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Sur des propriétés de certains modèles en mécanique des fluides près d'une surface

El Mehdi Said

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Normandie Université

THÈSE

Pour obtenir le diplôme de doctorat

Spécialité MATHÉMATIQUES

Préparée au sein de l'Université de Rouen Normandie

Sur des propriétés de certains modèles en mécanique des fluides près d'une surface

Présentée et soutenue par
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**Thèse soutenue le 03/10/2022
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Chapter 1

Introduction

In this thesis, we are interested on some model in fluid mechanics :

- In part I, we are interested in some systems of PDEs which describe the motion of turbulent flows arising in some atmospheric and oceanographic models. More precisely, we study the α -Navier Stokes system in a thin strip, which is a relevant simplification of the Navier Stokes equation for turbulent flows from a Direct Numerical Simulation and physical points of view.

We prove the globally existence and uniqueness of a solution of the Cauchy problem for the α -Navier Stokes system and also for its formal hydrostatic limit system in some analytic spaces for small initial data. Then we justify this hydrostatic limit.

- In part II, we are concerned with the Prandtl system for non-Newtonian fluids. We first consider the case in dimension one, which is the shear flow problem. We prove the local existence and the uniqueness in the parabolic Hölder and weighted Sobolev spaces for monotone initial data.

We then consider in dimension two an approximated system of the Non-newtonian Prandtl equation with an extra dissipation term $\varepsilon \partial_x^2$ in the horizontal variable and we derive some uniform estimates independent of ε for the monotone solutions of the Cauchy problem in some weighted Sobolev spaces and for a small enough uniform time.

1.1 Navier Stokes alpha system

1.1.1 Turbulent model for the Navier Stokes equation

The Navier-Stokes alpha system (NS- α) (also called the viscous Camassa-Holm equations) is a mathematical model for the dynamics of appropriately averaged turbulent fluid flows. It was introduced by Holm and al. (see [41, 42]) for the mean motion of an ideal incompressible fluid as a generalization of the inviscid 1D Camassa-Holm equation (see also [10, 23, 44, 57, 60] and the references therein).

The Navier-Stokes alpha system is written as follows :

$$\begin{cases} \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{v} + v_1 \nabla u_1 + v_2 \nabla u_2 = \nu \Delta \mathbf{v} - \nabla q \\ \nabla \cdot \mathbf{u} = 0, \end{cases} \quad (1.1)$$

where

$$\mathbf{v} = (v_1, v_2) = (1 - \alpha^2 \Delta) \mathbf{u} = ((1 - \alpha^2 \Delta) u_1, (1 - \alpha^2 \Delta) u_2)$$

and \mathbf{u} is the velocity of the fluid, q the modified pressure and α is a length independent of the position related to the Lagrangian fluctuations (see [41, 42]).

The global existence and uniqueness of a solution in a Sobolev space has been proved by Foias, Holm and Titi in [29] for a periodic domain. Moreover, they study the convergence when $\alpha \rightarrow 0$ of the solution to a weak solution of 3D NS equation. The Cauchy problem in a case of a bounded domain with Dirichlet-Neumann boundary conditions has been treated by Busuioc in [9]

In this thesis we consider the following Navier Stokes alpha system (1.1) in a thin strip

$$\mathcal{S}^\varepsilon = \{(x, y) \in \mathbb{R}^2, 0 < y < \varepsilon\},$$

where the width $\varepsilon > 0$ is supposed to be very small. Equipped with no-slip boundary conditions, we rewrite (1.1) as

$$(NS-\alpha)_\varepsilon \begin{cases} \partial_t U_1^\varepsilon + u_1^\varepsilon \partial_x U_1^\varepsilon + u_2^\varepsilon \partial_y U_1^\varepsilon + U_1^\varepsilon \partial_x u_1^\varepsilon + U_2^\varepsilon \partial_x u_2^\varepsilon = \nu \Delta U_1^\varepsilon - \partial_x q^\varepsilon \\ \partial_t U_2^\varepsilon + u_1^\varepsilon \partial_x U_2^\varepsilon + u_2^\varepsilon \partial_y U_2^\varepsilon + U_1^\varepsilon \partial_y u_1^\varepsilon + U_2^\varepsilon \partial_y u_2^\varepsilon = \nu \Delta U_2^\varepsilon - \partial_y q^\varepsilon \\ \partial_x u_1^\varepsilon + \partial_y u_2^\varepsilon = 0 \\ u_1^\varepsilon = u_2^\varepsilon = 0 \quad \text{on} \quad \{y = 0\} \cup \{y = \varepsilon\} \\ \partial_z u_1^\varepsilon = \partial_z u_2^\varepsilon = 0 \quad \text{on} \quad \{y = 0\} \cup \{y = \varepsilon\} \\ (u_1^\varepsilon, u_2^\varepsilon)|_{t=0} = (u_{1,0}, u_{2,0})(x, y), \end{cases} \quad (1.2)$$

where, for $i = 1, 2$,

$$U_i^\varepsilon = u_i^\varepsilon - \alpha^2 \Delta u_i^\varepsilon.$$

Now we consider the following rescaling

$$\begin{aligned} \alpha &= \varepsilon \alpha_1, \\ u_1^\varepsilon(t, x, y) &= u^\varepsilon \left(t, x, \frac{y}{\varepsilon} \right), \\ u_2^\varepsilon(t, x, y) &= \varepsilon v^\varepsilon \left(t, x, \frac{y}{\varepsilon} \right), \\ q^\varepsilon(t, x, y) &= \tilde{q}^\varepsilon \left(t, x, \frac{y}{\varepsilon} \right), \end{aligned}$$

and we perform the change of variable $z = \frac{y}{\varepsilon}$ and we set

$$\mathcal{S} = \{(x, z) \in \mathbb{R}^2, 0 < z < 1\}.$$

The system (1.2) becomes

$$\begin{cases} \partial_t \omega^\varepsilon - \alpha_1^2 \varepsilon^2 \partial_t \partial_x^2 u^\varepsilon + u^\varepsilon \partial_x \omega^\varepsilon - \alpha_1^2 \varepsilon^2 u^\varepsilon \partial_x^3 u^\varepsilon + v^\varepsilon \partial_z \omega^\varepsilon - \alpha_1^2 \varepsilon^2 v^\varepsilon \partial_z \partial_x^2 u^\varepsilon + \omega^\varepsilon \partial_x u^\varepsilon - \alpha_1^2 \varepsilon^2 \partial_x u^\varepsilon \partial_x^2 u^\varepsilon \\ \quad + \varepsilon^2 \gamma^\varepsilon \partial_x v^\varepsilon - \varepsilon^4 \alpha_1^2 \partial_x^2 v^\varepsilon \partial_x v^\varepsilon = \partial_z^2 \omega^\varepsilon + \varepsilon^2 \partial_x^2 \omega^\varepsilon - \varepsilon^2 \alpha_1^2 \partial_z^2 \partial_x^2 u^\varepsilon - \alpha_1^2 \varepsilon^4 \partial_x^4 u^\varepsilon - \partial_x \tilde{q}^\varepsilon \\ \varepsilon^2 \partial_t \gamma^\varepsilon - \alpha_1^2 \varepsilon^4 \partial_t \partial_x^2 v^\varepsilon + \varepsilon^2 u^\varepsilon \partial_x \gamma^\varepsilon - \alpha_1^2 \varepsilon^4 u^\varepsilon \partial_x^3 v^\varepsilon + \varepsilon^2 v^\varepsilon \partial_z \gamma^\varepsilon - \alpha_1^2 \varepsilon^4 v^\varepsilon \partial_z \partial_x^2 v^\varepsilon + \omega \partial_z u^\varepsilon - \alpha_1^2 \varepsilon^2 \partial_x^2 u^\varepsilon \partial_z u^\varepsilon \\ \quad + \varepsilon^2 \gamma^\varepsilon \partial_z v^\varepsilon - \alpha_1^2 \varepsilon^4 \partial_x^2 v^\varepsilon \partial_z v^\varepsilon = \varepsilon^2 \partial_z^2 \gamma^\varepsilon + \varepsilon^4 \partial_x^2 \gamma^\varepsilon - \alpha_1^2 \varepsilon^4 \partial_z^2 \partial_x^2 v^\varepsilon - \alpha_1^2 \varepsilon^6 \partial_x^4 v^\varepsilon - \partial_z \tilde{q}^\varepsilon \\ \partial_x u^\varepsilon + \partial_z v^\varepsilon = 0, \end{cases} \quad (1.3)$$

with the boundary conditions

$$(u^\varepsilon, v^\varepsilon)|_{z=0} = (u^\varepsilon, v^\varepsilon)|_{z=1} = 0 \quad \text{and} \quad (\nabla u^\varepsilon, \nabla v^\varepsilon)|_{z=0} = (\nabla u^\varepsilon, \nabla v^\varepsilon)|_{z=1} = 0 \quad (1.4)$$

where we denote $\omega^\varepsilon = u^\varepsilon - \alpha_1^2 \partial_x^2 u^\varepsilon$ and $\gamma^\varepsilon = v^\varepsilon - \alpha_1^2 \partial_z^2 v^\varepsilon$ to lighten the notations.

Taking $\varepsilon \rightarrow 0$, we obtain formally the ‘‘hydrostatic limit system’’ for the α -Navier Stoke equation :

$$(NS-\alpha)_0 \begin{cases} \partial_t \omega + u \partial_x \omega + v \partial_z \omega + \alpha_1^2 (\partial_z u \partial_x \partial_z u - \partial_z^2 u \partial_x u) = \partial_z^2 \omega - \partial_x p \\ \partial_z p = 0 \\ \partial_x u + \partial_y v = 0 \\ (u, v) = 0, \quad \partial_z u = 0 \quad \text{on} \quad z = 0 \\ (u, v) = 0, \quad \partial_z u = 0 \quad \text{on} \quad z = 1 \\ u|_{t=0} = u_0. \end{cases} \quad (1.5)$$

where

$$\omega = u - \alpha_1^2 \partial_z^2 u \quad \text{and} \quad p = \tilde{q} + \frac{1}{2}u^2 - \frac{1}{2}\alpha_1^2(\partial_z u)^2.$$

The main difficulty is the loss of the dissipative term $\partial_x^2 \omega$ in this limit system. In order to overcome this difficulty, and following the method of Paicu, Zhang and Zhang [64], we will work in some analytic function frameworks. Therefore, we introduce some elements of Littlewood-Paley theory and some functional spaces.

1.1.2 Main results on the Navier Stokes alpha equation

We consider a even smooth function χ in $C_0^\infty(\mathbb{R})$ such that the support is contained in the ball $B_{\mathbb{R}}(0, \frac{4}{3})$ and ψ is equal to 1 on a neighborhood of the ball $B_{\mathbb{R}}(0, \frac{3}{4})$ and we set $\psi(z) = \chi(\frac{z}{2}) - \chi(z)$. We denote $\mathcal{F}_h f = \widehat{f}$ and $\mathcal{F}_h^{-1} f$ be the Fourier transform and the inverse Fourier transform respectively in the horizontal direction of a function $f(x)$. We recall the following definitions of the homogeneous dyadic cut-off operators.

Definition 1.1.1. For all tempered distributions in the horizontal direction and for all $q \in \mathbb{Z}$ we set

$$\begin{aligned} \Delta_q^h u(x, y) &= \mathcal{F}_h^{-1}(\psi(2^{-q}|\xi|)\mathcal{F}_h u(\xi, y)), \\ S_q^h u(x, y) &= \mathcal{F}_h^{-1}(\chi(2^{-q}|\xi|)\mathcal{F}_h u(\xi, y)) = \sum_{q' \leq q-1} \Delta_{q'}^h u. \end{aligned}$$

As in [64], we consider the Besov-type spaces with respect to the x variable :

Definition 1.1.2. Let $s \in \mathbb{R}$ and $\mathcal{S} = \mathbb{R} \times]0, 1[$. For any $u \in S'_h(\mathcal{S})$, i.e,

$$u \in S'(\mathcal{S}) \quad \text{with} \quad \lim_{q \rightarrow -\infty} \|S_q^h u\|_{L^\infty} = 0,$$

we set

$$\|u\|_{B^s} = \sum_{q \in \mathbb{Z}} 2^{qs} \|\Delta_q^h u\|_{L^2}.$$

- For $s \leq \frac{1}{2}$, we define

$$B^s(\mathcal{S}) = \{u \in S'_h(\mathcal{S}), \quad \|u\|_{B^s} < \infty\}.$$

- For $s \in]k - \frac{1}{2}, k + \frac{1}{2}]$, with $k \in \mathbb{N}^*$, we define $B^s(\mathcal{S})$ as the subset of distributions $u \in S'_h(\mathcal{S})$ such that $\partial_x^k u \in B^{s-k}(\mathcal{S})$.

We also need the following time-weighted Chemin-Lerner-type spaces (see [22]).

Definition 1.1.3. Let $p \in [1, +\infty]$ and $T \in]0, \infty]$ and let $f \in L^1_{loc}(\mathbb{R}^+)$ be non-negative function. We define space $\widetilde{L}^p_{T,f}(B^s(\mathcal{S}))$ as the closure of $C([0, T], B^s(\mathcal{S}))$ with respect to the norm

$$\|u\|_{\widetilde{L}^p_{T,f}(B^s(\mathcal{S}))} = \sum_{q \in \mathbb{Z}} 2^{qs} \left(\int_0^T f(t) \|\Delta_q^h u\|_{L^2}^p dt \right)^{\frac{1}{p}}$$

with the usual change if $p = +\infty$. In the case of $f(t) \equiv 1$, we will simply use the notations $\widetilde{L}^p_T(B^s(\mathcal{S}))$ and $\|u\|_{\widetilde{L}^p_T(B^s(\mathcal{S}))}$.

The main difficulty consists in controlling the nonlinear terms, using the smoothing effect given by the above function spaces. The idea is to define the following auxiliary functions, using the method introduced by Chemin in [25] (see also [21] or [64]): for any $f \in L^2(\mathcal{S})$, we set

$$f_\phi(t, x, y) = e^{\phi(t, D_x)} f(t, x, y) := \mathcal{F}_{\xi \rightarrow x}^{-1}(e^{\phi(t, \xi)} \widehat{f}(t, \xi, y)) \quad \text{with} \quad \phi(t, \xi) = (a - \lambda\theta(t))|\xi|, \quad (1.6)$$

where the quantity $\theta(t)$ describes the evolution of the analytic band of u , satisfies

$$\theta'(t) = \|\partial_z^2 u_\phi(t)\|_{B^{\frac{1}{2}}} \quad \text{and} \quad \theta(0) = 0. \quad (1.7)$$

We first prove the global existence and uniqueness of the system (1.5) if for the initial data is such that $e^{a|D_x|}u_0$ and $e^{a|D_x|}\partial_z u_0$ belong to the analytic spaces $B^{\frac{1}{2}}$ and $B^{\frac{3}{2}}$ and are small enough. Moreover, we show that the solution is controlled in the space $B^{\frac{3}{2}}$:

Theorem 1.1. *Let $a > 0$ be fixed. There exist positive constants c and C such that if $e^{a|D_x|}u_0 \in B^{\frac{3}{2}}$, $e^{a|D_x|}\partial_z u_0 \in B^{\frac{3}{2}}$,*

$$\|e^{a|D_x|}u_0\|_{B^{\frac{1}{2}}} + \|e^{a|D_x|}\partial_z u_0\|_{B^{\frac{1}{2}}} \leq \frac{ca}{1 + \|e^{a|D_x|}u_0\|_{B^{\frac{3}{2}}} + \|e^{a|D_x|}\partial_z u_0\|_{B^{\frac{3}{2}}}} \quad (1.8)$$

and the compatibility condition $\int_0^1 u_0 dz = 0$, then there exists a unique global solution for the system (1.5) which satisfies

$$\begin{aligned} & \|e^{Rt}u_\phi\|_{\tilde{L}^\infty(\mathbb{R}^+, B^{\frac{3}{2}})} + \|e^{Rt}\partial_z u_\phi\|_{\tilde{L}^\infty(\mathbb{R}^+, B^{\frac{3}{2}})} + \|e^{Rt}\partial_z u_\phi\|_{\tilde{L}^2(\mathbb{R}^+, B^{\frac{3}{2}})} + \|e^{Rt}\partial_z^2 u_\phi\|_{\tilde{L}^2(\mathbb{R}^+, B^{\frac{3}{2}})} \\ & \leq C(\|e^{a|D_x|}u_0\|_{B^{\frac{3}{2}}} + \|e^{a|D_x|}\partial_z u_0\|_{B^{\frac{3}{2}}}). \end{aligned} \quad (1.9)$$

Furthermore, if $e^{a|D_x|}u_0 \in B^{\frac{5}{2}}$, $e^{a|D_x|}\partial_z u_0 \in B^{\frac{5}{2}}$, $e^{a|D_x|}\partial_z^2 u_0 \in B^{\frac{3}{2}}$, then

$$\begin{aligned} & \|e^{Rt}(\partial_t u)_\phi\|_{\tilde{L}^2(\mathbb{R}^+, B^{\frac{3}{2}})} + \|e^{Rt}(\partial_t \partial_z u)_\phi\|_{\tilde{L}^2(\mathbb{R}^+, B^{\frac{3}{2}})} + \|e^{Rt}\partial_z u_\phi\|_{\tilde{L}^\infty(\mathbb{R}^+, B^{\frac{3}{2}})} + \|e^{Rt}\partial_z^2 u_\phi\|_{\tilde{L}^\infty(\mathbb{R}^+, B^{\frac{3}{2}})} \\ & \leq C\left(\|e^{a|D_x|}\partial_z u_0\|_{B^{\frac{3}{2}}} + \|e^{a|D_x|}\partial_z^2 u_0\|_{B^{\frac{3}{2}}} + \|e^{a|D_x|}u_0\|_{B^{\frac{5}{2}}} + \|e^{a|D_x|}\partial_z u_0\|_{B^{\frac{5}{2}}}\right). \end{aligned}$$

We then prove the global well-posedness of the system (1.3)-(1.4) for small initial analytic data in some Besov spaces.

Theorem 1.2. *Let $a > 0$ and $0 < s < 1$ be fixed. For some $\tilde{c} > 0$ small enough, if the initial data satisfies*

$$\|e^{a|D_x|}(u_0^\varepsilon, \varepsilon v_0^\varepsilon)\|_{B^{\frac{1}{2}}} + \|e^{a|D_x|}(\varepsilon u_0^\varepsilon, \varepsilon^2 v_0^\varepsilon)\|_{B^{\frac{3}{2}}} + \|e^{a|D_x|}(\partial_z u_0^\varepsilon, \varepsilon \partial_z v_0^\varepsilon)\|_{B^{\frac{1}{2}}} \leq \tilde{c}a, \quad (1.10)$$

$e^{a|D_x|}(u_0^\varepsilon, v_0^\varepsilon) \in B^s \cap B^{s+1}$ and $e^{a|D_x|}(\partial_z u_0^\varepsilon, \partial_z v_0^\varepsilon) \in B^s$, then there exists a unique global solution for the system (1.3)-(1.4) which satisfies

$$\begin{aligned} & \|e^{\mathcal{R}t'}(u_\varphi^\varepsilon, \varepsilon v_\varphi^\varepsilon)\|_{\tilde{L}^\infty(\mathbb{R}^+, B^s)} + \|e^{\mathcal{R}t'}(\varepsilon u_\varphi^\varepsilon, \varepsilon^2 v_\varphi^\varepsilon)\|_{\tilde{L}^\infty(\mathbb{R}^+, B^{s+1})} + \|e^{\mathcal{R}t'}(\partial_z u_\varphi^\varepsilon, \varepsilon \partial_z \Delta_k^h v_\varphi^\varepsilon)\|_{\tilde{L}^\infty(\mathbb{R}^+, B^s)} \\ & \quad + \|e^{\mathcal{R}t'}(\varepsilon \partial_x u_\varphi^\varepsilon, \varepsilon^2 \partial_x v_\varphi^\varepsilon)\|_{\tilde{L}^2(\mathbb{R}^+, B^s)} \\ & \quad + \|e^{\mathcal{R}t'}(\partial_z u_\varphi^\varepsilon, \varepsilon \partial_z v_\varphi^\varepsilon)\|_{\tilde{L}^2(\mathbb{R}^+, B^s)} + \|e^{\mathcal{R}t'}(\varepsilon \partial_z \partial_x u_\varphi^\varepsilon, \varepsilon^2 \partial_z \partial_x v_\varphi^\varepsilon)\|_{\tilde{L}^2(\mathbb{R}^+, B^s)} \\ & \quad + \|e^{\mathcal{R}t'}(\varepsilon \partial_x^2 u_\varphi^\varepsilon, \varepsilon^3 \partial_x^2 v_\varphi^\varepsilon)\|_{\tilde{L}^2(\mathbb{R}^+, B^s)} + \|e^{\mathcal{R}t'}(\partial_z^2 u_\varphi^\varepsilon, \varepsilon \partial_z^2 v_\varphi^\varepsilon)\|_{\tilde{L}^2(\mathbb{R}^+, B^s)} \\ & \leq C\left(\|e^{a|D_x|}(u_0^\varepsilon, \varepsilon v_0^\varepsilon)\|_{B^s} + \|e^{a|D_x|}(\varepsilon u_0^\varepsilon, \varepsilon^2 v_0^\varepsilon)\|_{B^{s+1}} + \|e^{a|D_x|}(\partial_z u_0^\varepsilon, \varepsilon \partial_z v_0^\varepsilon)\|_{B^s}\right). \end{aligned}$$

Finally we justify the hydrostatic limit of the system (1.3)-(1.4).

Theorem 1.3. *Let $a > 0$ and $0 < s < \frac{1}{2}$ be fixed and $(u_0^\varepsilon, v_0^\varepsilon)$ satisfy (1.10). Let u_0 satisfy $e^{a|D_x|}u_0 \in B^{s+\frac{1}{2}} \cap B^{s+3}$, $e^{a|D_x|}\partial_z u_0 \in B^{s+\frac{1}{2}} \cap B^{s+3}$, $e^{a|D_x|}\partial_z^2 u_0 \in B^{s+1} \cap B^{s+2}$ and there holds (1.8) for some constant c small, and the compatibility condition $\int_0^1 u_0 dz = 0$. We consider $(u^\varepsilon, v^\varepsilon)$ a solution of the*

system (1.3)-(1.4) and (u, v) a solution of the hydrostatic system (1.5). Then we have

$$\begin{aligned}
& \|e^{\mathcal{R}t'}(w_{\varepsilon\psi}^1, \varepsilon w_{\varepsilon\psi}^2)\|_{\tilde{L}_T^\infty(B^s)} + \|e^{\mathcal{R}t'}(\varepsilon w_{\varepsilon\psi}^1, \varepsilon^2 w_{\varepsilon\psi}^2)\|_{\tilde{L}_T^\infty(B^{s+1})} + \|e^{\mathcal{R}t'}(\partial_z w_{\varepsilon\psi}^1, \varepsilon \partial_z w_{\varepsilon\psi}^2)\|_{\tilde{L}_T^\infty(B^s)} \\
& + \|e^{\mathcal{R}t'}(\varepsilon \partial_x w_{\varepsilon\psi}^1, \varepsilon^2 \partial_x w_{\varepsilon\psi}^2)\|_{\tilde{L}_T^2(B^s)} + \|e^{\mathcal{R}t'}(\partial_z w_{\varepsilon\psi}^1, \varepsilon \partial_z w_{\varepsilon\psi}^2)\|_{\tilde{L}_T^2(B^s)} \\
& + \|e^{\mathcal{R}t'}(\varepsilon \partial_x^2 w_{\varepsilon\psi}^1, \varepsilon^3 \partial_x^2 w_{\varepsilon\psi}^1)\|_{\tilde{L}_T^2(B^s)} + \|e^{\mathcal{R}t'}(\partial_z^2 w_{\varepsilon\psi}^1, \varepsilon \partial_z^2 w_{\varepsilon\psi}^1)\|_{\tilde{L}_T^2(B^s)} \\
& + \|e^{\mathcal{R}t'}(\varepsilon \partial_z \partial_x w_{\varepsilon\psi}^1, \varepsilon^2 \partial_z \partial_x w_{\varepsilon\psi}^2)\|_{\tilde{L}_T^2(B^s)} \\
& \leq C \left[\|(w_{\varepsilon\psi}^1(0), \varepsilon w_{\varepsilon\psi}^2(0))\|_{B^s} + \|(\varepsilon w_{\varepsilon\psi}^1(0), \varepsilon^2 w_{\varepsilon\psi}^2(0))\|_{B^{s+1}} \right. \\
& \quad \left. + \|(\partial_z w_{\varepsilon\psi}^1(0), \varepsilon \partial_z w_{\varepsilon\psi}^2(0))\|_{B^s} + M\varepsilon \right],
\end{aligned}$$

where $w_\varepsilon^1 = u^\varepsilon - u$, $w_\varepsilon^2 = v^\varepsilon - v$ and $(w_{\varepsilon\psi}^1, w_{\varepsilon\psi}^2)$ will be given in (3.45) and we have used the following notation

$$\|e^{\mathcal{R}t'}(w_{\varepsilon\psi}^1, \varepsilon w_{\varepsilon\psi}^2)\| = \|e^{\mathcal{R}t'} w_{\varepsilon\psi}^1\| + \varepsilon \|e^{\mathcal{R}t'} w_{\varepsilon\psi}^2\|.$$

This theorems justify the well-posedness of the hydrostatic α -Navier Stokes system in the analytic spaces and its hydrostatic limit.

1.2 Non newtonian prandtl equation

1.2.1 Prandtl boundary layer theory

The classical theoretical hydrodynamics is based on the Euler equations describing the motion of an ideal fluid. Despite the numerous achievements, some gaps between theoretical and experimental results need to be filled, which are mainly due to how to determine the force of resistance acting on a moving solid in a fluid. A classical attempt to close the gaps is to consider the Navier-Stokes equations, which are based on Newton's model of a viscous fluid. According to that model, tangential stresses in a fluid are linearly dependent on the respective shear deformation velocities and so the principal part of the Navier-Stokes equations consists in second order partial derivative diffusion term.

Unfortunately, many problems arise in this consideration. First of all, we encounter great mathematical difficulties to explain the connection between the viscous model and the theoretical ideal non-viscous model. For instance, it is extremely difficult to prove that small viscous friction forces may perceptibly affect the motion of a fluid. In 1904, Prandtl suggested an explanation of this phenomenon using his theory of boundary layer. He theoretically showed that the flow around a solid body could be divided into two zones: a thin layer around the body where viscous forces play crucial part and the outer region far from the body where friction could be neglected. Thus, for weakly viscous fluids, the influence of the small viscosity can only be perceptible in this very thin layer adjacent to the surface of the body, which is called boundary layer according to Prandtl. Outside this region, the fluid can be considered as frictionless and can be described by the Euler system.

The second question comes from the consideration of tangential stresses which are linearly dependent on the shear deformation velocities in Newton's model. This linear dependence turns out to be unacceptable for many media, for example melted and dissolved polymers, concentrated suspensions, emulsions, dye-stuffs, etc ... Many different models have been proposed for the description of non-newtonian flows (see, for instance, Litvinvo (1982), Shulman (1975)). Among these non-newtonian flows, the Ostwald model for nonlinear viscous fluids are particularly adapted to many physical fluids, where the stress tensor components depend on the powers of the components of the deformation velocity tensor.

Concerning Prandtl theory of boundary layers for Newtonian fluids, the mathematical justification of Prandtl equations is still a challenging problem in fluid mechanics. Under a monotonic assumption on the tangential velocity of the outer flow, the local existence of classical solutions for Prandtl equations was first proved in the pioneer works of Oleinik [61] (see also [62] for a complete survey), using the so-called Crocco transformation. In Xin and Zhang [76], the existence of global weak solutions was proved under Oleinik's monotonicity hypothesis and a favorable condition on the pressure. Recently, the well-posedness

of Prandtl equations under monotonic hypothesis in Sobolev framework was proved independently in [1] and [59] using direct energy estimate methods.

Without the monotonicity condition, phenomena of the boundary layer separation are expected and actually well observed. In this non-monotonicity framework, E and Engquist [28] constructed a finite time blow-up solution to the Prandtl equations. More recently, when the background shear flow has a non-degenerate critical point, some interesting instability phenomena of solutions to both the linear and nonlinear Prandtl equations around the shear flow were put in evidence, for example in [30, 33, 35, 36]. These instability results show that the monotonic assumption on the tangential velocity is very important for well-posedness, except in the framework of analytic functions. Indeed, in [68, 69], Sammartino and Caffisch proved the local existence of analytic solutions to the Prandtl equation and established a rigorous theory on the stability of boundary layers with analytic data using the abstract Cauchy-Kowaleskaya theory. This result was extended to the function space which is only analytic in the tangential variable in [55].

In the case of non-Newtonian Prandtl equations, same phenomena are expected to happen. To the best of our knowledge, the problem is still open. The goal of our work is to establish the similar results as in the Newtonian case. In the first step, we prove the well-posedness of the shear flow system and in the second step, we expect to obtain the same results for the full 2D non-Newtonian Prandtl system.

1.2.2 Main results on the non-Newtonian fluids

In this chapter, we are interested in the the Non Newtonian Prandtl equation

$$\begin{cases} \partial_t u + u \partial_x u + v \partial_y u + \partial_x p = n |\partial_y u|^{n-1} \partial_y^2 u & \text{in } [0, T] \times \mathbb{T} \times \mathbb{R}^+ \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 & \text{in } [0, T] \times \mathbb{T} \times \mathbb{R}^+ \\ u|_{t=0} = u_0 & \text{on } \mathbb{T} \times \mathbb{R}^+ \\ u|_{y=0} = 0 & \text{on } [0, T] \times \mathbb{T} \\ v|_{y=0} = 0 & \text{on } [0, T] \times \mathbb{T} \\ \lim_{y \rightarrow +\infty} u = U & \text{for all } (t, x) \in [0, T] \times \mathbb{T}, \end{cases}$$

where the velocity field $(u, v)(t, x, y)$ is an unknown, the initial data $u_0(x, y)$ and the outer flow $U(t, x)$ are given and satisfy the compatibility conditions :

$$u_0|_{y=0} = 0 \quad \text{and} \quad \lim_{y \rightarrow +\infty} u_0 = U|_{t=0}.$$

Furthermore, the given scalar pressure $p(t, x)$ and the outer flow $U(t, x)$ satisfy the well-known Bernoulli's law :

$$\partial_t U + U \partial_x U = -\partial_x p.$$

In this work, we will consider the Oleinik's monotonicity assumption :

$$\omega := \partial_y u > 0.$$

where ω is the vorticity.

Throughout this paper, we consider the homogeneous case with respect to the x variable : The solution u does not depend on x and the outer flow U is constant. For simplicity we take $U = 1$. Moreover, we will focus on the special case $n = 2$, which corresponds to a dilatant fluid. That is, we are interested in the solutions of the following equation, which is the so-called "shear-flow" problem.

$$\begin{cases} \partial_t u = 2 |\partial_y u| \partial_y^2 u & \text{in } [0, T] \times \mathbb{R}^+ \\ u|_{t=0} = u_0 & \text{on } \mathbb{T} \times \mathbb{R}^+ \\ u|_{y=0} = 0 & \text{on } [0, T] \\ \lim_{y \rightarrow +\infty} u = 1 & \text{for all } t \in [0, T], \end{cases} \quad (1.11)$$

This problem is equivalent to the following equation in ω :

$$\begin{cases} \partial_t \omega = 2\omega \partial_y^2 \omega + 2(\partial_y \omega)^2 & \text{in } [0, T] \times \mathbb{R}^+, \\ \omega|_{t=0} = \omega_0(y) & \text{on } \mathbb{R}^+ \\ \partial_y \omega|_{y=0} = 0 & \text{on } [0, T] \end{cases} \quad (1.12)$$

where $\omega_0 = \partial_y u_0 > 0$.

Previous results

For $n = 1$, the problem corresponds to the Prandtl system for a Newtonian fluid and has been widely studied [62], [XU], [Li].

The case $0 < n < 1$ describes the motion of a pseudo-plastic fluid near a boundary [62], [70]. The case $n > 1$ has been studied in [71] in the framework of a dilatant fluid for MHD boundary layer system.

The problem of existence and uniqueness of solutions of the boundary layer problem for non-newtonian fluids has been little studied in the case $0 < n < 1$ in [70], [72], [27], [61].

The problem for the both cases has been studied in [77], using an adapted method of [62]: The authors have used a transformation of the problem $\eta = u$ and $w(\tau, \xi, \eta) = |\partial_y u|^{n-1} \partial_y u$.

There are many papers on the parabolic p-Laplacian (see for example [26] and the references included, but our problem is different : the boundary conditions are different, and consequently the regularity of the solutions.

In this part, we will use some methods in [56] and [59].

We now introduce some function spaces. For $I = \mathbb{R}^+$ or $I = \mathbb{R}$, $s \in \mathbb{N}$ and $\gamma \geq 1$, we pose

$$H^{s,\gamma}(I) := \{\omega : I \rightarrow \mathbb{R}; \|\omega\|_{H^{s,\gamma}} < +\infty\}$$

where the weighted H^s norm $\|\cdot\|_{H^{s,\gamma}}$ is defined by

$$\|\omega\|_{H^{s,\gamma}}^2 := \sum_{k=0}^s \|\langle y \rangle^{\gamma+k} \partial_y^k \omega\|_{L^2(I)}^2 \quad (1.13)$$

where $\langle y \rangle = \sqrt{1+y^2}$. For $s \geq 4$, $\gamma \geq 1$, $\sigma > \gamma + 1$, $\delta \in (0, 1)$, we also define the following convex space

$$H_{\sigma,\delta}^{s,\gamma} := \left\{ \omega \in H^{s,\gamma} \mid \forall y \in I, \quad \langle y \rangle^\sigma \omega \geq \frac{\delta}{2} \text{ and } \sum_{k=0}^2 |\langle y \rangle^{\sigma+k} \partial_y^k \omega| \leq \frac{6}{\delta} \right\}$$

This is a metric space equipped with the distance $d(u, v) = \|u - v\|_{H^{s,\gamma}}$.

Here, the main idea is adding an extra weight for each y -derivative. This corresponds to the weight $\frac{1}{y}$ in the Hardy type inequality.

our result is the following theorem:

Theorem 1.4. (*Local Existence and Uniqueness to the non-newtonian shear flow equation (5.1)*). *Let $s \geq 4$ an even integer, $\gamma > 1$, $\sigma > \gamma + 1$ and $\delta \in]0, 1[$. Assume that $u_0 \in H^{s,\gamma-1}$ and the initial vorticity $\omega_0 = \partial_y u_0 \in H_{\sigma,2\delta}^{s,\gamma}$ and satisfies the compatibility conditions $\partial_y^{2k+1} \omega_0|_{y=0} = 0$ for k integer such that $k \leq s/2 + 1$. Then there exist a time $T = T(s, \gamma, \sigma, \delta, \|\omega_0\|_{H^{s,\gamma}}) > 0$ and a unique classical solution u to the non-newtonian shear flow equations (5.1) such that*

$$u \in L^\infty([0, T]; H^{s,\gamma-1}) \cap C([0, T]; H_{weak}^s)$$

and the vorticity

$$\omega = \partial_y u \in L^\infty([0, T]; H_{\sigma,\delta}^{s,\gamma}) \cap C([0, T]; H_{weak}^s)$$

where H_{weak}^s is the space H^s endowed with its weak topology.

In the case of 2-d problem we have the following a priori estimate, First we regularize the system (6.5) Then we have for any $\varepsilon > 0$,

$$\begin{cases} \partial_t u^\varepsilon + u^\varepsilon \partial_x u^\varepsilon + v^\varepsilon \partial_y u^\varepsilon = \varepsilon^2 \partial_x^2 u^\varepsilon + 2 |\partial_y u^\varepsilon| \partial_y^2 u^\varepsilon & \text{in } [0, T] \times \mathbb{T} \times \mathbb{R}^+ \\ \partial_x u^\varepsilon + \partial_y v^\varepsilon = 0 & \text{in } [0, T] \times \mathbb{T} \times \mathbb{R}^+ \\ u^\varepsilon|_{t=0} = u_0(x, y) & \text{on } \mathbb{T} \times \mathbb{R}^+ \\ u^\varepsilon|_{y=0} = 0 & \text{on } [0, T] \times \mathbb{T} \\ v^\varepsilon|_{y=0} = 0 & \text{on } [0, T] \times \mathbb{T} \\ \lim_{y \rightarrow +\infty} u^\varepsilon = 1 & \text{for all } (t, x) \in [0, T] \times \mathbb{T}. \end{cases} \quad (1.14)$$

Or equivalently, the regularized vorticity $\omega^\varepsilon := \partial_y u^\varepsilon$ satisfies the following regularized vorticity system : for any $\varepsilon > 0$

$$\partial_t \omega^\varepsilon + u^\varepsilon \partial_x \omega^\varepsilon + v^\varepsilon \partial_y \omega^\varepsilon = \varepsilon^2 \partial_x^2 \omega^\varepsilon + 2 \omega^\varepsilon \partial_y^2 \omega^\varepsilon + 2 (\partial_y \omega^\varepsilon)^2 \quad \text{in } [0, T] \times \mathbb{T} \times \mathbb{R}^+ \quad (1.15)$$

with the boundary condition

$$2 \omega^\varepsilon \partial_y \omega^\varepsilon|_{y=0} = 0 \quad \text{on } [0, T] \times \mathbb{T} \quad (1.16)$$

and the initial condition

$$\omega^\varepsilon|_{t=0} = \omega_0(x, y) \quad \text{on } \mathbb{T} \times \mathbb{R}^+. \quad (1.17)$$

where the velocity field (u, v) is given by

$$\begin{cases} u^\varepsilon(t, x, y) := 1 - \int_y^{+\infty} \omega^\varepsilon(t, x, \tilde{y}) d\tilde{y} \\ v^\varepsilon(t, x, y) := - \int_0^y \partial_x u^\varepsilon(t, x, \tilde{y}) d\tilde{y} \end{cases} \quad (1.18)$$

The objective of this section is to derive an uniform (in ε) weighted H^s estimate for ω . This estimate, include weighted L^2 estimates for $D^\alpha \omega$ for $|\alpha| \leq s$ and $\alpha_1 \leq s - 1$.

For the case $\alpha_1 = s$, we do not estimate directly ω : we have to use a weighted L^2 estimate of the cancellation function g_s which is defined (see (6.8)) by

$$g_s := \partial_x^s \omega - \frac{\partial_y \omega}{\omega} \partial_x^s (u - U).$$

We will combine these two estimates to obtain an uniform weighted energy estimate. of the pseudo-norm (see (6.7)) $\|\omega\|_{H_g^{s,\gamma}}^2$

$$\|\omega\|_{H_g^{s,\gamma}}^2 := \|(1+y)^\gamma g_s\|_{L^2(\mathbb{T} \times \mathbb{R}^+)}^2 + \sum_{\substack{|\alpha| \leq s \\ \alpha_1 \leq s-1}} \|(1+y)^{\gamma+\alpha_2} D^\alpha \omega\|_{L^2(\mathbb{T} \times \mathbb{R}^+)}^2$$

Theorem 1.5. *Let $s \geq 6$ be an even integer, $\gamma \geq 1$, $\sigma > \gamma + \frac{1}{2}$, $\delta \in (0, 1)$, and $\varepsilon \in (0, 1]$. We suppose there exists a solution $\omega^\varepsilon \in C([0, T]; H_{\sigma,\delta}^{s+4,\gamma} \cap C^1([0, T]; H^{s+2,\gamma}))$ of (6.11).*

Therefore we have the following uniform (in ε) estimate : There exists some constants $M_s, C_{s,\gamma,\sigma,\delta}, C_{s,\sigma,\delta}, C_s$ such that

$$\begin{aligned} \|\omega^\varepsilon\|_{H_g^{s,\gamma}} &\leq \left\{ \|\omega_0\|_{H_g^{s,\gamma}}^2 + \int_0^t F(\tau) d\tau \right\}^{\frac{1}{2}} \\ &\times \left\{ 1 - (2M_s - 1) C_{s,\gamma,\sigma,\delta} \left(\|\omega_0\|_{H_g^{s,\gamma}}^2 + \int_0^t F(\tau) d\tau \right)^{2M_s-1} t \right\}^{-\frac{1}{4M_s-2}} \end{aligned}$$

Where

$$F = C_{s,\gamma,\sigma,\delta} \{1 + \|\partial_x^{s+1} U\|_{L^\infty}^4\} + C_{s,\sigma,\delta} \sum_{r=0}^s \|\partial_t^r \partial_x^{s+1-2r} p\|_{L^\infty(\mathbb{T})}^4$$

The proof of this theorem is based on the following proposition:

Proposition 1.2.1. *Under the assumptions and notations of Theorem 6.1, we have*

$$\begin{aligned} \frac{d}{dt} \|\omega\|_{H_g^{s,\gamma}}^2 &\leq C_{s,\gamma,\sigma,\delta} \|\omega\|_{H_g^{s,\gamma}}^{4M_s} + C_{s,\gamma,\sigma,\delta} \{1 + \|\partial_x^{s+1} U\|_{L^\infty}^4\} \\ &\quad + C_s \sum_{r=0}^{\frac{s}{2}} \|\partial_t^r \partial_x^{s+1-2r} p\|_{L^\infty(\mathbb{T})}^4. \end{aligned} \tag{1.19}$$

Part I

Navier-Stokes-alpha model

Chapter 2

Hydrostatic limit of the Navier-Stokes-alpha model

2.1 Introduction

2.1.1 Motivation

The characteristic feature of a turbulent fluid according to Kolmogorov's theory is that the energy cascades from large scales to small scales until it reaches the dissipation scale and then turns into heat. This feature leads to important costs of calculations in numerical simulation because the grid resolutions cannot keep up to the dissipation scale, which is extremely small when the Reynolds number is large (which corresponds to turbulent flows). So the idea is to consider the effects of smaller scales on larger scales instead of capturing all scales and as a consequence, one can achieve a balance between computational costs and precision.

One way to model turbulent flows is the so-called Large Eddy Simulation method. It consists in filtering the small scales and directly calculating the large scales of the turbulent cascade (see [37] for instance). Another approach is the Reynolds Averaged Navier-Stokes, based on Reynolds decomposition, which provides mean quantities of turbulent flow while fluctuations will be modeled. This second method is used in the industry due to its small computational costs. However, both approaches meet a common problem which is the closure of the model systems where there are more unknowns than equations.

In order to overcome this difficulty, the Navier-Stokes-alpha model was introduced, where an energy "penalty" inhibits the creation of small excitations below a certain length scale α (also called the viscous Camassa-Holm equations, see [10, 23, 42, 44, 57, 60] and the references therein for a survey of Camassa-Holm equations). This "alpha-modification" leads to a change in the convection term of the Navier-Stokes equations. More precisely, this is the following system

$$\begin{cases} \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{v} + v_1 \nabla u_1 + v_2 \nabla u_2 = \nu \Delta \mathbf{v} - \nabla q \\ \nabla \cdot \mathbf{u} = 0, \end{cases} \quad (2.1)$$

where

$$\mathbf{v} = (v_1, v_2) = (1 - \alpha^2 \Delta) \mathbf{u} = ((1 - \alpha^2 \Delta) u_1, (1 - \alpha^2 \Delta) u_2)$$

and \mathbf{u} is the velocity of the fluid and q the modified pressure. In [29], Foias, Holm and Titi prove the global existence and uniqueness of the solution of (2.1) in a periodic domain for H^1 -initial data and the convergence of the solution of 3D Camassa-Holm equation (NS- α) towards a weak solution of 3D NS equations when α tends to zero and in [9], Busuioc gives a simple proof of the global existence and uniqueness of (2.1) for H_0^1 -initial data in bounded domains with Dirichlet boundary conditions.

In this paper, we consider the system (2.1) in the thin strip $\mathcal{S}^\varepsilon = \{(x, y) \in \mathbb{R}^2, 0 < y < \varepsilon\}$, where the width ε is supposed to be very small. This consideration is relevant for planetary-scale oceanic and atmospheric flows (see [65]), for which, the vertical scale (a few kilometers for oceans, 10-20 kilometers for the atmosphere) is much smaller than the horizontal scales (thousands of kilometers). In this framework, the fluid behaviors are approximated by the so-called hydrostatic model, in which the conservation momentum in the vertical direction is replaced by a simple hydrostatic equation. In the case of Navier-Stokes equations for a viscous fluid in a thin strip, the hydrostatic limit leads to the following rescaled system in domain $\mathbb{R} \times]0, 1[$

$$\begin{cases} \partial_t u + u \partial_x u + v \partial_z u = \partial_y^2 u - \partial_x p \\ \partial_y p = 0 \\ \partial_x u + \partial_y v = 0 \\ (u, v)|_{y=0} = (u, v)|_{y=1} = 0 \\ u|_{t=0} = u_0. \end{cases} \quad (2.2)$$

This model and its three-dimensional counterpart are very important in oceanography and meteorology (see [4, 52, 65]). It is known that without any structural assumption on the initial data, real-analyticity is both necessary [67] and sufficient [48] for the local well-posedness of the system (2.2). In [32] the authors proved that for convex initial data, the local well-posedness holds under simple Gevrey regularity. In [64], Paicu *et al.* proved the existence and uniqueness of global analytic solution with small analytic initial data with respect to variable x .

In this paper, we will study the hydrostatic limit of the Navier-Stokes-alpha equations as the strip width $\varepsilon \rightarrow 0$. Using the techniques of [64], we prove the global well-posedness of the limit system in appropriate analytic function spaces. We will justify this limit by proving the convergence of the Navier-Stokes-alpha system in a forthcoming paper. In the next subsection, we will give brief derivation of the hydrostatic limit of the system (2.1) as $\varepsilon \rightarrow 0$.

2.1.2 Hydrostatic limit of the Navier-Stokes-alpha model

We consider the system (2.1) in a thin strip

$$\mathcal{S}^\varepsilon = \{(x, y) \in \mathbb{R}^2, 0 < y < \varepsilon\},$$

where the width $\varepsilon > 0$ is supposed to be very small. Equipped with no-slip boundary conditions, we rewrite (2.1) as

$$\begin{cases} \partial_t U_1^\varepsilon + u_1^\varepsilon \partial_x U_1^\varepsilon + u_2^\varepsilon \partial_y U_1^\varepsilon + U_1^\varepsilon \partial_x u_1^\varepsilon + U_2^\varepsilon \partial_x u_2^\varepsilon = \nu \Delta U_1^\varepsilon - \partial_x q^\varepsilon \\ \partial_t U_2^\varepsilon + u_1^\varepsilon \partial_x U_2^\varepsilon + u_2^\varepsilon \partial_y U_2^\varepsilon + U_1^\varepsilon \partial_y u_1^\varepsilon + U_2^\varepsilon \partial_y u_2^\varepsilon = \nu \Delta U_2^\varepsilon - \partial_y q^\varepsilon \\ \partial_x u_1^\varepsilon + \partial_y u_2^\varepsilon = 0 \\ u_1^\varepsilon = u_2^\varepsilon = 0 \quad \text{on} \quad \{y = 0\} \cup \{y = \varepsilon\} \\ \partial_z u_1^\varepsilon = \partial_z u_2^\varepsilon = 0 \quad \text{on} \quad \{y = 0\} \cup \{y = \varepsilon\} \\ (u_1^\varepsilon, u_2^\varepsilon)|_{t=0} = (u_{1,0}, u_{2,0})(x, y), \end{cases} \quad (2.3)$$

where, for $i = 1, 2$,

$$U_i^\varepsilon = u_i^\varepsilon - \alpha^2 \Delta u_i^\varepsilon.$$

We consider the following rescaling

$$\begin{aligned} u_1^\varepsilon(t, x, y) &= u^\varepsilon\left(t, x, \frac{y}{\varepsilon}\right), \\ u_2^\varepsilon(t, x, ye) &= \varepsilon v^\varepsilon\left(t, x, \frac{y}{\varepsilon}\right), \\ q^\varepsilon(t, x, y) &= \tilde{q}^\varepsilon\left(t, x, \frac{y}{\varepsilon}\right), \\ \alpha &= \alpha_1 \varepsilon \end{aligned}$$

we perform the change of variable $z = \frac{y}{\varepsilon}$ and we set

$$\mathcal{S} = \{(x, z) \in \mathbb{R}^2, 0 < z < 1\}.$$

We remark that for any function $\varphi(t, x, y) = \psi(t, x, \frac{y}{\varepsilon})$, we have

$$\partial_y^{(k)}(\varphi(t, x, y)) = \varepsilon^{-k} \partial_z^{(k)} \psi(t, x, z).$$

Then the following calculations are immediate

$$\begin{cases} U_1^\varepsilon(t, x, y) = u^\varepsilon(t, x, z) - \alpha_1^2 \partial_z^2 u^\varepsilon(t, x, z) - \varepsilon^2 \alpha_1^2 \partial_x^2 u^\varepsilon(t, x, z) \\ U_2^\varepsilon(t, x, y) = \varepsilon(v^\varepsilon(t, x, z) - \alpha_1^2 \partial_z^2 v^\varepsilon(t, x, z)) - \varepsilon^3 \alpha_1^2 \partial_x^2 v^\varepsilon(t, x, z), \end{cases}$$

The system (2.3) becomes

$$\begin{cases} \partial_t \omega^\varepsilon - \alpha_1^2 \varepsilon^2 \partial_t \partial_x^2 u^\varepsilon + u^\varepsilon \partial_x \omega^\varepsilon - \alpha_1^2 \varepsilon^2 u^\varepsilon \partial_x^3 u^\varepsilon + v^\varepsilon \partial_z \omega^\varepsilon - \alpha_1^2 \varepsilon^2 v^\varepsilon \partial_z \partial_x^2 u^\varepsilon + \omega^\varepsilon \partial_x u^\varepsilon - \alpha_1^2 \varepsilon^2 \partial_x u^\varepsilon \partial_x^2 u^\varepsilon \\ \quad + \varepsilon^2 \gamma^\varepsilon \partial_x v^\varepsilon - \varepsilon^4 \alpha_1^2 \partial_x^2 v^\varepsilon \partial_x v^\varepsilon = \partial_z^2 \omega^\varepsilon + \varepsilon^2 \partial_x^2 \omega^\varepsilon - \varepsilon^2 \alpha_1^2 \partial_z^2 \partial_x^2 u^\varepsilon - \alpha_1^2 \varepsilon^4 \partial_x^4 u^\varepsilon - \partial_x \tilde{q}^\varepsilon \\ \varepsilon^2 \partial_t \gamma^\varepsilon - \alpha_1^2 \varepsilon^4 \partial_t \partial_x^2 v^\varepsilon + \varepsilon^2 u^\varepsilon \partial_x \gamma^\varepsilon - \alpha_1^2 \varepsilon^4 u^\varepsilon \partial_x^3 v^\varepsilon + \varepsilon^2 v^\varepsilon \partial_z \gamma^\varepsilon - \alpha_1^2 \varepsilon^4 v^\varepsilon \partial_z \partial_x^2 v^\varepsilon + \omega \partial_z u^\varepsilon - \alpha_1^2 \varepsilon^2 \partial_x^2 u^\varepsilon \partial_z u^\varepsilon \\ \quad + \varepsilon^2 \gamma^\varepsilon \partial_z v^\varepsilon - \alpha_1^2 \varepsilon^4 \partial_x^2 v^\varepsilon \partial_z v^\varepsilon = \varepsilon^2 \partial_z^2 \gamma^\varepsilon + \varepsilon^4 \partial_x^2 \gamma^\varepsilon - \alpha_1^2 \varepsilon^4 \partial_z \partial_x^2 v^\varepsilon - \alpha_1^2 \varepsilon^6 \partial_x^4 v^\varepsilon - \partial_z \tilde{q}^\varepsilon \\ \partial_x u^\varepsilon + \partial_z v^\varepsilon = 0, \end{cases} \quad (2.4)$$

where we denote $\omega^\varepsilon = u^\varepsilon - \alpha_1^2 \partial_z^2 u^\varepsilon$ and $\gamma^\varepsilon = v^\varepsilon - \alpha_1^2 \partial_z^2 v^\varepsilon$ to lighten the notations.

Formally taking $\varepsilon \rightarrow 0$ we obtain

$$\begin{cases} \partial_t(u - \alpha_1^2 \partial_z^2 u) + u \partial_x(u - \alpha_1^2 \partial_z^2 u) + v \partial_z(u - \alpha_1^2 \partial_z^2 u) + (u - \alpha_1^2 \partial_z^2 u) \partial_x u = \partial_z^2(u - \alpha_1^2 \partial_z^2 u) - \partial_x \tilde{q} \\ (u - \alpha_1^2 \partial_z^2 u) \partial_z u = -\partial_z \tilde{q} \\ \partial_x u + \partial_y v = 0 \\ (u, v) = 0, \quad \partial_z u = 0 \quad \text{on} \quad z = 0 \\ (u, v) = 0, \quad \partial_z u = 0 \quad \text{on} \quad z = 1 \\ u|_{t=0} = u_0. \end{cases} \quad (2.5)$$

We consider the modified pressure in the strip

$$p = \tilde{q} + \frac{1}{2} u^2 - \frac{1}{2} \alpha_1^2 (\partial_z u)^2$$

and we set

$$\omega = u - \alpha_1^2 \partial_z^2 u.$$

Then we can rewrite the system (2.5) as

$$\begin{cases} \partial_t \omega + u \partial_x \omega + v \partial_z \omega + \alpha_1^2 (\partial_z u \partial_x \partial_z u - \partial_z^2 u \partial_x u) = \partial_z^2 \omega - \partial_x p \\ \partial_z p = 0 \\ \partial_x u + \partial_y v = 0 \\ (u, v) = 0, \quad \partial_z u = 0 \quad \text{on} \quad z = 0 \\ (u, v) = 0, \quad \partial_z u = 0 \quad \text{on} \quad z = 1 \\ u|_{t=0} = u_0. \end{cases} \quad (2.6)$$

Remark 1. 1. The no-slip boundary conditions $(u, v)|_{y=0} = (u, v)|_{y=1} = 0$ and the incompressibility $\partial_x u + \partial_y v = 0$ imply

$$v(t, x, y) = \int_0^y \partial_{\tilde{y}} v(t, x, \tilde{y}) d\tilde{y} = - \int_0^y \partial_x u(t, x, \tilde{y}) d\tilde{y}.$$

2. From the incompressibility condition we deduce that

$$\partial_x \int_0^1 u(t, x, y) dy = - \int_0^1 \partial_y v(t, x, y) dy = v(t, x, 1) - v(t, x, 0) = 0,$$

which together with the fact that $u(t, x, y) \rightarrow 0$ as $|x| \rightarrow +\infty$, ensures that

$$\int_0^1 u(t, x, y) dy = 0. \quad (2.7)$$

Since $\partial_z p = 0$, integrating the first equation of system (2.6) and using the boundary conditions, we get

$$\partial_x p(t, x) = \alpha_1^2 \partial_z^3 u|_{z=0} - \alpha_1^2 \partial_z^3 u|_{z=1} - \partial_x \int_0^1 u^2(t, x, y) dz - \alpha_1^2 \partial_x \int_0^1 (\partial_z u)^2(t, x, y) dz.$$

We emphasize that, similar to Prandtl equation, the nonlinear term $v\partial_z \omega$ in the system (2.6) creates the loss of one derivative in x variable in energy-type estimates. So, in order to overcome this difficulty, it is natural to work in analytic function frameworks. In the next paragraph, we will introduce some elements of Littlewood-Paley theory and functional spaces that we are going to use in this paper.

2.1.3 Littlewood-Paley theory and functional spaces

We consider a even smooth function χ in $C_0^\infty(\mathbb{R})$ such that the support is contained in the ball $B_{\mathbb{R}}(0, \frac{4}{3})$ and ψ is equal to 1 on a neighborhood of the ball $B_{\mathbb{R}}(0, \frac{3}{4})$. We set $\psi(z) = \chi(\frac{z}{2}) - \chi(z)$ then the support of ψ is contained in the ring $\{z \in \mathbb{R} : \frac{3}{4} \leq |z| \leq \frac{8}{3}\}$, and ψ is identically equal to 1 on the ring $\{z \in \mathbb{R} : \frac{4}{3} \leq |z| \leq \frac{3}{2}\}$. Moreover, the functions χ and ψ verify the following important properties

$$\begin{aligned} \sum_q \psi(2^{-q}z) &= 1, \quad \forall z \in \mathbb{R}^*, \\ \chi(z) + \sum_{q \geq 0} \psi(2^{-q}z) &= 1, \quad \forall \eta \in \mathbb{R} \end{aligned}$$

and

$$\forall j, j' \in \mathbb{N}, |j - j'| \geq 2, \quad \text{supp } \psi(2^{-j} \cdot) \cap \text{supp } \psi(2^{-j'} \cdot) = \emptyset.$$

Let \mathcal{F}_h and \mathcal{F}_h^{-1} be the Fourier transform and the inverse Fourier transform respectively in the horizontal direction. We will also use the notation $\widehat{f} = \mathcal{F}_h f$. We introduce the following definitions of the homogeneous dyadic cut-off operators.

Definition 2.1.1. For all tempered distributions in the horizontal direction and for all $q \in \mathbb{Z}$ we set

$$\begin{aligned} \Delta_q^h u(x, y) &= \mathcal{F}_h^{-1}(\psi(2^{-q}|\xi|)\mathcal{F}_h u(\xi, y)), \\ S_q^h u(x, y) &= \mathcal{F}_h^{-1}(\chi(2^{-q}|\xi|)\mathcal{F}_h u(\xi, y)) = \sum_{q' \leq q-1} \Delta_{q'}^h u. \end{aligned}$$

In our paper, we will use the same functional spaces as in [64], the definition of which is given in what follows.

Definition 2.1.2. Let $s \in \mathbb{R}$ and $\mathcal{S} = \mathbb{R} \times]0, 1[$. For any $u \in S'_h(\mathcal{S})$, i.e.,

$$u \in S'(\mathcal{S}) \quad \text{with} \quad \lim_{q \rightarrow -\infty} \|S_q^h u\|_{L^\infty} = 0,$$

we set

$$\|u\|_{B^s} = \sum_{q \in \mathbb{Z}} 2^{qs} \|\Delta_q^h u\|_{L^2}.$$

- For $s \leq \frac{1}{2}$, we define

$$B^s(\mathcal{S}) = \{u \in S'_h(\mathcal{S}), \quad \|u\|_{B^s} < \infty\}.$$

- For $s \in [k - \frac{1}{2}, k + \frac{1}{2}]$, with $k \in \mathbb{N}^*$, we define $B^s(\mathcal{S})$ as the subset of distributions $u \in S'_h(\mathcal{S})$ such that $\partial_x^k u \in B^{s-k}(\mathcal{S})$.

Remark 2. For $u \in B^s$, there exists a summable sequence of positive numbers $d_q(u)$ with $\sum_q d_q(u) = 1$, such that

$$\|\Delta_q^h u\|_{L^2} \lesssim d_q(u) 2^{-qs} \|u\|_{B^s}.$$

We also need the following time-weighted Chemin-Lerner-type spaces (see [22]).

Definition 2.1.3. Let $p \in [1, +\infty]$ and $T \in]0, \infty]$ and let $f \in L^1_{loc}(\mathbb{R}^+)$ be non-negative function. We define space $\widetilde{L}^p_{T,f}(B^s(\mathcal{S}))$ as the closure of $C([0, T], B^s(\mathcal{S}))$ with respect to the norm

$$\|u\|_{\widetilde{L}^p_{T,f}(B^s(\mathcal{S}))} = \sum_{q \in \mathbb{Z}} 2^{qs} \left(\int_0^T f(t) \|\Delta_q^h u\|_{L^2}^p dt \right)^{\frac{1}{p}}$$

with the usual change if $p = +\infty$. In the case of $f(t) \equiv 1$, we will simply use the notations $\widetilde{L}^p_T(B^s(\mathcal{S}))$ and $\|u\|_{\widetilde{L}^p_T(B^s(\mathcal{S}))}$.

Remark 3. For $u \in \widetilde{L}^p_T(B^s)$, there exists a summable sequence of positive numbers $d_q(u, f)$ with $\sum_q d_q(u, f) = 1$, such that

$$\|\Delta_q^h u\|_{L^p_T(L^2)} \lesssim d_q(u, f) 2^{-pqs} \|u\|_{\widetilde{L}^p_T(B^s)}.$$

2.1.4 Main results

The main difficulty consists in controlling the nonlinear terms, using the smoothing effect given by the above function spaces. The idea is to define the following auxiliary functions, using the method introduced by Chemin in [25] (see also [21] or [64]): for any $f \in L^2(\mathcal{S})$, we set

$$f_\phi(t, x, y) = e^{\phi(t, D_x)} f(t, x, y) := \mathcal{F}_{\xi \rightarrow x}^{-1}(e^{\phi(t, \xi)} \widehat{f}(t, \xi, y)) \quad \text{with} \quad \phi(t, \xi) = (a - \lambda\theta(t))|\xi|, \quad (2.8)$$

where the quantity $\theta(t)$ describes the evolution of the analytic band of u , satisfies

$$\theta'(t) = \|\partial_z^2 u_\phi(t)\|_{B^{\frac{1}{2}}} \quad \text{and} \quad \theta(0) = 0. \quad (2.9)$$

We remark that if we differentiate a function of the type $e^{\phi(t, D_x)} f(t, x, y)$ with respect to the time variable, we have

$$\partial_t(e^{\phi(t, D_x)} f(t, x, y)) = -\lambda\theta'(t)|D_x|e^{\phi(t, D_x)} f(t, x, y) + e^{\phi(t, D_x)} \partial_t f(t, x, y),$$

where $-\lambda\theta'(t)|D_x|e^{\phi(t, D_x)} f(t, x, y)$ plays the role of a ‘‘smoothing term’’, provided that $\theta'(t) > 0$, and allows to ‘‘absorb’’ the loss of one derivative in the nonlinear terms when we perform energy-type estimates.

Our main result is the following.

Theorem 2.1. Let $a > 0$ be fixed. We assume that $e^{a|D_x|} u_0 \in B^{\frac{3}{2}}$ and $e^{a|D_x|} \partial_z u_0 \in B^{\frac{3}{2}}$. There exist positive constants c and C such that, if we suppose that

$$\|e^{a|D_x|} u_0\|_{B^{\frac{1}{2}}} + \|e^{a|D_x|} \partial_z u_0\|_{B^{\frac{1}{2}}} \leq \frac{ca}{1 + \|e^{a|D_x|} u_0\|_{B^{\frac{3}{2}}} + \|e^{a|D_x|} \partial_z u_0\|_{B^{\frac{3}{2}}}} \quad (2.10)$$

and the compatibility condition $\int_0^1 u_0 dz = 0$, then a unique global solution for the system (2.6) exists and satisfies

$$\begin{aligned} \|e^{Rt} u_\phi\|_{\widetilde{L}^\infty(\mathbb{R}^+, B^{\frac{3}{2}})} + \|e^{Rt} \partial_z u_\phi\|_{\widetilde{L}^\infty(\mathbb{R}^+, B^{\frac{3}{2}})} + \|e^{Rt} \partial_z u_\phi\|_{\widetilde{L}^2(\mathbb{R}^+, B^{\frac{3}{2}})} + \|e^{Rt} \partial_z^2 u_\phi\|_{\widetilde{L}^2(\mathbb{R}^+, B^{\frac{3}{2}})} \\ \leq C(\|e^{a|D_x|} u_0\|_{B^{\frac{3}{2}}} + \|e^{a|D_x|} \partial_z u_0\|_{B^{\frac{3}{2}}}). \end{aligned} \quad (2.11)$$

Furthermore, if $e^{a|D_x|}u_0 \in B^{\frac{5}{2}}$, $e^{a|D_x|}\partial_z u_0 \in B^{\frac{5}{2}}$, $e^{a|D_x|}\partial_z^2 u_0 \in B^{\frac{3}{2}}$, then

$$\begin{aligned} & \|e^{Rt}(\partial_t u)_\phi\|_{\tilde{L}^2(\mathbb{R}^+, B^{\frac{3}{2}})} + \|e^{Rt}(\partial_t \partial_z u)_\phi\|_{\tilde{L}^2(\mathbb{R}^+, B^{\frac{3}{2}})} + \|e^{Rt}\partial_z u_\phi\|_{\tilde{L}^\infty(\mathbb{R}^+, B^{\frac{3}{2}})} + \|e^{Rt}\partial_z^2 u_\phi\|_{\tilde{L}^\infty(\mathbb{R}^+, B^{\frac{3}{2}})} \\ & \leq C \left(\|e^{a|D_x|}\partial_z u_0\|_{B^{\frac{3}{2}}} + \|e^{a|D_x|}\partial_z^2 u_0\|_{B^{\frac{3}{2}}} + \|e^{a|D_x|}u_0\|_{B^{\frac{5}{2}}} + \|e^{a|D_x|}\partial_z u_0\|_{B^{\frac{5}{2}}} \right). \end{aligned} \quad (2.12)$$

Remark 4. The compatibility condition $\int_0^1 u_0 dz = 0$ ensures that $\int_0^1 u(t, \cdot) dz$ is conserved by the system (2.6).

The rest of the paper is arranged as follows : The proof of the main Theorem 2.1 is presented in the next Section 2.2. and the Appendices 2.3, 2.4 and 2.5 are devoted to the proof of estimates used in Section 2.2.

2.2 Global existence and uniqueness of the hydrostatic limit system

In this section, we prove the existence of a unique global solution of the system (2.6) using the method introduced by Chemin in [20]. We recall that, to be able to deal with the lost of one derivative in the tangential direction x , we work in the same functional settings as in [64].

2.2.1 Energy-type *a priori* estimates

The aim of the first part of this section is to prove the energy-type estimate (2.11). Our proof is based on the following important estimates of the nonlinear terms. We recall that ϕ, θ are the auxiliary functions defined as in (2.8) and (2.9) and we set

$$T^* = \sup \left\{ t > 0, \quad \theta(t) < \frac{a}{\lambda} \right\}.$$

Lemma 2.2.1. Let $s > 0$, $0 < T < T^*$, $R > 0$. There exist a generic constant $C \geq 1$ and some square root summable positive sequences $(\sum_q \check{d}_q^{\frac{1}{2}} = 1, \sum_q \check{d}_q^{\frac{1}{2}} = 1)$ such that

$$\int_0^T \left| \left\langle e^{Rt'} \Delta_q^h (u \partial_x u)_\phi, e^{Rt'} \Delta_q^h u_\phi \right\rangle \right| dt' \leq C 2^{-2qs} (\check{d}_q + \check{d}_q) \|e^{Rt'} u_\phi\|_{\tilde{L}_{T, \theta'(t)}^2(B^{s+\frac{1}{2}})}^2, \quad (2.13)$$

$$\int_0^T \left| \left\langle e^{Rt'} \Delta_q^h (u \partial_x \partial_z u)_\phi, e^{Rt'} \Delta_q^h \partial_z u_\phi \right\rangle \right| dt' \leq C 2^{-2qs} (\check{d}_q + \check{d}_q) \|e^{Rt'} \partial_z u_\phi\|_{\tilde{L}_{T, \theta'(t)}^2(B^{s+\frac{1}{2}})}^2, \quad (2.14)$$

$$\int_0^T \left| \left\langle e^{Rt'} \Delta_q^h (\partial_z u \partial_x u)_\phi, e^{Rt'} \Delta_q^h \partial_z u_\phi \right\rangle \right| dt' \leq C 2^{-2qs} (\check{d}_q + \check{d}_q) \|e^{Rt'} \partial_z u_\phi\|_{\tilde{L}_{T, \theta'(t)}^2(B^{s+\frac{1}{2}})}^2, \quad (2.15)$$

$$\int_0^T \left| \left\langle e^{Rt'} \Delta_q^h (\partial_z u \partial_x \partial_z u)_\phi, e^{Rt'} \Delta_q^h u_\phi \right\rangle \right| dt' \leq C 2^{-2qs} (\check{d}_q + \check{d}_q) \|e^{Rt'} \partial_z u_\phi\|_{\tilde{L}_{T, \theta'(t)}^2(B^{s+\frac{1}{2}})}^2, \quad (2.16)$$

$$\int_0^T \left| \left\langle e^{Rt'} \Delta_q^h (v \partial_z u)_\phi, e^{Rt'} \Delta_q^h u_\phi \right\rangle \right| dt' \leq C 2^{-2qs} (\check{d}_q + \check{d}_q) \|e^{Rt'} \partial_z u_\phi\|_{\tilde{L}_{T, \theta'(t)}^2(B^{s+\frac{1}{2}})}^2 \quad (2.17)$$

and

$$\begin{aligned} & \int_0^T \left| \left\langle e^{Rt'} \Delta_q^h (v \partial_z^2 u)_\phi, e^{Rt'} \Delta_q^h \partial_z u_\phi \right\rangle \right| dt' \\ & \leq C 2^{-2qs} (\check{d}_q + \check{d}_q) \|e^{Rt'} \partial_z u_\phi\|_{\tilde{L}_{T, \theta'(t)}^2(B^{s+\frac{1}{2}})} \left(\|e^{Rt'} \partial_z u_\phi\|_{\tilde{L}_{T, \theta'(t)}^2(B^{s+\frac{1}{2}})} + \|\partial_z u_\phi\|_{L_T^\infty(B^{\frac{3}{2}})}^{\frac{1}{2}} \|e^{Rt'} \partial_z^2 u_\phi\|_{\tilde{L}_T^2(B^s)} \right). \end{aligned} \quad (2.18)$$

The proof of this lemma is given in appendix 2.3. For (u, v) solution of (2.6), we define (u_ϕ, v_ϕ) be as in (2.8) and (2.9). Direct calculations show that (u_ϕ, v_ϕ) satisfies the following system

$$\begin{cases} \partial_t \omega_\phi + \lambda \theta'(t) |D_x| \omega_\phi + (u \partial_x \omega)_\phi + (v \partial_z \omega)_\phi + \alpha_1^2 (\partial_z u \partial_x \partial_z u)_\phi - \alpha_1^2 (\partial_z^2 u \partial_x u)_\phi = \partial_z^2 \omega_\phi - \partial_x p_\phi \\ \partial_z p_\phi = 0 \\ \partial_x u_\phi + \partial_y v_\phi = 0 \\ (u_\phi, v_\phi) = 0, \quad \partial_z u_\phi = 0 \quad \text{on } z = 0 \\ (u_\phi, v_\phi) = 0, \quad \partial_z u_\phi = 0 \quad \text{on } z = 1 \\ u_\phi|_{t=0} = e^{a|D_x|} u_0, \end{cases} \quad (2.19)$$

where $|D_x|$ denotes the Fourier multiplier of symbol $|\xi|$. For any $q \in \mathbb{Z}$, applying the dyadic operator Δ_q^h to the first equation of the system (2.19) and taking $L^2(S)$ inner product of the obtained equation with $\Delta_q^h u_\phi$, we get

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} (\|\Delta_q^h u_\phi\|_{L^2}^2 + \alpha_1^2 \|\Delta_q^h \partial_z u_\phi\|_{L^2}^2) + \lambda \theta'(t) \left(\| |D_x|^{\frac{1}{2}} \Delta_q^h u_\phi \|_{L^2}^2 + \alpha_1^2 \lambda \theta'(t) \left\| |D_x|^{\frac{1}{2}} \Delta_q^h \partial_z u_\phi \right\|_{L^2}^2 \right. \\ \left. + \|\Delta_q^h \partial_z u_\phi\|_{L^2}^2 + \alpha_1^2 \|\Delta_q^h \partial_z^2 u_\phi\|_{L^2}^2 \right) = - \langle \Delta_q^h (u \partial_x \omega)_\phi, \Delta_q^h u_\phi \rangle - \langle \Delta_q^h (v \partial_z \omega)_\phi, \Delta_q^h u_\phi \rangle \\ - \alpha_1^2 \langle \Delta_q^h (\partial_z u \partial_x \partial_z u)_\phi, \Delta_q^h u_\phi \rangle + \alpha_1^2 \langle \Delta_q^h (\partial_z^2 u \partial_x u)_\phi, \Delta_q^h u_\phi \rangle. \end{aligned} \quad (2.20)$$

Multiplying (2.20) by e^{2Rt} and integrating with respect to the time variable, we obtain

$$\begin{aligned} \|e^{Rt} \Delta_q^h u_\phi\|_{L^2}^2 + \alpha_1^2 \|e^{Rt} \Delta_q^h \partial_z u_\phi\|_{L^2}^2 + 2\lambda \int_0^t \theta'(t') \left\| e^{Rt'} |D_x|^{\frac{1}{2}} \Delta_q^h u_\phi(t') \right\|_{L^2}^2 dt' + 2 \int_0^t \|e^{Rt'} \Delta_q^h \partial_z u_\phi(t')\|_{L^2}^2 dt' \\ + 2\lambda \alpha_1^2 \int_0^t \theta'(t') \left\| e^{Rt'} |D_x|^{\frac{1}{2}} \Delta_q^h \partial_z u_\phi(t') \right\|_{L^2}^2 dt' + 2\alpha_1^2 \int_0^t \|e^{Rt'} \Delta_q^h \partial_z^2 u_\phi(t')\|_{L^2}^2 dt' \\ = \|\Delta_q^h u_\phi(0)\|_{L^2}^2 + \alpha_1^2 \|\Delta_q^h \partial_z u_\phi(0)\|_{L^2}^2 + 2D_{1,q} + 2D_{2,q} + 2D_{3,q} + 2D_{4,q}, \end{aligned}$$

where

$$\begin{aligned} D_{1,q} &= - \int_0^t \left\langle e^{Rt'} \Delta_q^h (u \partial_x \omega)_\phi, e^{Rt'} \Delta_q^h u_\phi \right\rangle dt', \\ D_{2,q} &= - \int_0^t \left\langle e^{Rt'} \Delta_q^h (v \partial_z \omega)_\phi, e^{Rt'} \Delta_q^h u_\phi \right\rangle dt', \\ D_{3,q} &= - \alpha_1^2 \int_0^t \left\langle e^{Rt'} \Delta_q^h (\partial_z u \partial_x \partial_z u)_\phi, e^{Rt'} \Delta_q^h u_\phi \right\rangle dt', \\ D_{4,q} &= \alpha_1^2 \int_0^t \left\langle e^{Rt'} \Delta_q^h (\partial_z^2 u \partial_x u)_\phi, e^{Rt'} \Delta_q^h u_\phi \right\rangle dt'. \end{aligned}$$

Using the definition of ω , we can write

$$D_{1,q} = - \int_0^t \left\langle e^{Rt'} \Delta_q^h (u \partial_x u)_\phi, e^{Rt'} \Delta_q^h u_\phi \right\rangle dt' + \alpha_1^2 \int_0^t \left\langle e^{Rt'} \Delta_q^h (u \partial_x \partial_z^2 u)_\phi, e^{Rt'} \Delta_q^h u_\phi \right\rangle dt'.$$

Since

$$\Delta_q^h (u \partial_x \partial_z^2 u)_\phi = \partial_z \Delta_q^h (u \partial_x \partial_z u)_\phi - \Delta_q^h (\partial_z u \partial_x \partial_z u)_\phi,$$

by integration by parts with respect to z variable, we have

$$\begin{aligned} \int_0^t \left\langle e^{Rt'} \Delta_q^h (u \partial_x \partial_z^2 u)_\phi, e^{Rt'} \Delta_q^h u_\phi \right\rangle dt' \\ = \int_0^t \left\langle e^{Rt'} \partial_z \Delta_q^h (u \partial_x \partial_z u)_\phi, e^{Rt'} \Delta_q^h u_\phi \right\rangle dt' - \int_0^t \left\langle e^{Rt'} \Delta_q^h (\partial_z u \partial_x \partial_z u)_\phi, e^{Rt'} \Delta_q^h u_\phi \right\rangle dt' \\ = - \int_0^t \left\langle e^{Rt'} \Delta_q^h (u \partial_x \partial_z u)_\phi, e^{Rt'} \Delta_q^h \partial_z u_\phi \right\rangle dt' - \int_0^t \left\langle e^{Rt'} \Delta_q^h (\partial_z u \partial_x \partial_z u)_\phi, e^{Rt'} \Delta_q^h u_\phi \right\rangle dt'. \end{aligned}$$

Thus

$$D_{1,q} = A_q + B_q + D_{3,q},$$

where

$$\begin{aligned} A_q &= - \int_0^t \left\langle e^{Rt'} \Delta_q^h (u \partial_x u)_\phi, e^{Rt'} \Delta_q^h u_\phi \right\rangle dt', \\ B_q &= -\alpha_1^2 \int_0^t \left\langle e^{Rt'} \Delta_q^h (u \partial_x \partial_z u)_\phi, e^{Rt'} \Delta_q^h \partial_z u_\phi \right\rangle dt'. \end{aligned}$$

In a similar way, we can write

$$D_{4,q} = C_q + D_{3,q},$$

where

$$C_q = -\alpha_1^2 \int_0^t \left\langle e^{Rt'} \Delta_q^h (\partial_z u \partial_x u)_\phi, e^{Rt'} \Delta_q^h \partial_z u_\phi \right\rangle dt',$$

and

$$D_{2,q} = E_q + F_q + C_q + D_{3,q}$$

and where

$$\begin{aligned} E_q &= - \int_0^t \left\langle e^{Rt'} \Delta_q^h (v \partial_z u)_\phi, e^{Rt'} \Delta_q^h u_\phi \right\rangle dt', \\ F_q &= -\alpha_1^2 \int_0^t \left\langle e^{Rt'} \Delta_q^h (v \partial_z^2 u)_\phi, e^{Rt'} \Delta_q^h \partial_z u_\phi \right\rangle dt'. \end{aligned}$$

To summarize, we have

$$\begin{aligned} & \|e^{Rt} \Delta_q^h u_\phi\|_{L^2}^2 + \alpha_1^2 \|e^{Rt} \Delta_q^h \partial_z u_\phi\|_{L^2}^2 + 2\lambda \alpha_1^2 \int_0^t \theta'(t') \left\| e^{Rt'} |D_x|^{\frac{1}{2}} \Delta_q^h \partial_z u_\phi(t') \right\|_{L^2}^2 dt' \\ & + 2\lambda \int_0^t \theta'(t') \left\| e^{Rt'} |D_x|^{\frac{1}{2}} \Delta_q^h u_\phi(t') \right\|_{L^2}^2 dt' + 2 \int_0^t \left\| e^{Rt'} \Delta_q^h \partial_z u_\phi(t') \right\|_{L^2}^2 dt' \\ & + 2\alpha_1^2 \int_0^t \left\| e^{Rt'} \Delta_q^h \partial_z^2 u_\phi(t') \right\|_{L^2}^2 dt' \\ & = \|\Delta_q^h u_\phi(0)\|_{L^2}^2 + \alpha_1^2 \|\Delta_q^h \partial_z u_\phi(0)\|_{L^2}^2 + 2A_q + 2B_q + 2C_q + 8D_{3,q} + 2E_q + 2F_q, \quad (2.21) \end{aligned}$$

where

$$\begin{aligned} A_q &= - \int_0^t \left\langle e^{Rt'} \Delta_q^h (u \partial_x u)_\phi, e^{Rt'} \Delta_q^h u_\phi \right\rangle dt', \\ B_q &= -\alpha_1^2 \int_0^t \left\langle e^{Rt'} \Delta_q^h (u \partial_x \partial_z u)_\phi, e^{Rt'} \Delta_q^h \partial_z u_\phi \right\rangle dt', \\ C_q &= -\alpha_1^2 \int_0^t \left\langle e^{Rt'} \Delta_q^h (\partial_z u \partial_x u)_\phi, e^{Rt'} \Delta_q^h \partial_z u_\phi \right\rangle dt', \\ D_{3,q} &= -\alpha_1^2 \int_0^t \left\langle e^{Rt'} \Delta_q^h (\partial_z u \partial_x \partial_z u)_\phi, e^{Rt'} \Delta_q^h u_\phi \right\rangle dt', \\ E_q &= - \int_0^t \left\langle e^{Rt'} \Delta_q^h (v \partial_z u)_\phi, e^{Rt'} \Delta_q^h u_\phi \right\rangle dt', \\ F_q &= -\alpha_1^2 \int_0^t \left\langle e^{Rt'} \Delta_q^h (v \partial_z^2 u)_\phi, e^{Rt'} \Delta_q^h \partial_z u_\phi \right\rangle dt'. \end{aligned}$$

We recall that we set

$$T^* = \sup \left\{ t > 0, \theta(t) < \frac{a}{\lambda} \right\}.$$

Lemma 2.2.1 yields that, for any $0 < T < T^*$, we have

$$\begin{aligned}
 & \|e^{Rt} \Delta_q^h u_\phi\|_{\tilde{L}_T^\infty(L^2)}^2 + \alpha_1^2 \|e^{Rt} \Delta_q^h \partial_z u_\phi\|_{\tilde{L}_T^\infty(L^2)}^2 + \lambda \alpha_1^2 2^q \int_0^T \theta'(t') \left\| e^{Rt'} \Delta_q^h \partial_z u_\phi(t') \right\|_{L^2}^2 dt' \\
 & + \alpha_1^2 \int_0^T \left\| e^{Rt'} \Delta_q^h \partial_z^2 u_\phi(t') \right\|_{L^2}^2 dt' + \int_0^T \left\| e^{Rt'} \Delta_q^h \partial_z u_\phi(t') \right\|_{L^2}^2 dt' + \lambda 2^q \int_0^T \theta'(t') \left\| e^{Rt'} \Delta_q^h u_\phi(t') \right\|_{L^2}^2 dt' \\
 & \leq \|\Delta_q^h u_\phi(0)\|_{L^2}^2 + \alpha_1^2 \|\Delta_q^h \partial_z u_\phi(0)\|_{L^2}^2 + C 2^{-2qs} d_q^2 \|e^{Rt'} u_\phi\|_{\tilde{L}_{T, \theta'(t)}^2(B^{s+\frac{1}{2}})}^2 \\
 & + C 2^{-2qs} d_q^2 \|e^{Rt'} \partial_z u_\phi\|_{\tilde{L}_{T, \theta'(t)}^2(B^{s+\frac{1}{2}})} \left(\|e^{Rt'} \partial_z u_\phi\|_{\tilde{L}_{T, \theta'(t)}^2(B^{s+\frac{1}{2}})} + \|\partial_z u_\phi\|_{L_T^\infty(B^{\frac{3}{2}})}^{\frac{1}{2}} \|e^{Rt'} \partial_z^2 u_\phi\|_{\tilde{L}_T^2(B^s)} \right).
 \end{aligned}$$

Multiplying the previous inequality by 2^{2qs} , taking square root of resulting inequality and then summing with respect to $q \in \mathbb{Z}$, we get

$$\begin{aligned}
 & \|e^{Rt} u_\phi\|_{\tilde{L}_T^\infty(B^s)} + \alpha_1^2 \|e^{Rt} \partial_z u_\phi\|_{\tilde{L}_T^\infty(B^s)} + \sqrt{\lambda} \|e^{Rt'} u_\phi\|_{\tilde{L}_{T, \theta'(t)}^2(B^{s+\frac{1}{2}})} + \sqrt{\lambda} \alpha_1 \|e^{Rt'} \partial_z u_\phi\|_{\tilde{L}_{T, \theta'(t)}^2(B^{s+\frac{1}{2}})} \\
 & + \|e^{Rt} \partial_z u_\phi\|_{\tilde{L}_T^2(B^s)} + \alpha_1 \|e^{Rt} \partial_z^2 u_\phi\|_{\tilde{L}_T^2(B^s)} \\
 & \leq 6 \|u_\phi(0)\|_{B^s} + 6 \alpha_1 \|\partial_z u_\phi(0)\|_{B^s} + C \|e^{Rt'} u_\phi\|_{\tilde{L}_{T, \theta'(t)}^2(B^{s+\frac{1}{2}})} + C \alpha_1 \|e^{Rt'} \partial_z u_\phi\|_{\tilde{L}_{T, \theta'(t)}^2(B^{s+\frac{1}{2}})} \\
 & + C \|\partial_z u_\phi\|_{L_T^\infty(B^{\frac{3}{2}})}^{\frac{1}{4}} \|e^{Rt'} \partial_z^2 u_\phi\|_{\tilde{L}_T^2(B^s)}^{\frac{1}{2}} \|e^{Rt'} \partial_z u_\phi\|_{\tilde{L}_{T, \theta'(t)}^2(B^{s+\frac{1}{2}})}^{\frac{1}{2}}.
 \end{aligned}$$

Therefore, there exists a constant C_1 such that

$$\begin{aligned}
 & \|e^{Rt} u_\phi\|_{\tilde{L}_T^\infty(B^s)} + \|e^{Rt} \partial_z u_\phi\|_{\tilde{L}_T^\infty(B^s)} + \sqrt{\lambda} \|e^{Rt'} u_\phi\|_{\tilde{L}_{T, \theta'(t)}^2(B^{s+\frac{1}{2}})} + \sqrt{\lambda} \|e^{Rt'} \partial_z u_\phi\|_{\tilde{L}_{T, \theta'(t)}^2(B^{s+\frac{1}{2}})} \\
 & + \|e^{Rt} \partial_z u_\phi\|_{\tilde{L}_T^2(B^s)} + \|e^{Rt} \partial_z^2 u_\phi\|_{\tilde{L}_T^2(B^s)} \\
 & \leq C_1 (\|u_\phi(0)\|_{B^s} + \|\partial_z u_\phi(0)\|_{B^s}) + C_1 \|e^{Rt'} u_\phi\|_{\tilde{L}_{T, \theta'(t)}^2(B^{s+\frac{1}{2}})} + C_1 \|e^{Rt'} \partial_z u_\phi\|_{\tilde{L}_{T, \theta'(t)}^2(B^{s+\frac{1}{2}})} \\
 & + C_1 \|\partial_z u_\phi\|_{L_T^\infty(B^{\frac{3}{2}})}^{\frac{1}{4}} \|e^{Rt'} \partial_z^2 u_\phi\|_{\tilde{L}_T^2(B^s)}^{\frac{1}{2}} \|e^{Rt'} \partial_z u_\phi\|_{\tilde{L}_{T, \theta'(t)}^2(B^{s+\frac{1}{2}})}^{\frac{1}{2}}.
 \end{aligned}$$

We remark that Young's inequality implies

$$\begin{aligned}
 C_1 \|\partial_z u_\phi\|_{L_T^\infty(B^{\frac{3}{2}})}^{\frac{1}{4}} \|e^{Rt'} \partial_z^2 u_\phi\|_{\tilde{L}_T^2(B^s)}^{\frac{1}{2}} \|e^{Rt'} \partial_z u_\phi\|_{\tilde{L}_{T, \theta'(t)}^2(B^{s+\frac{1}{2}})}^{\frac{1}{2}} \\
 \leq C_2 \|\partial_z u_\phi\|_{L_T^\infty(B^{\frac{3}{2}})}^{\frac{1}{2}} \|e^{Rt'} \partial_z u_\phi\|_{\tilde{L}_{T, \theta'(t)}^2(B^{s+\frac{1}{2}})} + \frac{1}{2} \|e^{Rt'} \partial_z^2 u_\phi\|_{\tilde{L}_T^2(B^s)}.
 \end{aligned}$$

So, putting $C_3 = \max(C_1, C_2)$ and choosing

$$\lambda \geq C_3^2 (1 + \|\partial_z u_\phi\|_{L_T^\infty(B^{\frac{3}{2}})}), \quad (2.22)$$

we have for any $s > 0$

$$\begin{aligned}
 \|e^{Rt} u_\phi\|_{\tilde{L}_T^\infty(B^s)} + \|e^{Rt} \partial_z u_\phi\|_{\tilde{L}_T^\infty(B^s)} + \|e^{Rt} \partial_z u_\phi\|_{\tilde{L}_T^2(B^s)} + \|e^{Rt} \partial_z^2 u_\phi\|_{\tilde{L}_T^2(B^s)} \\
 \leq C_3 (\|e^{a|D_x|} u_0\|_{B^s} + \|e^{a|D_x|} \partial_z u_0\|_{B^s}).
 \end{aligned} \quad (2.23)$$

In particular, under the condition (2.22), we have

$$\|\partial_z u_\phi\|_{\tilde{L}_T^\infty(B^{\frac{3}{2}})} \leq C_3 (\|e^{a|D_x|} u_0\|_{B^{\frac{3}{2}}} + \|e^{a|D_x|} \partial_z u_0\|_{B^{\frac{3}{2}}}).$$

Then, by taking $\lambda = C_3^2 (1 + C_3 (\|e^{a|D_x|} u_0\|_{B^{\frac{3}{2}}} + \|e^{a|D_x|} \partial_z u_0\|_{B^{\frac{3}{2}}}))$, the inequality (2.23) holds for any $s > 0$. In particular for $s = \frac{1}{2}$, we have

$$\begin{aligned}
 \|e^{Rt} u_\phi\|_{\tilde{L}_T^\infty(B^{\frac{1}{2}})} + \|e^{Rt} \partial_z u_\phi\|_{\tilde{L}_T^\infty(B^{\frac{1}{2}})} + \|e^{Rt} \partial_z u_\phi\|_{\tilde{L}_T^2(B^{\frac{1}{2}})} + \|e^{Rt} \partial_z^2 u_\phi\|_{\tilde{L}_T^2(B^{\frac{1}{2}})} \\
 \leq C_3 (\|e^{a|D_x|} u_0\|_{B^{\frac{1}{2}}} + \|e^{a|D_x|} \partial_z u_0\|_{B^{\frac{1}{2}}}).
 \end{aligned} \quad (2.24)$$

Now we want to estimate the function $\theta(t)$ on $[0, T^*]$. Using Cauchy-Schwarz inequality, we can write

$$\begin{aligned}\theta(t) &= \int_0^t e^{-Rt'} \times e^{Rt'} \|\partial_z^2 u_\phi(t')\|_{B^{\frac{1}{2}}} dt' \\ &\leq \left(\int_0^T e^{-2Rt'} dt' \right)^{\frac{1}{2}} \left(\int_0^T \|e^{Rt'} \partial_z^2 u_\phi(t')\|_{B^{\frac{1}{2}}}^2 dt' \right)^{\frac{1}{2}} \\ &\leq C_R \|e^{Rt} \partial_z^2 u_\phi\|_{\tilde{L}_T^2(B^{\frac{1}{2}})}.\end{aligned}$$

Then, we deduce from (2.24) the existence of a constant $C_4 > 0$ such that

$$\theta(t) \leq C_4 \left(\|e^{a|D_x|} u_0\|_{B^{\frac{1}{2}}} + \|e^{a|D_x|} \partial_z u_0\|_{B^{\frac{1}{2}}} \right).$$

Taking c small enough (see the assumption on the initial data (2.10)), we obtain, for any $t \in [0, T^*]$ that

$$\theta(t) \leq C_4 \left(\|e^{a|D_x|} u_0\|_{B^{\frac{1}{2}}} + \|e^{a|D_x|} \partial_z u_0\|_{B^{\frac{1}{2}}} \right) \leq \frac{a}{2\lambda}.$$

Using the definition of T^* and a continuity argument, we deduce $T^* = +\infty$ and that the inequality (2.23) holds for any $T > 0$.

Conclusion : Taking $s = \frac{3}{2}$ in (2.23), we obtain Estimate (2.11) in Theorem 2.1.

2.2.2 Estimate of the time derivative of the solution

In this paragraph, we give brief ideas of the proof of Estimate (2.12). Applying Δ_q^h to (2.19) and taking the L^2 inner product of the obtained equation with $e^{2Rt} \Delta_q^h(\partial_t u)_\phi$, we have

$$\begin{aligned}\|e^{Rt} \Delta_q^h(\partial_t u)_\phi\|_{L^2}^2 + \alpha_1^2 \|e^{Rt} \Delta_q^h(\partial_t \partial_z u)_\phi\|_{L^2}^2 & \\ = e^{2Rt} \langle \Delta_q^h \partial_z^2 u_\phi, \Delta_q^h(\partial_t u)_\phi \rangle - \alpha_1^2 e^{2Rt} \langle \Delta_q^h \partial_z^4 u_\phi, \Delta_q^h(\partial_t u)_\phi \rangle & \\ - e^{2Rt} \langle \Delta_q^h(u \partial_x u)_\phi, \Delta_q^h(\partial_t u)_\phi \rangle - e^{2Rt} \langle \Delta_q^h(v \partial_z u)_\phi, \Delta_q^h(\partial_t u)_\phi \rangle & \\ - e^{2Rt} \alpha_1^2 \langle \Delta_q^h(\partial_z u \partial_x \partial_z u)_\phi, \Delta_q^h(\partial_t u)_\phi \rangle + e^{2Rt} \alpha_1^2 \langle \Delta_q^h(\partial_z^2 u \partial_x u)_\phi, \Delta_q^h(\partial_t u)_\phi \rangle. &\end{aligned}\tag{2.25}$$

Since $(\partial_t u)_\phi = \partial_t u_\phi + \lambda \theta'(t) |D_x| u_\phi$, then

$$\begin{aligned}e^{2Rt} \langle \Delta_q^h \partial_z^2 u_\phi, \Delta_q^h(\partial_t u)_\phi \rangle &= e^{2Rt} \langle \Delta_q^h \partial_z^2 u_\phi, \Delta_q^h \partial_t u_\phi \rangle + \theta'(t) |D_x| e^{2Rt} \langle \Delta_q^h \partial_z^2 u_\phi, \Delta_q^h u_\phi \rangle \\ &= -\frac{1}{2} \frac{d}{dt} \|e^{Rt} \Delta_q^h \partial_z u_\phi\|_{L^2}^2 - \lambda \theta'(t) \|e^{Rt} |D_x|^{\frac{1}{2}} \Delta_q^h \partial_z u_\phi\|_{L^2}^2 \\ &\leq -\frac{1}{2} \frac{d}{dt} \|e^{Rt} \Delta_q^h \partial_z u_\phi\|_{L^2}^2.\end{aligned}\tag{2.26}$$

Similar calculations give

$$-e^{2Rt} \langle \Delta_q^h \partial_z^4 u_\phi, \Delta_q^h(\partial_t u)_\phi \rangle \leq -\frac{1}{2} \frac{d}{dt} \|e^{Rt} \Delta_q^h \partial_z^2 u_\phi\|_{L^2}^2.\tag{2.27}$$

Plugging the estimates (2.26) and (2.27) into (2.25), we get

$$\begin{aligned}\|e^{Rt} \Delta_q^h(\partial_t u)_\phi\|_{L^2}^2 + \alpha_1^2 \|e^{Rt} \Delta_q^h(\partial_t \partial_z u)_\phi\|_{L^2}^2 + \frac{1}{2} \frac{d}{dt} \|e^{Rt} \Delta_q^h \partial_z u_\phi\|_{L^2}^2 + \frac{1}{2} \alpha_1^2 \frac{d}{dt} \|e^{Rt} \Delta_q^h \partial_z^2 u_\phi\|_{L^2}^2 & \\ \leq -e^{2Rt} \langle \Delta_q^h(u \partial_x u)_\phi, \Delta_q^h(\partial_t u)_\phi \rangle + \alpha_1^2 e^{2Rt} \langle \Delta_q^h(u \partial_x \partial_z^2 u)_\phi, \Delta_q^h(\partial_t u)_\phi \rangle & \\ - e^{2Rt} \langle \Delta_q^h(v \partial_z u)_\phi, \Delta_q^h(\partial_t u)_\phi \rangle - \alpha_1^2 e^{2Rt} \langle \Delta_q^h(v \partial_z^2 u)_\phi, \Delta_q^h(\partial_t \partial_z u)_\phi \rangle & \\ - e^{2Rt} \alpha_1^2 \langle \Delta_q^h(\partial_z u \partial_x \partial_z u)_\phi, \Delta_q^h(\partial_t u)_\phi \rangle + 2e^{2Rt} \alpha_1^2 \langle \Delta_q^h(\partial_z^2 u \partial_x u)_\phi, \Delta_q^h(\partial_t u)_\phi \rangle &\end{aligned}$$

and so

$$\begin{aligned} & \|e^{Rt} \Delta_q^h(\partial_t u)_\phi\|_{L^2}^2 + \alpha_1^2 \|e^{Rt} \Delta_q^h(\partial_t \partial_z u)_\phi\|_{L^2}^2 + \frac{1}{2} \frac{d}{dt} \|e^{Rt} \Delta_q^h \partial_z u_\phi\|_{L^2}^2 + \frac{1}{2} \alpha_1^2 \frac{d}{dt} \|e^{Rt} \Delta_q^h \partial_z^2 u_\phi\|_{L^2}^2 \\ & \leq C \left(\|e^{Rt} \Delta_q^h(u \partial_x u)_\phi\|_{L^2}^2 + \|e^{Rt} \Delta_q^h(u \partial_x \partial_z^2 u)_\phi\|_{L^2}^2 + \|e^{Rt} \Delta_q^h(v \partial_z u)_\phi\|_{L^2}^2 \right. \\ & \quad \left. + \|e^{Rt} \Delta_q^h(v \partial_z^2 u)_\phi\|_{L^2}^2 + \|e^{Rt} \Delta_q^h(\partial_z u \partial_x \partial_z u)_\phi\|_{L^2}^2 + \|e^{Rt} \Delta_q^h(\partial_z^2 u \partial_x u)_\phi\|_{L^2}^2 \right). \end{aligned}$$

Integrating with respect to time yields

$$\begin{aligned} & \|e^{Rt} \Delta_q^h(\partial_t u)_\phi\|_{L_T^2(L^2)} + \alpha_1^2 \|e^{Rt} \Delta_q^h(\partial_t \partial_z u)_\phi\|_{L_T^2(L^2)} + \|e^{Rt} \Delta_q^h \partial_z u_\phi\|_{L_T^\infty(L^2)} + \|e^{Rt} \Delta_q^h \partial_z^2 u_\phi\|_{L_T^\infty(L^2)} \quad (2.28) \\ & \leq C \left(\|e^{a|D_x|} \Delta_q^h \partial_z u_0\|_{L^2} + \|e^{a|D_x|} \Delta_q^h \partial_z^2 u_0\|_{L^2} + \|e^{Rt} \Delta_q^h(u \partial_x u)_\phi\|_{L_T^2(L^2)} \right. \\ & \quad + \|e^{Rt} \Delta_q^h(u \partial_x \partial_z^2 u)_\phi\|_{L_T^2(L^2)} + \|e^{Rt} \Delta_q^h(v \partial_z u)_\phi\|_{L_T^2(L^2)} + \|e^{Rt} \Delta_q^h(v \partial_z^2 u)_\phi\|_{L_T^2(L^2)} \\ & \quad \left. + \|e^{Rt} \Delta_q^h(\partial_z u \partial_x \partial_z u)_\phi\|_{L_T^2(L^2)} + \|e^{Rt} \Delta_q^h(\partial_z^2 u \partial_x u)_\phi\|_{L_T^2(L^2)} \right). \end{aligned}$$

To be able to obtain Estimate (2.12), we need the following lemma, the proof of which will be given in the appendix 2.4.

Lemma 2.2.2. *We have the following estimates for the nonlinear terms :*

$$\|e^{Rt}(u \partial_x u)_\phi\|_{\tilde{L}_T^2(B^{\frac{3}{2}})} \lesssim \|\partial_z u_\phi\|_{\tilde{L}_T^\infty(B^{\frac{1}{2}})} \|e^{Rt} \partial_z u_\phi\|_{\tilde{L}_T^2(B^{\frac{5}{2}})}, \quad (2.29)$$

$$\|e^{Rt}(u \partial_x \partial_z^2 u)_\phi\|_{\tilde{L}_T^2(B^{\frac{3}{2}})} \lesssim \|\partial_z u_\phi\|_{\tilde{L}_T^\infty(B^{\frac{1}{2}})} \|e^{Rt} \partial_z^2 u_\phi\|_{\tilde{L}_T^2(B^{\frac{5}{2}})} + \|\partial_z u_\phi\|_{\tilde{L}_T^\infty(B^{\frac{5}{2}})} \|e^{Rt} \partial_z^2 u_\phi\|_{\tilde{L}_T^2(B^{\frac{1}{2}})}, \quad (2.30)$$

$$\|e^{Rt}(v \partial_z u)_\phi\|_{\tilde{L}_T^2(B^{\frac{3}{2}})} \lesssim \|\partial_z u_\phi\|_{\tilde{L}_T^\infty(B^{\frac{1}{2}})} \|e^{Rt} \partial_z u_\phi\|_{\tilde{L}_T^2(B^{\frac{5}{2}})}, \quad (2.31)$$

$$\|e^{Rt}(v \partial_z^2 u)_\phi\|_{\tilde{L}_T^2(B^{\frac{3}{2}})} \lesssim \|\partial_z u_\phi\|_{\tilde{L}_T^\infty(B^{\frac{1}{2}})} \|e^{Rt} \partial_z^2 u_\phi\|_{\tilde{L}_T^2(B^{\frac{5}{2}})} + \|\partial_z u_\phi\|_{\tilde{L}_T^\infty(B^{\frac{5}{2}})} \|e^{Rt} \partial_z^2 u_\phi\|_{\tilde{L}_T^2(B^{\frac{1}{2}})}, \quad (2.32)$$

$$\|e^{Rt}(\partial_z u \partial_x \partial_z u)_\phi\|_{\tilde{L}_T^2(B^{\frac{3}{2}})} \lesssim \|\partial_z u_\phi\|_{\tilde{L}_T^\infty(B^{\frac{1}{2}})} \|e^{Rt} \partial_z^2 u_\phi\|_{\tilde{L}_T^2(B^{\frac{5}{2}})} + \|\partial_z u_\phi\|_{\tilde{L}_T^\infty(B^{\frac{5}{2}})} \|e^{Rt} \partial_z^2 u_\phi\|_{\tilde{L}_T^2(B^{\frac{1}{2}})}, \quad (2.33)$$

$$\|e^{Rt} \Delta_q^h(\partial_x u \partial_z^2 u)_\phi\|_{\tilde{L}_T^2(B^{\frac{3}{2}})} \lesssim \|\partial_z u_\phi\|_{\tilde{L}_T^\infty(B^{\frac{1}{2}})} \|e^{Rt} \partial_z^2 u_\phi\|_{\tilde{L}_T^2(B^{\frac{5}{2}})} + \|\partial_z u_\phi\|_{\tilde{L}_T^\infty(B^{\frac{5}{2}})} \|e^{Rt} \partial_z^2 u_\phi\|_{\tilde{L}_T^2(B^{\frac{1}{2}})}. \quad (2.34)$$

Conclusion : Multiplying (2.28) by $2^{\frac{3q}{2}}$, summing up with respect to $q \in \mathbb{Z}$, and then using Lemma 2.2.2, assumption on initial data (2.10) and Estimate (2.23), we obtain Estimate (2.12) in Theorem 2.1.

2.2.3 Uniqueness of the solution

We suppose that u_1, u_2 are two solutions of (2.6) on $[0, T]$ with the same initial data. Let $U = u_1 - u_2$, $V = v_1 - v_2$, $P = p_1 - p_1$ then, (U, V, P) satisfies the following equations

$$\begin{cases} \partial_t(U - \alpha_1^2 \partial_z^2 U) + u_1 \partial_x(U - \alpha_1^2 \partial_z^2 U) + U \partial_x \omega_2 + v_1 \partial_z(U - \alpha_1^2 \partial_z^2 U) + V \partial_z \omega_2 + B \\ \quad = \partial_z^2(U - \alpha_1^2 \partial_z^2 U) - \partial_x P \\ \partial_z P = 0 \\ \partial_x U + \partial_z V = 0 \\ (U, V) = 0, \quad \partial_z U = 0 \quad \text{on } z = 0 \\ (U, V) = 0, \quad \partial_z U = 0 \quad \text{on } z = 1 \\ U|_{t=0} = 0, \end{cases} \quad (2.35)$$

where we set $\omega_2 = u_2 - \alpha_1^2 \partial_z^2 u_2$ and where

$$B = \alpha_1^2 \partial_z u_1 \partial_x \partial_z U + \alpha_1^2 \partial_z U \partial_x \partial_z u_2 - \alpha_1^2 \partial_z^2 u_1 \partial_x U - \alpha_1^2 \partial_z^2 U \partial_x u_2.$$

We now consider the following auxiliary functions

$$\Phi(t, \xi) = (a - \mu \Theta(t)) |\xi| \quad \text{with} \quad \Theta'(t) = \|\partial_z^2 u_{1\phi}(t)\|_{B^{\frac{1}{2}}} + \|\partial_z^2 u_{2\phi}(t)\|_{B^{\frac{1}{2}}}$$

and following (2.8) we define

$$f_{\Phi}(t, x, y) = e^{\Phi(t, D_x)} f(t, x, y) := \mathcal{F}_{\xi \rightarrow x}^{-1} (e^{\Phi(t, \xi)} \widehat{f}(t, \xi, y)),$$

where $\mu \geq \lambda$ will be determined later. From Theorem 2.1, we deduce that

$$\|u_{1\phi}\|_{L^\infty(B^{\frac{3}{2}})} + \|u_{2\phi}\|_{L^\infty(B^{\frac{3}{2}})} + \|\partial_z u_{1\phi}\|_{L^\infty(B^{\frac{3}{2}})} + \|\partial_z u_{2\phi}\|_{L^\infty(B^{\frac{3}{2}})} \leq M,$$

where $M \geq 1$ is a constant. Following the proof of (2.21), we get from (2.35) the following estimate

$$\begin{aligned} & \|e^{Rt} \Delta_q^h U_{\Phi}\|_{L^2}^2 + \alpha_1^2 \|e^{Rt} \Delta_q^h \partial_z U_{\Phi}\|_{L^2}^2 + 2\eta \alpha_1^2 \int_0^t \Theta'(t') \left\| e^{Rt'} |D_x|^{\frac{1}{2}} \Delta_q^h \partial_z U_{\Phi}(t') \right\|_{L^2}^2 dt' \\ & + 2\mu \int_0^t \Theta'(t') \left\| e^{Rt'} |D_x|^{\frac{1}{2}} \Delta_q^h U_{\Phi}(t') \right\|_{L^2}^2 dt' + 2 \int_0^t \left\| e^{Rt'} \Delta_q^h \partial_z U_{\Phi}(t') \right\|_{L^2}^2 dt' \\ & \quad + 2\alpha_1^2 \int_0^t \left\| e^{Rt'} \Delta_q^h \partial_z^2 U_{\Phi}(t') \right\|_{L^2}^2 dt' \\ & = 2 I_{1,q} + 2 I_{2,q} + 2 I_{3,q} + 2 I_{4,q} + 2 I_{5,q} + 2 I_{6,q} + 2 I_{7,q} + 2 I_{8,q}, \end{aligned} \quad (2.36)$$

where

$$\begin{aligned} I_{1,q} &= -\alpha_1^2 \int_0^t \left\langle e^{Rt'} \Delta_q^h (\partial_z u_1 \partial_x \partial_z U)_{\Phi}, e^{Rt'} \Delta_q^h U_{\Phi} \right\rangle dt', \\ I_{2,q} &= -\int_0^t \left\langle e^{Rt'} \Delta_q^h (u_1 \partial_x (U - \alpha_1^2 \partial_z^2 U))_{\Phi}, e^{Rt'} \Delta_q^h U_{\Phi} \right\rangle dt', \\ I_{3,q} &= \alpha_1^2 \int_0^t \left\langle e^{Rt'} \Delta_q^h (\partial_z^2 u_1 \partial_x U)_{\Phi}, e^{Rt'} \Delta_q^h U_{\Phi} \right\rangle dt', \\ I_{4,q} &= -\int_0^t \left\langle e^{Rt'} \Delta_q^h (V \partial_z \omega_2)_{\Phi}, e^{Rt'} \Delta_q^h U_{\Phi} \right\rangle dt', \\ I_{5,q} &= -\int_0^t \left\langle e^{Rt'} \Delta_q^h (v_1 \partial_z (U - \alpha_1^2 \partial_z^2 U))_{\Phi}, e^{Rt'} \Delta_q^h U_{\Phi} \right\rangle dt', \\ I_{6,q} &= \alpha_1^2 \int_0^t \left\langle e^{Rt'} \Delta_q^h (\partial_z^2 U \partial_x u_2)_{\Phi}, e^{Rt'} \Delta_q^h U_{\Phi} \right\rangle dt', \\ I_{7,q} &= -\alpha_1^2 \int_0^t \left\langle e^{Rt'} \Delta_q^h (\partial_z U \partial_x \partial_z u_2)_{\Phi}, e^{Rt'} \Delta_q^h U_{\Phi} \right\rangle dt', \\ I_{8,q} &= -\int_0^t \left\langle e^{Rt'} \Delta_q^h (U \partial_x \omega_2)_{\Phi}, e^{Rt'} \Delta_q^h U_{\Phi} \right\rangle dt'. \end{aligned}$$

The terms $I_{j,q}$, $j \in \{1, \dots, 8\}$ can be controlled as in the following lemma. The proof of this lemma is very close to the proof of Lemma 2.2.1. We will give a brief version of this proof in the appendix 2.5.

Lemma 2.2.3. *Let $s \in]0, 1[$, $T > 0$, $R > 0$. There exists a generic constant $C \geq 1$ such that, for any $0 < t < T$, we have*

$$|I_{1,q}| \leq C 2^{-2qs} d_q^2 \|e^{Rt'} \partial_z U_{\Phi}\|_{\tilde{L}_{T, \Theta'(t)}^2(B^{s+\frac{1}{2}})}^2, \quad (2.37)$$

$$|I_{2,q}| \leq C 2^{-2qs} d_q^2 \|e^{Rt'} \partial_z U_{\Phi}\|_{\tilde{L}_{T, \Theta'(t)}^2(B^{s+\frac{1}{2}})}^2, \quad (2.38)$$

$$|I_{3,q}| \leq C 2^{-2qs} d_q^2 \|e^{Rt'} \partial_z U_{\Phi}\|_{\tilde{L}_{T, \Theta'(t)}^2(B^{s+\frac{1}{2}})}^2, \quad (2.39)$$

$$|I_{4,q}| \leq C 2^{-2qs} d_q^2 \|e^{Rt'} \partial_z U_{\Phi}\|_{\tilde{L}_{T, \Theta'(t)}^2(B^{s+\frac{1}{2}})}^2, \quad (2.40)$$

$$|I_{5,q}| \leq C 2^{-2qs} d_q^2 \|u_{1\Phi}\|_{L_T^\infty(B^{\frac{3}{2}})}^{\frac{1}{2}} \|e^{Rt'} \partial_z^2 U_{\Phi}\|_{\tilde{L}_T^2(B^s)} \|e^{Rt'} \partial_z U_{\Phi}\|_{\tilde{L}_{T, \Theta'(t)}^2(B^{s+\frac{1}{2}})}^2, \quad (2.41)$$

$$|I_{6,q}| \leq C 2^{-2qs} d_q^2 \|u_{2\Phi}\|_{L_T^\infty(B^{\frac{3}{2}})}^{\frac{1}{2}} \|e^{Rt'} \partial_z^2 U_{\Phi}\|_{\tilde{L}_T^2(B^s)} \|e^{Rt'} \partial_z U_{\Phi}\|_{\tilde{L}_{T, \Theta'(t)}^2(B^{s+\frac{1}{2}})}^2, \quad (2.42)$$

$$\begin{aligned}
 |I_{7,q}| &\leq C2^{-2qs}d_q^2\|e^{Rt'}\partial_zU_\Phi\|_{\tilde{L}_{T,\Theta'(t)}^2(B^{s+\frac{1}{2}})}, \\
 &\quad \times \left(\|e^{Rt'}\partial_zU_\Phi\|_{\tilde{L}_{T,\Theta'(t)}^2(B^{s+\frac{1}{2}})} + \|\partial_zu_{2\Phi}\|_{L_T^\infty(B^{\frac{3}{2}})}^{\frac{1}{2}} \|e^{Rt'}\partial_zU_\Phi\|_{\tilde{L}_T^2(B^s)} \right),
 \end{aligned} \tag{2.43}$$

$$\begin{aligned}
 |I_{8,q}| &\leq C2^{-2qs}d_q^2\|e^{Rt'}\partial_zU_\Phi\|_{\tilde{L}_{T,\Theta'(t)}^2(B^{s+\frac{1}{2}})} \\
 &\quad \times \left(\|e^{Rt'}\partial_zU_\Phi\|_{\tilde{L}_{T,\Theta'(t)}^2(B^{s+\frac{1}{2}})} + \|\partial_zu_{2\Phi}\|_{L_T^\infty(B^{\frac{3}{2}})}^{\frac{1}{2}} \|e^{Rt'}\partial_zU_\Phi\|_{\tilde{L}_T^2(B^s)} \right).
 \end{aligned} \tag{2.44}$$

Now, using Lemma 2.2.3 and choosing $\mu = C^2 M^2$, we deduce from (2.36) that, for any $s \in]0, 1[$

$$\begin{aligned}
 \|e^{Rt}U_\Phi\|_{\tilde{L}_T^\infty(B^s)} + \|e^{Rt}\partial_zU_\Phi\|_{\tilde{L}_T^\infty(B^s)} + \|e^{Rt}\partial_zU_\Phi\|_{\tilde{L}_T^2(B^s)} + \|e^{Rt}\partial_z^2U_\Phi\|_{\tilde{L}_T^2(B^s)} \\
 \leq C(\|e^{a|D_x|}U_0\|_{B^s} + \|e^{a|D_x|}\partial_zU_0\|_{B^s}) = 0,
 \end{aligned}$$

which implies the uniqueness of the solution.

2.2.4 Construction of approximate solutions

Before introducing the construction methods, we remark that if (u, v, p) is a solution of (2.6) such that all the following expressions make sense, then we have :

Lemma 2.2.4.

$$\frac{1}{2} \frac{d}{dt} (\|u\|_{L^2}^2 + \alpha_1^2 \|\partial_z u\|_{L^2}^2) + \|\partial_z u\|_{L^2}^2 + \alpha_1^2 \|\partial_z^2 u\|_{L^2}^2 = 0. \tag{2.45}$$

We also remark that Estimate (2.45) is also true for the approximate solutions.

Proof. We set $w = u - \alpha_1^2 \partial_z^2 u$. Taking L^2 scalar product of (2.6) with u , we have

$$\langle \partial_t \omega, u \rangle + \langle u \partial_x \omega, u \rangle + \langle v \partial_z \omega, u \rangle + \alpha_1^2 \langle \partial_z u \partial_x \partial_z u, u \rangle - \alpha_1^2 \langle \partial_z^2 u \partial_x u, u \rangle = \langle \partial_z^2 \omega, u \rangle - \langle \partial_x p, u \rangle.$$

We first consider the linear terms. Performing integration by parts in z -variable and remarking that

$$\langle \partial_x p, u \rangle = -\langle p, \partial_x u \rangle = \langle p, \partial_z v \rangle = -\langle \partial_z p, v \rangle = 0.$$

We obtain

$$\frac{1}{2} \frac{d}{dt} (\|u\|_{L^2}^2 + \alpha_1^2 \|\partial_z u\|_{L^2}^2) + \|\partial_z u\|_{L^2}^2 + \alpha_1^2 \|\partial_z^2 u\|_{L^2}^2 + \text{NL} = 0,$$

where the non-linear term NL is

$$\text{NL} = \langle u \partial_x \omega, u \rangle + \langle v \partial_z \omega, u \rangle + \alpha_1^2 \langle \partial_z u \partial_x \partial_z u, u \rangle - \alpha_1^2 \langle \partial_z^2 u \partial_x u, u \rangle.$$

Performing integration by parts in x or z variable and using the incompressibility condition, we can write

$$\langle u \partial_x \omega, u \rangle = -\frac{1}{2} \int_{\mathbb{R} \times]0,1[} (\partial_x u) u^2 + 2\alpha_1^2 \int_{\mathbb{R} \times]0,1[} (\partial_z^2 u) (\partial_x u) u,$$

$$\begin{aligned}
 \langle v \partial_z \omega, u \rangle &= \frac{1}{2} \int_{\mathbb{R} \times]0,1[} v \partial_z (u^2) - \alpha_1^2 \int_{\mathbb{R} \times]0,1[} \partial_z (\partial_z^2 u) (uv) \\
 &= -\frac{1}{2} \int_{\mathbb{R} \times]0,1[} (\partial_z v) u^2 - \frac{1}{2} \alpha_1^2 \int_{\mathbb{R} \times]0,1[} (\partial_z u)^2 (\partial_z v) - \alpha_1^2 \int_{\mathbb{R} \times]0,1[} (\partial_z^2 u) (\partial_x u) u
 \end{aligned}$$

and

$$\alpha_1^2 \langle \partial_z u \partial_x \partial_z u, u \rangle = -\frac{1}{2} \alpha_1^2 \int_{\mathbb{R} \times]0,1[} (\partial_z u)^2 \partial_x u.$$

Putting these identities into NL, we get

$$\text{NL} = 0$$

and this concludes the proof of the lemma. \square

We now introduce the following Hilbert spaces

$$\begin{aligned} H_0^2 &= \{f \in H^2(]0, 1[) \mid f(0) = f(1) = f'(0) = f'(1) = 0\}, \\ H_0^1 &= \{f \in H^1(]0, 1[) \mid f(0) = f(1) = 0\} \end{aligned}$$

equipped with their respective norms

$$\begin{aligned} \|f\|_{H_0^2} &= \left(\alpha_1^2 \int_0^1 (f''(z))^2 dz + \int_0^1 (f'(z))^2 dz \right)^{\frac{1}{2}}, \\ \|f\|_{H_0^1} &= \left(\alpha_1^2 \int_0^1 (f'(z))^2 dz + \int_0^1 (f(z))^2 dz \right)^{\frac{1}{2}} \end{aligned}$$

and let be a common Hilbert basis $\{\tilde{e}_k\}_{k \geq 1}$ such that

$$\forall v \in H_0^2, \quad \langle \tilde{e}_k, v \rangle_{H_0^2} = \lambda_k \langle \tilde{e}_k, v \rangle_{H_0^1}.$$

For any (u, v, p) sufficiently smooth on $[0, T] \times \mathcal{S}$, $T > 0$, such that

$$\begin{aligned} \partial_x p &= \alpha_1^2 \partial_z^3 u|_{z=0} - \alpha_1^2 \partial_z^3 u|_{z=1} - \partial_x \int_0^1 (u)^2(t, x, y) dz - \alpha_1^2 \partial_x \int_0^1 (\partial_z u)^2(t, x, y) dz, \\ v(t, x, z) &= - \int_0^z \partial_x u(t, x, \tilde{z}) d\tilde{z}. \end{aligned}$$

We set

$$R(u) = -(u \partial_x u) + \alpha_1^2 (u \partial_x \partial_z^2 u) - (v \partial_z u) + \alpha_1^2 (v \partial_z \partial_z^2 u) - \alpha_1^2 (\partial_z u \partial_x \partial_z u) + \alpha_1^2 (\partial_z^2 u \partial_x u) - \partial_x p. \quad (2.46)$$

For any $n \in \mathbb{N}$, we will look for an approximate solution u^n of (2.6) of the form

$$u^n(t, x, z) = \sum_{i=1}^n \tilde{u}_{in}(t, x) \tilde{e}_i(z),$$

where $\{\tilde{u}_{in}\}$ is solution of the following system of n equations

$$\begin{cases} \langle \partial_t (u^n - \alpha_1^2 \partial_z^2 u^n), \tilde{e}_k \rangle_{L_z^2} = \langle \partial_z^2 (u^n - \alpha_1^2 \partial_z^2 u^n), \tilde{e}_k \rangle_{L_z^2} + \langle R(u^n), \tilde{e}_k \rangle_{L_z^2}, & k = 1, \dots, n \\ u_n|_{t=0} = \sum_{k=1}^n \langle u_0, \tilde{e}_k \rangle_{H_0^1} \tilde{e}_k, \end{cases}$$

which also means that, for $1 \leq k \leq n$,

$$\partial_t \tilde{u}_{kn}(t, x) = -\lambda_k \tilde{u}_{kn}(t, x) + \langle R(u^n)(t, x, \cdot), \tilde{e}_k \rangle_{L_z^2}.$$

Next, we define the following frequency cut-off operators J_n in x -variable. For $f \in L^2(\mathbb{R})$, we set

$$J_n f(x) = \mathcal{F}_h^{-1} (\mathbb{1}_{[-n, n]} \mathcal{F}_h f)(x).$$

The operator J_n is continuous from L^2 to L^2 , satisfies $J_n^2 f = J_n f$ and for any positive integers n and p , we have

$$\begin{aligned} |\partial_x^p J_n f|_{L^2(\mathbb{R})} &\leq n^p |f|_{L^2(\mathbb{R})}, \\ |\partial_x^p J_n f|_{L^\infty(\mathbb{R})} &\leq c_{np} |f|_{L^2(\mathbb{R})}. \end{aligned}$$

Now, we consider the following approximate system

$$\begin{cases} \partial_t \tilde{u}_{kn}(t, x) = -\lambda_k \tilde{u}_{kn}(t, x) + \langle J_n R(J_n u^n)(t, x, \cdot), \tilde{e}_k \rangle_{L_z^2}, & k \in \{1, \dots, n\} \\ \tilde{u}_{kn}|_{t=0} = \langle J_n u_0, \tilde{e}_k \rangle_{H_0^1}. \end{cases} \quad (2.47)$$

Since $\tilde{u}_{kn} \mapsto \langle J_n R(J_n u^n)(t, x, \cdot), \tilde{e}_k \rangle_{L^2}$ is locally Lipschitz in L^2 , (2.47) is a system of ordinary differential equations in L^2 . Then, Picard's theorem implies the existence of a unique maximal solution $\{\tilde{u}_{kn}(t, x)\}$ of (2.47) on $[0, T_n[$. Since $J_n^2 = J_n$, we deduce that $\{J_n \tilde{u}_{kn}\}$ is also a solution of (2.47) on $[0, T_n[$. Thus, the uniqueness of the solution implies that $\tilde{u}_{kn} = J_n \tilde{u}_{kn}$, for any $k \in \{1, \dots, n\}$. We remark that $\{\tilde{e}_k\}$ is not necessarily orthogonal in $L^2_z([0, 1])$ but using the Gram-Schmidt process, we can always construct a orthogonal family $\{e_k\}$ of $L^2_z([0, 1])$ such that

$$\text{Vect}\{e_1, \dots, e_n\} = \text{Vect}\{\tilde{e}_1, \dots, \tilde{e}_n\}.$$

We denote \mathbb{P}_n the orthogonal projection in $L^2_z([0, 1])$ onto $\text{Vect}\{e_1, \dots, e_n\}$

$$\mathbb{P}_n f = \sum_{k=1}^n \langle f, e_k \rangle_{L^2_z} e_k.$$

Then, we remark that u^n is also the solution of the system

$$\begin{cases} \partial_t(u^n - \alpha_1^2 \partial_z^2 u^n) = \partial_z^2(u^n - \alpha_1^2 \partial_z^2 u^n) + \mathbb{P}_n R(u^n) \\ u_n|_{t=0} = \mathbb{P}_n u_0. \end{cases} \quad (2.48)$$

Same calculations as in the proof of Lemma 2.2.4 and the fact that \mathbb{P}_n the orthogonal projection in $L^2_z([0, 1])$ onto $\text{Vect}\{e_1, \dots, e_n\}$ imply that

$$\frac{1}{2} \frac{d}{dt} (\|u^n\|_{L^2}^2 + \alpha_1^2 \|\partial_z u^n\|_{L^2}^2) + \|\partial_z u^n\|_{L^2}^2 + \alpha_1^2 \|\partial_z^2 u^n\|_{L^2}^2 = 0,$$

for any $t \in [0, T_n[$, which means $T_n = +\infty$.

Remark 5. Estimates (2.11) and (2.12) also apply to the approximate solutions u^n .

2.2.5 Passage to the limit

Now, we want to take the limit of the sequence of approximate solutions (2.48). We already see that $\{u^n\}$ is bounded in $L^\infty(\mathbb{R}_+, L^2_{loc})$ and due to (2.12), we remark that $\{\partial_t u^n\}$ is bounded in $L^2(\mathbb{R}_+, H^{-1}_{loc})$. Applying Aubin-Lions lemma, we deduce the existence of a subsequence, always noted by $\{u^n\}$ for the sake of simplicity, such that $u^n \rightharpoonup u$ in $L^\infty(\mathbb{R}_+, H^{-1}_{loc})$. By interpolation, we obtain

$$u^n \rightharpoonup u \quad \text{in} \quad L^\infty(\mathbb{R}_+, H^{-\delta}_{loc}), \quad \text{for all } 0 < \delta < 1. \quad (2.49)$$

We recall that u^n is solution of the following system

$$\begin{cases} \langle \partial_t(u^n - \alpha_1^2 \partial_z^2 u^n), e_k \rangle_{L^2_z} = \langle \partial_z^2(u^n - \alpha_1^2 \partial_z^2 u^n), e_k \rangle_{L^2_z} + \langle R(u^n), e_k \rangle_{L^2_z}, & k = 1, \dots, n \\ u_n|_{t=0} = \sum_{k=1}^n \langle u_0, e_k \rangle_{L^2_z} e_k, \end{cases}$$

where R is defined in (2.46). So the main point of this paragraph is to prove the following lemma on the convergence of the nonlinear term $R(u^n)$.

