(\mathsf{F}_n)|\geq k)\to 1.$$

Proof. Let $k\geq 1$ be fixed. From Theorem 5.22,

$$\mu_p(|\mathcal{C}(F)|\geq k)=1.$$

Let $\varepsilon>0$. Since $\{|\mathcal{C}(F)|\geq k\}=\cup_m\{|\mathcal{C}(F\cap G_m)|\geq k\}$, the monotone convergence theorem implies the existence of m large enough such that $\mu_p(|\mathcal{C}(F\cap G_m)|\geq k)\geq 1-\varepsilon$..

Since $\{|\mathcal{C}(F \cap G_m)| \geq k\}$ depends on finitely many edges, the weak convergence of $(\mu^W_{p,n})$ gives the existence of $n \geq m$ large enough such that

$$\mu^W_{p,n}(|\mathcal{C}(F_n \cap G_m)| \geq k) \geq 1 - 2\varepsilon,$$

and then, for n large enough,

$$\mu^W_{p,n}(|\mathcal{C}(F_n)| \geq k) \geq \mu^W_{p,n}(|\mathcal{C}(F_n \cap G_m)| \geq k) \geq 1 - 2\varepsilon.$$

Therefore, when $n \to \infty$, $\mu^W_{p,n}(|\mathcal{C}(F_n)| \geq k) \to 1$.

According to Theorem 6.2, for every n,

$$\mu^W_{h,n}(|\mathcal{C}(F_n)| \geq k) \geq \mu^W_{p,n}(|\mathcal{C}(F_n)| \geq k).$$

Therefore, when $n \to \infty$, $\mu^W_{h,n}(|\mathcal{C}(\mathsf{F}_n)| \geq k) \to 1$. □

Nevertheless, we cannot deduce directly that for every $k \geq 1$, almost surely for the determinantal wired measure μ^W_h in infinite volume, which is the weak limit of the sequence $(\mu^W_{h,n})$,

$$\mu^W_h(|\mathcal{C}(\mathsf{F})| \geq k) = 1,$$

and that μ^W_h-almost surely, there are infinitely many connected components.

Indeed, the convergence of $(\mu^W_{h,n})$ towards μ^W_h is only a weak convergence. If we consider, thanks to the Skorokhod's representation theorem, a sequence F_n such that F_n converges almost surely towards F with F following the law μ^W_h and F_n following the law $\mu^W_{h,n}$ for every n, then if $\varepsilon > 0$, we have m large enough such that

$$\mu^W_{h,m}(|\mathcal{C}(\mathsf{F}_m)| \geq k) \geq 1 - \varepsilon.$$

Then, almost surely, for n large enough, F_n coincides with F on the graph G_m. Nevertheless, $\mathsf{F}_n \cap G_m$ does not have the same law as F_m and therefore we cannot deduce that

$$\mu^W_{h,n}(|\mathcal{C}(\mathsf{F}_n \cap G_m)| \geq k) \geq 1 - \varepsilon,$$

which would have proved that when $m \to \infty$,

$$\mu^W_h(|\mathcal{C}(\mathsf{F} \cap G_m)| \geq k) \to 1,$$

and would have implied from the monotone convergence theorem that

$$\mu^W_h(|\mathcal{C}(\mathsf{F})| \geq k) = 1.$$

We will prove in Theorem 6.25 that under a stronger assumption on the weight function w_h coming from a unitary connection h, Assumption 6.17, the free and wired measures are equal, and using the determinantal aspect of this measure μ_h, we will show that μ_h-almost surely there are infinitely many finite unicycles and therefore infinitely many finite connected components.

Recall that we know from Proposition 3.3 that every finite connected component contains a unique cycle which has non-trivial holonomy. In the next subsection, we prove that under Assumption 5.18 on the cycle weights, every connected component with a cycle is finite.

6.2.2 All the connected components with cycles are finite

In this subsection, let w_- be the weight function obtained from w as in Definition 6.6 and assume that there exists a family of cycles $C \subset \mathcal{C}_-(G)$ such that Assumption 5.18 holds for w_-.

We denote by (μ_n) the sequence of free measures on CRSF in finite volume associated to the weight function w and we assume that this sequence converges weakly towards a measure in infinite volume, denoted by μ. Let us emphasize that this assumption is satisfied whenever the weight function is provided by a unitary connection h or when the weight function takes values in $[0, 1]$ and $w_- = w$ satisfies Assumption 5.18.

We will show in this subsection that under those assumptions, every connected component with a cycle is finite.

We will use the following Lemma on the exponential decay of the tail distribution of ending times, which is a corollary of Lemma 5.8 (Section 5.1).

Lemma 6.9. *Let $m \in \mathbb{N}$, $n \geq m$. Let $C \subset G_n$ be a subgraph of G_n. Let $\mathbb{P}_x$ be the law of a w_--loop erased random walk (X_n) starting from x and T_r be the rooting time of (X_n). Let T_C and $T_{m,x}$ be the hitting times of C and ∂B_m^x. Under Assumption 5.18, the following inequality holds*

$$\mathbb{P}_x(\min(T_C, T_r) \geq T_{m,x}) \leq \delta^m,$$

which concludes the proof.

Proof. Applying Lemma 5.8 to the random walk (X_n) with weight function w_- which satisfies Assumption 5.1, we get

$$\mathbb{P}_x(T_r \geq T_{m,x}) \leq \delta^m.$$

But since $\min(T_C, T_r) \leq T_r$,

$$\mathbb{P}_x(\min(T_C, T_r) \geq T_{m,x}) \leq \mathbb{P}_x(T_r \geq T_{m,x}) \leq \delta^m,$$

which concludes the proof. □

Let $x \in G$ be fixed. We introduce some events with compact support which depend on x and prove an upper bound on the probability of those events.

Definition 6.10. For $m \in \mathbb{N}$, we denote by $A_m = \{x \leftrightarrow \partial B_{2m}^x\}$ the event that x and ∂B_{2m}^x are connected in F, that is to say that there exists a path between x and ∂B_{2m}^x in B_{2m}^x. If $n \in \mathbb{N}$, we denote by $\Gamma_n^- = \{x \leftrightarrow \mathcal{C}_-(F_{G_n})\}$ the event that x is connected to a closed cycle in G_n of weight less than 1.

Lemma 6.11. *Let $m \in \mathbb{N}$. For $n \geq m$ large enough, if F_n is distributed according to the free measure μ_n on G_n,*

$$\mu_n(A_m \cap \Gamma_n^-) \leq (|\partial B_{2m}^x| + 1)\delta^m.$$

Proof. Let n large enough such that for every $y \in \partial B_{2m}^x$, $B_m^y \subset G_n$. Let $C \subset \mathcal{C}_+(G_n)$. From Theorem 6.7, conditional on $\mathcal{C}_+(F_n) = C$, F_n is given by an algorithm of w_--loop erased random walks with boundary conditions on C. The proof relies on the same ideas as the proof of Lemma 5.21 and Theorem 5.22.

The event $A_m \cap \Gamma_n^-$ is satisfied if there exists $y \in \{x\} \cup \partial B_{2m}^x$ such that the w_--loop erased random walk starting from y has left B_m^y before being rooted to a cycle in $\mathcal{C}_-(G_n)$ and before touching C.

From Lemma 6.9, for every $y \in \{x\} \cup \partial B_{2m}^x$,

$$\mathbb{P}_y(\min(T_C, T_r) \geq T_{m,y}) \leq \delta^m.$$

Then, the union bound concludes the proof. □

Lemma 6.12. *For every $x \in V$,*

$$\mu(\{x \leftrightarrow C_-(F)\} \cap \{|cc(x)| = \infty\}) = 0.$$

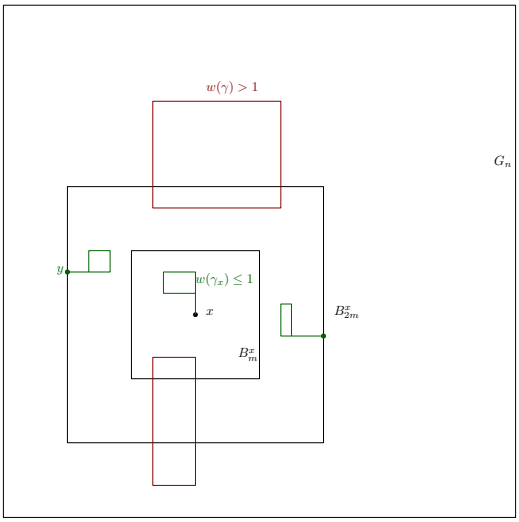

Figure 6.1 – Idea of proof of Lemma 6.12: if a vertex x is connected to a cycle of weight smaller than 1, the cycle is not too large and not too far from x, thus x is not connected to the boundary of a large enough neighborhood.

Proof. Let $\varepsilon > 0$. Let $m \in \mathbb{N}$ fixed, large enough such that $(|\partial B_{2m}^x| + 1)\delta^m < \varepsilon$. We consider the notations from Definition 6.10.

Since (Γ_n^-) is increasing, if we let $\Gamma^- := \{x \leftrightarrow \mathcal{C}_-(F)\} = \cup \Gamma_n^-$, then

$$\mu(A_m \cap \Gamma^-) = \mu(A_m \cap \cup \Gamma_n^-) = \mu(\cup_n (A_m \cap \Gamma_n^-)) = \lim_n \mu(A_m \cap \Gamma_n^-).$$

Let us consider n_0 large enough such that the inequality from Lemma 6.11 holds. Since for $n \geq n_0$, we have $\mu_n(A_m \cap \Gamma_{n_0}^-) \leq \mu_n(A_m \cap \Gamma_n^-) \leq \varepsilon$, we obtain

$$\mu(A_m \cap \Gamma_{n_0}^-) = \lim_n \mu_n(A_m \cap \Gamma_{n_0}^-) \leq \varepsilon.$$

It holds for every n_0 large enough and therefore, $\mu(A_m \cap \Gamma^-) \leq \varepsilon$. Since this inequality holds for m large enough, we have when $m \to \infty$, $\mu(A_m \cap \Gamma^-) \to 0$.

