



HAL
open science

New Physics with the Higgs boson and particles of spin up to two

Wenqi Ke

► **To cite this version:**

Wenqi Ke. New Physics with the Higgs boson and particles of spin up to two. High Energy Physics - Theory [hep-th]. Sorbonne Université, 2023. English. NNT : 2023SORUS208 . tel-04202532

HAL Id: tel-04202532

<https://theses.hal.science/tel-04202532v1>

Submitted on 11 Sep 2023

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

**THÈSE DE DOCTORAT
DE SORBONNE UNIVERSITÉ**

Spécialité : Physique Théorique

École doctorale n°564: Physique en Île-de-France

réalisée

au Laboratoire de Physique Théorique et Hautes Énergies

sous la direction de Karim BENAKLI

présentée par

Wenqi KE

pour obtenir le grade de

DOCTEUR DE SORBONNE UNIVERSITÉ

Sujet de la thèse :

**New Physics with the Higgs boson and particles of spin up
to two**

soutenue le 27/06/2023

devant le jury composé de :

M.	Karim BENAKLI	Directeur de Thèse
M^{me}	Sacha DAVIDSON	Rapporteur
M.	Pierre FAYET	Examineur
M.	Yann MAMBRINI	Examineur (Président du jury)
M.	Michel TYTGAT	Rapporteur

Remerciements

Cette thèse serait très différente, voire n'existerait pas, sans le soutien et l'accompagnement de mon directeur, mes collègues, mes amis ainsi que ma famille. Tout d'abord, je tiens à remercier mon directeur, Karim Benakli, de m'avoir acceptée en thèse et pour son aide et son encouragement tout au long de mon doctorat. À mes questions, il a toujours une réponse, une liste de références ou des pistes potentielles. Sa disponibilité, son intuition sur de nombreux sujets en physique et sa bienveillance (que demander de plus ?) ont fait de mon doctorat une période très stimulante et agréable. En plus de mon directeur, je remercie Pierre Fayet et Yann Mambrini, ainsi que mes rapporteurs Sacha Davidson et Michel Tytgat, d'avoir accepté de faire partie de mon jury.

Je remercie Pietro Slavich. Ses remarques et conseils sont toujours pertinents. Ce fut un véritable plaisir de collaborer et de discuter avec lui. Je le remercie de ses précieux conseils et de son aide, que ce soit lors de la préparation de mon premier séminaire ou à d'autres moments de ma thèse. Je remercie également Ignatios Antoniadis pour la collaboration et les discussions très intéressantes sur les gravitinos.

Je suis reconnaissante envers mes professeurs de l'ENS pour leurs cours exceptionnels. J'aimerais en particulier adresser ma gratitude à Pierre Fayet. Il a eu la gentillesse et la patience de m'expliquer toutes les bases du Modèle standard, quand j'entrais en master et que je connaissais à peine la QFT. J'ai beaucoup appris grâce à lui.

Tout au long de mes trois années de thèse, j'ai eu la chance de bénéficier d'un environnement de travail agréable. Je souhaite exprimer ma reconnaissance envers les chercheurs de mon laboratoire pour les discussions scientifiques enrichissantes et l'organisation des journal clubs et des séminaires. Mes remerciements vont également à Benoît Estienne et Guillaume Bossard pour leurs conseils lors des comités de suivi de thèse, ainsi qu'à Françoise Got et Carole Kurzydowski pour leur réactivité dans le traitement de mes demandes.

Je remercie les autres doctorants de mon laboratoire, qui sont mes collègues et aussi mes amis, pour les bons moments que nous avons partagés : Andriani, Anthony, Ari, Carlo, Damien, Diyar, Francesco M., François, Gaëtan, Grégoire, Greivin, Jordan, Jules, Léo, Manuel, Mathis, Osmin, Pierre, Simon, Tom, Vincent, Yann, Yehudi, et plus spécialement mes collègues de bureau: Andrea, Andrei (pour les inoubliables compétitions du morning club), Francesco C., et Maxl – ensemble, nous avons fait de notre bureau le plus sérieux de tous.

En dehors de mon laboratoire, je remercie mes amis et camarades de l'ENS : Edwan, Lucas, Magdy, Marc, Matthieu, Senwen, Thomas C., Thomas H., avec qui j'ai partagé les meilleurs moments de ma scolarité, allant de la révision intensive au cocktail chez Magdy. Je leur souhaite plein succès dans leur avenir scientifique.

À ma famille en France: Anne-Marie, Valérie, François. Ils m'accueillent toujours avec la

plus grande gentillesse et bienveillance, et grâce à eux, j'ai pu travailler dans les meilleures conditions. Je leur en suis profondément reconnaissante.

À Pierre-Emmanuel, la personne qui m'est la plus chère. Sans lui, je serais quelqu'un de différent. Je le remercie pour toutes ces choses qu'il m'a apprises, qu'il m'a fait découvrir en dehors de la physique, et pour son soutien émotionnel pendant les périodes difficiles. À ma vie ascétique de thésarde, il apporte la joie.

À mon père, qui malheureusement ne peut pas me voir finir ma thèse. Je lui dois mes premières aspirations scientifiques et c'est grâce à lui que j'ai pu poursuivre mes études en France. Je le remercie d'avoir soutenu mes études longues – un choix pas très féminin pour la Chine rurale dont je suis originaire. S'il était là, il serait fier de moi.

Abstract

Elementary particles are distinguished by their spin and their mass. In particular, the $\text{spin} \leq 1$ fields, corresponding to the Standard Model content, are theoretically well-known and have been observed in nature. Going to higher spins, however, serious issues often arise as one attempts to construct a Lagrangian. This is the case for charged massive higher spins, which are plagued by acausality and loss of hyperbolicity. Even the simplest examples of the spin-3/2 and spin-2 turn out to be highly challenging, despite the intense investigation over the past decades. In this thesis, we present consistent Lagrangians, the subsequent equations of motion, and constraints for the charged massive spin-3/2 and spin-2 states. These Lagrangians are derived from the superspace action for the first massive level of charged superstring in four dimensions.

The motivations for higher spin studies are not limited to a pure field-theoretic interest. It is postulated by many authors that higher spins may be produced in early universe, and could be a Dark Matter candidate. The gravitinos, which are spin-3/2 particles in supergravity, have received a particular attention. Some authors have recently pointed out that, when the gravitino sound speed vanishes, its production will diverge. We will show that for linear $\mathcal{N} = 1$, $d = 4$ supergravity, coupled to two fermions, the gravitino sound speed never vanishes, hence avoiding pathological production.

After discussing higher spins, we move on to the second part of this thesis, which focuses on physics beyond the Standard Model. While the Standard Model has achieved great success in numerous experiments, several significant anomalies, along with theoretical issues, suggest the need for new physics. In this thesis, we investigate supersymmetric candidates. First, we consider scenarios where supersymmetry lies at high energy scales that are currently inaccessible to colliders. Through an example, we show how the Higgs mass can unravel the parameter space of such supersymmetric models. Second, we study Dirac gaugino models. These models have interesting properties in Higgs alignment and can also accommodate the recently reported 7σ deviation in the W boson mass. Through numerical analysis, we highlight the importance of quantum corrections.

Keywords:

higher spin, supersymmetry, supergravity, gravitino, Higgs mass, Dirac gauginos

Résumé

Les particules élémentaires se distinguent par leur spin et leur masse. En particulier, les champs de $\text{spin} \leq 1$, correspondant au contenu du Modèle standard, sont bien connus théoriquement et ont été observés dans la nature. Cependant, s'agissant des spins supérieurs, des pathologies sérieuses apparaissent lors de la construction d'un lagrangien. C'est typiquement le cas pour les spins supérieurs massifs chargés, où persistent l'acausalité ainsi que la perte de l'hyperbolicité. Même les exemples les plus simples, tels que le spin-3/2 et le spin-2, se révèlent extrêmement difficiles à étudier, malgré de nombreuses tentatives durant les décennies passées. Dans cette thèse, nous présentons des lagrangiens cohérents, les équations du mouvement et les contraintes pour spin-3/2 et spin-2 massifs chargés. Ces lagrangiens sont déduits de l'action dans le superspace décrivant le premier niveau massif de la supercorde chargée en quatre dimensions.

Outre son intérêt pour la théorie des champs, l'étude des spins supérieurs a des applications phénoménologiques. Il a été postulé par plusieurs auteurs que des spins supérieurs pourraient avoir été produits dans l'univers primordial et pourraient constituer un candidat pour la matière noire. Les gravitinos, qui sont des particules de spin-3/2 dans la supergravité, ont reçu une attention particulière. Certains auteurs ont récemment remarqué que lorsque la vitesse du son du gravitino s'annule, sa production diverge. Nous montrons que pour la supergravité $\mathcal{N} = 1$, $d = 4$, qui contient deux fermions, la vitesse du son du gravitino ne s'annule jamais. Par conséquent, une telle production divergente est évitée.

Après avoir discuté des spins supérieurs, nous entrons dans la deuxième partie de cette thèse, autour de la physique au-delà du Modèle standard. Si de nombreuses expériences ont prouvé la robustesse du Modèle standard, certaines anomalies importantes, ainsi que des problèmes théoriques, suggèrent le besoin d'une nouvelle physique. Dans cette thèse, nous investiguons des candidats supersymétriques. Premièrement, nous considérons des scénarios où la supersymétrie réside dans des échelles de hautes énergies qui sont inaccessibles aux collisionneurs actuels. À travers un exemple, nous étudions comment la masse du boson de Higgs peut dévoiler l'espace de paramètres de tels modèles supersymétriques. Deuxièmement, nous étudions des modèles de jauginos de Dirac. Ces modèles présentent des propriétés intéressantes rendant possible l'alignement de Higgs, et pourraient expliquer la déviation de 7σ récemment rapportée dans la masse du boson W . Par une analyse numérique, nous mettons en évidence l'importance des corrections quantiques.

Mots clés :

Spin supérieur, supersymétrie, supergravité, gravitino, la masse du Higgs, jauginos de Dirac

Abbreviations

2HDM	Two-Higgs Doublet Model
AdS	Anti-de Sitter
BSM	beyond the Standard Model
DoF	degree of freedom
dS	de Sitter
EFT	Effective Field Theory
EoM	equation of motion
EW	electroweak
EWSB	electroweak symmetry breaking
FLRW	Friedman-Lemaître-Robertson-Walker
FSSM	Flavourful Supersymmetric Standard Model
GR	General Relativity
GUT	Grand Unification Theory
LHC	Large Hadron Collider
MSSM	Minimal Supersymmetric Standard Model
NLL	next-to-leading-logarithmic
NMSSM	Next-to-Minimal Supersymmetric Standard Model
QCD	Quantum Chromodynamics
QED	Quantum Electrodynamics
QFT	Quantum Field Theory
RGE	Renormalization Group Equation
SM	Standard Model
SSB	Spontaneous Symmetry Breaking
SUSY	supersymmetry
UV	ultraviolet
vev	vacuum expectation value
WFR	wave-function-renormalization

Introduction

The mathematical formulation of elementary particles, on which is built modern Quantum Field Theory (QFT), dates back to less than a century ago. Since the work by Wigner [1] and his contemporaries, it is understood that all consistent and relativistic quantum fields can be classified by their spin s and mass m . Several decades after the discovery of the electron by the end of the 19th century, Dirac [2] proposed the celebrated spin-1/2 equation of motion, which is linear in momentum and mass. The corresponding theory for bosons of spin-0 and spin-1 are also known to be described by the Klein-Gordon equation, which is of second order in spacetime derivative. These theories are readily generalized in Quantum Electrodynamics (QED) to charged particles, with the introduction of the *minimal coupling*: $\partial_\mu \rightarrow \mathcal{D}_\mu - ieA_\mu$, and more general non-abelian gauge symmetries find their framework in Yang-Mills theory. The massless limit is well-established as well, where one usually recovers an additional gauge invariance.

Obviously this is not the end of the story. Though to date, no elementary particle of spin $s > 1$ has been detected, constructing a “higher spin theory” is still of great significance. Going beyond the pure mathematical curiosity, higher spin states play a crucial role in gravity. In General Relativity (GR), the mediator of gravitational interactions, the *graviton*, has spin-2. And in supergravity, the superpartner of the graviton – the *gravitino*, is of spin-3/2. It is important that a field theory of these states be ready for applications. Beside the hypothetical elementary particles, higher spin fields do exist in nature, and they are observed in composite states. According to the $SU(3)$ classification, the “baryon decuplet” is formed by three-quark (u, d, s) states with a total spin $s = 3/2$. An example of them is the Ω^- baryon carrying a unit charge: discovered in bubble chamber in 1964 [3], its spin is confirmed to be 3/2 by BaBar in 2006 [4]. In a typical experimental setup, these baryons travel a distance which is much larger than their own size, and therefore the point particle approximation is well justified, calling for the use of a local action.

It was not long after the development of the spin-1/2 equation that the higher spin studies were initiated by Dirac [5] in 1936. Later, the spin-3/2 and spin-2 massive free Lagrangians have been constructed by Rarita, Schwinger [6] and Fierz, Pauli [7] respectively. Several decades after these two works, Singh and Hagen [8, 9] proposed a systematic Lagrangian formulation for *any* spin, necessitating lower spin auxiliary fields to enhance the constraint equations. The well-studied massive free theories open new avenues for generalizations and applications.

However, the seemingly immediate generalization to the charged higher spins turns out to be a highly challenging task. One may naïvely replace all the partial derivatives by covariant derivatives as in the spin $s \leq 1$ case, but this operation inevitably results in a pathological action – manifesting acausality and loss of hyperbolicity – due to the non-commutativity of

the covariant derivatives. Already in the simplest setup with $U(1)$ charge and constant small electromagnetic background, finding a consistent higher spin effective Lagrangian proves to be very difficult. The attempts mainly consist in extending the Lagrangian with non-minimal terms, such as the Ferderbush Lagrangian [10] for spin-2 or the non-minimal spin-3/2 Lagrangians that Deser, Pascalutsa and Waldron [11] proposed, but the former still suffers from high energy unitarity problem and the latter does not escape from acausality. In recent contributions, an interesting approach is to start from String Theory, which features higher spin states in its spectrum. The past few decades have witnessed the development of charged bosonic string [12], then charged superstring theory [13] in a constant electromagnetic background. Based on the charged string theory, the investigation of the first massive states of open bosonic string in $26d$ has led Argyres and Nappi [14, 15] to write up to now the only charged massive spin-2 Lagrangian, having the correct gyromagnetic ratio, without ghost or coupling to lower spin fields, though this Lagrangian is not valid away from $26d$. One may naturally expect such construction to be performed on superstring as well, so as to derive a charged massive spin-3/2 theory along with a spin-2 in the same multiplet. This topic will be addressed in Chapter 2.

On the other hand, the phenomenological aspects of higher spins are not limited to the composite states, and when formulated in a curved spacetime, they have interesting cosmological implications. For instance, it is possible, using curved spacetime QFT, to analyze the gravitational particle production and investigate higher spin dark matter candidates. Indispensable in supergravity theories, the gravitino and its production are studied by various authors around 2000 [16–19] in the context of $\mathcal{N} = 1$ supergravity, in particular in the Freedman-Lemaître-Robertson-Walker (FLRW) metric. Recently, a pathological phenomenon in gravitino production is pointed out by Kolb, Long and McDonough [20]. It is shown that in the FLRW background, the gravitino *sound speed* may vanish which results in a divergent gravitino production, and several non-linear supersymmetric models indeed exhibit such behaviour. In $\mathcal{N} = 1$ supergravity with one chiral multiplet, the gravitino sound speed is equal to the speed of light [16–18] and therefore safe from divergent production. What happens if there is more than one chiral multiplet? Would divergent gravitino production occur? In this case, the question becomes more subtle due the mixing of different fermions, and the definition of a *physical* sound speed shall be clarified. Chapter 3 of this thesis deals with this question.

After venturing into theories of higher spins, let us go back to the “more usual” territory of $s \leq 1$, which is far from being fully understood, either. The Standard Model (SM) of particle physics postulates three families of leptons and quarks, of spin-1/2, along with the spin-0 Higgs boson and spin-1 gauge bosons. The last particle discovered in this catalogue is the Higgs boson in 2012 by ATLAS and CMS [21, 22], and so far the SM predictions have successfully gone through a majority of the experiments.

Yet several theoretical issues and experimental anomalies constantly raise the necessity for going beyond the SM (BSM). In addition to the long-standing neutrino mass and oscillation problems, the SM is also challenged by other measurements, such as the muon $g - 2$ anomaly and W boson mass, where significant deviations are reported. On the theoretical side, the SM does not account for gravitational interactions, and is not a suitable theory at the reduced Planck mass scale $M_P \sim 10^{18}\text{GeV}$, characterizing the realm of Quantum Gravity. The mere observation that the electroweak (EW) scale is much lower than the

Planck scale poses the “hierarchy problem”, which becomes even more compelling in view of, *e.g.*, the sensitivity of the Higgs boson mass to the ultraviolet (UV) cutoff due to quantum corrections. Shortly after the observation of the Higgs boson and the measurement of its mass, an analysis is performed in [23], evolving SM parameters using Renormalization Group Equations (RGEs), which surprisingly reveals an instability of the Higgs potential at an energy scale lower than M_P . Numerous evidences call for BSM physics.

A well-motivated BSM candidate is supersymmetry (SUSY), which features an extended particle content, and consequently an enlarged parameter space. A number of theoretical or phenomenological issues related to the SM find a satisfying explanation with SUSY. The aforementioned UV-divergence of the quantum corrections to the scalar mass, can be exactly cancelled by the *superpartner* diagrams, leaving an additional quadratic term in the SUSY breaking mass scale. SUSY models are also interesting candidates to accommodate the experimental discrepancies, in particular, it can provide a sufficient enhancement to muon $g-2$ so as to attain the measured value. But in absence of strong evidence of superparticles in direct collider searches, it is plausible that the SUSY scale is beyond the current experimental capacity. In this case, how do we constrain the parameter space of a given model? A specific property in SUSY is that, the Higgs quartic coupling which is extracted from the Higgs mass, is not a free parameter contrary to the SM. In the example of the MSSM, at tree level, the quartic coupling is determined by the “ D -term” and radiative corrections receive contribution from all superparticles. Therefore, the SM-like Higgs mass contains information about masses and couplings of these new particles. The goal of Chapter 4 is to illustrate, through an explicit example, how Higgs mass can constrain the parameter space of a high-scale SUSY model.

As the SUSY particle content involves more than one scalar, the question that must be addressed is how the SM-like Higgs boson is incorporated. The common requirement is that the observed Higgs be aligned with the scalar mixing matrix eigenstate acquiring a non-zero vacuum expectation value (vev) – this is the so-called *Higgs alignment*. The previous high-scale SUSY scenario easily realizes the alignment, because the new scalars are much heavier than the SM-like one, thus they decouple, and the remaining light scalar is naturally aligned with the SM Higgs boson. If one allows lighter SUSY masses, Higgs alignment without decoupling may also be achieved, thanks to some underlying symmetries of the model.

We will explore in Chapter 5 *Dirac gaugino models*, where gauginos have Dirac masses. Such models were first introduced in [24] to provide mass to gluinos while preserving R -symmetry. Interestingly, Dirac gaugino models can achieve alignment without decoupling, in addition, the singlet scalar in its particle content generically enhances the W -boson mass, which may accommodate the recently reported 7 standard deviations from the CDF experiment [25].

This manuscript is structured as follows: In Chapter 1, we briefly introduce supersymmetry as well as related topics, providing necessary tools for the following content. In Chapter 2, we review the main results in free massive higher spin theories, and introduce the pathologies persisting in charged higher spins. We then present the calculation to obtain explicitly the consistent Lagrangian, equations of motion and constraints of charged massive spin-3/2 and spin-2 states. In Chapter 3, we introduce the divergent gravitino production related to the gravitino sound speed, and then investigate the gravitino propagation in the case of $\mathcal{N} = 1$, $D = 4$ supergravity with more than one chiral multiplet. In Chapter 4, we study the Higgs mass constraints on a specific supersymmetric model, that solves the muon $g - 2$

anomaly. In Chapter 5, we start by reviewing the Dirac gaugino models, and then discuss their alignment properties and how they accommodate the W boson mass deviations. We conclude in the last Chapter. Conventions and useful results are presented in appendices.

Contents

Introduction	7
1 Super-preliminaries	13
1.1 Supersymmetry	13
1.1.1 Superalgebra	13
1.1.2 Superspace, superfields, supersymmetric Lagrangians	15
1.1.3 Supersymmetry breaking	20
1.1.4 R-symmetries	21
1.2 Supergravity	22
1.2.1 Local supersymmetry	22
1.2.2 Minimal supergravity	23
1.2.3 Matter couplings and scalar potential	25
1.2.4 Super-Higgs mechanism	27
2 Charged massive spin-3/2 and spin-2	30
2.1 Free massive higher spins: a well-established theory	30
2.1.1 Dirac, Rarita-Schwinger, and Fierz-Pauli	30
2.1.2 A Lagrangian for any spin	32
2.2 Charged massive higher spins: why are they so difficult?	33
2.2.1 The Velo-Zwanziger problem	33
2.2.2 Attempts and main results	35
2.3 Implementation of String Theory	36
2.4 Summary of the main results	39
2.4.1 Bosons	39
2.4.2 Fermions	41
2.5 Publication	43
3 (Too) slow gravitinos	115
3.1 Spin-3/2 in a flat spacetime	117
3.2 Spin-3/2 in a curved spacetime: catastrophic production?	118
3.2.1 Rarita-Schwinger action and equations	118
3.2.2 The sound speed	119
3.3 Publication	122

4 Higgs mass constraints on a supersymmetric solution of the muon $g-2$ anomaly	139
4.1 Muon $g-2$ and supersymmetry	139
4.2 Higgs mass in supersymmetry	142
4.3 FSSM model content and constraints from the Higgs mass	144
4.4 Publication	145
5 Higgs alignment and W boson mass in Dirac gaugino models	172
5.1 Higgs alignment	172
5.2 Dirac gaugino models	174
5.3 The setup and main results	176
5.4 Publication	177
Conclusions	212
Appendix A Conventions in Chapter 2	214
Appendix B Conventions in Chapter 3	216

Chapter 1

Super-preliminaries

This chapter provides basic elements and tools on supersymmetry as well as related subjects, which is not intended to be an extensive review. There exists a plethora of introductions to supersymmetry and/or supergravity, such as [26–31], and the lessons on superspace [32]. This chapter is based on these references.

1.1 Supersymmetry

1.1.1 Superalgebra

The supersymmetry algebra generalizes the Poincaré algebra of spacetime symmetry, by introducing a spinorial generator¹ Q that obeys schematically the following commutation and anticommutation relations:

$$\begin{aligned} [Q, P^\mu] &= [\bar{Q}, P^\mu] = 0 \\ \{\bar{Q}, \bar{Q}\} &= \{Q, Q\} = 0 \\ \{Q, \bar{Q}\} &= P^\mu \end{aligned} \tag{1.1}$$

where P^μ generates spacetime translations. Such algebra involving both commutation and anticommutation relations is also called a *graded Lie algebra*, or *superalgebra*. The possible symmetries of the S -matrix are subject to the Coleman-Mandula no-go theorem [33], which forbids any non-trivial combination of internal symmetry and spacetime symmetry, but supersymmetry evades this restriction by generalizing Lie algebra to graded Lie algebra. It has been proven by Haag, Łopuszański and Sohnius [34], as an extension to the Coleman-Mandula theorem, that the maximal symmetry of the S -matrix is the direct product of an internal symmetry with the supersymmetry algebra, in other words, supersymmetry algebra is the only non-trivial graded Lie algebra that is consistent with S -matrix symmetries, with a possible extension to include central charges.

The irreducible representations of supersymmetry are called *supermultiplets* or simply *multiplets*, which contain both fermionic and bosonic states that are related by supersymmetric transformations. The fermions and bosons in the same supermultiplet are said to

¹We consider the case with only one generator, which is often referred to as $\mathcal{N} = 1$ supersymmetry.

be *superpartners* of each other. The above graded Lie algebra obeys the following crucial properties:

1. Since the squared-mass operator $-P^2$ commutes with Q, \bar{Q} as well as spacetime rotation and translation generators, all particles in the same supermultiplet then must have the same eigenvalue under $-P^2$, namely the superpartners have the same mass.
2. The energy P_0 in supersymmetric theories is always positive or zero. This is demonstrated by using the anticommutator $\{Q, \bar{Q}\}$ in (1.1), with indices restored. In fact, a supersymmetry-preserving state always has *zero* vacuum energy.
3. A supermultiplet contains an equal number of bosonic and fermionic degrees of freedom.

The first point implies that supersymmetry must be broken, because superparticles having degenerate masses with their observed SM partners have never been detected. The second point adds that supersymmetry is *spontaneously* broken if and only if the vacuum has non-zero energy.

The simplest supersymmetric model is the free massless Wess-Zumino model, describing a complex scalar ϕ with its superpartner, a two-component Weyl fermion ψ . The Lagrangian has kinetic terms only:

$$\mathcal{L} = -\partial^\mu \bar{\phi} \partial_\mu \phi - i \bar{\psi} \bar{\sigma}^\mu \partial_\mu \psi \quad (1.2)$$

We followed the conventions in Appendix A. Denoting the infinitesimal, constant, supersymmetry transformation parameter by a two-component Weyl fermion ϵ_α , the above Lagrangian is invariant under

$$\delta_\epsilon \psi_\alpha = \sqrt{2}i (\sigma^\mu \bar{\epsilon})_\alpha \partial_\mu \phi, \quad \delta_\epsilon \phi = \sqrt{2} \epsilon \psi \quad (1.3)$$

To see if the supersymmetry algebra closes, one should check if the commutator of two transformations, $(\delta_{\epsilon_2} \delta_{\epsilon_1} - \delta_{\epsilon_1} \delta_{\epsilon_2})$, is another symmetry of the theory. For the scalar, this commutator results in a spacetime derivative $\partial_\mu \phi$ which is indeed a symmetry corresponding to the generator P^μ . But for the fermion, this is true only if we use the equation of motion $\bar{\sigma}^\mu \partial_\mu \psi = 0$, in other words, the supersymmetry algebra is closed *on-shell* in this case, which may be problematic for the symmetry to hold at quantum level. The remedy is to introduce an *auxiliary field*, a complex scalar F , with a Lagrangian

$$\mathcal{L}_F = \bar{F} F \quad (1.4)$$

It is trivial that the equation of motion of F , is just $F = 0$. By construction, auxiliary fields do not possess dynamics, and therefore they can be integrated out using their equations of motion. Adding a term (1.4) amounts to a physically equivalent theory. Let F transform as

$$\delta_\epsilon F = \sqrt{2}i \bar{\epsilon} \bar{\sigma}^\mu \partial_\mu \psi \quad (1.5)$$

and modify the transformation of ψ

$$\delta_\epsilon \psi_\alpha = \sqrt{2}i (\sigma^\mu \bar{\epsilon})_\alpha \partial_\mu \phi + \sqrt{2} \epsilon_\alpha F \quad (1.6)$$

It follows that the total Lagrangian $\mathcal{L} + \mathcal{L}_F$ is still invariant under supersymmetry transformations, but now the commutator $(\delta_{\epsilon_2} \delta_{\epsilon_1} - \delta_{\epsilon_1} \delta_{\epsilon_2})$ acting on any field in this theory, yields the spacetime derivative of the same field, without the need for equations of motion. As

a result, thanks to the auxiliary field, the supersymmetry algebra is closed *off-shell*. As is stated before, in a supermultiplet we must have the same bosonic and fermionic degrees of freedom, which holds both on shell and off shell. In this example, we have two bosonic (ϕ) and two fermionic (ψ) degrees of freedom on shell, whereas off-shell we have four bosonic (ϕ, F) and four fermionic (ψ) degrees of freedom.

