Some problems of statistical mechanics for Coulomb and Riesz gases

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Fluctuat nec mergitur,

Quelqu'un dans la Rome antique

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Introduction

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Bilan des travaux

1) J. Boursier, D. Chafaï, C. Labbé, Universal cutoff for Dyson Ornstein Uhlenbeck process, *accepté pour publication dans PTRF,* (2021).

2) J. Boursier, Optimal local laws and CLT for 1D long-range Riesz gases (2021), arXiv:2112.05881.

3) J. Boursier, Thermodynamic limit and decay of the correlations for the circular long-range Riesz gas (2022), arXiv:2209.00396

4) J. Boursier, S. Serfaty, Dipole transition for the two-component plasma, *bientôt sur arXiv* (2022).

1.1 Contexte et motivations

Cette thèse est consacrée à l'étude mathématique de certains systèmes de particules en interaction appelés gaz de Riesz. Ceux-ci permettent de modéliser des particules aléatoires chargées (sans énergie cinétique) interagissant par paires au travers du noyau $g_s(x) = |x|^{-s}$ où s est un paramètre strictement positif. Par extension, la famille des gaz de Riesz comprend également les systèmes à interaction logarithmique en dimension une et deux, i.e le log-gaz uni-dimensionnel et le gaz de Coulomb bi-dimensionnel, deux modèles particulièrement importants en physique et en mathématiques. En tant que systèmes de particules à longue portée, les gaz de Riesz forment une famille de modèles particulièrement riche, dont nous nous proposons d'étudier quelques aspects.

Ce manuscrit s'insère dans une vaste littérature à l'intersection des probabilités et de la physique mathématique. Le comportement microscopique des gaz de Riesz à longue portée a fait l'objet d'une importante série de travaux qui ont permis de décrire, au moyen d'un principe de grande déviations, le comportement microscopique de ces systèmes de particules. Si cette description microscopique est bien valide en toute dimension, l'analyse des propriétés probabilistes de ces systèmes - comme les fluctuations, les corrélations, la rapidité de convergence à l'équilibre - est complètement différente en dimension une et en dimension supérieure. C'est en effet seulement en dimension une qu'il est possible d'exploiter la convexité de l'interaction, ce qui donne accès à toute une gamme d'inégalités de concentration et d'inégalités fonctionnelles pour étudier les fluctuations et le temps de relaxation de la dynamique.

Au fil de cette introduction, nous précisons d'abord le contexte mathématique de notre travail puis donnons un aperçu de quelques méthodes fondamentales du domaine avant de présenter les contributions principales de ce manuscrit.

1.1.1 Dynamique de Langevin

On présente la dynamique de Langevin suramortie qui permet de modéliser une dynamique moléculaire de particules sans énergie cinétique. On considère N particules $x_1, \ldots, x_N$ dans $\mathbb{R}^d$ interagissant selon une énergie générale (disons régulière) $H_N : (\mathbb{R}^d)^N \to \mathbb{R}$. On suppose que les particules évolue en cherchant à minimiser l'énergie, tout en étant agitées par un petit bruit Brownien. Pour un paramètre $\beta > 0$, qui joue le rôle de température inverse, on examine alors l'équation différentielle stochastique

$$\mathrm{d}X_t = -\alpha \nabla H_N(X_t)\mathrm{d}t + \sqrt{\frac{2\alpha}{\beta}}\mathrm{d}B_t, \tag{1.1}$$

où $(B_t)_{t\geqslant 0}$ est un mouvement Brownien standard sur $(\mathbb{R}^d)^N$ et $\alpha > 0$ un paramètre correspondant à un changement déterministe de temps. L'équation (1.1) définit un processus de Markov sur $(\mathbb{R}^d)^N$ dont le générateur infinitésimal du semi-groupe est donné par l'opérateur différentiel linéaire du second-ordre

$$\mathcal{L} := \alpha(\beta \nabla H_N \cdot \nabla - \Delta). \tag{1.2}$$

On peut de plus vérifier que la mesure invariante de ce processus de Markov est donnée par la mesure de probabilité ci-dessous, appelée distribution de Boltzmann-Gibbs :

$$\mathrm{d}\mathbb{P}_{N,\beta} = \frac{1}{Z_{N,\beta}} e^{-\beta H_N(X_N)}\mathrm{d}X_N, \tag{1.3}$$

où $Z_{N,\beta}$ est la fonction de partition

$$Z_{N,\beta} = \int_{(\mathbb{R}^d)^N} e^{-\beta H_N(X_N)}\mathrm{d}X_N.$$

Il se trouve que (1.3) est la solution d'un « principe variationnel de Gibbs ». On vérifie aisément que la mesure (1.3) minimise la fonctionnelle

$$\mu \in \mathcal{P}((\mathbb{R}^d)^N) \mapsto \beta \mathbb{E}_\mu[H_N] + \mathrm{Ent}(\mu),$$

où Ent désigne l'entropie sur $(\mathbb{R}^d)^N$:

$$\mathrm{Ent}(\mu) = \int \log \frac{\mathrm{d}\mu}{\mathrm{d}x} \mathrm{d}\mu, \tag{1.4}$$

si μ est absolument continue par rapport à la mesure de Lebesgue sur $(\mathbb{R}^d)^N$ et $\mathrm{Ent}(\mu) = +\infty$ sinon. On peut montrer que sous des hypothèses générales, par exemple si la Hessienne de H_N est minorée par une constante négative [220], alors la dynamique (1.1) est ergodique et qu'en particulier la loi de X_t converge en temps long vers la distribution de Boltzmann-Gibbs (1.3). Une question intéressante du point de vue de la physique statistique est de déterminer si cette propriété d'unicité de le mesure reste vraie lorsque la taille du système tend vers l'infini, autrement s'il y a un unique état de Gibbs.

1.1.2 La famille des gaz de Riesz

Spécifions à présent la classe d'énergie qui nous intéresse. Étant données N particules $x_1, \dots, x_N$ dans $\mathbb{R}^d$, on considère

$$\sum_{i \neq j} g_s(N^{\frac{s}{d}}(x_i - x_j)), \tag{1.5}$$

où g_s est le noyau de Riesz sur $\mathbb{R}^d$, associé à un certain paramètre $s \geqslant 0$, donné par la formule

$$g_s(x) = \begin{cases} \frac{1}{|x|^s} & \text{si } s > 0 \\ -\log|x| & \text{si } s = 0 \text{ et } d \in \{1, 2\}. \end{cases} \tag{1.6}$$

Le noyau de Riesz définit une interaction classique qui est correspond à la solution de l'équation de Laplace fractionnaire

$$(-\Delta)^{\frac{d-s}{2}} g_s = c_{s,d}\delta_0.$$

Ainsi l'interaction de Riesz recoupe l'interaction logarithmique en dimension 1 et 2 ainsi que l'interaction coulombienne pour $d \geqslant 2$. On utilise la terminologie suivante :

- Pour $s = 0$ et $d = 1$, on parle de log-gaz 1D ou de β-ensemble.
- Pour $s = 0$ et $d = 2$, on parle de log-gaz 2D, qui correspond aussi au gaz de Coulomb 2D.
- De façon plus générale, pour $s = d - 2$ et $d \geqslant 2$, on parle de gaz de Coulomb.

Le paramètre s détermine la singularité de l'interaction ainsi que sa portée. En effet pour $s > d$, l'interaction d'une configuration périodique bien espacée sur un domaine compact devient sommable et l'énergie est dite à courte portée. A l'opposé, pour $s \leqslant d$, les termes principaux dans l'énergie correspondent aux interactions à longue portée. Afin que les particules ne s'échappent pas à l'infini, il convient de compactifier le domaine, ce qui peut se faire de deux façons : la première consiste à ajouter à (1.5) un potentiel extérieur confinant, c'est-à-dire un terme de la forme $\sum_{i=1}^N V(x_i)$ avec $V : \mathbb{R}^d \to \mathbb{R}$ suffisamment régulière et croissant assez vite à l'infini, ce qui amène à considérer le Hamiltonien

$$\mathcal{H}_N^V : X_N \in (\mathbb{R}^d)^N \mapsto \sum_{i \neq j} g_s(N^{\frac{s}{d}}(x_i - x_j)) + N^{\frac{s}{d}} \sum_{i=1}^N V(x_i). \tag{1.7}$$

Dans ce cas on notera $\mathbb{P}_{N,\beta}^V$ la mesure de Gibbs

$$\mathrm{d}\mathbb{P}_{N,\beta}^V := \frac{1}{Z_{N,\beta}^V} e^{-\beta \mathcal{H}_N^V(X_N)} \mathrm{d}X_N, \tag{1.8}$$

où $Z_{N,\beta}^V$ est la fonction de partition

$$Z_{N,\beta}^V := \int e^{-\beta \mathcal{H}_N^V(X_N)} \mathrm{d}X_N.$$

Une autre façon de confiner les particules est de remplacer $\mathbb{R}^d$ par un domaine compact, par exemple par $[0,1]^d$ ou par le tore de dimension d, noté $\mathbb{T}^d$. Notons que dans le régime longue portée (à la différence du régime courte portée), si on confine un gaz de Riesz dans un domaine compact, alors la mesure d'équilibre se concentre sur la frontière du domaine et ce choix n'est donc pas pertinent. Enfin si les particules vivent sur le tore $\mathbb{T}^d$, soulignons que le noyau (1.6) doit être remplacé par le noyau de Riesz periodisé, voir sous-section (1.3.2).

1.1.3 Motivations

Donnons à présent quelques motivations physiques et mathématiques pour l'étude des gaz de Riesz. On se réfère aux comptes-rendus de littérature très complets [238, 77, 191].

- Les gaz de Riesz définissent pour $s < d$ une famille de modèles à longue portée, qui sont intéressants en tant que tels. En effet, ceux-ci échappent à la théorie classique de la physique statistique élaborée dans les années 70, 80 et 90 (Ruelle, Giorgii, Dobrushin...) [122, 127, 138, 207], qui donne des résultats généraux pour des interactions à courte portée, notamment en dimension 2. Les gaz de Riesz forment donc un cadre particulièrement riche pour développer de nouvelles méthodes pour comprendre les fluctuations et corrélations des systèmes à longue portée, y compris en dimension 1.

- Une motivation importante pour l'étude des log-gaz en dimension 1 vient des matrices aléatoires. En effet comme observé dans les papiers fondateurs [255, 106], le log-gaz 1D avec $\beta \in \{1,2,4\}$ apparaît comme loi jointe des valeurs propres de matrices aléatoires Gaussiennes symétriques/hermitiennes/symplectiques à entrées indépendantes. Les β-ensembles ont ainsi été abondamment étudiés en utilisant des techniques extrêmement variées. Les β-ensembles s'étendent également en des modèles discrets [42] qui modélisent alors les losanges horizontaux dans les pavages aléatoires de domaines hexogonaux par exemple. En dimension 2 pour $\beta = 2$, le log-gaz correspond à la loi jointe des valeurs propres d'un modèle de matrices non-hermitiennes, appelé ensemble de Ginibre [134]. De plus celui-ci a une structure déterminentale qui en fait un système intégrable [161, 119, 104].

- Le gaz de Coulomb en dimension 2 et 3 est un modèle particulièrement important puisque les interactions électrostatiques et gravitationnelles sont coulombiennes. En dimension 3, le gaz de Coulomb permet par exemple de modéliser les plasmas en astrophysique [25]. Le gaz de Coulomb bi-dimensionnel surgit quand à lui dans de nombreux domaines de la physique : il permet de décrire les vortex dans les modèles de Ginzburg-Landau (supraconductivité) et Gross-Pitaevskii (superfluidité) [227, 228], les vortex dans le modèle XY (magnétisme, mécanique du solide) [34], la fonction d'onde de Laughlin dans l'effet de Hall quantique fractionnaire en mécanique quantique. Le gaz de Riesz est une extension naturelle du gaz de Coulomb et a qui a également de nombreuses motivations physiques, voir par exemple [197, 18, 66, 249].

- L'étude de minimiseurs de (1.5) sur $\mathbb{T}^d$ est également un problème majeur en mathématiques et tout à fait d'actualité. Pour $s = +\infty$, ce problème n'est rien d'autre que le problème d'empilement compact, où il s'agit d'ordonner des sphères dures de telle sorte que la proportion d'espace occupé soit la plus grande possible. Il est conjecturé que pour certaines dimensions, les minimiseurs de (1.5) sont donnés par des réseaux périodiques : c'est la conjecture de cristallisation, qui explique la formation spontanée de structures très ordonnées. Plus précisément, il est conjecturé dans [87] que le réseau triangulaire en dimension 2, le réseau E_8 en dimension 8 et le réseau de Leech en dimension 24 sont les minimiseurs universels d'énergie de la forme (1.5) pour des interactions générales et « complètement monotones », dont les interactions de Riesz. Cette conjecture a été démontrée en dimension 8 et 24 dans [88] suite à l'avancée spectaculaire [252]. Le lecteur peut se référer à [36] pour un compte-rendu de littérature sur la conjecture de cristallisation. L'étude des minimiseurs de (1.5) joue également un rôle clé en théorie de l'approximation [174, 222, 57, 58, 82].

1.1.4 Comportement macroscopique

Considérons une énergie de la forme (1.7) sur $(\mathbb{R}^d)^N$ avec $s < d$ et $V : \mathbb{R}^d \to \mathbb{R}$ lisse et croissant suffisamment vite à l'infini. Le potentiel de Riesz étant à longue portée, le comportement macroscopique du système est dicté par une énergie de type champ-moyen. En effet, la force principale s'exerçant sur une particule est donnée par la force générée par la distribution moyenne de charges, qui est donc dominante devant la force exercée par les particules voisines. Ceci peut se formaliser avec un énoncé de Gamma-convergence ou bien avec un énoncé de grandes déviations. Pour étudier la densité macroscopique de particules, il est naturel de considérer la mesure empirique, définie par

$$\mu_N = \frac{1}{N}\sum_{i=1}^N \delta_{x_i}. \tag{1.9}$$

Le Hamiltonien (1.7) se réécrit comme une fonction de la mesure empirique, comme suit :

$$\mathcal{H}_N^V(X_N) = N^{2-\frac{s}{d}} \iint_{\Delta^c} g(N^{\frac{s}{d}}(x-y))\mathrm{d}\mu_N(x)\mathrm{d}\mu_N(y) + N^{2-\frac{s}{d}}\int V(x)\mathrm{d}\mu_N(x),$$

où Δ désigne la diagonale de $\mathbb{R}^d \times \mathbb{R}^d$. Ainsi, lorsque μ_N avoisine une certaine mesure μ, on s'attend à ce qu'en un certain sens, l'énergie $H_N^V(X_N)$ soit comparable à $N^{2-\frac{s}{d}} I_V(\mu)$ où I_V désigne la fonctionnelle

$$I_V : \mu \in \mathcal{P}(\mathbb{R}^2) \mapsto \iint g(x-y)\mathrm{d}\mu(x)\mathrm{d}\mu(y) + \int V(x)\mathrm{d}\mu(x). \tag{1.10}$$

Comme montré dans [237], la suite de fonctions $\{N^{\frac{s}{d}-2}H_N^V\}$ vues comme des fonctions sur $\mathcal{P}(\mathbb{R}^d)$ Γ-converge vers $I_V : \mathcal{P}(\mathbb{R}^d) \to (-\infty, +\infty)$ au sens de la convergence faible des mesures. D'autre part, on peut montrer que pour toute mesure $\nu \in \mathcal{P}(\mathbb{R}^d)$ et pour ε assez petit, on a

$$\int_{\mu_N \in B(\nu,\varepsilon)} e^{-\beta \mathcal{H}_N^V(X_N)}\mathrm{d}X_N = e^{-\beta N^{2-\frac{s}{d}} I_V(\nu) + o(N^2)}.$$

En particulier, la mesure de Gibbs se concentre autour de l'unique minimiseur μ_V de I_V, appelé mesure d'équilibre. Ainsi la suite des mesures images de $\mathbb{P}_{N,\beta}$ par l'application μ_N satisfait à un principe de grandes déviations (PGD) avec fonction de taux $\beta(I_V - I_V(\mu_V))$, [11, 80, 237], voir sous-section 1.2.1 pour une définition précise. En conséquence, μ_N converge vers une limite déterministe donnée par la mesure d'équilibre μ_V.

La mesure d'équilibre se caractérise au moyen d'équations d'Euler-Lagrange mais son support est délicat à déterminer. On utilisera la terminologie *bulk* pour parler de l'intérieur du support. Donnons deux cas particuliers importants. Dans le cas du log-gaz en dimension 1 avec potentiel quadratique (β-ensemble Gaussien), la mesure d'équilibre est donnée après mise à échelle par la loi du semi-cercle de densité $\sqrt{4-x^2}\mathbb{1}_{|x|<2}$, ce qui est cohérent pour $\beta \in \{1,2,4\}$ avec le théorème de Wigner [5]. Dans le cas du log-gaz en dimension 2 avec potentiel extérieur quadratique, la mesure d'équilibre est uniforme sur un disque dont le rayon dépend de β. On renvoie à [77, Table 1.2] pour une liste plus complète des mesures d'équilibre pour les log-gaz.

1.1.5 Mouvement Brownien de Dyson et limite hydrodynamique

Nous avons vu comment le système à l'équilibre peut être approché par sa limite de champ moyen. Il est également possible de dériver une limite de champ moyen dans l'équation de Langevin (1.1). Sur $\{(x_1,\ldots,x_N) \in \mathbb{R}^N : x_1 < \ldots < x_N\}$, on considère

$$\mathrm{d}X_i(t) = \frac{1}{N}\sum_{j:j\neq i}\frac{\mathrm{d}t}{X_i(t)-X_j(t)} - \frac{1}{2}V'(X_i(t))\mathrm{d}t + \sqrt{\frac{2}{\beta N}}\mathrm{d}B_i(t), \quad i=1,\ldots,N, \tag{1.11}$$

avec $\beta > 0$ et $V : \mathbb{R}^d \to \mathbb{R}$ lisse et croissant à l'infini. Ce processus stochastique est appelé mouvement Brownien de Dyson. Rappelons que comme observé par Dyson en 1962 [106], pour $\beta \in \{1,2,4\}$ et V quadratique, (1.11) correspond à l'évolution des valeurs propres de matrices $N\times N$ symétriques/hermitiennes/symplectiques à entrées Browniennes indépendantes. Par ailleurs comme nous l'avons vu, la loi invariante de $X(t)$ est donné par le β-ensemble (1.8).

Comme dans le paragraphe précédent, considérons la mesure empirique

$$\mu_N(t) = \frac{1}{N}\sum_{i=1}^N \delta_{X_i(t)}.$$

Comme montré par exemple dans [215], si la distribution initiale $\mu_N(0)$ converge faiblement vers une mesure μ_0 lorsque N tend vers l'infini, alors le processus $(\mu_N(t)_{t\geqslant 0})$ converge faiblement vers le processus déterministe $(\mu_t)_{t\geqslant 0}$, donné par l'unique solution de

$$\langle \mu_t, f\rangle = \langle \mu_0, f\rangle - \int_0^t \int V'(x)f'(x)\mu_s(dx)ds + \frac{\beta}{2}\int_0^t\int_{\mathbb{R}^2}\frac{f'(x)-f'(y)}{x-y}\mu_s(dx)\mu_s(dy)ds \tag{1.12}$$

pour tout $t \geqslant 0$ et $f \in \mathcal{C}_b^3(\mathbb{R},\mathbb{R})$. Cette équation d'évolution non-linéaire est l'équation de MacKean-Vlasov associée au mouvement Brownien de Dyson. En dimension supérieure le passage à la limite dans l'équation

$$\mathrm{d}X(t) = -\nabla\mathcal{H}_N^V(X(t))\mathrm{d}t + \sqrt{\frac{2}{\beta}}\mathrm{d}B_t, \quad X(t) \in (\mathbb{R}^d)^N$$

avec $\beta \in \mathbb{R}$ (équation de Langevin) ou $\beta = \infty$ (descente de gradient) constitue un problème très délicat. Celui-ci a été résolu pour la descente de gradient pour $d=1$ et $s \in (0,1)$ dans [30], pour $d=2$ et $s=0$ dans [136, 235] pour pour $s \in (0,d-2)$ dans [146, 71], puis pour $d-2 \leqslant s < d$ dans [240]. En incorporant la méthode d'énergie modulée de [240] à des techniques antérieures [155], une limite de champ moyen pour l'équation de Langevin avec bruit et interaction singulière a également été obtenu dans [61].

Ainsi le temps de relaxation à l'échelle macroscopique du mouvement Brownien de Dyson est dicté par l'équation d'évolution (1.12) et satisfait $t \gg 1$. Comme conjecturé par Dyson, la relaxation locale est beaucoup plus rapide et se produit à l'échelle mésoscopique ou microscopique $1/N \ll \eta \ll 1$ en un temps $t \gg \eta$. Le mouvement Brownien de Dyson est ainsi un outil particulièrement efficace pour démontrer l'universalité des statistiques des β-ensembles et matrices de Wigner [46, 47, 49, 116, 50, 175].

1.1.6 Description du comportement miscroscopique

Le système (1.8) peut être vu à l'échelle macroscopique comme un milieu continu qui se décrit à l'aide de la fonctionnelle d'énergie (1.10). Si l'on zoome d'un facteur $N^{\frac{1}{d}}$ autour d'un point $x \in \mathrm{supp}(\mu_V)$, on observe un nuage de points aléatoire avec en moyenne $\mu_V(x)$ particules par unité de volume. Au vu du scaling imposé dans (1.5), l'interaction entre deux particules voisines est de taille 1. Ainsi, à l'échelle microscopique, on observe un processus aléatoire non-trivial (distinct du processus de Poisson), que l'on peut alors tenter de décrire avec le formalisme des grandes déviations. La description de ce processus microscopique a été entreprise dans l'article important [182], qui couvre le cas régime longue portée $\max(0, d-2) \leqslant s < d$.

Notons $\mathcal{C}_N(x)$ la configuration centrée en x et mise à échelle d'un facteur $N^{\frac{1}{d}}$, i.e $\mathcal{C}_N(x) = \sum_{i=1}^N \delta_{\sqrt{N}(x_i - x)}$. L'observable adéquate [182, 129] pour décrire le processus de points est obtenue en moyennant $\mathcal{C}_N(x)$ pour x dans le support de la mesure d'équilibre :

$$i_N := \int_{\mathrm{supp}(\mu_V)} \delta_{(x, \mathcal{C}_N(x))} \mathrm{d}x. \tag{1.13}$$

Cet objet est appelé champ empirique. Le processus de points observé étant aléatoire, son comportement ne peut pas être décrit uniquement avec une fonction de type énergie. La fonctionnelle de grandes déviations, qui indique la probabilité d'observer un certain processus de point doit ainsi mesurer l'aléa présent dans ce processus de points. Cette fonction, qui mesure le volume de configurations, est appelée entropie relative spécifique et est l'analogue de (1.4) pour les processus en volume infini. Il est ainsi montré dans [182] que le processus moyenné i_N satisfait à un PGD avec une fonction de taux de la forme $\mathcal{F}_\beta = \beta \mathrm{Energie} + \mathrm{Entropie}$.

La difficulté pour étudier la limite thermodynamique du gaz de Riesz dans le régime longue portée est de montrer que l'énergie (1.7) peut se réécrire, après factorisation autour de l'équilibre et mise échelle, comme une fonctionnelle typiquement additive. Dans la série de travaux [230, 232, 218, 203, 182, 184, 186, 206] un ensemble de techniques inspirées de l'analyse du modèle de Ginzburg-Landau [228] ont été développées pour traiter les interactions de type Riesz. Le point de départ est de réécrire l'énergie comme la norme L^2 du champ coulombien généré par le jellium [230, 232] et une technique d'écrantage inspirée de [2] puis développée dans [182, 9, 239] permet de recoller des champs électriques entre eux et ainsi de montrer que l'énergie est typiquement additive. Ceci a permis d'écrire dans [182] un PGD pour l'observable (1.13) puis dans [180] pour (1.13) mais moyenné à de petites échelles. A l'aide d'approximations sous-additives et super-additives de l'énergie inspirées de techniques d'homogénisation stochastique, [9] améliore la méthode de [180] pour obtenir des lois locales valables à l'échelle microscopique ainsi que des estimées d'additivité optimales sur l'énergie et les fonctions de partition. Pour les β-ensembles, on renvoie également à [44, 43] qui donnent des développements à tout ordre des fonctions de partition.

1.1.7 Le plasma à deux composantes

On introduit à présent une variante du gaz de Coulomb bi-dimensionnel où deux types de particules de charge positive et négative coexistent. Ce système est appelé gaz de Coulomb (ou plasma) à deux composantes ou 2CP. Considérons N particules de charge positive $x_1, \ldots, x_N$ et N particules de charge négative $y_1, \ldots, y_N$ dans $\Lambda = [0,1]^2$ interagissant selon l'énergie

$$\mathcal{H}_N = \frac{1}{2} \sum_{i \neq j} g(x_i - x_j) + \frac{1}{2} \sum_{i \neq j} g(y_i - y_j) - \sum_{i,j} g(x_i - y_j)$$

où g est le noyau coulombien en dimension 2, c'est-à-dire

$$g(x) = -\log |x|, \quad x \in \mathbb{R}^2.$$

Le 2CP est alors donné par la mesure

$$d\mathbb{P}_{N,\beta} = \frac{1}{Z_{N,\beta}} \exp(-\beta \mathcal{H}_N(X_N, Y_N)) \mathbb{1}_{(X_N,Y_N)\in\Lambda^{2N}} dx_1 \dots dx_N dy_1 \dots dy_N \tag{1.14}$$

où $Z_{N,\beta}$ désigne la fonction de partition

$$Z_{N,\beta} = \int_{\Lambda^{2N}} \exp(-\beta \mathcal{H}_N(X_N, Y_N)) dx_1 \dots dx_N dy_1 \dots dy_N. \tag{1.15}$$

L'énergie tend vers $-\infty$ lorsque deux particules de charge opposées colisionnent avec un poids relatif en $\mathrm{distance}^{-\beta}$ dans la fonction de partition. Lorsque $\beta \in (0,2)$ cette singularité est intégrable : le bruit suffit à repousser les particules. Lorsque $\beta \geqslant 2$, la singularité devient non-intégrable et la fonction de partition (1.15) diverge. Ainsi, dans le régime de température $\beta \geqslant 2$, il convient de renormaliser l'interaction en la tronquant à une échelle $\eta = \frac{\lambda}{\sqrt{N}}$ pour donner un sens à la distribution (1.14).

L'une des motivations importantes à l'étude du gaz de Coulomb à deux composantes vient de la physique de la matière condensée et plus précisément du modèle XY. Le modèle XY est un modèle de ferromagnétisme dans le plan où des spins à valeurs dans le cercle unité interagissent selon le cosinus de l'angle des plus proches voisins. Le champ formé par les angles peut se décomposer en une onde de spin régulière et un ensemble de singularités composé de tourbillons d'indice 1 et -1, appelés vortex et antivortex. De plus, de façon remarquable, on peut monter que ces vortex correspondent à des charges positives et négatives interagissant selon le potentiel de Coulomb. Dans les années 70, Kosterlitz, Thouless et indépendemment Berezinsky [165, 164, 29] montrent qu'une transition de phase «d'ordre infini» a lieu dans le modèle XY. Cette transition se manifeste notamment par un passage d'une décroissance exponentielle des corrélations des spins à une décroissance algébrique en deçà de la température critique. La transition KT est liée à l'appariement en dipôles des vortex et antivortex et plus précisément à la variation du nombre de charges non neutres à une certaine échelle lorsque l'échelle augmente. Il existe des preuves mathématiques de l'existence de cette transition et celles-ci reposent sur des outils sophistiqués utilisant notamment la représentation de Sine-Gordon et des arguments de groupe de renormalisation [120, 121, 198]. Nous renvoyons à [35] pour une synthèse sur le sujet.

Par ailleurs dans le régime $\beta \in (0,2)$, une description variationnelle de la limite thermodynamique du plasma à deux composantes a été obtenue dans [186]. Le résultat de [186] se base notamment sur la formulation électrique de l'énergie, une technique d'écrantage inspirée de [183] ainsi que des techniques de grandes déviations empruntés quelques arguments combinatoires de [139].

1.1.8 Fluctuations des log-gaz

Une fois le comportement macroscopique du système établi, une question naturelle est de quantifier les fluctuations de la mesure $\sum_{i=1}^N \delta_{x_i} - N\mu_V$ contre des fonctions-test. Étant donnée une certaine fonction borélienne $\xi : \mathbb{R} \to \mathbb{R}$, on considère alors l'objet $\mathrm{Fluct}_N[\xi] := \sum_{i=1}^N \xi(x_i) - N\int \xi d\mu_V$, appelé statistique linéaire. Une question importante venue des matrices aléatoires est d'obtenir un théorème central limite (TCL) pour de telles quantités. Il apparaît que lorsque ξ est suffisamment régulière, $\mathrm{Fluct}_N[\xi]$ est d'ordre 1 avec un comportement asymptotique Gaussien, voir notamment [157, 241, 44, 46, 49, 27, 51, 145]. Lorsque la fonction-test ξ est singulière, un autre comportement peut être observé. Par exemple pour $\xi = \mathbb{1}_{(a,b)}$ avec (a,b) dans le support de μ_V, $\mathrm{Fluct}_N[\xi]$ fluctue en $\sqrt{\log N}$ et un TCL de type log-corrélé est satisfait. Ainsi, l'ordre de grandeur des fluctuations dépend de la régularité de la fonction-test. Notons que dans le cas des β-ensembles, le potentiel logarithmique satisfait à un TCL log-corrélé [46] et l'étude des extrêmes de ce champ s'inscrit dans un sujet d'actualité relié aux marches aléatoires branchantes [159, 102, 172]. Il existe de nombreuses motivations à l'étude des fluctuations des log-gaz et l'une d'elles est reliée à la conjecture KLS [195] et plus précisément à une conjecture plus faible appelée conjecture de variance généralisée. Comme montré dans [90], pour $p > 3$, la boule unité des matrices auto-adjointes pour la norme de $p-$Schatten satisfait à la conjecture de variance généralisée et ce résultat est obtenu en étudiant les fluctuations d'une certaine statistique linéaire sous le log-gaz avec potentiel $|x|^p$.

Les méthodes utilisées pour obtenir ces TCL exploitent le caractère longue portée de l'interaction, avec les équations de boucles ou la méthode de transport de [241, 184] inspirée de [157]. Notons que la preuve du TCL pour des fonctions-test lisses peut se passer du caractère 1D du modèle [27] et cette preuve fonctionne également pour le gaz de Coulomb bi-dimensionnel [184, 24, 187] malgré de nombreuses difficultés additionnelles. En revanche lorsque la fonction-test est singulière il semble nécessaire d'utiliser la convexité et les propriétés de concentration qui en découlent [46, 51]. Pour le gaz de Coulomb en dimension supérieure $d \geqslant 3$ ou bien pour le gaz de Riesz avec $s \in (0, d)$ et $d \geqslant 2$, le problème se complique car les termes à longue portée dans l'énergie deviennent moins dominants : il est alors nécessaire de contrôler avec une meilleure précision les *fluctuations* des variations locales de l'énergie et des termes d'angles. On peut citer [239] pour un TCL valide pour le gaz de Coulomb en dimension 3, sous une hypothèse d'absence de transition de phase. Un modèle simplifié du gaz de Coulomb en dimension 3, appelé gaz de Coulomb hiérarchique, est également étudié dans [83, 124].

1.1.9 Le processus Sinus beta

Le comportement microscopique du log-gaz peut être décrit au moyen du PDG [182], qui exhibe une fonctionelle d'énergie $\mathcal{F}_\beta$ sur les processus ponctuels. La question de l'unicité des minimiseurs de $\mathcal{F}_\beta$ paraît difficilement accessible en dimension supérieure à 1. En dimension 1 et pour $s = 0$, l'unicité a été obtenue dans [112] avec un argument de convexité par déplacement. Le minimiseur est alors identifié au processus Sine_β, qui correspond à la limite universelle du processus microscopique dans le bulk des β-ensembles.

Etant donné x dans l'intérieur support de μ_V, considérons la configuration aléatoire non moyennée centrée en x,

$$\mathcal{C}_N = \sum_{i=1}^N \delta_{\sqrt{N}(x_i - x)}.$$

La variable $\mathcal{C}_N$ est une variable aléatoire sur l'espace des configurations de points. Montrer l'universalité pour les β-ensembles à énergie fixée revient à montrer que la loi de $\mathcal{C}_N$ sous $\mathbb{P}_{N,\beta}$ converge vers un certain processus ponctuel universel. Cette universalité a été démontrée dans la série de travaux [46, 49, 116]. Le processus limite, appelé Sine_β avait déjà introduit dans [162] comme limite du β-ensemble circulaire (d'où son nom, puisque sur le cercle $g_0 = -\log|\frac{\sin x}{2}|$!) et dans [250] comme limite des β-ensembles Gaussiens. En outre, le processus Sine_β se décrit au moyen d'un système d'équations différentielles stochastiques [250] et également comme le spectre d'un opérateur aléatoire en dimension infinie [251].

Le processus Sine_β peut également être étudié à travers le formalisme de Dubroshin-Landford-Ruelle (DLR), développé dans les années 70-80 pour décrire les mesures de Gibbs en volume infini. Les équations DLR décrivent la loi du processus en donnant à l'intérieur d'un compact conditionnellement à l'extérieur, [93, 128]. Ceci suppose de pouvoir donner un sens à l'interaction d'un point avec une infinité d'autres, ce qui est délicat dans le cas des interactions à longue portée. La description DLR du processus Sine_β a été obtenue dans [94] en utilisant des estimées précises de rigidité. Une propriété intéressante qui peut être étudié dans le formalisme DLR est celle de la cardinale-rigidité. Suivant [131], on dit qu'un processus ponctuel est cardinal-rigide si pour tout compact, la donnée du processus restreint à l'extérieur du compact suffit à reconstituer le nombre de points à l'intérieur.

Cette propriété surprenante, qui traduit une forme forte de longue portée, a été démontrée pour le processus Sine_β dans [94, 86].

1.1.10 Temps de mélange et phénomène de cutoff

Nous avons discuté dans le paragraphe 1.1.5 la limite de (1.11) à t fixé, lorsque N tend vers l'infini. On se pose à présent une question différente (mais reliée) : « Quel temps faut-il attendre pour que le système (1.11) soit proche de l'équilibre $\mathbb{P}^V_{N,\beta}$? » En particulier, comment ce temps critique dépend-il de N ? Pour traduire cet énoncé, on se donne une certaine distance (ou divergence) dist sur l'espace des mesures de probabilité sur $\mathbb{R}^N$, prenant ses valeurs dans $[0, \max]$ et l'on se demande au bout de quel temps t_N,

$$\lim_{N\to\infty} \mathrm{dist}(\mathrm{Loi}(X^N_{t_N}), \mathbb{P}^V_{N,\beta}) = 0.$$

Cette question est motivée par un sujet d'actualité en probabilité, qui consiste à établir un phénomène de cutoff pour les processus de Markov. Rappelons-en la définition. Considérons une famille de processus de Markov indexée par N, $X^N = (X^N_t)_{t\geqslant 0}$, à valeurs dans un certain espace S^N et de loi invariante μ^N. Supposons qu'à N fixé et que pour toute condition initiale $x^N_0 \in S^N$, l'on ait

$$\lim_{N\to\infty} \mathrm{dist}(\mathrm{Loi}(X^N_t), \mu^N) = 0.$$

Alors on dit qu'un phénomène de cutoff a lieu si la convergence de X^N vers l'équilibre s'effectue de façon abrupte, c'est-à-dire s'il existe un temps critique c_N tel que pour tout $\varepsilon > 0$ assez petit,

$$\lim_{N\to\infty} \sup_{x^N_0 \in S^N} \mathrm{dist}(\mathrm{Loi}(X^N_{t_N}), \mu^N) = \begin{cases} \max & \text{si } t_N = c_N(1-\varepsilon) \\ 0 & \text{si } t_N = c_N(1+\varepsilon). \end{cases} \tag{1.16}$$

Le phénomène de cutoff est typiquement associé à un effet de grande dimension. Par exemple, X^N peut être une marche aléatoire sur le groupe symétrique, le mouvement Brownien sur la sphère de dimension N, etc.

Le cutoff pour le temps de mélange des processus de Markov a été mis en lumière par David Aldous et Persi Diaconis [3, 99, 101, 190] et est l'objet de nombreux travaux [224, 226, 199, 168, 68, 22].

Pour établir un résultat du type (1.16), il convient de conjecturer quelle est l'observable qui converge *le plus lentement* vers l'équilibre. Pour le mouvement Brownien de Dyson, comme expliqué en 1.1.5 ce sont les observables macroscopiques qui jouent ce rôle. Ainsi la limite hydrodynamique (1.12) donne une borne inférieure crédible sur le temps de mélange, qu'il convient alors de faire coïncider avec une borne supérieure.

1.2 Quelques méthodes pour les systèmes de particules

On donne à présent quelques point de repères sur les méthodes mathématiques, pour certaines classiques, utilisées dans la littérature sur les systèmes de particules en interactions.

1.2.1 Principes de grandes déviations

On rappelle la notion de principe de grandes déviations. Soit (μ_n) une suite de mesures de probabilité sur un espace topologique χ muni de sa tribu borélienne $\mathcal{B}$, (a_n) une suite de réels positifs tendant vers l'infini et $I : \chi \to [0, \infty]$ une fonction s.c.i. On dit que (μ_n) suit un principe de grandes déviations (PGD) de vitesse (a_n) et de fonction de taux I si pour tout $B \in \mathcal{B}$,

$$-\inf_{\mathring{B}} I \leqslant \liminf_{n\to\infty} \frac{1}{a_n} \log \mu_n(B) \leqslant \limsup_{n\to\infty} \frac{1}{a_n} \log \mu_n(B) \leqslant -\inf_{\bar{B}} I \tag{1.17}$$

Cette définition formalise le fait qu'en un certain sens, pour tout $B \in \mathcal{B}$,

$$\mu_n(B) \simeq e^{-a_n \inf_B I}. \tag{1.18}$$

La définition (1.17) peut paraître peu naturelle mais il s'agit en fait de la bonne formalisation de (1.18). La présence d'un infimum dans (1.17) reflète le fait qu'à l'échelle exponentielle la probabilité d'un événement rare partitionné en sous-événements est environ égale à la probabilité du moins rare de ces sous-événements. En effet si $\mu_n(B)$ décroît en $e^{-J(B)}$ avec $J : \chi \to [0, \infty]$ alors nécessairement $J(A \cup B) = \inf(J(A), J(B))$ car

$$\max(\mu_n(A), \mu_n(B)) \leqslant \mu_n(A \cup B) \leqslant 2 \max(\mu_n(A), \mu_n(B)).$$

On renvoie à [92, 212] pour une introduction générale sur le sujet. Les grandes déviations offrent un cadre naturel pour étudier le comportement asymptotique de systèmes de particules du type (1.8). Supposons que l'on ait trouvé une certaine observable i_N à valeurs dans χ qui satisfait selon nous à un PGD avec fonction de taux I et de vitesse (a_n). Par exemple i_N peut être la mesure empirique (1.9), le champ empirique (1.13). Dans ce cadre [80, 183] un PGD se prouve souvent en deux temps :

- Montrer une borne supérieure en donnant une minoration précise de l'énergie. On se restreint à un événement où notre suite d'observable i_N est proche d'une certaine observable limite $x \in \chi$ et on minore $a_n^{-1} H_N(X_N)$ à la limite par $I(x)$.
- Pour établir la borne inférieure du LDP on ne peut pas minorer l'énergie mais on peut toutefois se restreindre à un certain sous-espace de bonnes configurations. Il faut alors construire à la main des configurations telles que ι_N est proche de x et $a_n^{-1} H_N(X_N)$ proche de $I(x)$, en faisant attention à ce que le volume de configurations soit suffisant.

Ces techniques de grandes déviations sont basées sur des méthodes énergétiques et permettent de développer les fonctions de partition [182, 9] et dans certains cas d'obtenir des estimées sur les fluctuations du système [181, 24, 239].

1.2.2 Méthodes énergétiques

Nous introduisons quelques outils au fondement de la série de travaux [230, 232, 218, 203, 182, 184, 186, 206] qui permettent de manier efficacement l'énergie (1.7). Pour simplifier on se restreint au cas coulombien $s = d - 2$ et $d \geqslant 2$. Soit μ_V la mesure d'équilibre donnée par le minimiseur de la fonctionnelle (1.10). En notant $\mu_N := \frac{1}{N} \sum_{i=1}^N \delta_{x_i}$, on peut écrire (1.10) sous la forme

$$\mathcal{H}_N^V(X_N) = N^{-\frac{s}{d}} \iint g(x-y) \mathrm{d}(N\mu_N)(x) \mathrm{d}(N\mu_N)(y) + N \int V(x) \mathrm{d}\mu_N(x). \tag{1.19}$$

On pose $\mathrm{fluct}_N = N(\mu_N - \mu_V)$. En développant l'expression ci-dessus autour de μ_V, on obtient

$$\mathcal{H}_N^V(X_N) = N^{-\frac{s}{d}} \iint g(x-y) \mathrm{d}(N\mu_V)(x) \mathrm{d}(N\mu_V)(y) + 2N \sum_{i=1}^N \zeta_V(x_i) + F_N(X_N, \mu_V),$$

avec ζ_V le potentiel de confinement effectif

$$\zeta_V := g * \mu + \frac{V}{2} - \int \Big(g * \mu + \frac{V}{2} \Big) \mathrm{d}\mu_V$$

et $F_N(\cdot, \mu_V)$ l'énergie du second-ordre

$$F_N(X_N, \mu_V) := \iint g(x-y)\mathrm{dfluct}_N(x)\mathrm{dfluct}_N(y). \tag{1.20}$$

Par minimalité de μ_V, la fonction ζ_V s'annule sur le support de la mesure d'équilibre. Le système $N(\mu_N - \mu_V)$ peut être vu comme un ensemble de particules ponctuelles baignant dans un fond constant de densité $-N\mu_V$, ce qui correspond à un jellium en physique mathématique, voir par exemple [78, 79]. Il est ensuite possible de réécrire l'énergie (1.20) en fonction du champ électrique généré par le jellium fluct_N. Notons H_N le potentiel généré par le jellium, i.e $H_N = g * \mathrm{fluct}_N$. Puisque g est le noyau coulombien, on peut observer que

$$F_N(X_N, \mu_V) = -\frac{1}{c_{d,s}} \int H_N \Delta H_N.$$

En intégrant formellement par partie on trouverait

$$F_N(X_N, \mu_V) \simeq \frac{1}{c_{d,s}} \int |\nabla H_N|^2. \tag{1.21}$$

Le champ électrique ∇H_N divergeant en $\frac{1}{|x-x_i|}$ autour d'une charge ponctuelle x_i, celui-ci n'est pas dans L^2. Pour donner un sens à (1.21), il convient de régulariser le système. Au lieu de régularise le potentiel g lui-même, on peut désingulariser ∇H_N en étalant les masses de Dirac δ_{x_i} en des mesures uniformes $\delta_{x_i}^{(\eta_i)}$ sur la sphère $\partial B(x_i, \eta_i)$. Pour $\eta = (\eta_1, \ldots, \eta_N)$, on note alors H_N^η le potentiel généré par ce jellium régularisé :

$$H_{N,\eta} = g * \Big(\sum_{i=1}^N \delta_{x_i}^{(\eta)} - N\mu_V \Big).$$

On peut alors montrer que

$$F_N(X_N, \mu_V) = \frac{1}{c_{d,s}} \lim_{\eta \to 0} \Big(\int_{\mathbb{R}^d} |\nabla H_{N,\eta}|^2 - c_{d,s} \sum_{i=1}^N g(\eta_i) \Big). \tag{1.22}$$

Une observation fondamentale est que le membre de droite dans (1.22) est décroissant en le paramètre de troncature η à petite erreur près. De par le théorème de Newton, on peut montrer que si les boules $B(x_i, \eta_i)$ sont disjointes alors

$$F_N(X_N, \mu_V) = \frac{1}{c_{d,0}} \Big(\int_{\mathbb{R}^d} |\nabla H_{N,\eta}|^2 - c_{d,0} \sum_{i=1}^N g(\eta_i) \Big) + \text{petite erreur explicite}.$$

Cette réécriture de l'énergie, ainsi que la propriété de monotonie par rapport au paramètre de troncation η, permet de minorer l'énergie de façon très efficace. D'autre part, cette formulation donne des outils pour montrer que l'énergie $F_N(X_N, \mu_V)$ est typiquement additive : c'est la technique de screening.

1.2.3 Théorème de fluctuation-dissipation

Ce paragraphe décrit une méthode pour obtenir un TCL pour les statistiques linéaires en présence d'un système de particules à interaction longue portée du type (1.5) avec $s < d$. Considérons la mesure (1.8) et une fonction régulière $F : \mathbb{R}^d \to \mathbb{R}$. Une façon d'étudier les fluctuations de F sous $\mathbb{P}^V_{N,\beta}$ est de considérer la transformée de Laplace

$$\mathbb{E}_{\mathbb{P}^V_{N,\beta}}[e^{tF}], \quad t \in \mathbb{R}. \tag{1.23}$$

Cette transformée de Laplace peut se réécrire comme un ratio de fonction de partition

$$\mathbb{E}_{\mathbb{P}^V_{N,\beta}}[e^{tF}] = \frac{Z_{N,\beta}(t)}{Z^V_{N,\beta}} e^{-tN\int \xi \mathrm{d}\mu_V},$$

où

$$Z_{N,\beta}(t) := Z^{V-\frac{t}{\beta}F}_{N,\beta} = \int e^{tF-\beta\mathcal{H}^V_N(X_N)} \mathrm{d}X_N.$$

Notons $\mathbb{P}^V_{N,\beta}(t)$ la mesure de probabilité $\propto e^{tF}\mathbb{P}^V_{N,\beta}$. Sous $\mathbb{P}^V_{N,\beta}(t)$, les particules sont soumises à une force additionnelle en $-tF$ qui modifie la distribution des charges. Pour remettre les particules à l'équilibre, il faut appliquer un transport qui envoie $\mathbb{P}^V_{N,\beta}(t)$ sur $\mathbb{P}^V_{N,\beta}$ et par des résultats classiques d'analyse [59], ce transport peut être cherché sous la forme du gradient d'une fonction convexe $\nabla\Phi_t$. Ce gradient est solution de l'équation de Monge-Ampère

$$-\log\det D\Phi_t + \beta(\mathcal{H}^V_N \circ \nabla\Phi_t - \mathcal{H}^V_N) = tF - \log \mathbb{E}_{\mathbb{P}^V_{N,\beta}}[e^{tF}]. \tag{1.24}$$

En linéarisant l'équation en t et en écrivant $\nabla\Phi_t = \mathrm{Id} + t\nabla\phi + o(t)$, on se ramène alors à étudier l'équation linéaire

$$(\mathcal{L}\phi =) - \Delta\phi + \beta\nabla\mathcal{H}_N \cdot \nabla\phi = F - \mathbb{E}_{\mathbb{P}^V_{N,\beta}}[F]. \tag{1.25}$$

On reconnaît le générateur du semi-groupe de Markov (1.2). L'équation ci-dessus est appelée équation de Poisson, en analogie avec le cas sans interaction où $\mathcal{L}$ est un Laplacien. En effectuant le changement de variables $\mathrm{Id} + t\nabla\phi$ dans la transformée de Laplace (1.23) on peut observer que les termes linéaires s'annulent et que les termes quadratiques se regroupent en

$$\mathrm{Var}_{\mathbb{P}^V_{N,\beta}}[F] = \mathbb{E}_{\mathbb{P}^V_{N,\beta}}[\nabla F \cdot \nabla\phi]. \tag{1.26}$$

Cette formule est appelée formule de représentation de Helffer-Sjöstrand et peut aussi se montrer par une simple intégration par partie sous $\mathbb{P}^V_{N,\beta}$. Le passage par la transformée de Laplace donne de plus une interprétation mécanique de la solution de (1.25) : $\nabla\phi$ correspond au transport infinitésimal à appliquer aux particules pour les remettre à l'équilibre lorsque celles-ci ont été perturbées par $-tF$. Ce transport est aussi appelé *réponse linéaire* dans le langage des systèmes dynamiques.

Il est souvent intéressant d'étudier de (1.25) sous sa forme différenciée. On peut remarquer que le commutateur de $\mathcal{L}$ et de l'opérateur gradient fait apparaître la Hessienne de l'énergie :

$$\nabla\mathcal{L}\phi = \mathcal{L}\nabla\phi + \beta\nabla^2\mathcal{H}^V_N\nabla := A_1\nabla\phi$$

où A_1 est l'opérateur

$$A_1 = \mathcal{L} \otimes I_N + \beta\nabla^2\mathcal{H}^V_N. \tag{1.27}$$

Notons que ceci est l'équivalent du théorème de Bochner, qui exprime le défaut de commutation entre le Laplacien sur une variété riemanienne et le gradient en fonction du tenseur de Ricci [189].

Il existe au moins deux cas de figure où la représentation (1.26) est effective et permet de contrôler la variance de F : lorsque l'interaction est à longue portée et F est une statistique linéaire ou lorsque l'énergie est convexe. En effet si $F = \sum_{i=1}^N \xi(x_i) - N\int \xi \mathrm{d}\mu_V$ avec $\xi : \mathbb{R}^d \to \mathbb{R}$ et si $s \in (0, d)$ (mais pas pour $s > d$!), on peut chercher une solution approchée de (1.25) sous la forme d'un transport qui agit diagonalement $\nabla\Phi : X_N \mapsto (\psi(x_1), \ldots, \psi(x_N))$ avec $\psi : \mathbb{R}^d \to \mathbb{R}^d$. Le fonction ψ est de plus donnée par

$$g_s' * \psi = \frac{\xi - \int \xi \mathrm{d}\mu_V}{2\beta c_{s,d} N^{1-\frac{s}{d}}}. \tag{1.28}$$

Le transport ψ solution de (1.28) peut s'interpréter comme la solution d'un problème de transport en dimension d. En effet rappelons que la distribution macroscopique des charge sous $\mathbb{P}_{N,\beta}^V$ est donnée dans la limite où $N \to \infty$ par μ_V qui minimise I_V (1.10). Sous $\mathbb{P}_{N,\beta}(t)$ la distribution de macroscopique est tiltée et est donnée par μ_{V_t} qui minimise $I_{V_t} = I_V - \frac{t}{\beta}\mathrm{Fluct}_N[\xi]$. En écrivant les conditions d'optimalité de μ_{V_t} et en linéarisant formellement en t, on trouve que la solution de (1.28) transporte μ_{V_t} sur μ_V. Ceci illustre une commutation entre le fait de linéariser (1.24) et de passer à la limite de champ moyen.

1.2.4 Inégalités fonctionnelles

Dans ce paragraphe on rappelle quelques inégalités de concentration très classiques pour les mesures log-concaves. On considère une mesure μ sur $\mathbb{R}^d$ log-concave par rapport à la Gaussienne de variance $\sigma^2 > 0$, c'est-à-dire que μ s'écrit $\mathrm{d}\mu = e^{-f}\mathrm{d}\nu$ où $\nu = \mathcal{N}(0, \sigma^2)$ et $f : \mathbb{R}^d \to \mathbb{R}$ convexe. Alors pour $F \in H^1$, $\nu \in \mathcal{P}(\mathbb{R}^d)$ et $t \in \mathbb{R}$, on a

$$\mathrm{Var}_\mu[F] \leqslant \frac{1}{\sigma^2}\mathbb{E}_\mu[|\nabla F|^2], \quad \text{(Poincaré)} \tag{1.29}$$

$$\mathrm{Ent}[\nu \mid \mu] \leqslant \frac{1}{2\sigma^2}\mathrm{Fisher}[\nu, \mu], \quad \text{(log-Sobolev)} \tag{1.30}$$

$$\log \mathbb{E}_\mu[e^{tF}] \leqslant t\mathbb{E}_\mu[F] + \frac{t^2}{2\sigma^2}\sup|\nabla F|, \quad \text{(concentration Gaussienne)}, \tag{1.31}$$

La première inégalité est un cas particulier de l'inégalité de Brascamp-Lieb et peut s'obtenir par linéarisation de (1.30), qui suit du critère de Bakry-Emery (voir [37]). La preuve de la concentration Gaussienne s'obtient par exemple en appliquant (1.30) et l'argument de Herbst [189] ou par le théorème de contraction de Caffarelli [64]. Ces inégalités ont également des conséquences dynamiques. Si on considère la dynamique de Langevin (1.1) et que l'on note $\mu(t)$ la loi de X_t à l'instant t, on peut voir en dérivant l'entropie relative par rapport à μ le long du semi-groupe et un utilisant (1.30) que

$$\mathrm{Ent}[\mu_t \mid \mu] \leqslant \mathrm{Ent}[\mu_0 \mid \mu]e^{-\frac{2}{\sigma^2}t}. \tag{1.32}$$

Ceci fournit alors un moyen de quantifier le temps de convergence à l'équilibre dans le problème mentionné dans le paragraphe 1.1.10. Des généralisations importantes de (1.30) sont utilisées dans la littérature sur les β-ensembles. Dans le cas où μ est donnée par (1.8) avec $d = 1$, $s = 0$ et V convexe, alors comme observé dans [46], on peut écrire une inégalité de log-Sobolev qui exploite la convexité de l'interaction en se restreignant à des fonctions de divergence nulle.

1.2.5 Représentation des corrélations

On peut observer par polarisation que la formule (1.26) permet d'exprimer la covariance de deux fonctions $F : \mathbb{R}^d \to \mathbb{R}$ et $G : \mathbb{R}^d \to \mathbb{R}$ suffisamment lisse :

$$\mathrm{Cov}_{\mathbb{P}_{N,\beta}^V}[F, G] = \mathbb{E}_{\mathbb{P}_{N,\beta}^V}[\nabla\phi \cdot \nabla G],$$

où $\nabla\phi$ est solution de l'équation d'Helffer-Sjöstrand

$$A_1 \nabla\phi = \nabla F, \tag{1.33}$$

avec A_1 défini en (1.27). Ainsi, pour déterminer la décroissance de la corrélation de $F = N(x_{i+1} - x_i)$ et de $G = N(x_{j+1} - x_j)$, il convient d'étudier la décroissance des incréments de $\nabla\phi$. En présence d'une interaction convexe (i.e dans le cas $d = 1$), les inégalités de concentration (1.29) et (1.31) peuvent se réécrire sous la forme d'estimées L^2 et uniformes sur $\nabla\phi$, voir [149]. En revanche, obtenir des estimées de décroissance suppose une analyse plus fine. Il existe de nombreux travaux dans la littérature qui s'attaquent à la question de la décroissance des solutions de (1.27), en particulier lorsque l'énergie est convexe. Une première méthode consiste à réécrire (1.33) au moyen d'une représentation de Fenyman-Kac [12, 96, 132, 201, 149, 116] qui suppose de contrôler des marches aléatoires dans un environnement aléatoire. Il existe également d'autres points de vue plus analytiques qui s'inspirent des techniques d'homogénisation stochastique [201, 10, 91, 247].

1.3 Résultats obtenus et perspectives

Cette thèse est divisée en trois parties. Dans le chapitre 2 on étudie un problème de convergence vers l'équilibre pour le mouvement Brownien de Dyson, dans le second on étudie les fluctuations et les corrélations à l'équilibre pour le gaz de Riesz circulaire et dans le troisième on s'attachera à décrire l'équilibre pour le gaz de Coulomb à deux composantes dans le régime $\beta \in [2, +\infty)$. Ainsi les deux premières parties exploiteront la structure uni-dimensionelle du modèle et la convexité sous-jacente tandis que la troisième partie se base sur des inégalités énergétiques et des méthodes de grandes déviations.

1.3.1 Cutoff pour le temps de mélange du mouvement brownien de Dyson

Dans la première partie, en collaboration avec D. Chafaï et C. Labbé, on étudie le problème du temps de mélange du mouvement Brownien de Dyson présenté dans le paragraphe 1.1.10. On se restreint à un cas particulier où le potentiel extérieur V dans (1.7) est quadratique, ce qui donne un aspect intégrable au modèle [81, 177]. Lorsqu'il n'y a pas d'interaction, (1.11) est alors un processus de Ornstein-Uhlenbeck en dimension N $(= n)$. De façon à pouvoir éteindre l'interaction, on écrit l'énergie sous la forme

$$\mathcal{H}_n = \beta \sum_{i \neq j} \log \frac{1}{|x_i - x_j|} + n \sum_{i=1}^{n} V(x_i),$$

où $\beta \geqslant 0$ est désormais un paramètre qui contrôle la force de l'interaction et $V(x) = \frac{|x|^2}{2}$. On considère ensuite la dynamique de Langevin mise à échelle en temps,

$$X_0^n = x_0^n \in \mathbb{R}^n, \quad \mathrm{d}X_t^{n,i} = \sqrt{\frac{2}{n}} \mathrm{d}B_t^i - V'(X_t^{n,i})\mathrm{d}t + \frac{\beta}{n} \sum_{j \neq i} \frac{\mathrm{d}t}{X_t^{n,i} - X_t^{n,j}}, \quad 1 \leqslant i \leqslant n. \tag{1.34}$$

Dans la suite on parlera de (1.34) comme du processus de Dyson-Ornstein-Uhlenbeck (DOU) pour $\beta > 0$ et du processus de Ornstein-Uhlenbeck (OU) pour $\beta = 0$. A cause de la singularité de l'interaction, il peut être délicat de donner un sens à (1.34). C'est pourquoi on se restreint au cas $\beta = 0$ et $\beta \geqslant 1$, où l'équation est bien posée (la basse température rend les collisions improbables, comme pour le gaz de Coulomb à deux composantes). Pour $\beta = 2$, le système (1.34) correspond aux valeurs propres du processus de Ornstein Uhlenbeck matriciel. L'article [54] se propose d'étudier le phénomène de cutoff pour le temps de mélange de (1.34) et dans une grande variété de distances et de divergences.

Nous commençons par analyser le cas simple du processus OU. La mesure invariante P_n^0 est ici donnée par une Gaussienne en dimension n centrée de matrice de covariance $\frac{1}{n}I_n$, $P_n^0 = \mathcal{N}(0, \frac{1}{n}I_n)$. On peut établir un cutoff à condition initiale fixée : pour tout z_0^n, il y a cutoff avec un temps critique dépendant $|z_0^n|$, hormis dans le cas de la distance de Wasserstein où la norme de $|z_0^n|$ doit être assez grande.

Théorème 1 (Cutoff pour OU). *Soit* $Z^n = (Z_t^n)_{t\geqslant 0}$ *un OU donné par* (1.34) *avec* $\beta = 0$ *et* P_n^0 *sa loi invariante. Soit* $\mathrm{dist} \in \{\mathrm{TV}, \mathrm{Hellinger}, \mathrm{Entropy}, \chi^2, \mathrm{Fisher}\}$*, prenant ses valeurs dans* $[0, \max]$*. Alors, pour tout* $\varepsilon \in (0,1)$*,*

$$\lim_{n\to\infty} \mathrm{dist}(\mathrm{Law}(Z_{t_n}^n) \mid P_n^0) = \begin{cases} \max & \textit{si } t_n = (1-\varepsilon)c_n, \\ 0 & \textit{si } t_n = (1+\varepsilon)c_n \end{cases}$$

où

$$c_n = \begin{cases} \log(\sqrt{n}|z_0^n|) \vee \frac{1}{4}\log(n) & \textit{if } \mathrm{dist} \in \{\mathrm{TV}, \mathrm{Hellinger}, \mathrm{Entropy}, \chi^2\}, \\ \log(n|z_0^n|) \vee \frac{1}{2}\log(n) & \textit{if } \mathrm{dist} = \mathrm{Fisher}. \end{cases}$$

Pour la distance de Wasserstein on a la dichotomie suivante :

- *si* $\lim_{n\to\infty} |z_0^n| = +\infty$*, alors pour tout* $\varepsilon \in (0,1)$*, avec* $c_n = \log|z_0^n|$*,*

$$\lim_{n\to\infty} \mathrm{Wasserstein}(\mathrm{Law}(Z_{t_n}), P_n^0) = \begin{cases} +\infty & \textit{si } t_n = (1-\varepsilon)c_n, \\ 0 & \textit{si } t_n = (1+\varepsilon)c_n, \end{cases}$$

- *si* $\lim_{n\to\infty} |z_0^n| = \alpha \in [0,\infty)$ *alors il n'y a* pas de phénomène de cutoff *c'est-à-dire pour tout* $t > 0$

$$\lim_{n\to\infty} \mathrm{Wasserstein}^2(\mathrm{Law}(Z_t), P_n^0) = (\alpha^2 - 1)\mathrm{e}^{-2t} + 2(1 - \sqrt{1 - \mathrm{e}^{-2t}}).$$

Le théorème 1 met en lumière un phénomène intéressant : pour $\mathrm{dist} \in \{\mathrm{TV}, \mathrm{Hellinger}, \mathrm{Entropy}, \chi^2, \mathrm{Fisher}\}$, lorsque la condition initiale est très proche de l'équilibre, i.e ici $|z_0^n| \leqslant n^{-\frac{1}{4}}$, alors le temps de mélange est indépendant de z_0^n. Ce temps critique correspond au temps minimal pour étaler des conditions initiales ponctuelles.

Vient ensuite l'étude du processus DOU pour $\beta \geqslant 1$. Cette fois-ci on établit un cutoff au pire cas (1.16) comme dans la plupart des travaux sur le cutoff. Nous obtenons des bornes inférieures et supérieures sur le temps de mélange qui donnent (entre autres) le résultat suivant :

Théorème 2 (Cutoff pour DOU). *Soit* $(X_t^n)_{t\geqslant 0}$ *le processus DOU* (1.34) *avec* $\beta = 0$ *or* $\beta \geqslant 1$ *de loi invariante* P_n^β*. Prenons* $\mathrm{dist} \in \{\mathrm{TV}, \mathrm{Hellinger}, \mathrm{Wasserstein}\}$*. Soit* $(a_n)_n$ *une suite de réels satisfaisant* $\inf_n a_n > 0$*. Alors, pour tout* $\varepsilon \in (0,1)$*, on a*

$$\lim_{n\to\infty} \sup_{x_0^n \in [-a_n, a_n]^n} \mathrm{dist}(\mathrm{Law}(X_{t_n}^n) \mid P_n^\beta) = \begin{cases} \max & \textit{if } t_n = (1-\varepsilon)c_n \\ 0 & \textit{if } t_n = (1+\varepsilon)c_n \end{cases}$$

où

$$c_n := \begin{cases} \log(na_n) & \textit{if } \mathrm{dist} \in \{\mathrm{TV}, \mathrm{Hellinger}\} \\ \log(\sqrt{n}a_n) & \textit{if } \mathrm{dist} = \mathrm{Wasserstein} \end{cases}.$$

Ainsi le temps critique apparaît comme étant indépendant de l'intensité de l'interaction β. Pour $\beta = 2$, le résultat ci-dessus peut s'obtenir par contraction du temps de mélange du processus de Ornstein-Uhlenbeck matriciel qui suit du Théorème 1. Notons que les conditions initiales z_0^n qui réalisent la borne inférieure sont celles pour lesquelles la trace est loin de l'équilibre à l'instant initial, i.e $\liminf_{n\to\infty} \frac{1}{n}|x_0^{n,1} + \ldots + x_0^{n,n}| > 0$. Pour de telles conditions initiales, c'est donc le temps de mélange de la trace qui donne le temps critique. La preuve de la borne inférieure exploite un aspect intégrable du DOU et la borne supérieure utilise la décroissance exponentielle de l'entropie relative (1.32) ainsi que des arguments de monotonie et de couplage inspirés de [169]. Dans la sous-section 1.3.4, nous mentionnons quelques prolongements possibles à ce travail.

1.3.2 Fluctuations, corrélations et limite thermodynamique pour le gaz de Riesz circulaire

Les chapitres 3 et 4 de ce manuscrit sont consacrés à l'étude du gaz de Riesz sur le cercle dans le régime longue portée. Comme mentionné précédemment, le gaz de Riesz sur le cercle correspond pour $s = 0$ au log-gaz circulaire ou β-ensemble circulaire ou $C_\beta E$. Les β-ensembles sont des modèles fondamentaux de la physique statistique, abondamment étudiés dans la littérature probabiliste en raison de leur liens avec les matrices aléatoires. Il existe un très grand nombre de résultats sur ces modèles et l'une des richesses du sujet réside dans la multiplicité des approches et outils possibles pour les aborder : probabilités intégrables, modèles tridiagonaux, méthodes de gaz de Coulomb, approche dynamique avec le mouvement de Dyson, représentation avec des diffusions stochastiques, des spectres d'opérateurs aléatoires, etc. Pour $s \in (0,1)$, il semble exister moins de structures mathématiques sous-jacentes (pas de bon modèle matriciel a priori). On se propose alors d'étudier ce modèle avec une approche purement « physique statistique ». Plus précisément on poursuit le programme suivant :

1. Donner des estimées quasi-optimales sur les fluctuations des espacements entre particules avec des probabilités sous-exponentielles de déviation.
2. Énoncer un TCL pour les statistiques linéaires valables à toute échelle et pour des fonctions-test aussi singulières que possible.
3. Monter une estimée optimale de décroissance des corrélations pour les variables des gaps $N(x_{i+1} - x_i)$. Comparer le résultat à celui obtenu dans le régime courte portée $s > 1$.
4. Montrer que le processus microscopique converge vers un certain processus limite $\mathrm{Riesz}_{s,\beta}$ qui généralise alors Sine_β à des valeurs $s \in (0,1)$.

Précisons un peu le modèle sur lequel on travaille avant d'énoncer les résultats principaux. Sur le cercle $\mathbb{T} := \mathbb{R}/\mathbb{Z}$ et pour un paramètre $s \in (0,1)$, on considère le noyau de Riesz, solution de l'équation de Laplace fractionnaire

$$(-\Delta)^{\frac{1-s}{2}} g_s = c_s(\delta_0 - 1).$$

Le noyau g_s est donné par la périodisation du noyau de Riesz réel :

$$g_s : x \in \mathbb{T} \mapsto \lim_{n\to\infty}\Big(\sum_{k=-n}^{n} \frac{1}{|x+k|^s} - \frac{2}{1-s} n^{1-s}\Big) = \zeta(s,x) + \zeta(s,1-x),$$

où $\zeta(s,x)$ est la fonction zêta de Hurwitz. On considère alors l'énergie d'interaction par paires

$$\mathcal{H}_N : X_N \in \mathbb{T}^N \mapsto N^{-s} \sum_{i\neq j} g_s(x_i - x_j).$$

L'un des avantages majeurs de la dimension 1 est qu'il est possible d'ordonner les particules. Pour $x, y \in \mathbb{T}$, on dit que $x < y$ si $x = x' + k$, $y = y' + k'$ avec $k, k' \in \mathbb{Z}$, $x', y' \in [0,1)$ et $x' < y'$. Définissons alors D_N l'ensemble des particules ordonnées (x_1 étant libre)

$$D_N = \{X_N = (x_1, \ldots, x_N) \in \mathbb{T}^N : x_2 - x_1 < \ldots < x_N - x_1\}.$$

Le gaz de Riesz circulaire est alors donné par la mesure de probabilité

$$\mathrm{d}\mathbb{P}_{N,\beta} = \frac{1}{Z_{N,\beta}} \exp(-\beta \mathcal{H}_N(X_N)) \mathbb{1}_{D_N}(X_N) \mathrm{d}X_N.$$

Lorsque le nombre de particules tend vers l'infini, la distribution macroscopique de charge converge vers la mesure uniforme sur le cercle. On s'attend ainsi à ce que l'espacement (ou gap) $N(x_{i+k} - x_i)$ se concentre autour de k, lorsque k est suffisamment large. Si les variables étaient i.i.d, alors pour k suffisant grand cette quantité flucterait en $O(k^{\frac{1}{2}})$. Dans le cas du $C_\beta E$, il est connu que l'amplitude des fluctuations de $N(x_{i+k} - x_i)$ est d'ordre $O(\sqrt{\log k})$. Le problème (1) consiste alors à déterminer l'amplitude des fluctuations de ces gaps et nous obtenons dans [52] le résultat suivant :

Théorème 3 (Rigidité des gaps)**.** *Soit $\varepsilon > 0$ et $\delta = \frac{\varepsilon}{4(s+2)}$. Il existe deux constantes $C(\beta) > 0$ et $c(\beta) > 0$ localement uniformes en β telles que pour tout $i \in \{1, \ldots, N\}$ et $1 \leqslant k \leqslant \frac{N}{2}$, on a*

$$\mathbb{P}_{N,\beta}(|N(x_{i+k} - x_i) - k| \geqslant k^{\frac{s}{2}+\varepsilon}) \leqslant C(\beta) e^{-c(\beta) k^\delta}.$$

Ce résultat indique que le nombre de points dans un arc de cercle (a, b) fluctue au plus en $O(N^{\frac{s}{2}+\varepsilon})$. Notre second résultat affine cette asymptotique. On prouve un TCL pour la statistique linéaire $\mathrm{Fluct}_N[\xi(\ell_N^{-1} \cdot)] = \sum_{i=1}^N \xi(\ell_N^{-1} x_i) - N\ell_N \int \xi$ lorsque ξ satisfait aux hypothèses suivantes :

Hypothèses 1.

(i) (Régularité) ξ est $\mathcal{C}^{-s+\varepsilon}$ pour un certain $\varepsilon > 0$.

(ii) (Régularité par morceaux) Soit $\psi = (-\Delta)^{-\frac{s}{2}} \xi$. La fonction ψ est $\mathcal{C}^2$ par morceaux : il existe $a_1 < \ldots < a_p$ ($p \in \mathbb{N}$) telle que sur (a_i, a_{i+1}), ψ est $\mathcal{C}^2$, pour tout $i \in \{1, \ldots, p\}$.

(iii) (Singularité) Pour tout $i \in \{1, \ldots, p\}$, il existe $\alpha_i \in (0, 1 - \frac{s}{2})$ tel que

$$|\psi''|(x) \leqslant \frac{C}{|x - a_i|^{1+\alpha_i}}.$$

(iv) (Support) Soit $\{\ell_N\}$ une suite dans $[0,1]$. Supposons ξ supportée sur $(-\frac{1}{2}, \frac{1}{2})$ ou $\ell_N = 1$. Dans le premier cas, on note $\xi_0 : \mathbb{R} \to \mathbb{R}$

$$\xi_0(x) = \begin{cases} \xi(x) & \text{if } |x| \leqslant \frac{1}{2} \\ 0 & \text{if } |x| > \frac{1}{2}. \end{cases} \tag{1.35}$$

Ces hypothèses signifient que la fonction-test ξ est lisse par morceaux avec un nombre fini de singularités dominées par $|x|^{-\frac{s}{2}+\varepsilon}$ dans $\mathcal{C}^{1-\frac{s}{2}}(\mathbb{T}, \mathbb{R})$. Pour toutes mesures μ et ν sur $\mathbb{R}$ on note $d(\mu, \nu)$ la distance

$$\mathsf{d}(\mu, \nu) = \sup\Big\{ \int f \mathrm{d}(\mu - \nu) : |f|_\infty \leqslant 1, |f'|_\infty \leqslant 1 \Big\}.$$

On montre dans [52] le résultat suivant :

Théorème 4 (TCL pour les statistiques linéaires). *Soit ξ et ℓ_N satisfait les hypothèses 1.*

- *La suite de variables aléatoires $(N\ell_N)^{-\frac{s}{2}}\mathrm{Fluct}_N[\xi(\ell_N^{-1}\cdot)]$ converge en loi vers une Gaussienne centrée de variance σ_ξ^2 donnée par*

$$\sigma_\xi^2 = \frac{1}{2\beta c_s}\begin{cases} |\xi|^2_{H^{\frac{1-s}{2}}} & \text{if } \ell_N = 1 \\ |\xi_0|^2_{H^{\frac{1-s}{2}}} & \text{if } \ell_N \to 0, \text{ avec } \xi_0 \text{ comme dans (1.35)}. \end{cases}$$

- *Soit $Z \sim \mathcal{N}(0,\sigma_\xi^2)$ avec σ_ξ^2. Alors pour tout $\varepsilon > 0$, on a*

$$\mathrm{d}(\mathrm{Loi}((N\ell_N)^{-\frac{s}{2}}\mathrm{Fluct}_N[\xi(\ell_N^{-1}\cdot)]), \mathrm{Loi}(Z)) = O\Big((N\ell_N)^{-\frac{1-s}{2}} + (N\ell_N)^{-(1-\frac{s}{2}-\max\alpha_l-\varepsilon)}\Big).$$

Le théorème ci-dessus s'applique notamment à la fonction indicatrice $\mathbb{1}_{(a,b)}$ où (a,b) est un arc de cercle et à la fonction puissance inverse $|x|^{-\alpha}$ pour $\alpha \in (0,\frac{s}{2})$. En effet, $\mathbb{1}_{(a,b)}$ est dans $H^{\frac{1-s}{2}}$ pour tout $s \in (0,1)$ mais $\mathbb{1}_{(a,b)} \notin H^{\frac{1}{2}}$. Ainsi pour $s \in (0,1)$ la statistique linéaire associée est du même ordre de grandeur que pour des fonctions-test lisse contrairement au cas $s = 0$. Enfin notons que $|x|^{-\frac{s}{2}}$ est la puissance critique qui n'est pas dans $H^{\frac{1-s}{2}}$ et on s'attend alors à ce qu'un TCL de type log-corrélé soit vérifié à l'instar de $\mathbb{1}_{(a,b)}$ pour $s = 0$.

Une fois ces questions de fluctuations élucidées, on aborde le problème (3) sur la décroissance de la corrélations des gaps. Le théorème 4 montre que les particules $x_i - x_1$ et $x_k - x_1$ sont très corrélées avec une covariance de taille $|i-k|^s$. Ainsi pour montrer une forme de décroissance des corrélations il est très naturel de considérer les variables $N(x_{i+1}-x_i)$ plutôt que les variables x_i. On note d la distance symétrique sur $\{1,\dots,N\}$, i.e pour tout $1 \leqslant i,j \leqslant N$, $d(i,j) = \min(|j-i|, N-|j-i|)$. Nous obtenons dans [53] le résultat ci-dessous :

Théorème 5 (Décroissance des corrélations). *Soit $s \in (0,1)$. Pour tout $\varepsilon > 0$, il existe une constant $C > 0$ telle que pour tout $f,g : \mathbb{R} \to \mathbb{R}$ dans H^1 et pour tout $i,j \in \{1,\dots,N\}$,*

$$\begin{aligned}&|\mathrm{Cov}_{\mathbb{P}_{N,\beta}}[f(N(x_{i+1}-x_i)), h(N(x_{j+1}-x_j))]| \\ &\quad \leqslant C(\mathbb{E}_{\mathbb{P}_{N,\beta}}[f'(x_i)^2]^{\frac{1}{2}} + |f'|_\infty e^{-c(\beta)d(i,j)^\delta})(\mathbb{E}_{\mathbb{P}_{N,\beta}}[h'(x_j)^2]^{\frac{1}{2}} + |h'|_\infty e^{-c(\beta)d(i,j)^\delta})\frac{1}{d(i,j)^{2-s-\varepsilon}}.\end{aligned} \tag{1.36}$$

De plus, étant donné $\varepsilon > 0$ assez petit et $n \in \{1,\dots,N\}$, il existe i,j tels que $\frac{n}{2} \leqslant |i-j| \leqslant n$ et

$$|\mathrm{Cov}_{\mathbb{P}_{N,\beta}}[N(x_{i+1}-x_i), N(x_{j+1}-x_j)]| \geqslant \varepsilon\frac{1}{d(i,j)^{2-s}}.$$

Soit $s \in (1,+\infty)$. Alors pour tout $\varepsilon > 0$, il existe une constant $C > 0$ telle que pour tout $f,g : \mathbb{R} \to \mathbb{R}$ dans H^1 et pour tout $i,j \in \{1,\dots,N\}$,

$$\begin{aligned}&|\mathrm{Cov}_{\mathbb{P}_{N,\beta}}[f(N(x_{i+1}-x_i)), h(N(x_{j+1}-x_j))]| \\ &\leqslant C(\mathbb{E}_{\mathbb{P}_{N,\beta}}[f'(x_i)^2]^{\frac{1}{2}} + |f'|_\infty e^{-c(\beta)d(i,j)^\delta})(\mathbb{E}_{\mathbb{P}_{N,\beta}}[h'(x_j)^2]^{\frac{1}{2}} + |h'|_\infty e^{-c(\beta)d(i,j)^\delta})\Big(\frac{1}{d(i,j)^{s-\varepsilon}} + \frac{1}{N}\Big).\end{aligned} \tag{1.37}$$

En particulier la covariance entre $N(x_{i+1}-x_i)$ et $N(x_{j+1}-x_j)$ décroît en $d(i,j)^{-(2-s)}$, ce qui coïncide dans la limite où s tend vers 0 avec le cas du log-gaz étudié dans [116]. Par ailleurs ce résultat est cohérent avec le théorème 4 : étant donnés deux points à distance d, la corrélation point-point est de taille d^s, la corrélation point-gap de taille d^{s-1} (une dérivée de prise) et la corrélation gap-gap de taille d^{s-2} (deux dérivées de prises). Le théorème 5 met en lumière un phénomène quelque peu surprenant : la corrélation entre les gaps est croissante en s dans le régime longue portée bien que lorsque s augmente, la portée de l'énergie diminue. En physique statistique il est fréquent que la présence de grandes fluctuations impliquent une décroissance rapide des corrélations, ce qui est au coeur de l'argument de Mermin-Wagner sur l'absence de transition de phase du premier ordre pour les systèmes à symétries continues en dimension 2. Dans notre modèle à longue portée c'est bien le contraire qui se produit : plus les fluctuations sont grandes, plus la décroissance des corrélations est lente.

L'approche proposée dans [53] est de donner une preuve du Théorème 5 reposant uniquement sur l'analyse de l'équation de Helffer-Sjöstrand dans sa version statique (1.33). L'une des spécificités de cette équation est que l'on ne peut utiliser que des inégalités adimensionnelles (puisque N tend vers l'infini). En d'autres termes les seules opérations licites sont les intégrations par parties et les principes du maximum. On donne ainsi dans [53] une preuve simple de la décroissance des corrélations dans le cas $s \in (0,1)$ qui n'utilise pas de représentation en environnement aléatoire [116] ni de méthode d'homogénisation [10].

Avec des estimées de décorrélation du type du théorème 5, il est relativement aisé d'obtenir l'unicité des processus limite, comme énoncé dans le résultat ci-dessous :

Théorème 6 (Unicité de la mesure limite)**.** *Soit* $s \in (0,1) \cup (1,+\infty)$. *Il existe un processus ponctuel* $\mathrm{Riesz}_{s,\beta}$ *tel que pour tout* $x \in \mathbb{T}$*, la suite des processus ponctuels* $(\mathbb{Q}_{N,\beta}(x))$ *converge vers* $\mathrm{Riesz}_{s,\beta}$ *dans la topologie de la convergence locale : pour toute fonction borélienne et locale* $\phi : \mathrm{Conf}(\mathbb{R}) \to \mathbb{R}$*, on a*

$$\lim_{N\to\infty} \mathbb{E}_{\mathbb{Q}_{N,\beta}(x)}[\phi] = \mathbb{E}_{\mathrm{Riesz}_{s,\beta}}[\phi].$$

Ceci permet alors de définir un processus en volume infini $\mathrm{Riesz}_{s,\beta}$ qui généralise le processus Sine_β, voir 1.1.9. A la différence du processus Sine_β, il semble que le processus $\mathrm{Riesz}_{s,\beta}$ ne puisse être défini que comme l'unique limite des processus ponctuels.

1.3.3 L'énergie libre du gaz de Coulomb à deux composantes

La dernière partie de ce manuscrit réalisée en collaboration avec S. Serfaty a pour objet le gaz de Coulomb bi-dimensionnel à deux composantes introduit dans le paragraphe 1.1.7. On s'intéresse au régime de température $\beta \in (2,+\infty)$. Comme on l'a vu, la fonction de partition $Z_{N,\beta}$ n'est pas convergente et il convient de tronquer l'interaction à une certaine échelle $\eta := \frac{\lambda}{\sqrt{N}}$ où $\lambda > 0$ est un paramètre petit. Pour manipuler l'énergie, il est préférable, au lieu de tronquer, d'étaler les charges ponctuelles δ_x en des mesures uniformes de masse 1 sur des disques de rayon η, notées $\delta_x^{(\eta)}$.

On se donne N charges positives $x_1, \ldots, x_N$ et N charges négatives $y_1, \ldots, y_N$ dans $\Lambda := [0,\sqrt{N}]^2$, que l'on notera $z_1, \ldots, z_{2N}$ avec d_i le signe de z_i, i.e $d_i = 1$ si $1 \leqslant i \leqslant N$ et $d_i = -1$ si $N+1 \leqslant i \leqslant 2N$. On considère alors l'énergie

$$H_{N,\lambda} = \frac{1}{2}\sum_{i\neq j} \iint d_i d_j g(x-y)\delta_{z_i}^{(\eta)}(x)\delta_{z_j}^{(\eta)}(y) \tag{1.38}$$

ainsi que la mesure de probabilité

$$\mathbb{P}_{N,\beta}^{\lambda} = \frac{1}{Z_{N,\beta}^{\lambda}} e^{-\beta H_{N,\lambda}(X_N,Y_N)} \mathrm{d}x_1 \ldots, \mathrm{d}x_N \mathrm{d}y_1 \ldots \mathrm{d}y_N, \tag{1.39}$$

où

$$Z_{N,\beta}^{\lambda} = \int e^{-\beta H_{N,\lambda}(X_N,Y_N)} \mathrm{d}x_1 \ldots, \mathrm{d}x_N \mathrm{d}y_1 \ldots \mathrm{d}y_N.$$

On notera $g_\lambda(z)$ l'interaction effective de points à distance z :

$$g_\lambda(z) = \iint g(x-y)\delta_0^{(\lambda)}(x)\delta_z^{(\lambda)}(y). \tag{1.40}$$

Le gaz de Coulomb à deux composantes (2CP) est notamment étudié dans le régime $\beta \in (0,2)$ dans [186]. Un PGD pour le champ empirique y est donné, ainsi qu'un développement de la fonction de partition. L'objectif de notre travail est de décrire microscopiquement le plasma à deux composantes dans le régime $\beta \in (2,+\infty)$ et de montrer de façon quantitative la formation de dipôles. Plus précisément on s'intéresse aux questions suivantes :

1. Montrer que sous $\mathbb{P}_{N,\beta}^{\lambda}$ le système s'organise en une majorité de dipôles neutres de taille η. Quantifier cette proportion de bons dipôles.
2. Monter que l'interaction de l'assemblée de dipôles est bornée par une quantité proportionelle au nombre de points, avec un facteur qui tend vers 0 quand λ tend vers 0.
3. En déduire qu'à l'ordre principal lorsque $N \to \infty$ et $\lambda \to 0$, le système se concentre autour d'un processus Poissonien de dipôles, où les charges positives sont tirées indépendamment, avec une charge négative accrochée à chacune d'elles à distance typique η.

On introduit $(z_1,\ldots,z_{2N}) = (x_1,\ldots,x_N,y_1,\ldots,y_{2N})$ et pour tout $i = \{1,\ldots,2N\}$, on note d_i le signe de z_i, i.e $d_i = 1$ si $i \in \{1,\ldots,N\}$ et $d_i = -1$ si $i \in \{N+1,\ldots,2N\}$. Pour tout $i \in \{1,\ldots,2N\}$, on notera également $\phi_1(i)$ l'indice du plus proche voisin de z_i. Pour $\lambda > 0$ on définit

$$\gamma_\lambda := \begin{cases} \frac{1}{|\log \lambda|} & \text{if } \beta = 2 \\ \lambda^{\beta-2} & \text{if } \beta \in (2,4) \\ \lambda^2 |\log\lambda|^2 & \text{if } \beta = 4 \\ \lambda^2|\log\lambda| & \text{if } \beta > 4. \end{cases}$$

Théorème 7. *Soit $\beta \in [2,+\infty)$.*

1. *Il existe une constante explicite $C_\beta > 0$ telle que*

$$\log Z_{N,\beta}^{\lambda} = 2N\log N + N((2-\beta)\log\lambda \mathbf{1}_{\beta>2} + \log|\log\lambda|\mathbf{1}_{\beta=2}) - N + N\log C_\beta \mathbf{1}_{\beta>2} + O(N\gamma_\lambda).$$

2. *Soit*

$$I := \{1 \leqslant i \leqslant N : \phi_1 \circ \phi_1(i) = i, d_i d_{\phi_1(i)} = -1\}.$$

Pour tout $|t| \leqslant \frac{\beta}{2}$, on a

$$\log \mathbb{E}_{\mathbb{P}_{N,\beta}^{\lambda}}\Big[\exp\Big(t\mathsf{F}_\lambda - t\sum_{i\in I} \mathsf{g}_\lambda(z_i - z_{\phi_1(i)})\Big)\Big] \leqslant CN\gamma_\lambda.$$

3. *La mesure de Gibbs se concentre sur des dipôles de taille λ. En effet en notant*

$$D := \Big\{(X_N,Y_N),\ |I| \geqslant N(1-c\gamma_\lambda) - \sum_{i\in I} g_\lambda(\lambda^{-1}|z_i - z_{\phi_1(i)}|) \leqslant MN\Big\},$$

on a

$$\mathbb{P}_{N,\beta}^{\lambda}(D^c) \leqslant \exp(-CN),$$

pour $c > 0$, $C > 0$ et $M > 0$ indépendantes de N et λ.

La preuve de ce résultat repose sur des techniques de grandes déviations inspirées de [139] et de [186] et sur une méthode de minoration de l'énergie (1.38). Dans le cas du gaz de Coulomb à une composante, comme expliqué dans le paragraphe 1.2.2, l'énergie est monotone par rapport au paramètre de troncature : elle décroît quand le rayon des disques des charges étalées augmente. Cette propriété de monotonie n'est plus vraie pour le plasma à deux composantes mais en utilisant le théorème de Newton, on peut toutefois calculer de façon explicite l'erreur faite en changeant le rayon des disques dans (1.38). En augmentant ces rayons jusqu'à la semi-distance aux seconds plus proche voisins (resp. $2p$ plus proches voisins), on peut ainsi isoler les interactions à plus proches voisins qui sont dominantes dans l'énergie. En suivant [139] on peut ensuite décomposer l'espace des phases en fonction de la classe d'isomorphisme du graphe des plus proches voisins et montrer que le système se concentre sur des configurations ayant une majorité de dipôles neutres.

Nous établissons ensuite une inégalité énergétique donnant un contrôle uniforme sur les fluctuations des statsitiques linéaires pour des fonction-tests lipschitziennes.

Théorème 8. *Etant donnée une fonction lipschitzienne $\xi : \Lambda \to \mathbb{R}$, on note*

$$\mathrm{Fluct}_N(\xi) := \int_\Lambda \xi \Big(\sum_{i=1}^N (\delta_{x_i} - \delta_{y_i}) \Big).$$

Soit

$$\alpha_\lambda = \begin{cases} \lambda^{\frac{2(\beta-2)}{\beta}} & \textit{si } \beta \in (2,4) \\ \lambda |\log \lambda|^{1/2} & \textit{si } \beta = 4. \\ \lambda & \textit{si } \beta \in (4,\infty) \end{cases} \tag{1.41}$$

Alors, il existe une constante $C > 0$ telle que

$$\log \mathbb{E}_{\mathbb{P}^\lambda_{N,\beta}}[\exp((\mathrm{Fluct}_N(\xi))^2)] \leqslant C N \alpha_\lambda \|\nabla \xi\|^2_{L^\infty}.$$

L'estimée ci-dessus est obtenue en contrôlant les moments exponentiels de l'énergie (1.38) pour des charges étalées sur des disques de rayons aléatoires de taille typique $\alpha_\lambda^{1/2}$. A nouveau, nous calculons la variation d'énergie résultant de cet étalement des charges et concluons avec les développement des fonctions de partition obtenus dans le Théorème 7.

1.3.4 Perspectives de recherche

.

Une question naturelle est d'étendre le cutoff du Théorème 2 aux β-ensembles non-Gaussiens, en commençant par exemple par le cas où le potentiel extérieur V est uniformément convexe mais non quadratique. La borne supérieure donnée par l'inégalité de Log-Sobolev (1.32) ne coïncide a priori plus avec le trou spectral du générateur et il convient d'utiliser d'autres arguments. Au vu du théorème 1, il semble intéressant d'étudier le temps de mélange du mouvement Brownien de Dyson lorsque la condition initiale est très proche de l'équilibre, par exemple lorsque chaque z_0^i est placé en un quantile de la mesure d'équilibre. Il paraît crédible que le temps de mélange soit minoré par $c\log(n)$ pour une constant $c > 0$ indépendante de z_0^n et peut-être même universelle dans la classe des β-ensembles. Ce temps critique correspondrait au temps nécessaire pour étaler des conditions initiales ponctuelles.

En ce qui concerne l'analyse des gaz de Riesz, il serait naturel et intéressant d'étudier les dimensions supérieures. Une première étape est d'obtenir des lois locales dans le cas longue portée non coulombien. En raison du caractère non local du Laplacien fractionnaire, les méthodes de [180, 9] ne s'adaptent pas de façon immédiate. Une question particulièrement intrigante est celle de la décroissance des corrélations en dimension strictement plus grande que 1. Au vu de la preuve de [53] il est envisageable que certains arguments se passent de convexité et que le caractère longue portée de l'interaction, ainsi que la positivité de la transformée de Fourier du noyau de Riesz, permettent d'établir un résultat faible de décroissance des corrélations. Le cas du gaz de Riesz hypersingulier pourrait également être traité en adaptant les méthodes de [213] développées pour le gaz de sphères dures.

Le chapitre 5 de ce manuscrit peut se prolonger de nombreuses manières. Une première piste est d'affiner les asymptotiques du théorème 7 de façon à pouvoir observer les transitions liées à l'apparition des multipôles [170]. Une deuxième direction serait de mieux comprendre mathématiquement la transition KT, qui a lieu à $\beta = 4$. Cette transition devrait se voir dans la variation du nombre de dipôles non-neutres à une certaine échelle lorsque l'échelle augmente. Une étape importante dans cette direction est ainsi de comprendre la fluctuation de la charge dans le plasma à deux composantes. Pour commencer il serait intéressant d'étudier les fluctuations des statistiques linéaires lisses dans le cas $\beta \in (0, 2)$ et de montrer que celles-ci ont une amplitude en $o(\sqrt{N})$. Cette question est assez originale puisqu'à la différence du plasma à une composante [24, 181], la réponse linéaire associée à une statistique linéaire $\sum_{i=1}^N \xi(x_i) - \sum_{i=1}^N \xi(y_i)$ ne peut pas se chercher sous la forme d'un transport qui agit diagonalement, voir le paragraphe (1.2.3), et il convient alors d'imaginer une autre stratégie de preuve.

Chapter 2

Universal cutoff for Dyson Ornstein Uhlenbeck process

This chapter is based on the article Universal cutoff for Dyson Ornstein Uhlenbeck process, written in collaboration with D. Chafaï and C. Labbé, accepted for publication in *Probability Theory and Related Fields*.

Contents

2.1 Introduction and main results

Let us consider a Markov process $X = (X_t)_{t\geqslant 0}$ with state space S and invariant law μ for which

$$\lim_{t\to\infty} \mathrm{dist}(\mathrm{Law}(X_t) \mid \mu) = 0$$

where $\mathrm{dist}(\cdot \mid \cdot)$ is a distance or divergence on the probability measures on S. Suppose now that $X = X^n$ depends on a dimension, size, or complexity parameter n, and let us set $S = S^n$, $\mu = \mu^n$, and $X_0 = x_0^n \in S^n$. For example X^n can be a random walk on the symmetric group of permutations of $\{1, \dots, n\}$, Brownian motion on the group of $n \times n$ unitary matrices, Brownian motion on the n-dimensional sphere, etc. In many of such examples, it has been proved that when n is large enough, the supremum over some set of initial conditions x_0^n of the quantity $\mathrm{dist}(\mathrm{Law}(X_t^n) \mid \mu^n)$ collapses abruptly to 0 when t passes a critical value $c = c_n$ which may depend on n. This is often referred to as a *cutoff phenomenon*. More precisely, if dist ranges from 0 to $\max$, then, for some subset $S_0^n \subset S^n$ of initial conditions, some critical value $c = c_n$ and for all $\varepsilon \in (0, 1)$,

$$\lim_{n\to\infty} \sup_{x_0^n \in S_0^n} \mathrm{dist}(\mathrm{Law}(X_{t_n}^n) \mid \mu^n) = \begin{cases} \max & \text{if } t_n = (1-\varepsilon)c_n \\ 0 & \text{if } t_n = (1+\varepsilon)c_n \end{cases}.$$

It is standard to introduce, for an arbitrary small threshold $\eta > 0$, the quantity $\inf\{t \geqslant 0 : \sup_{x_0 \in S_0^n} \mathrm{dist}(\mathrm{Law}(X_t^n) \mid \mu^n) \leqslant \eta\}$ known as the *mixing time* in the literature. Of course such a definition fully makes sense as soon as $t \mapsto \sup_{x_0 \in S_0^n} \mathrm{dist}(\mathrm{Law}(X_t^n) \mid \mu^n)$ is non-increasing.

When S^n is finite, it is customary to take $S_0^n = S^n$. When S^n is infinite, it may happen that the supremum over the whole set S^n of the distance to equilibrium remains equal to $\max$ at all times, in which case one has to consider strict subspaces of initial conditions. For some processes, it is possible to restrict S_0^n to a single state in which case one obtains a very precise description of the convergence to equilibrium starting from this initial condition. Note that the constraint over the initial condition can be made compatible with a limiting dynamics, for instance a mean-field limit when the process describes an exchangeable interacting particle system.

The *cutoff phenomenon* was put forward by Aldous and Diaconis at the origin for random walks on finite sets, see for instance [3, 99, 84, 190] and references therein. The analysis of the cutoff phenomenon is the subject of an important activity, still seeking for a complete theory: let us mention that, for the total variation distance, Peres proposed the so-called product condition (the mixing time must be much larger than the inverse of the spectral gap) as a necessary and sufficient condition for a cutoff phenomenon to hold, but counter-examples were exhibited [190, Sec. 18.3] and the product condition is only necessary.

The study of the cutoff phenomenon for Markov diffusion processes goes back at least to the works of Saloff-Coste [224, 226] in relation notably with Nash–Sobolev type functional inequalities, heat kernel analysis, and Diaconis–Wilson probabilistic techniques. We also refer to the more recent work [199] for the case of diffusion processes on compact groups and symmetric spaces, in relation with group invariance and representation theory, a point of view inspired by the early works of Diaconis on Markov chains and of Saloff-Coste on diffusion processes. Even if most of the available results in the literature on the cutoff phenomenon are related to compact state spaces, there are some notable works devoted to non-compact spaces such as [168, 68, 22, 20, 21, 23].

Our contribution is an exploration of the cutoff phenomenon for the Dyson–Ornstein–Uhlenbeck diffusion process, for which the state space is $\mathbb{R}^n$. This process is an interacting particle system. When the interaction is turned off, we recover the Ornstein–Uhlenbeck process, a special case that has been considered previously in the literature but for which we also provide new results.

2.1.1 Distances

As for dist we use several standard distances or divergences between probability measures: total variation (denoted TV), Hellinger, relative entropy (denoted Kullback), relative variance (denoted χ^2), Wasserstein of order 2, and Fisher information, surveyed in Appendix 2.8.1. We take the following convention for probability measures μ and ν on the same space:

$$\mathrm{dist}(\mu \mid \nu) = \begin{cases} \|\mu-\nu\|_{\mathrm{TV}} & \text{when } \mathrm{dist} = \mathrm{TV} \\ \mathrm{Hellinger}(\mu,\nu) & \text{when } \mathrm{dist} = \mathrm{Hellinger} \\ \mathrm{Kullback}(\mu \mid \nu) & \text{when } \mathrm{dist} = \mathrm{Kullback} \\ \chi^2(\mu \mid \nu) & \text{when } \mathrm{dist} = \chi^2 \\ \mathrm{Wasserstein}(\mu,\nu) & \text{when } \mathrm{dist} = \mathrm{Wasserstein} \\ \mathrm{Fisher}(\mu \mid \nu) & \text{when } \mathrm{dist} = \mathrm{Fisher} \end{cases}, \tag{2.1}$$

see Appendix 2.8.1 for precise definitions. The maximal value $\max$ taken by dist is given by

$$\max = \begin{cases} 1 & \text{if } \mathrm{dist} \in \{\mathrm{TV}, \mathrm{Hellinger}\}, \\ +\infty & \text{if } \mathrm{dist} \in \{\mathrm{Kullback}, \chi^2, \mathrm{Wasserstein}, \mathrm{Fisher}\}. \end{cases} \tag{2.2}$$

2.1.2 The Dyson–Ornstein–Uhlenbeck (DOU) process and preview of main results

The DOU process is the solution $X^n = (X^n_t)_{t\geqslant 0}$ on $\mathbb{R}^n$ of the stochastic differential equation

$$X^n_0 = x^n_0 \in \mathbb{R}^n, \quad \mathrm{d}X^{n,i}_t = \sqrt{\frac{2}{n}}\mathrm{d}B^i_t - V'(X^{n,i}_t)\mathrm{d}t + \frac{\beta}{n}\sum_{j\neq i}\frac{\mathrm{d}t}{X^{n,i}_t - X^{n,j}_t}, \quad 1 \leqslant i \leqslant n, \tag{2.3}$$

where $(B_t)_{t\geqslant 0}$ is a standard n-dimensional Brownian motion (BM), and where

- $V(x) = \frac{x^2}{2}$ is a "confinement potential" acting through the drift $-V'(x) = -x$
- $\beta \geqslant 0$ is a parameter tuning the interaction strength.

The notation $X^{n,i}_t$ stands for the i-th coordinate of the vector X^n_t. The process X^n can be thought of as an interacting particle system of n one-dimensional Brownian particles $X^{n,1}, \ldots, X^{n,n}$, subject to confinement and singular pairwise repulsion when $\beta > 0$ (respectively first and second term in the drift). We take an inverse temperature of order n in (2.3) in order to obtain a mean-field limit without time-changing the process, see Section 2.2.5. The spectral gap is 1 for all $n \geqslant 1$, see Section 2.2.6. We refer to Section 2.2.9 for other parametrizations or choices of inverse temperature.

In the special cases $\beta \in \{0,1,2\}$, the cutoff phenomenon for the DOU process can be established by using Gaussian analysis and stochastic calculus, see Sections 2.1.4 and 2.1.5. For $\beta = 0$, the process reduces to the Ornstein–Uhlenbeck process (OU) and its behavior serves as a benchmark for the interaction case $\beta \neq 0$, while when $\beta \in \{1,2\}$, the approach involves a lift to unitary invariant ensembles of random matrix theory. For a general $\beta \geqslant 1$, our main results regarding the cutoff phenomenon for the DOU process are given in Sections 2.1.6 and 2.1.7. We are able, in particular, to prove the following: for all $\mathrm{dist} \in \{\mathrm{TV}, \mathrm{Hellinger}, \mathrm{Wasserstein}\}$, $a > 0$, $\varepsilon \in (0,1)$, we have

$$\lim_{n\to\infty} \sup_{x_0^n \in [-a,a]^n} \mathrm{dist}(\mathrm{Law}(X_{t_n}^n) \mid P_n^\beta) = \begin{cases} \max & \text{if } t_n = (1-\varepsilon)c_n \\ 0 & \text{if } t_n = (1+\varepsilon)c_n \end{cases},$$

where P_n^β is the invariant law of the process, and where

$$c_n := \begin{cases} \log(\sqrt{n}a) & \text{if dist} = \mathrm{Wasserstein} \\ \log(na) & \text{if dist} \in \{\mathrm{TV}, \mathrm{Hellinger}\} \end{cases}.$$

This result is stated in a slightly more general form in Corollary 2.1.7. Our proof relies crucially on an exceptional exact solvability of the dynamics, notably the fact that we know explicitly the optimal long time behavior in entropy and coupling distance, as well as the eigenfunction associated to the spectral gap which turns out to be linear and optimal. This comes from the special choice of V as well as the special properties of the Coulomb interaction. We stress that such an exact solvability is no longer available for a general strongly convex V, even for instance in the simple example $V(x) = \frac{x^2}{2} + x^4$ or for general linear forces. Nevertheless, and as usual, two other special classical choices of V could be explored, related to Laguerre and Jacobi weights, see Section 2.2.8.

2.1.3 Analysis of the Dyson–Ornstein–Uhlenbeck process

The process X^n was essentially discovered by Dyson in [106], in the case $\beta \in \{1,2,4\}$, because it describes the dynamics of the eigenvalues of $n \times n$ symmetric/Hermitian/symplectic random matrices with independent Ornstein–Uhlenbeck entries, see Lemma 2.5.1 and Lemma 2.5.2 below for the cases $\beta = 1$ and $\beta = 2$ respectively.

- Case $\beta = 0$ (interaction turned off). The particles become n independent one-dimensional Ornstein–Uhlenbeck processes, and the DOU process X^n becomes exactly the n-dimensional Ornstein–Uhlenbeck process Z^n solving (2.8). The process lives in $\mathbb{R}^n$. The particles collide but since they do not interact, this does not raise any issue.

- Case $0 < \beta < 1$. Then with positive probability the particles collide producing a blow up of the drift, see for instance [74, 81] for a discussion. Nevertheless, it is possible to define the process for all times, for instance by adding a local time term to the stochastic differential equation, see [81] and references therein. It is natural to expect that the cutoff universality works as for $\beta \notin (0,1)$, but for simplicity we do not consider this case here.

- Case $\beta \geqslant 1$. If we order the coordinates by defining the convex domain

  $$D_n = \{x \in \mathbb{R}^n : x_1 < \cdots < x_n\},$$

  and if $x_0^n \in D_n$ then the equation (2.3) admits a unique strong solution that never exits D_n, in other words the particles never collide and the order of the initial particles is preserved at all times, see [216]. Moreover if

  $$\overline{D}_n = \{x \in \mathbb{R}^n : x_1 \leqslant \cdots \leqslant x_n\}$$

then it is possible to start the process from the boundary $\overline{D}_n \backslash D_n$, in particular from x_0^n such that $x_0^{n,1} = \cdots = x_0^{n,n}$, and despite the singularity of the drift, it can be shown that with probability one, $X_t^n \in D_n$ for all $t > 0$. We refer to [6, Th. 4.3.2] for a proof in the Dyson Brownian Motion case that can be adapted *mutatis mutandis*.

In the sequel, we will only consider the cases $\beta = 0$ with $x_0^n \in \mathbb{R}^n$ and $\beta \geqslant 1$ with $x_0^n \in \overline{D}_n$. The drift in (2.3) is the gradient of a function, and (2.3) rewrites

$$X_0^n = x_0^n \in D_n, \quad \mathrm{d}X_t^n = \sqrt{\frac{2}{n}}\mathrm{d}B_t - \frac{1}{n}\nabla E(X_t^n)\mathrm{d}t, \tag{2.4}$$

where

$$E(x_1, \ldots, x_n) = n\sum_{i=1}^{n} V(x_i) + \beta\sum_{i>j}\log\frac{1}{|x_i - x_j|} \tag{2.5}$$

can be interpreted as the energy of the configuration of particles $x_1, \ldots, x_n$.

- If $\beta = 0$, then the Markov process X^n is an Ornstein–Uhlenbeck process, irreducible with unique invariant law $P_n^0 = \mathcal{N}(0, \frac{1}{n}I_n)$ which is reversible.
- If $\beta \geqslant 1$, then the Markov process X^n is not irreducible, but D_n is a recurrent class carrying a unique invariant law P_n^β, which is reversible and given by

$$P_n^\beta = \frac{\mathrm{e}^{-E(x_1,\ldots,x_n)}}{C_n^\beta}\mathbf{1}_{(x_1,\ldots,x_n)\in\overline{D}_n}\mathrm{d}x_1\cdots\mathrm{d}x_n, \tag{2.6}$$

where C_n^β is the normalizing factor given by

$$C_n^\beta = \int_{\overline{D}_n} \mathrm{e}^{-E(x_1,\ldots,x_n)}\mathrm{d}x_1\cdots\mathrm{d}x_n. \tag{2.7}$$

In terms of geometry, it is crucial to observe that since $-\log$ is convex on $(0, +\infty)$, the map

$$(x_1, \ldots, x_n) \in D_n \mapsto \mathrm{Interaction}(x_1, \ldots, x_n) = \beta\sum_{i>j}\log\frac{1}{x_i - x_j},$$

is convex. Thus, since V is convex on $\mathbb{R}$, it follows that E is convex on D_n. For all $\beta \geqslant 0$, the law P_n^β is log-concave with respect to the Lebesgue measure as well as with respect to $\mathcal{N}(0, \frac{1}{n}I_n)$.

2.1.4 Non-interacting case and Ornstein–Uhlenbeck benchmark

When we turn off the interaction by taking $\beta = 0$ in (2.3), the DOU process becomes an Ornstein–Uhlenbeck process (OU) $Z^n = (Z_t^n)_{t\geqslant 0}$ on $\mathbb{R}^n$ solving the stochastic differential equation

$$Z_0^n = z_0^n \in \mathbb{R}^n, \quad \mathrm{d}Z_t^n = \sqrt{\frac{2}{n}}\mathrm{d}B_t^n - Z_t^n\mathrm{d}t, \tag{2.8}$$

where B^n is a standard n-dimensional BM. The invariant law of Z^n is the product Gaussian law $P_n^0 = \mathcal{N}(0, \frac{1}{n}I_n) = \mathcal{N}(0, \frac{1}{n})^{\otimes n}$. The explicit Gaussian nature of $Z_t^n \sim \mathcal{N}(z_0^n\mathrm{e}^{-t}, \frac{1-\mathrm{e}^{-2t}}{n}I_n)$, valid for all $t \geqslant 0$, allows for a fine analysis of convergence to equilibrium, as in the following theorem.

Theorem 2.1.1 (Cutoff for OU: mean-field regime). *Let $Z^n = (Z^n_t)_{t\geqslant 0}$ be the OU process* (2.8) *and let P^0_n be its invariant law. Suppose that*

$$\varliminf_{n\to\infty} \frac{|z^n_0|^2}{n} > 0 \quad \textit{and} \quad \varlimsup_{n\to\infty} \frac{|z^n_0|^2}{n} < \infty$$

where $|z| = \sqrt{z_1^2 + \cdots + z_n^2}$ is the Euclidean norm. Then for all $\varepsilon \in (0,1)$,

$$\lim_{n\to\infty} \mathrm{dist}(\mathrm{Law}(Z^n_{t_n}) \mid P^0_n) = \begin{cases} \max & \textit{if } t_n = (1-\varepsilon)c_n \\ 0 & \textit{if } t_n = (1+\varepsilon)c_n \end{cases}$$

where

$$c_n = \begin{cases} \frac{1}{2}\log(n) & \textit{if } \mathrm{dist} = \mathrm{Wasserstein}, \\ \log(n) & \textit{if } \mathrm{dist} \in \{\mathrm{TV}, \mathrm{Hellinger}, \mathrm{Kullback}, \chi^2\}, \\ \frac{3}{2}\log(n) & \textit{if } \mathrm{dist} = \mathrm{Fisher}. \end{cases}$$

Theorem 2.1.1 is proved in Section 2.3. See Figure 2.1 and Figure 2.2 for a numerical experiment.

Theorem 2.1.1 constitutes a very natural benchmark for the cutoff phenomenon for the DOU process. Theorem 2.1.1 is not a surprise, and actually the TV and Hellinger cases are already considered in [168], see also [19]. Let us mention that in [21], a cutoff phenomenon for TV, entropy and Wasserstein is proven for the OU process of *fixed* dimension d and vanishing noise. This is to be compared with our setting where the dimension is sent to infinity: the results (and their proofs) are essentially the same in these two situations, however we will see below that if one considers more general initial conditions, there are some substantial differences according to whether the dimension is fixed or sent to infinity.

The restriction over the initial condition in Theorem 2.1.1 is spelled out in terms of the second moment of the empirical distribution, a natural choice suggested by the mean-field limit discussed in Section 2.2.5. It yields a mixing time of order $\log(n)$, just like for Brownian motion on compact Lie groups, see [226, 199]. For the OU process and more generally for overdamped Langevin processes, the non-compactness of the space is replaced by the confinement or tightness due to the drift.

Actually, Theorem 2.1.1 is a particular instance of the following, much more general result that reveals that, except for the Wasserstein distance, a cutoff phenomenon *always* occurs.

Theorem 2.1.2 (General cutoff for OU). *Let $Z^n = (Z^n_t)_{t\geqslant 0}$ be the OU process* (2.8) *and let P^0_n be its invariant law. Let $\mathrm{dist} \in \{\mathrm{TV}, \mathrm{Hellinger}, \mathrm{Kullback}, \chi^2, \mathrm{Fisher}\}$. Then, for all $\varepsilon \in (0,1)$,*

$$\lim_{n\to\infty} \mathrm{dist}(\mathrm{Law}(Z^n_{t_n}) \mid P^0_n) = \begin{cases} \max & \textit{if } t_n = (1-\varepsilon)c_n, \\ 0 & \textit{if } t_n = (1+\varepsilon)c_n \end{cases}$$

where

$$c_n = \begin{cases} \log(\sqrt{n}|z^n_0|) \vee \frac{1}{4}\log(n) & \textit{if } \mathrm{dist} \in \{\mathrm{TV}, \mathrm{Hellinger}, \mathrm{Kullback}, \chi^2\}, \\ \log(n|z^n_0|) \vee \frac{1}{2}\log(n) & \textit{if } \mathrm{dist} = \mathrm{Fisher}. \end{cases}$$

Regarding the Wasserstein distance, the following dichotomy occurs:

- *if $\lim_{n\to\infty} |z^n_0| = +\infty$, then for all $\varepsilon \in (0,1)$, with $c_n = \log|z^n_0|$,*

$$\lim_{n\to\infty} \mathrm{Wasserstein}(\mathrm{Law}(Z_{t_n}), P^0_n) = \begin{cases} +\infty & \textit{if } t_n = (1-\varepsilon)c_n, \\ 0 & \textit{if } t_n = (1+\varepsilon)c_n, \end{cases}$$

- *if* $\lim_{n\to\infty}|z_0^n| = \alpha \in [0,\infty)$ *then there is* no cutoff phenomenon *namely for any* $t > 0$

$$\lim_{n\to\infty} \mathrm{Wasserstein}^2(\mathrm{Law}(Z_t), P_n^0) = \alpha^2 \mathrm{e}^{-2t} + 2\Big(1 - \sqrt{1-\mathrm{e}^{-2t}} - \tfrac{1}{2}\mathrm{e}^{-2t}\Big).$$

Theorem 2.1.2 is proved in Section 2.3.

The observation that for every distance or divergence, except for the Wasserstein distance, a cutoff phenomenon occurs *generically* seems to be new.

Let us make a few comments. First, in terms of convergence to equilibrium the relevant observable in Theorem 2.1.2 appears to be the Euclidean norm $|z_0^n|$ of the initial condition. This quantity differs from the eigenfunction associated to the spectral gap of the generator, which is given by $z_1 + \cdots + z_n$ as we will recall later on. This is also related to the equality of (2.20) and (2.41). Second, cutoff occurs at a time that is *independent* of the initial condition provided that its Euclidean norm is small enough: this cutoff time appears as the time required to regularize the initial condition (a Dirac mass) into a sufficiently spread out absolutely continuous probability measure; in particular this cutoff phenomenon would not hold generically if we allowed for spread out (non-Dirac) initial conditions. Note that, for the OU process of *fixed* dimension and vanishing noise, we would not observe a cutoff phenomenon when starting from initial conditions with small enough Euclidean norm: this is a high dimensional phenomenon. In this respect, the Wasserstein distance is peculiar since it is much less stringent on the local behavior of the measures at stake: for instance $\lim_{n\to\infty} \mathrm{Wasserstein}(\delta_0, \delta_{1/n}) = 0$ while for all other distances or divergences considered here, the corresponding quantity would remain equal to $\max$. This explains the absence of *generic* cutoff phenomenon for Wasserstein. Third, the explicit expressions provided in our proof allow to extract the cutoff profile in each case, but we prefer not to provide them in our statement and refer the interested reader to the end of Section 2.3.

2.1.5 Exactly solvable intermezzo

When $\beta \neq 0$, the law of the DOU process is no longer Gaussian nor explicit. However several exactly solvable aspects are available. Let us recall that a Cox–Ingersoll–Ross process (CIR) of parameters a, b, σ is the solution $R = (R_t)_{t\geqslant 0}$ on $\mathbb{R}_+$ of

$$R_0 = r_0 \in \mathbb{R}_+, \quad \mathrm{d}R_t = \sigma\sqrt{R_t}\mathrm{d}W_t + (a - bR_t)\mathrm{d}t, \tag{2.9}$$

where W is a standard BM. Its invariant law is $\mathrm{Gamma}(2a/\sigma^2, 2b/\sigma^2)$ with density proportional to $r \geqslant 0 \mapsto r^{2a/\sigma^2-1}\mathrm{e}^{-2br/\sigma^2}$, with mean a/b, and variance $a\sigma^2/(2b^2)$. It was proved by William Feller in [118] that the density of R_t at an arbitrary t can be expressed in terms of special functions.

If $(Z_t)_{t\geqslant 0}$ is a d-dimensional OU process of parameters $\theta \geqslant 0$ and $\rho \in \mathbb{R}$, weak solution of

$$\mathrm{d}Z_t = \theta \mathrm{d}W_t - \rho Z_t \mathrm{d}t \tag{2.10}$$

where W is a d-dimensional BM, then $R = (R_t)_{t\geqslant 0}$, $R_t := |Z_t|^2$, is a CIR process with parameters $a = \theta^2 d$, $b = 2\rho$, $\sigma = 2\theta$. When $\rho = 0$ then Z is a BM while $R = |Z|^2$ is a squared Bessel process.

The following theorem gathers some exactly solvable aspects of the DOU process for general $\beta \geqslant 1$, which are largely already in the statistical physics folklore, see [209]. It is based on our knowledge of eigenfunctions associated to the first spectral values of the dynamics, see (2.23), and their remarkable properties. As in (2.23), we set $\pi(x) := x_1 + \cdots + x_n$ when $x \in \mathbb{R}^n$.

Theorem 2.1.3 (From DOU to OU and CIR). *Let* $(X_t^n)_{t\geqslant 0}$ *be the DOU process* (2.3)*, with* $\beta = 0$ *or* $\beta \geqslant 1$*, and let* P_n^β *be its invariant law. Then:*

- $(\pi(X_t^n))_{t\geqslant 0}$ *is a one-dimensional OU process weak solution of* (2.8) *with* $\theta = \sqrt{2}$, $\rho = 1$. *Its invariant law is* $\mathcal{N}(0,1)$. *It does not depend on* β, *and* $\pi(X_t^n) \sim \mathcal{N}(\pi(x_0^n)\mathrm{e}^{-t}, 1-\mathrm{e}^{-2t})$, $t \geqslant 0$. *Furthermore* $\pi(X_t^n)^2$ *is a CIR process of parameters* $a = 2$, $b = 2$, $\sigma = 2\sqrt{2}$.
- $(|X_t^n|^2)_{t\geqslant 0}$ *is a CIR process, weak solution of* (2.9) *with* $a = 2+\beta(n-1)$, $b = 2$, $\sigma = \sqrt{8/n}$. *Its invariant law is* $\mathrm{Gamma}(\frac{1}{2}(n+\beta\frac{n(n-1)}{2}), \frac{n}{2})$ *of mean* $1+\frac{\beta}{2}(n-1)$ *and variance* $\beta + \frac{2-\beta}{n}$. *Furthermore, if* $d = n+\beta\frac{n(n-1)}{2}$ *is a positive integer, then* $(|X_t^n|^2)_{t\geqslant 0}$ *has the law of* $(|Z_t|^2)_{t\geqslant 0}$ *where* $(Z_t)_{t\geqslant 0}$ *is a* d*-dimensional OU process, weak solution of* (2.8) *with* $\theta = \sqrt{2/n}$, $\rho = 1$, *and* $Z_0 = z_0^n$ *for an arbitrary* $z_0^n \in \mathbb{R}^d$ *such that* $|z_0^n| = |x_0^n|$.

At this step it is worth noting that Theorem 2.1.3 gives in particular, denoting $\beta_n := 1+\frac{\beta}{2}(n-1)$,

$$\mathbb{E}[\pi(X_t^n)] = \pi(x_0^n)\mathrm{e}^{-t} \underset{t\to\infty}{\longrightarrow} 0 \quad \text{and} \quad \mathbb{E}[|X_t^n|^2] = \beta_n + (|x_0^n|^2-\beta_n)\mathrm{e}^{-2t} \underset{t\to\infty}{\longrightarrow} \beta_n. \tag{2.11}$$

Following [81, Sec. 2.2], the limits can also be deduced from the Dumitriu–Edelman tridiagonal random matrix model [105] isospectral to β-Hermite. These formulas for the "transient" first two moments $\mathbb{E}[\pi(X_t^n)]$ and $\mathbb{E}[|X_t^n|^2]$ reveal an abrupt convergence to their equilibrium values :

- If $\lim_{n\to\infty} \frac{\pi(x_0^n)}{n} = \alpha \neq 0$ then for all $\varepsilon \in (0,1)$,

$$\lim_{n\to\infty} |\mathbb{E}[\pi(X_{t_n}^n)]| = \begin{cases} +\infty & \text{if } t_n = (1-\varepsilon)\log(n) \\ 0 & \text{if } t_n = (1+\varepsilon)\log(n) \end{cases}. \tag{2.12}$$

- If $\lim_{n\to\infty} \frac{|x_0^n|^2}{n} = \alpha \neq \frac{\beta}{2}$ then for all $\varepsilon \in (0,1)$, denoting $\beta_n := 1+\frac{\beta}{2}(n-1)$,

$$\lim_{n\to\infty} |\mathbb{E}[|X_{t_n}^n|^2]-\beta_n| = \begin{cases} +\infty & \text{if } t_n = (1-\varepsilon)\frac{1}{2}\log(n) \\ 0 & \text{if } t_n = (1+\varepsilon)\frac{1}{2}\log(n) \end{cases}. \tag{2.13}$$

These critical times are universal with respect to β. The first two transient moments are related to the eigenfunctions (2.23) associated to the first two non-zero eigenvalues of the dynamics. Higher order transient moments are related to eigenfunctions associated to higher order eigenvalues. Note that $\mathbb{E}[\pi(X_t^n)]$ and $\mathbb{E}[|X_t^n|^2]$ are the first two moments of the non-normalized mean empirical measure $\mathbb{E}[\sum_{i=1}^n \delta_{X_t^{n,i}}]$, and this lack of normalization is responsible of the critical times of order $\log(n)$. In contrast, the first two moments of the normalized mean empirical measure $\mathbb{E}[\frac{1}{n}\sum_{i=1}^n \delta_{X_t^{n,i}}]$, given by $\frac{1}{n}\mathbb{E}[\pi(X_t^n)]$ and $\frac{1}{n}\mathbb{E}[|X_t^n|^2]$ respectively, do not exhibit a critical phenomenon. This is related to the exponential decay of the first two moments in the mean-field limit (2.29), as well as the lack of cutoff for Wasserstein already revealed for OU by Theorem 2.1.2. This also reminds the high dimension behavior of norms in the field of the asymptotic geometric analysis of convex bodies. In another direction, this elementary observation on the moments also illustrates that the cutoff phenomenon for a given quantity is not stable under rather simple transformations of this quantity.

From the first part of Theorem 2.1.3 and contraction properties available for *some* distances or divergences, see Lemma 2.8.2, we obtain the following lower bound on the mixing time for the DOU, which is independent of β:

Corollary 2.1.4 (Lower bound on the mixing time). *Let* $(X_t^n)_{t\geqslant 0}$ *be the DOU process* (2.3) *with* $\beta = 0$ *or* $\beta \geqslant 1$, *and invariant law* P_n^β. *Let* $\mathrm{dist} \in \{\mathrm{TV}, \mathrm{Hellinger}, \mathrm{Kullback}, \chi^2, \mathrm{Wasserstein}\}$. *Set*

$$c_n := \begin{cases} \log(|\pi(x_0^n)|) & \textit{if } \mathrm{dist} \in \{\mathrm{TV}, \mathrm{Hellinger}, \mathrm{Kullback}, \chi^2\} \\ \log\left(\frac{|\pi(x_0^n)|}{\sqrt{n}}\right) & \textit{if } \mathrm{dist} = \mathrm{Wasserstein} \end{cases},$$

and assume that $\lim_{n\to\infty} c_n = \infty$. *Then, for all* $\varepsilon \in (0,1)$, *we have*

$$\lim_{n\to\infty} \mathrm{dist}(\mathrm{Law}(X^n_{(1-\varepsilon)c_n}) \mid P_n^\beta) = \max .$$

Theorem 2.1.3 and Corollary 2.1.4 are proved in Section 2.4.

The derivation of an upper bound on the mixing time is much more delicate: once again recall that the case $\beta = 0$ covered by Theorem 2.1.2 is specific as it relies on exact Gaussian computations which are no longer available for $\beta \geqslant 1$. In the next subsection, we will obtain results for general values of $\beta \geqslant 1$ via more elaborate arguments.

In the specific cases $\beta \in \{1,2\}$, there are some exactly solvable aspects that one can exploit to derive, in particular, precise upper bounds on the mixing times. Indeed, for these values of β, the DOU process is the process of eigenvalues of the matrix-valued OU process:

$$M_0 = m_0, \quad \mathrm{d}M_t = \sqrt{\frac{2}{n}}\mathrm{d}B_t - M_t \mathrm{d}t,$$

where B is a BM on the symmetric $n \times n$ matrices if $\beta = 1$ and on Hermitian $n \times n$ matrices if $\beta = 2$, see (2.62) and (2.74) for more details. Based on this observation, we can deduce an upper bound on the mixing times by contraction (for *most* distances or divergences).

Theorem 2.1.5 (Upper bound on mixing time in matrix case). *Let* $(X^n_t)_{t\geqslant 0}$ *be the DOU process* (2.3) *with* $\beta \in \{0,1,2\}$, *and invariant law* P_n^β, *and* $\mathrm{dist} \in \{\mathrm{TV}, \mathrm{Hellinger}, \mathrm{Kullback}, \chi^2, \mathrm{Wasserstein}\}$. *Set*

$$c_n := \begin{cases} \log(\sqrt{n}|x_0^n|) \vee \log(\sqrt{n}) & \text{if } \mathrm{dist} \in \{\mathrm{TV}, \mathrm{Hellinger}, \mathrm{Kullback}, \chi^2\} \\ \log(|x_0^n|) & \text{if } \mathrm{dist} = \mathrm{Wasserstein} \end{cases},$$

and assume that $\lim_{n\to\infty} c_n = \infty$ *if* $\mathrm{dist} = \mathrm{Wasserstein}$. *Then, for all* $\varepsilon \in (0,1)$, *we have*

$$\lim_{n\to\infty} \mathrm{dist}(\mathrm{Law}(X^n_{(1+\varepsilon)c_n}) \mid P_n^\beta) = 0.$$

Combining this upper bound with the lower bound already obtained above, we derive a cutoff phenomenon in this particular matrix case.

Corollary 2.1.6 (Cutoff for DOU in the matrix case). *Let* $(X^n_t)_{t\geqslant 0}$ *be the DOU process* (2.3), *with* $\beta \in \{0,1,2\}$, *and invariant law* P_n^β. *Let* $\mathrm{dist} \in \{\mathrm{TV}, \mathrm{Hellinger}, \mathrm{Kullback}, \chi^2, \mathrm{Wasserstein}\}$. *Let* $(a_n)_n$ *be a real sequence satisfying* $\inf_n \sqrt{n}a_n > 0$, *and assume further that* $\lim_{n\to\infty} \sqrt{n}a_n = \infty$ *if* $\mathrm{dist} = \mathrm{Wasserstein}$. *Then, for all* $\varepsilon \in (0,1)$, *we have*

$$\lim_{n\to\infty} \sup_{x_0^n \in [-a_n,a_n]^n} \mathrm{dist}(\mathrm{Law}(X^n_{t_n}) \mid P_n^\beta) = \begin{cases} \max & \text{if } t_n = (1-\varepsilon)c_n \\ 0 & \text{if } t_n = (1+\varepsilon)c_n \end{cases}$$

where

$$c_n := \begin{cases} \log(na_n) & \text{if } \mathrm{dist} \in \{\mathrm{TV}, \mathrm{Hellinger}, \mathrm{Kullback}, \chi^2\} \\ \log(\sqrt{n}a_n) & \text{if } \mathrm{dist} = \mathrm{Wasserstein} \end{cases}.$$

Theorem 2.1.5 and Corollary 2.1.6 are proved in Section 2.5.

It is worth noting that $d = n + \beta\frac{n(n-1)}{2}$ in Theorem 2.1.3 is indeed an integer in the "random matrix" cases $\beta \in \{1,2\}$, and corresponds then exactly to the degree of freedom of the Gaussian random matrix models GOE and GUE respectively. More precisely, if we let $X^n_\infty \sim P_n^\beta$ then:

- If $\beta = 1$ then P_n^β is the law of the eigenvalues of $S \sim \mathrm{GOE}_n$, and $|X_\infty^n|^2 = \sum_{j,k=1}^n S_{jk}^2$ which is the sum of n squared Gaussians of variance $v = 1/n$ (diagonal) plus twice the sum of $\frac{n^2-n}{2}$ squared Gaussians of variance $\frac{v}{2}$ (off-diagonal) all being independent. The duplication has the effect of renormalizing the variance from $\frac{v}{2}$ to v. All in all we have the sum of $d = \frac{n^2+n}{2}$ independent squared Gaussians of same variance v. See Section 2.5.
- If $\beta = 2$ then P_n^β is the law of the eigenvalues of $H \sim \mathrm{GUE}_n$, and $|X_\infty^n|^2 = \sum_{j,k=1}^n |H_{jk}|^2$ is the sum of n squared Gaussians of variance $v = 1/n$ (diagonal) plus twice the sum of $n^2 - n$ squared Gaussians of variance $\frac{v}{2}$ (off-diagonal) all being independent. All in all we have the sum of $d = n^2$ independent squared Gaussians of same variance v. See Section 2.5.

Another manifestation of exact solvability lies at the level of functional inequalities. Indeed, and following [81], the optimal Poincaré constant of P_n^β is given by $1/n$ and does not depend on β, and the extremal functions are tranlations/dilations of $x \mapsto \pi(x) = x_1 + \cdots + x_n$. This corresponds to a spectral gap of the dynamics equal to 1 and its associated eigenfunction. Moreover, the optimal logarithmic Sobolev inequality of P_n^β (Lemma 2.8.6) is given by $2/n$ and does not depend on β, and the extremal functions are of the form $x \mapsto \mathrm{e}^{c(x_1+\cdots+x_n)}$, $c \in \mathbb{R}$. This knowledge of the optimal constants and extremal functions and their independence with respect to β is truly remarkable. It plays a crucial role in the results presented in this article. More precisely, the optimal Poincaré inequality is used for the lower bound via the first eigenfunctions while the optimal logarithmic Sobolev inequality is used for the upper bound via exponential decay of the entropy.

2.1.6 Cutoff in the general interacting case

Our main contribution consists in deriving an upper bound on the mixing times in the general case $\beta \geqslant 1$: the proof relies on the logarithmic Sobolev inequality, some coupling arguments and a regularization procedure.

Theorem 2.1.7 (Upper bound on the mixing time: the general case). *Let $(X_t^n)_{t\geqslant 0}$ be the DOU process (2.3), with $\beta = 0$ or $\beta \geqslant 1$ and invariant law P_n^β. Take $\mathrm{dist} \in \{\mathrm{TV}, \mathrm{Hellinger}, \mathrm{Wasserstein}\}$. Set*

$$c_n := \begin{cases} \log(\sqrt{n}|x_0^n|) \vee \log(n) & \text{if } \mathrm{dist} \in \{\mathrm{TV}, \mathrm{Hellinger}\} \\ \log(|x_0^n|) \vee \log(\sqrt{n}) & \text{if } \mathrm{dist} = \mathrm{Wasserstein} \end{cases}.$$

Then, for all $\varepsilon \in (0,1)$, we have

$$\lim_{n\to\infty} \mathrm{dist}(\mathrm{Law}(X_{(1+\varepsilon)c_n}^n) \mid P_n^\beta) = 0.$$

Combining this upper bound with the general lower bound that we obtained in Corollary 2.1.4, we deduce the following cutoff phenomenon. Observe that it holds both for $\beta = 0$ and $\beta \geqslant 1$, and that the expression of the mixing time does not depend on β.

Corollary 2.1.8 (Cutoff for DOU in the general case). *Let $(X_t^n)_{t\geqslant 0}$ be the DOU process (2.3) with $\beta = 0$ or $\beta \geqslant 1$ and invariant law P_n^β. Take $\mathrm{dist} \in \{\mathrm{TV}, \mathrm{Hellinger}, \mathrm{Wasserstein}\}$. Let $(a_n)_n$ be a real sequence satisfying $\inf_n a_n > 0$. Then, for all $\varepsilon \in (0,1)$, we have*

$$\lim_{n\to\infty} \sup_{x_0^n \in [-a_n, a_n]^n} \mathrm{dist}(\mathrm{Law}(X_{t_n}^n) \mid P_n^\beta) = \begin{cases} \max & \text{if } t_n = (1-\varepsilon)c_n \\ 0 & \text{if } t_n = (1+\varepsilon)c_n \end{cases}$$

where

$$c_n := \begin{cases} \log(n a_n) & \text{if } \mathrm{dist} \in \{\mathrm{TV}, \mathrm{Hellinger}\} \\ \log(\sqrt{n} a_n) & \text{if } \mathrm{dist} = \mathrm{Wasserstein} \end{cases}.$$

The proofs of Theorem 2.1.7 and Corollary 2.1.8 for the TV and Hellinger distances are presented in Section 2.6. The Wasserstein distance is treated in Section 2.7. Let us make a comment on the assumptions made on a_n in Corollaries 2.1.6 and 2.1.8. They are dictated by the upper bounds established in Theorems 2.1.5 and 2.1.7, which take the form of maxima of two terms: one that depends on the initial condition, and another one which is a power of a logarithm of n. The logarithmic term is an upper bound on the time required to regularize a pointwise initial condition, its precise expression varies according to the method of proof we rely on: in the matrix case, it is the time required to regularize a larger object, the matrix-valued OU process; in the general case, it is related to the time it takes to make the entropy of a pointwise initial condition small. These bounds are not optimal for $\beta = 0$ (compare with Theorem 2.1.2), and probably neither for $\beta \geqslant 1$.

A natural, but probably quite difficult, goal would be to establish a cutoff phenomenon in the situation where the set of initial conditions is reduced to *any* given singleton, as in Theorem 2.1.2 for the case $\beta = 0$. Recall that in that case, the asymptotic of the mixing time is dictated by the Euclidean norm of the initial condition. In the case $\beta \geqslant 1$, this *cannot* be the right observable since the Euclidean norm does not measure the distance to equilibrium. Instead one should probably consider the Euclidean norm $|x_0^n - \rho_n|$, where ρ_n is the vector of the quantiles of order $1/n$ of the semi-circle law that arises in the mean-field limit equilibrium (see Subsection 2.2.5). More precisely

$$\rho_{n,i} = \inf\left\{t \in \mathbb{R} : \int_{-\infty}^{t} \frac{\sqrt{2\beta - x^2}}{\beta\pi}\mathbf{1}_{x\in[-\sqrt{2\beta},\sqrt{2\beta}]}\mathrm{d}x \geqslant \frac{i}{n}\right\}, \quad i \in \{1,\dots,n\}. \tag{2.14}$$

Note that $\rho_n = 0$ when $\beta = 0$.

A first step in this direction is given by the following result:

Theorem 2.1.9 (DOU in the general case and pointwise initial condition). *Let $(X_t^n)_{t\geqslant 0}$ be the DOU process (2.3) with $\beta = 0$ or $\beta \geqslant 1$, and invariant law P_n^β. There hold*

- *If $\lim_{n\to\infty} |x_0^n - \rho_n| = +\infty$, then, denoting $t_n = \log(|x_0^n - \rho_n|)$, for all $\varepsilon \in (0,1)$,*

$$\lim_{n\to\infty} \mathrm{Wasserstein}(\mathrm{Law}(X_{(1+\varepsilon)t_n}), P_n^\beta) = 0.$$

- *If $\lim_{n\to\infty} |x_0^n - \rho_n| = \alpha \in [0,\infty)$, then, for all $t > 0$,*

$$\varlimsup_{n\to\infty} \mathrm{Wasserstein}(\mathrm{Law}(X_t), P_n^\beta)^2 \leqslant \alpha^2 \mathrm{e}^{-2t}.$$

Theorem 2.1.9 is proved in Section 2.7.

2.1.7 Non-pointwise initial conditions

It is natural to ask about the cutoff phenomenon when the initial conditions X_0^n is not pointwise. Even if we turn off the interaction by taking $\beta = 0$, the law of the process at time t is then no longer Gaussian in general, which breaks the method of proof used for Theorem 2.1.1 and Theorem 2.1.2. Nevertheless, Theorem 2.1.10 below provides a universal answer, that is both for $\beta = 0$ and $\beta \geqslant 1$, at the price however of introducing several objects and notations. More precisely, for any probability measure μ on $\mathbb{R}^n$, we introduce

$$S(\mu) = \begin{cases} \displaystyle\int \frac{\mathrm{d}\mu}{\mathrm{d}x}\log\frac{\mathrm{d}\mu}{\mathrm{d}x}\mathrm{d}x = \text{``Kullback}(\mu \mid \mathrm{d}x)\text{''} & \text{if } \displaystyle\frac{\mathrm{d}\mu}{\mathrm{d}x}\log\frac{\mathrm{d}\mu}{\mathrm{d}x} \in L^1(\mathrm{d}x) \\ +\infty & \text{otherwise} \end{cases}. \tag{2.15}$$

Note that S takes its values in the whole $(-\infty,+\infty]$, and when $S(\mu)<+\infty$ then $-S(\mu)$ is the *Boltzmann–Shannon entropy* of the law μ. For all $x\in\mathbb{R}^n$ with $x_i\neq x_j$ for all $i\neq j$, we have

$$E(x_1,\dots,x_n)=n^2\iint\Phi(x,y)\mathbf{1}_{\{x\neq y\}}L_n(\mathrm{d}x)L_n(\mathrm{d}y) \tag{2.16}$$

where $L_n:=\frac{1}{n}\sum_{i=1}^n\delta_{x_i}$ and where $\Phi(x,y):=\frac{n}{n-1}\frac{V(x)+V(y)}{2}+\frac{\beta}{2}\log\frac{1}{|x-y|}$.

Let us define the map $\Psi:\mathbb{R}^n\mapsto\overline{D}_n$ by

$$\Psi(x_1,\dots,x_n):=(x_{\sigma(1)},\dots,x_{\sigma(n)}). \tag{2.17}$$

where σ is any permutation of $\{1,\dots,n\}$ that reorders the particles non-decreasingly.

Theorem 2.1.10 (Cutoff for DOU with product smooth initial conditions). *Let $(X_t^n)_{t\geqslant0}$ be the DOU process (2.3) with $\beta=0$ or $\beta\geqslant1$, and invariant law P_n^β. Let S, Φ, and Ψ be as in (2.15), (2.16), and (2.17). Let us assume that $\mathrm{Law}(X_0^n)$ is the image law or push forward of a product law $\mu_1\otimes\cdots\otimes\mu_n$ by Ψ where $\mu_1,\dots,\mu_n$ are laws on $\mathbb{R}$. Then:*

1. *If $\varlimsup_{n\to\infty}\left|\frac{1}{n}\sum_{i=1}^n\int x\mu_i(\mathrm{d}x)\right|\neq0$ then, for all $\varepsilon\in(0,1)$,*

$$\lim_{n\to\infty}\mathrm{Kullback}(\mathrm{Law}(X_{(1-\varepsilon)\log(n)})\mid P_n^\beta)=+\infty.$$

2. *If $\varlimsup_{n\to\infty}\frac{1}{n^2}\sum_{i=1}^n S(\mu_i)<\infty$ and $\varlimsup_{n\to\infty}\frac{1}{n^2}\sum_{i\neq j}\iint\Phi\,\mathrm{d}\mu_i\otimes\mathrm{d}\mu_j<\infty$, then, for all $\varepsilon\in(0,1)$,*

$$\lim_{n\to\infty}\mathrm{Kullback}(\mathrm{Law}(X_{(1+\varepsilon)\log(n)})\mid P_n^\beta)=0.$$

Theorem 2.1.10 is proved in Section 2.6.3.

It is likely that Theorem 2.1.10 can be extended to the case $\mathrm{dist}\in\{\mathrm{Wasserstein},\mathrm{Hellinger},\mathrm{Fisher}\}$.

2.1.8 Structure of the paper

- Section 2.2 provides additional comments and open problems.
- Section 2.3 focuses on the OU process ($\beta=0$) and gives the proofs of Theorems 2.1.1 and 2.1.2.
- Section 2.4 concerns the exact solvability of the DOU process for all β, and provides the proofs of Theorem 2.1.3 and Corollary 2.1.4.
- Section 2.5 is about random matrices and gives the proofs of Theorem 2.1.5 and Corollary 2.1.6.
- Section 2.6 deals with the DOU process for all β with the TV and Hellinger distances, and provides the proofs of Theorem 2.1.7 and Corollary 2.1.8.
- Section 2.7 gives the Wasserstein counterpart of Section 2.6 and the proof of Theorem 2.1.9.
- Appendix 2.8.1 provides a survey on distances and divergences, with new results.
- Appendix 2.8.2 gathers useful dynamical consequences of convexity.

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2.2 Additional comments and open problems

2.2.1 About the results and proofs

The proofs of our results rely among other ingredients on convexity and optimal functional inequalities, exact solvability, exact Gaussian formulas, coupling arguments, stochastic calculus, variational formulas, contraction properties and regularization.

The proofs of Theorems 2.1.1 and 2.1.2 are based on the explicit Gaussian nature of the OU process, which allows to use Gaussian formulas for all the distances and divergences that we consider (the Gaussian formula for Fisher seems to be new). Our analysis of the convergence to equilibrium of the OU process seems to go beyond what is already known, see for instance [168] and [22, 20, 21, 23].

Theorem 2.1.3 is a one-dimensional analogue of [39, Th. 1.2]. The proof exploits the explicit knowledge of eigenfunctions of the dynamics (2.23), associated with the first two non-zero spectral values, and their remarkable properties. The first one is associated to the spectral gap and the optimal Poincaré inequality. It implies Corollary 2.1.4, which is the provider of all our lower bounds on the mixing time for the cutoff.

The proof of Theorem 2.1.5 is based on a contraction property and the upper bound for matrix OU processes. It is not available beyond the matrix cases. All the other upper bounds that we establish are related to an optimal exponential decay which comes from convexity and involves sometimes coupling, the simplest instance being Theorem 2.1.7 about the Wasserstein distance. The usage of the Wasserstein metrics for Dyson dynamics is quite natural, see for instance [32].

The proof of Theorem 2.1.7 for the TV and $\mathrm{Hellinger}$ relies on the knowledge of the optimal exponential decay of the entropy (with respect to equilibrium) related to the optimal logarithmic Sobolev inequality. Since pointwise initial conditions have infinite entropy, the proof proceeds in three steps: first we regularize the initial condition to make its entropy finite, second we use the optimal exponential decay of the entropy of the process starting from this regularized initial condition, third we control the distance between the processes starting from the initial condition and its regularized version. This last part is inspired by a work of Lacoin [169] for the simple exclusion process on the segment, subsequently adapted to continuous state-spaces [67, 68], where one controls an *area* between two versions of the process.

The (optimal) exponential decay of the entropy (Lemma 2.8.7) is equivalent to the (optimal) logarithmic Sobolev inequality (Lemma 2.8.6). For the DOU process, the optimal logarithmic Sobolev inequality provided by Lemma 2.8.6 achieves also the universal bound with respect to the spectral gap, just like for Gaussians. This sharpness between the best logarithmic Sobolev constant and the spectral gap also holds for instance for the random walk on the hypercube, a discrete process for which a cutoff phenomenon can be established with the optimal logarithmic Sobolev inequality, and which can be related to the OU process, see for instance [101, 100] and references therein. If we generalize the DOU process by adding an arbitrary convex function to V, then we will still have a logarithmic Sobolev inequality – see [81] for several proofs including the proof via the Bakry–Émery criterion – however the optimal logarithmic Sobolev constant will no longer be explicit nor sharp with respect to the spectral gap, and the spectral gap will no longer be explicit.

The proof of Theorem 2.1.10 relies crucially on the tensorization property of $\mathrm{Kullback}$ and on the asymptotics on the normalizing constant C_n^β at equilibrium.

2.2.2 Analysis and geometry of the equilibrium

The full space $\mathbb{R}^n$ is, up to a bunch of hyperplanes, covered with $n!$ disjoint isometric copies of the convex domain D_n obtained by permuting the coordinates (simplices or Weyl chambers). Following [81], for all $\beta \geqslant 0$ let us define the law P_{*n}^β on $\mathbb{R}^n$ with density proportional to e^{-E}, just like for P_n^β in (2.6) but without the $\mathbf{1}_{(x_1,\dots,x_n)\in\overline{D}_n}$.