Therefore, since (A_m) is decreasing and $\cap_m A_m = \{|cc(x)| = \infty\}$, the monotone convergence theorem concludes the proof. □

Definition 6.13. For $l \in \mathbb{N}$, let $\Gamma_l^+ = \{x \leftrightarrow_{B_l^x} C_+(F_{B_l^x})\}$ be the event that x is connected inside B_l^x to a cycle with weight larger than 1 which is inside B_l^x, that is to say the event that in $F_{B_l^x}$, the connected component of x contains a cycle with weight larger than 1. For $m \in \mathbb{N}$, let $A_m := \{x \leftrightarrow \partial B_m^x\}$ be the event that x is connected to the boundary of B_m^x.

Lemma 6.14. *Let $m_0 \in \mathbb{N}$. For m large enough, there exists n_0 such that if $n \geq n_0$ and if F_n is distributed according to μ_n, then,*

$$\mu_n(A_m \cap \Gamma_l^+) \leq |\partial B_{m_0}^x| \delta^{m_0}.$$

Proof. Let $m_0, m \in \mathbb{N}$. Assume that m is large enough such that for every $y \in \partial B_m^x$, the equality $B_{m_0}^y \cap B_l^x = \emptyset$ holds. Let n be large enough such that for every $y \in \partial B_m^x$, the inclusion $B_{m_0}^y \subset G_n$ holds.

Let $C \in \mathcal{C}_+(G_n)$. Conditional on $\mathcal{C}_+(F_n) = C$, F_n is given by an algorithm of w_--loop erased random walks with wired conditions on C.

The event $\Gamma_l^+ \cap A_m$ is satisfied if the w_--loop erased random walk starting from x hits a cycle in $C_+(F_{B_l^x})$ before leaving B_l^x and before being rooted to another cycle and if one of the w_--loop erased random walks starting from points of ∂B_m^x reaches B_l^x before being rooted to a cycle or hitting C. Therefore, from Lemma 6.9 and from the union bound,

$$\mu_n(A_m \cap \Gamma_l^+) \leq |\partial B_{m_0}^x| \delta^{m_0},$$

which concludes the proof. □

Lemma 6.15. *For every $x \in V$,*

$$\mu(\{x \leftrightarrow C_+(F)\} \cap \{|cc(x)| = \infty\}) = 0.$$

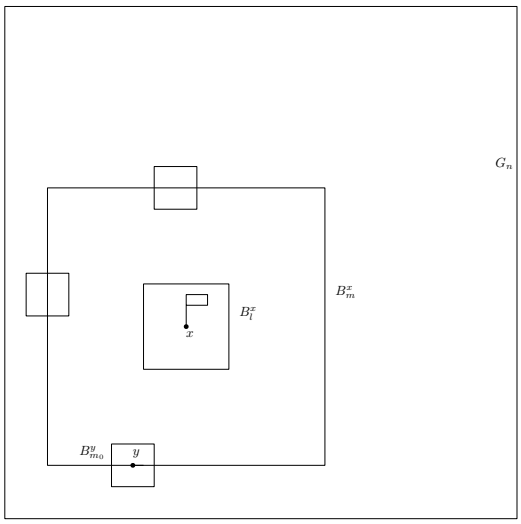

Figure 6.2 – Idea of proof of Lemma 6.15: if a vertex x is connected to a fixed finite cycle of weight larger than 1, vertices which are far enough from this cycle are not connected to x.

Proof. Let $x \in G$ and let

$$A := \{|cc(x)| = \infty\} = \cap_m A_m$$

be the event that the connected component of 0 is infinite.

Let $l \in \mathbb{N}$. Let $\varepsilon > 0$. Let m_0 large enough such that $|\partial B^x_{m_0}|\delta^{m_0} \leq \varepsilon$. Let m be large enough such that for every $y \in \partial B^x_m$, $B^y_{m_0} \cap B^x_l = \emptyset$. Let n be large enough such that for every $y \in \partial B^x_m$, $B^y_{m_0} \subset G_n$.

Let F_n distributed according μ_n. Then, from Lemma 6.2.2,

$$\mu_n(A_m \cap \Gamma^+_l) \leq |\partial B^x_{m_0}|\delta^{m_0} \leq \varepsilon.$$

Since this inequality holds for every n large enough and $A_m \cap \Gamma^+_l$ depends on finitely many edges,

$$\mu(A_m \cap \Gamma^+_l) = \lim_n \mu_n(A_m \cap \Gamma^+_l) \leq \varepsilon.$$

Since this inequality holds for every ε for m large enough, we obtain when $m \to \infty$,

$$\mu(A_m \cap \Gamma^+_l) \to 0.$$

Since A_m is decreasing,

$$\mu(A \cap \Gamma^+_l) = \mu(\cap_m A_m \cap \Gamma^+_l) = \lim_m \mu(A_m \cap \Gamma^+_l) = 0.$$

Since (Γ^+_l) is increasing and $\Gamma^+ = \{x \leftrightarrow C_+(F)\} = \cup_l \Gamma^+_l$,

$$\mu(A \cap \Gamma^+) = \mu(A \cap (\cup_l \Gamma^+_l)) = \mu(\cup_l (A \cap \Gamma^+_l)) = \lim_l \mu(A \cap \Gamma^+_l) = 0.$$

which is precisely what we wanted to prove. □

Let us emphasize that Lemma 6.15 and Lemma 6.12 show that for every $x \in G$, almost surely, if x is connected to a cycle in the random configuration F, the connected component of x is finite. Therefore, since G is countable, we immediately deduce the following theorem.

Theorem 6.16. *Under an infinite volume measure μ_w such that w_- satisfies Assumption 5.18, every connected component with a cycle is finite.*

Let us emphasize that Theorem 6.16 holds for measures which are not necessarily determinantal. From Proposition 3.3 and Theorem 6.16, we know that a connected component does not have a cycle if and only if it is an infinite connected component. We believe that when the weight function w is provided by a unitary connection h, the determinantal aspect of the limit measure μ_h avoids the existence of a connected component without a cycle but it is still a conjecture.

In the next section (6.3), we present some results based on the determinantal aspect of the measure μ_h for a weight function w_h which is provided by a unitary connection h. We will study the probability of missing some edges sets to show that, under a stronger assumption, almost surely, some finite unicycles are observed infinitely many times. Under this stronger assumption, the decay of edge-to-edge correlations is exponential. We believe that in that case, there are no infinite connected component.

6.3 Determinantal measures associated to a unitary complex connection.

In this section, we assume that the graph G is invariant under translations and that G is endowed with a unitary complex connection h which is also invariant under translations. Let w_h be the weight function associated to this connection.

Recall from [KL23] (see subsection 3.2.2) that the following decomposition holds:

$$\Omega^1_{\ell^2}(F) = \bigstar_{\ell^2} \oplus d_h(\mathcal{H}) \oplus \diamondsuit_{\ell^2},$$

where $\mathcal{H} = \{f \in \Omega^0 : \Delta_h f = 0 \text{ and } d_h f \in \Omega^1_{\ell^2}(F)\}$.

6.3.1 Acceptable unicycles are observed infinitely many times for the determinantal measure.

We assume that Assumption 5.18 holds for w_{h_-} and we also make the following assumption.

Assumption 6.17. *There exists some constants $m, N, l > 0$ such that for every $x \in G$, there exists a cycle γ_x of length $\|\gamma_x\| \leq l$ and holonomy $q_x \neq 1$ such that $|1 - q_x| > m$ and if $e \in E$, the inequality $|\{x | e \in \gamma_x\}| \leq N$ holds.*

From Subsection 3.2.2, the sequences of measures $(\mu^W_{h,n})$, $(\mu^F_{h,n})$ on CRSF on the sequence of growing subgraphs (G_n) with wired or free boundary conditions converge towards thermodynamic limits μ^F_h and μ^W_h, which are determinantal measures whose kernels are orthogonal projections on spaces $\diamondsuit^{\perp}_{\ell^2}$ and $\bigstar_{\ell^2}$.

The following lemma, from [KL23] shows that under Assumption 6.17, the following orthogonal decomposition holds:

$$\Omega^1_{\ell^2}(F) = \bigstar_{\ell^2} \oplus \diamondsuit_{\ell^2}.$$

Lemma 6.18. *[KL23] Under Assumption 6.17, $d_h(\mathcal{H}) = \{0\}$ and therefore spaces $\diamondsuit^{\perp}_{\ell^2}$ and $\bigstar_{\ell^2}$ are equal.*

Proof. Let $C = \frac{l}{m}$. Let $f \in \Omega^0(G)$, such that $d_h f \in \Omega^1_{\ell^2}(G)$. Assume that f is harmonic that is to say that $\Delta_h(f) = 0$. Let x in G and γ_x a cycle as in Assumption 6.17. Then, we have

$$|f(x)||1 - q_x| \leq \sum_{e \in \gamma_x} |d_h f(e)| \leq ||\gamma_x|| \max_{e \in \gamma_x} |d_h f(e)|,$$

and therefore,

$$|f(x)|^2 \leq \left(\frac{l}{1 - q_x} \max_{e \in \gamma_x} |d_h f(e)| \right)^2 \leq C^2 \max_{e \in \gamma_x} |d_h f(e)|^2.$$

Therefore,

$$\sum_x |f(x)|^2 \leq C^2 \sum_x \max_{e \in \gamma_x} |d_h f(e)|^2 \leq C^2 \sum_x \sum_{e \in \gamma_x} |d_h f(e)|^2 \leq C^2 \sum_e \sum_{x | e \in \gamma_x} |d_h f(e)|^2.$$

From Assumption 6.17, $|\{x | e \in \gamma_x\}| \leq N$. Therefore,

$$\sum_x |f(x)|^2 \leq C^2 N^2 \sum_e |d_h f(e)|^2 < \infty.$$

Therefore $f \in \ell^2$, and $||d_h f||^2 = \langle d_h f, d_h f \rangle = \langle f, \Delta_h f \rangle = 0$. Thus, we have $d_h f = 0$. □

Therefore, under Assumption 6.17, the measures μ_h^W and μ_h^F are equal. In the remainder of this subsection, we will denote by μ_h the determinantal measure in infinite volume associated to the connection h, whose kernel is the orthogonal projection on $\bigstar_{\ell^2}$.