Lemma 2.2.5. *We have the following convergence as $n \rightarrow \infty$*

$$R(u^n) \rightharpoonup R(u) \quad \text{in} \quad L^2_{loc}(H^{-4}_{loc}).$$

Proof of Lemma 2.2.5. We will only prove the convergence of the term $u^n \partial_x u^n$. The other terms of $R(u^n)$ can be treated in a similar way. We first write

$$u^n \partial_x u^n - u \partial_x u = (u^n - u) \partial_x u^n + u \partial_x (u^n - u).$$

From Estimates (2.11) and (2.49), we have $\{\partial_x u^n\}$ is uniformly bounded in $L^2_{loc}(H^{\frac{1}{2}}_{loc})$. Choosing $\delta < \frac{1}{2}$ in (2.49) and using the product law in Sobolev spaces on \mathbb{R}^2 , we get

$$\|(u^n - u) \partial_x u^n\|_{L^2_{loc}(H^{-\frac{1}{2}-\delta}_{loc})} \leq \|u^n - u\|_{L^\infty(H^{-\delta}_{loc})} \|\partial_x u^n\|_{L^2_{loc}(H^{\frac{1}{2}}_{loc})} \longrightarrow 0.$$

Now, using again Estimate (2.11), we have that u is bounded in $L^2_{loc}(H^{\frac{1}{2}}_{loc})$. Then, (2.49) and the product law in Sobolev spaces on \mathbb{R}^2 yield

$$\|u\partial_x(u^n - u)\|_{L^2_{loc}(H^{-\frac{1}{2}-\delta}_{loc})} \leq \|u\|_{L^2_{loc}(H^{\frac{1}{2}}_{loc})} \|\partial_x(u^n - u)\|_{L^\infty_{loc}(H^{-\delta}_{loc})} \longrightarrow 0.$$

□

Now, coming back to (2.48), Lemma 2.2.5 proves that the limit u is solution of (2.6) in the sense of distributions. This concludes the proof of Theorem 2.1.

2.3 Appendix - proof of the Lemma 2.2.1

We first introduce some notations and classical mathematical tools. Then we will prove the estimations of Lemma 2.2.1. We will only detail the proof of estimates (2.13) and (2.17). The same procedure can be followed to prove the other estimates.

We recall the following Bernstein-type lemma, which states that derivatives act almost as multipliers on distributions whose Fourier transforms are supported in a ball or an annulus. We refer the reader to [3] for a proof of this lemma.

Lemma 2.3.1. *Let $k \in \mathbb{N}$, $d \in \mathbb{N}^*$ and $r_1, r_2 \in \mathbb{R}$ satisfy $0 < r_1 < r_2$. There exists a constant $C > 0$ such that, for any $a, b \in \mathbb{R}$, $1 \leq a \leq b \leq +\infty$, for any $\lambda > 0$ and for any $u \in L^a(\mathbb{R}^d)$, we have*

$$\text{supp}(\widehat{u}) \subset \{\xi \in \mathbb{R}^d \mid |\xi| \leq r_1\lambda\} \implies \sup_{|\alpha|=k} \|\partial^\alpha u\|_{L^b} \leq C^k \lambda^{k+d(\frac{1}{a}-\frac{1}{b})} \|u\|_{L^a}$$

and

$$\text{supp}(\widehat{u}) \subset \{\xi \in \mathbb{R}^d \mid r_1\lambda \leq |\xi| \leq r_2\lambda\} \implies C^{-k} \lambda^k \|u\|_{L^a} \leq \sup_{|\alpha|=k} \|\partial^\alpha u\|_{L^a} \leq C^k \lambda^k \|u\|_{L^a}.$$

In order to prove the estimates of Lemma 2.2.1, we need the Bony decomposition of a product of two functions a and b in the horizontal direction (see [3])

$$ab = T_a^h b + T_b^h a + R^h(a, b), \quad (2.50)$$

where

$$T_a^h b = \sum_{q \in \mathbb{Z}} S_{q-1}^h a \Delta_q^h b \quad \text{and} \quad R^h(a, b) = \sum_{|q-q'| \leq 1} \Delta_q^h a \Delta_{q'}^h b = \sum_{q \in \mathbb{Z}} \widetilde{\Delta}_q^h a \Delta_q^h b$$

and

$$\widetilde{\Delta}_q^h f = \sum_{|q-q'| \leq 1} \Delta_{q'}^h f.$$

From the support properties to the Fourier transform of the terms $\Delta_q^h f$, we can verify

$$\Delta_q^h(S_{q'-1}^h a \Delta_{q'}^h b) = 0 \quad \text{if} \quad |q' - q| \geq 5 \quad \text{and} \quad \Delta_q^h(S_{q'+2}^h a \Delta_{q'}^h b) = 0 \quad \text{if} \quad q' \leq q - 4. \quad (2.51)$$

Then, for a function f (we suppose that all the expressions below make sense), we write

$$\int_0^T |\langle e^{Rt'} \Delta_q^h(a b)_\phi, e^{Rt'} \Delta_q^h f \rangle| dt' \leq \mathcal{A}_q + \mathcal{B}_q + \mathcal{R}_q, \quad (2.52)$$

where

$$\begin{aligned} \mathcal{A}_q &= \int_0^T |\langle e^{Rt'} \Delta_q^h(T_a^h b)_\phi, e^{Rt'} \Delta_q^h f \rangle| dt', \\ \mathcal{B}_q &= \int_0^T |\langle e^{Rt'} \Delta_q^h(T_b^h a)_\phi, e^{Rt'} \Delta_q^h f \rangle| dt', \\ \mathcal{R}_q &= \int_0^T |\langle e^{Rt'} \Delta_q^h(R^h(a, b))_\phi, e^{Rt'} \Delta_q^h f \rangle| dt'. \end{aligned}$$

Then we have from the definition of the operator T_a^h and (2.51)

$$\begin{aligned}\mathcal{A}_q &\leq \sum_{|q'-q|\leq 4} \int_0^T e^{2Rt'} \left| \langle \Delta_q^h (S_{q'-1}^h a \Delta_{q'}^h b)_\phi, \Delta_q^h f_\phi \rangle \right| dt', \\ \mathcal{B}_q &\leq \sum_{|q'-q|\leq 4} \int_0^T e^{2Rt'} \left| \langle \Delta_q^h (S_{q'-1}^h b \Delta_{q'}^h a)_\phi, \Delta_q^h f_\phi \rangle \right| dt', \\ \mathcal{R}_q &\leq \sum_{q \geq q-3} \int_0^T e^{2Rt'} \left| \langle \Delta_q^h (\Delta_{q'}^h a \Delta_{q'}^h b)_\phi, \Delta_q^h f_\phi \rangle \right| dt' .\end{aligned}$$

Throughout this section, we will keep these notations (which can however be modified when needed). Following the remark 2, we define For any $f \in L^2(R)$, we define

$$f^+ = \mathcal{F}_h^{-1}(|\widehat{f}|).$$

We remark that, for any $q \in \mathbb{Z}$, we have

$$\Delta_q^h f^+ = (\Delta_q^h f)^+, \quad S_q^h f^+ = (S_q^h f)^+ \quad \text{and} \quad \|f^+\|_{L^2} = \|f\|_{L^2}. \quad (2.53)$$

We will use the following lemma (see [21]):

Lemma 2.3.2. *For smooth functions we have*

$$\begin{aligned}|\mathcal{F}_h((S_{q'-1}^h a \Delta_{q'}^h b)_\phi)(\xi, y)| &\leq |\mathcal{F}_h(S_{q'-1}^h a_\phi^+ \Delta_{q'}^h b_\phi^+)(\xi, y)|, \\ |\mathcal{F}_h((\Delta_{q'}^h a \Delta_{q'}^h b)_\phi)(\xi, y)| &\leq |\mathcal{F}_h(\Delta_{q'}^h a_\phi^+ \Delta_{q'}^h b_\phi^+)(\xi, y)|.\end{aligned}$$

Proof. Setting $\tilde{a} = S_{q'-1}^h a$ and $\tilde{b} = \Delta_{q'}^h b$, we have from the inequality $\phi(t, \xi) \leq \phi(t, \xi - \eta) + \phi(t, \eta)$

$$\begin{aligned}|\mathcal{F}_h((\tilde{a} \tilde{b})_\phi)(\xi, y)| &= \left| e^{\phi(t, \xi)} \int_{\mathbb{R}} \mathcal{F}_h(\tilde{a})(\xi - \eta, y) \mathcal{F}_h(\tilde{b})(\eta, y) d\eta \right| \\ &\leq \int_{\mathbb{R}} \left| e^{\phi(t, \xi - \eta)} \mathcal{F}_h(\tilde{a})(\xi - \eta, y) \right| \left| e^{\phi(t, \eta)} \mathcal{F}_h(\tilde{b})(\eta, y) \right| d\eta \\ &\leq \int_{\mathbb{R}} \mathcal{F}_h(\tilde{a}_\phi^+)(\xi - \eta, y) \mathcal{F}_h(\tilde{b}_\phi^+)(\eta, y) d\eta,\end{aligned}$$

and this concludes the proof of the first inequality, and the same procedure can be followed to deduce the second inequality. \square

Therefore from this lemma, we derive the following corollary from Plancherel formula, Fubini's theorem and the Hölder's inequality.

Corollary 1. *For smooth functions we have*

$$\|\Delta_q^h f_\phi\|_{L^2} \lesssim \|f_\phi\|_{L^2},$$

$$\begin{aligned}|\langle \Delta_q^h (S_{q'-1}^h a \Delta_{q'}^h b)_\phi, \Delta_q^h f_\phi \rangle| &\lesssim \|S_{q'-1}^h a_\phi^+\|_{L^\infty} \|\Delta_{q'}^h b_\phi\|_{L^2} \|\Delta_q^h f_\phi\|_{L^2}, \\ &\lesssim \|S_{q'-1}^h a_\phi^+\|_{L_x^2(L_x^\infty)} \|\Delta_{q'}^h b_\phi\|_{L_x^\infty(L_x^2)} \|\Delta_q^h f_\phi\|_{L^2}, \\ &\lesssim \|S_{q'-1}^h a_\phi^+\|_{L_x^2(L_x^\infty)} \|\Delta_{q'}^h b_\phi\|_{L^2} \|\Delta_q^h f_\phi\|_{L_x^\infty(L_x^2)}\end{aligned}$$

and

$$\begin{aligned}|\langle \Delta_q^h (\Delta_{q'}^h a \Delta_{q'}^h b)_\phi, \Delta_q^h f_\phi \rangle| &\lesssim \|\Delta_{q'}^h a_\phi^+\|_{L^\infty} \|\Delta_{q'}^h b_\phi\|_{L^2} \|\Delta_q^h f_\phi\|_{L^2}, \\ &\lesssim \|\Delta_{q'}^h a_\phi^+\|_{L_x^2(L_x^\infty)} \|\Delta_{q'}^h b_\phi\|_{L_x^\infty(L_x^2)} \|\Delta_q^h f_\phi\|_{L^2}, \\ &\lesssim \|\Delta_{q'}^h a_\phi^+\|_{L_x^\infty(L_x^2)} \|\Delta_{q'}^h b_\phi\|_{L^2} \|\Delta_q^h f_\phi\|_{L_x^2(L_x^\infty)}.\end{aligned}$$

We will also use the following Poincaré inequality : For a function $u \in H^2$ such that u and u' vanish for $z = 0, 1$ we have

$$\|u\|_{L^\infty} \lesssim \|\partial_z u\|_{L^2} \quad \text{and} \quad \|u\|_{L^2} \lesssim \|\partial_z u\|_{L^2} \lesssim \|\partial_z^2 u\|_{L^2}. \quad (2.54)$$

Proof of Estimate (2.13). We apply the Bony's decomposition with $a = u$, $b = \partial_x u$ and $f = u_\phi$. Then following the notations of (2.52) we can write

$$\int_0^T | \langle e^{Rt'} \Delta_q^h (u \partial_x u)_\phi, e^{Rt'} \Delta_q^h u_\phi \rangle | dt' \leq \mathcal{A}_q + \mathcal{B}_q + \mathcal{R}_q.$$

• Estimate of $\mathcal{A}_q = \int_0^T | \langle e^{Rt'} \Delta_q^h (T_u^h \partial_x u)_\phi, e^{Rt'} \Delta_q^h u_\phi \rangle | dt'$. We have from (2.51)

$$\begin{aligned} \mathcal{A}_q &= \int_0^T \left| \left\langle e^{Rt'} \Delta_q^h \left(\sum_{q' \in \mathbb{Z}} S_{q'-1}^h u \Delta_{q'}^h \partial_x u \right)_\phi, e^{Rt'} \Delta_q^h u_\phi \right\rangle \right| dt' \\ &\leq \sum_{|q'-q| \leq 4} \int_0^T e^{2Rt'} \left| \left\langle \Delta_q^h (S_{q'-1}^h u \Delta_{q'}^h \partial_x u)_\phi, \Delta_q^h u_\phi \right\rangle \right| dt'. \end{aligned}$$

From the previous inequality and corollary 1 we get

$$\mathcal{A}_q \lesssim \sum_{|q'-q| \leq 4} \int_0^T e^{2Rt'} \|S_{q'-1}^h u_\phi^+\|_{L^\infty} \|\Delta_{q'}^h \partial_x u_\phi\|_{L^2} \|\Delta_q^h u_\phi\|_{L^2} dt'.$$

Since the support of $\Delta_q^h u_\phi$ is in a ring and using the first inequality of Lemma 2.3.1 with $\lambda = 2^q$, we get

$$\|\Delta_q^h u_\phi^+\|_{L^\infty} \lesssim 2^{\frac{q}{2}} \|\Delta_q^h u_\phi^+\|_{L_x^\infty(L_z^2)} = 2^{\frac{q}{2}} \|\Delta_q^h u_\phi\|_{L_x^\infty(L_z^2)}.$$

Therefore from Poincaré inequality (2.54) and remark 2 we get

$$\begin{aligned} \|\Delta_q^h u_\phi^+\|_{L^\infty} &\lesssim 2^{\frac{q}{2}} \|\Delta_q^h u_\phi\|_{L_x^2(L_z^\infty)} \lesssim 2^{\frac{q}{2}} \|\Delta_q^h \partial_z^2 u_\phi\|_{L^2} \\ &\lesssim d_q(\partial_z^2 u_\phi) \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}}. \end{aligned}$$

Then from the definition 2.1.1 of $S_{q'-1}$ and using $\sum_q d_q(\partial_z^2 u_\phi) = 1$,

$$\|S_{q'-1}^h u_\phi^+\|_{L^\infty} \lesssim \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}}. \quad (2.55)$$

We deduce from the previous inequalities, lemma 2.3.1 and the cauchy-schwarz inequality that

$$\begin{aligned} \mathcal{A}_q &\lesssim \sum_{|q'-q| \leq 4} \int_0^T e^{2Rt'} \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}} \left(2^{q'} \|\Delta_{q'}^h u_\phi\|_{L^2} \right) \|\Delta_q^h u_\phi\|_{L^2} dt' \\ &\lesssim \sum_{|q'-q| \leq 4} 2^{q'} \left(\int_0^T e^{2Rt'} \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}} \|\Delta_{q'}^h u_\phi\|_{L^2}^2 dt' \right)^{\frac{1}{2}} \times \left(\int_0^T e^{2Rt'} \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}} \|\Delta_q^h u_\phi\|_{L^2}^2 dt' \right)^{\frac{1}{2}}. \end{aligned}$$

Since $\theta'(t) = \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}}$, we have from remark 3

$$\left(\int_0^T e^{2Rt'} \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}} \|\Delta_{q'}^h u_\phi\|_{L^2}^2 dt' \right)^{\frac{1}{2}} \lesssim 2^{-q'(s+1/2)} d_{q'}(u_\phi, \theta') \|e^{Rt'} u_\phi\|_{\tilde{L}_{T, \theta'(t)}^2(B^{s+\frac{1}{2}})}$$

and we finally obtain

$$\mathcal{A}_q \lesssim 2^{-2qs} \tilde{d}_q \|e^{Rt'} u_\phi\|_{\tilde{L}_{T, \theta'(t)}^2(B^{s+\frac{1}{2}})}^2, \quad (2.56)$$

where \tilde{d}_q is the square root summable sequence (since $\sum_q d_q(u_\phi) = \sum_{q'} d_{q'}(u_\phi, \theta') = 1$) defined by

$$\tilde{d}_q = d_q(u_\phi) \left(\sum_{|q'-q| \leq 4} d_{q'}(u_\phi, \theta') 2^{(q-q')(s-1/2)} \right). \quad (2.57)$$

• Estimate of $\mathcal{B}_q = \int_0^T | \langle e^{Rt'} \Delta_q^h (T_{\partial_x}^h u)_\phi, e^{Rt'} \Delta_q^h u_\phi \rangle | dt'$. From the notations (2.52),

$$\mathcal{B}_q \lesssim \sum_{|q'-q| \leq 4} \int_0^T e^{2Rt'} \|S_{q'-1}^h \partial_x u_\phi^+\|_{L^\infty} \|\Delta_{q'}^h u_\phi\|_{L^2} \|\Delta_q^h u_\phi\|_{L^2} dt'.$$

Then, following the proof of Estimate (2.55) and the definition 2.1.1 of S_q^h , we have

$$\|S_{q'-1}^h \partial_x u_\phi^+\|_{L^\infty} \lesssim 2^{q'} \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}}.$$

Therefore using the Cauchy-Schwarz inequality and remark 2 we derive

$$\begin{aligned} \mathcal{B}_q &\lesssim \sum_{|q'-q| \leq 4} \int_0^T e^{2Rt'} \left(2^{q'} \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}} \right) \|\Delta_{q'}^h u_\phi\|_{L^2} \|\Delta_q^h u_\phi\|_{L^2} dt' \\ &\lesssim \sum_{|q'-q| \leq 4} 2^{q'} \left(\int_0^T e^{2Rt'} \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}} \|\Delta_{q'}^h u_\phi\|_{L^2}^2 \right)^{\frac{1}{2}} \times \left(\int_0^T e^{2Rt'} \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}} \|\Delta_q^h u_\phi\|_{L^2}^2 \right)^{\frac{1}{2}} \\ &\lesssim 2^{-2qs} d_q(u_\phi) \left(\sum_{|q'-q| \leq 4} d_{q'}(u_\phi) 2^{(q-q')(s-1/2)} \right) \|e^{Rt'} u_\phi\|_{\tilde{L}_{T, \theta'(t)}^2(B^{s+\frac{1}{2}})}. \end{aligned}$$

Then from the definition (2.57) of \tilde{d}_q

$$\mathcal{B}_q \lesssim 2^{-2qs} \tilde{d}_q \|e^{Rt'} u_\phi\|_{\tilde{L}_{T, \theta'(t)}^2(B^{s+\frac{1}{2}})}. \quad (2.58)$$

• Estimate of \mathcal{R}_q . We have from the definition (2.52)

$$\mathcal{R}_q \leq \sum_{q' \geq q-3} \int_0^T e^{2Rt'} \left| \langle \Delta_q^h (\Delta_{q'}^h u \Delta_{q'}^h \partial_x u)_\phi, \Delta_q^h u_\phi \rangle \right| dt'.$$

Following the proof of (2.55) and using corollary 1 and lemma 2.3.1 we get

$$\begin{aligned} \mathcal{R}_q &\lesssim \sum_{q' \geq q-3} \int_0^T e^{2Rt'} \|\tilde{\Delta}_{q'}^h u_\phi^+\|_{L_x^\infty(L_x^2)} \|\Delta_{q'}^h \partial_x u_\phi\|_{L^2} \|\Delta_q^h u_\phi\|_{L_x^2(L_x^\infty)} dt' \\ &\lesssim \sum_{q' \geq q-3} \int_0^T e^{2Rt'} \left(2^{-\frac{q'}{2}} d_{q'}(\partial_z^2 u_\phi) \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}} \right) \left(2^{q'} \|\Delta_{q'}^h u_\phi\|_{L^2} \right) \left(2^{\frac{q}{2}} \|\Delta_q^h u_\phi\|_{L^2} \right) dt'. \end{aligned}$$

Them using $d_{q'}(\partial_z^2 u_\phi) \leq 1$, the Cauchy-Schwarz inequality and remark 2 we get

$$\begin{aligned} \mathcal{R}_q &\lesssim 2^{\frac{q}{2}} \sum_{q' \geq q-3} 2^{\frac{q'}{2}} \left(\int_0^T e^{2Rt'} \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}} \|\Delta_{q'}^h u_\phi\|_{L^2}^2 \right)^{\frac{1}{2}} \times \left(\int_0^T e^{2Rt'} \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}} \|\Delta_q^h u_\phi\|_{L^2}^2 \right)^{\frac{1}{2}} \\ &\lesssim 2^{\frac{q}{2}} \sum_{q' \geq q-3} 2^{\frac{q'}{2}} \left(2^{-q'(s+1/2)} d_{q'}(u_\phi, \theta') \|e^{Rt'} u_\phi\|_{\tilde{L}_{T, \theta'(t)}^2(B^{s+\frac{1}{2}})} \right) \left(2^{-q(s+1/2)} d_q(u_\phi, \theta') \|e^{Rt'} u_\phi\|_{\tilde{L}_{T, \theta'(t)}^2(B^{s+\frac{1}{2}})} \right). \end{aligned}$$

That is

$$\mathcal{R}_q \lesssim 2^{-2qs} \tilde{d}_q \|e^{Rt'} u_\phi\|_{\tilde{L}_{T, \theta'(t)}^2(B^{s+\frac{1}{2}})}, \quad (2.59)$$

where \check{d}_q is the square root summable sequence defined by

$$\check{d}_q = d_q(u_\phi, \theta') \sum_{q' \geq q-3} 2^{(q-q')s} d_{q'}(u_\phi, \theta').$$

- By Summing the estimates (2.56), (2.58) and (2.59) we deduce (2.13). \square

Proof of Estimate (2.14). We apply the Bony's decomposition with $a = u$, $b = \partial_x \partial_z u$ and $f = \partial_z u_\phi$. Then following the notations of (2.52) we can write

$$\int_0^T | \langle e^{Rt'} \Delta_q^h (u \partial_x \partial_z u)_\phi, e^{Rt'} \Delta_q^h \partial_z u_\phi \rangle | dt' \leq \mathcal{A}_q + \mathcal{B}_q + \mathcal{R}_q.$$

- Estimate of $\mathcal{A}_q = \int_0^T | \langle e^{Rt'} \Delta_q^h (T_u^h \partial_x \partial_z u)_\phi, e^{Rt'} \Delta_q^h \partial_z u_\phi \rangle | dt'$.

$$\begin{aligned} \mathcal{A}_q &\lesssim \sum_{|q'-q| \leq 4} \int_0^T e^{2Rt'} \|S_{q'-1}^h u_\phi^+\|_{L^\infty} \|\Delta_{q'}^h \partial_x \partial_z u_\phi\|_{L^2} \|\Delta_q^h \partial_z u_\phi\|_{L^2} \\ &\lesssim \sum_{|q'-q| \leq 4} \int_0^T e^{2Rt'} \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}} 2^{q'} \|\Delta_{q'}^h \partial_x \partial_z u_\phi\|_{L^2} \|\Delta_q^h \partial_z u_\phi\|_{L^2} \\ &\lesssim \sum_{|q'-q| \leq 4} 2^{q'} \left(\int_0^T e^{2Rt'} \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}} \|\Delta_{q'}^h \partial_x \partial_z u_\phi\|_{L^2}^2 \right)^{\frac{1}{2}} \times \left(\int_0^T e^{2Rt'} \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}} \|\Delta_q^h \partial_z u_\phi\|_{L^2}^2 \right)^{\frac{1}{2}} \\ &\lesssim 2^{-2qs} \check{d}_q \|e^{Rt'} \partial_z u_\phi\|_{\tilde{L}_{T, \theta'(t)}^2(B^{s+\frac{1}{2}})}, \end{aligned}$$

where

$$\check{d}_q = d_q(\partial_z u_\phi, \theta') \left(\sum_{|q'-q| \leq 4} d_{q'}(\partial_z u_\phi, \theta') 2^{(q-q')(s-\frac{1}{2})} \right). \quad (2.60)$$

- Estimate of $\mathcal{B}_q = \int_0^T | \langle e^{Rt'} \Delta_q^h (T_{\partial_x \partial_z u}^h u)_\phi, e^{Rt'} \Delta_q^h \partial_z u_\phi \rangle | dt'$.

$$\begin{aligned} \mathcal{B}_q &\lesssim \sum_{|q'-q| \leq 4} \int_0^T e^{2Rt'} \|S_{q'-1}^h \partial_x \partial_z u_\phi^+\|_{L^\infty} \|\Delta_{q'}^h u_\phi\|_{L^2} \|\Delta_q^h \partial_z u_\phi\|_{L^2} dt \\ &\lesssim \sum_{|q'-q| \leq 4} \int_0^T e^{2Rt'} 2^{q'} \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}} \|\Delta_{q'}^h \partial_x \partial_z u_\phi\|_{L^2} \|\Delta_q^h \partial_z u_\phi\|_{L^2} \\ &\lesssim \sum_{|q'-q| \leq 4} 2^{q'} \left(\int_0^T e^{2Rt'} \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}} \|\Delta_{q'}^h \partial_x \partial_z u_\phi\|_{L^2}^2 \right)^{\frac{1}{2}} \times \left(\int_0^T e^{2Rt'} \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}} \|\Delta_q^h \partial_z u_\phi\|_{L^2}^2 \right)^{\frac{1}{2}} \\ &\lesssim 2^{-2qs} \check{d}_q \|e^{Rt'} \partial_z u_\phi\|_{\tilde{L}_{T, \theta'(t)}^2(B^{s+\frac{1}{2}})}, \end{aligned}$$

with the same previous definition (2.60) of \check{d}_q .

- Estimate of $\mathcal{R}_q = \int_0^T | \langle e^{Rt'} \Delta_q^h (R^h(u, \partial_x \partial_z u))_\phi, e^{Rt'} \Delta_q^h \partial_z u_\phi \rangle | dt'$.

$$\begin{aligned} \mathcal{R}_q &\lesssim \sum_{q' \geq q-3} \int_0^T e^{2Rt'} \|\tilde{\Delta}_{q'}^h u_\phi^+\|_{L_x^\infty(L_x^2)} \|\Delta_{q'}^h \partial_x \partial_z u_\phi\|_{L^2} \|\Delta_q^h \partial_z u_\phi\|_{L_x^2(L_x^\infty)} dt' \\ &\lesssim \sum_{q' \geq q-3} \int_0^T e^{2Rt'} 2^{-\frac{q'}{2}} \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}} 2^{q'} \|\Delta_{q'}^h \partial_x \partial_z u_\phi\|_{L^2} 2^{\frac{q'}{2}} \|\Delta_q^h \partial_z u_\phi\|_{L^2} dt' \\ &\lesssim 2^{\frac{q}{2}} \sum_{q' \geq q-3} 2^{\frac{q'}{2}} \left(\int_0^T e^{2Rt'} \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}} \|\Delta_{q'}^h \partial_x \partial_z u_\phi\|_{L^2}^2 \right)^{\frac{1}{2}} \times \left(\int_0^T e^{2Rt'} \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}} \|\Delta_q^h \partial_z u_\phi\|_{L^2}^2 \right)^{\frac{1}{2}} \\ &\lesssim 2^{-2qs} \check{d}_q \|e^{Rt'} \partial_z u_\phi\|_{\tilde{L}_{T, \theta'(t)}^2(B^{s+\frac{1}{2}})}, \end{aligned}$$

where

$$\check{d}_q = d_q(\partial_z u_\phi, \theta') \sum_{q' \geq q-3} 2^{(q-q')s} d_{q'}(\partial_z u_\phi, \theta'). \quad (2.61)$$

• By Summing the above estimates for \mathcal{A}_q , \mathcal{B}_q and \mathcal{R}_q , we deduce (2.14). \square

Proof of Estimate (2.15). We apply the Bony's decomposition with $a = \partial_z u$, $b = \partial_x u$ and $f = \partial_z u_\phi$. Then following the notations of (2.52) we can write

$$\int_0^T | \langle e^{Rt'} \Delta_q^h (\partial_z u \partial_x u)_\phi, e^{Rt'} \Delta_q^h \partial_z u_\phi \rangle | dt' \leq \mathcal{A}_q + \mathcal{B}_q + \mathcal{R}_q.$$

• Estimate of $\mathcal{A}_q = \int_0^T | \langle e^{Rt'} \Delta_q^h (T_{\partial_z u}^h \partial_x u)_\phi, e^{Rt'} \Delta_q^h \partial_z u_\phi \rangle | dt'$.

$$\mathcal{A}_q \lesssim \sum_{|q'-q| \leq 4} \int_0^T e^{2Rt'} \|S_{q'-1}^h \partial_z u_\phi^+\|_{L_z^2(L_x^\infty)} \|\Delta_{q'}^h \partial_x u_\phi\|_{L_z^\infty(L_x^2)} \|\Delta_q^h \partial_z u_\phi\|_{L^2}.$$

Using Lemma 2.3.1, Poincaré inequality we have

$$\begin{aligned} \|\Delta_q^h \partial_z u_\phi^+\|_{L_z^2(L_x^\infty)} &\lesssim 2^{\frac{q}{2}} \|\Delta_q^h \partial_z u_\phi\|_{L^2} \\ &\lesssim d_q \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}}. \end{aligned}$$

Then

$$\|S_{q'-1}^h \partial_z u_\phi^+\|_{L_z^2(L_x^\infty)} \lesssim \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}}.$$

Using Poincaré inequality and Lemma 2.3.1, we have

$$\begin{aligned} \|\Delta_{q'}^h \partial_x u_\phi\|_{L_z^\infty(L_x^2)} &\lesssim \|\Delta_{q'}^h \partial_x \partial_z u_\phi\|_{L^2} \\ &\lesssim 2^{q'} \|\Delta_{q'}^h \partial_z u_\phi\|_{L^2}. \end{aligned}$$

Then

$$\begin{aligned} \mathcal{A}_q &\lesssim \sum_{|q'-q| \leq 4} 2^{q'} \int_0^T e^{2Rt'} \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}} \|\Delta_{q'}^h \partial_z u_\phi\|_{L^2} \|\Delta_q^h \partial_z u_\phi\|_{L^2} dt' \\ &\lesssim \sum_{|q'-q| \leq 4} 2^{q'} \left(\int_0^T e^{2Rt'} \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}} \|\Delta_{q'}^h \partial_z u_\phi\|_{L^2}^2 \right)^{\frac{1}{2}} \times \left(\int_0^T e^{2Rt'} \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}} \|\Delta_q^h \partial_z u_\phi\|_{L^2}^2 \right)^{\frac{1}{2}} \\ &\lesssim 2^{-2qs} \check{d}_q \|e^{Rt'} \partial_z u_\phi\|_{\tilde{L}_{T, \theta'(t)}^2(B^{s+\frac{1}{2}})}^2, \end{aligned}$$

where \check{d}_q is given by (2.60). • Estimate of $\mathcal{B}_q = \int_0^T | \langle e^{Rt'} \Delta_q^h (T_{\partial_x u}^h \partial_z u)_\phi, e^{Rt'} \Delta_q^h \partial_z u_\phi \rangle | dt'$.

$$\begin{aligned} \mathcal{B}_q &\lesssim \sum_{|q'-q| \leq 4} \int_0^T e^{2Rt'} \|S_{q'-1}^h \partial_x u_\phi^+\|_{L^\infty} \|\Delta_{q'}^h \partial_z u_\phi\|_{L^2} \|\Delta_q^h \partial_z u_\phi\|_{L^2} dt \\ &\lesssim \sum_{|q'-q| \leq 4} \int_0^T e^{2Rt'} 2^{q'} \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}} \|\Delta_{q'}^h \partial_z u_\phi\|_{L^2} \|\Delta_q^h \partial_z u_\phi\|_{L^2} dt \\ &\lesssim \sum_{|q'-q| \leq 4} 2^{q'} \left(\int_0^T e^{2Rt'} \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}} \|\Delta_{q'}^h \partial_z u_\phi\|_{L^2}^2 \right)^{\frac{1}{2}} \times \left(\int_0^T e^{2Rt'} \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}} \|\Delta_q^h \partial_z u_\phi\|_{L^2}^2 \right)^{\frac{1}{2}} \\ &\lesssim 2^{-2qs} \check{d}_q \|e^{Rt'} \partial_z u_\phi\|_{\tilde{L}_{T, \theta'(t)}^2(B^{s+\frac{1}{2}})}^2. \end{aligned}$$

- Estimate of $\mathcal{R}_q = \int_0^T | \langle e^{Rt'} \Delta_q^h (R^h(\partial_z u, \partial_x u))_\phi, e^{Rt'} \Delta_q^h \partial_z u_\phi \rangle | dt'$.

$$\begin{aligned}
\mathcal{R}_q &\lesssim \sum_{q' \geq q-3} \int_0^T e^{2Rt'} \|\tilde{\Delta}_{q'}^h \partial_z u_\phi^+\|_{L^2} \|\Delta_{q'}^h \partial_x u_\phi\|_{L^\infty(L_x^2)} \|\Delta_{q'}^h \partial_z u_\phi\|_{L^2(L_x^\infty)} dt' \\
&\lesssim \sum_{q' \geq q-3} \int_0^T 2^{-\frac{q'}{2}} d_{q'}(\partial_z u_\phi) e^{2Rt'} \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}} 2^{q'} \|\Delta_{q'}^h \partial_z u_\phi\|_{L^2} 2^{\frac{q'}{2}} \|\Delta_{q'}^h \partial_z u_\phi\|_{L^2} dt' \\
&\lesssim 2^{\frac{q}{2}} \sum_{q' \geq q-3} \int_0^T 2^{\frac{q'}{2}} \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}} \|\Delta_{q'}^h \partial_z u_\phi\|_{L^2} \|\Delta_{q'}^h \partial_z u_\phi\|_{L^2} dt' \\
&\lesssim 2^{\frac{q}{2}} \sum_{q' \geq q-3} 2^{\frac{q'}{2}} \left(\int_0^T e^{2Rt'} \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}} \|\Delta_{q'}^h \partial_z u_\phi\|_{L^2}^2 \right)^{\frac{1}{2}} \times \left(\int_0^T e^{2Rt'} \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}} \|\Delta_{q'}^h \partial_z u_\phi\|_{L^2}^2 \right)^{\frac{1}{2}} \\
&\lesssim 2^{-2qs} \tilde{d}_q \|e^{Rt'} \partial_z u_\phi\|_{\tilde{L}_{T, \theta'(t)}^2(B^{s+\frac{1}{2}})}^2,
\end{aligned}$$

where \tilde{d}_q is given by (2.61).

- Summing up the above estimates for \mathcal{A}_q , \mathcal{B}_q and \mathcal{R}_q , we obtain (2.15). \square

Proof of Estimate (2.16). We apply the Bony's decomposition with $a = \partial_z u$, $b = \partial_x \partial_z u$ and $f = u_\phi$. Then following the notations of (2.52) we can write

$$\int_0^T | \langle e^{Rt'} \Delta_q^h (\partial_z u \partial_x \partial_z u)_\phi, e^{Rt'} \Delta_q^h u_\phi \rangle | dt' \leq \mathcal{A}_q + \mathcal{B}_q + \mathcal{R}_q.$$

- Estimate of $\mathcal{A}_q = \int_0^T | \langle e^{Rt'} \Delta_q^h (T_{\partial_z u}^h \partial_x \partial_z u)_\phi, e^{Rt'} \Delta_q^h u_\phi \rangle | dt'$.

$$\mathcal{A}_q \lesssim \sum_{|q'-q| \leq 4} \int_0^T e^{2Rt'} \|S_{q'-1}^h \partial_z u_\phi^+\|_{L^2(L_x^\infty)} \|\Delta_{q'}^h \partial_x \partial_z u_\phi\|_{L^2} \|\Delta_{q'}^h u_\phi\|_{L^\infty(L_x^2)}.$$

Using Lemma 2.3.1 and Poincaré inequality, we have

$$\begin{aligned}
\|\Delta_q^h \partial_z u_\phi^+\|_{L^2(L_x^\infty)} &\lesssim 2^{\frac{q}{2}} \|\Delta_q^h \partial_z u_\phi^+\|_{L^2} \\
&\lesssim d_q(\partial_z^2 u_\phi) \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}}.
\end{aligned}$$

Then

$$\|S_{q'-1}^h \partial_z u_\phi^+\|_{L^2(L_x^\infty)} \lesssim \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}}.$$

By Poincaré inequality we have

$$\|\Delta_q^h u_\phi\|_{L^\infty(L_x^2)} \lesssim \|\Delta_q^h \partial_z u_\phi\|_{L^2}.$$

Then

$$\begin{aligned}
&\int_0^T | \langle e^{Rt'} \Delta_q^h (T_{\partial_z u}^h \partial_x \partial_z u)_\phi, e^{Rt'} \Delta_q^h u_\phi \rangle | dt \\
&\lesssim \sum_{|q'-q| \leq 4} \int_0^T e^{2Rt'} \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}} \left(2^{q'} \|\Delta_{q'}^h \partial_z u_\phi\|_{L^2} \right) \|\Delta_{q'}^h \partial_z u_\phi\|_{L^2} \\
&\lesssim \sum_{|q'-q| \leq 4} 2^{q'} \left(\int_0^T e^{2Rt'} \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}} \|\Delta_{q'}^h \partial_z u_\phi\|_{L^2}^2 \right)^{\frac{1}{2}} \times \left(\int_0^T e^{2Rt'} \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}} \|\Delta_{q'}^h \partial_z u_\phi\|_{L^2}^2 \right)^{\frac{1}{2}} \\
&\lesssim 2^{-2qs} \tilde{d}_q \|e^{Rt'} \partial_z u_\phi\|_{\tilde{L}_{T, \theta'(t)}^2(B^{s+\frac{1}{2}})}^2,
\end{aligned}$$

where \tilde{d}_q is defined by (2.60).

- Estimate of $\mathcal{B}_q = \int_0^T | \langle e^{Rt'} \Delta_q^h (T_{\partial_x \partial_z u}^h \partial_z u)_\phi, e^{Rt'} \Delta_q^h u_\phi \rangle | dt'$.

$$\mathcal{B}_q \lesssim \sum_{|q'-q| \leq 4} \int_0^T e^{2Rt'} \|S_{q'-1}^h \partial_x \partial_z u_\phi^+\|_{L_x^2(L_x^\infty)} \|\Delta_{q'}^h \partial_z u_\phi\|_{L^2} \|\Delta_{q'}^h u_\phi\|_{L_x^\infty(L_x^2)} dt'.$$

We have

$$\begin{aligned} \|S_{q'-1}^h \partial_x \partial_z u_\phi^+\|_{L_x^2(L_x^\infty)} &\lesssim \sum_{l \leq q'-2} 2^{\frac{3l}{2}} \|\Delta_l^h \partial_z u_\phi^+\|_{L^2} \\ &= \sum_{l \leq q'-2} 2^{\frac{3l}{2}} \|\Delta_l^h \partial_z u_\phi\|_{L^2} \\ &\lesssim \sum_{l \leq q'-2} 2^l 2^{\frac{l}{2}} \|\Delta_l^h \partial_z^2 u_\phi\|_{L^2} \\ &\lesssim \sum_{l \leq q'-2} 2^l d_l(\partial_z^2 u_\phi) \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}} \\ &\lesssim 2^{q'} \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}}, \end{aligned}$$

and by the Poincaré inequality, we get

$$\|\Delta_q^h u_\phi\|_{L_x^\infty(L_x^2)} \lesssim \|\Delta_q^h \partial_z u_\phi\|_{L^2}.$$

From the previous estimates,

$$\begin{aligned} \mathcal{B}_q &\lesssim \sum_{|q'-q| \leq 4} \int_0^T e^{2Rt'} \|S_{q'-1}^h \partial_x \partial_z u_\phi^+\|_{L_x^2(L_x^\infty)} \|\Delta_{q'}^h \partial_z u_\phi\|_{L^2} \|\Delta_{q'}^h u_\phi\|_{L_x^\infty(L_x^2)} dt' \\ &\lesssim \sum_{|q'-q| \leq 4} \int_0^T e^{2Rt'} 2^{q'} \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}} \|\Delta_{q'}^h \partial_z u_\phi\|_{L^2} \|\Delta_{q'}^h \partial_z u_\phi\|_{L^2} dt' \\ &\lesssim \sum_{|q'-q| \leq 4} 2^{q'} \left(\int_0^T e^{2Rt'} \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}} \|\Delta_{q'}^h \partial_z u_\phi\|_{L^2}^2 \right)^{\frac{1}{2}} \times \left(\int_0^T e^{2Rt'} \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}} \|\Delta_{q'}^h \partial_z u_\phi\|_{L^2}^2 \right)^{\frac{1}{2}} \\ &\lesssim 2^{-2qs} \tilde{d}_q \|e^{Rt'} \partial_z u_\phi\|_{\tilde{L}_{T, \theta'(t)}^2(B^{s+\frac{1}{2}})}^2. \end{aligned}$$

Then

$$\mathcal{B}_q \lesssim 2^{-2qs} \tilde{d}_q \|e^{Rt'} \partial_z u_\phi\|_{\tilde{L}_{T, \theta'(t)}^2(B^{s+\frac{1}{2}})}^2.$$

- Estimate of $\mathcal{R}_q = \int_0^T | \langle e^{Rt'} \Delta_q^h (R^h(\partial_z u, \partial_x \partial_z u))_\phi, e^{Rt'} \Delta_q^h u_\phi \rangle | dt'$.

$$\begin{aligned} \mathcal{R}_q &\lesssim \sum_{q' \geq q-3} \int_0^T e^{2Rt'} \|\tilde{\Delta}_{q'}^h \partial_z u_\phi^+\|_{L^2} \|\Delta_{q'}^h \partial_x \partial_z u_\phi\|_{L^2} \|\Delta_{q'}^h u_\phi\|_{L^\infty} dt' \\ &\lesssim \sum_{q' \geq q-3} \int_0^T 2^{-\frac{q'}{2}} d_{q'}(\partial_z u_\phi) e^{2Rt'} \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}} 2^{q'} \|\Delta_{q'}^h \partial_z u_\phi\|_{L^2} 2^{\frac{q'}{2}} \|\Delta_{q'}^h \partial_z u_\phi\|_{L^2} dt' \\ &\lesssim 2^{\frac{q}{2}} \sum_{q' \geq q-3} \int_0^T 2^{\frac{q'}{2}} \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}} \|\Delta_{q'}^h \partial_z u_\phi\|_{L^2} \|\Delta_{q'}^h \partial_z u_\phi\|_{L^2} dt' \\ &\lesssim 2^{\frac{q}{2}} \sum_{q' \geq q-3} 2^{\frac{q'}{2}} \left(\int_0^T e^{2Rt'} \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}} \|\Delta_{q'}^h \partial_z u_\phi\|_{L^2}^2 \right)^{\frac{1}{2}} \times \left(\int_0^T e^{2Rt'} \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}} \|\Delta_{q'}^h \partial_z u_\phi\|_{L^2}^2 \right)^{\frac{1}{2}} \\ &\lesssim 2^{-2qs} \tilde{d}_q \|e^{Rt'} \partial_z u_\phi\|_{\tilde{L}_{T, \theta'(t)}^2(B^{s+\frac{1}{2}})}^2, \end{aligned}$$

where \tilde{d}_q is given by (2.61).

- By summing the three estimates for \mathcal{A}_q , \mathcal{B}_q and \mathcal{R}_q , we obtain (2.16). \square

Proof of Estimate (2.17). We apply the Bony's decomposition with $a = v$, $b = \partial_z u$ and $f = u_\phi$. Then following the notations of (2.52) we can write

$$\int_0^T | \langle e^{Rt'} \Delta_q^h (v \partial_z u)_\phi, e^{Rt'} \Delta_q^h u_\phi \rangle | dt' \leq \mathcal{A}_q + \mathcal{B}_q + \mathcal{R}_q.$$

- Estimate of $\mathcal{A}_q = \int_0^T | \langle e^{Rt'} \Delta_q^h (T_v^h \partial_z u)_\phi, e^{Rt'} \Delta_q^h u_\phi \rangle | dt'$. From corollary 1,

$$\mathcal{A}_q \lesssim \sum_{|q'-q| \leq 4} \int_0^T e^{2Rt'} \|S_{q'-1}^h v_\phi^+\|_{L^\infty} \|\Delta_{q'}^h \partial_z u_\phi\|_{L^2} \|\Delta_q^h u_\phi\|_{L^2}.$$

From the incompressibility condition (see remark 1) and the definition of f_ϕ we get

$$\partial_x u_\phi + \partial_z v_\phi = 0.$$

Since $v_\phi|_{z=0} = 0$, then

$$v_\phi(t, x, z) = - \int_0^z \partial_x u_\phi(t, x, z') dz' \quad \text{and} \quad \Delta_q^h v_\phi(t, x, z) = - \int_0^z \partial_x \Delta_q^h u_\phi(t, x, z') dz'.$$

Therefore from lemma 2.3.1 we get

$$\begin{aligned} \|\Delta_q^h v_\phi\|_{L_z^\infty(L_x^2)} &\leq \int_0^1 \|\partial_x \Delta_q^h u_\phi(t, \cdot, z')\|_{L_x^2} dz' \\ &\lesssim \int_0^1 2^q \|\Delta_q^h u_\phi(t, \cdot, z')\|_{L_x^2} dz' \\ &\lesssim 2^q \|\Delta_q^h u_\phi\|_{L^2}. \end{aligned} \tag{2.62}$$

Again by Lemma 2.3.1 and the properties of (2.53) of f^+ , and the previous inequality we have

$$\|\Delta_q^h v_\phi^+\|_{L^\infty} \lesssim 2^{\frac{q}{2}} \|\Delta_q^h v_\phi\|_{L_z^\infty(L_x^2)} \lesssim 2^{\frac{3q}{2}} \|\Delta_q^h u_\phi\|_{L^2}. \tag{2.63}$$

Then from the Poincaré inequality and the remark 2,

$$\|\Delta_q^h v_\phi^+\|_{L^\infty} \lesssim 2^{\frac{3q}{2}} \|\Delta_q^h \partial_z^2 u_\phi\|_{L^2} \lesssim 2^q d_q(\partial_z^2 u_\phi) \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}}.$$

Then

$$\|S_{q'-1}^h v_\phi^+\|_{L^\infty} \lesssim \sum_{l \leq q'-2} 2^l d_l(\partial_z^2 u_\phi) \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}} \lesssim 2^{q'} \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}}.$$

From the previous estimates and lemma 2.3.1,

$$\begin{aligned} \mathcal{A}_q &\lesssim \sum_{|q'-q| \leq 4} \int_0^T e^{2Rt'} \left(2^{q'} \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}} \right) \|\Delta_{q'}^h \partial_z u_\phi\|_{L^2} \|\Delta_q^h u_\phi\|_{L^2} \\ &\lesssim \sum_{|q'-q| \leq 4} 2^{q'} \left(\int_0^T e^{2Rt'} \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}} \|\Delta_{q'}^h \partial_z u_\phi\|_{L^2}^2 \right)^{\frac{1}{2}} \times \left(\int_0^T e^{2Rt'} \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}} \|\Delta_q^h \partial_z u_\phi\|_{L^2}^2 \right)^{\frac{1}{2}} \\ &\lesssim 2^{-2qs} \tilde{d}_q \|e^{Rt'} \partial_z u_\phi\|_{\tilde{L}_{T, \theta'(t)}^2(B^{s+\frac{1}{2}})}, \end{aligned}$$

where \tilde{d}_q is given by (2.60).

- Estimate of $\mathcal{B}_q = \int_0^T | \langle e^{Rt'} \Delta_q^h (T_{\partial_z u}^h v)_\phi, e^{Rt'} \Delta_q^h u_\phi \rangle | dt'$.

$$\begin{aligned}
\mathcal{B}_q &\lesssim \sum_{|q'-q|\leq 4} \int_0^T e^{2Rt'} \|S_{q'-1}^h \partial_z u_\phi^+\|_{L_z^\infty(L_x^\infty)} \|\Delta_{q'}^h v_\phi\|_{L_z^\infty(L_x^2)} \|\Delta_q^h u_\phi\|_{L^2} dt' \\
&\lesssim \sum_{|q'-q|\leq 4} \int_0^T e^{2Rt'} \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}} \left(2^{q'} \|\Delta_{q'}^h u_\phi\|_{L^2}\right) \|\Delta_q^h u_\phi\|_{L^2} \\
&\lesssim \sum_{|q'-q|\leq 4} 2^{q'} \left(\int_0^T e^{2Rt'} \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}} \|\Delta_{q'}^h \partial_z u_\phi\|_{L^2}^2\right)^{\frac{1}{2}} \times \left(\int_0^T e^{2Rt'} \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}} \|\Delta_q^h \partial_z u_\phi\|_{L^2}^2\right)^{\frac{1}{2}} \\
&\lesssim 2^{-2qs} \tilde{d}_q \|e^{Rt'} \partial_z u_\phi\|_{\tilde{L}_{T, \theta'(t)}^2(B^{s+\frac{1}{2}})}^2.
\end{aligned}$$

- Estimate of $\mathcal{R}_q = \int_0^T | \langle e^{Rt'} \Delta_q^h (R^h(v, \partial_z u))_\phi, e^{Rt'} \Delta_q^h u_\phi \rangle | dt'$.

$$\mathcal{R}_q \lesssim \sum_{q' \geq q-3} \int_0^T e^{2Rt'} \|\Delta_{q'}^h v_\phi\|_{L_z^\infty(L_x^2)} \|\tilde{\Delta}_{q'}^h \partial_z u_\phi\|_{L^2} \|\Delta_q^h u_\phi\|_{L_z^2(L_x^\infty)} dt'.$$

From Estimate (2.62) we derive

$$\begin{aligned}
\|\Delta_{q'}^h v_\phi\|_{L_z^\infty(L_x^2)} &\leq \int_0^1 \|\partial_x \Delta_{q'}^h u_\phi(t, \cdot, z')\|_{L_x^2} dz' \\
&\lesssim 2^{q'} \int_0^1 \|\Delta_{q'}^h u_\phi(t, \cdot, z')\|_{L_x^2} dz' \\
&\lesssim 2^{q'} \|\Delta_{q'}^h u_\phi\|_{L^2}.
\end{aligned}$$

Then, we have

$$\begin{aligned}
\mathcal{R}_q &\lesssim \sum_{q' \geq q-3} \int_0^T e^{2Rt'} \|\Delta_{q'}^h v_\phi\|_{L_z^\infty(L_x^2)} \|\tilde{\Delta}_{q'}^h \partial_z u_\phi\|_{L^2} 2^{\frac{q}{2}} \|\Delta_q^h u_\phi\|_{L^2} dt' \\
&\lesssim 2^{\frac{q}{2}} \sum_{q' \geq q-3} \int_0^T e^{2Rt'} 2^{q'} \|\Delta_{q'}^h u_\phi\|_{L^2} 2^{-\frac{q'}{2}} d_{q'}(\partial_z^2 u_\phi) \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}} \|\Delta_q^h u_\phi\|_{L^2} dt' \\
&\lesssim 2^{\frac{q}{2}} \sum_{q' \geq q-3} \int_0^T e^{2Rt'} 2^{\frac{q'}{2}} \|\Delta_{q'}^h u_\phi\|_{L^2} \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}} \|\Delta_q^h u_\phi\|_{L^2} dt' \\
&\lesssim 2^{\frac{q}{2}} \sum_{q' \geq q-3} 2^{\frac{q'}{2}} \left(\int_0^T e^{2Rt'} \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}} \|\Delta_{q'}^h \partial_z u_\phi\|_{L^2}^2\right)^{\frac{1}{2}} \times \left(\int_0^T e^{2Rt'} \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}} \|\Delta_q^h \partial_z u_\phi\|_{L^2}^2\right)^{\frac{1}{2}} \\
&\lesssim 2^{-2qs} \tilde{d}_q \|e^{Rt'} \partial_z u_\phi\|_{\tilde{L}_{T, \theta'(t)}^2(B^{s+\frac{1}{2}})}^2,
\end{aligned}$$

where \tilde{d}_q is given by (2.61).

- By summing up the above three estimates for \mathcal{A}_q , \mathcal{B}_q and \mathcal{R}_q , we obtain (2.17). \square

Proof of Estimate (2.18). We apply the Bony's decomposition with $a = v$, $b = \partial_z^2 u$ and $f = \partial_z u_\phi$. Then following the notations of (2.52) we can write

$$\int_0^T | \langle e^{Rt'} \Delta_q^h (v \partial_z^2 u)_\phi, e^{Rt'} \Delta_q^h \partial_z u_\phi \rangle | dt' \leq \mathcal{A}_q + \mathcal{B}_q + \mathcal{R}_q.$$

- Estimate of $\mathcal{A}_q = \int_0^T | \langle e^{Rt'} \Delta_q^h (T_v^h \partial_z^2 u)_\phi, e^{Rt'} \Delta_q^h \partial_z u_\phi \rangle | dt'$. By Poincaré inequality we have

$$\|\Delta_q^h u_\phi\|_{L^2} \lesssim \|\Delta_q^h \partial_z u_\phi\|_{L^2}^{\frac{1}{2}} \|\Delta_q^h \partial_z^2 u_\phi\|_{L^2}^{\frac{1}{2}}.$$

Then from (2.63) and the previous estimate, and remark 2 we get

$$\begin{aligned} \|\Delta_q^h v_\phi^+\|_{L^\infty} &\lesssim 2^{\frac{3q}{2}} \|\Delta_q^h \partial_z u_\phi\|_{L^2}^{\frac{1}{2}} \|\Delta_q^h \partial_z^2 u_\phi\|_{L^2}^{\frac{1}{2}} \\ &\lesssim 2^{\frac{q}{2}} d_q (\partial_z u_\phi)^{\frac{1}{2}} \|\partial_z u_\phi\|_{B^{\frac{3}{2}}}^{\frac{1}{2}} d_q (\partial_z^2 u_\phi)^{\frac{1}{2}} \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}}^{\frac{1}{2}}. \end{aligned}$$

Then we have

$$\|S_{q'-1}^h v_\phi^+\|_{L^\infty} \lesssim 2^{\frac{q'}{2}} \|\partial_z u_\phi\|_{B^{\frac{3}{2}}}^{\frac{1}{2}} \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}}^{\frac{1}{2}}.$$

Therefore

$$\begin{aligned} \mathcal{A}_q &\lesssim \sum_{|q'-q|\leq 4} \int_0^T e^{2Rt'} \|S_{q'-1}^h v_\phi^+\|_{L^\infty} \|\Delta_{q'}^h \partial_z^2 u_\phi\|_{L^2} \|\Delta_{q'}^h \partial_z u_\phi\|_{L^2} \\ &\lesssim \sum_{|q'-q|\leq 4} 2^{\frac{q'}{2}} \int_0^T e^{Rt'} \|\partial_z u_\phi\|_{B^{\frac{3}{2}}}^{\frac{1}{2}} \|\Delta_{q'}^h \partial_z^2 u_\phi\|_{L^2} e^{Rt'} \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}}^{\frac{1}{2}} \|\Delta_{q'}^h \partial_z u_\phi\|_{L^2} \\ &\lesssim \sum_{|q'-q|\leq 4} 2^{\frac{q'}{2}} \left(\int_0^T e^{2Rt'} \|\partial_z u_\phi\|_{B^{\frac{3}{2}}} \|\Delta_{q'}^h \partial_z^2 u_\phi\|_{L^2}^2 \right)^{\frac{1}{2}} \times \left(\int_0^T e^{2Rt'} \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}} \|\Delta_{q'}^h \partial_z u_\phi\|_{L^2}^2 \right)^{\frac{1}{2}} \\ &\lesssim \sum_{|q'-q|\leq 4} 2^{\frac{q'}{2}} \|\partial_z u_\phi\|_{L_T^\infty(B^{\frac{3}{2}})}^{\frac{1}{2}} \left(\int_0^T e^{2Rt'} \|\Delta_{q'}^h \partial_z^2 u_\phi\|_{L^2}^2 \right)^{\frac{1}{2}} \times \left(\int_0^T e^{2Rt'} \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}} \|\Delta_{q'}^h \partial_z u_\phi\|_{L^2}^2 \right)^{\frac{1}{2}} \\ &\lesssim 2^{-2qs} \hat{d}_q \|\partial_z u_\phi\|_{L_T^\infty(B^{\frac{3}{2}})}^{\frac{1}{2}} \|e^{Rt'} \partial_z^2 u_\phi\|_{\tilde{L}_T^2(B^s)} \|e^{Rt'} \partial_z u_\phi\|_{\tilde{L}_{T,\theta'(t)}^2(B^{s+\frac{1}{2}})}, \end{aligned}$$

where

$$\hat{d}_q = d_q(\partial_z^2 u_\phi, 1) \left(\sum_{|q'-q|\leq 4} d_{q'}(\partial_z u_\phi, \theta') 2^{(q-q')(s-\frac{1}{2})} \right).$$

- Estimate of $\mathcal{B}_q = \int_0^T | \langle e^{Rt'} \Delta_q^h (T_{\partial_z^2 u}^h v)_\phi, e^{Rt'} \Delta_q^h \partial_z u_\phi \rangle | dt'$.

$$\begin{aligned} \mathcal{B}_q &\lesssim \sum_{|q'-q|\leq 4} \int_0^T e^{2Rt'} \|S_{q'-1}^h \partial_z^2 u_\phi^+\|_{L_z^2(L_x^\infty)} \|\Delta_{q'}^h v_\phi\|_{L_z^\infty(L_x^2)} \|\Delta_{q'}^h \partial_z u_\phi\|_{L^2} dt' \\ &\lesssim \sum_{|q'-q|\leq 4} \int_0^T e^{2Rt'} \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}} \left(2^{q'} \|\Delta_{q'}^h \partial_z u_\phi\|_{L^2} \right) \|\Delta_{q'}^h \partial_z u_\phi\|_{L^2} dt' \\ &\lesssim \sum_{|q'-q|\leq 4} 2^{q'} \left(\int_0^T e^{2Rt'} \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}} \|\Delta_{q'}^h \partial_z u_\phi\|_{L^2}^2 \right)^{\frac{1}{2}} \times \left(\int_0^T e^{2Rt'} \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}} \|\Delta_{q'}^h \partial_z u_\phi\|_{L^2}^2 \right)^{\frac{1}{2}} \\ &\lesssim 2^{-2qs} \tilde{d}_q \|e^{Rt'} \partial_z u_\phi\|_{\tilde{L}_{T,\theta'(t)}^2(B^{s+\frac{1}{2}})}^2, \end{aligned}$$

where \tilde{d}_q is given by (2.60).

- Estimate of $\mathcal{R}_q = \int_0^T | \langle e^{Rt'} \Delta_q^h (R^h(v, \partial_z^2 u))_\phi, e^{Rt'} \Delta_q^h \partial_z u_\phi \rangle | dt'$. From corollary 1, we have

$$\mathcal{R}_q \lesssim \sum_{q' \geq q-3} \int_0^T e^{2Rt'} \|\Delta_{q'}^h v_\phi\|_{L_z^\infty(L_x^2)} \|\tilde{\Delta}_{q'}^h \partial_z^2 u_\phi\|_{L^2} \|\Delta_{q'}^h \partial_z u_\phi\|_{L_z^2(L_x^\infty)} dt'.$$

From (2.62), we have

$$\|\Delta_{q'}^h v_\phi\|_{L_z^\infty(L_x^2)} \lesssim 2^{q'} \|\Delta_{q'}^h \partial_z u_\phi\|_{L^2}.$$

Then, using remark 2 and lemma 2.3.1, remark 3, we have for any $s > 0$

$$\begin{aligned}
\mathcal{R}_q &\lesssim \sum_{q' \geq q-3} \int_0^T e^{2Rt'} 2^{q'} \|\Delta_{q'}^h \partial_z u_\phi\|_{L^2} \left(2^{-\frac{q'}{2}} d_{q'}(\partial_z^2 u_\phi) \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}} \right) 2^{\frac{q}{2}} \|\Delta_q^h \partial_z u_\phi\|_{L^2} dt' \\
&\lesssim 2^{\frac{q}{2}} \sum_{q' \geq q-3} 2^{\frac{q'}{2}} \int_0^T e^{2Rt'} \|\Delta_{q'}^h \partial_z u_\phi\|_{L^2} \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}} \|\Delta_q^h \partial_z u_\phi\|_{L^2} dt' \\
&\lesssim 2^{\frac{q}{2}} \sum_{q' \geq q-3} 2^{\frac{q'}{2}} \left(\int_0^T e^{2Rt'} \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}} \|\Delta_{q'}^h \partial_z u_\phi\|_{L^2}^2 \right)^{\frac{1}{2}} \times \left(\int_0^T e^{2Rt'} \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}} \|\Delta_q^h \partial_z u_\phi\|_{L^2}^2 \right)^{\frac{1}{2}} \\
&\lesssim 2^{-2qs} \check{d}_q \|e^{Rt'} \partial_z u_\phi\|_{\tilde{L}_{T, \theta'(t)}^2(B^{s+\frac{1}{2}})}^2,
\end{aligned}$$

where \check{d} is given by (2.61).

• Summing up the above three estimates, we obtain (2.18). \square

2.4 Appendix - Proof of the lemma 2.2.2

Proof of Estimate (2.29). We first recall the Bony's decomposition (2.50)

$$u \partial_x u = T_u^h \partial_x u + T_{\partial_x u}^h u + R^h(u, \partial_x u).$$

Then

$$\begin{aligned}
&\left(\int_0^T \|e^{Rt} \Delta_q^h (u \partial_x u)_\phi\|_{L^2}^2 dt \right)^{\frac{1}{2}} \\
&\lesssim \left(\int_0^T \|e^{Rt} \Delta_q^h (T_u^h \partial_x u)_\phi\|_{L^2}^2 dt \right)^{\frac{1}{2}} + \left(\int_0^T \|e^{Rt} \Delta_q^h (T_{\partial_x u}^h u)_\phi\|_{L^2}^2 dt \right)^{\frac{1}{2}} + \left(\int_0^T \|e^{Rt} \Delta_q^h (R^h(u, \partial_x u))_\phi\|_{L^2}^2 dt \right)^{\frac{1}{2}}.
\end{aligned}$$

• Estimate of $\|e^{Rt} \Delta_q^h (T_u^h \partial_x u)_\phi\|_{L^2}$. From the definition of T_h we have

$$\|e^{Rt} \Delta_q^h (T_u^h \partial_x u)_\phi\|_{L^2} = \|e^{Rt} \Delta_q^h \left(\sum_{q' \in \mathbb{Z}} S_{q'-1}^h u \Delta_{q'}^h \partial_x u \right)_\phi\|_{L^2} \lesssim \sum_{|q-q'| \leq 4} \left\| e^{Rt} \left(S_{q'-1}^h u \Delta_{q'}^h \partial_x u \right)_\phi \right\|_{L^2}.$$

From corollary 1, Poincarre inequality and lemma 2.3.1 we get

$$\begin{aligned}
\int_0^T \|e^{Rt} \Delta_q^h (T_u^h \partial_x u)_\phi\|_{L^2}^2 dt &\lesssim \sum_{|q-q'| \leq 4} \int_0^T \|S_{q'-1}^h u_\phi^+\|_{L^\infty}^2 \|e^{Rt} \Delta_{q'}^h \partial_x u_\phi\|_{L^2}^2 dt \\
&\lesssim \sum_{|q-q'| \leq 4} \int_0^T \|\Delta_{q'}^h \partial_z u_\phi\|_{B^{\frac{1}{2}}}^2 \left(2^{2q'} \|e^{Rt} \Delta_{q'}^h \partial_z u_\phi\|_{L^2}^2 \right) dt \\
&\lesssim \sum_{|q-q'| \leq 4} \|\Delta_{q'}^h \partial_z u_\phi\|_{L_T^\infty(B^{\frac{1}{2}})}^2 \left(2^{2q'} \int_0^T \|e^{Rt} \Delta_{q'}^h \partial_z u_\phi\|_{L^2}^2 dt \right).
\end{aligned}$$

Then from the remark 3

$$\begin{aligned}
\left(\int_0^T \|e^{Rt} \Delta_q^h (T_u^h \partial_x u)_\phi\|_{L^2}^2 dt \right)^{\frac{1}{2}} &\lesssim \sum_{|q-q'| \leq 4} \|\Delta_{q'}^h \partial_z u_\phi\|_{L_T^\infty(B^{\frac{1}{2}})} 2^{q'} \left(\int_0^T \|e^{Rt} \Delta_{q'}^h \partial_z u_\phi\|_{L^2}^2 dt \right)^{\frac{1}{2}} \\
&\lesssim \sum_{|q-q'| \leq 4} \|\partial_z u_\phi\|_{\tilde{L}_T^\infty(B^{\frac{1}{2}})} 2^{q'} \left(2^{-\frac{5q'}{2}} d_{q'}(\partial_z u_\phi, 1) \|e^{Rt} \Delta_{q'}^h \partial_z u_\phi\|_{\tilde{L}_T^2(B^{\frac{5}{2}})} \right) \\
&\lesssim \check{d}_q 2^{-\frac{3q}{2}} \|\partial_z u_\phi\|_{\tilde{L}_T^\infty(B^{\frac{1}{2}})} \|e^{Rt} \partial_z u_\phi\|_{\tilde{L}_T^2(B^{\frac{5}{2}})},
\end{aligned} \tag{2.64}$$

where

$$\tilde{d}_q = \sum_{|q-q'|\leq 4} d_{q'} (\partial_z u_\phi, 1) 2^{\frac{3}{2}(q-q')}. \quad (2.65)$$

- Estimate of $\|e^{Rt} \Delta_q^h (T_{\partial_x u}^h u)_\phi\|_{L^2}$. Following the previous estimates we get

$$\left(\int_0^T \|e^{Rt} \Delta_q^h (T_{\partial_x u}^h u)_\phi\|_{L^2}^2 dt \right)^{\frac{1}{2}} \lesssim \tilde{d}_q 2^{-\frac{3q}{2}} \|\partial_z u_\phi\|_{\tilde{L}_T^\infty(B^{\frac{1}{2}})} \|e^{Rt} \partial_z u_\phi\|_{\tilde{L}_T^2(B^{\frac{5}{2}})}. \quad (2.66)$$

- Estimate of $\|e^{Rt} \Delta_q^h (R^h(u, \partial_x u))_\phi\|_{L^2}$. From corollary 1, we have

$$\begin{aligned} \left(\int_0^T \|e^{Rt} \Delta_q^h (R^h(u, \partial_x u))_\phi\|_{L^2}^2 dt \right)^{\frac{1}{2}} &\lesssim \sum_{q' \geq q-3} \left(\int_0^T \|\tilde{\Delta}_{q'}^h u_\phi\|_{L_x^\infty}^2 \|e^{Rt} \Delta_{q'}^h \partial_x u_\phi\|_{L_x^\infty}^2 dt \right)^{\frac{1}{2}} \\ &\lesssim \sum_{|q-q'|\leq 4} \left(\int_0^T \|\Delta_{q'}^h \partial_z u_\phi\|_{B^{\frac{1}{2}}}^2 \left(2^{2q'} \|e^{Rt} \Delta_{q'}^h \partial_z u_\phi\|_{L^2}^2 \right) dt \right)^{\frac{1}{2}} \\ &\lesssim \sum_{|q-q'|\leq 4} \|\Delta_{q'}^h \partial_z u_\phi\|_{L_T^\infty(B^{\frac{1}{2}})} 2^{q'} \left(\int_0^T \|e^{Rt} \Delta_{q'}^h \partial_z u_\phi\|_{L^2}^2 dt \right)^{\frac{1}{2}}. \end{aligned}$$

Then from remark 3

$$\left(\int_0^T \|e^{Rt} \Delta_q^h (R^h(u, \partial_x u))_\phi\|_{L^2}^2 dt \right)^{\frac{1}{2}} \lesssim \check{d}_q 2^{-\frac{3q}{2}} \|\partial_z u_\phi\|_{\tilde{L}_T^\infty(B^{\frac{1}{2}})} \|e^{Rt} \partial_z u_\phi\|_{\tilde{L}_T^2(B^{\frac{5}{2}})}, \quad (2.67)$$

where

$$\check{d}_q = \sum_{q' \geq q-3} d_{q'} (\partial_z u_\phi, 1) 2^{\frac{3}{2}(q-q')}. \quad (2.68)$$

- By summing estimates (2.64), (2.66) and (2.67) we get

$$\left(\int_0^T \|e^{Rt} \Delta_q^h (u \partial_x u)_\phi\|_{L^2}^2 dt \right)^{\frac{1}{2}} \lesssim (2\check{d}_q + \check{q}_q) 2^{-\frac{3q}{2}} \|\partial_z u_\phi\|_{\tilde{L}_T^\infty(B^{\frac{1}{2}})} \|e^{Rt} \partial_z u_\phi\|_{\tilde{L}_T^2(B^{\frac{5}{2}})}.$$

Multiplying the previous inequality by $2^{\frac{3q}{2}}$ and taking the sum over \mathbb{Z} we obtain (2.29). \square

Proof of Estimate (2.30). We have

$$\begin{aligned} \left(\int_0^T \|e^{Rt} \Delta_q^h (u \partial_x \partial_z^2 u)_\phi\|_{L^2}^2 dt \right)^{\frac{1}{2}} &\lesssim \left(\int_0^T \|e^{Rt} \Delta_q^h (T_u^h \partial_x \partial_z^2 u)_\phi\|_{L^2}^2 dt \right)^{\frac{1}{2}} \\ &\quad + \left(\int_0^T \|e^{Rt} \Delta_q^h (T_{\partial_x \partial_z^2 u}^h u)_\phi\|_{L^2}^2 dt \right)^{\frac{1}{2}} + \left(\int_0^T \|e^{Rt} \Delta_q^h (R^h(u, \partial_x \partial_z^2 u))_\phi\|_{L^2}^2 dt \right)^{\frac{1}{2}}. \end{aligned}$$

By adapting the proof of Estimate (2.64) we get

$$\begin{aligned} \left(\int_0^T \|e^{Rt} \Delta_q^h (T_u^h \partial_x \partial_z^2 u)_\phi\|_{L^2}^2 dt \right)^{\frac{1}{2}} &\lesssim \tilde{d}_q 2^{-\frac{3q}{2}} \|\partial_z u_\phi\|_{\tilde{L}_T^\infty(B^{\frac{1}{2}})} \|e^{Rt} \partial_z^2 u_\phi\|_{\tilde{L}_T^2(B^{\frac{5}{2}})}, \\ \left(\int_0^T \|e^{Rt} \Delta_q^h (T_{\partial_x \partial_z^2 u}^h u)_\phi\|_{L^2}^2 dt \right)^{\frac{1}{2}} &\lesssim \tilde{d}_q 2^{-\frac{3q}{2}} \|e^{Rt} \partial_z^2 u_\phi\|_{\tilde{L}_T^2(B^{\frac{1}{2}})} \|\partial_z u_\phi\|_{\tilde{L}_T^\infty(B^{\frac{5}{2}})}, \end{aligned}$$

where \tilde{d}_q is given by (2.65). By adapting the proof of Estimate (2.67) we get

$$\left(\int_0^T \|e^{Rt} \Delta_q^h (R^h(u, \partial_x \partial_z^2 u))_\phi\|_{L^2}^2 dt \right)^{\frac{1}{2}} \lesssim \tilde{d}_q 2^{-\frac{3q}{2}} \|\partial_z u_\phi\|_{\tilde{L}_T^\infty(B^{\frac{1}{2}})} \|e^{Rt} \partial_z^2 u_\phi\|_{\tilde{L}_T^2(B^{\frac{5}{2}})},$$

where \tilde{d}_q is given by (2.68). Thus by summing the previous estimates we derive

$$\begin{aligned} & \left(\int_0^T \|e^{Rt} \Delta_q^h (u \partial_x \partial_z^2 u)_\phi\|_{L^2}^2 dt \right)^{\frac{1}{2}} \\ & \lesssim 2^{-\frac{3q}{2}} \left((\tilde{d}_q + \check{d}_q) \|\partial_z u_\phi\|_{\tilde{L}_T^\infty(B^{\frac{1}{2}})} \|e^{Rt} \partial_z^2 u_\phi\|_{\tilde{L}_T^2(B^{\frac{5}{2}})} + \tilde{d}_q \|e^{Rt} \partial_z^2 u_\phi\|_{\tilde{L}_T^2(B^{\frac{1}{2}})} \|\partial_z u_\phi\|_{\tilde{L}_T^\infty(B^{\frac{5}{2}})} \right). \end{aligned}$$

Multiplying the previous inequality by $2^{\frac{3q}{2}}$ and taking the sum over \mathbb{Z} , we obtain (2.30). \square

Proof of Estimate (2.31). We have by Bony's decomposition in x variable

$$\begin{aligned} \left(\int_0^T \|e^{Rt} \Delta_q^h (v \partial_z u)_\phi\|_{L^2}^2 dt \right)^{\frac{1}{2}} & \lesssim \left(\int_0^T \|e^{Rt} \Delta_q^h (T_v^h \partial_z u)_\phi\|_{L^2}^2 dt \right)^{\frac{1}{2}} \\ & + \left(\int_0^T \|e^{Rt} \Delta_q^h (T_{\partial_z u}^h v)_\phi\|_{L^2}^2 dt \right)^{\frac{1}{2}} + \left(\int_0^T \|e^{Rt} \Delta_q^h (R^h(v, \partial_z u))_\phi\|_{L^2}^2 dt \right)^{\frac{1}{2}}. \end{aligned}$$

- Estimate of $\left(\int_0^T \|e^{Rt} \Delta_q^h (T_v^h \partial_z u)_\phi\|_{L^2}^2 dt \right)^{\frac{1}{2}}$.

$$\begin{aligned} \int_0^T \|e^{Rt} \Delta_q^h (T_v^h \partial_z u)_\phi\|_{L^2}^2 dt & \leq \sum_{|q-q'|\leq 4} \int_0^T \|S_{q'-1}^h v_\phi^+\|_{L^\infty}^2 \|e^{Rt} \Delta_{q'}^h \partial_z u_\phi\|_{L^2}^2 dt \\ & \lesssim \sum_{|q-q'|\leq 4} \int_0^T \|\partial_z u_\phi\|_{B^{\frac{1}{2}}}^2 2^{2q'} \|e^{Rt} \Delta_{q'}^h \partial_z u_\phi\|_{L^2}^2 dt \\ & \lesssim \sum_{|q-q'|\leq 4} \|\partial_z u_\phi\|_{L_T^\infty(B^{\frac{1}{2}})}^2 \left(2^{2q'} \int_0^T \|e^{Rt} \Delta_{q'}^h \partial_z u_\phi\|_{L^2}^2 dt \right). \end{aligned}$$

Similar to (2.64) we get

$$\left(\int_0^T \|e^{Rt} \Delta_q^h (T_v^h \partial_z u)_\phi\|_{L^2}^2 dt \right)^{\frac{1}{2}} \lesssim \tilde{d}_q 2^{-\frac{3q}{2}} \|\partial_z u_\phi\|_{\tilde{L}_T^\infty(B^{\frac{1}{2}})} \|e^{Rt} \partial_z u_\phi\|_{\tilde{L}_T^2(B^{\frac{5}{2}})}, \quad (2.69)$$

where \tilde{d}_q is given by (2.65).

- Estimate of $\left(\int_0^T \|e^{Rt} \Delta_q^h (T_{\partial_z u}^h v)_\phi\|_{L^2}^2 dt \right)^{\frac{1}{2}}$.

$$\begin{aligned} \int_0^T \|e^{Rt} \Delta_q^h (T_{\partial_z u}^h v)_\phi\|_{L^2}^2 dt & \lesssim \sum_{|q-q'|\leq 4} \int_0^T \|S_{q'-1}^h \partial_z u_\phi^+\|_{L_z^2(L_x^\infty)}^2 \|e^{Rt} \Delta_{q'}^h v_\phi\|_{L_z^\infty(L_x^2)}^2 dt \\ & \lesssim \sum_{|q-q'|\leq 4} \int_0^T \|\partial_z u_\phi\|_{B^{\frac{1}{2}}}^2 2^{2q'} \|e^{Rt} \Delta_{q'}^h \partial_z u_\phi\|_{L^2}^2 dt \\ & \lesssim \sum_{|q-q'|\leq 4} \|\partial_z u_\phi\|_{L_T^\infty(B^{\frac{1}{2}})}^2 \left(2^{2q'} \int_0^T \|e^{Rt} \Delta_{q'}^h \partial_z u_\phi\|_{L^2}^2 dt \right). \end{aligned}$$

Similar to (2.64) we get

$$\left(\int_0^T \|e^{Rt} \Delta_q^h (T_{\partial_z u}^h v)_\phi\|_{L^2}^2 dt \right)^{\frac{1}{2}} \lesssim \tilde{d}_q 2^{-\frac{3q}{2}} \|\partial_z u_\phi\|_{\tilde{L}_T^\infty(B^{\frac{1}{2}})} \|e^{Rt} \partial_z u_\phi\|_{\tilde{L}_T^2(B^{\frac{5}{2}})}, \quad (2.70)$$

where \tilde{d}_q is given by (2.65).

- Estimate of $\|e^{Rt} \Delta_q^h (R^h(v, \partial_z u))_\phi\|_{L^2}$.

$$\begin{aligned} \int_0^T \|e^{Rt} \Delta_q^h (R^h(v, \partial_z u))_\phi\|_{L^2}^2 dt &\lesssim \sum_{q' \geq q-3} \int_0^T \|\Delta_{q'}^h v_\phi\|_{L_x^\infty(L_x^2)}^2 \|e^{Rt} \tilde{\Delta}_{q'}^h \partial_z u_\phi\|_{L_x^2(L_x^\infty)}^2 dt \\ &\lesssim \sum_{q' \geq q-3} \int_0^T \left(2^{2q'} \|\Delta_{q'}^h \partial_z u_\phi\|_{L^2}^2 \right) \|\partial_z u_\phi\|_{B^{\frac{1}{2}}}^2 dt \\ &\lesssim \sum_{q' \geq q-3} 2^{2q'} \left(\int_0^T \|\Delta_{q'}^h \partial_z u_\phi\|_{L^2}^2 dt \right) \|\partial_z u_\phi\|_{\tilde{L}_T^\infty(B^{\frac{1}{2}})}^2. \end{aligned}$$

Similar to (2.64) we get

$$\left(\int_0^T \|e^{Rt} \Delta_q^h (R^h(v, \partial_z u))_\phi\|_{L^2}^2 dt \right)^{\frac{1}{2}} \lesssim \check{d}_q 2^{-\frac{3q}{2}} \|e^{Rt} \partial_z u_\phi\|_{\tilde{L}_T^2(B^{\frac{5}{2}})} \|e^{Rt} \partial_z u_\phi\|_{\tilde{L}_T^\infty(B^{\frac{1}{2}})}, \quad (2.71)$$

where \check{d}_q is given by (2.68).

- By summing the previous estimates (2.69), (2.70) and (2.71) we get

$$\left(\int_0^T \|e^{Rt} \Delta_q^h (v \partial_z u)_\phi\|_{L^2}^2 dt \right)^{\frac{1}{2}} \lesssim (2\tilde{d}_q + \check{d}_q) 2^{-\frac{3q}{2}} \left(\|\partial_z u_\phi\|_{\tilde{L}_T^\infty(B^{\frac{1}{2}})} \|e^{Rt} \partial_z u_\phi\|_{\tilde{L}_T^2(B^{\frac{5}{2}})} \right).$$

Multiplying the previous inequality by $2^{\frac{3q}{2}}$ and taking the sum over \mathbb{Z} , we obtain (2.31). \square

Proof of Estimate (2.32). We have by Bony's decomposition in x variable

$$\begin{aligned} \left(\int_0^T \|e^{Rt} \Delta_q^h (v \partial_z^2 u)_\phi\|_{L^2}^2 dt \right)^{\frac{1}{2}} &\lesssim \left(\int_0^T \|e^{Rt} \Delta_q^h (T_v^h \partial_z^2 u)_\phi\|_{L^2}^2 dt \right)^{\frac{1}{2}} \\ &\quad + \left(\int_0^T \|e^{Rt} \Delta_q^h (T_{\partial_z^2 u}^h v)_\phi\|_{L^2}^2 dt \right)^{\frac{1}{2}} + \left(\int_0^T \|e^{Rt} \Delta_q^h (R^h(v, \partial_z^2 u))_\phi\|_{L^2}^2 dt \right)^{\frac{1}{2}}. \end{aligned}$$

By adapting the proof of Estimate (2.69) we obtain

$$\begin{aligned} \left(\int_0^T \|e^{Rt} \Delta_q^h (T_v^h \partial_z^2 u)_\phi\|_{L^2}^2 dt \right)^{\frac{1}{2}} &\lesssim \sum_{|q'-q| \leq 4} \left(\int_0^T \|S_{q'-1}^h v_\phi^+\|_{L^\infty}^2 \|\Delta_{q'}^h \partial_z^2 u_\phi\|_{L^2}^2 dt \right)^{\frac{1}{2}} \\ &\lesssim \tilde{d}_q 2^{-\frac{3q}{2}} \|\partial_z u_\phi\|_{\tilde{L}_T^\infty(B^{\frac{1}{2}})} \|e^{Rt} \partial_z^2 u_\phi\|_{\tilde{L}_T^2(B^{\frac{5}{2}})}. \end{aligned}$$

By adapting the proof of Estimate (2.70) we obtain

$$\begin{aligned} \left(\int_0^T \|e^{Rt} \Delta_q^h (T_{\partial_z^2 u}^h v)_\phi\|_{L^2}^2 dt \right)^{\frac{1}{2}} &\lesssim \sum_{|q-q'| \leq 4} \left(\|S_{q'-1}^h e^{Rt} \partial_z^2 u_\phi^+\|_{L_x^2(L_x^\infty)}^2 \|Delta_{q'}^h v_\phi\|_{L_x^\infty(L_x^2)}^2 \right)^{\frac{1}{2}} \\ &\lesssim \tilde{d}_q 2^{-\frac{3q}{2}} \|e^{Rt} \partial_z^2 u_\phi\|_{\tilde{L}_T^2(B^{\frac{1}{2}})} \|e^{Rt} \partial_z u_\phi\|_{\tilde{L}_T^\infty(B^{\frac{1}{2}})}. \end{aligned}$$

By adapting the proof of Estimate (2.71) we obtain

$$\begin{aligned} \left(\int_0^T \|e^{Rt} \Delta_q^h (R^h(v, \partial_z^2 u))_\phi\|_{L^2}^2 dt \right)^{\frac{1}{2}} &\lesssim \sum_{q' \geq q-3} \left(\int_0^T \|\Delta_{q'}^h v_\phi\|_{L^\infty(L_z^2)}^2 2^{\frac{q'}{2}} \|e^{Rt} \tilde{\Delta}_{q'}^h \partial_z^2 u_\phi\|_{L^2}^2 dt \right)^{\frac{1}{2}} \\ &\lesssim \tilde{d}_q 2^{-\frac{3q}{2}} \|\partial_z u_\phi\|_{\tilde{L}_T^\infty(B^{\frac{5}{2}})} \|e^{Rt} \partial_z^2 u_\phi\|_{\tilde{L}_T^2(B^{\frac{1}{2}})}. \end{aligned}$$

Then by summing the previous estimates we get

$$\begin{aligned} \left(\int_0^T \|e^{Rt} \Delta_q^h (v \partial_z^2 u)_\phi\|_{L^2}^2 dt \right)^{\frac{1}{2}} \\ \lesssim 2^{-\frac{3q}{2}} \left((\tilde{d}_q + \check{d}_q) \left(\|\partial_z u_\phi\|_{\tilde{L}_T^\infty(B^{\frac{5}{2}})} \|e^{Rt} \partial_z^2 u_\phi\|_{\tilde{L}_T^2(B^{\frac{1}{2}})} \right) + \tilde{d}_q \|\partial_z u_\phi\|_{\tilde{L}_T^\infty(B^{\frac{1}{2}})} \|e^{Rt} \partial_z^2 u_\phi\|_{\tilde{L}_T^2(B^{\frac{5}{2}})} \right). \end{aligned}$$

Multiplying the previous inequality by $2^{\frac{3q}{2}}$ and taking the sum over \mathbb{Z} , we obtain (2.32). \square

Proof of Estimate (2.33). We have by Bony's decomposition in x variable

$$\begin{aligned} \|e^{Rt} \Delta_q^h (\partial_z u \partial_x \partial_z u)_\phi\|_{L^2} \\ \leq \|e^{Rt} \Delta_q^h (T_{\partial_z u}^h \partial_x \partial_z u)_\phi\|_{L^2} + \|e^{Rt} \Delta_q^h (T_{\partial_x \partial_z u}^h \partial_z u)_\phi\|_{L^2} + \|e^{Rt} \Delta_q^h (R^h(\partial_z u, \partial_x \partial_z u))_\phi\|_{L^2}. \end{aligned}$$

By adapting the proof of Estimate (2.64) we get

$$\left(\int_0^T \|e^{Rt} \Delta_q^h (T_{\partial_z u}^h \partial_x \partial_z u)_\phi\|_{L^2}^2 dt \right)^{\frac{1}{2}} \lesssim \tilde{d}_q 2^{-\frac{3q}{2}} \|e^{Rt} \partial_z^2 u_\phi\|_{\tilde{L}_T^2(B^{\frac{1}{2}})} \|\partial_z u_\phi\|_{\tilde{L}_T^\infty(B^{\frac{5}{2}})}$$

and

$$\left(\int_0^T \|e^{Rt} \Delta_q^h (T_{\partial_x \partial_z u}^h \partial_z u)_\phi\|_{L^2}^2 dt \right)^{\frac{1}{2}} \lesssim \check{d}_q 2^{-\frac{3q}{2}} \|e^{Rt} \partial_z^2 u_\phi\|_{\tilde{L}_T^2(B^{\frac{1}{2}})} \|\partial_z u_\phi\|_{\tilde{L}_T^\infty(B^{\frac{5}{2}})}.$$

By adapting the proof of Estimate (2.67) we get

$$\left(\int_0^T \|e^{Rt} \Delta_q^h (R^h(\partial_z u, \partial_x \partial_z u))_\phi\|_{L^2}^2 dt \right)^{\frac{1}{2}} \lesssim \check{d}_q 2^{-\frac{3q}{2}} \|\partial_z u_\phi\|_{\tilde{L}_T^\infty(B^{\frac{1}{2}})} \|e^{Rt} \partial_z^2 u_\phi\|_{\tilde{L}_T^2(B^{\frac{5}{2}})}.$$

Then by summing the previous estimates we get

$$\begin{aligned} \left(\int_0^T \|e^{Rt} \Delta_q^h (\partial_z u \partial_x \partial_z u)_\phi\|_{L^2}^2 dt \right)^{\frac{1}{2}} \\ \lesssim 2^{-\frac{3q}{2}} \left(2\tilde{d}_q \|e^{Rt} \partial_z^2 u_\phi\|_{\tilde{L}_T^2(B^{\frac{1}{2}})} \|\partial_z u_\phi\|_{\tilde{L}_T^\infty(B^{\frac{5}{2}})} + \check{d}_q \|\partial_z u_\phi\|_{\tilde{L}_T^\infty(B^{\frac{1}{2}})} \|e^{Rt} \partial_z^2 u_\phi\|_{\tilde{L}_T^2(B^{\frac{5}{2}})} \right). \end{aligned}$$

Multiplying the previous inequality by $2^{\frac{3q}{2}}$ and taking the sum over \mathbb{Z} , we obtain (2.33). \square

Proof of Estimate (2.34). We have by Bony's decomposition in x variable

$$\|e^{Rt} \Delta_q^h (\partial_x u \partial_z^2 u)_\phi\|_{L^2} \leq \|e^{Rt} \Delta_q^h (T_{\partial_x u}^h \partial_z^2 u)_\phi\|_{L^2} + \|e^{Rt} \Delta_q^h (T_{\partial_z^2 u}^h \partial_x u)_\phi\|_{L^2} + \|e^{Rt} \Delta_q^h (R^h(\partial_x u, \partial_z^2 u))_\phi\|_{L^2}.$$

By adapting the proof of Estimate (2.64) we get

$$\begin{aligned} \left(\int_0^T \|e^{Rt} \Delta_q^h (T_{\partial_x u}^h \partial_z^2 u)_\phi\|_{L^2}^2 dt \right)^{\frac{1}{2}} &\lesssim \sum_{|q'-q| \leq 4} \left(\int_0^T \|S_{q'-1}^h \partial_x u_\phi^+\|_{L^\infty}^2 \|\Delta_{q'}^h \partial_z^2 u_\phi\|_{L^2}^2 dt \right)^{\frac{1}{2}} \\ &\lesssim \tilde{d}_q 2^{-\frac{3q}{2}} \|\partial_z u_\phi\|_{\tilde{L}_T^\infty(B^{\frac{1}{2}})} \|e^{Rt} \partial_z^2 u_\phi\|_{\tilde{L}_T^2(B^{\frac{5}{2}})}, \end{aligned}$$

where $\tilde{d}_{q,2} = \sum_{|q-q'|\leq 4} d_{q'}(\partial_z^2 u_\phi, 1)2^{\frac{3}{2}(q-q')}$.

- Estimate of $\left(\int_0^T \|e^{Rt}\Delta_q^h(T_{\partial_z^2 u}^h \partial_x u)_\phi\|_{L^2} dt\right)^{\frac{1}{2}}$.

$$\begin{aligned} \left(\int_0^T \|e^{Rt}\Delta_q^h(T_{\partial_z^2 u}^h \partial_x u)_\phi\|_{L^2} dt\right)^{\frac{1}{2}} &\lesssim \sum_{|q-q'|\leq 4} \left(\int_0^T \|S_{q'-1}^h e^{Rt}\partial_z^2 u_\phi^+\|_{L^2_x(L^\infty)}^2 \|\Delta_{q'}^h \partial_x u_\phi\|_{L^2_x(L^\infty)}^2 dt\right)^{\frac{1}{2}} \\ &\lesssim \sum_{|q-q'|\leq 4} \left(\int_0^T \|e^{Rt}\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}}^2 2^{2q'} \|\Delta_{q'}^h \partial_x u_\phi\|_{L^2}^2 dt\right)^{\frac{1}{2}} \\ &\lesssim \tilde{d}_q 2^{-\frac{3q}{2}} \|e^{Rt}\partial_z^2 u_\phi\|_{\tilde{L}_T^2(B^{\frac{1}{2}})} \|\partial_x u_\phi\|_{\tilde{L}_T^\infty(B^{\frac{5}{2}})}. \end{aligned}$$

By adapting the proof of Estimate (2.67) we get

$$\begin{aligned} \left(\int_0^T \|e^{Rt}\Delta_q^h(R^h(\partial_x u, \partial_z^2 u))_\phi\|_{L^2}^2 dt\right)^{\frac{1}{2}} &\lesssim \sum_{q'\geq q-3} \left(\int_0^T \|\Delta_{q'}^h \partial_x u_\phi\|_{L^\infty_x(L^2_x)}^2 \|e^{Rt}\tilde{\Delta}_{q'}^h \partial_z^2 u_\phi\|_{L^2_x(L^\infty_x)}^2 dt\right)^{\frac{1}{2}} \\ &\lesssim \tilde{d}_q 2^{-\frac{3q}{2}} \|\partial_x u_\phi\|_{\tilde{L}_T^\infty(B^{\frac{5}{2}})} \|e^{Rt}\partial_z^2 u_\phi\|_{\tilde{L}_T^2(B^{\frac{1}{2}})}. \end{aligned}$$

Then by summing up the previous estimates, we get

$$\begin{aligned} \left(\int_0^T \|e^{Rt}\Delta_q^h(\partial_x u \partial_z^2 u)_\phi\|_{L^2}^2 dt\right)^{\frac{1}{2}} &\lesssim 2^{-\frac{3q}{2}} \left(\tilde{d}_{q,2} \|\partial_x u_\phi\|_{\tilde{L}_T^\infty(B^{\frac{5}{2}})} \|e^{Rt}\partial_z^2 u_\phi\|_{\tilde{L}_T^2(B^{\frac{1}{2}})} + (\tilde{d}_q + \check{d}_q) \|\partial_x u_\phi\|_{\tilde{L}_T^\infty(B^{\frac{1}{2}})} \|e^{Rt}\partial_z^2 u_\phi\|_{\tilde{L}_T^2(B^{\frac{5}{2}})}\right). \end{aligned}$$

Multiplying the previous inequality by $2^{\frac{3q}{2}}$ and taking the sum over \mathbb{Z} , we obtain (2.34). \square

2.5 Appendix - Proof of the lemma 2.2.3

In this Appendix, we give a brief proof of estimates used to prove the uniqueness of the solution.

- Estimate of $I_{1,q}$. From Lemma 3.1 in [64] we obtain (2.37).
- Estimate of $I_{2,q}$. From Lemma 3.1 in [64] we have

$$\left|\int_0^t \left\langle e^{Rt'} \Delta_q^h(u_1 \partial_x U)_\Phi, e^{Rt'} \Delta_q^h U_\Phi \right\rangle dt'\right| \leq C 2^{-2qs} d_q^2 \|e^{Rt'} U_\Phi\|_{\tilde{L}_{T,\Theta'(t)}^2(B^{s+\frac{1}{2}})}^2.$$

By using integration by parts, and from Lemma 3.1 in [64] we have

$$\begin{aligned} &\left|\int_0^t \left\langle e^{Rt'} \Delta_q^h(u_1 \partial_x \partial_z^2 U)_\Phi, e^{Rt'} \Delta_q^h U_\Phi \right\rangle dt'\right| \\ &\leq \left|\int_0^t \left\langle e^{Rt'} \Delta_q^h(\partial_z u_1 \partial_x \partial_z U)_\Phi, e^{Rt'} \Delta_q^h U_\Phi \right\rangle dt'\right| + \left|\int_0^t \left\langle e^{Rt'} \Delta_q^h(u_1 \partial_x \partial_z U)_\Phi, e^{Rt'} \Delta_q^h \partial_z U_\Phi \right\rangle dt'\right| \\ &\leq C 2^{-2qs} d_q^2 \|e^{Rt'} \partial_z U_\Phi\|_{\tilde{L}_{T,\Theta'(t)}^2(B^{s+\frac{1}{2}})}^2. \end{aligned}$$

By summing up the above estimates we obtain (2.38).

- Estimate of $I_{3,q}$. By integration by parts in z variable and from Lemma 3.1 in [64], we obtain (2.39).
- Estimate of $I_{4,q}$. It follows from the proof of Estimate (5.14) in [64] that

$$\int_0^t \left|\left\langle e^{Rt'} \Delta_q^h(V \partial_z u_2)_\Phi, e^{Rt'} \Delta_q^h U_\Phi \right\rangle\right| dt' \lesssim 2^{-2qs} d_q^2 \|e^{Rt'} \partial_z U_\Phi\|_{\tilde{L}_{T,\Theta'(t)}^2(B^{s+\frac{1}{2}})}^2. \quad (2.72)$$

By integration by parts

$$\begin{aligned}
& \int_0^t \left| \left\langle e^{Rt'} \Delta_q^h (V \partial_z^3 u_2)_\Phi, e^{Rt'} \Delta_q^h U_\Phi \right\rangle \right| dt' \\
& \leq \int_0^t \left| \left\langle e^{Rt'} \Delta_q^h (V \partial_z^2 u_2)_\Phi, e^{Rt'} \Delta_q^h \partial_z U_\Phi \right\rangle \right| dt' + \int_0^t \left| \left\langle e^{Rt'} \Delta_q^h (\partial_x U \partial_z^2 u_2)_\Phi, e^{Rt'} \Delta_q^h U_\Phi \right\rangle \right| dt' \\
& \leq \int_0^t \left| \left\langle e^{Rt'} \Delta_q^h (V \partial_z^2 u_2)_\Phi, e^{Rt'} \Delta_q^h \partial_z U_\Phi \right\rangle \right| dt' + \int_0^t \left| \left\langle e^{Rt'} \Delta_q^h (\partial_z u_2 \partial_x U)_\Phi, e^{Rt'} \Delta_q^h \partial_z U_\Phi \right\rangle \right| dt' \\
& \quad + \int_0^t \left| \left\langle e^{Rt'} \Delta_q^h (\partial_z u_2 \partial_x \partial_z U)_\Phi, e^{Rt'} \Delta_q^h U_\Phi \right\rangle \right| dt'.
\end{aligned}$$

Following the prove of Lemma 3.2 in [64] we deduce for any $s \in]0, 1[$ that

$$\int_0^t \left| \left\langle e^{Rt'} \Delta_q^h (V \partial_z^2 u_2)_\Phi, e^{Rt'} \Delta_q^h \partial_z U_\Phi \right\rangle \right| dt' \lesssim 2^{-2qs} d_q^2 \|e^{Rt'} \partial_z U_\Phi\|_{\tilde{L}_{T, \Theta'(t)}^2(B^{s+\frac{1}{2}})}^2.$$

From Lemma 3.1 in [64] we deduce that

$$\begin{aligned}
& \int_0^t \left| \left\langle e^{Rt'} \Delta_q^h (\partial_z u_2 \partial_x U)_\Phi, e^{Rt'} \Delta_q^h \partial_z U_\Phi \right\rangle \right| dt' + \int_0^t \left| \left\langle e^{Rt'} \Delta_q^h (\partial_z u_2 \partial_x \partial_z U)_\Phi, e^{Rt'} \Delta_q^h U_\Phi \right\rangle \right| dt' \\
& \lesssim 2^{-2qs} d_q^2 \|e^{Rt'} \partial_z U_\Phi\|_{\tilde{L}_{T, \Theta'(t)}^2(B^{s+\frac{1}{2}})}^2.
\end{aligned}$$

Then

$$\int_0^t \left| \left\langle e^{Rt'} \Delta_q^h (V \partial_z^3 u_2)_\Phi, e^{Rt'} \Delta_q^h U_\Phi \right\rangle \right| dt' \lesssim 2^{-2qs} d_q^2 \|e^{Rt'} \partial_z U_\Phi\|_{\tilde{L}_{T, \Theta'(t)}^2(B^{s+\frac{1}{2}})}^2. \quad (2.73)$$

By summing the estimates (2.72) and (2.73) we obtain (2.40).

• Estimate of $I_{5,q}$. Following the proof of Estimate (5.13) in [64], we deduce that

$$\begin{aligned}
& \int_0^t \left| \left\langle e^{Rt'} \Delta_q^h (v_1 \partial_z U)_\Phi, e^{Rt'} \Delta_q^h U_\Phi \right\rangle \right| dt' \\
& \leq C 2^{-2qs} d_q^2 \|u_{1\Phi}\|_{L^\infty(B^{\frac{3}{2}})}^{\frac{1}{2}} \|e^{Rt'} \partial_z U_\Phi\|_{\tilde{L}_T^2(B^{\frac{1}{2}})} \|e^{Rt'} \partial_z U_\Phi\|_{\tilde{L}_{T, \Theta'(t)}^2(B^{s+\frac{1}{2}})}^2.
\end{aligned}$$

By integration by parts in z variable, we have

$$\begin{aligned}
& \left| \int_0^t \left\langle e^{Rt'} \Delta_q^h (v_1 \partial_z^3 U)_\Phi, e^{Rt'} \Delta_q^h U_\Phi \right\rangle dt' \right| \\
& \leq \left| \int_0^t \left\langle e^{Rt'} \Delta_q^h (v_1 \partial_z^2 U)_\Phi, e^{Rt'} \Delta_q^h \partial_z U_\Phi \right\rangle dt' \right| + \left| \int_0^t \left\langle e^{Rt'} \Delta_q^h (\partial_x u_1 \partial_z^2 U)_\Phi, e^{Rt'} \Delta_q^h U_\Phi \right\rangle dt' \right| \\
& \leq C 2^{-2qs} d_q^2 \|u_{1\Phi}\|_{L^\infty(B^{\frac{3}{2}})}^{\frac{1}{2}} \|e^{Rt'} \partial_z^2 U_\Phi\|_{\tilde{L}_T^2(B^{\frac{1}{2}})} \|e^{Rt'} \partial_z U_\Phi\|_{\tilde{L}_{T, \Theta'(t)}^2(B^{s+\frac{1}{2}})}^2.
\end{aligned}$$

This conclude the proof of (2.41).

• Estimate of $I_{6,q}$. Following the proof of Estimate (5.13) in [64], we get (2.42).

• Estimate of $I_{7,q}$. Following the proof of Estimate (5.11) in [64], we obtain (2.43).

• Estimate of $I_{8,q}$. Following the proof of Estimate (5.11) in [64], we deduce that

$$\begin{aligned}
& \int_0^t \left| \left\langle e^{Rt'} \Delta_q^h (U \partial_x u_2)_\Phi, e^{Rt'} \Delta_q^h U_\Phi \right\rangle \right| dt' \\
& \lesssim 2^{-2qs} d_q^2 \|e^{Rt'} U_\Phi\|_{\tilde{L}_{T, \Theta'(t)}^2(B^{s+\frac{1}{2}})} \left(\|e^{Rt'} U_\Phi\|_{\tilde{L}_{T, \Theta'(t)}^2(B^{s+\frac{1}{2}})} + \|u_{2\Theta}\|_{L^\infty(B^{\frac{3}{2}})}^{\frac{1}{2}} \|e^{Rt'} \partial_z U_\Phi\|_{\tilde{L}_T^2(B^{s+\frac{1}{2}})} \right).
\end{aligned}$$

By integration by parts

$$\begin{aligned}
& \int_0^t \left| \left\langle e^{Rt'} \Delta_q^h (U \partial_x \partial_z^2 u_2)_\Phi, e^{Rt'} \Delta_q^h U_\Phi \right\rangle \right| dt' \\
& \leq \int_0^t \left| \left\langle e^{Rt'} \Delta_q^h (U \partial_x \partial_z u_2)_\Phi, e^{Rt'} \Delta_q^h \partial_z U_\Phi \right\rangle \right| dt' + \int_0^t \left| \left\langle e^{Rt'} \Delta_q^h (\partial_z U \partial_x \partial_z u_2)_\Phi, e^{Rt'} \Delta_q^h U_\Phi \right\rangle \right| dt' \\
& \lesssim 2^{-2qs} d_q^2 \|e^{Rt'} \partial_z U_\Phi\|_{\tilde{L}_{T, \Theta'(t)}^2(B^{s+\frac{1}{2}})} \\
& \quad \times \left(\|e^{Rt'} \partial_z U_\Phi\|_{\tilde{L}_{T, \Theta'(t)}^2(B^{s+\frac{1}{2}})} + \|\partial_z u_{2\Theta}\|_{L^\infty(B^{\frac{3}{2}})}^{\frac{1}{2}} \|e^{Rt'} \partial_z U_\Phi\|_{\tilde{L}_T^2(B^{s+\frac{1}{2}})} \right).
\end{aligned}$$

Then by summing up the previous estimates, we get (2.44).

2.6 Appendix - additional scaling computations

We write

$$\begin{aligned}
u_1(t, x, y) &= u\left(t, x, \frac{y}{\varepsilon}\right) \\
u_2(t, x, y) &= \varepsilon v\left(t, x, \frac{y}{\varepsilon}\right) \\
q_1(t, x, y) &= q\left(t, x, \frac{y}{\varepsilon}\right)
\end{aligned}$$

Then the derivative of u with respect to time

$$\begin{aligned}
\partial_t u_1(t, x_1, x_2) &= \partial_t u(t, x, z) \\
\partial_t u_2(t, x_1, x_2) &= \varepsilon \partial_t v(t, x, z)
\end{aligned}$$

The derivatives of the first component u_1

$$\begin{aligned}
\partial_x u_1(t, x, y) &= \partial_x u(t, x, z) \\
\partial_y u_1(t, x, y) &= \varepsilon^{-1} \partial_z u(t, x, z) \\
\partial_{xy} u_1(t, x, y) &= \varepsilon^{-1} \partial_{xz}^2 u(t, x, z) \\
\partial_y^2 u_1(t, x, y) &= \varepsilon^{-2} \partial_z^2 u(t, x, z) \\
\partial_x^2 u_1(t, x, y) &= \partial_x^2 u(t, x, z)
\end{aligned}$$

The derivatives of the second component u_2

$$\partial_x u_2(t, x, y) = \varepsilon \partial_x v(t, x, z)$$

$$\partial_{yx}^2 u_2(t, x, y) = \partial_{zx}^2 v(t, x, z)$$

$$\partial_y u_2(t, x, y) = \partial_z v(t, x, z)$$

$$\partial_y^2 u_2(t, x, y) = \varepsilon^{-1} \partial_z^2 v(t, x, z)$$

$$\partial_x^2 u_2(t, x, y) = \varepsilon \partial_x^2 v(t, x, z)$$

The derivatives of the pressure q

$$\partial_x q_1(t, x, y) = \partial_x q(t, x, z)$$

$$\partial_y q_1(t, x, y) = \varepsilon^{-1} \partial_z q(t, x, z)$$

Definition of v_1

$$\begin{aligned} v_1(t, x, y) &= u_1(t, x, y) - \alpha^2 \partial_x^2 u_1(t, x, y) - \alpha^2 \partial_y^2 u_1(t, x, y) \\ &= u(t, x, z) - \alpha^2 \partial_x^2 u(t, x, z) - \frac{\alpha^2}{\varepsilon^2} \partial_z^2 u \end{aligned}$$

Definition of v_2

$$\begin{aligned} v_2(t, x, y) &= u_2(t, x, y) - \alpha^2 \partial_x^2 u_2(t, x, y) - \alpha^2 \partial_y^2 u_2(t, x, y) \\ &= \varepsilon v(t, x, z) - \alpha^2 \varepsilon \partial_x^2 v(t, x, z) - \frac{\alpha^2}{\varepsilon} \partial_z^2 v \end{aligned}$$

We set $\alpha_1 = \frac{\alpha}{\varepsilon}$ then $\alpha^2 = \alpha_1^2 \varepsilon^2$. Then we have

$$v_1 = u(t, x, z) - \alpha_1^2 \partial_z^2 u - \alpha_1^2 \varepsilon^2 \partial_x^2 u(t, x, z), \quad v_2 = \varepsilon(v(t, x, z) - \alpha_1^2 \partial_z^2 v) - \alpha_1^2 \varepsilon^3 \partial_x^2 v(t, x, z)$$

The derivatives of v_1

$$\partial_t v_1 = \partial_t(u - \alpha_1^2 \partial_z^2 u) - \alpha_1^2 \varepsilon^2 \partial_t \partial_x^2 u$$

$$\partial_x v_1 = \partial_x(u - \alpha_1^2 \partial_z^2 u) - \alpha_1^2 \varepsilon^2 \partial_x^3 u$$

$$\partial_x^2 v_1 = \partial_x^2(u - \alpha_1^2 \partial_z^2 u) - \alpha_1^2 \varepsilon^2 \partial_x^4 u$$

$$\partial_y v_1 = \varepsilon^{-1} \partial_z(u - \alpha_1^2 \partial_z^2 u) - \alpha_1^2 \varepsilon \partial_z \partial_x^2 u$$

$$\partial_y^2 v_1 = \varepsilon^{-2} \partial_z^2(u - \alpha_1^2 \partial_z^2 u) - \alpha_1^2 \partial_z^2 \partial_x^2 u$$

The derivatives of v_2

$$\partial_t v_2 = \varepsilon \partial_t (v - \alpha_1^2 \partial_z^2 v) - \alpha_1^2 \varepsilon^3 \partial_t \partial_x^2 v$$

$$\partial_x v_2 = \varepsilon \partial_x (v - \alpha_1^2 \partial_z^2 v) - \alpha_1^2 \varepsilon^3 \partial_x^3 v$$

$$\partial_x^2 v_2 = \varepsilon \partial_x^2 (v - \alpha_1^2 \partial_z^2 v) - \alpha_1^2 \varepsilon^3 \partial_x^4 v$$

$$\partial_y v_2 = \partial_z (v - \alpha_1^2 \partial_z^2 v) - \alpha_1^2 \varepsilon^2 \partial_z \partial_x^2 v$$

$$\partial_y^2 v_2 = \varepsilon^{-1} \partial_z^2 (v - \alpha_1^2 \partial_z^2 v) - \alpha_1^2 \varepsilon \partial_z^2 \partial_x^2 v$$

The Laplacian of v_1

$$\begin{aligned} \varepsilon^2 \Delta v_1 &= \varepsilon^2 \partial_x^2 v_1 + \varepsilon^2 \partial_y^2 v_1 \\ &= \varepsilon^2 (\partial_x^2 (u - \alpha_1^2 \partial_z^2 u) - \alpha_1^2 \varepsilon^2 \partial_x^4 u) + \varepsilon^2 (\varepsilon^{-2} \partial_z^2 (u - \alpha_1^2 \partial_z^2 u) - \alpha_1^2 \partial_z^2 \partial_x^2 u) \\ &= \partial_z^2 (u - \alpha_1^2 \partial_z^2 u) + \varepsilon^2 \partial_x^2 (u - \alpha_1^2 \partial_z^2 u) - \varepsilon^2 \alpha_1^2 \partial_z^2 \partial_x^2 u - \alpha_1^2 \varepsilon^4 \partial_x^4 u. \end{aligned}$$

The Laplacian of v_2

$$\begin{aligned} \varepsilon^2 \Delta v_2 &= \varepsilon^2 \partial_x^2 v_2 + \varepsilon^2 \partial_y^2 v_2 \\ &= \varepsilon^2 (\varepsilon \partial_x^2 (v - \alpha_1^2 \partial_z^2 v) - \alpha_1^2 \varepsilon^3 \partial_x^4 v) + \varepsilon^2 (\varepsilon^{-1} \partial_z^2 (v - \alpha_1^2 \partial_z^2 v) - \alpha_1^2 \varepsilon \partial_z^2 \partial_x^2 v) \\ &= \varepsilon \partial_z^2 (v - \alpha_1^2 \partial_z^2 v) + \varepsilon^3 \partial_x^2 (v - \alpha_1^2 \partial_z^2 v) - \alpha_1^2 \varepsilon^3 \partial_z^2 \partial_x^2 v - \alpha_1^2 \varepsilon^5 \partial_x^4 v \end{aligned}$$

Computation of non-linear terms (First equation)

$$u_1 \partial_x v_1 = u \partial_x (u - \alpha_1^2 \partial_z^2 u) - \alpha_1^2 \varepsilon^2 u \partial_x^3 u$$

$$\begin{aligned} u_2 \partial_y v_1 &= \varepsilon v (\varepsilon^{-1} \partial_z (u - \alpha_1^2 \partial_z^2 u) - \alpha_1^2 \varepsilon \partial_z \partial_x^2 u) \\ &= v \partial_z (u - \alpha_1^2 \partial_z^2 u) - \alpha_1^2 \varepsilon^2 v \partial_z \partial_x^2 u \end{aligned}$$

$$v_1 \partial_x u_1 = (u - \alpha_1^2 \partial_z^2 u) \partial_x u - \alpha_1^2 \varepsilon^2 \partial_x u \partial_x^2 u,$$

$$v_2 \partial_x u_2 = \varepsilon^2 (v - \alpha_1^2 \partial_z^2 v) \partial_x v - \varepsilon^4 \alpha_1^2 \partial_x^2 v(t, x, z) \partial_x v$$

Computation of non-linear terms (Second equation)

$$\begin{aligned} u_1 \partial_x v_2 &= u (\varepsilon \partial_x (v - \alpha_1^2 \partial_z^2 v) - \alpha_1^2 \varepsilon^3 \partial_x^3 v) \\ &= \varepsilon u \partial_x (v - \alpha_1^2 \partial_z^2 v) - \alpha_1^2 \varepsilon^3 u \partial_x^3 v \end{aligned}$$

$$u_2 \partial_y v_2 = \varepsilon v \partial_z (v - \alpha_1^2 \partial_z^2 v) - \alpha_1^2 \varepsilon^3 v \partial_z \partial_x^2 v$$

$$\begin{aligned} v_1 \partial_y u_1 &= ((u - \alpha_1^2 \partial_z^2 u) - \alpha_1^2 \varepsilon^2 \partial_x^2 u) \varepsilon^{-1} \partial_z u \\ &= \varepsilon^{-1} (u - \alpha_1^2 \partial_z^2 u) \partial_z u - \alpha_1^2 \varepsilon \partial_x^2 u \partial_z u \end{aligned}$$

$$v_2 \partial_y u_2 = \varepsilon (v - \alpha_1^2 \partial_z^2 v) \partial_z v - \alpha_1^2 \varepsilon^3 \partial_x^2 v \partial_z v$$

Incompressibility and boundary conditions. The divergence-free property of the velocity field is rewritten as follows

$$\partial_x u_1 + \partial_y u_2 = 0$$

then

$$\partial_x u + \partial_z v = 0. \quad (2.74)$$

We plug this terms in the system (3.2) and we get the system (3.3)

$$\begin{cases} \partial_t \omega - \alpha_1^2 \varepsilon^2 \partial_t \partial_x^2 u + u \partial_x \omega_1 - \alpha_1^2 \varepsilon^2 u \partial_x^3 u + v \partial_z \omega - \alpha_1^2 \varepsilon^2 v \partial_z \partial_x^2 u + \omega \partial_x u - \alpha_1^2 \varepsilon^2 \partial_x u \partial_x^2 u \\ + \varepsilon^2 w \partial_x v - \varepsilon^4 \alpha_1^2 \partial_x^2 v \partial_x v = \partial_z^2 \omega + \varepsilon^2 \partial_x^2 \omega - \varepsilon^2 \alpha_1^2 \partial_z^2 \partial_x^2 u - \alpha_1^2 \varepsilon^4 \partial_x^4 u - \partial_x q. \\ \varepsilon^2 \partial_t w - \alpha_1^2 \varepsilon^4 \partial_t \partial_x^2 v + \varepsilon^2 u \partial_x w - \alpha_1^2 \varepsilon^4 u \partial_x^3 v + \varepsilon^2 v \partial_z w - \alpha_1^2 \varepsilon^4 v \partial_z \partial_x^2 v + \omega \partial_z u - \alpha_1^2 \varepsilon^2 \partial_x^2 u \partial_z u \\ + \varepsilon^2 w \partial_z v - \alpha_1^2 \varepsilon^4 \partial_x^2 v \partial_z v = \varepsilon^2 \partial_z^2 w + \varepsilon^4 \partial_x^2 w - \alpha_1^2 \varepsilon^4 \partial_z^2 \partial_x^2 v - \alpha_1^2 \varepsilon^6 \partial_x^4 v - \partial_z q \\ \partial_x u + \partial_z v = 0. \end{cases}$$

$$\text{where } \omega = u - \alpha_1^2 \partial_z^2 u, \quad w = v - \alpha_1^2 \partial_z^2 v.$$

Chapter 3

Approximation of (NS- α) model

3.1 Introduction

The energy in a turbulent fluid flow cascades down to smaller and smaller scales until it reaches the dissipation scale, where it can be transformed into heat. This cascade, which creates fluid movements on smaller and smaller scales, is a characteristic of turbulence. This characteristic is also the main difficulty in the numerical simulation of turbulence. Indeed, all numerical simulations have a finite resolution and will eventually be unable to follow the cascade to the dissipation scale. The effects of fluid motion at the dissipation scale are not always easy to understand, especially for complex flows, for example near walls and interfaces. The effects of fluid motion at the sub-grid scale, which occur below the available resolution of numerical simulations, must be modelled.

There are several numerical models have been proposed for modeling turbulence (RANS, LES). The common difficulty with these models is the closure problem, i.e., how to model the effect of small eddies on the large ones. In order to overcome this difficulty Joseph Valentin Boussinesq introduced the turbulent eddy viscosity and proposed to relate the turbulence constraints to the mean flow. The Navier-Stokes alpha (NS- α) model provide an alternative way to reach closure by modifying the nonlinearity of the Navier-Stokes equation, instead of its dissipation. In the next paragraph we recall the history of this model

In 1993, Camassa and Holm proposed a new equation for the study of fluid motion in shallow water [10]. These authors consider the Euler equation in a domain with a fixed lower edge fixed plane, and an upper edge free surface extending to infinity in the horizontal directions. In this equation a particular form of solution is introduced, form motivated by the usual approximation of motion in shallow water. This leads to the Green-Nahdi equations, which are Hamiltonian. Further asymptotic expansions can be made in this equation; they lead, for example, to the Korteweg de Vries equation, or to the Benjamin-Bona-Mahony equation. The new idea of Camassa and Holm was to make an asymptotic expansion in the Hamiltonian of the Green-Naghdi equation, instead of doing it in the equation itself. Then, the equation is recovered from the new Hamiltonian. They obtain so the following model for the motion in shallow water in one dimension :

$$\partial_t u + 2\kappa \partial_x u + 3u \partial_x u - \alpha^2 \partial_t \partial_x^2 u = 2\alpha^2 \partial_x u \partial_x^2 u + \alpha^2 u \partial_x^3 u,$$

where the constant κ and α have units of speed and length. The same authors and Kouranbaeva [44] note that for $\kappa = 0$, the previous equation describes the motion along the geodesics of the group of diffeomorphisms associated to the metric given by the norm $\|u\|_{H_x^1} = \int (u^2 + \alpha^2 \partial_x u^2) dx$. From this remark Holm, Marsden and Ratiu [42] generalize to the higher dimensional case and the incompressible case and the resulting system is the following α -Euler equation :

$$\begin{cases} \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{v} + v_1 \nabla u_1 + v_2 \nabla u_2 = -\nabla q \\ \nabla \cdot \mathbf{u} = 0, \end{cases}$$

where $\mathbf{v} = \mathbf{u} - \alpha^2 \Delta \mathbf{u}$ and $q = \pi - \frac{1}{2}|u|^2 - \frac{\alpha^2}{2}|\nabla u|^2$ is the modified pressure, while π is the pressure.

Chen et al [24, 23], adding the viscosity term $\nu \Delta v$ into the previous α -Euler equation, proposed the resulting Navier Stokes- α model as a closure approximation for the Reynolds-averaged equations of the incompressible Navier-Stokes fluid :

$$\begin{cases} \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{v} + v_1 \nabla u_1 + v_2 \nabla u_2 = \nu \Delta \mathbf{v} - \nabla q \\ \nabla \cdot \mathbf{u} = 0, \end{cases} \quad (3.1)$$

where

$$\mathbf{v} = \mathbf{u} - \alpha^2 \Delta \mathbf{u} \quad \text{and} \quad q = \pi - \frac{1}{2}|u|^2 - \frac{\alpha^2}{2}|\nabla u|^2.$$

They tested this approximation on turbulent channel and pipe flows with a steady mean, by finding analytical solutions of the Navier Stokes - α (NS- α) model for the mean velocity and Reynolds equation and comparing them with experiments [9]. They found that the steady NS- α profiles are consistent with data obtained from turbulence measurements in most of the flow region for channels and pipes at moderate to high Reynolds numbers.

In this paper we consider the Navier stokes- α system (3.1) in a thin strip

$$\mathcal{S}^\varepsilon = \{(x, y) \in \mathbb{R}^2, 0 < y < \varepsilon\}$$

equipped with no-slip boundary conditions :

$$\begin{cases} \partial_t v_1^\varepsilon + u_1^\varepsilon \partial_x v_1^\varepsilon + u_2^\varepsilon \partial_y v_1^\varepsilon + v_1^\varepsilon \partial_x u_1^\varepsilon + v_2^\varepsilon \partial_x u_2^\varepsilon = \nu \Delta v_1^\varepsilon - \partial_x q^\varepsilon \\ \partial_t v_2^\varepsilon + u_1^\varepsilon \partial_x v_2^\varepsilon + u_2^\varepsilon \partial_y v_2^\varepsilon + v_1^\varepsilon \partial_y u_1^\varepsilon + v_2^\varepsilon \partial_y u_2^\varepsilon = \nu \Delta v_2^\varepsilon - \partial_y q^\varepsilon \\ \partial_x u_1^\varepsilon + \partial_y u_2^\varepsilon = 0 \\ u_1^\varepsilon = u_2^\varepsilon = 0 \quad \text{on} \quad \{y = 0\} \cup \{y = \varepsilon\} \\ \partial_z u_1^\varepsilon = \partial_z u_2^\varepsilon = 0 \quad \text{on} \quad \{y = 0\} \cup \{y = \varepsilon\} \\ (u_1^\varepsilon, u_2^\varepsilon)|_{t=0} = (u_{1,0}, u_{2,0})(x, y), \end{cases} \quad (3.2)$$

where, for $i = 1, 2$,

$$v_i^\varepsilon = u_i^\varepsilon - \alpha^2 \Delta u_i^\varepsilon.$$

We study the behavior of the previous system for small viscosity $\nu = \varepsilon^2$.

In [29], Foias, Holm and Titi prove the global existence and uniqueness of the solution of (3.2) in a periodic domain for H^1 -initial data and the convergence of the solution of 3D Camassa-Holm equation (NS- α) towards a weak solution of 3D NS equations when α tends to zero and in [9], Busuioc gives a simple proof of the global existence and uniqueness of (3.2) for H_0^1 -initial data in bounded domains with Dirichlet boundary conditions.

3.2 Rescaling of the system and main results

We consider the following rescaling

$$\begin{aligned} u_1^\varepsilon(t, x, y) &= u^\varepsilon \left(t, x, \frac{y}{\varepsilon} \right), \\ u_2^\varepsilon(t, x, y) &= \varepsilon v^\varepsilon \left(t, x, \frac{y}{\varepsilon} \right), \\ q^\varepsilon(t, x, y) &= \tilde{q}^\varepsilon \left(t, x, \frac{y}{\varepsilon} \right), \end{aligned}$$

we perform the change of variable $z = \frac{y}{\varepsilon}$ and we set

$$\mathcal{S} = \{(x, z) \in \mathbb{R}^2, 0 < z < 1\}.$$

We remark that for any function $\varphi(t, x, y) = \psi(t, x, \frac{y}{\varepsilon})$, we have

$$\partial_y^{(k)}(\varphi(t, x, y)) = \varepsilon^{-k} \partial_z^{(k)} \psi(t, x, z).$$

Then the following calculations are immediate

$$\begin{cases} v_1^\varepsilon(t, x, y) = u^\varepsilon(t, x, z) - \alpha_1^2 \partial_z^2 u^\varepsilon(t, x, z) - \varepsilon^2 \alpha_1^2 \partial_x^2 u^\varepsilon(t, x, z) \\ v_2^\varepsilon(t, x, y) = \varepsilon(v^\varepsilon(t, x, z) - \alpha_1^2 \partial_z^2 v^\varepsilon(t, x, z)) - \varepsilon^3 \alpha_1^2 \partial_x^2 v^\varepsilon(t, x, z), \end{cases}$$

where $\alpha_1 = \frac{\alpha}{\varepsilon}$. The system (3.2) becomes

$$\begin{cases} \partial_t \omega^\varepsilon - \alpha_1^2 \varepsilon^2 \partial_t \partial_x^2 u^\varepsilon + u^\varepsilon \partial_x \omega^\varepsilon - \alpha_1^2 \varepsilon^2 u^\varepsilon \partial_x^3 u^\varepsilon + v^\varepsilon \partial_z \omega^\varepsilon - \alpha_1^2 \varepsilon^2 v^\varepsilon \partial_z \partial_x^2 u^\varepsilon + \omega^\varepsilon \partial_x u^\varepsilon - \alpha_1^2 \varepsilon^2 \partial_x u^\varepsilon \partial_x^2 u^\varepsilon \\ \quad + \varepsilon^2 \gamma^\varepsilon \partial_x v^\varepsilon - \varepsilon^4 \alpha_1^2 \partial_x^2 v^\varepsilon \partial_x v^\varepsilon = \partial_z^2 \omega^\varepsilon + \varepsilon^2 \partial_x^2 \omega^\varepsilon - \varepsilon^2 \alpha_1^2 \partial_z^2 \partial_x^2 u^\varepsilon - \alpha_1^2 \varepsilon^4 \partial_x^4 u^\varepsilon - \partial_x \tilde{q}^\varepsilon \\ \varepsilon^2 \partial_t \gamma^\varepsilon - \alpha_1^2 \varepsilon^4 \partial_t \partial_x^2 v^\varepsilon + \varepsilon^2 u^\varepsilon \partial_x \gamma^\varepsilon - \alpha_1^2 \varepsilon^4 u^\varepsilon \partial_x^3 v^\varepsilon + \varepsilon^2 v^\varepsilon \partial_z \gamma^\varepsilon - \alpha_1^2 \varepsilon^4 v^\varepsilon \partial_z \partial_x^2 v^\varepsilon + \omega \partial_z u^\varepsilon - \alpha_1^2 \varepsilon^2 \partial_x^2 u^\varepsilon \partial_z u^\varepsilon \\ \quad + \varepsilon^2 \gamma^\varepsilon \partial_z v^\varepsilon - \alpha_1^2 \varepsilon^4 \partial_x^2 v^\varepsilon \partial_z v^\varepsilon = \varepsilon^2 \partial_z^2 \gamma^\varepsilon + \varepsilon^4 \partial_x^2 \gamma^\varepsilon - \alpha_1^2 \varepsilon^4 \partial_z^2 \partial_x^2 v^\varepsilon - \alpha_1^2 \varepsilon^6 \partial_x^4 v^\varepsilon - \partial_z \tilde{q}^\varepsilon \\ \partial_x u^\varepsilon + \partial_z v^\varepsilon = 0, \end{cases} \quad (3.3)$$

where we denote $\omega^\varepsilon = u^\varepsilon - \alpha_1^2 \partial_z^2 u^\varepsilon$ and $\gamma^\varepsilon = v^\varepsilon - \alpha_1^2 \partial_z^2 v^\varepsilon$ to lighten the notations.