If $\beta = 0$ then $P_{*n}^0 = P_n^0 = \mathcal{N}(0, \frac{1}{n}I_n)$ according to our definition of P_n^0.

If $\beta > 0$ then P_{*n}^β has density $(C_{*n}^\beta)^{-1}\mathrm{e}^{-E}$ with $C_{*n}^\beta = n!C_n^\beta$ where C_n^β is the normalization of P_n^β. Moreover P_{*n}^β is a mixture of $n!$ isometric copies of P_n^β, while P_n^β is the image law or push forward of P_{*n}^β by the map $\Psi_n : \mathbb{R}^n \to \overline{D}_n$ defined in (2.17). Furthermore for all bounded measurable $f : \mathbb{R}^n \to \mathbb{R}$, denoting Σ_n the symmetric group of permutations of $\{1, \dots, n\}$,

$$\int f \mathrm{d}P_{*n}^\beta = \int f_{\mathrm{sym}} \mathrm{d}P_n^\beta \quad \text{with} \quad f_{\mathrm{sym}}(x_1, \dots, x_n) := \frac{1}{n!} \sum_{\sigma\in\Sigma_n} f(x_{\sigma(1)}, \dots, x_{\sigma(n)}).$$

Regarding log-concavity, it is important to realize that if $\beta = 0$ then E is convex on $\mathbb{R}^n$, while if $\beta > 0$ then E is convex on D_n but is not convex on $\mathbb{R}^n$ and has $n!$ isometric local minima.

- The law P_{*n}^β is centered but is not log-concave when $\beta > 0$ since E is not convex on $\mathbb{R}^n$. As $\beta \to 0^+$ the law P_{*n}^β tends to $P_{*n}^0 = P_n^0 = \mathcal{N}(0, \frac{1}{n}I_n)$ which is log-concave.

- The law P_n^β is not centered but is log-concave for all $\beta \geqslant 0$. Its density vanishes at the boundary of D_n if $\beta > 0$. As $\beta \to 0^+$ the law P_n^β tends to the law of the order statistics of n i.i.d. $\mathcal{N}(0, \frac{1}{n})$.

2.2.3 Spectral analysis of the generator: the non-interacting case

This subsection and the next deal with analytical aspects of our dynamics. We start with the OU process ($\beta = 0$) for which everything is explicit; the next subsection deals with the DOU process ($\beta \geqslant 1$).

The infinitesimal generator of the OU process is given by

$$\mathsf{G}f = \frac{1}{n}\Big(\Delta - \nabla E \cdot \nabla\Big) = \frac{1}{n}\sum_{i=1}^n \partial_i^2 - \sum_{i=1}^n V'(x_i)\partial_i. \tag{2.18}$$

It is a self-adjoint operator on $L^2(\mathbb{R}^n, P_n^0)$ that leaves globally invariant the set of polynomials. Its spectrum is the set of all non-positive integers, that is, $\lambda_0 = 0 > \lambda_1 = -1 > \lambda_2 = -2 > \cdots$. The corresponding eigenspaces $F_0, F_1, F_2, \cdots$ are finite dimensional: F_m is spanned by the multivariate Hermite polynomials of degree m, in other words tensor products of univariate Hermite polynomials. In particular, F_0 is the vector space of constant functions while F_1 is the n-dimensional vector space of all linear functions.

Let us point out that G can be restricted to the set of P_n^0 square integrable *symmetric* functions: it leaves globally invariant the set of *symmetric* polynomials, its spectrum is unchanged but the associated eigenspaces E_m are the restrictions of the vector spaces F_m to the set of symmetric functions, in other words, E_m is spanned by the multivariate *symmetrized* Hermite polynomials of degree m. Note that E_1 is the one-dimensional space generated by $\pi(x) = x_1 + \cdots + x_n$.

The Markov semigroup $(\mathrm{e}^{t\mathsf{G}})_{t\geqslant 0}$ generated by G admits P_n^0 as a reversible invariant law since G is self-adjoint in $L^2(P_n^0)$. Following [224], let us introduce the *heat kernel* $p_t(x, y)$ which is the density of $\mathrm{Law}(X_t^n \mid X_0^n = x)$ with respect to the invariant law P_n^0. The long-time behavior reads

$\lim_{t\to\infty} p_t(x,\cdot) = 1$ for all $x \in \mathbb{R}^n$. Let $\|\cdot\|_p$ be the norm of $L^p = L^p(P_n^0)$. For all $1 \leqslant p \leqslant q$, $t \geqslant 0$, $x \in \mathbb{R}^n$, we have

$$2\|\mathrm{Law}(X_t^n \mid X_0^n = x) - P_n^0\|_{\mathrm{TV}} = \|p_t(x,\cdot) - 1\|_1 \leqslant \|p_t(x,\cdot) - 1\|_p \leqslant \|p_t(x,\cdot) - 1\|_q. \tag{2.19}$$

In the particular case $p = 2$ we can write

$$\|p_t(x,\cdot) - 1\|_2^2 = \sum_{m=1}^{\infty} \mathrm{e}^{-2mt} \sum_{\psi \in B_m} |\psi(x)|^2. \tag{2.20}$$

where B_m is an orthonormal basis of $F_m \subset L^2(P_n^0)$, hence

$$\|p_t(x,\cdot) - 1\|_2^2 \geqslant \mathrm{e}^{-2t} \sum_{\psi \in B_1} |\psi(x)|^2, \tag{2.21}$$

which leads to a lower bound on the χ^2 (in other words L^2) cutoff, provided one can estimate $\sum_{\psi\in B_1} |\psi(x)|^2$ which is the square of the norm of the projection of δ_x on B_1.

Following [224, Th. 6.2], an upper bound would follow from a Bakry–Émery curvature–dimension criterion $\mathrm{CD}(\rho, d)$ with a finite dimension d, in relation with Nash–Sobolev inequalities and dimensional pointwise estimates on the heat kernel $p_t(x,\cdot)$ or ultracontractivity of the Markov semigroup, see for instance [225, Sec. 4.1]. The OU process satisfies to $\mathrm{CD}(\rho, \infty)$ but never to $\mathrm{CD}(\rho, d)$ with d finite and is not ultracontractive. Actually the OU process is a critical case, see [7, Ex. 2.7.3].

2.2.4 Spectral analysis of the generator: the interacting case

We now assume that $\beta \geqslant 1$. The infinitesimal generator of the DOU process is the operator

$$\mathsf{G}f = \frac{1}{n}\Big(\Delta - \nabla E \cdot \nabla\Big) = \frac{1}{n}\sum_{i=1}^{n} \partial_i^2 - \sum_{i=1}^{n} V'(x_i)\partial_i + \frac{\beta}{2n}\sum_{j\neq i} \frac{\partial_i - \partial_j}{x_i - x_j}. \tag{2.22}$$

Despite the interaction term, the operator leaves globally invariant the set of *symmetric* polynomials. Following Lassalle in [176, 13], see also [81], the operator G is a self-adjoint operator on the space of P_{*n}^{β} square integrable *symmetric* functions of n variables, its spectrum does not depend on β and matches the spectrum of the OU process case $\beta = 0$. In particular the spectral gap is 1. The eigenspaces E_m are spanned by the generalized symmetrized Hermite polynomials of degree m. For instance, E_1 is the one-dimensional space generated by $\pi(x) = x_1 + \cdots + x_n$ while E_2 is the two-dimensional space spanned by

$$(x_1 + \cdots + x_n)^2 - 1 \quad \text{and} \quad x_1^2 + \cdots + x_n^2 - 1 - \frac{\beta}{2}(n-1). \tag{2.23}$$

From the isometry between $L^2(\overline{D}_n, P_n^{\beta})$ and $L^2_{\mathrm{sym}}(\mathbb{R}^n, P_{*n}^{\beta})$, the above explicit spectral decomposition applies to the semigroup of the DOU on $\overline{D}_n$. Formally, the discussion presented at the end of the previous subsection still applies. However, in the present interacting case the integrability properties of the heat kernel are not known: in particular, we do not know whether $p_t(x,\cdot)$ lies in $L^p(P_n^{\beta})$ for $t > 0$, $x \in \overline{D}_n$ and $p > 1$. This leads to the question, of independent interest, of pointwise upper and lower Gaussian bounds for heat kernels similar to the OU process, with explicit dependence of the constants over the dimension. We refer for example to [245, 109, 137] for some results in this direction.

2.2.5 Mean-field limit

The measure P_n^β is log-concave since E is convex, and its density writes

$$x \in \mathbb{R}^n \mapsto \frac{\mathrm{e}^{-\frac{n}{2}|x|^2}}{C_n^\beta} \prod_{i>j}(x_i - x_j)^\beta \mathbf{1}_{x_1 \leqslant \cdots \leqslant x_n}. \tag{2.24}$$

See [81, Sec. 2.2] for a high-dimensional analysis. The Boltzmann–Gibbs measure P_n^β is known as the β-Hermite ensemble or HβE. When $\beta = 2$, it is better known as the Gaussian Unitary Ensemble (GUE). If $X^n \sim P_n^\beta$ then the Wigner theorem states that the empirical measure with atoms distributed according to P_n^β converges in distribution to a semi-circle law, namely

$$\frac{1}{n}\sum_{i=1}^n \delta_{X^{n,i}} \xrightarrow[n\to\infty]{\text{weak}} \frac{\sqrt{2\beta - x^2}}{\beta\pi}\mathbf{1}_{x\in[-\sqrt{2\beta},\sqrt{2\beta}]}\mathrm{d}x, \tag{2.25}$$

and this can be deduced in this Coulomb gas context from a large deviation principle as in [28].

Let $(X^n)_{t\geqslant 0}$ be the process solving (2.3) with $\beta \geqslant 0$ or $\beta \geqslant 1$, and let

$$\mu_t^n = \frac{1}{n}\sum_{k=1}^n \delta_{X_t^{n,i}} \tag{2.26}$$

be the empirical measure of the particles at time t. Following notably [216, 33, 70, 69, 192, 103], if the sequence of initial conditions $(\mu_0^n)_{n\geqslant 1}$ converges weakly as $n \to \infty$ to a probability measure μ_0, then the sequence of measure valued processes $((\mu_t^n)_{t\geqslant 0})_{n\geqslant 1}$ converges weakly to the unique probability measure valued deterministic process $(\mu_t)_{t\geqslant 0}$ satisfying the evolution equation

$$\langle \mu_t, f\rangle = \langle \mu_0, f\rangle - \int_0^t\int V'(x)f'(x)\mu_s(\mathrm{d}x)\mathrm{d}s + \frac{\beta}{2}\int_0^t\int_{\mathbb{R}^2}\frac{f'(x)-f'(y)}{x-y}\mu_s(\mathrm{d}x)\mu_s(\mathrm{d}y)\mathrm{d}s \tag{2.27}$$

for all $t \geqslant 0$ and $f \in \mathcal{C}_b^3(\mathbb{R},\mathbb{R})$. The equation (2.27) is a weak formulation of a McKean–Vlasov equation or free Fokker–Planck equation associated to a free OU process. Moreover, if μ_0 has all its moments finite, then for all $t \geqslant 0$, we have the free Mehler formula

$$\mu_t = \mathrm{dil}_{\mathrm{e}^{-2t}}\mu_0 \boxplus \mathrm{dil}_{\sqrt{1-\mathrm{e}^{-2t}}}\mu_\infty, \tag{2.28}$$

where $\mathrm{dil}_\sigma\mu$ is the law of σX when $X \sim \mu$, where "$\boxplus$" stands for the free convolution of probability measures of Voiculescu free probability theory, and where μ_∞ is the semi-circle law of variance $\frac{\beta}{2}$. In particular, if μ_0 is a semi-circle law then μ_t is a semi-circle law for all $t \geqslant 0$.

Let us introduce the k-th moment $m_k(t) := \displaystyle\int x^k\mu_t(\mathrm{d}x)$ of μ_t. The first and second moments satisfy the differential equations $m_1' = -m_1$ and $m_2' = -2m_2 + \beta$ respectively, which give

$$m_1(t) = \mathrm{e}^{-t}m_1(0) \xrightarrow[t\to\infty]{} 0 \quad\text{and}\quad m_2(t) = m_2(0)\mathrm{e}^{-2t} + \frac{\beta}{2}(1-\mathrm{e}^{-2t}) \xrightarrow[t\to\infty]{} \frac{\beta}{2}. \tag{2.29}$$

More generally, beyond the first two moments, the Cauchy–Stieltjes transform

$$z \in \mathbb{C}_+ = \{z \in \mathbb{C} : \Im z > 0\} \mapsto s_t(z) = \int_{\mathbb{R}}\frac{\mu_t(\mathrm{d}x)}{x-z} \tag{2.30}$$

of μ_t is the solution of the following complex Burgers equation

$$\partial_t s_t(z) = s_t(z) + z\partial_z s_t(z) + \beta s_t(z)\partial_z s_t(z), \quad t \geqslant 0, z \in \mathbb{C}_+. \tag{2.31}$$

The semi-circle law on $[-c,c]$ has density $\frac{2\sqrt{c^2-x^2}}{\pi c^2}\mathbf{1}_{x\in[-c,c]}$, mean 0, second moment or variance $\frac{c^2}{4}$, and Cauchy–Stieltjes transform $s_t(z)=\frac{\sqrt{4z^2-4c^2}-2z}{c^2}$, $t\geqslant 0, z\in\mathbb{C}_+$.

The cutoff phenomenon is in a sense a diagonal (t,n) estimate, melting long time behavior and high dimension. When $|z_0^n|$ is of order n, cutoff occurs at a time of order $\approx\log(n)$: this informally corresponds to taking $t\to\infty$ in $(\mu_t)_{t\geqslant 0}$.

When μ_0 is centered with same second moment $\frac{\beta}{2}$ as μ_∞, then there is a Boltzmann H-theorem interpretation of the limiting dynamics as $n\to\infty$: the steady-state is the Wigner semi-circle law μ_∞, the second moment is conserved by the dynamics, the Voiculescu entropy is monotonic along the dynamics, grows exponentially, and is maximized by the steady-state.

2.2.6 L^p cutoff

Following [84], we can deduce an L^p cutoff started from x from an L^1 cutoff by showing that the heat kernel $p_t(x,\cdot)$ is in $L^p(P_n^\beta)$ for some $t>0$. Thanks to the Mehler formula, it can be checked that this holds for the OU case, despite the lack of ultracontractivity. The heat kernel of the DOU process is less accessible.

In another exactly solvable direction, the L^p cutoff phenomenon has been studied for instance in [224, 226] for Brownian motion on compact simple Lie groups, and in [226, 199] for Brownian motion on symmetric spaces, in relation with representation theory, an idea which goes back at the origin to the works of Diaconis on random walks on groups.

2.2.7 Cutoff window and profile

Once a cutoff phenomenon is established, one can ask for a finer description of the pattern of convergence to equilibrium. The *cutoff window* is the order of magnitude of the transition time from the value $\max$ to the value 0: more precisely, if cutoff occurs at time c_n then we say that the cutoff window is w_n if

$$\begin{aligned}
\varlimsup_{b\to+\infty}\varlimsup_{n\to\infty}\operatorname{dist}(\operatorname{Law}(X_{c_n+bw_n})\mid P_n^\beta)&=0,\\
\varliminf_{b\to-\infty}\varliminf_{n\to\infty}\operatorname{dist}(\operatorname{Law}(X_{c_n+bw_n})\mid P_n^\beta)&=\max,
\end{aligned}$$

and for any $b\in\mathbb{R}$

$$0<\varliminf_{n\to\infty}\operatorname{dist}(\operatorname{Law}(X_{c_n+bw_n})\mid P_n^\beta)\leqslant\varlimsup_{n\to\infty}\operatorname{dist}(\operatorname{Law}(X_{c_n+bw_n})\mid P_n^\beta)<\max.$$

Note that necessarily $w_n=o(c_n)$ by definition of the cutoff phenomenon. Note also that w_n is unique in the following sense: w_n' is a cutoff window if and only if w_n/w_n' remains bounded from above and below as $n\to\infty$.
We say that the *cutoff profile* is given by $\varphi:\mathbb{R}\to[0,1]$ if

$$\lim_{n\to\infty}\operatorname{dist}(\operatorname{Law}(X_{c_n+bw_n})\mid P_n^\beta)=\varphi(b).$$

The analysis of the OU process carried out in Theorems 2.1.1 and 2.1.2 can be pushed further to establish the so-called cutoff profiles, we refer to the end of Section 2.3 for details.

Regarding the DOU process, such a detailed description of the convergence to equilibrium does not seem easily accessible. However it is straightforward to deduce from our proofs that the cutoff window is of order 1, in other words the inverse of the spectral gap, in the setting of Corollary 2.1.6.

This is also the case in the setting of Corollary 2.1.8 for the Wasserstein distance.
We believe that this remains true in the setting of Corollary 2.1.8 for the TV and Hellinger distances: actually, a lower bound of the required order can be derived from the calculations in the proof of Corollary 2.1.4; on the other hand, our proof of the upper bound on the mixing time does not allow to give a precise enough upper bound on the window.

2.2.8 Other potentials

It is natural to ask about the cutoff phenomenon for the process solving (2.3) when V is a more general $\mathcal{C}^2$ function. The invariant law P_n^β of this Markov diffusion writes

$$\frac{\mathrm{e}^{-n\sum_{i=1}^n V(x_i)}}{C_n^\beta}\prod_{i>j}(x_i-x_j)^\beta \mathbf{1}_{(x_1,\dots,x_n)\in\overline{D}_n}\mathrm{d}x_1\cdots\mathrm{d}x_n. \tag{2.32}$$

The case where $V-\frac{\rho}{2}\left|\cdot\right|^2$ is convex for some constant $\rho\geqslant 0$ generalizes the DOU case and has exponential convergence to equilibrium, see [81]. Three exactly solvable cases are known:

- $\mathrm{e}^{-V(x)}=\mathrm{e}^{-\frac{x^2}{2}}$: the DOU process associated to the Gaussian law weight and the β-Hermite ensemble including HOE/HUE/HSE when $\beta\in\{1,2,4\}$,
- $\mathrm{e}^{-V(x)}=x^{a-1}\mathrm{e}^{-x}\mathbf{1}_{x\in[0,\infty)}$: the Dyson–Laguerre process associated to the Gamma law weight and the β-Laguerre ensembles including LOE/LUE/LSE when $\beta\in\{1,2,4\}$,
- $\mathrm{e}^{-V(x)}=x^{a-1}(1-x)^{b-1}\mathbf{1}_{x\in[0,1]}$: the Dyson–Jacobi process associated to the Beta law weight and the β-Jacobi ensembles including JOE/JUE/JSE when $\beta\in\{1,2,4\}$,

up to a scaling. Following Lassalle [176, 178, 177, 13] and Bakry [14], in these three cases, the multivariate orthogonal polynomials of the invariant law P_n^β are the eigenfunctions of the dynamics of the process. We refer to [108, 105, 194] for more information on (H/L/J)βE random matrix models.

The contraction property or spectral projection used to pass from a matrix process to the Dyson process can be used to pass from BM on the unitary group to the Dyson circular process for which the invariant law is the Circular Unitary Ensemble (CUE). This provides an upper bound for the cutoff phenomenon. The cutoff for BM on the unitary group is known and holds at critical time or order $\log(n)$, see for instance [226, 224, 199].

More generally, we could ask about the cutoff phenomenon for a McKean–Vlasov type interacting particle system $(X_t^n)_{t\geqslant 0}$ in $(\mathbb{R}^d)^n$ solution of the stochastic differential equation of the form

$$\mathrm{d}X_t^{n,i}=\sigma_{n,t}(X^n)\mathrm{d}B_t^n-\sum_{i=1}^n\nabla V_{n,t}(X_t^{n,i})\mathrm{d}t-\sum_{j\neq i}\nabla W_{n,t}(X_t^{n,i}-X_t^{n,j})\mathrm{d}t,\quad 1\leqslant i\leqslant n, \tag{2.33}$$

for various types of confinement V and interaction W (convex, repulsive, attractive, repulsive-attractive, etc), and discuss the relation with the propagation of chaos. The case where V and W are both convex and constant in time is already very well studied from the point of view of long-time behavior and mean-field limit in relation with convexity, see for instance [70, 69, 192].

Regarding universality, it is worth noting that if $V=\left|\cdot\right|^2$ and if W is convex then the proof by factorization of the optimal Poincaré and logarithmic Sobolev inequalities and their extremal functions given in [81] remains valid, paving the way to the generalization of many of our results in this spirit. On the other hand, the convexity of the limiting energy functional in the mean-field limit is of Bochner type and suggests to take for W a power, in other words a Riesz type interaction.

2.2.9 Alternative parametrization

If $(X^n_t)_{t\geqslant 0}$ is the process solution of the stochastic differential equation (2.3), then for all real parameters $\alpha > 0$ and $\sigma > 0$, the space scaled and time changed stochastic process $(Y^n_t)_{t\geqslant 0} = (\sigma X^n_{\alpha t})_{t\geqslant 0}$ solves the stochastic differential equation

$$Y^n_0 = \sigma x^n_0, \quad \mathrm{d}Y^{n,i}_t = \sqrt{\frac{2\alpha\sigma^2}{n}}\mathrm{d}B^i_t - \alpha Y^{n,i}_t \mathrm{d}t + \frac{\alpha\beta\sigma^2}{n}\sum_{j\neq i}\frac{\mathrm{d}t}{Y^{n,i}_t - Y^{n,j}_t}, \quad 1 \leqslant i \leqslant n, \tag{2.34}$$

where $(B_t)_{t\geqslant 0}$ is a standard n-dimensional BM. The invariant law of $(Y^n_t)_{t\geqslant 0}$ is

$$\frac{\mathrm{e}^{-\frac{n}{2\sigma^2}|y|^2}}{C^\beta_n}\prod_{i>j}(y_i - y_j)^\beta \mathbf{1}_{(y_1,\ldots,y_n)\in\overline{D}_n}\mathrm{d}y_1\cdots\mathrm{d}y_n \tag{2.35}$$

where C^β_n is the normalizing constant. This law and its normalization C^β_n depend on the "shape parameter" β, the "scale parameter" σ, and does not depend on the "speed parameter" α. When $\beta > 0$, taking $\sigma^2 = \beta^{-1}$, the stochastic differential equation (2.34) boils down to

$$Y^n_0 = \frac{x^n_0}{\sqrt{\beta}}, \quad \mathrm{d}Y^{n,i}_t = \sqrt{\frac{2\alpha}{n\beta}}\mathrm{d}B^i_t - \alpha Y^{n,i}_t\mathrm{d}t + \frac{\alpha}{n}\sum_{j\neq i}\frac{\mathrm{d}t}{Y^{n,i}_t - Y^{n,j}_t}, \quad 1 \leqslant i \leqslant n \tag{2.36}$$

while the invariant law becomes

$$\frac{\mathrm{e}^{-\frac{n\beta}{2}|y|^2}}{C^\beta_n}\prod_{i>j}(y_i - y_j)^\beta \mathbf{1}_{(y_1,\cdots,y_n)\in\overline{D}_n}\mathrm{d}y_1\cdots\mathrm{d}y_n. \tag{2.37}$$

The equation (2.36) is the one considered in [113, Eq. (12.4)] and in [154, Eq. (1.1)]. The advantage of (2.36) is that β can be now truly interpreted as an inverse temperature and the right-hand side in the analogue of (2.25) does not depend on β, while the drawback is that we cannot turn off the interaction by setting $\beta = 0$ and recover the OU process as in (2.3). It is worthwhile mentioning that for instance Theorem 2.1.7 remains the same for the process solving (2.36) in particular the cutoff threshold is at critical time $\frac{c_n}{\alpha}$ and does not depend on β.

2.2.10 Discrete models

There are several discrete space Markov processes admitting the OU process as a scaling limit, such as for instance the random walk on the discrete hypercube, related to the Ehrenfest model, for which the cutoff has been studied in [101, 100], and the M/M/∞ queuing process, for which a discrete Mehler formula is available [76]. Certain discrete space Markov processes incorporate a singular repulsion mechanism, such as for instance the exclusion process on the segment, for which the study of the cutoff in [169] shares similarities with our proof of Theorem 2.1.7. It is worthwhile noting that there are discrete Coulomb gases, related to orthogonal polynomials for discrete measures, suggesting to study discrete Dyson processes. More generally, it could be natural to study the cutoff phenomenon for Markov processes on infinite discrete state spaces, under curvature condition, even if the subject is notoriously disappointing in terms of high-dimensional analysis. We refer to the recent work [223] for the finite state space case.

2.3 Cutoff phenomenon for the OU

In this section, we prove Theorems 2.1.1 and 2.1.2: actually we only prove the latter since it implies the former. We start by recalling a well-known fact.

Lemma 2.3.1 (Mehler formula). *If $(Y_t)_{t\geqslant 0}$ is an OU process in $\mathbb{R}^d$ solution of the stochastic differential equation $Y_0 = y_0 \in \mathbb{R}^d$ and $\mathrm{d}Y_t = \sigma \mathrm{d}B_t - \mu Y_t \mathrm{d}t$ for parameters $\sigma > 0$ and $\mu > 0$ where B is a standard d-dimensional Brownian motion then*

$$(Y_t)_{t\geqslant 0} = \Big(y_0 \mathrm{e}^{-\mu t} + \sigma \int_0^t \mathrm{e}^{\mu(s-t)} \mathrm{d}B_s\Big)_{t\geqslant 0} \quad \textit{hence } Y_t \sim \mathcal{N}\Big(y_0 \mathrm{e}^{-\mu t}, \frac{\sigma^2}{2}\frac{1-\mathrm{e}^{-2\mu t}}{\mu}\mathrm{I}_d\Big) \textit{ for all } t \geqslant 0.$$

Moreover its coordinates are independent one-dimensional OU processes with initial condition y_0^i and invariant law $\mathcal{N}(0, \frac{\sigma^2}{2\mu})$, $1 \leqslant i \leqslant d$.

Proof of Theorem 2.1.1 and Theorem 2.1.2. By using Lemma 2.3.1, for all $n \geqslant 1$ and $t \geqslant 0$,

$$Z_t^n \sim \mathcal{N}\Big(z_0^n \mathrm{e}^{-t}, \frac{1-\mathrm{e}^{-2t}}{n} I_n\Big) = \otimes_{i=1}^n \mathcal{N}\Big(z_0^{n,i} \mathrm{e}^{-t}, \frac{1-\mathrm{e}^{-2t}}{n}\Big),\ P_n^0 = \mathcal{N}\Big(0, \frac{I_n}{n}\Big) = \mathcal{N}\Big(0, \frac{1}{n}\Big)^{\otimes n}. \tag{2.38}$$

Hellinger, Kullback, χ^2, Fisher, and Wasserstein cutoffs. A direct computation from (2.38) or Lemma 2.8.5 either from multivariate Gaussian formulas or univariate via tensorization gives

$$\mathrm{Hellinger}^2(\mathrm{Law}(Z_t^n), P_n^0) = 1 - \exp\Big(-\frac{n}{4}\frac{|z_0^n|^2 \mathrm{e}^{-2t}}{2-\mathrm{e}^{-2t}} + \frac{n}{4}\log\Big(4\frac{1-\mathrm{e}^{-2t}}{(2-\mathrm{e}^{-2t})^2}\Big)\Big), \tag{2.39}$$

$$2\mathrm{Kullback}(\mathrm{Law}(Z_t^n) \mid P_n^0) = n|z_0^n|^2 \mathrm{e}^{-2t} - n\mathrm{e}^{-2t} - n\log(1-\mathrm{e}^{-2t}), \tag{2.40}$$

$$\chi^2(\mathrm{Law}(Z_t^n) \mid P_n^0) = -1 + \frac{1}{(1-\mathrm{e}^{-4t})^{n/2}} \exp\Big(n|z_0^n|^2 \frac{\mathrm{e}^{-2t}}{1+\mathrm{e}^{-2t}}\Big), \tag{2.41}$$

$$\mathrm{Fisher}(\mathrm{Law}(Z_t^n) \mid P_n^0) = n^2|z_0^n|^2 \mathrm{e}^{-2t} + n^2 \frac{\mathrm{e}^{-4t}}{1-\mathrm{e}^{-2t}}, \tag{2.42}$$

$$\mathrm{Wasserstein}^2(\mathrm{Law}(Z_t^n), P_n^0) = |z_0^n|^2 \mathrm{e}^{-2t} + 2(1 - \sqrt{1-\mathrm{e}^{-2t}} - \frac{1}{2}\mathrm{e}^{-2t}), \tag{2.43}$$

which gives the desired lower and upper bounds as before by using the hypothesis on z_0^n.

Total variation cutoff. By using the comparison between total variation and Hellinger distances (Lemma 2.8.1) we deduce from (2.39) the cutoff in total variation distance at the same critical time. The upper bound for the total variation distance can alternatively be obtained by using the $\mathrm{Kullback}$ estimate (2.40) and the Pinsker–Csiszár–Kullback inequality (Lemma 2.8.1). Since both distributions are tensor products, we could use alternatively the tensorization property of the total variation distance (Lemma 2.8.4) together with the one-dimensional version of the Gaussian formula for $\mathrm{Kullback}$ (Lemma 2.8.1) to obtain the result for the total variation. □

Remark 1 (Competition between bias and variance mixing). *From the computations of the proof of Theorem 2.1.2, we can show that for* $\mathrm{dist} \in \{\mathrm{TV}, \mathrm{Hellinger}, \chi^2\}$

$$A_t := \mathrm{dist}(\mathrm{Law}(Z_t^n) \mid \mathrm{Law}(Z_t^n - z_0^n \mathrm{e}^{-t}))$$

has a cutoff at time $c_n^A = \log(\sqrt{n}|z_0^n|)$, while

$$B_t := \mathrm{dist}(\mathrm{Law}(Z_t^n - z_0^n \mathrm{e}^{-t}) \mid P_n^0)$$

admits a cutoff at time $c_n^B = \frac{1}{4}\log(n)$*. The triangle inequality for* dist *yields*

$$|A_t - B_t| \leqslant \mathrm{dist}(\mathrm{Law}(Z_t^n) \mid P_n^0) \leqslant A_t + B_t.$$

Therefore the critical time of Theorem 2.1.2 is dictated by either A_t *or* B_t*, according to whether* $c_n^A \gg c_n^B$ *or* $c_n^A \ll c_n^B$*. This can be seen as a competition between bias and variance mixing.*

Remark 2 (Total variation discriminating event for small initial conditions). *Let us introduce the random variable* $Z_\infty^n \sim P_n^0 = \mathcal{N}(0, \frac{1}{n}I_n) = \mathcal{N}(0, \frac{1}{n})^{\otimes n}$*, in accordance with (2.38). There holds*

$$S_t^n := \sum_{i=1}^{n}(Z_t^{n,i} - z_0^{n,i}\mathrm{e}^{-t})^2 \sim \mathrm{Gamma}\Big(\frac{n}{2}, \frac{n}{2(1-\mathrm{e}^{-2t})}\Big) \quad \textit{and} \quad |Z_\infty^n|^2 \sim \mathrm{Gamma}\Big(\frac{n}{2}, \frac{n}{2}\Big).$$

We can check, using an explicit computation of Hellinger and Kullback between Gamma distributions and the comparison between total variation and Hellinger distances (Lemma 2.8.1), that

$$C_t := \mathrm{dist}(\mathrm{Law}(S_t^n) \mid \mathrm{Law}(|Z_\infty^n|^2))$$

admits a cutoff at time $c_n^C = c_n^B = \frac{1}{4}\log(n)$*. Moreover, one can exhibit a discriminating event for the TV distance. Namely, we can observe that*

$$\Big\|\mathrm{Gamma}\Big(\frac{n}{2}, \frac{n}{2(1-\mathrm{e}^{-2t})}\Big) - \mathrm{Gamma}\Big(\frac{n}{2}, \frac{n}{2}\Big)\Big\|_{\mathrm{TV}} = \mathbb{P}(|Z_\infty^n|^2 \geqslant \alpha_t) - \mathbb{P}(S_t^n \geqslant \alpha_t)$$

with α_t *the unique point where the two densities meet, which happens to be*

$$\alpha_t = -\mathrm{e}^{2t}\log(1-\mathrm{e}^{-2t})(1-\mathrm{e}^{-2t}).$$

From the explicit expressions (2.39), (2.40), (2.41), (2.42), (2.43), it is immediate to extract the cutoff profile associated to the convergence of $\mathrm{Law}(Z_t^n)$ to P_n^0 in Hellinger, Kullback, χ^2, Fisher and Wasserstein. For Wasserstein we already know by Theorem 2.1.2 that a cutoff occurs if and only if $|z_0^n| \underset{n\to\infty}{\to} \infty$. In this case, regarding the profile, we have

$$\lim_{n\to\infty} \mathrm{Wasserstein}(\mathrm{Law}(Z_t^n), P_n^0) = \phi(b), \tag{2.44}$$

where for all $b \in \mathbb{R}$,

$$t_{n,b} = \log|z_0^n| + b \quad \text{and} \quad \phi(b) = \mathrm{e}^{-b}. \tag{2.45}$$

For the other distances and divergences, let us assume that the following limit exists

$$a := \lim_{n\to\infty} \sqrt{n}|z_0^n|^2 \in [0, +\infty]. \tag{2.46}$$

This quantity can be related with

$$c_n^A := \log(|z_0^n|\sqrt{n}) \quad \text{and} \quad c_n^B := \frac{\log n}{4} \tag{2.47}$$

which were already introduced in Remark 1. Indeed

$$a = 0 \Longleftrightarrow c_n^A \ll c_n^B, \quad a = +\infty \Longleftrightarrow c_n^A \gg c_n^B,$$

while $a \in (0, \infty)$ is equivalent to $c_n^A \asymp c_n^B$.
Then, for $\mathrm{dist} \in \{\mathrm{Hellinger}, \mathrm{Kullback}, \chi^2, \mathrm{Fisher}\}$, we have, for all $b \in \mathbb{R}$,

$$\lim_{n\to\infty} \mathrm{dist}(\mathrm{Law}(Z_{t_{n,b}}) \mid P_n^0) = \phi(b), \tag{2.48}$$

where $t_{n,b}$ and $\phi(b)$ are as in Table 2.1. The cutoff window is always of size 1.

Since the total variation distance is not expressed in a simple explicit manner, further computations are needed to extract the precise cutoff profile, which is given in the following lemma:

	$a=+\infty$	$a=0$	$a\in(0,+\infty)$
$t_{n,b}$			
Hellinger	$\log(\lvert z_0^n\rvert\sqrt{n})+b$	$\frac{\log n}{4}+b$	$\frac{\log n}{4}+b$
Kullback	$\log(\lvert z_0^n\rvert\sqrt{n})+b$	$\frac{\log n}{4}+b$	$\frac{\log n}{4}+b$
χ^2	$\log(\lvert z_0^n\rvert\sqrt{n})+b$	$\frac{\log n}{4}+b$	$\frac{\log n}{4}+b$
Fisher	$\log(\lvert z_0^n\rvert n)+b$	$\frac{\log n}{2}+b$	$\frac{\log n}{2}+b$
$\phi(b)$			
Hellinger	$\sqrt{1-\mathrm{e}^{-\frac{1}{8}\mathrm{e}^{-2b}}}$	$\sqrt{1-\mathrm{e}^{-\frac{1}{16}\mathrm{e}^{-4b}}}$	$\sqrt{1-\mathrm{e}^{-\frac{1}{8}a\mathrm{e}^{-2b}-\frac{1}{16}\mathrm{e}^{-4b}}}$
Kullback	$\frac{1}{2}\mathrm{e}^{-2b}$	$\frac{1}{4}\mathrm{e}^{-4b}$	$\frac{1}{2}a\mathrm{e}^{-2b}+\frac{1}{4}\mathrm{e}^{-4b}$
χ^2	$\mathrm{e}^{\mathrm{e}^{-2b}}-1$	$\mathrm{e}^{\frac{1}{2}\mathrm{e}^{-4b}}-1$	$\mathrm{e}^{a\mathrm{e}^{-2b}+\frac{1}{2}\mathrm{e}^{-4b}}-1$
Fisher	e^{-2b}	e^{-4b}	$a\mathrm{e}^{-2b}+\mathrm{e}^{-4b}$

Table 2.1: Values of $t_{n,b}$ and $\phi(b)$ for the cutoff profile of the OU process in (2.48).

Lemma 2.3.2 (Cutoff profile in TV for OU)**.** *Let $Z^n=(Z_t^n)_{t\geqslant 0}$ be the OU process (2.8), started from $z_0^n\in\mathbb{R}^n$, and let P_n^0 be its invariant law. Assume as in (2.46) that $a:=\lim_{n\to\infty}|z_0^n|^2\sqrt{n}\in[0,+\infty]$, and let $t_{n,b}$ be as in Table (2.1) for Hellinger. Then, for all $b\in\mathbb{R}$, we have*

$$\lim_{n\to\infty}\|\mathrm{Law}(Z^n_{t_{n,b}})-P_n^0\|_{\mathrm{TV}}=\phi(b),$$

where

$$\phi(b):=\begin{cases}\mathrm{erf}\Big(\dfrac{\mathrm{e}^{-b}}{2\sqrt{2}}\Big) & \text{if } a=+\infty\\[2mm] \mathrm{erf}\Big(\dfrac{\mathrm{e}^{-2b}}{4}\Big) & \text{if } a=0\\[2mm] \mathrm{erf}\Big(\dfrac{\sqrt{2a\mathrm{e}^{-2b}+\mathrm{e}^{-4b}}}{4}\Big) & \text{if } a\in(0,+\infty)\end{cases},$$

where $\mathrm{erf}(u):=\dfrac{1}{\sqrt{\pi}}\displaystyle\int_{|t|\leqslant u}\mathrm{e}^{-t^2}\mathrm{d}t=\mathbb{P}(|X|\leqslant\sqrt{2}u)$ *with $X\sim\mathcal{N}(0,1)$ is the* error function.

Proof of Lemma 2.3.2. The idea is to exploit the fact that we consider Gaussian product measures (the covariance matrices are multiple of the identity), which allows a finer analysis than for instance in [98, Le. 3.1]. We begin with a rather general step. Let μ and ν be two probability measures on $\mathbb{R}^n$ with densities f and g with respect to the Lebesgue measure $\mathrm{d}x$. We have then

$$\|\mu-\nu\|_{\mathrm{TV}}=\frac{1}{2}\int|f-g|\mathrm{d}x=\frac{1}{2}\Big(\int(f-g)\mathbf{1}_{g\leqslant f}\mathrm{d}x-\int(f-g)\mathbf{1}_{f\leqslant g}\mathrm{d}x\Big), \tag{2.49}$$

and since

$$-\int(f-g)\mathbf{1}_{f\leqslant g}\mathrm{d}x=-\int(f-g)(1-\mathbf{1}_{g<f})\mathrm{d}x=\int(f-g)\mathbf{1}_{g<f}\mathrm{d}x=\int(f-g)\mathbf{1}_{g\leqslant f}\mathrm{d}x,$$

we obtain

$$\|\mu-\nu\|_{\mathrm{TV}}=\int(f-g)\mathbf{1}_{g\leqslant f}\mathrm{d}x=\mu(g\leqslant f)-\nu(g\leqslant f). \tag{2.50}$$

In particular, when $\mu=\mathcal{N}(m_1,\sigma_1^2 I_n)$ and $\nu=\mathcal{N}(m_2,\sigma_2^2 I_n)$ then $g(x)\leqslant f(x)$ is equivalent to

$$\psi(x):=\frac{|x-m_1|^2}{\sigma_1^2}-\frac{|x-m_2|^2}{\sigma_2^2}\leqslant n\log\frac{\sigma_2^2}{\sigma_1^2}, \tag{2.51}$$

for all $x\in\mathbb{R}^n$, and therefore, with $Z_1\sim\mu$ and $Z_2\sim\nu$, we get

$$\|\mu-\nu\|_{\mathrm{TV}}=\mathbb{P}\Big(\psi(Z_1)\leqslant n\log\frac{\sigma_2^2}{\sigma_1^2}\Big)-\mathbb{P}\Big(\psi(Z_2)\leqslant n\log\frac{\sigma_2^2}{\sigma_1^2}\Big). \tag{2.52}$$

Let us assume from now on that $m_2=0$ and $\sigma_1\neq\sigma_2$. We can then gather the quadratic terms as

$$\psi(x)=\Big(1-\frac{\sigma_1^2}{\sigma_2^2}\Big)\frac{|x-\tilde{m}_1|^2}{\sigma_1^2}+\Big(\frac{1}{\sigma_1^2}-\frac{1}{(1-\frac{\sigma_1^2}{\sigma_2^2})\sigma_1^2}\Big)|m_1|^2 \quad\text{where}\quad \tilde{m}_1:=\frac{1}{1-\frac{\sigma_1^2}{\sigma_2^2}}m_1. \tag{2.53}$$

We observe at this step that the random variable $\frac{|Z_1-\tilde{m}_1|^2}{\sigma_1^2}$ follows a noncentral chi-squared distribution, which depends only on n and on the noncentrality parameter

$$\lambda_1:=\frac{|m_1-\tilde{m}_1|^2}{\sigma_1^2}=\frac{\sigma_1^2}{(\sigma_2^2-\sigma_1^2)^2}|m_1|^2. \tag{2.54}$$

Similarly, the random variable $\frac{|Z_2-\tilde{m}_1|^2}{\sigma_2^2}$ follows a noncentral chi-squared distribution, which depends only on n and on the noncentrality parameter

$$\lambda_2:=\frac{|\tilde{m}_1|^2}{\sigma_1^2}=\frac{\sigma_2^4}{\sigma_1^2(\sigma_2^2-\sigma_1^2)^2}|m_1|^2. \tag{2.55}$$

It follows that the law of $\psi(Z_1)$ and the law of $\psi(Z_2)$ depend over m_1 only via $|m_1|$. Hence

$$\psi(Z_1)\overset{\mathrm{d}}{=}X_1+\cdots+X_n \quad\text{and}\quad \psi(Z_2)\overset{\mathrm{d}}{=}Y_1+\cdots+Y_n \tag{2.56}$$

where $X_1,\ldots,X_n$ and $Y_1,\ldots,Y_n$ are two sequences of i.i.d. random variables for which the mean and variance depends only (and explicitly) on $|m_1|$, σ_1, σ_2. Note in particular that these means and variances are given by $\frac{1}{n}$ the ones of $\psi(Z_1)$ and $\psi(Z_2)$. Now we specialize to the case where $\mu=\mathrm{Law}(Z_t^n)=\mathcal{N}(z_0^n\mathrm{e}^{-t},\frac{1-\mathrm{e}^{-2t}}{n}I_n)$ and $\nu=\mathrm{Law}(Z_\infty^n)=\mathcal{N}(0,\frac{1}{n}I_n)=P_n^0$, and we find

$$\mathbb{E}[\psi(Z_1)]=n\Big(1-\frac{\sigma_t^2}{\sigma_\infty^2}\Big)-\frac{|z_0^n|^2\mathrm{e}^{-2t}}{\sigma_\infty^2},\quad \mathbb{E}[\psi(Z_2)]=n\Big(\frac{\sigma_\infty^2}{\sigma_t^2}-1\Big)+\frac{|z_0^n|^2\mathrm{e}^{-2t}}{\sigma_t^2}$$

while

$$\mathrm{Var}[\psi(Z_1)]=2n\Big(\frac{1}{\sigma_t^2}-\frac{1}{\sigma_\infty^2}\Big)^2\sigma_t^4+4\frac{\sigma_t^2}{\sigma_\infty^4}|z_0^n|^2\mathrm{e}^{-2t},\quad \mathrm{Var}[\psi(Z_2)]=2n\Big(\frac{1}{\sigma_t^2}-\frac{1}{\sigma_\infty^2}\Big)^2\sigma_\infty^4+4\frac{\sigma_\infty^2}{\sigma_t^4}|z_0^n|^2\mathrm{e}^{-2t}.$$

Let $t=t_{n,b}$ be as in Table 2.1 for Hellinger. Using (2.52) and the central limit theorem for the i.i.d. random variables $X_1,\ldots,X_n$ and $Y_1,\ldots,Y_n$, we get, with $Z\sim\mathcal{N}(0,1)$,

$$\|\mathrm{Law}(Z_t^n)-P_n^0\|_{\mathrm{TV}}=\mathbb{P}(Z\leqslant\gamma_{n,t})-\mathbb{P}(Z\leqslant\tilde{\gamma}_{n,t})+o_n(1),$$

where

$$\gamma_{n,t} := \frac{-n\log(1-\mathrm{e}^{-2t}) - \mathbb{E}[\psi(Z_t^n)]}{\sqrt{\mathrm{Var}[\psi(Z_t^n)]}}, \quad \tilde{\gamma}_{n,t} := \frac{-n\log(1-\mathrm{e}^{-2t}) - \mathbb{E}[\psi(Z_\infty^n)]}{\sqrt{\mathrm{Var}[\psi(Z_\infty^n)]}}.$$

Expanding $\gamma_{n,t_{n,b}}$ gives the cutoff profile. Let us detail the computations in the most involved case $\lim_{n\to\infty}|z_0^n|^2\sqrt{n} = a \in (0,+\infty)$. For all $b \in \mathbb{R}$, recall $t_{n,b} = \frac{\log n}{4} + b$. One may check that

$$-n\log(1-\mathrm{e}^{-2t_{n,b}}) - \mathbb{E}[\psi(Z_{t_{n,b}}^n)] = \frac{1}{2}\mathrm{e}^{-4b} + a\mathrm{e}^{-2b} + o_n(1),$$

$$-n\log(1-\mathrm{e}^{-2t_{n,b}}) - \mathbb{E}[\psi(Z_\infty^n)] = -\frac{1}{2}\mathrm{e}^{-4b} - a\mathrm{e}^{-2b} + o_n(1),$$

$$\mathrm{Var}[\psi(Z_{t_{n,b}}^n)] = 2\mathrm{e}^{-4b} + 4a\mathrm{e}^{-2b} + o_n(1), \quad \mathrm{Var}[\psi(Z_\infty^n)] = 2\mathrm{e}^{-4b} + 4a\mathrm{e}^{-2b} + o_n(1).$$

It follows that

$$\lim_{n\to\infty}\|\mathrm{Law}(Z_{t_{n,b}}^n) - P_n^0\|_{\mathrm{TV}} = \mathbb{P}\Big(|Z| \leqslant \frac{1}{2\sqrt{2}}\sqrt{\mathrm{e}^{-4b}+2a\mathrm{e}^{-2b}}\Big) = \mathrm{erf}\Big(\frac{1}{4}\sqrt{\mathrm{e}^{-4b}+2a\mathrm{e}^{-2b}}\Big).$$

The other cases are similar. □

2.4 General exactly solvable aspects

In this section, we prove Theorem 2.1.3 and Corollary 2.1.4.

The proof of Theorem 2.1.3 is based on the fact that the polynomial functions $\pi(x) = x_1 + \cdots + x_n$ and $|x|^2 = x_1^2 + \cdots + x_n^2$ are, up to an additive constant for the second, eigenfunctions of the dynamics associated to the spectral values -1 and -2 respectively, and that their "carré du champ" is affine. In the matrix cases $\beta \in \{1,2\}$, these functions correspond to the dynamics of the trace, the dynamics of the squared Hilbert–Schmidt trace norm, and the dynamics of the squared trace. It is remarkable that this phenomenon survives beyond these matrix cases, yet another manifestation of the Gaussian "ghosts" concept due to Edelman, see for instance [107].

Proof of Theorem 2.1.3. The process $Y_t := \pi(X_t^n)$ solves

$$\mathrm{d}Y_t = \sum_{i=1}^n \mathrm{d}X_t^{n,i} = \sqrt{\frac{2}{n}}\sum_{i=1}^n \mathrm{d}B_t^i - \sum_{i=1}^n X_t^{n,i}\mathrm{d}t + \frac{\beta}{n}\sum_{j\neq i}\frac{\mathrm{d}t}{X_t^{n,i}-X_t^{n,j}}.$$

By symmetry, the double sum vanishes. Note that the process $W_t := \frac{1}{\sqrt{n}}\sum_{i=1}^n B_t^i$ is a standard one dimensional BM, so that $\mathrm{d}Y_t = \sqrt{2}\mathrm{d}W_t - Y_t\mathrm{d}t$. This proves the first part of the statement.
We turn to the second part. Recall that $X_t \in D_n$ for all $t > 0$. By Itô's formula

$$\mathrm{d}(X_t^{n,i})^2 = \sqrt{\frac{8}{n}}X_t^{n,i}\mathrm{d}B_t^i - 2(X_t^{n,i})^2\mathrm{d}t + 2\frac{\beta}{n}X_t^{n,i}\sum_{j:j\neq i}\frac{\mathrm{d}t}{X_t^{n,i}-X_t^{n,j}} + \frac{2}{n}\mathrm{d}t.$$

Set $W_t := \sum_{i=1}^n\int_0^t \frac{X_s^{n,i}}{|X_s^n|}\mathrm{d}B_s^i$. The process $(W_t)_{t\geqslant 0}$ is a BM by the Lévy characterization since

$$\langle W\rangle_t = \int_0^t \frac{\sum_{i=1}^n (X_s^{n,i})^2}{|X_s^n|^2}\mathrm{d}s = t.$$

Furthermore, a simple computation shows that

$$\sum_{i=1}^{n} X_t^{n,i} \sum_{j:j\neq i} \frac{1}{X_t^{n,i} - X_t^{n,j}} = \frac{n(n-1)}{2}.$$

Consequently the process $R_t := |X_t^n|^2$ solves

$$\mathrm{d}R_t = \sqrt{\frac{8}{n}R_t}\mathrm{d}W_t + \Big(2 + \beta(n-1) - 2R_t\Big)\mathrm{d}t,$$

and is therefore a CIR process of parameters $a = 2 + \beta(n-1)$, $b = 2$, and $\sigma = \sqrt{8/n}$.
When $d = \frac{\beta}{2}n^2 + (1 - \frac{\beta}{2})n$ is a positive integer, the last property of the statement follows from the connection between OU and CIR recalled right before the statement of the theorem. □

The last proof actually relies on the following general observation. Let X be an n-dimensional continuous semi-martingale solution of

$$\mathrm{d}X_t = \sigma(X_t)\mathrm{d}B_t + b(X_t)\mathrm{d}t$$

where B is a n-dimensional standard BM, and where

$$x \in \mathbb{R}^n \mapsto \sigma(x) \in \mathcal{M}_{n,n}(\mathbb{R}) \quad \text{and} \quad x \in \mathbb{R}^n \mapsto b(x) \in \mathbb{R}^n$$

are Lipschitz. The infinitesimal generator of the Markov semigroup is given by

$$\mathsf{G}(f)(x) = \frac{1}{2}\sum_{i,j=1}^{n} a_{i,j}(x)\partial_{i,j}f(x) + \sum_{i=1}^{n} b_i(x)\partial_i f(x), \quad \text{where} \quad a(x) = \sigma(x)(\sigma(x))^\top,$$

for all $f \in \mathcal{C}^2(\mathbb{R}^n, \mathbb{R})$ and $x \in \mathbb{R}^n$. Then, by Itô's formula, the process $M^f = (M_t^f)_{t\geqslant 0}$ given by

$$M_t^f = f(X_t) - f(X_0) - \int_0^t (\mathsf{G}f)(X_s)\mathrm{d}s = \sum_{i,k=1}^{n} \int_0^t \partial_i f(X_s)\sigma_{i,k}(X_s)\mathrm{d}B_s^k$$

is a local martingale, and moreover, for all $t \geqslant 0$,

$$\langle M^f \rangle_t = \int_0^t \Gamma(f)(X_s)\mathrm{d}s \quad \text{where} \quad \Gamma(f)(x) = |\sigma(x)^\top \nabla f(x)|^2 = a(x)\nabla f \cdot \nabla f.$$

The functional quadratic form Γ is known as the "carré du champ" operator.
If f is an eigenfunction of G associated to the spectral value λ in the sense that $\mathsf{G}f = \lambda f$ (note by the way that $\lambda \leqslant 0$ since G generates a Markov process), then we get

$$f(X_t) = f(X_0) + \lambda \int_0^t f(X_s)\mathrm{d}s + M_t^f, \quad \text{in other words} \quad \mathrm{d}f(X_t) = \mathrm{d}M_t^f + \lambda f(X_t)\mathrm{d}t.$$

Now if $\Gamma(f) = c$ (as in the first part of the theorem), then by the Lévy characterization of Brownian motion, the continuous local martingale $W := \frac{1}{\sqrt{c}}M^f$ starting from the origin is a standard BM and we recover the result of the first part of the theorem. On the other hand, if $\Gamma(f) = cf$ (as in the second part of the theorem), then by the Lévy characterization of BM the local martingale

$$W := \int_0^t \frac{1}{\sqrt{cf(X_s)}}\mathrm{d}M_s^f$$

is a standard BM and we recover the result of the second part.

At this point, we observe that the infinitesimal generator of the CIR process R is the Laguerre partial differential operator

$$L(f)(x) = \frac{4}{n} x f''(x) + (2 + \beta(n-1) - 2x) f'(x). \tag{2.57}$$

This operator leaves invariant the set of polynomials of degree less than or equal to k, for all integer $k \geqslant 0$, a property inherited from (2.22). We will use this property in the following proof.

2.4.1 Proof of Corollary 2.1.4

By Theorem 2.1.3, $Z = \pi(X^n)$ is an OU process in $\mathbb{R}$ solution of the stochastic differential equation

$$Z_0 = \pi(X_0^n), \quad \mathrm{d}Z_t = \sqrt{2}\mathrm{d}B_t - Z_t \mathrm{d}t,$$

where B is a standard one-dimensional BM. By Lemma 2.3.1, $Z_t \sim \mathcal{N}(Z_0 \mathrm{e}^{-t}, 1 - \mathrm{e}^{-2t})$ for all $t \geqslant 0$ and the equilibrium distribution is $P_n^\beta \circ \pi^{-1} = \mathcal{N}(0,1)$. Using the contraction property stated in Lemma 2.8.2, the comparison between Hellinger and TV of Lemma 2.8.1 and the explicit expressions for Gaussian distributions of Lemma 2.8.5, we find

$$\begin{aligned}\|\mathrm{Law}(X_t^n) - P_n^\beta\|_{\mathrm{TV}} &\geqslant \|\mathrm{Law}(Z_t) - P_n^\beta \circ \pi^{-1}\|_{\mathrm{TV}} \\ &\geqslant \mathrm{Hellinger}^2(\mathrm{Law}(Z_t), P_n^\beta \circ \pi^{-1}) \\ &= 1 - \frac{(1-\mathrm{e}^{-2t})^{1/4}}{(1-\frac{1}{2}\mathrm{e}^{-2t})^{1/2}} \exp\Big(-\frac{\pi(X_0^n)^2 \mathrm{e}^{-2t}}{4(2-\mathrm{e}^{-2t})}\Big).\end{aligned}$$

Setting $c_n := \log(|\pi(X_0^n)|)$ and assuming that $\lim_{n\to\infty} c_n = \infty$, we deduce that for all $\varepsilon \in (0,1)$

$$\lim_{n\to\infty} \|\mathrm{Law}(X_{c_n(1-\varepsilon)}^n) - P_n^\beta\|_{\mathrm{TV}} = 1.$$

The comparison between $\mathrm{Hellinger}$ and TV of Lemma 2.8.1 allows to deduce that this remains true for the Hellinger distance.

We turn to Kullback. The contraction property stated in Lemma 2.8.2 and the explicit expressions for Gaussian distributions of Lemma 2.8.5 yield

$$\begin{aligned}2\mathrm{Kullback}(\mathrm{Law}(X_t^n) \mid P_n^\beta) &\geqslant 2\mathrm{Kullback}(\mathrm{Law}(Z_t) \mid P_n^\beta \circ \pi^{-1}) \\ &= \pi(X_0^n)^2 \mathrm{e}^{-2t} - \mathrm{e}^{-2t} - \log(1-\mathrm{e}^{-2t}).\end{aligned}$$

This is enough to deduce that

$$\lim_{n\to\infty} \mathrm{Kullback}(\mathrm{Law}(X_{(1-\varepsilon)c_n}^n) \mid P_n^\beta) = +\infty.$$

The situation is similar for χ^2: the contraction property stated in Lemma 2.8.2 and the explicit expressions for Gaussian distributions of Lemma 2.8.5 yield

$$\begin{aligned}\chi^2(\mathrm{Law}(X_t^n) \mid P_n^\beta) &\geqslant \chi^2(\mathrm{Law}(Z_t) \mid P_n^\beta \circ \pi^{-1}) \\ &= -1 + \frac{1}{\sqrt{1-\mathrm{e}^{-4t}}} \exp\left(\frac{1}{1+\mathrm{e}^{-2t}}(1-\pi(X_0^n)\mathrm{e}^{-t})^2\right),\end{aligned}$$

so that

$$\lim_{n\to\infty} \chi^2(\mathrm{Law}(X_{(1-\varepsilon)c_n}^n) \mid P_n^\beta) = +\infty.$$

Regarding the Wasserstein distance, we have $\|\pi\|_{\mathrm{Lip}} := \sup_{x\neq y}\frac{|\pi(x)-\pi(y)|}{|x-y|} \leqslant \sqrt{n}$ from the Cauchy–Schwarz inequality, and by Lemma 2.8.2, for all probability measures μ and ν on $\mathbb{R}^n$,

$$\mathrm{Wasserstein}(\mu\circ\pi^{-1},\nu\circ\pi^{-1}) \leqslant \sqrt{n}\mathrm{Wasserstein}(\mu,\nu). \tag{2.58}$$

Using the explicit expressions for Gaussian distributions of Lemma 2.8.5, we thus find

$$\begin{aligned}\mathrm{Wasserstein}^2(\mathrm{Law}(X_t^n),P_n^\beta) &\geqslant \frac{1}{n}\mathrm{Wasserstein}^2(\mathrm{Law}(Z_t),P_n^\beta\circ\pi^{-1})\\ &= \frac{1}{n}\Big(\pi(X_0^n)^2\mathrm{e}^{-2t}+2-\mathrm{e}^{-2t}-2\sqrt{1-\mathrm{e}^{-2t}}\Big).\end{aligned}$$

Setting $c_n := \log\left(\frac{|\pi(x_0^n)|}{\sqrt{n}}\right)$ and assuming $c_n\to\infty$ as $n\to\infty$, we thus deduce that for all $\varepsilon\in(0,1)$

$$\lim_{n\to\infty}\mathrm{Wasserstein}(\mathrm{Law}(X^n_{(1-\varepsilon)c_n}),P_n^\beta) = +\infty.$$

2.5 The random matrix cases

In this section, we prove Theorem 2.1.5 and Corollary 2.1.6 that cover the matrix cases $\beta\in\{1,2\}$. For these values of β, the DOU process is the image by the spectral map of a matrix OU process, connected to the random matrix models GOE and GUE. We could consider the case $\beta=4$ related to GSE. Beyond these three algebraic cases, it could be possible for an arbitrary $\beta\geqslant 1$ to use random tridiagonal matrices dynamics associated to β Dyson processes, see for instance [151].

The next two subsections are devoted to the proof of Theorem 2.1.5 in the $\beta=2$ and $\beta=1$ cases respectively. The third section provides the proof of Corollary 2.1.6.

2.5.1 Hermitian case ($\beta=2$)

Let Herm_n be the set of $n\times n$ complex Hermitian matrices, namely the set of $h\in\mathcal{M}_{n,n}(\mathbb{C})$ with $h_{i,j}=\overline{h_{j,i}}$ for all $1\leqslant i,j\leqslant n$. An element $h\in\mathrm{Herm}_n$ is parametrized by the n^2 real variables $(h_{i,i})_{1\leqslant i\leqslant n}$, $(\Re h_{i,j})_{1\leqslant i<j\leqslant n}$, $(\Im h_{i,j})_{1\leqslant i<j\leqslant n}$. We define, for $h\in\mathrm{Herm}_n$ and $1\leqslant i,j\leqslant n$,

$$\pi_{i,j}(h) = \begin{cases} h_{i,i} & \text{if } i=j\\ \sqrt{2}\,\Re h_{i,j} & \text{if } i<j\\ \sqrt{2}\,\Im h_{j,i} & \text{if } i>j\end{cases}. \tag{2.59}$$

Note that

$$\mathrm{Tr}(h^2) = \sum_{i,j=1}^n |h_{i,j}|^2 = \sum_{i=1}^n h_{i,i}^2 + 2\sum_{i<j}(\Re h_{i,j})^2 + 2\sum_{i<j}(\Im h_{i,j})^2 = \sum_{i,j}\pi_{i,j}(h)^2.$$

We thus identify Herm_n with $\mathbb{R}^n\times\mathbb{R}^{2\frac{n^2-n}{2}}=\mathbb{R}^{n^2}$, this identification is isometrical provided Herm_n is endowed with the norm $\sqrt{\mathrm{Tr}(h^2)}$ and $\mathbb{R}^{n^2}$ with the Euclidean norm.

The Gaussian Unitary Ensemble GUE_n is the Gaussian law on Herm_n with density

$$h\in\mathrm{Herm}_n\mapsto\frac{\mathrm{e}^{-\frac{n}{2}\mathrm{Tr}(h^2)}}{C_n}\quad\text{where}\quad C_n := \int_{\mathbb{R}^{n^2}}\mathrm{e}^{-\frac{n}{2}\mathrm{Tr}(h^2)}\prod_{i=1}^n\mathrm{d}h_{i,i}\prod_{i<j}\mathrm{d}\Re h_{i,j}\prod_{i<j}\mathrm{d}\Im h_{i,j}. \tag{2.60}$$

If H is a random $n\times n$ Hermitian matrix then $H\sim\mathrm{GUE}_n$ if and only if the n^2 real random variables $\pi_{i,j}(H)$, $1\leqslant i,j\leqslant n$, are independent Gaussian random variables with

$$\pi_{i,j}(H)\sim\mathcal{N}\Big(0,\frac{1}{n}\Big),\quad 1\leqslant i,j\leqslant n. \tag{2.61}$$

The law GUE_n is the unique invariant law of the Hermitian matrix OU process $(H_t)_{t\geqslant 0}$ on Herm_n solution of the stochastic differential equation

$$H_0=h_0\in\mathrm{Herm}_n,\quad \mathrm{d}H_t=\sqrt{\frac{2}{n}}\mathrm{d}B_t-H_t\mathrm{d}t, \tag{2.62}$$

where $B=(B_t)_{t\geqslant 0}$ is a Brownian motion on Herm_n, in the sense that the stochastic processes $(\pi_{i,j}(B_t))_{t\geqslant 0}$, $1\leqslant i\neq j\leqslant n$, are independent standard one-dimensional BM. The coordinates stochastic processes $(\pi_{i,j}(H_t))_{t\geqslant 0}$, $1\leqslant i,j\leqslant n$, are independent real OU processes.

For any h in Herm_n, we denote by $\Lambda(h)$ the vector of the eigenvalues of h ordered in non-decreasing order. Lemma 2.5.1 below is an observation which dates back to the seminal work of Dyson [106], hence the name DOU for X^n. We refer to [113, Ch. 12] and [6, Sec. 4.3] for a mathematical approach using modern stochastic calculus.

Lemma 2.5.1 (From matrix OU to DOU). *The image of GUE_n by the map Λ is the Coulomb gas P_n^β given by (2.6) with $\beta=2$. Moreover the stochastic process $X^n=(X^n_t)_{t\geqslant 0}=(\Lambda(H_t))_{t\geqslant 0}$ is well-defined and solves the stochastic differential equation (2.3) with $\beta=2$ and $x_0^n=\Lambda(h_0)$.*

Let $\beta=2$. Let us assume from now on that the initial value $h_0\in\mathrm{Herm}_n$ of $(H_t)_{t\geqslant 0}$ has eigenvalues x_0^n where x_0^n is as in Theorem 2.1.5. We start by proving the upper bound on the χ^2 distance stated in Theorem 2.1.5: it will be an adaptation of the proof of the upper bound of Theorem 2.1.1 applied to the Hermitian matrix OU process $(H_t)_{t\geqslant 0}$ combined with the contraction property of the χ^2 distance. Indeed, by Lemma 2.5.1 and the contraction property of Lemma 2.8.2

$$\chi^2(\mathrm{Law}(X_t^n)\mid P_n^\beta)\leqslant\chi^2(\mathrm{Law}(H_t)\mid\mathrm{GUE}_n). \tag{2.63}$$

We claim now that the right-hand side tends to 0 as $n\to\infty$ when $t=t_n$ is well chosen. Indeed, using the identification between Herm_n and $\mathbb{R}^{n^2}$ mentioned earlier, we have $\mathrm{GUE}_n=\mathcal{N}(m_2,\Sigma_2)$ where $m_2=0$ and where Σ_2 is an $n^2\times n^2$ diagonal matrix with

$$(\Sigma_2)_{(i,j),(i,j)}=\frac{1}{n}. \tag{2.64}$$

On the other hand, the Mehler formula (Lemma 2.3.1) gives $\mathrm{Law}(H_t)=\mathcal{N}(m_1,\Sigma_1)$ where $m_1=\mathrm{e}^{-t}h_0$ and where Σ_1 is an $n^2\times n^2$ diagonal matrix with

$$(\Sigma_1)_{(i,j),(i,j)}=\frac{1-\mathrm{e}^{-2t}}{n}. \tag{2.65}$$

Therefore, using Lemma 2.8.5, the analogue of (2.40) reads

$$\chi^2(\mathrm{Law}(H_t)\mid\mathrm{GUE}_n)=-1+\frac{1}{(1-\mathrm{e}^{-4t})^{n^2/2}}\exp\left(n|h_0|^2\frac{\mathrm{e}^{-2t}}{1+\mathrm{e}^{-2t}}\right). \tag{2.66}$$

where

$$|h_0|^2=\sum_{1\leqslant i,j\leqslant n}\pi_{i,j}(h_0)^2=\sum_{1\leqslant i,j\leqslant n}|(h_0)_{i,j}|^2=\mathrm{Tr}(h_0^2)=|x_0^n|^2. \tag{2.67}$$

Taking now $c_n := \log(\sqrt{n}|x_0^n|) \vee \log(\sqrt{n})$, for any $\varepsilon \in (0,1)$, we get

$$\chi^2(\mathrm{Law}(X^n_{(1+\varepsilon)c_n}) \mid P_n^\beta) \leqslant \chi^2(\mathrm{Law}(H_{(1+\varepsilon)c_n}) \mid \mathrm{GUE}_n) \underset{n\to\infty}{\longrightarrow} 0. \tag{2.68}$$

In the right-hand side of (2.66), the factor n^2 is the dimension of the $\mathbb{R}^{n^2}$ to which Herm_n is identified, while the factor n in the first term is due to the $1/n$ scaling in the stochastic differential equation of the process. This explains the difference with the analogue (2.40) in dimension n.

From the comparison between TV, Hellinger, Kullback and χ^2 stated in Lemma 2.8.1, we easily deduce that the previous convergence remains true upon replacing χ^2 by TV, $\mathrm{Hellinger}$ or $\mathrm{Kullback}$.

It remains to cover the upper bound for the Wasserstein distance. This distance is more sensitive to contraction arguments: according to Lemma 2.8.2, one needs to control the Lipschitz norm of the "contraction map" at stake. It happens that the spectral map, restricted to the set Herm_n of $n \times n$ Hermitian matrices, is 1-Lipschitz: more precisely, the Hoffman–Wielandt inequality, see [150] and [152, Th. 6.3.5], asserts that for any two such matrices A and B, denoting $\Lambda(A) = (\lambda_i(A))_{1\leqslant i\leqslant n}$ and $\Lambda(B) = (\lambda_i(B))_{1\leqslant i\leqslant n}$ the ordered sequences of their eigenvalues, we have

$$\sum_{i=1}^n |\lambda_i(A) - \lambda_i(B)|^2 \leqslant \sum_{i,j} |A_{i,j} - B_{i,j}|^2.$$

Applying Lemma 2.8.2, we thus deduce that

$$\mathrm{Wasserstein}(\mathrm{Law}(X_t^n), P_n^\beta) \leqslant \mathrm{Wasserstein}(\mathrm{Law}(H_t), \mathrm{GUE}_n). \tag{2.69}$$

Following the Gaussian computations in the proof of Theorem 2.1.2, we obtain

$$\mathrm{Wasserstein}^2(\mathrm{Law}(H_t), \mathrm{GUE}_n) = |x_0^n|^2 \mathrm{e}^{-2t} + 2 - \mathrm{e}^{-2t} - 2\sqrt{1-\mathrm{e}^{-2t}}. \tag{2.70}$$

Set $c_n := \log(|x_0^n|)$. If $c_n \to \infty$ as $n \to \infty$ then for all $\varepsilon \in (0,1)$ we find

$$\mathrm{Wasserstein}(\mathrm{Law}(X^n_{(1+\varepsilon)c_n}), P_n^\beta) \underset{n\to\infty}{\longrightarrow} 0.$$

This completes the proof of Theorem 2.1.5.

2.5.2 Symmetric case ($\beta = 1$)

The method is similar to the case $\beta = 2$. Let us focus only on the differences. Let Sym_n be the set of $n \times n$ real symmetric matrices, namely the set of $s \in \mathcal{M}_{n,n}(\mathbb{R})$ with $s_{i,j} = s_{j,i}$ for all $1 \leqslant i, j \leqslant n$. An element $s \in \mathrm{Sym}_n$ is parametrized by the $n + \frac{n^2-n}{2} = \frac{n(n+1)}{2}$ real variables $(s_{i,j})_{1\leqslant i\leqslant j\leqslant n}$. We define, for $s \in \mathrm{Sym}_n$ and $1 \leqslant i \leqslant j \leqslant n$,

$$\pi_{i,j}(s) = \begin{cases} s_{i,i} & \text{if } i = j \\ \sqrt{2}\, s_{i,j} & \text{if } i < j \end{cases}. \tag{2.71}$$

Note that

$$\mathrm{Tr}(s^2) = \sum_{i,j=1}^n s_{i,j}^2 = \sum_{i=1}^n s_{i,i}^2 + 2\sum_{i<j} s_{i,j}^2 = \sum_{1\leqslant i\leqslant j\leqslant n} \pi_{i,j}(s)^2.$$

We thus identify isometrically Sym_n, endowed with the norm $\sqrt{\mathrm{Tr}(h^2)}$, with $\mathbb{R}^n \times \mathbb{R}^{\frac{n^2-n}{2}} = \mathbb{R}^{\frac{n(n+1)}{2}}$ endowed with the Euclidean norm.

The Gaussian Orthogonal Ensemble GOE_n is the Gaussian law on Sym_n with density

$$s \in \mathrm{Sym}_n \mapsto \frac{\mathrm{e}^{-\frac{n}{2}\mathrm{Tr}(s^2)}}{C_n} \quad \text{where} \quad C_n := \int_{\mathbb{R}^{\frac{n(n+1)}{2}}} \mathrm{e}^{-\frac{n}{2}\mathrm{Tr}(s^2)} \prod_{1\leqslant i\leqslant j\leqslant n} \mathrm{d}s_{i,j}. \tag{2.72}$$

If S is a random $n \times n$ real symmetric matrix then $S \sim \mathrm{GOE}_n$ if and only if the $\frac{n(n+1)}{2}$ real random variables $\pi_{i,j}(S)$, $1 \leqslant i \leqslant j \leqslant n$, are independent Gaussian random variables with

$$\pi_{i,j}(S) \sim \mathcal{N}\Big(0, \frac{1}{n}\Big), \quad 1 \leqslant i \leqslant j \leqslant n. \tag{2.73}$$

The law GOE_n is the unique invariant law of the real symmetric matrix OU process $(S_t)_{t\geqslant 0}$ on Sym_n solution of the stochastic differential equation

$$S_0 = s_0 \in \mathrm{Sym}_n, \quad \mathrm{d}S_t = \sqrt{\frac{2}{n}}\mathrm{d}B_t - S_t\mathrm{d}t \tag{2.74}$$

where $B = (B_t)_{t\geqslant 0}$ is a Brownian motion on Sym_n, in the sense that the stochastic processes $(\pi_{i,j}(B_t))_{t\geqslant 0}$, $1 \leqslant i \leqslant j \leqslant n$, are independent standard one-dimensional BM. The coordinates stochastic processes $(\pi_{i,j}(S_t))_{t\geqslant 0}$, $1 \leqslant i \leqslant j \leqslant n$, are independent real OU processes.

For any s in Sym_n, we denote by $\Lambda(s)$ the vector of the eigenvalues of s ordered in non-decreasing order. Lemma 2.5.2 below is the real symmetric analogue of Lemma 2.5.1.

Lemma 2.5.2 (From matrix OU to DOU). *The image of* GOE_n *by the map* Λ *is the Coulomb gas* P_n^β *given by* (2.6) *with* $\beta = 1$. *Moreover the stochastic process* $X^n = (X_t^n)_{t\geqslant 0} = (\Lambda(S_t))_{t\geqslant 0}$ *is well-defined and solves the stochastic differential equation* (2.3) *with* $\beta = 1$ *and* $x_0^n = \Lambda(s_0)$.

As for the case $\beta = 2$, the idea now is that the DOU process is sandwiched between a real OU process and a matrix OU process.

By similar computations to the case $\beta = 2$, the analogue of (2.66) becomes

$$\chi^2(\mathrm{Law}(H_t) \mid \mathrm{GOE}_n) = -1 + \frac{1}{(1-\mathrm{e}^{-4t})^{\frac{(n(n+1))^2}{8}}} \exp\Big(n|h_0|^2 \frac{\mathrm{e}^{-2t}}{1+\mathrm{e}^{-2t}}\Big). \tag{2.75}$$

This allows to deduce the upper bound for TV, Hellinger, Kullback and χ^2. Regarding the Wasserstein distance, the analogue of (2.70) reads

$$\mathrm{Wasserstein}^2(\mathrm{Law}(S_t), \mathrm{GOE}_n) = |x_0^n|^2\mathrm{e}^{-2t} + 2 - \mathrm{e}^{-2t} - 2\sqrt{1-\mathrm{e}^{-2t}}. \tag{2.76}$$

If $\lim_{n\to\infty} \log(|x_0^n|) = \infty$ then we deduce the asserted result, concluding the proof of Theorem 2.1.5.

2.5.3 Proof of Corollary 2.1.6

Let $\beta \in \{1, 2\}$. Recall the definitions of a_n and c_n from the statement. Take $x_0^{n,i} = a_n$ for all i, and note that $\pi(x_0^n) = na_n$. Given our assumptions on a_n, Corollary 2.1.4 yields for this particular choice of initial condition and for any $\varepsilon \in (0, 1)$

$$\lim_{n\to\infty} \mathrm{dist}(\mathrm{Law}(X^n_{(1-\varepsilon)c_n}) \mid P_n^\beta) = \max.$$

On the other hand, in the proof of Theorem 2.1.5 we saw that

$$\chi^2(\mathrm{Law}(X_t^n) \mid P_n^\beta) \leqslant -1 + \frac{1}{(1-\mathrm{e}^{-4t})^{b_n/2}} \exp\Big(n|x_0^n|^2 \frac{\mathrm{e}^{-2t}}{1+\mathrm{e}^{-2t}}\Big),$$

where $b_n = n^2$ for $\beta = 2$ and $b_n = (n(n+1)/2)^2$ for $\beta = 1$. Since $|x_0^n| \leqslant \sqrt{n}a_n$ for all $x_0^n \in [-a_n, a_n]^n$, and given the comparison between TV, Hellinger, Kullback and χ^2 stated in Lemma 2.8.1 we obtain for $\mathrm{dist} \in \{\mathrm{TV}, \mathrm{Hellinger}, \mathrm{Kullback}, \chi^2\}$ and for all $\varepsilon \in (0,1)$

$$\lim_{n\to\infty} \sup_{x_0^n \in [-a_n,a_n]^n} \mathrm{dist}(\mathrm{Law}(X^n_{(1+\varepsilon)c_n}) \mid P_n^\beta) = 0,$$

thus concluding the proof of Corollary 2.1.6 regarding theses distances.
Concerning Wasserstein, the proof of Theorem 2.1.5 shows that for any $x_0^n \in [-a_n, a_n]^n$ we have

$$\begin{aligned}\mathrm{Wasserstein}^2(\mathrm{Law}(X_t^n), P_n^\beta) &\leqslant |x_0^n|^2 \mathrm{e}^{-2t} + 2 - \mathrm{e}^{-2t} - 2\sqrt{1-\mathrm{e}^{-2t}}\\ &\leqslant n a_n^2 \mathrm{e}^{-2t} + 2 - \mathrm{e}^{-2t} - 2\sqrt{1-\mathrm{e}^{-2t}}.\end{aligned}$$

If $\sqrt{n}a_n \to \infty$, then for $c_n = \log(\sqrt{n}a_n)$ we deduce that for all $\varepsilon \in (0,1)$

$$\lim_{n\to\infty} \sup_{x_0^n \in [-a_n,a_n]^n} \mathrm{dist}(\mathrm{Law}(X^n_{(1+\varepsilon)c_n}) \mid P_n^\beta) = 0.$$

2.6 Cutoff phenomenon for the DOU in TV and Hellinger

In this section, we prove Theorem 2.1.7 and Corollary 2.1.8 for the TV and Hellinger distances. We only consider the case $\beta \geqslant 1$, although the arguments could be adapted *mutatis mutandis* to cover the case $\beta = 0$: note that the result of Theorem 2.1.7 and Corollary 2.1.8 for $\beta = 0$ can be deduced from Theorem 2.1.2. At the end of this section, we also provide the proof of Theorem 2.1.10.

2.6.1 Proof of Theorem 2.1.7 in TV and Hellinger

By the comparison between TV and Hellinger stated in Lemma 2.8.1, it suffices to prove the result for the TV distance, so we concentrate on this distance until the end of this section. Our proof is based on the exponential decay of the relative entropy at an explicit rate given by the optimal logarithmic Sobolev constant. However, this requires the relative entropy of the initial condition to be *finite*. Consequently, we proceed in three steps. First, given an arbitrary initial condition $x_0^n \in \overline{D}_n$, we build an absolutely continuous probability measure $\mu_{x_0^n}$ on D_n that approximates $\delta_{x_0^n}$ and whose relative entropy is not too large. Second, we derive a decay estimate starting from this regularized initial condition. Third, we control the total variation distance between the two processes starting respectively from $\delta_{x_0^n}$ and $\mu_{x_0^n}$.

2.6.1.1 Regularization

In order to have a finite relative entropy at time 0, we first regularize the initial condition by smearing out each particle in a ball of radius bounded below by $n^{-(\kappa+1)}$, for some $\kappa > 0$. Let us first introduce the regularization at scale η of a Dirac distribution δ_z, $z \in \mathbb{R}$ by

$$\delta_z^{(\eta)}(\mathrm{d}u) = \mathrm{Uniform}([z, z+\eta])(\mathrm{d}u) = \eta^{-1}\mathbf{1}_{[z,z+\eta]}\mathrm{d}u.$$

Given $x \in \overline{D}_n$ and $\kappa > 0$, we define a regularized version of δ_x at scale $n^{-\kappa}$, that we denote μ_x, by setting

$$\mu_x = \otimes_{i=1}^n \delta^{(\eta)}_{x_i+3i\eta}, \tag{2.77}$$

where $\eta := n^{-(\kappa+1)}$. The parameters have been tuned in such a way that, independently of the choice of $x \in \overline{D}_n$, the following properties hold. The supports of the Dirac masses $\delta^{(\eta)}_{x_i+3i\eta}$, $i \in \{1,\ldots,n\}$, lie at distance at least η from each other. The volume of the support of μ_x is equal to η^n, and therefore the relative entropy of μ_x with respect to the Lebesgue measure is not too large. Finally, provided $X_0^n = x$ and Y_0^n is distributed according to μ_x, almost surely $|X_0^n - Y_0^n|_\infty \leqslant (3n+1)\eta$.

2.6.1.2 Convergence of the regularized process to equilibrium

Lemma 2.6.1 (Convergence of regularized process). *Let $(Y_t^n)_{t\geqslant 0}$ be a DOU process solution of (2.3), $\beta \geqslant 1$, and let P_n^β be its invariant law. Assume that $\mathrm{Law}(Y_0^n)$ is the regularized measure $\mu_{x_0^n}$ in (2.77) associated to some initial condition $x_0^n \in \overline{D}_n$. Then there exists a constant $C > 0$, only depending on κ, such that for all $t \geqslant 0$, all $n \geqslant 2$ and all $x_0^n \in \overline{D}_n$*

$$\mathrm{Kullback}(\mathrm{Law}(Y_t^n) \mid P_n^\beta) \leqslant C(n|x_0^n|^2 + n^2\log(n))\mathrm{e}^{-2t}.$$

Proof of Lemma 2.6.1. By Lemma 2.8.7 and since $\mathrm{Law}(Y_0^n) = \mu_{x_0^n}$, for all $t \geqslant 0$, there holds

$$\mathrm{Kullback}(\mathrm{Law}(Y_t^n) \mid P_n^\beta) \leqslant \mathrm{Kullback}(\mu_{x_0^n} \mid P_n^\beta)\mathrm{e}^{-2t}. \tag{2.78}$$

Now we have

$$\mathrm{Kullback}(\mu_{x_0^n} \mid P_n^\beta) = \mathbb{E}_{\mu_{x_0^n}}\left[\log \frac{\mathrm{d}\mu_{x_0^n}}{\mathrm{d}P_n^\beta}\right].$$

Recall the definition of S in (2.15). As P_n^β has density $\frac{\mathrm{e}^{-E}}{C_n^\beta}$, we may re-write this as

$$\mathrm{Kullback}(\mu_{x_0^n} \mid P_n^\beta) = S(\mu_{x_0^n}) + \mathbb{E}_{\mu_{x_0^n}}[E] + \log C_n^\beta. \tag{2.79}$$

Recall the partition function $C_{*n}^\beta = n! C_n^\beta$ from Subsection 2.2.2. It is proved in [28], using explicit expressions involving Gamma functions via a Selberg integral, that for some constant $C > 0$

$$\log C_n^\beta \leqslant \log C_{*n}^\beta \leqslant Cn^2. \tag{2.80}$$

Next, we claim that $S(\mu_{x_0^n}) \leqslant n\log(n^{1+\kappa})$. Indeed since $\mu_{x_0^n}$ is a product measure, the tensorization property of entropy recalled in Lemma 2.8.4 gives

$$\mathrm{Kullback}(\mu_{x_0^n} \mid \mathrm{d}x) = \sum_{i=1}^n \mathrm{Kullback}(\delta_0^{(\eta)} \mid \mathrm{d}x).$$

Moreover an immediate computation yields $\mathrm{Kullback}(\delta_0^{(\eta)} \mid \mathrm{d}x) = \log(\eta^{-1})$ so that, given the definition of η, we get

$$\mathrm{Kullback}(\mu_{x_0^n} \mid \mathrm{d}x) = n\log(n^{\kappa+1}). \tag{2.81}$$

We turn to the estimation of the term $\mathbb{E}_{\mu_{x_0^n}}[E]$. The confinement term can be easily bounded:

$$\mathbb{E}_{\mu_{x_0^n}}\left[\frac{n}{2}\sum_{i=1}^n x_i^2\right] \leqslant (n|x_0^n|^2 + n^2\eta^2).$$

Let us now estimate the logarithmic energy of $\mu_{x_0^n}$. Using the fact that the logarithmic function is increasing, together with the fact the supports of $\delta_{x_i+3i\eta}^{(\eta)}$ lie at distance at least η from each other, we notice that for any $i > j$ there holds

$$\begin{aligned}\mathbb{E}_{\mu_{x_0^n}}[\log|x_i - x_j|] &= \iint \log|x-y|\delta_{x_i+3i\eta}^{(\eta)}(\mathrm{d}x)\delta_{x_j+3j\eta}^{(\eta)}(\mathrm{d}y)\\ &\geqslant \iint \log|x-y|\delta_{3\eta}^{(\eta)}(\mathrm{d}x)\delta_0^{(\eta)}(\mathrm{d}y)\\ &\geqslant \log\eta.\end{aligned}$$

It follows that the initial logarithmic energy cannot be much larger than $n^2 \log n$:

$$\mathbb{E}_{\mu_{x_0^n}}\left[\sum_{i>j} \log \frac{1}{|x_i - x_j|}\right] \leqslant \frac{n(n-1)}{2} \log n^{\kappa+1}.$$

This implies that there exists a constant $C > 0$, only depending on κ, such that for all $n \geqslant 2$

$$\mathbb{E}_{\mu_{x_0^n}}[E] = \mathbb{E}_{\mu_{x_0^n}}\left[\frac{n}{2}\sum_{i=1}^{n} |x_i|^2 + \beta \sum_{i>j} \log \frac{1}{|x_i - x_j|}\right] \leqslant C\big(n|x_0^n|^2 + n^2 \log n\big). \tag{2.82}$$

Inserting (2.80), (2.81) and (2.82) into (2.79) we obtain (for a different constant $C > 0$)

$$\mathrm{Kullback}(\mu_{x_0^n} \mid P_n^\beta) \leqslant C\big(n|x_0^n|^2 + n^2 \log n\big).$$

This bound, combined with (2.78), concludes the proof of Lemma 2.6.1. □

2.6.1.3 Convergence to the regularized process in total variation distance

Let $(X_t^n)_{t\geqslant 0}$ and $(Y_t^n)_{t\geqslant 0}$ be two DOU processes with $X_0^n = x_0^n$ and $\mathrm{Law}(Y_0^n) = \mu_{x_0^n}$, where the measure $\mu_{x_0^n}$ is defined in (2.77). Below we prove that, as soon as the parameter κ is large enough, the total variation distance between $\mathrm{Law}(X_t^n)$ and $\mathrm{Law}(Y_t^n)$ tends to 0, for any fixed $t > 0$.

Note that at time 0, almost surely, there holds $X_0^{n,i} \leqslant Y_0^{n,i}$, for every $i \in \{1, \ldots, n\}$. We now introduce a coupling of the processes $(X_t^n)_{t\geqslant 0}$ and $(Y_t^n)_{t\geqslant 0}$ that preserves this ordering at all times. Consider two independent standard BM B^n and W^n in $\mathbb{R}^n$. Let X^n be the solution of (2.3) driven by B^n, and let Y^n be the solution of

$$\mathrm{d}Y_t^{n,i} = \sqrt{\frac{2}{n}}\Big(\mathbf{1}_{\{Y_t^{n,i} \neq X_t^{n,i}\}}\mathrm{d}W_t^i + \mathbf{1}_{\{Y_t^{n,i} = X_t^{n,i}\}}\mathrm{d}B_t^i\Big) - Y_t^{n,i}\mathrm{d}t + \frac{\beta}{n}\sum_{j\neq i} \frac{\mathrm{d}t}{Y_t^{n,i} - Y_t^{n,j}}, \quad 1 \leqslant i \leqslant n.$$

We denote by $\mathbb{P}$ the probability measure under which these two processes are coupled. Let us comment on the driving noise in the equation satisfied by Y^n. When the i-th coordinates of X^n and Y^n equal, we take the same driving Brownian motion and the difference $Y^{n,i} - X^{n,i}$ remains non-negative due to the convexity of $-\log$, see the monotoncity result stated in Lemma 2.8.9. On the other hand, when these two coordinates differ, we take independent driving Brownian motions in order for their difference to have non-zero quadratic variation (this allows to increase their merging probability). Under this coupling, the ordering of X^n and Y^n is thus preserved at all times, and if $X_s^n = Y_s^n$ for some $s \geqslant 0$, then it remains true at all times $t \geqslant s$. Note however that if $X_s^{n,i} = Y_s^{n,i}$, then this equality does not remain true at all times except if all the coordinates match.

As in (2.91), the total variation distance between the laws of X_t^n and Y_t^n may be bounded by

$$\|\mathrm{Law}(Y_t^n) - \mathrm{Law}(X_t^n)\|_{\mathrm{TV}} \leqslant \mathbb{P}(X_t^n \neq Y_t^n),$$

for all $t \geqslant 0$. We wish to establish that for any given $t > 0$,

$$\lim_{n\to\infty} \mathbb{P}(X_t^n \neq Y_t^n) = 0.$$

To do so, we work with the *area* between the two processes X^n and Y^n, defined by

$$A_t^n := \sum_{i=1}^{n} \big(Y_t^{n,i} - X_t^{n,i}\big) = \pi(Y_t^n) - \pi(X_t^n), \quad t \geqslant 0.$$

As the two processes are ordered at any time, this is nothing but the geometric area between the two discrete interfaces $i \mapsto X_t^{n,i}$ and $i \mapsto Y_t^{n,i}$ associated to the configurations X_t^n and Y_t^n. We deduce that the merging time of the two processes coincide with the hitting time of 0 by this area, that we denote by $\tau = \inf\{t \geqslant 0 : A_t^n = 0\}$.

The process A^n has a very simple structure: it is a semimartingale that behaves like an OU process with a randomly varying quadratic variation. Let N_t be the number of coordinates that do not coincide at time t, that is

$$N_t := \#\{i \in \{1, \ldots, n\} : X_t^{n,i} \neq Y_t^{n,i}\}.$$

Then A^n satisfies

$$\mathrm{d}A_t^n = -A_t^n \mathrm{d}t + \mathrm{d}M_t,$$

where M is a centered martingale with quadratic variation

$$\mathrm{d}\langle M\rangle_t = \frac{2}{n} N_t \mathrm{d}t. \tag{2.83}$$

Note that whenever $t < \tau$ we have

$$\mathrm{d}\langle M\rangle_t \geqslant \frac{2}{n}.$$

This *a priori* lower bound on the quadratic variation of M, combined with the Dubins–Schwarz theorem, allows to check that $\tau < \infty$ almost surely. Note that in view of the coupling between X_t^n and Y_t^n, we have $X_t^n = Y_t^n$ for all $t \geqslant \tau$.

Recall the following informal fact: with large probability, a Brownian motion starting from a hits b by a time of order $(a-b)^2$. For a continuous martingale, this becomes: with large probability, a continuous martingale starting from a accumulates a quadratic variation of order $(a-b)^2$ up to its first hitting time of b. Our next lemma states such a bound on the supermartingale A^n.

Lemma 2.6.2. *Let $a > b \geqslant 0$. Let $\tau_b = \inf\{t > 0 : A_t = b\} < \infty$ almost surely. Then, for all $u \geqslant 1$,*

$$\mathbb{P}(\langle A\rangle_{\tau_b} \geqslant (a-b)^2 u \mid A_0 = a) \leqslant 4u^{-1/2}.$$

Proof. Without loss of generality one can assume that $A_0 = a$ almost surely.
By Itô's formula, for all $\lambda \geqslant 0$, the process

$$S_t = \exp\Big(-\lambda A_t - \frac{\lambda^2}{2}\langle A\rangle_t\Big),$$

defines a submartingale (taking its values in $[0,1]$). Doob's stopping theorem yields

$$\mathbb{E}[\mathrm{e}^{-\frac{\lambda^2}{2}\langle A\rangle_{\tau_b}}] = \mathrm{e}^{\lambda b}\mathbb{E}[S_{\tau_b}] \geqslant \mathrm{e}^{\lambda b}\mathbb{E}[S_0] = \mathrm{e}^{-\lambda(a-b)}.$$

On the other hand, for $\lambda = 2(a-b)^{-1}u^{-1/2}$, there holds

$$\begin{aligned}
\mathbb{E}[\mathrm{e}^{-\frac{\lambda^2}{2}\langle A\rangle_{\tau_b}}] &\leqslant \mathbb{P}\big(\langle A\rangle_{\tau_b} < (a-b)^2 u\big) + \mathrm{e}^{-\frac{\lambda^2}{2}(a-b)^2 u}\,\mathbb{P}\big(\langle A\rangle_{\tau_b} \geqslant (a-b)^2 u\big)\\
&\leqslant 1 - (1 - \mathrm{e}^{-\frac{\lambda^2}{2}(a-b)^2 u})\mathbb{P}\big(\langle A\rangle_{\tau_b} \geqslant (a-b)^2 u\big)\\
&\leqslant 1 - \frac{1}{2}\mathbb{P}\big(\langle A\rangle_{\tau_b} \geqslant (a-b)^2 u\big).
\end{aligned}$$

Consequently one deduces that

$$\mathbb{P}(\langle A\rangle_{\tau_b} \geqslant (a-b)^2 u) \leqslant 2(1 - \mathrm{e}^{-\lambda(a-b)}) \leqslant 4u^{-1/2}.$$

□

We are now ready to prove the following lemma:

Lemma 2.6.3. *If $\kappa > \frac{3}{2}$, then for every sequence of times $(t_n)_n$ with $\underline{\lim}_{n\to\infty} t_n > 0$, we have*

$$\lim_{n\to\infty} \sup_{x_0^n \in \overline{D}_n} \|\mathrm{Law}(Y_{t_n}^n) - \mathrm{Law}(X_{t_n}^n)\|_{\mathrm{TV}} = 0.$$

Proof of Lemma 2.6.3. Let $(t_n)_n$ be a sequence of times such that $\underline{\lim}_{n\to\infty} t_n > 0$. In view of the definition of $\mu_{x_0^n}$ and η, the initial area satisfies almost surely

$$A_0^n \leqslant 4n^{1-\kappa}.$$

According to Lemma 2.6.2, with a probability that goes to 1, one has

$$\langle A^n \rangle_\tau - \langle A^n \rangle_0 < 16 n^{2-2\kappa} \log n.$$

On the other hand, by (2.83), we have the following control on the quadratic variation:

$$\langle A \rangle_\tau - \langle A \rangle_0 \geqslant \frac{2}{n}\tau.$$

One deduces that, with a probability that goes to 1,

$$\tau \leqslant \frac{16}{2} n^{3-2\kappa} \log n,$$

and this quantity goes to 0 as $n \to \infty$, whenever $\kappa > \frac{3}{2}$. Therefore for $\kappa > \frac{3}{2}$, there holds

$$\lim_{n\to\infty} \sup_{x_0^n \in \overline{D}_n} \mathbb{P}(X_{t_n}^n \neq Y_{t_n}^n) = 0,$$

thus concluding the proof of Lemma 2.6.3. □

Proof of Theorem 2.1.7 in TV and Hellinger. Let $\kappa > \frac{3}{2}$ and fix some initial condition $x_0^n \in \overline{D}_n$. By the triangle inequality for TV, there holds

$$\|\mathrm{Law}(X_t^n) - P_n^\beta\|_{\mathrm{TV}} \leqslant \|\mathrm{Law}(Y_t^n) - P_n^\beta\|_{\mathrm{TV}} + \|\mathrm{Law}(X_t^n) - \mathrm{Law}(Y_t^n)\|_{\mathrm{TV}}. \tag{2.84}$$

Taking $t = t_n(1+\varepsilon)$ with $t_n = \log(\sqrt{n}|x_0^n|) \vee \log(n)$, one deduces from Lemma 2.6.1 and the Pinsker inequality stated in Lemma 2.8.1 that the first term in the right-hand side of (2.84) vanishes as n tends to infinity. Meanwhile Lemma 2.6.3 guaranties that the second term tends to 0 as n tends to infinity. We also conclude using the comparison between TV and Hellinger (see Lemma 2.8.1) that

$$\lim_{n\to\infty} \mathrm{Hellinger}(\mathrm{Law}(X_{t_n}^n), P_n^\beta) = 0.$$

□

2.6.2 Proof of Corollary 2.1.8 in TV and Hellinger

Proof of Corollary 2.1.8 in TV and Hellinger. By Lemma 2.8.1 and the triangle inequality for TV, we have

$$\begin{aligned}\sup_{x_0^n \in [-a_n, a_n]^n} \|\mathrm{Law}(X_t^n) - P_n^\beta\|_{\mathrm{TV}} \leqslant & \sup_{x_0^n \in [-a_n, a_n]^n} \|\mathrm{Law}(Y_t^n) - \mathrm{Law}(X_t^n)\|_{\mathrm{TV}} \\ & + \sup_{x_0^n \in [-a_n, a_n]^n} \sqrt{2\,\mathrm{Kullback}(\mathrm{Law}(Y_t^n) \mid P_n^\beta)}.\end{aligned}$$

Take $t = (1+\varepsilon)c_n$ with $c_n = \log(na_n)$. Lemmas 2.6.1 and 2.6.3, combined with the assumption made on (a_n), show that the two terms on the right-hand side vanish as $n \to \infty$. Using Lemma 2.8.1, the same result holds for $\mathrm{Hellinger}$.
On the other hand, take $x_0^{n,i} = a_n$ for all i and note that $\pi(x_0^n) = na_n$ goes to $+\infty$ as $n \to \infty$. By Corollary 2.1.4 we find

$$\lim_{n\to\infty} \sup_{x_0^n \in [-a_n,a_n]^n} \mathrm{dist}(\mathrm{Law}(X^n_{(1-\varepsilon)c_n}) \mid P_n^\beta) = 1$$

whenever $\mathrm{dist} \in \{\mathrm{TV}, \mathrm{Hellinger}\}$. □

2.6.3 Proof of Theorem 2.1.10

Proof of Theorem 2.1.10. Lower bound. The contraction property provided by Lemma 2.8.2 gives

$$\mathrm{Kullback}(\mathrm{Law}(X_t^n) \mid P_n^\beta) \geqslant \mathrm{Kullback}(\mathrm{Law}(\pi(X_t^n)) \mid P_n^\beta \circ \pi^{-1}).$$

By Theorem 2.1.3 $P_n \circ \pi^{-1} = \mathcal{N}(0,1)$ and $Y = \pi(X^n)$ is an OU process weak solution of $Y_0 = \pi(X_0^n)$ and $\mathrm{d}Y_t = \sqrt{2}\mathrm{d}B_t - Y_t \mathrm{d}t$. In particular for all $t \geqslant 0$, $\mathrm{Law}(Y_t)$ is a mixture of Gaussian laws in the sense that for any measurable test function g with polynomial growth,

$$\mathbb{E}_{\mathrm{Law}(Y_t)}[g] = \mathbb{E}[g(Y_t)] = \mathbb{E}[G_t(Y_0)] \quad \text{where} \quad G_t(y) = \mathbb{E}_{\mathcal{N}(y\mathrm{e}^{-t},1-\mathrm{e}^{-2t})}[g].$$

Now we use (again) the variational formula used in the proof of Lemma 2.8.2 to get

$$\mathrm{Kullback}(\mathrm{Law}(\pi(X_t^n)) \mid P_n^\beta \circ \pi^{-1}) = \sup_g \{\mathbb{E}_{\mathrm{Law}(\pi(X_t^n))}[g] - \log \mathbb{E}_{\mathcal{N}(0,1)}[\mathrm{e}^g]\},$$

and taking for g the linear function defined by $g(x) = \lambda x$ for all $x \in \mathbb{R}$ and for some $\lambda \neq 0$ yields

$$\mathrm{Kullback}(\mathrm{Law}(\pi(X_t^n)) \mid P_n^\beta \circ \pi^{-1}) \geqslant \lambda \mathrm{e}^{-t} \sum_{i=1}^n \int x \mu_i(\mathrm{d}x) - \frac{\lambda^2}{2}.$$

Finally, by using the assumption on first moment and taking λ small enough we get, for all $\varepsilon \in (0,1)$,

$$\lim_{n\to\infty} \mathrm{Kullback}(\mathrm{Law}(\pi(X^n_{(1-\varepsilon)\log(n)})) \mid P_n^\beta \circ \pi^{-1}) = +\infty,$$

Upper bound. From Lemma 2.8.7 we have, for all $t \geqslant 0$,

$$\mathrm{Kullback}(\mathrm{Law}(X_t^n) \mid P_n^\beta) \leqslant \mathrm{Kullback}(\mathrm{Law}(X_0^n) \mid P_n^\beta)\mathrm{e}^{-2t}.$$

Arguing like in the proof of Lemma 2.6.1 and using the contraction property of $\mathrm{Kullback}$ provided by Lemma 2.8.2 for the map Ψ defined in (2.17), we can write the following decomposition

$$\begin{aligned}
\mathrm{Kullback}(\mathrm{Law}(X_0^n) \mid P_n^\beta) &\leqslant \mathrm{Kullback}(\otimes_{i=1}^n \mu_i \mid P_{*n}^\beta) \\
&= S(\otimes_{i=1}^n \mu_i) + \mathbb{E}_{\otimes_{i=1}^n \mu_i}[E] + \log C_{*n}^\beta \\
&\leqslant \sum_{i=1}^n S(\mu_i) + \sum_{i\neq j} \iint \Phi \mathrm{d}\mu_i \otimes \mathrm{d}\mu_j + Cn^2.
\end{aligned}$$

Combining (2.80) with the assumptions on the μ_i's yields for some constant $C > 0$

$$\mathrm{Kullback}(\mathrm{Law}(X_0^n) \mid P_n^\beta) \leqslant Cn^2$$

and it follows finally that for all $\varepsilon \in (0,1)$,

$$\lim_{n\to\infty} \mathrm{Kullback}(\mathrm{Law}(X_{(1+\varepsilon)\log(n)}) \mid P_n^\beta) = 0.$$

□

2.7 Cutoff phenomenon for the DOU in Wasserstein

2.7.1 Proofs of Theorem 2.1.7 and Corollary 2.1.8 in Wasserstein

Let $(X_t)_{t\geqslant 0}$ be the DOU process. By Lemma 2.8.7, for all $t \geqslant 0$ and all initial conditions $X_0 \in \overline{D}_n$,

$$\mathrm{Wasserstein}^2(\mathrm{Law}(X_t), P_n^\beta) \leqslant \mathrm{e}^{-2t}\mathrm{Wasserstein}^2(\mathrm{Law}(X_0), P_n^\beta).$$

Suppose now that $\mathrm{Law}(X_0^n) = \delta_{x_0^n}$. Then the triangle inequality for the Wasserstein distance gives

$$\mathrm{Wasserstein}^2(\delta_{x_0^n}, P_n^\beta) = \int |x_0^n - x|^2 \, P_n^\beta(\mathrm{d}x) \leqslant 2|x_0^n|^2 + 2\int |x|^2 \, P_n^\beta(\mathrm{d}x).$$

By Theorem 2.1.3, the mean at equilibrium of $|X_t^n|^2$ equals $1 + \frac{\beta}{2}(n-1)$ and therefore

$$\int |x|^2 \, P_n^\beta(\mathrm{d}x) = 1 + \frac{\beta}{2}(n-1).$$

We thus get

$$\mathrm{Wasserstein}^2(\mathrm{Law}(X_t^n), P_n^\beta) \leqslant 2(|x_0^n|^2 + 1 + \frac{\beta}{2}(n-1))\mathrm{e}^{-2t}.$$

Set $c_n := \log(|x_0^n|) \vee \log(\sqrt{n})$. For any $\varepsilon \in (0,1)$, we have

$$\lim_{n\to\infty} \mathrm{Wasserstein}(\mathrm{Law}(X^n_{(1+\varepsilon)c_n}), P_n^\beta) = 0$$

and this concludes the proof of Theorem 2.1.7 in the Wasserstein distance.
Regarding the proof of Corollary 2.1.8, if $x_0^n \in [-a_n, a_n]^n$ then $|x_0^n| \leqslant \sqrt{n}a_n$. Therefore if $\inf_n a_n > 0$, setting $c_n = \log(\sqrt{n}a_n)$ we find, as required,

$$\lim_{n\to\infty} \sup_{x_0^n \in [-a_n, a_n]^n} \mathrm{Wasserstein}(\mathrm{Law}(X^n_{(1+\varepsilon)c_n}), P_n^\beta) = 0.$$

2.7.2 Proof of Theorem 2.1.9

This is an adaptation of the previous proof. We compute

$$\begin{aligned}\mathrm{Wasserstein}^2(\delta_{x_0^n}, P_n^\beta) &= \int |x_0^n - x|^2 \, P_n^\beta(\mathrm{d}x) \\ &\leqslant 2\,|x_0^n - \rho_n|^2 + 2\int |\rho_n - x|^2 \, P_n^\beta(\mathrm{d}x),\end{aligned}$$

where $\rho_n \in D_n$ is the vector of the quantiles of order $1/n$ of the semi-circle law as in (2.14). The rigidity estimates established in [48, Th. 2.4] justify that

$$\lim_{n\to\infty} \int |\rho_n - x|^2 \, P_n^\beta(\mathrm{d}x) = 0.$$

If $|x_0^n - \rho_n|$ diverges with n, we deduce that for all $\varepsilon \in (0,1)$, with $t_n = \log(|x_0^n - \rho_n|)$,

$$\lim_{n\to\infty} \mathrm{Wasserstein}(\mathrm{Law}(X^n_{(1+\varepsilon)t_n}), P_n^\beta) = 0.$$

On the other hand, if $|x_0^n - \rho_n|$ converges to some limit α then we easily get, for any $t \geqslant 0$,

$$\varlimsup_{n\to\infty} \mathrm{Wasserstein}^2(\mathrm{Law}(X_t^n), P_n^\beta) \leqslant \alpha^2 \mathrm{e}^{-2t}.$$

Remark 3 (High-dimensional phenomena). *With $X_n \sim P_n^\beta$, in the bias-variance decomposition*

$$\int |\rho_n - x|^2 \, P_n^\beta(\mathrm{d}x) = |\mathbb{E}X_n - \rho_n|^2 + \mathbb{E}(|X_n - \mathbb{E}X_n|^2),$$

the second term of the right hand side is a variance term that measures the concentration of the log-concave random vector X_n around its mean $\mathbb{E}X_n$, while the first term in the right hand side is a bias term that measures the distance of the mean $\mathbb{E}X_n$ to the mean-field limit ρ_n. Note also that $\mathbb{E}(|X_n - \mathbb{E}X_n|^2) = \mathbb{E}(|X_n|^2) - |\mathbb{E}X_n|^2 = 1 + \frac{\beta}{2}(n-1) - |\mathbb{E}X_n|^2$, reducing the problem to the mean. We refer to [140] for a fine asymptotic analysis in the determinantal case $\beta = 2$.

2.8 Appendix

2.8.1 Distances and divergences

We use the following standard distances and divergences to quantify the trend to equilibrium of Markov processes and to formulate the cutoff phenomena.

The *Wasserstein–Kantorovich–Monge transportation distance* of order 2 and with respect to the underlying Euclidean distance is defined for all probability measures μ and ν on $\mathbb{R}^n$ by

$$\mathrm{Wasserstein}(\mu, \nu) = \Big(\inf_{(X,Y)} \mathbb{E}[|X - Y|^2] \Big)^{1/2} \in [0, +\infty] \tag{2.85}$$

where $|x| = \sqrt{x_1^2 + \cdots + x_n^2}$ and where the inf runs over all couples (X, Y) with $X \sim \mu$ and $Y \sim \nu$.

The *total variation distance* between probability measures μ and ν on the same space is

$$\|\mu - \nu\|_{\mathrm{TV}} = \sup_A |\mu(A) - \nu(A)| \in [0, 1] \tag{2.86}$$

where the supremum runs over Borel subsets. If μ and ν are absolutely continuous with respect to a reference measure λ with densities f_μ and f_ν then $\|\mu - \nu\|_{\mathrm{TV}} = \frac{1}{2}\int |f_\mu - f_\nu| \mathrm{d}\lambda = \frac{1}{2}\|f_\mu - f_\nu\|_{L^1(\lambda)}$.

The *Hellinger distance* between probability measures μ and ν with densities f_μ and f_ν with respect to the same reference measure λ is

$$\mathrm{Hellinger}(\mu, \nu) = \Big(\int \frac{1}{2}(\sqrt{f_\mu} - \sqrt{f_\nu})^2 \mathrm{d}\lambda \Big)^{1/2} = \Big(1 - \int \sqrt{f_\mu f_\nu} \mathrm{d}\lambda \Big)^{1/2} \in [0, 1]. \tag{2.87}$$

This quantity does not depend on the choice of λ. We have $\mathrm{Hellinger}(\mu, \nu) = \frac{1}{\sqrt{2}}\|\sqrt{f_\mu} - \sqrt{f_\nu}\|_{L^2(\lambda)}$. Note that an alternative normalization is sometimes considered in the literature, making the maximal value of the Hellinger distance equal $\sqrt{2}$.

The *Kullback–Leibler divergence or relative entropy* is defined by

$$\mathrm{Kullback}(\nu \mid \mu) = \int \log \tfrac{\mathrm{d}\nu}{\mathrm{d}\mu} \mathrm{d}\nu = \int \tfrac{\mathrm{d}\nu}{\mathrm{d}\mu} \log \tfrac{\mathrm{d}\nu}{\mathrm{d}\mu} \mathrm{d}\mu \in [0, +\infty] \tag{2.88}$$

if ν is absolutely continuous with respect to μ, and $\mathrm{Kullback}(\nu \mid \mu) = +\infty$ otherwise.

The χ^2 *divergence* or relative variance is given by

$$\chi^2(\nu \mid \mu) = \Big\| \tfrac{\mathrm{d}\nu}{\mathrm{d}\mu} - 1 \Big\|_{L^2(\mu)}^2 = \int \Big| \frac{\mathrm{d}\nu}{\mathrm{d}\mu} - 1 \Big|^2 \mathrm{d}\mu = \|\frac{\mathrm{d}\nu}{\mathrm{d}\mu}\|_{L^2(\mu)}^2 - 1 \in [0, +\infty]. \tag{2.89}$$

We set it to $+\infty$ if ν is not absolutely continuous with respect to μ. If μ and ν have densities f_μ and f_ν with respect to a reference measure λ then $\chi^2(\nu \mid \mu) = \int (f_\nu^2/f_\mu)\mathrm{d}\lambda - 1$.

The (logarithmic) *Fisher information or divergence* is defined by

$$\mathrm{Fisher}(\nu \mid \mu) = \int \left|\nabla \log \tfrac{\mathrm{d}\nu}{\mathrm{d}\mu}\right|^2 \mathrm{d}\nu = \int \frac{|\nabla \frac{\mathrm{d}\nu}{\mathrm{d}\mu}|^2}{\frac{\mathrm{d}\nu}{\mathrm{d}\mu}} \mathrm{d}\mu = 4\int \left|\nabla \sqrt{\tfrac{\mathrm{d}\nu}{\mathrm{d}\mu}}\right|^2 \mathrm{d}\mu \in [0, +\infty] \tag{2.90}$$

if ν is absolutely continuous with respect to μ, and $\mathrm{Fisher}(\nu \mid \mu) = +\infty$ otherwise.

Each of these distances or divergences has its advantages and drawbacks. In some sense, the most sensitive is Fisher due to its Sobolev nature, then χ^2, then Kullback which can be seen as a sort of $L^{1+} = L \log L$ norm, then TV and Hellinger which are comparable, then Wasserstein, but this rough hierarchy misses some subtleties related to some scales and nature of the arguments.

Some of these distances or divergences can generically be compared as the following result shows.

Lemma 2.8.1 (Inequalities). *For any probability measures μ and ν on the same space,*

$$\begin{aligned} \|\mu - \nu\|_{\mathrm{TV}}^2 &\leqslant 2\mathrm{Kullback}(\nu \mid \mu) \\ 2\mathrm{Hellinger}^2(\mu,\nu) &\leqslant \mathrm{Kullback}(\nu \mid \mu) \\ \mathrm{Kullback}(\nu \mid \mu) &\leqslant 2\chi(\nu \mid \mu) + \chi^2(\nu \mid \mu) \\ \mathrm{Hellinger}^2(\mu,\nu) \leqslant \|\mu - \nu\|_{\mathrm{TV}} &\leqslant \mathrm{Hellinger}(\mu,\nu)\sqrt{2 - \mathrm{Hellinger}(\mu,\nu)^2}. \end{aligned}$$

We refer to [208, p. 61-62] for a proof. The inequality between the total variation distance and the relative entropy is known as the Pinsker or Csiszár–Kullback inequality, while the inequalities between the total variation distance and the Hellinger distance are due to Kraft. There are many other metrics between probability measures, see for instance [211, 133] for a discussion.

The total variation distance can also be seen as a special Wasserstein distance of order 1 with respect to the atomic distance, namely

$$\|\mu - \nu\|_{\mathrm{TV}} = \inf_{(X,Y)} \mathbb{P}(X \neq Y) = \inf_{(X,Y)} \mathbb{E}[\mathbf{1}_{X \neq Y}] \in [0,1] \tag{2.91}$$

where the infimum runs over all couplings $X \sim \mu$ and $Y \sim \nu$. This explains in particular why TV is more sensitive than $\mathrm{Wasserstein}$ at short scales but less sensitive at large scales, a consequence of the sensitivity difference between the underlying atomic and Euclidean distances. The probabilistic representations of TV and $\mathrm{Wasserstein}$ make them compatible with techniques of coupling, which play an important role in the literature on convergence to equilibrium of Markov processes.

We gather now useful results on distances and divergences.

Lemma 2.8.2 (Contraction properties). *Let μ and ν be two probability measures on a same measurable space S. Let $f : S \mapsto T$ be a measurable function, where T is another measurable space.*

- *If $\mathrm{dist} \in \{\mathrm{TV}, \mathrm{Kullback}, \chi^2\}$ then*

$$\mathrm{dist}(\nu \circ f^{-1} \mid \mu \circ f^{-1}) \leqslant \mathrm{dist}(\nu \mid \mu).$$

- *If $S = \mathbb{R}^n$, $T = \mathbb{R}^k$ then, denoting $\|f\|_{\mathrm{Lip}} = \sup_{x \neq y} \frac{|f(x)-f(y)|}{|x-y|}$,*

$$\mathrm{Wasserstein}(\mu \circ f^{-1}, \nu \circ f^{-1}) \leqslant \|f\|_{\mathrm{Lip}} \mathrm{Wasserstein}(\mu,\nu).$$

The notation f^{-1} stands for the reciprocal map $f^{-1}(A) = \{y \in S : f(x) \in A\}$ and $\mu \circ f^{-1}$ is the image measure or push-forward of μ by the map f, defined by $(\mu \circ f^{-1})(A) = \mu(f^{-1}(A))$. In terms of random variables we have $Y \sim \mu \circ f^{-1}$ if and only $Y = f(X)$ where $X \sim \mu$.

The proof of the contraction properties of Lemma 2.8.2 are all based on variational formulas. Note that following [253, Ex. 22.20 p. 588], there is a variational formula for Fisher that comes from its dual representation as an inverse Sobolev norm. We do not develop this idea in this work.

Proof. The proof of the contraction property for Wasserstein comes from the fact that every coupling of μ and ν produces a coupling for $\mu \circ f^{-1}$ and $\nu \circ f^{-1}$. Regarding TV, the contraction property is a consequence of the definition of this distance and of measurability. In the case of Kullback, the property can be proved using the following well known variational formula:

$$\mathrm{Kullback}(\nu \mid \mu) = \sup_g \{\mathbb{E}_\nu[g] - \log \mathbb{E}_\mu[\mathrm{e}^g]\}$$

where the supremum runs over all $g \in L^1(\nu)$, or by approximation when the supremum runs over all bounded measurable g. This variational formula can be derived for instance by applying Jensen's inequality to $-\log \mathbb{E}_\nu[\mathrm{e}^g \frac{\mathrm{d}\mu}{\mathrm{d}\nu}]$. Equality is achieved for $g = \log(\mathrm{d}\nu/\mathrm{d}\mu)$. Now, taking $g = h \circ f$ gives

$$\mathrm{Kullback}(\nu \mid \mu) \geqslant \mathbb{E}_{\nu\circ f^{-1}}[h] - \log \mathbb{E}_{\mu \circ f^{-1}}[\mathrm{e}^h],$$

and it remains to take the supremum over h to get

$$\mathrm{Kullback}(\nu \mid \mu) \geqslant \mathrm{Kullback}(\nu \circ f^{-1} \mid \mu \circ f^{-1}).$$

The variational formula for $\mathrm{Kullback}(\cdot \mid \mu)$ is a manifestation of its convexity, it expresses this functional as the envelope of its tangents, its Fenchel–Legendre transform or convex dual is the log-Laplace transform. Such a variational formula is equivalent to tensorization, and is available for all Φ-entropies such that $(u,v) \mapsto \Phi''(u)v^2$ is convex, see [76, Th. 4.4]. In particular, the analogous variational formula as well as the consequence in terms of contraction are also available for χ^2 which corresponds to the Φ-entropy with $\Phi(u) = u^2 - 1$ (variance as a Φ-entropy). $\square$

Lemma 2.8.3 (Scale invariance versus homogeneity)**.** *The total variation distance is scale invariant while the Wasserstein distance is homogeneous just like a norm, namely for all probability measures μ and ν on $\mathbb{R}^n$ and all scaling factor $\sigma \in (0,\infty)$, denoting $\mu_\sigma = \mathrm{Law}(\sigma X)$ where $X \sim \mu$, we have*

$$\|\mu_\sigma - \nu_\sigma\|_{\mathrm{TV}} = \|\mu - \nu\|_{\mathrm{TV}} \quad \textit{while} \quad \mathrm{Wasserstein}(\mu_\sigma, \nu_\sigma) = \sigma \mathrm{Wasserstein}(\mu, \nu).$$

Proof. For the Wasserstein distance, the result follows from

$$\mathrm{Wasserstein}(\mu_\sigma, \nu_\sigma) = \Big(\inf_{(X,Y)} \mathbb{E}[|\sigma X - \sigma Y|^2]\Big)^{1/2} = \sigma \mathrm{Wasserstein}(\mu, \nu)\ ,$$

while for the TV distance, it comes from the fact that $A \mapsto A_\sigma := \{\sigma x : x \in A\}$ is a bijection. $\square$

We turn to the behavior of the distances/divergences under tensorization.

Lemma 2.8.4 (Tensorization). *For all probability measures $\mu_1,\dots,\mu_n$ and $\nu_1,\dots,\nu_n$ on $\mathbb{R}$, we have*

$$\begin{aligned}
\mathrm{Hellinger}^2(\otimes_{i=1}^n\mu_i,\otimes_{i=1}^n\nu_i) &= 1-\prod_{i=1}^n\Big(1-\mathrm{Hellinger}^2(\mu_i,\nu_i)\Big),\\
\mathrm{Kullback}(\otimes_{i=1}^n\nu_i\mid\otimes_{i=1}^n\mu_i) &= \sum_{i=1}^n\mathrm{Kullback}(\nu_i\mid\mu_i),\\
\chi^2(\otimes_{i=1}^n\mu_i\mid\otimes_{i=1}^n\nu_i) &= -1+\prod_{i=1}^n(\chi^2(\mu_i,\nu_i)+1),\\
\mathrm{Fisher}(\otimes_{i=1}^n\nu_i\mid\otimes_{i=1}^n\mu_i) &= \sum_{i=1}^n\mathrm{Fisher}(\nu_i\mid\mu_i),\\
\mathrm{Wasserstein}^2(\otimes_{i=1}^n\mu_i,\otimes_{i=1}^n\nu_i) &= \sum_{i=1}^n\mathrm{Wasserstein}^2(\mu_i,\nu_i),\\
\max_{1\leqslant i\leqslant n}\|\mu_i-\nu_i\|_{\mathrm{TV}}\leqslant\|\otimes_{i=1}^n\mu_i-\otimes_{i=1}^n\nu_i\|_{\mathrm{TV}} &\leqslant \sum_{i=1}^n\|\mu_i-\nu_i\|_{\mathrm{TV}}.
\end{aligned}$$

The equality for the Wasserstein distance comes by taking the product of optimal couplings. The first inequality for the total variation distance comes from its contraction property (Lemma 2.8.2), while the second comes from $|(a_1\cdots a_n)-(b_1\cdots b_n)|\leqslant\sum_{i=1}^n|a_i-b_i|(a_1\cdots a_{i-1})(b_{i+1}\cdots b_n)$, $a_1,\dots,a_n,b_1,\dots,b_n\in[0,+\infty)$, which comes itself from the triangle inequality on the telescoping sum $\sum_{i=1}^n(c_i-c_{i-1})$ where $c_i=(a_1\cdots a_i)(b_{i+1}\cdots b_n)$ via $c_i-c_{i-1}=(a_i-b_i)(a_1\cdots a_{i-1})(b_{i+1}\cdots b_n)$.

Lemma 2.8.5 (Explicit formulas for Gaussian distributions). *For all $n\geqslant1$, $m_1,m_2\in\mathbb{R}^n$, and all $n\times n$ covariance matrices Σ_1,Σ_2, denoting $\Gamma_1=\mathcal{N}(\mu_1,\Sigma_1)$ and $\Gamma_2=\mathcal{N}(\mu_2,\Sigma_2)$, we have*

$$\begin{aligned}
\mathrm{Hellinger}^2(\Gamma_1,\Gamma_2) &= 1-\frac{\det(\Sigma_1\Sigma_2)^{1/4}}{\det(\frac{\Sigma_1+\Sigma_2}{2})^{1/2}}\mathrm{e}^{-\frac14(\Sigma_1+\Sigma_2)^{-1}(m_2-m_1)\cdot(m_2-m_1)},\\
2\mathrm{Kullback}(\Gamma_1\mid\Gamma_2) &= \Sigma_2^{-1}(m_1-m_2)\cdot(m_1-m_2)+\mathrm{Tr}(\Sigma_2^{-1}\Sigma_1-\mathrm{I}_n)+\log\det(\Sigma_2\Sigma_1^{-1}),\\
\chi^2(\Gamma_1\mid\Gamma_2) &= -1+\frac{\det(\Sigma_2)}{\sqrt{\det(\Sigma_1)\det(2\Sigma_2-\Sigma_1)}}\mathrm{e}^{\frac12(\Sigma_2^{-1}+(2\Sigma_2\Sigma_1^{-1}\Sigma_2-\Sigma_2)^{-1})(m_2-m_1)\cdot(m_2-m_1)},\\
\mathrm{Fisher}(\Gamma_1\mid\Gamma_2) &= |\Sigma_2^{-1}(m_1-m_2)|^2+\mathrm{Tr}(\Sigma_2^{-2}\Sigma_1-2\Sigma_2^{-1}+\Sigma_1^{-1})\\
\mathrm{Wasserstein}^2(\Gamma_1,\Gamma_2) &= |m_1-m_2|^2+\mathrm{Tr}\Big(\Sigma_1+\Sigma_2-2\sqrt{\sqrt{\Sigma_1}\Sigma_2\sqrt{\Sigma_1}}\Big),
\end{aligned}$$

where the formula for $\chi^2(\Gamma_1\mid\Gamma_2)$ holds if $2\Sigma_2>\Sigma_1$, and $\chi^2(\Gamma_1\mid\Gamma_2)=+\infty$ otherwise. Moreover the formulas for Fisher and Wasserstein rewrite, if Σ_1 and Σ_2 commute, $\Sigma_1\Sigma_2=\Sigma_2\Sigma_1$, to

$$\begin{aligned}
\mathrm{Fisher}(\Gamma_1\mid\Gamma_2) &= |\Sigma_2^{-1}(m_1-m_2)|^2+\mathrm{Tr}(\Sigma_2^{-2}(\Sigma_2-\Sigma_1)^2\Sigma_1^{-1})\\
\mathrm{Wasserstein}^2(\Gamma_1,\Gamma_2) &= |m_1-m_2|^2+\mathrm{Tr}((\sqrt{\Sigma_1}-\sqrt{\Sigma_2})^2).
\end{aligned}$$

Regarding the total variation distance, there is no general simple formula for Gaussian laws, but we can use for instance the comparisons with Kullback and Hellinger (Lemma 2.8.1), see [98] for a discussion.

Proof of Lemma 2.8.5. We refer to [204, p. 47 and p. 51] for Kullback and Hellinger, and to [135] for Wasserstein, a far more subtle case. The formula for $\chi^2(\Gamma_1\mid\Gamma_2)$ follows easily from a direct computation. We have not found in the literature a formula for Fisher. Let us give it here for the sake of completeness. Using $\mathbb{E}[X_iX_j] = \Sigma_{ij} + m_im_j$ when $X \sim \mathcal{N}(m,\Sigma)$ we get, for all $n \times n$ symmetric matrices A and B

$$\mathbb{E}[AX \cdot BX] = \sum_{i,j,k=1}^{n} A_{ij}B_{ik}\mathbb{E}[X_jX_k] = \sum_{i,j,k=1}^{n} A_{ij}B_{ik}(\Sigma_{jk} + m_jm_k) = \mathrm{Trace}(A\Sigma B) + Am \cdot Bm$$

and thus for all n-dimensional vectors a and b,

$$\begin{aligned}\mathbb{E}[A(X-a)\cdot B(X-b)] &= \mathbb{E}[AX\cdot BX] + A(m-a)\cdot B(m-b) - Am\cdot Bm\\ &= \mathrm{Trace}(A\Sigma B) + A(m-a)\cdot B(m-b).\end{aligned}$$

Now, using the notation $q_i(x) = \Sigma_i^{-1}(x-m_i)\cdot(x-m_i)$ and $|\Sigma_i| = \det(\Sigma_i)$,

$$\begin{aligned}\mathrm{Fisher}(\Gamma_1 \mid \Gamma_2) &= 4\frac{\sqrt{|\Sigma_2|}}{\sqrt{|\Sigma_1|}}\int\Big|\nabla \mathrm{e}^{-\frac{q_1(x)}{4}+\frac{q_2(x)}{4}}\Big|^2 \frac{\mathrm{e}^{-\frac{q_2(x)}{2}}}{\sqrt{2\pi|\Sigma_2|}}\mathrm{d}x\\ &= \int|\Sigma_2^{-1}(x-m_2) - \Sigma_1^{-1}(x-m_1)|^2\frac{\mathrm{e}^{-\frac{q_1(x)}{2}}}{\sqrt{2\pi|\Sigma_1|}}\mathrm{d}x\\ &= \int(|\Sigma_2^{-1}(x-m_2)|^2 - 2\Sigma_2^{-1}(x-m_2)\cdot\Sigma_1^{-1}(x-m_1) + |\Sigma_1^{-1}(x-m_1)|^2)\frac{\mathrm{e}^{-\frac{q_1(x)}{2}}}{\sqrt{2\pi|\Sigma_1|}}\mathrm{d}x\\ &= \mathrm{Trace}(\Sigma_2^{-1}\Sigma_1\Sigma_2^{-1}) + |\Sigma_2^{-1}(m_1-m_2)|^2 - 2\mathrm{Trace}(\Sigma_2^{-1}) + \mathrm{Trace}(\Sigma_1^{-1})\\ &= \mathrm{Trace}(\Sigma_2^{-2}\Sigma_1 - 2\Sigma_2^{-1} + \Sigma_1^{-1}) + |\Sigma_2^{-1}(m_1-m_2)|^2.\end{aligned}$$

The formula when $\Sigma_1\Sigma_2 = \Sigma_2\Sigma_1$ follows immediately. □

2.8.2 Convexity and its dynamical consequences

We gather useful dynamical consequences of convexity. We start with functional inequalities.

Lemma 2.8.6 (Logarithmic Sobolev inequality). *Let P_n^β be the invariant law of the DOU process solving (2.3). Then, for all law ν on $\mathbb{R}^n$, we have*

$$\mathrm{Kullback}(\nu \mid P_n^\beta) \leqslant \frac{1}{2n}\mathrm{Fisher}(\nu \mid P_n^\beta).$$

Moreover the constant $\frac{1}{2n}$ is optimal.
Furthermore, finite equality is achieved if and only if $\mathrm{d}\nu/\mathrm{d}P_n^\beta$ is of the form $\mathrm{e}^{\lambda(x_1+\cdots+x_n)}$, $\lambda \in \mathbb{R}$.

Linearizing the log-Sobolev inequality above with $\mathrm{d}\nu/\mathrm{d}P_n^\beta = 1+\varepsilon f$ gives the Poincaré inequality

$$\mathrm{Var}_{P_n^\beta}(f) \leqslant -\int f\mathsf{G}f\mathrm{d}P_n^\beta. \tag{2.92}$$

It can be extended by truncation and regularization from the case where f is smooth and compactly supported to the case where f is in the Sobolev space $H^1(P_n^\beta)$. Finite equality is achieved when f is an eigenfunction associated to the eigenvalue -1 of G, namely $f(x) = a(x_1+\cdots+x_n)+b$, $a,b\in\mathbb{R}$, hence the other name *spectral gap inequality*. It rewrites in terms of χ^2 divergence as

$$\chi^2(\nu \mid P_n^\beta) \leqslant \frac{1}{n}\int\Big|\nabla\frac{\mathrm{d}\nu}{\mathrm{d}P_n^\beta}\Big|^2\mathrm{d}P_n^\beta. \tag{2.93}$$

The right-hand side plays for the χ^2 divergence the role played by Fisher for Kullback.

We refer to [113, 81] for a proof of Lemma 2.8.6. This logarithmic Sobolev inequality is a consequence of the log-concavity of P_n^β with respect to $\mathcal{N}(0, \frac{1}{n}\mathrm{I}_n)$. A slightly delicate aspect lies in the presence of the restriction to D_n, which can be circumvented by using a regularization procedure.

There are many other functional inequalities which are a consequence of this log-concavity, for instance the Talagrand transportation inequality that states that when ν has finite second moment,

$$\mathrm{Wasserstein}^2(\nu, P_n^\beta) \leqslant \frac{1}{n}\mathrm{Kullback}(\nu \mid P_n^\beta)$$

and the HWI inequality[1] that states that when ν has finite second moment,

$$\mathrm{Kullback}(\nu \mid P_n^\beta) \leqslant \mathrm{Wasserstein}(\nu, P_n^\beta)\sqrt{\mathrm{Fisher}(\nu \mid P_n^\beta)} - \frac{n}{2}\mathrm{Wasserstein}^2(\nu \mid P_n^\beta),$$

and we refer to [253] for this couple of functional inequalities, that we do not use here.

Lemma 2.8.7 (Sub-exponential convergence to equilibrium). *Let $(X_t^n)_{t\geqslant 0}$ be the DOU process solution of (2.3) with $\beta = 0$ or $\beta \geqslant 1$, and let P_n^β be its invariant law. Then for all $t \geqslant 0$, we have the sub-exponential convergences*

$$\begin{aligned}
\chi^2(\mathrm{Law}(X_t^n) \mid P_n^\beta) &\leqslant \mathrm{e}^{-2t}\chi^2(\mathrm{Law}(X_0^n) \mid P_n^\beta),\\
\mathrm{Kullback}(\mathrm{Law}(X_t^n) \mid P_n^\beta) &\leqslant \mathrm{e}^{-2t}\mathrm{Kullback}(\mathrm{Law}(X_0^n) \mid P_n^\beta),\\
\mathrm{Fisher}(\mathrm{Law}(X_t^n) \mid P_n^\beta) &\leqslant \mathrm{e}^{-2t}\mathrm{Fisher}(\mathrm{Law}(X_0^n) \mid P_n^\beta),\\
\mathrm{Wasserstein}^2(\mathrm{Law}(X_t^n), P_n^\beta) &\leqslant \mathrm{e}^{-2t}\mathrm{Wasserstein}^2(\mathrm{Law}(X_0^n), P_n^\beta).
\end{aligned}$$

Recall that when $\beta > 0$ the initial condition X_0^n is always taken in D_n.

For each inequality, if the right-hand side is infinite then the inequality is trivially satisfied. This is in particular the case for $\mathrm{Kullback}$ and Fisher when $\mathrm{Law}(X_0^n)$ is not absolutely continuous with respect to the Lebesgue measure, and for Wasserstein when $\mathrm{Law}(X_0^n)$ has infinite second moment.

Elements of proof of Lemma 2.8.7. The idea is that an exponential decay for $\mathrm{Kullback}$, χ^2, Fisher, and $\mathrm{Wasserstein}$ can be established by taking the derivative, using a functional inequality, and using the Grönwall lemma. More precisely, for $\mathrm{Kullback}$ it is a log-Sobolev inequality, for χ^2 a Poincaré inequality, for $\mathrm{Wasserstein}$ a transportation type inequality, and for Fisher a Bakry–Émery Γ_2 inequality, see for instance [7, 16, 253]. It is a rather standard piece of probabilistic functional analysis, related to the log-concavity of P_n^β. We recall the crucial steps for the reader convenience. Let us set $\mu_t = \mathrm{Law}(X_t^n)$ and $\mu = P_n^\beta$. For $t > 0$ the density $p_t = \mathrm{d}\mu_t/\mathrm{d}\mu$ exists and solves the evolution equation $\partial_t p_t = \mathsf{G}p_t$ where G is as in (2.22). We have the integration by parts

$$\int f\mathsf{G}g\mathrm{d}\mu = \int g\mathsf{G}f\mathrm{d}\mu = -\frac{1}{n}\int \nabla f\cdot\nabla g\mathrm{d}\mu.$$

For $\mathrm{Kullback}$, we find using these tools, for all $t > 0$, denoting $\Phi(u) := u\log(u)$,

$$\begin{aligned}
\partial_t\mathrm{Kullback}(\mu_t \mid \mu) = \int \Phi'(f_t)\mathsf{G}f_t\mathrm{d}\mu &= -\frac{1}{n}\int \Phi''(f_t)|\nabla f_t|^2\mathrm{d}\mu\\
&= -\frac{1}{n}\mathrm{Fisher}(\mu_t \mid \mu) \leqslant -2\mathrm{Kullback}(\mu_t \mid \mu), \quad (2.94)
\end{aligned}$$

[1] Here "H" is the capital η used by Boltzmann for entropy, "W" is for Wasserstein, "I" is for Fisher information.

where the inequality comes from the logarithmic Sobolev inequality of Lemma 2.8.6. It remains to use the Grönwall lemma to get the exponential decay of $\mathrm{Kullback}$.

The derivation of the exponential decay of the Fisher divergence follows the same lines by differentiating again with respect to time. Indeed, after a sequence of differential computations and integration by parts, we find, see for instance [7, Ch. 5], [16], or [253],

$$\partial_t \mathrm{Fisher}(\mu_t \mid \mu) = -2n \int f_t \Gamma_2(\log(f_t)) \mathrm{d}\mu, \tag{2.95}$$

where $\Gamma_2(f) := \frac{1}{n^2} f''^2 + \frac{1}{n} V'' f'^2$ is the Bakry–Émery "Gamma-two" operator of the dynamics. Now using the convexity of V, we get, by the Grönwall lemma, for all $t > 0$,

$$\partial_t \mathrm{Fisher}(\mu_t \mid \mu) \leqslant -2 \mathrm{Fisher}(\mu_t \mid \mu). \tag{2.96}$$

This can be used to prove the log-Sobolev inequality, see [7, Ch. 5], [16], and [253]. This differential approach goes back at least to Boltzmann (statistical physics) and Stam (information theory) and was notably extensively developed later on by Bakry, Ledoux, Villani and their followers.

For the Wasserstein distance, we proceed by coupling. Indeed, since the diffusion coefficient is constant in space, we can simply use a *parallel coupling*. Namely, let $(X'_t)_{t\geqslant 0}$ be the process started from another possibly random initial condition X'_0, and satisfying to the same stochastic differential equation, with the same BM. We get

$$\mathrm{d}(X_t - X'_t) = -\frac{1}{n}(\nabla E(X_t) - \nabla E(X'_t))\mathrm{d}t,$$

hence

$$\mathrm{d}(X_t - X'_t) \cdot (X_t - X'_t) = -\frac{1}{n}((\nabla E(X_t) - \nabla E(X'_t)) \cdot (X_t - X'_t))\mathrm{d}t. \tag{2.97}$$

Now since E is uniformly convex with $\nabla^2 E \geqslant n I_n$, we get, for all $x, y \in \mathbb{R}^n$,

$$(\nabla E(x) - \nabla E(y)) \cdot (x - y) \geqslant n|x - y|^2,$$

which gives

$$\mathrm{d}|X_t - X'_t|^2 \leqslant -2|X_t - X'_t|^2 \mathrm{d}t$$

and by the Grönwall lemma,

$$|X_t - X'_t|^2 \leqslant \mathrm{e}^{-2t}|X_0 - X'_0|^2.$$

It follows that

$$\mathrm{Wasserstein}^2(\mathrm{Law}(X_t), \mathrm{Law}(X'_t)) \leqslant \mathrm{e}^{-2t}\mathbb{E}[|X_0 - X'_0|^2].$$

By taking the infimum over all couplings of X_0 and X'_0 we get

$$\mathrm{Wasserstein}^2(\mathrm{Law}(X_t), \mathrm{Law}(X'_t)) \leqslant \mathrm{e}^{-2t}\mathrm{Wasserstein}^2(\mathrm{Law}(X_0), \mathrm{Law}(X'_0)).$$

Taking $X'_0 \sim P_n^\beta$ we get, by invariance, for all $t \geqslant 0$,

$$\mathrm{Wasserstein}^2(\mathrm{Law}(X_t), P_n^\beta) \leqslant \mathrm{e}^{-2t}\mathrm{Wasserstein}^2(\mathrm{Law}(X_0), P_n^\beta).$$

□

Lemma 2.8.8 (Monotonicity). *Let $(X_t^n)_{t\geqslant 0}$ be the DOU process (2.3), with $\beta = 0$ or $\beta \geqslant 1$ and invariant law P_n^β. Then for all $\mathrm{dist} \in \{\mathrm{TV}, \mathrm{Hellinger}, \mathrm{Kullback}, \chi^2, \mathrm{Fisher}, \mathrm{Wasserstein}\}$, the function $t \geqslant 0 \mapsto \mathrm{dist}(\mathrm{Law}(X_t^n) \mid P_n^\beta)$ is non-increasing.*

Elements of proof of Lemma 2.8.8. The monotonicity for $\mathrm{TV}, \mathrm{Hellinger}, \mathrm{Kullback}, \chi^2$ comes from the Markov nature of the process and the convexity of

$$u \mapsto \Phi(u) = \begin{cases} \frac{1}{2}|u-1| & \text{if dist} = \mathrm{TV} \\ 1-\sqrt{u} & \text{if dist} = \mathrm{Hellinger}^2 \\ u\log(u) & \text{if dist} = \mathrm{Kullback} \\ u^2-1 & \text{if dist} = \chi^2 \end{cases}.$$

This is known as the Φ-entropy dissipation of Markov processes, see [75, 253, 16]. This can also be seen from (2.94). The monotonicity for TV follows also from the contraction property of the total variation with respect to general Markov kernels, see [190, Ex. 4.2].

The monotonicity for Fisher comes from the identity (2.95) and the convexity of V. By (2.94) this monotonicity is also equivalent to the convexity of $\mathrm{Kullback}$ along the dynamics. The monotonicity for $\mathrm{Wasserstein}$ can be obtained by computing the derivative along the dynamics starting from (2.97), but this is more subtle due to the variational nature of this distance and involves the convexity of V, see for instance [40, Bottom of p. 2442 and Lem. 3.2].

The monotonicities can also be extracted from the exponential decays of Lemma 2.8.7 thanks to the Markov property and the profile $\mathrm{e}^{-t} = 1-t+o(t)$ of the prefactor in the right hand side. $\square$

The convexity of the interaction $-\log$ as well as the constant nature of the diffusion coefficient in the evolution equation (2.3) allows to use simple "maximum principle" type arguments to prove that the dynamic exhibits a monotonous behavior and an exponential decay.

Lemma 2.8.9 (Monotonicity and exponential decay). *Let $(X_t^n)_{t\geq 0}$ and $(Y_t^n)_{t\geq 0}$ be a pair of DOU processes solving (2.3), $\beta \geqslant 1$, driven by the same Brownian motion $(B_t)_{t\geq 0}$ on $\mathbb{R}^n$ and with respective initial conditions $X_0^n \in \overline{D}_n$ and $Y_0^n \in \overline{D}_n$. If for all $i \in \{1,\dots,n\}$*

$$X_0^{n,i} \leqslant Y_0^{n,i}$$

then the following properties hold true:

- *(Monotonicity property) for all $t \geqslant 0$ and $i \in \{1,\dots,n\}$,*

$$X_t^{n,i} \leqslant Y_t^{n,i},$$

- *(Decay estimate) for all $t \geqslant 0$,*

$$\max_{i\in\{1,\dots,n\}}(Y_t^{n,i} - X_t^{n,i}) \leqslant \max_{i\in\{1,\dots,n\}}(Y_0^{n,i} - X_0^{n,i})\mathrm{e}^{-t}.$$

Proof of Lemma 2.8.9. The difference of $Y_t^n - X_t^n$ satisfies

$$\partial_t(Y_t^{n,i} - X_t^{n,i}) = \frac{\beta}{n}\sum_{j:j\neq i}\frac{(Y_t^{n,j} - X_t^{n,j}) - (Y_t^{n,i} - X_t^{n,i})}{(Y_t^{n,j} - Y_t^{n,i})(X_t^{n,j} - X_t^{n,i})} - (Y_t^{n,i} - X_t^{n,i}). \tag{2.98}$$

Since there are almost surely no collisions between the coordinates of X^n, resp. of Y^n, the right-hand side is almost surely finite for all $t > 0$ and every process $Y_t^{n,i} - X_t^{n,i}$ is $\mathcal{C}^1$ on $(0,\infty)$. Note that at time 0 some derivatives may blow up as two coordinates of X^n or Y^n may coincide.