Lemma 6.19. *Let $U \subset E$ be a finite subgraph of G. Let A be the set of vertices of U and let $C_U = (E\backslash U) \cap (A \times V \cup V \times A)$ be the finite set of edges which have at least an extremity in A and which are not in U. Then, if F is a random configuration distributed according μ_h, the following conditions are equivalent :*

- *Almost surely, F intersect C_U, that is to say,*

$$\mu_h(\mathsf{F} \cap C_U = \emptyset) = 0.$$

- *There exists $f \neq 0$ such that $d_h(f) \in Vect(w_e)_{e \in C_U}$.*

Proof. From the determinantal aspect of μ_h, we have

$$\mu_h(F \cap C_U = \emptyset) = \det(I - \Pi^{\bigstar})_{C_U}^{C_U} = \det(\Pi^{\diamondsuit})_{C_U}^{C_U} = \det(\langle \Pi^{\diamondsuit}(w_e), \Pi^{\diamondsuit}(w_{e'}) \rangle)_{e,e' \in C_U}.$$

The previous determinant vanishes if and only if the family $(\Pi^{\diamondsuit}(w_e))_{w_e \in C_U}$ is not independent. Let us consider $(\alpha_e)_{e \in C_U}$. By linearity of $\Pi^{\diamondsuit}$, we have

$$\sum \Pi^{\diamondsuit}(w_e)\alpha_e = \Pi^{\diamondsuit}(\sum w_e \alpha_e).$$

Therefore, the family $(\Pi^{\diamondsuit}(w_e))_{w_e \in C_U}$ is not independent if and only if there exists $(\alpha_e)_{e \in C_U}$ such that

$$\sum w_e \alpha_e \in im(d_h) \backslash \{0\},$$

which concludes the proof. □

The equivalence of Lemma 6.19 implies the following corollary, which gives another proof of Proposition 3.3 in the case of the determinantal measure μ_h.

Corollary 6.20. *If F is distributed according to μ_h and if T is a finite connected component without any cycle of non trivial holonomy, then,*

$$\mu_h(C_T \cap \mathsf{F} = \emptyset) = 0.$$

Therefore, under μ_h, almost surely, every finite connected component of F contains a cycle of non trivial holonomy.

Proof. Let $x \in V$ and let F_x be the connected component of x in F. Let T be a finite subgraph which contains x and which does not contain a cycle of non-trivial holonomy.

Let r be a vertex of T. Let us define f by $f(r) = 1$ and for every edge $e \in T$, if $f(e^-)$ is defined, let $f(e^+) = h_e f(e^-)$. Since every cycle of T has trivial holonomy, f is well-defined. Therefore, for every $e \in T$, $d_h f(e) = 0$. Then, define f by $f_{V \backslash V(T)} = 0$. Then, if e has both extremities in $V \backslash V(T)$, $d_h f(e) = f(e^+) - h_e f(e^-) = 0$. Therefore, there exists $f \neq 0$ such that $d_h(f) \in Vect(w_e)_{e \in C_T}$ and Lemma 6.19 implies that

$$\mu_h(C_T \cap \mathsf{F} = \emptyset) = 0.$$

Since the set $\mathcal{T}_x$ of finite subgraphs which contain x and which do not contain a cycle of non trivial holonomy is countable, we deduce that

$$\mu_h(F_x \in \mathcal{T}_x) = 0.$$

Since the graph G is countable, almost surely, every finite connected component of F contains a cycle of non trivial holonomy. □

Notice that this new proof of Proposition 3.3 for μ_h does not use the weak convergence of the sequence of measures in finite volume but the determinantal property of the limit measure μ_h.

Definition 6.21. Let $U_1, \dots, U_p$ be a finite family of finite disjoints unicycles, all of whose cycles γ_i have holonomies $q_i \neq 1$. Let $A = \{v \in V | v \in U_1 \cup \dots \cup U_p\}$ be the set of vertices of $U := \cup_i U_i$. We say that $U_1, \dots, U_p$ is an admissible unicycles family if in every connected component of the induced subgraph by $A^{\complement}$, $G_{A^{\complement}} = (A^{\complement}, \{e : e^-, e^+ \in A^{\complement}\})$, there exists a cycle of non trivial holonomy.

Lemma 6.22. *Let $U_1, \dots, U_p$ be an admissible unicycles family. Let C_U be the finite set of edges which have at least an extremity in A and which are not in $U := \cup_i U_i$. Then, if F is a random configuration distributed according to μ_h, we have*

$$\mu_h(\mathsf{F} \cap C_U = \emptyset) > 0.$$

Proof. Let f be such that $\sum w_e \alpha_e = d_h(f)$, with $(\alpha_e)_{e \in C_U} \in \mathbb{C}^{C_U}$. Let us prove that $f = 0$. Let $i \in [1, p]$ and let $e \in U_i$. By definition of C_U, $e \notin C_U$. Therefore, $d_h f(e) = 0$. In particular, for every $e \in \gamma_i$, $d_h f(e) = 0$. If x is a vertex of γ_i,

$$(1 - q_i) f(x) = \sum_{e \in \gamma_i} d_h f(e) = 0.$$

Therefore, since $q_i \neq 1$, $f(x) = 0$. If $v \in A$, there exists a path in U between v and a vertex $x_i \in V(\gamma_i)$ and therefore $f(v) = 0$.

Now, if $v \notin A$, the connected component of v in $A^{\complement}$ contains a cycle of non trivial holonomy in $E(A^{\complement})$ and since $E(A^{\complement}) \cap C_U = \emptyset$, we also have $f(v) = 0$ for every $v \in A^{\complement}$.

Therefore, $f = 0$ and if $e \in C_U$, we have

$$\alpha_e = d_h f(e) = f(e^+) - h_e f(e^-) = 0,$$

and the family $(\Pi^{\Diamond}(w_e))_{w_e \in C_A}$ is independent. Therefore

$$\mu_h(\mathsf{F} \cap C_U = \emptyset) > 0,$$

which is the desired inequality. □

Let us underline that the admissible condition on the family of unicycles is a necessary one. First of all, from Corollary 6.20, every U_i must contain a cycle of non-trivial holonomy. Moreover, the following Lemma shows that the condition on $A^{\complement}$ is a necessary one.

Lemma 6.23. *Let $U_1, \dots, U_p$ a finite family of disjoint unicycles such that there exists $x \in A^{\complement}$ whose connected component in the induced subgraph by $A^{\complement}$ does not contain any cycle of non trivial holonomy. Then,*

$$\mu_h(\mathsf{F} \cap C_U = \emptyset) = 0.$$

Proof. Let T be the connected component in the induced subgraph by $A^{\complement}$ of x. If the event $\{\mathsf{F} \cap C_U = \emptyset\}$ is satisfied, then the connected component $cc(x)$ of x in F is included in T. Indeed, if an edge e connects a point of $A^{\complement}$ to a point of A, it is in C_U and therefore not in F. Since T does not contain any cycle of non trivial holonomy, neither does $cc(x)$, which cannot happen from Proposition 3.3. □

Theorem 6.24. *Let $p \in \mathbb{N}^*$ and let $U_1, \dots, U_p$ be an admissible unicycles family. Let us denote by K_1 a finite box containing $U_1, \dots, U_p$, $(K_k)_k$ a family of disjoint boxes translated from K_1, and $U_1^k, \dots, U_p^k$ the translated cycle-rooted trees from $U_1, \dots, U_p$. Let B_k be the event*

$$B_k = \{\mathsf{F} \cap (\cup_i C_{U_i^k}) = \emptyset\}.$$

Then,

$$\mu_h(\limsup_{k\to\infty} B_k) = 1.$$

Proof. From invariance under translation,

$$\mu_h(B_k) = \mu_h(F \cap (\cup_i C_{U_i^k}) = \emptyset) > 0,$$

and therefore

$$\sum_k \mu_h(B_k) = \infty.$$

The events $(B_k^{\complement})_{k\geq 1}$ are not independent but they are negatively associated since the measure is determinantal and the events are growing and have disjoint supports. Therefore, for every $l \leq m$,

$$\mu_h(\cap_{l\leq k\leq m} {B_k}^{\complement}) \leq \prod_{l\leq k\leq m} \mu_h(B_k^{\complement}).$$

Therefore we can adapt the proof of the lemma of Borel-Cantelli to see that

$$\mu_h(\limsup B_k) = 1.$$

Indeed, for every $l \leq m$,

$$\begin{aligned}\mu_h(\cup_{l\leq k\leq m} B_k) &= 1 - \mu_h(\cap_{l\leq k\leq m} B_k^{\complement}) \\ &\geq 1 - \prod_{l\leq k\leq m} \mu_h(B_k^{\complement}) \\ &= 1 - \exp(\sum_{l\leq k\leq m} \ln(1-\mu_h(B_k))).\end{aligned}$$

Since $\sum_k \mu_h(B_k) = \infty$, when $m \to \infty$, $\sum_{l\leq k\leq m} \ln(1-\mu_h(B_k)) \to -\infty$,

$$\mu_h(\cup_{l\leq k\leq m} B_k) \to 1.$$

Therefore, for every $l \in \mathbb{N}$,

$$\mu_h(\cup_{l\leq k} B_k) = 1.$$

By countable intersection,

$$\mu_h(\limsup B_k) = 1.$$

which is precisely the claim of the theorem. □

Notice that if the event B_k is satisfied, for every $i \in [1,p]$, for every $x \in V(U_i^k)$, the connected component of x in F is included in U_i^k. From Proposition 3.3, every finite connected component of F contains exactly one cycle, then, for every $x \in V(U_i^k)$, the connected component of x contains the cycle γ_i^k and therefore the connected component of x in F is precisely the unicycle U_i^k.

