



**HAL**  
open science

# Two categorical studies : relative homological algebra of R-modules, fixpoint theorems for Q-categories

Arij Benkhadra

► **To cite this version:**

Arij Benkhadra. Two categorical studies : relative homological algebra of R-modules, fixpoint theorems for Q-categories. Rings and Algebras [math.RA]. Université du Littoral Côte d'Opale; Université Mohammed V (Rabat). Faculté des sciences, 2023. English. NNT : 2023DUNK0662 . tel-04199889

**HAL Id: tel-04199889**

**<https://theses.hal.science/tel-04199889>**

Submitted on 8 Sep 2023

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



## Thèse de Doctorat

*Mention : Mathématiques*  
présentée

à l'Ecole Doctorale en Sciences Technologie et Santé (ED 585) de  
**l'Université du Littoral-Côte d'Opale**

et

au Centre d'Etudes Doctorales Sciences et Technologies de  
**l'Université Mohammed V de Rabat**

par

**Arij Benkhadra**

pour obtenir le grade de Docteur  
de l'Université du Littoral-Côte d'Opale

*Deux études catégoriques :  
algèbre homologique relative pour  $R$ -modules,  
théorème du point fixe pour  $Q$ -categories*

Soutenue le 10/02/2023, après avis des rapporteurs, devant le jury d'examen :

A. Khelif, Maître de Conférences HDR, Sorbonne Université	Président de jury
H. Mouanis, Professeure, Université de Fès	Rapporteur
U. Höhle, Professeur Ret., Université de Wuppertal	Rapporteur
Z. E. A. Abdelali, Professeur, Université de Rabat	Examineur
Z. E. A. Guennoun, Professeur, Université de Rabat	Invité
D. Bennis, Professeur, Université de Rabat	Co-directeur de thèse
I. Stubbe, Maître de Conférences HDR, Université du Littoral	Co-directeur de thèse





Laboratoire de Mathématiques Pures et Appliquées

Centre Universitaire de la Mi-Voix

Maison de la Recherche Blaise Pascal

50, Rue Ferdinand Buisson

CS 80699

Calais Cedex, France.

**Fax:** (33) 03 21 46 55 86.

**Site:** [www.lmpa.univ-littoral.fr](http://www.lmpa.univ-littoral.fr)



Laboratoire Mathématiques, statistiques et applications

Centre d'Etudes Doctorales en Sciences et Technologies de Rabat

Faculté des Sciences Rabat

4 Avenue Ibn Batouta

B.P. 1014

Rabat, Maroc.

**Fax:** (212) 37 77 54 61.

**Site:** [www.fsr.ac.ma](http://www.fsr.ac.ma)



# Table des matières

<b>Remerciements</b>	<b>vii</b>
<b>Résumé</b>	<b>ix</b>
<b>Abstract</b>	<b>xi</b>
<b>Introduction aux articles sélectionnés</b>	<b>1</b>
1    Algèbre homologique relative des modules sur un anneau . . . . .	2
1.1    Cohérence relative . . . . .	4
1.2    Dimensions homologiques relatives . . . . .	5
2    Un théorème de point fixe pour les catégories enrichies sur un quantale	8
3    Références . . . . .	19
<b>Introduction to the selected articles</b>	<b>23</b>
1    Relative homological algebra for ring-modules . . . . .	24
1.1    Relative coherence . . . . .	25
1.2    Relative homological dimensions . . . . .	27
2    A fixpoint theorem for quantale-enriched categories . . . . .	29
3    References . . . . .	40
<b>Articles sélectionnés / Selected articles</b>	<b>43</b>
<b>Relative coherent modules</b>	<b>45</b>
by M. Amini, A. Benkhadra, Driss Bennis and Mohammed El Ha-	
joui, Asian-European Journal of Mathematics <b>14</b> (2021), 1–15.	

<b>Gorenstein <math>n</math>-<math>\mathcal{X}</math>-injective and <math>n</math>-<math>\mathcal{X}</math>-flat modules with respect to a special finitely presented module</b>	<b>61</b>
by M. Amini, A. Benkhadra and Driss Bennis, Algebra Colloquium <b>28</b> (2021), 673–688.	
<b>A logical analysis of fixpoint theorems</b>	<b>79</b>
by A. Benkhadra and I. Stubbe. This text is a longer version of the article with the same title and by the same authors that will appear in the Cahiers de Topologie et Géométrie Différentielle Catégoriques (2022).	

# Remerciements

Nous tenons à adresser nos remerciements à toutes les personnes qui ont contribué de près ou de loin à l'aboutissement de ce travail :

Pour commencer, j'aimerais adresser mes remerciements et ma reconnaissance à mes directeurs de thèse les professeurs Driss Bennis et Isar Stubbe pour leurs soutiens inconditionnel, patience, disponibilité et surtout leurs qualités humaines à côté de celles scientifiques. J'ai eu une grande chance d'être leur étudiante. L'autonomie qu'ils m'ont offert m'a permis d'apprendre beaucoup. Je remercie particulièrement le professeur A. Khélif d'avoir accepté d'être président du jury. Je suis très reconnaissant aux professeurs Zine El Abbidine Guenoun et Nadia Boudi qui m'ont supporté au cours de cette thèse et n'ont jamais refusé de me tenir la main quand j'en avais besoin. Je tiens aussi à remercier les professeurs U.Höle et M. Hakima qui ont accepté la tâche de rapporter cette thèse. Je remercie également le professeur Z.E.A Abdelali d'avoir accepté d'examiner ce travail. La présence de tous les membres du Jury à la soutenance de cette thèse m'a gracieusement honoré. Je tiens à remercier tous les membres du Laboratoire de Mathématiques Pures et Appliquées (LMPA, ULCO) et le Laboratoire d'Analyse Mathématique et Applications (LAMA, UM5). Je remercie le CNRST (Maroc) et l'ULCO pour m'avoir attribuée le financement qui m'a permis de travailler dans des bonnes conditions.

Je remercie tout particulièrement Manal, Maha, Karima, Fadwa et toutes les personnes de l'administration de la FSR et Isabelle Buchard pour le soutien, l'aide, la patience et l'encouragement qu'ils m'avaient octroyé dès mes premiers jours et surtout durant les périodes difficiles à L'Université Mohammed V et à Calais.

Ces années de thèse, ainsi que mes années d'études auraient été impossible et infiniment moins agréables et enrichissantes sans mes parents, Abdelkader et Oum El Mohassanat qui malgré la distance ont su me soutenir et me supporter en permanence.



Leurs amours inconditionnels, optimisme et la confiance qu'ils me témoignent m'ont portée tout au long de mes études. Mes parents m'ont mené là où je suis aujourd'hui. A vous mes frères et sœurs, du fond du cœur Merci, et que Dieu vous protège.

Un remerciement particulier à mon grand-frère et mon premier ami, pour m'avoir encouragé et épaulé durant toutes mes études et depuis toujours.

Et enfin, un grand merci à mes ami(e)s pour leur soutien, leur gentillesse et avec qui j'ai partagé de bons moments : Yasmina Bendidi, Ayoub Harrat, Ferial Bouhadjera, Fatima Bouyghf, Ibtissam Ben Amara, Zakaria Hmad.

À tous ces intervenants, et à tous ceux que j'aurais oublié de citer, je présente mes remerciements, mon respect et ma gratitude.

# Résumé

Cette thèse de doctorat est composée de trois articles de recherche : les deux premiers portent sur la théorie des modules sur un anneau  $R$ , le troisième porte sur les catégories enrichies dans un quantale  $Q$ .

Le premier article (en collaboration avec M. Amini, D. Bennis et M. El Hajoui) est une continuation du travail fait par Bennis [4], unifiant plusieurs notions d'anneaux cohérents. Nous introduisons les modules de type cohérent unifiant les concepts de cohérence relative existants.

Dans le deuxième article (en collaboration avec M. Amini et D. Bennis), nous généralisons l'article d'Enochs et Jenda [17] sur les  $R$ -modules injectifs et plats de Gorenstein. En considérant une classe  $\mathcal{X}$  de  $R$ -modules et un entier  $n \geq 1$ , nous introduisons, via des modules spéciaux finiment présentés, les concepts de module injectif et plat de  $n$ - $\mathcal{X}$ -Gorenstein, et nous obtenons quelques propriétés équivalentes de ces modules sur des anneaux  $n$ - $\mathcal{X}$ -cohérents. Nous étudions ensuite les relations entre les modules  $n$ - $\mathcal{X}$ -injectives de Gorenstein et les modules  $n$ - $\mathcal{X}$ -plats de Gorenstein sur les  $\mathcal{X}$ - $FC$ -anneaux (c'est-à-dire les anneaux auto- $n$ - $\mathcal{X}$ -injectifs et  $n$ - $\mathcal{X}$ -cohérents). Plusieurs résultats connus sont généralisés à ce nouveau contexte.

Le troisième article (en collaboration avec I. Stubbe) fournit une analyse logique de certains théorèmes de points fixes, en les replaçant dans le contexte des catégories  $Q$ -enrichies (largement étudiées par Stubbe [36]). Plus précisément, nous démontrons un théorème du point fixe pour des contractions sur des  $Q$ -catégories Cauchy-complètes, qui est valable pour tout quantale  $Q$  dont le treillis sous-jacent est continu, et qui s'applique à une notion spécifique de contraction. Des conditions suffisantes pour l'unicité du point fixe sont établies. Tout au long de l'étude, nous soulignons l'importance de la continuité du quantale sous-jacent. Les exemples comprennent des théorèmes de points fixes connus et nouveaux pour les espaces métriques, les es-

paces métriques flous et les espaces métriques probabilistes.

**Keywords :** Anneaux, Modules, Algèbre Homologique ; Quantales, Catégories Enrichies, Théorèmes du Point Fixe.

**AMS Subject Classification (2020) :** 06F07, 16D80, 16E10, 16E65, 16P70, 18D20, 54E35, 54H25.

# Abstract

This doctoral dissertation is composed of three research articles: the first two are on  $R$ -module theory (for  $R$  a ring), the third is on  $Q$ -enriched categories (for  $Q$  a quantale).

The first article (joint work with M. Amini, D. Bennis and M. El Hajoui) continues the work done by Bennis [4], who unified several coherent-like rings. We introduce coherent-like modules which also emphasizes our point of view by unifying the existed relative coherent concepts. Several classical results are generalized and some new results are given.

In the second article (joint work with M. Amini and D. Bennis), we generalize the work of Enochs and Jenda [17] on Gorenstein injective and Gorenstein flat  $R$ -modules. Letting  $\mathcal{X}$  be a class of  $R$ -modules and  $n \geq 1$  an integer, we introduce, via special finitely presented modules, the concepts of Gorenstein  $n$ - $\mathcal{X}$ -injective and  $n$ - $\mathcal{X}$ -flat modules, and we obtain some equivalent properties of these modules on  $n$ - $\mathcal{X}$ -coherent rings. We then investigate the relations among Gorenstein  $n$ - $\mathcal{X}$ -injective,  $n$ - $\mathcal{X}$ -flat, injective and flat modules on  $\mathcal{X}$ - $FC$ -rings (i.e., self  $n$ - $\mathcal{X}$ -injective and  $n$ - $\mathcal{X}$ -coherent rings). Several known results are generalized to this new context.

The third article (joint work with I. Stubbe) provides a logical analysis of some fixed point theorems, by putting these in the context of  $Q$ -enriched categories (extensively studied by Stubbe [36]). More precisely, we prove a fixpoint theorem for contractions on Cauchy-complete  $Q$ -enriched categories that holds for any quantale  $Q$  whose underlying lattice is continuous, and applies to a specific notion of contraction. Sufficient conditions for the uniqueness of the fixpoint are established. Throughout, we stress the importance of the continuity of the underlying quantale. Examples include known and new fixpoint theorems for metric spaces, fuzzy metric spaces, and probabilistic metric spaces.

**Keywords:** Rings, Modules, Homological Algebra; Quantales, Enriched Categories, Fixpoint Theorems.

**AMS Subject Classification (2020):** 06F07, 16D80, 16E10, 16E65, 16P70, 18D20, 54E35, 54H25.

# Introduction aux articles sélectionnés

Cette thèse est essentiellement constituée de trois articles retranscrits à partir de la page p. 43, qui retracent mes recherches doctorales effectuées entre janvier 2018 et novembre 2022, sous la direction des Prof. Driss Bennis (Université Mohammed V, Maroc) et Prof. Isar Stubbe (Université du Littoral, France). Comme il est courant pour les articles de recherche, leur présentation est plutôt laconique et technique : ils sont destinés à un lectorat de spécialistes. Néanmoins, je suis persuadé que les idées principales développées dans ces articles peuvent être comprises sans en donner tous les détails. Dans les sections qui suivent, j'essaierai de présenter en toute simplicité quelques principaux résultats obtenus au cours de mes études doctorales.

Comme mes études doctorales ont été supervisées par deux directeurs de thèse différents dans deux institutions différentes (dans le cadre d'une convention de cotutelle), les articles inclus abordent deux sujets différents. Les deux premiers articles (avec D. Bennis) ont pour sujets les anneaux et les modules ; le troisième article (avec I. Stubbe) porte sur les catégories enrichies sur des quantales. Cependant, ces articles font tous partie du domaine plus vaste de la théorie générale des catégories : en effet, nous intéressons à la *catégorie* des  $R$ -modules (pour un certain anneau  $R$ ) et à la *catégorie* des catégories  $Q$ -enrichies (pour un certain quantale  $Q$ ). De plus, un anneau  $R$  est lui-même un monoïde dans la catégorie monoïdale  $\text{Ab}$  des groupes abéliens, alors qu'un quantale  $Q$  est un monoïde dans la catégorie monoïdale  $\text{Sup}$  des treillis sup-complets. Donc, la théorie des catégories est le thème central de cette thèse de doctorat.

Dans la première section ci-dessous, je discuterai très brièvement les articles que

l'on peut trouver sur p. 45 et p. 61 : leur sujet est l'étude de l'algèbre homologique relative dans la catégorie des  $R$ -modules. Dans la section 2, je discuterai plus en détail l'article, présenté à partir de p. 79, sur un théorème de points fixes pour les catégories enrichies dans  $\mathcal{Q}$ . Une dernière section rassemble toutes les références utilisées dans cette introduction.

## 1 Algèbre homologique relative des modules sur un anneau

Dans tout ce qui suit,  $R$  désignera un anneau associatif unitaire (non nécessairement commutatif) et tous les modules seront considérés des  $R$ -modules unitaires à gauche (sauf indication contraire).

Le but de l'introduction des modules sur un anneau est de généraliser le concept d'espaces vectoriels en affaiblissant le champ des scalaires pour devenir qu'un anneau  $R$  (toujours supposé unitaire). Au début, l'étude des  $R$ -modules s'est concentrée sur la généralisation, autant que possible, des propriétés des espaces vectoriels. Mais contrairement aux espaces vectoriels, les modules sur un anneau sont avérés plus compliqués : tous les modules n'ont pas une base ; même ceux qui l'ont, c'est-à-dire les modules libres, n'ont pas nécessairement un rang unique ; les sous-modules de modules de type fini ne sont pas nécessairement de type fini.

En 1960, Chase [9] a introduit et étudié les anneaux et modules cohérents. Le terme "cohérent" a été utilisé pour la première fois, en 1953, par Serre [33] pour étudier les faisceaux cohérents. Pour être précis, on dit qu'un anneau est cohérent si chaque idéal du type fini est de présentation finie. De nombreux résultats concernant les anneaux cohérents ont été influencés par l'un de ses plus importants exemples : Les anneaux noethériens (c'est-à-dire les anneaux dans lesquels *chaque* idéal est du type fini).

Rappelons quelques résultats importants concernant les (modules sur) anneaux cohérents :

1. Chase [9] : un anneau  $R$  est cohérent à gauche si et seulement si tout produit direct de  $R$ -modules (à droite) plats est aussi plat.
2. Cheatham and Stone [10] : les assertions suivantes sont équivalentes :
  - (a)  $R$  est cohérent à gauche,

- (b) un  $R$ -module  $M$  est injectif si et seulement si  $M^* := \text{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$  est plat,
  - (c) un  $R$ -module  $M$  est injectif si et seulement si  $M^{**}$  est injectif,
  - (d) un  $R$ -module  $M$  est plat si et seulement si  $M^{**}$  est plat.
3. Stenström [34] : un anneau  $R$  est cohérent à gauche si et seulement si toute limite directe de modules FP-injectif<sup>1</sup> est aussi FP-injectif.
  4. Enochs [16] : un anneau  $R$  est cohérent à gauche si et seulement si tout  $R$ -module à gauche admet une pré-enveloppe plat<sup>2</sup>.

Au cours des dernières années, différents types d’anneaux cohérents ont été introduits : Ding et al. [13] ont défini un anneau  $J$ -cohérent comme étant un anneau sur lequel tous les idéaux de type fini du radical de Jacobson sont de présentation finie ; Ouyng et al. [31] ont introduit de manière similaire les anneaux nil-cohérents, en remplaçant le radical de Jacobson par le nil-radical ; Mao [26] a défini les anneaux min-cohérents en exigeant que seuls les idéaux simples de type fini soient de présentations finies ; Camillo [7] a défini les anneaux  $\Pi$ -cohérents comme étant ceux pour lesquels chaque  $R$ -modules sans torsion de type fini est de présentation finie. Dans cet esprit, un anneau cohérent est exactement un anneau  $\text{Idl}$ -cohérent.

Une autre extension de la notion d’anneau cohérent a été introduite par Costa dans [11], et développé e.g. par Dobbs et al. [14]. Rappelons d’abord qu’un  $R$ -module  $M$  est dit  $n$ -présenté (pour  $n \geq 0$ ) s’il existe une suite exacte de  $R$ -modules

$$F_n \rightarrow F_{n-1} \rightarrow \cdots \rightarrow F_1 \rightarrow F_0 \rightarrow M \rightarrow 0$$

où chaque  $R_i$  est un module libre de type fini. (Donc “0-présenté” veut dire finiment engendré, et “1-présenté” veut dire finiment présenté.) Cette définition s’applique aux idéaux dans un anneau  $R$ , puisqu’ils sont exactement les sous-modules de  $R$  en tant que  $R$ -module. Maintenant on peut définir : un anneau  $R$  est  $n$ -cohérent (pour  $n \geq 1$ ) si chaque idéal  $(n - 1)$ -présenté est  $n$ -présenté.

---

1. un  $R$ -module est FP-injective  $M$ , aussi dit absolument pure, si  $\text{Ext}^1(N, M) = 0$ , pour tout  $R$ -module  $N$  de présentation finie.

2. Un  $R$ -module  $F$  contenu dans une classe de  $R$ -modules  $\mathcal{X}$  est dit être une  $\mathcal{X}$ -pré-enveloppe d’un  $R$ -module  $M$ , s’il existe un homomorphisme  $\varphi: M \rightarrow F$  tel que, pour tout homomorphisme  $\varphi': M \rightarrow F'$  avec  $F' \in \mathcal{X}$ , il existe un homomorphisme  $f: F \rightarrow F'$  tel que  $\varphi' = f\varphi$ . En considérant  $\mathcal{X}$  la classe des  $R$ -modules plats, on retrouve la notion de pré-enveloppe plat de  $M$ .



En 2010, Bennis [4] a unifié les généralisations mentionnées ci-dessus, en introduisant ce qu'on appelle les *anneaux  $n$ - $\mathcal{X}$ -cohérents*, où  $\mathcal{X}$  est une classe d'idéaux et  $n$  un entier non négatif.

## 1.1 Cohérence relative

Dans notre article, présenté à partir de p. 45, nous définissons et étudions les  *$R$ -modules  $n$ - $\mathcal{X}$ -cohérents*. Explicitement, soit  $M$  un  $R$ -module et  $\mathcal{X}$  une classe de sous-modules de  $M$ . Nous désignons par  $\mathcal{X}_n$  la classe des modules  $n$ -présentés de  $\mathcal{X}$ . Nous disons que  $M$  est  $n$ - $\mathcal{X}$ -cohérent (pour  $n \geq 1$ ) si  $\mathcal{X}_{n-1}$  est non-vide et  $\mathcal{X}_{n-1} \subseteq \mathcal{X}_n$ .

**Exemple 1.1** Soient  $R$  un anneau et  $M$  un  $R$ -module.

- Pour  $\mathcal{X}$  la classe des idéaux (à gauche) de  $R$ , l'anneau  $R$  est  $n$ - $\mathcal{X}$ -cohérent si et seulement si il est  $n$ -cohérent (à gauche) au sens de [14].
- Pour  $\mathcal{X}$  la classe des tous les sous-modules de  $M$ , alors le module  $M$  est 1- $\mathcal{X}$ -cohérent si et seulement s'il est pseudo-cohérent au sens de [22].
- Si  $\mathcal{X}$  est la classe des sous-modules de  $Nil(R)M$ , alors le module  $M$  1- $\mathcal{X}$ -cohérent si et seulement s'il est  $Nil_*$ -cohérent au sens de [1].
- Si  $\mathcal{X}$  est le radical de Jacobson de  $R$ , alors  $R$  est 1- $\mathcal{X}$ -cohérents si et seulement s'il est  $J$ -cohérent au sens de [13].

Notre objectif principal est de montrer comment certains des résultats connus (donnée ci-dessus) sur les anneaux et les modules cohérents restent valables dans le contexte des modules  $n$ - $\mathcal{X}$ -cohérent. On a montré, en particulier, la généralisation du théorème de Chase [9] (voir aussi [1]) :

**Theorem 1.2** Soient  $M$  un  $R$ -module et  $\mathcal{X}$  une classe de sous-module de  $M$  telle que  $\mathcal{X}_n$  est stable par sommes finies, et par sous-modules cycliques. Alors,  $M$  est 1- $\mathcal{X}$ -cohérent à gauche si et seulement si les deux conditions suivantes sont vérifiées :

1. Pour tout  $a \in M$  tel que  $Ra$  est dans  $\mathcal{X}_0$ ,  $(0 : Ra)$  est un sous-module de type fini de  $M$ ,
2. l'intersection de deux sous-modules quelconques de  $M$  dans  $\mathcal{X}_0$  est de type fini.

Nous avons également établi des conditions nécessaires et suffisantes pour avoir la  $n$ - $\mathcal{X}$ -cohérence d'un pullback dans la catégorie des anneaux<sup>3</sup> :

**Theorem 1.3** *Supposons que  $R$  soit le pullback de  $R_1$  et  $R_2$  dans la catégorie des anneaux commutatifs unitaires. Soit  $\mathcal{X}$  une classe d'idéaux de  $R$  telle que pour chaque  $I \in \mathcal{X}_1$  il existe une suite exacte  $0 \rightarrow K \rightarrow R^k \rightarrow I \rightarrow 0$  avec  $K \in \mathcal{X}$ . Soit  $\mathcal{Y}^i$  une classe d'idéaux de  $R_i$  pour  $i = 1, 2$ . Supposons que pour chaque  $N \in \mathcal{X}_n$ ,  $\text{Tor}_j^R(R_i, N)$  est dans  $\mathcal{Y}_{n+1-j}^i$  pour  $1 \leq j \leq n+1$  et  $i = 1, 2$ . Supposons que pour tout  $Y_i \in \mathcal{Y}_n^i$ , il existe  $X_i \in \mathcal{X}_n$  tel que  $R_i \otimes_R X_i \simeq Y_i$ . Alors,  $R$  est  $n$ - $\mathcal{X}$ -cohérent si chaque  $R_i$  est  $n$ - $\mathcal{Y}^i$ -cohérent.*

## 1.2 Dimensions homologiques relatives

Notre deuxième article, qui se trouve à la page 61, se concentre sur l'étude des dimensions relatives des  $R$ -modules en la liant à la notion de cohérence relative.

En 1956, Eilenberg et Cartan [8] ont introduit l'algèbre homologique (en particulier les foncteurs Hom et Ext) pour classer et étudier les anneaux et les modules via leurs invariants et dimensions homologiques. La dite dimension de Gorenstein d'un  $R$ -module est une application importante de cette théorie en géométrie algébrique (voir par exemple [24]). En 1955, Enochs et Jenda [17] ont introduit et étudié les modules de Gorenstein, menant à l'*algèbre homologique relative* : une théorie généralisant les concepts de l'algèbre homologique standard en utilisant une classe spéciale de modules comme paramètre.

Plus précisément, soit  $\mathcal{X}$  une classe de  $R$ -modules à droite, et on suppose, pour tout  $k \geq 0$ , la sous-classe  $\mathcal{X}_k$  des modules  $k$ -présentés de  $\mathcal{X}$  est non-vide. On dit qu'un  $R$ -module à droite  $M$  est  $n$ - $\mathcal{X}$ -injectif si  $\text{Ext}_R^n(U, M) = 0$  pour tout  $U \in \mathcal{X}_n$ . Un  $R$ -module à gauche  $N$  est dit  $n$ - $\mathcal{X}$ -plat si  $\text{Tor}_R^n(N, U) = 0$  pour tout  $U \in \mathcal{X}_n$ . Nous utilisons les notations  $\mathcal{X}\mathcal{I}_n$  (resp.  $\mathcal{X}\mathcal{F}_n$ ) pour désigner la classe de tous les  $R$ -modules  $n$ - $\mathcal{X}$ -injectifs (resp.  $n$ - $\mathcal{X}$ -plat à droite).

On dit qu'un anneau  $R$  est  $n$ - $\mathcal{X}$ -cohérent à gauche si pour tout  $R$ -module à gauche  $n$ -présenté de  $\mathcal{X}$  est  $(n+1)$ -présenté. Cette définition (étudiée en détail dans [4])

---

3. Dans l'article, l'énoncé de ce théorème contient quelques ambiguïtés. La classe  $\mathcal{X}$  a été considérée comme une classe de modules  $R$ , ce qui n'est pas correct. La même remarque vaut pour la classe  $\mathcal{Y}^i$ . Dans cette introduction, nous donnons l'énoncé correct du théorème.

reproduit celle donnée dans notre premier article – discutée dans la sous-section précédente – lorsque l’on restreint  $\mathcal{X}$  à une classe d’idéaux à gauche de  $R$ . Si, cependant, nous considérons que  $\mathcal{X}$  est la classe de tous les  $R$ -modules à gauche, alors cela coïncide avec la notion d’“anneau cohérent fort” de [14], et est un “anneau cohérent” de [11]. Nous avons démontré ce qui suit :

**Theorem 1.4** *Soit  $R$  un anneau  $n$ - $\mathcal{X}$ -cohérent à gauche pour une certaine classe  $\mathcal{X}$  de  $\mathbb{R}$ -module. Les assertions suivantes sont équivalentes :*

1.  $R$  est auto  $n$ - $\mathcal{X}$ -injectif, i.e.  $n$ - $\mathcal{X}$ -injectif autant qu’un module sur lui-même,
2. tout  $R$ -module admet une  $\mathcal{X}$   $\mathcal{F}_n$ -couverture<sup>4</sup> épimorphique,
3. tout  $R$ -module à droite admet une  $\mathcal{X}$   $\mathcal{F}_n$  pré-enveloppe monomorphique,
4. tout  $R$ -module à droite injectif est  $n$ - $\mathcal{X}$ -plat,
5. tout  $R$ -module 1- $\mathcal{X}$ -injectif à droite est  $n$ - $\mathcal{X}$ -plat,
6. tout  $R$ -module  $n$ - $\mathcal{X}$ -injectif à droite est  $n$ - $\mathcal{X}$ -plat,
7. tout  $R$ -module plat est  $n$ - $\mathcal{X}$ -injectif.

Pour énoncer notre prochain résultat, nous devons introduire quelques notions. En utilisant les précédentes notations. Pour  $U \in \mathcal{X}_n \subseteq \mathcal{X}$ , il existe –par définition– une suite exacte

$$F_n \longrightarrow F_{n-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow U \longrightarrow 0,$$

où chaque  $F_i$  est un  $R$ -module libre de type fini. Soit  $K_{n-1} = \text{Im}(F_{n-1} \rightarrow F_{n-2})$  et  $K_n = \text{Im}(F_n \rightarrow F_{n-1})$ . La suite exacte courte  $0 \rightarrow K_n \rightarrow F_n \rightarrow K_{n-1} \rightarrow 0$  est appelée une suite exacte spéciale de  $U$ . Il est clair que  $K_n$  et  $K_{n-1}$  sont respectivement de type fini et aussi de présentation finie. Nous appelons  $K_n$  un  $R$ -module spécialement généré par  $\mathcal{X}$  (ou  $\mathcal{X}$ -généralisé), et  $K_{n-1}$  est un  $R$ -module de spécialement présenté par  $\mathcal{X}$  (ou  $\mathcal{X}$ -présenté).

Par analogie avec la définition des dimensions de Gorenstein (voir [17]), nous définissons un  $R$ -module Gorenstein  $n$ - $\mathcal{X}$ -injectif s’il existe une suite exacte de  $R$ -modules  $n$ - $\mathcal{X}$ -injectifs

$$\mathbf{A} = \cdots \longrightarrow A_1 \longrightarrow A_0 \longrightarrow A^0 \longrightarrow A^1 \longrightarrow \cdots$$

---

4. C’est la notion duale de la notion de pré-enveloppe

où  $G = \ker(A^0 \rightarrow A^1)$ , tel que  $\text{Hom}_R(K_{n-1}, \mathbf{A})$  est une suite exacte chaque fois que  $K_{n-1}$  est spécialement  $\mathcal{X}$ -présenté avec  $\text{pd}_R(K_{n-1}) < \infty$ <sup>5</sup>. De même, un  $R$ -module à droite est dit Gorenstein  $n$ - $\mathcal{X}$ -plat s'il existe une suite exacte de  $R$ -modules (à droite)  $n$ - $\mathcal{X}$ -plats

$$\mathbf{F} = \cdots \rightarrow F_1 \rightarrow F_0 \rightarrow F^0 \rightarrow F^1 \rightarrow \cdots$$

avec  $G = \ker(F^0 \rightarrow F^1)$ , tel que  $\mathbf{F} \otimes_R K_{n-1}$  est une suite exacte à chaque fois que  $K_{n-1}$  est spécialement  $\mathcal{X}$ -présenté avec  $\text{fd}_R(K_{n-1}) < \infty$ <sup>6</sup>.

Nous présentons notre résultat principal caractérisant les anneaux auto-injectifs relatifs via les modules injectifs et projectifs de Gorenstein :

**Theorem 1.5** *Soit  $R$  un anneau  $n$ - $\mathcal{X}$ -cohérent à gauche pour une certaine classe  $\mathcal{X}$  de  $R$ -modules. Les affirmations suivantes sont équivalentes :*

1.  *$R$  est auto  $n$ - $\mathcal{X}$ -injectif à gauche,*
2. *tout  $R$ -module Gorenstein  $n$ - $\mathcal{X}$ -plat est Gorenstein  $n$ - $\mathcal{X}$ -injectif,*
3. *tout  $R$ -module Gorenstein plat est Gorenstein  $n$ - $\mathcal{X}$ -injectif,*
4. *tout  $R$ -module plat est Gorenstein  $n$ - $\mathcal{X}$ -injectif,*
5. *tout  $R$ -module Gorenstein projectif est Gorenstein  $n$ - $\mathcal{X}$ -injectif,*
6. *tout  $R$ -module projectif est Gorenstein  $n$ - $\mathcal{X}$ -injectif,*
7. *tout  $R$ -module Gorenstein injectif à droite est Gorenstein  $n$ - $\mathcal{X}$ -plat,*
8. *tout  $R$ -module injectif à droite est Gorenstein  $n$ - $\mathcal{X}$ -plat,*
9. *tout  $R$ -module Gorenstein 1- $\mathcal{X}$ -injectif est Gorenstein  $n$ - $\mathcal{X}$ -plat,*
10. *tout  $R$ -module Gorenstein  $n$ - $\mathcal{X}$ -injectif est Gorenstein  $n$ - $\mathcal{X}$ -plat.*

Les résultats de ce papier unifient des notions homologiques telles que : Gorenstein relative rings,  $FP$ -rings, quasi-frobenuis rings (voir [17], [16], [28] et [29]).

Dans les démonstrations des caractérisations homologiques de ces anneaux, les outils sont toujours les mêmes, l'élément clé étant les propriétés de stabilité d'une classe bien choisie de modules (stable par extension, par somme directe, par composante directe...). Nous avons cherché à donner une classe la plus générale possible

---

5. Ceci veut dire que  $\text{Ext}^i(M, N) = 0$  pour tout  $R$ -module  $N$  et tout  $i \geq n + 1$ .

6. Ceci veut dire que  $\text{Tor}_i(M, N) = 0$  pour tout  $R$ -module à droite  $N$  et tout  $i \geq n + 1$ .

de module, avec le minimum de conditions, pour supporter l'essentiel de la structure homologique qu'on tente de généraliser.

**Exemple 1.6** Soient  $R = k[X]$ , où  $k$  est un corps et  $\mathcal{X}$  une classe de  $R$ -modules gradué. Il s'avère que tout  $R$ -module gradué à gauche (resp. à droite) est Gorenstein  $2$ - $\mathcal{X}$ -injectif (resp.  $2$ - $\mathcal{X}$ -plat). Néanmoins, il existe des  $R$ -modules gradué à gauche (resp. à droite) qui ne sont pas Gorenstein  $1$ - $\mathcal{X}$ -injectif (resp.  $1$ - $\mathcal{X}$ -plat). Ceci montre que, pour  $m > n$ , Gorenstein  $m$ - $\mathcal{X}$ -injectif (resp.  $m$ - $\mathcal{X}$ -plat) n'implique pas, en général, Gorenstein  $n$ - $\mathcal{X}$ -injectif (resp.  $n$ - $\mathcal{X}$ -plat).

## 2 Un théorème de point fixe pour les catégories enrichies sur un quantale

C'est une idée très répandue en mathématiques d'étudier les points fixes d'un endomorphisme donné : pour  $f: X \rightarrow X$  trouver  $x \in X$  tel que  $f(x) = x$ . Le premier théorème de point fixe remonte à Brouwer [6] en 1912, qui a montré que les applications continues sur la boule unité fermée dans  $\mathbb{R}^n$  ont un point fixe ; ce théorème a été généralisé aux espaces de dimension infinie par Schauder [32] en 1930. Les preuves utilisaient des outils topologiques très puissants, mais le résultat ne fournissait que l'existence du point invariant.

En 1922, Banach [2] a montré le désormais célèbre théorème du point fixe de Banach :

**Theorem 2.1 (Banach, 1922)** *Soient  $(X, d)$  un espace métrique complet non-vidé et  $f: X \rightarrow X$  une contraction, c'est-à-dire  $d(fx, fy) \leq k \cdot d(x, y)$  pour un certain  $0 < k < 1$ . Alors  $f$  a un point fixe unique.*

La preuve, étonnamment simple, est la suivante : prenez n'importe quel  $x$  dans  $X$ , itérez  $f$  pour obtenir une suite de Cauchy  $x, f(x); f(f(x)), \dots$ , et montrez qu'elle converge vers un point fixe unique de  $f$ .

De nombreux auteurs ont généralisé le théorème de Banach à d'autres espaces, en affaiblissant les conditions tout en conservant l'essentiel de la preuve : la convergence de l'itération dite de Picard  $x, f(x); f(f(x)), \dots$ . Mais une de ses généralisations se

distingue du fait qu'elle change la notion de contraction : elle est due à Boyd et Wong [5] :

**Theorem 2.2 (Boyd and Wong, 1969)** Soient  $(X, d)$  un espace métrique complet non-vide et  $\varphi: \overline{\{d(x, y) \mid x, y \text{ dans } X\}} \rightarrow \mathbb{R}^+$  une fonction semi-continue supérieurement telle que  $\varphi(0) = 0$  et  $\varphi(t) < t$  pour tout  $t \neq 0$ . Alors toute application satisfaisant  $d(fx, fy) \leq \varphi(d(x, y))$  pour tout  $x, y \in X$  a un unique point fixe.

Dans notre papier qui peut être trouvé à la page 79, nous faisons une *analyse logique* des théorèmes ci-dessus : en présentant une généralisation appropriée aux catégories enrichies dans un quantale, comme nous l'expliquerons ci-dessous.

## Des espaces métriques aux catégories enrichies

Dans son célèbre article, Lawvere [25] a observé que les espaces métriques sont particulièrement des catégories enrichies. En effet, en notant  $[0, \infty] = \mathbb{R}^+ \cup \{+\infty\}$  l'espace des nombres réels positifs étendus, on peut voir un espace métrique  $(X, d)$  comme une application

$$d: X \times X \rightarrow [0, \infty]$$

qui, pour chaque couple de points  $(x, y)$ , calcule la distance  $d(x, y)$  qui les sépare ; cette fonction satisfait certainement aux deux assertions suivantes :

$$d(x, y) + d(y, z) \geq d(x, z) \quad \text{et} \quad 0 \geq d(x, x)$$

pour tout  $x, y, z \in X$ . Lawvere appelle cette structure un *espace métrique généralisé* ; on obtient la notion habituelle d'espace métrique en imposant la symétrie ( $d(x, y) = d(y, x)$ ), la finitude ( $d(x, y) \neq \infty$ ) et la séparation ( $d(x, y) = 0$  implique  $x = y$ ).

Remarquons maintenant que  $([0, \infty], \wedge)$  est un treillis complet (que l'on prend avec l'opposé de l'ordre naturel), que  $([0, \infty], +, 0)$  est un monoïde, et qu'il existe une compatibilité particulière entre la structure du treillis et celle du monoïde :

$$a + \left(\bigwedge_i b_i\right) = \bigwedge_i (a + b_i) \quad \text{et} \quad \left(\bigwedge_i b_i\right) + a = \bigwedge_i (b_i + a)$$

pour tout  $a, (b_i)_{i \in I}$  dans  $[0, \infty]$ .

En procédant à l'abstraction de cette structure algébrique, nous pouvons maintenant remplacer les nombres réels positifs par tout autre *quantale*  $Q$ , c'est-à-dire un

treillis complet  $(Q, \vee)$  doté d'un monoïde  $(Q, \circ, 1)$  tel que le produit se distribue sur les suprema arbitraires. Les quantales ont été définis pour la première fois par Mulvey [30] pour servir de contrepartie non-commutative aux locales, cette dernière notion remontant aux travaux d'Ehresmann [15].

La définition ci-dessus de l'espace métrique (généralisé) devient alors ce qu'on appelle maintenant une *Q-catégorie*  $\mathbb{C}$  : Cela est constitué d'un ensemble  $\mathbb{C}_0$  et d'une fonction

$$\mathbb{C} : \mathbb{C}_0 \times \mathbb{C}_0 \rightarrow Q : (x, y) \mapsto \mathbb{C}(x, y)$$

qui calcule, pour chaque couple  $(x, y)$  d'"objets", l'"hom"  $\mathbb{C}(x, y)$  entre eux ; et doit satisfaire aux assertions suivantes :

$$\mathbb{C}(x, y) \circ \mathbb{C}(y, z) \leq \mathbb{C}(x, z) \quad \text{and} \quad 1 \leq \mathbb{C}(x, x)$$

pour tout  $x, y, z \in \mathbb{C}_0$ . On peut définir les *Q-catégories* symétriques, finitaires et séparées de manière évidente.

En fait, de nombreuses autres structures mathématiques familières peuvent être considérées comme des catégories *Q* : les ensembles ordonnés sont des catégories enrichies dans l'algèbre de Boole à deux éléments  $(\{0, 1\}, \vee, \wedge, 1)$  ; les ordres dits flous sont des catégories enrichies dans une norme *t* continue à gauche, c'est-à-dire un quantale (commutative et intégral) sur l'intervalle  $[0, 1]$  (voir par exemple [12, 37]) ; et même les espaces métriques probabilistes s'avèrent être des exemples de catégories enrichies dans un quantale (voir par exemple [23]). Nous reviendrons sur ces exemples à la fin de cette section.

Lawvere a fortement insisté sur le fait que de nombreux résultats concernant les espaces métriques doivent être considérés comme des cas particuliers de la théorie générale (enrichie) des catégories. Il a donné l'exemple des suites de Cauchy et la complétude (c'est-à-dire la convergence des suites de Cauchy), que nous allons maintenant expliquer brièvement.

### **Pré-faisceaux et suites de Cauchy**

Un *pré-faisceau* (contravariant)  $\phi$  sur une *Q-catégorie*  $\mathbb{C}$  est une application

$$\phi : \mathbb{C}_0 \rightarrow Q : x \mapsto \phi(x)$$

qui satisfait  $\mathbb{C}(y, x) \circ \phi(x) \leq \phi(y)$ , pour tout  $x, y \in \mathbb{C}_0$ . Tout élément  $c \in \mathbb{C}_0$  détermine un pré-faisceau particulier :

$$\mathbb{C}(-, c): \mathbb{C}_0 \rightarrow Q: x \rightarrow \mathbb{C}(x, c).$$

On dit qu'un pré-faisceau donné  $\phi: \mathbb{C} \rightarrow Q$  est *représentable* par  $c \in \mathbb{C}_0$  si  $\phi(x) = \mathbb{C}(x, c)$  pour tout  $x \in \mathbb{C}_0$ ; il est facile de montrer qu'un tel  $c \in \mathbb{C}_0$  est alors (essentielllement) unique.

Il s'avère ainsi (voir également [23]) que toute suite  $\underline{x} = (x_n)_{n \in \mathbb{N}}$  d'éléments dans  $\mathbb{C}_0$  détermine un pré-faisceau sur  $\mathbb{C}$ , à savoir :

$$\phi_{\underline{x}}: \mathbb{C}_0 \rightarrow Q: y \mapsto \bigvee_{N \in \mathbb{N}} \bigwedge_{n \geq N} \mathbb{C}(y, x_n).$$

La représentativité du pré-faisceau  $\phi_{\underline{x}}$  par un certain  $c \in \mathbb{C}_0$  signifie exactement que

$$\bigvee_{N \in \mathbb{N}} \bigwedge_{n \geq N} \mathbb{C}(y, x_n) = \mathbb{C}(y, c) \quad \text{for all } y \in \mathbb{C}_0.$$

De plus, on dit qu'une telle suite est de *Cauchy* si :

$$1 \leq \bigvee_{N \in \mathbb{N}} \bigwedge_{m \geq N} \bigwedge_{n \geq N} \mathbb{C}(x_m, x_n).$$

On voit facilement que, si  $\phi_{\underline{x}}$  est représentable, alors il doit être de Cauchy; mais l'inverse n'est pas en vrai en général.