Let us define

$$M(t) = \max_{i\in\{1,\dots,N\}}(Y_t^{n,i} - X_t^{n,i}) \quad \text{and} \quad m(t) = \min_{i\in\{1,\dots,N\}}(Y_t^{n,i} - X_t^{n,i}).$$

Elementary considerations imply that M and m are themselves $\mathcal{C}^1$ on $(0,\infty)$ and that at all times $t > 0$, there exist i, j such that

$$\partial_t M(t) = \partial_t (Y_t^{n,i} - X_t^{n,i}) \quad \text{and} \quad \partial_t m(t) = \partial_t (Y_t^{n,j} - X_t^{n,j}).$$

This would not be true if there were infinitely many processes of course. Now observe that if at time $t > 0$ we have $Y_t^{n,i} - X_t^{n,i} = M(t)$, then

$$\partial_t (Y_t^{n,i} - X_t^{n,i}) \leqslant -(Y_t^{n,i} - X_t^{n,i}).$$

This implies that $\partial_t M(t) \leqslant -M(t)$. Similarly, we can deduce that $\partial_t m(t) \geqslant -m(t)$. Integrating these differential equations, we get for all $t \geqslant t_0 > 0$

$$M(t) \leqslant \mathrm{e}^{-(t-t_0)} M(t_0), \quad m(t) \geqslant \mathrm{e}^{-(t-t_0)} m(t_0).$$

Since all processes are continuous on $[0,\infty)$, we can pass to the limit $t_0 \downarrow 0$ and get for all $t \geqslant 0$,

$$\min_{i\in\{1,\dots,N\}} (Y_t^{n,i} - X_t^{n,i}) \geqslant 0, \quad \max_{i\in\{1,\dots,N\}} (Y_t^{n,i} - X_t^{n,i}) \leqslant \mathrm{e}^{-t} \max_{i\in\{1,\dots,N\}} (Y_0^{n,i} - X_0^{n,i}).$$

$\square$

Remark 4 (Beyond DOU dynamics). *The monotonicity property of Lemma 2.8.9 relies on the convexity of the interaction $-\log$, and has nothing to do with the long-time behavior and the strength of V. In particular, this monotonicity property remains valid for the process solving (2.3) with an arbitrary V provided that it is $\mathcal{C}^1$ and there is no explosion, even in the situation where V is not strong enough to ensure that the process has an invariant law. If V is $\mathcal{C}^2$ then the decay estimate of Lemma 2.8.9 survives in the following decay or growth form:*

$$\max_{i\in\{1,\dots,n\}} (Y_t^{n,i} - X_t^{n,i}) \leqslant \max_{i\in\{1,\dots,n\}} (Y_0^{n,i} - X_0^{n,i}) \mathrm{e}^{t(-\inf_{\mathbb{R}} V'')}, \quad t \geqslant 0.$$

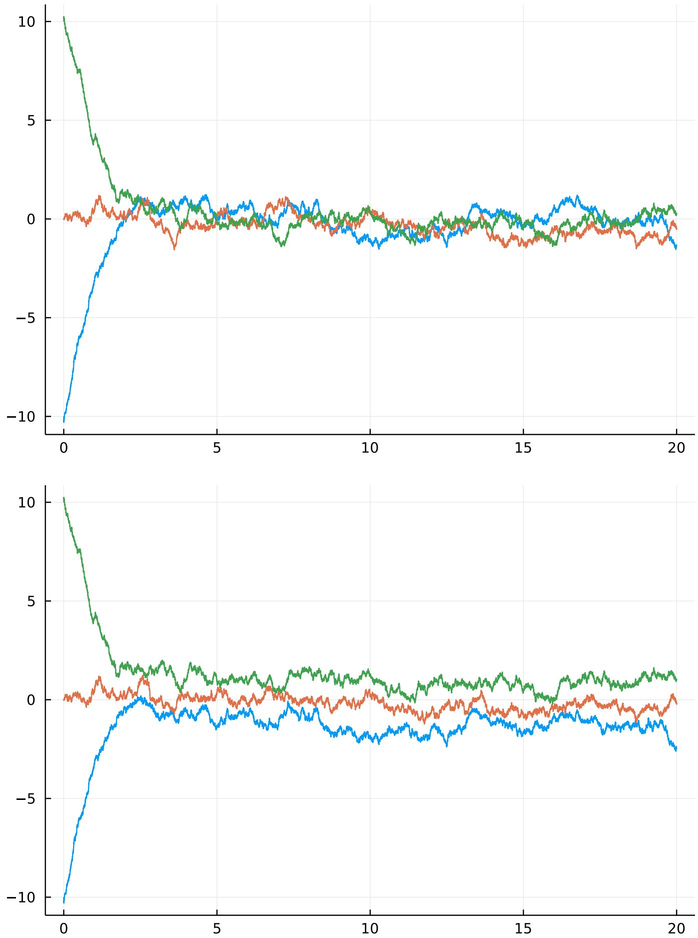

Figure 2.1: A trajectory of a single DOU with $n = 3$ and $x_0^n = (-10, 0, 10)$, $\beta = 0$ on top and $\beta = 2$ on bottom. The driving Brownian motions are the same.

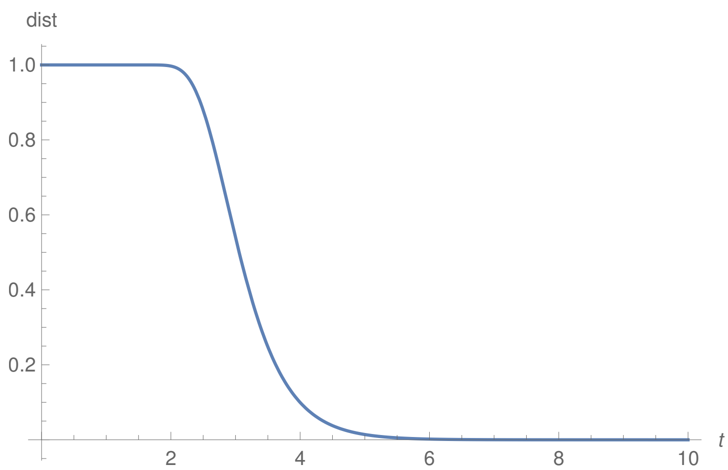

Figure 2.2: Plot of the function $t \mapsto \mathrm{Hellinger}(\mathrm{Law}(X_t^n) \mid P_n^\beta)$ (see (2.39) for the explicit formula) with $n = 50$, $\beta = 0$, and $\frac{|x_0^n|^2}{n} = 1$. Note that $\log(50) \approx 3.9$.

Chapter 3

Optimal local laws and CLT for the long-range Riesz gas

This chapter is based on the article Optimal local laws and CLT for the circular Riesz gas, *arXiv preprint arXiv:2112.05881.*

Contents

3.1 Introduction

3.1.1 Setting of the problem

In this paper, we study the one-dimensional Riesz gas on the circle. We denote $\mathbb{T} := \mathbb{R}/\mathbb{Z}$. For a parameter $s \in (0,1)$, let us consider the Riesz s-kernel on $\mathbb{T}$, defined by

$$g_s(x) = \lim_{n\to\infty}\Big(\sum_{k=-n}^{n}\frac{1}{|k+x|^s} - \frac{2}{1-s}n^{1-s}\Big) = \zeta(s,x)+\zeta(s,1-x), \tag{3.1}$$

where $\zeta(s,x)$ stands for the Hurwitz zeta function [31]. Note that g is the fundamental solution of the fractional Laplace equation on the circle

$$(-\Delta)^{\frac{1-s}{2}} g = c_s(\delta_0 - 1), \tag{3.2}$$

with c_s given by

$$c_s = \frac{\Gamma(\frac{1-s}{2})}{\Gamma(\frac{s}{2})}\frac{\sqrt{\pi}}{2^{1-s}}. \tag{3.3}$$

We endow $\mathbb{T}$ with the natural order $x < y$ if $x = x' + k$, $y = y' + k'$ with $k, k' \in \mathbb{Z}$, $x', y' \in [0,1)$ and $x' < y'$ and work on the set of ordered configurations

$$D_N = \{X_N = (x_1,\dots,x_N) \in \mathbb{T}^N : x_2 - x_1 < \dots < x_N - x_1\}.$$

On D_N let us consider the energy

$$\mathcal{H}_N : X_N \in D_N \mapsto N^{-s}\sum_{i\neq j} g_s(x_i - x_j), \tag{3.4}$$

where g is given by (3.2). The circular Riesz gas, at the inverse temperature $\beta > 0$, is defined by the Gibbs measure

$$\mathrm{d}\mathbb{P}_{N,\beta} = \frac{1}{Z_{N,\beta}}\exp(-\beta\mathcal{H}_N(X_N))\mathbb{1}_{D_N}(X_N)\mathrm{d}X_N,$$

where $Z_{N,\beta}$ is the normalizing constant, called the *partition function*, given by

$$Z_{N,\beta} = \int_{D_N} e^{-\beta\mathcal{H}_N(X_N)}\mathrm{d}X_N.$$

Throughout the paper, s is a fixed parameter in $(0,1)$.

The choice of the normalization in the definition of the energy (3.4) appears to be a natural choice, making β the effective inverse temperature governing the microscopic scale behavior.

The model described above belongs to a family of interacting particle systems named *Riesz gases*. On $\mathbb{R}^{\mathrm{d}}$, those are associated to a kernel of the form $|x|^{-s}$ with $s > 0$. The Riesz family also contains in dimensions 1 and 2 the so-called *log-gases* with kernel $-\log|x|$. For $\mathrm{d} \geqslant 2$ and $s = \mathrm{d}-2$, $|x|^{-s}$ is the fundamental solution of the Laplace equation on $\mathbb{R}^d$ and therefore corresponds to the Coulombian interaction. The parameter s determines the singularity as well as the *range* of the interaction. When $s > \mathrm{d}$, the interaction is short-range and the system, referred to as *hypersingular Riesz gas*, resembles a nearest-neighbour model. For $s \in (0,\mathrm{d})$ or $s = 0$ and $\mathrm{d} = 1, 2$, Riesz gases are long-range particle systems, which have, as such, attracted much attention in both mathematical and physical contexts.

The 1D log-gas, also called β-ensemble has been extensively studied in the last decades, partly for its connection to random matrix theory. Indeed, it corresponds, in the cases $\beta \in \{1,2,4\}$, to the distribution of the eigenvalues of $N \times N$ symmetric/Hermitian/symplectic random matrices with independent Gaussian entries (see the original paper of Dyson [106]). The 1D log-gas also appears in many other contexts such as zeros of random polynomials, zeros of the Riemann function and is conjectured to be related for instance to the eigenvalues of random Schrödinger operators [191]. The 2D Coulomb gas is another fundamental model, which has raised considerable attention in the last decades. Among many other examples, it is connected to non-unitary random matrices, Ginzburg-Landau vertices, Fekete points, complex geometry, the XY model and the KT transition [238]. For other values of s, let us mention that the case $s = 2$ in dimension 1 is an integrable system, called *classical Calogero-Sutherland model*. The study of minimizers of Riesz interactions is also a dynamic topic [143, 82] and is the object of long-standing conjectures related to sphere packing problems [36]. From a statistical physics perspective, even in dimension 1, the Riesz gas is not fully elucidated since the classical theory of the 60-70s [221] fails to be applied due the long-range nature of the interaction. The reader may refer to the nice review [191], where an account of the literature and many open problems on Riesz gases are given.

As the number of particles N tends to infinity, the empirical measure

$$\mu_N := \frac{1}{N}\sum_{i=1}^N \delta_{x_i}$$

converges almost surely under $\mathbb{P}_{N,\beta}$ (in a suitable topology) to the uniform measure on the circle. This result can be obtained through standard large deviations techniques (see for instance [236, Ch. 2] for the case of Riesz gases on the real line, which adapts readily to the periodic setting). In the large N limit, particles tend to spread uniformly on the circle, which suggests that particles spacing (or gaps) $N(x_{i+k} - x_i)$ concentrate around the value k. The first goal of this paper is to quantify the fluctuations of the gaps around their mean. We establish the optimal size of the fluctuations of $N(x_{i+k} - x_i)$, which turns out to be in $O(\beta^{-\frac{1}{2}} k^{\frac{s}{2}})$, as conjectured in the recent physics paper [234]. This type of result is referred to in the literature as a rigidity estimate. It was intensively investigated for β-ensembles (see for instance [46, 49, 47, 51]), but the correct observable in that case is $x_i - \gamma_i$, where x_i is the i-th particle and γ_i the classical location of the i-th particle, that is the corresponding quantile of the equilibrium measure arising in the mean-field limit.

A complementary way to study the rigidity of the system is to investigate the fluctuations of linear statistics of the form

$$\mathrm{Fluct}_N[\xi(\ell_N^{-1}\cdot)] := \sum_{i=1}^N \xi(\ell_N^{-1} x_i) - N\ell_N \int_{\mathbb{T}} \xi, \tag{3.5}$$

where $\xi : \mathbb{T} \to \mathbb{R}$ is a given measurable test-function and $\{\ell_N\}$ a sequence of numbers in $(0, 1]$. For smooth test-functions, many central limit theorem (CLT) results are available in the literature on 1D-log gases, including [157, 241, 44, 46, 49, 27, 51, 145]. For 1D Riesz gases with $s \in (0, 1)$, to our knowledge, no prior results on CLT for linear statistics are known. In this paper we obtain a *quantitative* CLT for (3.5), which is valid at all scales $\{\ell_N\}$ down to microscopic scales $\ell_N \gg \frac{1}{N}$. A major direction in random matrix theory is to establish CLTs for (3.5) allowing test-functions which are *as singular as possible*. Indeed it is a natural question to capture the fluctuations of the number of points and of the logarithmic potential, which are key observables for the log-gas. The question of the optimal regularity on the test-function has also drawn a lot of interest because it encapsulates non-universality features in the context of Wigner matrices. In this paper, the stake for us is to provide a robust method allowing to treat singular test-functions in a systematic way. The main question we investigate is therefore a *regularity issue*. Using new concentration inequalities we are able to treat singular test-functions, including characteristic functions of intervals and inverse power function up to the critical power $\frac{s}{2}$. In particular we obtain a CLT for the number of points, thus extending to a Riesz (periodic and fully convex) setting some of the recent results of [51].

Let us now introduce the main tools and objects used in the proofs. For any reasonable Gibbs measure on D_N (or $\mathbb{R}^N$), the fluctuations of any (smooth) statistics $F : D_N \to \mathbb{R}$ are related to the properties of a partial differential equation called the Helffer-Sjöstrand (H.-S.) equation, which is sometimes referred to as a Witten Laplacian (on 1-forms). This equation appears in [242, 243, 149]. It is more substantially studied in [148, 147, 201], where it is used to establish correlation decay, uniqueness of the limiting measure and log-Sobolev inequalities for models with convex interactions. The purpose of the present paper is to show how the analysis of Helffer-Sjöstrand equations provides powerful tools to study the fluctuations of linear statistics with singular test-functions.

The proof of the near-optimal rigidity is essentially similar to [46]. It exploits the convexity of the interaction and is thus very specific to 1D systems. The method is mainly based on a concentration inequality for divergence-free functions and on a key convexity result due to Brascamp [55].

The method of proof of the CLT for linear statistics starts by performing the mean-field transport argument usually attributed to Johansson [157]. When studying the Laplace transform of linear statistics $\mathrm{Fluct}_N[\xi]$, this consists in applying a well-chosen change of variables on each point, depending only on its position, to transport the uniform measure on the circle to the perturbed equilibrium measure (perturbed by the effect of adding $t\mathrm{Fluct}_N[\xi]$ to the energy). In this paper, one computes variances instead of Laplace transforms and the implementation of the transport of [157] takes the form of an integration by parts. This argument is a variation of the so-called loop equations (see [46] for various comments on this topic). It is the starting point of many CLTs on β-ensembles and Coulomb gases but it received a more systematic analysis in the series of works [184, 27, 181, 187, 239]. One can also interpret this transport as a mean-field approximation of the H.-S. equation associated to (the gradient of) linear statistics.

Since the transport is an approximate solution (a mean-field approximation) of the H.-S. equation, it creates an error term, sometimes called loop equation term, which is essentially a local weighted energy and the heart of the problem is to estimate its fluctuations. In contrast, in [184, 27, 181, 239] the typical size in a large deviation sense of this error term is evaluated, rather than the size of its fluctuations. Let us point out however that the later seems untractable in dimension $\mathrm{d} \geqslant 2$ because of the lack of convexity. The core of this paper is about the control on the fluctuations of this loop equation term through the analysis of the related H.-S. equation. Our proof is based on the following three technical inputs:

- The near-optimal rigidity estimates on gaps and nearest-neighbour distances,
- A Poincaré inequality in gap coordinates,
- A comparison principle for the Helffer-Sjöstrand equation.

The use of the comparison principle mentioned above (also known in [72, 149]) is one of the main novelties of the paper. This is the key technical tool to be able to treat linear statistics with singular test-functions. Indeed, after performing loop equations techniques, we will study singular local quantities, for which standard concentration inequalities, such as the Brascamp-Lieb or log-Sobolev inequalities, do not give the right order of fluctuations.

The central limit theorem is then obtained from a rather straightforward application of Stein's method. We show how the mean-field transport naturally leads to an approximate Gaussian integration by parts formula. As a result, quantifying normality boils down to controlling the variance of the loop equation term. The CLT then follows from the variance bound discussed in the above comments.

3.1.2 Main results

The following results are valid for all parameter $s \in (0,1)$, thus covering the entire long-range regime.

Our first result concerns the fluctuations of gaps and discrepancies. We establish the following near-optimal decay estimate:

Theorem 3.1.1 (Near-optimal rigidity). *Let $\varepsilon \in (0,1)$ and $\delta = \frac{\varepsilon}{4(s+2)}$. There exists two constants $C(\beta) > 0$ and $c(\beta) > 0$ locally uniform in β such that for each $i \in \{1, \dots, N\}$ and $1 \leqslant k \leqslant \frac{N}{2}$, we have*

$$\mathbb{P}_{N,\beta}(|N(x_{i+k} - x_i) - k| \leqslant k^{\frac{s}{2}+\varepsilon}) \geqslant 1 - C(\beta)e^{-c(\beta)k^{\delta}}. \tag{3.6}$$

Similarly for all $\varepsilon \in (0,1)$, setting $\delta = \frac{\varepsilon}{2(s+2)}$, there exist two constants $C(\beta) > 0$ and $c(\beta) > 0$ locally uniform in β such that for all $a \in \mathbb{T}$ and $\ell_N \in (0, \frac{1}{4})$, we have

$$\mathbb{P}_{N,\beta}\Big(\Big|\sum_{i=1}^{N} \mathbb{1}_{(a-\ell_N, a+\ell_N)}(x_i) - 2N\ell_N\Big| \leqslant (N\ell_N)^{\frac{s}{2}+\varepsilon}\Big) \geqslant 1 - C(\beta)e^{-c(\beta)(N\ell_N)^{\delta}}. \tag{3.7}$$

Theorem 3.1.1 is the natural extension of the rigidity result of [46, Th. 3.1] in the Riesz setting. As in [46], the controls on the deviations of the considered quantities have exponentially small probability. This exponential estimate then allows one to control the maximum of the deviations of the gaps and therefore reduces the phase space to an event where all gaps are close to their standard value. Theorem 3.1.1 is proved in Section 3.4.

The purpose of the next result is to show that $k^{\frac{s}{2}}$ is the exact fluctuation scale of the gap $N(x_{i+k} - x_i)$. This is equivalent to proving that $\sum_{i=1}^{N} \mathbb{1}_{(0,\ell_N)}(x_i)$ fluctuates at scale $(N\ell_N)^{\frac{s}{2}}$, for all $\ell_N \in (0,1)$. In fact we consider a larger class of linear statistics with singular test-functions. These are defined by

$$\mathrm{Fluct}_N[\xi] = \sum_{i=1}^{N} \xi(x_i) - N\int_{\mathbb{T}} \xi,$$

where $\xi : \mathbb{T} \to \mathbb{R}$ is a singular but piecewise smooth test-function. We are also able to treat linear statistics $\mathrm{Fluct}_N[\xi(\ell_N^{-1}\cdot)]$ with test-functions supported at any scales $\{\ell_N\}$, including the microscopic scale. We make the following assumptions on ξ and on the sequence $\{\ell_N\}$:

Assumptions 3.1.1.

(i) (Global regularity) The map ξ is in $\mathcal{C}^{-s+\varepsilon}(\mathbb{T}, \mathbb{R})$ for some $\varepsilon > 0$.

(ii) (Piecewise regularity) Let $\psi = (-\Delta)^{-\frac{s}{2}}\xi$. The map ψ is piecewise $\mathcal{C}^2$: there exist $a_1 < \ldots < a_p$ ($p \in \mathbb{N}$) such that on (a_i, a_{i+1}), ψ is $\mathcal{C}^2$, for each $i \in \{1, \dots, p\}$ (with the convention that $a_{p+1} = a_1$).

(iii) (Singularity) For each $i \in \{1, \dots, p\}$, there exists $\alpha_i \in (0, 1 - \frac{s}{2})$ such that

$$|\psi''|(x) \leqslant \frac{C}{|x - a_i|^{1+\alpha_i}}. \tag{3.8}$$

(iv) (Support) Let $\{\ell_N\}$ be a sequence in $[0,1)$. Assume either that ξ is supported on $(-\frac{1}{2}, \frac{1}{2})$ or that $\ell_N = 1$. In the first case, we let $\xi_0 : \mathbb{R} \to \mathbb{R}$ given by

$$\xi_0(x) = \begin{cases} \xi(x) & \text{if } |x| \leqslant \frac{1}{2} \\ 0 & \text{if } |x| > \frac{1}{2}. \end{cases} \tag{3.9}$$

Let us first comment upon the above assumptions.

Remark 5 (Comments on the Assumptions 3.1.1).

- *Assumptions (ii) and (iii) compare the singularities of* ξ *with the singularity at* 0 *of* $|x|^{-\frac{s}{2}+\varepsilon}$*: the derivative of order* $1-\frac{s}{2}$ *of* ξ *near a singularity* $a \in \mathbb{T}$ *is bounded by the derivative of order* $1-\frac{s}{2}$ $x \mapsto |x-a|^{-\frac{s}{2}+\varepsilon}$*. Note that the function* $x \mapsto |x|^{-\frac{s}{2}}$ *is the critical inverse power which does not lie in* $H^{\frac{1-s}{2}}(\mathbb{T},\mathbb{R})$*.*

- *When the scale* ℓ_N *tends to* 0*, Assumption (iv) ensures that* $\mathrm{Fluct}_N[\xi(\ell_N^{-1}\cdot)]$ *is at most of order* $O(N\ell_N)$*.*

- *The characteristic function* $\xi = \mathbb{1}_{(-a,a)}$ *satisfies Assumptions 3.1.1. Indeed the map* $\psi = (-\Delta)^{-\frac{s}{2}}\xi$ *is piecewise* $\mathcal{C}^2$ *with two singularities at* $-a$ *and* a *and* ψ'' *satisfies (3.8) with* $\alpha_1 = \alpha_2 = 1-s$*:* $\psi''(x) \underset{x\to -a}{\sim} c_0\frac{1}{|x+a|^{2-s}}$ *and* $\psi''(x) \underset{x\to a}{\sim} c_0\frac{1}{|x-a|^{2-s}}$ *for some constant* $c_0 \neq 0$*.*

Before stating the theorem, recall the definition of the fractional Sobolev seminorm on the circle $|\cdot|_{H^\alpha}$, for $\alpha > 0$. Let $h : \mathbb{T} \to \mathbb{R}$ in $L^2(\mathbb{T},\mathbb{R})$ of Fourier coefficient $\hat{f}(k)$, $k \in \mathbb{Z}$. Whenever it is finite we call $|h|_{H^\alpha}^2$ the quantity

$$|h|_{H^\alpha}^2 = \sum_{k\in\mathbb{Z}} |k|^{2\alpha}|\hat{h}|^2(k).$$

Similarly, the fractional Sobolev seminorm of a function $h : \mathbb{R} \to \mathbb{R}$ in $L^2(\mathbb{R},\mathbb{R})$, that we also denote $|h|_{H^\alpha}$, is defined by

$$|h|_{H^\alpha}^2 = \int |\xi|^{2\alpha}|\hat{h}|^2(\xi)\mathrm{d}\xi, \tag{3.10}$$

where $\hat{h}$ stands for the Fourier transform of h.

The variance of $\mathrm{Fluct}_N[\xi(\cdot\ell_N^{-1})]$ under $\mathbb{P}_{N,\beta}$ may be expanded as follows:

Theorem 3.1.2 (Variance of singular linear statistics). *Let* ξ *and* $\{\ell_N\}$ *satisfying Assumptions 3.1.1. Let* $\psi = (-\Delta)^{-\frac{s}{2}}\xi$*. Let* $a_1 < \ldots < a_p$ *be the singularities of* ψ'' *and denote* $1+\alpha_1, \ldots, 1+\alpha_p$ *their order as in (3.8). Let* $\alpha = \max_{i=1}^p \alpha_i$*. Let us define*

$$\sigma_\xi^2 = \frac{1}{2\beta c_s}\begin{cases} |\xi|^2_{H^{\frac{1-s}{2}}} & \text{if } \ell_N = 1 \\ |\xi_0|_{H^{\frac{1-s}{2}}} & \text{if } \ell_N \to 0, \text{ with } \xi_0 \text{ as in (3.9).}\end{cases} \tag{3.11}$$

For all $\varepsilon > 0$*, there holds*

$$\begin{aligned}\mathrm{Var}_{\mathbb{P}_{N,\beta}}[\mathrm{Fluct}_N[\xi(\ell_N^{-1}\cdot)]] &= N^s\sigma^2_{\xi(\cdot\ell_N^{-1})} + O((N\ell_N)^{2s-2+\max(1,2\alpha)+\varepsilon}) \\ &= (N\ell_N)^s(\sigma_\xi^2 + \mathbb{1}_{\ell_N\to 0}O(\ell_N^2|\xi|_{L^2}^2)) + O((N\ell_N)^{2s-2+\max(1,2\alpha)+\varepsilon}).\end{aligned} \tag{3.12}$$

Note that since $\max\alpha_l < 1-\frac{s}{2}$, $\xi \in H^{\frac{1-s}{2}}(\mathbb{T},\mathbb{R})$ and moreover the remaining term in the expansion (3.12) is always $o((N\ell_N)^s)$.

Remark 6 (On the adaptation to β-ensembles). *We expect that our method can also give a (log-correlated) CLT for the test-functions* $\mathbb{1}_{(-a,a)}$ *and* $x \mapsto \log|x-a|$ *for 1D log-gases on the circle or on the real line when the external potential is convex.*

By Remark 5, the number-variance (i.e variance of the number of points) of the Riesz gas grows in $O(N^s)$, like the variance of smooth linear statistics. In comparison, for the 1D log-gas, smooth linear statistics fluctuate in $O(1)$ with an asymptotic variance proportional to the squared Sobolev norm $|\cdot|^2_{H^{\frac{1}{2}}}$ (see [27] for instance) whereas the number of points in an interval $(-a,a)$ fluctuate in $O(\log N)$. This distinct behavior is due to the fact that the characterize function $\mathbb{1}_{(-\frac{1}{2},\frac{1}{2})}$ is not in $H^{\frac{1}{2}}(\mathbb{T},\mathbb{R})$ but is in $H^{\frac{1-s}{2}}(\mathbb{T},\mathbb{R})$ for all $s\in(0,1)$. Theorem 3.1.2 shows that concerning the fluctuations, the Riesz gas with $s\in(0,1)$ interpolates between the 1D log-gas case $s=0$ and the Poisson-type case $s=1$. Moreover Theorem 3.1.2 makes the 1D long-range Riesz gas a hyperuniform particle system in the sense of [249] (meaning that the number-variance is much smaller than for i.i.d variables).

As mentioned in the beginning of the introduction, the next-order term in the expansion (3.12) corresponds to the variance of a local energy arising from the mean-field transport of [157], sometimes referred in the literature to as a *loop equation term*. One could extract the leading-order of this variance and relate it to the second derivative of the free energy of the infinite Riesz gas.

The next question we address concerns the asymptotic behavior of rescaled linear statistics. We show that under the Assumptions 3.1.1 and provided $\ell_N \gg \frac{1}{N}$, the linear statistics converges after rescaling to a Gaussian random variable. For any probability measures μ and ν on $\mathbb{R}$ let us denote $\mathrm{d}(\mu,\nu)$ the distance

$$\mathsf{d}(\mu,\nu)=\sup\Big\{\int f\mathrm{d}(\mu-\nu) : |f|_\infty\leqslant 1, |f'|_\infty\leqslant 1\Big\}.$$

We establish the following result:

Theorem 3.1.3 (CLT for singular linear statistics). *Let ξ and ℓ_N satisfying Assumptions 3.1.1.*

- *The sequence of random variables $(N\ell_N)^{-\frac{s}{2}}\mathrm{Fluct}_N[\xi(\ell_N^{-1}\cdot)]$ converges in distribution to a centered Gaussian random variable with variance σ_ξ^2 given by*

$$\sigma_\xi^2=\frac{1}{2\beta c_s}\begin{cases}|\xi|^2_{H^{\frac{1-s}{2}}} & \text{if } \ell_N=1\\ |\xi_0|^2_{H^{\frac{1-s}{2}}} & \text{if } \ell_N\to 0, \text{ with } \xi_0 \text{ as in (3.9).}\end{cases} \tag{3.13}$$

- *Let Z be a centered Gaussian random variable with variance σ_ξ^2. Then for all $\varepsilon>0$, we have*

$$\mathsf{d}(\mathrm{Law}((N\ell_N)^{-\frac{s}{2}}\mathrm{Fluct}_N[\xi(\ell_N^{-1}\cdot)]),\mathrm{Law}(Z))=O\Big((N\ell_N)^{-\frac{1-s}{2}}+(N\ell_N)^{-(1-\frac{s}{2}-\max\alpha_l-\varepsilon)}\Big).$$

Theorem 3.1.3 can be interpreted as the convergence of the field $\sum_{i=1}^N g_s(x_i-\cdot)-N\int g_s(x-\cdot)\mathrm{d}x$ to a fractional Gaussian field for the weak topology. Observe that if ξ_i and ξ_j have disjoint support then if $s\in(0,1)$, $\Gamma_{i,j}$ is not, in general, equal to 0, which shows that the corresponding fractional field does not exhibit spatial independence. This reflects the non-local nature of the fractional Laplacian $(-\Delta)^{\frac{1-s}{2}}$ for $s\in(0,1)$. Following Remark 5, Theorem 3.1.3 gives a CLT for gaps and discrepancies:

Corollary 3.1.4 (CLT for the number of points). *For all $a\in\mathbb{T}$ and $\{\ell_N\}$ such that $\ell_N\gg\frac{1}{N}$, the sequence of random variables*

$$N^{-\frac{s}{2}}\zeta(-s,2\ell_N)^{-\frac{1}{2}}\Big(\sum_{i=1}^N \mathbb{1}_{(-\ell_N,\ell_N)}(x_i)-2N\ell_N\Big)$$

converges in distribution to a centered Gaussian random variables with variance

$$\sigma^2(a) := \frac{\operatorname{cotan}(\frac{\pi}{2}s)}{\beta\frac{\pi}{2}s}.$$

For each $i \in \{1, \ldots, N\}$ and any sequence of integers $\{k_N\}$ with values in $\{1, \ldots, \frac{N}{2}\}$ such that $k_N \to \infty$ as $N \to \infty$, the sequence of random variables

$$k_N^{-\frac{s}{2}}\zeta(-s, \frac{k_N}{N})^{-\frac{1}{2}}(N(x_{i+k_N} - x_i) - k_N)$$

converges in distribution to a centered Gaussian random variables with variance $\sigma^2(\frac{1}{2})$.

Note that $\zeta(-s, 2\ell_N) = O(\ell_N^s)$ with $\zeta(s, 2\ell_N) \sim \ell_N^s$ when $\ell_N \to 0$. Corollary 3.1.4 is an extension of the results on the fluctuations of single particles in the bulk for β-ensembles, see [141] for the GUE and [51]. Theorem 3.1.3 can also be applied to power-type functions of the form $x \in \mathbb{T} \mapsto |x|^{-\alpha}$ with $\alpha \in (0, \frac{s}{2})$.

Corollary 3.1.5 (CLT for power-type functions). *Let $a \in \mathbb{T}$ and $\alpha \in (0, \frac{s}{2})$. Let $\ell_N \gg \frac{1}{N}$. Then*

$$N^{-\frac{s}{2}}\Big(\sum_{i=1}^N \frac{1}{|x_i - a|^\alpha} - 2^\alpha N\Big)$$

converges in distribution to a centered Gaussian random variables with a variance given by (3.11).

The test-function $x \in \mathbb{T} \mapsto |x|^{-\frac{s}{2}}$ is the critical inverse power which does not lie in $H^{\frac{1-s}{2}}$. This should be compared in the case $s = 0$ to the test-functions $\mathbb{1}_{(-a,a)}$ and $-\log|x|$, for which the associated linear statistics satisfy a log-correlated central limit theorem as shown for instance in [51].

3.1.3 Context, related results, open questions

Rigidity of β-ensembles As mentioned in the introduction, Theorems 3.1.1 and 3.1.3 are the natural extensions to the circular Riesz gas of some known results on the fluctuations of β-ensembles. We refer again to [46, 49, 47, 51] for rigidity estimates, to [157, 241, 44, 27, 173, 144, 205] for CLTs for linear statistics with smooth test-functions and to [144, 172] for the case of the circular β-ensemble. In the case of the GUE, that is for $\beta = 2$ with a quadratic potential, a CLT for test-functions in $H^{\frac{1}{2}}$ is obtained in [244] using a Littlewood-Paley type decomposition argument. However as observed in [172, Rem. 1.3], the minimal regularity of the test-function depends on β. Indeed for $\beta = 4$, leveraging on variances expansions of [156], [172] exhibits a test-function in $H^{\frac{1}{2}}$ such that the associated linear statistics does not have a finite limit. Since the characteristic function of a given interval is not is $H^{\frac{1}{2}}$, the asymptotic scaling of discrepancies in intervals is not of order 1. It is proved in [141] that for the GUE, eigenvalues x_i in the bulk fluctuate in $O(\sqrt{\log i})$ and that discrepancies are of order $\sqrt{\log N}$. A general CLT for the characteristic functions of intervals and for the logarithm function is given in the recent paper [51]. Concerning the method of proof, let us point out a very similar variation on Stein's method developed in [173], see also [144] for a high-temperature regime.

Local laws and fluctuations for the Langevin dynamics A related and much studied question concerns the rigidity of the Dyson Brownian motion, an evolving gas of particles whose invariant distribution is given by β-ensemble. The time to equilibrium at the microscopic scale of Dyson Brownian motion was studied in many papers including [114, 115], see also [45] for optimal relaxation estimates. A central limit theorem at mesoscopic scale for linear statistics of the Dyson Brownian Motion is established in [153], thus exhibiting a time dependent covariance structure.

Decay of the correlations and Helffer-Sjöstrand representation The decay of the gaps correlations of β-ensembles have been extensively studied in [116], where a power-law decay in the inverse squared distance is established. The starting point of [116] is based on a a representation of the correlation function by a random walk in a dynamic random environment or in other words on a dynamic interpretation of the Helffer-Sjöstrand operator. The paper [116] then develops a sophisticated homogenization theory for a system of discrete parabolic equations. In a different context, a more direct analysis of the Helffer-Sjöstrand operator has been developed in the groundwork [201] to characterize the scaling limit of the gradient interface model in arbitrary dimension $\mathrm{d} \geqslant 1$. Combining ideas from [201] and from quantitative stochastic homogenization, the paper [10] then shows that the free energy associated to this model is at least $\mathcal{C}^{2,\alpha}$ for some $\alpha > 0$. We also refer to the recent paper [247] which studies in a similar framework the scaling limit of the non-Gaussian membrane model. In non-convex settings, much fewer results are available in the literature. One can mention the work [91] which establishes the optimal decay for the two-point correlation function of the Villain rotator model in $\mathbb{Z}^d$, for $\mathrm{d} \geqslant 3$ at low temperature. It could be interesting to develop a direct method to analyze the large scale decay of the Helffer-Sjöstrand equation in the context of one-dimensional Riesz gases. We plan to address this question in future work.

Uniqueness of the limiting point process The question of the decay of the correlations mentioned above is related to property of uniqueness of the limiting measure. One expects that after rescaling, chosen so that the typical distance between consecutive points is of order 1, the point process converges, in a suitable topology, to a certain point process Riesz_β. For $s = 0$, the limiting point process called Sine_β, is unique and universal as proved in [46, 49]. The existence of a limit was first established in [250] for β-ensembles with quadratic exterior potential, together with a sophisticated description and in [163] for the circular β-ensemble. The Sine_β process has also been characterized as the unique minimizer of the free energy functional governing the microscopic behavior in [111] using a displacement convexity argument. In Chapter 4, we prove the existence of a limiting point process Riesz_β for the circular Riesz ensemble.

1D hypersingular Riesz gases Although the 1D hypersingular Riesz gas (i.e $s > 1$) is not hyperuniform, its fluctuations are also of interest. In such a system, the macroscopic and microscopic behaviors are *coupled*, a fact which translates into the linear response associated to linear statistics (in contrast with long-range particle systems, the linear response is a combination of a mean-field change of variables, moving each point according to its position only, and of local perturbations). Simple heuristic computations shows that the limiting variance is then proportional to a L^2 norm (after subtraction of the mean) with a factor depending on the second order derivative of the free energy.

Fluctuations of Riesz gases in higher dimension For $\mathrm{d} \geqslant 1$ and s smaller than d, the proof of existence of a thermodynamic limit for the Riesz gas is delicate as the energy is long-range. It was obtained in [182] for $s \in (\min(\mathrm{d}-2, 0), \mathrm{d})$, leveraging among many other ingredients on an electric formulation of the Riesz energy, see [206], and on a screening procedure introduced in [228] and then improved in [218, 206]. The first task to study the fluctuations of higher dimensional long-range Riesz gases is to establish local laws, that is to control the number of points and the energy in cubes of small scales. This was done for the Coulomb gas in arbitrary dimension down to the microscopic scale in the paper [9] using subadditive and supperadditive approximate energies. Due to the lack of convexity, establishing a CLT or even a sub-poissonian rigidity estimate for linear statistics of Riesz gases in arbitrary dimension is a very delicate task. In dimension 2, since long-range interactions are overwhelmingly dominant, a CLT for linear statistics with smooth test-functions can be proved, see [184, 24, 180], without proving any "probabilistic cancellation" on local quantities, but only a "quenched cancellation" on some angle term. Let us finally mention the work [188] where the 2D Coulomb gas is shown to be hyperuniform, meaning that the variance of the number of points in a ball scales much smaller than the volume. The paper [188] establishes an important quantitative translation invariance property based on refinements of Mermin-Wagner type arguments, see also [248]. In higher dimension much fewer results are available. One can mention the result of [239] which treats the 3D Coulomb gas at high temperature "under a no phase transition assumption". A simpler variation of the 3D Colomb gas, named *hierarchical* Coulomb gas, has also been investigated in the work [83], followed by [124].

3.1.4 Outline of the main proofs

We now explain the general strategy to obtain the variance expansion formula of Theorem 3.1.2 and the CLT of Theorem 3.1.3. Since the proof of Theorem 3.1.1 is similar to the proof of [46, Th. 3.1] we do not detail it here.

To simplify assume that $\ell_N = 1$. We are interested in the fluctuations of the linear statistics $\mathrm{Fluct}_N[\xi]$, where $\xi : \mathbb{T} \to \mathbb{R}$ is a piecewise smooth function satisfying Assumptions 3.1.1.

The Helffer-Sjöstrand equation Let $F : D_N \to \mathbb{R}$ smooth enough. The fluctuations of F are related to a partial differential equation through the representation

$$\mathrm{Var}_{\mathbb{P}_{N,\beta}}[F] = \mathbb{E}_{\mathbb{P}_{N,\beta}}[\nabla F \cdot \nabla \phi], \tag{3.14}$$

where $\phi : D_N \to \mathbb{R}$ solves the Poisson equation

$$\begin{cases} \mathcal{L}\phi = F - \mathbb{E}_{\mathbb{P}_{N,\beta}}[F] & \text{on } D_N \\ \nabla \phi \cdot \vec{n} = 0 & \text{on } \partial D_N, \end{cases} \tag{3.15}$$

where $\mathcal{L}$ stands for the generator

$$\mathcal{L} = \beta \nabla \mathcal{H}_N \cdot \nabla - \Delta.$$

Note that (3.14) directly follows by integration by parts once it is known that (3.15) has a solution. Differentiating (3.15), one obtains the so-called Helffer-Sjöstrand equation which reads

$$\begin{cases} A_1 \nabla \phi = \nabla F & \text{on } D_N \\ \nabla \phi \cdot \vec{n} = 0 & \text{on } \partial D_N, \end{cases} \tag{3.16}$$

with A_1 formally given by

$$A_1 = \beta \nabla^2 \mathcal{H}_N + \mathcal{L} \otimes I_N.$$

We will use various tools to analyze the solution of (3.16) based on mean-field approximations, convexity and monotonicity.

Since $\partial_{ij}\mathcal{H}_N \leqslant 0$ for each $i \neq j$, it is standard that $\mathbb{P}_{N,\beta}$ satisfies the FKG inequality, meaning that the covariance between two increasing functions is non-negative. This can be formulated by saying that $\mathcal{L}^{-1}$ preserves the cone of increasing functions: if $\nabla F \geqslant 0$ (coordinate wise), then $\nabla\phi \geqslant 0$. A nice consequence is the following: if $F, G : D_N \to \mathbb{R}$ are such that $|\nabla F| \leqslant \nabla G$, then

$$\mathrm{Var}_{\mathbb{P}_{N,\beta}}[F] \leqslant \mathrm{Var}_{\mathbb{P}_{N,\beta}}[G]. \tag{3.17}$$

This comparison principle can be extended to non-gradient vector-fields, which will be used as a key argument to handle the fluctuations of some complicated singular functions.

Mean-field transport It turns out that when F is a linear statistics, i.e $F = \mathrm{Fluct}_N[\xi]$ for some smooth enough test-function $\xi : \mathbb{T} \to \mathbb{R}$, then the solution $\nabla\phi$ of (3.16) can be approximated by a transport Ψ_N in the form $X_N \in D_N \mapsto \frac{1}{N^{1-s}}(\psi(x_1), \ldots, \psi(x_N))$ for some well-chosen map $\psi : \mathbb{T} \to \mathbb{R}$. Letting $\Delta := \{(x, y) \in \mathbb{T}^2 : x = y\}$, one may write

$$\nabla H_N \cdot \Psi_N = N \iint_{\Delta^c} g_s'(x-y)(\psi(x) - \psi(y))\mathrm{d}\mu_N(x)\mathrm{d}\mu_N(y),$$

where $\mu_N := \frac{1}{N}\sum_{i=1}^N \delta_{x_i}$. Let us expand μ_N around the Lebesgue measure $\mathrm{d}x$ on $\mathbb{T}$ and denote $\mathrm{fluct}_N = N(\mu_N - \mathrm{d}x)$. Noting that the constant term vanishes, one can check that

$$\nabla H_N \cdot \Psi_N = 2\int(-g_s' * \psi)\mathrm{fluct}_N + \frac{1}{N^{1-s}}\mathrm{A}[\psi] \tag{3.18}$$

with

$$\mathrm{A}[\psi] = \iint_{\Delta^c}(\psi(x) - \psi(y))N^{-s}g_s'(x-y)\mathrm{dfluct}_N(x)\mathrm{dfluct}_N(y). \tag{3.19}$$

The leading-order of (3.18) being a linear statistics, one can choose ψ such that $-\beta\nabla H_N \cdot \Psi_N + \mathrm{div}\,\Psi_N \simeq F$ by letting ψ such that

$$\psi' = -\frac{1}{2\beta c_s}(-\Delta)^{\frac{1-s}{2}}\xi$$

with $\int \psi = 0$. The central task of the paper is to show that for a large class of singular maps ψ and all $\varepsilon > 0$,

$$\mathrm{Var}_{\mathbb{P}_{N,\beta}}[\mathrm{A}[\psi]] \leqslant CN^{1+\varepsilon}|\psi'_{\mathrm{reg}}|^2_{L^2}. \tag{3.20}$$

Splitting the variance of the next-order term Denote

$$\zeta : (x, y) \in \mathbb{T}^2 \mapsto \frac{\psi(x) - \psi(y)}{x - y}$$

so that

$$\mathrm{A}[\psi] = \iint_{\Delta^c}\zeta(x, y)N^{-s}\tilde{g}(x-y)\mathrm{dfluct}_N(x)\mathrm{dfluct}_N(y), \tag{3.21}$$

where $\tilde{g} : x \in \mathbb{T}\backslash\{0\} \mapsto xg_s'(x)$. Note that for each $i = 1, \ldots, N$

$$\partial_i\mathrm{A}[\psi] = \underbrace{2\int_{y\neq x_i}\partial_1\zeta(x_i, y)N^{-s}\tilde{g}(x-y)\mathrm{dfluct}_N(y)}_{\simeq \mathrm{V}_i} + \underbrace{2\int_{y\neq x_i}\zeta(x_i, y)N^{-s}\tilde{g}'(x-y)\mathrm{dfluct}_N(y)}_{\simeq \mathrm{W}_i}.$$

We have thus split $\nabla \mathrm{A}[\psi]$ into a macroscopic force V and a microscopic force W (the splitting is in fact slightly different). By subadditivity, it follows that

$$\mathrm{Var}_{\mathbb{P}_{N,\beta}}[\mathrm{A}[\psi]] \leqslant 2\underbrace{\mathbb{E}_{\mathbb{P}_{N,\beta}}[\mathrm{V}\cdot A_1^{-1}\mathrm{V}]}_{(I)} + 2\underbrace{\mathbb{E}_{\mathbb{P}_{N,\beta}}[\mathrm{W}\cdot A_1^{-1}\mathrm{W}]}_{(II)}.$$

Control on (II) with Poincaré inequality in gap coordinates In gap coordinates the microscopic force W behaves well: there exists $\tilde{\mathrm{W}}$ such that for all $U_N \in \mathbb{R}^N$,

$$\mathrm{W}\cdot U_N = -\sum_{i=1}^N \tilde{\mathrm{W}}_i N(u_{i+1}-u_i)$$

satisfying typically (i.e with overwhelming probability) the estimate

$$|\tilde{\mathrm{W}}|^2 \leqslant CN^{1+\varepsilon}|\psi'|_{L^2}^2 \tag{3.22}$$

for all $\varepsilon > 0$. By penalizing configurations with large nearest-neighbor distances, one can modify the Gibbs measure into a new one being uniformly log-concave with respect to the variables $N(x_{i+1} - x_i), i = 1,\dots,N$. Applying the Poincaré inequality in gap coordinates therefore gives using (3.22),

$$(I) \leqslant CN^{1+\varepsilon}|\psi'|_{L^2}^2. \tag{3.23}$$

Control on (I) with the comparison principle In substance, one should think of V as satisfying for each $i = 1,\dots,N$

$$|\mathrm{V}_i| \leqslant C|\psi''(x_i)| + \text{“Lower order terms”}. \tag{3.24}$$

Note that for instance if $\xi = \mathbb{1}_{(a,b)}$, ψ''_{reg} blows like $|x|^{-(2-s)}$ near a and b. Therefore for such singular ψ, the Poincaré inequality does not provide satisfactory estimates for (I).

If V_i was exactly given by $\psi''(x_i)$ for each $i = 1,\dots,N$, one could upper bound $\mathbb{E}_{\mathbb{P}_{N,\beta}}[\mathrm{V}\cdot A_1^{-1}\mathrm{V}]$ by $N|\psi'_{\mathrm{reg}}|_{L^2}^2$ since for all $f \in L^2(\mathbb{T})$,

$$\mathrm{Var}_{\mathbb{P}_{N,\beta}}[\mathrm{Fluct}_N[f]] \leqslant N|f|_{L^2}^2. \tag{3.25}$$

The idea is to use the comparison principle (3.17) to compare the Dirichlet energy of V with respect to the variance of a linear statistics, which are easier to handle using for instance (3.25). Let $\zeta_N : \mathbb{T} \to \mathbb{R}$ such that $\zeta_N' = C|\psi''|$. Equation (3.24) can be put in the form

$$|\mathrm{V}| \leqslant \nabla \mathrm{Fluct}_N[\zeta_N].$$

It then follows from (3.17) that

$$\mathbb{E}_{\mathbb{P}_{N,\beta}}[\mathrm{V}\cdot A_1^{-1}\mathrm{V}] \leqslant \mathrm{Var}_{\mathbb{P}_{N,\beta}}[\mathrm{Fluct}_N[\zeta_N]] + \text{“Lower order terms”}$$

and the variance of $\mathrm{Fluct}_N[\zeta_N]$ is then roughly bounded by

$$\mathrm{Var}_{\mathbb{P}_{N,\beta}}[\mathrm{Fluct}_N[\zeta_N]] \leqslant N|\zeta_N|_{L^2}^2 \leqslant CN^{\max(1,2\max\alpha_i)},$$

which yields

$$(II) \leqslant CN^{\max(1,2\max\alpha_i)} \leqslant CN|\psi'|_{L^2}^2. \tag{3.26}$$

Let us emphasize that ζ_N is in fact slightly more complicated: its fluctuations are therefore studied as an auxiliary linear statistics by rerunning the previous steps.

Combining (3.23) and (3.26) gives (3.20), which easily concludes the proof of Theorem 3.1.2.

Central limit theorem The starting point for the proof of the CLT of Theorem 3.1.3 is very similar to [173] and proceeds by Stein's method. Let $G_N = N^{-\frac{s}{2}}\mathrm{Fluct}_N[\xi]$. We shall prove that for all $\eta \in \mathcal{C}^1(\mathbb{R},\mathbb{R})$ such that $|\eta'|_\infty \leqslant 1$, up to a small error term,

$$\mathbb{E}_{\mathbb{P}_{N,\beta}}[\eta(G_N)G_N] = \sigma_\xi^2 \mathbb{E}_{\mathbb{P}_{N,\beta}}[\eta'(G_N)] + \mathsf{Error}_N, \tag{3.27}$$

with σ_ξ^2 as in (3.13). The fundamental observation of Stein is that this approximate integration by parts formula quantifies a distance to normality. Indeed letting Z be a centered random variable with variance σ_ξ^2 and $h : \mathbb{R} \to \mathbb{R}$ smooth, one can solve the ODE

$$x\eta(x) - \eta'(x) = h(x) - \mathbb{E}[h(Z)] \tag{3.28}$$

and (3.27) can be written in the form

$$\mathbb{E}_{\mathbb{P}_{N,\beta}}[h(G_N)] - \mathbb{E}[h(Z)] = \mathsf{Error}_N,$$

showing that G_N is approximately Gaussian. Let us explain how to obtain (3.27). Let $\nabla\phi = A_1^{-1}\nabla G_N$. By integration by parts we have

$$\mathbb{E}_{\mathbb{P}_{N,\beta}}[\eta(G_N)G_N] = \mathbb{E}_{\mathbb{P}_{N,\beta}}[\eta'(G_N)\nabla G_N \cdot \nabla\phi]. \tag{3.29}$$

The goal is then to prove that $\nabla G_N \cdot \nabla \phi$ concentrates around σ_ξ^2. As explained in the second paragraph, $\nabla\phi$ may be approximated by the transport $N^{-1+\frac{s}{2}}\Psi$ with $\Psi : X_N \in D_N \mapsto (\psi(x_1), \dots, \psi(x_N))$ for some well-chosen map $\psi : \mathbb{T} \to \mathbb{R}$. Performing this approximate transport allows one to replace (3.29) by

$$\begin{aligned}\mathbb{E}_{\mathbb{P}_{N,\beta}}[\eta(G_N)G_N] - \sigma_\xi^2\mathbb{E}_{\mathbb{P}_{N,\beta}}[\eta'(G_N)] &= \underbrace{\mathbb{E}_{\mathbb{P}_{N,\beta}}\Big[\eta'(G_N)\Big(\frac{1}{N}\sum_{i=1}^N \xi'(x_i)\psi(x_i) - \sigma_\xi^2\Big)\Big]}_{(I)} \\ &\quad + \underbrace{\frac{1}{N^{1-\frac{s}{2}}}\mathrm{Cov}_{\mathbb{P}_{N,\beta}}[\eta(G_N), \beta\mathrm{A}[\psi] - \mathrm{Fluct}_N[\psi']]}_{(II)}.\end{aligned} \tag{3.30}$$

The error term (I) is handled with the local laws, the error term (II) by inserting (3.20) which concludes the proof of the CLT.

3.1.5 Structure of the paper

- Section 3.2 records some useful preliminaries on the fractional Laplacian on the circle.
- Section 3.3 shows the well-posedness of the Helffer-Sjöstrand equation and gives various consequences of convexity and monotonicity.
- Section 3.4 completes the proof of the near-optimal rigidity result of Theorem 3.1.1.
- Section 3.5 provides a proof of the variance expansion of Theorem 3.1.2.
- Section 3.6 contains the proof of the CLT of Theorem 3.1.3.

3.1.6 Notation

We denote $d : \{1,\dots,N\}^2 \to \mathbb{N}$ the symmetric distance $d(i,j) = \min(|j-i|, N-|j-i|)$ for each $1 \leqslant i,j \leqslant N$, Δ the diagonal $\Delta = \{(x,y) \in \mathbb{T}^2 : x = y\}$.

For all $\alpha \in (0,1)$ we let $\mathcal{C}^{\alpha}(\mathbb{T},\mathbb{R})$ be the space of α-Hölder continuous functions from $\mathbb{T}$ to $\mathbb{R}$ $\mathcal{C}^{-\alpha}(\mathbb{T},\mathbb{R})$ the dual of $\mathcal{C}^{\alpha}(\mathbb{T},\mathbb{R})$. We write $\nabla^2 f$ for the Hessian of a real-valued function f.

For future works, we keep track of the dependency of the constants in β. For all $A,B \geqslant 0$, we write $A \leqslant C(\beta)B$ whenever there exists a constant $C \in \mathbb{R}$ (which may depend on s) locally uniform in β such that $A \leqslant CB$. Similarly, we write $A = O_{\beta}(1)$ whenever there exists a constant C locally uniform in β such that $|A| \leqslant C$.

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3.2 Preliminaries

3.2.1 The fundamental solution of the fractional Laplacian on circle

We begin by justifying that the fundamental solution of the fractional Laplace equation on the circle is given by (3.1). The formula (3.1) can be expected since it corresponds to the periodic summation of the inverse power function $x \in \mathbb{R} \mapsto |x|^{-s}$, which is the fundamental solution of the fractional Laplace equation on the real line. For all complex variables s and a such that $\mathrm{Re}(s) > 1$ and $a \neq 0,-1,-2,\dots$, set

$$\zeta(s,a) = \sum_{n=0}^{\infty} \frac{1}{(n+a)^s}.$$

Given $a \neq 0,-1,-2,\dots$, one can extend in a unique manner $\zeta(\cdot,a)$ into a meromorphic function on the whole complex plane with a unique pole at $s=1$, which is simple with a residue equal to 1. This function is called the Hurwitz zeta function [31].

Lemma 3.2.1 (Fundamental solution). *Let g be the solution of the fractional Laplace equation on the circle*

$$(-\Delta)^{\frac{1-s}{2}} g = c_s(\delta_0 - 1), \tag{3.31}$$

with c_s as in (3.3). Then for all $x \in \mathbb{T}\backslash\{0,-1\}$,

$$g_s(x) = \zeta(s,x) + \zeta(s,1-x) = \lim_{n\to\infty}\Big(\sum_{k=-n}^{n} \frac{1}{|k+x|^s} - \frac{2}{1-s} n^{1-s}\Big). \tag{3.32}$$

Moreover for all $p \geqslant 1$ and all $x \in \mathbb{T}$

$$\begin{aligned} g^{(p)}(x) &= (-1)^p s \dots (s+p-1)(\zeta(s+k,x) + (-1)^p \zeta(s+k,1-x)) \\ &= (-1)^p s \dots (s+p-1) \sum_{k\in\mathbb{Z}} \frac{1}{|x+k|^{s+p}}. \end{aligned} \tag{3.33}$$

Proof. Let g be the fundamental solution of (3.31). Following [217], one first derives the semi-group representation for $(-\Delta)^{-\frac{1-s}{2}}$. Let $\alpha \in (0,1)$. Recall

$$\lambda^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^{\infty} e^{-\lambda t} \frac{\mathrm{d}t}{t^{1-\alpha}}, \quad \text{for all } \lambda > 0. \tag{3.34}$$

For an integrable function on the torus on the torus and $k \in \mathbb{Z}$, we let $\hat{f}(k)$ be k-th component of the Fourier series of f, namely

$$\hat{f}(k) = \frac{1}{2\pi} \int_{\mathbb{T}} f(y) e^{-iky} \mathrm{d}y.$$

The fractional Laplacian $(-\Delta)^{-\alpha}$ on the torus is defined by the Fourier multiplier

$$\widehat{(-\Delta)^{-\alpha} f}(k) = |k|^{-2\alpha} \widehat{f}(k), k \in \mathbb{Z} \quad \text{for all } f \in \mathcal{S}(\mathbb{T}, \mathbb{R}) \text{ such that } \textstyle\int f = 0,$$

where $\mathcal{S}(\mathbb{T}, \mathbb{R})$ is the space of real-valued Schwartz functions on $\mathbb{T}$. Applying (3.34) to $\lambda = |k|^2$ then gives

$$\widehat{(-\Delta)^{-\alpha} f}(k) = \frac{1}{\Gamma(\alpha)} \int_0^{\infty} e^{-|k|^2 t} \widehat{f}(k) \frac{\mathrm{d}t}{t^{1-\alpha}}. \tag{3.35}$$

Let $(W_t)_{t \geqslant 0}$ be the heat kernel on $\mathbb{T}$, defined by its Fourier coefficients

$$\widehat{W_t}(k) = e^{-|k| t^2}, \quad \text{for all } k \in \mathbb{Z},\, t \geqslant 0.$$

The heat kernel on the circle is then given by its Fourier series or alternatively by the periodization of the heat kernel on $\mathbb{R}$ [217]:

$$W_t(x) = \frac{1}{2\pi} \sum_{k \in \mathbb{Z}} e^{-t|k|^2} e^{ikx} = \frac{1}{\sqrt{4\pi t}} \sum_{k \in \mathbb{Z}} e^{-\frac{|x-k|^2}{4t}}. \tag{3.36}$$

One may rewrite (3.35) as

$$\widehat{(-\Delta)^{-\alpha} f}(k) = \frac{1}{\Gamma(\alpha)} \int_0^{\infty} \widehat{f * W_t}(k) \frac{\mathrm{d}t}{t^{1-\alpha}}.$$

For $f \in \mathcal{S}(\mathbb{T})$ such that $\int f = 0$, $(-\Delta)^{-\alpha} f$ equals its Fourier series and one obtains

$$(-\Delta)^{-\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^{\infty} f * W_t(x) \frac{\mathrm{d}t}{t^{1-\alpha}}. \tag{3.37}$$

Moreover by (3.36),

$$\int_{\mathbb{T}} W_t(y) \mathrm{d}y = 1 = \frac{1}{\sqrt{4\pi t}} \sum_{k \in \mathbb{Z}} \int_{\mathbb{T}} e^{-\frac{|y-k|^2}{4t}} \mathrm{d}y.$$

Applying (3.37) to $\alpha = \frac{1-s}{2} \in (0,1)$ and $f = c_s(\delta_0 - 1)$ therefore gives

$$g_s(x) = \frac{c_s}{\Gamma(\frac{1-s}{2})} \int_0^{\infty} (W_t(x) - 1) \frac{\mathrm{d}t}{t^{\frac{1+s}{2}}} = \frac{c_s}{\Gamma(\frac{1-s}{2})} \frac{1}{\sqrt{4\pi}} \int_0^{\infty} \sum_{k \in \mathbb{Z}} \Big(e^{-\frac{|x-k|^2}{4t}} - \int_{\mathbb{T}} e^{-\frac{|x-k|^2}{4t}} \mathrm{d}x \Big) \frac{\mathrm{d}t}{t^{1+\frac{s}{2}}}. \tag{3.38}$$

Define the sequence of functions

$$u_k : t \in \mathbb{R}^{+*} \mapsto \frac{1}{t^{1+\frac{s}{2}}} \Big(e^{-\frac{|x-k|^2}{4t}} - \int_{\mathbb{T}} e^{-\frac{|y-k|^2}{4t}} \mathrm{d}y \Big), \quad k \in \mathbb{Z}^*.$$

First observe that when $t \geqslant 1$,

$$\sum_{k \in \mathbb{Z}^*} e^{-|k|^2 t} \leqslant e^{-\frac{t}{2}} \sum_{k \in \mathbb{Z}^*} e^{-\frac{1}{2}|k|^2 t} \leqslant C e^{-\frac{t}{2}}.$$

It follows that

$$\sum_{k\in\mathbb{Z}^*}\int_0^1 |u_k|(t)\mathrm{d}t \leqslant C\sum_{k\in\mathbb{Z}^*}\int_1^\infty \frac{1}{t^{1-\frac{s}{2}}}e^{-|k|^2 t}\mathrm{d}t \leqslant C\int_1^\infty \frac{1}{t^{1-\frac{s}{2}}}e^{-\frac{t}{2}} < \infty. \tag{3.39}$$

To treat the other part of the integral, we can write

$$\begin{aligned}u_k(t) = \frac{1}{t^{1+\frac{s}{2}}}\int_{\mathbb{T}}\Big(e^{-\frac{|x-k|^2}{4t}} - e^{-\frac{|y-k|^2}{4t}}\Big)\mathrm{d}y &= \frac{1}{t^{1+\frac{s}{2}}}e^{-\frac{|k-x|^2}{4t}}\int_{|k-y|>|k-x|}\Big(1-e^{\frac{|k-x|^2-|k-y|^2}{4t}}\Big)\mathrm{d}y\\ &+\frac{1}{t^{1+\frac{s}{2}}}\int_{|k-y|<|k-x|}e^{-\frac{|k-y|^2}{4t}}\Big(1-e^{\frac{|k-y|^2-|k-x|^2}{4t}}\Big)\mathrm{d}y.\end{aligned}$$

As a consequence, there exists a constant $C>0$ such that for each $k\in\mathbb{Z}$ and all $t\in\mathbb{R}$,

$$|u_k(t)| \leqslant \frac{Ck}{t^{2+\frac{s}{2}}}e^{-\frac{(k-1)^2}{4t}}.$$

When $u \geqslant 1$, by comparison to a Gaussian integral, one may check that

$$\sum_{k\in\mathbb{Z}^*} ke^{-\frac{|k|^2}{4u^2}} \leqslant Cu^2,$$

which leads to

$$\int_1^\infty \sum_{k\in\mathbb{Z}} |u_k(t)|\mathrm{d}t \leqslant C\int_1^\infty \frac{1}{t^{1+\frac{s}{2}}}\mathrm{d}t < \infty. \tag{3.40}$$

Combining (3.39) and (3.40), we deduce by Fubini's theorem that the order of integration and summation in (3.38) can be inverted and we find

$$\begin{aligned}g_s(x) &= \frac{c_s}{\Gamma(\frac{1-s}{2})\sqrt{4\pi}}\sum_{k\in\mathbb{Z}}\int_0^\infty u_k(t)\mathrm{d}t\\ &= \frac{c_s}{\Gamma(\frac{1-s}{2})\sqrt{4\pi}}\sum_{k\in\mathbb{Z}}\Big(\int_0^\infty e^{-\frac{|x-k|^2}{4}t}\frac{\mathrm{d}t}{t^{1+\frac{s}{2}}} - \int_0^\infty\int_{\mathbb{T}} e^{-\frac{|y-k|^2}{4}t}\frac{\mathrm{d}t}{t^{1+\frac{s}{2}}}\mathrm{d}y\Big)\\ &= \frac{\Gamma(\frac{s}{2})c_s}{\Gamma(\frac{1-s}{2})\sqrt{4\pi}}\sum_{k\in\mathbb{Z}}\Big(\frac{1}{|\frac{x-k}{2}|^s} - \int_{\mathbb{T}}\frac{\mathrm{d}y}{|\frac{y-k}{2}|^s}\Big)\\ &= \lim_{n\to\infty}\Big(\sum_{k=-n}^{n}\frac{1}{|x-k|^s} - \frac{2}{1-s}n^{1-s}\Big)\\ &= \lim_{n\to\infty}\sum_{k=0}^{n}\Big(\frac{1}{(k+1-x)^s} - \frac{1}{1-s}n^{1-s}\Big) + \lim_{n\to\infty}\sum_{k=0}^{n}\Big(\frac{1}{(k+x)^s} - \frac{1}{1-s}n^{1-s}\Big),\end{aligned}$$

where we have used the change of variables $t\in\mathbb{R}^{+*}\mapsto\frac{1}{t}$ and (3.34) in the third equality. According to the Euler-Maclaurin formula, one may rewrite each sum as follows using the fact that for all $a\in(0,1)$,

$$\lim_{n\to\infty}\Big(\sum_{k=0}^{n}\frac{1}{(k+a)^s} - \frac{1}{1-s}n^{1-s}\Big) = -\frac{1}{1-s}a^{1-s} + \frac{1}{a^s} - s\int_0^\infty \frac{t-\lfloor t\rfloor-\frac{1}{2}}{(t+a)^{1+s}}\mathrm{d}t. \tag{3.41}$$

If s is complex-valued with $\sigma := \mathrm{Re}(s) > 1$, then the left-hand side of the last display is given $\zeta(s,a)$. Using an analytic continuation argument, it is argued in [31, Eq. (5.2)] that for $\sigma > -1$ and $a\in(0,1)$, $\zeta(s,a)$ coincides with the right-hand side of (3.41). As a consequence we find that for all $x\in\mathbb{T}$ and $s\in(0,1)$

$$g_s(x) = \zeta(s,1-x) + \zeta(s,x).$$

The identity (3.33) is then standard. □

3.2.2 Inverse Riesz transform

In Section 3.5, when implementing loop equations techniques, we will consider the solution ψ of a convolution equation in the form $g * \psi' = \xi - \int \xi$ with $\int \psi = 0$, where $\xi : \mathbb{T} \to \mathbb{R}$ is a smooth test-function. The map ψ' is given by the fractional Laplacian of order $\frac{1-s}{2}$ of ξ. Proceeding as in the proof of Lemma 3.2.1, one can easily derive a pointwise formula for ψ when ξ is smooth enough. We add some other useful simple formulas.

Lemma 3.2.2 (Inversion of the Riesz transform). *Let $\xi \in \mathcal{C}^{-s+\varepsilon}(\mathbb{T}, \mathbb{R})$ for some $\varepsilon > 0$. Let $\psi \in \mathcal{C}^{\varepsilon}(\mathbb{T}, \mathbb{R})$ given by*

$$\psi' = -\frac{1}{2c_s}(-\Delta)^{\frac{1-s}{2}}\xi \quad \text{and} \quad \int \psi = 0.$$

Then

$$\psi(x) = \frac{1}{\pi \tan(\frac{\pi}{2}s)} \int \sum_{k\in\mathbb{Z}} \frac{\xi(y) - \int \xi}{|x-y-k|^{1-s}} \mathrm{sgn}(x-y-k)\mathrm{d}y, \quad \text{for all } x \in \mathbb{T} \tag{3.42}$$

$$\frac{1}{2c_s}|\xi|^2_{H^{\frac{1-s}{2}}} = \iint g_s''(x-y)(\psi(x)-\psi(y))^2 \mathrm{d}x\mathrm{d}y - 2\int \xi'(x)\psi(x)\mathrm{d}x = -\int \xi'(x)\psi(x)\mathrm{d}x. \tag{3.43}$$

Assume that ξ is supported on $(-\frac{1}{2}, \frac{1}{2})$. Let $\ell_N \in (0, 1]$. Let $\xi_0 : \mathbb{R} \to \mathbb{R}$ such that

$$\xi_0(x) = \begin{cases} \xi(x) & \text{if } |x| \leqslant \frac{1}{2} \\ 0 & \text{if } |x| > \frac{1}{2}. \end{cases} \tag{3.44}$$

Then

$$|\xi(\ell_N^{-1}\cdot)|_{H^{\frac{1-s}{2}}} = \ell_N^s |\xi_0|^2_{H^{\frac{1-s}{2}}} + O(\ell_N^2 |\xi_0|^2_{L^2}). \tag{3.45}$$

Proof. Let $\xi \in \mathcal{C}^{-s+\varepsilon}(\mathbb{T}, \mathbb{R})$ for some $\varepsilon > 0$ and let ψ such that $\psi' = -\frac{1}{2c_s}(-\Delta)^{\frac{1-s}{2}}\xi$ with $\int \psi = 0$. If $\xi \in \mathcal{C}^{1-s+\varepsilon}(\mathbb{T}, \mathbb{R})$ it is well-known, see for instance [246, Th. 2], that for all $x \in \mathbb{T}$,

$$\psi'(x) = -\frac{c_s'}{2c_s} \int \sum_{k\in\mathbb{Z}} \frac{\xi(y) - \xi(x)}{|x-y-k|^{2-s}} \mathrm{d}y, \quad \text{where} \quad c_s' = \frac{2^{1-s}\Gamma(1-\frac{s}{2})}{|\Gamma(-\frac{1-s}{2})|\pi^{\frac{1}{2}}}. \tag{3.46}$$

For completeness, let us sketch the main arguments. Let $\alpha \in (0, 1)$. We have

$$\lambda^\alpha = \frac{1}{\Gamma(-\alpha)} \int_0^\infty (e^{-t\lambda} - 1) \frac{\mathrm{d}t}{t^{1+\alpha}}, \quad \text{for all } \lambda > 0.$$

Arguing as in the proof of Lemma 3.2.1, we find that for any $f \in L^2(\mathbb{T})$ and $k \in \mathbb{Z}$,

$$\widehat{(-\Delta)^\alpha f}(k) = |\xi|^{2\alpha} \hat{f}(k) = \frac{1}{\Gamma(-\alpha)} \int_0^\infty (e^{-|k|^2 t}\hat{f}(k) - 1) \frac{\mathrm{d}t}{t^{1+\alpha}} \mathrm{d}t,$$

which gives by taking the inverse Fourier transform

$$(-\Delta)^\alpha f(x) = \frac{1}{\Gamma(-\alpha)} \int_0^\infty (f * W_t(x) - f(x)) \frac{\mathrm{d}t}{t^{1+\alpha}} \mathrm{d}t, \tag{3.47}$$

where W_t is as in (3.36). If $f \in \mathcal{C}^{1-s+\varepsilon}(\mathbb{T},\mathbb{R})$, then, as shown in [217], one can invert the order of integration. We then compute

$$\int_0^\infty W_t(x)\frac{\mathrm{d}t}{t^{1+\alpha}} = \frac{1}{\sqrt{4\pi}}\sum_{k\in\mathbb{Z}}\int_0^\infty \frac{1}{t^{\frac{3}{2}+\alpha}}e^{-\frac{|x-k|^2}{4t}}\mathrm{d}t = \frac{1}{\sqrt{4\pi}}\sum_{k\in\mathbb{Z}}\int_0^\infty \frac{1}{t^{\frac{1}{2}-\alpha}}e^{-\frac{|x-k|^2}{4}t}\mathrm{d}t$$
$$= \frac{\Gamma(\frac{1}{2}+\alpha)}{\sqrt{4\pi}}\sum_{k\in\mathbb{Z}}\frac{1}{(\frac{|x-k|}{2})^{1+2\alpha}}.$$

Inserting this into (3.47) gives the representation

$$(-\Delta)^{\frac{1-s}{2}}f(x) = \frac{2^{1-s}\Gamma(1-\frac{s}{2})}{|\Gamma(-\frac{1-s}{2})\pi^{\frac{1}{2}}}\sum_{k\in\mathbb{Z}}\int_{\mathbb{T}}\frac{f(y)-f(x)}{|x-y-k|^{2-s}}\mathrm{d}y,$$

which is accordance with (3.46). We also compute

$$\frac{c_s'}{c_s} = \frac{1}{\pi}\frac{\Gamma(1-\frac{s}{2})\Gamma(\frac{s}{2})}{|\Gamma(-\frac{1-s}{2})|\Gamma(\frac{1-s}{2})} = \frac{c_s'}{c_s} = \frac{1-s}{\frac{\pi}{2}\tan(\frac{\pi}{2}s)}.$$

Integrating the above formula with the condition that $\int\psi = 0$ yields

$$\psi(x) = \frac{1}{\pi\tan(\frac{\pi}{2}s)}\int\sum_{k\in\mathbb{Z}}\frac{\xi(y)-\int\xi}{|x-y-k|^{1-s}}\mathrm{sgn}(x-y)\mathrm{d}y, \tag{3.48}$$

Therefore (3.42) holds when $\xi \in \mathcal{C}^{1-s+\varepsilon}(\mathbb{T},\mathbb{R})$, for some $\varepsilon > 0$. By density, we conclude that (3.42) holds as soon as $\xi \in \mathcal{C}^{-s+\varepsilon}$ for some $\varepsilon > 0$. Equation (3.43) follows by integration by parts.

Assume that ξ is supported on $(-\frac{1}{2},\frac{1}{2})$. Let $\xi_0 : \mathbb{R} \to \mathbb{R}$ such that

$$\xi_0(x) = \begin{cases} \xi(x) & \text{if } |x| \leqslant \frac{1}{2} \\ 0 & \text{if } |x| > \frac{1}{2}. \end{cases}$$

We have

$$\begin{aligned}
|\xi(\ell_N^{-1}\cdot)|^2_{H^{\frac{1-s}{2}}} &= \int_{\mathbb{T}}\mathbb{1}_{|x|<\frac{1}{2}\ell_N}\xi(\ell_N^{-1}x)\sum_{k\in\mathbb{Z}}\int_{\mathbb{T}}\frac{\xi(\ell_N^{-1}x)-\xi(\ell_N^{-1}y)}{|x-y+k|^{2-s}}\mathbb{1}_{|y|<\frac{1}{2}\ell_N}\mathrm{d}y\mathrm{d}x \\
&= \int_{\mathbb{R}}\xi_0(\ell_N^{-1}x)\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}}\frac{\xi_0(\ell_N^{-1}x)-\xi_0(\ell_N^{-1}y)}{|x-y+k|^{2-s}}\mathrm{d}y\mathrm{d}x \\
&= \ell_N^s|\xi_0|^2_{H^{\frac{1-s}{2}}} + \sum_{k\in\mathbb{Z}*}\int_{\mathbb{R}}\xi_0(\ell_N^{-1}x)\frac{\xi_0(\ell_N^{-1}x)-\xi_0(\ell_N^{-1}y)}{|x-y+k|^{2-s}}\mathrm{d}y\mathrm{d}x \\
&= \ell_N^s|\xi_0|^2_{H^{\frac{1-s}{2}}} + O\Big(\ell_N^2\int|\xi_0(x)||\xi_0(x)-\xi_0(y)|\mathrm{d}x\mathrm{d}y\Big) \\
&= \ell_N^s|\xi_0|^2_{H^{\frac{1-s}{2}}} + O(\ell_N^2|\xi|^2_{L^2}).
\end{aligned}$$

□

Next, we apply the pointwise formula (3.42) to indicator and inverse power functions.

Lemma 3.2.3 (Explicit formulas). *Let $\zeta(s,a)$ be the Hurwitz zeta function. Let $\xi = \mathbb{1}_{(-a,a)} - 2a$ for some $0 < a < \frac{1}{2}$ and ψ be given by (3.42). We have*

$$\psi(x) = -\frac{\operatorname{cotan}(\frac{\pi}{2}s)}{\pi s}(\zeta(-s, x+a) - \zeta(-s, x-a)), \quad \forall x \in \mathbb{T}.$$

Let $\alpha \in (0,s)$, $\xi = \zeta(\alpha,\cdot) + \zeta(\alpha, 1-\cdot)$ and ψ be given by (3.42). Then with $c_s, c_\alpha, c_{c-\alpha}$ as in (3.3), we have

$$\psi(x) = \frac{c_\alpha}{2c_{s-\alpha}c_s}(\zeta(s-\alpha, x) + \zeta(s-\alpha, 1-x)), \quad \forall x \in \mathbb{T}.$$

Lemma 3.2.4 (Decay at infinity). *Let $\xi \in \mathcal{C}^{-s+\varepsilon}(\mathbb{T},\mathbb{R})$ such that $\int \xi = 0$. Assume that ξ is supported on $(-a,a)$ for some $0 < a < \frac{1}{4}$. Then $\psi := (-\Delta)^{\frac{1-s}{2}}\xi$ is $\mathcal{C}^\infty$ on $\mathbb{T}\backslash[-a,a]$ and for each $k \geqslant 1$, there exists a constant $C_k > 0$ such that for all $x \in \mathbb{T}\backslash[-2a,2a]$,*

$$|\psi^{(k)}|(x) \leqslant C_k |\xi|_{L^1} \frac{1}{|x|^{1-s+k}}. \tag{3.49}$$

Proof. Since $\xi \in L^1(\mathbb{T},\mathbb{R})$, one may differentiate (3.42) under the integral sign. Using the fact that $\int(\xi - \int \xi) = 0$, we obtain (3.49). □

In Section 3.5 we will consider test-functions ξ with poor regularity, that one should regularize at a small scale $\ell > 0$ to obtain a transport with bounded second derivative. For any $\ell > 0$, define the smoothing kernel

$$K_\ell : x \mapsto \frac{1}{\ell}\begin{cases} \frac{1}{\ell}(x+\ell) & \text{if } x \in (-\ell, 0) \\ -\frac{1}{\ell}(x-\ell) & \text{if } x \in (0,\ell) \\ 0 & \text{if } |x| \geqslant \ell. \end{cases} \tag{3.50}$$

Lemma 3.2.5 (Regularization). *Let $\xi \in \mathcal{C}^{-s+\varepsilon}(\mathbb{T},\mathbb{R})$, piecewise $\mathcal{C}^{2-s}(\mathbb{T},\mathbb{R})$. Fix $y_0 \in \operatorname{supp}(\xi)$. Let $\psi \in \mathcal{C}^{\varepsilon}(\mathbb{T},\mathbb{R})$ be given by $\psi' = (-\Delta)^{\frac{1-s}{2}}\xi$ with $\int \psi = 0$. Assume that ξ is supported on $(y_0 - a, y_0 + a)$ for some $0 \leqslant a \leqslant \frac{1}{2}$. Assume that there exist $\alpha_i \geqslant 0$, $i = 1, \ldots, l$ and $\chi_0 > 0$ such that*

$$|\psi''|(x) \leqslant \chi_0 \Big(\sum_{l=1}^{p} \frac{1}{|x-a_l|^{1+\alpha_l}} \mathbb{1}_{|x-y_0|<2a} + \frac{1}{(a+|x-y_0|)^{3-s}} \Big), \quad \forall x \in \mathbb{T} \tag{3.51}$$

Let $\ell \in (0,1)$ and $\psi_\ell = \psi * K_\ell$ with K_ℓ given by (3.50). There exists a constant $C > 0$ such that

$$|\psi_\ell''|(x) \leqslant C\chi_0 \Big(\sum_{l=1}^{p} \frac{1}{(|x-a_l| \vee \ell)^{1+\alpha_l}} \mathbb{1}_{|x-y_0|<2a} + \frac{1}{(a+|x-y_0|)^{3-s}} \Big), \quad \forall x \in \mathbb{T}. \tag{3.52}$$

3.3 The Helffer-Sjöstrand equation

In this section we introduce some results on the solutions of Helffer-Sjöstrand equations. We first state existence and uniqueness results valid for energies with convex pairwise interactions and derive a known comparison principle that one adapts to the circular setting. We then study an important change of coordinates which leads to the study of H.-S. equations on affine hyperplanes of $\mathbb{R}^N$. Finally we give a maximum principle on the solution and recall some standard results for log-concave probability measures.

3.3.1 Well-posedness and first properties

In this subsection we introduce the H.-S. equation and state some standard existence and uniqueness results. We follow partly the presentation of [10]. Let μ be a probability measure on D_N in the form

$$\mathrm{d}\mu = e^{-H(X_N)}\mathbb{1}_{D_N}(X_N)\mathrm{d}X_N,$$

with $H : D_N \to \mathbb{R}$ measurable. We make the following assumptions on H:

Assumptions 3.3.1. *Assume $H : D_N \to \mathbb{R}$ is in the form*

$$H : X_N \mapsto \sum_{i \neq j} \chi(|x_i - x_j|),$$

with $\chi : \mathbb{R}^{+*} \to \mathbb{R}$ satisfying

$$\chi \in \mathcal{C}^{\infty}(\mathbb{R}^{+*}, \mathbb{R}), \quad \chi'' \geqslant c > 0.$$

Let $F : D_N \to \mathbb{R}$ be a smooth enough function. We seek to express the variance of F under μ in terms of a partial differential equation. Let us recall the integration by parts formula for μ. Define on $\mathcal{C}_c^{\infty}(D_N)$ the Langevin operator

$$\mathcal{L}^{\mu} = \beta \nabla \mathcal{H}_N \cdot \nabla - \Delta,$$

with ∇ and Δ the gradient and Laplace operators of $\mathbb{T}^N$. Recall that the operator $\mathcal{L}$ is the generator of a Langevin dynamics for which μ is the invariant measure. By integration by parts under μ, for any functions $\phi, \psi \in \mathcal{C}^{\infty}(D_N)$ such that $\nabla\phi \cdot \vec{n} = 0$ a.e on ∂D_N, we have

$$\mathbb{E}_{\mu}[\psi \mathcal{L}^{\mu}\phi] = \mathbb{E}_{\mu}[\nabla\psi \cdot \nabla\phi]. \tag{3.53}$$

Moreover, let us emphasize that whenever $\lim_{x \to 0} \chi(x) = \infty$, the Neumann boundary condition $\nabla\phi \cdot \vec{n} = 0$ is not necessary. Let us now assume that the Poisson equation

$$\begin{cases} \mathcal{L}^{\mu}\phi = F - \mathbb{E}_{\mu}[F] & \text{on } D_N \\ \nabla\phi \cdot \vec{n} = 0 & \text{on } \partial D_N \end{cases} \tag{3.54}$$

admits a solution in a weak sense. Then, by (3.53), one may rewrite the variance of F as

$$\mathrm{Var}_{\mu}[F] = \mathbb{E}_{\mu}[\nabla F \cdot \nabla\phi].$$

The above formula is called the Helffer-Sjöstrand representation formula. Let us formally differentiate the equation (3.54). Formally we have

$$\nabla \mathcal{L}^{\mu}\phi = A_1^{\mu}\nabla\phi,$$

with A_1 the so-called Helffer-Sjöstrand operator

$$A_1^{\mu} = \nabla^2 H + \mathcal{L}^{\mu} \otimes I_N.$$

Therefore the solution ϕ of $\mathcal{L}^{\mu}\phi = F - \mathbb{E}_{\mu}[F]$ formally satisfies

$$\begin{cases} A_1^{\mu}\nabla\phi = \nabla F & \text{on } D_N \\ \nabla\phi \cdot \vec{n} = 0 & \text{on } \partial D_N. \end{cases} \tag{3.55}$$

This PDE is called the Helffer-Sjöstrand equation. To make the above statements rigorous one should work on the appropriate functional spaces. Let us define the norm

$$\|F\|_{H^1(\mu)} = \mathbb{E}_\mu[F^2]^{\frac{1}{2}} + \mathbb{E}_\mu[|\nabla F|^2]^{\frac{1}{2}}.$$

Define $H^1(\mu)$ the completion of $\mathcal{C}^\infty(D_N)$ with respect to the norm $\|\cdot\|_{H^1(\mu)}$. Also define the norm

$$\|F\|_{H^{-1}(\mu)} = \sup\{|\mathbb{E}_\mu[FG]| : G \in H^1(\mu), \|G\|_{H^1(\mu)} \leqslant 1\}.$$

We let $H^{-1}(\mu)$ be the dual of $H^1(\mu)$, defined as the completion of $\mathcal{C}^\infty(D_N)$ with respect to the norm $\|\cdot\|_{H^{-1}(\mu)}$. We are interested in the well-posedness of (3.55) when ∇F is replaced by any vector-field (possibly non-gradient) $v \in L^2(\{1,\dots,N\}, H^{-1}(\mu))$.

Since the density of μ with respect to the Lebesgue measure on D_N is not bounded from below, the existence of a solution to the Helffer-Sjöstrand equation (3.55) is not straightforward. To circumvent this difficulty we prove that (3.55) is well-posed when F is a function of the gaps. Define the map

$$\Pi : X_N \in D_N \mapsto (x_2 - x_1, x_3 - x_1, \dots, x_N - x_1) \in \mathbb{R}^{N-1} \tag{3.56}$$

and the push-forward of μ by the map Π

$$\mu' = \mu \circ \Pi^{-1}. \tag{3.57}$$

Proposition 3.3.1 (Existence and representation). *Let μ satisfying Assumptions 3.3.1. Assume that F is in the form $F = G \circ \Pi$, $G \in H^1(\mu')$ or that the kernel χ is bounded. Then there exists a unique $\nabla\phi \in L^2(\{1,\dots,N\}, H^1(\mu))$ such that*

$$\begin{cases} A_1^\mu \nabla\phi = \nabla F & \text{in } D_N \\ \nabla\phi \cdot \vec{n} = 0 & \text{on } \partial D_N, \end{cases} \tag{3.58}$$

with the first identity being, for each coordinate, an identity on elements of $H^{-1}(\mu)$. Moreover the solution of (3.58) is the unique minimizer of the functional

$$\psi \mapsto \mathbb{E}_\mu[\psi \cdot \nabla^2 H\psi + |D\psi|^2 - 2\nabla F \cdot \psi], \tag{3.59}$$

on maps $\psi \in L^2(\{1,\dots,N\}, H^1(\mu))$ such that $\nabla\phi \cdot \vec{n} = 0$ on ∂D_N. The variance of F may be represented as

$$\mathrm{Var}_\mu[F] = \mathbb{E}_\mu[\nabla\phi \cdot \nabla F] \tag{3.60}$$

and the covariance between F any function $G \in H^1(\mu)$ as

$$\mathrm{Cov}_\mu[F, G] = \mathbb{E}_\mu[\nabla\phi \cdot \nabla G].$$

The identity (3.60) is called the Helffer-Sjöstrand formula. The proof of Proposition 3.3.1 is postponed to the Appendix (see Section 3.7.1).

Remark 7 (Remark on the boundary condition). *Let $\phi \in H^1(\mu)$. The map $\nabla\phi$ satisfies the boundary condition $\nabla\phi \cdot \vec{n} = 0$ on ∂D_N if and only if $\partial_i\phi = \partial_j\phi$ whenever $i = j$, for each $i, j \in \{1,\dots,N\}$.*

Remark 8 (Link to the Monge-Ampère equation). *We formally discuss the link between (3.58) and the Monge-Ampère equation. Let $F : D_N \to \mathbb{R}$ be a smooth enough test-function. For all $t \geqslant 0$,* *consider the measure* $\mathrm{d}\mu_t = \frac{1}{\mathbb{E}_\mu[e^{tF}]} e^{tF} \mathrm{d}\mu$. *The measure* μ_t *can be written* $\mu_t = \mu \circ \Phi_t^{-1}$ *with* $\Phi_t : D_N \to D_N$ *such that* $\int \Phi_t = 1$ *solution of the Monge-Ampère equation*

$$-\log \det D\Phi_t + H \circ \Phi_t - H = tF - \log \mathbb{E}_\mu[e^{tF}].$$

Formally, since $\mu_t = \mu + t\nu + o(t)$, *one expects that* $\Phi_t = \mathrm{Id} + t\phi + o(t)$. *Linearizing the above equation in* t *formally gives*

$$\mathcal{L}^\mu \phi = F - \mathbb{E}_\mu[F],$$

which is the Poisson equation (3.54).

When ∇F is replaced by a non-gradient vector-field v, the solutions ψ of the equation $A_1\psi = v$ with a Neumann boundary condition are in general non-unique. In order to have a well-posed equation one assumes additionally that $\sum_{i=1}^N v_i = 0$ and that each coordinate v_i is a function of the gaps.

Proposition 3.3.2 (Well-posedness for non-gradient vector-fields). *Let* μ *satisfying Assumptions 3.3.1. Let* $v \in L^2(\{1,\ldots,N\}, H^{-1}(\mu))$ *such that* $v\cdot(e_1+\ldots+e_N) = 0$ *and for each* $i \in \{1,\ldots,N\}$, $v_i = w_i \circ \Pi$ *for some* $w_i \in H^{-1}(\mu')$. *There exists a unique* $\psi \in L^2(\{1,\ldots,N\}, H^1(\mu))$ *such that*

$$\begin{cases} A_1^\mu \psi = v & \text{on } D_N \\ \psi \cdot (e_1 + \ldots + e_N) = 0 & \text{on } D_N \\ \psi \cdot \vec{n} = 0 & \text{on } \partial D_N. \end{cases} \tag{3.61}$$

Moreover the solution of (3.61) is also the unique minimizer of

$$\psi \mapsto \mathbb{E}_\mu[\psi \cdot \nabla^2 H \psi + |D\psi|^2 - 2v \cdot \psi],$$

on maps $\psi \in L^2(\{1,\ldots,N\}, H^1(\mu))$ *such that* $\psi \cdot \vec{n} = 0$ *on* ∂D_N.

We postpone the proof of Lemma 3.3.2 to the Appendix (see Section 3.7.1). When v satisfies the assumptions of Lemma 3.3.2 we can denote non-ambiguously $\psi = (A_1^\mu)^{-1} v$ the solution of (3.61).

Lemma 3.3.3. *Let* μ *satisfying Assumptions 3.3.1. Let* $v, w \in L^2(\{1,\ldots,N\}, H^{-1}(\mu))$ *satisfying the assumptions of Proposition 3.3.2. We have*

$$\mathbb{E}_\mu[(v+w)\cdot(A_1^\mu)^{-1}(w+w)] \leqslant 2\Big(\mathbb{E}_\mu[v\cdot(A_1^\mu)^{-1}v] + \mathbb{E}_\mu[w\cdot(A_1^\mu)^{-1}w]\Big). \tag{3.62}$$

Let $F = G \circ \Pi$ *with* $\nabla G \in L^2(\{1,\ldots,N\}, H^{-1}(\mu'))$. *Then the solution of (3.61) is the solution* $\nabla\phi$ *of (3.58).*

Proof. Since $v - w$ satisfies $(e_1 + \ldots + e_N)\cdot(v-w) = 0$, one can define $(A_1^\mu)^{-1}(v-w)$. Moreover note that by integration by parts

$$\mathbb{E}_\mu[(v-w)\cdot(A_1^\mu)^{-1}(v-w)] \geqslant 0.$$

By linearity this implies (3.62).

Let us prove the second part of the statement. Let $F = G\circ\Pi$ with $\nabla G \in L^2(\{1,\ldots,N\}, H^{-1}(\mu'))$. Let $\nabla\phi$ be the solution of (3.58) with F function of the gaps. Observe that $\sum_{i=1}^N \partial_i F = 0$. Taking the scalar product of (3.58) with $(e_1 + \ldots + e_N)$ yields

$$(e_1 + \ldots + e_N)\cdot \nabla^2 H\psi + \mathcal{L}^\mu(\psi\cdot(e_1+\ldots+e_N)) = 0.$$

By symmetry, $(e_1 + \ldots + e_N) \cdot \nabla^2 H \psi = 0$ and therefore

$$\mathcal{L}^\mu(\psi \cdot (e_1 + \ldots + e_N)) = 0.$$

Since $\nabla(\psi \cdot (e_1 + \ldots + e_N)) \cdot \vec{n} = 0$ on ∂D_N, this implies that

$$\nabla \psi \cdot (e_1 + \ldots + e_N) = 0.$$

□

3.3.2 Monotonicity and its consequences

In this subsection, we give some monotonicity results related to the FKG inequality.

Let μ be a probability measure on $\mathbb{R}^N$ in the form

$$\mathrm{d}\mu = e^{-H(X_N)} \mathrm{d}X_N$$

with $H : D_N \to \mathbb{R}$ smooth verifying $\partial_{ij} H \leqslant 0$ for each $i \neq j$. The FKG inequality [15] then states that the covariance between two increasing functions (i.e increasing in all their coordinates) is non-negative. This can be reformulated by saying that for all increasing function $F : D_N \to \mathbb{R}$ smooth enough, the solution ϕ of (3.54) is also increasing. In our case, we have $\partial_{ij}\mathcal{H}_N \leqslant 0$ on D_N for each $i \neq j$ and since the Langevin dynamics is conservative in D_N (i.e the process does not hit the boundary a.s), the FKG inequality holds true.

In the next proposition we show that a maximum principle for solutions of non-gradient Helffer-Sjöstrand equations (3.61) holds, at the condition of fixing an origin. Let

$$D_{N-1}^x = \{(x_1, \ldots, x_{N-1} \in \mathbb{T}^{N-1} : x_1 - x \leqslant x_2 - x \ldots x_{N-1} - x\}.$$

Denote μ^x the law of $(x_2, \ldots, x_N)$ conditionally on $x_1 = x$ when $(x_1, \ldots, x_N)$ is distributed according to μ. Let $H^x : y \in D_{N-1}^x \mapsto H(x, y)$. Let $\mathcal{L}^x = \mathcal{L}^{\mu^x}$ acting on $H^1(\mu^x)$ and $A_1^x = A_1^{\mu_x}$ acting on $L^2(\{1, \ldots, N-1\}, H^1(\mu^x))$.

Proposition 3.3.4 (Existence with a fixed point). *Let μ satisfying Assumptions 3.3.1. Let $x \in \mathbb{T}$. Let $v \in L^2(\{1, \ldots, N-1\}, H^{-1}(\mu^x))$. There exists a unique $\psi \in L^2(\{1, \ldots, N-1\}, H^1(\mu^x))$ solution of*

$$\begin{cases} A_1^x \psi = v & \text{on } D_N^x \\ \psi_1 = 0 & \text{on } D_N^x \\ \psi \cdot \vec{n} = 0 & \text{on } \partial D_N^x. \end{cases} \tag{3.63}$$

If $F \in H^1(\mu^x)$, then the solution of (3.63) is in the form $\psi = \nabla \phi \in L^2(\{1, \ldots, N-1\}, H^1(\mu^x))$. Moreover the variance of F under μ^x may be represented as

$$\mathrm{Var}_{\mu^x}[F] = \mathbb{E}_{\mu^x}[\nabla \phi \cdot \nabla F].$$

In the sequel given $v \in L^2(\{1, \ldots, N-1\}, H^{-1}(\mu^x))$, we denote $A_1^x v$ the solution of (3.63). The proof of Proposition 3.3.4 is entirely similar to the proof of Proposition 3.3.2. We can now state the maximum principle for (3.63), derived for instance in [72, 149].

Lemma 3.3.5 (Monotonicity). *Let μ satisfying Assumptions 3.3.1. Assume additionally that $\lim_{x \to 0} \chi(x) = +\infty$. Let $v \in L^2(\{1, \ldots, N-1\}, H^{-1}(\mu^x))$. Let $\psi \in L^2(\{1, \ldots, N-1\}, H^1(\mu^x))$ be the solution of (3.63). Assume that*

$$v_i \geqslant 0 \quad \text{a.e on } D_N^x, \quad \text{for each } i \in \{1, \ldots, N-1\}.$$

Then

$$\psi_i \geqslant 0 \quad \text{a.e on } D_N^x, \quad \text{for each } i \in \{1, \ldots, N-1\}.$$

Proof. Let $\psi \in L^2(\{1,\ldots,N-1\}, H^1(\mu^x))$ be the solution of (3.63). Let us prove that for each $i \in \{1,\ldots,N-1\}$, $\psi_i \geqslant 0$ a.e on D_N^x. Let ψ^+ and ψ^- be the positive and negative parts of ψ. Taking the scalar product of the equation $A_1^x\psi = v$ with ψ^- gives

$$\psi^- \cdot \nabla^2 H^x \psi + \psi^- \cdot \mathcal{L}^x \psi = \psi^- \cdot v \geqslant 0.$$

By integration by parts under μ^x, one can observe that

$$\mathbb{E}_{\mu^x}\Big[\psi^- \cdot \nabla^2 H^x \psi + \sum_{i=1}^N \nabla \psi_i^- \cdot \nabla \psi_i\Big] \geqslant 0.$$

Indeed since $\lim_{x\to 0}\chi(x) = +\infty$, the boundary term in the above integration by parts vanishes. Note that $\nabla\psi_i^- \cdot \nabla\psi_i = -|\nabla\psi_i^-|^2$ and

$$\psi^- \cdot \nabla^2 H^x \psi^+ = \sum_{i\neq j} \chi''(x_j - x_i)(\psi_i^-\psi_i^+ - \psi_i^-\psi_j^+) = -\sum_{i\neq j}\chi''(x_j-x_i)\psi_i^-\psi_j^+ \leqslant 0. \tag{3.64}$$

One deduces that

$$\mathbb{E}_{\mu^x}[\psi^- \cdot \nabla^2 H^x \psi^- + |D\psi^-|^2] = 0.$$

This implies that $\psi^- = 0$ and concludes the proof. Let us emphasize that we have crucially used the assumption that the interaction blows up when two particles collide, which somehow puts the boundary of the domain at infinity. $\square$

As a crucial consequence of Lemma 3.3.5, one can compare the variances of two functions under μ^x by comparison of their gradients. We derive the following new observation:

Lemma 3.3.6 (Energy comparison). *Let μ satisfying Assumptions 3.3.1 and assume $\lim_{x\to 0}\chi(x) = +\infty$. Let $x \in \mathbb{T}$ and $v, w \in L^2(\{1,\ldots,N-1\}, H^{-1}(\mu^x))$. Assume that for each $i \in \{1,\ldots,N-1\}$,*

$$|v_i| \leqslant w_i, \quad \text{a.e on } D_N^x. \tag{3.65}$$

Then

$$\mathbb{E}_{\mu^x}[v \cdot (A_1^x)^{-1} v] \leqslant \mathbb{E}_{\mu^x}[w \cdot (A_1^x)^{-1} w].$$

In particular if $F, G \in H^1(\mu)$ satisfy for each $i \in \{1,\ldots,N-1\}$,

$$|\partial_i F| \leqslant \partial_i G, \quad \text{a.e on } D_N^x,$$

then

$$\mathrm{Var}_{\mu^x}[F] \leqslant \mathrm{Var}_{\mu^x}[G].$$

Proof. For $x = (x_1,\ldots,x_N) \in \mathbb{R}^N$, we use the notation $x \geqslant 0$ whenever for each $i \in \{1,\ldots,N-1\}$, $x_i \geqslant 0$. Let $v, w \in L^2(\{1,\ldots,N\}, H^{-1}(\mu^x))$ as in the statement of Lemma 3.3.6. Let v^+ and v^- be the positive and negative parts of v. Using the fact that A_1^x is self adjoint on $L^2(\{1,\ldots,N\}, H^1(\mu^x))$, one finds that

$$\mathbb{E}_{\mu^x}[w \cdot (A_1^x)^{-1} w] - \mathbb{E}_{\mu^x}[v \cdot (A_1^x)^{-1} v] = \mathbb{E}_{\mu^x}[(v+w) \cdot (A_1^x)^{-1}(w-v)].$$

Note that since $w - v \geqslant 0$, by Lemma 3.3.5, $(A_1^x)^{-1}(w-v) \geqslant 0$ and that $w + v \geqslant 0$, one gets

$$\mathbb{E}_{\mu^x}[(v+w) \cdot (A_1^x)^{-1}(w-v)] \geqslant 0,$$

which gives the desired result. The second part of statement is straightforward. $\square$

Lemma 3.3.5 and Lemma 3.3.6 allow a comparisons between general vector-field $\psi \in L^2(\{1,\ldots,N\}, H^1(\mu))$. However if one restricts the comparison to gradients, much less is required on the measure μ, as shown in the following:

Lemma 3.3.7. *Let μ be a probability measure on D_N in the form $\mathrm{d}\mu = e^{-H}\mathrm{d}X_N$ with $H : D_N \to \mathbb{R}$ in $\mathcal{C}^2$ such that the dynamics is conservative and*

$$\partial_{ij} H \leqslant 0 \quad \text{for each } i \neq j.$$

Let $F, G \in H^1(\mu)$ such that for each $i \in \{1,\ldots,N\}$,

$$|\partial_i F| \leqslant \partial_i G. \tag{3.66}$$

Then

$$\mathrm{Var}_\mu[F] \leqslant \mathrm{Var}_\mu[G]. \tag{3.67}$$

Proof. It is standard that μ satisfies the FKG inequality meaning that for all measurable non-decreasing functions f and g, the covariance between f and g under μ is non-negative. We refer to [17, Th. 1.3] in the $\mathbb{R}^N$ case.

Let $F, G \in H^1(\mu)$ be as in (3.66). One may write

$$\mathrm{Var}_\mu[G] = \mathrm{Var}_\mu[F] + \mathrm{Cov}_\mu[G+F, G-F]$$

Since $G - F$ and $F + G$ are non-decreasing, their covariance is non-negative, concluding the proof of (3.67). $\square$

3.3.3 Variances upper bounds

We recall some well-known consequences of convexity regarding variances.

Lemma 3.3.8 (Brascamp-Lieb inequality)**.** *Let μ satisfying Assumptions 3.3.1. Let $\mathcal{A} \subset D_N$ be a convex domain with a piecewise smooth boundary. Let $F : D_N \to \mathbb{R}$ in the form $F = G \circ \Pi$ with $\nabla G \in L^2(\{1,\ldots,N-1\}, H^{-1}(\mu'))$. There holds*

$$\mathrm{Var}_\mu[F \mid \mathcal{A}] \leqslant -\mathbb{E}_\mu\Big[\min_{U_N \in \mathbb{R}^N} \Big(U_N \cdot \nabla^2 H U_N - 2\nabla F \cdot U_N\Big) \mid \mathcal{A}\Big] = \mathbb{E}_\mu[\nabla F \cdot (\nabla^2 H)^{-1} \nabla F \mid \mathcal{A}]. \tag{3.68}$$

Elements of proof. Let us illustrate the proof in the case $\mathcal{A} = D_N$. By Proposition 3.3.1, the variance of F may be expressed as

$$\mathrm{Var}_\mu[F] = -\min_{\phi \in H^1(\mu)} \mathbb{E}_\mu[\nabla\phi \cdot \nabla^2 H \nabla\phi + |\nabla^2\phi|^2 - 2\nabla F \cdot \nabla\phi].$$

Since $\mathbb{E}_\mu[|\nabla^2\phi|^2] \geqslant 0$, one gets

$$\begin{aligned}
\mathrm{Var}_\mu[F] &\leqslant -\min_{\phi \in H^1(\mu)} \mathbb{E}_\mu[\nabla\phi \cdot \nabla^2 H \nabla\phi - 2\nabla F \cdot \nabla\phi] \\
&= -\mathbb{E}_\mu\Big[\min_{U_N \in \mathbb{R}^N} U_N \cdot \nabla^2 H U_N - 2\nabla H \cdot U_N\Big] \\
&= \mathbb{E}_\mu[\nabla F \cdot (\nabla^2 H)^{-1} \nabla F].
\end{aligned}$$

$\square$

The Brascamp-Lieb inequality requires some regularity on the function F. We now give a simple concentration property for linear statistics, which depends only on the L^2 norm of the test-function. We obtain the following sub-poissonian estimate:

Lemma 3.3.9. *Let μ satisfying Assumptions 3.3.1. Let $\xi \in L^2$. We have*

$$\mathrm{Var}_\mu\Big[\sum_{i=1}^N \xi(\ell_N^{-1}x_i)\Big] \leqslant N\ell_N\Big(\int_{\mathbb{T}}\xi^2 - \Big(\int_{\mathbb{T}}\xi\Big)^2\Big). \tag{3.69}$$

Proof. Let $\xi \in L^2(\mathbb{T},\mathbb{R})$. Let (ξ_k) be a sequence of elements of $\mathcal{C}^2(\mathbb{T},\mathbb{R})$ such that (ξ_k) converge to ξ in $L^2(\mathbb{T},\mathbb{R})$. Let us prove that (3.69) holds for ξ_k. Since we have not proved that the H.-S. equation (3.58) is well posed only for gradients of functions of the gaps, we proceed by regularizing μ. For $\eta > 0$, let χ_η be such that χ_η is bounded by η^{-1}, $\chi_\eta'' \geqslant 0$ and $\chi_\eta \leqslant \chi$. Define

$$\mathrm{d}\mu_\eta = \frac{1}{Z_\eta}e^{-H_\eta}\mathrm{d}X_N, \quad \text{where} \quad H_\eta = \sum_{i\neq j}\chi_\eta(x_i - x_j).$$

Denote $\mathcal{L}^{\mu_\eta}$ the operator acting on $H^{-1}(D_N,\mathbb{R})$,

$$\mathcal{L}^{\mu_\eta} = \nabla H_\eta \cdot \nabla - \Delta.$$

Since the density of μ_η is bounded from below and from above with respect to the Lebesgue measure on D_N, one may apply Proposition 3.3.1, which allows to express the variance of ξ_k under μ_η as

$$\mathrm{Var}_{\mu_\eta}[\mathrm{Fluct}_N[\xi_k]] = -\min \mathbb{E}_{\mu_\eta}[\nabla\phi\cdot\nabla^2 H\nabla\phi + |\nabla^2\phi|^2 - 2\nabla\phi\cdot\nabla\mathrm{Fluct}_N[\xi_k]],$$

where the minimum is taken over maps $\phi : D_N \to \mathbb{R}$ such that $\nabla\phi \in L^2(\{1,\dots,N\}, H^1(D_N,\mathbb{R}))$ and $\nabla\phi\cdot\vec{n} = 0$ on ∂D_N. Since $\nabla^2 H_\eta$ is non-negative, one may bound this by

$$\mathrm{Var}_{\mu_\eta}[\mathrm{Fluct}_N[\xi_k]] \leqslant -\min \mathbb{E}_{\mu_\eta}[|\nabla^2\phi|^2 - 2\nabla\phi\cdot\nabla\mathrm{Fluct}_N[\xi_k]], \tag{3.70}$$

where the minimum is taken over maps $\phi \in H^1$. The variational problem (3.70) has a minimum, attained at a certain $\phi_k \in H^1$, which satisfies

$$\begin{cases} \mathcal{L}^{\mu_\eta}(\partial_i\phi_k)(X_N) = \xi_k'(x_i) & \text{for each } i \in \{1,\dots,N\} \text{ and } X_N \in D_N \\ \nabla\phi\cdot\vec{n} = 0 & \text{on } \partial D_N. \end{cases} \tag{3.71}$$

Let $\theta_k : \mathbb{T} \to \mathbb{R}$ be such that $\theta_k' = \xi_k - \int\xi_k$. Set

$$\phi_k : X_N \in D_N \mapsto \theta_k(x_1) + \ldots + \theta_k(x_N).$$

Recalling Remark 7, one can observe that ϕ_k is a solution to (3.71) and by convexity ϕ_k is a minimizer of (3.70), which yields

$$\begin{aligned}\mathrm{Var}_{\mu_\eta}[\mathrm{Fluct}_N[\xi_k]] \leqslant \mathbb{E}_{\mu_\eta}\Big[\sum_{i=1}^N \theta_k''(x_i)^2 - 2\sum_{i=1}^N \xi_k'(x_i)\theta_k'(x_i)\Big] &= \mathbb{E}_{\mu_\eta}\Big[\sum_{i=1}^N \xi_k^2(x_i)\Big] \\ &= N\Big(\int_{\mathbb{T}}\xi_k^2 - \Big(\int_{\mathbb{T}}\xi_k\Big)^2\Big).\end{aligned}$$

Letting k go to $+\infty$ yields

$$\mathrm{Var}_{\mu_\eta}[\mathrm{Fluct}_N[\xi]] \leqslant N\Big(\int_{\mathbb{T}}\xi^2 - \Big(\int_{\mathbb{T}}\xi\Big)^2\Big).$$

Then, letting η tend to 0, we deduce by dominated convergence that (3.69) holds. $\square$

3.3.4 Log-Sobolev inequalities and Gaussian concentration

In this subsection we gather results of Log-Sobolev inequalities and Gaussian concentration. We first recall a crucial convexity result proved in [55].

Lemma 3.3.10. *Let μ satisfying Assumptions 3.3.1. Let $I \subset \{1, \ldots, N\}$, $|I| = K$. Denote π_I and π_{I^c} the projections on the coordinates $(x_i)_{i\in I}$ and $(x_i)_{i\in I^c}$. Split H into $H = H_1 \circ \pi_I + H_2$ with*

$$H_1 = X_K \in D_K \mapsto \sum_{i\neq j\in I} \chi(|x_i - x_j|), \quad H_2 : X_N \in D_N \mapsto \sum_{i\in I^c}\sum_{j\neq i} \chi(|x_i - x_j|).$$

Let ν be the push-forward of μ by the map π_I. Then $\widetilde{\nu}$ may be written

$$\mathrm{d}\widetilde{\nu}(x) \propto e^{-(H_1+\widetilde{H})}\mathbb{1}_{D_K}(x)\mathrm{d}x,$$

with

$$\nabla^2 \widetilde{H} \geqslant 0.$$

Proof. Let $I \subset \{1, \ldots, N\}$ and $K = |I|$. On D_N, introduce the coordinates $(z, y) = ((x_i)_{i\in I}, (x_i)_{i\in I^c})$. Fix $z \in D_K$, $v \in \mathbb{R}^K$ and denote $h : t \mapsto \widetilde{H}(z + tv)$. One can check that

$$h''(t) = \mathbb{E}_{P(x+tv)}[v \cdot \partial_{11}H_2 v] - \mathrm{Var}_{P(x+tv)}[v \cdot \partial_1 H_2], \tag{3.72}$$

where $P(z)$ is the probability measure

$$\mathrm{d}P(z) = \frac{1}{Z(z)} e^{-H_2(z,y)}\mathbb{1}_{(z,y)\in D_N}\mathrm{d}y.$$

Since $y \mapsto H_2(x, y)$ is convex, one may apply the Brascamp-Lieb inequality stated in Lemma 3.3.8, which gives

$$\mathrm{Var}_{P(x+tv)}[v \cdot \partial_1 H_2] \leqslant \mathbb{E}_{P(x+tv)}[v \cdot (\partial_{12}H_2)(\partial_{22}H_2)^{-1}(\partial_{12}H_2)v].$$

Furthermore since $\nabla^2 H_2$ is non-negative, its Schur complement is non-negative, which gives

$$\partial_{11}H_2 - \partial_{12}H_2(\partial_{22}H_2)^{-1}\partial_{12}H_2 \geqslant 0.$$

Inserting this into (3.72), this justifies that $\nabla^2 \widetilde{H} \geqslant 0$. $\square$

We pause to state the Log-Sobolev inequality for uniformly log-concave measures on convex domains of $\mathbb{R}^N$, which is a special case of the Bakry-Emery criterion [15]. Recall the relative entropy of a probability measure μ with respect to ν, defined by

$$\mathrm{Ent}(\mu \mid \nu) = \int \log \frac{\mathrm{d}\mu}{\mathrm{d}\nu}\mathrm{d}\mu \in [0, +\infty],$$

if ν is absolutely continuous with respect to μ and $\mathrm{Ent}(\nu \mid \mu) = +\infty$ otherwise. Let also recall the Fisher information of μ with respect to ν, given by

$$\mathrm{Fisher}(\mu \mid \nu) = \left|\nabla \log \frac{\mathrm{d}\nu}{\mathrm{d}\mu}\right|^2 \mathrm{d}\nu,$$

if ν is absolutely continuous with respect to μ and $\mathrm{Fisher}(\nu \mid \mu) = +\infty$ otherwise.

Lemma 3.3.11. *Let K be a convex domain of $\mathbb{R}^N$. Let $w > 0$ and γ^w be a centered Gaussian distribution on $\mathbb{R}^N$ with covariance matrix $\frac{1}{w}\mathrm{I}_n$. Let γ_K^w defined by conditioning γ^w into K. Assume that μ is a measure on K in the form $\mathrm{d}\mu = f\mathrm{d}\gamma_K^w$ with $f : K \to \mathbb{R}$ Borel and log-concave. Then ν satisfies a log-Sobolev inequality with constant $2w$, meaning for all probability measure $\mu \in \mathcal{P}(\mathcal{A})$,*

$$\mathrm{Ent}(\mu \mid \nu) \leqslant 2w\mathrm{Fisher}(\mu \mid \nu).$$

Lemma 3.3.12. *Let K be a convex domain of $\mathbb{R}^N$. Let $w > 0$ and γ^w be a centered Gaussian distribution on $\mathbb{R}^N$ with covariance matrix $\frac{1}{w}\mathrm{I}_n$. Let γ_K^w defined by conditioning γ^w into K. Assume that μ is a measure on K in the form $\mathrm{d}\mu = f\mathrm{d}\gamma_K^w$ with $f : K \to \mathbb{R}$ Borel and log-concave. Then μ satisfies Gaussian concentration: for all $F \in H^1$, we have*

$$\log \mathbb{E}_\mu[e^{tF}] \leqslant t\mathbb{E}_\mu[F] + \frac{w}{2}t^2 \sup_K |\nabla F|^2, \quad \text{for all } t \in \mathbb{R}. \tag{3.73}$$

We now state a concentration result which can be applied to divergence-free test-functions for measures on the form given by Assumptions 3.3.1. Recall

$$U_N \cdot \nabla^2 H U_N \geqslant c \sum_{i \neq j} (u_i - u_j)^2, \quad \text{for all } U_N \in \mathbb{R}^N.$$

The crucial observation is that when $\sum_{i=1}^N u_i = 0$, the Hessian of the energy controls $N-1$ times the Euclidean norm of u:

$$U_N \cdot \nabla^2 H U_N \geqslant (N-1)c \sum_{i=1}^N u_i^2. \tag{3.74}$$

Furthermore one can observe that the solution ϕ of the equation $\mathcal{L}\phi = F - \mathbb{E}_\mu[F]$ is divergence-free whenever F is divergence-free. Combining this with (3.74) gives the following Gaussian estimate:

Lemma 3.3.13. *Let $I \subset \{1, \dots, N\}$ and π_I the projection on the coordinates $(x_i)_{i \in I}$. Let μ satisfying Assumptions 3.3.1. Let $F = G \circ \pi_I \in H^1(\mu)$. Assume that F is independent of $\sum_{i \in I} x_i$, i.e $\sum_{i \in I} \partial_i F = 0$. For all $t \in \mathbb{R}$ we have*

$$\log \mathbb{E}_\mu[e^{tF}] \leqslant t\mathbb{E}_\mu[F] + \frac{t^2}{2c(|I|-1)} \sup |\nabla F|^2.$$

The proof of Lemma 3.3.13 can be found in [46, Le. 3.9]. It can be adapted readily to our circular setting. For completeness we sketch the main arguments below and follow line by line the proof of [46].

Proof. Let $I \in \{1, \dots, N\}$ of cardinal m. To simplify the notation assume that $I = \{1, \dots, m\}$. On D_N introduce the coordinates (x, x') with $x = (x_i)_{i \in I} \in D_m$ and $x' = (x_i)_{i \in I'} \in D_{N-m}$. The energy H can be split into $H(x, x') = H_1(x) + H_2(x, x')$ with H_1 uniformly convex, H_2 convex and H_1 independent of $\sum_{i \in I} x_i$, i.e $\sum_{i \in I} \partial_i H_1 = 0$. On D_m, introduce the coordinates $x = (z, w)$ with $z = (x_1, \dots, x_{m-1}) \in D_{m-1}$ and $w = m^{-\frac{1}{2}} \sum_{i=1}^m x_i$. Observe that this change of variables can be written in the form $(z, w) = M^*(x_1, \dots, x_m)$, with M an orthogonal matrix. Since H_1 is independent of w, one can write it in the form $H_1 = \widetilde{H}_1(z)$. Similarly F can be written in the form $F = g(z)$. Let ν be the law of the push-forward of μ by $x \in \mathbb{R}^N \mapsto z$. Namely, $\mathrm{d}\nu = \frac{1}{Z}e^{-\widetilde{H}(z)}\mathrm{d}z$, with

$$\widetilde{H}(z) = -\log Z + \widetilde{H}_1(z) - \log \int e^{-H(x,y)} \mathrm{d}w\mathrm{d}y. \tag{3.75}$$

The point is that $\widetilde{H}$ is convex. We claim that

$$\partial_{zz}\widetilde{H} \geqslant \frac{1}{Z}\int_{\mathbb{R}\times\mathbb{R}^{N-m}} e^{-H(x,y)}(\partial_{zz}H - \partial_{zq}H(\partial_{qq}H)^{-1}\partial_{zq}H)\mathrm{d}w\mathrm{d}y. \tag{3.76}$$

The proof of (3.76) is similar to the proof of Lemma 3.3.10. Since H_1 is independent of q, one has

$$\partial_{zq}H(\partial_{qq}H)^{-1}\partial_{zq}H = \partial_{zq}H_2(\partial_{qq}H_2)^{-1}\partial_{zq}H_2.$$

Hence, by positivity of $\partial_{zz}H_2$, its Schur complement is positive and

$$\partial_{zz}H - \partial_{zq}H(\partial_{qq}H)^{-1}\partial_{zq}H = \partial_{zz}H_1 + \partial_{zz}H_2 - \partial_{zq}H_2(\partial_{qq}H_2)^{-1}\partial_{zq}H_2 \geqslant \partial_{zz}H_1.$$

Inserting this into (3.76) we deduce that for all $u \in \mathbb{R}^{m-1}$,

$$u\cdot\partial_{zz}\widetilde{H}u \geqslant u\cdot\partial_{zz}H_1 u = \widetilde{M}u\cdot\partial_{xx}H_1\widetilde{M}u \geqslant c\sum_{i\neq j}((\widetilde{M}u)_i - (\widetilde{M}u)_j)^2,$$

where $\widetilde{M}$ denotes the first $m-1$ columns of M. Moreover we can observe that

$$\sum_{i\neq j}((\widetilde{M}u)_i - (\widetilde{M}u)_j)^2 = (m-1)\sum_{i=1}^{m-1}u_i^2.$$

Since ν is uniformly log-concave with a lower bound on the Hessian equal to $(m-1)c$, we can apply the Gaussian concentration of Lemma 3.3.12, which gives for all $t \in \mathbb{R}$,

$$\mathbb{E}_\mu[e^{tF}] = \mathbb{E}_\nu[e^{tg}] \leqslant e^{t\mathbb{E}_\nu[g]+\frac{t^2}{2c(m-1)}\sup|\nabla_z g|^2}.$$

We can now observe that, since M is orthogonal, $|\nabla_z g|^2 = |\nabla F|^2$. This concludes the proof. □

3.4 Near-optimal rigidity

This section is devoted to the proof of the rigidity result of Theorem 3.1.1. The method uses various techniques invented in the seminal paper [46, Th. 3.1]. Being working on the circle instead of the real line, some simplifications can be made: among other things, the expectation of gaps under the Gibbs measure is known and one does not need to estimate the accuracy of standard positions, which was one of the main issues of [46]. The first task for us is to obtain a local law on gaps saying that for each $i \in \{1,\ldots,N\}$ and $1 \leqslant k \leqslant \frac{N}{2}$, $N(x_{i+k}-x_i)$ is typically of order k. To this end we perform the mutliscale analysis of [46] allowing one bootstrap this local law down to microscale. The argument is based on a convexifying procedure that we first detail.

3.4.1 Comparison to a constrained Gibbs measures

Because the Hessian of the energy degenerates when particles are far away from each other, one cannot directly derive Gaussian concentration estimates for $\mathbb{P}_{N,\beta}$. Following [46], one may add to the Hamiltonian a convexifying term, which penalizes configurations with large gaps. Let θ be a smooth cutoff function $\theta : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\theta(x) = x^2$ for $x > 1$, $\theta = 0$ on $[0, \frac{1}{2}]$ and $\theta'' \geqslant 0$ on $\mathbb{R}^+$. Let $I = \{j : i \leqslant j \leqslant i+k\}$ and $K > 0$. Define

$$\mathrm{F} = k^2\theta\Big(\frac{N}{K}(x_{i+k}-x_i)\Big) \tag{3.77}$$

and the locally constrained Gibbs measure

$$\mathrm{d}\mathbb{Q}_{N,\beta} = \frac{1}{K_{N,\beta}} e^{-\beta(\mathcal{H}_N+\mathrm{F})(X_N)} \mathbb{1}_{D_N}(X_N)\mathrm{d}X_N. \tag{3.78}$$

In the sequel we will often take $K = \lfloor k^{1+\varepsilon} \rfloor$ for some $\varepsilon > 0$. The measure $\mathbb{Q}_{N,\beta}$ is more concentrated than $\mathbb{P}_{N,\beta}$ in the directions e_i for $i \in I$. Recall the total variation distance between two measures μ and ν on D_N:

$$\mathrm{TV}(\mu,\nu) = \sup_{\mathcal{A}\in\mathcal{B}(D_N)} |\mu(\mathcal{A}) - \nu(\mathcal{A})|.$$

The Pinsker inequality, see [7, Ch. 5] for a proof, asserts that

$$\mathrm{TV}(\mu,\nu)^2 \leqslant 2\mathrm{Ent}(\mu \mid \nu), \tag{3.79}$$

where $\mathrm{Ent}(\cdot \mid \nu)$ is the relative entropy with respect to ν. Using (3.79) and the log-concavity of the constrained measure (3.78), one may derive the following control:

Lemma 3.4.1. *Let $i \in \{1,\dots,N\}$, $1 \leqslant k \leqslant \frac{N}{2}$, $I = \{j : i \leqslant j \leqslant i+k\}$ and $K > 0$. Let $\mathbb{Q}_{N,\beta}$ be the measure (3.78). Denote π_I the projection $\pi_I : X_N \in D_N \mapsto (x_i)_{i\in I} \in D_{k+1}$. There exists a constant $C(\beta)$ depending only on β and s and locally uniform in β such that*

$$\mathrm{TV}(\mathbb{P}_{N,\beta}\circ\pi_I^{-1}, \mathbb{Q}_{N,\beta}\circ\pi_I^{-1})^2 \leqslant C(\beta)k^5 K^s \mathbb{E}_{\mathbb{P}_{N,\beta}}\Big[\Big(\frac{N}{K}(x_{i+k}-x_i)\Big)^2 \mathbb{1}_{x_{i+k}-x_i \geqslant \frac{K}{2N}}\Big].$$

Proof. Applying Pinsker's inequality (3.79) to $\mu = \mathbb{P}_{N,\beta}\circ\pi_I^{-1}$ and $\nu = \mathbb{Q}_{N,\beta}\circ\pi_I^{-1}$ reads

$$\mathrm{TV}(\mathbb{P}_{N,\beta}\circ\pi_I^{-1}, \mathbb{Q}_{N,\beta}\circ\pi_I^{-1})^2 \leqslant 2\mathrm{Ent}(\mathbb{P}_{N,\beta}\circ\pi_I^{-1} \mid \mathbb{Q}_{N,\beta}\circ\pi_I^{-1}). \tag{3.80}$$

Note

$$\mathrm{Ent}(\mathbb{P}_{N,\beta}\circ\pi_I^{-1} \mid \mathbb{Q}_{N,\beta}\circ\pi_I^{-1}) = \mathrm{Ent}(\mathbb{P}_{N,\beta}\circ(\mathrm{Gap}_{k+1}\circ\pi_I)^{-1} \mid \mathbb{Q}_{N,\beta}\circ(\mathrm{Gap}_{k+1}\circ\pi_I)^{-1}).$$

Indeed under $\mathbb{P}_{N,\beta}$ and $\mathbb{Q}_{N,\beta}$, the law of x_i is uniformly distributed on the circle and independent of the law of $\mathrm{Gap}_{k+1}\circ\pi_I(X_N)$. By Lemma 3.3.10, the Hamiltonian H of $\mathrm{Gap}_{k+1}\circ\pi_I(X_N)$ satisfies

$$U_{k+1}\cdot\nabla^2 H U_{k+1} \geqslant C\sum_{i<j\in I}\frac{(N(u_j-u_i))^2}{K^{s+2}} \geqslant C\sum_{i\in I\setminus\max I}\frac{(N(u_{i+1}-u_i))^2}{K^{s+2}}, \quad \text{for all } U_{k+1}\in\mathbb{R}^{k+1}.$$

Consequently the measure $\nu := \mathbb{Q}_{N,\beta}\circ(\mathrm{Gap}_{k+1}\circ\pi_I)^{-1}$ is c-uniformly log-concave for $c = \frac{C\beta}{K^{s+2}}$ and by Lemma 3.3.11 it satisfies a log-Sobolev inequality with constant $2c^{-1}$. Writing $\mathrm{F} = \mathrm{G}\circ \mathrm{Gap}_{k+1}\circ\pi_I$, this gives

$$\mathrm{Ent}(\mathbb{P}_{N,\beta}\circ(\mathrm{Gap}_{k+1}\circ\pi_I)^{-1} \mid \nu) \leqslant C(\beta)K^{s+2}\mathbb{E}_{\mathbb{P}_{N,\beta}}[|\nabla(\mathrm{G}\circ\mathrm{Gap}_{k+1}\circ\pi_I)|^2]. \tag{3.81}$$

One can next upper bound the Fisher information by

$$\begin{aligned}\mathbb{E}_{\mathbb{P}_{N,\beta}}[|\nabla \mathrm{G}|^2\circ\mathrm{Gap}_{k+1}\circ\pi_I] &\leqslant C(\beta)k^5K^{-2}\mathbb{E}_{\mathbb{P}_{N,\beta}}\Big[\Big(\theta'\Big(\frac{N}{K}(x_{i+k}-x_i)\Big)\Big)^2\Big] \\ &\leqslant C(\beta)k^5K^{-2}\mathbb{E}_{\mathbb{P}_{N,\beta}}\Big[\Big(\frac{N}{K}(x_{i+k}-x_i)\Big)\Big)^2\mathbb{1}_{x_{i+k}-x_i\geqslant\frac{K}{2N}}\Big].\end{aligned}$$

Inserting this into (3.81) and using (3.80) concludes the proof of Lemma 3.4.1. $\square$

3.4.2 First local law

We establish a local law, saying that each gap $N(x_{i+k}-x_i)$ is typically of order k with an exponentially small probability of deviations.

Lemma 3.4.2. *Let $\delta>0$. There exist two constants $c(\beta)>0$ and $C(\beta)>0$ locally uniform in β such that for each $i\in\{1,\ldots,N\}$ and $1\leqslant k\leqslant \frac{N}{2}$,*

$$\mathbb{P}_{N,\beta}(N(x_{i+1}-x_i)\geqslant k^{1+\delta})\leqslant C(\beta)\exp\Big(-c(\beta)k^{2\min(\delta,\frac{1-s}{2(2+s)})}\Big). \tag{3.82}$$

The proof of Lemma 3.4.2, inspired from the multiscale analysis of [46], proceeds by a bootstrap on scales: if the local law (3.82) is assumed to hold for $1\leqslant k\leqslant\frac{N}{2}$, then in view of Lemma 3.4.1, one may convexify the measure in a window of size k without changing much the measure. Moreover, the convexified measure satisfies better concentration estimates, allowing one to prove through Lemma 3.3.13 that (3.82) holds at a slightly smaller scale.

Proof.

Step 1: setting the bootstrap Let $\delta_0:=\frac{1-s}{2(2+s)}$. We wish to prove that there exist two constants $c_0(\beta)$ and $C_0(\beta)>0$ locally uniform in β such that for each $i\in\{1,\ldots,N\}$, $1\leqslant k\leqslant\frac{N}{2}$ and all $\delta\in(0,\delta_0]$,

$$\mathbb{P}_{N,\beta}(N(x_{i+k}-x_i)\geqslant k^{1+\delta})\leqslant C_0(\beta)e^{-c_0(\beta)k^{2\delta}}. \tag{3.83}$$

Let $K\geqslant 1$. Assume that (3.83) holds for each $k\geqslant K$. Note that this easily implies that for all $\delta>0$ and $k\geqslant K$,

$$\mathbb{P}_{N,\beta}(N(x_{i+k}-x_i)\geqslant k^{1+\delta})\leqslant C_0(\beta)e^{-c_0(\beta)k^{2\frac{1+\delta}{1+\delta_0}}}. \tag{3.84}$$

Let us prove that there exists some $\alpha_0\in(0,1)$ such that (3.83) holds for each $k\geqslant K^{1-\alpha_0}$. Fix $\alpha_0\in(0,1)$, $i\in\{1,\ldots,N\}$, $k\geqslant K^{1-\alpha_0}$, $\gamma>0$ and

$$I=\{j: i\leqslant j\leqslant i+K\}.$$

Let θ be a smooth cutoff function $\theta:\mathbb{R}^+\to\mathbb{R}^+$ such that $\theta(x)=x^2$ for $x>1$, $\theta=0$ on $[0,\frac{1}{2}]$ and $\theta''\geqslant 0$ on $\mathbb{R}^+$. Let $\gamma\in(0,\delta_0]$ be a constant to be carefully chosen. As in (3.78) set

$$\mathrm{F}=K^2\theta\Big(\frac{N}{K^{1+\gamma}}(x_{i+K}-x_i)\Big). \tag{3.85}$$

Let $\mathbb{Q}_{N,\beta}$ be the constrained Gibbs measure

$$\mathrm{d}\mathbb{Q}_{N,\beta}=\frac{1}{K_{N,\beta}}e^{-\beta(\mathcal{H}_N+\mathrm{F})(X_N)}\mathbb{1}_{D_N}(X_N)\mathrm{d}X_N.$$

Let $\delta\in(0,\frac{1-s}{2(2+s)}]$. Since $x_{i+k}-x_i$ is a function of $(x_i)_{i\in I}$, one can write

$$\mathbb{P}_{N,\beta}(N(x_{i+k}-x_i)\geqslant k^{1+\delta})\leqslant\mathbb{Q}_{N,\beta}(N(x_{i+k}-x_i)\geqslant k^{1+\delta})+\mathrm{TV}(\mathbb{P}_{N,\beta}\circ\pi_I^{-1},\mathbb{Q}_{N,\beta}\circ\pi_I^{-1}). \tag{3.86}$$

Step 2: upper bound on the total variation distance. Let us control the total variation distance between the push-forwards of $\mathbb{P}_{N,\beta}$ and $\mathbb{Q}_{N,\beta}$ onto the coordinates $(x_i)_{i\in I}$. By Lemma 3.4.1, we have

$$\mathrm{TV}(\mathbb{P}_{N,\beta}\circ\pi_I^{-1},\mathbb{Q}_{N,\beta}\circ\pi_I^{-1})\leqslant C(\beta)K^{5+s(1+\gamma)}\mathbb{E}_{\mathbb{P}_{N,\beta}}\Big[\theta\Big(\frac{1}{K^{1+\gamma}}N(x_{i+K}-x_i)\Big)\Big].$$

One can upper bound the right-hand side of the last display by

$$\mathbb{E}_{\mathbb{P}_{N,\beta}}\Big[\theta\Big(\frac{1}{K^{1+\gamma}}N(x_{i+K}-x_i)\Big)\Big]\leqslant C(\beta)K^{5+s(1+\gamma)}\sum_{j\geqslant K^{1+\gamma}}\frac{j^2}{K^{2(1+\gamma)}}\mathbb{P}_{N,\beta}(N(x_{i+K}-x_i)\geqslant j).$$

Using the the induction hypothesis (3.83), one finds that for all $\kappa<2\min(\gamma,\delta_0)=2\gamma$,

$$\mathbb{E}_{\mathbb{P}_{N,\beta}}\Big[\theta\Big(\frac{1}{K^{1+\gamma}}N(x_{i+K}-x_i)\Big)\Big]\leqslant C(\beta)e^{-c_0(\beta)K^{\kappa}}.$$

Recalling that $k\geqslant K^{1-\alpha_0}$, we deduce that for all $\kappa'<\frac{2\gamma}{1-\alpha_0}$,

$$\mathrm{TV}(\mathbb{P}_{N,\beta}\circ\pi_I^{-1},\mathbb{Q}_{N,\beta}\circ\pi_I^{-1})\leqslant C(\beta)e^{-c_0(\beta)k^{\kappa'}}.$$

Therefore, provided

$$\gamma>\delta(1-\alpha_0), \tag{3.87}$$

there exists $k_0(\beta)$ locally constant in β such that for all $k\geqslant k_0(\beta)$,

$$\mathrm{TV}(\mathbb{P}_{N,\beta}\circ\pi_I^{-1},\mathbb{Q}_{N,\beta}\circ\pi_I^{-1})\leqslant \frac{1}{2}C_0(\beta)e^{-c_0(\beta)k^{2\delta}}. \tag{3.88}$$

Step 3: accuracy under $\mathbb{Q}_{N,\beta}$. One shall first study the expectation of $N(x_{i+k}-x_i)$ under $\mathbb{Q}_{N,\beta}$. Since it is not bounded, one cannot directly apply (3.88) and one needs to prove a tightness result. Let $\varepsilon'>0$. Observe that

$$\begin{aligned}\log\mathbb{E}_{\mathbb{Q}_{N,\beta}}[e^{(N(x_{i+k}-x_i))^{\varepsilon'}}]&=\log\mathbb{E}_{\mathbb{P}_{N,\beta}}[e^{(N(x_{i+k}-x_i))^{\varepsilon'}-\beta\mathrm{F}}]-\log\mathbb{E}_{\mathbb{P}_{N,\beta}}[e^{-\beta\mathrm{F}}]\\&\leqslant\log\mathbb{E}_{\mathbb{P}_{N,\beta}}[e^{(N(x_{i+k}-x_i))^{\varepsilon'}-\beta\mathrm{F}}]\leqslant\log\mathbb{E}_{\mathbb{P}_{N,\beta}}[e^{(N(x_{i+k}-x_i))^{\varepsilon'}}].\end{aligned}$$

By Jensen's inequality, in view of (3.83), we have

$$\log\mathbb{E}_{\mathbb{P}_{N,\beta}}[e^{-\beta\mathrm{F}}]\geqslant-\beta\mathbb{E}_{\mathbb{P}_{N,\beta}}[\mathrm{F}]\geqslant-C(\beta)K. \tag{3.89}$$

There remains to upper bound the exponential moment of $(N(x_{i+k}-x_i))^{\varepsilon'}$ under $\mathbb{P}_{N,\beta}$. For all $\alpha>0$, one may write

$$\mathbb{E}_{\mathbb{P}_{N,\beta}}[e^{(N(x_{i+k}-x_i))^{\varepsilon'}}]\leqslant\mathbb{E}_{\mathbb{P}_{N,\beta}}[e^{(N(x_{i+K}-x_i))^{\varepsilon'}}]\leqslant e^{K^{\varepsilon'}}\sum_{j\geqslant K}e^{(j+1)^{\varepsilon'}}\mathbb{P}_{N,\beta}(N(x_{i+K}-x_i)\geqslant j).$$

Using (3.84), one finds that for $\varepsilon'>0$ small enough depending on s,

$$\log\mathbb{E}_{\mathbb{Q}_{N,\beta}}[e^{(N(x_{i+k}-x_i))^{\varepsilon'}}]\leqslant C(\beta)K.$$

It follows that

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}_{N,\beta}}[N(x_{i+k}-x_i)\mathbb{1}_{N(x_{i+k}-x_i)>K^{2/\varepsilon'}}] &\leqslant \mathbb{E}_{\mathbb{Q}_{N,\beta}}[e^{\frac{1}{2}(N(x_{i+k}-x_i))^{\varepsilon'}}\mathbb{1}_{N(x_{i+k}-x_i)>K^{2/\varepsilon'}}]+O_\beta(1)\\ &\leqslant \mathbb{E}_{\mathbb{Q}_{N,\beta}}[e^{(N(x_{i+k}-x_i))^{\varepsilon'}}]^{\frac{1}{2}}\mathbb{Q}_{N,\beta}(N(x_{i+k}-x_i)>K^{2/\varepsilon'})^{\frac{1}{2}}+O_\beta(1)\\ &\leqslant \mathbb{E}_{\mathbb{Q}_{N,\beta}}[e^{(N(x_{i+k}-x_i))^{\varepsilon'}}]^{\frac{1}{2}}e^{-K^2}\mathbb{E}_{\mathbb{Q}_{N,\beta}}[e^{(N(x_{i+k}-x_i))^{\varepsilon'}}]\\ &\leqslant C(\beta)e^{c(\beta)K-K^2}+O_\beta(1)=O_\beta(1).\end{aligned} \tag{3.90}$$

Similar computations show that

$$\mathbb{E}_{\mathbb{P}_{N,\beta}}[N(x_{i+k}-x_i)\mathbb{1}_{N(x_{i+k}-x_i)>K^{2/\varepsilon'}}]=O_\beta(1). \tag{3.91}$$

Having this tightness property, we can now compare the expectations of $N(x_{i+k}-x_i)$ under $\mathbb{P}_{N,\beta}$ and $\mathbb{Q}_{N,\beta}$. One may write

$$\begin{aligned}&|\mathbb{E}_{\mathbb{P}_{N,\beta}}[N(x_{i+k}-x_i)\mathbb{1}_{N(x_{i+k}-x_i)\leqslant K^{2/\varepsilon'}}]-\mathbb{E}_{\mathbb{Q}_{N,\beta}}[N(x_{i+k}-x_i)\mathbb{1}_{N(x_{i+k}-x_i)\leqslant K^{2/\varepsilon'}}]|\\ &\qquad\leqslant K^{2/\varepsilon'}\mathrm{TV}(\mathbb{P}_{N,\beta}\circ\pi_I^{-1},\mathbb{Q}_{N,\beta}\circ\pi_I^{-1})\leqslant\frac{1}{2}K^{2/\varepsilon'}C_0(\beta)e^{-c_0(\beta)k^{2\delta}},\end{aligned}$$

where we have used (3.88) in the last inequality. Besides, combining (3.90) and (3.91), one gets

$$\mathbb{E}_{\mathbb{P}_{N,\beta}}[N(x_{i+k}-x_i)\mathbb{1}_{N(x_{i+k}-x_i)>K^{2/\varepsilon'}}]-\mathbb{E}_{\mathbb{Q}_{N,\beta}}[N(x_{i+k}-x_i)\mathbb{1}_{N(x_{i+k}-x_i)>K^{2/\varepsilon'}}]=O_\beta(1).$$

One deduces that

$$\mathbb{E}_{\mathbb{Q}_{N,\beta}}[N(x_{i+k}-x_i)]=\mathbb{E}_{\mathbb{P}_{N,\beta}}[N(x_{i+k}-x_i)]+O_\beta(1)=k+O_\beta(1). \tag{3.92}$$

Step 4: fluctuations under $\mathbb{Q}_{N,\beta}$. One shall study the fluctuations of $N(x_{i+k}-x_i)$ under $\mathbb{Q}_{N,\beta}$ by applying the concentration estimate for divergence-free functions stated in Lemma 3.3.13. Denote

$$G: X_N\in D_N\mapsto N(x_{i+k}-x_i).$$

Observe that $\sum_{j=1}^N\partial_i G=0$, $\partial_i G=0$ for each $i\in I^c$ and $\sup|\nabla G|^2=2N^2$. Moreover $\mathbb{Q}_{N,\beta}$ satisfies the assumptions of Lemma 3.3.13: $\mathbb{Q}_{N,\beta}$ may be written $\mathbb{Q}_{N,\beta}=e^{-(H_1+H_2)}\mathrm{d}X_N$ with $\nabla^2 H_2\geqslant 0$ and $H_1(x)=\widetilde{H}(\pi(x))$ such that

$$U_{2k+1}\cdot\widetilde{H}U_{2k+1}\geqslant c|U_{2k+1}|^2,\quad\text{for all } U_{2k+1}\in\mathbb{R}^{2k+1}\text{ where } c=\beta N^2K^{-(1+\gamma)(s+2)}.$$

Lemma 3.3.13 therefore gives

$$|\log\mathbb{E}_{\mathbb{Q}_{N,\beta}}[e^{tG}]-t\mathbb{E}_{\mathbb{Q}_{N,\beta}}[G]|\leqslant C(\beta)t^2K^{(1+\gamma)(s+2)-1},\quad\text{for all } t\in\mathbb{R}.$$

It follows that

$$\log\mathbb{Q}_{N,\beta}(|N(x_{i+k}-x_i)-\mathbb{E}_{\mathbb{Q}_{N,\beta}}[N(x_{i+k}-x_i)]|\geqslant k^{1+\delta})\leqslant C(\beta)-c(\beta)\frac{k^{2(1+\delta)}}{K^{(1+\gamma)(s+2)-1}}.$$

Recalling that $k\geqslant K^{1-\alpha_0}$, one can rewrite this as

$$\log\mathbb{Q}_{N,\beta}(|N(x_{i+k}-x_i)-\mathbb{E}_{\mathbb{Q}_{N,\beta}}[N(x_{i+k}-x_i)]|\geqslant k^{1+\delta})\leqslant C(\beta)-c(\beta)k^{2(1+\delta)-\frac{1}{1-\alpha_0}((1+\gamma)(s+2)-1)},$$

which gives by (3.92) the estimate

$$\log\mathbb{Q}_{N,\beta}(|N(x_{i+k}-x_i)-k|\geqslant k^{1+\delta})\leqslant C(\beta)-c(\beta)k^{2(1+\delta)-\frac{1}{1-\alpha_0}((1+\gamma)(s+2)-1)}. \tag{3.93}$$

Step 5: conclusion. The exponent in the last display is larger than 2δ if and only if

$$2(1+\delta)-\frac{1}{1-\alpha_0}((1+\gamma)(s+2)-1)>2\delta \Longleftrightarrow \gamma<\frac{2(1-\alpha_0)-(1+s)}{2+s}. \tag{3.94}$$

Observe that the conditions (3.87) and (3.94) can be satisfied if and only if

$$(1-\alpha_0)\delta<\frac{2(1-\alpha_0)-(1+s)}{2+s} \Longleftrightarrow 1-\alpha_0>\frac{(2+s)\delta+(1+s)}{2}. \tag{3.95}$$

Having chosen $\delta<2\delta_0=\frac{1-s}{2+s}$, we deduce that there exists $\alpha_0\in(0,1)$ such that (3.95) is satisfied, which yields the existence of $\gamma>0$ satisfying both (3.87) and (3.94). For such constants $\alpha_0>0$ and $\gamma>0$, we deduce from (3.88) and (3.93) that there exists $k_0(\beta)$ locally constant in β such that for each $k\geqslant k_0(\beta)$

$$\mathbb{Q}_{N,\beta}(N(x_{i+k}-x_i)\geqslant k^{1+\delta})\leqslant\frac{1}{2}C_0(\beta)e^{-c_0(\beta)k^{2\delta}}. \tag{3.96}$$

In combination with (3.88) this implies the existence of a number $k_0(\beta)$ locally constant in β such that for each $k\geqslant k_0(\beta)$

$$\mathbb{P}_{N,\beta}(N(x_{i+k}-x_i)\geqslant k^{1+\delta})\leqslant C_0(\beta)e^{-c_0(\beta)k^{2\delta}},$$

thus concluding the bootstrap (3.83). Observe that (3.83) trivially holds for $K=N$. One concludes that (3.83) holds for each $k\geqslant k_0(\beta)$. At the cost of changing $C_0(\beta)$, (3.83) also holds for each $k\geqslant 1$. □

3.4.3 Reduction to a block average

In this subsection we implement a method of [46], consisting in replacing the position of a point x_i by a block average at a certain scale. For each $i\in\{1,\dots,N\}$ and $1\leqslant k\leqslant\frac{N}{2}$, let $I_k(i)$ stand for the interval of indices

$$I_k(i)=\{j\in\{1,\dots,N\}:d(i,j)\leqslant k\}.$$

Define the block average

$$x_i^{[k]}=\frac{1}{2k+1}\sum_{j\in I_k(i)}x_j. \tag{3.97}$$

Lemma 3.4.3 (Comparison to a block average). *Let $\varepsilon\in(0,1)$. There exist two constants $C(\beta)>0$ and $c(\beta)>0$ locally uniform in β such that for each $i\in\{1,\dots,N\}$ and $1\leqslant k\leqslant\frac{N}{2}$,*

$$\mathbb{P}_{N,\beta}(|N(x_i-x_i^{[k]})|\geqslant k^{\frac{s}{2}+\varepsilon})\leqslant C(\beta)e^{-c(\beta)k^{\frac{\varepsilon}{4(s+2)}}}.$$

Proof. Let $i\in\{1,\dots,N\}$ and $1\leqslant k\leqslant\frac{N}{2}$. Fix $\varepsilon>0$. Let $\alpha=\frac{1}{p}$ with $p\in\mathbb{N}$. Since $x_i^{[0]}=x_i$, one can break $N(x_i-x_i^{[k]})$ into

$$N(x_i-x_i^{[k]})=\sum_{m=0}^{p-1}N(x_i^{[[k^{m\alpha}]]}-x_i^{[[k^{(m+1)\alpha}]]}).$$

For each $m\in\{0,\dots,p-1\}$, denote

$$G_m=N(x_i^{[[k^{m\alpha}]]}-x_i^{[[k^{(m+1)\alpha}]]})\quad\text{and}\quad I_m=I_{\lfloor k^{(m+1)\alpha}\rfloor}(i).$$

The function G_m only depends on the variables $(x_j)_{j\in I_m}$ and $\sum_{j\in I_m}\partial_j G_m = 0$. Let $\varepsilon' > 0$. Let $\mathbb{Q}_{N,\beta}$ be the constrained Gibbs measure (3.78) with $I = I_m$ and $K = k^{(m+1)\alpha+\varepsilon'}$. The measure $\mathbb{Q}_{N,\beta}$ satisfies Assumption 3.3.1 with $c = \beta N^2 k^{-((m+1)\alpha+\varepsilon')(s+2)}$. Note that $\sup|\nabla G_m|^2 = O(N^2k^{-\alpha m})$. Thus, by Lemma 3.3.13, for all $t\in\mathbb{R}$, we have

$$\log \mathbb{E}_{\mathbb{Q}_{N,\beta}}[e^{tG_m}] = t\mathbb{E}_{\mathbb{Q}_{N,\beta}}[G_m] + O_\beta(t^2k^{\alpha ms+\alpha(1+s)+\varepsilon'(s+2)}).$$

Fix ε' and α such that $\alpha(1+s)+\varepsilon'(s+2) = \frac{\varepsilon}{2}$, say $\alpha = \frac{\varepsilon}{2(1+s)}$ and $\varepsilon' = \frac{\varepsilon}{4(s+2)}$. Since $\alpha m \leqslant 1$, one sees that

$$\mathbb{Q}_{N,\beta}(|G_m - \mathbb{E}_{\mathbb{Q}_{N,\beta}}[G_m]| \geqslant k^{\frac{s}{2}+\varepsilon}) \leqslant C(\beta)e^{-c(\beta)k^\varepsilon}. \tag{3.98}$$

Arguing as in the proof of Lemma 3.4.2, see Step 3, one finds that the expectation of G_m under $\mathbb{Q}_{N,\beta}$ satisfies

$$\mathbb{E}_{\mathbb{Q}_{N,\beta}}[G_m] = O_\beta(1). \tag{3.99}$$

Consequently by (3.98) and (3.99), one has

$$\mathbb{Q}_{N,\beta}(|G_m| \geqslant k^{\frac{s}{2}+\varepsilon}) \leqslant C(\beta)e^{-c(\beta)k^\varepsilon}.$$

Meanwhile by Lemma 3.4.1 and Lemma 3.4.2, we have

$$\mathrm{TV}(\mathbb{P}_{N,\beta}\circ\pi_{I_m}^{-1}, \mathbb{Q}_{N,\beta}\circ\pi_{I_m}^{-1}) \leqslant C(\beta)e^{-c(\beta)k^{\varepsilon'}}.$$

It follows that

$$\mathbb{P}_{N,\beta}(|G_m| \geqslant k^{\frac{s}{2}+\varepsilon}) \leqslant C(\beta)e^{-c(\beta)k^{\frac{\varepsilon}{4(s+2)}}}.$$

As a consequence there exist constants depending on ε such that

$$\mathbb{P}_{N,\beta}\Big(\sum_{m=0}^{p-1}|G_m| \geqslant \alpha^{-1}k^\varepsilon\Big) \leqslant C(\beta)e^{-c(\beta)k^{\frac{\varepsilon}{4(s+2)}}}.$$

This concludes the proof of Lemma 3.4.3. □

3.4.4 Proof of Theorem 3.1.1

The proof of Theorem 3.1.1 immediately follows from Lemma 3.4.3. Indeed when studying the fluctuations of $N(x_j - x_i)$, up to a small error, one can replace x_i and x_j by their block average at scale $d(j,i)$. Furthermore, the difference of these block averages may be bounded using Lemma 3.3.13 since its gradient has a small enough Euclidian norm.