By countable intersection over $p \in \mathbb{N}^*$ and $U_1, \dots, U_p$ families of p disjoints cycle-rooted trees which are in an accepted configuration, we deduce the following result.

Theorem 6.25. *μ_h almost surely, we observe whatever finite admissible family of unicycles infinitely many times.*

6.3.2 Examples of periodic complex connections

Assume that h is a periodic connection h of rank 1 with fundamental domain of size 1 which is given by some unitary complex numbers $(u_1, \dots, u_d) \in \mathbb{C}^d$, in the sense that for every $j \in [1,d]$,

$$h(x, x+t_j) = u_j,$$

where (t_i) is the set of translation-vectors, that is here the canonical basis of $\mathbb{R}^d$.

Notice that every cycle has a trivial holonony since complex numbers commute. Therefore, Assumptions 5.18 and 6.17 do not hold here. Moreover, the function $z \mapsto P(z)$ vanishes at $(u_1^{-1}, \dots, u_d^{-1})$ and therefore, Corollary 4.24 does not hold here.

From Theorem 4.26, the measure μ_h coincides with μ_{id} and the model has the same law as the uniform spanning forest. In particular, the edge-to-edge correlations decay at polynomial rate, no connected component has a cycle and all connected components are infinite trees.

We may also consider a connection which is 2-periodic in the following sense. Let u be a unitary complex and let us consider in this subsection the graph $G = \mathbb{Z}^2$ endowed with the following connection

$$h((x,y),(x+1,y)) = \begin{cases} u \text{ if } y = 0 \ [2] \\ 1 \text{ else} \end{cases} \quad , \quad h((x,y),(x,y+1)) = 1$$

The graph is $(2\mathbb{Z})^2$-periodic with fundamental domain $[-1,1]^2$, and the weight function w_h is invariant under translations.

Lemma 6.26. *If $u \notin \{1,-1\}$, Assumptions 5.18 and 6.17 hold for the graph G and the weight function w_{h_-} obtained from the connection h.*

Proof. We first assume that $\Re(u) \in [0,1[$. From invariance under translation, and since the exhaustion (G_n) satisfies $d(0,\partial G_n) \sim 4n$, it is enough to show the existence of $\alpha > 0$ and $\beta > 0$ and a family of loops $C \subset \mathcal{C}_-(\mathbb{Z}^2)$ such that for every loop $\gamma \in C, \alpha \leq w(\gamma) \leq 1$ and for every $x \in \partial G_n$, there exists a loop $\gamma_x \in C \cap (G_{n+1} \backslash (G_n \cup \partial G_{n+1}))$ such that the inequality $\mathbb{P}_x(\gamma_x) \geq \beta$ holds.

Let us construct a family of loops $(\gamma_j^+, \gamma_j^-)_{j\in[1,2]}$ such that for every $x \in \partial G_n$, there exists a loop starting from x which stays in $G_{n+1} \backslash (G_n \cup \partial G_{n+1})$ and which is a loop translated from a loop of this family.

We still denote by (t_i) the canonical basis of $\mathbb{Z}^2$. Let $j \in [1,2]$ and define $\gamma_j^{+,-}$ by

$$\begin{cases} \gamma_j^+ = (0, t_j, 2t_j, 2t_j + t_{j[2]+1}, t_j + t_{j[2]+1}, t_j) \\ \gamma_j^- = (0, -t_j, -2t_j, -2t_j + t_{j[2]+1}, -t_j + t_{j[2]+1}, -t_j) \end{cases}$$

Let $x \in \partial G_n$ be a fixed vertex and let j be such that $x_j = \pm n$. If $x_j = n$, the loop defined by $\gamma_x := x + \gamma_j^+$ satisfies the assumption and if $x_j = -n$, the loop $\gamma_x := x + \gamma_j^-$ satisfies the assumption.

If (X_n) is a simple random walk starting from x, it makes the loop γ_x of length 5 with probability

$$\mathbb{P}_x(\gamma_x) = \left(\frac{1}{2d}\right)^5.$$

Notice that from invariance under translation, the weight of the loop γ_x is

$$w_h(\gamma_x) = 1 - \Re(u^{\pm 1}).$$

Denoting by $\alpha = \min(1 - \Re(u), 1 - \Re(u^{-1}))$, Assumption 5.18 is satisfied.

If u is such that $\Re(u) \in]-1, 0]$, then $\Re(u^2), \Re(u^{-2}) \in [0,1[$. Then, the family of loops defined by $(x + \gamma_j^{+,-})_{x\in G, j\in[1,d]}$ with

$$\begin{cases} \gamma_j^+ = (0, t_j, 2t_j, 3t_j, 3t_j + t_{j[2]+1}, 2t_j + t_{j[2]+1}, t_j + t_{j[2]+1}, t_j) \\ \gamma_j^- = (0, -t_j, -2t_j, -3t_j, -3t_j + t_{j[2]+1}, -2t_j + t_{j[2]+1}, -t_j + t_{j[2]+1}, -t_j) \end{cases}$$

satisfy Assumption 5.18 with $\beta = \left(\frac{1}{2d}\right)^7$ and $\alpha = \min(1 - \Re(u^2), 1 - \Re(u^{-2}))$.

We also check that Assumption 6.17 holds. For every $v \in \mathbb{Z}^2$, there exists a cycle γ_v rooted at v, of length 4 and of holonomy $u \neq 1$. More precisely, if $v = (x, y)$, $\gamma_v = v + l_r$ where $l_r = ((0,0), (1,0), (1,1), (0,1), (0,0))$. Notice that if $y = 0$ [2], γ_v has holonomy u, whereas if $y = 1$ [2], γ_v has holonomy u^{-1}. For every edge $e \in E(\mathbb{Z}^d)$, $|\{v | e \in \gamma_v\}| = 2$. Therefore, Assumption 6.17 is satisfied for $m = \min(|1 - u|, |1 - u^{-1}|)$ and $N = 2$. □

We deduce from Theorem 6.25 that μ_h almost-surely every finite admissible family of unicycles is observed infinitely many times and in particular, there are almost surely infinitely many finite connected components. Moreover, every connected component with a cycle is finite.

Chapter 7

Q-determinantal measures associated to a unitary quaternionic connection.

In this chapter, we denote by $\mathbb{H}$ the algebra of quaternions which is an associative normed division algebra over the real numbers. Quaternions are generally represented in the form $a + bi + cj + dk$, where a, b, c, d are real numbers and $1, i, j, k$ are the basis vectors. Recall that the multiplication of quaternions is non commutative but i, j, k are defined such that the following formula for quaternion multiplication hold:

$$ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j, \quad i^2 = j^2 = k^2 = ijk = -1.$$

Conjugation of quaternions is analogous to conjugation of complex numbers. To define it, let $q = a + bi + cj + dk$ be a quaternion. Then, the conjugate of q is the quaternion $q^* = a - bi - cj - dk$. The norm over quaternions is defined for every quaternion $q = a + bi + cj + dk$ by

$$|q| = \sqrt{qq^*} = \sqrt{a^2 + b^2 + c^2 + d^2}.$$

We denote by $U(\mathbb{H})$ the subgroup of unit quaternions.

Just as complex numbers can be represented as matrices, so can quaternions. There is an injective homomorphism from $\mathbb{H}$ to the matrix ring $M(2, \mathbb{C})$ defined as follows. The quaternion $q = a + bi + cj + dk$ can be represented as

$$\begin{pmatrix} a + bi & c + di \\ -c + di & a - bi \end{pmatrix}$$

The norm of a quaternion is the square root of the determinant of the corresponding matrix. By restriction, this representation yields an isomorphism between the subgroup of unit quaternions $U(\mathbb{H})$ and their image $SU_2(\mathbb{C})$.

In 1922, Moore ([Moo22]) introduced determinants of quaternion Hermitian matrices, called Q-determinants. The Q-determinant of a self-dual matrix M is defined as

$$Q\det(M) = \sum_{\sigma \in S_n} sgn(\sigma) \prod_{cycles} \frac{1}{2} Tr(w)$$

where each permutation σ is written as a product of disjoint cycles and $tr(w)$ is the trace of the product of matrix entries in that cycle.

In [Meh04], this notion of Q-determinant is extended to self-dual matrices with entries in $GL_2(\mathbb{C})$ and the following formula from [Meh04] allows us to compute Q-determinants explicitly. If M is an $n \times n$ self-dual matrix with entries in $GL_2(\mathbb{C})$ and $\hat{M}$ is the associated $2n \times 2n$ matrix obtained by replacing each entry with the 2×2 block of its entries, then

$$Qdet(M) = Pf(Z\hat{M}),$$

the Pfaffian of the antisymmetric matrix $Z\hat{M}$ where Z is the matrix with 2×2 diagonal blocks $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and zeros elsewhere.