We pose

$$p^\varepsilon = \tilde{q}^\varepsilon + \frac{1}{2}(u^\varepsilon)^2 - \frac{1}{2}\alpha_1^2(\partial_z u^\varepsilon)^2$$

Then the previous system become

$$\begin{cases} \partial_t \omega^\varepsilon - \alpha_1^2 \varepsilon^2 \partial_t \partial_x^2 u^\varepsilon + u^\varepsilon \partial_x \omega^\varepsilon - \alpha_1^2 \varepsilon^2 u^\varepsilon \partial_x^3 u^\varepsilon + v^\varepsilon \partial_z \omega^\varepsilon - \alpha_1^2 \varepsilon^2 v^\varepsilon \partial_z \partial_x^2 u^\varepsilon + \alpha_1^2 (\partial_z u^\varepsilon \partial_x \partial_z u^\varepsilon - \partial_z^2 u^\varepsilon \partial_x u^\varepsilon) \\ \quad - \alpha_1^2 \varepsilon^2 \partial_x u^\varepsilon \partial_x^2 u^\varepsilon + \varepsilon^2 \gamma^\varepsilon \partial_x v^\varepsilon - \varepsilon^4 \alpha_1^2 \partial_x^2 v^\varepsilon \partial_x v^\varepsilon = \partial_z^2 \omega^\varepsilon + \varepsilon^2 \partial_x^2 \omega^\varepsilon - \varepsilon^2 \alpha_1^2 \partial_z^2 \partial_x^2 u^\varepsilon - \alpha_1^2 \varepsilon^4 \partial_x^4 u^\varepsilon - \partial_x p^\varepsilon \\ \varepsilon^2 \partial_t \gamma^\varepsilon - \alpha_1^2 \varepsilon^4 \partial_t \partial_x^2 v^\varepsilon + \varepsilon^2 u^\varepsilon \partial_x \gamma^\varepsilon - \alpha_1^2 \varepsilon^4 u^\varepsilon \partial_x^3 v^\varepsilon + \varepsilon^2 v^\varepsilon \partial_z \gamma^\varepsilon - \alpha_1^2 \varepsilon^4 v^\varepsilon \partial_z \partial_x^2 v^\varepsilon - \alpha_1^2 \varepsilon^2 \partial_x^2 u^\varepsilon \partial_z u^\varepsilon \\ \quad + \varepsilon^2 \gamma^\varepsilon \partial_z v^\varepsilon - \alpha_1^2 \varepsilon^4 \partial_x^2 v^\varepsilon \partial_z v^\varepsilon = \varepsilon^2 \partial_z^2 \gamma^\varepsilon + \varepsilon^4 \partial_x^2 \gamma^\varepsilon - \alpha_1^2 \varepsilon^4 \partial_z^2 \partial_x^2 v^\varepsilon - \alpha_1^2 \varepsilon^6 \partial_x^4 v^\varepsilon - \partial_z p^\varepsilon \\ \partial_x u^\varepsilon + \partial_z v^\varepsilon = 0 \end{cases} \quad (3.4)$$

with the boundary conditions

$$(u^\varepsilon, v^\varepsilon)|_{z=0} = (u^\varepsilon, v^\varepsilon)|_{z=1} = 0 \quad \text{and} \quad (\nabla u^\varepsilon, \nabla v^\varepsilon)|_{z=0} = (\nabla u^\varepsilon, \nabla v^\varepsilon)|_{z=1} = 0. \quad (3.5)$$

We remark that formally the limit when $\varepsilon \rightarrow 0$ of the previous system is the following hydrostatic system

$$\begin{cases} \partial_t \omega + u \partial_x \omega + v \partial_z \omega + \alpha_1^2 (\partial_z u \partial_x \partial_z u - \partial_z^2 u \partial_x u) = \partial_z^2 \omega - \partial_x p \\ \partial_z p = 0 \\ \partial_x u + \partial_y v = 0 \\ (u, v) = 0, \quad \partial_z u = 0 \quad \text{on} \quad z = 0 \\ (u, v) = 0, \quad \partial_z u = 0 \quad \text{on} \quad z = 1 \\ u|_{t=0} = u_0 \end{cases} \quad (3.6)$$

where $\omega = u - \alpha_1^2 \partial_z^2 u$ and $p = \tilde{q} + \frac{1}{2}u^2 - \frac{1}{2}\alpha_1^2(\partial_z u)^2$.

The following theorem is about the global well-posedness of the system (3.4)-(3.5) for small initial analytic data in some Besov spaces (see subsection 2.1.3).

Theorem 3.1. *Let $a > 0$ be fixed. We assume that the initial data satisfies*

$$\|e^{a|D_x|}(u_0^\varepsilon, \varepsilon v_0^\varepsilon)\|_{B^{\frac{1}{2}}} + \|e^{a|D_x|}(\varepsilon u_0^\varepsilon, \varepsilon^2 v_0^\varepsilon)\|_{B^{\frac{3}{2}}} + \|e^{a|D_x|}(\partial_z u_0^\varepsilon, \varepsilon \partial_z v_0^\varepsilon)\|_{B^{\frac{1}{2}}} \leq \tilde{c}a \quad (3.7)$$

For some \tilde{c} small, and for any $0 < s < 1$ if $e^{a|D_x|}(u_0^\varepsilon, v_0^\varepsilon) \in B^s \cap B^{s+1}$, $e^{a|D_x|}(\partial_z u_0^\varepsilon, \partial_z v_0^\varepsilon) \in B^s$, then a unique global solution for the system (3.4)-(3.5) exists and satisfies

$$\begin{aligned} & \|e^{\mathcal{R}t'}(u_\varphi^\varepsilon, \varepsilon v_\varphi^\varepsilon)\|_{\tilde{L}^\infty(\mathbb{R}^+, B^s)} + \|e^{\mathcal{R}t'}(\varepsilon u_\varphi^\varepsilon, \varepsilon^2 v_\varphi^\varepsilon)\|_{\tilde{L}^\infty(\mathbb{R}^+, B^{s+1})} + \|e^{\mathcal{R}t'}(\partial_z u_\varphi^\varepsilon, \varepsilon \partial_z \Delta_k^h v_\varphi^\varepsilon)\|_{\tilde{L}^\infty(\mathbb{R}^+, B^s)} \\ & \quad + \|e^{\mathcal{R}t'}(\varepsilon \partial_x u_\varphi^\varepsilon, \varepsilon^2 \partial_x v_\varphi^\varepsilon)\|_{\tilde{L}^2(\mathbb{R}^+, B^s)} \\ & + \|e^{\mathcal{R}t'}(\partial_z u_\varphi^\varepsilon, \varepsilon \partial_z v_\varphi^\varepsilon)\|_{\tilde{L}^2(\mathbb{R}^+, B^s)} + \|e^{\mathcal{R}t'}(\varepsilon \partial_z \partial_x u_\varphi^\varepsilon, \varepsilon^2 \partial_z \partial_x v_\varphi^\varepsilon)\|_{\tilde{L}^2(\mathbb{R}^+, B^s)} \\ & \quad + \|e^{\mathcal{R}t'}(\varepsilon \partial_x^2 u_\varphi^\varepsilon, \varepsilon^3 \partial_x^2 v_\varphi^\varepsilon)\|_{\tilde{L}^2(\mathbb{R}^+, B^s)} + \|e^{\mathcal{R}t'}(\partial_z^2 u_\varphi^\varepsilon, \varepsilon \partial_z^2 v_\varphi^\varepsilon)\|_{\tilde{L}^2(\mathbb{R}^+, B^s)} \\ & \leq C \left(\|e^{a|D_x|}(u_0^\varepsilon, \varepsilon v_0^\varepsilon)\|_{B^s} + \|e^{a|D_x|}(\varepsilon u_0^\varepsilon, \varepsilon^2 v_0^\varepsilon)\|_{B^{s+1}} + \|e^{a|D_x|}(\partial_z u_0^\varepsilon, \varepsilon \partial_z v_0^\varepsilon)\|_{B^s} \right). \end{aligned} \quad (3.8)$$

The next theorem concerns the convergence of the previous solution to the solution of the hydrostatic system (3.6).

Theorem 3.2. *Let $a > 0$ and $0 < s < \frac{1}{2}$ be fixed and $(u_0^\varepsilon, v_0^\varepsilon)$ satisfy (3.7). Let u_0 satisfy $e^{a|D_x|}u_0 \in B^{s+\frac{1}{2}} \cap B^{s+3}$, $e^{a|D_x|}\partial_z u_0 \in B^{s+\frac{1}{2}} \cap B^{s+3}$, $e^{a|D_x|}\partial_z^2 u_0 \in B^{s+1} \cap B^{s+2}$ and there holds (2.10) for some constant c small, and the compatibility condition $\int_0^1 u_0 dz = 0$. Then we have*

$$\begin{aligned} & \|e^{\mathcal{R}t'}(w_{\varepsilon_\psi}^1, \varepsilon w_{\varepsilon_\psi}^2)\|_{\tilde{L}_T^\infty(B^s)} + \|e^{\mathcal{R}t'}(\varepsilon w_{\varepsilon_\psi}^1, \varepsilon^2 w_{\varepsilon_\psi}^2)\|_{\tilde{L}_T^\infty(B^{s+1})} + \|e^{\mathcal{R}t'}(\partial_z w_{\varepsilon_\psi}^1, \varepsilon \partial_z w_{\varepsilon_\psi}^2)\|_{\tilde{L}_T^\infty(B^s)} \\ & \quad + \|e^{\mathcal{R}t'}(\varepsilon \partial_x w_{\varepsilon_\psi}^1, \varepsilon^2 \partial_x w_{\varepsilon_\psi}^2)\|_{\tilde{L}_T^2(B^s)} + \|e^{\mathcal{R}t'}(\partial_z w_{\varepsilon_\psi}^1, \varepsilon \partial_z w_{\varepsilon_\psi}^2)\|_{\tilde{L}_T^2(B^s)} \\ & \quad + \|e^{\mathcal{R}t'}(\varepsilon \partial_x^2 w_{\varepsilon_\psi}^1, \varepsilon^3 \partial_x^2 w_{\varepsilon_\psi}^1)\|_{\tilde{L}_T^2(B^s)} + \|e^{\mathcal{R}t'}(\partial_z^2 w_{\varepsilon_\psi}^1, \varepsilon \partial_z^2 w_{\varepsilon_\psi}^1)\|_{\tilde{L}_T^2(B^s)} \\ & \quad + \|e^{\mathcal{R}t'}(\varepsilon \partial_z \partial_x w_{\varepsilon_\psi}^1, \varepsilon^2 \partial_z \partial_x w_{\varepsilon_\psi}^2)\|_{\tilde{L}_T^2(B^s)} \\ & \leq C \left[\|(w_{\varepsilon_\psi}^1(0), \varepsilon w_{\varepsilon_\psi}^2(0))\|_{B^s} + \|(\varepsilon w_{\varepsilon_\psi}^1(0), \varepsilon^2 w_{\varepsilon_\psi}^2(0))\|_{B^{s+1}} \right. \\ & \quad \left. + \|(\partial_z w_{\varepsilon_\psi}^1(0), \varepsilon \partial_z w_{\varepsilon_\psi}^2(0))\|_{B^s} + M\varepsilon \right], \end{aligned} \quad (3.9)$$

where $w_\varepsilon^1 = u^\varepsilon - u$, $w_\varepsilon^2 = v^\varepsilon - v$, $(w_{\varepsilon_\psi}^1, w_{\varepsilon_\psi}^2)$ will be given in (3.45) and we have used the following notation

$$\|e^{\mathcal{R}t'}(w_{\varepsilon_\psi}^1, \varepsilon w_{\varepsilon_\psi}^2)\| = \|e^{\mathcal{R}t'} w_{\varepsilon_\psi}^1\| + \varepsilon \|e^{\mathcal{R}t'} w_{\varepsilon_\psi}^2\|.$$

3.3 Fundamental a-priori energy estimates

In this section, we will provide some energy estimates which are needed for the construction of solutions of the system (3.2). To be precised, we will prove the following lemmas.

Lemma 3.3.1. *Let (u_1, u_2, q) be a smooth solution of the system (3.2) on $[0, T]$. Then,*

$$\begin{aligned} & \|u_1^\varepsilon\|_{L_T^\infty(L^2)}^2 + \|u_2^\varepsilon\|_{L_T^\infty(L^2)}^2 + \alpha^2 \left(\|\nabla u_1^\varepsilon\|_{L_T^\infty(L^2)}^2 + \|\nabla u_2^\varepsilon\|_{L_T^\infty(L^2)}^2 \right) \\ & \quad + 2\nu \int_0^T (\|\nabla u_1^\varepsilon\|_{L^2}^2 + \|\nabla u_2^\varepsilon\|_{L^2}^2) ds + \nu \alpha^2 \int_0^T (\|\Delta u_1^\varepsilon\|_{L^2}^2 + \|\Delta u_2^\varepsilon\|_{L^2}^2) ds \\ & \quad = \|u_1^\varepsilon(0)\|_{L^2}^2 + \|u_2^\varepsilon(0)\|_{L^2}^2 + \alpha^2 (\|\nabla u_1^\varepsilon(0)\|_{L^2}^2 + \|\nabla u_2^\varepsilon(0)\|_{L^2}^2). \end{aligned} \quad (3.10)$$

Proof. We take the L^2 -product of the first and second equations of (3.2) respectively with u_1^ε and u_2^ε and then the sum of the two obtained equations. Performing an integration by parts and using the incompressibility condition, we get

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} (\|u_1^\varepsilon\|_{L^2}^2 + \|u_2^\varepsilon\|_{L^2}^2 + \alpha^2 (\|\nabla u_1^\varepsilon\|_{L^2}^2 + \|\nabla u_2^\varepsilon\|_{L^2}^2)) \\ & \quad + \nu (\|\nabla u_1^\varepsilon\|_{L^2}^2 + \|\nabla u_2^\varepsilon\|_{L^2}^2) + \nu \alpha^2 (\|\Delta u_1^\varepsilon\|_{L^2}^2 + \|\Delta u_2^\varepsilon\|_{L^2}^2) = 0. \end{aligned}$$

Estimate (3.10) can be obtained by integrating the above identity with respect to time variable and then taking the supremum over $[0, T]$. \square

Lemma 3.3.2. *Let (u_1, u_2, q) be a smooth solution of the system (3.2) on $[0, T]$. There exists a constant $C > 0$ such that for any $t \in [0, T]$, we have*

$$\|\nabla u_1^\varepsilon\|_{L^2}^2 + \|\nabla u_2^\varepsilon\|_{L^2}^2 + \alpha^2 (\|\Delta u_1^\varepsilon\|_{L^2}^2 + \|\Delta u_2^\varepsilon\|_{L^2}^2) \leq C.$$

Proof. We follow the same calculations as in the proof of Lemma 3.3.1 and the L^2 -product of the first and second equations of (3.2) respectively with $\partial_t u_1^\varepsilon$ and $\partial_t u_2^\varepsilon$. Summing the two obtained equations and integrating by parts, we deduce that

$$\begin{aligned} & \|\partial_t u_1^\varepsilon\|_{L^2}^2 + \|\partial_t u_2^\varepsilon\|_{L^2}^2 + \alpha^2 (\|\nabla(\partial_t u_1^\varepsilon)\|_{L^2}^2 + \|\nabla(\partial_t u_2^\varepsilon)\|_{L^2}^2) \\ & + \nu \frac{1}{2} \frac{d}{dt} (\|\nabla u_1^\varepsilon\|_{L^2}^2 + \|\nabla u_2^\varepsilon\|_{L^2}^2 + \alpha^2 (\|\Delta u_1^\varepsilon\|_{L^2}^2 + \|\Delta u_2^\varepsilon\|_{L^2}^2)) = \sum_{k,l} I_{kl}, \end{aligned} \quad (3.11)$$

where

$$\begin{aligned} I_{11} + I_{12} + I_{13} + I_{14} &= \int v_1^\varepsilon u_1^\varepsilon \partial_x (\partial_t u_1^\varepsilon) dx dz + \int v_1^\varepsilon u_2^\varepsilon \partial_y (\partial_t u_1^\varepsilon) dx dz \\ & + \int v_2^\varepsilon u_1^\varepsilon \partial_x (\partial_t u_2^\varepsilon) dx dz + \int v_2^\varepsilon u_2^\varepsilon \partial_y (\partial_t u_2^\varepsilon) dx dz \end{aligned} \quad (3.12)$$

and

$$\begin{aligned} I_{21} + I_{22} + I_{23} + I_{24} &= \alpha_1^2 \int \partial_x u_2^\varepsilon \Delta u_2^\varepsilon \partial_t u_1^\varepsilon dx dz + \alpha_1^2 \int \partial_y u_1^\varepsilon \Delta u_1^\varepsilon \partial_t u_2^\varepsilon dx dz \\ & + \alpha_1^2 \int \partial_x u_1^\varepsilon v_2^\varepsilon \partial_t u_2^\varepsilon dx dz + \alpha_1^2 \int \partial_y u_2^\varepsilon v_1^\varepsilon \partial_t u_1^\varepsilon dx dz. \end{aligned} \quad (3.13)$$

We will now estimate the four terms I_{kl} in (3.12). From Hölder inequality, we first have

$$|I_{11}| \leq \|u_1^\varepsilon\|_{L^4} \|u_1^\varepsilon\|_{L^4} \|\partial_x(\partial_t u_1^\varepsilon)\|_{L^2} + \alpha^2 \|\Delta u_1^\varepsilon\|_{L^2} \|u_1^\varepsilon\|_{L^\infty} \|\partial_x(\partial_t u_1^\varepsilon)\|_{L^2}.$$

Using Poincaré inequality and the classical Sobolev imbedding theorem, we derive

$$\|u_1^\varepsilon\|_{L^\infty} \leq C \|\Delta u_1^\varepsilon\|_{L^2}$$

and using Gagliardo–Nirenberg inequality, we deduce

$$\|u_1^\varepsilon\|_{L^4} \leq C \|u_1^\varepsilon\|_{L^2}^{\frac{1}{2}} \|\nabla u_1^\varepsilon\|_{L^2}^{\frac{1}{2}} \leq C \|\nabla u_1^\varepsilon\|_{L^2}.$$

Then, Young inequality yields

$$|I_{11}| \leq C \|\nabla u_1^\varepsilon\|_{L^2}^4 + C \alpha^2 \|\Delta u_1^\varepsilon\|_{L^2}^4 + \frac{1}{4} \alpha_1^2 \|\partial_x \partial_t u_1^\varepsilon\|_{L^2}^2.$$

Similar calculations show that,

$$\begin{aligned} |I_{12}| &\leq C \|\nabla u_1^\varepsilon\|_{L^2}^2 \|\nabla u_2^\varepsilon\|_{L^2}^2 + C \alpha^2 \|\Delta u_1^\varepsilon\|_{L^2}^2 \|\Delta u_2^\varepsilon\|_{L^2}^2 + \frac{1}{4} \alpha^2 \|\nabla \partial_t u_1^\varepsilon\|_{L^2}^2, \\ |I_{13}| &\leq C \|\nabla u_1^\varepsilon\|_{L^2}^2 \|\nabla u_2^\varepsilon\|_{L^2}^2 + C \alpha^2 \|\Delta u_1^\varepsilon\|_{L^2}^2 \|\Delta u_2^\varepsilon\|_{L^2}^2 + \frac{1}{4} \alpha^2 \|\nabla \partial_t u_2^\varepsilon\|_{L^2}^2, \\ |I_{14}| &\leq C \|\nabla u_2^\varepsilon\|_{L^2}^4 + C \alpha^2 \|\Delta u_2^\varepsilon\|_{L^2}^4 + \frac{1}{4} \alpha^2 \|\nabla \partial_t u_2^\varepsilon\|_{L^2}^2. \end{aligned}$$

The estimate of the four terms I_{kl} in (3.13) can be obtained using Poincaré, Sobolev and Gagliardo–Nirenberg inequalities and Young inequality. Indeed,

$$|I_{21}| \leq \|\partial_x u_2^\varepsilon\|_{L^4} \|\Delta u_2^\varepsilon\|_{L^2} \|\partial_t u_1^\varepsilon\|_{L^4} \leq C \alpha^2 \|\Delta u_2^\varepsilon\|_{L^2}^4 + \frac{1}{4} \alpha^2 \|\nabla \partial_t u_1^\varepsilon\|_{L^2}^2.$$

In the same way, we get

$$\begin{aligned} |I_{22}| &\leq C\alpha^2 \|\Delta u_1^\varepsilon\|_{L^2}^4 + \frac{1}{4}\alpha^2 \|\nabla \partial_t u_2^\varepsilon\|_{L^2}^2, \\ |I_{23}| &\leq C\alpha^2 \|\Delta u_1^\varepsilon\|_{L^2}^2 \|\Delta u_2^\varepsilon\|_{L^2}^2 + \frac{1}{4}\alpha^2 \|\nabla \partial_t u_1^\varepsilon\|_{L^2}^2, \\ |I_{24}| &\leq C\alpha^2 \|\Delta u_1^\varepsilon\|_{L^2}^2 \|\Delta u_2^\varepsilon\|_{L^2}^2 + \frac{1}{4}\alpha^2 \|\nabla \partial_t u_2^\varepsilon\|_{L^2}^2. \end{aligned}$$

Now, putting the previous estimates of I_{kl} into (3.11), we obtain

$$\begin{aligned} &2 \left(\|\partial_t u_1^\varepsilon\|_{L^2}^2 + \|\partial_t u_2^\varepsilon\|_{L^2}^2 \right) + \alpha^2 \left(\|\nabla(\partial_t u_1^\varepsilon)\|_{L^2}^2 + \|\nabla(\partial_t u_2^\varepsilon)\|_{L^2}^2 \right) \\ &\quad \nu \frac{d}{dt} \left(\|\nabla u_1^\varepsilon\|_{L^2}^2 + \|\nabla u_2^\varepsilon\|_{L^2}^2 + \alpha^2 \left(\|\Delta u_1^\varepsilon\|_{L^2}^2 + \|\Delta u_2^\varepsilon\|_{L^2}^2 \right) \right) \\ &\leq C \left(\|\nabla u_1^\varepsilon\|_{L^2}^2 + \|\nabla u_2^\varepsilon\|_{L^2}^2 + \alpha^2 \left(\|\Delta u_1^\varepsilon\|_{L^2}^2 + \|\Delta u_2^\varepsilon\|_{L^2}^2 \right) \right) \\ &\quad \times \left(\|\nabla u_1^\varepsilon\|_{L^2}^2 + \|\nabla u_2^\varepsilon\|_{L^2}^2 + \alpha^2 \left(\|\Delta u_1^\varepsilon\|_{L^2}^2 + \|\Delta u_2^\varepsilon\|_{L^2}^2 \right) \right). \end{aligned} \quad (3.14)$$

Let

$$\|u\|_{H_0^2}^2 = \|\nabla u_1^\varepsilon\|_{L^2}^2 + \|\nabla u_2^\varepsilon\|_{L^2}^2 + \alpha^2 \left(\|\Delta u_1^\varepsilon\|_{L^2}^2 + \|\Delta u_2^\varepsilon\|_{L^2}^2 \right),$$

then, we can rewrite (3.14) as follows

$$\frac{d}{dt} \|u\|_{H_0^2}^2 \leq C\nu \|u\|_{H_0^2}^2 \times \|u\|_{H_0^2}^2. \quad (3.15)$$

Standard Gronwall lemma implies

$$\|u\|_{H_0^2}^2 \leq \|u(0)\|_{H_0^2}^2 \exp \left(C\nu \int_0^T \|u(s)\|_{H_0^2}^2 ds \right)$$

and then, from Estimate (3.10), we conclude that $\|u(t)\|_{H_0^2} \leq C$, or any $t \in [0, T]$. \square

From the above lemmas, we can also prove that

Lemma 3.3.3. *Let (u_1, u_2, q) be a smooth solution of the system (3.2). Then we have :*

$$\begin{aligned} &2 \int_0^T \left(\|\partial_t u_1^\varepsilon\|_{L^2}^2 + \|\partial_t u_2^\varepsilon\|_{L^2}^2 \right) ds + \alpha^2 \int_0^T \left(\|\nabla(\partial_t u_1^\varepsilon)\|_{L^2}^2 + \|\nabla(\partial_t u_2^\varepsilon)\|_{L^2}^2 \right) ds \\ &\quad \nu \left(\|\nabla u_1^\varepsilon\|_{L_T^\infty(L^2)}^2 + \|\nabla u_2^\varepsilon\|_{L_T^\infty(L^2)}^2 + \alpha^2 \left(\|\Delta u_1^\varepsilon\|_{L_T^\infty(L^2)}^2 + \|\Delta u_2^\varepsilon\|_{L_T^\infty(L^2)}^2 \right) \right) \\ &\leq \nu \left(\|\nabla u_1^\varepsilon(0)\|_{L^2}^2 + \|\nabla u_2^\varepsilon(0)\|_{L^2}^2 + \alpha^2 \left(\|\Delta u_1^\varepsilon(0)\|_{L^2}^2 + \|\Delta u_2^\varepsilon(0)\|_{L^2}^2 \right) \right) + C. \end{aligned} \quad (3.16)$$

Proof. Integrating Estimate (3.14) with respect to time and using Lemmas 3.3.1-3.3.2, we can easily deduce Estimate (3.16). \square

Application : Uniqueness of the smooth solution of the system (3.2).

Proposition 3.3.1. *Let u, \tilde{u} two smooth solutions of (3.2). Then, setting $\delta u = u - \tilde{u}$, we have the following estimate*

$$\frac{d}{dt} \|\delta u\|_{H_0^1}^2 + \|\delta u\|_{H_0^2}^2 \leq C \|\delta u\|_{H_0^1}^2 \left(\|u\|_{H_0^2}^2 + \|\tilde{u}\|_{H_0^2}^2 \right)$$

where

$$\begin{aligned} \|\delta u\|_{H_0^1}^2 &= \|\delta u\|_{L^2}^2 + \alpha^2 \|\nabla \delta u\|_{L^2}^2, \\ \|\delta u\|_{H_0^2}^2 &= \|\nabla \delta u\|_{L^2}^2 + \alpha^2 \|\Delta \delta u\|_{L^2}^2. \end{aligned}$$

Therefore, if $u(0) = \tilde{u}(0)$, then $u = \tilde{u}$ and we obtain the uniqueness of the smooth solution of the system (3.2).

Proof. Let u, \tilde{u} two smooth solutions of (3.2). We denote

$$\begin{aligned} \delta u &= u - \tilde{u}, \quad v = u - \alpha^2 \Delta u, \quad \tilde{v} = \tilde{u} - \alpha^2 \Delta \tilde{u} \\ \delta v &= v - \tilde{v} = \delta u - \alpha^2 \Delta \delta u, \quad \delta q = q - \tilde{q} \end{aligned}$$

then, δu verifies the following system

$$\begin{cases} \frac{\partial}{\partial t} \delta v + (\delta u \cdot \nabla) v + v_1 \nabla \delta u_1 + v_2 \nabla \delta u_2 + (\tilde{u} \cdot \nabla) \delta v + \delta v_1 \nabla \tilde{u}_1 + \delta v_2 \nabla \tilde{u}_2 = \nu \Delta \delta v - \nabla \delta q \\ \nabla \cdot \delta u = 0, \end{cases} \quad (3.17)$$

Taking the L^2 -scalar product of the first equation of (3.17) with δu , we get

$$\int \frac{\partial}{\partial t} \delta v \cdot \delta u + \int (\delta u \cdot \nabla v + v_1 \nabla \delta u_1 + v_2 \nabla \delta u_2) \cdot \delta u + \int (\tilde{u} \cdot \nabla \delta v + \delta v_1 \nabla \tilde{u}_1 + \delta v_2 \nabla \tilde{u}_2) \cdot \delta u - \nu \int \Delta \delta v \cdot \delta u = 0.$$

Performing integration by parts and using boundary conditions, we derive

$$\int \frac{\partial}{\partial t} \delta v \cdot \delta u = \int \partial_t (\delta u - \alpha^2 \Delta \delta u) \cdot \delta u = \frac{1}{2} \frac{d}{dt} \|\delta u\|_{H_0^1}^2,$$

and

$$- \int \Delta \delta v \cdot \delta u = \int \Delta \delta u \cdot \delta u - \alpha^2 \int \Delta \Delta \delta v \cdot \delta u = \|\delta u\|_{H_0^2}^2,$$

Using integration by parts and incompressibility condition, we get

$$\int (\delta u \cdot \nabla v + v_1 \nabla \delta u_1 + v_2 \nabla \delta u_2) \cdot \delta u = 0$$

and

$$\begin{aligned} & \int (\tilde{u} \cdot \nabla \delta v + \delta v_1 \nabla \tilde{u}_1 + \delta v_2 \nabla \tilde{u}_2) \cdot \delta u \\ &= - \int \delta v_1 u_1 \partial_x \delta u_1 - \int \delta v_2 \tilde{u}_2 \partial_y \delta u_2 - \int \delta v_2 \tilde{u}_2 \partial_y \delta u_1 - \int \delta v_2 \tilde{u}_1 \partial_x \delta u_2 \\ & \quad - \int \delta v_2 \partial_y \tilde{u}_2 \delta u_1 - \int \delta v_2 \partial_x \tilde{u}_1 \delta u_2 - \alpha^2 \int \Delta \delta u_2 \partial_x \tilde{u}_2 \delta u_1 - \alpha^2 \int \Delta \delta u_1 \partial_y \tilde{u}_1 \delta u_2 \end{aligned}$$

Then

$$\frac{d}{dt} \|\delta u\|_{H_0^1}^2 + 2 \|\delta u\|_{H_0^2}^2 = -2 \int (\tilde{u} \cdot \nabla \delta v + \delta v_1 \nabla \tilde{u}_1 + \delta v_2 \nabla \tilde{u}_2) \cdot \delta u.$$

Similar to the estimates of the terms $(I_{kl})_{1 \leq k, l \leq 4}$, we derive that

$$\left| \int (\tilde{u} \cdot \nabla \delta v + \delta v_1 \nabla \tilde{u}_1 + \delta v_2 \nabla \tilde{u}_2) \cdot \delta u \right| \leq C \|\delta u\|_{H_0^1}^2 (\|u\|_{H_0^2}^2 + \|\tilde{u}\|_{H_0^2}^2) + \|\delta u\|_{H_0^2}^2$$

Then

$$\frac{d}{dt} \|\delta u\|_{H_0^1}^2 + \|\delta u\|_{H_0^2}^2 \leq C \|\delta u\|_{H_0^1}^2 (\|u\|_{H_0^2}^2 + \|\tilde{u}\|_{H_0^2}^2)$$

Therefore by Gronwall lemma and Estimate (3.10), if $u(0) = \tilde{u}(0)$ then, we obtain $\|\delta u\|_{H_0^1} = 0$, which proves the uniqueness of the smooth solution. \square

3.4 Existence of a weak solution of (3.4) – (3.5)

We construct a weak solution of (3.4)-(3.5) using the Galerkin method. We introduce the following Hilbert spaces

$$\begin{aligned} V_2 &= \left\{ f \in (H^2(\mathbb{R} \times]0, 1[))^2 \mid \nabla \cdot f = 0, \quad f|_{z=0,1} = \nabla f|_{z=0,1} = 0 \right\}, \\ V_1 &= \left\{ f \in (H^1(\mathbb{R} \times]0, 1[))^2 \mid \nabla \cdot f = 0, \quad f|_{z=0,1} = 0 \right\}, \\ H &= \left\{ f \in (L^2(\mathbb{R} \times]0, 1[))^2 \mid \nabla \cdot f = 0, \quad f|_{z=0,1} = 0 \right\} \end{aligned}$$

and we define the respective norms

$$\begin{aligned} \|u\|_{H_0^2} &= \left(\|\partial_z u\|_{L^2}^2 + \varepsilon^2 \|\partial_x u\|_{L^2}^2 + \alpha_1^2 \|\partial_z^2 u\|_{L^2}^2 + \alpha_1^2 \varepsilon^4 \|\partial_x^2 u\|_{L^2}^2 + 2\alpha_1^2 \varepsilon^2 \|\partial_x \partial_z u\|_{L^2}^2 \right)^{\frac{1}{2}}, \\ \|u\|_{H_0^1} &= \left(\|u\|_{L^2}^2 + \alpha_1^2 \|\partial_z u\|_{L^2}^2 + \alpha_1^2 \varepsilon^2 \|\partial_x u\|_{L^2}^2 \right)^{\frac{1}{2}} \end{aligned}$$

$$\begin{aligned} \|f\|_{V_2} &= \|f_1\|_{H_0^2} + \varepsilon^2 \|f_2\|_{H_0^2} \\ \|f\|_{V_1} &= \|f_1\|_{H_0^1} + \varepsilon^2 \|f_2\|_{H_0^1}. \end{aligned}$$

We then consider a common Hilbert basis $\{\tilde{e}_k\}_{k \geq 1}$ such that

$$\forall v \in V_2, \quad \langle \tilde{e}_k, v \rangle_{V_2} = \lambda_k \langle \tilde{e}_k, v \rangle_{V_1}.$$

Proposition 3.4.1. *We consider an initial data $(u_0, v_0) \in V_2$. Then there a solution $(u, v) \in L^\infty(0, \infty; V_1) \cap L^2(0, \infty; V_2)$ which verify the weak formulation of (3.4)-(3.5).*

Proof. We first construct an approximate solution using the Cauchy-Lipschitz theorem and then we pass to the limit.

- *Galerkin method for the proof of the existence of a weak solution of (3.4)-(3.5).* We set

$$\begin{aligned} Q_1(u, v) &= u^\varepsilon \partial_x \omega^\varepsilon - \alpha_1^2 \varepsilon^2 u^\varepsilon \partial_x^3 u^\varepsilon + v^\varepsilon \partial_z \omega^\varepsilon - \alpha_1^2 \varepsilon^2 v^\varepsilon \partial_z \partial_x^2 u^\varepsilon + \alpha_1^2 (\partial_z u^\varepsilon \partial_x \partial_z u^\varepsilon - \partial_z^2 u^\varepsilon \partial_x u^\varepsilon) \\ &\quad - \alpha_1^2 \varepsilon^2 \partial_x u^\varepsilon \partial_x^2 u^\varepsilon + \varepsilon^2 \gamma^\varepsilon \partial_x v^\varepsilon - \varepsilon^4 \alpha_1^2 \partial_x^2 v^\varepsilon \partial_x v^\varepsilon. \end{aligned} \quad (3.18)$$

and

$$\begin{aligned} Q_2(u, v) &= \varepsilon^2 u^\varepsilon \partial_x \gamma^\varepsilon - \alpha_1^2 \varepsilon^4 u^\varepsilon \partial_x^3 v^\varepsilon + \varepsilon^2 v^\varepsilon \partial_z \gamma^\varepsilon - \alpha_1^2 \varepsilon^4 v^\varepsilon \partial_z \partial_x^2 v^\varepsilon - \alpha_1^2 \varepsilon^2 \partial_x^2 u^\varepsilon \partial_z v^\varepsilon \\ &\quad + \varepsilon^2 \gamma^\varepsilon \partial_z v^\varepsilon - \alpha_1^2 \varepsilon^4 \partial_x^2 v^\varepsilon \partial_z v^\varepsilon. \end{aligned} \quad (3.19)$$

For any $n \in \mathbb{N}$, we will look for an approximate solution (u^n, v^n) of (3.4) of the form

$$u_n(t, x, z) = \sum_{i=1}^n \tilde{u}_{in}(t) \tilde{e}_{i1}(x, z),$$

and

$$v_n(t, x, z) = \sum_{i=1}^n \tilde{u}_{in}(t) \tilde{e}_{i2}(x, z),$$

where $(\tilde{u}_{kn})_{1 \leq k \leq n}$ is solution of the following system of n equations

$$\begin{cases} \langle \partial_t \omega_n^\varepsilon, \tilde{e}_{k1} \rangle_{L^2} - \alpha_1^2 \varepsilon^2 \langle \partial_t \partial_x^2 u^\varepsilon, \tilde{e}_{k1} \rangle_{L^2} + \varepsilon^2 \langle \partial_t \gamma_n^\varepsilon, \tilde{e}_{k2} \rangle_{L^2} - \alpha_1^2 \varepsilon^4 \langle \partial_t \partial_x^2 v^\varepsilon, \tilde{e}_{k2} \rangle_{L^2} \\ = \langle \partial_z^2 \omega^\varepsilon + \varepsilon^2 \partial_x^2 \omega^\varepsilon - \varepsilon^2 \alpha_1^2 \partial_z^2 \partial_x^2 u^\varepsilon - \alpha_1^2 \varepsilon^4 \partial_x^4 u^\varepsilon - \partial_x \tilde{q}^\varepsilon, \tilde{e}_{k1} \rangle_{L^2} - \langle Q_1(u^n, v^n), \tilde{e}_{k1} \rangle_{L^2} \\ + \langle \varepsilon^2 \partial_z^2 \gamma^\varepsilon + \varepsilon^4 \partial_x^2 \gamma^\varepsilon - \alpha_1^2 \varepsilon^4 \partial_z^2 \partial_x^2 v^\varepsilon - \alpha_1^2 \varepsilon^6 \partial_x^4 v^\varepsilon - \partial_z p^\varepsilon, \tilde{e}_{k2} \rangle_{L^2} - \langle Q_2(u^n, v^n), \tilde{e}_{k2} \rangle_{L^2}, \end{cases} \quad (3.20)$$

which also means that

$$\begin{cases} \frac{d}{dt} \tilde{u}_{kn}(t) = -\lambda_k \tilde{u}_{kn}(t) - \langle Q_1(u_n, v_n), \tilde{e}_{k1} \rangle_{L^2} - \langle Q_2(u_n, v_n), \tilde{e}_{k2} \rangle_{L^2}, \quad k \in \{1, \dots, n\} \\ \tilde{u}_{kn}|_{t=0} = \langle u_0, \tilde{e}_k \rangle_{V'}. \end{cases} \quad (3.21)$$

Since $\tilde{u}_{kn} \mapsto \langle Q_1(u_n, v_n), \tilde{e}_{k1} \rangle_{L^2} - \langle Q_2(u_n, v_n), \tilde{e}_{k2} \rangle_{L^2}$ is locally Lipschitz in \mathbb{R} , (3.21) is a system of ordinary differential equations. Then, Cauchy-Lipschitz theorem implies the existence of a unique maximal solution $\{\tilde{u}_{kn}(t, x)\}$ of (3.21) on $[0, T_n[$.

We now recall the two fundamental a-priori estimates (3.10) and (3.16) of part 2.2 in the rescaled variables.

Lemma 3.4.1. For $(u^\varepsilon, v^\varepsilon, p^\varepsilon)$ smooth solution to the (3.4)

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} (\|u\|_{L^2}^2 + \alpha_1^2 \|\partial_z u\|_{L^2}^2 + \alpha_1^2 \varepsilon^2 \|\partial_x u\|_{L^2}^2 + \varepsilon^2 \|v\|_{L^2}^2 + \alpha_1^2 \varepsilon^2 \|\partial_z v\|_{L^2}^2 + \alpha_1^2 \varepsilon^4 \|\partial_x v\|_{L^2}^2) \\ & + \|\partial_z u\|_{L^2}^2 + \varepsilon^2 \|\partial_x u\|_{L^2}^2 + \alpha_1^2 \|\partial_z^2 u\|_{L^2}^2 + \alpha_1^2 \varepsilon^4 \|\partial_x^2 u\|_{L^2}^2 + 2\alpha_1^2 \varepsilon^2 \|\partial_x \partial_z u\|_{L^2}^2 \\ & + \varepsilon^2 \|\partial_z v\|_{L^2}^2 + \varepsilon^4 \|\partial_x v\|_{L^2}^2 + \alpha_1^2 \varepsilon^2 \|\partial_z^2 v\|_{L^2}^2 + \alpha_1^2 \varepsilon^6 \|\partial_x^2 v\|_{L^2}^2 + 2\alpha_1^2 \varepsilon^4 \|\partial_x \partial_z v\|_{L^2}^2 = 0 \end{aligned}$$

Lemma 3.4.2. For $(u^\varepsilon, v^\varepsilon, p^\varepsilon)$ smooth solution to the (3.4)

$$\begin{aligned} & 2 \int_0^T (\|\partial_t u^\varepsilon\|_{L^2}^2 + \varepsilon^2 \|\partial_t v^\varepsilon\|_{L^2}^2) ds + \alpha_1^2 \int_0^T (\|\partial_z \partial_t u\|^2 + \varepsilon^2 \|\partial_x \partial_t u\|_{L^2}^2) ds \tag{3.22} \\ & + \alpha_1^2 \int_0^T (\varepsilon^2 \|\partial_z \partial_t v\|_{L^2}^2 + \varepsilon^4 \|\partial_x \partial_t v\|_{L^2}^2) ds + \left(\|\partial_z u\|_{L_T^\infty(L^2)}^2 + \varepsilon^2 \|\partial_x u\|_{L_T^\infty(L^2)}^2 \right) \\ & + \varepsilon^2 \|\partial_z v\|_{L_T^\infty(L^2)}^2 + \varepsilon^4 \|\partial_x v\|_{L_T^\infty(L^2)}^2 + \alpha_1^2 \|\partial_z^2 u\|_{L_T^\infty(L^2)}^2 + \alpha_1^2 \varepsilon^4 \|\partial_x^2 u\|_{L_T^\infty(L^2)}^2 + 2\alpha_1^2 \varepsilon^2 \|\partial_x \partial_z u\|_{L_T^\infty(L^2)}^2 \\ & + \alpha_1^2 \varepsilon^2 \|\partial_z^2 v\|_{L_T^\infty(L^2)}^2 + \alpha_1^2 \varepsilon^6 \|\partial_x^2 v\|_{L_T^\infty(L^2)}^2 + 2\alpha_1^2 \varepsilon^4 \|\partial_x \partial_z v\|_{L_T^\infty(L^2)}^2 \\ & \leq \nu (\|\nabla u_1^\varepsilon(0)\|_{L^2}^2 + \|\nabla u_2^\varepsilon(0)\|_{L^2}^2 + \alpha^2 (\|\Delta u_1^\varepsilon(0)\|_{L^2}^2 + \|\Delta u_2^\varepsilon(0)\|_{L^2}^2)) + C \end{aligned}$$

Following the proof of the two previous lemmas, we can show that the approximate solution u_n verifies the same energy identity and energy estimate of these lemmas: we multiply (3.20) by \tilde{u}_{kn} and we sum up over k , then from (3.4.1) we obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} (\|u_n\|_{L^2}^2 + \alpha_1^2 \|\partial_z u_n\|_{L^2}^2 + \alpha_1^2 \varepsilon^2 \|\partial_x u_n\|_{L^2}^2 + \varepsilon^2 \|v_n\|_{L^2}^2 + \alpha_1^2 \varepsilon^2 \|\partial_z v_n\|_{L^2}^2 + \alpha_1^2 \varepsilon^4 \|\partial_x v_n\|_{L^2}^2) \\ & + \|\partial_z u_n\|_{L^2}^2 + \varepsilon^2 \|\partial_x u_n\|_{L^2}^2 + \alpha_1^2 \|\partial_z^2 u_n\|_{L^2}^2 + \alpha_1^2 \varepsilon^4 \|\partial_x^2 u_n\|_{L^2}^2 + 2\alpha_1^2 \varepsilon^2 \|\partial_x \partial_z u_n\|_{L^2}^2 \\ & + \varepsilon^2 \|\partial_z v_n\|_{L^2}^2 + \varepsilon^4 \|\partial_x v_n\|_{L^2}^2 + \alpha_1^2 \varepsilon^2 \|\partial_z^2 v_n\|_{L^2}^2 + \alpha_1^2 \varepsilon^6 \|\partial_x^2 v_n\|_{L^2}^2 + 2\alpha_1^2 \varepsilon^4 \|\partial_x \partial_z v_n\|_{L^2}^2 = 0 \end{aligned}$$

for any $t \in [0, T_n[$, which means $T_n = +\infty$. Furthermore, multiplying (3.20) by $\partial_t \tilde{u}_{kn}$ and summing up over k , then from (3.4.2) we obtain

$$2 \int_0^T (\|\partial_t u_n^\varepsilon\|_{L^2}^2 + \varepsilon^2 \|\partial_t v_n^\varepsilon\|_{L^2}^2) ds + \alpha_1^2 \int_0^T (\|\partial_z \partial_t u_n\|^2 + \varepsilon^2 \|\partial_x \partial_t u_n\|_{L^2}^2) ds \tag{3.23}$$

$$+ \alpha_1^2 \int_0^T (\varepsilon^2 \|\partial_z \partial_t v_n\|_{L^2}^2 + \varepsilon^4 \|\partial_x \partial_t v_n\|_{L^2}^2) ds + \left(\|\partial_z u_n\|_{L_T^\infty(L^2)}^2 + \varepsilon^2 \|\partial_x u_n\|_{L_T^\infty(L^2)}^2 \right) \tag{3.24}$$

$$\begin{aligned} & + \varepsilon^2 \|\partial_z v_n\|_{L_T^\infty(L^2)}^2 + \varepsilon^4 \|\partial_x v_n\|_{L_T^\infty(L^2)}^2 + \alpha_1^2 \|\partial_z^2 u_n\|_{L_T^\infty(L^2)}^2 + \alpha_1^2 \varepsilon^4 \|\partial_x^2 u_n\|_{L_T^\infty(L^2)}^2 + 2\alpha_1^2 \varepsilon^2 \|\partial_x \partial_z u_n\|_{L_T^\infty(L^2)}^2 \\ & + \alpha_1^2 \varepsilon^2 \|\partial_z^2 v_n\|_{L_T^\infty(L^2)}^2 + \alpha_1^2 \varepsilon^6 \|\partial_x^2 v_n\|_{L_T^\infty(L^2)}^2 + 2\alpha_1^2 \varepsilon^4 \|\partial_x \partial_z v_n\|_{L_T^\infty(L^2)}^2 \end{aligned} \tag{3.25}$$

$$\leq \nu (\|\nabla u_1^\varepsilon(0)\|_{L^2}^2 + \|\nabla u_2^\varepsilon(0)\|_{L^2}^2 + \alpha^2 (\|\Delta u_1^\varepsilon(0)\|_{L^2}^2 + \|\Delta u_2^\varepsilon(0)\|_{L^2}^2)) + C$$

• *Passage to the limit.* We have $u_n, v_n \in L_{loc}^\infty(\mathbb{R}, V_{1,loc})$, and $\partial_t u_n, \partial_t v_n \in L^2(\mathbb{R}, H)$ By Aubin-Lions lemma, the exist a subsequences of u_{n_k}, v_{n_k} respectively of u_n, v_n , such that $u_{n_k} \rightarrow u, v_{n_k} \rightarrow v$ in $L^2(\mathbb{R}, H)$. Then we pass to the limit when n_k goes to infinity and we obtain a solution $(u, v) \in L^\infty(0, \infty; V_1) \cap L^2(0, \infty; V_2)$ which verify the weak formulation of (3.4)-(3.5). \square

3.5 Global well-posedness of system (3.4)–(3.5) in analytic spaces in x direction

The goal of this section is to prove Theorem 3.1. Using the same technique as in the previous chapter, we define

$$f_\varphi^\varepsilon(t, x, z) = e^{\varphi(t, D_x)} f^\varepsilon(t, x, z) := \mathcal{F}_{\xi \rightarrow x}^{-1} (e^{\varphi(t, \xi)} \hat{f}^\varepsilon(t, \xi, z)) \quad \text{with} \quad \varphi(t, \xi) = (a - \lambda\eta(t))|\xi|, \tag{3.26}$$

where the quantity $\eta(t)$ describes the evolution of the analytic band, satisfies

$$\eta'(t) = \varepsilon^2 \|\partial_x v_\varphi^\varepsilon(t)\|_{B^{\frac{1}{2}}} + \varepsilon \|\partial_z \partial_x u_\varphi^\varepsilon(t)\|_{B^{\frac{1}{2}}} + \varepsilon^2 \|\partial_x^2 u_\varphi^\varepsilon(t)\|_{B^{\frac{1}{2}}} + \varepsilon^3 \|\partial_x^2 v_\varphi^\varepsilon(t)\|_{B^{\frac{1}{2}}} + \|\partial_z^2 u_\varphi^\varepsilon(t)\|_{B^{\frac{1}{2}}} \quad \text{and} \quad \eta(0) = 0, \tag{3.27}$$

and where (u, v) is ...

Simple calculations imply that (u_φ, v_φ) satisfies to the following system

$$\left\{ \begin{array}{l} \partial_t \omega_\varphi - \alpha_1^2 \varepsilon^2 \partial_t \partial_x^2 u_\varphi + \lambda \eta'(t) |D_x| (u_\varphi - \alpha_1^2 \partial_z^2 u_\varphi) - \alpha_1^2 \varepsilon^2 \lambda \eta'(t) |D_x| \partial_x^2 u_\varphi + (u \partial_x \omega)_\varphi - \alpha_1^2 \varepsilon^2 (u \partial_x^3 u)_\varphi + (v \partial_z \omega)_\varphi \\ - \alpha_1^2 \varepsilon^2 (v \partial_z \partial_x^2 u)_\varphi + \alpha_1^2 (\partial_z u \partial_x \partial_z u)_\varphi - \alpha_1^2 (\partial_z^2 u \partial_x u)_\varphi - \alpha_1^2 \varepsilon^2 (\partial_x u \partial_x^2 u)_\varphi + \varepsilon^2 (\gamma \partial_x v)_\varphi - \varepsilon^4 \alpha_1^2 (\partial_x^2 v \partial_x v)_\varphi \\ = \partial_z^2 \omega_\varphi + \varepsilon^2 \partial_x^2 \omega_\varphi - \varepsilon^2 \alpha_1^2 \partial_z^2 \partial_x^2 u_\varphi - \alpha_1^2 \varepsilon^4 \partial_x^4 u_\varphi - \partial_x p_\varphi. \\ \\ \varepsilon^2 \partial_t \gamma_\varphi - \alpha_1^2 \varepsilon^4 \partial_t \partial_x^2 v_\varphi + \lambda \varepsilon^2 \eta'(t) |D_x| (v_\varphi - \alpha_1^2 \partial_z^2 v_\varphi) - \alpha_1^2 \varepsilon^4 \lambda \eta'(t) |D_x| \partial_x^2 v_\varphi + \varepsilon^2 (u \partial_x \gamma)_\varphi - \alpha^2 \varepsilon^4 (u \partial_x^3 v)_\varphi \\ + \varepsilon^2 (v \partial_z \gamma)_\varphi - \alpha_1^2 \varepsilon^4 (v \partial_z \partial_x^2 v)_\varphi - \alpha_1^2 \varepsilon^2 (\partial_x^2 u \partial_z u)_\varphi + \varepsilon^2 (\gamma \partial_z v)_\varphi - \alpha_1^2 \varepsilon^4 (\partial_x^2 v \partial_z v)_\varphi \\ = \varepsilon^2 \partial_z^2 \gamma_\varphi + \varepsilon^4 \partial_x^2 \gamma_\varphi - \alpha_1^2 \varepsilon^4 \partial_z^2 \partial_x^2 v_\varphi - \alpha^2 \varepsilon^6 \partial_x^4 v_\varphi - \partial_z p_\varphi \\ \\ \partial_x u_\varphi + \partial_z v_\varphi = 0, \end{array} \right. \quad (3.28)$$

where $\omega_\varphi = u_\varphi - \alpha^2 \partial_z^2 u_\varphi$, $\gamma_\varphi = v_\varphi - \alpha^2 \partial_z^2 v_\varphi$, $|D_x|$ denotes the Fourier multiplier with symbol $|\xi|$. With the boundary conditions

$$(u_\varphi, v_\varphi)|_{z=0} = (u_\varphi, v_\varphi)|_{z=1} = 0 \quad \text{and} \quad (\nabla u_\varphi, \nabla v_\varphi)|_{z=0} = (\nabla u_\varphi, \nabla v_\varphi)|_{z=1} = 0 \quad (3.29)$$

Applying the dyadic cut-off operator Δ_k^h to (3.28) and then taking the L^2 inner product with $(\Delta_k^h u_\varphi, \Delta_k^h v_\varphi)$, we have

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|(\Delta_k^h u_\varphi, \varepsilon \Delta_k^h v_\varphi)\|_{L^2}^2 + \frac{1}{2} \alpha_1^2 \varepsilon^2 \frac{d}{dt} \|(\partial_x \Delta_k^h u_\varphi, \varepsilon \partial_x \Delta_k^h v_\varphi)\|_{L^2}^2 + \frac{1}{2} \alpha_1^2 \frac{d}{dt} \|(\partial_z \Delta_k^h u_\varphi, \varepsilon \partial_z \Delta_k^h v_\varphi)\|_{L^2}^2 \\ & + \lambda \eta'(t) \langle |D_x| (\Delta_k^h u_\varphi, \varepsilon \Delta_k^h v_\varphi), (\Delta_k^h u_\varphi, \varepsilon \Delta_k^h v_\varphi) \rangle - \lambda \eta'(t) \alpha_1^2 \langle |D_x| (\Delta_k^h \partial_z^2 u_\varphi, \varepsilon \Delta_k^h \partial_z^2 v_\varphi), (\Delta_k^h u_\varphi, \varepsilon \Delta_k^h v_\varphi) \rangle \\ & - \alpha_1^2 \varepsilon^2 \lambda \eta'(t) \langle |D_x| (\Delta_k^h \partial_x^2 u_\varphi, \varepsilon \Delta_k^h \partial_x^2 v_\varphi), (\Delta_k^h u_\varphi, \varepsilon \Delta_k^h v_\varphi) \rangle + 2 \|(\varepsilon \partial_x \Delta_k^h u_\varphi, \varepsilon^2 \partial_x \Delta_k^h v_\varphi)\|_{L^2}^2 \\ & + \|(\partial_z \Delta_k^h u_\varphi, \varepsilon \partial_z \Delta_k^h v_\varphi)\|_{L^2}^2 + \alpha^2 \|(\varepsilon^2 \partial_x^2 \Delta_k^h u_\varphi, \varepsilon^3 \partial_x^2 \Delta_k^h v_\varphi)\|_{L^2}^2 + \alpha^2 \|(\partial_z^2 \Delta_k^h u_\varphi, \varepsilon \partial_z^2 \Delta_k^h v_\varphi)\|_{L^2}^2 \\ & + 2 \varepsilon^2 \alpha^2 \|(\partial_x \partial_z \Delta_k^h u_\varphi, \varepsilon \partial_x \partial_z \Delta_k^h v_\varphi)\|_{L^2}^2 \\ & = A_{1,k} + A_{2,k} + A_{3,k} + A_{4,k} + A_{5,k} + A_{6,k} + A_{7,k} + A_{8,k} + A_{9,k} + A_{10,k} + A_{11,k}, \end{aligned} \quad (3.30)$$

where

$$\begin{aligned} A_{1,k} &= \langle \Delta_k^h (u \partial_x \omega)_\varphi, \Delta_k^h u_\varphi \rangle + \varepsilon^2 \langle \Delta_k^h (u \partial_x \gamma)_\varphi, \Delta_k^h v_\varphi \rangle \\ A_{2,k} &= \alpha_1^2 \varepsilon^2 \langle \Delta_k^h (u \partial_x^3 u)_\varphi, \Delta_k^h u_\varphi \rangle \\ A_{3,k} &= \alpha_1^2 \varepsilon^4 \langle \Delta_k^h (u \partial_x^3 v)_\varphi, \Delta_k^h v_\varphi \rangle \\ A_{4,k} &= - \langle \Delta_k^h (v \partial_z \omega)_\varphi, \Delta_k^h u_\varphi \rangle - \varepsilon^2 \langle \Delta_k^h (v \partial_z \gamma)_\varphi, \Delta_k^h v_\varphi \rangle \\ A_{5,k} &= \alpha_1^2 \varepsilon^2 \langle \Delta_k^h (v \partial_z \partial_x^2 u)_\varphi, \Delta_k^h u_\varphi \rangle + \alpha_1^2 \varepsilon^4 \langle (v \partial_z \partial_x^2 v)_\varphi, \Delta_k^h v_\varphi \rangle \\ A_{6,k} &= -\alpha_1^2 \langle \Delta_k^h (\partial_z u \partial_x \partial_z u)_\varphi, \Delta_k^h u_\varphi \rangle - \alpha_1^2 \langle \Delta_k^h (\partial_z^2 u \partial_x u)_\varphi, \Delta_k^h u_\varphi \rangle \\ A_{7,k} &= \alpha_1^2 \varepsilon^2 \langle \Delta_k^h (\partial_x u \partial_x^2 u)_\varphi, \Delta_k^h u_\varphi \rangle \\ A_{8,k} &= \alpha_1^2 \varepsilon^2 \langle \Delta_k^h (\partial_x^2 u \partial_z u)_\varphi, \Delta_k^h v_\varphi \rangle \\ A_{9,k} &= -\varepsilon^2 \langle \Delta_k^h (\gamma \partial_x v)_\varphi, \Delta_k^h u_\varphi \rangle - \varepsilon^2 \langle \Delta_k^h (\gamma \partial_z v)_\varphi, \Delta_k^h v_\varphi \rangle \\ A_{10,k} &= -\varepsilon^4 \alpha_1^2 \langle \Delta_k^h (\partial_x^2 v \partial_x v)_\varphi, \Delta_k^h u_\varphi \rangle \\ A_{11,k} &= \alpha_1^2 \varepsilon^4 \langle \Delta_k^h (\partial_x^2 v \partial_z v)_\varphi, \Delta_k^h v_\varphi \rangle \end{aligned}$$

Direct integration by parts allows to write

$$A_{2,k} = -A_{7,k} + \tilde{A}_{2,k}.$$

where

$$\tilde{A}_{2,k} = -\varepsilon^2 \alpha_1^2 \langle \Delta_k^h (u \partial_x^2 u)_\varphi, \partial_x \Delta_k^h u_\varphi \rangle.$$

Likewise, we have

$$A_{3,k} = A_{11,k} + \tilde{A}_{3,k},$$

where

$$\tilde{A}_{3,k} = -\alpha_1^2 \varepsilon^4 \langle \Delta_k^h (u \partial_x^2 v)_\varphi, \Delta_k^h \partial_x v_\varphi \rangle.$$

Using Lemma 2.3.1, multiplying (3.30) by $e^{2\mathcal{R}t}$ and then integrating the resulting inequality over $[0, t]$, we obtain

$$\begin{aligned} & \frac{1}{2} \|e^{\mathcal{R}t'} (\Delta_k^h u_\varphi, \varepsilon \Delta_k^h v_\varphi)\|_{L^2}^2 + \frac{1}{2} \alpha_1^2 \varepsilon^2 c 2^{2k} \|e^{\mathcal{R}t'} (\Delta_k^h u_\varphi, \varepsilon \Delta_k^h v_\varphi)\|_{L^2}^2 + \frac{1}{2} \alpha_1^2 \|e^{\mathcal{R}t'} (\partial_z \Delta_k^h u_\varphi, \varepsilon \partial_z \Delta_k^h v_\varphi)\|_{L^2}^2 \\ & + \frac{3}{4} \lambda 2^k \int_0^t \eta'(t) \|e^{\mathcal{R}t'} (\Delta_k^h u_\varphi, \varepsilon \Delta_k^h v_\varphi)\|_{L^2}^2 + \frac{3}{4} \alpha_1^2 \lambda 2^k \int_0^t \eta'(t) \|e^{\mathcal{R}t'} \partial_z (\Delta_k^h u_\varphi, \varepsilon \Delta_k^h v_\varphi)\|_{L^2}^2 \\ & + \frac{3}{4} \alpha_1^2 \varepsilon^2 \lambda 2^k \int_0^t \eta'(t) \|e^{\mathcal{R}t'} \partial_x (\Delta_k^h u_\varphi, \varepsilon \Delta_k^h v_\varphi)\|_{L^2}^2 + 2\varepsilon^2 \int_0^t e^{2\mathcal{R}t'} \|(\Delta_k^h \partial_x u_\varphi, \varepsilon \Delta_k^h \partial_x v_\varphi)\|_{L^2}^2 dt' \\ & + \int_0^t e^{2\mathcal{R}t'} \|(\partial_z \Delta_k^h u_\varphi, \varepsilon \partial_z \Delta_k^h v_\varphi)\|_{L^2}^2 dt' + 2\varepsilon^2 \alpha_1^2 \int_0^t e^{2\mathcal{R}t'} \|(\partial_z \partial_x \Delta_k^h u_\varphi, \varepsilon \partial_z \partial_x \Delta_k^h v_\varphi)\|_{L^2}^2 dt' \\ & + \alpha_1^2 \int_0^t e^{2\mathcal{R}t'} (\|(\varepsilon^2 \Delta_k^h \partial_x^2 u_\varphi, \varepsilon^3 \Delta_k^h \partial_x^2 v_\varphi)\|_{L^2}^2 + \|(\partial_z^2 \Delta_k^h u_\varphi, \varepsilon \partial_z^2 \Delta_k^h v_\varphi)\|_{L^2}^2) dt' \\ & \leq \frac{1}{2} \|(\Delta_k^h u_\varphi(0), \varepsilon \Delta_k^h v_\varphi(0))\|_{L^2}^2 + \frac{1}{2} \alpha_1^2 \varepsilon^2 c 2^{2k} \|(\Delta_k^h u_\varphi(0), \varepsilon \Delta_k^h v_\varphi(0))\|_{L^2}^2 \\ & \quad + \frac{1}{2} \alpha_1^2 \|(\partial_z \Delta_k^h u_\varphi(0), \varepsilon \partial_z \Delta_k^h v_\varphi(0))\|_{L^2}^2 + I_k. \end{aligned} \quad (3.31)$$

where

$$I_k = \int_0^t e^{2\mathcal{R}t'} (A_{1,k} + \tilde{A}_{2,k} + \tilde{A}_{3,k} + A_{4,k} + A_{5,k} + A_{6,k} + A_{8,k} + A_{9,k} + A_{10,k} + 2A_{11,k}) dt'.$$

We remark that the term $A_{7,k}$ does not appear in the expression of I_k due to the fact that

$$A_{2,k} + A_{7,k} = \tilde{A}_{2,k}.$$

In what follows, without loss of generality, we suppose that $\mathcal{R} = 0$ and the case where $\mathcal{R} > 0$ can be treated in an exactly same way. We also assume that $T < T^*$, with T^* being determined by

$$T^* = \sup \left\{ t > 0, \eta(t) < \frac{a}{\lambda} \right\}.$$

Remark 6. Similarly to the proof of Estimate (2.13), we have, for $0 < s < 1$

$$\int_0^T \left| \left\langle e^{\mathcal{R}t'} \Delta_q^h (u \partial_x w)_\phi, e^{\mathcal{R}t'} \Delta_q^h w_\phi \right\rangle \right| dt' \leq C 2^{-2qs} (\tilde{d}_q + \check{d}_q) \|e^{\mathcal{R}t'} w_\phi\|_{\tilde{L}_{T, \theta'(t)}^2(B^{s+\frac{1}{2}})}^2.$$

Indeed, Bony's decomposition (2.50) with respect to x variable applied to $u \partial_x w$ implies

$$u \partial_x w = T_u^h \partial_x w + T_{\partial_x w}^h u + R^h(u, \partial_x w).$$

We remark that we can estimate the first and the third term of the above sum exactly as we do for the terms $\mathcal{A}_q, \mathcal{R}_q$ (see proof of Estimate (2.13)). However, we need to control the second term in a slightly different way from what we do to estimate the term \mathcal{B}_q in the proof of (2.13). We write

$$\begin{aligned} & \int_0^T | \langle \Delta_k^h (T_{\partial_x w}^h u)_\phi, \Delta_k^h u_\phi \rangle | dt \\ & \lesssim \sum_{|q'-q| \leq 4} \int_0^T \|S_{k'-1}^h \partial_x w_\phi^+\|_{L_z^2(L_x^\infty)} \| \Delta_{k'}^h u_\phi \|_{L_z^\infty(L_x^2)} \| \Delta_k^h u_\phi \|_{L^2} \\ & \lesssim \sum_{|k'-k| \leq 4} \int_0^T \|S_{k'-1}^h \partial_x w_\phi^+\|_{L_z^2(L_x^\infty)} 2^{-\frac{k'}{2}} d_{k'} (\partial_z^2 u_\phi) \| \partial_z^2 u_\phi \|_{B^{\frac{1}{2}}} \| \Delta_q^h u_\phi \|_{L^2}. \end{aligned}$$

Since $d_{k'}(\partial_z^2 u_\phi) \leq 1$, Cauchy-Schwarz inequality yields

$$\begin{aligned} & \int_0^T | \langle \Delta_k^h (T_{\partial_x w}^h u)_\phi, \Delta_k^h u_\phi \rangle | dt \\ & \lesssim \sum_{|k'-k| \leq 4} 2^{-\frac{k'}{2}} \left(\int_0^T \|S_{q'-1}^h \partial_x w_\phi^+\|_{L_z^2(L_x^\infty)} \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}} dt \right)^{\frac{1}{2}} \times \left(\int_0^T \|\Delta_k^h w_\phi\|_{L^2}^2 \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}} dt \right)^{\frac{1}{2}} \end{aligned}$$

We observe from the definition of S_k^h see (2.1.1) and remark (3) for $s < 1$ that

$$\begin{aligned} \left(\int_0^T \|S_{k'-1}^h \partial_x w_\phi^+\|_{L_z^2(L_x^\infty)} \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}} dt \right)^{\frac{1}{2}} & \lesssim \left(\int_0^T \left(\sum_{l \leq k'-2} 2^{\frac{l}{2}} \|\Delta_l^h \partial_x w_\phi^+\|_{L^2} \sqrt{\|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}}} \right)^2 dt \right)^{\frac{1}{2}} \\ & \lesssim \sum_{l \leq k'-2} 2^{\frac{3l}{2}} \left(\int_0^T \|\Delta_l^h w_\phi\|_{L^2}^2 \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}} dt \right)^{\frac{1}{2}} \\ & \lesssim \sum_{l \leq k'-2} 2^{\frac{3l}{2}} 2^{-l(s+\frac{1}{2})} d_l(\partial_z^2 u_\phi, \eta') \|w_\phi\|_{\tilde{L}_{T, \eta'(t)}^2(B^{s+\frac{1}{2}})} \\ & \lesssim 2^{k'(1-s)} d_{k'} \|w_\phi\|_{\tilde{L}_{T, \eta'(t)}^2(B^{s+\frac{1}{2}})}, \end{aligned}$$

where

$$d_{k'} = \sum_{l \leq k'-2} 2^{(l-k')(1-s)} d_l(\partial_z^2 u_\phi, \eta')$$

is a summable sequence because $s < 1$. Then

$$\int_0^T | \langle \Delta_k^h (T_{\partial_x w}^h u)_\phi, \Delta_k^h u_\phi \rangle | dt \lesssim \tilde{d}_k 2^{-2ks} \|w_\phi\|_{\tilde{L}_{T, \eta'(t)}^2(B^{s+\frac{1}{2}})}^2,$$

where

$$\tilde{d}_k = d_k(w_\phi, \eta') \sum_{|k-k'| \leq 4} 2^{(k-k')(s-1/2)} d_{k'}.$$

- Estimate of $\int_0^T |A_{1,k}| dt$. We write

$$\int_0^T |A_{1,k}| dt \leq \int_0^T (| \langle \Delta_k^h (u \partial_x \omega)_\phi, \Delta_k^h u_\phi \rangle | + |\varepsilon^2 \langle \Delta_k^h (u \partial_x \gamma)_\phi, \Delta_k^h v_\phi \rangle |) dt$$

For the first term, we have

$$\int_0^T | \langle \Delta_k^h (u \partial_x \omega)_\phi, \Delta_k^h u_\phi \rangle | \leq \int_0^T | \langle \Delta_k^h (u \partial_x u)_\phi, \Delta_k^h u_\phi \rangle | dt + \alpha_1^2 \int_0^T | \langle \Delta_k^h (u \partial_x \partial_z^2 u)_\phi, \Delta_k^h u_\phi \rangle | dt.$$

We recall from Estimate (2.13) that

$$\int_0^T | \langle \Delta_k^h (u \partial_x u)_\phi, \Delta_k^h u_\phi \rangle | dt \lesssim 2^{-2ks} (\tilde{d}_k + \check{d}_k) \|e^{Rt'} u_\phi\|_{\tilde{L}_{T, \eta'(t)}^2(B^{s+\frac{1}{2}})}^2.$$

Next, integration by parts with respect to z variable yields

$$\begin{aligned} \alpha_1^2 \int_0^T | \langle \Delta_k^h (u \partial_x \partial_z^2 u)_\phi, \Delta_k^h u_\phi \rangle | dt & \leq \alpha_1^2 \int_0^T | \langle \Delta_k^h (\partial_z u \partial_x \partial_z u)_\phi, \Delta_k^h u_\phi \rangle | \\ & \quad + \alpha_1^2 \int_0^T | \langle \Delta_k^h (u \partial_x \partial_z u)_\phi, \Delta_k^h \partial_z u_\phi \rangle | dt. \end{aligned}$$

and Lemma 2.2.1 gives

$$\int_0^T \left| \left\langle e^{Rt'} \Delta_k^h (\partial_z u \partial_x \partial_z u)_\varphi, \Delta_k^h u_\varphi \right\rangle \right| dt' \leq C 2^{-2ks} (\tilde{d}_k + \check{d}_k) \|e^{Rt'} \partial_z u_\phi\|_{\tilde{L}_{T, \theta'(t)}^2(B^{s+\frac{1}{2}})}^2,$$

and

$$\int_0^T \left| \left\langle e^{Rt'} \Delta_k^h (u \partial_x \partial_z u)_\varphi, \Delta_k^h \partial_z u_\varphi \right\rangle \right| dt' \leq C 2^{-2ks} (\tilde{d}_k + \check{d}_k) \|e^{Rt'} \partial_z u_\phi\|_{\tilde{L}_{T, \theta'(t)}^2(B^{s+\frac{1}{2}})}^2.$$

For the second term, we have

$$\varepsilon^2 \int_0^T |\langle \Delta_k^h (u \partial_x \gamma)_\varphi, \Delta_k^h v_\varphi \rangle| dt \leq \varepsilon^2 \int_0^T |\langle \Delta_k^h (u \partial_x v)_\varphi, \Delta_k^h v_\varphi \rangle| dt + \varepsilon^2 \alpha_1^2 \int_0^T |\langle \Delta_k^h (u \partial_x \partial_z^2 v)_\varphi, \Delta_k^h v_\varphi \rangle| dt.$$

Following the proof of Estimate (2.13), we can write

$$\varepsilon^2 \int_0^T |\langle \Delta_k^h (u \partial_x v)_\varphi, \Delta_k^h v_\varphi \rangle| dt \lesssim 2^{-2ks} (\tilde{d}_k + \check{d}_k) \varepsilon^2 \|e^{Rt'} \partial_z v_\phi\|_{\tilde{L}_{T, \eta'(t)}^2(B^{s+\frac{1}{2}})}^2.$$

Next, integration by parts in z variable implies

$$\int_0^T |\langle \Delta_k^h (u \partial_x \partial_z^2 v)_\varphi, \Delta_k^h v_\varphi \rangle| dt = \int_0^T |\langle \Delta_k^h (u \partial_x \partial_z v)_\varphi, \Delta_k^h \partial_z v_\varphi \rangle| dt + \int_0^T |\langle \Delta_k^h (\partial_z u \partial_x \partial_z v)_\varphi, \Delta_k^h v_\varphi \rangle| dt.$$

By adapting Estimates (2.14) and (2.16), we obtain

$$\int_0^T \left| \left\langle e^{Rt'} \Delta_q^h (u \partial_x \partial_z v)_\phi, e^{Rt'} \Delta_q^h \partial_z v_\phi \right\rangle \right| dt' \leq C 2^{-2ks} (\tilde{d}_k + \check{d}_k) \|e^{Rt'} \partial_z v_\phi\|_{\tilde{L}_{T, \theta'(t)}^2(B^{s+\frac{1}{2}})}^2,$$

and

$$\varepsilon^2 \int_0^T \left| \left\langle e^{Rt'} \Delta_q^h (\partial_z u \partial_x \partial_z v)_\phi, e^{Rt'} \Delta_q^h v_\phi \right\rangle \right| dt' \lesssim 2^{-2ks} (\tilde{d}_k + \check{d}_k) \varepsilon^2 \|e^{Rt'} \partial_z v_\phi\|_{\tilde{L}_{T, \theta'(t)}^2(B^{s+\frac{1}{2}})}^2.$$

Combining all the above inequalities gives

$$\int_0^T |A_{1,k}| dt \lesssim 2^{-2ks} (\tilde{d}_k + \check{d}_k) \|e^{Rt'} \partial_z u_\phi\|_{\tilde{L}_{T, \eta'(t)}^2(B^{s+\frac{1}{2}})}^2 + 2^{-2ks} (\tilde{d}_k + \check{d}_k) \varepsilon^2 \|e^{Rt'} \partial_z v_\phi\|_{\tilde{L}_{T, \theta'(t)}^2(B^{s+\frac{1}{2}})}^2.$$

Multiplying the previous inequality by 2^{2qs} , taking square root of the resulting inequality and then summing with respect to $q \in \mathbb{Z}$, we finally get

$$\sum_{k \in \mathbb{Z}} 2^{ks} \left(\int_0^T |A_{1,k}| dt \right)^{\frac{1}{2}} \lesssim \|e^{Rt'} \partial_z u_\phi\|_{\tilde{L}_{T, \eta'(t)}^2(B^{s+\frac{1}{2}})} + \varepsilon \|e^{Rt'} \partial_z v_\phi\|_{\tilde{L}_{T, \theta'(t)}^2(B^{s+\frac{1}{2}})}$$

- Estimate of $\int_0^T |\tilde{A}_{2,k}| dt$ is a direct consequence of Remark 6. We have

$$\sum_{k \in \mathbb{Z}} 2^{ks} \left(\int_0^T |\tilde{A}_{2,k}| dt \right)^{\frac{1}{2}} \lesssim \varepsilon \|\partial_x u_\phi\|_{\tilde{L}_{T, \eta'(t)}^2(B^{s+\frac{1}{2}})}.$$

- Estimate of $\int_0^T |\tilde{A}_{3,k}| dt = \int_0^T \alpha_1^2 \varepsilon^4 |\langle \Delta_k^h (u \partial_x^2 v)_\varphi, \Delta_k^h \partial_x v_\varphi \rangle| dt$. Following the proof of Lemma 3.1 in [64] with $w = \partial_x v$, we obtain for any $s \in]0, 1[$ that

$$\int_0^T |\langle \Delta_k^h (u \partial_x^2 v)_\varphi, \Delta_k^h \partial_x v_\varphi \rangle| dt \lesssim (\tilde{d}_k + \check{d}_k) 2^{-2ks} \|\partial_x v_\phi\|_{\tilde{L}_{T, \eta'(t)}^2(B^{s+\frac{1}{2}})}^2.$$

Thus,

$$\sum_{k \in \mathbb{Z}} 2^{ks} \left(\int_0^T |\tilde{A}_{3,k}| dt \right)^{\frac{1}{2}} \lesssim \varepsilon^2 \|\partial_x v_\varphi\|_{\tilde{L}_{T,\eta'(t)}^2(B^{s+\frac{1}{2}})}. \quad (3.32)$$

- Estimate of $\int_0^T |A_{4,k}| dt$. We write

$$\int_0^T |A_{4,k}| dt \leq \int_0^T |\langle \Delta_k^h(v \partial_z \omega)_\varphi, \Delta_k^h u_\varphi \rangle| dt + \varepsilon^2 \int_0^T |\langle \Delta_k^h(v \partial_z \gamma)_\varphi, \Delta_k^h v_\varphi \rangle| dt \quad (3.33)$$

- For the first term on the right-hand side of (3.33), we have

$$\int_0^T |\langle \Delta_k^h(v \partial_z \omega)_\varphi, \Delta_k^h u_\varphi \rangle| dt \leq \int_0^T |\langle \Delta_k^h(v \partial_z u)_\varphi, \Delta_k^h u_\varphi \rangle| dt + \alpha_1^2 \int_0^T |\langle \Delta_k^h(v \partial_z^3 u)_\varphi, \Delta_k^h u_\varphi \rangle| dt.$$

Using Estimate (2.17), we obtain

$$\int_0^T |\langle \Delta_k^h(v \partial_z u)_\varphi, \Delta_k^h u_\varphi \rangle| dt \lesssim 2^{-2ks} (\tilde{d}_k + \check{d}_k) \|e^{Rt'} \partial_z u_\phi\|_{\tilde{L}_{T,\eta'(t)}^2(B^{s+\frac{1}{2}})}^2.$$

Then, an integration by parts implies

$$\int_0^T |\langle \Delta_k^h(v \partial_z^3 u)_\varphi, \Delta_k^h u_\varphi \rangle| dt \leq \int_0^T |\langle \Delta_k^h(\partial_x u \partial_z^2 u)_\varphi, \Delta_k^h u_\varphi \rangle| dt + \int_0^T |\langle \Delta_k^h(v \partial_z^2 u)_\varphi, \Delta_k^h \partial_z u_\varphi \rangle| dt.$$

Adapting the proof of [64, Lemma 3.2], we get, for any $0 < s < 1$

$$\int_0^T |\langle \Delta_k^h(v \partial_z^3 u)_\varphi, \Delta_k^h u_\varphi \rangle| dt \lesssim (\tilde{d}_k + \check{d}_k) 2^{-2ks} \|\partial_z u_\phi\|_{\tilde{L}_{T,\eta'(t)}^2(B^{s+\frac{1}{2}})}^2,$$

which implies

$$\sum_{k \in \mathbb{Z}} 2^{ks} \left(\int_0^T |\langle \Delta_k^h(v \partial_z \omega)_\varphi, \Delta_k^h u_\varphi \rangle| dt \right)^{\frac{1}{2}} \lesssim \|\partial_z u_\phi\|_{\tilde{L}_{T,\eta'(t)}^2(B^{s+\frac{1}{2}})}.$$

- For the second term on the right-hand side of (3.33), we write

$$\varepsilon^2 \int_0^T |\langle \Delta_k^h(v \partial_z \gamma)_\varphi, \Delta_k^h v_\varphi \rangle| dt \leq \varepsilon^2 \int_0^T |\langle \Delta_k^h(v \partial_z v)_\varphi, \Delta_k^h v_\varphi \rangle| dt + \varepsilon^2 \alpha_1^2 \int_0^T |\langle \Delta_k^h(v \partial_z^3 v)_\varphi, \Delta_k^h v_\varphi \rangle| dt$$

From Estimate (3.12) of Lemma 3.3 in [64], we have

$$\varepsilon \sum_{k \in \mathbb{Z}} 2^{ks} \left(\int_0^T |\langle \Delta_k^h(v \partial_z v)_\varphi, \Delta_k^h v_\varphi \rangle| dt \right)^{\frac{1}{2}} \lesssim \|(u_\varphi, \varepsilon v_\varphi)\|_{\tilde{L}_{T,\eta'(t)}^2(B^{s+\frac{1}{2}})}.$$

So, for any $0 < s < 1$

$$\varepsilon^2 \int_0^T |\langle \Delta_k^h(v \partial_z v)_\varphi, \Delta_k^h v_\varphi \rangle| dt \lesssim (\tilde{d}_k + \check{d}_k) 2^{-2ks} \|(u_\varphi, \varepsilon v_\varphi)\|_{\tilde{L}_{T,\eta'(t)}^2(B^{s+\frac{1}{2}})}^2$$

Next, performing an integration by parts and using the incompressibility condition, we get

$$\int_0^T |\langle \Delta_k^h(v \partial_z^3 v)_\varphi, \Delta_k^h v_\varphi \rangle| dt \leq \int_0^T |\langle \Delta_k^h(\partial_x u \partial_z^2 v)_\varphi, \Delta_k^h v_\varphi \rangle| dt + \int_0^T |\langle \Delta_k^h(v \partial_z^2 v)_\varphi, \Delta_k^h \partial_z v_\varphi \rangle| dt.$$

Following the proof of (3.38), we can show that

$$\begin{aligned} \varepsilon^2 \sum_{k \in \mathbb{Z}} 2^{ks} \left(\int_0^t |\langle \Delta_k^h(\partial_x u \partial_z^2 v)_\varphi, \Delta_k^h v_\varphi \rangle| dt' \right)^{\frac{1}{2}} \\ \lesssim \sqrt{\|\partial_z u_\varphi\|_{\tilde{L}_{T, \eta'(t)}^2(B^{s+\frac{1}{2}})}} \sqrt{\varepsilon \|v_\varphi\|_{\tilde{L}_{T, \eta'(t)}^2(B^{s+\frac{1}{2}})}} \\ \lesssim \|\partial_z u_\varphi\|_{\tilde{L}_{T, \eta'(t)}^2(B^{s+\frac{1}{2}})} + \varepsilon \|v_\varphi\|_{\tilde{L}_{T, \eta'(t)}^2(B^{s+\frac{1}{2}})}. \end{aligned}$$

Now, we can again adapt the proof of Estimate (3.12) in [64, Lemma 3.3] to prove that, for any $0 < s < 1$,

$$\varepsilon \sum_{k \in \mathbb{Z}} 2^{ks} \left(\int_0^T |\langle \Delta_k^h(v \partial_z^2 v)_\varphi, \Delta_k^h \partial_z v_\varphi \rangle| dt \right)^{\frac{1}{2}} \lesssim \|(\partial_z u_\varphi, \varepsilon \partial_z v_\varphi)\|_{\tilde{L}_{T, \eta'(t)}^2(B^{s+\frac{1}{2}})}$$

Summing the above estimates yields

$$\varepsilon \sum_{k \in \mathbb{Z}} 2^{ks} \left(\int_0^T |\langle \Delta_k^h(v \partial_z \gamma)_\varphi, \Delta_k^h u_\varphi \rangle| dt \right)^{\frac{1}{2}} \lesssim \|\partial_z u_\varphi\|_{\tilde{L}_{T, \eta'(t)}^2(B^{s+\frac{1}{2}})} + \varepsilon \|\partial_z v_\varphi\|_{\tilde{L}_{T, \eta'(t)}^2(B^{s+\frac{1}{2}})}$$

which implies

$$\sum_{k \in \mathbb{Z}} 2^{ks} \left(\int_0^T |A_{4,k}| dt \right)^{\frac{1}{2}} \lesssim \|\partial_z u_\varphi\|_{\tilde{L}_{T, \eta'(t)}^2(B^{s+\frac{1}{2}})} + \varepsilon \|\partial_z v_\varphi\|_{\tilde{L}_{T, \eta'(t)}^2(B^{s+\frac{1}{2}})}.$$

- Estimate of $\int_0^T |A_{5,k}| dt$. As what precedes, we cut this integral into two parts

$$\int_0^T |A_{5,k}| dt \leq \int_0^T |\alpha_1^2 \varepsilon^2 \langle \Delta_k^h(v \partial_z \partial_x^2 u)_\varphi, \Delta_k^h u_\varphi \rangle| dt + \int_0^T \alpha_1^2 \varepsilon^4 |\langle (v \partial_z \partial_x^2 v)_\varphi, \Delta_k^h v_\varphi \rangle| dt.$$

- We perform an integration by parts on the first term on the right-hand side

$$\int_0^T |\langle \Delta_k^h(v \partial_z \partial_x^2 u)_\varphi, \Delta_k^h u_\varphi \rangle| dt \leq \int_0^T |\langle \Delta_k^h(\partial_x u \partial_x^2 u)_\varphi, \Delta_k^h u_\varphi \rangle| dt + \int_0^T |\langle \Delta_k^h(v \partial_x^2 u)_\varphi, \Delta_k^h \partial_z u_\varphi \rangle| dt.$$

By adapting the proof of Estimate (3.11) in [64, Lemma 3.2], we obtain, for any $0 < s < 1$

$$\varepsilon^2 \sum_{k \in \mathbb{Z}} 2^{ks} \left(\int_0^T |\langle \Delta_k^h(\partial_x u \partial_x^2 u)_\varphi, \Delta_k^h u_\varphi \rangle| dt' \right)^{\frac{1}{2}} \lesssim \|\partial_z u_\varphi\|_{\tilde{L}_{T, \eta'(t)}^2(B^{s+\frac{1}{2}})}$$

and

$$\varepsilon^2 \sum_{k \in \mathbb{Z}} 2^{ks} \left(\int_0^T |\langle \Delta_k^h(v \partial_x^2 u)_\varphi, \Delta_k^h \partial_z u_\varphi \rangle| dt' \right)^{\frac{1}{2}} \lesssim \|\partial_z u_\varphi\|_{\tilde{L}_{T, \eta'(t)}^2(B^{s+\frac{1}{2}})}.$$

Then, we deduce that

$$\varepsilon^2 \sum_{k \in \mathbb{Z}} 2^{ks} \left(\int_0^T |\langle \Delta_k^h(v \partial_z \partial_x^2 u)_\varphi, \Delta_k^h u_\varphi \rangle| dt' \right)^{\frac{1}{2}} \lesssim \|\partial_z u_\varphi\|_{\tilde{L}_{T, \eta'(t)}^2(B^{s+\frac{1}{2}})}.$$

- The second term on the right-hand side can also be controlled by integration by part

$$\int_0^T |\langle \Delta_k^h(v \partial_z \partial_x^2 v)_\varphi, \Delta_k^h v_\varphi \rangle| dt \leq \int_0^T |\langle \Delta_k^h(\partial_x u \partial_x^2 v)_\varphi, \Delta_k^h v_\varphi \rangle| dt + \int_0^T |\langle \Delta_k^h(v \partial_x^2 v)_\varphi, \Delta_k^h \partial_z v_\varphi \rangle| dt.$$

The estimate of the first term in the above inequality is given by (3.38). For the other term, we can follow the proof of (3.38) with some minor changes and we obtain

$$\begin{aligned} & \varepsilon^2 \sum_{k \in \mathbb{Z}} 2^{ks} \left(\int_0^T |\langle \Delta_k^h(v \partial_x^2 v)_\varphi, \Delta_k^h \partial_z v_\varphi \rangle| dt' \right)^{\frac{1}{2}} \\ & \lesssim \sqrt{\|\partial_z u_\varphi\|_{\tilde{L}_{T, \eta'(t)}^2(B^{s+\frac{1}{2}})}} \sqrt{\varepsilon \|\partial_z v_\varphi\|_{\tilde{L}_{T, \eta'(t)}^2(B^{s+\frac{1}{2}})}} \lesssim \|(\partial_z u_\varphi, \varepsilon \partial_z v_\varphi)\|_{\tilde{L}_{T, \eta'(t)}^2(B^{s+\frac{1}{2}})}. \end{aligned}$$

Then

$$\sum_{k \in \mathbb{Z}} 2^{ks} \left(\int_0^T \alpha_1^2 \varepsilon^4 |\langle (v \partial_z \partial_x^2 v)_\varphi, \Delta_k^h v_\varphi \rangle| dt \right)^{\frac{1}{2}} \lesssim \|(\partial_z u_\varphi, \varepsilon \partial_z v_\varphi)\|_{\tilde{L}_{T, \eta'(t)}^2(B^{s+\frac{1}{2}})},$$

which implies

$$\sum_{k \in \mathbb{Z}} 2^{ks} \left(\int_0^T |A_{5,k}| dt \right)^{\frac{1}{2}} \lesssim \|\partial_z u_\varphi\|_{\tilde{L}_{T, \eta'(t)}^2(B^{s+\frac{1}{2}})} + \varepsilon \|\partial_z v_\varphi\|_{\tilde{L}_{T, \eta'(t)}^2(B^{s+\frac{1}{2}})}.$$

- Estimate of $\int_0^T |A_{6,k}| dt$. We have

$$\int_0^T |A_{6,k}| dt \leq \int_0^T \alpha_1^2 |\langle \Delta_k^h(\partial_z u \partial_x \partial_z u)_\varphi, \Delta_k^h u_\varphi \rangle| dt + \int_0^T \alpha_1^2 |\langle \Delta_k^h(\partial_z^2 u \partial_x u)_\varphi, \Delta_k^h u_\varphi \rangle| dt.$$

Lemma 2.2.1 implies

$$\int_0^T |\langle \Delta_k^h(\partial_z u \partial_x \partial_z u)_\varphi, \Delta_k^h u_\varphi \rangle| dt' \leq C 2^{-2ks} (\tilde{d}_k + \check{d}_k) \|\partial_z u_\varphi\|_{\tilde{L}_{T, \theta'(t)}^2(B^{s+\frac{1}{2}})}^2.$$

Adapting the proof of (3.38), we obtain

$$\sum_{k \in \mathbb{Z}} 2^{ks} \left(\int_0^T |\langle \Delta_k^h(\partial_x u \partial_z^2 u)_\varphi, \Delta_k^h u_\varphi \rangle| dt \right)^{\frac{1}{2}} \lesssim \|\partial_z u_\varphi\|_{\tilde{L}_{T, \eta'(t)}^2(B^{s+\frac{1}{2}})}$$

So,

$$\sum_{k \in \mathbb{Z}} 2^{ks} \left(\int_0^T |A_{6,k}| dt \right)^{\frac{1}{2}} \lesssim \|\partial_z u_\varphi\|_{\tilde{L}_{T, \eta'(t)}^2(B^{s+\frac{1}{2}})}.$$

- Estimate of $\int_0^T A_{8,k}$. We perform an integration par parts with respect to x variable

$$\int_0^T |\langle \Delta_k^h(\partial_x^2 u \partial_z u)_\varphi, \Delta_k^h v_\varphi \rangle| dt \leq \int_0^T |\langle \Delta_k^h(\partial_x u \partial_x \partial_z u)_\varphi, \Delta_k^h v_\varphi \rangle| dt + \int_0^T |\langle \Delta_k^h(\partial_x u \partial_z u)_\varphi, \Delta_k^h \partial_x v_\varphi \rangle| dt.$$

For any $0 < s < 1$, we have

$$\varepsilon^2 \int_0^T |\langle \Delta_k^h(\partial_x u \partial_x \partial_z u)_\varphi, \Delta_k^h v_\varphi \rangle| dt \lesssim (\tilde{d}_k + \check{d}_k) 2^{-2ks} \|(\partial_z u_\varphi, \varepsilon v_\varphi)\|_{\tilde{L}_{T, \eta'(t)}^2(B^{s+\frac{1}{2}})}^2$$

and

$$\varepsilon^2 \int_0^T |\langle \Delta_k^h(\partial_x u \partial_z u)_\varphi, \Delta_k^h \partial_x v_\varphi \rangle| dt \lesssim (\tilde{d}_k + \check{d}_k) 2^{-2ks} \|(\partial_z u_\varphi, \varepsilon^2 \partial_x v_\varphi)\|_{\tilde{L}_{T, \eta'(t)}^2(B^{s+\frac{1}{2}})}^2$$

Thus,

$$\sum_{k \in \mathbb{Z}} 2^{ks} \left(\int_0^T |A_{8,k}| dt \right)^{\frac{1}{2}} \lesssim \|(\partial_z u_\varphi, \varepsilon v_\varphi)\|_{\tilde{L}_{T, \eta'(t)}^2(B^{s+\frac{1}{2}})} + \|(\partial_z u_\varphi, \varepsilon^2 \partial_x v_\varphi)\|_{\tilde{L}_{T, \eta'(t)}^2(B^{s+\frac{1}{2}})}.$$

- Estimate of $\int_0^T |A_{9,k}| dt$. Performing an integration by parts, we have

$$\begin{aligned} \langle \Delta_k^h(\gamma \partial_x v)_\varphi, \Delta_k^h u_\varphi \rangle &= \langle \Delta_k^h(v \partial_x v)_\varphi, \Delta_k^h u_\varphi \rangle - \alpha_1^2 \langle \Delta_k^h(\partial_z^2 v \partial_x v)_\varphi, \Delta_k^h u_\varphi \rangle \\ &= -\frac{1}{2} \langle \Delta_k^h(v^2)_\varphi, \Delta_k^h \partial_x u_\varphi \rangle dt - \alpha_1^2 \langle \Delta_k^h(\partial_x v \partial_z^2 v)_\varphi, \Delta_k^h u_\varphi \rangle \end{aligned}$$

and

$$\begin{aligned} \langle \Delta_k^h(\gamma \partial_z v)_\varphi, \Delta_k^h v_\varphi \rangle &= \langle \Delta_k^h(v \partial_z v)_\varphi, \Delta_k^h v_\varphi \rangle - \alpha_1^2 \langle \Delta_k^h(\partial_z^2 v \partial_z v)_\varphi, \Delta_k^h v_\varphi \rangle \\ &= -\frac{1}{2} \langle \Delta_k^h(v^2)_\varphi, \Delta_k^h \partial_z v_\varphi \rangle dt - \alpha_1^2 \langle \Delta_k^h(\partial_z v \partial_z^2 v)_\varphi, \Delta_k^h v_\varphi \rangle. \end{aligned}$$

Then, the incompressibility condition implies

$$A_{9,k} = \varepsilon^2 \alpha_1^2 \langle \Delta_k^h(\partial_x v \partial_z^2 v)_\varphi, \Delta_k^h u_\varphi \rangle + \varepsilon^2 \alpha_1^2 \langle \Delta_k^h(\partial_z v \partial_z^2 v)_\varphi, \Delta_k^h v_\varphi \rangle.$$

Adapting the proof of Estimate (3.11) in [64, Lemma 3.2] we get

$$\begin{aligned} \alpha_1^2 \varepsilon^2 \sum_{k \in \mathbb{Z}} 2^{ks} \left(\int_0^T |\langle \Delta_k^h(\partial_x v \partial_z^2 v)_\varphi, \Delta_k^h u_\varphi \rangle| dt \right)^{\frac{1}{2}} \\ \lesssim \left(\varepsilon \|v_\varphi\|_{\tilde{L}_{T,\eta'(t)}^2(B^{s+\frac{1}{2}})} \|u_\varphi\|_{\tilde{L}_{T,\eta'(t)}^2(B^{s+\frac{1}{2}})} \right)^{\frac{1}{2}} + \left(\varepsilon \|\partial_x u_\varphi\|_{\tilde{L}_{T,\eta'(t)}^2(B^{s+\frac{1}{2}})} \|u_\varphi\|_{\tilde{L}_{T,\eta'(t)}^2(B^{s+\frac{1}{2}})} \right)^{\frac{1}{2}} \\ \lesssim \varepsilon \|v_\varphi\|_{\tilde{L}_{T,\eta'(t)}^2(B^{s+\frac{1}{2}})} + \varepsilon \|\partial_x u_\varphi\|_{\tilde{L}_{T,\eta'(t)}^2(B^{s+\frac{1}{2}})} + \|u_\varphi\|_{\tilde{L}_{T,\eta'(t)}^2(B^{s+\frac{1}{2}})} \end{aligned}$$

and

$$\begin{aligned} \alpha_1^2 \varepsilon^2 \sum_{k \in \mathbb{Z}} 2^{ks} \left(\int_0^T |\langle \Delta_k^h(\partial_x u \partial_z^2 v)_\varphi, \Delta_k^h v_\varphi \rangle| dt \right)^{\frac{1}{2}} \\ \lesssim \left(\varepsilon \|v_\varphi\|_{\tilde{L}_{T,\eta'(t)}^2(B^{s+\frac{1}{2}})} \|u_\varphi\|_{\tilde{L}_{T,\eta'(t)}^2(B^{s+\frac{1}{2}})} \right)^{\frac{1}{2}} + \left(\varepsilon \|\partial_x u_\varphi\|_{\tilde{L}_{T,\eta'(t)}^2(B^{s+\frac{1}{2}})} \|u_\varphi\|_{\tilde{L}_{T,\eta'(t)}^2(B^{s+\frac{1}{2}})} \right)^{\frac{1}{2}} \\ \lesssim \varepsilon \|v_\varphi\|_{\tilde{L}_{T,\eta'(t)}^2(B^{s+\frac{1}{2}})} + \varepsilon \|\partial_x u_\varphi\|_{\tilde{L}_{T,\eta'(t)}^2(B^{s+\frac{1}{2}})} + \|u_\varphi\|_{\tilde{L}_{T,\eta'(t)}^2(B^{s+\frac{1}{2}})}, \end{aligned}$$

which means

$$\sum_{k \in \mathbb{Z}} 2^{ks} \left(\int_0^T |A_{9,k}| dt \right)^{\frac{1}{2}} \lesssim \varepsilon \|v_\varphi\|_{\tilde{L}_{T,\eta'(t)}^2(B^{s+\frac{1}{2}})} + \varepsilon \|\partial_x u_\varphi\|_{\tilde{L}_{T,\eta'(t)}^2(B^{s+\frac{1}{2}})} + \|u_\varphi\|_{\tilde{L}_{T,\eta'(t)}^2(B^{s+\frac{1}{2}})}.$$

- Estimate of $\int_0^T |A_{10,k}| dt$. Similarly to the above estimates, adapting the proof of Estimate (3.11) in [64, Lemma 3.2] we get

$$\begin{aligned} \sum_{k \in \mathbb{Z}} 2^{ks} \left(\int_0^T |A_{10,k}| dt \right)^{\frac{1}{2}} &\lesssim \left(\varepsilon \|v_\varphi\|_{\tilde{L}_{T,\eta'(t)}^2(B^{s+\frac{1}{2}})} \|u_\varphi\|_{\tilde{L}_{T,\eta'(t)}^2(B^{s+\frac{1}{2}})} \right)^{\frac{1}{2}} \\ &\quad + \left(\varepsilon \|\partial_x u_\varphi\|_{\tilde{L}_{T,\eta'(t)}^2(B^{s+\frac{1}{2}})} \|u_\varphi\|_{\tilde{L}_{T,\eta'(t)}^2(B^{s+\frac{1}{2}})} \right)^{\frac{1}{2}} \\ &\lesssim \varepsilon \|v_\varphi\|_{\tilde{L}_{T,\eta'(t)}^2(B^{s+\frac{1}{2}})} + \varepsilon \|\partial_x u_\varphi\|_{\tilde{L}_{T,\eta'(t)}^2(B^{s+\frac{1}{2}})} + \|u_\varphi\|_{\tilde{L}_{T,\eta'(t)}^2(B^{s+\frac{1}{2}})}. \end{aligned}$$

- Estimate of $\int_0^T |A_{11,k}| dt$. Applying Bony's decomposition 2.50 with respect to the horizontal variable to $\partial_x u \partial_x^2 v$, we get

$$\partial_x u \partial_x^2 v = T_{\partial_x u}^h \partial_x^2 v + T_{\partial_x^2 v}^h \partial_x u + R^h(\partial_x u, \partial_x^2 v). \quad (3.34)$$

We first have

$$\begin{aligned}
& \varepsilon^4 \int_0^T |\langle \Delta_k^h (T_{\partial_x u}^h \partial_x^2 v)_\varphi, \Delta_k^h v_\varphi \rangle| dt \\
& \lesssim \varepsilon^4 \sum_{|k-k'|\leq 4} \int_0^T \|S_{k'-1}^h \partial_x u_\varphi\|_{L^\infty} \|\Delta_{k'}^h \partial_x^2 v_\varphi\|_{L^2} \|\Delta_k^h v_\varphi\|_{L^2} dt \\
& \lesssim \varepsilon \sum_{|k-k'|\leq 4} 2^{\frac{k'}{2}} d_{k'} (\partial_x^2 v_\varphi) \int_0^t \|S_{k'-1}^h \partial_x u_\varphi\|_{L^\infty} \varepsilon^3 \|\partial_x^2 v_\varphi\|_{B^{\frac{1}{2}}} \|\Delta_k^h v_\varphi\|_{L^2} dt \\
& \lesssim \varepsilon \sum_{|k-k'|\leq 4} 2^{\frac{k'}{2}} \left(\int_0^T \|S_{k'-1}^h \partial_x u_\varphi\|_{L^\infty}^2 \varepsilon^3 \|\partial_x^2 v_\varphi\|_{B^{\frac{1}{2}}}^2 dt \right)^{\frac{1}{2}} \times \left(\int_0^T \varepsilon^3 \|\partial_x^2 v_\varphi\|_{B^{\frac{1}{2}}}^2 \|\Delta_k^h v_\varphi\|_{L^2}^2 dt \right)^{\frac{1}{2}}.
\end{aligned}$$

Using Bernstein Lemma 2.3.1 and Poincaré inequality, we deduce

$$\begin{aligned}
\|S_{k'-1}^h \partial_x u_\varphi\|_{L^\infty} & \leq \sum_{l \leq k'-2} \|\Delta_l^h \partial_x u_\varphi\|_{L^\infty} \\
& \lesssim \sum_{l \leq k'-2} 2^{\frac{3l}{2}} \|\Delta_l^h u_\varphi\|_{L_v^\infty(L_h^2)} \\
& \lesssim \sum_{l \leq k'-2} 2^{\frac{3l}{2}} \|\Delta_l^h \partial_z u_\varphi\|_{L^2}.
\end{aligned}$$

Then, from Remark 3, we derive for $0 < s < 1$,

$$\begin{aligned}
\left(\int_0^T \|S_{k'-1}^h \partial_x u_\varphi\|_{L^\infty}^2 \varepsilon^3 \|\partial_x^2 v_\varphi\|_{B^{\frac{1}{2}}}^2 dt \right)^{\frac{1}{2}} & \lesssim \left(\int_0^T \left(\sum_{l \leq k'-2} 2^{\frac{3l}{2}} \|\Delta_l^h \partial_z u_\varphi\|_{L^2} \sqrt{\varepsilon^3 \|\partial_x^2 v_\varphi\|_{B^{\frac{1}{2}}}} \right)^2 dt \right)^{\frac{1}{2}} \\
& \lesssim \sum_{l \leq k'-2} 2^{\frac{3l}{2}} \left(\int_0^T \|\Delta_l^h \partial_z u_\varphi\|_{L^2}^2 \varepsilon^3 \|\partial_x^2 v_\varphi\|_{B^{\frac{1}{2}}}^2 dt \right)^{\frac{1}{2}} \\
& \lesssim \sum_{l \leq k'-2} 2^{\frac{3l}{2}} 2^{-l(s+\frac{1}{2})} d_l (\partial_z u_\varphi, \eta') \|\partial_z u_\varphi\|_{\tilde{L}_{T, \eta'(t)}^2(B^{s+\frac{1}{2}})} \\
& \lesssim 2^{k'(1-s)} d_{k'} \|\partial_z u_\varphi\|_{\tilde{L}_{T, \eta'(t)}^2(B^{s+\frac{1}{2}})},
\end{aligned}$$

where

$$d_{k'} = \sum_{l \leq k'-2} 2^{(l-k')(1-s)} d_l (\partial_z u_\varphi, \eta')$$

is summable sequence. Then

$$\varepsilon^4 \int_0^T |\langle \Delta_k^h (T_{\partial_x u}^h \partial_x^2 v)_\varphi, \Delta_k^h v_\varphi \rangle| dt \lesssim \tilde{d}_k 2^{-2ks} \|\partial_z u_\varphi\|_{\tilde{L}_{T, \eta'(t)}^2(B^{s+\frac{1}{2}})} \varepsilon \|v_\varphi\|_{\tilde{L}_{T, \eta'(t)}^2(B^{s+\frac{1}{2}})}, \quad (3.35)$$

where

$$\tilde{d}_k = d_k(v_\varphi, \eta') \sum_{|k-k'|\leq 4} 2^{(q-q')(s-\frac{1}{2})} d_{k'}.$$

Next, using Bernstein Lemma 2.3.1 and Poincaré inequality, we get

$$\begin{aligned}
 & \varepsilon^4 \int_0^T \left| \left\langle \Delta_k^h (T_{\partial_x^2 v}^h \partial_x u)_\varphi, \Delta_k^h v_\varphi \right\rangle \right| dt \\
 & \lesssim \varepsilon \sum_{|k-k'| \leq 4} \int_0^T \varepsilon^3 \|S_{k'-1}^h \partial_x^2 v_\varphi\|_{L_z^2(L_x^\infty)} \|\Delta_{k'}^h \partial_x u_\varphi\|_{L_z^\infty(L_x^2)} \|\Delta_k^h v_\varphi\|_{L^2} dt \\
 & \lesssim \varepsilon \sum_{|k-k'| \leq 4} \int_0^T \varepsilon^3 \|\partial_x^2 v_\varphi\|_{B^{\frac{1}{2}}} 2^{k'} \|\Delta_{k'}^h \partial_z u_\varphi\|_{L^2} \|\Delta_k^h v_\varphi\|_{L^2} dt \\
 & \lesssim \varepsilon \sum_{|k-k'| \leq 4} \left(\int_0^T \varepsilon^3 \|\partial_x^2 v_\varphi\|_{B^{\frac{1}{2}}} \|\Delta_{k'}^h \partial_z u_\varphi\|_{L^2}^2 dt \right)^{\frac{1}{2}} \times \left(\int_0^T \varepsilon^3 \|\partial_x^2 v_\varphi\|_{B^{\frac{1}{2}}} \|\Delta_k^h v_\varphi\|_{L^2}^2 dt \right)^{\frac{1}{2}}
 \end{aligned}$$

Remark 3 implies

$$\varepsilon^4 \int_0^T \left| \left\langle \Delta_k^h (T_{\partial_x^2 v}^h \partial_x u)_\varphi, \Delta_k^h v_\varphi \right\rangle \right| dt \lesssim \tilde{d}_k 2^{-2ks} \|\partial_z u_\varphi\|_{\tilde{L}_{T, \eta'(t)}^2(B^{s+\frac{1}{2}})} \varepsilon \|v_\varphi\|_{\tilde{L}_{T, \eta'(t)}^2(B^{s+\frac{1}{2}})}, \quad (3.36)$$

where

$$\tilde{d}_k = d_k(v_\varphi, \eta') \sum_{|k-k'| \leq 4} 2^{(q-q')(s-\frac{1}{2})} d_{k'}(\partial_z u_\varphi, \eta').$$

Finally, for the term $R^h(\partial_x u, \partial_x^2 v)$, we write

$$\begin{aligned}
 & \varepsilon^4 \int_0^T \left| \left\langle \Delta_k^h (R^h(\partial_x u, \partial_x^2 v))_\varphi, \Delta_k^h v_\varphi \right\rangle \right| dt \\
 & \lesssim 2^{\frac{q}{2}} \varepsilon^4 \sum_{k' \geq k-3} \int_0^T e^{2Rt'} \|\Delta_{k'}^h \partial_x u_\varphi\|_{L_z^\infty(L_x^2)} \|\tilde{\Delta}_{k'}^h \partial_x^2 v_\varphi\|_{L^2} \|\Delta_k^h u_\varphi\|_{L_z^2(L_x^\infty)} dt' \\
 & \lesssim \varepsilon \sum_{k' \geq k-3} \int_0^T 2^{k'} \|\Delta_{k'}^h \partial_z u_\varphi\|_{L^2} 2^{-\frac{k'}{2}} d_{k'}(\partial_x^2 v_\varphi) \varepsilon^3 \|\partial_x^2 v_\varphi\|_{B^{\frac{1}{2}}} 2^{\frac{k'}{2}} \|\Delta_k^h \partial_z u_\varphi\|_{L^2} dt' \\
 & \lesssim \varepsilon 2^{\frac{k}{2}} \sum_{k' \geq k-3} 2^{\frac{k'}{2}} \left(\int_0^T \|\Delta_{k'}^h \partial_z u_\varphi\|_{L^2} \varepsilon^3 \|\partial_x^2 v_\varphi\|_{B^{\frac{1}{2}}} dt \right)^{\frac{1}{2}} \times \left(\int_0^T \varepsilon^3 \|\partial_x^2 v_\varphi\|_{B^{\frac{1}{2}}} \|\Delta_k^h \partial_z u_\varphi\|_{L^2} dt \right)^{\frac{1}{2}}
 \end{aligned}$$

Using Remark 3, we have

$$\varepsilon^4 \int_0^T \left| \left\langle \Delta_k^h (R^h(\partial_x u, \partial_x^2 v))_\varphi, \Delta_k^h v_\varphi \right\rangle \right| dt \lesssim \tilde{d}_k 2^{-2ks} \|\partial_z u_\varphi\|_{\tilde{L}_{T, \eta'(t)}^2(B^{s+\frac{1}{2}})} \varepsilon \|v_\varphi\|_{\tilde{L}_{T, \eta'(t)}^2(B^{s+\frac{1}{2}})}, \quad (3.37)$$

where

$$\tilde{d}_k = d_k(v_\varphi, \eta') \sum_{k' \geq k-3} 2^{(k-k')s} d_{k'}(\partial_z u_\varphi, \eta').$$

Combining (3.35)–(3.37), we obtain

$$\begin{aligned}
 \sum_{k \in \mathbb{Z}} 2^{ks} \left(\int_0^T |A_{11,k}| dt \right)^{\frac{1}{2}} & \lesssim \sqrt{\|\partial_z u_\varphi\|_{\tilde{L}_{T, \eta'(t)}^2(B^{s+\frac{1}{2}})}} \sqrt{\varepsilon \|v_\varphi\|_{\tilde{L}_{T, \eta'(t)}^2(B^{s+\frac{1}{2}})}} \\
 & \lesssim \|\partial_z u_\varphi\|_{\tilde{L}_{T, \eta'(t)}^2(B^{s+\frac{1}{2}})} + \varepsilon \|v_\varphi\|_{\tilde{L}_{T, \eta'(t)}^2(B^{s+\frac{1}{2}})}. \quad (3.38)
 \end{aligned}$$

Conclusion. Multiplying the estimate (3.31) by 2^{2ks} , taking square root of resulting inequality then summing with respect to $q \in \mathbb{Z}$ and using all the previous estimates, we get for any $0 < s < 1$

$$\begin{aligned}
& \|e^{\mathcal{R}t'}(u_\varphi, \varepsilon v_\varphi)\|_{\tilde{L}_T^\infty(B^s)} + \|e^{\mathcal{R}t'}(\varepsilon u_\varphi, \varepsilon^2 v_\varphi)\|_{\tilde{L}_T^\infty(B^{s+1})} + \|e^{\mathcal{R}t'}(\partial_z u_\varphi, \varepsilon \partial_z \Delta_k^h v_\varphi)\|_{\tilde{L}_T^\infty(B^s)} \\
& + \sqrt{\lambda} \|e^{\mathcal{R}t'}(u_\varphi, \varepsilon v_\varphi)\|_{\tilde{L}_{T, \eta'(t)}^2(B^{s+\frac{1}{2}})} + \sqrt{\lambda} \|e^{\mathcal{R}t'} \partial_z(\Delta_k^h u_\varphi, \varepsilon \Delta_k^h v_\varphi)\|_{\tilde{L}_{T, \eta'(t)}^2(B^{s+\frac{1}{2}})} \\
& + \sqrt{\lambda} \|e^{\mathcal{R}t'}(\varepsilon \partial_x u_\varphi, \varepsilon^2 \partial_x v_\varphi)\|_{\tilde{L}_{T, \eta'(t)}^2(B^{s+\frac{1}{2}})} + \|e^{\mathcal{R}t'}(\varepsilon \partial_x u_\varphi, \varepsilon^2 \partial_x v_\varphi)\|_{\tilde{L}_T^2(B^s)} \\
& + \|e^{\mathcal{R}t'}(\partial_z u_\varphi, \varepsilon \partial_z v_\varphi)\|_{\tilde{L}_T^2(B^s)} dt' + \|e^{\mathcal{R}t'}(\varepsilon \partial_z \partial_x u_\varphi, \varepsilon^2 \partial_z \partial_x v_\varphi)\|_{\tilde{L}_T^2(B^s)} \\
& + \|e^{\mathcal{R}t'}(\varepsilon \partial_x^2 u_\varphi, \varepsilon^3 \partial_x^2 v_\varphi)\|_{\tilde{L}_T^2(B^s)} + \|e^{\mathcal{R}t'}(\partial_z^2 u_\varphi, \varepsilon \partial_z^2 v_\varphi)\|_{\tilde{L}_T^2(B^s)} \\
& \leq C \left[\|(u_\varphi(0), \varepsilon v_\varphi(0))\|_{B^s} + \|(\varepsilon u_\varphi(0), \varepsilon^2 v_\varphi(0))\|_{B^{s+1}} + \|(\partial_z u_\varphi(0), \varepsilon \partial_z v_\varphi(0))\|_{B^s} \right. \\
& \left. + \|(\partial_z u_\varphi, \varepsilon \partial_x v_\varphi)\|_{\tilde{L}_{T, \eta'(t)}^2(B^{s+\frac{1}{2}})} + \varepsilon \|\partial_x u_\varphi\|_{\tilde{L}_{T, \eta'(t)}^2(B^{s+\frac{1}{2}})} + \varepsilon^2 \|\partial_x v_\varphi\|_{\tilde{L}_{T, \eta'(t)}^2(B^{s+\frac{1}{2}})} \right]. \quad (3.39)
\end{aligned}$$

Taking $\lambda = C^2$ and $s = \frac{1}{2}$, we obtain

$$\begin{aligned}
& \|e^{\mathcal{R}t'}(u_\varphi, \varepsilon v_\varphi)\|_{\tilde{L}_T^\infty(B^{\frac{1}{2}})} + \|e^{\mathcal{R}t'}(\varepsilon u_\varphi, \varepsilon^2 v_\varphi)\|_{\tilde{L}_T^\infty(B^{\frac{3}{2}})} + \|e^{\mathcal{R}t'}(\partial_z u_\varphi, \varepsilon \partial_z \Delta_k^h v_\varphi)\|_{\tilde{L}_T^\infty(B^{\frac{1}{2}})} \\
& + \|e^{\mathcal{R}t'}(\varepsilon \partial_x u_\varphi, \varepsilon^2 \partial_x v_\varphi)\|_{\tilde{L}_T^2(B^{\frac{1}{2}})} \\
& + \|e^{\mathcal{R}t'}(\partial_z u_\varphi, \varepsilon \partial_z v_\varphi)\|_{\tilde{L}_T^2(B^{\frac{1}{2}})} + \|e^{\mathcal{R}t'}(\varepsilon \partial_z \partial_x u_\varphi, \varepsilon^2 \partial_z \partial_x v_\varphi)\|_{\tilde{L}_T^2(B^{\frac{1}{2}})} \\
& + \|e^{\mathcal{R}t'}(\varepsilon \partial_x^2 u_\varphi, \varepsilon^3 \partial_x^2 v_\varphi)\|_{\tilde{L}_T^2(B^{\frac{1}{2}})} + \|e^{\mathcal{R}t'}(\partial_z^2 u_\varphi, \varepsilon \partial_z^2 v_\varphi)\|_{\tilde{L}_T^2(B^{\frac{1}{2}})} \\
& \leq C \left(\|e^{a|D_x|}(u_0, \varepsilon v_0)\|_{B^{\frac{1}{2}}} + \|e^{a|D_x|}(\varepsilon u_0, \varepsilon^2 v_0)\|_{B^{\frac{3}{2}}} + \|e^{a|D_x|}(\partial_z u_0, \varepsilon \partial_z v_0)\|_{B^{\frac{1}{2}}} \right). \quad (3.40)
\end{aligned}$$

Now we give a control of the analyticity bandwidth. We deduce from (3.27) that

$$\begin{aligned}
\eta(t) &= \int_0^t (\varepsilon^2 \|\partial_x v_\varphi^\varepsilon(t)\|_{B^{\frac{1}{2}}} + \varepsilon \|\partial_z \partial_x u_\varphi^\varepsilon(t)\|_{B^{\frac{1}{2}}} + \varepsilon \|\partial_x^2 u_\varphi^\varepsilon(t)\|_{B^{\frac{1}{2}}} + \varepsilon^3 \|\partial_x^2 v_\varphi^\varepsilon(t)\|_{B^{\frac{1}{2}}} + \|\partial_z^2 u_\varphi^\varepsilon(t)\|_{B^{\frac{1}{2}}}) dt' \\
&\leq \left(\int_0^t e^{-2\mathcal{R}t'} dt' \right)^{\frac{1}{2}} \left(\int_0^t (\|e^{\mathcal{R}t'}(\varepsilon \partial_x^2 u_\varphi, \varepsilon^3 \partial_x^2 v_\varphi)\|_{B^{\frac{1}{2}}} + \|e^{\mathcal{R}t'}(\partial_z^2 u_\varphi, \varepsilon \partial_z \partial_x u_\varphi)\|_{B^{\frac{1}{2}}} + \varepsilon^2 \|\partial_x v_\varphi^\varepsilon(t)\|_{B^{\frac{1}{2}}})^2 dt' \right)^{\frac{1}{2}} \\
&\leq C_{\mathcal{R}} \left(\|e^{\mathcal{R}t'}(\varepsilon \partial_x^2 u_\varphi, \varepsilon^3 \partial_x^2 v_\varphi)\|_{\tilde{L}_T^2(B^{\frac{1}{2}})} + \|e^{\mathcal{R}t'}(\partial_z^2 u_\varphi, \varepsilon \partial_z \partial_x u_\varphi)\|_{\tilde{L}_T^2(B^{\frac{1}{2}})} + \varepsilon^2 \|e^{\mathcal{R}t'} \partial_x v_\varphi\|_{\tilde{L}_T^2(B^{\frac{1}{2}})} \right) \\
&\leq C \left(\|e^{a|D_x|}(u_0, \varepsilon v_0)\|_{B^{\frac{1}{2}}} + \|e^{a|D_x|}(\varepsilon u_0, \varepsilon^2 v_0)\|_{B^{\frac{3}{2}}} + \|e^{a|D_x|}(\partial_z u_0, \varepsilon \partial_z v_0)\|_{B^{\frac{1}{2}}} \right),
\end{aligned}$$

where $C = C_{\mathcal{R}} C$. If we take \tilde{c} to be small enough then, for any $t \in [0, T^*[$, we have

$$C \left(\|e^{a|D_x|}(u_0, \varepsilon v_0)\|_{B^{\frac{1}{2}}} + \|e^{a|D_x|}(\varepsilon u_0, \varepsilon^2 v_0)\|_{B^{\frac{3}{2}}} + \|e^{a|D_x|}(\partial_z u_0, \varepsilon \partial_z v_0)\|_{B^{\frac{1}{2}}} \right) \leq \frac{a}{2\lambda}$$

Using the definition of T^* and a continuity argument, we deduce $T^* = +\infty$. This concludes the proof of Theorem 3.1.