1.1.2 Superspace, superfields, supersymmetric Lagrangians

Superfields, first introduced by Salam and Strathdee [35], are very powerful and elegant tools to treat supersymmetric theories. And thanks the notion of *superspace*, one acquires a geometrical picture of supersymmetry transformations, where the usual spacetime coordinates are supplemented by anticommuting (Grassmann) coordinates θ_α , with $\alpha = 1, 2$. The hermitian conjugate of θ_α has a dotted index, $\bar{\theta}_{\dot{\alpha}}$, with $\dot{\alpha} = 1, 2$. Points in superspace are labelled by $(x^\mu, \theta_\alpha, \bar{\theta}_{\dot{\alpha}})$. We can then define the group elements

$$G(x, \theta, \bar{\theta}) = e^{i(-x^\mu P_\mu + \theta Q + \bar{\theta} \bar{Q})} \quad (1.7)$$

The group action induces a translation on the enlarged coordinate space. Superfields are functions of superspace, which can be seen as a Taylor expansion in the Grassmann coordinates $(\theta, \bar{\theta})$. Remarkably, due to the anticommuting property of θ , this expansion is finite, and we have

$$\begin{aligned} F(x, \theta, \bar{\theta}) = & f(x) + \theta\phi(x) + \bar{\theta}\bar{\chi}(x) \\ & + \theta\theta m(x) + \bar{\theta}\bar{\theta}n(x) + \theta\sigma^\mu\bar{\theta}v_\mu(x) \\ & + \theta\theta\bar{\theta}\bar{\lambda}(x) + \bar{\theta}\bar{\theta}\theta\psi(x) + \theta\theta\bar{\theta}\bar{\theta}d(x) \end{aligned} \quad (1.8)$$

Among the components, there are 16 bosonic degrees of freedom in total, and 16 fermionic degrees of freedom. The superfield transforms as

$$\begin{aligned} \delta_\epsilon F(x, \theta, \bar{\theta}) \equiv & (\epsilon Q + \bar{\epsilon} \bar{Q}) F \\ \text{with } \epsilon Q + \bar{\epsilon} \bar{Q} = & \epsilon^\alpha \left(\frac{\partial}{\partial \theta^\alpha} - i\sigma_{\alpha\dot{\alpha}}^\mu \bar{\theta}^{\dot{\alpha}} \partial_\mu \right) + \bar{\epsilon}_{\dot{\alpha}} \left(\frac{\partial}{\partial \bar{\theta}^{\dot{\alpha}}} + i\bar{\sigma}^{\mu\dot{\alpha}\alpha} \theta_\alpha \partial_\mu \right) \end{aligned} \quad (1.9)$$

In addition, it is easy to see that linear combination of superfields, and product of superfields, are still superfields. The superfields are highly reducible, under constraints that are preserved by the transformation (1.9). The reduced superfields have less components, but no new dynamical content. To find such constraints, it is convenient to introduce the covariant derivatives

$$\begin{aligned} D_\alpha &= \frac{\partial}{\partial \theta^\alpha} + i\sigma_{\alpha\dot{\alpha}}^\mu \bar{\theta}^{\dot{\alpha}} \partial_\mu \\ \bar{D}_{\dot{\alpha}} &= -\frac{\partial}{\partial \bar{\theta}^{\dot{\alpha}}} - i\theta^\alpha \sigma_{\alpha\dot{\alpha}}^\mu \partial_\mu \end{aligned} \quad (1.10)$$

satisfying the following anticommutation relations

$$\{D_\alpha, \bar{D}_{\dot{\alpha}}\} = -2i\sigma_{\alpha\dot{\alpha}}^\mu \partial_\mu, \quad \{D_\alpha, D_\beta\} = \{\bar{D}_{\dot{\alpha}}, \bar{D}_{\dot{\beta}}\} = 0 \quad (1.11)$$

They also anticommute with Q and \bar{Q} , so indeed these derivatives transform covariantly under supersymmetry:

$$D_\alpha (\delta F) = \delta (D_\alpha F), \quad \bar{D}_{\dot{\alpha}} (\delta F) = \delta (\bar{D}_{\dot{\alpha}} F) \quad (1.12)$$

The first supersymmetry-preserving constraint one can apply is simply the covariant derivative

$$\bar{D}_{\dot{\alpha}}\Phi = 0 \quad (1.13)$$

Superfields obeying this condition are called *scalar superfields*, or *chiral superfields*. The most general solution to this equation is determined by three components, with

$$\begin{aligned} \Phi = & A(x) + \sqrt{2}\theta\psi(x) + \theta\theta F(x) \\ & + i\theta\sigma^{\mu}\bar{\theta}\partial_{\mu}A(x) - \frac{i}{\sqrt{2}}\theta\theta\partial_{\mu}\psi(x)\sigma^{\mu}\bar{\theta} + \frac{1}{4}\theta\theta\bar{\theta}\bar{\theta}\partial^2 A(x) \end{aligned} \quad (1.14)$$

The components (A, ψ, F) form a *scalar* or *chiral multiplet*. The fields in the Wess-Zumino model mentioned in Section 1.1.1, consist precisely of a chiral multiplet, and actually their supersymmetry transformations are exactly the components of (1.9). Products and linear combinations of chiral superfields are again chiral superfields.

The second allowed constraint is a reality constraint

$$V = V^{\dagger} \quad (1.15)$$

The corresponding superfields are called *vector superfields* (because of the presence of a vector component) or *real superfields*. We write their expansion as

$$\begin{aligned} V(x, \theta, \bar{\theta}) = & C(x) + i\theta\chi(x) - i\bar{\theta}\bar{\chi}(x) \\ & + \frac{i}{2}\theta\theta[M(x) + iN(x)] - \frac{i}{2}\bar{\theta}\bar{\theta}[M(x) - iN(x)] \\ & - \theta\sigma^{\mu}\bar{\theta}v_{\mu}(x) + i\theta\theta\bar{\theta}\left[\bar{\lambda}(x) + \frac{i}{2}\bar{\sigma}^{\mu}\partial_{\mu}\chi(x)\right] \\ & - i\bar{\theta}\bar{\theta}\theta\left[\lambda(x) + \frac{i}{2}\sigma^{\mu}\partial_{\mu}\bar{\chi}(x)\right] + \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}\left[D(x) + \frac{1}{2}\partial^2 C(x)\right]. \end{aligned} \quad (1.16)$$

where the scalars are all real. Notice that the vector superfield is invariant under the *supergauge transformation* $V \rightarrow V + \Phi + \Phi^{\dagger}$ where Φ is a chiral superfield. There is a gauge choice of Φ called the *Wess-Zumino gauge*, which eliminates C, χ, M, N , resulting in

$$V_{\text{WZ}} = -\theta\sigma^{\mu}\bar{\theta}v_{\mu} + i\theta\theta\bar{\theta}\bar{\lambda} - i\bar{\theta}\bar{\theta}\theta\lambda + \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}D \quad (1.17)$$

This gauge breaks linear supersymmetry, but preserves the gauge invariance of the vector component $v_{\mu} \rightarrow v_{\mu} + \partial_{\mu}\Lambda$. One can view this as a partial supergauge fixing. One may also attach an extra spacetime index to the real superfield $V \rightarrow V_m$, so in components, instead of a vector (spinor) we will have a rank-2 tensor (vector-spinor). This superfield will be crucial to construct spin-2 and spin-3/2 theories in Chapter 2.

We have already seen in Φ and V necessary ingredients for any theory containing scalars, spinors, or vectors. In fact, the superfields allow to construct supersymmetric Lagrangians in a rather simple way. The approach is essentially looking for components of some superfield combinations, that transform under supersymmetry as a total derivative. This component is then a piece of the Lagrangian, because the supersymmetry transformation of the Lagrangian gives zero up to a total derivative, that we drop. To begin with, *the $\theta\theta\bar{\theta}\bar{\theta}$ component of a*

superfield always transforms as a total derivative, as can be inferred from the component expansion of (1.9). For chiral superfields, the kinetic terms and the auxiliary mass term arise from the $\theta\theta\bar{\theta}$ component of $\Phi^\dagger\Phi$

$$\begin{aligned}\mathcal{L}_K &= \Phi_j^\dagger\Phi_j\Big|_{\theta\theta\bar{\theta}} \\ &= -i\bar{\psi}_j\bar{\sigma}^\mu\partial_\mu\psi_j + \bar{A}_j\partial^2 A_j + \bar{F}_j F_j\end{aligned}\tag{1.18}$$

For $j = 1$, we recover the Wess-Zumino model in Section 1.1.1.

Moreover, note that the $\theta\theta$ component of Φ transforms as a total derivative, given by (1.5). This property also holds for Φ^n , $n > 1$. Then one proposes the following renormalizable Lagrangian describing masses and interactions along with linear terms:

$$\mathcal{L}_I = \left(\frac{1}{2}m_{ij}\Phi_i\Phi_j + \frac{1}{3}g_{ijk}\Phi_i\Phi_j\Phi_k + \lambda_i\Phi_i\right)\Big|_{\theta\theta} + \text{h.c.}\tag{1.19}$$

The complex scalars F_j are always auxiliary fields. Integrating them out, we obtain finally the general renormalizable supersymmetric Lagrangian of chiral multiplets:

$$\begin{aligned}\mathcal{L} &= -i\bar{\psi}_j\bar{\sigma}^\mu\partial_\mu\psi_j + \bar{A}_j\partial^2 A_j - \frac{1}{2}m_{jk}\psi_j\psi_k - \frac{1}{2}m_{jk}^*\bar{\psi}_j\bar{\psi}_k \\ &\quad - g_{ijk}\psi_i\psi_j A_k - g_{ijk}^*\bar{\psi}_i\bar{\psi}_j\bar{A}_k - V(A_i, \bar{A}_j)\end{aligned}\tag{1.20}$$

The last line contains Yukawa couplings and the scalar potential. For the expression of the scalar potential, it is convenient to introduce the *superpotential*, which is a holomorphic function corresponding to the piece in parentheses in (1.19):

$$W(x) = \frac{1}{2}m_{ij}x_i x_j + \frac{1}{3}g_{ijk}x_i x_j x_k + \lambda_i x_i\tag{1.21}$$

and the scalar potential takes the form

$$V(A_i, \bar{A}_j) = \sum_i \bar{F}_i F_i = \sum_i \left|\frac{\partial W(A)}{\partial A_i}\right|^2\tag{1.22}$$

Now let us turn to the Lagrangian of vector superfields, which is expected to describe gauge theories. Contrary to the previous case, the kinetic terms of the vector field v_m and the spinor λ cannot be extracted from the $\theta\theta\bar{\theta}$ component, and in fact they will not arise from any V^n , $n > 1$. A natural way to introduce spacetime derivative is to act the covariant derivatives D_α and $\bar{D}_{\dot{\alpha}}$ on vector superfields. More concretely, one defines

$$\mathcal{W}_\alpha = -\frac{1}{4}\bar{D}\bar{D}D_\alpha V, \quad \bar{\mathcal{W}}_{\dot{\alpha}} = -\frac{1}{4}DD\bar{D}_{\dot{\alpha}} V\tag{1.23}$$

They are chiral superfields (dressed with a spinor index), due the properties $D^3 = \bar{D}^3 = 0$, and are also *supergauge invariant* under $V \rightarrow V + \Phi + \Phi^\dagger$. As before, a supersymmetric Lagrangian can be obtained from the $\theta\theta$ component of products of \mathcal{W}_α , $\bar{\mathcal{W}}_{\dot{\alpha}}$:

$$\mathcal{L} = \frac{1}{4}\left(\mathcal{W}^\alpha\mathcal{W}_\alpha\Big|_{\theta\theta} + \bar{\mathcal{W}}_{\dot{\alpha}}\bar{\mathcal{W}}^{\dot{\alpha}}\Big|_{\theta\theta}\right)\tag{1.24}$$

To see what terms this Lagrangian gives rise to, it is more clear to fix the Wess-Zumino gauge, using the expression (1.17). We find

$$\mathcal{L} = \frac{1}{2}D^2 - \frac{1}{4}v^{mn}v_{mn} - i\lambda\sigma^\mu\partial_\mu\bar{\lambda} \quad (1.25)$$

where $v_{mn} = \partial_m v_n - \partial_n v_m$. Indeed, we recover the kinetic terms of v_m and λ . Besides, an additional $V^2|_{\theta\theta\bar{\theta}\bar{\theta}}$ complements (1.24) with a mass term of v_m and kinetic terms for χ, C . In total, the supersymmetric Lagrangian of a vector superfield

$$\mathcal{L} = \frac{1}{4} \left(\mathcal{W}^\alpha \mathcal{W}_\alpha|_{\theta\theta} + \bar{\mathcal{W}}_{\dot{\alpha}} \bar{\mathcal{W}}^{\dot{\alpha}}|_{\theta\theta} \right) + V^2|_{\theta\theta\bar{\theta}\bar{\theta}} \quad (1.26)$$

describes a vector multiplet: one vector field (v_m), two spin-1/2 fields (χ, λ), and one scalar field (C). Finally, note that the $\theta\theta\bar{\theta}\bar{\theta}$ component of V itself is both supersymmetry and supergauge invariant, thus one may add to (1.26) the *Fayet-Iliopoulos term*:

$$\mathcal{L}_{\text{FI}} = -2\kappa V|_{\theta\theta\bar{\theta}\bar{\theta}} = -\kappa D \quad (1.27)$$

where we dropped the total derivative. This term plays an important role in spontaneous supersymmetry breaking.

Before ending this subsection, we briefly discuss supersymmetric gauge theories, and elaborate the theory of interacting chiral superfields Φ_i that transform under the representation R of a gauge group G . Starting from the global symmetry, the chiral superfields transform according to

$$\Phi^i \rightarrow \left[e^{i\Lambda^a(T^a)} \right]^i_j \Phi^j \quad \text{or} \quad \delta\Phi^i = i\Lambda^a (T^a)^i_j \Phi^j \quad (1.28)$$

where T^a are the generators of the representation. The parameters Λ^a are real constants, hence chiral superfields obeying (1.13). The kinetic term of the Lagrangian $\Phi_j^\dagger \Phi_j|_{\theta\theta\bar{\theta}\bar{\theta}}$ is obviously invariant under the supergauge transformation, as for the interacting terms, $W(\Phi)|_{\theta\theta} + \text{h.c.}$, the supergauge invariance imposes constraints on the superpotential.

Going to local supergauge symmetries, scalar superfields will not suffice, and it is necessary to introduce a vector superfield. The reason is that the gauge parameters $\Lambda(x)$ are no longer constant chiral superfields, and the kinetic term loses supergauge invariance. The invariance can be restored by vector superfields V_a belonging to the adjoint representation of G

$$V = V^a T^a \quad (1.29)$$

transforming as

$$e^V \rightarrow e^{i\Lambda^\dagger} e^V e^{-i\Lambda} \quad (1.30)$$

To the first order, $\delta V = -i(\Lambda - \Lambda^\dagger)$ and we recognize the previously encountered transformation $V \rightarrow V + \Phi + \Phi^\dagger$ with $\Phi = -i\Lambda$. Higher orders can be evaluated by Hausdorff's formula. The kinetic term is modified so that it stays invariant

$$\mathcal{L}_K = \Phi^\dagger e^V \Phi|_{\theta\theta\bar{\theta}\bar{\theta}} \quad (1.31)$$

The form of the superpotential $W(\Phi)$ is always subject to the requirement of supergauge invariance. Turning our attention to the vector superfield, we know that its Lagrangian is

determined by the chiral superfields $\mathcal{W}_\alpha, \bar{\mathcal{W}}_{\dot{\alpha}}$ defined in (1.23). Now, these superfields are not supergauge invariant anymore, neither are the subsequent kinetic terms, and one shall use a new definition

$$\mathcal{W}_\alpha = -\frac{1}{4}\bar{D}\bar{D}e^{-V}D_\alpha e^V, \quad \bar{\mathcal{W}}_{\dot{\alpha}} = -\frac{1}{4}DDe^{-V}\bar{D}_{\dot{\alpha}}e^V \quad (1.32)$$

with transformations

$$\mathcal{W}_\alpha \rightarrow e^{i\Lambda}\mathcal{W}_\alpha e^{-i\Lambda}, \quad \bar{\mathcal{W}}_{\dot{\alpha}} \rightarrow e^{i\Lambda}\bar{\mathcal{W}}_{\dot{\alpha}} e^{-i\Lambda} \quad (1.33)$$

Analogous to the ordinary Yang-Mills theory, the kinetic term will be a trace of $\mathcal{W}^\alpha\mathcal{W}_\alpha$. Finally, the full supersymmetric Lagrangian is

$$\mathcal{L} = \frac{1}{4k}\text{Tr}\left(\mathcal{W}^\alpha\mathcal{W}_\alpha|_{\theta\theta} + \bar{\mathcal{W}}_{\dot{\alpha}}\bar{\mathcal{W}}^{\dot{\alpha}}|_{\bar{\theta}\bar{\theta}}\right) + \Phi^\dagger e^V\Phi|_{\theta\theta\bar{\theta}\bar{\theta}} + [W(\Phi)|_{\theta\theta} + \text{h.c.}] \quad (1.34)$$

where k is a normalization factor to be determined later. The superpotential shall satisfy supergauge invariance.

In the presence of abelian $U(1)$ factors in the gauge group, the transformation law of the corresponding vector superfields is $\delta V_A = -i(\Lambda - \Lambda^\dagger)$, thus the $\theta\theta\bar{\theta}\bar{\theta}$ component of V_A^a is supergauge invariant. We also know that it is supersymmetry invariant as it transforms as a total derivative. In this case, we can add to (1.34) the Fayet-Iliopoulos terms:

$$\mathcal{L}_{\text{FI}} = \sum_a -2\kappa^a V_A^a|_{\theta\theta\bar{\theta}\bar{\theta}} = \sum_a -\kappa^a D_A^a \quad (1.35)$$

One may use again the Wess-Zumino gauge to expand (1.34) for a simple and explicit form of the component Lagrangian. One expects to recover a Lagrangian describing chiral multiplets (A_i, ψ_i, F_i) charged under the gauge group, along with vector multiplets (v_m^a, λ^a, D^a) where v_m^a are the gauge bosons and λ^a their superpartners, the *gauginos*. The integration of the auxiliary F_i and D^a yields for the scalar potential, F -terms and D -terms respectively. To summarize, ignoring for the moment \mathcal{L}_{FI} , the scalar potential is:

$$\begin{aligned} V(A^i, \bar{A}_j) &= \sum_i \left| \frac{dW}{dA^i} \right|^2 + \frac{1}{2} \sum_a g^2 (\bar{A}_i T_j^{ai} A^j)^2 \\ &= \sum_i |F^i|^2 + \frac{1}{2} \sum_a (D^a)^2 \end{aligned} \quad (1.36)$$

where g is the gauge coupling. Clearly, the scalar potential is positive or zero. According to (1.20), the Yukawa Lagrangian can be expressed in terms of the superpotential as

$$\mathcal{L}_{\text{Yuk}} = -\frac{1}{2} \frac{d^2 W(A^k)}{dA^i dA^j} (\psi^i \psi^j) + \text{h.c.} \quad (1.37)$$

We also have gauge invariant interactions between the chiral fermion ψ_i , the scalars A_i and the gauginos λ^a :

$$\mathcal{L}_{\text{int}} = \sqrt{2}ig (\bar{\psi}_i \bar{\lambda}^a) T^{ai}{}_j A^j + \text{h.c.} \quad (1.38)$$

1.1.3 Supersymmetry breaking

For applications in real world, supersymmetry must be broken. As is evidenced by (1.36), the scalar potential has a global minimum $V = 0$ when

$$\langle F^i \rangle = \langle D^a \rangle = 0 \quad \Leftrightarrow \quad \left\langle \frac{dW}{dA_i} \right\rangle = \langle D^a \rangle = 0 \quad (1.39)$$

has a solution. Any other vacuum with $\langle F^i \rangle \neq 0$ or $\langle D^a \rangle \neq 0$ breaks supersymmetry. The vacuum expectation values $\langle A_i \rangle$ satisfy the minimum conditions

$$\frac{dV}{dA^i} = F^j \frac{d^2 W}{dA^i dA^j} - g D^a \left(\bar{A}_j (T^a)^j{}_i \right) = 0 \quad (1.40)$$

In addition, the superpotential must be gauge invariant, namely,

$$\bar{A}_j (T^a)^j{}_i F^i = i \frac{d\bar{W}(\bar{A}_i)}{d\bar{A}_i} \delta_a \bar{A}_i = 0 \quad (1.41)$$

(1.40) together with (1.41) can be put in a matrix form:

$$\begin{pmatrix} F^i & D^a \end{pmatrix} \begin{pmatrix} \frac{d^2 W}{dA^i dA^j} & -g \bar{A}_j (T^a)^j{}_i \\ -g \bar{A}_j (T^a)^i{}_j & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \end{pmatrix} \quad (1.42)$$

Interestingly, the 2×2 matrix above coincides with the mass matrix of the fermions (ψ, λ) , from the Lagrangians (1.37)-(1.38):

$$\mathcal{L}_{\text{Yuk}} + \mathcal{L}_{\text{int}} = -\frac{1}{2} \begin{pmatrix} \psi^i & \sqrt{2}i\lambda^b \end{pmatrix} \begin{pmatrix} \frac{d^2 W}{dA^i dA^j} & -g \bar{A}_j (T^a)^j{}_i \\ -g \bar{A}_j (T^a)^i{}_j & 0 \end{pmatrix} \begin{pmatrix} \psi^j \\ \sqrt{2}i\lambda^a \end{pmatrix} \quad (1.43)$$

Hence if a vacuum breaks supersymmetry, with $\langle F^i \rangle \neq 0$ or $\langle D^a \rangle \neq 0$, then the 2×2 matrix has a zero eigenvalue. In other words, there exists a massless fermion:

$$v = \langle F_i \rangle \psi^i - \frac{i}{\sqrt{2}} g \langle D^a \rangle \lambda^a \quad (1.44)$$

This is the supersymmetric version of the Goldstone theorem: spontaneous supersymmetry breaking gives rise to a massless *Goldstone fermion* v , also called a *Goldstino*. When every vacuum state possesses a positive energy, it indicates that the supersymmetric gauge theory has undergone spontaneous symmetry breaking. Such scenario can be achieved by an appropriate choice of the superpotential W or the D -term D^a , more concretely, there exist two realizations, constructed respectively by O’Raifeartaigh [36] and Fayet, Iliopoulos [37, 38]. We review below the two mechanisms of spontaneous supersymmetry breaking.

The O’Raifeartaigh mechanism or the F -term breaking consists in specific choices of a superpotential W and three chiral multiplets Φ_i , $i = 0, 1, 2$, such that there is no solution to $\frac{dW}{dA^i} = 0$. One of the possibilities is to pick the superpotential

$$W = \lambda \Phi_0 + m \Phi_1 \Phi_2 + g \Phi_0 \Phi_1^2 \quad (1.45)$$

The linear term implies that Φ_0 must be a gauge singlet. In fact, this linear term is necessary for F -term breaking at tree-level in a cubic polynomial superpotential. The scalar potential is

$$V(A_0, A_1, A_2) = \left| \lambda + gA_1^2 \right|^2 + |mA_2 + 2gA_0A_1|^2 + |mA_1|^2 + \frac{1}{2} \sum_a (D^a)^2 \quad (1.46)$$

Assume that $D^a = 0$ has a solution. We also suppose $m^2 > 2\lambda g$ and all parameters are real, the minimum of (1.46) then corresponds to $A_1 = A_2 = 0$, independent of A_0 , and the vacuum energy at this point is $V = \lambda^2 > 0$. Thus supersymmetry is spontaneously broken.

The Fayet-Iliopoulos mechanism or the D -term breaking may occur when the gauge group includes at least one abelian $U(1)$ factor. As is mentioned in the previous subsection, in this case the $\theta\theta\bar{\theta}\bar{\theta}$ component of a vector superfield is both supersymmetry and supergauge invariant. The additional Fayet-Iliopoulos terms (1.35) would force the D -term to acquire a non-zero vacuum expectation value. To see this, let us consider a simple model with $U(1)$ -charged scalar fields. The superpotential is $W = \frac{1}{2}m_i\Phi_i^2$ and the charge of A_i is given by q_i . This configuration results in the following scalar potential

$$V = \sum_i |m_i|^2 |A_i|^2 + \frac{1}{2} \left(\kappa - g \sum_i q_i |A_i|^2 \right)^2 \quad (1.47)$$

This potential never vanishes for $\kappa \neq 0$, therefore supersymmetry is spontaneously broken.

In realistic models that incorporate the SM, one oftentimes introduces *soft terms* that explicitly break supersymmetry while preserving some of its desirable properties. These terms do not add quadratic divergences to the Higgs mass, and allow to obtain superparticles that are heavier than their SM partners. The values of the soft-breaking parameters are *a priori* not predicted by any fundamental theories, and are subject to theoretical as well as experimental constraints. They can be viewed as a low energy effect due to an unknown spontaneous breaking at a higher scale.

1.1.4 R-symmetries

We briefly mention in this last subsection the *R-symmetries*, that are relevant for the introduction of Dirac gaugino models in Chapter 5. In $\mathcal{N} = 1$ supersymmetry that we are focusing on, the R-symmetry is associated with a global $U(1)_R$ transformation rotating the Grassmann coordinates:

$$\theta \rightarrow e^{i\alpha}\theta, \quad \bar{\theta} \rightarrow e^{-i\alpha}\bar{\theta} \quad (1.48)$$

where $\theta, \bar{\theta}$ are said to carry R-charges $+1, -1$. For a theory invariant under R-symmetry, the superfield carrying r_S R-charge will transform as

$$S(x, \theta, \bar{\theta}) \rightarrow e^{ir_S\alpha} S(x, e^{-i\alpha}\theta, e^{i\alpha}\bar{\theta}) \quad (1.49)$$

R-symmetry does not commute with the supersymmetry generator and therefore different components in the superfield can have different R-charges. For a chiral superfield in the form (1.14) with R-charge r_Φ , the R-charges of the components A (complex scalar), ψ (chiral fermion), F (auxiliary scalar) are then $r_\Phi, r_\Phi - 1, r_\Phi - 2$. The vector superfield, on the other hand, carries no R-charge due to its reality. In the Wess-Zumino gauge (1.17), we infer the

R-charges of v_μ (vector boson), λ (gaugino), D (auxiliary scalar), that are 0, +1, 0. If a theory is to be R-symmetric, then necessarily *the R-charge of the superpotential is +2*, as can be observed from (1.19). This requirement is not automatically satisfied: for a general renormalizable superpotential

$$W(\Phi) = \frac{1}{2}m_{ij}\Phi_i\Phi_j + \frac{1}{3}g_{ijk}\Phi_i\Phi_j\Phi_k + \lambda_i\Phi_i, \quad (1.50)$$

only one of three terms can stay, if the total R-charge of W is equal to +2. R-symmetry is not necessary for a Lagrangian to be consistent, and can be violated in many situations. However, it is particularly significant for supersymmetry breaking. In fact, the Nelson-Seiberg theorem [39] states that in the case of spontaneous supersymmetry breaking by a non-zero F -term, with a generic superpotential, then the theory must have an exact R-symmetry that remains unbroken. In the simple example of the O’Raifeartaigh mechanism, Eq. (1.45), we find the charge assignments:

$$r_{\Phi_0} = +2, \quad r_{\Phi_1} = 0, \quad r_{\Phi_2} = +2 \quad (1.51)$$

for the superpotential to possess an R-charge +2. Besides, R-symmetry cannot be spontaneously broken at electroweak scale because otherwise it would lead to a massless “R-axion” that is experimentally problematic. Consequently, R-symmetry is either conserved or explicitly broken.

1.2 Supergravity

1.2.1 Local supersymmetry

The previous section covers global supersymmetry where the transformation parameter ϵ is constant. While global supersymmetry is a mathematically consistent theory, it is incompatible with gravity because the latter requires diffeomorphism invariance, which will be broken by a constant spinorial parameter ϵ . From another perspective, if supersymmetry is to be promoted to a local symmetry, then it must couple to gravity. This fact can be observed through the example of the massless free Wess-Zumino model:

$$\mathcal{L} = -\partial^\mu\bar{\phi}\partial_\mu\phi - i\bar{\psi}\bar{\sigma}^\mu\partial_\mu\psi \quad (1.52)$$

We have seen in the previous section that this Lagrangian is invariant under global supersymmetry transformations (1.3), but when the parameter ϵ becomes spacetime-dependent, $\epsilon \rightarrow \epsilon(x)$, the transformations leave residual terms

$$\delta_\epsilon\mathcal{L} = (\partial_\mu\bar{\epsilon})j_L^\mu + (\partial_\mu\epsilon)j_R^\mu, \quad j_L^\mu = \sqrt{2}\partial_\nu\phi\bar{\sigma}^\nu\sigma^\mu\bar{\psi}, \quad j_R^\mu = \sqrt{2}\partial_\nu\bar{\phi}\sigma^\nu\bar{\sigma}^\mu\psi \quad (1.53)$$

The super-Noether current $j^\mu = j_L^\mu + j_R^\mu$ is a conserved current, namely $\partial^\mu j_\mu = 0$ if one uses the equations of motion. In order to cancel the residual terms in $\delta_\epsilon\mathcal{L}$, we shall introduce a gauge field that, same as j^μ , behaves as a vector-spinor. Denoting this gauge field as χ_μ , we have

$$\delta_\epsilon\chi_\mu = M_P\partial_\mu\epsilon \quad (1.54)$$

where the factor M_P is added to ensure the correct mass dimension. The Noether’s method then indicates the addition of a new piece to the Lagrangian

$$\mathcal{L}' = -\frac{1}{M_P}(\bar{\chi}_\mu j_L^\mu + \chi_\mu j_R^\mu) \quad (1.55)$$

The variation of \mathcal{L}' cancels $\delta_\epsilon \mathcal{L}$, but gives rise to the variation of the currents j_L^μ, j_R^μ that is in general non-vanishing. By calculating the terms bilinear in ϕ and ψ one recovers precisely their energy momentum tensors contracted with χ_μ , namely, these terms take the form

$$\begin{aligned} \delta_\epsilon (\mathcal{L} + \mathcal{L}') &= -\frac{1}{M_P} (\bar{\chi}_\mu \delta_\epsilon j_L^\mu + \chi_\mu \delta_\epsilon j_R^\mu) \\ &\sim \frac{1}{M_P} \bar{\epsilon} \gamma_\mu \chi_\nu T^{\mu\nu} + \dots \end{aligned} \quad (1.56)$$

which suggests adding a new symmetric rank-2 field $g_{\mu\nu}$ that transforms as

$$\delta g_{\mu\nu} \sim \frac{1}{M_P} \bar{\epsilon} \bar{\sigma}_{(\mu} \chi_{\nu)} \quad (1.57)$$

and modifying the Lagrangian with an extra piece so as to cancel (1.56)

$$\mathcal{L}'' \sim -g_{\mu\nu} T^{\mu\nu} \quad (1.58)$$

The only candidate that can couple to the energy momentum tensor is the dynamical spacetime metric. Therefore, we have an overview of the necessity of gravity, *i.e.* the introduction of the spin-2 graviton and its spin-3/2 superpartner gravitino², sitting in a *supergravity multiplet*, when promoting supersymmetry to a local symmetry. This is actually not limited to the Wess-Zumino model discussed here, and holds in general cases.