Finally, we know (see for instance [KL20c, Proposition 6.4]) that if M is a quaternion Hermitian matrix, and $\hat{M}$ is the associated matrix obtained by replacing each quaternionic entry with the 2×2 block of its complex entries, then we have the following relation:

$$Qdet(M)^2 = \det(\hat{M}).$$

The Laplacian Δ associated to a unitary quaternionic connection is a self-dual operator. According to [Ken11], its Q-determinant has the following expansion:

$$Q\det\Delta = \sum_{F\ CRSF} \prod_{\gamma\ cycle} (2 - Tr(hol(\gamma)))$$

where $Tr(hol(\gamma))$ is the trace of the holonomy of the cycle γ (see also [Kas15, KL20a] for alternative proofs).

On a finite graph denoted by $G = (V, E)$, the model of random CRSF can also be defined for a quaternionic unitary connection. In this case, the model is a Q-determinantal process. Indeed, the expansion of the Q-determinant of the Laplacian defines a natural Q-determinantal measure μ_Q on CRSF whose partition function is given by the Q-determinant of the Laplacian. This measure is studied in [Kas15] and [KL20c] and satisfies some similar properties to the determinantal measures.

The generating polynomial g_{μ_Q} of the measure μ_Q is the square-root of a generating polynomial associated to a kernel $\hat{K}$ obtained by replacing unitary quaternions of the connection by SU_2 matrices:

$$g_{\mu_Q}(z_1, \dots, z_n) = \sqrt{\det(I + (diag(z_1, z_1, \dots, z_n, z_n) - I)\hat{K})_{e_1,\dots,e_n}}.$$

This kernel $\hat{K}$ is the kernel of the determinantal linear process, so-called quantum spanning forest, associated to a rank 2 complex connection, as defined in Section 2.3. More generally, the model of quantum spanning forests associated to a quaternionic unitary connection on a vector-bundle of larger rank can be defined as in [KL20c] and Section 2.3, and we have a correspondence between quaternionic and complex connections.

7.1 Correspondence between quaternionic and complex connections

According to [KL20c], the superposition of two independent samples of this quantum spanning forests has the same distribution as a quantum spanning forests associated to a complex unitary connection on a larger rank vector-bundle. This correspondence is established for finite graphs in [KL20c, Proposition 6.13], and we extend this result to the case of infinite graphs in the following subsection.

7.1.1 Superposition of two samples associated to a quaternionic unitary connection.

Assume that G is a countably infinite graph and that (G_n) is an exhaustion of G by finite graphs. Recall the following result from [KL20c] which gives a way to prove Theorem 7.2 for infinite graphs.

Theorem 7.1. *[KL20c, Proposition 6.13] For every $n \in \mathbb{N}$, if $Q^{(1)}$ and $Q^{(2)}$ are independent quantum spanning forests associated to a kernel k_n on a finite graph G_n with values in the quaternionic field and Q is the quantum spanning forest associated to k_n on G_n with values in the complex field, we have the following equality in law:*

$$(\dim_{\mathbb{C}} Q_e)_{e\in G_n} \overset{\mathcal{L}}{=} (\dim_{\mathbb{H}} Q^1_e + \dim_{\mathbb{H}} Q^2_e)_{e\in G_n}.$$

Theorem 7.2. *If $Q^{(1)}$ and $Q^{(2)}$ are independent quantum spanning forests of kernel k on G associated to a unitary connection h of rank N with values in the quaternionic field and Q is the quantum spanning forest of kernel $\hat{k}$ associated to the corresponding connection $\hat{h}$ of rank $2N$ with values in the complex field, we have for every $(e_1, ..., e_m) \in G$,*

$$(\dim_{\mathbb{C}} Q_e)_{e\in(e_1,...,e_m)} \overset{\mathcal{L}}{=} (\dim_{\mathbb{H}} Q^1_e + \dim_{\mathbb{H}} Q^2_e)_{e\in(e_1,...,e_m)}.$$

Proof. We give two proofs of this theorem, one which is based on the convergence of k_n to k and on Theorem [KL20c, Proposition 6.13] for finite graphs and the other one which is based on a direct computation of Laplace transform, with the same ideas as in the proof of Theorem [KL20c, Proposition 6.13].

1. Let n large enough such that $(e_1, ..., e_m) \in G_n$. Let μ_n be the determinantal measure whose kernel, denoted by k_n, is associated to the unitary quaternionic connection h of rank N and $\hat{\mu}_n$ be the determinantal measure whose kernel, denoted by $\hat{k}_n$, is associated to the corresponding unitary complex connection $\hat{h}$ of rank $2N$.

   Then, if $((Q_n)_e)_{e\in G_n}$ and $((Q^1_n)_e)_{e\in G_n}, ((Q^2_n)_e)_{e\in G_n}$ are determinantal linear processes of respective kernels $\hat{k}_n$ and $(k_n \bigotimes k_n)$, according to [KL20c, Proposition 6.13], we have the following equality in law:

   $$(\dim_{\mathbb{C}}(Q_n)_e)_{e\in G_n} \overset{\mathcal{L}}{=} (\dim_{\mathbb{H}}(Q^1_n)_e + \dim_{\mathbb{H}}(Q^2_n)_e)_{e\in G_n},$$

   and in particular,

   $$(\dim_{\mathbb{C}}(Q_n)_e)_{e\in(e_1,...,e_m)} \overset{\mathcal{L}}{=} (\dim_{\mathbb{H}}(Q^1_n)_e + \dim_{\mathbb{H}}(Q^2_n)_e)_{e\in(e_1,...,e_m)}.$$

Since we have the weak convergence of the determinantal measures associated to k_n and $\hat{k}_n$ to the determinantal measure associated to k and $\hat{k}$, we deduce that

$$(\dim_{\mathbb{C}} Q_e)_{e\in(e_1,\dots,e_m)} \overset{\mathcal{L}}{=} (\dim_{\mathbb{H}} Q_e^1 + \dim_{\mathbb{H}} Q_e^2)_{e\in(e_1,\dots,e_m)}.$$

2. For every $e \in E$, let $(b_i^e)_{i\leq N}$ be an orthonormal basis a the fiber F_e, for instance the canonical basis of $\mathbb{C}^N$, and let K be the matrix in those bases of the kernel associated to a connection with values in the quaternionic field and $\hat{K}$ be the matrix in those bases of the same kernel seen as an operator with values in the complex field, which corresponds to a complex-valued connection of rank $2N$.

   Let $\Lambda \subset E$ be a finite subset of edges. Let $B = ((b_i^e)_{i\leq N})_{e\in\Lambda}$ be the finite collections of the bases of the fibers $(F_e)_{e\in\Lambda}$ over the edges of the finite set Λ.

   We can compute the Laplace transform for the quantum spanning forest with quaternionic connexion. If X is a random quantum spanning forest of kernel K adapted to the fibers $(F_e)_{e\in E}$. From properties of determinantal or Q-determinantal point processes, we have for every $(w(b))_{b\in B}$,

$$\mathbb{E}\left[\prod_{b\in X\cap B}(1+w(b))\right] = \det(1+wK_{|B}) = \sum_{J\subset B}\left(\prod_{b\in J} w(b)\right)\det K_J^J.$$

   Writing $\det_{\mathbb{H}}$ instead of $Qdet$, we have the following relation between the determinant of an operator with values in the quaternionic field and the same operator seen as an operator with values in the complex field:

$$\det_{\mathbb{C}}(1+w\hat{K}) = \det_{\mathbb{H}}(1+wK)^2.$$

   Then if $\hat{B} = B \times B$, since $\hat{K}$ is the operator K seen as an operator with values in the complex field, we have,

$$\begin{aligned}\det_{\mathbb{C}}(1+w\hat{K}_{\hat{B}}) &= \det_{\mathbb{H}}(1+wK_B)^2 \\ &= \sum_{J_1,J_2\subset B}\left(\prod_{b_1\in J_1} w(b_1)\right)\left(\prod_{b_2\in J_2} w(b_2)\right)\det_{\mathbb{H}} K_{J_1}^{J_1}\det_{\mathbb{H}} K_{J_2}^{J_2}.\end{aligned}$$

   Since $\hat{K}$ is also the kernel of a determinantal linear process on a complex vector bundle of rank $2N$, we also have the relation

$$\det_{\mathbb{C}}(1+w\hat{K}_{\hat{B}}) = \sum_{J_1\times J_2\subset\hat{B}}\left(\prod_{(b_1,b_2)\in J_1\times J_2} w(b_1)w(b_2)\right)\det_{\mathbb{C}}\hat{K}_{J_1\times J_2}^{J_1\times J_2}.$$

   Notice that if $J_1 \subset B$, $J_2 \subset B$, we have

$$\left(\prod_{b_1\in J_1} w(b_1)\right)\left(\prod_{b_2\in J_2} w(b_2)\right) = \prod_{(b_1,b_2)\in J_1\times J_2} w(b_1)w(b_2).$$

   Therefore, for every $J_1 \subset B$, $J_2 \subset B$,

$$\det_{\mathbb{C}}\hat{K}_{J_1\times J_2}^{J_1\times J_2} = \det_{\mathbb{H}} K_{J_1}^{J_1}\det_{\mathbb{H}} K_{J_2}^{J_2},$$

which means that for every R_1, R_2 in $\oplus_{e\in\Lambda}F_e$ adapted to the bases $((b_i^e)_{i\leq N})_{e\in E}$, we have

$$\mathbb{P}(R_1 \oplus R_2 \subset Q) = \mathbb{P}(R_1 \subset Q^{(1)})\mathbb{P}(R_2 \subset Q^{(2)}).$$

Since it holds for every choice of bases over the fibers of the vector bundle, we have for every $R_1, R_2 \in \oplus_{e\in\Lambda}F_e$,

$$\mathbb{P}(R_1 \oplus R_2 \subset Q) = \mathbb{P}(R_1 \subset Q^{(1)})\mathbb{P}(R_2 \subset Q^{(2)}).$$

In particular,

$$(\dim_{\mathbb{C}} Q_e)_{e\in\Lambda} \overset{\mathcal{L}}{=} (\dim_{\mathbb{H}} Q_e^1 + \dim_{\mathbb{H}} Q_e^2)_{e\in\Lambda},$$

which concludes the proof. □

An immediate consequence of this theorem is that in the case of $N = 1$, on a infinite graph, the superposition of two independent samples of the associated Q-determinantal probability measure on CRSF gives a random $\{0, 1, 2\}$-valued fields of occupation number which has the same distribution as the occupation number of a rank 2 quantum spanning forest with a SU_2-connection.