3.6 Convergence to the hydrostatic Navier-Stokes-alpha system

This section is devoted to the proof of Theorem 3.2. We justify the limit of the rescaled Navier-Stokes-alpha system (3.42) towards the hydrostatic Navier-Stokes-alpha system (2.6) when $\varepsilon \rightarrow 0$. We introduce the following difference terms

$$w_\varepsilon^1 = u^\varepsilon - u, \quad w_\varepsilon^2 = v^\varepsilon - v, \quad \tilde{p}_\varepsilon = p^\varepsilon - p. \quad (3.41)$$

We recall that $(u^\varepsilon, v^\varepsilon, p^\varepsilon)$ is a solution of following system

$$\begin{cases} \partial_t \omega^\varepsilon - \alpha_1^2 \varepsilon^2 \partial_t \partial_x^2 u^\varepsilon + u^\varepsilon \partial_x \omega^\varepsilon - \alpha_1^2 \varepsilon^2 u^\varepsilon \partial_x^3 u^\varepsilon + v^\varepsilon \partial_z \omega^\varepsilon - \alpha_1^2 \varepsilon^2 v^\varepsilon \partial_z \partial_x^2 u^\varepsilon + \alpha_1^2 (\partial_z u^\varepsilon \partial_x \partial_z u^\varepsilon - \partial_z^2 u^\varepsilon \partial_x u^\varepsilon) \\ - \alpha_1^2 \varepsilon^2 \partial_x u^\varepsilon \partial_x^2 u^\varepsilon + \varepsilon^2 \gamma^\varepsilon \partial_x v^\varepsilon - \varepsilon^4 \alpha_1^2 \partial_x^2 v^\varepsilon \partial_x v^\varepsilon = \partial_z^2 \omega^\varepsilon + \varepsilon^2 \partial_x^2 \omega^\varepsilon - \varepsilon^2 \alpha_1^2 \partial_z^2 \partial_x^2 u^\varepsilon - \alpha_1^2 \varepsilon^4 \partial_x^4 u^\varepsilon - \partial_x p^\varepsilon \\ \varepsilon^2 \partial_t \gamma^\varepsilon - \alpha_1^2 \varepsilon^4 \partial_t \partial_x^2 v^\varepsilon + \varepsilon^2 u^\varepsilon \partial_x \gamma^\varepsilon - \alpha_1^2 \varepsilon^4 u^\varepsilon \partial_x^3 v^\varepsilon + \varepsilon^2 v^\varepsilon \partial_z \gamma^\varepsilon - \alpha_1^2 \varepsilon^4 v^\varepsilon \partial_z \partial_x^2 v^\varepsilon - \alpha_1^2 \varepsilon^2 \partial_x^2 u^\varepsilon \partial_z u^\varepsilon \\ + \varepsilon^2 \gamma^\varepsilon \partial_z v^\varepsilon - \alpha_1^2 \varepsilon^4 \partial_x^2 v^\varepsilon \partial_z v^\varepsilon = \varepsilon^2 \partial_z^2 \gamma^\varepsilon + \varepsilon^4 \partial_x^2 \gamma^\varepsilon - \alpha_1^2 \varepsilon^4 \partial_z^2 \partial_x^2 v^\varepsilon - \alpha_1^2 \varepsilon^6 \partial_x^4 v^\varepsilon - \partial_z p^\varepsilon \\ \partial_x u^\varepsilon + \partial_z v^\varepsilon = 0, \end{cases}$$

and (u, v, p) is a solution of the limit system

$$\begin{cases} \partial_t \omega + u \partial_x \omega + v \partial_z \omega + \alpha_1^2 (\partial_z u \partial_x \partial_z u - \partial_z^2 u \partial_x u) = \partial_z^2 \omega - \partial_x p \\ \partial_z p = 0 \\ \partial_x u + \partial_y v = 0 \\ (u, v) = 0, \quad \partial_z u = 0 \quad \text{on } z = 0 \\ (u, v) = 0, \quad \partial_z u = 0 \quad \text{on } z = 1 \\ u|_{t=0} = u_0. \end{cases}$$

Therefore $(w_\varepsilon^1, w_\varepsilon^2, \tilde{p}_\varepsilon)$ verifies

$$\begin{cases} \partial_t (w_\varepsilon^1 - \alpha_1^2 \partial_z^2 w_\varepsilon^1) - \alpha_1^2 \varepsilon^2 \partial_t \partial_x^2 w_\varepsilon^1 - \partial_z^2 (w_\varepsilon^1 - \alpha_1^2 \partial_z^2 w_\varepsilon^1) - \varepsilon^2 \partial_x^2 (w_\varepsilon^1 - \alpha_1^2 \partial_z^2 w_\varepsilon^1) \\ + \varepsilon^2 \alpha_1^2 \partial_z^2 \partial_x^2 w_\varepsilon^1 + \alpha_1^2 \varepsilon^4 \partial_x^4 w_\varepsilon^1 + \partial_x \tilde{p}_\varepsilon = R_\varepsilon^1 \\ \varepsilon^2 \partial_t (w_\varepsilon^2 - \alpha_1^2 \partial_z^2 w_\varepsilon^2) - \alpha_1^2 \varepsilon^4 \partial_t \partial_x^2 w_\varepsilon^2 - \varepsilon^2 \partial_z^2 (w_\varepsilon^2 - \alpha_1^2 \partial_z^2 w_\varepsilon^2) - \varepsilon^4 \partial_x^2 (w_\varepsilon^2 - \alpha_1^2 \partial_z^2 w_\varepsilon^2) \\ + \alpha_1^2 \varepsilon^4 \partial_z^2 \partial_x^2 w_\varepsilon^2 + \alpha_1^2 \varepsilon^6 \partial_x^4 w_\varepsilon^2 + \partial_z \tilde{p}_\varepsilon = R_\varepsilon^2 \\ \partial_x w_\varepsilon^1 + \partial_z w_\varepsilon^2 = 0, \\ (w_\varepsilon^1, w_\varepsilon^2)|_{z=0} = (w_\varepsilon^1, w_\varepsilon^2)|_{z=1} = 0 \\ (\nabla w_\varepsilon^1, \nabla w_\varepsilon^2)|_{z=0} = (\nabla w_\varepsilon^1, \nabla w_\varepsilon^2)|_{z=1} = 0 \\ (w_\varepsilon^1, w_\varepsilon^2)|_{t=0} = (u_0^\varepsilon - u_0, v_0^\varepsilon - v_0), \end{cases} \quad (3.42)$$

where

$$\begin{aligned} R_\varepsilon^1 &= (\varepsilon^2 \partial_x^2 \omega + \alpha_1^2 \varepsilon^2 \partial_t \partial_x^2 u - \varepsilon^2 \alpha_1^2 \partial_z^2 \partial_x^2 u - \alpha_1^2 \varepsilon^4 \partial_x^4 u) \\ &\quad - (u^\varepsilon \partial_x \omega^\varepsilon - u \partial_x \omega) - (v^\varepsilon \partial_z \omega^\varepsilon - v \partial_z \omega) \\ &\quad - \alpha_1^2 (\partial_z u^\varepsilon \partial_x \partial_z u^\varepsilon - \partial_z u \partial_x \partial_z u) + \alpha_1^2 (\partial_z^2 u^\varepsilon \partial_x u^\varepsilon - \partial_z^2 u \partial_x u) \\ &\quad + (\alpha_1^2 \varepsilon^2 u^\varepsilon \partial_x^3 u^\varepsilon + \alpha_1^2 \varepsilon^2 \partial_x u^\varepsilon \partial_x^2 u^\varepsilon) + \alpha_1^2 \varepsilon^2 v^\varepsilon \partial_z \partial_x^2 u^\varepsilon - \varepsilon^2 \gamma^\varepsilon \partial_x v^\varepsilon + \varepsilon^4 \alpha_1^2 \partial_x^2 v^\varepsilon \partial_x v^\varepsilon, \end{aligned} \quad (3.43)$$

and

$$\begin{aligned} R_\varepsilon^2 &= -\varepsilon^2 \partial_t (v - \alpha_1^2 \partial_z^2 v) + \varepsilon^4 \partial_x^2 (v - 2\alpha_1^2 \partial_z^2 v) + \varepsilon^2 \partial_z^2 (v - \alpha_1^2 \partial_z^2 v) + \alpha_1^2 \varepsilon^4 \partial_t \partial_x^2 v - \alpha_1^2 \varepsilon^6 \partial_x^4 v \\ &\quad - \varepsilon^2 u^\varepsilon \partial_x \gamma^\varepsilon + \alpha_1^2 \varepsilon^4 u^\varepsilon \partial_x^3 v^\varepsilon + \alpha_1^2 \varepsilon^2 \partial_x^2 u^\varepsilon \partial_z v^\varepsilon - 2\varepsilon^2 v^\varepsilon \partial_z v^\varepsilon + \varepsilon^2 \alpha_1^2 v^\varepsilon \partial_z^3 v^\varepsilon + \alpha_1^2 \varepsilon^4 v^\varepsilon \partial_z \partial_x^2 v^\varepsilon \\ &\quad - \varepsilon^2 \alpha_1^2 \partial_z^2 v^\varepsilon \partial_z v^\varepsilon + \alpha_1^2 \varepsilon^4 \partial_x^2 v^\varepsilon \partial_z v^\varepsilon. \end{aligned} \quad (3.44)$$

Proof of Theorem 3.2. We recall from (3.26)

$$f_\psi(t, x, z) = e^{\psi(t, D_x)} f(t, x, z) := \mathcal{F}_{\xi \rightarrow x}^{-1}(e^{\psi(t, \xi)} \hat{f}(t, \xi, z)) \quad \text{with} \quad \psi(t, \xi) = (a - \beta \zeta(t))|\xi|, \quad (3.45)$$

where $\beta \geq \lambda$ will be determined later and ζ is defined by

$$\begin{cases} \zeta'(t) = \|(\varepsilon^2 \partial_x^2 u_\varphi^\varepsilon, \partial_z^2 u_\varphi^\varepsilon)(t)\|_{B^{\frac{1}{2}}} + \|(\varepsilon^3 \partial_x^2 v_\varphi^\varepsilon, \varepsilon \partial_z \partial_x u_\varphi^\varepsilon)(t)\|_{B^{\frac{1}{2}}} + \varepsilon^2 \|\partial_x v_\varphi^\varepsilon(t)\|_{B^{\frac{1}{2}}} + \|\partial_z^2 u_\phi(t)\|_{B^{\frac{1}{2}}} \\ \zeta(0) = 0. \end{cases} \quad (3.46)$$

We observe that, if we take c in (2.10) and \tilde{c} in (3.7) small enough then $\psi(t) > 0$ and

$$\psi(t, \xi) \leq \min(\phi(t, \xi), \varphi(t, \xi)).$$

Estimate (2.12) implies, for any $s > 0$,

$$\begin{aligned} & \|e^{Rt}(\partial_t u)_\phi\|_{\tilde{L}^2(\mathbb{R}^+, B^s)} + \|e^{Rt}(\partial_t \partial_z u)_\phi\|_{\tilde{L}^2(\mathbb{R}^+, B^s)} + \|e^{Rt} \partial_z u_\phi\|_{\tilde{L}^\infty(\mathbb{R}^+, B^s)} + \|e^{Rt} \partial_z^2 u_\phi\|_{\tilde{L}^\infty(\mathbb{R}^+, B^s)} \\ & \leq C \left(\|e^{a|D_x|} \partial_z u_0\|_{B^s} + \|e^{a|D_x|} \partial_z^2 u_0\|_{B^s} + \|e^{a|D_x|} u_0\|_{B^{s+1}} + \|e^{a|D_x|} \partial_z u_0\|_{B^{s+1}} \right). \end{aligned} \quad (3.47)$$

From the previous estimate and Estimates (3.8), (2.23), we deduce that there exist $M \geq 1$ independent of ε such that, for any $0 < s < 1$, we have

$$\begin{aligned} & \|\partial_z^2 u_\phi\|_{\tilde{L}^2(\mathbb{R}^+, B^{s+\frac{1}{2}} \cap B^{s+3})} + \|\partial_t u_\phi\|_{\tilde{L}^2(\mathbb{R}^+, B^{s+1})} + \|\partial_z u_\phi\|_{L^\infty(\mathbb{R}^+, B^{\frac{3}{2}})} + \varepsilon \|\partial_x u_\varphi^\varepsilon\|_{L^\infty(\mathbb{R}^+, B^{\frac{1}{2}})} \\ & \quad + \|u_\varphi^\varepsilon\|_{L^\infty(\mathbb{R}^+, B^{\frac{1}{2}})} + \|u_\phi\|_{L^\infty(\mathbb{R}^+, B^{\frac{3}{2}})} + \|u_\phi\|_{L^\infty(\mathbb{R}^+, B^{\frac{7}{2}})} \\ & \quad + \varepsilon \|\partial_z \partial_x u_\varphi^\varepsilon\|_{\tilde{L}^2(\mathbb{R}^+, B^s)} + \varepsilon^3 \|\partial_x^2 v_\varphi^\varepsilon\|_{\tilde{L}^2(\mathbb{R}^+, B^s)} + \|(\partial_t \partial_z u)_\phi\|_{\tilde{L}^2(\mathbb{R}^+, B^{s+1})} \\ & \quad + \|\partial_z u_\varphi^\varepsilon\|_{L^\infty(\mathbb{R}^+, B^{\frac{1}{2}})} + \|u_\phi\|_{L^\infty(\mathbb{R}^+, B^{\frac{3}{2}})} \leq M \end{aligned} \quad (3.48)$$

where u_ϕ and u_φ^ε are defined respectively by (2.8) and (?). Performing a similar derivation as for (3.31), we get

$$\begin{aligned} X_{1,k} & \leq \frac{1}{2} \|(\Delta_k^h w_{\varepsilon_\psi}^1(0), \varepsilon \Delta_k^h w_{\varepsilon_\psi}^2(0))\|_{L^2}^2 + \frac{1}{2} \alpha_1^2 \varepsilon^2 c 2^{2k} \|(\Delta_k^h w_{\varepsilon_\psi}^1(0), \varepsilon \Delta_k^h w_{\varepsilon_\psi}^2(0))\|_{L^2}^2 \\ & \quad + \frac{1}{2} \alpha_1^2 \|(\partial_z \Delta_k^h w_{\varepsilon_\psi}^1(0), \varepsilon \partial_z \Delta_k^h w_{\varepsilon_\psi}^2(0))\|_{L^2}^2 + R_k^1 + R_k^2. \end{aligned} \quad (3.49)$$

where $X_{1,k}$, R_k^1 and R_k^2 are defined as follows

$$\begin{aligned} X_{1,k} & = \frac{1}{2} \|e^{\mathcal{R}t'} (\Delta_k^h w_{\varepsilon_\psi}^1, \varepsilon \Delta_k^h w_{\varepsilon_\psi}^2)\|_{L^2}^2 + \frac{1}{2} \alpha_1^2 \varepsilon^2 c 2^{2k} \|e^{\mathcal{R}t'} (\Delta_k^h w_{\varepsilon_\psi}^1, \varepsilon \Delta_k^h w_{\varepsilon_\psi}^2)\|_{L^2}^2 + \frac{1}{2} \alpha_1^2 \|e^{\mathcal{R}t'} (\partial_z \Delta_k^h w_{\varepsilon_\psi}^1, \varepsilon \partial_z \Delta_k^h w_{\varepsilon_\psi}^2)\|_{L^2}^2 \\ & \quad + \frac{3}{4} \beta 2^k \int_0^t \eta'(t') \|e^{\mathcal{R}t'} (\Delta_k^h w_{\varepsilon_\psi}^1, \varepsilon \Delta_k^h w_{\varepsilon_\psi}^2)\|_{L^2}^2 dt' + \frac{3}{4} \alpha_1^2 \beta 2^k \int_0^t \eta'(t') \|e^{\mathcal{R}t'} \partial_z (\Delta_k^h w_{\varepsilon_\psi}^1, \varepsilon \Delta_k^h w_{\varepsilon_\psi}^2)\|_{L^2}^2 dt' \\ & \quad + \frac{3}{4} \alpha_1^2 \varepsilon^2 \beta 2^k \int_0^t \eta'(t') \|e^{\mathcal{R}t'} (\Delta_k^h \partial_x w_{\varepsilon_\psi}^1, \varepsilon \Delta_k^h \partial_x w_{\varepsilon_\psi}^2)\|_{L^2}^2 dt' + \varepsilon^2 \int_0^t e^{2\mathcal{R}t'} \|(\Delta_k^h \partial_x w_{\varepsilon_\psi}^1, \varepsilon \Delta_k^h \partial_x w_{\varepsilon_\psi}^2)\|_{L^2}^2 dt' \\ & \quad + \int_0^t e^{2\mathcal{R}t'} \|(\partial_z \Delta_k^h w_{\varepsilon_\psi}^1, \varepsilon \partial_z \Delta_k^h w_{\varepsilon_\psi}^2)\|_{L^2}^2 dt' + \varepsilon^2 \alpha_1^2 \int_0^t e^{2\mathcal{R}t'} \|(\partial_z \partial_x \Delta_k^h w_{\varepsilon_\psi}^1, \varepsilon \partial_z \partial_x \Delta_k^h w_{\varepsilon_\psi}^2)\|_{L^2}^2 dt' \\ & \quad + \alpha_1^2 \int_0^t e^{2\mathcal{R}t'} \left(\varepsilon^2 \|(\Delta_k^h \partial_x^2 w_{\varepsilon_\psi}^1, \varepsilon^2 \Delta_k^h \partial_x^2 w_{\varepsilon_\psi}^2)\|_{L^2}^2 + \|(\partial_z^2 \Delta_k^h w_{\varepsilon_\psi}^1, \varepsilon \partial_z^2 \Delta_k^h w_{\varepsilon_\psi}^2)\|_{L^2}^2 \right) dt', \end{aligned}$$

$$\begin{aligned}
R_k^1 &= \int_0^T \left| \left\langle \Delta_k^h (\varepsilon^2 \partial_x^2 \omega + \alpha_1^2 \varepsilon^2 \partial_t \partial_x^2 u - \varepsilon^2 \alpha_1^2 \partial_z^2 \partial_x^2 u - \alpha_1^2 \varepsilon^4 \partial_x^4 u)_\psi, \Delta_k^h w_{\varepsilon_\psi}^1 \right\rangle \right| dt' \\
&+ \int_0^T \left| \left\langle \Delta_k^h ((u^\varepsilon \partial_x \omega^\varepsilon - u \partial_x \omega))_\psi, \Delta_k^h w_{\varepsilon_\psi}^1 \right\rangle \right| dt' + \int_0^T \left| \left\langle \Delta_k^h (v^\varepsilon \partial_z \omega^\varepsilon - v \partial_z \omega)_\psi, \Delta_k^h w_{\varepsilon_\psi}^1 \right\rangle \right| dt' \\
&+ \alpha_1^2 \int_0^T \left| \left\langle \Delta_k^h (\partial_z u^\varepsilon \partial_x \partial_z u^\varepsilon - \partial_z u \partial_x \partial_z u)_\psi, \Delta_k^h w_{\varepsilon_\psi}^1 \right\rangle \right| dt' + \alpha_1^2 \int_0^T \left| \left\langle \Delta_k^h (\partial_z^2 u^\varepsilon \partial_x u^\varepsilon - \partial_z^2 u \partial_x u)_\psi, \Delta_k^h w_{\varepsilon_\psi}^1 \right\rangle \right| dt' \\
&+ \alpha_1^2 \varepsilon^2 \int_0^T \left| \left\langle \Delta_k^h (u^\varepsilon \partial_x^2 u^\varepsilon)_\psi, \Delta_k^h \partial_x w_{\varepsilon_\psi}^1 \right\rangle \right| dt' + \alpha_1^2 \varepsilon^2 \int_0^T \left| \left\langle \Delta_k^h (v^\varepsilon \partial_z \partial_x^2 u^\varepsilon)_\psi, \Delta_k^h w_{\varepsilon_\psi}^1 \right\rangle \right| dt' \\
&+ \alpha_1^2 \varepsilon^2 \int_0^T \left| \left\langle \Delta_k^h (-\varepsilon^2 \gamma^\varepsilon \partial_x v^\varepsilon + \varepsilon^4 \alpha_1^2 \partial_x^2 v^\varepsilon \partial_x v^\varepsilon)_\psi, \Delta_k^h w_{\varepsilon_\psi}^1 \right\rangle \right| dt',
\end{aligned}$$

and

$$\begin{aligned}
R_k^2 &= \int_0^T \left| \left\langle \Delta_k^h (\partial_t (v - \alpha_1^2 \partial_z^2 v) + \varepsilon^4 \partial_x^2 (v - 2\alpha_1^2 \partial_z^2 v) + \varepsilon^2 \partial_z^2 (v - \alpha_1^2 \partial_z^2 v) + \alpha_1^2 \varepsilon^4 \partial_t \partial_x^2 v - \alpha_1^2 \varepsilon^6 \partial_x^4 v)_\psi, \Delta_k^h w_{\varepsilon_\psi}^2 \right\rangle \right| dt' \\
&+ \int_0^T \left| \left\langle \Delta_k^h (-\varepsilon^2 u^\varepsilon \partial_x \gamma^\varepsilon + \alpha_1^2 \varepsilon^4 u^\varepsilon \partial_x^3 v^\varepsilon + \alpha_1^2 \varepsilon^2 \partial_x^2 u^\varepsilon \partial_z u^\varepsilon - 2\varepsilon^2 v^\varepsilon \partial_z v)_\psi, \Delta_k^h w_{\varepsilon_\psi}^2 \right\rangle \right| dt' \\
&+ \int_0^T \left| \left\langle \Delta_k^h (\varepsilon^2 \alpha_1^2 v^\varepsilon \partial_z^2 v + \alpha_1^2 \varepsilon^4 v^\varepsilon \partial_x^2 v^\varepsilon)_\psi, \Delta_k^h \partial_z w_{\varepsilon_\psi}^2 \right\rangle \right| dt'.
\end{aligned}$$

Multiplying $\|X_{1,k}\|_{L^\infty([0,T])}$ by 2^{2ks} , taking square root and then summing with respect to $k \in \mathbb{Z}$, we deduce from (3.49) that

$$\begin{aligned}
X_1 &\leq C \left[\|(w_{\varepsilon_\psi}^1(0), \varepsilon w_{\varepsilon_\psi}^2(0))\|_{B^s} + \|(\varepsilon w_{\varepsilon_\psi}^1(0), \varepsilon^2 w_{\varepsilon_\psi}^2(0))\|_{B^{s+1}} \right. \\
&\quad \left. + \|(\partial_z w_{\varepsilon_\psi}^1(0), \varepsilon \partial_z w_{\varepsilon_\psi}^2(0))\|_{B^s} + \sum_{k \in \mathbb{Z}} 2^{ks} (R_k^1)^{\frac{1}{2}} + \sum_{k \in \mathbb{Z}} 2^{ks} (R_k^2)^{\frac{1}{2}} \right] \quad (3.50)
\end{aligned}$$

where

$$\begin{aligned}
X_1 &= \|e^{\mathcal{R}t'} (w_{\varepsilon_\psi}^1, \varepsilon w_{\varepsilon_\psi}^2)\|_{\tilde{L}_T^\infty(B^s)} + \|e^{\mathcal{R}t'} (\varepsilon w_{\varepsilon_\psi}^1, \varepsilon^2 w_{\varepsilon_\psi}^2)\|_{\tilde{L}_T^\infty(B^{s+1})} + \|e^{\mathcal{R}t'} (\partial_z w_{\varepsilon_\psi}^1, \varepsilon \partial_z w_{\varepsilon_\psi}^2)\|_{\tilde{L}_T^\infty(B^s)} \\
&\quad + \sqrt{\beta} \|e^{\mathcal{R}t'} (w_{\varepsilon_\psi}^1, \varepsilon w_{\varepsilon_\psi}^2)\|_{\tilde{L}_{T, \zeta'(t)}^2(B^{s+\frac{1}{2}})} + \sqrt{\beta} \|e^{\mathcal{R}t'} \partial_z (\Delta_k^h w_{\varepsilon_\psi}^1, \varepsilon \Delta_k^h w_{\varepsilon_\psi}^2)\|_{\tilde{L}_{T, \zeta'(t)}^2(B^{s+\frac{1}{2}})} \\
&\quad + \sqrt{\beta} \|e^{\mathcal{R}t'} (\varepsilon \partial_x w_{\varepsilon_\psi}^1, \varepsilon^2 \partial_x w_{\varepsilon_\psi}^2)\|_{\tilde{L}_{T, \zeta'(t)}^2(B^{s+\frac{1}{2}})} + \|e^{\mathcal{R}t'} (\varepsilon \partial_x w_{\varepsilon_\psi}^1, \varepsilon^2 \partial_x w_{\varepsilon_\psi}^2)\|_{\tilde{L}_T^2(B^s)} \\
&\quad + \|e^{\mathcal{R}t'} (\partial_z w_{\varepsilon_\psi}^1, \varepsilon \partial_z w_{\varepsilon_\psi}^2)\|_{\tilde{L}_T^2(B^s)} + \|e^{\mathcal{R}t'} (\varepsilon \partial_z \partial_x w_{\varepsilon_\psi}^1, \varepsilon^2 \partial_z \partial_x w_{\varepsilon_\psi}^2)\|_{\tilde{L}_T^2(B^s)} \\
&\quad + \|e^{\mathcal{R}t'} (\varepsilon \partial_x^2 w_{\varepsilon_\psi}^1, \varepsilon^3 \partial_x^2 w_{\varepsilon_\psi}^2)\|_{\tilde{L}_T^2(B^s)} + \|e^{\mathcal{R}t'} (\partial_z^2 w_{\varepsilon_\psi}^1, \varepsilon \partial_z^2 w_{\varepsilon_\psi}^2)\|_{\tilde{L}_T^2(B^s)}.
\end{aligned}$$

In order to complete the proof of Theorem 3.2, we need the following estimates.

Proposition 3.6.1. *We have*

$$\begin{aligned}
\sum_{k \in \mathbb{Z}} \sqrt{2^{2ks} R_k^1} &\leq C \left[M\varepsilon + M \|\partial_z w_{\varepsilon_\psi}^1\|_{\tilde{L}_{T, \zeta'(t)}^2(B^{s+\frac{1}{2}})} + M\varepsilon \|\partial_x w_{\varepsilon_\psi}^1\|_{\tilde{L}_{T, \zeta'(t)}^2(B^{s+\frac{1}{2}})} + M \|w_{\varepsilon_\psi}^1\|_{\tilde{L}_{T, \zeta'(t)}^2(B^{s+\frac{1}{2}})} \right] \\
&\quad + \frac{1}{2} \varepsilon \|\partial_z \partial_x w_{\varepsilon_\psi}^1\|_{\tilde{L}_T^2(B^s)} + \frac{1}{2} \varepsilon^2 \|\partial_x w_{\varepsilon_\psi}^2\|_{\tilde{L}_T^2(B^s)} \\
&\quad + \frac{1}{2} \varepsilon \|w_{\varepsilon_\psi}^1\|_{\tilde{L}_T^2(B^{s+1})} + \frac{1}{2} \varepsilon^2 \|w_{\varepsilon_\psi}^1\|_{\tilde{L}_T^2(B^{s+2})} + \frac{1}{2} \|\partial_z^2 w_{\varepsilon_\psi}^1\|_{\tilde{L}_T^2(B^s)} \quad (3.51)
\end{aligned}$$

Proposition 3.6.2. *We have*

$$\begin{aligned}
& \sum_{k \in \mathbb{Z}} \sqrt{2^{2ks} R_k^2} \\
& \leq C \left[M\varepsilon + M\varepsilon \|\partial_z w_{\varepsilon\psi}^2\|_{\tilde{L}_{T,\zeta'(t)}^2(B^{s+\frac{1}{2}})} + M\varepsilon^2 \|\partial_x w_{\varepsilon\psi}^1\|_{\tilde{L}_{T,\zeta'(t)}^2(B^{s+\frac{1}{2}})} + M\varepsilon \|w_{\varepsilon\psi}^2\|_{\tilde{L}_{T,\zeta'(t)}^2(B^{s+\frac{1}{2}})} \right. \\
& \quad \left. + M\varepsilon^2 \|\partial_x w_{\varepsilon\psi}^2\|_{\tilde{L}_{T,\zeta'(t)}^2(B^{s+\frac{1}{2}})} + M \|w_{\varepsilon\psi}^1\|_{\tilde{L}_{T,\zeta'(t)}^2(B^{s+\frac{1}{2}})} \right] \\
& \quad + \frac{1}{2}\varepsilon \|\partial_z w_{\varepsilon\psi}^2\|_{\tilde{L}_T^2(B^s)} + \frac{1}{2}\varepsilon^2 \|w_{\varepsilon\psi}^2\|_{\tilde{L}_T^2(B^{s+1})} + \frac{1}{2}\varepsilon^3 \|\partial_x^2 w_{\varepsilon\psi}^2\|_{\tilde{L}_T^2(B^s)} + \frac{1}{2}\varepsilon^3 \|\partial_x^2 w_{\varepsilon\psi}^2\|_{\tilde{L}_T^2(B^s)} \\
& \quad + \frac{1}{2}\varepsilon \|\partial_z \partial_x w_{\varepsilon\psi}^1\|_{\tilde{L}_T^2(B^s)} \quad (3.52)
\end{aligned}$$

We skip the proof of these above propositions for now. These proof will be given in Appendices 3.7 and 3.8. We continue the proof of Theorem 3.2 as follows. We inject the estimates of Propositions 3.6.1 and 3.6.2 in Estimate (3.50) to get

$$\begin{aligned}
X_1 & \leq C \left[\|(w_{\varepsilon\psi}^1(0), \varepsilon w_{\varepsilon\psi}^2(0))\|_{B^s} + \|(\varepsilon w_{\varepsilon\psi}^1(0), \varepsilon^2 w_{\varepsilon\psi}^2(0))\|_{B^{s+1}} + \|(\partial_z w_{\varepsilon\psi}^1(0), \varepsilon \partial_z w_{\varepsilon\psi}^2(0))\|_{B^s} + M\varepsilon \right] \\
& \quad + C \left[M \|\partial_z w_{\varepsilon\psi}^1\|_{\tilde{L}_{T,\zeta'(t)}^2(B^{s+\frac{1}{2}})} + M\varepsilon \|\partial_x w_{\varepsilon\psi}^1\|_{\tilde{L}_{T,\zeta'(t)}^2(B^{s+\frac{1}{2}})} \right. \\
& \quad \left. + M\varepsilon \|\partial_z w_{\varepsilon\psi}^2\|_{\tilde{L}_{T,\zeta'(t)}^2(B^{s+\frac{1}{2}})} + M\varepsilon \|w_{\varepsilon\psi}^2\|_{\tilde{L}_{T,\zeta'(t)}^2(B^{s+\frac{1}{2}})} \right. \\
& \quad \left. + M\varepsilon^2 \|\partial_x w_{\varepsilon\psi}^2\|_{\tilde{L}_{T,\zeta'(t)}^2(B^{s+\frac{1}{2}})} + M \|w_{\varepsilon\psi}^1\|_{\tilde{L}_{T,\zeta'(t)}^2(B^{s+\frac{1}{2}})} \right]. \quad (3.53)
\end{aligned}$$

Choosing $\beta = C^2 M^2$, we obtain

$$\begin{aligned}
& \|e^{\mathcal{R}t'}(w_{\varepsilon\psi}^1, \varepsilon w_{\varepsilon\psi}^2)\|_{\tilde{L}_T^\infty(B^s)} + \|e^{\mathcal{R}t'}(\varepsilon w_{\varepsilon\psi}^1, \varepsilon^2 w_{\varepsilon\psi}^2)\|_{\tilde{L}_T^\infty(B^{s+1})} + \|e^{\mathcal{R}t'}(\partial_z w_{\varepsilon\psi}^1, \varepsilon \partial_z w_{\varepsilon\psi}^2)\|_{\tilde{L}_T^\infty(B^s)} \\
& \quad + \|e^{\mathcal{R}t'}(\varepsilon \partial_x w_{\varepsilon\psi}^1, \varepsilon^2 \partial_x w_{\varepsilon\psi}^2)\|_{\tilde{L}_T^2(B^s)} + \|e^{\mathcal{R}t'}(\partial_z w_{\varepsilon\psi}^1, \varepsilon \partial_z w_{\varepsilon\psi}^2)\|_{\tilde{L}_T^2(B^s)} \\
& \quad + \|e^{\mathcal{R}t'}(\varepsilon \partial_x^2 w_{\varepsilon\psi}^1, \varepsilon^3 \partial_x^2 w_{\varepsilon\psi}^1)\|_{\tilde{L}_T^2(B^s)} + \|e^{\mathcal{R}t'}(\partial_z^2 w_{\varepsilon\psi}^1, \varepsilon \partial_z^2 w_{\varepsilon\psi}^1)\|_{\tilde{L}_T^2(B^s)} \\
& \quad \quad + \|e^{\mathcal{R}t'}(\varepsilon \partial_z \partial_x w_{\varepsilon\psi}^1, \varepsilon^2 \partial_z \partial_x w_{\varepsilon\psi}^2)\|_{\tilde{L}_T^2(B^s)} \\
& \leq C \left[\|(w_{\varepsilon\psi}^1(0), \varepsilon w_{\varepsilon\psi}^2(0))\|_{B^s} + \|(\varepsilon w_{\varepsilon\psi}^1(0), \varepsilon^2 w_{\varepsilon\psi}^2(0))\|_{B^{s+1}} \right. \\
& \quad \left. + \|(\partial_z w_{\varepsilon\psi}^1(0), \varepsilon \partial_z w_{\varepsilon\psi}^2(0))\|_{B^s} + M\varepsilon \right], \quad (3.54)
\end{aligned}$$

which concludes the proof of Theorem 3.2. \square

3.7 Appendix. Proof of proposition 3.6.1

We first estimate R_k^1 . We recall

$$\begin{aligned}
R_k^1 &= \int_0^T \left| \left\langle \Delta_k^h (\varepsilon^2 \partial_x^2 \omega + \alpha_1^2 \varepsilon^2 \partial_t \partial_x^2 u - \varepsilon^2 \alpha_1^2 \partial_z^2 \partial_x^2 u - \alpha_1^2 \varepsilon^4 \partial_x^4 u)_\psi, \Delta_k^h w_{\varepsilon\psi}^1 \right\rangle \right| dt' \\
&+ \int_0^T \left| \left\langle \Delta_k^h ((u^\varepsilon \partial_x \omega^\varepsilon - u \partial_x \omega))_\psi, \Delta_k^h w_{\varepsilon\psi}^1 \right\rangle \right| dt' + \int_0^T \left| \left\langle \Delta_k^h (v^\varepsilon \partial_z \omega^\varepsilon - v \partial_z \omega)_\psi, \Delta_k^h w_{\varepsilon\psi}^1 \right\rangle \right| dt' \\
&+ \alpha_1^2 \int_0^T \left| \left\langle \Delta_k^h (\partial_z u^\varepsilon \partial_x \partial_z u^\varepsilon - \partial_z u \partial_x \partial_z u)_\psi, \Delta_k^h w_{\varepsilon\psi}^1 \right\rangle \right| dt' + \alpha_1^2 \int_0^T \left| \left\langle \Delta_k^h (\partial_z^2 u^\varepsilon \partial_x u^\varepsilon - \partial_z^2 u \partial_x u)_\psi, \Delta_k^h w_{\varepsilon\psi}^1 \right\rangle \right| dt' \\
&+ \alpha_1^2 \varepsilon^2 \int_0^T \left| \left\langle \Delta_k^h (u^\varepsilon \partial_x^2 u^\varepsilon)_\psi, \Delta_k^h \partial_x w_{\varepsilon\psi}^1 \right\rangle \right| dt' + \alpha_1^2 \varepsilon^2 \int_0^T \left| \left\langle \Delta_k^h (v^\varepsilon \partial_z \partial_x^2 u^\varepsilon)_\psi, \Delta_k^h w_{\varepsilon\psi}^1 \right\rangle \right| dt' \\
&+ \alpha_1^2 \varepsilon^2 \int_0^T \left| \left\langle \Delta_k^h (-\varepsilon^2 \gamma^\varepsilon \partial_x v^\varepsilon + \varepsilon^4 \alpha_1^2 \partial_x^2 v^\varepsilon \partial_x v^\varepsilon)_\psi, \Delta_k^h w_{\varepsilon\psi}^1 \right\rangle \right| dt',
\end{aligned}$$

Since R_k^1 is a sum of several terms, we will break down this estimate by treating each of the components of R_k^1 separately.

$$\begin{aligned}
&1. \text{ Estimate of the terms } \int_0^T \left| \left\langle \Delta_k^h (\varepsilon^2 \partial_x^2 \omega + \alpha_1^2 \varepsilon^2 \partial_t \partial_x^2 u - \varepsilon^2 \alpha_1^2 \partial_z^2 \partial_x^2 u - \alpha_1^2 \varepsilon^4 \partial_x^4 u)_\psi, \Delta_k^h w_{\varepsilon\psi}^1 \right\rangle \right| dt' \\
&\varepsilon^2 \int_0^T \left| \left\langle \Delta_k^h \partial_x^2 \omega_\psi, \Delta_k^h w_{\varepsilon\psi}^1 \right\rangle \right| \leq \varepsilon^2 \int_0^T \left| \left\langle \Delta_k^h \partial_x^2 u_\psi, \Delta_k^h w_{\varepsilon\psi}^1 \right\rangle \right| + \alpha_1^2 \varepsilon^2 \int_0^T \left| \left\langle \Delta_k^h \partial_x^2 \partial_z^2 u_\psi, \Delta_k^h w_{\varepsilon\psi}^1 \right\rangle \right|
\end{aligned}$$

We have

$$\begin{aligned}
\varepsilon^2 \int_0^T \left| \left\langle \Delta_k^h \partial_x^2 u_\psi, \Delta_k^h w_{\varepsilon\psi}^1 \right\rangle \right| dt' &\leq \int_0^T \varepsilon \|\Delta_k^h \partial_x^2 u_\psi\|_{L^2} \|\Delta_k^h w_{\varepsilon\psi}^1\|_{L^2} dt' \\
&\lesssim 2^{-2ks} \int_0^T 2^{k(s+1)} \varepsilon \|\Delta_k^h \partial_z u_\psi\|_{L^2} 2^{k(s+1)} \varepsilon \|\Delta_k^h w_{\varepsilon\psi}^1\|_{L^2} dt' \\
&\lesssim 2^{-2ks} \tilde{d}_k \varepsilon \|\partial_z u_\psi\|_{\tilde{L}_T^2(B^{s+1})} \|\varepsilon w_{\varepsilon\psi}^1\|_{\tilde{L}_T^2(B^{s+1})};
\end{aligned}$$

where

$$\tilde{d}_k = d_k(\partial_z u_\psi, 1) d_k(w_{\varepsilon\psi}^1, 1)$$

and d_k is defined in Remark 3 and $\sum_k d_k = 1$. And in similar way we get

$$\varepsilon^2 \int_0^T \left| \left\langle \Delta_k^h \partial_x^2 \partial_z^2 u_\psi, \Delta_k^h w_{\varepsilon\psi}^1 \right\rangle \right| \lesssim 2^{-2ks} \tilde{d}_k \varepsilon \|\partial_z^2 u_\psi\|_{\tilde{L}_T^2(B^{s+1})} \|\varepsilon w_{\varepsilon\psi}^1\|_{\tilde{L}_T^2(B^{s+1})};$$

where

$$\tilde{d}_k = d_k(\partial_z^2 u_\psi, 1) d_k(w_{\varepsilon\psi}^1, 1)$$

Then

$$\varepsilon^2 \int_0^T \left| \left\langle \Delta_k^h \partial_x^2 \omega_\psi, \Delta_k^h w_{\varepsilon\psi}^1 \right\rangle \right| \lesssim 2^{-2ks} \tilde{d}_k \varepsilon \|\partial_z^2 u_\psi\|_{\tilde{L}_T^2(B^{s+1})} \|\varepsilon w_{\varepsilon\psi}^1\|_{\tilde{L}_T^2(B^{s+1})}.$$

We have also

$$\varepsilon^2 \int_0^T \left| \left\langle \Delta_k^h \partial_t \partial_x^2 u_\psi, \Delta_k^h w_{\varepsilon\psi}^1 \right\rangle \right| \lesssim 2^{-2ks} \tilde{d}_k \varepsilon \|\partial_t u_\psi\|_{\tilde{L}_T^2(B^{s+1})} \|\varepsilon w_{\varepsilon\psi}^1\|_{\tilde{L}_T^2(B^{s+1})}$$

and

$$\begin{aligned}
\varepsilon^2 \int_0^T \left| \left\langle \Delta_k^h \partial_z^2 \partial_x^2 u_\psi, \Delta_k^h w_{\varepsilon\psi}^1 \right\rangle \right| &\lesssim 2^{-2ks} \tilde{d}_k \varepsilon \|\partial_z^2 u_\psi\|_{\tilde{L}_T^2(B^{s+1})} \|\varepsilon w_{\varepsilon\psi}^1\|_{\tilde{L}_T^2(B^{s+1})} \\
\varepsilon^2 \int_0^T \left| \left\langle \Delta_k^h \partial_x^4 u_\psi, \Delta_k^h w_{\varepsilon\psi}^1 \right\rangle \right| &\lesssim 2^{-2ks} \tilde{d}_k \varepsilon \|\partial_z^2 u_\psi\|_{\tilde{L}_T^2(B^{s+2})} \|\varepsilon w_{\varepsilon\psi}^1\|_{\tilde{L}_T^2(B^{s+2})}.
\end{aligned}$$

2. Estimate of $\int_0^T \left| \left\langle \Delta_k^h (u^\varepsilon \partial_x \omega^\varepsilon - u \partial_x \omega)_\psi, \Delta_k^h w_{\varepsilon\psi}^1 \right\rangle \right|$. We have

$$\begin{aligned} u^\varepsilon \partial_x \omega^\varepsilon - u \partial_x \omega &= (u^\varepsilon - u) \partial_x \omega^\varepsilon + u \partial_x (\omega^\varepsilon - \omega). \\ &= w_\varepsilon^1 \partial_x u - \alpha_1^2 w_\varepsilon^1 \partial_x \partial_z^2 u + u \partial_x w_\varepsilon^1 - \alpha_1^2 u \partial_x \partial_z^2 w_\varepsilon^1 \end{aligned}$$

Following the proof of (2.13), we get

$$\int_0^T \left| \left\langle \Delta_k^h (u \partial_x w_\varepsilon^1)_\psi, \Delta_k^h w_{\varepsilon\psi}^1 \right\rangle \right| \lesssim 2^{-2ks} (\tilde{d}_k + \check{d}_k) \|w_{\varepsilon\psi}^1\|_{\tilde{L}_{T,\eta'}^2(B^{s+\frac{1}{2}})}^2$$

By integration by part, we obtain

$$\int_0^T \left| \left\langle \Delta_k^h (u \partial_x \partial_z^2 w_\varepsilon^1)_\psi, \Delta_k^h w_{\varepsilon\psi}^1 \right\rangle \right| \leq \int_0^T \left| \left\langle \Delta_k^h (\partial_z u \partial_x \partial_z w_\varepsilon^1)_\psi, \Delta_k^h w_{\varepsilon\psi}^1 \right\rangle \right| + \int_0^T \left| \left\langle \Delta_k^h (u \partial_x \partial_z w_\varepsilon^1)_\psi, \Delta_k^h \partial_z w_{\varepsilon\psi}^1 \right\rangle \right|$$

Following the proof of (2.14), (2.15) we get

$$\int_0^T \left| \left\langle \Delta_k^h (\partial_z u \partial_x \partial_z w_\varepsilon^1)_\psi, \Delta_k^h w_{\varepsilon\psi}^1 \right\rangle \right| \lesssim 2^{-2ks} (\tilde{d}_k + \check{d}_k) \|\partial_z w_{\varepsilon\psi}^1\|_{\tilde{L}_{T,\eta'}^2(B^{s+\frac{1}{2}})}^2$$

and

$$\int_0^T \left| \left\langle \Delta_k^h (u \partial_x \partial_z w_\varepsilon^1)_\psi, \Delta_k^h \partial_z w_{\varepsilon\psi}^1 \right\rangle \right| \lesssim 2^{-2ks} (\tilde{d}_k + \check{d}_k) \|\partial_z w_{\varepsilon\psi}^1\|_{\tilde{L}_{T,\eta'}^2(B^{s+\frac{1}{2}})}^2$$

We have from the estimate (5.11) in [64]

$$\int_0^T \left| \left\langle \Delta_k^h (w_\varepsilon^1 \partial_x u)_\psi, \Delta_k^h w_{\varepsilon\psi}^1 \right\rangle \right| \lesssim 2^{-2ks} (\tilde{d}_k + \check{d}_k) \|w_{\varepsilon\psi}^1\|_{\tilde{L}_{T,\eta'}^2(B^{s+\frac{1}{2}})} \left(\|w_{\varepsilon\psi}^1\|_{\tilde{L}_{T,\eta'}^2(B^{s+\frac{1}{2}})} + \|u_\psi\|_{L_T^\infty(B^{\frac{3}{2}})}^{\frac{1}{2}} \|\partial_z w_{\varepsilon\psi}^1\|_{\tilde{L}_T^2(B^s)} \right)$$

and

$$\begin{aligned} &\int_0^T \left| \left\langle \Delta_k^h (w_\varepsilon^1 \partial_x \partial_z^2 u)_\psi, \Delta_k^h w_{\varepsilon\psi}^1 \right\rangle \right| \\ &\lesssim 2^{-2ks} (\tilde{d}_k + \check{d}_k) \|\partial_z w_{\varepsilon\psi}^1\|_{\tilde{L}_{T,\eta'}^2(B^{s+\frac{1}{2}})} \left(\|\partial_z w_{\varepsilon\psi}^1\|_{\tilde{L}_{T,\eta'}^2(B^{s+\frac{1}{2}})} + \|\partial_z u_\psi\|_{L_T^\infty(B^{\frac{3}{2}})}^{\frac{1}{2}} \|\partial_z w_{\varepsilon\psi}^1\|_{\tilde{L}_{T,\eta'}^2(B^s)} \right) \end{aligned}$$

Conclusion :

$$\begin{aligned} &\int_0^T \left| \left\langle \Delta_k^h (u^\varepsilon \partial_x \omega^\varepsilon - u \partial_x \omega)_\psi, \Delta_k^h w_{\varepsilon\psi}^1 \right\rangle \right| \\ &\lesssim 2^{-2ks} (\tilde{d}_k + \check{d}_k) \|\partial_z w_{\varepsilon\psi}^1\|_{\tilde{L}_{T,\eta'}^2(B^{s+\frac{1}{2}})} \left(\|\partial_z w_{\varepsilon\psi}^1\|_{\tilde{L}_{T,\eta'}^2(B^{s+\frac{1}{2}})} + \|\partial_z u_\psi\|_{L_T^\infty(B^{\frac{3}{2}})}^{\frac{1}{2}} \|\partial_z w_{\varepsilon\psi}^1\|_{\tilde{L}_{T,\eta'}^2(B^s)} \right) \\ &\quad + 2^{-2ks} (\tilde{d}_k + \check{d}_k) \|\partial_z w_{\varepsilon\psi}^1\|_{\tilde{L}_{T,\eta'}^2(B^{s+\frac{1}{2}})}^2 \end{aligned}$$

3. Estimate of $\int_0^T \left| \left\langle \Delta_k^h (v^\varepsilon \partial_z \omega^\varepsilon - v \partial_z \omega)_\psi, \Delta_k^h w_{\varepsilon\psi}^1 \right\rangle \right|$. We have

$$\begin{aligned} v^\varepsilon \partial_z \omega^\varepsilon - v \partial_z \omega &= v^\varepsilon \partial_z (\omega^\varepsilon - \omega) + w_\varepsilon^2 \partial_z \omega. \\ &= v^\varepsilon \partial_z w_\varepsilon^1 - \alpha_1^2 v^\varepsilon \partial_z^3 w_\varepsilon^1 + w_\varepsilon^2 \partial_z \omega. \end{aligned}$$

Therefore the integral is estimate by three terms.

• Estimate of $\int_0^T \left| \left\langle \Delta_k^h (v^\varepsilon \partial_z w_\varepsilon^1)_\psi, \Delta_k^h w_{\varepsilon\psi}^1 \right\rangle \right|$. We write

$$v^\varepsilon \partial_z w_\varepsilon^1 = w_\varepsilon^2 \partial_z w_\varepsilon^1 + v \partial_z w_\varepsilon^1$$

We deduce first from Lemma 3.2 in [64] that

$$\int_0^T \left| \left\langle \Delta_k^h(w_\varepsilon^2 \partial_z w_\varepsilon^1)_\psi, \Delta_k^h w_{\varepsilon\psi}^1 \right\rangle \right| \lesssim 2^{-2ks} (\tilde{d}_k + \check{d}_k) \|w_{\varepsilon\psi}^1\|_{\tilde{L}_{T,\eta'}^2(B^{s+\frac{1}{2}})}$$

and by adapting the proof of Estimate (5.13) in [64] we get

$$\int_0^T \left| \left\langle \Delta_k^h(v \partial_z w_\varepsilon^1)_\psi, \Delta_k^h w_{\varepsilon\psi}^1 \right\rangle \right| \lesssim 2^{-2ks} (\tilde{d}_k + \check{d}_k) \|u_\psi\|_{L_T^\infty(B^{\frac{3}{2}})}^{\frac{1}{2}} \|\partial_z w_\varepsilon^1\|_{\tilde{L}_T^2(B^s)} \|\partial_z w_{\varepsilon\psi}^1\|_{\tilde{L}_{T,\eta'}^2(B^{s+\frac{1}{2}})}$$

- Estimate of $\int_0^T \left| \left\langle \Delta_k^h(v^\varepsilon \partial_z^3 w_\varepsilon^1)_\psi, \Delta_k^h w_{\varepsilon\psi}^1 \right\rangle \right|$. We write also

$$v^\varepsilon \partial_z^3 w_\varepsilon^1 = w_\varepsilon^2 \partial_z^3 w_\varepsilon^1 + v \partial_z^3 w_\varepsilon^1$$

By integration by parts, we have

$$\int_0^T \left| \left\langle \Delta_k^h(w_\varepsilon^2 \partial_z^3 w_\varepsilon^1)_\psi, \Delta_k^h w_{\varepsilon\psi}^1 \right\rangle \right| dt \leq \int_0^T \left| \left\langle \Delta_k^h(\partial_x w_\varepsilon^1 \partial_z^2 w_\varepsilon^1)_\psi, \Delta_k^h w_{\varepsilon\psi}^1 \right\rangle \right| dt + \int_0^T \left| \left\langle \Delta_k^h(v \partial_z^2 w_\varepsilon^1)_\psi, \Delta_k^h \partial_z w_{\varepsilon\psi}^1 \right\rangle \right| dt$$

Adapting the proof of Lemma 3.2 in [64], we get for any $0 < s < 1$

$$\int_0^T \left| \left\langle \Delta_k^h(w_\varepsilon^2 \partial_z^3 w_\varepsilon^1)_\psi, \Delta_k^h w_{\varepsilon\psi}^1 \right\rangle \right| dt \lesssim (\tilde{d}_k + \check{d}_k) 2^{-2ks} \|\partial_z w_\varepsilon^1\|_{\tilde{L}_{T,\eta'(t)}^2(B^{s+\frac{1}{2}})}^2$$

Using integration by part we obtain

$$\int_0^T \left| \left\langle \Delta_k^h(v \partial_z^3 w_\varepsilon^1)_\psi, \Delta_k^h w_{\varepsilon\psi}^1 \right\rangle \right| dt \leq \int_0^T \left| \left\langle \Delta_k^h(\partial_x u \partial_z^2 w_\varepsilon^1)_\psi, \Delta_k^h w_{\varepsilon\psi}^1 \right\rangle \right| dt + \int_0^T \left| \left\langle \Delta_k^h(v \partial_z^2 w_\varepsilon^1)_\psi, \Delta_k^h \partial_z w_{\varepsilon\psi}^1 \right\rangle \right| dt$$

Then by adapting the proof of Estimate (5.13) in [64], we derive for any $0 < s < \frac{1}{2}$ that

$$\int_0^T \left| \left\langle \Delta_k^h(v \partial_z^3 w_\varepsilon^1)_\psi, \Delta_k^h w_{\varepsilon\psi}^1 \right\rangle \right| \lesssim 2^{-2ks} (\tilde{d}_k + \check{d}_k) \|u_\psi\|_{L_T^\infty(B^{\frac{3}{2}})}^{\frac{1}{2}} \|\partial_z^2 w_\varepsilon^1\|_{\tilde{L}_T^2(B^s)} \|\partial_z w_{\varepsilon\psi}^1\|_{\tilde{L}_{T,\eta'}^2(B^{s+\frac{1}{2}})}$$

- Estimate of the term $\int_0^T \left| \left\langle \Delta_k^h(w_\varepsilon^2 \partial_z w_\varepsilon^1)_\psi, \Delta_k^h w_{\varepsilon\psi}^1 \right\rangle \right| dt$. We have

$$\int_0^T \left| \left\langle \Delta_k^h(w_\varepsilon^2 \partial_z w_\varepsilon^1)_\psi, \Delta_k^h w_{\varepsilon\psi}^1 \right\rangle \right| dt \leq \int_0^T \left| \left\langle \Delta_k^h(w_\varepsilon^2 \partial_z u)_\psi, \Delta_k^h w_{\varepsilon\psi}^1 \right\rangle \right| dt + \alpha_1^2 \int_0^T \left| \left\langle \Delta_k^h(w_\varepsilon^2 \partial_z^3 u)_\psi, \Delta_k^h w_{\varepsilon\psi}^1 \right\rangle \right| dt$$

Adapting the proof of Estimate (5.14) in [64]. We obtain

$$\int_0^T \left| \left\langle \Delta_k^h(w_\varepsilon^2 \partial_z u)_\psi, \Delta_k^h w_{\varepsilon\psi}^1 \right\rangle \right| dt \lesssim (\tilde{d}_k + \check{d}_k) 2^{-2ks} \|w_{\varepsilon\psi}^1\|_{\tilde{L}_{T,\eta'(t)}^2(B^{s+\frac{1}{2}})}^2$$

By integration by part we have

$$\int_0^T \left| \left\langle \Delta_k^h(w_\varepsilon^2 \partial_z^3 u)_\psi, \Delta_k^h w_{\varepsilon\psi}^1 \right\rangle \right| dt \leq \int_0^T \left| \left\langle \Delta_k^h(\partial_x w_\varepsilon^1 \partial_z^2 u)_\psi, \Delta_k^h w_{\varepsilon\psi}^1 \right\rangle \right| dt + \int_0^T \left| \left\langle \Delta_k^h(w_\varepsilon^2 \partial_z^2 u)_\psi, \Delta_k^h \partial_z w_{\varepsilon\psi}^1 \right\rangle \right| dt$$

Adapting the proof of Estimate (5.14) in [64].

$$\int_0^T \left| \left\langle \Delta_k^h(\partial_x w_\varepsilon^1 \partial_z^2 u)_\psi, \Delta_k^h w_{\varepsilon\psi}^1 \right\rangle \right| dt \lesssim (\tilde{d}_k + \check{d}_k) 2^{-2ks} \|w_{\varepsilon\psi}^1\|_{\tilde{L}_{T,\eta'(t)}^2(B^{s+\frac{1}{2}})}^2$$

and

$$\int_0^T \left| \left\langle \Delta_k^h(w_\varepsilon^2 \partial_z u)_\psi, \Delta_k^h \partial_z w_{\varepsilon\psi}^1 \right\rangle \right| dt \lesssim (\tilde{d}_k + \check{d}_k) 2^{-2ks} \|\partial_z w_{\varepsilon\psi}^1\|_{\tilde{L}_{T,\eta'(t)}^2(B^{s+\frac{1}{2}})}^2$$

Then

$$\int_0^T \left| \left\langle \Delta_k^h (w_\varepsilon^2 \partial_z^3 u)_\psi, \Delta_k^h w_{\varepsilon\psi}^1 \right\rangle \right| dt \lesssim (\tilde{d}_k + \check{d}_k) 2^{-2ks} \|\partial_z w_{\varepsilon\psi}^1\|_{\tilde{L}_{T,\eta'(t)}^2(B^{s+\frac{1}{2}})}^2$$

• Conclusion : by summing the three previous estimates, we conclude

$$\begin{aligned} & \int_0^T \left| \left\langle \Delta_k^h (v^\varepsilon \partial_z \omega^\varepsilon - v \partial_z \omega)_\psi, \Delta_k^h w_{\varepsilon\psi}^1 \right\rangle \right| \\ & \lesssim 2^{-2ks} (\tilde{d}_k + \check{d}_k) \left(\|u_\psi\|_{L_T^\infty(B^{\frac{3}{2}})}^{\frac{1}{2}} \|\partial_z^2 w_{\varepsilon\psi}^1\|_{\tilde{L}_T^2(B^s)} \|\partial_z w_{\varepsilon\psi}^1\|_{\tilde{L}_{T,\eta'}^2(B^{s+\frac{1}{2}})} + \|\partial_z w_{\varepsilon\psi}^1\|_{\tilde{L}_{T,\eta'(t)}^2(B^{s+\frac{1}{2}})}^2 \right) \end{aligned}$$

4. Estimate of $\int_0^T \left| \left\langle \Delta_k^h (\partial_z u^\varepsilon \partial_x \partial_z u^\varepsilon - \partial_z u \partial_x \partial_z u)_\psi, \Delta_k^h w_{\varepsilon\psi}^1 \right\rangle \right| dt$. We write

$$\partial_z u^\varepsilon \partial_x \partial_z u^\varepsilon - \partial_z u \partial_x \partial_z u = \partial_z u^\varepsilon \partial_x \partial_z w_\varepsilon^1 + \partial_z w_\varepsilon^1 \partial_x \partial_z u$$

Adapting the proof of Estimate (2.16), we get

$$\int_0^T \left| \left\langle \Delta_k^h (\partial_z u^\varepsilon \partial_x \partial_z w_\varepsilon^1)_\psi, \Delta_k^h w_{\varepsilon\psi}^1 \right\rangle \right| dt \lesssim (\tilde{d}_k + \check{d}_k) 2^{-2ks} \|\partial_z w_{\varepsilon\psi}^1\|_{\tilde{L}_{T,\eta'(t)}^2(B^{s+\frac{1}{2}})}^2$$

By Bony's decomposition, we have

$$\partial_z w_\varepsilon^1 \partial_x \partial_z u = T_{\partial_z w_\varepsilon^1}^h \partial_x \partial_z u + T_{\partial_x \partial_z u}^h \partial_z w_\varepsilon^1 + R^h(\partial_z w_\varepsilon^1, \partial_x \partial_z u)$$

By adapting the proof of estimate (5.11) in [64]

$$\begin{aligned} & \int_0^T \left| \left\langle \Delta_k^h (\partial_z w_\varepsilon^1 \partial_x \partial_z u)_\psi, \Delta_k^h w_{\varepsilon\psi}^1 \right\rangle \right| \\ & \lesssim 2^{-2ks} (\tilde{d}_k + \check{d}_k) \|\partial_z w_{\varepsilon\psi}^1\|_{\tilde{L}_{T,\eta'}^2(B^{s+\frac{1}{2}})} \left(\|\partial_z w_{\varepsilon\psi}^1\|_{\tilde{L}_{T,\eta'}^2(B^{s+\frac{1}{2}})} + \|\partial_z u_\psi\|_{L_T^\infty(B^{\frac{3}{2}})}^{\frac{1}{2}} \|\partial_z w_{\varepsilon\psi}^1\|_{\tilde{L}_T^2(B^s)} \right) \end{aligned}$$

Conclusion :

$$\begin{aligned} & \int_0^T \left| \left\langle \Delta_k^h (\partial_z u^\varepsilon \partial_x \partial_z u^\varepsilon - \partial_z u \partial_x \partial_z u)_\psi, \Delta_k^h w_{\varepsilon\psi}^1 \right\rangle \right| dt \\ & \lesssim 2^{-2ks} (\tilde{d}_k + \check{d}_k) \left(\|\partial_z w_{\varepsilon\psi}^1\|_{\tilde{L}_{T,\eta'}^2(B^{s+\frac{1}{2}})}^2 + \|\partial_z u_\psi\|_{L_T^\infty(B^{\frac{3}{2}})}^{\frac{1}{2}} \|\partial_z w_{\varepsilon\psi}^1\|_{\tilde{L}_T^2(B^s)} \|\partial_z w_{\varepsilon\psi}^1\|_{\tilde{L}_{T,\eta'}^2(B^{s+\frac{1}{2}})} \right) \end{aligned}$$

5. Estimate of $\int_0^T \left| \left\langle \Delta_k^h (\partial_z^2 u^\varepsilon \partial_x u^\varepsilon - \partial_z^2 u \partial_x u)_\psi, \Delta_k^h w_{\varepsilon\psi}^1 \right\rangle \right| dt$. We write

$$\partial_z^2 u^\varepsilon \partial_x u^\varepsilon - \partial_z^2 u \partial_x u = \partial_z^2 u^\varepsilon \partial_x w_\varepsilon^1 + \partial_z^2 w_\varepsilon^1 \partial_x u$$

By integration by parts, and following the proof of Lemma 3.1 in [64], we get

$$\begin{aligned} \int_0^T \left| \left\langle \Delta_k^h (\partial_z^2 u^\varepsilon \partial_x w_\varepsilon^1)_\psi, \Delta_k^h w_{\varepsilon\psi}^1 \right\rangle \right| & \leq \int_0^T \left| \left\langle \Delta_k^h (\partial_z u^\varepsilon \partial_x \partial_z w_\varepsilon^1)_\psi, \Delta_k^h w_{\varepsilon\psi}^1 \right\rangle \right| + \int_0^T \left| \left\langle \Delta_k^h (\partial_z u^\varepsilon \partial_x w_\varepsilon^1)_\psi, \Delta_k^h \partial_z w_{\varepsilon\psi}^1 \right\rangle \right| \\ & \lesssim 2^{-2ks} (\tilde{d}_k + \check{d}_k) \|\partial_z w_{\varepsilon\psi}^1\|_{\tilde{L}_{T,\eta'}^2(B^{s+\frac{1}{2}})}^2 \end{aligned}$$

By adapting the proof of estimate (5.11) in [64]

$$\begin{aligned} & \int_0^T \left| \left\langle \Delta_k^h (\partial_z^2 w_\varepsilon^1 \partial_x u)_\psi, \Delta_k^h w_{\varepsilon\psi}^1 \right\rangle \right| \\ & \lesssim 2^{-2ks} (\tilde{d}_k + \check{d}_k) \|\partial_z w_{\varepsilon\psi}^1\|_{\tilde{L}_{T,\eta'}^2(B^{s+\frac{1}{2}})} \left(\|\partial_z w_{\varepsilon\psi}^1\|_{\tilde{L}_{T,\eta'}^2(B^{s+\frac{1}{2}})} + \|\partial_z u_\psi\|_{L_T^\infty(B^{\frac{3}{2}})}^{\frac{1}{2}} \|\partial_z^2 w_{\varepsilon\psi}^1\|_{\tilde{L}_T^2(B^s)} \right) \end{aligned}$$

Conclusion :

$$\int_0^T \left| \left\langle \Delta_k^h (\partial_z^2 u^\varepsilon \partial_x u^\varepsilon - \partial_z^2 u \partial_x u)_\psi, \Delta_k^h w_{\varepsilon_\psi}^1 \right\rangle \right| dt \\ \lesssim 2^{-2ks} (\tilde{d}_k + \check{d}_k) \left(\|\partial_z w_{\varepsilon_\psi}^1\|_{\tilde{L}_{T,\eta'}^2(B^{s+\frac{1}{2}})}^2 + \|\partial_z u_\psi\|_{L_T^\infty(B^{\frac{3}{2}})}^{\frac{1}{2}} \|\partial_z^2 w_{\varepsilon_\psi}^1\|_{\tilde{L}_T^2(B^s)} \|\partial_z w_{\varepsilon_\psi}^1\|_{\tilde{L}_{T,\eta'}^2(B^{s+\frac{1}{2}})} \right)$$

6. Estimate of $\varepsilon^2 \int_0^T \left| \left\langle \Delta_k^h (u^\varepsilon \partial_x^2 u^\varepsilon)_\psi, \Delta_k^h \partial_x w_{\varepsilon_\psi}^1 \right\rangle \right| dt$.

We recall the following estimates (see (2.23), (3.8)). For any $s > 0$

$$\|e^{Rt} u_\phi\|_{\tilde{L}_T^\infty(B^s)} + \|e^{Rt} \partial_z u_\phi\|_{\tilde{L}_T^\infty(B^s)} + \|e^{Rt} \partial_z u_\phi\|_{\tilde{L}_T^2(B^s)} + \|e^{Rt} \partial_z^2 u_\phi\|_{\tilde{L}_T^2(B^s)} \\ \leq C_3 (\|e^{a|D_x|} u_0\|_{B^s} + \|e^{a|D_x|} \partial_z u_0\|_{B^s}). \quad (3.55)$$

and for any $0 < s < 1$

$$\|e^{\mathcal{R}t'} (u_\varphi, \varepsilon v_\varphi)\|_{\tilde{L}_T^\infty(B^s)} + \|e^{\mathcal{R}t'} (\varepsilon u_\varphi, \varepsilon^2 v_\varphi)\|_{\tilde{L}_T^\infty(B^{s+1})} + \|e^{\mathcal{R}t'} (\partial_z u_\varphi, \varepsilon \partial_z \Delta_k^h v_\varphi)\|_{\tilde{L}_T^\infty(B^s)} \\ + \|e^{\mathcal{R}t'} (\varepsilon \partial_x u_\varphi, \varepsilon^2 \partial_x v_\varphi)\|_{\tilde{L}_T^2(B^s)} \\ + \|e^{\mathcal{R}t'} (\partial_z u_\varphi, \varepsilon \partial_z v_\varphi)\|_{\tilde{L}_{T,\eta'(t)}^2(B^s)} + \|e^{\mathcal{R}t'} (\varepsilon \partial_z \partial_x u_\varphi, \varepsilon^2 \partial_z \partial_x v_\varphi)\|_{\tilde{L}_T^2(B^s)} \\ + \|e^{\mathcal{R}t'} (\varepsilon \partial_x^2 u_\varphi, \varepsilon^3 \partial_x^2 v_\varphi)\|_{\tilde{L}_T^2(B^s)} + \|e^{\mathcal{R}t'} (\partial_z^2 u_\varphi, \varepsilon \partial_z^2 v_\varphi)\|_{\tilde{L}_T^2(B^s)} \\ \leq C_1 \left(\|e^{a|D_x|} (u_0, \varepsilon v_0)\|_{B^s} + \|e^{a|D_x|} (\varepsilon u_0, \varepsilon^2 v_0)\|_{B^{s+1}} + \|e^{a|D_x|} (\partial_z u_0, \varepsilon \partial_z v_0)\|_{B^s} \right). \quad (3.56)$$

We write :

$$u^\varepsilon \partial_x^2 u^\varepsilon = u^\varepsilon \partial_x^2 w_\varepsilon^1 + u^\varepsilon \partial_x^2 u.$$

• Estimate of $\varepsilon^2 \int_0^T \left| \left\langle \Delta_k^h (u^\varepsilon \partial_x^2 w_\varepsilon^1)_\psi, \Delta_k^h \partial_x w_{\varepsilon_\psi}^1 \right\rangle \right|$: By adapting the proof of Lemma 3.1 in [64], we get

$$\varepsilon^2 \int_0^T \left| \left\langle \Delta_k^h (u^\varepsilon \partial_x^2 w_\varepsilon^1)_\psi, \Delta_k^h \partial_x w_{\varepsilon_\psi}^1 \right\rangle \right| \lesssim (\tilde{d}_k + \check{d}_k) 2^{-2ks} \varepsilon^2 \|\partial_x w_{\varepsilon_\psi}^1\|_{\tilde{L}_{T,\zeta'(t)}^2(B^{s+\frac{1}{2}})}^2$$

• Estimate of the term $\varepsilon^2 \int_0^T \left| \left\langle \Delta_k^h (u^\varepsilon \partial_x^2 u)_\psi, \Delta_k^h \partial_x w_{\varepsilon_\psi}^1 \right\rangle \right|$:

By Bony's decomposition, we have

$$u^\varepsilon \partial_x^2 u = T_{u^\varepsilon}^h \partial_x^2 u + T_{\partial_x^2 u}^h u^\varepsilon + R^h(u^\varepsilon, \partial_x^2 u).$$

We have,

$$\int_0^T \left| \left\langle \Delta_k^h (T_{u^\varepsilon}^h \partial_x^2 u)_\psi, \Delta_k^h \partial_x w_{\varepsilon_\psi}^1 \right\rangle \right| \lesssim \sum_{|k'-k| \leq 4} \int_0^t \|S_{k'-1}^h u^\varepsilon\|_{L^\infty} \|\Delta_{k'}^h \partial_x^2 u\|_{L^2} \|\Delta_k^h \partial_x w_\varepsilon^1\|_{L^2}$$

Using Lemma 2.3.1, Poincaré inequality we have

$$\|\Delta_l^h u^\varepsilon\|_{L^\infty} \lesssim 2^{\frac{l}{2}} \|\Delta_l^h u^\varepsilon\|_{L_x^\infty(L_x^2)} \\ \lesssim 2^{\frac{l}{2}} \|\Delta_l^h u^\varepsilon\|_{L^2}^{\frac{1}{2}} \|\Delta_l^h \partial_z^2 u^\varepsilon\|_{L^2}^{\frac{1}{2}} \\ \lesssim d_l \|u^\varepsilon\|_{B^{\frac{1}{2}}}^{\frac{1}{2}} \|\partial_z^2 u^\varepsilon\|_{B^{\frac{1}{2}}}^{\frac{1}{2}}.$$

Then

$$\|S_{k'-1}^h u^\varepsilon\|_{L^\infty} \lesssim \|u^\varepsilon\|_{B^{\frac{1}{2}}}^{\frac{1}{2}} \|\partial_z^2 u^\varepsilon\|_{B^{\frac{1}{2}}}^{\frac{1}{2}}.$$

Then

$$\begin{aligned}
& \varepsilon^2 \int_0^T \left| \left\langle \Delta_k^h (T_{u^\varepsilon}^h \partial_x^2 u)_\psi, \Delta_k^h \partial_x w_{\varepsilon^\psi}^1 \right\rangle \right| \\
& \lesssim \varepsilon^2 \sum_{|k'-k| \leq 4} \int_0^t \|u^\varepsilon\|_{B^{\frac{1}{2}}}^{\frac{1}{2}} \|\partial_z^2 u^\varepsilon\|_{B^{\frac{1}{2}}}^{\frac{1}{2}} (2^{2k'} \|\Delta_{k'}^h u\|_{L^2}) \|\Delta_k^h \partial_x w_{\varepsilon^\psi}^1\|_{L^2} \\
& \lesssim \varepsilon^2 \sum_{|k'-k| \leq 4} \|u^\varepsilon\|_{L_T^\infty(B^{\frac{1}{2}})}^{\frac{1}{2}} 2^{2k'} \|\Delta_{k'}^h u\|_{L_T^2(L^2)} \left(\int_0^T \|\partial_z^2 u^\varepsilon\|_{B^{\frac{1}{2}}} \|\Delta_k^h \partial_x w_{\varepsilon^\psi}^1\|_{L^2}^2 \right)^{\frac{1}{2}} \\
& \lesssim \varepsilon 2^{-2ks} \tilde{d}_k \|u^\varepsilon\|_{L_T^\infty(B^{\frac{1}{2}})}^{\frac{1}{2}} \|\partial_z u\|_{L_T^2(B^{s+\frac{3}{2}})} \varepsilon \|\partial_x w_{\varepsilon^\psi}^1\|_{\tilde{L}_{T,\zeta'(t)}^2(B^{s+\frac{1}{2}})}
\end{aligned}$$

where

$$\tilde{d}_k = d_k(w_{\varepsilon^\psi}^1, \zeta') \sum_{|k'-k| \leq 4} 2^{(k-k')(s-1/2)} d_{k'}(\partial_z u_\psi, 1)$$

For any $0 < s < 1$, we have

$$\|S_{k'-1}^h \partial_x^2 u_\psi\|_{L_T^2(L_z^2(L_x^\infty))} \lesssim 2^{k'(1-s)} d_{k'} \|\partial_z u_\psi\|_{\tilde{L}_T^2(B^{s+\frac{3}{2}})}$$

where $d_{k'} = \sum_{l \leq k'-2} 2^{-(k'-l)(1-s)} d_l(\partial_z u_\psi, 1)$. Then

$$\begin{aligned}
& \varepsilon^2 \int_0^T \left| \left\langle \Delta_k^h (T_{\partial_x^2 u}^h u^\varepsilon)_\psi, \Delta_k^h \partial_x w_{\varepsilon^\psi}^1 \right\rangle \right| \\
& \lesssim \sum_{|k'-k| \leq 4} \varepsilon^2 \int_0^T \|S_{k'-1}^h \partial_x^2 u_\psi\|_{L_z^2(L_x^\infty)} \|\Delta_{k'}^h u_\psi^\varepsilon\|_{L_z^\infty(L_x^2)} \|\Delta_k^h \partial_x w_{\varepsilon^\psi}^1\|_{L^2} \\
& \lesssim \sum_{|k'-k| \leq 4} \varepsilon^2 \int_0^T \|S_{k'-1}^h \partial_x^2 u_\psi\|_{L_z^2(L_x^\infty)} 2^{-\frac{k'}{2}} \|u_\psi^\varepsilon\|_{B^{\frac{1}{2}}}^{\frac{1}{2}} \|\partial_z^2 u_\psi^\varepsilon\|_{B^{\frac{1}{2}}}^{\frac{1}{2}} \|\Delta_k^h \partial_x w_{\varepsilon^\psi}^1\|_{L^2} \\
& \lesssim \varepsilon^2 \sum_{|k'-k| \leq 4} 2^{-\frac{k'}{2}} \|u_\psi^\varepsilon\|_{L_T^\infty(B^{\frac{1}{2}})}^{\frac{1}{2}} \|S_{k'-1}^h \partial_x^2 u_\psi\|_{L_T^2(L_z^2(L_x^\infty))} \left(\int_0^T \|\partial_z^2 u^\varepsilon\|_{B^{\frac{1}{2}}} \|\Delta_k^h \partial_x w_{\varepsilon^\psi}^1\|_{L^2}^2 \right)^{\frac{1}{2}} \\
& \lesssim \varepsilon 2^{-2ks} \tilde{d}_k \|u_\psi^\varepsilon\|_{\tilde{L}^\infty(B^{\frac{1}{2}})}^{\frac{1}{2}} \|\partial_z u_\psi\|_{\tilde{L}_T^2(B^{s+\frac{3}{2}})} \varepsilon \|\partial_x w_{\varepsilon^\psi}^1\|_{\tilde{L}_{T,\zeta'(t)}^2(B^{s+\frac{1}{2}})},
\end{aligned}$$

where

$$\tilde{d}_k = d_k(w_{\varepsilon^\psi}^1, \zeta') \sum_{|k'-k| \leq 4} 2^{(k-k')(s-1/2)} d_{k'}(\partial_x u_\psi^\varepsilon, 1)$$

$$\begin{aligned}
& \varepsilon^2 \int_0^T \left| \left\langle \Delta_k^h (R^h(u^\varepsilon, \partial_x^2 u))_\psi, \Delta_k^h \partial_x w_{\varepsilon^\psi}^1 \right\rangle \right| dt \\
& \lesssim \varepsilon^2 \sum_{k' \geq k-3} \int_0^T \|\Delta_{k'}^h u_\psi^\varepsilon\|_{L_z^\infty(L_x^2)} \|\tilde{\Delta}_{k'}^h \partial_x^2 u_\psi\|_{L^2} \|\Delta_k^h \partial_x w_{\varepsilon^\psi}^1\|_{L_z^2(L_x^\infty)} dt \\
& \lesssim \varepsilon^2 \sum_{k' \geq k-3} \int_0^T 2^{-\frac{k'}{2}} \|u_\psi^\varepsilon\|_{B^{\frac{1}{2}}}^{\frac{1}{2}} \|\partial_z^2 u_\psi^\varepsilon\|_{B^{\frac{1}{2}}}^{\frac{1}{2}} 2^{2k'} \|\tilde{\Delta}_{k'}^h u_\psi\|_{L^2} 2^{\frac{k}{2}} \|\Delta_k^h \partial_x w_{\varepsilon^\psi}^1\|_{L^2} dt \\
& \lesssim \varepsilon^2 2^{\frac{k}{2}} \sum_{k' \geq k-3} 2^{-\frac{k'}{2}} \|u_\psi^\varepsilon\|_{L_T^\infty(B^{\frac{1}{2}})}^{\frac{1}{2}} 2^{2k'} \|\tilde{\Delta}_{k'}^h u_\psi\|_{L_T^2(L^2)} \left(\int_0^T \|\partial_z^2 u^\varepsilon\|_{B^{\frac{1}{2}}} \|\Delta_k^h \partial_x w_{\varepsilon^\psi}^1\|_{L^2}^2 \right)^{\frac{1}{2}} \\
& \lesssim \varepsilon 2^{-2ks} \tilde{d}_k \|u_\psi^\varepsilon\|_{L_T^\infty(B^{\frac{1}{2}})}^{\frac{1}{2}} \|\partial_z u_\psi\|_{\tilde{L}_T^2(B^{s+\frac{3}{2}})} \varepsilon \|\partial_x w_{\varepsilon^\psi}^1\|_{\tilde{L}_{T,\zeta'(t)}^2(B^{s+\frac{1}{2}})},
\end{aligned}$$

where

$$\check{d}_k = d_k(w_{\varepsilon\psi}^1, \zeta') \sum_{|k'-k|\leq 4} 2^{(k-k')s} d_{k'}(\partial_z u_\psi, 1)$$

By summing the three previous estimates, we have :

$$\begin{aligned} \varepsilon^2 \int_0^T \left| \left\langle \Delta_k^h(u^\varepsilon \partial_x^2 u)_\psi, \Delta_k^h \partial_x w_{\varepsilon\psi}^1 \right\rangle \right| & \\ \lesssim 2^{-2ks} \left[(\check{d}_k + \check{d}_k) \varepsilon \|u_\psi^\varepsilon\|_{L_T^\infty(B^{\frac{1}{2}})}^{\frac{1}{2}} \|\partial_z u_\psi\|_{\tilde{L}_T^2(B^{s+\frac{3}{2}})} \varepsilon \|\partial_x w_{\varepsilon\psi}^1\|_{\tilde{L}_{T,\zeta'(t)}^2(B^{s+\frac{1}{2}})} \right], \end{aligned} \quad (3.57)$$

• Conclusion. By summing the previous estimates, we get

$$\begin{aligned} \varepsilon^2 \int_0^T \left| \left\langle \Delta_k^h(\partial_x u^\varepsilon \partial_x^2 u^\varepsilon)_\psi, \Delta_k^h \partial_x w_{\varepsilon\psi}^1 \right\rangle \right| & \\ \lesssim 2^{-2ks} (\check{d}_k + \check{d}_k) \left[\varepsilon^2 \|\partial_x w_{\varepsilon\psi}^1\|_{\tilde{L}_{T,\zeta'(t)}^2(B^{s+\frac{1}{2}})}^2 + \varepsilon \|u_\psi^\varepsilon\|_{L_T^\infty(B^{\frac{1}{2}})}^{\frac{1}{2}} \|\partial_z u_\psi\|_{\tilde{L}_T^2(B^{s+\frac{3}{2}})} \varepsilon \|\partial_x w_{\varepsilon\psi}^1\|_{\tilde{L}_{T,\zeta'(t)}^2(B^{s+\frac{1}{2}})} \right]. \end{aligned}$$

7. Estimate of $\varepsilon^2 \int_0^T \left| \left\langle \Delta_k^h(v^\varepsilon \partial_z \partial_x^2 u^\varepsilon)_\psi, \Delta_k^h w_{\varepsilon\psi}^1 \right\rangle \right|$. By integration by part we have

$$\int_0^T \left| \left\langle \Delta_k^h(v^\varepsilon \partial_z \partial_x^2 u^\varepsilon)_\psi, \Delta_k^h w_{\varepsilon\psi}^1 \right\rangle \right| \lesssim \int_0^T \left| \left\langle \Delta_k^h(v^\varepsilon \partial_z \partial_x u^\varepsilon)_\psi, \Delta_k^h \partial_x w_{\varepsilon\psi}^1 \right\rangle \right| + \int_0^T \left| \left\langle \Delta_k^h(\partial_x v^\varepsilon \partial_z \partial_x u^\varepsilon)_\psi, \Delta_k^h w_{\varepsilon\psi}^1 \right\rangle \right|$$

• Estimate of $\int_0^T \left| \left\langle \Delta_k^h(v^\varepsilon \partial_z \partial_x u^\varepsilon)_\psi, \Delta_k^h \partial_x w_{\varepsilon\psi}^1 \right\rangle \right|$. We write,

$$v^\varepsilon \partial_z \partial_x u^\varepsilon = w_\varepsilon^2 \partial_z \partial_x u^\varepsilon + v \partial_z \partial_x u^\varepsilon$$

and

$$\partial_x v^\varepsilon \partial_z \partial_x u^\varepsilon = \partial_x w_\varepsilon^2 \partial_z \partial_x u^\varepsilon + \partial_x v \partial_z \partial_x u^\varepsilon$$

Then we estimate the term $\int_0^T \left| \left\langle \Delta_k^h(w_\varepsilon^2 \partial_z \partial_x u^\varepsilon)_\psi, \Delta_k^h \partial_x w_{\varepsilon\psi}^1 \right\rangle \right| dt$. By adapting the proof of Lemma 3.2 in [64], we get

$$\varepsilon^2 \int_0^T \left| \left\langle \Delta_k^h(w_\varepsilon^2 \partial_z \partial_x u^\varepsilon)_\psi, \Delta_k^h \partial_x w_{\varepsilon\psi}^1 \right\rangle \right| dt \lesssim 2^{-2ks} (\check{d}_k + \check{d}_k) \|w_{\varepsilon\psi}^1\|_{\tilde{L}_{T,\zeta'(t)}^2(B^{s+\frac{1}{2}})} \varepsilon \|\partial_x w_{\varepsilon\psi}^1\|_{\tilde{L}_{T,\zeta'(t)}^2(B^{s+\frac{1}{2}})}$$

Estimate of the term $\int_0^T \left| \left\langle \Delta_k^h(v \partial_z \partial_x u^\varepsilon)_\psi, \Delta_k^h \partial_x w_{\varepsilon\psi}^1 \right\rangle \right| dt$. We write

$$v \partial_z \partial_x u^\varepsilon = v \partial_z \partial_x w_{\varepsilon\psi}^1 + v \partial_z \partial_x u.$$

By adapting the proof of Estimate (5.13), in [64], we get

$$\varepsilon^2 \int_0^T \left| \left\langle \Delta_k^h(v \partial_z \partial_x w_{\varepsilon\psi}^1)_\psi, \Delta_k^h \partial_x w_{\varepsilon\psi}^1 \right\rangle \right| dt \lesssim 2^{-2ks} (\check{d}_k + \check{d}_k) \|u_\psi\|_{L_T^\infty(B^{\frac{3}{2}})}^{\frac{1}{2}} \varepsilon \|\partial_z \partial_x w_{\varepsilon\psi}^1\|_{\tilde{L}_T^2(B^s)} \varepsilon \|\partial_x w_{\varepsilon\psi}^1\|_{\tilde{L}_{T,\zeta'(t)}^2(B^{s+\frac{1}{2}})},$$

and

$$\varepsilon^2 \int_0^T \left| \left\langle \Delta_k^h(v \partial_z \partial_x u)_\psi, \Delta_k^h \partial_x w_{\varepsilon\psi}^1 \right\rangle \right| dt \lesssim 2^{-2ks} (\check{d}_k + \check{d}_k) \varepsilon \|u_\psi\|_{L_T^\infty(B^{\frac{3}{2}})}^{\frac{1}{2}} \|\partial_z u_\psi\|_{\tilde{L}_T^2(B^{s+1})} \varepsilon \|\partial_x w_{\varepsilon\psi}^1\|_{\tilde{L}_{T,\zeta'(t)}^2(B^{s+\frac{1}{2}})}.$$

Then

$$\begin{aligned} \int_0^T \left| \left\langle \Delta_k^h(v^\varepsilon \partial_z \partial_x u^\varepsilon)_\psi, \Delta_k^h \partial_x w_{\varepsilon\psi}^1 \right\rangle \right| & \\ \lesssim 2^{-2ks} (\check{d}_k + \check{d}_k) \left[\varepsilon \|\partial_x w_{\varepsilon\psi}^1\|_{\tilde{L}_{T,\zeta'(t)}^2(B^{s+\frac{1}{2}})} \|w_{\varepsilon\psi}^1\|_{\tilde{L}_{T,\zeta'(t)}^2(B^{s+\frac{1}{2}})} \right. & \\ + \|u_\psi\|_{L_T^\infty(B^{\frac{3}{2}})}^{\frac{1}{2}} \varepsilon \|\partial_z \partial_x w_{\varepsilon\psi}^1\|_{\tilde{L}_T^2(B^s)} \|\partial_x w_{\varepsilon\psi}^1\|_{\tilde{L}_{T,\zeta'(t)}^2(B^{s+\frac{1}{2}})} & \\ \left. + \varepsilon \|u_\psi\|_{L_T^\infty(B^{\frac{3}{2}})}^{\frac{1}{2}} \|\partial_z u_\psi\|_{\tilde{L}_T^2(B^{s+1})} \varepsilon \|\partial_x w_{\varepsilon\psi}^1\|_{\tilde{L}_{T,\zeta'(t)}^2(B^{s+\frac{1}{2}})} \right] & \end{aligned}$$

- Estimate of $\int_0^T \left| \left\langle \Delta_k^h(\partial_x v^\varepsilon \partial_z \partial_x u^\varepsilon)_\psi, \Delta_k^h w_{\varepsilon\psi}^1 \right\rangle \right|$. We write

$$\partial_x v^\varepsilon \partial_z \partial_x u^\varepsilon = \partial_x w_\varepsilon^2 \partial_z \partial_x u^\varepsilon + \partial_x v \partial_z \partial_x u^\varepsilon$$

Estimate of $\int_0^T \left| \left\langle \Delta_k^h(\partial_x w_\varepsilon^2 \partial_z \partial_x u^\varepsilon)_\psi, \Delta_k^h w_{\varepsilon\psi}^1 \right\rangle \right| dt$. By adapting the proof of Lemma 3.2 in [64], we get

$$\varepsilon^2 \int_0^T \left| \left\langle \Delta_k^h(\partial_x w_\varepsilon^2 \partial_z \partial_x u^\varepsilon)_\psi, \Delta_k^h w_{\varepsilon\psi}^1 \right\rangle \right| dt \lesssim 2^{-2ks} (\tilde{d}_k + \check{d}_k) \varepsilon \|\partial_x w_{\varepsilon\psi}^1\|_{\tilde{L}_{T,\zeta'(t)}^2(B^{s+\frac{1}{2}})} \|w_{\varepsilon\psi}^1\|_{\tilde{L}_{T,\zeta'(t)}^2(B^{s+\frac{1}{2}})}$$

Estimate of the term $\int_0^T \left| \left\langle \Delta_k^h(\partial_x v \partial_z \partial_x u^\varepsilon)_\psi, \Delta_k^h w_{\varepsilon\psi}^1 \right\rangle \right| dt$. By adapting the proof of Estimate (5.13), in [64], we get

$$\varepsilon^2 \int_0^T \left| \left\langle \Delta_k^h(\partial_x v \partial_z \partial_x u^\varepsilon)_\psi, \Delta_k^h w_{\varepsilon\psi}^1 \right\rangle \right| dt \lesssim 2^{-2ks} (\tilde{d}_k + \check{d}_k) \varepsilon \|u_\psi\|_{L_T^\infty(B^{\frac{7}{2}})}^{\frac{1}{2}} \varepsilon \|\partial_z \partial_x u_\psi^\varepsilon\|_{\tilde{L}_T^2(B^s)} \|\partial_z w_{\varepsilon\psi}^1\|_{\tilde{L}_{T,\zeta'(t)}^2(B^{s+\frac{1}{2}})},$$

Therefore

$$\begin{aligned} & \int_0^T \left| \left\langle \Delta_k^h(\partial_x v^\varepsilon \partial_z \partial_x u^\varepsilon)_\psi, \Delta_k^h w_{\varepsilon\psi}^1 \right\rangle \right| \\ & \lesssim 2^{-2ks} (\tilde{d}_k + \check{d}_k) \left[\varepsilon \|\partial_x w_{\varepsilon\psi}^1\|_{\tilde{L}_{T,\zeta'(t)}^2(B^{s+\frac{1}{2}})} \|w_{\varepsilon\psi}^1\|_{\tilde{L}_{T,\zeta'(t)}^2(B^{s+\frac{1}{2}})} \right. \\ & \quad \left. + \varepsilon \|u_\psi\|_{L_T^\infty(B^{\frac{7}{2}})}^{\frac{1}{2}} \varepsilon \|\partial_z \partial_x u_\psi^\varepsilon\|_{\tilde{L}_T^2(B^s)} \|\partial_z w_{\varepsilon\psi}^1\|_{\tilde{L}_{T,\zeta'(t)}^2(B^{s+\frac{1}{2}})} \right] \end{aligned}$$

- Conclusion : By summing up the two previous estimates we get

$$\begin{aligned} & \varepsilon^2 \int_0^T \left| \left\langle \Delta_k^h(v^\varepsilon \partial_z \partial_x^2 u^\varepsilon)_\psi, \Delta_k^h w_{\varepsilon\psi}^1 \right\rangle \right| \\ & \lesssim 2^{-2ks} (\tilde{d}_k + \check{d}_k) \left[\varepsilon \|\partial_x w_{\varepsilon\psi}^1\|_{\tilde{L}_{T,\zeta'(t)}^2(B^{s+\frac{1}{2}})} \|w_{\varepsilon\psi}^1\|_{\tilde{L}_{T,\zeta'(t)}^2(B^{s+\frac{1}{2}})} \right. \\ & \quad + \|u_\psi\|_{L_T^\infty(B^{\frac{3}{2}})}^{\frac{1}{2}} \varepsilon \|\partial_z \partial_x w_{\varepsilon\psi}^1\|_{\tilde{L}_T^2(B^s)} \|\partial_x w_{\varepsilon\psi}^1\|_{\tilde{L}_{T,\zeta'(t)}^2(B^{s+\frac{1}{2}})} \\ & \quad + \varepsilon \|u_\psi\|_{L_T^\infty(B^{\frac{3}{2}})}^{\frac{1}{2}} \|\partial_z u_\psi\|_{\tilde{L}_T^2(B^{s+1})} \varepsilon \|\partial_x w_{\varepsilon\psi}^1\|_{\tilde{L}_{T,\zeta'(t)}^2(B^{s+\frac{1}{2}})} \\ & \quad \left. + \varepsilon \|u_\psi\|_{L_T^\infty(B^{\frac{7}{2}})}^{\frac{1}{2}} \varepsilon \|\partial_z \partial_x u_\psi^\varepsilon\|_{\tilde{L}_T^2(B^s)} \|\partial_z w_{\varepsilon\psi}^1\|_{\tilde{L}_{T,\zeta'(t)}^2(B^{s+\frac{1}{2}})} \right] \end{aligned}$$

8. Estimate of $\varepsilon^2 \int_0^T \left| \left\langle \Delta_k^h(\gamma^\varepsilon \partial_x v^\varepsilon)_\psi, \Delta_k^h w_{\varepsilon\psi}^1 \right\rangle \right|$.

$$\varepsilon^2 \int_0^T \left| \left\langle \Delta_k^h(\gamma^\varepsilon \partial_x v^\varepsilon)_\psi, \Delta_k^h w_{\varepsilon\psi}^1 \right\rangle \right| \leq \varepsilon^2 \int_0^T \left| \left\langle \Delta_k^h(v^\varepsilon \partial_x v^\varepsilon)_\psi, \Delta_k^h w_{\varepsilon\psi}^1 \right\rangle \right| + \varepsilon^2 \alpha_1^2 \int_0^T \left| \left\langle \Delta_k^h(\partial_z^2 v^\varepsilon \partial_x v^\varepsilon)_\psi, \Delta_k^h w_{\varepsilon\psi}^1 \right\rangle \right|$$

- The term $\varepsilon^2 \alpha_1^2 \int_0^T \left| \left\langle \Delta_k^h(\partial_z^2 v^\varepsilon \partial_x v^\varepsilon)_\psi, \Delta_k^h w_{\varepsilon\psi}^1 \right\rangle \right|$ is treated see the estimate (??).
- Estimate of the term $\varepsilon^2 \int_0^T \left| \left\langle \Delta_k^h(v^\varepsilon \partial_x v^\varepsilon)_\psi, \Delta_k^h w_{\varepsilon\psi}^1 \right\rangle \right|$. We write

$$v^\varepsilon \partial_x v^\varepsilon = w_\varepsilon^2 \partial_x v^\varepsilon + v \partial_x w_\varepsilon^2 + v \partial_x v$$

By adapting the proof of Lemma 3.2 in [64], we obtain

$$\varepsilon^2 \int_0^T \left| \left\langle \Delta_k^h(w_\varepsilon^2 \partial_x v^\varepsilon)_\psi, \Delta_k^h w_{\varepsilon\psi}^1 \right\rangle \right| \lesssim 2^{-2ks} (\tilde{d}_k + \check{d}_k) \|w_{\varepsilon\psi}^1\|_{\tilde{L}_{T,\zeta'(t)}^2(B^{s+\frac{1}{2}})}^2$$

By adapting the proof of Estimate (5.13), in [64], we get

$$\varepsilon^2 \int_0^T \left| \left\langle \Delta_k^h(v \partial_x w_\varepsilon^2)_\psi, \Delta_k^h w_{\varepsilon\psi}^1 \right\rangle \right| \lesssim 2^{-2ks} (\tilde{d}_k + \check{d}_k) \|u_\psi\|_{L_T^\infty(B^{\frac{3}{2}})}^{\frac{1}{2}} \varepsilon^2 \|\partial_x w_{\varepsilon\psi}^2\|_{\tilde{L}_T^2(B^s)} \|w_{\varepsilon\psi}^1\|_{\tilde{L}_{T,\zeta'(t)}^2(B^{s+\frac{1}{2}})},$$

- Estimate of $\varepsilon^2 \int_0^T \left| \left\langle \Delta_k^h (v \partial_x v)_\psi, \Delta_k^h w_{\varepsilon_\psi}^1 \right\rangle \right|$. By Bony's decomposition, we have

$$v \partial_x v = T_v^h \partial_x w_\varepsilon^2 + T_{\partial_x v}^h v + R^h(v, \partial_x v)$$

Let us estimate the following three terms : Estimate of $\varepsilon^2 \int_0^t \left| \left\langle \Delta_k^h (T_v^h \partial_x v)_\psi, \Delta_k^h w_{\varepsilon_\psi}^1 \right\rangle \right| dt'$

$$\begin{aligned} & \varepsilon^2 \int_0^t \left| \left\langle \Delta_k^h (T_v^h \partial_x v)_\psi, \Delta_k^h w_{\varepsilon_\psi}^1 \right\rangle \right| dt' \\ & \lesssim \varepsilon^2 \sum_{|k-k'| \leq 4} \int_0^T \|S_{k'-1}^h v\|_{L^\infty} \|\Delta_{k'}^h \partial_x v_\psi\|_{L^2} \|\Delta_k^h w_{\varepsilon_\psi}^1\|_{L^2} dt \\ & \lesssim \varepsilon^2 \sum_{|k-k'| \leq 4} \int_0^T 2^{\frac{k'}{2}} \|u_\psi\|_{B^{\frac{3}{2}}}^{\frac{1}{2}} \|\partial_z^2 u_\psi\|_{B^{\frac{1}{2}}}^{\frac{1}{2}} \|\Delta_{k'}^h \partial_x^2 u_\psi\|_{L^2} \|\Delta_k^h w_{\varepsilon_\psi}^1\|_{L^2} dt \\ & \lesssim \varepsilon^2 \sum_{|k-k'| \leq 4} \|u_\psi\|_{L_T^\infty(B^{\frac{3}{2}})}^{\frac{1}{2}} (2^{\frac{k'}{2}} 2^{2k'} \|\Delta_{k'}^h \partial_x u_\psi\|_{L_T^2(L^2)}) \left(\int_0^T \|\partial_z^2 u_\psi\|_{B^{\frac{1}{2}}} \|\Delta_k^h w_{\varepsilon_\psi}^1\|_{L^2}^2 dt \right)^{\frac{1}{2}} \\ & \lesssim 2^{-2ks} \varepsilon^2 \tilde{d}_k \|u_\psi\|_{L_T^\infty(B^{\frac{3}{2}})}^{\frac{1}{2}} \|\partial_z u_\psi\|_{\tilde{L}_T^2(B^{s+2})} \|w_{\varepsilon_\psi}^1\|_{\tilde{L}_{T, \zeta'(t)}^2(B^{s+\frac{1}{2}})}, \end{aligned}$$

where

$$\tilde{d}_k = d_k(w_{\varepsilon_\psi}^1, \xi') \sum_{|k-k'| \leq 4} 2^{(k-k')(s-1/2)} d_{k'}(\partial_x^2 u_\psi, 1)$$

$$\begin{aligned} & \varepsilon^2 \int_0^t \left| \left\langle \Delta_k^h (T_{\partial_x v^\varepsilon}^h v)_\psi, \Delta_k^h w_{\varepsilon_\psi}^1 \right\rangle \right| dt' \\ & \lesssim \varepsilon^2 \sum_{|k-k'| \leq 4} \int_0^t \|S_{k'-1}^h \partial_x v_\psi^\varepsilon\|_{L_x^2(L_x^\infty)} \|\Delta_{k'}^h v_\psi\|_{L_x^\infty(L_x^2)} \|\Delta_k^h w_{\varepsilon_\psi}^1\|_{L^2} dt' \end{aligned}$$

Estimate of $\varepsilon^2 \int_0^T \left| \left\langle \Delta_k^h (T_{\partial_x v^\varepsilon}^h v)_\psi, \Delta_k^h w_{\varepsilon_\psi}^1 \right\rangle \right| dt'$. We have

$$\|\Delta_{k'}^h \partial_x v_\psi\|_{L_x^\infty(L_x^2)} \lesssim \|u_\psi\|_{B^{\frac{3}{2}}}^{\frac{1}{2}} \|\partial_z^2 u_\psi\|_{B^{\frac{1}{2}}}^{\frac{1}{2}}$$

Then

$$\begin{aligned} & \varepsilon^2 \int_0^T \left| \left\langle \Delta_k^h (T_{\partial_x v^\varepsilon}^h v)_\psi, \Delta_k^h w_{\varepsilon_\psi}^1 \right\rangle \right| dt' \\ & \lesssim \varepsilon^2 \sum_{|k-k'| \leq 4} \int_0^T \|S_{k'-1}^h \partial_x v_\psi\|_{L_x^2(L_x^\infty)} \|u_\psi\|_{B^{\frac{3}{2}}}^{\frac{1}{2}} \|\partial_z^2 u_\psi\|_{B^{\frac{1}{2}}}^{\frac{1}{2}} \|\Delta_k^h w_{\varepsilon_\psi}^1\|_{L^2} dt \\ & \lesssim \varepsilon \sum_{|k-k'| \leq 4} \|u_\psi\|_{L_T^\infty(B^{\frac{3}{2}})}^{\frac{1}{2}} \|S_{k'-1}^h \partial_x v_\psi\|_{L_T^2(L_x^2(L_x^\infty))} \left(\int_0^T \|\partial_z^2 u_\psi\|_{B^{\frac{1}{2}}} \|\Delta_k^h w_{\varepsilon_\psi}^1\|_{L^2}^2 dt \right)^{\frac{1}{2}} \\ & \lesssim 2^{-2ks} \tilde{d}_k \varepsilon^2 \|u_\psi\|_{L_T^\infty(B^{\frac{3}{2}})}^{\frac{1}{2}} \|\partial_z u_\psi\|_{\tilde{L}_T^2(B^{s+2})} \|w_{\varepsilon_\psi}^1\|_{\tilde{L}_{T, \zeta'(t)}^2(B^{s+\frac{1}{2}})}, \end{aligned}$$

where

$$\tilde{d}_k = d_k(w_{\varepsilon_\psi}^1, \xi') \sum_{|k-k'| \leq 4} 2^{(k-k')(s-1/2)} d_{k'}$$

Estimate of $\varepsilon^2 \int_0^T \left| \left\langle \Delta_k^h (R^h(v, \partial_x v))_\psi, \Delta_k^h w_{\varepsilon\psi}^1 \right\rangle \right| dt$. We have

$$\begin{aligned}
& \varepsilon^2 \int_0^T \left| \left\langle \Delta_k^h (R^h(v, \partial_x v))_\psi, \Delta_k^h w_{\varepsilon\psi}^1 \right\rangle \right| dt \\
& \lesssim \varepsilon^2 \sum_{k' \geq k-3} \int_0^T \|\Delta_{k'}^h v_\psi\|_{L_z^\infty(L_x^2)} \|\tilde{\Delta}_{k'}^h \partial_x v_\psi\|_{L^2} \|\Delta_{k'}^h w_{\varepsilon\psi}^1\|_{L_z^2(L_x^\infty)} dt \\
& \lesssim \varepsilon^2 \sum_{k' \geq k-3} \int_0^T \|u_\psi\|_{B^{\frac{3}{2}}}^{\frac{1}{2}} \|\partial_z^2 u_\psi\|_{B^{\frac{1}{2}}}^{\frac{1}{2}} \|\Delta_{k'}^h \partial_x^2 u_\psi\|_{L^2} 2^{\frac{k}{2}} \|\Delta_{k'}^h w_{\varepsilon\psi}^1\|_{L^2} dt \\
& \lesssim \varepsilon^2 \sum_{k' \geq k-3} 2^{\frac{k}{2}} \|u_\psi\|_{L_T^\infty(B^{\frac{3}{2}})}^{\frac{1}{2}} \|\Delta_{k'}^h \partial_x^2 u_\psi\|_{L_T^2(L^2)} \left(\int_0^T \|\partial_z^2 u_\psi\|_{B^{\frac{1}{2}}} \|\Delta_{k'}^h w_{\varepsilon\psi}^1\|_{L^2}^2 dt \right)^{\frac{1}{2}} \\
& \lesssim \varepsilon^2 2^{-2ks} \check{d}_k \|u_\psi\|_{L_T^\infty(B^{\frac{3}{2}})}^{\frac{1}{2}} \|\partial_z u_\psi\|_{\tilde{L}_T^2(B^{s+2})} \|w_{\varepsilon\psi}^1\|_{\tilde{L}_{T, \zeta'(t)}^2(B^{s+\frac{1}{2}})},
\end{aligned}$$

where

$$\check{d}_k = d_k(w_{\varepsilon\psi}^1, \xi') \sum_{k' \geq k-3} 2^{(k-k')s} d_{k'}(\partial_z u_\psi, 1)$$

By summing the previous three estimates we get :

$$\begin{aligned}
& \varepsilon^2 \int_0^T \left| \left\langle \Delta_k^h (v \partial_x v)_\psi, \Delta_k^h w_{\varepsilon\psi}^1 \right\rangle \right| \\
& \lesssim 2^{-2ks} (\tilde{d}_k + \check{d}_k) \varepsilon^2 \|u_\psi\|_{L_T^\infty(B^{\frac{3}{2}})}^{\frac{1}{2}} \|\partial_z u_\psi\|_{\tilde{L}_T^2(B^{s+2})} \|w_{\varepsilon\psi}^1\|_{\tilde{L}_{T, \zeta'(t)}^2(B^{s+\frac{1}{2}})},
\end{aligned}$$

As a consequence

$$\begin{aligned}
& \varepsilon^2 \int_0^T \left| \left\langle \Delta_k^h (v^\varepsilon \partial_x v^\varepsilon)_\psi, \Delta_k^h w_{\varepsilon\psi}^1 \right\rangle \right| \\
& \lesssim 2^{-2ks} (\tilde{d}_k + \check{d}_k) \left[\varepsilon^2 \|u_\psi\|_{L_T^\infty(B^{\frac{3}{2}})}^{\frac{1}{2}} \|\partial_z u_\psi\|_{\tilde{L}_T^2(B^{s+2})} \|w_{\varepsilon\psi}^1\|_{\tilde{L}_{T, \zeta'(t)}^2(B^{s+\frac{1}{2}})} \right. \\
& \quad \left. + \|u_\psi\|_{L_T^\infty(B^{\frac{3}{2}})}^{\frac{1}{2}} \varepsilon^2 \|\partial_x w_{\varepsilon\psi}^2\|_{\tilde{L}_T^2(B^s)} \|w_{\varepsilon\psi}^1\|_{\tilde{L}_{T, \zeta'(t)}^2(B^{s+\frac{1}{2}})} + \|w_{\varepsilon\psi}^1\|_{\tilde{L}_{T, \zeta'(t)}^2(B^{s+\frac{1}{2}})}^2 \right]
\end{aligned}$$

• Conclusion :

$$\begin{aligned}
& \varepsilon^2 \int_0^T \left| \left\langle \Delta_k^h (\gamma^\varepsilon \partial_x v^\varepsilon)_\psi, \Delta_k^h w_{\varepsilon\psi}^1 \right\rangle \right| \\
& \lesssim 2^{-2ks} (\tilde{d}_k + \check{d}_k) \left[\varepsilon^2 \|u_\psi\|_{L_T^\infty(B^{\frac{3}{2}})}^{\frac{1}{2}} \|\partial_z u_\psi\|_{\tilde{L}_T^2(B^{s+2})} \|w_{\varepsilon\psi}^1\|_{\tilde{L}_{T, \zeta'(t)}^2(B^{s+\frac{1}{2}})} \right. \\
& \quad + \|u_\psi\|_{L_T^\infty(B^{\frac{3}{2}})}^{\frac{1}{2}} \varepsilon^2 \|\partial_x w_{\varepsilon\psi}^2\|_{\tilde{L}_T^2(B^s)} \|w_{\varepsilon\psi}^1\|_{\tilde{L}_{T, \zeta'(t)}^2(B^{s+\frac{1}{2}})} + \|w_{\varepsilon\psi}^1\|_{\tilde{L}_{T, \zeta'(t)}^2(B^{s+\frac{1}{2}})}^2 \\
& \quad + \varepsilon \|\partial_x w_{\varepsilon\psi}^1\|_{\tilde{L}_{T, \zeta'(t)}^2(B^{s+\frac{1}{2}})} \|w_{\varepsilon\psi}^1\|_{\tilde{L}_{T, \zeta'(t)}^2(B^{s+\frac{1}{2}})} \\
& \quad \left. + \varepsilon \|u_\psi\|_{L_T^\infty(B^{\frac{3}{2}})}^{\frac{1}{2}} \varepsilon \|\partial_z \partial_x u_\psi^\varepsilon\|_{\tilde{L}_T^2(B^s)} \|\partial_z w_{\varepsilon\psi}^1\|_{\tilde{L}_{T, \zeta'(t)}^2(B^{s+\frac{1}{2}})} \right]
\end{aligned}$$

9. Estimate of $\varepsilon^4 \alpha_1^2 \int_0^T \left| \left\langle \Delta_k^h (\partial_x^2 v^\varepsilon \partial_x v^\varepsilon)_\psi, \Delta_k^h w_{\varepsilon\psi}^1 \right\rangle \right|$. We write

$$\partial_x v^\varepsilon \partial_x^2 v^\varepsilon = \partial_x w_\varepsilon^2 \partial_x^2 v^\varepsilon + \partial_x v \partial_x^2 v^\varepsilon$$

• Estimate of $\varepsilon^4 \int_0^T \left| \left\langle \Delta_k^h (\partial_x w_\varepsilon^2 \partial_x^2 v^\varepsilon)_\psi, \Delta_k^h w_{\varepsilon\psi}^1 \right\rangle \right|$. By adapting the proof of Lemma 3.2 in [64], we obtain

$$\varepsilon^4 \int_0^T \left| \left\langle \Delta_k^h (\partial_x w_\varepsilon^2 \partial_x^2 v^\varepsilon)_\psi, \Delta_k^h w_{\varepsilon\psi}^1 \right\rangle \right| \lesssim 2^{-2ks} (\tilde{d}_k + \check{d}_k) \varepsilon \|\partial_x w_{\varepsilon\psi}^1\|_{\tilde{L}_{T, \zeta'(t)}^2(B^{s+\frac{1}{2}})} \|w_{\varepsilon\psi}^1\|_{\tilde{L}_{T, \zeta'(t)}^2(B^{s+\frac{1}{2}})}$$

• Estimate of $\varepsilon^4 \int_0^T \left| \left\langle \Delta_k^h (\partial_x v \partial_x^2 v^\varepsilon)_\psi, \Delta_k^h w_{\varepsilon_\psi}^1 \right\rangle \right|$. By applying Bony's decomposition (2.50) for x variable to $\partial_x v \partial_x^2 v^\varepsilon$. We get that

$$\partial_x v \partial_x^2 v^\varepsilon = T_{\partial_x v}^h \partial_x^2 v^\varepsilon + T_{\partial_x^2 v^\varepsilon}^h \partial_x v + R^h(\partial_x v, \partial_x^2 v^\varepsilon)$$

Let us estimate the following three terms :

Estimate of $\varepsilon^4 \int_0^t \left| \left\langle \Delta_k^h (T_{\partial_x v}^h \partial_x^2 v^\varepsilon)_\varphi, \Delta_k^h w_{\varepsilon_\psi}^1 \right\rangle \right| dt'$. For any $0 < s < \frac{1}{2}$, we have

$$\begin{aligned} & \varepsilon^4 \int_0^t \left| \left\langle \Delta_k^h (T_{\partial_x v}^h \partial_x^2 v^\varepsilon)_\varphi, \Delta_k^h w_{\varepsilon_\psi}^1 \right\rangle \right| dt' \\ & \lesssim \varepsilon^4 \sum_{|k-k'| \leq 4} \int_0^T \|S_{k'}^{h'} \partial_x v\|_{L^\infty} \|\Delta_{k'}^h \partial_x^2 v^\varepsilon\|_{L^2} \|\Delta_k^h w_{\varepsilon_\psi}^1\|_{L^2} dt \\ & \lesssim \varepsilon^4 \sum_{|k-k'| \leq 4} \int_0^T 2^{\frac{k'}{2}} \|u_\psi\|_{B^{\frac{7}{2}}}^{\frac{1}{2}} \|\partial_z^2 u_\psi\|_{B^{\frac{1}{2}}}^{\frac{1}{2}} \|\Delta_{k'}^h \partial_x^2 v^\varepsilon\|_{L^2} \|\Delta_k^h w_{\varepsilon_\psi}^1\|_{L^2} dt \\ & \lesssim \varepsilon^4 \sum_{|k-k'| \leq 4} \|u_\psi\|_{L_T^\infty(B^{\frac{7}{2}})}^{\frac{1}{2}} (2^{\frac{k'}{2}} \|\Delta_{k'}^h \partial_x^2 v^\varepsilon\|_{L_T^2(L^2)}) \left(\int_0^T \|\partial_z^2 u_\psi\|_{B^{\frac{1}{2}}} \|\Delta_k^h w_{\varepsilon_\psi}^1\|_{L^2}^2 dt \right)^{\frac{1}{2}} \\ & \lesssim 2^{-2ks} \varepsilon \tilde{d}_k \|u_\psi\|_{L_T^\infty(B^{\frac{7}{2}})}^{\frac{1}{2}} \varepsilon^3 \|\partial_x^2 v^\varepsilon\|_{\tilde{L}_T^2(B^s)} \|w_{\varepsilon_\psi}^1\|_{\tilde{L}_{T, \zeta'(t)}^2(B^{s+\frac{1}{2}})}, \end{aligned}$$

where

$$\tilde{d}_k = d_k(w_{\varepsilon_\psi}^1, \xi') \sum_{|k-k'| \leq 4} 2^{(k-k')(s-1/2)} d_{k'}(\partial_x^2 v^\varepsilon, 1)$$

Estimate of $\varepsilon^4 \int_0^t \left| \left\langle \Delta_k^h (T_{\partial_x^2 v^\varepsilon}^h \partial_x v)_\varphi, \Delta_k^h w_{\varepsilon_\psi}^1 \right\rangle \right| dt'$: For any $0 < s < \frac{1}{2}$, we have

$$\|S_{k'}^{h'} \partial_x^2 v^\varepsilon\|_{L_T^2(L_x^\infty)} \lesssim 2^{-k'(s-1/2)} d_{k'} \|\partial_x^2 v^\varepsilon\|_{\tilde{L}_T^2(B^s)}$$

where where $d_{k'} = \sum_{l \leq k'-2} 2^{-(k'-l)(1/2-s)} d_l(\partial_x^2 v^\varepsilon, 1)$. Then

$$\begin{aligned} & \varepsilon^4 \int_0^t \left| \left\langle \Delta_k^h (T_{\partial_x^2 v^\varepsilon}^h \partial_x v)_\varphi, \Delta_k^h w_{\varepsilon_\psi}^1 \right\rangle \right| dt' \\ & \lesssim \varepsilon^4 \sum_{|k-k'| \leq 4} \int_0^t \|S_{k'}^{h'} \partial_x^2 v^\varepsilon\|_{L_x^2(L_x^\infty)} \|\Delta_{k'}^h \partial_x v\|_{L_x^\infty(L_x^2)} \|\Delta_k^h w_{\varepsilon_\psi}^1\|_{L^2} dt' \\ & \lesssim \varepsilon^4 \sum_{|k-k'| \leq 4} \int_0^t \|S_{k'}^{h'} \partial_x^2 v^\varepsilon\|_{L_x^2(L_x^\infty)} \|u_\psi\|_{B^{\frac{7}{2}}}^{\frac{1}{2}} \|\partial_z^2 u_\psi\|_{B^{\frac{1}{2}}}^{\frac{1}{2}} \|\Delta_k^h w_{\varepsilon_\psi}^1\|_{L^2} dt' \\ & \lesssim \varepsilon^4 \sum_{|k-k'| \leq 4} \|u_\psi\|_{L_T^\infty(B^{\frac{7}{2}})}^{\frac{1}{2}} \|S_{k'}^{h'} \partial_x^2 v^\varepsilon\|_{L_T^2(L_x^2(L_x^\infty))} \left(\int_0^T \|\partial_z^2 u\|_{B^{\frac{1}{2}}} \|\Delta_k^h w_{\varepsilon_\psi}^1\|_{L^2}^2 dt \right)^{\frac{1}{2}} \\ & \lesssim 2^{-2ks} \varepsilon \tilde{d}_k \|u_\psi\|_{L_T^\infty(B^{\frac{7}{2}})}^{\frac{1}{2}} \varepsilon^3 \|\partial_x^2 v^\varepsilon\|_{\tilde{L}_T^2(B^s)} \|w_{\varepsilon_\psi}^1\|_{\tilde{L}_{T, \zeta'(t)}^2(B^{s+\frac{1}{2}})}, \end{aligned}$$

where

$$\tilde{d}_k = d_k(w_{\varepsilon_\psi}^1, \zeta') \sum_{|k-k'| \leq 4} 2^{(k-k')(s-1/2)} d_{k'}$$

Estimate of $\varepsilon^4 \int_0^T \left| \left\langle \Delta_k^h (R^h(\partial_x v, \partial_x^2 v^\varepsilon))_\psi, \Delta_k^h w_{\varepsilon_\psi}^1 \right\rangle \right| dt$

$$\begin{aligned}
& \varepsilon^4 \int_0^T \left| \left\langle \Delta_k^h (R^h(\partial_x v, \partial_x^2 v^\varepsilon))_\psi, \Delta_k^h w_{\varepsilon_\psi}^1 \right\rangle \right| dt \\
& \lesssim \varepsilon^4 \sum_{k' \geq k-3} \int_0^T \|\Delta_{k'}^h \partial_x v_\psi\|_{L^\infty(L_x^\infty)} \|\tilde{\Delta}_{k'}^h \partial_x^2 v_\psi^\varepsilon\|_{L^2} \|\Delta_{k'}^h w_{\varepsilon_\psi}^1\|_{L^2(L_x^\infty)} dt \\
& \lesssim \varepsilon^4 \sum_{k' \geq k-3} \int_0^T \|u_\psi\|_{B^{\frac{7}{2}}}^{\frac{1}{2}} \|\partial_z^2 u_\psi\|_{B^{\frac{1}{2}}}^{\frac{1}{2}} \|\Delta_{k'}^h \partial_x^2 v_\psi^\varepsilon\|_{L^2} 2^{\frac{k}{2}} \|\Delta_{k'}^h w_{\varepsilon_\psi}^1\|_{L^2} dt \\
& \lesssim \varepsilon^4 \sum_{k' \geq k-3} 2^{\frac{k}{2}} \|u_\psi\|_{L_T^\infty(B^{\frac{7}{2}})}^{\frac{1}{2}} \|\Delta_{k'}^h \partial_x^2 v_\psi^\varepsilon\|_{L_T^2(L^2)} \left(\int_0^T \|\partial_z^2 u_\psi\|_{B^{\frac{1}{2}}} \|\Delta_{k'}^h w_{\varepsilon_\psi}^1\|_{L^2}^2 dt \right)^{\frac{1}{2}} \\
& \lesssim \varepsilon 2^{-2ks} \check{d}_k \|u_\psi\|_{L_T^\infty(B^{\frac{7}{2}})}^{\frac{1}{2}} \varepsilon^3 \|\partial_x^2 v_\psi^\varepsilon\|_{\tilde{L}_T^2(B^s)} \|w_{\varepsilon_\psi}^1\|_{\tilde{L}_{T, \zeta'(t)}^2(B^{s+\frac{1}{2}})},
\end{aligned}$$

where

$$\check{d}_k = d_k(w_{\varepsilon_\psi}^1, \zeta') \sum_{k' \geq k-3} 2^{(k-k')s} d_{k'}(\partial_x^2 v_\psi^\varepsilon, 1).$$

By summing up the previous three estimates we obtain

$$\varepsilon^4 \int_0^T \left| \left\langle \Delta_k^h (\partial_x v \partial_x^2 v^\varepsilon)_\psi, \Delta_k^h w_{\varepsilon_\psi}^1 \right\rangle \right| \lesssim \varepsilon 2^{-2ks} c_k \|u_\psi\|_{L_T^\infty(B^{\frac{7}{2}})}^{\frac{1}{2}} \varepsilon^3 \|\partial_x^2 v_\psi^\varepsilon\|_{\tilde{L}_T^2(B^s)} \|w_{\varepsilon_\psi}^1\|_{\tilde{L}_{T, \zeta'(t)}^2(B^{s+\frac{1}{2}})}$$

where

$$c_k = \tilde{d}_k + \check{d}_k + \check{d}_k.$$

• Conclusion :

$$\begin{aligned}
& \varepsilon^4 \alpha_1^2 \int_0^T \left| \left\langle \Delta_k^h (\partial_x^2 v^\varepsilon \partial_x v^\varepsilon)_\psi, \Delta_k^h w_{\varepsilon_\psi}^1 \right\rangle \right| \\
& \lesssim \varepsilon 2^{-2ks} c_k \|u_\psi\|_{L_T^\infty(B^{\frac{7}{2}})}^{\frac{1}{2}} \varepsilon^3 \|\partial_x^2 v_\psi^\varepsilon\|_{\tilde{L}_T^2(B^s)} \|w_{\varepsilon_\psi}^1\|_{\tilde{L}_{T, \zeta'(t)}^2(B^{s+\frac{1}{2}})} \\
& \quad + 2^{-2ks} (\tilde{d}_k + \check{d}_k) \varepsilon \|\partial_x w_{\varepsilon_\psi}^1\|_{\tilde{L}_{T, \zeta'(t)}^2(B^{s+\frac{1}{2}})} \|w_{\varepsilon_\psi}^1\|_{\tilde{L}_{T, \zeta'(t)}^2(B^{s+\frac{1}{2}})},
\end{aligned}$$

10. Final conclusion: By summing up the previous estimates and using (3.48) Young's inequality, we obtain

$$\begin{aligned}
& \sum_{k \in \mathbb{Z}} \sqrt{2^{2ks} R_k^1} \\
& \leq C \left[M\varepsilon + M \|\partial_z w_{\varepsilon_\psi}^1\|_{\tilde{L}_{T, \eta'}^2(B^{s+\frac{1}{2}})} + M\varepsilon \|\partial_x w_{\varepsilon_\psi}^1\|_{\tilde{L}_{T, \zeta'(t)}^2(B^{s+\frac{1}{2}})} + M \|w_{\varepsilon_\psi}^1\|_{\tilde{L}_{T, \zeta'(t)}^2(B^{s+\frac{1}{2}})} \right] \\
& \quad + \frac{1}{2} \varepsilon \|\partial_z \partial_x w_{\varepsilon_\psi}^1\|_{\tilde{L}_T^2(B^s)} + \frac{1}{2} \varepsilon^2 \|\partial_x w_{\varepsilon_\psi}^2\|_{\tilde{L}_T^2(B^s)} \\
& \quad + \frac{1}{2} \varepsilon \|w_{\varepsilon_\psi}^1\|_{\tilde{L}_T^2(B^{s+1})} + \frac{1}{2} \varepsilon^2 \|w_{\varepsilon_\psi}^1\|_{\tilde{L}_T^2(B^{s+2})} + \frac{1}{2} \|\partial_z^2 w_{\varepsilon_\psi}^1\|_{\tilde{L}_{T, \eta'}^2(B^s)} \quad (3.58)
\end{aligned}$$

This conclude the proof of estimate on R_ε^1 and the proof of Proposition 3.6.1.