Appliqué au quantale spécifique  $Q = ([0, \infty], \wedge, +, 0)$  des nombres réels positifs étendus (notez encore que nous inversons l'ordre, donc les suprema deviennent des infimums, et vice-versa), en prenant une suite  $\underline{x} = (x_n)_{n \in \mathbb{N}}$  dans un espace métrique  $(X, d)$ , les définitions habituelles d'une suite qui converge vers un point  $c$ ,

$$\lim_{n \rightarrow +\infty} d(y, x_n) = d(y, c) \quad \text{pour tout } y \in X,$$

et d'une suite de Cauchy,

$$0 \geq \lim_{m, n \rightarrow +\infty} d(x_n, x_m).$$

Bien évidemment, pour les espaces métriques, nous savons très bien que toute suite convergente est de Cauchy, mais que la réciproque n'est pas nécessairement vraie; et nous savons également que la limite d'une suite est unique. Nous voyons donc maintenant comment il s'agit d'une instance particulière des  $Q$ -catégories.



## Distributeurs et complétude de Cauchy

On dit qu'un espace métrique  $(X, d)$  est *complet* si toute suite de Cauchy converge. En général, pour une  $Q$ -catégorie, la situation est un peu complexe. Pour faire valoir notre point de vue, nous devons rappeler la notion de *distributeur*, introduite par Bénabou.

Un  $Q$ -distributeur  $\Phi: \mathbb{A} \dashv\vdash \mathbb{B}$  entre deux  $Q$ -catégories est une fonction

$$\Phi: \mathbb{B}_0 \times \mathbb{A}_0 \rightarrow Q: (y, x) \mapsto \Phi(y, x)$$

telle que

$$\mathbb{B}(y', y) \circ \Phi(y, x) \circ \Phi(x, x') \leq \Phi(y', x')$$

pour tout  $x, x' \in \mathbb{A}_0$  et  $y, y' \in \mathbb{B}_0$ . Deux tels distributeurs, disons  $\Phi: \mathbb{A} \dashv\vdash \mathbb{B}$  et  $\Psi: \mathbb{B} \dashv\vdash \mathbb{C}$ , se composent via un "produit matriciel" :

$$(\Psi \circ \Phi): \mathbb{C}_0 \times \mathbb{A}_0 \mapsto Q: (z, x) \mapsto \bigvee_{y \in \mathbb{B}_0} \Psi(z, y) \circ \Phi(y, x).$$

Il n'est pas difficile de vérifier qu'avec cette loi de composition, les  $Q$ -catégories et les  $Q$ -distributeurs sont les objets et les morphismes d'une catégorie, que nous notons par  $\text{Dist}(Q)$ . Mieux encore, cette catégorie est localement ordonnée : pour deux distributeurs parallèles, disons  $\Phi: \mathbb{A} \dashv\vdash \mathbb{B}$  et  $\Phi': \mathbb{A} \dashv\vdash \mathbb{B}$ , nous définissons l'ordre sur  $\text{Dist}(Q)$ , simplement par  $\Phi \leq \Phi'$  chaque fois que cette inégalité tient élément par élément (en considérant  $\Phi$  et  $\Phi'$  comme des fonctions dans le treillis complet  $Q$ ). Comme l'ordre local des  $Q$ -distributeurs est compatible avec leurs compositions, nous aurons que  $\text{Dist}(Q)$  est une 2-catégorie.

En appliquant l'algèbre générale des 2-catégories au cas particulier  $\text{Dist}(Q)$ , nous pouvons maintenant dire que deux distributeurs de  $Q$   $\Phi: \mathbb{A} \dashv\vdash \mathbb{B}$  et  $\Psi: \mathbb{B} \dashv\vdash \mathbb{A}$  sont *adjoint à (gauche/droite)*, noté  $\Phi \dashv \Psi$ , si

$$\mathbb{A} \leq \Psi \circ \Phi \quad \text{et} \quad \Phi \circ \Psi \leq \mathbb{B}.$$

Il est maintenant acquis que les *pré-faisceaux* introduits précédemment, sont des distributeurs particuliers. En effet, en écrivant  $\mathbb{1}$  pour la catégorie  $Q$  définie par  $\mathbb{1}_0 = \{*\}$  et  $\mathbb{1}(*, *) = 1$ , il existe une correspondance exacte entre les pré-faisceaux (contravariants) sur une  $Q$ -catégorie  $\mathbb{C}$  d'une part, et les  $Q$ -distributeurs  $\mathbb{1} \dashv\vdash \mathbb{C}$

d'autre part. En d'autres termes, les pré-faisceaux peuvent être considérés comme des *morphismes particuliers* dans la 2-catégories  $\text{Dist}(Q)$ .

En partant de cette perspective, nous adoptons la notion de Lawvere et disons qu'un pré-faisceau (contravariant)  $\phi: \mathbb{C}_0 \rightarrow Q$  est un *pré-faisceau de Cauchy* si, vu comme distributeur  $\phi: \mathbb{1} \dashv\rightarrow \mathbb{C}$ , il est un adjoint à gauche dans  $\text{Dist}(Q)$ . Une constatation importante (présentée dans [23]) est alors qu'une suite  $\underline{x} = (x_n)_{n \in \mathbb{N}}$  d'éléments dans  $\mathbb{C}_0$  est de Cauchy (selon la définition rappelée précédemment) si et seulement si

$$\phi_{\underline{x}}: \mathbb{1} \dashv\rightarrow \mathbb{C} \text{ avec } \phi_{\underline{x}}(y) = \bigvee_{N \in \mathbb{N}} \bigwedge_{n \geq N} \mathbb{C}(y, x_n)$$

(c'est-à-dire le distributeur qui correspond au pré-faisceau introduit précédemment) est adjoint à gauche au distributeur

$$\psi_{\underline{x}}: \mathbb{C} \dashv\rightarrow \mathbb{1} \text{ avec } \psi_{\underline{x}}(y) = \bigvee_{N \in \mathbb{N}} \bigwedge_{n \geq N} \mathbb{C}(x_n, y).$$

Finalement, une  $Q$ -catégorie  $\mathbb{C}$  est dite *Cauchy-complète* si tout pré-faisceau de Cauchy sur  $\mathbb{C}$  est représentable. Il s'ensuit en particulier que, dans une  $Q$ -catégorie (catégoriquement) Cauchy-complète, toute suite de Cauchy est convergente (représentable).

### Itération de Picard

Soit  $\mathbb{C}$  une  $Q$ -catégorie, et considérons les images itérées d'un objet  $x \in \mathbb{C}_0$  par une fonction  $f: \mathbb{C}_0 \rightarrow \mathbb{C}_0$ : cela produit une suite  $(f^n(x))_{n \in \mathbb{N}}$ . Cette suite détermine deux distributeurs (comme expliqué ci-dessus), que nous allons écrire sous la forme

$$\phi_{x,f}: \mathbb{1} \rightarrow \mathbb{C} \quad \text{et} \quad \psi_{x,f}: \mathbb{C} \dashv\rightarrow \mathbb{1}.$$

Ces distributeurs sont adjoints (à gauche/droite) si et seulement si la suite  $(f^n(x))_{n \in \mathbb{N}}$  est de Cauchy (dans le sens catégorique approprié). Ainsi, si la catégorie  $\mathbb{C}$  est complète au sens de Cauchy, alors (le pré-faisceau correspondant)  $\phi_{x,f}$  est représentable. Nous pouvons alors montrer :

**Proposition 2.3** *Soient  $Q$  un quantale,  $\mathbb{C}$  une  $Q$ -catégorie Cauchy-complète, et  $f: \mathbb{C} \rightarrow \mathbb{C}$  un foncteur sur  $\mathbb{C}$ , c'est-à-dire une application  $f: \mathbb{C}_0 \rightarrow \mathbb{C}_0$  telle que  $\mathbb{C}(fx, fy) \geq \mathbb{C}(x, y)$ . Supposons qu'il existe un  $x \in \mathbb{C}_0$  pour lequel les distributeurs  $\phi_{x,f}: \mathbb{1} \rightarrow \mathbb{C}$*

et  $\psi_{x,f}: \mathbb{C} \rightarrow \mathbb{1}$  sont des adjoints. Alors  $f$  a un point fixe, à savoir,  $f(c) \cong c$  où  $c \in \mathbb{C}_0$  est l'objet qui représente  $\phi_{x,f}$ .

Ceci montre vraiment comment l'argument classique – qui fait intervenir les itérations de Picard – peut être adapté de manière appropriée aux catégories enrichies dans  $Q$  : on trouve un point fixe pour  $f$  par la “convergence” de la suite de Cauchy  $x, f(x), f(f(x)), \dots$ , en d'autres termes, par la représentativité du pré-faisceau de Cauchy  $\phi_{x,f}$ .

Toutefois, il nous reste encore à trouver les conditions qui font d'une fonction  $f: \mathbb{C}_0 \rightarrow \mathbb{C}_0$  produit une suite de Cauchy  $x, f(x), f(f(x)), \dots$  ! Lors de nos tentatives pour généraliser les théorèmes de point fixe de Banach, et de Boyd et Wong, nous avons remarqué que nous avons besoin de conditions supplémentaires sur le quantale sous-jacent  $Q$  et la fonction de contrôle  $\varphi$  – que nous allons expliquer par la suite.

### Treillis continus

La théorie des *treillis continus* fait partie de la théorie des domaines ; la référence standard sur le sujet est [19]. Dans cette courte introduction, nous ne pouvons rendre pleinement justice à la complexité de ce sujet et à ses applications en informatique théorique. Nous nous contenterons de rappeler certaines des définitions clé afin que le lecteur puisse apprécier leur caractère algébrique. Nous supposons que  $L$  est un treillis *complet*.

Un sous-ensemble  $D \subseteq L$  est dit ordonné filtrant s'il est non-vide et, pour tout  $x, y \in D$  il existe  $a \in D$  tel que  $x \vee y \subseteq z$ . Pour tout deux éléments  $a, b \in L$ , on écrit  $a \ll b$ , et on dit que  $a$  est *largement inférieur* à  $b$ , si, pour tout sous-ensemble filtrant  $D \subseteq L$ ,  $b \leq \bigvee D$  implique l'existence d'un  $d \in D$  tel que  $a \leq d$ . Notons que le sous-ensemble  $\{u \in L \mid u \ll a\}$  est lui-même ordonné filtrant. Et on note  $a \lll b$ , pour dire que  $a$  est *totalemment inférieur* à  $b$ , si, pour tout sous-ensemble  $S \subseteq L$ ,  $b \leq \bigvee S$  exige l'existence d'un  $s \in S$  tel que  $a \leq s$ . Il est facile de voir que  $a \lll b$  implique  $a \ll b$ , qui à son tour implique que  $a \leq b$ .

Le treillis complet  $L$  est dit :

- $\wedge$ -continue si, pour tout  $a \in L$  et tout ensemble filtrant  $D \subseteq L$ ,

$$a \wedge (\bigvee D) = \bigvee_{d \in D} a \wedge d,$$

— *continu* si, pour tout  $a \in L$ ,

$$a = \bigvee \{u \in L \mid u \ll a\},$$

— *totalelement continu* si, pour tout  $a \in L$ ,

$$a = \bigvee \{u \in L \mid u \ll\ll a\}.$$

Un treillis totalelement continu est également continu, et un treillis continu est également  $\wedge$ -continu. Supposons maintenant que  $L$  et  $M$  sont des treillis complets, alors une fonction  $f: L \rightarrow M$  est dite *semi-continue inférieurement* si

$$f\left(\bigvee D\right) = \bigvee_{d \in D} f(d)$$

pour tout ensemble ordonné filtrant  $D \subseteq L$ . Ceci est, en fait, équivalent à la condition (en analyse réelle peut-être plus familière) que

$$f\left(\bigvee_{j \in J} \bigwedge_{j' \geq j} x_{j'}\right) \leq \bigvee_{j \in J} \bigwedge_{j' \geq j} f(x_{j'}).$$

pour toute suite généralisée  $(x_j)_{j \in J}$  in  $L$ . Si  $L$  et  $M$  sont des treillis continus, alors ceci est aussi équivalent à

$$f(x) = \bigvee \{f(y) \mid y \ll x\}$$

pour tout  $x \in L$ . En effet, une telle fonction  $f: L \rightarrow M$  est toujours monotone.

Bien évidemment, toute suite  $(x_i)_{i \in \mathbb{N}}$  dans  $L$  est une suite généralisée (aussi dite, filet), mais l'inverse n'est pas vrai. Il est donc logique de définir la variante plus faible de la semi-continuité inférieure :

**Definition 2.4** Une fonction  $f: L \rightarrow M$  est dite *sequentiellement semi-continue inférieurement* si

$$f\left(\bigvee_{N \in \mathbb{N}} \bigwedge_{n \geq N} x_n\right) \leq \bigvee_{N \in \mathbb{N}} \bigwedge_{n \geq N} \varphi(x_n)$$

pour toute suite  $(x_n)_{n \in \mathbb{N}}$  dans  $L$ . Une telle fonction  $f$  est toujours monotone.

Nous avons, maintenant, toutes les notions nécessaires pour parler de notre théorème de point fixe.

## Un théorème de point fixe pour les $Q$ -catégories

Le quantale de Lawvere  $([0, \infty], \wedge, +, 0)$  possède de nombreuses propriétés fortes. Par exemple, comme son treillis complet sous-jacent est linéaire, il est totalement continu, donc également  $\wedge$ -continu, local, etc. La structure monoïdale est également très particulière : elle est commutative, intégrale (ce qui signifie que l'unité est le plus grand élément), annulable, etc. Pour montrer des résultats dans les espaces métriques, on s'appuie souvent sur l'une ou l'autre propriété des nombres réels (positifs étendus).

Par conséquent, lorsqu'on généralise les théorèmes du point fixe de Banach et de Boyd et Wong (ou tout autre résultat dans les espaces métriques, d'ailleurs) aux  $Q$ -catégories, on doit isoler les propriétés (algébriques) de  $Q$  qui sont pertinentes pour la preuve. Pour se rapprocher des résultats originaux pour les espaces métriques, nous avons envisagé de travailler avec des  $Q$ -catégories Cauchy-complètes, et avons cherché à formuler une notion de contraction qui produise une suite de Cauchy par les itérations de Picard. Nous avons ainsi été amenés à ce qui suit :

**Definition 2.5** Si  $\varphi: Q \rightarrow Q$  et  $f: \mathbb{C}_0 \rightarrow \mathbb{C}_0$  sont deux applications telles que

1.  $\varphi(t) \geq t$  pour tout  $t \in Q$ ,
2. si  $\varphi(t) = t$  alors  $t = 0$  ou  $1 \leq t$ ,
3.  $\mathbb{C}(fx, fy) \geq \varphi(\mathbb{C}(x, y))$  pour tout  $x, y \in \mathbb{C}_0$ ,

alors on dit que  $f$  est une  $\varphi$ -contraction, et on dit aussi que  $\varphi$  est une fonction de contrôle de  $f$ .

On voit facilement qu'une telle contraction est un  $Q$ -foncteur, et donc la Proposition 2.3 s'applique facilement. Mieux encore, il existe des conditions simples sur une  $Q$ -catégorie  $\mathbb{C}$  pour qu'une contraction  $\varphi$  ait un point fixe *unique* :

**Proposition 2.6** Soient  $Q$  un quantale, et  $f: \mathbb{C} \rightarrow \mathbb{C}$  une  $\varphi$ -contraction. Si  $\mathbb{C}(x, y) \neq 0$  pour tout  $x, y \in \mathbb{C}_0$ , alors deux points fixes quelconques de  $f$  sont isomorphes. Si  $\mathbb{C}(x, y) = \mathbb{C}(y, x)$  pour tout les  $x, y \in \mathbb{C}_0$ , alors deux point fixes quelconques de  $f$  sont soit isomorphes, soit dans des composantes différentes de  $\mathbb{C}$ .

Notre résultat principal, cependant, est l'énoncé suivant de conditions suffisantes explicites sur une  $\varphi$ -contraction, pour qu'elle produise une suite de Cauchy par les

itérations de Picard :

**Proposition 2.7** *Soient  $Q$  un quantale dont le treillis sous-jacent est continu,  $\mathbb{C}$  une  $Q$ -catégorie, et  $f: \mathbb{C} \rightarrow \mathbb{C}$  est une  $\varphi$ -contraction, pour laquelle la fonction de contrôle  $\varphi: Q \rightarrow Q$  est sequentiellment semi-continues inférieurement. Pour tout  $x \in \mathbb{C}_0$  tel que  $\mathbb{C}(x, fx) \neq 0 \neq \mathbb{C}(fx, x)$ , la suite  $x, f(x), f(f(x)), \dots$  est de Cauchy.*

La preuve complète est élaborée et technique, mais nous allons en donner un bref aperçu pour expliquer où les hypothèses sont utilisées.

Supposons que la suite  $x, f(x), f(f(x)), \dots$  n'est pas de Cauchy, c'est-à-dire

$$1 \not\leq \bigvee_{N \in \mathbb{N}} \bigwedge_{m \geq N} \bigwedge_{n \geq N} \mathbb{C}(f^m(x), f^n(x)).$$

Par la continuité de  $Q$ , nous savons que  $1 = \bigvee\{\epsilon \in Q \mid \epsilon \ll 1\}$ , ce qui équivaut l'existence d'un  $\epsilon \ll 1$  tel que  $\epsilon \not\leq \bigvee_{N \in \mathbb{N}} \bigwedge_{m, n \geq N} \mathbb{C}(f^m x, f^n x)$ . Ainsi,  $0 \neq \epsilon \ll 1$  est tel que

$$\text{for all } k \in \mathbb{N} \exists m_k, n_k \geq k \text{ tel que } \epsilon \not\leq \mathbb{C}(f^{m_k} x, f^{n_k} x).$$

Un "choix subtil" de tels indices  $m_k, n_k \geq k$  peut être fait, à savoir

$$d_k := \mathbb{C}(f^{m_k} x, f^{n_k} x),$$

de sorte que, en utilisant l'hypothèse que  $\mathbb{C}(fx, x) \neq 0 \neq \mathbb{C}(x, fx)$  ainsi que la semi-continuité de  $\varphi$  et la  $\wedge$ -continuité de  $Q$ , on aura que

$$\varphi\left(\bigvee_{N \in \mathbb{N}} \bigwedge_{k \geq N} d_k\right) = \bigvee_{N \in \mathbb{N}} \bigwedge_{k \geq N} d_k \neq 0.$$

À partir de notre définition 2.5, il s'ensuit que  $1 \leq \bigvee_{N \in \mathbb{N}} \bigwedge_{k \geq N} d_k$ . Comme ce supremum est évidemment filtrant,  $\epsilon \ll 1$  étant dans le treillis continu  $Q$  implique que  $\epsilon \leq \bigwedge_{k \geq N_0} d_k$  pour un certain  $N_0$ . Ce qui contredit notre précédente observation que  $\epsilon \not\leq d_k$  pour tout  $k$ .

En rassemblant les propositions ci-dessus 2.3, 2.6 et 2.7, nous pouvant maintenant synthétiser :

**Theorem 2.8** Soit  $Q$  un quantale dont le treillis sous-jacent est continue, et supposons que  $f: \mathbb{C} \rightarrow \mathbb{C}$  est une  $\varphi$ -contraction sur une  $Q$ -catégorie Cauchy-complète, telle que la fonction de contrôle  $\varphi: Q \rightarrow Q$  est sequentiellement semi-continue inférieurement.

S'il existe  $x \in \mathbb{C}_0$  tel que  $\mathbb{C}(x, fx) \neq 0 \neq \mathbb{C}(fx, x)$  alors  $f$  admet un point fixe. Si pour tout  $x, y \in \mathbb{C}_0$ ,  $\mathbb{C}(x, y) \neq 0$ , alors ce point fixe est (essentiellement) unique. Si  $\mathbb{C}$  est symétrique, alors tous deux point fixes quelconques sont soit isomorphes, soit situés dans des composantes différentes de  $\mathbb{C}$ .

L'idée de la preuve est donc –précisément comme pour les espaces métriques! – que la contractivité de  $f$  force la suite des itérations  $x, f(x), f(f(x)), \dots$  à être de Cauchy (au sens approprié aux  $Q$ -catégories) et, de plus, elle “converge” vers un point fixe de  $f$ .

## Exemples

Tout treillis complet linéaire est totalement continu, et donc également continu. Ceci nous permet d'appliquer le théorème 2.8 : pour le quantale  $([0, \infty], \wedge, +, 0)$ , on retrouve le théorème du point fixe de Banach en considérant la fonction de contrôle (évidemment, semi-continue supérieurement)

$$\varphi: [0, \infty] \rightarrow [0, \infty]: t \mapsto k \cdot t$$

pour un certain  $0 < k < 1$ . Ainsi, nous pouvons dire a posteriori que le théorème du point fixe de Banach tient *parce que* le quantale des nombres réels positifs est continu et que la fonction de contrôle  $\varphi(t) = k \cdot t$  est semi-continue supérieurement.

Une analyse attentive de la démonstration détaillée du théorème 2.8 montre que  $\varphi$  n'a pas besoin d'être définie sur  $Q$  tout entier, mais qu'il suffit de considérer l'“adhérence” (dans le sens adéquat) de l'image de  $\mathbb{C}: \mathbb{C}_0 \times \mathbb{C}_0 \rightarrow Q$ . Nous retrouvons donc aussi le théorème du point fixe de Boyd et Wong pour les espaces métriques.

Une *t-norme continue à gauche* est un quantale commutatif et intégral sur l'intervalle  $[0, 1]$ . Parce que  $[0, 1]$  est un ordre linéaire complet, le treillis sous-jacent du quantale  $Q = ([0, 1], \vee, *, 1)$  est aussi continu. Ces quantales sont utilisés en logique multivaluée (ou “floues”) : les  $Q$ -catégories sont appelées “ordres fous” [37]. La for-

mulation du théorème 2.8 pour ce quantale produit un théorème de point fixe pour de tels ordres flous.

Le produit tensoriel (dans la catégorie Sup des treillis complets et applications préservant les suprema entre treillis continus) de treillis continu est aussi un treillis continu. Ceci est vrai en particulier pour le treillis complet

$$\Delta = \{f: [0, \infty] \rightarrow [0, 1] \mid \forall t: f(t) = \bigvee_{s < t} f(s)\}$$

(en considérant l'ordre ponctuel) car il s'agit exactement du produit tensoriel de  $[0, \infty]$  et de  $[0, 1]$  (voir [21]).

Considérons une  $t$ -norme continue à gauche, disons  $([0, 1], \bigvee, *, 1)$ , alors  $\Delta$  est aussi un quantale pour l'opération

$$(f * g)(t) = \bigvee_{r+s=t} f(r) * g(s)$$

ayant comme unité

$$e(t) = \begin{cases} 0 & \text{si } t = 0, \\ 1 & \text{sinon} \end{cases}$$

Une catégorie enrichie sur  $\Delta$  est, exactement, un espace métrique *probabiliste* (généralisé), [23]. Ainsi, l'application du théorème 2.8 produit un théorème de point fixe pour de tels espaces (et il existe effectivement des fonctions de contrôle  $\varphi: \Delta \rightarrow \Delta$  qui remplissent les conditions du théorème).

### 3 Références

- [1] K. Alaoui Ismaili, D. E. Dobbs and N. Mahdou, *Commutative rings and modules that are Nil<sub>\*</sub>-coherent or special Nil<sub>\*</sub>-coherent*, J. Algebra Appl. **16** (2017), 1750187.
- [2] S. Banach, *Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales*, Fund. Math. **3** (1922), 133—181.
- [3] J. Bénabou, *Les distributeurs*, Rapport no. 33 des Séminaires de Mathématique Pure, Université Catholique de Louvain, Louvain-la-Neuve (1973).
- [4] D. Bennis, *n-X-Coherent rings*, Int. Electron. J. Algebra **7** (2010), 128–139.



- [5] D. W. Boyd, and J.S.W. Wong, *On nonlinear contractions*, Proceedings of the American Mathematical Society **20** (1969), 458–464.
- [6] L. E. J. Brouwer, *Über Abbildung von Mannigfaltigkeiten*, Math. Ann. **71** (1911), 97–115.
- [7] V. Camillo, *Coherence for polynomial rings*, J. Algebra **132** (1990), 72–76.
- [8] H. Cartan and S. Eilenberg, *Homological Algebra*, Princeton Math. Series Princeton University Press, (1956).
- [9] S. U. Chase, *Direct product of modules*, Trans. Amer. Math. Soc. **97** (1960), 457–473.
- [10] T. J. Cheatham and D. R. Stone, *Flat and projective character modules*, Proc. Amer. Math. Soc. **81** (1981), 175–177.
- [11] D. L. Costa, *Parameterizing families of non-Noetherian rings*, Comm. Algebra **22** (1994), 3997–4011.
- [12] C. Coppola, G. Giangiacomo and P. Tiziana, *Convergence and fixed points by fuzzy orders*, Fuzzy Sets and systems **159** (2008), 1178–1190.
- [13] N. Ding, Y. Li and L. Mao, *J-coherent rings*, J. Algebra Appl. **8** (2009), 139–155.
- [14] Dobbs, Kabbaj, Mahdou, *n-Coherent rings and modules*, Dekker Lect. Notes Pure Appl. Math. **185** (1997), 269–282.
- [15] Ch. Ehresmann, *Gattungen von lokalen Strukturen*, Jahresbericht der Deutschen Mathematiker-Vereinigung **60** (1958), 49–77.
- [16] E. E. Enochs and O. M. G. Jenda, *Relative homological algebra*, Walter de Gruyter, Berlin-New York (2000).
- [17] E. E. Enochs and O. M. G. Jenda, *Gorenstein injective and projective modules*, Math. Z. **220** (1995), 611–633.
- [18] Z. Gao and F. Wang, *Coherent rings and Gorenstein FP-injective modules*, Comm. Algebra **40** (2012), 1669–1679.
- [19] G. Gierz, K. H. Hofmann, K. Keimel, J. D. Lawson, M. W. Mislove, D. S. Scott, *A Compendium of Continuous Lattices*, Springer-Verlag, Berlin, New York, (1980).

- [20] S. Glaz, *Commutative coherent rings*, Lecture Notes in Math., Springer-Verlag, Berlin (1989).
- [21] J. Gutiérrez García, U. Höhle, T. Kubiak, Tensor products of complete lattices and their application in constructing quantales, *Fuzzy Sets Syst.* 313 (2017), 43–60.
- [22] M. E. Harris, *Some results on coherent rings*, *Proc. Amer. Math. Soc.* **17** (1966), 474–479.
- [23] D. Hofmann and C. Reis, *Probabilistic metric spaces as enriched categories*, *Fuzzy Sets and Systems* **210** (2013), 1–21.
- [24] C. Huneke, *Hyman Bass and ubiquity : Gorenstein rings*, *Algebra, k-theory, groups, and education* (1999), 55–78.
- [25] F. W. Lawvere, *Metric spaces, generalized logic and closed categories*, *Rendiconti del Seminario Matematico e Fisico di Milano*, XLIII (1973), 135–166. Also in : *Reprints in Theory and Applications of Categories*, **1** (2002), 1–37.
- [26] L. Mao, *Min-Flat Modules and Min-Coherent Rings*, *Comm. Algebra* **35** (2007), 635–650.
- [27] L. Mao,  *$\Pi$ -Coherent dimension and  $\Pi$ -Coherent rings*, *J. Korean Math. Soc.* **44** (2007), 719–731.
- [28] W. K. Nicholson and M. F. Yousif, *FP-RINGS*, *Communications in Algebra*, (29), No.1, (2001), 415–425.
- [29] W. K. Nicholson and M. F. Yousif, *Quasi-frobenius rings*, Cambridge University Press, No.158 (2003).
- [30] C. J. Mulvey, *Suppl. Rend. Circ. Mat. Palermo II* **12** (1986), 99–104.
- [31] L. Ouyng and Y. Xiang, *Nil<sub>\*</sub>-Coherent rings*, *Bull. Korean Math. Soc.* **51** (2014), 579–594.
- [32] J. Schauder, *Der Fixpunktsatz in Funktionalräumen*, *Studia Math.* **2** (1930), 171—180.
- [33] J.P. Serre, *Faisceaux algébriques cohérents*, *Annals of Mathematics* **61** (1955), 197–278.
- [34] B. Stenström, *Coherent rings and FP-injective modules*, *J. London Math. Soc.* **2** (1970), 323–329.

- [35] B. Stenström, *Rings of Quotients*, Springer-Verlag, Berlin, Heidelberg, New York (1975).
- [36] I. Stubbe, *Categorical structures enriched in a quantaloid : categories, distributors and functors*, Theory Appl. Categ. **14** (2005), 1–45.
- [37] I. Stubbe, *An introduction to quantaloid-enriched categories*, Fuzzy Sets and Systems **256** (2014), 95–116.
- [38] X. Y. Yang and Z. K. Liu, *FP-gr-injective modules*, Math. J. Okayama Univ. **53** (2011), 83–100.
- [39] T. Zhao, Z. Gao and Z. Huang, *Relative FP-gr-injective and gr-flat modules*, Int. Algebra and Computation **28** (2018), 959–977.

# Introduction to the selected articles

This dissertation is in essence composed of three articles, reproduced from p. 43 on, that reflect the doctoral research that I have done between January 2018 and November 2022, under supervision of Prof. Driss Bennis (Université Mohammed V, Morocco) and Prof. Isar Stubbe (Université du Littoral, France). As is usual for research articles, their setup is rather terse and technical: they are intended for a readership of specialists. Yet, I am convinced that the main ideas developed in these articles can be understood without giving all the details. In the paragraphs that follow, I shall attempt to give a gentle introduction to some of the main results that were obtained in the course of my doctoral studies.

As my doctoral studies were supervised by two different supervisors at two different institutions (under a *convention de cotutelle*), the included articles treat two different subjects. The first two articles (with D. Bennis) have rings and modules as their subject; the third article (with I. Stubbe) is on quantale-enriched categories. However, these articles are all part of the larger field of general Category Theory: indeed, we are interested in the *category* of  $R$ -modules (for some ring  $R$ ) and in the *category* of  $Q$ -enriched categories (for some quantale  $Q$ ). Moreover, a ring  $R$  itself is a monoid in the monoidal category  $\text{Ab}$  of abelian groups, whereas a quantale  $Q$  is a monoid in the monoidal category  $\text{Sup}$  of sup-complete lattices. So Category Theory is the federating theme of this doctoral dissertation.

In the first Section below, I shall briefly discuss the articles that can be found on p. 45 and p. 61: their subject is the study of relative homological algebra in the category of  $R$ -modules. In Section 2, I shall discuss in somewhat more detail the

article, to be found on p. 79, on a fixpoint theorem for  $Q$ -enriched categories. A final section gathers all the references used in this Introduction.

## 1 Relative homological algebra for ring-modules

In all the following,  $R$  will be an associative (not necessarily commutative) ring with identity, and all modules will be unital left  $R$ -modules (unless specified otherwise).

The idea behind ring-modules is to generalize the concept of vector spaces by weakening the field of scalars to be only a ring  $R$  (always assumed with a unit). At first, the study of  $R$ -modules focused on generalizing as much as possible the properties of vector spaces. But unlike vector spaces, ring-modules are more complicated: not all modules have a basis; free modules (i.e. those which do have a basis) need not have a unique rank; submodules of finitely generated modules are not necessarily finitely generated.

In 1960, Chase [9] introduced and studied coherent rings and modules. (The term “coherent” was used for the first time, in 1953 by Serre [33] to study coherent sheaves.) To be precise, a ring is said to be (left) coherent if every finitely generated (left) ideal is finitely presented. Many results for coherent rings were influenced by one of its most important examples: (left) Noetherian rings (i.e. rings in which *every* (left) ideal is finitely generated).

Let us recall some important results involving (modules on) coherent rings:

1. Chase [9]: a ring  $R$  is left coherent if and only if every direct product of flat right  $R$ -modules is flat.
2. Cheatham and Stone [10]: the following assertions are equivalent:
  - (a)  $R$  is left coherent,
  - (b) any  $R$ -module  $M$  is injective if and only if  $M^* := \text{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$  is flat,
  - (c) any  $R$ -module  $M$  is injective if and only if  $M^{**}$  is injective,
  - (d) any  $R$ -module  $M$  is flat if and only if  $M^{**}$  is flat.
3. Stenström [34]: a ring  $R$  is left coherent if and only if every direct limit of FP-injective<sup>7</sup>  $R$ -modules is also FP-injective.

---

7. An  $R$ -module  $M$  is FP-injective, also called absolutely pure, if  $\text{Ext}^1(N, M) = 0$  for every

4. Enochs [16]: a ring  $R$  is left coherent if and only if every right  $R$ -module has a flat preenvelope<sup>8</sup>.

In the last few years, various coherent-like rings were introduced: Ding et al. [13] defined a  $J$ -coherent ring to be a ring for which all the finitely generated ideals of the Jacobson radical are finitely presented; Ouyng and at. [31] similarly introduced nil-coherent rings, replacing the Jacobson radical by the nilradical; Mao [26] defined min-coherent rings by requiring only finitely generated simple ideals to be finitely presented; Camillo [7] defined  $\Pi$ -coherent rings as those for which every finitely generated torsionless  $R$ -module is finitely presented. In this spirit, a coherent ring is precisely an  $\text{Idl}(R)$ -coherent ring.

A quite different extension of the notion of coherent ring was introduced by Costa in [11], and further developed by e.g. Dobbs et al. [14]. Recall first that an  $R$ -module  $M$  is  $n$ -presented (for some  $n \geq 0$ ) if there is an exact sequence of  $R$ -modules

$$F_n \rightarrow F_{n-1} \rightarrow \cdots \rightarrow F_1 \rightarrow F_0 \rightarrow M \rightarrow 0$$

where each  $F_i$  is a finitely generated free module. (So “0-presented” means finitely generated, and “1-presented” means finitely presented.) This definition readily applies to ideals in a ring  $R$ , since these are exactly the submodules of  $R$  seen as  $R$ -module. Now we can define: a ring  $R$  is  $n$ -coherent (for  $n \geq 1$ ) if every  $(n - 1)$ -presented ideal is  $n$ -presented. Thus 1-coherent rings are exactly coherent rings.

In 2010, Bennis [4] unified the aforementioned generalizations by introducing so-called  $n$ - $\mathcal{X}$ -coherent rings, where  $\mathcal{X}$  is a class of ideals and  $n$  is a non-negative integer.

## 1.1 Relative coherence

In our paper, that can be found from p. 45 onwards, we define and study  $n$ - $\mathcal{X}$ -coherent  $R$ -modules. Explicitly, let  $M$  be an  $R$ -module and  $\mathcal{X}$  a class of submodules of  $M$ . We

---

finitely presented  $R$ -module  $N$ .

8. An  $R$ -module  $F$  in some class of  $R$ -modules  $\mathcal{X}$  is said to be an  $\mathcal{X}$ -preenvelope of an  $R$ -module  $M$ , if there exist an homomorphism  $\varphi: M \rightarrow F$  such that, for any homomorphism  $\varphi': M \rightarrow F'$  with  $F' \in \mathcal{X}$ , there is an homomorphism  $f: F \rightarrow F'$  such that  $\varphi' = f\varphi$ . Taking  $\mathcal{X}$  to be the class of flat  $R$ -modules, we speak simply of the flat preenvelope of  $M$ .

denote by  $\mathcal{X}_n$  the class of  $n$ -presented modules in  $\mathcal{X}$ . We say that  $M$  is  $n$ - $\mathcal{X}$ -coherent (for  $n \geq 1$ ) if  $\mathcal{X}_{n-1}$  is non-empty and  $\mathcal{X}_{n-1} \subseteq \mathcal{X}_n$ .

**Example 1.1** Let  $R$  be a ring and  $M$  an  $R$ -module.

- For  $\mathcal{X}$  the class of all (left) ideals in  $R$ , the ring  $R$  is  $n$ - $\mathcal{X}$ -coherent if and only if it is (left)  $n$ -coherent in the sense of [14].
- For  $\mathcal{X}$  the class of all submodules of  $M$ , the module  $M$  is  $1$ - $\mathcal{X}$ -coherent if and only if it is pseudo-coherent in the sense of [22].
- If  $\mathcal{X}$  is the class of all submodules of  $\text{Nil}(R)M$ , then the module  $M$  is  $1$ - $\mathcal{X}$ -coherent if and only if it is  $\text{Nil}_*$ -coherent in the sense of [1].
- If  $\mathcal{X}$  is the Jacobson radical of  $R$ , then  $R$  is  $1$ - $\mathcal{X}$ -coherent if and only if it is  $J$ -coherent in the sense of [13].

Our main aim is to show how some of the well-known results on coherent rings and modules given above still hold true in the  $n$ - $\mathcal{X}$ -coherent case. We proved, in particular, a generalization of Chase's Theorem [9] (see also [1]):

**Theorem 1.2** *Let  $M$  be an  $R$ -module and let  $\mathcal{X}$  be a class of submodules of  $M$  such that  $\mathcal{X}_n$  is closed under finite sum and closed under cyclic submodules. Then  $M$  is left  $1$ - $\mathcal{X}$ -coherent if and only if the following two conditions hold:*

1. *For any  $a \in M$  such that  $Ra$  is in  $\mathcal{X}_0$ ,  $(0 :_R a)$  is a finitely generated submodule of  $M$ ,*
2. *the intersection of any two submodules of  $M$  in  $\mathcal{X}_0$  is finitely generated.*

We also established a necessary and sufficient condition for the  $n$ - $\mathcal{X}$ -coherence of a pullback in the category of rings<sup>9</sup>:

**Theorem 1.3** *Suppose that  $R$  is the pullback of  $R_1$  and  $R_2$  in the category of commutative rings with unit. Let  $\mathcal{X}$  be a class of ideals of  $R$  such that for every  $I \in \mathcal{X}_1$  there exists an exact sequence  $0 \rightarrow K \rightarrow R^k \rightarrow I \rightarrow 0$  with  $K \in \mathcal{X}$ . Let  $\mathcal{Y}^i$  be a class of ideals of  $R_i$  for  $i = 1, 2$ . Suppose that for each  $N \in \mathcal{X}_n$ ,  $\text{Tor}_j^R(R_i, N)$  is in*

---

<sup>9</sup>. In the paper, the statement of this theorem contains some ambiguities. The class  $\mathcal{X}$  was considered to be a class of  $R$ -modules, which is not correct. The same remark goes for the class  $\mathcal{Y}^i$ . In this introduction, we give the correct statement of the theorem.

$\mathcal{Y}_{n+1-j}^i$  for  $1 \leq j \leq n+1$  and  $i = 1, 2$ . Suppose that for any  $Y_i \in \mathcal{Y}_n^i$ , there exists an  $X_i \in \mathcal{X}_n$  such that  $R_i \otimes_R X_i \simeq Y_i$ . Then,  $R$  is  $n$ - $\mathcal{X}$ -coherent if and only if  $R_i$  is  $n$ - $\mathcal{Y}^i$ -coherent.

## 1.2 Relative homological dimensions

Our second paper, which can be found on p. 61, focuses on the study of relative dimensions of  $R$ -modules, connecting this to the relative coherence discussed above.

In 1956, Eilenberg and Cartan [8] introduced homological algebra (in particular the functors  $\text{Hom}$  and  $\text{Ext}$ ) to classify and study rings and modules via their homological invariants and dimensions. The so-called Gorenstein dimension of  $R$ -modules is an important application of this theory in algebraic geometry (see for instance [24]). In 1995, Enochs and Jenda [17] introduced and studied Gorenstein modules, leading to *relative homological algebra*: a theory generalizing concepts from standard homological algebra by using a special class of modules as a parameter.

More precisely, let  $\mathcal{X}$  be a class of left  $R$ -modules, and suppose that, for any  $k \geq 0$ , the subclass  $\mathcal{X}_k$  of the  $k$ -presented  $R$ -modules in  $\mathcal{X}$  is non-empty. A left  $R$ -module  $M$  is said to be  $n$ - $\mathcal{X}$ -injective if  $\text{Ext}_R^n(U, M) = 0$  for any  $U \in \mathcal{X}_n$ . A right  $R$ -module  $N$  is called  $n$ - $\mathcal{X}$ -flat if  $\text{Tor}_R^n(N, U) = 0$  for any  $U \in \mathcal{X}_n$ . We use  $\mathcal{X}\mathcal{I}_n$  (resp.  $\mathcal{X}\mathcal{F}_n$ ) to denote the class of all  $n$ - $\mathcal{X}$ -injective left  $R$ -modules (resp.  $n$ - $\mathcal{X}$ -flat right  $R$ -modules).

We now define a ring  $R$  to be left  $n$ - $\mathcal{X}$ -coherent if every  $n$ -presented left  $R$ -module in  $\mathcal{X}$  is  $(n+1)$ -presented. This definition (studied in detail in [4]) reproduces the definition given in our first paper – discussed in the previous Subsection – when restricting  $\mathcal{X}$  to be a class of left  $R$ -ideals. If, however, we let  $\mathcal{X}$  be the class of all (left)  $R$ -modules, then this coincides with the notion of “strong (left) coherent ring” in [14], and is a “(left) coherent ring” in [11]. We proved the following:

**Theorem 1.4** *Let  $R$  be a left  $n$ - $\mathcal{X}$ -coherent ring for some class  $\mathcal{X}$  of  $R$ -modules. The following statements are equivalent:*

1.  $R$  is self left  $n$ - $\mathcal{X}$ -injective, i.e.  $n$ - $\mathcal{X}$ -injective as a module over itself,
2. every left  $R$ -module has an epimorphic  $\mathcal{X}\mathcal{I}_n$ -cover<sup>10</sup>,

---

10. This notions is a dual notion of pre-envelope



3. every right  $R$ -module has a monomorphic  $\mathcal{X}\mathcal{F}$ -preenvelope,
4. every injective right  $R$ -module is  $n$ - $\mathcal{X}$ -flat,
5. every 1- $\mathcal{X}$ -injective right  $R$ -module is  $n$ - $\mathcal{X}$ -flat,
6. every  $n$ - $\mathcal{X}$ -injective right  $R$ -module is  $n$ - $\mathcal{X}$ -flat,
7. every left flat  $R$ -module is  $n$ - $\mathcal{X}$ -injective.