1.2.2 Minimal supergravity

We now come to the minimal $\mathcal{N} = 1$ supersymmetric extension of General Relativity in four dimensions, which involves one supergravity multiplet in absence of matter multiplet. This form allows for various extensions, such as in different dimensions or with extended supersymmetry or coupling to matter multiplets. The minimal content is the curvature tensor, and a gravitino ψ_μ which is a necessary ingredient to render supersymmetry local, so the basic approach is to start from the Einstein-Hilbert action along with a curved-space Rarita-Schwinger action describing a massless gravitino:

$$S = \int d^4x (\mathcal{L}_{EH} + \mathcal{L}_{RS}) \int d^4x \sqrt{-g} \left(\frac{M_P^2}{2} R - \frac{1}{2} \bar{\psi}_\mu \gamma^{\mu\nu\rho} D_\nu \psi_\rho \right) \quad (1.59)$$

From now on, we adopt the four-component notation for the fermions. The Lorentz covariant derivative of a fermion field is given by $D_\mu \psi = \partial_\mu \psi + \frac{1}{4} \omega_\mu^{ab} \bar{\gamma}_{ab} \psi$ and the curved-space γ matrices are related to the flat-space ones ($\bar{\gamma}$'s) through $\gamma_\mu \equiv e^a{}_\mu \bar{\gamma}_a$ where $e^a{}_\mu$ denotes the frame field (*vierbein*) describing the graviton. The spin connection is introduced through the action of the full covariant derivative on the vierbein: $\nabla_\mu e_a = \omega_\mu{}^b{}_a e_b$, with $e_a \equiv e_a^\mu \partial_\mu$. Imposing furthermore that the spin connection preserve the metric tensor and be torsion-free, ω_μ^{ab} can be expressed in terms of $e^a{}_\mu$. We will encounter again these definitions in Section 3.2. Strictly speaking, as we are coupling the fermion to gravity, the full covariant derivative ∇ should also contain the Christoffel symbol Γ , but in the above action it appears only in an antisymmetrized form $\Gamma_{[\nu\rho]}^\sigma = 0$, and therefore does not contribute:

$$\nabla_{[\nu} \psi_{\rho]} = \partial_{[\nu} \psi_{\rho]} + \frac{1}{4} \omega_{[\nu}^{ab} \gamma_{ab} \psi_{\rho]} - \Gamma_{[\nu\rho]}^\sigma \psi_\sigma = D_{[\nu} \psi_{\rho]} \quad (1.60)$$

²We will give more technical details about the gravitino in Chapters 2 and 3.

The next step is to supersymmetrize (1.59), keeping the assumption that the gravitino is the gauge field of supersymmetry, as is suggested by the example of the Wess-Zumino model. A reasonable guess for the gauge transformation is to covariantize (1.54):

$$\delta_\epsilon \psi_\mu = M_P D_\mu \epsilon = M_P \left(\partial_\mu + \frac{1}{4} \omega_\mu^{ab} \bar{\gamma}_{ab} \right) \epsilon \quad (1.61)$$

In accordance with the four-component notation, ϵ and ψ_μ are now Majorana spinors. As for the gauge transformation of the vierbein, one may either draw inspiration from (1.57), or derive it from the transformation law of the gravitino, by cancelling the terms in $\delta_\epsilon S$ generated by $\delta_\epsilon \psi_\mu$. We give directly the result

$$\delta_\epsilon e_\mu^a = \frac{1}{2M_P} \bar{\epsilon} \gamma^a \psi_\mu \quad (1.62)$$

At this point, the action is not yet invariant, because the variation of the spin connection $\delta_\epsilon \omega_\mu^{ab}$ also contributes. Demanding that the residual terms in $\delta_\epsilon S$ vanish results in multiple possibilities, and $\delta_\epsilon \omega_\mu^{ab}$ depends on the formalism one chooses. Two different solutions have been derived by Deser, Zumino [40]:

$$\delta \omega_\mu^{bc} = B_\mu^{bc} - \frac{1}{2} e_\mu^c B_e^{be} + \frac{1}{2} e_\mu^b B_e^{ce}, \quad (1.63)$$

with

$$B_\mu^{bc} = \frac{i}{2M_P} \bar{\epsilon} \gamma_\mu \gamma_5 D_\rho \psi_\sigma \epsilon^{\rho\sigma bc}, \quad (1.64)$$

and by Ferrara, Freedman, van Nieuwenhuizen [41]:

$$\delta \hat{\omega}_\mu^{ab} = \frac{1}{M_P} \bar{\epsilon} \gamma^\rho (D^\sigma \psi^\tau) \left(2e_\rho^{[a} e_\tau^{b]} g_{\mu\sigma} - e_\tau^{[a} e_\sigma^{b]} g_{\mu\rho} \right), \quad (1.65)$$

where $\omega_\mu^{ab} = \hat{\omega}_\mu^{ab}(e, \psi)$ is the solution to $De^a = \bar{\psi} \wedge \bar{\gamma}^a \psi / (4M_P^2)$.

We have had a glimpse on the minimal $\mathcal{N} = 1$, $D = 4$ supergravity with only one supergravity multiplet (e_μ^a, ψ_μ) , constructed around a Minkowski background $\eta^{ab} \sim (-1, 1, 1, 1)$. In GR, one can add to the Einstein-Hilbert action a *cosmological constant* term

$$S = \int d^4x \sqrt{-g} \left(\frac{M_P^2}{2} R - M_P^2 \Lambda \right) \quad (1.66)$$

and according to the sign of Λ , one defines the *de Sitter* (dS) vacua ($\Lambda > 0$), as well as the *anti-de Sitter* (AdS) vacua ($\Lambda < 0$). An important question is then whether the above action can be supersymmetrized as before, which is equivalent to finding a supergroup that incorporates the symmetry group of dS or AdS spacetime. In the pure supergravity theory that we are discussing, it is established that only the AdS group can be consistently extended to a supergroup, and a *positive* cosmological constant always breaks supersymmetry. The vierbein still transforms as Eq. (1.62), whereas the transformation rule of the gravitino is shifted by a term proportional to γ :

$$\delta \psi_\mu = M_P D_\mu \epsilon - \frac{g}{2} M_P^2 \gamma_\mu \epsilon \quad (1.67)$$

The supersymmetry invariant action of a supergravity multiplet in AdS is written as

$$S_{\text{AdS}} = \int d^4x \sqrt{-g} \left(\frac{M_P^2}{2} R - M_P^2 \Lambda - \frac{1}{2} \bar{\psi}_{\mu R} \gamma^{\mu\nu\rho} D_\nu \psi_{\rho L} - \frac{1}{2} \bar{\psi}_{\mu L} \gamma^{\mu\nu\rho} D_\nu \psi_{\rho R} \right. \\ \left. - M_P \frac{g}{2} \bar{\psi}_{\mu R} \gamma^{\mu\nu} \psi_{\nu R} - M_P \frac{g^*}{2} \bar{\psi}_{\mu L} \gamma^{\mu\nu} \psi_{\nu L} \right) \quad (1.68)$$

where $\Lambda = -3M_P^2|g|^2 < 0$ and recall that g is the determinant of the metric $g_{\mu\nu}$. The factor $\sqrt{-g}$ in the action can also be replaced by the determinant of the vierbein $e = \sqrt{-g}$. Remarkably, in the supergravity multiplet, we now have a massless graviton and a “massive” gravitino while preserving supersymmetry. In fact, the statement in the previous section that all particles in the same multiplet should have degenerate mass, does not always hold in a curved spacetime, where extra mass-like terms may appear. In reality, the concrete definition of “mass” in AdS is subtle and needs much more careful treatments, which is not of our focus here.

1.2.3 Matter couplings and scalar potential

In order to describe matter and Yang-Mills interactions, we must couple the supergravity multiplet to chiral multiplets or vector multiplets. While a primitive sketch of chiral multiplet coupling was drawn in Section 1.2.1, with the simplest Wess-Zumino model, the discussion shall be extended.

To this end, let us momentarily return to global supersymmetry, and add some material that has not been covered in Section 1.1. Consider a Wess-Zumino model with n_C chiral multiplets (ϕ^m, χ^m) , where χ^m are Majorana fermions and $m = 1, \dots, n_C$. In fact, just as in non-linear σ -models, the kinetic terms of the complex scalars can be non-minimal. In general cases, they are derived from a real function, called the *Kähler potential* $\mathcal{K}(\phi, \bar{\phi})$:

$$\mathcal{L} \supset -g_{m\bar{n}} \left[\partial_\mu \phi^m \partial^\mu \bar{\phi}^{\bar{n}} + \bar{\chi}_L^m \not{D} \chi_R^{\bar{n}} + \bar{\chi}_R^{\bar{n}} \not{D} \chi_L^m \right], \quad \text{with} \quad g_{m\bar{n}} = \partial_m \partial_{\bar{n}} \mathcal{K}(\phi, \bar{\phi}) \quad (1.69)$$

where the barred indices represent the complex conjugate of ϕ , or the right-handed part of the fermion, namely:

$$\phi^{\bar{n}} \equiv \bar{\phi}^n, \quad \partial_m \equiv \frac{\partial}{\partial \phi^m}, \quad \partial_{\bar{n}} \equiv \frac{\partial}{\partial \bar{\phi}^{\bar{n}}} \\ \chi_R^{\bar{m}} \equiv P_R \chi^m, \quad \chi_L^m \equiv P_L \chi^m \quad (1.70)$$

Note that the $\{mn\}$, $\{\bar{m}\bar{n}\}$ components of the Kähler metric $g_{m\bar{n}}$ are vanishing. The covariant derivative \mathcal{D} acting on the scalars trivially reduces to a partial derivative, while its action on the fermion is given by

$$\mathcal{D}_\mu \chi_R^{\bar{n}} \equiv \partial_\mu \chi_R^{\bar{n}} + \left(\partial_\mu \phi^{\bar{m}} \right) \Gamma_{\bar{m}\bar{l}}^{\bar{n}} \bar{\chi}_R^{\bar{l}}, \quad \mathcal{D}_\mu \chi_L^n \equiv \partial_\mu \chi_L^n + \left(\partial_\mu \phi^m \right) \Gamma_{ml}^n \chi_L^l \quad (1.71)$$

with Γ the Christoffel symbol related to $g_{m\bar{n}}$. It is useful to note that the Kähler potential is invariant upon adding the real part of a holomorphic function:

$$\mathcal{K}(\phi^n, \bar{\phi}^{\bar{n}}) \rightarrow \mathcal{K}(\phi^n, \bar{\phi}^{\bar{n}}) + h(\phi^n) + h^*(\bar{\phi}^{\bar{n}}) \quad (1.72)$$

The scalar potential, the Yukawa Lagrangian, and the supersymmetry transformations that we presented in Section 1.1.2 are deformed by $g_{m\bar{n}}$ correspondingly.

Next, we couple n_V vector multiplets (A_μ^I, λ^I) , $I = 1, \dots, n_V$, to the chiral multiplets above. The non-minimal kinetic terms of the vector multiplets are encoded in a set of holomorphic functions $f_{IJ}(\phi^m)$, called the *gauge kinetic function*:

$$\mathcal{L} \supset \text{Re}(f_{IJ}(\phi^m)) \left[-\frac{1}{4} F_{\mu\nu}^I F^{\mu\nu J} - \frac{1}{2} \bar{\lambda}^I \hat{\not{\partial}} \lambda^J \right] \quad (1.73)$$

where $F_{\mu\nu}^I$ are the field strengths of the gauge bosons, and $\hat{\not{\partial}}$ represents some suitable gauge covariant derivative. To guarantee the correct sign of the kinetic terms, $g_{m\bar{n}}$ and $\text{Re}(f_{IJ})$ must be positive definite.

Both in global and local supersymmetry, the most general action of n_C chiral multiplets coupled to n_V vector multiplets is determined by the following input data:

1. The Kähler potential \mathcal{K} and the gauge kinetic function f_{IJ} , related to the kinetic terms of the chiral and vector multiplets.
2. The holomorphic superpotential W . It contributes the scalar potential, the Yukawa Lagrangian and the fermion mass terms.
3. The gauge group and the representation under which the chiral multiplet transforms.
4. The Fayet-Iliopoulos terms in the presence of Abelian gauge groups.

The full supergravity Lagrangian for arbitrary number of chiral and vector multiplets is rather tedious and can be found, *e.g.*, in [30], Section 18. We will instead focus on the scalar potential:

$$V = e^{\frac{\mathcal{K}}{M_P^2}} g^{m\bar{n}} \mathcal{D}_m W \mathcal{D}_{\bar{n}} \bar{W} - 3e^{\frac{\mathcal{K}}{M_P^2}} \frac{|W|^2}{M_P^2} + \frac{1}{2} \text{Re} f_{IJ} D^I D^J \quad (1.74)$$

where the Kähler-covariant derivative acting on the superpotential is defined as

$$\mathcal{D}_m W \equiv \left[\partial_m + \frac{(\partial_m \mathcal{K})}{M_P^2} \right] W \quad (1.75)$$

For $\langle W \rangle \neq 0$, (1.74) can be put in a simpler form by introducing a function \mathcal{G} that is left invariant under the transformation (1.72):

$$\begin{aligned} \mathcal{G} &\equiv \mathcal{K} + M_P^2 \log(|W|^2/M_P^6), \\ \text{with } \mathcal{G}_m &\equiv \partial_m \mathcal{G} = M_P^2 \frac{\mathcal{D}_m W}{W}, \quad \mathcal{G}_{m\bar{n}} \equiv \partial_m \partial_{\bar{n}} \mathcal{G} = g_{m\bar{n}} \end{aligned} \quad (1.76)$$

so that the scalar potential becomes

$$V = e^{\frac{\mathcal{G}}{M_P^2}} \left(M_P^2 \mathcal{G}^{m\bar{n}} \mathcal{G}_m \mathcal{G}_{\bar{n}} - 3M_P^4 \right) + \frac{1}{2} \text{Re} f_{IJ} D^I D^J \quad (1.77)$$

The first and third terms correspond to the F - and D - terms that are also present in global supersymmetry. The D -term takes the same form in global and local supersymmetry, but one of the differences is that, in global supersymmetry, the D -term is independent of the

superpotential W . In the local case, these two are related, and the D -term can be expressed in terms of the Kähler invariant function \mathcal{G} (for $\langle W \rangle \neq 0$):

$$D^I = i (\text{Re}f)^{-1I\bar{J}} \xi(\phi^n)_J^m \mathcal{G}_m \quad (1.78)$$

where $\xi(\phi^n)_J^m$ is a holomorphic Killing vector relevant to the Kähler manifold.

The second term in (1.77)³ is new in local supersymmetry, which endows supergravity with a crucial property – the scalar potential can also be negative. The earlier statement in global supersymmetry that supersymmetry-preserving vacua always have zero energy, needs to be rectified. The order parameter of the supersymmetry breaking is vev of fermionic transformations, in other words, supersymmetry is preserved if and only if $\langle \delta\psi_\mu \rangle = \langle \delta\chi^m \rangle = \langle \delta\lambda^I \rangle = 0$, with $m = 1, \dots, n_C$ and $I = 1, \dots, n_V$. In supergravity, this condition translates into:

1. $\langle \mathcal{D}_m W \rangle = 0$
2. $\langle D^I \rangle = 0$

They are equivalent to (1.39) in global supersymmetry. In supergravity, however, these two conditions are not independent: when $\langle W \rangle \neq 0$, according to (1.78), the first condition implies the second one. The supersymmetry vacua satisfy

$$\langle V \rangle = -3 \left\langle e^{\frac{\kappa}{M_P^2} |W|^2} \frac{|W|^2}{M_P^2} \right\rangle \leq 0 \quad (1.79)$$

We see another remarkable difference with regard to global supersymmetry. In supergravity, the supersymmetric vacua are always *negative semi-definite*, and conversely, vanishing or negative vacua does not necessarily mean that supersymmetry is unbroken. A supersymmetric vacuum has zero energy if and only if $\langle W \rangle = \langle D^I \rangle = 0$. Besides, note that a direct consequence of (1.78) is that, pure D -term breaking occurs only when $\langle W \rangle = 0$.

Finally, the negative contribution to the scalar potential is also significant in cosmological applications, because a positive vev of the scalar potential corresponds to a positive cosmological constant, and most models encounter the problem of a cosmological constant that is way too large. In supergravity, this negative term comes to compensate the positive part, and eventually may result in a sufficiently small but positive cosmological constant, compatible with realistic scenarios.

1.2.4 Super-Higgs mechanism

In Section 1.1.3 we reviewed the basic aspects of supersymmetry breaking, which features the appearance of a massless Goldstone fermion – the Goldstino. If global supersymmetry were to hold in nature at some UV scale, it is puzzling that the Goldstino signaling the spontaneous breaking has not been observed yet. Supergravity provides an elegant answer to this question, because supersymmetry is now a gauge symmetry. In gauge theories that we are familiar with, the gauge bosons acquire masses (*viz.* the helicity 0 mode) by absorbing the

³Conventionally, the “ F -term” in supergravity is the sum of the first and the second terms.

Goldstone boson, through the Higgs mechanism. Analogously in supergravity, the gauge field gravitino will also become massive (*viz.* obtain helicities $\pm 1/2$) by eating up the Goldstino. This is the *super-Higgs mechanism* [42–44], that will be discussed next.

In global supersymmetry, we have derived the Goldstino expression (1.44) in terms of the chiral fermions and gauginos in the theory. In supergravity, one can find a generalization of this expression

$$v_L = M_P \frac{\mathcal{D}_m W}{W} \chi_L^m + \frac{e^{-\mathcal{K}/(2M_P^2)} M_P}{W} \frac{i}{2} \lambda_L^I \mathcal{P}_I \quad (1.80)$$

where $\mathcal{P}_I \equiv \text{Re}(f_{IJ}) D^J$. The Goldstino-gravitino mixing originates from the gravitino couplings to all other fermions in the supergravity Lagrangian:

$$e^{-1} \mathcal{L}_{\text{mix}} = \frac{1}{M_P} \bar{\psi}_{\mu R} \gamma^\mu \left(\frac{i}{2} \lambda_L^I \mathcal{P}_I + \chi_L^m e^{\mathcal{K}/(2M_P^2)} \mathcal{D}_m W \right) + \text{h.c.} \quad (1.81)$$

The Goldstino transformation demonstrates a shift symmetry $\delta v_L \propto \epsilon_L + \dots$, much like the Goldstone boson in conventional gauge theories. As a result, a field redefinition can be worked out to achieve the *unitary gauge* $v = 0$, which eliminates the Goldstino. This field redefinition will result in new terms in the Lagrangian, including the gravitino mass term, which takes on the standard Rarita-Schwinger form. In Minkowski spacetime, this term corresponds to the physical mass of the gravitino, and this is how the gravitino acquires mass in the super-Higgs mechanism. The detailed calculation for the general case can be found in [45]. Here, we will show a simplified scenario with chiral multiplets only, and constant scalars (namely, their derivative vanishes). A further simplification is to consider Minkowski vacua, where the definition of the “mass” is unambiguous. Starting from the scalar potential:

$$V = e^{\frac{\mathcal{G}}{M_P^2}} \left(M_P^2 g^{m\bar{n}} \mathcal{G}_m \mathcal{G}_{\bar{n}} - 3M_P^4 \right) \quad (1.82)$$

A Minkowski vacuum $V = 0$ clearly breaks supersymmetry. It implies:

$$g^{m\bar{n}} \mathcal{G}_m \mathcal{G}_{\bar{n}} - 3M_P^2 = 0 \quad (1.83)$$

and the minimum condition is then:

$$\partial_m V|_{V=0} = e^{\frac{\mathcal{G}}{M_P^2}} M_P^2 [\mathcal{G}^n \mathcal{D}_m \mathcal{G}_n + \mathcal{G}_m] = 0 \quad \Rightarrow \quad \mathcal{G}^n (\mathcal{D}_m \mathcal{G}_n) = -\mathcal{G}_m \quad (1.84)$$

In this case, the fermionic Lagrangian takes the form

$$e^{-1} \mathcal{L}_F = -\frac{1}{2} \bar{\psi}_\mu \gamma^{\mu\nu\rho} \partial_\nu \psi_\rho - g_{m\bar{n}} \left[\bar{\chi}_L^m \not{\partial} \chi_R^{\bar{n}} + \bar{\chi}_R^{\bar{n}} \not{\partial} \chi_L^m \right] + \left[-\mathcal{M}_{mn} \bar{\chi}_L^m \chi_L^n + \frac{2S}{M_P^2} \bar{\psi}_{\mu R} \gamma^\mu \chi_L^m \mathcal{G}_m + \frac{S}{M_P} \bar{\psi}_{\mu R} \gamma^{\mu\nu} \psi_{\nu R} + \text{h.c.} \right] \quad (1.85)$$

with

$$\mathcal{M}_{mn} = e^{\mathcal{K}/(2M_P^2)} (\mathcal{D}_m \mathcal{D}_n W) \quad , \quad S \equiv \frac{1}{2M_P} e^{\mathcal{K}/(2M_P^2)} W \quad (1.86)$$

Now we are going to look for a fermionic state corresponding to the Goldstino. In global supersymmetry in absence of D -terms, the Goldstino is a linear combination of chiral

fermions, lying in the direction of F -term breaking. The situation is similar here, as can be seen in the supersymmetry transformations of the fermions:

$$\delta\psi_{\mu L} = M_P\partial_\mu\epsilon_L + S\gamma_\mu\epsilon_R, \quad \delta\chi_L^m = -\frac{\bar{S}}{M_P}\mathcal{G}^m\epsilon_L \quad (1.87)$$

One easily deduces the Goldstino expression:

$$v_L = \frac{\mathcal{G}_m\chi_L^m}{\sqrt{\mathcal{G}_n\mathcal{G}^n}} = \frac{\mathcal{G}_m\chi_L^m}{\sqrt{3}M_P} \quad (1.88)$$

where we used (1.83) in the second equality. Its variation gives

$$\delta v_L = \frac{\mathcal{G}_m\delta\chi_L^m}{\sqrt{3}M_P} = -\sqrt{3}\bar{S}\epsilon_L \quad (1.89)$$

By a redefinition

$$\chi_L^m \rightarrow \chi_L^{\prime m} + \frac{\mathcal{G}^m}{\sqrt{3}M_P}v_L \quad (1.90)$$

analogously for v_R , the new chiral fermions are then invariant under supersymmetry: $\delta\chi^{\prime m} = 0$. Consequently, the fermionic Lagrangian becomes

$$\begin{aligned} e^{-1}\mathcal{L}_F = & -\frac{1}{2}\bar{\psi}_\mu\gamma^{\mu\nu\rho}\partial_\nu\psi_\rho - [\bar{v}_L\cancel{\partial}v_R + \bar{v}_R\cancel{\partial}v_L] - g_{m\bar{n}} [\bar{\chi}_L^{\prime m}\cancel{\partial}\chi_R^{\prime\bar{n}} + \bar{\chi}_R^{\prime\bar{n}}\cancel{\partial}\chi_L^{\prime m}] \\ & + \left[-\frac{4S}{M_P}\bar{v}_Lv_L + \frac{2\sqrt{3}S}{M_P}\bar{\psi}_{\mu R}\gamma^\mu v_L + \frac{S}{M_P}\bar{\psi}_{\mu R}\gamma^{\mu\nu}\psi_{\nu R} + \text{h.c.} \right] \\ & - [\mathcal{M}_{mn}\bar{\chi}_L^{\prime m}\chi_L^{\prime n} + \text{h.c.}] \end{aligned} \quad (1.91)$$

In absence of gravitino, as is the case of global supersymmetry, the redefinition (1.89) simply decouples the Goldstino from the chiral fermions, leaving the Goldstino as a physical massless fermion. Here, the introduction of the gravitino allows for an additional redefinition:

$$\psi_L^\mu \rightarrow \psi_L^{\prime\mu} - \frac{1}{\sqrt{3}\bar{S}}\partial^\mu v_L - \frac{1}{\sqrt{3}}\gamma^\mu v_R \quad (1.92)$$

analogously for ψ_R^μ . This redefinition sets the gravitino transformation to zero: $\delta\psi^{\prime\mu} = 0$. After this step, we have fixed the unitary gauge $v = 0$, and the Goldstino is completely eliminated from the Lagrangian:

$$\begin{aligned} e^{-1}\mathcal{L}_F = & -\frac{1}{2}\bar{\psi}'_\mu\gamma^{\mu\nu\rho}\partial_\nu\psi'_\rho - g_{m\bar{n}} [\bar{\chi}_L^{\prime m}\cancel{\partial}\chi_R^{\prime\bar{n}} + \bar{\chi}_R^{\prime\bar{n}}\cancel{\partial}\chi_L^{\prime m}] \\ & + \left[\frac{S}{M_P}\bar{\psi}'_{\mu R}\gamma^{\mu\nu}\psi'_{\nu R} + \text{h.c.} \right] - [\mathcal{M}_{mn}\bar{\chi}_L^{\prime m}\chi_L^{\prime n} + \text{h.c.}] \end{aligned} \quad (1.93)$$

We immediately recognize the Rarita-Schwinger Lagrangian for the gravitino, and read off its mass

$$m_{3/2} = \frac{2S}{M_P} \quad (1.94)$$

Chapter 2

Charged massive spin-3/2 and spin-2

An important example of non-trivial background in field theories is the electromagnetic field, where acausality is known to persist for charged higher spins. For spins lower than $3/2$, the minimal coupling $\partial \rightarrow \mathfrak{D} \equiv \partial - iA$ gives the correct theory, but this replacement results in pathologies once the spin goes higher. This is the Velo-Zwanziger problem. Many attempts have been made in the past, in the search for a consistent charged massive higher spin Lagrangian, but even in the simplest cases of spin-3/2 and spin-2, finding a Lagrangian turns out to be a highly challenging task. In this Chapter based on [46–48], we will address this issue for spin-3/2 and spin-2, exploring a recent result of superstring theory [13] where a consistent superspace action for charged massive spin-3/2 and spin-2, coupled to lower spins, was obtained. We will expand and simplify this action, so as to derive a Lagrangian involving these higher spin states along with lower spins. In addition, we will find the decoupled equations of motion and constraints for charged massive spin-3/2 and spin-2 that are free of pathologies.

In this Chapter, the terminology *higher spin* signifies spin higher than 1, *i.e.* higher than the spins of all observed elementary particles. Besides, we will be focusing exclusively on the massive case. We will be working in the Minkowski space, with the metric convention $\eta \sim (-, +, +, \dots)$. Different from the previous Chapter, Latin letters are used as spacetime indices.

2.1 Free massive higher spins: a well-established theory

2.1.1 Dirac, Rarita-Schwinger, and Fierz-Pauli

It is a well-known fact in QFT that irreducible unitary representations of the Poincaré group can be classified by their spin (integer or half-integer) and their mass [1, 49]. In particular, the observed elementary particles consist of those in the Standard Model, including spin-1/2 quarks and leptons, spin-1 gauge bosons, as well as spin-0 Higgs boson¹. The Lagrangians of free spin $s \leq 1$ fields, massive or massless, can be found in a large amount of literature. The massive spin-1/2 is described by the Dirac Lagrangian:

$$\mathcal{L}_{1/2} = \bar{\psi} \not{\partial} \psi - iM \bar{\psi} \psi \tag{2.1}$$

¹We are not raising the question of whether they are truly elementary. Relevant investigations exist, which are not the focus of this manuscript.

And we have the Proca Lagrangian for the massive spin-1:

$$\mathcal{L}_1 = -\frac{1}{4}F^{mn}F_{mn} - \frac{1}{2}M^2V^mV_m, \quad \text{with} \quad F_{mn} \equiv \partial_mV_n - \partial_nV_m \quad (2.2)$$

Taking the variation with regard to V_m , one obtains

$$\frac{\delta\mathcal{L}_1}{\delta V^m} = \partial^2V_m - \partial_m\partial_nV^n - M^2V_m = 0 \quad (2.3)$$

Applying furthermore a partial derivative ∂^m on the above equation, a constraint equation arises

$$\partial^mV_m = 0 \quad (2.4)$$

which in turn implies the Klein-Gordon equation $(\partial^2 - M^2)V_m = 0$, and results in 3 on-shell DoFs for the massive spin-1, as is expected from its 3 helicity states. Contrary to the massless case, the Proca Lagrangian is not gauge invariant, but gauge invariance can be created through the introduction of a *Stückelberg field*, with the redefinition $V_m \rightarrow V_m + \partial_m\phi/M$. Subsequently, the Proca Lagrangian becomes

$$\mathcal{L}'_1 = -\frac{1}{4}F^{mn}F_{mn} - \frac{1}{2}\partial^m\phi\partial_m\phi - \frac{1}{2}M^2V^mV_m - MV^m\partial_m\phi \quad (2.5)$$

which is now invariant under the gauge transformations $\delta V_m = \partial_m\Lambda$, $\delta\phi = -M\Lambda$. Conversely, the Stückelberg field ϕ can be eliminated in the unitary gauge, thereby bringing back the Lagrangian (2.2).

Higher spin studies have first been undertaken by Dirac, in an attempt to generalize the spin-1/2 equation. The immediate fermionic neighbor of spin-1/2 – the spin-3/2 field – is given by Ψ_m transforming in the spinor-vector representation of the Lorentz group. Its free Lagrangian, derived by Rarita and Schwinger [6], is conventionally written as²

$$\mathcal{L}_{3/2} = -\bar{\Psi}_m\gamma^{mnk}\partial_n\Psi_k - iM\bar{\Psi}_m\gamma^{mn}\Psi_n \quad (2.6)$$

(2.6) yields a Dirac equation as well as two constraints

$$(i\not{\partial} + M)\Psi_m = 0, \quad \gamma^m\Psi_m = 0, \quad \partial^m\Psi_m = 0 \quad (2.7)$$

which result in four propagating DoFs.

Around the same period, Fierz and Pauli [7] have obtained a Lagrangian describing a massive spin-2 particle, represented by a real rank-2 symmetric tensor h_{mn} :

$$\begin{aligned} \mathcal{L}_2 = & -\frac{1}{2}\partial^k h^{mn}\partial_k h_{mn} + \frac{1}{2}\partial^m h\partial_m h - \frac{1}{2}M^2 h^{mn}h_{mn} + \frac{1}{2}M^2 h^2 \\ & + h^{mn}\partial_m\partial_n h + \partial^n h_{mn}\partial_k h^{mk} \end{aligned} \quad (2.8)$$

where h stands for the trace of h_{mn} : $h \equiv h^m{}_m$. The Fierz-Pauli Lagrangian is precisely the sum of the linear expansion of the Einstein-Hilbert action, and two mass terms. The signs in front of these mass terms are crucial in order to avoid ghosts. Similar to spin-3/2, (2.8) gives rise to two constraint equations (also called the *transverse-traceless* constraints), in addition

²The coefficients in this Lagrangian differs from the one presented in Chapter 3, due to the different choices of the γ -matrices.

to the Klein-Gordon equation. To reach this system, we begin with the Euler-Lagrange equation of h_{mn} in (2.8):

$$\left(\partial^2 - M^2\right) h_{mn} - \left(\partial^2 - M^2\right) \eta_{mn} h + \partial_m \partial_n h + \eta_{mn} \partial_k \partial_l h^{kl} - \partial_n \partial^k h_{mk} - \partial_m \partial^k h_{nk} = 0 \quad (2.9)$$

which yields the following three equations:

1. Trace

$$-2\partial^2 h + 3M^2 h + 2\partial^m \partial^n h_{mn} = 0 \quad (2.10)$$

2. Divergence

$$\partial_m h - \partial^n h_{mn} = 0 \quad (2.11)$$

3. Double Divergence

$$\partial^2 h - \partial^m \partial^n h_{mn} = 0 \quad (2.12)$$

Using (2.12) to replace $\partial^m \partial^n h_{mn}$ in (2.10), we obtain the trace constraint $h = 0$, which according to (2.11) implies the divergence constraint $\partial^n h_{mn} = 0$. Inserting these two conditions back to the Euler-Lagrange equation, we obtain the Klein Gordon equation for the spin-2: $(\partial^2 - M^2) h_{mn} = 0$. In summary,

$$\left(\partial^2 - M^2\right) h_{mn} = 0, \quad h = 0, \quad \partial^n h_{mn} = 0 \quad (2.13)$$

The above constraints remove five DoFs on-shell, so the massive spin-2 has five propagating DoFs.