3.8 Appendix. Proof of proposition 3.6.2

Estimate of R_k^2 . we recall that

$$\begin{aligned} R_k^2 &= \int_0^T \left| \left\langle \Delta_k^h (\partial_t (v - \alpha_1^2 \partial_z^2 v) + \varepsilon^4 \partial_x^2 (v - 2\alpha_1^2 \partial_z^2 v) + \varepsilon^2 \partial_z^2 (v - \alpha_1^2 \partial_z^2 v) + \alpha_1^2 \varepsilon^4 \partial_t \partial_x^2 v - \alpha_1^2 \varepsilon^6 \partial_x^4 v)_\psi, \Delta_k^h w_{\varepsilon_\psi}^2 \right\rangle \right| dt' \\ &+ \int_0^T \left| \left\langle \Delta_k^h (-\varepsilon^2 u^\varepsilon \partial_x \gamma^\varepsilon + \alpha_1^2 \varepsilon^4 u^\varepsilon \partial_x^3 v^\varepsilon + \alpha_1^2 \varepsilon^2 \partial_x^2 u^\varepsilon \partial_z u^\varepsilon - 2\varepsilon^2 v^\varepsilon \partial_z v)_\psi, \Delta_k^h w_{\varepsilon_\psi}^2 \right\rangle \right| dt' \\ &+ \int_0^T \left| \left\langle \Delta_k^h (\varepsilon^2 \alpha_1^2 v^\varepsilon \partial_z^2 v + \alpha_1^2 \varepsilon^4 v^\varepsilon \partial_x^2 v^\varepsilon)_\psi, \Delta_k^h \partial_z w_{\varepsilon_\psi}^2 \right\rangle \right| dt'. \end{aligned}$$

1. Estimate of $\varepsilon^2 \int_0^T \left| \left\langle \Delta_k^h (\partial_t (v - \alpha_1^2 \partial_z^2 v))_\psi, \Delta_k^h w_{\varepsilon_\psi}^2 \right\rangle \right|$

$$\varepsilon^2 \int_0^T \left| \left\langle \Delta_k^h (\partial_t v)_\psi, \Delta_k^h w_{\varepsilon_\psi}^2 \right\rangle \right| \lesssim 2^{-2ks} \varepsilon^2 d_k(\partial_t u_\psi, 1) d_k(\partial_z w_{\varepsilon_\psi}^2, 1) \|(\partial_t u)_\psi\|_{\tilde{L}_T^2(B^{s+1})} \|\partial_z w_{\varepsilon_\psi}^2\|_{\tilde{L}_T^2(B^s)}$$

and

$$\varepsilon^2 \int_0^T \left| \left\langle \Delta_k^h (\partial_t \partial_z^2 v)_\psi, \Delta_k^h w_{\varepsilon_\psi}^2 \right\rangle \right| \lesssim 2^{-2ks} \varepsilon^2 d_k(\partial_t \partial_z u_\psi, 1) d_k(\partial_z w_{\varepsilon_\psi}^2, 1) \|(\partial_t \partial_z u)_\psi\|_{\tilde{L}_T^2(B^{s+1})} \|\partial_z w_{\varepsilon_\psi}^2\|_{\tilde{L}_T^2(B^s)}.$$

Then

$$\varepsilon^2 \int_0^T \left| \left\langle \Delta_k^h (\partial_t (v - \alpha_1^2 \partial_z^2 v))_\psi, \Delta_k^h w_{\varepsilon_\psi}^2 \right\rangle \right| \lesssim 2^{-2ks} \varepsilon^2 d_k(\partial_t \partial_z u_\psi, 1) d_k(\partial_z w_{\varepsilon_\psi}^2, 1) \|(\partial_t \partial_z u)_\psi\|_{\tilde{L}_T^2(B^{s+1})} \|\partial_z w_{\varepsilon_\psi}^2\|_{\tilde{L}_T^2(B^s)}$$

2. Estimate of $\varepsilon^4 \int_0^T \left| \left\langle \Delta_k^h \partial_x^2 (v - \alpha_1^2 \partial_z^2 v), \Delta_k^h w_{\varepsilon_\psi}^2 \right\rangle \right|$. We have

$$\varepsilon^4 \int_0^T \left| \left\langle \Delta_k^h (\partial_x^2 v)_\psi, \Delta_k^h w_{\varepsilon_\psi}^2 \right\rangle \right| \lesssim 2^{-2ks} \varepsilon^4 d_k(\partial_z u_\psi, 1) d_k(w_{\varepsilon_\psi}^2, 1) \|\partial_z u_\psi\|_{\tilde{L}_T^2(B^{s+2})} \|w_{\varepsilon_\psi}^2\|_{\tilde{L}_T^2(B^{s+1})}$$

and

$$\varepsilon^4 \int_0^T \left| \left\langle \Delta_k^h (\partial_x^2 \partial_z^2 v)_\psi, \Delta_k^h w_{\varepsilon_\psi}^2 \right\rangle \right| \lesssim 2^{-2ks} \varepsilon^4 d_k(\partial_z u_\psi, 1) d_k(w_{\varepsilon_\psi}^2, 1) \|\partial_z u_\psi\|_{\tilde{L}_T^2(B^{s+2})} \|w_{\varepsilon_\psi}^2\|_{\tilde{L}_T^2(B^{s+1})}.$$

Then

$$\varepsilon^4 \int_0^T \left| \left\langle \Delta_k^h \partial_x^2 (v - \alpha_1^2 \partial_z^2 v), \Delta_k^h w_{\varepsilon_\psi}^2 \right\rangle \right| \lesssim 2^{-2ks} \varepsilon^4 d_k(\partial_z u_\psi, 1) d_k(w_{\varepsilon_\psi}^2, 1) \|\partial_z u_\psi\|_{\tilde{L}_T^2(B^{s+2})} \|w_{\varepsilon_\psi}^2\|_{\tilde{L}_T^2(B^{s+1})}$$

3. Estimate of $\varepsilon^2 \int_0^T \left| \left\langle \Delta_k^h \partial_z^2 (v - \alpha_1^2 \partial_z^2 v)_\psi, \Delta_k^h w_{\varepsilon_\psi}^2 \right\rangle \right|$. We have

$$\varepsilon^2 \int_0^T \left| \left\langle \Delta_k^h (\partial_z^2 v)_\psi, \Delta_k^h w_{\varepsilon_\psi}^2 \right\rangle \right| \lesssim 2^{-2ks} \varepsilon^2 d_k(\partial_z^2 u_\psi, 1) d_k(\partial_z^2 w_{\varepsilon_\psi}^2, 1) \|\partial_z^2 u_\psi\|_{\tilde{L}_T^2(B^{s+1})} \|\partial_z^2 w_{\varepsilon_\psi}^2\|_{\tilde{L}_T^2(B^s)}$$

$$\varepsilon^2 \int_0^T \left| \left\langle \Delta_k^h (\partial_z^4 v)_\psi, \Delta_k^h w_{\varepsilon_\psi}^2 \right\rangle \right| \lesssim 2^{-2ks} \varepsilon^2 d_k(\partial_z^2 u_\psi, 1) d_k(\partial_z^2 w_{\varepsilon_\psi}^2, 1) \|\partial_z^2 u_\psi\|_{\tilde{L}_T^2(B^{s+1})} \|\partial_z^2 w_{\varepsilon_\psi}^2\|_{\tilde{L}_T^2(B^s)}$$

Then

$$\varepsilon^2 \int_0^T \left| \left\langle \Delta_k^h \partial_z^2 (v - \alpha_1^2 \partial_z^2 v)_\psi, \Delta_k^h w_{\varepsilon_\psi}^2 \right\rangle \right| \lesssim 2^{-2ks} \varepsilon^2 d_k(\partial_z^2 u_\psi, 1) d_k(\partial_z^2 w_{\varepsilon_\psi}^2, 1) \|\partial_z^2 u_\psi\|_{\tilde{L}_T^2(B^{s+1})} \|\partial_z^2 w_{\varepsilon_\psi}^2\|_{\tilde{L}_T^2(B^s)}$$

4. Estimate of $\varepsilon^4 \int_0^T \left| \left\langle \Delta_k^h (\partial_t \partial_x^2 v)_\psi, \Delta_k^h w_{\varepsilon_\psi}^2 \right\rangle \right|$. We have by integration by parts in x variable

$$\varepsilon^4 \int_0^T \left| \left\langle \Delta_k^h (\partial_t \partial_x^2 v)_\psi, \Delta_k^h w_{\varepsilon_\psi}^2 \right\rangle \right| \lesssim 2^{-2ks} \varepsilon^4 d_k(\partial_t \partial_z u_\psi, 1) d_k(\partial_z w_{\varepsilon_\psi}^2, 1) \|(\partial_t \partial_z u)_\psi\|_{\tilde{L}_T^2(B^{s+1})} \|\partial_x^2 w_{\varepsilon_\psi}^2\|_{\tilde{L}_T^2(B^s)}$$

5. Estimate of $\alpha_1^2 \varepsilon^6 \int_0^T \left| \left\langle \Delta_k^h \partial_x^4 v, \Delta_k^h w_{\varepsilon_\psi}^2 \right\rangle \right|$. We have

$$\varepsilon^6 \int_0^T \left| \left\langle \Delta_k^h (\partial_x^4 v)_\psi, \Delta_k^h w_{\varepsilon_\psi}^2 \right\rangle \right| \lesssim 2^{-2ks} \varepsilon^6 d_k(\partial_z u_\psi, 1) d_k(\partial_x^2 w_{\varepsilon_\psi}^2, 1) \|\partial_z u_\psi\|_{\tilde{L}_T^2(B^{s+3})} \|\partial_x^2 w_{\varepsilon_\psi}^2\|_{\tilde{L}_T^2(B^s)}$$

6. Estimate of $\varepsilon^2 \int_0^T \left| \left\langle \Delta_k^h (u^\varepsilon \partial_x \gamma^\varepsilon)_\psi, \Delta_k^h w_{\varepsilon_\psi}^2 \right\rangle \right|$. We have

$$\varepsilon^2 \int_0^T \left| \left\langle \Delta_k^h (u^\varepsilon \partial_x \gamma^\varepsilon)_\psi, \Delta_k^h w_{\varepsilon_\psi}^2 \right\rangle \right| \leq \varepsilon^2 \int_0^T \left| \left\langle \Delta_k^h (u^\varepsilon \partial_x v^\varepsilon)_\psi, \Delta_k^h w_{\varepsilon_\psi}^2 \right\rangle \right| + \varepsilon^2 \alpha_1^2 \int_0^T \left| \left\langle \Delta_k^h (u^\varepsilon \partial_x \partial_z^2 v^\varepsilon)_\psi, \Delta_k^h w_{\varepsilon_\psi}^2 \right\rangle \right|$$

By adapting the proof of (5.16) and (5.17) in [64], we deduce for any $0 < s < 1$ that

$$\begin{aligned} & \varepsilon^2 \int_0^T \left| \left\langle \Delta_k^h (u^\varepsilon \partial_x v^\varepsilon)_\psi, \Delta_k^h w_{\varepsilon_\psi}^2 \right\rangle \right| \\ & \lesssim c_k 2^{-2ks} \varepsilon^2 \left(\|w_{\varepsilon_\psi}^2\|_{\tilde{L}_{T, \zeta'(t)}^2(B^{s+\frac{1}{2}})}^2 + \|u_\psi^\varepsilon\|_{L_T^\infty(B^{\frac{1}{2}})}^{\frac{1}{2}} \|\partial_z u_\psi\|_{\tilde{L}_T^2(B^{s+\frac{3}{2}})} \|w_{\varepsilon_\psi}^2\|_{\tilde{L}_{T, \zeta'(t)}^2(B^{s+\frac{1}{2}})} \right), \end{aligned}$$

where (c_k) is a generic sequence square root summable on \mathbb{Z} . By integration by part we have

$$\begin{aligned} & \varepsilon^2 \alpha_1^2 \int_0^T \left| \left\langle \Delta_k^h (u^\varepsilon \partial_x \partial_z^2 v^\varepsilon)_\psi, \Delta_k^h w_{\varepsilon_\psi}^2 \right\rangle \right| \\ & \leq \varepsilon^2 \alpha_1^2 \int_0^T \left| \left\langle \Delta_k^h (u^\varepsilon \partial_x \partial_z v^\varepsilon)_\psi, \Delta_k^h \partial_z w_{\varepsilon_\psi}^2 \right\rangle \right| + \varepsilon^2 \alpha_1^2 \int_0^T \left| \left\langle \Delta_k^h (\partial_z u^\varepsilon \partial_x \partial_z v^\varepsilon)_\psi, \Delta_k^h w_{\varepsilon_\psi}^2 \right\rangle \right| \end{aligned}$$

We write :

$$u^\varepsilon \partial_x \partial_z v^\varepsilon = u^\varepsilon \partial_x \partial_z w_\varepsilon^2 + u^\varepsilon \partial_x \partial_z v$$

It follows from Lemma 3.1 in [64] that

$$\varepsilon^2 \int_0^T \left| \left\langle \Delta_k^h (u^\varepsilon \partial_x \partial_z w_\varepsilon^2)_\psi, \Delta_k^h \partial_z w_{\varepsilon_\psi}^2 \right\rangle \right| \lesssim c_k 2^{-2ks} \varepsilon^2 \|\partial_z w_{\varepsilon_\psi}^2\|_{\tilde{L}_{T, \zeta'(t)}^2(B^{s+\frac{1}{2}})}^2$$

Estimate of $\varepsilon^2 \int_0^T \left| \left\langle \Delta_k^h (u^\varepsilon \partial_x \partial_z v)_\psi, \Delta_k^h \partial_z w_{\varepsilon_\psi}^2 \right\rangle \right| dt$. Because of

$$\|S_{k'-1}^h u_\psi^\varepsilon\|_{L^\infty} \lesssim \|u_\psi^\varepsilon\|_{B^{\frac{1}{2}}} \|\partial_z^2 u_\psi^\varepsilon\|_{B^{\frac{1}{2}}},$$

we have

$$\begin{aligned} & \varepsilon^2 \int_0^t \left| \left\langle \Delta_k^h (T_{u^\varepsilon}^h \partial_x \partial_z v)_\psi, \Delta_k^h \partial_z w_{\varepsilon_\psi}^2 \right\rangle \right| dt' \\ & \lesssim \varepsilon^2 \sum_{|k-k'| \leq 4} \int_0^T \|S_{k'-1}^h u_\psi^\varepsilon\|_{L^\infty} \|\Delta_{k'}^h \partial_x^2 u_\psi\|_{L^2} \|\Delta_k^h \partial_z w_{\varepsilon_\psi}^2\|_{L^2} dt \\ & \lesssim \varepsilon^2 \sum_{|k-k'| \leq 4} \int_0^T \|u_\psi^\varepsilon\|_{B^{\frac{1}{2}}} \|\partial_z^2 u_\psi^\varepsilon\|_{B^{\frac{1}{2}}} 2^{2k'} \|\Delta_{k'}^h \partial_x^2 u_\psi\|_{L^2} \|\Delta_k^h \partial_z w_{\varepsilon_\psi}^2\|_{L^2} dt \\ & \lesssim \varepsilon^2 \sum_{|k-k'| \leq 4} \|u_\psi^\varepsilon\|_{L_T^\infty(B^{\frac{1}{2}})}^{\frac{1}{2}} (2^{2k'} \|\Delta_{k'}^h \partial_x^2 u_\psi\|_{L_T^2(L^2)}) \left(\int_0^T \|\partial_z^2 u_\psi^\varepsilon\|_{B^{\frac{1}{2}}} \|\Delta_k^h \partial_z w_{\varepsilon_\psi}^2\|_{L^2}^2 dt \right)^{\frac{1}{2}} \\ & \lesssim 2^{-2ks} \varepsilon \tilde{d}_k \|u_\psi^\varepsilon\|_{L_T^\infty(B^{\frac{1}{2}})}^{\frac{1}{2}} \|\partial_z u_\psi\|_{\tilde{L}_T^2(B^{s+\frac{3}{2}})} \varepsilon \|\partial_z w_{\varepsilon_\psi}^2\|_{\tilde{L}_{T, \zeta'(t)}^2(B^{s+\frac{1}{2}})}, \end{aligned}$$

where

$$\tilde{d}_k = d_k(\partial_z w_{\varepsilon_\psi}^2, \xi') \sum_{|k-k'| \leq 4} 2^{(k-k')(s-1/2)} d_{k'}(\partial_z u_\psi, 1).$$

Or because of for any $0 < s < 1$

$$\|S_{k'-1}^h \partial_x^2 u_\psi\|_{L_T^2(L_x^\infty)} \lesssim 2^{k'(1-s)} d_{k'} \|\partial_z u_\psi\|_{\tilde{L}_T^2(B^{s+\frac{3}{2}})}$$

where $d_{k'} = \sum_{l \leq k'-2} 2^{-(k'-l)(1-s)} d_l(\partial_z u_\psi, 1)$. We get

$$\begin{aligned} & \varepsilon^2 \int_0^t \left| \left\langle \Delta_k^h (T_{\partial_x \partial_z v}^h u^\varepsilon)_\varphi, \Delta_k^h \partial_z w_{\varepsilon_\psi}^2 \right\rangle \right| dt' \\ & \lesssim \varepsilon^2 \sum_{|k-k'| \leq 4} \int_0^T \|S_{k'-1}^h \partial_x^2 u_\psi\|_{L_x^\infty} \|\Delta_{k'}^h u_\psi^\varepsilon\|_{L_x^\infty} \|\Delta_k^h \partial_z w_{\varepsilon_\psi}^2\|_{L^2} dt \\ & \lesssim \varepsilon^2 \sum_{|k-k'| \leq 4} \int_0^T \|S_{k'-1}^h \partial_x^2 u_\psi\|_{L_x^\infty} 2^{-\frac{k'}{2}} \|u_\psi^\varepsilon\|_{B^{\frac{1}{2}}}^{\frac{1}{2}} \|\partial_z^2 u_\psi^\varepsilon\|_{B^{\frac{1}{2}}}^{\frac{1}{2}} \|\Delta_k^h \partial_z w_{\varepsilon_\psi}^2\|_{L^2} dt \\ & \lesssim \varepsilon^2 \sum_{|k-k'| \leq 4} \|u_\psi^\varepsilon\|_{L_T^\infty(B^{\frac{1}{2}})}^{\frac{1}{2}} 2^{-\frac{k'}{2}} \|S_{k'-1}^h \partial_x^2 u_\psi\|_{L_T^2(L_x^\infty)} \left(\int_0^T \|\partial_z^2 u_\psi\|_{B^{\frac{1}{2}}} \|\Delta_k^h \partial_z w_{\varepsilon_\psi}^2\|_{L^2}^2 dt \right)^{\frac{1}{2}} \\ & \lesssim 2^{-2ks} \varepsilon \tilde{d}_k \|u_\psi^\varepsilon\|_{\tilde{L}_T^\infty(B^{\frac{1}{2}})}^{\frac{1}{2}} \|\partial_z u_\psi\|_{\tilde{L}_T^2(B^{s+\frac{3}{2}})} \varepsilon \|\partial_z w_{\varepsilon_\psi}^2\|_{\tilde{L}_{T,\zeta'(t)}^2(B^{s+\frac{1}{2}})}, \end{aligned}$$

where

$$\tilde{d}_k = d_k(\partial_z w_{\varepsilon_\psi}^2, \xi') \sum_{|k-k'| \leq 4} 2^{(k-k')(s-1/2)} d_{k'}.$$

In this way

$$\begin{aligned} & \varepsilon^2 \int_0^T \left| \left\langle \Delta_k^h (R^h(u^\varepsilon, \partial_x \partial_z v))_\psi, \Delta_k^h \partial_z w_{\varepsilon_\psi}^2 \right\rangle \right| dt \\ & \lesssim \varepsilon^2 \sum_{k' \geq k-3} \int_0^T \|\Delta_{k'}^h u_\psi^\varepsilon\|_{L_x^\infty} \|\tilde{\Delta}_{k'}^h \partial_x^2 u_\psi\|_{L^2} \|\Delta_k^h \partial_z w_{\varepsilon_\psi}^2\|_{L_x^\infty} dt \\ & \lesssim \varepsilon^2 \sum_{k' \geq k-3} 2^{\frac{k}{2}} \int_0^T 2^{-\frac{k'}{2}} \|u_\psi^\varepsilon\|_{B^{\frac{1}{2}}}^{\frac{1}{2}} \|\partial_z^2 u_\psi^\varepsilon\|_{B^{\frac{1}{2}}}^{\frac{1}{2}} 2^{2k'} \|\Delta_{k'}^h \partial_z u_\psi\|_{L^2} \|\Delta_k^h \partial_z w_{\varepsilon_\psi}^2\|_{L^2} dt \\ & \lesssim \varepsilon^2 \sum_{k' \geq k-3} 2^{\frac{1}{2}(k-k')} \|u_\psi^\varepsilon\|_{L_T^\infty(B^{\frac{1}{2}})}^{\frac{1}{2}} (2^{2k'} \|\Delta_{k'}^h \partial_z u_\psi\|_{L_T^2(L^2)}) \left(\int_0^T \|\partial_z^2 u_\psi\|_{B^{\frac{1}{2}}} \|\Delta_k^h \partial_z w_{\varepsilon_\psi}^2\|_{L^2}^2 dt \right)^{\frac{1}{2}} \\ & \lesssim \varepsilon 2^{-2ks} \check{d}_k \|u_\psi^\varepsilon\|_{L_T^\infty(B^{\frac{1}{2}})}^{\frac{1}{2}} \|\partial_z u_\psi\|_{\tilde{L}_T^2(B^{s+\frac{3}{2}})} \varepsilon \|\partial_z w_{\varepsilon_\psi}^2\|_{\tilde{L}_{T,\zeta'(t)}^2(B^{s+\frac{1}{2}})}, \end{aligned}$$

where

$$\check{d}_k = d_k(\partial_z w_{\varepsilon_\psi}^2, \zeta') \sum_{k' \geq k-3} 2^{(k-k')s} d_{k'}(\partial_z u_\psi, 1).$$

By summing up the previous estimate, we obtain

$$\begin{aligned} & \varepsilon^2 \alpha_1^2 \int_0^T \left| \left\langle \Delta_k^h (u^\varepsilon \partial_x \partial_z v^\varepsilon)_\psi, \Delta_k^h w_{\varepsilon_\psi}^2 \right\rangle \right| \tag{3.59} \\ & \lesssim c_k 2^{-2ks} \alpha_1^2 \left[\varepsilon^2 \|\partial_z w_{\varepsilon_\psi}^2\|_{\tilde{L}_{T,\zeta'(t)}^2(B^{s+\frac{1}{2}})}^2 + \varepsilon \|u_\psi^\varepsilon\|_{L_T^\infty(B^{\frac{1}{2}})}^{\frac{1}{2}} \|\partial_z u_\psi\|_{\tilde{L}_T^2(B^{s+\frac{3}{2}})} \varepsilon \|\partial_z w_{\varepsilon_\psi}^2\|_{\tilde{L}_{T,\zeta'(t)}^2(B^{s+\frac{1}{2}})} \right], \end{aligned}$$

And similar to the the previous estimate, we obtain

$$\begin{aligned} & \varepsilon^2 \alpha_1^2 \int_0^T \left| \left\langle \Delta_k^h (\partial_z u^\varepsilon \partial_x \partial_z v^\varepsilon)_\psi, \Delta_k^h \partial_z w_{\varepsilon_\psi}^2 \right\rangle \right| \\ & \lesssim c_k 2^{-2ks} \alpha_1^2 \left[\varepsilon^2 \|\partial_z w_{\varepsilon_\psi}^2\|_{\tilde{L}_{T,\zeta'(t)}^2(B^{s+\frac{1}{2}})}^2 + \varepsilon \|\partial_z u_\psi^\varepsilon\|_{L_T^\infty(B^{\frac{1}{2}})}^{\frac{1}{2}} \|\partial_z u_\psi\|_{\tilde{L}_T^2(B^{s+\frac{3}{2}})} \varepsilon \|\partial_z w_{\varepsilon_\psi}^2\|_{\tilde{L}_{T,\zeta'(t)}^2(B^{s+\frac{1}{2}})} \right], \end{aligned}$$

Conclusion :

$$\begin{aligned} & \varepsilon^2 \int_0^T \left| \left\langle \Delta_k^h (u^\varepsilon \partial_x \gamma^\varepsilon)_\psi, \Delta_k^h w_{\varepsilon\psi}^2 \right\rangle \right| \\ & \lesssim c_k 2^{-2ks} \alpha_1^2 \left[\varepsilon^2 \|\partial_z w_{\varepsilon\psi}^2\|_{\tilde{L}_{T,\zeta'(t)}^2(B^{s+\frac{1}{2}})}^2 + \varepsilon \|\partial_z u_\psi^\varepsilon\|_{L_T^\infty(B^{\frac{1}{2}})}^{\frac{1}{2}} \|\partial_z u_\psi\|_{\tilde{L}_T^2(B^{s+\frac{5}{2}})} \varepsilon \|\partial_z w_{\varepsilon\psi}^2\|_{\tilde{L}_{T,\zeta'(t)}^2(B^{s+\frac{1}{2}})} \right], \end{aligned}$$

7. Estimate of $\alpha_1^2 \varepsilon^4 \int_0^T \left| \left\langle \Delta_k^h (u^\varepsilon \partial_x^3 v^\varepsilon)_\psi, \Delta_k^h w_{\varepsilon\psi}^2 \right\rangle \right|$. By integration by part, we have

$$\int_0^T \left| \left\langle \Delta_k^h (u^\varepsilon \partial_x^3 v^\varepsilon)_\psi, \Delta_k^h w_{\varepsilon\psi}^2 \right\rangle \right| \leq \int_0^T \left| \left\langle \Delta_k^h (\partial_x u^\varepsilon \partial_x^2 v^\varepsilon)_\psi, \Delta_k^h w_{\varepsilon\psi}^2 \right\rangle \right| + \int_0^T \left| \left\langle \Delta_k^h (u^\varepsilon \partial_x^2 v^\varepsilon)_\psi, \Delta_k^h \partial_x w_{\varepsilon\psi}^2 \right\rangle \right|$$

We write

$$\partial_x u^\varepsilon \partial_x^2 v^\varepsilon = \partial_x u^\varepsilon \partial_x^2 w_\varepsilon^2 + \partial_x u^\varepsilon \partial_x^2 v$$

and

$$u^\varepsilon \partial_x^2 v^\varepsilon = u^\varepsilon \partial_x^2 w_\varepsilon^2 + u^\varepsilon \partial_x^2 v$$

By adapting the proof of Lemma (3.1) in [64], Because $0 < \varepsilon < 1$ we get

$$\begin{aligned} \varepsilon^4 \int_0^T \left| \left\langle \Delta_k^h (\partial_x u^\varepsilon \partial_x^2 w_\varepsilon^2)_\psi, \Delta_k^h w_{\varepsilon\psi}^2 \right\rangle \right| & \lesssim 2^{-2ks} c_k \varepsilon^2 \|\partial_x w_{\varepsilon\psi}^1\|_{\tilde{L}_{T,\zeta'(t)}^2(B^{s+\frac{1}{2}})} \varepsilon \|w_{\varepsilon\psi}^2\|_{\tilde{L}_{T,\zeta'(t)}^2(B^{s+\frac{1}{2}})} \\ & \lesssim 2^{-2ks} c_k \varepsilon \|\partial_x w_{\varepsilon\psi}^1\|_{\tilde{L}_{T,\zeta'(t)}^2(B^{s+\frac{1}{2}})} \varepsilon \|w_{\varepsilon\psi}^2\|_{\tilde{L}_{T,\zeta'(t)}^2(B^{s+\frac{1}{2}})} \end{aligned}$$

and

$$\varepsilon^4 \int_0^T \left| \left\langle \Delta_k^h (u^\varepsilon \partial_x^2 w_\varepsilon^2)_\psi, \Delta_k^h \partial_x w_{\varepsilon\psi}^2 \right\rangle \right| \lesssim 2^{-2ks} c_k \varepsilon \|\partial_x w_{\varepsilon\psi}^1\|_{\tilde{L}_{T,\zeta'(t)}^2(B^{s+\frac{1}{2}})} \varepsilon^2 \|\partial_x w_{\varepsilon\psi}^2\|_{\tilde{L}_{T,\zeta'(t)}^2(B^{s+\frac{1}{2}})}$$

Similar to (3.57), we obtain

$$\begin{aligned} \varepsilon^4 \int_0^T \left| \left\langle \Delta_k^h (\partial_x u^\varepsilon \partial_x^2 v)_\psi, \Delta_k^h w_{\varepsilon\psi}^2 \right\rangle \right| & \tag{3.60} \\ & \lesssim 2^{-2ks} \left[(\tilde{d}_k + \check{d}_k) \varepsilon \varepsilon^{\frac{1}{2}} \|\partial_x u_\psi^\varepsilon\|_{L_T^\infty(B^{\frac{1}{2}})}^{\frac{1}{2}} \|\partial_z u_\psi\|_{\tilde{L}_T^2(B^{s+\frac{5}{2}})} \varepsilon^2 \|w_{\varepsilon\psi}^2\|_{\tilde{L}_{T,\zeta'(t)}^2(B^{s+\frac{1}{2}})} \right], \end{aligned}$$

and

$$\begin{aligned} \varepsilon^4 \int_0^T \left| \left\langle \Delta_k^h (u^\varepsilon \partial_x^2 v)_\psi, \Delta_k^h \partial_x w_{\varepsilon\psi}^2 \right\rangle \right| & \tag{3.61} \\ & \lesssim 2^{-2ks} \left[(\tilde{d}_k + \check{d}_k) \varepsilon^2 \|u_\psi^\varepsilon\|_{L_T^\infty(B^{\frac{1}{2}})}^{\frac{1}{2}} \|\partial_z u_\psi\|_{\tilde{L}_T^2(B^{s+\frac{5}{2}})} \varepsilon^2 \|\partial_x w_{\varepsilon\psi}^2\|_{\tilde{L}_{T,\zeta'(t)}^2(B^{s+\frac{1}{2}})} \right], \end{aligned}$$

Conclusion :

$$\begin{aligned} \varepsilon^4 \int_0^T \left| \left\langle \Delta_k^h (u^\varepsilon \partial_x^3 v^\varepsilon)_\psi, \Delta_k^h w_{\varepsilon\psi}^2 \right\rangle \right| & \tag{3.62} \\ & \lesssim 2^{-2ks} c_k \left[\varepsilon \|\partial_x w_{\varepsilon\psi}^1\|_{\tilde{L}_{T,\zeta'(t)}^2(B^{s+\frac{1}{2}})} \varepsilon \|w_{\varepsilon\psi}^2\|_{\tilde{L}_{T,\zeta'(t)}^2(B^{s+\frac{1}{2}})} + \varepsilon \|\partial_x w_{\varepsilon\psi}^1\|_{\tilde{L}_{T,\zeta'(t)}^2(B^{s+\frac{1}{2}})} \varepsilon^2 \|\partial_x w_{\varepsilon\psi}^2\|_{\tilde{L}_{T,\zeta'(t)}^2(B^{s+\frac{1}{2}})} \right. \\ & \quad + \varepsilon^2 \|u_\psi^\varepsilon\|_{L_T^\infty(B^{\frac{1}{2}})}^{\frac{1}{2}} \|\partial_z u_\psi\|_{\tilde{L}_T^2(B^{s+\frac{5}{2}})} \varepsilon^2 \|\partial_x w_{\varepsilon\psi}^2\|_{\tilde{L}_{T,\zeta'(t)}^2(B^{s+\frac{1}{2}})} \\ & \quad \left. + \varepsilon \varepsilon^{\frac{1}{2}} \|\partial_x u_\psi^\varepsilon\|_{L_T^\infty(B^{\frac{1}{2}})}^{\frac{1}{2}} \|\partial_z u_\psi\|_{\tilde{L}_T^2(B^{s+\frac{5}{2}})} \varepsilon^2 \|w_{\varepsilon\psi}^2\|_{\tilde{L}_{T,\zeta'(t)}^2(B^{s+\frac{1}{2}})} \right], \end{aligned}$$

8. Estimate of $\alpha_1^2 \varepsilon^2 \int_0^T \left| \left\langle \Delta_k^h (\partial_x^2 u^\varepsilon \partial_z u^\varepsilon)_\psi, \Delta_k^h w_{\varepsilon\psi}^2 \right\rangle \right|$. Similar to (3.59), we obtain

$$\begin{aligned} & \varepsilon^2 \alpha_1^2 \int_0^T \left| \left\langle \Delta_k^h (\partial_z u^\varepsilon \partial_x^2 u^\varepsilon)_\psi, \Delta_k^h w_{\varepsilon\psi}^2 \right\rangle \right| \\ & \lesssim c_k 2^{-2ks} \alpha_1^2 \left[\varepsilon \|\partial_x w_{\varepsilon\psi}^1\|_{\tilde{L}_{T,\zeta'(t)}^2(B^{s+\frac{1}{2}})} \varepsilon \|w_{\varepsilon\psi}^2\|_{\tilde{L}_{T,\zeta'(t)}^2(B^{s+\frac{1}{2}})} \right. \\ & \quad \left. + \varepsilon \|\partial_z u_\psi^\varepsilon\|_{L_T^\infty(B^{\frac{1}{2}})}^{\frac{1}{2}} \|\partial_z u_\psi\|_{\tilde{L}_T^2(B^{s+\frac{3}{2}})} \varepsilon \|\partial_z w_{\varepsilon\psi}^2\|_{\tilde{L}_{T,\zeta'(t)}^2(B^{s+\frac{1}{2}})} \right], \end{aligned} \quad (3.63)$$

9. Estimate of $\varepsilon^2 \int_0^T \left| \left\langle \Delta_k^h (v^\varepsilon \partial_x u^\varepsilon)_\psi, \Delta_k^h w_{\varepsilon\psi}^2 \right\rangle \right|$. We write

$$\begin{aligned} v^\varepsilon \partial_x u^\varepsilon &= w_\varepsilon^2 \partial_x u^\varepsilon + v \partial_x u^\varepsilon \\ &= w_\varepsilon^2 \partial_x u^\varepsilon + v \partial_x w_\varepsilon^1 + v \partial_x u \end{aligned}$$

By adapting the proof of Lemma 3.2 in [64], we obtain

$$\varepsilon^2 \int_0^T \left| \left\langle \Delta_k^h (w_\varepsilon^2 \partial_x u^\varepsilon)_\psi, \Delta_k^h w_{\varepsilon\psi}^2 \right\rangle \right| \lesssim 2^{-2ks} (\tilde{d}_k + \check{d}_k) \|w_{\varepsilon\psi}^1\|_{\tilde{L}_{T,\zeta'(t)}^2(B^{s+\frac{1}{2}})} \varepsilon \|w_{\varepsilon\psi}^2\|_{\tilde{L}_{T,\zeta'(t)}^2(B^{s+\frac{1}{2}})}$$

• Estimate of $\varepsilon^2 \int_0^T \left| \left\langle \Delta_k^h (v \partial_x w_\varepsilon^1)_\psi, \Delta_k^h w_{\varepsilon\psi}^2 \right\rangle \right|$. By Bony's decomposition, we have

$$v \partial_x w_\varepsilon^1 = T_v^h \partial_x w_\varepsilon^1 + T_{\partial_x w_\varepsilon^1}^h v + R^h(v, \partial_x w_\varepsilon^1).$$

Let us handle the following three terms :

Estimate of $\int_0^T \left| \left\langle \Delta_k^h (T_{\partial_x w_\varepsilon^1}^h v)_\psi, \Delta_k^h w_{\varepsilon\psi}^2 \right\rangle \right|$

$$\int_0^T \left| \left\langle \Delta_k^h (T_{\partial_x w_\varepsilon^1}^h v)_\psi, \Delta_k^h w_{\varepsilon\psi}^2 \right\rangle \right| \lesssim \sum_{|k'-k| \leq 4} \int_0^t \|S_{k'-1}^h \partial_x w_{\varepsilon\psi}^1\|_{L_z^2(L_x^\infty)} \|\Delta_{k'}^h v_\psi\|_{L_z^\infty(L_x^2)} \|\Delta_k^h w_{\varepsilon\psi}^2\|_{L^2}$$

We have

$$\|\Delta_{k'}^h v_\psi\|_{L_z^\infty(L_x^2)} \lesssim \|u_\psi\|_{B^{\frac{3}{2}}}^{\frac{1}{2}} \|\partial_z^2 u_\psi\|_{B^{\frac{1}{2}}}^{\frac{1}{2}},$$

and for any $0 < s < \frac{1}{2}$

$$\|S_{k'-1}^h \partial_x w_{\varepsilon\psi}^1\|_{L_T^2(L_z^2(L_x^\infty))} \lesssim 2^{k'(\frac{1}{2}-s)} d_{k'} \|\partial_x w_\varepsilon^1\|_{\tilde{L}_T^2(B^s)}.$$

where

$$d_{k'} = \sum_{l \leq k'-2} 2^{(l-k')(\frac{1}{2}-s)} d_l(\partial_x w_\varepsilon^1, 1).$$

Then

$$\begin{aligned} & \varepsilon^2 \int_0^T \left| \left\langle \Delta_k^h (T_{\partial_x w_\varepsilon^1}^h v)_\psi, \Delta_k^h w_{\varepsilon\psi}^2 \right\rangle \right| \\ & \lesssim \varepsilon^2 \sum_{|k'-k| \leq 4} \int_0^t \|S_{k'-1}^h \partial_x w_{\varepsilon\psi}^1\|_{L_z^2(L_x^\infty)} \|u_\psi\|_{B^{\frac{3}{2}}}^{\frac{1}{2}} \|\partial_z^2 u_\psi\|_{B^{\frac{1}{2}}}^{\frac{1}{2}} \|\Delta_k^h w_\varepsilon^2\|_{L^2} \\ & \lesssim \varepsilon^2 \sum_{|k'-k| \leq 4} \|u\|_{L_T^\infty(B^{\frac{3}{2}})}^{\frac{1}{2}} \|S_{k'-1}^h \partial_x w_{\varepsilon\psi}^1\|_{L_T^2(L_z^2(L_x^\infty))} \left(\int_0^T \|\partial_z^2 u_\psi\|_{B^{\frac{1}{2}}} \|\Delta_k^h w_\varepsilon^2\|_{L^2}^2 \right)^{\frac{1}{2}} \\ & \lesssim 2^{-2ks} \tilde{d}_k \|u\|_{L_T^\infty(B^{\frac{1}{2}})}^{\frac{3}{2}} \varepsilon \|\partial_x w_\varepsilon^1\|_{\tilde{L}_T^2(B^s)} \varepsilon \|w_{\varepsilon\psi}^2\|_{\tilde{L}_{T,\zeta'(t)}^2(B^{s+\frac{1}{2}})} \end{aligned}$$

where

$$\tilde{d}_k = d_k(w_{\varepsilon\psi}^1, \zeta') \sum_{|k'-k| \leq 4} 2^{(k-k')(s-1/2)} d_{k'}$$

Estimate of $\varepsilon^2 \int_0^T \left| \left\langle \Delta_k^h (T_v^h \partial_x w_\varepsilon^1)_\psi, \Delta_k^h w_\varepsilon^2 \right\rangle \right|$. We have

$$\|S_{k'-1}^h v\|_{L^\infty} \lesssim 2^{\frac{k'}{2}} \|u_\psi\|_{B^{\frac{3}{2}}}^{\frac{1}{2}} \|\partial_z^2 u_\psi\|_{B^{\frac{1}{2}}}^{\frac{1}{2}}.$$

Then

$$\begin{aligned} & \varepsilon^2 \int_0^T \left| \left\langle \Delta_k^h (T_v^h \partial_x w_\varepsilon^1)_\psi, \Delta_k^h w_\varepsilon^2 \right\rangle \right| \\ & \lesssim \sum_{|k'-k| \leq 4} \varepsilon^2 \int_0^t \|S_{k'-1}^h v_\psi\|_{L^\infty} \|\Delta_{k'}^h \partial_x w_\varepsilon^1\|_{L^2} \|\Delta_k^h w_\varepsilon^2\|_{L^2} \\ & \lesssim \varepsilon \sum_{|k'-k| \leq 4} 2^{\frac{k'}{2}} \|u_\psi\|_{L_T^\infty(B^{\frac{3}{2}})}^{\frac{1}{2}} \varepsilon \|\Delta_{k'}^h \partial_x w_\varepsilon^1\|_{L_T^2(L^2)} \left(\int_0^T \|\partial_z^2 u^\varepsilon\|_{B^{\frac{1}{2}}} \|\Delta_k^h w_\varepsilon^2\|_{L^2}^2 \right)^{\frac{1}{2}} \\ & \lesssim 2^{-2ks} \tilde{d}_k \|u_\psi\|_{L_T^\infty(B^{\frac{3}{2}})}^{\frac{1}{2}} \varepsilon \|\partial_x w_\varepsilon^1\|_{\tilde{L}_T^2(B^s)} \varepsilon \|w_\varepsilon^2\|_{\tilde{L}_{T,\zeta'(t)}^2(B^{s+\frac{1}{2}})}, \end{aligned}$$

where

$$\tilde{d}_k = d_k(w_\varepsilon^1, \zeta') \sum_{|k'-k| \leq 4} 2^{(k-k')(s-1/2)} d_{k'}(\partial_x w_\varepsilon^1, 1)$$

Estimate of $\varepsilon^2 \int_0^T \left| \left\langle \Delta_k^h (T_v^h \partial_x w_\varepsilon^1)_\psi, \Delta_k^h w_\varepsilon^2 \right\rangle \right|$

$$\begin{aligned} & \varepsilon^2 \int_0^T \left| \left\langle \Delta_k^h (R^h(v, \partial_x w_\varepsilon^1)_\psi, \Delta_k^h w_\varepsilon^2 \right\rangle \right| dt \\ & \lesssim \varepsilon^2 \sum_{k' \geq k-3} \int_0^T \|\Delta_{k'}^h v_\psi\|_{L_z^\infty(L_x^2)} \|\tilde{\Delta}_{k'}^h \partial_x w_\varepsilon^1\|_{L^2} \|\Delta_k^h w_\varepsilon^2\|_{L_z^2(L_x^\infty)} dt \\ & \lesssim \varepsilon \sum_{k' \geq k-3} \int_0^T \|u_\psi\|_{B^{\frac{3}{2}}}^{\frac{1}{2}} \|\partial_z^2 u_\psi\|_{B^{\frac{1}{2}}}^{\frac{1}{2}} \|\tilde{\Delta}_{k'}^h \partial_x w_\varepsilon^1\|_{L^2} 2^{\frac{k}{2}} \|\Delta_k^h w_\varepsilon^2\|_{L^2} dt \\ & \lesssim 2^{\frac{k}{2}} \sum_{k' \geq k-3} \|u_\psi\|_{L_T^\infty(B^{\frac{1}{2}})}^{\frac{1}{2}} \|\tilde{\Delta}_{k'}^h \partial_x w_\varepsilon^1\|_{L_T^2(L^2)} \left(\int_0^T \|\partial_z^2 u^\varepsilon\|_{B^{\frac{1}{2}}} \|\Delta_k^h w_\varepsilon^2\|_{L^2}^2 \right)^{\frac{1}{2}} \\ & \lesssim 2^{-2ks} \check{d}_k \|u_\psi\|_{L_T^\infty(B^{\frac{3}{2}})}^{\frac{1}{2}} \varepsilon \|\partial_x w_\varepsilon^1\|_{\tilde{L}_T^2(B^s)} \varepsilon \|w_\varepsilon^2\|_{\tilde{L}_{T,\zeta'(t)}^2(B^{s+\frac{1}{2}})} \end{aligned}$$

where

$$\check{d}_k = d_k(w_\varepsilon^1, \zeta') \sum_{|k'-k| \leq 4} 2^{(k-k')s} d_{k'}(\partial_x w_\varepsilon^1, 1)$$

Therefore summing the three previous estimates, we get

$$\begin{aligned} & \varepsilon^2 \int_0^T \left| \left\langle \Delta_k^h (v \partial_x w_\varepsilon^1)_\psi, \Delta_k^h w_\varepsilon^2 \right\rangle \right| \\ & \lesssim 2^{-2ks} \left[(\tilde{d}_k + \check{d}_k) \|u_\psi\|_{L_T^\infty(B^{\frac{3}{2}})}^{\frac{1}{2}} \varepsilon \|\partial_x w_\varepsilon^1\|_{\tilde{L}_T^2(B^s)} \varepsilon \|w_\varepsilon^2\|_{\tilde{L}_{T,\zeta'(t)}^2(B^{s+\frac{1}{2}})} \right], \end{aligned} \quad (3.64)$$

• Estimate of $\varepsilon^2 \int_0^T \left| \left\langle \Delta_k^h (v \partial_x u)_\psi, \Delta_k^h w_\varepsilon^2 \right\rangle \right|$. From the previous estimate we have

$$\begin{aligned} & \varepsilon^2 \int_0^T \left| \left\langle \Delta_k^h (v \partial_x u)_\psi, \Delta_k^h w_\varepsilon^2 \right\rangle \right| \\ & \lesssim 2^{-2ks} \left[(\tilde{d}_k + \check{d}_k) \varepsilon \|u_\psi\|_{L_T^\infty(B^{\frac{3}{2}})}^{\frac{1}{2}} \|u\|_{\tilde{L}_T^2(B^{s+1})} \varepsilon \|w_\varepsilon^2\|_{\tilde{L}_{T,\zeta'(t)}^2(B^{s+\frac{1}{2}})} \right], \end{aligned} \quad (3.65)$$

Conclusion :

$$\begin{aligned} \varepsilon^2 \int_0^T \left| \left\langle \Delta_k^h(v^\varepsilon \partial_x u^\varepsilon)_\psi, \Delta_k^h w_{\varepsilon\psi}^2 \right\rangle \right| \\ \lesssim 2^{-2ks} (\tilde{d}_k + \check{d}_k) \left[\varepsilon \|u_\psi\|_{L_T^\infty(B^{\frac{3}{2}})}^{\frac{1}{2}} \|u\|_{\tilde{L}_T^2(B^{s+1})} \varepsilon \|w_{\varepsilon\psi}^2\|_{\tilde{L}_{T,\zeta'(t)}^2(B^{s+\frac{1}{2}})} \right. \\ \left. + \|u_\psi\|_{L_T^\infty(B^{\frac{3}{2}})}^{\frac{1}{2}} \varepsilon \|\partial_x w_{\varepsilon\psi}^1\|_{\tilde{L}_T^2(B^s)} \varepsilon \|w_{\varepsilon\psi}^2\|_{\tilde{L}_{T,\zeta'(t)}^2(B^{s+\frac{1}{2}})} \right. \\ \left. + \|w_{\varepsilon\psi}^1\|_{\tilde{L}_{T,\zeta'(t)}^2(B^{s+\frac{1}{2}})} \varepsilon \|w_{\varepsilon\psi}^2\|_{\tilde{L}_{T,\zeta'(t)}^2(B^{s+\frac{1}{2}})} \right], \end{aligned} \quad (3.66)$$

10. Estimate of $\varepsilon^2 \alpha_1^2 \int_0^T \left| \left\langle \Delta_k^h(v^\varepsilon \partial_z^2 v^\varepsilon)_\psi, \Delta_k^h \partial_z w_{\varepsilon\psi}^2 \right\rangle \right|$. We write

$$v^\varepsilon \partial_z \partial_x u^\varepsilon = w_\varepsilon^2 \partial_x \partial_z u^\varepsilon + v \partial_z \partial_x w_{\varepsilon\psi}^1 + v \partial_z \partial_x u.$$

By adapting the proof of Lemma 3.2 in [64], we obtain

$$\varepsilon^2 \int_0^T \left| \left\langle \Delta_k^h(w_\varepsilon^2 \partial_x \partial_z u^\varepsilon)_\psi, \Delta_k^h \partial_z w_{\varepsilon\psi}^2 \right\rangle \right| \lesssim 2^{-2ks} (\tilde{d}_k + \check{d}_k) \|w_{\varepsilon\psi}^1\|_{\tilde{L}_{T,\zeta'(t)}^2(B^{s+\frac{1}{2}})} \varepsilon \|\partial_z w_{\varepsilon\psi}^2\|_{\tilde{L}_{T,\zeta'(t)}^2(B^{s+\frac{1}{2}})}$$

By adapting the proof of Estimate (3.64), we get

$$\varepsilon^2 \int_0^T \left| \left\langle \Delta_k^h(v \partial_z \partial_x w_{\varepsilon\psi}^1)_\psi, \Delta_k^h \partial_z w_{\varepsilon\psi}^2 \right\rangle \right| dt \lesssim 2^{-2ks} (\tilde{d}_k + \check{d}_k) \|u_\psi\|_{L_T^\infty(B^{\frac{3}{2}})}^{\frac{1}{2}} \varepsilon \|\partial_z \partial_x w_{\varepsilon\psi}^1\|_{\tilde{L}_T^2(B^s)} \varepsilon \|\partial_z w_{\varepsilon\psi}^2\|_{\tilde{L}_{T,\zeta'(t)}^2(B^{s+\frac{1}{2}})},$$

and

$$\varepsilon^2 \int_0^T \left| \left\langle \Delta_k^h(v \partial_z \partial_x u)_\psi, \Delta_k^h \partial_z w_{\varepsilon\psi}^2 \right\rangle \right| dt \lesssim 2^{-2ks} (\tilde{d}_k + \check{d}_k) \varepsilon \|u_\psi\|_{L_T^\infty(B^{\frac{3}{2}})}^{\frac{1}{2}} \|\partial_z u_\psi\|_{\tilde{L}_T^2(B^{s+1})} \varepsilon \|\partial_z w_{\varepsilon\psi}^2\|_{\tilde{L}_{T,\zeta'(t)}^2(B^{s+\frac{1}{2}})}.$$

Conclusion :

$$\begin{aligned} \varepsilon^2 \alpha_1^2 \int_0^T \left| \left\langle \Delta_k^h(v^\varepsilon \partial_z^2 v^\varepsilon)_\psi, \Delta_k^h \partial_z w_{\varepsilon\psi}^2 \right\rangle \right| \\ \lesssim 2^{-2ks} (\tilde{d}_k + \check{d}_k) \left[\|w_{\varepsilon\psi}^1\|_{\tilde{L}_{T,\zeta'(t)}^2(B^{s+\frac{1}{2}})} \varepsilon \|\partial_z w_{\varepsilon\psi}^2\|_{\tilde{L}_{T,\zeta'(t)}^2(B^{s+\frac{1}{2}})} \right. \\ \left. + \|u_\psi\|_{L_T^\infty(B^{\frac{3}{2}})}^{\frac{1}{2}} \varepsilon \|\partial_z \partial_x w_{\varepsilon\psi}^1\|_{\tilde{L}_T^2(B^s)} \varepsilon \|\partial_z w_{\varepsilon\psi}^2\|_{\tilde{L}_{T,\zeta'(t)}^2(B^{s+\frac{1}{2}})} \right. \\ \left. + \varepsilon \|u_\psi\|_{L_T^\infty(B^{\frac{3}{2}})}^{\frac{1}{2}} \|\partial_z u_\psi\|_{\tilde{L}_T^2(B^{s+1})} \varepsilon \|\partial_z w_{\varepsilon\psi}^2\|_{\tilde{L}_{T,\zeta'(t)}^2(B^{s+\frac{1}{2}})} \right] \end{aligned}$$

11. Estimate of $\alpha_1^2 \varepsilon^4 \int_0^T \left| \left\langle \Delta_k^h(v^\varepsilon \partial_x^2 v^\varepsilon)_\psi, \Delta_k^h \partial_z w_{\varepsilon\psi}^2 \right\rangle \right|$. We write

$$v^\varepsilon \partial_x^2 v^\varepsilon = w_\varepsilon^2 \partial_x^2 v^\varepsilon + v \partial_x^2 w_\varepsilon^2 + v \partial_x^2 v$$

By adapting the proof of Lemma 3.2 in [64], we obtain

$$\varepsilon^4 \int_0^T \left| \left\langle \Delta_k^h(w_\varepsilon^2 \partial_x^2 v^\varepsilon)_\psi, \Delta_k^h \partial_z w_{\varepsilon\psi}^2 \right\rangle \right| \lesssim 2^{-2ks} (\tilde{d}_k + \check{d}_k) \|w_{\varepsilon\psi}^1\|_{\tilde{L}_{T,\zeta'(t)}^2(B^{s+\frac{1}{2}})} \varepsilon \|\partial_z w_{\varepsilon\psi}^2\|_{\tilde{L}_{T,\zeta'(t)}^2(B^{s+\frac{1}{2}})}$$

By adapting the proof of Estimate (3.64), we get

$$\varepsilon^4 \int_0^T \left| \left\langle \Delta_k^h(v \partial_x^2 w_{\varepsilon\psi}^2)_\psi, \Delta_k^h \partial_z w_{\varepsilon\psi}^2 \right\rangle \right| dt \lesssim 2^{-2ks} (\tilde{d}_k + \check{d}_k) \|u_\psi\|_{L_T^\infty(B^{\frac{3}{2}})}^{\frac{1}{2}} \varepsilon^3 \|\partial_x^2 w_{\varepsilon\psi}^2\|_{\tilde{L}_T^2(B^s)} \varepsilon \|\partial_z w_{\varepsilon\psi}^2\|_{\tilde{L}_{T,\zeta'(t)}^2(B^{s+\frac{1}{2}})},$$

and

$$\varepsilon^4 \int_0^T \left| \left\langle \Delta_k^h(v \partial_x^2 v)_\psi, \Delta_k^h \partial_z w_{\varepsilon\psi}^2 \right\rangle \right| dt \lesssim 2^{-2ks} (\tilde{d}_k + \check{d}_k) \varepsilon \|u_\psi\|_{L_T^\infty(B^{\frac{3}{2}})}^{\frac{1}{2}} \|\partial_z u_\psi\|_{\tilde{L}_T^2(B^{s+3})} \varepsilon^3 \|\partial_z w_{\varepsilon\psi}^2\|_{\tilde{L}_{T,\zeta'(t)}^2(B^{s+\frac{1}{2}})}.$$

Conclusion :

$$\begin{aligned}
& \alpha_1^2 \varepsilon^4 \int_0^T \left| \left\langle \Delta_k^h (v^\varepsilon \partial_x^2 v^\varepsilon)_\psi, \Delta_k^h \partial_z w_{\varepsilon\psi}^2 \right\rangle \right| \\
& \lesssim 2^{-2ks} (\tilde{d}_k + \check{d}_k) \left[\|w_{\varepsilon\psi}^1\|_{\tilde{L}_{T,\zeta'(t)}^2(B^{s+\frac{1}{2}})} \varepsilon \|\partial_z w_{\varepsilon\psi}^2\|_{\tilde{L}_{T,\zeta'(t)}^2(B^{s+\frac{1}{2}})} \right. \\
& \quad + \|u_\psi\|_{L_T^\infty(B^{\frac{3}{2}})}^{\frac{1}{2}} \varepsilon^3 \|\partial_x^2 w_{\varepsilon\psi}^2\|_{\tilde{L}_T^2(B^s)} \varepsilon \|\partial_z w_{\varepsilon\psi}^2\|_{\tilde{L}_{T,\zeta'(t)}^2(B^{s+\frac{1}{2}})} \\
& \quad \left. + \varepsilon \|u_\psi\|_{L_T^\infty(B^{\frac{3}{2}})}^{\frac{1}{2}} \|\partial_z u_\psi\|_{\tilde{L}_T^2(B^{s+3})} \varepsilon^3 \|\partial_z w_{\varepsilon\psi}^2\|_{\tilde{L}_{T,\zeta'(t)}^2(B^{s+\frac{1}{2}})} \right]
\end{aligned}$$

12. Final conclusion : Then from the previous estimates and the estimate (3.48), then Young inequality, we get

$$\begin{aligned}
& \sum_{k \in \mathbb{Z}} \sqrt{2^{2ks} R_k^2} \\
& \leq C \left[M\varepsilon + M\varepsilon \|\partial_z w_{\varepsilon\psi}^2\|_{\tilde{L}_{T,\zeta'(t)}^2(B^{s+\frac{1}{2}})} + M\varepsilon^2 \|\partial_x w_{\varepsilon\psi}^1\|_{\tilde{L}_{T,\zeta'(t)}^2(B^{s+\frac{1}{2}})} + M\varepsilon \|w_{\varepsilon\psi}^2\|_{\tilde{L}_{T,\zeta'(t)}^2(B^{s+\frac{1}{2}})} \right. \\
& \quad \left. + M\varepsilon^2 \|\partial_x w_{\varepsilon\psi}^2\|_{\tilde{L}_{T,\zeta'(t)}^2(B^{s+\frac{1}{2}})} + M \|w_{\varepsilon\psi}^1\|_{\tilde{L}_{T,\zeta'(t)}^2(B^{s+\frac{1}{2}})} \right] \\
& \quad + \frac{1}{2} \varepsilon \|\partial_z w_{\varepsilon\psi}^2\|_{\tilde{L}_T^2(B^s)} + \frac{1}{2} \varepsilon^2 \|w_{\varepsilon\psi}^2\|_{\tilde{L}_T^2(B^{s+1})} + \frac{1}{2} \varepsilon^3 \|\partial_x^2 w_{\varepsilon\psi}^2\|_{\tilde{L}_T^2(B^s)} + \frac{1}{2} \varepsilon^3 \|\partial_x^2 w_{\varepsilon\psi}^2\|_{\tilde{L}_T^2(B^s)} \\
& \quad + \frac{1}{2} \varepsilon \|\partial_z \partial_x w_{\varepsilon\psi}^1\|_{\tilde{L}_T^2(B^s)} \quad (3.67)
\end{aligned}$$

This estimates on R_2 conclude the proof of proposition 3.6.2.

Part II

Boundary layer for the power-law model

Chapter 4

Introduction to the power-law non-Newtonian Prandtl equations

4.1 Boundary layer equations for non-newtonian fluids

The classical theoretical hydrodynamics is based on the Euler equations describing the motion of an ideal fluid. Despite the numerous achievements, some gaps between theoretical and experimental results need to be filled, which are mainly due to how to determine the force of resistance acting on a moving solid in a fluid. A classical attempt to close the gaps is to consider the Navier-Stokes equations, which are based on Newton's model of a viscous fluid. According to that model, tangential stresses in a fluid are linearly dependent on the respective shear deformation velocities and so the principal part of the Navier-Stokes equations consists in second order partial derivative diffusion term.

Unfortunately, many problems arise in this consideration. First of all, we encounter great mathematical difficulties to explain the connection between the viscous model and the theoretical ideal non-viscous model. For instance, it is extremely difficult to prove that small viscous friction forces may perceptibly affect the motion of a fluid. In 1904, Prandtl suggested an explanation of this phenomenon using his theory of boundary layer. He theoretically showed that the flow around a solid body could be divided into two zones: a thin layer around the body where viscous forces play crucial part and the outer region far from the body where friction could be neglected. Thus, for weakly viscous fluids, the influence of the small viscosity can only be perceptible in this very thin layer adjacent to the surface of the body, which is called boundary layer according to Prandtl. Outside this region, the fluid can be considered as frictionless and can be described by the Euler system.

The second question comes from the consideration of tangential stresses which are linearly dependent on the shear deformation velocities in Newton's model. This linear dependence turns out to be unacceptable for many media, for example melted and dissolved polymers, concentrated suspensions, emulsions, dye-stuffs, etc . . . Many different models have been proposed for the description of non-newtonian flows (see, for instance, Litvinvo (1982), Shulman (1975)). Among these non-newtonian flows, the Ostwald model for nonlinear viscous fluids are particularly adapted to many physical fluids, where the stress tensor components depend on the powers of the components of the deformation velocity tensor.

In this work, we want to combine the two above considerations and study the behavior near a surface of non-newtonian fluids which follow Ostwald power-law. Let us be more specific. We suppose that u_1, u_2 be the fluid velocity components with respect to the coordinates x_1, x_2 , and let

$$e_{ij} = \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}$$

be the components of the deformation velocity tensor. In the power-law model of a non-newtonian fluid, the stress tensor (τ_{ij}) is assumed to be of the form

$$\tau_{ij} = -\delta_{ij}p + k \left(\frac{1}{2} e_{ml} e_{lm} \right)^{\frac{n-1}{2}} e_{ij}, \quad n > 0.$$

Here, p denotes the pressure, k is the index of consistency of the fluid and δ_{ij} the usual Kronecker symbol. Here and in what follows, we also suppose that summation is taken over repeated indices from 1 to 2. For $0 < n < 1$, the fluid is called pseudo-plastic and for $n > 1$, the fluid is dilatant. In the case $n = 1$, we get the usual newtonian fluid. Using Newton's second law and supposing that the fluid is incompressible, the motion of an incompressible fluid following the above power-law type of dependence is described by the equations

$$\begin{cases} \frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho} \frac{\partial p}{\partial x_i} + \frac{k}{\rho} \frac{\partial}{\partial x_j} \left[\left(\frac{1}{2} e_{ml} e_{lm} \right)^{\frac{n-1}{2}} e_{ij} \right], & i = 1, 2 \\ \frac{\partial u_j}{\partial x_j} = 0, \end{cases} \quad (4.1)$$

which represent a generalization of the Navier-Stokes equations. Here, the quantity $k \left(\frac{1}{2} e_{ml} e_{lm} \right)^{\frac{n-1}{2}}$ is called apparent viscosity, and for $n \neq 1$, it depends on shear deformation velocities. Experiments show that, for nonlinearly viscous flows or for ordinary fluids past solid surfaces, there is a wall-adjacent region with the predominance of the internal friction forces over the inertial ones and therefore, a viscous boundary layer exists. In the next section, we will formally derive the Prandtl-like equations for power-law non-newtonian fluids.

4.2 Derivation of power-law non-newtonian Prandtl equations

We recall that, if U_0 and X are the characteristic velocity and the characteristic dimension of a given nonlinear viscous flow, then the Reynolds number is given by

$$R = U_0^{2-n} X^n \rho / k.$$

The dimensionless equations for power-law non-newtonian fluids can be written as

$$\begin{cases} \frac{\partial u_1}{\partial t} + u_1 \frac{\partial u_1}{\partial x_1} + u_2 \frac{\partial u_1}{\partial x_2} = -\frac{\partial p}{\partial x_1} + \frac{2}{R} \frac{\partial}{\partial x_1} \left(\mu \frac{\partial u_1}{\partial x_1} \right) + \frac{1}{R} \frac{\partial}{\partial x_2} \left(\mu \left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \right) \right) \\ \frac{\partial u_2}{\partial t} + u_1 \frac{\partial u_2}{\partial x_1} + u_2 \frac{\partial u_2}{\partial x_2} = -\frac{\partial p}{\partial x_2} + \frac{1}{R} \frac{\partial}{\partial x_1} \left(\mu \left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \right) \right) + \frac{2}{R} \frac{\partial}{\partial x_2} \left(\mu \frac{\partial u_2}{\partial x_2} \right) \\ \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} = 0, \end{cases} \quad (4.2)$$

where the dimensionless apparent viscosity μ is now given by

$$\mu = \left(\frac{1}{2} e_{ml} e_{lm} \right)^{\frac{n-1}{2}} = \left(2 \left(\frac{\partial u_1}{\partial x_1} \right)^2 + \left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \right)^2 + 2 \left(\frac{\partial u_2}{\partial x_2} \right)^2 \right)^{\frac{n-1}{2}}.$$

We set $\varepsilon = R^{-\frac{1}{n+1}}$, which in fact represents the boundary-layer thickness and our goal is to study the limit $\varepsilon \rightarrow 0$. In order to simplify the notation, we will not use index ε on the solution (u_1, u_2, p) but it is clear that the solution depends on ε . We rewrite the system (4.2) as follows

$$\begin{cases} \frac{\partial u_1}{\partial t} + u_1 \frac{\partial u_1}{\partial x_1} + u_2 \frac{\partial u_1}{\partial x_2} = -\frac{\partial p}{\partial x_1} + 2\varepsilon^{n+1} \frac{\partial}{\partial x_1} \left(\mu \frac{\partial u_1}{\partial x_1} \right) + \varepsilon^{n+1} \frac{\partial}{\partial x_2} \left(\mu \left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \right) \right) \\ \frac{\partial u_2}{\partial t} + u_1 \frac{\partial u_2}{\partial x_1} + u_2 \frac{\partial u_2}{\partial x_2} = -\frac{\partial p}{\partial x_2} + \varepsilon^{n+1} \frac{\partial}{\partial x_1} \left(\mu \left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \right) \right) + 2\varepsilon^{n+1} \frac{\partial}{\partial x_2} \left(\mu \frac{\partial u_2}{\partial x_2} \right) \\ \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} = 0, \\ u_1 = 0, \text{ on } x_2 = 0 \\ u_2 = 0, \text{ on } x_2 = 0 \\ u|_{t=0} = u_0(x, y). \end{cases} \quad (4.3)$$

For the sake of simplicity, we shall consider the problem (4.3) in the half plane

$$\Omega = \{x_1 \in \mathbb{R}, x_2 > 0\} \quad \text{with} \quad \partial\Omega = \{x_1 \in \mathbb{R}, x_2 = 0\}.$$

Following Prandtl's typical approach for boundary layers, we look for solutions of (4.3) in the following asymptotic expansion form

$$\begin{aligned} u_i &= u_i^{I,0} + u_i^{B,0} + \varepsilon \left(u_i^{I,1} + u_i^{B,1} \right) + \varepsilon^2 \left(u_i^{I,2} + u_i^{B,2} \right) + \dots \\ p &= p^{I,0} + p^{B,0} + \varepsilon \left(p^{I,1} + p^{B,1} \right) + \varepsilon^2 \left(p^{I,2} + p^{B,2} \right) + \dots \end{aligned}$$

where $i = 1, 2$. The ‘‘interior part’’ functions with the index ‘‘I’’ describe the outer fluids far from the surface and depend on (t, x_1, x_2) while the ‘‘boundary layer part’’ with the index ‘‘B’’ consists in smooth functions of dependind on $(t, x_1, \frac{x_2}{\varepsilon})$ and rapidly (exponentially) decreasing to zero when the normal space variable $y = \frac{x_2}{\varepsilon}$ goes to infinity. In what follows, we will study the leading order expansions of the system (4.3) with respect to ε .

4.2.1 Asymptotic expansion of linear terms

The following asymptotic expansion of the linear terms can be easily obtained by derivation with respect to the time and space variables. We recall that the index i stands for 1 or 2 and indicates the tangential and normal components of u .

- Derivative of u with respect to the time variable

$$\partial_t u_i = \partial_t (u_i^{I,0} + u_i^{B,0}) + \varepsilon \partial_t (u_i^{I,1} + u_i^{B,1}) + \mathcal{O}(\varepsilon^2).$$

- Derivatives u with respect to the space variables: for any $k \geq 0$, we have

$$\begin{aligned} \partial_{x_1}^k u_i &= \partial_{x_1}^k (u_i^{I,0} + u_i^{B,0}) + \varepsilon \partial_{x_1}^k (u_i^{I,1} + u_i^{B,1}) + \mathcal{O}(\varepsilon^2) \\ \partial_{x_2}^k u_i &= (\partial_{x_2}^k u_i^{I,0} + \varepsilon \partial_{x_2}^k u_i^{I,1}) + \sum_{l=0}^{k+1} \varepsilon^{l-k} \partial_y^l u_i^{B,l} + \mathcal{O}(\varepsilon^2), \end{aligned}$$

and for any $j, k \geq 0$, we have

$$\partial_{x_1}^j \partial_{x_2}^k u_i = (\partial_{x_1}^j \partial_{x_2}^k u_i^{I,0} + \varepsilon \partial_{x_1}^j \partial_{x_2}^k u_i^{I,1}) + \sum_{l=0}^{k+1} \varepsilon^{l-k} \partial_{x_1}^j \partial_y^l u_i^{B,l} + \mathcal{O}(\varepsilon^2).$$

In particular, we need the following derivatives of u

$$\begin{aligned} \partial_{x_1 x_2}^2 u_1 &= \varepsilon^{-1} \partial_{x_1 x_2}^2 u_1^{B,0} + (\partial_{x_1 x_2}^2 u_1^{I,0} + \partial_{x_1 y}^2 u_1^{B,1}) + \varepsilon (\partial_{x_1 x_2}^2 u_1^{I,1} + \partial_{x_1 y}^2 u_1^{B,2}) + \mathcal{O}(\varepsilon^2) \\ \partial_{x_1 x_2}^2 u_2 &= \varepsilon^{-1} \partial_{x_1} \partial_y u_2^{B,0} + (\partial_{x_1 x_2}^2 u_2^{I,0} + \partial_{x_1} \partial_y u_2^{B,1}) + \varepsilon (\partial_{x_1 x_2}^2 u_2^{I,1} + \partial_{x_1} \partial_y u_2^{B,2}) + \mathcal{O}(\varepsilon^2) \end{aligned}$$

- Derivatives of the pressure p :

$$\begin{aligned} \partial_{x_1} p &= \partial_{x_1} (p^{I,0} + p^{B,0}) + \varepsilon \partial_{x_1} (p^{I,1} + p^{B,1}) + \mathcal{O}(\varepsilon^2) \\ \partial_{x_2} p &= \varepsilon^{-1} \partial_y p^{B,0} + (\partial_{x_2} p^{I,0} + \partial_y p^{B,1}) + \varepsilon (\partial_{x_2} p^{I,1} + \partial_y p^{B,2}) + \mathcal{O}(\varepsilon^2) \end{aligned}$$

4.2.2 Asymptotic expansion of the nonlinear terms

Using the above expression of u , we easily obtain the following expansion for the nonlinear terms

$$u_1 \partial_{x_1} u_1 = (u_1^{I,0} + u_1^{B,0}) \partial_{x_1} u_1^{I,0} + (u_1^{I,0} + u_1^{B,0}) \partial_{x_1} u_1^{B,0} + \mathcal{O}(\varepsilon)$$

$$u_2 \partial_{x_2} u_1 = \varepsilon^{-1} (u_2^{I,0} + u_2^{B,0}) \partial_y u_1^{B,0} + (u_2^{I,0} + u_2^{B,0}) (\partial_{x_2} u_1^{I,0} + \partial_y u_1^{B,1}) + (u_2^{I,1} + u_2^{B,1}) \partial_y u_1^{B,0} + \mathcal{O}(\varepsilon)$$

$$u_1 \partial_{x_1} u_2 = (u_1^{I,0} + u_1^{B,0}) \partial_{x_1} u_2^{I,0} + (u_1^{I,0} + u_1^{B,0}) \partial_{x_1} u_2^{B,0} + \mathcal{O}(\varepsilon)$$

$$u_2 \partial_{x_2} u_2 = \varepsilon^{-1} (u_2^{I,0} + u_2^{B,0}) \partial_y u_2^{B,0} + (u_2^{I,0} + u_2^{B,0}) (\partial_{x_2} u_2^{I,0} + \partial_y u_2^{B,1}) + (u_2^{I,1} + u_2^{B,1}) \partial_y u_2^{B,0} + \mathcal{O}(\varepsilon).$$

4.2.3 Incompressibility and boundary conditions

The divergence-free property of the velocity field can be rewritten as follows

$$0 = \nabla \cdot u = \varepsilon^{-1} \partial_y u_2^{B,0} + (\partial_1 u_1^{I,0} + \partial_2 u_2^{I,0}) + (\partial_1 u_1^{B,0} + \partial_y u_2^{B,1}) + \mathcal{O}(\varepsilon).$$

Away from the boundary, all the ‘‘boundary terms’’ rapidly decrease to zero. Then, at the leading order, the incompressibility of the outer fluids simply writes

$$\nabla \cdot u^{I,0} = 0.$$

Now, near the boundary, by identification with respect to the order of ε , we get

$$\partial_y u_2^{B,0} = 0,$$

which implies

$$u_2^{B,0} = 0$$

by noting that $u_2^{B,0}(t, x_1, y) \rightarrow 0$ as $y \rightarrow +\infty$. Then, the incompressibility of the fluids inside the boundary layer becomes

$$\partial_1 u_1^{B,0} + \partial_y u_2^{B,1} = 0. \quad (4.4)$$

4.2.4 Estimate of the viscosity

We recall that the apparent viscosity is given by the following expression

$$\mu = [2(\partial_1 u_1)^2 + 2(\partial_2 u_2)^2 + (\partial_2 u_1)^2 + 2\partial_2 u_1 \partial_1 u_2 + (\partial_1 u_2)^2]^{\frac{n-1}{2}}$$

Using the expansion of u given above, we obtain

$$(\partial_1 u_1)^2 = (\partial_1 (u_1^{I,0} + u_1^{B,0}))^2 + \mathcal{O}(\varepsilon)$$

$$(\partial_2 u_2)^2 = \varepsilon^{-2} (\partial_y u_2^{B,0})^2 + \varepsilon^{-1} 2\partial_y u_2^{B,0} (\partial_2 u_2^{I,0} + \partial_y u_2^{B,1}) + [(\partial_2 u_2^{I,0} + \partial_y u_2^{B,1})^2 + 2\partial_y u_2^{B,0} \partial_2 u_2^{I,1}] + \mathcal{O}(\varepsilon)$$

$$(\partial_2 u_1)^2 = \varepsilon^{-2} (\partial_y u_1^{B,0})^2 + \varepsilon^{-1} 2\partial_y u_1^{B,0} (\partial_2 u_1^{I,0} + \partial_y u_1^{B,1}) + [(\partial_2 u_1^{I,0} + \partial_y u_1^{B,1})^2 + 2\partial_y u_1^{B,0} \partial_2 u_1^{I,1}] + \mathcal{O}(\varepsilon)$$

$$(\partial_1 u_2)^2 = (\partial_1 (u_2^{I,0} + u_2^{B,0}))^2 + \mathcal{O}(\varepsilon)$$

and

$$\partial_2 u_1 \partial_1 u_2 = \varepsilon^{-1} \partial_y u_1^{B,0} \partial_1 (u_2^{I,0} + u_2^{B,0}) + [\partial_y u_1^{B,0} \partial_1 (u_2^{I,1} + u_2^{B,1}) + (\partial_2 u_1^{I,0} + \partial_y u_1^{B,1}) \partial_1 (u_2^{I,0} + u_2^{B,0})] + \mathcal{O}(\varepsilon).$$

Putting the above identities into the expression of μ , we get

$$\begin{aligned} \mu = & \left[\varepsilon^{-2} \left((\partial_y u_1^{B,0})^2 + 2(\partial_y u_2^{B,0})^2 \right) \right. \\ & + \varepsilon^{-1} \left(4\partial_y u_2^{B,0} (\partial_2 u_2^{I,0} + \partial_y u_2^{B,1}) + 2\partial_y u_1^{B,0} (\partial_2 u_1^{I,0} + \partial_y u_1^{B,1}) + 2\partial_y u_1^{B,0} \partial_1 (u_2^{I,0} + u_2^{B,0}) \right) \\ & + 2(\partial_1 (u_1^{I,0} + u_1^{B,0}))^2 + 2((\partial_2 u_2^{I,0} + \partial_y u_2^{B,1})^2 + 2\partial_y u_2^{B,0} \partial_2 u_2^{B,1}) + (\partial_2 u_1^{I,0} + \partial_y u_1^{B,1})^2 \\ & + 2\partial_y u_1^{B,0} \partial_2 u_1^{I,1} + (\partial_1 (u_2^{I,0} + u_2^{B,0}))^2 + 2\partial_y u_1^{B,0} \partial_1 (u_2^{I,1} + u_2^{B,1}) \\ & \left. + 2(\partial_2 u_1^{I,0} + \partial_y u_1^{B,1}) \partial_1 (u_2^{I,0} + u_2^{B,0}) + O(\varepsilon) \right]^{\frac{n-1}{2}} \end{aligned}$$

Therefore

$$\begin{aligned} \mu = & \varepsilon^{-(n-1)} \left[\left((\partial_y u_1^{B,0})^2 + 2(\partial_y u_2^{B,0})^2 \right) \right. \\ & \left. + \varepsilon \left(4\partial_y u_2^{B,0} (\partial_2 u_2^{I,0} + \partial_y u_2^{B,1}) + 2\partial_y u_1^{B,0} (\partial_2 u_1^{I,0} + \partial_y u_1^{B,1}) + 2\partial_y u_1^{B,0} \partial_1 (u_2^{I,0} + u_2^{B,0}) \right) + O(\varepsilon^2) \right]^{\frac{n-1}{2}}. \end{aligned}$$

Since $\partial_y u_1^{B,0} = 0$, we deduce that

$$\mu = \varepsilon^{-(n-1)} [(\partial_y u_1^{B,0})^2]^{(n-1)/2} + O(\varepsilon^{-(n-2)}).$$

Lemma 4.2.1. *We have the following equivalences*

- (i) $\varepsilon^{n+1} \partial_2 (\mu \partial_2 u_1) = n [(\partial_y u_1^{B,0})^2]^{(n-1)/2} \partial_y^2 u_1^{B,0} + O(\varepsilon)$,
- (ii) $\varepsilon^{n+1} \partial_1 (\mu \partial_1 u) = O(\varepsilon^2)$,
- (iii) $\varepsilon^{n+1} \partial_2 (\mu \partial_1 u_2) = O(\varepsilon)$,
- (iv) $\varepsilon^{n+1} \partial_1 (\mu \partial_2 u_1) = O(\varepsilon)$,
- (v) $\varepsilon^{n+1} \partial_2 (\mu \partial_2 u_2) = O(\varepsilon)$.

Proof.

(i) The fact that

$$\mu = \varepsilon^{-(n-1)} [(\partial_y u_1^{B,0})^2]^{(n-1)/2} + O(\varepsilon^{-(n-2)})$$

implies

$$\partial_2 \mu = \varepsilon^{-n} (n-1) [(\partial_y u_1^{B,0})^2]^{\frac{n-3}{2}} \partial_y u_1^{B,0} \partial_y^2 u_1^{B,0} + O(\varepsilon^{-(n-1)}). \quad (4.5)$$

Since

$$\begin{aligned} \partial_2 u_1 &= \varepsilon^{-1} \partial_y u_1^{B,0} + (\partial_{x_2} u_1^{I,0} + \partial_y u_1^{B,1}) + \varepsilon (\partial_{x_2} u_1^{I,1} + \partial_y u_1^{B,2}) + O(\varepsilon^2) \\ \partial_2^2 u_1 &= \varepsilon^{-2} \partial_y^2 u_1^{B,0} + \varepsilon^{-1} \partial_y^2 u_1^{B,1} + (\partial_{x_2}^2 u_1^{I,0} + \partial_y^2 u_1^{B,2}) + \varepsilon (\partial_{x_2}^2 u_1^{I,1} + \partial_y^2 u_1^{B,3}) + O(\varepsilon^2), \end{aligned}$$

we deduce that

$$\begin{aligned} \partial_2 \mu \partial_2 u_1 &= \varepsilon^{-(n+1)} (n-1) [(\partial_y u_1^{B,0})^2]^{(n-1)/2} \partial_y^2 u_1^{B,0} + O(\varepsilon^{-n}) \\ \mu \partial_2^2 u_1 &= \varepsilon^{-(n+1)} [(\partial_y u_1^{B,0})^2]^{(n-1)/2} \partial_y^2 u_1^{B,0} + O(\varepsilon^{-n}). \end{aligned}$$

Therefore,

$$\partial_2 (\mu \partial_2 u_1) = \varepsilon^{-(n+1)} \left[n [(\partial_y u_1^{B,0})^2]^{(n-1)/2} \partial_y^2 u_1^{B,0} \right] + O(\varepsilon^{-n})$$

and

$$\varepsilon^{n+1} \partial_2 (\mu \partial_2 u_1) = n [(\partial_y u_1^{B,0})^2]^{(n-1)/2} \partial_y^2 u_1^{B,0} + O(\varepsilon).$$

(ii) We write

$$\partial_1 (\mu \partial_1 u) = \partial_1 \mu \partial_1 u + \mu \partial_1^2 u.$$

We have

$$\begin{aligned}\partial_1 u &= \partial_1(u^{I,0} + u^{B,0}) + \varepsilon \partial_1(u^{I,1} + u^{B,1}) + O(\varepsilon^2) \\ \partial_1^2 u &= \partial_1^2(u^{I,0} + u^{B,0}) + \varepsilon \partial_1^2(u^{I,1} + u^{B,1}) + O(\varepsilon^2)\end{aligned}$$

and so

$$\partial_1 \mu = \varepsilon^{-(n-1)}(n-1)[(\partial_y u_1^{B,0})^2]^{\frac{n-1}{3}} \partial_y u_1^{B,0} \partial_{x_1 y} u_1^{B,0} + O(\varepsilon^{-n}).$$

Then we obtain

$$\begin{aligned}\partial_1 \mu \partial_1 u &= \varepsilon^{-(n-1)}(n-1)[(\partial_y u_1^{B,0})^2]^{\frac{n-3}{2}} \partial_y u_1^{B,0} \partial_{1y} u_1^{B,0} \partial_1(u^{I,0} + u^{B,0}) + O(\varepsilon^{-(n-2)}) \\ \mu \partial_1^2 u &= \varepsilon^{-(n-1)}[(\partial_y u_1^{B,0})^2]^{\frac{n-1}{2}} \partial_{x_1}^2(u^{I,0} + u^{B,0}) + O(\varepsilon^{-(n-2)}),\end{aligned}$$

and

$$\begin{aligned}\partial_1(\mu \partial_1 u) &= \varepsilon^{-(n-1)} \left[(n-1)[(\partial_y u_1^{B,0})^2]^{\frac{n-3}{2}} \partial_y u_1^{B,0} \partial_{1y} u_1^{B,0} \partial_1(u^{I,0} + u^{B,0}) \right. \\ &\quad \left. + [(\partial_y u_1^{B,0})^2]^{\frac{n-1}{2}} \partial_{x_1}^2(u^{I,0} + u^{B,0}) \right] + O(\varepsilon^{-(n-2)})\end{aligned}$$

which means that

$$\varepsilon^{n+1} \partial_1(\mu \partial_1 u) = O(\varepsilon^2).$$

(iii) Similarly to (ii), we write

$$\partial_2(\mu \partial_1 u_2) = \partial_2 \mu \partial_1 u_2 + \mu \partial_{21}^2 u.$$

We have

$$\begin{aligned}\partial_{x_1} u_2 &= \partial_{x_1}(u_2^{I,0} + u_2^{B,0}) + \varepsilon \partial_{x_1}(u_2^{I,1} + u_2^{B,1}) + O(\varepsilon^2) \\ \partial_{x_2 x_1}^2 u_2 &= \varepsilon^{-1} \partial_{y x_1}^2 u_2^{B,0} + (\partial_{x_2 x_1}^2 u_2^{I,0} + \partial_{y x_1}^2 u_2^{B,1}) + \varepsilon (\partial_{x_2 x_1}^2 u_2^{I,1} + \partial_{y x_1}^2 u_2^{B,2}) + O(\varepsilon^2),\end{aligned}$$

which imply

$$\begin{aligned}\partial_2 \mu \partial_1 u_2 &= \varepsilon^{-n}(n-1)[(\partial_y u_1^{B,0})^2]^{\frac{n-3}{2}} \partial_y u_1^{B,0} \partial_y^2 u_1^{B,0} \partial_{x_1} u_2^{I,0} + O(\varepsilon^{-(n-1)}) \\ \mu \partial_{21}^2 u &= \varepsilon^{-(n-1)}[(\partial_y u_1^{B,0})^2]^{(n-1)/2} (\partial_{x_2 x_1}^2 u_2^{I,0} + \partial_{y x_1}^2 u_2^{B,1}) + O(\varepsilon^{-(n-2)}).\end{aligned}$$

Therefore

$$\partial_2(\mu \partial_1 u_2) = \varepsilon^{-n}(n-1)[(\partial_y u_1^{B,0})^2]^{\frac{n-3}{2}} \partial_y u_1^{B,0} \partial_y^2 u_1^{B,0} \partial_{x_1} u_2^{I,0} + O(\varepsilon^{-(n-1)})$$

and

$$\varepsilon^{n+1} \partial_2(\mu \partial_1 u_2) = \varepsilon \left[(n-1)[(\partial_y u_1^{B,0})^2]^{\frac{n-3}{2}} \partial_y u_1^{B,0} \partial_y^2 u_1^{B,0} \partial_{x_1} u_2^{I,0} \right] + O(\varepsilon^2).$$

(iv) We have

$$\partial_1(\mu \partial_2 u_1) = \partial_1 \mu \partial_2 u_1 + \mu \partial_{12}^2 u$$

We recall that

$$\begin{aligned}\partial_{x_2} u_1 &= \varepsilon^{-1} \partial_y u_1^{B,0} + (\partial_{x_2} u_1^{I,0} + \partial_y u_1^{B,1}) + \varepsilon (\partial_{x_2} u_1^{I,1} + \partial_y u_1^{B,2}) + O(\varepsilon^2) \\ \partial_{x_1 x_2}^2 u_1 &= \varepsilon^{-1} \partial_{x_1 x_2}^2 u_1^{B,0} + (\partial_{x_1 x_2}^2 u_1^{I,0} + \partial_{x_1 y}^2 u_1^{B,1}) + \varepsilon (\partial_{x_1 x_2}^2 u_1^{I,1} + \partial_{x_1 y}^2 u_1^{B,2}) + O(\varepsilon^2)\end{aligned}$$

and for the viscosity

$$\begin{aligned}\mu &= \varepsilon^{-(n-1)}[(\partial_y u_1^{B,0})^2]^{(n-1)/2} + O(\varepsilon^{-(n-2)}) \\ \partial_1 \mu &= \varepsilon^{-(n-1)}(n-1)[(\partial_y u_1^{B,0})^2]^{\frac{n-1}{3}} \partial_y u_1^{B,0} \partial_{1y} u_1^{B,0} + O(\varepsilon^{-n}).\end{aligned}$$

Then,

$$\begin{aligned}\partial_1 \mu \partial_2 u_1 &= \varepsilon^{-n} (n-1) [(\partial_y u_1^{B,0})^2]^{\frac{n-1}{2}} \partial_{1y} u_1^{B,0} + O(\varepsilon^{-(n-1)}) \\ \mu \partial_{12}^2 u &= \varepsilon^{-n} [(\partial_y u_1^{B,0})^2]^{(n-1)/2} \partial_{x_1 x_2}^2 u_1^{B,0} + O(\varepsilon^{-(n-1)}).\end{aligned}$$

As a consequence, we obtain

$$\partial_1 (\mu \partial_2 u_1) = \varepsilon^{-n} [(\partial_y u_1^{B,0})^2]^{\frac{n-1}{2}} \left[(n-1) \partial_{1y} u_1^{B,0} + \partial_{x_1 x_2}^2 u_1^{B,0} \right] + O(\varepsilon^{-(n-1)})$$

and

$$\varepsilon^{n+1} \partial_1 (\mu \partial_2 u_1) = \varepsilon [(\partial_y u_1^{B,0})^2]^{\frac{n-1}{2}} \left[(n-1) \partial_{1y} u_1^{B,0} + \partial_{x_1 x_2}^2 u_1^{B,0} \right] + O(\varepsilon^{-(n-1)}).$$

(v) We write

$$\partial_2 (\mu \partial_2 u_2) = \partial_2 \mu \partial_2 u_2 + \mu \partial_{22}^2 u.$$

We recall that

$$\begin{aligned}\mu &= \varepsilon^{-(n-1)} [(\partial_y u_1^{B,0})^2]^{(n-1)/2} + O(\varepsilon^{-(n-2)}) \\ \partial_2 \mu &= \varepsilon^{-n} (n-1) [(\partial_y u_1^{B,0})^2]^{\frac{n-3}{2}} \partial_y u_1^{B,0} \partial_y^2 u_1^{B,0} + O(\varepsilon^{-(n-1)})\end{aligned}$$

and the derivatives of u_2

$$\begin{aligned}\partial_{x_2} u_2 &= \varepsilon^{-1} \partial_y u_2^{B,0} + (\partial_{x_2} u_2^{I,0} + \partial_y u_2^{B,1}) + \varepsilon (\partial_{x_2} u_2^{I,1} + \partial_y u_2^{B,2}) + O(\varepsilon^2) \\ \partial_{x_2}^2 u_2 &= \varepsilon^{-2} \partial_y^2 u_2^{B,0} + \varepsilon^{-1} \partial_y^2 u_2^{B,1} + (\partial_{x_2}^2 u_2^{I,0} + \partial_y^2 u_2^{B,2}) + \varepsilon (\partial_{x_2}^2 u_2^{I,1} + \partial_y^2 u_2^{B,3}) + O(\varepsilon^2).\end{aligned}$$

Then,

$$\begin{aligned}\partial_2 \mu \partial_2 u_2 &= \varepsilon^{-n+1} (n-1) [(\partial_y u_1^{B,0})^2]^{\frac{n-3}{2}} \partial_y u_1^{B,0} \partial_y^2 u_1^{B,0} (\partial_{x_2} u_2^{I,1} + \partial_y u_2^{B,2}) + O(\varepsilon^{-(n-2)}) \\ \mu \partial_{22}^2 u &= \varepsilon^{-n} [(\partial_y u_1^{B,0})^2]^{(n-1)/2} \partial_y^2 u_2^{B,1} + O(\varepsilon^{-(n-1)}).\end{aligned}$$

As a consequence, we have

$$\partial_2 (\mu \partial_2 u_2) = \varepsilon^{-n} [(\partial_y u_1^{B,0})^2]^{(n-1)/2} \partial_y^2 u_2^{B,1} + O(\varepsilon^{-(n-1)})$$

and

$$\varepsilon^{n+1} \partial_2 (\mu \partial_2 u_2) = \varepsilon [(\partial_y u_1^{B,0})^2]^{(n-1)/2} \partial_y^2 u_2^{B,1} + O(\varepsilon^2).$$

□

4.2.5 Behaviors of the fluid far from the boundary

Taking the leading term of $O(\varepsilon^0)$, we simply obtain the Euler equation :

$$\begin{cases} \partial_t u_1^{I,0} + u_1^{I,0} \frac{\partial u_1^{I,0}}{\partial x_1} + u_2^{I,0} \frac{\partial u_1^{I,0}}{\partial x_2} = -\frac{\partial p^{I,0}}{\partial x_1} \\ \frac{\partial u_2^{I,0}}{\partial t} + u_1^{I,0} \frac{\partial u_2^{I,0}}{\partial x_1} + u_2^{I,0} \frac{\partial u_2^{I,0}}{\partial x_2} = -\frac{\partial p^{I,0}}{\partial x_2} \\ \frac{\partial u_1^{I,0}}{\partial x_1} + \frac{\partial u_2^{I,0}}{\partial x_2} = 0, \\ u_2^{I,0}|_{x_2=0} = 0 \\ u^{I,0}|_{t=0} = u_0^{I,0}. \end{cases} \quad (4.6)$$

4.2.6 Boundary layer equations

Inside the boundary (in the domain $0 < x_2 < \varepsilon$), we consider the Taylor expansions with respect to the normal variable x_2 .

$$\begin{aligned} u_i^{I,j}(t, x_1, x_2) &= u_i^{I,j}(t, x_1, 0) + x_2 \partial_2 u_i^{I,j}(t, x_1, 0) + \frac{x_2^2}{2} \partial_2^2 u_i^{I,j}(t, x_1, 0) + \dots \\ p_i^{I,j}(t, x_1, x_2) &= p_i^{I,j}(t, x_1, 0) + x_2 \partial_2 p_i^{I,j}(t, x_1, 0) + \frac{x_2^2}{2} \partial_2^2 p_i^{I,j}(t, x_1, 0) + \dots \end{aligned}$$

Performing the change of variable $y = \frac{x_2}{\varepsilon}$, we can write

$$\begin{cases} u_i^{I,j}(t, x_1, x_2) = \overline{u_i^{I,j}} + \varepsilon y \overline{\partial_2 u_i^{I,j}} + \frac{\varepsilon^2 y^2}{2} \overline{\partial_2^2 u_i^{I,j}} + \dots \\ p_i^{I,j}(t, x_1, x_2) = \overline{p_i^{I,j}} + \varepsilon y \overline{\partial_2 p_i^{I,j}} + \frac{\varepsilon^2 y^2}{2} \overline{\partial_2^2 p_i^{I,j}} + \dots \end{cases} \quad (4.7)$$

where $\overline{f} = f(t, x_1, 0)$ is the trace of f on $\{x_2 = 0\}$. Now, we recall that

$$\begin{aligned} u(t, x_1, x_2) &= u^{I,0}(t, x_1, x_2) + u^{B,0}(t, x_1, \frac{x_2}{\varepsilon}) + \varepsilon \left[u^{I,1}(t, x_1, x_2) + u^{B,1}(t, x_1, \frac{x_2}{\varepsilon}) \right] + \dots \\ p(t, x_1, x_2) &= p^{I,0}(t, x_1, x_2) + p^{B,0}(t, x_1, \frac{x_2}{\varepsilon}) + \varepsilon \left[p^{I,1}(t, x_1, x_2) + p^{B,1}(t, x_1, \frac{x_2}{\varepsilon}) \right] + \dots \end{aligned}$$

Taking into account the expression (4.7), the velocity u becomes

$$u(t, x_1, x_2) = (\overline{u^{I,0}} + u^{B,0}) + \varepsilon [\overline{u^{I,1}} + u^{B,1} + y \overline{\partial_2 u^{I,0}}] + O(\varepsilon^2)$$

which means that

$$\begin{aligned} u_1(t, x_1, x_2) &= (\overline{u_1^{I,0}} + u_1^{B,0}) + \varepsilon [\overline{u_1^{I,1}} + u_1^{B,1} + y \overline{\partial_2 u_1^{I,0}}] + O(\varepsilon^2) \\ u_2(t, x_1, x_2) &= \underbrace{(\overline{u_2^{I,0}} + u_2^{B,0})}_{\text{(will be proved to be 0)}} + \varepsilon [\overline{u_2^{I,1}} + u_2^{B,1} + y \overline{\partial_2 u_2^{I,0}}] + O(\varepsilon^2) \end{aligned}$$

The derivatives of u are

$$\begin{aligned} \partial_t u &= \partial_t (\overline{u^{I,0}} + u^{B,0}) + \varepsilon \partial_t [\overline{u^{I,1}} + u^{B,1} + y \overline{\partial_2 u^{I,0}}] + O(\varepsilon^2) \\ \partial_1 u &= \partial_1 (\overline{u^{I,0}} + u^{B,0}) + \varepsilon \partial_1 [\overline{u^{I,1}} + u^{B,1} + y \overline{\partial_2 u^{I,0}}] + O(\varepsilon^2) \\ \partial_2 u &= \varepsilon^{-1} \partial_y u^{B,0} + (\partial_y u^{B,1} + \overline{\partial_2 u^{I,0}}) + O(\varepsilon). \end{aligned}$$

The non linear terms are computed as follows

$$\begin{aligned} u_1 \partial_1 u &= (\overline{u_1^{I,0}} + u_1^{B,0}) \partial_1 (\overline{u^{I,0}} + u^{B,0}) + O(\varepsilon) \\ u_2 \partial_2 u &= \varepsilon^{-1} (\overline{u_2^{I,0}} + u_2^{B,0}) \partial_y u^{B,0} + \left[\overline{u_2^{I,1}} + u_2^{B,1} + y \overline{\partial_2 u_2^{I,0}} \right] \partial_y u^{B,0} \\ &\quad + \underbrace{(\overline{u_2^{I,0}} + u_2^{B,0})}_{\text{(will be proved to be 0)}} (\partial_y u^{B,1} + \overline{\partial_2 u^{I,0}}) + O(\varepsilon). \end{aligned}$$

Concerning the pressure, we have

$$\begin{aligned} p(t, x_1, x_2) &= (\overline{p^{I,0}} + p^{B,0}) + \varepsilon [\overline{p^{I,1}} + p^{B,1} + y \overline{\partial_2 p^{I,0}}] + O(\varepsilon^2) \\ \partial_1 p &= \partial_1 (\overline{p^{I,0}} + p^{B,0}) + \varepsilon \partial_1 [\overline{p^{I,1}} + p^{B,1} + y \overline{\partial_2 p^{I,0}}] + O(\varepsilon^2) \\ \partial_2 p &= \varepsilon^{-1} \partial_y p^{B,0} + (\partial_y p^{B,1} + \overline{\partial_2 p^{I,0}}) + O(\varepsilon). \end{aligned}$$

- Now, we put all the above asymptotic expressions into the system (4.3). At the leading order ε^{-1} , we get

$$(\overline{u_2^{I,0}} + u_2^{B,0}) \partial_y u^{B,0} + \begin{pmatrix} 0 \\ \partial_y p^{B,0} \end{pmatrix} = 0.$$

We recall that $u_2^{B,0} = 0$ and that $\overline{u_2^{I,0}} = 0$ from the boundary conditions. We deduce that $\partial_y p^{B,0} = 0$ and then $p^{B,0} = 0$ since $p^{B,0} \rightarrow 0$ as $y \rightarrow +\infty$. That is the classical principle of fluid mechanics, which states that the pressure does not change in the boundary layer.

- At the next order ε^0 , we obtain

$$\left\{ \begin{array}{l} \partial_t(\overline{u^{I,0}} + u^{B,0}) + (\overline{u^{I,0}} + u^{B,0}) \cdot \overline{\nabla u^{I,0}} + (\overline{u_1^{I,0}} + u_1^{B,0}) \partial_{x_1} u^{B,0} + y \overline{\partial_{x_2} u_2^{I,0}} \partial_y u^{B,0} \\ + (\overline{u_2^{I,1}} + u_2^{B,1}) \partial_y u^{B,0} + \overline{\nabla p^{I,0}} + \left(\begin{array}{c} \partial_{x_1} p^{B,0} \\ \partial_y p^{B,1} \end{array} \right) = n \left[(\partial_y u_1^{B,0})^2 \right]^{(n-1)/2} \partial_y^2 u^{B,0} \end{array} \right. \quad (4.8)$$

The second component of the above vector equation reads

$$\overline{\partial_{x_2} p^{I,0}} + \partial_y p^{B,1} = 0,$$

which implies

$$p^{B,1} = 0, \quad (4.9)$$

using the fact that $\overline{\partial_{x_2} p^{I,0}} = 0$ from Euler system (4.6).

Now, for any $j \geq 0$, we define

$$u^{p,j}(t, x_1, y) = u^{B,j}(t, x_1, y) + \sum_{k=0}^j \frac{y^k}{k!} \overline{\partial_y^k u^{I,j-k}}(t, x_1). \quad (4.10)$$

Under this consideration, it is easy to prove that

$$\left\{ \begin{array}{l} u_1^{p,0} = u_1^{I,0}(t, x_1, 0) + u_1^{B,0}(t, x_1, y) \\ u_2^{p,1} = u_2^{I,1}(t, x_1, 0) + u_2^{B,1}(t, x_1, y) + y \partial_{x_2} u_2^{I,0}(t, x_1, 0) \end{array} \right.$$

satisfy the following non-Newtonian Prandtl-like system

$$\left\{ \begin{array}{l} \partial_t u_1^{p,0} + u_1^{p,0} \partial_{x_1} u_1^{p,0} + u_2^{p,1} \partial_y u_1^{p,0} + \overline{\partial_{x_1} p^{I,0}} = n \left| \partial_y u_1^{p,0} \right|^{n-1} \partial_y^2 u_1^{p,0} \\ \frac{\partial u_1^{p,0}}{\partial x_1} + \frac{\partial u_2^{p,1}}{\partial y} = 0, \\ u_1^{p,0} = 0 \quad \text{on } \{y = 0\} \\ u_2^{p,1} = 0 \quad \text{on } \{y = 0\} \\ u_1^{p,0} \rightarrow \overline{u_1^{I,0}} \quad \text{as } y \rightarrow +\infty, \end{array} \right.$$

where $\overline{u_1^{I,0}}$ and $\overline{\partial_{x_1} p^{I,0}}$ satisfies the so-called Bernoulli's law :

$$\overline{\partial_t u_1^{I,0}} + \overline{u_1^{I,0}} \overline{\partial_{x_1} u_1^{I,0}} + \overline{\partial_{x_1} p^{I,0}} = 0. \quad (4.11)$$

Chapter 5

Shear flow for Non-newtonian Prandtl equations

5.1 Introduction

In this chapter, we are interested in the the Non Newtonian Prandtl equation

$$\begin{cases} \partial_t u + u \partial_x u + v \partial_y u + \partial_x p = n |\partial_y u|^{n-1} \partial_y^2 u & \text{in } [0, T] \times \mathbb{T} \times \mathbb{R}^+ \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 & \text{in } [0, T] \times \mathbb{T} \times \mathbb{R}^+ \\ u|_{t=0} = u_0 & \text{on } \mathbb{T} \times \mathbb{R}^+ \\ u|_{y=0} = 0 & \text{on } [0, T] \times \mathbb{T} \\ v|_{y=0} = 0 & \text{on } [0, T] \times \mathbb{T} \\ \lim_{y \rightarrow +\infty} u = U & \text{for all } (t, x) \in [0, T] \times \mathbb{T}, \end{cases}$$

where the velocity field $(u, v)(t, x, y)$ is an unknown, the initial data $u_0(x, y)$ and the outer flow $U(t, x)$ are given and satisfy the compatibility conditions

$$u_0|_{y=0} = 0 \quad \text{and} \quad \lim_{y \rightarrow +\infty} u_0 = U|_{t=0}.$$

Furthermore, the given scalar pressure $p(t, x)$ and the outer flow $U(t, x)$ satisfy the well-known Bernoulli's law

$$\partial_t U + U \partial_x U = -\partial_x p.$$

In this work, we will consider the Oleinik's monotonicity assumption :

$$\omega := \partial_y u > 0$$

where ω is the vorticity.

Throughout this paper, we consider the homogeneous case with respect to the x variable : The solution u does not depend on x and the outer flow U is constant. For simplicity we take $U = 1$. Moreover, we will focus on the special case $n = 2$. That is, we are interested in the solutions of the following equation, which is the so-called "shear-flow" problem :

$$\begin{cases} \partial_t u = 2 |\partial_y u| \partial_y^2 u & \text{in } [0, T] \times \mathbb{R}^+ \\ u|_{t=0} = u_0 & \text{on } \mathbb{T} \times \mathbb{R}^+ \\ u|_{y=0} = 0 & \text{on } [0, T] \\ \lim_{y \rightarrow +\infty} u = 1 & \text{for all } t \in [0, T]. \end{cases} \quad (5.1)$$

The general case of dilatable fluids with $n > 1$ will be treated in a forthcoming work. This problem is equivalent to the following vorticity equation

$$\begin{cases} \partial_t \omega = 2\omega \partial_y^2 \omega + 2(\partial_y \omega)^2 & \text{in } [0, T] \times \mathbb{R}^+, \\ \omega|_{t=0} = \omega_0(y) & \text{on } \mathbb{R}^+ \\ \partial_y \omega|_{y=0} = 0 & \text{on } [0, T]. \end{cases} \quad (5.2)$$

For $n = 1$, the problem corresponds to the Prandtl system for a Newtonian fluid and has been widely studied. See the pioneering works of Oleinik, Samokhin [62] using the Croco transform. Recently some authors have used some energy estimates to study the Cauchy problem in Sobolev spaces : see Alexandre, Wang, Xu, Yang in [1] and Masmoudi, Wong in [59] for Sobolev spaces. The case of the Cauchy problem in Gevrey spaces has been seen by several authors, *e.g.* Gerard-Varet, Maekawa, Masmoudi in [31], Li, Wu, Xu in [50], ...

The case $0 < n < 1$ describes the motion of a pseudo-plastic fluid near a boundary [62], [70]. The case $n > 1$ has been studied in [71] in the framework of a dilatable fluid for MHD boundary layer system. The problem of existence and uniqueness of solutions of the boundary layer problem for non-newtonian fluids has also been partially studied in the case $0 < n < 1$ in [70], [72], [27], [61]. The problem for the both cases has been studied in [77], using an adapted method of [62]: The authors have used a transformation of the problem $\eta = u$ and $w(\tau, \xi, \eta) = |\partial_y u|^{n-1} \partial_y u$.

We also want to mention several results on the parabolic p-Laplacian but in a different setting concerning the boundary conditions and consequently the regularity of the solutions. We send the reader to DiBenedetto [26] and the references therein for more details.

In this paper, we will follow the idea of Masmoudi-Wong [59]. We need to introduce the function spaces which will be used in this chapter. For $I = \mathbb{R}^+$ or $I = \mathbb{R}$, $s \in \mathbb{N}$ and $\gamma \geq 1$, we set

$$H^{s,\gamma}(I) := \{\omega : I \rightarrow \mathbb{R}; \|\omega\|_{H^{s,\gamma}} < +\infty\}$$

where the weighted H^s norm $\|\cdot\|_{H^{s,\gamma}}$ is defined by

$$\|\omega\|_{H^{s,\gamma}}^2 := \sum_{k=0}^s \|\langle y \rangle^{\gamma+k} \partial_y^k \omega\|_{L^2(I)}^2 \quad (5.3)$$

with $\langle y \rangle = \sqrt{1+y^2}$. For $s \geq 4$, $\gamma \geq 1$, $\sigma > \gamma + 1$, $\delta \in (0, 1)$, we also define the following convex space

$$H_{\sigma,\delta}^{s,\gamma} := \left\{ \omega \in H^{s,\gamma} \mid \forall y \in I, \quad \langle y \rangle^\sigma \omega \geq \frac{\delta}{2} \text{ and } \sum_{k=0}^2 |\langle y \rangle^{\sigma+k} \partial_y^k \omega| \leq \frac{6}{\delta} \right\},$$

which is a metric space, equipped with the distance $d(u, v) = \|u - v\|_{H^{s,\gamma}}$. The main idea consists in adding an extra weight for each y -derivative. This corresponds to the weight $\frac{1}{y}$ in the Hardy type inequality.

In this paper we prove the following theorem:

Theorem 5.1. (*Local Existence and Uniqueness to the non-newtonian shear flow equation (5.1)*). *Let $s \geq 4$ an even integer, $\gamma > 1$, $\sigma > \gamma + 1$ and $\delta \in]0, 1[$. Assume that $u_0 \in H^{s,\gamma-1}$ and the initial vorticity $\omega_0 = \partial_y u_0 \in H_{\sigma,2\delta}^{s,\gamma}$ and satisfies the compatibility conditions $\partial_y^{2k+1} \omega_0|_{y=0} = 0$ for k integer such that $k \leq s/2 + 1$. Then there exist a time $T = T(s, \gamma, \sigma, \delta, \|\omega_0\|_{H^{s,\gamma}}) > 0$ and a unique classical solution u to the non-newtonian shear flow equations (5.1) such that*

$$u \in L^\infty([0, T]; H^{s,\gamma-1}) \cap C([0, T]; H_{weak}^s)$$

and the vorticity

$$\omega = \partial_y u \in L^\infty([0, T]; H_{\sigma,\delta}^{s,\gamma}) \cap C([0, T]; H_{weak}^s)$$

Where H_{weak}^s is the space H^s endowed with its weak topology.

In what follows, we will consider the equivalent symmetrized problem: We define $\tilde{\omega}(t, y) = \omega(t, |y|)$ and $\tilde{\omega}_0(y) = \omega_0(|y|)$ and for the sake of simplicity, we will still denote $\tilde{\omega}$ by ω and $\tilde{\omega}_0$ by ω_0 . From the parity of the coefficients of the equation on ω , it is easy to check that ω is a solution of

$$\begin{cases} \partial_t \omega = 2\omega \partial_y^2 \omega + 2(\partial_y \omega)^2 & \text{in } [0, T] \times \mathbb{R} \\ \omega|_{t=0} = \omega_0(y) & \text{on } \mathbb{R}. \end{cases} \quad (5.4)$$

The paper is organized as follows. We first prove the existence and the regularity of the solution ω_ε of this Cauchy problem, then we pass to the limit $n \rightarrow \infty$ then $\varepsilon \rightarrow 0$, to get a regular solution of (5.5).

5.2 Regularized problem and existence result in Hölder spaces

The main purpose of this section is to prove the existence of the approximate regularized system associated to (5.4). Throughout this section, we mainly use to the parabolic techniques in Hölder spaces in Lunardi [56] (see also appendix 5.8). For any $\varepsilon \in]0, 1]$, we consider the following regularized vorticity system

$$\begin{cases} \partial_t \omega^\varepsilon = 2\varepsilon^2 \partial_y^2 \omega^\varepsilon + 2\omega^\varepsilon \partial_y^2 \omega^\varepsilon + 2(\partial_y \omega^\varepsilon)^2 & \text{in } [0, T] \times \mathbb{R} \\ \omega^\varepsilon|_{t=0} = \omega_0(y) & \text{on } \mathbb{R}. \end{cases} \quad (5.5)$$

We prove the existence of a solution of (5.5) in some Hölder spaces.

Theorem 5.2. *For a fixed $\varepsilon \in]0, 1]$ and $\alpha \in]0, 1[$ we suppose that $\omega_0 \in C^{2+\alpha}(\mathbb{R})$ and $\omega_0 > 0$. Then there exists $T_\varepsilon > 0$ (which depends also on α and $|\omega_0|_{C^{2+\alpha}}$) and a regular solution $\omega_\varepsilon \in C^{1,2+\alpha}([0, T_\varepsilon] \times \mathbb{R})$ of the regularized vorticity equation (5.5).*

Moreover, for a fixed integer k , if we suppose that $\omega_0 \in C^{2+k+\alpha}(\mathbb{R})$, then the solution ω_ε belongs to the space $C^{1,2+k+\alpha}([0, T_\varepsilon] \times \mathbb{R})$.

Proof. To solve the equation (5.5) we will consider the approximate iterative scheme : We consider a small time $0 < \delta \leq 1$ which will be fixed later and the complete metric set

$$Y = \{w \in C^{1,2+\alpha}([0, \delta] \times \mathbb{R}) \mid w|_{t=0} = \omega_0\}$$

with the distance $d(u, v) = |u - v|_{1,2+\alpha}$. We set $\omega_0(t, y) \equiv \omega_0(y)$ and we suppose that for $n \in \mathbb{N}$ we have constructed $\omega_n \in Y$, $\omega_n > 0$ and we define ω_{n+1} the solution of the linear Cauchy problem

$$\begin{cases} \partial_t \omega_{n+1} = 2(\omega_n + \varepsilon) \partial_y^2 \omega_{n+1} + 2\partial_y \omega_n \partial_y \omega_{n+1} & \text{in } [0, \delta] \times \mathbb{R} \\ \omega_{n+1}|_{t=0} = \omega_0(x, y) & \text{on } \mathbb{R}. \end{cases} \quad (5.6)$$

Since w_0 is time constant, the function w_1 is the solution of the following Cauchy problem :

$$\begin{cases} \partial_t \omega_1 = 2(\omega_0(x) + \varepsilon) \partial_y^2 \omega_1 + 2\partial_y \omega_0(x) \partial_y \omega_1 & \text{in } [0, \delta] \times \mathbb{R} \\ \omega_1|_{t=0} = \omega_0(x, y) & \text{on } \mathbb{R}. \end{cases}$$

Since $\omega_0 > 0$, the assumptions of Theorem 5.3 (see appendix 5.8) are fulfilled and there exists a unique solution ω_1 which belongs to the space $C^{1,2+\alpha}([0, \delta] \times \mathbb{R})$. Therefore ω_1 belongs to Y . Since the initial data ω_0 is positive, then by the maximum principle the solution ω_1 remains positive.

Now for a fixed $n \in \mathbb{N}$ and for a positive function $\omega_n \in C^{1,2+\alpha}([0, \delta] \times \mathbb{R})$, we first prove the existence of $\omega_{n+1} \in C^{1,2+\alpha}([0, \delta] \times \mathbb{R})$ a positive solution of (5.6). We derive some uniform estimates on the sequence $(\omega_n)_n$, we then prove that $(\omega_n)_n$ is a Cauchy sequence and finally we pass to the limit $n \rightarrow \infty$.

- Existence of a solution $\omega_{n+1} \in Y$ of the linear problem (5.6) by a fixed point theorem.

We first remark that, if we suppose that ω_n is positive, then the linear operator defined (5.6) is parabolic. Therefore, since $\omega_{n+1}|_{t=0}$ is positive, by the Maximum Principle, the solution ω_{n+1} , if it exists, remains also positive, and by recurrence the sequence $(\omega_{n+1})_n$ is positive.

We define for smooth function $\omega > 0$ the following elliptic operator

$$L_\omega = 2\omega \partial_y^2 + 2\partial_y \omega \partial_y$$

and for $\omega = \omega_0 + \varepsilon$ the following operator

$$L_0 = L_{(\omega_0 + \varepsilon)} = 2(\omega_0 + \varepsilon) \partial_y^2 + 2\partial_y \omega_0 \partial_y$$

which is an autonomous elliptic operator whose coefficients belong to $C^\alpha(\mathbb{R})$. We reformulate the problem : ω_{n+1} is a solution of the problem (5.6) if and only if ω_{n+1} is a solution of the following problem :

$$\begin{cases} \partial_t \omega_{n+1} = L_0 \omega_{n+1} + L_{(\omega_n - \omega_0)} \omega_{n+1} & \text{in } [0, \delta] \times \mathbb{R} \\ \omega_{n+1}|_{t=0} = \omega_0 & \text{on } \mathbb{R}. \end{cases}$$

Consequently, for $V \in Y$, we consider the following Cauchy problem :

$$\begin{cases} \partial_t U = L_0 U + L_{(\omega_n - \omega_0)} V & \text{in } [0, \delta] \times \mathbb{R} \\ U|_{t=0} = \omega_0 & \text{on } \mathbb{R}. \end{cases} \quad (5.7)$$

We now use the following technical lemma, which proof will be given below.

Lemma 5.2.1. *For some fixed $0 < \alpha < 1$, $0 < \delta \leq 1$ and $\omega \in C^{1,2+\alpha}([0, \delta] \times \mathbb{R})$, we consider the following differential operator*

$$L_\omega = 2\omega \partial_y^2 + 2\partial_y \omega \partial_y.$$

There exists a positive constant K_α such that for every functions ω and V in $C^{1,2+\alpha}([0, \delta] \times \mathbb{R})$, ω verifying $\omega|_{t=0} = 0$, we have

$$|L_\omega V|_{C^{0,\alpha}} \leq K_\alpha \delta^{\frac{\alpha}{2}} |\omega|_{C^{1,2+\alpha}} |V|_{C^{1,2+\alpha}}. \quad (5.8)$$

Moreover, the function $L_\omega f$ belongs to the spaces $C^{0,\alpha}([0, \delta] \times \mathbb{R}) \cap UC([0, \delta] \times \mathbb{R})$.

From this lemma, the second member $f = L_{(\omega_n - \omega_0)} V$ belongs to the spaces $C^{1,2+\alpha}([0, \delta] \times \mathbb{R})$ and $UC([0, \delta] \times \mathbb{R})$. From Theorem 5.3 (see also 5.1.4 of Lunardi [Lunardi]), there exists a unique solution $U \in Y$ of the Cauchy problem (5.6) and we set $\Gamma^n V = U$. Therefore ω_{n+1} is a solution of (5.6) if and only if ω_{n+1} is a fixed point of the operator Γ^n . We now prove that Γ^n is a contraction.

From Theorem 5.3 there exists a constant $c_{\alpha,\varepsilon}$ such that

$$|\Gamma^n V|_{C^{1,2+\alpha}} \leq c_{\alpha,\varepsilon} |L_{(\omega_n - \omega_0)} V|_{C^{0,\alpha}}$$

and from lemma 5.2.1

$$|L_{(\omega_n - \omega_0)} V|_{C^{0,\alpha}} \leq K_\alpha \delta^{\frac{\alpha}{2}} |\omega_n - \omega_0|_{C^{1,2+\alpha}} |V|_{C^{1,2+\alpha}}.$$

Therefore, setting $C_{\alpha,\varepsilon} = c_{\alpha,\varepsilon} K_\alpha$, we have for all function $V \in Y$

$$|\Gamma^n V|_{C^{1,2+\alpha}} \leq C_{\alpha,\varepsilon} \delta^{\frac{\alpha}{2}} |\omega_n - \omega_0|_{C^{1,2+\alpha}} |V|_{C^{1,2+\alpha}}. \quad (5.9)$$

We suppose that we have the following estimate :

$$\text{(Hypothesis)} \quad C_{\alpha,\varepsilon} \delta^{\frac{\alpha}{2}} |\omega_n - \omega_0|_{C^{1,2+\alpha}} \leq 1/2. \quad (5.10)$$

Therefore, we conclude that for all $V_1, V_2 \in Y$ we have

$$|\Gamma^n V_2 - \Gamma^n V_1|_{C^{1,2+\alpha}} \leq \frac{1}{2} |V_2 - V_1|_{C^{1,2+\alpha}}$$

and the operator Γ^n is a contraction in the space Y , so there exists a unique fixed point ω_{n+1} of Γ^n in Y , which is the unique solution of (5.7) in Y .

• Uniform estimate on $|\omega_n - \omega_0|_{C^{1,2+\alpha}}$. We first establish a recurrence formula on $|\omega_n - \omega_0|_{C^{1,2+\alpha}}$. Since $\Gamma^n \omega_{n+1} = \omega_{n+1}$ and Γ^n is a contraction, we have

$$|\omega_{n+1} - \Gamma^n \omega_0|_{C^{1,2+\alpha}([0,t] \times \mathbb{R})} \leq \frac{1}{2} |\omega_{n+1} - \omega_0|_{C^{1,2+\alpha}}.$$

Therefore,

$$\begin{aligned} |\omega_{n+1} - \omega_0|_{C^{1,2+\alpha}} &\leq |\omega_{n+1} - \Gamma^n \omega_0|_{C^{1,2+\alpha}} + |\Gamma^n \omega_0 - \omega_0|_{C^{1,2+\alpha}} \\ &\leq \frac{1}{2} |\omega_{n+1} - \omega_0|_{C^{1,2+\alpha}} + |\Gamma^n \omega_0|_{C^{1,2+\alpha}} + |\omega_0|_{C^{2+\alpha}}, \end{aligned}$$

that is

$$|\omega_{n+1} - \omega_0|_{C^{1,2+\alpha}} \leq 2 |\Gamma^n \omega_0|_{C^{1,2+\alpha}} + 2 |\omega_0|_{C^{2+\alpha}}.$$

Then using (5.9) we get

$$|\Gamma^n \omega_0|_{C^{1,2+\alpha}} \leq (C_{\alpha,\varepsilon} \delta^{\frac{\alpha}{2}} |\omega_0|_{C^{1,2+\alpha}}) |\omega_n - \omega_0|_{C^{1,2+\alpha}}$$

and from the two previous estimates

$$|\omega_{n+1} - \omega_0|_{C^{1,2+\alpha}} \leq (2 C_{\alpha,\varepsilon} |\omega_0|_{C^{2+\alpha}} \delta^{\frac{\alpha}{2}}) |\omega_n - \omega_0|_{C^{1,2+\alpha}} + 2 |\omega_0|_{C^{2+\alpha}}.$$

We now choose δ such that $(2 C_{\alpha,\varepsilon} |\omega_0|_{C^{2+\alpha}} \delta^{\frac{\alpha}{2}}) \leq \frac{1}{2}$, that is

$$\delta \leq \delta_1 \triangleq \left(\frac{1}{4 C_{\alpha,\varepsilon} |\omega_0|_{C^{2+\alpha}}} \right)^{2/\alpha}.$$

Then from the two previous inequalities,

$$|\omega_{n+1} - \omega_0|_{C^{1,2+\alpha}} \leq \frac{1}{2} |\omega_n - \omega_0|_{C^{1,2+\alpha}} + 2 |\omega_0|_{C^{2+\alpha}}$$

and by induction we directly have

$$|\omega_n - \omega_0|_{C^{1,2+\alpha}} \leq 4 |\omega_0|_{C^{2+\alpha}}, \quad \forall n \in \mathbb{N}. \quad (5.11)$$

Now, for $0 < \delta < \delta_1$, we examine the condition (5.10). From the previous estimate, we have

$$C_{\alpha,\varepsilon} \delta^{\frac{\alpha}{2}} |\omega_n - \omega_0|_{C^{1,2+\alpha}} \leq (4 C_{\alpha,\varepsilon} |\omega_0|_{C^{2+\alpha}} \delta^{\frac{\alpha}{2}})$$

and if we choose δ small enough such that $(4 C_{\alpha,\varepsilon} |\omega_0|_{C^{2+\alpha}} \delta^{\frac{\alpha}{2}}) \leq \frac{1}{2}$, that is

$$\delta \leq \delta_2 \triangleq \left(\frac{1}{8 C_{\alpha,\varepsilon} |\omega_0|_{C^{2+\alpha}}} \right)^{2/\alpha},$$

the condition (5.10) is fulfilled. Now we choose $0 < \delta \leq \min(1, \delta_1, \delta_2)$. Under this condition, we prove by recurrence that the condition (5.10) is fulfilled for all integer n and that the sequence $(\omega_n)_n$ is well-defined in the space Y and is bounded in Y .

• Convergence of the sequence $(\omega_n)_n$. We now prove that $(\omega_n)_n$ is a Cauchy sequence. We write

$$\partial_t \omega_{n+1} = L_0 \omega_{n+1} + L_{(\omega_n - \omega_0)} \omega_{n+1} \quad \text{and} \quad \partial_t \omega_n = L_0 \omega_n + L_{(\omega_{n-1} - \omega_0)} \omega_n.$$

Then

$$\partial_t (\omega_{n+1} - \omega_n) = L_0 (\omega_{n+1} - \omega_n) + L_{(\omega_{n-1} - \omega_0)} (\omega_{n+1} - \omega_n) + L_{(\omega_n - \omega_{n-1})} \omega_{n+1}.$$

That is $U = \omega_{n+1} - \omega_n$ is solution of the problem

$$\partial_t U = L_0 U + f(t, \cdot) \quad \text{where} \quad f = L_{(\omega_{n-1} - \omega_0)} (\omega_{n+1} - \omega_n) + L_{(\omega_n - \omega_{n-1})} \omega_{n+1}.$$

Again by Theorem 5.3 since $(\omega_{n-1} - \omega_0)|_{t=0} = 0$ and $(\omega_n - \omega_{n-1})|_{t=0} = 0$, we have

$$|\omega_{n+1} - \omega_n|_{C^{1,2+\alpha}} \leq c_{\alpha,\varepsilon} |L_{(\omega_{n-1} - \omega_0)} (\omega_{n+1} - \omega_n)|_{C^{0,\alpha}} + c_{\alpha,\varepsilon} |L_{(\omega_n - \omega_{n-1})} \omega_{n+1}|_{C^{0,\alpha}}.$$

From lemma 5.2.1 and the estimate (5.11) we derive

$$\begin{aligned} & |\omega_{n+1} - \omega_n|_{C^{1,2+\alpha}} \\ & \leq C_{\alpha,\varepsilon} \delta^{\alpha/2} |\omega_{n-1} - \omega_0|_{C^{1,2+\alpha}} |\omega_{n+1} - \omega_n|_{C^{1,2+\alpha}} + C_{\alpha,\varepsilon} \delta^{\alpha/2} |\omega_n - \omega_{n-1}|_{C^{1,2+\alpha}} |\omega_{n+1}|_{C^{1,2+\alpha}} \\ & \leq \left(4 C_{\alpha,\varepsilon} |\omega_0|_{C^{2+\alpha}} \delta^{\alpha/2}\right) |\omega_{n+1} - \omega_n|_{C^{1,2+\alpha}} + \left(5 C_{\alpha,\varepsilon} |\omega_0|_{C^{2+\alpha}} \delta^{\alpha/2}\right) |\omega_n - \omega_{n-1}|_{C^{1,2+\alpha}}. \end{aligned}$$

If we choose

$$\delta \leq \delta_3 \triangleq \left(\frac{1}{20 C_{\alpha,\varepsilon} |\omega_0|_{C^{2+\alpha}}} \right)^{2/\alpha}.$$

then we derive

$$|\omega_{n+1} - \omega|_{C^{1,2+\alpha}} \leq \frac{1}{2} |\omega - \omega_{n-1}|_{C^{1,2+\alpha}} + \frac{1}{4} |\omega - \omega_{n-1}|_{C^{1,2+\alpha}},$$

that is

$$|\omega_{n+1} - \omega|_{C^{1,2+\alpha}} \leq \frac{1}{2} |\omega - \omega_{n-1}|_{C^{1,2+\alpha}}.$$

Finally we set

$$\delta \triangleq \min(1, \delta_1, \delta_2, \delta_3).$$

Therefore (ω_n) is a Cauchy sequence in $C^{1,2+\alpha}([0, \delta] \times \mathbb{R})$ and converges to a function $\omega \in Y$ such that

$$\partial_t \omega = L_0 \omega + L_{(\omega - \omega_0)} \omega$$

and therefore is a solution of (5.5). This concludes the proof of the first point of the theorem.

• **Regularity of the solution.** We suppose that $\omega_0 \in C^{3+\alpha}$. Then setting $U = \partial_y \omega$ we have $U \in C^{0,1+\alpha}$ and from lemma 5.8.1 $U_1 \in C^{(1+\alpha)/2,0}$ and formally is solution of the following equation.

$$\begin{cases} \partial_t U = 2(\omega + \varepsilon) \partial_y^2 U + 6 \partial_y \omega \partial_y U & \text{in } [0, T] \times \mathbb{R} \\ U|_{t=0} = \partial_y \omega_0(y) & \text{on } \mathbb{R}. \end{cases} \quad (5.12)$$

From the first point of the theorem, changing in the proof the differential operator $2\omega \partial_y^2 + 2\partial_y \omega \partial_y$ by the differential operator $2\omega \partial_y^2 + 6\partial_y \omega \partial_y$, there exists a unique solution \tilde{U} of (5.12) and $\tilde{U} \in C^{1,2+\alpha}$. Therefore, setting

$$Z(t, y) = \left(\omega(t, 0) + \int_0^y \tilde{U}(t, z) dz \right) - \omega(t, y),$$

we can check that $Z \in C^{1,2+\alpha}$ and Z is a solution of

$$\begin{cases} \partial_t Z = 2(\omega + \varepsilon) \partial_y^2 Z + 2 \partial_y \omega \partial_y Z \\ Z|_{t=0} = 0. \end{cases}$$

Therefore by the maximum principle $Z \equiv 0$ and $U \equiv \tilde{U}$. We have proved that $\omega \in C^{1,3+\alpha}$. We can proceed in the same way for each order of derivatives k . This concludes the proof of Theorem □

We now prove Lemma 5.2.1.

Proof of lemma 5.2.1. For some fixed $0 < \alpha < 1$ and $0 < \delta \leq 1$ and we consider some function ω, V bellowing to $C^{1,2+\alpha}([0, \delta] \times \mathbb{R})$ such that $\omega|_{t=0} = 0$. We will prove an estimate on

$$|L_\omega V|_{C^{0,\alpha}} = \|L_\omega V\|_\infty + [L_\omega V]_{C^{0,\alpha}}.$$

We now will estimate each terms.

- Estimate of $\|L_\omega V\|_\infty \leq 2\|\omega \partial_y^2 V\|_\infty + 2\|\partial_y \omega \partial_y V\|_\infty$. We first remark that

$$\begin{aligned} |\omega(t, y)| &= |\omega(t, y) - \omega(0, y)| \\ &\leq |t - 0| \|\partial_t \omega\|_\infty \end{aligned}$$

Therefore

$$\|\omega(t, y)\|_\infty \leq \delta |\omega|_{C^{1,2+\alpha}}. \quad (5.13)$$

We have also

$$\begin{aligned} |\partial_y \omega(t, y)| &= |t|^{(1+\alpha)/2} \left(\frac{|\partial_y \omega(t, y) - \partial_y \omega(0, y)|}{|t - 0|^{(1+\alpha)/2}} \right) \\ &\leq \delta^{(1+\alpha)/2} |\omega|_{C^{(1+\alpha)/2,0}}. \end{aligned}$$

Therefore from lemma 5.8.1

$$\|\partial_y \omega\|_\infty \leq \delta^{(1+\alpha)/2} c_{1,\alpha} |\omega|_{C^{1,2+\alpha}}. \quad (5.14)$$

We then have from the previous estimates

$$\begin{aligned} \|2\omega \partial_y^2 V\|_\infty &\leq 2\|\omega\|_\infty \|\partial_y^2 V\|_\infty \\ &\leq 2\delta |\omega|_{C^{1,2+\alpha}} |V|_{C^{1,2+\alpha}} \end{aligned}$$

and

$$\begin{aligned} \|2\partial_y \omega \partial_y V\|_\infty &\leq 2\|\partial_y \omega\|_\infty \|\partial_y V\|_\infty \\ &\leq 2\delta^{(1+\alpha)/2} c_{1,\alpha} |\omega|_{C^{1,2+\alpha}} |V|_{C^{1,2+\alpha}}. \end{aligned}$$

Then

$$\|L_\omega V\|_\infty \leq 2(1 + c_{1,\alpha}) \delta^{(1+\alpha)/2} |\omega|_{C^{1,2+\alpha}} |V|_{C^{1,2+\alpha}}. \quad (5.15)$$

- Estimate of $[L_\omega V]_{C^{0,\alpha}} \leq 2[\omega \partial_y^2 V]_{C^{0,\alpha}} + 2[\partial_y \omega \partial_y V]_{C^{0,\alpha}}$. We use the classic inequality

$$[fg]_{C^{0,\alpha}} \leq \|f\|_\infty [g]_{C^{0,\alpha}} + [f]_{C^{0,\alpha}} \|g\|_\infty.$$

We first estimate $[\omega]_{C^{0,\alpha}}$: Since $\partial_y \omega|_{t=0} = 0$ we have

$$\begin{aligned} |\omega(t, y_2) - \omega(t, y_1)| &= \left| \int_{y_1}^{y_2} (\partial_y \omega(t, z) - \partial_y \omega(0, z)) dz \right| \\ &\leq \int_{y_1}^{y_2} |s - 0|^{(1+\alpha)/2} |\partial_y \omega|_{C^{(1+\alpha)/2,0}} dz. \end{aligned}$$

Therefore from lemma 5.8.1

$$|\omega(t, y_2) - \omega(t, y_1)| \leq |y_2 - y_1| \delta^{(1+\alpha)/2} c_{1,\alpha} |\omega|_{C^{1,2+\alpha}}.$$

That is

$$\frac{1}{|y_2 - y_1|^\alpha} |\omega(t, y_2) - \omega(t, y_1)| \leq \begin{cases} \delta^{(1+\alpha)/2} c_{1,\alpha} |\omega|_{C^{1,2+\alpha}} & \text{if } 0 < |y_2 - y_1| \leq 1 \\ 2\|\omega\|_\infty & \text{if } |y_2 - y_1| \geq 1 \end{cases}$$

and finally from (5.13)

$$[\omega]_{C^{0,\alpha}} \leq \max(c_{1,\alpha}, 2) \delta^{(1+\alpha)/2} |\omega|_{C^{1,2+\alpha}}. \quad (5.16)$$

In a same way, we estimate $[\partial_y \omega]_{C^{0,\alpha}}$: Since $\partial_y^2 \omega|_{t=0} = 0$ we have

$$\begin{aligned} |\partial_y \omega(t, y_2) - \partial_y \omega(t, y_1)| &= \left| \int_{y_1}^{y_2} (\partial_y^2 \omega(t, z) - \partial_y^2 \omega(0, z)) dz \right| \\ &\leq \int_{y_1}^{y_2} |s - 0|^{\alpha/2} |\partial_y^2 \omega|_{C^{\alpha/2,0}} dz. \end{aligned}$$

Therefore from lemma 5.8.1

$$|\partial_y \omega(t, y_2) - \partial_y \omega(t, y_1)| \leq |y_2 - y_1| \delta^{\alpha/2} c_{2,\alpha} |\omega|_{C^{1,2+\alpha}}.$$

That is

$$\frac{1}{|y_2 - y_1|^\alpha} |\partial_y \omega(t, y_2) - \partial_y \omega(t, y_1)| \leq \begin{cases} \delta^{\alpha/2} c_{2,\alpha} |\omega|_{C^{1,2+\alpha}} & \text{if } 0 < |y_2 - y_1| \leq 1 \\ 2 \|\partial_y \omega\|_\infty & \text{if } |y_2 - y_1| \geq 1. \end{cases}$$

Finally we get from (5.14)

$$|\partial_y \omega|_{C^{0,\alpha}} \leq \max(c_{2,\alpha}, 2c_{1,\alpha}) \delta^{\alpha/2} |\omega|_{C^{1,2+\alpha}}. \quad (5.17)$$

We then derive from (5.13), (5.14), (5.16), (5.17)

$$\begin{aligned} [\omega \partial_y^2 V]_{C^{0,\alpha}} &\leq \|\omega\|_\infty [\partial_y^2 V]_{C^{0,\alpha}} + [\omega]_{C^{0,\alpha}} \|\partial_y^2 V\|_\infty \\ &\leq (\delta |\omega|_{C^{1,2+\alpha}}) |V|_{C^{1,2+\alpha}} + \left(\max(C'_\alpha, 2) \delta^{(1+\alpha)/2} |\omega|_{C^{1,2+\alpha}} \right) |V|_{C^{1,2+\alpha}} \end{aligned}$$

and

$$\begin{aligned} [\partial_y \omega \partial_y V]_{C^{0,\alpha}} &\leq \|\partial_y \omega\|_\infty [\partial_y V]_{C^{0,\alpha}} + [\partial_y \omega]_{C^{0,\alpha}} \|\partial_y V\|_\infty \\ &\leq \left(\delta^{(1+\alpha)/2} C'_\alpha |\omega|_{C^{1,2+\alpha}} \right) |V|_{C^{1,2+\alpha}} + \left(\delta^{\alpha/2} \max(C_\alpha, C'_\alpha) |\omega|_{C^{1,2+\alpha}} \right) |V|_{C^{1,2+\alpha}}. \end{aligned}$$

• Conclusion : From the two previous estimates, we have

$$|L_\omega V|_{C^{0,\alpha}} \leq C_\alpha \delta^{\alpha/2} |\omega|_{C^{1,2+\alpha}} |V|_{C^{1,2+\alpha}}$$

and using (5.15) we prove the estimate (5.8) of the lemma. It remains to prove that the function $L_\omega V$ belongs in the space $UC([0, \delta] \times \mathbb{R})$. We remark that for $y, \Delta y \in \mathbb{R}$ and $t, t + \Delta t \in [0, \delta]$, and, following the proof of the estimate (5.15), we derive

$$\begin{aligned} |(L_\omega V)(t + \Delta t, y + \Delta y) - (L_\omega V)(t, y)| \\ &\leq |(L_\omega V)(t + \Delta t, y + \Delta y) - (L_\omega V)(t, y + \Delta y)| + |(L_\omega V)(t, y + \Delta y) - (L_\omega V)(t, y)| \\ &\leq C(|\Delta t|^{(1+\alpha)/2} + |\Delta y|^\alpha). \end{aligned}$$

This concludes the proof of lemma 5.2.1. □

5.3 Upper and lower solutions

Proposition 5.3.1. *We consider a initial data $\omega_0 \in C^2(\mathbb{R})$ such that*

$$c_0 \langle y \rangle^{-\sigma} \leq \omega_0(y) \leq c'_0 \langle y \rangle^{-\sigma}$$

for some positive constants c_0, c'_0 . We suppose that there exists a solution $\omega \in C^{1,2}([0, T] \times \mathbb{R})$ of equation (5.5). Then there exists a positive constants A_0 such that

$$\forall t \in [0, T], \forall y \in \mathbb{R}, \quad c_0 \langle y \rangle^{-\sigma} e^{-A_0 t} \leq \omega(t, y) \leq c'_0 \langle y \rangle^{-\sigma} e^{A_0 t}.$$

where $A_0 \leq C_{0,\sigma}(1 + |\omega|_\infty + |\partial_y \omega|_\infty)$.