Proof of Theorem 3.1.1. Let $i\in\{1,\ldots,N\}$, $1\leqslant k\leqslant \frac{N}{2}$ and $\varepsilon>0$. Let us split the gap $N(x_{i+k}-x_i)$ into

$$N(x_{i+k}-x_i) = N(x_{i+k}-x_{i+k}^{[k]}) - N(x_i - x_i^{[k]}) + N(x_{i+k}^{[k]} - x_i^{[k]}). \tag{3.100}$$

By Lemma 3.4.3, letting $\delta = \frac{\varepsilon}{4(s+2)}$, we have

$$\mathbb{P}_{N,\beta}(|N(x_i^{[k]} - x_i)| \geqslant k^{\frac{s}{2}+\varepsilon}) \leqslant C(\beta)e^{-c(\beta)k^\delta}, \tag{3.101}$$

$$\mathbb{P}_{N,\beta}(|N(x_{i+k}^{[k]} - x_{i+k})| \geqslant k^{\frac{s}{2}+\varepsilon}) \leqslant C(\beta)e^{-c(\beta)k^\delta}, \tag{3.102}$$

Let

$$G: X_N\in D_N \mapsto N(x_{i+k}^{[k]} - x_i^{[k]}).$$

Let $\mathbb{Q}_{N,\beta}$ be the constrained Gibbs measure (3.77) with $I = \{j : d(i,j) \leqslant k\}$ and $K = k^{(1+\gamma)}$ for some $\gamma > 0$ to fix later. Note that $\sup |\nabla G|^2 = O_\beta(\frac{N^2}{k})$. Moreover observe that $\mathbb{Q}_{N,\beta}$ satisfies Assumptions 3.3.1 with $c = \beta N^2 k^{-(1+\gamma)(s+2)}$. Consequently Lemma 3.3.13 gives

$$\log \mathbb{E}_{\mathbb{Q}_{N,\beta}}[e^{tG}] = t\mathbb{E}_{\mathbb{Q}_{N,\beta}}[G] + O_\beta(t^2 k^{(1+\gamma)(s+2)-2}), \quad \text{for all } t \in \mathbb{R}. \tag{3.103}$$

Arguing like in the proof of Lemma 3.4.2, one gets

$$\mathbb{E}_{\mathbb{Q}_{N,\beta}}[G] = O_\beta(1). \tag{3.104}$$

Fix $\gamma = \frac{\varepsilon}{s+2}$. By (3.103) and (3.104) one finds

$$\mathbb{Q}_{N,\beta}(|G| \geqslant k^{\frac{s}{2}+\varepsilon}) \leqslant C(\beta) e^{-c(\beta)k^\varepsilon}. \tag{3.105}$$

Again, by Lemma 3.4.1 and Lemma 3.4.2, one has

$$\mathrm{TV}(\mathbb{P}_{N,\beta} \circ \pi_I^{-1}, \mathbb{Q}_{N,\beta} \circ \pi_I^{-1}) \leqslant C(\beta) e^{-c(\beta)k^{\frac{\varepsilon}{s+2}}}. \tag{3.106}$$

Combining (3.105) and (3.106) one deduces that

$$\mathbb{Q}_{N,\beta}(|G| \geqslant k^{\frac{s}{2}+\varepsilon}) \leqslant C(\beta) e^{-c(\beta)k^{\frac{\varepsilon}{s+2}}}.$$

Together with (3.101) and (3.102), this proves (3.6). Since for each $i \in \{1, \ldots, N\}$, x_i is uniformly distributed on $\mathbb{T}$, one easily concludes the proof of (3.7). □

3.4.5 Control on the probability of near collisions

Let us control the probability of having two particles very close to each other. One may fix a single gap and show that there exists an inverse power of this gap with a finite exponential moment, which gives via Markov's inequality the following bound:

Lemma 3.4.4. *Let $\alpha \in (0, \frac{s}{2})$. There exist two constants $C(\beta) > 0$ and $c(\beta) > 0$ locally uniform in β such that for each $i \in \{1, \ldots, N\}$ and $\varepsilon > 0$ small enough, there holds*

$$\mathbb{P}_{N,\beta}(N(x_{i+1} - x_i) \leqslant \varepsilon) \leqslant C(\beta) e^{-c(\beta)\varepsilon^{-\alpha}}.$$

Proof. Let $\alpha \in (0, \frac{s}{2})$. Let $\eta : \mathbb{R}^+ \to \mathbb{R}^+$ be a smooth function such that

$$\eta(x) = \begin{cases} 1 & \text{if } x \in [0,1] \\ 0 & \text{if } x \in [2, +\infty]. \end{cases}$$

One shall study the fluctuations of $X_N \in D_N \mapsto \xi(N(x_2 - x_1))$ where

$$\xi : x \in \mathbb{R}^+ \mapsto \eta(x)|x|^{-\alpha}.$$

Let μ be the push-forward of $\mathbb{P}_{N,\beta}$ by the map $X_N \in D_N \mapsto N(x_2 - x_1)$, i.e the law of a single gap under $\mathbb{P}_{N,\beta}$. By Lemma 3.3.10, the measure μ is of the form

$$\mathrm{d}\mu = e^{-\beta(g+H)(y)} \mathbb{1}_{(0,N)}(y)\mathrm{d}y,$$

with $H : (0, N) \to \mathbb{R}$ convex. Consider $\psi \in L^2(\mu)$ solution of

$$\begin{cases} \beta(g+H)'\psi - \psi' = \xi - \int \xi & \text{on } (0, N) \\ \psi(0) = \psi(N) = 0. \end{cases} \tag{3.107}$$

We claim that

$$\sup_{y\in(0,N)}\frac{|\psi(y)|}{|y|^{1+s-\alpha}}\leqslant C(\beta). \tag{3.108}$$

First, there exists a constant $c=c(\beta)>0$ such that for $\psi\leqslant 0$ on $[0,c)$. Second, note that

$$-(g+H)'\leqslant g_s'\leqslant 0.$$

It follows that for all $x\in(0,c)$,

$$0\leqslant -\psi(x)\leqslant e^{\beta g_s(x)}\int_0^x e^{\beta g_s(y)}(\xi(y)-\textstyle\fint\xi)\mathrm{d}y. \tag{3.109}$$

In view of (3.109), one may compute that

$$\sup_{y\in(0,c)}\frac{|\psi(y)|}{|y|^{1+s-\alpha}}\leqslant C(\beta).$$

Third, since ψ is bounded on $(0,+\infty)$, we deduce that (3.108) holds.

From (3.108) one can derive a Gaussian concentration estimate by considering the Laplace transform of ξ under μ. For a small $t\in\mathbb{R}$, let us perform the change of variables $\mathrm{Id}+t\psi$ with ψ given by (3.107). For t small enough, this defines a valid change of variables and therefore

$$\mathbb{E}_\mu[\exp(t\xi)]=\mathbb{E}_\mu\Big[\exp\Big(t\xi\circ(\mathrm{Id}+t\psi)-\beta((g+H)\circ(\mathrm{Id}+t\psi)-(g+H))+\log(\mathrm{Id}+t\psi')\Big)\Big].$$

By convexity, we have

$$(g+H)\circ(\mathrm{Id}+t\psi)-(g+H)\geqslant t(g+H)'\psi,$$

$$\log(\mathrm{Id}+t\psi')\leqslant t\psi'.$$

Taylor-expanding $\xi_k\circ(\mathrm{Id}+t\psi)$ and using that ψ solves (3.107), one gets that for t small enough

$$\log\mathbb{E}_\mu[\exp(t\xi)]-t\mathbb{E}_\mu[\xi]\leqslant\log\mathbb{E}_\mu\Big[\exp\Big(t^2\sup_{y\in(0,N)}\frac{|\psi(y)|}{|y|^{1+s-\alpha}}\Big)\Big]\leqslant C(\beta)t^2, \tag{3.110}$$

where we have used that $\alpha\in(0,\frac{s}{2})$ in the last inequality. To control the expectation, one may write by symmetry

$$\mathbb{E}_\mu[\xi]=\frac{1}{N}\mathbb{E}_{\mathbb{P}_{N,\beta}}\Big[\sum_{i=1}^N\xi(N(x_{i+1}-x_i))\Big]=O_\beta(1). \tag{3.111}$$

The proof of Lemma 3.4.4 then follows from (3.110), (3.111) and Markov's inequality. □

3.5 Optimal rigidity for singular linear statistics

In this section, we give the optimal scaling of gaps and discrepancies and improve the fluctuation results of Theorem 3.1.1. We will consider statistics with test-functions having poor regularity. In contrast with Section 3.4, we give controls on variances rather than exponential moments. We however believe that our method can be upgraded to get Gaussian concentration.

3.5.1 Mean-field transport

We now present the transportation argument of [157, 241]. As mentioned in the introduction, this transport is the starting point of many CLTs on β-ensembles and Coulomb gases including the series of papers [184, 27, 181, 239]. The method consists in moving each particle according to its position only, so that at the first order, the main term of the linear variation of the energy compensates the linear statistics. This transport, which can be interpreted as a mean-field approximation of the solution of the Helffer-Sjöstrand equation, creates a local error term, sometimes called "loop equation term". For a measurable map $\psi : \ell_N^{-1}\mathbb{T} \to \mathbb{R}$, we denote $\mathrm{A}_{\ell_N}[\psi]$ the quantity

$$\mathrm{A}_{\ell_N}[\psi] = \iint_{\Delta^c} N\ell_N(\psi(\ell_N^{-1}x) - \psi(\ell_N^{-1}y))N^{-(1+s)}g_s'(x-y)\mathrm{dfluct}_N(x)\mathrm{dfluct}_N(y). \tag{3.112}$$

Remark 9. *The loop equation term (3.112) appears in many proofs of CLTs for log-gases. For the 2D Coulomb gas, it is replaced by an angle term, as seen in [184] and [24]. For β-ensembles, the corresponding quantity is smooth and may therefore be controlled using the local laws by bounding the measure fluct_N. In the Riesz case $s \in (0,1)$, (3.112) is as singular as the energy, which makes this term more delicate to treat.*

Proposition 3.5.1. *Let $\xi \in \mathcal{C}^{-s+\varepsilon}(\mathbb{T},\mathbb{R})$ for some $\varepsilon > 0$ and $\ell_N \in (0,1]$. Assume either that ξ is supported on $(-\frac{1}{2},\frac{1}{2})$ or that $\ell_N = 1$. Let $\psi \in \mathcal{C}^{\varepsilon}(\ell_N^{-1}\mathbb{T},\mathbb{R})$ given by*

$$\psi' = -\frac{1}{2\beta c_s}\ell_N^{1-s}(-\Delta)^{\frac{1-s}{2}}(\xi(\ell_N^{-1}\cdot))(\ell_N\cdot) \quad \text{and} \quad \int \psi = 0 \tag{3.113}$$

and $\Psi \in \mathcal{C}^{\varepsilon}(D_N,\mathbb{R}^N)$ given by

$$\Psi : X_N \in D_N \mapsto \ell_N(\psi(\ell_N^{-1}x_1),\ldots,\psi(\ell_N^{-1}x_N)).$$

We have

$$\begin{aligned}\mathrm{Var}_{\mathbb{P}_{N,\beta}}[\mathrm{Fluct}_N[\xi(\ell_N^{-1}\cdot)]] &= -\frac{1}{(N\ell_N)^{2(1-s)}}\mathbb{E}_{\mathbb{P}_{N,\beta}}[\beta\Psi\cdot\nabla^2\mathcal{H}_N\Psi + |D\Psi|^2] \\ &+ \frac{2}{(N\ell_N)^{1-s}}\mathbb{E}_{\mathbb{P}_{N,\beta}}\Big[\sum_{i=1}^N \xi'(\ell_N^{-1}x_i)\psi(\ell_N^{-1}x_i)\Big] + \frac{1}{(N\ell_N)^{2(1-s)}}\mathrm{Var}_{\mathbb{P}_{N,\beta}}\Big[\beta\mathrm{A}_{\ell_N}[\psi] - \sum_{i=1}^N\psi'(\ell_N^{-1}x_i)\Big].\end{aligned} \tag{3.114}$$

Remark 10 (Scaling relation)**.** *On the circle, the Riesz kernel does not satisfy any nice scaling relation, as opposed to the Riesz kernel on the real line. As a consequence the fractional Laplacian of $\xi(\ell_N^{-1}\cdot)$ cannot be expressed as a dilatation of the fractional Laplacian of ξ, hence (3.113). If ξ is replaced by a function $\xi : \mathbb{R} \to \mathbb{R}$ and if $(-\Delta)^{\frac{1-s}{2}}$ denotes the fractional Laplacian on the real line, we have*

$$\ell_N^{1-s}(-\Delta)^{\frac{1-s}{2}}(\xi(\ell_N^{-1}\cdot) = ((-\Delta)^{\frac{1-s}{2}}\xi)(\ell_N^{-1}\cdot)$$

and (3.113) would be given by $\psi' = -\frac{1}{2\beta c_s}(-\Delta)^{\frac{1-s}{2}}\xi$. When the scale $\ell_N \to 0$, then arguing as in Lemma 3.2.2, we can see that the solution of (3.113) approaches $-\frac{1}{2\beta c_s}(-\Delta)^{\frac{1-s}{2}}\xi_0$ where $\xi_0 : \mathbb{R} \to \mathbb{R}$ is as in (3.44). Therefore in the limit where $\ell_N \to 0$, (3.113) should be understand as a function supported on $\mathbb{R}$ and independent of ℓ_N.

Proof. Let $\psi \in \mathcal{C}^\varepsilon(\mathbb{T},\mathbb{R})$ for some $\varepsilon > 0$ be such that $\int \psi = 0$. Define the transport

$$\Psi : X_N \in D_N \mapsto (\psi(x_1), \dots, \psi(x_N)).$$

Let us expand $\nabla \mathcal{H}_N \cdot \Psi$. Let $\mu_N = \frac{1}{N}\sum_{i=1}^N \delta_{x_i}$ be the empirical measure. Almost surely under $\mathbb{P}_{N,\beta}$, there holds

$$\nabla \mathcal{H}_N \cdot \Psi = \iint_{\Delta^c} N^{-(1+s)} g_s'(x-y) N(\psi(x)-\psi(y))\mathrm{d}(N\mu_N)(x)\mathrm{d}(N\mu_N)(y),$$

where Δ stands for the diagonal $\{(x,y) \in \mathbb{T}^2 : x = y\}$. By decomposing μ_N into $\mu_N = \mathrm{d}x + \frac{1}{N}\mathrm{fluct}_N$, one can break $\nabla \mathcal{H}_N \cdot \Psi_{\ell_N}$ into

$$\begin{aligned}\nabla \mathcal{H}_N \cdot \Psi_{\ell_N} = N^2 &\iint N(\psi(x)-\psi(y))N^{-(1+s)}g_s'(x-y)\mathrm{d}x\mathrm{d}y \\ &+ 2N\int\Big(\int N(\psi(x)-\psi(y))N^{-(1+s)}g_s'(x-y)\mathrm{d}y\Big)\mathrm{dfluct}_N(x) + \mathrm{A}[\psi], \end{aligned} \tag{3.115}$$

with $\mathrm{A}[\psi]$ as defined in (3.112) with $\ell_N = 1$. For the crossed term we can write

$$N\int N(\psi(x)-\psi(y))N^{-(1+s)}g_s'(x-y)\mathrm{d}y = -N^{1-s}g_s' * \psi. \tag{3.116}$$

Let $\psi \in \mathcal{C}^\varepsilon(\mathbb{T},\mathbb{R})$ be the solution of the convolution equation $-2\beta g_s' * \psi = \xi - \int \xi$ with $\int \psi = 0$. Since g is the fundamental solution of the fractional Laplace equation $(-\Delta)^{\frac{1-s}{2}} g = c_s(\delta_0 - 1)$, ψ is the unique solution of

$$\psi' = -\frac{1}{2\beta c_s}(-\Delta)^{\frac{1-s}{2}}\xi \quad \text{with} \quad \int \psi = 0. \tag{3.117}$$

For this map, one can observe that the constant term in the splitting (3.115) vanishes:

$$N\iint N(\psi(x)-\psi(y))N^{-(1+s)}g_s'(x-y)\mathrm{d}x\mathrm{d}y = -N^{1-s}\int g_s' * \psi = 0.$$

By (3.115) and (3.116), there holds

$$\beta \nabla \mathcal{H}_N \cdot \frac{1}{N^{1-s}}\Psi = \mathrm{Fluct}_N[\xi] + \frac{1}{N^{1-s}}\mathrm{A}[\psi]. \tag{3.118}$$

Since $\int \psi = 0$, there exists $\phi \in \mathcal{C}^{1,\varepsilon}(\mathbb{T},\mathbb{R})$ such that $\phi' = \phi$. Let $\Phi \in \mathcal{C}^{1,\varepsilon}(D_N,\mathbb{R})$ be such that $\nabla \Phi = \Psi$, i.e $\Phi : X_N \in D_N \mapsto \phi(x_1) + \ldots + \phi(x_N)$. One can write

$$\begin{aligned}\mathrm{Var}_{\mathbb{P}_{N,\beta}}[\mathrm{Fluct}_N[\xi]] = \mathrm{Var}_{\mathbb{P}_{N,\beta}}&\Big[\mathrm{Fluct}_N[\xi] - \frac{1}{N^{1-s}}\mathcal{L}\Phi\Big] \\ &- \frac{1}{N^{2(1-s)}}\mathrm{Var}_{\mathbb{P}_{N,\beta}}[\mathcal{L}\Phi] + \frac{2}{N^{1-s}}\mathrm{Cov}_{\mathbb{P}_{N,\beta}}[\mathrm{Fluct}_N[\xi(\ell_N^{-1}\cdot)], \mathcal{L}\Phi].\end{aligned}$$

By (3.118), we have

$$\mathrm{Var}_{\mathbb{P}_{N,\beta}}\Big[\mathrm{Fluct}_N[\xi] - \frac{1}{N^{1-s}}\mathcal{L}\Phi\Big] = \frac{1}{N^{2(1-s)}}\mathrm{Var}_{\mathbb{P}_{N,\beta}}[\beta\mathrm{A}[\psi] - \mathrm{Fluct}_N[\psi']].$$

For the two other terms, observing that Ψ satisfies the boundary condition $\Psi \cdot \vec{n} = 0$ on ∂D_N, we get by integration by parts

$$\mathrm{Var}_{\mathbb{P}_{N,\beta}}[\mathcal{L}\Phi] = \mathbb{E}_{\mathbb{P}_{N,\beta}}[\beta\Psi \cdot \nabla^2\mathcal{H}_N\Psi + |D\Psi|^2]$$

and

$$\mathrm{Cov}_{\mathbb{P}_{N,\beta}}[\mathrm{Fluct}_N[\xi],\mathcal{L}\Phi]=\mathbb{E}_{\mathbb{P}_{N,\beta}}\Big[\sum_{i=1}^N\xi'(x_i)\psi(x_i)\Big].$$

We thus obtain

$$\begin{aligned}\mathrm{Var}_{\mathbb{P}_{N,\beta}}[\mathrm{Fluct}_N[\xi]]&=-\frac{1}{N^{2(1-s)}}\mathbb{E}_{\mathbb{P}_{N,\beta}}[\beta\Psi\cdot\nabla^2\mathcal{H}_N\Psi+|D\Psi|^2]\\&+\frac{2}{N^{1-s}}\mathbb{E}_{\mathbb{P}_{N,\beta}}\Big[\sum_{i=1}^N\xi'(x_i)\psi(x_i)\Big]+\frac{1}{N^{2(1-s)}}\mathrm{Var}_{\mathbb{P}_{N,\beta}}\Big[\beta\mathrm{A}[\psi]-\sum_{i=1}^N\psi'(x_i)\Big].\end{aligned}\tag{3.119}$$

Let ξ supported on $(-\frac{1}{2},\frac{1}{2})$ and $\ell_N\in(0,1]$. Let $\psi\in\mathcal{C}^\varepsilon(\ell_N^{-1}\mathbb{T},\mathbb{R})$ such that

$$\psi'=-\frac{1}{2\beta c_s}\ell_N^{1-s}(-\Delta)^{\frac{1-s}{2}}(\xi(\ell_N^{-1}\cdot))(\ell_N\cdot)\quad\text{with}\quad\int\psi=0\tag{3.120}$$

and $\Psi\in\mathcal{C}^\varepsilon(D_N,\mathbb{R}^N)$ given by

$$\Psi:X_N\in D_N\mapsto\ell_N(\psi(\ell_N^{-1}x_1),\ldots,\psi(\ell_N^{-1}x_N)).$$

Applying (3.119) to $\xi(\ell_N^{-1}\cdot)$ allows one to write

$$\begin{aligned}\mathrm{Var}_{\mathbb{P}_{N,\beta}}[\mathrm{Fluct}_N[\xi(\ell_N^{-1}\cdot)]]&=-\frac{1}{(N\ell_N)^{2(1-s)}}\mathbb{E}_{\mathbb{P}_{N,\beta}}[\beta\Psi\cdot\nabla^2\mathcal{H}_N\Psi+|D\Psi|^2]\\+\frac{2}{(N\ell_N)^{1-s}}\mathbb{E}_{\mathbb{P}_{N,\beta}}\Big[\sum_{i=1}^N\xi'(\ell_N^{-1}x_i)\psi(\ell_N^{-1}x_i)\Big]&+\frac{1}{(N\ell_N)^{2(1-s)}}\mathrm{Var}_{\mathbb{P}_{N,\beta}}\Big[\beta\mathrm{A}_{\ell_N}[\psi]-\sum_{i=1}^N\psi'(\ell_N^{-1}x_i)\Big],\end{aligned}$$

Indeed, letting $\phi_N:=\ell_N\psi(\ell_N^{-1}\cdot)$ with ψ as in (3.120), we have

$$\phi_N'=\psi'(\ell_N^{-1}\cdot)=-\frac{1}{2\beta c_s}\ell_N^{1-s}(-\Delta)^{\frac{1-s}{2}}(\xi(\ell_N^{-1}\cdot)).$$

Therefore $\ell_N^{1/(1-s)}\phi_N$ is the solution of (3.117) and by inserting this into (3.119) applied to $\xi(\ell_N^{-1}\cdot)$, we deduce that (3.129) holds. □

Remark 11. *Proposition 3.5.1 can be interpreted as a mean-field approximate solution of*

$$\begin{cases}A_1\nabla\phi=\nabla\mathrm{Fluct}_N[\xi] & \text{on } D_N\\ \nabla\phi\cdot\vec{n}=0 & \text{on } \partial D_N.\end{cases}$$

The existence of an approximate solution in the class of "diagonal transports", $\Psi:X_N\in D_N\mapsto(\psi(x_1),\ldots,\psi(x_N))$ is a consequence of the long-range nature of the system and more precisely of the mean-field approximation of the energy. For the hypersingular Riesz gas, i.e the Riesz gas with $g_s(x)\sim|x|^{-s}$ for $s>1$, one cannot approximate the solution of the above equation within this class.

3.5.2 Splitting of the loop equation term

In view of Proposition 3.5.1, expanding the variance of a linear statistic reduces to controlling the loop equation term (3.112). Let us first discard a strategy based on local laws only. Recall that for all $\psi \in \mathcal{C}^{\varepsilon}(\ell_N^{-1}\mathbb{T},\mathbb{R})$,

$$\mathrm{A}_{\ell_N}[\psi] = \iint_{\Delta^c} N\ell_N(\psi(\ell_N^{-1}x) - \psi(\ell_N^{-1}y))N^{-(1+s)}g_s'(x-y)\mathrm{dfluct}_N(x)\mathrm{dfluct}_N(y).$$

By using local laws on gaps, one may control the above integral away from the boundary. Nevertheless $\mathrm{A}_{\ell_N}[\psi]$ contains among other terms the quantity

$$\sum_{i=1}^N N(\psi(\ell_N^{-1}x_{i+1}) - \psi(\ell_N^{-1}x_i))N^{-(1+s)}g_s'(x_{i+1}-x_i),$$

which is in $O(N\ell_N|\psi'|_\infty)$ with overwhelming probability. Therefore applying a local law estimate will give in the best case, the bound

$$\mathrm{Var}_{\mathbb{P}_{N,\beta}}[\mathrm{A}_{\ell_N}[\psi]] = O((N\ell_N)^2|\psi'|_\infty^2).$$

Inserting this into Proposition 3.5.1 gives an error term of order $O((N\ell_N)^{2s})$, which is larger than the expected order of fluctuations of linear statistics. Instead one shall exploit the convexity of the interaction and bound the fluctuations of $\mathrm{A}_{\ell_N}[\psi]$ using various concentration inequalities. As emphasized in Section 3.3, the variance of a smooth function under a log-concave probability measure is related to the norm of its gradient and one should therefore first differentiate (3.112).

Before entering into the main computations, we first define a localized version of $\mathrm{A}_{\ell_N}[\psi]$. We will assume that 0 is in the support of ξ and then we then let i_0 be the index (defined almost surely) such that x_i is the closest point to 0:

$$i_0 = \underset{1\leqslant i\leqslant N}{\mathrm{argmin}}|x_i|.$$

Fix $\gamma > 1$ and let

$$I_N = \{i \in \{1,\dots,N\} : d(i,i_0) \leqslant (N\ell_N)^\gamma\}. \tag{3.121}$$

For $\psi \in \mathcal{C}^{\varepsilon}(\ell_N^{-1}\mathbb{T},\mathbb{R})$, we define a localized version of $\mathrm{A}_{\ell_N}[\psi]$ by letting

$$\begin{aligned}\widetilde{\mathrm{A}}_{\ell_N}[\psi] = &\sum_{i\neq j\in I_N} N\ell_N(\psi(\ell_N^{-1}x_i) - \psi(\ell_N^{-1}x_j))N^{-(1+s)}g_s'(x_i-x_j)\\ &- 2N\sum_{i\in I_N}\int_{|y|\leqslant\frac{(N\ell_N)^\gamma}{N}} N\ell_N(\psi(\ell_N^{-1}x) - \psi(\ell_N^{-1}y))N^{-(1+s)}g_s'(x_i-y)\mathrm{d}y. \end{aligned}\tag{3.122}$$

For $\varepsilon > 0$, define the good event

$$\begin{aligned}\mathcal{A} = \{&X_N \in D_N : \forall i,k : i, i+k \in I_N, N|x_{i+k}-x_i-Nk| \leqslant (N\ell_N)^\varepsilon k^{\frac{s}{2}}\}\\ &\cap \{X_N \in D_N : \forall i \in I_N, (N\ell_N)^{-\varepsilon} \leqslant N|x_{i+1}-x_i| \leqslant (N\ell_N)^\varepsilon\}.\end{aligned}\tag{3.123}$$

Lemma 3.5.2. *Let ξ satisfying Assumptions 3.1.1. Let $\psi_0 \in \mathcal{C}^\delta(\mathbb{T},\mathbb{R})$ such that $g_s' * \psi_0 = \xi - \int\xi$ with $\int\psi_0 = 0$. Assume that ψ_0'' has p singularities in $a_1,\dots,a_p$ of order $1+\alpha_1,\dots,1+\alpha_p$, with*

$\alpha_i \in (0,1)$, as defined in (3.8). Assume that ξ is supported on $(-\frac{1}{2},\frac{1}{2})$ or that $\ell_N = 1$. Let $\psi \in \mathcal{C}^{\delta}(\ell_N^{-1}\mathbb{T},\mathbb{R})$ given by

$$\psi' = -\frac{1}{2c_s}\ell_N^{1-s}(-\Delta)^{\frac{1-s}{2}}(\xi(\ell_N^{-1}\cdot))(\ell_N\cdot) \quad \textit{and} \quad \int \psi = 0. \tag{3.124}$$

Let $\psi_{\mathrm{reg}} = \psi * K_\ell$ with K_ℓ defined in (3.50) and $\ell = 1/(N\ell_N)^{1-\varepsilon'}$ with $\varepsilon' > 0$. Denote $I = (-\frac{(N\ell_N)^\gamma}{N},\frac{(N\ell_N)^\gamma}{N})\cap\mathbb{T}$. One can break $\nabla\widetilde{\mathrm{A}}_{\ell_N}[\psi_{\mathrm{reg}}]$ into $\mathrm{V}+\mathrm{W}$ with $\mathrm{V},\mathrm{W} \in L^2(\{1,\ldots,N\},H^1(\mathbb{P}_{N,\beta}))$ satisfying

- *For each $i \in I_N^c$, $\mathrm{V}_i = \mathrm{W}_i = 0$.*
- *Uniformly on $i \in I_N$,*

$$\begin{aligned}\sup_{\mathcal{A}}|\mathrm{V}_i| &\leqslant C(N\ell_N)^{\kappa\varepsilon}\Big(\ell_N^{-1} + \sum_{l=1}^{p}\ell_N^{\alpha_l}\frac{1}{(|x_i-\ell_N a_l|\vee\frac{1}{N})^{1+\alpha_l}}\Big)\mathbb{1}_{|x_i|<2\ell_N}\\ &+C(N\ell_N)^{\kappa\varepsilon}\sum_{l:\alpha_l>\frac{s}{2}}(N\ell_N)^{\alpha_l}N^{-\frac{s}{2}}\frac{1}{(|x_i-\ell_N a_l|\vee\frac{(N\ell_N)^{\varepsilon'}}{N})^{1+\frac{s}{2}}}+C(N\ell_N)^{\kappa\varepsilon}N^{-\frac{s}{2}}\frac{1}{(\ell_N+|x_i|)^{1+\frac{s}{2}}}\\ &+\frac{C}{(N\ell_N)^{\gamma s}}\frac{\ell_N^{-1}}{(\ell_N^{-1}|x_i|+1)^{2-s}}+C(N\ell_N)^{\kappa\varepsilon-(2-s)(1-\gamma)}N^{-\frac{s}{2}}\frac{1}{(d(x_i,\partial I)\wedge\frac{1}{N})^{1+\frac{s}{2}}}.\end{aligned} \tag{3.125}$$

- *There exists $\tilde{\mathrm{W}} \in L^2(\{1,\ldots,N\},H^1(\mathbb{P}_{N,\beta}))$ such that for all $U_N \in \mathbb{R}^N$,*

  $$\mathrm{W}\cdot U_N = -\sum_{i=1}^{N}\tilde{\mathrm{W}}N(u_{i+1}-u_i)$$

  with

  $$\sup_{\mathcal{A}}|\tilde{\mathrm{W}}|^2 \leqslant C(N\ell_N)^{\kappa\varepsilon}(N\ell_N + (N\ell_N)^{2\max\alpha_l}). \tag{3.126}$$

The proof of Lemma 3.5.2 is deferred to the Appendix.

Remark 12 (Remarks on the decomposition). *Conditionally on $x_1 = x$, F becomes a function of the gaps, i.e $F = \tilde{F}\circ\mathrm{Gap}_N$ for some $\tilde{F}:\mathbb{R}^N\to\mathbb{R}$. However, due to the (nonintegrable) singularity of ψ', the norm $|\nabla\tilde{F}|$ is too large for the Log-Sobolev inequality to give sharp bounds on the variance of $\widetilde{\mathrm{A}}_{\ell_N}[\psi_{\mathrm{reg}}]$.*

3.5.3 Variance quantitative expansion

We proceed to the proof of Theorem 3.1.2. The first step is to perform the mean-field transport of Proposition 3.5.1, which reduces the problem to approximating the variance of the loop equation term (3.112). We then multiply the gradient of (3.112) by a cutoff function supported on a good event (of overwhelming probability) on which gaps are close to their standard values. Using the uniform controls of the last subsection, we then deduce from a Poincaré inequality in gap coordinates and from the comparison principle of Lemma 3.3.5 a sharp control on the variance of (3.112).

Proof of Theorem 3.1.2. Let ξ satisfying Assumptions 3.1.1. Let $\psi \in \mathcal{C}^{\delta}(\mathbb{T},\mathbb{R})$ defined by

$$\psi' = -\frac{1}{2\beta c_s}(-\Delta)^{\frac{1-s}{2}}\xi \quad \text{with} \int\psi = 0.$$

By assumption, ψ'' has singularities in $a_1 < \ldots < a_p$ of order $1+\alpha_1,\ldots,1+\alpha_p$ with $\alpha_1,\ldots,\alpha_p \in (0,1-\frac{s}{2})$. Let $\{\ell_N\}$ be a sequence of positive numbers in $(0,1]$. Assume either that ξ is supported on $(-\frac{1}{2},\frac{1}{2})$ or that $\ell_N = 1$.

Step 1: regularization. Let $\varepsilon' > 0$. Define

$$\xi_{\mathrm{reg}} = \xi * K_\ell \quad \text{with } K_\ell \text{ defined in (3.50) and } \ell = 1/(N\ell_N)^{1-\varepsilon'}. \tag{3.127}$$

By Lemma 3.3.9, there holds

$$\mathrm{Var}_{\mathbb{P}_{N,\beta}}[\mathrm{Fluct}_N[\xi_{\mathrm{reg}}(\ell_N^{-1}\cdot) - \xi(\ell_N^{-1}\cdot)]] \leqslant N\ell_N|\xi_{\mathrm{reg}} - \xi|_{L^2}^2.$$

Moreover one can check that

$$|\xi_{\mathrm{reg}} - \xi|_{L^2}^2 \leqslant C(N\ell_N)^{\kappa\varepsilon'-1}\Big(1 + (N\ell_N)^{2(\max\alpha_l-(1-s))}\Big).$$

Since $\max\alpha_l < 1 - \frac{s}{2}$, for ε' small enough, the above quantity is $o((N\ell_N)^{-(1-s)})$. We deduce that, up to a lower order term, one can replace the test-function ξ be its regularized version ξ_{reg}:

$$\begin{aligned}\mathrm{Var}_{\mathbb{P}_{N,\beta}}[\mathrm{Fluct}_N[\xi]] = \mathrm{Var}_{\mathbb{P}_{N,\beta}}[\mathrm{Fluct}_N[\xi_{\mathrm{reg}}]] + O\Big(\mathrm{Var}_{\mathbb{P}_{N,\beta}}[\mathrm{Fluct}_N[\xi_{\mathrm{reg}}]]^{\frac{1}{2}}(1+(N\ell_N)^{\max\alpha_l-(1-s)})\\ + (N\ell_N)^{\kappa\varepsilon'+2\max\alpha_l-2(1-s)}\Big).\end{aligned} \tag{3.128}$$

Step 2: mean-field transport. Let $\psi_{\mathrm{reg}} \in \mathcal{C}^2(\ell_N^{-1}\mathbb{T}, \mathbb{T})$ be such that

$$\psi'_{\mathrm{reg}} = -\frac{1}{2\beta c_s}(-\Delta)^{\frac{1-s}{2}}(\xi_{\mathrm{reg}}(\ell_N^{-1}\cdot))(\ell_N\cdot) \quad \text{with} \int \psi_{\mathrm{reg}} = 0.$$

Let us now define the map

$$\Psi_{\ell_N} : X_N \in D_N \mapsto \ell_N(\psi_{\mathrm{reg}}(\ell_N^{-1}x_1), \ldots, \psi_{\mathrm{reg}}(\ell_N^{-1}x_N)).$$

By Proposition 3.5.1 we have

$$\begin{aligned}\mathrm{Var}_{\mathbb{P}_{N,\beta}}[\mathrm{Fluct}_N[\xi_{\mathrm{reg}}(\ell_N^{-1}\cdot)]] = -\frac{1}{(N\ell_N)^{2(1-s)}}\mathbb{E}_{\mathbb{P}_{N,\beta}}[\beta\Psi_{\ell_N}\cdot\nabla^2\mathcal{H}_N\Psi_{\ell_N} + |D\Psi_{\ell_N}|^2]\\ + \frac{2}{(N\ell_N)^{1-s}}\mathbb{E}_{\mathbb{P}_{N,\beta}}\Big[\sum_{i=1}^N(\xi'_{\mathrm{reg}}\psi_{\mathrm{reg}})(\ell_N^{-1}x_i)\Big] + \frac{1}{(N\ell_N)^{2(1-s)}}\mathrm{Var}_{\mathbb{P}_{N,\beta}}\Big[\beta\mathrm{A}_{\ell_N}[\psi_{\mathrm{reg}}] - \sum_{i=1}^N\psi'_{\mathrm{reg}}(\ell_N^{-1}x_i)\Big].\end{aligned} \tag{3.129}$$

By Lemma 3.3.9, the variance of $\mathrm{Fluct}_N[\psi'_{\mathrm{reg}}(\ell_N^{-1}\cdot)]$ is bounded by

$$\mathrm{Var}_{\mathbb{P}_{N,\beta}}[\mathrm{Fluct}_N[\psi'_{\mathrm{reg}}(\ell_N^{-1}\cdot)]] \leqslant N\ell_N|\psi'_{\mathrm{reg}}|_{L^2}^2.$$

Since $|\psi'_{\mathrm{reg}}|_{L^2}^2 \leqslant C(1 + (N\ell_N)^{2\max\alpha_l-1})$, this implies that

$$\mathrm{Var}_{\mathbb{P}_{N,\beta}}\Big[\beta\mathrm{A}_{\ell_N}[\psi_{\mathrm{reg}}] - \sum_{i=1}^N\psi'_{\mathrm{reg}}(\ell_N^{-1}x_i)\Big] \leqslant C(\beta)\Big(\mathrm{Var}_{\mathbb{P}_{N,\beta}}[\mathrm{A}_{\ell_N}[\psi_{\mathrm{reg}}]] + (N\ell_N)^{\max(1,2\max\alpha_l)}\Big).$$

Step 3: asymptotic of the mean-field terms. Define

$$\mathrm{B}_{\ell_N}[\psi_{\mathrm{reg}}] = \iint_{\Delta^c} N^{-(s+2)} g_s''(x-y)(N\ell_N)^2(\psi_{\mathrm{reg}}(\ell_N^{-1}x) - \psi_{\mathrm{reg}}(\ell_N^{-1}y))^2 \mathrm{dfluct}_N(x)\mathrm{dfluct}_N(y). \tag{3.130}$$

By splitting the empirical measure μ_N into $\mu_N = \mathrm{dx} + \frac{1}{N}\mathrm{fluct}_N$ like in the proof of Proposition 3.5.1 and using (3.43), we can easily show that

$$\begin{aligned}
&-\frac{1}{(N\ell_N)^{2(1-s)}}\mathbb{E}_{\mathbb{P}_{N,\beta}}[\beta\Psi_{\ell_N}\cdot\nabla^2\mathcal{H}_N\Psi_{\ell_N} + |D\Psi_{\ell_N}|^2] \\
&+\frac{2}{(N\ell_N)^{2(1-s)}}\mathbb{E}_{\mathbb{P}_{N,\beta}}\Big[\sum_{i=1}^N \xi_{\mathrm{reg}}'(\ell_N^{-1}x_i)\ell_N^s\psi_{\mathrm{reg}}(\ell_N^{-1}x_i)\Big] = \frac{1}{2\beta c_s}N^s|\xi_{\mathrm{reg}}(\ell_N^{-1}\cdot)|^2_{H^{\frac{1-s}{2}}} - (N\ell_N)^{2s-1}\int(\psi_{\mathrm{reg}}')^2 \\
&\qquad - \frac{\beta}{(N\ell_N)^{2(1-s)}}\mathbb{E}_{\mathbb{P}_{N,\beta}}[\mathrm{B}_{\ell_N}[\psi_{\mathrm{reg}}]].
\end{aligned} \tag{3.131}$$

When ℓ_N tends to 0, by (3.45), we have

$$|\xi_{\mathrm{reg}}(\ell_N^{-1}\cdot)|^2_{H^{\frac{1-s}{2}}} = \ell_N^s|\widetilde{\xi}_{\mathrm{reg}}|^2_{H^{\frac{1-s}{2}}} + O(\ell_N^2|\xi|^2_{L^2}).$$

Moreover one can easily prove that

$$\mathbb{E}_{\mathbb{P}_{N,\beta}}[\mathrm{B}_{\ell_N}[\psi_{\mathrm{reg}}]] \leqslant C(\beta)(N\ell_N)^{\kappa\varepsilon}((N\ell_N + (N\ell_N)^{2\max\alpha_l}), \tag{3.132}$$

see Lemma 3.7.4 in Appendix 3.7.2.

Step 4: reduction to a finite-range quantity We now reduce to a localized version of $\mathrm{A}_{\ell_N}[\psi_{\mathrm{reg}}]$. Let $\gamma > \frac{3-s}{2-s} \vee \frac{1}{s} \vee 1 + \frac{1}{1-s}$ and $y_0 \in \mathrm{supp}(\xi)$. As in Subsection 3.5.2, define

$$i_0 = \underset{1\leqslant i\leqslant N}{\mathrm{argmin}}|x_i| \quad \text{and} \quad I_N = \{j : d(j,i_0) \leqslant (N\ell_N)^\gamma\}.$$

Let us split $\mathrm{A}_{\ell_N}[\psi_{\mathrm{reg}}]$ into $\mathrm{A}_{\ell_N}[\psi_{\mathrm{reg}}] = \widetilde{\mathrm{A}}_{\ell_N}[\psi_{\mathrm{reg}}] + \mathrm{A}^{\mathrm{ext}}$ with $\widetilde{\mathrm{A}}_{\ell_N}[\psi_{\mathrm{reg}}]$ as defined in (3.122):

$$\begin{aligned}
\widetilde{\mathrm{A}}_{\ell_N}[\psi_{\mathrm{reg}}] = &\sum_{i\in I_N, d(i,j)\leqslant(N\ell_N)^\gamma} N\ell_N(\psi(\ell_N^{-1}x_i) - \psi_{\mathrm{reg}}(\ell_N^{-1}x_j))N^{-(1+s)}g_s'(x_i-x_j) \\
&- 2N\sum_{i\in I_N}\int_{|y|\leqslant\frac{(N\ell_N)^\gamma}{N}} N\ell_N(\psi_{\mathrm{reg}}(\ell_N^{-1}x) - \psi_{\mathrm{reg}}(\ell_N^{-1}y))N^{-(1+s)}g_s'(x_i-y)\mathrm{d}y.
\end{aligned}$$

In Appendix 3.7.2, we show that the remaining term $\mathrm{A}^{\mathrm{ext}}$ is $o_\beta((N\ell_N)^{\frac{1}{2}})$, since γ has been chosen large enough: there exist $C(\beta) > 0$ and $\delta > 0$ such that

$$\mathbb{P}_{N,\beta}(|\mathrm{A}^{\mathrm{ext}}| > (N\ell_N)^{\frac{1}{2}}) \leqslant C(\beta)e^{-c(N\ell_N)^\delta}. \tag{3.133}$$

We thus deduce that

$$\mathrm{Var}_{\mathbb{P}_{N,\beta}}[\mathrm{A}^{\mathrm{ext}}] = o_\beta(N\ell_N). \tag{3.134}$$

The estimate (3.133) is proved in Lemma 3.7.5.

Step 5: fixing an origin In order to apply the comparison principle of Lemma 3.3.5, one needs to fix an origin. Recall that x_1 is uniformly distributed on the circle. Conditioning by x_1 allows one to split the variance of $\widetilde{\mathrm{A}}_{\ell_N}[\psi_{\mathrm{reg}}]$ in the following way:

$$\mathrm{Var}_{\mathbb{P}_{N,\beta}}[\widetilde{\mathrm{A}}_{\ell_N}[\psi_{\mathrm{reg}}]]=\mathbb{E}_{\mathbb{P}_{N,\beta}}[\mathrm{Var}_{\mathbb{P}_{N,\beta}}[\widetilde{\mathrm{A}}_{\ell_N}[\psi_{\mathrm{reg}}]\mid x_1=x]]+\mathrm{Var}_{\mathbb{P}_{N,\beta}}[\mathbb{E}_{\mathbb{P}_{N,\beta}}[\widetilde{\mathrm{A}}_{\ell_N}[\psi_{\mathrm{reg}}]\mid x_1=x]].$$

We claim that

$$\mathrm{Var}_{\mathbb{P}_{N,\beta}}[\mathbb{E}_{\mathbb{P}_{N,\beta}}[\widetilde{\mathrm{A}}_{\ell_N}[\psi_{\mathrm{reg}}]\mid x_1]]\leqslant C(\beta)((N\ell_N)^{2(1-s)}+(N\ell_N)^{2\max\alpha_l}). \tag{3.135}$$

The proof of (3.135) uses the fact that the law of $x_2,\ldots,x_N$ under $\mathbb{P}_{N,\beta}(\cdot\mid x_1=x)$ is the law of $x_2-x_1+x,\ldots,x_N-x_1+x$ under $\mathbb{P}_{N,\beta}$ as well as the rigidity estimate of Theorem 3.1.1. We postpone the details to the Appendix, see Lemma 3.7.6.

Step 6: convexification and reduction to a good event Let us first define a convexication of $\mathbb{P}_{N,\beta}$ by penalizing large nearest-neighbor gaps in the window I_N. We proceed as in Section 3.4. Let $\theta:\mathbb{R}^+\to\mathbb{R}^+$ be a smooth cutoff function such that $\theta(x)=|x|^2$ for $x>1$, $\theta=0$ on $[0,\frac{1}{2}]$ and $\theta''\geqslant 0$ on $\mathbb{R}^+$. Fix $\varepsilon>0$ and $x\in\mathbb{T}$. Define

$$\mathrm{F}=\sum_{i,i+1\in I_N}\theta\Big(\frac{N(x_{i+1}-x_i)}{(N\ell_N)^{\varepsilon}}\Big)$$

and the locally constrained Gibbs measure

$$\mathrm{d}\mathbb{Q}_{N,\beta}=\frac{1}{K_{N,\beta}}e^{-\beta(\mathcal{H}_N+\mathrm{F})}\mathbb{1}_{D_N}(X_N)\mathrm{d}X_N. \tag{3.136}$$

In view of Lemma 3.4.1 and Theorem 3.1.1, the total variation distance between $\mathbb{P}_{N,\beta}(\cdot\mid x_1=x)$ and $\mathbb{Q}_{N,\beta}(\cdot\mid x_1=x)$ satisfies

$$\mathrm{TV}(\mathbb{P}_{N,\beta}(\cdot\mid x_1=x),\mathbb{Q}_{N,\beta}(\cdot\mid x_1=x))\leqslant C(\beta)e^{-c(N\ell_N)^{\delta}}, \tag{3.137}$$

for some constants $C>0$ and $c>0$ depending on β and some constant $\delta>0$ depending on ε. The above estimate together with (3.135) can be summarized into

$$\mathrm{Var}_{\mathbb{P}_{N,\beta}}[\widetilde{\mathrm{A}}_{\ell_N}[\psi_{\mathrm{reg}}]]=\mathbb{E}_{\mathbb{P}_{N,\beta}}[\mathrm{Var}_{\mathbb{Q}_{N,\beta}(\cdot|x_1=x)}[\widetilde{\mathrm{A}}_{\ell_N}[\psi_{\mathrm{reg}}]\eta]]+O_\beta((N\ell_N)^{2(1-s)}+(N\ell_N)^{2\max\alpha_l}). \tag{3.138}$$

Recall the good event (3.123). Let $\tilde{\theta}:\mathbb{R}\to\mathbb{R}$ such that $\tilde{\theta}(x)=1$ for $|x|>1$, $\tilde{\theta}(x)=0$ for $|x|<\frac{1}{2}$ and $\tilde{\theta}(x)=2|x|-1$ for $\frac{1}{2}\leqslant|x|\leqslant 1$. Define the cutoff function

$$\eta=\prod_{i\in I_N}\prod_{-\frac{N}{2}\leqslant k\leqslant\frac{N}{2}:i+k\in I_N}\tilde{\theta}\Big(\frac{N(x_{i+k}-x_i)-k}{|k|^{\frac{s}{2}}(N\ell_N)^{\varepsilon}}\Big)\prod_{i\in I_N}\tilde{\theta}\Big(\frac{(N\ell_N)^{\varepsilon}}{N(x_{i+1}-x_i)}\Big) \tag{3.139}$$

By subadditivity one can write

$$\begin{aligned}\mathrm{Var}_{\mathbb{Q}_{N,\beta}(\cdot|x_1=x)}[\widetilde{\mathrm{A}}_{\ell_N}[\psi_{\mathrm{reg}}]]&\leqslant 2\mathbb{E}_{\mathbb{Q}_{N,\beta}(\cdot|x_1=x)}[(\eta\nabla\widetilde{\mathrm{A}}_{\ell_N}[\psi_{\mathrm{reg}}])(A_1^x)^{-1}(\eta\nabla\widetilde{\mathrm{A}}_{\ell_N})[\psi_{\mathrm{reg}}]]\\&\quad+2\mathbb{E}_{\mathbb{Q}_{N,\beta}(\cdot|x_1=x)}[(\nabla\eta\widetilde{\mathrm{A}}_{\ell_N}[\psi_{\mathrm{reg}}])(A_1^x)^{-1}(\nabla\eta\widetilde{\mathrm{A}}_{\ell_N}[\psi_{\mathrm{reg}}])].\end{aligned} \tag{3.140}$$

Let us split $\nabla\widetilde{\mathrm{A}}_{\ell_N}[\psi_{\mathrm{reg}}]$ into

$$\nabla\widetilde{\mathrm{A}}_{\ell_N}[\psi_{\mathrm{reg}}]=\mathrm{V}+\mathrm{W}$$

with V, W as in Lemma 3.5.2. Using subadditivity again we find

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}_{N,\beta}(\cdot|x_1=x)}[(\eta\nabla\widetilde{\mathrm{A}}_{\ell_N}[\psi_{\mathrm{reg}}])(A_1^x)^{-1}(\eta\nabla\widetilde{\mathrm{A}}_{\ell_N})[\psi_{\mathrm{reg}}]]&\leqslant 2(\mathbb{E}_{\mathbb{Q}_{N,\beta}(\cdot|x_1=x)}[(\eta\mathrm{V})\cdot(A_1^x)^{-1}(\eta\mathrm{V})]\\&\quad+2\mathbb{E}_{\mathbb{Q}_{N,\beta}(\cdot|x_1=x)}[(\eta\mathrm{W})\cdot(A_1^x)^{-1}(\eta\mathrm{W})]).\end{aligned} \tag{3.141}$$

Step 7: using Poincaré in gap coordinates for $\eta\mathrm{W}$ and $\widetilde{\mathrm{A}}_{\ell_N}[\psi_{\mathrm{reg}}]\nabla\eta$ To estimate the Dirichlet energy of $\eta\mathrm{W}$ one can take advantage of the fact that $\mathbb{Q}_{N,\beta}$ is uniformly log-concave in gap coordinates. Indeed proceeding as in the proof of Lemma 3.3.8 one can write

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}_{N,\beta}(\cdot|x_1=x)}[\eta\mathrm{W}\cdot(A_1^x)^{-1}(\eta\mathrm{W})] &\leqslant \beta^{-1}\mathbb{E}_{\mathbb{Q}_{N,\beta}(\cdot|x_1=x)}[\eta^2\mathrm{W}\cdot(\nabla^2(\mathcal{H}_N+\mathrm{F}))^{-1}\mathrm{W}]\\ &= -\beta^{-1}\mathbb{E}_{\mathbb{Q}_{N,\beta}}\Big[\min_{U_N\in\mathbb{R}^N} U_N\cdot\nabla^2(\mathcal{H}_N+\mathrm{F})U_N-2(\eta\mathrm{W})\cdot U_N\Big]. \end{aligned}\tag{3.142}$$

By definition of $\mathbb{Q}_{N,\beta}$, for all $U_N\in\mathbb{R}^N$,

$$U_N\cdot\nabla^2(\mathcal{H}_N+\mathrm{F})U_N\geqslant (N\ell_N)^{-(s+2)\varepsilon}\sum_{i=1}^N(N(u_{i+1}-u_i))^2$$

and besides

$$\eta\mathrm{W}\cdot U_N=-\sum_{i=1}^N\eta\tilde{\mathrm{W}}_iN(u_{i+1}-u_i)$$

with $|\eta\tilde{\mathrm{W}}|$ satisfying (3.126). Inserting these into (3.142) we deduce that there exist constants $C>0$ and $\kappa>0$ such that

$$\mathbb{E}_{\mathbb{Q}_{N,\beta}(\cdot|x_1=x)}[\eta\mathrm{W}\cdot(A_1^x)^{-1}(\eta\mathrm{W})]\leqslant C\beta^{-1}(N\ell_N)^{\kappa\varepsilon}\mathbb{E}_{\mathbb{Q}_{N,\beta}(\cdot|x_1=x)}[\eta^2|\tilde{\mathrm{W}}|^2]\leqslant C\beta^{-1}(N\ell_N)^{\kappa\varepsilon+\max(1,2\max\alpha_l))}.$$

Similarly for the vector-field $\widetilde{\mathrm{A}}_{\ell_N}[\psi_{\mathrm{reg}}]\nabla\eta$, one gets

$$\mathbb{E}_{\mathbb{Q}_{N,\beta}(\cdot|x_1=x)}[(\nabla\eta\widetilde{\mathrm{A}}_{\ell_N}[\psi_{\mathrm{reg}}])(A_1^x)^{-1}(\nabla\eta\widetilde{\mathrm{A}}_{\ell_N}[\psi_{\mathrm{reg}}])]\leqslant C\beta^{-1}(N\ell_N)^{\kappa\varepsilon}\mathbb{E}_{\mathbb{Q}_{N,\beta}(\cdot|x_1=x)}[|\widetilde{\mathrm{A}}_{\ell_N}[\psi_{\mathrm{reg}}]\nabla\eta|^2].$$

Since $\nabla\eta=0$ on $\mathcal{A}^c$ with $\widetilde{\mathrm{A}}_{\ell_N}[\psi_{\mathrm{reg}}]$ uniformly bounded on $\mathcal{A}$ by $(N\ell_N)^\kappa$ for some $\kappa>0$ we find that

$$\mathbb{E}_{\mathbb{Q}_{N,\beta}(\cdot|x_1=x)}[|\widetilde{\mathrm{A}}_{\ell_N}[\psi_{\mathrm{reg}}]\nabla\eta|^2]\leqslant C(N\ell_N)^\kappa\mathbb{E}_{\mathbb{Q}_{N,\beta}(\cdot|x_1=x)}[|\nabla\eta|^2]\leqslant C(\beta)e^{-c(\beta)(N\ell_N)^\delta},\tag{3.143}$$

where we have used Theorem 3.1.1 and Lemma 3.4.4 in the last inequality.

Step 8: using the comparison principle for V By Lemma 3.5.2, there exist constants $C>0$ and $\kappa>0$ such that for each $i\in I_N$,

$$\begin{aligned}\sup|\eta\mathrm{V}_i|\leqslant C(N\ell_N)^{\kappa\varepsilon}&\sum_{l:\alpha_l>\frac{s}{2}}(N\ell_N)^{\alpha_l}N^{-\frac{s}{2}}\frac{1}{(|x_i-\ell_Na_l|\vee\frac{(N\ell_N)^{\varepsilon'}}{N})^{1+\frac{s}{2}}}\mathbb{1}_{|x_i|\leqslant 2\ell_N}\\ &+C(N\ell_N)^{\kappa\varepsilon}\sum_{l=1}^p\ell_N^{\alpha_l}\frac{1}{(|x_i-\ell_Na_l|\vee\frac{1}{N})^{1+\alpha_l}}\mathbb{1}_{|x_i|\leqslant 2\ell_N}+C(N\ell_N)^{\kappa\varepsilon}N^{-\frac{s}{2}}\frac{1}{(\ell_N+|x_i|)^{1+\frac{s}{2}}}\\ &+\frac{C}{(N\ell_N)^{\gamma s}}\frac{\ell_N^{-1}}{(\ell_N^{-1}|x_i|+1)^{2-s}}+C(N\ell_N)^{\kappa\varepsilon-(2-s)(1-\gamma)}N^{-\frac{s}{2}}\frac{1}{(d(x_i,\partial I)\wedge\frac{1}{N})^{1+\frac{s}{2}}}.\end{aligned}\tag{3.144}$$

The main term in the right-hand side of (3.144) corresponds to the gradient of a linear statistics that we now define. Let $\zeta_N:\mathbb{T}\to\mathbb{R}$ piecewise smooth such that

$$\begin{aligned}\zeta_N':x\in\mathbb{T}\mapsto \ell_N^{-1}\eta_N(\ell_N^{-1}x)+N^{-\frac{s}{2}}\frac{1}{(|x|+\ell_N)^{1+\frac{s}{2}}}\\ +C(N\ell_N)^{\kappa\varepsilon}\sum_{l:\alpha_l>\frac{s}{2}}(N\ell_N)^{\alpha_l}N^{-\frac{s}{2}}\frac{1}{(|x-a_l\ell_N|\vee\frac{(N\ell_N)^{\varepsilon'}}{N})^{1+\frac{s}{2}}}.\end{aligned}\tag{3.145}$$

Note that ζ_N is a discontinuous function, increasing on $[0,1)$. There exist constants $C>0$ and $\kappa>0$ such that

$$|\mathrm{V}|\leqslant C(N\ell_N)^{\kappa\varepsilon}\nabla\mathrm{Fluct}_N[\zeta_N].$$

One can now apply the comparison principle of Lemma 3.3.5 and more precisely its consequence stated in Lemma 3.3.6 to get

$$\mathbb{E}_{\mathbb{Q}_{N,\beta}(\cdot|x_1=x)}[(\eta\mathrm{V})\cdot(A_1^x)^{-1}(\eta\mathrm{V})]\leqslant 4C^2(N\ell_N)^{2\kappa\varepsilon}\,\mathrm{Var}_{\mathbb{Q}_{N,\beta}(\cdot|x_1=x)}\Big[\sum_{i=1}^N\zeta_N(x_i)\Big].$$

Using (3.137), we can write

$$\mathrm{Var}_{\mathbb{Q}_{N,\beta}(\cdot|x_1=x)}[\mathrm{Fluct}_N[\zeta_N]]\leqslant\mathrm{Var}_{\mathbb{P}_{N,\beta}(\cdot|x_1=x)}[\mathrm{Fluct}_N[\zeta_N]]+O_\beta(e^{-c(\beta)(N\ell_N)^\kappa}).\tag{3.146}$$

Moreover,

$$\mathbb{E}_{\mathbb{P}_{N,\beta}}[\mathrm{Var}_{\mathbb{Q}_{N,\beta}(\cdot|x_1=x)}[\mathrm{Fluct}_N[\zeta_N]]\leqslant\mathrm{Var}_{\mathbb{P}_{N,\beta}}[\mathrm{Fluct}_N[\zeta_N]]+O_\beta(e^{-c(\beta)(N\ell_N)^\kappa}).$$

The test-function ζ_N is the sum of the five test-functions appearing when integrating (3.145). For the two first terms, one may use the sub-Poissonian estimate of Lemma 3.3.9. For the three other terms, this estimate is not precise enough, and one shall study the variance of the corresponding linear statistics separately. Let us define

$$f_\eta(x):=f(x)\mathbb{1}_{x\notin(0,\eta)}+f'(\eta)(x-\eta)+f(\eta),\quad\text{where}\quad f(x):=x^{-\frac{s}{2}},\quad\eta:=1/N^{1-\varepsilon'}.\tag{3.147}$$

By Lemma 3.3.9, we have

$$\begin{aligned}&\mathrm{Var}_{\mathbb{Q}_{N,\beta}(\cdot|x_1=x)}\Big[\sum_{i=1}^N\zeta_N(x_i)\Big]\\&\qquad\leqslant C(\beta)(N\ell_N)^{\kappa\varepsilon}(N\ell_N)^{\max(1,2\max\alpha_l)}+C(\beta)(N\ell_N)^{\kappa\varepsilon}N^{-s}\,\mathrm{Var}_{\mathbb{P}_{N,\beta}}\Big[\sum_{i=1}^N f_\eta(x_i)\Big].\end{aligned}$$

Combining the above estimates one gets

$$\mathbb{E}_{\mathbb{Q}_{N,\beta}(\cdot|x_1=x)}[\mathrm{V}\cdot(A_1^x)^{-1}\mathrm{V}]\leqslant C(\beta)(N\ell_N)^{\kappa\varepsilon}(N\ell_N)^{\max(1,2\max\alpha_l)}+C(\beta)(N\ell_N)^{\kappa\varepsilon}N^{-s}\,\mathrm{Var}_{\mathbb{P}_{N,\beta}}\Big[\sum_{i=1}^N f_\eta(x_i)\Big].\tag{3.148}$$

There remains to bound from above the variance of the singular linear statistics f_η defined in (3.147). Let us observe that the test-function f is the critical case of Assumptions 3.1.1, since f is not in $H^{\frac{1-s}{2}}$. Let us highlight that the cutoff $\ell=1/(N\ell_N)^{1-\varepsilon'}$ has been added to circumvent this criticality. Indeed it is enough to prove that for any fixed $\varepsilon'>0$, there exists a constant $C(\beta)>0$ such that

$$\mathrm{Var}_{\mathbb{P}_{N,\beta}}[\mathrm{Fluct}_N[f_\eta]]\leqslant C(\beta)N^s.\tag{3.149}$$

Having established (3.149) will show that the right-hand side of (3.148) is negligible.

Step 9: bound on the auxiliary linear statistics Let f_η be as in (3.147). To establish (3.149), we apply to f_η the first steps of the current proof. Let $\phi \in \mathcal{C}^\varepsilon(\mathbb{T},\mathbb{R})$ be such that

$$\phi' = -\frac{1}{2\beta c_s}(-\Delta)^{\frac{1-s}{2}} f_\eta \quad \text{with} \quad \int \phi = 0.$$

Observe that by Lemma 3.2.5, the map ϕ satisfies

$$|\phi'| \leqslant C(\beta)\frac{1}{|x|^{1-\frac{s}{2}} \vee \frac{1}{N^{1-\varepsilon'}}}.$$

Denote

$$\Psi_N : X_N \in D_N \mapsto (\phi(x_1),\ldots,\phi(x_N)).$$

By Proposition 3.5.1 we have

$$\begin{aligned}\mathrm{Var}_{\mathbb{P}_{N,\beta}}[\mathrm{Fluct}_N[f_\eta]] &= -\frac{1}{N^{2(1-s)}}\mathbb{E}_{\mathbb{P}_{N,\beta}}[\beta\Psi_N\cdot\nabla^2\mathcal{H}_N\Psi_N + |\nabla^2\Psi_N|^2] \\ &\quad + \frac{2}{N^{1-s}}\mathbb{E}_{\mathbb{P}_{N,\beta}}[\mathrm{Fluct}_N[f_\eta'\phi] + \frac{1}{N^{2(1-s)}}\mathrm{Var}_{\mathbb{P}_{N,\beta}}[\beta\mathrm{A}[\phi] - \mathrm{Fluct}_N[\phi']].\end{aligned}$$

By Lemma 3.3.9, there holds

$$\mathrm{Var}_{\mathbb{P}_{N,\beta}}[\mathrm{Fluct}_N[\phi']] \leqslant N|\phi'|_{L^2}^2 = O(N^{2-s}).$$

Expanding the mean-field terms like in Step 4, one deduces that for all $\varepsilon > 0$,

$$\begin{aligned}\mathrm{Var}_{\mathbb{P}_{N,\beta}}[\mathrm{Fluct}_N[f_\eta]] &\leqslant C(\beta)\Big(N^s + \frac{1}{N^{2(1-s)}}\mathrm{Var}_{\mathbb{P}_{N,\beta}}[\mathrm{A}[\phi]] + N^{1-2(1-s)+\varepsilon}|\phi'|_{L^2}^2\Big) \\ &\leqslant C(\beta)\Big(N^s + \frac{1}{N^{2(1-s)}}\mathrm{Var}_{\mathbb{P}_{N,\beta}}[\mathrm{A}[\phi]] + N^{s+\varepsilon-\varepsilon'(1-s)}\Big) \\ &\leqslant C(\beta)\Big(N^s + \frac{1}{N^{2(1-s)}}\mathrm{Var}_{\mathbb{P}_{N,\beta}}[\mathrm{A}[\phi]]\Big),\end{aligned} \tag{3.150}$$

where the last inequality is obtained by choosing ε small enough compared to ε'. For $I_N = \{1,\ldots,N\}$ and $\ell_N = 1$ and $\mathbb{Q}_{N,\beta}$ be the constrained Gibbs measure (3.136) for $I_N = \{1,\ldots,N\}$ and $\ell_N = 1$. Arguing like in Step 4, one justifies that

$$\mathrm{Var}_{\mathbb{P}_{N,\beta}}[\mathrm{A}[\phi]] = \mathbb{E}_{\mathbb{P}_{N,\beta}}[\mathrm{Var}_{\mathbb{Q}_{N,\beta}}[\mathrm{A}[\phi] \mid x_1 = x]] + O_\beta(N^{2-s-\varepsilon'(1-s)+\kappa\varepsilon}).$$

Let $x \in \mathbb{T}$ and η^0 the cutoff (3.139) for $I_N = \{1,\ldots,N\}$. Let us decompose $\nabla\mathrm{A}[\phi]$ into

$$\nabla\mathrm{A}[\phi] = \mathrm{V}^0 + \mathrm{W}^0$$

with $\mathrm{V}^0, \mathrm{W}^0$ as in Lemma 3.5.2. By subadditivity and proceeding as in Step 7, one can write

$$\begin{aligned}\mathrm{Var}_{\mathbb{Q}_{N,\beta}(\cdot|x_1=x)}[\mathrm{A}[\phi]] \leqslant 3(\mathbb{E}_{\mathbb{Q}_{N,\beta}(\cdot|x_1=x)}[(\eta^0\mathrm{V}^0)\cdot(A_1^x)^{-1}(\eta^0\mathrm{V}^0)] + \mathbb{E}_{\mathbb{Q}_{N,\beta}(\cdot|x_1=x)}[(\eta^0\mathrm{W}^0)\cdot(A_1^x)^{-1}(\eta^0\mathrm{W}^0)]) \\ + O_\beta(e^{-c(N\ell_N)^\kappa}).\end{aligned}$$

In view of Lemma 3.5.2, there exists $\tilde{\mathrm{W}}^0 \in L^2(\{1,\ldots,N\}, H^1(\mathbb{P}_{N,\beta}))$ such that for all $U_N \in \mathbb{R}^N$,

$$\mathrm{W}^0 \cdot U_N = -\sum_{i=1}^N \tilde{\mathrm{W}}_i^0 N(u_{i+1} - u_i)$$

with
$$\sup|\eta^0\tilde{\mathrm{W}}^0|^2 \leqslant CN^{\kappa\varepsilon+1}\int_{\frac{1}{N^{1-\varepsilon'}}}^1 \frac{1}{x^{2-s}}\mathrm{d}x \leqslant CN^{\kappa\varepsilon+2-s-\varepsilon'(1-s)}.$$

Therefore proceeding as in Step 7, one finds
$$\mathbb{E}_{\mathbb{Q}_{N,\beta}(\cdot|x_1=x)}[(\eta^0\mathrm{W}^0)\cdot(A_1^x)^{-1}(\eta^0\mathrm{W}^0)] \leqslant C(\beta)N^{\kappa\varepsilon+2-s-\varepsilon'(1-s)}. \tag{3.151}$$

Now for $\eta^0\mathrm{V}^0$, following the line of reasoning of Step 8, we obtain
$$\mathbb{E}_{\mathbb{Q}_{N,\beta}(\cdot|x_1=x)}[\eta^0\mathrm{V}^0\cdot(A_1^x)^{-1}(\eta^0\mathrm{V}^0)] \leqslant C(\beta)N^{\kappa\varepsilon-s}\,\mathrm{Var}_{\mathbb{P}_{N,\beta}}[\mathrm{Fluct}_N[f]] + C(\beta)N^{\kappa\varepsilon+2-s-\varepsilon'(1-s)}, \tag{3.152}$$
with f defined in (3.147). At this point, the sub-poissonian estimate of Lemma 3.3.9 is sharp enough: one can write
$$\mathrm{Var}_{\mathbb{P}_{N,\beta}}[\mathrm{Fluct}_N[f]] = O_\beta(N).$$
Inserting this into (3.152) using (3.151) and (3.150) and choosing ε small enough, one obtains
$$\mathrm{Var}_{\mathbb{P}_{N,\beta}}[\mathrm{Fluct}_N[f_\eta]] = O_\beta(N^{s+\kappa\varepsilon-\varepsilon'(1-s)}).$$
By choosing ε small enough, we deduce that there exists a constant C depending on ε' such that
$$\mathrm{Var}_{\mathbb{P}_{N,\beta}}[\mathrm{Fluct}_N[f_\eta]] \leqslant C(\beta)N^s. \tag{3.153}$$

Step 10: conclusion. Inserting (3.153) into (3.148) yields
$$\mathbb{E}_{\mathbb{Q}_{N,\beta}(\cdot|x_1=x)}[(\eta\mathrm{V})\cdot(A_1^x)^{-1}(\eta\mathrm{V})] \leqslant C(\beta)(N\ell_N)^{\kappa\varepsilon+\max(1,2\max\alpha_l)}. \tag{3.154}$$
Therefore recalling (3.134), (3.138) and using (3.141) one gets
$$\mathrm{Var}_{\mathbb{P}_{N,\beta}}[\mathrm{A}_{\ell_N}[\psi_{\mathrm{reg}}]] \leqslant C(\beta)(N\ell_N)^{\kappa\varepsilon+\max(1,2\max\alpha_l)}+C(\beta)(N\ell_N)^{2(1-s)}(1+(N\ell_N)^{s+2\max\alpha_l-(2-s)}). \tag{3.155}$$
Inserting this into (3.129) and using (3.131), (3.132) we conclude
$$\mathrm{Var}_{\mathbb{P}_{N,\beta}}[\mathrm{Fluct}_N[\xi_{\mathrm{reg}}(\ell_N^{-1}\cdot)]] = \sigma_{\xi_{\mathrm{reg}}}^2(N\ell_N)^s + O_\beta((N\ell_N)^{\kappa\varepsilon+\max(1,2\max\alpha_l)-2(1-s)}).$$
Finally one can check that
$$\sigma_{\xi_{\mathrm{reg}}}^2 = \sigma_\xi^2 + O\Big(\frac{1}{N\ell_N}+\frac{1}{(N\ell_N)^{1-\frac{s}{2}-\max\alpha_l+\kappa\varepsilon'}}\Big). \tag{3.156}$$

By choosing ε' small enough we can absorb the error terms (3.128) and (3.156) into the error arising from (3.155). This concludes the proof of Theorem 3.1.2. □

3.6 Central Limit Theorem

3.6.1 Proof of the CLT

In this subsection we give a proof of the CLT for singular linear statistics stated in Theorem 3.1.3. One could proceed by considering the Laplace transform of the fluctuations but instead we deduce the CLT from an application of Stein's method. The starting point of the method shares many similarities with [173]. In the sequel, we leverage on variance estimates obtained in the last subsection and obtain a CLT in a weak topology. We believe that one could obtain with the same approach a *local* CLT for smooth test-functions.

Proof of Theorem 3.1.3. Let ξ be a test-function satisfying Assumptions 3.1.1 and $\{\ell_N\}$ such that $\ell_N \gg \frac{1}{N}$. Assume either that ξ is supported on $(-\frac{1}{2}, \frac{1}{2})$ or that $\ell_N = 1$. Define

$$G_N = (N\ell_N)^{-\frac{s}{2}}\mathrm{Fluct}_N[\xi(\ell_N^{-1}\cdot)].$$

Let $\eta : \mathbb{R} \to \mathbb{R}$ be a smooth function. The principle of Stein method is to prove that up to a small error term Error_N, the following integration by parts formula holds:

$$\mathbb{E}_{\mathbb{P}_{N,\beta}}[\eta(G_N)G_N] = \sigma_\xi^2\mathbb{E}_{\mathbb{P}_{N,\beta}}[\eta'(G_N)] + \mathsf{Error}_N, \tag{3.157}$$

with

$$\sigma_\xi^2 = \frac{1}{2\beta c_s}|\xi|^2_{H^{\frac{1-s}{2}}}. \tag{3.158}$$

Indeed (3.157) controls a certain distance to a normal distribution. Let $Z \sim \mathcal{N}(0, \sigma_\xi^2)$. If $h : \mathbb{R} \to \mathbb{R}$ is smooth enough, then one can solve the ODE

$$\eta'(x) - \eta(x)x = h(x) - \mathbb{E}[h(Z)], \quad x \in \mathbb{R}.$$

The fundamental observation is that

$$\mathbb{E}_{\mathbb{P}_{N,\beta}}[h(G_N)] - \mathbb{E}[h(Z)] = \mathbb{E}_{\mathbb{P}_{N,\beta}}[\eta(G_N)G_N - \sigma_\xi^2\eta'(G_N)].$$

This allows one to prove the following standard inequality (see for instance [85]):

$$\sup_{t\in\mathbb{R}} |\mathbb{P}_{N,\beta}(G_N \leqslant t) - \mathbb{P}(Z \leqslant t)| \leqslant 2\Big(\sup_{\eta\in\mathcal{D}} |\mathbb{E}_{\mathbb{P}_{N,\beta}}[\eta'(G_N) - G_N\eta(G_N)]|\Big)^{\frac{1}{2}},$$

where $\mathcal{D}$ is the set of functions $\eta : \mathbb{R} \to \mathbb{R}$ such that $|\eta|_\infty \leqslant 1$, $|\eta'|_\infty \leqslant 1$. Let us now prove that (3.157) holds for $\eta \in \mathcal{D}$ up to a small error term.

Step 1: regularization Let $\varepsilon' > 0$. Consider ξ_{reg} the regularization of ξ at scale $1/(N\ell_N)^{(1-\varepsilon')}$ as in (3.127). Set

$$\tilde{g}_N = (N\ell_N)^{-\frac{s}{2}}\mathrm{Fluct}_N[\xi_{\mathrm{reg}}(\ell_N^{-1}\cdot)]$$

Observe that

$$\mathbb{E}_{\mathbb{P}_{N,\beta}}[\eta(G_N)G_N - \eta'(G_N)] = \mathbb{E}_{\mathbb{P}_{N,\beta}}[\eta(\tilde{g}_N)\tilde{g}_N - \eta'(\tilde{g}_N)] + O_\beta(1 + (N\ell_N)^{\max\alpha_l - (1-\frac{s}{2})}). \tag{3.159}$$

In the sequel we establish that $\tilde{g}_N$ satisfies the approximate Gaussian formula (3.157).

Step 2: main computation Let us first discuss the relation between (3.159) and the H.-S. equation $A_1\nabla\phi = \nabla\tilde{G}_N$. The discussion is only formal since we have not proved the well-posedness of the H.-S. equation under $\mathbb{P}_{N,\beta}$ for test-functions which are not functions of the gaps. Let $\nabla\phi$ be the solution of

$$\begin{cases} A_1\nabla\phi = \nabla\tilde{G}_N & \text{on } D_N \\ \nabla\phi\cdot\vec{n} = 0 & \text{on } \partial D_N. \end{cases}$$

By integration by parts under $\mathbb{P}_{N,\beta}$, one can write

$$\mathbb{E}_{\mathbb{P}_{N,\beta}}[\eta(\tilde{g}_N)\tilde{g}_N] = \mathbb{E}_{\mathbb{P}_{N,\beta}}[\eta(\tilde{G}_N)\mathcal{L}\tilde{G}_N] = \mathbb{E}_{\mathbb{P}_{N,\beta}}[\eta'(\tilde{G}_N)\nabla\tilde{G}_N\cdot\nabla\phi]. \tag{3.160}$$

The point is to show that under $\mathbb{P}_{N,\beta}$, the quantity $\nabla \tilde{G}_N \cdot \nabla \phi$ concentrates around a constant, which turns out to be σ_ξ^2. Let $\psi_{\mathrm{reg}} \in \mathcal{C}^2(\ell_N^{-1}\mathbb{T}, \mathbb{R})$ satisfying

$$\psi_{\mathrm{reg}}' = -\frac{1}{2\beta c_s}\ell_N^{1-s}(-\Delta)^{\frac{1-s}{2}}(\xi_{\mathrm{reg}}(\ell_N^{-1}\cdot))(\ell_N \cdot) \quad \text{and} \quad \int \psi_{\mathrm{reg}} = 0.$$

By Remark 11, $\nabla \phi$ may be decomposed into

$$\nabla \phi = \frac{1}{(N\ell_N)^{1-\frac{s}{2}}}(\ell_N \psi_{\mathrm{reg}}(\ell_N^{-1}x_1), \dots, \ell_N \psi_{\mathrm{reg}}(\ell_N^{-1}x_N)) \\ + \frac{1}{(N\ell_N)^{1-\frac{s}{2}}} A_1^{-1}[\beta \nabla \mathrm{A}_{\ell_N}[\psi_{\mathrm{reg}}] - \nabla \mathrm{Fluct}_N[\psi_{\mathrm{reg}}'(\ell_N^{-1}\cdot)], \quad (3.161)$$

with $\mathrm{A}_{\ell_N}[\psi_{\mathrm{reg}}]$ defined in (3.112). Inserting (3.161) into (3.160) leads to

$$\mathbb{E}_{\mathbb{P}_{N,\beta}}[\eta(\tilde{G}_N)\tilde{G}_N] = \mathbb{E}_{\mathbb{P}_{N,\beta}}\Big[\eta'(\tilde{G}_N)\sum_{i=1}^N \frac{1}{N\ell_N}\psi_{\mathrm{reg}}(\ell_N^{-1}x_i)\xi_{\mathrm{reg}}'(\ell_N^{-1}x_i)\Big] \\ + \frac{1}{(N\ell_N)^{1-\frac{s}{2}}}\mathbb{E}_{\mathbb{P}_{N,\beta}}\Big[\eta'(\tilde{G}_N)\nabla \tilde{G}_N \cdot A_1^{-1}(\beta \nabla \mathrm{A}_{\ell_N}[\psi_{\mathrm{reg}}] - \nabla \mathrm{Fluct}_N[\psi_{\mathrm{reg}}'(\ell_N^{-1}\cdot)])\Big]$$

One can reformulate this into

$$\mathbb{E}_{\mathbb{P}_{N,\beta}}[\eta(\tilde{G}_N)\tilde{G}_N] = \mathbb{E}_{\mathbb{P}_{N,\beta}}\Big[\eta'(\tilde{G}_N)\sum_{i=1}^N \frac{1}{N\ell_N}\psi_{\mathrm{reg}}(\ell_N^{-1}x_i)\xi_{\mathrm{reg}}'(\ell_N^{-1}x_i)\Big] \\ + \frac{1}{(N\ell_N)^{1-\frac{s}{2}}}\mathrm{Cov}_{\mathbb{P}_{N,\beta}}[\eta(\tilde{G}_N), (\beta \mathrm{A}_{\ell_N}[\psi_{\mathrm{reg}}] - \mathrm{Fluct}_N[\psi_{\mathrm{reg}}'(\ell_N^{-1}\cdot)]]$$

and the above identity follows from a rigorous integration by parts, as in the proof of Proposition 3.5.1. It follows that $\tilde{g}_N$ satisfies the approximate Gaussian identity

$$\mathbb{E}_{\mathbb{P}_{N,\beta}}[\eta(\tilde{G}_N)\tilde{G}_N] = \sigma_\xi^2 \mathbb{E}_{\mathbb{P}_{N,\beta}}[\eta'(\tilde{G}_N)] + \mathrm{Error}_N^1 + \mathrm{Error}_N^2, \quad (3.162)$$

with

$$\mathrm{Error}_N^1 = \mathbb{E}_{\mathbb{P}_{N,\beta}}\Big[\eta'(\tilde{G}_N)\Big(\sum_{i=1}^N \frac{1}{N\ell_N}\psi_{\mathrm{reg}}(\ell_N^{-1}x_i)\xi_{\mathrm{reg}}'(\ell_N^{-1}x_i) - \sigma_\xi^2\Big)\Big],$$

$$\mathrm{Error}_N^2 = \frac{1}{(N\ell_N)^{1-\frac{s}{2}}}\mathrm{Cov}_{\mathbb{P}_{N,\beta}}[\eta(\tilde{G}_N), (\beta \mathrm{A}_{\ell_N}[\psi_{\mathrm{reg}}] - \mathrm{Fluct}_N[\psi_{\mathrm{reg}}'(\ell_N^{-1}\cdot)]].$$

Step 3: the error term Error_N^1 One can bound Error_N^1 by

$$|\mathrm{Error}_N^1| \leqslant |\eta'|_\infty \frac{1}{N\ell_N}\mathbb{E}_{\mathbb{P}_{N,\beta}}\Big[|\sum_{i=1}^N \xi_{\mathrm{reg}}'(\ell_N^{-1}x_i)\psi_{\mathrm{reg}}(\ell_N^{-1}x_i) - N\ell_N\sigma_\xi^2|\Big].$$

Since $\psi_{\mathrm{reg}}' = -\frac{1}{2\beta c_s}(-\Delta)^{\frac{1-s}{2}}\xi_{\mathrm{reg}}$, observe that

$$\mathbb{E}_{\mathbb{P}_{N,\beta}}\Big[\sum_{i=1}^N \xi_{\mathrm{reg}}'(\ell_N^{-1}x_i)\psi_{\mathrm{reg}}(\ell_N^{-1}x_i)\Big] = N\ell_N\sigma_{\xi_{\mathrm{reg}}}^2,$$

with $\sigma_{\xi_{\rm reg}}$ as in (3.158). Moreover from the definition of $\xi_{\rm reg}$,

$$\sigma_{\xi_{\rm reg}}^2 = \sigma_\xi^2 + O\Big(\frac{1}{N\ell_N} + \frac{1}{(N\ell_N)^{1-\frac{s}{2}-\max\alpha_l+\kappa\varepsilon'}}\Big).$$

We therefore obtain the following bias-variance decomposition:

$$|\mathsf{Error}_N^1| \leqslant C|\eta'|_\infty \frac{1}{N\ell_N}\Big(1+\mathrm{Var}_{\mathbb{P}_{N,\beta}}\Big[\sum_{i=1}^N \xi_{\rm reg}'(\ell_N^{-1}x_i)\psi_{\rm reg}(\ell_N^{-1}x_i)\Big]^{\frac{1}{2}}\Big).$$

By the sub-poissonian estimate of Lemma 3.3.9,

$$\mathrm{Var}_{\mathbb{P}_{N,\beta}}\Big[\sum_{i=1}^N \xi_{\rm reg}'(\ell_N^{-1}x_i)\psi_{\rm reg}(\ell_N^{-1}x_i)\Big]^{\frac{1}{2}} \leqslant (N\ell_N)^{\frac{1}{2}}|\psi_{\rm reg}\ \xi_{\rm reg}'|_{L^2}.$$

By Assumptions 3.1.1, $\psi_{\rm reg}$ is of order $O(|x-a|^{1-\max\alpha_l})$ around a singularity a of $\xi_{\rm reg}$ while $\xi_{\rm reg}'$ grows in $O(|x-a|^{-\max_l\alpha_l-s})$. It follows that

$$|\psi_{\rm reg}\ \xi_{\rm reg}'|_{L^2}^2 \leqslant C\Big(1+\int_{\frac{1}{N\ell_N}}^1 \frac{1}{|x|^{4\max\alpha_l-2+2s}}\mathrm{d}x\Big) \leqslant C(1+(N\ell_N)^{4\max\alpha_l+2s-3}).$$

Hence

$$\mathbb{E}_{\mathbb{P}_{N,\beta}}\Big[|\sum_{i=1}^N \xi_{\rm reg}'(\ell_N^{-1}x_i)\psi_{\rm reg}(\ell_N^{-1}x_i) - N\ell_N\sigma_\xi^2|\Big] \leqslant C(N\ell_N)^{\frac{1}{2}} + C(N\ell_N)^{2\max\alpha_l+s-1}.$$

and

$$\mathsf{Error}_N^1 = O((N\ell_N)^{-\frac{1-s}{2}} + (N\ell_N)^{-(2-2\max\alpha_l-s)}). \tag{3.163}$$

Since $\max\alpha_l < 1-\frac{s}{2}$, note that $\mathsf{Error}_N^1 = o_{N\ell_N}(1)$.

Step 4: the error term Error_N^2 To upper bound Error_N^2, one can write for instance

$$|\mathsf{Error}_N^2| \leqslant \frac{2\beta}{(N\ell_N)^{1-\frac{s}{2}}}|\eta|_\infty \mathrm{Var}_{\mathbb{P}_{N,\beta}}[\mathrm{A}_{\ell_N}[\psi_{\rm reg}]]^{\frac{1}{2}} + \frac{2}{(N\ell_N)^{1-\frac{s}{2}}}|\eta|_\infty \mathrm{Var}_{\mathbb{P}_{N,\beta}}[\mathrm{Fluct}_N[\psi_{\rm reg}'(\ell_N^{-1}\cdot)]]^{\frac{1}{2}}. \tag{3.164}$$

A quantitative bound on the variance of $\mathrm{A}_\ell[\psi_{\rm reg}]$ has been obtained in the proof of Theorem 3.1.2. The estimate (3.155) asserts that

$$\mathrm{Var}_{\mathbb{P}_{N,\beta}}[\mathrm{A}_{\ell_N}[\psi_{\rm reg}]] \leqslant C(\beta)(N\ell_N)^{\kappa\varepsilon+\max(1,2\max\alpha_l)}.$$

Moreover from the sub-poissonian of Lemma 3.3.9,

$$\mathrm{Var}_{\mathbb{P}_{N,\beta}}[\mathrm{Fluct}_N[\psi_{\rm reg}'(\ell_N^{-1}\cdot)] \leqslant N\ell_N\int(\psi_{\rm reg}')^2 \leqslant C(\beta)(N\ell_N)^{\max(1,2\max\alpha_l)}.$$

It follows that

$$|\mathsf{Error}_N^2| \leqslant \frac{C(\beta)}{(N\ell_N)^{\frac{1-s}{2}}}|\eta|_\infty(N\ell_N)^{\frac{\kappa\varepsilon}{2}+\max(\frac{1}{2},\max\alpha_l)}. \tag{3.165}$$

Step 5: conclusion Inserting (3.163), (3.165) into (3.162) and using (3.159) one obtains

$$|\mathbb{E}_{\mathbb{P}_{N,\beta}}[\eta(G_N)G_N - \sigma_\xi^2\eta'(G_N)]| \leqslant C\Big((N\ell_N)^{-(1-\frac{s}{2}-\max\alpha_l-\kappa\varepsilon)} + (N\ell_N)^{-\frac{1-s}{2}}\Big).$$

Since $\max\alpha_l < 1-\frac{s}{2}$, this error term is $o_{N\ell_N}(1)$ and this concludes the proof of Theorem 3.1.3. □

3.6.2 Proof of Corollary 3.1.4

Proof of Corollary 3.1.4. By Lemma 3.2.3, the function $\xi := \mathbb{1}_{(-a,a)}$ satisfies Assumptions 3.1.1 and one may apply Theorem 3.1.3. Let us define

$$\psi = -\frac{1}{2c_s}(-\Delta)^{\frac{1-s}{2}}\xi.$$

By integration by parts, the asymptotic variance σ_ξ^2 may be expressed as

$$\sigma_\xi^2 = \frac{1}{2\beta c_s}|\xi|^2_{H^{\frac{1-s}{2}}} = -\frac{1}{\beta}\int_{(-a,a)}\psi' = -\frac{1}{\beta}(\psi(a)-\psi(-a)).$$

Furthermore, from the explicit computation of Lemma 3.2.3, we have

$$\psi(x) = -\frac{\mathrm{cotan}(\frac{\pi}{2}s)}{\pi s}(\zeta(-s,x+a)-\zeta(-s,x-a)).$$

It follows that

$$\sigma_\xi^2 = \frac{\mathrm{cotan}(\frac{\pi}{2}s)}{\beta\frac{\pi}{2}s}\zeta(-s,2a). \tag{3.166}$$

Now for $\ell_N \to 0$, we have

$$\lim_{N\to\infty}\frac{1}{(N\ell_N)^s}\lim_{N\to\infty}\mathrm{Var}_{\mathbb{P}_{N,\beta}}[\mathrm{Fluct}_N[\xi(\ell_N^{-1}\cdot)]] = \tilde{\sigma}_\xi^2 := |\xi_0|^2_{H^{\frac{1-s}{2}}},$$

where $\xi_0 : \mathbb{R}\to\mathbb{R}, \xi_0 = \mathbb{1}_{(-a,a)}$. In this case, by expanding (3.166) as a tends to 0, we find

$$\tilde{\sigma}_\xi^2 = \frac{\mathrm{cotan}(\frac{\pi}{2}s)}{\beta\frac{\pi}{2}s}(2a)^s.$$

Let $\{k_N\}$ be a sequence in $\{1,\ldots,\frac{N}{2}\}$ such that $k_N\to\infty$. Let

$$i_0 = \underset{1\leqslant i\leqslant N}{\mathrm{argmin}}|x_i|.$$

Let us prove that $k_N^{-\frac{s}{2}}(N(x_{i_0+k_N}-x_{i_0})-k_N)$ converges in distribution. Let $\varepsilon>0$. Let $\pi:\mathbb{Z}\mapsto]-\frac{N}{2},\frac{N}{2}]\cap\mathbb{Z}$ such that $\pi(n)=n \bmod N$ and $\pi(n)\in]-\frac{N}{2},\frac{N}{2}]\cap\mathbb{Z}$. Define the event

$$\mathcal{A} = \{X_N\in D_N : |Nx_{i_0}|\leqslant k_N^\varepsilon, \forall i,j\in\{k: d(k,i_0)\leqslant 2k_N\}, |N(x_j-x_i)-N\pi(j-i)|\leqslant d(j,i)^{\frac{s}{2}+\varepsilon}\}.$$

Since i_0 is the index of the smallest point, by Theorem 3.1.1, the event $\mathcal{A}$ has overwhelming probability: there exists $\delta>0$ depending on ε such that

$$\mathbb{P}_{N,\beta}(\mathcal{A}^c)\leqslant C(\beta)e^{-c(\beta)k_N^\delta}.$$

Moreover note that on $\mathcal{A}$, we have

$$N(x_{k_N+i_0}-x_{i_0})-k_N = -\Big(\sum_{i=1}^N \mathbb{1}_{(0,\ell_N)}(x_i)-N\ell_N\Big)+O(k_N^\varepsilon+k_N^{\frac{s}{2}(\frac{s}{2}+\varepsilon)}),$$

where $\ell_N=\frac{k_N}{N}$. Since $\mathcal{A}$ has overwhelming probability, choosing $\varepsilon>0$ small enough, one deduces from Theorem 3.1.3 and the above computations that

$$N^{-\frac{s}{2}}\zeta(-s,\frac{k_N}{N})^{-\frac{1}{2}}(N(x_{k_N+i_0}-x_{i_0})-k_N)\underset{\mathrm{Law}}{\Longrightarrow}\mathcal{N}(0,\sigma^2), \quad \text{where} \quad \sigma^2=\frac{\mathrm{cotan}(\frac{\pi}{2}s)}{\beta\frac{\pi}{2}s}. \tag{3.167}$$

By symmetry, (3.167) holds for each $i\in\{1,\ldots,N\}$. $\square$

3.7 Appendix

3.7.1 Well-posedness of the H.-S. equation

3.7.1.1 Well-posedness for gradients

Let μ satisfying Assumptions 3.3.1. The formal adjoint with respect to μ of the derivation ∂_i, $i \in \{1, \ldots, N\}$ is given by

$$\partial_i^* w = \partial_i w - (\partial_i H) w,$$

meaning that for all $v, w \in \mathcal{C}^\infty(D_N)$ such that $v \cdot \vec{n} = 0$ on ∂D_N, the following identity holds

$$\mathbb{E}_\mu[(\partial_i v) w] = \mathbb{E}_\mu[v \partial_i^* w]. \tag{3.168}$$

The above identity can be shown by integration by parts under the Lebesgue measure on D_N. Recall the map

$$\Pi : X_N \in D_N \mapsto (x_2 - x_1, \ldots, x_N - x_1) \in \mathbb{T}^{N-1}$$

and $\mu' = \mu \circ \Pi^{-1}$.

Lemma 3.7.1. *Assume that μ satisfies Assumptions 3.3.1. Let $F \in H^{-1}(\mu)$. Assume either that F is in the form $F = G \circ \Pi$ with $G \in H^{-1}(\mu')$ or that χ is bounded. Then there exists a unique $\phi \in H^1(\mu)$ such that that*

$$\begin{cases} \mathcal{L}^\mu \phi = F - \mathbb{E}_\mu[F] & \text{on } D_N \\ \nabla \phi \cdot \vec{n} = 0 & \text{on } \partial D_N \\ \mathbb{E}_\mu[\phi] = 0. \end{cases} \tag{3.169}$$

Moreover the solution ϕ of (3.169) is the unique minimizer of

$$\phi \mapsto \mathbb{E}_\mu[|\nabla \phi|^2 - 2\phi F],$$

over functions $\phi \in H^1(\mu)$ such that $\mathbb{E}_\mu[\phi] = 0$.

Lemma 3.7.1 is a variation on Lax-Milgram's lemma. When the interaction kernel χ is bounded, a uniform Poincaré inequality holds. If one does not assume that χ is bounded, then one can observe that the Poincaré inequality holds for all functions of the gaps.

Proof. Assume that $F = G \circ \Pi$ with $G \in H^{-1}(\mu')$. Let

$$E = \{\phi \in H^1(\mu) : \phi = \psi \circ \Pi, \psi \in H^1(\mu'), \mathbb{E}_\mu[\phi] = 0\}$$

and

$$J : \phi \in E \mapsto \mathbb{E}_\mu[|\nabla \phi|^2] - 2\mathbb{E}_\mu[F\phi]. \tag{3.170}$$

One can write

$$|\mathbb{E}_\mu[F\phi]| \leqslant \|F\|_{H^{-1}(\mu)} \|\phi\|_{H^1(\mu)}.$$

By assumptions $\nabla^2 H \geqslant c$ on the subspace $\{x \in \mathbb{R}^N : x_1 = 0\}$ for some constant $c > 0$ and therefore

$$\mathbb{E}_\mu[\phi^2] = \mathrm{Var}_\mu[\phi] = \mathrm{Var}_{\mu'}[\psi] \leqslant \mathbb{E}_{\mu'}[\nabla \psi \cdot (\nabla^2 H)^{-1} \nabla \psi] \leqslant c^{-1} \mathbb{E}_{\mu'}[|\nabla \psi|^2] = \frac{1}{4c} \mathbb{E}_\mu[|\nabla \phi|^2]. \tag{3.171}$$

Consequently there exists some constant $C > 0$ such that for all $\phi \in E$,

$$|\mathbb{E}_\mu[F\phi]| \leqslant C \|F\|_{H^{-1}(\mu)} \mathbb{E}_\mu[|\nabla \phi|^2]^{\frac{1}{2}}. \tag{3.172}$$

It follows that J is coercive with respect to the $H^1(\mu)$ norm and that J is bounded from below. Let (ϕ_k) be a sequence of elements of E such that $(J(\phi_k))$ converges to $\inf J$. Since (ϕ_k) is bounded in $H^1(\mu)$, there exists a sub-sequence converging weakly to a certain $\phi \in E$. It follows from (3.172) that J is l.s.c on $H^1(\mu)$. Since J is convex, J is l.s.c for the weak topology on $H^1(\mu)$. Therefore ϕ is a minimizer of J on E. The first-order minimality condition for ϕ reads

$$\mathbb{E}_\mu[\nabla\phi \cdot \nabla h] = \mathbb{E}_\mu[Fh],$$

for all $h \in E$. By integration by parts one may rewrite the above quantity as

$$\mathbb{E}_\mu[\nabla\phi \cdot \nabla h] = \mathbb{E}_\mu[\mathcal{L}^\mu \phi h] + \int_{\partial D_N} (\nabla\phi \cdot \vec{n}) h e^{-H}.$$

It follows that $\nabla\phi \cdot \vec{n} = 0$ on ∂D_N and that for all $h \in E$,

$$\mathbb{E}_\mu[(\mathcal{L}^\mu \phi - F + \mathbb{E}_\mu[F])h] = 0.$$

If $h \in H^1(\mu)$, letting $\tilde{h} = \int h(x_1, \ldots, x_N)\mathrm{d}x_1$, one may note that

$$\mathbb{E}_\mu[\mathcal{L}^\mu \phi - F + \mathbb{E}_\mu[F])h] = \mathbb{E}_\mu[(\mathcal{L}^\mu \phi - F + \mathbb{E}_\mu[F])\tilde{h}] = 0.$$

We deduce that ϕ satisfies

$$\mathcal{L}^\mu \phi = F - \mathbb{E}_\mu[F],$$

as elements of $H^{-1}(\mu)$ and $\nabla\phi \cdot \vec{n} = 0$ on ∂D_N. The uniqueness is straightforward.

When the density of μ is bounded from below by a positive constant, it is standard that μ satisfies a Poincaré inequality. We deduce from the same arguments the proof of existence and uniqueness of a solution to (3.169). □

We can now complete the proof of Proposition 3.3.1.

Proof of Proposition 3.3.1. Let $F = G \circ \Pi$ with $G \in H^1(\mu')$. Recall that if $F \in H^1(\mu)$, $\nabla F \in L^2(\{1, \ldots, N\}, H^{-1}(\mu))$. Indeed

$$\sum_{i=1}^N \|\partial_i F\|^2_{H^{-1}(\mu)} \leqslant \sum_{i=1}^N \|\partial_i F\|^2_{L^2} \leqslant \|\phi\|^2_{H^1(\mu)}.$$

By Lemma 3.7.1, there exists $\phi \in H^1(\mu)$ such that $\mathcal{L}\phi = F - \mathbb{E}_\mu[F]$ as elements of $H^{-1}(\mu)$ and $\nabla\phi \cdot \vec{n} = 0$ on ∂D_N. Let $w \in \mathcal{C}^\infty(D_N)$ such that $w \cdot \nabla n = 0$ on ∂D_N. For each $i \in \{1, \ldots, N\}$, we have

$$\begin{aligned}\mathbb{E}_\mu[w\partial_i F] = \mathbb{E}_\mu[\partial_i^* w(F - \mathbb{E}_\mu[F])] = \mathbb{E}_\mu[\partial_i^* w \mathcal{L}^\mu \phi] = \mathbb{E}_\mu[\nabla \partial_i^* w \cdot \nabla\phi] \\ = \sum_{j=1}^N \mathbb{E}_\mu[(\partial_i^* \partial_j w)\partial_j \phi] + \sum_{j=1}^N \mathbb{E}_\mu[([\partial_j, \partial_i^*]w)\partial_j\phi].\end{aligned}$$

For the first term in the sum above, we have

$$\sum_{j=1}^N \mathbb{E}_\mu[(\partial_i^* \partial_j w)\partial_j\phi] = \sum_{j=1}^N \mathbb{E}_\mu[(\partial_j w)\partial_i\partial_j\phi] = \mathbb{E}_\mu[\nabla w \cdot \nabla(\partial_i\phi)] = \mathbb{E}_\mu[w\mathcal{L}^\mu(\partial_i\phi)].$$

For the second term, using the identity $[\partial_j, \partial_i^*] = (\nabla^2 H)_{i,j}$, one can write

$$\sum_{j=1}^{N} \mathbb{E}_\mu[([\partial_j, \partial_i^*]w)\partial_j \phi] = \mathbb{E}_\mu[we_i \cdot \nabla^2 H \nabla \phi].$$

We conclude by density that, in the sense of $H^{-1}(\mu)$, for each $i \in \{1, \ldots, N\}$,

$$\mathcal{L}^\mu(\partial_i \phi) + (\nabla^2 H \nabla \phi)_i = \partial_i F.$$

This concludes the proof of existence of a solution to (3.58). The Helffer-Sjöstrand formula (3.60) then easily follows from an integration by parts: letting $\nabla\phi$ be the solution of (3.58), we can write

$$\mathrm{Var}_\mu[F] = \mathbb{E}_\mu[(F - \mathbb{E}_\mu[F])\mathcal{L}^\mu \phi] = \mathbb{E}_\mu[\nabla F \cdot \nabla \phi].$$

When the density of μ is bounded from below we conclude likewise.

Let us no prove that the variational characterisation of the solution of (3.58). Let

$$E = \{\psi \in L^2(\{1, \ldots, N\}, H^1(\mu)) : \psi = v \circ \mathrm{Gap}_N, v \in L^2(\{1, \ldots, N\}, H^1(\mu'))\}$$

and

$$J : \psi \in E \mapsto \mathbb{E}_\mu[|D\psi|^2 + \psi \cdot \nabla^2 H \psi - 2\nabla F \cdot \psi].$$

By standard arguments, one can easily prove that J admits a unique minimizer ψ, which satisfies the Euler-Lagrange equation

$$\begin{cases} A_1^\mu \psi = \nabla F & \text{on } D_N \\ \psi \cdot \vec{n} = 0 & \text{on } \partial D_N. \end{cases} \tag{3.173}$$

Since J is convex on E, if $\nabla\phi$ verifies (3.173), then $\psi = \nabla\phi$. □

3.7.2 Auxiliary estimates

3.7.2.1 Discrete convolution products

Lemma 3.7.2. *Let α, β be such that $\alpha + \beta > 1$. Let $k_0 \in \mathbb{N}$.*

(i) If $\alpha \in (0,1)$ and $\beta \in (0,1)$,

$$\sum_{k \in \mathbb{N}, k \neq k_0} \frac{1}{k^\alpha} \frac{1}{|k_0 - k|^\beta} \leqslant \frac{C}{k_0^{\alpha+\beta-1}}.$$

(ii) If $\alpha = 1$ and $\beta \in (0,1)$,

$$\sum_{k \in \mathbb{N}, k \neq k_0} \frac{1}{k^\alpha} \frac{1}{|k_0 - k|^\beta} \leqslant \frac{C \log k_0}{k_0^\beta},$$

(iii) If $\alpha > 1$ and $\beta > 0$,

$$\sum_{k \in \mathbb{N}, k \neq k_0} \frac{1}{k^\alpha} \frac{1}{|k_0 - k|^\beta} \leqslant \frac{C}{k_0^{\min(\alpha,\beta)}}.$$

The above estimates follow from straightforward computations, see for instance [200, 91]. Let us now adapt Lemma 3.7.2 to truncated convolution products.

Lemma 3.7.3. *Let α, β be such that $\alpha + \beta > 1$. Let $k_0 \in \mathbb{N}$.*

(i) If $\alpha > 1$ and $\beta \in (0,1)$,

$$\sum_{k \geqslant K, k \neq k_0} \frac{1}{k^\alpha} \frac{1}{|k-k_0|^\beta} \lesssim \begin{cases} \frac{1}{K^{\alpha+\beta-1}} & \text{if } k_0 \leqslant \frac{K}{2} \\ \frac{1}{k_0^\beta} \frac{1}{K^{\alpha-1}} & \text{if } k_0 \geqslant \frac{K}{2}. \end{cases} \tag{3.174}$$

(ii) If $\alpha > 1$ and $\beta = 1$,

$$\sum_{k \geqslant K, k \neq k_0} \frac{1}{k^\alpha} \frac{1}{|k_0-k|^\beta} \lesssim \begin{cases} \frac{1}{K^{\alpha+\beta-1}} & \text{if } k_0 \leqslant \frac{K}{2} \\ \frac{1}{k_0^{\alpha+\beta-1}} + \frac{1}{k_0^\beta} \log k_0 & \text{if } k_0 \geqslant \frac{K}{2}. \end{cases}$$

(iii) If $\alpha > 1$ and $\beta > 1$,

$$\sum_{k \geqslant K, k \neq k_0} \frac{1}{k^\alpha} \frac{1}{|k-k_0|^\beta} \lesssim \begin{cases} \frac{1}{K^{\alpha+\beta-1}} & \text{if } k_0 \leqslant \frac{K}{2} \\ \frac{1}{k_0^\alpha} \frac{1}{(K-k_0)^{\beta-1}} & \text{if } \frac{K}{2} \leqslant k_0 < K \\ \frac{1}{k_0^\alpha} + \frac{1}{k_0^\beta} \frac{1}{K^{\alpha-1}} & \text{if } k_0 \geqslant K. \end{cases}$$

Proof of Lemma 3.7.3. Let us prove the three statements together. Let $\alpha > 1$. We split the sum over k along the condition $k \geqslant 2k_0$. If $k \geqslant 2k_0$, $|k-k_0| \geqslant \frac{k}{2}$ and therefore

$$\sum_{k \geqslant 2k_0, k \geqslant K} \frac{1}{k^\alpha} \frac{1}{|k-k_0|^\beta} \lesssim \sum_{k \geqslant 2k_0, k \geqslant K} \frac{1}{k^{\alpha+\beta}} \lesssim \frac{1}{\max(K,k_0)^{\alpha+\beta-1}}.$$

If $k_0 \leqslant \frac{K}{2}$, the remaining part is empty and therefore

$$\sum_{k \geqslant K, k \neq k_0} \frac{1}{k^\alpha} \frac{1}{|k-k_0|^\beta} \lesssim \frac{1}{K^{\alpha+\beta-1}}.$$

Assume now that $k_0 \geqslant \frac{K}{2}$. We split the remaining term into two parts according to whether $|k-k_0| \geqslant \frac{k_0}{2}$ or not. For the first contribution one has

$$\sum_{K \leqslant k \leqslant 2k_0, |k-k_0| \geqslant \frac{k_0}{2}} \frac{1}{k^\alpha} \frac{1}{|k-k_0|^\beta} \lesssim \frac{1}{k_0^\beta} \sum_{k \geqslant K} \frac{1}{k^\alpha} \lesssim \frac{1}{k_0^\beta} \frac{1}{K^{\alpha-1}}.$$

For the second contribution we may write

$$\sum_{\substack{K \leqslant k \leqslant 2k_0, |k-k_0| \leqslant \frac{k_0}{2} \\ k \neq k_0}} \frac{1}{k^\alpha} \frac{1}{|k-k_0|^\beta} \lesssim \frac{1}{k_0^\alpha} \sum_{\substack{|k-k_0| \leqslant \frac{k_0}{2}, k \geqslant K \\ k \neq k_0}} \frac{1}{|k-k_0|^\beta}.$$

One can bound the sum in the right-hand side by

$$\sum_{\substack{k \geqslant K, |k-k_0| \leqslant \frac{k_0}{2} \\ k \neq k_0}} \frac{1}{|k-k_0|^\beta} \leqslant \begin{cases} k_0^{1-\beta} & \text{if } \beta \in (0,1) \\ \log k_0 & \text{if } \beta = 1 \\ 1 & \text{if } \beta > 1 \text{ and } k_0 \geqslant K \\ \frac{1}{(K-k_0)^{\beta-1}} & \text{if } \beta > 1 \text{ and } \frac{K}{2} \leqslant k_0 < K. \end{cases}$$

Combining the above estimates concludes the proof of Lemma 3.7.3. ☐

3.7.2.2 Proof of Lemma 3.5.2

Proof of Lemma 3.5.2. We suppose that ξ is supported on $(-\frac{1}{2},\frac{1}{2})$ and that $\ell_N \in (0,1)$, the proof in the macroscopic case being similar. Let $i_0 = \underset{1\leqslant i\leqslant N}{\operatorname{argmin}} |x_i|$, $\gamma > 1$ and

$$I_N = \{i \in \{1,\dots,N\} : d(i,i_0) \leqslant (N\ell_N)^\gamma\}.$$

To lighten the notation let us write $I_N = \{1,\dots,K\}$ with $K = 1 + \lfloor (N\ell_N)^\gamma \rfloor$. Let $\psi \in \mathcal{C}^\delta(\ell_N^{-1}\mathbb{T},\mathbb{R})$ given by (3.124) and $\psi_{\mathrm{reg}} = \psi * K_\ell$ with $\ell = 1/(N\ell_N)^{\varepsilon'}$ for some $\varepsilon' > 0$.

Denote u^N the regular grid of spacing $\frac{1}{N}$ on $\mathbb{T}$:

$$\mathrm{u}_i^N = \frac{i}{N} \in \mathbb{T}, \quad i = 1,\dots,N. \tag{3.175}$$

Step 1: splitting Let $\tilde{g} : x \in \mathbb{T}\backslash\{0\} \mapsto g_s'(x)x$. One may express $\mathrm{A}_{\ell_N}[\psi_{\mathrm{reg}}]$ as

$$\mathrm{A}_{\ell_N}[\psi_{\mathrm{reg}}] = \iint_{\Delta^c} \frac{\ell_N(\psi_{\mathrm{reg}}(\ell_N^{-1}y) - \psi_{\mathrm{reg}}(\ell_N^{-1}x))}{y-x} N^{-s}\tilde{g}(y-x)\mathrm{dfluct}_N(x)\mathrm{dfluct}_N(y).$$

Denote

$$\zeta : (x,y) \in (\ell_N^{-1}\mathbb{T})^2 \mapsto \frac{\ell_N(\psi_{\mathrm{reg}}(\ell_N^{-1}y) - \psi_{\mathrm{reg}}(\ell_N^{-1}x))}{y-x}.$$

One can split the gradient of $\widetilde{\mathrm{A}}_{\ell_N}[\psi_{\mathrm{reg}}]$ into $\mathrm{V}+\mathrm{W}$ with $\mathrm{V}_i, \mathrm{W}_i$ given for each $i \in I_N$ by

$$\begin{aligned}\mathrm{V}_i = 2\sum_{k:i+k\in I_N} \partial_1\zeta(x_i,x_{i+k})N^{-s}\tilde{g}(x_{i+k}-x_i) - 2N\int_{|y|\leqslant\frac{(N\ell_N)^\gamma}{N}} \partial_1\zeta(x_i,y)N^{-s}\tilde{g}(y-x_i)\mathrm{d}y \\ + 2\sum_{k:i+k\in I_N} \zeta(x_i,x_{i+k})N^{-s}\tilde{g}'(\mathrm{u}_k^N) - 2N\int_{|y|\leqslant\frac{(N\ell_N)^\gamma}{N}} \zeta(x_i,y)N^{-s}\tilde{g}'(y-x_i)\mathrm{d}y\end{aligned}$$

and

$$\mathrm{W}_i = 2\sum_{k:i+k\in I_N} \zeta(x_i,x_{i+k})N^{-s}(\tilde{g}'(x_{i+k}-x_i) - \tilde{g}'(\mathrm{u}_k^N))$$

with $\mathrm{V}_i = \mathrm{W}_i = 0$ for each $i \in I_N^c$. Let us isolate from V_i the discretization errors: for each $i \in I_N$ write $\mathrm{V}_i = \mathrm{V}_i^1 + \mathrm{V}_i^2$ with

$$\begin{aligned}\mathrm{V}_i^1 = 2\sum_{k:i+k\in I_N} \partial_1\zeta(x_i,x_{i+k})N^{-s}\tilde{g}(x_{i+k}-x_i) - 2\sum_{k:i+k\in I_N} \partial_1\zeta(x_i,x_i+\mathrm{u}_k^N)N^{-s}\tilde{g}(\mathrm{u}_k^N) \\ + 2\sum_{k:i+k\in I_N} (\zeta(x_i,x_{i+k}) - \zeta(x_i,x_i+\mathrm{u}_k^N))N^{-s}\tilde{g}'(\mathrm{u}_k^N)\end{aligned}$$

and

$$\mathrm{V}_i^2 = 2\sum_{k:i+k\in I_N} (\partial_1\zeta(x_i,\mathrm{u}_k^N)\tilde{g}(\mathrm{u}_k^N) + \zeta(x_i,\mathrm{u}_k^N)\tilde{g}'(\mathrm{u}_k^N)) - 2N\int_{|y|\leqslant\frac{(N\ell_N)^\gamma}{N}} (\partial_1\zeta(x_i,y)\tilde{g}(y) + \zeta(x_i,y)\tilde{g}'(y))\mathrm{d}y.$$

Step 2: control on V^1 For each $i \in I_N$, one may write

$$\mathrm{V}_i^1 = 2 \sum_{k:i+k\in I_N} \partial_1 \zeta(x_i, x_{i+k}) N^{-s}(\tilde{g}(x_{i+k} - x_i) - \tilde{g}(\mathrm{u}_k^N))$$
$$+2 \sum_{k:i+k\in I_N} (\partial_1\zeta(x_i, x_{i+k}) - \partial_1\zeta(x_i, x_i + \mathrm{u}_k^N))N^{-s}\tilde{g}(\mathrm{u}_k^N) + 2 \sum_{k:i+k\in I_N} (\zeta(x_i, x_{i+k}) - \zeta(x_i, x_i + \mathrm{u}_k^N))N^{-s}\tilde{g}'(\mathrm{u}_k^N). \tag{3.176}$$

Denote $\mathrm{V}_i^{1,1}$, $\mathrm{V}_i^{1,2}$ and $\mathrm{V}_i^{1,3}$ the three terms in (3.176). By assumption, ψ_{reg} satisfies $|\psi_{\mathrm{reg}}''| \leqslant C\eta_N$ where

$$\eta_N : x \in \ell_N^{-1}\mathbb{T} \mapsto \Big(1 + \sum_{l=1}^p \frac{1}{(|x - a_l| \vee \frac{1}{(N\ell_N)^{1-\varepsilon'}})^{1+\alpha_l}}\Big)\mathbb{1}_{|x|<2} + \frac{1}{(1+|x|)^{3-s}}. \tag{3.177}$$

Observe that there exists a constant $C > 0$ such that for all $x, y \in \mathbb{T}$,

$$|\partial_1\zeta(x,y)| \leqslant C\ell_N^{-1}(\eta_N(\ell_N^{-1}x) + \eta_N(\ell_N^{-1}y)). \tag{3.178}$$

Therefore, on the good event $\mathcal{A}$ defined in (3.123), one can bound $\mathrm{V}^{1,1}$ uniformly in $i \in I_N$ by

$$|\mathrm{V}_i^{1,1}| \leqslant C\ell_N^{-1}(N\ell_N)^{\kappa\varepsilon} \sum_{j\in I_N: j\neq i} (\eta_N(\ell_N^{-1}x_j) + \eta_N(\ell_N^{-1}(x_i + \mathrm{u}_{j-i}^N)))\frac{1}{|j-i|^{1+\frac{s}{2}}}. \tag{3.179}$$

Let us keep track of the indices near the singularities of ξ: for each $i = 1, \ldots, p$ let

$$k^l := \underset{1\leqslant i\leqslant N}{\mathrm{argmin}} |x_i - \ell_N a_l|. \tag{3.180}$$

Note that there exists $C_0 > 0$ such that for all $X_N \in \mathcal{A}$,

$$|j - \frac{K}{2}| \geqslant C_0 N\ell_N \Longrightarrow |x_j| > 2\ell_N, \quad \text{for each } j \in I_N.$$

Moreover we can write

$$\frac{1}{|\ell_N^{-1}(x - \ell_N a_l)| \vee (N\ell_N)^{-(1-\varepsilon')}} = \frac{\ell_N}{|x - \ell_N a_l| \vee \ell_N(N\ell_N)^{-(1-\varepsilon')}} = \frac{N\ell_N}{|Nx - N\ell_N a_l| \vee (N\ell_N)^{\varepsilon'}}.$$

It follows that

$$\eta_N(\ell_N^{-1}x_j) + \eta_N(\ell_N^{-1}(x_i + \mathrm{u}_{j-i}^N)) \leqslant C(N\ell_N)^{\kappa\varepsilon}\mathbb{1}_{|j-\frac{K}{2}|\leqslant C_0 N\ell_N}$$
$$\times\Big(1 + \sum_{l=1}^p (N\ell_N)^{1+\alpha_l}\frac{1}{(1+|j-k^l|)^{1+\alpha_l}} \wedge \frac{1}{(N\ell_N)^{\varepsilon'(1+\alpha_l)}}\Big) + C(N\ell_N)^{\kappa\varepsilon}\frac{(N\ell_N)^{3-s}}{(|j-\frac{K}{2}| + N\ell_N)^{3-s}}.$$

Inserting this into (3.179) gives

$$|\mathrm{V}_i^{1,1}| \leqslant C\ell_N^{-1}(N\ell_N)^{\kappa\varepsilon} \sum_{|j-\frac{K}{2}|\leqslant C_0 N\ell_N} \Big(1 + \sum_{l=1}^p (N\ell_N)^{1+\alpha_l}\frac{1}{1+|j-k^l|^{1+\alpha_l}} \wedge \frac{1}{(N\ell_N)^{\varepsilon'(1+\alpha_l)}}\frac{1}{1+|j-i|^{1+\frac{s}{2}}}\Big)$$
$$+ C\ell_N^{-1}(N\ell_N)^{\kappa\varepsilon}(N\ell_N)^{3-s}\sum_{j=1}^N \frac{1}{(N\ell_N + |j-\frac{K}{2}|)^{3-s}}\frac{1}{1+|j-i|^{1+\frac{s}{2}}}. \tag{3.181}$$

One can first check that

$$\sum_{|j-\frac{K}{2}|\leqslant C_0N\ell_N} \frac{1}{|j-i|^{1+\frac{s}{2}}} \leqslant \begin{cases} 1 & \text{if } |i-\frac{K}{2}| \leqslant 2C_0N\ell_N \\ \frac{N\ell_N}{|i-\frac{K}{2}|^{1+\frac{s}{2}}} & \text{if } |i-\frac{K}{2}| \geqslant 2C_0N\ell_N. \end{cases} \tag{3.182}$$

The behavior of the discrete convolution product depends on whether $\alpha_l < \frac{s}{2}$ or not. By Lemma 3.7.3, we have

$$\begin{aligned} &\sum_{|j-\frac{K}{2}|\leqslant C_0N\ell_N} \frac{1}{1+|j-k^l|^{1+\alpha_l}} \wedge \frac{1}{(N\ell_N)^{\varepsilon'(1+\alpha_l)}} \frac{1}{1+|j-i|^{1+\frac{s}{2}}} \\ &\leqslant C \begin{cases} \frac{1}{|i-k^l|^{1+\alpha_l}} + \frac{1}{(|i-k^l|\vee(N\ell_N)^{\varepsilon'})^{1+\frac{s}{2}}} & \text{if } |i-\frac{K}{2}| \leqslant 2C_0N\ell_N \text{ and } \alpha_l > \frac{s}{2} \\ \frac{1}{|i-k^l|^{1+\alpha_l}} & \text{if } |i-\frac{K}{2}| \leqslant 2C_0N\ell_N \text{ and } \alpha_l \leqslant \frac{s}{2} \\ \frac{1}{|i-\frac{K}{2}|^{1+\frac{s}{2}+\alpha_l}} & \text{if } |i-\frac{K}{2}| \geqslant 2C_0N\ell_N. \end{cases} \end{aligned} \tag{3.183}$$

On the event $\mathcal{A}$, there exist constants $C>0$ and $\kappa>0$ such that

$$|i-k^l| \geqslant C(N\ell_N)^{-\kappa\varepsilon}|Nx_i - N\ell_N a_l|, \quad \text{for each } i \in I_N,\ 1 \leqslant l \leqslant p.$$

Besides we can check that

$$\sum_{j=1}^{N} \frac{1}{(N\ell_N + |j-\frac{N}{2}|)^{3-s}} \frac{1}{|j-i|^{1+\frac{s}{2}}} \leqslant C \begin{cases} \frac{1}{(N\ell_N)^{3-\frac{s}{2}}} & \text{if } |i-\frac{K}{2}| \leqslant 2N\ell_N \\ \frac{1}{|i-\frac{K}{2}|^{3-\frac{s}{2}}} & \text{if } |i-\frac{K}{2}| \geqslant 2N\ell_N. \end{cases}$$

Observe that the term (3.182) is dominant at infinity. One can therefore gather these expressions into

$$\begin{aligned} |\mathrm{V}_i^{1,1}| \leqslant{}& C(N\ell_N)^{\kappa\varepsilon} \sum_{l:\alpha_l>\frac{s}{2}} (N\ell_N)^{\alpha_l} N^{-\frac{s}{2}} \frac{1}{(|x_i-\ell_N a_l| \vee \frac{(N\ell_N)^{\varepsilon'}}{N})^{1+\frac{s}{2}}} \mathbb{1}_{|x_i|<2\ell_N} \\ &+ C(N\ell_N)^{\kappa\varepsilon} \sum_{l=1}^{p} \ell_N^{\alpha_l} \frac{1}{(|x_i - N\ell_N a_l| \vee \frac{1}{N})^{1+\alpha_l}} \mathbb{1}_{|x_i|<2\ell_N} + C(N\ell_N)^{\kappa\varepsilon} N^{-\frac{s}{2}} \frac{1}{(|x_i|+\ell_N)^{1+\frac{s}{2}}}. \end{aligned} \tag{3.184}$$

We turn to the second term of (3.176). Observe that there exists a constant $C>0$ such that for all $a,x,y \in \mathbb{T}$,

$$|\partial_1\zeta(x,y+a) - \partial_1\zeta(x,y)| \leqslant C\ell_N^{-1}(\eta_N(\ell_N^{-1}y) + \eta_N(\ell_N^{-1}(y+a)) + \eta_N(\ell_N^{-1}x))\frac{|a|}{|x-y|}.$$

Applying this to $x=x_i$, $y=x_j$, $a = x_j - x_i - \mathrm{u}_{j-i}^N$, we find that there exist constants $C>0$ and $\kappa>0$ such that uniformly on $\mathcal{A}$ and for each $i,j \in I_N$,

$$|\partial_1\zeta(x_i,x_j) - \partial_1\zeta(x_i,x_i+\mathrm{u}_{j-i}^N)| \leqslant C\ell_N^{-1}(N\ell_N)^{\kappa\varepsilon}(\eta_N(\ell_N^{-1}x_i) + \eta_N(\ell_N^{-1}(x_i+\mathrm{u}_{j-i}^N)) + \eta_N(\ell_N^{-1}x_j))\frac{1}{|j-i|^{1+\frac{s}{2}}}.$$

As a consequence we get that $\mathrm{V}_i^{1,2}$ is bounded uniformly on $\mathcal{A}$ by

$$|\mathrm{V}_i^{1,2}| \leqslant C\ell_N^{-1}(N\ell_N)^{\kappa\varepsilon} \sum_{j\in I_N: j\neq i} (\eta_N(\ell_N^{-1}x_j) + \eta_N(\ell_N^{-1}(x_i+\mathrm{u}_{j-i}^N)))\frac{1}{|j-i|^{1+\frac{s}{2}}}.$$

In view of the previous computation, one deduces that $\mathrm{V}_i^{1,2}$ verifies the estimate (3.184). With similar arguments one can check that $\mathrm{V}_i^{1,3}$ also satisfies (3.184) and therefore

$$\begin{aligned}|\mathrm{V}_i^1| \leqslant C(N\ell_N)^{\kappa\varepsilon} \sum_{l:\alpha_l>\frac{s}{2}} (N\ell_N)^{\alpha_l} N^{-\frac{s}{2}} \frac{1}{(|x_i-\ell_N a_l| \vee \frac{(N\ell_N)^{\varepsilon'}}{N})^{1+\frac{s}{2}}} \mathbb{1}_{|x_i|<2\ell_N} \\ + C(N\ell_N)^{\kappa\varepsilon} \sum_{l=1}^p \ell_N^{\alpha_l} \frac{1}{(|x_i-N\ell_N a_l| \vee \frac{1}{N})^{1+\alpha_l}} \mathbb{1}_{|x_i|<2\ell_N} + C(N\ell_N)^{\kappa\varepsilon} N^{-\frac{s}{2}} \frac{1}{(|x_i|+\ell_N)^{1+\frac{s}{2}}}.\end{aligned} \tag{3.185}$$

Step 3: control on V^2 Fix $x \in \mathbb{T}$ and define

$$f: y \in N\mathbb{T} \mapsto \zeta(x, x+\frac{y}{N}) N^{-s} \tilde{g}'(\frac{y}{N}) + \partial_1 \zeta(x, x+\frac{y}{N}) N^{-s} \tilde{g}(\frac{y}{N}).$$

For each $K_1, K_2 \in \{1, \ldots, \frac{N}{2}\}$, one may write using the Euler-Maclaurin formula

$$\sum_{-K_1 \leqslant k \leqslant K_2, k \neq 0} f(k) = \int_{[-K_1,K_2]\setminus[-1,1]} f(y)\mathrm{d}y + \frac{f(-K_1)+f(K_2)}{2} + O\Big(\int_{-\frac{N}{2}}^{-1} |f'(y)|\mathrm{d}y + \int_1^{\frac{N}{2}} |f'(y)|\mathrm{d}y\Big).$$

Let us thus upper bound the L^1 norm of f'. First, note as in (3.178) that for all $y \in N\mathbb{T}$,

$$\Big|\partial_2 \zeta(x, x+\frac{y}{N})\Big| \leqslant C\ell_N^{-1}\Big(\eta_N(\ell_N^{-1}x) + \eta_N(\ell_N^{-1}(x+\frac{y}{N})\Big).$$

It follows that

$$\int_1^{\frac{N}{2}} \Big|\partial_2 \zeta(x+\frac{y}{N}) N^{-(1+s)} \tilde{g}'(\frac{y}{N})\Big| \mathrm{d}y \leqslant C\ell_N^{-1} \int_1^{\frac{N}{2}} \Big(\eta_N(\ell_N^{-1}x) + \eta_N(\ell_N^{-1}(x+\frac{y}{N}))\Big) \frac{1}{y^{1+s}} \mathrm{d}y.$$

We recognize an expression similar to (3.179) and after performing some computations we find

$$\begin{aligned}\int_1^{\frac{N}{2}} \Big|\partial_2 \zeta(x, N^{-(1+s)} \tilde{g}'(\frac{y}{N})\Big| \mathrm{d}y \leqslant C(N\ell_N)^{\kappa\varepsilon} \sum_{l=1}^p \ell_N^{\alpha_l} \frac{1}{(|x-\ell_N a_l| \vee \frac{1}{N})^{1+\alpha_l}} \mathbb{1}_{|x|<2\ell_N} \\ + C(N\ell_N)^{\kappa\varepsilon} \sum_{l:\alpha_l>s} (N\ell_N)^{\alpha_l} N^{-s} \frac{1}{(|x-\ell_N a_l| \vee \frac{(N\ell_N)^{\varepsilon'}}{N})^{1+s}} + C(N\ell_N)^{\kappa\varepsilon} N^{-s} \frac{1}{(|x|+\ell_N)^{1+s}}.\end{aligned} \tag{3.186}$$

Using that for all $x \in \mathbb{T}$, $y \in (-\frac{N}{2}, \frac{N}{2})$,

$$|\zeta(x, x+\frac{y}{N})| \leqslant C\ell_N^{-1}\Big(\eta_N(\ell_N^{-1}x) + \eta_N(\ell_N^{-1}(x+\frac{y}{N}))\Big)\frac{|y|}{N},$$

one gets

$$\int_1^{\frac{N}{2}} |\zeta(x, x+\frac{y}{N})| N^{-(s+1)} \tilde{g}''(\frac{y}{N}) \mathrm{d}y \leqslant C\ell_N - 1 \int_1^{\frac{N}{2}} \Big(\eta_N(\ell_N^{-1}x) + \eta_N(\ell_N^{-1}(x+\frac{y}{N}))\Big) \frac{1}{y^{1+s}} \mathrm{d}y.$$

Finally, noting that for all $x \in \mathbb{T}$, $y \in (-\frac{N}{2}, \frac{N}{2})$,

$$|\partial_{21} \zeta(x, x+\frac{y}{N})| \leqslant C\ell_N^{-1}\Big(\eta_N(\ell_N^{-1}x) + \eta_N(\ell_N^{-1}(x+\frac{y}{N}))\Big)\frac{N}{|y|},$$

one may check that

$$\int_1^{\frac{N}{2}} \frac{1}{N}|\partial_{21}\zeta(x,x+\frac{y}{N})|N^{-s}\tilde{g}(\frac{y}{N})\mathrm{d}y \leqslant C\ell_N^{-1}\int_1^{\frac{N}{2}}\Big(\eta_N(\ell_N^{-1}x)+\eta_N(\ell_N^{-1}(x+\frac{y}{N}))\Big)\frac{1}{y^{1+s}}\mathrm{d}y.$$

Combining these estimates we find that $\int_1^{\frac{N}{2}}|f'(y)|\mathrm{d}y$ is bounded by the right-hand side of (3.186) and so is $\int_{-\frac{N}{2}}^{-1}|f'(y)|\mathrm{d}y$. Inserting this into (3.188) we conclude that V_i^2 is bounded by

$$\begin{aligned}|\mathrm{V}_i^2| &\leqslant C(N\ell_N)^{\kappa\varepsilon}\sum_{l=1}^p \ell_N^{\alpha_l}\frac{1}{(|x-\ell_N a_l|\vee\frac{1}{N})^{1+\alpha_l}}\mathbb{1}_{|x|<2\ell_N}\\ &+C(N\ell_N)^{\kappa\varepsilon}\sum_{l:\alpha_l>s}(N\ell_N)^{\alpha_l}N^{-s}\frac{1}{(|x-\ell_N a_l|\vee\frac{(N\ell_N)^{\varepsilon'}}{N})^{1+s}}+C(N\ell_N)^{\kappa\varepsilon}N^{-s}\frac{1}{(|x|+\ell_N)^{1+s}}+Cf(d(i,\partial I_N)).\end{aligned} \tag{3.187}$$

Split f into f_1+f_2 with $f_1: y\in N\mathbb{T}\mapsto \zeta(x,x+\frac{y}{N})N^{-s}\tilde{g}'(\frac{y}{N})$. First recall that

$$\Big|\zeta(x_i,x_i+\frac{d(i,\partial I_N)}{N})\Big| \leqslant C\frac{\ell_N^{2-s}}{(\ell_N+|x_i|)^{2-s}}.$$

Let $i\in I_N$ such that $d(i,i_0)\leqslant \frac{1}{2}d(i,\partial I_N)$. One can write

$$|f_1(d(i,\partial I_N))| \leqslant \frac{CN}{(N\ell_N)^{\gamma(1+s)}}\frac{1}{(\ell_N^{-1}|x_i|+1)^{2-s}} = \frac{C}{(N\ell_N)^{\gamma s}}\frac{\ell_N^{-1}}{(\ell_N^{-1}|x_i|+1)^{2-s}}.$$

Let $i\in I_N$ such that $d(i,i_0)\geqslant \frac{1}{2}d(i,\partial I_N)$. Then

$$|f_1(d(i,\partial I_N))| \leqslant \frac{CN}{d(i,\partial I_N)^{1+s}}(N\ell_N)^{-(2-s)(1-\gamma)}.$$

Let $I=(-\frac{(N\ell_N)^\gamma}{N},\frac{(N\ell_N)^\gamma}{N})\cap\mathbb{T}$. On the event $\mathcal{A}$ there holds

$$\frac{1}{d(i,\partial I_N)^{1+\frac{s}{2}}} \leqslant C\frac{(N\ell_N)^{\kappa\varepsilon}}{(Nd(x_i,\partial I)\wedge 1)^{1+\frac{s}{2}}}.$$

It follows that on $\mathcal{A}$,

$$|f_1(d(i,\partial I_N))| \leqslant C(N\ell_N)^{\kappa\varepsilon-(2-s)(1-\gamma)}N^{-\frac{s}{2}}\frac{1}{(d(x_i,\partial I)\wedge\frac{1}{N})^{1+\frac{s}{2}}}.$$

Besides we also have that for each $i\in I_N$,

$$|f_2(d(i,\partial I_N))| \leqslant C(N\ell_N)^{\kappa\varepsilon}\ell_N^{-1}\eta_N(\ell_N^{-1}x).$$

Since (3.187) can be absorbed into (3.185), we have obtained that uniformly on $i\in I_N$

$$\begin{aligned}|\mathrm{V}_i| &\leqslant C(N\ell_N)^{\kappa\varepsilon}\sum_{l:\alpha_l>\frac{s}{2}}(N\ell_N)^{\alpha_l}N^{-\frac{s}{2}}\frac{1}{(|x_i-\ell_N a_l|\vee\frac{(N\ell_N)^{\varepsilon'}}{N})^{1+\frac{s}{2}}}\mathbb{1}_{|x_i|<2\ell_N}\\ &\quad + C(N\ell_N)^{\kappa\varepsilon}\sum_{l=1}^p\ell_N^{\alpha_l}\frac{1}{(|x_i-N\ell_N a_l|\vee\frac{1}{N})^{1+\alpha_l}}\mathbb{1}_{|x_i|<2\ell_N}+C(N\ell_N)^{\kappa\varepsilon}N^{-\frac{s}{2}}\frac{1}{(|x_i|+\ell_N)^{1+\frac{s}{2}}}\\ &\qquad +\frac{C}{(N\ell_N)^{\gamma s}}\frac{\ell_N^{-1}}{(\ell_N^{-1}|x_i|+1)^{2-s}}+C(N\ell_N)^{\kappa\varepsilon-(2-s)(1-\gamma)}N^{-\frac{s}{2}}\frac{1}{(d(x_i,\partial I)\wedge\frac{1}{N})^{1+\frac{s}{2}}}.\end{aligned}$$

Step 4: control on W in gap coordinates Let $\tilde{\mathrm{W}} \in L^2(\{1,\ldots,N\}, H^1(\mathbb{P}_{N,\beta}))$ be the vector-field given for each $i \in I_N$ by

$$\tilde{\mathrm{W}}_i = \sum_{k=1}^{N} \sum_{\substack{l \in I_N: \\ l+k \in I_N, i-k<l\leqslant i}} \zeta(x_{l+k}, x_l) N^{-(1+s)}(\tilde{g}'(x_{l+k}-x_l) - \tilde{g}'(\mathrm{u}_k^N))(\delta_{k\neq N/2} + \frac{1}{2}\delta_{k=N/2})$$

and $\tilde{\mathrm{W}}_i = 0$ for $i \notin I_N$. For all $U_N \in \mathbb{R}^N$, we have

$$\mathrm{W}\cdot U_N = -\sum_{i=1}^{N} \tilde{\mathrm{W}}_i N(u_{i+1}-u_i).$$

There exist constants $C, \kappa > 0$ such that uniformly on $\mathcal{A}$, for each $l, l+k \in I_N$

$$N^{-(1+s)}|\tilde{g}'(x_{l+k}-x_k) - \tilde{g}'(\mathrm{u}_k^N)| \leqslant \frac{C(N\ell_N)^{\kappa\varepsilon}}{k^{2+\frac{s}{2}}}. \tag{3.188}$$

By assumption $|\psi_{\mathrm{reg}}'| \leqslant C\gamma_N$ where

$$\gamma_N : x \in \ell_N^{-1}\mathbb{T} \mapsto \Big(1 + \sum_{l=1}^{p} \frac{1}{|x-a_l|^{\alpha_l}}\Big)\mathbb{1}_{|x|<2} + \frac{1}{1+|x|^{2-s}}.$$

One may thus bound $\tilde{\mathrm{W}}_i$ uniformly on $\mathcal{A}$ by

$$|\tilde{\mathrm{W}}_i| \leqslant C(N\ell_N)^{\kappa\varepsilon} \sum_{k=1}^{K}\sum_{j=i+1}^{K} (|\gamma_N(\ell_N^{-1}x_k)| + |\gamma_N(\ell_N^{-1}x_j)|)\frac{1}{|j-k|^{2+\frac{s}{2}}}.$$

Reindexing this sum gives

$$|\tilde{\mathrm{W}}_i| \leqslant C(N\ell_N)^{\kappa\varepsilon} \sum_{j=1}^{K} |\gamma_N(\ell_N^{-1}x_j)|\frac{1}{|j-i|^{1+\frac{s}{2}}}. \tag{3.189}$$