In the following subsection, we justify that Q determinantal measures are negatively associated just like the determinantal measures. This lemma will be useful to apply results such as Theorem 6.25 for Q-determinantal measures on CRSF.

7.1.2 Q-determinantal measures are negatively associated.

Let K be the kernel of a Q-determinantal measure μ_Q and denote by $\hat{K}$ the operator which is obtained from K by replacing quaternions by matrices in $M_2(\mathbb{C})$.

Lemma 7.3. *The measure μ_Q is negatively associated.*

Proof. Recall that if $E = \{e_1, \dots, e_n\}$, the generating polynomial of the Q-determinantal measure μ_Q, associated to a kernel K is given by

$$g_{\mu_Q}(z_1, \dots, z_n) = \mathbb{E}\left[\prod_{i|e_i\in F} z_i\right] = Q\det(I + (diag(z_1, \dots, z_n) - I)K).$$

From [BBL09], a measure which is strongly Rayleigh, which means that its generating polynomial is real stable, is negatively associated. Therefore, we just need to check that the polynomial g_{μ_Q} is real stable, that is to say that for every complex numbers $z_1, \dots, z_n$ such that the inequality $\Im(z_j) > 0$ holds for every $j \leq n$, we have $g_{\mu_Q}(z_1, \dots, z_n) \neq 0$.

According to Q-determinant properties, if $z_1, \dots, z_n$ are complex numbers such that the inequality $\Im(z_j) > 0$ holds for every $j \leq n$, then

$$\begin{aligned} g_{\mu_Q}(z_1, \dots, z_n)^2 &= Q\det(I + (diag(z_1, \dots, z_n) - I)K)^2 \\ &= \det(I + (diag(z_1, z_1, \dots, z_n, z_n) - I)\hat{K}), \end{aligned}$$

where $\hat{K}$ is obtained from K replacing quaternions by matrices in $M_2(\mathbb{C})$.

The operator $\hat{K}$ is the kernel of a determinantal linear process which is Strongly Rayleigh and therefore the function

$$(z_1, \hat{z}_1, \dots, z_n, \hat{z}_n) \mapsto \det(I + (diag(z_1, \hat{z}_1, \dots, z_n, \hat{z}_n) - I)\hat{K})$$

is a real stable polynomial. Then, for $z_1, \dots, z_n$ such that the inequality $\Im(z_j) > 0$ holds for every $j \leq n$, we have

$$\det(I + (diag(z_1, z_1, \dots, z_n, z_n) - I)\hat{K}) \neq 0,$$

and finally g_{μ_Q} is strongly Rayleigh. □

In the following subsection, we compute the kernel of a quantum spanning forest associated to a quaternionic connection on a vector bundle of rank N in the case where the graph and the connection are periodic.

7.1.3 Kernel for a periodic quaternionic unitary connection

Let $d \geq 2$. Let us consider in this subsection the graph $G = \mathbb{Z}^d$ endowed with the following connection. Let $M_1, \dots, M_d \in U_N(\mathbb{H})$ be unitary matrices with quaternionic values and (t_i) be the canonical basis of $\mathbb{Z}^d$. For every $j \in [1, d]$, define for every $x \in \mathbb{Z}^d$,

$$h(x, x + t_j) = M_j.$$

For every n, we consider the quaternionic operator of orthogonal projection K_n on $im(\tilde{d}_{h,n})$ with periodic boundary conditions;

$$K_n : \Omega^1(\tilde{G}_n) \to \Omega^1(\tilde{G}_n).$$

From the correspondence between $U_N(\mathbb{H})$ and $U_{2N}(\mathbb{C})$, for every $i \in \{1, \dots, d\}$, define $\hat{M}_i \in U_{2N}(\mathbb{C})$ as the associated matrix to M_i, and denote by $\hat{K}_n$ the operator of orthogonal projection with complex values, which corresponds to the connection defined by $\hat{M}_1, \dots, \hat{M}_d \in U_{2N}(\mathbb{C})$. Notice that for every edges e, e', the matrix $(\hat{K}_n)_{e,e'} \in M_{2N}(\mathbb{C})$ is the corresponding matrix to $(K_n)_{e,e'} \in M_N(\mathbb{H})$.

From Lemma 4.15, the sequence of kernels $(\hat{K}_n)_n$, associated to a complex periodic connection $(\hat{M}_1, \dots, \hat{M}_d)$ on a vector bundle of rank $2N$, with periodic boundary conditions, converges weakly in the sense that for every edges e, e', we have the following convergence in $M_{2N}(\mathbb{C})$.

$$(\hat{K}_n)_{e,e'} \to \hat{K}_{e,e'} = \int_{|z_1|=1,\dots,|z_d|=1} \hat{K}_{[e],[e']}(z_1, ..., z_d) z_1^{x_1 - y_1} ... z_d^{x_d - y_d} \prod \frac{dz_i}{2i\pi z_i}$$

where $\hat{K}(z_1, ..., z_d) : \Omega^1(\tilde{G}_1) \to \Omega^1(\tilde{G}_1)$ is the kernel associated to the connection defined by $(z_1\hat{M}_1, \dots, z_d\hat{M}_d)$.

For every edges e, e', we know that for every n, $(K_n)_{e,e'} \in M_N(\mathbb{H})$ and therefore, from the identification between $\mathbb{H}$ et $M_2(\mathbb{C})$, the following convergence holds in $M_N(\mathbb{H})$:

$$(K_n)_{e,e'} \to K_{e,e'} \in M_N(\mathbb{H}).$$

with the identification $K_{e,e'} \mapsto \hat{K}_{e,e'}$ between $M_N(\mathbb{H})$ et $M_{2N}(\mathbb{C})$.

Therefore, the sequence of kernels K_n with quaternionic values converges towards a kernel K which is the kernel of the infinite volume measure and the complex-valued operator $\hat{K}$ which corresponds to K has the following integral expression for every couple of edges e, e':

$$\hat{K}_{e,e'} = \int_{|z_1|=1,...,|z_d|=1} \hat{K}_{[e],[e']}(z_1, ..., z_d) z_1^{x_1-y_1} ... z_d^{x_d-y_d} \prod \frac{dz_i}{2i\pi z_i}.$$

Recall that the norm of a quaternion is given by $|q|^2 = a^2 + b^2 + c^2 + d^2$ where the corresponding matrix to q is

$$M_q = \begin{pmatrix} a+ib & -c-di \\ c-di & a-ib \end{pmatrix}.$$

Therefore, edge-to-edge correlations decay for a quaternionic unitary connection are given by decay of coefficients of the corresponding complex-valued operator. If each entry of the matrix $\hat{K}_{e,e'} \in M_{2N}(\mathbb{C})$ has exponential decay with the distance $|e - e'| \to \infty$, then the norms of the entries of $K_{e,e'} \in M_N(\mathbb{H})$ have exponential decay.

In the next section, we will study the case where $N = 1$ and the measure associated to a unitary quaternionic connection has support in CRSF and we will still assume that the graph and the connection are periodic.

7.2 Periodic unitary quaternionic connection of rank 1 on the graph $\mathbb{Z}^d$

Let $d \geq 2$. Let us consider in this subsection the graph $G = \mathbb{Z}^d$ endowed with the following connection. Let $q_1, \ldots, q_d \in U(\mathbb{H})$ be unitary quaternions and (t_i) be the canonical basis of $\mathbb{Z}^d$. For every $j \in [1, d]$, define for every $x \in \mathbb{Z}^d$,

$$h(x, x + t_j) = q_j.$$

From subsection 7.1.3, if for every $i \in \{1, \ldots, d\}$, we denote by $A_i \in SU_2(\mathbb{C})$ the corresponding matrix to the unitary quaternion q_i and by $\hat{K} \in M_2(\mathbb{C})$ the operator of orthogonal projection with complex values, which corresponds to the connection defined by $A_1, ..., A_d \in SU_2(\mathbb{C})$, then for every couple of edges e, e', we have

$$\hat{K}_{e,e'} = \int_{|z_1|=1,...,|z_d|=1} \hat{K}_{[e],[e']}(z_1, ..., z_d) z_1^{x_1-y_1} ... z_d^{x_d-y_d} \prod \frac{dz_i}{2i\pi z_i}.$$

where $\hat{K}(z_1, ..., z_d) : \Omega^1(\tilde{G}_1) \to \Omega^1(\tilde{G}_1)$ is the kernel associated to a U_2-valued connection, obtained from the connection multiplied by $z_1, \ldots, z_d$, that is to say the connection defined by $(z_1 A_1, \ldots, z_d A_d)$. Therefore, for every couple of edges e, e', the matrix $\tilde{\hat{K}}_{[e],[e']}(z_1, \ldots, z_d)$ is a U_2 matrix whose entries are rational fractions in $(z_1, \ldots, z_d)$.

In particular, edge-to-edge decay of correlations are determined by the existence of common eigenvectors of the matrices $A_1, \ldots, A_d$. Since the matrices $A_1, \ldots, A_d$ are in $SU_2(\mathbb{C})$, we know that there exists a common eigenvector of $A_1, \ldots, A_d$ if and only if $A_1, \ldots, A_d$ commute which is equivalent to the condition that $q_1, \ldots, q_d$ commute in the quaternionic field. We will distinguish in the following the case where all quaternions $q_1, \ldots, q_d$ commute and the case where at least two of them do not commute.