Note that the analogous Stückelberg fields can be introduced as well, so that the new Lagrangians become gauge invariant. For spin-3/2, one may redefine (2.6) with $\Psi_m \rightarrow \Psi_m + \partial_m v$. The result is invariant under the transformations $\delta\Psi_m = \partial_m \Upsilon$, $\delta v = -\Upsilon$, where v and Υ are Dirac spinors. For spin-2, one can shift h_{mn} with $h_{mn} \rightarrow h_{mn} + \partial_{(m} A_{n)} + \partial_m \partial_n \phi$, and the gauge invariance is manifest with *e.g.* $\delta h_{mn} = \partial_{(m} \Lambda_{n)} + 2\partial_m \partial_n \tilde{\Lambda}$, $\delta A_m = -\Lambda_m - \partial_m \tilde{\Lambda}$, $\delta\phi = -\tilde{\Lambda}$. In the unitary gauge, we recover again (2.6) and (2.8).

2.1.2 A Lagrangian for any spin

A Lagrangian formulation for massive free fields of arbitrary spin has first been obtained by Singh and Hagen [8, 9], starting from the very basic requirement of Lorentz invariance. In this subsection, we review briefly the main arguments and results in their work.

For elementary bosons and fermions, one can choose Lorentz group representations where the corresponding fields are given by *symmetric traceless* tensors, respectively, $\Phi_{n_1 \dots n_s}$ and $\Psi_{n_1 \dots n_t}$, with $t = s - \frac{1}{2}$ and s represents the spin. The (γ -)traceless condition is:

$$\eta^{n_1 n_2} \Phi_{n_1 n_2 \dots n_s} = 0, \quad \gamma^{n_1} \Psi_{n_1 \dots n_t} = 0 \quad (2.14)$$

Lorentz invariance imposes the Fierz-Pauli system:

$$\begin{aligned} \left(\partial^2 - M^2\right) \Phi_{n_1 \dots n_s} &= 0, & (i\cancel{\partial} + M) \Psi_{n_1 \dots n_t} &= 0, \\ \partial^{n_1} \Phi_{n_1 \dots n_s} &= 0, & \partial^{n_1} \Psi_{n_1 \dots n_t} &= 0. \end{aligned} \quad (2.15)$$

The key point in the construction *à la* Singh-Hagen is the use of *auxiliary fields*, that enhance the constraint equations and vanish on-shell. Indeed, it is very complicated, if not impossible, to guess a Lagrangian starting directly from the on-shell equation (2.15). The authors of [7] have already pointed out the necessity of additional fields in the construction of a higher spin Lagrangian, and later, in search of $s = 2, 3, 4$ Lagrangians, Fronsdal [50] and Chang [51] have also suggested introducing auxiliary fields. In the following, we will see how they are introduced in a systematic way.

The primary goal is to find the variation of the Lagrangian with regard to the higher spin field, namely $\delta\mathcal{L}/\delta\Phi^{(s)}$, so that one can easily reconstruct the Lagrangian from this expression. In addition, $\delta\mathcal{L}/\delta\Phi^{(s)}$ has the same symmetries as $\Phi^{(s)}$ itself, *i.e.* traceless and totally symmetric, and this equation must yield the EoM and constraints (2.15). For instance, in the spin-1 case, we had (2.3) which gave rise to the divergence constraint as well as the Klein-Gordon equation. Going to spin-2, one might be tempted to generalize (2.3) and try

$$\frac{\delta\mathcal{L}}{\delta\Phi_{mn}} = \partial^2\Phi_{mn} - 2\{\partial_m(\partial\Phi)_n\}_{\text{S.T.}} - M^2\Phi_{mn} = 0 \quad (2.16)$$

where S.T. means the symmetric traceless part of the tensor: $\{T_{mn}\}_{\text{S.T.}} = T_{(mn)} - \frac{1}{4}\eta_{mn}T^k{}_k$. The coefficient of the second term is chosen such that, applying ∂^n on (2.16), one would obtain $\partial^n\Phi_{mn} = 0$. However, this result requires in addition $\partial^m\partial^n\Phi_{mn} = 0$, which cannot be obtained from (2.16). In fact, an equation of the form (2.16) alone does not realize the Fierz-Pauli system (2.15), and more equations are needed, hence the necessity of new fields. For spin-2, it is precisely the auxiliary field that imposes $\partial^m\partial^n\Phi_{mn} = 0$.

The simplest candidate to impose the scalar equation $\partial\partial\Phi = 0$, is obviously a scalar field φ . Its only possible coupling to Φ_{mn} in the Lagrangian is $\varphi\partial^m\partial^n\Phi_{mn}$, which yields the system

$$\begin{aligned} \partial^2\Phi_{mn} - 2\{\partial_m(\partial\Phi)_n\}_{\text{S.T.}} - M^2\Phi_{mn} + c\{\partial_m\partial_n\varphi\}_{\text{S.T.}} &= 0 \\ \partial^m\partial^n\Phi_{mn} &= (\partial^2 - aM^2)\varphi \end{aligned} \quad (2.17)$$

where a and c encode the unknown φ mass term and the coupling constant of $\varphi\partial^m\partial^n\Phi_{mn}$. It turns out that there is a unique choice $c = 2/3$, $a = 2$, that results in

$$\varphi = 0 = \partial^m\partial^n\Phi_{mn} \quad (2.18)$$

We then get the desired equations $\partial^n\Phi_{mn} = 0$, $(\partial^2 - M^2)\Phi_{mn} = 0$, and the Lagrangian can be immediately inferred from (2.17). The higher spins $s > 2$ follow the same procedure, necessitating a series of lower spin auxiliary fields $s - 2, \dots, 1, 0$, that vanish on-shell. The fermionic case is analogous, thus will not be detailed here. The most general form of the Lagrangian, for any given spin, is provided in [8, 9].

2.2 Charged massive higher spins: why are they so difficult?

2.2.1 The Velo-Zwanziger problem

We have seen in the previous section that the Lagrangian formulation for massive free states of arbitrary spin has been very well constructed. Taking one step forward, we may consider an *interacting* massive higher spin theory, typically with an electromagnetic field,

and search for a corresponding Lagrangian for it. Far from a pure academic exercise, such a setup is of phenomenological interest, because charged higher spin composite particles do exist in Nature, and one example is the Ω^- baryon whose spin is determined to be $s = 3/2$. In a typical experimental context, it is reasonable to take the point particle limit and describe the behavior of these composite states by a local action. In the following, we will be focusing on massive higher spin fields charged under $U(1)$, interacting with an electromagnetic field.

A consistent Lagrangian formulation for charged higher spins is a problem with a long history. Already back in 1939, Fierz and Pauli [7] have pointed out that the simplest replacement $\partial_m \rightarrow \mathfrak{D}_m \equiv \partial_m + ieA_m$ in the on-shell equations – attempted by Dirac [5] – will result in algebraically inconsistent equations for higher spins. In the spin-2 case, the minimal coupling $\partial_m \rightarrow \mathfrak{D}_m$ gives rise to:

$$\begin{aligned} (\mathfrak{D}^2 - M^2) h_{mn} &= 0 \\ h^m{}_m &= 0, \quad \mathfrak{D}^n h_{mn} = 0 \end{aligned} \tag{2.19}$$

The above two lines are obviously incompatible, because taking the divergence of the EoM, one obtains $\mathfrak{D}^n \mathfrak{D}^2 h_{mn} = 0$, implying $[\mathfrak{D}^n \mathfrak{D}^2, \mathfrak{D}^2 \mathfrak{D}^n] h_{mn} = 0$, which contradicts with the property of the $U(1)$ covariant derivative. Fierz and Pauli have then proposed that in order to avoid this algebraic inconsistency, the equation of motion and constraints must be derivable from a Lagrangian by variational principle. They first evoked the use of auxiliary fields in the higher spin Lagrangian, that later was generalized by Singh and Hagen (see Section 2.1.2). This construction has been successful in the uncharged case. Switching on the electromagnetic field, they suggested in [7] introducing minimal coupling in the uncharged Lagrangian and the consequent equation of motion.

However, pathologies persist. Three decades later, Velo and Zwanziger [52–54] (see also [55]) have investigated in great detail this inconsistency of charged higher spin propagation, and analyzed the EoM as well as the constraints for spin $s > 1$ by using the method of characteristics. Their results led to what is known today as the *Velo-Zwanziger problem*, that we shall clarify shortly.

Indeed, for massive states of spin $s \leq 1$, the minimal coupling is exempt from any pathology. Replacing the partial derivatives by $U(1)$ -covariant derivatives, one obtains:

$$\begin{aligned} \text{spin-0: } (\partial^2 - M^2) \phi &= 0 & \implies (\mathfrak{D}^2 - M^2) \phi &= 0 \\ \text{spin-1/2: } (i\not{\partial} + M) \psi &= 0 & \implies (i\not{\mathfrak{D}} + M) \psi &= 0 \\ \text{spin-1: } (\partial^2 - M^2) V_m &= 0, \quad \partial^m V_m = 0 & \implies (\mathfrak{D}^2 - M^2) V_m &= 0, \quad \mathfrak{D}^m V_m = 0 \end{aligned} \tag{2.20}$$

which are manifestly consistent from our knowledge on QED. The spin-1 corresponds to the marginal case, from which constraint equations arise and the non-commutativity of the covariant derivative starts to have an impact on the consistency of the on-shell system, as we have seen in the example of (2.19).

For higher spins, it is convenient to study their propagation properties through the characteristic determinant $\Delta(\mathbf{n})$ [56]. The latter is obtained *via* the substitution $i\partial_m \rightarrow \mathbf{n}_m$ in the EoM, and then recover the determinant of the coefficient matrix, keeping only the leading terms in \mathbf{n}_m . For instance, given an EoM $\left[i \left(\Gamma^k \right)_m{}^n \partial_k + B_m{}^n \right] \Psi_n = 0$, we have $\Delta(\mathbf{n}) = \left| \left(\Gamma^k \right)_{mn} \mathbf{n}_k \right|$. The *normals* N_m to the characteristic surface are the solutions of

$\Delta(\mathbf{n}) = 0$, and the maximal propagation speed of the signal is then $N^0/|\vec{N}|$. This method enables us to check the following two consistency conditions:

- **Hyperbolicity:** if N^0 is real for any \vec{N}
- **Causality:** if all solutions N_m are space-like, namely, the characteristic surface is always time-like

Additionally, the absence of propagating ghosts requires the on-shell DoF to be $(2s + 1)$. A careful analysis with this method [52–55], along with the DoF counting, shows that massive charged states with spin $s > 1$ minimally coupled to electromagnetism, suffer from various pathologies: faster-than-light propagation for spin-3/2, and incorrect DoF for spin-2.

Such a phenomenon is usually called the Velo-Zwanziger problem. This problem persists even in an effective theory with a small constant electromagnetic background, and actually a large class of non-minimal couplings still fail to restore the consistency, as we will see next.

2.2.2 Attempts and main results

Given that the pathological propagation of charged massive higher spins arises as a consequence of the minimal coupling, an obvious attempt to cure the problem, would be to introduce *non-minimal couplings*. Following the work by Federbush [10], one first complexifies the Fierz-Pauli Lagrangian (2.8) and replaces the partial derivatives by $U(1)$ -covariant derivatives. The subsequent Lagrangian clearly exhibits the Velo-Zwanziger problem. The next step is to add a non-minimal coupling term, containing the gyromagnetic ratio g , and the electromagnetic field strength ϵ_{mn} , so we obtain:

$$\begin{aligned} \mathcal{L} = & -|\mathcal{D}_m h_{nk}|^2 + 2|\mathcal{D}_m h^{mn}|^2 + |\mathcal{D}_m h|^2 + (\bar{h}_{mn} \mathcal{D}^m \mathcal{D}^n h + \text{h.c.}) - M^2 (\bar{h}_{mn} h^{mn} - \bar{h}h) \\ & + 2ieg \text{Tr} (h \cdot \epsilon \cdot \bar{h}) \end{aligned} \quad (2.21)$$

The field strength is taken to be constant here, and (2.21) is called the *Federbush Lagrangian* for a charged massive spin-2 field. The calculation of the EoM can be carried out analogously. In particular, the trace equation in the free case, $h = 0$, now takes the form:

$$h \propto M^{-4} ie (2g - 1) \epsilon^{mn} \mathcal{D}_m \mathcal{D}^k h_{kn} + \dots \quad (2.22)$$

The ellipsis represent additional terms, without derivatives, that disappear in the free case. We immediately see the crucial requirement here: if the first term containing derivatives does not vanish, then (2.22) will not be a constraint, and consequently some ghost DoFs will be propagating. To avoid ghosts, the gyromagnetic ratio must be $g = 1/2$.

Then comes the first issue. In order to guarantee high energy unitarity of the scattering amplitudes, it is necessary that the gyromagnetic ratio is $g = 2$ at tree level for charged elementary particles of arbitrary spin [57]³. This means that the Federbush Lagrangian will not have a good high energy behavior. The second and more problematic issue of (2.21) is that it allows for superluminal propagation even for very small background magnetic field [58]. Indeed, using the method of characteristics as in [52–55], there always exists a Lorentz frame with a space-like characteristic surface, which implies acausality. Nevertheless, the

³We can see that the minimal coupling in (2.19) also leads to an EoM whose gyromagnetic ratio is not 2.

Federbush Lagrangian remains up to date, the only $4d$ Lagrangian for charged massive spin-2, without ghosts (if $g = 1/2$), and without coupling to lower spin fields.

On the other hand, significant effort has also been made for the charged massive spin-3/2 case, but no explicit consistent Lagrangian has been obtained yet. In 2000, Deser, Pascualutsa and Waldron [11] have investigated a broad class of non-minimal couplings modifying the minimal spin-3/2 Lagrangian, analysing in detail their low energy behavior, causality, unitarity as well as existence of ghosts. But in the end, none of them escaped from inconsistency. Later, Porrati and Rahman [59] have extended the class of non-minimal couplings considered in [11], and in particular, their studies were based on the requirement that the γ -trace constraint $\gamma^m \Psi_m = 0$ be enforced to *all orders*. This condition actually ensures simultaneously hyperbolicity, causality, and correct DoFs, as long as the Lagrangian is hermitian, and has a canonical kinetic term. The coefficients of the final spin-3/2 Lagrangian are given, implicitly, by a recursive relation.

As a side comment, one does know a consistent example of charged massive spin-3/2 Lagrangian, which is the gravitino charged under $U(1)$ -graviphoton in $\mathcal{N} = 2$ supergravity. But in the zero cosmological constant limit, the causality bound implies planckian gravitino mass [60], also the charge and the mass shall obey the Kaluza-Klein relation $e = m/M_{Pl}$. Beyond these conditions, the same pathologies as before still persist.

2.3 Implementation of String Theory

Our excursion into various attempts has revealed the significant difficulty in the construction of charged massive higher spin Lagrangians. On one hand, the Federbush Lagrangian for spin-2 has the undesired gyromagnetic ratio $g = 1/2$, as well as the more problematic superluminal propagation. On the other hand, even though a causal spin-3/2 Lagrangian *a priori* exists [59], its explicit form is still unknown. Indeed, starting from the basic requirements – causality, hyperbolicity, correct DoFs, and $g = 2$ – to guess a Lagrangian turns out to be a very complex task.

An alternative way to tackle the problem is to use String Theory. Originally proposed as a model of hadronic resonances, String Theory naturally contains an infinite tower of higher spin states, providing an intriguing opportunity to study higher spins. Moreover, these states can carry electric charge, and in this case it is shown that the gyromagnetic ratio is $g = 2$ for *all* states in the spectrum [57]. The formalism of charged open bosonic string in a *constant* electromagnetic background has been constructed in [12], which led Argyres and Nappi [14, 15] to derive the Lagrangian of the first mass level of charged open bosonic string. This level features one massive spin-2 field along with a massive vector boson, but fortunately enough, the gauge transformations of these two fields imply that the vector boson is a Stückelberg field, therefore can be eliminated in the unitary gauge. The Argyres-Nappi Lagrangian describes the only physical field at the first mass level – the massive charged spin-2. It takes the form

$$\begin{aligned} \mathcal{L}_{AN} = & \bar{\mathcal{H}}_{mn} \left(\mathcal{D}^2 - 2 - \frac{1}{2} \text{Tr } \epsilon^2 \right) h^{mn} + 2i \bar{\mathcal{H}}_{mn} (\epsilon \cdot h - h \cdot \epsilon)^{mn} - \bar{\mathcal{H}} \left(\mathcal{D}^2 - 2 - \frac{1}{2} \text{Tr } \epsilon^2 \right) \mathcal{H} \\ & - \bar{\mathcal{H}}_{mn} \left\{ \mathcal{D}^m \mathcal{D}^k [(1 - i\epsilon) h]_k^n - \frac{1}{2} \mathcal{D}^m \mathcal{D}^n \mathcal{H} + (m \leftrightarrow n) \right\} + \bar{\mathcal{H}} \mathcal{D}^m \mathcal{D}^n \mathcal{H}_{mn} \end{aligned} \quad (2.23)$$

where we defined the rescaled tensor $\mathcal{H}_{mn} \equiv (1 - i\epsilon)_{mk} (1 - i\epsilon)_{nl} h^{kl}$ and $\mathcal{H} \equiv \mathcal{H}^m{}_m$. The inverse string tension is fixed to $\alpha' = 1/2$. The field strength is given by $i\epsilon_{mn} = [\mathcal{D}_m, \mathcal{D}_n]$, with the electric charge set to $e = 1$. In $26d$, the Argyres-Nappi Lagrangian yields the following EoM and constraints:

$$\begin{aligned} \left(\mathcal{D}^2 - 2 - \frac{1}{2} \text{Tr } \epsilon^2 \right) \mathcal{H}_{mn} + 2i (\epsilon \cdot \mathcal{H} - \mathcal{H} \cdot \epsilon)_{mn} &= 0 \\ \mathcal{D}^n \mathcal{H}_{mn} &= 0 \\ \mathcal{H} &= 0 \end{aligned} \tag{2.24}$$

The first line is a deformation of the Klein-Gordon equation, with an electromagnetic field-dependent contribution to the mass. The second and third lines correspond to the constraints: one can immediately notice that they are the charged version of the transverse-traceless constraints $\partial^n h_{mn} = 0$, $h = 0$. (2.24) is in effect a deformed Fierz-Pauli system, which is manifestly consistent, with correct gyromagnetic ratio $g = 2$, correct DoF, hyperbolic and causal EoM. Besides, this Fierz-Pauli system (2.24) is also generalized to arbitrary integer spin using Virasoro algebra in [58]. In addition, one recovers exactly the Fierz-Pauli Lagrangian from (2.23) in the free case $\mathcal{D} \rightarrow \partial$, $\epsilon = 0$.

The Velo-Zwanziger problem seems solved for spin-2. However, this is true only in $26d$: away from this critical dimension, the trace becomes dynamical and (2.24) no longer holds [61]. A lower-dimensional Lagrangian has been obtained by Porrati and Rahman [61] by compactifying (2.23). This process introduces an extra scalar ϕ which is singlet of the internal coordinates, and assumes that the electromagnetic field is zero in the compactification directions. As a result, the dimensionally reduced Argyres-Nappi Lagrangian, in $d < 26$ dimensions, reads

$$\begin{aligned} \mathcal{L}_d = & \bar{\mathfrak{h}}_{mn} \left(\mathcal{D}^2 - 2 - \frac{1}{2} \text{Tr } \epsilon^2 \right) h^{mn} + 2i \bar{\mathfrak{h}}_{mn} (\epsilon \cdot h - h \cdot \epsilon)^{mn} - \bar{\mathfrak{h}} \left(\mathcal{D}^2 - 2 - \frac{1}{2} \text{Tr } \epsilon^2 \right) \mathfrak{h} \\ & - \bar{\mathfrak{h}}_{mn} \left\{ \mathcal{D}^m \mathcal{D}^n [(1 - i\epsilon) \cdot h]_p^n - \frac{1}{2} \mathcal{D}^m \mathcal{D}^n \mathfrak{h} + (m \leftrightarrow n) \right\} + \bar{\mathfrak{h}} \mathcal{D}^m \mathcal{D}^n \mathfrak{h}_{mn} \\ & + \left[\bar{\mathfrak{h}}_{mn} \mathcal{D}^m \mathcal{D}^n \phi - \left\{ \bar{\mathfrak{h}} + \frac{1}{2} \left(\frac{D-d-1}{D-d} \right) \bar{\phi} \right\} \left(\mathcal{D}^2 - 2 - \frac{1}{2} \text{Tr } \epsilon^2 \right) \phi + \text{h.c.} \right] \end{aligned} \tag{2.25}$$

The same rescaling as before is used $\mathfrak{h}_{mn} \equiv (1 - i\epsilon)_{mk} (1 - i\epsilon)_{nl} h^{kl}$. Some algebraic manipulations lead to the (deformed-)Klein-Gordon EoMs

$$\begin{aligned} \left(\mathcal{D}^2 - 2 - \frac{1}{2} \text{Tr } \epsilon^2 \right) \phi &= 0 \\ \left(\mathcal{D}^2 - 2 - \frac{1}{2} \text{Tr } \epsilon^2 \right) \mathfrak{h}_{mn} + 2i (\epsilon \cdot \mathfrak{h} - \mathfrak{h} \cdot \epsilon)_{mn} &= 0, \end{aligned} \tag{2.26}$$

whereas both the divergence and trace constraints of \mathfrak{h}_{mn} are coupled to the scalar, but they can be decoupled *on-shell* by a redefinition $\mathfrak{h}_{mn} \rightarrow \mathfrak{h}'_{mn}$, so that \mathfrak{h}'_{mn} satisfies the Fierz-Pauli system (2.24), making the consistency manifest. It is unclear though, whether ϕ and \mathfrak{h}_{mn} can be decoupled directly at the level of the Lagrangian.

The construction of charged massive spin-2 Lagrangian using bosonic open string theory has been successful. In $26d$, the Argyres-Nappi Lagrangian contains a spin-2 only, giving rise to the Fierz-Pauli system. By dimensional reduction, the new Lagrangian can describe $4d$ as

well, at the cost of coupling an additional scalar ϕ to the spin-2 field. The same approach as in [14, 15] can also be realized at the second mass level of open bosonic string, as is done by Klishevich [62], and this yields a Lagrangian for charged massive spin-3 coupled to spin-1.

Now it is natural to wonder whether String Theory helps to derive a Lagrangian for charged massive spin-3/2. Indeed, such fermionic states do exist in superstring theory. In [63], it is shown that the free open superstring compactified to $4d$ (in a way that preserves spacetime supersymmetry) features one spin-2 multiplet – with spin-2, spin-3/2 and spin-1 – as well as two scalar multiplets on its first mass level. And one year later, the same authors have derived the superspace action of these multiplets [64], from which it is possible to get a Lagrangian. Then, it is reasonable to expect that the superspace action can be obtained for charged superstring first mass level as well. The generalization of [63, 64] to the charged case is achieved in [13], resulting in a superspace action describing, among others, a charged massive spin-2 and spin-3/2 that propagate consistently.

The above results are very encouraging. The next question will be: now we have a superspace action, what does the Lagrangian look like? The answer to this question requires many non-trivial algebraic manipulations. It is straightforward, though tedious, to expand the superspace action in components, but after expansion the Lagrangian involves numerous auxiliary DoFs. As is pointed out in [65], the presence of auxiliary fields is one of the main drawbacks of the superspace action in [64], thus the authors of [65] found it simpler to begin with a free Lagrangian then supersymmetrize it, than to identify the physical DoFs directly from the superspace action. Here, our goal is to remedy this problem and show how these auxiliary fields can be totally eliminated. The final Lagrangian should contain physical DoFs only, with charged massive spin-2, spin-3/2, spin-1, spin-1/2 and complex scalars [13]. There are more states at the first mass level than in the bosonic string case, due to the spacetime supersymmetry, and therefore we expect to have a more complicated spin-2 Lagrangian than (2.23).

The following sections are devoted to the development of the superspace action in [13]. To illustrate the basic ideas, we will start from the neutral superspace action, already presented in [64], and proceed to expand, simplify and eventually decouple the Lagrangian. It is sensible to tackle first the free case, because the result is already known, *viz.*, we shall find in the end the Fierz-Pauli Lagrangian for spin-2, Rarita-Schwinger Lagrangian for spin-3/2, and free Lagrangians for spin-1, spin-1/2 and spin-0. In a second time, we move to the charged case, and carry out a similar computation for the charged superspace action in [13]. The Lagrangians will be presented in different forms: one compact form which allows for a simpler derivation of on-shell equations, and another one as deformed Fierz-Pauli or Rarita-Schwinger Lagrangian. The latter reduces to the well-known free Lagrangian when the external field vanishes, but is more complicated and contains higher derivatives. In addition, we derive the EoM and constraints for charged massive spin-2 and spin-3/2, that despite a mixing with lower spins, can be decoupled on shell. The former is precisely a four-dimensional version of the Fierz-Pauli system (2.24), and the latter, perhaps our principal result, has not been obtained before in the literature.

2.4 Summary of the main results

We summarize in this section the main results obtained after the expansion of the super-space action in [13], including the Lagrangian of physical states and the equations of motion for spin-2 and spin-3/2. Detailed calculations, as well as alternative forms of the Lagrangian, are presented in Section 2.5.

Note that we consider constant electromagnetic background, with covariant derivative \mathfrak{D}_m satisfying:

$$[\mathfrak{D}_m, \mathfrak{D}_n] = iQ\epsilon_{mn} \quad (2.27)$$

All the fields have unit charge $Q = \pm 1$.