Recalling (3.180), for each $l = 1,\ldots,p$ and $j \in I_N$ we have that uniformly on $\mathcal{A}$,

$$N|x_j - \ell_N a_l| \geqslant (N\ell_N)^{-\varepsilon}|j-k^l|.$$

Furthermore there exists a constant $C_0 > 0$ such that for each $i, j \in I_N$ and uniformly on $\mathcal{A}$

$$|j-i| \geqslant C_0 N\ell_N \Longrightarrow |x_j| > 2\ell_N.$$

Inserting this into (3.189) we find that

$$\begin{aligned}|\tilde{\mathrm{W}}_i| \leqslant C(N\ell_N)^{\kappa\varepsilon} &\sum_{|j-\frac{K}{2}|\leqslant C_0N\ell_N} \sum_{l=1}^{p} (N\ell_N)^{\alpha_l}\frac{1}{|j-k^l|^{\alpha_l}}\frac{1}{|j-i|^{1+\frac{s}{2}}} \\ &+ C(N\ell_N)^{2-s}\sum_{j=1}^{N}\frac{1}{(N\ell_N + |j-\frac{K}{2}|)^{2-s}}\frac{1}{|j-i|^{1+\frac{s}{2}}}.\end{aligned} \tag{3.190}$$

Let $1 \leqslant l \leqslant p$. Since $\alpha_l \in (0,1)$, one can observe that

$$\sum_{|j-\frac{K}{2}|\leqslant C_0N\ell_N} \frac{1}{|j-k^l|^{\alpha_l}}\frac{1}{|j-i|^{1+\frac{s}{2}}} \leqslant C\begin{cases} \frac{1}{|i-k^l|^{\alpha_l}} & \text{if } |i-\frac{K}{2}| \leqslant 2C_0N\ell_N \\ \frac{N\ell_N}{|i-\frac{K}{2}|^{1+\frac{s}{2}+\alpha_l}} & \text{if } |i-\frac{K}{2}| \geqslant 2C_0N\ell_N. \end{cases}$$

Summing the squares of these over $1 \leqslant i \leqslant K-1$ therefore gives

$$(N\ell_N)^{2\alpha_l} \sum_{1\leqslant i\leqslant K} \Big(\sum_{|j-\frac{K}{2}|\leqslant C_0 N\ell_N} \frac{1}{|j-k^l|^{\alpha_l}} \frac{1}{|j-i|^{1+\frac{s}{2}}} \Big)^2 \leqslant C \begin{cases} N\ell_N & \text{if } \alpha_l \in (0,\frac{1}{2}) \\ (N\ell_N)\log(N\ell_N) & \text{if } \alpha_l = \frac{1}{2} \\ (N\ell_N)^{2\alpha_l} & \text{if } \alpha_l \in (\frac{1}{2},1). \end{cases} \tag{3.191}$$

Besides we can check that

$$\sum_{j=1}^N \frac{1}{(N\ell_N + |j-\frac{N}{2}|)^{2-s}} \frac{1}{|j-i|^{1+\frac{s}{2}}} \leqslant C \begin{cases} \frac{1}{(N\ell_N)^{2-s}} & \text{if } |i-\frac{K}{2}| \leqslant 2N\ell_N \\ \frac{N\ell_N}{|i-\frac{K}{2}|^{3-s}} & \text{if } |i-\frac{K}{2}| \geqslant 2N\ell_N. \end{cases}$$

Summing the squares of these over $1 \leqslant i \leqslant K-1$ gives

$$(N\ell_N)^{2(2-s)} \sum_{i=1}^K \Big(\sum_{j=1}^N \frac{1}{N\ell_N + |j-\frac{N}{2}|^{2-s}} \frac{1}{|j-i|^{1+\frac{s}{2}}} \Big)^2 \leqslant CN\ell_N. \tag{3.192}$$

Inserting (3.191) and (3.192) into (3.190) we concludes that there exist constants $C>0$ and $\kappa>0$ such

$$\sup_{\mathcal{A}} |\tilde{\mathrm{W}}|^2 \leqslant C(N\ell_N)^{\kappa\varepsilon}(N\ell_N)^{\max(1,2\max\alpha_l)}. \tag{3.193}$$

□

3.7.2.3 Additional useful estimates

Lemma 3.7.4. *Let ξ satisfying Assumptions 3.1.1. Let $\psi_0 \in \mathcal{C}^\delta(\mathbb{T},\mathbb{R})$ such that $g_s' * \psi_0 = \xi$ with $\int \psi_0 = 0$. Assume that ψ_0'' has p singularities in $a_1,\ldots,a_p$ of order $1+\alpha_1,\ldots,1+\alpha_p$, with $\alpha_i \in (0,1)$, as defined in (3.8). Assume that ξ is supported on $(-\frac{1}{2},\frac{1}{2})$ or that $\ell_N = 1$. Let $\psi \in \mathcal{C}^\delta(\ell_N^{-1}\mathbb{T},\mathbb{R})$ given by*

$$\psi' = -\frac{1}{2c_s}\ell_N^{1-s}(-\Delta)^{\frac{1-s}{2}}(\xi(\ell_N^{-1}\cdot))(\ell_N\cdot) \quad \textit{and} \quad \int \psi = 0.$$

Let $\psi_{\mathrm{reg}} = \psi * K_\ell$ with K_ℓ defined in (3.50) and $\ell = 1/(N\ell_N)^{1-\varepsilon'}$ with $\varepsilon'>0$. We have

$$\mathbb{E}_{\mathbb{P}_{N,\beta}}[\mathrm{B}_{\ell_N}[\psi_{\mathrm{reg}}]] \leqslant C(\beta)(N\ell_N)^{\kappa\varepsilon}(N\ell_N + (N\ell_N)^{2\max\alpha_l}). \tag{3.194}$$

Proof. Let us recall

$$\mathrm{B}_{\ell_N}[\psi_{\mathrm{reg}}] = \iint_{\Delta^c} N^{-(s+2)} g_s''(x-y)(N\ell_N)^2(\psi_{\mathrm{reg}}(\ell_N^{-1}x) - \psi_{\mathrm{reg}}(\ell_N^{-1}y))^2 \mathrm{dfluct}_N(x)\mathrm{dfluct}_N(y).$$

Denote

$$h : (x,y) \in (\ell_N^{-1}\mathbb{T})^2\backslash\Delta \mapsto N^{-(s+2)} g_s''(x-y)(N\ell_N)^2(\psi_{\mathrm{reg}}(\ell_N^{-1}x) - \psi_{\mathrm{reg}}(\ell_N^{-1}y))^2.$$

Since the first marginal of $\mathbb{P}_{N,\beta}$ is the Lebesgue measure $\mathrm{d}x$ on $\mathbb{T}$, one can simplify the expectation of $\mathrm{B}_{\ell_N}[\psi_{\mathrm{reg}}]$ into

$$\mathbb{E}_{\mathbb{P}_{N,\beta}}[\mathrm{B}_{\ell_N}[\psi_{\mathrm{reg}}]] = \sum_{i=1}^N \mathbb{E}_{\mathbb{P}_{N,\beta}}\Big[\sum_{j:j\neq i} h(x_j,x_i) - N\int h(y,x_i)\mathrm{d}y\Big].$$

Let u^N be the regular grid as in (3.175). One may split the above term into

$$\sum_{j:j\neq i} h(x_j,x_i) - N\int h(y,x_i)\mathrm{d}y = \sum_{j:j\neq i}(h(x_j,x_i)-h(x_i+\mathrm{u}^N_{j-i},x_i)) + \sum_{j:j\neq i} h(x_i+\mathrm{u}^N_{j-i},x_i) - N\int h(x_i+y,x_i)\mathrm{d}y := \mathrm{E}_i^1+\mathrm{E}_i^2. \quad (3.195)$$

Let us factorize h into $h(x,y)=\tilde h(x,y)N^{-(s+2)}g_s''(x-y)$ by setting

$$\tilde h : (x,y)\mapsto (N\ell_N)^2(\psi_{\mathrm{reg}}(\ell_N^{-1}x)-\psi_{\mathrm{reg}}(\ell_N^{-1}y))^2.$$

Let

$$\gamma_N : x\in \ell_N^{-1}\mathbb{T}\mapsto \Big(1+\sum_{l=1}^p \frac{1}{|x-a_l|^{\alpha_l}}\Big)\mathbb{1}_{|x|<1}+\frac{1}{1+|x|^{2-s}}.$$

By assumption, one can bound the difference of $\tilde h(x,y+a)$ and $\tilde h(x,y)$ by

$$\begin{aligned}|\tilde h(x,y+a)-\tilde h(x,y)| &\leqslant CN^2\ell_N|\psi(\ell_N^{-1}x)-\psi(\ell_N^{-1}y)|(\gamma_N(\ell_N^{-1}(y+a))+\gamma_N(\ell_N^{-1}y))|a|\\ &\leqslant CN^2(\gamma_N(\ell_N^{-1}x)+\gamma_N(\ell_N^{-1}y))(\gamma_N(\ell_N^{-1}(y+a))+\gamma_N(\ell_N^{-1}y))|a||x-y|.\end{aligned}$$

It follows that

$$\begin{aligned}&\mathbb{E}_{\mathbb{P}_{N,\beta}}[|\mathrm{E}_i^1|]\\ &\leqslant C(\beta)\mathbb{E}_{\mathbb{P}_{N,\beta}}\Big[\sum_{j:j\neq i}(\gamma_N(\ell_N^{-1}x_i)+\gamma_N(\ell_N^{-1}x_j))(\gamma_N(\ell_N^{-1}x_j)+\gamma_N(\ell_N^{-1}(x_i+\mathrm{u}^N_{j-i})))\frac{|N(x_j-x_i)-N\mathrm{u}^N_{j-i}|}{|N(x_j-x_i)|^{1+s}}\Big]\end{aligned}$$

By Theorem 3.1.1, one can thus write

$$|\mathrm{E}_i^1|\leqslant C(\beta)(N\ell_N)^{\kappa\varepsilon}\sum_{\substack{j:j\neq i\\ 1\leqslant j\leqslant C_0N\ell_N}}\sum_{l=1}^p (N\ell_N)^{2\alpha_l}\frac{1}{|j-N\ell_N a_l|^{2\alpha_l}}\frac{1}{|i-j|^{1+\frac{s}{2}}} + C(N\ell_N)^{\kappa\varepsilon}\sum_{j:j\neq i}(N\ell_N)^{2(2-s)}\frac{1}{(j+N\ell_N)^{2-s}}\frac{1}{|j-i|^{1+\frac{s}{2}}}.$$

By assumption, $\alpha<1-\frac{s}{2}$ and therefore $2\alpha_l<2-s<2-\frac{s}{2}$. Consequently one may use directly (3.183). After some computations one finds

$$\mathbb{E}_{\mathbb{P}_{N,\beta}}\Big[\sum_{i=1}^N|\mathrm{E}_i^1|\Big]\leqslant C(\beta)(N\ell_N)^{\kappa\varepsilon+\max(2\max\alpha_l,1)}. \quad (3.196)$$

For the discretization error, proceeding as in the proof of Lemma 3.5.2, one can write

$$\begin{aligned}\Big|\sum_{j:j\neq i} h(x_i+\mathrm{u}^N_{j-i},x_i)-N\int h(x_i+y,x_i)\mathrm{d}y\Big| &\leqslant \frac{1}{N}\int_1^{\frac{N}{2}}|\partial_1 h(x_i,\frac{y}{N},x_i)|\mathrm{d}y\\ &\leqslant C\int_1^{\frac{N}{2}}\frac{1}{y^{1+s}}(\gamma_N(\ell_N^{-1}x_i)^2+\gamma_N(\ell_N^{-1}(x_i+\frac{y}{N}))^2)\leqslant C\gamma_N(\ell_N^{-1}x_i)^2.\end{aligned}$$

Summing the above estimate yields

$$\mathbb{E}_{\mathbb{P}_{N,\beta}}\Big[\sum_{i=1}^N|\mathrm{E}_i^2|\Big]\leqslant C(\beta)(N\ell_N)^{\kappa\varepsilon+\max(2\max\alpha_l,1)}. \quad (3.197)$$

In combination with (3.195) and (3.196), this gives (3.194). $\square$

Lemma 3.7.5. *Let ξ satisfying Assumptions 3.1.1 and* $\mathrm{A}^{\mathrm{ext}} := \mathrm{A}_{\ell_N}[\psi_{\mathrm{reg}}] - \widetilde{\mathrm{A}}_{\ell_N}[\psi_{\mathrm{reg}}]$ *with* $\mathrm{A}_{\ell_N}[\psi_{\mathrm{reg}}]$ *and* $\widetilde{\mathrm{A}}_{\ell_N}[\psi_{\mathrm{reg}}]$ *as in (3.112), (3.122). Let* $\gamma > \frac{3-s}{2-s} \vee \frac{1}{s}$. *There exist constants* $C(\beta) > 0, c(\beta) > 0$ *and* $\delta > 0$ *such that*

$$\mathbb{P}_{N,\beta}(|\mathrm{A}^{\mathrm{ext}}| > (N\ell_N)^{\frac{1}{2}}) \leqslant C(\beta) e^{-c(\beta)(N\ell_N)^{\delta}}.$$

Proof. Let $\varepsilon > 0$ be a small number, with $\varepsilon \leqslant \frac{s}{4} \wedge \frac{1-s}{2}$. Define the good event

$$\mathcal{B} = \{X_N \in D_N : \forall 1 \leqslant i \leqslant N, 1 \leqslant k \leqslant N/2, |N(x_{i+k} - x_i) - k| \leqslant k^{\frac{s}{2}+\varepsilon} \vee (N\ell_N)^{\frac{s}{2}+\varepsilon}\} \\ \cap \{X_N \in D_N : \forall 1 \leqslant i \leqslant N, (N\ell_N)^{-\varepsilon} \leqslant N(x_{i+1} - x_i) \leqslant (N\ell_N)^{\varepsilon}\}.$$

In view of Theorem 3.1.1, the event $\mathcal{B}$ has overwhelming probability: there exist $\delta > 0$ depending on ε and $C(\beta) > 0, c(\beta) > 0$ such that

$$\mathbb{P}_{N,\beta}(\mathcal{B}^c) \leqslant C(\beta) e^{-c(\beta)(N\ell_N)^{\delta}}. \tag{3.198}$$

Let us now upper bound $\mathrm{A}^{\mathrm{ext}}$ on $\mathcal{B}$. By assumption the map ψ'_{reg} is bounded by $C\gamma_N$ with

$$\gamma_N : x \in \ell_N^{-1}\mathbb{T} \mapsto \Big(1 + \sum_{l=1}^{p} \frac{1}{|x - a_l|^{\alpha_l}}\Big) \mathbb{1}_{|x|<1} + \frac{1}{1 + |x|^{2-s}}.$$

On can therefore bound $\mathrm{A}^{\mathrm{ext}}$ on $\mathcal{B}$ by

$$\begin{aligned} |\mathrm{A}^{\mathrm{ext}}| &\leqslant C(N\ell_N)^{\kappa\varepsilon} \sum_{i=1}^{N} \sum_{j \in I_N^c} (\gamma_N(\ell_N^{-1}x_i) + \gamma_N(\ell_N^{-1}x_j)) \frac{1}{1 + d(j,i)^{1+\frac{s}{2}-\varepsilon}} \\ &\leqslant C(N\ell_N)^{\kappa\varepsilon} \sum_{i=1}^{N} \gamma_N(\ell_N^{-1}x_i) \frac{1}{1 + d(i, I_N^c)^{\frac{s}{2}-\varepsilon}} + C(N\ell_N)^{\kappa\varepsilon} \sum_{j \in I_N^c} \gamma_N(\ell_N^{-1}x_j) \\ &\leqslant C(N\ell_N)^{\kappa\varepsilon} \sum_{i \in I_N} \gamma_N(\ell_N^{-1}x_i) \frac{1}{d(i, \partial I_N^c)^{\frac{s}{2}-\varepsilon}} + C(N\ell_N)^{\kappa\varepsilon} \sum_{j \in I_N^c} \gamma_N(\ell_N^{-1}x_j). \end{aligned} \tag{3.199}$$

The second sum of the last display is bounded by

$$\Big|\sum_{j \in I_N^c} \gamma_N(\ell_N^{-1}x_j)\Big| \leqslant C(N\ell_N)^{\kappa\varepsilon + (2-s)} \sum_{k \geqslant (N\ell_N)^{\gamma}} \frac{1}{k^{2-s-\varepsilon}} \leqslant C'(N\ell_N)^{\kappa'\varepsilon + (2-s) - \gamma(1-s)}. \tag{3.200}$$

Recall $|I_N| = 1 + 2\lfloor (N\ell_N)^{\gamma} \rfloor$. One may split the first sum into

$$\begin{aligned} &\sum_{i \in I_N} \gamma_N(\ell_N^{-1}x_i) \frac{1}{d(i, \partial I_N^c)^{\frac{s}{2}-\varepsilon}} \\ &= \sum_{i \in I_N, d(i, \partial I_N) > \frac{1}{2}(N\ell_N)^{\gamma}} \gamma_N(\ell_N^{-1}x_i) \frac{1}{d(i, \partial I_N^c)^{\gamma(\frac{s}{2}-\varepsilon)}} + \sum_{i \in I_N, d(i, \partial I_N) \leqslant \frac{1}{2}(N\ell_N)^{\gamma}} \gamma_N(\ell_N^{-1}x_i) \frac{1}{d(i, \partial I_N^c)^{\frac{s}{2}+\varepsilon}} \\ &\leqslant C \sum_{i \in I_N} \gamma_N(\ell_N^{-1}x_i) \frac{1}{(N\ell_N)^{\gamma(\frac{s}{2}-\varepsilon)}} + C \sum_{i \in I_N, d(i, I_N^c) \leqslant \frac{1}{2}(N\ell_N)^{\gamma}} \gamma(\ell_N^{-1}x_i). \end{aligned}$$

Since the singularities of ψ' are in $L^1(\ell_N^{-1}\mathbb{T}, \mathbb{R})$, one can check that on $\mathcal{B}$,

$$\sum_{i \in I_N} \gamma_N(\ell_N^{-1}x_i) \leqslant C(N\ell_N)^{\kappa\varepsilon + 1}. \tag{3.201}$$

Moreover arguing as in (3.200), one finds that on $\mathcal{B}$,

$$\sum_{i\in I_N, d(i,I_N^c)\leqslant \frac{1}{2}(N\ell_N)^\gamma} \gamma_N(\ell_N^{-1}x_i) \leqslant C(N\ell_N)^{\kappa\varepsilon+(2-s)-\gamma(1-s)}.$$

Combining this with (3.199), (3.200) and (3.201), one finally gets that on $\mathcal{B}$,

$$|\mathrm{A}^{\mathrm{ext}}| \leqslant C(N\ell_N)^{\kappa\varepsilon}((N\ell_N)^{2-s-\gamma(1-s-\varepsilon)} + (N\ell_N)^{1-\gamma(\frac{s}{2}-\varepsilon)}).$$

Choosing $\gamma > \frac{3-s}{2-s} \vee \frac{1}{s}$, one thus gets that for $\varepsilon > 0$ small enough, $\mathrm{A}^{\mathrm{ext}}$ satisfies

$$\sup_{\mathcal{B}} |\mathrm{A}^{\mathrm{ext}}| \leqslant C(N\ell_N)^{\frac{1}{2}}.$$

Together with (3.198), this finishes the proof. □

Lemma 3.7.6. *Let ξ satisfying Assumptions 3.1.1. Let $\psi_0 \in \mathcal{C}^\delta(\mathbb{T},\mathbb{R})$ such that $g_s' * \psi_0 = \xi$ with $\int \psi_0 = 0$. Assume that ψ_0'' has p singularities in $a_1,\ldots,a_p$ of order $1+\alpha_1,\ldots,1+\alpha_p$, with $\alpha_i \in (0,1)$, as defined in (3.8). Assume that ξ is supported on $(-\frac{1}{2},\frac{1}{2})$ or that $\ell_N = 1$. Let $\psi \in \mathcal{C}^\delta(\ell_N^{-1}\mathbb{T},\mathbb{R})$ given by*

$$\psi' = -\frac{1}{2c_s}\ell_N^{1-s}(-\Delta)^{\frac{1-s}{2}}(\xi(\ell_N^{-1}\cdot))(\ell_N\cdot) \quad \textit{and} \quad \int \psi = 0.$$

Let $\psi_{\mathrm{reg}} = \psi * K_\ell$ with K_ℓ defined in (3.50) and $\ell = 1/(N\ell_N)^{1-\varepsilon'}$ with $\varepsilon' > 0$. Let $\widetilde{\mathrm{A}}_{\ell_N}[\psi_{\mathrm{reg}}]$ given by (3.122). We have

$$\mathrm{Var}_{\mathbb{P}_{N,\beta}}[\mathbb{E}_{\mathbb{P}_{N,\beta}}[\widetilde{\mathrm{A}}_{\ell_N}[\psi_{\mathrm{reg}}] \mid x_1]] \leqslant C(\beta)((N\ell_N)^{2(1-s)} + N\ell_N).$$

Proof. First recall that for any $\psi : \mathbb{T} \to \mathbb{R}$ smooth enough such that $\int \psi = 0$,

$$\mathbb{E}_{\mathbb{P}_{N,\beta}}[\mathrm{A}_{\ell_N}[\psi]] = 0. \tag{3.202}$$

Indeed, letting $\xi = -2\beta g * \psi$ and $\nabla \Phi : X_N \mapsto \frac{1}{(N\ell_N)^{1-s}}\ell_N(\psi(\ell_N^{-1}x_1),\ldots,\psi(\ell_N^{-1}x_N))$, we have shown in the proof of Proposition 3.5.1 that

$$0 = \mathbb{E}_{\mathbb{P}_{N,\beta}}[\mathcal{L}\Phi] = \mathbb{E}_{\mathbb{P}_{N,\beta}}\Big[\mathrm{Fluct}_N[\xi(\ell_N^{-1}\cdot)] - \frac{1}{(N\ell_N)^{1-s}}\mathrm{Fluct}_N[\psi'(\ell_N^{-1}\cdot)] + \beta \mathrm{A}_{\ell_N}[\psi]\Big].$$

The first marginal of $\mathbb{P}_{N,\beta}$ being the Lebesgue measure on the circle, one obtains (3.202). Let $i_0 = \operatorname{argmin}_{1\leqslant i\leqslant N}|x_i|$. The point is that

$$\begin{aligned}
\mathrm{Law}_{\mathbb{P}_{N,\beta}}(x_1,\ldots,x_N \mid x_1 = x) &= \mathrm{Law}_{\mathbb{P}_{N,\beta}}(x_1+x,\ldots,x_N+x \mid x_1 = 0)\\
&= \mathrm{Law}_{\mathbb{P}_{N,\beta}}(x_1+x,\ldots,x_N+x \mid x_{i_0} = 0)\\
&= \mathrm{Law}_{\mathbb{P}_{N,\beta}}(x_1-x_{i_0}+x,\ldots,x_N-x_{i_0}+x).
\end{aligned}$$

Fix $x_0 \in \mathbb{T}$ and let us denote $\psi_{x_0} = \psi_{\mathrm{reg}}(x_0+\cdot)$. In view of the preceding remark,

$$\begin{aligned}
&\mathbb{E}_{\mathbb{P}_{N,\beta}}[\mathrm{A}_{\ell_N}[\psi_{\mathrm{reg}}] \mid x_1 = x]\\
&= \mathbb{E}_{\mathbb{P}_{N,\beta}}\Big[\iint_{\Delta^c} N^{-(1+s)}g_s'(x-y)N\ell_N(\psi_{x_0}(\ell_N^{-1}(x-x_{i_0}))-\psi_{x_0}(\ell_N^{-1}(y-x_{i_0}))\mathrm{dfluct}_N(x)\mathrm{dfluct}_N(y)\Big].
\end{aligned}$$

By Theorem 3.1.1, $x_{i_0} = O(\frac{1}{N})$ with overwhelming probability. Since the singularities of ψ' are in $L^1(\mathbb{T}, \mathbb{R})$, one obtains by Taylor expansion that

$$\mathbb{E}_{\mathbb{P}_{N,\beta}}[\mathrm{A}_{\ell_N}[\psi_{\mathrm{reg}}] \mid x_1 = x_0] = \mathbb{E}_{\mathbb{P}_{N,\beta}}[\mathrm{A}_{\ell_N}[\psi_{x_0}]] + O_\beta((N\ell_N)^{1-s}).$$

Applying (3.202) to ψ_{x_0} one thus gets

$$\mathbb{E}_{\mathbb{P}_{N,\beta}}[\mathrm{A}_{\ell_N}[\psi_{x_0}]] = 0,$$

which gives

$$\mathrm{Var}_{\mathbb{P}_{N,\beta}}[\mathbb{E}_{\mathbb{P}_{N,\beta}}[\mathrm{A}_{\ell_N}[\psi_{\mathrm{reg}}] \mid x_1 = x_0]] = O_\beta((N\ell_N)^{2(1-s)}). \tag{3.203}$$

Besides, by Lemma 3.7.5, one can write

$$\mathbb{E}_{\mathbb{P}_{N,\beta}}[\mathrm{Var}_{\mathbb{P}_{N,\beta}}[\mathrm{A}^{\mathrm{ext}} \mid x_1 = x_0]] \leqslant \mathrm{Var}_{\mathbb{P}_{N,\beta}}[\mathrm{A}_{\ell_N}] \leqslant C(\beta) N\ell_N. \tag{3.204}$$

Combining (3.203) and (3.204) therefore gives

$$\mathrm{Var}_{\mathbb{P}_{N,\beta}}[\widetilde{\mathrm{A}}_{\ell_N}[\psi_{\mathrm{reg}}] \mid x_1 = x_0] \leqslant C(\beta)((N\ell_N)^{2(1-s)} + N\ell_N).$$

□

Chapter 4

Decay of correlations and thermodynamic limit for the circular Riesz gas

This chapter is based on the article Decay of correlations and thermodynamic limit for the circular Riesz gas, *arXiv preprint arXiv:2209.00396*.

Contents

4.1 Introduction

4.1.1 Setting of the problem

The circular Riesz gas This paper aims to study an interacting particles system on the circle $\mathbb{T} := \mathbb{R}/\mathbb{Z}$, named circular Riesz gas. Let us note that given a parameter $s > 0$, the Riesz s-kernel on $\mathbb{T}$ is defined by

$$g_s : x \in \mathbb{T} \mapsto \lim_{n\to\infty} \Big(\sum_{k=-n}^{n} \frac{1}{|x+k|^s} - \frac{2}{1-s} n^{1-s} \Big). \tag{4.1}$$

Also note that for $s \in (0,1)$, g_s is the fundamental solution of the fractional Laplace equation

$$(-\Delta)^{\frac{1-s}{2}} g_s = c_s(\delta_0 - 1), \tag{4.2}$$

where $(-\Delta)^{\frac{1-s}{2}}$ is the fractional Laplacian on $\mathbb{T}$. Let us now endow $\mathbb{T}$ with the natural order $x < y$ if $x = x' + k$, $y = y' + k'$ with $k, k' \in \mathbb{Z}$, $x', y' \in [0,1)$ and $x' < y'$, allowing one to define the set of ordered configurations

$$D_N = \{X_N = (x_1, \ldots, x_N) \in \mathbb{T}^N : x_2 - x_1 < \ldots < x_N - x_1\}.$$

And let us also consider the pairwise energy

$$\mathcal{H}_N : X_N \in D_N \mapsto N^{-s} \sum_{i \neq j} g_s(x_i - x_j). \tag{4.3}$$

Finally, the circular Riesz gas at inverse temperature $\beta > 0$ corresponds to the probability measure

$$\mathrm{d}\mathbb{P}_{N,\beta} = \frac{1}{Z_{N,\beta}} e^{-\beta \mathcal{H}_N(X_N)} \mathbb{1}_{D_N}(X_N) \mathrm{d}X_N. \tag{4.4}$$

One of the main motivations for studying such an ensemble stems from random matrix theory. For $s = 0$, the Riesz kernel on $\mathbb{R}$, i.e the solution of $(-\Delta)^{\frac{1}{2}} g = \delta_0$, is given up to a multiplicative constant, by the logarithm kernel $-\log|x|$ and by $\log|\sin(x/2)|$ on the circle. Interacting particles systems such as (4.4) on $\mathbb{R}$ with logarithmic interaction and external potential are called *1D log-gases* or *β-ensembles* and the *circular log-gas* or *circular β-ensemble* corresponds to (4.4) with the log kernel on $\mathbb{T}$. As observed by Dyson [106], for some special values of β, namely $\beta \in \{1, 2, 4\}$, the β-ensemble matches the joint law of the N eigenvalues of symmetric/hermitian/symplectic random matrices with independent Gaussian entries and there are numerous results on β-ensembles including results on fluctuations, correlations, infinite volume limit, edge behavior, dynamical properties, relaxation time, etc.

The one-dimensional Riesz gas is a natural extension of β-ensembles and a fundamental model on which to understand the properties of *long-range* particles systems. The interaction (4.1) is indeed long-range when $s \in (0, 1)$ while short-range (or hyper-singular, following the terminology of [41]) when $s \in (1, +\infty)$. The long-range Riesz gas is to this extent a particularly rich model in which interesting phenomena occur, falling outside the classical theory of statistical mechanics (Ruelle, Dobruhsin, Georgii, etc). Riesz gases, as a family of power-law interacting particles systems on $\mathbb{R}^d$, have also received much attention in the physics literature. Apart from the log and Coulomb cases, which are ubiquitous in both mathematical and physics contexts [238], Riesz gases have been found out to be natural models in solid state physics, ferrofluids, elasticity, see for instance [197, 18, 66, 249]. We refer to the nice review [191] which presents a comprehensive account of the literature with many open problems.

The first-order asymptotic of long-range Riesz gases is governed by a mean-field energy functional, which prescribes the macroscopic distribution of particles [80, 237], corresponding in our circular setting (4.4) to the uniform measure of the circle. In Chapter 3, we have investigated the fluctuations of the system and shown that gaps (large spacing between particles) fluctuate much less than for i.i.d variables and much more than in the log-gas case. Additionally we have established a central limit theorem for linear statistics with singular test-functions, which can be applied in particular to characteristic functions of intervals, thus proving rigorously the predictions of the physics literature [191, 234]. The purpose of this very paper is to investigate another class of problems, related to the question of decay of correlations. More precisely we work at proving the optimal decay of gap correlations as in [116] which considers this question for β-ensembles and at proving the uniqueness of the limiting measure. We will show that after rescaling, chosen so that the typical spacing between particles is of order 1, the point process converges in the large N limit to a certain point process $\mathrm{Riesz}_{s,\beta}$.

Infinite volume limit Let $(x_1, \ldots, x_N)$ be distributed according to (4.4). Fix a centering point on $\mathbb{T}$, say $x = 0$, and consider the rescaled point configuration

$$\mathcal{C}_N = \sum_{i=1}^{N} \delta_{N x_i} \mathbb{1}_{|x_i| < \frac{1}{4}}.$$

With a slight abuse of notation, $\mathcal{C}_N$ can be seen as a random variable on point configurations on $\mathbb{R}$. Our goal is to prove that the law of $\mathcal{C}_N$ converges as N tends to infinity, in a suitable topology, to a certain point process $\mathrm{Riesz}_{s,\beta}$. This property is known in statistical physics as the uniqueness of the Gibbs state and is related to phase transitions. Note that while the existence of limiting point processes is standard [129, 94], uniqueness is not expected to hold for general interactions even in dimension one. In the cases of the Gaussian and circular β-ensembles, a unique limit has been exhibited in the seminal works [250, 166] and then shown to be universal in the bulk of β-ensembles for a large class of smooth external potentials in [46, 49], see also [26]. The limiting measure, called the Sine_β process, can be described using a system of coupled stochastic differential equations [250] or alternatively as the spectrum of an infinite-dimensional random operator [251]. In contrast, the one-dimensional Coulomb gas, i.e with kernel $|x|^{-s}$ for $s = -1$, is not translation invariant in infinite volume as proved in [167] and Gibbs states are therefore non-unique. As a consequence, the proof of uniqueness for the long-range gas should use both convexity arguments and the decay of the (effective) interaction. In higher dimension, let us mention that the existence of a limit, up to an extraction, for the microcopic process has been proved for the Coulomb gas in [9], but the uniqueness of such a limit is still a completely open problem.

Decay of the correlations A proof of uniqueness for the limiting measure of the averaged microscopic process is obtained for the log-gas in [111] using a displacement convexity argument showing that the free energy of the infinite gas has a unique minimizer. The strategy of [111] could possibly be applied to the circular Riesz setting, but this method does not provide convergence without averaging nor a speed of convergence. Instead, we propose to examine the rate of decay of correlations, which is much related to this uniqueness problem. Since points are very correlated (fluctuations being small), the appropriate observables to examine are the nearest-neighbor variables. For 1D log-gases, the correlation between $N(x_{i+1} - x_i)$ and $N(x_{j+1} - x_j)$ is proven in [116] to decay in $|i - j|^{-2}$. In this paper we give for the first time a proof of the optimal decay of gap correlations for the circular Riesz gas, which matches the case $s = 0$ found in [116] as well as the predictions of the physics literature [1, 196, 191]. Moreover we establish that this gap correlation exhibits a discontinuity at $s = 1$ with a much faster decay for $s = 1^+$ than $s = 1^-$.

The Helffer-Sjöstrand equation For generic Gibbs measure on D_N (or $\mathbb{R}^N$), the covariance between two smooth enough test-functions is connected to the decay of the solution of a partial differential equation, named the Helffer-Sjöstrand (H.-S.) equation. This equation appears in [242, 243, 149] and is more substantially studied in [148, 147, 201], where it is used to establish correlation decay, uniqueness of the limiting measure and Log-Sobolev inequalities for models with convex interactions. Different approaches to obtaining decay estimates on the solutions of Helffer-Sjöstrand equations have been developed in the statistical physics literature, mainly for Gibbs measure with convex interactions. The random walk representation of [116], already pointed out in [149], [201] and used priorly in [12, 96, 132] for instance, corresponds to a Feynman-Kac representation of the solution of the H.-S. equation. The work [116] then develops a sophisticated homogenization theory for a system of coupled partial differential equations. There are also more analytic methods relying on ideas from stochastic homogenization, see for instance [201, 10, 91, 247].

As aforementioned, the method available in the literature [116] to prove the decay of correlations for the 1D log-gas requires that one controls random walks in random environments, which can be quite technical. The gamble of the present paper is to develop a method relying *only on integration by parts* to treat the long-range Riesz gas with $s \in (0, 1)$. We will first consider as a landmark the hypersingular case $s > 1$ and work with a known distortion argument, used for instance in

[147] or in older techniques to study the decay of eigenfunctions of Schrödinger operators [89]. We will then adapt the method to the long-range case using substantial new inputs including discrete elliptic regularity estimates. Let us emphasize that as it stands, our method cannot be applied to the logarithmic case since it requires to have nearest-neighbor gaps *all* bounded from above by a large N-dependent constant much smaller than N, with overwhelming probability. Note that this was also one of the crucial difficulty in [116] preventing a simple implementation of the techniques of Caffarelli, Chan and Vasseur [65].

4.1.2 Main results

Let us denote d the symmetric distance of $\{1,\dots,N\}$, i.e $d(i,j)=\min(|j-i|,N-|j-i|)$ for each $1\leqslant i,j\leqslant N$. Our first result, which concerns the correlations between gaps in the long-range regime $s\in(0,1)$, is the following:

Theorem 4.1.1 (Decay of the correlations for the long-range Riesz gas)**.** *Let $s\in(0,1)$. For all $\varepsilon>0$, there exists a constant $C>0$ such that for all $\xi,\chi:\mathbb{R}\to\mathbb{R}$ in H^1 and for each $i,j\in\{1,\dots,N\}$,*

$$\begin{aligned}&|\operatorname{Cov}_{\mathbb{P}_{N,\beta}}[\xi(N(x_{i+1}-x_i)),\chi(N(x_{j+1}-x_j))]|\\&\leqslant C(\beta)(\mathbb{E}_{\mathbb{P}_{N,\beta}}[\xi'(x_i)^2]^{\frac12}+|\xi'|_\infty e^{-c(\beta)d(i,j)^\delta})(\mathbb{E}_{\mathbb{P}_{N,\beta}}[\chi'(x_j)^2]^{\frac12}+|\chi'|_\infty e^{-c(\beta)d(i,j)^\delta})\frac{1}{d(i,j)^{2-s-\varepsilon}}.\end{aligned}\tag{4.5}$$

Moreover, given $\varepsilon>0$ small enough and any $n\in\{1,\dots,N\}$, there exist i,j such that $\frac{n}{2}\leqslant|i-j|\leqslant n$ and

$$|\operatorname{Cov}_{\mathbb{P}_{N,\beta}}[N(x_{i+1}-x_i),N(x_{j+1}-x_j)]|\geqslant\varepsilon\frac{1}{d(i,j)^{2-s}}.\tag{4.6}$$

Theorem 4.1.1 is the natural extension of [116], which proves that that for β-ensembles the correlation between $N(x_{i+1}-x_i)$ and $N(x_{j+1}-x_j)$ decays in $|i-j|^{-2}$. The lower bound (4.6) is obtained by using a result from Chapter 3 which gives the leading-order asymptotic of the correlation between $N(x_i-x_1)$ and $N(x_j-x_i)$. Theorem 4.1.1 is in accordance with the expected decay of the truncated correlation function in the mathematical physics and physics literature, see [191].

Let us comment on the norms appearing in (4.5). Our method is mainly based on L^2 arguments for a distortion of the Helffer-Sjöstrand equation system which is captured by the L^2 norm of ξ' and χ'. Besides by assuming that ξ' and χ' are uniformly bounded, we can control the solution on a bad event of exponentially small probability by carrying out a maximum principle argument.

Theorem 4.1.1 should be compared to the decay of correlations in the short-range case, that we quantify in the next theorem:

Theorem 4.1.2 (Decay of correlations for the short-range Riesz gas)**.** *Let $s\in(1,+\infty)$. There exists a constant $\kappa>0$ such that for all $\xi,\chi:\mathbb{R}\to\mathbb{R}$ in H^1 and each $i,j\in\{1,\dots,N\}$, we have*

$$\begin{aligned}&|\operatorname{Cov}_{\mathbb{P}_{N,\beta}}[\xi(N(x_{i+1}-x_i)),\chi(N(x_{j+1}-x_j))]|\\&\leqslant C(\beta)(\mathbb{E}_{\mathbb{P}_{N,\beta}}[\xi'(x_i)^2]^{\frac12}+|\xi'|_\infty e^{-c(\beta)d(i,j)^\delta})(\mathbb{E}_{\mathbb{P}_{N,\beta}}[\chi'(x_j)^2]^{\frac12}+|\chi'|_\infty e^{-c(\beta)d(i,j)^\delta})\Big(\frac{1}{d(i,j)^{s-\varepsilon}}+\frac{1}{N}\Big).\end{aligned}\tag{4.7}$$

Remark 13 (Lagrange multiplier and finite volume correlations)**.** *The factor $\frac{1}{N}$ reflects correlations due to fact that the total number of points in system is fixed, see [117, 210, 38]. In fact, in the framework of Helffer-Sjöstrand equations, it can be interpreted as a Lagrange multiplier associated to the constraint $\sum_{j=1}^N N(x_{j+1}-x_j)=N$, with the convention that $x_{N+1}=x_1$. Interestingly, this correction does not appear in the long-range case (see Theorem 4.1.1).*

It would be interesting to establish the rate of decay of correlations in the case $s = 1$. We believe that for $s = 1$, the situation is similar to the long-range case stated in Theorem 4.1.1 and that correlations decays in $d(i,j)^{-1}\log d(i,j)^{-\kappa}$ for some $\kappa > 0$. Our next result concerns the limit as N tends to infinity of the law of the configuration

$$\sum_{i=1}^{N} \delta_{Nx_i} \mathbb{1}_{|x_i|<\frac{1}{4}}, \tag{4.8}$$

Since $\mathbb{P}_{N,\beta}$ is translation invariant, this is equivalent to centering the configuration around any point $x \in \mathbb{T}$. Let $\mathrm{Conf}(\mathbb{R})$ be the set of locally finite, simple point configurations in $\mathbb{R}$. Given a Borel set $B \subset \mathbb{R}$, we let $N_B : \mathrm{Conf}(\mathbb{R}) \to \mathbb{N}$ be the number of points lying in B. The set $\mathrm{Conf}(\mathbb{R})$ is endowed with the σ-algebra generated by the maps $\{N_B : B \text{ Borel}\}$. A point process is then a probability measure on $\mathrm{Conf}(\mathbb{R})$. Let $(x_1, \ldots, x_N)$ distributed according to $\mathbb{P}_{N,\beta}$. For all $x \in \mathbb{T}$, denote

$$\mathbb{Q}_{N,\beta} = \mathrm{Law}\left(\sum_{i=1}^{N} \delta_{Nx_i} \mathbb{1}_{|x_i|<\frac{1}{4}}\right) \in \mathcal{P}(\mathrm{Conf}(\mathbb{R})). \tag{4.9}$$

Theorem 4.1.3 (Uniqueness of the limiting measure). *Let $s \in (0,1) \cup (1,+\infty)$. There exists a translation invariant point process $\mathrm{Riesz}_{s,\beta}$ such that the sequence of point processes $(\mathbb{Q}_{N,\beta})$ converges to $\mathrm{Riesz}_{s,\beta}$ in the topology of local convergence: for any bounded, Borel and local test function $\phi : \mathrm{Conf}(\mathbb{R}) \to \mathbb{R}$, we have*

$$\lim_{N\to\infty} \mathbb{E}_{\mathbb{Q}_{N,\beta}}[\phi] = \mathbb{E}_{\mathrm{Riesz}_{s,\beta}}[\phi].$$

Theorem 4.1.3 extends the known convergence results for β-ensembles, see [46, 47, 250, 179, 94]. Additionally we are able to give a quantitative bound on the convergence of $\mathbb{Q}_{N,\beta}(x)$ to $\mathrm{Riesz}_{s,\beta}$ for smooth test-functions.

Theorem 4.1.4 (Quantitative convergence). *Let $s \in (0,1) \cup (1,+\infty)$. Let $K \in \{1, \ldots, \frac{N}{2}\}$ and $G : \mathbb{R}^K \to \mathbb{R}$ in H^1. Let $F : X_N \to D_N \mapsto G(N(x_2 - x_1), \ldots, N(x_K - x_{K-1}))$. Fix $x \in \mathbb{R}$ and let us denote $z_1 = \mathrm{argmin}_{z\in\mathcal{C}}|z_i - x|$. Then for all $\varepsilon > 0$, there holds*

$$\mathbb{E}_{\mathbb{P}_{N,\beta}}[F] = \mathbb{E}_{\mathrm{Riesz}_{s,\beta}}[G(z_2 - z_1, \ldots, z_K - z_{K-1})] + O_\beta\left(N^{-\frac{s}{2}+\varepsilon} \sup |\nabla G|^2\right).$$

Combining the CLT of Chapter 3 and the convergence result of Theorem 4.1.3, we can additionally prove a CLT for gaps and discrepancies under the $\mathrm{Riesz}_{s,\beta}$ process. Let $\zeta(s,x)$ the Hurwitz zeta function (see for instance [31]).

Theorem 4.1.5 (Hyperuniformity of the $\mathrm{Riesz}_{s,\beta}$ process). *Let $s \in (0,1)$. Under the process $\mathrm{Riesz}_{s,\beta}$, the sequence of random variables*

$$K^{-\frac{s}{2}}(z_K - z_1 - K)$$

converges in distribution to $Z \sim \mathcal{N}(0, \sigma^2)$ as K tends to infinity with

$$\sigma^2 = \frac{1}{\beta \frac{\pi}{2} s} \mathrm{cotan}\left(\frac{\pi}{2} s\right).$$

Moreover, the variance of $z_K - z_1$ under $\mathrm{Riesz}_{s,\beta}$ may be expanded as

$$\mathrm{Var}_{\mathrm{Riesz}(\beta)}[z_k - z_1] = K^s \sigma^2 + o(K^s). \tag{4.10}$$

In particular, Theorem 4.1.5 implies that the fluctuations of the number of points in a given interval under $\mathrm{Riesz}_{s,\beta}$ is much smaller than for the Poisson process. In the language of [249], this says that $\mathrm{Riesz}_{s,\beta}$ is hyperuniform when $s \in (0,1)$. Our techniques, combined with the method of Chapter 3, can also give a central limit theorem for linear statistics under the $\mathrm{Riesz}_{s,\beta}$ process, as done in [181, 172] for Sine_β.

We conclude this set of results by studying the repulsion of the $\mathrm{Riesz}_{s,\beta}$ process at 0. We show that the probability of having two particles very close to each other decays exponentially.

Proposition 4.1.6. *Fix $\alpha \in (0, \frac{s}{2})$. Let $\varepsilon \in (0,1)$. There exist constants $c(\beta) > 0$ and $C(\beta) > 0$ depending on α and locally uniformly in β such that*

$$\mathbb{P}_{\mathrm{Riesz}_{s,\beta}}(|z_{i+1} - z_i| \geqslant \varepsilon) \geqslant 1 - C(\beta) e^{-c(\beta)\varepsilon^{-\alpha}}.$$

4.1.3 Related questions and perspective

DLR equations and number-rigidity Having proved the existence of an infinite volume limit for the circular Riesz gas, a natural question is then to study the $\mathrm{Riesz}_{s,\beta}$ process from a statistical physics perspective. The first step in that direction is to establish the Dubroshin-Landford-Ruelle (DLR) equations for the $\mathrm{Riesz}_{s,\beta}$ process as was done for the Sine_β process in [94]. We refer to [128] for a presentation of DLR equations in the context of lattice gases and to [93] in the context of point processes. A question of interest is then to study the number-rigidity property within the family of long-range Riesz gases. Number-rigidity is a qualitative property, recently put forward in [131] which says the following: a point process is number-rigid whenever given any compact domain of $\mathbb{R}^d$, the knowledge of the exterior determines in a deterministic fashion the number of points inside the domain. Number-rigidity is a quite surprising phenomenon, which has been proved to occur for the 1D log-gas independently in [86] and in [94] using DLR equations. The recent work [95] also provides a strategy to rule out number-rigidity. Together with the local laws of Chapter 3, the result of [95] should say that the $\mathrm{Riesz}_{s,\beta}$ process is not number-rigid for $s \in (0,1)$. This reflects a major difference between the log-gas which is purely long-range and the Riesz gas for which the effective energy is short-range.

Regularity of the free energy A natural question is to investigate the regularity with respect to β of the infinite volume process $\mathrm{Riesz}_{s,\beta}$. A way to address this problem is to study the regularity of the free energy of the infinite Riesz gas, which is defined by

$$f : \beta \in (0, +\infty) \mapsto \lim_{N \to \infty} -\frac{1}{N}\Big(\log Z_{N,\beta} - \frac{1}{2}\beta N^{2-s} \iint g_s(x-y)\mathrm{d}x\mathrm{d}y\Big). \tag{4.11}$$

The existence of such a limit was obtained in [182] for Riesz gases in arbitrary dimension $d \geqslant 1$ with $\max(0, d-2) < s < d$. In dimension one, one expects that no phase transition occurs for the circular Riesz gas and that the free energy is smooth and even analytic. To prove that f is twice differentiable, a standard approach is to prove that the rescaled variance of the energy under (4.4) converges locally uniformly in β as N tends to infinity. This should be an easy consequence of Theorems 4.1.1 and 4.1.3.

Coulomb and Riesz gases in $d \geqslant 2$ Because the Hamiltonian of the Riesz gas in dimension $d \geqslant 2$ is not convex, it is not clear how one could obtain a result on the decay of correlations. In fact, even showing local laws in the long-range setting is still open, except in the Coulomb case $s = d-2$ tackled into the series of papers [182, 180] culminating into the optimal local law result of [9]. Nevertheless, a quantitative estimate on the translation invariance of the 2D Coulomb gas has been recently obtained in the work [188], building on a Mermin-Wagner's-type argument in the spirit of [127], see also [248] for related considerations. Concerning other Riesz gases, the hypersingular Riesz gas [142] is seemingly a more tractable model to look at since it resembles, as s becomes large, the hard-core model, for which some results are known. For the latter, the translation invariance of the infinite volume Gibbs measures has been proved in [213] by constructing approximate translations avoiding particles collapses.

4.1.4 Outline of the proofs

As mentioned, the heart of the paper is about the analysis of a partial differential equation related to the correlations, in the context of long-range Riesz gases. Given a reasonable Gibbs measure $\mathrm{d}\mu = e^{-H(X_N)}\mathrm{d}X_N$ on D_N (or $\mathbb{R}^N$), the well-known fluctuation-dissipation relation asserts that the covariance between any smooth functions $F, G : D_N \to \mathbb{R}$ may be expressed as

$$\mathrm{Cov}_\mu[F,G] = \mathbb{E}_\mu[\nabla\phi \cdot \nabla G], \tag{4.12}$$

where $\nabla\phi$ solves

$$\begin{cases} -\Delta\phi + \nabla H \cdot \nabla\phi = F - \mathbb{E}_\mu[F] & \text{on } D_N \\ \nabla\phi \cdot \vec{n} = 0 & \text{on } \partial D_N, \end{cases} \tag{4.13}$$

One may recognize the operator $\mathcal{L}^\mu = -\Delta + \nabla H \cdot \nabla$ which is the infinitesimal generator of the Markov semigroup associated to the Langevin dynamics with energy H. The Helffer-Sjöstrand equation then corresponds to the equation obtained by differentiating (4.13), which reads

$$\begin{cases} A_1^\mu \nabla\phi = \nabla F & \text{on } D_N \\ \nabla\phi \cdot \vec{n} = 0 & \text{on } \partial D_N. \end{cases} \quad \text{where} \quad A_1^\mu := \nabla^2 H + \mathcal{L}^\mu \otimes I_N. \tag{4.14}$$

When the Hessian of the energy is uniformly positive-definite, then one can derive by integration by parts a weighted L^2 estimate on $|\nabla\phi|$, which yields a Brascamp-Lieb inequality. Additionally a maximum principle argument can also give a uniform bound on $|\nabla\phi|$ as seen in [149].

The Hamiltonian we are interested in is rather a convex function of the gaps than of the points. Henceforth it is very convenient to rewrite Equation (4.14) in a new of system of coordinates. We define the change of variables

$$\mathrm{Gap}_N : X_N \in D_N \mapsto (N|x_2 - x_1|, \ldots, N|x_N - x_1|) \in \mathbb{R}^N$$

and work on the polyhedron

$$\mathcal{M}_N := \{(y_1, \ldots, y_N) \in (\mathbb{R}^{+*})^N : y_1 + \ldots + y_N = N\}.$$

Assume that the measure of interest μ can be written $\mathrm{d}\mu = e^{-H^{\mathrm{g}} \circ \mathrm{Gap}_N(X_N)} \mathbb{1}_{D_N}(X_N)\mathrm{d}X_N$ and that the test-functions in (4.12) are of the form $F = \xi \circ \mathrm{Gap}_N$ and $G = \chi \circ \mathrm{Gap}_N$. Set $\nu = \mathrm{Gap}_N \# \mu$. Then letting

$$\mathcal{L}^\nu = \nabla H^{\mathrm{g}} \cdot \nabla - \Delta \quad \text{and} \quad A_1^\nu = \nabla^2 H^{\mathrm{g}} + \mathcal{L}^\nu \otimes I_N,$$

one may check that the variance of F under μ can also be represented as

$$\mathrm{Cov}_\mu[F,G] = \mathbb{E}_\nu[\nabla\psi\cdot\nabla\chi], \tag{4.15}$$

where $\nabla\psi$ solves

$$\begin{cases} A_1^\nu\nabla\psi = \nabla\xi + \lambda(e_1+\ldots+e_N) & \text{on } \mathcal{M}_N \\ \nabla\psi\cdot(e_1+\ldots+e_N) = 0 & \text{on } \mathcal{M}_N \\ \nabla\psi\cdot\vec{n} = 0 & \text{on } \partial\mathcal{M}_N. \end{cases} \tag{4.16}$$

Let us mention that the coefficient λ in (4.16) can be seen as a Lagrange multiplier associated to the linear constraint $y_1+\ldots+y_N = N$. Our main problem is to understand how $\partial_j\psi$ decays when $\nabla\xi = e_1$. A first important insight comes from expanding the Hessian of the energy (4.3) in gap coordinates, that we denote $\mathcal{H}_N^{\mathrm{g}}$. Using some rigidity estimates obtained in Chapter 3, one can show that for all $\varepsilon>0$, there exists $\delta>0$ such that

$$\mathbb{P}_{N,\beta}\Big(\Big|\partial_{ij}\mathcal{H}_n^{\mathrm{g}} - \frac{1}{1+d(i,j)^s}\Big| \geqslant \frac{1}{1+d(i,j)^{1+\frac{s}{2}-\varepsilon}}\Big) \leqslant Ce^{-d(i,j)^\delta},$$

where d stands for the symmetric distance on $\{1,\ldots,N\}$, i.e $d(i,j)=\min(|i-j|,N-|i-j|)$. In other words, the interaction matrix in the system (4.16) concentrates around a constant long-range matrix. This already gives a first heuristic to understand the decay of gap correlations stated in Theorem 4.1.1, which is consistent with the decay of $h:=(-\Delta)^{\frac{1-s}{2}}\delta_0$.

Due to the long-range nature of the interaction, the analysis of (4.16) is rather delicate. Let us present an idea of the proof in the short-range case $s>1$ as it will be a model for the long-range case also. To simplify assume that there exist $s>1$ and $c>0$ such that uniformly

$$\nabla^2\mathcal{H}_N^{\mathrm{g}} \geqslant c^{-1}\mathrm{Id} \quad \text{with} \quad |\partial_{ij}\mathcal{H}_N^{\mathrm{g}}| \leqslant \frac{C}{d(i,j)^s} \text{for each } 1\leqslant i,j\leqslant N. \tag{4.17}$$

The matrix $\mathcal{H}_N^{\mathrm{g}}$ then looks like a diagonally dominant matrix. The idea to obtain a decay estimate on the solution of (4.16) is to multiply $\partial_i\psi$ by $d(i,1)^\alpha$ for some well-chosen $\alpha>0$. Let $\mathrm{L}_\alpha = \mathrm{diag}((1+d(j,1)^\alpha)_j)\in\mathcal{M}_N(\mathbb{R})$ be the distortion matrix and $\psi^{\mathrm{dis}} := \mathrm{L}_\alpha\nabla\psi$ which solves

$$\beta(\nabla^2\mathcal{H}_N^{\mathrm{g}} + \delta_{\mathrm{L}_\alpha})\psi^{\mathrm{dis}} + \mathcal{L}^\nu\psi^{\mathrm{dis}} = e_1 + \lambda\mathrm{L}_\alpha(e_1+\ldots+e_N),$$

where $\delta_{\mathrm{L}_\alpha}$ is the commutator

$$\delta_{\mathrm{L}_\alpha} := \mathrm{L}_\alpha\nabla^2\mathcal{H}_N^{\mathrm{g}}\mathrm{L}_\alpha^{-1} - \nabla^2\mathcal{H}_N^{\mathrm{g}}. \tag{4.18}$$

A first key is that the more $\nabla^2\mathcal{H}_N^{\mathrm{g}}$ is diagonal, the more it will commute with diagonal matrices. In fact one can check that for $\alpha < s-\frac{1}{2}$, the commutator (4.18) is small compared to the identity, in the sense of quadratic forms. By integration by parts and using the convexity of $\mathcal{H}_N^{\mathrm{g}}$, this entails an L^2 estimate on ψ^{dis} and therefore a hint on the global decay of $\nabla\psi$. This idea of studying a distorted vector-field is well known in statistical physics, see for instance [147, 89]. By projecting (4.16) in a smaller window we can then improve through a bootstrap argument this first decay estimate.

In the long-range regime $s\in(0,1)$, the above argument no longer works. A natural way of proceeding is to factorize Equation (4.16) around the ground state by multiplying the system by a matrix A close to the inverse of the Riesz matrix $\mathbb{H}_s := (\frac{\mathbb{1}_{i\neq j}}{d(i,j)^s})_{1\leqslant i,j\leqslant N}$. A simple construction can ensure that $\mathrm{A}\nabla^2\mathcal{H}_n^{\mathrm{g}}$ remains uniformly positive-definite but the drawback of the operation is that the differential term $D\psi$ can no longer be controlled. The main novelty of the paper is a method based on the comparison of the two distorted norms

$$\mathbb{E}_\nu\Big[\sum_{i=1}^n d(i,1)^{2\alpha}(\partial_i\psi)^2\Big] \quad \text{and} \quad \mathbb{E}_\nu\Big[\sum_{i=1}^n d(i,1)^{2\gamma}|\nabla(\partial_i\psi)|^2\Big], \tag{4.19}$$

for well-chosen constants $\alpha > 0$ and $\gamma > 0$. The first step is to derive an elliptic regularity estimate on the solution of (4.16). We prove that the solution has a discrete fractional primitive of order $\frac{3}{2} - s$ in L^2 (up to some $n^{\kappa\varepsilon}$ multiplicative factor) provided ψ_i decays fast enough. In a second step we will control $|\mathrm{L}_\gamma \nabla^2 \psi|$ by $|\mathrm{L}_{\frac{\gamma}{2}+\frac{1}{4}} \nabla \psi|$ (up to a residual term that we do not comment here). The proof uses the distortion argument presented in the short-range case, the elliptic regularity estimate and a discrete Gagliardo-Nirenberg inequality. In a third step we control $|\mathrm{L}_\alpha \nabla \psi|$ by $|\mathrm{L}_{\alpha - \frac{1-s}{2}} \nabla^2 \psi|$ by implementing the factorization trick aforementioned. Combining these two inequalities we deduce that for $\alpha = \frac{3}{2} - s$ and $\gamma = 1 - \frac{s}{2}$, each of the terms in (4.19) are small. This gives the optimal global decay on the solution of (4.16), which we then seek to localize.

The proof of localization, which allows one to go from (4.19) to an estimate on a single $\partial_i \psi$, is also quite delicate. Fix an index $j \in \{1, \ldots, N\}$ and let

$$J = \Big\{ i \in \{1, \ldots, N\} : d(i,j) \leqslant \frac{1}{2} d(j,1) \Big\}.$$

Projecting Equation (4.16) on the window J makes an exterior field appear, which takes the form

$$\mathrm{V}_l := -\beta \sum_{i \in J^c} \partial_{il} \mathcal{H}_N^{\mathbf{g}} \partial_i \psi, \quad l \in J. \tag{4.20}$$

We then break V into the sum of an almost constant field $\mathrm{V}^{(1)}$ (looking like $\mathrm{V}_j \sum_{l \in J} e_l$) and a smaller field $\mathrm{V}^{(2)}$. A key is that the equation (4.49) associated to a vector-field proportional to $(e_1 + \ldots + e_N)$ is much easier to analyze. It indeed admits a mean-field approximation, quite similar to the mean-field approximation of (4.14) when F is a linear statistics, see Chapter 3. We then bootstrap the decay of solutions of (4.16). Applying the induction hypothesis to bound (4.20) *and* to bound the decay of (4.16) in the window J, one finally obtains after a finite number of iterations, the optimal result of Theorem 4.1.1.

The uniqueness of the limiting point process stated in Theorem 4.1.3 is then a routine application of our result on decay of correlations (in fact stated for slightly more general systems than (4.16)). Because the existence of an accumulation point of (4.9) in the local topology is standard, the problem can be rephrased into a uniqueness question. We will prove that the sequence (4.9) defines, in some informal sense, a Cauchy sequence. We let $I = \{1, \ldots, n\}$ be the active window and draw the exterior configurations under $\mathbb{P}_{N,\beta}$ and $\mathbb{P}_{N',\beta}$ for distinct values of N and N' which satisfy $n \ll N, N'$. We then let μ_n and ν_n the conditioned measures in gap coordinates, which we try to compare. To allow such a comparison, the strategy is to define a continuous path $\nu(t)$ from μ_n to ν_n by linear interpolation of the exterior energies. Given a test-function $F : \mathbb{R}^n \to \mathbb{R}$ depending on a finite number of coordinates, we can then write

$$\mathbb{E}_{\mu_n}[F] - \mathbb{E}_{\nu_n}[F] = \int_0^1 \mathrm{Cov}_{\nu(t)}[\nabla F, \nabla E(t)] \mathrm{d}t, \tag{4.21}$$

where $E(t)$ corresponds to the exterior energy term. By applying our result on the decay of correlations to the measure $\nu(t)$, we find that (4.21) is small, which easily concludes the proof of Theorem 4.1.3.

4.1.5 Organization of the paper

- Section 4.2 records various preliminary results, such as rigidity estimates on circular Riesz gases and controls on the discrete fractional Laplacian.

- Section 4.3 focuses on the well-posedness of the Helffer-Sjöstrand equation and states some of its basic properties.

- In Section 4.4 we introduce our distortion techniques to prove the decay of correlations in the long-range case.
- Section 4.5 is the heart of the paper. It develops a more involved method to be able to treat the decay of correlations for the long-range Riesz gas.
- Section 4.6 concludes the proof of uniqueness of the limiting measure of Theorem 4.1.3.

4.1.6 Notation

We let d be a distance on $\{1,\ldots,N\}$ defined for each $i,j\in\{1,\ldots,N\}$ by

$$d(i,j)=\min(|j-i|,N-|j-i|).$$

For $x\in\mathbb{R}^n$, we let $|x|$ be the Euclidian norm of x and for $M\in\mathcal{M}_n(\mathbb{R})$, $\|M\|$ be the Hilbert-Schmidt norm of M, i.e

$$\|M\|=\sup_{v\in\mathbb{R}^n\setminus\{0\}}\frac{|Mv|}{|v|}.$$

We let $(e_1,\ldots,e_N)$ be the standard orthonormal basis of $\mathbb{R}^N$.

We either use the notation $\nabla^2 f$ for the Hessian of a real-valued function $f:\mathbb{R}^n\to\mathbb{R}$.

For $A,B\geqslant 0$, we write $A\leqslant C(\beta)B$ or $A=O_\beta(B)$ whenever there exists a constant $C\in\mathbb{R}^+$ locally uniform in β (which might depend on s) such that $A\leqslant CB$.

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4.2 Preliminaries

We begin by recording some useful preliminary results that will be used throughout the paper.

4.2.1 Discrepancy estimates

One shall first state a control on the probability of having two particles very close to each other. According to [52, Lem. 4.5], the following holds:

Lemma 4.2.1. *Let $s\in(0,1)$ and $\alpha\in(0,\frac{s}{2})$. There exist constants $C(\beta)>0$ and $c(\beta)>0$ locally uniform in β such that for each $i\in\{1,\ldots,N\}$ and $\varepsilon>0$,*

$$\mathbb{P}_{N,\beta}(N(x_{i+1}-x_i)\leqslant\varepsilon)\leqslant C(\beta)e^{-c(\beta)\varepsilon^{-\alpha}}.$$

In addition, in view of [52, Th. 1], the fluctuations of large gaps satisfy the following estimate:

Theorem 4.2.2 (Near-optimal rigidity)**.** *Let $s\in(0,1)$. There exists a constant $C(\beta)$ locally uniform in β such that for all $\varepsilon>0$, setting $\delta=\frac{\varepsilon}{4(s+2)}$, for each $i\in\{1,\ldots,N\}$ and $1\leqslant k\leqslant\frac{N}{2}$, we have*

$$\mathbb{P}_{N,\beta}(|N(x_{i+k}-x_i)-k|\geqslant k^{\frac{s}{2}+\varepsilon})\leqslant C(\beta)e^{-c(\beta)k^\delta}.$$

Let us highlight that the variance of $N(x_{i+k}-x_i)$ can in fact be shown to be of order k^s, together with a central limit theorem, see [52, Cor. 1.1].

4.2.2 Fractional Laplacian on the circle

In this subsection we justify the expression of the fundamental solution of the fractional Laplace equation on the circle (4.2). Recall the Hurwitz zeta function [8].

Lemma 4.2.3 (Fundamental solution). *Let g_s be the solution of (4.2). Let $s \in (0,1)$. For all $x \in \mathbb{T}$, we have*

$$g_s(x) = \zeta(s,x) + \zeta(s,1-x) = \lim_{n\to\infty}\Big(\sum_{k=-n}^{n} \frac{1}{|k+x|^s} - \frac{2}{1-s} n^{1-s}\Big). \tag{4.22}$$

Moreover for all $p \geqslant 1$ and all $x \in \mathbb{T}$, we have

$$g_s^{(p)}(x) = (-1)^p s \cdots (s+p-1) \sum_{k\in\mathbb{Z}} \frac{1}{|x+k|^{s+p}}.$$

Proof. We only sketch the main arguments and refer to [52, Sec. 2] for a more detailed proof. Using the Fourier characterization of the fractional Laplacian and applying the formula

$$\lambda^{-\frac{1-s}{2}} = \frac{1}{\Gamma(\frac{1-s}{2})}\int_0^{\infty} e^{-\lambda t} \frac{\mathrm{d}t}{t^{1-\frac{1-s}{2}}},$$

valid for all $\lambda > 0$, one can express g_s as

$$g_s(x) = \frac{c_s}{\Gamma(\frac{1-s}{2})}\int_0^{\infty} (W_t(x)-1)\frac{\mathrm{d}t}{t^{\frac{1+s}{2}}},$$

where W_t is the heat kernel on $\mathbb{T}$, namely

$$W_t(x) = \frac{1}{2\pi}\sum_{k\in\mathbb{Z}} e^{-t|k|^2} e^{ikx} = \frac{1}{\sqrt{4\pi t}}\sum_{k\in\mathbb{Z}} e^{-\frac{|x-k|^2}{4t}}.$$

The proof of (4.22) then follows from Fubini's theorem which allows one to invert the order of integration and summation. $\square$

The kernel g_s can be identified with a periodic function on $\mathbb{R}$ and a crucial consequence of (4.22) is that the restriction of this function to $(0,1)$ is convex, thus allowing the use of various consequences of convexity, such as concentration and functional inequalities.

4.2.3 Discrete and semi-discrete Fourier transforms

In the sequel we will need to consider the discrete Fourier transform of functions defined on the discrete circle $\mathbb{Z}/N\mathbb{Z}$. The Fourier and inverse Fourier transforms on $\mathbb{Z}/N\mathbb{Z}$ are defined by

$$\mathcal{F}_d(f)(\theta) = \sum_{n=0}^{N-1} f(n) e^{in\theta}, \quad \text{for } f:\{1,\dots,N\}\to\mathbb{R}, \quad \theta = \frac{2\pi k}{N}, \quad k \in \{0,\dots,N-1\}, \tag{4.23}$$

$$\mathcal{F}_d^{-1}(\phi)(n) = \frac{1}{N}\sum_{k=0}^{N-1} \phi\Big(\frac{2\pi k}{N}\Big) e^{\frac{-2i\pi k}{N}} \mathrm{d}\theta, \quad \text{for } \phi : \{2k\pi/N : 0 \leqslant k \leqslant N-1\} \to \mathbb{R}, \quad n \in \{1,\dots,N\}. \tag{4.24}$$

Recall that for all f defined on $\{1,\dots,N\}$, $f = \mathcal{F}_d^{-1}\circ\mathcal{F}_d(f)$.

Besides if $f : \mathbb{Z} \to \mathbb{R}$ is in L^2, then the semi-discrete Fourier of $\mathbb{Z}$ defined by

$$\mathcal{F}_{\mathbb{Z}}(f)(\theta) = \sum_{n=0}^{+\infty} f(n)e^{in\theta}, \quad \theta \in [0, 2\pi],$$

belongs to $L^2([0, 2\pi])$ and one can recover f by the Fourier inverse transform

$$f = \mathcal{F}_{\mathbb{Z}}^{-1}(\mathcal{F}_{\mathbb{Z}}(f)),$$

where

$$\mathcal{F}_{\mathbb{Z}}^{-1}(\phi)(n) = \frac{1}{2\pi}\int_0^{2\pi} \phi(\theta)e^{-in\theta}\mathrm{d}\theta, \quad \text{for } \phi \in L^2([0, 2\pi]), \quad n \in \mathbb{Z}.$$

4.2.4 Inversion of the Riesz matrix

We study the inverse of two discrete convolution equations on $\mathbb{Z}/N\mathbb{Z}$. Let us denote $\tilde{g}_s$ the Riesz kernel on $\mathbb{R}$, i.e

$$\tilde{g}_s : x \in \mathbb{R} \mapsto \frac{1}{|x|^s} \in \mathbb{R}^{+\infty} \cup \{+\infty\}.$$

We will be studying the inverses of

$$\mathbb{H}_s = \Big(g_s(d(i,j)\mathbb{1}_{i\neq j})\Big)_{1\leqslant i,j\leqslant N} \in \mathcal{M}_N(\mathbb{R}), \tag{4.25}$$

$$\widetilde{\mathbb{H}}_s = \Big(\tilde{g}_s(d(i,j))\mathbb{1}_{i\neq j}\Big)_{1\leqslant i,j\leqslant N} \in \mathcal{M}_N(\mathbb{R}). \tag{4.26}$$

Lemma 4.2.4 (Decay of the inverse Riesz matrix). *Let* $\mathrm{M} \in \{\mathbb{H}_s, \widetilde{\mathbb{H}}_s\}$. *There exists a constant* $C > 0$ *such that for each* $1 \leqslant i, j \leqslant N$,

$$|(\mathrm{M}^{-1})_{i,j}| \leqslant \frac{C}{1 + d(j,i)^{2-s}}. \tag{4.27}$$

In addition we have

$$\Big|\sum_{i=1}^{N} (\mathrm{M}^{-1})_{i,1}\Big| \leqslant \frac{C}{N^{1-s}}. \tag{4.28}$$

Let us observe that (4.27) is consistent with the decay of the fundamental solution of the fractional Laplacian. Indeed the coefficient $(\mathbb{H}_s)_{i,1}^{-1}$ is given by the i-th coordinate of the solution v of the convolution equation $v * g_s = \delta(1)$ on $\mathbb{Z}/N\mathbb{Z}$. The continuous counterpart of this equation is $g_s * \psi = \delta_0$ on the real line and it is well-known that the solution ψ decays in $|x|^{-(2-s)}$ near the origin.

Proof.

Step 1: the aliasing formula We first consider the case $\mathrm{M} = \mathbb{H}_s$. Let $\psi : \{1, \ldots, N\} \to \mathbb{R}$ be the solution of the convolution equation $g_s * \psi = \delta(1)$ on $\{1, \ldots, N\}$. One can express ψ as the solution of

$$\mathcal{F}_d(\psi)\mathcal{F}_d(g_s) = 1,$$

where $\mathcal{F}_d$ stands for the discrete Fourier transform on $\mathbb{Z}/N\mathbb{Z}$, as defined in (4.23). For shortcut, for all $k \in \{0, \ldots, N-1\}$, we denote $\theta_k = \frac{2\pi k}{N}$. We claim that $\mathcal{F}_d(g_s)$ is non-vanishing, which we will prove afterwards. Let $h \in L^2([0, 2\pi])$ such that for all $\theta \in \{\theta_0, \ldots, \theta_{N-1}\}$,

$$\frac{1}{\mathcal{F}_d(g_s)} = h.$$

The function h shall be specified later. Let $\phi : \mathbb{Z} \to \mathbb{R}$ such that

$$\mathcal{F}_{\mathbb{Z}}(\phi) = h. \tag{4.29}$$

The point is that one can recover ψ from ϕ: for each $1 \leqslant n \leqslant N$, there holds

$$\psi(n) = \sum_{k=0}^{\infty} \phi(n + kN). \tag{4.30}$$

Indeed by computing the discrete Fourier transform of the right-hand side, we find that for all $\theta \in \{\theta_0, \dots, \theta_{N-1}\}$,

$$\begin{aligned}\sum_{n=0}^{N-1} \sum_{k=0}^{\infty} \phi(n + kN) e^{in\theta} &= \sum_{n=0}^{N-1} \sum_{k=0}^{\infty} \phi(n + kN) e^{i(n+kN)\theta} \\ &= \sum_{n=0}^{\infty} \phi(n) e^{in\theta} = h(\theta) = \mathcal{F}_d(\psi)(\theta).\end{aligned}$$

By Fourier inversion, this concludes the proof of the aliasing formula (4.30).

Step 2: discrete and semi-discrete Fourier transform of g_s Let us now compute the discrete Fourier transform of g_s on $\mathbb{Z}/N\mathbb{Z}$. First one can observe that for each $0 \leqslant k \leqslant N-1$,

$$\mathcal{F}_d(g_s)(\theta_k) = \sum_{n=1}^{+\infty} \frac{1}{n^s} e^{in\theta_k} + \sum_{n=1}^{+\infty} \frac{1}{n^s} e^{-in\theta_k}. \tag{4.31}$$

Let us emphasize that the above identity is only true for $\theta \in \{\theta_0, \dots, \theta_{N-1}\}$. The above sum is related to a well-known function called *periodic zeta function* [8], defined by

$$F(x, s) = \sum_{n=1}^{\infty} \frac{e^{2i\pi nx}}{n^s},$$

where $s \in \mathbb{C}$ and $x \in \mathbb{R}$ satisfy $\mathrm{Re}(s) > 1$ if x is an integer and $\mathrm{Re}(s) > 0$ otherwise. One can express (4.31) as

$$\mathcal{F}_d(g_s)(\theta_k) = F\Big(\frac{\theta_k}{2\pi}, s\Big) + F\Big(-\frac{\theta_k}{2\pi}, s\Big), \quad \text{for each } 0 \leqslant k \leqslant N-1.$$

Also, when $\mathrm{Re}(s) > 0$ and $0 < x < 1$, it is known, see [8], that

$$F(x, s) = \frac{\Gamma(1-s)}{(2\pi)^{1-s}} \Big(e^{i\pi\frac{1-s}{2}} \Gamma(1-s, x) + e^{-i\pi\frac{1-s}{2}} \Gamma(1-s, 1-x) \Big).$$

Consequently we have the identity $\mathcal{F}_d(g_s) = S$ on $\{\theta_0, \dots, \theta_{N-1}\}$, where

$$S(\theta) = \frac{2^s \Gamma(1-s)}{\pi^{1-s}} \cos\Big(\frac{\pi(1-s)}{2}\Big) \Big(\Gamma(1-s, \frac{\theta}{2\pi}) + \Gamma(1-s, 1-\frac{\theta}{2\pi}) \Big). \tag{4.32}$$

One can observe that there exists a constant $c > 0$ such that for all $\theta \in [0, 2\pi]$,

$$S(\theta) \geqslant \frac{c}{|\theta|^{1-s}}. \tag{4.33}$$

Step 3: conclusion for $\mathrm{M} = \mathbb{H}_s$ We have shown that the discrete Fourier transform of g_s on $\mathbb{Z}/N\mathbb{Z}$ does not vanish, thus allowing to use (4.30). We now specify $h = S$. Let us define

$$\phi : n \in \mathbb{Z} \mapsto \int_0^1 S(\theta) e^{-in\theta} \mathrm{d}\theta.$$

One can check using (4.31) that

$$|\phi(n)| \leqslant \frac{C}{n^{2-s}}.$$

Since $\phi \in L^2$, by Fourier inversion, one can observe that $\mathcal{F}_d(\phi) = S$. Consequently, applying (4.30), we find that there exists a constant $C > 0$ such that for each $1 \leqslant n \leqslant N$,

$$|\psi(n)| \leqslant C \sum_{k=0}^{\infty} \frac{1}{|n + kN|^{2-s}} \leqslant \frac{C}{n^{2-s}},$$

which proves (4.27) in the case $\mathrm{M} = \mathbb{H}_s$.

Step 4: discrete Fourier transform of $\tilde{g}_s$ We wish to show that the discrete Fourier transform fo $\tilde{g}_s$ is non-vanishing. Let us define the function

$$S_N : \theta \in [0, 2\pi] \mapsto \sum_{n=0}^{N-1} \tilde{g}_s(n) e^{in\theta}. \tag{4.34}$$

One can note that for each $1 \leqslant k \leqslant N$, $\mathcal{F}_d(g_s)(\theta_k) = S_N(\theta_k)$. Moreover (S_N) converges pointwise to the function defined in (4.32). In addition, using Abel's summation formula, we get that for all $\theta \in [0, 2\pi]$,

$$R_N(\theta) := S_N(\theta) - S(\theta) = O\Big(\frac{1}{N^s|\theta|}\Big). \tag{4.35}$$

Consequently there exists a constant $c > 0$ such that for $|\theta| > \frac{c}{N}$, $S_N(\theta)$ is non zero. Let us check that S_N does not cancel on $[0, \frac{c}{N}]$. The point is that for $\theta = \frac{\alpha}{N}$ with $|\alpha| \leqslant c$,

$$S_N(\theta) = N^{1-s} \frac{1}{N} \sum_{k=0}^{N-1} \frac{1}{(\frac{k}{N})^s} e^{2i\pi\alpha \frac{k}{N}} = N^{1-s}(c_\alpha + o_N(1))$$

with $c_\alpha > 0$. We thus deduce that S_N has no zero on $[0, 2\pi]$ and one may apply the aliasing formula (4.30). Let us define

$$\phi : n \in \mathbb{Z} \mapsto \int S_N(\theta) e^{in\theta}.$$

Step 5: bound on ϕ In view of (4.33) and (4.35), there exists $c_1 > 0$ such that for $|\theta| > \frac{c_1}{N}$, $0 < \frac{R_N(\theta)}{S(\theta)} < \frac{1}{2}$ and

$$\frac{1}{S_N(\theta)} = \frac{1}{S(\theta)} \Big(1 + \sum_{k=1}^{\infty} (-1)^k \Big(\frac{R_N(\theta)}{S(\theta)}\Big)^k\Big).$$

For $k = 1$ using (4.35), we have

$$\begin{aligned} &\int_{[-\pi,\pi]\setminus[-\frac{c_1}{N}, \frac{c_1}{N}]} \frac{R_N(\theta)}{S(\theta)^2} e^{-in\theta} \mathrm{d}\theta \\ &\quad = - \sum_{l=N+1}^{\infty} \Big(\frac{1}{(l+1)^s} - \frac{1}{l^s}\Big) \int_{[-\pi,\pi]\setminus[-\frac{c_1}{N}, \frac{c_1}{N}]} \frac{1}{S(\theta)^2} \frac{\cos(\frac{(l+1)\theta}{2}) \sin(\frac{l\theta}{2})}{\sin(\frac{\theta}{2})} e^{-in\theta} \mathrm{d}\theta. \end{aligned} \tag{4.36}$$

Let $l \geqslant N+1$. Let us define $h, G_{n,l} : [-\pi,\pi] \to \mathbb{R}$ such that for all $\theta \in [-\pi,\pi]$

$$h(\theta) = \frac{1}{S(\theta)^2 \sin(\frac{\theta}{2})}, \quad G''_{n,l}(\theta) = \cos\Big(\frac{(l+1)\theta}{2}\Big)\sin\Big(\frac{l\theta}{2}\Big)e^{-in\theta}.$$

Noting

$$|h(\theta)| \leqslant C|\theta|^{1-2s}, \quad |h''(\theta)| \leqslant \frac{C}{|\theta|^{1+2s}}, \quad |G_{n,l}|(\theta) \leqslant \frac{C}{l}, \quad |\widetilde{G}_{n,l}|(\theta) \leqslant \frac{C}{l^2},$$

one gets by integration by parts,

$$\int_{[-\pi,\pi]\setminus[-\frac{c_1}{N},\frac{c_1}{N}]} \frac{1}{S(\theta)^2}\frac{\cos(\frac{(l+1)\theta}{2})\sin(\frac{l\theta}{2})}{\sin(\frac{\theta}{2})}e^{-in\theta}\mathrm{d}\theta = -\int_{-\pi}^{\pi} h'(\theta)G'_{n,l}(\theta)\mathrm{d}\theta + O\Big(\frac{1}{N^{2-2s}}\Big).$$

Integrating by parts again gives

$$-\int_{-\pi}^{\pi} h'_l(\theta)G'_{n,l}(\theta)\mathrm{d}\theta = \int_{-\pi l}^{\pi l} h''(\theta)G_{n,l}(\theta)\mathrm{d}\theta = \frac{1}{l}\int_{-\pi}^{\pi} h''(\theta/l)G_{n,l}(\theta/l)\mathrm{d}\theta = O\Big(\frac{1}{l^{2(1-s)}}\Big).$$

Inserting this into (4.36) and summing this over l yields

$$\int_{[-\pi,\pi]\setminus[-\frac{c_1}{N},\frac{c_1}{N}]} \frac{R_N(\theta)}{S(\theta)^2}e^{-in\theta}\mathrm{d}\theta = O\Big(\frac{1}{N^{2-s}}\Big).$$

Let $2 \leqslant k \leqslant \frac{2}{s}-1$. By performing iterative integration by parts as in the foregoing computations, we find that

$$\Big|\int_{[-\pi,\pi]\setminus[-\frac{c_1}{N},\frac{c_1}{N}]} \frac{R_N(\theta)^k}{S(\theta)^{k+1}}e^{-in\theta}\mathrm{d}\theta\Big| = O\Big(\frac{1}{N^{2-s}}\Big).$$

Finally if $k \geqslant \frac{2}{s}$, the integral at hand is convergent at infinity and by (4.33), (4.35) we have

$$\Big|\int_{[-\pi,\pi]\setminus[-\frac{c_1}{N},\frac{c_1}{N}]} \frac{R_N(\theta)^k}{S(\theta)^{k+1}}e^{-in\theta}\mathrm{d}\theta\Big| \leqslant \frac{C}{N^{ks}}\int_{[-\pi,\pi]\setminus[-\frac{c_1}{N},\frac{c_1}{N}]} \frac{1}{|\theta|^{s(k+1)-1}} + O\Big(\frac{1}{N^{2-s}}\Big) = O\Big(\frac{1}{N^{2-s}}\Big).$$

We conclude that

$$\int_{[-\pi,\pi]\setminus[-\frac{c_1}{N},\frac{c_1}{N}]} \frac{\mathrm{d}\theta}{S_N(\theta)} = \int_{[-\pi,\pi]\setminus[-\frac{c_1}{N},\frac{c_1}{N}]} \frac{\mathrm{d}\theta}{S(\theta)} + O\Big(\frac{1}{N^{2-s}}\Big). \tag{4.37}$$

Furthermore one can easily check that

$$\int_{[-\frac{c_1}{N},\frac{c_1}{N}]} \frac{1}{S_N(\theta)}\mathrm{d}\theta \leqslant CN^s\int_{[-\frac{c_1}{N},\frac{c_1}{N}]} |\theta|\mathrm{d}\theta = O\Big(\frac{1}{N^{2-s}}\Big) \quad \text{and} \quad \int_{[-\frac{c_1}{N},\frac{c_1}{N}]} \frac{1}{S(\theta)}\mathrm{d}\theta = O\Big(\frac{1}{N^{2-s}}\Big). \tag{4.38}$$

Combining (4.37) and (4.38) we get

$$\int_{-\pi}^{\pi} \frac{1}{S_N(\theta)}e^{-in\theta}\mathrm{d}\theta = \int_{-\pi}^{\pi} \frac{1}{S(\theta)}e^{-in\theta}\mathrm{d}\theta + O\Big(\frac{1}{N^{2-s}}\Big). \tag{4.39}$$

We deduce that there exists a constant $C > 0$ such that for each $n \in \mathbb{Z}$,

$$|\phi(n)| \leqslant \frac{C}{n^{2-s}}.$$

In particular, $\phi \in L^2$ and $\mathcal{F}_d(\phi) = \frac{1}{S_N}$. Consequently using (4.30), one deduces that there exists a constant $C > 0$ such that for each $n \in \mathbb{Z}$,

$$|\psi(n)| \leqslant \frac{C}{n^{2-s}}.$$

The estimate (4.28) is straightforward. □

Remark 14 (Discrete fractional primitive). *In view of (4.32) the convolution of $f:\mathbb{Z}/N\mathbb{Z}\to\mathbb{R}$ with g_s formally corresponds to a fractional primitive of f of order $1-s$.*

4.3 The Helffer-Sjöstrand equation

In this section we introduce some standard results on Helffer-Sjöstrand equations. We first recall basic properties valid for a certain class of convex Gibbs measures. We then study an important change of variables and rewrite the Helffer-Sjöstrand in gap coordinates. For the class of Gibbs measures we are interested in, the energy is a convex function of the gaps. This allows one to derive a maximum principle for solutions, which will be a central tool in the rest of the paper.

4.3.1 Well-posedness

We start by explaining the principle of Helffer-Sjöstrand representation and give some existence and uniqueness results. The subsection is similar to Chapter 3 and follows partly the presentation of [10]. Let μ be a probability measure on D_N in the form

$$\mathrm{d}\mu = e^{-H(X_N)}\mathbb{1}_{D_N}(X_N)\mathrm{d}X_N,$$

where $H:D_N\to\mathbb{R}$ is a smooth and convex function. Given a smooth test-function $F:D_N\to\mathbb{R}$, we wish to rewrite its variance in a convenient and effective way. Let us recall the integration by parts formula for μ. Let $\mathcal{L}^\mu$ be the operator acting on $\mathcal{C}^\infty(D_N,\mathbb{R})$ given by

$$\mathcal{L}^\mu = \nabla H\cdot\nabla - \Delta,$$

where ∇ and Δ are the standard gradient and Laplace operators on $\mathbb{T}^N$. The operator $\mathcal{L}^\mu$ is the generator of the Langevin dynamics associated to the energy H of which μ is the unique invariant measure. By integration by parts under μ, for any functions $\phi,\psi\in\mathcal{C}^\infty(D_N,\mathbb{R})$ such that $\nabla\phi\cdot\vec{n}=0$ on ∂D_N, we can write

$$\mathbb{E}_\mu[\psi\mathcal{L}^\mu\phi] = \mathbb{E}_\mu[\nabla\psi\cdot\nabla\phi]. \tag{4.40}$$

This formula may be proved by integration by parts under the Lebesgue measure on D_N.

Assume that the Poisson equation

$$\begin{cases} \mathcal{L}^\mu\phi = F-\mathbb{E}_\mu[F] & \text{on } D_N \\ \nabla\phi\cdot\vec{n}=0 & \text{on } \partial D_N \end{cases} \tag{4.41}$$

admits a weak solution in a certain functional space. Then, by (4.40), the variance of F under μ can be expressed as

$$\mathrm{Var}_\mu[F] = \mathbb{E}_\mu[\nabla F\cdot\nabla\phi].$$

The above identity is called the Helffer-Sjöstrand representation formula. Let us differentiate (4.41). Formally, for all $\phi\in\mathcal{C}^\infty(D_N,\mathbb{R})$, we have

$$\nabla\mathcal{L}^\mu\phi = A_1\nabla\phi,$$

where A_1^μ is the so-called Helffer-Sjöstrand operator given by

$$A_1^\mu = \nabla^2 H + \mathcal{L}^\mu\otimes I_N,$$

with $\mathcal{L}^\mu\otimes I_N$ acting diagonally on $L^2(\{1,\ldots,N\},\mathcal{C}^\infty(D_N,\mathbb{R}))$. Therefore the solution $\nabla\phi$ of (4.41) formally satisfies

$$\begin{cases} A_1^\mu\nabla\phi = \nabla F & \text{on } D_N \\ \nabla\phi\cdot\vec{n}=0 & \text{on } \partial D_N. \end{cases} \tag{4.42}$$

This partial differential equation is called the Helffer-Sjöstrand equation. Let us now introduce the appropriate functional spaces to make these derivations rigorous. Let us define the norm

$$\|F\|_{H^1(\mu)} = \mathbb{E}_\mu[F^2]^{\frac{1}{2}} + \mathbb{E}_\mu[|\nabla F|^2]^{\frac{1}{2}}.$$

Let $H^1(\mu)$ be the completion of $\mathcal{C}^\infty(D_N)$ with respect to the norm $\|\cdot\|_{H^1(\mu)}$. Let also define the norm

$$\|F\|_{H^{-1}(\mu)} = \sup\{|\mathbb{E}_\mu[FG]| : G \in H^1(\mu), \|G\|_{H^1(\mu)} \leqslant 1\}.$$

We denote $H^{-1}(\mu)$ the dual of $H^1(\mu)$, that is the completion of $\mathcal{C}^\infty(D_N)$ with respect to the norm $\|\cdot\|_{H^{-1}(\mu)}$. We wish to prove that under mild assumptions on F, Equation (4.42) is well-posed, in the sense of $L^2(\{1,\dots,N\}, H^{-1}(\mu))$. Let us now make the following assumptions on μ:

Assumptions 4.3.1. *Assume that μ is a probability measure on D_N written*

$$\mathrm{d}\mu = e^{-H(X_N)}\mathbb{1}_{D_N}(X_N)\mathrm{d}X_N,$$

with $H : D_N \to \mathbb{R}$ in the form

$$H : X_N \mapsto \sum_{i\neq j} \chi(|x_i - x_j|),$$

with $\chi : \mathbb{R}^{+*} \to \mathbb{R}$ satisfying

$$\chi \in \mathcal{C}^2(\mathbb{R}^{+*}, \mathbb{R}) \quad \text{and} \quad \chi'' \geqslant c > 0.$$

In our applications, χ is often given by g_s or a variant of it and the density of the measure μ is not necessarily bounded from below with respect to the Lebesgue measure on D_N. Additionally, the measure μ does not satisfies a uniform Poincaré inequality. Due to these limitations, to prove the well-posedness of (4.42), we further assume that F is a function of the gaps. We denote

$$\Pi : X_N \in D_N \mapsto (x_2 - x_1, \dots, x_N - x_1) \in \mathbb{T}^{N-1}. \tag{4.43}$$

We also let μ' be the push-forward of μ by the map Π:

$$\mu' = \mu \circ \Pi^{-1}.$$

We can now state the following well-posedness result:

Proposition 4.3.1 (Existence and representation). *Let μ satisfying Assumptions 4.3.1. Let $F \in H^1(\mu)$. Assume that F is in the form $F = G \circ \Pi$, $G \in H^1(\mu')$ or that $\mu \geqslant c > 0$. Then there exists a unique $\nabla\phi \in L^2(\{1,\dots,N\}, H^1(\mu))$ such that*

$$\begin{cases} A_1^\mu \nabla\phi = \nabla F & \text{on } D_N \\ \nabla\phi \cdot \vec{n} = 0 & \text{on } \partial D_N, \end{cases} \tag{4.44}$$

with the first identity being, for each coordinate, an identity on elements of $H^{-1}(\mu)$. Moreover the solution of (4.44) is the unique minimizer of the functional

$$\nabla\phi \mapsto \mathbb{E}_\mu[\nabla\phi \cdot \nabla^2 H \nabla\phi + |\nabla^2\phi|^2 - 2\nabla F \cdot \nabla\phi],$$

over maps $\nabla\phi \in L^2(\{1,\dots,N\}, H^1(\mu))$. The variance of F may be represented as

$$\mathrm{Var}_\mu[F] = \mathbb{E}_\mu[\nabla\phi \cdot \nabla F] \tag{4.45}$$

and the covariance between F any function $G \in H^1(\mu)$ as

$$\mathrm{Cov}_\mu[F, G] = \mathbb{E}_\mu[\nabla\phi \cdot \nabla G].$$

The identity (4.45) is called the Helffer-Sjöstrand formula. The proof of Proposition 4.3.1 is postponed to the Appendix, see Section 4.7.2.

Remark 15 (On the boundary condition). *The boundary condition $\nabla\phi \cdot \vec{n} = 0$ on ∂D_N means that if $x_i = x_j$, then $\partial_i \phi(X_N) = \partial_j \phi(X_N)$.*

Remark 16 (Link to the Monge-Ampère equation). *We formally discuss the link between (4.44) and the Monge-Ampère equation. Let $F : D_N \to \mathbb{R}$ be a smooth test-function. For all $t \geqslant 0$, consider the measure $\mathrm{d}\mu_t = \frac{e^{tF}}{\mathbb{E}_\mu[e^{tF}]}\mathrm{d}\mu$. According to well-known optimal transportation results [60], the measure μ_t can be written $\mu_t = \mu \circ \nabla\Phi_t^{-1}$ with $\Phi_t : D_N \to \mathbb{R}$ solution of the Monge-Ampère equation*

$$-\log\det D\nabla\Phi_t + H \circ \nabla\Phi_t - H = tF - \log \mathbb{E}_\mu[e^{tF}].$$

Formally, since $\nu(t) = \mu + t\nu + o(t)$, one expects that $\Phi_t = \mathrm{Id} + t\phi + o(t)$. Linearizing the above equation in t formally gives

$$\mathcal{L}^\mu \phi = F - \mathbb{E}_\mu[F],$$

which is the Poisson equation (4.41). The boundary condition in (4.44) reflects the fact that for all $t \geqslant 0$, $\nabla\Phi_t$ maps D_N on itself.

Proposition 4.3.2. *Let μ satisfying Assumptions 4.3.1. Let $v \in L^2(\{1,\ldots,N\}, H^{-1}(\mu'))$ such that*

$$v \cdot (e_1 + \ldots + e_N) = 0.$$

There exists a unique $\psi \in L^2(\{1,\ldots,N\}, H^1(\mu))$ such that

$$\begin{cases} A_1^\mu \psi = v & \text{on } D_N \\ \psi \cdot \vec{n} = 0 & \text{on } \partial D_N. \end{cases} \tag{4.46}$$

In addition if $v = \nabla F \in L^2(\{1,\ldots,N\}, H^{-1}(\mu'))$, then the solution of (4.46) is given by the solution of (4.44).

The proof of Proposition 4.3.2 is also given in the Appendix.

4.3.2 Change of coordinates

In the sequel we will study the decay of correlations in gap coordinates. Define the map

$$\mathrm{Gap}_N : X_N \in D_N \mapsto (N(x_2 - x_1), \ldots, N(x_1 - x_N)) \in \mathcal{M}_N,$$

where

$$\mathcal{M}_N = \mathrm{Gap}_N(D_N) = \{Y_N \in (\mathbb{R}^{+*})^N : y_1 + \ldots + y_N\}. \tag{4.47}$$

Since $\mathcal{M}_N$ is not an open subset of D_N, Proposition 4.3.1 should be slightly adapted. Let μ satisfying Assumptions 4.3.1 and $H^{\mathrm{g}} : \mathcal{M}_N \to \mathbb{R}$ be such that

$$H = H^{\mathrm{g}} \circ \mathrm{Gap}_N.$$

Define the generator acting on $\mathcal{C}^\infty(\mathcal{M}_N, \mathbb{R})$,

$$\mathcal{L}^\nu = \nabla H^{\mathrm{g}} \cdot \nabla - \Delta,$$

with ∇ and Δ the standard gradient and Laplace operator on $\mathcal{M}_N$. Also let A_1^ν be the Helffer-Sjöstrand operator acting on $L^2(\{1,\ldots,N\}, \mathcal{C}^\infty(\mathcal{M}_N, \mathbb{R}))$:

$$A_1^\nu = \nabla^2 H^{\mathrm{g}} + \mathcal{L}^\nu \otimes I_N.$$

Let $F : D_N \to \mathbb{R}$ in the form $F = G \circ \mathrm{Gap}_N$ with $G : \mathbb{R}^N \to \mathbb{R}$ smooth. Let us rewrite Equation (4.44) in gap coordinates. One can expect that the solution $\nabla\phi$ of (4.44) can be factorized into $\phi = \psi \circ \mathrm{Gap}_N$ with $\nabla\psi \in L^2(\{1, \dots, N\}, H^1(\nu))$. Let us derive some formal computation to conjecture the equation satisfied by $\nabla\psi$. For all $t \geqslant 0$, let $\mathrm{d}\nu_t = \frac{e^{tG}}{\mathbb{E}_\nu[e^{tG}]}\mathrm{d}\nu$. In view of Remark 16, we wish to find a map $\nabla\psi \in L^1(\{1, \dots, N\}, H^1(\nu))$ such that in a certain sense,

$$\nu \circ (\mathrm{Id} + t\nabla\psi) = \nu_t + o(t). \tag{4.48}$$

Since ν and ν_t are both measures on $\mathcal{M}_N$, one can observe that $\sum_{i=1}^N \partial_i\psi = 0$. It is standard the the Gibbs measure ν_t is the minimizer of the functional

$$\nu \in \mathcal{P}(\mathcal{M}_N) \mapsto \mathbb{E}_\nu[H^{\mathrm{g}} + tG] + \mathrm{Ent}(P),$$

where Ent stands for the entropy on $\mathcal{M}_N$. Equation (4.48) is compatible with the variational characterization if $\nabla\psi$ minimizes

$$\nabla\psi \mapsto \mathbb{E}_\nu[\nabla\psi \cdot \nabla^2\mathcal{H}^{\mathrm{g}}\nabla\psi + |\nabla^2\psi|^2 - 2\nabla G \cdot \nabla\psi],$$

over maps $\nabla\psi \in L^2(\{1, \dots, N\}, H^1(\nu))$ such that $\sum_{i=1}^N \partial_i\psi = 0$ and $\nabla\psi \cdot \vec{n} = 0$ on $\partial\mathcal{M}_N$. The Lagrange equation associated for the minimality of $\nabla\psi$ reads

$$A_1^\nu \nabla\psi = \nabla G + \lambda(e_1 + \dots + e_N),$$

where $\lambda : \mathcal{M}_N \to \mathbb{R}$ is a smooth function. We now state this result in the following proposition:

Proposition 4.3.3. *Let μ satisfying Assumptions 4.3.1. Let $F \in H^1(\mu)$ in the form $F = G \circ \mathrm{Gap}_N$ with $G \in H^1(\nu)$. There exists a unique $\nabla\psi \in L^2(\{1, \dots, N\}, H^1(\nu))$ solution of*

$$\begin{cases} A_1^\nu \nabla\psi = \nabla G + \lambda(e_1 + \dots + e_N) & \text{on } \mathcal{M}_N \\ \nabla\psi \cdot (e_1 + \dots + e_N) = 0 & \text{on } \mathcal{M}_N \\ \nabla\psi \cdot \vec{n} = 0 & \text{on } \partial\mathcal{M}_N, \end{cases} \tag{4.49}$$

with λ satisfying

$$\lambda = \frac{1}{N}(e_1 + \dots + e_N) \cdot (\nabla^2 H^{\mathrm{g}} \nabla\psi - \nabla G). \tag{4.50}$$

The variance of F can be represented as

$$\mathrm{Var}_\mu[F] = \mathbb{E}_\nu[\nabla G \cdot \nabla\psi].$$

Furthermore, $\nabla\psi$ is the unique minimizer of

$$\nabla\psi \mapsto \mathbb{E}_\nu[\nabla\psi \cdot \nabla^2 H^{\mathrm{g}}\nabla\psi + |\nabla^2\psi|^2 - 2\nabla G \cdot \nabla\psi],$$

over maps $\nabla\psi \in L^2(\{1, \dots, N\}, H^1(\nu))$ such that $\nabla\psi \cdot (e_1 + \dots + e_N) = 0$.

The proof of Proposition 4.3.3 is postponed to the Appendix, see Section 4.7.2.

Remark 17. *There are several manners to factorize the energy (4.3) since we are working on the circle. We choose the more natural one and set*

$$\mathcal{H}_N^{\mathrm{g}} : Y_N \in \mathcal{M}_N \mapsto N^{-s} \sum_{i=1}^N \sum_{k=1}^{N/2} g_s(y_i + \dots y_{i+k})(2\mathbb{1}_{k \neq N/2} + \mathbb{1}_{k=N/2}).$$

One may check that for each $i \in \{1, \ldots, N\}$ and $Y_N \in \mathcal{M}_N$,

$$\partial_i \mathcal{H}_N^{\mathrm{g}}(Y_N) = \sum_{k=1}^{N/2} \sum_{l: i-k<l\leqslant i} N^{-(1+s)} g_s'\Big(\frac{y_i + \ldots + y_{i+l}}{N}\Big)(2\mathbb{1}_{k\neq N/2} + \mathbb{1}_{k=N/2}) \tag{4.51}$$

and for each $i, j \in \{1, \ldots, N\}$ and $Y_N \in \mathcal{M}_N$,

$$\partial_{ij} \mathcal{H}_N^{\mathrm{g}}(Y_N) = \sum_{\substack{1\leqslant k,k'\leqslant N/2 \\ |k-k'|\leqslant N/2}} N^{-(1+s)} g_s''\Big(\frac{y_{i-k} + \ldots + y_{j+k'}}{N}\Big)(2\mathbb{1}_{|k-k'|\neq N/2} + \mathbb{1}_{|k-k'|=N/2}). \tag{4.52}$$

Recall that under the Gibbs measure (4.4), for large k, the spacing $N(x_{i+k} - x_i)$ concentrates around k. The expression (4.52) then tells us that the Hessian of the energy in gap coordinates concentrates around a constant matrix with off-diagonal entries decaying in $d(i,j)^{-s}$, similar to (4.25) or (4.26).

4.3.3 The Brascamp-Lieb inequality

We now recall the Brascamp-Lieb inequality, a basic concentration inequality for strictly convex log-concave measures [55]. In our context, the measure μ is not strictly log-concave, but its pushforward ν is, therefore allowing one to upper bound the variance of any smooth function of the gaps in the following way:

Lemma 4.3.4. *Let $\mathcal{A} \subset D_N$ be a convex domain with a piecewise smooth boundary. Let $F = G \circ \mathrm{Gap}_N$ with $G \in H^1(\nu)$. There holds*

$$\mathrm{Var}_\mu[F \mid \mathcal{A}] \leqslant \mathbb{E}_\mu[\nabla F \cdot (\nabla^2 H)^{-1} \nabla F \mid \mathcal{A}].$$

4.3.4 Localization

In this subsection we record a crucial convexity Lemma, which is due to Brascamp, see [55]. This lemma is based on the Brascamp-Lieb inequality for log-concave measures on D_N, originally derived in [56] on $\mathbb{R}^N$, see also Lemma 4.3.4.

Lemma 4.3.5. *Let μ be a measure on D_N in the form $\mathrm{d}\mu = e^{-H}\mathrm{d}X_N$, with H smooth enough. On D_N let us introduce the coordinates $x = (x_1, \ldots, x_n)$ and $y = (x_{n+1}, \ldots, x_N)$. Assume that H may be written in the form $H(x,y) = H_1(x) + H_2(x,y)$ with $\nabla^2 H_2$ non-negative. Let $\widetilde{\mu}$ be the push forward of μ by the map $X_N \mapsto (x_1, \ldots, x_n)$. Then, the measure $\widetilde{\mu}$ may be written in the form $\mathrm{d}\widetilde{\mu}(x) = e^{-\widetilde{H}(x)}\mathrm{d}x$, with*

$$\widetilde{H}(x) = -\log \int e^{-H(x,y)} \mathrm{d}y$$

and $\widetilde{H}$ satisfies

$$\nabla^2 \widetilde{H} \geqslant \nabla^2 H_1.$$

Moreover, we have

$$\partial_i \widetilde{H}(x) = \partial_i H(x) - \mathbb{E}_{\mu(\cdot|x)}[\partial_i H_2], \quad \textit{for each} \quad 1 \leqslant i \leqslant n, \quad x \in D_n, \tag{4.53}$$

$$\partial_{ij} \widetilde{H}(x) = \partial_{ij} H(x) - \mathrm{Cov}_{\mu(\cdot|x)}[\partial_i H_2, \partial_j H_2], \quad \textit{for each} \quad 1 \leqslant i,j \leqslant n, \quad x \in D_n. \tag{4.54}$$

4.3.5 Maximum principle

In this subsection we derive a useful maximum principle, which allows one to bound the supremum of the L^2 norm of the solution in presence of a uniformly convex Hamiltonian. This maximum principle is fairly standard on $\mathbb{R}^N$, see for instance [149, Sec. 10]. We adapt the proof to make it work on D_N and $\mathcal{M}_N$. A more subtle analysis could perhaps permit to treat general convex domains.

Proposition 4.3.6. *Let μ satisfying Assumptions 4.3.1 and $\nu = \mathrm{Gap}_N\#\mu$. Assume additionally that $\lim_{x\to 0}\chi'(x) = -\infty$. Let $\mathsf{M} : \mathcal{M}_N \to \mathcal{S}_N(\mathbb{R})$ be a measurable map. Assume that there exists a constant $c > 0$ such that for for all $U_N \in \mathbb{R}^N$,*

$$U_N \cdot \mathsf{M}U_N \geqslant c|U_N|^2. \tag{4.55}$$

Let $v \in L^2(\{1,\dots,N\}, H^1(\nu))$ and $\psi \in L^2(\{1,\dots,N\}, H^1(\nu))$ be the solution of

$$\begin{cases} \mathsf{M}\psi + (\mathcal{L}^\nu \otimes I_N)\psi = v + \lambda(e_1 + \ldots + e_N) & \text{on } \mathcal{M}_N \\ \psi \cdot (e_1 + \ldots + e_N) = 0 & \text{on } \mathcal{M}_N \\ \psi \cdot \vec{n} = 0 & \text{on } \partial\mathcal{M}_N. \end{cases} \tag{4.56}$$

Then ψ satisfies the following uniform estimate:

$$\sup|\psi| \leqslant c^{-1}\sup|v|. \tag{4.57}$$

We give a proof of Proposition 4.3.6 via stochastic flow following the approach of [73, Th. 2.1].

Proof. We wish to give a Feynman-Kac representation for solutions of (4.56). Let

$$X_t^x = x - \int_0^t \nabla H^{\mathfrak{g}}(X_s^x)\mathrm{d}s + \sqrt{2}\mathrm{d}B_t.$$

Note that since $\lim_{x\to 0}\xi'(x) = -\infty$, the dynamics is conservative: for all $x \in D_N$, the process X_t^x does not hit the boundary of $\mathcal{M}_N$ a.s.

Let $A : L^2(\{1,\dots,N\}, H^1(\nu)) \mapsto L^2(\{1,\dots,N\}, H^{-1}(\nu))$ be the operator (4.56). Given a source vector-field $v \in L^2(\{1,\dots,N\}, H^1(\nu))$, one may represent the solution of (4.56) as

$$\psi = \int_0^{+\infty} e^{tA}v\mathrm{d}t.$$

This follows from the fact that A has a spectral gap in $L^2(\{1,\dots,N\}, H^1(\nu))$. One can then represent $e^{tA}v$ as

$$e^{tA}v = \mathbb{E}_\nu[v(X_t^x)e^{-\int_0^t \mathsf{M}(X_s^x)\mathrm{d}s}].$$

Using Assumption (4.55), one gets

$$\sup|e^{tA}v| \leqslant \sup|v|e^{-tc}$$

Integrating this with respect to t gives (4.57). □

The proof of Proposition 4.3.6 is an adaptation in a more involved case of a known maximum principle for the Helffer-Sjöstrand equation, see for instance [149].

Let us emphasize that the above proof crucially relies on the fact that $\lim_{x\to 0}\chi'(x) = -\infty$. We now give the standard Gaussian concentration lemma for uniformly log-concave measures on convex bodies.

Lemma 4.3.7. *Let μ satisfying Assumptions 4.3.1 and $\nu = \mathrm{Gap}_N \# \mu$. Let c_N be the constant in (4.55). Let $\mathcal{A} \subset D_N$ be a convex domain with a piecewise smooth boundary. Let $F = G \circ \mathrm{Gap}_N$ with $G \in H^1(\nu)$. For all $t \in \mathbb{R}$, we have*

$$\log \mathbb{E}_\mu[e^{tF} \mid \mathcal{A}] \leqslant t\mathbb{E}_\mu[F \mid \mathcal{A}] + \frac{t^2}{2c_N} \sup_{\mathcal{A}} |\nabla G|^2.$$

Lemma 4.3.7 can be derived using Log-Sobolev inequality and Herbst argument. When a measure μ is uniformly log-concave on a convex domain on $\mathbb{R}^n$, it follows from the Bakry-Emery criterion [15] that μ satisfies a Log-Sobolev inequality.

Lemma 4.3.8. *Let μ be a uniformly log-concave measure on a convex domain of $\mathbb{R}^N$, with a convexity constant larger than $c > 0$. Then μ satisfies the Log-Sobolev inequality with constant $2c^{-1}$.*

4.3.6 Concentration inequality for divergence free functions

If μ is of the form of Assumptions 4.3.1, μ is not uniformly log-concave and on cannot apply directly Lemma 4.3.7. However, one can observe that

$$U_N \cdot \nabla^2 H U_N \geqslant c_N \sum_{i \neq j} (N(u_i - u_j))^2 = c_N(N-1) \sum_{i=1}^N u_i^2,$$
$$\text{for all } U_N \in \mathbb{R}^N \text{ such that } u_1 + \ldots + u_N = 0. \quad (4.58)$$

Using this observation and the particular structure of μ, one can give a concentration estimate for divergence free functions F, i.e for F verifying $\partial_1 \phi + \ldots + \partial_N \phi = 0$. We now state this crucial concentration result found in [46].

Lemma 4.3.9. *Let μ satisfying Assumptions 4.3.1. Assume that $\chi'' \geqslant c_N$. Let $I \subset \{1, \ldots, N\}$, $\mathrm{card}(I) = K$. Let $F \in H^1(\mu)$ such that $\sum_{i=1}^N \partial_i F = 0$ and $\partial_i F = 0$ for each $i \in I^c$. We have*

$$\mathrm{Var}_\mu[F] \leqslant \frac{1}{(K-1)c_N} \mathbb{E}_\mu[|\nabla F|^2]. \quad (4.59)$$

Furthermore, for all $t \in \mathbb{R}$,

$$\log \mathbb{E}_\mu[e^{tF}] \leqslant t\mathbb{E}_\mu[F] + \frac{t^2}{2(K-1)c_N} \sup |\nabla F|^2.$$

We refer to [46] for a proof, see also [52, Lem. 3.13] for a transcription.

4.4 Decay of correlations for the HS Riesz gas

This section considers the hypersingular Riesz gas, i.e the Riesz gas with the kernel (4.22) for a parameter $s > 1$. We show that the covariance between $N(x_{i+1} - x_j)$ and $N(x_{j+1} - x_j)$ decays at least in $d(i, j)^{-(s+1)}$. To this end we will be studying the Helffer-Sjöstrand equation in gap coordinates (4.49). Advantaged by that the Hessian of the energy in gap coordinates has typically summable entries, one may implement a simple distortion argument inspired from [147] to obtain decay estimates.

4.4.1 Study of a commutator

Let us begin by introducing the distortion argument. Given $s>1$, let ν be the measure (4.4) in gap coordinates or a slight variant of it. We will be studying the equation

$$\begin{cases} A_1^\nu \nabla\psi = e_1 + \lambda(e_1+\ldots+e_N) & \text{on } \mathcal{M}_N \\ \nabla\psi\cdot(e_1+\ldots+e_N)=0 & \text{on } \mathcal{M}_N \\ \nabla\psi\cdot\vec{n}=0 & \text{on } \partial\mathcal{M}_N. \end{cases} \tag{4.60}$$

By Remark 17, if $\nu=\mathbb{P}^{\mathrm{g}}_{N,\beta}$ there exists an event of overwhelming probability on which the Hessian of the energy in gap coordinates decays in $d(i,j)^{-s}$ away from the diagonal. The idea is to study the equation satisfied by $\mathrm{L}_\alpha\nabla\psi$, where L_α stands for the following distortion matrix:

$$\mathrm{L}_\alpha=\mathrm{diag}(\gamma_1,\ldots,\gamma_N),\quad \text{where } \gamma_i=1+d(i,i_0)^\alpha \text{ for each } 1\leqslant i\leqslant N. \tag{4.61}$$

Let us denote

$$\psi^{\mathrm{dis}}=\mathrm{L}_\alpha\nabla\psi \quad \in L^2(\{1,\ldots,N\},H^1(\nu)).$$

One can check that ψ^{dis} solves

$$A_1^\nu\nabla\psi+\beta\delta_{\mathrm{L}_\alpha}\nabla\psi=e_1+\lambda\mathrm{L}_\alpha(e_1+\ldots+e_N),\quad \text{where}\quad \delta_{\mathrm{L}_\alpha}:=\mathrm{L}_\alpha\nabla^2\mathcal{H}_N^{\mathrm{g}}\mathrm{L}_\alpha^{-1}-\mathcal{H}_N^{\mathrm{g}}.$$

Note that when $\mathrm{M}\in\mathcal{M}_N(\mathbb{R})$ is a matrix with off-diagonal entries decaying fast enough, then the commutator $\mathrm{L}_\alpha\mathrm{M}\mathrm{L}_\alpha^{-1}-\mathrm{M}$ is in some sense small compared to the identity, as shown in the next lemma.

Lemma 4.4.1 (Commutation lemma). *Let $s>1$ and $\mathrm{M}\in\mathcal{M}_N(\mathbb{R})$. Assume that there exists a constant $\varepsilon>0$ such that*

$$|\mathrm{M}_{i,j}|\leqslant\frac{N^\varepsilon}{1+d(i,j)^s},\quad \textit{for each } 1\leqslant i,j\leqslant N. \tag{4.62}$$

Let $\alpha\in(\frac12,s-\frac12)$ and L_α be as in (4.61). There exist constants $C>0$ and $c>0$ such that for all $\varepsilon_0>0$ small enough, letting $\varepsilon'=\frac{\varepsilon+\varepsilon_0}{\min(s-1,s-\frac{1}{2-\alpha})}$, we have that for all $U_N\in\mathbb{R}^N$,

$$|U_N\cdot(\mathrm{L}_\alpha\mathrm{M}\mathrm{L}_\alpha^{-1}-\mathrm{M})U_N|\leqslant\frac12 N^{-\varepsilon_0}|U_N|^2+CC_N^\kappa|U_N|\Big(\sum_{i:d(i,1)\leqslant cN^{\varepsilon'}}u_i^2\Big)^{\frac12}. \tag{4.63}$$

Proof. Let $\mathrm{M}\in\mathcal{M}_N(\mathbb{R})$ satisfying (4.62), $\alpha>0$, L_α be as in (4.61) and $U_N\in\mathbb{R}^N$. We denote

$$\delta_{\mathrm{L}_\alpha}=\mathrm{L}_\alpha\mathrm{M}\mathrm{L}_\alpha^{-1}-\mathrm{M}\in\mathcal{M}_N(\mathbb{R}).$$

For each $1\leqslant i\leqslant N$, one may split $(\delta_{\mathrm{L}_\alpha}U_N)_i$ into

$$(\delta_{\mathrm{L}_\alpha}U_N)_i=\underbrace{\sum_{l:d(i,l)\leqslant\frac12 d(i,1)}(\delta_{\mathrm{L}_\alpha})_{i,l}u_l}_{(I)_i}+\underbrace{\sum_{l:d(i,l)>\frac12 d(i,1)}(\delta_{\mathrm{L}_\alpha})_{i,l}u_l}_{(II)_i}. \tag{4.64}$$

If $d(i,l)\leqslant\frac12 d(i,1)$, then

$$\Big|\frac{\gamma_i-\gamma_l}{\gamma_l}\Big|\leqslant C\frac{d(i,l)}{1+d(i,1)}$$

and it follows from Cauchy-Schwarz inequality that

$$|(I)_i| \leqslant \frac{Cn^{\varepsilon}}{d(i,1)^{s-\frac{1}{2}}}|U_N|. \tag{4.65}$$

Let us choose $\alpha \in (\frac{1}{2}, s-\frac{1}{2})$. If $d(i,l) \geqslant \frac{1}{2}d(i,1)$, then

$$\left|\frac{\gamma_i - \gamma_l}{\gamma_l}\right| \leqslant C\frac{\gamma_i}{\gamma_l},$$

which gives, since $\alpha > \frac{1}{2}$,

$$|(II)_i| \leqslant Cn^{\varepsilon}d(i,1)^{\alpha-s}\sum_{l:d(i,l)>\frac{1}{2}d(i,1)}\frac{1}{d(l,1)^{\alpha}}|u_l| \leqslant \frac{Cn^{\varepsilon}}{d(i,1)^{s-\alpha}}|U_N|. \tag{4.66}$$

Let $K_0 \geqslant 1$. Combining (4.65) and (4.66) one obtains

$$\begin{aligned}|U_N\cdot\delta_{\mathrm{L}_\alpha}U_N| &\leqslant CN^{\varepsilon}|U_N|^2\Big(\sum_{i:d(i,1)\geqslant K_0}\frac{1}{d(i,1)^{2\min(s-\frac{1}{2},s-\alpha)}}\Big)^{\frac{1}{2}} + CN^{\varepsilon}|U_N|\Big(\sum_{i:d(i,1)\leqslant K_0}u_i^2\Big)^{\frac{1}{2}}\\ &\leqslant CN^{\varepsilon}|U_N|^2\frac{1}{K_0^{\min(s-1,\mathrm{s}-\frac{1}{2}-\alpha)}} + CN^{\varepsilon}|U_N|\Big(\sum_{i:d(i,1)\leqslant K_0}u_i^2\Big)^{\frac{1}{2}}.\end{aligned}$$

Therefore by choosing $K_0 = cN^{\varepsilon'}$ with $\varepsilon' = \frac{\varepsilon+\varepsilon_0}{\min(s-1,s-\frac{1}{2}-\alpha)}$, we find that

$$|U_N\cdot\delta_{\mathrm{L}_\alpha}U_N| \leqslant \frac{1}{2}N^{-\varepsilon_0}|U_N|^2 + CN^{\varepsilon}|U_N|\Big(\sum_{i:d(i,1)\leqslant K_0}u_i^2\Big)^{\frac{1}{2}}. \tag{4.67}$$

□

4.4.2 Localization in a smaller window

Due to the degeneracy of the interaction at infinity, the system lacks of uniform convexity and one shall sometimes restrict the system to a smaller window. Fix n to be the size of a subsystem, say $n = N$ or $n \leqslant N/2$. One may add some convexity within the window $I := \{1,\ldots,n\}$ without changing much the measure. Denote $\pi : \mathcal{M}_N \to \pi(\mathcal{M}_N) \subset \mathbb{R}^n$ the projection on the coordinates $(x_i)_{i\in I}$. For $\varepsilon > 0$ and $\theta : [0,+\infty) \to (0,+\infty)$ smooth such that $\theta = 0$ on $(1,+\infty)$, $\theta'' \geqslant 1$ on $[0,\frac{1}{2})$, $\theta'' \geqslant 0$ on $[0,+\infty]$, let us define

$$\mathrm{F}^{\mathrm{g}} : X_n \in \mathbb{R}^n \mapsto \sum_{i=1}^n \theta(n^{-\varepsilon}x_i) \tag{4.68}$$

and the constrained measures

$$\mathrm{d}\mathbb{Q}^{\mathrm{g}}_{N,\beta} \propto e^{-\beta\mathrm{F}^{\mathrm{g}}\circ\pi}\mathrm{d}\mathbb{P}^{\mathrm{g}}_{N,\beta}. \tag{4.69}$$

Note that the forcing (4.68) is tuned so that the total variation distance between $\mathbb{P}_{N,\beta}$ and $\mathbb{Q}_{N,\beta}$ decays exponentially in n using the Log-Sobolev inequality. We now define

$$\nu := \mathbb{Q}^{\mathrm{g}}_{N,\beta}\circ\pi^{-1}. \tag{4.70}$$

Define

$$\widetilde{E}_{n,N} : x \in \pi(\mathcal{M}_N) \mapsto -\frac{1}{\beta}\log\int e^{-\beta(\mathcal{H}^{\mathrm{g}}_{N-n}(y)+\mathcal{H}^{\mathrm{g}}_{n,N}(x,y))}\mathrm{d}y, \tag{4.71}$$

where

$$\mathcal{H}^{\mathrm{g}}_{n,N} : (x,y) \in (\mathbb{R}^n \times \mathbb{R}^{N-n}) \cap \mathcal{M}_N \mapsto \mathcal{H}^{\mathrm{g}}_N(x,y) - \mathcal{H}^{\mathrm{g}}_n(x) - \mathcal{H}^{\mathrm{g}}_{N-n}(y). \tag{4.72}$$

By Lemma 4.3.5, ν may be written in the form

$$\mathrm{d}\nu(x) \propto e^{-\beta\widetilde{\mathcal{H}}^{\mathrm{g}}_n(x)}\mathbb{1}_{\pi(\mathcal{M}_N)}(x)\mathrm{d}x \tag{4.73}$$

where

$$\widetilde{\mathcal{H}}^{\mathrm{g}}_n := \mathcal{H}^{\mathrm{g}}_n + \mathrm{F}^{\mathrm{g}} + \widetilde{E}_{n,N}. \tag{4.74}$$

In the sequel one will be studying the decay of the covariance between x_i and x_j under ν through the analysis of the associated Helffer-Sjöstrand equation. Define the good event

$$\begin{aligned}\mathcal{A} = \big\{X_n \in \pi(\mathcal{M}_N) : \forall i \in \{1,\ldots,n\}, n^{-\varepsilon} \leqslant x_i \leqslant n^{\varepsilon}\big\} \\ \cap \left\{\forall i \in \{1,\ldots,n\}, k \in \{1,\ldots,n-i\}, |x_i + \ldots + x_{i+k-1} - k| \leqslant n^{\varepsilon}k^{\frac{1}{2}}\right\}.\end{aligned} \tag{4.75}$$

Let $A = \nabla^2 F(U_n) \in \mathcal{M}_n(\mathbb{R})$ for some $U_n \in \mathbb{R}^n$ where F is the quadratic form

$$F : X_n \in \mathbb{R}^n \mapsto \sum_{i,j\in I} g_s''(|j-i|)(x_i + \ldots + x_j)^2.$$

Let us decompose $\nabla^2\widetilde{\mathcal{H}}^{\mathrm{g}}_n$ into $\nabla^2\widetilde{\mathcal{H}}^{\mathrm{g}}_n = \mathrm{M} + M$ with

$$\mathrm{M} = \nabla^2\mathrm{F}^{\mathrm{g}} + \nabla^2\mathcal{H}^{\mathrm{g}}_n\mathbb{1}_{\mathcal{A}} + A\mathbb{1}_{\mathcal{A}^c} \quad \text{and} \quad M = \nabla^2\mathcal{H}^{\mathrm{g}}_n\mathbb{1}_{\mathcal{A}^c} - A\mathbb{1}_{\mathcal{A}^c} + \nabla^2\widetilde{E}_{n,N}. \tag{4.76}$$

In the case $n \leqslant N/2$, we will replace $\nabla^2\widetilde{\mathcal{H}}^{\mathrm{g}}_n$ in (4.60) by M and derive some decay estimates on the solution, which will be transferred to the solution of (4.60) using a convexity argument. One can check that uniformly on the event (4.75) and for each $1 \leqslant i,j \leqslant n$, we have

$$|M_{i,j}| \leqslant \frac{Cn^{\kappa\varepsilon}}{1 + d(i,\partial I)^{s-1/2}d(j,\partial I)^{s-1/2}}. \tag{4.77}$$

For the purpose of Section 4.6 it is convenient to work with a general measure ν on $\pi(\mathcal{M}_N)$ satisfying the following:

Assumptions 4.4.1. *Let ν be a probability measure on $\pi(\mathcal{M}_N)$ in the form $\mathrm{d}\nu = e^{-\beta H^{\mathrm{g}}(x)}\mathrm{d}x$ with $H^{\mathrm{g}} : \pi(\mathcal{M}_N) \to \mathbb{R}$ in $\mathcal{C}^2$ and such that*

$$\lim_{d(x,\pi(\mathcal{M}_N))\to 0} \nabla H^{\mathrm{g}}(x) \cdot \vec{n} = -\infty.$$

Let $\mathcal{A}$ be the good event (4.75). Assume that there exist $C > 0, \delta > 0$ (depending on ε) such that

$$\nu(\mathcal{A}^c) \leqslant e^{-n^{\delta}}.$$

Note that the above condition ensures first that no boundary term appears in the computations and second that the Langevin dynamics is conservative, implying that the maximum principle of Proposition 4.3.6 holds true.

Instead of the specific interaction matrix defined in (4.76) we will be working with a more general measurable function M from $\pi(\mathcal{M}_N)$ to $\mathcal{S}_{n_0}(\mathbb{R})$ with $n_0 \leqslant n$ satisfying the following:

Assumptions 4.4.2. *Let $n_0 \leqslant n$. Let M be a measurable map from $\pi(\mathcal{M}_N)$ to $\mathcal{S}_{n_0}(\mathbb{R})$.*

1. *There exists $\kappa > 0$ such that uniformly on $\pi(\mathcal{M}_N)$,*

$$\mathsf{M} \geqslant n^{-\kappa\varepsilon} I_{n_0}.$$

2. *There exist $\kappa > 0$ and $C > 0$ such that uniformly on $\pi(\mathcal{M}_N)$ and for each $1 \leqslant i, j \leqslant n_0$,*

$$|\mathsf{M}_{i,j}| \leqslant \frac{C n^{\kappa\varepsilon}}{1 + |i-j|^s}.$$

4.4.3 The initial decay estimate

In this subsection we introduce a simple perturbation argument, which gives a first estimate on the decay of correlations for the constrained hypersingular Riesz gas. The method can be applied to other convex models for which the Hessian of the energy satisfies some decay assumption. This technique follows from an adaptation of a rather classical argument in statistical physics [149, 89].

Lemma 4.4.2. *Let $s \in (1, +\infty)$. Let ν and M satisfying Assumptions 4.4.1 and 4.4.2. Let $\chi_n \in H^1(\nu), i_0 \in \{1, \ldots, n\}$ and $\psi \in L^2(I, H^1(\nu))$ be the solution of*

$$\begin{cases} \beta \mathsf{M}\psi + \mathcal{L}^\nu \psi = \chi_n e_{i_0} & \text{on } \pi(\mathcal{M}_N) \\ \psi \cdot \vec{n} = 0 & \text{on } \partial\pi(\mathcal{M}_N). \end{cases} \tag{4.78}$$

Then, for all $\alpha \in (\frac{1}{2}, s - \frac{1}{2})$, there exist a constant $C(\beta)$ locally uniform in β and $\kappa > 0$ such that

$$\mathbb{E}_\nu\Big[\sum_{i=1}^n d(i, i_0)^{2\alpha} \psi_i^2\Big]^{\frac{1}{2}} \leqslant C(\beta) n^{\kappa\varepsilon} \mathbb{E}_\nu[\chi_n^2]^{\frac{1}{2}}. \tag{4.79}$$

Proof. Let $\psi \in L^2(I, H^1(\nu))$ be in the solution of (4.78). Taking the scalar product of (4.78) with ψ and integrating by parts, one may show that there exist constants $\kappa > 0$ and $C > 0$ such that

$$\mathbb{E}_\nu[|\nabla\psi|^2] + \beta\mathbb{E}_\nu[|\psi|^2] \leqslant C\beta^{-1} n^{\kappa\varepsilon} \mathbb{E}_\nu[\chi_n^2]. \tag{4.80}$$

Fix $\alpha \in (\frac{1}{2}, s - \frac{1}{2})$ and consider as in (4.61) the distortion matrix

$$\mathrm{L}_\alpha = \mathrm{diag}(\gamma_1, \ldots, \gamma_n), \quad \text{where } \gamma_i = 1 + d(i, i_0)^\alpha \text{ for each } 1 \leqslant i \leqslant n.$$

Let us define u^{dis} the distorted vector-field

$$u^{\mathrm{dis}} := \mathrm{L}_\alpha u \in L^2(I, H^1(\nu)). \tag{4.81}$$

Observing that $\mathrm{L}_\alpha e_{i_0} = e_{i_0}$, we can check that u^{dis} solves

$$A_1^\nu u^{\mathrm{dis}} + \beta \delta_{\mathrm{L}_\alpha} u^{\mathrm{dis}} = \chi_n e_{i_0}, \tag{4.82}$$

where

$$\delta_{\mathrm{L}_\alpha} := \mathrm{L}_\alpha \mathsf{M} \mathrm{L}_\alpha^{-1} - \mathsf{M}.$$

Taking the scalar product of (4.82) with u^{dis} and integrating by parts under ν gives

$$\mathbb{E}_\nu[\beta u^{\mathrm{dis}} \cdot (\mathsf{M} + \delta_{\mathrm{L}_\alpha}) u^{\mathrm{dis}}] + \mathbb{E}_\nu[|\nabla u^{\mathrm{dis}}|^2] = \mathbb{E}_\nu[u_{i_0} \chi_n], \tag{4.83}$$

where we have used the fact that $u_{i_0}^{\mathrm{dis}} = u_{i_0}$. This gives

$$\mathbb{E}_\nu[\beta u^{\mathrm{dis}} \cdot (\mathsf{M} + \delta_{\mathrm{L}_\alpha}) u^{\mathrm{dis}}] + \mathbb{E}_\nu[|\nabla u^{\mathrm{dis}}|^2] \leqslant C(\beta) n^{\kappa_0 \varepsilon} \mathbb{E}_\nu[\chi_n^2].$$

By assumption, there exist constants $C > 0, \kappa > 0$ such that uniformly on D_N and for each $i \neq j$,

$$|\mathsf{M}_{i,j}| \leqslant \frac{C n^{\kappa\varepsilon}}{1 + d(i,j)^s}.$$

One may therefore apply Lemma 4.4.1 to the matrix $\mathrm{M} = \nabla^2 \mathcal{H}_n^{\mathrm{g}}(X_n)$, which gives the existence of $\kappa > 0$ and $\kappa' > 0$ independent of X_n such that, letting

$$K_0 = \lfloor n^{\kappa\varepsilon} \rfloor,$$

there holds

$$|\mathbb{E}_\nu[u^{\mathrm{dis}} \cdot \delta_{\mathrm{L}_\alpha} u^{\mathrm{dis}}]| \leqslant \frac{n^{-\varepsilon(s+2)}}{2} \mathbb{E}_\nu[|u^{\mathrm{dis}}|^2] - C(\beta) n^{\kappa'\varepsilon} \mathbb{E}_\nu[|u^{\mathrm{dis}}|^2]^{\frac{1}{2}} \mathbb{E}_\nu\Big[\sum_{i: d(i,1) \leqslant K_0} (u_i^{\mathrm{dis}})^2\Big]^{\frac{1}{2}}. \tag{4.84}$$

Furthermore, using the definition of u^{dis} (4.81) and the a priori bound (4.80), we find that

$$\mathbb{E}_\nu\Big[\sum_{i: d(i,1) \leqslant K_0} (u_i^{\mathrm{dis}})^2\Big]^{\frac{1}{2}} \leqslant K_0^\alpha \mathbb{E}_\nu[|u|^2]^{\frac{1}{2}} \leqslant C(\beta) n^{\kappa''\varepsilon} \mathbb{E}_\nu[\chi_n^2]^{\frac{1}{2}}.$$

Combining these we deduce that there exists $\kappa > 0$ such that

$$\frac{\beta}{2} n^{-\varepsilon(s+2)} \mathbb{E}_\nu\Big[\sum_{i=1}^n d(i,i_0)^{2\alpha} (\psi_i^{(1)})^2\Big]^{\frac{1}{2}} + \mathbb{E}_\nu\Big[\sum_{i=1}^n d(i,i_0)^{2\alpha} |\nabla \psi_i^{(l)}|^2\Big]^{\frac{1}{2}} \leqslant C(\beta) n^{\kappa\varepsilon} \mathbb{E}_\nu[\chi_n^2]^{\frac{1}{2}}. \tag{4.85}$$

□

4.4.4 Bootstrap on the decay exponent

This subsection introduce a an iterative argument to improve the decay estimate of Lemma 4.4.2. The method consists in studying the projection of Equation (4.78) in a small window. By controlling the field outside the window with the a priori decay estimate, one obtains through the distortion argument of Lemma 4.4.2 a better decay estimate on the solution. After a finite number of iterations one gets the following result:

Proposition 4.4.3. *Let $s \in (1, +\infty)$. Let ν and M satisfying Assumptions 4.4.1 and 4.4.2. Let $\chi_n \in H^1(\nu), i_0 \in \{1, \ldots, n\}$ and $\psi \in L^2(I, H^1(\nu))$ be the solution of*

$$\begin{cases} \beta \mathsf{M}\psi + \mathcal{L}^\nu \psi = \chi_n e_{i_0} & \text{on } \pi(\mathcal{M}_N) \\ \psi \cdot \vec{n} = 0 & \text{on } \partial\pi(\mathcal{M}_N). \end{cases} \tag{4.86}$$

There exist $\kappa > 0$ and $C(\beta) > 0$ locally uniform in β such that for each $1 \leqslant j \leqslant n$,

$$\mathbb{E}_\nu[\psi_j^2]^{\frac{1}{2}} \leqslant C(\beta) n^{\kappa\varepsilon} \frac{1}{1 + d(j,i_0)^s} \mathbb{E}_\nu[\chi_n^2]^{\frac{1}{2}}. \tag{4.87}$$

Proof.

Step 1: setting the bootstrap Assume that for any $n_0 \leqslant n$ and all M taking values and in $\mathcal{S}_{n_0}(\mathbb{R})$ satisfying Assumptions 4.4.2, each $i_0 \in \{1, \ldots, n\}$ and $\chi_n \in H^1(\nu)$, the solution $\psi \in L^2(I, H^1(\nu))$ of

$$\begin{cases} \mathsf{M}\psi + \mathcal{L}^\nu \psi = \chi_n e_{i_0} & \text{on } \pi(\mathcal{M}_N) \\ \psi \cdot \vec{n} = 0 & \text{on } \partial\pi(\mathcal{M}_N) \end{cases} \tag{4.88}$$

satisfies for some $\alpha \geqslant s - \frac{1}{2}$, $\kappa > 0$ and $\delta > 0$ the estimate

$$\mathbb{E}_\nu[\psi_j^2]^{\frac{1}{2}} \leqslant C(\beta) n^{\kappa\varepsilon} \Big(\frac{1}{1 + d(j, i_0)^\alpha} + \frac{1}{n} \Big) \mathbb{E}_\nu[\chi_n^2]^{\frac{1}{2}}. \tag{4.89}$$

We wish to prove that (4.89) holds for $\alpha = s$. Without loss of generality one may assume that $n = n_0$. Fix $i_0 \in \{1, \ldots, n\}$, $\chi_n \in H^1(\nu)$ and ψ solution of (4.88).

Step 2: localization Fix an index $j \in \{1, \ldots, n\}$ and define the window

$$J = \{ i \in \{1, \ldots, n\} : d(j, i) \leqslant d(j, i_0)/2 \}. \tag{4.90}$$

Our aim is to study the equation satisfied by $\psi^J := (\psi_j)_{j \in J} \in L^2(J, H^1(\nu))$. Projecting Equation (4.116) on the l-th coordinate for $l \in J$ reads

$$\beta \sum_{i \in J} \mathsf{M}_{i,l} \psi_i + \mathcal{L}^\nu \psi_l = -\beta \sum_{i \in J^c} \mathsf{M}_{i,l} \psi_i.$$

Let us denote $\mathsf{M}^J = (\mathsf{M}_{i,j})_{i,j \in J}$ and $\mathrm{V} \in L^2(J, H^1(\nu))$ given for each $l \in J$ by

$$\mathrm{V}_l = -\beta \sum_{i \in J^c} \mathsf{M}_{i,l} \psi_i, \tag{4.91}$$

so that ψ^J solves

$$\begin{cases} \beta \mathsf{M}^J \psi^J + \mathcal{L}^\nu \psi^J = \mathrm{V} & \text{on } \pi(\mathcal{M}_N) \\ \psi^J \cdot \vec{n} = 0 & \text{on } \partial\pi(\mathcal{M}_N). \end{cases} \tag{4.92}$$

Step 3: bound on the exterior field Fix $l \in J$ and split V_l into

$$\mathrm{V}_l = \underbrace{\sum_{i \in J^c, d(i, i_0) \leqslant \frac{1}{2} d(j, i_0)} \mathsf{M}_{i,l} \psi_i}_{(I)_l} + \underbrace{\sum_{i \in J^c, d(i, i_0) > \frac{1}{2} d(j, i_0)} \mathsf{M}_{i,l} \psi_i}_{(II)_l}. \tag{4.93}$$

Using Cauchy-Schwarz inequality and Lemma 4.4.2, we find

$$\mathbb{E}_\nu[(I)_l^2]^{\frac{1}{2}} \leqslant C(\beta) n^{\kappa\varepsilon} \frac{1}{d(j, i_0)^{s - \frac{1}{2}}} \frac{1}{d(l, \partial J)^{s - \frac{1}{2}}} \mathbb{E}_\nu[\chi_n^2]^{\frac{1}{2}}.$$

On the other hand using Cauchy-Schwarz inequality and Lemma 4.4.2 again, one gets

$$\mathbb{E}_\nu[(II)_l^2]^{\frac{1}{2}} \leqslant \frac{C(\beta) n^{\kappa\varepsilon}}{d(j, i_0)^s} \mathbb{E}_\nu[\chi_n^2]^{\frac{1}{2}}.$$

Step 4: optimal decay for the auxiliary system Let us split $\psi = \sum_{l\in J}\psi^{(l)}$, where for each $l \in J$ $\psi^{(l)} \in L^2(J, H^1(\nu))$ solves

$$\begin{cases} \beta \mathsf{M}^J\psi^{(l)} + \mathcal{L}^\nu \psi^{(l)} = \mathrm{V}_l e_l & \text{on } \pi(\mathcal{M}_N) \\ \psi^{(l)}\cdot \vec{n} = 0 & \text{on } \partial\pi(\mathcal{M}_N) \end{cases}$$

One may apply the bootstrap assumption (4.89) to M^J and $\psi^{(l)}$, which gives the bound

$$\mathbb{E}_\nu[(\psi_j^{(l)})^2]^{\frac{1}{2}} \leqslant C(\beta)n^{\kappa\varepsilon}\frac{1}{d(j,l)^\alpha}\Big(\frac{1}{d(j,i_0)^{s-\frac{1}{2}}}\frac{1}{d(l,\partial J)^{s-\frac{1}{2}}} + \frac{1}{d(j,i_0)^s}\Big)\mathbb{E}_\nu[\chi_n^2]^{\frac{1}{2}}.$$

Summing this over $l \in J$ yields

$$\mathbb{E}_\nu[\psi_j^2]^{\frac{1}{2}} \leqslant \frac{C(\beta)n^{\kappa\varepsilon}}{d(j,i_0)^{\alpha'}}\mathbb{E}_\nu[\chi_n^2]^{\frac{1}{2}},$$

where

$$\alpha' = \min(s, s+\alpha-1, 3s-\alpha).$$

Since $\alpha \geqslant s - \frac{1}{2}$ and $s > 1$, $\alpha' > \alpha$. After a finite number of iterations, we find that (4.89) holds for $\alpha = s$. □

4.4.5 Conclusion in the case $n = N$

In view of Proposition 4.3.3, the H.-S. equation contains when $n = N$ a Lagrange multiplier associated to the linear constraints that $y_1 + \ldots + y_N = N$ on $\mathcal{M}_N$. By controlling this multiplier one obtains the following result:

Lemma 4.4.4. *Let $\chi_n \in H^1(\nu), i_0 \in \{1,\ldots,n\}$. Let $\psi \in L^2(I, H^1(\nu))$ solution of*

$$\begin{cases} \beta \mathsf{M}\psi + \mathcal{L}^\nu\psi = \chi_n e_{i_0} + \lambda(e_1 + \ldots + e_n) & \text{on } \pi(\mathcal{M}_N) \\ \psi\cdot(e_1+\ldots+e_n) = 0 & \text{on } \pi(\mathcal{M}_N) \\ \psi\cdot\vec{n} = 0 & \text{on } \partial\pi(\mathcal{M}_N). \end{cases} \tag{4.94}$$

There exists constants $C(\beta) > 0, \delta > 0$ such that for each $1 \leqslant j \leqslant n$,

$$\mathbb{E}_\nu[\psi_j^2]^{\frac{1}{2}} \leqslant C(\beta)n^{\kappa\varepsilon}\Big(\frac{1}{1+d(j,i_0)^s} + \frac{1}{n}\Big)\mathbb{E}_\nu[\chi_n^2]^{\frac{1}{2}}. \tag{4.95}$$

Proof. Let us first prove that the Lagrange multiplier λ in (4.94) satisfies

$$\mathbb{E}_\nu[\lambda^2]^{\frac{1}{2}} \leqslant \frac{C(\beta)}{n^{1-\kappa\varepsilon}}\mathbb{E}_\nu[\chi_n^2]^{\frac{1}{2}}, \tag{4.96}$$

for some constants $C(\beta) > 0, \kappa > 0$. By linearity one can split ψ into $\psi = \psi^{(1)} + \psi^{(2)}$ where $\psi^{(1)} \in L^2(I, H^1(\nu))$ solves

$$\begin{cases} \beta\mathsf{M}\psi^{(1)} + \mathcal{L}^\nu\psi^{(1)} = \chi_n e_{i_0} & \text{on } \pi(\mathcal{M}_N) \\ \psi^{(1)}\cdot\vec{n} = 0 & \text{on } \partial\pi(\mathcal{M}_N). \end{cases}$$

In view of Proposition 4.4.3, we have uniformly in $j \in I$,

$$\mathbb{E}_\nu[(\psi_j^{(1)})^2]^{\frac{1}{2}} \leqslant C(\beta)n^{\kappa\varepsilon}\frac{1}{1+d(j,i_0)^s}\mathbb{E}_\nu[\chi_n^2]^{\frac{1}{2}}, \tag{4.97}$$

$$\mathbb{E}_\nu[(\psi_j^{(2)})^2]^{\frac{1}{2}} \leqslant C(\beta)n^{\kappa\varepsilon}\mathbb{E}_\nu[\lambda^2]^{\frac{1}{2}}. \tag{4.98}$$

Let $K_0 \geqslant 1$. Split M into $\mathsf{M}^{(1)} + \mathsf{M}^{(2)}$ with $\mathsf{M}^{(1)}$ given for each $i, j \in I$ by

$$\mathsf{M}_{i,j}^{(1)} = \mathsf{M}_{i,j}\mathbb{1}_{d(i,j)\leqslant K_0}.$$

Let $u = \sum_{j\in I} e_j$. Recall from (4.50) that

$$n\lambda = \beta u \cdot \mathsf{M}\psi - \chi_n = \beta u \cdot \mathsf{M}^{(1)}\psi + \beta u \cdot \mathsf{M}^{(2)}\psi - \chi_n.$$

First note that there exists $C(\beta) > 0, \kappa > 0$ such that

$$\mathbb{E}_\nu[(u \cdot \mathsf{M}^{(1)}\psi)^2]^{\frac{1}{2}} \leqslant C(\beta)n^{\kappa\varepsilon}K_0^{\kappa}\mathbb{E}_\nu[|\psi|^2]^{\frac{1}{2}}.$$

Moreover taking the scalar product of (4.94) with ψ and integrating by parts under ν yields the energetic estimate

$$\mathbb{E}_\nu[|\psi|^2]^{\frac{1}{2}} \leqslant C(\beta)n^{\kappa\varepsilon}\mathbb{E}_\nu[\chi_n^2]^{\frac{1}{2}}.$$

Consequently there exists constants $C(\beta) > 0, \kappa > 0$ such that

$$\mathbb{E}_\nu[(u \cdot \mathsf{M}^{(1)}\psi)^2]^{\frac{1}{2}} \leqslant C(\beta)n^{\kappa\varepsilon}K_0^{\kappa}\mathbb{E}_\nu[\chi_n^2]^{\frac{1}{2}}. \tag{4.99}$$

Besides, employing (4.97), we find

$$\mathbb{E}_\nu[(u \cdot \mathsf{M}^{(2)}\psi^{(1)})^2]^{\frac{1}{2}} \leqslant C(\beta)n^{\kappa\varepsilon}\mathbb{E}_\nu[\chi_n^2]^{\frac{1}{2}}. \tag{4.100}$$

Finally, note

$$|u \cdot \mathsf{M}^{(2)}\psi^{(2)}| \leqslant \sum_{i,k:d(i,k)\geqslant K_0} |\mathsf{M}_{i,k}||\psi_k^{(2)}| \leqslant n^{\kappa\varepsilon}K_0^{-s}\sum_k |\psi_k^{(2)}|.$$

Using the bound (4.98) one can see that

$$\mathbb{E}_\nu[(u \cdot \mathsf{M}^{(2)}\psi^{(2)})^2]^{\frac{1}{2}} \leqslant C(\beta)n^{\kappa\varepsilon}K_0^{-s}n\mathbb{E}_\nu[\lambda^2]^{\frac{1}{2}}. \tag{4.101}$$

Taking K_0 large with enough with respect to n^ε, one can make the left-hand side of (4.101) smaller than $\frac{n}{2}\mathbb{E}_\nu[\lambda^2]^{\frac{1}{2}}$. Combining this with (4.99) and (4.100) one obtains

$$n\mathbb{E}_\nu[\lambda^2]^{\frac{1}{2}} \leqslant \frac{n}{2}\mathbb{E}_\nu[\lambda^2]^{\frac{1}{2}} + C(\beta)n^{\kappa\varepsilon}\mathbb{E}_\nu[\chi_n^2]^{\frac{1}{2}},$$

which proves (4.96). Combined with (4.97) and (4.98) this concludes the proof of (4.95). □

4.4.6 Estimate on the main equation

There remains to compare the solution of (4.60) to the solution $\psi^{(1)}$ of the simplified equation (4.78). This supposes to estimate the quantity $M\psi^{(1)}$ where M is the perturbation in (4.76).