To state our other main result, we need to introduce some more notions. With notations as before, for  $U \in \mathcal{X}_n \subseteq \mathcal{X}$  there is – by definition – an exact sequence

$$F_n \longrightarrow F_{n-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow U \longrightarrow 0$$

with each  $F_i$  a finitely generated free  $R$ -module. Let  $K_{n-1} = \text{Im}(F_{n-1} \rightarrow F_{n-2})$  and  $K_n = \text{Im}(F_n \rightarrow F_{n-1})$ . The short exact sequence  $0 \rightarrow K_n \rightarrow F_n \rightarrow K_{n-1} \rightarrow 0$  is called a special short exact sequence of  $U$ . It is clear that  $K_n$  and  $K_{n-1}$  are finitely generated and finitely presented, respectively. We call  $K_n$  a special  $\mathcal{X}$ -generated  $R$ -module and  $K_{n-1}$  a special  $\mathcal{X}$ -presented  $R$ -module.

Analogously to the definition of Gorenstein dimensions (see [17]), we define an  $R$ -module  $G$  to be Gorenstein  $n$ - $\mathcal{X}$ -injective if there exists an exact sequence of  $n$ - $\mathcal{X}$ -injective  $R$ -modules

$$\mathbf{A} = \cdots \longrightarrow A_1 \longrightarrow A_0 \longrightarrow A^0 \longrightarrow A^1 \longrightarrow \cdots$$

where  $G = \ker(A^0 \rightarrow A^1)$ , such that  $\text{Hom}_R(K_{n-1}, \mathbf{A})$  is an exact sequence whenever  $K_{n-1}$  is special  $\mathcal{X}$ -presented with  $\text{pd}_R(K_{n-1}) < \infty$ <sup>11</sup>. Similarly, a right  $R$ -module  $G$  is called Gorenstein  $n$ - $\mathcal{X}$ -flat if there exists an exact sequence of  $n$ - $\mathcal{X}$ -flat right  $R$ -modules

$$\mathbf{F} = \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow F^0 \longrightarrow F^1 \longrightarrow \cdots$$

with  $G = \ker(F^0 \rightarrow F^1)$ , such that  $\mathbf{F} \otimes_R K_{n-1}$  is an exact sequence whenever  $K_{n-1}$  is special  $\mathcal{X}$ -presented with  $\text{fd}_R(K_{n-1}) < \infty$ <sup>12</sup>.

Our main result characterizes relative self-injective rings via the Gorenstein injective and projective modules:

- 
11. This means that  $\text{Ext}^i(M, N) = 0$  for every  $R$ -module  $N$  and every  $i \geq n + 1$ .
  12. This means that  $\text{Tor}_i(M, N) = 0$  for every right  $R$ -module  $N$  and every  $i \geq n + 1$ .

**Theorem 1.5** *Let  $R$  be a left  $n$ - $\mathcal{X}$ -coherent ring for some class  $\mathcal{X}$  of  $R$ -modules. The following statements are equivalent:*

1.  *$R$  is self left  $n$ - $\mathcal{X}$ -injective,*
2. *every Gorenstein  $n$ - $\mathcal{X}$ -flat  $R$ -module is Gorenstein  $n$ - $\mathcal{X}$ -injective,*
3. *every Gorenstein flat  $R$ -module is Gorenstein  $n$ - $\mathcal{X}$ -injective,*
4. *every flat  $R$ -module is Gorenstein  $n$ - $\mathcal{X}$ -injective,*
5. *every Gorenstein projective  $R$ -module is Gorenstein  $n$ - $\mathcal{X}$ -injective,*
6. *every projective  $R$ -module is Gorenstein  $n$ - $\mathcal{X}$ -injective,*
7. *every Gorenstein injective right  $R$ -module is Gorenstein  $n$ - $\mathcal{X}$ -flat,*
8. *every injective right  $R$ -module is Gorenstein  $n$ - $\mathcal{X}$ -flat,*
9. *every Gorenstein 1- $\mathcal{X}$ -injective right  $R$ -module is Gorenstein  $n$ - $\mathcal{X}$ -flat,*
10. *every Gorenstein  $n$ - $\mathcal{X}$ -injective right  $R$ -module is Gorenstein  $n$ - $\mathcal{X}$ -flat.*

The results of this paper unify some known results for such notions – amongst others – as Gorenstein relative rings,  $FP$ -rings, and quasi-frobenius rings (see [17], [16], [28] and [29]).

In the proofs of the homological characterizations of these rings, the tools are always the same, the one crucial idea being the stability properties of some well-chosen classes of modules (closed by extensions, by direct sum, by direct component...). We tried to give the most general class of modules possible, with a minimum of conditions to support the essential homological structure that we are trying to generalize.

**Example 1.6** Let  $R = k[X]$  with  $k$  a field, and take for  $\mathcal{X}$  the class of graded  $R$ -modules. It so turns out every graded left (resp., right)  $R$ -module is Gorenstein 2- $\mathcal{X}$ -injective (resp., flat). However, there exist graded left (resp., right)  $R$ -modules that are not Gorenstein 1- $\mathcal{X}$ -injective (resp., flat). This shows that, for  $m > n$ , Gorenstein  $m$ - $\mathcal{X}$ -injectivity (resp., flatness) does not imply, in general, Gorenstein  $n$ - $\mathcal{X}$ -injectivity (resp., flatness).

## 2 A fixpoint theorem for quantale-enriched categories

It is a very general idea in mathematics to study fixpoints of a given endomorphism: for  $f: X \rightarrow X$ , find  $x \in X$  such that  $f(x) = x$ . The first fixpoint theorem dates

back to Brouwer [6] in 1912, who showed that continuous maps on the closed unit ball in  $\mathbb{R}^n$  have a fixpoint; this theorem was generalized to infinite dimensional spaces by Schauder [32] in 1930. The proofs used very strong topological tools, yet the outcome provided only the existence of the invariant point.

In 1922, Banach [2] proved the now famous Banach Fixpoint Theorem:

**Theorem 2.1 (Banach, 1922)** *Let  $(X, d)$  be a non-empty complete metric space and  $f: X \rightarrow X$  a contraction, that is,  $d(fx, fy) \leq k \cdot d(x, y)$  for some  $0 < k < 1$ . Then  $f$  has a unique fixpoint.*

The surprisingly simple proof goes as follows: take any  $x \in X$ , iterate  $f$  to obtain a Cauchy sequence  $x, f(x), f(f(x)), \dots$ , and show that it converges to a unique fixpoint of  $f$ . Many authors generalized Banach's Theorem to other spaces, weakening the conditions while maintaining the gist of the proof: the convergence of the so-called Picard iteration  $x, f(x), f(f(x)), \dots$ . But one generalization stands out as it changes the notion of contraction—it is due to Boyd and Wong [5]:

**Theorem 2.2 (Boyd and Wong, 1969)** *Let  $(X, d)$  be a non-empty complete metric space and  $\varphi: \{\overline{d(x, y)} \mid x, y \in X\} \rightarrow \mathbb{R}^+$  an upper-semicontinuous function such that  $\varphi(0) = 0$  and  $\varphi(t) < t$  for all  $t \neq 0$ . Then any map  $f: X \rightarrow X$  satisfying  $d(fx, fy) \leq \varphi(d(x, y))$  for all  $x, y \in X$  has a unique fixpoint.*

In our article which can be found at p. 79, we make a *logical analysis* of the above theorems: by showing an appropriate generalization to quantale-enriched categories, as we explain next.

## From metric spaces to enriched categories

Lawvere [25] famously observed that metric spaces are particular enriched categories. Indeed, writing  $[0, \infty] = \mathbb{R}^+ \cup \{+\infty\}$  for the extended positive real numbers, we may regard a metric space  $(X, d)$  as a map

$$d: X \times X \rightarrow [0, \infty]$$

that computes, for every pair  $(x, y)$  of points, the distance  $d(x, y)$  between them; this function certainly satisfies

$$d(x, y) + d(y, z) \geq d(x, z) \quad \text{and} \quad 0 \geq d(x, x)$$

for all  $x, y, z \in X$ . Lawvere calls this a *generalized metric space*; one gets the usual notion of metric space when imposing symmetry ( $d(x, y) = d(y, x)$ ), finiteness ( $d(x, y) \neq \infty$ ) and separatedness ( $d(x, y) = 0$  implies  $x = y$ ).

Note now that  $([0, \infty], \wedge)$  is a complete lattice (that we take with the opposite of the natural order), that  $([0, \infty], +, 0)$  is a monoid, and that there is a particular compatibility between the lattice structure and the monoid structure:

$$a + \left(\bigwedge_i b_i\right) = \bigwedge_i (a + b_i) \quad \text{and} \quad \left(\bigwedge_i b_i\right) + a = \bigwedge_i (b_i + a)$$

for all  $a, (b_i)_{i \in I}$  in  $[0, \infty]$ . Making abstraction of this algebraic structure, we can now replace the positive real numbers by any other *quantale*  $Q$ , that is, a complete lattice  $(Q, \vee)$  endowed with a monoid  $(Q, \circ, 1)$  such that the product distributes over arbitrary suprema. Quantales were first defined by Mulvey [30] to serve as a non-commutative counterpart of locales, the latter notion going back to the work of Ehresmann [15].

The above definition of (generalized) metric space then becomes what is now called a  *$Q$ -category*  $\mathbb{C}$ : it consists of a set  $\mathbb{C}_0$  together with a function

$$\mathbb{C}: \mathbb{C}_0 \times \mathbb{C}_0 \rightarrow Q: (x, y) \mapsto \mathbb{C}(x, y)$$

that computes, for every pair  $(x, y)$  of “objects”, the “hom”  $\mathbb{C}(x, y)$  between them; and it must satisfy

$$\mathbb{C}(x, y) \circ \mathbb{C}(y, z) \leq \mathbb{C}(x, z) \quad \text{and} \quad 1 \leq \mathbb{C}(x, x)$$

for all  $x, y, z \in \mathbb{C}_0$ . One can define symmetric, finitary and separated  $Q$ -categories in the obvious manner.

In fact, many other familiar mathematical structures can be seen as  $Q$ -categories: ordered sets are categories enriched in the two-element Boolean algebra  $(\{0, 1\}, \vee, \wedge, 1)$ ; so-called fuzzy orders are categories enriched in a left-continuous  $t$ -norm, i.e. a (commutative and integral) quantale on the unit interval  $[0, 1]$  (see e.g. [12, 37]); and even probabilistic metric spaces turn out to be examples of quantale-enriched categories (see e.g. [23]). We shall come back to these examples at the end of this Section.

Lawvere strongly advocated that many results for metric spaces ought to be seen as special cases of general (enriched) category theory. He gave the example of Cauchy sequences and completeness (i.e. the convergence of Cauchy sequences), which we shall briefly explain now.

## Presheaves and Cauchy sequences

A (contravariant) presheaf  $\phi$  on a  $Q$ -category  $\mathbb{C}$  is a map

$$\phi: \mathbb{C}_0 \rightarrow Q: x \mapsto \phi(x)$$

satisfying  $\mathbb{C}(y, x) \circ \phi(x) \leq \phi(y)$  for all  $x, y \in \mathbb{C}_0$ . Any single element  $c \in \mathbb{C}_0$  determines a particular presheaf:

$$\mathbb{C}(-, c): \mathbb{C}_0 \rightarrow Q: x \mapsto \mathbb{C}(x, c).$$

A given presheaf  $\phi: \mathbb{C} \rightarrow Q$  is said to be *representable* by  $c \in \mathbb{C}_0$  if  $\phi(x) = \mathbb{C}(x, c)$  for all  $x \in \mathbb{C}_0$ ; it is straightforward to show that such a  $c \in \mathbb{C}_0$  is then (essentially) unique.

It so turns out (see also [23]) that any sequence  $\underline{x} = (x_n)_{n \in \mathbb{N}}$  of elements in  $\mathbb{C}_0$  determines a presheaf on  $\mathbb{C}$ , namely:

$$\phi_{\underline{x}}: \mathbb{C}_0 \rightarrow Q: y \mapsto \bigvee_{N \in \mathbb{N}} \bigwedge_{n \geq N} \mathbb{C}(y, x_n).$$

Representability of the presheaf  $\phi_{\underline{x}}$  by some  $c \in \mathbb{C}_0$  now means exactly that

$$\bigvee_{N \in \mathbb{N}} \bigwedge_{n \geq N} \mathbb{C}(y, x_n) = \mathbb{C}(y, c) \quad \text{for all } y \in \mathbb{C}_0.$$

Furthermore, we say that such a sequence is *Cauchy* if

$$1 \leq \bigvee_{N \in \mathbb{N}} \bigwedge_{m \geq N} \bigwedge_{n \geq N} \mathbb{C}(x_m, x_n).$$

It is easily seen that, if  $\phi_{\underline{x}}$  is representable, then it must be Cauchy; but the converse does not hold in general.

Applied to the specific quantale  $Q = ([0, \infty], \wedge, +, 0)$  of extended positive real numbers (note again that we reverse the order, so suprema become infima, and vice versa), taking a sequence  $\underline{x} = (x_n)_{n \in \mathbb{N}}$  in a metric space  $(X, d)$ , the above definitions precisely correspond with the usual definitions of a sequence that converges to a point  $c$ ,

$$\lim_{n \rightarrow +\infty} d(y, x_n) = d(y, c) \quad \text{for all } y \in X,$$

and of a Cauchy sequence,

$$0 \geq \lim_{m, n \rightarrow +\infty} d(x_n, x_m).$$

Of course, for metric spaces, we know very well that every convergent sequence is Cauchy, but that the converse need not hold; and we also know that a convergence point is unique. So we now see how this is a particular instance of general  $Q$ -enriched category theory.

## Distributors and Cauchy-completeness

For metric spaces, we say that  $(X, d)$  is *complete* if every Cauchy sequence converges. For general  $Q$ -categories, the situation is slightly more involved. To make our point, we must recall Bénabou's [3] notion of *distributor*.

A  $Q$ -distributor  $\Phi: \mathbb{A} \dashv\vdash \mathbb{B}$  between  $Q$ -categories is a function

$$\Phi: \mathbb{B}_0 \times \mathbb{A}_0 \rightarrow Q: (y, x) \mapsto \Phi(y, x)$$

such that

$$\mathbb{B}(y', y) \circ \Phi(y, x) \circ \Phi(x, x') \leq \Phi(y', x')$$

holds for all  $x, x' \in \mathbb{A}_0$  and  $y, y' \in \mathbb{B}_0$ . Two such distributors, say  $\Phi: \mathbb{A} \dashv\vdash \mathbb{B}$  and  $\Psi: \mathbb{B} \dashv\vdash \mathbb{C}$ , compose with a “matrix-product”:

$$(\Psi \circ \Phi): \mathbb{C}_0 \times \mathbb{A}_0 \mapsto Q: (z, x) \mapsto \bigvee_{y \in \mathbb{B}_0} \Psi(z, y) \circ \Phi(y, x).$$

It is not difficult to verify that, with this composition law,  $Q$ -categories and  $Q$ -distributors are the objects and arrows of a category, that we shall write as  $\text{Dist}(Q)$ . Better still, this category is locally ordered: for two parallel distributors, say  $\Phi: \mathbb{A} \dashv\vdash \mathbb{B}$  and  $\Phi': \mathbb{A} \dashv\vdash \mathbb{B}$ , we simply define  $\Phi \leq \Phi'$  whenever this inequality holds element-wise (viewing  $\Phi$  and  $\Phi'$  as functions into the complete lattice  $Q$ ). As the local order of  $Q$ -distributors is compatible with their composition, we find here that  $\text{Dist}(Q)$  is a 2-category.

Applying general 2-categorical algebra to the particular case of  $\text{Dist}(Q)$ , we may now say that two  $Q$ -distributors  $\Phi: \mathbb{A} \dashv\vdash \mathbb{B}$  and  $\Psi: \mathbb{B} \dashv\vdash \mathbb{A}$  are (*left/right adjoint*), denoted  $\Phi \dashv \Psi$ , if

$$\mathbb{A} \leq \Psi \circ \Phi \quad \text{and} \quad \Phi \circ \Psi \leq \mathbb{B}.$$

It is now a matter of fact that the *presheaves* introduced previously, are particular distributors. Indeed, writing  $\mathbb{1}$  for the  $Q$ -category defined by  $\mathbb{1}_0 = \{*\}$  and  $\mathbb{1}(*, *) = 1$ ,

there is an exact correspondence between (contravariant) presheaves on a  $Q$ -category  $\mathbb{C}$  on the one hand, and  $Q$ -distributors  $\mathbb{1} \dashv\!\!\dashv \mathbb{C}$  on the other. That is to say, presheaves can be considered as *particular arrows* in the 2-category  $\text{Dist}(Q)$ .

With this insight, we follow Lawvere and say that a (contravariant) presheaf  $\phi: \mathbb{C}_0 \rightarrow Q$  is a *Cauchy presheaf* if, viewed as a distributor  $\phi: \mathbb{1} \dashv\!\!\dashv \mathbb{C}$ , it is a left adjoint in  $\text{Dist}(Q)$ . An important observation (made in [23]) is then that a sequence  $\underline{x} = (x_n)_{n \in \mathbb{N}}$  of elements in  $\mathbb{C}_0$  is Cauchy (as per the definition recalled earlier on) if and only if the distributor

$$\phi_{\underline{x}}: \mathbb{1} \dashv\!\!\dashv \mathbb{C} \text{ with elements } \phi_{\underline{x}}(y) = \bigvee_{N \in \mathbb{N}} \bigwedge_{n \geq N} \mathbb{C}(y, x_n)$$

(i.e. the distributor that corresponds with the presheaf introduced earlier on) is left adjoint to the distributor

$$\psi_{\underline{x}}: \mathbb{C} \dashv\!\!\dashv \mathbb{1} \text{ with elements } \psi_{\underline{x}}(y) = \bigvee_{N \in \mathbb{N}} \bigwedge_{n \geq N} \mathbb{C}(x_n, y).$$

Finally, a  $Q$ -category  $\mathbb{C}$  is said to be *Cauchy-complete* if any Cauchy presheaf on  $\mathbb{C}$  is representable. It follows in particular that, in a (categorically) Cauchy-complete  $Q$ -category, every Cauchy sequence is representable.

### Picard iteration

Let  $\mathbb{C}$  be a  $Q$ -category, and consider the iterated images of a function  $f: \mathbb{C}_0 \rightarrow \mathbb{C}_0$  on some  $x \in \mathbb{C}_0$ : this produces a sequence  $(f^n(x))_{n \in \mathbb{N}}$ . This sequence determines two distributors (as explained above), that we shall write as

$$\phi_{x,f}: \mathbb{1} \rightarrow \mathbb{C} \quad \text{and} \quad \psi_{x,f}: \mathbb{C} \dashv\!\!\dashv \mathbb{1}.$$

These distributors are (left/right) adjoint if and only if the sequence  $(f^n(x))_{n \in \mathbb{N}}$  is Cauchy (in the appropriate categorical sense). Thus, if the category  $\mathbb{C}$  is Cauchy complete, then (the presheaf corresponding with)  $\phi_{x,f}$  is representable. We can then prove:

**Proposition 2.3** *Let  $Q$  be any quantale,  $\mathbb{C}$  any Cauchy complete  $Q$ -category, and  $f: \mathbb{C} \rightarrow \mathbb{C}$  any  $Q$ -functor, that is, a map  $f: \mathbb{C}_0 \rightarrow \mathbb{C}_0$  such that  $\mathbb{C}(fx, fy) \geq \mathbb{C}(x, y)$ . Suppose that there exists an  $x \in \mathbb{C}_0$  for which the distributors  $\phi_{x,f}: \mathbb{1} \rightarrow \mathbb{C}$*

and  $\psi_{x,f}: \mathbb{C} \dashv \Rightarrow \mathbf{1}$  are (left/right) adjoint. Then  $f$  has a fixpoint, namely,  $f(c) \cong c$  where  $c \in \mathbb{C}_0$  represents  $\phi_{x,f}$ .

This really says how the classical argument – involving the Picard iteration – can be suitably adapted to general  $Q$ -categories: we find a fixpoint for  $f$  by “convergence” of the Cauchy sequence  $x, f(x), f(f(x)), \dots$ , i.e. by representability of the Cauchy presheaf  $\phi_{x,f}$ .

However, we still need to find conditions that make a function  $f: \mathbb{C}_0 \rightarrow \mathbb{C}_0$  produce a Cauchy sequence  $x, f(x), f(f(x)), \dots$  in the first place! In our attempts to generalize Banach’s, and Boyd and Wong’s, Fixpoint Theorems, we noticed that we needed extra conditions on the underlying quantale  $Q$  and the control function  $\varphi$ —that we shall explain next.

## Continuous lattices

The theory of *continuous lattices* is part of Domain Theory; the standard reference on the subject is [19]. In this short Introduction we cannot do full justice to the complexity of this subject and its applications in theoretical computer science. Instead, we shall only recall some of the key definitions so that the reader can appreciate their algebraic character. Throughout we assume that  $L$  is a *complete lattice*.

A subset  $D \subseteq L$  is directed if it is non-empty and, for any  $x, y \in D$  there exists a  $z \in D$  such that  $x \vee y \subseteq z$ . For two elements  $a, b \in L$  we write  $a \ll b$ , and we say that  $a$  is *way below*  $b$ , if, for every directed subset  $D \subseteq L$ ,  $b \leq \bigvee D$  implies the existence of a  $d \in D$  such that  $a \leq d$ . Note that the subset  $\{u \in L \mid u \ll a\}$  is itself directed.

We write  $a \lll b$ , and say that  $a$  is *totally below*  $b$ , if, for every subset  $S \subseteq L$ ,  $b \leq \bigvee S$  implies the existence of a  $s \in S$  such that  $a \leq s$ . It is easy to see that  $a \lll b$  implies  $a \ll b$ , which in turn implies  $a \leq b$ .

The complete lattice  $L$  is said to be

— *meet-continuous* if, for every  $a \in L$  and every directed  $D \subseteq L$ ,

$$a \wedge (\bigvee D) = \bigvee_{d \in D} a \wedge d,$$

— *continuous* if, for every  $a \in L$ ,

$$a = \bigvee \{u \in L \mid u \lll a\},$$



— *totally continuous* if, for every  $a \in L$ ,

$$a = \bigvee \{u \in L \mid u \lll a\}.$$

A totally continuous lattice is also continuous, and a continuous lattice is also meet-continuous.

Suppose now that  $L$  and  $M$  are both complete lattices, then a function  $f: L \rightarrow M$  is said to be *lower-semicontinuous* if

$$f(\bigvee D) = \bigvee_{d \in D} f(d)$$

for all directed  $D \subseteq L$ . This is, in fact, equivalent to the (in analysis perhaps more familiar) condition that

$$f(\bigvee_{j \in J} \bigwedge_{j' \geq j} x_j) \leq \bigvee_{j \in J} \bigwedge_{j' \geq j} f(x_j).$$

for every net  $(x_j)_{j \in J}$  in  $L$ . If  $L$  and  $M$  are continuous lattices, then this is further equivalent to

$$f(x) = \bigvee \{f(y) \mid y \ll x\}$$

for all  $x \in L$ . By the way, such an  $f: L \rightarrow M$  is always monotone.

Of course, any sequence  $(x_i)_{i \in \mathbb{N}}$  in  $L$  is a net, but the converse is not true. It therefore it makes sense to define the following weaker variant of lower-semicontinuity:

**Definition 2.4** A function  $f: L \rightarrow M$  is *sequentially lower-semicontinuous* if

$$f(\bigvee_{N \in \mathbb{N}} \bigwedge_{n \geq N} x_n) \leq \bigvee_{N \in \mathbb{N}} \bigwedge_{n \geq N} \varphi(x_n)$$

for any sequence  $(x_n)_{n \in \mathbb{N}}$  in  $L$ .

Such an  $f$  is always monotone.

We now have everything ready to come to our Fixpoint Theorem.

### Fixpoint theorem for $Q$ -categories

Lawvere's quantale of extended positive real numbers,  $([0, \infty], \wedge, +, 0)$  has many strong properties. For instance, because its underlying complete lattice is linear, it

is totally continuous, hence also (meet-)continuous, localic, etc. Also the monoid structure is quite particular: it is commutative, integral (meaning that the unit is the greatest element), cancellable, etc. When proving results for metric spaces, one often relies on one or the other property of the (extended positive) real numbers.

Of course, such strong properties certainly do not hold for quantales in general. Therefore, when generalizing Banach's and Boyd and Wong's Fixpoint Theorems (or any other results for metric spaces, for that matter) to  $Q$ -categories, one has to single out those (algebraic) properties of  $Q$  that are relevant for the proof. To say close to the original results for metric spaces, we set out to work with Cauchy-complete  $Q$ -categories, and aimed at formulating a notion of contraction that produces a Cauchy sequence by Picard iteration. We were thus led to the following:

**Definition 2.5** *If  $\varphi: Q \rightarrow Q$  and  $f: \mathbb{C}_0 \rightarrow \mathbb{C}_0$  are maps such that*

1.  $\varphi(t) \geq t$  for all  $t \in Q$ ,
2. if  $\varphi(t) = t$  then  $t = 0$  or  $1 \leq t$ ,
3.  $\mathbb{C}(fx, fy) \geq \varphi(\mathbb{C}(x, y))$  for all  $x, y \in \mathbb{C}_0$ ,

*then we say that  $f$  is a  $\varphi$ -contraction, and we say that  $\varphi$  is a control function for  $f$ .*

Any such contraction is easily seen to be a  $Q$ -functor, and therefore Proposition 2.3 readily applies. Better still, there are simple conditions on a  $Q$ -category  $\mathbb{C}$  for a  $\varphi$ -contraction to have a *unique* fixpoint:

**Proposition 2.6** *Let  $Q$  be any quantale, and  $f: \mathbb{C} \rightarrow \mathbb{C}$  any  $\varphi$ -contraction. If  $\mathbb{C}(x, y) \neq 0$  for all  $x, y \in \mathbb{C}_0$ , then any two fixpoints of  $f$  are isomorphic. If  $\mathbb{C}(x, y) = \mathbb{C}(y, x)$  for all  $x, y \in \mathbb{C}_0$ , then any two fixpoints of  $f$  are either isomorphic or in different summands of  $\mathbb{C}$ .*

Our main result, though, is the following statement of explicit sufficient conditions on a  $\varphi$ -contraction, for it to produce a Cauchy sequence by Picard iteration:

**Proposition 2.7** *Let  $Q$  be a quantale whose underlying lattice is continuous and let  $f: \mathbb{C} \rightarrow \mathbb{C}$  be a  $\varphi$ -contraction on a  $Q$ -category for which the control function  $\varphi: Q \rightarrow Q$  is sequentially lower-semicontinuous. For any  $x \in \mathbb{C}_0$  such that  $\mathbb{C}(x, fx) \neq 0 \neq \mathbb{C}(fx, x)$ , the sequence  $x, f(x), f(f(x)), \dots$  is Cauchy.*

The full proof is elaborate and technical, yet let us give a quick sketch to explain where the hypotheses are used.

Suppose that the sequence  $x, f(x), f(f(x)), \dots$  is *not* Cauchy, i.e. that

$$1 \not\leq \bigvee_{N \in \mathbb{N}} \bigwedge_{m \geq N} \bigwedge_{n \geq N} \mathbb{C}(f^m(x), f^n(x)).$$

By continuity of  $Q$  we know that  $1 = \bigvee \{ \epsilon \in Q \mid \epsilon \ll 1 \}$ , so the above is equivalent to the existence of an  $\epsilon \ll 1$  such that  $\epsilon \not\leq \bigvee_{N \in \mathbb{N}} \bigwedge_{m, n \geq N} \mathbb{C}(f^m x, f^n x)$ . Thus  $0 \neq \epsilon \ll 1$  is such that

for all  $k \in \mathbb{N}$  there exist  $m_k, n_k \geq k$  such that  $\epsilon \not\leq \mathbb{C}(f^{m_k} x, f^{n_k} x)$ .

A “clever choice” of such indices  $m_k, n_k \geq k$  can be made, say

$$d_k := \mathbb{C}(f^{m_k} x, f^{n_k} x),$$

so that, using the hypothesis that  $\mathbb{C}(f x, x) \neq 0 \neq \mathbb{C}(x, f x)$  together with lower-semicontinuity of  $\varphi$  and (meet-)continuity of  $Q$ , the sequence  $(d_k)_{k \in \mathbb{N}}$  in  $Q$  satisfies

$$\varphi\left(\bigvee_{N \in \mathbb{N}} \bigwedge_{k \geq N} d_k\right) = \bigvee_{N \in \mathbb{N}} \bigwedge_{k \geq N} d_k \neq 0.$$

From our Definition 2.5 it then follows that  $1 \leq \bigvee_{N \in \mathbb{N}} \bigwedge_{k \geq N} d_k$ . As this supremum is obviously directed, the fact that  $\epsilon \ll 1$  in the continuous lattice  $Q$  implies that  $\epsilon \leq \bigwedge_{k \geq N_0} d_k$  for some  $N_0$ . This is in contradiction with our earlier observation that  $\epsilon \not\leq d_k$  for all  $k$ !

Putting the above Propositions 2.3, 2.6 and 2.7 together, we can now summarize:

**Theorem 2.8** *Let  $Q$  be quantale whose underlying lattice is continuous, and suppose that  $f: \mathbb{C} \rightarrow \mathbb{C}$  is a  $\varphi$ -contraction on a Cauchy-complete  $Q$ -category for which the control function  $\varphi: Q \rightarrow Q$  is sequentially lower-semicontinuous. If there exists an  $x \in \mathbb{C}_0$  such that  $\mathbb{C}(x, f x) \neq 0 \neq \mathbb{C}(f x, x)$  then  $f$  has a fixpoint. If  $\mathbb{C}$  has no non-zero homs, then this fixpoint is (essentially) unique. If  $\mathbb{C}$  is symmetric, then any two fixpoints are either isomorphic or lie in different summands.*

The idea for the proof is thus – precisely as for metric spaces! – that the contractivity of  $f$  forces the sequence  $x, f(x), f(f(x)), \dots$  to be Cauchy (in the appropriate sense for  $Q$ -categories), and that moreover it “converges” to a fixpoint of  $f$ .

## Examples

Any linear complete lattice is totally continuous, and therefore also continuous. This opens the way for the “classical” application of Theorem 2.8: for the quantale  $([0, \infty], \wedge, +, 0)$ , we recover the Banach Fixpoint Theorem by regarding the (obviously upper-semicontinuous) control function

$$\varphi: [0, \infty] \rightarrow [0, \infty]: t \mapsto k \cdot t$$

for some  $0 < k < 1$ . Thus, we may say in retrospect that the Banach Fixpoint Theorem holds *because* the quantale of positive real numbers is continuous and the control function  $\varphi(t) = k \cdot t$  is upper-semicontinuous.

A careful analysis of the detailed proof of Theorem 2.8 shows that  $\varphi$  need not be defined on the whole of  $Q$ , but that it suffices to have it on the “closure” (in an appropriate sense) of the image of  $\mathbb{C}: \mathbb{C}_0 \times \mathbb{C}_0 \rightarrow Q$ . So we recover also the Boyd and Wong Fixpoint Theorem for metric spaces.

A *left-continuous  $t$ -norm* is a commutative and integral quantale on the unit interval  $Q = ([0, 1], \vee, *, 1)$ . Because  $[0, 1]$  is a complete linear order, this quantale again has a continuous underlying lattice. These quantales are used in many-valued (or “fuzzy”) logic:  $Q$ -categories are called “fuzzy orders” [37]. Spelling out Theorem 2.8 produces a fixpoint theorem for such fuzzy orders.

The tensor product (in the category  $\text{Sup}$  of complete lattices and supremum-preserving maps) of continuous lattices is again a continuous lattice. This holds in particular for the complete lattice

$$\Delta = \{f: [0, \infty] \rightarrow [0, 1] \mid \forall t: f(t) = \bigvee_{s < t} f(s)\}$$

(with pointwise order) because it is exactly the tensor product of  $[0, \infty]$  and  $[0, 1]$  (see [21]). Suppose that a left-continuous  $t$ -norm is given, say  $([0, 1], \vee, *, 1)$ , then  $\Delta$  too is a quantale, for the binary operation

$$(f * g)(t) = \bigvee_{r+s=t} f(r) * g(s)$$

and with unit element

$$e(t) = \begin{cases} 0 & \text{if } t = 0, \\ 1 & \text{else} \end{cases}$$

A category enriched in  $\Delta$  is, exactly, a (generalized) *probabilistic metric space* [23], so application of Theorem 2.8 produces a fixpoint theorem for such spaces (and there are indeed control functions  $\varphi: \Delta \rightarrow \Delta$  that meet the conditions in the Theorem).

### 3 References

- [1] K. Alaoui Ismaili, D. E. Dobbs and N. Mahdou, *Commutative rings and modules that are  $Nil_*$ -coherent or special  $Nil_*$ -coherent*, J. Algebra Appl. **16** (2017), 1750187.
- [2] S. Banach, *Sur les operations dans les ensembles abstraits et leur application aux equations integrales*, Fund. Math. **3** (1922), 133—181.
- [3] J. Bénabou, *Les distributeurs*, Rapport no. 33 des Séminaires de Mathématique Pure, Université Catholique de Louvain, Louvain-la-Neuve (1973).
- [4] D. Bennis,  *$n$ - $\mathcal{X}$ -Coherent rings*, Int. Electron. J. Algebra **7** (2010), 128–139.
- [5] D. W. Boyd, and J.S.W. Wong, *On nonlinear contractions*, Proceedings of the American Mathematical Society **20** (1969), 458–464.
- [6] L. E. J. Brouwer, *Über Abbildung von Mannigfaltigkeiten*, Math. Ann. **71** (1911), 97—115.
- [7] V. Camillo, *Coherence for polynomial rings*, J. Algebra **132** (1990), 72–76.
- [8] H. Cartan and S. Eilenberg, *Homological Algebra*, Princeton Math. Series Princeton University Press, (1956).
- [9] S. U. Chase, *Direct product of modules*, Trans. Amer. Math. Soc. **97** (1960), 457–473.
- [10] T. J. Cheatham and D. R. Stone, *Flat and projective character modules*, Proc. Amer. Math. Soc. **81** (1981), 175–177.
- [11] D. L. Costa, *Parameterizing families of non-Noetherian rings*, Comm. Algebra **22** (1994), 3997–4011.
- [12] C. Coppola, G. Giangiacomo and P. Tiziana, *Convergence and fixed points by fuzzy orders*, Fuzzy Sets and systems **159** (2008), 1178–1190.
- [13] N. Ding, Y. Li and L. Mao, *J-coherent rings*, J. Algebra Appl. **8** (2009), 139–155.

- [14] Dobbs, Kabbaj, Mahdou, *n-Coherent rings and modules*, Dekker Lect. Notes Pure Appl. Math. 185 (1997), 269–282.
- [15] Ch. Ehresmann, *Gattungen von lokalen Strukturen*, Jahresbericht der Deutschen Mathematiker-Vereinigung 60 (1958), 49–77.
- [16] E. E. Enochs and O. M. G. Jenda, *Relative homological algebra*, Walter de Gruyter, Berlin-New York (2000).
- [17] E. E. Enochs and O. M. G. Jenda, *Gorenstein injective and projective modules*, Math. Z. **220** (1995), 611–633.
- [18] Z. Gao and F. Wang, *Coherent rings and Gorenstein FP-injective modules*, Comm. Algebra **40** (2012), 1669–1679.
- [19] G. Gierz, K. H. Hofmann, K. Keimel, J. D. Lawson, M. W. Mislove, D. S. Scott, *A Compendium of Continuous Lattices*, Springer-Verlag, Berlin, New York, (1980).
- [20] S. Glaz, *Commutative coherent rings*, Lecture Notes in Math., Springer-Verlag, Berlin (1989).
- [21] J. Gutiérrez García, U. Höhle, T. Kubiak, *Tensor products of complete lattices and their application in constructing quantales*, Fuzzy Sets Syst. 313 (2017), 43–60.
- [22] M. E. Harris, *Some results on coherent rings*, Proc. Amer. Math. Soc. **17** (1966), 474–479.
- [23] D. Hofmann and C. Reis, *Probabilistic metric spaces as enriched categories*, Fuzzy Sets and Systems **210** (2013), 1–21.
- [24] C. Huneke, *Hyman Bass and ubiquity: Gorenstein rings*, Algebra, k-theory, groups, and education (1999), 55–78.
- [25] F. W. Lawvere, *Metric spaces, generalized logic and closed categories*, Rendiconti del Seminario Matematico e Fisico di Milano, XLIII (1973), 135–166. Also in: Reprints in Theory and Applications of Categories, **1** (2002), 1–37.
- [26] L. Mao, *Min-Flat Modules and Min-Coherent Rings*, Comm. Algebra **35** (2007), 635–650.
- [27] L. Mao,  *$\Pi$ -Coherent dimension and  $\Pi$ -Coherent rings*, J. Korean Math. Soc. **44** (2007), 719–731.

- [28] W. K. Nicholson and M. F. Yousif, *FP-RINGS*, Communications in Algebra, (29), No.1, (2001), 415–425.
- [29] W. K. Nicholson and M. F. Yousif, *Quasi-frobenius rings*, Cambridge University Press, No.158 (2003).
- [30] C. J. Mulvey, *Suppl. Rend. Circ. Mat. Palermo II* **12** (1986), 99–104.
- [31] L. Ouyng and Y. Xiang, *Nil<sub>\*</sub>-Coherent rings*, Bull. Korean Math. Soc. **51** (2014), 579–594.
- [32] J. Schauder, *Der Fixpunktsatz in Funktionalräumen*, Studia Math. **2** (1930), 171—180.
- [33] J.P. Serre, *Faisceaux algébriques cohérents*, Annals of Mathematics **61** (1955), 197–278.
- [34] B. Stenström, *Coherent rings and FP-injective modules*, J. London Math. Soc. **2** (1970), 323–329.
- [35] B. Stenström, *Rings of Quotients*, Springer-Verlag, Berlin, Heidelberg, New York (1975).
- [36] I. Stubbe, *Categorical structures enriched in a quantaloid: categories, distributors and functors*, Theory Appl. Categ. **14** (2005), 1–45.
- [37] I. Stubbe, *An introduction to quantaloid-enriched categories*, Fuzzy Sets and Systems **256** (2014), 95–116.
- [38] X. Y. Yang and Z. K. Liu, *FP-gr-injective modules*, Math. J. Okayama Univ. **53** (2011), 83–100.
- [39] T. Zhao, Z. Gao and Z. Huang, *Relative FP-gr-injective and gr-flat modules*, Int. Algebra and Computation **28** (2018), 959–977.

**Articles sélectionnés**  
**Selected articles**





## Relative coherent modules

Mostafa Amini

*Department of Mathematics, Payame Noor University, Tehran, Iran.  
amini.pnu1356@gmail.com*

Arij Benkhadra<sup>a</sup>, Driss Bennis<sup>b</sup>

*Faculty of Sciences,  
Mohammed V University in Rabat, Rabat, Morocco.*

*a. benkhadra.arij@gmail.com*

*b. driss.bennis@um5.ac.ma; driss\_bennis@hotmail.com*

Mohammed El Hajoui

*Faculté Polydisciplinaire de Taroudant, Université Ibn Zohr, Agadir, Maroc.  
hajoui4@yahoo.fr*

Received (Day Month Year)

Revised (Day Month Year)

Several authors have introduced various type of coherent-like rings and proved analogous results on these rings. It appears that all these relative coherent rings and all the used techniques can be unified. In [2], several coherent-like rings are unified. In this manuscript we continue this work and we introduce coherent-like module which also emphasizes our point of view by unifying the existed relative coherent concepts. Several classical results are generalized and some new results are given.

*Keywords:*  $n$ - $\mathcal{X}$ -coherent modules;  $n$ - $\mathcal{X}$ -coherent rings.

AMS Subject Classification: 16D80, 16E05, 16E30, 16E65, 16P70

### 1. Introduction

Throughout this paper  $R$  will be an associative (non necessarily commutative) ring with identity, and all modules will be unital left  $R$ -modules (unless specified otherwise). In this section, first some fundamental concepts and notations are stated. Let  $n$  be a non-negative integer and  $M$  an  $R$ -module. Then  $M$  is said to be  $n$ -presented if there is an exact sequence of  $R$ -modules  $F_n \rightarrow F_{n-1} \rightarrow \cdots \rightarrow F_1 \rightarrow F_0 \rightarrow M \rightarrow 0$ , where each  $F_i$  is a finitely generated free module. In particular, 0-presented and 1-presented modules are finitely generated and finitely presented modules, respectively.  $M$  is said to be infinitely presented if it is  $n$ -presented for every positive integer  $n$ . A ring  $R$  is called (left) *coherent*, if every finitely generated (left) ideal is finitely presented, equivalently every finitely presented  $R$ -module is

2-presented and so infinitely presented. The coherent rings were first appeared in Chase's paper [5] without being mentioned by name. The term coherent was first used by Bourbaki in [3]. Since then, coherent rings have become a vigorously active area of research. For background on coherence for commutative rings, we refer the reader to [9]. A ring  $R$  is called (left)  $n$ -coherent ring if every  $(n-1)$ -presented (left) ideal is  $n$ -presented, equivalently every  $n$ -presented  $R$ -module is  $(n+1)$ -presented. Also, it is clear that 0-coherent (resp., 1-coherent) rings are just Noetherian (resp., coherent) rings. The  $n$ -coherent rings by Costa in [6] introduced, for more details see [2, 8, 11, 16, 17]. In [8], Kabbaj et al. introduced the concept of  $n$ -coherent modules, and  $M$  is called  $n$ -coherent module if it is  $(n-1)$ -presented and every  $(n-1)$ -presented submodule of  $M$  is  $n$ -presented, the 1-coherent modules are just the coherent modules, see [3].