2.4.1 Bosons

The Lagrangian of the bosonic sector reads:

$$\begin{aligned} \mathcal{L}_B = & \bar{\mathcal{M}}_1 \left(-2 + \mathfrak{D}^2 \right) \mathcal{M}_1 + \bar{\mathcal{N}}_1 \left(-2 + \mathfrak{D}^2 \right) \mathcal{N}_1 \\ & + \bar{\mathcal{C}}^m \mathfrak{D}^2 \mathcal{C}_m + \mathfrak{D}^m \bar{\mathcal{C}}_m \mathfrak{D}^n \mathcal{C}_n - 2\bar{\mathcal{C}}^m (\eta_{mn} - i\epsilon_{mn}) \mathcal{C}^n \\ & + 2\bar{a}^m a_m - i\epsilon_{mn} \bar{a}^m a^n + \mathfrak{D}^m \bar{a}_m \mathfrak{D}^n a_n + \frac{1}{\sqrt{2}} \left[\bar{F}^{mn}(a) \left(F_{mn}(c) - \mathcal{H}_{[mn]} \right) + \text{h.c.} \right] \\ & - 2\bar{c}^m c_m - \frac{2}{5} \mathfrak{D}^m \bar{c}_m \mathfrak{D}^n c_n + \left[\bar{c}^m \left(-\frac{2}{5} \mathfrak{D}_m \mathcal{H} + \mathfrak{D}^n \mathcal{H}_{nm} \right) + \text{h.c.} \right] \\ & + \frac{1}{2} \bar{\mathcal{H}}_{mn} \mathfrak{D}^2 h^{mn} + \frac{1}{2} \mathfrak{D}^n \bar{\mathcal{H}}_{mn} \mathfrak{D}_k h^{mk} - \bar{\mathcal{H}}^{(mn)} \mathcal{H}_{(mn)} + i\epsilon^{nk} \bar{\mathcal{H}}_{mn} h_k^m + \frac{1}{10} \bar{\mathcal{H}} \mathcal{H} \end{aligned} \quad (2.28)$$

Different from the Argyres-Nappi Lagrangian (2.23), the rescaled field \mathcal{H}_{mn} has only one $(\eta - i\epsilon)$ factor:

$$\mathcal{H}_{mn} \equiv (\eta_{mk} - i\epsilon_{mk}) h^k_n, \quad \mathcal{H} = h \quad (2.29)$$

The field strength and its dual are defined as

$$F_{mn}(a) \equiv \mathfrak{D}_m a_n - \mathfrak{D}_n a_m, \quad \tilde{F}_{mn}(a) \equiv \frac{1}{2} \epsilon_{mnpq} F^{pq}(a) \quad (2.30)$$

same for c_m . This Lagrangian yields on shell the following equations:

- **Equations of motion**

$$\begin{aligned} (\mathfrak{D}^2 - 2) \mathcal{M}_1 &= 0, \quad (\mathfrak{D}^2 - 2) \mathcal{N}_1 = 0 \\ (\mathfrak{D}^2 - 2) h_{mn} &= 2i (\epsilon_{km} h^k_n + \epsilon_{kn} h^k_m), \quad (\mathfrak{D}^2 - 2) \mathcal{C}_m + 2i\epsilon_{mn} \mathcal{C}^n = 0 \\ (\mathfrak{D}^2 - 2) a_m + 2i\epsilon_{mn} a^n &= 0, \quad (\mathfrak{D}^2 - 2) c_m + 2i\epsilon_{mn} c^n = 0 \end{aligned} \quad (2.31)$$

- **Constraints**

$$\begin{aligned} \mathfrak{D}^m \mathcal{C}_m &= 0, \quad \mathcal{H}_{[mn]} = \frac{1}{\sqrt{2}} \tilde{F}_{mn}(a) + F_{mn}(c) \\ h + 4\mathfrak{D}^m c_m &= 0, \quad c_m + \frac{1}{2} \mathfrak{D}^n h_{mn} = 0 \end{aligned} \quad (2.32)$$

The constraints (2.32) remove 12 degrees of freedom, resulting in 12 degrees of freedom on shell for the field content $\{a_m, c_m, \mathcal{C}_m, \mathcal{M}_1, \mathcal{N}_1, h_{mn}\}$, which is in agreement with [13]. However, the fields $\{a_m, c_m, h_{mn}\}$ are coupled on shell. They can be decoupled by the following redefinitions

$$\begin{aligned} a'_m &\equiv a_m - \frac{i}{2}\epsilon_{mn}a^n - \frac{i}{2\sqrt{2}}\tilde{\epsilon}^{nk}\mathcal{D}_k h_{mn} + \frac{i}{2\sqrt{2}}\tilde{\epsilon}_{mn}\mathcal{D}^n h + \sqrt{2}i\tilde{\epsilon}_{mn}c^n \\ c'_m &\equiv c_m - \frac{\sqrt{2}i}{4}\tilde{\epsilon}_{mn}a^n + \frac{i}{4}\epsilon^{nk}\mathcal{D}_n h_{mk} \end{aligned} \quad (2.33)$$

and

$$\begin{aligned} \mathfrak{h}_{mn} &\equiv \frac{2}{3}h_{mn} - \frac{1}{6}\eta_{mn}h - \frac{i}{2}\epsilon_m{}^k h_{kn} + \frac{1}{3}\mathcal{D}_m c_n \\ &\quad - \frac{i}{2}\left(\epsilon_{mk}\mathcal{D}^k c_n - \epsilon_{mk}\mathcal{D}_n c^k + \frac{1}{2}\eta_{mn}\epsilon^{kl}\mathcal{D}_k c_l\right) \\ &\quad - \frac{1}{4}\left(\epsilon_{mk}\epsilon^{lk}h_{nl} + \epsilon_{mk}\epsilon_{nl}h^{kl} - \frac{1}{2}\eta_{mn}\epsilon^{kl}\epsilon^p{}_l h_{kp}\right) \\ &\quad + \frac{1}{2-\epsilon\tilde{\epsilon}}\left[\frac{1}{12}\mathcal{D}_m\mathcal{D}_n h - \frac{1}{16}\epsilon_{mk}\epsilon^k{}_n h + \frac{i}{8}\epsilon_{mk}\mathcal{D}^k\mathcal{D}_n h - \frac{5\epsilon\tilde{\epsilon}}{96}\eta_{mn}h\right] \\ &\quad - \frac{1}{\sqrt{2}}\frac{1}{2+\epsilon\tilde{\epsilon}}\left[-\frac{i}{2}\tilde{\epsilon}_{mk}\mathcal{D}^k\mathcal{D}_n\mathcal{D}_l a^l + \frac{5}{16}(\epsilon\tilde{\epsilon})\eta_{mn}\mathcal{D}^k a_k \right. \\ &\quad \left. - \frac{1}{4}(\epsilon\tilde{\epsilon})\mathcal{D}_m\mathcal{D}_n\mathcal{D}_k a^k + \tilde{\epsilon}_{mk}\epsilon_{ln}\mathcal{D}^k\mathcal{D}^l\mathcal{D}^p a_p\right] + (m \leftrightarrow n) \end{aligned} \quad (2.34)$$

In consequence, the new equations are completely decoupled:

$$\begin{aligned} (\mathcal{D}^2 - 2)\mathfrak{h}_{mn} &= 2i\left(\epsilon_{km}\mathfrak{h}^k{}_n + \epsilon_{kn}\mathfrak{h}^k{}_m\right) \\ \mathcal{D}^n\mathfrak{h}_{mn} &= 0, \quad \mathfrak{h} = 0 \\ \mathcal{D}_m\mathcal{D}_n a'^n &= 2a'_m, \quad \mathcal{D}_m\mathcal{D}_n c'^n = 2c'_m, \end{aligned} \quad (2.35)$$

The first two lines for the spin-2 are the same as (2.24) but the latter is derived from the Argyres-Nappi Lagrangian in 26d. The third line does not resemble the conventional equations of motion for massive vectors. In reality, vector fields obeying the equation $\partial_m\partial_n V^n = 2V_m$ are dual to scalars as is shown in [66], so they each count one degree of freedom. The generalization of this result to the charged case is immediate, so a_m, c_m are dual to complex scalars (see Sec. 4.4 in the next section). In fact, only the longitudinal component of these vectors is physical. It is possible to replace a_m, c_m by their dual scalars in (2.28), so the new Lagrangian contains physical degrees of freedom only, but the latter has a more complicated form provided in the next section. In both cases, we are able to decouple on shell all the fields yielding a Fierz-Pauli system for the spin-2.

The gyromagnetic ratio is rather transparent from the equation of motion. For the sake of clarity, let us restore the mass and the charge:

$$(\mathcal{D}^2 - M^2)\mathfrak{h}_{mn} = 2iQ\left(\epsilon_{km}\mathfrak{h}^k{}_n + \epsilon_{kn}\mathfrak{h}^k{}_m\right) \quad (2.36)$$

For any particle of integer spin s , mass m , and charge Q , the gyromagnetic ratio g appears in the equation of motion as [11, 60]:

$$(\mathcal{D}^2 - M^2)\varphi^{m_1\cdots m_s} = iQg s \epsilon^{n(m_1}\varphi_n{}^{m_2\cdots m_s)} \quad (2.37)$$

A direct comparison with (2.36) indicates $g = 2$.

2.4.2 Fermions

Following the two component notation, the Lagrangian of the physical fermionic states in the superspace action is

$$\begin{aligned}
\mathcal{L}_F = & -\frac{i}{2} \left[2 \left(\lambda_1^m \sigma^n \mathcal{D}_n \bar{\lambda}_{1m} \right) + \left(\bar{\chi}_{1m} \bar{\sigma}^n \sigma^k \bar{\sigma}^m \mathcal{D}_k \chi_{1n} \right) \right] - \sqrt{2} [(\lambda_1^m \chi_{1m}) + \text{h.c.}] \\
& + \left[-\frac{i}{4} \left(\psi_1 \sigma^m \mathcal{D}_m \bar{\psi}_1 \right) + 2i (\gamma_1 \sigma^m \mathcal{D}_m \bar{\gamma}_1) \right] \\
& + \left[\frac{3}{\sqrt{2}} (\chi_1^m \sigma_{mn} \mathcal{D}^n \psi_1) - \frac{1}{2\sqrt{2}} (\chi_1^m \mathcal{D}_m \psi_1) \right. \\
& \quad \left. - \frac{i}{2} (\lambda_1^m \sigma_m \bar{\psi}_1) - 2i (\bar{\chi}_1^m \bar{\sigma}_m \gamma_1) - \sqrt{2} (\lambda_1^m \mathcal{D}_m \gamma_1) + \text{h.c.} \right] \\
& + \left[\frac{1}{\sqrt{2}} (\psi_1 \gamma_1) + \text{h.c.} \right] + (1 \leftrightarrow 2) - \left[\frac{1}{2} \bar{\chi}_1^m (\epsilon \cdot \bar{\sigma}) \bar{\sigma}_m \gamma_1 + \frac{1}{2} \chi_2^m (\epsilon \cdot \sigma) \sigma_m \bar{\gamma}_2 + \text{h.c.} \right]
\end{aligned} \tag{2.38}$$

where we introduced the rescalings

$$\begin{aligned}
\bar{\lambda}_{1m} & \equiv (\eta_{mn} - i\epsilon_{mn}) \bar{\lambda}_1^n, & \lambda_{2m} & \equiv (\eta_{mn} - i\epsilon_{mn}) \lambda_2^n \\
\chi_{1m} & \equiv (\eta_{mn} - i\epsilon_{mn}) \chi_1^n, & \bar{\chi}_{2m} & \equiv (\eta_{mn} - i\epsilon_{mn}) \bar{\chi}_2^n
\end{aligned} \tag{2.39}$$

and notice that the Weyl fermions χ_{1m}, χ_{2m} appear only under the rescaled form, whereas the rescaled and non-rescaled $\lambda_{1m}, \lambda_{2m}$ are mixed through the kinetic terms. In the above Lagrangian, the spin-3/2 states $\{\chi_{jm}, \lambda_{jm}\}$ are coupled to spin-1/2 $\{\gamma_j, \bar{\psi}_j\}$, $j = 1, 2$. On shell, it gives the constraints

$$\begin{aligned}
\mathcal{D}^m \bar{\lambda}_{1m} & = \frac{3}{2} \bar{\psi}_1 - \frac{\sqrt{2}}{4} \bar{\sigma}^m (\epsilon \cdot \sigma) \chi_{1m}, & \mathcal{D}^m \chi_{1m} & = -\frac{i}{2} (\epsilon \cdot \sigma) \gamma_1 \\
\sigma^m \bar{\lambda}_{1m} & = \frac{3}{\sqrt{2}} i \gamma_1 - \frac{1}{\sqrt{2}} (\epsilon \cdot \sigma) \gamma_1, & \bar{\sigma}^m \chi_{1m} & = 0
\end{aligned} \tag{2.40}$$

Same as for the bosons, we can also find a redefinition that decouples spin-3/2 from spin-1/2 on shell:

$$\begin{aligned}
\bar{\lambda}'_{1m} & \equiv \bar{\lambda}_{1m} + \frac{i}{2\sqrt{2}} [1 - i(\epsilon \cdot \bar{\sigma})] \bar{\sigma}_m \gamma_1 - \frac{1}{2} [\eta_{mn} - i(\epsilon_{mn} + i\tilde{\epsilon}_{mn})] \mathcal{D}^n \bar{\psi}_1 \\
\chi'_{1m} & \equiv \chi_{1m} + \frac{1}{2\sqrt{2}} (\epsilon \cdot \sigma) \sigma_m \bar{\psi}_1
\end{aligned} \tag{2.41}$$

so that we obtain a Fierz-Pauli system, deformed by the electromagnetic background, that describes the correct degrees of freedom for spin-3/2:

$$\begin{aligned}
i\sigma^n \mathcal{D}_n \bar{\lambda}'_{1m} & = -\sqrt{2} (\eta_{mn} - i\epsilon_{mn}) \chi_1'^m \\
i\bar{\sigma}^n \mathcal{D}_n \chi'_{1m} & = -\sqrt{2} \bar{\lambda}_{1m}, \\
\mathcal{D}^m \chi'_{1m} & = 0, & \mathcal{D}^m \bar{\lambda}'_{1m} & = -\frac{\sqrt{2}}{4} \bar{\sigma}^m (\epsilon \cdot \sigma) \chi'_{1m} \\
\bar{\sigma}^m \chi'_{1m} & = 0, & \sigma^m \bar{\lambda}'_{1m} & = 0
\end{aligned} \tag{2.42}$$

as well as the Dirac equations for the spin-1/2 fields:

$$i\bar{\sigma}^m \mathfrak{D}_m \gamma_1 = -\sqrt{2}\bar{\psi}_1, \quad i\sigma^m \mathfrak{D}_m \bar{\psi}_1 = -\sqrt{2}\gamma_1 \quad (2.43)$$

Equations for the fermions with index 2 are obtained by $1 \leftrightarrow 2$, $\epsilon \leftrightarrow -\epsilon$. Remarkably, the γ -traceless constraint is satisfied even in the presence of the background, whereas the divergence constraint receives a contribution proportional to the γ -matrix. Taking into account the constraints, we again have 12 complex degrees of freedom for the fermions, equal to that of the bosons.

We may also convert the two-component notation into four-component notation (see Appendix A), introducing:

$$\Phi_1 \equiv \begin{pmatrix} \gamma_{1\alpha} \\ \bar{\psi}_1^{\dot{\alpha}} \end{pmatrix}, \quad \Psi_{1m} \equiv \begin{pmatrix} \chi'_{1m\alpha} \\ \bar{\lambda}'_{1m\dot{\alpha}} \end{pmatrix} \quad (2.44)$$

The equations of motion and constraints become

- **Equations of motion**

$$(i\mathfrak{D} + \sqrt{2}) \Phi_1 = 0, \quad (i\mathfrak{D} + \sqrt{2}) \Psi_{1m} = \sqrt{2}i\epsilon_{mn} \Psi_{1L}^n \quad (2.45)$$

- **Constraints**

$$\left[\mathfrak{D}^m - \frac{\sqrt{2}}{4} (\epsilon^{mn} + i\tilde{\epsilon}^{mn}) \gamma_n \right] \Psi_{1m} = 0, \quad \gamma^m \Psi_{1m} = 0 \quad (2.46)$$

A quick analysis of the characteristic determinant shows that the spin-3/2 equation of motion is causal. In the meantime, showing the gyromagnetic ratio $g = 2$ needs more manipulations. To start with, we put (2.45) in a second order form, for clarity we also restore the charge Q and the mass M :

$$\left[(\mathfrak{D}^2 - M^2) \eta_{mn} + 2Qi\epsilon_{mn} - \frac{1}{2}iQ\gamma^{pq}\epsilon_{pq}\eta_{mn} \right] \Psi_1^n = 0 \quad (2.47)$$

Since we are considering constant electromagnetic field, the potential is parameterized as $A_n = -\frac{1}{2}(\epsilon X)_n$ and the covariant derivative is given by $\mathfrak{D}_m = \partial_m + iQA_m$.

Without loss of generality, we choose vanishing electric field $\epsilon_{0i} = 0$, while the magnetic field is given by $\epsilon_{ij} = -\epsilon_{ijk}B^k$, where ϵ is a 3d Levi-Civita symbol ($i, j, k = 1, 2, 3$). Going to momentum space, the above equation reads:

$$\begin{aligned} (\mathcal{H} - QA_0)^2 \Psi_{1m} &= \left[(\vec{p} - Q\vec{A})^2 + M^2 \right] \Psi_{1m} + Q\epsilon_{rs} (\mathcal{M}^{rs})_{mn} \Psi_1^n \\ &= \left[(\vec{p} - Q\vec{A})^2 + M^2 \right] \Psi_{1m} - Q\epsilon_{ijk} B^k (\mathcal{M}^{ij})_{mn} \Psi_1^n \end{aligned} \quad (2.48)$$

where $\mathcal{H} \equiv i\partial_0$, and $(\mathcal{M}^{rs})_{mn} = (J^{rs})_{mn} + S^{rs}\eta_{mn}$ are the generators of the Lorentz group, with $(J^{rs})_{mn} = -2i\eta_m^{[r}\eta^{s]n}$, $S^{rs} = \frac{i}{2}\gamma^{rs}$. The generator of rotations for the spin-3/2 representation is $(S_k)_{mn} = \frac{1}{2}\epsilon_{ijk} (\mathcal{M}^{ij})_{mn}$, so

$$(\mathcal{H} - QA_0)^2 \Psi_{1m} = \left[(\vec{p} - Q\vec{A})^2 + M^2 \right] \Psi_{1m} - 2Q\vec{B} \cdot \vec{S}_{mn} \Psi_1^n \quad (2.49)$$

By definition, the gyromagnetic ratio is extracted from the last term on the right hand side, which is $g = 2$, as is expected.

2.5 Publication

Spin-3/2 and spin-2 charged massive states in a constant electromagnetic background

Karim Benakli,^a Cassiano A. Daniel,^b Wenqi Ke^a

^a*Sorbonne Université, CNRS, Laboratoire de Physique Théorique et Hautes Energies, LPTHE, F-75005 Paris, France.*

^b*ICTP South American Institute for Fundamental Research Instituto de Física Teórica, Universidade Estadual Paulista, Rua Dr. Bento Teobaldo Ferraz 271, 01140-070, São Paulo - SP, Brasil*

E-mail: kbenakli@lpthe.jussieu.fr, c.daniel@unesp.br,
wke@lpthe.jussieu.fr

ABSTRACT: We develop in components the superspace action obtained in [1] describing the first massive level of the open charged superstring in a flat four-dimensional spacetime. In the absence of an electromagnetic background, we show how the Rarita-Schwinger and Fierz-Pauli Lagrangians are retrieved for spin-3/2 and 2, respectively. We then write different forms of the action in the presence of the electromagnetic background. The resulting equations of motion describe the propagation of fields of charged spin-3/2 and spin-1/2 on the one hand, and spin-2, 1 and 0 on the other.

Contents

1	Introduction	2
2	Superspace action in absence of electromagnetic background	5
2.1	The superfields	5
2.2	Bosons	8
2.2.1	Gauge transformations	8
2.2.2	The Bosonic Lagrangian	9
2.2.3	Recovering the Fierz-Pauli Lagrangian	13
2.2.4	Summary	15
2.3	Fermions	16
2.3.1	Gauge transformations	16
2.3.2	The fermionic Lagrangian	17
2.3.3	Recovering the Rarita-Schwinger Lagrangian	19
3	Superspace action in the presence of an electromagnetic background	21
3.1	The superspace action	21
3.2	Decoupled equations of motion in superspace	24
4	Charged massive bosons	25
4.1	Gauge transformations	26
4.2	A compact bosonic Lagrangian	27
4.3	The equations of motion	33
4.4	Decoupling the equations of motion	37
4.5	A deformed Fierz-Pauli bosonic Lagrangian	40
5	Charged massive fermions	45
5.1	Gauge transformations	45
5.2	A compact fermionic Lagrangian	47
5.3	Equations of motion and new spin-3/2	51
5.4	A deformed Rarita-Schwinger Lagrangian	59
6	Conclusions	63
A	Conventions	64
B	Spinor algebra results	65
C	Inverse matrices with the field strength ϵ	66
D	Physical and non-physical fields	67

1 Introduction

The field theory of high spin particles is an old and difficult problem. For the case of massive particles, a challenge arises as soon as one tries to propagate states of spin greater than 1 in an electromagnetic background. In a 1936 paper [2], Dirac called for the problem of writing the equations of motion for such states to be addressed. He wrote: “It is desirable to have the equation ready for a possible future discovery of an elementary particle with a spin greater than a half, or for approximate application to composite particles. Further, the underlying theory is of considerable mathematical interest.” This problem was quickly taken up by Fierz and Pauli [3] who showed the difficulty of it, and in passing wrote in this paper their famous Lagrangian for a massive uncharged spin-2. But the most striking aspect of the difficulty of the problem will only be known a few decades later thanks, in particular, to the works of Johnson and Sudarshan [4], Velo and Zwanziger [5–7]. The problem has remained to this day, but the massive fundamental particles known to current physics do not have such spins. Yet several works have allowed first to understand well the difficulties, then to make some notable progress.

Johnson and Sudarshan tried to canonically quantize minimally coupled spin-3/2 fields and found that the equal-time switches are not compatible with the relativistic covariance of the theory [4]. Later, Velo and Zwanziger found that the minimally coupled Lagrangians for spin-3/2 and spin-2 already exhibit pathological behaviour at the classical level: the former allows faster-than-light propagation while the latter suffers from the loss of a constraint leading to the propagation of six degrees of freedom instead of the five physical ones [5, 6]. Both problems appear for a certain particular value of the electromagnetic field strength. The observation that this value is the same for both cases was an indication that they have a common origin. In fact, it was later shown, for spin-2 in [8] and for spin-3/2 in [9, 10], that the source of the problems is that the set of secondary constraints becomes degenerate. This signals the appearance of a loss of invertibility, *i.e.* the constraints no longer determine all the components of the fields. When the secondary constraints are degenerate, a tertiary constraint can be obtained for spin-2 [11] as well as for spin-3/2 with, in this case, a loss of degrees of freedom [10]. This new constraint then leads to acausality and loss of hyperbolicity.

These original analyses use systems of Fierz-Pauli equations, which imply that, to describe a field of spin s , one must introduce additional fields of lower spin $s - 1, s - 2, \dots$. In the free case, these clearly appear as non-propagating fields that are projected by constraints. However, the introduction of interaction seems to mix the different components of the fields, originally easily decoupled between physical and auxiliary, in a non-trivial way to give combinations that propagate as new physical degrees of freedom. In the historical attempts mentioned above, one ends up with the wrong number of degrees of freedom for the considered field of spin-3/2 or 2. The culprit of non-causal propagation can be traced back to the form of the interaction and it is concluded that it is necessary to go beyond the minimal coupling. The authors of [12] considered adding a set of non-minimal terms to the Lagrangian and proved that it is not easy to restore in this way causal propagation in these theories. In fact, the present work involves another nonlinear modification of the theory,

in particular of the kinetic terms, and relies heavily on the presence of lower spin fields. Moreover, when we try to decouple these different fields, we will not be able to present a fully satisfactory Lagrangian with only the fields of higher spins but only fully decoupled equations of motion.

The Federbush Lagrangian [13] is the only four-dimensional Lagrangian with the correct number of on-shell d.o.f. describing an isolated charged massive spin-2 state, thanks to the presence of a non-minimal coupling term. Unfortunately, the equations of motion derived from this Lagrangian also allow for superluminal propagation and thus suffer from the causality loss problem. Note that, here, the non-minimal coupling induces a gyromagnetic ratio $g = 1/2$, instead of the expected value of $g = 2$, which raises the question of a necessary modification, a completeness, in the ultra-violet (UV), due to the violation of unitarity at high energies [14]. This means that the Lagrangian does not provide a satisfactory answer to the problem at hand.

String theory, which was originally proposed to model hadronic resonances, contains arbitrarily high spin states in its spectrum of massive oscillators. These form the Regge turns. In particular, the first excited level of the open string contains massive spin-2, as well as spin-3/2 in the supersymmetric case. It is therefore no surprise that soon after the solution describing the propagation of the string in an electromagnetic field was given [15–17], it was used to study the propagation problem of charged spin > 1 states. Using string field theory, Argyres and Nappi studied the first massive level of the open bosonic string [18, 19]. They derived a Lagrangian for the massive charged spin-2 field. The obtained form is free of any pathologies mentioned above only in dimension $d = 26$. Its reduction to four dimensions has been studied by Porrati and Rahman [20], who showed that it results in a spin-2 field coupled to a scalar. The study of the second mass level of bosonic strings has led to the action describing a charged massive spin-3 coupled to lower spin states.

A number of points need to be highlighted here. First, in the critical dimensional bosonic open string ($d = 26$), the content of the first massive level is simply a massive spin-2 state, with the other states playing the role of Stückelberg fields. This is fortunate since it implies that the Lagrangian derived for this mass level will give the desired Lagrangian for a spin-2 particle. Secondly, the Euler-Lagrange equations can be triturated to give a Fierz-Pauli system which has a simple form. It was shown in [21] that this system could be obtained directly by the Virasoro algebra and has been generalised for fields with integer spin greater than 2. Finally, this Fierz-Pauli system is consistent in different dimensions, including 4 dimensions. In particular, a redefinition of the fields allows to see it from the equations obtained from the dimensional reduction of the Argyres-Nappi Lagrangian.

An ansatz has been proposed by Porrati and Rahman in [22] for a Lagrangian describing a charged massive state with spin-3/2 propagating in a constant electromagnetic background. In front of the different terms of this Lagrangian, coefficients, functions of the electromagnetic field strength, are introduced which satisfy recursive equations that can then be solved order by order in the electromagnetic field strength (divided by powers of the particle mass to make dimensionless quantities). This proves the existence of a Lagrangian and the equations of motion sought, but it remains to be seen whether a compact and explicit expression exists for them. On the other hand, a consistent example of a

charged massive spin-3/2 Lagrangian is known in supergravity $\mathcal{N} = 2$ where the gravitino is charged under $U(1)$ -graviphoton. But in the zero cosmological constant limit, the causality of the equations of motion in this model implies a Planckian particle mass [23].

An obvious question is then whether an effective Lagrangian calculation can be performed for massive superstring modes, similar to the one for bosonic strings described above, and what results can be derived. Such work has been done in [1] using open superstring field theory in a constant electromagnetic background, for a four-dimensional compactification preserving the $\mathcal{N} = 1$ supersymmetry. A four-dimensional superspace action for the first massive level states was obtained and the equations of motion were derived in the Lorenz gauge.

Compared to the bosonic case, the supersymmetric case is more complex. The first level of massive states does not only contain a state with spin-2 but also states with lower spin. All the corresponding fields appeared coupled both in the equations of motion and in the constraints. It is the same for fermionic states with the appearance of couplings between the spin-3/2 and spin-1/2 fields. One of the aims of the present work is to study whether these equations can be decoupled. It will be shown that this is indeed the case.

The main purpose of this work is to write the effective Lagrangian bilinear in the fields representing the massive first level states of the superstring. The action is already written in [1] for the superfields, but here we want to have the expression with only physical fields without the auxiliary fields. It is straightforward, though tedious, to develop this action into components. Then it is necessary to make appropriate, not always obvious, choices of redefining the fields to lead to useful forms of the final Lagrangian. It is therefore useful to present these steps in some detail.

Obviously, it is a priori convenient to be guided by the simplest case without electromagnetic field for which the final Lagrangian is known: Fierz-Pauli for the spin-2 field, Rarita-Schwinger for the spin-3/2 fields and the free Lagrangians of the spin-1/2 and 0 fields. The action in the superspace of the first massive state and the corresponding equations of motion have already been obtained in [24]. However, the transition from this Lagrangian to the Fierz-Pauli and Rarita-Schwinger Lagrangians has never been performed. In fact, [25] found it simpler to start from the Fierz-Pauli bosonic Lagrangian and supersymmetrise it than to start from the Lagrangian of [24] and show how it describes the spin-2. This is because of the large number of Stückelberg and auxiliary fields involved. We will first remedy this situation. We will show how the Fierz-Pauli and Rarita-Schwinger Lagrangians can be obtained from superstring field theory, and incidentally, we will be able to identify through the necessary redefinition of the physical fields the purpose of various other fields, such as the Stückelberg or auxiliaries.

The original problem posed in the 1930s was to find the equations of motion and constraints, *i.e.* a Fierz-Pauli system, governing the propagation of charged particles with spin greater than 1 [3]. In this work, we will recover for spin-2 the Argyres-Nappi result, but we will also be able to write explicit equations describing the case with spin-3/2, perhaps our main result. The other problem is to write a Lagrangian describing these systems, the variational principle being originally introduced only as a means to find a consistent system of equations, has become the main subject of research. On this point, our

results are not totally satisfactory. Indeed, one would have hoped that the electromagnetic field strength would only introduce deformations of the Fierz-Pauli and Rarita-Schwinger Lagrangians. However, our results show that it also introduces a coupling between fields of different spins. To be more precise, our results consist of several forms of Lagrangians, corresponding to various redefinitions of the physical fields and choices of integration order of the Stückelberg and auxiliary fields. One of these forms is a deformation of Fierz-Pauli and Rarita-Schwinger but long and containing higher order derivatives. Other forms we present are more compact and manageable, but do not automatically identify with the known free Lagrangians in the absence of electromagnetic fields. Despite our many attempts, we have not found a form where the fields with different spins are decoupled, without being able to definitively exclude this possibility.

This work is organised as follows. In section 2, we consider the superspace action in the vanishing limit of the electromagnetic field, corresponding to that derived in [24] for the neutral case, and develop it into components. We show how all non-propagating degrees of freedom are eliminated. An appropriate field redefinition is performed to recover Fierz-Pauli and Rarita-Schwinger Lagrangians. In section 3, the electromagnetic background is turned on, giving rise to the superspace action for charged states in [1]. We exhibit an on-shell redefinition of the superfields that decouples the original equations of motion and constraints obtained in [1]. Next, in sections 4 and 5, we proceed to expand this action separately for the bosonic and fermionic fields. After simplification, the bosonic and fermionic Lagrangians are presented in two forms, firstly a compact form and secondly a deformation of the Fierz-Pauli or Rarita-Schwinger Lagrangian. From these Lagrangians, we derive explicitly the equations of motion as well as the constraints for the spin-3/2 and spin-2 states. In section 6, we draw our conclusions. Finally, conventions and useful results are detailed in the appendices.

Part of the tensor calculation is carried out with the help of the `xAct` package [26] for `Mathematica`. In this work, we use natural units. Moreover, we take the usual string theory convention $\alpha' = 1/2$. An arbitrary mass parameter has been explicitly restored in the letters [27, 28] that contain some of our main results.

2 Superspace action in absence of electromagnetic background

We start by studying the much simpler case of the Lagrangian for neutral fields.