Moreover, if we suppose that $\omega \in \mathcal{C}^{1,3}([0, T] \times \mathbb{R})$ and

$$|\partial_y \omega_0(y)| \leq c_1 \langle y \rangle^{-(\sigma+1)},$$

for some positive constant c_1 , then there exists a positive constant A_1 such that

$$\forall t \in [0, T], \forall y \in \mathbb{R}, \quad |\partial_y \omega(t, y)| \leq c_1 \langle y \rangle^{-(\sigma+1)} e^{A_1 t}.$$

where $A_1 \leq C_{1,\sigma}(1 + |\omega|_\infty + |\partial_y \omega|_\infty)$

In addition, if we suppose that $\omega \in \mathcal{C}^{1,4}([0, T] \times \mathbb{R})$ and

$$|\partial_y^2 \omega_0(y)| \leq c_2 \langle y \rangle^{-(\sigma+2)},$$

for some positive constant c_2 , then there exists a positive constant A_2 such that

$$\forall t \in [0, T], \forall y \in \mathbb{R}, \quad |\partial_y^2 \omega(t, y)| \leq c_2 \langle y \rangle^{-(\sigma+2)} e^{A_2 t}.$$

where $A_2 \leq C_{2,\sigma}(1 + |\omega|_\infty + |\partial_y \omega|_\infty + |\partial_y^2 \omega|_\infty)$.

Proof. We split the proof into three parts :

- Estimate of ω . We define the elliptic operator

$$L = \partial_t - 2(\omega + \varepsilon) \partial_y^2 - 2\partial_y \omega \partial_y.$$

and we consider the following functions

$$\underline{\omega}(t, y) = c_0 \langle y \rangle^{-\sigma} e^{-At} \quad \text{and} \quad \bar{\omega}(t, y) = c'_0 \langle y \rangle^{-\sigma} e^{At}$$

We compute

$$\begin{aligned} L\underline{\omega} &= -\underline{\omega} \left(A + 2(\omega + \varepsilon) \frac{\partial_y^2 \langle y \rangle^{-\sigma}}{\langle y \rangle^{-\sigma}} + 2\partial_y \omega \frac{\partial_y \langle y \rangle^{-\sigma}}{\langle y \rangle^{-\sigma}} \right) \\ L\bar{\omega} &= \bar{\omega} \left(A - 2(\omega + \varepsilon) \frac{\partial_y^2 \langle y \rangle^{-\sigma}}{\langle y \rangle^{-\sigma}} + 2\partial_y \omega \frac{\partial_y \langle y \rangle^{-\sigma}}{\langle y \rangle^{-\sigma}} \right). \end{aligned}$$

and we deduce that $L\underline{\omega} < 0$ and $L\bar{\omega} > 0$ and for

$$A = 1 + 2c_\sigma(1 + |\omega|_\infty) + 2c_\sigma |\partial_y \omega|_\infty$$

where

$$c_\sigma = \max(|\partial_y^2 \langle y \rangle^{-\sigma} / \langle y \rangle^{-\sigma}|_\infty, |\partial_y \langle y \rangle^{-\sigma} / \langle y \rangle^{-\sigma}|_\infty).$$

Since $\underline{\omega} \leq \omega \leq \bar{\omega}$ for $t = 0$ and $\lim_{y \rightarrow \pm\infty}(\underline{\omega}) = \lim_{y \rightarrow \pm\infty}(\omega) = \lim_{y \rightarrow \pm\infty}(\bar{\omega}) = 0$, we can apply the the maximum principle.

- Estimate of $\partial_y \omega$. We compute from (5.5)

$$\partial_t \partial_y \omega - 2(\omega + \varepsilon) \partial_y^3 \omega - 6\partial_y \omega \partial_y^2 \omega = 0$$

and we set $U = \partial_y \omega$. Therefore

$$\partial_t U - 2(\omega + \varepsilon) \partial_y^2 U - 6\partial_y \omega \partial_y U = 0$$

and we define the elliptic differential operator

$$L_1 = \partial_t - 2(\omega + \varepsilon) \partial_y^2 - 6\partial_y \omega \partial_y.$$

We have

$$|\partial_y \omega_0| \leq c_1 \langle y \rangle^{-(\sigma+1)}$$

and we set

$$\bar{U}(t, y) = c_1 \langle y \rangle^{-(\sigma+1)} e^{At}.$$

Therefore

$$\begin{aligned} L_1 \bar{U} &= -A \langle y \rangle^{-(\sigma+1)} e^{-At} - 2(\omega + \varepsilon) \partial_y^2 \langle y \rangle^{-(\sigma+1)} e^{-At} - 2\partial_y \omega \partial_y \langle y \rangle^{-(\sigma+1)} e^{-At} \\ &= \bar{U} \left(A - 2(\omega + \varepsilon) \frac{\partial_y^2 \langle y \rangle^{-(\sigma+1)}}{\langle y \rangle^{-(\sigma+1)}} + 2\partial_y \omega \frac{\partial_y \langle y \rangle^{-(\sigma+1)}}{\langle y \rangle^{-(\sigma+1)}} \right) \end{aligned}$$

We conclude as previously .

- Estimate of $\partial_y^2 \omega$. We compute from (5.5)

$$\partial_t \partial_y^2 \omega - 2\omega \partial_y^4 \omega - 8\partial_y \omega \partial_y^3 \omega - 6(\partial_y^2 \omega)^2 = 0$$

and we set $U = \partial_y^2 \omega$. We then have

$$\partial_t U - 2\omega \partial_y^2 U - 8\partial_y \omega \partial_y U + (A - 6\partial_y^2 \omega)U = 0$$

and we define the elliptic differential operator

$$L_2 = \partial_t - 2\omega \partial_y^2 - 8\partial_y \omega \partial_y + (A - 6\partial_y^2 \omega).$$

We have

$$|\partial_y^2 \omega_0| \leq c_1 \langle y \rangle^{-(\sigma+2)}$$

and we set

$$\bar{U}(t, y) = c_1 \langle y \rangle^{-(\sigma+2)}.$$

Therefore

$$L_2 \bar{U} = \bar{U} \left(A - 2(\omega + \varepsilon) \frac{\partial_y^2 \langle y \rangle^{-(\sigma+2)}}{\langle y \rangle^{-(\sigma+2)}} - 8\partial_y \omega \frac{\partial_y \langle y \rangle^{-(\sigma+2)}}{\langle y \rangle^{-(\sigma+2)}} - 6\partial_y^2 \omega \right).$$

We conclude as previously.

This conclude the proof of the proposition. \square

Proposition 5.3.2. *We consider a fixed integer $k \geq 3$ and an initial data $\omega_0 \in \mathcal{C}(\mathbb{R})$ such that for all $y \in \mathbb{R}$*

$$\begin{aligned} c_0 \langle y \rangle^{-\sigma} \leq \omega_0(y) \leq c'_0 \langle y \rangle^{-\sigma} \\ \sum_{\ell=3}^k \langle y \rangle^{\sigma+\ell} |\partial_y^\ell \omega_0(y)| \leq M_k \end{aligned}$$

for some positive constants c_1, c_1, M_k . We suppose that there exists a solution $\omega \in \mathcal{C}^{1,k}([0, T] \times \mathbb{R})$ of equation (5.5). Then there exists some positive constants A_k and C_k such that for $t \in [0, T]$ and $y \in \mathbb{R}$

$$\sum_{\ell=3}^k \langle y \rangle^{\sigma+\ell} |\partial_y^\ell \omega(t, y)| \leq C_k e^{At} \quad (5.18)$$

where $A_k \lesssim (1 + |\omega|_\infty + |\partial_y \omega|_\infty + |\partial_y^2 \omega|_\infty)$.

Proof. We will prove the proposition by induction.

- Estimate of $\partial_y^3 \omega$. We set $U = \partial_y^3 \omega e^{-At}$. Therefore

$$\partial_t U - 2\omega \partial_y^2 U - 10\partial_y \omega \partial_y U + (A - 20\partial_y^2 \omega)U = 0$$

and we define the elliptic differential operator

$$L_3 = \partial_t - 2\omega\partial_y^2 - 10\partial_y\omega\partial_y + (A - 20\partial_y^2\omega).$$

We have

$$|\partial_y^3\omega_0| \leq c_1\langle y\rangle^{-(\sigma+3)}$$

and we set

$$\bar{U}(t, y) = c_1\langle y\rangle^{-(\sigma+3)}.$$

Therefore

$$L_3\bar{U} = \bar{U} \left(A - 2(\omega + \varepsilon) \frac{\partial_y^2\langle y\rangle^{-(\sigma+3)}}{\langle y\rangle^{-(\sigma+3)}} - 10\partial_y\omega \frac{\partial_y\langle y\rangle^{-(\sigma+3)}}{\langle y\rangle^{-(\sigma+3)}} - 20\partial_y^2\omega \right).$$

We conclude by following the proof of the estimation of $\partial_y^2\omega$ in Proposition 5.3.1.

• We consider an integer $k \geq 4$ and we assume that the estimate (5.18) is true for $k - 1$ and we will prove it for k . We compute

$$\partial_t\partial_k\omega - \partial_y^{k+2}(\omega\omega) = 0.$$

By the Leibniz formula, we get

$$\partial_t\partial_y^k\omega - \sum_{\ell=0}^{k+2} \binom{k+2}{\ell} \partial_y^\ell\omega \partial_y^{k+2-\ell}\omega = 0.$$

We then rewrite

$$\partial_t\partial_y^k\omega - 2\omega\partial_y^{k+2}\omega - b_k\partial_y\omega\partial_y^{k+1}\omega = R_k$$

where $b_k = 2(k+2)$, $c_k = (k+2)(k+1)$ and

$$R_k = \sum_{\ell=3}^{k-1} \binom{k+2}{\ell} \partial_y^\ell\omega \partial_y^{k+2-\ell}\omega.$$

We set $U = \partial_y^k\omega e^{-At}$ and we derive

$$\partial_t U - 2\omega\partial_y^2 U - b_k\partial_y\omega\partial_y U + (A - c_k\partial_y^2\omega)U = R_k e^{-At}$$

We then define the elliptic differential operator

$$L_k = \partial_t - 2\omega\partial_y^2 - b_k\partial_y\omega\partial_y + (A - c_k\partial_y^2\omega).$$

We have from the assumption on the initial data

$$|\partial_y^k\omega_0| \leq c_1\langle y\rangle^{-(\sigma+k)}$$

and we set

$$\bar{U}(t, y) = C\langle y\rangle^{-(\sigma+k)}.$$

where $C \geq c_1$ is a constant which will be prescribed later. Therefore

$$L_k\bar{U} = \bar{U} \left(A - 2(\omega + \varepsilon) \frac{\partial_y^2\langle y\rangle^{-(\sigma+k)}}{\langle y\rangle^{-(\sigma+k)}} - b_k\partial_y\omega \frac{\partial_y\langle y\rangle^{-(\sigma+k)}}{\langle y\rangle^{-(\sigma+k)}} - c_k\partial_y^2\omega \right).$$

- We first choose the parameter A such that $L_k\bar{U} \geq 1\bar{U}$ and the coefficient $(A - c_k\partial_y^2\omega)$ of order 0 of L_k is positive :

$$A = 2(|\omega|_\infty + 1) \left| \frac{\partial_y^2\langle y\rangle^{-(\sigma+k)}}{\langle y\rangle^{-(\sigma+k)}} \right|_\infty + b_k|\partial_y\omega|_\infty \left| \frac{\partial_y\langle y\rangle^{-(\sigma+k)}}{\langle y\rangle^{-(\sigma+k)}} \right|_\infty + c_k|\partial_y^2\omega|_\infty.$$

- We then choose the parameter C : By the induction hypothesis, we have the estimates $|\partial_y^\ell \omega| \lesssim \langle y \rangle^{-(\sigma+\ell)}$ for $0 \leq \ell \leq k-1$. Then we compute

$$|R_k| \leq \sum_{\ell=3}^{k-1} \binom{k+2}{\ell} |\partial_y^\ell \omega| |\partial_y^{k+2-\ell} \omega| \lesssim \langle y \rangle^{-(2\sigma+k+2)}.$$

Since $2\sigma + k + 2 \geq \sigma + k$, there exists a constant c_2 such that

$$|R| \leq c_2 \langle y \rangle^{-(\sigma+k)}.$$

Then we set $C = \max(c_1, c_2)$. We then have

$$L_k \bar{U} \geq C \langle y \rangle^{-(\sigma+k)} \geq |R| \geq R e^{-At} = L_k U \quad \text{and} \quad \bar{U}|_{t=0} = C \langle y \rangle^{-(\sigma+k)} \geq U|_{t=0}.$$

We apply the maximum principle and the function \bar{U} is an upper solution and this concludes the proof of Proposition 5.3.2. \square

5.4 Uniform Estimates in the weighted Sobolev spaces $H^{s,\gamma}$

The objective of this section is to derive an uniform (in ε) weighted H^s estimate for ω . We recall

$$\|\omega\|_{H^{s,\gamma}}^2 := \sum_{k=0}^s \|\langle y \rangle^{\gamma+k} \partial_y^k \omega\|_{L^2(\mathbb{R})}^2 \quad (5.19)$$

We first prove the existence of a solution in the Sobolev space $H^{s,\gamma}$.

Proposition 5.4.1. *Let $s \geq 4$ be an integer, $\gamma \geq 1$, $\sigma > \gamma + 1$, $\delta \in (0, 1)$ and $\varepsilon \in (0, 1]$. If $\omega_0 \in H_{\sigma, 2\delta}^{s,\gamma} \cap H_{\sigma, 2\delta}^{s+4,\gamma}$ and verifies*

$$\sum_{k=3}^{s+4} \sup_{y \in \mathbb{R}} (\langle y \rangle^{\sigma+k} |\partial_y^k \omega_0(y)|) < \infty,$$

then there exist a maximal time of existence

$$T^* := T(s, \gamma, \sigma, \delta, \varepsilon, \|\omega_0\|_{H^{s+4}}) > 0$$

and a solution $\omega^\varepsilon \in C([0, T^*]; H^{s,\gamma}) \cap C^1([0, T^*]; H^{s+2,\gamma})$ to the system (5.5) and such that for $t \in [0, T^*[$ we have $\omega(t) \in H_{\sigma, \delta}^{s+4,\gamma}$.

Proof. From Propositions 5.2 there exists $T_\varepsilon > 0$ ($T_\varepsilon = T_\varepsilon(\|\omega_0\|_{C^{s+4+\alpha}})$) and a regular solution $\omega_\varepsilon \in C^{1, s+4+\alpha}([0, T_\varepsilon] \times \mathbb{R})$ of the regularized vorticity equation 5.5.

Then from Proposition 5.3.1 and 5.3.2, this solution verifies

$$\forall t \in [0, T_\varepsilon], \forall y \in \mathbb{R}, \quad \frac{\delta}{2} \langle y \rangle^{-\sigma} \leq \omega^\varepsilon(t, y) \quad \text{and} \quad \sum_{k=0}^2 \langle y \rangle^{\sigma+k} |\partial_y^k \omega(t, y)| \leq \frac{6}{\delta}$$

and

$$\sum_{\ell=3}^{s+4} \langle y \rangle^{\sigma+\ell} |\partial_y^\ell \omega^\varepsilon(t, y)| \leq 2C_s.$$

Since $\sigma > \gamma + 1$, then using the previous estimates and the definition of the norm (5.19) we prove that the solution $\omega^\varepsilon \in C([0, T_\varepsilon]; H_{\sigma, \delta}^{s,\gamma}) \cap C([0, T_\varepsilon]; H_{\sigma, \delta}^{s+4,\gamma})$. \square

According to Propostion 5.4.1, the maximal time of existence $T_{s,\gamma,\sigma,\delta,\varepsilon,\|\omega_0\|_{H^{s+4}}}$ of the solution ω^ε may depend on ε . The aim of the following Proposition is to remove the ε -dependence by deriving uniform in ε estimate for ω^ε .

Proposition 5.4.2. *Let $s \geq 3$ be an even integer, $\gamma \geq 1$, $\sigma > \gamma + 1$, $\delta \in (0,1)$ and $\varepsilon \in (0,1]$. If $\omega^\varepsilon \in C([0,T]; H_{\sigma,\delta}^{s+4,\gamma}) \cap C^1([0,T]; H^{s+2,\gamma})$ solve (5.5), then we have the following uniform (in ε) estimate :*

$$\forall t \in [0, \min(T, T_1)[, \quad \|\omega^\varepsilon(t)\|_{H^{s,\gamma}}^2 + 1 \leq \frac{\|\omega_0\|_{H^{s,\gamma}}^2 + 1}{\left(1 - \frac{1}{2}t C_{s,\gamma,\sigma,\delta} \sqrt{\|\omega_0\|_{H^{s,\gamma}}^2 + 1}\right)^2} \quad (5.20)$$

where $T_1 = \frac{2}{C_{s,\gamma,\sigma,\delta} \sqrt{\|\omega_0\|_{H^{s,\gamma}}^2 + 1}}$.

Proof. We claim that

$$\frac{d}{dt} \|\omega^\varepsilon\|_{H^{s,\gamma}}^2 \leq C_{s,\gamma,\sigma,\delta} \|\omega^\varepsilon\|_{H^{s,\gamma}}^3 + C_{s,\gamma,\sigma,\delta} \quad (5.21)$$

We prove below the previous estimate. If we suppose the estimate, therefore, from the comparison principle of ordinary differential equations,

$$\frac{d}{dt} (\|\omega^\varepsilon\|_{H^{s,\gamma}}^2 + 1) \leq C_{s,\gamma,\sigma,\delta} (\|\omega^\varepsilon\|_{H^{s,\gamma}}^2 + 1)^{\frac{3}{2}}$$

we get the estimate (5.20).

We now prove estimate (6.15). In a first step we give an estimate for each term $\|\langle y \rangle^{\gamma+k} \partial_y^k \omega\|_{L^2}$ for $0 \leq k \leq s$, and in a second step we sum the estimates.

1. Estimate of $\|\langle y \rangle^{\gamma+k} \partial_y^k \omega\|_{L^2}$ for $0 \leq k \leq s$. Differentiating the vorticity equation with respect to y k times, we obtain the evolution equation for $\partial_y^k \omega^\varepsilon$

$$\partial_y^k (\partial_t \omega) - 2\varepsilon \partial_y^{k+2} \omega - \partial_y^{k+2} (\omega \omega) = 0. \quad (5.22)$$

From the Leibnitz formula, we get

$$\partial_t (\partial_y^k \omega) - 2\varepsilon \partial_y^{k+2} \omega - 2\omega \partial_y^{k+2} \omega = R_k \quad \text{where} \quad R_k = \sum_{\ell=1}^{k+1} \binom{k+2}{\ell} \partial_y^\ell \omega \partial_y^{k+2-\ell} \omega.$$

Multiplying the previous relation by $\langle y \rangle^{2\gamma+2k} \partial_y^k \omega$, and then integrating over \mathbb{R}^+

$$\frac{1}{2} \frac{d}{dt} \|\langle y \rangle^{\gamma+k} \partial_y^k \omega\|_{L^2(\mathbb{T} \times \mathbb{R}^+)}^2 - 2\varepsilon \int \langle y \rangle^{2\gamma+2k} \partial_y^k \omega \partial_y^{k+2} \omega - 2 \int \langle y \rangle^{2\gamma+2k} \omega \partial_y^k \omega \partial_y^{k+2} \omega = \sum_{\ell=1}^{k+1} \tilde{R}_{k,\ell}$$

where

$$\tilde{R}_{k,\ell} = \binom{k+2}{\ell} \int \langle y \rangle^{2\gamma+2k} \partial_y^\ell \omega \partial_y^{k+2-\ell} \omega \partial_y^k \omega.$$

By integration by parts in the y -variable we have :

$$\begin{aligned} -2\varepsilon \int \langle y \rangle^{2\gamma+2k} \partial_y^k \omega \partial_y (\partial_y^{k+1} \omega) &= \varepsilon \|\langle y \rangle^{\gamma+k} \partial_y^{k+1} \omega\|_{L^2}^2 + A_{1,k} \\ -2 \int \langle y \rangle^{2\gamma+2k} \omega \partial_y^k \omega \partial_y (\partial_y^{k+1} \omega) &= 2 \|\langle y \rangle^{\gamma+k} \sqrt{\omega} \partial_y^{k+1} \omega\|_{L^2}^2 + A_{2,k} + A_{3,k} \end{aligned}$$

where

$$\begin{aligned} A_{1,k} &= 2\varepsilon \int \partial_y (\langle y \rangle^{2\gamma+2k}) \partial_y^k \omega \partial_y^{k+1} \omega \\ A_{2,k} &= 2 \int \partial_y (\langle y \rangle^{2\gamma+2k}) \omega \partial_y^k \omega \partial_y^{k+1} \omega \\ A_{3,k} &= 2 \int \langle y \rangle^{2\gamma+2k} \partial_y \omega \partial_y^k \omega \partial_y^{k+1} \omega. \end{aligned}$$

Then we have :

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\langle y \rangle^{\gamma+k} \partial_y^k \omega\|_{L^2}^2 + 2\varepsilon \|\langle y \rangle^{\gamma+k} \partial_y^{k+1} \omega\|_{L^2}^2 + 2 \|\langle y \rangle^{\gamma+k} \sqrt{\omega} \partial_y^{k+1} \omega\|_{L^2}^2 \\ & = -A_{1,k} - A_{2,k} - A_{3,k} + \sum_{\ell=1}^{k+1} \tilde{R}_{k,\ell} \end{aligned} \quad (5.23)$$

We now estimate each previous terms $A_{i,k}$ and \tilde{R}_ℓ .

- Estimate of the term $A_{1,k}$:

$$\begin{aligned} |A_{1,k}| & \leq 2\varepsilon \int \langle y \rangle^{2\gamma+2k} (|\partial_y(\langle y \rangle^{2\gamma+2k})/(\langle y \rangle^{2\gamma+2k})|_\infty |\partial_y^k \omega|) |\partial_y^{k+1} \omega| \\ & \leq \varepsilon \|\langle y \rangle^{(\gamma+k)} \partial_y^{k+1} \omega\|_{L^2}^2 + C_{s,\gamma} \|\omega\|_{H^{s,\gamma}}^2. \end{aligned}$$

- Estimate of the term $A_{2,k}$:

$$\begin{aligned} |A_{2,k}| & \leq 2 \int \langle y \rangle^{2\gamma+2k} (|\partial_y(\langle y \rangle^{2\gamma+2k})/(\langle y \rangle^{2\gamma+2k})|_\infty \sqrt{\omega} |\partial_y^k \omega|) \sqrt{\omega} |\partial_y^{k+1} \omega| \\ & \leq \frac{1}{2} \|\langle y \rangle^{(\gamma+k)} \sqrt{\omega} \partial_y^{k+1} \omega\|_{L^2}^2 + C_{s,\gamma,\sigma,\delta} \|\omega\|_{H^{s,\gamma}}^2. \end{aligned}$$

- Estimate of the term $A_{3,k} = \frac{2}{k+1} \tilde{R}_{1,k} = \frac{2}{k+1} \tilde{R}_{k+1,k}$:

$$\begin{aligned} |A_{3,k}| + |\tilde{R}_{1,k}| + |\tilde{R}_{k+1,k}| & \leq (2k+4) \int \langle y \rangle^{2\gamma+2k} (|\partial_y \omega / \omega|_\infty \sqrt{\omega} |\partial_y^k \omega|) \sqrt{\omega} |\partial_y^{k+1} \omega| \\ & \leq \frac{1}{2} \|\langle y \rangle^{(\gamma+k)} \sqrt{\omega} \partial_y^{k+1} \omega\|_{L^2}^2 + C_{s,\gamma,\sigma,\delta} \|\omega\|_{H^{s,\gamma}}^2. \end{aligned}$$

- Estimate of $\tilde{R}_{\ell,k}$ for $k \geq 2$ and $2 \leq \ell \leq k$. Since $\tilde{R}_{\ell,k} = \tilde{R}_{k+2-\ell,k}$, we can consider the case $2 \leq \ell \leq (k+2-\ell)$. Next we have $\ell \leq k/2+1 \leq s/2+1$. Since $s \geq 4$ we have $\ell \leq s-1$ therefore

$$|\partial_y^\ell \omega|_\infty \leq \|\omega\|_{H^s} \leq \|\omega\|_{H^{s,\gamma}}$$

and since $k+2-\ell \leq k \leq s$

$$\begin{aligned} |\tilde{R}_{\ell,k}| & \leq \binom{k+2}{\ell} \int \langle y \rangle^{2\gamma+2k} |\partial_y^\ell \omega|_\infty |\partial_y^{k+2-\ell} \omega| |\partial_y^k \omega| \\ & \leq C_{s,\gamma} \|\omega\|_{H^{s,\gamma}}^3 \end{aligned}$$

- Estimate of $\tilde{R}_{k,\ell}$ for $k=0,1$:

$$\begin{aligned} |\tilde{R}_{0,1}| & \leq 2 \int \langle y \rangle^{2\gamma} |\partial_y \omega| |\partial_y \omega| |\omega| \leq C_{s,\gamma} \|\omega\|_{H^{s,\gamma}}^3, \\ |\tilde{R}_{1,1}| = |\tilde{R}_{1,2}| & \leq 3 \int \langle y \rangle^{2\gamma+2} |\partial_y^2 \omega| |\partial_y \omega| |\partial_y \omega| \leq C_{s,\gamma} \|\omega\|_{H^{s,\gamma}}^3. \end{aligned}$$

By injecting the previous estimates of $A_{i,k}$ and $\tilde{R}_{k,\ell}$ in (5.23), we get

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\langle y \rangle^{\gamma+k} \partial_y^k \omega\|_{L^2(\mathbb{T} \times \mathbb{R}^+)}^2 + \varepsilon \|\langle y \rangle^{\gamma+k} \partial_y^{k+1} \omega\|_{L^2}^2 + \|\langle y \rangle^{\gamma+k} \sqrt{\omega} \partial_y^{k+1} \omega\|_{L^2}^2 \\ & \leq C_{s,\gamma,\sigma,\delta} \|\omega\|_{H^{s,\gamma}}^3 + C_{s,\gamma,\sigma,\delta}. \end{aligned}$$

- 2) Estimate of $\|\omega\|_{H^{s,\gamma}}^2$. We take the sum from $k=0$ to $k=s$ of the previous inequality, then we get

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\omega\|_{H^{s,\gamma}}^2 + \varepsilon \|\partial_y \omega\|_{H^{s,\gamma}}^2 + \|\sqrt{\omega} \partial_y \omega\|_{H^{s,\gamma}}^2 \\ & \leq C_{s,\gamma,\sigma,\delta} \|\omega\|_{H^{s,\gamma}}^3 + C_{s,\gamma,\sigma,\delta}. \end{aligned}$$

This concludes the proof of (6.15) and proposition 5.4.2. \square

5.5 Uniform life span of approximate solutions

According to propositions 5.4.1, a solutions ω^ε to the vorticity equations (5.5) exists up to a time $[0, T_{s,\gamma,\sigma,\delta,\omega_0}^* [$ which may depend on ε . However in proposition 5.4.2 we have already derived uniform in ε estimate for ω^ε , then one can apply the standard continuous induction argument to solve the vorticity (5.5) up to time interval that is independent of ε . Then we have the following Proposition :

Proposition 5.5.1. *Let $s \geq 4$ be an even integer, $\gamma \geq 1$, $\sigma > \gamma + 1$, $\delta \in (0, 1)$ and $\varepsilon \in (0, 1]$, $\omega_0 \in H_{\sigma,2\delta}^{s,\gamma}$. We suppose there exists a solution $\omega \in \mathcal{C}^{1,4}([0, T] \times \mathbb{R})$ and $\omega^\varepsilon \in C([0, T]; H_{\sigma,\delta}^{s+4,\gamma}) \cap C^1([0, T]; H^{s+2,\gamma})$ of the vorticity equation. Then there exists a uniform life span $T_0 := T_0(s, \gamma, \sigma, \delta, \|\omega_0\|_{H^{s,\gamma}}) > 0$, which is independent of ε such that with the following uniforme in ε : For any $\varepsilon \in [0, 1]$ and any $t \in [0, \min(T, T_0)]$, we have*

$$\|\omega^\varepsilon\|_{H^{s,\gamma}} \leq 2(\|\omega_0\|_{H^{s,\gamma}} + 1), \quad (5.24)$$

$$\sup_{y \in \mathbb{R}} \left(\sum_{k=0}^2 |\langle y \rangle^{\sigma+k} \partial_y^k \omega^\varepsilon(t)| \right) \leq \frac{6}{\delta}, \quad (5.25)$$

$$\inf_{y \in \mathbb{R}} \langle y \rangle^\sigma \omega^\varepsilon(t, y) \geq \frac{\delta}{2}. \quad (5.26)$$

Moreover, the solution ω can be extended up to the time T_0 .

Proof. The uniforme time of existence $T_0 := T_0(s, \gamma, \sigma, \delta, \|\omega_0\|_{H^{s,\gamma}})$ can be guaranteed by using the induction continuity argument and the uniform estimates (5.24)-(5.26). We now prove this estimates.

- From Proposition 5.4.2 (estimate (5.20)), we recall :

$$\forall t \in [0, \min(T, T_1)[, \quad \|\omega(t)\|_{H^{s,\gamma}}^2 + 1 \leq \frac{\|\omega_0\|_{H^{s,\gamma}}^2 + 1}{\left(1 - \frac{1}{2}t C_{s,\gamma,\sigma,\delta} \sqrt{\|\omega_0\|_{H^{s,\gamma}}^2 + 1}\right)^2}$$

where

$$T_1 = \frac{2}{C_{s,\gamma,\sigma,\delta} \sqrt{\|\omega_0\|_{H^{s,\gamma}}^2 + 1}}.$$

Therefore for $t \leq T_1/2$ we have

$$\|\omega(t)\|_{H^{s,\gamma}} \leq 2(\|\omega_0\|_{H^{s,\gamma}} + 1). \quad (5.27)$$

- From Proposition 5.3.1 there exists a positive constant C_σ such that for $t \leq T_1/2$

$$\forall y \in \mathbb{R}, \quad \delta \langle y \rangle^{-\sigma} e^{-C_\sigma A t} \leq \omega(t, y) \quad \text{and} \quad \sum_{k=0}^2 \langle y \rangle^{\sigma+k} |\partial_y^k \omega(t, y)| \leq \frac{3}{\delta} e^{C_\sigma A t} \quad (5.28)$$

where

$$A = 1 + \sup_{t \in [0, T_1/2]} \left(\sum_{k=0}^2 |\partial_y^k \omega(\cdot, \cdot)|_\infty \right).$$

- From the assumption on ω_0 , we have for $y \in \mathbb{R}$

$$\delta \leq \langle y \rangle^\sigma \omega_0(y) \quad \text{and} \quad \sum_{k=0}^2 \langle y \rangle^{\sigma+k} |\partial_y^k \omega_0(y)| \leq \frac{3}{\delta}$$

and we derive

$$A \leq 1 + a_0 \sup_{0 \leq t \leq T_1/2} \|\omega\|_{H^{s,\gamma}}$$

where a_0 is a constant such that $(\sum_{k=0}^2 |\partial_y^k \omega|_\infty) \leq a_0 \|\omega\|_{H^{3,\gamma}}$. Then applying (5.27) we get

$$A \leq 1 + 2a_0(\|\omega_0\|_{H^{s,\gamma}} + 1).$$

• Conclusion : We set

$$T_0 = \min \left\{ \frac{T_1}{2}, \frac{\ln 2}{C_\sigma(1 + 2a_0(\|\omega_0\|_{H^{s,\gamma}} + 1))} \right\}.$$

We obtain from (5.28) that for $t \leq T_0$,

$$\forall y \in \mathbb{R}, \quad \frac{\delta}{2} \langle y \rangle^{-\sigma} \leq \omega(t, y) \quad \text{and} \quad \sum_{k=0}^2 \langle y \rangle^{\sigma+k} |\partial_y^k \omega(t, y)| \leq \frac{6}{\delta}.$$

□

5.6 Proof of the main theorem

5.6.1 Convergence of the constructed approximate solutions

From Proposition 5.5.1, for $\varepsilon > 0$ there exist a solution ω^ε of (5.5) defined on a fixed interval $[0, T]$ (for $T = T_0$) and we have

$$\sup_{0 \leq t \leq T} \|\omega\|_{H^{s,\gamma}} \leq 2(\|\omega_0\|_{H^{s,\gamma}} + 1) < +\infty. \quad (5.29)$$

Furthermore, using evolution equations (5.5), uniform H^s bound (5.24), one also finds that $\partial_t \omega$ is uniformly (in ε) bounded in $L^\infty([0, T]; H^{s-2,\gamma})$. By the Lions-Aubin lemma and the compact embedding of $H^{s,\gamma}$ in $H_{loc}^{s'}$ stated in Lemma (5.6.1), we have after taking a subsequence, as $\varepsilon_k \rightarrow 0^+$,

$$\begin{cases} \omega^{\varepsilon_k} \xrightarrow{*} \omega & \text{in } L^\infty([0, T]; H^{s,\gamma}) \\ \omega^{\varepsilon_k} \rightarrow \omega & \text{in } C([0, T]; H_{loc}^{s'}) \end{cases} \quad (5.30)$$

for all $s' < s$, where $\omega = \partial_y u \in L^\infty([0, T]; H^{s,\gamma}) \cap \bigcap_{s < s'} C([0, T]; H_{loc}^{s'})$. Using the local uniform convergence of $\partial_x u^{\varepsilon_k}$, we also have the pointwise convergence. Combining (5.30), (5.29), one may justify the pointwise convergence of all terms in the regularized non Newtonian shear flow equations (5.5). Then, passing to the limit $\varepsilon_k \rightarrow 0^+$ in (5.5), we know that the limit u solve the non-Newtonian shear flow equation (5.1) in the classical sense. Lastly, in order to complete the proof of convergence, it remain to justify that $\omega \in L^\infty([0, T]; H_{\sigma,\delta}^{s,\gamma})$ and the matching condition (5.1)₄. Since $\partial_y^k \omega^{\varepsilon_k}$ converges pointwise to $\partial_y^k \omega$ for $k = 0, 1, 2$ and ω^{ε_k} satisfies

$$(1+y)^\sigma \omega^{\varepsilon_k} \geq \frac{\delta}{2} \quad \text{and} \quad \sum_{k=0}^2 |(1+y)^{\sigma+\alpha_2} \partial_y^k \omega^{\varepsilon_k}| \leq \frac{3}{\delta}, \quad (5.31)$$

we deduce that (5.31) still holds for ω , and hence $\omega \in L^\infty([0, T]; H_{\sigma,\delta}^{s,\gamma})$. Also, by Lebesgue's dominated convergence theorem,

$$\int_0^{+\infty} \omega dy = \lim_{\varepsilon_k \rightarrow 0^+} \int_0^{+\infty} \omega^{\varepsilon_k} dy = 1,$$

which is equivalent to the matching condition (5.1)₄.

Lemma 5.6.1. *Let s be a positive integer, $\gamma' \geq 0$, and $M < +\infty$. Assume*

$$\|f^\varepsilon\|_{H^{s,\gamma'}} \leq M \quad (5.32)$$

for all $\varepsilon \in (0, 1]$. Then there exist a function $f \in H^{s,\gamma'}$ and a sequence $\{\varepsilon_k\}_{k \in \mathbb{N}} \subseteq (0, 1]$ with $\lim_{k \rightarrow \infty} \varepsilon_k = 0^+$ such that as $\varepsilon_k \rightarrow 0^+$,

$$f^{\varepsilon_k} \xrightarrow{H^{s,\gamma'}} f \quad \text{and} \quad f^{\varepsilon_k} \xrightarrow{H_{loc}^{s'}} f \quad \text{for all } s' < s. \quad (5.33)$$

Proof. First of all, let us mention that $H^{s,\gamma'}$ has an inner product structure :

$$\langle \phi, \psi \rangle_{H^{s,\gamma'}} = \sum_{k=0}^s \int_0^{+\infty} (1+y)^{2\gamma'+2k} \partial_y^k \phi \partial_y^k \psi,$$

so the uniform bound (5.32) implies the weak convergence of f^{ε_k} in (5.33) via the Banach-Alaoglu theorem. Next, by the definition of $\|\cdot\|_{H^{s,\gamma'}}$,

$$\|f^\varepsilon\|_{H^s} \leq \|f^\varepsilon\|_{H^{s,\gamma'}} \leq M.$$

This implies the local $H^{s'}$ -norm convergence in (5.33) for all $s' < s$ because of the standard compactness of $H^s(\mathbb{R}^+)$. \square

5.6.2 Uniqueness of the solution to the non-newtonian shear flow equations

Proposition 5.6.1. *For any $s \geq 4, \gamma \geq 1, \sigma > \gamma + 1$, and $\delta \in]0, 1[$, let (u_i, v_i) solve the non-newtonian prandtl equations (6.1) with same initial data and the vorticity $\omega_i = \partial_y u_i \in C([0, T]; H_{\sigma, \delta}^{s,\gamma}) \cap C^1([0, T]; H^{s-2,\gamma})$ for $i = 1, 2$. Define $\tilde{u} = u_1 - u_2$. Then we have $\tilde{u} = 0$.*

Proof. One can check that \tilde{u} satisfies

$$\begin{cases} \partial_t \tilde{u} = 2\omega_1 \partial_y^2 \tilde{u} + \partial_y \omega_2 \partial_y \tilde{u} & \text{in } [0, T] \times \mathbb{R}^+ \\ \tilde{u}|_{t=0} = 0 & \text{on } \mathbb{T} \times \mathbb{R}^+ \\ \tilde{u}|_{y=0} = 0 & \text{on } [0, T] \\ \lim_{y \rightarrow +\infty} \tilde{u} = 0 & \text{for all } t \in [0, T], \end{cases}$$

Then applying the maximum principle, we get $\tilde{u} = 0$ which conclude the proof of the uniqueness. \square

5.7 Uniform estimates on the solution

In this section, we establish some uniform estimates in time and with respect to ε of the solutions of (5.5). We denote \mathcal{C}_0^k the space of function u of class \mathcal{C}^k such that $\partial_y^\ell u(y) \rightarrow 0$ uniformly as $|y| \rightarrow \infty$ for $0 \leq \ell \leq k$.

Proposition 5.7.1. *We consider a initial data $\omega_0 \in \mathcal{C}_0^2(\mathbb{R})$ such that $\omega_0 > 0$ and $\varepsilon \in [0, 1]$.*

1. *We suppose that there exists a solution $\omega \in \mathcal{C}_0^{1,2}([0, T] \times \mathbb{R})$ of equation (5.5). Then we have the following uniform estimates :*

$$\forall t \in [0, T], \forall y \in \mathbb{R}, \quad 0 < \omega(t, y) \leq |\omega_0|_\infty.$$

2. *We suppose that there exists a solution $\omega \in \mathcal{C}_0^{1,3}([0, T] \times \mathbb{R})$ of equation (5.5). Then we have the following uniform estimates :*

$$\forall t \in [0, T], \forall y \in \mathbb{R}, \quad |\partial_y \omega(t, y)| \leq |\partial_y \omega_0|_\infty.$$

Proof. We construct some upper and lower solutions.

1. We define the following parabolic differential operator

$$\mathcal{L}_0 = \partial_t - 2(\varepsilon + \omega) \partial_y^2 - 2(\partial_y \omega) \partial_y.$$

Then ω verifies $\mathcal{L}_0 \omega = 0$, and it is straight-ward to verify that $\underline{\omega}(t, y) \equiv 0$ is a lower solution and $\bar{\omega}(t, y) \equiv |\omega_0|_\infty$ is an upper solution :

$$\begin{aligned} \mathcal{L}_0 \underline{\omega} &\leq \mathcal{L}_0 \omega \leq \mathcal{L}_0 \bar{\omega} \quad \text{on } [0, T] \times \mathbb{R}, \\ \underline{\omega} &\leq \omega \leq \bar{\omega} \quad \text{for } t = 0 \text{ and } y \rightarrow \pm\infty. \end{aligned}$$

Therefore from Maximum Principle we have $\underline{\omega} \leq \omega \leq \bar{\omega}$ on $[0, T] \times \mathbb{R}$.

2. We define the following parabolic differential operator

$$\mathcal{L}_1 = \partial_t - 2(\varepsilon + \omega)\partial_y^2 - 6(\partial_y \omega)\partial_y.$$

Then it is also immediate to verify $\underline{\omega}(t, y) \equiv -|\partial_y \omega_0|_\infty$ is a lower solution and $\bar{\omega}(t, y) \equiv |\partial_y \omega_0|_\infty$ is an upper solution. □

We now derive some estimates in Sobolev spaces of the solution which are uniform in time and with respect to ε .

Proposition 5.7.2. *We consider a initial data $\omega_0 \in \mathcal{C}_0^2(\mathbb{R})$ such that*

$$c_0 \langle y \rangle^{-\sigma} \leq \omega_0(y) \leq c'_0 \langle y \rangle^{-\sigma} \quad \text{and} \quad |\partial_y \omega_0(y)| \leq c_1 \langle y \rangle^{-(\sigma+1)}$$

for some positive constants c_0, c'_0, c_1 and we suppose that there exists a solution ω of equation (5.5) such that $\omega \in \mathcal{C}^{1,3}([0, T] \times \mathbb{R})$ and $\omega \in \mathcal{C}^1([0, T], H^6)$.

Then we have the following uniform estimates : there exist some positive constants C_0, C'_0, a_0 which depend only on ω_0 such that for $\varepsilon \in [0, 1]$ and $t \in [0, T]$:

$$\frac{1}{2} \int |\omega(t, y)|^2 dy + \int_0^t \left(\int 2\omega(s, t) |\partial_y \omega(s, t)|^2 dy \right) ds \leq \frac{1}{2} \int |\omega_0(y)|^2 dy, \quad (5.34)$$

$$\frac{1}{2} \int |\partial_y \omega(t, y)|^2 dy + \int_0^t \left(\int 2\omega(s, t) |\partial_y^2 \omega(s, t)|^2 dy \right) ds \leq \frac{1}{2} \int |\partial_y \omega_0(y)|^2 dy, \quad (5.35)$$

$$\frac{1}{2} \int |\partial_y^2 \omega(t, y)|^2 dy + \int_0^t \left(\int \omega(s, t) |\partial_y^3 \omega(s, t)|^2 dy \right) ds \leq C_0 e^{2a_0 t}, \quad (5.36)$$

$$\frac{1}{2} \int |\partial_y^3 \omega(t, y)|^2 dy + \int_0^t \left(\int \omega(s, t) |\partial_y^4 \omega(s, t)|^2 dy \right) ds \leq C'_0 e^{4a_0 t}. \quad (5.37)$$

Proof. We first derive some energy-type estimates, and we use a bootstrap method.

• We first claim that we have the following inequalities :

$$\frac{1}{2} \frac{d}{dt} \int |\omega|^2 dy + \int 2\omega |\partial_y \omega|^2 dy \leq 0, \quad (5.38)$$

$$\frac{1}{2} \frac{d}{dt} \int |\partial_y \omega|^2 dy + \int 2\omega |\partial_y^2 \omega|^2 dy \leq 0, \quad (5.39)$$

$$\frac{1}{2} \frac{d}{dt} \int |\partial_y^2 \omega|^2 dy + \int 2\omega |\partial_y^3 \omega|^2 dy \leq - \int 6 \partial_y \omega \partial_y^2 \omega \partial_y^3 \omega dy, \quad (5.40)$$

$$\frac{1}{2} \frac{d}{dt} \int |\partial_y^3 \omega|^2 dy + \int 2\omega |\partial_y^4 \omega|^2 dy \leq - \int 32 \partial_y \omega \partial_y^3 \omega \partial_y^4 \omega dy. \quad (5.41)$$

Indeed, multiplying the first equation of (5.1), that is

$$\partial_t \omega - 2\varepsilon \partial_y^2 \omega - \partial_y(\omega^2) = 0$$

respectively by ω , $\partial_y^2\omega$, $\partial_y^4\omega$ and $\partial_y^6\omega$, and using some integration by parts, we get

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int |\omega|^2 dy + \int \partial_y(\omega^2) \partial_y \omega dy &\leq 0, \\ \frac{1}{2} \frac{d}{dt} \int |\partial_y \omega|^2 dy + \int \partial_y^2(\omega^2) \partial_y^2 \omega dy &\leq 0, \\ \frac{1}{2} \frac{d}{dt} \int |\partial_y^2 \omega|^2 dy + \int \partial_y^3(\omega^2) \partial_y^3 \omega dy &\leq 0, \\ \frac{1}{2} \frac{d}{dt} \int |\partial_y^3 \omega|^2 dy + \int \partial_y^4(\omega^2) \partial_y^4 \omega dy &\leq 0. \end{aligned}$$

We expand the second term of each previous equations and we get

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int |\omega|^2 dy + \int 2\omega |\partial_y \omega|^2 dy &\leq 0, \\ \frac{1}{2} \frac{d}{dt} \int |\partial_y \omega|^2 dy + \int 2\omega |\partial_y^2 \omega|^2 dy + \int 2(\partial_y \omega)^2 \partial_y^2 \omega dy &\leq 0, \\ \frac{1}{2} \frac{d}{dt} \int |\partial_y^2 \omega|^2 dy + \int 2\omega |\partial_y^3 \omega|^2 dy + \int 6\partial_y \omega \partial_y^2 \omega \partial_y^3 \omega dy &\leq 0, \\ \frac{1}{2} \frac{d}{dt} \int |\partial_y^3 \omega|^2 dy + \int 2\omega |\partial_y^4 \omega|^2 dy + \int 8\partial_y \omega \partial_y^3 \omega \partial_y^4 \omega dy + \int 6\partial_y^2 \omega \partial_y^2 \omega \partial_y^4 \omega dy &\leq 0. \end{aligned}$$

1. The first equation of the previous system is directly (5.38).
2. We compute $\int 2(\partial_y \omega)^2 \partial_y^2 \omega dy = \int \frac{2}{3} \partial_y((\partial_y \omega)^3) dy = 0$, and we get (5.39).
3. The third equation of the previous system gives directly (5.40).
4. We integrate by part

$$\int 6(\partial_y^2 \omega)^2 \partial_y^4 \omega dy = - \int 12 \partial_y^2 \omega (\partial_y^3 \omega)^2 dy = \int 24 \partial_y \omega \partial_y^3 \omega \partial_y^4 \omega dy,$$

and we derive (5.41).

- Proof of the estimates (5.34) and (5.35). We integrate (5.38) and (5.39) with respect to the variable t and we get directly these estimates.
- Upper and lower solution argument. From proposition 5.3.1, there exists some constants a_0 and b_0 which depends only on ω_0 such that

$$\forall (t, y) \in [0, T] \times \mathbb{R}, \quad |\partial_y \omega(t, y)| \leq b_0 e^{a_0 t} \omega(t, y).$$

Injecting this estimate in equations (5.40)-(5.41) and using the Young's inequality for products, we derive

$$\frac{1}{2} \frac{d}{dt} \int |\partial_y^2 \omega|^2 dy + \int 2\omega |\partial_y^3 \omega|^2 dy \leq b'_0 e^{2a_0 t} \int \omega |\partial_y^2 \omega|^2 dy + \int \omega |\partial_y^3 \omega|^2 dy, \quad (5.42)$$

$$\frac{1}{2} \frac{d}{dt} \int |\partial_y^3 \omega|^2 dy + \int 2\omega |\partial_y^4 \omega|^2 dy \leq b''_0 e^{2a_0 t} \int \omega |\partial_y^3 \omega|^2 dy + \int \omega |\partial_y^4 \omega|^2 dy, \quad (5.43)$$

where $b'_0 = \frac{1}{4}(6b_0)^2$ and $b''_0 = \frac{1}{4}(32b_0)^2$.

- Proof of the estimate (5.36). We integrate the estimate (5.42) with respect to the variable t , and then using estimate (5.35), we derive

$$\begin{aligned} \frac{1}{2} \int |\partial_y^2 \omega(t, y)|^2 dy + \int_0^t \left(\int \omega(s, t) |\partial_y^3 \omega(s, t)|^2 dy \right) ds \\ \leq \frac{1}{2} \int |\partial_y^2 \omega_0(y)|^2 dy + \int_0^t \left(b'_0 e^{2a_0 s} \int \omega(s, y) |\partial_y^2 \omega(s, y)|^2 dy \right) ds, \\ \leq \frac{1}{2} \int |\partial_y^2 \omega_0(y)|^2 dy + b'_0 e^{2a_0 t} \left(\frac{1}{4} \int |\partial_y \omega_0(y)|^2 dy \right) \\ \leq C_0 e^{2a_0 t} \end{aligned}$$

for some constant C_0 which depends only on ω_0 .

• Proof of the estimate (5.37). We integrate the estimate (5.43) with respect to the variable t , and using estimate (5.36), we get

$$\begin{aligned} & \frac{1}{2} \int |\partial_y^3 \omega(t, y)|^2 dy + \int_0^t \left(\int \omega(s, t) |\partial_y^4 \omega(s, t)|^2 dy \right) ds \\ & \leq \frac{1}{2} \int |\partial_y^3 \omega_0(y)|^2 dy + \int_0^t \left(b_0'' e^{2a_0 s} \int \omega(s, y) |\partial_y^3 \omega(s, y)|^2 dy \right) ds \\ & \leq \frac{1}{2} \int |\partial_y^3 \omega_0(y)|^2 dy + b_0'' e^{2a_0 t} \left(\int_0^t C_0 e^{2a_0 s} ds \right) \\ & \leq C_0' e^{4a_0 t} \end{aligned}$$

for some constant C_0' which depends only on ω_0 . This concludes the proof of proposition 5.7.2. \square

As a consequences, we derive some uniform estimates in L^∞ of the terms ω , $\partial_y \omega$ and $\partial_y^2 \omega$.

Corollary 2. *Under the hypothesis of the previous propositions 6.1.1 we have the following estimates: There exists a constant C_0'' which depends only on ω_0 such that for all $t \in [0, T]$ and $y \in \mathbb{R}$ we have :*

$$0 < \omega(t, y) \leq |\omega_0|_\infty, \quad |\partial_y \omega(t, y)| \leq |\partial_y \omega_0|_\infty, \quad |\partial_y^2 \omega(t, y)| \leq C_0'' e^{4a_0 t}.$$

Remark 7. *Under the assumptions of the previous corollary, we could check that the solution is global in time.*

5.8 Appendix : Parabolic Hölder spaces

We present below some classical results on the Chauchy problem for linear parabolic equations in parabolic Hölder spaces (see for instance [56]).

Let Ω be either \mathbb{R}^n or an open set of \mathbb{R}^n with uniformly C^2 boundary. We denote for $x \in \Omega$ the partial derivatives of a real defined function $u(x)$ as $\partial_i u = \partial_{x_i} u$, $\partial_{ij} u = \partial_{x_i} \partial_{x_j} u$ for all $1 \leq i, j \leq n$. For a multi index $\gamma \in \mathbb{N}$ we denote also $D^\gamma = \partial_1^{\gamma_1} \dots \partial_n^{\gamma_n}$.

We note $C^k(\bar{\Omega})$ the set of all real valued functions u which are continuous and bounded together with all their derivatives up to the order k and $UC(\bar{\Omega})$ the set of uniformly bounded continuous real valued functions.

We recall the definition of the Hölder space : for $0 < \alpha < 1$ the Banach space $C^\alpha(\bar{\Omega})$ is the space of all functions continuous and bounded u for which where

$$|u|_{C^\alpha(\bar{\Omega})} = \|u\|_\infty + [u]_{\alpha; \bar{\Omega}} < \infty$$

where

$$\|u\|_\infty = \sup_{x \in \bar{\Omega}} |u(x)| \quad \text{and} \quad [u]_{\alpha; \bar{\Omega}} = \sup_{\substack{x_1 \neq x_2 \\ x_1, x_2 \in \bar{\Omega}}} \frac{|u(x_1) - u(x_2)|}{|x_1 - x_2|^\alpha}.$$

If $k \geq 0$ we define in the same way the Hölder space $C^{k+\alpha}(\bar{\Omega})$ the set of all functions u which are continuous and bounded together with all their derivatives up to the order k and such that the norm

$$|u|_{C^{k+\alpha}(\bar{\Omega})} = |u|_{C^k(\bar{\Omega})} + [u]_{k+\alpha; \bar{\Omega}} < \infty$$

where

$$|u|_{C^k(\bar{\Omega})} = \sum_{|\gamma| \leq k} \|D^\gamma u\|_\infty \quad \text{and} \quad [u]_{k+\alpha; \bar{\Omega}} = \sum_{|\gamma|=k} [D^\gamma u]_{\alpha; \bar{\Omega}}.$$

We consider $I \subset \mathbb{R}$ an interval and $\Omega \subset \mathbb{R}^n$ as before and $Q = I \times \Omega \subset \mathbb{R}^{n+1}$ and we define the following anisotropic Hölder spaces of functions defined in Q . For $0 < \alpha, \beta < 1$ and $k \in \mathbb{N}$ be fixed. In the definitions which follow, u will be a bounded continuous real valued functions $u(t, x)$ defined on Q and we drop the symbol of the domains Ω or Q in the norm if the context is clear.

- $C^{\alpha,0}(I \times \bar{\Omega})$ is the Banach space of all bounded continuous functions $u(t, x)$ such that

$$|u|_{C^{\alpha,0}} = \sup_{x \in \bar{\Omega}} |u(\cdot, x)|_{C^\alpha(I)} < \infty.$$

- $C^{0,\beta}(I \times \bar{\Omega})$ is the Banach space of all bounded continuous functions $u(t, x)$ such that

$$|u|_{C^{0,\beta}} = \sup_{t \in I} |u(t, \cdot)|_{C^\alpha(\bar{\Omega})} < \infty.$$

- $C^{\alpha,\beta}(I \times \bar{\Omega})$ is the Banach space of all bounded continuous functions $u(t, x)$ such that

$$|u|_{C^{\alpha,\beta}} = |u|_{C^{\alpha,0}} + |u|_{C^{0,\beta}} < \infty.$$

We also recall the definition of the parabolic Hölder spaces. First we define in \mathbb{R}^{n+1} the parabolic distance between the points $z_1 = (t_1, x_1)$, $z_2 = (t_2, x_2)$ as

$$\rho(z_1, z_2) = |t_1 - t_2|^{\frac{1}{2}} + |x_1 - x_2|.$$

If u is a function in a domain Q , we denote

$$[u]_{\alpha/2,\alpha} = \sup_{\substack{z_1 \neq z_2 \\ z_1, z_2 \in Q}} \frac{|u(z_1) - u(z_2)|}{(\rho(z_1, z_2))^\alpha}.$$

We define the classical Hölder parabolic spaces. For $0 < \alpha < 2$ we set

$$\begin{aligned} C^{\alpha/2,\alpha}(\bar{Q}) &= C^{\alpha/2,0}(\bar{Q}) \cap C^{0,\alpha}(\bar{Q}), \\ |u|_{C^{\alpha/2,\alpha}} &= |u|_{C^{\alpha/2,0}} + |u|_{C^{0,\alpha}} \end{aligned}$$

(an equivalent norm is $\|u\|_\infty + [u]_{\alpha/2,\alpha}$) and

$$\begin{aligned} C^{1+\alpha/2,2+\alpha}(\bar{Q}) &= \left\{ u \in C^{1,2}(\bar{Q}) \mid \partial_t u, \partial_{ij} u \in C^{\alpha/2,\alpha}(\bar{Q}) \forall i, j \right\}, \\ |u|_{C^{1+\alpha/2,2+\alpha}} &= \|u\|_\infty + \sum_{i=1}^n \|\partial_i u\|_\infty + |\partial_t u|_{C^{\alpha/2,\alpha}} + \sum_{i,j=1}^n |\partial_{ij} u|_{C^{\alpha/2,\alpha}}. \end{aligned}$$

We will use the following lemma (see for example Lemma 5.1.1 in Lunardi [56]) :

Lemma 5.8.1. *Let $0 < \alpha < 1$ and u belongs to $C^{1,2+\alpha}([a, b] \times \bar{\Omega})$, then the function $\tilde{u}(t) = u(t, \cdot)$ belongs to $C^{1/2}([a, b]; C^{1+\alpha}(\bar{\Omega}))$ and to $Lip([a, b]; C^\alpha(\bar{\Omega}))$. Moreover, the derivatives $\partial_{ij} u$ belongs to $C^{\alpha/2,0}([a, b] \times \bar{\Omega})$, the derivatives $\partial_i u$ belongs to $C^{(1+\alpha)/2,0}([a, b] \times \bar{\Omega})$ and there exist some positive constants $c_{i,\alpha}$ (depending only on $(b-a), n, \alpha$) such that*

$$\begin{aligned} |u|_{C^{1/2}([a,b]; C^{1+\alpha}(\bar{\Omega}))} + |u|_{Lip([a,b]; C^\alpha(\bar{\Omega}))} &\leq c_{0,\alpha} |u|_{C^{1,2+\alpha}([a,b] \times \bar{\Omega})} \\ \sum_{i=1}^n |\partial_i u|_{C^{(1+\alpha)/2,0}([a,b] \times \bar{\Omega})} &\leq c_{1,\alpha} |u|_{C^{1,2+\alpha}([a,b] \times \bar{\Omega})} \\ \sum_{i,j=1}^n |\partial_{ij} u|_{C^{\alpha/2,0}([a,b] \times \bar{\Omega})} &\leq c_{2,\alpha} |u|_{C^{1,2+\alpha}([a,b] \times \bar{\Omega})}. \end{aligned}$$

We consider the general parabolic equation in $Q = [0, T] \times \mathbb{R}^n$

$$\begin{cases} \partial_t u = Lu + f(t, \cdot) \\ u|_{t=0} = u_0 \end{cases} \quad (5.44)$$

where

$$Lu(x) = \sum_{i,j} a_{ij}(x) \partial_{ij} u + \sum_i b_i(x) \partial_i u + c(x) u.$$

We give the classical result in the Hölder spaces (see for example Theorem 5.1.4 in Lunardi [56]).

Theorem 5.3. *We suppose that the coefficients a_{ij} , b_i , c belong to $C^\alpha(\mathbb{R}^n)$, and there exists a constant $\kappa > 0$ and $K > 0$ such that $|a_{ij}|_\alpha$, $|b_i|_\alpha$, $|c|_\alpha \leq K$ and*

$$\sum_{i,j} a^{ij} \xi^i \xi^j \geq \kappa |\xi|^2, \quad \forall \xi \in \mathbb{R}^n. \quad (5.45)$$

In addition we assume that $u_0 \in C^{2+\alpha}(\mathbb{R}^n)$. For $T > 0$ we note $Q = [0, T] \times \mathbb{R}^n$.

• *Let $f \in C^{\alpha/2, \alpha}(Q)$. Then there exists a unique function $u \in C^{1+\alpha/2, 2+\alpha}(Q)$ such that u is a solution of the Cauchy problem (5.44) in Q . Moreover, there exists a constant c depending only on n, α, κ, K, T such that*

$$|u|_{1+\alpha/2, 2+\alpha} \leq c (|f|_{\alpha/2, \alpha} + |u_0|_{C^{2+\alpha}}).$$

• *Let $f \in C^{0, \alpha}(Q) \cap UC(Q)$. Then there exists a unique function $u \in C^{1, 2+\alpha}(Q)$ such that u is a solution of the Cauchy problem (5.44) in Q . Moreover, there exists a constant c depending only on n, α, κ, K, T such that*

$$|u|_{C^{1, 2+\alpha}} \leq c (|f|_{C^{0, \alpha}} + |u_0|_{C^{2+\alpha}}).$$

Chapter 6

Non-newtonian Prandtl equation in two dimension

6.1 Introduction

Throughout this paper, we consider the special case $n = 2$ and the periodic domain

$$\mathbb{T} \times \mathbb{R}^+ = \{(x, y); \quad x \in \mathbb{R}/\mathbb{Z}, \quad 0 \leq y < +\infty\}.$$

There fore we are concerned with the two-dimensional equations in this domain and the problem (??) become

$$\begin{cases} \partial_t u + u \partial_x u + v \partial_y u + \partial_x p = 2 |\partial_y u| \partial_y^2 u & \text{in } [0, T] \times \mathbb{T} \times \mathbb{R}^+ \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 & \text{in } [0, T] \times \mathbb{T} \times \mathbb{R}^+ \\ u|_{t=0} = u_0 & \text{on } \mathbb{T} \times \mathbb{R}^+ \\ u|_{y=0} = 0 & \text{on } [0, T] \times \mathbb{T} \\ v|_{y=0} = 0 & \text{on } [0, T] \times \mathbb{T} \\ \lim_{y \rightarrow +\infty} u = U & \text{for all } (t, x) \in [0, T] \times \mathbb{T}, \end{cases} \quad (6.1)$$

where the velocity field $(u, v)(t, x, y)$ is an unknown, the initial data $u_0(x, y)$ and the outer flow $U(t, x)$ are given and satisfy the compatibility conditions :

$$u_0|_{y=0} = 0 \quad \text{and} \quad \lim_{y \rightarrow +\infty} u_0 = U|_{t=0}. \quad (6.2)$$

Furthermore, the given scalar pressure $p(t, x)$ and the outer flow $U(t, x)$ satisfy the well-known Bernoulli's law :

$$\partial_t U + U \partial_x U = -\partial_x p. \quad (6.3)$$

In this work, we will consider the Oleinik's monotonicity assumption :

$$\omega := \partial_y u > 0. \quad (6.4)$$

Under this hypothesis, one must further assume $U > 0$.

$$\begin{cases} \partial_t u + u \partial_x u + v \partial_y u = 2 |\partial_y u| \partial_y^2 u & \text{in } [0, T] \times \mathbb{T} \times \mathbb{R}^+ \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 & \text{in } [0, T] \times \mathbb{T} \times \mathbb{R}^+ \\ u|_{t=0} = u_0 & \text{on } \mathbb{T} \times \mathbb{R}^+ \\ u|_{y=0} = 0 & \text{on } [0, T] \times \mathbb{T} \\ v|_{y=0} = 0 & \text{on } [0, T] \times \mathbb{T} \\ \lim_{y \rightarrow +\infty} u = 1 & \text{for all } (t, x) \in [0, T] \times \mathbb{T}, \end{cases} \quad (6.5)$$

We now introduce some function spaces. For $s \in \mathbb{N}$ and $\gamma \geq 1$, we pose

$$H^{s,\gamma} := \{\omega : \mathbb{T} \times \mathbb{R}^+ \rightarrow \mathbb{R}; \|\omega\|_{H^{s,\gamma}} < +\infty\}$$

where the weighted H^s norm $\|\cdot\|_{H^{s,\gamma}}$ is defined by $(\alpha = (\alpha_1, \alpha_2))$, $D^\alpha := \partial_x^{\alpha_1} \partial_y^{\alpha_2}$

$$\|\omega\|_{H^{s,\gamma}}^2 := \sum_{|\alpha| \leq s} \|(1+y)^{\gamma+\alpha_2} D^\alpha \omega\|_{L^2(\mathbb{T} \times \mathbb{R}^+)}^2. \quad (6.6)$$

We define the space $H_{\sigma,\delta}^{s,\gamma}$ for ω by

$$H_{\sigma,\delta}^{s,\gamma} := \left\{ \omega \in H^{s,\gamma} \mid \forall x \in \mathbb{T}, \quad \forall y > 0, \quad (1+y)^\sigma \omega \geq \delta \right. \\ \left. \text{and} \quad \sum_{|\alpha| \leq 2} |(1+y)^{\sigma+\alpha_2} D^\alpha \omega|^2 \leq \frac{1}{\delta^2} \right\}$$

where $s \geq 4, \gamma \geq 1, \sigma > \gamma + \frac{1}{2}, \delta \in (0, 1)$. Here, the main idea is adding an extra weight for each y -derivative. This corresponds to the weight $\frac{1}{y}$ in the Hardy type inequality. Judging from nonlinear cancellations, the weighted norm (6.6) is not suitable for estimating solutions of the equations (6.1). Thus, we introduce another weighted pseudo-norm for the vorticity ω , namely

$$\|\omega\|_{H_g^s}^2 := \|(1+y)^\gamma g_s\|_{L^2(\mathbb{T} \times \mathbb{R}^+)}^2 + \sum_{\substack{|\alpha| \leq s \\ \alpha_1 \leq s-1}} \|(1+y)^{\gamma+\alpha_2} D^\alpha \omega\|_{L^2(\mathbb{T} \times \mathbb{R}^+)}^2 \quad (6.7)$$

where

$$g_s := \partial_x^s \omega - \frac{\partial_y \omega}{\omega} \partial_x^s (u - U), \quad \lim_{y \rightarrow \infty} u = 1 \quad \text{and} \quad u(t, x, y) := \int_0^y \omega(t, x, \tilde{y}) d\tilde{y} \quad (6.8)$$

provided that $\omega := \partial_y u > 0$. The difference between norms (6.6) and (6.7) is that we replace the weighted L^2 norm of $\partial_x^s \omega$ by that of g_s , which is a better quantity because g_s can avoid the loss of x -derivative. The first important observation is that as long as $\omega \in H_{\sigma,\delta}^{s,\gamma}$, we can show that the new weighted norm (6.7) is almost equivalent to the weighted $H^{s,\gamma}$ norm (6.6), that is,

$$\|\omega\|_{H_g^s} \lesssim \|\omega\|_{H^{s,\gamma}} + \|u - U\|_{H^{s,\gamma-1}} \lesssim \|\omega\|_{H_g^s} + \|\partial_x^s U\|_{L^2} \quad (6.9)$$

provided that $\omega = \partial_y u$, $u|_{y=0} = 0$ and $\lim_{y \rightarrow +\infty} u = 1$.

The main purpose of this section is to describe the approximate systems that we use to prove existence. Since our a priori weighted H^s estimates are somewhat more nonlinear than usual, the approximate scheme is slightly more complicated.

We denote u, v, ω, p instead $u^\varepsilon, v^\varepsilon, \omega^\varepsilon, p^\varepsilon$. Our approximate scheme has three different levels and will be explained as follows. The first approximation of (6.5) is the regularized generalized Prandtl equations : for any $\varepsilon > 0$,

$$\begin{cases} \partial_t u^\varepsilon + u^\varepsilon \partial_x u^\varepsilon + v^\varepsilon \partial_y u^\varepsilon = \varepsilon^2 \partial_x^2 u^\varepsilon + 2 |\partial_y u^\varepsilon| \partial_y^2 u^\varepsilon & \text{in } [0, T] \times \mathbb{T} \times \mathbb{R}^+ \\ \partial_x u^\varepsilon + \partial_y v^\varepsilon = 0 & \text{in } [0, T] \times \mathbb{T} \times \mathbb{R}^+ \\ u^\varepsilon|_{t=0} = u_0(x, y) & \text{on } \mathbb{T} \times \mathbb{R}^+ \\ u^\varepsilon|_{y=0} = 0 & \text{on } [0, T] \times \mathbb{T} \\ v^\varepsilon|_{y=0} = 0 & \text{on } [0, T] \times \mathbb{T} \\ \lim_{y \rightarrow +\infty} u^\varepsilon = 1 & \text{for all } (t, x) \in [0, T] \times \mathbb{T}, \end{cases} \quad (6.10)$$

Or equivalently, the regularized vorticity $\omega^\varepsilon := \partial_y u^\varepsilon$ satisfies the following regularized vorticity system : for any $\varepsilon > 0$

$$\partial_t \omega^\varepsilon + u^\varepsilon \partial_x \omega^\varepsilon + v^\varepsilon \partial_y \omega^\varepsilon = \varepsilon^2 \partial_x^2 \omega^\varepsilon + 2 \omega^\varepsilon \partial_y^2 \omega^\varepsilon + 2 (\partial_y \omega^\varepsilon)^2 \quad \text{in } [0, T] \times \mathbb{T} \times \mathbb{R}^+ \quad (6.11)$$

with the boundary condition

$$2\omega^\varepsilon \partial_y \omega^\varepsilon|_{y=0} = 0 \quad \text{on} \quad [0, T] \times \mathbb{T} \quad (6.12)$$

and the initial condition

$$\omega^\varepsilon|_{t=0} = \omega_0(x, y) \quad \text{on} \quad \mathbb{T} \times \mathbb{R}^+. \quad (6.13)$$

where the velocity field (u, v) is given by

$$\begin{cases} u^\varepsilon(t, x, y) := 1 - \int_y^{+\infty} \omega^\varepsilon(t, x, \tilde{y}) d\tilde{y} \\ v^\varepsilon(t, x, y) := - \int_0^y \partial_x u^\varepsilon(t, x, \tilde{y}) d\tilde{y} \end{cases} \quad (6.14)$$

The objective of this section is to derive an uniform (in ε) weighted H^s estimate for ω . This estimate, include weighted L^2 estimates for $D^\alpha \omega$ for $|\alpha| \leq s$ and $\alpha_1 \leq s-1$.

For the case $\alpha_1 = s$, we do not estimate directly ω : we have to use a weighted L^2 estimate of the cancellation function g_s which is defined (see (6.8)) by

$$g_s := \partial_x^s \omega - \frac{\partial_y \omega}{\omega} \partial_x^s (u - U).$$

We will combine these two estimates to obtain an uniform weighted energy estimate. of the pseudo-norm (see (6.7)) $\|\omega\|_{H_g^{s,\gamma}}^2$

$$\|\omega\|_{H_g^{s,\gamma}}^2 := \|(1+y)^\gamma g_s\|_{L^2(\mathbb{T} \times \mathbb{R}^+)}^2 + \sum_{\substack{|\alpha| \leq s \\ \alpha_1 \leq s-1}} \|(1+y)^{\gamma+\alpha_2} D^\alpha \omega\|_{L^2(\mathbb{T} \times \mathbb{R}^+)}^2$$

Theorem 6.1. *Let $s \geq 6$ be an even integer, $\gamma \geq 1$, $\sigma > \gamma + \frac{1}{2}$, $\delta \in (0, 1)$, and $\varepsilon \in (0, 1]$. We suppose there exists a solution $\omega^\varepsilon \in C([0, T_\varepsilon]; H_{\sigma,\delta}^{s+4,\gamma} \cap C^1([0, T_\varepsilon]; H^{s+2,\gamma}))$ of (6.11)-(6.14).*

Therefore we have the following uniform with respect to ε estimate : There exists some constants $M_s, C_{s,\gamma,\sigma,\delta}, C_{s,\sigma,\delta}, C_s$ such that for $t \in [0, T_\varepsilon]$

$$\begin{aligned} \|\omega^\varepsilon(t)\|_{H_g^{s,\gamma}} &\leq \left\{ \|\omega_0\|_{H_g^{s,\gamma}}^2 + \int_0^t F(\tau) d\tau \right\}^{\frac{1}{2}} \\ &\times \left\{ 1 - (2M_s - 1) C_{s,\gamma,\sigma,\delta} \left(\|\omega_0\|_{H_g^{s,\gamma}}^2 + \int_0^t F(\tau) d\tau \right)^{2M_s-1} t \right\}^{-\frac{1}{4M_s-2}} \end{aligned}$$

where

$$F(t) = C_{s,\gamma,\sigma,\delta} \{1 + \|\partial_x^{s+1} U(t)\|_{L^\infty}^4\} + C_{s,\sigma,\delta} \sum_{r=0}^{\frac{s}{2}} \|\partial_t^r \partial_x^{s+1-2r} p(t)\|_{L^\infty(\mathbb{T})}^4.$$

The proof of this theorem is based on the following proposition:

Proposition 6.1.1. *Under the assumptions and notations of Theorem 6.1, we have for $t \in [0, T_\varepsilon]$*

$$\begin{aligned} \frac{d}{dt} \|\omega^\varepsilon(t)\|_{H_g^{s,\gamma}}^2 &\leq C_{s,\gamma,\sigma,\delta} \|\omega^\varepsilon(t)\|_{H_g^{s,\gamma}}^{4M_s} + C_{s,\gamma,\sigma,\delta} \{1 + \|\partial_x^{s+1} U(t)\|_{L^\infty}^4\} \\ &+ C_s \sum_{r=0}^{\frac{s}{2}} \|\partial_t^r \partial_x^{s+1-2r} p(t)\|_{L^\infty(\mathbb{T})}^4. \end{aligned} \quad (6.15)$$

Sections S1, S2, S3 are devoted to the proof of this proposition.

Proof of Theorem 6.1. From Proposition 6.1.1, there exists some constants $C_s, C_{s,\gamma,\sigma,\delta}, M_s$ such that :

$$\begin{aligned} \frac{d}{dt} \|\omega\|_{H_g^{s,\gamma}}^2 &\leq C_{s,\gamma,\sigma,\delta} \|\omega\|_{H_g^{s,\gamma}}^{4M_s} + C_{s,\gamma,\sigma,\delta} \{1 + \|\partial_x^{s+1} U\|_{L^\infty}^4\} \\ &\quad + C_s \sum_{r=0}^{\frac{s}{2}} \|\partial_t^r \partial_x^{s+1-2r} p\|_{L^\infty(\mathbb{T})}^4. \end{aligned}$$

Therefore, from the comparison principle of ordinary differential equations, we get

$$\begin{aligned} \|\omega\|_{H_g^{s,\gamma}}^2 &\leq \left\{ \|\omega_0\|_{H_g^{s,\gamma}}^2 + \int_0^t F(\tau) d\tau \right\} \\ &\quad \times \left\{ 1 - \left(\frac{4M_s}{2} - 1 \right) C_{s,\gamma,\sigma,\delta} \left(\|\omega_0\|_{H_g^{s,\gamma}}^2 + \int_0^t F(\tau) d\tau \right)^{\frac{4M_s-2}{2}} t \right\}^{-\frac{2}{4M_s-2}} \end{aligned}$$

where

$$F = C_{s,\gamma,\sigma,\delta} \{1 + \|\partial_x^{s+1} U\|_{L^\infty}^4\} + C_{s,\sigma,\delta} \sum_{r=0}^{\frac{s}{2}} \|\partial_t^r \partial_x^{s+1-2r} p\|_{L^\infty(\mathbb{T})}^4.$$

This conclude the proof of the Theorem. □

The rest of this part is devoted to the proof of Proposition 6.1.1, which is based on propositions (prop estim oo) + (prop estim g). Some very long and technical sections are devoted to the proof of this two propositions (prop estim oo) and (prop estim g).

6.2 Uniform estimates in time

In this section, we prove some uniform estimates in time of the solutions of (6.10).

Proposition 6.2.1. *We consider an initial data $\omega_0 \in H_{\sigma, 2\delta}^{s, \gamma}$. Then there exists a positive constant T (which not depends on ε) such for all ω^ε solution of (6.11)-(6.14), $\omega^\varepsilon \in C([0, T_\varepsilon]; H_{\sigma, \delta}^{s+4, \gamma} \cap C^1([0, T_\varepsilon]; H^{s+2, \gamma}))$, then we have the uniform estimates with respect to ε : For all $t \in [0, \min(T, T_\varepsilon)]$ we have $\omega^\varepsilon(t) \in H_{\sigma, \delta}^{s, \gamma}$ and*

$$\begin{aligned} \|\omega^\varepsilon(t)\|_{H_{\sigma, \delta}^{s, \gamma}} &\leq 4\|\omega_0\|_{H_{\sigma, \delta}^{s, \gamma}}, \\ \max_{(x, y) \in \mathbb{T} \times \mathbb{R}^+} \sum_{|\alpha| \leq 2} |(1+y)^{\sigma+\alpha_2} D^\alpha \omega^\varepsilon(t, x, y)|^2 &\leq \frac{1}{\delta^2}, \\ \min_{(x, y) \in \mathbb{T} \times \mathbb{R}^+} (1+y)^\sigma \omega^\varepsilon(t, x, y) &\geq \delta. \end{aligned}$$

Remark 8. *From this proposition and using classical arguments of continuity, we can solve the system up to the time T .*

Proof. We consider the term F defined in Theorem 6.1. Then we can prove that

$$\|F\|_{L^\infty} \leq C_{s, \gamma, \sigma, \delta} M_U$$

where M_U is a constant which depends only on U . We recall from Theorem 6.1 the estimate :

$$\begin{aligned} \|\omega\|_{H_{\sigma, \delta}^{s, \gamma}} &\leq \left\{ \|\omega_0\|_{H_{\sigma, \delta}^{s, \gamma}}^2 + \int_0^t F(\tau) d\tau \right\}^{\frac{1}{2}} \\ &\times \left\{ 1 - \left(\frac{4M_s}{2} - 1 \right) C_{s, \gamma, \sigma, \delta} \left(\|\omega_0\|_{H_{\sigma, \delta}^{s, \gamma}}^2 + \int_0^t F(\tau) d\tau \right)^{\frac{4M_s-2}{2}} t \right\}^{-\frac{1}{4M_s-2}}. \end{aligned}$$

So if we choose

$$T_1 = \min \left\{ \frac{3\|\omega_0\|_{H_{\sigma, \delta}^{s, \gamma}}^2}{C_{s, \gamma, \sigma, \delta} M_U}, \frac{1 - 2^{-4M_s+2}}{2^{4M_s-2} C_{s, \gamma, \sigma, \delta} \|\omega_0\|_{H_{\sigma, \delta}^{s, \gamma}}^{4M_s-2}} \right\}$$

we have

$$\|\omega\|_{H_{\sigma, \delta}^{s, \gamma}} \leq 4\|\omega_0\|_{H_{\sigma, \delta}^{s, \gamma}} \quad \forall t \in [0, T_1].$$

We know from definition (6.46), (6.47) of Ω and G that for any $t \in [0, T_1]$

$$\Omega \leq 4\|\omega_0\|_{H_{\sigma, \delta}^{s, \gamma}} \quad \text{and} \quad G \leq 4\|\omega_0\|_{H_{\sigma, \delta}^{s, \gamma}} + M_U = K$$

Thus, if we choose

$$T_2 = \min \left\{ T_1, \frac{1}{64\delta^2 C_{s, \gamma} (1 + 4\|\omega_0\|_{H_{\sigma, \delta}^{s, \gamma}}) \|\omega_0\|_{H_{\sigma, \delta}^{s, \gamma}}^2}, \frac{\ln 2}{C_{s, \gamma, \sigma, \delta} (1 + K)} \right\}$$

and using (6.49) from Proposition (6.7.1) and the assumption on the initial data

$$\sum_{|\alpha| \leq 2} |(1+y)^{\sigma+\alpha_2} D^\alpha \omega_0|^2 \leq \frac{1}{4\delta^2},$$

we have

$$\sum_{|\alpha| \leq 2} |(1+y)^{\sigma+\alpha_2} D^\alpha \omega|^2 \leq \frac{1}{\delta^2} \quad \forall t \in [0, T_2].$$

Let us choose

$$T_3 = \min \left\{ T_1, \frac{\delta}{8C_{s, \gamma} (1 + 4\|\omega_0\|_{H_{\sigma, \delta}^{s, \gamma}}) \|\omega_0\|_{H_{\sigma, \delta}^{s, \gamma}}}, \frac{1}{6C_{s, \gamma, \sigma, \delta} (1 + K)}, \frac{\ln 2}{C_{s, \gamma, \sigma, \delta} (1 + K)} \right\}.$$

Then using (6.50) and (6.46), (6.47) and again the assumption on the initial data

$$(1+y)^\sigma \omega_0 \geq 2\delta,$$

we obtain

$$\min_{(x,y) \in \mathbb{T} \times \mathbb{R}^+} (1+y)^\sigma \omega(t, x, y) \geq \delta \quad \forall t \in [0, T_3].$$

Finally, if we choose

$$T = \min \{T_1, T_2, T_3\}$$

we conclude that $\omega \in H_{\sigma, \delta}^{s, \gamma}$.

□

6.3 Proof of Proposition 6.1.1

In this section, we start to prove the energy type estimate of Proposition 6.1.1.

Proof. We now prove estimate (6.15).

We have from proposition (prop estim oo) (6.4.1)

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \sum_{\substack{|\alpha| \leq s \\ \alpha_1 \leq s-1}} \|(1+y)^{2\gamma+2\alpha_2} D^\alpha \omega\|_{L^2(\mathbb{T} \times \mathbb{R}^+)}^2 \\ & \leq -\varepsilon^2 \sum_{\substack{|\alpha| \leq s \\ \alpha_1 \leq s-1}} \|(1+y)^{\gamma+\alpha_2} \partial_x D^\alpha \omega\|_{L^2}^2 \\ & \quad - \frac{1}{3} \sum_{\substack{|\alpha| \leq s \\ \alpha_1 \leq s-1}} \|(1+y)^{\gamma+\alpha_2} \omega^{\frac{1}{2}} \partial_y D^\alpha \omega\|_{L^2}^2 \\ & \quad + \frac{1}{2} \|(1+y)^\gamma \sqrt{\omega} \partial_y g_s\|_{L^2}^2 \\ & \quad + C_{s, \delta} \|\omega\|_{H_g^{s, \gamma}}^{4M_s} + C_{s, \delta, \gamma} \|\omega\|_{H_g^{s, \gamma}}^{2M_s+2} \\ & \quad + C_{s, \gamma} \|\omega\|_{H_g^{s, \gamma}}^4 + C_{s, \gamma} \|\omega\|_{H_g^{s, \gamma}}^3 + C_{s, \delta, \gamma} \|\omega\|_{H_g^{s, \gamma}}^2 \\ & \quad + C_{s, \gamma} \|\omega\|_{H_g^{s, \gamma}}^3 + C_{s, \gamma, \sigma, \delta} \left\{ \|\omega\|_{H_g^{s, \gamma}} + \|\partial_x^s U\|_{L^2} \right\} \|\omega\|_{H_g^{s, \gamma}}^2 \\ & \quad + C_{s, \gamma, \sigma, \delta} \left\{ \|\omega\|_{H_g^{s, \gamma}} + \|\partial_x^s U\|_{L^2(\mathbb{T})} \right\}^2 \|\omega\|_{H_g^{s, \gamma}}^2 \\ & \quad + C_{s, \gamma, \sigma, \delta} \left\{ \|\omega\|_{H_g^{s, \gamma}} + \|\partial_x^s U\|_{L^2(\mathbb{T})} \right\} \|\omega\|_{H_g^{s, \gamma}} \\ & \quad + C_{s, \delta} \sum_{r=0}^{\frac{s}{2}} \|\partial_t^r \partial_x^{s+1-2r} p\|_{L^\infty(\mathbb{T})}^2 + C_{s, \delta} \sum_{r=0}^{\frac{s}{2}} \|\partial_t^r \partial_x^{s+1-2r} p\|_{L^\infty(\mathbb{T})}^4 \end{aligned}$$

We simplify some terms

$$\begin{aligned}
\frac{1}{2} \frac{d}{dt} \sum_{\substack{|\alpha| \leq s \\ \alpha_1 \leq s-1}} \|(1+y)^{2\gamma+2\alpha_2} D^\alpha \omega\|_{L^2(\mathbb{T} \times \mathbb{R}^+)}^2 \\
\leq -\varepsilon^2 \sum_{\substack{|\alpha| \leq s \\ \alpha_1 \leq s-1}} \|(1+y)^{\gamma+\alpha_2} \partial_x D^\alpha \omega\|_{L^2}^2 \\
- \frac{1}{3} \sum_{\substack{|\alpha| \leq s \\ \alpha_1 \leq s-1}} \|(1+y)^{\gamma+\alpha_2} \omega^{\frac{1}{2}} \partial_y D^\alpha \omega\|_{L^2}^2 \\
+ \frac{1}{2} \|(1+y)^\gamma \sqrt{\omega} \partial_y g_s\|_{L^2}^2 \\
+ C_{s,\gamma,\sigma,\delta} \|\omega\|_{H_g^{s,\gamma}}^{4M_s} + C_{s,\gamma,\sigma,\delta} + C_{s,\gamma,\sigma,\delta} \|\partial_x^{s+1} U\|_{L^\infty(\mathbb{T})}^4 \\
+ C_s \sum_{r=0}^{\frac{s}{2}} \|\partial_t^r \partial_x^{s+1-2r} p\|_{L^\infty(\mathbb{T})}^4
\end{aligned} \tag{6.16}$$

We use the Propositions (prop estim g) (6.5.1) :

$$\begin{aligned}
\frac{1}{2} \frac{d}{dt} \|(1+y)^\gamma g_s\|_{L^2}^2 &\leq -\frac{\varepsilon^2}{2} \|(1+y)^\gamma \partial_x g_s\|_{L^2}^2 - \frac{5}{8} \|(1+y)^\gamma \omega^{\frac{1}{2}} \partial_y g_s\|_{L^2}^2 \\
&+ \frac{1}{4} \|(1+y)^{\gamma+1} \sqrt{\omega} \partial_y^2 \partial_x^{s-1} \omega\|_{L^2}^2 \\
&+ C_{s,\gamma,\delta} \|\omega\|_{H_g^{s,\gamma}}^4 + C_{s,\gamma} \|\omega\|_{H_g^{s,\gamma}}^3 + C_{s,\gamma,\delta} \|\omega\|_{H_g^{s,\gamma}}^2 \\
&+ C_{\delta,\gamma} \|(1+y)^\gamma g_s\|_{L^2}^2 \\
&+ C_{s,\gamma,\sigma,\delta} \{ \|\omega\|_{H_g^{s,\gamma}} + \|\partial_x^s U\|_{L^2(\mathbb{T})} \} \|\omega\|_{H_g^{s,\gamma}}^2 \\
&+ C_{s,\gamma,\sigma,\delta} \{ 1 + \|\omega\|_{H_g^{s,\gamma}} + \|\partial_x^{s+1} U\|_{L^2(\mathbb{T})} \} \\
&\times \{ \|\omega\|_{H_g^{s,\gamma}} + \|\partial_x^{s+1} U\|_{L^2(\mathbb{T})} \} \{ \|\omega\|_{H_g^{s,\gamma}} + \|\omega\|_{H_g^{s,\gamma}}^2 \} \\
&+ C_{s,\gamma,\sigma,\delta} \|\partial_x^{s+1} U\|_{L^\infty} \{ \|\omega\|_{H_g^{s,\gamma}} + \|\partial_x^s U\|_{L^2(\mathbb{T})} \} \|\omega\|_{H_g^{s,\gamma}} \\
&+ C_\delta \|\omega\|_{H_g^{s,\gamma}}^2 \|\partial_x^s U\|_{L^\infty(\mathbb{T})} \\
&+ C_{\sigma,\delta} \|\partial_x^{s+1} p\|_{L^2(\mathbb{T})}
\end{aligned}$$

We simplify

$$\begin{aligned}
\frac{1}{2} \frac{d}{dt} \|(1+y)^\gamma g_s\|_{L^2}^2 &\leq -\frac{\varepsilon^2}{2} \|(1+y)^\gamma \partial_x g_s\|_{L^2}^2 - \frac{5}{8} \|(1+y)^\gamma \omega^{\frac{1}{2}} \partial_y g_s\|_{L^2}^2 \\
&+ \frac{1}{4} \|(1+y)^{\gamma+1} \sqrt{\omega} \partial_y^2 \partial_x^{s-1} \omega\|_{L^2}^2 \\
&+ C_{s,\gamma,\sigma,\delta} \|\omega\|_{H_g^{s,\gamma}}^6 + C_{s,\gamma,\sigma,\delta} \\
&+ C_{s,\gamma,\sigma,\delta} \|\partial_x^{s+1} U\|_{L^\infty(\mathbb{T})}^4 + C_{\sigma,\delta} \|\partial_x^{s+1} p\|_{L^2(\mathbb{T})}
\end{aligned} \tag{6.17}$$

We add the previous estimates (6.16) and (6.17) and from the definition of the pseudo-norm $\|\cdot\|_{H_g^{s,\gamma}}^2$ we get since $M_s \geq 5$

$$\begin{aligned}
\frac{d}{dt} \|\omega\|_{H_g^{s,\gamma}}^2 &\leq -\varepsilon^2 \sum_{\substack{|\alpha| \leq s \\ \alpha_1 \leq s-1}} \|(1+y)^{\gamma+\alpha_2} \partial_x D^\alpha \omega\|_{L^2}^2 - \frac{1}{2} \varepsilon^2 \|(1+y)^\gamma \partial_x g_s\|_{L^2}^2 \\
&\quad - \frac{1}{3} \sum_{\substack{|\alpha| \leq s \\ \alpha_1 \leq s-1}} \|(1+y)^{\gamma+\alpha_2} \omega^{\frac{1}{2}} \partial_y D^\alpha \omega\|_{L^2}^2 - \frac{5}{8} \|(1+y)^\gamma \omega^{\frac{1}{2}} \partial_y g_s\|_{L^2}^2 \\
&\quad + \frac{1}{4} \|(1+y)^{\gamma+1} \omega^{\frac{1}{2}} \partial_y^2 \partial_x^{s-1} \omega\|_{L^2}^2 + \frac{1}{2} \|(1+y)^\gamma \omega^{\frac{1}{2}} \partial_y g_s\|_{L^2}^2 \\
&\quad + C_{s,\gamma,\sigma,\delta} \|\omega\|_{H_g^{s,\gamma}}^{4M_s} + C_{s,\gamma,\sigma,\delta} \|\omega\|_{H_g^{s,\gamma}}^6 + C_{s,\gamma,\sigma,\delta} \\
&\quad + C_{s,\gamma,\sigma,\delta} \|\partial_x^{s+1} U\|_{L^\infty(\mathbb{T})}^4 \\
&\quad + C_s \sum_{r=0}^{\frac{s}{2}} \|\partial_t^r \partial_x^{s+1-2r} p\|_{L^\infty(\mathbb{T})}^4 + C_{\sigma,\delta} \|\partial_x^{s+1} p\|_{L^2(\mathbb{T})}
\end{aligned}$$

Then

$$\begin{aligned}
\frac{1}{2} \frac{d}{dt} \|\omega\|_{H_g^{s,\gamma}}^2 &\leq -\varepsilon^2 \sum_{\substack{|\alpha| \leq s \\ \alpha_1 \leq s-1}} \|(1+y)^{\gamma+\alpha_2} \partial_x D^\alpha \omega\|_{L^2}^2 - \frac{1}{2} \varepsilon^2 \|(1+y)^\gamma \partial_x g_s\|_{L^2}^2 \\
&\quad - \frac{1}{12} \sum_{\substack{|\alpha| \leq s \\ \alpha_1 \leq s-1}} \|(1+y)^{\gamma+\alpha_2} \omega^{\frac{1}{2}} \partial_y D^\alpha \omega\|_{L^2}^2 - \frac{1}{8} \|(1+y)^\gamma \omega^{\frac{1}{2}} \partial_y g_s\|_{L^2}^2 \\
&\quad + C_{s,\gamma,\sigma,\delta} \|\omega\|_{H_g^{s,\gamma}}^{4M_s} + C_{s,\gamma,\sigma,\delta} \|\omega\|_{H_g^{s,\gamma}}^6 + C_{s,\gamma,\sigma,\delta} \\
&\quad + C_{s,\gamma,\sigma,\delta} \|\partial_x^{s+1} U\|_{L^\infty(\mathbb{T})}^4 \\
&\quad + C_s \sum_{r=0}^{\frac{s}{2}} \|\partial_t^r \partial_x^{s+1-2r} p\|_{L^\infty(\mathbb{T})}^4 + C_{\sigma,\delta} \|\partial_x^{s+1} p\|_{L^2(\mathbb{T})}
\end{aligned}$$

Then we conclude that

$$\begin{aligned}
\frac{d}{dt} \|\omega\|_{H_g^{s,\gamma}}^2 &\leq C_{s,\gamma,\sigma,\delta} \|\omega\|_{H_g^{s,\gamma}}^{4M_s} + C_{s,\gamma,\sigma,\delta} \{1 + \|\partial_x^{s+1} U\|_{L^\infty}^4\} \\
&\quad + C_{s,\sigma,\delta} \sum_{r=0}^{\frac{s}{2}} \|\partial_t^r \partial_x^{s+1-2r} p\|_{L^\infty(\mathbb{T})}^4
\end{aligned}$$

This concludes the proof of theorem 6.1. □

6.4 Estimate for the vorticity term

On an energy type identity for $\|(1+y)^{2\gamma+2\alpha_2} D^\alpha \omega\|_{L^2}^2$

Proposition 6.4.1. VORTICITY TERM

We have the following estimate

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \sum_{\substack{|\alpha| \leq s \\ \alpha_1 \leq s-1}} \|(1+y)^{2\gamma+2\alpha_2} D^\alpha \omega\|_{L^2(\mathbb{T} \times \mathbb{R}^+)}^2 \\
& \leq -\varepsilon^2 \sum_{\substack{|\alpha| \leq s \\ \alpha_1 \leq s-1}} \|(1+y)^{\gamma+\alpha_2} \partial_x D^\alpha \omega\|_{L^2}^2 \\
& \quad - \frac{1}{3} \sum_{\substack{|\alpha| \leq s \\ \alpha_1 \leq s-1}} \|(1+y)^{\gamma+\alpha_2} \omega^{\frac{1}{2}} \partial_y D^\alpha \omega\|_{L^2}^2 \\
& \quad + C_{s,\gamma} \|\omega\|_{H_g^{s,\gamma}}^4 + C_{s,\gamma} \|\omega\|_{H_g^{s,\gamma}}^3 + C_{s,\delta,\gamma} \|\omega\|_{H_g^{s,\gamma}}^2 \\
& \quad + C_{s,\gamma} \|\omega\|_{H_g^{s,\gamma}}^3 + C_{s,\gamma,\sigma,\delta} \left\{ \|\omega\|_{H_g^{s,\gamma}} + \|\partial_x^s U\|_{L^2} \right\} \|\omega\|_{H_g^{s,\gamma}}^2 \\
& \quad + C_{s,\delta} \sum_{r=0}^{\frac{s}{2}} \|\partial_t^r \partial_x^{s+1-2r} p\|_{L^\infty(\mathbb{T})}^2 + C_{s,\delta} \sum_{r=0}^{\frac{s}{2}} \|\partial_t^r \partial_x^{s+1-2r} p\|_{L^\infty(\mathbb{T})}^4 \quad (6.18) \\
& \quad + C_{s,\delta,\gamma} \|\omega\|_{H_g^{s,\gamma}}^{2M_s+2} + C_{s,\delta} \|\omega\|_{H_g^{s,\gamma}}^{4M_s} \\
& \quad + C_{s,\gamma,\sigma,\delta} \left\{ \|\omega\|_{H_g^{s,\gamma}} + \|\partial_x^s U\|_{L^2(\mathbb{T})} \right\}^2 \|\omega\|_{H_g^{s,\gamma}}^2 \\
& \quad + C_{s,\gamma,\sigma,\delta} \left\{ \|\omega\|_{H_g^{s,\gamma}} + \|\partial_x^s U\|_{L^2(\mathbb{T})} \right\} \|\omega\|_{H_g^{s,\gamma}} \\
& \quad + \frac{1}{2} \|(1+y)^\gamma \sqrt{\omega} \partial_y g_s\|_{L^2}^2
\end{aligned}$$

We first give the main steps of the proof of proposition 6.4.1 We first establish some identity energy on some weighted norm of ω . Differentiating the vorticity equation with respect to x α_1 times and y α_2 times, we obtain the evolution equation for $D^\alpha \omega^\varepsilon$

$$D^\alpha(\partial_t \omega) + D^\alpha(u \partial_x \omega) + D^\alpha(v \partial_y \omega) = \varepsilon^2 D^\alpha(\partial_x^2 \omega) + 2D^\alpha(\omega \partial_y^2 \omega) + 2D^\alpha((\partial_y \omega)^2)$$

and

$$\left\{ \begin{array}{l}
D^\alpha(\partial_t \omega) = \partial_t(D^\alpha \omega^\varepsilon) \\
D^\alpha(u \partial_x \omega) = u \partial_x D^\alpha \omega + \sum_{0 < \beta \leq \alpha} \binom{\alpha}{\beta} D^\beta u \partial_x D^{\alpha-\beta} \omega \\
D^\alpha(v \partial_y \omega) = v \partial_y D^\alpha \omega + \sum_{0 < \beta \leq \alpha} \binom{\alpha}{\beta} D^\beta v \partial_y D^{\alpha-\beta} \omega \\
D^\alpha(\partial_x^2 \omega) = \partial_x^2(D^\alpha \omega) \\
D^\alpha(\omega \partial_y^2 \omega) = \omega \partial_y^2(D^\alpha \omega) + \sum_{0 < \beta \leq \alpha} \binom{\alpha}{\beta} D^\beta \omega \partial_y^2(D^{\alpha-\beta} \omega) \\
D^\alpha(\partial_y \omega \partial_y \omega) = \partial_y \omega \partial_y(D^\alpha \omega) + \sum_{0 < \beta \leq \alpha} \binom{\alpha}{\beta} \partial_y(D^\beta \omega) \partial_y(D^{\alpha-\beta} \omega).
\end{array} \right.$$

Then

$$\left\{ \begin{aligned} & \partial_t(D^\alpha \omega^\varepsilon) + u \partial_x D^\alpha \omega + \sum_{0 < \beta \leq \alpha} \binom{\alpha}{\beta} D^\beta u \partial_x D^{\alpha-\beta} \omega + v \partial_y D^\alpha \omega \\ & + \sum_{0 < \beta \leq \alpha} \binom{\alpha}{\beta} D^\beta v \partial_y D^{\alpha-\beta} \omega \\ & = \varepsilon^2 \partial_x^2 (D^\alpha \omega) + 2\omega \partial_y^2 (D^\alpha \omega) + 2 \sum_{0 < \beta \leq \alpha} \binom{\alpha}{\beta} D^\beta \omega \partial_y^2 (D^{\alpha-\beta} \omega) \\ & + 2\partial_y \omega \partial_y (D^\alpha \omega) + 2 \sum_{0 < \beta \leq \alpha} \binom{\alpha}{\beta} \partial_y (D^\beta \omega) \partial_y (D^{\alpha-\beta} \omega) \end{aligned} \right. \quad (6.19)$$

Multiplying the previous relation by $(1+y)^{2\gamma+2\alpha_2} D^\alpha \omega$, and then integrating over $\mathbb{T} \times \mathbb{R}^+$

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|(1+y)^{2\gamma+2\alpha_2} D^\alpha \omega\|_{L^2(\mathbb{T} \times \mathbb{R}^+)}^2 \\ & = \varepsilon^2 \iint (1+y)^{2\gamma+2\alpha_2} D^\alpha \omega \partial_x^2 D^\alpha \omega + 2 \iint (1+y)^{2\gamma+2\alpha_2} D^\alpha \omega \omega \partial_y^2 D^\alpha \omega \\ & + 2 \sum_{0 < \beta \leq \alpha} \binom{\alpha}{\beta} \iint (1+y)^{2\gamma+2\alpha_2} D^\beta \omega D^\alpha \omega \partial_y^2 D^{\alpha-\beta} \omega \\ & + 2 \iint (1+y)^{2\gamma+2\alpha_2} D^\alpha \omega \partial_y \omega \partial_y D^\alpha \omega \\ & + 2 \sum_{0 < \beta \leq \alpha} \binom{\alpha}{\beta} \iint (1+y)^{2\gamma+2\alpha_2} D^\alpha \omega \partial_y D^\beta \omega \partial_y D^{\alpha-\beta} \omega \\ & - \iint (1+y)^{2\gamma+2\alpha_2} D^\alpha \omega \{u \partial_x D^\alpha \omega + v \partial_y D^\alpha \omega\} \\ & - \sum_{0 < \beta \leq \alpha} \binom{\alpha}{\beta} \iint (1+y)^{2\gamma+2\alpha_2} D^\alpha \omega \{D^\beta u \partial_x D^{\alpha-\beta} \omega + D^\beta v \partial_y D^{\alpha-\beta} \omega\} \end{aligned}$$

By integration by parts in the x -variable we have :

$$\varepsilon^2 \iint (1+y)^{2\gamma+2\alpha_2} D^\alpha \omega \partial_x^2 D^\alpha \omega = -\varepsilon^2 \|(1+y)^{\gamma+\alpha_2} \partial_x D^\alpha \omega\|_L^2.$$

Integrating by parts in the y -variable, we have

$$\begin{aligned} & 2 \iint (1+y)^{2\gamma+2\alpha_2} \omega D^\alpha \omega \partial_y^2 D^\alpha \omega \\ & = -2 \iint \partial_y ((1+y)^{2\gamma+2\alpha_2} \omega D^\alpha \omega) \partial_y D^\alpha \omega - 2 \int_{\mathbb{T}} \omega D^\alpha \omega \partial_y D^\alpha \omega|_{y=0} \\ & = -2(2\gamma+2\alpha_2) \iint (1+y)^{2\gamma+2\alpha_2-1} \omega D^\alpha \omega \partial_y D^\alpha \omega - 2 \int_{\mathbb{T}} \omega D^\alpha \omega \partial_y D^\alpha \omega|_{y=0} \\ & - 2 \iint (1+y)^{2\gamma+2\alpha_2} \omega \partial_y D^\alpha \omega \partial_y D^\alpha \omega - 2 \iint (1+y)^{2\gamma+2\alpha_2} D^\alpha \omega \partial_y \omega \partial_y D^\alpha \omega \end{aligned}$$

Then we have :

$$\frac{1}{2} \frac{d}{dt} \|(1+y)^{2\gamma+2\alpha_2} D^\alpha \omega\|_{L^2(\mathbb{T} \times \mathbb{R}^+)}^2 \quad (6.20)$$

$$+ \varepsilon^2 \|(1+y)^{\gamma+\alpha_2} \partial_x D^\alpha \omega\|_{L^2}^2 + 2 \|(1+y)^{\gamma+\alpha_2} \omega^{\frac{1}{2}} \partial_y D^\alpha \omega\|_{L^2}^2 \quad (6.21)$$

$$= -A_1 + A_2 + A_3 - A_4 - A_5 - A_6 \quad (6.22)$$

where

$$\begin{aligned}
A_1 &= 4(\gamma + \alpha_2) \iint (1+y)^{2\gamma+2\alpha_2-1} \omega D^\alpha \omega \partial_y D^\alpha \omega \\
A_2 &= 2 \sum_{0 < \beta \leq \alpha} \binom{\alpha}{\beta} \iint (1+y)^{2\gamma+2\alpha_2} D^\beta \omega D^\alpha \omega \partial_y^2 D^{\alpha-\beta} \omega \\
A_3 &= 2 \sum_{0 < \beta \leq \alpha} \binom{\alpha}{\beta} \iint (1+y)^{2\gamma+2\alpha_2} D^\alpha \omega \partial_y D^\beta \omega \partial_y D^{\alpha-\beta} \omega \\
A_4 &= \iint (1+y)^{2\gamma+2\alpha_2} D^\alpha \omega \{u \partial_x D^\alpha \omega + v \partial_y D^\alpha \omega\} \\
A_5 &= \sum_{0 < \beta \leq \alpha} \binom{\alpha}{\beta} \iint (1+y)^{2\gamma+2\alpha_2} D^\alpha \omega \{D^\beta u \partial_x D^{\alpha-\beta} \omega + D^\beta v \partial_y D^{\alpha-\beta} \omega\} \\
A_6 &= 2 \int_{\mathbb{T}} \omega D^\alpha \omega \partial_y D^\alpha \omega|_{y=0}
\end{aligned}$$

We now estimate each previous terms A_i in the nex subsection.

6.4.1 Estimate of A_1 to A_4

- Estimate of the term A_1 :

$$\begin{aligned}
|A_1| &= 4(\gamma + \alpha_2) \left| \iint (1+y)^{2\gamma+2\alpha_2-1} \omega D^\alpha \omega \partial_y D^\alpha \omega \right| \\
&\leq \frac{1}{4} \|(1+y)^{(\gamma+\alpha_2)} \omega^{\frac{1}{2}} \partial_y D^\alpha \omega\|_{L^2}^2 + C_{\delta,s,\gamma} \|\omega\|_{H_y^{s,\gamma}}^2.
\end{aligned}$$

- Estimate of A_2

$$A_2 = 2 \sum_{0 < \beta \leq \alpha} \binom{\alpha}{\beta} \iint (1+y)^{2\gamma+2\alpha_2} D^\beta \omega D^\alpha \omega \partial_y^2 D^{\alpha-\beta} \omega$$

if $\beta = (1, 0)$ i.e $\alpha_1 \geq 1$ and $\alpha_2 \geq 0$

$$\begin{aligned}
&\left| 2 \iint (1+y)^{2\gamma+2\alpha_2} D^\beta \omega D^\alpha \omega \partial_y^2 D^{\alpha-\beta} \omega \right| \\
&\leq \iint |(1+y)^{\gamma+\alpha_2+1} \partial_x^{\alpha_1-1} \partial_y^{\alpha_2+2} \omega (1+y)^\gamma \partial_x \omega (1+y)^{\gamma+\alpha_2} D^\alpha \omega| \\
&\leq \frac{1}{4} \|(1+y)^{\gamma+\alpha_2+1} \partial_x^{\alpha_1-1} \partial_y^{\alpha_2+2} \omega\|_{L^2}^2 + 4 \|(1+y)^\gamma \partial_x \omega\|_{L^\infty}^2 \|(1+y)^{\gamma+\alpha_2} D^\alpha \omega\|_{L^2}^2 \\
&\leq \frac{1}{4} \|(1+y)^{\gamma+\alpha_2+1} \partial_x^{\alpha_1-1} \partial_y^{\alpha_2+2} \omega\|_{L^2}^2 + C_{s,\gamma} \|\omega\|_{H_y^{s,\gamma}}^4
\end{aligned}$$

if $\beta = (0, 1)$ for $\alpha_1 = 0$ and $\alpha_2 \geq 1$

$$\begin{aligned}
&\left| 2 \iint (1+y)^{2\gamma+2\alpha_2} D^\beta \omega D^\alpha \omega \partial_y^2 D^{\alpha-\beta} \omega \right| \\
&\leq \iint |(1+y)^{\gamma+\alpha_2} \partial_x^{\alpha_1} \partial_y^{\alpha_2+1} \omega (1+y)^{\gamma+1} \partial_y \omega (1+y)^{\gamma+\alpha_2} D^\alpha \omega| \\
&\leq \frac{1}{4} \|(1+y)^{\gamma+\alpha_2} \partial_x^{\alpha_1} \partial_y^{\alpha_2+1} \omega\|_{L^2}^2 + 4 \|(1+y)^{\gamma+1} \partial_y \omega\|_{L^\infty}^2 \|(1+y)^{\gamma+\alpha_2} D^\alpha \omega\|_{L^2}^2 \\
&\leq \frac{1}{4} \|(1+y)^{\gamma+\alpha_2} \partial_x^{\alpha_1} \partial_y^{\alpha_2+1} \omega\|_{L^2}^2 + C_{s,\gamma} \|\omega\|_{H_y^{s,\gamma}}^4
\end{aligned}$$

if $1 < |\beta| \leq |\alpha| - 2$

$$\begin{aligned} & \left| 2 \iint (1+y)^{2\gamma+2\alpha_2} D^\beta \omega D^\alpha \omega \partial_y^2 D^{\alpha-\beta} \omega \right| \\ & \leq 2 \iint |(1+y)^{\gamma+\beta_2} D^\beta \omega (1+y)^{\gamma+\alpha_2} D^\alpha \omega (1+y)^{\gamma+\alpha_2-\beta_2+2} \partial_y^2 D^{\alpha-\beta} \omega| \\ & \leq 2 \| (1+y)^{\gamma+\beta_2} D^\beta \omega \|_{L^\infty} \| (1+y)^{\gamma+\alpha_2} D^\alpha \omega \|_{L^2} \| (1+y)^{\gamma+\alpha_2-\beta_2+2} \partial_y^2 D^{\alpha-\beta} \omega \|_{L^2} \\ & \leq C_{s,\gamma} \|\omega\|_{H_g^{s,\gamma}}^3 \end{aligned}$$

if $|\alpha| - 2 < |\beta| \leq s$ ($s \geq 6$)

$$\begin{aligned} & \left| 2 \iint (1+y)^{2\gamma+2\alpha_2} D^\beta \omega D^\alpha \omega \partial_y^2 D^{\alpha-\beta} \omega \right| \\ & \leq 2 \iint |(1+y)^{\gamma+\beta_2} D^\beta \omega (1+y)^{\gamma+\alpha_2} D^\alpha \omega (1+y)^{\gamma+\alpha_2-\beta_2+2} \partial_y^2 D^{\alpha-\beta} \omega| \\ & \leq 2 \| (1+y)^{\gamma+\beta_2} D^\beta \omega \|_{L^2} \| (1+y)^{\gamma+\alpha_2} D^\alpha \omega \|_{L^2} \| (1+y)^{\gamma+\alpha_2-\beta_2+2} \partial_y^2 D^{\alpha-\beta} \omega \|_{L^\infty} \\ & \leq C_{s,\gamma} \|\omega\|_{H_g^{s,\gamma}}^3 \end{aligned}$$

Then

$$\begin{aligned} |A_2| & \leq C_{s,\gamma} \|\omega\|_{H_g^{s,\gamma}}^4 + C_{s,\gamma} \|\omega\|_{H_g^{s,\gamma}}^3 + \frac{1}{4} \| (1+y)^{\gamma+\alpha_2} \partial_x^{\alpha_1} \partial_y^{\alpha_2+1} \omega \|_{L^2}^2 \\ & \quad + \frac{1}{4} \| (1+y)^{\gamma+\alpha_2+1} \partial_x^{\alpha_1-1} \partial_y^{\alpha_2+2} \omega \|_{L^2}^2 \end{aligned}$$

- Estimate of A_3

$$A_3 = \sum_{0 < \beta \leq \alpha} \binom{\alpha}{\beta} \iint (1+y)^{2\gamma+2\alpha_2} \partial_y D^\beta \omega D^\alpha \omega \partial_y D^{\alpha-\beta} \omega$$

if $|\beta| + 1 \leq |\alpha| - 2 \leq s - 2$

$$\begin{aligned} & \left| \iint (1+y)^{2\gamma+2\alpha_2} \partial_y D^\beta \omega D^\alpha \omega \partial_y D^{\alpha-\beta} \omega \right| \\ & \leq \iint |(1+y)^{\gamma+\beta_2+1} \partial_y D^\beta \omega (1+y)^{\gamma+\alpha_2} D^\alpha \omega (1+y)^{\gamma_2+\alpha_2-\beta_2+1} \partial_y D^{\alpha-\beta} \omega| \\ & \leq \| (1+y)^{\gamma+\beta_2+1} \partial_y D^\beta \omega \|_{L^\infty} \| (1+y)^{\gamma+\alpha_2} D^\alpha \omega \|_{L^2} \| (1+y)^{\gamma+\alpha_2-\beta_2+1} \partial_y D^{\alpha-\beta} \omega \|_{L^2} \\ & \leq C_{s,\gamma} \|\omega\|_{H_g^{s,\gamma}}^3 \end{aligned}$$

if $s > |\beta| + 1 > |\alpha| - 2$ ($s \geq 6$)

$$\begin{aligned} & \left| \iint \partial_y D^\beta \omega D^\alpha \omega \partial_y D^{\alpha-\beta} \omega \right| \\ & \leq \iint |(1+y)^{\gamma+\beta_2+1} \partial_y D^\beta \omega (1+y)^{\gamma+\alpha_2} D^\alpha \omega (1+y)^{\gamma_2+\alpha_2-\beta_2+1} \partial_y D^{\alpha-\beta} \omega| \\ & \leq \| (1+y)^{\gamma+\beta_2+1} \partial_y D^\beta \omega \|_{L^2} \| (1+y)^{\gamma+\alpha_2} D^\alpha \omega \|_{L^2} \| (1+y)^{\gamma+\alpha_2-\beta_2+1} \partial_y D^{\alpha-\beta} \omega \|_{L^\infty} \\ & \leq C_{s,\gamma} \|\omega\|_{H_g^{s,\gamma}}^3. \end{aligned}$$

Then

$$|A_3| \leq C_{s,\gamma} \|\omega\|_{H_g^{s,\gamma}}^3$$

- Estimate of A_4 : Integrating by parts and using $\partial_x u + \partial_y v = 0$, we have

$$\begin{aligned} |A_4| & = \left| \iint (1+y)^{2\gamma+2\alpha_2} D^\alpha \omega \{ u \partial_x D^\alpha \omega + v \partial_y D^\alpha \omega \} \right| \\ & = \left| -(\gamma + \alpha_2) \iint (1+y)^{2\gamma+2\alpha_2-1} v |D^\alpha \omega|^2 \right| \\ & \leq C_{s,\gamma,\sigma,\delta} \{ \|\omega\|_{H_g^{s,\gamma}} + \|\partial_x^s U\|_{L^2} \} \|\omega\|_{H_g^{s,\gamma}}^2 \end{aligned}$$

by using inequality (6.84).

6.4.2 Estimate of the nonlinear term A_5

We set

$$A_5 = \sum_{0 < \beta \leq \alpha} \binom{\alpha}{\beta} \iint (1+y)^{2\gamma+2\alpha_2} D^\alpha \omega \{ D^\beta u \partial_x D^{\alpha-\beta} \omega + D^\beta v \partial_y D^{\alpha-\beta} \omega \}$$

Using the facts that $\partial_y v = -\partial_x u$ and $\partial_y u = \omega$, one may check that all terms are one of the following three types, we denote $e_1 := (1, 0)$, and $e_2 := (0, 1)$, for $\eta \in \mathbb{N}$ and $\kappa, \theta \in \mathbb{N}^2$:

Type I :

$$J_1 := \iint (1+y)^{2\gamma+2\alpha_2} D^\alpha \omega \partial_x^\eta v D^\kappa \omega$$

where $1 \leq \eta \leq s-1$ and $\eta e_1 + \kappa = \alpha + e_2$;

Type II :

$$J_2 := \iint (1+y)^{2\gamma+2\alpha_2} D^\alpha \omega \partial_x^\eta u D^\kappa \omega$$

where $1 \leq \eta \leq s$ and $\eta e_1 + \kappa = \alpha + e_1$; and

Type III :

$$J_3 := \iint (1+y)^{2\gamma+2\alpha_2} D^\alpha \omega D^\theta \omega D^\kappa \omega$$

where $|\theta| \leq s-2$ and $\theta + \kappa = \alpha + e_1 - e_2$.