Proposition 4.4.5. *Let $s \in (1, +\infty)$. Let $\widetilde{\mathcal{H}}_n^{\mathfrak{g}}$ be as in (4.74). Let $i_0 \in \{1, \dots, n\}$ such that $|i_0 - n/2| \leqslant n/4$. Let $\chi_n \in H^1(\nu)$ and $\psi \in L^2(I, H^1(\nu))$ be the solution of*

$$\begin{cases} \beta\nabla^2\widetilde{\mathcal{H}}_n^{\mathfrak{g}}\psi + \mathcal{L}^\nu\psi = \chi_n e_{i_0} & \text{on } \pi(\mathcal{M}_N) \\ \psi \cdot \vec{n} = 0 & \text{on } \partial\pi(\mathcal{M}_N). \end{cases} \tag{4.102}$$

Then, uniformly in $1 \leqslant j \leqslant n$, *we have*

$$\mathbb{E}_\nu[\psi_j^2]^{\frac{1}{2}} \leqslant C(\beta) n^{\kappa\varepsilon}\Big(\frac{1}{1+d(i_0,j)^s} + \frac{1}{n^{\min(s-1/2,2s-2)}}\Big)(\mathbb{E}_\nu[\chi_n^2]^{\frac{1}{2}} + \sup|\chi_n| e^{-c(\beta)n^\delta}). \tag{4.103}$$

Similarly let $w \in L^2(\{1,\ldots,N\}, H^1(\nu))$ *be the solution of*

$$\begin{cases} \beta\nabla^2(\mathcal{H}_N^{\mathrm{g}} + \mathrm{F}^{\mathrm{g}})w + \mathcal{L}^\nu w = \chi_N e_{i_0} + \lambda(e_1 + \ldots + e_N) & \textit{on } \mathcal{M}_N \\ w\cdot(e_1+\ldots+e_N) = 0 & \textit{on } \mathcal{M}_N \\ w \cdot \vec{n} = 0 & \textit{on } \partial\mathcal{M}_N \end{cases} \tag{4.104}$$

Then, uniformly in $1 \leqslant j \leqslant N$, *we have*

$$\mathbb{E}_\nu[w_j^2]^{\frac{1}{2}} \leqslant C(\beta) n^{\kappa\varepsilon}\Big(\frac{1}{1+d(i_0,j)^s} + \frac{1}{N}\Big)(\mathbb{E}_\nu[\chi_n^2]^{\frac{1}{2}} + \sup|\chi_n| e^{-c(\beta)n^\delta}). \tag{4.105}$$

Proof. Let $M^{(1)} = \nabla^2\mathcal{H}_n^{\mathrm{g}}\mathbb{1}_{\mathcal{A}^c} - A\mathbb{1}_{\mathcal{A}^c}$ and $M^{(2)} = \nabla^2 \widetilde{E}_{n,N}$. Let $\psi \in L^2(I, H^1(\nu))$ be the solution of (4.102), $\psi^{(1)} \in L^2(I, H^1(\nu))$ solving

$$\begin{cases} \beta \mathsf{M}\psi^{(1)} + \mathcal{L}^\nu \psi^{(1)} = \chi_n e_{i_0} & \text{on } \pi(\mathcal{M}_N) \\ \psi^{(1)}\cdot\vec{n} = 0 & \text{on } \partial\pi(\mathcal{M}_N). \end{cases}$$

Define $\psi^{(2)} = \psi - \psi^{(1)}$. One can check that $\psi^{(2)}$ is solution of

$$\begin{cases} \beta\nabla^2\widetilde{\mathcal{H}}_n^{\mathrm{g}}\psi^{(2)} + \mathcal{L}^\nu\psi^{(2)} = \beta M^{(1)}\psi & \text{on } \pi(\mathcal{M}_N) \\ \psi^{(2)}\cdot\vec{n} = 0 & \text{on } \partial\pi(\mathcal{M}_N). \end{cases}$$

Taking the scalar product of the above equation with $\psi^{(2)}$ and integrating by parts under ν we obtain, recalling the definition of the good event (4.75),

$$\beta\mathbb{E}_\nu[\psi^{(2)}\cdot\mathsf{M}\psi^{(2)}] \leqslant C(\beta)(\sup|\psi^{(1)}|\mathbb{E}_\nu[|M^{(1)}|^2]^{\frac{1}{2}} + \mathbb{E}_\nu[|M^{(2)}|^2\mathbb{1}_{\mathcal{A}^c}]^{\frac{1}{2}} + \mathbb{E}_\nu[\mathbb{1}_{\mathcal{A}}\psi^{(1)}\cdot M^{(2)}\psi^{(1)}]) \tag{4.106}$$

Applying the maximum principle of Proposition 4.3.6 we find that there exist $C(\beta), \kappa > 0$ such that

$$\sup|\psi^{(1)}| \leqslant C(\beta) n^{\kappa\varepsilon}\sup|\chi_n|.$$

By Assumption 4.4.1 there exist constants $C(\beta) > 0, c(\beta) > 0, \delta > 0$ such that

$$\mathbb{E}_\nu[|M^{(1)}|^2]^{\frac{1}{2}} \leqslant C(\beta)e^{-c(\beta)n^\delta},$$

$$\mathbb{E}_\nu[\mathbb{1}_{\mathcal{A}^c}|M^{(2)}|^2]^{\frac{1}{2}} \leqslant C(\beta)e^{-c(\beta)n^\delta}.$$

Let us now estimate the vector-field $M^{(2)}\psi^{(1)}$. We claim that for uniformly in $1 \leqslant j \leqslant n$,

$$\mathbb{E}_\nu[\mathbb{1}_{\mathcal{A}}(M^{(2)}\psi^{(1)})_j^2]^{\frac{1}{2}} \leqslant \frac{C(\beta)n^{\kappa\varepsilon}}{1+d(j,\partial I)^{\frac{s}{2}}}\frac{1}{n^{s-1/2}}\mathbb{E}_\nu[\chi_n^2]^{\frac{1}{2}}. \tag{4.107}$$

Fix $1 \leqslant j \leqslant n$. Recall that for any x in the interior of $\mathcal{A}$ and for each $1 \leqslant k, l \leqslant n$,

$$M_{k,l}^{(2)}(x) = \partial_{kl}\widetilde{E}_{n,N}(x) = \mathbb{E}_{\mathbb{Q}_{N,\beta}^{\mathrm{g}}(\cdot|x)}[\partial_{kl}\mathcal{H}_{n,N}^{\mathrm{g}}] - \mathrm{Cov}_{\mathbb{Q}_{N,\beta}^{\mathrm{g}}(\cdot|x)}[\partial_k\mathcal{H}_{n,N}^{\mathrm{g}}, \partial_l\mathcal{H}_{n,N}^{\mathrm{g}}].$$

In view of (4.77) we have that for each $1 \leqslant k, l \leqslant n$,

$$\mathbb{E}_\nu[(M_{k,l}^{(2)})^2]^{\frac{1}{2}} \leqslant \frac{C(\beta)n^{\kappa\varepsilon}}{1+d(k,\partial I)^{s-1/2}d(l,\partial I)^{s-1/2}}. \tag{4.108}$$

One can then split the quantity $(M^{(2)}\psi^{(1)})_j$ into

$$(M^{(2)}\psi^{(1)})_j = \underbrace{\sum_{k:d(k,\partial I)\leqslant n/4} M^{(2)}_{j,k}\psi^{(1)}_k}_{(I)_j} + \underbrace{\sum_{k:d(k,\partial I)>n/4} M^{(2)}_{j,k}\psi^{(1)}_k}_{(II)_j}.$$

For the first quantity, using (4.108) and (4.87), we can write

$$\begin{aligned}\mathbb{E}_\nu[\mathbb{1}_{\mathcal{A}}(I)_j^2]^{\frac12} &\leqslant \frac{C(\beta)n^{\kappa\varepsilon}}{1+d(j,\partial I)^{s-1/2}}\sum_{k:d(k,\partial I)\leqslant n/4}\frac{1}{|k-\frac{n}{2}|^s}\frac{1}{1+d(k,\partial I)^{s-1/2}}\mathbb{E}_\nu[\chi_n^2]^{\frac12}\\ &\leqslant \frac{C(\beta)n^{\kappa\varepsilon}}{1+d(j,\partial I)^{s-1/2}}\frac{1}{n^{\min(s,2s-3/2)}}\mathbb{E}_\nu[\chi_n^2]^{\frac12}.\end{aligned} \tag{4.109}$$

For the second quantity using the bound on the increments of M given in (4.77), we find

$$\begin{aligned}\mathbb{E}_\nu\Big[\mathbb{1}_{\mathcal{A}}\Big(\sum_{k:d(k,\partial I)>n/4} M^{(2)}_{j,k}\psi^{(1)}_k\Big)^2\Big]^{\frac12} &\leqslant \frac{C(\beta)n^{\kappa\varepsilon}}{1+d(j,\partial I)^{s-1/2}}\sum_{k:d(k,\partial I)>n/4}\frac{1}{|k-\frac{n}{2}|^s}\frac{1}{n^{s-1/2}}\mathbb{E}_\nu[\chi_n^2]^{\frac12}\\ &\leqslant \frac{C(\beta)n^{\kappa\varepsilon}}{1+d(j,\partial I)^{s-1/2}}\frac{1}{n^{s-1/2}}\mathbb{E}_\nu[\chi_n^2]^{\frac12}.\end{aligned} \tag{4.110}$$

Putting (4.109) and (4.110) together we obtain (4.107). Summing this over j yields

$$\mathbb{E}_\nu[|M^{(2)}\psi^{(1)}|^2]^{1/2} \leqslant \frac{C(\beta)n^{\kappa\varepsilon}}{n^{\min(s-1/2,2s-2)}}\mathbb{E}_\nu[\chi_n^2]^{\frac12}. \tag{4.111}$$

Using the uniform convexity of $\widetilde{\mathcal{H}}^{\mathrm{g}}_n$, we then obtain from (4.106) the bound

$$\mathbb{E}_\nu[|\psi^{(2)}|^2]^{1/2} \leqslant \frac{C(\beta)n^{\kappa\varepsilon}}{n^{\min(s-1/2,2s-2)}}(\mathbb{E}_\nu[\chi_n^2]^{\frac12} + \sup|\chi_n|e^{-c(\beta)n^\delta}).$$

In particular, together with (4.87), this yields (4.103). The proof of (4.105) follows from similar considerations by making use of Lemma 4.4.4. □

4.4.7 Decay of gaps correlations

We are now ready to conclude the proof of the decay of correlations for the hypersingular Riesz gas. When x_i and x_j are at macroscopic or large mesosopic distance, one can take $n=N$ and use the estimate of Proposition 4.4.3. Otherwise we choose n to be a power of $|i-j|$ and apply the estimate of Proposition 4.4.5 for such a number n. This will complete proof of Theorem 4.1.2.

Proof of of Theorem 4.1.2. Let ν be the constrained measure on $\{1,\dots,N\}$ defined in (4.70) with $n=N$. Using the Pinsker inequality, the fact that ν satisfies a Log-Sobolev inequality (see Lemma 4.3.8) and the local law of Lemma 4.7.2, one can observe that

$$\mathrm{TV}(\mathbb{P}^{\mathrm{g}}_{N,\beta},\nu) \leqslant (2\mathrm{Ent}(\mathbb{P}^{\mathrm{g}}_{N,\beta}\mid\nu))^{\frac12} \leqslant C(\beta)N^{\kappa\varepsilon}\mathbb{E}_{\mathbb{P}^{\mathrm{g}}_{N,\beta}}[|\nabla \mathrm{F}^{\mathrm{g}}|^2]^{\frac12} \leqslant C(\beta)e^{-c(\beta)N^\delta}.$$

In particular, it follows that

$$\mathrm{Cov}_{\mathbb{P}_{N,\beta}}[\xi(N(x_{j+1}-x_j)),\chi(N(x_{i+1}-x_i))] = \mathrm{Cov}_\nu[\xi(x_j),\chi(x_i)] + O_\beta(e^{-N^\delta}\sup|\xi|\sup|\chi|). \tag{4.112}$$

Moreover, by Proposition 4.3.3, the covariance term in the last display may be expressed as

$$\mathrm{Cov}_\nu[\xi(x_j), \chi(x_i)] = \mathbb{E}_\nu[\xi'(x_j)\partial_j\phi],$$

with $\nabla\phi \in L^2(\{1,\ldots,N\}, H^1(\nu))$ solution of

$$\begin{cases} A_1^\nu \nabla\phi = \chi'(x_i)e_i + \lambda(e_1 + \ldots + e_N) & \text{on } \pi(\mathcal{M}_N) \\ \nabla\phi \cdot (e_1 + \ldots + e_N) = 0 & \text{on } \pi(\mathcal{M}_N) \\ \nabla\phi \cdot \vec{n} = 0 & \text{on } \partial\pi(\mathcal{M}_N). \end{cases}$$

Using the estimate of Proposition 4.4.5, Hölder's inequality and (4.112), one obtains (4.5) in the case where $d(j,i) \geqslant N^{\varepsilon_0}$.

We now consider the case where $d(i,j)$ is much smaller than a power of N. Let $n \in \{1,\ldots,N\}$ be the smallest number such that

$$\frac{1}{n^{\min(s-1/2,2s-2)}} \leqslant \frac{1}{d(i,j)^{1+s}}.$$

Without loss of generality, one can assume that $1 \leqslant \frac{n}{3} \leqslant i,j \leqslant \frac{2n}{3}$. Since $N(x_{i+1}-x_i)$ and $N(x_{j+1}-x_1)$ are functions of $x_1,\ldots,x_n$ and since $\mathcal{A}$ has overwhelming probability, one may write

$$\mathrm{Cov}_{\mathbb{P}_{N,\beta}}[\xi(N(x_{j+1}-x_j)), \chi(N(x_{i+1}-x_i))] = \mathrm{Cov}_\nu[\xi(x_j),\chi(x_i)] + O_\beta(e^{-c(\beta)n^\delta}\sup|\xi|\sup|\chi|). \tag{4.113}$$

By Proposition 4.3.3 again one can express this covariance term as

$$\mathrm{Cov}_\nu[x_j, x_i] = \mathbb{E}_\nu[\xi'(x_j)\partial_j\phi], \tag{4.114}$$

where $\nabla\phi \in L^2(I, H^1(\nu))$ is solution of

$$\begin{cases} A_1^\nu \nabla\phi = \chi'(x_i)e_i & \text{on } \pi(\mathcal{M}_N) \\ \nabla\phi \cdot \vec{n} = 0 & \text{on } \partial\pi(\mathcal{M}_N). \end{cases} \tag{4.115}$$

Inserting the result of Proposition 4.4.5 we find that

$$\mathbb{E}_\nu[(\partial_j\phi)^2]^{\frac{1}{2}} \leqslant C(\beta)n^{\kappa\varepsilon}\Big(\frac{1}{d(j,i)^s} + \frac{1}{n^{\min(s-1/2,2s-2)}}\Big)(\mathbb{E}_\nu[\chi'(x_i)^2]^{\frac{1}{2}} + \sup|\chi'|e^{-c(\beta)n^\delta}).$$

Inserting this into (4.114) and using (4.113) completes the proof of (4.5) by choosing n large enough. □

4.5 Decay of correlations for the long-range Riesz gas

This section is the core of the paper and aims to develop a method to study the decay of correlations in the long-range case $s \in (0,1)$. Because the Hessian of the energy in gap coordinates concentrates around the Riesz matrix (4.25) which has slowly decaying entries, it is not clear how the strategy of Section 4.4 can be adapted. Indeed the commutation result of Lemma 4.4.1 cannot be applied to (4.25). The trick is to exploit the fact that the Hessian is not only positive-definite but actually controls a fractional primitive of the solution. This should be compared with the method of [52, Sec. 4] adapted from [46] which exploits the long-range nature of the interaction to have sharp concentration estimates.

4.5.1 Periodization

We begin by performing the following series of reductions, which will lead to the study of a simplified equation:

1. Convexification and reduction to $(x_1, \ldots, x_n)$,
2. Adding of a Schur complement to the energy of the n points and splitting of the H.-S. operator,
3. Embedding the system into a periodic system of $2n$ points,
4. Control on the perturbation operator.

As pointed out in Section 4.4, due to the lack of uniform convexity, the study of the correlations at microscopic distance requires to localize the system at a smaller scale. Let $n \in \{1, \ldots, N\}$ be the active scale, I the window $I = \{1, \ldots, n\}$ and $\pi : \mathcal{M}_N \to \pi(\mathcal{M}_N) \subset \mathbb{R}^n$ be the projection on the coordinates $(x_i)_{i \in I}$. Let $\theta : \mathbb{R}^+ \to \mathbb{R}^+$ smooth such that $\theta = 0$ on $(1, +\infty)$, $\theta'' \geqslant 1$ on $[0, \frac{1}{2})$ and $\theta'' \geqslant 0$ on $[0, +\infty]$. Let $\varepsilon > 0$ and F^{g} be the forcing

$$\mathrm{F}^{\mathrm{g}} : X_n \in \mathbb{R}^n \mapsto \sum_{i=1}^n \theta(n^{-\varepsilon} x_i)$$

and the constrained measure

$$\mathrm{d}\mathbb{Q}^{\mathrm{g}}_{N,\beta} \propto e^{-\beta \mathrm{F}^{\mathrm{g}} \circ \pi} \mathrm{d}\mathbb{P}^{\mathrm{g}}_{N,\beta}.$$

Let $\nu = \mathbb{P}^{\mathrm{g}}_{N,\beta} \circ \pi^{-1}$. We will be studying the solution $\psi \in L^2(I, H^1(\nu))$ of

$$\begin{cases} A_1^{\nu} \psi = \chi_n e_{i_0} & \text{on } \pi(\mathcal{M}_N) \\ \psi \cdot \vec{n} = 0 & \text{on } \partial \pi(\mathcal{M}_N), \end{cases} \tag{4.116}$$

One would like to work with a periodic system of size $\bar{n} \geqslant 2n$ instead of (4.116). The idea is to subtract from A_1^{ν} the appropriate quantity to identify the equation with the projection on the coordinates $(x_i)_{i \in I}$ of a larger system of size $\bar{n}$. Let $\bar{n} \geqslant 2n$ and $\bar{I} = \{1, \ldots, 2n\}$. Let K_0 be a large power of $\lfloor n^{\varepsilon} \rfloor$. Consider $\mathrm{M} \in \mathcal{M}_{\bar{n}}(\mathbb{R})$ the truncated Riesz matrix at distance K_0, i.e $\mathrm{M} = \nabla^2 F(x)$ for some $x \in \mathbb{R}^{\bar{n}}$ where

$$F : X_{\bar{n}} \in \mathbb{R}^{\bar{n}} \mapsto \sum_{i,j \in \bar{I} : d(i,j) \geqslant K_0} g_s''(d(i,j))(x_i + \ldots + x_j)^2.$$

Consider the block decomposition of M on $\mathbb{R}^n \times \mathbb{R}^{\bar{n}-n}$,

$$\mathrm{M} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad A \in \mathcal{M}_n(\mathbb{R}). \tag{4.117}$$

Also let

$$G : X_{\bar{n}} \in \mathbb{R}^{\bar{n}} \mapsto \sum_{i \in I, j \in I^c : d(i,j) \geqslant K_0} g_s''(d(i,j))(x_i + \ldots + x_j)^2.$$

Let $A^{(2)} = (\partial_{ij} G)_{i,j \in I}$ and $A^{(1)} = A - A^{(2)}$. Let

$$\mathrm{M}^{(1)} = \begin{pmatrix} A^{(1)} & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \mathrm{M}^{(2)} = \begin{pmatrix} A^{(2)} & B \\ C & D \end{pmatrix}. \tag{4.118}$$

Since $\nabla^2 F \geqslant 0$, $\mathrm{M}^{(2)} \geqslant 0$ and therefore $A^{(1)} - BD^{-1}C \geqslant 0$. Furthermore we also have $A^{(1)} \geqslant 0$. Noting that D is positive-definite, we will subtract from A_1^ν the operator

$$B(D+\beta^{-1}\mathcal{L}^\nu \otimes I_{\bar{n}-n})^{-1}C.$$

The measure ν can be written

$$\mathrm{d}\nu(x) = \exp(-\beta\widetilde{\mathcal{H}}_n^{\mathrm{g}}(x))\mathbb{1}_{\pi(\mathcal{M}_N)}(x)\mathrm{d}x,$$

where for any $x \in \pi(\mathcal{M}_N)$ and $1 \leqslant i,j \leqslant n$,

$$\partial_{ij}\widetilde{\mathcal{H}}_n^{\mathrm{g}}(x) = \partial_{ij}\mathrm{F}^{\mathrm{g}}(x) + \partial_{ij}\mathcal{H}_n^{\mathrm{g}}(x) + \mathbb{E}_{\mathbb{Q}^{\mathrm{g}}_{N,\beta}(\cdot|x)}[\partial_{ij}\mathcal{H}^{\mathrm{g}}_{n,N}(x,\cdot)] - \mathrm{Cov}_{\mathbb{Q}^{\mathrm{g}}_{N,\beta}(\cdot|x)}[\partial_i\mathcal{H}^{\mathrm{g}}_{n,N}(x,\cdot), \partial_j\mathcal{H}^{\mathrm{g}}_{n,N}(x,\cdot)], \tag{4.119}$$

with

$$\mathcal{H}^{\mathrm{g}}_{n,N} : (x,y) \in (\mathbb{R}^n \times \mathbb{R}^{N-n}) \cap \mathcal{M}_N \mapsto \mathcal{H}^{\mathrm{g}}_N(x,y) - \mathcal{H}^{\mathrm{g}}_n(x) - \mathcal{H}^{\mathrm{g}}_{N-n}(y).$$

For $\varepsilon > 0$, define the good event

$$\mathcal{A} = \{X_n \in \pi(\mathcal{M}_N) : \forall i, i+k \in \{1,\ldots,n\}, n^{-\varepsilon} \leqslant x_i \leqslant n^{\varepsilon}, |x_i + \ldots + x_{i+k} - k| \leqslant n^{\varepsilon} k^{\frac{s}{2}}\}. \tag{4.120}$$

Let us split A_1^ν into

$$A_1^\nu = \bar{A}_1^\nu + M, \tag{4.121}$$

where $\bar{A}_1^\nu, M : L^2(I, H^1(\nu)) \to L^2(I, H^{-1}(\nu))$ are given by

$$\bar{A}_1^\nu := \beta\nabla^2\mathrm{F}^{\mathrm{g}} + \beta(\nabla^2\mathcal{H}_n^{\mathrm{g}} + \mathbb{E}_{\mathbb{Q}^{\mathrm{g}}_{N,\beta}(\cdot|x)}[\nabla^2\mathcal{H}^{\mathrm{g}}_{n,N}(x,\cdot)])\mathbb{1}_{\mathcal{A}} + \beta A\mathbb{1}_{\mathcal{A}^c} - \beta B(D+\beta^{-1}\mathcal{L}^\nu \otimes I_{\bar{n}-n})^{-1}C + \mathcal{L}^\nu \otimes I_n, \tag{4.122}$$

$$\begin{aligned} M := \beta(\nabla^2\mathcal{H}_n^{\mathrm{g}} + \mathbb{E}_{\mathbb{Q}^{\mathrm{g}}_{N,\beta}(\cdot|x)}[\nabla^2\mathcal{H}^{\mathrm{g}}_{n,N}(x,\cdot)])\mathbb{1}_{\mathcal{A}^c} - \beta A\mathbb{1}_{\mathcal{A}^c} - \beta\,\mathrm{Cov}_{\mathbb{Q}^{\mathrm{g}}_{N,\beta}(\cdot|x)}[\nabla\mathcal{H}^{\mathrm{g}}_{n,N}(x,\cdot), \nabla\mathcal{H}^{\mathrm{g}}_{n,N}(x,\cdot)] \\ + \beta B(D+\beta^{-1}\mathcal{L}^\nu \otimes I_{\bar{n}-n})^{-1}C. \end{aligned} \tag{4.123}$$

One can prove that the operator $\bar{A}_1^\nu$ has a spectral gap, resulting in the uniqueness of the solution $\psi \in L^2(I, H^1(\nu))$ of

$$\begin{cases} \bar{A}_1^\nu \psi = v & \text{on } \pi(\mathcal{M}_N) \\ \psi \cdot \vec{n} = 0 & \text{on } \partial\pi(\mathcal{M}_N), \end{cases}$$

for any $v \in L^2(I, H^1(\nu))$. As in Section 4.4, we work with general measures ν on $\pi(\mathcal{M}_N)$.

Assumptions 4.5.1. *Let ν be a probability measure on $\pi(\mathcal{M}_N)$ in the form $\mathrm{d}\nu = e^{-\beta H^{\mathrm{g}}(x)}\mathrm{d}x$ with $H^{\mathrm{g}} : \pi(\mathcal{M}_N) \to \mathbb{R}$ $\mathcal{C}^2$ and such that*

$$\lim_{d(x,\partial\pi(\mathcal{M}_N))\to 0} \nabla H^{\mathrm{g}}(x) \cdot \vec{n} = -\infty.$$

Let $\mathcal{A}$ be the good event (4.120). Assume that there exist $C > 0, \delta > 0$ (depending on ε) such that

$$\nu(\mathcal{A}^c) \leqslant e^{-n^\delta}.$$

As in Section 4.4, one shall work with a slightly more general system, for the sake of the bootstrap argument to come. Let $\widetilde{A} : \pi(\mathcal{M}_N) \to \mathcal{S}_n(\mathbb{R})$ be a measurable map. Let $\mathsf{M} : \pi(\mathcal{M}_N) \to \mathcal{M}_{\bar{n}}(\mathbb{R})$ be given by

$$\mathsf{M} = \begin{pmatrix} \widetilde{A} & B \\ C & D \end{pmatrix}, \tag{4.124}$$

with B, C, D constants matrices as in (4.117). One shall impose the following assumptions on M:

Assumptions 4.5.2. *Let $\mathsf{M} : \pi(\mathcal{M}_N) \to \mathcal{S}_n(\mathbb{R})$ be as (4.124). Assume that*

1. *There exists a positive constant $\kappa > 0$ such that uniformly on $\pi(\mathcal{M}_N)$,*

$$\mathsf{M} \geqslant n^{-\kappa\varepsilon} I_{\bar{n}}.$$

2. *There exists a family on non-negative functions $(\alpha_{i,k})$ such that for all $U_{\bar{n}} \in \mathbb{R}^{\bar{n}}$,*

$$U_{\bar{n}} \cdot \mathsf{M} U_{\bar{n}} = \sum_{i,k} \alpha_{i,k}(u_i + \ldots + u_k)^2.$$

3. *Let $A \in \mathcal{M}_n(\mathbb{R})$ be as in (4.117). There exists a positive constant $\kappa > 0$ such that uniformly on (4.120) and for each $1 \leqslant i, j \leqslant n$,*

$$\widetilde{A}_{i,j} = A_{i,j} + O\Big(\frac{n^{\kappa\varepsilon}}{d(i,j)^{1+\frac{s}{2}}}\Big). \tag{4.125}$$

Finally let $\bar{A}_1 : L^2(I, H^1(\nu)) \to L^2(I, H^{-1}(\nu))$ in the form

$$\bar{A}_1 = \beta\widetilde{A} - B(\beta D + \mathcal{L}^\nu \otimes I_{\bar{n}-n})^{-1}C + \mathcal{L}^\nu \otimes I_n. \tag{4.126}$$

Lemma 4.5.1. *Let M be in the form (4.124) for $\widetilde{A}$ satisfying Assumptions 4.5.2. Let $\bar{A}_1$ be given by (4.126). Let $\psi \in L^2(\bar{I}, H^1(\nu))$ be the solution of*

$$\begin{cases} \beta\mathsf{M}\psi + (\mathcal{L}^\nu \otimes I_{\bar{n}})\psi = \chi_n e_{i_0} & \text{on } \pi(\mathcal{M}_N) \\ \psi \cdot \vec{n} = 0 & \text{on } \partial\pi(\mathcal{M}_N). \end{cases} \tag{4.127}$$

Let $\psi^{(1)} \in L^2(I, H^1(\nu))$ be the solution of

$$\begin{cases} \bar{A}_1\psi^{(1)} = \chi_n e_{i_0} & \text{on } \pi(\mathcal{M}_N) \\ \psi^{(1)} \cdot \vec{n} = 0 & \text{on } \partial\pi(\mathcal{M}_N). \end{cases} \tag{4.128}$$

We have the identity

$$\psi_j = \psi^{(1)}_j \quad \text{for each } j \in I. \tag{4.129}$$

Proof. Uniqueness and existence of solutions of (4.127) and (4.128) follow from the Lax-Migram's theorem. Let us indeed prove that the quadratic forms

$$v \in L^2(I, H^1(\nu)) \mapsto \mathbb{E}_\nu[v \cdot \bar{A}_1^\nu v],$$

$$w \in L^2(\bar{I}, H^1(\nu)) \mapsto \mathbb{E}_\nu[w \cdot (\beta\mathsf{M} + \mathcal{L}^\nu \otimes I_{\bar{n}-n})w]$$

are coercive. Let us split $\mathcal{H}_n^{\mathsf{g}}$ into $H^{(1)} + H^{(2)}$ with

$$H^{(1)} : X_n \in \pi(\mathcal{M}_N) \mapsto N^{-s} \sum_{i \neq j : |i-j| \leqslant K_0} g_s(x_i + \ldots + x_j). \tag{4.130}$$

Denote

$$\mathrm{M}^{(1)} = \begin{pmatrix} \nabla^2 H^{(1)} + \nabla^2 \mathrm{F}^{\mathrm{g}} & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \mathrm{M}^{(2)} = \begin{pmatrix} \nabla^2 H^{(2)}(x) + \mathbb{E}_{\mathbb{Q}_{N,\beta}(\cdot|x)}[\nabla^2 \mathcal{H}^{\mathrm{g}}_{n,N}(x,\cdot)] & B \\ C & D \end{pmatrix}.$$

Let M be as in (4.117). Observe that there exists $\kappa > 0$ such that for all $U_{\bar{n}} = (U_n, V_{\bar{n}-n}) \in \mathbb{R}^{\bar{n}}$

$$\left| U_{\bar{n}} \cdot (\mathrm{M}^{(2)} - \mathrm{M}) U_{\bar{n}} \right| \leqslant n^{\kappa\varepsilon} K_0^{-\frac{s}{2}} |U_n|^2$$
$$U_n \cdot \nabla^2 (\mathrm{F}^{\mathrm{g}} + H^{(1)}) U_n \geqslant n^{-\kappa\varepsilon} |U_n|^2.$$

Let us choose $K_0 = \lfloor n^{\varepsilon} \rfloor^m$ for m large enough. In view of the last displays one can see that there exists $\kappa > 0$ such that for all $U_{\bar{n}} = (U_n, V_{\bar{n}-n}) \in \mathbb{R}^{\bar{n}}$,

$$U_{\bar{n}} \cdot \mathrm{M} U_{\bar{n}} \geqslant U_{\bar{n}} \cdot \mathrm{M} U_{\bar{n}} + n^{-\kappa\varepsilon} |U_n|^2 \geqslant n^{-\kappa'\varepsilon} |U_{\bar{n}}|^2.$$

Since $\mathcal{L}^{\nu}$ is non-negative we obtain that for all $v \in L^2(\bar{I}, H^1(\nu))$,

$$\mathbb{E}_{\nu}[v \cdot (\beta \mathrm{M} + \mathcal{L}^{\nu} \otimes I_{\bar{n}}) v] \geqslant n^{-\kappa\varepsilon} \mathbb{E}_{\nu}[|v|^2].$$

Because $\mathrm{M}^{(2)} \geqslant 0$, we also have that $A^{(2)} - BD^{-1}C \geqslant 0$. Then note

$$A(x) - BD^{-1}C \geqslant \nabla^2 H^{(1)} + A^{(2)} - BD^{-1}C - O(n^{\kappa\varepsilon} K_0^{-\frac{s}{2}} I_n) \geqslant \nabla^2 H^{(1)} - O(n^{\kappa\varepsilon} K_0^{-\frac{s}{2}} I_n) \geqslant n^{-\kappa\varepsilon} I_n. \tag{4.131}$$

Let $w \in L^2(I, H^1(\nu))$. One can observe that

$$w \cdot B(D + \beta^{-1} \mathcal{L}^{\nu} \otimes I_{\bar{n}-n})^{-1}(Cw) = (Cw) \cdot (D + \beta^{-1} \mathcal{L}^{\nu} \otimes I_{\bar{n}-n})^{-1}(Cw).$$

Integrating this over ν and using the fact that D is positive shows that for all $w \in L^2(I, H^1(\nu))$,

$$0 \leqslant \mathbb{E}_{\nu}[w \cdot B(D + \beta^{-1} \mathcal{L}^{\nu} \otimes I_{\bar{n}-n})^{-1}(Cw)] \leqslant \mathbb{E}_{\nu}[w \cdot BD^{-1}Cw]. \tag{4.132}$$

Consequently, inserting (4.131), we find

$$\mathbb{E}_{\nu}[w \cdot \bar{A}_1^{\nu} w] \geqslant n^{-\kappa\varepsilon} \mathbb{E}_{\nu}[|w|^2].$$

□

Let us next explain how to compare $(A_1^{\nu})^{-1}$ to $(\bar{A}_1^{\nu})^{-1}$. Let $\psi \in L^2(I, H^1(\nu))$ be the solution of (4.116) and $\psi^{(1)} \in L^2(I, H^1(\nu))$ of

$$\begin{cases} \bar{A}_1^{\nu} \psi^{(1)} = \chi_n e_{i_0} & \text{on } \pi(\mathcal{M}_N) \\ \psi^{(1)} \cdot \vec{n} = 0 & \text{on } \partial\pi(\mathcal{M}_N). \end{cases}$$

Let $w := \psi^{(1)} - \psi \in L^2(I, H^1(\nu))$, which solves

$$\begin{cases} A_1^{\nu} w = M \psi^{(1)} & \text{on } \pi(\mathcal{M}_N) \\ w \cdot \vec{n} = 0 & \text{on } \partial\pi(\mathcal{M}_N). \end{cases}$$

Taking the scalar product of the first line of the last display with w and integrating by parts with respect to ν yields

$$\beta n^{-\kappa\varepsilon} \mathbb{E}_{\nu}[|w|^2] \leqslant \beta \mathbb{E}_{\nu}[w \cdot \nabla^2 \widetilde{\mathcal{H}}_n^{\mathrm{g}} w] \leqslant \mathbb{E}_{\nu}[w \cdot M \psi^{(1)}]. \tag{4.133}$$

We will prove in Lemma 4.5.3 that

$$|\mathbb{E}_\nu[w_i(M\psi^{(1)})_i]| \leqslant C(\beta)n^{\kappa\varepsilon}\mathbb{E}_\nu[w_i^2]^{\frac{1}{2}}\sum_{j\in I}\frac{1}{1+d(i,\partial I)^{\frac{s}{2}}d(j,\partial I)^{\frac{s}{2}}}\mathbb{E}_\nu[(\psi_j^{(1)})^2]^{\frac{1}{2}}.$$

Inserting the last display into (4.133) will then give

$$\mathbb{E}_\nu[|w|^2]^{\frac{1}{2}} \leqslant C(\beta)n^{\kappa\varepsilon}n^{\frac{1-s}{2}}\sum_{j=1}^{n}\frac{\mathbb{E}_\nu[(\psi_j^{(1)})^2]^{\frac{1}{2}}}{1+d(j,\partial I)^{\frac{s}{2}}}. \tag{4.134}$$

Our main task is to establish that $\psi_j^{(1)}$ typically decays in $d(j,i_0)^{-(2-s)}$, making the left-hand side of (4.134) bounded by $n^{-1/2}$. This will show that the increments of ψ are bounded by $d(j,i_0)^{-(2-s)}+O(n^{-1/2})$, allowing to conclude the proof of Theorem 4.1.1 by choosing n large enough.

We finally complete Step 4 and control the operator (4.123). Recall that $B^\intercal = C$.

Lemma 4.5.2. *Let ν satisfying Assumptions 4.5.1. Let $s\in(0,1)$. Let B,C,D be as in (4.117). Recall $I=\{1,\ldots,n\}$. Let η, $\phi\in L^2(\nu)$. Then for each $1\leqslant i,j\leqslant n$, we have*

$$|\mathbb{E}_\nu[(\eta Ce_j)^\intercal(\beta D+\mathcal{L}^\nu\otimes I_{\bar{n}-n})^{-1}(\phi Ce_i)]| \leqslant \frac{C(\beta)n^{\kappa\varepsilon}}{d(i,\partial I)^{\frac{s}{2}}d(j,\partial I)^{\frac{s}{2}}}\mathbb{E}_\nu[\eta^2]^{\frac{1}{2}}\mathbb{E}_\nu[\phi^2]^{\frac{1}{2}}. \tag{4.135}$$

In addition for each $1\leqslant i,j,l\leqslant n$, we have

$$|\mathbb{E}_\nu[(\eta C)^\intercal(\beta D+\mathcal{L}^\nu\otimes I_{\bar{n}-n})^{-1}(\phi Ce_i)]| \leqslant \frac{C(\beta)n^{\kappa\varepsilon}|j-l|}{\min(d(j,\partial I)^{1+\frac{s}{2}},d(l,\partial I)^{1+\frac{s}{2}})}\frac{1}{d(j,\partial I)^{\frac{s}{2}}}\mathbb{E}_\nu[\eta^2]^{\frac{1}{2}}\mathbb{E}_\nu[\phi^2]^{\frac{1}{2}}. \tag{4.136}$$

The term in the left-hand side of (4.135) is comparable to the covariance between $\partial_i\mathcal{H}^{\mathrm{g}}_{n,N}$ and $\partial_j\mathcal{H}^{\mathrm{g}}_{n,N}$ under a Gaussian measure. This analogy suggests us to proceed as if we were trying to control the variances of $\partial_i\mathcal{H}^{\mathrm{g}}_{n,N}$ and $\partial_j\mathcal{H}^{\mathrm{g}}_{n,N}$, which would require to control the fluctuations of large gaps. We will thus import a method of [46] which starts by decomposing a given gap into a sum of block averaged statistics.

Proof. First note that since $\beta D'+\mathcal{L}^\nu\otimes I_{\bar{n}-n}$ is a positive operator on $L^2(I,H^1(\nu))$, we find that

$$\begin{aligned}&|\mathbb{E}_\nu[(\eta Ce_j)^\intercal(\beta D+\mathcal{L}^\nu\otimes I_{\bar{n}-n})^{-1}(\phi Ce_i)]|\\&\quad\leqslant \mathbb{E}_\nu[(\eta Ce_j)\cdot(\beta D'+\mathcal{L}^\nu\otimes I_{\bar{n}-n})^{-1}(\eta Ce_j)]^{\frac{1}{2}}\mathbb{E}_\nu[(\phi Ce_i)\cdot(\beta D+\mathcal{L}^\nu\otimes I_{\bar{n}-n})^{-1}(\phi Ce_i)]^{\frac{1}{2}}.\end{aligned} \tag{4.137}$$

Using the positivity of $\mathcal{L}^\nu\otimes I_{\bar{n}-n}$ and D, one can write

$$\mathbb{E}_\nu[(\eta Ce_j\eta)\cdot(\beta D+\mathcal{L}^\nu\otimes I_{\bar{n}-n})^{-1}(\eta Ce_j)] \leqslant \beta^{-1}\mathbb{E}_\nu[(\eta Ce_j)\cdot D^{-1}(\eta Ce_j)] = \beta^{-1}\mathbb{E}_\nu[\eta^2](Ce_j)\cdot D^{-1}(Ce_j).$$

The right-hand side of the last display may be identified with the variance of $(CZ')_j$ where Z' is a Gaussian vector $Z'\sim\mathcal{N}(0,D)$. Let Z be the random vector defined for each $k\in\{1,\ldots,N\}\backslash I$ by $Z_k=Z_1'+\ldots+Z_k'$.

One may check that

$$(C'Z)_j = \sum_{i\in\{1,\ldots,N\}\backslash I}\ \sum_{k\in\{1,\ldots,N\}\backslash I:d(k,j)\geqslant d(k,i)}\frac{1}{|i-k|^{s+2}}N(Z_k-Z_j). \tag{4.138}$$

We claim that there exists $C > 0$ and $\kappa > 0$ such that for each $1 \leqslant i \leqslant n$ and $1 \leqslant i + k \leqslant n$,

$$\mathrm{Var}[N(Z_{i+k} - Z_i)] \leqslant C k^{s+\kappa\varepsilon}. \tag{4.139}$$

Combining (4.138) and (4.139) entails, modulo (4.139),

$$|e_j \cdot BD^{-1}Ce_j)| \leqslant \frac{C}{1 + d(j, \partial I)^{s/2}}.$$

Let us now prove the claim (4.139). Fix $1 \leqslant i \leqslant i + k \leqslant n$. One shall split $N(Z_{i+k} - Z_i)$ into a sum of block average statistics. For each $1 \leqslant k \leqslant n/2$ and $i \in \{1, \ldots, n\}$, let $I_k(k)$ be an interval of integers in $\{n+1, \ldots, n\}$ of cardinal $k+1$ such that $i \in I_k(i)$. Define the block average

$$Z_i^{[k]} = \frac{1}{k+1} \sum_{j \in I_k(i)} Z_k.$$

Let $\alpha > 0$ be a small number, $\alpha = \frac{1}{p}$ with $p \in \mathbb{N}^*$. One may write

$$N(Z_i - Z_i^{[k]}) = \sum_{m=0}^{p-1} N(Z_i^{[[k^{m\alpha}]]} - Z_i^{[[k^{(m+1)\alpha}]]}). \tag{4.140}$$

For each $m \in \{0, \ldots, p-1\}$, denote $G_m = N(Z_i^{[[k^{m\alpha}]]} - Z_i^{[[k^{(m+1)\alpha}]]})$ and $I_m = I_{\lfloor k^{(m+1)\alpha} \rfloor}(i)$. Let us define the matrix $D^{(1)} = (D^{(1)})_{i,j \in I_m}$ by

$$D^{(1)}_{i,j} = \begin{cases} D_i & \text{if } i \neq j \\ -\sum_{k \in I_m, k \neq i} D_j & \text{if } i = j. \end{cases}$$

Let $u = (\partial_i G_m)_{i \in I_m}$ and $D^{I_m} = (D_{i,j})_{i,j \in I_m}$. Since G_m depends only on the variables in I_m, we have the identity

$$\mathrm{Var}[G_m] = u \cdot D^{I_m} u.$$

Moreover, since $D^{I_m} \geqslant D^{(1)}$, there holds

$$\mathrm{Var}[G_m] \leqslant u \cdot D^{(1)} u.$$

Let $v = (D^{(1)})^{-1} u$. Using the fact that $\sum_{i \in I_m} \partial_i G_m = 0$ and $D^{(1)} \sum_{i \in I_m} e_i = 0$, one may check that $\sum_{i \in I_m} v_i = 0$. It follows that

$$v \cdot D^{(1)} v \geqslant \sum_{i \neq j \in I_m} \frac{1}{|i-j|^{s+2}} (N(v_i - v_j))^2 \geqslant \frac{N^2}{|I_m|^{s+1}} |v|^2.$$

Furthermore observe that

$$|\nabla G_m|^2 \leqslant \frac{CN^2}{|I_m|}.$$

The two last displays give by integration by parts the series of inequalities

$$\beta \frac{N^2}{|I_m|^{s+1}} |v|^2 \leqslant v \cdot D^{(1)} v \leqslant C|v| \frac{N^2}{|I_m|}.$$

It follows that

$$\mathrm{Var}[G_m] \leqslant C(\beta) |I_m|^s \mathbb{E}_\mu[\eta^2]^{\frac{1}{2}}. \tag{4.141}$$

Summing (4.141) over m and using (4.140), one finds that

$$\mathrm{Var}[N(Z_{i+k}-Z_i)]\leqslant Ck^{s+\kappa\varepsilon},$$

which yields (4.139), thus concluding the proof of (4.135).

The proof of (4.136) is similar. □

Let us now control the operator M appearing in (4.123).

Lemma 4.5.3. *Let $\mathcal{A}$ be the good event (4.120). Uniformly in $x\in\mathcal{A}$, $1\leqslant i\leqslant j\leqslant n$ and N, we have*

$$\mathrm{Var}_{\mathbb{Q}^{\mathrm{g}}_{N,\beta}(\cdot|x)}[\partial_i\mathcal{H}^{\mathrm{g}}_{n,N},\partial_j\mathcal{H}^{\mathrm{g}}_{n,N}]\leqslant\frac{C(\beta)n^{\kappa\varepsilon}}{d(i,\partial I)^{\frac{s}{2}}d(j,\partial I)^{\frac{s}{2}}}. \tag{4.142}$$

Let ν satisfying Assumptions 4.4.1. Then for all $\phi,\eta\in L^2(\nu)$ and $1\leqslant i,j\leqslant n$,

$$\mathbb{E}_\nu[\phi e_i\cdot M(\eta e_j)]\leqslant\frac{C(\beta)n^{\kappa\varepsilon}}{d(i,\partial I)^{\frac{s}{2}}d(j,\partial I)^{\frac{s}{2}}}\mathbb{E}_\nu[\eta^2]^{\frac12}\mathbb{E}_\nu[\phi^2]^{\frac12}. \tag{4.143}$$

In addition, for all $\phi,\eta\in L^2(\nu)$ and $1\leqslant i,l,j\leqslant n$,

$$\mathbb{E}_\nu[(\phi e_i)\cdot M(\eta(e_j-e_l))]\leqslant\frac{C(\beta)n^{\kappa\varepsilon}}{d(j,\partial I)^{\frac{s}{2}}}\frac{d(i,l)}{(d(i,\partial I)\wedge d(l,\partial I))^{1+\frac{s}{2}}}\mathbb{E}_\nu[\phi^2]^{\frac12}\mathbb{E}_\nu[\eta^2]^{\frac12}+C(\beta)e^{-c(\beta)n^{\delta}}\sup|\phi|\sup|\eta|. \tag{4.144}$$

Proof. The control (4.142) is a direct consequence a rigidity estimate under $\mathbb{Q}_{N,\beta}(\cdot\mid x)$ that we defer to Lemma 4.6.2, which proof can be found in the Appendix. Regarding the definition of (4.123), the bound on the Schur complement (4.143) follows from (4.142) and Lemma 4.5.2. Since $\mathcal{A}$ has overwhelming probability one may bound the contribution involving the Hessian of $\mathcal{H}_n^{(2)}$ and $\mathbb{E}_{\mathbb{Q}_{N,\beta}(\cdot|x)}[\mathcal{H}_{n,N}(x,\cdot)]$ (in gap coordinates) by $\sup|\phi|\sup|\eta|C(\beta)e^{-c(\beta)n^{\delta}}$. □

Note that (4.142) one could refine (4.142) and show that this term concentrates around the quantity $\mathrm{Cov}[CZ')_j,(CZ')_i]$, where $Z'\sim\mathcal{N}(0,D)$. One expects that there exists some $\alpha>0$ such that

$$\begin{aligned}\mathrm{Var}_{\mathbb{Q}^{\mathrm{g}}_{N,\beta}(\cdot|x)}[\partial_i\mathcal{H}^{\mathrm{g}}_{n,N},\partial_j\mathcal{H}^{\mathrm{g}}_{n,N}]&=(P_nCe_i)\cdot(D+\beta^{-1}\mathcal{L}^\nu\otimes I_{\bar n-n})^{-1}(P_nCe_j)\\&\quad+n^{\kappa\varepsilon}O_\beta\Big(\frac{1}{1+d(i,\partial I)^{\frac{s+\alpha}{2}}}\frac{1}{1+d(j,\partial I)^{\frac{s+\alpha}{2}}}\Big),\end{aligned}$$

where B, C and D are as in (4.117). Having such an expansion could refine our control on the solution of (4.116) through (4.133).

4.5.2 Elliptic regularity estimate

The stake for us is to obtain a decay estimate on the solution of (4.128). We first derive an elliptic regularity estimate and give an L^2 bound on the discrete primitive of order $\frac32-s$ of ψ in terms of $|\mathrm{L}_{1/2}\psi|$. We then state a straightforward control on the L^1 norm on the discrete primitive of order $1-s$ of ψ with respect to $|\mathrm{L}_{3/2-s}\psi|$. By interpolation, this yields via a discrete (1D) Gagliardo-Nirenberg inequality a control on the L^p norm with $p=\frac{1}{1-s/2}$ of the fractional primitive of order $1-\frac{s}{2}$ of ψ. Throughout the section, for all $\alpha>0$, L_α stands for the distortion matrix

$$\mathrm{L}_\alpha=\mathrm{diag}(\gamma_1,\ldots,\gamma_{\bar n})\quad\text{with}\quad\gamma_i=1+d(i,i_0)^\alpha\quad\text{for each }1\leqslant i\leqslant\bar n. \tag{4.145}$$

Lemma 4.5.4. *Let $s \in (0,1)$. Let ν and M satisfying Assumptions 4.5.1 and 4.5.2. Let $\chi_n \in H^1(\nu)$, $i_0 \in \{1,\ldots,n\}$ and $\psi \in L^2(\bar{I}, H^1(\nu))$ be the solution of*

$$\begin{cases} \beta\mathsf{M}\psi + \mathcal{L}^\nu\psi = \chi_n e_{i_0} + \lambda(e_1 + \ldots + e_{\bar{n}}) & \text{on } \pi(\mathcal{M}_N) \\ \psi\cdot(e_1+\ldots+e_{\bar{n}}) = 0 & \text{on } \pi(\mathcal{M}_N) \\ \psi\cdot\vec{n} = 0 & \text{on } \partial\pi(\mathcal{M}_N). \end{cases} \tag{4.146}$$

Recalling (4.145), there exists $\kappa > 0$ such that letting $p = \frac{1}{1-s/2}$,

$$\mathbb{E}_\nu\Big[\Big(\sum_{i=1}^{\bar{n}}|(g_{s/2}*\psi)_i|^p\Big)^{2/p}\Big]^{\frac12} \leqslant C(\beta)n^{\kappa\varepsilon}\Big(\mathbb{E}_\nu[\chi_n^2]^{\frac12}+\sup|\chi_n|e^{-c(\beta)n^\delta}+\mathbb{E}_\nu[|\mathrm{L}_{1/2}\psi|^2]^{\frac12}+n\mathbb{E}_\nu[\lambda^2]^{\frac12}\Big)^s \times \mathbb{E}_\nu[|\mathrm{L}_{3/2-s}\psi|^2]^{\frac{1-s}{2}}. \tag{4.147}$$

Proof. Let us denote $v = \chi_n e_{i_0} + \lambda(e_1+\ldots+e_{\bar{n}})$. Let $\psi \in L^2(\bar{I}, H^1(\nu))$ be the solution of (4.146). In view of (4.125), the matrix M may be split into $\mathsf{M} = \mathsf{M}^{(1)} + \mathsf{M}^{(2)}$ where $\mathsf{M}^{(1)} \in \mathcal{M}_{\bar{n}}(\mathbb{R})$ is the constant Toeplitz matrix with the Riesz kernel g_s and $\mathsf{M}^{(2)}$ satisfying

$$|\mathsf{M}^{(2)}_{i,j}| \leqslant \frac{n^{\kappa\varepsilon}}{d(i,j)^{1+\frac{s}{2}}}, \quad \text{for each } i,j \in \bar{I}.$$

Taking the convolution of (4.146) with g_{s-1} and the scalar product with ψ easily gives

$$\mathbb{E}_\nu\Big[\sum_{i=1}^{\bar{n}}(g_{s-1/2}*\psi)_i^2\Big]^{\frac12} \leqslant C(\beta)n^{\kappa\varepsilon}\Big(\mathbb{E}_\nu[\chi_n^2]^{\frac12} + \sup|\chi_n|e^{-c(\beta)n^\delta} + \mathbb{E}_\nu[|\mathrm{L}_{1/2}\psi|^2]^{\frac12} + n\mathbb{E}_\nu[\lambda^2]^{\frac12}\Big). \tag{4.148}$$

Indeed, the differential terms satisfies

$$\sum_{i=1}^{\bar{n}}\mathbb{E}_\nu[\mathcal{L}^\nu((g_{s-1}*\psi)_i)\psi_i] = \sum_{i=1}^{\bar{n}}\mathbb{E}_\nu[\nabla(g_{s-1}*\psi)_i)\cdot\nabla\psi_i] = \sum_{i,j,k}\mathbb{E}_\nu[g_{s-1}(d(i,k))\partial_j\psi_k\cdot\partial_j\psi_i].$$

Since g_{s-1} is a positive kernel, for each $j \in \{1,\ldots,\bar{n}\}$, setting $u_k = \partial_j\psi_k$, we have

$$\sum_{i,k} g_{s-1}(d(i,k))u_i u_k \geqslant 0,$$

which justifies the claim (4.148).

Recall that by Remark 14, the convolution of a discrete function $f : \mathbb{Z}/n\mathbb{Z}$ with g_α for $\alpha > -1$ corresponds to a fractional primitive of order $1-\alpha$ of f. One can now interpolate between the L^1 norm of the primitive of ψ of order $1-s$ and the L^2 norm of the primitive of order $1-\frac{s}{2}$. Let $\phi : \mathbb{T} \to \mathbb{R}$ smooth enough. Applying Lemma 4.7.1 to $u := g_{s-1/2}*\psi$ with $s_1 = 0$, $s_2 = \frac12$, $s_0 = \frac12 - \frac{s}{2} \in (s_1,s_2)$, $\theta = s$, $p_1 = 2$, $p_2 = 1$ and $p = \frac{1}{1-\frac{s}{2}}$ gives

$$\|g_{s/2}*\psi\|_{L^{\frac{1}{1-s/2}}(\mathbb{T})} \leqslant C\|g_s*\psi\|^\theta_{L^1(\mathbb{T})}\|g_{s-1/2}*\psi\|^{1-\theta}_{L^2(\mathbb{T})}. \tag{4.149}$$

Let $\phi_0 : \mathbb{T} \to \mathbb{R}$ smooth such that $\phi(\frac{i}{n}) = \psi_i$ for each $i \in \{1,\ldots,\bar{n}\}$. Using (4.149) and making ϕ_0 slightly vary, we deduce that

$$\Big(\sum_{i=1}^{\bar{n}}|(g_{s/2}*\psi)_i|^{\frac{1}{1-s/2}}\Big)^{1-s/2} \leqslant C\Big(\sum_{i=1}^{\bar{n}}|(g_s*\psi)_i|\Big)^{1-s}\Big(\sum_{i=1}^{\bar{n}}(g_{s-1/2}*\psi)_i^2\Big)^{\frac{s}{2}}. \tag{4.150}$$

Besides, by Cauchy-Schwarz inequality, it is straightforward to check that

$$\sum_{i=1}^{\bar{n}}|(g_s*\psi)_i| \leqslant C(\beta)n^{\kappa\varepsilon}|\mathrm{L}_{3/2-s}\psi|. \tag{4.151}$$

Inserting (4.151) and (4.148) into (4.150) one obtains (4.147). □

4.5.3 Control on derivatives

The aim is now to control the decay of $\nabla\psi_i$ with respect the (global) decay of ψ_i. The proof relies on the distortion argument of Lemma 4.4.2, the central task being to bound a variant of the commutator $\mathrm{L}_\alpha \mathsf{M} \mathrm{L}_\alpha^{-1} - \mathsf{M}$ from above.

Let us pause to explain the strategy of this proof. At first let us fix a small parameter $\varepsilon_0 > 0$. In view of its specific positive-definiteness structure, M can be bounded from below by a matrix $\tilde{\mathsf{M}}$ where interactions are cut off for $d(i,k) > d(i,i_0)^{1-\varepsilon_0}$. We then seek to control $(\mathrm{L}_\alpha \mathsf{M} \mathrm{L}_\alpha^{-1}\psi^{\mathrm{dis}} - \tilde{\mathsf{M}}\psi^{\mathrm{dis}})_i$ for each $1 \leqslant i \leqslant \bar{n}$. By construction, $(\tilde{\mathsf{M}}\psi^{\mathrm{dis}})_i$ may be bounded by $|\mathrm{L}_{3/2-s-\varepsilon_0}\psi|$. Similarly one can bound the left and right tails of $(\mathrm{L}_\alpha \mathsf{M} \mathrm{L}_\alpha^{-1}\psi^{\mathrm{dis}})_i$ by $|\mathrm{L}_{3/2-s-\varepsilon_0}\psi|$. We are thus left to estimate

$$\sum_{k\in A(i)} \psi_k \tilde{g}_s(d(i,k)) \quad \text{where} \quad A(i) := \{k \neq i : d(i,i_0)^{1-\varepsilon_0} \leqslant d(i,k) \leqslant d(i,i_0)^{1+\varepsilon_0}\}. \tag{4.152}$$

The point is to express this sum with respect $w := \mathbb{H}_{s/2}\psi$, the discrete primitive w of order $1-s/2$ of ψ, which gives

$$\sum_{k\in A(i)} \psi_k \tilde{g}_s(d(i,k)) = \sum_{l=1}^{\bar{n}} \sum_{k\in A(i)} g_{s/2}^{-1}(d(k,l)) \tilde{g}_s(d(i,k)) \mathbb{1}_{i\neq k} w_l, \tag{4.153}$$

where $g_{s/2}^{-1} := \mathbb{H}_{s/2}^{-1} e_1$. Given an index l, one shall therefore estimate a *truncated* convolution product between $\tilde{g}_s$ and $g_{s/2}^{-1}$. If l lies away from the boundary of $A(i)$, this product almost equals $\tilde{g}_s * g_{s/2}^{-1}(l) \simeq g_{1-s/2}$. Fixing a threshold of size $d(i,i_0)^{1-2\varepsilon_0}$, one can decompose (4.153) according to whether $d(l, \partial A(i)) \geqslant d(i,i_0)^{1-2\varepsilon_0}$. Owing to the previous remark and by Hölder's inequality, one can bound the first contribution by the L^p norm of w with $p = \frac{1}{1-s/2}$ and insert (4.147). On the other hand, the second contribution can be controlled by $|\mathrm{L}_{3/2-s-\varepsilon_0}\psi|$.

We finally obtain a control on $|\mathrm{L}_{1-s/2} D\psi|$ depending on $|\mathrm{L}_{1-s/2-\varepsilon_0}\psi|$ and on $n^{\varepsilon_0}|\mathrm{L}_{1/2}\psi|$. A reversed inequality will be proved in the next subsection allowing one to control $|\mathrm{L}_{3/2-s}\psi|$ by $|\mathrm{L}_{1-s/2}\psi|$. Since $\varepsilon_0 > 0$ and $3/2 - s > 1/2$, this will provide a bound on $|\mathrm{L}_{3/2-s}\psi|$ and $|\mathrm{L}_{1-s/2} D\psi|$.

Lemma 4.5.5. *Let $s \in (0,1)$. Let ν and M satisfying Assumptions 4.5.1 and 4.5.2. Let $\chi_n \in H^1(\nu)$, $i_0 \in \{1,\ldots,n\}$ and $\psi \in L^2(\bar{I}, H^1(\nu))$ be the solution of*

$$\begin{cases} \beta \mathsf{M}\psi + \mathcal{L}^\nu \psi = \chi_n e_{i_0} + \lambda(e_1 + \ldots + e_{\bar{n}}) & \text{on } \pi(\mathcal{M}_N) \\ \psi \cdot (e_1 + \ldots + e_{\bar{n}}) = 0 & \text{on } \pi(\mathcal{M}_N) \\ \psi \cdot \vec{n} = 0 & \text{on } \partial\pi(\mathcal{M}_N). \end{cases} \tag{4.154}$$

Let $\alpha_0 \in (\frac{1-2s}{1-s}, 1)$ as in Lemma 4.5.4. Let $\gamma \geqslant \frac{1}{2}$. There exist $C(\beta)$ locally uniform in β, $\kappa > 0$, $\delta > 0$ and $\varepsilon_0 > 0$ such that

$$\begin{aligned} \mathbb{E}_\nu\Big[\sum_{i=1}^{\bar{n}} d(i,i_0)^{2(\frac{\gamma}{2}+\frac{1}{4})} |\nabla\psi_i|^2\Big] \leqslant C(\beta) n^{\kappa\varepsilon} \mathbb{E}_\nu[|\mathrm{L}_\gamma \psi|^2]^{\frac{1}{2}} \Big(n^{\kappa\varepsilon_0} \mathbb{E}_\nu[|\mathrm{L}_{1/2}\psi|^2]^{\frac{1-\alpha_0}{2}} \mathbb{E}_\nu[|\mathrm{L}_{3/2-s}\psi|^2]^{\frac{\alpha_0}{2}} \\ + n^{-\varepsilon_0} \mathbb{E}_\nu[|\mathrm{L}_{3/2-s}\psi|^2]^{\frac{1}{2}} + n^{\kappa\varepsilon_0+1} (\mathbb{E}_\nu[\lambda^2]^{\frac{1}{2}} \Big) + n^{\kappa(\varepsilon_0+\varepsilon)} \mathbb{E}_\nu[\chi_n^2]. \end{aligned} \tag{4.155}$$

Proof. Let $\psi \in L^2(\bar{I}, H^1(\nu))$ be the solution of (4.154).

Step 1: a priori estimates and distortion First note that ψ satisfies the energetic estimate

$$\mathbb{E}_\nu[|\psi|^2]^{\frac{1}{2}} + \mathbb{E}_\nu[|D\psi|^2]^{\frac{1}{2}} \leqslant C(\beta) n^{\kappa\varepsilon} \mathbb{E}_\nu[\chi_n^2]^{\frac{1}{2}}. \tag{4.156}$$

For $\alpha \geqslant \frac{1}{2}$, let $\mathrm{L}_\alpha \in \mathcal{M}_{\bar{n}}(\mathbb{R})$ be as in (4.145). Let $\psi^{\mathrm{dis}} = \mathrm{L}_\alpha \psi$. Multiplying (4.175) by L_α, one can see that ψ^{dis} solves

$$\beta \mathrm{L}_\alpha \mathsf{M} \mathrm{L}_\alpha^{-1} \psi^{\mathrm{dis}} + \mathcal{L}^\nu \psi^{\mathrm{dis}} = \chi_n e_{i_0} + \lambda \mathrm{L}_\alpha (e_1 + \ldots + e_{\bar{n}}).$$

In contrast with the short-range case, one cannot expect $|\mathsf{M}\psi^{\mathrm{dis}}|$ to be of order $n^{\kappa\varepsilon}$ under ν if $\alpha = \frac{3}{2} - s$ and one should therefore not split $\mathrm{L}_\alpha \mathsf{M} \psi^{\mathrm{dis}}$ into $\mathsf{M}\psi^{\mathrm{dis}} + (\mathrm{L}_\alpha \mathsf{M} \mathrm{L}_\alpha^{-1} - \mathsf{M})\psi^{\mathrm{dis}}$. We will instead isolate short-range interactions. Fix a small parameter $\varepsilon_0 > 0$. By Assumptions 4.4.1, there exists a family of non-negative functions $(\alpha_{i,j})_{i,j \in \bar{I}}$ such that

$$U_N \cdot \mathsf{M} U_N = \sum_{k \neq l} \alpha_{k,l} (u_k + \ldots + u_l)^2 \geqslant \sum_{k \neq l : d(k,l) \leqslant d(k,i_0)^{1-\varepsilon_0}} \alpha_{k,l} (u_k + \ldots + u_l)^2 := U_N \cdot \widetilde{\mathsf{M}} U_N.$$

By construction, we therefore have $\mathsf{M} \geqslant \widetilde{\mathsf{M}}$, where

$$\widetilde{\mathsf{M}}_{i,j} := \sum_{k \geqslant j, l \leqslant i : d(k,l) \leqslant d(i,i_0)^{1-\varepsilon_0}} \alpha_{k,l}.$$

Denoting $l_0 := \lfloor d(i,i_0)^{1-\varepsilon_0} \rfloor$, let us define the matrix valued-function given for each $i, j \in \bar{I}$ by

$$\widetilde{\mathsf{M}}^{(1)}_{i,j} = \begin{cases} g_s(j-i) - g_s(l_0) - h_s(l_0)(d(j,i) - l_0) & \text{if } d(j,i) \leqslant l_0 \\ 0 & \text{if } d(j,i) > l_0. \end{cases} \tag{4.157}$$

Finally let $\mathsf{M}^{(2)} = \mathsf{M} - \mathbb{H}_s$ be the random part of M and set

$$\delta^{(1)}_{\mathrm{L}_\alpha} = \mathrm{L}_\alpha \mathbb{H}_s \mathrm{L}_\alpha^{-1} - \widetilde{\mathsf{M}}^{(1)} \quad \text{and} \quad \delta^{(2)}_{\mathrm{L}_\alpha} = \mathrm{L}_\alpha \mathsf{M}^{(2)} \mathrm{L}_\alpha^{-1} - \mathsf{M}^{(2)},$$

so that ψ^{dis} is solution of

$$\beta \widetilde{\mathsf{M}} \psi^{\mathrm{dis}} + \beta \delta^{(1)}_{\mathrm{L}_\alpha} \psi^{\mathrm{dis}} + \beta \delta^{(2)}_{\mathrm{L}_\alpha} \psi^{\mathrm{dis}} + \mathcal{L}^\nu \psi^{\mathrm{dis}} = \chi_n e_{i_0} + \lambda \mathrm{L}_\alpha (e_1 + \ldots + e_{\bar{n}}). \tag{4.158}$$

Step 2: integration by parts We proceed as in the proof of Lemma 4.4.2. Taking the scalar product of (4.158) with ψ^{dis} reads

$$\mathbb{E}_\nu[\beta \psi^{\mathrm{dis}} \cdot (\widetilde{\mathsf{M}} + \delta^{(1)}_{\mathrm{L}_\alpha} + \delta^{(2)}_{\mathrm{L}_\alpha}) \psi^{\mathrm{dis}}] + \mathbb{E}_\nu[|\nabla \psi^{\mathrm{dis}}|^2] = \mathbb{E}_\nu[\psi_{i_0} \chi_n + \mathrm{L}_{2\alpha} \psi \cdot (e_1 + \ldots + e_{\bar{n}}) \lambda]. \tag{4.159}$$

By construction, there exists a constant $\kappa_0 > 0$ such that

$$\widetilde{\mathsf{M}} \geqslant n^{-\kappa_0 \varepsilon} I_{\bar{n}}. \tag{4.160}$$

It therefore remains to control the commutators $\delta^{(1)}_{\mathrm{L}_\alpha}$ and $\delta^{(2)}_{\mathrm{L}_\alpha}$.

Step 3: control on the long-range commutator This step is the most important of the proof. Recalling that $\mathrm{L}_\alpha\mathbb{H}_s\mathrm{L}_\alpha^{-1}\psi^{\mathrm{dis}}=\mathrm{L}_\alpha\mathbb{H}_s\psi$, one may split $\delta_{\mathrm{L}_\alpha}^{(1)}\psi^{\mathrm{dis}}$ into

$$(\delta_{\mathrm{L}_\alpha}^{(1)}\psi^{\mathrm{dis}})_i=\underbrace{d(i,i_0)^\alpha\sum_{k:d(i,k)\geqslant d(i,i_0)^{1-\varepsilon_0}}g_s(i-k)\psi_k}_{(I)_i}+\underbrace{\sum_{k:d(i,k)\leqslant d(i,i_0)^{1-\varepsilon_0}}g_s(i-k)\Big(\frac{d(i,i_0)^\alpha}{d(k,i_0)^\alpha}-1\Big)\psi_k^{\mathrm{dis}}}_{(II)_i}+(III)_i, \tag{4.161}$$

with

$$(III)_i=h_s(d(i,i_0)^{1-\varepsilon_0})\sum_{k:d(i,k)\leqslant d(i,i_0)^{1-\varepsilon_0}}(d(i,k)-d(i,i_0)^{1-\varepsilon_0})\psi_k^{\mathrm{dis}}-g_s(d(i,i_0)^{1-\varepsilon_0})\sum_{k:d(i,k)\leqslant d(i,i_0)^{1-\varepsilon_0}}\psi_k^{\mathrm{dis}}.$$

Let us split $(I)_i$ further into

$$(I)_i=\underbrace{d(i,i_0)^\alpha\sum_{k:d(i,i_0)^{1-\varepsilon_0}\leqslant d(i,k)\leqslant d(i,i_0)^{1+\varepsilon_0}}g_s(i-k)\psi_k}_{(I)_i'}+\underbrace{d(i,i_0)^\alpha\sum_{k:d(i,k)>d(i,i_0)^{1+\varepsilon_0}}g_s(i-k)\psi_k}_{(I)_i''}.$$

First note that by Cauchy-Schwarz inequality,

$$|(I)_i''|\leqslant C\Big(\sum_{k:d(i,k)>d(i,i_0)^{1+\varepsilon_0}}\frac{1}{d(i,k)^{2s}}\frac{1}{d(i_0,k)^{3-2s}}\Big)^{\frac12}|\mathrm{L}_{3/2-s}\psi|\leqslant\frac{C}{d(i,i_0)^{1+\varepsilon_0}}|\mathrm{L}_{3/2-s}\psi|.$$

We turn to the term $(I)_i'$. The idea is to express it with respect to the primitive of order $1-s/2$ of ψ and to use the $L^{\frac{1}{1-s/2}}$ control of Lemma 4.5.4. Let $w=\mathbb{H}_{s/2}\psi$ and $g_{s/2}^{-1}=\mathbb{H}_{s/2}^{-1}e_1$. One may write

$$(I)_i'=\sum_{l=1}^{\bar n}\Big(\sum_{k:d(i,i_0)^{1-\varepsilon_0}\leqslant d(i,k)\leqslant d(i,i_0)^{1+\varepsilon_0}}\frac{1}{d(i,k)^s}g_{s/2}^{-1}(k-l)\Big)w_l. \tag{4.162}$$

The value of the truncated convolution product in front of w_l depends on whether l lies close to the boundary of $A(i):=\{k:d(i,i_0)^{1-\varepsilon_0}\leqslant d(i,k)\leqslant d(i,i_0)^{1+\varepsilon_0}\}$. We claim that there exists a constant $C>0$ such that for each $l\in\{1,\ldots,\bar n\}$,

$$\Big|\sum_{k\in A(i)}\frac{1}{d(i,k)^s}g_{s/2}^{-1}(k-l)\Big|\leqslant C\Big(\frac{1}{d(i,l)^s}\frac{1}{d(l,\partial A(i))^{1-s/2}}+\frac{1}{d(i,l)^{1+\frac{s}{2}-\kappa\varepsilon_0}}\Big). \tag{4.163}$$

Let us prove (4.163). First, in view of Lemma 4.2.4, the kernel $g_{s/2}^{-1}$ satisfies

$$|g_{s/2}^{-1}|(k)\leqslant\frac{C}{d(k,1)^{2-s/2}}\quad\text{for each }1\leqslant k\leqslant\bar n, \tag{4.164}$$

with

$$\Big|\sum_{k=1}^{\bar n}g_{s/2}^{-1}(k)\Big|\leqslant\frac{C}{n^{1-\frac{s}{2}}}. \tag{4.165}$$

If $d(l,A(i))\geqslant d(i,i_0)$, then by (4.164), the result if straightforward. Now if $l\in A(i)$ with $d(l,\partial A(i))\geqslant d(i,i_0)$, one can write

$$\sum_{k\in A(i)}\frac{1}{d(i,k)^s}g_{s/2}^{-1}(k-l)=-\sum_{k\in A(i)}\frac{1}{d(i,k)^s}g_{s/2}^{-1}(k-l)=O\Big(\frac{1}{d(i,i_0)^{1+\frac{s}{2}-\kappa\varepsilon_0}}\Big).$$

Finally let l such that $d(l,\partial A(i)) \leqslant d(i,i_0)$. One has

$$\sum_{k\in A(i)} \frac{1}{d(i,k)^s} g_{s/2}^{-1}(k-l) = \sum_{k\in A(i):d(k,l)\leqslant \frac{3}{4}d(i,i_0)} \frac{1}{d(i,k)^s} g_{s/2}^{-1}(k-l) + \sum_{k\in A(i):d(k,l)> \frac{3}{4}d(i,i_0)} \frac{1}{d(i,k)^s} g_{s/2}^{-1}(k-l).$$

In view of (4.164) there holds

$$\Big| \sum_{k\in A(i):d(k,l)> \frac{3}{4}d(i,i_0)} \frac{1}{d(i,k)^s} g_{s/2}^{-1}(k-l)\Big| \leqslant \frac{C}{d(i,i_0)^{1+\frac{s}{2}-\kappa\varepsilon_0}}.$$

Let us split the first term by writing

$$\frac{1}{d(i,k)^s} = \frac{1}{d(i,l)^s} + \frac{1}{d(i,k)^s} - \frac{1}{d(i,l)^s}.$$

Since $d(l,A(i)) \leqslant d(i,i_0)$ and $d(k,l) \leqslant \frac{3}{4}d(i,i_0)$ one has

$$\Big|\frac{1}{d(i,k)^s} - \frac{1}{d(i,l)^s}\Big| \leqslant \frac{Cd(k,l)}{d(i,i_0)^{1+s}}. \tag{4.166}$$

Using in turn (4.164) and (4.165), one can see that

$$\begin{aligned}\sum_{k\in A(i):d(k,i)\leqslant d(i,i_0)} g_{s/2}^{-1}(k-l) &= \sum_{k\in A(i)} g_{s/2}^{-1}(k-l) + O\Big(\frac{1}{d(i,i_0)^{1-s/2}}\Big)\\ &= O\Big(\frac{1}{d(l,\partial A(i))^{1-s/2}} + \frac{1}{d(i,i_0)^{1-s/2}}\Big).\end{aligned}$$

Finally inserting (4.166) we have

$$\Big|\sum_{k\in A(i):d(k,l)\leqslant d(i,i_0)} \Big(\frac{1}{d(i,k)^s} - \frac{1}{d(i,l)^s}\Big)\frac{1}{d(k,l)^{2-s/2}}\Big| \leqslant C\frac{d(i,i_0)^{s/2}}{d(i,l)^{1+s}} \leqslant \frac{C}{d(i,l)^{1+\frac{s}{2}-\kappa\varepsilon_0}}.$$

Combining the two last displays, one obtains the claimed estimate (4.163).

Let us split the sum over l in (4.162) according to whether $d(l,\partial A(i)) \geqslant d(i,i_0)^{1-2\varepsilon_0}$. For the first contribution one can write

$$\begin{aligned}&\Big|\sum_{l:d(l,\partial A(i))\geqslant d(i,i_0)^{1-2\varepsilon_0}} \frac{1}{d(i,l)^s}\frac{1}{d(l,\partial A(i))^{1-s/2}} w_l\Big| \leqslant Cd(i,i_0)^{\kappa\varepsilon_0} \sum_{l:d(i,l)\geqslant d(i,i_0)^{1-2\varepsilon_0}} \frac{1}{d(i,l)^{1+\frac{s}{2}}}|w_l|\\ &\leqslant Cd(i,i_0)^{\kappa\varepsilon_0}\Big(\sum_{l=1}^{\bar n}|w_l|^{\frac{1}{1-s/2}}\Big)^{1-s/2}\Big(\sum_{l:d(i,l)\geqslant d(i,i_0)^{1-2\varepsilon_0}} \frac{1}{d(i,l)^{\frac{2}{s}(1+\frac{s}{2})}}\Big)^{\frac{s}{2}} \leqslant \frac{C}{d(i,i_0)^{1-\kappa\varepsilon_0}}\Big(\sum_{l=1}^{\bar n}|w_l|^{\frac{1}{1-s/2}}\Big)^{1-s/2}.\end{aligned}$$

Inserting the estimate (4.147) of Lemma 4.5.4 then yields

$$\begin{aligned}&\mathbb{E}_\nu\Big[\Big|\sum_{l:d(l,\partial A(i))\geqslant d(i,i_0)^{1-2\varepsilon_0}} \frac{1}{d(i,l)^s}\frac{1}{d(l,\partial A(i))^{1-s/2}} w_l\Big|^2\Big]^{\frac{1}{2}}\\ &\qquad\leqslant C(\beta)n^{\kappa\varepsilon}\frac{1}{d(i,i_0)^{1-\kappa\varepsilon_0}}(\mathbb{E}_\nu[|\mathrm{L}_{1/2}\psi|^2]^{\frac{1}{2}} + \mathbb{E}_\nu[\chi_n^2]^{\frac{1}{2}})^s\mathbb{E}_\nu[|\mathrm{L}_{3/2-s}\psi|^2]^{\frac{1-s}{2}}.\end{aligned}$$

For the second contribution, one can check via Cauchy-Schwarz inequality that

$$|w_l| \leqslant \frac{C}{d(l,i_0)^{1-s/2}} |\mathrm{L}_{3/2-s}\psi|.$$

It follows that

$$\begin{aligned}\Big|\sum_{l:d(l,\partial A(i))\leqslant d(i,i_0)^{1-2\varepsilon_0}} \frac{1}{d(i,l)^s}\frac{1}{d(l,\partial A(i))^{1-\frac{s}{2}}} w_l\Big| &\leqslant C\frac{1}{d(i,i_0)^{1+\frac{s}{2}}} \sum_{l:d(l,\partial A(i))\leqslant d(i,i_0)^{1-2\varepsilon_0}} \frac{1}{d(l,\partial A(i))^{1-\frac{s}{2}}}|\mathrm{L}_{3/2-s}\psi| \\ &\leqslant \frac{C}{d(i,i_0)^{1+s\varepsilon_0}}|\mathrm{L}_{3/2-s}\psi|.\end{aligned} \tag{4.167}$$

We have crucially used the fact that in (4.167), the series $\sum_k \frac{1}{k^{1-s/2}}$ is diverging, in order to have an error in the last display much smaller than $d(i,i_0)^{-1}$, when $\varepsilon_0>0$. This justifies our choice of considering a fractional primitive of order $1-s/2$ (rather than $3/2-s$ for instance). One can gather these estimates into

$$\begin{aligned}\mathbb{E}_\nu[(I)_i^2]^{\frac12} \leqslant \frac{C(\beta)n^{\kappa\varepsilon}}{d(i,i_0)^{1-\alpha}}\Big(\Big(\mathbb{E}_\nu[\chi_n^2]^{\frac12} + \sup|\chi_n|e^{-c(\beta)n^\delta} + n^{\kappa\varepsilon_0}\mathbb{E}_\nu[|\mathrm{L}_{1/2}\psi|^2]^{\frac12}\Big)^s \\ + \mathbb{E}_\nu[|\mathrm{L}_{3/2-s}\psi|^2]^{\frac{1-s}{2}} + \frac{1}{d(i,i_0)^{1+s\varepsilon_0}}\mathbb{E}_\nu[|\mathrm{L}_{3/2-s}\psi|^2]^{\frac12}\Big).\end{aligned} \tag{4.168}$$

We now control the terms $(II)_i$ and $(III)_i$. Let us write $(II)_i$ as

$$\begin{aligned}(II)_i &= \sum_{k:d(i,k)\leqslant d(i,i_0)^{1-\varepsilon_0}} \frac{1}{d(i,k)^s}(d(i,i_0)^\alpha - d(k,i_0)^\alpha)\psi_k \\ &= d(i,i_0)^\alpha \sum_{k:d(i,k)\leqslant d(i,i_0)^{1-\varepsilon_0}} \frac{1}{d(i,k)^s}\Big(1-\frac{d(i,k)^\alpha}{d(i,i_0)^\alpha}\Big)\psi_k.\end{aligned}$$

One can Taylor expand the weight in the above equation when $d(i,k)\leqslant d(i,i_0)^{1-\varepsilon_0}$ into

$$\Big|1-\frac{d(i,k)^\alpha}{d(i,i_0)^\alpha}\Big| \leqslant C\frac{d(i,k)}{d(i,i_0)}.$$

This allows one to upper bound $(II)_i$ by

$$|(II)_i| \leqslant d(i,i_0)^{\alpha-1}\sum_{k:d(i,k)\leqslant d(i,i_0)^{1-\varepsilon_0}} d(i,k)^{1-s}|\psi_k| \leqslant \frac{C}{d(i,i_0)^{1-\alpha+(1-s)\varepsilon_0}}|\mathrm{L}_{3/2-s}\psi|. \tag{4.169}$$

Similarly, by expanding $d(k,i_0)^\alpha$ for k close to i, one obtains

$$|(III)_i| \leqslant \frac{C}{d(i,i_0)^{1-\alpha+\varepsilon_0(1-s)}}|\mathrm{L}_{3/2-s}\psi|. \tag{4.170}$$

Putting (4.168), (4.169) and (4.170) together, one obtains that for $\varepsilon_0>0$ large enough with respect to ε, there exists $\kappa>0$ such that

$$\begin{aligned}|\mathbb{E}_\nu[\psi^{\mathrm{dis}}\cdot\delta^{(1)}_{\mathrm{L}_\alpha}\psi^{\mathrm{dis}}]| \leqslant C(\beta)n^{\kappa\varepsilon}\mathbb{E}_\nu[|\mathrm{L}_{2\alpha-1/2}\psi|^2]^{\frac12}\Big(n^{\kappa\varepsilon_0}\mathbb{E}_\nu[|\mathrm{L}_{1/2}\psi|^2]^{\frac12} + n^{-\varepsilon_0}\mathbb{E}_\nu[|\mathrm{L}_{3/2-s}\psi|^2]^{\frac12} \\ n^{\kappa\varepsilon_0}\mathbb{E}_\nu[|\mathrm{L}_{1/2}\psi|^2]^{\frac{s}{2}}\mathbb{E}_\nu[|\mathrm{L}_{3/2-s}\psi|^2]^{\frac{1-s}{2}} + n^{\kappa\varepsilon_0}\mathbb{E}_\nu[\lambda^2]^{\frac12}\Big) + C(\beta)n^{\kappa\varepsilon_0}\mathbb{E}_\nu[\chi_n^2].\end{aligned} \tag{4.171}$$

Step 4: control on the short-range commutator It remains to upper bound $\delta^{(2)}_{\mathrm{L}_\alpha}$. Recall that by (4.124), the off-diagonal entries of $\mathsf{M}^{(2)}$ typically decays in $d(i,j)^{-(1+\frac{s}{2})}$. One may write

$$(\delta^{(2)}_{\mathrm{L}_\alpha}\psi^{\mathrm{dis}})_i = \underbrace{\sum_{k:d(i,k)\leqslant\frac{1}{2}d(i,i_0)} \mathsf{M}^{(2)}_{i,k}\Big(\frac{d(i,i_0)^\alpha}{d(k,1)^\alpha}-1\Big)\psi^{\mathrm{dis}}_k}_{(I)_i} + \underbrace{\sum_{k:d(i,k)>\frac{1}{2}d(i,i_0)} \mathsf{M}^{(2)}_{i,k}\Big(\frac{d(i,i_0)^\alpha}{d(k,1)^\alpha}-1\Big)\psi^{\mathrm{dis}}_k}_{(II)_i}.$$

The first term can be bounded for any value of α by

$$\mathbb{E}_\nu[(I)_i^2]^{\frac{1}{2}} \leqslant \frac{C(\beta)n^{\kappa\varepsilon}}{d(i,i_0)^{\frac{1}{2}+\frac{s}{2}}}\mathbb{E}_\nu[|\psi^{\mathrm{dis}}|^2]^{\frac{1}{2}},$$

with $C(\beta)$ depending on α. For the second term we have

$$\mathbb{E}_\nu[(II)_i^2]^{\frac{1}{2}} \leqslant \frac{C(\beta)n^{\kappa\varepsilon}}{d(i,i_0)^{1+\frac{s}{2}-\alpha}}\mathbb{E}_\nu[|\mathrm{L}_{1/2}\psi|^2]^{\frac{1}{2}}.$$

Consequently arguing as in the short-range case (see the proof of Lemma 4.4.2) we obtain

$$\Big|\mathbb{E}_\nu\Big[\sum_{i=1}^{\bar n}\psi_i^{\mathrm{dis}}(I)_i\Big]\Big| \leqslant \frac{\beta}{2}n^{-\varepsilon(s+2)}\mathbb{E}_\nu[|\psi^{\mathrm{dis}}|^2] + C(\beta)n^{\kappa\varepsilon}\mathbb{E}_\nu[|\psi^{\mathrm{dis}}|^2]\mathbb{E}_\nu[\chi_n^2]^{\frac{1}{2}}.$$

By construction, we have

$$\mathbb{E}_\nu\Big[\psi^{\mathrm{dis}}\cdot\tilde{\mathsf{M}}\psi^{\mathrm{dis}} + \sum_{i=1}^{\bar n}\psi_i^{\mathrm{dis}}(I)_i\Big] \geqslant 0. \tag{4.172}$$

For the second term, the point is to give a control in term of $\mathrm{L}_{2\alpha-1/2}\psi$:

$$\Big|\mathbb{E}_\nu\Big[\sum_{i=1}^{\bar n}\psi_i^{\mathrm{dis}}(II)_i\Big]\Big| \leqslant C(\beta)n^{\kappa\varepsilon}\mathbb{E}_\nu[|\mathrm{L}_{2\alpha-1/2}\psi|^2]^{\frac{1}{2}}\mathbb{E}_\nu[|\mathrm{L}_{1/2}\psi|^2]^{\frac{1}{2}}. \tag{4.173}$$

Step 5: conclusion Note that for $\alpha\geqslant\frac{1}{2}$, $2\alpha-\frac{1}{2}\geqslant\alpha$. Therefore in view of (4.171), (4.172) and (4.173) we obtain from (4.159) that for $\alpha\geqslant\frac{1}{2}$,

$$\begin{aligned}\mathbb{E}_\nu\Big[\sum_{i=1}^{\bar n}d(i,i_0)^{2\alpha}|\nabla\psi_i|^2\Big] \leqslant C(\beta)n^{\kappa\varepsilon}\mathbb{E}_\nu[|\mathrm{L}_{2\alpha-1/2}\psi|^2]^{\frac{1}{2}}\Big(n^{\kappa\varepsilon_0}\mathbb{E}_\nu[|\mathrm{L}_{1/2}\psi|^2]^{\frac{s}{2}}\mathbb{E}_\nu[|\mathrm{L}_{3/2-s}\psi|^2]^{\frac{1-s}{2}}\\ + n^{-\varepsilon_0}\mathbb{E}_\nu[|\mathrm{L}_{3/2-s}\psi|^2]^{\frac{1}{2}} + n^{\kappa\varepsilon_0} + n\mathbb{E}_\nu[\lambda^2]^{\frac{1}{2}}\Big) + n^{\kappa\varepsilon_0}\mathbb{E}_\nu[\chi_n^2]\Big).\end{aligned} \tag{4.174}$$

This completes the proof of Lemma 4.5.5. □

4.5.4 Global decay estimate

Leveraging on the a priori estimate of Lemma 4.5.5, we establish a global decay estimate on the solution. The method uses a factorization of the system around its ground state to reduce the problem to the well-understood short-range situation of Section 4.4. Let us emphasize that due to the degeneracy of the inverse of Riesz matrix (4.25), it is unavoidable to have an a priori control on $D\psi$ such as (4.155).

Lemma 4.5.6. *Let $s \in (0,1)$. Let ν and M satisfying Assumptions 4.5.1 and 4.5.2. Let $\chi_n \in H^1(\nu)$, $i_0 \in \{1, \ldots, n\}$ and $\psi \in L^2(\bar{I}, H^1(\nu))$ be the solution of*

$$\begin{cases} \beta \mathsf{M}\psi + \mathcal{L}^\nu \psi = \chi_n e_{i_0} + \lambda(e_1 + \ldots + e_{\bar{n}}) & \text{on } \pi(\mathcal{M}_N) \\ \psi \cdot (e_1 + \ldots + e_{\bar{n}}) = 0 & \text{on } \pi(\mathcal{M}_N) \\ \psi \cdot \vec{n} = 0 & \text{on } \partial\pi(\mathcal{M}_N). \end{cases} \tag{4.175}$$

There exists a constant $C(\beta)$ locally uniform in β and $\kappa > 0$ such that

$$\mathbb{E}_\nu\Big[\sum_{i=1}^{\bar{n}} d(i,i_0)^{2-s}|\nabla\psi_i|^2\Big]^{\frac{1}{2}} + \mathbb{E}_\nu\Big[\sum_{i=1}^{\bar{n}} d(i,i_0)^{3-2s}\psi_i^2\Big]^{\frac{1}{2}} \leqslant C(\beta) n^{\kappa\varepsilon}\mathbb{E}_\nu[\chi_n^2]^{\frac{1}{2}}. \tag{4.176}$$

In addition, there exist a constant $C(\beta)$ locally uniform in β and $\kappa > 0$ such that

$$\mathbb{E}_\nu[\lambda^2]^{\frac{1}{2}} \leqslant \frac{C(\beta)}{n^{1-\kappa\varepsilon}}\mathbb{E}_\nu[\chi_n^2]^{\frac{1}{2}}. \tag{4.177}$$

Proof. The proof builds on the estimate (4.155). The strategy is to multiply the system (4.175) by a constant matrix close to the inverse of $\mathbb{H}_s$, so that the system becomes short-range. There are two difficulties: first one should keep a positive-definite matrix and second one should control the differential terms involving $\mathcal{L}^\nu$.

Step 1: factorization around the ground state To solve the first issue, the idea is to define a kernel f which is vanishing outside a certain grid centered at 1 and of length $K_1 = \lfloor n^\varepsilon \rfloor^{\kappa_0}$ for some $\kappa_0 \in \mathbb{N}^*$. Assume first that $m := \frac{\bar{n}}{K_1} \in \mathbb{N}$. Define

$$f(l) = \begin{cases} g_s^{-1}(k) & \text{if } l = 1 + kK_1,\ 0 \leqslant k \leqslant m-1 \\ 0 & \text{otherwise} \end{cases}, \tag{4.178}$$

where $g_s^{-1} = \mathbb{H}_s^{-1} e_1$. Also let A be the Toeplitz matrix associated to h:

$$\mathsf{A} := (f(j-i))_{i,j} \in \mathcal{M}_{\bar{n}}(\mathbb{R}). \tag{4.179}$$

Let us first show that f is a positive-definite kernel on $\{1, \ldots, \bar{n}\}$. Let $\theta \in \{\frac{2k\pi}{\bar{n}} : 0 \leqslant k \leqslant \bar{n}-1\}$. One may notice that

$$\sum_{k=0}^{\bar{n}-1} f(k)e^{ik\theta} = \sum_{k=0}^{m-1} g_s^{-1}(k)e^{ik\theta K_1}.$$

Since $K_1\theta \in \{\frac{2k\pi}{m} : 0 \leqslant k \leqslant m-1\}$, the above sum is positive. It follows that (4.178) defines a positive-definite kernel and (4.179) a positive-definite matrix.

Assume that $\frac{\bar{n}}{K_1} \notin \mathbb{N}$. Let $m = \lfloor \frac{\bar{n}}{K_1} \rfloor$ and $v \in \mathbb{R}^{mK_1}$ with $v_k = u_k$ for each $1 \leqslant k \leqslant mK_1$. Let also $\mathsf{A}' = (f(i-j))_{1 \leqslant i,j \leqslant mK_1}$. One can observe that for all $U_{\bar{n}} \in \mathbb{R}^{\bar{n}}$,

$$|U_{\bar{n}} \cdot \mathsf{A}U_{\bar{n}} - V_{mK_1} \cdot \mathsf{A}'V_{mK_1}| \leqslant Cn^{\kappa\varepsilon}\Big(\sum_{k=1}^{K_1} |u_k|^2\Big)^{\frac{1}{2}} |\mathrm{L}_{3/2-s}U_N|.$$

We now argue that for K_1 large enough, the matrix AM is positive-definite. This is quite delicate since as is well known, the product of two positive-definite matrix is not in general positive-definite. Assume first that $\frac{\bar{n}}{K_1} \in \mathbb{N}$. The idea is to separate M into the sum of a Toeplitz matrix associated to a positive kernel and a random "diagonally dominant" positive matrix. As in Subsection 4.5.1, we first isolate small-range interactions, which do not concentrate around a constant, but provide some near-uniform convexity. Following Assumptions 4.4.1, there exists a family of non-negative functions $(\alpha_{i,j})_{i,j\in \bar{I}}$ such that

$$U_N \cdot \mathsf{M} U_N = \sum_{k\neq l} \alpha_{k,l}(u_k + \ldots + u_l)^2.$$

For K_1 as above, let us split M into $\mathsf{M} = \mathsf{M}^{(1)} + \mathsf{M}^{(2)}$ with for each $1 \leqslant i, j \leqslant \bar{n}$,

$$\mathsf{M}^{(1)}_{i,j} = \sum_{(k,l)\in I_{i,j}} \alpha_{k,l} \mathbb{1}_{d(k,l)\leqslant K_1},$$

where $I_{i,j} := \{k \in \bar{I} : d(k, \frac{i+j}{2}) > \frac{1}{2} d(i, j)\}$. Since $\mathsf{M}^{(1)}_{i,j} = 0$ if $d(i, j) > K_1$ observe that $A\mathsf{M}^{(1)} = \mathsf{M}^{(1)}$. Consequently there exists $\kappa_0 > 0$ (independent of K_1) such that

$$\mathsf{A}\mathsf{M}^{(1)} \geqslant n^{-\kappa_0 \varepsilon} I_{\bar{n}}. \tag{4.180}$$

Let us now control the product of A with the long-range matrix $\mathsf{M}^{(2)}$. To this end, we split $\mathsf{M}^{(2)}$ into the sum of a Toeplitz matrix and of a random part. Let us h be the Riesz kernel truncated at K_1 defined for each $k \in \{1, \ldots, \bar{n}\}$ by

$$h(k) := \sum_{(i,j)\in I_{1,k}} g''_s(j - i)\mathbb{1}_{d(j,i)\geqslant K_1}. \tag{4.181}$$

Observe that h is a non-negative kernel since for all $U_N \in \mathbb{R}^N$,

$$\sum_{i,j} h(i - j)u_i u_j = \sum_{i,j} g''_s(i - j)\mathbb{1}_{d(i,j)\geqslant K_1}(u_i - u_j)^2.$$

Now let $\mathsf{M}^{(2,1)}$ be the Toeplitz matrix associated to h and $\mathsf{M}^{(2,2)} := \mathsf{M}^{(2)} - \mathsf{M}^{(2,1)}$. Since Toeplitz matrices do commute, the product of A and $\mathsf{M}^{(2,1)}$ is non-negative. For the random part $\mathsf{M}^{(2,2)}$, note

$$|\mathsf{M}^{(2,2)}_{i,j}| \leqslant \frac{Cn^{\kappa\varepsilon}}{d(i, j)^{1+\frac{s}{2}}}\mathbb{1}_{d(i,j)\geqslant K_1},$$

uniformly for $1 \leqslant i, j \leqslant \bar{n}$. Therefore denoting $\|\cdot\|$ the spectral norm on $\mathcal{M}_{\bar{n}}(\mathbb{R})$, we find that on (4.120),

$$\|\mathsf{A}\mathsf{M}^{(2,2)}\| \leqslant Cn^{\kappa\varepsilon} K_1^{-\frac{s}{2}}. \tag{4.182}$$

This can be made much smaller than the lower bound in (4.180) by choosing K_1 large enough, thus proving that $\mathsf{A}\mathsf{M}^{(2)}$ is positive-definite. In conclusion, if $\frac{\bar{n}}{K_1} \in \mathbb{N}$, there exists $\kappa > 0$ such that on (4.120),

$$\mathsf{A}\mathsf{M} \geqslant n^{-\kappa\varepsilon} I_{\bar{n}}.$$

To summarize, on the first hand, the positivity of $\mathsf{A}\mathsf{M}^{(1)}$ follows from the construction (4.178), the positivity of $\mathsf{M}^{(1)}$ and (4.182). On the one hand the positivity of $\mathsf{A}\mathsf{M}^{(2,1)}$ follows from the fact A and $\mathsf{M}^{(2,1)}$ are positive and commute. Now, if $\frac{\bar{n}}{K_1} \notin \mathbb{N}$, then for all $U_{\bar{n}} \in \mathbb{R}^{\bar{n}}$,

$$U_{\bar{n}} \cdot \mathsf{A}\mathsf{M} U_{\bar{n}} \geqslant n^{-\kappa\varepsilon} I_{\bar{n}} - Cn^{\kappa\varepsilon}\Big(\sum_{i=1}^{K_1} |u_i|^2\Big)^{\frac{1}{2}} |\mathrm{L}_{3/2-s} U_{\bar{n}}|. \tag{4.183}$$

We will apply (4.183) to $\psi^{\mathrm{dis}} := \mathrm{L}_\alpha \psi$ and control $\sum_{i=1}^{K_1} (\psi^{\mathrm{dis}}_i)^2$ by $K_1^{2\alpha} |\psi|^2$.

Finally, the kernel (4.178) defines an approximation of g_s^{-1}: choosing K_1 to be a large power of $\lfloor n^{\varepsilon} \rfloor$ as above, one can check that there exists a constant $\kappa > 0$ such that for each $k \in \{1, \ldots, \bar{n}\}$,

$$|h * f|(k) \leqslant \frac{C n^{\kappa\varepsilon}}{1 + d(k,1)^{2-s}}. \tag{4.184}$$

Indeed, if $i = 1 + (K_1 - 1)l \in \{1, \ldots, \bar{n}\}$, then

$$\sum_{k=1}^{\bar{n}} g_s(k-i) f(k) = K_1^s \sum_{k=1}^{\frac{\bar{n}}{K_1}} g_s(k) f(k-l) = K_1^{-s} \mathbb{1}_{l=1}.$$

Now if $i \in \{1, \ldots, \bar{n}\}$, one can decompose is it into $i = i_0 + (i - i_0)$ with $i_0 \in \{1 + (K_1 - 1)\mathbb{Z}\} \cap \{1, \ldots, \bar{n}\}$ and $|i - i_0| \leqslant K_1$. Therefore, by Taylor expansion,

$$\Big| \sum_{k=1}^{\bar{n}} g_s(k-i) f(k) - \sum_{k=1}^{\bar{n}} g_s(k-i_0) f(k) - O(K_1) \sum_{k=1}^{\bar{n}} g_s'(k-i) f(k) \Big| \leqslant C K_1^2 \sum_{k=1}^{\bar{n}} \frac{1}{d(i,k)^{2+s}} \frac{1}{d(k,1)^{2-s}} \leqslant \frac{C K_1^2}{d(i,1)^{2-s}}.$$

In addition, one can check that the first-order term verifies

$$\Big| \sum_{k=1}^{\bar{n}} g_s'(k-i) f(k) \Big| \leqslant \frac{C}{d(i,1)^{2-s}},$$

thus implying that

$$\Big| \sum_{k=1}^{\bar{n}} g_s(k-i) f(k) \Big| \leqslant \frac{C n^{\kappa\varepsilon}}{d(i,1)^{2-s}}.$$

By comparing g_s to h, we conclude the proof of (4.184).

Step 2: distortion For $\alpha \geqslant \frac{1}{2}$, let $\mathrm{L}_\alpha \in \mathcal{M}_{\bar{n}}(\mathbb{R})$ be as in (4.145). The argument proceeds by multiplying Equation (4.175) by $\mathrm{L}_\alpha \mathsf{A}$. Set $\psi^{\mathrm{dis}} = \mathrm{L}_\alpha \psi$, which solves

$$\beta \mathrm{L}_\alpha \mathsf{AM} \mathrm{L}_\alpha^{-1} \psi^{\mathrm{dis}} + (\mathrm{L}_\alpha \mathsf{A} \mathrm{L}_\alpha^{-1} - \mathsf{A}) \mathcal{L}^\nu \psi^{\mathrm{dis}} + \mathsf{A} \mathcal{L}^\nu \psi^{\mathrm{dis}} = \mathrm{L}_\alpha \mathsf{A}(\chi_n e_{i_0} + \lambda(e_1 + \ldots + e_{\bar{n}})). \tag{4.185}$$

Set

$$\delta_{\mathrm{L}_\alpha} = \mathrm{L}_\alpha \mathsf{AM} \mathrm{L}_\alpha^{-1} - \mathsf{AM}.$$

Taking the scalar product of (4.185) with ψ^{dis} and integrating over ν yields

$$\beta \mathbb{E}_\nu \Big[\psi^{\mathrm{dis}} \cdot (\mathsf{AM} + \delta_{\mathrm{L}_\alpha}) \psi^{\mathrm{dis}} + \sum_{i,k} \mathsf{A}_{i,k} \nabla \psi_i^{\mathrm{dis}} \cdot \nabla \psi_k^{\mathrm{dis}} + \sum_{i,k} (\mathrm{L}_\alpha \mathsf{A} \mathrm{L}_\alpha^{-1} - \mathsf{A})_{i,k} \nabla \psi_i^{\mathrm{dis}} \cdot \nabla \psi_k^{\mathrm{dis}} \Big]$$
$$= \mathbb{E}_\nu [\chi_n \psi_{i_0} + \lambda \mathrm{L}_\alpha \psi \cdot \mathrm{L}_\alpha \mathsf{A}(e_1 + \ldots + e_{\bar{n}})]. \tag{4.186}$$

Step 3: control on the commutator with $\mathcal{L}^\nu$ We give a control on the quantity $\mathrm{L}_\alpha \mathsf{A} \mathrm{L}_\alpha^{-1} - \mathsf{A}$. Recall that the matrix A fails to be uniformly positive-definite (in n). Consequently one cannot bound the differential term in (4.186) by the norm $|D\psi^{\mathrm{dis}}|$. However as we have seen in Lemma 4.5.5 the gradient of ψ satisfies a global decay estimate whenever ψ does. Let us first split the quantity of interest into

$$\sum_k (\mathrm{L}_\alpha \mathsf{A} \mathrm{L}_\alpha^{-1} - \mathsf{A})_{i,k} \nabla \psi_i^{\mathrm{dis}} \cdot \nabla \psi_k^{\mathrm{dis}} = \sum_k f(i-k) \Big(\frac{d(i,i_0)^\alpha}{d(k,i_0)^\alpha} - 1 \Big) \nabla \psi_i^{\mathrm{dis}} \cdot \nabla \psi_k^{\mathrm{dis}}$$
$$= \underbrace{\sum_{k: d(k,i) \leqslant \frac{1}{2} d(i,i_0)} f(i-k) \Big(\frac{d(i,i_0)^\alpha}{d(k,i_0)^\alpha} - 1 \Big) \nabla \psi_i^{\mathrm{dis}} \cdot \nabla \psi_k^{\mathrm{dis}}}_{(I)_i} + \underbrace{\sum_{k: d(k,i) > \frac{1}{2} d(i,i_0)} f(i-k) \Big(\frac{d(i,i_0)^\alpha}{d(k,i_0)^\alpha} - 1 \Big) \nabla \psi_i^{\mathrm{dis}} \cdot \nabla \psi_k^{\mathrm{dis}}}_{(II)_i}.$$

We seek to control the expectation of $(I)_i$ and $(II)_i$ in term of $\mathbb{E}_\nu[|\mathrm{L}_\gamma D\psi|^2]$. For the second term, using (4.184) and the fact that $\sum_{k=1}^{\bar{n}}\psi_k=0$, we find

$$\mathbb{E}_\nu[|(II)_i|]\leqslant C(\beta)n^{\kappa\varepsilon}\frac{\mathbb{E}_\nu[|\nabla\psi_i^{\mathrm{dis}}|^2]^{\frac{1}{2}}}{d(i,i_0)^{2-s-\alpha}}\mathbb{E}_\nu\Big[\sum_{k:d(k,i)>\frac{1}{2}d(i,i_0)}d(k,i_0)^{2\gamma}|\nabla\psi_k|^2\Big]^{\frac{1}{2}}$$
$$\times\Big(\sum_{k:d(i,k)\geqslant\frac{1}{2}d(i,i_0)}\frac{1}{d(k,1)^{2\gamma}}\Big)^{\frac{1}{2}}\leqslant C(\beta)n^{\kappa\varepsilon}\frac{\mathbb{E}_\nu[|\nabla\psi_i^{\mathrm{dis}}|^2]^{\frac{1}{2}}}{d(i,i_0)^{\frac{3}{2}-s-\alpha+\gamma}}\mathbb{E}_\nu[|\mathrm{L}_\gamma D\psi|^2]^{\frac{1}{2}}.\quad(4.187)$$

For the first term, using Cauchy-Schwarz inequality one can first write

$$|(I)_i|\leqslant\frac{C}{d(i,i_0)}\Big(\sum_{k:d(i,k)\leqslant\frac{1}{2}d(i,i_0)}\frac{1}{d(i,k)}|\nabla\psi_k^{\mathrm{dis}}|^2\Big)^{\frac{1}{2}}\Big(\sum_{k:d(i,k)\leqslant\frac{1}{2}d(i,i_0)}\frac{1}{d(i,k)^{1-2s}}|\nabla\psi_k^{\mathrm{dis}}|^2\Big)^{\frac{1}{2}}.$$

Summing this over i yields

$$\sum_{i=1}^{\bar{n}}|(I)_i|\leqslant C\Big(\sum_{i=1}^{\bar{n}}\sum_{k:d(i,k)\leqslant\frac{1}{2}d(i,i_0)}\frac{1}{d(i,k)}d(k,i_0)^{2\gamma}|\nabla\psi_k|^2\Big)^{\frac{1}{2}}$$
$$\times\Big(\sum_{i=1}^{\bar{n}}\frac{1}{d(i,i_0)^{2-4(\alpha-\gamma)}}\sum_{k:d(i,k)\leqslant\frac{1}{2}d(i,i_0)}\frac{1}{d(i,k)^{1-2s}}d(k,i_0)^{2\gamma}|\nabla\psi_k|^2\Big)^{\frac{1}{2}}$$
$$\leqslant Cn^{\kappa\varepsilon}\Big(\sum_{i=1}^{\bar{n}}d(i,i_0)^{2\gamma}|\nabla\psi_i|^2\Big)^{\frac{1}{2}}\Big(\sum_{k=1}^{\bar{n}}d(k,i_0)^{2\gamma}|\nabla\psi_k|^2\frac{1}{d(k,i_0)^{2-2s-4(\alpha-\gamma)}}\Big)^{\frac{1}{2}}.\quad(4.188)$$

Combining (4.188) and (4.187), one can see that if $\alpha\leqslant\gamma+\frac{1-s}{2}$, then

$$\Big|\mathbb{E}_\nu\Big[\sum_{i,k}(\mathrm{L}_\alpha\mathsf{A}\mathrm{L}_\alpha^{-1}-\mathsf{A})_{i,k}\nabla\psi_i^{\mathrm{dis}}\cdot\nabla\psi_k^{\mathrm{dis}}\Big]\Big|\leqslant C(\beta)n^{\kappa\varepsilon}\mathbb{E}_\nu\Big[\sum_{i=1}^{n}d(i,i_0)^{2\gamma}|\nabla\psi_i|^2\Big].\quad(4.189)$$

Step 4: control on the commutator $\delta_{\mathrm{L}_\alpha}$ One should now control the commutator $\delta_{\mathrm{L}_\alpha}$ appearing in (4.186). Let us recall the decay estimate on $f*h$ stated in (4.184). By analyzing $\mathsf{AM}^{(2)}$, one can see that the off-diagonals entries of AM typically decay in

$$\mathbb{E}_\nu[(\mathsf{AM})_{i,j}^2]^{\frac{1}{2}}\leqslant\frac{Cn^{\kappa\varepsilon}}{d(i,j)^{2-s}}.$$

As a consequence one may apply Lemma 4.4.1 which tells us that for $\alpha\in(0,\frac{3}{2}-s]$,

$$\mathbb{E}_\nu[\psi^{\mathrm{dis}}\cdot\delta_{\mathrm{L}_\alpha}^{(1)}\psi^{\mathrm{dis}}]\leqslant\frac{n^{-\kappa_0\varepsilon}}{2}\mathbb{E}_\nu[|\psi^{\mathrm{dis}}|^2]+C(\beta)n^{\kappa\varepsilon}\mathbb{E}_\nu[|\psi^{\mathrm{dis}}|^2]^{\frac{1}{2}}\mathbb{E}_\nu[|\psi|^2]^{\frac{1}{2}}.$$

From the positivity of AM stated in (4.183) this gives

$$\mathbb{E}_\nu[\psi^{\mathrm{dis}}(\mathsf{AM}+\delta_{\mathrm{L}_\alpha})\psi^{\mathrm{dis}}]\geqslant\frac{n^{-\kappa_0\varepsilon}}{2}\mathbb{E}_\nu[|\psi^{\mathrm{dis}}|^2]-n^{\kappa\varepsilon}\mathbb{E}_\nu[|\psi^{\mathrm{dis}}|^2]^{\frac{1}{2}}\mathbb{E}_\nu[|\psi|^2]^{\frac{1}{2}}.\quad(4.190)$$

Step 5: conclusion Combining (4.186), (4.189) and (4.190) one gets that for $\alpha \in (0, \frac{3}{2} - s]$,

$$\mathbb{E}_\nu[|\mathrm{L}_\alpha \psi|^2]^{\frac{1}{2}} \leqslant C(\beta) n^{\kappa\varepsilon}\Big(\mathbb{E}_\nu[\chi_n^2]^{\frac{1}{2}} + n^{\alpha-\frac{1}{2}+s}\mathbb{E}_\nu[\lambda^2]^{\frac{1}{2}} + \mathbb{E}_\nu\Big[\sum_{i=1}^{\bar{n}} d(i,i_0)^{2(\alpha-\frac{1-s}{2})}|\nabla\psi_i|^2\Big]^{\frac{1}{2}}\Big). \quad (4.191)$$

In particular taking $\alpha = \frac{3}{2} - s$, one obtains

$$\mathbb{E}_\nu[|\mathrm{L}_{3/2-s}\psi|^2]^{\frac{1}{2}} \leqslant C(\beta) n^{\kappa\varepsilon}\Big(\mathbb{E}_\nu[\chi_n^2]^{\frac{1}{2}} + n\mathbb{E}_\nu[\lambda^2]^{\frac{1}{2}} + \mathbb{E}_\nu\Big[\sum_{i=1}^{\bar{n}} d(i,i_0)^{2(1-\frac{s}{2})}|\nabla\psi_i|^2\Big]^{\frac{1}{2}}\Big). \quad (4.192)$$

Furthermore applying the estimate (4.155) with $\gamma = 1 - \frac{s}{2}$, we recognize

$$\mathbb{E}_\nu\Big[\sum_{i=1}^{\bar{n}} d(i,i_0)^{2(1-\frac{s}{2})}|\nabla\psi_i|^2\Big] \leqslant Cn^{\kappa\varepsilon}\Big(n^{-\varepsilon_0}\mathbb{E}_\nu[|\mathrm{L}_{3/2-s}\psi|^2] + n^{\kappa\varepsilon_0}\mathbb{E}_\nu[|\mathrm{L}_{3/2-s}\psi|^2]^{\frac{1-s}{2}}\mathbb{E}_\nu[|\mathrm{L}_{1/2}\psi|^2]^{\frac{s}{2}}$$
$$+ n^{\kappa\varepsilon_0}\mathbb{E}_\nu[|\mathrm{L}_{3/2-s}|^2]^{\frac{1}{2}} n\mathbb{E}_\nu[\lambda^2]^{\frac{1}{2}} + \mathbb{E}_\nu[\chi_n^2]\Big). \quad (4.193)$$

Since $s \in (0,1)$, combining (4.192) and (4.193) one gets

$$\mathbb{E}_\nu[|\mathrm{L}_{3/2-s}\psi|^2]^{\frac{1}{2}} + \mathbb{E}_\nu[|\mathrm{L}_{1-s/2}D\psi|^2]^{\frac{1}{2}} \leqslant C(\beta)n^{\kappa\varepsilon}(n^{\kappa\varepsilon_0}\mathbb{E}_\nu[\chi_n^2]^{\frac{1}{2}} + n^{-\varepsilon_0}\mathbb{E}_\nu[|\mathrm{L}_{3/2-s}\psi|^2]^{\frac{1}{2}} + n\mathbb{E}_\nu[\lambda^2]^{\frac{1}{2}}).$$

Taking $\varepsilon_0 > 0$ large enough with respect to ε, one obtains the existence of a constant $\kappa > 0$ such that

$$\mathbb{E}_\nu[|\mathrm{L}_{3/2-s}\psi|^2]^{\frac{1}{2}} + \mathbb{E}_\nu[|\mathrm{L}_{1-s/2}D\psi|^2]^{\frac{1}{2}} \leqslant C(\beta)n^{\kappa\varepsilon}(\mathbb{E}_\nu[\chi_n^2]^{\frac{1}{2}} + n\mathbb{E}_\nu[\lambda^2]^{\frac{1}{2}}). \quad (4.194)$$

Using the expression (4.50), one can also see that

$$\mathbb{E}_\nu[\lambda^2]^{\frac{1}{2}} \leqslant C(\beta)n^{\kappa\varepsilon-1}\mathbb{E}_\nu[|\mathrm{L}_{1/2}\psi|^2]^{\frac{1}{2}}. \quad (4.195)$$

Since $\frac{3}{2} - s > \frac{1}{2}$, one gets from (4.194) and (4.195) the estimates (4.176) and (4.177). □

One shall extend the global decay estimate of Lemma 4.5.6 to the H.-S. equation without linear constraint.

Lemma 4.5.7. *Let $s \in (0,1)$. Let ν and* M *satisfying Assumptions 4.5.1 and 4.5.2. Let $\chi_n \in H^1(\nu)$, $i_0 \in \{1, \ldots, n\}$ and $\psi \in L^2(\bar{I}, H^1(\nu))$ be the solution of*

$$\begin{cases} \beta \mathrm{M}\psi + \mathcal{L}^\nu \psi = \chi_n e_{i_0} & \text{on } \pi(\mathcal{M}_N) \\ \psi \cdot \vec{n} = 0 & \text{on } \partial\pi(\mathcal{M}_N). \end{cases} \quad (4.196)$$

There exist a constant $C(\beta)$ locally uniform in β and $\kappa > 0$ such that

$$\mathbb{E}_\nu\Big[\sum_{i=1}^{\bar{n}} d(i,i_0)^{2-s}|\nabla\psi_i|^2\Big]^{\frac{1}{2}} + \mathbb{E}_\nu\Big[\sum_{i=1}^{\bar{n}} d(i,i_0)^{3-2s}\psi_i^2\Big]^{\frac{1}{2}} \leqslant C(\beta)n^{\kappa\varepsilon}\mathbb{E}_\nu[\chi_n^2]^{\frac{1}{2}}.$$

Proof. Let $\psi \in L^2(\bar{I}, H^1(\nu))$ be the solution of (4.196). One can decompose ψ into $\psi = v + w$ where $v, w \in L^2(\bar{I}, H^1(\nu))$ solve

$$\begin{cases} \beta \mathrm{M}v + \mathcal{L}^\nu v = \chi_n e_{i_0} + \lambda(e_1 + \ldots + e_{\bar{n}}) & \text{on } \pi(\mathcal{M}_N) \\ v \cdot (e_1 + \ldots + e_{\bar{n}}) = 0 & \text{on } \pi(\mathcal{M}_N) \\ v \cdot \vec{n} = 0 & \text{on } \partial\pi(\mathcal{M}_N), \end{cases} \quad (4.197)$$

$$\begin{cases} \beta \mathsf{M} w + \mathcal{L}^\nu w = \lambda(e_1 + \ldots + e_{\bar{n}}) & \text{on } \pi(\mathcal{M}_N) \\ w \cdot \vec{n} = 0 & \text{on } \partial\pi(\mathcal{M}_N). \end{cases} \tag{4.198}$$

For the vector-field v, one may apply Lemma 4.5.6 which gives

$$\mathbb{E}_\nu\Big[\sum_{i=1}^{\bar{n}} d(i,i_0)^{2-s}|\nabla v_i|^2\Big]^{\frac{1}{2}} + \mathbb{E}_\nu\Big[\sum_{i=1}^{\bar{n}} d(i,i_0)^{3-2s} v_i^2\Big]^{\frac{1}{2}} \leqslant C(\beta) n^{\kappa\varepsilon} \mathbb{E}_\nu[\chi_n^2]^{\frac{1}{2}} \tag{4.199}$$

as well as

$$\mathbb{E}_\nu[\lambda^2]^{\frac{1}{2}} \leqslant \frac{C(\beta)}{n^{1-\kappa\varepsilon}} \mathbb{E}_\nu[\chi_n^2]^{\frac{1}{2}}. \tag{4.200}$$

It remains to address Equation (4.198). One can write a mean-field approximation for (4.198) in the form $f(e_1 + \ldots + e_{\bar{n}})$ where $f \in H^1(\nu)$ is the solution of

$$\beta f + \frac{1}{\bar{n}^{1-s}} \mathcal{L}^\nu f = \lambda. \tag{4.201}$$

By integration by parts this implies together with the control (4.200) that

$$\mathbb{E}_\nu[f^2]^{\frac{1}{2}} \leqslant \frac{C(\beta)}{n^{2-s-\kappa\varepsilon}} \mathbb{E}_\nu[\chi_n^2]^{\frac{1}{2}} \tag{4.202}$$

and

$$\mathbb{E}_\nu[|\nabla f|^2]^{\frac{1}{2}} \leqslant \frac{C(\beta)}{n^{\frac{3}{2}-s-\kappa\varepsilon}} \mathbb{E}_\nu[\chi_n^2]^{\frac{1}{2}}. \tag{4.203}$$

Define $w^{(1)} = f \times (e_1 + \ldots + e_{\bar{n}})$ and $w^{(2)} = w - w^{(1)}$ which is solution of

$$\begin{cases} \beta \mathsf{M} w^{(2)} + \mathcal{L}^\nu w^{(2)} = -\beta \mathsf{M}^{(2)} w^{(1)} & \text{on } \pi(\mathcal{M}_N) \\ w \cdot \vec{n} = 0 & \text{on } \partial\pi(\mathcal{M}_N). \end{cases}$$

By (4.202), there holds

$$\mathbb{E}_\nu[|\mathsf{M}^{(2)} w^{(1)}|^2]^{\frac{1}{2}} \leqslant \frac{C(\beta)}{n^{\frac{3}{2}-s-\kappa\varepsilon}} \mathbb{E}_\nu[\chi_n^2]^{\frac{1}{2}}.$$

In particular

$$\mathbb{E}_\nu[|w|^2]^{\frac{1}{2}} \leqslant \frac{C(\beta)}{n^{\frac{3}{2}-s-\kappa\varepsilon}} \mathbb{E}_\nu[\chi_n^2]^{\frac{1}{2}} \tag{4.204}$$

and similarly

$$\mathbb{E}_\nu[|\nabla w|]^{\frac{1}{2}} \leqslant \frac{C(\beta)}{n^{1-\frac{s}{2}-\kappa\varepsilon}} \mathbb{E}_\nu[\chi_n^2]^{\frac{1}{2}}. \tag{4.205}$$

It follows from (4.204) and (4.205) that w satisfies the estimate (4.199) and so does ψ. □

4.5.5 Localization and optimal decay

Let us now adapt the localization argument of Subsection 4.4.4 to derive the near-optimal decay of the solution of (4.175). Having proved Lemma 4.5.6, it remains to control the decay of ψ_j for a single $j \in \bar{I}$. To this end, we project the periodized equation (4.175) into a small window centered around j. After isolating an exterior field, one can see that the projected equation has a similar structure as the equation one is starting from. By splitting the external field in a suitable manner, one can then decompose the solution into two parts, that we control separately.

Proposition 4.5.8. *Let $s \in (0,1)$. Let ν and* M *satisfying Assumptions 4.5.1 and 4.5.2. Let $\chi_n \in H^1(\nu)$, $i_0 \in \{1,\ldots,n\}$ and $\psi \in L^2(\bar{I}, H^1(\nu))$ be the solution of*

$$\begin{cases} \beta \mathsf{M}\psi + \mathcal{L}^\nu \psi = \chi_n e_{i_0} + \lambda(e_1 + \ldots + e_{\bar{n}}) & \text{on } \pi(\mathcal{M}_N) \\ \psi \cdot (e_1 + \ldots + e_{\bar{n}}) = 0 & \text{on } \pi(\mathcal{M}_N) \\ \psi \cdot \vec{n} = 0 & \text{on } \partial\pi(\mathcal{M}_N). \end{cases} \tag{4.206}$$

There exist $C(\beta)$ locally uniform in β and $\kappa > 0$ such that for each $1 \leqslant i \leqslant n$,

$$\mathbb{E}_\nu[\psi_i^2]^{\frac{1}{2}} \leqslant \frac{C(\beta) n^{\kappa\varepsilon}}{1 + d(i,i_0)^{2-s}} \mathbb{E}_\nu[\chi_n^2]^{\frac{1}{2}}, \tag{4.207}$$

$$\mathbb{E}_\nu[|\nabla\psi_i|^2]^{\frac{1}{2}} \leqslant \frac{C(\beta) n^{\kappa\varepsilon}}{1 + d(i,i_0)^{\frac{3}{2}-\frac{s}{2}}} \mathbb{E}_\nu[\chi_n^2]^{\frac{1}{2}}. \tag{4.208}$$

Proof. We proceed by bootstrapping the decay exponent on solutions of (4.206) and (4.196) for *all* M satisfying Assumptions 4.5.2. Assume that there exist $\alpha \geqslant \frac{3}{2} - s$ and $\gamma \geqslant 1 - \frac{s}{2}$ with $\gamma \leqslant \alpha$ such that for M satisfying Assumptions 4.5.2 and all $\chi_n \in H^{-1}(\nu)$, $i_0 \in \{1,\ldots,n\}$, if $\psi \in L^2(\bar{I}, H^1(\nu))$ solves (4.206) or (4.196), then there exists $C(\beta)$ and $\kappa > 0$ such that for each $1 \leqslant j \leqslant n$,

$$\mathbb{E}_\nu[\psi_j^2]^{\frac{1}{2}} \leqslant \frac{C(\beta) n^{\kappa\varepsilon}}{d(j,i_0)^{\alpha}} \mathbb{E}_\nu[\chi_n^2]^{\frac{1}{2}}, \tag{4.209}$$

$$\mathbb{E}_\nu[|\nabla\psi_j|^2]^{\frac{1}{2}} \leqslant \frac{C(\beta) n^{\kappa\varepsilon}}{d(j,i_0)^{\gamma}} \mathbb{E}_\nu[\chi_n^2]^{\frac{1}{2}}. \tag{4.210}$$

In addition to (4.209) and (4.210), we will also make a systematic use of the global estimates of Lemma 4.5.6 and Lemma 4.5.7.

Step 1: projection and embedding Let $\chi_n \in H^1(\nu)$, $i_0 \in \{1,\ldots,n\}$ and $\psi \in L^2(\bar{I}, H^1(\nu))$ be the solution of (4.206). Fix an index $j \in \{1,\ldots,\bar{n}\}$ and define the window

$$J := \{i \in \{1,\ldots,\bar{n}\} : d(i,j) \leqslant d(i_0,j)/2\}. \tag{4.211}$$

Let $n_0 = |J|$. Let $\psi^J := (\psi_i)_{i\in J} \in L^2(J, H^1(\nu))$. Projecting (4.206) onto (4.211) reads

$$\begin{cases} \beta \mathsf{M}^J \psi^J + \mathcal{L}^\nu \psi^0 = -\beta \Big(\sum_{l \in J^c} \mathsf{M}_{i,l}\psi_l \Big)_{i\in J} & \text{on } \pi(\mathcal{M}_N) \\ \psi^0 \cdot \vec{n} = 0 & \text{on } \partial\pi(\mathcal{M}_N). \end{cases} \tag{4.212}$$

Let us operate the series of reductions of Subsection 4.5.1 to reduce the study to a periodic system of size $\bar{n}_0 = 2n_0$. One may assume that $d(j,i_0) \geqslant n^{\kappa\varepsilon}$ for some large $\kappa > 0$, otherwise the statements (4.207) and (4.208) are straightforward. Let us denote $\bar{J} = \{1,\ldots,\bar{n}_0\}$. We now let d stand for the symmetric distance on $\bar{J}$. Consider the Riesz matrix on $\bar{J}$ truncated at $K_0 = \lfloor n^{\kappa\varepsilon} \rfloor$ chosen as in (4.117), namely $\mathsf{M}_0 = \nabla^2 F(x) \in \mathcal{M}_{\bar{n}_0}$ for some $x \in \mathbb{R}^{\bar{n}_0}$ where

$$F : X_{\bar{n}_0} \in \mathbb{R}^{\bar{n}_0} \mapsto \sum_{i,j \in \bar{J} : d(i,j) \geqslant K_0} g_s''(d(i,j))(x_i + \ldots + x_j)^2.$$

Consider the block decomposition of M on $\mathbb{R}^{n_0} \times \mathbb{R}^{\bar{n}_0 - n_0}$,

$$\mathrm{M} = \begin{pmatrix} A_0 & B_0 \\ C_0 & D_0 \end{pmatrix}, \quad A_0 \in \mathcal{M}_{n_0}(\mathbb{R}). \tag{4.213}$$

Let us add and subtract to the first line of (4.212) the quantity $B_0(D_0+\beta^{-1}\mathcal{L}^\nu \otimes I_{n_0})C_0$. Defining

$$\mathsf{M}_0=\begin{pmatrix}\mathsf{M}^J & B_0\\ C_0 & D_0\end{pmatrix},$$

with B_0, C_0 and D_0 as in (4.213), this allows one to identify ψ_j^J with ψ_j^0 for each $j\in\{1,\ldots,n_0\}$, where $\psi^0\in L^2(\bar{J},H^1(\nu))$ solves

$$\begin{cases}\beta\mathsf{M}_0\psi^0+\mathcal{L}^\nu\psi^0=\mathrm{V} & \text{on } \pi(\mathcal{M}_N)\\ \psi^0\cdot\vec{n}=0 & \text{on } \partial\pi(\mathcal{M}_N).\end{cases}$$

Moreover, the external field $\mathrm{V}\in L^2(\bar{J},H^1(\nu))$ satisfies $\mathrm{V}_l=0$ if $l\in\{n_0+1,\ldots,\bar{n}_0\}$ and for each $l\in\{1,\ldots,n_0\}$,

$$\mathrm{V}_l=-\beta\sum_{i\in J^c}\mathsf{M}_{i,l}\psi_i-\sum_{i\in J}e_l\cdot B_0(\beta D_0+\mathcal{L}^\nu\otimes I_{\bar{n}_0-n_0})^{-1}(C_0e_i\psi_i)+\lambda.$$

Note that M_0 satisfies Assumptions 4.5.2.

Step 2: splitting of the exterior potential Fix $\varepsilon'>0$ and partition $\bar{J}$ into $K:=\lfloor d(j,i_0)^{\varepsilon'}\rfloor$ intervals $I_1,\ldots,I_K$ of equal size, up to a $O(d(j,i_0)^{1-\varepsilon'})$ for the last one. For each $k\in\{1,\ldots,K\}$, let i_k be an index in the center of I_k. One can split the external potential into $\mathrm{V}=\mathrm{V}^{(1)}+\mathrm{V}^{(2)}$, where

$$\mathrm{V}_l^{(2)}=\mathrm{V}_{i_k}\quad \text{if } l\in I_k.$$

Note that $\mathrm{V}^{(2)}$ is piecewise constant on the partition $\bar{J}=\cup_{k=1}^K I_k$. By linearity, ψ^0 can be decomposed into $\psi^0=v+w$ with $v,w\in L^2(\bar{J},H^1(\nu))$ solving

$$\begin{cases}\beta\mathsf{M}_0v+\mathcal{L}^\nu v=\sum_{l\in J}\mathrm{V}_l^{(1)}e_l & \text{on } \pi(\mathcal{M}_N)\\ v\cdot\vec{n}=0 & \text{on } \partial\pi(\mathcal{M}_N),\end{cases}\tag{4.214}$$

$$\begin{cases}\beta\mathsf{M}_0w+\mathcal{L}^\nu w=\sum_{l\in J}\mathrm{V}_l^{(2)}e_l & \text{on } \pi(\mathcal{M}_N)\\ w\cdot\vec{n}=0 & \text{on } \partial\pi(\mathcal{M}_N).\end{cases}\tag{4.215}$$

Step 3: study of v By using Cauchy-Schwarz inequality, Equation (4.135), the fact that $\sum_{k=1}^{\bar{n}}\psi_k=0$, the estimates (4.176) and (4.177) and Lemma 4.5.2, one may check that for each $l\in J$,

$$\mathbb{E}_\nu[(\mathrm{V}^{(1)})_l^2]^{\frac12}\leqslant C(\beta)n^{\kappa\varepsilon}\frac{d(j,l)^{1-\varepsilon'}}{d(j,i_0)^{\frac32-s}}\frac{1}{d(l,\partial J)^{\frac12+s}}\mathbb{E}_\nu[\chi_n^2]^{\frac12}.$$

Note that we have not made use of the bootstrap assumption for this last estimate but rather of the global estimate (4.176). Let us decompose v into $v=\sum_{l\in J}v^{(l)}$ where for each $l\in J$, $v^{(l)}\in L^2(\bar{J},H^1(\nu))$ solves

$$\begin{cases}\beta\mathsf{M}_0v^{(l)}+\mathcal{L}^\nu v^{(l)}=\mathrm{V}_l^{(1)}e_l & \text{on } \pi(\mathcal{M}_N)\\ v^{(l)}\cdot\vec{n}=0 & \text{on } \partial\pi(\mathcal{M}_N),\end{cases}\tag{4.216}$$

By applying the bootstrap assumption (4.209) in the window $\bar{J}$, one can see that for each $l\in J$ and $j\in\bar{J}$,

$$\mathbb{E}_\nu[(v_j^{(l)})^2]^{\frac12}\leqslant C(\beta)n^{\kappa\varepsilon}\frac{d(j,l)^{1-\varepsilon'-\alpha}}{d(j,i_0)^{\frac32-s}}\frac{1}{d(l,\partial J)^{\frac12+s}}\mathbb{E}_\nu[\chi_n^2]^{\frac12}.$$

Summing this over $l \in J$ yields

$$\mathbb{E}_\nu[v_j^2]^{\frac{1}{2}} \leqslant \frac{C(\beta)n^{\kappa\varepsilon}}{d(j,i_0)^{\alpha+\varepsilon'}}\mathbb{E}_\nu[\chi_n^2]^{\frac{1}{2}}. \tag{4.217}$$

In a similar manner, using the induction hypothesis (4.210), one also obtains

$$\mathbb{E}_\nu[|\nabla v_j|^2]^{\frac{1}{2}} \leqslant \frac{C(\beta)n^{\kappa\varepsilon}}{d(j,i_0)^{\gamma+\varepsilon'}}\mathbb{E}_\nu[\chi_n^2]^{\frac{1}{2}}. \tag{4.218}$$

Step 4: study of w It remains to study the solution w associated to the piecewise constant vector-field $\mathrm{V}^{(2)}$. The argument is inspired from the mean-field approximation of the linear response associated to a linear statistics, see for instance Chapter 3. We will construct an approximation of w by replacing M_0 by the constant Riesz matrix on the window $\bar{J}$. For each $k \in \{1,\ldots,K\}$, we let $w^{(k)} \in L^2(\bar{J}, H^1(\nu))$ be the solution of

$$\begin{cases} \beta \mathsf{M}_0 w^{(k)} + \mathcal{L}^\nu w^{(k)} = \mathrm{V}_{i_k}^{(2)} \sum_{l\in I_k} e_l & \text{on } \pi(\mathcal{M}_N) \\ w^{(k)} \cdot \vec{n} = 0 & \text{on } \partial\pi(\mathcal{M}_N). \end{cases}$$

Let $\phi^{(k)} \in L^2(\bar{J}, H^1(\nu))$ be the solution of

$$\beta g_s * \phi^{(k)} + \mathcal{L}^\nu \phi^{(k)} = \mathrm{V}_{i_k}^{(2)} \sum_{l\in I_k} e_l. \tag{4.219}$$

We let $\mathsf{M}_0^{(2)}$ be the difference between M_0 and the Toeplitz matrix associated to g_s. Let also $\eta^{(k)} \in L^2(\bar{J}, H^1(\nu))$ defined by $\eta_i^{(k)} = \phi_{i+1}^{(k)} - \phi_i^{(k)}$ for each $i \in \bar{J}$. One shall observe that

$$\beta g_s * \eta^{(k)} + \mathcal{L}^\nu \eta^{(k)} = \mathrm{V}_{i_k}^{(2)}(e_{i_{k+1}} - e_{i_k}).$$

Using the bootstrap assumption we find that for each $i \in \bar{J}$,

$$\mathbb{E}_\nu[(\eta_i^{(k)})^2]^{\frac{1}{2}} \leqslant \frac{C(\beta)n^{\kappa\varepsilon}}{d(j,i_0)^{\alpha}}\mathbb{E}_\nu[(\mathrm{V}_{i_k}^{(2)})]^{\frac{1}{2}},$$

$$\mathbb{E}_\nu[|\nabla \eta_i^{(k)}|^2]^{\frac{1}{2}} \leqslant \frac{C(\beta)n^{\kappa\varepsilon}}{d(j,i_0)^{\gamma}}\mathbb{E}_\nu[(\mathrm{V}_{i_k}^{(2)})]^{\frac{1}{2}}.$$

In view of Lemma 4.5.6, we also have

$$\mathbb{E}_\nu[(\mathrm{V}_{i_k}^{(2)})]^{\frac{1}{2}} \leqslant \frac{C(\beta)n^{\kappa\varepsilon}}{d(j,i_0)}\mathbb{E}_\nu[\chi_n^2]^{\frac{1}{2}}.$$

It thus follows that

$$\mathbb{E}_\nu[(\eta_i^{(k)})^2]^{\frac{1}{2}} \leqslant \frac{C(\beta)n^{\kappa\varepsilon}}{d(j,i_0)^{\alpha+1}}\mathbb{E}_\nu[\chi_n^2]^{\frac{1}{2}}, \tag{4.220}$$

$$\mathbb{E}_\nu[|\nabla \eta_i^{(k)}|^2]^{\frac{1}{2}} \leqslant \frac{C(\beta)n^{\kappa\varepsilon}}{d(j,i_0)^{\gamma+1}}\mathbb{E}_\nu[\chi_n^2]^{\frac{1}{2}}.$$

Besides, from the global estimate of Lemma 4.5.6, letting $S = g_s * \phi^{(k)}$, we have

$$\mathbb{E}_\nu[S_i^2]^{\frac{1}{2}} \leqslant \frac{C(\beta)n^{\kappa\varepsilon}}{d(j,i_0)}\mathbb{E}_\nu[\chi_n^2]^{\frac{1}{2}}. \tag{4.221}$$

Let $\varepsilon_0 \in (0,1)$ be a small number. One may then write $\phi_j^{(k)}$ as

$$\phi_j^{(k)} = \sum_{l \in \bar{J}} g_s^{-1}(d(j,l)) S_l = \underbrace{\sum_{l \in \bar{J}: d(j,l) > d(j,i_0)^{1-\varepsilon_0}} g_s^{-1}(d(j,l)) S_l}_{(I)_j} + \underbrace{\sum_{l \in \bar{J}: d(j,l) \leqslant d(j,i_0)^{1-\varepsilon_0}} g_s^{-1}(d(j,l)) S_l}_{(II)_j}.$$

For the first term using (4.221) we find

$$\mathbb{E}_\nu[(I)_j^2]^{\frac{1}{2}} \leqslant \frac{C(\beta) n^{\kappa(\varepsilon+\varepsilon_0)}}{d(j,i_0)^{2-s}} \mathbb{E}_\nu[\chi_n^2]^{\frac{1}{2}}.$$

One may then split the second term into

$$(II)_j = \underbrace{\sum_{l \in \bar{J}: d(j,l) \leqslant d(j,i_0)^{1-\varepsilon_0}} g_s^{-1}(d(j,l))(S_j - S_l)}_{(II)_j'} + \underbrace{\sum_{l \in \bar{J}: d(j,l) > d(j,i_0)^{1-\varepsilon_0}} g_s^{-1}(d(j,l)) S_j}_{(II)_j''}. \tag{4.222}$$

In view of (4.221), $(II)_j''$ is bounded by

$$\mathbb{E}_\nu[((II)_j'')^2]^{\frac{1}{2}} \leqslant \frac{C(\beta) n^{\kappa(\varepsilon+\varepsilon_0)}}{d(j,i_0)^{2-s}} \mathbb{E}_\nu[\chi_n^2]^{\frac{1}{2}}.$$

For $(II)_j'$ we can note that

$$S_l - S_j = \sum_{i \in \bar{J}} \phi_i^{(k)} \Big(\frac{1}{d(l,i)^s} - \frac{1}{d(j,i)^s} \Big) = \sum_{i \in \bar{J}} (\phi_i^{(k)} - \phi_j^{(k)}) \Big(\frac{1}{d(l,i)^s} - \frac{1}{d(j,i)^s} \Big).$$

At this point one may use the bound on the increments of $\phi^{(k)}$ stated in (4.220), which gives

$$\mathbb{E}_\nu[|S_l - S_j|^2]^{\frac{1}{2}} \leqslant C(\beta) n^{\kappa\varepsilon} d(l,j) d(j,i_0)^{1-s} \frac{1}{d(j,1)^{\alpha+1}} \mathbb{E}_\nu[\chi_n^2]^{\frac{1}{2}}.$$

Plugging this into (4.222) leads to

$$\mathbb{E}_\nu[(\phi_j^{(k)})^2]^{\frac{1}{2}} \leqslant C(\beta) n^{\kappa\varepsilon} \Big(\frac{n^{-\varepsilon_0}}{d(j,i_0)^\alpha} + \frac{n^{\kappa\varepsilon_0}}{d(j,i_0)^{2-s}} \Big) \mathbb{E}_\nu[\chi_n^2]^{\frac{1}{2}}. \tag{4.223}$$

A similar computation shows that

$$\mathbb{E}_\nu[|\nabla \phi_j^{(k)}|^2]^{\frac{1}{2}} \leqslant C(\beta) n^{\kappa\varepsilon} \Big(\frac{n^{-\varepsilon_0}}{d(j,i_0)^\gamma} + \frac{n^{\kappa\varepsilon_0}}{d(j,i_0)^{\frac{3}{2}-\frac{s}{2}}} \Big) \mathbb{E}_\nu[\chi_n^2]^{\frac{1}{2}}. \tag{4.224}$$

Let us emphasize that $\phi^{(k)}$ differs from $w^{(k)}$.