2.1 The superfields

We are interested in the fields corresponding to the states of the first massive level of the open superstring, compactified on a Calabi-Yau space, thus with a four-dimensional $\mathcal{N} = 1$ supersymmetry. The use of the hybrid formalism for open superstring field theory [29] allows to obtain the corresponding superspace action. This was done in the neutral case in [24, 30] and recently for charged fields in [1].¹ The superspace action in the neutral case reads

¹We follow the same conventions as in [1], the only difference being a global minus sign in front of the action.

$$\begin{aligned}
S_{\text{free}} = & -\frac{1}{16} \int d^4x p_0^2 \bar{p}_0^2 \{ V^m [-\{d_0^2, \bar{d}_0^2\} V_m + 16\Pi_0^n \Pi_{n0} V_m - 32V_m \\
& + 16\bar{\sigma}_m^{\dot{\alpha}\alpha} (d_{\alpha 0} \bar{V}_{\dot{\alpha}} - \bar{d}_{\dot{\alpha}0} V_\alpha) + 64\Pi_{m0} \mathcal{B} + 48\bar{\sigma}_m^{\dot{\alpha}\alpha} [\bar{d}_{\dot{\alpha}0}, d_{\alpha 0}] \mathcal{C}] \\
& + V^\alpha [8\bar{d}_{\dot{\alpha}0} d_{\alpha 0} \bar{V}^{\dot{\alpha}} - 4\bar{d}_0^2 V_\alpha + 2d_{\alpha 0} \bar{d}_0^2 (-2i\mathcal{B} + 18\mathcal{C}) - 96i\Pi_{\alpha\dot{\alpha}0} \bar{d}_0^{\dot{\alpha}} \mathcal{C}] \\
& + \bar{V}_{\dot{\alpha}} [-4d_0^2 \bar{V}^{\dot{\alpha}} + 2\bar{d}_0^{\dot{\alpha}} d_0^2 (2i\mathcal{B} + 18\mathcal{C}) - 96i\Pi_0^{\dot{\alpha}\alpha} d_{\alpha 0} \mathcal{C}] \\
& + \mathcal{B} [\{d_0^2, \bar{d}_0^2\} \mathcal{B} - 64\mathcal{B} + 6i [d_0^2, \bar{d}_0^2] \mathcal{C}] + 3\mathcal{C} [11 \{d_0^2, \bar{d}_0^2\} \mathcal{C} - 128\Pi_0^n \Pi_{n0} \mathcal{C} + 64\mathcal{C}] \} \\
\end{aligned} \tag{2.1}$$

where p_0, \bar{p}_0 are the derivatives with regard to the Grassmann coordinates θ and $\bar{\theta}$, respectively. The superderivatives $d_{\alpha 0}, \bar{d}_{\dot{\alpha}0}$ act as

$$\begin{aligned}
d_{\alpha 0} &= \frac{\partial}{\partial \theta^\alpha} + i(\sigma^m \bar{\theta})_\alpha \partial_m \\
\bar{d}_{\dot{\alpha}0} &= -\frac{\partial}{\partial \bar{\theta}^{\dot{\alpha}}} - i(\theta \sigma^m)_{\dot{\alpha}} \partial_m
\end{aligned} \tag{2.2}$$

whereas Π_0^m reduces here to the usual partial derivative: $\Pi_0^m \equiv -\partial^m$. The real superfields $V_m, \mathcal{B}, \mathcal{C}$ and the spinor superfield V_α can be expanded in components as

$$\begin{aligned}
V_m &= C_m + i(\theta \chi_m) - i(\bar{\theta} \bar{\chi}_m) + i(\theta \theta) M_m - i(\bar{\theta} \bar{\theta}) \bar{M}_m + (\theta \sigma^n \bar{\theta}) h_{mn} \\
&+ i(\theta \theta) (\bar{\theta} \bar{\lambda}_m) - i(\bar{\theta} \bar{\theta}) (\theta \lambda_m) + (\theta \theta) (\bar{\theta} \bar{\theta}) D_m \\
\mathcal{B} &= \varphi + i(\theta \gamma) - i(\bar{\theta} \bar{\gamma}) + i(\theta \theta) N - i(\bar{\theta} \bar{\theta}) \bar{N} + (\theta \sigma^m \bar{\theta}) c_m \\
&+ i(\theta \theta) (\bar{\theta} \bar{\rho}) - i(\bar{\theta} \bar{\theta}) (\theta \rho) + (\theta \theta) (\bar{\theta} \bar{\theta}) G \\
\mathcal{C} &= \phi + i(\theta \xi) - i(\bar{\theta} \bar{\xi}) + i(\theta \theta) M - i(\bar{\theta} \bar{\theta}) \bar{M} + (\theta \sigma^m \bar{\theta}) a_m \\
&+ i(\theta \theta) (\bar{\theta} \bar{\psi}) - i(\bar{\theta} \bar{\theta}) (\theta \psi) + (\theta \theta) (\bar{\theta} \bar{\theta}) D \\
V_\alpha &= v_\alpha + \theta_\alpha s - (\sigma^{mn} \theta)_\alpha s_{mn} + (\sigma^m \bar{\theta})_\alpha w_m + (\theta \theta) \eta_\alpha + (\bar{\theta} \bar{\theta}) \zeta_\alpha + (\theta \sigma^m \bar{\theta}) r_{m\alpha} \\
&+ (\theta \theta) (\sigma^m \bar{\theta})_\alpha q_m + (\bar{\theta} \bar{\theta}) \theta_\alpha t - (\bar{\theta} \bar{\theta}) (\sigma^{mn} \theta)_\alpha t_{mn} + (\theta \theta) (\bar{\theta} \bar{\theta}) \mu_\alpha \\
\bar{V}^{\dot{\alpha}} &= \bar{v}^{\dot{\alpha}} + \bar{\theta}^{\dot{\alpha}} \bar{s} - (\bar{\sigma}^{mn} \bar{\theta})^{\dot{\alpha}} \bar{s}_{mn} - (\bar{\sigma}^m \theta)^{\dot{\alpha}} \bar{w}_m + (\bar{\theta} \bar{\theta}) \bar{\eta}^{\dot{\alpha}} + (\theta \theta) \bar{\zeta}^{\dot{\alpha}} + (\theta \sigma^m \bar{\theta}) \bar{r}_{m\dot{\alpha}} \\
&- (\bar{\theta} \bar{\theta}) (\bar{\sigma}^m \theta)^{\dot{\alpha}} \bar{q}_m + (\theta \theta) \bar{\theta}^{\dot{\alpha}} \bar{t} - (\theta \theta) (\bar{\sigma}^{mn} \bar{\theta})^{\dot{\alpha}} \bar{t}_{mn} + (\theta \theta) (\bar{\theta} \bar{\theta}) \bar{\mu}^{\dot{\alpha}}
\end{aligned} \tag{2.3}$$

Many of the component fields are auxiliary and must be eliminated via redefinition to obtain a Lagrangian containing only the physical fields, *i.e.* those representing propagating degrees of freedom. Note that here s_{mn} and t_{mn} are self-dual anti-symmetric tensors, namely, they satisfy:

$$\begin{aligned}
\varepsilon^{mnpq} t_{pq} &= -2i t^{mn}, & \varepsilon^{mnpq} \bar{t}_{pq} &= 2i \bar{t}^{mn} \\
\varepsilon^{mnpq} s_{pq} &= -2i s^{mn}, & \varepsilon^{mnpq} \bar{s}_{pq} &= 2i \bar{s}^{mn}
\end{aligned} \tag{2.4}$$

where ε_{mnpq} is the Levi-Civita tensor. For future convenience, we introduce the following real combinations:

$$\begin{aligned}
\tau_1 &\equiv t + \bar{t}, & \tau_2 &\equiv i(t - \bar{t}) \\
\omega_{1m} &\equiv w_m + \bar{w}_m, & \omega_{2m} &\equiv i(w_m - \bar{w}_m) \\
\tau_{1mn} &\equiv t_{mn} + \bar{t}_{mn}, & \tau_{2mn} &\equiv i(t_{mn} - \bar{t}_{mn})
\end{aligned} \tag{2.5}$$

where τ_{1mn} and τ_{2mn} are not independent, as Eq. (2.4) implies

$$\varepsilon^{mnpq}\tau_{1pq} = -2\tau_2^{mn}, \quad \varepsilon^{mnpq}\tau_{2pq} = 2\tau_1^{mn}. \quad (2.6)$$

Here h_{mn} is a generic rank 2 real tensor which can be decomposed into

$$h_{mn} = v_{mn} + f_{mn} + \frac{1}{4}\eta_{mn}h \quad (2.7)$$

where v_{mn} is symmetric and traceless. After redefinition, it contains the degrees of freedom of a massive spin-2 field. f_{mn} is anti-symmetric, and h is the trace of h_{mn} .

In bosonic string field theory, the action of the string field is obtained from the worldsheet correlator $\langle VQ_{\text{BRST}}V \rangle$, where V is the vertex operator associated to a string state and Q_{BRST} the BRST charge. The nilpotency of Q_{BRST} implies that this action is invariant under a gauge transformation of the form $\delta V = Q_{\text{BRST}}\Lambda$, with Λ the gauge parameter. Since we are using the hybrid formalism for the superstring [31], we have the BRST-like charges $\{G^+, \tilde{G}^+\}$ and the superstring field theory action, $\langle VG^+\tilde{G}^+V \rangle$, is then invariant under a gauge transformation of the form $\delta V = G^+\Lambda + \tilde{G}^+\tilde{\Lambda}$. For the string states we consider here, V is a linear combination of the superfields appearing in the action (2.1). The gauge transformations that leave this action invariant can then be written as [1] :

$$\begin{aligned} \delta V^m &= -4i\sigma_{\alpha\dot{\alpha}}^m \bar{d}_0^{\dot{\alpha}} E^\alpha - 4i\sigma_{\alpha\dot{\alpha}}^m d_0^\alpha \bar{E}^{\dot{\alpha}}, \\ \delta V_\alpha &= -2d_0^2 \Pi_{\alpha\dot{\alpha}0} \bar{E}^{\dot{\alpha}} + \frac{i}{2} d_0^2 \bar{d}_0^2 E_\alpha + 16iE_\alpha, \\ \delta \mathcal{B} &= -\frac{1}{2} (\bar{d}_{\dot{\alpha}0} d_0^2 \bar{E}^{\dot{\alpha}} + d_0^\alpha \bar{d}_0^2 E_\alpha), \\ \delta \mathcal{C} &= \frac{i}{2} (\bar{d}_{\dot{\alpha}0} d_0^2 \bar{E}^{\dot{\alpha}} - d_0^\alpha \bar{d}_0^2 E_\alpha). \end{aligned} \quad (2.8)$$

where E_α and its conjugate $\bar{E}^{\dot{\alpha}}$ are the superfields of gauge parameters. They will be expanded as:

$$\begin{aligned} E_\alpha &= \Lambda_{1\alpha} + \theta_\alpha \Lambda_2 - (\sigma^{mn}\theta)_\alpha \Lambda_{2mn} + (\sigma^m\bar{\theta})_\alpha \Lambda_{3m} + (\theta\theta)\Lambda_{4\alpha} + (\bar{\theta}\bar{\theta})\Lambda_{5\alpha} + (\theta\sigma^m\bar{\theta})\Lambda_{6m\alpha} \\ &\quad + (\theta\theta)(\sigma^m\bar{\theta})_\alpha \Lambda_{7m} + (\bar{\theta}\bar{\theta})\theta_\alpha \Lambda_8 - (\bar{\theta}\bar{\theta})(\sigma^{mn}\theta)_\alpha \Lambda_{8mn} + (\theta\theta)(\bar{\theta}\bar{\theta})\Lambda_{9\alpha} \\ \bar{E}^{\dot{\alpha}} &= \bar{\Lambda}_1^{\dot{\alpha}} + \bar{\theta}^{\dot{\alpha}} \bar{\Lambda}_2 - (\bar{\sigma}^{mn}\bar{\theta})^{\dot{\alpha}} \bar{\Lambda}_{2mn} - (\bar{\sigma}^m\theta)^{\dot{\alpha}} \bar{\Lambda}_{3m} + (\bar{\theta}\bar{\theta})\bar{\Lambda}_4^{\dot{\alpha}} + (\theta\theta)\bar{\Lambda}_5^{\dot{\alpha}} + (\theta\sigma^m\bar{\theta})\bar{\Lambda}_{6m}^{\dot{\alpha}} \\ &\quad - (\bar{\theta}\bar{\theta})(\bar{\sigma}^m\theta)^{\dot{\alpha}} \bar{\Lambda}_{7m} + (\theta\theta)\bar{\theta}^{\dot{\alpha}} \bar{\Lambda}_8 - (\theta\theta)(\bar{\sigma}^{mn}\bar{\theta})^{\dot{\alpha}} \bar{\Lambda}_{8mn} + (\theta\theta)(\bar{\theta}\bar{\theta})\bar{\Lambda}_9^{\dot{\alpha}} \end{aligned} \quad (2.9)$$

and, for convenience, we define the real gauge parameters

$$\begin{aligned} \hat{\Lambda}_{3m} &\equiv \Lambda_{3m} + \bar{\Lambda}_{3m}, & \tilde{\Lambda}_{3m} &\equiv i(\Lambda_{3m} - \bar{\Lambda}_{3m}) \\ \hat{\Lambda}_8 &\equiv \Lambda_8 + \bar{\Lambda}_8, & \tilde{\Lambda}_8 &\equiv i(\Lambda_8 - \bar{\Lambda}_8) \\ \hat{\Lambda}_{8mn} &\equiv \Lambda_{8mn} + \bar{\Lambda}_{8mn}, & \tilde{\Lambda}_{8mn} &\equiv i(\Lambda_{8mn} - \bar{\Lambda}_{8mn}) \end{aligned} \quad (2.10)$$

where $\hat{\Lambda}_{8mn}$ and $\tilde{\Lambda}_{8mn}$ follow relations analogous to those of Eq. (2.6), since Λ_{8mn} is self-dual.

In the next subsections, we will first develop the action and the components gauge transformations, for bosons and fermions, respectively. Then, we will perform suitable field

redefinitions in the Lagrangian to eliminate the gauge parameters. In [1, 30], the *Lorenz gauge* is fixed on shell, requiring for example $d_0^2 \mathcal{C} = 0$. Here, we will instead adopt the *unitary gauge*, which is more appropriate for our purpose of keeping the physical fields only; it is then systematic to work out the corresponding field redefinitions at the Lagrangian level. In the end, both gauge choices, up to some additional field redefinitions, lead to the same (decoupled) on-shell equations for the fields with spin-3/2 and with spin-2, as we will show in section 3.

2.2 Bosons

We start with the bosonic part of the action, and we present in detail the manipulations which allow to recover the Fierz-Pauli action for a massive field with spin-2.

2.2.1 Gauge transformations

The gauge transformations (2.8) lead for the case of the bosonic components of the superfields to:

Fields in \mathcal{B} :

$$\begin{aligned}
\delta c_m &= -2\partial_m \partial^n \hat{\Lambda}_{3n} + 4\varepsilon_{mnpq} \partial^n \hat{\Lambda}_8^{pq} + 2\partial^2 \hat{\Lambda}_{3m} \\
\delta N &= \delta \bar{N} = 0 \\
\delta G &= \frac{1}{2} \partial^2 \partial^m \tilde{\Lambda}_{3m} + \partial^2 \hat{\Lambda}_8 \\
\delta \varphi &= -2\partial^m \tilde{\Lambda}_{3m} - 4\hat{\Lambda}_8
\end{aligned} \tag{2.11}$$

Fields in \mathcal{C} :

$$\begin{aligned}
\delta a_m &= -2\partial_m \partial^n \tilde{\Lambda}_{3n} + 8\partial^n \hat{\Lambda}_{8mn} + 2\partial^2 \tilde{\Lambda}_{3m} \\
\delta M &= \delta \bar{M} = 0 \\
\delta D &= -\frac{1}{2} \partial^2 \partial^m \hat{\Lambda}_{3m} + \partial^2 \tilde{\Lambda}_8 \\
\delta \phi &= 2\partial^m \hat{\Lambda}_{3m} - 4\tilde{\Lambda}_8
\end{aligned} \tag{2.12}$$

Fields in V_m :

$$\begin{aligned}
\delta M_m &= -4i\partial_m \Lambda_2 - 8i\partial^n \Lambda_{2mn} - 8\Lambda_{7m} \\
\delta D_m &= 4\partial_m \hat{\Lambda}_8 + 8\partial^n \hat{\Lambda}_{8mn} \\
\delta C_m &= -8\tilde{\Lambda}_{3m} \\
\delta h &= -8\partial^m \hat{\Lambda}_{3m} + 32\tilde{\Lambda}_8 \\
\delta v_{mn} &= 4(\partial_m \hat{\Lambda}_{3n} + \partial_n \hat{\Lambda}_{3m}) - 2\eta_{mn} \partial^k \hat{\Lambda}_{3k} \\
\delta f_{mn} &= 4\varepsilon_{mnpq} \partial^p \tilde{\Lambda}^{3q} + 16\tilde{\Lambda}_{8mn}
\end{aligned} \tag{2.13}$$

Fields in $V_\alpha, \bar{V}^{\dot{\alpha}}$:

$$\begin{aligned}
\delta \omega_{1m} &= 8\partial^2 \tilde{\Lambda}_{3m} - 8\partial_m \partial^n \tilde{\Lambda}_{3n} + 16\tilde{\Lambda}_{3m} + 32\partial^n \hat{\Lambda}_{8mn} \\
\delta \omega_{2m} &= -8\partial_m \partial^n \hat{\Lambda}_{3n} - 16\hat{\Lambda}_{3m} + 16\partial_m \tilde{\Lambda}_8 \\
\delta \tau_1 &= -4\partial^2 \partial^m \hat{\Lambda}_{3m} + 16\tilde{\Lambda}_8 + 8\partial^2 \tilde{\Lambda}_8, \quad \delta \tau_2 = -16\hat{\Lambda}_8 \\
\delta \tau_{2mn} &= 8(\partial_m \partial^k \hat{\Lambda}_{8nk} - \partial_n \partial^k \hat{\Lambda}_{8mk}) + 2\partial^2 (\partial_m \tilde{\Lambda}_{3n} - \partial_n \tilde{\Lambda}_{3m}) - 16\hat{\Lambda}_{8mn} \\
\delta q_m &= 16i\Lambda_{7m}, \quad \delta s = 16i\Lambda_2, \quad \delta s_{mn} = 16i\Lambda_{2mn}
\end{aligned} \tag{2.14}$$

while the gauge transformation of τ_{1mn} is determined by $\varepsilon^{mnpq}\delta\tau_{2pq} = 2\delta\tau_1^{mn}$. Note that all bosonic gauge parameters appear algebraically in the transformations at least once. It follows that the unitary gauge eliminates all “gauge degrees of freedom” from the Lagrangian.

2.2.2 The Bosonic Lagrangian

The bosonic part of the neutral Lagrangian can be separated into two decoupled parts which do not involve the same gauge parameters, namely each of them is separately gauge invariant.

$$\mathcal{L}_B \equiv \mathcal{L}_1 + \mathcal{L}_2 \quad (2.15)$$

In the first place, we have²

$$\begin{aligned} \mathcal{L}_1 = & -6\bar{M}(4+3\partial^2)M + 6\left[(2iM_m + q_m)\partial^m\bar{M} - i\left(N + \frac{1}{2}s\right)\partial^2\bar{M} + \text{h.c.}\right] \\ & + 4M_m\bar{M}^m + [M_m(4\partial^m\bar{N} + \partial^m\bar{s} + 2\partial_n\bar{s}^{mn} + 2i\bar{q}^m) + \text{h.c.}] \\ & - \frac{1}{4}s\partial^2\bar{s} + q_m\bar{q}^m - 2\bar{N}(\partial^2 - 4)N - (2iq_m\partial^m\bar{N} + s\partial^2\bar{N} + \text{h.c.}) \\ & + (\partial_k s^{mk})(\partial^n\bar{s}_{mn}) - \frac{1}{2}i(q_m\partial^m\bar{s} + 2q_m\partial_n\bar{s}^{mn} + \text{h.c.}) \end{aligned} \quad (2.16)$$

The first step is to integrate M_m , with

$$M_m = \frac{1}{4}[-4\partial_m N - \partial_m s - 2\partial^n s_{mn} + 2i(6\partial_m M + q_m)] \quad (2.17)$$

The integration of M_m also eliminates s , s_{mn} and q_m . In particular, s could be seen as a Stückelberg field for M_m , in the sense that a redefinition $M_m \rightarrow M'_m - \frac{1}{4}\partial_m s$ makes s disappear completely from the Lagrangian. Note however that in the gauge transformations of our Lagrangian, $(M_m + \frac{1}{4}\partial_m s)$ itself is not gauge invariant because other fields interfere with its gauge transformation. For the remaining fields we have $\delta M = \delta N = 0$, so the Lagrangian is totally gauge-fixed. It reads,

$$\begin{aligned} \mathcal{L}_1 = & -\frac{1}{2}M_1(12-9\partial^2)M_1 - \frac{1}{2}M_2(12-9\partial^2)M_2 - 3M_2\partial^2N_1 + 3M_1\partial^2N_2 \\ & + \frac{1}{2}N_1(4+\partial^2)N_1 + \frac{1}{2}N_2(4+\partial^2)N_2 \end{aligned} \quad (2.18)$$

where we defined the real and imaginary parts of $\{M, N\}$ as

$$\begin{aligned} N_1 &= N + \bar{N}, & N_2 &= i(N - \bar{N}) \\ M_1 &= M + \bar{M}, & M_2 &= i(M - \bar{M}) \end{aligned} \quad (2.19)$$

After the redefinition

$$N_1 \rightarrow \frac{1}{4}N_1 + 3M_2, \quad M_1 \rightarrow \frac{1}{12}M_1 - \frac{N_2}{3} \quad (2.20)$$

²The possibility of integration by parts in the Lagrangian, *i.e.* that the boundary contributions from the total derivatives vanish, is assumed throughout this work.

M_2 and N_2 lose their kinetic terms (*i.e.* become auxiliary). Then integrating them out, we obtain a Lagrangian of two free massive real scalars N_1, M_1 :

$$\mathcal{L}_1 = \frac{1}{2}M_1(-2 + \partial^2)M_1 + \frac{1}{2}N_1(-2 + \partial^2)N_1 \quad (2.21)$$

On the other hand, \mathcal{L}_2 contains only real fields, with

$$\begin{aligned} \mathcal{L}_2 = & -\frac{1}{2}v^{mn}(2 - \partial^2)v_{mn} + \frac{1}{2}\partial^n v_{mn}\partial_k v^{mk} + \partial^n v_{mn}\partial_k f^{mk} - \frac{1}{2}f^{mn}(2 - \partial^2)f_{mn} + \partial^n v_{mn}\omega_2^m \\ & + \frac{1}{2}\partial^n f_{mn}\partial_k f^{mk} - 2c^m(\partial^n f_{mn} - \partial^n v_{mn}) + \varepsilon_{mnpq}f^{mn}\tau_2^{pq} - (\partial_n\partial_m v^{mn})\left(\frac{1}{4}h + 6\phi\right) \\ & - 2c^m c_m - \frac{1}{2}(\partial^m c_m)^2 + \frac{1}{8}\omega_2^m\partial^2\omega_{2m} - \frac{1}{8}(\partial_m\omega_2^m)^2 - \partial^m c_m\left(6D + \frac{1}{2}h + \frac{3}{2}\partial^2\phi - \tau_1\right) \\ & + \partial^m\omega_{2m}\left(3D - \frac{1}{2}\partial_n c^n + \frac{1}{4}h - \frac{9}{4}\partial^2\phi - \frac{1}{2}\tau_1\right) - \Omega^m\left(6a_m - \frac{1}{2}\omega_{1m}\right) - 66D^2 \\ & - 3D(4h - 6\tau_1 + 8\phi - 5\partial^2\phi) - \frac{1}{4}h\left(1 - \frac{3}{8}\partial^2\right)h - \tau_1^2 - \tau_1\left(\frac{3}{2}\partial^2\phi - h\right) - \frac{33}{8}\phi\partial^4\phi \\ & + \frac{3}{2}h\partial^2\phi + 2D_m^2 - \frac{1}{8}\varphi\partial^4\varphi - G(-8\varphi + \partial^2\varphi) - 2G^2 - \frac{1}{8}\omega_1^m\partial^2\omega_{1m} - \frac{1}{8}(\partial^m\omega_{1m})^2 \\ & - D^m(12a_m - 4C_m + \partial^2C_m - 4\partial_m\varphi - 2\omega_{1m}) - \varphi\left(\frac{1}{2}\partial^2\tau_2 - \frac{3}{2}\partial^2\partial^m a_m + \frac{1}{4}\partial^2\partial^m\omega_{1m}\right) \\ & - G(-6\partial^m a_m + 4\partial^m C_m + \partial^m\omega_{1m} + 2\tau_2) + 6(\partial^m a_m)(\partial^n C_n) - \tau_2(\partial^m C_m - 3\partial^m a_m) \\ & + \omega_1^m(2\partial^n\tau_{2mn} + 3\partial^2 a_m) - \frac{1}{2}\tau_2\partial^m\omega_{1m} + \frac{9}{2}(\partial^m a_m)(\partial^n\omega_{1n}) + \tau_{2mn}\tau_2^{mn} \\ & - 2\partial^n\tau_{2mn}(6a^m - C^m) + \frac{1}{8}C^m\partial^4 C_m - 6a_m(2\partial^2 - 1)a^m - \frac{33}{2}a_m a^m + 3a_m\partial^2 C^m \end{aligned} \quad (2.22)$$

Notice that the fields $D, D_m, G, \tau_1, \tau_2, \tau_{2mn}$ are auxiliary, and can be integrated out before the gauge fixing process. The scalar field φ disappears subsequently.³ We are now left with

$$\begin{aligned} \mathcal{L}_2 = & 6(a^m\partial^2 a_m + \partial^m a_m\partial^n a_n - 2a^m a_m) + \left(\frac{3}{2}C^m\partial^2 C_m - 2C^m C_m + 2\partial^m C_m\partial^n C_n\right) \\ & + 6(2a^m C_m - a^m\partial^2 C_m - \partial^m a_m\partial^n C_n) - C^m\Omega_m - 2\left(\frac{1}{5}\partial^m c_m\partial^n c_n + c^m c_m\right) \\ & + \frac{1}{8}(3\omega_1^m\partial^2\omega_{1m} + \omega_2^m\partial^2\omega_{2m}) - \frac{1}{2}\omega_1^m\omega_{1m} + \frac{1}{2}\partial^m\omega_{1m}\partial^n\omega_{1n} - \frac{1}{10}\partial^m\omega_{2m}\partial^n\omega_{2n} \\ & + \partial^m\omega_{1m}(2\partial^n C_n - 3\partial^n a_n) + \omega_1^m\left(6a_m - 3\partial^2 a_m - 2C_m + \frac{3}{2}\partial^2 C_m - \frac{1}{2}\Omega_m\right) \\ & - \frac{6}{5}(3\phi\partial^4\phi - \phi\partial^2\phi + 8\phi^2) + \left(\frac{3}{32}h\partial^2 h - \frac{3}{20}h^2\right) + \left(\frac{9}{10}\phi\partial^2 h - \frac{12}{5}h\phi\right) \end{aligned} \quad (2.23)$$

³Without going into the details of the calculation, it can be seen from the fact that φ is not an independent degree of freedom but a pure gauge field associated with the gauge transformation with the parameter $\hat{\Lambda}_8$. The other fields transforming with $\hat{\Lambda}_8$ are $\{G, D_m, \tau_2, \tau_{2mn}\}$. When these fields are integrated, φ must naturally disappear from the Lagrangian.

$$\begin{aligned}
& + 6\phi \left(\frac{2}{5} \partial^m c_m - \frac{2}{5} \partial^2 \partial^m c_m - \frac{1}{5} \partial^m \omega_{2m} - \frac{3}{10} \partial^2 \partial^m \omega_{2m} - \partial^m \partial^n v_{mn} \right) \\
& + h \left(\frac{3}{10} \partial^m c_m - \frac{3}{20} \partial^m \omega_{2m} - \frac{1}{4} \partial^m \partial^n v_{mn} \right) + 2c^m (\partial^n v_{mn} - \partial^n f_{mn}) - \frac{3}{5} \partial^m c_m \partial^n \omega_{2n} \\
& + \omega_2^m \partial^n v_{mn} + \frac{1}{2} \partial^n f_{mn} \partial_k f^{mk} + \frac{1}{2} \partial^n v_{mn} \partial_k v^{mk} + \partial^n v_{mn} \partial_k f^{mk} + \frac{1}{2} v^{mn} (\partial^2 - 2) v_{mn}
\end{aligned}$$

with the dual field strength of f_{mn} given by

$$\Omega_m \equiv \varepsilon_{mnpq} \partial^n f^{pq} \quad (2.24)$$

The Lagrangian Eq. (2.23) is our starting point here. It is obviously not in a desirable form as it contains terms with more than two derivatives as well as several non-propagating, pure gauge or auxiliary fields. To find the gauge-fixed Lagrangian, it is useful to construct gauge invariant field combinations:

$$\begin{aligned}
\delta \left(h + 8\phi + 2\partial^2 \phi + \frac{1}{2} \partial_m \omega_2^m \right) &= 0 \\
\delta \left(v_{mn} + \frac{1}{4} \partial_m \omega_{2n} + \frac{1}{4} \partial_n \omega_{2m} - \frac{1}{8} \eta_{mnp} \partial_k \omega_2^k + 2\partial_m \partial_n \phi - \frac{1}{2} \eta_{mnp} \partial^2 \phi \right) &= 0 \\
\delta \left(c_m + \frac{1}{2} \partial^n f_{mn} + \frac{1}{8} \partial^2 \omega_{2m} - \frac{1}{8} \partial_m \partial_n \omega_2^n \right) &= 0 \\
\delta \left(C_m + \frac{1}{2} \omega_{1m} - \frac{1}{2} \Omega_m \right) &= 0 \\
\delta \left(a_m - \frac{1}{2} C_m - \frac{1}{4} \omega_{1m} \right) &= 0
\end{aligned} \quad (2.25)$$

Note that ω_{2m} and ϕ can be considered as Stückelberg fields for the symmetric part of h_{mn} (describing a field with spin-2). This can be seen from the following definition of a new massive field, which no longer has a transformation under gauge symmetry:

$$\tilde{h}_{mn} \equiv v_{mn} + \frac{1}{4} \eta_{mnp} (h + 8\phi) + \frac{1}{4} \partial_m (\omega_{2n} + 4\partial_n \phi) + \frac{1}{4} \partial_n (\omega_{2m} + 4\partial_m \phi), \quad \delta \tilde{h}_{mn} = 0 \quad (2.26)$$

It is then straightforward to eliminate the Stückelberg fields by setting the unitary gauge, so that eventually ϕ and ω_{2m} will disappear from the Lagrangian. By a similar reasoning, we can algebraically gauge away ω_{1m} and f_{mn} . More precisely, the unitary gauge is fixed by the two-step redefinition:

1. $\omega_{1m} \rightarrow \omega_{1m} + 4a_m, \omega_{2m} \rightarrow \omega_{2m} - 4\partial_m \phi$

As a consequence, the vector fields ω_{1m}, ω_{2m} become pure gauges, whose new gauge transformations are $\delta \omega_{1m} = 16\tilde{\Lambda}_{3m}, \delta \omega_{2m} = -16\hat{\Lambda}_{3m}$, without dependence of the other gauge parameters. The next step consists in redefining the other fields so that these two gauge parameters $\{\hat{\Lambda}_{3m}, \tilde{\Lambda}_{3m}\}$ do not appear in their transformations, and so when ω_{1m} and ω_{2m} become the only fields depending on $\tilde{\Lambda}_{3m}$ and $\hat{\Lambda}_{3m}$, they can be eliminated, *i.e.* algebraically gauged away.

$$\begin{aligned}
2. \quad c_m &\rightarrow c_m - \frac{1}{2}\partial^n f_{mn} - \frac{1}{8}\partial^2 \omega_{2m} + \frac{1}{8}\partial_m \partial_n \omega_2^m \\
h &\rightarrow h - 8\phi - \frac{1}{2}\partial^m \omega_{2m}, \quad v_{mn} \rightarrow v_{mn} + \frac{1}{8}\eta_{mn}\partial^k \omega_{2k} - \frac{1}{4}\partial_m \omega_{2n} - \frac{1}{4}\partial_n \omega_{2m} \\
C_m &\rightarrow C_m - \frac{1}{2}\omega_{1m}, \quad a_m \rightarrow a_m + \frac{1}{4}\varepsilon_{mnpq}\partial^n f^{pq}
\end{aligned}$$

The resulting Lagrangian is totally gauge-fixed: $\delta a_m = \delta C_m = \delta c_m = \delta v_{mn} = \delta h = 0$. We can furthermore decouple C_m by the shift

$$a_m \rightarrow a_m - \frac{1}{2}C_m \quad (2.27)$$

which gives rise to

$$\begin{aligned}
\mathcal{L}_2 &= \frac{3}{2}C^m (\partial^2 - 2) C_m + \frac{3}{2}\partial^m C_m \partial^n C_n + 4a^m a_m + 2\partial^m a_m \partial^n a_n \\
&\quad - 2c^m c_m - \frac{2}{5}\partial^m c_m \partial^n c_n - \frac{3}{20}h^2 + \frac{3}{32}h\partial^2 h - v^{mn}v_{mn} + \frac{3}{10}h\partial^m c_m \\
&\quad + 2c^m \partial^n v_{mn} - \frac{1}{4}h\partial^m \partial^n v_{mn} + \frac{1}{2}\partial^n v_{mn}\partial_k v^{mk} + \frac{1}{2}v^{mn}\partial^2 v_{mn}
\end{aligned} \quad (2.28)$$

After normalisation $C_m \rightarrow C_m/\sqrt{3}$, C_m has a Proca Lagrangian for massive spin-1 particles with equation of motion and constraint:

$$(\partial^2 - 2) C_m = 0, \quad \partial^m C_m = 0 \quad (2.29)$$

The Lagrangian of a_m seems to have a wrong sign in front of the mass term, and a kinetic term of the form $a^m \partial^2 a_m$ is absent. In fact, the equation of motion of a_m is

$$2a_m = \partial^n \partial_m a_n \quad (2.30)$$

taking its divergence, we have

$$(\partial^2 - 2)\partial^m a_m = 0 \quad (2.31)$$

The equation (2.30) also implies $\partial_m a_n - \partial_n a_m = 0$, namely, a_m is curl-free and can be written as the gradient of a scalar. Therefore, a_m has only one longitudinal degree of freedom. In effect, a_m is precisely the *Curtright-Freund field* [32] describing the dual theory of a free massive scalar.⁴ This duality has been shown in [32] by introducing a totally anti-symmetric tensor v_{abc} and a parent Lagrangian

$$\mathcal{L}_{\text{parent}} = \frac{1}{6}\varepsilon^{mnlk}v_{mnlk}\partial_l \phi + \frac{1}{12}(v_{mnlk})^2 + \phi^2 \quad (2.32)$$

Both fields are auxiliary and can be integrated out. If one eliminates v_{mnlk} , a free massive scalar is recovered, otherwise, one finds a Lagrangian of a new vector $v^m \equiv \frac{1}{6}\varepsilon^{mnlk}v_{nkl}$ which is proportional to $(\partial^m v_m)^2 + 2v_m^2$. Alternatively, a straightforward way to convert the vector into the dual scalar, is to add a decoupled auxiliary scalar field A to the a_m Lagrangian, and the new Lagrangian is physically equivalent:

$$\mathcal{L} = a^m a_m + \frac{1}{2}(\partial^m a_m)^2 - \frac{1}{2}A^2 \quad (2.33)$$

⁴For a detailed study on the duality between tensors of different ranks, see for example [33].