It suffices to control J_1, J_2 and J_3

- Estimates for type I : When $1 \leq \eta \leq s-2$, applying proposition (6.10.1), we have, since $\kappa_2 = \alpha_2 + 1$,

$$\begin{aligned} |J_1| &\leq \|(1+y)^{\gamma+\alpha_2} D^\alpha \omega\|_{L^2} \left\| \frac{\partial_x^\eta v}{1+y} \right\|_{L^\infty} \|(1+y)^{\gamma+\kappa_2} D^\kappa \omega\|_{L^2} \\ &\leq C_{s,\gamma,\sigma,\delta} \{ \|\omega\|_{H_g^{s,\gamma}} + \|\partial_x^s U\|_{L^2(\mathbb{T})} \} \|\omega\|_{H_g^{s,\gamma}}^2 \end{aligned}$$

when $\eta = s-1$, by the triangle inequality and proposition (6.10.1), we have, since $\kappa_2 = \alpha_2 + 1$,

$$\begin{aligned} |J_1| &\leq \|(1+y)^{\gamma+\alpha_2} D^\alpha \omega\|_{L^2} \left\| \frac{\partial_x^{s-1} v + y \partial_x^s U}{1+y} \right\|_{L^2} \|(1+y)^{\gamma+\kappa_2} D^\kappa \omega\|_{L^\infty} \\ &\leq C_{s,\gamma,\sigma,\delta} \{ \|\omega\|_{H_g^{s,\gamma}} + \|\partial_x^s U\|_{L^2(\mathbb{T})} \} \|\omega\|_{H_g^{s,\gamma}}^2 \end{aligned}$$

- Estimates for type II : When $1 \leq \eta \leq s-1$, applying Proposition (6.10.1), we have, since $\kappa_2 = \alpha_2$,

$$\begin{aligned} |J_2| &\leq \|(1+y)^{\gamma+\alpha_2} D^\alpha \omega\|_{L^2} \|\partial_x^\eta u\|_{L^\infty} \|(1+y)^{\gamma+\kappa_2} D^\kappa \omega\|_{L^2} \\ &\leq C_{s,\gamma,\sigma,\delta} \{ \|\omega\|_{H_g^{s,\gamma}} + \|\partial_x^s U\|_{L^2(\mathbb{T})} \} \|\omega\|_{H_g^{s,\gamma}}^2 \end{aligned}$$

when $\eta = s$, by the triangle inequality and proposition (6.10.1), we have, since $\kappa = (0, \alpha_2)$ and $0 \leq \alpha_2 \leq 1$,

$$\begin{aligned} |J_2| &\leq \|(1+y)^{\gamma+\alpha_2} D^\alpha \omega\|_{L^2} \|\partial_x^s (u - U)\|_{L^2} \|(1+y)^{\gamma+\kappa_2} D^\kappa \omega\|_{L^\infty} \\ &\quad + \|(1+y)^{\gamma+\alpha_2} D^\alpha \omega\|_{L^2} \|\partial_x^s U\|_{L^\infty} \|(1+y)^{\gamma+\kappa_2} D^\kappa \omega\|_{L^2} \\ &\leq C_{s,\gamma,\sigma,\delta} \{ \|\omega\|_{H_g^{s,\gamma}} + \|\partial_x^s U\|_{L^2(\mathbb{T})} \} \|\omega\|_{H_g^{s,\gamma}}^2 \end{aligned}$$

- Estimate for type III : Applying proposition (6.10.1), we have, since $\theta_2 + \kappa_2 = \alpha_2 - 1$ and $|\theta| \leq s-2$,

$$\begin{aligned} |J_3| &\leq \|(1+y)^{\gamma+\alpha_2} D^\alpha \omega\|_{L^2} \|(1+y)^{1+\theta_2} D^\theta \omega\|_{L^\infty} \|(1+y)^{\gamma+\kappa_2} D^\kappa \omega\|_{L^2} \\ &\leq C_{s,\gamma,\sigma,\delta} \{ \|\omega\|_{H_g^{s,\gamma}} + \|\partial_x^s U\|_{L^2(\mathbb{T})} \} \|\omega\|_{H_g^{s,\gamma}}^2 \end{aligned}$$

Combining all estimates for types I through III, we prove :

$$A_5 \leq C_{s,\gamma,\sigma,\delta} \{ \|\omega\|_{H_g^{s,\gamma}} + \|\partial_x^s U\|_{L^2(\mathbb{T})} \} \|\omega\|_{H_g^{s,\gamma}}^2$$

6.4.3 Estimate of A_6 for $|\alpha| \leq s - 1$

We consider the term

$$A_6 = 2 \int_{\mathbb{T}} \omega D^\alpha \omega \partial_y D^\alpha \omega|_{y=0}$$

and when $|\alpha| \leq s - 1$ we estimate this term using the simple trace following estimate :

Lemma 6.4.1. *If $f \in \mathcal{C}^1(\mathbb{T} \times [0, 1])$, then*

$$\left| \int_{\mathbb{T}} f dx|_{y=0} \right| \leq 2 \left\{ \int_0^1 \int_{\mathbb{T}} |f| dx dy + \int_0^1 \int_{\mathbb{T}} |\partial_y f| dx dy \right\} \quad (6.23)$$

Proof. We have

$$\begin{aligned} \int_0^1 \int_{\mathbb{T}} \partial_y f dx dy &= \int_{\mathbb{T}} \int_0^1 \partial_y f dy dx \\ &= \int_{\mathbb{T}} f|_{y=1} dx - \int_{\mathbb{T}} f|_{y=0} dx \end{aligned}$$

Then

$$\int_{\mathbb{T}} f|_{y=0} dx = \int_{\mathbb{T}} f|_{y=1} dx - \int_0^1 \int_{\mathbb{T}} \partial_y f dx dy$$

And

$$\begin{aligned} \left| \int_{\mathbb{T}} f|_{y=0} dx \right| &\leq \left| \int_{\mathbb{T}} f|_{y=1} dx \right| + \left| \int_0^1 \int_{\mathbb{T}} \partial_y f dx dy \right| \\ &\leq \left| \int_{\mathbb{T}} f|_{y=1} dx \right| + \int_0^1 \int_{\mathbb{T}} |\partial_y f| dx dy \end{aligned}$$

We have

$$\begin{aligned} \left| \int_{\mathbb{T}} f|_{y=1} dx \right| &= \left| \int_{\mathbb{T}} \left(\int_0^1 \partial_y (yf) dy \right) dx \right| \\ &\leq \int_0^1 \int_{\mathbb{T}} |f| dx dy + \int_0^1 \int_{\mathbb{T}} |y| |\partial_y f| dx dy \\ &\leq \int_0^1 \int_{\mathbb{T}} |f| dx dy + \int_0^1 \int_{\mathbb{T}} |\partial_y f| dx dy \end{aligned}$$

So

$$\begin{aligned} \left| \int_{\mathbb{T}} f|_{y=0} dx \right| &\leq \int_0^1 \int_{\mathbb{T}} |f| dx dy + 2 \int_0^1 \int_{\mathbb{T}} |\partial_y f| dx dy \\ &\leq 2 \left\{ \int_0^1 \int_{\mathbb{T}} |f| dx dy + \int_0^1 \int_{\mathbb{T}} |\partial_y f| dx dy \right\} \end{aligned}$$

This concludes the lemma. □

We now apply the previous trace estimate and we get

$$\begin{aligned} |A_6| &\leq C \left\{ \int_0^1 \int_{\mathbb{T}} |\omega D^\alpha \omega \partial_y D^\alpha \omega| dx dy + \int_0^1 \int_{\mathbb{T}} |\partial_y (\omega D^\alpha \omega \partial_y D^\alpha \omega)| dx dy \right\} \\ &\leq C \left\{ \int_0^{+\infty} \int_{\mathbb{T}} |\omega D^\alpha \omega \partial_y D^\alpha \omega| dx dy + \int_0^{+\infty} \int_{\mathbb{T}} |\partial_y (\omega D^\alpha \omega \partial_y D^\alpha \omega)| dx dy \right\} \end{aligned}$$

We have for the first right-hand side term

$$\begin{aligned}
& \int_0^{+\infty} \int_{\mathbb{T}} |\omega D^\alpha \omega \partial_y D^\alpha \omega| dx dy \\
& \leq \int_0^{+\infty} \int_{\mathbb{T}} (1+y)^\gamma |\omega| (1+y)^{\gamma+\alpha_2} |D^\alpha \omega| (1+y)^{\gamma+\alpha_2+1} |\partial_y D^\alpha \omega| dx dy \\
& \leq \|(1+y)^\gamma \omega\|_{L^\infty} \|(1+y)^{\gamma+\alpha_2} D^\alpha \omega\|_{L^2} \|(1+y)^{\gamma+\alpha_2+1} \partial_y D^\alpha \omega\|_{L^2} \\
& \leq C_{s,\gamma} \|\omega\|_{H_y^{s,\gamma}}^3
\end{aligned}$$

and for the second term we compute

$$\partial_y(\omega D^\alpha \omega \partial_y D^\alpha \omega) = \partial_y \omega D^\alpha \omega \partial_y D^\alpha \omega + \omega \partial_y D^\alpha \omega \partial_y D^\alpha \omega + \omega D^\alpha \omega \partial_y^2 D^\alpha \omega.$$

Therefore

$$\int_0^{+\infty} \int_{\mathbb{T}} |\partial_y(\omega D^\alpha \omega \partial_y D^\alpha \omega)| dx dy \leq X_1 + X_2 + X_3$$

where

$$\begin{aligned}
X_1 &= \iint |\partial_y \omega D^\alpha \omega \partial_y D^\alpha \omega|, \\
X_2 &= \iint |\omega \partial_y D^\alpha \omega \partial_y D^\alpha \omega|, \\
X_3 &= \iint |\omega D^\alpha \omega \partial_y^2 D^\alpha \omega|.
\end{aligned}$$

We estimate each terms

$$\begin{aligned}
X_1 &\leq \int_0^{+\infty} \int_{\mathbb{T}} (1+y)^{\gamma+1} |\partial_y \omega| (1+y)^{\gamma+\alpha_2} |D^\alpha \omega| (1+y)^{\gamma+\alpha_2+1} |\partial_y D^\alpha \omega| dx dy \\
&\leq \|(1+y)^{\gamma+1} \partial_y \omega\|_{L^\infty} \|(1+y)^{\gamma+\alpha_2} D^\alpha \omega\|_{L^2} \|(1+y)^{\gamma+\alpha_2+1} \partial_y D^\alpha \omega\|_{L^2} \\
&\leq C_{s,\gamma} \|\omega\|_{H_y^{s,\gamma}}^3
\end{aligned}$$

and

$$\begin{aligned}
X_2 &\leq \int_0^{+\infty} \int_{\mathbb{T}} (1+y)^\gamma |\omega| (1+y)^{\gamma+\alpha_2+1} |\partial_y D^\alpha \omega| (1+y)^{\gamma+\alpha_2+1} |\partial_y D^\alpha \omega| dx dy \\
&\leq \|(1+y)^\gamma \omega\|_{L^\infty} \|(1+y)^{\gamma+\alpha_2+1} \partial_y D^\alpha \omega\|_{L^2} \|(1+y)^{\gamma+\alpha_2+1} \partial_y D^\alpha \omega\|_{L^2} \\
&\leq C_{s,\gamma} \|\omega\|_{H_y^{s,\gamma}}^3
\end{aligned}$$

and(by cauchy inequality)

$$\begin{aligned}
X_3 &= \iint \sqrt{6} |\sqrt{\omega} D^\alpha \omega| \frac{\sqrt{\omega}}{\sqrt{6}} |\partial_y^2 D^\alpha \omega| \\
&\leq 3 \iint \omega |D^\alpha \omega|^2 + \frac{1}{12} \iint \omega |\partial_y^2 D^\alpha \omega| \\
&\leq 3 \iint (1+y)^\gamma \omega (1+y)^{2(\gamma+\alpha_2)} |D^\alpha \omega|^2 + \frac{1}{12} \iint (1+y)^{2(\gamma+\alpha_2+1)} \omega |\partial_y^2 D^\alpha \omega|^2 \\
&\leq 3 \|(1+y)^\gamma \omega\|_{L^\infty} \|(1+y)^{\gamma+\alpha_2} D^\alpha \omega\|_{L^2}^2 + \frac{1}{12} \|(1+y)^{\gamma+\alpha_2+1} \sqrt{\omega} \partial_y^2 D^\alpha \omega\|_{L^2}^2 \\
&\leq C_{s,\gamma} \|\omega\|_{H_y^{s,\gamma}}^3 + \frac{1}{12} \|(1+y)^{\gamma+\alpha_2+1} \sqrt{\omega} \partial_y^2 D^\alpha \omega\|_{L^2}^2
\end{aligned}$$

In conclusion from the previous estimates, we deduce

$$|A_6| \leq C_{s,\gamma} \|\omega\|_{H_y^{s,\gamma}}^3 + \frac{1}{12} \|(1+y)^{\gamma+\alpha_2+1} \sqrt{\omega} \partial_y^2 D^\alpha \omega\|_{L^2}^2$$

6.4.4 Estimate of A_6 for $|\alpha| = s$

We recall the definition of the term

$$A_6 = 2 \int_{\mathbb{T}} \omega D^\alpha \omega \partial_y D^\alpha \omega|_{y=0}.$$

When $|\alpha| = s$, the order of derivative of $\partial_y D^\alpha \omega|_{y=0}$ is too high so that we cannot control the boundary integral $2 \int_{\mathbb{T}} \omega D^\alpha \omega \partial_y D^\alpha \omega|_{y=0}$ by the trace estimate (6.23). In order to make use of (6.23), we must reduce the order of the problematic term $\partial_y D^\alpha \omega|_{y=0}$.

Proposition 6.4.2. *We have the following estimate on the term A_6 :*

$$\begin{aligned} \sum_{\substack{|\alpha| \leq s \\ \alpha_1 \leq s-1}} |A_6| &\leq C_{s,\delta} \sum_{r=0}^{\frac{s}{2}} \|\partial_t^r \partial_x^{s+1-2r} p\|_{L^\infty(\mathbb{T})}^2 + C_{s,\delta} \sum_{r=0}^{\frac{s}{2}} \|\partial_t^r \partial_x^{s+1-2r} p\|_{L^\infty(\mathbb{T})}^4 \\ &+ C_{s,\delta,\gamma} \|\omega\|_{H_g^{s,\gamma}}^{2M_s+2} + C_{s,\delta,\gamma} \|\omega\|_{H_g^{s,\gamma}}^2 + C_{s,\delta} \|\omega\|_{H_g^{s,\gamma}}^{4M_s} \\ &+ C_{s,\gamma,\sigma,\delta} \left\{ \|\omega\|_{H_g^{s,\gamma}} + \|\partial_x^s U\|_{L^2(\mathbb{T})} \right\}^2 \|\omega\|_{H_g^{s,\gamma}}^2 \\ &+ C_{s,\gamma,\sigma,\delta} \left\{ \|\omega\|_{H_g^{s,\gamma}} + \|\partial_x^s U\|_{L^2(\mathbb{T})} \right\} \left\{ \|\omega\|_{H_g^{s,\gamma}} + \|\omega\|_{H_g^{s,\gamma}}^2 \right\} \\ &+ \frac{1}{2} \|(1+y)^\gamma \sqrt{\omega} \partial_y g_s\|_{L^2}^2 + C_{s,\gamma} \|\omega\|_{H_g^{s,\gamma}}^3 + C_{s,\gamma} \|\omega\|_{H_g^{s,\gamma}}^4 \\ &+ \frac{5}{6} \sum_{\substack{|\alpha| \leq s \\ \alpha_1 \leq s-1}} \|(1+y)^{\gamma+\alpha_2} \sqrt{\omega} \partial_y D^\alpha \omega\|_{L^2}^2 \end{aligned}$$

To prove this result, we need to consider two cases : α_2 even and α_2 odd. This proposition will be a direct consequence of the propositions 6.4.3 and 6.4.4 in the following two subsections.

In both cases, we will use the following lemma : On utilise le fait que ω est solution de l'équation,; et la structure de l'équation fait que l'on peut remplacer des termes en ω problématiques à estimer par des coefficients $(\partial_t - \varepsilon^2 \partial_x^2)^{m_1(j)} \partial_x^{m_2(j)} p$ qui sont bien maîtrisés.

Lemma 6.4.2. *Let $s \geq 6$ an even integer. There exists an increasing sequence $N(n_2) \in \mathbb{N}^*$ such that, for any integer $0 \leq n_2 \leq s/2$ and for $0 \leq n_1 \leq s - 2n_2$, we have*

$$\begin{aligned} \partial_x^{n_1} (\omega \partial_y^{2n_2+1} \omega)|_{y=0} &= \sum_{j=1}^{N(n_2)} C(j) \varepsilon^{2\ell(j)} \omega^{-k(j)} \left((\partial_t - \varepsilon^2 \partial_x^2)^{m_1(j)} \partial_x^{m_2(j)} p \right) \prod_{i=1}^{M(j)} D^{\beta(i,j)} \omega \Big|_{y=0}, \quad (6.24) \end{aligned}$$

where

$$\begin{cases} 0 \leq M(j) \leq N(n_2) \\ C(j) \in \mathbb{R} \\ \ell(j) \in \{0, \dots, n_2\} \\ m_1(j) \in \{0, \dots, n_2\} \\ m_2(j) \in \{0, \dots, n_1 + 2n_2 + 1\} \\ k(j) \in \{0, \dots, n_1 + 2n_2\}, \end{cases} \quad (6.25)$$

and where, for any $j \in \{1, \dots, N(n_2)\}$ and $i \in \{1, \dots, M(j)\}$, we have

$$0 \leq |\beta(i, j)| \leq n_1 + 2n_2, \quad (6.26)$$

$$0 \leq \sum_{i=1}^{M(j)} |\beta(i, j)| \leq n_1 + 2n_2 + 1. \quad (6.27)$$

Remark 9. *On the left-hand side, we can have a boundary term with $s+1$ space derivatives of ω , which cannot be controlled by the trace estimate (6.23). On the contrary, on the right-hand side, all the terms have at most s space derivatives of ω , which allows us to apply (6.23).*

Avec la relation (6.26), on réduit de 1 l'ordre des dérivées à estimer.

Avec la relation (6.27), on limite le nombre de termes dérivées d'ordre $\geq s-1$.

And when α_2 is odd, on fait une intégration par partie et on se ramène au cas α_2 is even.

6.4.5 Proof of the estimate of A_6 in the even case

We prove in this section the following proposition.

Proposition 6.4.3. *We have the following estimate on the term A_6 :*

When α_2 is even we have :

For $\alpha = (0, s)$

$$\begin{aligned} |A_6| &\leq C_{s,\delta} \sum_{r=0}^{\frac{s}{2}} \|\partial_t^r \partial_x^{s+1-2r} p\|_{L^\infty(\mathbb{T})}^2 + C_{s,\delta} \sum_{r=0}^{\frac{s}{2}} \|\partial_t^r \partial_x^{s+1-2r} p\|_{L^\infty(\mathbb{T})}^4 \\ &\quad + \frac{1}{24} \|(1+y)^{\gamma+s} \sqrt{\omega} \partial_y^{s+1} \omega\|_{L^2}^2 \\ &\quad + C_{s,\delta} \|\omega\|_{H_g^{s,\gamma}}^{2M_s} \|\omega\|_{H_g^{s,\gamma}}^2 + C_{s,\delta} \|\omega\|_{H_g^{s,\gamma}}^{4M_s} \\ &\quad + \frac{1}{24} \|(1+y)^{\gamma+s} \sqrt{\omega} \partial_y^{s+1} \omega\|_{L^2}^2 \end{aligned}$$

For $\alpha_1 \geq 2$

$$\begin{aligned} |A_6| &\leq C_{s,\delta} \sum_{r=0}^{\frac{s}{2}} \|\partial_t^r \partial_x^{s+1-2r} p\|_{L^\infty(\mathbb{T})}^2 + C_{s,\delta} \sum_{r=0}^{\frac{s}{2}} \|\partial_t^r \partial_x^{s+1-2r} p\|_{L^\infty(\mathbb{T})}^4 \\ &\quad + \frac{1}{24} \|(1+y)^{\gamma+\alpha_2} \sqrt{\omega} \partial_y D^\alpha \omega\|_{L^2}^2 \\ &\quad + C_{s,\delta} \|\omega\|_{H_g^{s,\gamma}}^{2M_s} \|\omega\|_{H_g^{s,\gamma}}^2 + C_{s,\delta} \|\omega\|_{H_g^{s,\gamma}}^{4M_s} \\ &\quad + \frac{1}{24} \|(1+y)^{\gamma+\alpha_2} \sqrt{\omega} \partial_x^{\alpha_1} \partial_y^{\alpha_2+1} \omega\|_{L^2}^2 \\ &\quad + C_{s,\gamma} \|\omega\|_{H_g^{s,\gamma}}^3 + C_{s,\gamma} \|\omega\|_{H_g^{s,\gamma}}^4 + \frac{1}{12} \|(1+y)^{\gamma+\alpha_2} \sqrt{\omega} \partial_y D^\alpha \omega\|_{L^2}^2 \\ &\quad + \frac{1}{12} \|(1+y)^{\gamma+\alpha_2+1} \sqrt{\omega} \partial_x^{\alpha_1-1} \partial_y^{\alpha_2+2} \omega\|_{L^2}^2 \end{aligned}$$

Proof. We have by Leibniz formula :

$$\begin{aligned} \partial_x^{\alpha_1} (\omega \partial_y^{\alpha_2+1} \omega) &= \sum_{k=0}^{\alpha_1} \binom{\alpha_1}{k} \partial_x^k \omega \partial_x^{\alpha_1-k} \partial_y^{\alpha_2+1} \omega \\ &= \omega \partial_x^{\alpha_1} \partial_y^{\alpha_2+1} \omega + \sum_{k=1}^{\alpha_1} \binom{\alpha_1}{k} \partial_x^k \omega \partial_x^{\alpha_1-k} \partial_y^{\alpha_2+1} \omega \end{aligned}$$

Then

$$\omega \partial_x^{\alpha_1} \partial_y^{\alpha_2+1} \omega = \partial_x^{\alpha_1} (\omega \partial_y^{\alpha_2+1} \omega) - \sum_{k=1}^{\alpha_1} \binom{\alpha_1}{k} \partial_x^k \omega \partial_x^{\alpha_1-k} \partial_y^{\alpha_2+1} \omega$$

So the term A_6 becomes :

Case 1. α_2 is even. When $\alpha_2 = 2n_2$ for some $n_2 \in \mathbb{N}$
For $\alpha = (0, s)$

$$\begin{aligned} A_6 &= 2 \int_{\mathbb{T}} \omega D^\alpha \omega \partial_y D^\alpha \omega|_{y=0} \\ &= 2 \int_{\mathbb{T}} \partial_y^s \omega \omega \partial_y^{s+1} \omega|_{y=0} \\ &= \sum_{j=1}^{N(n_2)} C(j) \varepsilon^{2\ell(j)} \int_{\mathbb{T}} \omega^{-k(j)} \partial_y^s \omega \left((\partial_t - \varepsilon^2 \partial_x^2)^{m_1(j)} \partial_x^{m_2(j)} p \right) \prod_{i=1}^{M(j)} D^{\beta(i,j)} \omega \Big|_{y=0} \end{aligned}$$

Using trace estimate ((6.23))

$$\begin{aligned} |A_6| &\leq C_{s,\delta} \sum_{r=0}^{\frac{s}{2}} \|\partial_t^r \partial_x^{s+1-2r} p\|_{L^\infty(\mathbb{T})}^2 + C_{s,\delta} \sum_{r=0}^{\frac{s}{2}} \|\partial_t^r \partial_x^{s+1-2r} p\|_{L^\infty(\mathbb{T})}^4 \\ &\quad + \frac{1}{24} \|(1+y)^{\gamma+s} \sqrt{\omega} \partial_y^{s+1} \omega\|_{L^2}^2 \\ &\quad + C_{s,\delta} \|\omega\|_{H_g^{s,\gamma}}^{2M_s} \|\omega\|_{H_g^{s,\gamma}}^2 + C_{s,\delta} \|\omega\|_{H_g^{s,\gamma}}^{4M_s} \\ &\quad + \frac{1}{24} \|(1+y)^{\gamma+s} \sqrt{\omega} \partial_y^{s+1} \omega\|_{L^2}^2 \end{aligned}$$

For $\alpha_1 \geq 2$ even because we assume s even.

$$\begin{aligned} A_6 &= 2 \int_{\mathbb{T}} \omega D^\alpha \omega \partial_y D^\alpha \omega|_{y=0} \\ &= A_{61} + A_{62} \end{aligned}$$

where

$$\begin{aligned} A_{61} &= 2 \int_{\mathbb{T}} D^\alpha \omega \partial_x^{\alpha_1} (\omega \partial_y^{\alpha_2+1} \omega)|_{y=0} \\ A_{62} &= -2 \sum_{k=1}^{\alpha_1} \binom{\alpha_1}{k} \int_{\mathbb{T}} \partial_x^k \omega D^\alpha \omega \partial_x^{\alpha_1-k} \partial_y^{\alpha_2+1} \omega|_{y=0} \end{aligned}$$

- Estimate of A_{61}

$$\begin{aligned} A_{61} &= 2 \int_{\mathbb{T}} D^\alpha \omega \partial_x^{\alpha_1} (\omega \partial_y^{\alpha_2+1} \omega)|_{y=0} \\ &= \sum_{j=1}^{N(n_2)} C(j) \varepsilon^{2\ell(j)} \int_{\mathbb{T}} \omega^{-k(j)} D^\alpha \omega \left((\partial_t - \varepsilon^2 \partial_x^2)^{m_1(j)} \partial_x^{m_2(j)} p \right) \prod_{i=1}^{M(j)} D^{\beta(i,j)} \omega \Big|_{y=0} \end{aligned}$$

We apply the trace estimate (6.23)

$$\left| \int_{\mathbb{T}} \omega^{-k(j)} D^\alpha \omega \left((\partial_t - \varepsilon^2 \partial_x^2)^{m_1(j)} \partial_x^{m_2(j)} p \right) \prod_{i=1}^{M(j)} D^{\beta(i,j)} \omega \Big|_{y=0} \right| \leq C \{A_{61,1} + A_{61,2}\}$$

Where

$$\begin{aligned} A_{61,1} &= \int_0^1 \int_{\mathbb{T}} \left| \omega^{-k(j)} D^\alpha \omega \left((\partial_t - \varepsilon^2 \partial_x^2)^{m_1(j)} \partial_x^{m_2(j)} p \right) \prod_{i=1}^{M(j)} D^{\beta(i,j)} \omega \right| dx dy \\ A_{61,2} &= \int_0^1 \int_{\mathbb{T}} \left| \partial_y \left(\omega^{-k(j)} D^\alpha \omega \left((\partial_t - \varepsilon^2 \partial_x^2)^{m_1(j)} \partial_x^{m_2(j)} p \right) \prod_{i=1}^{M(j)} D^{\beta(i,j)} \omega \right) \right| dx dy \end{aligned}$$

- Estimate of $A_{61,1}$

$$A_{61,1} \leq C_{m_1(s),\delta} \sum_{r=0}^{m_1(j)} \left\| \partial_t^r \partial_x^{2m_1(j)+m_2(j)-2r} p \right\|_{L^\infty(\mathbb{T})} \left\| (1+y)^{\gamma+\beta_2(i,j)} D^{\beta(i,j)} \omega \right\|_{L^\infty}^{M(j)-1} \\ \times \left\| (1+y)^{\gamma+\beta_2} D^\beta \omega \right\|_{L^2} \left\| (1+y)^{\gamma+\alpha_2} D^\alpha \omega \right\|_{L^2}$$

by young inequality

$$A_{61,1} \leq C_{m_1(s),\delta} \sum_{r=0}^{\frac{s}{2}} \left\| \partial_t^r \partial_x^{s+1-2r} p \right\|_{L^\infty(\mathbb{T})}^2 + C_{m_1(s),\delta} \left\| (1+y)^{\gamma+\beta_2(i,j)} D^{\beta(i,j)} \omega \right\|_{L^\infty}^{2(M(j)-1)} \\ \times \left\| (1+y)^{\gamma+\beta_2} D^\beta \omega \right\|_{L^2}^2 \left\| (1+y)^{\gamma+\alpha_2} D^\alpha \omega \right\|_{L^2}^2$$

Then

$$\sum_{j=1}^{N(s)} A_{61,1} \leq C_{m_1(s),\delta} \sum_{r=0}^{\frac{s}{2}} \left\| \partial_t^r \partial_x^{s+1-2r} p \right\|_{L^\infty(\mathbb{T})}^2 \\ + C_{m_1(s),\delta} \left\| (1+y)^{\gamma+\beta_2} D^\beta \omega \right\|_{L^2}^2 \left\| (1+y)^{\gamma+\alpha_2} D^\alpha \omega \right\|_{L^2}^2 \\ \times \sum_{j=1}^{N(s)} \left\| (1+y)^{\gamma+\beta_2(i,j)} D^{\beta(i,j)} \omega \right\|_{L^\infty}^{2(M(j)-1)}$$

Which gives

$$\sum_{j=1}^{N(s)} A_{61,1} \leq C_{m_1(s),\delta} \sum_{r=0}^{\frac{s}{2}} \left\| \partial_t^r \partial_x^{s+1-2r} p \right\|_{L^\infty(\mathbb{T})}^2 \\ + C_{m_1(s),\delta} \left\| (1+y)^{\gamma+\beta_2} D^\beta \omega \right\|_{L^2}^2 \left\| (1+y)^{\gamma+\alpha_2} D^\alpha \omega \right\|_{L^2}^2 \\ \times N(s) \left(1 + \left\| (1+y)^{\gamma+\beta_2(s)} D^{\beta(s)} \omega \right\|_{L^\infty}^{2(M_s-1)} \right)$$

We develop we obtain

$$\sum_{j=1}^{N(s)} A_{61,1} \leq C_{m_1(s),\delta} \sum_{r=0}^{\frac{s}{2}} \left\| \partial_t^r \partial_x^{s+1-2r} p \right\|_{L^\infty(\mathbb{T})}^2 \\ + C_{s,\delta} \left\| (1+y)^{\gamma+\beta_2} D^\beta \omega \right\|_{L^2}^2 \left\| (1+y)^{\gamma+\alpha_2} D^\alpha \omega \right\|_{L^2}^2 \\ + \left\| (1+y)^{\gamma+\beta_2} D^\beta \omega \right\|_{L^2}^2 \left\| (1+y)^{\gamma+\alpha_2} D^\alpha \omega \right\|_{L^2}^2 \\ \times C_{s,\delta} \left\| (1+y)^{\gamma+\beta_2(s)} D^{\beta(s)} \omega \right\|_{L^\infty}^{2(M_s-1)}$$

where $\beta = s - 1$ or $\beta = s$ and $\beta(i, j) \leq s - 2$. Then

$$\sum_{j=1}^{N(s)} A_{61,1} \leq C_{s,\delta} \sum_{r=0}^{\frac{s}{2}} \left\| \partial_t^r \partial_x^{2s-2r} p \right\|_{L^\infty(\mathbb{T})}^2 + C_{s,\delta} \left\| \omega \right\|_{H_g^{s,\gamma}}^{2(M_s-1)} \left\| \omega \right\|_{H_g^{s,\gamma}}^4 \\ + C_{s,\delta} \left\| \omega \right\|_{H_g^{s,\gamma}}^4$$

- Estimate of $A_{61,2}$

$$A_{61,2} \leq A_{61,21} + A_{61,22} + A_{61,23}$$

Where

$$A_{61,21} = \int_0^1 \int_{\mathbb{T}} \left| -k(j) \omega^{-(k(j)+1)} \partial_y \omega D^\alpha \omega \left((\partial_t - \varepsilon^2 \partial_x^2)^{m_1(j)} \partial_x^{m_2(j)} p \right) \prod_{i=1}^{M(j)} D^{\beta(i,j)} \omega \right| dx dy$$

$$A_{61,22} = \int_0^1 \int_{\mathbb{T}} \left| \omega^{-k(j)} \partial_y D^\alpha \omega \left((\partial_t - \varepsilon^2 \partial_x^2)^{m_1(j)} \partial_x^{m_2(j)} p \right) \prod_{i=1}^{M(j)} D^{\beta(i,j)} \omega \right| dx dy$$

$$A_{61,23} = \int_0^1 \int_{\mathbb{T}} \left| \omega^{-k(j)} D^\alpha \omega \left((\partial_t - \varepsilon^2 \partial_x^2)^{m_1(j)} \partial_x^{m_2(j)} p \right) \partial_y \left(\prod_{i=1}^{M(j)} D^{\beta(i,j)} \omega \right) \right| dx dy$$

• Estimate of $A_{61,21}$

$$A_{61,21} \leq C_{m_1(s),\delta} \sum_{r=0}^{m_1(j)} \left\| \partial_t^r \partial_x^{2m_1(j)+m_2(j)-2r} p \right\|_{L^\infty(\mathbb{T})} \left\| (1+y)^{\gamma+\beta_2(i,j)} D^{\beta(i,j)} \omega \right\|_{L^\infty}^{M(j)-1}$$

$$\times \left\| (1+y)^{\gamma+\beta_2} D^\beta \omega \right\|_{L^2} \left\| (1+y)^{\gamma+\alpha_2} D^\alpha \omega \right\|_{L^2}$$

Because

$$\delta^{(k(j)+1)} (1+y)^{(k(j)+1)\sigma} \leq \frac{1}{\omega^{(k(j)+1)}} \leq \frac{1}{\delta^{k(j)+1}} (2)^{(k(j)+1)\sigma}$$

$$|\partial_y \omega| \leq \delta^{-1} (1+y)^{-\sigma-1}$$

by young inequality

$$A_{61,21} \leq C_{m_1(s),\delta} \sum_{r=0}^{m_1(j)} \left\| \partial_t^r \partial_x^{2m_1(j)+m_2(j)-2r} p \right\|_{L^\infty(\mathbb{T})}^2 + C_{m_1(s),\delta} \left\| (1+y)^{\gamma+\beta_2(i,j)} D^{\beta(i,j)} \omega \right\|_{L^\infty}^{2(M(j)-1)}$$

$$\times \left\| (1+y)^{\gamma+\beta_2} D^\beta \omega \right\|_{L^2}^2 \left\| (1+y)^{\gamma+\alpha_2} D^\alpha \omega \right\|_{L^2}^2$$

where $\beta = s-1$ or $\beta = s$ and $\beta(i,j) \leq s-2$. Then

$$\sum_{j=1}^{N(s)} A_{61,21} \leq C_{s,\delta} \sum_{r=0}^{\frac{s}{2}} \left\| \partial_t^r \partial_x^{2s-2r} p \right\|_{L^\infty(\mathbb{T})}^2 + C_{s,\delta} \|\omega\|_{H_g^{s,\gamma}}^{2(M_s-1)} \|\omega\|_{H_g^{s,\gamma}}^4 + C_{s,\delta} \|\omega\|_{H_g^{s,\gamma}}^4$$

• Estimate of $A_{61,22}$

$$A_{61,22} \leq C_{m_1(s),\delta} \sum_{r=0}^{m_1(j)} \left\| \partial_t^r \partial_x^{2m_1(j)+m_2(j)-2r} p \right\|_{L^\infty(\mathbb{T})} \left\| (1+y)^{\gamma+\beta_2(i,j)} D^{\beta(i,j)} \omega \right\|_{L^\infty}^{M(j)-1}$$

$$\times \left\| (1+y)^{\gamma+\beta_2} D^\beta \omega \right\|_{L^2} \left\| (1+y)^{\gamma+\alpha_2} \sqrt{\omega} \partial_y D^\alpha \omega \right\|_{L^2}$$

by young inequality

$$A_{61,22} \leq C_{m_1(s),\delta} \sum_{r=0}^{m_1(j)} \left\| \partial_t^r \partial_x^{2m_1(j)+m_2(j)-2r} p \right\|_{L^\infty(\mathbb{T})}^4 + C_{m_1(s),\delta} \left\| (1+y)^{\gamma+\beta_2(i,j)} D^{\beta(i,j)} \omega \right\|_{L^\infty}^{4(M(j)-1)}$$

$$\times \left\| (1+y)^{\gamma+\beta_2} D^\beta \omega \right\|_{L^2}^4 + \frac{1}{24} \left\| (1+y)^{\gamma+\alpha_2} \sqrt{\omega} \partial_y D^\alpha \omega \right\|_{L^2}^2$$

where $\beta = s-1$ or $\beta = s$ and $\beta(i,j) \leq s-2$. Then

$$\sum_{j=1}^{N(s)} A_{61,22} \leq C_{s,\delta} \sum_{r=0}^{\frac{s}{2}} \left\| \partial_t^r \partial_x^{2s-2r} p \right\|_{L^\infty(\mathbb{T})}^4$$

$$+ C_{s,\delta} \|\omega\|_{H_g^{s,\gamma}}^{4(M_s-1)} \|\omega\|_{H_g^{s,\gamma}}^4 + C_{s,\delta} \|\omega\|_{H_g^{s,\gamma}}^4$$

$$+ \frac{1}{24} \left\| (1+y)^{\gamma+\alpha_2} \sqrt{\omega} \partial_y D^\alpha \omega \right\|_{L^2}^2$$

- Estimate of $A_{61,23}$.

$$A_{61,23} \leq C_{m_1(s),\delta} \sum_{r=0}^{m_1(j)} \left\| \partial_t^r \partial_x^{2m_1(j)+m_2(j)-2r} p \right\|_{L^\infty(\mathbb{T})} \left\| (1+y)^{\gamma+\beta_2(i,j)} D^{\beta(i,j)} \omega \right\|_{L^\infty}^{M(j)-1} \\ \times \left\| (1+y)^{\gamma+\alpha_2} \sqrt{\omega} \partial_x^{\alpha_1} \partial_y^{\alpha_2+1} \omega \right\|_{L^2} \left\| (1+y)^{\gamma+\alpha_2} D^\alpha \omega \right\|_{L^2}$$

by young inequality

$$A_{61,23} \leq C_{m_1(s),\delta} \sum_{r=0}^{m_1(s)} \left\| \partial_t^r \partial_x^{2m_1(j)+m_2(j)-2r} p \right\|_{L^\infty(\mathbb{T})}^4 \\ + C_{m_1(s),\delta} \left\| (1+y)^{\gamma+\beta_2(i,j)} D^{\beta(i,j)} \omega \right\|_{L^\infty}^{4(M(j)-1)} \\ \times \left\| (1+y)^{\gamma+\alpha_2} D^\alpha \omega \right\|_{L^2}^4 \\ + \frac{1}{24} \left\| (1+y)^{\gamma+\alpha_2} \sqrt{\omega} \partial_x^{\alpha_1} \partial_y^{\alpha_2+1} \omega \right\|_{L^2}^2$$

where $\beta = s-1$ or $\beta = s$ and $\beta(i,j) \leq s-2$. Then

$$\sum_{j=1}^{N(s)} A_{61,23} \leq C_{s,\delta} \sum_{r=0}^{\frac{s}{2}} \left\| \partial_t^r \partial_x^{2s-2r} p \right\|_{L^\infty(\mathbb{T})}^4 \\ + C_{s,\delta} \|\omega\|_{H_g^{s,\gamma}}^{4(M_s-1)} \|\omega\|_{H_g^{s,\gamma}}^4 + C_{s,\delta} \|\omega\|_{H_g^{s,\gamma}}^4 \\ + \frac{1}{24} \left\| (1+y)^{\gamma+\alpha_2} \sqrt{\omega} \partial_x^{\alpha_1} \partial_y^{\alpha_2+1} \omega \right\|_{L^2}^2$$

We conclude that

$$|A_{61}| \leq C_{s,\delta} \sum_{r=0}^{\frac{s}{2}} \left\| \partial_t^r \partial_x^{2s-2r} p \right\|_{L^\infty(\mathbb{T})}^2 + C_{s,\delta} \sum_{r=0}^{\frac{s}{2}} \left\| \partial_t^r \partial_x^{2s-2r} p \right\|_{L^\infty(\mathbb{T})}^4 \\ + \frac{1}{24} \left\| (1+y)^{\gamma+\alpha_2} \sqrt{\omega} \partial_y D^\alpha \omega \right\|_{L^2}^2 \\ + C_{s,\delta} \|\omega\|_{H_g^{s,\gamma}}^{2M_s+2} + C_{s,\delta} \|\omega\|_{H_g^{s,\gamma}}^{4M_s} + C_{s,\delta} \|\omega\|_{H_g^{s,\gamma}}^4 \\ + \frac{1}{24} \left\| (1+y)^{\gamma+\alpha_2} \sqrt{\omega} \partial_x^{\alpha_1} \partial_y^{\alpha_2+1} \omega \right\|_{L^2}^2$$

- Estimate of A_{62}

We can apply (6.23) then

$$\left| \int_{\mathbb{T}} \partial_x^k \omega D^\alpha \omega \partial_x^{\alpha_1-k} \partial_y^{\alpha_2+1} \omega \Big|_{y=0} \right| \\ \leq C \left\{ \int_0^1 \int_{\mathbb{T}} |\partial_x^k \omega D^\alpha \omega \partial_x^{\alpha_1-k} \partial_y^{\alpha_2+1} \omega| dx dy + \int_0^1 \int_{\mathbb{T}} |\partial_y (\partial_x^k \omega D^\alpha \omega \partial_x^{\alpha_1-k} \partial_y^{\alpha_2+1} \omega)| dx dy \right\} \\ \leq C \left\{ \iint |\partial_x^k \omega D^\alpha \omega \partial_x^{\alpha_1-k} \partial_y^{\alpha_2+1} \omega| dx dy + \int_0^{+\infty} \int_{\mathbb{T}} |\partial_y (\partial_x^k \omega D^\alpha \omega \partial_x^{\alpha_1-k} \partial_y^{\alpha_2+1} \omega)| dx dy \right\}$$

and

For $1 \leq k \leq \alpha_1 \leq s-2$

$$\iint |\partial_x^k \omega D^\alpha \omega \partial_x^{\alpha_1-k} \partial_y^{\alpha_2+1} \omega| dx dy \\ \leq \iint (1+y)^\gamma |\partial_x^k \omega| (1+y)^{\gamma+\alpha_2} |D^\alpha \omega| (1+y)^{\gamma+\alpha_2+1} |\partial_x^{\alpha_1-k} \partial_y^{\alpha_2+1} \omega| dx dy \\ \leq \|(1+y)^\gamma \partial_x^k \omega\|_{L^\infty} \|(1+y)^{\gamma+\alpha_2} D^\alpha \omega\|_{L^2} \|(1+y)^{\gamma+\alpha_2+1} \partial_x^{\alpha_1-k} \partial_y^{\alpha_2+1} \omega\|_{L^2} \\ \leq C_{s,\gamma} \|\omega\|_{H_g^{s,\gamma}}^3$$

we have

$$\begin{aligned} & \partial_y(\partial_x^k \omega D^\alpha \omega \partial_x^{\alpha_1-k} \partial_y^{\alpha_2+1} \omega) \\ &= \partial_y \partial_x^k \omega D^\alpha \omega \partial_x^{\alpha_1-k} \partial_y^{\alpha_2+1} \omega + \partial_x^k \omega \partial_y D^\alpha \omega \partial_x^{\alpha_1-k} \partial_y^{\alpha_2+1} \omega + \partial_x^k \omega D^\alpha \omega \partial_x^{\alpha_1-k} \partial_y^{\alpha_2+2} \omega \end{aligned}$$

and

For $1 \leq k \leq \alpha_1 \leq s-3$

$$\begin{aligned} & \iint |\partial_y \partial_x^k \omega D^\alpha \omega \partial_x^{\alpha_1-k} \partial_y^{\alpha_2+1} \omega| dx dy \\ & \leq \iint (1+y)^{\gamma+1} |\partial_x^k \partial_y \omega| (1+y)^{\gamma+\alpha_2} |D^\alpha \omega| (1+y)^{\gamma+\alpha_2+1} |\partial_x^{\alpha_1-k} \partial_y^{\alpha_2+1} \omega| dx dy \\ & \leq \|(1+y)^\gamma \partial_x^k \partial_y \omega\|_{L^\infty} \|(1+y)^{\gamma+\alpha_2} D^\alpha \omega\|_{L^2} \|(1+y)^{\gamma+\alpha_2+1} \partial_x^{\alpha_1-k} \partial_y^{\alpha_2+1} \omega\|_{L^2} \\ & \leq C_{s,\gamma} \|\omega\|_{H_g^{s,\gamma}}^3 \end{aligned}$$

and

For $k = \alpha_1 = s-2$

$$\begin{aligned} & \iint |\partial_y \partial_x^k \omega D^\alpha \omega \partial_x^{\alpha_1-k} \partial_y^{\alpha_2+1} \omega| dx dy \\ & \leq \iint (1+y)^{\gamma+1} |\partial_x^k \partial_y \omega| (1+y)^{\gamma+\alpha_2} |D^\alpha \omega| (1+y)^{\gamma+\alpha_2+1} |\partial_x^{\alpha_1-k} \partial_y^{\alpha_2+1} \omega| dx dy \\ & \leq \|(1+y)^\gamma \partial_x^k \partial_y \omega\|_{L^2} \|(1+y)^{\gamma+\alpha_2} D^\alpha \omega\|_{L^2} \|(1+y)^{\gamma+\alpha_2+1} \partial_x^{\alpha_1-k} \partial_y^{\alpha_2+1} \omega\|_{L^\infty} \\ & \leq C_{s,\gamma} \|\omega\|_{H_g^{s,\gamma}}^3 \end{aligned}$$

and

$$\begin{aligned} & \iint |\partial_x^k \omega \partial_x^{\alpha_1-k} \partial_y^{\alpha_2+1} \omega \partial_y D^\alpha \omega| dx dy \\ &= \iint \frac{\sqrt{6}}{\sqrt{\omega}} |\partial_x^k \omega| |\partial_x^{\alpha_1-k} \partial_y^{\alpha_2+1} \omega| \frac{\sqrt{\omega}}{\sqrt{6}} |\partial_y D^\alpha \omega| dx dy \\ & \leq 3 \iint \frac{1}{\omega} |\partial_x^k \omega|^2 |\partial_x^{\alpha_1-k} \partial_y^{\alpha_2+1} \omega|^2 dx dy + \frac{1}{12} \iint \omega |\partial_y D^\alpha \omega|^2 dx dy \\ & \leq \frac{3}{\delta} \iint (1+y)^\sigma |\partial_x^k \omega|^2 |\partial_x^{\alpha_1-k} \partial_y^{\alpha_2+1} \omega|^2 dx dy + \frac{1}{12} \iint \omega |\partial_y D^\alpha \omega|^2 dx dy \\ & \leq \frac{3}{\delta} X + \frac{1}{12} \iint (1+y)^{2(\gamma+\alpha_2)} \omega |\partial_y D^\alpha \omega|^2 dx dy \end{aligned}$$

where

$$X = \iint (1+y)^\sigma |\partial_x^k \omega|^2 |\partial_x^{\alpha_1-k} \partial_y^{\alpha_2+1} \omega|^2 dx dy$$

For $1 \leq k \leq \alpha_1 \leq s-2$

$$\begin{aligned} X & \leq \iint |(1+y)^\sigma \partial_x^k \omega|^2 (1+y)^{2(\gamma+\alpha_2+1)} |\partial_x^{\alpha_1-k} \partial_y^{\alpha_2+1} \omega|^2 dx dy \\ & \leq \|(1+y)^\sigma \partial_x^k \omega\|_{L^\infty}^2 \|(1+y)^{\gamma+\alpha_2+1} \partial_x^{\alpha_1-k} \partial_y^{\alpha_2+1} \omega\|_{L^2}^2 \\ & \leq C_{s,\gamma} \|\omega\|_{H_g^{s,\gamma}}^4 \end{aligned}$$

Then

$$\iint |\partial_x^k \omega \partial_x^{\alpha_1-k} \partial_y^{\alpha_2+1} \omega \partial_y D^\alpha \omega| dx dy \leq C_{s,\gamma} \|\omega\|_{H_g^{s,\gamma}}^4 + \frac{1}{12} \|(1+y)^{\gamma+\alpha_2} \sqrt{\omega} \partial_y D^\alpha \omega\|_{L^2}^2$$

Estimate of the last term $\iint \partial_x^k \omega D^\alpha \omega \partial_x^{\alpha_1-k} \partial_y^{\alpha_2+2} \omega$
For $k=1$

$$\begin{aligned}
& \iint |\partial_x \omega D^\alpha \omega \partial_x^{\alpha_1-1} \partial_y^{\alpha_2+2} \omega| \\
&= \iint \frac{\sqrt{\omega}}{\sqrt{6}} |\partial_x^{\alpha_1-1} \partial_y^{\alpha_2+2} \omega| \frac{\sqrt{6}}{\sqrt{\omega}} |\partial_x \omega| |D^\alpha \omega| dx dy \\
&\leq \frac{1}{12} \iint \omega |\partial_x^{\alpha_1-1} \partial_y^{\alpha_2+2} \omega|^2 dx dy + 3 \iint \frac{1}{\omega} |\partial_x \omega|^2 |D^\alpha \omega|^2 dx dy \\
&\leq \frac{3}{\delta} \iint (1+y)^\sigma |\partial_x \omega|^2 |D^\alpha \omega|^2 dx dy + \frac{1}{12} \iint \omega |\partial_x^{\alpha_1-1} \partial_y^{\alpha_2+2} \omega|^2 dx dy \\
&\leq C_{s,\gamma} \|\omega\|_{H_g^{s,\gamma}}^4 + \frac{1}{12} \|(1+y)^{\gamma+\alpha_2+1} \sqrt{\omega} \partial_x^{\alpha_1-1} \partial_y^{\alpha_2+2} \omega\|_{L^2}^2
\end{aligned}$$

For $2 \leq k \leq \alpha_1 \leq s-2$

$$\begin{aligned}
& \iint |\partial_x^k \omega D^\alpha \omega \partial_x^{\alpha_1-k} \partial_y^{\alpha_2+2} \omega| dx dy \\
&\leq \iint (1+y)^\gamma |\partial_x^k \omega| (1+y)^{\gamma+\alpha_2} |D^\alpha \omega| (1+y)^{\gamma+\alpha_2+2} |\partial_x^{\alpha_1-k} \partial_y^{\alpha_2+2} \omega| dx dy \\
&\leq \|(1+y)^\gamma \partial_x^k \omega\|_{L^\infty} \|(1+y)^{\gamma+\alpha_2} D^\alpha \omega\|_{L^2} \|(1+y)^{\gamma+\alpha_2+2} \partial_x^{\alpha_1-k} \partial_y^{\alpha_2+2} \omega\|_{L^2} \\
&\leq C_{s,\gamma} \|\omega\|_{H_g^{s,\gamma}}^3
\end{aligned}$$

In conclusion

$$\begin{aligned}
A_{62} &\leq C_{s,\gamma} \|\omega\|_{H_g^{s,\gamma}}^3 + C_{s,\gamma} \|\omega\|_{H_g^{s,\gamma}}^4 + \frac{1}{12} \|(1+y)^{\gamma+\alpha_2} \sqrt{\omega} \partial_y D^\alpha \omega\|_{L^2}^2 \\
&\quad + \frac{1}{12} \|(1+y)^{\gamma+\alpha_2+1} \sqrt{\omega} \partial_x^{\alpha_1-1} \partial_y^{\alpha_2+2} \omega\|_{L^2}^2
\end{aligned}$$

This conclude the proof of the estimate on A_6 for the even case. \square

6.4.6 Proof of the estimate of A_6 in the odd case

α_2 is odd, that is $\alpha_2 = 2q + 1$ for some $q \in \mathbb{N}$. We recall

$$A_6 = 2 \int_{\mathbb{T}} \omega D^\alpha \omega \partial_y D^\alpha \omega|_{y=0}.$$

Proposition 6.4.4. *We have the following estimate on the term A_6 :
When α_2 is odd we have :*

$$\begin{aligned}
|A_6| &\leq C_{s,\delta} \sum_{r=0}^{\frac{s}{2}} \|\partial_t^r \partial_x^{s+1-2r} p\|_{L^\infty(\mathbb{T})}^2 + C_{s,\delta} \sum_{r=0}^{\frac{s}{2}} \|\partial_t^r \partial_x^{s+1-2r} p\|_{L^\infty(\mathbb{T})}^4 \\
&\quad + \frac{1}{12} \|(1+y)^{\gamma+\alpha_2+1} \sqrt{\omega} \partial_x^{\alpha_1-1} \partial_y^{\alpha_2+2} \omega\|_{L^2}^2 \\
&\quad + C_{s,\delta,\gamma} \{ \|\omega\|_{H_g^{s,\gamma}}^{2M_s} + 1 \} \|\omega\|_{H_g^{s,\gamma}}^2 + C_{s,\delta} \|\omega\|_{H_g^{s,\gamma}}^{4M_s} \\
&\quad + \frac{1}{6} \|(1+y)^{\gamma+\alpha_2} \sqrt{\omega} \partial_x^{\alpha_1} \partial_y^{\alpha_2+1} \omega\|_{L^2}^2 \\
&\quad + C_{s,\gamma,\sigma,\delta} \left\{ \|\omega\|_{H_g^{s,\gamma}} + \|\partial_x^s U\|_{L^2(\mathbb{T})} \right\}^2 \|\omega\|_{H_g^{s,\gamma}}^2 \\
&\quad + C_{s,\gamma,\sigma,\delta} \left\{ \|\omega\|_{H_g^{s,\gamma}} + \|\partial_x^s U\|_{L^2(\mathbb{T})} \right\} \{ \|\omega\|_{H_g^{s,\gamma}} + \|\omega\|_{H_g^{s,\gamma}}^2 \} \\
&\quad + \frac{1}{4} \|(1+y)^{\gamma+\alpha_2+1} \sqrt{\omega} \partial_x^{\alpha_1-1} \partial_y^{\alpha_2+2} \omega\|_{L^2}^2 \\
&\quad + \frac{1}{2c_s} \|(1+y)^\gamma \sqrt{\omega} \partial_y g_s\|_{L^2}^2 + C_{s,\gamma} \|\omega\|_{H_g^{s,\gamma}}^3 + C_{s,\gamma} \|\omega\|_{H_g^{s,\gamma}}^4
\end{aligned}$$

Proof. Using integration by parts in the x-variable, we have

$$A_6 = -2 \int_{\mathbb{T}} \partial_x (\omega \partial_x^{\alpha_1} \partial_y^{\alpha_2} \omega) \partial_x^{\alpha_1-1} \partial_y^{\alpha_2+1} \omega \Big|_{y=0}$$

Directly by Leibniz formula

$$\begin{aligned} \partial_x^{\alpha_1} (\omega \partial_y^{\alpha_2} \omega) &= \sum_{k=0}^{\alpha_1} \binom{\alpha_1}{k} \partial_x^k \omega \partial_x^{\alpha_1-k} \partial_y^{\alpha_2} \omega \\ &= \omega \partial_x^{\alpha_1} \partial_y^{\alpha_2} \omega + \sum_{k=1}^{\alpha_1} \binom{\alpha_1}{k} \partial_x^k \omega \partial_x^{\alpha_1-k} \partial_y^{\alpha_2} \omega \end{aligned}$$

Then

$$\omega \partial_x^{\alpha_1} \partial_y^{\alpha_2} \omega = \partial_x^{\alpha_1} (\omega \partial_y^{\alpha_2} \omega) - \sum_{k=1}^{\alpha_1} \binom{\alpha_1}{k} \partial_x^k \omega \partial_x^{\alpha_1-k} \partial_y^{\alpha_2} \omega$$

And

$$\begin{aligned} \partial_x (\omega \partial_x^{\alpha_1} \partial_y^{\alpha_2} \omega) &= \partial_x^{\alpha_1+1} (\omega \partial_y^{\alpha_2} \omega) - \sum_{k=1}^{\alpha_1} \binom{\alpha_1}{k} \partial_x (\partial_x^k \omega \partial_x^{\alpha_1-k} \partial_y^{\alpha_2} \omega) \\ &= \partial_x^{\alpha_1+1} (\omega \partial_y^{\alpha_2} \omega) - \sum_{k=1}^{\alpha_1} \binom{\alpha_1}{k} (\partial_x^{k+1} \omega \partial_x^{\alpha_1-k} \partial_y^{\alpha_2} \omega + \partial_x^k \omega \partial_x^{\alpha_1+1-k} \partial_y^{\alpha_2} \omega) \end{aligned}$$

Then

$$A_6 = \tilde{A}_{61} + \tilde{A}_{62} + \tilde{A}_{63}$$

Where

$$\begin{aligned} \tilde{A}_{61} &= -2 \int_{\mathbb{T}} \partial_x^{\alpha_1+1} (\omega \partial_y^{\alpha_2} \omega) \partial_x^{\alpha_1-1} \partial_y^{\alpha_2+1} \omega \Big|_{y=0} \\ \tilde{A}_{62} &= -2 \sum_{k=1}^{\alpha_1} \binom{\alpha_1}{k} \int_{\mathbb{T}} \partial_x^{k+1} \omega \partial_x^{\alpha_1-k} \partial_y^{\alpha_2} \omega \partial_x^{\alpha_1-1} \partial_y^{\alpha_2+1} \omega \Big|_{y=0} \\ \tilde{A}_{63} &= -2 \sum_{k=1}^{\alpha_1} \binom{\alpha_1}{k} \int_{\mathbb{T}} \partial_x^k \omega \partial_x^{\alpha_1+1-k} \partial_y^{\alpha_2} \omega \partial_x^{\alpha_1-1} \partial_y^{\alpha_2+1} \omega \Big|_{y=0} \end{aligned}$$

- Estimate of \tilde{A}_{61}

$$\begin{aligned} \tilde{A}_{61} &= -2 \int_{\mathbb{T}} \partial_x^{\alpha_1+1} (\omega \partial_y^{\alpha_2} \omega) \partial_x^{\alpha_1-1} \partial_y^{\alpha_2+1} \omega \Big|_{y=0} \\ &= \sum_{j=1}^{N(q)} C(j) \varepsilon^{2\ell(j)} \int_{\mathbb{T}} \omega^{-k(j)} \partial_x^{\alpha_1-1} \partial_y^{\alpha_2+1} \omega \left((\partial_t - \varepsilon^2 \partial_x^2)^{m_1(j)} \partial_x^{m_2(j)} p \right) \prod_{i=1}^{M(j)} D^{\beta(i,j)} \omega \Big|_{y=0} \end{aligned}$$

We apply the trace estimate (6.23) and proposition (6.10.1), the same in the case 1 we obtain

$$\begin{aligned} |\tilde{A}_{61}| &\leq C_{s,\delta} \sum_{r=0}^{\frac{s}{2}} \|\partial_t^r \partial_x^{2s+1-2r} p\|_{L^\infty(\mathbb{T})}^2 + C_{s,\delta} \sum_{r=0}^{\frac{s}{2}} \|\partial_t^r \partial_x^{2s+1-2r} p\|_{L^\infty(\mathbb{T})}^4 \\ &\quad + \frac{1}{12} \|(1+y)^{\gamma+\alpha_2+1} \sqrt{\omega} \partial_x^{\alpha_1-1} \partial_y^{\alpha_2+2} \omega\|_{L^2}^2 \\ &\quad + C_{s,\delta} \|\omega\|_{H_g^{s,\gamma}}^{2M_s} \|\omega\|_{H_g^{s,\gamma}}^2 + C_{s,\delta} \|\omega\|_{H_g^{s,\gamma}}^{4M_s} + C_{s,\delta} \|\omega\|_{H_g^{s,\gamma}}^4 \\ &\quad + \frac{1}{12} \|(1+y)^{\gamma+\alpha_2} \sqrt{\omega} \partial_x^{\alpha_1} \partial_y^{\alpha_2+1} \omega\|_{L^2}^2 \end{aligned}$$

- Estimate of \tilde{A}_{62}

We can apply (6.23) then

$$\left| \int_{\mathbb{T}} \partial_x^{k+1} \omega \partial_x^{\alpha_1 - k} \partial_y^{\alpha_2} \omega \partial_x^{\alpha_1 - 1} \partial_y^{\alpha_2 + 1} \omega|_{y=0} \right| \leq C \{X_1 + X_2\}$$

Where

$$\begin{aligned} X_1 &= \iint |\partial_x^{k+1} \omega \partial_x^{\alpha_1 - k} \partial_y^{\alpha_2} \omega \partial_x^{\alpha_1 - 1} \partial_y^{\alpha_2 + 1} \omega| dx dy \\ X_2 &= \int_0^{+\infty} \int_{\mathbb{T}} |\partial_y (\partial_x^{k+1} \omega \partial_x^{\alpha_1 - k} \partial_y^{\alpha_2} \omega \partial_x^{\alpha_1 - 1} \partial_y^{\alpha_2 + 1} \omega)| dx dy \end{aligned}$$

For $1 \leq k \leq \alpha_1 \leq s - 3$

$$\begin{aligned} X_1 &\leq \iint (1+y)^\gamma |\partial_x^{k+1} \omega| (1+y)^{\gamma + \alpha_2} |\partial_x^{\alpha_1 - k} \partial_y^{\alpha_2} \omega| (1+y)^{\gamma + \alpha_2 + 1} |\partial_x^{\alpha_1 - 1} \partial_y^{\alpha_2 + 1} \omega| dx dy \\ &\leq \|(1+y)^\gamma \partial_x^{k+1} \omega\|_{L^\infty} \|(1+y)^{\gamma + \alpha_2} \partial_x^{\alpha_1 - k} \partial_y^{\alpha_2} \omega\|_{L^2} \|(1+y)^{\gamma + \alpha_2 + 1} \partial_x^{\alpha_1 - 1} \partial_y^{\alpha_2 + 1} \omega\|_{L^2} \\ &\leq C_{s, \gamma} \|\omega\|_{H_g^{s, \gamma}}^3 \end{aligned}$$

If $\alpha_1 = s - 1$

For $k = s - 2$

$$\begin{aligned} X_1 &\leq \iint (1+y)^\gamma |\partial_x^{s-1} \omega| (1+y)^{\gamma+1} |\partial_x \partial_y \omega| (1+y)^{\gamma+2} |\partial_x^{s-2} \partial_y^2 \omega| dx dy \\ &\leq \|(1+y)^\gamma \partial_x^{s-1} \omega\|_{L^2} \|(1+y)^{\gamma+1} \partial_x \partial_y \omega\|_{L^\infty} \|(1+y)^{\gamma+2} \partial_x^{s-2} \partial_y^2 \omega\|_{L^2} \\ &\leq C_{s, \gamma, \sigma, \delta} \left\{ \|\omega\|_{H_g^{s, \gamma}} + \|\partial_x^s U\|_{L^2(\mathbb{T})} \right\} \|\omega\|_{H_g^{s, \gamma}}^2 \end{aligned}$$

For $k = s - 1$

$$\begin{aligned} X_1 &\leq \iint (1+y)^\gamma |\partial_x^s \omega| (1+y)^{\gamma+1} |\partial_y \omega| (1+y)^{\gamma + \alpha_2 + 1} |\partial_x^{s-2} \partial_y^2 \omega| dx dy \\ &\leq \|(1+y)^\gamma \partial_x^s \omega\|_{L^2} \|(1+y)^{\gamma+1} \partial_y \omega\|_{L^\infty} \|(1+y)^{\gamma+2} \partial_x^{s-2} \partial_y^2 \omega\|_{L^2} \\ &\leq C_{s, \gamma, \sigma, \delta} \left\{ \|\omega\|_{H_g^{s, \gamma}} + \|\partial_x^s U\|_{L^2(\mathbb{T})} \right\} \|\omega\|_{H_g^{s, \gamma}}^2 \end{aligned}$$

and

$$X_2 \leq X_2' + X_2'' + X_2'''$$

Where

$$\begin{aligned} X_2' &= \int_0^{+\infty} \int_{\mathbb{T}} |\partial_y \partial_x^{k+1} \omega \partial_x^{\alpha_1 - k} \partial_y^{\alpha_2} \omega \partial_x^{\alpha_1 - 1} \partial_y^{\alpha_2 + 1} \omega| dx dy \\ X_2'' &= \int_0^{+\infty} \int_{\mathbb{T}} |\partial_x^{k+1} \omega \partial_x^{\alpha_1 - k} \partial_y^{\alpha_2 + 1} \omega \partial_x^{\alpha_1 - 1} \partial_y^{\alpha_2 + 1} \omega| dx dy \\ X_2''' &= \int_0^{+\infty} \int_{\mathbb{T}} |\partial_x^{k+1} \omega \partial_x^{\alpha_1 - k} \partial_y^{\alpha_2} \omega \partial_x^{\alpha_1 - 1} \partial_y^{\alpha_2 + 2} \omega| dx dy \end{aligned}$$

We estimate each terms :

For $1 \leq k \leq \alpha_1 \leq s - 5$

$$\begin{aligned} X_2' &\leq \iint (1+y)^{\gamma+1} |\partial_y \partial_x^{k+1} \omega| (1+y)^{\gamma + \alpha_2} |\partial_x^{\alpha_1 - k} \partial_y^{\alpha_2} \omega| (1+y)^{\gamma + \alpha_2 + 1} |\partial_x^{\alpha_1 - 1} \partial_y^{\alpha_2 + 1} \omega| dx dy \\ &\leq \|(1+y)^{\gamma+1} \partial_y \partial_x^{k+1} \omega\|_{L^\infty} \|(1+y)^{\gamma + \alpha_2} \partial_x^{\alpha_1 - k} \partial_y^{\alpha_2} \omega\|_{L^2} \|(1+y)^{\gamma + \alpha_2 + 1} \partial_x^{\alpha_1 - 1} \partial_y^{\alpha_2 + 1} \omega\|_{L^2} \\ &\leq C_{s, \gamma} \|\omega\|_{H_g^{s, \gamma}}^3 \end{aligned}$$

If $\alpha_1 = s - 3$,
For $s - 4 \leq k \leq s - 3$

$$\begin{aligned} X'_2 &\leq \iint (1+y)^{\gamma+1} |\partial_y \partial_x^{k+1} \omega| (1+y)^{\gamma+\alpha_2} |\partial_x^{\alpha_1-k} \partial_y^{\alpha_2} \omega| (1+y)^{\gamma+\alpha_2+1} |\partial_x^{\alpha_1-1} \partial_y^{\alpha_2+1} \omega| dx dy \\ &\leq \|(1+y)^{\gamma+1} \partial_y \partial_x^{k+1} \omega\|_{L^2} \|(1+y)^{\gamma+\alpha_2} \partial_x^{\alpha_1-k} \partial_y^{\alpha_2} \omega\|_{L^\infty} \|(1+y)^{\gamma+\alpha_2+1} \partial_x^{\alpha_1-1} \partial_y^{\alpha_2+1} \omega\|_{L^2} \\ &\leq C_{s,\gamma} \|\omega\|_{H_g^{s,\gamma}}^3 \end{aligned}$$

If $\alpha_1 = s - 1$
For $k = s - 2$

$$\begin{aligned} X'_2 &\leq \iint (1+y)^{\gamma+1} |\partial_y \partial_x^{s-1} \omega| (1+y)^{\gamma+1} |\partial_x \partial_y \omega| (1+y)^{\gamma+2} |\partial_x^{s-2} \partial_y^2 \omega| dx dy \\ &\leq \|(1+y)^{\gamma+1} \partial_y \partial_x^{s-1} \omega\|_{L^2} \|(1+y)^{\gamma+1} \partial_x \partial_y \omega\|_{L^\infty} \|(1+y)^{\gamma+2} \partial_x^{s-2} \partial_y^2 \omega\|_{L^2} \\ &\leq C_{s,\gamma} \|\omega\|_{H_g^{s,\gamma}}^3 \end{aligned}$$

For $k = s - 1$
We have

$$g_s = \partial_x^s \omega - \frac{\partial_y \omega}{\omega} \partial_x^s (u - U)$$

Then

$$\partial_x^s \omega = g_s + \frac{\partial_y \omega}{\omega} \partial_x^s (u - U)$$

and

$$\partial_y \partial_x^s \omega = \partial_y g_s + \partial_y (a) \partial_x^s (u - U) + a \partial_x^s \omega$$

and

$$X'_2 = \int_0^{+\infty} \int_{\mathbb{T}} |\partial_y \partial_x^s \omega \partial_y \omega \partial_x^{s-2} \partial_y^2 \omega| dx dy$$

$$\begin{aligned} |X'_2| &\leq \int_0^{+\infty} \int_{\mathbb{T}} |\partial_y g_s \partial_y \omega \partial_x^{s-2} \partial_y^2 \omega| dx dy + \int_0^{+\infty} \int_{\mathbb{T}} |a \partial_y \omega \partial_x^s \omega \partial_x^{s-2} \partial_y^2 \omega| dx dy \\ &\quad + \int_0^{+\infty} \int_{\mathbb{T}} |\partial_y a \partial_y \omega \partial_x^s (u - U) \partial_x^{s-2} \partial_y^2 \omega| dx dy \end{aligned}$$

We have :

$$\begin{aligned} |\partial_y \omega| &\leq \delta^{-1} (1+y)^{-\sigma-1} \\ \|(1+y)a\|_{L^\infty} &\leq \frac{1}{\delta} \\ \|\partial_y a\|_{L^\infty} &\leq \delta^{-2} + \delta^{-4} \end{aligned}$$

Then

$$\begin{aligned} &\int_0^{+\infty} \int_{\mathbb{T}} |a \partial_y \omega \partial_x^s \omega \partial_x^{s-2} \partial_y^2 \omega| dx dy \\ &\leq \int_0^{+\infty} \int_{\mathbb{T}} |(1+y)a| |\partial_y \omega| (1+y)^\gamma |\partial_x^s \omega| (1+y)^{\gamma+2} |\partial_x^{s-2} \partial_y^2 \omega| dx dy \\ &\leq \frac{1}{\delta} \|(1+y)a\|_{L^\infty} \|(1+y)^\gamma \partial_x^s \omega\|_{L^2} \|(1+y)^{\gamma+2} \partial_x^{s-2} \partial_y^2 \omega\|_{L^2} \\ &\leq C_{s,\gamma,\sigma,\delta} \left\{ \|\omega\|_{H_g^{s,\gamma}} + \|\partial_x^s U\|_{L^2(\mathbb{T})} \right\} \|\omega\|_{H_g^{s,\gamma}} \end{aligned}$$

and

$$\begin{aligned}
& \int_0^{+\infty} \int_{\mathbb{T}} |\partial_y a \partial_y \omega \partial_x^s \omega \partial_x^{s-2} \partial_y^2 \omega| dx dy \\
& \leq \int_0^{+\infty} \int_{\mathbb{T}} |\partial_y a| |\partial_y \omega| (1+y)^\gamma |\partial_x^s \omega| (1+y)^{\gamma+2} |\partial_x^{s-2} \partial_y^2 \omega| dx dy \\
& \leq \frac{1}{\delta} \|\partial_y a\|_{L^\infty} \|(1+y)^\gamma \partial_x^s \omega\|_{L^2} \|(1+y)^{\gamma+2} \partial_x^{s-2} \partial_y^2 \omega\|_{L^2} \\
& \leq C_{s,\gamma,\sigma,\delta} \left\{ \|\omega\|_{H_g^{s,\gamma}} + \|\partial_x^s U\|_{L^2(\mathbb{T})} \right\} \|\omega\|_{H_g^{s,\gamma}}
\end{aligned}$$

We denote $C_{\alpha_1} = \sum_{k=1}^{\alpha_1} \binom{\alpha_1}{k}$ and $c_s = \sum_{\substack{|\alpha| \leq s \\ \alpha_1 \leq s-1}} 1$. Then

$$\begin{aligned}
& \left| \iint \partial_y g_s \partial_y \omega \partial_x^{s-2} \partial_y^2 \omega \right| \\
& \leq \frac{1}{2C_{\alpha_1} c_s} \iint (1+y)^{2\gamma} \omega |\partial_y g_s|^2 + \frac{C_{\alpha_1} c_s}{2} \iint \frac{1}{\omega} (1+y)^{2\gamma} |\partial_y \omega|^2 |\partial_x^{s-2} \partial_y^2 \omega|^2 \\
& \leq \frac{1}{2C_{\alpha_1} c_s} \|(1+y)^\gamma \sqrt{\omega} \partial_y g_s\|_{L^2}^2 + \frac{C_{\alpha_1} c_s}{2\delta} \iint (1+y)^{2\gamma+\sigma} |\partial_y \omega|^2 |\partial_x^{s-2} \partial_y^2 \omega|^2 \\
& \leq \frac{1}{2C_{\alpha_1} c_s} \|(1+y)^\gamma \sqrt{\omega} \partial_y g_s\|_{L^2}^2 + \frac{C_{\alpha_1} c_s}{2\delta} \iint (1+y)^{\sigma+1} |\partial_y \omega|^2 (1+y)^{2(\gamma+2)} |\partial_x^{s-2} \partial_y^2 \omega|^2 \\
& \leq \frac{1}{2C_{\alpha_1} c_s} \|(1+y)^\gamma \sqrt{\omega} \partial_y g_s\|_{L^2}^2 + \frac{C_{\alpha_1} c_s}{2\delta^2} \|(1+y)^{\gamma+2} \partial_x^{s-2} \partial_y^2 \omega\|_{L^2}^2
\end{aligned}$$

and

$$\begin{aligned}
& \left| \sum_{k=1}^{\alpha_1} \binom{\alpha_1}{k} \iint \partial_y g_s \partial_y \omega \partial_x^{s-2} \partial_y^2 \omega \right| \\
& \leq \frac{1}{2c_s} \|(1+y)^\gamma \sqrt{\omega} \partial_y g_s\|_{L^2}^2 + \frac{C_{\alpha_1}^2 c_s}{2\delta^2} \|(1+y)^{\gamma+2} \partial_x^{s-2} \partial_y^2 \omega\|_{L^2}^2 \\
& \leq \frac{1}{2c_s} \|(1+y)^\gamma \sqrt{\omega} \partial_y g_s\|_{L^2}^2 + C_{s,\delta} \|\omega\|_{H_g^{s,\gamma}}^2.
\end{aligned}$$

In the proof of the proposition, since we consider the sum $\sum_{\substack{|\alpha| \leq s \\ \alpha_1 \leq s-1}}$, we will get the term

$$\frac{1}{2} \|(1+y)^\gamma \sqrt{\omega} \partial_y g_s\|_{L^2}^2$$

Estimate of X_2''

For $1 \leq k \leq \alpha_1 \leq s-3$

$$\begin{aligned}
X_1 & \leq \iint (1+y)^\gamma |\partial_x^{k+1} \omega| (1+y)^{\gamma+\alpha_2+1} |\partial_x^{\alpha_1-k} \partial_y^{\alpha_2+1} \omega| (1+y)^{\gamma+\alpha_2+1} |\partial_x^{\alpha_1-1} \partial_y^{\alpha_2+1} \omega| dx dy \\
& \leq \|(1+y)^\gamma \partial_x^{k+1} \omega\|_{L^\infty} \|(1+y)^{\gamma+\alpha_2+1} \partial_x^{\alpha_1-k} \partial_y^{\alpha_2+1} \omega\|_{L^2} \|(1+y)^{\gamma+\alpha_2+1} \partial_x^{\alpha_1-1} \partial_y^{\alpha_2+1} \omega\|_{L^2} \\
& \leq C_{s,\gamma} \|\omega\|_{H_g^{s,\gamma}}^3
\end{aligned}$$

If $\alpha_1 = s-1$

For $k = s-2$

$$\begin{aligned}
X_1 & \leq \iint (1+y)^\gamma |\partial_x^{s-1} \omega| (1+y)^{\gamma+1} |\partial_x \partial_y^2 \omega| (1+y)^{\gamma+2} |\partial_x^{s-2} \partial_y^2 \omega| dx dy \\
& \leq \|(1+y)^\gamma \partial_x^{s-1} \omega\|_{L^2} \|(1+y)^{\gamma+1} \partial_x \partial_y^2 \omega\|_{L^\infty} \|(1+y)^{\gamma+2} \partial_x^{s-2} \partial_y^2 \omega\|_{L^2} \\
& \leq C_{s,\gamma,\sigma,\delta} \left\{ \|\omega\|_{H_g^{s,\gamma}} + \|\partial_x^s U\|_{L^2(\mathbb{T})} \right\} \|\omega\|_{H_g^{s,\gamma}}^2
\end{aligned}$$

For $k = s - 1$

$$\begin{aligned} X_1 &\leq \iint (1+y)^\gamma |\partial_x^s \omega| (1+y)^{\gamma+1} |\partial_y^2 \omega| (1+y)^{\gamma+2} |\partial_x^{s-2} \partial_y^2 \omega| dx dy \\ &\leq \|(1+y)^\gamma \partial_x^s \omega\|_{L^2} \|(1+y)^{\gamma+1} \partial_y^2 \omega\|_{L^\infty} \|(1+y)^{\gamma+2} \partial_x^{s-2} \partial_y^2 \omega\|_{L^2} \\ &\leq C_{s,\gamma,\sigma,\delta} \left\{ \|\omega\|_{H_g^{s,\gamma}} + \|\partial_x^s U\|_{L^2(\mathbb{T})} \right\} \|\omega\|_{H_g^{s,\gamma}}^2 \end{aligned}$$

Estimate of X_2''' :

$$\begin{aligned} &\int_0^{+\infty} \int_{\mathbb{T}} |\partial_x^{k+1} \omega \partial_x^{\alpha_1-k} \partial_y^{\alpha_2} \omega \partial_x^{\alpha_1-1} \partial_y^{\alpha_2+2} \omega| dx dy \\ &= \iint \frac{\sqrt{\omega}}{\sqrt{6C_{\alpha_1}}} |\partial_x^{\alpha_1-1} \partial_y^{\alpha_2+2} \omega| \frac{\sqrt{6C_{\alpha_1}}}{\sqrt{\omega}} |\partial_x^{k+1} \omega| |\partial_x^{\alpha_1-k} \partial_y^{\alpha_2} \omega| dx dy \\ &\leq \frac{1}{12C_{\alpha_1}} \iint \omega |\partial_x^{\alpha_1-1} \partial_y^{\alpha_2+2} \omega|^2 dx dy + 3C_{\alpha_1} \iint \frac{1}{\omega} |\partial_x^{k+1} \omega|^2 |\partial_x^{\alpha_1-k} \partial_y^{\alpha_2} \omega|^2 dx dy \\ &\leq \frac{3C_{\alpha_1}}{\delta} \iint |\partial_x^{k+1} \omega|^2 |\partial_x^{\alpha_1-k} \partial_y^{\alpha_2} \omega|^2 dx dy + \frac{1}{12C_{\alpha_1}} \iint \omega |\partial_x^{\alpha_1-1} \partial_y^{\alpha_2+2} \omega|^2 dx dy \\ &\leq \frac{3C_s}{\delta} \iint |\partial_x^{k+1} \omega|^2 |\partial_x^{\alpha_1-k} \partial_y^{\alpha_2} \omega|^2 dx dy + \frac{1}{12C_{\alpha_1}} \|(1+y)^{\gamma+\alpha_2+1} \sqrt{\omega} \partial_x^{\alpha_1-1} \partial_y^{\alpha_2+2} \omega\|_{L^2}^2 \end{aligned}$$

Then

$$\begin{aligned} &\sum_{k=1}^{\alpha_1} \binom{\alpha_1}{k} \int_0^{+\infty} \int_{\mathbb{T}} |\partial_x^{k+1} \omega \partial_x^{\alpha_1-k} \partial_y^{\alpha_2} \omega \partial_x^{\alpha_1-1} \partial_y^{\alpha_2+2} \omega| dx dy \\ &\leq \frac{3C_s}{\delta} \sum_{k=1}^{\alpha_1} \binom{\alpha_1}{k} \iint |\partial_x^{k+1} \omega|^2 |\partial_x^{\alpha_1-k} \partial_y^{\alpha_2} \omega|^2 dx dy + \frac{1}{12} \|(1+y)^{\gamma+\alpha_2+1} \sqrt{\omega} \partial_x^{\alpha_1-1} \partial_y^{\alpha_2+2} \omega\|_{L^2}^2 \end{aligned}$$

Estimate of $\sum_{k=1}^{\alpha_1} \binom{\alpha_1}{k} \iint |\partial_x^{k+1} \omega|^2 |\partial_x^{\alpha_1-k} \partial_y^{\alpha_2} \omega|^2 dx dy$

For $1 \leq k \leq \alpha_1 < s - 2$

$$\begin{aligned} &\iint |\partial_x^{k+1} \omega|^2 |\partial_x^{\alpha_1-k} \partial_y^{\alpha_2} \omega|^2 dx dy \\ &\leq \iint |(1+y)^\gamma \partial_x^{k+1} \omega|^2 (1+y)^{2(\gamma+\alpha_2)} |\partial_x^{\alpha_1-k} \partial_y^{\alpha_2} \omega|^2 dx dy \\ &\leq \|(1+y)^\gamma \partial_x^{k+1} \omega\|_{L^\infty} \|(1+y)^{\gamma+\alpha_2} \partial_x^{\alpha_1-k} \partial_y^{\alpha_2} \omega\|_{L^2}^2 \\ &\leq C_{s,\gamma} \|\omega\|_{H_g^{s,\gamma}}^4 \end{aligned}$$

For $\alpha_1 = s - 1$ ($k = s - 2, s - 1$)

$$\begin{aligned} &\iint |\partial_x^{k+1} \omega|^2 |\partial_x^{\alpha_1-k} \partial_y^{\alpha_2} \omega|^2 dx dy \\ &\leq \iint |\partial_x^{\alpha_1-k} \partial_y^{\alpha_2} \omega|^2 |(1+y)^{2\gamma} \partial_x^{k+1} \omega|^2 dx dy \\ &\leq \|(1+y)^{\gamma+\alpha_2} \partial_x^{\alpha_1-k} \partial_y^{\alpha_2} \omega\|_{L^\infty}^2 \|(1+y)^\gamma \partial_x^{k+1} \omega\|_{L^2}^2 \\ &\leq C_{s,\gamma,\sigma,\delta} \left\{ \|\omega\|_{H_g^{s,\gamma}} + \|\partial_x^s U\|_{L^2(\mathbb{T})} \right\}^2 \|\omega\|_{H_g^{s,\gamma}}^2 \end{aligned}$$

• Estimate of \tilde{A}_{63}

We can apply (6.23) then

$$\left| 2 \sum_{k=1}^{\alpha_1} \binom{\alpha_1}{k} \int_{\mathbb{T}} \partial_x^k \omega \partial_x^{\alpha_1+1-k} \partial_y^{\alpha_2} \omega \partial_x^{\alpha_1-1} \partial_y^{\alpha_2+1} \omega|_{y=0} \right| \leq C \{W_1 + W_2\}$$

Where

$$W_1 = \iint |\partial_x^k \omega \partial_x^{\alpha_1+1-k} \partial_y^{\alpha_2} \omega \partial_x^{\alpha_1-1} \partial_y^{\alpha_2+1} \omega| dx dy$$

$$W_2 = \int_0^{+\infty} \int_{\mathbb{T}} |\partial_y (\partial_x^k \omega \partial_x^{\alpha_1+1-k} \partial_y^{\alpha_2} \omega \partial_x^{\alpha_1-1} \partial_y^{\alpha_2+1} \omega)| dx dy$$

Estimate of W_1
 $1 \leq k \leq \alpha_1 < s - 2$

$$W_1 \leq \iint (1+y)^\gamma |\partial_x^k \omega| (1+y)^{\gamma+\alpha_2} |\partial_x^{\alpha_1+1-k} \partial_y^{\alpha_2} \omega| (1+y)^{\gamma+\alpha_2+1} |\partial_x^{\alpha_1-1} \partial_y^{\alpha_2+1} \omega| dx dy$$

$$\leq \|(1+y)^\gamma \partial_x^k \omega\|_{L^\infty} \|(1+y)^{\gamma+\alpha_2+1} \partial_x^{\alpha_1+1-k} \partial_y^{\alpha_2} \omega\|_{L^2} \|(1+y)^{\gamma+\alpha_2+1} \partial_x^{\alpha_1-1} \partial_y^{\alpha_2+1} \omega\|_{L^2}$$

$$\leq C_{s,\gamma} \|\omega\|_{H_g^{s,\gamma}}^3$$

For $\alpha_1 = s - 1$ ($k = s - 2, s - 1$)

$$W_1 \leq \iint (1+y)^\gamma |\partial_x^k \omega| (1+y)^{\gamma+\alpha_2} |\partial_x^{\alpha_1+1-k} \partial_y^{\alpha_2} \omega| (1+y)^{\gamma+\alpha_2+1} |\partial_x^{\alpha_1-1} \partial_y^{\alpha_2+1} \omega| dx dy$$

$$\leq \|(1+y)^\gamma \partial_x^k \omega\|_{L^2} \|(1+y)^{\gamma+\alpha_2+1} \partial_x^{\alpha_1+1-k} \partial_y^{\alpha_2} \omega\|_{L^\infty} \|(1+y)^{\gamma+\alpha_2+1} \partial_x^{\alpha_1-1} \partial_y^{\alpha_2+1} \omega\|_{L^2}$$

$$\leq C_{s,\gamma} \|\omega\|_{H_g^{s,\gamma}}^3$$

And

$$W_2 \leq W_2' + W_2'' + W_2'''$$

Where

$$W_2' = \int_0^{+\infty} \int_{\mathbb{T}} |\partial_y \partial_x^k \omega \partial_x^{\alpha_1+1-k} \partial_y^{\alpha_2} \omega \partial_x^{\alpha_1-1} \partial_y^{\alpha_2+1} \omega| dx dy$$

$$W_2'' = \int_0^{+\infty} \int_{\mathbb{T}} |\partial_x^k \omega \partial_x^{\alpha_1+1-k} \partial_y^{\alpha_2+1} \omega \partial_x^{\alpha_1-1} \partial_y^{\alpha_2+1} \omega| dx dy$$

$$W_2''' = \int_0^{+\infty} \int_{\mathbb{T}} |\partial_x^k \omega \partial_x^{\alpha_1+1-k} \partial_y^{\alpha_2} \omega \partial_x^{\alpha_1-1} \partial_y^{\alpha_2+2} \omega| dx dy$$

Estimate of W_2' :
For $1 \leq k \leq \alpha_1 \leq s - 3$,

$$W_2' \leq \iint (1+y)^{\gamma+1} |\partial_y \partial_x^k \omega| (1+y)^{\gamma+\alpha_2} |\partial_x^{\alpha_1+1-k} \partial_y^{\alpha_2} \omega| (1+y)^{\gamma+\alpha_2+1} |\partial_x^{\alpha_1-1} \partial_y^{\alpha_2+1} \omega| dx dy$$

$$\leq \|(1+y)^{\gamma+1} \partial_y \partial_x^k \omega\|_{L^\infty} \|(1+y)^{\gamma+\alpha_2} \partial_x^{\alpha_1+1-k} \partial_y^{\alpha_2} \omega\|_{L^2} \|(1+y)^{\gamma+\alpha_2+1} \partial_x^{\alpha_1-1} \partial_y^{\alpha_2+1} \omega\|_{L^2}$$

$$\leq C_{s,\gamma} \|\omega\|_{H_g^{s,\gamma}}^3$$

If $\alpha_1 = s - 1$
For $s - 2 \leq k \leq s - 1$,

$$W_2' \leq \iint (1+y)^{\gamma+1} |\partial_y \partial_x^k \omega| (1+y)^{\gamma+\alpha_2} |\partial_x^{\alpha_1+1-k} \partial_y^{\alpha_2} \omega| (1+y)^{\gamma+\alpha_2+1} |\partial_x^{\alpha_1-1} \partial_y^{\alpha_2+1} \omega| dx dy$$

$$\leq \|(1+y)^{\gamma+1} \partial_y \partial_x^k \omega\|_{L^2} \|(1+y)^{\gamma+\alpha_2} \partial_x^{\alpha_1+1-k} \partial_y^{\alpha_2} \omega\|_{L^\infty} \|(1+y)^{\gamma+\alpha_2+1} \partial_x^{\alpha_1-1} \partial_y^{\alpha_2+1} \omega\|_{L^2}$$

$$\leq C_{s,\gamma} \|\omega\|_{H_g^{s,\gamma}}^3$$

Estimate of W_2''

For $k = 1$

$$\begin{aligned}
W_2'' &= \iint \frac{\sqrt{\omega}}{\sqrt{6}} |\partial_x^{\alpha_1} \partial_y^{\alpha_2+1} \omega| \frac{\sqrt{6}}{\sqrt{\omega}} |\partial_x \omega| |\partial_x^{\alpha_1-1} \partial_y^{\alpha_2+1} \omega| dx dy \\
&\leq \frac{1}{12} \iint \omega |\partial_x^{\alpha_1} \partial_y^{\alpha_2+1} \omega|^2 + 3 \iint \frac{1}{\omega} |\partial_x \omega|^2 |\partial_x^{\alpha_1-1} \partial_y^{\alpha_2+1} \omega|^2 \\
&\leq \frac{1}{12} \iint (1+y)^{2(\gamma+\alpha_2)} \omega |\partial_x^{\alpha_1} \partial_y^{\alpha_2+1} \omega|^2 + \frac{3}{\delta} \iint (1+y)^{2\sigma} |\partial_x \omega|^2 |\partial_x^{\alpha_1-1} \partial_y^{\alpha_2+1} \omega|^2 \\
&\leq \frac{1}{12} \|(1+y)^{\gamma+\alpha_2} \sqrt{\omega} \partial_x^{\alpha_1} \partial_y^{\alpha_2+1} \omega\|_{L^2}^2 + C_\delta \|(1+y)^{\gamma+\alpha_2+1} \partial_x^{\alpha_1-1} \partial_y^{\alpha_2+1} \omega\|_{L^2}^2 \\
&\leq \frac{1}{12} \|(1+y)^{\gamma+\alpha_2} \sqrt{\omega} \partial_x^{\alpha_1} \partial_y^{\alpha_2+1} \omega\|_{L^2}^2 + C_\delta \|\omega\|_{H_g^{s,\gamma}}^2
\end{aligned}$$

For $2 \leq k \leq \alpha_1 \leq s-3$

$$\begin{aligned}
W_2'' &\leq \int_0^{+\infty} \int_{\mathbb{T}} |\partial_x^k \omega \partial_x^{\alpha_1+1-k} \partial_y^{\alpha_2+1} \omega \partial_x^{\alpha_1-1} \partial_y^{\alpha_2+1} \omega| dx dy \\
&\leq \|(1+y)^{\gamma+1} \partial_x^k \omega\|_{L^\infty} \|(1+y)^{\gamma+\alpha_2} \partial_x^{\alpha_1+1-k} \partial_y^{\alpha_2+1} \omega\|_{L^2} \|(1+y)^{\gamma+\alpha_2+1} \partial_x^{\alpha_1-1} \partial_y^{\alpha_2+1} \omega\|_{L^2} \\
&\leq C_{s,\gamma} \|\omega\|_{H_g^{s,\gamma}}^3
\end{aligned}$$

If $\alpha_1 = s-1$

For $s-2 \leq k \leq s-1$

$$\begin{aligned}
W_2'' &\leq \int_0^{+\infty} \int_{\mathbb{T}} |\partial_x^k \omega \partial_x^{\alpha_1+1-k} \partial_y^{\alpha_2+1} \omega \partial_x^{\alpha_1-1} \partial_y^{\alpha_2+1} \omega| dx dy \\
&\leq \|(1+y)^{\gamma+1} \partial_x^k \omega\|_{L^2} \|(1+y)^{\gamma+\alpha_2} \partial_x^{\alpha_1+1-k} \partial_y^{\alpha_2+1} \omega\|_{L^\infty} \|(1+y)^{\gamma+\alpha_2+1} \partial_x^{\alpha_1-1} \partial_y^{\alpha_2+1} \omega\|_{L^2} \\
&\leq C_{s,\gamma} \|\omega\|_{H_g^{s,\gamma}}^3
\end{aligned}$$

Estimate of W_2''' :

$$\begin{aligned}
&\int_0^{+\infty} \int_{\mathbb{T}} |\partial_x^k \omega \partial_x^{\alpha_1+1-k} \partial_y^{\alpha_2} \omega \partial_x^{\alpha_1-1} \partial_y^{\alpha_2+2} \omega| dx dy \\
&= \iint \frac{\sqrt{\omega}}{\sqrt{6C_{\alpha_1}}} |\partial_x^{\alpha_1-1} \partial_y^{\alpha_2+2} \omega| \frac{\sqrt{6C_{\alpha_1}}}{\sqrt{\omega}} |\partial_x^k \omega| |\partial_x^{\alpha_1+1-k} \partial_y^{\alpha_2} \omega| dx dy \\
&\leq \frac{1}{12C_{\alpha_1}} \iint \omega |\partial_x^{\alpha_1-1} \partial_y^{\alpha_2+2} \omega|^2 dx dy + 3C_{\alpha_1} \iint \frac{1}{\omega} |\partial_x^k \omega|^2 |\partial_x^{\alpha_1+1-k} \partial_y^{\alpha_2} \omega|^2 dx dy \\
&\leq \frac{3C_{\alpha_1}}{\delta} \iint |\partial_x^k \omega|^2 |\partial_x^{\alpha_1+1-k} \partial_y^{\alpha_2} \omega|^2 dx dy + \frac{1}{12C_{\alpha_1}} \iint \omega |\partial_x^{\alpha_1-1} \partial_y^{\alpha_2+2} \omega|^2 dx dy \\
&\leq \frac{3C_s}{\delta} \iint |\partial_x^k \omega|^2 |\partial_x^{\alpha_1+1-k} \partial_y^{\alpha_2} \omega|^2 dx dy + \frac{1}{12C_{\alpha_1}} \|(1+y)^{\gamma+\alpha_2+1} \sqrt{\omega} \partial_x^{\alpha_1-1} \partial_y^{\alpha_2+2} \omega\|_{L^2}^2
\end{aligned}$$

Then

$$\begin{aligned}
&\sum_{k=1}^{\alpha_1} \binom{\alpha_1}{k} \int_0^{+\infty} \int_{\mathbb{T}} |\partial_x^k \omega \partial_x^{\alpha_1+1-k} \partial_y^{\alpha_2} \omega \partial_x^{\alpha_1-1} \partial_y^{\alpha_2+2} \omega| dx dy \\
&\leq \frac{3C_s}{\delta} \sum_{k=1}^{\alpha_1} \binom{\alpha_1}{k} \iint |\partial_x^k \omega|^2 |\partial_x^{\alpha_1+1-k} \partial_y^{\alpha_2} \omega|^2 dx dy + \frac{1}{12} \|(1+y)^{\gamma+\alpha_2+1} \sqrt{\omega} \partial_x^{\alpha_1-1} \partial_y^{\alpha_2+2} \omega\|_{L^2}^2
\end{aligned}$$

Estimate of $\sum_{k=1}^{\alpha_1} \binom{\alpha_1}{k} \iint |\partial_x^k \omega|^2 |\partial_x^{\alpha_1+1-k} \partial_y^{\alpha_2} \omega|^2 dx dy$

For $1 \leq k \leq \alpha_1 < s - 2$

$$\begin{aligned}
& \iint |\partial_x^k \omega|^2 |\partial_x^{\alpha_1+1-k} \partial_y^{\alpha_2} \omega|^2 dx dy \\
& \leq \iint |(1+y)^\gamma \partial_x^k \omega|^2 (1+y)^{2(\gamma+\alpha_2)} |\partial_x^{\alpha_1+1-k} \partial_y^{\alpha_2} \omega|^2 dx dy \\
& \leq \|(1+y)^\gamma \partial_x^k \omega\|_{L^\infty} \|(1+y)^{\gamma+\alpha_2} \partial_x^{\alpha_1+1-k} \partial_y^{\alpha_2} \omega\|_{L^2}^2 \\
& \leq C_{s,\gamma} \|\omega\|_{H_g^{s,\gamma}}^4
\end{aligned}$$

For $\alpha_1 = s - 1$ ($k = s - 2, s - 1$)

$$\begin{aligned}
& \iint |\partial_x^k \omega|^2 |\partial_x^{\alpha_1+1-k} \partial_y^{\alpha_2} \omega|^2 dx dy \\
& \leq \iint |\partial_x^{\alpha_1+1-k} \partial_y^{\alpha_2} \omega|^2 (1+y)^{2\gamma} |\partial_x^k \omega|^2 dx dy \\
& \leq \|(1+y)^{\gamma+\alpha_2} \partial_x^{\alpha_1+1-k} \partial_y^{\alpha_2} \omega\|_{L^\infty}^2 \|(1+y)^\gamma \partial_x^k \omega\|_{L^2}^2 \\
& \leq C_{s,\gamma,\sigma,\delta} \left\{ \|\omega\|_{H_g^{s,\gamma}} + \|\partial_x^s U\|_{L^2(\mathbb{T})} \right\}^2 \|\omega\|_{H_g^{s,\gamma}}^2
\end{aligned}$$

This concludes the proof of proposition 6.4.4 (the odd case), and therefore the proof of the main proposition 6.4.2. \square

6.4.7 Conclusion

Proof. of proposition 6.4.1 From the energy equality 6.20 and the estimates of each term A_i of the left hand side, we get the estimate 6.18 This concludes the proof of proposition 6.4.1. \square

6.5 Estimate for the cancellation term

In this section, we derive the weighted L^2 estimate for the cancellation term g_s^ε (see (6.8))

$$g_s^\varepsilon := \partial_x^s \omega^\varepsilon - \frac{\partial_y \omega}{\omega} \partial_x^s (u - U)$$

by using the standard energy methods. Indeed, this term allows to eliminate the problematic term $v \partial_y \omega$, which cannot be directly estimate.

Proposition 6.5.1. (L^2 Estimate for g_s)

$$\begin{aligned}
\frac{1}{2} \frac{d}{dt} \|(1+y)^\gamma g_s\|_{L^2}^2 & \leq -\frac{\varepsilon^2}{2} \|(1+y)^\gamma \partial_x g_s\|_{L^2}^2 - \frac{5}{8} \|(1+y)^\gamma \omega^{\frac{1}{2}} \partial_y g_s\|_{L^2}^2 \\
& + C_{\delta,\gamma} \|(1+y)^\gamma g_s\|_{L^2}^2 + \frac{1}{4} \|(1+y)^{\gamma+1} \sqrt{\omega} \partial_y^2 \partial_x^{s-1} \omega\|_{L^2}^2 \\
& + C_{s,\gamma,\sigma,\delta} \{ \|\omega\|_{H_g^{s,\gamma}} + \|\partial_x^s U\|_{L^2(\mathbb{T})} \} \|\omega\|_{H_g^{s,\gamma}}^2 \\
& + C_{s,\gamma,\sigma,\delta} \{ 1 + \|\omega\|_{H_g^{s,\gamma}} + \|\partial_x^{s+1} U\|_{L^2(\mathbb{T})} \} \\
& \times \{ \|\omega\|_{H_g^{s,\gamma}} + \|\partial_x^{s+1} U\|_{L^2(\mathbb{T})} \} \{ \|\omega\|_{H_g^{s,\gamma}} + \|\omega\|_{H_g^{s,\gamma}}^2 \} \\
& + C_{s,\gamma,\sigma,\delta} \|\partial_x^{s+1} U\|_{L^\infty} \{ \|\omega\|_{H_g^{s,\gamma}} + \|\partial_x^s U\|_{L^2(\mathbb{T})} \} \|\omega\|_{H_g^{s,\gamma}} \\
& + C_\delta \|\omega\|_{H_g^{s,\gamma}}^2 \|\partial_x^s U\|_{L^\infty(\mathbb{T})} + C_{s,\gamma,\delta} \|\omega\|_{H_g^{s,\gamma}}^2 \\
& + C_{s,\gamma} \|\omega\|_{H_g^{s,\gamma}}^3 + C_{s,\gamma,\delta} \|\omega\|_{H_g^{s,\gamma}}^4 + C_{\sigma,\delta} \|\partial_x^{s+1} p\|_{L^2(\mathbb{T})}
\end{aligned}$$

6.5.1 Derivation of equations for g_s^ε

Let us recall the evolution equations for the vorticity

$$\partial_t \omega + u \partial_x \omega + v \partial_y \omega = \varepsilon^2 \partial_x^2 \omega + 2\omega \partial_y^2 \omega + 2(\partial_y \omega)^2 \quad (6.11)$$

and the velocity

$$\begin{aligned} \partial_t(u - U) + u \partial_x(u - U) + v \partial_y(u - U) \\ = \varepsilon^2 \partial_x^2(u - U) + 2\partial_y(u - U) \partial_y^2(u - U) - (u - U) \partial_x U \end{aligned} \quad (6.28)$$

where we applied the regularized Bernoulli's law (??) LABEL in the derivation of (6.28). Since our aim is to control the H^s -norm of ω (or $u-U$), let us differentiate (6.11) s times with respect to x . Then we have

$$\begin{aligned} \partial_t(\partial_x^s \omega) + u \partial_x(\partial_x^s \omega) + v \partial_y(\partial_x^s \omega) + \partial_x^s v \partial_y \omega \\ = 2\omega \partial_y^2(\partial_x^s \omega) + 2 \sum_{j=1}^s \binom{s}{j} \partial_x^j \omega \partial_y^2(\partial_x^{s-1} \omega) + 2\partial_x^s((\partial_y \omega)^2) + \varepsilon^2 \partial_x^2(\partial_x^s \omega) \\ - \sum_{j=0}^{s-1} \binom{s}{j} \partial_x^{s-j} u \partial_x^{j+1} \omega - \sum_{j=0}^{s-1} \binom{s}{j} \partial_x^{s-j} v \partial_x^j \partial_y \omega \end{aligned} \quad (6.29)$$

and

$$\begin{aligned} \partial_t(\partial_x^s(u - U)) + u \partial_x(\partial_x^s(u - U)) + v \partial_y(\partial_x^s(u - U)) + \partial_x^s v \omega \\ = \varepsilon^2 \partial_x^2(\partial_x^s(u - U)) + 2\omega \partial_y^2(\partial_x^s(u - U)) + 2 \sum_{j=1}^s \binom{s}{j} \partial_x^j \omega \partial_x^{s-j} \partial_y \omega \\ - \sum_{j=1}^s \binom{s}{j} \partial_x^{s-j} u \partial_x^{s+j}(u - U) - \sum_{j=1}^s \binom{s}{j} \partial_x^{s-j} v \partial_x^j \omega \\ - \sum_{j=1}^s \binom{s}{j} \partial_x^j(u - U) \partial_x^{s-j+1} U \end{aligned} \quad (6.30)$$

where we applied the fact that $\omega = \partial_y(u - U)$. The main obstacle in (6.29) and (6.30) is the term $\partial_x^s v = -\partial_x^{s+1} \partial_y^{-1} u$, which has $s+1$ x -derivatives so that the standard energy estimates cannot apply. However, since there are two equations (6.29) and (6.30), we can eliminate the problematic term $\partial_x^s v$ by subtracting them in an appropriate way.

Subtracting $\frac{\partial_y \omega}{\omega} \times$ (6.30) from (6.29), we obtain

$$\begin{aligned} \partial_t g_s^\varepsilon + u \partial_x g_s + v \partial_y g_s + (\partial_t a^\varepsilon + u \partial_x a + v \partial_y a - \varepsilon^2 \partial_x^2 a - 2\omega \partial_y^2 a) \partial_x^s(u - U) \\ = \varepsilon^2 \partial_x^2 g_s + 2\varepsilon^2 \partial_x a \partial_x(\partial_x^s(u - U)) + 2\omega \partial_y^2 g_s \\ + 4\omega \partial_y a \partial_y(\partial_x^s(u - U)) + 2 \sum_{j=1}^s \binom{s}{j} \partial_x^j \omega \partial_y^2(\partial_x^{s-1} \omega) - 2a \sum_{j=1}^s \binom{s}{j} \partial_x^j \omega \partial_x^{s-j} \partial_y \omega \\ - \sum_{j=0}^{s-1} \binom{s}{j} g_{j+1} \partial_x^{s-j} u + 2\partial_x^s((\partial_y \omega)^2) - \sum_{j=1}^s \binom{s}{j} \partial_x^{s-j} v \{ \partial_x^j \partial_y \omega - a \partial_x^j \omega \} \\ + a \sum_{j=1}^s \binom{s}{j} \partial_x^j(u - U) \partial_x^{s-j+1} U. \end{aligned} \quad (6.31)$$

We recall the vorticity equation (6.11)

$$\partial_t \omega + u \partial_x \omega + v \partial_y \omega = \varepsilon^2 \partial_x^2 \omega + 2\omega \partial_y^2 \omega + 2(\partial_y \omega)^2 \quad (6.11)$$

Differentiating (6.11) with respect to y once, we obtain :

$$\begin{aligned} \partial_t(\partial_y\omega) + u\partial_x(\partial_y\omega) + v\partial_y(\partial_y\omega) \\ = 4(\partial_y\omega)\partial_y^2\omega + 2\omega\partial_y^3\omega + 2\partial_y\omega\partial_y^2\omega + \varepsilon^2\partial_x^2(\partial_y\omega) - \omega\partial_x\omega + \partial_x u\partial_y\omega. \end{aligned}$$

Using the previous identity and (6.11) we can compute :

$$\begin{aligned} (\partial_t + u\partial_x + v\partial_y)a &= \frac{(\partial_t + u\partial_x + v\partial_y)\partial_y\omega}{\omega} - \frac{\partial_y\omega(\partial_t + u\partial_x + v\partial_y)\omega}{\omega^2} \\ &= \varepsilon^2 \left[\frac{\partial_x^2(\partial_y\omega)}{\omega} - a \frac{\partial_x^2\omega}{\omega} \right] + 2\omega \left[\frac{\partial_y^3\omega}{\omega} - a \frac{\partial_y^2\omega}{\omega} \right] \\ &\quad + 6a\partial_y^2\omega - 2a^2\partial_y\omega - \partial_x\omega + a\partial_x u \end{aligned}$$

and we have by direct differentiation of a

$$\begin{cases} \partial_x^2 a = \frac{\partial_x^2(\partial_y\omega)}{\omega} - a \frac{\partial_x^2\omega}{\omega} - 2 \frac{\partial_x\omega}{\omega} \partial_x a \\ \partial_y^2 a = \frac{\partial_y^3\omega}{\omega} - a \frac{\partial_y^2\omega}{\omega} - 2 a \partial_y a. \end{cases}$$

Then

$$\begin{aligned} (\partial_t + u\partial_x + v\partial_y)a &= \varepsilon^2 \left(\partial_x^2 a + 2 \frac{\partial_x\omega}{\omega} \partial_x a \right) + 2\omega (\partial_y^2 a + 2 a \partial_y a) + 6a\partial_y^2\omega \\ &\quad - 2a^2\partial_y\omega - \partial_x\omega + a\partial_x u. \end{aligned}$$

It follows that

$$\begin{aligned} (\partial_t + u\partial_x + v\partial_y - \varepsilon^2\partial_x^2 - 2\omega\partial_y^2)a &= 2\varepsilon^2 \frac{\partial_x\omega}{\omega} \partial_x a + 4a \partial_y a \omega + 6a\partial_y^2\omega \\ &\quad - 2a^2\partial_y\omega - \partial_x\omega + a\partial_x u. \end{aligned}$$

The equation (6.31) become

$$\begin{aligned} \partial_t g_s^\varepsilon + u\partial_x g_s + v\partial_y g_s \\ + \left(2\varepsilon^2 \frac{\partial_x\omega}{\omega} \partial_x a + 4a \partial_y a \omega + 6a\partial_y^2\omega - 2a^2\partial_y\omega - \partial_x\omega + a\partial_x u \right) \partial_x^s (u - U) \\ = \varepsilon^2 \partial_x^2 g_s + 2\varepsilon^2 \partial_x a \partial_x^{s+1} (u - U) + 2\omega \partial_y^2 g_s \\ + 4\omega \partial_y a \partial_y (\partial_x^s (u - U)) + 2 \sum_{j=1}^s \binom{s}{j} \partial_x^j \omega \partial_y^2 (\partial_x^{s-1} \omega) - 2a \sum_{j=1}^s \binom{s}{j} \partial_x^j \omega \partial_x^{s-j} \partial_y \omega \\ - \sum_{j=0}^{s-1} \binom{s}{j} g_{j+1} \partial_x^{s-j} u + 2\partial_x^s ((\partial_y\omega)^2) - \sum_{j=1}^s \binom{s}{j} \partial_x^{s-j} v \{ \partial_x^j \partial_y \omega - a \partial_x^j \omega \} \\ + a \sum_{j=1}^s \binom{s}{j} \partial_x^j (u - U) \partial_x^{s-j+1} U. \end{aligned}$$

We modify the previous equation

$$\begin{aligned}
& \partial_t g_s^\varepsilon + u \partial_x g_s + v \partial_y g_s \\
& + \left(4a \partial_y a \omega + 6a \partial_y^2 \omega - 2a^2 \partial_y \omega - \partial_x \omega + a \partial_x u \right) \partial_x^s (u - U) \\
& = \varepsilon^2 \partial_x^2 g_s + 2\omega \partial_y^2 g_s + 4\omega \partial_y a \partial_x^s \omega \\
& + 2\varepsilon^2 \partial_x a \left(\partial_x^{s+1} (u - U) - \frac{\partial_x \omega}{\omega} \partial_x^s (u - U) \right) \\
& + 2 \sum_{j=1}^s \binom{s}{j} \partial_x^j \omega \partial_y^2 \partial_x^{s-j} \omega - 2a \sum_{j=1}^s \binom{s}{j} \partial_x^j \omega \partial_x^{s-j} \partial_y \omega \\
& - \sum_{j=0}^{s-1} \binom{s}{j} g_{j+1} \partial_x^{s-j} u + 2 \partial_x^s ((\partial_y \omega)^2) \\
& - \sum_{j=1}^s \binom{s}{j} \partial_x^{s-j} v \{ \partial_x^j \partial_y \omega - a \partial_x^j \omega \} + a \sum_{j=1}^s \binom{s}{j} \partial_x^j (u - U) \partial_x^{s-j+1} U. \quad (6.32)
\end{aligned}$$

6.5.2 Derivation of the energy term for g_s^ε

Multiplying the evolution equation 6.32 by $(1+y)^{2\gamma} g_s^\varepsilon$, and then integrating over $\mathbb{T} \times \mathbb{R}^+$, we derive the following energy identity :

$$\frac{1}{2} \frac{d}{dt} \|(1+y)^\gamma g_s\|_{L^2}^2 = \varepsilon^2 \iint (1+y)^{2\gamma} g_s \partial_x^2 g_s + 2 \iint (1+y)^{2\gamma} \omega g_s \partial_y^2 g_s + R \quad (6.33)$$

$$R = \sum_{k=1}^5 B_k + \sum_{k=1}^6 I_k + J.$$

where

$$\begin{aligned}
B_1 &= - \iint (1+y)^{2\gamma} g_s \{ u \partial_x g_s + v \partial_y g_s \} \\
B_2 &= 2\varepsilon^2 \iint (1+y)^{2\gamma} g_s \partial_x a \left\{ \partial_x^{s+1} (u - U) - \frac{\partial_x \omega}{\omega} \partial_x^s (u - U) \right\} \\
B_3 &= - \sum_{j=0}^{s-1} \binom{s}{j} \iint (1+y)^{2\gamma} g_s g_{j+1} \partial_x^{s-j} u \\
B_4 &= - \sum_{j=0}^{s-1} \binom{s}{j} \iint (1+y)^{2\gamma} g_s \partial_x^{s-j} v \{ \partial_x^j \partial_y \omega - a \partial_x^j \omega \} \\
B_5 &= \sum_{j=0}^{s-1} \binom{s}{j} \iint (1+y)^{2\gamma} g_s \partial_x^j (u - U) \partial_x^{s-j+1} U
\end{aligned}$$

and

$$\begin{aligned}
I_1 &= -4 \iint (1+y)^{2\gamma} g_s a \omega \partial_y a \partial_x^s \omega \\
I_2 &= -2 \sum_{j=1}^s \binom{s}{j} \iint (1+y)^{2\gamma} g_s a \partial_x^j \omega \partial_x^{s-j} \partial_y \omega \\
I_3 &= -6 \iint (1+y)^{2\gamma} g_s a \partial_y^2 \omega \partial_x^s (u-U) \\
I_4 &= 2 \iint (1+y)^{2\gamma} g_s a^2 \partial_y \omega \partial_x^s (u-U) \\
I_5 &= + \iint (1+y)^{2\gamma} g_s \partial_x \omega \partial_x^s (u-U) \\
I_6 &= - \iint (1+y)^{2\gamma} g_s a \partial_x u \partial_x^s (u-U).
\end{aligned}$$

$$J = 2 \iint (1+y)^{2\gamma} g_s \partial_x^s ((\partial_y \omega)^2) + 2 \sum_{j=1}^s \binom{s}{j} \iint (1+y)^{2\gamma} g_s \partial_x^j \omega \partial_y^2 (\partial_x^{s-j} \omega).$$

6.5.3 Estimate of the control terms

Lemma 6.5.1. *We have*

$$\begin{aligned}
&\frac{1}{2} \frac{d}{dt} \|(1+y)^\gamma g_s\|_{L^2}^2 + \varepsilon^2 \|(1+y)^\gamma \partial_x g_s\|_{L^2}^2 + 2 \|(1+y)^\gamma (\omega)^{\frac{1}{2}} \partial_y g_s\|_{L^2}^2 \\
&\leq R - 4\gamma \iint (1+y)^{2\gamma-1} \omega g_s \partial_y g_s - 2 \iint (1+y)^{2\gamma} \partial_y \omega g_s \partial_y g_s + 2 \int_{\mathbb{T}} \omega g_s \partial_y g_s|_{y=0}.
\end{aligned} \tag{6.34}$$

• We estimate the two control terms in the second right hand-side of the energy identity (6.33). For the first term, we get

$$\begin{aligned}
\varepsilon^2 \iint (1+y)^{2\gamma} g_s \partial_x^2 g_s &= -\varepsilon^2 \iint (1+y)^{2\gamma} \partial_x g_s \partial_x g_s \\
&= -\varepsilon^2 \|(1+y)^\gamma \partial_x g_s\|_{L^2}^2.
\end{aligned}$$

For the second term, we have

$$\begin{aligned}
2 \iint (1+y)^{2\gamma} \omega g_s \partial_y^2 g_s &= -2 \iint \partial_y ((1+y)^{2\gamma} \omega g_s) \partial_y g_s + 2 \int_{\mathbb{T}} \omega g_s \partial_y g_s|_{y=0} \\
&= -4\gamma \iint (1+y)^{2\gamma-1} \omega g_s \partial_y g_s - 2 \iint (1+y)^{2\gamma} \partial_y \omega g_s \partial_y g_s \\
&\quad - 2 \iint (1+y)^{2\gamma} \omega \partial_y g_s \partial_y g_s + 2 \int_{\mathbb{T}} \omega g_s \partial_y g_s|_{y=0} \\
&= -2 \|(1+y)^\gamma (\omega)^{\frac{1}{2}} \partial_y g_s\|_{L^2}^2 - 4\gamma \iint (1+y)^{2\gamma-1} \omega g_s \partial_y g_s \\
&\quad - 2 \iint (1+y)^{2\gamma} \partial_y \omega g_s \partial_y g_s + 2 \int_{\mathbb{T}} \omega g_s \partial_y g_s|_{y=0}.
\end{aligned}$$

We then estimate

$$\begin{aligned}
2 \iint (1+y)^{2\gamma} \omega g_s \partial_y^2 g_s &\leq - \|(1+y)^\gamma (\omega)^{\frac{1}{2}} \partial_y g_s\|_{L^2}^2 + \frac{1}{2} \|(1+y)^\gamma (\omega)^{\frac{1}{2}} \partial_y g_s\|_{L^2}^2 \\
&\quad + 8\gamma^2 \iint (1+y)^{2\gamma-1} \omega (g_s)^2 + \frac{1}{4} \|(1+y)^\gamma (\omega)^{\frac{1}{2}} \partial_y g_s\|_{L^2}^2 \\
&\quad + \iint (1+y)^{2\gamma} \frac{(\partial_y \omega)^2}{\omega} (g_s)^2 + \int_{\mathbb{T}} \omega g_s \partial_y g_s|_{y=0} \\
&\leq -2 \|(1+y)^\gamma (\omega)^{\frac{1}{2}} \partial_y g_s\|_{L^2}^2 + \frac{1}{2} \|(1+y)^\gamma (\omega)^{\frac{1}{2}} \partial_y g_s\|_{L^2}^2 \\
&\quad + C_{\delta,\gamma} \|(1+y)^\gamma (g_s)\|_{L^2}^2 + \frac{1}{4} \|(1+y)^\gamma (\omega)^{\frac{1}{2}} \partial_y g_s\|_{L^2}^2 \\
&\quad + \left\| \frac{(\partial_y \omega)^2}{\omega} \right\|_{L^\infty} \|(1+y)^\gamma (g_s)\|_{L^2}^2 + 2 \int_{\mathbb{T}} \omega g_s \partial_y g_s|_{y=0} \\
&\leq -\frac{5}{4} \|(1+y)^\gamma (\omega)^{\frac{1}{2}} \partial_y g_s\|_{L^2}^2 + C_{\delta,\gamma} \|(1+y)^\gamma g_s\|_{L^2}^2 + 2 \int_{\mathbb{T}} \omega g_s \partial_y g_s|_{y=0}.
\end{aligned}$$

6.5.4 Estimate of the boundary term $2 \int_{\mathbb{T}} \omega g_s \partial_y g_s|_{y=0}$

We have :

$$g_s = \partial_x^s \omega - \frac{\partial_y \omega}{\omega} \partial_x^s (u - U)$$

Then

$$\begin{aligned}
\partial_y g_s &= \partial_x^s \partial_y \omega - \partial_y \left(\frac{\partial_y \omega}{\omega} \right) \partial_x^s (u - U) - \frac{\partial_y \omega}{\omega} \partial_x^s \omega \\
&= \partial_x^s \partial_y \omega - \frac{\partial_y^2 \omega \omega - (\partial_y \omega)^2}{\omega^2} \partial_x^s (u - U) - \frac{\partial_y \omega}{\omega} \partial_x^s \omega \\
&= \partial_x^s \partial_y \omega - \frac{\partial_y^2 \omega}{\omega} \partial_x^s (u - U) + \left(\frac{\partial_y \omega}{\omega} \right)^2 \partial_x^s (u - U) - \frac{\partial_y \omega}{\omega} \partial_x^s \omega
\end{aligned}$$

So

$$\begin{aligned}
\partial_y g_s|_{y=0} &= \partial_x^s \partial_y \omega + \frac{\partial_y^2 \omega}{\omega} \partial_x^s (U) - \frac{\partial_y \omega}{\omega} g_s|_{y=0} \\
\partial_x^s (\omega \partial_y \omega) &= \sum_{j=0}^{j=s} \binom{s}{j} \partial_x^j \omega \partial_x^{s-j} \partial_y \omega \\
&= \omega \partial_x^s \partial_y \omega + \sum_{j=1}^{j=s} \binom{s}{j} \partial_x^j \omega \partial_x^{s-j} \partial_y \omega
\end{aligned}$$

Therefore

$$\partial_x^s \partial_y \omega = \frac{1}{\omega} \left[\partial_x^s (\omega \partial_y \omega) - \sum_{j=1}^{j=s} \binom{s}{j} \partial_x^j \omega \partial_x^{s-j} \partial_y \omega \right]$$

Then

$$\partial_y g_s|_{y=0} = \frac{1}{2\omega} \partial_x^{s+1} p^\varepsilon - \frac{1}{\omega} \partial_x^s \omega \partial_y \omega - \frac{1}{\omega} \sum_{j=1}^{j=s-1} \binom{s}{j} \partial_x^j \omega \partial_x^{s-j} \partial_y \omega + \frac{\partial_y^2 \omega}{\omega} \partial_x^s (U) - \frac{\partial_y \omega}{\omega} g_s|_{y=0}$$

Thus

$$\begin{aligned}
2 \int_{\mathbb{T}} \omega g_s \partial_y g_s|_{y=0} &= \int_{\mathbb{T}} g_s \partial_x^{s+1} p^\varepsilon|_{y=0} - 2 \int_{\mathbb{T}} g_s \partial_x^s \omega \partial_y \omega|_{y=0} - 2 \sum_{j=1}^{j=s-1} \binom{s}{j} \int_{\mathbb{T}} g \partial_x^j \omega \partial_x^{s-j} \partial_y \omega|_{y=0} \\
&\quad + 2 \int_{\mathbb{T}} g_s \partial_y^2 \omega \partial_x^s U|_{y=0} - 2 \int_{\mathbb{T}} g_s^2 \partial_y \omega|_{y=0}
\end{aligned}$$

Using the simple trace estimate : We have

$$\left| 2 \int_{\mathbb{T}} g_s \partial_x^{s+1} p^\varepsilon dx \Big|_{y=0} \right| \leq C \|\omega\|_{H_g^{s,\gamma}}^2 + \frac{1}{4} \|(1+y)^\gamma \sqrt{\omega} \partial_y g_s\|_{L^2}^2 + C_{\sigma,\delta} \|\partial_x^{s+1} p\|_{L^2(\mathbb{T})}$$

and

$$\left| 2 \int_{\mathbb{T}} g_s \partial_y^2 \omega \partial_x^s U \Big|_{y=0} \right| \leq C_\delta \|\omega\|_{H_g^{s,\gamma}}^2 \|\partial_x^s U\|_{L^\infty(\mathbb{T})} + \frac{1}{4} \|(1+y)^\gamma \sqrt{\omega} \partial_y g_s\|_{L^2}^2$$

and

$$\left| 2 \int_{\mathbb{T}} g_s^2 \partial_y \omega \Big|_{y=0} \right| \leq C_{s,\gamma} \|\omega\|_{H_g^{s,\gamma}}^2 + \frac{1}{4} \|(1+y)^\gamma \sqrt{\omega} \partial_y g_s\|_{L^2}^2 + C_{s,\gamma,\delta} \|\omega\|_{H_g^{s,\gamma}}^2$$

and

$$\left| 2 \int_{\mathbb{T}} g_s \partial_x^s \omega \partial_y \omega \Big|_{y=0} \right| \leq C_{s,\gamma,\sigma,\delta} \left\{ \|\omega\|_{H_g^{s,\gamma}} + \|\partial_x^s U\|_{L^2} \right\} \|\omega\|_{H_g^{s,\gamma}}^2 + \|(1+y)^\gamma \sqrt{\omega} \partial_y g_s\|_{L^2}^2 + C_{s,\gamma,\delta} \|\omega\|_{H_g^{s,\gamma}}^2$$

Estimation of the term $2 \sum_{j=1}^s \binom{s}{j} \int_{\mathbb{T}} g_s \partial_x^j \omega \partial_y \partial_x^{s-j} \omega \Big|_{y=0}$ We use the trace estimate we have

$$\int_{\mathbb{T}} g_s \partial_x^j \omega \partial_y \partial_x^{s-j} \omega \Big|_{y=0} \leq C \int_0^1 \int_{\mathbb{T}} |g_s \partial_x^j \omega \partial_y \partial_x^{s-j} \omega| dx dy + C \int_0^1 \int_{\mathbb{T}} |\partial_y (g_s \partial_x^j \omega \partial_y \partial_x^{s-j} \omega)| dx dy$$

Estimation of the term

$$\sum_{j=1}^s \binom{s}{j} \iint g_s \partial_x^j \omega \partial_y (\partial_x^{s-j} \omega)$$

if $1 \leq j \leq s-2$

$$\begin{aligned} \iint g_s \partial_x^j \omega \partial_y (\partial_x^{s-j} \omega) &\leq \iint (1+y)^\gamma g_s (1+y)^\gamma \partial_x^j \omega (1+y)^{\gamma+1} \partial_y (\partial_x^{s-j} \omega) \\ &\leq \|(1+y)^\gamma \partial_x^j \omega\|_{L^\infty} \|(1+y)^\gamma g_s\|_{L^2} \|(1+y)^{\gamma+1} \partial_y \partial_x^{s-j} \omega\|_{L^2} \\ &\leq C_{s,\gamma} \|\omega\|_{H_g^{s,\gamma}}^3 \end{aligned}$$

for $j = s-1$

$$\begin{aligned} \iint g_s \partial_x^j \omega \partial_y (\partial_x^{s-j} \omega) &\leq \iint (1+y)^\gamma g_s (1+y)^\gamma \partial_x^j \omega (1+y)^{\gamma+1} \partial_y (\partial_x^{s-j} \omega) \\ &\leq \|(1+y)^\gamma \partial_x^j \omega\|_{L^2} \|(1+y)^\gamma g_s\|_{L^2} \|(1+y)^{\gamma+1} \partial_y \partial_x^{s-j} \omega\|_{L^\infty} \\ &\leq C_{s,\gamma} \|\omega\|_{H_g^{s,\gamma}}^3 \end{aligned}$$

for $j = s$

$$\begin{aligned} \iint g_s \partial_x^j \omega \partial_y (\partial_x^{s-j} \omega) &\leq \iint (1+y)^\gamma g_s (1+y)^\gamma \partial_x^j \omega (1+y)^{\gamma+1} \partial_y (\partial_x^{s-j} \omega) \\ &\leq \|(1+y)^\gamma \partial_x^j \omega\|_{L^2} \|(1+y)^\gamma g_s\|_{L^2} \|(1+y)^{\gamma+1} \partial_y \partial_x^{s-j} \omega\|_{L^\infty} \\ &\leq C_{s,\gamma,\sigma,\delta} \left\{ \|\omega\|_{H_g^{s,\gamma}} + \|\partial_x^s U\|_{L^2} \right\} \|\omega\|_{H_g^{s,\gamma}}^2 \end{aligned}$$

Then

$$\sum_{j=1}^s \binom{s}{j} \iint g_s \partial_x^j \omega \partial_y (\partial_x^{s-j} \omega) \leq C_{s,\gamma} \|\omega\|_{H_g^{s,\gamma}}^3 + C_{s,\gamma,\sigma,\delta} \left\{ \|\omega\|_{H_g^{s,\gamma}} + \|\partial_x^s U\|_{L^2} \right\} \|\omega\|_{H_g^{s,\gamma}}^2$$

we have

$$\partial_y (g_s \partial_x^j \omega \partial_y \partial_x^{s-j} \omega) = \partial_y g_s \partial_x^j \omega \partial_y \partial_x^{s-j} \omega + g_s \partial_y \partial_x^j \omega \partial_y \partial_x^{s-j} \omega + g_s \partial_x^j \omega \partial_y^2 \partial_x^{s-j} \omega$$

Then

$$\int_0^1 \int_{\mathbb{T}} |\partial_y (g_s \partial_x^j \omega \partial_y \partial_x^{s-j} \omega)| dx dy \leq V_1 + V_2 + V_3$$

where

$$\begin{aligned} V_1 &= \int_0^1 \int_{\mathbb{T}} |\partial_y g_s \partial_x^j \omega \partial_y \partial_x^{s-j} \omega| \\ V_2 &= \int_0^1 \int_{\mathbb{T}} |g_s \partial_y \partial_x^j \omega \partial_y \partial_x^{s-j} \omega| \\ V_3 &= \int_0^1 \int_{\mathbb{T}} |g_s \partial_x^j \omega \partial_y^2 \partial_x^{s-j} \omega| \end{aligned}$$

• **Estimate of the Term**

$$V_1 = \sum_{j=1}^s \binom{s}{j} \iint (1+y)^{2\gamma} \partial_y g_s \partial_x^j \omega \partial_y (\partial_x^{s-j} \omega)$$

We denote $C_s = \sum_{j=1}^s \binom{s}{j}$

$$\begin{aligned} & \left| \iint (1+y)^{2\gamma} \partial_y g_s \partial_x^j \omega \partial_y (\partial_x^{s-j} \omega) \right| \\ & \leq \frac{1}{4C_s} \iint (1+y)^{2\gamma} \omega |\partial_y g_s|^2 + C_s \iint \frac{1}{\omega} (1+y)^{2\gamma} |\partial_x^j \omega|^2 |\partial_y (\partial_x^{s-j} \omega)|^2 \\ & \leq \frac{1}{4C_s} \|(1+y)^\gamma \sqrt{\omega} \partial_y g_s\|_{L^2}^2 + \frac{C_s}{\delta} \iint (1+y)^{2\gamma+\sigma} |\partial_x^j \omega|^2 |\partial_y (\partial_x^{s-j} \omega)|^2 \\ & \leq \frac{1}{4C_s} \|(1+y)^\gamma \sqrt{\omega} \partial_y g_s\|_{L^2}^2 + \frac{C_s}{2\delta} \iint (1+y)^{2\gamma} |\partial_x^j \omega|^2 (1+y)^{2(\sigma+1)} |\partial_y (\partial_x^{s-j} \omega)|^2 \end{aligned}$$

Then

$$\begin{aligned} |V_1| & \leq \frac{1}{4} \|(1+y)^\gamma \sqrt{\omega} \partial_y g_s\|_{L^2}^2 \\ & \quad + \frac{C_s}{\delta} \sum_{j=1}^s \binom{s}{j} \iint (1+y)^{2\gamma} |\partial_x^j \omega|^2 (1+y)^{2(\sigma+1)} |\partial_y (\partial_x^{s-j} \omega)|^2 \\ & \leq \frac{1}{4} \|(1+y)^\gamma \sqrt{\omega} \partial_y g_s\|_{L^2}^2 + C_{s,\sigma} \|\omega\|_{H_s^{\sigma,\gamma}}^4 \\ & \quad + C_{s,\gamma,\sigma,\delta} \left\{ \|\omega\|_{H_s^{\sigma,\gamma}} + \|\partial_x^s U\|_{L^2} \right\}^2 \|\omega\|_{H_s^{\sigma,\gamma}}^2 \end{aligned}$$

• **Estimate of the term**

$$V_2 = 2 \sum_{j=1}^{s-1} \binom{s}{j} \iint (1+y)^{2\gamma} g_s \partial_x^j \partial_y \omega \partial_x^{s-j} \partial_y \omega$$

If $j \leq s-3$ then $j+1 \leq s-2$

$$\begin{aligned} & \left| \iint (1+y)^{2\gamma} g_s \partial_x^j \partial_y \omega \partial_x^{s-j} \partial_y \omega \right| \\ & \leq \iint (1+y)^\gamma g_s (1+y)^{\gamma+1} \partial_x^j \partial_y \omega (1+y)^{\gamma+1} \partial_x^{s-j} \partial_y \omega \\ & \leq \|(1+y)^{\gamma+1} \partial_x^j \partial_y \omega\|_{L^\infty} \|(1+y)^\gamma g_s\|_{L^2} \|(1+y)^{\gamma+1} \partial_x^{s-j} \partial_y \omega\|_{L^2} \\ & \leq C_{s,\gamma} \|\omega\|_{H_s^{\sigma,\gamma}}^3 \end{aligned}$$

if $j > s-3$ ($s \geq 6$) ($s-j+1 < s-2$)

$$\begin{aligned} & \left| \iint (1+y)^{2\gamma} g_s \partial_x^j \partial_y \omega \partial_x^{s-j} \partial_y \omega \right| \\ & \leq \iint (1+y)^\gamma g_s (1+y)^{\gamma+1} \partial_x^j \partial_y \omega (1+y)^{\gamma+1} \partial_x^{s-j} \partial_y \omega \\ & \leq \|(1+y)^{\gamma+1} \partial_x^j \partial_y \omega\|_{L^2} \|(1+y)^\gamma g_s\|_{L^2} \|(1+y)^{\gamma+1} \partial_x^{s-j} \partial_y \omega\|_{L^\infty} \\ & \leq C_{s,\gamma} \|\omega\|_{H_s^{\sigma,\gamma}}^3. \end{aligned}$$

Then

$$|V_2| \leq C_{s,\gamma} \|\omega\|_{H_g^{s,\gamma}}^3$$

• Estimate of the term

$$V_3 = \sum_{j=1}^s \binom{s}{j} \iint (1+y)^{2\gamma} g_s \partial_x^j \omega \partial_y^2 (\partial_x^{s-j} \omega)$$

For $j = 1$

$$\begin{aligned} & \left| \iint (1+y)^{2\gamma} g_s \partial_x \omega \partial_y^2 (\partial_x^{s-1} \omega) \right| \\ & \leq \frac{1}{4} \iint (1+y)^{2(\gamma+1)} \omega |\partial_y^2 \partial_x^{s-1} \omega|^2 + \iint \frac{1}{\omega} (1+y)^\gamma |\partial_x \omega|^2 (1+y)^{2\gamma} |g_s|^2 \\ & \leq \frac{1}{4} \|(1+y)^{\gamma+1} \sqrt{\omega} \partial_y^2 \partial_x^{s-1} \omega\|_{L^2}^2 + \frac{1}{\delta} \iint (1+y)^{\gamma+\sigma} |\partial_x \omega|^2 (1+y)^{2\gamma} |g_s|^2 \\ & \leq \frac{1}{4} \|(1+y)^{\gamma+1} \sqrt{\omega} \partial_y^2 \partial_x^{s-1} \omega\|_{L^2}^2 + \frac{1}{\delta} \|(1+y)^{\gamma+\sigma} \partial_x \omega\|_{L^\infty}^2 \|(1+y)^\gamma g_s\|_{L^2}^2 \\ & \leq \frac{1}{4} \|(1+y)^{\gamma+1} \sqrt{\omega} \partial_y^2 \partial_x^{s-1} \omega\|_{L^2}^2 + C_{s,\gamma,\delta} \|\omega\|_{H_g^{s,\gamma}}^4 \end{aligned}$$

for $1 < j \leq s-2$

$$\begin{aligned} & \left| \iint (1+y)^{2\gamma} g_s \partial_x^j \omega \partial_y^2 (\partial_x^{s-j} \omega) \right| \\ & \leq \iint (1+y)^\gamma g_s (1+y)^\gamma \partial_x^j \omega (1+y)^{\gamma+2} \partial_y^2 (\partial_x^{s-j} \omega) \\ & \leq \|(1+y)^\gamma \partial_x^j \omega\|_{L^\infty} \|(1+y)^\gamma g_s\|_{L^2} \|(1+y)^{\gamma+2} \partial_x^{s-j} \partial_y^2 \omega\|_{L^2} \\ & \leq C_{s,\gamma} \|\omega\|_{H_g^{s,\gamma}}^3 \end{aligned}$$

if $s-2 < j \leq s$

$$\begin{aligned} & \left| \iint (1+y)^{2\gamma} g_s \partial_x^j \omega \partial_y^2 (\partial_x^{s-j} \omega) \right| \\ & \leq \iint (1+y)^\gamma |g_s (1+y)^\gamma \partial_x^j \omega (1+y)^{\gamma+2} \partial_y^2 (\partial_x^{s-j} \omega)| \\ & \leq \|(1+y)^\gamma \partial_x^j \omega\|_{L^2} \|(1+y)^\gamma g_s\|_{L^2} \|(1+y)^{\gamma+2} \partial_x^{s-j} \partial_y^2 \omega\|_{L^\infty} \\ & \leq C_{s,\gamma} \|\omega\|_{H_g^{s,\gamma}}^3. \end{aligned}$$

Then

$$|V_3| \leq C_{s,\gamma} \|\omega\|_{H_g^{s,\gamma}}^3 + C_{s,\gamma,\delta} \|\omega\|_{H_g^{s,\gamma}}^4 + \frac{1}{4} \|(1+y)^{\gamma+1} \sqrt{\omega} \partial_y^2 \partial_x^{s-1} \omega\|_{L^2}^2$$

We conclude that

$$\begin{aligned} \left| 2 \int_{\mathbb{T}} \omega g_s \partial_y g_s |_{y=0} \right| & \leq \frac{5}{8} \|(1+y)^\gamma \sqrt{\omega} \partial_y g_s\|_{L^2}^2 + C_{\sigma,\delta} \|\partial_x^{s+1} p\|_{L^2(\mathbb{T})} \\ & \quad + C_\delta \|\omega\|_{H_g^{s,\gamma}}^2 \|\partial_x^s U\|_{L^\infty(\mathbb{T})} + C_{s,\gamma,\delta} \|\omega\|_{H_g^{s,\gamma}}^2 \\ & \quad + C_{s,\gamma,\sigma,\delta} \left\{ \|\omega\|_{H_g^{s,\gamma}} + \|\partial_x^s U\|_{L^2} \right\} \|\omega\|_{H_g^{s,\gamma}}^2 \\ & \quad + C_{s,\gamma} \|\omega\|_{H_g^{s,\gamma}}^3 + C_{s,\gamma,\delta} \|\omega\|_{H_g^{s,\gamma}}^4 \\ & \quad + C_{s,\gamma,\sigma,\delta} \left\{ \|\omega\|_{H_g^{s,\gamma}} + \|\partial_x^s U\|_{L^2} \right\}^2 \|\omega\|_{H_g^{s,\gamma}}^2 \\ & \quad + \frac{1}{4} \|(1+y)^{\gamma+1} \sqrt{\omega} \partial_y^2 \partial_x^{s-1} \omega\|_{L^2}^2 \end{aligned}$$

6.5.5 Estimate of the terms B_i

We have the following estimates (Terms identical to Masmoudi):

Proposition 6.5.2. *There exist positive constants $C, C_\delta, C_{\gamma, \delta}$ and $C_{s, \gamma, \sigma, \delta}$*

$$\begin{aligned} |B_1| &\leq C_{s, \gamma, \sigma, \delta} \{ \|\omega\|_{H_g^{s, \gamma}} + \|\partial_x^s U\|_{L^2(\mathbb{T})} \} \|\omega\|_{H_g^{s, \gamma}}^2, \\ |B_2| &\leq \frac{1}{2} \varepsilon^2 \|(1+y)^\gamma \partial_x g_s\|_{L^2}^2 + \varepsilon^2 C_{s, \gamma, \sigma, \delta} \{ \|\omega\|_{H_g^{s, \gamma}} + \|\partial_x^{s+1} U\|_{L^2(\mathbb{T})} \} \|\omega\|_{H_g^{s, \gamma}}, \\ |B_3| &\leq C_{s, \gamma, \sigma, \delta} \{ \|\omega\|_{H_g^{s, \gamma}} + \|\partial_x^s U\|_{L^2(\mathbb{T})} \}^2 \|\omega\|_{H_g^{s, \gamma}}, \\ |B_4| &\leq C_{s, \gamma, \sigma, \delta} \{ \|\omega\|_{H_g^{s, \gamma}} + \|\partial_x^s U\|_{L^2(\mathbb{T})} \} \|\omega\|_{H_g^{s, \gamma}}^2, \\ |B_5| &\leq C_{s, \gamma, \sigma, \delta} \|\partial_x^{s+1} U\|_{L^\infty} \{ \|\omega\|_{H_g^{s, \gamma}} + \|\partial_x^s U\|_{L^2(\mathbb{T})} \} \|\omega\|_{H_g^{s, \gamma}}. \end{aligned}$$

Proof. • Estimate of B_1 We recall

$$B_1 = - \iint (1+y)^{2\gamma} g_s \{ u \partial_x g_s + v \partial_y g_s \}.$$

Integrating by parts (cf remark), and using $\partial_x u + \partial_y v = 0$, we have

$$\begin{aligned} |B_1| &= \left| \gamma \iint (1+y)^{2\gamma-1} v |g_s|^2 \right| \\ &= \left| \gamma \iint (1+y)^{2\gamma} \frac{v}{1+y} |g_s|^2 \right| \\ &\leq \gamma \left\| \frac{v}{1+y} \right\|_{L^\infty} \|(1+y)^\gamma g_s\|_{L^2}^2 \\ &\leq C_{s, \gamma, \sigma, \delta} \{ \|\omega\|_{H_g^{s, \gamma}} + \|\partial_x^s U\|_{L^2(\mathbb{T})} \} \|\omega\|_{H_g^{s, \gamma}}^2. \end{aligned}$$

• Estimate of B_2 We recall

$$B_2 = 2\varepsilon^2 \iint (1+y)^{2\gamma} g_s \partial_x a \{ \partial_x^{s+1}(u-U) - \frac{\partial_x \omega}{\omega} \partial_x^s(u-U) \}.$$

We have $(1+y)^\sigma \geq \delta$ and $|(1+y)^{\sigma+\alpha_2} D^\alpha \omega| \leq \delta^{-1}$ for all $|\alpha| \leq 2$. Thus, we have $\|(1+y) \partial_x a\|_{L^\infty} \leq \delta^{-2} + \delta^{-4}$ and $\|\frac{\partial_x \omega}{\omega}\|_{L^\infty} \leq \delta^{-2}$ and hence

$$\begin{aligned} &\left| 2\varepsilon^2 \iint (1+y)^{2\gamma} g_s \partial_x a \{ \partial_x^{s+1}(u-U) - \frac{\partial_x \omega}{\omega} \partial_x^s(u-U) \} \right| \\ &\leq 2\varepsilon^2 C_\delta \|(1+y)^\gamma g_s\|_{L^2}^2 \|(1+y)^{\gamma-1} \partial_x^{s+1}(u-U)\|_{L^2}^2 + \|(1+y)^{\gamma-1} \partial_x^s(u-U)\|_{L^2}^2 \end{aligned}$$

Now, we require the following inequality :

$$\begin{aligned} \|(1+y)^{\gamma-1} \partial_x^{s+1}(u-U)\|_{L^2}^2 &\leq C_{\gamma, \sigma, \delta} \left\{ \|\partial_x^{s+1} U\|_{L^2(\mathbb{T})} \right. \\ &\quad \left. + \|(1+y)^\gamma \partial_x g_s\|_{L^2} + \|(1+y)^{\gamma-1} \partial_x^s(u-U)\|_{L^2}^2 \right\} \end{aligned} \quad (6.35)$$

Assuming this for the moment, we can apply it and Proposition (6.10.1) we obtain

$$\begin{aligned} &\left| 2\varepsilon^2 \iint (1+y)^{2\gamma} g_s \partial_x a \{ \partial_x^{s+1}(u-U) - \frac{\partial_x \omega}{\omega} \partial_x^s(u-U) \} \right| \\ &\leq \varepsilon^2 C_{s, \gamma, \sigma, \delta} \{ \|\omega\|_{H_g^{s, \gamma}} + \|\partial_x^s U\|_{L^2(\mathbb{T})} + \|\partial_x^{s+1} U\|_{L^2} + \|(1+y)^\gamma \partial_x g_s\|_{L^2(\mathbb{T})} \} \|\omega\|_{H_g^{s, \gamma}}, \\ &\leq \frac{1}{2} \varepsilon^2 \|(1+y)^\gamma \partial_x g_s\|_{L^2}^2 + \varepsilon^2 C_{s, \gamma, \sigma, \delta} \{ \|\omega\|_{H_g^{s, \gamma}} + \|\partial_x^{s+1} U\|_{L^2(\mathbb{T})} \} \|\omega\|_{H_g^{s, \gamma}} \end{aligned}$$

- Estimate of B_3 . We recall

$$B_3 = - \sum_{j=0}^{s-1} \binom{s}{j} \iint (1+y)^{2\gamma} g_s g_{j+1} \partial_x^{s-j} u.$$

By Cauchy's inequality and the inequality

$$\|\partial_x^s U\|_{L^2(\mathbb{T})} \leq \frac{1}{\pi} \|\partial_x^{s+1} U\|_{L^2(\mathbb{T})}$$

To complete the justification, we have to verify (6.35) as follows. Since $\delta \leq (1+y)^\sigma \omega \leq \delta^{-1}$ and $u|_{y=0} = 0$ applying part (ii) of Lemma B.1, we have

$$\begin{aligned} & \|(1+y)^{\gamma-1} \partial_x^{s+1}(u-U)\|_{L^2}^2 \\ & \leq \delta^{-1} \left\| (1+y)^{\gamma-\sigma-1} \frac{\partial_x^{s+1}(u-U)}{\omega} \right\|_{L^2} \\ & \leq C_{\gamma,\sigma,\delta} \left\{ \|\partial_x^{s+1} U\|_{L^2(\mathbb{T})} + \left\| (1+y)^\gamma \partial_y \left(\frac{\partial_x^{s+1}(u-U)}{\omega} \right) \right\|_{L^2} \right\} \end{aligned}$$

It is worth noting that

$$\begin{aligned} \omega \partial_y \left(\frac{\partial_x^{s+1}(u-U)}{\omega} \right) &= g_{s+1} \\ &= \partial_x g + \partial_x a \partial_x^s (u-U) \end{aligned}$$

so we have :

$$\begin{aligned} & \|(1+y)^{\gamma-1} \partial_x^{s+1}(u-U)\|_{L^2}^2 \\ & \leq C_{\gamma,\sigma,\delta} \left\{ \|\partial_x^{s+1} U\|_{L^2(\mathbb{T})} + \|(1+y)^\gamma \partial_x g_s\|_{L^2} + \|(1+y) \partial_x a\|_{L^\infty} \|(1+y)^{\gamma-1} \partial_x^s (u-U)\|_{L^2} \right\}, \\ & \leq C_{\gamma,\sigma,\delta} \left\{ \|\partial_x^{s+1} U\|_{L^2(\mathbb{T})} + \|(1+y)^\gamma \partial_x g_s\|_{L^2} + \|(1+y)^{\gamma-1} \partial_x^s (u-U)\|_{L^2}^2 \right\}. \end{aligned}$$

because $\|(1+y) \partial_x a\|_{L^\infty} \leq \delta^{-2} + \delta^{-4}$.