7.2.1 When the quaternions do not commute

Let us assume that there exists k, l such that q_k and q_l do not commute, that is to say that $q_k q_l q_k^{-1} q_l^{-1} \neq 1$. Then, the matrices $A_1, \dots, A_d$ do not have a common eigenvector. It implies the following result.

Proposition 7.4. *If there exist k, l such that q_k and q_l do not commute, edge-to-edge correlations decrease at exponential rate.*

Proof. The kernel of the model associated to the SU_2-valued connection $A_1, \dots, A_d$ can be written as

$$\hat{K}_{e,e'} = \int_{|z_1|=1,\dots,|z_d|=1} \hat{K}_{[e],[e']}(z_1, \dots, z_d) z_1^{x_1-y_1} \dots z_d^{x_d-y_d} \frac{dz_1}{2i\pi z_1} \dots \frac{dz_d}{2i\pi z_d}$$

where $\hat{K}(z_1, ..., z_d) : \Omega^1(\tilde{G}_1) \to \Omega^1(\tilde{G}_1)$ is the kernel associated to the connection defined by $(z_1 A_1, \dots, z_d A_d)$. From equation (4.6), there exists a U_2 matrix Q whose entries are Laurent polynomials in $(\xi_1, \dots, \xi_d)$ such that

$$\hat{K}_{[e],[e']}(z_1, \dots, z_d) = \frac{(Q(z))_{[e],[e']}}{P(\underline{z})}.$$

where $P(z_1, \dots, z_d)$ is the characteristic polynomial associated to $(A_1, \dots, A_d)$.

Since the matrices $A_1, \dots, A_d$ do not have a common eigenvector, from Theorem 4.12, the characteristic polynomial $z \mapsto P(z)$ does not vanish on $\mathbb{T}_d$, and from Corollary 4.24, if $e, e' \in E(\mathbb{Z}^d)$, $||\hat{K}_{e,e'}||$ decays at exponential rate when $|e - e'| \to \infty$. □

Since a Q-determinantal measure is negatively associated, under Assumption 6.17, Theorem 6.25 holds for Q-determinantal measures on CRSF. Let us justify that both Assumptions 6.17 and 5.18 hold in the case where quaternions do not commute.

Lemma 7.5. *If there exist two indices k, l such that q_k and q_l do not commute, and such that the inequality $\Re(q_k q_l q_k^{-1} q_l^{-1}) \geq 0$ holds, then Assumptions 6.17 and 5.18 hold for the weight function w_- obtained from w.*

Proof. Let us construct a family of loops $(\gamma_j^+, \gamma_j^-)_{j\in[1,d]}$ such that for every vertex $x \in \partial G_n$, there exists a loop starting from x which stays in $G_{n+1} \backslash (G_n \cup \partial G_{n+1})$ and which is a loop translated from a loop of this family.

Let $j \in [1, d]$. Define $\gamma_j^{+,-}$ by

$$\begin{cases} \gamma_j^+ = (0, t_j, t_j + t_k, t_j + t_k + t_l, t_j + t_l, t_j) \\ \gamma_j^- = (0, -t_j, -t_j - t_k, -t_j - t_k - t_l, -t_j - t_l, -t_j) \end{cases}$$

Let $x \in \partial G_n$ be a fixed vertex and let j be such that $x_j = \pm n$. If $x_j = n$, the loop defined by $\gamma_x := x + \gamma_j^+$ satisfies the assumption and if $x_j = -n$, the loop $\gamma_x := x + \gamma_j^-$ satisfies the assumption.

If (X_n) is a simple random walk starting from x, it makes the loop γ_x of length 5 with probability

$$\mathbb{P}_x(\gamma_x) = \left(\frac{1}{2d}\right)^5.$$

Notice that from the invariance under translation, the weight of the loop γ_x is

$$w_h(\gamma_x) = 1 - \Re((q_k q_l q_k^{-1} q_l^{-1})^{\pm 1}).$$

Therefore, if there exist k, l such that q_k and q_l do not commute, and such that the inequality $\Re(q_k q_l q_k^{-1} q_l^{-1}) \geq 0$ holds, then Assumptions 6.17 and 5.18 hold. □

Under those assumptions, every connected component with a cycle is finite and there exist infinitely many finite connected components, since almost surely every finite admissible family of unicycles is observed infinitely many times and in particular every finite unicycle of non trivial holonomy is observed infinitely many times.

In the following subsection, we study the case where all quaternions commute.

7.2.2 Commuting quaternions and uniform spanning forest

Assume with the same notations as in the previous subsection that for every k, l,

$$q_k q_l = q_l q_k.$$

Theorem 7.6. *Under this assumption, the model has the same law as the uniform spanning forest of [BLPS01]. In particular, some specific edge-to-edge correlations decrease at polynomial rate and all connected components are infinite trees.*

Proof. Recall that the generating polynomial of the model of rank 1 with a unitary quaternionic connection is the square-root of the generating polynomial of the model associated to the complex connection of rank 2 obtained by replacing unitary quaternions of the connection by SU_2 matrices.

Using the correspondence between $U(\mathbb{H})$ and SU_2, under the assumption that the quaternions q_j commute, the matrices in SU_2 which are associated to the quaternions q_j can be all diagonalized in a fixed orthonormal basis.

$$q_j \leftrightarrow M_{q_j} \begin{pmatrix} u_j & 0 \\ 0 & \bar{u}_j \end{pmatrix}$$

Then, if we denote by $\hat{K}$ the kernel obtained by replacing unitary quaternions by SU_2 matrices, then, for every $z \in \mathbb{T}^d$, for every couple of edges e, e', the matrix denoted by $\hat{K}_{[e],[e']}(z_1, \dots, z_d)$ is, in the fixed orthonormal basis, a diagonal matrix whose entries are equal to the quantities obtained for the complex-valued connections defined by unitary complex numbers $(z_1 u_1, \dots, z_d u_d)$ and $(z_1 \bar{u}_1, \dots, z_d \bar{u}_d)$.

Therefore, the kernel obtained by replacing unitary quaternions by SU_2 matrices can be written as

$$\hat{K}_{e,e'} = \begin{pmatrix} u_1^{y_1 - x_1} \dots u_d^{y_d - x_d} K^{id}_{e,e'} & 0 \\ 0 & \bar{u}_1^{y_1 - x_1} \dots \bar{u}_d^{y_d - x_d} K^{id}_{e,e'} \end{pmatrix}.$$

Therefore, the kernel of the model is

$$\hat{K} = \begin{pmatrix} U^* & 0 \\ 0 & U \end{pmatrix} \begin{pmatrix} K^{id} & 0 \\ 0 & K^{id} \end{pmatrix} \begin{pmatrix} U & 0 \\ 0 & U^* \end{pmatrix},$$

where U is a diagonal matrix in U_2 and K^{id} is the kernel associated to a trivial connection, that is the kernel of the model of uniform spanning forests, as defined in [BP93, BLPS01], whose integral expression, obtained by [BP93], is recalled in Equation (4.1).

Then, the generating polynomial of the model of rank 1 with a unitary quaternionic connection $q_1, \dots, q_d$ is given by

$$\begin{aligned} g_{\mu_Q}(z_1, \dots, z_n) &= \sqrt{\det(I + (diag(z_1, z_1, \dots, z_n, z_n) - I)\hat{K})_{e_1,\dots,e_n}} \\ &= \det(I + (diag(z_1, \dots, z_n) - I)K^{id}_{e_1,\dots,e_n}) \\ &= g_{\mu_{id}}(z_1, \dots, z_n) \end{aligned}$$

which concludes the proof. □

Conclusion

In this thesis, we presented two families of Boltzmann probability measures on cycle-rooted spanning forests of finite graphs associated to weight functions on cycles which are symmetric under orientation reversal.

The first one, which is the family of probability measures associated to a complex or quaternionic unitary connection of rank 1 of the graph, is a family of determinantal measures whereas the second one, which is the family of probability measures associated to a weight function smaller than 1 is a family of measures which are sampled by a Wilson type algorithm. We justified that for both families, under some assumptions, sequences of probability measures on cycle-rooted spanning forests of finite growing subgraphs converge towards a probability measure on subgraphs of an infinite graph and described properties of random configurations under this limit measure.

We proved that under the Wilson algorithm measure in infinite volume, under an assumption of minoration satisfied by the weight function, all connected components are finite. For determinantal measures in infinite volume, we proved under similar assumptions that every connected component is finite if and only if it has a cycle and that there are almost surely infinitely many unicycles. In particular, the following question naturally arises:

Question 1. *Under a determinantal measure associated to a rank 1 connection such that there exists a family of cycles, invariant under translations, all of whose cycles have non-trivial holonomies, can we observe with positive probability an infinite tree?*

In other words, could we observe in infinite volume a component without any cycle which arises from growing cycles? When the connection h is such that the residual space $d_h(\mathcal{H})$ is null, we know that edge-to-edge correlations decrease at exponential rate and therefore we believe that every connected component is finite.

Nevertheless, Kenyon introduced in [Ken19, Sun16] some Gibbs determinantal measures on spanning forests of an infinite planar graph which are determinantal associated to kernels which are not self-adjoint operators. Under those measures, all connected components are infinite almost surely, even if correlations can decrease at exponential rate.