In this paper, we introduce the  $n$ - $\mathcal{X}$ -coherent modules. Let  $n$  be an integer,  $M$  an  $R$ -module and  $\mathcal{X}$  a class of submodules of  $M$ . Then,  $M$  is said to be  $n$ - $\mathcal{X}$ -coherent if  $\mathcal{X}_{n-1}$  is non empty and every submodule of  $\mathcal{X}_{n-1}$  is in  $\mathcal{X}_n$ , where  $\mathcal{X}_{n-1}$  and  $\mathcal{X}_n$  are two classes of  $(n-1)$ -presented modules and  $n$ -presented modules in  $\mathcal{X}$ , respectively. In particular, if  $\mathcal{X}$  is a class of  $R$ -modules and  $M = R$ , then  $R$  is said to be an  $n$ - $\mathcal{X}$ -coherent ring if every  $R$ -module of  $\mathcal{X}_n$  is in  $\mathcal{X}_{n+1}$  (see [2]). Our main aim is to show that the well-known Glaz, Smaili, Dobbles, Mahdou, Kabbaj, Chase, Greenberg and Scrivanti characterization of coherent modules and coherent rings hold true for any  $n$ - $\mathcal{X}$ -coherent module and any  $n$ - $\mathcal{X}$ -coherent ring. So, in Section 2, first we study some results of  $n$ - $\mathcal{X}$ -coherent modules on short exact sequences, factor modules, homomorphism of  $R$ -modules and direct sum of  $R$ -modules. Also in this section, several results on transfer of  $n$ - $\mathcal{X}$ -coherence are developed and then in end, another characterizations of  $n$ - $\mathcal{X}$ -coherence using the notion of thickness are given (see Theorems 2.1, 2.2, 2.3, 2.5, 2.6, 2.7, 2.8 and Proposition 2.2). Finally, in Section 3, with considering pullback diagram, some characterizations of  $n$ - $\mathcal{X}$ -coherent rings are studied (see Theorems 3.1 and 3.2).

## 2. $n$ - $\mathcal{X}$ -coherent modules

Among the many generalizations of the notion of a coherent ring, we recall the following one:  $R$  is said to be (left)  $J$ -coherent, if every finitely generated (left) ideal of  $R$  contained in  $Rad(R)$ , the radical of  $R$ , is finitely presented [7]. Also,  $R$  is said to be (left)  $Nil_*$ -coherent, if every finitely generated (left) ideal of  $R$  contained in  $Nil(R)$ , the nilradical of  $R$ , is finitely presented [13]. Here, we introduce the following definition of coherence which generalizes all the definitions above.

**Definition 2.1.** Let  $n$  be an integer,  $M$  an  $R$ -module and  $\mathcal{X}$  a class of submodules of  $M$ . Let  $\mathcal{X}_{n-1}$  and  $\mathcal{X}_n$  be two classes of  $(n-1)$ -presented modules and  $n$ -presented modules in  $\mathcal{X}$ , respectively. We say that  $M$  is (left)  $n$ - $\mathcal{X}$ -coherent, if  $\mathcal{X}_{n-1}$  is non empty and every module of  $\mathcal{X}_{n-1}$  is in  $\mathcal{X}_n$ .

**Example 2.1.**

- (1) If  $\mathcal{X}$  is the class of all submodules of  $M$  and  $n = 1$ , then  $M$  is  $n$ - $\mathcal{X}$ -coherent if and only if it is pseudo coherent. If, in addition,  $M$  is finitely generated then  $M$  is  $n$ - $\mathcal{X}$ -coherent if and only if it is coherent (see [12]).
- (2) If  $\mathcal{X}$  is the class of all submodules of  $M$  contained in  $\text{Nil}(R)M$  and  $n = 1$ , then  $M$  is  $n$ - $\mathcal{X}$ -coherent if and only if it is  $\text{Nil}_*$ -coherent.
- (3) Let  $R$  be a semisimple ring and let  $\mathcal{X}$  be any non empty class of submodules of an  $R$ -module  $M$ . Then,  $M$  is  $n$ - $\mathcal{X}$ -coherent for every integer  $n$ .
- (4) Let  $M$  be an  $R$ -module and let  $\mathcal{X}$  be a class of all finitely generated projective submodules of  $M$ . Then,  $M$  is  $n$ - $\mathcal{X}$ -coherent.
- (5) Let  $K$  be a field and  $E$  be a  $k$ -vector space with infinite rank. Consider  $R = K \times E$  the trivial extension of  $K$  by  $E$ . If  $\mathcal{X}$  is the class of all 2-presented  $R$ -submodules of  $M$ , then  $M$  is  $n$ - $\mathcal{X}$ -coherent, since every 2-presented  $R$ -submodule of  $M$  is projective. But, if  $\mathcal{X}^i$  is the class of all 1-presented  $R$ -submodules of any desirable  $R$ -module  $M_i$ , then there is an  $R$ -module  $M_i$  such that  $M_i$  is not 2- $\mathcal{X}^i$ -coherent, since if any  $M_i$  is 2- $\mathcal{X}^i$ -coherent, then  $R$  is regular, a contradiction (see [15]).
- (6) Let  $R_{n+1} = R_n \times M_n$  be the trivial extension, where  $R_i$  is a non-noetherian commutative ring for any  $i \geq 0$ . Consider  $M_0 = \frac{R_0}{I}$  for a finitely generated ideal  $I$  of  $R_0$ . If  $\mathcal{X}$  is the class of all  $R$ -submodules of  $M_{n+1} = \frac{R_{n+1}}{M_n}$ , then  $M_{n+1}$  is not  $(n+2)$ - $\mathcal{X}$ -coherent for every  $n \geq 0$  (see [17]).

For a morphism  $\varphi : A \rightarrow B$  and a class  $\mathcal{X}$  of submodules of  $A$ , we denote by  $\varphi(\mathcal{X})$  the class of submodules of  $B$  of the form  $\varphi(N)$  with  $N$  in  $\mathcal{X}$ . The following theorem is a generalization of [9].

**Theorem 2.1.** *Let  $0 \rightarrow M_1 \xrightarrow{h} M_2 \xrightarrow{s} M_3 \rightarrow 0$  be an exact sequence of  $R$ -modules and  $\mathcal{X}$  and  $\mathcal{Y}$  two classes of submodules of  $M_1$  and  $M_2$ , respectively. Then,*

- (1)  $M_2$  is  $n$ - $\mathcal{Y}$ -coherent, if  $M_3$  is  $n$ - $s(\mathcal{Y})$ -coherent and  $M_1$  is  $n$ -coherent.
- (2)  $M_1$  is  $n$ - $\mathcal{X}$ -coherent if  $M_2$  is  $n$ - $h(\mathcal{X})$ -coherent.
- (3)  $M_3$  is  $n$ - $s(\mathcal{Y})$ -coherent if  $M_2$  is  $n$ - $\mathcal{Y}$ -coherent and  $h(M_1)$  is  $(n-1)$ -presented in  $\mathcal{Y}$ .

**Proof.** (1) Let  $N_2$  be an  $(n-1)$ -presented submodule in  $\mathcal{Y}$ . Our aim is to prove that  $N_2$  is  $n$ -presented. For that, consider the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & K & \longrightarrow & N_2 & \longrightarrow & s(N_2) \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & M_1 & \longrightarrow & M_2 & \longrightarrow & M_3 \longrightarrow 0
 \end{array}$$

In view of the exactness of the first row, it suffices to show that both  $K$  and  $s(N_2)$

are  $n$ -presented. Since  $N_2$  is an  $(n-1)$ -presented module in  $\mathcal{Y}$ ,  $s(N_2)$  is an  $(n-1)$ -presented module in  $s(\mathcal{Y})$ , so  $s(N_2)$  is  $n$ -presented. By the exactness of the top row,  $K$  is an  $(n-1)$ -presented submodule of  $M_1$  which is  $n$ -coherent. Then  $K$  is  $n$ -presented.

(2) Let  $N_1$  be an  $(n-1)$ -presented submodule in  $\mathcal{X}$ . Since  $h$  is injective,  $h(N_1) \cong N_1$  which is  $(n-1)$ -presented, so  $h(N_1)$  is an  $(n-1)$ -presented module of  $h(\mathcal{X})$ , and then by hypothesis,  $h(N_1)$  is  $n$ -presented, so is  $N_1$ .

(3) Let  $N_3$  be an  $(n-1)$ -presented submodule in  $s(\mathcal{Y})$ . Notice that  $\Im(h) = \ker(s) \subseteq s^{-1}(N_3)$ .  $M_1$  is  $n$ -presented, since  $M_2$  is  $n$ - $\mathcal{Y}$ -coherent. So, we get using the horseshoe lemma to the following diagram is commutative with exact rows and columns:

$$\begin{array}{ccccccc}
& & 0 & & 0 & & 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & K_1 & \longrightarrow & K_2 & \longrightarrow & K_3 \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & F_{n-1} & \longrightarrow & F_{n-1} \oplus F'_{n-1} & \longrightarrow & F'_{n-1} \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
& & \vdots & & \vdots & & \vdots \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & F_0 & \longrightarrow & F_0 \oplus F'_0 & \longrightarrow & F'_0 \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & M_1 & \longrightarrow & s^{-1}(N_3) & \longrightarrow & N_3 \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
& & 0 & & 0 & & 0
\end{array}$$

Where  $F_i$  and  $F'_i$  are finitely generated free modules for every  $i \in \{0, \dots, n-1\}$ . Due to the exactness of the right vertical sequence, it suffices to prove that  $K_3$  is finitely generated. For that, it is sufficient to prove that  $K_2$  is finitely generated. Since the middle vertical sequence is exact, it suffices to show that  $s^{-1}(N_3)$  is  $n$ -presented. We have that  $N_3$  is in  $s(\mathcal{Y})$ , then  $s^{-1}(N_3)$  is in  $s^{-1}(s(\mathcal{Y}))$  and so it is in  $\mathcal{Y}$ . And since  $N_3$  is  $(n-1)$ -presented, so is  $s^{-1}(N_3)$  and consequently, it is  $n$ -presented. This implies that  $K_2$  is finitely generated as desired.  $\square$

Consider a short exact sequence as in Theorem 2.1. If for some class  $\mathcal{Y}$  of submodules of  $M_2$ , we have that  $s(\mathcal{Y}) = 0$ , then  $M_2$  is  $n$ - $\mathcal{Y}$ -coherent if  $M_1$  is  $n$ -coherent. For example, for  $I = \text{ann}(M_3)$  and  $\mathcal{Y}$  the class of submodules of  $IM_2$ , it is evident that  $M_2$  is  $n$ - $\mathcal{Y}$ -coherent if and only if  $IM_2$  is  $n$ -coherent. This can be seen just by the definition of  $n$ -coherence and also if we take in Theorem 2.1, the short exact sequence  $0 \longrightarrow IM_2 \xrightarrow{h} M_2 \xrightarrow{s} \frac{M_2}{IM_2} \longrightarrow 0$ .

In what follows, for a submodule  $N$  of an  $R$ -module  $M$  and a class  $\mathcal{X}$  of submodules of  $M$ , we will denote by  $\frac{\mathcal{X}}{N}$  the class of quotient modules  $\frac{L}{N}$ , where  $L \in \mathcal{X}$

and contains  $N$ . The following corollary generalizes [1] .

**Corollary 2.1.** *Let  $M$  be an  $R$ -module,  $N$  a submodule of  $M$  and  $\mathcal{X}$  a class of submodules of  $M$ . Then, the following assertions hold:*

- (1) *If  $M$  is  $n$ - $\mathcal{X}$ -coherent,  $N$  is  $(n-1)$ -presented and each module in  $\mathcal{X}$  contains  $N$ , then  $\frac{M}{N}$  is  $n$ - $\frac{\mathcal{X}}{N}$ -coherent.*
- (2) *Assume that  $\frac{\mathcal{X}}{N}$  is non empty. Then  $M$  is  $n$ - $\mathcal{X}$ -coherent if  $\frac{M}{N}$  is  $n$ - $\frac{\mathcal{X}}{N}$ -coherent and  $N$  is  $n$ -coherent.*

**Proof.** Let  $\pi : M \rightarrow \frac{M}{N}$  be the canonical surjection. It is evident that, if  $\mathcal{X}$  is the class of submodules  $K$  of  $M$  containing  $N$ , then  $\frac{\mathcal{X}}{N}$  is the class of quotient modules  $\frac{K}{N}$  with  $K$  in  $\mathcal{X}$ . Applying Theorem 2.1 to exact sequence:  $0 \rightarrow N \rightarrow M \rightarrow \frac{M}{N} \rightarrow 0$ , we get the following results.  $\square$

**Lemma 2.1.** *Let  $\mathcal{X}$  and  $\mathcal{Y}$  two classes of submodules of  $M$  with  $\mathcal{X} \subseteq \mathcal{Y}$ . If  $M$  is  $n$ - $\mathcal{Y}$ -coherent, then  $M$  is  $n$ - $\mathcal{X}$ -coherent.*

For some submodule  $K$  of an  $R$ -module  $M$  and a class  $\mathcal{Y}$  of submodules of  $M$ , we denote by  $tr_K(\mathcal{Y})$  the class of submodules of  $K$  of the form  $K \cap Y$  with  $Y \in \mathcal{Y}$ . Also, we denote by  $f(\mathcal{Y})$  the class of submodules of the form  $f(Y)$  with  $Y \in \mathcal{Y}$ . The following proposition is a generalization of [9] .

**Proposition 2.1.** *Let  $f : M \rightarrow N$  be a homomorphism of  $R$ -modules and  $\mathcal{X}$  and  $\mathcal{Y}$  two classes of submodules of  $M$  and  $N$ , respectively. Then, The following assertions hold:*

- (1) *If  $M$  is  $n$ - $\mathcal{X}$ -coherent, then  $\text{Ker}(f)$  is  $n$ - $tr_{\text{Ker}(f)}(\mathcal{X})$ -coherent.*
- (2) *If  $N$  is  $n$ - $\mathcal{Y}$ -coherent, then  $\text{Im}(f)$  is  $n$ - $tr_{\text{Im}(f)}(\mathcal{Y})$ -coherent.*
- (3) *If  $M$  is  $n$ - $\mathcal{X}$ -coherent and  $\text{ker}(f)$  is an  $(n-1)$ -presented module in  $\mathcal{X}$ , then  $\text{Im}(f)$  is  $n$ - $f(\mathcal{X})$ -coherent.*
- (4) *If  $N$  is  $n$ - $\mathcal{Y}$ -coherent and  $\text{Im}(f)$  is an  $(n-1)$ -presented module in  $\mathcal{Y}$ , then  $\text{Coker}(f)$  is  $n$ - $\frac{\mathcal{Y}}{\text{Im}(f)}$ -coherent.*

**Proof.** The two first assertions follow by applying (2) of Theorem 2.1 and Lemma 2.1 to the following exact sequences:

$$0 \longrightarrow \text{Ker}(f) \longrightarrow M \longrightarrow \frac{M}{\text{Ker}(f)} \longrightarrow 0$$

and

$$0 \longrightarrow \text{Im}(f) \xrightarrow{i} N \longrightarrow \text{Coker}(i) \longrightarrow 0.$$

The two last assertions follow by applying (3) of Theorem 2.1 to the following exact sequences:

$$0 \longrightarrow \text{Ker}(f) \longrightarrow M \longrightarrow \text{Im}(f) \longrightarrow 0$$

and

$$0 \longrightarrow \text{Im}(f) \longrightarrow N \longrightarrow \text{Coker}(f) \longrightarrow 0. \quad \square$$

Now, we set the result concerning the coherence of the direct sum of modules. It generalizes [9]. Let  $(M_i)_{i \in I}$  be a family of  $R$ -modules and  $\mathcal{X}^i$  a class of submodules of  $M_i$ , for each  $i \in I$ . We will denote by  $\bigoplus_{i \in I} \mathcal{X}^i$  the class of modules of the form  $\bigoplus_{i \in I} N_i$  with each  $N_i$  is in  $\mathcal{X}^i$ .

**Theorem 2.2.** *Let  $(M_i)_{i \in \{1, \dots, m\}}$  be a finite family of  $R$ -modules and  $\mathcal{X}^i$  a class of submodules of  $M_i$  for every  $i = 1, \dots, m$ . Then,  $\bigoplus_{i=1}^m M_i$  is an  $n$ - $(\bigoplus_{i=1}^m \mathcal{X}^i)$ -coherent  $R$ -module if and only if  $M_i$  is  $n$ - $\mathcal{X}_i$ -coherent for all  $i = 1, \dots, m$ .*

**Proof.** The “only if” part follows easily using Lemma 2.1, (2) of Theorem 2.1 and the following exact sequence:  $0 \rightarrow M_i \rightarrow \bigoplus_{j=1}^m M_j \rightarrow \frac{\bigoplus_{j=1}^m M_j}{M_i} \rightarrow 0$ . For the “if” part, consider an  $(n-1)$ -presented submodule  $N$  of  $\bigoplus_{i=1}^m M_i$  and the canonical projection  $\pi_i : \bigoplus_{i=1}^m M_i \rightarrow M_i$ . Thus,  $\pi_i(N) \in \mathcal{X}_{n-1}^i$  for all  $i = 1, \dots, m$ . Then,  $\pi_i(N)$  is  $n$ -presented for all  $i = 1, \dots, m$ . We have the following exact sequence:

$$0 \longrightarrow \bigoplus_{j=1, j \neq i}^m \pi_j(N) \longrightarrow N \longrightarrow \pi_i(N) \longrightarrow 0.$$

Consequently by [17],  $N$  is  $n$ -presented, which completes the proof.  $\square$

The following result is a generalization of [9].

**Corollary 2.2.** *Let  $R$  be a commutative ring,  $M$  a finitely generated  $R$ -module and  $N$  an  $n$ - $\mathcal{X}$ -coherent module for some class  $\mathcal{X}$  of submodules of  $N$ . Then,  $\text{Hom}_R(M, N)$  is  $n$ - $\mathcal{Y}$ -coherent, where  $\mathcal{Y}$  is the class of submodules of  $\text{Hom}_R(M, N)$  which are isomorphic to a module in  $\text{tr}_A(\sum_{i=1}^k \mathcal{X})$ .*

**Proof.** Since  $M$  is finitely generated, there is an exact sequence of  $R$ -modules  $0 \rightarrow K \rightarrow R^k \rightarrow M \rightarrow 0$  for some non-negative integer  $k$ . As  $\text{Hom}_R(-, N)$  is a left-exact functor,  $\text{Hom}_R(M, N) \cong A$ , where  $A$  is a submodule of  $\text{Hom}_R(R^k, N) \cong N^k$ . By Theorem 2.2,  $N^k$  is  $n$ - $\bigoplus_{i=1}^k \mathcal{X}$ -coherent, and so by Proposition 2.1,  $A$  is  $n$ - $\text{tr}_A(\sum_{i=1}^k \mathcal{X})$ -coherent, which completes the proof.  $\square$

We finish this section with some transfer results. First, we present a generalization of [8].

**Theorem 2.3.** *Let  $I$  be an  $(n-1)$ -presented two-sided ideal of a ring  $R$ ,  $M$  an  $\frac{R}{I}$ -module and  $\mathcal{X}$  a class of submodules of  $M$ . Then,  $M$  is  $n$ - $\mathcal{X}$ -coherent as an  $R$ -module if and only if  $M$  is  $n$ - $\mathcal{X}$ -coherent as an  $\frac{R}{I}$ -module.*

For the proof, we need the following lemma.

**Lemma 2.2** ([8], lemmas 2.6 and 2.7). *Let  $R \rightarrow S$  be a ring homomorphism and  $m$  a non-negative integer. Then, the following assertions hold:*

- (1) *If  $S$  is  $m$ -presented as an  $R$ -module, then every  $m$ -presented  $S$ -module is an  $m$ -presented  $R$ -module.*
- (2) *If  $S$  is  $(m - 1)$ -presented as an  $R$ -module and  $M$  a  $S$ -module, then  $M$  is  $m$ -presented  $S$ -module if  $M$  is an  $m$ -presented  $R$ -module.*

*Proof of Theorem 2.3.* Let  $R \rightarrow \frac{R}{I}$  be the canonical homomorphism and  $N$  a submodule of  $M$ . Using Lemma 2.2, we get the following equivalences:  $N$  is an  $(n - 1)$ -presented  $\frac{R}{I}$ -submodule of  $\mathcal{X}$  if and only if it is  $(n - 1)$ -presented  $R$ -submodule of  $\mathcal{X}$ . Consequently,  $M$  is an  $n$ - $\mathcal{X}$ -coherent as an  $\frac{R}{I}$ -module if and only if it is  $n$ - $\mathcal{X}$ -coherent as an  $R$ -module.  $\square$

Assume that  $S \geq R$  is a unitary ring extension. Then, the ring  $S$  is called right  $R$ -projective in case, for any right  $S$ -module  $M_S$  with an  $S$ -module  $N_S$ ,  $N_R \mid M_R$  implies  $N_S \mid M_S$ , where  $N \mid M$  means that  $N$  is a direct summand of  $M$ . The ring extension  $S \geq R$  is called a *finite normalizing extension* in case there is a finite subset  $\{s_1, \dots, s_n\} \subseteq S$  such that  $S = \sum_{i=1}^n s_i R$  and  $s_i R = R s_i$  for  $i = 1, \dots, n$ . A finite normalizing extension  $S \geq R$  is called an *almost excellent extension* in case  ${}_R S$  is flat,  $S_R$  is projective, and the ring  $S$  is right  $R$ -projective (see [17]).

In the following, we mainly consider the properties of  $n$ - $\mathcal{X}$ -coherent modules and  $n$ - $\mathcal{X}$ -coherent rings under an almost excellent extension of commutative rings.

**Theorem 2.4.** *Let  $S \geq R$  be an almost excellent extension,  $M$  a  $S$ -module and  $\mathcal{X}$  a class of submodules of  $M$ . Then, the following statements are equivalent:*

- (1)  *$M$  is  $n$ - $\mathcal{X}$ -coherent as an  $R$ -module;*
- (2)  *$\text{Hom}_R(S, M)$  is  $n$ - $\mathcal{Y}$ -coherent, where  $\mathcal{Y}$  is the class of submodules of  $\text{Hom}_R(S, M)$ ;*
- (3)  *$M$  is  $n$ - $\mathcal{X}$ -coherent as an  $S$ -module.*

**Proof.** (1)  $\implies$  (2)  $S$  is a finitely generated  $R$ -module. So (2) follows from Corollary 2.2.

(2)  $\implies$  (3) Assume that  $N$  is an  $(n - 1)$ -presented submodule of  $M$  in  $\mathcal{X}$ . We show that  $N$  is  $n$ -presented. By [18],  $M \cong K$ , where  $K$  is a direct summand of  $\text{Hom}_R(S, M)$ . Therefore by hypothesis and Corollary 2.1,  $K$  is  $n$ - $\text{tr}_K(\mathcal{Y})$ -coherent. So, we deduce that  $N$  is  $n$ -presented.

(3)  $\implies$  (1) By [17],  $S$  is an  $n$ -presented  $R$ -module. So, if  $N$  is an  $(n - 1)$ -presented submodule of  $R$ -module  $M$  in  $\mathcal{X}$ , then by Lemma 2.2 and (3),  $N$  is an  $n$ -presented submodule of  $R$ -module  $M$ , and hence  $M$  is  $n$ - $\mathcal{X}$ -coherent as an  $R$ -module.  $\square$

In what follows, we will denote by  $\mathcal{X}_k$ , for some class of  $R$ -modules  $\mathcal{X}$  and an integer  $k$ , the subclass of  $k$ -presented submodules of  $\mathcal{X}$  (which we assume they



exist). For an  $R$ -module  $M$  and a class  $\mathcal{X}$  of submodules of  $M$ , we will denote by  ${}^{S^\otimes}\mathcal{X}$  the class  $\{S \otimes N, \text{ where } N \text{ is a module of } \mathcal{X}\}$ .

**Theorem 2.5.** *Let  $S \geq R$  be an almost excellent extension,  $M$  a  $S$ -module and  $\mathcal{X}$  a class of submodules of  $M$ . Then, the following statements are equivalent:*

- (1)  $M$  is  $n$ - $\mathcal{X}$ -coherent as an  $R$ -module;
- (2)  $S \otimes_R M$  is  $n$ - ${}^{S^\otimes}\mathcal{X}$ -coherent;
- (3)  $M$  is  $n$ - $\mathcal{X}$ -coherent as an  $S$ -module.

**Proof.** (1)  $\implies$  (2) Assume that  $N$  is an  $(n-1)$ -presented submodule of  $S \otimes_R M$  in  ${}^{S^\otimes}\mathcal{X}$ . So, there is a submodule  $I$  in  $\mathcal{X}$  such that  $N = S \otimes_R I$ . By [17],  $I \in \mathcal{X}_{n-1}$  as an  $R$ -module, and so by (1),  $I \in \mathcal{X}_n$ . Hence by Lemma 2.2, we deduce that  $N$  is  $n$ -presented.

(2)  $\implies$  (1) Assume that  $N$  is an  $(n-1)$ -presented submodule of  $M$  in  $\mathcal{X}$ . Then by [17] and (2),  $N \in \mathcal{X}_n$ .

(1)  $\implies$  (3) and (3)  $\implies$  (1) are trivial.  $\square$

**Corollary 2.3.** *Let  $S \geq R$  be an almost excellent extension and  $\mathcal{X}$  a class of  $R$ -modules. Then,  $R$  is  $n$ - $\mathcal{X}$ -coherent if and only if  $S$  is  $n$ - ${}^{S^\otimes}\mathcal{X}$ -coherent.*

**Proof.** It is particular of Theorem 2.5.  $\square$

The next result generalizes [8] and [9].

**Theorem 2.6.** *Let  $R \rightarrow S$  be a ring homomorphism making  $S$  a faithfully flat right  $R$ -module,  $M$  an  $R$ -module and  $\mathcal{X}$  a class of submodules of  $M$ . Then*

- (1)  $M$  is  $n$ - $\mathcal{X}$ -coherent, if  $S \otimes_R M$  is an  $n$ - ${}^{S^\otimes}\mathcal{X}$ -coherent  $S$ -module.
- (2)  $M$  is  $n$ - $\mathcal{X}$ -coherent if and only if  $S \otimes_R M$  is an  $n$ - ${}^{S^\otimes}\mathcal{X}_{n-1}$ -coherent  $S$ -module.

**Proof.** (1) Let  $N$  be an  $(n-1)$ -presented module of  $\mathcal{X}$ , then  $S \otimes N$  is an  $(n-1)$ -presented module of  ${}^{S^\otimes}\mathcal{X}$  (since  $S$  is flat). Then,  $S \otimes N$  is  $n$ -presented, so is  $N$  since  $S$  is faithfully flat.

(2)( $\Leftarrow$ ) This is a direct consequence of (1).

( $\Rightarrow$ ) Assume that  $K$  is an  $(n-1)$ -presented submodule of  $S \otimes_R M$  in  ${}^{S^\otimes}\mathcal{X}_{n-1}$ . So, there is an  $(n-1)$ -presented submodule  $N$  in  $\mathcal{X}_{n-1}$  such that  $K = S \otimes_R N$ . By hypothesis,  $N$  is in  $\mathcal{X}_n$ , and so by [9],  $K$  is  $n$ -presented  $\square$

**Corollary 2.4.** *Let  $R \rightarrow S$  be a ring homomorphism making  $S$  a faithfully flat right  $R$ -module and  $\mathcal{X}$  a class of ideals of  $R$ . Then,  $R$  is an  $n$ - $\mathcal{X}$ -coherent ring, if  $S$  is an  $n$ - ${}^{S^\otimes}\mathcal{X}$ -coherent ring.*

**Proof.** It is enough to take  $M = R$ .  $\square$

**Question :** Let  $R \rightarrow S$  be a ring homomorphism making  $S$  a faithfully flat right  $R$ -module and  $\mathcal{X}$  a class of ideals of  $R$ . If  $R$  is  $n$ - $\mathcal{X}$ -coherent, then what conditions on the fibers  $R \rightarrow S$  are required in order that  $S$  is  $n$ - $S^\otimes \mathcal{X}$ -coherent?

Now, we give a generalization of the classical result due to Chase in [5] stating that  $R$  is coherent if and only if the annihilator of any element  $a$  of  $R$  is finitely generated and the intersection of any two finitely generated ideals in  $R$  is also finitely generated.

We say that a class of modules  $\mathcal{X}$  is said to be closed under finite sums if, for every finite family of modules  $\{M_i\}_{i \in I}$  in  $\mathcal{X}$ ,  $\sum_{i \in I} M_i$  is also in  $\mathcal{X}$ . A class  $\mathcal{X}$  is said to be closed under cyclic submodules if, whenever  $N$  is a cyclic submodules of a module in  $\mathcal{X}$ , it is also in  $\mathcal{X}$ .

The following theorem generalizes [5].

**Theorem 2.7.** *Let  $M$  be an  $R$ -module and let  $\mathcal{X}$  be a class of submodules of  $M$  such that  $\mathcal{X}_n$  is closed under finite sum and closed under cyclic submodules. Then,  $M$  is left 1- $\mathcal{X}$ -coherent if and only if  $(0 :_R a)$  is a finitely generated of  $M$  for any  $a \in M$  such that  $Ra$  is in  $\mathcal{X}_0$  and the intersection of any two submodules of  $M$  in  $\mathcal{X}_0$  is finitely generated.*

**Proof.** Suppose that  $M$  is 1- $\mathcal{X}$ -coherent and let  $a$  be in  $M$  such that  $Ra$  is in an element  $N$  of  $\mathcal{X}_0$ , then  $Ra \in \mathcal{X}_0$ . Then,  $Ra$  is in  $\mathcal{X}_1$ . Considering the exact sequence:  $0 \rightarrow (0 :_R a) \rightarrow R \rightarrow Ra \rightarrow 0$ , we get that  $(0 :_R a)$  is a finitely generated ideal of  $R$ .

Now, let  $N$  and  $L$  be in  $\mathcal{X}_0$ . Then,  $N + L \in \mathcal{X}_0$ . So by hypothesis,  $N + L$  is in  $\mathcal{X}_1$  and  $N \oplus L$  is finitely generated as an  $R$ -module. Via the exact sequence  $0 \rightarrow N \cap L \rightarrow N \oplus L \rightarrow N + L \rightarrow 0$ , we get that  $N \cap L$  is a finitely generated submodule of  $M$ .

Conversely, let  $N \in \mathcal{X}_0$ . Then, there exist  $a_1, \dots, a_p \in M$  such that  $N = \sum_{i=1}^p Ra_i$ . We prove by induction on  $p$  that  $N$  is 1-presented. If  $p = 1$ ,  $(0 :_R a_1)$  is finitely generated submodule of  $M$ . Hence,  $N$  is 1-presented by the exactness of the sequence  $0 \rightarrow (0 :_R a_1) \rightarrow R \rightarrow N \rightarrow 0$ . For the induction step (with  $p > 1$ ), consider the following exact sequence  $0 \rightarrow (\sum_{i=1}^{p-1} Ra_i) \cap Ra_p \rightarrow (\sum_{i=1}^{p-1} Ra_i) \oplus Ra_p \rightarrow N \rightarrow 0$ . By hypothesis on  $\mathcal{X}_0$ , we have  $Ra_p$  and  $\sum_{i=1}^{p-1} Ra_i$  are in  $\mathcal{X}_0$ , then they are in  $\mathcal{X}_1$ , thus  $(\sum_{i=1}^{p-1} Ra_i) \oplus Ra_p$  is 1-presented. Therefore,  $(\sum_{i=1}^{p-1} Ra_i) \cap Ra_p$  is a finitely generated ideal of  $M$ , and thus  $N$  is 1-presented.  $\square$

Let  $I$  be an ideal of  $R$  and  $\mathcal{X}$  be the class of ideals  $J$  of  $R$  contained in  $I$ . Then,  $R$  is 1- $\mathcal{X}$ -coherent if and only if  $I$  is quasi-coherent.

**Corollary 2.5.** *Let  $\mathcal{X}$  be a class of ideals of  $R$  such that  $\mathcal{X}_n$  is closed under finite sum and closed under cyclic submodules. Then,  $R$  is left 1- $\mathcal{X}$ -coherent if and only*

if  $(0 :_R a)$  is a finitely generated ideal of  $R$  for any  $a \in R$  such that  $Ra$  is in  $\mathcal{X}_0$  and the intersection of any two ideals in  $\mathcal{X}_0$  is finitely generated.

**Corollary 2.6.** *Let  $I$  be an ideal of  $R$ . Then,  $I$  is quasi-coherent if and only if  $(0 :_R a)$  is a finitely generated ideal of  $R$  for any  $a \in I$  and the intersection of any two left (resp., right) ideals contained in  $I$  is finitely generated.*

As an application of the previous results established in this section, we get the following result on the coherence of the amalgamated algebra along an ideal which is proved differently in [1].

**Corollary 2.7.** *Let  $R_1$  and  $R_2$  be two unitary associative rings and let  $f : R_1 \rightarrow R_2$  be a ring homomorphism. Let  $J$  be a finitely generated ideal of  $Nil(R_2)$  such  $f^{-1}(J)$  is a finitely generated ideal of  $R_1$ . Then,  $R_1 \bowtie^f J$  is 1- $Nil$ -coherent  $R_1$ -module if and only if  $R_1$  and  $f(R_1) + J$  are 1- $Nil$ -coherent.*

**Proof.** The direct implication is proved directly using corollary 2.1 and the fact that  $p_{R_2}(A \bowtie^f J) = f(A) + J$  and  $p_{R_1}(A \bowtie^f J) = A$  for any ideal  $A$  of  $R_1$ , where  $p_{R_1}$  and  $p_{R_2}$  are respectively the projection of  $R_1 \bowtie^f J$  on  $R_1$  and  $R_2$ . For the inverse, in light of theorem , it sufficient to prove that  $(0 : (a, f(a) + j))$  is a finitely generated ideal of  $R_1 \bowtie^f J$  for any  $a \in R_1$  and  $j \in J$  such that  $R(a, f(a) + j)$  is in the nil-radical of  $R_1 \bowtie^f J$  and the intersection of any two submodules of  $R_1 \bowtie^f J$  in the nilradical is finitely generated. For that, it is easy to prove that  $(0 :_{R_1} (a, f(a) + j)) = (0 :_{R_1} a) \cap (0 :_{R_1} f(a) + j)$  and for any two ideals of  $R_1$ , we have  $(N \bowtie^f J) \cap (I \bowtie^f J) = (N \cap I) \bowtie^f J$ .  $\square$

Now, we give some transfer results. We start with a generalization of [8] and [9]. Let  $(M_i, i \in \{1, \dots, p\})$  be a family of modules and  $\mathcal{X}^i$  a class of submodules of  $M_i$  for  $i \in \{1, \dots, p\}$ . We will denote by  $\prod_{i=1}^p \mathcal{X}^i$  the class of the submodules  $\prod_{i=1}^p N_i$  with each  $N_i$  is in  $\mathcal{X}^i$ .

**Theorem 2.8.** *Let  $(R_i, 1 \leq i \leq p)$  be a family of rings. Let  $(M_i, 1 \leq i \leq p)$  be a family of  $R_i$ -modules,  $p \geq 1$  an integer,  $\mathcal{X}^i$  a class of submodules of  $M_i$  for any integer  $i \in \{1, \dots, p\}$  and  $\mathcal{X} = \prod_{i=1}^p \mathcal{X}^i$ . Then,  $M = \prod_{i=1}^p M_i$  is  $n$ - $\mathcal{X}$ -coherent  $R$ -module if and only if  $M_i$  is  $n$ - $\mathcal{X}^i$ -coherent  $R_i$ -module for each  $i = 1, \dots, p$ , where  $R = \prod_{i=1}^p R_i$ .*

**Proof.** ( $\implies$ ) Let  $p = 2$ . Consider the short exact sequence  $0 \rightarrow R_2 \rightarrow R_1 \times R_2 \rightarrow R_1 \rightarrow 0$ , where  $R_1$  is an  $n$ -presented  $R_1 \times R_2$ -module, since by [8],  $R_1$  is an infinitely presented  $R_1 \times R_2$ -module. So, if  $N_1$  is a submodule of  $M_1$  in  $\mathcal{X}_{n-1}^1$ , then by Lemma 2.2(1),  $N_1$  is in  $\mathcal{X}_{n-1}$ , and so  $N_1$  is in  $\mathcal{X}_n$ . Therefore by Lemma 2.2(2),  $N_1$  is in  $\mathcal{X}_n^1$ . Similarly, if  $N_2$  is a submodule of  $M_2$  in  $\mathcal{X}_{n-1}^2$ , then  $N_2$  is in  $\mathcal{X}_n^2$ .

( $\Leftarrow$ ) Suppose that, for every  $i \in \{1, \dots, p\}$ ,  $M_i$  is  $n$ - $\mathcal{X}^i$ -coherent. Let  $N$  be a module of  $\mathcal{X}_{n-1}$ . Then there exist  $N_1, \dots, N_p$  in  $\prod_{i=1}^p \mathcal{X}^i$  such that  $N = \prod_{i=1}^p N_i$ . For each  $i \in \{1, \dots, p\}$ ,  $N_i$  is in  $\mathcal{X}_{n-1}$ . And so,  $N_i$  is in  $\mathcal{X}_{n-1}^i$ . Hence, it is in  $\mathcal{X}_n^i$  by  $n$ - $\mathcal{X}_i$ -coherence of  $M_i$ . Consequently,  $N$  is also in  $\mathcal{X}_n$ , and so we deduce that  $M$  is an  $n$ - $\mathcal{X}$ -coherent  $R$ -module.  $\square$

**Example 2.2.** Let  $(R_i, 1 \leq i \leq n)$  be a family of rings in Example 2.1(6), and also let  $(M_i, 1 \leq i \leq n)$  be a family of  $R_i$ -modules in Example 2.1(6). Consider  $R = \prod_{i=1}^n R_i$  and  $M = \prod_{i=1}^n M_i$ . If  $\mathcal{X}^i$  is a class of submodules of  $M_i$  for any integer  $i \in \{1, \dots, n\}$  and  $\mathcal{X} = \prod_{i=1}^n \mathcal{X}^i$ , then by Example 2.1(6) and Theorem 2.8,  $M$  is not  $(n+1)$ - $\mathcal{X}$ -coherent  $R$ -module.

**Corollary 2.8.** Let  $(R_i, 1 \leq i \leq p)$  be a family of rings,  $p \geq 1$  an integer, . Let  $\mathcal{X}^i$  be a class of ideals of  $R_i$  for any integer  $i \in \{1, \dots, p\}$  and  $\mathcal{X} = \prod_{i=1}^p \mathcal{X}^i$ . Then,  $R = \prod_{i=1}^p R_i$  is  $n$ - $\mathcal{X}$ -coherent if only if  $R_i$  is  $n$ - $\mathcal{X}^i$ -coherent for each  $i = 1, \dots, p$ .

We end this section by establishing another characterization of  $n$ - $\mathcal{X}$ -coherence using the notion of thickness. A class of modules  $\mathcal{Y}$  is said to be thick if it is closed under direct summands and whenever we are given a short exact sequence  $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$  with two out of the three terms  $A, B, C$  in  $\mathcal{Y}$ , so is the third module. In [9] and [4], it is proved that when  $R$  is coherent, the class of  $n$ -presented  $R$ -modules is thick. Here, we set the following generalization.

**Proposition 2.2.** Let  $n$  be a non negative integer and  $\mathcal{X}$  a class of  $R$ -modules which is closed under direct summand and kernels of epimorphisms. Then, the following assertions are equivalent:

- (1)  $R$  is  $n$ - $\mathcal{X}$ -coherent;
- (2) The class  $\mathcal{X}_{n-1}$  is thick;
- (3)  $\mathcal{X}_{n-1} = \mathcal{X}_\infty$ .

**Proof.** (3)  $\implies$  (2). It suffices to show that  $\mathcal{X}_\infty$  is thick which is easily deduced using [9], since  $\mathcal{X}_\infty = \bigcap_{k \geq 0} \mathcal{X}_k$ .

(2)  $\implies$  (1). Let  $I \in \mathcal{X}_{n-1}$ , then there is an exact sequence of  $R$ -modules  $0 \rightarrow K \rightarrow F_0 \rightarrow I \rightarrow 0$ , where  $K \in \mathcal{X}_{n-2}$  and  $F_0$  is finitely generated and free. Since  $\mathcal{X}_{n-1}$  is thick and both  $I$  and  $F_0$  are in  $\mathcal{X}_{n-1}$ , we get that  $K \in \mathcal{X}_{n-1}$  and so  $I \in \mathcal{X}_n$ .

(1)  $\implies$  (3). Let  $I \in \mathcal{X}_{n-1}$ . By the coherence of  $R$ ,  $I \in \mathcal{X}_n$ . Using the same argument as in (2)  $\implies$  (1), we can obtain an  $(n+1)$ -presentation of  $I$ . Iterating this procedure yields a finite  $m$ -presentation of  $I$  for all  $m \geq n$ . Hence  $I \in \bigcap_{m \geq 0} \mathcal{X}_m = \mathcal{X}_\infty$ .  $\square$

### 3. On the coherence of pullbacks

By a ring, we mean a commutative ring with identity. Considering a commutative square of rings and ring homomorphisms of the following form :

$$\begin{array}{ccc} R & \xrightarrow{i_1} & R_1 \\ \downarrow i_2 & & \downarrow j_1 \\ R_2 & \xrightarrow{j_2} & R_0 \end{array} \quad (1)$$

Recall that (1) is called a pullback diagram if  $R = \ker(j_1 \circ p_1 - j_2 \circ p_2)$ , where  $p_1$  be the projection of  $R$  on  $R_1$  and  $p_2$  be the projection of  $R$  on  $R_2$ .

In the following, we say that a class  $\mathcal{X}$  of modules satisfies the property  $(*)$  proper if for every module  $M \in \mathcal{X}_1$ , there exists an exact sequence  $0 \rightarrow K \rightarrow R^k \rightarrow M \rightarrow 0$  with  $K \in \mathcal{X}$ . We, also, consider a pullback diagram (1) with  $i_1$  is surjective. The following lemma can be found in [14] .

**Lemma 3.1.** *An  $R$ -module  $M$  is finitely generated if and only if  $R_i \otimes_R M$  is a finitely generated  $R_i$ -module for  $i = 1, 2$ .*

The following proposition generalizes [10] and [14] .