Then do the field redefinition

$$A \rightarrow A + \partial^m a_m \quad (2.34)$$

The $(\partial^m a_m)^2$ term is shifted away, and we are able to integrate out a_m , yielding the Lagrangian of a free massive scalar. After rescaling A , we have

$$\mathcal{L} = \frac{1}{2}A(\partial^2 - 2)A \quad (2.35)$$

Conversely, the Lagrangian of a_m can be recovered from (2.35) by adding to the latter an auxiliary vector, with a single mass term $B^m B_m$, then shifting B_m by $\partial_m A$ allows to eliminate A 's kinetic term, and integrating out A , we obtain $B_m B^m + \frac{1}{2}(\partial^m B_m)^2$.

2.2.3 Recovering the Fierz-Pauli Lagrangian

The second and third lines of (2.28) provide terms of the Lagrangian of a massive spin-2 field. However, we do not recognize the Fierz-Pauli Lagrangian because of the couplings of v_{mn} to the c_m vector field.

To acquire a better understanding of these couplings, we shall investigate this Lagrangian on shell. The equations of motion are

$$\begin{aligned} (2 - \partial^2)v_{mn} &= -(\partial_m c_n + \partial_n c_m) - \frac{1}{2}(\partial_m \partial^k v_{nk} + \partial_n \partial^k v_{mk}) - \frac{1}{4}\partial_m \partial_n h \\ \left(\partial^2 - \frac{8}{5}\right)h &= \frac{4}{3}\partial_m \partial_n v^{mn} - \frac{8}{5}\partial_m c^m \\ c_m &= \frac{1}{5}\partial_m \partial^n c_n - \frac{3}{40}\partial_m h + \frac{1}{2}\partial^n v_{mn} \end{aligned} \quad (2.36)$$

They imply

$$\begin{aligned} h &= -4\partial_m c^m, \quad \partial^n v_{mn} = 0 \\ (\partial^2 - 2)c_m &= 0, \quad (\partial^2 - 2)v_{mn} = 0 \\ (\partial^2 - 2)h &= 0, \quad c_m = -\frac{1}{8}\partial_m h \end{aligned} \quad (2.37)$$

The constraints $c_m = -\frac{1}{8}\partial_m h$ and $\partial^n v_{mn} = 0$ remove 8 degrees of freedom, meaning that $\{v_{mn}, h, c_m\}$ count in total 6 degrees of freedom on shell. Taking into account the to-be spin-2 particle which itself has 5 degrees of freedom, the remaining field then must be a scalar. In fact, the above equations can be decoupled by introducing a new symmetric rank-2 tensor, whose trace is shifted by the divergence of c_m :

$$\mathfrak{h}_{mn} \equiv v_{mn} + \eta_{mn}\left(\frac{1}{4}h + \partial_k c^k\right) \quad (2.38)$$

After rewriting (2.37), the field \mathfrak{h}_{mn} is found to satisfy a Fierz-Pauli system of equation of motion and constraints:

$$\begin{aligned} (\partial^2 - 2)c_m &= 0, \quad 2c_m - \partial_m \partial_n c^n = 0 \\ (\partial^2 - 2)\mathfrak{h}_{mn} &= 0, \quad \partial^n \mathfrak{h}_{mn} = 0, \quad \mathfrak{h}^m_m = 0 \end{aligned} \quad (2.39)$$

Clearly, c_m has the same equation of motion as a_m , thus is dual to a massive scalar. However, at the Lagrangian level, a naive redefinition of the field $h \rightarrow h - 4\partial^m c_m$ to absorb the divergence in the trace does not decouple c_m , and in addition gives rise to higher derivative couplings.

To deal with this issue, the same technique (2.33)-(2.34) can be employed. We add to (2.28) an auxiliary scalar term $+\frac{2}{5}B^2$, then shift B with

$$B \rightarrow B + \partial^m c_m - \frac{3}{8}h \quad (2.40)$$

where the second term cancels the kinetic term of c_m , and the third one is to decouple h and c_m on shell. As a result, c_m becomes auxiliary, with equation of motion

$$c_m = \frac{1}{2}\partial^n v_{mn} - \frac{1}{5}\partial_m B \quad (2.41)$$

Integrating out c_m and also substituting a_m with its dual massive scalar A , the Lagrangian \mathcal{L}_2 becomes

$$\begin{aligned} \mathcal{L}_2 = & \frac{1}{2}C^m (\partial^2 - 2) C_m + \frac{1}{2}\partial^m C_m \partial^n C_n + \frac{1}{2}A (\partial^2 - 2) A + \frac{2}{5}B^2 - \frac{2}{25}B\partial^2 B \\ & - \frac{3}{10}hB + \frac{2}{5}B\partial^m \partial^n v_{mn} - \frac{3}{32}h^2 + \frac{3}{32}h\partial^2 h - \frac{1}{4}h\partial^m \partial^n v_{mn} \\ & - v^{mn}v_{mn} + \frac{1}{2}v^{mn}\partial^2 v_{mn} + \partial^n v_{mn}\partial_k v^{mk} \end{aligned} \quad (2.42)$$

Only physical degrees of freedom remain in the Lagrangian, but the new scalar B is not yet decoupled. This will be remedied by an additional field redefinition

$$B \rightarrow \frac{5\sqrt{2}}{4}B + \frac{15}{8}h, \quad h \rightarrow h + 2\sqrt{2}B \quad (2.43)$$

Finally, we express the spin-2 field in terms of a symmetric tensor $h'_{mn} \equiv v_{mn} + \frac{1}{4}\eta_{mn}h$, and the bosonic Lagrangian is written in the following

$$\begin{aligned} \mathcal{L}_B = & \frac{1}{2}h'^{mn} (\partial^2 - 2) h'_{mn} - \frac{1}{2}h (\partial^2 - 2) h + h'_{mn}\partial^m \partial^n h + \partial^n h'_{mn}\partial_k h'^{mk} \\ & + \frac{1}{2}C^m (\partial^2 - 2) C_m + \frac{1}{2}(\partial^m C_m)^2 + \frac{1}{2}A (\partial^2 - 2) A + \frac{1}{2}B (\partial^2 - 2) B \\ & + \frac{1}{2}M_1 (\partial^2 - 2) M_1 + \frac{1}{2}N_1 (\partial^2 - 2) N_1 \end{aligned} \quad (2.44)$$

One recognises the Fierz-Pauli Lagrangian in the first line, complemented with a set of decoupled real scalars $\{A, B, M_1, N_1\}$ and one massive vector C_m . This Lagrangian of the bosonic sector contains 12 degrees of freedom on shell as it should.

2.2.4 Summary

The bosonic action after expansion of the superfields into components has 80 degrees of freedom off-shell. Several degrees of freedom are non-physical:

- Auxiliary fields, which are integrated out before starting to perform appropriate field redefinitions.
- Gauge degrees of freedom, that are totally fixed by the unitary gauge.
- Non propagating fields as the transverse components of a_m, c_m .

After getting rid of these, further field redefinitions are needed to decouple fields in the Lagrangian.

For \mathcal{L}_1 , the auxiliary field M_m is integrated out yielding (2.18). The real scalars N_2, M_2 are first rendered auxiliary by the redefinition

$$N_1 \rightarrow \frac{1}{4}N_1 + 3M_2, \quad M_1 \rightarrow \frac{1}{12}M_1 - \frac{N_2}{3} \quad (2.45)$$

then integrated out to give:

$$\mathcal{L}_1 = \frac{1}{2}M_1 (-2 + \partial^2) M_1 + \frac{1}{2}N_1 (-2 + \partial^2) N_1 \quad (2.46)$$

For \mathcal{L}_2 , the auxiliary fields $\{D, D_m, G, \tau_1, \tau_2, \tau_{2mn}\}$ are integrated first, which leads to (2.23). The process of fixing the gauge amounts to the following one-step redefinitions

$$\begin{aligned} \omega_{1m} &\rightarrow \omega_{1m} + 2a_m + \Omega_m - \frac{2}{\sqrt{3}}C_m, & \omega_{2m} &\rightarrow \omega_{2m} - 4\partial_m\phi \\ c_m &\rightarrow c_m - \frac{1}{2}\partial^n f_{mn} - \frac{1}{8}\partial^2\omega_{2m} + \frac{1}{8}\partial_m\partial_n\omega_2^n \\ h &\rightarrow h - 8\phi - \frac{1}{2}\partial^m\omega_{2m}, & v_{mn} &\rightarrow v_{mn} + \frac{1}{8}\eta_{mn}\partial^k\omega_{2k} - \frac{1}{4}\partial_m\omega_{2n} - \frac{1}{4}\partial_n\omega_{2m} \\ C_m &\rightarrow \frac{1}{\sqrt{3}}C_m - \frac{1}{2}\omega_{1m}, & a_m &\rightarrow \frac{1}{2}a_m + \frac{1}{4}\Omega_m - \frac{1}{2\sqrt{3}}C_m \end{aligned} \quad (2.47)$$

which take us to

$$\begin{aligned} \mathcal{L}_2 &= \frac{1}{2}C^m (\partial^2 - 2) C_m + \frac{1}{2}\partial^m C_m \partial^n C_n + a^m a_m + \frac{1}{2}\partial^m a_m \partial^n a_n - \frac{1}{2}A^2 + \frac{2}{5}B^2 \\ &\quad - 2c^m c_m - \frac{2}{5}\partial^m c_m \partial^n c_n - \frac{3}{20}h^2 + \frac{3}{32}h\partial^2 h - v^{mn}v_{mn} + \frac{3}{10}h\partial^m c_m \\ &\quad + 2c^m \partial^n v_{mn} - \frac{1}{4}h\partial^m \partial^n v_{mn} + \frac{1}{2}\partial^n v_{mn} \partial_k v^{mk} + \frac{1}{2}v^{mn} \partial^2 v_{mn} \end{aligned} \quad (2.48)$$

when auxiliary scalars A, B are added. Then, making the field redefinitions

$$A \rightarrow A + \partial^m a_m, \quad B \rightarrow B + \partial^m c_m - \frac{3}{8}h \quad (2.49)$$

and eliminating a_m, c_m , we are left with their dual scalars. The last redefinition

$$B \rightarrow \frac{5\sqrt{2}}{4}B + \frac{15}{8}h, \quad h \rightarrow h + 2\sqrt{2}B \quad (2.50)$$

gets us in the end to the decoupled bosonic Lagrangian (2.44) with a Fierz-Pauli part for the massive spin-2 field.

2.3 Fermions

This subsection is structured in a similar way to the corresponding bosonic part 2.2. First, we present below the gauge transformations under which the fermionic Lagrangian is invariant. Then, we will perform a series of redefinitions and fix the gauge to be the unitary gauge, thus eliminating the non-physical fields to end up with the desired Lagrangian where the field with spin-3/2 is described by a Rarita-Schwinger Lagrangian.

2.3.1 Gauge transformations

The gauge transformations (2.8) lead for the case of the fermionic components of the superfields to:

Fields in \mathcal{B} :

$$\begin{aligned}\delta\gamma_\alpha &= -i\partial^2\Lambda_{1\alpha} - 4(\sigma^m\partial_m\bar{\Lambda}_5)_\alpha + 2\partial^m\Lambda_{6m\alpha} - 4i\Lambda_{9\alpha} \\ \delta\rho_\alpha &= \frac{1}{2}(\sigma^m\partial_m\partial^2\bar{\Lambda}_1)_\alpha + 2i\partial^2\Lambda_{5\alpha} - i(\sigma^m\partial_m\partial_n\bar{\Lambda}_6^n)_\alpha + 2(\sigma^m\partial_m\bar{\Lambda}_9)_\alpha\end{aligned}\quad (2.51)$$

Fields in \mathcal{C} :

$$\begin{aligned}\delta\xi_\alpha &= \partial^2\Lambda_{1\alpha} + 4i(\sigma^m\partial_m\bar{\Lambda}_5)_\alpha + 2i\partial^m\Lambda_{6m\alpha} + 4\Lambda_{9\alpha} \\ \delta\psi_\alpha &= -\frac{1}{2}i(\sigma^m\partial_m\partial^2\bar{\Lambda}_1)_\alpha - 2\partial^2\Lambda_{5\alpha} - (\sigma^m\partial_m\partial_n\bar{\Lambda}_6^n)_\alpha - 2i(\sigma^m\partial_m\bar{\Lambda}_9)_\alpha\end{aligned}\quad (2.52)$$

Fields in V_m :

$$\begin{aligned}\delta\chi_{m\alpha} &= -4[2\sigma_m\bar{\Lambda}_5 + \sigma^n\bar{\sigma}_m(\Lambda_{6n} - i\partial_n\Lambda_1)]_\alpha \\ \delta\lambda_{m\alpha} &= -4i(\sigma_n\bar{\sigma}_m\partial^n\Lambda_5)_\alpha + 8(\sigma_m\bar{\Lambda}_9)_\alpha + 2i\left(\sigma^n\bar{\sigma}^k\sigma_m\partial_k\bar{\Lambda}_{6n}\right)_\alpha\end{aligned}\quad (2.53)$$

Fields in $V_\alpha, \bar{V}^{\dot{\alpha}}$:

$$\begin{aligned}\delta v_\alpha &= 2i(8\Lambda_{1\alpha} + \partial^2\Lambda_{1\alpha}) - 8(\sigma_m\partial^m\bar{\Lambda}_5)_\alpha - 4\partial^m\Lambda_{6m\alpha} + 8i\Lambda_{9\alpha} \\ \delta\eta_\alpha &= 16i\Lambda_{4\alpha} \\ \delta\zeta_\alpha &= -2(\sigma_m\partial^m\partial^2\bar{\Lambda}_1)_\alpha + 8i(2\Lambda_{5\alpha} + \partial^2\Lambda_{5\alpha}) + 4i(\sigma_m\partial^m\partial^n\bar{\Lambda}_{6n})_\alpha - 8(\sigma_m\partial^m\bar{\Lambda}_9)_\alpha \\ \delta r_{m\alpha} &= 2\partial^2\partial_m\Lambda_{1\alpha} + 8i(\sigma^n\partial_n\partial_m\bar{\Lambda}_5)_\alpha + 4i(4\Lambda_{6m\alpha} + \partial_m\partial^n\Lambda_{6n\alpha}) + 8\partial_m\Lambda_{9\alpha} \\ \delta\mu_\alpha &= \frac{1}{2}i\partial^4\Lambda_{1\alpha} - 2(\sigma_m\partial^m\partial^2\bar{\Lambda}_5)_\alpha - \partial^2\partial^m\Lambda_{6m\alpha} + 2i(8\Lambda_{9\alpha} + \partial^2\Lambda_{9\alpha})\end{aligned}\quad (2.54)$$

It will be useful to note that the following field combinations are gauge invariant:

$$\delta\left(\rho_\alpha + \frac{1}{2}i(\sigma^m\partial_m\bar{\gamma})_\alpha\right) = 0, \quad \delta\left(\psi_\alpha + \frac{1}{2}i(\sigma^m\partial_m\bar{\xi})_\alpha\right) = 0\quad (2.55)$$

Before developing the superfields into their components in the action, it is already possible to glimpse some characteristics of the fermionic Lagrangian. To begin with, we note that the gauge transformation $\delta\eta_\alpha$ is algebraic in a gauge parameter that does not occur anywhere else, so η_α will not appear in the Lagrangian. The remaining fields share the gauge parameters $\{\Lambda_{1\alpha}, \Lambda_{5\alpha}, \Lambda_{6m\alpha}, \Lambda_{9\alpha}\}$ as well as their Hermitian conjugates, so it follows that we will be able to eliminate three spin-1/2 and one spin-3/2 field per gauge. More precisely,

we have (i) $\delta v_\alpha, \delta \gamma_\alpha, \delta \xi_\alpha, \delta \lambda_{m\alpha}, \delta \mu_\alpha$ are algebraic in $\Lambda_{9\alpha}$ (ii) $\delta \chi_{m\alpha}, \delta \zeta_\alpha$ are algebraic in $\Lambda_{5\alpha}$ (iii) $\delta r_{m\alpha}$ and $\delta \bar{\chi}_{m\alpha}$ are algebraic in $\Lambda_{6\alpha}$ (iv) δv_α is algebraic in $\Lambda_{1\alpha}$. Our strategy is the same as in the bosonic case: we gauge away algebraically $v_\alpha, r_{m\alpha}, \zeta_\alpha, \xi_\alpha$, and in the gauge-fixed Lagrangian, we integrate out auxiliary degrees of freedom. Finally, we perform further redefinitions necessary to decouple the physical fermions. Our final Lagrangian contains the spin-3/2 $\{\chi_m, \bar{\lambda}_m\}$ as well as the spin-1/2 $\{\gamma, \bar{\psi}\}$ fields. This differs from [1], where ξ instead of γ is kept as a physical fermion. Of course, here and in the case of charged fields, either choice leads to an equivalent Lagrangian related to the other by field redefinitions. In the same way, one can also eliminate χ_m as a pure gauge instead of r_m .

2.3.2 The fermionic Lagrangian

The expansion of the superfields into components and the integration over the Grassmannian coordinates lead to the following Lagrangian for fermionic fields:

$$\begin{aligned}
-\mathcal{L}_F = & i(\lambda^m \sigma^n \partial_n \bar{\lambda}_m) + \frac{i}{4} (\bar{\chi}^m \bar{\sigma}^n \partial_n \partial^2 \chi_m) - \frac{1}{2} [(\lambda^m \partial^2 \chi_m) + (\bar{\chi}^m \partial^2 \bar{\lambda}_m)] + 2[\lambda^m \chi_m + \bar{\chi}^m \bar{\lambda}_m] \\
& - \frac{33}{4} i [(\bar{\xi} \bar{\sigma}^m \partial_m \partial^2 \xi) + 4(\psi \sigma^m \partial_m \bar{\psi})] + \frac{15}{2} [(\psi \partial^2 \xi) + (\bar{\xi} \partial^2 \bar{\psi})] - 12[(\psi \xi) + (\bar{\xi} \bar{\psi})] \\
& + 3 \left[i(\chi^m \partial_m \psi) - i(\lambda^m \partial_m \xi) + 2(\lambda^m \sigma_m \bar{\psi}) + \frac{1}{2}(\chi^m \sigma_m \partial^2 \bar{\xi}) + \text{h.c.} \right] \\
& - 6 \left[i(\partial^m \psi \sigma_{mn} \chi^n) + \frac{1}{2}(\chi^m \sigma^n \partial_m \partial_n \bar{\xi}) + i(\lambda^m \sigma_{mn} \partial^n \xi) + \text{h.c.} \right] \\
& + \frac{3}{4} i [(v \partial^2 \psi) - (\bar{\psi} \partial^2 \bar{v})] + \frac{9}{8} [(v \sigma^m \partial_m \partial^2 \bar{\xi}) + \text{h.c.}] + 9i [(\mu \psi) - (\bar{\psi} \bar{\mu})] \\
& + \frac{3}{2} [(\mu \sigma^m \partial_m \bar{\xi}) + \text{h.c.}] - 3i [(\zeta \partial^2 \xi) - (\bar{\xi} \partial^2 \bar{\zeta})] - 6 [(\zeta \sigma^m \partial_m \bar{\psi}) + \text{h.c.}] \\
& + \frac{3}{4} [i(r_m \sigma^m \partial^2 \bar{\xi}) - 3i(r_m \sigma^n \partial_n \partial^m \bar{\xi}) - 4(r_m \sigma^{mn} \partial_n \psi) + 4(r_m \partial^m \psi) + \text{h.c.}] \\
& + \frac{1}{2} [(\lambda^m \partial_m v) + 2(\lambda_m \sigma^{mn} \partial_n v) - 2i(\chi^m \sigma_m \bar{\mu}) + \text{h.c.}] \\
& + \frac{1}{2} [-(\chi^m \partial_m \zeta) - 2(\chi_m \sigma^{mn} \partial_n \zeta) + 2i(\lambda^m \sigma_m \bar{\zeta}) + \text{h.c.}] + \frac{1}{4} [(\chi^m \sigma^n \partial_n \bar{r}_m) \\
& \quad + (\bar{\chi}^m \bar{\sigma}^n \partial_n r_m) + (\bar{\chi}^m \bar{\sigma}^n \partial_m r_n) - i \epsilon_{mlkn} (\chi^m \sigma^l \partial^k \bar{r}^n) - 2i(\lambda_m \sigma^n \bar{\sigma}^m r_n) + \text{h.c.}] \\
& + \frac{1}{2} [(v \partial^2 \zeta) + (\bar{v} \partial^2 \bar{\zeta})] + 2[(\mu \zeta) + (\bar{\zeta} \bar{\mu})] + i[(\zeta \partial^m r_m) + (\bar{r}^m \partial_m \bar{\zeta})] \\
& + \frac{1}{8} [-4i(v \sigma^m \partial_m \bar{\mu}) + (r_m \sigma^m \partial^2 \bar{v}) - 2(v \sigma^m \partial_m \partial_n \bar{r}^n) + \text{h.c.}] - i(\zeta \sigma^m \partial_m \bar{\zeta}) \\
& - \frac{1}{2} [(r_m \sigma^m \bar{\mu}) + \text{h.c.}] - \frac{1}{8} i [(r_m \sigma^k \bar{\sigma}^n \sigma^m \partial_k \bar{r}_n) + (r_m \sigma^k \bar{\sigma}^m \sigma^n \partial_n \bar{r}_k)] \\
& + \frac{3}{2} \left[\left(\rho + \frac{1}{2} i \bar{\gamma} \bar{\sigma}^m \partial_m \right) (i \partial^2 \xi - 2 \sigma^n \partial_n \bar{\psi}) + \text{h.c.} \right] + 2 [(\chi^m \partial_m \rho) + (\lambda^m \partial_m \gamma) + \text{h.c.}] \\
& + \left[\left(\mu - \frac{1}{4} \partial^2 v \right) \left(\rho + \frac{1}{2} i \sigma^m \partial_m \bar{\gamma} \right) + \text{h.c.} \right] + \left[i r_m \sigma^{mn} \partial_n \left(\rho + \frac{1}{2} i \sigma^k \partial_k \bar{\gamma} \right) + \text{h.c.} \right] \\
& - i \left(\rho + \frac{1}{2} i \bar{\gamma} \bar{\sigma}^m \partial_m \right) \sigma^n \partial_n \left(\bar{\rho} + \frac{1}{2} i \bar{\sigma}^k \partial_k \gamma \right) + 4[(\gamma \rho) + (\bar{\rho} \bar{\gamma})]
\end{aligned} \tag{2.56}$$

As expected, the η_α component is absent. Moreover, μ appears as a Lagrange multiplier, giving rise to a constraint that will be applied at the end of the gauge fixing. Another feature of (2.56) is the presence of higher derivative terms, requiring an additional redefinition of the fields. We proceed in several steps:

1. Eliminate higher derivative kinetic terms of $\chi_{m\alpha}$:

$$\bar{\lambda}_m^{\dot{\alpha}} \rightarrow \bar{\lambda}_m^{\dot{\alpha}} + \frac{1}{2}i(\bar{\sigma}^n \partial_n \chi_m)^{\dot{\alpha}}$$

2. Algebraically gauge away v_α , which is in effect the Stückelberg field of r_m :

$$\mu_\alpha \rightarrow \mu_\alpha + \frac{1}{4}\partial^2 v_\alpha, \quad r_{m\alpha} \rightarrow r_{m\alpha} + i\partial_m v_\alpha$$

3. Eliminate higher derivative kinetic terms of ξ_α :

$$\zeta_\alpha \rightarrow \zeta_\alpha - 2(\sigma^m \partial_m \bar{\xi})_\alpha, \quad r_{m\alpha} \rightarrow r_{m\alpha} + 4\partial_m \xi_\alpha, \quad \psi_\alpha \rightarrow \psi_\alpha - \frac{1}{2}i(\sigma^m \partial_m \bar{\xi})_\alpha$$

4. Algebraically gauge away $r_{m\alpha}$:

$$\chi_{m\alpha} \rightarrow \chi_{m\alpha} + \frac{1}{4}i(\sigma_n \bar{\sigma}_m r^n)_\alpha, \quad \mu_\alpha \rightarrow \mu_\alpha - \frac{1}{2}i\partial^m r_{m\alpha}$$

5. Algebraically gauge away ζ_α :

$$\begin{aligned} \chi_{m\alpha} &\rightarrow \chi_{m\alpha} - \frac{1}{2}i(\sigma_m \bar{\zeta})_\alpha, \quad \mu_\alpha \rightarrow \mu_\alpha + i(\sigma^m \partial_m \bar{\zeta})_\alpha \\ \lambda_{m\alpha} &\rightarrow \lambda_{m\alpha} + \partial_m \zeta_\alpha, \quad \gamma_\alpha \rightarrow \gamma_\alpha - \frac{1}{2}i(\sigma^m \partial_m \bar{\zeta})_\alpha, \quad \rho_\alpha \rightarrow \rho_\alpha + \frac{1}{4}\partial^2 \zeta_\alpha \end{aligned}$$

6. Eliminate higher derivative kinetic terms of γ_α :

$$\rho_\alpha \rightarrow \rho_\alpha - \frac{1}{2}i(\sigma^m \partial_m \bar{\gamma})_\alpha$$

7. Algebraically gauge away ξ_α :

$$\gamma_\alpha \rightarrow \gamma_\alpha - i\xi_\alpha, \quad \mu_\alpha \rightarrow \mu_\alpha + 4i\xi_\alpha, \quad \lambda_{m\alpha} \rightarrow \lambda_{m\alpha} + 2(\sigma_m \bar{\xi})_\alpha$$