Step 5: conclusion for $\mathsf{M}_0^{(2)} = 0$ Assume that $\mathsf{M}_0^{(2)} = 0$. Then $\phi^{(k)} = w^{(k)}$ and one may infer from (4.217) and (4.223) that there exists a small $\eta > 0$ such that

$$\mathbb{E}_\nu[\psi_j^2]^{\frac{1}{2}} \leqslant C(\beta) n^{\kappa\varepsilon} \Big(\frac{1}{d(j,i_0)^{\alpha+\eta}} + \frac{1}{d(j,i_0)^{2-s}} \Big) \mathbb{E}_\nu[\chi_n^2]^{\frac{1}{2}},$$

$$\mathbb{E}_\nu[|\nabla \psi_j|^2]^{\frac{1}{2}} \leqslant C(\beta) n^{\kappa\varepsilon} \Big(\frac{1}{d(j,i_0)^{\gamma+\eta}} + \frac{1}{d(j,i_0)^{\frac{3}{2}-\frac{s}{2}}} \Big) \mathbb{E}_\nu[\chi_n^2]^{\frac{1}{2}}.$$

One concludes after a finite number of steps that

$$\mathbb{E}_\nu[\psi_j^2]^{\frac{1}{2}} \leqslant \frac{C(\beta) n^{\kappa\varepsilon}}{d(j,i_0)^{2-s}} \mathbb{E}_\nu[\chi_n^2]^{\frac{1}{2}}, \tag{4.225}$$

$$\mathbb{E}_\nu[|\nabla \psi_j|^2]^{\frac{1}{2}} \leqslant \frac{C(\beta) n^{\kappa\varepsilon}}{d(j,i_0)^{\frac{3}{2}-\frac{s}{2}}} \mathbb{E}_\nu[\chi_n^2]^{\frac{1}{2}}. \tag{4.226}$$

Step 6: control of w in the general case We go back to the general case. Let us define $e^{(k)}=w^{(k)}-\phi^{(k)}$ where $\phi^{(k)}$ is as in (4.219). Note that $e^{(k)}$ solves

$$\beta \mathsf{M}_0 e^{(k)}+\mathcal{L}^\nu e^{(k)}=-\beta \mathsf{M}_0^{(2)}\phi^{(k)}.$$

According to the estimates (4.225) and (4.226) of Step 6, the vector-field $\mathsf{M}_0^{(2)}\phi^{(k)}$ satisfies for each $1\leqslant i\leqslant n$,

$$\mathbb{E}_\nu[((\mathsf{M}_0^{(2)}\phi)_i^{(k)})^2]^{\frac{1}{2}}\leqslant \frac{C(\beta)n^{\kappa\varepsilon}}{d(j,i_0)^{2-s}}\mathbb{E}_\nu[\chi_n^2]^{\frac{1}{2}},$$

$$\mathbb{E}_\nu[|\nabla(\mathsf{M}_0^{(2)}\phi)_i^{(k)}|^2]^{\frac{1}{2}}\leqslant \frac{C(\beta)n^{\kappa\varepsilon}}{d(j,i_0)^{\frac{3}{2}-\frac{s}{2}}}\mathbb{E}_\nu[\chi_n^2]^{\frac{1}{2}}.$$

It follows from the bootstrap assumptions (4.209) and (4.210) that for each $1\leqslant i\leqslant n$,

$$\mathbb{E}_\nu[(e_i^{(k)})^2]^{\frac{1}{2}}\leqslant C(\beta)n^{\kappa\varepsilon}\Big(\frac{1}{d(j,i_0)^{2-s}}+\frac{1}{d(j,i_0)^{\alpha+1-s}}\Big)\mathbb{E}_\nu[\chi_n^2]^{\frac{1}{2}},$$

$$\mathbb{E}_\nu[|\nabla e_i^{(k)}|^2]^{\frac{1}{2}}\leqslant C(\beta)n^{\kappa\varepsilon}\Big(\frac{1}{d(j,i_0)^{\frac{s}{2}-\frac{s}{2}}}+\frac{1}{d(j,i_0)^{\gamma+1-s}}\Big)\mathbb{E}_\nu[\chi_n^2]^{\frac{1}{2}}.$$

Consequently the same estimate holds for $w^{(k)}$. Summing this over k yields this existence of a constant $\kappa>0$ such that

$$\mathbb{E}_\nu[w_j^2]^{\frac{1}{2}}\leqslant C(\beta)n^{\kappa(\varepsilon+\varepsilon')}\Big(\frac{1}{d(j,i_0)^{2-s}}+\frac{1}{d(j,i_0)^{\alpha+1-s}}\Big)\mathbb{E}_\nu[\chi_n^2]^{\frac{1}{2}},$$

$$\mathbb{E}_\nu[|\nabla w_j|^2]^{\frac{1}{2}}\leqslant C(\beta)n^{\kappa(\varepsilon+\varepsilon')}\Big(\frac{1}{d(j,i_0)^{\frac{3}{2}-\frac{s}{2}}}+\frac{1}{d(j,i_0)^{\gamma+1-s}}\Big)\mathbb{E}_\nu[\chi_n^2]^{\frac{1}{2}}.$$

Combined with (4.217) and (4.218), this improves the induction hypotheses (4.209) and (4.210) provided $\varepsilon'>0$ is chosen small enough. After a finite number of iterations, one finally gets (4.207) and (4.208).

Step 7: conclusion for equation (4.196) In view of the bootstrap assumption, it remains to consider the solution ψ of (4.196). Let us split ψ as in the proof of Lemma 4.5.7 into $\psi=v+w$ where $v,w\in L^2(\bar{I},H^1(\nu))$ are solutions of (4.197) and (4.198). By applying the result of Step 6 to v, one can see that there exists a positive $\eta>0$ such that for each $i\in\{1,\ldots,\bar{n}\}$,

$$\mathbb{E}_\nu[v_i^2]^{\frac{1}{2}}\leqslant C(\beta)n^{\kappa\varepsilon}\Big(\frac{1}{d(i,i_0)^{\alpha+\eta}}+\frac{1}{d(i,i_0)^{2-s}}\Big)\mathbb{E}_\nu[\chi_n^2]^{\frac{1}{2}}, \tag{4.227}$$

$$\mathbb{E}_\nu[|\nabla v_i|^2]^{\frac{1}{2}}\leqslant C(\beta)n^{\kappa\varepsilon}\Big(\frac{1}{d(i,i_0)^{\gamma+\eta}}+\frac{1}{d(i,i_0)^{\frac{3}{2}-\frac{s}{2}}}\Big)\mathbb{E}_\nu[\chi_n^2]^{\frac{1}{2}}. \tag{4.228}$$

As in the proof of Lemma 4.5.7 one shall split w into $w=w^{(1)}+w^{(2)}$ with

$$w^{(1)}=f\times(e_1+\ldots+e_{\bar{n}}),$$

where f is given by (4.201). Let $\mathsf{M}^{(2)}$ be the difference between M and the Toeplitz matrix associated to g_s. Observe that $w^{(2)}$ solves

$$\begin{cases}\beta \mathsf{M}w^{(2)}+\mathcal{L}^\nu w^{(2)}=-\beta\mathsf{M}^{(2)}w^{(1)} & \text{on } \pi(\mathcal{M}_N)\\ w\cdot\vec{n}=0 & \text{on } \partial\pi(\mathcal{M}_N).\end{cases}$$

Using (4.202) we find that for each $i \in \{1, \ldots, \bar{n}\}$,

$$\mathbb{E}_\nu[(\mathsf{M}^{(2)} w^{(1)})_i^2]^{\frac{1}{2}} \leqslant \frac{C(\beta) n^{\kappa\varepsilon}}{d(j, i_0)^{2-s}} \mathbb{E}_\nu[\chi_n^2]^{\frac{1}{2}}.$$

By applying the bootstrap assumption to upper bound $w^{(2)}$, we find that for each $i \in \{1, \ldots, \bar{n}\}$,

$$\mathbb{E}_\nu[(w_i^{(2)})^2]^{\frac{1}{2}} \leqslant C(\beta) n^{\kappa\varepsilon} \Big(\frac{1}{d(j, i_0)^{2-s}} + \frac{1}{d(j, i_0)^{\alpha+1-s}} \Big) \mathbb{E}_\nu[\chi_n^2]^{\frac{1}{2}}.$$

Similarly, applying (4.203), one gets

$$\mathbb{E}_\nu[|\nabla w_i^{(1)}|^2]^{\frac{1}{2}} \leqslant C(\beta) n^{\kappa\varepsilon} \Big(\frac{1}{d(j, i_0)^{\frac{3}{2}-\frac{s}{2}}} + \frac{1}{d(j, i_0)^{\gamma+1-s}} \Big) \mathbb{E}_\nu[\chi_n^2]^{\frac{1}{2}}.$$

Combining the two last displays with (4.227) and (4.228) improves the recursion hypothesis when ψ is solution of (4.196). □

Remark 18. *Even though the Lagrange multiplier in (4.206) is of order $1/n$, there is no correction of order $1/n$ in (4.207), contrarily to the case $s > 1$. This is related to the fact that $u := \mathbb{H}_s^{-1}(e_1 + \ldots + e_n)$ satisfies $u_i \sim c/n^{1-s}$ for each $1 \leqslant i \leqslant n$. Note that in the above proof, the Lagrange multiplier is contained in $\mathrm{V}^{(2)}$ and the smallness of the associated solution shown in (4.223).*

4.5.6 Decay estimate for solutions of (4.116)

In the case $n \leqslant N/2$, one shall now deduce from Proposition 4.5.8 a control on the solution of (4.116).

Proposition 4.5.9. *Let $s \in (0, 1)$. Let $i_0 \in \{1, \ldots, n\}$, $\chi_n \in H^1(\nu)$ and $\psi \in L^2(I, H^1(\nu))$ solution of*

$$\begin{cases} A_1^\nu \psi = \chi_n e_{i_0} & \text{on } \pi(\mathcal{M}_N) \\ \psi \cdot \vec{n} = 0 & \text{on } \partial\pi(\mathcal{M}_N)), \end{cases} \tag{4.229}$$

Assume that $|i_0 - n/2| \leqslant n/4$. There exist constants $C(\beta) > 0, c(\beta) > 0, \delta > 0$ and $\kappa > 0$ such that for each $j \in \{1, \ldots, n\}$,

$$\mathbb{E}_\nu[\psi_j^2]^{\frac{1}{2}} \leqslant C(\beta) n^{\kappa\varepsilon} \Big(\frac{1}{d(j, i_0)^{2-s}} + \frac{1}{\sqrt{n}} \Big) (\mathbb{E}_\nu[\chi_n^2]^{\frac{1}{2}} + \sup |\chi_n| e^{-c(\beta) n^\delta}). \tag{4.230}$$

Proof. The proof is similar to that of Proposition 4.4.5. Let $\psi \in L^2(I, H^1(\nu))$ be the solution of (4.229) and $\psi^{(1)}$ solution of

$$\begin{cases} \bar{A}_1^\nu \psi^{(1)} = \chi_n e_{i_0} & \text{on } \pi(\mathcal{M}_N) \\ \psi^{(1)} \cdot \vec{n} = 0 & \text{on } \partial\pi(\mathcal{M}_N), \end{cases} \tag{4.231}$$

Let $\psi^{(2)} := \psi - \psi^{(1)}$, which solves

$$\begin{cases} A_1^\nu \psi^{(2)} = -\beta M \psi^{(1)} & \text{on } \pi(\mathcal{M}_N) \\ \psi^{(2)} \cdot \vec{n} = 0 & \text{on } \partial\pi(\mathcal{M}_N). \end{cases}$$

Taking the scalar product of the above equation with $\psi^{(2)}$ and integrating by parts under ν yields

$$\mathbb{E}_\nu[|\psi^{(2)}|^2] \leqslant C(\beta) n^{\kappa\varepsilon} \mathbb{E}_\nu[\psi^{(2)} \cdot M\psi^{(1)}]. \tag{4.232}$$

We claim that uniformly in $1 \leqslant j \leqslant n$,

$$\mathbb{E}_\nu[\psi^{(2)} \cdot M\psi^{(1)}] \leqslant \frac{C(\beta)}{n^{1-\kappa\varepsilon}} \mathbb{E}_\nu[|\psi^{(2)}|^2]^{\frac{1}{2}} \mathbb{E}_\nu[\chi_n^2]^{\frac{1}{2}}. \tag{4.233}$$

Let $\mathcal{A}$ be the good event (4.120). Fix $1 \leqslant j \leqslant n$. One can split the quantity $(M\psi^{(1)}) \cdot e_j$ into

$$(M\psi^{(1)}) \cdot e_j = \underbrace{\sum_{k:d(k,\partial I)\leqslant n/4} e_j \cdot M(e_k \psi_k^{(1)})}_{(I)_j} + \underbrace{\sum_{k:d(k,\partial I)>n/4} e_j \cdot M(e_k \psi_k^{(1)})}_{(II)_j}.$$

By (4.143) and (4.144), one may upper bound the first quantity by

$$\begin{aligned}\mathbb{E}_\nu[\mathbb{1}_{\mathcal{A}} \psi_j^{(2)} (I)_j] &\leqslant \frac{C(\beta) n^{\kappa\varepsilon}}{d(j,\partial I)^{\frac{s}{2}}} \sum_{k:d(k,\partial I)\leqslant n/4} \frac{1}{d(k,i_0)^{2-s}} \frac{1}{d(k,\partial I)^{\frac{s}{2}}} \mathbb{E}_\nu[(\psi_j^{(2)})^2]^{\frac{1}{2}} \mathbb{E}_\nu[\chi_n^2]^{\frac{1}{2}} \\ &\leqslant \frac{C(\beta) \mathbb{E}_\nu[(\psi_j^{(2)})^2]^{\frac{1}{2}}}{n^{1-\frac{s}{2}-\kappa\varepsilon} d(j,\partial I)^{\frac{s}{2}}} \mathbb{E}_\nu[\chi_n^2]^{\frac{1}{2}}.\end{aligned}$$

For the second quantity, we can write

$$(I)_j = \sum_{k:d(k,\partial I)>n/4} e_j \cdot M((e_k - e_{i_0})\psi_k^{(1)}) + \sum_{k:d(k,\partial I)\leqslant n/4} e_j \cdot M(e_{i_0} \psi_k^{(1)}). \tag{4.234}$$

For the first term of the last display, using the bound on the increments of M given in (4.253), we find that

$$\begin{aligned}&\mathbb{E}_\nu\Big[\mathbb{1}_{\mathcal{A}} \psi_j^{(2)} e_j \cdot \Big(\sum_{k:d(k,\partial I)>n/4} M((e_k - e_{i_0})\psi_k^{(1)}) \Big) \Big] \\ &\leqslant \frac{C(\beta) n^{\kappa\varepsilon}}{d(j,\partial I)^{\frac{s}{2}}} \sum_{k:d(k,\partial I)>n/4} \frac{1}{d(i_0,k)^{1-s}} \frac{1}{n^{1+\frac{s}{2}}} \mathbb{E}_\nu[(\psi_j^{(2)})^2]^{\frac{1}{2}} \mathbb{E}_\nu[\chi_n^2]^{\frac{1}{2}} \leqslant \frac{C(\beta) \mathbb{E}_\nu[(\psi_j^{(2)})^2]^{\frac{1}{2}}}{n^{1-\frac{s}{2}+\kappa\varepsilon} d(j,\partial I)^{\frac{s}{2}}} \mathbb{E}_\nu[\chi_n^2]^{\frac{1}{2}}.\end{aligned} \tag{4.235}$$

Because $\psi^{(1)} \cdot (e_1 + \ldots + e_n) = 0$, the second term of (4.234) satisfies

$$\mathbb{E}_\nu\Big[\mathbb{1}_{\mathcal{A}} \psi_j^{(2)} e_j \cdot \Big(\sum_{k:d(k,\partial I)>n/4} M(e_{i_0} \psi_k^{(1)}) \Big) \Big] \leqslant \frac{C(\beta) \mathbb{E}_\nu[(\psi_j^{(2)})^2]^{\frac{1}{2}}}{n^{1-\kappa\varepsilon}} \mathbb{E}_\nu[\chi_n^2]^{\frac{1}{2}}. \tag{4.236}$$

Putting (4.234), (4.235) and (4.236) together we obtain (4.233). Summing this over j yields

$$\mathbb{E}_\nu[\mathbb{1}_{\mathcal{A}} \psi^{(2)} \cdot M\psi^{(1)}] \leqslant \frac{C(\beta) \mathbb{E}_\nu[|\psi^{(2)}|^2]^{\frac{1}{2}}}{n^{1-\kappa\varepsilon}} \mathbb{E}_\nu[\chi_n^2].$$

Finally, inserting the maximum principle of Proposition 4.3.6 we obtain

$$\mathbb{E}_\nu[\mathbb{1}_{\mathcal{A}^c} \psi^{(2)} \cdot M\psi^{(1)}] \leqslant C(\beta) n^{\kappa\varepsilon} \sup |\chi_n| \mathbb{E}_\nu[|\psi^{(2)}|^2]^{\frac{1}{2}} \mathbb{E}_\nu[\mathbb{1}_{\mathcal{A}^c} |M|^2]^{\frac{1}{2}} \leqslant C(\beta) e^{-c(\beta) n^\delta} \sup |\chi_n| \mathbb{E}_\nu[|\psi^{(2)}|^2]^{\frac{1}{2}}.$$

Inserting the last displays into (4.232) we find

$$\mathbb{E}_\nu[|\psi^{(2)}|^2] \leqslant \frac{C(\beta)}{n^{1-\kappa\varepsilon}} (\mathbb{E}_\nu[\chi_n^2] + e^{-c(\beta) n^\delta} \sup |\chi_n|^2). \tag{4.237}$$

In particular, for each $1 \leqslant j \leqslant n$, there holds

$$\mathbb{E}_\nu[(\psi_j^{(2)})^2] \leqslant \frac{C(\beta)}{n^{1-\kappa\varepsilon}} (\mathbb{E}_\nu[\chi_n^2] + e^{-c(\beta) n^\delta} \sup |\chi_n|^2)$$

and the estimate (4.230) follows. □

4.5.7 Proof of Theorem 4.1.1

Proof of Theorem 4.1.1. Arguing as in the proof of Theorem 4.1.2, one may deduce Theorem 4.1.1 from the decay estimate of Propositions 4.5.8 and 4.5.9. Note that for gaps $N(x_{i+1}-x_i)$ and $N(x_{j+1}-x_j)$ at macroscopic distance, one may directly apply Proposition 4.5.8, whereas for gaps at small microscopic or microscopic distance, one can import the result of Proposition 4.5.9, which yields (4.5) by coosing n large enough with respect to $d(i,j)$. □

4.6 Uniqueness of the limiting measure

In this section we show that the sequence of the laws of microscopic processes converges, in a suitable topology, to a certain point process $\mathrm{Riesz}_{s,\beta}$, as claimed in Theorem 4.1.3. The existence of an accumulation point being a routine argument, Theorem 4.1.3 is in fact a uniqueness result. To establish uniqueness of the accumulation point, one should prove that in a certain sense, the sequence of the microscopic point processes forms a Cauchy sequence. In the following subsection, we further explain the strategy of proof and reduce the problem to a statement on the decay of correlations.

4.6.1 Reduction to a correlation estimate

To prove Theorem 4.1.4, we seek to compare the two following quantities:

$$\mathbb{E}_{\mathbb{P}^{\mathrm{g}}_{N,\beta}}[F(x_1,\ldots,x_n)] \quad \text{and} \quad \mathbb{E}_{\mathbb{P}^{\mathrm{g}}_{N',\beta}}[F(x_1,\ldots,x_n)], \quad \text{with} \quad F:\mathbb{R}^N\to\mathbb{R} \quad \text{smooth}, \tag{4.238}$$

where $1\leqslant n\leqslant N'\leqslant N$. Let us denote $I=\{1,\ldots,n\}$ and $\pi: D_N\to\pi(D_N)$ the projection on the coordinates $(x_1,\ldots,x_n)$. Let also $I'=\{1,\ldots,n\}$ and $\pi':\mathcal{M}_N\to\pi(\mathcal{M}_N)$ the projection on the coordinates $(x_1,\ldots,x_n)$. We claim that if F depends on variables in the bulk of $\{1,\ldots,n\}$, then the expectation of F under $\mathbb{P}^{\mathrm{g}}_{N,\beta}$ and $\mathbb{P}^{\mathrm{g}}_{N',\beta}$ approximately coincide, whenever N and N' are chosen large enough. We will draw an exterior configuration $y=(y_{n+1},\ldots,y_N)\in\pi_{I^c}(D_N)$ from $\mathbb{P}_{N,\beta}$ and an exterior configuration $z=(z_{n+1},\ldots,z_{N'})\in\pi_{I^c}(D_{N'})$ from $\mathbb{P}_{N',\beta}$ and compare the conditioned measures $\mathbb{P}_{N,\beta}(\cdot\mid y)$ and $\mathbb{P}_{N',\beta}(\cdot\mid z)$. Let us slightly modify the measures $\mathbb{P}_{N,\beta}$ and $\mathbb{P}_{N',\beta}$ by adding the following quantity to the Hamiltonian:

$$\mathrm{F}=\sum_{i=1}^n\theta\Big(\frac{N(x_{i+1}-x_i)}{n^{\varepsilon}}\Big). \tag{4.239}$$

Define F^{g} such that $\mathrm{F}=\mathrm{F}^{\mathrm{g}}\circ\mathrm{Gap}_N$ and the constrained measures

$$\mathrm{d}\mathbb{Q}_{N,\beta}\propto e^{-\beta\mathrm{F}}\mathrm{d}\mathbb{P}_{N,\beta} \quad \mathrm{d}\mathbb{Q}_{N',\beta}\propto e^{-\beta\mathrm{F}}\mathrm{d}\mathbb{P}_{N',\beta} \tag{4.240}$$

$$\mathrm{d}\mathbb{Q}^{\mathrm{g}}_{N,\beta}\propto e^{-\beta\mathrm{F}^{\mathrm{g}}}\mathrm{d}\mathbb{P}^{\mathrm{g}}_{N,\beta} \quad \mathrm{d}\mathbb{Q}^{\mathrm{g}}_{N',\beta}\propto e^{-\beta\mathrm{F}^{\mathrm{g}}}\mathrm{d}\mathbb{P}^{\mathrm{g}}_{N',\beta}. \tag{4.241}$$

We say that a configuration $y=(y_{n+1},\ldots,y_N)\in\pi_{I^c}(D_N)$ is admissible if

$$|N(y_{i+k}-y_i)-k|\leqslant Cn^{\varepsilon}k^{\frac{s}{2}} \quad \text{for each } n+1\leqslant i,i+k\leqslant N \tag{4.242}$$

and that $y\in\pi_{I^c}(D_N)$ and $z\in\pi_{I^c}(D_{N'})$ are compatible if

$$N-N(y_N-y_{n+1})=N'-N(z_N-z_{n+1}). \tag{4.243}$$

Given $y\in\pi_{I^c}(D_N)$ and $z\in\pi_{I^c}(D_{N'})$ two admissible and compatible configurations, we define the conditioned measures

$$\mu_n^y=\mathbb{Q}_{N,\beta}(\cdot\mid y) \quad \text{and} \quad \mu_n^z=\mathbb{Q}_{N,\beta}(\cdot\mid z). \tag{4.244}$$

Letting

$$\mathcal{A}_n = \{(x_1,\ldots,x_n) \in \pi(D_n) : N(x_n - x_1) \leqslant N - N(y_N - y_{n+1})\},$$

we can write

$$\mathrm{d}\mu_n^y(x) \propto e^{-\beta(\mathcal{H}_n(x)+\mathcal{H}_{n,N}(x,y)+\mathrm{F}(x))}\mathbb{1}_{\mathcal{A}_n}(x)\mathrm{d}x \tag{4.245}$$

$$\mathrm{d}\mu_n^z(x) \propto e^{-\beta(\mathcal{H}_n(x)+\mathcal{H}_{n,N'}(x,z)+\mathrm{F}(x))}\mathbb{1}_{\mathcal{A}_n}(x)\mathrm{d}x, \tag{4.246}$$

where $\mathcal{H}_{n,N}(x,y)$ stands for the interaction between x and y. To compare μ_n^y and μ_n^z, a first possibility is to transport one measure onto the other and to study the decay of the solution of the Monge-Ampere equation. Instead, we interpolate between μ_n^y and μ_n^z and consider a continuous path $\mu(t)$ in the space of probability measures on $\pi(D_N)$. There are several ways of interpolating, one of them consisting in running the Langevin dynamics as in [10]. A simple way of proceeding is to consider a convex combination of $\mathcal{H}_{n,N}$ and $\mathcal{H}_{n,N'}$. For $t \in [0,1]$, we define

$$E(t)(x) = (1-t)\mathcal{H}_{n,N}(x,y) + t\mathcal{H}_{n,N'}(x,z) \quad \text{and} \quad \mathcal{H}_n(t) = \mathcal{H}_n + \mathrm{F} + E(t) \tag{4.247}$$

and the probability measure

$$\mathrm{d}\mu(t)(x) \propto e^{-\beta\mathcal{H}_n(t)(x)}\mathbb{1}_{\mathcal{A}_n}(x)\mathrm{d}x. \tag{4.248}$$

Observe that $\mu(0) = \mu_n^y$ and $\mu(1) = \mu_n^z$.

Let $G : \mathbb{R}^n \to \mathbb{R}$ be a measurable bounded function. Define

$$h : t \in [0,1] \mapsto \mathbb{E}_{\mu(t)}[G].$$

It is straightforward to check that h is smooth and that for all $t \in (0,1)$,

$$h'(t) = \beta\,\mathrm{Cov}_{\mu(t)}[G, \mathcal{H}_{n,N}(\cdot,y) - \mathcal{H}_{n,N}(\cdot,z)].$$

Integrating this between 0 and 1, we obtain the following integral representation of the difference of the expectations of G under μ_n and ν_n:

Lemma 4.6.1. *Let $G : \mathbb{R}^n \to \mathbb{R}$ be a measurable bounded function in the form $G = \widetilde{G} \circ \phi$ where $\phi : X_n \in \mathbb{R}^{n+1} \mapsto (N(x_2 - x_1), \ldots, N(x_n - x_{n-1}))$. Let also $\mu(t)$ be the measure defined in (4.248), $\nu(t) = \phi\#\mu(t)$, $\tilde{y} = \mathrm{Gap}_{N-n}(y)$, $\tilde{z} = \mathrm{Gap}_{N-n}(z)$ and $\alpha_n \in (0, N')$. We have*

$$\mathbb{E}_{\mu_n^z}[G] = \mathbb{E}_{\mu_n^y}[G] + \beta\int_0^1 \mathrm{Cov}_{\nu(t)}[\widetilde{G}, \mathcal{H}^{\mathrm{g}}_{n,N}(\cdot,\tilde{y}) - \mathcal{H}^{\mathrm{g}}_{n,N}(\cdot,\tilde{z})]\mathrm{d}t. \tag{4.249}$$

We will consider functions $\widetilde{G}$ depending on a small number of coordinates in the bulk of $\{1,\ldots,n\}$. Let us emphasize that $\partial_i(\mathcal{H}^{\mathrm{g}}_{n,N}(\cdot,\tilde{y}) - \mathcal{H}^{\mathrm{g}}_{n,N}(\cdot,\tilde{z}))$ typically decays in $d(i,\partial I)^{-\frac{s}{2}}$ under $\nu(t)$. One should therefore prove that the decay of correlations under $\nu(t)$ is fast enough in order to compensate the long-range of the interaction and conclude that the covariance term in (4.249) is small. One shall apply the general result of Proposition 4.4.5 to the measure $\mu(t)$. This first requires to prove that $\mu(t)$ satisfies Assumption 4.4.1. The main task is to obtain rigidity estimates under $\mu(t)$.

4.6.2 Rigidity estimates under the perturbed measure

We control the expectation and the fluctuations of gaps under the measure $\mu(t)$.

Lemma 4.6.2. *Let $s \in (0,1)$. Let $1 \leqslant n \leqslant N \leqslant N'$ with $N \gg n^{\frac{2}{s}}$. Let $y \in \pi_{I^c}(D_N)$ and $z \in \pi_{I^c}(D_{N'})$ be two admissible and compatible configurations in the sense of (4.242) and (4.243). Let $\mu(t)$ be the probability measure (4.248). There a exists constants $\kappa > 0$, $C(\beta) > 0$ and $c(\beta) > 0$ locally uniform in β such that*

$$\mu(t)(N(x_{i+1} - x_i)) \geqslant n^{\kappa\varepsilon}) \leqslant C(\beta)e^{-c(\beta)n^\delta}, \quad \text{for each } 1 \leqslant i \leqslant n, \tag{4.250}$$

$$\mu(t)(|N(x_{i+k} - x_i) - k| \geqslant n^{\kappa\varepsilon}k^{\frac{s}{2}}) \leqslant C(\beta)e^{-c(\beta)n^\delta}, \quad \text{for each } 1 \leqslant i \leqslant i+k \leqslant n. \tag{4.251}$$

4.6.3 Decomposition of the operator

To compare $\mu_n^{(y)}$ and $\mu_n^{(z)}$, we study the decay of correlations under the measure $\mu(t)$ defined in (4.248). Following the procedure of Subsection 4.5.1, one may split $A_1^{\nu(t)}$ into $A_1^{\nu(t)} = \bar{A}_1^{\nu(t)} + M(t)$ with

$$\bar{A}_1^{\nu(t)} := \beta\nabla^2\mathrm{F}^{\mathrm{g}} + \beta A\mathbb{1}_{\mathcal{A}^c} + \beta\nabla^2(\mathcal{H}_n^{\mathrm{g}}(x) + E(t))\mathbb{1}_{\mathcal{A}} - \beta B(D + \beta^{-1}\mathcal{L}^{\nu(t)} \otimes I_n)^{-1}C + \mathcal{L}^{\nu(t)} \otimes I_n,$$

$$M(t) : \beta\nabla^2(\mathcal{H}_n^{\mathrm{g}} + E(t))\mathbb{1}_{\mathcal{A}^c} - \beta A\mathbb{1}_{\mathcal{A}^c} + \beta B(D + \beta^{-1}\mathcal{L}^{\nu(t)} \otimes I_n)^{-1}C,$$

where A, B, C, D are as in (4.117). In view of Lemmas 4.6.2 and 4.5.2, for $s \in (0,1)$, there exists $C(\beta) > 0, c(\beta) > 0, \delta > 0$ and $\kappa > 0$ such that for each $1 \leqslant i, j, l \leqslant n$, η, $\phi \in L^2(\nu(t))$,

$$\mathbb{E}_{\nu(t)}[(\phi e_j) \cdot M(t)(\eta e_i)]^{\frac{1}{2}} \leqslant \frac{C(\beta)n^{\kappa\varepsilon}}{d(i,\partial I)^{\frac{s}{2}}d(j,\partial I)^{\frac{s}{2}}}\mathbb{E}_{\nu(t)}[\phi^2]^{\frac{1}{2}}\mathbb{E}_{\nu(t)}[\eta^2]^{\frac{1}{2}} + C(\beta)e^{-c(\beta)n^\delta}\sup|\phi|\sup|\eta|, \tag{4.252}$$

$$\mathbb{E}_{\nu(t)}\Big[(\phi e_j)M(t)(\eta(e_i - e_l))\Big]^{\frac{1}{2}} \leqslant \frac{C(\beta)n^{\kappa\varepsilon}|i-l|}{\min(d(i,\partial I)^{1+\frac{s}{2}}, d(j,\partial I)^{1+\frac{s}{2}})d(j,\partial I)^{\frac{s}{2}}}\mathbb{E}_{\nu(t)}[\phi^2]^{\frac{1}{2}}\mathbb{E}_{\nu(t)}[\eta^2]^{\frac{1}{2}} + C(\beta)e^{-c(\beta)n^\delta}\sup|\phi|\sup|\eta|. \tag{4.253}$$

Similarly in the case $s \in (1, +\infty)$, for each $1 \leqslant i, j, l \leqslant n$, η, $\phi \in L^2(\nu(t))$,

$$\mathbb{E}_{\nu(t)}[(\phi e_j) \cdot M(t)(\eta e_i)]^{\frac{1}{2}} \leqslant \frac{C(\beta)n^{\kappa\varepsilon}}{d(i,\partial I)^{s-\frac{1}{2}}d(j,\partial I)^{s-\frac{1}{2}}}\mathbb{E}_{\nu(t)}[\phi^2]^{\frac{1}{2}}\mathbb{E}_{\nu(t)}[\eta^2]^{\frac{1}{2}} + C(\beta)e^{-c(\beta)n^\delta}\sup|\phi|\sup|\eta|, \tag{4.254}$$

$$\mathbb{E}_{\nu(t)}\Big[(\phi e_j)M(t)(\eta(e_i - e_l))\Big]^{\frac{1}{2}} \leqslant \frac{C(\beta)n^{\kappa\varepsilon}|i-l|}{\min(d(i,\partial I)^{\frac{3}{2}+s}, d(j,\partial I)^{\frac{3}{2}+s})d(j,\partial I)^{\frac{1}{2}+s}}\mathbb{E}_{\nu(t)}[\phi^2]^{\frac{1}{2}}\mathbb{E}_{\nu(t)}[\eta^2]^{\frac{1}{2}} + C(\beta)e^{-c(\beta)n^\delta}\sup|\phi|\sup|\eta|. \tag{4.255}$$

4.6.4 Decay of the approximate solution

One may apply the estimate (4.207) of Section 4.5 to the measure $\nu(t)$.

Lemma 4.6.3. *Let $s \in (0,1) \cup (1,+\infty)$. Let $y \in \pi_{I^c}(\mathcal{M}_N)$ be an admissible configuration in the sense of (4.242) and $\nu(t)$ be the measure defined in (4.248). Let $\chi_n \in H^1$, $i_0 \in \{1,\dots,n\}$ and $\psi \in L^2(I, H^1(\mu(t)))$ solution of*

$$\begin{cases} \bar{A}_1^{\nu(t)}\psi = \chi_n e_{i_0} + \lambda(e_1 + \ldots + e_n) & \text{on } \mathcal{A}_n \\ \psi \cdot (e_1 + \ldots + e_n) = 0 & \text{on } \mathcal{A}_n \\ \psi \cdot \vec{n} = 0 & \text{on } \partial\mathcal{A}_n. \end{cases} \tag{4.256}$$

There exist constants $\kappa > 0$ and $C(\beta) > 0$ such that

$$\mathbb{E}_{\nu(t)}[\psi_j^2]^{\frac{1}{2}} \leqslant C(\beta) n^{\kappa\varepsilon} (\mathbb{E}_{\nu(t)}[\chi_n^2]^{\frac{1}{2}} + \sup|\chi_n| e^{-c(\beta)n^\delta}) \Big(\frac{\mathbb{1}_{n<N}}{\sqrt{n}} + \frac{\mathbb{1}_{s\in(0,1)}}{d(i_0,j)^{2-s}} + \frac{\mathbb{1}_{s\in(1,+\infty)}}{d(i_0,j)^{1+s}} \Big). \tag{4.257}$$

We establish the well-posedness of (4.257) in the Appendix, see Section 4.7.2.

Proof. In view of Lemma 4.6.2, one may observe that $\nu(t)$ satisfies Assumptions 4.5.1 if $s \in (1, +\infty)$ and Assumptions 4.4.1 if $s \in (0, 1)$. The estimate of Proposition 4.4.3 can therefore be applied to $\nu(t)$, which gives (4.257). □

4.6.5 Estimate on the main equation

It remains to study the decay of the solution of the Helffer-Sjöstrand equation associated to $\nu(t)$ when the source vector-field is localized on a small number of coordinates in the bulk of $\{1, \ldots, n\}$. To this end we study the difference between the solution of the main equation ψ and the solution $\psi^{(1)}$ of the approximate equation (4.256). By convexity, we obtain a satisfactory bound on ψ and conclude that the correlation under $\nu(t)$ between a gap in the bulk of $\{1, \ldots, n\}$ and the interaction energy $E(t)$ tends to 0 as n tends to infinity.

Lemma 4.6.4. *Let $s \in (0, 1) \cup (1, +\infty)$. Let $y \in \pi_{I^c}(D_n)$ be an admissible configuration in the sense of (4.242) and $\nu(t)$ be the measure defined in (4.248). Let $\chi_n \in H^1$, $i_0 \in \{1, \ldots, n\}$ such that $|i_0 - \frac{n}{2}| \leqslant \frac{n}{4}$. Let $\psi \in L^2(I', H^1(\nu(t)))$ solving*

$$\begin{cases} \beta \nabla^2 (\mathcal{H}_n^{\mathrm{g}}(t) + \mathrm{F}^{\mathrm{g}})\psi + \mathcal{L}^{\nu(t)}\psi = \chi_n e_{i_0} + \lambda(e_1 + \ldots + e_n) & \text{on } \mathcal{A}_n \\ \psi \cdot (e_1 + \ldots + e_n) = 0 & \text{on } \mathcal{A}_n \\ \psi \cdot \vec{n} = 0 & \text{on } \partial\mathcal{A}_n. \end{cases} \tag{4.258}$$

There exist a constant $C(\beta) > 0$ and a constant $\kappa > 0$ such that

$$\sum_{j=1}^n \frac{\mathbb{E}_{\nu(t)}[\psi_j^2]^{\frac{1}{2}}}{d(j, \partial I)^{\frac{s}{2}}} \leqslant C(\beta) n^{\kappa\varepsilon} (\mathbb{E}_{\nu(t)}[\chi_n^2]^{\frac{1}{2}} + \sup|\chi_n| e^{-c(\beta)n^\delta})(n^{-\frac{s}{2}} \mathbb{1}_{s\in(0,1)} + n^{-\frac{1}{2}} \mathbb{1}_{s\in(1,+\infty)}).$$

Proof. Let $s \in (0, 1)$. Let $\psi \in L^2(I, H^1(\nu(t)))$ be the solution of (4.258). Let $\psi^{(1)} \in L^2(I, H^1(\nu(t)))$ be the solution of

$$\begin{cases} \beta \mathrm{M} \psi^{(1)} + \mathcal{L}^{\nu(t)}\psi^{(1)} = \chi_n e_{i_0} + \lambda(e_1 + \ldots + e_n) & \text{on } \mathcal{A}_n \\ \psi^{(1)} \cdot (e_1 + \ldots + e_n) = 0 & \text{on } \mathcal{A}_n \\ \psi^{(1)} \cdot \vec{n} = 0 & \text{on } \partial\mathcal{A}_n. \end{cases}$$

Set $\psi^{(2)} = \psi - \psi^{(1)} \in L^2(I, H^1(\nu(t)))$. One can observe that $\psi^{(2)}$ is solution of

$$\begin{cases} \beta \nabla^2 (\mathcal{H}_n^{\mathrm{g}}(t) + \mathrm{F}^{\mathrm{g}})\psi^{(2)} + \mathcal{L}^{\nu(t)}\psi^{(2)} = -\beta M(t)\psi^{(1)} + \lambda(e_1 + \ldots + e_n) & \text{on } \mathcal{A}_n \\ \psi^{(2)} \cdot (e_1 + \ldots + e_n) = 0 & \text{on } \mathcal{A}_n \\ \psi^{(2)} \cdot \vec{n} = 0 & \text{on } \partial\mathcal{A}_n. \end{cases}$$

Using the bounds (4.252) and (4.253) and arguing as in the proof of Proposition 4.5.9, we get

$$\mathbb{E}_{\nu(t)}[|\psi^{(2)}|^2] \leqslant \frac{C(\beta)}{n^{1-\kappa\varepsilon}} (\mathbb{E}_{\nu(t)}[\chi_n^2] + \sup|\chi_n|^2 e^{-c(\beta)n^\delta}).$$

By Cauchy-Schwarz inequality, this yields

$$\sum_{j=1}^n \frac{1}{d(j, \partial I)^{\frac{s}{2}}} \mathbb{E}_{\nu(t)}[(\psi_j^{(2)})^2]^{\frac{1}{2}} \leqslant C(\beta) n^{\kappa\varepsilon - \frac{s}{2}} (\mathbb{E}_{\nu(t)}[\chi_n^2]^{\frac{1}{2}} + \sup|\chi_n| e^{-c(\beta)n^\delta})$$

and the same estimate holds for ψ. We conclude likewise in the case $s \in (1, +\infty)$. □

4.6.6 Proof of Theorem 4.1.3 and Theorem 4.1.4

Inserting the decay estimate of Lemma 4.6.3 into the identity (4.249), one may easily compare the measure μ_n^y and μ_n^z. Integrating y and z in the set of admissible configurations gives in particular the following comparison between the measure $\mathbb{P}_{N,\beta}$ and $\mathbb{P}_{N',\beta}$:

Proposition 4.6.5. *Let $s \in (0,1) \cup (1,+\infty)$. Let $G : \mathbb{R}^n \to \mathbb{R}$ in H^1 such that $\sup |\nabla G| < \infty$. Assume that G depends only on the variables x_i for $i \in J := \{\lfloor \frac{n}{2} \rfloor - K, \dots, \lfloor \frac{n}{2} \rfloor + K\}$ with $K \leqslant n/5$. Let $\mathcal{A}$ be the good event (4.120). We have*

$$\begin{aligned}&\Big|\mathbb{E}_{\mathbb{P}^{\mathrm{g}}_{N,\beta}}[G(x_1,\dots,x_n)] - \mathbb{E}_{\mathbb{P}^{\mathrm{g}}_{N,\beta}}[G(x_1,\dots,x_n)]\Big| \\ &\qquad \leqslant C(\beta) n^{\kappa\varepsilon}(n^{-\frac{s}{2}}\mathbb{1}_{s\in(0,1)} + n^{-\frac{1}{2}}\mathbb{1}_{s\in(1,+\infty)})\Big(\sup_{\mathcal{A}} \sum_{i\in J} |\partial_i G| + e^{-c(\beta)n^{\delta}} \sup \sum_{i\in J}|\partial_i G|\Big). \end{aligned} \tag{4.259}$$

Proof. The proof follows from Lemma 4.6.3 and from the local laws. Let us define

$$\mathcal{A}_n = \{(x_1,\dots,x_n) : \pi(\mathcal{M}_N) : x_1 + \ldots + x_n \leqslant 2n\}.$$

By restricting the domain of integration to $\mathcal{A}_n$, which has overwhelming probability by Theorem 4.2.2, one can write

$$\mathbb{E}_{\mathbb{P}^{\mathrm{g}}_{N,\beta}}[G\circ\pi] = \mathbb{E}_{\mathbb{P}^{\mathrm{g}}_{N,\beta}}\Big[\mathbb{E}_{\mathbb{P}^{\mathrm{g}}_{N,\beta}}[G\circ\pi \mid \mathcal{A}_n]\Big] + C(\beta)\sup|G|e^{-c(\beta)n^{\delta}}, \tag{4.260}$$

$$\mathbb{E}_{\mathbb{P}^{\mathrm{g}}_{N',\beta}}[G\circ\pi] = \mathbb{E}_{\mathbb{P}^{\mathrm{g}}_{N,\beta}}\Big[\mathbb{E}_{\mathbb{P}^{\mathrm{g}}_{N,\beta}}[G\circ\pi \mid \mathcal{A}_n\Big] + C(\beta)\sup|G|e^{-c(\beta)n^{\delta}}. \tag{4.261}$$

Let F^{g} be the forcing (4.239) and $\mathbb{Q}^{\mathrm{g}}_{N,\beta}$, $\mathbb{Q}^{\mathrm{g}}_{N',\beta}$ as in (4.241). The measure $\mathbb{Q}^{\mathrm{g}}_{N,\beta}\circ\pi^{-1}(\cdot\mid\mathcal{A}_n)$ being uniformly log-concave with constant $c = \beta n^{-\varepsilon(s+2)}$ on the convex set $\mathcal{A}_n$, it follows from the Barky-Emery criterion (see Lemma 4.3.8) that $\mathbb{Q}^{\mathrm{g}}_{N,\beta}\circ\pi^{-1}(\cdot\mid\mathcal{A}_n)$ satisfies a Log-Sobolev inequality with constant $2c^{-1}$. In particular,

$$\mathrm{Ent}[\mathbb{P}^{\mathrm{g}}_{N,\beta}\circ\pi^{-1}(\cdot\mid\mathcal{A}_n)\mid \mathbb{Q}^{\mathrm{g}}_{N,\beta}\circ\pi^{-1}(\cdot\mid\mathcal{A}_n)] \leqslant 2c^{-1}\mathbb{E}_{\mathbb{P}_{N,\beta}}[|\nabla \mathrm{F}|^2].$$

Using Theorem 4.2.1, one can upper bound the relative entropy by

$$\mathrm{Ent}[\mathbb{P}^{\mathrm{g}}_{N,\beta}\circ\pi^{-1}(\cdot\mid\mathcal{A}_n)\mid \mathbb{Q}^{\mathrm{g}}_{N,\beta}\circ\pi^{-1}(\cdot\mid\mathcal{A}_n)] \leqslant C(\beta)e^{-c(\beta)n^{\delta}}, \quad \text{for some } \delta>0.$$

It follows from the Pinsker inequality that

$$\mathrm{TV}(\mathbb{P}^{\mathrm{g}}_{N,\beta}\circ\pi^{-1}(\cdot\mid\mathcal{A}_n), \mathbb{Q}^{\mathrm{g}}_{N,\beta}\circ\pi^{-1}(\cdot\mid\mathcal{A}_n)) \leqslant C(\beta)e^{-c(\beta)n^{\delta}}.$$

Similarly we find

$$\mathrm{TV}(\mathbb{P}^{\mathrm{g}}_{N',\beta}\circ\pi^{-1}(\cdot\mid\mathcal{A}_n), \mathbb{Q}^{\mathrm{g}}_{N',\beta}\circ\pi^{-1}(\cdot\mid\mathcal{A}_n)) \leqslant C(\beta)e^{-c(\beta)n^{\delta}}.$$

One may therefore replace the expressions in (4.260) and (4.261) by

$$\mathbb{E}_{\mathbb{P}^{\mathrm{g}}_{N,\beta}}[G\circ\pi\mid\mathcal{A}_n] = \mathbb{E}_{\mathbb{Q}^{\mathrm{g}}_{N,\beta}}[G\circ\pi\mid\mathcal{A}_n] + O_\beta(\sup|G|e^{-c(\beta)n^{\delta}}), \tag{4.262}$$

$$\mathbb{E}_{\mathbb{P}^{\mathrm{g}}_{N',\beta}}[G\circ\pi\mid\mathcal{A}_n] = \mathbb{E}_{\mathbb{Q}^{\mathrm{g}}_{N',\beta}}[G\circ\pi\mid\mathcal{A}_n] + O_\beta(\sup|G|e^{-c(\beta)n^{\delta}}). \tag{4.263}$$

Fix two exterior admissible (in the sense of (4.242)) configurations $y \in \pi_{I^c}(D_N)$ and $z \in \pi_{I^c}(D_{N'})$ and let $\tilde{y} = \mathrm{Gap}_{N-n}(y)$ and $\tilde{z} = \mathrm{Gap}_{N-n}(z)$. Let $\mu(t)$ $(= \mu(t,y,z))$ be interpolating between

μ_n^y and μ_n^z as in (4.248) and let $\nu(t)$ be the push-forward of $\mu(t)$ be $X_n \in \pi(D_n) \mapsto (N(x_2 - x_1), \ldots, N(x_n - x_n))$. Assume that G depends only on x_i for $i \in J := \{\lfloor \frac{n}{2} \rfloor - K, \ldots, \lfloor \frac{n}{2} \rfloor + K\}$. For each $i \in J$, let $\psi^{(t,i)} \in L^2(I', H^1(\nu(t)))$ be the solution of

$$\begin{cases} \beta A_1^{\nu(t)} \psi^{(t,i)} = (\partial_i G) e_i & \text{on } \mathcal{A}_n \\ \psi^{(t,i)} \cdot (e_1 + \ldots + e_n) = 0 & \text{on } \mathcal{A}_n \\ \psi^{(t,i)} \cdot \vec{n} = 0 & \text{on } \partial \mathcal{A}_n. \end{cases}$$

By applying the estimates of Lemmas 4.6.1, 4.6.2 and 4.6.4, we find

$$\begin{aligned} |\mathbb{E}_{\mathbb{P}^{\mathrm{g}}_{N,\beta}}[G \circ \pi \mid \tilde{y}] - \mathbb{E}_{\mathbb{P}^{\mathrm{g}}_{N',\beta}}[G \circ \pi \mid \tilde{z}| &\leqslant \int_0^1 |\operatorname{Cov}_{\nu(t)}[G, \mathcal{H}^{\mathrm{g}}_{n,N'} - \mathcal{H}^{\mathrm{g}}_{n,N}]| \mathrm{d}t \\ &= \sum_{i \in J} \int_0^1 |\mathbb{E}_{\nu(t)}[\nabla(\mathcal{H}^{\mathrm{g}}_{n,N'} - \mathcal{H}^{\mathrm{g}}_{n,N}) \cdot \psi^{(t,i)}]| \mathrm{d}t \\ &\leqslant C(\beta) n^{\kappa\varepsilon - \frac{s}{2}} \sum_{i \in J} (\mathbb{E}_{\nu(t)}[(\partial_i G)^2]^{\frac{1}{2}} + \sup |\partial_i G| e^{-c(\beta) n^\delta}) \\ &\leqslant C(\beta) n^{\kappa\varepsilon - \frac{s}{2}} \sup_{\mathcal{A}} \sum_{i \in J} (|\partial_i G| + \sup |\partial_i G| e^{-c(\beta) n^\delta}), \end{aligned} \tag{4.264}$$

where we have used the fact that the event (4.120) has overwhelming probability under $\nu(t)$ (see Lemma 4.6.2). Moreover, under $\mathbb{Q}_{N,\beta}$ (resp $\mathbb{Q}_{N',\beta}$), the exterior configuration y (resp z) is admissible with overwhelming probability. Therefore, integrating (4.264) over y and z in the set of admissible configurations, one obtains from (4.262) and (4.263) the claimed result. □

We are now ready to conclude the proof of the uniqueness of the limiting measure. We will consider random variables in the space of configurations on $\mathbb{R}$ and one should first define a σ-algebra on it. We let $\mathrm{Conf}(\mathbb{R})$ be the set of locally finite and simple point configurations in $\mathbb{R}$. Given a Borel set $B \subset \mathbb{R}$, we denote $N_B : \mathrm{Conf}(\mathbb{R}) \to \mathbb{N}$ the number of points lying in B. We then endow $\mathrm{Conf}(\mathbb{R})$ with the σ-algebra generated by the maps $\{N_B : B \text{ Borel}\}$. We call point process a probability measure on $\mathrm{Conf}(\mathbb{R})$. We then say that a sequence P_N of point processes converges to P for the local topology on $\mathrm{Conf}(\mathbb{R})$ whenever for any bounded, Borel and local function $f : \mathrm{Conf}(\mathbb{R}) \to \mathbb{R}$, the following convergence holds:

$$\lim_{n \to \infty} \mathbb{E}_{P_N}[f] = \mathbb{E}_P[f].$$

Proof of Theorems 4.1.3 and 4.1.4.

Step 1: compactness. Let $(x_1, \ldots, x_N)$ distributed according to $\mathbb{P}_{N,\beta}$. Denote

$$Q^N = \mathrm{Law}\left(\sum_{i=1}^N \delta_{Nx_i} \mathbb{1}_{|x_i| < \frac{1}{4}}\right) \in \mathcal{P}(\mathrm{Conf}(\mathbb{R})).$$

Let us show that the sequence (Q^N) has an accumulation point in the local topology on $\mathcal{P}(\mathrm{Conf}(\mathbb{R}))$. We follow the strategy of [94, Prop. 2.9]. For all $R > 0$ denote $\Lambda_R = [-R, R]$ and for all $Q \in \mathcal{P}(\mathrm{Conf}(\mathbb{R}))$, Q_R the law of $\mathcal{C}|_{\Lambda_R}$ when $\mathcal{C}$ is distributed according to Q. For two point processes P and Q, define the relative specific entropy of P with respect to Q by

$$\mathrm{Ent}[P \mid Q] = \limsup_{R \to \infty} \mathrm{Ent}[P_R \mid Q_R].$$

Let Π be a Poisson point process on $\mathbb{R}$. According to [129, Prop. 2.6], the level sets of $\mathrm{Ent}[\cdot \mid \Pi]$ are sequentially compact for the local topology. As a consequence it is enough to check that

$$\sup_{N\in\mathbb{N}^*} \sup_{K\in\mathbb{N}^*} \frac{1}{K}\mathrm{Ent}[Q_K^N, \Pi_{\Lambda_K}] < \infty. \tag{4.265}$$

Let B_{K,Λ_K} be a Bernoulli process on Λ_K. Following [94], one can split the relative entropy into

$$\begin{aligned}\mathrm{Ent}[Q_K^N \mid \Pi_{\Lambda_K}] &= \int \log \frac{\mathrm{d}Q_K^N}{\mathrm{d}B_{K,\lambda_K}} \mathrm{d}Q_K^N + \int \log \frac{\mathrm{d}B_{K,\Lambda_K}}{\mathrm{d}\Pi_{\Lambda_K}} \mathrm{d}Q_K^N \\ &= -\log K_{N,\beta}(\Lambda_K) - \beta \mathbb{E}_{Q_K^N}\Big[\sum_{x_i \neq x_j \in \mathcal{C}} g_s(x_i - x_j)\Big] - \log\left(e^{-N}\frac{N^N}{N!}\right),\end{aligned} \tag{4.266}$$

where

$$K_{N,\beta}(\Lambda_K) = \int \exp\Big(-\beta \sum_{x_i\neq x_j \in \mathcal{C}\cap\Lambda_K} g_s(x_i - x_j)\Big)\mathbb{1}_{\frac{N}{4}D_N(X_N)} \mathrm{d}X_N.$$

From the rigidity estimates of Theorem 4.2.2, we have

$$\log K_{N,\beta}(\Lambda_K) = -\beta \mathbb{E}_{Q_K^N}\Big[\sum_{x_i\neq x_j\in\mathcal{C}\cap\Lambda_K} g_s(x_i - x_j)\Big] + O_\beta(K).$$

Inserting this into (4.266), we deduce that (4.265) holds. It follows that (Q^N) has an accumulation point in the local topology.

Step 2: uniqueness. Let us now prove that this accumulation point is unique. Let $P, Q \in \mathcal{P}(\mathrm{Conf}(\mathbb{R}))$ be two accumulation points of (Q^N) in the local topology. Note that P and Q are necessarily translation invariant. Let $k_0 \geqslant 1$. Set

$$F : \mathcal{C} \in \mathrm{Conf}(\mathbb{R}) \mapsto G(z_2 - z_1, \dots, z_{k_0} - z_1),$$

with $G : \mathbb{R}^{k_0} \to \mathbb{R}$ smooth. In view of Proposition 4.6.5, we can see that

$$\mathbb{E}_P[F] = \mathbb{E}_Q[F].$$

This implies that for each $k_0 \in \mathbb{N}$, the law of $(z_2 - z_1, \dots, z_{k_0} - z_1)$ under P equals the law of $(z_2 - z_1, \dots, z_{k_0} - z_1)$ under Q. Since P and Q are translation invariant, we conclude that $P = Q$. □

The proof of Theorem 4.1.4 is now straightforward.

Proof of Theorem 4.1.4. By Theorem 4.1.3,

$$\lim_{N\to\infty} \mathbb{E}_{\mathbb{P}_{N,\beta}}[F\circ\pi] = \mathbb{E}_{\mathrm{Riesz}_{s,\beta}}[G(z_2 - z_1, \dots, z_{k_0} - z_1)].$$

Since the error term in (4.259) is uniform in N, this concludes the proof of Theorem 4.1.3. □

4.6.7 Proof of the hyperuniformity result

Having already established in Chapter 3 that the N-Riesz gas is hyperuniform and that $N(x_K - x_1)$ is of order $O(K^s)$ under $\mathbb{P}_{N,\beta}$ with a Gaussian asymptotic behavior, it is now immediate using the convergence result of Theorem 4.1.3 to prove that $\mathrm{Riesz}_{s,\beta}$ is also hyperuniform.

Proof of Theorem 4.1.5. Let $1 \leqslant K \leqslant \frac{N}{2}$. Set $\ell_N = \frac{N}{K}$. Let

$$F_N = (N\ell_N)^{-\frac{s}{2}} \left(\sum_{i=1}^N \mathbb{1}_{(0,\ell_N)(x_i)} - \ell_N \right).$$

Let $Z \sim \mathcal{N}(0, \sigma^2)$ with

$$\sigma^2 = \frac{1}{\beta \frac{\pi}{2} s} \mathrm{cotan}\left(\frac{\pi}{2} s\right).$$

Let $\eta : \mathbb{R} \to \mathbb{R}$ such that $|\eta'|_\infty \leqslant 1$. In Chapter 3, we have proved that

$$\mathbb{E}_{\mathbb{P}_{N,\beta}}[\eta(F_N)] = \mathbb{E}[\eta(Z)] + o_K(1), \tag{4.267}$$

with a $o_K(1)$ uniform in N. Set

$$\widetilde{F}_N = K^{-\frac{s}{2}} N(x_K - x_1).$$

Using Theorem 4.2.2, we can prove that

$$\mathbb{E}_{\mathbb{P}_{N,\beta}}[\eta(\widetilde{F}_N)] = \mathbb{E}_{\mathbb{P}_{N,\beta}}[\eta(F_N)] + o_K(1), \tag{4.268}$$

with a $o_K(1)$ uniform in N. Now by Theorem 4.1.4, we have

$$\lim_{N\to\infty} \mathbb{E}_{\mathbb{P}_{N,\beta}}[\eta(\widetilde{F}_N)] = \mathbb{E}_{\mathrm{Riesz}_{s,\beta}}[\eta(K^{-\frac{s}{2}}(z_K - z_1 - K))]. \tag{4.269}$$

Combining (4.267), (4.268) and (4.269), one deduces that

$$\mathbb{E}_{\mathrm{Riesz}_{s,\beta}}[\eta(K^{-\frac{s}{2}}(z_K - z_1 - K))] = \mathbb{E}[\eta(Z)] + o_K(1).$$

We deduce that under the process $\mathrm{Riesz}_{s,\beta}$, the sequence $K^{-\frac{s}{2}}(z_K - z_1 - K)$ converges in distribution to $Z \sim \mathcal{N}(0, \sigma^2)$. Moreover by Chapter 3,

$$\mathrm{Var}_{\mathbb{P}_{N,\beta}}[F_N] = \mathrm{Var}[Z] + o_N(K^s),$$

with a $o_N(K^s)$ uniform in N. Proceeding as above, one easily prove the variance estimate (4.10). □

4.6.8 Proof of the repulsion estimate

Proof of Proposition 4.1.6. Let $\alpha \in (0, \frac{s}{2})$. We have proved in [52, Lemma 4.5] that there exist two constants $C(\beta) > 0$ and $c(\beta) > 0$ locally uniform in β such that for each $i \in \{1, \dots, N\}$ and $\varepsilon > 0$ small enough,

$$\mathbb{P}_{N,\beta}(N(x_{i+1} - x_i) \leqslant \varepsilon) \leqslant C(\beta) e^{-c(\beta)\varepsilon^{-\alpha}}.$$

Since $(\mathbb{P}^{\mathrm{g}}_{N,\beta})$ converges to $\mathrm{Riesz}_{s,\beta}$ in the local topology, we can pass the above inequality to the limit as $N \to \infty$ and we obtain

$$\mathbb{P}_{\mathrm{Riesz}_{s,\beta}}(z_{i+1} - z_i \leqslant \varepsilon) \leqslant C(\beta) e^{-c(\beta)\varepsilon^{-\alpha}}.$$

□

4.7 Appendix

4.7.1 Discrete Gagliardo-Nirenberg inequality

The Gagliardo-Nirenberg inequality, originally proved independently in [123, 202], is an interpolation inequality between different weak derivatives in L^p spaces. The result was at first stated for derivatives of integer order and then extended to derivatives of fractional order in the rather recent paper [62]. The main result of [62] gives sufficient and necessary conditions on the orders and exponents for an interpolation inequality to hold on $\mathbb{R}^n$. For shortcut, we only present one of the cases where the interpolation inequality is valid.

Lemma 4.7.1 (Brezis-Mironescu). *Let $1 \leqslant p, p_1, p_2 \leqslant \infty$. Let $s_1, s_2 \geqslant 0$ and $\theta \in (0,1)$ such that*

$$s_1 \leqslant s_2, \quad s_0 = \theta s_1 + (1-\theta)s_2, \quad \frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}.$$

Assume that $s_2 < 1$. Then, there exists a constant $C > 0$ depending on $p_1, p_2, s_1, s_2, \theta$ such that for all $u \in W^{s_1,p_1}(\mathbb{R}) \cap W^{s_2,p_2}(\mathbb{R})$,

$$\|u\|_{W^{s_0,p}(\mathbb{R})} \leqslant C\|u\|^{\theta}_{W^{s_1,p_1}(\mathbb{R})}\|u\|^{1-\theta}_{W^{s_2,p_2}(\mathbb{R})}.$$

By taking a periodic function of period 1 on $(-n,n)$, one can show by letting n tend to infinity that Lemma 4.7.1 also holds for functions defined on the circle.

4.7.2 Well-posedness results

The proofs of Propositions 4.3.1 and 4.3.3 can be found in [52, Appendix A]. For completeness we sketch the main arguments below.

Let μ satisfying Assumptions 4.3.1. The formal adjoint with respect to μ of the derivation ∂_i, $i \in \{1,\dots,N\}$ is given by

$$\partial_i^* w = \partial_i w - (\partial_i H)w,$$

meaning that for all $v, w \in \mathcal{C}^\infty(D_N, \mathbb{R})$ such that $\nabla w \cdot \vec{n} = 0$, the following identity holds

$$\mathbb{E}_\mu[(\partial_i v)w] = \mathbb{E}_\mu[v\partial_i^* w]. \tag{4.270}$$

The above identity can be shown by integration by parts under the Lebesgue measure on D_N. Recall the map

$$\Pi : X_N \in D_N \mapsto (x_2 - x_1, \dots, x_N - x_1) \in \mathbb{T}^{N-1}$$

and

$$\mu' = \mu \circ \Pi^{-1}.$$

Proof of Proposition 4.3.1. Let $F = G \circ \Pi$ with $G \in H^1(\mu)$. Recall that if $F \in H^1(\mu)$, then $\nabla F \in L^2(\{1,\dots,N\}, H^{-1}(\mu))$. Let

$$E = \{\phi \circ \Pi : \phi \in H^1(\mu'), \mathbb{E}_\mu[\phi \circ \Pi] = 0\}.$$

Consider the functional

$$J : \phi \in E \mapsto \mathbb{E}_\mu[|\nabla \phi|^2 - 2\phi F].$$

One may easily check that J admits a unique minimizer. Indeed for all $\phi = \psi \circ \Pi \in E$, one can write

$$|\mathbb{E}_\mu[\phi F]| \leqslant \|F\|_{H^{-1}(\mu)}\|\phi\|_{H^1(\mu)}.$$

Moreover since $\phi \in E$, one can observe that

$$\mathbb{E}_\mu[|\phi|^2] = \mathbb{E}_{\mu'}[|\psi|^2] \leqslant c^{-1}\mathbb{E}_{\mu'}[|\nabla \psi|^2] = \frac{1}{2c}\mathbb{E}_\mu[|\nabla \phi|^2].$$

It follows that J is bounded from below. Since J is convex and l.s.c, by standard arguments, it is l.s.c for the weak topology of $H^1(\mu)$ and therefore J admits a minimizer ϕ.

One can then easily check by integration by parts that the Euler-Lagrange equations for ϕ state that a.e on D_N,

$$\mathcal{L}^\mu \phi = F - \mathbb{E}_\mu[F], \tag{4.271}$$

with the boundary condition

$$\nabla \phi \cdot \vec{n} = 0, \tag{4.272}$$

a.e on ∂D_N. Equations (4.271) and (4.272) easily imply that J admits a unique minimizer.

Let us now differentiate rigorously Equation (4.271). Let $w \in \mathcal{C}_c^\infty(D_N)$ and $i \in \{1, \ldots, N\}$. By integration by parts, we have

$$\begin{aligned}\mathbb{E}_\mu[w\partial_i F] &= \mathbb{E}_\mu[\partial_i^* w(F - \mathbb{E}_\mu[F])] = \mathbb{E}_\mu[\partial_i^* w \mathcal{L}\phi] = \mathbb{E}_\mu[\nabla \partial_i^* w \cdot \nabla \phi] \\ &= \sum_{j=1}^N \mathbb{E}_\mu[(\partial_i^* \partial_j w)\partial_j \phi] + \sum_{j=1}^N \mathbb{E}_\mu[([\partial_j, \partial_i^*]\, w)\partial_j \phi].\end{aligned}$$

The first term of the right-hand side of the last display may be expressed as

$$\sum_{j=1}^N \mathbb{E}_\mu[(\partial_i^* \partial_j w)\partial_j \phi] = \sum_{j=1}^N \mathbb{E}_\mu[(\partial_j w)\partial_i \partial_j \phi] = \mathbb{E}_\mu[\nabla w \cdot \nabla(\partial_i \phi)] = \mathbb{E}_\mu[w\mathcal{L}^\mu(\partial_i \phi)].$$

For the second term, recalling the identity $[\partial_j, \partial_i^*] = (\nabla^2 H)_{i,j}$, one may write

$$\sum_{j=1}^N \mathbb{E}_\mu[([\partial_j, \partial_i^*]\, w)\partial_j \phi] = \mathbb{E}_\mu[(w \cdot \nabla^2 H \nabla \phi)_i].$$

One deduces that, in the sense $H^{-1}(\mu)$, for each $i \in \{1, \ldots, N\}$,

$$(\nabla^2 H \nabla \phi)_i + \mathcal{L}^\mu(\partial_i \phi) = \partial_i F.$$

Together with the boundary condition (4.272), this concludes the proof of existence and uniqueness of a solution to (4.44). We turn to the proof the variational characterization of the solution of (4.44). Let

$$J : L^2(\{1, \ldots, N\}, H^1(\mu)) \mapsto \mathbb{E}_\mu[|D\psi|^2 + \psi \cdot \nabla^2 H \psi - 2\psi \cdot \nabla F]. \tag{4.273}$$

By standard arguments, one can prove that J admits a minimizer ψ, which satisfies the Euler-Lagrange equation

$$A_1^\mu \psi = \nabla F.$$

Moreover, one may assume that $\psi \cdot \vec{n} = 0$ on ∂D_N. By integration by parts, we conclude that $\psi = \nabla \phi$. □

Let us now prove Proposition 4.3.3. Recall the notation

$$\mathrm{Gap}_N : X_N \in D_N \mapsto (N|x_2 - x_1|, N|x_3 - x_2|, \ldots, N|x_N - x_1|) \in \mathbb{R}^N,$$

$$\mathcal{M}_N = \mathrm{Gap}_N(D_N) \quad \text{and} \quad \nu = \mathrm{Gap}_N \# \mu.$$

Proof of Proposition 4.3.3. Let $G \in H^{-1}(\nu)$. Denote $E = \{\phi \in H^1(\nu) : \mathbb{E}_\nu[\phi]\} = 0$ and J the functional

$$J : \phi \in E \mapsto \mathbb{E}_\nu[|\nabla\phi|^2 - 2\phi G].$$

By standard arguments (see the proof of Proposition 4.3.1), we can show that J admits a unique minimizer ϕ. Since ϕ is a minimizer of J, for all $h \in E$,

$$\mathbb{E}_\nu[\nabla\phi \cdot \nabla h] = \mathbb{E}_\nu[Gh].$$

By integration by parts, one can observe that for all $h \in E$,

$$\mathbb{E}_\nu[\nabla\phi \cdot \nabla h] = \mathbb{E}_\nu[\mathcal{L}^\nu \phi h] + \int_{\partial\mathcal{M}_N} (\nabla\phi \cdot \vec{n}) h e^{-H}.$$

By density, it then follows that

$$\begin{cases} \mathcal{L}^\nu\phi = G - \mathbb{E}_\nu[G] & \text{on } \mathcal{M}_N \\ \nabla\phi \cdot \vec{n} = 0 & \text{on } \partial\mathcal{M}_N. \end{cases}$$

To prove that $\nabla\phi$ satisfies the Helffer-Sjöstrand equation (4.49), we need to adapt the integration by parts formula (4.270). One may easily show that for all $v \in \mathcal{C}^\infty(\mathcal{M}_N)$ such that $\nabla v \cdot \vec{n} = 0$ on ∂D_N and $\psi \in L^2(\{1, \ldots, N\}, \mathcal{C}^\infty(\mathcal{M}_N)$ such that $\psi \cdot (e_1 + \ldots + e_N) = 0$, there holds

$$\mathbb{E}_\nu\left[\psi \cdot \nabla v\right] = \mathbb{E}_\nu\left[v(-\nabla H^{\mathrm{g}} \cdot \psi + \mathrm{div}\psi)\right]. \tag{4.274}$$

Let $w \in L^2(\{1, \ldots, N\}, \mathcal{C}_c^\infty(\mathcal{M}_N))$ such that $\sum_{i=1}^N w_i = 0$. In view of (4.274),

$$\mathbb{E}_\nu[w \cdot \nabla G] = \mathbb{E}_\nu[(G - \mathbb{E}_\nu[G])(-\nabla H^{\mathrm{g}} \cdot w + \mathrm{div} w)] = \mathbb{E}_\nu[\mathcal{L}^\nu\phi(-\nabla H^{\mathrm{g}} \cdot w + \mathrm{div} w)].$$

Integrating part the last equation gives

$$\mathbb{E}_\nu[w \cdot \nabla G] = \mathbb{E}_\nu[\nabla\phi \cdot \nabla(-\nabla H^{\mathrm{g}} \cdot w + \mathrm{div} w)] = \mathbb{E}_\nu[w \cdot (\mathcal{L}^\nu\nabla\phi + \nabla^2 H^{\mathrm{g}}\nabla\phi)].$$

By density, we deduce that there exists a Lagrange multiplier $\lambda \in H^{-1}(\nu)$ such that

$$\nabla^2 H^{\mathrm{g}}\nabla\phi + \mathcal{L}^\nu\nabla\phi = \nabla G + \lambda(e_1 + \ldots + e_N).$$

Recalling that $\nabla\phi \cdot \vec{n} = 0$ on $\partial\mathcal{M}_N$, this yields the existence of a solution to (4.49). Since $\sum_{i=1}^N \partial_i\phi = 0$, taking the scalar product of the above equation with $e_1 + \ldots + e_N$ yields

$$\lambda = \frac{1}{N}(e_1 + \ldots + e_N) \cdot \nabla^2 H^{\mathrm{g}}\nabla\phi.$$

The uniqueness of the solution to (4.49) is straightforward. The proof of the variational characterization comes from arguments similar to the proof of Proposition 4.3.1. □

4.7.3 Local laws for the HS Riesz gas

Lemma 4.7.2. *Let $s > 1$. For all $\varepsilon > 0$ small enough, there exists $\delta > 0$ such that*

$$\mathbb{P}_{N,\beta}(N(x_{i+1} - x_i) \geqslant k^\varepsilon) \leqslant C(\beta)e^{-c(\beta)k^\delta}, \quad \text{for each } 1 \leqslant i \leqslant N. \tag{4.275}$$

For all $\varepsilon > 0$ small enough, there exists $\delta > 0$ such that

$$\mathbb{P}_{N,\beta}(|N(x_{i+k} - x_i) - k| \geqslant k^{\frac{1}{2}+\varepsilon}) \leqslant C(\beta)e^{-c(\beta)k^\delta}, \quad \text{for each } 1 \leqslant i \leqslant N \text{ and } 1 \leqslant k \leqslant \frac{N}{2}. \tag{4.276}$$

Proof. We consider the case $1 < s < 2$. The case $s \geqslant 2$ is simpler and will be sketched afterwards. One shall proceed by a bootstrap on scales. Consider the statement $\mathcal{P}(k)$: for all $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\mathbb{P}_{N,\beta}(N(x_{i+k} - x_i) \geqslant k^{1+\varepsilon}) \leqslant C(\beta)e^{-c(\beta)k^{\delta}}. \tag{4.277}$$

Assume that $\mathcal{P}(K)$ holds. Let us prove that $\mathcal{P}(K^{1-\alpha_0})$ holds for $\alpha_0 \in (0,1)$ small enough. Let $k = \lfloor K^{1-\alpha_0} \rfloor$. Let $i \in \{1, \ldots, N\}$ and

$$I = \{j \in \{1, \ldots, N\}, d(j,i) \leqslant k\}.$$

Let θ be a smooth cutoff function $\theta : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\theta(x) = x^2$ for $x > 1$, $\theta = 0$ on $[0, \frac{1}{2}]$ and $\theta'' \geqslant 0$ on $\mathbb{R}^+$. For $\gamma > 0$ to determine later define the forcing

$$\mathrm{F} = \sum_{i \neq j \in I} \theta\Big(\frac{N(x_j - x_i)}{K^{1+\gamma}}\Big)$$

and the constrained probability measure

$$\mathrm{d}\mathbb{Q}_{N,\beta} = \frac{1}{K_{N,\beta}} e^{-\beta \mathrm{F}} \mathrm{d}\mathbb{P}_{N,\beta}.$$

One can write

$$\mathbb{P}_{N,\beta}(N(x_{i+k} - x_i) \geqslant k^{1+\delta}) \leqslant \mathbb{Q}_{N,\beta}(N(x_{i+k} - x_i) \geqslant k^{1+\delta}) + \mathrm{TV}(\mathbb{P}_{N,\beta}, \mathbb{Q}_{N,\beta}).$$

By choosing $\gamma > \delta(1 - \alpha_0)$, one can show that

$$\mathrm{TV}(\mathbb{P}_{N,\beta}, \mathbb{Q}_{N,\beta}) \leqslant C(\beta)e^{-c(\beta)k^{2\delta}}.$$

Using Lemma 4.3.9, one has

$$\log \mathbb{E}_{\mathbb{Q}_{N,\beta}}[e^{tN(x_{i+k}-x_i)}] \leqslant t\mathbb{E}_{\mathbb{Q}_{N,\beta}}[N(x_{i+k} - x_i)] + \frac{t^2}{2\beta}K^{s+\gamma}, \quad \text{for all } t \in \mathbb{R}.$$

Moreover with computations similar to Chapter 3, we find

$$\mathbb{E}_{\mathbb{Q}_{N,\beta}}[N(x_{i+k} - x_i)] = k + O_\beta(1).$$

Combining the two last display we find

$$\mathbb{Q}_{N,\beta}(N(x_{i+k} - x_i) \geqslant k^{1+\delta}) \leqslant C(\beta)e^{-c(\beta)k^{\frac{s+\gamma}{1-\alpha_0} - 2(1+\delta)}}.$$

The exponent in the right-hand side of the last display is strictly smaller than -2δ if and only if

$$\gamma < 2(1 - \alpha_0) - s.$$

Since $s < 2$, there exists $\alpha_0 > 0$ small enough such that

$$\delta(1 - \alpha_0) < 2(1 - \alpha_0) - s.$$

One concludes that (4.277) holds for each $k \geqslant K^{1-\alpha_0}$, for some constant α_0 depending only on s. After a finite number of steps, one concludes that $\mathcal{P}(1)$ holds. The estimate 4.275 immediately follows.

For $s \geqslant 2$, the proof of Lemma 4.7.2 can be run without making use of convexity arguments. One can establish (4.275) by showing that the log-Laplace transform of the energy of K consecutive points is of order K. This can be done recursively by controlling the interaction energy oft two intervals of points. For this short-range model, one can control this interaction by shrinking configurations as in [142, Proof of Prop. 4.4].

Let us now justify (4.275). One can constrain small gaps and define a new measure uniformly log-concave in gap coordinates. By rewriting $N(x_{i+k} - x_i)$ into $N(x_{i+k} - x_i) = \sum_{j=i}^{i+k-1} N(x_{j+1} - x_j)$, one easily concludes the proof of (4.276). $\square$

4.7.4 Local laws for the interpolating measure

We provide some useful rigidity estimates for the conditioned measure (4.244) and adapt the proofs of Chapter 3 which are based on techniques of [46].

Proof of Lemma 4.6.2. Let $y \in \pi_{I^c}(D_N)$ and $z \in \pi_{I^c}(D_N)$ be as in the statement of Lemma 4.6.2 and $\mu(t)$ as in (4.248). The first bound (4.250) is immediate in view of the forcing (4.239). Let us prove (4.251).

Step 1: control of the fluctuations Let $i \in \{1, \ldots, n\}$ and $k \in \{1, \ldots, N/2\}$ such that $1 \leqslant i + k \leqslant n$. We wish to prove that for $\varepsilon' > 0$ large enough with respect to ε, there exists $\delta > 0$ depending on $\varepsilon' > 0$ such that

$$\mu(t)\Big(|N(x_{i+k} - x_i) - \mathbb{E}_{\mu(t)}[N(x_{i+k} - x_i)]| \geqslant k^{\frac{s}{2}+\varepsilon} n^{\varepsilon}\Big) \leqslant C(\beta)e^{-c(\beta)k^{\delta}}. \tag{4.278}$$

We will make use of a method of [46], which consists in splitting decomposing the gap $N(x_{i+k}-x_i)$ into a sum of block average statistics. For each $i \in J^c$, we define $I_k(i)$ be an interval of integers of cardinal $k+1$ such that $i \in I_k(i)$ and define the block average

$$x_i^{[k]} = \frac{1}{k+1} \sum_{j \in I_k(i)} x_j.$$

Let $\alpha > 0$ be a small number, $\alpha = \frac{1}{p}$ with $p \in \mathbb{N}^*$. Since $x_i^{[0]} = x_i$, one can break $x_i - x_i^{[k]}$ into

$$N(x_i - x_i^{[k]}) = \sum_{m=0}^{p-1} N(x_i^{[[k^{m\alpha}]]} - x_i^{[[k^{(m+1)\alpha}]]}). \tag{4.279}$$

For each $m \in \{0, \ldots, p-1\}$, denote $G_m = N(x_i^{[[k^{m\alpha}]]} - x_i^{[[k^{(m+1)\alpha}]]})$ and $I_m = I_{\lfloor k^{(m+1)\alpha}\rfloor}(i)$. We study the fluctuations of G_m. Because G_m depends only on the variables in I_m and since $\sum_{i \in I_m} \partial_i G_m$, one can use the Gaussian concentration result for divergence free test-functions stated in Lemma 4.3.9. Fix $m \in \{0, \ldots, p-1\}$ and introduce the coordinates $x = (x_i)_{i \in I_m}$ and $y = (x_i)_{i \in I \setminus I_m}$ on $\pi(D_N)$. The measure $\mu(t)$ satisfies the assumptions of Lemma 4.3.9 in the window I_m. It can indeed be written

$$\mathrm{d}\mu(t) = e^{-\beta H(x,y)} \mathbb{1}_{\pi(D_N)(x,y)} \mathrm{d}x\mathrm{d}y \quad \text{with} \quad H(x,y) = H_1(x) + H_2(x,y)$$

where H_2 is convex and H_1 satisfies $\sum_{i \in I_m} \partial_i H_1 = 0$ with

$$\nabla^2 H_1 \geqslant N^2 k^{-(m+1)\alpha(s+2-\varepsilon)}.$$

As a consequence, one may use Lemma 4.3.9, which entails

$$\begin{aligned}\log \mathbb{E}_{\mu(t)}[e^{tG_m}] &\leqslant t\mathbb{E}_{\mu(t)}[G_m] + \frac{t^2}{2\beta} N^{-2} k^{(m+1)\alpha(s+2+\varepsilon)} |I_m|^{-1} \sup |\nabla G_m|^2 \\ &\leqslant t\mathbb{E}_{\nu(t)}[G_m] + \frac{t^2}{2\beta} k^{\alpha(s+1)+ms\alpha+\varepsilon(s+2)}.\end{aligned}$$

We conclude that for ε' large enough with respect to ε, there exists $\delta > 0$ depending on ε' such that

$$\mu(t)(|G_m - \mathbb{E}_{\mu}[G_m]| \geqslant k^{\frac{s}{2}+\varepsilon'}) \leqslant C(\beta)e^{-c(\beta)k^{\delta}}.$$

Inserting this in (4.279), one deduces that for ε' large enough with respect to ε, there exists $\delta > 0$ depending on ε' such that

$$\mu(t)(|N(x_i - x_i^{[k]}) - \mathbb{E}_{\mu(t)}[N(x_i - x_i^{[k]})]| \geqslant k^{\frac{s}{2}+\varepsilon}) \leqslant C(\beta)e^{-c(\beta)k^\delta}. \tag{4.280}$$

One can finally check that the variable $N(x_{i+k}^{[k]} - x_{i+k})$ verifies the same estimate: proceeding as for G_m with $m = p-1$, we obtain that for $\varepsilon' > 0$ large enough with respect to ε, there exists $\delta > 0$ depending on ε' such that

$$\mu(t)(|N(x_i^{[k]} - x_{i+k}^{[k]}) - \mathbb{E}_{\mu(t)}[N(x_i^{[k]} - x_{i+k}^{[k]})]| \geqslant k^{\frac{s}{2}+\varepsilon}) \leqslant C(\beta)e^{-c(\beta)k^\delta}. \tag{4.281}$$

Combining (4.280) applied to i and $i+k$ and (4.281), one finally gets the claim (4.278).

Step 2: accuracy estimate It remains to control the expectation of $N(x_{i+k} - x_i)$ under $\mu(t)$. By construction we can write

$$\mathbb{E}_{\mu(t)}[N(x_{i+k}-x_i)]-\mathbb{E}_{\mu(0)}[N(x_{i+k}-x_i)] = \beta\int_0^t \mathrm{Cov}_{\mu(s)}[N(x_{i+k}-x_i), \mathcal{H}_{n,N}(\cdot,z)-\mathcal{H}_{n,N}(\cdot,y)]\mathrm{d}s.$$

By Cauchy-Schwarz inequality and using (4.278) one can write

$$|\mathbb{E}_{\mu(t)}[N(x_{i+k}-x_i)]-\mathbb{E}_{\mu(0)}[N(x_{i+k}-x_i)]| \leqslant C(\beta)n^{\kappa\varepsilon}k^{\frac{s}{2}}\int_0^t \mathrm{Var}_{\mu(s)}[\mathcal{H}_{n,N}(\cdot,z)-\mathcal{H}_{n,N}(\cdot,y)]^{\frac{1}{2}}\mathrm{d}s. \tag{4.282}$$

First of all, let us use the fact that there exists a constant $C > 0$ such that for all $x \in \mathbb{T}$,

$$|N^{-s}g_s(\frac{x}{N}) - \tilde{g}_s(x)| \leqslant \frac{C}{N^s}, \tag{4.283}$$

where $\tilde{g}_s : x \in \mathbb{T} \mapsto \frac{1}{|x|^s}$. Let us denote

$$\widetilde{\mathcal{H}}_{n,N}(x,y) = \sum_{i\in I, j\in\{1,\ldots,N\}\setminus I} \frac{1}{|N(x_i - y_j)|^s}$$

and

$$\widetilde{\mathcal{H}}_{n,N'}(x,z) = \sum_{i\in I, j\in\{1,\ldots,N'\}\setminus I} \frac{1}{|N(x_i - z_j)|^s}.$$

Recall that $N' \leqslant N$. To begin the comparison let us restrict the sum as follows:

$$\begin{aligned}\widetilde{\mathcal{H}}_{n,N}(x,y) - \widetilde{\mathcal{H}}_{n,N'}(x,z) = &\sum_{i\in I}\sum_{j\in\{1,\ldots,N'\}\setminus I}\Big(\frac{1}{|N(x_i - y_j)|^s} - \frac{1}{|N(x_i - z_j)|^s}\Big)\\ &+ \sum_{i\in I}\sum_{j\in\{1,\ldots,N\}\setminus\{1,\ldots,N'\}} \frac{1}{|N(x_i - z_j)|^s}\end{aligned} \tag{4.284}$$

Let us control the first sum, say the terms at the right-hand side of I. Fix $k \in I$. By Taylor expansion, one may write

$$\sum_{j=n+1}^{N'/2}\Big(\frac{1}{|N(x_k - y_j)|^s} - \frac{1}{|N(x_k - z_j)|^s}\Big) = \sum_{j=n+1}^{N'/2} \tilde{g}_s'(N(x_k - y_j))N(y_j - z_j) + (I)_k \tag{4.285}$$

where the error term $(I)_k$ satisfies

$$\mathrm{Var}_{\mu(s)}[(I)_k]^{\frac{1}{2}} \leqslant C(\beta) n^{\kappa\varepsilon} \sum_{j=n+1}^{N'/2} \frac{|j-n|^{s+\varepsilon'}}{|j-k|^{2+s}} \leqslant \frac{C(\beta) n^{\kappa\varepsilon}}{d(k,\partial I)^{1-\varepsilon'}}, \tag{4.286}$$

for some $\varepsilon' > 0$. By Taylor expansion again and using Lemma 4.6.2, one can write

$$\mathrm{Var}_{\mu(s)}[\tilde{g}_s'(N(x_k - y_j))] \leqslant C(\beta) n^{\kappa\varepsilon} \tilde{g}_s''(j-k)(n-k)^{s/2+\kappa\varepsilon}.$$

The leading-order of the right-hand side of (4.285) therefore satisfies

$$\sum_{j=n+1}^{N'/2} \tilde{g}_s'(N(x_k - y_j)) N(z_j - y_j) = \sum_{j=n+1}^{N'/2} \tilde{g}_s'(|j-k|) N(z_j - y_j) + (II)_k \tag{4.287}$$

with

$$\mathrm{Var}_{\mu(s)}[(II)_k]^{\frac{1}{2}} \leqslant C(\beta) n^{\kappa\varepsilon} \sum_{j=n+1}^{N'/2} \frac{|j-n|^{s/2}|n-k|^{s/2+\kappa\varepsilon}}{|j-k|^{s+2}} \leqslant \frac{C(\beta) n^{\kappa\varepsilon}}{d(k,\partial I)^{1-\varepsilon'}}. \tag{4.288}$$

The point is that leading order term in (4.287) is constant with respect to x and its variance is therefore 0 under $\mu(s)$. It follows that uniformly in s,

$$\mathrm{Var}_{\mu(s)}\Big[\sum_{j=n+1}^{N'/2} \Big(\frac{1}{|N(x_k - y_j)|^s} - \frac{1}{|N(x_k - z_j)|^s} \Big)\Big] \leqslant C(\beta) n^{\kappa\varepsilon}. \tag{4.289}$$

One may proceed similarly for the terms at the left-hand side of I and one concludes that (4.289) also holds for the first quantity in (4.284). It remains to upper bound the second term in (4.284). By assumptions on z, one has

$$\mathrm{Var}_{\mu(s)}\Big[\sum_{i\in I} \sum_{j\in\{1,\dots,N\}\setminus\{1,\dots,N'\}} \frac{1}{|N(x_i - z_j)|^s}\Big]^{\frac{1}{2}} \leqslant C(\beta) n^{\kappa\varepsilon+1} N^{-\frac{s}{2}+\varepsilon}. \tag{4.290}$$

Combining (4.283), (4.286), (4.289) and its similar estimate, we find that uniformly in s,

$$\mathrm{Var}_{\mu(s)}[\mathcal{H}_{n,N}(\cdot, y) - \mathcal{H}_{n,N}(\cdot, z)] \leqslant C(\beta) n^{\kappa\varepsilon}.$$

Inserting this into (4.282) one obtains

$$|\mathbb{E}_{\mu(t)}[N(x_{i+k} - x_i)] - \mathbb{E}_{\mu(0)}[N(x_{i+k} - x_i)]| \leqslant C(\beta) n^{\kappa\varepsilon} k^{\frac{s}{2}+\kappa\varepsilon}. \tag{4.291}$$

Let us denote $\mathcal{B} \subset \pi_{I^c}(D_N)$ the set of admissible configurations as defined in (4.242). By taking $t = 1$ and $N = N'$, we find that for all $y, z \in \mathcal{B}$,

$$|\mathbb{E}_{\mathbb{Q}_{N,\beta}(\cdot|y)}[N(x_{i+k} - x_i)] - \mathbb{E}_{\mathbb{Q}_{N,\beta}(\cdot|z)}[N(x_{i+k} - x_i)]| \leqslant C(\beta) n^{\kappa\varepsilon} k^{\frac{s}{2}+\kappa\varepsilon}. \tag{4.292}$$

Since by Theorem 4.2.2

$$|\mathbb{E}_{\mathbb{Q}_{N,\beta}}[N(x_{i+k} - x_i)\mathbb{1}_{\mathcal{B}}] - k| \leqslant C(\beta) n^{\kappa\varepsilon},$$

we deduce from (4.292) that for all admissible configuration $y \in \pi(D_N)$,

$$|\mathbb{E}_{\mathbb{Q}_{N,\beta}(\cdot|y)}[N(x_{i+k} - x_i)] - k| \leqslant C(\beta) n^{\kappa\varepsilon} k^{\frac{s}{2}+\kappa\varepsilon}.$$

Inserting this into (4.291) concludes the proof of Lemma 4.6.2. $\square$

Chapter 5

Dipole transition for the two-component plasma

This chapter is a joint work with Sylvia Serfaty and will appear on the arXiv soon.

Contents

5.1 Introduction

5.1.1 Setting of the problem

In the 1970's, Kosterlitz and Thouless [164, 165] and independently Berezinsky [29] predicted a completely new type of phase transitions without long range order in two-dimensional systems, now called Berezinsky-Kosterlitz-Thouless (BKT) transition. This celebrated transition (see [34] for a review) was predicted to happen in a whole range of models which exhibit quantized vortices in a neutral ensemble, more specifically the XY or "rotator" spin model, models of dislocations and superfluids, and it has important consequences for condensed matter physics.

The transition in the XY model is probably the one that has attracted the most attention in the mathematical physics community. In this model unit spins are sampled on a lattice, constituting a $\mathbb{U}(1)$ analogue of the Ising model. The BKT transition consists in that the correlation function between distant spins decays exponentially above the transition temperature, and decays in power law below [198, 121, 35, 63]. This transition is explained by the formation of topological vortices, which are points around which the spin field has a nonzero degree or winding number. Below the transition temperature, vortices are bound into dipole pairs (i.e. pairs of vortices with opposite winding numbers), while above the transition temperature, vortices are like "free particles".

In the original papers, as well as in subsequent research, it is expected that in the XY model (or its simplified variant, the Villain model) the energy of the system can be split into a vortex-gas energy and a spin-wave contribution, corresponding to the fluctuations around the vortex configurations [164, 165, 160]. This statement turns out to be delicate to prove rigorously, and this has attracted the attention of researchers, even recently [126, 125].

Once the spin-wave contribution can be separated, the model reduces to a (2D) gas of dipoles with logarithmic interaction, which can also be called a two-component plasma, or (two-component) Coulomb gas. The Coulomb gas is thus a fundamental model on which to understand the BKT transition, as seen in the original paper of Kosterlitz [165].

The lattice (two-component) Coulomb gas was studied in the seminal work of Fröhlich-Spencer [121] via the sine-Gordon representation and expansions into multipole ensembles, allowing to analyze the decay rate of correlation functions, thus giving the first proof of the BKT transition, see also [198, 35].

The Coulomb gas may as well be studied in the continuum rather than on a lattice, and is expected to exhibit the same transition between a situation with free vortices and a situation with vortices of opposite sign bound in dipole pairs. There is a subtlety however, due to the fact that this "dipole transition" should happen at inverse temperature $\beta = 2$ in the units we use, while the KT transition between exponentially and algebraically decaying correlations is expected to happen at $\beta = 4$ in this setting. Also, it is a little delicate to directly compare the situation of the Coulomb gas in the continuum where one takes a fixed number N of charges of each sign, corresponding to a canonical ensemble, and the situation of the XY model, corresponding to a grand-canonical ensemble where the number of vortices is not prescribed.

Here we will focus on the continuum Coulomb gas or "two component plasma" in the canonical case and we will provide a proof of the "dipole transition" based simply on the analysis of dipoles pairs via large deviations techniques that allow to weigh their energy and entropy costs, in some sense very close to the arguments and computations found in the original papers [165, 164] and also in the seminal paper [139].

5.1.2 Model

We first consider the continuum Coulomb gas, defined as an ensemble with configurations (X_N, Y_N) (with $X_N = (x_1, \ldots, x_N) \in \Lambda^N$ and $Y_N = (y_1, \ldots, y_N) \in (\Lambda^N$ of N positive and N negative particles (or vortices with degrees $+1$ or -1) in the blown-up cube $\Lambda = [0, \sqrt{N}]^2$ of $\mathbb{R}^2$, having energy

$$\mathsf{F}(X_N, Y_N) = \frac{1}{2}\left(\sum_{i \neq j} -\log|x_i - x_j| - \log|y_i - y_j| + 2\sum_{i,j} \log|x_i - y_j|\right), \tag{5.1}$$

and consider the (canonical) ensemble

$$\frac{1}{Z_{N,\beta}} \exp\left(-\beta \mathsf{F}(X_N, Y_N)\right) \mathrm{d}X_N \mathrm{d}Y_N, \tag{5.2}$$

with $\mathrm{d}X_N$ and $\mathrm{d}Y_N$ the uniform Lebesgue measures on Λ^N. This model was studied in particular in [139, 97], and more recently in [186]. The integral defining $Z_{N,\beta}$ diverges as soon as $\beta \geqslant 2$, due to the fact that the energy of very short dipoles diverges in a nonintegrable way, which corresponds to the dipole transition. The ensemble (5.2) was thus studied only in the regime $\beta < 2$ in the aforementioned works [139, 97, 186]. The latest results of [186], building on important insights from [139] and techniques developed for the study of the one-component Coulomb gas in [233, 219, 183, 184], show an expansion of $\log Z_{N,\beta}$ as $N \to \infty$, as well as a large deviations principle on point processes, which characterize a situation with free interacting particles, with competition between the attraction of opposite charges and the entropic repulsion. This corresponds to the situation of temperature larger than the critical temperature.

In order to study such a system for $\beta \geqslant 2$, a truncation of the interaction is needed, as already recognized in [164, 165] and analyzed in [121], see also the discussion in [170]. Let us for shortcut always denote

$$\mathsf{g}(x) = -\log|x|, \tag{5.3}$$

and we will abuse notation by considering g as either of function of $\mathbb{R}^2$ or of $\mathbb{R}$ depending on the context.

Truncating the interaction involves introducing a small lengthscale λ and *renormalizing* the divergent part of the free energy as $\lambda \to 0$. A natural proposed way is to truncate the energy at a distance λ and consider

$$\frac{1}{2}\sum_{i,j} \mathsf{g}(x_i - x_j) \wedge \mathsf{g}(\lambda) + \mathsf{g}(y_i - y_j) \wedge \mathsf{g}(\lambda) - 2\mathsf{g}(x_i - y_j) \wedge \mathsf{g}(\lambda), \tag{5.4}$$

where $\wedge$ denotes the minimum of two numbers.

The precise method of truncation of the interaction is not really important, and here we propose a variant of this which is convenient for our techniques: instead of truncating g we consider charges smeared on discs of radius λ, with λ small, interacting otherwise in the normal Coulomb fashion: letting $\delta_z^{(\lambda)}$ denote the uniform measure of mass 1 supported on $B(z, \lambda)$, we let

$$\kappa := \iint \mathsf{g}(x-y)\delta_0^{(1)}(x)\delta_0^{(1)}(y), \tag{5.5}$$

and observe, by scaling, that

$$\iint \mathsf{g}(x-y)\delta_0^{(\lambda)}(x)\delta_0^{(\lambda)}(y) = \mathsf{g}(\lambda) + \kappa. \tag{5.6}$$

We then consider the energy

$$\mathsf{F}_\lambda(X_N, Y_N) = \frac{1}{2}\iint \mathsf{g}(x-y)\mathrm{d}\left(\sum_{i=1}^N \delta_{x_i}^{(\lambda)} - \delta_{y_i}^{(\lambda)}\right)(x)\mathrm{d}\left(\sum_{i=1}^N \delta_{x_i}^{(\lambda)} - \delta_{y_i}^{(\lambda)}\right)(y) - N(\mathsf{g}(\lambda) + \kappa). \tag{5.7}$$

Here, compared to (5.1) we have reinserted the self-interaction terms which are no longer infinite but equal to $\mathsf{g}(\lambda) + \kappa$, and then subtracted them off.

We will denote by

$$\mathsf{g}_\lambda(z) = \iint \mathsf{g}(x-y)\delta_0^{(\lambda)}(x)\delta_z^{(\lambda)}(y), \tag{5.8}$$

the effective interaction between two points at distance $|z|$. Moreover, the convolution $\mathsf{g} * \delta_0^{(\lambda)}$ is harmonic outside of $B(0, \lambda)$ and it follows from the mean-value theorem (or Newton's theorem) that

$$\mathsf{g}_\lambda(z) = \int \mathsf{g} * \delta_0^{(\lambda)}\delta_z^{(\lambda)} = \mathsf{g}(z) \quad \text{for } |z| \geqslant 2\lambda. \tag{5.9}$$

Thus we see that F_λ is the same as (5.4) except with $\mathsf{g}(x_i - x_j) \wedge \mathsf{g}(\lambda)$ replaced by $\mathsf{g}_\lambda(x_i - x_j)$, and if the distances between points are larger than λ, the interactions coincide and F_λ coincides with F. Also if $\lambda = 0$ then the definition in (5.7) coincides with F of (5.1), as proved in [186] – this is essentially Newton's theorem and Green's formula. Let us point out that the choice of $\delta_z^{(\lambda)}$ to be the uniform measure in the unit ball is unimportant, we could replace it by any radial distribution of the form $\frac{1}{\lambda^2}\rho(\frac{x-z}{\lambda})$ with ρ radial, as was done in [219]. Newton's theorem would still apply and nothing else would change, except for the precise value of the constant κ. Finally, we could in principle use charges smeared on circles instead of discs as in previous works [206, 183, 184], it does make the initial computations simpler but the potential generated a circle is too singular for our needs here.

We will thus work with (5.7) and study

$$\mathrm{d}\mathbb{P}_{N,\beta}^{\lambda} = \frac{1}{Z_{N,\beta}^{\lambda}} \exp\left(-\beta \mathsf{F}_{\lambda}(X_N, Y_N)\right) \mathrm{d}X_N \mathrm{d}Y_N \tag{5.10}$$

in the limit where λ tends to zero, where

$$Z_{N,\beta}^{\lambda} = \int_{[0,\sqrt{N}]^{2N}\times[0,\sqrt{N}]^{2N}} \exp\left(-\beta \mathsf{F}_{\lambda}(X_N, Y_N)\right) \mathrm{d}X_N \mathrm{d}Y_N. \tag{5.11}$$

When $\beta < 2$ one can immediately set $\lambda = 0$ and recover the model studied in [186], but when $\beta > 2$ one expects $\log Z_{N,\beta}^{\lambda}$ to diverge as $\lambda \to 0$. The picture that emerges from the literature, mostly based on the sine-Gordon representation, is well-described in [171]: for $\beta > 2$, the divergence of the system as $\lambda \to 0$ corresponds to the pairing of short dipoles (of lengthscale λ), and the transition at $\beta = 2$ is followed as β increases by a sequence of transitions corresponding to the formation of a subdominant proportion of multipoles (quadrupoles, sextupoles etc) as the temperature is decreased and the entropic repulsion becomes less strong [121]. This is due to the dipole-dipole interaction which is weakly attractive. When β reaches 4, in the grand canonical setting (when the number of particles is not fixed) the system is expected to collapse under the attraction of the dipoles, as first shown in [120] via a Euclidean Field Theory approach, however we will see that it is not the case in the canonical setting here. As written in [171] the complete mathematical picture is far from complete from the mathematical angle, and most of the approaches rely on the sine-Gordon transformation and on sophisticated Renormalization Group techniques which require to assume translation invariance [158]. In [121], it is written "We believe that the techniques of Section 5 will eventually permit us to prove convergence of an expansion of the two-dimensional Coulomb gas in terms of neutral multipole configurations, at low density and low temperature, designed to imply the existence of the Kosterlitz-Thouless transition. But the required combinatorial and refined electrostatic estimates are still missing."

Our main goal here is to analyze (5.7)–(5.10) via a simple and direct approach based solely on energy and entropy, precisely via electrostatic estimates for the energy. We obtain below a precise free energy expansion as $N \to \infty$ and $\lambda \to 0$, and use it to prove that configurations mostly form free dipoles for all $\beta \in (2, +\infty)$, as characterized by convergence to a Poisson process of dipoles (this shows that the multipoles, though present, concern only a vanishing fraction of the particles). Combined with the description of [186], it constitutes a first proof of the dipole transition, and we hope this point of view will also inform the understanding of the BKT transition. We also address the important question of the fluctuations of the (two-component) Coulomb gas.

Note that the two-component Coulomb gas or plasma is quite different from the one-component Coulomb gas or plasma, which consists only of positively charged particles repelling each other and confined by an external potential, or equivalently a uniform negative background charge (this is then called a jellium). The one-component Coulomb gas never diverges or forms dipole pairs, but rather the particle density converges macroscopically to an equilibrium measure limit (dictated by the external potential) while at the microscopic scale the particles arrange themselves in more and more ordered point patterns as temperature decreases, in fact the system is expected to crystallize at zero temperature, at least in low dimensions. There has been much progress on the one-component plasma in recent years, including free energy expansion [193, 183, 9, 239], local laws for the distribution of points down to the microscale [180, 9], variational characterization of the limiting point processes [183], and CLTs for the fluctuations of linear statistics [4, 214, 24, 184, 239].

The main points in common with [233, 183, 180, 184, 9, 239] but also with [186] will be the general philosophy of electrostatic energy and large deviations techniques, as well as the *electric formulation* of the energy that we present just below.

5.1.3 Main results and method

The electric formulation mentioned above consists in reexpressing the energy in terms of the *electric potential* h_λ generated by the configuration (X_N, Y_N) and defined as a function *over all* $\mathbb{R}^2$ by

$$h_\lambda[X_N, Y_N] := \mathsf{g} * \left(\sum_{i=1}^N \delta_{x_i}^{(\lambda)} - \delta_{y_i}^{(\lambda)}\right), \tag{5.12}$$

where $*$ denotes the convolution. In the sequel, we will most often drop the $[X_N, Y_N]$ dependence in the notation.

Note that by definition of g, h_λ satisfies the Poisson equation

$$-\Delta h_\lambda[X_N, Y_N] = 2\pi\left(\sum_{i=1}^N \delta_{x_i}^{(\lambda)} - \delta_{y_i}^{(\lambda)}\right). \tag{5.13}$$

A direct insertion into (5.7) and integration by parts using (5.13) yields the following rewriting of the energy

$$\mathsf{F}_\lambda(X_N, Y_N) = \frac{1}{4\pi}\int_{\mathbb{R}^2} |\nabla h_\lambda[X_N, Y_N]|^2 - N(\mathsf{g}(\lambda) + \kappa). \tag{5.14}$$

Before stating our main result, let us introduce some more notation. Let us define the probability measure

$$\mathrm{d}\mu_\beta(r) := \frac{2\pi}{C_\beta}\exp\Big(\frac{\beta}{2}\mathsf{g}_1(r)\Big)\mathbf{1}_{\mathbb{R}^+}(r)r\mathrm{d}r, \tag{5.15}$$

where C_β stands for the normalizing constant

$$C_\beta := 2\pi\int_0^\infty \exp\Big(\frac{\beta}{2}\mathsf{g}_1(r)\Big)r\mathrm{d}r. \tag{5.16}$$

We will denote $\{z_1, \dots, z_{2N}\} = \{x_1, \dots, x_N, y_1, \dots, y_N\}$ the collection of all points (positive or negative) and denote their charge $d_i = 1$ if $i \in [1, N]$ $d_i = -1$ if $i \in [N+1, 2N]$. We will also denote by $\phi_1(i)$ the index for the/a first nearest neighbor to z_i among all the points $z_j, j \neq i$.

Our first theorem provides a free energy expansion and a concentration on dipoles configurations. The constant C_β is related to our precise way of smearing the Dirac charges and corresponds to the interaction of overlapping disc charges, it is defined in (5.16).

Theorem 5.1.1. *For all $\lambda > 0$, denote*

$$\gamma_\lambda := \begin{cases} \frac{1}{|\log\lambda|} & \text{if } \beta = 2 \\ \lambda^{\beta-2} & \text{if } \beta \in (2,4) \\ \lambda^2|\log\lambda|^2 & \text{if } \beta = 4 \\ \lambda^2|\log\lambda| & \text{if } \beta > 4. \end{cases} \tag{5.17}$$

For each $\beta \in (2, +\infty)$ the following holds

1.

$$\log Z^{\lambda}_{N,\beta} = 2N\log N + N((2-\beta)\log\lambda \mathbf{1}_{\beta>2} + \log|\log\lambda|\mathbf{1}_{\beta=2}) - N + N\log C_\beta \mathbf{1}_{\beta>2} + O(N\gamma_\lambda), \tag{5.18}$$

2. *Let*

$$I := \{1 \leqslant i \leqslant N : \phi_1 \circ \phi_1(i) = i, d_i d_{\phi_1(i)} = -1\}. \tag{5.19}$$

For all $|t| \leqslant \frac{\beta}{2}$,

$$\log \mathbb{E}_{\mathbb{P}^{\lambda}_{N,\beta}}\Big[\exp\Big(t\mathsf{F}_\lambda - t\sum_{i\in I}\mathsf{g}_\lambda(z_i - z_{\phi_1(i)})\Big)\Big] \leqslant CN\gamma_\lambda. \tag{5.20}$$

3. *the Gibbs measure is concentrated on mostly neutral* λ*-dipoles configurations in the sense that letting*

$$D := \Big\{Z_{2N} \in \Lambda^{2N},\ |I| \geqslant N(1-b\gamma_\lambda), \sum_{i\in I}\mathsf{g}_\lambda(|z_i - z_{\phi_1(i)}|) \leqslant N\log\lambda + MN\Big\}, \tag{5.21}$$

we have

$$\mathbb{P}^{\lambda}_{N,\beta}(D^c) \leqslant \exp(-CN), \tag{5.22}$$

for some $b > 0$, $C > 0$ *and* $M > 0$ *independent of* N *and* λ.

The formula (5.18) can be compared with the formula for $\beta < 2$ obtained in [186], and this already exhibits a transition at $\beta = 2$, since the divergence in λ is present only for $\beta \geqslant 2$. The *screening method* of [186] would in fact allow us to prove an almost additivity of the free energy $\log Z^{\lambda}_{N,\beta}$ — once the $\frac{1}{2}\beta N\log N$ corresponding to the interaction scaling has been removed — for any β, and the existence of a thermodynamic limit

$$f(\beta,\lambda) = \lim_{N\to\infty}\frac{\log Z^{\lambda}_{N,\beta} - \frac{\beta}{2}N\log N}{N} \tag{5.23}$$

with an explicit rate of convergence independent of λ. We believe the rate can be made to be $O(N^{-1/2}\log N)$ by analogy with [9] but we defer this to future work, in any case to obtain a rate independent of λ it suffices to apply almost verbatim the proof in [186]. For $\beta < 2$, this is proven in [186] (with $\lambda = 0$) and a variational characterization of f is also provided there:

$$f(\beta,0) = -\min_{P\in\mathcal{P}}\mathcal{F}_\beta(P) \tag{5.24}$$

where $\mathcal{P}$ corresponds to the space of stationary signed point processes of intensity 1 (each species has intensity 1), and $\mathcal{F}_\beta$ (the rate function in the Large Deviations Principle proved there) is the sum of β times a suitable "renormalized energy" of point processes (an infinite volume Coulomb interaction energy) and a specific relative entropy with respect to the Poisson point process of intensity 1. We see that in that regime we have free particles of positive and negative charges, whose positions is governed by the minimization of $\mathcal{F}_\beta$.

The result (5.18) then completes this picture by proving that for $\beta \in (2,+\infty)$

$$f(\beta,\lambda) = (2-\beta)\log\lambda - 1 + \log C_\beta + o_\lambda(1) \tag{5.25}$$

and that for $\beta = 2$,

$$f(\beta,\lambda) = \log|\log\lambda| - 1 + o_\lambda(1). \tag{5.26}$$

Here the energy is dominated by pure dipole energy, demonstrating the transition from free particles to bound pairs. The question of the sharp rate of convergence $o_\lambda(1)$ is very important as it encodes the multipole transitions. Here we obtain a power rate which exhibits a transition at $\beta = 4$, which we believe to be sharp and to correspond to the BKT transition. The proofs also show a form of transition at $\beta = 3$, which corresponds to the quadrupole transition (i.e. a transition in the proportion of still negligible quadrupoles).

We next show as a corollary that the averaged microscopic process concentrates as N tends to infinity and λ tends to 0 to a Poissonian dipole process. In addition we derive some large deviations asymptotic as λ tends to 0, which can be read as the limit of the large deviations principle at fixed λ. For each centering point in Λ, we will observe the particles x_j lying around x and the attached nearest-neighbor particles. To have a well-defined limit, one shall blow-up the nearest-neighbor distances by a factor λ^{-1}. Still using the notation (5.19) for positive charges belonging to a neutral dipole, let us consider the rescaled configuration centered at x denoted

$$\mathcal{C}_N(x) := \sum_{i\in I} \delta_{(\sqrt{N}(z_i - x), \lambda^{-1}\sqrt{N}(z_{\phi_1(i)} - z_i))} \in \mathcal{P}(\mathbb{R}^2 \times \mathbb{R}^2). \tag{5.27}$$

Let χ be set of simple locally finite point configurations on $\mathbb{R}^2$. We will work on the set

$$E = \Big\{ \sum_{i\in J} \delta_{(x_j,y_j)} : J \text{ finite or countable }, (x_j)_{j\in J} \in \chi, (y_j)_{j\in J} \in (\mathbb{R}^2)^{|J|} \Big\} \tag{5.28}$$

endowed with the coarsest topology $\mathcal{T}$ generated by

$$\Big\{ \sum_{j\in J} \delta_{(x_j,y_j)} \in E : x_j \in A, y_j \in B \Big\}, \tag{5.29}$$

where A and B are measurable subsets of $\mathbb{R}^2$ with A bounded. This topology is designed so that the y_i variable is observable only if x_i lies in a compact set. In Subsection 5.6 one will endow $(E, \mathcal{T})$ will a distance d. The variable (5.27) is a random variable on E. We next consider an averaging of (5.27) over the centering point in Λ:

$$i_N := \int_\Lambda \delta_{(x, \mathcal{C}_N(x))} \mathrm{d}x \in \mathcal{P}(\Lambda \times E). \tag{5.30}$$

The space $\mathcal{P}(E)$ is then endowed with the topology of weak convergence generated by the functions in $\mathcal{C}_b(E)$. Let us now define the Poissonian dipole process denoted $\mathcal{P}^{\mathrm{dip}}$. Given $X = (X_i)_{i\in\mathbb{N}}$ a Poisson point process of intensity 1 on $\mathbb{R}^2$, $(u_i)_{i\in\mathbb{N}}$ a sequence of i.i.d variable of law μ_β (5.15) independent of X, we let $\mathcal{P}^{\mathrm{dip}} \in \mathcal{P}(E)$ be

$$\mathcal{P}^{\mathrm{dip}} := \mathrm{Law}\Big(\sum_{i=1}^N \delta_{(X_i, u_i)} \Big). \tag{5.31}$$

Let also $\bar{\mathcal{P}}^{\mathrm{dip}}$ be the tensorization of $\mathcal{P}^{\mathrm{dip}}$ with the Lebesgue measure on Λ:

$$\bar{\mathcal{P}}^{\mathrm{dip}} := \mathrm{d}x|_\Lambda \otimes \mathcal{P}^{\mathrm{dip}} \in \mathcal{P}(\Lambda \times E). \tag{5.32}$$

We will show that (i_N) concentrates around (5.32) as N tends to infinity and $\lambda \to 0$ with a Large Deviations Principle. To define the rate function, we start by introducing a (specific) relative entropy for probability measures on E similar to [212]. For all $P, Q \in \mathcal{P}(E)$, let

$$\mathrm{Ent}(Q \mid P) := \lim_{R\to\infty} \frac{1}{R} \mathrm{Ent}(Q|_{\Lambda_R} \mid P|_{\Lambda_R}), \tag{5.33}$$

where $\Lambda_R := [0,\sqrt{R}]^2$. For all $P \in \mathcal{P}(\Lambda \times E)$ and $x \in \Lambda$ we let $P^x \in \mathcal{P}(E)$ be the desintegration of P with respect to x. Let $\overline{\mathrm{Ent}}$ be an averaging of Ent defined for all $\bar{P}, \bar{Q} \in \mathcal{P}(\Lambda \times E)$ by

$$\overline{\mathrm{Ent}}(\bar{Q} \mid \bar{P}) := \int_\Lambda \mathrm{Ent}(Q^x \mid P^x)\mathrm{d}x.$$

Let us emphasize that the entropy thus defined depends on the topology put on E.

Theorem 5.1.2. *For any measurable subset A of $\mathcal{P}(E \times \Lambda)$, we have*

$$\begin{aligned}- \inf_{Q \in \mathring{A}} \overline{\mathrm{Ent}}(Q \mid \bar{\mathcal{P}}^{\mathrm{dip}}) \leqslant \liminf_{\lambda \to 0} \liminf_{N \to \infty} \frac{1}{N} \log \mathbb{P}^\lambda_{N,\beta}(i_N \in A) \\ \leqslant \limsup_{N\to\infty} \frac{1}{N} \log \mathbb{P}^\lambda_{N,\beta}(i_N \in A) \leqslant - \inf_{Q \in \bar{A}} \overline{\mathrm{Ent}}(Q \mid \bar{\mathcal{P}}^{\mathrm{dip}}) \quad (5.34)\end{aligned}$$

In addition, for all $\varepsilon > 0$, we have

$$\lim_{\lambda\to 0} \limsup_{N\to\infty} \mathbb{P}^\lambda_{N,\beta}(i_N \notin B(\bar{\mathcal{P}}^{\mathrm{dip}}, \varepsilon)) = 0. \quad (5.35)$$

We finally address the important question of the fluctuations of linear statistics. We provide an energetic control, similar in spirit to [185, Prop 2.5], showing that Lipschitz functions fluctuate less than $O(N^{1/2}\alpha_\lambda^{1/2})$, for some α_λ tending to 0 as λ tends to 0. Note that one could obtain a better rate of fluctuations in $o(N^{1/2})$ at fixed λ, but this would require different arguments.

Proposition 5.1.3. *For a Lipschitz test-function $\xi : \Lambda \to \mathbb{R}$, let us define*

$$\mathrm{Fluct}_N(\xi) := \int_\Lambda \xi \left(\sum_{i=1}^N (\delta_{x_i} - \delta_{y_i}) \right).$$

Set

$$\alpha_\lambda = \begin{cases} |\log \lambda|^{-1} & \text{if } \beta = 2 \\ \lambda^{\frac{2(\beta-2)}{\beta}} & \text{if } \beta \in (2,4) \\ \lambda |\log \lambda|^{1/2} & \text{if } \beta = 4 \\ \lambda & \text{if } \beta \in (4,\infty). \end{cases} \quad (5.36)$$

Then, there exists a constant $C > 0$ such that

$$\log \mathbb{E}_{\mathbb{P}^\lambda_{N,\beta}}[\exp((\mathrm{Fluct}_N(\xi))^2)] \leqslant C N \alpha_\lambda \|\nabla \xi\|^2_{L^\infty}.$$

In order to prove Theorem 5.1.1 we need to obtain sharp upper and lower bounds on the energy of a configuration in terms of its dipoles and multipoles, and good corresponding “volume estimates". A generic configuration can be quite complicated, in particular it is not obvious how to extract its dipoles. As we did in [186] we follow important ideas of [139] which consists in examining the nearest neighbor graph of a configuration and its nearest neighbor distances, denoted for each point of the configuration z_i by $\mathsf{r}_1(z_i)$. Combining this description in terms of nearest neighbor distances with the electric formulation turns out to be a powerful way to quickly obtain energy lower bounds. This is done by a ball-growth method, which consists in expanding the circular charges $\delta_{z_i}^{(\lambda)}$ into a charge $\delta_{z_i}^{(\mathsf{r}_1(z_i))}$ of same mass but supported in the disc $B(z_i, \mathsf{r}_1(z_i))$. This way the discs remain distinct and Newton's theorem applies to show the interaction energy has essentially not changed during that growth process. This however misses an order 1 in the interaction energy of each dipole.

In order to avoid this loss, we push here this method further: we examine second nearest neighbor distances $\mathrm{r}_2(z_i)$ and second nearest neighbor graphs and expand the circular charges to size r_2. In the works on the one-component plasma [184, 9, 239] we were able to take advantage of the fact that when all the charges are positive, the interaction of disc charges decreases when the radii are increased. This is no longer the case in a situation with different signs, and so instead of using monotonicity, we proceed to a direct estimation of the change in the interaction when the discs are increased. We then obtain an estimate which bounds from below the energy *in terms of nearest neighbor interactions only* and which is more precise than that of [139] or [186], as the only error remaining in the estimate corresponds to dipole-dipole interaction.

An important feature of this dipole decomposition lower bound is that it is amenable to integration in phase-space with the method of Gunson-Panta (revisited in [186]) of separating the integral over types of nearest neighbor graphs. This part contains the most delicate estimates as we need to control the contributions of the dipole-dipole interaction errors, something not handled in [139, 186] and which requires new ideas.

A matching lower bound is provided, which leverages again on the electric formulation to compute the energy as a sum of noninteracting dipoles (they are made noninteracting by solving for local electric potentials satisfying zero Neumann boundary condition). Once matching upper and lower bounds are obtained, it must follow that the Gibbs measure did concentrate on dipole configurations, as deduced in (5.22).

Plan of the paper: Section 5.2 is devoted to the proof of the energy lower bound and reduction to a dipole energy by the ball-growth method. In Section 5.3, this lower bound is inserted into the Gibbs measure to produce, via suitable decomposition of the phase-space and large deviation estimates, the free energy upper bound. Section 5.4 provides a matching lower bound by explicit construction of configurations and estimates of their free energy. Section 5.5 present a first bound on linear statistics, and Section 5.6 proves Theorem 5.1.2 on the convergence to a Poisson dipole process.

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5.2 Nearest neighbors and dipole decomposition lower bound

5.2.1 Definitions

Signed point configurations

With the shortcut Z_{2N} for (X_N, Y_N), we are able to rewrite (5.12) as

$$h_\lambda = \mathrm{g} * \left(\sum_{i=1}^{2N} d_i \delta_{z_i}^{(\lambda)} \right)$$

and thus

$$-\Delta h_\lambda = 2\pi \sum_{i=1}^{2N} d_i \delta_{z_i}^{(\lambda)}. \tag{5.37}$$

When increasing the discs we will also denote similarly for any vector $\vec{\alpha} = (\alpha_1, \dots, \alpha_{2N})$ in $\mathbb{R}^{2N}$

$$h_{\vec{\alpha}} = \mathrm{g} * \left(\sum_{i=1}^{2N} d_i \delta_{z_i}^{(\alpha_i)} \right). \tag{5.38}$$

Successive nearest neighbor distances

First we set

$$\mathsf{r}_1(z_i) := \max\left(\lambda, \frac{1}{4}\min_{j\neq i}|z_i - z_j|\right), \tag{5.39}$$

then for each $p \geqslant 2$,

$$\mathsf{r}_p(z_i) := \max\left(\lambda, \frac{1}{4}\min_{j\notin\{i,\phi_1(i),\dots\phi_{p-1}(i)\}}|z_i - z_j|\right) \tag{5.40}$$

where $\phi_0(i) = i$ and for each $k \geqslant 1$, $z_{\phi_k(i)}$ denotes some point (it is in general not unique) of the configuration $\notin \{z_i, \dots, z_{\phi_{k-1}(i)}\}$ such that $|z_i - \phi_k(i)|$ achieves the min that arises in the definition of $r_k(z_i)$. We call $z_{\phi_k(i)}$ the k-th nearest neighbor to z_i. We note that we always have

$$\max\left(\lambda, \frac{1}{4}|z_i - z_{\phi_k(i)}|\right) = \mathsf{r}_k(z_i).$$

We also denote

$$\mathcal{N}_k(i) := \{i, \phi_1(i), \dots, \phi_k(i)\} \tag{5.41}$$

the set of the k first nearest neighbors indices.

Nearest neighbor graphs

As discussed in the introduction, the dipole decomposition estimate will be used in conjunction with the method of Gunson-Panta [139] which breaks the configuration into nearest neighbor graphs. It is worth noting however that [139] builds the nearest neighbor graphs of all particles, irrespective of their sign, whereas for us the sign will play an important role.

The nearest-neighbor graph $\gamma_{2N}(Z_{2N})$ of Z_{2N} is a directed graph on $\{1, \dots, 2N\}$, with an edge from p to q if z_q is the nearest-neighbor of z_p. The graph $\gamma_{2N}(Z_{2N})$ has between 1 and N connected components and each of its connected components contains a 2-cycle with trees attached to each vertex of the 2-cycle. We denote $D_{2N,K}$ the set of nearest-neighbor graphs on $\{1, \dots, 2N\}$ with K connected components. Note that each labeling of points gives rise to a different digraph. We also let $D_{2N,K,n}$, $1 \leqslant n \leqslant K$ be the set of nearest-neighbor graphs with n neutral 2-cycles.

Let $\gamma \in D_{2N,K,n}$. Let us denote $I_1, \dots, I_K$ the connected components of γ and for each $k \in \{1, \dots, K\}$, let us label m_k and m_k' the two vertices of the 2-cycle in I_k and call $\mathcal{C}_k = \{m_k, m_k'\}$ the corresponding 2-cycle. We also let

$$I^{\mathrm{dip}} := \{\cup_k \mathcal{C}_k, C_k = \{i, \phi_1(i)\}, d_i d_{\phi_1(i)} = -1\} \tag{5.42}$$

i.e. the indices corresponding to isolated neutral dipoles (i.e. whose 2-cycle form a connected component of the graph γ).

Additional results on g_λ

Returning to (5.8), we note that

$$\mathsf{g}_1(z) = \kappa + \iint \mathsf{g}(x-y)\delta_0^{(1)}(y)\left(\delta_z^{(1)} - \delta_0^{(1)}\right)(x) = \kappa + \fint_{\partial B(0,1)} z\cdot\nu + O(|z|^2)$$
$$= \kappa + O(|z|^2) \tag{5.43}$$

as $|z| \to 0$, where we used that $\mathsf{g} * \delta_0^{(1)} = \mathsf{g} = 0$ on $\partial B(0,1)$ by Newton's theorem.

By scaling, we also have

$$\mathsf{g}_\lambda(z) = \mathsf{g}(\lambda) + \mathsf{g}_1(z/\lambda), \qquad |\mathsf{g}_\lambda(z) - \mathsf{g}(\max(|z|, \lambda))| \leqslant C \tag{5.44}$$

where C is some universal constant.

5.2.2 Dipole expansion of the energy

For $i \in \mathcal{C}_k$ for some $k = 1, \ldots, K$, we let

$$D_i = d_i + d_{\phi_1(i)}. \tag{5.45}$$

If the 2-cycle is neutral, as in the most typical case, then $D_i = 0$.

Proposition 5.2.1 (Dipole decomposition of the energy). *Let (X_N, Y_N) be any configuration in Λ^{2N} and consider its nearest neighbor graph decomposition as above with n denoting the number of neutral 2-cycles. We have*

$$\begin{aligned}
&\mathsf{F}_\lambda(X_N, Y_N) \\
&\geqslant \frac{1}{2} \sum_{k=1}^{K} \Bigg(\sum_{i \in \mathcal{C}_k} d_i d_{\phi_1(i)} \mathsf{g}_\lambda(z_i - z_{\phi_1(i)}) - d_i D_i \left(\mathsf{g}\left(\min(\mathsf{r}_2(z_i), \mathsf{r}_2(z_{\phi_1(i)})\right) + \kappa \right) - C \left(\frac{\mathsf{r}_1(z_i)}{\mathsf{r}_2(z_i)} \right)^2 \\
&\qquad\qquad - \sum_{i \in I_k \setminus \mathcal{C}_k} (\mathsf{g}(\mathsf{r}_2(z_i)) + \kappa) \Bigg)
\end{aligned} \tag{5.46}$$

where C is universal.

Here the error term corresponds to dipole-dipole interaction and is small when a dipole is well-separated from other points so that $\mathsf{r}_1 \ll \mathsf{r}_2$, which we can expect for a large proportion of the small dipoles, but not for long dipoles nor dipoles which belong to a quadrupole or more generally a multipole. The error term in $d_i D_i$ corresponds to nonneutral dipoles, they are sharp, and not problematic since nonneutral dipoles carry an excess energy which can be retrieved from the main interaction term, but has to be limited by the distance to second nearest neighbors.

We now rephrase this inequality into one that is less sharp but will be more convenient for our purposes.

Corollary 5.2.2. *For any configuration (X_N, Y_N) in Λ^{2N}, using the above notation, we have*

$$\begin{aligned}
\mathsf{F}_\lambda(X_N, Y_N) \geqslant -\frac{1}{2} &\sum_{i \notin \{\cup_k \mathcal{C}_k, d_i d_{\phi_1(i)} = 1\}} \mathsf{g}_\lambda(z_i - z_{\phi_1(i)}) \\
&- C \sum_{i \in I^{\mathrm{dip}}, \phi_2(i) \in I^{\mathrm{dip}}} \left(\frac{\mathsf{r}_1(z_i)}{\mathsf{r}_2(z_i)} \right)^2 - C(N - n),
\end{aligned} \tag{5.47}$$

where C is universal, I^{dip} is as in (5.42) and n is the number of neutral 2-cycles in the graph.

The right-hand side thus reduces the interaction to just nearest-neighbor interactions, except those of non-neutral 2-cycles (which in fact contribute positively to the energy). The error term is restricted to the connected components that consist of just an isolated neutral 2-cycle.

Proof of the corollary. First we note that if $i \in \mathcal{C}_k$ and if $d_i d_{\phi_1(i)} = -1$, we have $D_i = 0$ and the corresponding term in the sum (5.47) reduces to $-\mathsf{g}_\lambda(z_i - z_{\phi_1(i)})$. If on the other hand $d_i d_{\phi_1(i)} = 1$, then $d_i D_i = 2$ and we observe that since $\min(\mathsf{r}_2(z_i), \mathsf{r}_2(z_{\phi_1(i)})) \geqslant \mathsf{r}_1(z_i) = \mathsf{r}_1(z_{\phi_1(i)})$ and by definition (5.39) and (5.44), we have

$$\mathsf{g}_\lambda(z_i - z_{\phi_1(i)}) - 2\left(\mathsf{g}\left(\min(\mathsf{r}_2(z_i), \mathsf{r}_2(z_{\phi_1(i)}))\right) + \kappa\right) - C\left(\frac{\mathsf{r}_1(z_i)}{\mathsf{r}_2(z_i)}\right)^2 \geqslant -C'$$

for some universal constant C'. Next, for $i \in I_k \backslash \mathcal{C}_k$ we use that $\mathsf{r}_2(z_i) \geqslant \mathsf{r}_1(z_i)$ and by definition (5.39) and (5.44), $\mathsf{g}(\mathsf{r}_2(z_i)) \geqslant \mathsf{g}_\lambda(z_i - z_{\phi_1(i)}) - C$ with again C a universal constant. We deduce that

$$\mathsf{F}_\lambda(X_N, Y_N) \geqslant -\frac{1}{2} \sum_{i \notin \{i \in \cup_k \mathcal{C}_k\}} \mathsf{g}_\lambda(z_i - z_{\phi_1(i)})$$
$$-\frac{1}{2} \sum_{i \in \cup_k \mathcal{C}_k, d_i d_{\phi_1(i)} = -1} \left(\mathsf{g}_\lambda(z_i - z_{\phi_1(i)}) - C\left(\frac{\mathsf{r}_1(z_i)}{\mathsf{r}_2(z_i)}\right)^2\right) - C|\{i : d_i d_{\phi_1(i)} = 1\}| - C\sum_{k=1}^K (|I_k| - 2).$$

We then note that by definition of n (the number of neutral 2-cycles), we have

$$|\{i : d_i d_{\phi_1(i)} = 1\}| \leqslant 2(N - n)$$

while the number of points not in a 2-cycle is bounded by the number of points not in a neutral 2-cycle, which is $2(N-n)$. Since $\mathsf{r}_1/\mathsf{r}_2$ is always bounded by 1, we may absorb the error term into $N - n$ for all points belonging to a connected component of γ_{2N} which is not reduced to a 2-cycle or whose second nearest-neighbor index $\phi_2(i)$ is not in a 2-cycle. Hence the result follows. □

We now turn to the proof of Proposition 5.2.1. As explained in the introduction, the proof relies on an enlargement of the disc charges. To evaluate the change of energy, we use the following lemma.

Lemma 5.2.3. *For any $2N$-tuples $\vec{\tau}$ and $\vec{\alpha}$, we have*

$$\frac{1}{2\pi}\left(\int_{\mathbb{R}^2} |\nabla h_{\vec{\tau}}|^2 - |\nabla h_{\vec{\alpha}}|^2\right) = \sum_{i,j} \int_{\mathbb{R}^2} d_i d_j \left(\mathsf{g} * \delta_{z_i}^{(\tau_i)} - \mathsf{g} * \delta_{z_i}^{(\alpha_i)}\right)\left(\delta_{z_j}^{(\tau_j)} + \delta_{z_j}^{(\alpha_j)}\right). \tag{5.48}$$

Proof. Observe that

$$h_{\vec{\tau}} - h_{\vec{\alpha}} = 2\pi \sum_{i=1}^{2N} d_i \left(\mathsf{g} * \delta_{z_i}^{(\tau_i)} - \mathsf{g} * \delta_{z_i}^{(\alpha_i)}\right),$$

and them expand using integrations by parts and $-\Delta h_{\vec{\alpha}} = 2\pi \sum_{i=1}^{2N} d_i \delta_{z_i}^{(\alpha_i)}$. □

Proof Proposition 5.2.1. We are going to define for each point z_i in the configuration, an appropriate radius τ_i. Each index i belongs to one connected component I_k of the nearest-neighbor graph γ of the configuration. We let

$$\tau_i = \begin{cases} \min\left(\mathsf{r}_2(z_i), \mathsf{r}_2(z_{\phi_1(i)})\right) & \text{if } i \in \mathcal{C}_k \\ \mathsf{r}_2(z_i) & \text{otherwise.} \end{cases} \tag{5.49}$$

We then apply Lemma 5.2.3 and increase the balls from $\alpha_i = \lambda$ to τ_i. We obtain that

$$\int_{\mathbb{R}^2} |\nabla h_\lambda|^2 - \int_{\mathbb{R}^2} |\nabla h_{\vec{\tau}}|^2 = 2\pi \sum_{i,j} d_i d_j \left(\iint \mathsf{g}(x-y)\delta_{z_i}^{(\lambda)}(x)\delta_{z_j}^{(\lambda)}(y) - \iint \mathsf{g}(x-y)\delta_{z_i}^{(\tau_i)}(x)\delta_{z_j}^{(\tau_j)}(y) \right). \quad (5.50)$$

First, if $\tau_i = \tau_j = \lambda$, the terms in parenthesis cancel. We may thus restrict the sum to the situation where $\max(\tau_i, \tau_j) > \lambda$, which also means that $\mathsf{r}_2(z_i)$ or $\mathsf{r}_2(z_j)$ is the true (quarter of the) second neighbor distance. Next, if $B(z_i, \lambda)$ and $B(z_j, \lambda)$ intersect, so do $B(z_i, \tau_i)$ and $B(z_j, \tau_j)$ since by definition and (5.40), $\tau_i \geqslant \lambda$, $\tau_j \geqslant \lambda$. If on the other hand $B(z_i, \lambda)$ and $B(z_j, \lambda)$ are disjoint, and $B(z_i, \tau_i)$ and $B(z_j, \tau_j)$ as well, then by Newton's theorem and mean value theorem the two terms in the parenthesis are equal to $\mathsf{g}(z_i - z_j)$ hence cancel. We may thus restrict the sum to the situation where $\max(\tau_i, \tau_j) > \lambda$ and $B(z_i, \tau_i)$ and $B(z_j, \tau_j)$ intersect, that is

$$|z_i - z_j| \leqslant \tau_i + \tau_j \leqslant \mathsf{r}_2(z_i) + \mathsf{r}_2(z_j) \leqslant 2\max(\mathsf{r}_2(z_i), \mathsf{r}_2(z_j)) \quad (5.51)$$

in view of the definitions (5.49). Since $\mathsf{r}_2(z_i)$ or $\mathsf{r}_2(z_j)$ is the true (quarter of the) second neighbor distance, this implies that $j \in \mathcal{N}_1(i)$ or $i \in \mathcal{N}_1(j)$. Moreover, let us show that (5.51) implies that we have both $i \in \mathcal{N}_1(i)$ and $j \in \mathcal{N}_1(j)$. If $i = j$ this is obvious. If not, then say $j \neq i$ and $j \notin \mathcal{N}_1(i)$, this means that $j = \phi_k(i)$ with $k \geqslant 2$. In particular $z_i, z_{\phi_1(i)}$ and z_j are distinct and we thus have, by definition (5.40)

$$\mathsf{r}_2(z_i) \leqslant \frac{1}{4}|z_i - z_j|.$$

On the other hand, we know that $z_i = z_{\phi_1(j)}$, and $z_{\phi_1(i)}$ is a point distinct from z_j and $z_{\phi_1(j)}$ thus by triangle inequality and definition (5.40)

$$\mathsf{r}_2(z_j) \leqslant \frac{1}{4}|z_j - z_{\phi_1(i)}| \leqslant \frac{1}{4}|z_j - z_i| + \frac{1}{4}|z_i - z_{\phi_1(i)}| \leqslant \frac{1}{4}|z_i - z_j| + \frac{1}{4}|z_i - z_j|$$

from which it follows that

$$\mathsf{r}_2(z_i) + \mathsf{r}_2(z_j) \leqslant \frac{3}{4}|z_i - z_j|$$

a contradiction with (5.51). Thus the sum reduces to terms for which $i = j$ or i and j are both nearest neighbor to each other, which we denote by $\sim$, i.e. i and j belong to a 2-cycle of the nearest neighbor graph.

With the definition (5.8) and (5.6), we thus get from (5.50) that

$$\int_{\mathbb{R}^2} |\nabla h_\lambda|^2 \geqslant 2\pi \sum_{i=1}^{2N} (\mathsf{g}_\lambda(0) - \mathsf{g}(\tau_i) - \kappa) + 2\pi \sum_{i \neq j: j \sim i} d_i d_j \Big(\mathsf{g}_\lambda(z_i - z_j) - \iint \mathsf{g}(x-y)\delta_{z_i}^{(\tau_i)}(x)\delta_{z_j}^{(\tau_j)}(y) \Big). \quad (5.52)$$

We examine the contribution of the 2-cycles. If $i \in \mathcal{C}_k$, then $\tau_i = \tau_j$ by definition and thus by (5.44) the contribution of the parenthesis in (5.52) is

$$2\pi d_i d_j \Big(\mathsf{g}_\lambda(z_i - z_j) - \mathsf{g}(\tau_i) - \mathsf{g}_1\big(\frac{z_i - z_j}{\tau_i}\big) \Big).$$

This term appears twice due to the two edges between the vertices of the cycle and to the equality $\tau_i = \tau_j$.

We also note that $B(z_i, 4\mathsf{r}_1(z_i))$ contains at least 2 points, hence by triangle inequality, we find that if $i \sim j$, we have

$$\mathsf{r}_2(z_j) \leqslant \frac{1}{4}|z_i - z_j| + \mathsf{r}_2(z_i). \tag{5.53}$$

Reversing the roles of i and j this implies that if $i \sim j$, we have

$$|\mathsf{r}_2(z_i) - \mathsf{r}_2(z_j)| \leqslant \frac{1}{4}|z_i - z_j| \leqslant \mathsf{r}_1(z_i). \tag{5.54}$$

We deduce that $\frac{z_i - z_j}{\tau_i} = O(\frac{\mathsf{r}_1(z_i)}{\mathsf{r}_2(z_i)})$.

In view of (5.43) we may replace $\mathsf{g}_1(\frac{z_i - z_j}{\tau_i})$ by $\kappa + O\Big(\frac{|z_i - z_j|^2}{\tau_i^2}\Big)$, and then by $\kappa + O\Big(\Big(\frac{\mathsf{r}_1(z_i)}{\mathsf{r}_2(z_i)}\Big)^2\Big)$. Inserting these facts into (5.52), we obtain that

$$\int_{\mathbb{R}^2} |\nabla h_\lambda|^2 \geqslant 4\pi N \mathsf{g}_\lambda(0) - 2\pi \sum_{i=1}^{2N} (\mathsf{g}(\tau_i) + \kappa) \\ + 2\pi \sum_{i \neq j : j \sim i} d_i d_j \left(\mathsf{g}_\lambda(z_i - z_j) - \mathsf{g}(\tau_i) - \kappa + O\Big(\Big(\frac{\mathsf{r}_1(z_i)}{\mathsf{r}_2(z_i)}\Big)^2\Big) \right). \tag{5.55}$$

We may now split this over the connected components of the nearest neighbor graph I_k, and obtain by regrouping terms

$$\int_{\mathbb{R}^2} |\nabla h_\lambda|^2 \geqslant 4\pi N \mathsf{g}_\lambda(0) \\ + 2\pi \sum_{k=1}^{K} \Bigg(\sum_{i \in \mathcal{C}_k} d_i d_{\phi_1(i)} \mathsf{g}_\lambda(z_i - z_{\phi_1(i)}) - d_i \left(d_i + d_{\phi_1(i)}\right) (\mathsf{g}(\tau_i) + \kappa) + O\Big(\Big(\frac{\mathsf{r}_1(z_i)}{\mathsf{r}_2(z_i)}\Big)^2\Big) \\ - \sum_{i \in I_k \setminus \mathcal{C}_k} \mathsf{g}(\mathsf{r}_2(\tau_i)) + \kappa \Bigg)$$

In view of the definition of F_λ (5.14) we obtain the result.

□

5.3 Free energy upper bound

This section is devoted to the proof of the free energy upper bound. This will be based on the energy lower bound of Corollary 5.2.2 which reduces the interaction to nearest neighbor terms, together with a quadratic error depending on second nearest neighbor distances. The main difficulty is to partition the phase-space efficiently to integrate the exponential of this reduced energy. This is based on refinements of the Gunson-Panta change of variables and approach [139]. We will start by proving upper bounds for simpler nearest neighbor models and build up to the upper bound for the full model including the second nearest neighbors errors.

5.3.1 Preliminaries

We start by presenting some tools needed to implement the integrations.

Dirichlet integrals

We first recall a result on computing "multiple Dirichlet integrals of type 1", see [254, p. 258].

Lemma 5.3.1 (Dirichlet integrals). *Given an integer $k \geqslant 1$ and $\alpha_1, \dots, \alpha_k > 0$, let*

$$I_k(\alpha_1, \dots, \alpha_k) := \int_{(\mathbb{R}^+)^k} \mathbf{1}_{0<t_1+\ldots+t_k<1} t_1^{\alpha_1 - 1} \dots t_k^{\alpha_k - 1} \mathrm{d}t_1 \dots \mathrm{d}t_k. \tag{5.56}$$

Then

$$I_k(\alpha_1, \dots, \alpha_k) = \frac{\Gamma(\alpha_1) \dots \Gamma(\alpha_k)}{\Gamma(\alpha_1 + \ldots + \alpha_k)} \frac{1}{\alpha_1 + \ldots + \alpha_k} \tag{5.57}$$

and if the k-tuple $(\alpha_i)_{i=1}^k$ is defined by $\alpha_i = \alpha > 0$ for $1 \leqslant i \leqslant k_0$ and $\alpha_i = 1$ for $k_0 + 1 \leqslant i \leqslant k$, then

$$\log I_k = -(\alpha k_0 + (k - k_0)) \log(\alpha k + k - k_0) + O(k). \tag{5.58}$$

Proof. The identity (5.57) can be checked by successive integration by parts or using a Fourier transform argument. For the particular case stated in the lemma, we find from (5.57) that

$$I_k = \frac{\Gamma(\alpha)^{k_0}}{\Gamma(\alpha k_0 + k - k_0)} \frac{1}{\alpha k_0 + k - k_0}. \tag{5.59}$$

The relation (5.58) is then derived using Stirling's formula. □

Gunson-Panta change of variables

The Gunson-Panta method relies on considering the nearest neighbor graph of a configuration and partitioning the phase-space accordingly. We use the notation introduced already in Section 5.2, but we can apply it more generally to p points, and not necessarily an even number of points. Given a set of p points $z_1, \dots, z_p$, let $I_1, \dots, I_K$ denote the connected components of its nearest neighbor graph, and denote by m_k and m'_k the two points of the 2-cycle $\mathcal{C}_k$. We then define the Gunson-Panta change of variables via the map

$$\Phi_p^{GP}(z_1, \dots, z_p) = (u_1, \dots, u_p) \tag{5.60}$$

where

$$u_i = \begin{cases} z_i - z_{\phi_1(i)} & \text{if } i \in I_k, j \neq m'_k \\ z_i & \text{if } i = m'_k. \end{cases} \tag{5.61}$$

Number of graphs

It will be important to count the number of nearest-neighbor graph types. The number of nearest neighbor graphs on $\{1, \dots, p\}$ with K connected components is given by

$$|D_{p,K}| = \frac{2(p-1)! p^{p-2K}}{2^K (K-1)! (p-2K)!}. \tag{5.62}$$

This identity can be found for instance in [139]. One can check that

$$\log |D_{p,K}| = p \log p - K \log K + (p - 2K)(\log p - \log(p - 2K)) - K - K \log 2 + O(\log p). \tag{5.63}$$

Remark 19 (Typical number of connected components). *Assume that $z_1, \dots, z_p$ are p i.i.d variables drawn uniformly on the square $\Lambda = [0,1]^2$. Then, the number of connected components of γ_p satisfies*

$$\mathbb{E}[K] = \frac{3\pi}{8\pi + 3\sqrt{3}} p + o(p).$$

We refer to [110, Theorem 2] for a proof of this statement.