**Proposition 3.1.** *Let  $M$  be an  $R$ -module and let  $\mathcal{X}$  be a class of  $R$ -modules satisfying the property  $(*)$  and let  $\mathcal{Y}^i$  be a class of  $R_i$ -modules such that  $R_i \otimes \mathcal{X}_k$  is a subclass of  $\mathcal{Y}_k^i$  for every integer  $k$  and  $i = 1, 2$ .*

*Suppose that  $\text{Tor}_j^R(R_i, M)$  is in  $\mathcal{Y}_{n-j}^i$  for  $1 \leq j \leq n$ , then  $M$  is in  $\mathcal{X}_n$  if and only if  $R_i \otimes_R M$  is in  $\mathcal{Y}_n^i$  for  $i = 1, 2$ .*

**Proof.** We use the induction on  $n$ . The case  $n = 0$  follows easily from Lemma 3.1.

Now, assume that  $\text{Tor}_j^R(R_i, M)$  is in  $\mathcal{Y}_{n-j}^i$  for  $1 \leq j \leq n$  and  $i = 1, 2$ .

Let  $M$  is in  $\mathcal{X}_n$ . Then, we have an exact sequence of  $R$ -modules

$$0 \longrightarrow K \xrightarrow{i} R^k \xrightarrow{f} M \longrightarrow 0. \quad (a)$$

We have that  $\text{Tor}_j^R(R_i, K) \cong \text{Tor}_{j+1}^R(R_i, M)$  for any  $j \geq 1$ . So,  $R_i \otimes K$  is in  $\mathcal{Y}_{n-1}^i$ . Now, we tensor the short exact sequence (a) with  $R_i$  over  $R$  and we obtain the following two exact sequences:

$$0 \longrightarrow K_i \longrightarrow R_i^k \longrightarrow R_i \otimes_R M \longrightarrow 0$$

and

$$0 \longrightarrow \text{Tor}_1^R(R_i, M) \longrightarrow R_i \otimes_R K \longrightarrow R_i \otimes_R M \longrightarrow 0$$

with  $K_i = \ker(\mathbf{1}_{R_i} \otimes f)$ .

From the previous two exact sequences, we can deduce that  $K_i$  is in  $\mathcal{Y}_{n-1}^i$  which implies that  $R_i \otimes_R M$  is in  $\mathcal{Y}_n^i$  for  $i = 1, 2$ .

For the converse, suppose that  $R_i \otimes_R M$  is in  $\mathcal{Y}_n^i$ , then  $K_i$  is in  $\mathcal{Y}_{n-1}^i$ . And so  $R_i \otimes K$  is in  $\mathcal{Y}_{n-1}^i$ . Hence  $M$  is in  $\mathcal{X}_n$ .  $\square$

The following theorem generalizes [14].

**Theorem 3.1.** *Let  $M$  be an  $R$ -module and let  $\mathcal{X}$  be a class of  $R$ -modules satisfying the property  $(*)$  and let  $\mathcal{Y}^i$  be a class of  $R_i$ -modules such that  ${}^{R_i} \otimes \mathcal{X}_k$  is a subclass of  $\mathcal{Y}_k^i$  for every integer  $k$  and  $i = 1, 2$ .*

*Suppose that for each  $M \in \mathcal{X}_n$ ,  $\text{Tor}_j^R(R_i, M)$  is in  $\mathcal{Y}_{n+1-j}^i$  for  $1 \leq j \leq n+1$  and  $i = 1, 2$ . Then,  $R$  is  $n$ - $\mathcal{X}$ -coherent if  $R_i$  is  $n$ - $\mathcal{Y}^i$ -coherent.*

**Proof.** Let  $M$  be an  $R$ -module in  $\mathcal{X}_n$ . By Proposition 3.1,  $R_i \otimes_R M$  is in  $\mathcal{Y}_n^i$ . Then, by the coherence of  $R_i$ ,  $R_i \otimes_R M$  is in  $\mathcal{Y}_{n+1}^i$ . Again, by Proposition 3.1,  $M$  is in  $\mathcal{X}_{n+1}$ , and hence  $R$  is  $n$ - $\mathcal{X}$ -coherent.  $\square$

**Corollary 3.1.** *Let  $M$  be an  $R$ -module and let  $\mathcal{X}$  be a class of  $R$ -modules satisfying the property  $(*)$  and let  $\mathcal{Y}^i$  be a class of  $R_i$ -modules such that  ${}^{R_i} \otimes \mathcal{X}_k$  is a subclass of  $\mathcal{Y}_k^i$  for every integer  $k$  and  $i = 1, 2$ .*

*Suppose that  $R_i$  is  $n$ - $\mathcal{Y}^i$ -coherent. Then,  $R$  is  $n$ - $\mathcal{X}$ -coherent if and only if for all  $I \in \mathcal{X}_n$ ,  $\text{Tor}_j^R(R_i, \frac{R}{I})$  is in  $\mathcal{Y}_{n+1-j}^i$  for  $1 \leq j \leq n+1$  and  $i = 1, 2$ .*

**Proof.** The only if assertion follows from Theorem 3.1, we will prove the converse. Let  $I \in \mathcal{X}_n$ , then we have an exact sequence of the form

$$0 \longrightarrow I \longrightarrow R \xrightarrow{\pi} \frac{R}{I} \longrightarrow 0 \quad (1)$$

Tensoring the sequence (1) with  $R_i$  ( $i = 1, 2$ ) over  $R$  and put  $H_i = \ker(\mathbb{1}_{R_i} \otimes \pi)$ , we obtain two exact sequences

$$0 \longrightarrow H_i \longrightarrow R_i \xrightarrow{\mathbb{1}_{R_i} \otimes \pi} R_i \otimes_R \frac{R}{I} \longrightarrow 0 \quad (2)$$

and

$$0 \longrightarrow \text{Tor}_1^R(R_i, \frac{R}{I}) \longrightarrow R_i \otimes_R I \longrightarrow H_i \longrightarrow 0. \quad (3)$$

Using the coherence of  $R$  and the exactness of the sequences (1), (2) and (3), we get that  $\text{Tor}_1^R(R_i, \frac{R}{I})$  is in  $\mathcal{Y}_n^i$ .

Now, since  $I$  is in  $\mathcal{X}_n$ , we have an exact sequence

$$0 \longrightarrow P \longrightarrow R^s \longrightarrow I \longrightarrow 0.$$

Using a similar argument, we get that  $\text{Tor}_1^R(R_i, I)$  is in  $\mathcal{Y}_{n-1}^i$ , and hence  $\text{Tor}_2^R(R_i, \frac{R}{I})$  is also in  $\mathcal{Y}_{n-1}^i$ . Since  $P$  is in  $\mathcal{X}_{n-1}$ , we have an exact sequence

$$0 \longrightarrow P_0 \longrightarrow R^r \longrightarrow P \longrightarrow 0.$$

Using a similar argument, we get that  $\text{Tor}_1^R(R_i, P)$  is in  $\mathcal{Y}_{n-2}^i$ , and hence  $\text{Tor}_2^R(R_i, I)$  is also in  $\mathcal{Y}_{n-2}^i$ . Iterating with the same argument, we get that for each  $I \in \mathcal{X}_n$ ,  $\text{Tor}_j^R(R_i, \frac{R}{I})$  is in  $\mathcal{Y}_{n+1-j}^i$  for  $1 \leq j \leq n+1$  and  $i = 1, 2$ .  $\square$

Now, we establish necessary and sufficient conditions for the coherence of the pullback diagram.

**Theorem 3.2.** *Let  $\mathcal{X}$  be a class of  $R$ -modules satisfying the property  $(*)$  and let  $\mathcal{Y}^i$  be a class of  $R_i$ -modules for  $i = 1, 2$ .*

*Suppose that for each  $M \in \mathcal{X}_n$ ,  $\text{Tor}_j^R(R_i, M)$  is in  $\mathcal{Y}_{n+1-j}^i$  for  $1 \leq j \leq n+1$  and  $i = 1, 2$ . Also, let for any module in  $\mathcal{Y}_n^i$ , there exists a module  $X_i \in \mathcal{X}_n$  such that  $R_i \otimes_R X_i \simeq Y_i$ . Then,  $R$  is  $n$ - $\mathcal{X}$ -coherent if and only if  $R_i$  is  $n$ - $\mathcal{Y}^i$ -coherent.*

**Proof.** The direct sense of the equivalence is proved in Theorem 3.1. For the converse, let  $N_i$  be a  $R_i$ -module of  $\mathcal{Y}_n^i$ , ( $i = 1, 2$ ). By hypothesis,  $N_i = R_i \otimes_R N'_i$ , where  $N'_i$  is in  $\mathcal{X}_n$ . Then, by the coherence of  $R$ ,  $N'_i$  is in  $\mathcal{X}_{n+1}$ . Hence,  $N_i$  is in  $\mathcal{Y}_{n+1}^i$  for  $i = 1, 2$ .  $\square$

### Acknowledgment

We thank the referee(s) for helpful suggestions on an earlier version of this paper. Arij Benkhadra's research reported in this publication was supported by a scholarship from the Graduate Research Assistantships in Developing Countries Program of the Commission for Developing Countries of the International Mathematical Union.

### References

1. K. Alaoui Ismaili, D. E. Dobbs and N. Mahdou, Commutative rings and modules that are  $\text{Nil}_*$ -coherent or special  $\text{Nil}_*$ -coherent, *J. Algebra Appl.* **16** (2017), 1750187.
2. D. Bennis,  $n$ - $\mathcal{X}$ -coherent rings, *Int. Electron. J. Algebra*, **7** (2010), 128-139.
3. N. Bourbaki, *Algre Homologique*, Chapitre 10, Masson, Paris (1980).
4. D. Bravo and M. A. Prez, Finitness conditions and cotorsion pairs, *J. Pure Appl. Algebra*, **221** (2017), 1249-1267.
5. S. U. Chase, Direct product of modules, *Trans. A.M.S.* **97** (1960), 457-473.
6. D. L. Costa, Parameterizing families of non-Noetherian rings, *Comm. Algebra*, **22** (1994), 3997-4011.
7. N. Ding, Y. Li and L. Mao, J-coherent rings, *J. Algebra App.* **8** (2009), 139-155.
8. D. E. Dobbs, S. Kabbaj and N. Mahdou,  $n$ -coherent rings and modules, *Lect. Notes Pure Appl. Math.* **185** (1997), 269-281.
9. S. Glaz, *Commutative coherent rings*, Lecture Notes in Math., Springer-Verlag, Berlin (1989).
10. B. Greenberg, Coherence in cartesian squares, *J. Algebra*, **5** (1978), 12-25.
11. W. Li, B. Ouyang and Changsha,  $(n, d)$ -injective covers,  $n$ -coherent rings, and  $(n, d)$ -rings, *Czechoslovak Mathematical Journal.* **64**(139) (2014), 289-304.
12. M. E. Harris, Some results on coherent rings, *Proc. Amer. Math. Soc.* **17** (1966), 474-479.
13. L. Ouyng and Y. Xiang,  $\text{Nil}_*$ -Coherent rings, *Bull. Korean Math. Soc.* **51** (2014), 579-594.
14. S. Scrivanti, On the  $\lambda$ -dimension of cartesian squares, *Math. Scand.* **71** (1992), 16-23.
15. Z. Zhu, On  $n$ -coherent rings,  $n$ -hereditary rings and  $n$ -regular rings, *Bull Iranian math. Soc.* **37**(2011), 251-267.
16. T. Zhao, Z. Gao and Z. Huang, Relative  $FP$ - $gr$ -injective and  $gr$ -flat modules, *Int.J. Algebra and Computation.* **28**(6)(2018), 959-977.

17. W. Xu, On  $n$ -presented modules and almost excellent extensions, *Comm. Algebra*, **27**(3)(2007), 1091-1102.
18. W. Xu, On almost excellent extensions, *Algebra. Colloq.*, **3**(1996), 125-134.





## Gorenstein $n$ - $\mathcal{X}$ -Injective and $n$ - $\mathcal{X}$ -Flat Modules with Respect to a Special Finitely Presented Module

**Mostafa Amini**

*Department of Mathematics, Faculty of Sciences  
Payame Noor University, Tehran, Iran  
E-mail: amini.pnu1356@gmail.com*

**Arij Benkhadra    Driss Bennis**

*Department of Mathematics, Faculty of Sciences  
Mohammed V University in Rabat, Rabat, Morocco  
E-mail: benkhadra.arij@gmail.com    driss.bennis@um5.ac.ma  
driss\_bennis@hotmail.com*

Received 20 October 2019  
Revised 31 August 2020

Communicated by Nanqing Ding

**Abstract.** Let  $R$  be a ring,  $\mathcal{X}$  a class of  $R$ -modules and  $n \geq 1$  an integer. We introduce the concepts of Gorenstein  $n$ - $\mathcal{X}$ -injective and  $n$ - $\mathcal{X}$ -flat modules via special finitely presented modules. Besides, we obtain some equivalent properties of these modules on  $n$ - $\mathcal{X}$ -coherent rings. Then we investigate the relations among Gorenstein  $n$ - $\mathcal{X}$ -injective,  $n$ - $\mathcal{X}$ -flat, injective and flat modules on  $\mathcal{X}$ -FC-rings (i.e., self  $n$ - $\mathcal{X}$ -injective and  $n$ - $\mathcal{X}$ -coherent rings). Several known results are generalized to this new context.

**2010 Mathematics Subject Classification:** 16D80, 16E05, 16E30, 16E65, 16P70

**Keywords:**  $n$ - $\mathcal{X}$ -coherent ring, Gorenstein  $n$ - $\mathcal{X}$ -injective module, Gorenstein  $n$ - $\mathcal{X}$ -flat module

### 1 Introduction

In 1995, Enochs et al. introduced the concept of Gorenstein injective and Gorenstein flat modules. These modules have become a vigorously active area of research. For background on Gorenstein modules, we refer the reader to [8, 9, 12]. In 2012, Gao and Wang introduced and studied in [11] Gorenstein FP-injective modules. They established various homological properties of Gorenstein FP-injective modules mainly over a coherent ring (for more details, see [14]).

Recall that coherent rings were first appeared in Chase's paper [3] without being mentioned by name. The term coherent was first used by Bourbaki in [2]. Later,  $n$ -coherent rings were introduced by Costa [5]. Let  $n$  be a non-negative integer. An

## Gorenstein $n$ - $\mathcal{X}$ -Injective and $n$ - $\mathcal{X}$ -Flat Modules with Respect to a Special Finitely Presented Module

**Mostafa Amini**

*Department of Mathematics, Faculty of Sciences  
Payame Noor University, Tehran, Iran  
E-mail: amini.pnu1356@gmail.com*

**Arij Benkhadra Driss Bennis**

*Department of Mathematics, Faculty of Sciences  
Mohammed V University in Rabat, Rabat, Morocco  
E-mail: benkhadra.arij@gmail.com driss.bennis@um5.ac.ma  
driss\_bennis@hotmail.com*

Received 20 October 2019  
Revised 31 August 2020

Communicated by Nanqing Ding

**Abstract.** Let  $R$  be a ring,  $\mathcal{X}$  a class of  $R$ -modules and  $n \geq 1$  an integer. We introduce the concepts of Gorenstein  $n$ - $\mathcal{X}$ -injective and  $n$ - $\mathcal{X}$ -flat modules via special finitely presented modules. Besides, we obtain some equivalent properties of these modules on  $n$ - $\mathcal{X}$ -coherent rings. Then we investigate the relations among Gorenstein  $n$ - $\mathcal{X}$ -injective,  $n$ - $\mathcal{X}$ -flat, injective and flat modules on  $\mathcal{X}$ -FC-rings (i.e., self  $n$ - $\mathcal{X}$ -injective and  $n$ - $\mathcal{X}$ -coherent rings). Several known results are generalized to this new context.

**2010 Mathematics Subject Classification:** 16D80, 16E05, 16E30, 16E65, 16P70

**Keywords:**  $n$ - $\mathcal{X}$ -coherent ring, Gorenstein  $n$ - $\mathcal{X}$ -injective module, Gorenstein  $n$ - $\mathcal{X}$ -flat module

### 1 Introduction

In 1995, Enochs et al. introduced the concept of Gorenstein injective and Gorenstein flat modules. These modules have become a vigorously active area of research. For background on Gorenstein modules, we refer the reader to [8, 9, 12]. In 2012, Gao and Wang introduced and studied in [11] Gorenstein FP-injective modules. They established various homological properties of Gorenstein FP-injective modules mainly over a coherent ring (for more details, see [14]).

Recall that coherent rings were first appeared in Chase's paper [3] without being mentioned by name. The term coherent was first used by Bourbaki in [2]. Later,  $n$ -coherent rings were introduced by Costa [5]. Let  $n$  be a non-negative integer. An

$R$ -module  $M$  is said to be  $n$ -presented if there is an exact sequence

$$F_n \longrightarrow F_{n-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

of  $R$ -modules, where each  $F_i$  is finitely generated free. Moreover, a ring  $R$  is called left  $n$ -coherent if every  $n$ -presented  $R$ -module is  $(n+1)$ -presented. Thus, for  $n = 1$ , left  $n$ -coherent rings are nothing but coherent rings (see [5, 7, 17]). Chen and Ding in [4] introduced, by using  $n$ -presented modules,  $n$ -FP-injective and  $n$ -flat modules. Bennis in [1] introduced  $n$ - $\mathcal{X}$ -injective and  $n$ - $\mathcal{X}$ -flat modules and  $n$ - $\mathcal{X}$ -coherent rings for any class  $\mathcal{X}$  of  $R$ -modules. In 2018, Zhao et al. [20] introduced  $n$ -FP-gr-injective graded modules,  $n$ -gr-flat graded right modules and  $n$ -gr-coherent graded rings on a class of graded  $R$ -modules. In addition, they defined special finitely presented graded modules via projective resolutions of  $n$ -presented graded left modules; specifically, if  $U$  is an  $n$ -presented graded module, then in the exact sequence  $F_n \rightarrow F_{n-1} \rightarrow \cdots \rightarrow F_1 \rightarrow F_0 \rightarrow U \rightarrow 0$ ,  $K_{n-1} = \text{im}(F_{n-1} \rightarrow F_{n-2})$  is called a special finitely presented module. In this paper, we unify and extend various homological notions, including the ones cited above, to a more general context. Namely, we define special finitely presented modules via projective resolutions of  $n$ -presented modules in a given class  $\mathcal{X}$  of  $R$ -modules. Then we introduce and study Gorenstein  $n$ - $\mathcal{X}$ -injective and  $n$ - $\mathcal{X}$ -flat modules with respect to special finitely presented modules.

The paper is organized as follows: In Section 2, some fundamental concepts and some preliminary results are stated. In Section 3, we give some characterizations of  $n$ - $\mathcal{X}$ -injective and  $n$ - $\mathcal{X}$ -flat modules. In Section 4, we introduce the notions of Gorenstein  $n$ - $\mathcal{X}$ -injective and  $n$ - $\mathcal{X}$ -flat modules. We generalize some results of [20] to the context of  $n$ - $\mathcal{X}$ -injective and  $n$ - $\mathcal{X}$ -flat modules as well as some results of [11] to the context of Gorenstein  $n$ - $\mathcal{X}$ -injective modules. Then we give some characterizations of Gorenstein  $n$ - $\mathcal{X}$ -injective and  $n$ - $\mathcal{X}$ -flat modules on  $n$ - $\mathcal{X}$ -coherent rings. In Section 5, we introduce and investigate  $\mathcal{X}$ -FC rings (i.e., self  $n$ - $\mathcal{X}$ -injective and  $n$ - $\mathcal{X}$ -coherent rings), every module of which is Gorenstein  $n$ - $\mathcal{X}$ -injective and every Gorenstein  $n$ - $\mathcal{X}$ -injective right module of which is Gorenstein  $n$ - $\mathcal{X}$ -flat. Furthermore, examples are given which show that the Gorenstein  $m$ - $\mathcal{X}$ -injectivity (resp., the  $m$ - $\mathcal{X}$ -flatness) does not imply, in general, the Gorenstein  $n$ - $\mathcal{X}$ -injectivity (resp., the  $n$ - $\mathcal{X}$ -flatness) for any  $m > n$ .

## 2 Preliminaries

Throughout this paper  $R$  will be an associative (not necessarily commutative) ring with identity, and all modules will be unital left  $R$ -modules (unless specified otherwise). In this section, some fundamental concepts and notations are stated.

Let  $n$  be a non-negative integer,  $M$  an  $R$ -module and  $\mathcal{X}$  a class of  $R$ -modules. Then  $M$  is said to be *Gorenstein injective* (resp., *Gorenstein flat*) [8, 9] if there is an exact sequence  $\cdots \rightarrow I_1 \rightarrow I_0 \rightarrow I^0 \rightarrow I^1 \rightarrow \cdots$  of injective (resp., flat) modules with  $M = \ker(I^0 \rightarrow I^1)$  such that  $\text{Hom}(U, -)$  (resp.,  $U \otimes_R -$ ) leaves the sequence exact whenever  $U$  is an injective left (resp., right) module. The Gorenstein projective modules are defined dually. Recall that  $M$  is said to be  $n$ -FP-injective [4] if  $\text{Ext}_R^n(U, M) = 0$  for any  $n$ -presented  $R$ -module  $U$ . In the case  $n = 1$ ,  $n$ -FP-

injective modules are nothing but the well-known FP-injective modules. A right module  $N$  is called  $n$ -flat if  $\text{Tor}_n^R(N, U) = 0$  for any  $n$ -presented  $R$ -module  $U$ . Also,  $M$  is said to be Gorenstein FP-injective [11] if there is an exact sequence  $\mathbf{E} = \cdots \rightarrow E_1 \rightarrow E_0 \rightarrow E^0 \rightarrow E^1 \rightarrow \cdots$  with  $M = \ker(E^0 \rightarrow E^1)$  such that  $\text{Hom}_R(P, \mathbf{E})$  is an exact sequence whenever  $P$  is finitely presented with  $\text{pd}_R(P) < \infty$ . A graded  $R$ -module  $M$  is called  $n$ -FP-gr-injective [20] if  $\text{Ext}_R^n(N, M) = 0$  for any  $n$ -presented graded  $R$ -module  $N$ . A graded right  $R$ -module  $M$  is called  $n$ -gr-flat [20] if  $\text{Tor}_R^n(M, N) = 0$  for any  $n$ -presented graded  $R$ -module  $N$  (for more details about graded modules, see [10, 13, 15]).

From now on,  $\mathcal{X}_k$  is a non-empty class of  $k$ -presented  $R$ -modules in a given class  $\mathcal{X}$  for any integer  $k \geq 0$ .

An  $R$ -module  $M$  is said to be  $n$ - $\mathcal{X}$ -injective [1] if  $\text{Ext}_R^n(U, M) = 0$  for any  $U \in \mathcal{X}_n$ . A right  $R$ -module  $N$  is called  $n$ - $\mathcal{X}$ -flat [1] if  $\text{Tor}_n^R(N, U) = 0$  for any  $U \in \mathcal{X}_n$ . We use  $\mathcal{X}\mathcal{I}_n$  (resp.,  $\mathcal{X}\mathcal{F}_n$ ) to denote the class of all  $n$ - $\mathcal{X}$ -injective left  $R$ -modules (resp.,  $n$ - $\mathcal{X}$ -flat right  $R$ -modules).

A ring  $R$  is called left  $n$ - $\mathcal{X}$ -coherent if every  $n$ -presented  $R$ -module in  $\mathcal{X}$  is  $(n+1)$ -presented. It is clear that when  $n = 0$  (resp.,  $n = 1$ ) and  $\mathcal{X}$  is the class of all cyclic  $R$ -modules, then  $R$  is Noetherian (resp., coherent).

### 3 $n$ - $\mathcal{X}$ -Injective, $n$ - $\mathcal{X}$ -Flat and Special $\mathcal{X}$ -Presented Modules

In this section, we state a relative version of [20, Definition 3.1] and provide several characterizations of  $n$ - $\mathcal{X}$ -injective and  $n$ - $\mathcal{X}$ -flat modules.

Let us introduce the following notions. Let  $n \geq 0$  be an integer and  $U \in \mathcal{X}_n$  for a class  $\mathcal{X}$  of  $R$ -modules. Then an exact sequence of the form

$$F_n \longrightarrow F_{n-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow U \longrightarrow 0,$$

where each  $F_i$  is a finitely generated free  $R$ -module, exists. Now let us define  $K_{n-1} = \text{im}(F_{n-1} \rightarrow F_{n-2})$  and  $K_n = \text{im}(F_n \rightarrow F_{n-1})$ . The short exact sequence  $0 \rightarrow K_n \rightarrow F_{n-1} \rightarrow K_{n-1} \rightarrow 0$  is called a special short exact sequence of  $U$ . It is clear that  $K_n$  and  $K_{n-1}$  are finitely generated and finitely presented, respectively. We call  $K_n$  a special  $\mathcal{X}$ -generated  $R$ -module and  $K_{n-1}$  a special  $\mathcal{X}$ -presented  $R$ -module.

Moreover, a short exact sequence  $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$  of  $R$ -modules is called special  $\mathcal{X}$ -pure if for every special  $\mathcal{X}$ -presented  $K_{n-1}$ , there exists the following exact sequence:

$$0 \longrightarrow \text{Hom}_R(K_{n-1}, A) \longrightarrow \text{Hom}_R(K_{n-1}, B) \longrightarrow \text{Hom}_R(K_{n-1}, C) \longrightarrow 0.$$

The  $R$ -module  $A$  is said to be special  $\mathcal{X}$ -pure in  $B$ . In addition, the exact sequence  $0 \rightarrow C^* \rightarrow B^* \rightarrow A^* \rightarrow 0$  is called a split special exact sequence. If  $M$  is an  $n$ - $\mathcal{X}$ -injective (resp., a flat)  $R$ -module, then  $\text{Ext}_R^1(K_{n-1}, M) \cong \text{Ext}_R^n(U, M) = 0$  (resp.,  $\text{Tor}_1^R(M, K_{n-1}) \cong \text{Tor}_n^R(M, U) = 0$ ) for any  $U \in \mathcal{X}_n$ .

In particular, if  $\mathcal{X}$  is a class of graded  $R$ -modules, every special  $\mathcal{X}$ -generated module is a special finitely generated  $R$ -module, and every special  $\mathcal{X}$ -presented module is a special finitely presented graded  $R$ -module. Also, every  $n$ - $\mathcal{X}$ -injective

$R$ -module is  $n$ -FP-gr-injective and every  $n$ - $\mathcal{X}$ -flat right  $R$ -module is  $n$ -gr-flat (see [20]).

**Proposition 3.1.** *Let  $\mathcal{X}$  be a class of  $R$ -modules and  $M$  an  $R$ -module. Then the following statements are equivalent:*

- (a)  $M$  is  $n$ - $\mathcal{X}$ -injective.
- (b) Every short exact sequence  $0 \rightarrow M \rightarrow A \rightarrow C \rightarrow 0$  is special  $\mathcal{X}$ -pure.
- (c)  $M$  is special  $\mathcal{X}$ -pure in any injective  $R$ -module containing it.
- (d)  $M$  is special  $\mathcal{X}$ -pure in  $E(M)$ .

*Proof.* (a) $\Rightarrow$ (b) Let  $U \in \mathcal{X}_n$ . Then there exists the following exact sequence:

$$F_n \longrightarrow F_{n-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow U \longrightarrow 0,$$

where each  $F_i$  is a finitely generated free  $R$ -module. If  $K_{n-1} = \text{im}(F_{n-1} \rightarrow F_{n-2})$  is special  $\mathcal{X}$ -presented, then  $\text{Ext}_R^1(K_{n-1}, M) \cong \text{Ext}_R^n(U, M) = 0$  since the  $R$ -module  $M$  is  $n$ - $\mathcal{X}$ -injective. Hence, (b) follows.

(b) $\Rightarrow$ (c) Consider the canonical short exact sequence  $0 \rightarrow M \rightarrow E \rightarrow E/M \rightarrow 0$  with  $E$  an injective  $R$ -module containing  $M$ . So by (b),  $M$  is special  $\mathcal{X}$ -pure in  $E$ .

(c) $\Rightarrow$ (d) is trivial.

(d) $\Rightarrow$ (a) Assume that  $U \in \mathcal{X}_n$  and  $K_{n-1}$  is special  $\mathcal{X}$ -presented. By (4), the short exact sequence  $0 \rightarrow M \rightarrow E(M) \rightarrow E(M)/M \rightarrow 0$  is special  $\mathcal{X}$ -pure. Therefore,  $\text{Ext}_R^1(K_{n-1}, M) = 0$ , and so from  $\text{Ext}_R^1(K_{n-1}, M) \cong \text{Ext}_R^n(U, M)$  we see that  $M$  is  $n$ - $\mathcal{X}$ -injective.  $\square$

The following lemma is a generalization of [18, Exercise 40].

**Lemma 3.2.** *Let  $\mathcal{X}$  be a class of  $R$ -modules and  $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$  be a short exact sequence of  $R$ -modules. Then the following statements are equivalent:*

- (a) The exact sequence  $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$  is special  $\mathcal{X}$ -pure.
- (b) The sequence  $0 \rightarrow \text{Hom}_R(K_{n-1}, A) \rightarrow \text{Hom}_R(K_{n-1}, B) \rightarrow \text{Hom}_R(K_{n-1}, C) \rightarrow 0$  is exact for every special  $\mathcal{X}$ -presented  $K_{n-1}$ .
- (c) The short exact sequence  $0 \rightarrow C^* \rightarrow B^* \rightarrow A^* \rightarrow 0$  is a split special exact sequence.

**Proposition 3.3.** *Let  $\mathcal{X}$  be a class of  $R$ -modules. Then the following hold:*

- (a) Every special  $\mathcal{X}$ -pure submodule of an  $n$ - $\mathcal{X}$ -flat right  $R$ -module is  $n$ - $\mathcal{X}$ -flat.
- (b) Every special  $\mathcal{X}$ -pure submodule of an  $R$ -module is  $n$ - $\mathcal{X}$ -injective.

*Proof.* (a) Let  $A$  be a special  $\mathcal{X}$ -pure submodule of an  $n$ - $\mathcal{X}$ -flat right  $R$ -module  $B$ . Then, by Lemma 3.2, the sequence  $0 \rightarrow (B/A)^* \rightarrow B^* \rightarrow A^* \rightarrow 0$  is a split special exact sequence. By [1, Lemma 2.8],  $B^*$  is  $n$ - $\mathcal{X}$ -injective. Thus, from [1, Lemma 2.7] and Lemma 3.2 we deduce that  $A$  is  $n$ - $\mathcal{X}$ -flat.

(b) Let  $A$  be a special  $\mathcal{X}$ -pure submodule of an  $R$ -module  $B$ . Then the exact sequence  $0 \rightarrow A \rightarrow B \rightarrow B/A \rightarrow 0$  is special  $\mathcal{X}$ -pure. So by Proposition 3.1,  $A$  is  $n$ - $\mathcal{X}$ -injective.  $\square$

**Remark 3.4.** (a) Every flat right  $R$ -module is  $n$ - $\mathcal{X}$ -flat.

(b) Every injective left (or right)  $R$ -module is  $n$ - $\mathcal{X}$ -injective.

(c) If  $U \in \mathcal{X}_m$ , then  $U \in \mathcal{X}_n$  for any  $m \geq n$ .

A ring  $R$  is called self left  $n$ - $\mathcal{X}$ -injective if  $R$  is an  $n$ - $\mathcal{X}$ -injective left  $R$ -module.

Let  $\mathcal{F}$  be a class of  $R$ -modules and  $M$  an  $R$ -module. Following [6], we say that a morphism  $f: F \rightarrow M$  is an  $\mathcal{F}$ -precover of  $M$  if  $F \in \mathcal{F}$  and  $\text{Hom}_R(F', F) \rightarrow \text{Hom}_R(F', M) \rightarrow 0$  is exact for all  $F' \in \mathcal{F}$ . An  $\mathcal{F}$ -precover  $f \in \text{Hom}_R(F, M)$  of  $M$  is called an  $\mathcal{F}$ -cover if every endomorphism  $g: F \rightarrow F$  with  $fg = f$  is an automorphism. Dually, the notions of  $\mathcal{F}$ -preenvelopes and  $\mathcal{F}$ -envelopes are defined.

**Theorem 3.5.** *Let  $R$  be a left  $n$ - $\mathcal{X}$ -coherent ring and  $\mathcal{X}$  be a class of  $R$ -modules. Then the following statements are equivalent:*

- (a)  $R$  is self left  $n$ - $\mathcal{X}$ -injective.
- (b) For any  $R$ -module, there is an epimorphic  $\mathcal{X}\mathcal{I}$ -cover.
- (c) For any right  $R$ -module, there is a monomorphic  $\mathcal{X}\mathcal{F}$ -preenvelope.
- (d) Every injective right  $R$ -module is  $n$ - $\mathcal{X}$ -flat.
- (e) Every 1- $\mathcal{X}$ -injective right  $R$ -module is  $n$ - $\mathcal{X}$ -flat.
- (f) Every  $n$ - $\mathcal{X}$ -injective right  $R$ -module is  $n$ - $\mathcal{X}$ -flat.
- (g) Every flat  $R$ -module is  $n$ - $\mathcal{X}$ -injective.

*Proof.* (a) $\Rightarrow$ (c) By [1, Theorem 2.16], every right  $R$ -module  $N$  has an  $n$ - $\mathcal{X}$ -flat preenvelope  $f: N \rightarrow F$ . By [1, Theorem 2.13],  $R^*$  is  $n$ - $\mathcal{X}$ -flat, and so  $\prod R^*$  is  $n$ - $\mathcal{X}$ -flat by [1, Theorem 2.6]. On the other hand,  $R^*$  is a cogenerator, so a monomorphism of the form  $g: N \rightarrow \prod R^*$  exists. Hence, there exists a homomorphism  $h: F \rightarrow \prod R^*$  such that  $hf = g$ , which implies that  $f$  is monic.

(c) $\Rightarrow$ (d) Let  $E$  be an injective right  $R$ -module. By (c), there is  $f: E \rightarrow F$ , a monic  $n$ - $\mathcal{X}$ -flat preenvelope of  $E$ . So the sequence  $0 \rightarrow E \rightarrow F \rightarrow F/E \rightarrow 0$  splits, and hence  $E$  is  $n$ - $\mathcal{X}$ -flat.

(c) $\Rightarrow$ (e) The proof is similar to that of (c) $\Rightarrow$ (d).

(d) $\Rightarrow$ (f) Let  $N$  be an  $n$ - $\mathcal{X}$ -injective right  $R$ -module. Then by Proposition 3.1, the exact sequence  $0 \rightarrow N \rightarrow E(N) \rightarrow E(N)/N \rightarrow 0$  is special  $\mathcal{X}$ -pure. Since by (c)  $E(N)$  is  $n$ - $\mathcal{X}$ -flat, we deduce from Proposition 3.3 that  $N$  is  $n$ - $\mathcal{X}$ -flat.

(e) $\Rightarrow$ (d) is clear by Remark 3.4.

(d) $\Rightarrow$ (a) By (d),  $R^*$  is  $n$ - $\mathcal{X}$ -flat since  $R^*$  is injective. So  $R$  is self left  $n$ - $\mathcal{X}$ -injective by [1, Theorem 2.13].

(f) $\Rightarrow$ (g) Let  $F$  be a flat  $R$ -module. Then  $F^*$  is injective, so  $F^*$  is  $n$ - $\mathcal{X}$ -flat by (f), and hence  $F$  is  $n$ - $\mathcal{X}$ -injective.

(g) $\Rightarrow$ (b) For any  $R$ -module  $M$ , there is an  $\mathcal{X}\mathcal{I}_n$ -cover  $f: C \rightarrow M$ . Notice that  $R$  is an  $n$ - $\mathcal{X}$ -injective  $R$ -module, so  $f$  is an epimorphism.

(b) $\Rightarrow$ (a) By hypothesis,  $R$  has an epimorphic  $\mathcal{X}\mathcal{I}_n$ -cover  $f: D \rightarrow R$ . Then we have a split exact sequence  $0 \rightarrow \ker(f) \rightarrow D \rightarrow R \rightarrow 0$  with  $D$  being  $n$ - $\mathcal{X}$ -injective. Thus,  $R$  is  $n$ - $\mathcal{X}$ -injective as a left  $R$ -module.  $\square$

**Proposition 3.6.** *Let  $R$  be a left  $n$ - $\mathcal{X}$ -coherent ring and  $\mathcal{X}$  a class of  $R$ -modules. If  $\{A_i\}_{i \in I}$  is a family of  $R$ -modules, then  $\bigoplus_{i \in I} A_i$  is  $n$ - $\mathcal{X}$ -injective if and only if every  $A_i$  is  $n$ - $\mathcal{X}$ -injective.*

*Proof.* Assume that  $U \in \mathcal{X}_n$ . Then there exists a special exact sequence  $0 \rightarrow K_n \rightarrow F_{n-1} \rightarrow K_{n-1} \rightarrow 0$  of  $\mathcal{X}_n$ . Since  $R$  is  $n$ - $\mathcal{X}$ -coherent, we conclude that  $U \in \mathcal{X}_{n+1}$

and  $K_n$  is special  $\mathcal{X}$ -presented. Accordingly, if  $\{A_i\}_{i \in I}$  is a family of  $n$ - $\mathcal{X}$ -injective  $R$ -modules, we have

$$\mathrm{Hom}\left(K_n, \bigoplus_{i \in I} A_i\right) \cong \bigoplus_{i \in I} \mathrm{Hom}(K_n, A_i).$$

One easily gets

$$\mathrm{Ext}_R^n\left(U, \bigoplus_{i \in I} A_i\right) \cong \mathrm{Ext}_R^1\left(K_n, \bigoplus_{i \in I} A_i\right) \cong \bigoplus_{i \in I} \mathrm{Ext}_R^1(K_n, A_i) \cong \bigoplus_{i \in I} \mathrm{Ext}_R^n(U, A_i).$$

The proof is completed.  $\square$

#### 4 Gorenstein $n$ - $\mathcal{X}$ -Injective and $n$ - $\mathcal{X}$ -Flat Modules

In this section, we investigate Gorenstein  $n$ - $\mathcal{X}$ -injective and Gorenstein  $n$ - $\mathcal{X}$ -flat modules, which are defined below. Then by using the results of Section 3, some characterizations of them are given.

**Definition 4.1.** Let  $R$  be a ring and  $\mathcal{X}$  be a class of  $R$ -modules.

- (a) An  $R$ -module  $G$  is said to be *Gorenstein  $n$ - $\mathcal{X}$ -injective* if there exists an exact sequence of  $n$ - $\mathcal{X}$ -injective  $R$ -modules  $\mathbf{A} = \cdots \rightarrow A_1 \rightarrow A_0 \rightarrow A^0 \rightarrow A^1 \rightarrow \cdots$  with  $G = \ker(A^0 \rightarrow A^1)$  such that  $\mathrm{Hom}_R(K_{n-1}, \mathbf{A})$  is an exact sequence whenever  $K_{n-1}$  is special  $\mathcal{X}$ -presented with  $\mathrm{pd}_R(K_{n-1}) < \infty$ .
- (b) An  $R$ -module  $G$  is called a *Gorenstein  $n$ - $\mathcal{X}$ -flat right  $R$ -module* if there exists an exact sequence of  $n$ - $\mathcal{X}$ -flat right  $R$ -modules:

$$\mathbf{F} = \cdots \rightarrow F_1 \rightarrow F_0 \rightarrow F^0 \rightarrow F^1 \rightarrow \cdots$$

with  $G = \ker(F^0 \rightarrow F^1)$  such that  $\mathbf{F} \otimes_R K_{n-1}$  is an exact sequence whenever  $K_{n-1}$  is special  $\mathcal{X}$ -presented with  $\mathrm{fd}_R(K_{n-1}) < \infty$ .

For example, if  $\mathcal{X}$  is the class of all cyclic  $R$ -modules, then every Gorenstein 1- $\mathcal{X}$ -injective  $R$ -module is Gorenstein FP-injective, and every Gorenstein 1- $\mathcal{X}$ -flat right  $R$ -module is Gorenstein flat (see [1, 11]).

**Remark 4.2.** (a) Every  $n$ - $\mathcal{X}$ -flat right  $R$ -module is Gorenstein  $n$ - $\mathcal{X}$ -flat.

(b) Every  $n$ - $\mathcal{X}$ -injective  $R$ -module is Gorenstein  $n$ - $\mathcal{X}$ -injective.

(c) In Definition 4.1, one easily finds that  $\ker(A_i \rightarrow A_{i-1})$ ,  $\ker(A^i \rightarrow A^{i+1})$  and  $\ker(F_i \rightarrow F_{i-1})$ ,  $K^i = \ker(F^i \rightarrow F^{i+1})$  are Gorenstein  $n$ - $\mathcal{X}$ -injective and Gorenstein  $n$ - $\mathcal{X}$ -flat, respectively.

**Lemma 4.3.** Let  $R$  be a left  $n$ - $\mathcal{X}$ -coherent ring and  $\mathcal{X}$  a class of  $R$ -modules. If  $K_{n-1}$  is a special  $\mathcal{X}$ -presented  $R$ -module with  $\mathrm{fd}_R(K_{n-1}) < \infty$ , then we have  $\mathrm{pd}_R(K_{n-1}) < \infty$ .

*Proof.* If  $\mathrm{fd}_R(K_{n-1}) = m < \infty$ , there exists  $U \in \mathcal{X}_n$  such that  $\mathrm{fd}_R(U) \leq n + m$ . We show that  $\mathrm{pd}_R(U) \leq n + m$ . Since  $R$  is  $n$ - $\mathcal{X}$ -coherent, the projective resolution  $\cdots \rightarrow F_{n+1} \rightarrow F_n \rightarrow \cdots \rightarrow F_0 \rightarrow U \rightarrow 0$ , where any  $F_i$  is finitely generated



free, exists. On the other hand, the exact sequence above is a flat resolution. So by [16, Proposition 8.17], its  $(n + m - 1)$ -syzygy is flat. Accordingly, the exact sequence  $0 \rightarrow K_{n+m-1} \rightarrow F_{n+m-1} \rightarrow \cdots \rightarrow F_0 \rightarrow U \rightarrow 0$  is a flat resolution. A simple observation shows that whether  $n \geq m$  or  $n < m$ ,  $K_{n+m-1}$  is always finitely presented. Consequently, by [16, Theorem 3.56],  $K_{n+m-1}$  is projective. Thus,  $\text{pd}_R(U) \leq n + m$  if and only if  $\text{pd}_R(K_{n-1}) \leq m$ .  $\square$

In the next theorem, we show that in the case of left  $n$ - $\mathcal{X}$ -coherent rings, Gorenstein  $n$ - $\mathcal{X}$ -flatness and Gorenstein  $n$ - $\mathcal{X}$ -injectivity are determined via only the existence of the corresponding exact complexes.