More precisely, those measures are defined as limits of determinantal measures, associated to a non unitary connection, on growing finite planar graphs which are seen as

graphs on growing tori, such that the only cycles whose holonomies are non trivial are cycles whose homotopy classes are non trivial, that are cycles winding around the torus. Under those measures, we observe almost surely large cycles winding around the torus which give rise to bi-infinite connected components which are characterized by their slopes. Intuitively, we believe that if a sequence of measures on cycle-rooted spanning forests on growing finite graphs converges towards a measure on subgraphs of an infinite graph, the only infinite connected components that we could observe are bi-infinite trees which arise from the limit of growing cycles in both directions. Following this idea, we naturally ask the following question:

Question 2. *If an infinite tree is observed, is it a bi-infinite tree?*

However, this intuition is not always correct. Indeed, as we mentioned in Chapter 6, the limit of the sequence of determinantal measures on cycle-rooted spanning forests associated to a periodic unitary complex connection on an exhaustion of the lattice $\mathbb{Z}^d$ is the uniform spanning forest measure on $\mathbb{Z}^d$ studied by [BP93, LP16, BLPS01], under which every connected component is almost surely a tree with only one end.

Furthermore, a theorem from Lyons ([Lyo03, Theorem 7.2]) which holds for every determinantal point process implies that for a random cycle-rooted spanning forest F distributed according to the determinantal measure μ_h, we have μ_h-almost surely,

$$\Omega^1_{\ell^2}(\mathsf{F}) = \overline{\Pi^{\ell^2(\mathsf{F})}(\bigstar_{h,\ell^2}(G))}^{\ell^2} = \bigstar_{h,\ell^2}(\mathsf{F}).$$

Let us assume for a while that F contains a bi-infinite tree T and an edge $e = (x, y)$ such that if we remove e then F is the disjoint union of two infinite trees T_x and T_y. The theorem of Lyons mentioned above states that almost surely, every 1-form $\theta \in \Omega^1_{\ell^2}(\mathsf{F})$ on F can be approximated in ℓ^2-norm by derivatives $(d^{\mathsf{F}}_h f_n)$ of functions (f_n) with compact support. We believe that choosing θ to be the indicator function of the edge e, the sequence of functions (f_n) is forced to converge towards a function which is constant non zero either on T_x or T_y. Nevertheless, even if the sequence (f_n) is a sequence of 0-forms with compact support, (f_n) does not necessarily converge towards a 0-form in ℓ^2 and unfortunately we cannot find a contradiction.

The proof that every connected component is finite under the Wilson measure relies on algorithms of loop-erased random walks. In the same way, to prove that every connected component with a cycle is finite, we used an algorithm of loop-erased random walks conditional on cycles of weights larger than 1, on a finite graph. Unfortunately, this algorithm does not hold on infinite graphs since we cannot consider the measure conditional on the infinite set of cycles of weights larger than 1. A natural question that we can ask is the following:

Question 3. *Could we construct an algorithm to sample determinantal measures from loop-erased random walks?*

This question is also studied by [FB22] who give an algorithm of loop-erased random walks to sample cycle-rooted spanning forests associated to the weight function $\min(1, w)$ and approximate determinantal measures associated to a magnetic Laplacian from measures associated to this weight function. Nevertheless, this method does not seem enough sharp

to show the finiteness of connected components under the determinantal measures in our case.

We emphasize that the Wilson measures on cycle-rooted spanning forests are not only a tool to study configurations under determinantal measures but measures of relevant interest to sample randomly a cycle-rooted spanning forests according to a loop-erased random walks algorithm. From that point of view, open questions about those measures seem to be worth paying attention to. In particular, we ask the following question:

Question 4. *Can we compute edge-to-edge correlations exactly and are the Wilson algorithm measures negatively correlated under some assumptions?*

We proved that under some assumptions, correlations between edges which are far enough appart are exponentially small. We have not succeeded yet in computing exactly edge-to-edge correlations, but it would be worth working on it further.

We also emphasize that one of the goal of the thesis is to study more general spanning forests in rank N, called quantum spanning forests, which correspond to a superposition of random cycle-rooted spanning forests, whose joint law is determinantal. We proved in Chapter 4 that under the determinantal measure associated to a $\mathbb{Z}^d$-periodic connection on the lattice $\mathbb{Z}^d$, defined by d unitary commutating matrices of size N, the joint law is the superposition of N independent uniform spanning forests. We also proved in Chapter 7 that the superposition of two independent samples of the Q-determinantal measure on cycle-rooted spanning forests has the same law as the occupation number of a rank 2 quantum spanning forest with a SU_2-connection.

More generally, the superposition of N independent samples of a probability measure on CRSF, non necessarily determinantal, gives a random $\{1, \dots, N\}$-valued fields of occupation number. Another natural question about the Wilson algorithm measure on cycle-rooted spanning forests is to understand the law of the superposition of several samples of this measure.

Question 5. *What is the law of the number of occupation of a superposition of independent random configurations distributed according to the Wilson measure? Can we observe a percolation phase transition according to the value of the cycle weights?*

The law of a superposition of cycle-rooted spanning forests is better understood in the case of determinantal measures associated to a unitary connection on a vector bundle of rank N over a graph, but even in the determinantal case, several questions of relevant interest remain open. As mentioned above, in the case of rank 1, Kenyon [Ken19, Sun16] introduced Gibbs determinantal measures associated to non-unitary connections of rank 1 over a planar periodic graph, which gave rise to random spanning forests, all of whose connected components are bi-infinite trees with slopes.

Those determinantal measures are in correspondence with dimer measures on planar graphs and their kernel have integral expression involving a characteristic polynomial. Different phases can be observed, depending on the rate of decay of edge-to-edge correlations and the corresponding phase diagram involves the spectral curve, that is the zero set of the characteristic polynomial. In the case of a unitary connections on a vector bundle of rank N over a periodic graph, we observed two phases of edge-to-edge correlations decay depending on the existence of zeros of the polynomial in the unit torus, that are

zeros whose coordinates have unit modulus. The following question naturally arises to go further in the study of the phase diagram of the model of quantum spanning forests.

Question 6. *Could we define measures associated to a non-unitary connection for arbitrary rank $N \geq 2$? Would we observe a phase diagram involving the zeros of the characteristic polynomial whose coordinates have modulus different than 1?*

The construction of the determinantal measures associated to a non-unitary connection of rank 1 in [Ken19, Sun16] relies on combinatorial formulas for the vector-bundle Laplacian that hold in rank 1 and in a particular case of rank 2, which is the case where the matrices are of unit determinant ([For93, Ken11, KL20a]). Unfortunately, even in rank 2, the combinatorial expression of the Laplacian determinant is not sufficient to extend the construction. Indeed, this construction relies on Fourier decompositions, as in Chapter 4, which involve multiplications of matrices by scalars, and combinatorial formulas do not hold anymore for those new matrices.

As mentioned above, in the case of a unitary periodic connection of rank N, the edge-to-edge correlations decay depends on the existence of zeros of the characteristic polynomial in the unit torus. We proved in Chapter 4 that this condition is equivalent to the existence of common eigenvectors of the unitary matrices. In particular, in the case of a special unitary periodic connection of rank 2, or equivalently in the case of a quaternionic connection of rank 1, as studied in Chapter 7, the edge-to-edge correlations rate of decay depends on the commutation of the parameters. We observe therefore a phase transition when the commutator of quaternions tends to 0. A natural quantity which characterizes a phase transition is the free energy of the model and its degree of non-analyticity is called the order of the phase transition.

Question 7. *What is the degree of non analyticity of the free-energy when the commutator of quaternionic parameters, defining a periodic unitary quaternionic connection, tends to 0?*

We tried to compute the free-energy of the model for some particular unitary quaternions, expressed as matrices in SU_2 but we have not obtained yet an exact asymptotic expansion of the free energy and its degree of non-analyticity when the commutator tends to 0.

On specific graphs called isoradial graphs, authors of [BdTR17] study a Z-invariant model (see [BD78]) of rooted spanning forests defined by a determinantal probability measure associated to a massive Laplacian. They compute the order of a phase transition, that is the degree of non-analyticity of the free-energy, when an elliptic parameter which defines the probability measure tends to 0. Those computations rely on local formulas for the Green function and the kernel of the model, which hold thanks to the Z-invariance. It raises the following question:

Question 8. *Do there exist unitary connections such that the Z-invariance property holds for the associated determinantal measures? For such connections, can we compute local formulas for the kernels and deduce the order of the phase transition?*

Notice that it makes sense to look for such connections since the model of rooted spanning forests associated to a massive Laplacian can be interpreted as a particular case of the model of cycle-rooted spanning forests associated to a unitary connection, by adding

on every vertex a small self-loop whose holonomy is equal to the mass of the vertex. We have not succeeded yet in computing connections which satisfy the Z-invariance property but we believe that this question is worth working on further.

Other questions of relevant interest can be raised, in particular in the case of large N. As mentioned in Chapter 2, a relevant quantity to understand quantum spanning forests is the trace, that is the set of dimensions of random subspaces of the fibers, whose marginal expectations and variances are given by the eigenvalues of the kernel. When the measure on quantum spanning forests is associated to a periodic unitary connection of rank N, this kernel has an integral expression involving unitary matrices of size N. Since the asymptotic behavior of eigenvalues of random matrices of large size is well understood, the following question naturally arises:

Question 9. *What is the asymptotic behavior when $N \to \infty$ of the trace of a quantum spanning forest distributed according to the determinantal probability measure associated to a random periodic unitary connection on a vector-bundle of size N over an infinite periodic graph, depending on the probability law on the set of unitary matrices $U_N(\mathbb{C})$?*

We also believe that the model of quantum spanning forests could be related to other statistical physics models such as loop soups [LW03], Gaussian free field [LW16, WP20], covariant fields [KL20b], matrix valued fields [Lup22], Fortuin-Kasteleyn percolation [FK72] for $q < 1$, also called the random cluster model which is believed to be negatively correlated (see [GW04]), in particular in the limit case $q \to 0$, and a recently studied model of random forests, so-called arboreal gas [BCHS21, BCH21, HH23]), and exploring precise links between those models seems to be worth paying attention to.

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