5.3.2 Upper bound for a reduced dipole model

Corollary 5.2.2 tells us that up to an error term involving ratios of nearest and second-nearest neighbor distances, one can bound from below F_λ by

$$\mathsf{F}_\lambda^{\mathrm{dip}}(Z_{2N}) := -\frac{1}{2}\sum_{i\notin\{i\in\cup_k\mathcal{C}_k, d_i d_{\phi_1(i)}=1\}} \mathsf{g}_\lambda(z_i - z_{\phi_1(i)}) - C(N-n), \tag{5.64}$$

where n stands for the number of neutral 2-cycles in the system. The energy $\mathsf{F}_\lambda^{\mathrm{dip}}$ defines a reduced dipole model that we now study.

Lemma 5.3.2 (Expansion for the reduced dipole model). *Let $\beta \in [2,+\infty)$ and C_β be the constant defined in (5.16). For $\beta > 2$, we have*

$$\log\int_{[0,\sqrt{N}]^{2N}\times[0,\sqrt{N}]^{2N}} \exp\left(-\beta \mathsf{F}_\lambda^{\mathrm{dip}}(X_N,Y_N)\right) \mathrm{d}X_N \mathrm{d}Y_N \\ \leqslant 2N\log N + N(2-\beta)\log\lambda - N + N\log C_\beta + O(\log N). \tag{5.65}$$

For $\beta = 2$, we have

$$\log\int_{[0,1]^{2N}\times[0,1]^{2N}} \exp\left(-\beta \mathsf{F}_\lambda^{\mathrm{dip}}(X_N,Y_N)\right) \mathrm{d}X_N \mathrm{d}Y_N \leqslant 2N\log N + N\log|\log\lambda| - N + O\Big(\frac{N}{|\log\lambda|}\Big). \tag{5.66}$$

Because $\beta \geqslant 2$, the free energy of a neutral dipole of size λ diverges in $\log(\lambda^{2-\beta}\mathbf{1}_{\beta>2} + |\log\lambda|\mathbf{1}_{\beta=2})$ as λ tends to 0. As a consequence, pairs of particles of opposite charges are formed and most of them are of size λ. We will see below that the the leading-order behavior of the system under $\mathbb{P}_{N,\beta}^\lambda$ is the same as under this reduced dipole model.

Proof. **Step 1: change of variables.** Let us denote

$$K_{N,\beta}^\lambda = \int_{[0,\sqrt{N}]^{2N}\times[0,\sqrt{N}]^{2N}} \exp\left(-\beta \mathsf{F}_\lambda^{\mathrm{dip}}(X_N,Y_N)\right) \mathrm{d}X_N \mathrm{d}Y_N. \tag{5.67}$$

Following [139], we expand the partition function by splitting the phase space according to the nearest-neighbor graph of the points γ_{2N}. Let γ be a graph in $D_{2N,K}$. Performing the Gunson-Panta change of variables (5.60) with $p = 2N$, and using (5.64) we may write

$$\int_{\{\gamma_{2N}=\gamma\}} \exp(-\beta \mathsf{F}_\lambda^{\mathrm{dip}})\mathrm{d}Z_{2N} \\ \leqslant e^{C\beta(N-n)} \int_{\Phi_{2N}^{GP}(\{\gamma_{2N}=\gamma\})} \prod_{k=1}^K \prod_{i\in I_k\setminus\mathcal{C}_k} \exp\Big(\frac{\beta}{2}\mathsf{g}_\lambda(u_i)\Big)\exp\Big(\beta \mathbf{1}_{\{d_{m_k}d_{m_k'}=-1\}}\mathsf{g}_\lambda(u_{m_k})\Big)\mathrm{d}U_N. \tag{5.68}$$

Indeed, if $\{m_k, m_k'\}$ is a 2-cycle, the interaction $\mathsf{g}_\lambda(u_{m_k})$ is counted twice, which explains why this term appears in the above integral with a factor β instead of $\beta/2$ in the above integral, which turns out to be crucial: since $\beta > 2$, when $\{m_k, m_k'\}$ is a 2-cycle, the term $\exp(\beta\mathsf{g}_\lambda(u_{m_k}))$ diverges. On the other hand the weight of pairs of points which are not in a dipole while converge if $\beta < 4$ and diverge if $\beta \geqslant 4$.

Step 2: integration in the case $\beta \in (2,4)$. If i is not in a 2-cycle, the weight associated with $\mathrm{g}_\lambda(u_i)$ is not divergent when $\beta < 4$, hence one can remove the cutoff λ. Using that $\mathrm{g}_\lambda \leqslant \mathrm{g} + C$ from (5.44) we find

$$\int_{\{\gamma_{2N}=\gamma\}} \exp(-\beta \mathsf{F}_\lambda^{\mathrm{dip}}) \leqslant e^{C\beta(N-n)} \\ \times \int_{\Phi(\{\gamma_{2N}=\gamma\})} \prod_{k=1}^{K} \prod_{i \in I_k \setminus \mathcal{C}_k} C \exp\Big(\frac{\beta}{2}\mathrm{g}(u_i)\Big) \exp\Big(\beta \mathbf{1}_{\{d_{m_k} d_{m'_k}=-1\}} \mathrm{g}_\lambda(u_{m_k})\Big) \mathrm{d}U_N. \tag{5.69}$$

The domain of integration is a complicated subset of $\mathbb{R}^{2N}$ but one can approximate it by a simple subset by keeping only a volume constraint. The balls $B(z_i, 2\mathrm{r}_1(z_i))$ being disjoint, one may check that

$$\Phi_{2N}^{GP}(\{\gamma_{2N}=\gamma\}) \subset D := \Big\{U_{2N} \in \mathbb{R}^{4N} : \sum_{k=1}^{K} \sum_{i \in I_k, i \neq m'_k} |u_i|^2 \leqslant \frac{N}{4\pi}, \forall k \in \{1,\dots,K\}, u_{m'_k} \in \Lambda\Big\}.$$

Using this approximation one can integrate separately the neutral dipole variables u_{m_k} and the variables u'_{m_k} on Λ and we get

$$\int_{\{\gamma_{2N}=\gamma\}} \exp(-\beta \mathsf{F}_\lambda^{\mathrm{dip}}) \leqslant e^{C\beta(N-n)} \Big(\int_{\mathbb{R}^2} \exp(\beta \mathrm{g}_\lambda(u)) \mathrm{d}u\Big)^n \\ \times C^{2(N-K)} N^K \int_{D'} \frac{1}{|u_1|^{\frac{\beta}{2}}} \cdots \frac{1}{|u_{2N-K-2n}|^{\frac{\beta}{2}}} \mathrm{d}U_{2N-K-n}, \tag{5.70}$$

where

$$D' = \Big\{U_{2N-K-n} \in (\mathbb{R}^2)^{2N-K-n} : |U_{2N-K-n}|^2 \leqslant \frac{N}{4\pi}\Big\}, \tag{5.71}$$

where $u_{2N-K-2n+1}, \dots, u_{2N-K-n}$ correspond to the non-neutral dipoles, which are not counted in the interaction.

Performing a polar change of coordinates for the first integral we recognize

$$\int_{\mathbb{R}^2} \exp(\beta \mathrm{g}_\lambda(u)) \mathrm{d}u = 2\pi \int_0^\infty \exp(\beta \mathrm{g}_\lambda(r)) r \mathrm{d}r = 2\pi \lambda^{\beta-2} \int_0^\infty \exp(\beta \mathrm{g}_1(r)) r \mathrm{d}r = \lambda^{\beta-2} C_\beta > 0 \tag{5.72}$$

by definition (5.16).

It remains to estimate the second integral in (5.70). By performing a polar change of coordinates again and the change of variables $r'_i = r_i^2$, we can rewrite this as an integral over a simplex of $\mathbb{R}^{2N-K-n}$. By scaling one obtains

$$\int_{D'} \frac{1}{|u_1|^{\frac{\beta}{2}}} \cdots \frac{1}{|u_{2N-K-2n}|^{\frac{\beta}{2}}} \mathrm{d}U_{2N-K-n} \\ \leqslant \frac{(2\pi)^{2N-K-n}}{2^{2N-K-n}} \Big(\frac{N}{4\pi(2N-K-n)}\Big)^{2N-n-K-\frac{\beta}{4}(2N-K-2n)} \int_{D''} r_1^{-\frac{\beta}{4}} \dots r_{2N-K-2n}^{-\frac{\beta}{4}} \mathrm{d}R_{2N-K-n}, \tag{5.73}$$

where

$$D'' = \{R_{2N-K-n} \in (\mathbb{R}^+)^{2N-K-n} : r_1 + \dots + r_{2N-K-n} \leqslant 2N-K-n\}.$$

Since $\beta \in (2,4)$, we can set $\alpha = 1 - \frac{\beta}{4} > 0$ and $k_0 = 2N - K - 2n$ and insert (5.58) into (5.73) to obtain, using $2N - 2K - 2n \leqslant 2(N-n)$,

$$\begin{aligned}\log \int_{D'} \prod_{i=1}^{2N-K-n} \exp\Big(\frac{\beta}{2}\mathsf{g}_\lambda(v_i)\Big) v_i \mathrm{d}v_i \\ \leqslant \Big(2N - n - K - \frac{\beta}{4}(2N-K-2n)\Big) \log\Big(\frac{2N}{2N-K-n}\Big) + c_0(N-n) \\ \leqslant (2N-K-n)\log\Big(\frac{2N}{2N-K-n}\Big) + c_0(N-n),\end{aligned} \tag{5.74}$$

for some constant c_0 depending on β. Combining with (5.70) and (5.72), this gives

$$\begin{aligned}\log \int_{\{\gamma_{2N}=\gamma\}} \exp(-\beta \mathsf{F}_\lambda^{\mathrm{dip}}) \leqslant K \log N + n(2-\beta)\log\lambda + N \log C_\beta \\ + (2N-K-n)\log\Big(\frac{2N}{2N-K-n}\Big) + c_0(N-n)\end{aligned} \tag{5.75}$$

Step 3: integration in the case $\beta = 2$. In the case $\beta = 2$, one may separate the variables $u_{m_k'}$ from the others but one should keep a volume constraint on the integral over dipole variables. Instead of (5.70), we write

$$\begin{aligned}\int_{\{\gamma_{2N}=\gamma\}} \exp(-\beta \mathsf{F}_\lambda^{\mathrm{dip}}) \leqslant e^{C\beta(N-n)} N^K \int_D \exp(\beta \mathsf{g}_\lambda(u_1)) \ldots \exp(\beta \mathsf{g}_\lambda(u_n)) \mathrm{d}U_n \\ \times C^{2(N-K)} \int_{D'} \frac{1}{|u_1|^{\frac{\beta}{2}}} \cdots \frac{1}{|u_{2N-2K-2n}|^{\frac{\beta}{2}}} \mathrm{d}U_{2N-K-n},\end{aligned} \tag{5.76}$$

where D' is as in (5.71) and D given by

$$D = \Big\{U_n \in (\mathbb{R}^2)^n : |u_1|^2 + \ldots + |u_n|^2 \leqslant \frac{N}{4\pi}\Big\}. \tag{5.77}$$

For the first integral, one may check that

$$\begin{aligned}\log \int_D \exp(\beta \mathsf{g}_\lambda(u_1)) \ldots \exp(\beta \mathsf{g}_\lambda(u_n)) \mathrm{d}U_n \leqslant n \log\Big(\int_0^{\sqrt{\frac{N}{n}}} \exp(\beta \mathsf{g}_\lambda(r))\mathrm{d}r\Big) \\ = n\log\Big(\int_0^\lambda \exp(\beta \mathsf{g}_\lambda(r))\mathrm{d}r + \int_\lambda^{\sqrt{\frac{N}{n}}} \frac{1}{r}\mathrm{d}r\Big) = n\log\Big(|\log\lambda| + \frac{1}{2}\log(\frac{N}{n}) + O(1)\Big).\end{aligned} \tag{5.78}$$

Let us emphasize that contrarily to the case $\beta > 2$, the integral of $\exp(\beta \mathsf{g}_\lambda)$ over $(0,\lambda)$ does not diverge as λ tends to 0. The second integral may be bounded as in Step 2, which gives together with (5.78),

$$\begin{aligned}\log \int_{\{\gamma_{2N}=\gamma\}} \exp(-\beta \mathsf{F}_\lambda^{\mathrm{dip}}) \leqslant K\log N + n\log|\log\lambda| + n\log\Big(|\log\lambda| + \frac{1}{2}\log(\frac{N}{n}) + O(1)\Big) \\ + (2N-K-n)\log\Big(\frac{2N}{2N-K-n}\Big) + c_0(N-n).\end{aligned}$$

Step 4: integration in the case $\beta \geqslant 4$. For $\beta > 4$ the variables with factor $-\beta/2$ in (5.70) turn to be also diverging and one instead writes

$$\int_{\{\gamma_{2N}=\gamma\}} \exp(-\beta \mathsf{F}_\lambda^{\mathrm{dip}}) \leqslant e^{C\beta(N-n)} N^K \Big(\int_{\mathbb{R}^2} \exp(\beta \mathsf{g}_\lambda(u))\mathrm{d}u\Big)^n \Big(\int_{\mathbb{R}^2} \exp\Big(\frac{\beta}{2}\mathsf{g}_\lambda(u)\Big)\mathrm{d}u\Big)^{2(N-K)}. \tag{5.79}$$

As a consequence, arguing as in (5.72)), one gets

$$\log \int_{\{\gamma_{2N}=\gamma\}} \exp(-\beta \mathsf{F}_\lambda^{\mathrm{dip}}) \leqslant K \log N + (\beta-2)n|\log\lambda| + n\log C_\beta + 2(N-K)(\frac{\beta}{2}-2)|\log\lambda| + C(N-n), \quad (5.80)$$

where the term $(N-K)\log C_{\beta/2}$ is absorbed into $N-n$. For $\beta=4$ one gets

$$\log \int_{\{\gamma_{2N}=\gamma\}} \exp(-\beta \mathsf{F}_\lambda^{\mathrm{dip}}) \leqslant K \log N + (\beta-2)n|\log\lambda| + n\log C_\beta + 2(N-K)\log|\log\lambda|\mathbf{1}_{\beta=4} + (2N-K-n)\log\Big(\frac{2N}{2N-K-n}\Big) + C(N-n), \quad (5.81)$$

Step 5: sum over graphs and optimization.

The total number of pairs of particles is $N(2N-1)$ and the number of pairs of neutral charge is N^2. It follows that

$$|D_{2N,K,n}| = |D_{2N,K}| \frac{\binom{N^2}{n}\binom{N(N-1)}{K-n}}{\binom{N(2N-1)}{K}}.$$

One can therefore check from (5.62) and (5.63) that

$$\log|D_{2N,K,n}| \leqslant \log|D_{2N,K}| - n\log 2 + O(\log N), \quad (5.82)$$

which yields using Stirling's formula

$$\log|D_{2N,K,n}| \leqslant 2N\log(2N) - K\log K + 2(N-K)(\log N - \log(N-K)) - n - 2n\log 2 + O(\log N). \quad (5.83)$$

Combining (5.75) and (5.83), we find that in the case $\beta \in (2,4)$,

$$\log \int_{\{\gamma_{2N}=\gamma\}} \exp(-\beta \mathsf{F}_\lambda^{\mathrm{dip}}) + \log|D_{N,K,n}| \leqslant n(2-\beta)\log\lambda + (2N-K-n)\log\Big(\frac{2N}{2N-K-n}\Big) + N\log C_\beta + 2N\log(2N) - K\log K - 2(N-K)\log\Big(1-\frac{K}{N}\Big) - n(1+2\log 2) + c_0(N-n) + O(\log N). \quad (5.84)$$

Using the fact the logarithm of a sum of N terms equals the logarithm of the maximum, up to an error smaller than $\log N$, we find

$$\log K_{N,\beta}^\lambda \leqslant \max_{1\leqslant n\leqslant K\leqslant N}\Big(\log \int_{\{\gamma_{2N}=\gamma^{n,K}\}} \exp(-\beta \mathsf{F}_\lambda^{\mathrm{dip}}) + \log|D_{2N,K,n}|\Big) \leqslant 2N\log(2N) + N(\log C_\beta - 1 - 2\log 2) + \max_{1\leqslant n\leqslant K\leqslant N} I(K,n) + O(\log N), \quad (5.85)$$

where

$$I(K,n) := n(2-\beta)\log\lambda + (2N-K-n)\log\Big(\frac{2N}{2N-K-n}\Big) + K\log N - K\log K - 2(N-K)\log\Big(1-\frac{K}{N}\Big) + c_0(N-n). \quad (5.86)$$

It remains to optimize $I(K,n)$ with respect to K and n under the condition that $1 \leqslant n \leqslant K \leqslant N$. Let us denote $x = \frac{K}{N}$ and $y = \frac{n}{N}$ which are such that $0 \leqslant y \leqslant x \leqslant 1$. The important point is that the terms in $\log N$ in the definition of (5.86) scale out: more precisely the term $K \log N$ coming from the integration of the roots of the K 2-cycles in Λ cancels out, at the leading order, with the term $-K\log K$ coming from the combinatorial factor (5.83). On $A := \{(x,y) \in [0,1]^2 : y \leqslant x\}$ define the function

$$\varphi_\beta(x,y) = y(2-\beta)\log\lambda + (2-x-y)\log\Big(\frac{2}{2-x-y}\Big) - x\log x - 2(1-x)\log(1-x) + c_0(1-x), \tag{5.87}$$

so that $I(K,n) = N\varphi_\beta(\frac{n}{N},\frac{K}{N})$. The term $y(2-\beta)\log\lambda$ imposes the maximum. Indeed the other terms are independent of λ and since $\log\lambda \to -\infty$ as $\lambda \to 0$ with $\beta > 2$, there exists a constant $c > 0$ depending on β such that for λ small enough, the maximum of φ_β is attained for $y \geqslant 1 - \frac{c}{|\log\lambda|}$. Fix $x \geqslant 1 - \frac{c}{|\log\lambda|}$. The function $y \mapsto \varphi_\beta(x,y)$ is increasing on $(1 - \frac{c}{|\log\lambda|}, 1)$ therefore

$$\sup_{(x,y)\in A} \varphi_\beta(x,y) = \sup_{x > 1 - \frac{c}{|\log\lambda|}} \varphi(x,x) = \varphi_\beta(1,1). \tag{5.88}$$

It follows that for λ small enough,

$$\max_{1\leqslant n\leqslant K\leqslant N} I(K,n) = I(N,N). \tag{5.89}$$

Inserting this into (5.85) we find that for $\beta \in (2,4)$.

$$\log K^\lambda_{N,\beta} \leqslant 2N\log N + (2-\beta)N\log\lambda + N\log C_\beta - N + O(\log N), \tag{5.90}$$

which concludes the proof of the lemma in the case $\beta \in (2,4)$. In the case $\beta = 2$, (5.85) is replaced by

$$\log K^\lambda_{N,\beta} \leqslant 2N\log(2N) - N(1+2\log 2) + \max_{0\leqslant n\leqslant K\leqslant N} I(K,n) + O\Big(\frac{N}{|\log\lambda|}\Big),$$

where

$$\begin{aligned} I(K,n) := n\log\log|\lambda| + n(\frac{\beta}{2}-1)\log N - (1-\frac{\beta}{4})(2N-2K)\log\Big((1-\frac{\beta}{4})2(N-K) + K - n\Big) \\ - K\log K - 2(N-K)\log\Big(1-\frac{K}{N}\Big) + c_0(N-n) = N\varphi_\beta(\frac{n}{N},\frac{K}{N}), \end{aligned} \tag{5.91}$$

where φ_β is defined by

$$\begin{aligned} \varphi_\beta : (x,y) \in A \mapsto y\log|\log\lambda| + y(\frac{\beta}{2}-1)\log N - (2-\frac{\beta}{2})(1-x)\log\Big(N\Big((2-\frac{\beta}{2})(1-x)+x-y\Big)\Big) \\ - x\log(Nx) - 2(1-x)\log(1-x) - y(1+2\log 2) + c_0(1-y), \end{aligned}$$

Arguing exactly like in the case $\beta \in (2,4)$, one may check that for λ small enough, ϕ_β is maximal for $x = y = 1$, which proves (5.66). Now assume that $\beta \geqslant 4$. In view of (5.80), the expression (5.85) is then replaced by

$$\log K^\lambda_{N,\beta} \leqslant \log K^\lambda_{N,\beta} \leqslant 2N\log(2N) + N\log(C_\beta - 1 - 2\log 2) + \max_{1\leqslant n\leqslant K\leqslant N} I(K,n) + O(\log N),$$

where

$$I(K,n) = n(2-\beta)\log\lambda + 2(N-K)\Big((\frac{\beta}{2}-2)|\log\lambda|\mathbf{1}_{\beta>4} + \log|\log\lambda|\mathbf{1}_{\beta=4}\Big) - K\log K \\ - 2(N-K)\log(1-\frac{K}{N}) + c_0(N-n), \quad (5.92)$$

leading to optimize the function

$$\varphi_\beta(x,y) = y(2-\beta)\log\lambda + 2(1-x)\Big((\frac{\beta}{2}-2)(\log\lambda)\mathbf{1}_{\beta>4} + \log|\log\lambda|\Big) \\ - 2(1-x)\log(1-x) + c_0(1-y). \quad (5.93)$$

One can check that for λ small enough, ϕ_β attains its maximum at $x=y=1$, thus showing that (5.90) also holds for $\beta \geqslant 4$.

Step 6: optimization under constraint.
For the rest of the paper, it will be useful to optimize $I(K,n)$ under constraint. In order to upper bound the probability of having less than n_0 neutral dipoles, we will need to consider the maximum of $I(K,n)$ under the constraint $n \leqslant n_0$ and $K \geqslant n$. Let us compute the maximum of φ_β over the event $A = \{(x,y) \in [0,1]^2 : y \leqslant \frac{n_0}{N}, x \geqslant y\}$. Proceeding as in Step 5, one can see that λ small enough,

$$\sup_{(x,y)\in A} \varphi_\beta(x,y) \leqslant \varphi_\beta(\frac{n_0}{N}, \frac{n_0}{N}). \quad (5.94)$$

As a consequence for λ small enough,

$$\max_{n\leqslant n_0, K\geqslant n} I(K,n) \leqslant I(n_0,n_0,N,N) \leqslant n_0((2-\beta)\log\lambda\mathbf{1}_{\beta>2} + \log|\log\lambda|\mathbf{1}_{\beta=2}) + C(N-n_0). \quad (5.95)$$

□

5.3.3 Upper bound on the energy of p points for a nearest-neighbor model

We now study a new nearest-neighbor model which will be useful for evaluating the contributions of the dipole-dipole interaction or quadrupole errors that appear in (5.47). We consider an integral of p variables $z_1,\dots,z_p$ living on Λ, where the nearest neighbor interaction energy is counted only for a small subgroup of k points $z_1,\dots,z_k$. In practice, this result will be applied to $p \leqslant cN\lambda^\alpha$ for some $\alpha > 0$. Let us emphasize that the computations differ significantly from those of Lemma 5.3.2 since the probability that both points of a 2-cycle in the nearest neighbor graph of $z_1,\dots,z_p$ belong to the subset $\{z_1,\dots,z_k\}$ is small.

Lemma 5.3.3. *Let $k \leqslant p$ and let us consider the energy over $[0,1]^{2p}$ defined by*

$$F_k(z_1,\dots,z_p) := \frac{1}{2}\sum_{i=1}^{k}\log\left(\mathsf{r}_1(z_i)\right) \quad (5.96)$$

with r_1 defined by

$$\mathsf{r}_1(z_i) = \min\Big(\frac{\lambda}{\sqrt{p}}, \frac{1}{4}\min_{j\neq i}|z_i - z_j|\Big). \quad (5.97)$$

Then

$$\log\Big(\int_{\Lambda^p}\exp\left(-\beta F_k(z_1,\dots,z_p)\right)\mathrm{d}Z_p\Big)$$
$$\leqslant \frac{\beta}{4}k\log p + Ck + \begin{cases} 0 & \textit{if } \beta<2\\ \frac{k}{2}\log\Big(1+|\log\lambda|\frac{k}{p}\Big) & \textit{if } \beta=2\\ \frac{k}{2}\log\Big(1+\lambda^{2-\beta}\frac{k}{p}\Big) & \textit{if } \beta\in(2,4)\\ \frac{k}{2}\log\Big(\lambda^{-2}\frac{k}{p}\vee|\log\lambda|^2\Big) & \textit{if } \beta=4\\ \frac{k}{2}\log\Big(\lambda^{2-\beta}\frac{k}{p}\vee\lambda^{4-\beta}\Big) & \textit{if } \beta>4.\end{cases} \quad (5.98)$$

Additionally there also holds

$$\log\Big(\int_{[0,1]^{2p}}\prod_{i=1}^k\frac{|\log\frac{\mathsf{r}_1(z_i)}{\lambda}|}{\mathsf{r}_1(z_i)^2}\mathrm{d}Z_p\Big)\leqslant k\log p+Ck+\frac{k}{2}\log\Big(\frac{1}{\lambda^2}\frac{k}{p}\vee|\log\lambda|^4\Big). \quad (5.99)$$

Proof. Let $C_0>1$. Let us consider the integral

$$\int_{[0,1]^{2p}}\mathbf{1}_{|z_1|^2+\ldots+|z_k|^2\leqslant C_0\frac{k}{p}}\exp(-\beta F_k(z_1,\dots,z_p))\mathrm{d}Z_p. \quad (5.100)$$

Let $l\in\{0,\dots,k\}$ be the number of points in $\{1,\dots,k\}$ with nearest-neighbor in $\{1,\dots,k\}$. There are $\binom{k}{l}$ ways of choosing such points. Assume that these points correspond to $z_1,\dots,z_l$. There remains to choose the nearest-neighbors of the points $z_{l+1},\dots,z_k$ among $z_{k+1},\dots,z_p$. The difficulty is that some of the points $z_{l+1},\dots,z_k$ might share the same nearest-neighbor. Let us denote A_l the event

$$A_l=\{(z_1,\dots,z_p)\in[0,1]^{2p}:\phi_1(z_1),\dots,\phi_1(z_l)\in\{1,\dots,l\},\phi_1(z_{l+1})=z_{k+1},\dots,\phi_1(z_k)=z_{2k-l}\}. \quad (5.101)$$

Given a partition $n_1+\ldots+n_m=k-l$ with $n_1,\dots,n_m\geqslant 1$ and $m\geqslant 1$, one shall choose m subsets of $\{1,\dots,p\}$ of respective cardinal $n_1,\dots,n_m$ and match all subset to a certain unique element of $\{k+1,\dots,n\}$. The number of choices is then equal to

$$\binom{k-l}{n_1}\binom{k-l-n_1}{n_2}\cdots\binom{n_{m-1}+n_m}{n_{m-1}}(p-k)(p-1)\ldots(p-k-m+1). \quad (5.102)$$

Let us remark that the number of groups m is bounded from below by $\frac{k-l}{6}$ since a single point can be the nearest-neighbor of at most 6 distinct points. Assume that these common nearest-neighbors are given by $z_{k+1},\dots,z_{k+m}$ $(m\leqslant k-l)$. For $1\leqslant m\leqslant\frac{k-l}{6}$ and $n:=(n_1,\dots,n_m)$ such that $n_1+\ldots+n_m=k-l$, let us denote $A_{l,m,n}$ the event

$$A_{l,m,n}=A_l\cap\{(z_1,\dots,z_p)\in[0,1]^{2p}:\phi_1(z_{l+1})=\ldots=\phi_1(z_{l+n_1})=k+1,\ldots$$
$$\phi_1(z_{k-n_m+1})=\ldots=\phi_1(z_k)=k+m\}. \quad (5.103)$$

One can see that

$$\int_{[0,1]^{2p}}\mathbf{1}_{|z_1|^2+\ldots+|z_k|^2\leqslant C_0\frac{k}{p}}\exp(-\beta F_k(z_1,\dots,z_p))\mathrm{d}Z_p$$
$$\leqslant\sum_{l=0}^k\binom{k}{l}\sum_{m,n_1+\ldots+n_m=k-l}\binom{k-l}{n_1}\binom{k-l-n_1}{n_2}\cdots\binom{n_{m-1}+n_m}{n_{m-1}}(p-k)(p-1)\ldots(p-k-m+1)$$
$$\times\int_{A_{l,m,n}\cap\{|z_1|^2+\ldots+|z_k|^2\leqslant C_0\frac{k}{p}\}}\exp(-\beta F_k(z_1,\dots,z_p))\mathrm{d}z_1\ldots\mathrm{d}z_p. \quad (5.104)$$

Assume that there are k_0 2-cycles in $\{1,\dots,l\}$, $k_0=1,\dots,\lfloor l/2\rfloor$. There is a number $\frac{l!}{(l-2k_0)!k_0!}2^{-k_0}$ of ways of choosing k_0 pairs among $\{1,\dots,l\}$. In addition, for each point in $\{1,\dots,l\}$ not in one of these 2-cycles, there are less than $(k-l)^{l-2k_0}$ number of choices for the nearest-neighbor. One may assume without loss of generality that $z_1,\dots,z_{2k_0}$ are the points belonging to the 2-cycles. Let $A^{k_0}_{l,m,n}\subset A_{l,m,n}$ be the set of points with a nearest-neighbor labelling satisfying the above constraints.

By integrating out the variables $z_{k+m+1},\dots,z_p$ on a subset of volume $1-C_0\frac{k}{p}$, we find that

$$\begin{aligned}&\int_{A_{l,m,n}\cap\{|z_1|^2+\ldots+|z_{k+m}|^2\leqslant C_0\frac{k}{p}\}}\exp(-\beta F_k(z_1,\dots,z_p))\mathrm{d}z_1\dots\mathrm{d}z_p\leqslant C^k\Big(1-C_0\frac{k}{p}\Big)^{p-k-m}\\&\times\sum_{k_0=0}^{\lfloor l/2\rfloor}\frac{l!(k-l)^{l-2k_0}}{(l-2k_0)!k_0!2^{k_0}}\times\int_{A^{k_0}_{l,m,n}\cap\{|z_1|^2+\ldots+|z_{k+m}|^2\leqslant C_0\frac{k}{p}\}}\exp(-\beta F_k(z_1,\dots,z_{k+m}))\mathrm{d}z_1\dots\mathrm{d}z_{k+m}.\end{aligned}\tag{5.105}$$

Note that although our way of counting these functional digraphs is very rough, it is precise enough for the optimization over k_0 (note that for $k=p$ it is not that different from formula (5.62)). The important point is that, when scaling out k_0 as $k_0=lx$, the terms in $l\log(l)$ in the combinatorial factors and in the integral cancel out.

Let us decompose $F_k(z_1,\dots,z_{k+m})$ into

$$F_k(z_1,\dots,z_{k+m})=\frac{1}{2}\sum_{i=1}^{2k_0}\log(\mathsf{r}_1(z_i))+\frac{1}{2}\sum_{i=2k_0+1}^{k}\log(\mathsf{r}_1(z_i)).\tag{5.106}$$

By construction, on the event $A^{k_0}_{l,m,n}$, the first term of (5.106) depends only on $z_1,\dots,z_{2k_0}$. One may therefore integrate the second term of (5.106) with respect to $z_{2k_0+1},\dots,z_{m+k}$. By performing a change of variables in $z_i-z_{\phi_1(i)}$ for $2k_0+1\leqslant i\leqslant k$, one may reduce the integral to a multiple Dirichlet integral as was done in the proof of Lemma 5.3.2. We find

$$\begin{aligned}&\int_{A^{k_0}_{l,m,n}\cap\{|z_1|^2+\ldots+|z_k|^2\leqslant C_0\frac{k}{p}\}}\exp\Big(-\frac{1}{2}\sum_{i=2k_0+1}^{k}\log(\mathsf{r}_1(z_i))\Big)\mathrm{d}z_{2k_0+1}\dots\mathrm{d}z_{k+m}\\&\leqslant C^k\left(\Big(\sqrt{\frac{C_0k}{p(k-2k_0)}}\Big)^{(2-\frac{\beta}{2})(k-2k_0)}\mathbf{1}_{\beta\in[2,4)}+\log\Big(\frac{\sqrt{\frac{C_0k}{k-2k_0}}+1}{\lambda}\Big)^{k-2k_0}\mathbf{1}_{\beta=4}\right.\\&\left.\qquad+\Big(\frac{\lambda}{\sqrt{p}}\Big)^{(2-\frac{\beta}{2})(k-2k_0)}\mathbf{1}_{\beta>4}\right),\end{aligned}\tag{5.107}$$

and the bound is independent of m and n. In addition it is easy to check that

$$\begin{aligned}&\int_{A^{k_0}_{l,m,n}\cap\{|z_1|^2+\ldots+|z_k|^2\leqslant C_0\frac{k}{p}\}}\exp\Big(-\frac{1}{2}\sum_{i=1}^{2k_0}\log(\mathsf{r}_1(z_i))\Big)\mathrm{d}z_1\dots\mathrm{d}z_{k+m}\\&\qquad\leqslant C^k\left(\Big(\frac{\lambda}{\sqrt{p}}\Big)^{(2-\beta)k_0}\mathbf{1}_{\beta>2}+\log\Big(\frac{\sqrt{C_0\frac{k}{k_0}}+1}{\lambda}\Big)^{k_0}\mathbf{1}_{\beta=2}\right).\end{aligned}$$

Putting the two last displays together, one obtains

$$\int_{A^{k_0}_{l,m,n}\cap\{|z_1|^2+\ldots+|z_k|^2\leqslant C_0\frac{k}{p}\}}\exp(-\beta F_k(z_1,\ldots,z_{k+m}))\mathrm{d}z_1\ldots\mathrm{d}z_{k+m}$$
$$\leqslant C^k\frac{l!}{(l-2k_0)!k_0!}(k-l)^{l-2k_0}\left(\Big(\frac{\lambda}{\sqrt{p}}\Big)^{(2-\beta)k_0}\mathbf{1}_{\beta>2}+\log\Big(\frac{\sqrt{\frac{C_0k}{k_0}}+1}{\lambda}\Big)^{k_0}\mathbf{1}_{\beta=2}\right)$$
$$\times\left(\Big(\sqrt{\frac{C_0k}{p(k-2k_0)}}\Big)^{(2-\frac{\beta}{2})(k-2k_0)}\mathbf{1}_{\beta\in(2,4)}+\log\Big(\frac{\sqrt{\frac{C_0k}{k-2k_0}}+1}{\lambda}\Big)^{k-2k_0}\mathbf{1}_{\beta=4}+\Big(\frac{\lambda}{\sqrt{p}}\Big)^{(2-\frac{\beta}{2})(k-2k_0)}\mathbf{1}_{\beta>4}\right).$$

Let us optimize the above function with respect to $k_0\in\{0,\ldots,\lfloor l/2\rfloor\}$. Assume $\beta\in(2,4)$. Using that $(\log(k-2k_0)-\log(k))(k-2k_0)=O(k)$ and $2-\frac{\beta}{2}\leqslant 1$, we have

$$\begin{aligned}
\log\int_{A^{k_0}_{l,m,n}\cap\{|z_1|^2+\ldots+|z_k|^2\leqslant C_0\frac{k}{p}\}}&\exp(-\beta F_k(z_1,\ldots,z_{k+m}))\mathrm{d}z_1\ldots\mathrm{d}z_{k+m}\\
&\leqslant l\log l-(l-2k_0)\log(l)-k_0\log l+(l-2k_0)\log l+(2-\beta)k_0|\log\lambda|\\
&\quad-((1-\frac{\beta}{2})k-k_0)\log p+k\log(C_0)+Ck\\
&=l\log l-(1-\frac{\beta}{2}k)\log p-((1-\frac{\beta}{2}k-k_0)\log(\frac{p}{l})+(2-\beta)k_0|\log\lambda|+k\log(C_0)+Ck.
\end{aligned}\tag{5.108}$$

We then argue that the maximum of the above function with respect to k_0 is attained for $k_0=l$ if $\frac{p}{l}\lambda^{\beta-2}$ is small enough and for $k_0=0$ if $\frac{p}{l}\lambda^{\beta-2}$ is large enough. When $\beta\in(2,4)$, we deduce that (5.105) is bounded by

$$\begin{aligned}
\log\int_{A_{l,m,n}\cap\{|z_1|^2+\ldots+|z_k|^2\leqslant C_0\frac{k}{p}\}}&\exp(-\beta F_k(z_1,\ldots,z_p))\mathrm{d}z_1\ldots\mathrm{d}z_p\\
&\leqslant C^k\Big(1-C_0\frac{k}{p}\Big)^{p-k-m}l^{l/2}C_0^k\begin{cases}(\frac{\lambda}{\sqrt{p}})^{(2-\beta)\frac{l}{2}}\Big(\sqrt{\frac{k}{p(k-l)}}\Big)^{(2-\frac{\beta}{2})(k-l)} & \text{if } \frac{p}{l}\lambda^{\beta-2}\leqslant 1\\ (k-l)^l(\frac{1}{\sqrt{p}})^{(2-\frac{\beta}{2})k} & \text{if } \frac{p}{l}\lambda^{\beta-2}\geqslant 1.\end{cases}
\end{aligned}\tag{5.109}$$

Similar computations show that for $\beta=2$,

$$\begin{aligned}
\log\int_{A_{l,m,n}\cap\{|z_1|^2+\ldots+|z_k|^2\leqslant C_0\frac{k}{p}\}}&\exp(-\beta F_k(z_1,\ldots,z_p))\mathrm{d}z_1\ldots\mathrm{d}z_p\\
&\leqslant C^k\Big(1-C_0\frac{k}{p}\Big)^{p-k-m}C_0^k\begin{cases}l^{l/2}|\log\lambda|^{\frac{l}{2}}\Big(\sqrt{\frac{k}{p(k-l)}}\Big)^{(2-\frac{\beta}{2})(k-\frac{l}{2})} & \text{if } \frac{p}{l}\frac{1}{|\log\lambda|}\leqslant 1\\ (k-l)^l(\frac{1}{\sqrt{p}})^{(2-\frac{\beta}{2})k} & \text{if } \frac{p}{l}\frac{1}{|\log\lambda|}\geqslant 1,\end{cases}
\end{aligned}\tag{5.110}$$

for $\beta=4$,

$$\begin{aligned}
\log\int_{A_{l,m,n}\cap\{|z_1|^2+\ldots+|z_k|^2\leqslant C_0\frac{k}{p}\}}&\exp(-\beta F_k(z_1,\ldots,z_p))\mathrm{d}z_1\ldots\mathrm{d}z_p\\
&\leqslant C^k\Big(1-C_0\frac{k}{p}\Big)^{p-k-m}C_0^k\begin{cases}l^{l/2}|\log\lambda|^{\frac{l}{2}}\Big(\sqrt{\frac{k}{p(k-l)}}\Big)^{(2-\frac{\beta}{2})(k-\frac{l}{2})} & \text{if } \frac{p}{l}\frac{\lambda^2}{|\log\lambda|^2}\leqslant 1\\ \Big(1-C_0\frac{k}{p}\Big)^{p-k-m}(k-l)^l|\log\lambda|^{(2-\frac{\beta}{2})k} & \text{if } \frac{p}{l}\frac{\lambda^2}{|\log\lambda|^2}\geqslant 1,\end{cases}
\end{aligned}\tag{5.111}$$

and for $\beta > 4$,

$$\log \int_{A_{l,m,n}\cap\{|z_1|^2+\ldots+|z_k|^2\leqslant C_0\frac{k}{p}\}} \exp(-\beta F_k(z_1,\ldots,z_p))\mathrm{d}z_1\ldots\mathrm{d}z_p$$
$$\leqslant C^k\Big(1-C_0\frac{k}{p}\Big)^{p-k-m}\begin{cases} l^{l/2}(\frac{\lambda}{\sqrt{p}})^{\frac{l}{2}}\Big(\sqrt{\frac{k}{p(2k-l)}}\Big)^{(2-\frac{\beta}{2})(k-\frac{l}{2})} & \text{if } \frac{p}{l}\lambda^2\leqslant 1\\ (\frac{\lambda}{\sqrt{p}})^{(2-\frac{\beta}{2})k} & \text{if } \frac{p}{l}\lambda^2\geqslant 1.\end{cases} \tag{5.112}$$

Notice that there exists a constant $C>0$ such that

$$\sup_{C_0>1}\Big(1-\frac{C_0k}{p}\Big)^{p-k-m}C_0^k\leqslant C^k. \tag{5.113}$$

Using (5.113) and the fact that

$$\sum_{m,n_1+\ldots+n_m=k-l}\binom{k-l}{n_1}\binom{k-l-n_1}{n_2}\cdots\binom{n_{m-1}+n_m}{n_{m-1}}(p-k)(p-1)\ldots(p-k-m+1)$$
$$=(p-k)^{k-l}, \tag{5.114}$$

one finds that if each of the first subcases of (5.109), (5.110), (5.111) and (5.112), there holds

$$\int_{[0,1]^{2p}}\exp(-\beta F_k(z_1,\ldots,z_p))\mathrm{d}Z_p\leqslant C^k\sum_{l=0}^k\binom{k}{l}p^{k-l}\Big(\frac{\lambda}{\sqrt{p}}\Big)^{(2-\beta)\frac{l}{2}}\Big(\mathbf{1}_{\beta>2}+|\log\lambda|^{\frac{l}{2}}\mathbf{1}_{\beta=2}\Big)l^{l/2}$$
$$\times\Bigg(\Big(\sqrt{\frac{k}{p(k-2l)}}\Big)^{(2-\frac{\beta}{2})(k-\frac{l}{2})}\mathbf{1}_{\beta\in(2,4)}+|\log\lambda|^{k-\frac{l}{2}}\mathbf{1}_{\beta=4}+\Big(\frac{\lambda}{\sqrt{p}}\Big)^{(2-\frac{\beta}{2})(k-\frac{l}{2})}\mathbf{1}_{\beta>4}\Bigg)$$
$$\leqslant C^kp^{\frac{\beta}{4}k}\sum_{l=0}^k\binom{k}{l}\Big(\frac{l}{p}\Big)^{\frac{l}{2}}(\lambda^{(1-\frac{\beta}{2})l}\mathbf{1}_{\beta>2}+|\log\lambda|^{l/2}\mathbf{1}_{\beta=2}\Big)\Big(\frac{k}{k-l}\Big)^{(1-\frac{\beta}{4})(k-l)}$$
$$\times\Big(\mathbf{1}_{\beta<4}+|\log\lambda|^{k-l}\mathbf{1}_{\beta=4}+\lambda^{(2-\frac{\beta}{2})(k-l)}\mathbf{1}_{\beta>4}\Big). \tag{5.115}$$

Otherwise in all of the second subcases of (5.109), (5.110), (5.111) and (5.112), the logarithm of the left-hand side of (5.115) is bounded by (5.98). It remains to optimize (5.115) over l. Let us write $l=kx$ and consider the function

$$\phi_\beta : x\in(0,1)\mapsto -x\log(x)-(1-x)\log(1-x)+\frac{x}{2}\log(\lambda^{2-\beta})+\frac{x}{2}\log(x\frac{k}{p})-(1-\frac{\beta}{4})(1-x)\log(1-x)$$
$$=-\frac{x}{2}\log(x)+\frac{\beta}{4}(1-x)\log(1-x)+\frac{x}{2}\log(\lambda^{2-\beta}\frac{k}{p}).$$

Notice that there exists a constant $C>0$ such that for all $x\in(0,1)$

$$|\phi_\beta(x)|\leqslant C+\frac{x}{2}\log(\lambda^{2-\beta}\frac{k}{p})\leqslant C+\frac{1}{2}\log(\lambda^{2-\beta}\frac{k}{p})\mathbf{1}_{\lambda^{2-\beta}\frac{k}{p}\geqslant 1}.$$

Inserting this into (5.115) shows that for $\beta\in(2,4)$

$$\sum_{l=0}^k\binom{k}{l}p^{k-l}\Big(\frac{1}{\sqrt{p}}\Big)^{(2-\frac{\beta}{2})(k-l)}\Big(\frac{\lambda}{\sqrt{p}}\Big)^{(2-\beta)\frac{l}{2}}l^{l/2}\leqslant C^k\Big(1+\lambda^{2-\beta}\frac{k}{p}\Big)^{k/2}.$$

If $\beta \geqslant 4$, the bound (5.115) may be written

$$\log \sum_{l=0}^{k} \binom{k}{l} p^{k-l} \Big(\frac{l}{p}\Big)^{\frac{l}{2}} \lambda^{-l} (\mathbf{1}_{\beta>4} + |\log \lambda|^{k-l} \mathbf{1}_{\beta=4}) \leqslant Ck + \sup_{x\in(0,1)} \phi_\beta(x),$$

where

$$\phi_\beta : x \in (0,1) \mapsto -x\log(x) - (1-x)\log(1-x) + \frac{x}{2}\log(\lambda^{2-\beta}) + \frac{x}{2}\log(x\frac{k}{p}) \\ + (1-x)(\log(\lambda^{2-\frac{\beta}{2}})\mathbf{1}_{\beta>4} + \log|\log\lambda|\mathbf{1}_{\beta=4}).$$

Again notice

$$|\phi_\beta(x)| \leqslant C + \frac{x}{2}\log\Big(\frac{k}{p}\lambda^{2-\beta}\Big) + (1-x)(\log(\lambda^{2-\frac{\beta}{2}})\mathbf{1}_{\beta>4} + \log|\log\lambda|\mathbf{1}_{\beta=4}).$$

The above function being linear in x, one can write

$$\sup|\phi_\beta| \leqslant C + \max\Big(\frac{1}{2}\log(\frac{k}{p}\lambda^{2-\beta}), \log(\lambda^{2-\frac{\beta}{2}})\mathbf{1}_{\beta>4} + \log|\log\lambda|\mathbf{1}_{\beta=4}\Big),$$

which coincides with (5.98). The proof of (5.99) is similar. □

Building on the last lemma, we treat the quadratic error terms arising from the energy lower bound of Corollary 5.2.2 when the nearest-neighbor graph consists only of isolated neutral dipoles.

Lemma 5.3.4. *Let $f(x) = -t|x|^2$ for some $t \geqslant 0$. Let γ be a nearest neighbor graph with p isolated neutral 2-cycle components. For every z_i, let $\mathsf{r}_1'(z_i)$ denote its nearest neighbor of same sign, i.e.*

$$\mathsf{r}_1'(z_i) := \max\Big(\lambda, \frac{1}{4}\min_{j\neq i, d_i d_j = 1}|z_i - z_j|\Big).$$

For any $\beta > 2$ we have

$$\log\int_{\gamma_{2p}=\gamma} \exp\Big(-\beta\Big(\mathsf{F}_\lambda^{\mathrm{dip}}(X_p,Y_p) + \sum_{i\in I^{\mathrm{dip}}, d_i=1} f\Big(\frac{\mathsf{r}_1(z_i)}{\mathsf{r}_1'(z_i)}\Big)\Big)\Big)\mathrm{d}X_p\mathrm{d}Y_p \\ \leqslant p\left(\log N + (2-\beta)\log\lambda\right) + \log C_\beta + C(\lambda^{\beta-2}\mathbf{1}_{\beta<4} + \lambda^2|\log\lambda|^2\mathbf{1}_{\beta=4} + \lambda^2|\log\lambda|\mathbf{1}_{\beta>4})\Big). \tag{5.116}$$

For $\beta = 2$ we have

$$\log\int_{\gamma_{2p}=\gamma} \exp\Big(-\beta\Big(\mathsf{F}_\lambda^{\mathrm{dip}}(X_p,Y_p) + \sum_{i\in I^{\mathrm{dip}}, d_i=1} f\Big(\frac{\mathsf{r}_1(z_i)}{\mathsf{r}_1'(z_i)}\Big)\Big)\Big)\mathrm{d}X_p\mathrm{d}Y_p \\ \leqslant p\Big(\log N + \log|\log\lambda| + \frac{C}{|\log\lambda|}\Big). \tag{5.117}$$

Proof. First, without loss of generality, we may assume that for each i, the nearest neighbor to x_i is y_i. Let us first make the change of variables

$$(X_p, Y_p) \mapsto (X_p, W_p := X_p - Y_p).$$

It is a valid change of variables on the set $\{\gamma_{2p} = \gamma\}$. We note that in the setting we are in,

$$\mathsf{r}_1(z_i) = \max(\lambda, \frac{1}{4}|w_i|) \leqslant \mathsf{r}_1'(z_i),$$

and so we may then rewrite

$$\int_{\gamma_{2p}=\gamma} \exp\Big(-\beta\Big(\mathsf{F}_\lambda^{\mathrm{dip}}(X_p,Y_p) + \sum_{i\in I^{\mathrm{dip}}, d_i=1} f\Big(\frac{\mathsf{r}_1(z_i)}{\mathsf{r}_1'(z_i)}\Big)\Big)\Big) \mathrm{d}X_p \mathrm{d}Y_p$$
$$= \int_{\gamma_{2p}=\gamma} \exp\left(\beta \sum_{i=1}^p \mathsf{g}_\lambda(w_i) - \beta\sum_{i=1}^p f\Big(\frac{\max(\frac14|w_i|,\lambda)}{\mathsf{r}_1'(x_i)}\Big)\right)\mathrm{d}X_p\mathrm{d}W_p. \quad (5.118)$$

Let us integrate over W_p first, and simplify the domain of integration by including it in the set $\{|w_i| \leqslant \mathsf{r}_1'(x_i)\}$. Using polar coordinates, (5.44), performing the change of variables $|w_i| = \lambda|w_i'|$, and recalling that by definition $\mathsf{r}_1'(x_i) \geqslant \lambda$ we may then bound each integral over w_i by

$$2\pi\lambda^{2-\beta}\int_0^{\mathsf{r}_1'(x_i)/\lambda} r\exp\left(\beta\mathsf{g}_1(r) - \beta f\Big(\frac{\lambda\max(\frac14 r,1)}{\mathsf{r}_1'(x_i)}\Big)\right)\mathrm{d}r$$
$$\leqslant 2\pi\lambda^{2-\beta}\int_0^{\mathsf{r}_1'(x_i)/\lambda} r\exp\beta\mathsf{g}_1(r)\left(1 - C\beta f\Big(\frac{\lambda\max(\frac14 r,1)}{\mathsf{r}_1'(x_i)}\Big)\right)\mathrm{d}r$$
$$\leqslant \lambda^{2-\beta}\left(C_\beta + C\beta t\int_0^1 r\Big(\frac{\lambda}{\mathsf{r}_1'(x_i)}\Big)^2\mathrm{d}r + C\beta t\int_1^{\mathsf{r}_1'(x_i)/\lambda} r^{1-\beta}\Big(\frac{\lambda r}{\mathsf{r}_1'(x_i)}\Big)^2\mathrm{d}r\right) \quad (5.119)$$

Note that here we have used the definition of C_β in (5.16) and the fact that g_1 is bounded in the unit ball. We then find, using $\mathsf{r}_1'(x_i) \geqslant \lambda$,

$$2\pi\lambda^{2-\beta}\int_0^{\mathsf{r}_1'(x_i)/\lambda} r\exp\left(\beta\mathsf{g}_1(r) - \beta f\Big(\frac{\lambda\max(\frac14 r,1)}{\mathsf{r}_1'(x_i)}\Big)\right)\mathrm{d}r$$
$$\leqslant \lambda^{2-\beta}\left(C_\beta + C\beta t\frac{\lambda^2}{\mathsf{r}_1'(x_i)^2} + C\beta t\lambda^{\beta-2}\mathsf{r}_1'(x_i)^{2-\beta} + C\beta t\frac{\lambda^2}{\mathsf{r}_1'(x_i)^2}\left(\log\frac{\mathsf{r}_1'(x_i)}{\lambda}\right)\mathbf{1}_{\beta=4}\right)$$
$$\leqslant \lambda^{2-\beta}C_\beta\left(1 + CC_\beta^{-1}\beta t\left(\left(\frac{\lambda}{\mathsf{r}_1'(x_i)}\right)^{\beta-2}\mathbf{1}_{\beta<4} + \left(\frac{\lambda}{\mathsf{r}_1'(x_i)}\right)^2\mathbf{1}_{\beta>4} + \frac{\lambda^2}{\mathsf{r}_1'(x_i)^2}\left(\log\frac{\mathsf{r}_1'(x_i)}{\lambda}\right)\mathbf{1}_{\beta=4}\right)\right). \quad (5.120)$$

Thus, defining

$$\varphi(x) := CC_\beta^{-1}\beta t\left(\left(\frac{\lambda}{x}\right)^{\beta-2}\mathbf{1}_{\beta\leqslant 4} + \left(\frac{\lambda}{x}\right)^2\mathbf{1}_{\beta>4} + \frac{\lambda^2}{x^2}\left(\log\frac{x}{\lambda}\right)\mathbf{1}_{\beta=4}\right), \quad (5.121)$$

we may bound the left-hand side of (5.118) by from above by

$$\int_{\gamma_{2p}=\gamma}\exp\Big(-\beta\Big(\mathsf{F}_\lambda^{\mathrm{dip}}(X_p,Y_p) + \sum_{i\in\cup_k\mathcal{C}_k, |I_k|=2, d_i d_{\phi_1(i)}=-1, d_i=1} f\Big(\frac{\mathsf{r}_1(z_i)}{\mathsf{r}_1'(z_i)}\Big)\Big)\Big)\mathrm{d}X_p\mathrm{d}Y_p$$
$$\leqslant \lambda^{p(2-\beta)}C_\beta^p\int_{[0,1]^{2p}}\prod_{i=1}^p(1+\varphi(\mathsf{r}_1'(x_i)))\mathrm{d}X_p. \quad (5.122)$$

We next turn to bounding the integral appearing in the right-hand side. Expanding the product and inserting the definition of φ, we find that for $\beta \in [2,4)$,

$$\log \int_{\Lambda^p} \prod_{i=1}^p (1+\varphi(\mathsf{r}_1'(x_i)))\mathrm{d}X_p \leqslant \log \sum_{k=0}^p \binom{p}{k} C^k \lambda^{k(\beta-2)} \int_{\Lambda^p} \mathsf{r}_1(x_1)^{2-\beta}\dots \mathsf{r}_1(x_k)^{2-\beta}\mathrm{d}x_1\dots\mathrm{d}x_p, \quad (5.123)$$

with a constant that depends on β, and where we return to the notation $\mathsf{r}_1(x_i)$ to denote the nearest neighbor of x_i within the system of the x_i's. To evaluate these integrals, we may apply Lemma 5.3.3 to $\beta' = 2(\beta-2)$. For $\beta \in [2,3)$ we then find (after rescaling the lemma)

$$\log \int_{\Lambda^p} \prod_{i=1}^p (1+\varphi(\mathsf{r}_1'(x_i)))\mathrm{d}X_p \leqslant \log \sum_{k=0}^p \binom{p}{k} \lambda^{k(\beta-2)} C^k + O(\log p).$$

By Newton's formula this implies that for $\beta \in (2,3)$,

$$\log \int_{\Lambda^p} \prod_{i=1}^p (1+\varphi(\mathsf{r}_1'(x_i)))\mathrm{d}X_p \leqslant \log(1+C\lambda^{\beta-2})^p \leqslant Cp\lambda^{\beta-2},$$

which shows (5.116). In the general case $\beta \in [2,+\infty)$, by applying Lemma 5.3.3, we find

$$\log \int_{\Lambda^p} \prod_{i=1}^p (1+\varphi(\mathsf{r}_1'(x_i)))\mathrm{d}X_p \leqslant p \sup_{(0,1)} \phi_\beta + O(\log p), \quad (5.124)$$

where ϕ_β is the function defined by

$$\begin{aligned}\phi_\beta : x \in (0,1) \mapsto -x\log(x) - (1-x)\log(1-x) + x(\log(\lambda^{\max(\beta-2,2)})\mathbf{1}_{\beta>2} + \log|\log\lambda|\mathbf{1}_{\beta=2}) \\ + \frac{x}{2}\begin{cases} 0 & \text{if } \beta \in [2,3) \\ \log(1+|\log\lambda|x) & \text{if } \beta = 3 \\ \log(1+\lambda^{6-2\beta}x) & \text{if } \beta \in (3,4) \\ \log\max(\lambda^{-2}x, |\log\lambda|^4) & \text{if } \beta = 4 \\ \log\max(\lambda^{-2}x, |\log\lambda|^2) & \text{if } \beta \in (4,+\infty). \end{cases} \quad (5.125)\end{aligned}$$

The function ϕ_β being concave, it has a unique maximizer on $[0,1]$. We claim that for $\beta \in [3,4)$, the maximizer x of ϕ_β satisfies

$$x(\lambda^{6-2\beta}\mathbf{1}_{\beta\in(2,3)} + |\log\lambda|\mathbf{1}_{\beta=3}) = O(1). \quad (5.126)$$

Assume by contradiction that (5.126) does not hold. Then, up to an extraction, one may assume that

$$x(\lambda^{6-2\beta}\mathbf{1}_{\beta\in(3,4)} + |\log\lambda|\mathbf{1}_{\beta=3}) \xrightarrow[\lambda\to 0]{} +\infty. \quad (5.127)$$

If $x = 1$, then $\phi_\beta(x) = \log(\lambda) + O(1) < 0 = \phi_\beta(0)$. Therefore $x \in (0,1)$ and by minimality $\phi_\beta'(x) = 0$ and therefore

$$\log(x) = \log(\lambda^{\beta-2}) + \frac{1}{2}\log((\lambda^{6-2\beta}\mathbf{1}_{\beta\in(3,4)} + |\log\lambda|\mathbf{1}_{\beta=3})x) + O(1),$$

which implies that $x = O(\lambda^2)$, thus contradicting (5.127). As a consequence (5.126) holds. Moreover if $x \neq 0$, then again $\phi'_\beta(x) = 0$ which gives, by Taylor expanding the logarithm that

$$\log(x) = \log(\lambda^{\beta-2}) + O(1)$$

and one can check that $\phi_\beta(\lambda^{\beta-2}) > 0$. Therefore we find that

$$\sup_{(0,1)} \phi_\beta(x) \leqslant C\lambda^{\beta-2},$$

which gives together with (5.124) the control (5.116) in the case $\beta \in [3,4)$.

We next turn to the case $\beta > 4$. Let x be the maximizer of ϕ_β. One can first dismiss the cases $x = 0$ and $x = 1$. Assume that $x \geqslant \lambda^2 |\log \lambda|^2$. We then have

$$\begin{aligned}\phi_\beta(x) &= -x\log x - (1-x)\log(1-x) + x\log(\lambda^2) + \frac{x}{2}\log(\lambda^{-2}x) \\ &= -\frac{x}{2}\log x - (1-x)\log(1-x) + x\log(\lambda) := \psi(x) = \sup_{y \geqslant \lambda^2|\log\lambda|} \{\psi(y)\}.\end{aligned}$$

It is easy to check that ψ is maximal for $y = C\lambda^2$ and decaying on $[y, 2\lambda^2|\log\lambda|^2]$. This implies that $x = \lambda^2|\log\lambda|^2$. Let us examine the case where $x > \lambda^2|\log\lambda|^2$. We then find by optimizing under constraint as before that $\log(x) = \log(\lambda^2|\log\lambda|) + O(1)$. We conclude by observing that

$$\phi_\beta(\lambda^2|\log\lambda|) > \phi_\beta(\lambda^2|\log\lambda|^2).$$

It follows that

$$\sup_{(0,1)} \phi_\beta \leqslant C\lambda^2|\log\lambda|,$$

which gives combined to (5.124) the proof of (5.116) in the case $\beta > 4$.

It remains to examine the case $\beta = 4$. As before, we dismiss the case $x = 0$ and $x = 1$. Consider first the case where the maximizer x of ϕ_β satisfies $x \geqslant \lambda^2|\log\lambda|^4$. We then find by optimization under constraint as for the case $\beta \in [3,4)$ that $x = \lambda^2|\log\lambda|^4$. Now assume that $x \leqslant \lambda^2|\log\lambda|^4$. Then

$$\phi_\beta(x) = -x\log(x) - (1-x)\log(1-x) + x\log(\lambda^2|\log\lambda|^2) := f(x).$$

Consequently x must be equal to the maximizer of f under the constraint $x \leqslant \lambda^2|\log\lambda|^2$, which turns out to satisfy $x = \log(\lambda^2|\log\lambda|^2) + O(1)$. We have thus observed that in both case, the maximizer of ϕ_β satisfies $x = O(\lambda^2|\log\lambda|^2)$, yielding

$$\sup_{(0,1)} \phi_\beta \leqslant \lambda^2|\log\lambda|^2,$$

concluding the proof of (5.116) in the case $\beta = 4$. Le us emphasize that for $\beta = 4$, both functions in the definition of the maximum in (5.125) give rise to the same maximizer, hinting a form of criticality at $\beta = 4$. □

5.3.4 Main result

We may now obtain the main upper bound.

Proposition 5.3.5 (Upper bound). *Let $\beta \in (2, +\infty)$ and C_β be the constant defined in (5.16). There holds*

$$\log Z^\lambda_{N,\beta} \leqslant 2N \log N + (2-\beta) N \log \lambda - N + N \log C_\beta \\ + O\Big(N(\lambda^{\beta-2}\mathbf{1}_{\beta<4} + \lambda^2 |\log \lambda|^2 \mathbf{1}_{\beta=4} + \lambda^2 |\log \lambda| \mathbf{1}_{\beta>4})\Big). \quad (5.128)$$

For $\beta = 2$, there holds

$$\log Z^\lambda_{N,\beta} \leqslant 2N \log N + N \log |\log \lambda| - N + O\Big(\frac{N}{|\log \lambda|}\Big). \quad (5.129)$$

Proof. To bound the partition function from above, we start by inserting the lower bound (5.47) into its definition. This way it suffices to bound from above

$$\log \int_{\Lambda^{2N}} \exp\Big(-\beta\Big(\mathsf{F}^{\mathrm{dip}}_\lambda(X_N, Y_N) - C \sum_{i \in I^{\mathrm{dip}}, \phi_2(i) \in I^{\mathrm{dip}}} \Big(\frac{\mathsf{r}_1(z_i)}{\mathsf{r}_2(z_i)}\Big)^2\Big)\Big) \mathrm{d}X_N \mathrm{d}Y_N. \quad (5.130)$$

Let us split the integrals over the functional digraphs. Let $\gamma \in D_{2N,K,n}$. For each configuration with graph γ, we may relabel the points so that the positive charges that form the neutral two-cycles $\mathcal{C}_k$ such that $|I_k| = 2$ (i.e. isolated neutral 2-cycles), are $x_1, \ldots x_p$ ($p \leqslant n$). We may assume that each x_i for $i \leqslant p$ forms a cycle with y_i. The remaining points are labelled $z_1, \ldots, z_{2N-2p}$. In view of (5.64), we may rewrite with obvious notation,

$$\mathsf{F}^{\mathrm{dip}}_\lambda(X_N, Y_N) - C \sum_{i \in I^{\mathrm{dip}}, \phi_2(i) \in I^{\mathrm{dip}}} \Big(\frac{\mathsf{r}_1(z_i)}{\mathsf{r}_2(z_i)}\Big)^2 \\ = \mathsf{F}^{\mathrm{dip}}_\lambda(Z_{2N-2p}) - \sum_{i=1}^{p} \Big(\mathsf{g}_\lambda(x_i - y_i) + C \sum_{i \in I^{\mathrm{dip}}, \phi_2(i) \in I^{\mathrm{dip}}} \Big(\frac{\mathsf{r}_1(z_i)}{\mathsf{r}_2(z_i)}\Big)^2\Big),$$

i.e. we split the energy between the contribution of the isolated neutral 2-cycles, which we can treat by Lemma 5.3.4, and that of the rest, which we can treat by Lemma 5.3.2. We denote by γ^1 the nearest neighbor graph of the first n x_i's and y_i's, and γ^2 that of the rest of the variables. Separating variables we obtain

$$\int_{\{\gamma_{2N} = \gamma\}} \exp\Big(-\beta\Big(\mathsf{F}^{\mathrm{dip}}_\lambda(X_N, Y_N) - C \sum_{i \in I^{\mathrm{dip}}, \phi_2(i) \in I^{\mathrm{dip}}} \Big(\frac{\mathsf{r}_1(z_i)}{\mathsf{r}_2(z_i)}\Big)^2\Big)\Big) \mathrm{d}X_N \mathrm{d}Y_N \\ \leqslant \int_{\Lambda^{2N-2p} \cap \{\gamma_{2N-2p}(Z_{2N-2p}) = \gamma^2\}} \exp\Big(-\beta \mathsf{F}^{\mathrm{dip}}_\lambda(Z_{2N-2p})\Big) \mathrm{d}Z_{2N-2p} \\ \times \int_{\Lambda^{2p} \cap \{\gamma_{2p}(X_p, Y_p) = \gamma^1\}} \exp\Big(-\beta\Big(\mathsf{F}^{\mathrm{dip}}_\lambda(X_p, Y_p) - C \sum_{i \in I^{\mathrm{dip}}, \phi_2(i) \in I^{\mathrm{dip}}} \Big(\frac{\mathsf{r}_1(z_i)}{\mathsf{r}_2(z_i)}\Big)^2\Big)\Big) \mathrm{d}X_p \mathrm{d}Y_p. \quad (5.131)$$

We next claim that

$$\sum_{i \in I^{\mathrm{dip}}, \phi_2(i) \in I^{\mathrm{dip}}} \Big(\frac{\mathsf{r}_1(z_i)}{\mathsf{r}_2(z_i)}\Big)^2 \leqslant -C \sum_{i \in I^{\mathrm{dip}}, d_i = 1} f\Big(\frac{\mathsf{r}_1(z_i)}{\mathsf{r}_1'(z_i)}\Big), \quad (5.132)$$

where $f(x) = -tx^2$ for some appropriate constant t, as above. This allows to replace in (5.131) the left-hand side term by the right-hand side. Assuming the claim, and inserting into (5.131) the results of Lemma 5.3.2 (applied with $N-n$), more precisely (5.75), and Lemma 5.3.4, we find

$$\begin{aligned}
&\log \int_{\{\gamma_{2N}=\gamma\}} \exp\Big(-\beta\Big(\mathsf{F}_\lambda^{\mathrm{dip}}(X_N,Y_N) - C \sum_{i\in\{\cup_k \mathcal{C}_k, |I_k|=2, d_i d_{\phi_1(i)}=-1\}} \Big(\frac{\mathsf{r}_1(z_i)}{\mathsf{r}_2(z_i)}\Big)^2\Big)\Big) \mathrm{d}X_N \mathrm{d}Y_N \\
&\leqslant (n-p)((2-\beta)\log\lambda \mathbf{1}_{\beta>2} + \log|\log\lambda|\mathbf{1}_{\beta=2}) + (K-p)\log N + (N-p)\log C_\beta \mathbf{1}_{\beta>2} \\
&\quad + (2N-K-n)\log\Big(\frac{2N}{2N-K-n}\Big) + c_0(N-n) \\
&\quad + p\log N + p((2-\beta)\log\lambda)\mathbf{1}_{\beta>2} + \log|\log\lambda|\mathbf{1}_{\beta=2} + \log C_\beta \mathbf{1}_{\beta>2} + CN\gamma_\lambda \\
&= n((2-\beta)\log\lambda\mathbf{1}_{\beta>2} + \log|\log\lambda|\mathbf{1}_{\beta=2}) + K\log N + N\log C_\beta \mathbf{1}_{\beta>2} \\
&\quad + (2N-K-n)\log\Big(\frac{2N}{2N-K-n}\Big) + c_0(N-n) + CN\gamma_\lambda, \qquad (5.133)
\end{aligned}$$

where γ_λ is as in (5.17). There remains to sum over the functional digraphs. Using (5.83), we first obtain that

$$\begin{aligned}
&\log\Big(|D_{2N,K,n}| \int_{\{\gamma_{2N}=\gamma\}} \exp\Big(-\beta\Big(\mathsf{F}_\lambda^{\mathrm{dip}}(X_N,Y_N) \\
&\qquad - C \sum_{i\in\{\cup_k \mathcal{C}_k, |I_k|=2, d_i d_{\phi_1(i)}=-1\}} \Big(\frac{\mathsf{r}_1(z_i)}{\mathsf{r}_2(z_i)}\Big)^2\Big)\Big) \mathrm{d}X_N \mathrm{d}Y_N\Big) \\
&\leqslant 2N\log(2N) + K\log N - K\log K + 2(N-K)(\log N - \log(N-K)) - n - 2n\log 2 + O(\log N) \\
&\quad + n((2-\beta)\log\lambda\mathbf{1}_{\beta>2} + \log|\log\lambda|\mathbf{1}_{\beta=2}) + N\log C_\beta\mathbf{1}_{\beta>2} \\
&\quad + (2N-K-n)\log\Big(\frac{2N}{2N-K-n}\Big) + c_0(N-n) + CN\gamma_\lambda \\
&\leqslant 2N\log(2N) + N(\log C_\beta \mathbf{1}_{\beta>2} - 1 - 2\log 2) + I(K,n) + CN\gamma_\lambda,
\end{aligned}$$

where $I(K,n)$ is as in (5.86) for $\beta \in (2,4)$, as in (5.91) for $\beta = 2$ and as in (5.92) for $\beta \geqslant 4$. Arguing as in the rest of the proof of Lemma 5.3.2, i.e. maximizing over K and n, we deduce the result.

We finish by proving the claim (5.132). Let us first consider the case of a positively charged point, say $z_1 = x_1$, belonging to an isolated neutral 2-cycle. If $\mathsf{r}_2(x_1) \geqslant \frac{1}{2}\mathsf{r}_1'(x_1)$ then the absorption is also obvious. So we may reduce to the case $\mathsf{r}_2(x_1) \leqslant \frac{1}{2}\mathsf{r}_1'(x_1)$, which implies that $\mathsf{r}_2(x_1)$ is achieved at a negative charge, say y_2, and since we consider only the case where $\phi_2(i) \in I^{\mathrm{dip}}$, this means that y_2 forms a neutral dipole with, say, x_2. We have

$$\mathsf{r}_2(x_1) = \max\left(\lambda, \frac{1}{4}|x_1 - y_2|\right) \leqslant \frac{1}{2}\mathsf{r}_1'(x_1) = \frac{1}{2}\max(\lambda, \frac{1}{4}|x_1 - x_2|)$$

and

$$|x_1 - y_2| \leqslant \frac{1}{2}|x_1 - x_2|$$

We may then write by reverse triangle inequality that

$$|x_2 - y_2| \geqslant |x_2 - x_1| - |x_1 - y_2|$$

so that

$$\mathsf{r}_1(x_2) \geqslant \mathsf{r}_1'(x_1) - |x_1 - y_2| \geqslant 2\mathsf{r}_2(x_1) - \frac{1}{4}|x_1 - y_2| \geqslant \mathsf{r}_2(x_1) \geqslant \mathsf{r}_1(x_1).$$

Moreover, by triangle inequality

$$|x_2 - x_1| \leqslant |x_1 - y_2| + |y_2 - x_2|$$

so that by definition of r_2 and the fact that the nearest neighbor to y_2 is x_2, we have

$$\mathsf{r}_1'(x_2) \leqslant \mathsf{r}_2(x_1) + \frac{1}{4}|y_2 - x_1| \leqslant 2\mathsf{r}_2(x_1).$$

It thus follows that

$$\frac{\mathsf{r}_1(x_1)}{\mathsf{r}_2(x_1)} \leqslant 2\frac{\mathsf{r}_1(x_2)}{\mathsf{r}_1'(x_2)}$$

so that the corresponding term in the sum in the left-hand side of (5.132) can be absorbed into the sum corresponding to x_2 in the right-hand side.

We next turn to the case of a negatively charged point, say y_1. We first assume that the min in the definition of $\mathsf{r}_2(y_1)$ is achieved by a positive charge, say x_2. By triangle inequality, and since x_1 is the nearest neighbor of y_1,

$$|x_1 - x_2| \leqslant |x_1 - y_1| + |y_1 - x_2| \leqslant 2|y_1 - x_2|$$

so that

$$\mathsf{r}_1'(x_1) \leqslant \max(\lambda, \frac{1}{2}|y_1 - x_2|) \leqslant 2\mathsf{r}_2(y_1).$$

Thus, since $\mathsf{r}_1(x_1) = \mathsf{r}_1(y_1)$,

$$\frac{\mathsf{r}_1(x_1)}{\mathsf{r}_1'(x_1)} \geqslant \frac{1}{2}\frac{\mathsf{r}_1(y_1)}{\mathsf{r}_2(y_1)}$$

and the absorption can be made as well. Secondly, we consider the case where the min in the definition of $\mathsf{r}_2(y_1)$ is achieved by a negative charge, say y_2, which forms an isolated neutral dipole with x_2. If $\frac{\mathsf{r}_1(x_1)}{\mathsf{r}_1'(x_1)} \geqslant \frac{1}{6}$ then the left-hand side term $\frac{\mathsf{r}_1(y_1)}{\mathsf{r}_2(y_1)}$ can be absorbed into the right-hand side term, up to a multiplicative constant. We may thus assume that $\mathsf{r}_1(x_1) \leqslant \frac{1}{6}\mathsf{r}_1'(x_1)$. In the same way, we may assume that $\mathsf{r}_1(x_2) \leqslant \frac{1}{6}\mathsf{r}_1'(x_2)$. In particular $\mathsf{r}_1'(x_1)$ and $\mathsf{r}_1'(x_2)$ are not equal to λ but to a true quarter minimal distance. Then we may write by triangle inequality that

$$|x_1 - x_2| \leqslant |y_1 - y_2| + |x_1 - y_1| + |x_2 - y_2| \leqslant |y_1 - y_2| + 8\max(\mathsf{r}_1(x_1), \mathsf{r}_1(x_2)).$$

If the max is achieved by x_1, we then deduce that

$$\frac{\mathsf{r}_1(y_1)}{\mathsf{r}_2(y_1)} \leqslant \frac{4\mathsf{r}_1(y_1)}{|y_1 - y_2|} \leqslant \frac{4\mathsf{r}_1(x_1)}{|x_1 - x_2| - 8\mathsf{r}_1(x_1)} \leqslant \frac{4\mathsf{r}_1(x_1)}{4\mathsf{r}_1'(x_1) - 8\mathsf{r}_1(x_1)}$$
$$\leqslant \frac{1}{\mathsf{r}_1'(x_1)/\mathsf{r}_1(x_1) - 2} \leqslant C\frac{\mathsf{r}_1(x_1)}{\mathsf{r}_1'(x_1)} \tag{5.134}$$

by using that $\mathsf{r}_1(x_1) \leqslant \frac{1}{6}\mathsf{r}_1'(x_1)$. If the max is achieved by x_2 then the reasoning is identical. We have thus proved the claim in all cases. □

Remark 20 (On the quadripole transition). *At $\beta = 3$, the free energy of quadripoles, i.e of two very close neutral dipoles, starts diverging as λ tends to 0. Consequently, points which are not in small well-separated dipoles prefer forming small quadripoles than being alone. The error term in $N - n$ in our computations, see for instance (5.133), contains these quadripole terms and should therefore be expanded in order to see the transition at $\beta = 3$. This would require a more precise cluster expansion of the energy.*

5.4 Free energy lower bound

We now derive a lower bound on the partition function. For that, we use a method inspired from previous work on the one-component plasma and on Ginzburg-Landau, starting in [229, 233], which allows, thanks to the electric formulation of the energy, to compute the interaction additively in terms of electric potentials defined in disjoint subregions of the space.

Proposition 5.4.1. *Assume $\beta \in (2, +\infty)$. We have*

$$\log Z^{\lambda}_{N,\beta} \geqslant 2N \log N + N(2-\beta)\log \lambda - N + N \log C_\beta + O\left(N(\lambda^{\beta-2}\mathbf{1}_{\beta<4} + \lambda^2|\log\lambda|^2\mathbf{1}_{\beta=4} + \lambda^2|\log\lambda|\mathbf{1}_{\beta>4})\right). \quad (5.135)$$

For $\beta = 2$, we have

$$\log Z^{\lambda}_{N,\beta} \geqslant 2N\log N + N\log|\log\lambda| - N + O\Big(\frac{N}{|\log\lambda|}\Big). \quad (5.136)$$

Proof. **Step 1: bounding the energy from above.** We are going to reduce the integral to configurations where $y_i \in B(x_i, \frac{1}{2}\mathsf{r}(x_i))$ where $\mathsf{r}(x_i) := \frac{1}{2}\min_{i\neq j}|x_j - x_i|$. For such configurations let us now bound the energy from above. First we recall (5.14). We note that for the configurations in the integration set, the balls $B(x_i, \mathsf{r}(x_i))$ are disjoint and contain only the points x_i and y_i. We then let, for each i, u_i solve

$$\begin{cases} -\Delta u_i = 2\pi(\delta^{(\lambda)}_{x_i} - \delta^{(\lambda)}_{y_i}) & \text{in } B(x_i, \mathsf{r}(x_i)) \\ \frac{\partial u_i}{\partial \nu} = 0 & \text{on } \partial B(x_i, \mathsf{r}(x_i)) \end{cases}$$

We then define a global “electric field" E by pasting together the electric fields defined over these disjoint balls:

$$E := \sum_{i=1}^N \mathbf{1}_{B(x_i,\mathsf{r}(x_i))}\nabla u_i.$$

Thanks to the crucial choice of zero Neumann boundary conditions on the boundary of the disjoint balls, this vector field satisfies

$$-\mathrm{div}\, E = 2\pi\left(\sum_{i=1}^N \delta^{(\lambda)}_{x_i} - \delta^{(\lambda)}_{y_i}\right) = -\Delta h_\lambda \quad (5.137)$$

where h_λ is the electric potential of the configuration as in (5.12). The trick is then to take advantage of the L^2 projection property onto gradients to show that the energy can be estimated from above by the L^2 norm of E: indeed

$$\int_{\mathbb{R}^2}|E|^2 = \int_{\mathbb{R}^2}|\nabla h_\lambda|^2 + \int_{\mathbb{R}^2}|E-\nabla h_\lambda|^2 + 2\int_{\mathbb{R}^2}(E-\nabla h_\lambda)\cdot\nabla h_\lambda$$

and the last term vanishes after integration by parts, in view of (5.137). It thus follows that

$$\int_{\mathbb{R}^2}|\nabla h_\lambda|^2 \leqslant \frac{1}{4\pi}\sum_{i=1}^N\int_{B(x_i,\mathsf{r}(x_i))}|\nabla u_i|^2 \quad (5.138)$$

that is, we can reduce the computation to a sum over the disjoint balls. We next bound the right-hand side. First we let $v_i := u_i - (\mathsf{g} * \delta^{(\lambda)}_{x_i} - \mathsf{g} * \delta^{(\lambda)}_{y_i})$. It solves

$$\begin{cases} -\Delta v_i = 0 & \text{in } B(x_i, \mathsf{r}(x_i)) \\ \frac{\partial v_i}{\partial \nu} = \left(-\frac{(x-x_i)}{|x-x_i|^2} + \frac{(x-y_i)}{|x-y_i|^2}\right)\cdot\nu & \text{on } \partial B(x_i, \mathsf{r}(x_i)) \end{cases}$$

and thus by elliptic regularity estimates we have

$$\|\nabla v_i\|_{L^\infty(B(x_i,\frac{3}{4}\mathsf{r}(x_i)))} \leqslant C\frac{|x_i-y_i|}{\mathsf{r}(x_i)^2}. \tag{5.139}$$

Now, using (5.139), we find

$$\begin{aligned}
\int_{B(x_i,\mathsf{r}(x_i))} |\nabla u_i|^2 &= 2\pi \int_{B(x_i,\mathsf{r}(x_i))} u_i \left(\delta_{x_i}^{(\lambda)} - \delta_{y_i}^{(\lambda)}\right) \\
&= 2\pi \left(2(\mathsf{g}(\lambda)+\kappa) - \int \mathsf{g} * \delta_{x_i}^{(\lambda)}\delta_{y_i}^{(\lambda)} - \int \mathsf{g} * \delta_{y_i}^{(\lambda)}\delta_{x_i}^{(\lambda)}\right) + \int v_i\left(\delta_{x_i}^{(\lambda)} - \delta_{y_i}^{(\lambda)}\right) \\
&= 4\pi(\mathsf{g}(\lambda)+\kappa) - 4\pi \iint \mathsf{g}(x-y)\delta_{x_i}^{(\lambda)}(x)\delta_{y_i}^{(\lambda)}(y) + O\Big(\frac{|x_i-y_i|^2}{\mathsf{r}(x_i)^2}\Big) \\
&= 4\pi(\mathsf{g}(\lambda)+\kappa) - 4\pi\mathsf{g}_\lambda(x_i-y_i) + O\Big(\frac{|x_i-y_i|^2}{\mathsf{r}(x_i)^2}\Big).
\end{aligned}$$

Inserting into (5.138) and (5.14) we deduce that

$$\mathsf{F}_\lambda(X_N,Y_N) \leqslant -\sum_{i=1}^N \mathsf{g}_\lambda(x_i-y_i) + O\Big(\frac{|x_i-y_i|^2}{\mathsf{r}(x_i)^2}\Big). \tag{5.140}$$

In all cases, since we have built the configurations so that $|x_i-y_i| \leqslant \frac{1}{2}\mathsf{r}(x_i)$ we can bound the error term by $O(N)$.

Step 2: bounding the free energy. Because of all the possible relabelling of the pairs, we may write

$$Z_{N,\beta}^\lambda \geqslant N! \int_{x_i\in[0,\sqrt{N}]^2, y_i\in B(x_i,\frac{1}{2}\mathsf{r}(x_i))} \exp(-\beta\mathsf{F}_\lambda(X_N,Y_N))\mathrm{d}y_1\ldots\mathrm{d}y_N\mathrm{d}x_1\ldots\mathrm{d}x_N \tag{5.141}$$

where as above $\mathsf{r}(x_i) = \frac{1}{2}\min_{j\neq i}|x_j-x_i|$. We may now insert (5.140) into (5.141) to obtain

$$\begin{aligned}
Z_{N,\beta}^\lambda &\geqslant N! \int_{\substack{x_i\in[0,\sqrt{N}]^2 \\ y_i\in B(x_i,\frac{1}{2}\mathsf{r}(x_i))}} \exp\left(\beta\sum_{i=1}^N \mathsf{g}_\lambda(x_i-y_i) + O\Big(\frac{|x_i-y_i|^2}{\mathsf{r}(x_i)^2}\Big)\right)\mathrm{d}y_1\ldots\mathrm{d}y_N\mathrm{d}x_1\ldots\mathrm{d}x_N \\
&\geqslant N!\int_{[0,\sqrt{N}]^{2N}} \prod_{i=1}^N \int_0^{\frac{1}{2}\mathsf{r}(x_i)} 2\pi r \exp\Big(\beta\mathsf{g}_\lambda(r) - C\frac{r^2}{\mathsf{r}(x_i)^2}\Big)\mathrm{d}r\mathrm{d}x_i
\end{aligned} \tag{5.142}$$

We have

$$\begin{aligned}
&\int_0^{\frac{1}{2}\mathsf{r}(x_i)} 2\pi r\exp\Big(\beta\mathsf{g}_\lambda(r) - C\frac{r^2}{\mathsf{r}(x_i)^2}\Big)\mathrm{d}r \\
&\quad = \int_0^{\frac{1}{2}\mathsf{r}(x_i)} 2\pi r\exp\Big(\beta(\mathsf{g}(\lambda)+\mathsf{g}_1(\frac{r}{\lambda}))\Big)\mathrm{d}r + O\Big(\int_0^{\frac{1}{2}\mathsf{r}(x_i)} \frac{r^3}{\mathsf{r}(x_i)^2(r\wedge\lambda)^\beta}\mathrm{d}r\Big)
\end{aligned}$$

Similarly to (5.119), we compute that

$$\begin{aligned}
&\int_0^{\frac{1}{2}\mathsf{r}(x_i)} \frac{r^3}{\mathsf{r}(x_i)^2(r\wedge\lambda)^\beta}\mathrm{d}r \\
&= O\left(\frac{\mathsf{r}(x_i)^2}{\lambda^\beta}\mathbf{1}_{\mathsf{r}(x_i)\leqslant 2\lambda} + \mathbf{1}_{\mathsf{r}(x_i)\geqslant 2\lambda}\left(\frac{\lambda^{4-\beta}}{\mathsf{r}(x_i)^2} + \mathsf{r}(x_i)^{2-\beta}\mathbf{1}_{\beta\in(2,4)} + \frac{\lambda^{4-\beta}}{\mathsf{r}(x_i)^2}\mathbf{1}_{\beta>4} + \frac{1}{\mathsf{r}(x_i)^2}\log\frac{\mathsf{r}(x_i)}{2\lambda}\mathbf{1}_{\beta=4}\right)\right) \\
&\leqslant C\left(\mathsf{r}(x_i)^{2-\beta}\mathbf{1}_{\beta\in[2,4)} + \frac{\lambda^{4-\beta}}{\mathsf{r}(x_i)^2}\wedge\frac{\mathsf{r}(x_i)^2}{\lambda^\beta}\mathbf{1}_{\beta\geqslant 4} + \frac{(\log\frac{\mathsf{r}(x_i)}{2\lambda})\wedge 0}{\mathsf{r}(x_i)^2}\mathbf{1}_{\beta=4}\right).
\end{aligned}$$

We also compute that, in view of (5.44)

$$\begin{aligned}0 \leqslant \int_{\frac{1}{2}\frac{\mathsf{r}(x_i)}{\lambda}}^{\infty} 2\pi s \exp(\beta \mathsf{g}_1(s))\mathrm{d}s &\leqslant \mathbf{1}_{\mathsf{r}(x_i)\geqslant 2\lambda} \int_{\frac{1}{2}\frac{\mathsf{r}(x_i)}{\lambda}}^{\infty} 2\pi s \exp(\beta \mathsf{g}(s)+C))\mathrm{d}s + C\mathbf{1}_{\mathsf{r}(x_i)\leqslant 2\lambda} \\ &\leqslant C\left(\left(\left(\frac{\mathsf{r}(x_i)}{2\lambda}\right)^{2-\beta}\mathbf{1}_{\beta>2} + \log\Big(\frac{\mathsf{r}(x_i)}{2\lambda}\Big)\mathbf{1}_{\beta=2}\right)\mathbf{1}_{\mathsf{r}(x_i)\geqslant 2\lambda} + \mathbf{1}_{\mathsf{r}(x_i)\leqslant 2\lambda}\right) \\ &= C\left(\Big(\frac{\mathsf{r}(x_i)^{2-\beta}}{\lambda^{2-\beta}}\wedge 1\Big)\mathbf{1}_{\beta>2} + \Big(\log\Big(\frac{\mathsf{r}(x_i)}{\lambda}\Big)\wedge 1\Big)\mathbf{1}_{\beta=2}\right).\end{aligned}$$

This error term can always be absorbed in the others, thus, in view of (5.16), we may write

$$\begin{aligned}&\int_0^{\frac{1}{2}\mathsf{r}(x_i)} 2\pi r \exp\Big(\beta \mathsf{g}_\lambda(r) - C\frac{r^2}{\mathsf{r}(x_i)^2}\Big)\mathrm{d}r \\ &= \lambda^{2-\beta}\left(C_\beta + O\left(\frac{\mathsf{r}(x_i)^{2-\beta}}{\lambda^{2-\beta}}\mathbf{1}_{\beta\in[2,4)} + \frac{\lambda^2}{\mathsf{r}(x_i)^2}\wedge\frac{\mathsf{r}(x_i)^2}{\lambda^2}\mathbf{1}_{\beta\geqslant 4} + \frac{\lambda^2}{\mathsf{r}(x_i)^2}\left((\log\frac{\mathsf{r}(x_i)}{2\lambda})\wedge 0\right)\mathbf{1}_{\beta=4}\right)\right),\end{aligned}$$

for $\beta > 2$ and

$$\int_0^{\frac{1}{2}\mathsf{r}(x_i)} 2\pi r \exp\Big(\beta \mathsf{g}_\lambda(r) - C\frac{r^2}{\mathsf{r}(x_i)^2}\Big)\mathrm{d}r = |\log\lambda|\Big(1 + O\Big(\frac{1}{|\log\lambda|}\Big)\Big), \tag{5.143}$$

for $\beta = 2$. Let us start with the case $\beta \geqslant 4$. Inserting this result into (5.142), we then find

$$\begin{aligned}\log Z_{N,\beta}^{\lambda} &\geqslant \log N! + N(2-\beta)\log\lambda + N\log C_\beta \\ &+ \log\int_{[0,\sqrt{N}]^{2N}}\prod_{i=1}^{N}\Big(1 - C\frac{\lambda^2}{\mathsf{r}(x_i)^2}\wedge\frac{\mathsf{r}(x_i)^2}{\lambda^2} - C\frac{\lambda^2}{\mathsf{r}(x_i)^2}\left((\log\frac{\mathsf{r}(x_i)}{2\lambda})\wedge 0\right)\mathbf{1}_{\beta=4}\Big)\mathrm{d}x_1\ldots\mathrm{d}x_N\end{aligned} \tag{5.144}$$

But

$$\begin{aligned}&\log\int_{[0,\sqrt{N}]^{2N}}\prod_{i=1}^{N}\Big(1 - C\frac{\lambda^2}{\mathsf{r}(x_i)^2}\wedge\frac{\mathsf{r}(x_i)^2}{\lambda^2} - C\frac{\lambda^2}{\mathsf{r}(x_i)^2}\Big((\log\frac{\mathsf{r}(x_i)}{2\lambda})\wedge 0\Big)\mathbf{1}_{\beta=4}\Big)\mathrm{d}x_1\ldots\mathrm{d}x_N \\ &\geqslant \log\int_{[0,\sqrt{N}]^{2N}}\prod_{i=1}^{N}\Big(1 - C\frac{\lambda^2}{\mathsf{r}_1(x_i)^2} - C\frac{\lambda^2}{\mathsf{r}_1(x_i)^2}\Big((\log\frac{\mathsf{r}_1(x_i)}{\lambda})\wedge 0\Big)\mathbf{1}_{\beta=4})\mathrm{d}x_1\ldots\mathrm{d}x_N \\ &= \log\int_{[0,\sqrt{N}]^{2N}}\prod_{i=1}^{N}\Big(1 - C\varphi(\mathsf{r}_1(x_i))\Big)\mathrm{d}x_1\ldots\mathrm{d}x_N\end{aligned} \tag{5.145}$$

where r_1 is as in (5.39) and φ is as in (5.121). We may then expand the product as in (5.123). Up to the sign, and the inequality being reversed, the terms in the product are identical to those found in (5.123). We thus deduce in the same way that

$$\begin{aligned}&\log\int_{[0,\sqrt{N}]^{2N}}\prod_{i=1}^{N}\Big(1 - C\frac{\lambda^2}{\mathsf{r}(x_i)^2}\wedge\frac{\mathsf{r}(x_i)^2}{\lambda^2} - C\frac{\lambda^2}{\mathsf{r}(x_i)^2}\left((\log\frac{\mathsf{r}(x_i)}{2\lambda})\wedge 0\right)\mathbf{1}_{\beta=4}\Big)\mathrm{d}x_1\ldots\mathrm{d}x_N \\ &\geqslant N(\lambda^2|\log\lambda|^2\mathbf{1}_{\beta=4} + \lambda^2|\log\lambda|\mathbf{1}_{\beta>4})\end{aligned} \tag{5.146}$$

and inserting into (5.144), we obtain the result.

Next, we turn to the case $\beta \in (2,4)$. In that case we have instead

$$\log Z^{\lambda}_{N,\beta} \geqslant \log N! + N((2-\beta)\log\lambda \mathbf{1}_{\beta>2} + \log|\log\lambda|\mathbf{1}_{\beta=2}) + N\log C_\beta \mathbf{1}_{\beta>2} + N\log N + \log\Big(N^{-N}\int_{[0,\sqrt{N}]^{2N}} \prod_{i=1}^{N}\Big(1 - C\frac{\mathsf{r}(x_i)^{2-\beta}}{\lambda^{2-\beta}}\Big)\mathrm{d}x_1\dots\mathrm{d}x_N\Big). \quad (5.147)$$

We may use Jensen's inequality to write

$$\begin{aligned}\log\Big(N^{-N}&\int_{[0,\sqrt{N}]^{2N}}\prod_{i=1}^{N}\Big(1-C\mathsf{r}(x_i)^2\lambda^{\beta-2}\Big)\mathrm{d}x_1\dots\mathrm{d}x_N\Big)\\
&\geqslant N^{-N}\int_{[0,\sqrt{N}]^{2N}}\sum_{i=1}^{N}\log\Big(1-C\mathsf{r}(x_i)^{2-\beta}\lambda^{\beta-2}\Big)\mathrm{d}x_1\dots\mathrm{d}x_N\\
&\geqslant -C\Big(\lambda^{\beta-2}\sum_{i=1}^{N}N^{-N}\int_{[0,\sqrt{N}]^{2N}}\mathsf{r}(x_i)^{2-\beta}\mathrm{d}x_1\dots\mathrm{d}x_N\Big)\\
&= -C\Big(N\lambda^{\beta-2}N^{-N}\int_{[0,\sqrt{N}]^{2N}}\mathsf{r}(x_1)^{2-\beta}\mathrm{d}x_1\dots\mathrm{d}x_N\Big).\end{aligned}$$

It remains to evaluate the expectation of $\mathsf{r}^{2-\beta}$ under the Lebesgue measure on $[0,1]^{2N}$. Let $\mathcal{P}$ be a Poisson point process of intensity 1. First, one may justify that

$$\lim_{N\to\infty} N^{-N}\int_{[0,\sqrt{N}]^{2N}}\mathsf{r}(x_1)^{2-\beta}\mathrm{d}x_1\dots\mathrm{d}x_N = \mathbb{E}_{\mathcal{P}}[\mathsf{r}^{2-\beta}]. \quad (5.148)$$

The distribution f of r is called nearest-neighbor function. It is related to the so-called spherical contact distribution function. The point is that a Poisson point process, conditioned to having one point at $x\in\mathbb{R}^2$ remains a Poisson point process. As a consequence, the probability that $\mathsf{r}\geqslant r$ is equal to the probability that the number of points in $B(0,r)$ equals to 0, i.e the probability of $X=0$ where X is a Poisson variable of parameter $\lambda=\pi r^2$. We deduce that

$$\mathbb{P}_{\mathcal{P}}(\mathsf{r}\leqslant r) = 1-e^{-\pi r^2},$$

which implies that

$$f(r) = 2\pi r e^{-\pi r^2}.$$

As a consequence we find that since $\beta<4$,

$$\int f(r)r^{2-\beta}\mathrm{d}r = 2\pi\int\frac{1}{r^{\beta-3}}e^{-\pi r^2}\mathrm{d}r<\infty.$$

We deduce that

$$\log\Big(N^{-N}\int_{[0,\sqrt{N}]^{2N}}\prod_{i=1}^{N}\Big(1-C\mathsf{r}(x_i)^2\lambda^{\beta-2}\Big)\mathrm{d}x_1\dots\mathrm{d}x_N\Big)\geqslant -CN\lambda^{\beta-2}.$$

Inserting into (5.147) and using Stirling's formula we find the result (5.135) in the case $\beta\in(2,4)$.

Finally, the bound (5.143) being independent of x_i's, (5.136) is straightforward. □

It is now immediate to complete the dipole description of Theorem 5.1.1.

Proof of Theorem 5.1.1. Denote

$$G = \sum_{i \in I} \mathsf{g}_\lambda(z_i - z_{\phi_1(i)}) \tag{5.149}$$

and let $K^\lambda_{N,\beta}$ be the reduced partition function

$$\tilde{K}^\lambda_{N,\beta} = \int_{[0,\sqrt{N}]^{2N} \times [0,\sqrt{N}]^{2N}} \exp\left(-\beta G(X_N, Y_N)\right) \mathrm{d}X_N \mathrm{d}Y_N.$$

Let γ_λ be the error rate defined in (5.17). First, one may observe that for $t = \beta$, the Laplace transform (5.20) is nothing but the ratio of partition function

$$\log \mathbb{E}_{\mathbb{P}^\lambda_{N,\beta}}\Big[\exp(\beta(\mathsf{F}_\lambda - G))\Big] = \frac{\tilde{K}^\lambda_{N,\beta}}{Z^\lambda_{N,\beta}},$$

which we can bound using Lemma 5.3.2 and Proposition 5.4.1 by

$$\log \mathbb{E}_{\mathbb{P}^\lambda_{N,\beta}}\Big[\exp(\beta(\mathsf{F}_\lambda - G))\Big] \leqslant CN\gamma_\lambda.$$

By Hölder's inequality, this proves the upper bound for all $0 \leqslant t \leqslant \beta$. It remains to prove the inequality for $-\frac{\beta}{2} \leqslant t \leqslant 0$. Let us upper bound (5.20) for $t = -\frac{\beta}{2}$. Applying Corollary 5.2.2, one may bound the energy F_λ from below, which gives

$$\begin{aligned}\log \mathbb{E}_{\mathbb{P}^\lambda_{N,\beta}}&\Big[\exp\Big(\frac{\beta}{2}(G - \mathsf{F}_\lambda)\Big)\Big] \\ &\leqslant \frac{1}{Z^\lambda_{N,\beta}} \int \exp\Big(-\beta \mathsf{F}^{\mathrm{dip}}_\lambda + C \sum_{i \in I^{\mathrm{dip}}, \phi_2(i) \in I^{\mathrm{dip}}} \Big(\frac{\mathsf{r}_1(z_i)}{\mathsf{r}_2(z_i)}\Big)^2 + C(N-n)\Big) \mathrm{d}Z_{2N}.\end{aligned}$$

In the proof of Proposition 5.3.5, we have shown that the above integral satisfies

$$\begin{aligned}\log \int \exp\Big(-\beta \mathsf{F}^{\mathrm{dip}}_\lambda + C \sum_{i \in I^{\mathrm{dip}}, \phi_2(i) \in I^{\mathrm{dip}}} \Big(\frac{\mathsf{r}_1(z_i)}{\mathsf{r}_2(z_i)}\Big)^2 + C(N-n)\Big) \mathrm{d}Z_{2N} \\ \leqslant 2N \log N + (2-\beta)\log\lambda + N \log C_\beta - N + CN\gamma_\lambda.\end{aligned}$$

Together with the lower bound of Proposition 5.4.1, this concludes the proof of (5.20) for $-\frac{\beta}{2} \leqslant t \leqslant 0$.

We turn to the proof of Item (3) of Theorem 5.1.1. Let I be the set of the indices of positive charges belonging to a neutral dipole:

$$I = \{1 \leqslant i \leqslant N : \phi_1 \circ \phi_1(i) = i, d_i d_{\phi_1(i)} = -1\}. \tag{5.150}$$

Fix $n_0 \in \{0, \dots, N\}$. We seek to upper bound the probability that of having less than n_0 neutral dipoles. For each $1 \leqslant n \leqslant K \leqslant N$, select a functional digraph $\gamma^{K,n} \in D_{2N,K,n}$. By decomposing the event $\{|I| \leqslant n_0\}$ as the disjoint union of the events $\{\gamma_{2N} = \gamma^{K,n}\}$ for $n \leqslant n_0$ and $K \geqslant n$, we can write

$$\begin{aligned}\log \mathbb{P}^\lambda_{N,\beta}(|I| \leqslant n_0) \leqslant \max_{(K,n): K \geqslant n, n \leqslant n_0} \Big(\log \int_{\{\gamma_{2N} = \gamma^{K,n}\}} \exp(-\beta \mathsf{F}_\lambda) + \log|D_{2N,K,n}|\Big) \\ - \log Z^\lambda_{N,\beta} + O(\log N).\end{aligned} \tag{5.151}$$

We have already controlled this quantity in the proof of Proposition 5.3.5. For instance, for $\beta \in (2,4)$ one may insert (5.133) and we get

$$\log \int_{\{\gamma_{2N}=\gamma^{K,n}\}} \exp(-\beta \mathsf{F}_\lambda) \leqslant n(2-\beta)\log\lambda + N\log C_\beta \\ - (2N-n-K)\log\Big(\frac{2N}{2N-K-n}\Big) + c_0(N-n) + CN\gamma_\lambda,$$

for some constant $c_0 > 0$. This leads to optimizing the same function as in (5.85):

$$\max_{(K,n):K\geqslant n, n\leqslant n_0} \Big(\log \int_{\{\gamma_{2N}=\gamma^{K,n}\}} \exp(-\beta \mathsf{F}_\lambda) + \log |D_{2N,K,n}|\Big) \\ \leqslant 2N\log(2N) + N(\log C_\beta - 1 - 2\log 2) + \max_{(K,n):1\leqslant n\leqslant n_0, K\geqslant n} I(K,n) + CN\gamma_\lambda,$$

where $I(K,n)$ is as in (5.86). We have already optimized $I(K,n)$ under this constraint in the proof of Lemma 5.3.2. Applying the estimate (5.95) and the lower bound on $\log Z^\lambda_{N,\beta}$ found in Proposition 5.2.1, we thus find that for λ small enough,

$$\max_{(K,n):K\geqslant n, n\leqslant n_0} \Big(\log \int_{\{\gamma_{2N}=\gamma^{K,n}\}} \exp(-\beta \mathsf{F}_\lambda) + \log |D_{2N,K,n}|\Big) - \log Z^\lambda_{N,\beta} \\ \leqslant 2N\log N + N\log C_\beta - N + (\beta-2)\log\lambda(N-n_0) + C(N-n_0).$$

Hence there exist λ_0, $c>0$ and $M_1>0$ depending on β such that for any $|\lambda| \leqslant \lambda_0$,

$$\mathbb{P}^\lambda_{N,\beta}(|I| \leqslant N(1-c\gamma_\lambda)) \leqslant \exp(-M_1 N\gamma_\lambda). \tag{5.152}$$

One can next observe that

$$\mathbb{E}_{\mathbb{P}^\lambda_{N,\beta}}\Big[\exp\Big(\frac{\beta}{2}\mathsf{F}^{\mathrm{dip}}_\lambda\Big)\Big] = \log \mathbb{E}_{\mathbb{P}^\lambda_{N,\beta/2}}\Big[\exp\Big(\frac{\beta}{2}(\mathsf{F}^{\mathrm{dip}}_\lambda - \mathsf{F}_\lambda)\Big)\Big]\frac{Z^\lambda_{N,\beta/2}}{Z^\lambda_{N,\beta}}.$$

Using Theorem 5.1.1 to expand the ratio of partition function together with (5.20), we find that

$$\log \mathbb{E}_{\mathbb{P}^\lambda_{N,\beta}}\Big[\exp\Big(\sum_{i\in I}\mathsf{g}_1\Big(\lambda^{-1}(z_i - z_{\phi_1(i)})\Big)\Big)\Big] = O(N).$$

Together with Markov's inequality this concludes the proof of (5.22). □

5.5 Energetic control on linear statistics

In this section, we leverage on our ball-growth method for electric energy lower bounds of Section 5.2 to derive an energetic control on the fluctuations of linear statistics, which is the equivalent of [184, Prop 2.5] for the one-component plasma. In the next proposition, we show that the log-Laplace transform of linear statistics is of order of a power of λ times $N^{\frac{1}{2}}$ for some constant depending on β, provided the test-function is smooth enough. Let us emphasize that linear statistics are in fact expected to fluctuate much less, i.e in rate $o_N(\sqrt{N})$ for small but fixed λ. Proving such a rigidity statement would require more involved techniques.

Proof of Proposition 5.1.3. **Step 1: the electric energy bounds the fluctuations.** As in the one-component case [231, 185] or in [186], the fluctuations are well bounded by the electric energy $\int |\nabla h_{\vec{\alpha}}|^2$, where $h_{\vec{\alpha}}$ is as in (5.38), as soon as $\vec{\alpha}$ is small enough. We recall the elementary argument.

Let ξ be a Lipschitz test-function from Λ to $\mathbb{R}$. Taking the Laplacian of (5.38), using Green's formula and the Cauchy-Schwartz inequality, we have

$$\left|\int_\Lambda \xi \, \mathrm{d}\left(2\pi \sum_{i=1}^{2N} d_i \delta_{z_i}^{(\alpha_i)}\right)\right| = \left|\int_{\mathbb{R}^2} \xi \Delta h_{\vec{\alpha}}\right| \leqslant \|\nabla \xi\|_{L^2(\Lambda)} \left(\int_{\mathbb{R}^2} |\nabla h_{\vec{\alpha}}|^2\right)^{\frac{1}{2}}.$$

On the other hand, by definition of the smeared charges, we may write

$$\left|\int_\Lambda \xi \, \mathrm{d}\left(2\pi \sum_{i=1}^{2N} d_i (\delta_{z_i} - \delta_{z_i}^{(\alpha_i)})\right)\right| \leqslant \|\nabla \xi\|_{L^\infty} \sum_{i=1}^{2N} \alpha_i.$$

Combining the two relations, we deduce that

$$|\mathrm{Fluct}_N(\xi)| \leqslant \frac{1}{2\pi} \|\nabla \xi\|_{L^\infty} \left(\left(\int_{\mathbb{R}^2} |\nabla h_{\vec{\alpha}}|^2\right)^{\frac{1}{2}} + \sum_{i=1}^{2N} \alpha_i\right),$$

hence

$$|\mathrm{Fluct}_N(\xi)|^2 \leqslant C \|\nabla \xi\|_{L^\infty}^2 \left(\int_{\mathbb{R}^2} |\nabla h_{\vec{\alpha}}|^2 + \left(\sum_{i=1}^{2N} \alpha_i\right)^2\right), \tag{5.153}$$

Step 2: upper bound for the electric energy. The proof consists in repeating the proof of Proposition 5.2.1. We define the radii τ_i as in that proposition and let

$$\alpha_i = \begin{cases} \lambda & \text{if } \mathsf{r}_1(z_i) > \gamma \\ \tau_i \wedge \gamma & \text{if } \mathsf{r}_1(z_i) \leqslant \gamma \end{cases} \tag{5.154}$$

for a $\gamma \in (0,1)$ to be chosen later. We then bound from below $\int |\nabla h_{\vec{\alpha}}|^2$ as in the proof of Proposition 5.2.1. The points such that $\mathsf{r}_1(z_i) > \gamma$ do not contribute any terms since the corresponding balls are not inflated. We obtain the same contributions for the other points as in Proposition 5.2.1, except with the $\mathsf{r}_2(z_i)$ replaced by $\gamma \wedge \mathsf{r}_2(z_i)$. Arguing also as in the proof of Corollary 5.2.2, we may obtain

$$\begin{aligned}
\frac{1}{4\pi}\int_{\mathbb{R}^2} |\nabla h_\lambda|^2 - N(\mathsf{g}(\lambda) + \kappa) &\geqslant \frac{1}{4\pi}\int_{\mathbb{R}^2} |\nabla h_{\vec{\alpha}}|^2 - \frac{1}{2} \sum_{i \notin \{\cup_k C_k, d_i d_{\phi_1(i)} = 1\}, \mathsf{r}_1(z_i) \leqslant \gamma} \mathsf{g}_\lambda(z_i - z_{\phi_1(i)}) \\
&\quad - C \sum_{i \in I^{\mathrm{dip}}, \phi_2(i) \in I^{\mathrm{dip}}} \left(\frac{\mathsf{r}_1(z_i)}{\mathsf{r}_2(z_i) \wedge \gamma}\right)^2 \mathbf{1}_{\mathsf{r}_1(z_i) \leqslant \gamma} - C(N-n) \\
&\geqslant \frac{1}{4\pi}\int_{\mathbb{R}^2} |\nabla h_{\vec{\alpha}}|^2 - \frac{1}{2} \sum_{i \notin \{\cup_k C_k, d_i d_{\phi_1(i)} = 1\}} \mathsf{g}_\lambda(z_i - z_{\phi_1(i)}) \\
&\quad - C \sum_{i \in I^{\mathrm{dip}}, \phi_2(i) \in I^{\mathrm{dip}}} \left(\frac{\mathsf{r}_1(z_i)}{\mathsf{r}_2(z_i)}\right)^2 + \left(\frac{\mathsf{r}_1(z_i)}{\gamma}\right)^2 \mathbf{1}_{\mathsf{r}_1(z_i) \leqslant \gamma} - C(N-n).
\end{aligned}$$

We may rewrite this as

$$\frac{1}{4\pi}\int_{\mathbb{R}^2} |\nabla h_{\vec{\alpha}}|^2 \leqslant A_1 + C(A_2 + A_3),$$

with

$$A_1 = \mathsf{F}_\lambda(Z_{2N}) - \mathsf{F}_\lambda^{\mathrm{dip}}(Z_{2N}),$$

$$A_2 = \sum_{i \in I^{\mathrm{dip}}, \phi_2(i) \in I^{\mathrm{dip}}} \left(\frac{\mathsf{r}_1(z_i)}{\mathsf{r}_2(z_i)}\right)^2 + (N-n),$$

$$A_3 = \sum_{i \in I^{\mathrm{dip}}, \phi_2(i) \in I^{\mathrm{dip}}} \left(\frac{\mathsf{r}_1(z_i)}{\gamma}\right)^2 \mathbf{1}_{\mathsf{r}_1(z_i) \leqslant \gamma}.$$

Step 3: bounding exponential moments of the electric energy. Let us estimate the exponential moments of A_1 and A_2 separately. One has

$$\mathbb{E}_{\mathbb{P}_{N,\beta}^\lambda}[\exp(\beta A_1)] = \frac{1}{Z_{N,\beta}^\lambda} \int \exp(-\beta \mathsf{F}_\lambda^{\mathrm{dip}}(Z_{2N})) \mathrm{d}Z_{2N}.$$

In view of Lemma 5.3.2 and Proposition 5.4.1 we have

$$\log \mathbb{E}_{\mathbb{P}_{N,\beta}^\lambda}[\exp(\beta A_1)] \leqslant CN(|\log \lambda|^{-1} \mathbf{1}_{\beta=2} + \lambda^{\beta-2} \mathbf{1}_{\beta \in (2,4)} + \lambda^2 |\log \lambda|^2 \mathbf{1}_{\beta=4} + \lambda^2 |\log \lambda| \mathbf{1}_{\beta>4}). \tag{5.155}$$

For the term A_2, we have already shown that

$$\log \mathbb{E}_{\mathbb{P}_{N,\beta}^\lambda}\Big[\exp\Big(\sum_{i \in I^{\mathrm{dip}}, \phi_2(i) \in I^{\mathrm{dip}}} \Big(\frac{\mathsf{r}_1(z_i)}{\mathsf{r}_2(z_i)}\Big)^2\Big)\Big] \leqslant CN(|\log \lambda|^{-1} \mathbf{1}_{\beta=2} + \lambda^{\beta-2} \mathbf{1}_{\beta \in (2,4)} + \lambda^2 |\log \lambda|^2 \mathbf{1}_{\beta=4} + \lambda^2 |\log \lambda| \mathbf{1}_{\beta>4}). \tag{5.156}$$

Using Corollary 5.2.2 we find that

$$\mathbb{E}_{\mathbb{P}_{N,\beta}^\lambda}[e^{tA_3}] \leqslant \frac{1}{Z_{N,\beta}^\lambda} \int \exp(tA_3 - \beta \mathsf{F}_\lambda^{\mathrm{dip}} + G),$$

where $\mathsf{F}_\lambda^{\mathrm{dip}}$ is as in (5.64) and G given by

$$G := C \sum_{i \in I^{\mathrm{dip}}, \phi_2(i) \in I^{\mathrm{dip}}} \left(\frac{\mathsf{r}_1(z_i)}{\mathsf{r}_2(z_i)}\right)^2 + C(N-n),$$

with n being the number of neutral dipoles. Let us denote $\mathbb{Q}$ the reduced dipole model

$$\mathrm{d}\mathbb{Q} = \frac{1}{K_{N,\beta}^\lambda} \exp(-\beta \mathsf{F}_\lambda^{\mathrm{dip}}) \mathrm{d}Z_{2N}.$$

Using Hölder's inequality, one may write

$$\mathbb{E}_{\mathbb{P}_{N,\beta}^\lambda}[\exp(tA_3)] \leqslant \mathbb{E}_{\mathbb{Q}}[\exp(2tA_3)]^{\frac{1}{2}} \mathbb{E}_{\mathbb{Q}}[e^{2G}]^{\frac{1}{2}} \frac{K_{N,\beta}^\lambda}{Z_{N,\beta}^\lambda}. \tag{5.157}$$

Inserting the upper bound on $K_{N,\beta}^\lambda$ given by Lemma 5.3.2, the lower bound on $Z_{N,\beta}^\lambda$ of Proposition 5.4.1, the auxiliary estimate of Lemma 5.3.4, we reduce to

$$\log \mathbb{E}_{\mathbb{P}_{N,\beta}^\lambda}[\exp(tA_3)] \leqslant \frac{1}{2} \log \mathbb{E}_{\mathbb{Q}}[\exp(2tA_3)] + CN(|\log \lambda|^{-1} \mathbf{1}_{\beta=2} + \lambda^{\beta-2} \mathbf{1}_{\beta \in (2,4)} + \lambda^2 |\log \lambda|^2 \mathbf{1}_{\beta=4} + \lambda^2 |\log \lambda| \mathbf{1}_{\beta>4}).$$

After some computations we find that

$$\begin{aligned}\log\int \exp(tA_3-\beta\mathsf{F}_\lambda^{\mathrm{dip}}) &\leqslant \frac{\beta}{2}N\log N - N + N\log\Big(\lambda^{2-\beta}C_\beta\mathbf{1}_{\beta>2}+|\log\lambda|\mathbf{1}_{\beta=2}\\ &\quad + C\mathbf{1}_{\beta=2}+\gamma^{2-\beta}\mathbf{1}_{\beta\in(2,4)}+\lambda^{4-\beta}\gamma^{-2}\mathbf{1}_{\beta\in(4,\infty)}+|\log\lambda|\gamma^{-2}\mathbf{1}_{\beta=4}\Big)+C\log N\\ &\leqslant \frac{\beta}{2}N\log N - N + N\log(\lambda^{2-\beta}C_\beta)\mathbf{1}_{\beta>2}+N\log|\log\lambda|\mathbf{1}_{\beta=2}\\ &\quad + CN\Big(|\log\lambda|^{-1}\mathbf{1}_{\beta=2}+(\lambda/\gamma)^{\beta-2}\mathbf{1}_{\beta\in(2,4)}+(\lambda/\gamma)^2|\log\lambda|\mathbf{1}_{\beta=4}+(\lambda/\gamma)^2\mathbf{1}_{\beta>4}\Big).\end{aligned}$$

We thus conclude combining the last display with (5.155) and (5.156) that

$$\begin{aligned}&\log\mathbb{E}_{\mathbb{P}^\lambda_{N,\beta}}\left[\exp\left(\int_{\mathbb{R}^2}|\nabla h_{\vec\alpha}|^2\right)\right]\\ &\quad\leqslant CN\Big(|\log\lambda|^{-1}\mathbf{1}_{\beta=2}+(\lambda/\gamma)^{\beta-2}\mathbf{1}_{\beta\in(2,4)}+(\lambda/\gamma)^2|\log\lambda|\mathbf{1}_{\beta=4}+(\lambda/\gamma)^2\mathbf{1}_{\beta>4}\Big).\end{aligned}\tag{5.158}$$

Step 4: conclusion. Combining (5.158) with (5.153), we obtain

$$\begin{aligned}&\frac12\log\mathbb{E}_{\mathbb{P}^\lambda_{N,\beta}}\left[\exp(|\mathrm{Fluct}_N(\xi)|^2)\right]\\ &\quad\leqslant C\|\nabla\xi\|_{L^\infty}\Big(N\gamma^2+CN\Big(|\log\lambda|^{-1}\mathbf{1}_{\beta=2}+(\lambda/\gamma)^{\beta-2}\mathbf{1}_{\beta\in(2,4)}+(\lambda/\gamma)^2|\log\lambda|\mathbf{1}_{\beta\in(4,\infty)}\\ &+(\lambda/\gamma)^2\mathbf{1}_{\beta>4}+N(|\log\lambda|^{-1}\mathbf{1}_{\beta=2}+\lambda^{\beta-2}\mathbf{1}_{\beta\in(2,4)}+\lambda^2|\log\lambda|^2\mathbf{1}_{\beta=4}+\lambda^2|\log\lambda|\mathbf{1}_{\beta>4}\Big)\Big).\end{aligned}\tag{5.159}$$

Optimizing over γ, we may then choose γ as follows:

$$\gamma=\begin{cases}|\log\lambda|^{-1/2} & \text{if } \beta=2\\ \lambda^{\frac{(\beta-2)}{\beta}} & \text{if } \beta\in(2,4)\\ \lambda^{1/2}|\log\lambda|^{1/4} & \text{if } \beta=4\\ \lambda^{1/2} & \text{if } \beta\in(4,\infty).\end{cases}\tag{5.160}$$

Inserting this into (5.159) concludes the proof of the proposition. □

5.6 Convergence to a Poisson dipole process

In this subsection we show that the empirical field defined in (5.30) satisfies in the large N limit and as λ tends to 0, a large deviations principle with rate function given by a certain entropy on point processes, which differs from the specific relative entropy of [183]. Recall the definitions of $(E,\mathcal{T})$, i_N, $\mathcal{P}^{\mathrm{dip}}$, $\bar{\mathcal{P}}^{\mathrm{dip}}$ from (5.28), (5.30), (5.31) and (5.32). Let us recall that the Borel σ-algebra on E can be defined by the σ-algebra generated by functions of the form

$$\mathbf{1}_{A_1}(x_{n_1})\mathbf{1}_{B_1}(y_{n_1})\dots\mathbf{1}_{A_p}(x_{n_p})\mathbf{1}_{B_p}(y_{n_p}),\tag{5.161}$$

where $n_1,\dots,n_p\in I$, $A_1,\dots,A_p$ are bounded measurable sets of $\mathbb{R}^2$, $B_1,\dots,B_p$ measurable sets of $\mathbb{R}^2$. We have thus defined a probability space $(E,\mathcal{A})$. One can check that the topological space $(E,\mathcal{T})$ can be endowed with a distance by setting

$$d(\mathcal{C},\mathcal{C}')=\sum_{k\geqslant1}\frac{1}{2^k}\frac{\sup\left\{|\sum_{i\in J}(f(x_i,y_i)-f(x_i',y_i'))\mathbf{1}_{x_i,x_i'\in\Lambda_k}|:f,g\in\mathrm{Lip}_1(\mathbb{R}^2\times\mathbb{R}^2)\right\}}{|\mathcal{C}|(\Lambda_k)+|\mathcal{C}'|(\Lambda_k)},\tag{5.162}$$

where $\Lambda_k = [0, \sqrt{k}]^2$.

Let $\mathcal{C}_b(E)$ be the set of continuous bounded functions from $(E, \mathcal{T})$ to $(\mathbb{R}, |\cdot|)$. The relative entropy (5.33) with respect to $\mathcal{P}^{\mathrm{dip}}$ can be expressed as

$$\mathrm{Ent}[P \mid \mathcal{P}^{\mathrm{dip}}] = \sup_{f \in \mathcal{C}_b(E)} (\mathbb{E}_P[f] - \log \mathbb{E}_{\mathcal{P}^{\mathrm{dip}}}[e^f]). \tag{5.163}$$

The proof of Theorem 5.1.2 follows the line of reasoning of the Gärtner-Ellis theorem, also used in [212, Chapter 6] to prove a process-level LDP for the empirical field in a discrete setting. The first step is to replace the large deviation principle of [130] for the Poisson process by an analogous statement for our Poissonian dipole process. For all $\mathcal{C} = \sum_{i \in I} \delta_{(x_i, y_i)} \in E$ and $x \in \Lambda$, we let

$$\theta_x \cdot \mathcal{C} := \sum_{i \in I} \delta_{(x_i - x, y_i)}. \tag{5.164}$$

We also define the maps

$$j_n : \mathcal{C} \in E \mapsto \frac{1}{|\Lambda_n|} \int_{\Lambda_n} \delta_{\theta_x \cdot \mathcal{C}} \mathrm{d}x \in \mathcal{P}(E), \tag{5.165}$$

$$\bar{j}_n : \mathcal{C} \in E \mapsto \frac{1}{|\Lambda_n|} \int_{\Lambda_n} \delta_{(x, \theta_x \cdot \mathcal{C})} \mathrm{d}x \in \mathcal{P}(\Lambda \times E). \tag{5.166}$$

Lemma 5.6.1. *Let (Λ_n) be an increasing sequence of cubes such that $\cup_n \Lambda_n = \mathbb{R}^2$. Let R_n be the push-forward of $\mathcal{P}^{\mathrm{dip}}$ by the map (5.165). Then (R_n) satisfies a large deviation principle at speed $|\Lambda_n|$ with rate function $\mathrm{Ent}(\cdot \mid \mathcal{P}^{\mathrm{dip}})$.*

To prove Lemma 5.6.1, we adapt almost line by line the proof of [212, Chapter 6]. We begin by showing that given a local continuous bounded function on E, the limit (5.167) is well-defined, thus defining the so-called pressure. Using the variational characterization of the entropy, this will prove the upper bound for all compact sets, which can be extended to an upper bound for all closed sets by exponential tightness. The proof of the lower bound is similar to Cramer's theorem in that it uses a change of measure but the law of large numbers is replaced by the ergodic theorem.

Proof. Without loss of generality one may assume that $\Lambda_n = [0, \sqrt{n}]^2$.
Step 1: study of the pressure. Let $f \in \mathcal{C}_{b,loc}(E)$. One shall first prove that the following limit is well-defined:

$$p(f) := \lim_{n \to \infty} \frac{1}{|\Lambda_n|} \log \mathbb{E}_{\mathcal{P}^{\mathrm{dip}}} \left[\exp \int_{\Lambda_n} f(\mathcal{C}_N(x)) \mathrm{d}x \right]. \tag{5.167}$$

The proof proceeds by a super-additivity argument. For each $n \in \mathbb{N}^*$, denote

$$p_n(f) = \frac{1}{|\Lambda_n|} \log \mathbb{E}_{\mathcal{P}^{\mathrm{dip}}} \left[\exp \int_{\Lambda_n} f(\mathcal{C}_N(x)) \mathrm{d}x \right]. \tag{5.168}$$

As in [212, Prop 6.14], we cover the set Λ_n with shifted well separated cubes of size m for some $m < N$. Let $m < N$ and $\Lambda_m^{(l)} \subset \Lambda_n$, for $l \in \{1, \ldots, k^2\}$ be k^2 shifted copies of Λ_l, chosen so that the distance between each consecutive subcube is at distance r in each direction. One may take $k = \lfloor \frac{2n-1}{2m++2n-1} \rfloor$. One can check that the volume not covered by the union of the $\Lambda_m^{[l]}$'s satisfies

$$|\Lambda_n| - k^2 |\Lambda_m| \leqslant |\Lambda_n| \kappa_{N,m}$$

with

$$\lim_{m \to \infty} \lim_{n \to \infty} \kappa_{n,m} = 0. \tag{5.169}$$

By construction the variables $\int_{\Lambda_m^{(l)}} f(\mathcal{C}_n(x))\mathrm{d}x$, $1 \leqslant l \leqslant k^2$ are independent. We then conclude with (5.169) and independence that

$$p_n(f) \leqslant \kappa_{N,m}\|f\|_\infty + \frac{|\Lambda_m|}{(2m+2r-1)^2} p_m(f).$$

We deduce that (5.167) is well-defined. In addition we find that if f is bounded and $\mathcal{F}_{\Lambda_m}$-measurable, then

$$p(f) \leqslant \frac{1}{|\Lambda_m|} \log \mathbb{E}\left[\exp(|\Lambda_m| f)\right]. \tag{5.170}$$

Step 2: duality. We now put the space $\mathcal{P}(E)$ in duality with $\mathcal{C}_{b,loc}(E)$. We claim that for all $Q \in \mathcal{P}(E)$,

$$p^*(Q) = \mathrm{Ent}(Q \mid \mathcal{P}^{\mathrm{dip}}), \tag{5.171}$$

where

$$p^*(Q) := \sup_{f \in \mathcal{C}_{b,loc}(E)} (\mathbb{E}_Q[f] - p(f)).$$

Let $f \in \mathcal{C}_{n,loc}(E)$. Let m such that $f = f|_{\Lambda_m}$. In view of (5.170), we have

$$p^*(Q) \geqslant \mathbb{E}_Q\Big[\frac{f}{|\Lambda_m|}\Big] - p\Big(\frac{f}{|\Lambda_m|}\Big) \geqslant \frac{1}{|\Lambda_m|}(\mathbb{E}_Q[f] - \log \mathbb{E}[e^f]).$$

Taking the supremum over $f \in \mathcal{C}_{b,loc}(E)$ yields

$$p^*(Q) \geqslant \mathrm{Ent}(Q|_{\Lambda_m} \mid P|_{\Lambda_m})$$

and therefore $p^*(Q) \geqslant \mathrm{Ent}(Q \mid P)$. Conversely, $\bar{f} = \int_{\Lambda_n} f(\mathcal{C}_n(x))\mathrm{d}x$ is $\mathcal{F}_{\Lambda_{m+n}}$-measurable and

$$\mathrm{Ent}(Q|_{\Lambda_{m+n}} \mid P|_{\Lambda_{m+n}}) \geqslant \mathbb{E}_Q[\bar{f}] - \log \mathbb{E}[e^{\bar{f}}] = |\Lambda_n|(\mathbb{E}_Q[f] - p_n(f)),$$

where $p_n(f)$ is as in (5.168). By letting n tend to infinity, we thus find

$$\mathrm{Ent}(Q \mid P) \geqslant \mathbb{E}_Q[f] - p(f)$$

and therefore $\mathrm{Ent}(Q \mid P) \geqslant p^*(Q)$.

Step 3: exponential tightness. Let us now show that (j_n) is tight, meaning that for all $b > 0$ there exists a compact $K_b \subset \mathcal{P}(E)$ such that $\mathcal{P}^{\mathrm{dip}}(j_n \notin K_b) \leqslant e^{-|\Lambda_n| b}$. For $R > 0$, let $\mathcal{N}_R : \Lambda \times E \to \mathbb{R}^+$ be the map such that $\mathcal{N}_R(r, \mathcal{C}) = |\mathcal{C}^+| \cap \Lambda_R$. One can observe that j_N is supported on the set

$$\bigcap_{R \in \mathbb{N}^*} \{P \in \mathcal{P}(\Lambda \times E) : \mathbb{E}_P[\mathcal{N}_R] \leqslant 2\pi R^2\}, \tag{5.172}$$

which is a compact set of $\mathcal{P}(E)$, see for instance [183, Lemma 7.7]. This proves that (j_n) is tight.

Step 4: upper bound. The upper bound can be first proved for compact subsets proceeding as in the proof of Cramer's theorem, see also [212, Theorem 4.24]. It then follows from (5.171) that for any compact set $F \subset \mathcal{P}(E)$,

$$\limsup_{n \to \infty} \frac{1}{n} \log \mathcal{P}^{\mathrm{dip}}(i_n \in F) \leqslant -\inf_{Q \in F} \mathrm{Ent}(Q \mid \mathcal{P}^{\mathrm{dip}}).$$

This upper bound can be generalized to closed sets by using the fact that (i_N) is exponentially tight in $\mathcal{P}(\Lambda \times E)$.

Step 5: density of ergodic processes. We denote $\mathcal{P}_{1,s}(E)$ the set of point processes on E which are translation invariant (or stationary) and $\mathcal{P}_{1,e}(E) \subset \mathcal{P}_{1,s}(E)$ the subset of extreme points of $\mathcal{P}_{1,s}(E)$. Note that $\mathcal{P}_{1,s}(E)$ is exactly the set of ergodic processes on E. Recall that a stationary point process P is ergodic if and only if for all $A \in \mathcal{T}$ which is invariant by translation, $P(A) \in \{0,1\}$. We claim that for all $Q \in \mathcal{P}_{1,s}(E)$, there is a sequence of ergodic processes (Q_k) which converges weakly to Q and such that

$$\lim_{k\to\infty} \mathrm{Ent}(Q_k \mid \mathcal{P}^{\mathrm{dip}}) = \mathrm{Ent}(Q \mid \mathcal{P}^{\mathrm{dip}}).$$

The proof can be adapted readily from [212, Lemma 6.9].

Step 6: lower bound. Let O be an open subset of $\mathcal{P}(E)$. Let $Q \in \mathcal{P}(E)$ be an ergodic process. One can assume that $\mathrm{Ent}(Q \mid \mathcal{P}^{\mathrm{dip}}) < \infty$, which implies that for all Λ_R, $Q|_{\Lambda_R}$ has a Radon-Nikodym derivative $f_R = \frac{\mathrm{d}Q|_{\Lambda_R}}{\mathrm{d}(\mathcal{P}^{\mathrm{dip}})|_{\Lambda_R}}$. One can thus perform the following change of variables:

$$\begin{aligned}\frac{1}{|\Lambda_n|}\log \mathcal{P}^{\mathrm{dip}}(j_n \in O) &= \frac{1}{|\Lambda_n|}\log \int \mathbf{1}_O(j_n) f_n^{-1}\mathrm{d}Q_n\\ &= \frac{1}{|V_n|}\log Q(j_n\in O) + \frac{1}{|V_n|}\log\Big[\frac{1}{Q(j_n\in O)}\int_{j_n\in O} f_n^{-1}\mathrm{d}Q_n\Big]\\ &\geqslant \frac{1}{|V_n|}\log Q(j_n \in O) - \frac{1}{|V_n| Q(j_n\in O)}\int_{j_n\in O}\log f_n \mathrm{d}Q_n,\end{aligned}$$

where we have used Jensen's inequality in the last line. Using the fact that $x\log x \geqslant -\frac{1}{e}$ for all $x>0$, we find that

$$\frac{1}{|\Lambda_n|}\log \mathcal{P}^{\mathrm{dip}}(j_n \in O) \geqslant \frac{1}{|V_n|}\log Q(j_n\in O) - \frac{1}{Q(j_n \in O)|V_n|}\mathrm{Ent}(Q|_{\Lambda_n} \mid P|_{\Lambda_n}) - \frac{1}{e|V_n|Q(j_n\in O)}.$$

One may then justify that

$$\lim_{n\to\infty} Q(j_n \in O) = 1$$

and therefore

$$\lim_{n\to\infty}\frac{1}{|V_n|}\log \mathcal{P}^{\mathrm{dip}}(j_n\in O) \geqslant -\mathrm{Ent}(Q\mid \mathcal{P}^{\mathrm{dip}}),$$

which gives

$$\lim_{n\to\infty}\frac{1}{|\Lambda_n|}\log \mathcal{P}^{\mathrm{dip}}(j_n\in O) \geqslant -\inf_{Q\in\mathcal{P}_{1,e}(E)}\mathrm{Ent}(Q\mid\mathcal{P}^{\mathrm{dip}}). \tag{5.173}$$

Arguing with the density result of Step 5, this concludes the proof of the lower bound for all closed sets. □

Next one extends the large deviations principle of Lemma 5.6.1 to the sequence of the push-forwards of $\mathbb{P}^{\lambda}_{N,\beta}$ by (5.30). There are three tasks to deal with: one should handle the tagged microscopic field instead of j_N, reduce the problem to an LDP under the dipole measure $\mathbb{Q}$ and finally deal with Bernoulli variables instead of Poissonian variables.

Proof of Theorem 5.1.2. **Step 1: reduction to the dipole measure.** To lighten the notation, set

$$\gamma_\lambda := \begin{cases} \frac{1}{|\log\lambda|} & \text{if } \beta = 2\\ \lambda^{\beta-2} & \text{if } \beta\in(2,4)\\ \lambda^2|\log\lambda|^2 & \text{if } \beta = 4\\ \lambda^2|\log\lambda| & \text{if } \beta>4.\end{cases} \tag{5.174}$$

The results of the previous sections show that there exists a constant $C > 0$ such that for every measurable subset B of Λ^{2N},

$$\left|\log \int_B \exp(-\beta \mathsf{F}_\lambda) - \log \int_B \exp(-\beta \mathsf{F}_\lambda^{\mathrm{dip}})\right| \leqslant CN\gamma_\lambda.$$

In addition, one also has the stronger statement

$$\left|\log \int_B \exp(-\beta \mathsf{F}_\lambda) - \log \int_{B\cap\{|I^c|\leqslant C_0 N\alpha_\lambda} \exp(-\beta \mathsf{F}_\lambda^{\mathrm{dip}}(Z_{2N}))\mathrm{d}Z_{2N}\right| \leqslant CN\gamma_\lambda,$$

where I is as in (5.19). Let $n \geqslant N(1-C_0\alpha_\lambda)$. Now assume that B is given by $B = \{i_N(X_N, Y_N) \in G\}$ where G is a measurable subset of $\mathcal{P}(\Lambda \times E)$. Let us denote

$$\tilde{B}_n = \{(X_N, Y_N) \in \Lambda_N^N \times (\lambda^{-1}\Lambda_N)^N : \bar{j}_N(X_N, Y_N) \in G\}.$$

One then reduces the integral over $\mathrm{d}Z_{2N}$ above as an integral on $2n$ variables. Given $Z_{2n} \in \Lambda^{2n}$, we let $\gamma_{2n}(Z_{2n})$ be the nearest-neighbor graph of Z_{2n} and let $\gamma \in D_{2n,n,n}$ be a graph with n neutral 2-cycles. Using the upper bound (5.116) of Lemma 5.3.4, we find

$$\begin{aligned}
&\log \int_{B\cap\{|I^c|\leqslant C_0 N\lambda^{\beta-2}} \exp(-\beta \mathsf{F}_\lambda^{\mathrm{dip}}) \\
&\quad = \log\Big(\sum_{n\geqslant N(1-C_0\lambda^{\beta-2})} n! \int_{B\cap\{\gamma_{2n}=\gamma\}} \prod_{i=1}^n \exp\Big(\frac{\beta}{2}\mathsf{g}_\lambda(u_i)\Big)\mathrm{d}u_1\ldots\mathrm{d}u_n\mathrm{d}x_1\ldots\mathrm{d}x_n\Big) + O(N\gamma_\lambda).
\end{aligned}$$

After a series of reduction one then estimates

$$\begin{aligned}
&\log \int_{B\cap\{|I^c|\leqslant C_0 N\lambda^{\beta-2}} \exp(-\beta \mathsf{F}_\lambda^{\mathrm{dip}}) \\
&\qquad = \log\Big(\sum_{n\geqslant N(1-C_0\lambda^{\beta-2})} n! \int_B \prod_{i=1}^n \exp\Big(\frac{\beta}{2}\mathsf{g}_\lambda(u_i)\Big)\mathrm{d}u_1\ldots\mathrm{d}u_n\mathrm{d}x_1\ldots\mathrm{d}x_n\Big) + O(N\gamma_\lambda).
\end{aligned}$$

Now by scaling,

$$\begin{aligned}
&\int_B \prod_{i=1}^n \exp\Big(\frac{\beta}{2}\mathsf{g}_\lambda(u_i)\Big)\mathrm{d}U_n\mathrm{d}X_n \\
&\quad = \lambda^{n(2-\beta)}N^{-n}\int_{\Lambda_N^n\times(\lambda^{-1}\Lambda_N)^n} \mathbf{1}_B(N^{1/2}X_n, N^{1/2}Y_N)\prod_{i=1}^n \exp\Big(\frac{\beta}{2}\mathsf{g}_1(u_i)\Big)\mathrm{d}U_n\mathrm{d}X_n \\
&\qquad = \lambda^{n(2-\beta)}N^{-n}\int_{\Lambda_n^n\times(\lambda^{-1}\Lambda_n)^n} \mathbf{1}_{\{\bar{j}_n(X_n,U_n)\in G\}}\mathrm{d}\mu_\beta^{\otimes n}(U_n)\mathrm{d}X_n + O(N\gamma_\lambda).
\end{aligned}$$

We finally obtain

$$\begin{aligned}
\log \int_{i_N\in G} \exp(-\beta \mathsf{F}_\lambda) &= N + N((2-\beta)\log\lambda \mathbf{1}_{\beta>2} + \log|\log\lambda|\mathbf{1}_{\beta=2}) \\
&\quad + \log \int_{\Lambda_N^N\times(\lambda^{-1}\Lambda_N)^N} \prod_{i=1}^N \mathbf{1}_{\bar{j}_N(X_N,U_N)\in G} \exp\Big(\frac{\beta}{2}\mathsf{g}_1(u_i)\Big)\mathrm{d}X_N\mathrm{d}U_N. \quad (5.175)
\end{aligned}$$

Step 2: from Poisson to Bernoulli. The only differences between the above integral and $\bar{\mathcal{P}}_d(\bar{j}_n \in G)$ are the fact that the number of positive charges falling into a given domain is not Poissonian but rather Bernoulli and the fact that the u_i's are distributed according to the law μ_β truncated at λ^{-1}. Arguing as in [183], we can prove that for all closed set $F \subset \mathcal{P}(\Lambda \times E)$ and open set $O \subset \mathcal{P}(\Lambda \times E)$,

$$\limsup_{\lambda \to 0} \limsup_{n \to \infty} \frac{1}{n} \log \int_{\Lambda_n^n \times (\lambda^{-1}\Lambda_n)^n} \mathbf{1}_{\{\bar{j}_n(X_n, U_n) \in F\}} \mathrm{d}\mu_\beta^{\otimes n}(U_n) \mathrm{d}X_n \leqslant - \inf_{\bar{Q} \in F} \overline{\mathrm{Ent}}(Q \mid \bar{\mathcal{P}}^{\mathrm{dip}}),$$

$$\liminf_{\lambda \to 0} \liminf_{n \to \infty} \frac{1}{n} \log \int_{\Lambda_n^n \times (\lambda^{-1}\Lambda_n)^n} \mathbf{1}_{\{\bar{j}_n(X_n, U_n) \in O\}} \mathrm{d}\mu_\beta^{\otimes n}(U_n) \mathrm{d}X_n \leqslant - \inf_{\bar{Q} \in O} \overline{\mathrm{Ent}}(Q \mid \bar{\mathcal{P}}^{\mathrm{dip}}).$$

Inserting this into (5.175) yields the claimed result. □

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RÉSUMÉ

Cette thèse se propose d'étudier divers problèmes de mécanique statistique pour une famille de systèmes de particules en interaction, appelés gaz de Coulomb et de Riesz.

Nous commençons par examiner le temps de mélange du mouvement Brownien de Dyson avec confinement quadratique, dont la mesure invariante est donnée par le beta-ensemble d'Hermite. Nous établissons un résultat de cutoff pour le temps de mélange du système dans une variété de distances et de divergences, lorsque le nombre de particules tend vers l'infini.

Nous considérons ensuite les fluctuations et corrélations du gaz de Riesz circulaire dans le régime longue portée. Tout d'abord, nous quantifions les fluctuations des espacements entre particules et énonçons un théorème central limite pour les statistiques linéaires valables pour des fonctions-tests possiblement très singulières. Puis nous montrons une estimée optimale sur la décroissance de la corrélation des gaps, qui nous permet de montrer l'unicité du processus limite en volume infini.

La suite de ce manuscrit est consacrée à l'étude du gaz de Coulomb bi-dimensionnel à deux composantes dans un régime de basse température où la fonction de partition diverge. Après avoir proposé une renormalisation efficace du modèle, nous donnons un développement asymptotique de la fonction de partition lorsque le paramètre de troncature tend vers zéro, des estimées sur nombre et la taille de dipôles neutres ainsi qu'un contrôle énergétique sur les fluctuations.

MOTS CLÉS

Systèmes de particules en interaction à longue portée, temps de mélange, grandes déviations, fluctuations, décroissance des corrélations, limite thermodynamique, transition de phase.

ABSTRACT

This thesis is devoted to the analysis of different problems concerning the statistical mechanics of a family of interacting particles systems, named Coulomb and Riesz gases.

We begin by studying the mixing time of the Dyson Brownian motion with quadratic confinement, whose invariant measure is the Hermite beta-ensemble. We establish a cutoff phenomenon for the mixing time in a variety of distances and divergences, when the number of particles tend to infinity.

We then consider the fluctuations and correlations of the circular Riesz gas in the long-range regime. First, we quantify the fluctuations of gaps and give a central limit theorem for linear statistics allowing very singular test-functions. Second, one shows an optimal estimate on the decay of gaps correlations, allowing one to prove the uniqueness of the infinite volume measure.

The rest of the manuscript is devoted to the study of the two-dimensional two-component plasma in a low temperature regime where the partition function diverges. After proposing an efficient way to renormalize the model, we derive an asymptotic expression for the partition function as the truncation parameter tends to zero, some estimates on the number and size of neutral dipoles and an energetic control on the fluctuations.

KEYWORDS

Long-range interacting particles system, mixing time, large deviations, fluctuations, fluctuations, decay of correlations, thermodynamic limit, phase transition.