**Theorem 4.4.** *Let  $R$  be a left  $n$ - $\mathcal{X}$ -coherent ring and  $\mathcal{X}$  be a class of  $R$ -modules. Then the following statements hold:*

- (a) *A right  $R$ -module  $G$  is Gorenstein  $n$ - $\mathcal{X}$ -flat if and only if there is an exact sequence  $\mathbf{F} = \cdots \rightarrow F_1 \rightarrow F_0 \rightarrow F^0 \rightarrow F^1 \rightarrow \cdots$  of  $n$ - $\mathcal{X}$ -flat right  $R$ -modules such that  $G = \ker(F^0 \rightarrow F^1)$ .*
- (b) *An  $R$ -module  $G$  is Gorenstein  $n$ - $\mathcal{X}$ -injective if and only if there is an exact sequence  $\mathbf{A} = \cdots \rightarrow A_1 \rightarrow A_0 \rightarrow A^0 \rightarrow A^1 \rightarrow \cdots$  of  $n$ - $\mathcal{X}$ -injective  $R$ -modules such that  $G = \ker(A^0 \rightarrow A^1)$ .*

*Proof.* (a)  $(\Rightarrow)$  It follows by definition.

$(\Leftarrow)$  By definition, it suffices to show that  $\mathbf{F} \otimes_R K_{n-1}$  is exact for every special  $\mathcal{X}$ -presented  $K_{n-1}$  with  $\text{fd}_R(K_{n-1}) < \infty$ . By Lemma 4.3,  $\text{pd}_R(K_{n-1}) < \infty$ . Let  $\text{pd}_R(K_{n-1}) = m$ . We prove by induction on  $m$ . The case  $m = 0$  is clear. Assume that  $m \geq 1$ . There exists a special exact sequence  $0 \rightarrow K_n \rightarrow P_{n-1} \rightarrow K_{n-1} \rightarrow 0$  of  $U \in \mathcal{X}_n$ , where  $P_{n-1}$  is projective. Now, from the  $n$ - $\mathcal{X}$ -coherence of  $R$ , we deduce that  $K_n$  is special  $\mathcal{X}$ -presented. Also,  $\text{pd}_R(K_n) \leq m - 1$ . So the following short exact sequence of complexes exists:

$$\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \vdots \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & F_1 \otimes_R K_n & \longrightarrow & F_1 \otimes_R P_{n-1} & \longrightarrow & F_1 \otimes_R K_{n-1} \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & F_0 \otimes_R K_n & \longrightarrow & F_0 \otimes_R P_{n-1} & \longrightarrow & F_0 \otimes_R K_{n-1} \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & F^0 \otimes_R K_n & \longrightarrow & F^0 \otimes_R P_{n-1} & \longrightarrow & F^0 \otimes_R K_{n-1} \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & F^1 \otimes_R K_n & \longrightarrow & F^1 \otimes_R P_{n-1} & \longrightarrow & F^1 \otimes_R K_{n-1} \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & \vdots & & \vdots & & \vdots \\
 & & \parallel & & \parallel & & \parallel \\
 0 & \longrightarrow & \mathbf{F} \otimes_R K_n & \longrightarrow & \mathbf{F} \otimes_R P_{n-1} & \longrightarrow & \mathbf{F} \otimes_R K_{n-1} \longrightarrow 0
 \end{array}$$

By induction,  $\mathbf{F} \otimes_R P_{n-1}$  and  $\mathbf{F} \otimes_R K_n$  are exact, and hence  $\mathbf{F} \otimes_R K_{n-1}$  is exact by [16, Theorem 6.10].

(b)  $(\Rightarrow)$  This is a direct consequence of the definition.

$(\Leftarrow)$  Let  $K_{n-1}$  be a special  $\mathcal{X}$ -presented  $R$ -module with  $\text{pd}_R(K_{n-1}) < \infty$ . Then a similar proof to that of (a) shows that  $\text{Hom}_R(K_{n-1}, \mathbf{A})$  is exact, and hence  $G$  is Gorenstein  $n$ - $\mathcal{X}$ -injective.  $\square$

**Corollary 4.5.** *Let  $R$  be a left  $n$ - $\mathcal{X}$ -coherent ring and  $\mathcal{X}$  be a class of  $R$ -modules. Then for any  $R$ -module  $G$ , the following assertions are equivalent:*

- (a)  $G$  is Gorenstein  $n$ - $\mathcal{X}$ -injective.
- (b) There is an exact sequence  $\cdots \rightarrow A_1 \rightarrow A_0 \rightarrow G \rightarrow 0$  of  $R$ -modules, where every  $A_i$  is  $n$ - $\mathcal{X}$ -injective.
- (c) There is a short exact sequence  $0 \rightarrow L \rightarrow M \rightarrow G \rightarrow 0$  of  $R$ -modules, where  $M$  is  $n$ - $\mathcal{X}$ -injective and  $L$  is Gorenstein  $n$ - $\mathcal{X}$ -injective.

*Proof.* The implications (a) $\Rightarrow$ (b) and (a) $\Rightarrow$ (c) follow from the definition.

(b) $\Rightarrow$ (a) There is an exact sequence  $0 \rightarrow G \rightarrow I^0 \rightarrow I^1 \rightarrow \cdots$ , where every  $I^i$  is injective for any  $i \geq 0$ . By Remark 3.4, each  $I^i$  is  $n$ - $\mathcal{X}$ -injective. So an exact sequence  $\cdots \rightarrow A_1 \rightarrow A_0 \rightarrow I^0 \rightarrow I^1 \rightarrow \cdots$  of  $n$ - $\mathcal{X}$ -injective modules exists, where  $G = \ker(I^0 \rightarrow I^1)$ . Therefore,  $G$  is Gorenstein  $n$ - $\mathcal{X}$ -injective by Theorem 4.4.

(c) $\Rightarrow$ (b) Assume that there is an exact sequence

$$0 \longrightarrow L \longrightarrow M \longrightarrow G \longrightarrow 0, \quad (1)$$

where  $M$  is  $n$ - $\mathcal{X}$ -injective and  $L$  is Gorenstein  $n$ - $\mathcal{X}$ -injective. Since  $L$  is Gorenstein  $n$ - $\mathcal{X}$ -injective, there is an exact sequence

$$\cdots \longrightarrow A'_2 \longrightarrow A'_1 \longrightarrow A'_0 \longrightarrow L \longrightarrow 0, \quad (2)$$

where every  $A'_i$  is  $n$ - $\mathcal{X}$ -injective. Assembling the sequences (1) and (2), we get the exact sequence  $\cdots \rightarrow A'_2 \rightarrow A'_1 \rightarrow A'_0 \rightarrow M \rightarrow G \rightarrow 0$ , where  $M$  and  $A'_i$  are  $n$ - $\mathcal{X}$ -injective, as desired.  $\square$

**Corollary 4.6.** *Let  $R$  be a left  $n$ - $\mathcal{X}$ -coherent ring and  $\mathcal{X}$  be a class of  $R$ -modules. Then for any right  $R$ -module  $G$ , the following assertions are equivalent:*

- (a)  $G$  is Gorenstein  $n$ - $\mathcal{X}$ -flat.
- (b) There is an exact sequence  $0 \rightarrow G \rightarrow B^0 \rightarrow B^1 \rightarrow \cdots$  of right  $R$ -modules, where every  $B^i$  is  $n$ - $\mathcal{X}$ -flat.
- (c) There is a short exact sequence  $0 \rightarrow G \rightarrow M \rightarrow L \rightarrow 0$  of right  $R$ -modules, where  $M$  is  $n$ - $\mathcal{X}$ -flat and  $L$  is Gorenstein  $n$ - $\mathcal{X}$ -flat.

*Proof.* The implications (a) $\Rightarrow$ (b) and (a) $\Rightarrow$ (c) follow from the definition.

(b) $\Rightarrow$ (a) For any right  $R$ -module  $G$ , there is an exact sequence

$$\cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow G \longrightarrow 0,$$

where  $P_i$  is flat for any  $i \geq 0$ . By Remark 3.4, every  $P_i$  is  $n$ - $\mathcal{X}$ -flat. Hence, there is an exact sequence  $\cdots \rightarrow P_1 \rightarrow P_0 \rightarrow B^0 \rightarrow B^1 \rightarrow \cdots$  of  $n$ - $\mathcal{X}$ -flat right modules, where  $G = \ker(B^0 \rightarrow B^1)$ . So by Theorem 4.4,  $G$  is Gorenstein  $n$ - $\mathcal{X}$ -flat.

(c) $\Rightarrow$ (b) Assume there is an exact sequence

$$0 \longrightarrow G \longrightarrow M \longrightarrow L \longrightarrow 0, \tag{3}$$

where  $M$  is  $n\mathcal{X}$ -flat and  $L$  is Gorenstein  $n\mathcal{X}$ -flat. Since  $L$  is Gorenstein  $n\mathcal{X}$ -flat, there is an exact sequence

$$0 \longrightarrow L \longrightarrow F^0 \longrightarrow F^1 \longrightarrow F^2 \longrightarrow \dots, \tag{4}$$

where every  $F^i$  is  $n\mathcal{X}$ -flat. Assembling the sequences (3) and (4), we get the exact sequence  $0 \rightarrow G \rightarrow M \rightarrow F^0 \rightarrow F^1 \rightarrow F^2 \rightarrow \dots$ , where  $M$  and all  $F^i$  are  $n\mathcal{X}$ -flat, as desired.  $\square$

**Proposition 4.7.** *Let  $\mathcal{X}$  be a class of  $R$ -modules. Then the following hold:*

- (a) *Every direct product of Gorenstein  $n\mathcal{X}$ -injective  $R$ -modules is a Gorenstein  $n\mathcal{X}$ -injective  $R$ -module.*
- (b) *Every direct sum of Gorenstein  $n\mathcal{X}$ -flat right  $R$ -modules is a Gorenstein  $n\mathcal{X}$ -flat  $R$ -module.*

*Proof.* (a) Let  $U \in \mathcal{X}_n$  and  $\{A_i\}_{i \in I}$  be a family of  $n\mathcal{X}$ -injective  $R$ -modules. By [1, Lemma 2.7],  $\prod A_i$  is  $n\mathcal{X}$ -injective. So if  $\{G_i\}_{i \in I}$  is a family of Gorenstein  $n\mathcal{X}$ -injective  $R$ -modules, then we can see that the corresponding exact sequences of  $n\mathcal{X}$ -injective  $R$ -modules  $\mathbf{A}_i = \dots \rightarrow (A_i)_1 \rightarrow (A_i)_0 \rightarrow (A_i)^0 \rightarrow (A_i)^1 \rightarrow \dots$ , with  $G_i = \ker((A_i)^0 \rightarrow (A_i)^1)$ , induce the following exact sequence of  $n\mathcal{X}$ -injective  $R$ -modules:

$$\prod_{i \in I} \mathbf{A}_i = \dots \longrightarrow \prod_{i \in I} (A_i)_1 \longrightarrow \prod_{i \in I} (A_i)_0 \longrightarrow \prod_{i \in I} (A_i)^0 \longrightarrow \prod_{i \in I} (A_i)^1 \longrightarrow \dots,$$

where  $\prod_{i \in I} G_i = \ker(\prod_{i \in I} (A_i)^0 \rightarrow \prod_{i \in I} (A_i)^1)$ . If  $K_{n-1}$  is special  $\mathcal{X}$ -presented, then

$$\text{Hom}_R(K_{n-1}, \prod_{i \in I} \mathbf{A}_i) \cong \prod_{i \in I} \text{Hom}_R(K_{n-1}, \mathbf{A}_i).$$

By hypothesis,  $\text{Hom}_R(K_{n-1}, \mathbf{A}_i)$  is exact, and consequently  $\prod_{i \in I} G_i$  is Gorenstein  $n\mathcal{X}$ -injective.

(b) Let  $U \in \mathcal{X}_n$  and  $\{I_i\}_{i \in J}$  be a family of  $n\mathcal{X}$ -flat right  $R$ -modules. By [1, Lemma 2.7],  $\bigoplus_{i \in J} I_i$  is  $n\mathcal{X}$ -flat. So if  $\{G_i\}_{i \in J}$  is a family of Gorenstein  $n\mathcal{X}$ -flat right  $R$ -modules, the corresponding exact sequences of  $n\mathcal{X}$ -flat right  $R$ -modules  $\mathbf{I}_i = \dots \rightarrow (I_i)_1 \rightarrow (I_i)_0 \rightarrow (I_i)^0 \rightarrow (I_i)^1 \rightarrow \dots$ , where  $G_i = \ker((I_i)^0 \rightarrow (I_i)^1)$ , induce the following exact sequence of  $n\mathcal{X}$ -flat right  $R$ -modules:

$$\bigoplus_{i \in J} \mathbf{I}_i = \dots \longrightarrow \bigoplus_{i \in J} (I_i)_1 \longrightarrow \bigoplus_{i \in J} (I_i)_0 \longrightarrow \bigoplus_{i \in J} (I_i)^0 \longrightarrow \bigoplus_{i \in J} (I_i)^1 \longrightarrow \dots,$$

where  $\bigoplus_{i \in J} G_i = \ker((\bigoplus_{i \in J} I_i)^0 \rightarrow (\bigoplus_{i \in J} I_i)^1)$ . If  $K_{n-1}$  is special  $\mathcal{X}$ -presented, then

$$\left(\bigoplus_{i \in J} \mathbf{I}_i \otimes_R K_{n-1}\right) \cong \bigoplus_{i \in J} (\mathbf{I}_i \otimes_R K_{n-1}).$$

By hypothesis,  $(\mathbf{I}_i \otimes_R K_{n-1})$  is exact, and consequently  $\bigoplus_{i \in J} G_i$  is Gorenstein  $n$ - $\mathcal{X}$ -flat.  $\square$

Now, we study the Gorenstein  $n$ - $\mathcal{X}$ -injectivity and Gorenstein  $n$ - $\mathcal{X}$ -flatness of modules in short exact sequences.

**Proposition 4.8.** *Let  $R$  be a left  $n$ - $\mathcal{X}$ -coherent ring and  $\mathcal{X}$  a class of  $R$ -modules.*

- (a) *Let  $0 \rightarrow A \rightarrow G \rightarrow N \rightarrow 0$  be an exact sequence of  $R$ -modules. If  $A$  and  $N$  are Gorenstein  $n$ - $\mathcal{X}$ -injective, then  $G$  is Gorenstein  $n$ - $\mathcal{X}$ -injective.*
- (b) *Let  $0 \rightarrow K \rightarrow G \rightarrow B \rightarrow 0$  be an exact sequence of right  $R$ -modules. If  $K$  and  $B$  are Gorenstein  $n$ - $\mathcal{X}$ -flat, then  $G$  is Gorenstein  $n$ - $\mathcal{X}$ -flat.*

*Proof.* (a) Since  $A$  and  $N$  are Gorenstein  $n$ - $\mathcal{X}$ -injective, by Corollary 4.5 there exist exact sequences  $0 \rightarrow K \rightarrow A_0 \rightarrow A \rightarrow 0$  and  $0 \rightarrow L \rightarrow N_0 \rightarrow N \rightarrow 0$  of  $R$ -modules, where  $A_0$  and  $N_0$  are  $n$ - $\mathcal{X}$ -injective and, moreover,  $K$  and  $L$  are Gorenstein  $n$ - $\mathcal{X}$ -injective. Now we consider the following commutative diagram:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & K & \longrightarrow & K \oplus L & \longrightarrow & L \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & A_0 & \longrightarrow & A_0 \oplus N_0 & \longrightarrow & N_0 \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & A & \longrightarrow & G & \longrightarrow & N \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
 \end{array}$$

The exactness of the middle horizontal sequence, where  $A_0$  and  $N_0$  are  $n$ - $\mathcal{X}$ -injective, implies that  $A_0 \oplus N_0$  is  $n$ - $\mathcal{X}$ -injective by [1, Lemma 2.7]. Also,  $K \oplus L$  is Gorenstein  $n$ - $\mathcal{X}$ -injective by Proposition 4.7(a). Hence, from the middle vertical sequence and Corollary 4.5, we deduce that  $G$  is Gorenstein  $n$ - $\mathcal{X}$ -injective.

(b) Since  $K$  and  $B$  are Gorenstein  $n$ - $\mathcal{X}$ -flat, by Corollary 4.6 there exist exact sequences  $0 \rightarrow K \rightarrow K_0 \rightarrow L_1 \rightarrow 0$  and  $0 \rightarrow B \rightarrow B_0 \rightarrow L'_1 \rightarrow 0$  of  $R$ -modules, where  $K_0, B_0$  are  $n$ - $\mathcal{X}$ -flat and  $L_1, L'_1$  are Gorenstein  $n$ - $\mathcal{X}$ -flat. Consider the following commutative diagram:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & K & \longrightarrow & G & \longrightarrow & B \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & K_0 & \longrightarrow & K_0 \oplus B_0 & \longrightarrow & B_0 \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & L_1 & \longrightarrow & L_1 \oplus L'_1 & \longrightarrow & L'_1 \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
 \end{array}$$

The exactness of the middle horizontal sequence with  $K_0$  and  $B_0$   $n\text{-}\mathcal{X}$ -flat implies that  $K_0 \oplus B_0$  is  $n\text{-}\mathcal{X}$ -flat by [1, Lemma 2.7]. Also,  $L_1 \oplus L'_1$  is Gorenstein  $n\text{-}\mathcal{X}$ -flat by Proposition 4.7(b). Hence, from the middle vertical sequence and Corollary 4.6, we deduce that  $G$  is Gorenstein  $n\text{-}\mathcal{X}$ -flat.  $\square$

The left  $n\text{-}\mathcal{X}$ -injective dimension of an  $R$ -module  $M$ , denoted by  $\text{id}_{\mathcal{X}_n}(M)$ , is defined to be the least non-negative integer  $m$  such that  $\text{Ext}_R^{n+m+1}(U, M) = 0$  for any  $U \in \mathcal{X}_n$ . The left  $n\text{-}\mathcal{X}$ -flat dimension of a right  $R$ -module  $M$ , which is denoted by  $\text{fd}_{\mathcal{X}_n}(M)$ , is defined to be the least non-negative integer  $m$  such that  $\text{Tor}_{n+m+1}^R(M, U) = 0$  for any  $U \in \mathcal{X}_n$ . If  $G$  is Gorenstein  $n\text{-}\mathcal{X}$ -injective, then  $\text{id}_{\mathcal{X}_n}(G) = m$  if there is an exact sequence  $0 \rightarrow A_m \rightarrow \dots \rightarrow A_1 \rightarrow A_0 \rightarrow G \rightarrow 0$  or  $0 \rightarrow G \rightarrow A^0 \rightarrow A^1 \rightarrow \dots \rightarrow A^m \rightarrow 0$  of  $n\text{-}\mathcal{X}$ -injective  $R$ -modules. Similarly, if  $G$  is Gorenstein  $n\text{-}\mathcal{X}$ -flat and  $\text{fd}_{\mathcal{X}_n}(G) = m$ , then the above exact sequences for  $n\text{-}\mathcal{X}$ -flat right  $R$ -modules exist.

The following theorems are generalizations of Corollaries 4.5 and 4.6 and Proposition 4.8.

**Theorem 4.9.** *Let  $R$  be a left  $n\text{-}\mathcal{X}$ -coherent ring and  $\mathcal{X}$  be a class of  $R$ -modules which is closed under kernels of epimorphisms. Then for every  $R$ -module  $G$ , the following statements are equivalent:*

- (a)  $G$  is Gorenstein  $n\text{-}\mathcal{X}$ -injective.
- (b) There exists an  $n\text{-}\mathcal{X}$ -injective resolution of  $G$

$$\dots \xrightarrow{f_3} A_2 \xrightarrow{f_2} A_1 \xrightarrow{f_1} A_0 \xrightarrow{f_0} G \longrightarrow 0$$

such that  $\bigoplus_{i=0}^{\infty} \text{im}(f_i)$  is Gorenstein  $n\text{-}\mathcal{X}$ -injective.

- (c) There exists an exact sequence  $\dots \xrightarrow{f_3} A_2 \xrightarrow{f_2} A_1 \xrightarrow{f_1} A_0 \xrightarrow{f_0} G \rightarrow 0$  of  $R$ -modules, where  $A_i$  has finite  $n\text{-}\mathcal{X}$ -injective dimension for any  $i \geq 0$ , such that  $\bigoplus_{i=0}^{\infty} \text{im}(f_i)$  is Gorenstein  $n\text{-}\mathcal{X}$ -injective.

*Proof.* (a) $\Rightarrow$ (b) By Corollary 4.5, there is an exact sequence

$$\dots \xrightarrow{f_3} A_2 \xrightarrow{f_2} A_1 \xrightarrow{f_1} A_0 \xrightarrow{f_0} G \longrightarrow 0,$$

where every  $A_i$  is  $n\text{-}\mathcal{X}$ -injective. Consider the following exact sequences:

$$\begin{array}{ccccccc} \dots & \longrightarrow & A_2 & \longrightarrow & A_1 & \longrightarrow & A_0 & \longrightarrow & \text{im}(f_0) & \longrightarrow & 0, \\ \dots & \longrightarrow & A_3 & \longrightarrow & A_2 & \longrightarrow & A_1 & \longrightarrow & \text{im}(f_1) & \longrightarrow & 0, \\ & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots \end{array}$$

By Proposition 3.6,  $\bigoplus_{i \in I} A_i$  is  $n\text{-}\mathcal{X}$ -injective. Thus, there exists an exact sequence

$$\dots \longrightarrow \bigoplus_{i \geq 2} A_i \longrightarrow \bigoplus_{i \geq 1} A_i \longrightarrow \bigoplus_{i \geq 0} A_i \longrightarrow \bigoplus_{i=0}^{\infty} \text{im}(f_i) \longrightarrow 0$$

of  $n\text{-}\mathcal{X}$ -injective  $R$ -modules. Consequently, Corollary 4.5 implies that  $\bigoplus_{i=0}^{\infty} \text{im}(f_i)$  is Gorenstein  $n\text{-}\mathcal{X}$ -injective.

- (b) $\Rightarrow$ (c) Trivial.

(c)⇒(a) Let  $\dots \xrightarrow{f_3} A_2 \xrightarrow{f_2} A_1 \xrightarrow{f_1} A_0 \xrightarrow{f_0} G \rightarrow 0$  be an exact sequence of  $R$ -modules, where  $A_i$  has finite  $n$ - $\mathcal{X}$ -injective dimension. By Corollary 4.5, it is sufficient to prove that  $A_i$  is  $n$ - $\mathcal{X}$ -injective for any  $i \geq 0$ . Consider the short exact sequence  $0 \rightarrow \text{im}(f_{i+1}) \rightarrow A_i \rightarrow \text{im}(f_i) \rightarrow 0$  for any  $i \geq 0$ . Therefore, the short exact sequence  $0 \rightarrow \bigoplus_{i=0}^{\infty} \text{im}(f_{i+1}) \rightarrow \bigoplus_{i=0}^{\infty} A_i \rightarrow \bigoplus_{i=0}^{\infty} \text{im}(f_i) \rightarrow 0$  exists. By (c) and Proposition 4.8(a),  $\bigoplus_{i=0}^{\infty} A_i$  is Gorenstein  $n$ - $\mathcal{X}$ -injective. Also,  $\bigoplus_{i=0}^{\infty} A_i$  has finite  $n$ - $\mathcal{X}$ -injective dimension. If  $\text{id}_{\mathcal{X}_n}(\bigoplus_{i=0}^{\infty} A_i) = k$ , then there exists an  $n$ - $\mathcal{X}$ -injective resolution of  $\bigoplus_{i=0}^{\infty} A_i$ :

$$0 \longrightarrow B_k \longrightarrow B_{k-1} \longrightarrow \dots \longrightarrow B_0 \longrightarrow \bigoplus_{i \in I} A_i \longrightarrow 0.$$

Let  $L_{k-1} = \ker(B_{k-1} \rightarrow B_{k-2})$  and  $U \in \mathcal{X}_n$ . Then the exact sequence  $0 \rightarrow B_k \rightarrow B_{k-1} \rightarrow L_{k-1} \rightarrow 0$  induces the following exact sequence:

$$0 = \text{Ext}_R^n(U, B_{k-1}) \longrightarrow \text{Ext}_R^n(U, L_{k-1}) \longrightarrow \text{Ext}_R^{n+1}(U, B_k) \longrightarrow \dots$$

By hypothesis,  $B_k$  is  $(n+1)$ - $\mathcal{X}$ -injective, and we also find that  $U \in \mathcal{X}_{n+1}$  since  $R$  is  $n$ - $\mathcal{X}$ -coherent. So  $\text{Ext}_R^{n+1}(U, B_k) = 0$ , and hence  $\text{Ext}_R^n(U, L_{k-1}) = 0$ . Thus,  $L_{k-1}$  is  $n$ - $\mathcal{X}$ -injective. Then, with the same process, we see that  $\bigoplus_{i=0}^{\infty} A_i$  is  $n$ - $\mathcal{X}$ -injective, and so by Proposition 3.6,  $A_i$  is  $n$ - $\mathcal{X}$ -injective for any  $i \geq 0$ . □

For the following theorem, the proof is similar to that of Theorem 4.9.

**Theorem 4.10.** *Let  $R$  be a left  $n$ - $\mathcal{X}$ -coherent ring and  $\mathcal{X}$  be a class of  $R$ -modules which is closed under kernels of epimorphisms. Then for every right  $R$ -module  $G$ , the following statements are equivalent:*

- (a)  $G$  is Gorenstein  $n$ - $\mathcal{X}$ -flat.
- (b) There exists the following right  $n$ - $\mathcal{X}$ -flat resolution of  $G$ :

$$0 \longrightarrow G \xrightarrow{f^0} I^0 \xrightarrow{f^1} I^1 \xrightarrow{f^2} \dots$$

such that  $\bigoplus_{i=0}^{\infty} \text{im}(f^i)$  is Gorenstein  $n$ - $\mathcal{X}$ -flat.

- (c) There exists an exact sequence

$$0 \longrightarrow G \xrightarrow{f^0} I^0 \xrightarrow{f^1} I^1 \xrightarrow{f^2} \dots$$

of right  $R$ -modules, where  $I_i$  has finite  $n$ - $\mathcal{X}$ -flat dimension for any  $i \geq 0$ , such that  $\bigoplus_{i=0}^{\infty} \text{im}(f^i)$  is Gorenstein  $n$ - $\mathcal{X}$ -flat.

### 5 $\mathcal{X}$ -FC-Rings

A ring  $R$  is called a left  $\mathcal{X}$ -FC-ring if  $R$  is self left  $n$ - $\mathcal{X}$ -injective and left  $n$ - $\mathcal{X}$ -coherent. In this section, we investigate properties of Gorenstein  $n$ - $\mathcal{X}$ -injective and  $n$ - $\mathcal{X}$ -flat modules over  $\mathcal{X}$ -FC-rings, thus generalizing several classical results. Notice that the notion of  $\mathcal{X}$ -FC-ring generalizes the classical notions of quasi-Frobenius and FC (i.e., IF) rings.

It is well known that quasi-Frobenius (resp., FC) rings can be viewed as rings over which all modules are Gorenstein injective (resp., Gorenstein FP-injective). Here we extend this fact as well as other known ones to our new context.

**Proposition 5.1.** *Let  $\mathcal{X}$  be a class of  $R$ -modules. Then every  $R$ -module is Gorenstein  $n$ - $\mathcal{X}$ -injective if and only if every projective  $R$ -module is  $n$ - $\mathcal{X}$ -injective and for any  $R$ -module  $N$ ,  $\text{Hom}_R(-, N)$  is exact with respect to all special short exact sequences of  $\mathcal{X}_n$  with modules of finite projective dimension.*

*Proof.* ( $\Rightarrow$ ) Let  $M$  be a projective  $R$ -module. Then by hypothesis  $M$  is Gorenstein  $n$ - $\mathcal{X}$ -injective. So the following  $n$ - $\mathcal{X}$ -injective resolution of  $M$  exists:

$$\cdots \longrightarrow A_1 \longrightarrow A_0 \longrightarrow M \longrightarrow 0.$$

Since  $M$  is projective,  $M$  is  $n$ - $\mathcal{X}$ -injective as a direct summand of  $A_0$ . Also, by hypothesis and Definition 4.1,  $\text{Hom}_R(-, N)$  is exact with respect to all special short exact sequences with modules of finite projective dimension since every  $R$ -module  $N$  is Gorenstein  $n$ - $\mathcal{X}$ -injective.

( $\Leftarrow$ ) Choose an injective resolution  $0 \rightarrow G \rightarrow E^0 \rightarrow E^1 \rightarrow \cdots$  of  $G$  and a projective resolution  $\cdots \rightarrow F_1 \rightarrow F_0 \rightarrow G \rightarrow 0$  of  $G$ , where by hypothesis every  $F_i$  is  $n$ - $\mathcal{X}$ -injective. Assembling these resolutions, we get by Remark 3.4 the following exact sequence of  $n$ - $\mathcal{X}$ -injective  $R$ -modules:

$$\mathbf{A} = \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow E^0 \longrightarrow E^1 \longrightarrow \cdots,$$

where  $G = \ker(E^0 \rightarrow E^1)$ ,  $K^i = \ker(E^i \rightarrow E^{i+1})$  and  $K_i = \ker(F_i \rightarrow F_{i-1})$  for any  $i \geq 1$ . Let  $K_{n-1}$  be a special  $\mathcal{X}$ -presented module with  $\text{pd}_R(K_{n-1}) < \infty$ . Then by hypothesis, we have  $\text{Ext}_R^1(K_{n-1}, G) = \text{Ext}_R^1(K_{n-1}, K_i) = \text{Ext}_R^1(K_{n-1}, K^i) = 0$ . So  $\text{Hom}_R(K_{n-1}, \mathbf{A})$  is exact, and hence  $G$  is Gorenstein  $n$ - $\mathcal{X}$ -injective.  $\square$

**Proposition 5.2.** *Let  $\mathcal{X}$  be a class of  $R$ -modules. Then every right  $R$ -module is Gorenstein  $n$ - $\mathcal{X}$ -flat if and only if every injective right  $R$ -module is  $n$ - $\mathcal{X}$ -flat and for any  $R$ -module  $N$ ,  $N \otimes_R -$  is exact with respect to all special short exact sequences of  $\mathcal{X}_n$  with modules of finite projective dimension.*

*Proof.* Similar to the proof of Proposition 5.1.  $\square$

**Theorem 5.3.** *Let  $R$  be a left  $n$ - $\mathcal{X}$ -coherent ring and  $\mathcal{X}$  be a class of  $R$ -modules. Then the following statements are equivalent:*

- (a) *Every  $R$ -module is Gorenstein  $n$ - $\mathcal{X}$ -injective.*
- (b) *Every projective  $R$ -module is  $n$ - $\mathcal{X}$ -injective.*
- (c)  *$R$  is self left  $n$ - $\mathcal{X}$ -injective.*

*Proof.* The implications (a) $\Rightarrow$ (b) $\Rightarrow$ (c) hold by Proposition 5.1.

(c) $\Rightarrow$ (a) Let  $G$  be an  $R$ -module and  $\cdots \rightarrow F_1 \rightarrow F_0 \rightarrow G \rightarrow 0$  be any free resolution of  $G$ . Then by Proposition 3.6, each  $F_i$  is  $n$ - $\mathcal{X}$ -injective. Hence, by Corollary 4.5 the proof completes.  $\square$

**Examples 5.4.** Let  $R = k[x^3, x^2, x^2y, xy^2, xy, y^2, y^3]$  be a ring and  $\mathcal{X}$  the class of all 1-presented  $R$ -modules. We claim that  $R$  is not 1- $\mathcal{X}$ -injective. Suppose on the contrary that  $R$  is 1- $\mathcal{X}$ -injective. We find that  $R/Rx^2$  is special  $\mathcal{X}$ -presented since  $Rx^2 \cong R$  is special  $\mathcal{X}$ -generated. In addition,  $\text{pd}_R(R/Rx^2) < \infty$ . Hence, by Proposition 5.1 and Theorem 5.3,  $R/Rx^2$  is projective. Therefore, the exact

sequence  $0 \rightarrow Rx^2 \rightarrow R \rightarrow R/Rx^2 \rightarrow 0$  splits. Thus,  $Rx^2$  is a direct summand of  $R$ , and so  $x^2$  is an idempotent, a contradiction.

Let  $\mathcal{X}$  be a class of graded  $R$ -modules. Then a graded ring  $R$  is said to be  $n$ -gr-regular if and only if  $R$  is  $n$ - $\mathcal{X}$ -regular if and only if every  $n$ -presented  $R$ -module in  $\mathcal{X}$  is projective if and only if every  $R$ -module in  $\mathcal{X}$  is  $n$ - $\mathcal{X}$ -injective if and only if every right  $R$ -module in  $\mathcal{X}$  is  $n$ - $\mathcal{X}$ -flat. This is a generalization of [19, Proposition 3.11]. Notice that, when  $n = 1$ ,  $R$  is gr-regular if and only if  $R$  is 1- $\mathcal{X}$ -regular (see [18]).

The example below shows that, for some class  $\mathcal{X}$  of  $R$ -modules and for any  $m > n$ , every Gorenstein  $n$ - $\mathcal{X}$ -injective (resp.,  $n$ - $\mathcal{X}$ -flat) module is Gorenstein  $m$ - $\mathcal{X}$ -injective (resp.,  $m$ - $\mathcal{X}$ -flat). But, Gorenstein  $m$ - $\mathcal{X}$ -injectivity (resp.,  $m$ - $\mathcal{X}$ -flatness) does not imply Gorenstein  $n$ - $\mathcal{X}$ -injectivity (resp.,  $n$ - $\mathcal{X}$ -flatness) in general.

*Examples 5.5.* (a) Let  $R$  be a graded ring and  $\mathcal{X}$  be a class of graded  $R$ -module. Then for any  $m > n$ , every Gorenstein  $n$ - $\mathcal{X}$ -injective (resp.,  $n$ - $\mathcal{X}$ -flat) module is Gorenstein  $m$ - $\mathcal{X}$ -injective (resp.,  $m$ - $\mathcal{X}$ -flat) since by [20, Remark 3.5] every  $n$ - $\mathcal{X}$ -injective (resp.,  $n$ - $\mathcal{X}$ -flat) module is  $m$ - $\mathcal{X}$ -injective (resp.,  $m$ - $\mathcal{X}$ -flat).

(b) Let  $R = k[X]$ , where  $k$  is a field and  $\mathcal{X}$  is a class of graded  $R$ -modules. Then by Remark 4.2, every graded left (resp., right)  $R$ -module is Gorenstein 2- $\mathcal{X}$ -injective (resp., 2- $\mathcal{X}$ -flat) as every 2-presented graded  $R$ -module is projective. We claim that there is a graded left (resp., right)  $R$ -module  $N$  that is not Gorenstein 1- $\mathcal{X}$ -injective (resp., 1- $\mathcal{X}$ -flat). Now suppose on the contrary that every graded left (resp., right)  $R$ -module is Gorenstein 1- $\mathcal{X}$ -injective (resp., 1- $\mathcal{X}$ -flat). If  $U$  is a finitely presented graded module, then the special exact sequence  $0 \rightarrow L \rightarrow F_0 \rightarrow U \rightarrow 0$  of graded modules exists. So by Proposition 5.1 (and Proposition 5.2),  $U$  is projective, and it follows that  $R$  is 1- $\mathcal{X}$ -regular or  $\mathcal{X}$ -regular, a contradiction; see [20, Example 3.6].

**Proposition 5.6.** *Let  $\mathcal{X}$  be a class of  $R$ -modules.*

- (a) *If  $G$  is a Gorenstein injective  $R$ -module, then  $\text{Hom}_R(-, G)$  is exact with respect to all special short exact sequences with modules of finite projective dimension.*
- (b) *If  $G$  is a Gorenstein flat right  $R$ -module, then  $G \otimes_R -$  is exact with respect to all special short exact sequences with modules of finite flat dimension.*

*Proof.* (a) Let  $0 \rightarrow K_n \rightarrow P_n \rightarrow K_{n-1} \rightarrow 0$  be a special short exact sequence of  $U \in \mathcal{X}_n$ . It is clear that  $\text{pd}_R(U) = m < \infty$  since  $\text{pd}_R(K_{n-1}) < \infty$ . Also, let  $G$  be Gorenstein injective. Then the following injective resolution of  $G$  exists:

$$0 \longrightarrow N \longrightarrow A_{m-1} \longrightarrow \cdots \longrightarrow A_0 \longrightarrow G \longrightarrow 0.$$

So  $\text{Ext}_R^{n+i}(U, A_j) = 0$  for all  $0 \leq j \leq m-1$  and any  $i \geq 0$ . Thus, we deduce that  $\text{Ext}_R^{n+i}(U, G) \cong \text{Ext}_R^{m+n+i}(U, N) = 0$  for any  $i \geq 0$ . Accordingly, we obtain  $\text{Ext}_R^1(K_{n-1}, G) \cong \text{Ext}_R^n(U, G) = 0$ .

- (b) The proof is similar to the one above. □

Now we can state the main result of this section.



**Theorem 5.7.** *Let  $R$  be a left  $n$ - $\mathcal{X}$ -coherent ring and  $\mathcal{X}$  be a class of  $R$ -modules. Then the following statements are equivalent:*

- (a)  $R$  is self left  $n$ - $\mathcal{X}$ -injective.
- (b) Every Gorenstein  $n$ - $\mathcal{X}$ -flat  $R$ -module is Gorenstein  $n$ - $\mathcal{X}$ -injective.
- (c) Every Gorenstein flat  $R$ -module is Gorenstein  $n$ - $\mathcal{X}$ -injective.
- (d) Every flat  $R$ -module is Gorenstein  $n$ - $\mathcal{X}$ -injective.
- (e) Every Gorenstein projective  $R$ -module is Gorenstein  $n$ - $\mathcal{X}$ -injective.
- (f) Every projective  $R$ -module is Gorenstein  $n$ - $\mathcal{X}$ -injective.
- (g) Every Gorenstein injective right  $R$ -module is Gorenstein  $n$ - $\mathcal{X}$ -flat.
- (h) Every injective right  $R$ -module is Gorenstein  $n$ - $\mathcal{X}$ -flat.
- (i) Every Gorenstein 1- $\mathcal{X}$ -injective right  $R$ -module is Gorenstein  $n$ - $\mathcal{X}$ -flat.
- (j) Every Gorenstein  $n$ - $\mathcal{X}$ -injective right  $R$ -module is Gorenstein  $n$ - $\mathcal{X}$ -flat.

*Proof.* The implications (a) $\Rightarrow$ (b)–(f) follow immediately from Theorem 5.3. The implications (c) $\Rightarrow$ (d) $\Rightarrow$ (f) and (e) $\Rightarrow$ (f) are trivial.

(c) $\Rightarrow$ (a) Assume that  $G$  is a projective  $R$ -module. Then  $G$  is flat, and so  $G$  is Gorenstein  $n$ - $\mathcal{X}$ -injective by (c). Thus, similarly to the proof of ( $\Rightarrow$ ) of Proposition 5.1, we see that  $G$  is  $n$ - $\mathcal{X}$ -injective. Hence, the assertion follows from Theorem 5.3.

(f) $\Rightarrow$ (a) This is similar to the proof of (c) $\Rightarrow$ (a).

(a) $\Rightarrow$ (i) By Theorem 3.5, every 1- $\mathcal{X}$ -injective right  $R$ -module is  $n$ - $\mathcal{X}$ -flat. Suppose that  $G$  is Gorenstein 1- $\mathcal{X}$ -injective. So an exact sequence

$$\mathbf{M} = \cdots \rightarrow M_1 \rightarrow M_0 \rightarrow M^0 \rightarrow M^1 \rightarrow \cdots$$

of  $n$ - $\mathcal{X}$ -flat right  $R$ -modules exists, where  $G = \ker(M^0 \rightarrow M^1)$ . Let  $K_{n-1}$  be special  $\mathcal{X}$ -presented with  $\text{fd}_R(K_{n-1}) < \infty$ . Then similarly to the proof of Theorem 4.4(a), we see that  $\mathbf{M} \otimes_R K_{n-1}$  is exact, and hence  $G$  is Gorenstein  $n$ - $\mathcal{X}$ -flat.

(i) $\Rightarrow$ (g) By Remark 3.4, every injective right  $R$ -module is 1- $\mathcal{X}$ -injective. So, if  $G$  is Gorenstein injective, then an exact sequence

$$\mathbf{E} = \cdots \rightarrow E_1 \rightarrow E_0 \rightarrow E^0 \rightarrow E^1 \rightarrow \cdots$$

of 1- $\mathcal{X}$ -injective right  $R$ -modules exists, where  $G = \ker(E^0 \rightarrow E^1)$ . Hence, if  $U \in \mathcal{X}_1$  with  $\text{pd}(U) < \infty$ , then  $U$  is special  $\mathcal{X}$ -presented, and by Proposition 5.6,  $\text{Hom}_R(U, \mathbf{E})$  is exact. Therefore,  $G$  is Gorenstein 1- $\mathcal{X}$ -injective.

(g) $\Rightarrow$ (h) is trivial since every injective  $R$ -module is Gorenstein injective.

(h) $\Rightarrow$ (a) Let  $M$  be an injective right  $R$ -module. Since  $M$  is Gorenstein  $n$ - $\mathcal{X}$ -flat, we have an exact sequence  $\mathbf{M} = \cdots \rightarrow M_1 \rightarrow M_0 \rightarrow M^0 \rightarrow M^1 \rightarrow \cdots$ , where any  $M_i$  is  $n$ - $\mathcal{X}$ -flat and  $M = \ker(M^0 \rightarrow M^1)$ . Then the split exact sequence  $0 \rightarrow M \rightarrow M^0 \rightarrow L \rightarrow 0$  implies that  $M$  is  $n$ - $\mathcal{X}$ -flat, and hence by Theorem 3.5 we deduce that  $R$  is self left  $n$ - $\mathcal{X}$ -injective.