For $j = 0, 1, \dots, s-1$

$$\begin{aligned} \left| \iint (1+y)^{2\gamma} g_s g_{j+1} \partial_x^{s-j} u \right| &\leq \|\partial_x^{s-j} u\|_{L^\infty} \|(1+y)^\gamma g_{j+1}\|_{L^2} \|(1+y)^\gamma g_s\|_{L^2} \\ &\leq C_{s,\gamma,\sigma,\delta} \left\{ \|\omega\|_{H_g^{s,\gamma}} + \|\partial_x^s U\|_{L^2(\mathbb{T})} \right\}^2 \|\omega\|_{H_g^{s,\gamma}} \end{aligned}$$

Then

$$|B_3| \leq C_{s,\gamma,\sigma,\delta} \left\{ \|\omega\|_{H_g^{s,\gamma}} + \|\partial_x^s U\|_{L^2(\mathbb{T})} \right\}^2 \|\omega\|_{H_g^{s,\gamma}}$$

- Estimate of B_4 . We recall

$$B_4 = - \sum_{j=0}^{s-1} \binom{s}{j} \iint (1+y)^{2\gamma} g_s \partial_x^{s-j} v \{ \partial_x^j \partial_y \omega - a \partial_x^j \omega \}$$

Follows from the Holder inequality and proposition (6.10.1), (6.82), (6.81) For $j = 2, 3, \dots, s-1$, using $\|(1+y)a\|_{L^\infty} \leq \delta^{-2}$ and proposition E.3, we have

$$\begin{aligned} & \left| \iint (1+y)^{2\gamma} g_s \partial_x^{s-j} v \{ \partial_x^j \partial_y \omega - a \partial_x^j \omega \} \right| \\ & \leq \left\| \frac{\partial_x^{s-j} v}{1+y} \right\|_{L^\infty} \|(1+y)^\gamma g_s\|_{L^2} \\ & \quad \times \left\{ \|(1+y)^{\gamma+1} \partial_x^j \partial_y \omega\|_{L^2} + \|(1+y)a\|_{L^\infty} \|(1+y)^\gamma \partial_x^j \omega\|_{L^2} \right\} \\ & \leq C_{s,\gamma,\sigma,\delta} \left\{ \|\omega\|_{H_g^{s,\gamma}} + \|\partial_x^s U\|_{L^2(\mathbb{T})} \right\} \|\omega\|_{H_g^{s,\gamma}}^2 \end{aligned}$$

When $j = 1$, using Proposition B.3 and $\|(1+y)a\|_{L^\infty} \leq \delta^{-2}$ again, we have

$$\begin{aligned} & \left| \iint (1+y)^{2\gamma} g_s \partial_x^{s-1} v \{ \partial_x \partial_y \omega - a \partial_x \omega \} \right| \\ & \leq \left[\left\| \frac{\partial_x^{s-1} v + y \partial_x^s U}{1+y} \right\|_{L^2} \left(\|(1+y)^{\gamma+1} \partial_x \partial_y \omega\|_{L^2} + \|(1+y)a\|_{L^\infty} \|(1+y)^\gamma \partial_x \omega\|_{L^2} \right) \right. \\ & \quad \left. + \|\partial_x^s U\|_{L^\infty} \left(\|(1+y)^{\gamma+1} \partial_x \partial_y \omega\|_{L^2} + \|(1+y)a\|_{L^\infty} \|(1+y)^\gamma \partial_x \omega\|_{L^2} \right) \right] \|(1+y)^\gamma g_s\|_{L^2} \\ & \leq C_{s,\gamma,\sigma,\delta} \{ \|\omega\|_{H_g^{s,\gamma}} + \|\partial_x^s U\|_{L^2(\mathbb{T})} \} \|\omega\|_{H_g^{s,\gamma}}^2 \end{aligned}$$

Then

$$|B_4| \leq C_{s,\gamma,\sigma,\delta} \{ \|\omega\|_{H_g^{s,\gamma}} + \|\partial_x^s U\|_{L^2(\mathbb{T})} \} \|\omega\|_{H_g^{s,\gamma}}^2$$

- Estimate of B_5 . We recall

$$B_5 = \sum_{j=0}^{s-1} \binom{s}{j} \iint (1+y)^{2\gamma} g_s a \partial_x^j (u-U) \partial_x^{s-j+1} U.$$

For any $j = 0, 1, \dots, s-1$, by proposition (6.10.1), $\|(1+y)a\|_{L^\infty} \leq \delta^{-2}$, and the fact that

$$\|\partial_x^2 U\|_{L^\infty(\mathbb{T})} \leq \|\partial_x^3 U\|_{L^\infty(\mathbb{T})} \leq \dots \leq \|\partial_x^{s+1} U\|_{L^\infty(\mathbb{T})},$$

we have

$$\begin{aligned} & \left| \iint (1+y)^{2\gamma} g_s a \partial_x^j (u-U) \partial_x^{s-j+1} U \right| \\ & \leq \|(1+y)^\gamma g_s\|_{L^2} \|(1+y)a\|_{L^\infty} \|(1+y)^{\gamma-1} \partial_x^j (u-U)\|_{L^2} \|\partial_x^{s-j+1} U\|_{L^\infty(\mathbb{T})} \\ & \leq C_{s,\gamma,\sigma,\delta} \|\partial_x^{s+1} U\|_{L^\infty} \{ \|\omega\|_{H_g^{s,\gamma}} + \|\partial_x^s U\|_{L^2(\mathbb{T})} \} \|\omega\|_{H_g^{s,\gamma}}, \end{aligned}$$

and this conclude the proposition. □

6.5.6 Estimates of the terms I_k .

We estimate the other terms. We recall

$$a = \frac{\partial_y \omega}{\omega}$$

- Estimate of I_1 . We recall

$$I_1 = -4 \iint (1+y)^{2\gamma} g_s a \omega \partial_y a \partial_x^s \omega.$$

$$\begin{aligned} |I_1| & \leq 4 \frac{1}{\delta} \iint (1+y)^{2\gamma} |a| |\partial_y a| |g_s| |\partial_x^s \omega| \\ & \leq \frac{4}{\delta} \|(1+y)a\|_{L^\infty} \|\partial_y a\|_{L^\infty} \|(1+y)^\gamma g_s\|_{L^2} \|(1+y)^\gamma \partial_x^s \omega\|_{L^2} \\ & \leq C_{s,\gamma,\sigma,\delta} \{ \|\omega\|_{H_g^{s,\gamma}} + \|\partial_x^s U\|_{L^2(\mathbb{T})} \} \|\omega\|_{H_g^{s,\gamma}} \end{aligned}$$

- Estimate of I_2 . We recall

$$I_2 = -2 \sum_{j=1}^s \binom{s}{j} \iint (1+y)^{2\gamma} g_s a \partial_x^j \omega \partial_x^{s-j} \partial_y \omega.$$

We have

$$|I_2| \leq C_{s,\gamma,\delta} \|\omega\|_{H_g^{s,\gamma}}^3.$$

Indeed, for $j = 1, 2, \dots, s-2$ (since $1 \leq 1+y$)

$$\begin{aligned} & \left| \iint (1+y)^{2\gamma} g_s a \partial_x^j \omega \partial_x^{s-j} \partial_y \omega \right| \\ & \leq \iint (1+y)^{2\gamma} \left| g_s (1+y) a (1+y)^\gamma \partial_x^j \omega (1+y)^{\gamma+1} \partial_x^{s-j} \partial_y \omega \right| \\ & \leq \|(1+y)a\|_{L^\infty} \|(1+y)^\gamma \partial_x^j \omega\|_{L^\infty} \|(1+y)^\gamma g_s\|_{L^2} \|(1+y)^{\gamma+1} \partial_x^{s-j} \partial_y \omega\|_{L^2} \\ & \leq C_{s,\gamma,\delta} \|\omega\|_{H_g^{s,\gamma}}^3. \end{aligned}$$

and for $j = s-1, \dots, s$.

$$\begin{aligned} & \left| \iint (1+y)^{2\gamma} g_s a \partial_x^j \omega \partial_x^{s-j} \partial_y \omega \right| \\ & \leq \iint \left| (1+y)^{2\gamma} g_s (1+y) a (1+y)^\gamma \partial_x^j \omega (1+y)^{\gamma+1} \partial_x^{s-j} \partial_y \omega \right| \\ & \leq \|(1+y)a\|_{L^\infty} \|(1+y)^{\gamma+1} \partial_x^{s-j} \partial_y \omega\|_{L^\infty} \|(1+y)^\gamma g_s\|_{L^2} \|(1+y)^\gamma \partial_x^j \omega\|_{L^2} \\ & \leq C_{s,\gamma,\delta} \|\omega\|_{H_g^{s,\gamma}}^3, \end{aligned}$$

and the result follows.

- Estimate of I_3 . We recall

$$I_3 = -6 \iint (1+y)^{2\gamma} g_s a \partial_y^2 \omega \partial_x^s (u-U)$$

$$\begin{aligned} |I_3| & \leq 4 \iint (1+y) |a| (1+y)^{\gamma+2} |\partial_y^2 \omega| (1+y)^\gamma |g_s| (1+y)^{\gamma-1} |\partial_x^s (u-U)| \\ & \leq \|(1+y)a\|_{L^\infty} \|(1+y)^{\gamma+2} \partial_y^2 \omega\|_{L^\infty} \|(1+y)^\gamma g_s\|_{L^2} \|(1+y)^{\gamma-1} \partial_x^s (u-U)\|_{L^2} \\ & \leq C_{s,\gamma,\sigma,\delta} \{ \|\omega\|_{H_g^{s,\gamma}} + \|\partial_x^s U\|_{L^2} \} \|\omega\|_{H_g^{s,\gamma}}^2 \end{aligned}$$

- Estimate of I_4 . We recall

$$I_4 = 2 \iint (1+y)^{2\gamma} g_s a^2 \partial_x^s (u-U).$$

$$\begin{aligned} |I_4| & \leq \left| 2 \iint (1+y)^\gamma g_s (1+y)^2 a^2 (1+y)^{\gamma-1} \partial_x^s (u-U) \right| \\ & \leq \|(1+y)a\|_{L^\infty}^2 \|(1+y)^\gamma g_s\|_{L^2} \|(1+y)^{\gamma-1} \partial_x^s (u-U)\|_{L^2} \\ & \leq C_{s,\gamma,\sigma,\delta} \{ \|\omega\|_{H_g^{s,\gamma}} + \|\partial_x^s U\|_{L^2} \} \|\omega\|_{H_g^{s,\gamma}} \end{aligned}$$

- Estimate of I_5 . We recall

$$I_5 = \iint (1+y)^{2\gamma} g_s \partial_x \omega \partial_x^s (u-U).$$

We estimate :

$$\begin{aligned} |I_5| & \leq \iint (1+y)^\gamma g_s (1+y)^\gamma \partial_x \omega (1+y)^{\gamma-1} \partial_x^s (u-U) \\ & \leq \|(1+y)^\gamma \partial_x \omega\|_{L^\infty} \|(1+y)^\gamma g_s\|_{L^2} \|(1+y)^{\gamma-1} \partial_x^s (u-U)\|_{L^2} \\ & \leq C_{s,\gamma,\sigma,\delta} \{ \|\omega\|_{H_g^{s,\gamma}} + \|\partial_x^s U\|_{L^2} \} \|\omega\|_{H_g^{s,\gamma}}^2 \end{aligned}$$

- Estimate of I_6 . We recall

$$I_6 = - \iint (1+y)^{2\gamma} g_s a \partial_x u \partial_x^s (u-U).$$

$$\begin{aligned} |I_6| &\leq \iint (1+y)^\gamma g_s (1+y) a \partial_x u (1+y)^{\gamma-1} \partial_x^s (u-U) \\ &\leq \|(1+y)a\|_{L^\infty} \|\partial_x u\|_{L^\infty} \|(1+y)^\gamma g_s\|_{L^2} \|(1+y)^{\gamma-1} \partial_x^s (u-U)\|_{L^2} \\ &\leq C_{s,\gamma,\sigma,\delta} \{\|\omega\|_{H_g^{s,\gamma}} + \|\partial_x^s U\|_{L^2}\}^2 \|\omega\|_{H_g^{s,\gamma}} \end{aligned}$$

We have to estimate each of these three terms. It is the goal of the following subsections.

6.5.7 Estimate of J

We recall that

$$J = 2 \iint (1+y)^{2\gamma} g_s \partial_x^s ((\partial_y \omega)^2) 2 \sum_{j=1}^s \binom{s}{j} \iint (1+y)^{2\gamma} g_s \partial_x^j \omega \partial_y^2 \partial_x^{s-j} \omega.$$

Estimate of the term $2 \iint (1+y)^{2\gamma} g_s \partial_x^s ((\partial_y \omega)^2)$

$$\begin{aligned} \partial_x^s (\partial_y \omega \cdot \partial_y \omega) &= \sum_{j=0}^s \binom{s}{j} \partial_x^j \partial_y \omega \partial_x^{s-j} \partial_y \omega \\ &= 2 \partial_y \omega \partial_x^s \partial_y \omega + \sum_{j=1}^{s-1} \binom{s}{j} \partial_x^j \partial_y \omega \partial_x^{s-j} \partial_y \omega \end{aligned}$$

$$\begin{aligned} 2 \iint (1+y)^{2\gamma} g_s \partial_x^s ((\partial_y \omega)^2) &= 4 \iint (1+y)^{2\gamma} g_s \partial_y \omega \partial_x^s \partial_y \omega \\ &\quad + 2 \sum_{j=1}^{s-1} \binom{s}{j} \iint (1+y)^{2\gamma} g_s \partial_x^j \partial_y \omega \partial_x^{s-j} \partial_y \omega \end{aligned}$$

Then we derive

$$\begin{aligned} J &= 4 \iint (1+y)^{2\gamma} g_s \partial_y \omega \partial_x^s \partial_y \omega \\ &\quad + 2 \sum_{j=1}^{s-1} \binom{s}{j} \iint (1+y)^{2\gamma} g_s \partial_x^j \omega \partial_y \partial_x^{s-j} \omega \\ &\quad + 2 \sum_{j=1}^s \binom{s}{j} \iint (1+y)^{2\gamma} g_s \partial_x^j \omega \partial_y^2 \partial_x^{s-j} \omega \end{aligned}$$

We have

$$g_s = \partial_x^s \omega - \frac{\partial_y \omega}{\omega} \partial_x^s (u-U)$$

Then

$$\partial_x^s \omega = g_s + \frac{\partial_y \omega}{\omega} \partial_x^s (u-U)$$

and

$$\partial_y \partial_x^s \omega = \partial_y g_s + \partial_y (a) \partial_x^s (u-U) + a \partial_x^s \omega$$

Then

$$g_s \partial_y \omega \partial_y \partial_x^s \omega = g_s \partial_y \omega \partial_y g_s + g_s \partial_y \omega \partial_y (a) \partial_x^s (u-U) + g_s \partial_y \omega a \partial_x^s \omega$$

Then

$$\begin{aligned} \left| 4 \int_0^{+\infty} \int_{\mathbb{T}} (1+y)^{2\gamma} g_s \partial_y \omega \partial_y \partial_x^s \omega dx dy \right| &\leq \left| 4 \int_0^{+\infty} \int_{\mathbb{T}} (1+y)^{2\gamma} g_s \partial_y \omega \partial_y g_s dx dy \right| \\ &\quad + \left| 4 \int_0^{+\infty} \int_{\mathbb{T}} (1+y)^{2\gamma} g_s \partial_y \omega \partial_y (a) \partial_x^s (u - U) dx dy \right| \\ &\quad + \left| 4 \int_0^{+\infty} \int_{\mathbb{T}} (1+y)^{2\gamma} g_s \partial_y \omega a \partial_x^s \omega dx dy \right| \end{aligned}$$

$$\begin{aligned} \left| 4 \int_0^{+\infty} \int_{\mathbb{T}} (1+y)^{2\gamma} g_s \partial_y \omega \partial_y g_s dx dy \right| &\leq 4 \int_0^{+\infty} \int_{\mathbb{T}} (1+y)^{2\gamma} |g_s \partial_y \omega \partial_y g_s| dx dy \\ &\leq 32 \int_0^{+\infty} \int_{\mathbb{T}} \frac{|\partial_y \omega|^2}{\omega} (1+y)^{2\gamma} |g_s|^2 \\ &\quad + \frac{1}{4} \int_0^{+\infty} \int_{\mathbb{T}} (1+y)^{2\gamma} \omega |\partial_y g_s|^2 \\ &\leq \frac{32}{\delta} \|(1+y)^{\sigma+1} \partial_y \omega\|_{L^\infty}^2 \|(1+y)^\gamma g_s\|_{L^2}^2 \\ &\quad + \frac{1}{4} \|(1+y)^\gamma \sqrt{\omega} \partial_y g_s\|_{L^2}^2 \\ &\leq C_{s,\gamma} \|\omega\|_{H_g^{s,\gamma}}^2 + \frac{1}{4} \|(1+y)^\gamma \sqrt{\omega} \partial_y g_s\|_{L^2}^2 \end{aligned}$$

$$\begin{aligned} \left| 4 \int_0^{+\infty} \int_{\mathbb{T}} (1+y)^{2\gamma} g_s \partial_y \omega \partial_y (a) \partial_x^s (u - U) dx dy \right| &\leq 4 \int_0^{+\infty} \int_{\mathbb{T}} (1+y)^{2\gamma} |g_s \partial_y \omega \partial_y (a) \partial_x^s (u - U)| dx dy \\ &\leq 4 \int_0^{+\infty} \int_{\mathbb{T}} (1+y)^{\gamma+1} |\partial_y \omega| |\partial_y a| (1+y)^\gamma |g_s| (1+y)^{\gamma-1} |\partial_x^s (u - U)| dx dy \\ &\leq 4 \|(1+y)^{\gamma+1} \partial_y \omega\|_{L^\infty} \|\partial_y a\|_{L^\infty} \|(1+y)^\gamma g_s\|_{L^2} \|(1+y)^{\gamma-1} \partial_x^s (u - U)\|_{L^2} \\ &\leq C_{s,\gamma,\sigma,\delta} \{ \|\omega\|_{H_g^{s,\gamma}} + \|\partial_x^s U\|_{L^2} \} \|\omega\|_{H_g^{s,\gamma}}^2 \end{aligned}$$

$$\begin{aligned} \left| 4 \int_0^{+\infty} \int_{\mathbb{T}} (1+y)^{2\gamma} g_s \partial_y \omega a \partial_x^s \omega dx dy \right| &\leq 4 \int_0^{+\infty} \int_{\mathbb{T}} (1+y)^{2\gamma} |g_s \partial_y \omega a \partial_x^s \omega| dx dy \\ &\leq 4 \int_0^{+\infty} \int_{\mathbb{T}} (1+y) |a| (1+y)^{\gamma+1} |\partial_y \omega| (1+y)^\gamma |g_s| (1+y)^\gamma |\partial_x^s \omega| \\ &\leq 4 \|(1+y) a\|_{L^\infty} \|(1+y)^{\gamma+1} \partial_y \omega\|_{L^\infty} \|(1+y)^\gamma g_s\|_{L^2} \|(1+y)^\gamma \partial_x^s \omega\|_{L^2} \\ &\leq C_{s,\gamma,\sigma,\delta} \{ \|\omega\|_{H_g^{s,\gamma}} + \|\partial_x^s U\|_{L^2} \} \|\omega\|_{H_g^{s,\gamma}}^2 \end{aligned}$$

Then

$$\begin{aligned} \left| 4 \int_0^{+\infty} \int_{\mathbb{T}} (1+y)^{2\gamma} g_s \partial_y \omega \partial_y \partial_x^s \omega dx dy \right| &\leq C_{s,\gamma} \|\omega\|_{H_g^{s,\gamma}}^2 + \frac{1}{4} \|(1+y)^\gamma \sqrt{\omega} \partial_y g_s\|_{L^2}^2 \\ &\quad + C_{s,\gamma,\sigma,\delta} \{ \|\omega\|_{H_g^{s,\gamma}} \|\partial_x^s U\|_{L^2} \} \|\omega\|_{H_g^{s,\gamma}}^2 \end{aligned}$$

• Estimation of the term

$$2 \sum_{j=1}^{s-1} \binom{s}{j} \iint (1+y)^{2\gamma} g_s \partial_x^j \omega \partial_y \partial_x^{s-j} \omega$$

If $1 \leq j \leq s-2$

$$\begin{aligned} \left| \iint (1+y)^{2\gamma} g_s \partial_x^j \omega \partial_y \partial_x^{s-j} \omega \right| &\leq \iint (1+y)^\gamma |g_s| (1+y)^\gamma |\partial_x^j \omega| (1+y)^{\gamma+1} |\partial_y \partial_x^{s-j} \omega| \\ &\leq \|(1+y)^\gamma \partial_x^j \omega\|_{L^\infty} \|(1+y)^\gamma g_s\|_{L^2} \|(1+y)^{\gamma+1} \partial_y \partial_x^{s-j} \omega\|_{L^2} \\ &\leq C_{s,\gamma} \|\omega\|_{H_g^{s,\gamma}}^3 \end{aligned}$$

for $j = s-1$

$$\begin{aligned} \left| \iint (1+y)^{2\gamma} g_s \partial_x^j \omega \partial_y \partial_x^{s-j} \omega \right| &\leq \iint (1+y)^\gamma |g_s| (1+y)^\gamma |\partial_x^j \omega| (1+y)^{\gamma+1} |\partial_y \partial_x^{s-j} \omega| \\ &\leq \|(1+y)^\gamma \partial_x^j \omega\|_{L^2} \|(1+y)^\gamma g_s\|_{L^2} \|(1+y)^{\gamma+1} \partial_y \partial_x^{s-j} \omega\|_{L^\infty} \\ &\leq C_{s,\gamma} \|\omega\|_{H_g^{s,\gamma}}^3 \end{aligned}$$

Then

$$\left| 2 \sum_{j=1}^{s-1} \binom{s}{j} \iint (1+y)^{2\gamma} g_s \partial_x^j \omega \partial_y \partial_x^{s-j} \omega \right| \leq C_{s,\gamma} \|\omega\|_{H_g^{s,\gamma}}^3$$

- Estimate of the term $2 \sum_{j=1}^s \binom{s}{j} \iint (1+y)^{2\gamma} g_s \partial_x^j \omega \partial_y^2 \partial_x^{s-j} \omega$

For $j = 1$

$$\begin{aligned} &\left| \iint (1+y)^{2\gamma} g_s \partial_x \omega \partial_y^2 \partial_x^{s-1} \omega \right| \\ &\leq \frac{1}{4} \iint (1+y)^{2(\gamma+1)} \omega |\partial_y^2 \partial_x^{s-1} \omega|^2 + \iint \frac{1}{\omega} (1+y)^\gamma |\partial_x \omega|^2 (1+y)^{2\gamma} |g_s|^2 \\ &\leq \frac{1}{4} \|(1+y)^{\gamma+1} \sqrt{\omega} \partial_y^2 \partial_x^{s-1} \omega\|_{L^2}^2 + \frac{1}{\delta} \iint (1+y)^{\gamma+\sigma} |\partial_x \omega|^2 (1+y)^{2\gamma} |g_s|^2 \\ &\leq \frac{1}{4} \|(1+y)^{\gamma+1} \sqrt{\omega} \partial_y^2 \partial_x^{s-1} \omega\|_{L^2}^2 + \frac{1}{\delta} \|(1+y)^{\gamma+\sigma} \partial_x \omega\|_{L^\infty}^2 \|(1+y)^\gamma g_s\|_{L^2}^2 \\ &\leq \frac{1}{4} \|(1+y)^{\gamma+1} \sqrt{\omega} \partial_y^2 \partial_x^{s-1} \omega\|_{L^2}^2 + C_{s,\gamma,\delta} \|\omega\|_{H_g^{s,\gamma}}^4 \end{aligned}$$

for $1 < j \leq s-2$

$$\begin{aligned} &\left| \iint (1+y)^{2\gamma} g_s \partial_x^j \omega \partial_y^2 \partial_x^{s-j} \omega \right| \\ &\leq \iint (1+y)^\gamma |g_s| (1+y)^\gamma |\partial_x^j \omega| (1+y)^{\gamma+2} |\partial_y^2 \partial_x^{s-j} \omega| \\ &\leq \|(1+y)^\gamma \partial_x^j \omega\|_{L^\infty} \|(1+y)^\gamma g_s\|_{L^2} \|(1+y)^{\gamma+2} \partial_y^2 \partial_x^{s-j} \omega\|_{L^2} \\ &\leq C_{s,\gamma} \|\omega\|_{H_g^{s,\gamma}}^3 \end{aligned}$$

if $s-2 < j \leq s$ ($s \geq 6$)

$$\begin{aligned} \left| \iint (1+y)^{2\gamma} g_s \partial_x^j \omega \partial_y^2 \partial_x^{s-j} \omega \right| &\leq \iint (1+y)^\gamma |g_s| (1+y)^\gamma |\partial_x^j \omega| (1+y)^{\gamma+2} |\partial_y^2 \partial_x^{s-j} \omega| \\ &\leq \|(1+y)^\gamma \partial_x^j \omega\|_{L^2} \|(1+y)^\gamma g_s\|_{L^2} \|(1+y)^{\gamma+2} \partial_y^2 \partial_x^{s-j} \omega\|_{L^\infty} \\ &\leq C_{s,\gamma} \|\omega\|_{H_g^{s,\gamma}}^3 \end{aligned}$$

Then

$$\begin{aligned} \left| 2 \sum_{j=1}^s \binom{s}{j} \iint (1+y)^{2\gamma} g_s \partial_x^j \omega \partial_y^2 \partial_x^{s-j} \omega \right| &\leq \frac{1}{2} \|(1+y)^{\gamma+1} \sqrt{\omega} \partial_y^2 \partial_x^{s-1} \omega\|_{L^2}^2 \\ &\quad + C_{s,\gamma,\delta} \|\omega\|_{H_g^{s,\gamma}}^4 + C_{s,\gamma} \|\omega\|_{H_g^{s,\gamma}}^3 \end{aligned}$$

We conclude

$$\begin{aligned}
|J| &\leq C_{s,\gamma} \|\omega\|_{H_g^{s,\gamma}}^2 + \frac{1}{4} \|(1+y)^\gamma \sqrt{\omega} \partial_y g_s\|_{L^2}^2 \\
&\quad + C_{s,\gamma,\sigma,\delta} \{ \|\omega\|_{H_g^{s,\gamma}} \|\partial_x^s U\|_{L^2} \} \|\omega\|_{H_g^{s,\gamma}}^2 \\
&\quad + \frac{1}{2} \|(1+y)^{\gamma+1} \sqrt{\omega} \partial_y^2 \partial_x^{s-1} \omega\|_{L^2}^2 \\
&\quad + C_{s,\gamma,\delta} \|\omega\|_{H_g^{s,\gamma}}^4 + C_{s,\gamma} \|\omega\|_{H_s^{s,\gamma}}^3
\end{aligned}$$

6.5.8 Conclusion of the proof of Proposition 6.5.1

From the Lemma 6.5.1 and the previous estimates B_k, I_k, J we conclude the proof of Proposition 6.5.1

6.6 Appendix : Proof of the reduction boundary lemma

In this section, we prove Lemma 6.4.2 which allows to reduce the number of y -derivatives of the boundary terms. First, we recall the following evolution equation of $D^\alpha \omega$

$$\begin{aligned}
&\partial_t(D^\alpha \omega) + u \partial_x D^\alpha \omega + v \partial_y D^\alpha \omega \\
&\quad + \sum_{0 < \beta \leq \alpha} \binom{\alpha}{\beta} D^\beta u \partial_x D^{\alpha-\beta} \omega + \sum_{0 < \beta \leq \alpha} \binom{\alpha}{\beta} D^\beta v \partial_y D^{\alpha-\beta} \omega \\
&= \varepsilon^2 \partial_x^2 (D^\alpha \omega) + 2\omega \partial_y^2 (D^\alpha \omega) + 2 \sum_{0 < \beta \leq \alpha} \binom{\alpha}{\beta} D^\beta \omega \partial_y^2 (D^{\alpha-\beta} \omega) \\
&\quad + 2\partial_y \omega \partial_y (D^\alpha \omega) + 2 \sum_{0 < \beta \leq \alpha} \binom{\alpha}{\beta} \partial_y (D^\beta \omega) \partial_y (D^{\alpha-\beta} \omega). \quad (6.19)
\end{aligned}$$

Remark 10. On the boundary $\{y = 0\}$, we have

- (a) The terms $u \partial_x D^\alpha \omega$ and $v \partial_y D^\alpha \omega$ vanish.
- (b) If $\beta_2 = 0$ then $\binom{\alpha}{\beta} D^\beta u \partial_x D^{\alpha-\beta} \omega$ and $\binom{\alpha}{\beta} D^\beta v \partial_y D^{\alpha-\beta} \omega$ vanish.
- (c) If $\beta_2 = 1$, using the incompressibility property $\partial_x u + \partial_y v = 0$, we deduce that the term

$$\binom{\alpha}{\beta} D^\beta v \partial_y D^{\alpha-\beta} \omega = - \binom{\alpha}{\beta} \partial_x^{\beta_1+1} u \partial_y D^{\alpha-\beta} \omega$$

vanish.

Remark 11. Using the fact that $\omega = \partial_y u$ and the incompressibility property $\partial_x u + \partial_y v = 0$, we have

- (a) If $\beta_2 \geq 1$, then

$$\binom{\alpha}{\beta} D^\beta u \partial_x D^{\alpha-\beta} \omega = \binom{\alpha}{\beta} D^{(\beta_1, \beta_2-1)} \omega \partial_x D^{\alpha-\beta} \omega.$$

- (b) If $\beta_2 \geq 2$, then

$$\binom{\alpha}{\beta} D^\beta v \partial_y D^{\alpha-\beta} \omega = - \binom{\alpha}{\beta} D^{(\beta_1+1, \beta_2-2)} \omega \partial_y D^{\alpha-\beta} \omega.$$

On the boundary $\{y = 0\}$, Equation (6.19) becomes

$$(\partial_t - \varepsilon^2 \partial_x^2) D^\alpha \omega|_{y=0} = \sum_{i=1}^{N(\alpha)} C(i) \varepsilon^{2\ell(i)} D^{\beta(i)} \omega D^{\gamma(i)} \omega|_{y=0}, \quad (6.36)$$

where $N(\alpha) \in \mathbb{N}^*$ and for any $1 \leq i \leq N(\alpha)$,

$$\begin{cases} C(i) \in \mathbb{R} \\ \ell(i) \in \{0, 1\} \\ 0 \leq |\beta(i)| + |\gamma(i)| \leq |\alpha| + 2. \end{cases}$$

Remark 12. Setting $L = \partial_t - \varepsilon^2 \partial_x^2$, we have

(a) For any U and V sufficiently smooth,

$$L(UV) = L(U)V + UL(V) - 2\varepsilon^2 \partial_x U \partial_x V. \quad (6.37)$$

(a) In the general case, for any $n \geq 2$, for any U_1, \dots, U_n sufficiently smooth,

$$L \prod_{k=1}^n U_k = \sum_{k=1}^n (LU_k) \prod_{i \neq k} U_i - \varepsilon^2 \sum_{k \neq \ell} (\partial_x U_k \partial_x U_\ell) \prod_{i \neq k, \ell} U_i. \quad (6.38)$$

Proof of Lemma 6.4.2. We first prove Lemma 6.4.2 in the case where $n_1 = 0$. For the sake of simplicity, we will omit the index and denote n_2 by n .

Considering the Prandtl equation (6.10) and replacing $\partial_y u$ by ω , we get

$$\partial_t u^\varepsilon + u^\varepsilon \partial_x u^\varepsilon + v^\varepsilon \partial_y u^\varepsilon + \partial_x p^\varepsilon = 2\omega \partial_y \omega + \varepsilon^2 \partial_x^2 u \quad (*)$$

Since u and v are equal to 0 on the boundary, we get the value

$$2\omega \partial_y \omega|_{y=0} = \partial_x p$$

We derive the previous equation (*) with respect to y and get

$$\partial_t \omega + \partial_y u \partial_x \omega + \partial_y v \partial_y \omega + u \partial_x \omega + v \partial_y \omega = 2(\partial_y \omega)^2 + 2\omega \partial_y^2 \omega + \varepsilon^2 \partial_x^2 \omega$$

from the incompressibility relation

$$\partial_y u \partial_x \omega + \partial_y v \partial_y \omega = 0$$

$$\partial_t \omega + u \partial_x \omega + v \partial_y \omega = 2(\partial_y \omega)^2 + 2\omega \partial_y^2 \omega + \varepsilon^2 \partial_x^2 \omega \quad (**)$$

We derive again the previous equation with respect to y and get

$$\begin{aligned} \partial_t \partial_y \omega + (\partial_y u \partial_x \omega + \partial_y v \partial_y \omega) + (u \partial_x \partial_y \omega + v \partial_y \partial_y \omega) \\ = 4\partial_y \omega \partial_y^2 \omega + 2\partial_y \omega \partial_y^3 \omega + 2\omega \partial_y^3 \omega + \varepsilon^2 \partial_x^2 \partial_y \omega \end{aligned}$$

From the incompressibility, we replace $\partial_y v = -\partial_x u$ and $\partial_y u = \omega$ and since the trace is equal to 0, we derive

$$\begin{aligned} \partial_t \partial_y \omega|_{y=0} + (\omega \partial_x \omega - \partial_x u \partial_y \omega) + (u \partial_x \partial_y \omega + v \partial_y \partial_y \omega)|_{y=0} \\ = 4\partial_y \omega \partial_y^2 \omega + 2\partial_y \omega \partial_y^3 \omega + 2\omega \partial_y^3 \omega + \varepsilon^2 \partial_x^2 \partial_y \omega|_{y=0} \end{aligned}$$

and then

$$\partial_t \partial_y \omega|_{y=0} + \omega \partial_x \omega|_{y=0} = 6\partial_y \omega \partial_y^2 \omega + 2\omega \partial_y^3 \omega + \varepsilon^2 \partial_x^2 \partial_y \omega|_{y=0}.$$

That is, we obtain the formulation for the term $\omega \partial_y^3 \omega$

$$2\omega \partial_y^3 \omega|_{y=0} = (\partial_t - \varepsilon^2 \partial_x^2)(\partial_y \omega) + \omega \partial_x \omega - 6\partial_y \omega \partial_y^2 \omega|_{y=0}.$$

We have some estimate on the the right hand side of the following equation

$$(\partial_t - \varepsilon^2 \partial_x^2)(\omega \partial_y \omega)|_{y=0} = (\partial_t - \varepsilon^2 \partial_x^2) \partial_x \mathbf{p}$$

Therefore we expand the left hand side of the previous evolution equation, we divide by ω , we use (6.6) the value of $\omega\partial_y\omega$ on the boundary and we get

$$\begin{aligned} (\partial_t - \varepsilon^2 \partial_x^2)(\partial_y \omega)|_{y=0} &= \frac{1}{2\omega}(\partial_t - \varepsilon^2 \partial_x^2)\partial_x p - 2\partial_y \omega \partial_y^2 \omega \\ &\quad - \frac{2\varepsilon^2}{\omega} \partial_x \omega \partial_x \partial_y \omega - \frac{2}{\omega}(\partial_y \omega)^3|_{y=0} \end{aligned}$$

From the previous expression, we get from

$$\begin{aligned} 2\omega \partial_y^3 \omega|_{y=0} &= \frac{1}{2\omega}(\partial_t - \varepsilon^2 \partial_x^2)\partial_x p - 2\partial_y \omega \partial_y^2 \omega \\ &\quad - \frac{2\varepsilon^2}{\omega} \partial_x \omega \partial_x \partial_y \omega - \frac{2}{\omega}(\partial_y \omega)^3 + \omega \partial_x \omega - 6\partial_y \omega \partial_y^2 \omega|_{y=0} \end{aligned}$$

In conclusion we have

$$\begin{cases} 2\omega \partial_y \omega|_{y=0} = \partial_x p \\ 2\omega \partial_y^3 \omega|_{y=0} = \frac{1}{2\omega}(\partial_t - \varepsilon^2 \partial_x^2)\partial_x p - 2\partial_y \omega \partial_y^2 \omega \\ \quad - \frac{2\varepsilon^2}{\omega} \partial_x \omega \partial_x \partial_y \omega - \frac{2}{\omega}(\partial_y \omega)^3 + \omega \partial_x \omega - 6\partial_y \omega \partial_y^2 \omega|_{y=0}. \end{cases}$$

which imply Identity (6.24) of Lemma 6.4.2 in the case where $n = 0$ and $n = 1$. We next proceed by recurrence with respect to n and we suppose that (6.24) is verified for $0 \leq n-1 < s/2$. Rewriting Equation (6.19) with $\alpha = (0, 2n-1)$, and using Remark 11, we obtain

$$\begin{aligned} &(\partial_t - \varepsilon^2 \partial_x^2)\partial_y^{2n-1} \omega + u \partial_x \partial_y^{2n-1} \omega + v \partial_y^{2n} \omega - (2n-1)\partial_x u \partial_y^{2n-k} \omega \\ &+ \sum_{k=1}^{2n-1} \binom{2n-1}{k} \partial_y^{k-1} \omega \partial_x \partial_y^{2n-1-k} \omega - \sum_{k=2}^{2n-1} \binom{2n-1}{k} \partial_x \partial_y^{k-2} \omega \partial_y^{2n-k} \omega \\ &= 2\omega \partial_y^{2n+1} \omega + 2 \sum_{k=1}^{2n-1} \binom{2n-1}{k} \partial_y^k \omega \partial_y^{2n+1-k} \omega \\ &\quad + 2\partial_y \omega \partial_y^{2n} \omega + 2 \sum_{k=1}^{2n-1} \binom{2n-1}{k} \partial_y^{k+1} \omega \partial_y^{2n-k} \omega. \quad (6.39) \end{aligned}$$

Identity (6.37) of Lemma 12 yields

$$\begin{aligned} (\partial_t - \varepsilon^2 \partial_x^2)(\omega \partial_y^{2n-1} \omega) \\ = \partial_y^{2n-1} \omega (\partial_t - \varepsilon^2 \partial_x^2) \omega + \omega (\partial_t - \varepsilon^2 \partial_x^2)(\partial_y^{2n-1} \omega) - 2\varepsilon^2 \partial_x \omega \partial_x \partial_y^{2n-1} \omega, \end{aligned}$$

which implies

$$\begin{aligned} (\partial_t - \varepsilon^2 \partial_x^2)(\partial_y^{2n-1} \omega) &= \omega^{-1}(\partial_t - \varepsilon^2 \partial_x^2)(\omega \partial_y^{2n-1} \omega) \\ &\quad - \omega^{-1} \partial_y^{2n-1} \omega (\partial_t - \varepsilon^2 \partial_x^2) \omega + 2\varepsilon^2 \omega^{-1} \partial_x \omega \partial_x \partial_y^{2n-1} \omega. \quad (6.40) \end{aligned}$$

Then, combining (6.39), (6.40) and Remark 11, on the boundary $\{y = 0\}$, we obtain

$$2\omega \partial_y^{2n+1} \omega|_{y=0} = \sum_{k=1}^{2n-1} B_k,$$

where

$$\begin{aligned} B_1 &= \omega^{-1}(\partial_t - \varepsilon^2 \partial_x^2)(\omega \partial_y^{2n-1} \omega)|_{y=0} \\ B_2 &= -\omega^{-1} \partial_y^{2n-1} \omega (\partial_t - \varepsilon^2 \partial_x^2) \omega|_{y=0} \\ B_3 &= \varepsilon^2 \omega^{-1} \partial_x \omega \partial_x \partial_y^{2n-1} \omega|_{y=0} \end{aligned}$$

$$\begin{aligned}
B_4 &= \sum_{k=1}^{2n-1} \binom{2n-1}{k} \partial_y^{k-1} \omega \partial_x \partial_y^{2n-1-k} \omega|_{y=0} \\
B_5 &= - \sum_{k=2}^{2n-1} \binom{2n-1}{k} \partial_x \partial_y^{k-2} \omega \partial_y^{2n-k} \omega|_{y=0} \\
B_6 &= -2 \sum_{k=1}^{2n-1} \binom{2n-1}{k} \partial_y^k \omega \partial_y^{2n+1-k} \omega|_{y=0} \\
B_7 &= -2 \partial_y \omega \partial_y^{2n} \omega|_{y=0}
\end{aligned}$$

and

$$B_8 = -2 \sum_{k=1}^{2n-1} \binom{2n-1}{k} \partial_y^{k+1} \omega \partial_y^{2n-k} \omega|_{y=0}.$$

We remark immediately that the terms B_3 to B_8 are all in the same form of the last term of Equation (6.24). For B_2 , using (6.36) with $\alpha = (0, 0)$, we can write

$$(\partial_t - \varepsilon^2 \partial_x) \omega|_{y=0} = \sum_{i=1}^{N(0)} C(i) \varepsilon^{2\ell(i)} D^{\beta(i)} \omega D^{\gamma(i)} \omega|_{y=0},$$

where $N(0) \in \mathbb{N}^*$ and for any $1 \leq i \leq N(0)$,

$$\begin{cases} C(i) \in \mathbb{R} \\ \ell(i) \in \{0, 1\} \\ 0 \leq |\beta(i)| + |\gamma(i)| \leq 2. \end{cases}$$

Thus, B_2 is also in the same form of the last term of Equation (6.24). It only remains to study the term B_1 .

Expression of the term B_1

We recall

$$B_1 = \omega^{-1} (\partial_t - \varepsilon^2 \partial_x^2) (\omega \partial_y^{2n-1} \omega)|_{y=0}.$$

Using the recurrence hypothesis (6.24) for $n-1$

$$\omega \partial_y^{2n-1} \omega|_{y=0} = \sum_{j=1}^{N(n-1)} C(j) \varepsilon^{2\tilde{\ell}(j)} \omega^{-2\tilde{k}(j)} \left((\partial_t - \varepsilon^2 \partial_x^2)^{\tilde{m}_1(j)} \partial_x^{\tilde{m}_2(j)} p \right) \prod_{i=1}^{\tilde{M}(j)} D^{\tilde{\beta}(i,j)} \omega|_{y=0},$$

where

$$\begin{cases} 0 \leq \tilde{M}(j) \leq N(n-1) \\ \tilde{\ell}(j) \in \{0, \dots, n-1\} \\ \tilde{k}(j) \in \{0, \dots, n-1\} \\ 0 \leq |\tilde{\beta}(i,j)| \leq 2(n-1) \\ \sum_{i=1}^{\tilde{M}(j)} |\tilde{\beta}(i,j)| \leq 2n-1, \end{cases} \quad (6.41)$$

we get

$$B_1 = \omega^{-1} L \sum_{j=1}^{N(n-1)} C(j) \varepsilon^{2\tilde{\ell}(j)} \omega^{-2\tilde{k}(j)} \left(L^{\tilde{m}_1(j)} \partial_x^{\tilde{m}_2(j)} p \right) \prod_{i=1}^{\tilde{M}(j)} D^{\tilde{\beta}(i,j)} \omega|_{y=0},$$

where we recall that $L = \partial_t - \varepsilon^2 \partial_x^2$. Applying the law (6.38) of Lemma 12, we will write

$$B_1 = (B_{11} + B_{12} + B_{13} + B_{14} + B_{15} + B_{16})|_{y=0}$$

where

$$\begin{aligned}
B_{11} &= \omega^{-1} \sum_{j=1}^{N(n-1)} C(j) \varepsilon^{2\tilde{\ell}(j)} L \left(\omega^{-2\tilde{k}(j)} \right) \left(L^{\tilde{m}_1(j)} \partial_x^{\tilde{m}_2(j)} p \right) \prod_{i=1}^{\tilde{M}(j)} D^{\tilde{\beta}(i,j)} \omega, \\
B_{12} &= \sum_{j=1}^{N(n-1)} C(j) \varepsilon^{2\tilde{\ell}(j)} \omega^{-(2\tilde{k}(j)+1)} \left(L^{\tilde{m}_1(j)+1} \partial_x^{\tilde{m}_2(j)} p \right) \prod_{i=1}^{\tilde{M}(j)} D^{\tilde{\beta}(i,j)} \omega, \\
B_{13} &= \omega^{-1} \sum_{j=1}^{N(n-1)} C(j) \varepsilon^{2\tilde{\ell}(j)} \omega^{-2\tilde{k}(j)} \left(L^{\tilde{m}_1(j)} \partial_x^{\tilde{m}_2(j)} p \right) L \left(\prod_{i=1}^{\tilde{M}(j)} D^{\tilde{\beta}(i,j)} \omega \right), \\
B_{14} &= -2\omega^{-1} \sum_{j=1}^{N(n-1)} C(j) \varepsilon^{2\tilde{\ell}(j)+2} \partial_x \left(\omega^{-2\tilde{k}(j)} \right) \left(L^{\tilde{m}_1(j)} \partial_x^{\tilde{m}_2(j)+1} p \right) \prod_{i=1}^{\tilde{M}(j)} D^{\tilde{\beta}(i,j)} \omega, \\
B_{15} &= -2\omega^{-1} \sum_{j=1}^{N(n-1)} C(j) \varepsilon^{2\tilde{\ell}(j)+2} \partial_x \left(\omega^{-2\tilde{k}(j)} \right) \left(L^{\tilde{m}_1(j)} \partial_x^{\tilde{m}_2(j)} p \right) \partial_x \prod_{i=1}^{\tilde{M}(j)} D^{\tilde{\beta}(i,j)} \omega,
\end{aligned}$$

and

$$B_{16} = -2 \sum_{j=1}^{N(n-1)} C(j) \varepsilon^{2\tilde{\ell}(j)+2} \omega^{-(2\tilde{k}(j)+1)} \left(L^{\tilde{m}_1(j)} \partial_x^{\tilde{m}_2(j)+1} p \right) \partial_x \prod_{i=1}^{\tilde{M}(j)} D^{\tilde{\beta}(i,j)} \omega,$$

Taking into account the conditions (6.41), we immediately find that $B_{12}|_{y=0}$, $B_{16}|_{y=0}$ are in the same form as in the right-hand side of (6.24), with the conditions (6.25)-(6.27). Next, since

$$\partial_x \left(\omega^{-2\tilde{k}(j)} \right) = -2\tilde{k}(j) \omega^{-(2\tilde{k}(j)+1)} \partial_x \omega,$$

the terms $B_{14}|_{y=0}$ and $B_{15}|_{y=0}$ can also write in the same form as in the right-hand side of (6.24), with the conditions (6.25)-(6.27).

For $B_{11}|_{y=0}$, we first use the following simple identity

$$\begin{aligned}
L \left(\omega^{-2\tilde{k}(j)} \right) &= \omega^{-(2\tilde{k}(j)+1)} L \omega + \varepsilon^2 (2\tilde{k}(j) + 1) \omega^{-(2\tilde{k}(j)+1)} \partial_x^2 \omega \\
&\quad - 2\varepsilon^2 \tilde{k}(j) (2\tilde{k}(j) + 1) \omega^{-(2\tilde{k}(j)+2)} \partial_x \omega.
\end{aligned}$$

Using (6.36) with $\alpha = (0, 0)$, we have

$$L \omega|_{y=0} = \sum_{i=1}^{N(0)} C(i) \varepsilon^{2\ell(i)} D^{\beta(i)} \omega D^{\gamma(i)} \omega|_{y=0},$$

where $N(0) \in \mathbb{N}^*$ and for any $1 \leq i \leq N(0)$,

$$\begin{cases} C(i) \in \mathbb{R} \\ \ell(i) \in \{0, 1\} \\ 0 \leq |\beta(i)| + |\gamma(i)| \leq 2. \end{cases}$$

We can deduce that $B_{11}|_{y=0}$ is in the same form as in the right-hand side of (6.24), with the conditions (6.25)-(6.27).

For the term $B_{13}|_{y=0}$, Identity (6.38) of Remark 12 gives

$$\begin{aligned}
L \prod_{i=1}^{\tilde{M}(j)} D^{\tilde{\beta}(i,j)} \omega &= \sum_{k=1}^n L \left(D^{\tilde{\beta}(k,j)} \omega \right) \prod_{i \neq k} D^{\tilde{\beta}(i,j)} \omega \\
&\quad - \varepsilon^2 \sum_{k \neq \ell} \left(\partial_x D^{\tilde{\beta}(k,j)} \omega \partial_x D^{\tilde{\beta}(\ell,j)} \omega \right) \prod_{i \neq k, \ell} D^{\tilde{\beta}(i,j)} \omega.
\end{aligned}$$

Using (6.36) with $\alpha = \tilde{\beta}(i, j)$, we have

$$L\left(D^{\tilde{\beta}(k,j)}\omega\right)|_{y=0} = \sum_{i=1}^{N(\tilde{\beta}(k,j))} C(i)\varepsilon^{2\ell(i)}D^{\beta(i)}\omega D^{\gamma(i)}\omega|_{y=0},$$

where $N(\tilde{\beta}(k, j)) \in \mathbb{N}^*$ and for any $1 \leq i \leq \tilde{\beta}(k, j)$,

$$\begin{cases} C(i) \in \mathbb{R} \\ \ell(i) \in \{0, 1\} \\ 0 \leq |\beta(i)| + |\gamma(i)| \leq |\tilde{\beta}(k, j)| + 2. \end{cases}$$

We then conclude that $B_{13}|_{y=0}$ is in the same form as in the right-hand side of (6.24), with the conditions (6.25)-(6.27).

Identity (6.24) is valid for n and Lemma 6.4.2 is proved for $n_1 = 0$ and $n_2 = n$. The case where $n_1 > 0$ can be obtained by explicitly differentiating n_1 times with respect to x the term $\omega \partial_y^{2n_2+1} \omega|_{y=0}$.

□

6.7 Appendix : Maximum principle

In this appendix, we derive some pointwise estimates of the solution from the maximum principle. We have adapted the proofs of Masmoudi and Wong [59] to the problem of non-Newtonian fluids.

We first recall the maximum principle for parabolic equations that we will use in our paper.

Definition 6.7.1. Let $n \in \mathbb{N}^*$ and $\Omega \in \mathbb{R}^n$ be a bounded open set. We set

$$\Omega_T = \Omega \times]0, T[,$$

and

$$C^{2,1}(\Omega_T) = \{u : \Omega_T \rightarrow \mathbb{R} \mid u(\cdot, t) \in C^2(\Omega), u(x, \cdot) \in C^1(]0, T[), \forall x \in \Omega, \forall t \in]0, T[\}.$$

Theorem 6.2. Let $\Omega \in \mathbb{R}^n$ an bounded open set, $T > 0$ and L the following elliptic operator

$$Lu(x, t) = -\operatorname{tr} \left(a_{ij}(x, t) \partial_{x_i, x_j}^2 u(x, t) \right)_{i,j} + b(x, t) \cdot \nabla_x u(x, t),$$

for any $u \in C^{2,1}(\Omega_T) \cap C(\overline{\Omega_T})$, where

- $a(x, t)$ is a positively definite matrix,
- $b : \Omega_T \rightarrow \mathbb{R}^n$ a bounded continuous function.

Assuming that for $u \in C^{2,1}(\Omega_T) \cap C(\overline{\Omega_T})$ and for any $(x, t) \in \Omega_T$, we have

$$\partial_t u(x, t) + Lu(x, t) + cu(x, t) \leq 0,$$

where $c : \Omega \rightarrow \mathbb{R}_+$ a bounded and positive function. Then,

$$\max_{\Omega_T} u \leq \max_{\partial\Omega \times [0, T] \cup \overline{\Omega} \times \{0\}} u^+,$$

where u^+ is the positive part of u .

Moreover, if $c \equiv 0$, then

$$\max_{\overline{\Omega_T}} u = \max_{\partial\Omega \times [0, T] \cup \overline{\Omega} \times \{0\}} u.$$

The proof of Theorem 6.2 (in a more general case) is classical. We will not directly apply Theorem 6.2 but we will use its following corollaries.

Corollary 3. Let $\varepsilon, \lambda \geq 0$. If $H \in C([0, T]; C^2(\mathbb{T} \times \mathbb{R}_+)) \cap C^1([0, T]; C^0(\mathbb{T} \times \mathbb{R}_+))$ is a bounded function satisfying the following inequality

$$(\partial_t + b_1 \partial_x + b_2 \partial_y - \varepsilon^2 \partial_x^2 - \omega \partial_y^2) H \leq fH,$$

where $w : [0, T] \times \mathbb{T} \times \mathbb{R}_+ \rightarrow \mathbb{R}_+^*$ and b_1, b_2, f are continuous function satisfying

$$B = \left\| \frac{b_2}{1+y} \right\|_{L^\infty([0, T] \times \mathbb{T} \times \mathbb{R}_+)} < +\infty, \quad (6.42)$$

and

$$\|f\|_{L^\infty([0, T] \times \mathbb{T} \times \mathbb{R}_+)} \leq \lambda. \quad (6.43)$$

Then, for any $t \in [0, T]$, we have

$$\sup_{\mathbb{T} \times \mathbb{R}_+} H(t) \leq \max \left\{ e^{\lambda t} \|H(0)\|_{L^\infty(\mathbb{T} \times \mathbb{R}_+)}, \max_{\tau \in [0, t]} e^{\lambda(t-\tau)} \|H(\tau)|_{y=0}\|_{L^\infty(\mathbb{T})} \right\}. \quad (6.44)$$

Proof of Corollary 3. For any $\mu > 0$, we define

$$\mathcal{H} = e^{-\lambda t} H - \mu B t - \mu \ln(1 + y).$$

Then the derivatives of \mathcal{H}

$$\begin{aligned}\partial_t \mathcal{H} &= e^{-\lambda t} (\partial_t H - \lambda H) - \mu B \\ \partial_x \mathcal{H} &= e^{-\lambda t} \partial_x H \\ \partial_x^2 \mathcal{H} &= e^{-\lambda t} \partial_x^2 H \\ \partial_y \mathcal{H} &= e^{-\lambda t} \partial_y H - \frac{\mu}{1+y} \\ \partial_y^2 \mathcal{H} &= e^{-\lambda t} \partial_y^2 H + \frac{\mu}{(1+y)^2}\end{aligned}$$

We denote

$$X = \{\partial_t + b_1 \partial_x + b_2 \partial_y - \varepsilon^2 \partial_x^2 - \omega \partial_y^2 + (\lambda - f)\} \mathcal{H}$$

and

$$\begin{aligned}X_1 &= \{\partial_t - \lambda + b_1 \partial_x + b_2 \partial_y - \varepsilon^2 \partial_x^2 - \omega \partial_y^2 + (\lambda - f)\} H \\ &\leq 0\end{aligned}$$

Then for any $\tilde{t} \in]0, T]$

$$\begin{aligned}X &= e^{-\lambda t} X_1 - \mu B - \frac{b_2 \mu}{1+y} - \frac{\omega \mu}{(1+y)^2} - (\lambda - f) \{\mu B t + \mu \ln(1 + y)\} \\ &= e^{-\lambda t} X_1 - \mu \left(B - \frac{b_2}{1+y} \right) - \frac{\omega \mu}{(1+y)^2} - (\lambda - f) \{\mu B t + \mu \ln(1 + y)\} \\ &\leq 0 \quad \text{in } [0, \tilde{t}] \times \mathbb{T} \times \mathbb{R}^+\end{aligned}$$

So by the Thoerm 6.2, we have

$$\max_{[0, \tilde{t}] \times \mathbb{T} \times [0, R]} \mathcal{H} \leq \max_{\{0\} \times \mathbb{T} \times [0, R] \cup [0, \tilde{t}] \times \mathbb{T} \times \{0, R\}} \mathcal{H}^+$$

for $0 < \tau < \tilde{t}$, $x \in \mathbb{T}$, $y = R$,

$$\begin{aligned}\mathcal{H}(\tau, x, R) &= e^{-\lambda \tau} H(\tau, x, R) - \mu B t - \mu \ln(1 + R). \\ &\leq \|H\|_{L^\infty} - \mu \ln(1 + R) \\ &\leq 0 \quad \text{for } R \geq \exp\left(\frac{1}{\mu} \|H\|_{L^\infty}\right) - 1\end{aligned}$$

which mean $\mathcal{H}(\tau, x, R)^+ = 0$. We denote

$$R_\mu = \exp\left(\frac{1}{\mu} \|H\|_{L^\infty}\right) - 1$$

Then for any $R \geq R_\mu$

$$\begin{aligned}\max_{[0, \tilde{t}] \times \mathbb{T} \times [0, R]} \mathcal{H} &\leq \max\left\{ \max_{\mathbb{T} \times [0, R]} \mathcal{H}(0)^+, \max_{[0, \tilde{t}] \times \mathbb{T}} \mathcal{H}|_{y=0}^+ \right\} \\ &\leq \max\left\{ \max_{\mathbb{T} \times \mathbb{R}^+} \mathcal{H}(0)^+, \max_{[0, \tilde{t}] \times \mathbb{T}} \mathcal{H}|_{y=0}^+ \right\}\end{aligned}$$

So

$$\begin{aligned}\max_{[0, \tilde{t}] \times \mathbb{T} \times \mathbb{R}^+} \mathcal{H} &\leq \max\left\{ \max_{\mathbb{T} \times \mathbb{R}^+} \mathcal{H}(0)^+, \max_{[0, \tilde{t}] \times \mathbb{T}} \mathcal{H}|_{y=0}^+ \right\} \\ &\leq \max\left\{ \max_{\mathbb{T} \times \mathbb{R}^+} H(0)^+, \max_{(\tau, x) \in [0, \tilde{t}] \times \mathbb{T}} e^{-\lambda \tau} H(\tau, x, 0)^+ \right\} \\ &\leq \max\left\{ \|H(0)\|_{L^\infty(\mathbb{T} \times \mathbb{R}^+)}, \max_{\tau \in [0, \tilde{t}]} e^{-\lambda \tau} \|H|_{y=0}\|_{L^\infty(\mathbb{T})} \right\}\end{aligned}$$

Therefore, for any $(x, y) \in \mathbb{T} \times \mathbb{R}^+$, we have

$$\begin{aligned} e^{-\lambda \tilde{t}} H(\tilde{t}, x, y) - \mu B \tilde{t} - \mu \ln(1 + y) \\ \leq \max\{\|H(0)\|_{L^\infty(\mathbb{T} \times \mathbb{R}^+)}, \max_{\tau \in [0, \tilde{t}]} e^{-\lambda \tau} \|H|_{y=0}\|_{L^\infty(\mathbb{T})}\} \end{aligned}$$

We pass to the limit $\mu \rightarrow 0^+$, and we replace the time \tilde{t} by t , we obtain 6.51 □

Corollary 4. *Let $\varepsilon, \lambda \geq 0$.*

If $H \in C([0, T]; C^2(\mathbb{T} \times \mathbb{R}_+)) \cap C^1([0, T]; C^0(\mathbb{T} \times \mathbb{R}_+))$ is a bounded function with

$$\kappa(t) = \min\left\{\min_{\mathbb{T} \times \mathbb{R}^+} H(0), \min_{[0, \tilde{t}] \times \mathbb{T}} H|_{y=0}\right\} \geq 0$$

and satisfying the following equality

$$(\partial_t + b_1 \partial_x + b_2 \partial_y - \varepsilon^2 \partial_x^2 - \omega \partial_y^2)H = fH,$$

where $w : [0, T] \times \mathbb{T} \times \mathbb{R}_+ \rightarrow \mathbb{R}_+^$ and b_1, b_2, f are continuous function satisfying (6.42), (6.43), then for any $t \in [0, T]$, we have*

$$\min_{\mathbb{T} \times \mathbb{R}_+} H(t) \geq (1 - \lambda t e^{\lambda t}) \kappa(t) \quad (6.45)$$

Proof of Corollary 4. For any fixed $\tilde{t} \in]0, T]$ and $\mu > 0$, we define

$$h = e^{-\lambda t} \{H - \kappa(\tilde{t})\} + \{\lambda \kappa(\tilde{t}) + \mu B\}t + \mu \ln(1 + y).$$

Then we can check that

$$\{\partial_t - \lambda + b_1 \partial_x + b_2 \partial_y - \varepsilon^2 \partial_x^2 - \omega \partial_y^2 + (\lambda - f)\}h > 0 \quad \text{in } [0, \tilde{t}] \times \mathbb{T} \times \mathbb{R}^+$$

So by the weak minimum principle

$$\begin{aligned} \min_{[0, \tilde{t}] \times \mathbb{T} \times [0, R]} h &\geq \min\left\{\min_{\mathbb{T} \times [0, R]} h(0), \min_{[0, \tilde{t}] \times \mathbb{T}} h|_{y=0}, \min_{[0, \tilde{t}] \times \mathbb{T}} h|_{y=R}\right\} \\ &\geq \min\left\{\min_{\mathbb{T} \times \mathbb{R}^+} h(0), \min_{[0, \tilde{t}] \times \mathbb{T}} h|_{y=0}, \min_{[0, \tilde{t}] \times \mathbb{T}} h|_{y=R}\right\} \end{aligned}$$

We have for any $0 \leq \tau \leq \tilde{t}$, $x \in \mathbb{T}$

$$\begin{aligned} h(\tau, x, R) &= e^{-\lambda \tau} H(\tau, x, R) - e^{-\lambda \tau} \kappa(\tilde{t}) + \{\lambda \kappa(\tilde{t}) + \mu B\}\tau + \mu \ln(1 + R) \\ &\geq -e^{-\lambda \tau} \|H\|_{L^\infty} - e^{-\lambda \tau} \kappa(\tilde{t}) + \mu \ln(1 + R) \\ &\geq -e^{-\lambda \tau} \{\|H\|_{L^\infty} + \kappa(\tilde{t})\} + \mu \ln(1 + R) \\ &\geq -\{\|H\|_{L^\infty} + \kappa(\tilde{t})\} + \mu \ln(1 + R) \end{aligned}$$

Then for any

$$R \geq \exp\left(\frac{1}{\mu} \{\|H\|_{L^\infty} + \kappa(\tilde{t})\}\right) - 1$$

We have

$$h(\tau, x, R) \geq 0$$

and

$$\min_{[0, \tilde{t}] \times \mathbb{T} \times [0, R]} h \geq 0$$

we can pass to the limit $R \rightarrow \infty$, We obtain

$$\min_{[0, \tilde{t}] \times \mathbb{T} \times \mathbb{R}^+} h \geq 0$$

Therefore, for any $(x, y) \in \mathbb{T} \times \mathbb{R}^+$, we have

$$h(\tilde{t}, x, y) \geq 0$$

It follows that

$$e^{-\lambda \tilde{t}} \{H(\tilde{t}, x, y) - \kappa(\tilde{t})\} + \{\lambda \kappa(\tilde{t}) + \mu B\} \tilde{t} + \mu \ln(1 + y) \geq 0$$

Which implies

$$e^{-\lambda \tilde{t}} \{H(\tilde{t}, x, y) - \kappa(\tilde{t})\} \geq -\lambda \kappa(\tilde{t}) \tilde{t} - (\mu B \tilde{t} + \mu \ln(1 + y))$$

which gives

$$H(\tilde{t}, x, y) \geq \kappa(\tilde{t}) - \lambda \tilde{t} e^{\lambda \tilde{t}} \kappa(\tilde{t}) - (\mu B \tilde{t} + \mu \ln(1 + y)) e^{\lambda \tilde{t}}$$

Letting $\mu \rightarrow 0$, and replacing \tilde{t} by t , we obtain (6.45). \square

The goal of this section is to prove the following proposition.

Proposition 6.7.1. *We define the real functions Ω and $G : [0, T] \rightarrow \mathbb{R}^+$ by*

$$\Omega(t) = \sup_{[0, t]} \|\omega\|_{H_y^{s, \gamma}} \quad (6.46)$$

$$G(t) = \Omega(t) + \sup_{[0, t]} \|\partial_x^s U\|_{L^2(\mathbb{T})}. \quad (6.47)$$

If $s \geq 6$, we define

$$I(t) = \sum_{|\alpha| \leq 2} (1 + y)^{2(\sigma + \alpha_2)} |D^\alpha \omega(t)|^2. \quad (6.48)$$

Then we have the following estimate

$$\begin{aligned} \|I(t)\|_{L^\infty(\mathbb{T} \times \mathbb{R}^+)} &\leq \left(\|I(0)\|_{L^\infty(\mathbb{T} \times \mathbb{R}^+)} + C_{s, \gamma} (1 + \Omega(t)) \Omega(t)^2 t \right) \\ &\quad \times \exp(C_{s, \gamma, \sigma, \delta} (1 + G(t)) t). \end{aligned} \quad (6.49)$$

and

$$\begin{aligned} \min_{(x, y) \in \mathbb{T} \times \mathbb{R}^+} (1 + y)^\sigma \omega(t, x, y) &\geq \left(1 - C_{s, \gamma, \sigma, \delta} (1 + G(t)) t e^{C_{s, \gamma, \sigma, \delta} (1 + G(t)) t} \right) \\ &\quad \times \left(\min_{(x, y) \in \mathbb{T} \times \mathbb{R}^+} (1 + y)^\sigma \omega_0(x, y) - C_{s, \gamma} \{1 + \Omega(t)\} \Omega(t) t \right). \end{aligned} \quad (6.50)$$

We give first the main steps of the proof :

1. We recall the regularized system (6.11)-(6.14):

$$\begin{cases} \partial_t \omega + u \partial_x \omega + (v - 2\partial_y \omega) \partial_y \omega = \varepsilon^2 \partial_x^2 \omega + 2\omega \partial_y^2 \omega & \text{in } [0, T] \times \mathbb{T} \times \mathbb{R} \\ \omega|_{t=0} = \omega_0(x, y) & \text{on } \mathbb{T} \times \mathbb{R}^+ \\ 2\omega \partial_y \omega|_{y=0} = \partial_x p & \text{on } [0, T] \times \mathbb{T} \end{cases}$$

We define the operator L by

$$L = \partial_t + u \partial_x + (v - 2\partial_y \omega) \partial_y - \varepsilon^2 \partial_x^2 - 2\omega \partial_y^2$$

and we have

$$L\omega = 0.$$

2. We set

$$B_\alpha = (1+y)^{\sigma+\alpha_2} D^\alpha \omega.$$

We get the following equation

$$\begin{aligned} \|\partial_t I|_{y=0}\|_{L^\infty(\mathbb{T})} &\leq C_{s,\gamma} \{1 + \|\omega\|_{H_g^{s,\gamma}}\} \|\omega\|_{H_g^{s,\gamma}}^2 \\ \partial_t B_\alpha + u \partial_x B_\alpha + (v - 2\partial_y \omega) \partial_y B_\alpha - \varepsilon^2 \partial_x^2 B_\alpha - 2\omega \partial_y^2 B_\alpha \\ &= S_\alpha + Q_\alpha \partial_y B_\alpha + R_\alpha B_\alpha \end{aligned}$$

where

$$\begin{aligned} Q_\alpha &= -\frac{4(\sigma + \alpha_2)}{1+y} \omega, \\ R_\alpha &= (\sigma + \alpha_2) \frac{(v - 2\partial_y \omega)}{1+y} + \frac{2(\sigma + \alpha_2)(\sigma + \alpha_2 + 1)}{(1+y)^2} \omega \\ S_\alpha &= -\sum_{0 < \beta \leq \alpha} \binom{\alpha}{\beta} \left\{ (1+y)^{\beta_2} D^\beta u B_{\alpha-\beta+e_1} + (1+y)^{\beta_2-1} D^\beta v B_{\alpha-\beta+e_2} \right\} \\ &\quad + 2 \sum_{0 < \beta \leq \alpha} \binom{\alpha}{\beta} \left\{ (1+y)^{\beta_2-2} D^\beta \omega B_{\alpha-\beta+2e_2} + (1+y)^{\beta_2-1} \partial_y D^\beta \omega B_{\alpha-\beta+e_2} \right\} \end{aligned}$$

and

$$\begin{aligned} |Q_\alpha| &\leq \frac{4(\sigma + \alpha_2)}{\delta(1+y)^{1+\sigma}} \\ &\leq C_{\delta,\sigma} \end{aligned}$$

$$\begin{aligned} |R_\alpha| &\leq (\sigma + \alpha_2) \left| \frac{v}{1+y} \right| + 2(\sigma + \alpha_2) |\partial_y \omega| + \frac{2(\sigma + \alpha_2)(\sigma + \alpha_2 + 1)}{(1+y)^2} \omega \\ &\leq C_{s,\gamma,\sigma,\delta} \{ \|\omega\|_{H_g^{s,\gamma}} + \|\partial_x^s U\|_{L^2(\mathbb{T})} \} + C_{\sigma,\delta} \\ &\leq C_{s,\gamma,\sigma,\delta} \{ 1 + \|\omega\|_{H_g^{s,\gamma}} + \|\partial_x^s U\|_{L^2(\mathbb{T})} \} \end{aligned}$$

$$\begin{aligned} |S_\alpha| &\leq C_{s,\gamma,\sigma,\delta} \{ \|\omega\|_{H_g^{s,\gamma}} + \|\partial_x^s U\|_{L^2(\mathbb{T})} \} \\ &\quad \times \sum_{0 < \beta \leq \alpha} \binom{\alpha}{\beta} \left\{ |B_{\alpha-\beta+e_1}| + 2 |B_{\alpha-\beta+e_2}| + |B_{\alpha-\beta+2e_2}| \right\} \end{aligned}$$

3. We set

$$I = \sum_{|\alpha| \leq 2} |B_\alpha|^2$$

The function I is solution of the following equation

$$LI = \partial_t I + u \partial_x I + (v - 2\partial_y \omega) \partial_y I - \varepsilon^2 \partial_x^2 I - 2\omega \partial_y^2 I$$

where the right hand side term is

$$\begin{aligned} LI &= -2 \sum_{|\alpha| \leq 2} \left(\varepsilon^2 |\partial_x B_\alpha|^2 + 2\omega |\partial_y B_\alpha|^2 \right) \\ &\quad + 2 \sum_{|\alpha| \leq 2} \left(S_\alpha B_\alpha + Q_\alpha B_\alpha \partial_y B_\alpha + R_\alpha |B_\alpha|^2 \right) \end{aligned}$$

4. We next prove the estimate

$$LI \leq C_{s,\gamma,\sigma,\delta} \{1 + \|\omega\|_{H_g^{s,\gamma}} + \|\partial_x^s U\|_{L^2(\mathbb{T})}\} I$$

We have

$$\begin{aligned} \left| \frac{v}{1+y} - 2 \frac{\partial_y \omega}{1+y} \right| &\leq \left| \frac{v}{1+y} \right| + 2 \left| \frac{\partial_y \omega}{1+y} \right| \\ &\leq \left\| \frac{v}{1+y} \right\|_{L^\infty} + 2(1+y)^{\sigma+1} |\partial_y \omega| \\ &\leq C_{s,\gamma,\sigma,\delta} \{ \|\omega\|_{H_g^{s,\gamma}} + \|\partial_x^s U\|_{L^2(\mathbb{T})} \} + \frac{2}{\delta} \end{aligned}$$

Then

$$\left\| \frac{v}{1+y} - 2 \frac{\partial_y \omega}{1+y} \right\|_{L^\infty} < \infty$$

Because we suppose that $\omega \in H_g^{s,\gamma}$. And

$$\begin{aligned} |f| &= C_{s,\gamma,\sigma,\delta} \{1 + \|\omega\|_{H_g^{s,\gamma}} + \|\partial_x^s U\|_{L^2(\mathbb{T})}\} \\ &\leq C_{s,\gamma,\sigma,\delta} \{1 + \sup_{[0,t]} \|\omega\|_{H_g^{s,\gamma}} + \sup_{[0,t]} \|\partial_x^s U\|_{L^2(\mathbb{T})}\} \\ &= C_{s,\gamma,\sigma,\delta} \{1 + G(t)\} \\ &= \lambda \end{aligned}$$

5. We apply Corollaries 3 to the previous parabolic equation and we get :

$$\begin{aligned} \sup_{\mathbb{T} \times \mathbb{R}_+} H(t) &\leq \max \left\{ e^{C_{s,\gamma,\sigma,\delta} \{1+G(t)\} t} \|I(0)\|_{L^\infty(\mathbb{T} \times \mathbb{R}_+)}, \right. \\ &\quad \left. \max_{\tau \in [0,t]} e^{C_{s,\gamma,\sigma,\delta} \{1+G(t)\} (t-\tau)} \|I(\tau)|_{y=0}\|_{L^\infty(\mathbb{T})} \right\}. \quad (6.51) \end{aligned}$$

6. Lower bound estimate for $B_{(0,0)} = (1+y)^\sigma \omega$

We know that $B_{(0,0)}$ satisfies

$$\{\partial_t + u\partial_x + (v - 2\partial_y \omega - Q_{(0,0)})\partial_y - \varepsilon^2 \partial_x^2 - 2\omega \partial_y^2\} B_{(0,0)} = R_{(0,0)} B_{(0,0)}$$

Where

$$R_{(0,0)} = \sigma \frac{(v - 2\partial_y \omega)}{1+y} + \frac{2\sigma(\sigma+1)}{(1+y)^2} \omega$$

We have

$$\begin{aligned} |R_{(0,0)}| &\leq C_{s,\gamma,\sigma,\delta} \{ \|\omega\|_{H_g^{s,\gamma}} + \|\partial_x^s U\|_{L^2(\mathbb{T})} \} + \frac{2}{\delta} + \frac{2\sigma(\sigma+1)}{\delta} \\ &\leq C_{s,\gamma,\sigma,\delta} \{1 + \|\omega\|_{H_g^{s,\gamma}} + \|\partial_x^s U\|_{L^2(\mathbb{T})}\} \\ &\leq C_{s,\gamma,\sigma,\delta} \{1 + \sup_{[0,t]} \|\omega\|_{H_g^{s,\gamma}} + \sup_{[0,t]} \|\partial_x^s U\|_{L^2(\mathbb{T})}\} \\ &= C_{s,\gamma,\sigma,\delta} \{1 + G(t)\} \\ &= \lambda \end{aligned}$$

and

$$\left\| \frac{v - 2\partial_y \omega}{1+y} + \frac{4\sigma}{(1+y)^2} \omega \right\|_{L^\infty} < \infty$$

Using Corollary 4 we obtain

$$\begin{aligned} & \min_{(x,y) \in \mathbb{T} \times \mathbb{R}^+} (1+y)^\sigma \omega(t, x, y) \\ & \geq \left(1 - C_{s,\gamma,\sigma,\delta} (1+G(t)) t e^{C_{s,\gamma,\sigma,\delta}(1+G(t))t}\right) \\ & \quad \times \min_{(x,y) \in \mathbb{T} \times \mathbb{R}^+} \left((1+y)^\sigma \omega_0(x, y), \right. \\ & \quad \left. \min_{(t,x) \in [0,t] \times \mathbb{T}} (1+y)^\sigma \omega(t, x, 0) \right). \end{aligned} \quad (6.52)$$

7. Controls on Boundary Values.

- Estimate of $I|_{y=0}$

$$I|_{y=0} = \sum_{|\alpha| \leq 2} |D^\alpha \omega|_{y=0}|^2$$

For this we write the evolution equation for $D^\alpha \omega$ at $y = 0$: for any $|\alpha| \leq 2$

$$\begin{aligned} \partial_t D^\alpha \omega|_{y=0} &= \varepsilon^2 \partial_x^2 D^\alpha \omega + 2\omega \partial_y^2 D^\alpha \omega + 2\partial_y \omega \partial_y D^\alpha \omega|_{y=0} \\ & \quad + E'_\alpha|_{y=0} + E''_\alpha|_{y=0} \end{aligned}$$

Where

$$\begin{aligned} E'_\alpha &= \sum_{0 < \beta \leq \alpha} \binom{\alpha}{\beta} \{D^\beta u \partial_x D^{\alpha-\beta} \omega + D^\beta v \partial_y D^{\alpha-\beta} \omega\} \\ E''_\alpha &= 2 \sum_{0 < \beta \leq \alpha} \binom{\alpha}{\beta} \{D^\beta \omega \partial_y^2 D^{\alpha-\beta} \omega + \partial_y D^\beta \omega \partial_y D^{\alpha-\beta} \omega\} \end{aligned}$$

We use the boundary condition $u|_{y=0} = v|_{y=0} = 0$, E'_α is given explicitly by

$$E'_\alpha = \begin{cases} 0 & \text{if } \alpha_2 = 0 \\ -\alpha_2 \omega \partial_x \partial_y^{\alpha_2-1} \omega & \text{if } \alpha_1 = 0 \text{ and } 1 \leq \alpha_2 \leq 2 \\ -\partial_x \omega \partial_x \omega - \omega \partial_x^2 \omega & \text{if } \alpha = (1, 1) \end{cases}$$

Furthermore, by proposition E.3 , if $s \geq 6$, then

$$\|E'_\alpha|_{y=0}\|_{L^\infty(\mathbb{T})} \leq C_{s,\gamma} \|\omega\|_{H_g^{s,\gamma}}^2 \quad (6.53)$$

$$\|E''_\alpha|_{y=0}\|_{L^\infty(\mathbb{T})} \leq C_{s,\gamma} \|\omega\|_{H_g^{s,\gamma}}^2 \quad (6.54)$$

In addition for $\varepsilon \in]0, 1]$

$$\begin{aligned} & \|\varepsilon^2 (\partial_x^2 D^\alpha \omega + 2\omega \partial_y^2 D^\alpha \omega)|_{y=0}\|_{L^\infty} \\ & \leq C_{s,\gamma} \|\omega\|_{H_g^{s,\gamma}} + C_{s,\gamma} \|\omega\|_{H_g^{s,\gamma}}^2 \end{aligned} \quad (6.55)$$

and

$$2\|\partial_y \omega \partial_y D^\alpha \omega|_{y=0}\|_{L^\infty(\mathbb{T})} \leq C_{s,\gamma} \|\omega\|_{H_g^{s,\gamma}}^2 \quad (6.56)$$

Therefore, using (6.53), (6.54), (6.55), (6.56) and inequality (B.10)(masmoudi), we have for $s \geq 6$

$$\|\partial_t I|_{y=0}\|_{L^\infty(\mathbb{T})} \leq C_{s,\gamma} \{1 + \|\omega\|_{H_g^{s,\gamma}}\} \|\omega\|_{H_g^{s,\gamma}}^2$$

and hence, by integration and definition of Ω , we obtain

$$\|I|_{y=0}\|_{L^\infty(\mathbb{T})} \leq \|I(0)|_{y=0}\|_{L^\infty(\mathbb{T})} + C_{s,\gamma} \{1 + \Omega(t)\} \Omega(t)^2 t \quad (6.57)$$

we put the estimate (6.57) in (6.51), we prove the desired estimate (6.7.1)

- Estimate of $\omega|_{y=0}$

Using (6.55) – (6.56) and the definition of (6.46), we have for any $s \geq 4$

$$\|\partial_t \omega|_{y=0}\|_{L^\infty(\mathbb{T})} \leq C_{s,\gamma} \{1 + \Omega(t)\} \Omega(t)$$

So by integration we obtain

$$\min_{\mathbb{T}} \omega(t)|_{y=0} \geq \min_{\mathbb{T}} \omega_0|_{y=0} - C_{s,\gamma} \{1 + \Omega(t)\} \Omega(t) t \quad (6.58)$$

Combining (6.52) and (6.58), we prove (6.50).

6.8 Appendix : Decay Rates for $H_{\sigma,\delta}^{S,\gamma}$ Functions

In this Appendix, we recall from [59] some pointwise estimates on the decay of the vorticity when $y \rightarrow \infty$.

Proposition 6.8.1 (Masmoudi - Wong [59]). *Let $s' \geq 4$ be an integer, $\gamma \geq 1$, $\sigma > \gamma + \frac{1}{2}$, and $\delta \in (0, 1)$. If $\omega \in H_{\sigma,\delta}^{s'+4,\gamma}$, then ω is $s' + 2$ times continuously differentiable and there exists a constant $C_{s',\gamma,\delta,\|\omega\|_{H^{s'+4,\gamma}}} > 0$ such that for all $|\alpha| \leq s' + 2$,*

$$|D^\alpha \omega| \leq C_{s',\gamma,\delta,\|\omega\|_{H^{s'+4,\gamma}}} (1+y)^{-b_\alpha} \quad (6.59)$$

where the exponent

$$b_\alpha = \begin{cases} \sigma + \alpha_2 & \text{if } |\alpha| \leq 2, \\ \frac{\sigma + (2^{|\alpha|-2} - 1)\gamma}{2^{|\alpha|-2}} & \text{if } 2 \leq |\alpha| \leq s' + 1, \\ \gamma + \alpha_2 & \text{if } |\alpha| = s' + 2. \end{cases} \quad (6.60)$$

Remark 13. (Decay Rates from Sobolev Embedding). *Using the standard sobolev embedding argument and the definition of $H_{\sigma,\delta}^{s'+4,\gamma}$, one may prove that if $\omega \in H_{\sigma,\gamma}^{s'+4,\gamma}$, then ω is $s' + 2$ times continuously differentiable and there exists a constant $C_{s,\gamma} > 0$ such that*

$$|D^\alpha \omega| \leq \begin{cases} \delta^{-1} (1+y)^{-\sigma-\alpha_2} & \text{if } |\alpha| \leq 2, \\ C_{s,\gamma} \|\omega\|_{H^{s'+4,\gamma}} (1+y)^{-\gamma-\alpha_2} & \text{if } 2 \leq |\alpha| \leq s' + 1, \end{cases} \quad (6.61)$$

Thus, the interesting part of Proposition C.1 is that the decay rate of (6.59) is better than that of (6.61). This slightly better pointwise decay will help us to deal with the boundary terms at $y = +\infty$ while we are integrating by parts in the y -variable.

Proof. OF Proposition C.1. According to Remark C.2, we are only required to justify the inequality (6.59) with the decay rate defined in (6.60). First of all, let us state without proof the following calculus lemma.

Lemma 6.8.1 (Masmoudi - Wong [59]). (Pointwise Interpolation). *Let $f : \mathbb{T} \times \mathbb{R}^+ \rightarrow \mathbb{R}$ be a twice differentiable function. Then we have the following:*

- If there exist constants C_0, C_2, b_0 , and b_2 such that $|\partial_x^i f| \leq C_i (1+y)^{-b_i}$ for all $i = 0, 2$, then

$$|\partial_x f| \leq 2\sqrt{C_0 C_2} (1+y)^{-\frac{b_0+b_2}{2}} \quad \text{in } \mathbb{T} \times \mathbb{R}^+ \quad (6.62)$$

- If there exist constants C_0, C_2, b_0 , and b_2 such that $|\partial_x^i f| \leq C_i (1+y)^{-b_i}$ for all $i = 0, 2$, then

$$|\partial_y f| \leq 2\sqrt{C_0 C_2} (1+y)^{-\frac{b_0+b_2}{2}} \quad \text{in } \mathbb{T} \times \mathbb{R}^+ \quad (6.63)$$

The proof of Lemma C.3 is based on a standard Taylor series expansion technique and will be omitted here. Now, applying Lemma C.3 to $D^\alpha \omega$ inductively on $|\alpha| = 3, 4, \dots, s' + 1$ with (6.61), we prove (6.59) with the exponent b_α defined in (6.60). \square

Remark 14. (Further Improvement on the Decay Rate). *The decay rate b_α defined in (6.60) is obviously not optimal because one can apply the pointwise interpolation Lemma C.3 again to further improve it. However, we do not intend to optimize it here, Indeed, repeatedly applying the pointwise interpolation Lemma C.3, One may improve the decay rate b_α as*

$$b_\alpha = \begin{cases} \sigma + \alpha_2 & \text{if } |\alpha| \leq 2, \\ \frac{(s'+2-|\alpha|)\sigma + (|\alpha|-2)\gamma}{s'} + \alpha_2^- & \text{if } 3 \leq |\alpha| \leq s' + 1, \\ \gamma + \alpha_2 & \text{if } |\alpha| = s' + 2. \end{cases} \quad (6.64)$$

6.9 Appendix : Almost Equivalence of weighted Norms

In this appendix, we recall from [59] the almost equivalence of the two norms $\|\omega\|_{H_g^{s,\gamma}}$ and $\|\omega\|_{H^{s,\gamma}}$. For the sake of completeness, we recall also its proof.

Proposition 6.9.1 (Masmoudi - Wong [59]). *] Let $s \geq 6$ be an integer, $\gamma \geq 1$, $\sigma > \gamma + \frac{1}{2}$ and $\delta \in (0, 1)$. For any $\omega \in H_{\sigma,\delta}^{s,\gamma}(\mathbb{T} \times \mathbb{R}^+)$, we have the following inequality: there exist positive constants c_δ and $C_{s,\gamma,\sigma,\delta}$ such that*

$$c_\delta \|\omega\|_{H_g^{s,\gamma}} \leq \|\omega\|_{H^{s,\gamma}} + \|u - U\|_{H^{s,\gamma-1}} \leq C_{s,\gamma,\sigma,\delta} \left\{ \|\omega\|_{H_g^{s,\gamma}} + \|\partial_x^s U\|_{L^2} \right\} \quad (6.65)$$

provided that $\omega = \partial_y u$, $u|_{y=0} = 0$ and $\lim_{y \rightarrow +\infty} u = U$, where the weighted H^s -norm $\|\cdot\|_{H^{s,\gamma}}$ and $\|\cdot\|_{H_g^{s,\gamma}}$ are defined by (6.6) and (6.7) respectively.

Proof. Without loss of generality, we only need to prove inequality (6.65) for any smooth function ω because the general case can be obtained by the standard density argument. First of all, it follows from the definitions of $\|u - U\|_{H^{s,\gamma-1}}$ and $\|\omega\|_{H^{s,\gamma}}$ that

$$\begin{aligned} \|\omega\|_{H^{s,\gamma}} + \sum_{k=0}^s \|(1+y)^{\gamma-1} \partial_x^k (u - U)\|_{L^2} & \leq \|\omega\|_{H^{s,\gamma}} + \|u - U\|_{H^{s,\gamma-1}} \\ & \leq 2 \left(\|\omega\|_{H^{s,\gamma}} + \sum_{k=0}^s \|(1+y)^{\gamma-1} \partial_x^k (u - U)\|_{L^2} \right) \end{aligned} \quad (6.66)$$

Furthermore, applying Wirtinger's inequality in the x-variable repeatedly and part (i) of Lemma E.1, we have

$$\begin{cases} \sum_{k=0}^s \|(1+y)^{\gamma-1} \partial_x^k (u - U)\|_{L^2} \leq \frac{1 - \pi^{-2s}}{1 - \pi^{-2}} \|(1+y)^{\gamma-1} \partial_x^s (u - U)\|_{L^2} \\ \|(1+y)^{\gamma-1} (u - U)\|_{L^2} \leq \frac{2}{2\gamma - 1} \|(1+y)^{\gamma-1} \omega\|_{L^2} \leq \frac{2}{2\gamma - 1} \|\omega\|_{H^{s,\gamma}} \end{cases}$$

and then there exists a constant $C_{s,\gamma} > 0$ such that

$$\begin{aligned} \|\omega\|_{H^{s,\gamma}} + \|(1+y)^{\gamma-1} \partial_x^s (u - U)\|_{L^2} & \leq \|\omega\|_{H^{s,\gamma}} + \sum_{k=0}^s \|(1+y)^{\gamma-1} \partial_x^k (u - U)\|_{L^2} \\ & \leq C_{s,\gamma} \left\{ \|\omega\|_{H^{s,\gamma}} + \|(1+y)^{\gamma-1} \partial_x^s (u - U)\|_{L^2} \right\} \end{aligned} \quad (6.67)$$

Therefore, according to inequality (6.66) and (6.67), it suffices to prove

$$\begin{aligned} c_\delta \|\omega\|_{H_g^{s,\gamma}} & \leq \|\omega\|_{H^{s,\gamma}} + \|(1+y)^{\gamma-1} \partial_x^s (u - U)\|_{L^2} \\ & \leq C_{s,\gamma,\sigma,\delta} \left\{ \|\omega\|_{H_g^{s,\gamma}} + \|\partial_x^s U\|_{L^2} \right\} \end{aligned} \quad (6.68)$$

Lemma 6.9.1. *Let $s \geq 6 \geq 4$ be an integer, $\gamma \geq 1$, $\sigma > \gamma + \frac{1}{2}$, and $\delta \in]0, 1[$. If $\omega \in H_{\sigma,\delta}^{s,\gamma}(\mathbb{T} \times \mathbb{R}^+)$, then for any $k = 1, 2, \dots, s$,*

$$\|(1+y)^\gamma g_k\|_{L^2} \leq \|(1+y)^\gamma \partial_x^k \omega\|_{L^2} + \delta^{-2} \|(1+y)^{\gamma-1} \partial_x^k (u - U)\|_{L^2} \quad (6.69)$$

where $g_k = \partial_x^k \omega - \frac{\partial_y \omega}{\omega} \partial_x^k (u - U)$. In addition, if $u|_{y=0} = 0$, then for any $k = 1, 2, \dots, s$,

$$\begin{aligned} \|(1+y)^\gamma \partial_x^k \omega\|_{L^2} + \|(1+y)^{\gamma-1} \partial_x^k (u - U)\|_{L^2} & \leq C_{\gamma,\sigma,\delta} \left\{ \|(1+y)^\gamma g_k\|_{L^2} + \|\partial_x^k U\|_{L^2(\mathbb{T})} \right\} \end{aligned} \quad (6.70)$$

Assuming Lemma D.2, which be shown at the end of this appendix, for the moment we can show inequality (6.68) as follows. Applying Lemma D.2, for $k = s$, we obtain, from (6.69) and (6.70)

$$\begin{aligned} & \frac{1}{2}\delta^4\|(1+y)^\gamma g_s\|_{L^2}^2 \\ & \leq \|(1+y)^\gamma \partial_x^s \omega\|_{L^2}^2 + \|(1+y)^{\gamma-1} \partial_x^s (u-U)\|_{L^2}^2 \\ & \leq C_{\gamma,\sigma,\delta} \left\{ \|(1+y)^\gamma g_s\|_{L^2}^2 + \|\partial_x^s U\|_{L^2(\mathbb{T})}^2 \right\} \end{aligned} \quad (6.71)$$

Adding

$$\sum_{\substack{|\alpha| \leq s \\ \alpha_1 \leq s-1}} \|(1+y)^{\gamma+\alpha_2} D^\alpha \omega\|_{L^2(\mathbb{T} \times \mathbb{R}^+)}^2$$

to (6.71), we have

$$\begin{aligned} \frac{1}{2}\delta^4 \|\omega\|_{H_g^{s,\gamma}}^2 & \leq \|\omega\|_{H^{s,\gamma}}^2 + \|(1+y)^{\gamma-1} \partial_x^s (u-U)\|_{L^2}^2 \\ & \leq C_{s,\gamma,\sigma,\delta} \left\{ \|\omega\|_{H_g^{s,\gamma}}^2 + \|\partial_x^s U\|_{L^2}^2 \right\} \end{aligned}$$

which implies inequality (6.68). \square

Finally, in order to complete the justification of the almost equivalence relation (6.65), we will prove Lemma D.2.

Proof. of Lemma D.2. To prove (6.69), let us recall from the definition of $H_{\sigma,\delta}^{s,\gamma}$ that

$$\begin{cases} \delta(1+y)^{-\sigma} \leq \omega \leq \delta^{-1}(1+y)^{-\sigma} \\ |\partial_y \omega| \leq \delta^{-1}(1+y)^{-\sigma-1} \end{cases} \quad (6.72)$$

then $\frac{|\partial_y \omega|}{\omega} \leq \delta^{-2}(1+y)^{-1}$, and hence, for any $k = 1, 2, \dots, s$,

$$\begin{aligned} \|(1+y)^\gamma g_k\|_{L^2} & \leq \|(1+y)^\gamma \partial_x^k \omega\|_{L^2} + \left\| (1+y) \frac{\partial_y \omega}{\omega} \right\|_{L^\infty} \|(1+y)^{\gamma-1} \partial_x^k (u-U)\|_{L^2} \\ & \leq \|(1+y)^\gamma \partial_x^k \omega\|_{L^2} + \delta^{-2} \|(1+y)^{\gamma-1} \partial_x^k (u-U)\|_{L^2} \end{aligned}$$

which is inequality (6.69). Next, we are going to show (6.70). the main observation is that we can rewrite $g_k = \omega \partial_y \left(\frac{\partial_x^k (u-U)}{\omega} \right)$. Thus, applying (6.72)₁ and part (ii) of Lemma E.1, we have

$$\begin{aligned} & \|(1+y)^{\gamma-1} \partial_x^k (u-U)\|_{L^2} \\ & \leq \delta^{-1} \left\| (1+y)^{\gamma-\sigma-1} \frac{\partial_x^k (u-U)}{\omega} \right\|_{L^2} \\ & \leq C_{\gamma,\sigma} \delta^{-1} \left\{ \left\| \frac{\partial_x^k U}{\omega|_{y=0}} \right\|_{L^2(\mathbb{T})} + \left\| (1+y)^{\gamma-\sigma} \partial_y \left(\frac{\partial_x^k (u-U)}{\omega} \right) \right\|_{L^2} \right\} \\ & \leq C_{\gamma,\sigma} \delta^{-1} \left\{ \|\partial_x^k U\|_{L^2(\mathbb{T})} + \|(1+y)^\gamma g_k\|_{L^2} \right\} \end{aligned} \quad (6.73)$$

Now, using the triangle inequality, (6.72), and (6.73), we also have

$$\begin{aligned} & \|(1+y)^\gamma \partial_x^k \omega\|_{L^2} \\ & \leq \|(1+y)^\gamma g_k\|_{L^2} + \delta^{-2} \|(1+y) \partial_x^k (u-U)\|_{L^2} \\ & \leq C_{\gamma,\sigma,\delta} \left\{ \|(1+y)^\gamma g_k\|_{L^2} + \|\partial_x^k U\|_{L^2(\mathbb{T})} \right\}. \end{aligned} \quad (6.74)$$

Summing (6.73) and (6.74), we prove (6.70). \square

6.10 Appendix : Calculus Inequalities

In this appendix, we recall from [59] some calculus inequalities for the incompressible velocity field (u, v) , the vorticity ω , and the quantity g_k . For the sake of completeness, we recall also its proof.

6.10.1 Basic Inequalities

In this subsection we will state two elementary inequalities (i.e; Lemma E.1 and Lemma E.2 below). Let us first introduce Hardy-type inequalities.

Lemma 6.10.1. [Masmoudi, Wong [59]] *Let $f : \mathbb{T} \times \mathbb{R}^+ \mapsto \mathbb{R}$. Then*

(i) *if $\lambda > -\frac{1}{2}$ and $\lim_{y \rightarrow +\infty} f(x, y) = 0$, then*

$$\|(1+y)^\lambda f\|_{L^2(\mathbb{T} \times \mathbb{R}^+)} \leq \frac{2}{2\lambda+1} \|(1+y)^{\lambda+1} \partial_y f\|_{L^2(\mathbb{T} \times \mathbb{R}^+)} \quad (6.75)$$

(ii) *if $\lambda < -\frac{1}{2}$, then*

$$\begin{aligned} & \|(1+y)^\lambda f\|_{L^2(\mathbb{T} \times \mathbb{R}^+)} \\ & \leq \sqrt{-\frac{1}{2\lambda+1}} \|f|_{y=0}\|_{L^2(\mathbb{T})} - \frac{2}{2\lambda+1} \|(1+y)^{\lambda+1} \partial_y f\|_{L^2(\mathbb{T} \times \mathbb{R}^+)} \end{aligned} \quad (6.76)$$

Next, we will state the following Sobolev-type inequality

Lemma 6.10.2. [Masmoudi, Wong [59]] *Let $f : \mathbb{T} \times \mathbb{R}^+ \mapsto \mathbb{R}$. Then there exists a universal constant $C > 0$ such that*

$$\begin{aligned} & \|f\|_{L^\infty(\mathbb{T} \times \mathbb{R}^+)} \\ & \leq C \{ \|f\|_{L^2(\mathbb{T} \times \mathbb{R}^+)} + \|\partial_x f\|_{L^2(\mathbb{T} \times \mathbb{R}^+)} + \|\partial_y^2 f\|_{L^2(\mathbb{T} \times \mathbb{R}^+)} \}. \end{aligned} \quad (6.77)$$

6.10.2 Estimates for $H_{\sigma, \delta}^{s, \gamma}$ Functions

In this subsection we recall from [59] some control of the solution in the L^2 and L^∞ norm by the weighted norm $\|\cdot\|_{H_g^{s, \gamma}}$. For the sake of completeness, we recall also its proof, which is a consequence of Lemma (6.10.1) and Lemma (6.10.2).

Proposition 6.10.1. [Masmoudi, Wong [59]] *Let the vector field (u, v) defined on $\mathbb{T} \times \mathbb{R}^+$ satisfy the incompressibility condition $\partial_x u + \partial_y v = 0$, the Dirichlet boundary condition $u|_{y=0} = v|_{y=0} = 0$, and $\lim_{y \rightarrow +\infty} u = U$. If the vorticity $\omega = \partial_y u \in H_{\sigma, \delta}^{s, \gamma}$ for some integer $s > 4$ and for some constants $\gamma \geq 1$, $\sigma > \gamma + \frac{1}{2}$, and $\delta \in (0, 1)$, then we have the following estimates: there exist a constant $C_{s, \gamma, \sigma, \delta} > 0$ such that the following hold:*

(i) *For all $k = 0, 1, \dots, s$,*

$$\|(1+y)^{\gamma-1} \partial_x^k (u - U)\|_{L^2} \leq C_{s, \gamma, \sigma, \delta} \{ \|\omega\|_{H_g^{s, \gamma}} + \|\partial_x^s U\|_{L^2(\mathbb{T})} \}. \quad (6.78)$$

(ii) *For all $k = 0, 1, \dots, s-1$,*

$$\left\| \frac{\partial_x^k v + y \partial_x^{k+1} U}{1+y} \right\|_{L^2} \leq C_{s, \gamma, \sigma, \delta} \{ \|\omega\|_{H_g^{s, \gamma}} + \|\partial_x^s U\|_{L^2(\mathbb{T})} \}. \quad (6.79)$$

(iii) *For all $|\alpha| \leq s$,*

$$\|(1+y)^{\gamma+\alpha_2} D^\alpha \omega\|_{L^2} \leq \begin{cases} C_{s,\gamma,\sigma,\delta} \{\|\omega\|_{H_g^{s,\gamma}} + \|\partial_x^s U\|_{L^2(\mathbb{T})}\} & \text{if } \alpha = (s, 0), \\ \|\omega\|_{H_g^{s,\gamma}} & \text{if } \alpha \neq (s, 0) \end{cases} \quad (6.80)$$

(iv) For all $k = 0, 1, \dots, s$,

$$\|(1+y)^\gamma g_k\|_{L^2(\mathbb{T})} \leq \begin{cases} C_{s,\gamma,\sigma,\delta} \{\|\omega\|_{H_g^{s,\gamma}} + \|\partial_x^s U\|_{L^2(\mathbb{T})}\} & \text{if } k = 1, 2, \dots, s-1, \\ \|\omega\|_{H_g^{s,\gamma}} & \text{if } k = s, \end{cases} \quad (6.81)$$

where the quantity $g_k = \partial_x^k - \frac{\partial_y \omega}{\omega} \partial_x^k (u - U)$.

(v) For all $k = 0, 1, \dots, s-1$,

$$\|\partial_x^k u\|_{L^\infty} \leq C_{s,\gamma,\sigma,\delta} \{\|\omega\|_{H_g^{s,\gamma}} + \|\partial_x^s U\|_{L^2(\mathbb{T})}\} \quad (6.82)$$

(vi) For all $k = 0, 1, \dots, s-2$,

$$\left\| \frac{\partial_x^k v}{1+y} \right\|_{L^\infty} \leq C_{s,\gamma,\sigma,\delta} \{\|\omega\|_{H_g^{s,\gamma}} + \|\partial_x^s U\|_{L^2(\mathbb{T})}\} \quad (6.83)$$

(vii) For all $|\alpha| \leq s-2$,

$$\|(1+y)^{\gamma+\alpha_2} D^\alpha \omega\|_{L^\infty} \leq C_{s,\gamma} \|\omega\|_{H_g^{s,\gamma}} \quad (6.84)$$

Proof. (i) It follows from the definition of $\|\cdot\|_{H^{s,\gamma-1}}$ that

$$\|(1+y)^{\gamma-1} \partial_x^k (u - U)\|_{L^2} \leq \|u - U\|_{H^{s,\gamma-1}},$$

so inequality (6.78) is a direct consequence of almost equivalence inequality (6.65).

(ii) Applying part (ii) of lemma E.1 and inequality (6.78), we have

$$\left\| \frac{\partial_x^k v + y \partial_x^{k+1} U}{1+y} \right\|_{L^2} \leq 2 \|\partial_x^{k+1} (u - U)\|_{L^2} \leq C_{s,\gamma,\sigma,\delta} \{\|\omega\|_{H_g^{s,\gamma}} + \|\partial_x^s U\|_{L^2(\mathbb{T})}\},$$

which is inequality (6.79).

(iii) Inequality (6.80) follows directly from the definition of $\|\cdot\|_{H_g^{s,\gamma}}$ and inequality (6.65).

(iv) Using inequality (6.69), (6.78), (6.80), we have

$$\begin{aligned} \|(1+y)^\gamma g_k\|_{L^2} &\leq \|(1+y)^\gamma \partial_x^k \omega\|_{L^2} + \delta^{-2} \|(1+y)^{\gamma-1} \partial_x^k (u - U)\|_{L^2} \\ &\leq C_{s,\gamma,\sigma,\delta} \{\|\omega\|_{H_g^{s,\gamma}} + \|\partial_x^s U\|_{L^2(\mathbb{T})}\}, \end{aligned}$$

which is inequality (6.81) for $k = 1, 2, \dots, s-1$. When $k = s$, the better upper bound in (6.81) follows directly from the definition of $\|\omega\|_{H_g^{s,\gamma}}$. (v) For any $k = 1, 2, \dots, s-2$, applying Lemma (6.10.2), inequality (6.78), and (6.80), we have

$$\begin{aligned} \|\partial_x^k (u - U)\|_{L^\infty} &\leq C \{\|\partial_x^k (u - U)\|_{L^2} + \|\partial_x^{k+1} (u - U)\|_{L^2} + \|\partial_x^k \partial_y \omega\|_{L^2}\} \\ &\leq C_{s,\gamma,\sigma,\delta} \{\|\omega\|_{H_g^{s,\gamma}} + \|\partial_x^s U\|_{L^2(\mathbb{T})}\}, \end{aligned}$$

and hence, by triangle inequality and $\|\partial_x^k U\|_{L^\infty(\mathbb{T})} \leq C_s \|\partial_x^s U\|_{L^2(\mathbb{T})}$, we justify (6.82). For the case $k = 0$, let us first recall from the hypothesis that $\omega = \partial_y u > 0$, so

$$0 \leq u < U = \int_0^{+\infty} \omega dy, \quad (6.85)$$

Thus, using the Cauchy-Schwartz inequality and estimate (6.80), we have

$$\|U\|_{L^2(\mathbb{T})}^2 = \int_{\mathbb{T}} \left| \int_0^\infty \omega dy \right|^2 \leq \frac{1}{2\gamma-1} \|(1+y)^\gamma \omega\|_{L^2}^2 \leq \frac{1}{2\gamma-1} \|\omega\|_{H_g^{s,\gamma}}^2,$$

and hence, by (6.85), the Sobolev inequality, and $\|\partial_x U\|_{L^2(\mathbb{T})} \leq C_s \|\partial_x^s U\|_{L^2(\mathbb{T})}$, we obtain

$$\begin{aligned} \|u\|_{L^\infty} &\leq \|U\|_{L^\infty(\mathbb{T})} \leq C\{\|U\|_{L^2(\mathbb{T})} + \|\partial_x U\|_{L^2(\mathbb{T})}\} \\ &\leq C_{s,\gamma,\sigma,\delta}\{\|\omega\|_{H_g^{s,\gamma}} + \|\partial_x^s U\|_{L^2(\mathbb{T})}\}, \end{aligned}$$

which is inequality (6.82). (vi) Applying the triangle inequality, sobolev inequality, Lemma (6.10.2), $\partial_x u + \partial_y v = 0$, and $\omega = \partial_y u$, we have

$$\begin{aligned} \left\| \frac{\partial_x^k v}{1+y} \right\|_{L^\infty} &\leq \left\| \frac{y \partial_x^{k+1} U}{1+y} \right\|_{L^\infty} + \left\| \frac{\partial_x^k v + y \partial_x^{k+1} U}{1+y} \right\|_{L^\infty} \\ &\leq C \left(\|\partial_x^{k+1} U\|_{L^2(\mathbb{T})} + \|\partial_x^{k+2} U\|_{L^2(\mathbb{T})} + \left\| \frac{\partial_x^k v + y \partial_x^{k+1} U}{1+y} \right\|_{L^2} \right. \\ &\quad + \left\| \frac{\partial_x^{k+1} v + y \partial_x^{k+2} U}{1+y} \right\|_{L^2} + \left\| \frac{\partial_x^k v + y \partial_x^{k+1} U}{(1+y)^3} \right\|_{L^2} \\ &\quad \left. + \left\| \frac{\partial_x^{k+1}(u-U)}{(1+y)^2} \right\|_{L^2} + \left\| \frac{\partial_x^{k+1} \omega}{1+y} \right\|_{L^2} \right) \end{aligned}$$

which implies inequality (6.83) because of Wirtinger's inequality and (6.78)-(6.80).

(vii) Inequality (6.84) follows directly from Lemma (6.10.2) and inequality (6.80). \square

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On some properties of particular models in fluid mechanics near a surface

Abstract : The first part of the thesis is devoted to the study of the α -Navier-Stokes 2D flows in a very thin infinite strip, and the second part to the study of boundary layer problem for the Ostwald model of non-Newtonian fluids for which, the stress tensor does not linearly depend on the deformation tensor.

In the first part, we first proved the global well-posedness of the hydrostatic limit system for small analytic data and then the global well-posedness of the Navier-Stokes alpha system in very thin strips of width ε for small analytic data. Finally, we proved the convergence to the limit system when $\varepsilon \rightarrow 0$. The main idea for the proof of the previous results is the control of the unknown function $\tilde{u} = \exp(a - \lambda\theta(t)|D_x|)u$, which is equivalent to the control of the solution u in analytic spaces in the x variable. The analyticity radius of the solution is expected to decrease in time but the auxiliary function $a - \lambda\theta(t)$ provides a lower bound of the analytical bandwidth. This lower bound away from zero yields the global existence of analytical solutions.

For the second part, we study a non-Newtonian Prandtl system which possesses a nonlinear diffusion term. The difficulty here lies in the fact that the associate parabolic problem is degenerate: it lacks a second derivative in the Laplacian operator. We first add a small smoothing term and then study the singular limit of the obtain regularized solutions when the smoothing term tends to zero. We emphasize that the full 2D problem is still open in Sobolev setting and so in the first step, we consider the associate 1D shear flow system. Using a smoothing process, a fixed point argument and the maximum principle, we prove the existence of solutions for the model in the Hölder space $C^{1,2+\alpha}([0, T] \times \mathbb{R})$. We remark that this solution enjoy good decay properties when y tends to infinity, which enables the use of Masmoudi-Wong's method (Comm. Pure Appl. Math. 2015) to prove the well-posedness of the shear flow system in Sobolev settings. We remark that both the maximum principle and the Masmoudi-Wong's method do not work directly to the full 2D system due to the lack of good understanding of the nonlinear diffusion term. However, we still can prove *a priori* energy-type estimates for the full 2D system in appropriate weighted Sobolev spaces. Our forthcoming work will then focus on finding a way to exploit these estimates.

Sur des propriétés de certains modèles en mécanique des fluides près d'une surface

Résumé : La première partie de la thèse est consacrée à l'étude d'écoulements 2D α -Navier-Stokes dans une bande infinie très mince, et la deuxième partie à l'étude du problème de la couche limite pour le modèle d'Ostwald pour des fluides non newtoniens pour lesquels le tenseur des contraintes ne dépend pas linéairement du tenseur des déformations.

Dans la première partie, nous montrons d'abord que le problème de Cauchy pour le système de la limite hydrostatique est globalement bien posé pour de petites données analytiques. Puis de même pour le problème de Cauchy du système α -Navier-Stokes dans une bande très mince de largeur ε pour de petites données analytiques. Enfin, nous prouvons la convergence vers le système limite lorsque $\varepsilon \rightarrow 0$. L'idée principale de la preuve des résultats précédents passe par le contrôle de la fonction inconnue $\tilde{u} = \exp(a - \lambda\theta(t)|D_x|)u$, qui est équivalente au contrôle de la solution u dans les espaces analytiques dans la variable x . On s'attend à ce que le rayon d'analyticité de la solution diminue avec le temps mais la fonction auxiliaire $a - \lambda\theta(t)$ fournit une borne inférieure dans la bande passante analytique. Cette borne inférieure loin de zéro donne l'existence globale de solutions analytiques.

Dans un deuxième temps, nous étudions un système de Prandtl non newtonien qui possède un terme de diffusion non linéaire. La difficulté réside ici dans le fait que le problème parabolique associé est dégénéré : il manque une dérivée seconde dans l'opérateur laplacien. Nous ajoutons d'abord un petit terme de diffusion puis nous étudions la limite singulière des solutions régularisées obtenues lorsque le terme de diffusion tend vers zéro. Nous soulignons que le problème 2D complet est toujours ouvert dans le cadre des espaces de Sobolev et donc dans la première étape, nous considérons le système d'écoulement 1D associé ("shear flow"). En utilisant une technique de régularisation, un argument de point fixe et le principe du maximum, nous montrons l'existence de solutions dans l'espace de Hölder $C^{1,2+\alpha}([0, T] \times \mathbb{R})$. Nous remarquons que cette solution possède de bonnes propriétés de décroissance lorsque y tend vers l'infini, ce qui permet d'utiliser la méthode de Masmoudi-Wong (Comm. Pure Appl. Math. 2015) pour montrer que le problème dit de "shear flow" est bien posé dans les espaces de Sobolev. Nous remarquons que le principe du maximum et la méthode de Masmoudi-Wong ne fonctionnent pas directement sur le système 2D complet en raison du manque de bonne compréhension du terme de diffusion non linéaire. Cependant, nous pouvons toujours prouver des estimations de type d'énergie *a priori* pour le système 2D complet dans des espaces de Sobolev pondérés appropriés. Nos prochains travaux se concentreront ensuite sur la recherche d'un moyen d'exploiter ces estimations.