(a) $\Rightarrow$ (j) Suppose that  $G$  is a Gorenstein  $n$ - $\mathcal{X}$ -injective right  $R$ -module. By Theorem 3.5(f), every  $n$ - $\mathcal{X}$ -injective right  $R$ -module is  $n$ - $\mathcal{X}$ -flat. Thus, an exact sequence  $\mathbf{N} = \cdots \rightarrow N_1 \rightarrow N_0 \rightarrow N^0 \rightarrow N^1 \rightarrow \cdots$  of  $n$ - $\mathcal{X}$ -flat right  $R$ -modules exists, where  $G = \ker(N^0 \rightarrow N^1)$ . Then similarly to the proof of Theorem 4.4(a), (j) follows.

(j) $\Rightarrow$ (g) is clear. □

*Acknowledgements.* The authors thank the referee for the very helpful comments and suggestions. Arij Benkhadra's research reported in this publication was supported by a scholarship from the Graduate Research Assistantships in Developing Countries Program of the Commission for Developing Countries of the International Mathematical Union.

## References

- [1] D. Bennis,  $n$ - $\mathcal{X}$ -Coherent rings, *Int. Electron. J. Algebra* **7** (2010) 128–139.
- [2] N. Bourbaki, *Éléments de Mathématiques. Algèbre: Chapitre 10, Algèbre Homologique*, Masson, Paris, 1980.
- [3] S.U. Chase, Direct product of modules, *Trans. Amer. Math. Soc.* **97** (1960) 457–473.
- [4] J. L. Chen, N. Ding, On  $n$ -coherent rings, *Comm. Algebra* **24** (1996) 3211–3216.
- [5] D.L. Costa, Parameterizing families of non-Noetherian rings, *Comm. Algebra* **22** (1994) 3997–4011.
- [6] S. Crivei, B. Torrecillas, On some monic covers and epic envelopes, *Arab. J. Sci. Eng. Sect. C Theme Issues* **33** (2) (2008) 123–135.
- [7] D.E. Dobbs, S. Kabbaj, N. Mahdou,  $n$ -Coherent rings and modules, in: *Commutative Ring Theory* (Fès, 1995), Lecture Notes in Pure and Appl. Math., 185, Dekker, New York, 1997, pp. 269–281.
- [8] E.E. Enochs, O.M.G. Jenda, Gorenstein injective and projective modules, *Math. Z.* **220** (1995) 611–633.
- [9] E.E. Enochs, O.M.G. Jenda, B. Torrecillas, Gorenstein flat modules, *Nanjing Daxue Xuebao Shuxue Bannian Kan* **10** (1) (1993) 1–9.
- [10] Z.H. Gao, J. Peng,  $n$ -Strongly Gorenstein graded modules, *Czech. Math. J.* **69** (2019) 55–73.
- [11] Z.H. Gao, F.G. Wang, Coherent rings and Gorenstein FP-injective modules, *Comm. Algebra* **40** (2012) 1669–1679.
- [12] H. Holm, Gorenstein homological dimensions, *J. Pure Appl. Algebra* **189** (2004) 167–193.
- [13] L.X. Mao, Strongly Gorenstein graded modules, *Front. Math. China* **12** (1) (2017) 157–176.
- [14] L.X. Mao, N.Q. Ding, Gorenstein FP-injective and Gorenstein flat modules, *J. Algebra Appl.* **7** (4) (2008) 491–506.
- [15] C. Năstăsescu, Some constructions over graded rings, *J. Algebra Appl.* **120** (1989) 119–138.
- [16] J. Rotman, *An Introduction to Homological Algebra*, 2nd ed., Universitext, Springer, New York, 2009.
- [17] B. Stenström, Coherent rings and FP-injective modules, *J. London Math. Soc.* **2** (1970) 323–329.
- [18] B. Stenström, *Rings of Quotients*, Springer-Verlag, Berlin, 1975.
- [19] X.Y. Yang, Z.K. Liu, FP-gr-injective modules, *Math. J. Okayama Univ.* **53** (2011) 83–100.
- [20] T.W. Zhao, Z.H. Gao, Z.Y. Huang, Relative FP-gr-injective and gr-flat modules, *Internat. J. Algebra Comput.* **28** (2018) 959–977.



# A logical analysis of fixpoint theorems

Arij Benkhadra\* and Isar Stubbe†

October 7, 2022‡

## Abstract

We prove a fixpoint theorem for contractions on Cauchy-complete quantale-enriched categories. It holds for any quantale whose underlying lattice is continuous, and applies to a specific notion of contraction. Sufficient conditions for the uniqueness of the fixpoint are established. Throughout, we stress the importance of the continuity of the underlying quantale. Examples include known and new fixpoint theorems for metric spaces, fuzzy metric spaces, and probabilistic metric spaces.

## Introduction

A beautiful and important result in metric space theory, is Banach's fixpoint theorem [3] from 1922: "Every contraction on a complete metric space admits a unique fixpoint." Many generalizations and applications of Banach's theorem have been, and are still, studied. In 1972, Lawvere [21] famously showed that metric spaces are a particular instance of enriched categories. It is thus natural to investigate whether fixpoint theorems still make sense in the vast context of enriched categories. This is precisely the subject of this paper.

More precisely, we shall take quantale-enriched categories as generalization of metric spaces. That is to say, we fix a quantale  $Q$ , and work with categories, functors and distributors enriched in  $Q$ . Our contribution shows that fixpoint theorems for  $Q$ -categories depend on the interplay between three essential

---

\*Laboratoire de Mathématiques Pures et Appliquées, Université du Littoral, Calais, France; Centre de Recherche de Mathématiques, Université Mohammed V, Rabat, Maroc; [arij.benkhadra@univ-littoral.fr](mailto:arij.benkhadra@univ-littoral.fr)

†Laboratoire de Mathématiques Pures et Appliquées, Université du Littoral, Calais, France; [isar.stubbe@univ-littoral.fr](mailto:isar.stubbe@univ-littoral.fr)

‡This text is a longer version of the article with the same title that the authors have submitted for publication in an international journal. In particular Subsections 1.2 and 1.4 are considerably more elaborate in this text. The sole purpose of this text is its inclusion in the PhD thesis of Arij Benkhadra.

parameters. Indeed, a given contraction must be “strong enough” (we shall measure its strength by means of a control function); the space on which it acts must be “complete enough” for the Picard iteration to converge to a fixpoint (we shall take this to be Cauchy-completeness in the sense of Lawvere); but we also need sufficiently strong algebraic properties of the underlying quantale  $Q$  to allow for the formulation of precisely that convergence.

In concreto, we shall prove a fixpoint theorem for Cauchy-complete  $Q$ -categories<sup>1</sup> that holds for any quantale  $Q$  whose underlying complete lattice is continuous and for a specific notion of contraction. Besides, we make plain when and why such a fixpoint is unique (up to isomorphism). As examples we find the classical Banach fixpoint theorem for metric spaces, and Boyd and Wong’s [5] generalization thereof (taking the underlying quantale to be the positive real numbers); but we also formulate new results for fuzzy ordered sets (when working over a left-continuous  $t$ -norm) and for probabilistic metric spaces (now the quantale is the tensor product of the positive reals with a left-continuous  $t$ -norm).

In Section 1 we shall provide all the necessary notions from quantale-enriched category theory to make this paper reasonably self-contained; we follow [29] for the general theory, and [16] specifically for the comparison between categorical and sequential Cauchy-completeness. In Section 2 we first introduce the contractions that we are interested in, then we show how these contractions determine Cauchy distributors under the appropriate algebraic condition on the quantale  $Q$ , and finally we formulate the resulting fixpoint theorem for Cauchy-complete  $Q$ -categories. The examples in Section 3 show how our fixpoint theorem generalizes known results from the literature, and provides for new results too. We end with a short conclusion in Section 4.

## 1. Quantale-enriched categories

### 1.1 $Q$ -enriched categories, functors and distributors

In this section we recall some key notions from [29] on quantale-enriched categories<sup>2</sup>; we encourage the reader to go back-and-forth to Subsection 1.3 for

<sup>1</sup>To stay faithful to Banach’s theorem in the metric case, we have chosen to study fixpoints for contractions on *Cauchy-complete*  $Q$ -categories. Let us mention, though, that other authors have studied other kinds of completeness, e.g. Wagner [31] chooses *liminf-complete*  $Q$ -categories, whereas Ackerman [1] works with *spherically complete*  $Q$ -categories (and both use a commutative quantale  $Q$ ).

<sup>2</sup>That reference actually treats the more general *quantaloid*-enriched category theory, but the reader will easily convert those results to the simpler quantale-enriched case. See also [30] for a gentle introduction to the subject.

the relevant examples.

Throughout, we fix a quantale  $Q = (Q, \vee, \circ, 1)$ : it is a complete sup-lattice  $(Q, \vee)$  endowed with a monoid<sup>3</sup> structure  $(Q, \circ, 1)$  such that the product distributes over arbitrary suprema:

$$s \circ \left( \bigvee_i t_i \right) = \bigvee_i (s \circ t_i) \quad \text{and} \quad \left( \bigvee_i s_i \right) \circ t = \bigvee_i (s_i \circ t).$$

In other words, but more abstractly, a quantale is a monoid in the symmetric monoidal closed category  $\mathbf{Sup}$  of complete lattices and supremum-preserving morphisms.

A  $Q$ -enriched category  $\mathbb{C}$  (or  $Q$ -category  $\mathbb{C}$  for short) consists of a set  $\mathbb{C}_0$  (of “objects”) together with a  $Q$ -valued (“hom”) precategory

$$\mathbb{C}: \mathbb{C}_0 \times \mathbb{C}_0 \rightarrow Q: (x, y) \mapsto \mathbb{C}(x, y)$$

satisfying, for all  $x, y, z \in \mathbb{C}_0$ , the following (“composition” and “identity”) conditions:

$$\mathbb{C}(x, y) \circ \mathbb{C}(y, z) \leq \mathbb{C}(x, z) \quad \text{and} \quad 1 \leq \mathbb{C}(x, x).$$

A  $Q$ -functor  $F: \mathbb{C} \rightarrow \mathbb{D}$  between two  $Q$ -categories is a function  $F: \mathbb{C}_0 \rightarrow \mathbb{D}_0: x \mapsto Fx$  satisfying, for all  $x, x' \in \mathbb{C}_0$ , the (“functoriality”) condition

$$\mathbb{C}(x', x) \leq \mathbb{D}(Fx', Fx).$$

Two such  $Q$ -functors  $F: \mathbb{A} \rightarrow \mathbb{B}$  and  $G: \mathbb{B} \rightarrow \mathbb{C}$  can be composed in the obvious way to produce a new functor  $G \circ F: \mathbb{A} \rightarrow \mathbb{C}$ , and the identity function on  $\mathbb{A}_0$  provides for the identity functor  $1_{\mathbb{A}}: \mathbb{A} \rightarrow \mathbb{A}$ . Thus  $Q$ -categories and  $Q$ -functors are the objects and morphisms of a (large) category  $\mathbf{Cat}(Q)$ .

A  $Q$ -distributor (also called *bimodule* or *profunctor*)  $\Phi: \mathbb{C} \multimap \mathbb{D}$  between two  $Q$ -categories is a  $Q$ -valued predicate

$$\Phi: \mathbb{D}_0 \times \mathbb{C}_0 \rightarrow Q: (y, x) \mapsto \Phi(y, x)$$

satisfying, for all  $x, x' \in \mathbb{C}_0$  and  $y, y' \in \mathbb{D}_0$ , the (“action”) condition

$$\mathbb{D}(y', y) \circ \Phi(y, x) \circ \mathbb{C}(x, x') \leq \Phi(y', x').$$

Two such distributors, say  $\Phi: \mathbb{A} \multimap \mathbb{B}$  and  $\Psi: \mathbb{B} \multimap \mathbb{C}$ , compose as

$$(\Psi \circ \Phi): \mathbb{C}_0 \times \mathbb{A}_0 \mapsto Q: (z, x) \mapsto \bigvee_{y \in \mathbb{B}_0} \Psi(z, y) \circ \Phi(y, x).$$

---

<sup>3</sup>We do *not* assume that 1, the unit of the monoid, is the top element of the lattice.

The identity distributor on  $\mathbb{C}$  is the “hom” predicate  $\mathbb{C}: \mathbb{C}_0 \times \mathbb{C}_0 \rightarrow Q$  itself, and so  $Q$ -categories and  $Q$ -distributors form a (large) category  $\text{Dist}(Q)$ . However, there is more: the elementwise ordering of distributors makes  $\text{Dist}(Q)$  a 2-category<sup>4</sup>.

Applying general 2-categorical algebra, we may now say that two  $Q$ -distributors  $\Phi: \mathbb{A} \dashv\vdash \mathbb{B}$  and  $\Psi: \mathbb{B} \dashv\vdash \mathbb{A}$  are (*left/right*) *adjoint*, denoted  $\Phi \dashv \Psi$ , if

$$\mathbb{A} \leq \Psi \circ \Phi \quad \text{and} \quad \Phi \circ \Psi \leq \mathbb{B}.$$

Every functor  $F: \mathbb{A} \rightarrow \mathbb{B}$  *represents* an adjoint pair of distributors  $F_* \dashv F^*$  defined by

$$F_*(b, a) = \mathbb{B}(b, Fa) \quad \text{and} \quad F^*(a, b) = \mathbb{B}(Fa, b).$$

With this, the inclusion functor

$$\text{Cat}(Q) \rightarrow \text{Dist}(Q): (F: \mathbb{A} \rightarrow \mathbb{B}) \mapsto (F_*: \mathbb{A} \dashv\vdash \mathbb{B})$$

naturally makes  $\text{Cat}(Q)$  a *locally ordered category* by defining, for  $F, G \in \text{Cat}(Q)$ ,

$$F \leq G \stackrel{\text{def}}{\iff} F_* \leq G_*.$$

Whenever  $F \leq G$  and  $G \leq F$ , we write  $F \cong G$  and say that these functors are *isomorphic*.

For a fixed  $Q$ -category  $\mathbb{C}$ , we may consider, for any other  $Q$ -category  $\mathbb{A}$ , the map which assigns to any functor  $F: \mathbb{A} \rightarrow \mathbb{C}$  the left adjoint distributor  $F_*: \mathbb{A} \dashv\vdash \mathbb{C}$ :

$$\text{Cat}(Q)(\mathbb{A}, \mathbb{C}) \rightarrow \text{LAdjDist}(Q)(\mathbb{A}, \mathbb{C}): F \mapsto F_*.$$

This map is (by definition of the local order in  $\text{Cat}(Q)$ ) order-preserving and order-reflecting. If, for each  $\mathbb{A}$ , this map is also surjective (in words: every left adjoint distributor into  $\mathbb{C}$  is representable by a functor), then we say that  $\mathbb{C}$  is *Cauchy-complete*.

Let  $\mathbf{1}$  be the  $Q$ -category defined by  $\mathbf{1}_0 = \{*\}$  and  $\mathbf{1}(*, *) = 1$ . A distributor  $\phi: \mathbf{1} \dashv\vdash \mathbb{C}$  is called a (contravariant) *presheaf* on  $\mathbb{C}$ . There is a natural bijection between  $Q$ -functors  $\mathbf{1} \rightarrow \mathbb{C}$  and elements of  $\mathbb{C}_0$ . In particular, for any  $c \in \mathbb{C}_0$  there is a  $Q$ -functor  $\Delta_c: \mathbf{1} \rightarrow \mathbb{C}: * \mapsto c$  which represents the left adjoint presheaf  $(\Delta_c)_*: \mathbb{C}_0 \times \mathbf{1}_0 \rightarrow Q: (x, *) \mapsto \mathbb{C}(x, c)$ . Therefore, by putting

$$c \leq c' \stackrel{\text{def}}{\iff} \Delta_c \leq \Delta_{c'} \iff \forall x \in \mathbb{C}_0: \mathbb{C}(x, c) \leq \mathbb{C}(x, c') \iff 1 \leq \mathbb{C}(c, c')$$

<sup>4</sup>Much better still:  $\text{Dist}(Q)$  is a *quantaloid*, i.e. a category enriched in  $\text{Sup}$ . Since we do not need this very rich structure in this paper, we shall not dwell on it here.

the set  $\mathbb{C}_0$  becomes an order  $(\mathbb{C}_0, \leq)$ . If both  $c \leq c'$  and  $c' \leq c$  hold, then we write  $c \cong c'$  and we say that these objects of  $\mathbb{C}$  are *isomorphic*. It is furthermore a result in  $Q$ -category theory (which holds in greater generality too) that  $\mathbb{C}$  is Cauchy-complete if and only if

$$\text{Cat}(Q)(\mathbf{1}, \mathbb{C}) \rightarrow \text{LAdjDist}(Q)(\mathbf{1}, \mathbb{C})$$

is surjective; in words,  $\mathbb{C}$  is Cauchy-complete if and only if each left adjoint presheaf on  $\mathbb{C}$  is representable. The importance of Cauchy-complete  $Q$ -categories was made very clear in Lawvere's seminal paper [21] on the subject.

## 1.2 Sequences vs. presheaves

In this subsection we review some results that can be found in [16, Section 4.3]: they relate “sequential” Cauchy-completeness to “categorical” Cauchy-completeness.

Given a sequence  $x = (x_n)_{n \in \mathbb{N}}$  in a  $Q$ -category  $\mathbb{C}$ , we define a quantity  $C_x$  that measures the “Cauchyness” of the sequence:

$$C_x = \bigvee_{N \in \mathbb{N}} \bigwedge_{n \geq N} \bigwedge_{m \geq N} \mathbb{C}(x_n, x_m).$$

**Definition 1.2.1** *We say that a sequence  $x = (x_n)_{n \in \mathbb{N}}$  is Cauchy if  $C_x \geq 1$ .*

Now, for any sequence  $x = (x_n)_{n \in \mathbb{N}}$  in a  $Q$ -category  $\mathbb{C}$ , we define

$$\phi_x(y) := \bigvee_{N \in \mathbb{N}} \bigwedge_{n \geq N} \mathbb{C}(y, x_n) \quad \text{and} \quad \psi_x(y) := \bigvee_{N \in \mathbb{N}} \bigwedge_{n \geq N} \mathbb{C}(x_n, y)$$

for all  $y \in \mathbb{C}_0$ , and show that these presheaves  $\phi_x : \mathbf{1} \rightarrow \mathbb{C}$  and  $\psi_x : \mathbb{C} \rightarrow \mathbf{1}$  on  $\mathbb{C}$  are adjoint if and only if the sequence  $x = (x_n)_{n \in \mathbb{N}}$  is Cauchy.

First, let us verify that:

**Lemma 1.2.2** *Both  $\phi_x$  and  $\psi_x$  are  $Q$ -enriched distributors.*

*Proof.* For  $\phi_x$  we must show that, for all  $a, b \in \mathbb{C}_0$ ,

$$\mathbb{C}(a, b) \circ \phi_x(b) \leq \phi_x(a) \quad \text{and} \quad \phi_x(b) \circ \mathbf{1}(*, *) \leq \phi_x(b).$$

The second inequality is trivial. For the first, we compute as follows:

$$\mathbb{C}(a, b) \circ \phi_x(b) = \mathbb{C}(a, b) \circ \left( \bigvee_{N \in \mathbb{N}} \bigwedge_{n \geq N} \mathbb{C}(b, x_n) \right)$$



$$\begin{aligned}
&= \bigvee_{N \in \mathbb{N}} \mathbb{C}(a, b) \circ \left( \bigwedge_{n \geq N} \mathbb{C}(b, x_n) \right) \\
&\leq \bigvee_{N \in \mathbb{N}} \bigwedge_{n \geq N} \mathbb{C}(a, b) \circ \mathbb{C}(b, x_n) \\
&\leq \bigvee_{N \in \mathbb{N}} \bigwedge_{n \geq N} \mathbb{C}(a, x_n) \\
&= \phi_x(a)
\end{aligned}$$

The reasoning for  $\psi_x$  is entirely analogous.  $\square$

Now, we recall the result, from [16], about the connection between Cauchy sequences and adjoint presheaves. We include the proof here, to indicate a reoccurring computation that we shall use later on too.

**Proposition 1.2.3** *A sequence  $x = (x_n)_{n \in \mathbb{N}}$  is Cauchy if and only if  $\phi_x \dashv \psi_x$ .*

*Proof.* The stated adjointness means precisely that:

- (i)  $\phi_x(b) \circ \psi_x(a) \leq \mathbb{C}(b, a)$ , for all  $a, b \in \mathbb{C}_0$ ,
- (ii)  $\bigvee_{y \in \mathbb{C}_0} \psi_x(y) \circ \phi_x(y) \geq 1$ .

Of these two required inequalities, the first is always true, regardless the Cauchy-ness of the sequence  $x = (x_n)_{n \in \mathbb{N}}$ . Indeed, for  $a, b \in \mathbb{C}_0$  we can compute:

$$\begin{aligned}
\phi_x(b) \circ \psi_x(a) &= \left( \bigvee_{N \in \mathbb{N}} \bigwedge_{n \geq N} \mathbb{C}(b, x_n) \right) \circ \left( \bigvee_{M \in \mathbb{N}} \bigwedge_{m \geq M} \mathbb{C}(x_m, a) \right) \\
&\stackrel{(*)}{=} \bigvee_{N \in \mathbb{N}} \left( \bigwedge_{n \geq N} \mathbb{C}(b, x_n) \right) \circ \left( \bigwedge_{m \geq N} \mathbb{C}(x_m, a) \right) \\
&\leq \bigvee_{N \in \mathbb{N}} \mathbb{C}(b, x_N) \circ \mathbb{C}(x_N, a) \\
&\leq \mathbb{C}(b, a).
\end{aligned}$$

In (\*) we use that the sequences

$$\left( \bigwedge_{m \geq M} \mathbb{C}(x_m, a) \right)_{M \in \mathbb{N}} \quad \text{and} \quad \left( \bigwedge_{n \geq N} \mathbb{C}(b, x_n) \right)_{N \in \mathbb{N}}$$

are increasing<sup>5</sup>.

<sup>5</sup>For two sequences  $(a_n)_{n \in \mathbb{N}}$  and  $(b_n)_{n \in \mathbb{N}}$  of elements in  $Q$ , distributivity assures that  $\bigvee_n a_n \circ \bigvee_m b_m = \bigvee_{n,m} a_n \circ b_m$ . However, when both sequences are increasing, i.e.  $n \leq n'$  implies  $a_n \leq a_{n'}$  and  $b_n \leq b_{n'}$ , then this is further equal to  $\bigvee_n a_n \circ b_n$ .

As for the second required inequality, we shall prove (again, regardless the Cauchyness of the sequence) that

$$C_x \circ C_x \stackrel{(1)}{\leq} \bigvee_{y \in \mathbb{C}_0} \psi_x(y) \circ \phi_x(y) \stackrel{(2)}{\leq} C_x;$$

the stated result then follows immediately. Inequality (1) can be verified as follows:

$$\begin{aligned} \bigvee_{y \in \mathbb{C}_0} \psi_x(y) \circ \phi_x(y) &= \bigvee_{y \in \mathbb{C}_0} \left( \bigvee_{N \in \mathbb{N}} \bigwedge_{n \geq N} \mathbb{C}(x_n, y) \right) \circ \left( \bigvee_{M \in \mathbb{N}} \bigwedge_{m \geq M} \mathbb{C}(y, x_m) \right) \\ &\stackrel{(*)}{=} \bigvee_{y \in \mathbb{C}_0} \bigvee_{N \in \mathbb{N}} \left( \bigwedge_{n \geq N} \mathbb{C}(x_n, y) \right) \circ \left( \bigwedge_{m \geq N} \mathbb{C}(y, x_m) \right) \\ &\geq \bigvee_{N \in \mathbb{N}} \left( \bigwedge_{n \geq N} \mathbb{C}(x_n, x_N) \right) \circ \left( \bigwedge_{m \geq N} \mathbb{C}(x_N, x_m) \right) \\ &\stackrel{(*)}{=} \left( \bigvee_{N \in \mathbb{N}} \bigwedge_{n \geq N} \mathbb{C}(x_n, x_N) \right) \circ \left( \bigvee_{M \in \mathbb{N}} \bigwedge_{m \geq M} \mathbb{C}(x_M, x_m) \right) \\ &\geq \left( \bigvee_{N \in \mathbb{N}} \bigwedge_{n \geq N} \bigwedge_{m \geq N} \mathbb{C}(x_n, x_m) \right) \circ \left( \bigvee_{M \in \mathbb{N}} \bigwedge_{n \geq M} \bigwedge_{m \geq M} \mathbb{C}(x_n, x_m) \right) \\ &= C_x \circ C_x \end{aligned}$$

In both equalities marked with (\*) we again use that we are dealing with (a product of suprema of) increasing sequences. Inequality (2) follows thusly:

$$\begin{aligned} \bigvee_{y \in \mathbb{C}_0} \psi_x(y) \circ \phi_x(y) &= \bigvee_{y \in \mathbb{C}_0} \left( \bigvee_{N \in \mathbb{N}} \bigwedge_{n \geq N} \mathbb{C}(x_n, y) \right) \circ \left( \bigvee_{M \in \mathbb{N}} \bigwedge_{m \geq M} \mathbb{C}(y, x_m) \right) \\ &\stackrel{(*)}{=} \bigvee_{y \in \mathbb{C}_0} \bigvee_{N \in \mathbb{N}} \left( \bigwedge_{n \geq N} \mathbb{C}(x_n, y) \right) \circ \left( \bigwedge_{m \geq M} \mathbb{C}(y, x_m) \right) \\ &\leq \bigvee_{y \in \mathbb{C}_0} \bigvee_{N \in \mathbb{N}} \bigwedge_{n \geq N} \bigwedge_{m \geq N} \mathbb{C}(x_n, y) \circ \mathbb{C}(y, x_m) \\ &\leq \bigvee_{N \in \mathbb{N}} \bigwedge_{n \geq N} \bigwedge_{m \geq N} \mathbb{C}(x_n, x_m) \\ &= C_x. \end{aligned}$$

For (\*) we use the argument on increasing sequences again. □

### 1.3 Examples of $Q$ -enriched categories

In the rest of the paper, our examples of  $Q$ -enriched categories will be:

**Example 1.3.1 (Ordered sets)** The simplest (non-trivial) example of a quantale is the two-element Boolean algebra  $Q = (\{0, 1\}, \vee, \wedge, 1)$ . In this case, a  $Q$ -category is precisely an ordered set  $(P, \leq)$ , i.e. a set  $P$  equipped with a binary relation  $\leq$  whose characteristic function  $P \times P \rightarrow \{0, 1\}: (x, y) \mapsto \llbracket x \leq y \rrbracket$  satisfies the following axioms:

- (1)  $\llbracket x \leq y \rrbracket \wedge \llbracket y \leq z \rrbracket \leq \llbracket x \leq z \rrbracket$ ,
- (2)  $1 \leq \llbracket x \leq x \rrbracket$ .

(This order-relation need not be anti-symmetric; some call this a “preorder”.) A  $Q$ -functor between such  $Q$ -categories is a monotone map between ordered sets. It is well-known (and easy to verify) that every ordered set is, viewed as an enriched category, Cauchy-complete.

**Example 1.3.2 (Metric spaces)** Let  $Q = ([0, \infty], \wedge, +, 0)$  be Lawvere’s quantale of extended positive real numbers, i.e. it is the segment  $[0, \infty]$  (with  $\infty$  included) with the converse (!) of the natural (linear) order, and with the sum as binary operation. As pointed out by Lawvere [21], a  $Q$ -category is precisely a *generalised metric space*  $(X, d)$ , that is, a set  $X$  together with a distance function  $d: X \times X \rightarrow [0, \infty]$  such that

- (1)  $d(x, y) + d(y, z) \geq d(x, z)$ ,
- (2)  $0 \geq d(x, x)$ .

The adjective “generalized” here indicates that such a metric need not be finitary (so  $d(x, y) = \infty$  is allowed) nor symmetric (so  $d(x, y) \neq d(y, x)$  is allowed), nor separated (so  $d(x, y) = 0 = d(y, x)$  for  $x \neq y$  is allowed). A  $Q$ -functor between such  $Q$ -categories is a non-expanding map between (generalized) metric spaces. Lawvere [21] famously showed that a metric space is Cauchy-complete as enriched category if and only if all Cauchy sequences (in the usual sense for metric spaces) converge.

**Example 1.3.3 (Fuzzy orders)** A so-called *left-continuous  $t$ -norm* is precisely a commutative and integral quantale whose underlying (linear) sup-lattice is  $([0, 1], \vee)$  (see e.g. [19, 30]); the multiplication of such a quantale is then typically written as  $x * y$ . Examples include  $x * y = xy$  (the “product  $t$ -norm”),  $x * y = \min\{x, y\}$  (the “minimum  $t$ -norm”) and  $x * y = \max\{x + y - 1, 0\}$  (the “Lukasiewicz  $t$ -norm”); in fact, every *continuous  $t$ -norm* (meaning that the multiplication is a continuous function) is in a precise sense an amalgamation of these three (see e.g. [13]). These quantales are the corner stone of “fuzzy” logic: the truth values in this logic can vary between 0 and 1, conjunction

is computed with  $*$ , and implication is computed with the adjoint to multiplication. A category enriched in a left-continuous  $t$ -norm  $([0, 1], \vee, *, 1)$  thus consists of a set  $P$  together with a map  $P \times P \rightarrow [0, 1]: (x, y) \mapsto \llbracket x \leq y \rrbracket$  satisfying

- (1)  $\llbracket x \leq y \rrbracket * \llbracket y \leq z \rrbracket \leq \llbracket x \leq z \rrbracket$ ,
- (2)  $1 \leq \llbracket x \leq x \rrbracket$ .

Following [32, 24, 7, 22], we call this a *fuzzy (pre)order*: the truth value  $\llbracket x \leq y \rrbracket \in [0, 1]$  is interpreted as “the extent to which  $x \leq y$  holds in  $P$ ”. A  $Q$ -functor between such  $Q$ -categories is a map between fuzzy preorders that does not decrease the value of the “fuzzy” order. By Theorem 4.19 of [16] (and the definition of Cauchy sequence in a  $Q$ -category) it follows that a fuzzy order is categorically Cauchy-complete if and only if all Cauchy sequences (in the usual sense for fuzzy orders, see Definition 4.1 in [7]) converge.

**Example 1.3.4 (Probabilistic metric space)** Fix a *left-continuous*  $t$ -norm  $([0, 1], \vee, *, 1)$ . It was shown by Hofmann and Reis [16], and further explained in [8], that the set

$$\Delta = \{f : [0, \infty] \rightarrow [0, 1] \mid f(t) = \bigvee_{s < t} f(s)\}$$

of so-called distance distributions<sup>6</sup> is a quantale for pointwise suprema, with the convolution product

$$(f * g)(t) = \bigvee_{r+s=t} f(r) * g(s)$$

as binary operation, and

$$e(t) = \begin{cases} 0 & \text{if } t = 0, \\ 1 & \text{else} \end{cases}$$

as two-sided unit. Indeed, it is shown in [11, Examples 2.1.10 and 2.3.36] that the quantale  $Q = (\Delta, \vee, *, e)$  is the tensor product in the category of suplattices, as well as the coproduct in the category of commutative quantales, of the Lawvere quantale  $([0, \infty], \wedge, +, 0)$  and the left-continuous  $t$ -norm  $([0, 1], \vee, *, 1)$ . A  $Q$ -category has been called a (*generalized*) *probabilistic metric space* by some [16, 14], and a (*generalized*) *fuzzy metric space* by others [20, 9]; it consists of a set  $X$  together with a probabilistic distance function  $d: X \times X \times [0, \infty] \rightarrow [0, 1]$  such that

<sup>6</sup>Because domain and codomain are continuous lattices, these are precisely the lower semicontinuous functions, see [10, Proposition II-2.1].

- (0)  $d(x, y, t) = \bigvee_{s < t} d(x, y, s)$ ,
- (1)  $d(x, x, t) = 1$  for  $t > 0$ ,
- (2)  $d(x, y, r) * d(y, z, s) \leq d(x, z, r + s)$ .

Such an object is often denoted  $(X, d, *)$ , to stress the importance of the  $t$ -norm. The intended meaning of  $d(x, y, t)$  is that it expresses “the probability that the distance from  $x$  to  $y$  is strictly less than  $t$ ”. (Again, we do not insist on finiteness, symmetry or separatedness for such a space, each of which can be expressed suitably; see also [28].) A  $Q$ -enriched functor is a map between such spaces that does not decrease such probabilistic distances. Hofmann and Reiss [16] proved that a probabilistic metric space is categorically Cauchy-complete if and only if all Cauchy sequences (as traditionally defined in probabilistic metric spaces, see [6, 16]) converge.

#### 1.4 Continuous lattices

For the reader’s convenience, we gather here a few well-known facts regarding continuous lattices; our reference on this subject is [10].

Throughout we assume that  $L$  is a *complete lattice*.

A subset  $D \subseteq L$  is directed if it is non-empty and, for any  $x, y \in D$  there exists a  $z \in D$  such that  $x \vee y \subseteq z$ . For two elements  $a, b \in L$  we write  $a \ll b$ , and we say that  $a$  is *way below*  $b$ , if, for every directed subset  $D \subseteq L$ ,  $b \leq \bigvee D$  implies the existence of a  $d \in D$  such that  $a \leq d$ . We write  $a \lll b$ , and say that  $a$  is *totally below*  $b$ , if, for every subset  $S \subseteq L$ ,  $b \leq \bigvee S$  implies the existence of a  $s \in S$  such that  $a \leq s$ . It is easy to see that  $a \lll b$  implies  $a \ll b$ , which in turn implies  $a \leq b$ . The complete lattice  $L$  is said to be

- *meet-continuous* if, for every  $a \in L$  and every directed  $D \subseteq L$ ,

$$a \wedge (\bigvee D) = \bigvee_{d \in D} a \wedge d, \quad (1)$$

- *continuous* if, for every  $a \in L$ ,

$$a = \bigvee \{u \in L \mid u \ll a\}, \quad (2)$$

- *totally continuous* if, for every  $a \in L$ ,

$$a = \bigvee \{u \in L \mid u \lll a\}. \quad (3)$$

A totally continuous lattice is also continuous, and a continuous lattice is also meet-continuous. Whereas the first of these assertions is trivial, we shall give some detail regarding the second. Thereto, we rephrase somewhat the definitions.

**Lemma 1.4.1** *L is meet-continuous if and only if, for every  $a \in L$  and directed  $D \subseteq L$ ,*

$$a \leq \bigvee D \implies a \leq \bigvee_{d \in D} a \wedge d. \quad (4)$$

(Note that  $a \geq \bigvee_{d \in D} a \wedge d$  always holds, so in effect we have an equality.)

*Proof.* First we show (1) $\Rightarrow$ (4). If  $a \leq \bigvee D$  then  $a = a \wedge \bigvee D = \bigvee_{d \in D} a \wedge d$  as wanted. Now for (4) $\Rightarrow$ (1), because  $a \wedge \bigvee D \leq \bigvee D$  we have by hypothesis that  $a \wedge \bigvee D \leq \bigvee_{d \in D} ((a \wedge \bigvee D) \wedge d)$  and the latter is equal to  $\bigvee_{d \in D} (a \wedge d)$ ; the opposite inequality always holds.  $\square$

**Lemma 1.4.2** *L is continuous if and only if, for every  $a, b \in L$ ,*

$$(\forall u \ll a : u \leq b) \implies a \leq b, \quad (5)$$

*if and only if, for every  $a, b \in L$ ,*

$$a \not\leq b \implies (\exists u \ll a : u \not\leq b). \quad (6)$$

*Proof.* (2) $\Rightarrow$ (5) is immediate, and (5) $\Leftrightarrow$ (6) holds by contraposition. Now suppose that (6) holds, yet that  $a \not\leq \bigvee \{u \in L \mid u \ll a\}$ . Then there must be a  $u \ll a$  such that  $u \not\leq \bigvee \{u \in L \mid u \ll a\}$  which is absurd. (Note that  $a \geq \bigvee \{u \in L \mid u \ll a\}$  always holds.) So (2) follows from (6).  $\square$

**Proposition 1.4.3** *Every continuous lattice is meet-continuous.*

*Proof.* We shall show that (2) implies (4). Suppose that  $a \leq \bigvee D$ , then for all  $u \ll a$  there must be a  $d_u \in D$  such that  $u \leq d_u$ . But  $u \ll a$  also implies  $u \leq a$ , so that  $u \leq a \wedge d_u$ . Therefore  $a = \bigvee \{u \in L \mid u \ll a\} \leq \bigvee_{d \in D} a \wedge d$ , as needed.  $\square$

Finally, we point out that:

**Proposition 1.4.4** *L is totally continuous if and only if it is completely distributive, that is, the supremum-map*

$$\bigvee : \text{Dwn}(L) \rightarrow L : S \mapsto \bigvee S \quad (7)$$

*preserves infima (where  $\text{Dwn}(L)$  denotes the complete lattice of downward closed subsets of  $L$ , with infima given by intersection).*

*Proof.* Suppose first that (3) holds. Then, for any downward closed subset  $S \subseteq L$  and any  $a \in L$ , one easily sees that

$$\{u \in L \mid u \lll a\} \subseteq S \quad \text{if and only if} \quad a \leq \bigvee S. \quad (8)$$

This shows a (Galois) adjunction between the order-preserving maps  $a \mapsto \{u \in L \mid u \lll a\}$  (the left adjoint) and  $S \mapsto \bigvee S$  (the right adjoint), and so  $S \mapsto \bigvee S$  preserves infima. Conversely, suppose that  $S \mapsto \bigvee S$  preserves infima. Because of the complete lattices involved, this is equivalent to saying that it has a left adjoint, say  $a \mapsto l(a)$ : that is, for any  $S \in \text{Dwn}(L)$ ,  $l(a) \subseteq S$  if and only if  $a \leq \bigvee S$ . Let us, for the sake of the argument, denote  $\delta(a) = \{u \in L \mid u \lll a\}$ ; then using the adjunction in (\*) we have the equivalences

$$\begin{aligned} a \leq \bigvee \delta(a) &\stackrel{(*)}{\iff} l(a) \subseteq \delta(a) \\ &\iff (x \in l(a) \Rightarrow x \in \delta(a)) \\ &\iff (x \in l(a) \Rightarrow x \lll a) \\ &\iff (x \in l(a) \Rightarrow (\forall S \in \text{Dwn}(L) : a \leq \bigvee S \Rightarrow x \in S)) \\ &\stackrel{(*)}{\iff} (x \in l(a) \Rightarrow (\forall S \in \text{Dwn}(L) : l(a) \subseteq S \Rightarrow x \in S)), \end{aligned}$$

the last of which clearly holds true. As  $a \geq \bigvee \delta(a)$  always holds, we have shown (3).  $\square$

Finally we recall the pertinent notion of morphism in this context:

**Proposition 1.4.5** *For any function  $f: L \rightarrow M$  between complete lattices, we have that*

$$f(\bigvee D) = \bigvee_{d \in D} f(d) \quad (9)$$

for all directed  $D \subseteq L$  if and only if

$$f(\bigvee_{j \in J} \bigwedge_{j' \geq j} x_j) \leq \bigvee_{j \in J} \bigwedge_{j' \geq j} f(x_j). \quad (10)$$

for every net<sup>7</sup>  $(x_j)_{j \in J}$  in  $L$ . If  $L$  and  $M$  are continuous lattices, then this is further equivalent to

$$f(x) = \bigvee \{f(y) \mid y \ll x\} \quad (11)$$

for all  $x \in K$ . Such an  $f$  is always monotone.

<sup>7</sup>A net is a family of elements indexed by a directed poset. Any sequence  $(x_i)_{i \in \mathbb{N}}$  is a net, but the converse is – obviously – not true.

*Proof.* For (9) $\Rightarrow$ (10), first observe that, whenever  $a \leq b$  in  $L$ , the set  $\{a, b\}$  is directed. By application of the hypothesis one easily sees that  $f(b) = f(a) \vee f(b)$  so that  $f(a) \leq f(b)$ . Whence  $f$  is monotone. Now, given a net  $(x_j)_{j \in J}$ , the subset

$$\{y_j := \bigwedge_{j' \geq j} x_{j'} \mid j \in J\} \subseteq L$$

is directed: for  $y_m$  and  $y_n$  there is always a  $k \in J$  which is larger than  $m$  and  $n$  (since  $J$  is directed), and  $y_k \geq y_m \vee y_n$ . Applying the hypothesis, and using that  $f$  is monotone, we find indeed that

$$f\left(\bigvee_{j \in J} \bigwedge_{j' \geq j} x_{j'}\right) = \bigvee_{j \in J} f\left(\bigwedge_{j' \geq j} x_{j'}\right) \leq \bigvee_{j \in J} \bigwedge_{j' \geq j} f(x_{j'}).$$

Conversely, suppose that (10) holds. It follows first and foremost that  $f$  is monotonous: suppose that  $a \leq b$ , then applying the hypotheses on the sequence  $a, b, a, b, \dots$  provides  $f(a) \leq f(a) \wedge f(b)$  so that  $f(a) \leq f(b)$ . Now, for a directed  $D \subseteq L$ , we can consider the net

$$(x_d := d)_{d \in D}$$

for which the hypothesis together with the monotonicity of  $f$  implies

$$f(\bigvee D) = f\left(\bigvee_{d \in D} \bigwedge_{d' \geq d} x_{d'}\right) \leq \bigvee_{d \in D} \bigwedge_{d' \geq d} f(x_{d'}) = \bigvee_{d \in D} f(d).$$

As the reverse inequality is always true (for monotonous  $f$ ), we get the required inequality.

From now on we suppose that  $L$  and  $M$  are continuous lattices. It follows directly from (2) and the directedness of  $\{u \in L \mid u \ll x\} \subseteq L$  that (9) $\Rightarrow$ (11). Conversely, suppose that (11) holds. Note first that  $f$  is monotonous: if  $a \leq b$  then  $\{u \in L \mid u \ll a\} \subseteq \{u \in L \mid u \ll b\}$  and so  $\{f(u) \in L \mid u \ll a\} \subseteq \{f(u) \in L \mid u \ll b\}$  from which  $f(a) \leq f(b)$  follows by application of the hypothesis. Next, if  $D \subseteq L$  is directed, then

$$u \ll \bigvee D \quad \text{if and only if} \quad u \leq d \text{ for some } d \in D$$

and so, using monotonicity,  $f(\bigvee D) = \bigvee \{f(u) \mid u \leq d \in D\} = \bigvee \{f(d) \mid d \in D\}$ , as wanted.  $\square$

Functions as in the previous Proposition are called *lower-semicontinuous*.

Any complete linear lattice  $L$  is completely distributive(= totally continuous). Because every subset of  $L$  is directed (if  $a, b \in S \subseteq L$  then either  $a = a \vee b$  or  $b = a \vee b$  and so in any case  $a \vee b \in S$ ), the notions of “way below”



and “totally below” coincide. In fact, we have  $a \ll b$  if and only if either  $a = 0$ , or  $a < b$ , or  $(a = b \text{ and } b \neq \bigvee\{x \in L \mid x < b\})$ , see [10]. We record  $([0, \infty], \wedge)$  and  $([0, 1], \vee)$  as important examples. Since the tensor product (in the category **Sup** of complete lattices and supremum-preserving morphisms) of two completely distributive lattices is again completely distributive [11, Theorem 2.1.17], we may also record the complete lattice of distance distributions,  $(\Delta, \vee) \cong ([0, \infty], \wedge) \otimes ([0, 1], \vee)$ , as an example of a completely distributive lattice. A description of the way-below relation (involving step functions) can be found in [16]. By the way, it follows from (11) in Proposition 1.4.5 that the elements of  $\Delta$  are indeed all the lower semicontinuous functions from  $[0, \infty]$  to  $[0, 1]$ .

## 2. Fixpoints for contractions on $Q$ -categories

### 2.1 Contractions on a $Q$ -enriched category

Let  $Q$  be any quantale (and write  $0$  for its bottom element), and  $\mathbb{C}$  any  $Q$ -enriched category.

**Definition 2.1.1** *If  $\varphi : Q \rightarrow Q$  and  $f : \mathbb{C}_0 \rightarrow \mathbb{C}_0$  are maps such that*

1.  $\varphi(t) \geq t$  for all  $t \in Q$ ,
2. if  $\varphi(t) = t$  then  $t = 0$  or  $1 \leq t$ ,
3. for all  $x, y \in \mathbb{C}$ ,  $\mathbb{C}(fx, fy) \geq \varphi(\mathbb{C}(x, y))$ ,

*then we say that  $f$  is a  $\varphi$ -contraction, and we say that  $\varphi$  is a control function for  $f$ .*

A control function  $\varphi$  is thus bigger than the identity function on the whole of  $Q$ , and strictly so except possibly in  $t = 0$  or  $t \geq 1$ . Note too that a  $\varphi$ -contraction  $f$  is always a  $Q$ -functor  $f : \mathbb{C} \rightarrow \mathbb{C}$ , but not every  $Q$ -functor is  $\varphi$ -contractive for some control function  $\varphi$ .

We wish to investigate the possible fixpoints of such contractions. Let us first make this formal:

**Definition 2.1.2** *Let  $f : \mathbb{C} \rightarrow \mathbb{C}$  be a  $Q$ -functor. A fixpoint for  $f$  is an  $u \in \mathbb{C}$  such that  $fu \cong u$  in  $\mathbb{C}$ , that is to say, we have both  $1 \leq \mathbb{C}(fu, u)$  and  $1 \leq \mathbb{C}(u, fu)$ .*

In general, such fixpoints are of course not unique. However, if  $f$  is a  $\varphi$ -contraction, and both  $fu \cong u$  and  $fu' \cong u'$  hold, then it follows from the

triangular inequality in  $\mathbb{C}$  that

$$\begin{aligned}
\mathbb{C}(u, u') &\geq \mathbb{C}(u, fu) \circ \mathbb{C}(fu, fu') \circ \mathbb{C}(fu', u') \\
&\geq 1 \circ \mathbb{C}(fu, fu') \circ 1 \\
&= \mathbb{C}(fu, fu') \\
&\geq \varphi(\mathbb{C}(u, u')) \\
&\geq \mathbb{C}(u, u')
\end{aligned}$$

Since  $\varphi(t) > t$  for all  $0 \neq t \not\geq 1$ , we must have  $\mathbb{C}(u, u') = 0$  or  $\mathbb{C}(u, u') \geq 1$ . Exchanging  $u$  and  $u'$  one sees that also  $\mathbb{C}(u', u) = 0$  or  $\mathbb{C}(u', u) \geq 1$ . Hence there are exactly four possibilities:

$$\left\{ \begin{array}{l} \mathbb{C}(u, u') \geq 1 \\ \mathbb{C}(u', u) \geq 1 \end{array} \right\} \text{ or } \left\{ \begin{array}{l} \mathbb{C}(u, u') \geq 1 \\ \mathbb{C}(u', u) = 0 \end{array} \right\} \text{ or } \left\{ \begin{array}{l} \mathbb{C}(u, u') = 0 \\ \mathbb{C}(u', u) \geq 1 \end{array} \right\} \text{ or } \left\{ \begin{array}{l} \mathbb{C}(u, u') = 0 \\ \mathbb{C}(u', u) = 0 \end{array} \right\}$$

Under mild assumptions on  $\mathbb{C}$  we can now formulate uniqueness results for fixpoints.

**Proposition 2.1.3** *Let  $\mathbb{C}$  be a  $Q$ -category all of whose homs are non-zero, and  $f: \mathbb{C} \rightarrow \mathbb{C}$  any  $\varphi$ -contraction. If  $fu \cong u$  and  $fu' \cong u'$  then  $u \cong u'$ .*

*Proof.* In the four possible cases above, only the first is compatible with non-zero homs in  $\mathbb{C}$ . □

For  $Q$ -categories with homs that can be equal to 0, we have a different result.

**Proposition 2.1.4** *Let  $\mathbb{C}$  be symmetric  $Q$ -category (meaning that  $\mathbb{C}(x, y) = \mathbb{C}(y, x)$  for all  $x, y \in \mathbb{C}$ ) and  $f: \mathbb{C} \rightarrow \mathbb{C}$  any  $\varphi$ -contraction. If  $fu \cong u$  and  $fu' \cong u'$  then either  $u \cong u'$  or  $\mathbb{C}(u, u') = 0$ .*

*Proof.* In the four possible cases above, only the first and the last are compatible with symmetry in  $\mathbb{C}$ . □

## 2.2 From contractions to adjoint presheaves

We shall be interested in a particular set of control functions for contractions on a  $Q$ -category, namely those that satisfy a weak variant<sup>8</sup> of *lower-semicontinuity* (see [10]):

---

<sup>8</sup>A function  $f: L \rightarrow M$  between complete lattices is lower-semicontinuous if the sup-inf condition in Definition 2.2.1 holds for all *nets* in  $L$  (i.e. a family of elements indexed by a directed poset); here we only require it for *sequences*.

**Definition 2.2.1** Let  $\Phi$  be the set of control functions  $\varphi : Q \rightarrow Q$  such that, for any  $(t_n)_{n \in \mathbb{N}}$  in  $Q$ ,

$$\varphi\left(\bigvee_{N \in \mathbb{N}} \bigwedge_{n \geq N} t_n\right) \leq \bigvee_{N \in \mathbb{N}} \bigwedge_{n \geq N} \varphi(t_n). \quad (12)$$

We shall speak of a  $\Phi$ -contraction  $f$  whenever  $f$  is a  $\varphi$ -contraction with  $\varphi \in \Phi$ .

Given any  $\varphi$ -contraction  $f$  on a  $Q$ -category  $\mathbb{C}$  and an object  $x \in \mathbb{C}_0$ , the sequence  $(f^n x)_{n \in \mathbb{N}}$  determines two distributors

$$\phi_{x,f}(y) = \bigvee_{N \in \mathbb{N}} \bigwedge_{n \geq N} \mathbb{C}(y, f^n x) \text{ and } \psi_{x,f}(y) = \bigvee_{N \in \mathbb{N}} \bigwedge_{n \geq N} \mathbb{C}(f^n x, y)$$

We shall now identify sufficient conditions on  $Q$  and  $\varphi$  in order to get an adjunction between these distributors.

**Proposition 2.2.2** Let  $Q$  be a quantale whose underlying lattice is continuous<sup>9</sup>, and  $f : \mathbb{C} \rightarrow \mathbb{C}$  a  $\Phi$ -contraction on a  $Q$ -category. For any  $x \in \mathbb{C}_0$  such that  $\mathbb{C}(x, fx) \neq 0 \neq \mathbb{C}(fx, x)$  we have  $\phi_{x,f} \dashv \psi_{x,f}$ .

*Proof.* Putting  $C_{x,f} := \bigvee_{N \in \mathbb{N}} \bigwedge_{n \geq N} \bigwedge_{m \geq N} \mathbb{C}(f^n x, f^m x) \in Q$ , we recall from Subsection 1.2 that  $\phi_{x,f} \dashv \psi_{x,f}$  if and only if  $C_{x,f} \geq 1$ . We shall show that  $C_{x,f} \not\geq 1$  leads to a contradiction.

(i) Picking an  $x \in \mathbb{C}_0$  such that  $\mathbb{C}(x, fx) \neq 0 \neq \mathbb{C}(fx, x)$ , we put  $c_n := \mathbb{C}(f^n x, f^{n+1} x) \in Q$  for all  $n \in \mathbb{N}$ . By assumption,  $0 < c_0 \leq 1$  and the conditions on  $\varphi$  imply that  $c_0 \leq \varphi(c_0) \leq c_1$ . Repeating the argument we find that  $c_n \leq \varphi(c_n) \leq c_{n+1}$ , so the sequence is increasing and strictly above 0. Therefore we can compute, using the conditions on  $\varphi$ , that:

$$\begin{aligned} \bigvee_{N \in \mathbb{N}} c_N &= \bigvee_{N \in \mathbb{N}} c_{N+1} \\ &= \bigvee_{N \in \mathbb{N}} \bigwedge_{n \geq N} c_{n+1} \\ &\geq \bigvee_{N \in \mathbb{N}} \bigwedge_{n \geq N} \varphi(c_n) \\ &\geq \varphi\left(\bigvee_{N \in \mathbb{N}} \bigwedge_{n \geq N} c_n\right) \\ &= \varphi\left(\bigvee_{N \in \mathbb{N}} c_N\right) \end{aligned}$$

---

<sup>9</sup>It is tempting to speak of a *continuous quantale*, yet this terminology is in conflict with that of *continuous t-norm*. Indeed, the underlying lattice of any *t-norm* is the continuous lattice  $[0, 1]$ , yet not every *t-norm* is continuous (as a function in two variables). So we shall stick to the somewhat cumbersome “quantale with underlying continuous lattice”.

$$\geq \bigvee_{N \in \mathbb{N}} c_N$$

We thus find a fixpoint of  $\varphi$  which is not 0, so it must satisfy  $1 \leq \bigvee_{N \in \mathbb{N}} c_N$ .

(ii) Similarly, the sequence  $(a_n := \mathbb{C}(f^{n+1}x, f^n x))_{n \in \mathbb{N}}$  must also satisfy  $1 \leq \bigvee_{n \in \mathbb{N}} a_n$ .

(iii) Next, suppose that  $1 \not\leq C_{f,x}$ ; by continuity of the underlying complete lattice of  $Q$ , this means that there exists an  $\epsilon \ll 1$  such that  $\epsilon \not\leq C_{f,x}$  (and so in particular  $\epsilon \neq 0$ ). Using the definition of  $C_{f,x}$  as a sup-inf, we may infer:

$$\begin{aligned} \epsilon \not\leq \bigvee_{k \in \mathbb{N}} \left( \bigwedge_{n \geq k} \bigwedge_{m \geq k} \mathbb{C}(f^n x, f^m x) \right) &\implies \forall k \in \mathbb{N} : \epsilon \not\leq \bigwedge_{n \geq k} \bigwedge_{m \geq k} \mathbb{C}(f^n x, f^m x) \\ &\implies \forall k \in \mathbb{N}, \exists n_k, m_k \geq k : \epsilon \not\leq \mathbb{C}(f^{n_k} x, f^{m_k} x) \end{aligned}$$

In the last line above, it cannot be the case that  $m_k = n_k$ , because otherwise  $\mathbb{C}(f^{n_k} x, f^{n_k} x) \geq 1$  (by the “identity” axiom for the  $Q$ -category  $\mathbb{C}$ ), which would then also be above  $\epsilon \ll 1$ . So suppose that  $n_k < m_k$ , then we can replace  $m_k$  by

$$m'_k := \min\{m > n_k \mid \epsilon \not\leq \mathbb{C}(f^{n_k} x, f^m x)\}$$

and so we still have  $\epsilon \not\leq \mathbb{C}(f^{n_k} x, f^{m'_k} x)$ , but now we know also that  $\epsilon \leq \mathbb{C}(f^{n_k} x, f^{m_k-1} x)$ . Similarly, if  $n_k > m_k$  then we may replace  $n_k$  by

$$n'_k := \min\{n > m_k \in \mathbb{N} \mid \epsilon \not\leq \mathbb{C}(f^n x, f^{m_k} x)\}$$

and we still have  $\epsilon \not\leq \mathbb{C}(f^{n'_k} x, f^{m_k} x)$ , but now we know also that  $\epsilon \leq \mathbb{C}(f^{n'_k-1} x, f^{m_k} x)$ . That is to say, we can always pick  $n_k, m_k \geq k$  to ensure that

$$\epsilon \not\leq \mathbb{C}(f^{n_k} x, f^{m_k} x) \text{ and } \begin{cases} \text{either } \mathbb{C}(f^{n_k} x, f^{m_k-1} x) \geq \epsilon & (A) \\ \text{or } \mathbb{C}(f^{n_k-1} x, f^{m_k} x) \geq \epsilon & (B) \end{cases}$$

Now denote, for each such pick of  $n_k, m_k \geq k \in \mathbb{N}$ ,

$$d_k := \mathbb{C}(f^{n_k} x, f^{m_k} x);$$

and let us insist that  $\epsilon \not\leq d_k$  for all  $k \in \mathbb{N}$ . In case condition (A) holds for  $d_k$ , then in particular  $m_k > n_k$  so  $m_k \geq 1$ , and we can use the “composition” axiom in  $\mathbb{C}$  to get

$$\begin{aligned} \epsilon \circ c_{m_k-1} &\leq \mathbb{C}(f^{n_k} x, f^{m_k-1} x) \circ \mathbb{C}(f^{m_k-1} x, f^{m_k} x) \\ &\leq \mathbb{C}(f^{n_k} x, f^{m_k} x) \\ &= d_k \end{aligned}$$

In case condition (B) holds for  $d_k$  we can similarly prove that

$$a_{n_k-1} \circ \epsilon \leq d_k.$$

Hence, using in (\*) that a continuous lattice is always meet-continuous, and that both sequences

$$\left( \bigwedge \{d_k \mid k \geq N \text{ and } (A) \text{ holds}\} \right)_{N \in \mathbb{N}} \text{ and } \left( \bigwedge \{d_k \mid k \geq N \text{ and } (B) \text{ holds}\} \right)_{N \in \mathbb{N}}$$

are increasing, we may compute that

$$\begin{aligned} \bigvee_{N \in \mathbb{N}} \bigwedge_{k \geq N} d_k &= \bigvee_{N \in \mathbb{N}_0} \bigwedge_{k \geq N} d_k \\ &= \bigvee_{N \in \mathbb{N}_0} \left( \bigwedge \{d_k \mid k \geq N \text{ and } (A) \text{ holds}\} \wedge \bigwedge \{d_k \mid k \geq N \text{ and } (B) \text{ holds}\} \right) \\ &\stackrel{(*)}{=} \left( \bigvee_{N \in \mathbb{N}_0} \bigwedge \{d_k \mid k \geq N \text{ and } (A) \text{ holds}\} \right) \\ &\quad \wedge \left( \bigvee_{N \in \mathbb{N}_0} \bigwedge \{d_k \mid k \geq N \text{ and } (B) \text{ holds}\} \right) \\ &\geq \left( \bigvee_{N \in \mathbb{N}_0} \bigwedge \{\epsilon \circ c_{m_k-1} \mid k \geq N \text{ and } (A) \text{ holds}\} \right) \\ &\quad \wedge \left( \bigvee_{N \in \mathbb{N}_0} \bigwedge \{a_{n_k-1} \circ \epsilon \mid k \geq N \text{ and } (B) \text{ holds}\} \right) \\ &\geq \left( \bigvee_{N \in \mathbb{N}} \bigwedge_{m \geq N} \epsilon \circ c_m \right) \wedge \left( \bigvee_{N \in \mathbb{N}} \bigwedge_{m \geq N} a_m \circ \epsilon \right) \\ &\geq \left( \epsilon \circ \left( \bigvee_{N \in \mathbb{N}} \bigwedge_{m \geq N} c_m \right) \right) \wedge \left( \left( \bigvee_{N \in \mathbb{N}} \bigwedge_{m \geq N} a_m \right) \circ \epsilon \right) \\ &= \left( \epsilon \circ \left( \bigvee_{N \in \mathbb{N}} c_N \right) \right) \wedge \left( \left( \bigvee_{N \in \mathbb{N}} a_N \right) \circ \epsilon \right) \\ &= (\epsilon \circ 1) \wedge (1 \circ \epsilon) \\ &= \epsilon \end{aligned}$$

So, even though  $\epsilon \not\leq d_k$  (for all  $k \in \mathbb{N}$ ), we do have that  $0 \neq \epsilon \leq \bigvee_{N \in \mathbb{N}} \bigwedge_{k \geq N} d_k$ .

(iv) Using the “composition” axiom in  $\mathbb{C}$ , we have for every  $k \geq N \in \mathbb{N}$  (recall that  $n_k, m_k \geq k$  too) that

$$d_k \geq c_{n_k} \circ \mathbb{C}(f^{n_k+1}x, f^{m_k+1}x) \circ a_{m_k} \geq c_{n_k} \circ \varphi(d_k) \circ a_{m_k} \geq c_N \circ \varphi(d_k) \circ a_N$$

and so we may compute that

$$\begin{aligned}
\bigvee_{N \in \mathbb{N}} \bigwedge_{k \geq N} d_k &\geq \bigvee_{N \in \mathbb{N}} \bigwedge_{k \geq N} (c_N \circ \varphi(d_k) \circ a_N) \\
&\geq \bigvee_{N \in \mathbb{N}} \left( c_N \circ \left( \bigwedge_{k \geq N} \varphi(d_k) \right) \circ a_N \right) \\
&\stackrel{(*)}{=} \left( \bigvee_{N \in \mathbb{N}} c_N \right) \circ \left( \bigvee_{N \in \mathbb{N}} \bigwedge_{n \geq N} \varphi(d_k) \right) \circ \left( \bigvee_{N \in \mathbb{N}} a_N \right) \\
&= 1 \circ \left( \bigvee_{N \in \mathbb{N}} \bigwedge_{k \geq N} \varphi(d_k) \right) \circ 1 \\
&\geq \varphi \left( \bigvee_{N \in \mathbb{N}} \bigwedge_{n \geq N} d_k \right) \\
&\geq \bigvee_{N \in \mathbb{N}} \bigwedge_{n \geq N} d_k
\end{aligned}$$

where in  $(*)$  we used once more the argument involving increasing sequences (explained in a previous footnote), but now for three sequences instead of two. This means that  $\bigvee_{N \in \mathbb{N}} \bigwedge_{k \geq N} d_k$  is a fixpoint of  $\varphi$  which – as we showed earlier – is not 0, so we must have  $1 \leq \bigvee_{N \in \mathbb{N}} \bigwedge_{k \geq N} d_k$ .

(v) Since  $\epsilon \ll 1 \leq \bigvee_{N \in \mathbb{N}} \bigwedge_{k \geq N} d_k$ , and the latter supremum is directed, there must exist an  $N_0 \in \mathbb{N}$  such that  $\epsilon \leq \bigwedge_{k \geq N_0} d_k$ . Yet, we established earlier that  $\epsilon \not\leq d_k$  for all  $k \in \mathbb{N}$ . This is the announced contradiction.  $\square$

**Remark 2.2.3** For simplicity’s sake, we asked in Definition 2.2.1 a control function  $\varphi \in \Phi$  to satisfy  $\varphi(\bigvee_{N \in \mathbb{N}} \bigwedge_{n \geq N} t_n) \leq \bigvee_{N \in \mathbb{N}} \bigwedge_{n \geq N} \varphi(t_n)$  for any sequence  $(t_n)_{n \in \mathbb{N}}$  in  $Q$ . Yet, in the proof above, given a  $\varphi$ -contraction  $f: \mathbb{C} \rightarrow \mathbb{C}$ , we really only need such inequalities for sequences whose elements are “homs” in  $\mathbb{C}$  (i.e. elements in the image of  $\mathbb{C}: \mathbb{C}_0 \times \mathbb{C}_0 \rightarrow Q$ ). Similarly, in Definition 2.1.1 we asked  $\varphi$  to be strictly increasing on all  $t \in Q$ , except possibly in  $t = 0$  or  $t \geq 1$ . But again, in the proof above we really only need this strictness in those particular elements of  $Q$  which are suprema, or suprema of infima, of “homs” in  $\mathbb{C}$ . This observation will be useful in Example 3.2.2.

### 2.3 Fixpoint for a contraction on a Cauchy-complete $Q$ -category

In the above Subsection we discovered sufficient conditions for a  $\varphi$ -contraction  $f: \mathbb{C}_0 \rightarrow \mathbb{C}_0$  to determine adjoint distributors. If the  $Q$ -category  $\mathbb{C}$  is Cauchy-complete, this adjoint pair is represented by an object of  $\mathbb{C}$ . We will now show that this representing object is a fixpoint for the contraction.

**Proposition 2.3.1** *Let  $Q$  be any quantale and  $f: \mathbb{C} \rightarrow \mathbb{C}$  any  $Q$ -functor on a Cauchy-complete  $Q$ -category. If there exists an  $x \in \mathbb{C}_0$  such that  $\phi_{x,f} \dashv \psi_{x,f}$  then  $f$  has a fixpoint.*

*Proof.* By Cauchy-completeness of  $\mathbb{C}$ , the presheaves  $\phi_{x,f}$  and  $\psi_{x,f}$  are representable:  $\phi_{x,f} = \mathbb{C}(-, u)$  and  $\psi_{x,f} = \mathbb{C}(u, -)$  for some  $u \in \mathbb{C}_0$ . Now we can compute that

$$\begin{aligned}
\mathbb{C}(fu, u) &= \phi_{x,f}(fu) \\
&= \bigvee_{N \in \mathbb{N}} \bigwedge_{n \geq N} \mathbb{C}(fu, f^n x) \\
&= \bigvee_{N \in \mathbb{N}_0} \bigwedge_{n \geq N} \mathbb{C}(fu, f^n x) \\
&\stackrel{(*)}{\geq} \bigvee_{N \in \mathbb{N}} \bigwedge_{n \geq N} \mathbb{C}(u, f^n x) \\
&= \phi_{x,f}(u) \\
&= \mathbb{C}(u, u) \\
&\geq 1
\end{aligned}$$

using the “functoriality” axiom for  $f$  in  $(*)$ . Similarly one computes that  $\mathbb{C}(u, fu) \geq 1$ . Therefore we have both  $u \geq fu$  and  $fu \geq u$  in (the underlying order of)  $\mathbb{C}$ , which means that  $u \cong fu$ , as wanted.  $\square$

Putting Propositions 2.2.2 and 2.3.1 together, we arrive at:

**Theorem 2.3.2 (Fixpoint theorem for a  $\Phi$ -contraction)** *Let  $Q$  be quantale whose underlying lattice is continuous, and  $f: \mathbb{C} \rightarrow \mathbb{C}$  a  $\Phi$ -contraction on a Cauchy-complete  $Q$ -category. If there exists an  $x \in \mathbb{C}_0$  such that  $\mathbb{C}(x, fx) \neq 0 \neq \mathbb{C}(fx, x)$  then  $f$  has a fixpoint.*

In the above Theorem, the obtained fixpoint depends on the element  $x \in \mathbb{C}$  chosen such that  $\mathbb{C}(x, fx) \neq 0 \neq \mathbb{C}(fx, x)$ . However, let us recall that Propositions 2.1.3 and 2.1.4 provide mild conditions on  $\mathbb{C}$  to make the fixpoint of a contraction unique.

### 3. Examples and counterexamples

The examples in this section show how Theorem 2.3.2 generalizes known fixpoint theorems from the literature, and provides new ones too. Also, we mention a counterexample to show that the conditions cannot be weakened unless supplementary conditions are considered.

### 3.1 Orders

The two-element boolean algebra being obviously a (totally) continuous lattice, the quantale  $Q = (\{0, 1\}, \vee, \wedge, 1)$  satisfies the condition in Theorem 2.3.2, so this Theorem can potentially say something about ordered sets. Note that the functions

$$\varphi_1: \{0, 1\} \rightarrow \{0, 1\}: 0 \mapsto 0, 1 \mapsto 1 \quad \text{and} \quad \varphi_2: \{0, 1\} \rightarrow \{0, 1\}: 0 \mapsto 1, 1 \mapsto 1$$

are precisely the elements of  $\Phi$ . A map  $f: (P, \leq) \rightarrow (P, \leq)$  is a  $\varphi_1$ -contraction if and only if  $f$  is monotone; and it is a  $\varphi_2$ -contraction if and only if  $f$  is essentially constant ( $fx \cong fy$  for all  $x, y \in P$ ). Any non-empty ordered set is Cauchy-complete as a  $Q$ -enriched category. It is part of the hypotheses in Theorem 2.3.2 that there exists an  $x \in P$  such that  $x \leq fx$  and  $fx \leq x$ ; in other words, *by hypothesis* there exists a fixpoint  $fx \cong x$ . Of course this makes the conclusion of the Theorem (namely, the existence of a fixpoint) trivial! Moreover, the fixpoint that is constructed in the proof (as an object representing a left adjoint presheaf) is in this particular case precisely isomorphic to the fixpoint given as hypothesis. So, for the two-element Boolean algebra, Theorem 2.3.2 does not give any result; the Theorem can thus only be meaningful for “richer” quantales. (We hasten to add that there exist of course very important fixpoint theorems for ordered sets; but these usually require more stringent completeness conditions on the ordered set and/or more stringent continuity conditions on the map. See e.g. [10].)

### 3.2 Metric spaces

Lawvere’s quantale  $Q = ([0, \infty], \wedge, +, 0)$  is linear, and therefore continuous<sup>10</sup>. It is also an integral quantale: the unit 0 for the monoid structure is the top element of the lattice (note again that the order on  $[0, \infty]$  is the reverse of the natural order!). This makes the notion of contraction in Definition 2.1.1 slightly simpler, so by application of Theorem 2.3.2 we can produce the following result:

**Corollary 3.2.1** *Let  $\varphi: [0, \infty] \rightarrow [0, \infty]$  be an upper-semicontinuous function so that  $\varphi(t) < t$  for any  $t \notin \{0, \infty\}$  and  $\varphi(0) = 0$ . Let  $f: X \rightarrow X$  be a map on a Cauchy-complete generalized metric space  $(X, d)$  such that  $d(fx, fy) \leq$*

<sup>10</sup>Any complete linear lattice  $L$  is completely distributive and (therefore) also totally continuous, continuous and meet-continuous. Because every subset of  $L$  is directed (if  $a, b \in S \subseteq L$  then either  $a = a \vee b$  or  $b = a \vee b$  and so in any case  $a \vee b \in S$ ), the notions of “way below” and “totally below” coincide. In fact, we have  $a \ll b$  if and only if either  $a = 0$ , or  $a < b$ , or  $(a = b \text{ and } b \neq \bigvee \{x \in L \mid x < b\})$ , see [10].



$\varphi(d(x, y))$  for all  $x, y \in X$ . If there is an  $x \in X$  such that  $d(x, fx) \neq \infty \neq d(fx, x)$  then the sequence  $(f^n x)_{n \in \mathbb{N}}$  converges to a fixpoint of  $f$ .

If  $(X, d)$  is a *finitary* generalized metric space, then any  $x \in X$  will produce a convergent sequence  $(f^n x)_{n \in \mathbb{N}}$ ; and Proposition 2.1.3 implies that all such sequences  $(f^n x)_{n \in \mathbb{N}}$  converge to an essentially unique fixpoint of  $f$  (unique if the space is also separated). If  $(X, d)$  is a *symmetric* generalized metric space, then any  $x \in X$  such that  $d(x, fx) \neq \infty$  will produce a convergent sequence  $(f^n x)_{n \in \mathbb{N}}$ ; and Proposition 2.1.4 implies that any two fixpoints of  $f$  are either isomorphic (equal if the space is also separated) or at distance  $\infty$  from each other (i.e. the space  $(X, d)$  decomposes as a categorical sum of two non-empty spaces, and the fixpoints are in different summands). Furthermore, for the reasons explained in Remark 2.2.3, in the above result it is actually enough to require that  $\varphi$  is (defined on and) upper-semicontinuous on the closure of  $\{d(x, y) \mid x, y \in X\}$ . This is how the above result is formulated by Boyd and Wong [5, Theorem 1] in the particular case that  $(X, d)$  is a finitary, symmetric, separated metric space (see also [2, 26]):

**Example 3.2.2** For  $(X, d)$  a Cauchy-complete metric space, let  $\varphi: \overline{\{d(x, y) \mid x, y \in X\}} \rightarrow [0, \infty]$  be an upper-semicontinuous function that maps 0 to 0 and so that  $\varphi(t) < t$  for any  $t \notin \{0, \infty\}$ . Then any map  $f: X \rightarrow X$  satisfying  $d(fx, fy) \leq \varphi(d(x, y))$  for all  $x, y \in X$  has a unique fixpoint, and for any  $x \in X$  the sequence  $(f^n x)_{n \in \mathbb{N}}$  converges to that fixpoint.

On the other hand, the control function defined by  $\varphi(t) = k \cdot t$  for  $0 < k < 1$  certainly satisfies the conditions in Corollary 3.2.1, so we find the following particular case:

**Example 3.2.3** Let  $f: X \rightarrow X$  be a map on a Cauchy-complete generalized metric space  $(X, d)$  for which there exists a  $0 < k < 1$  such that  $d(fx, fy) \leq k \cdot d(x, y)$  for all  $x, y \in X$ . If there is an  $x \in X$  such that  $d(x, fx) \neq \infty \neq d(fx, x)$  then the sequence  $(f^n x)_{n \in \mathbb{N}}$  converges to a fixpoint of  $f$ .

Obviously, if  $(X, d)$  is finitary, symmetric and separated (and hence an ordinary metric space), we find here the well-known Banach Fixpoint Theorem (see e.g. [18]). Finally, we mention that Ackerman [1] has produced an example of a *non-expansive* contraction – whose control function merely satisfies  $\varphi(t) \leq t$  instead of  $\varphi(t) < t$  for  $t \notin \{0, \infty\}$  – on a Cauchy-complete metric space which does not have a fixpoint. This shows that this condition on the control map cannot be weakened without strengthening some other conditions in Corollary 3.2.1.

### 3.3 Fuzzy orders

The quantale  $Q = ([0, 1], \vee, *, 1)$ , where  $*$  is a left-continuous t-norm, is linear (thus continuous, see previous footnote) and integral. Hence, by application of Theorem 2.3.2 we find:

**Corollary 3.3.1** *Let  $\varphi: [0, 1] \rightarrow [0, 1]$  be a lower-semicontinuous function satisfying  $\varphi(t) > t$  for all  $0 < t < 1$ , and  $\varphi(1) = 1$ . Suppose that  $(P, \llbracket \cdot \leq \cdot \rrbracket)$  is a complete fuzzy preorder, and that  $f: P \rightarrow P$  is a function such that  $\llbracket fx \leq fy \rrbracket \geq \varphi(\llbracket x \leq y \rrbracket)$ . If there exist  $x \in P$  such that  $\llbracket x \leq fx \rrbracket \neq 0 \neq \llbracket fx \leq x \rrbracket$ , then the sequence  $(f^n(x))_{n \in \mathbb{N}}$  converges to a fixpoint of  $f$ .*

This can be regarded as a (straightforward) generalization of Corollary 3.2.1 above, since Lawvere’s quantale  $([0, \infty], \wedge, +, 0)$  is isomorphic to the product t-norm  $([0, 1], \wedge, \cdot, 1)$  by the order-reversing map  $[0, \infty] \rightarrow [0, 1]: t \mapsto \exp(-t)$ .

### 3.4 Probabilistic metric spaces

The integral quantale  $(\Delta, \vee, *, e)$  of distance distributions (wrt. a left-continuous t-norm  $*$ ) is completely distributive<sup>11</sup>, hence continuous, so we can apply Theorem 2.3.2.

**Corollary 3.4.1** *Let  $\varphi: \Delta \rightarrow \Delta$  be a lower-semicontinuous function satisfying  $\varphi(u) > u$  for all  $0 < u < e$ , and  $\varphi(e) = e$ . Suppose that  $f: X \rightarrow X$  is a function on a Cauchy-complete generalized probabilistic metric space  $(X, d, *)$  such that  $d(fx, fy, t) \geq \varphi(d(x, y, t))$  for all  $t$ . If there exists an  $x \in X$  such that  $d(x, fx, t) \neq 0 \neq d(fx, x, t)$  then  $f$  has a fixpoint.*

It follows furthermore from Proposition 2.1.3 that, if  $d(x, y, \infty) = 1$  for all  $x, y \in X$  (i.e. the space is *finitary*), then the fixed point is unique.

There are indeed examples of control functions  $\varphi: \Delta \rightarrow \Delta$  that the above statement asks for, e.g.

$$\varphi(u)(t) := \begin{cases} \frac{1}{2}(u(t) + 1) & \text{if } 0 < t \leq \infty \\ 0 & \text{if } t = 0 \end{cases}$$

Unfortunately though, the “Banach control function” which is appropriate in the setting of probabilistic metric spaces<sup>12</sup>,

$$\varphi(u)(t) = u(Kt) \text{ for some } 1 < K < \infty,$$

<sup>11</sup>Indeed, the complete distributivity of the underlying suplattices follows from [11, Theorem 2.1.17], who show that the tensor product of completely distributive complete lattices is completely distributive.

<sup>12</sup>In [13] the contractions with this control function are called *probabilistic q-contractions*.

does not satisfy  $\varphi(u) \neq u$  for all  $0 \neq u \neq e$  (e.g. the “almost constant” functions, defined by  $u(t) = u_0$  for  $0 < u_0 < 1$  and  $0 < t \leq \infty$ , are fixpoints of  $\varphi$ ). One possible solution (hinted at by a result in [12]) would be to work with finitary probabilistic metric spaces. These can be seen as categories enriched in the subquantale

$$\Delta^+ = \{u \in \Delta \mid u(\infty) = 1\} \cup \{0\}$$

of  $\Delta$ . Restricted to  $\Delta^+$ , the Banach control function does not have fixpoints other than 0 and  $e$ : if  $u \in \Delta^+ \setminus \{0\}$  satisfies  $u(t) = u(Kt)$ , then for any  $0 < t_0 < \infty$ ,

$$1 = u(\infty) = u\left(\bigvee_{n \in \mathbb{N}} K^n t_0\right) = \bigvee_{n \in \mathbb{N}} u(K^n t_0) = u(t_0),$$

so indeed  $u = e$ . However, we do not know whether  $\Delta^+$  is continuous (we conjecture that it is not), so we do not know whether we can apply Theorem 2.3.2 without modifications: this will be a topic of further investigation.

## 4. Conclusion and further work

With our study of fixpoint theorems for quantale-enriched categories, we exemplified that such results depend not only on the strength of the contraction and the completeness of the space, but also on the algebraic properties of the underlying quantale: any fixpoint theorem results from an equilibrium between those three aspects.

In future work, we want to investigate how several examples of fixpoint theorems in the literature (see e.g. [7, 12, 13, 18]) fit – or, perhaps, do not fit – with our quantale-enriched approach. This could lead to variants on our Theorem 2.3.2, where different algebraic properties of  $Q$  are combined with different conditions on the control functions of contractions, or with different completeness conditions on the  $Q$ -categories (see e.g. [31]). We also intend to study fixpoint theorems for *quantaloid*-enriched categories. This generalization, far from trivial, has the benefit to include in particular the theory of partial metric spaces [30, 17] and of probabilistic partial metric spaces [14], two areas for which only few fixpoint theorems are known [23, 25, 27].

## Acknowledgement

The authors thank Z. A. Guenoun and especially U. Höhle for their comments on an earlier version of the manuscript.

## References

- [1] N. L. Ackerman, A fixed point theorem for contracting maps of symmetric continuity spaces, *Topol. Proc.* Vol. 47 (2016), 89–100.
- [2] I. D. Andelovic, On Boyd-Wong-type fixed point results, *Scientific Publications of the State University of Novi Pazar Series A: Applied Mathematics, Informatics and mechanics*, Vol. 7, no. 1 (2005), 41–46.
- [3] S. Banach, Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales, *Fundamenta Mathematicae*, Vol. 3 (1922), 133–181.
- [4] E. Barcz, A new proof and consequences of the fixed point theorem of Matkowski, *Annales Mathematicae Silesianae* Vol. 35 no. 2 (2021), 149–157.
- [5] D. W. Boyd, and J.S.W. Wong, On nonlinear contractions, *Proceedings of the American Mathematical Society* Vol. 20 No. 2 (1969), 458–464.
- [6] Y.-M. Chai, A note on the probabilistic quasi-metric spaces, *J. Sichuan Univ. (Nat. Sci. Ed.)* 46 (2009) 543–547.
- [7] C. Coppola, G. Giangiacomo and P. Tiziana, Convergence and fixed points by fuzzy orders, *Fuzzy Sets and systems* Vol. 159 no. 10 (2008), 1178–1190.
- [8] P. Eklund, J. Gutiérrez García, U. Höhle and J. Kortelainen, *Semi-groups in complete lattices: quantales, modules and related topics*, Vol. 54 (2018), ISBN : 978-3-319-78947-7.
- [9] A. George, P. Veeramani, On some results in fuzzy metric spaces, *Fuzzy Sets Syst.* 64 (1994) 395–399.
- [10] G. Gierz, K. H. Hofmann, K. Keimel, J. D. Lawson, M. W. Mislove, D. S. Scott, *A Compendium of Continuous Lattices*, Springer-Verlag, Berlin, New York, 1980.
- [11] J. Gutiérrez García, U. Höhle, T. Kubiak, Tensor products of complete lattices and their application in constructing quantales, *Fuzzy Sets Syst.* 313 (2017) 43–60.
- [12] M. Grabiec, Fixed points in fuzzy metric spaces, *Fuzzy Sets Syst.* 27 (1988) 385–389.

- [13] O. Hadžić and E. Pap, Fixed point theory in probabilistic metric spaces Vol. 536, Dordrecht: Kluwer Academic Publishers (2001).
- [14] J. He, H. Lai, L. Shen, Towards probabilistic partial metric spaces: Diagonals between distance distributions, *Fuzzy Sets and Systems*, 370 (2019), 99–119.
- [15] E. Hewitt, K.R. Stromberg, *Real and abstract analysis*, Springer (1965).
- [16] D. Hofmann and C. Reis, Probabilistic metric spaces as enriched categories, *Fuzzy Sets and Systems*, Vol. 210 (2013), 1–21.
- [17] D. Hofmann and I. Stubbe, Topology from enrichment: the curious case of partial metrics, *Cahiers de topologie et géométrie différentielle catégoriques* Vol. LIX-4 (2018), 307–353.
- [18] M. Jleli and B. Samet, A generalized metric space and related fixed point theorems, *Fixed point theory and Applications*, Vol. 2015.1 (2015), 1–14.
- [19] E.P. Klement, R. Mesiar and E. Pap, *Triangular norms*, Vol. 8, Springer Science and Business Media (2013).
- [20] I. Kramosil, J. Michálek, Fuzzy metrics and statistical metric spaces, *Kybernetika (Prague)* 11 (1975) 336–344.
- [21] F. W. Lawvere, Metric spaces, generalized logic and closed categories, *Reprints in Theory and Applications of Categories*, No. 1 (2002), 1–37.
- [22] H. Lai and D. Zhang, Fuzzy preorder and fuzzy topology, *Fuzzy Sets Syst.* 157 (2006), 1865–1885.
- [23] S. Oltra and O. Valero, Banach’s fixed point theorem for partial metric spaces, *Rendiconti dell’Istituto di Matematica dell’Università di Trieste* Vol.36 (2004), 17–26.
- [24] S. V. Ovchinnikov, Representations of transitive fuzzy relations, in: H. J. Skala *et al.* (eds.), *Aspects of vagueness*, D. Reidel, Dordrecht (1984), 105–118.
- [25] R. Pant, R. Shukla, H.K. Nashine and R. Panicker, Some new fixed point theorems in partial metric spaces with applications, *Journal of Function Spaces* (2017), 1–13.
- [26] L. Pasicki, The Boyd-Wong idea extended, *Fixed Point Theory and Applications* Vol. 2016 No. 1 (2016), 1–5.

- [27] S. Romaguera, Fixed point theorems for generalized contractions on partial metric spaces, *Topology and its Applications* 159 (2012), 194–199.
- [28] B. Schweizer and A. Sklar, *Probabilistic metric spaces*, Nord-Holland, Amsterdam (1983).
- [29] I. Stubbe, Categorical structures enriched in a quantaloid: categories, distributors and functors, *Theory and Applications of Categories* Vol. 14 (2005), 1–45.
- [30] I. Stubbe, An introduction to quantaloid-enriched categories, *Fuzzy Sets and Systems* Vol. 256 (2014), 95–116.
- [31] K.R. Wagner, Fundamental Study Liminf convergence in  $\Omega$ -categories, *Theoretical Computer Science* 184 (1997), 61–104.
- [32] L.A. Zadeh, Similarity relations and fuzzy orderings, *Information Sciences* 3 (1971), 177–200.