Overall, all these steps amount to a redefinition:

$$\begin{aligned} \lambda_{m\alpha} &\rightarrow \lambda_{m\alpha} + \frac{1}{2}i(\sigma^n \partial_n \bar{\chi}_m)_\alpha + \frac{1}{8}(\sigma^k \bar{\sigma}^n \sigma_m \partial_k \bar{r}_n)_\alpha + \frac{1}{4}(\sigma^n \bar{\sigma}_m \partial_n \zeta)_\alpha + \partial_m \zeta_\alpha + 2(\sigma_m \bar{\xi})_\alpha \\ \mu_\alpha &\rightarrow \mu_\alpha + \frac{1}{4}\partial^2 v_\alpha - \frac{1}{2}i\partial^m r_{m\alpha} + i(\sigma^m \partial_m \bar{\zeta})_\alpha + 4i\xi_\alpha \\ r_{m\alpha} &\rightarrow r_{m\alpha} + i\partial_m v_\alpha + 4\partial_m \xi_\alpha \\ \zeta_\alpha &\rightarrow \zeta_\alpha - 2(\sigma^m \partial_m \bar{\xi})_\alpha \\ \psi_\alpha &\rightarrow \psi_\alpha - \frac{1}{2}i(\sigma^m \partial_m \bar{\xi})_\alpha \\ \chi_{m\alpha} &\rightarrow \chi_{m\alpha} + \frac{1}{4}i(\sigma_n \bar{\sigma}_m r^n)_\alpha - \frac{1}{2}i(\sigma_m \bar{\zeta})_\alpha \\ \gamma_\alpha &\rightarrow \gamma_\alpha - \frac{1}{2}i(\sigma^m \partial_m \bar{\zeta})_\alpha - i\xi_\alpha \\ \rho_\alpha &\rightarrow \rho_\alpha + \frac{1}{4}\partial^2 \zeta_\alpha - \frac{1}{2}i(\sigma^m \partial_m \bar{\gamma})_\alpha + \frac{1}{2}(\sigma^m \partial_m \bar{\xi})_\alpha \end{aligned} \tag{2.57}$$

We obtain, subsequently, a gauge-fixed Lagrangian exempt from higher derivatives:

$$\begin{aligned}
\mathcal{L}_F = & -i(\lambda^m \sigma^n \partial_n \bar{\lambda}_m) - 2i(\bar{\chi}^m \bar{\sigma}^n \partial_n \chi_m) - 2[(\lambda^m \chi_m) + (\bar{\chi}^m \bar{\lambda}_m)] \\
& + 33i(\psi \sigma^m \partial_m \bar{\psi}) - [6i(\chi^m \partial_m \psi) + 6(\lambda^m \sigma_m \bar{\psi}) + \text{h.c.}] \\
& - 9i[(\mu \psi) - (\bar{\psi} \bar{\mu})] - [i(\mu \sigma_m \bar{\chi}^m) + \text{h.c.}] - [(\mu \rho) + \text{h.c.}] \\
& + i(\rho \sigma^m \partial_m \bar{\rho}) - 4[(\gamma \rho) + (\bar{\rho} \bar{\gamma})] + 4i(\gamma \sigma^m \partial_m \bar{\gamma}) \\
& + 3[(\rho \sigma^m \partial_m \bar{\psi}) + \text{h.c.}] - 2[(\chi^m \partial_m \rho) + (\lambda^m \partial_m \gamma) + \text{h.c.}]
\end{aligned} \tag{2.58}$$

μ_α as a Lagrange multiplier implies the additional constraint:

$$\rho_\alpha = -9i\psi_\alpha - i(\sigma_m \bar{\chi}^m)_\alpha \tag{2.59}$$

which can be used to eliminate ρ_α . The Lagrangian takes then the form:

$$\begin{aligned}
\mathcal{L}_F = & -i(\lambda^m \sigma^n \partial_n \bar{\lambda}_m) - i(\bar{\chi}_m \bar{\sigma}^n \sigma^k \bar{\sigma}^m \partial_k \chi_n) - 2[(\lambda^m \chi_m) + (\bar{\chi}^m \bar{\lambda}_m)] \\
& + 60i(\psi \sigma^m \partial_m \bar{\psi}) - [6i(\chi^m \sigma_n \bar{\sigma}_m \partial^n \psi) + 6(\lambda^m \sigma_m \bar{\psi}) + \text{h.c.}] \\
& + 4i(\gamma \sigma^m \partial_m \bar{\gamma}) + 36[i(\psi \gamma) + \text{h.c.}] + 4[i(\gamma \sigma^m \bar{\chi}_m) + \text{h.c.}] \\
& - 2[(\lambda^m \partial_m \gamma) + \text{h.c.}]
\end{aligned} \tag{2.60}$$

2.3.3 Recovering the Rarita-Schwinger Lagrangian

A quick investigation shows that (2.60) describes on-shell 12 fermionic degrees of freedom. While it is possible, we will not be restricted to the above compact form of the Lagrangian. Same as in the bosonic case, we will show that the different fields can be decoupled in the Lagrangian, and we will put it into a more usual form. For this purpose, we will make the following series of redefinitions, where for each step we present the resulting Lagrangian:

1. Put the kinetic term for $\lambda_{m\alpha}$ in the most usual Rarita-Schwinger form

$$\gamma_\alpha \rightarrow \gamma_\alpha - \frac{1}{2}i(\sigma^m \bar{\lambda}_m)_\alpha, \quad \lambda_{m\alpha} \rightarrow \lambda_{m\alpha} - 2i(\sigma_m \bar{\gamma})_\alpha \tag{2.61}$$

$$\begin{aligned}
\mathcal{L}_F = & -\varepsilon^{mkn\ell}(\lambda_m \sigma_\ell \partial_k \bar{\lambda}_n) - i(\bar{\chi}_m \bar{\sigma}^n \sigma^k \bar{\sigma}^m \partial_k \chi_n) - 4[(\chi^m \sigma_{mn} \lambda^n) + \text{h.c.}] \\
& + 60i(\psi \sigma^m \partial_m \bar{\psi}) - 12i(\gamma \sigma^m \partial_m \bar{\gamma}) - 12[i(\psi \gamma) + \text{h.c.}] \\
& - [6i(\chi^m \sigma_n \bar{\sigma}_m \partial^n \psi) - 12(\lambda^m \sigma_m \bar{\psi}) + 4(\gamma \sigma^{mn} \partial_m \lambda_n) + \text{h.c.}]
\end{aligned} \tag{2.62}$$

2. Recover the correct sign for the kinetic term of ψ_α

$$\chi_{m\alpha} \rightarrow \chi_{m\alpha} - 2(\sigma_m \bar{\psi})_\alpha \tag{2.63}$$

$$\begin{aligned}
\mathcal{L}_F = & -\varepsilon^{mkn\ell}(\lambda_m \sigma_\ell \partial_k \bar{\lambda}_n) - i(\bar{\chi}_m \bar{\sigma}^n \sigma^k \bar{\sigma}^m \partial_k \chi_n) - 4[(\chi^m \sigma_{mn} \lambda^n) + \text{h.c.}] \\
& - 4i(\psi \sigma^m \partial_m \bar{\psi}) - 12i(\gamma \sigma^m \partial_m \bar{\gamma}) - 12[i(\psi \gamma) + \text{h.c.}] \\
& - [6i(\chi^m \sigma_n \bar{\sigma}_m \partial^n \psi) - 8i(\psi \partial^m \chi_m) + 4(\gamma \sigma^{mn} \partial_m \lambda_n) + \text{h.c.}]
\end{aligned} \tag{2.64}$$

3. Put the kinetic term for $\chi_{m\alpha}$ in the most usual Rarita-Schwinger form

$$\lambda_{m\alpha} \rightarrow \lambda_{m\alpha} + i(\sigma^n \partial_m \bar{\chi}_n)_\alpha \quad \psi_\alpha \rightarrow \psi_\alpha + \frac{1}{2}(\sigma^m \bar{\chi}_m)_\alpha \quad (2.65)$$

$$\begin{aligned} \mathcal{L}_F = & -\varepsilon^{mknl}(\lambda_m \sigma_k \partial_n \bar{\lambda}_l) - 2\varepsilon^{mknl}(\chi_m \sigma_k \partial_n \bar{\chi}_l) - 4[(\chi^m \sigma_{mn} \lambda^n) + \text{h.c.}] \\ & - 4i(\psi \sigma^m \partial_m \bar{\psi}) - 12i(\gamma \sigma^m \partial_m \bar{\gamma}) - 12[i(\psi \gamma) + \text{h.c.}] \\ & - [8i(\psi \sigma^{mn} \partial_m \chi_n) + 4(\gamma \sigma^{mn} \partial_m \lambda_n) + 6i(\gamma \sigma^m \bar{\chi}_m) + \text{h.c.}] \end{aligned} \quad (2.66)$$

4. Decouple the spin-3/2 and spin-1/2 fields

$$\lambda_{m\alpha} \rightarrow \lambda_{m\alpha} - i(\sigma_m \bar{\gamma})_\alpha + 2i\partial_m \psi_\alpha \quad (2.67)$$

$$\begin{aligned} \mathcal{L}_F = & -\varepsilon^{mknl}(\lambda_m \sigma_k \partial_n \bar{\lambda}_l) - 2\varepsilon^{mknl}(\chi_m \sigma_k \partial_n \bar{\chi}_l) - 4[(\chi^m \sigma_{mn} \lambda^n) + \text{h.c.}] \\ & - 4i(\psi \sigma^m \partial_m \bar{\psi}) - 18i(\gamma \sigma^m \partial_m \bar{\gamma}) - 12[i(\psi \gamma) + \text{h.c.}] \end{aligned} \quad (2.68)$$

5. Rescale the fermions

$$\bar{\psi}^{\dot{\alpha}} \rightarrow \frac{i}{2}\bar{\psi}^{\dot{\alpha}}, \quad \gamma_\alpha \rightarrow \frac{1}{3\sqrt{2}}\gamma_\alpha, \quad \chi_{m\alpha} \rightarrow \frac{1}{\sqrt{2}}\chi_{m\alpha} \quad (2.69)$$

Overall (2.61)-(2.69) can be combined into a single step:

$$\begin{aligned} \chi_{m\alpha} & \rightarrow \sqrt{2}(\sigma_{mn} \chi^n)_\alpha - i(\sigma_m \bar{\psi})_\alpha \\ \bar{\lambda}_m^{\dot{\alpha}} & \rightarrow \bar{\lambda}_m^{\dot{\alpha}} - \frac{1}{\sqrt{2}}i(\bar{\sigma}_m \gamma)^{\dot{\alpha}} + \partial_m \bar{\psi}^{\dot{\alpha}} + \frac{1}{\sqrt{2}}i(\bar{\sigma}^n \partial_m \chi_n)^{\dot{\alpha}} \\ \gamma_\alpha & \rightarrow \frac{1}{\sqrt{2}}\gamma_\alpha - \frac{i}{2}(\sigma^m \partial_m \bar{\psi})_\alpha - \frac{1}{2}i(\sigma^m \bar{\lambda}_m)_\alpha + \frac{1}{2\sqrt{2}}(\sigma^m \bar{\sigma}^n \partial_m \chi_n)_\alpha \\ \bar{\psi}^{\dot{\alpha}} & \rightarrow \frac{i}{2}\bar{\psi}^{\dot{\alpha}} - \frac{1}{2\sqrt{2}}(\bar{\sigma}^m \chi_m)^{\dot{\alpha}} \end{aligned} \quad (2.70)$$

which results in

$$\begin{aligned} \mathcal{L}_F = & -\varepsilon^{mknl}(\lambda_m \sigma_n \partial_k \bar{\lambda}_l) + \varepsilon^{mknl}(\bar{\chi}_m \bar{\sigma}_n \partial_k \chi_l) - 2\sqrt{2}[(\lambda^m \sigma_{mn} \chi^n) + \text{h.c.}] \\ & - i(\psi \sigma^m \partial_m \bar{\psi}) - i(\gamma \sigma^m \partial_m \bar{\gamma}) - \sqrt{2}[(\psi \gamma) + \text{h.c.}] \end{aligned} \quad (2.71)$$

Eq. (2.71) is the sum of a Rarita-Schwinger Lagrangian for the massive spin-3/2 $(\chi_m, \bar{\lambda}_m)$ and a Dirac Lagrangian for the massive spin-1/2 $(\gamma, \bar{\psi})$. The corresponding equations of motion and constraints are

$$\begin{aligned} i\bar{\sigma}^{n\dot{\alpha}\alpha} \partial_n \gamma_\alpha & = -\sqrt{2}\bar{\psi}^{\dot{\alpha}}, \quad i\sigma_{\alpha\dot{\alpha}}^m \partial_m \bar{\psi}^{\dot{\alpha}} = -\sqrt{2}\gamma_\alpha \\ i\bar{\sigma}^{n\dot{\alpha}\alpha} \partial_n \chi_{m\alpha} & = -\sqrt{2}\bar{\lambda}_m^{\dot{\alpha}}, \quad i\sigma_{\alpha\dot{\alpha}}^n \partial_n \bar{\lambda}_m^{\dot{\alpha}} = -\sqrt{2}\chi_{m\alpha} \\ \bar{\sigma}^{m\dot{\alpha}\alpha} \chi_{m\alpha} & = 0, \quad \partial^m \chi_{m\alpha} = 0, \quad \sigma_{\alpha\dot{\alpha}}^m \bar{\lambda}_m^{\dot{\alpha}} = 0, \quad \partial^m \bar{\lambda}_m^{\dot{\alpha}} = 0 \end{aligned} \quad (2.72)$$

It is useful to observe that the physical degrees of freedom all reside in $\{\mathcal{B}, \mathcal{C}, V_m\}$, with \mathcal{B} and \mathcal{C} each providing a chiral multiplet, $\{\gamma_\alpha, B, N_1\}$ and $\{\bar{\psi}^{\dot{\alpha}}, A, M_1\}$, and V_m providing a massive spin-2 multiplet, $\{h'_{mn}, \chi_{m\alpha}, \bar{\lambda}_m^{\dot{\alpha}}, C_m\}$. When the background is turned on, these multiplets will be complex instead of real in the neutral case here. Moreover, we expect to find the same (but complex) fields corresponding to the physical degrees of freedom. Their Lagrangians then reduce to (2.44), (2.72) when the electromagnetic field is set to zero.

3 Superspace action in the presence of an electromagnetic background

In a constant electromagnetic background, the action in superspace, describing at the bilinear level the fields associated with the first massive level of the open superstring, has been derived in [1] using the four-dimensional hybrid formalism for open superstring field theory [29]. Compared to above, the physical bosons become complex and the number of fermions is doubled, as a result we have 12 *complex* degrees of freedom on-shell for the bosons and for the fermions.

In the bosonic sector, the physical degrees of freedom (d.o.f.'s) correspond to a massive spin-2 (5 d.o.f.'s), a massive vector C_m (3 d.o.f.'s) and four scalars (4 d.o.f.'s), all of which are complex (*i.e.* 12 complex d.o.f.'s). For its part, the fermionic sector includes two massive Dirac fermions of spin-3/2 (8 d.o.f.'s), and two massive Dirac fermions of spin-1/2 (4 d.o.f.'s) (*i.e.* 12 complex d.o.f.'s).

3.1 The superspace action

The action of [1]:

$$\begin{aligned}
S = & -\frac{1}{16} \int d^4 x p_0^2 \bar{p}_0^2 \left\{ V_n^\dagger (\eta^{nm} - i\epsilon^{nm}) [-\{d_0^2, \bar{d}_0^2\} V_m + 16\Pi_0^n \Pi_{n0} V_m - 32(\eta_{mp} - i\epsilon_{mp}) V^p \right. \\
& - 32((\partial\bar{\theta}_0 \bar{d}_0) V_m + (\partial\theta_0 d_0) V_m) + 8\bar{\sigma}_m^{\dot{\alpha}\alpha} (d_{\alpha 0} \bar{U}_{2\dot{\alpha}} - \bar{d}_{\dot{\alpha}0} U_{1\alpha}) + 32\Pi_{m0} \mathcal{B} \\
& + 24\bar{\sigma}_m^{\dot{\alpha}\alpha} [\bar{d}_{\dot{\alpha}0}, d_{\alpha 0}] \mathcal{C} + U_2^\alpha [-8\sigma_{\alpha\dot{\alpha}}^n (\eta_{nm} - i\epsilon_{nm}) \bar{d}_0^{\dot{\alpha}} V^m + 4\bar{d}_{\dot{\alpha}0} d_{\alpha 0} \bar{U}_2^\alpha - 4\bar{d}_0^2 U_{1\alpha} \\
& + d_{\alpha 0} \bar{d}_0^2 (-2i\mathcal{B} + 18\mathcal{C}) + \partial\theta_{\alpha 0} (-32i\mathcal{B} - 96\mathcal{C}) - 48i\Pi_{\alpha\dot{\alpha}0} \bar{d}_0^{\dot{\alpha}} \mathcal{C}] \\
& - \bar{U}_{1\dot{\alpha}} [-8\bar{\sigma}^{n\dot{\alpha}\alpha} (\eta_{nm} - i\epsilon_{nm}) d_{\alpha 0} V^m + 4d_0^2 \bar{U}_2^\alpha - 4d_0^\alpha \bar{d}_0^{\dot{\alpha}} U_{1\alpha} - \bar{d}_0^{\dot{\alpha}} d_0^2 (2i\mathcal{B} + 18\mathcal{C}) \\
& + \partial\bar{\theta}_0^{\dot{\alpha}} (-32i\mathcal{B} + 96\mathcal{C}) + 48i\Pi_0^{\dot{\alpha}\alpha} d_{\alpha 0} \mathcal{C}] + \mathcal{B}^\dagger [-32\Pi_0^n (\eta_{nm} - i\epsilon_{nm}) V^m \\
& + (\{d_0^2, \bar{d}_0^2\} - 64) \mathcal{B} + 3i [d_0^2, \bar{d}_0^2] \mathcal{C} - i(2d_0^2 \bar{d}_{\dot{\alpha}0} + 32\partial\bar{\theta}_{\dot{\alpha}0}) \bar{U}_2^\alpha + i(2\bar{d}_0^2 d_0^\alpha + 32\partial\theta_0^\alpha) U_{1\alpha}] \\
& + 3\mathcal{C}^\dagger [-8\bar{\sigma}^{n\dot{\alpha}\alpha} [d_{\alpha 0}, \bar{d}_{\dot{\alpha}0}] (\eta_{nm} - i\epsilon_{nm}) V^m - (6d_0^\alpha \bar{d}_0^2 + 8i\Pi_0^{\dot{\alpha}\alpha} \bar{d}_{\dot{\alpha}0}) U_{1\alpha} \\
& - (6\bar{d}_{\dot{\alpha}0} d_0^2 + 8i\Pi_{\alpha\dot{\alpha}0} d_0^\alpha) \bar{U}_2^\alpha - [d_0^2, \bar{d}_0^2] i\mathcal{B} \\
& \left. - (-11 \{d_0^2, \bar{d}_0^2\} + 128\Pi_0^n \Pi_{n0} - 256\partial\bar{\theta}_{\dot{\alpha}0} \bar{d}_0^{\dot{\alpha}} - 256\partial\theta_0^\alpha d_{\alpha 0} - 64) \mathcal{C} \right\}
\end{aligned} \tag{3.1}$$

is invariant under the gauge transformations:

$$\begin{aligned}
\delta V^m &= -4i\sigma_{\alpha\dot{\alpha}}^m \bar{d}_0^{\dot{\alpha}} E_1^\alpha - 4i\sigma_{\alpha\dot{\alpha}}^m d_0^\alpha \bar{E}_2^{\dot{\alpha}} \\
\delta \mathcal{B} &= -\frac{1}{2} (\bar{d}_{\dot{\alpha}0} d_0^2 \bar{E}_2^{\dot{\alpha}} + d_0^\alpha \bar{d}_0^2 E_{1\alpha}) \\
\delta \mathcal{C} &= \frac{i}{2} (\bar{d}_{\dot{\alpha}0} d_0^2 \bar{E}_2^{\dot{\alpha}} - d_0^\alpha \bar{d}_0^2 E_{1\alpha}) \\
\delta U_{1\alpha} &= -16i\partial\bar{\theta}_0^{\dot{\alpha}} (\bar{d}_{\dot{\alpha}0} E_{1\alpha} + d_{\alpha 0} \bar{E}_{2\dot{\alpha}}) - 2d_0^2 \Pi_{\alpha\dot{\alpha}0} \bar{E}_2^{\dot{\alpha}} + \frac{i}{2} d_0^2 \bar{d}_0^2 E_{1\alpha} + 16i\Delta_\alpha^\beta E_{1\beta}, \\
\delta \bar{U}_2^{\dot{\alpha}} &= 16i\partial\theta_{\alpha 0} (\bar{d}_0^{\dot{\alpha}} E_1^\alpha + d_0^\alpha \bar{E}_2^{\dot{\alpha}}) + 2\bar{d}_0^2 \Pi_0^{\dot{\alpha}\alpha} E_{1\alpha} - \frac{i}{2} \bar{d}_0^2 d_0^2 \bar{E}_2^{\dot{\alpha}} - 16i\bar{\Delta}_{\dot{\beta}}^{\dot{\alpha}} \bar{E}_2^{\dot{\beta}}
\end{aligned} \tag{3.2}$$

where we introduced the following notations:

$$\begin{aligned}
\alpha_0 &\equiv -i\mathfrak{D}, & \Pi_0^m &= -i\alpha_0^m + \frac{i}{2}\varepsilon^{mnr s}\epsilon_{rs}(\theta\sigma_n\bar{\theta}) \\
(\epsilon\cdot\sigma)_\alpha{}^\beta &= \epsilon_{mn}\sigma^{mn}{}_\alpha{}^\beta, & (\epsilon\cdot\bar{\sigma})^{\dot{\alpha}}{}_{\dot{\beta}} &= \epsilon_{mn}\bar{\sigma}^{mn\dot{\alpha}}{}_{\dot{\beta}} \\
\partial\theta_{\alpha 0} &= -\frac{i}{2}(\epsilon\cdot\sigma)_\alpha{}^\beta\theta_\beta, & \partial\bar{\theta}_{\dot{\alpha} 0} &= \frac{i}{2}(\epsilon\cdot\bar{\sigma})^{\dot{\beta}}{}_{\dot{\alpha}}\bar{\theta}_{\dot{\beta}} \\
\Delta_\alpha{}^\beta &= \delta_\alpha{}^\beta + \frac{i}{2}(\epsilon\cdot\sigma)_\alpha{}^\beta, & \bar{\Delta}^{\dot{\alpha}}{}_{\dot{\beta}} &= \delta^{\dot{\alpha}}{}_{\dot{\beta}} + \frac{i}{2}(\epsilon\cdot\bar{\sigma})^{\dot{\alpha}}{}_{\dot{\beta}} \\
d_{\alpha 0} &= \frac{\partial}{\partial\theta^\alpha} - (\sigma^m\bar{\theta})_\alpha\alpha_{0m} - \frac{i}{2}(\bar{\theta}\bar{\theta})(\epsilon\cdot\sigma)_\alpha{}^\beta\theta_\beta \\
\bar{d}_{\dot{\alpha} 0} &= -\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} + (\theta\sigma^m)_{\dot{\alpha}}\alpha_{0m} + \frac{i}{2}(\theta\theta)(\epsilon\cdot\bar{\sigma})^{\dot{\beta}}{}_{\dot{\alpha}}\bar{\theta}_{\dot{\beta}}
\end{aligned} \tag{3.3}$$

It can be verified that when the electromagnetic field strength cancels, the expression (3.1) leads to the same form for the action as the neutral case given by (2.1), except that now $V_m, \mathcal{B}, \mathcal{C}$ are *complex* superfields, and the spinor superfield V_α is doubled to $\{U_{1\alpha}, \bar{U}_2^{\dot{\alpha}}\}$. This superspace action describes the propagation of an open superstring carrying at its ends the charges q_0 and q_π , hence a total charge $Q = q_0 + q_\pi$. For the constant electromagnetic background, the covariant derivative is given by

$$D_n = (\partial - iQF \cdot X)_n \tag{3.4}$$

where X^n is a space-time coordinate. Dealing with strings, instead of point-like particles, we need to introduce the dressed form of the covariant derivative [15, 18]

$$\mathfrak{D}_m = -i\mathfrak{M}_{mn}D^n, \quad [\mathfrak{D}_m, \mathfrak{D}_n] = i\epsilon_{mn} \tag{3.5}$$

The matrix \mathfrak{M} has the property:

$$\mathfrak{M} \cdot \mathfrak{M}^T = \frac{\epsilon}{QF} \tag{3.6}$$

The stringy origin of the massive states manifests itself in the dependence of ϵ on the constant electromagnetic field strength F_{mn} through [15, 34]

$$\epsilon = \frac{\Lambda^2}{\pi} \left[\operatorname{arctanh}\left(\frac{\pi q_0 F}{\Lambda^2}\right) + \operatorname{arctanh}\left(\frac{\pi q_\pi F}{\Lambda^2}\right) \right] \tag{3.7}$$

with Λ the fundamental (string) scale of the theory.

As noted in [18] for the case of the bosonic open string, and as it appears through our manipulations in this work, the consistency of the Lagrangian and the derivation of the equations of motion make use of the anti-symmetric property of ϵ_{mn} , but nowhere does the explicit dependence of ϵ_{mn} on F_{mn} intervene. Therefore, our analysis continues to be valid if we take everywhere the limit of quantum field theory $\epsilon_{mn} \rightarrow QF_{mn}$ and $\mathfrak{D}_m \rightarrow D_m$. Following this, to ease the presentation, our working configuration can be reformulated, independently of a stringy framework or not, as follows: the superfields are charged under the $U(1)$ of the electromagnetic background, to which we associate a covariant derivative \mathfrak{D}_m , whose commutator gives a constant anti-symmetric tensor ϵ_{mn} , hereafter referred to

as the *electromagnetic field strength*, an obvious abuse of language. For convenience, we will assume that $\{V_m, \mathcal{B}, \mathcal{C}, U_{1\alpha}, \bar{U}_2^{\dot{\alpha}}\}$ carry a positive unit charge, so their conjugates are negatively charged. It is then easy to verify that the action (3.1) is $U(1)$ -invariant. For the covariant derivative we have $[\mathcal{D}_m, \mathcal{D}_n] = iq\epsilon_{mn}$, with $q = \pm 1$. For example, given a positively charged superfield component ϕ , we have

$$[\mathcal{D}_m, \mathcal{D}_n]\phi = i\epsilon_{mn}\phi, \quad [\mathcal{D}_m, \mathcal{D}_n]\bar{\phi} = -i\epsilon_{mn}\bar{\phi} \quad (3.8)$$

The expansion into components of the superfields reads:

$$\begin{aligned} V_m &= C_m + i(\theta\chi_{1m}) - i(\bar{\theta}\bar{\chi}_{2m}) + i(\theta\theta)M_{1m} - i(\bar{\theta}\bar{\theta})\bar{M}_{2m} + (\theta\sigma^n\bar{\theta})h_{mn} \\ &\quad + i(\theta\theta)(\bar{\theta}\bar{\lambda}_{1m}) - i(\bar{\theta}\bar{\theta})(\theta\lambda_{2m}) + (\theta\theta)(\bar{\theta}\bar{\theta})D_m \\ \mathcal{B} &= \varphi + i(\theta\gamma_1) - i(\bar{\theta}\bar{\gamma}_2) + i(\theta\theta)N_1 - i(\bar{\theta}\bar{\theta})\bar{N}_2 + (\theta\sigma^m\bar{\theta})c_m \\ &\quad + i(\theta\theta)(\bar{\theta}\bar{\rho}_1) - i(\bar{\theta}\bar{\theta})(\theta\rho_2) + (\theta\theta)(\bar{\theta}\bar{\theta})G \\ \mathcal{C} &= \phi + i(\theta\xi_1) - i(\bar{\theta}\bar{\xi}_2) + i(\theta\theta)M_1 - i(\bar{\theta}\bar{\theta})\bar{M}_2 + (\theta\sigma^m\bar{\theta})a_m \\ &\quad + i(\theta\theta)(\bar{\theta}\bar{\psi}_1) - i(\bar{\theta}\bar{\theta})(\theta\psi_2) + (\theta\theta)(\bar{\theta}\bar{\theta})D \\ U_{1\alpha} &= v_{1\alpha} + \theta_\alpha s_1 - (\sigma^{mn}\theta)_\alpha s_{1mn} + (\sigma^m\bar{\theta})_\alpha w_{1m} + (\theta\theta)\eta_{1\alpha} + (\bar{\theta}\bar{\theta})\zeta_{1\alpha} + (\theta\sigma^m\bar{\theta})r_{1m\alpha} \\ &\quad + (\theta\theta)(\sigma^m\bar{\theta})_\alpha q_{1m} + (\bar{\theta}\bar{\theta})\theta_\alpha t_1 - (\bar{\theta}\bar{\theta})(\sigma^{mn}\theta)_\alpha t_{1mn} + (\theta\theta)(\bar{\theta}\bar{\theta})\mu_{1\alpha} \\ \bar{U}_1^{\dot{\alpha}} &= \bar{v}_1^{\dot{\alpha}} + \bar{\theta}^{\dot{\alpha}}\bar{s}_1 - (\bar{\sigma}^{mn}\bar{\theta})^{\dot{\alpha}}\bar{s}_{1mn} - (\bar{\sigma}^m\theta)^{\dot{\alpha}}\bar{w}_{1m} + (\bar{\theta}\bar{\theta})\bar{\eta}_1^{\dot{\alpha}} + (\theta\theta)\bar{\zeta}_1^{\dot{\alpha}} + (\theta\sigma^m\bar{\theta})\bar{r}_{1m}^{\dot{\alpha}} \\ &\quad - (\bar{\theta}\bar{\theta})(\bar{\sigma}^m\theta)^{\dot{\alpha}}\bar{q}_{1m} + (\theta\theta)\bar{\theta}^{\dot{\alpha}}\bar{t}_1 - (\theta\theta)(\bar{\sigma}^{mn}\bar{\theta})^{\dot{\alpha}}\bar{t}_{1mn} + (\theta\theta)(\bar{\theta}\bar{\theta})\bar{\mu}_1^{\dot{\alpha}} \\ U_{2\alpha} &= v_{2\alpha} + \theta_\alpha s_2 - (\sigma^{mn}\theta)_\alpha s_{2mn} + (\sigma^m\bar{\theta})_\alpha w_{2m} + (\theta\theta)\eta_{2\alpha} + (\bar{\theta}\bar{\theta})\zeta_{2\alpha} + (\theta\sigma^m\bar{\theta})r_{2m\alpha} \\ &\quad + (\theta\theta)(\sigma^m\bar{\theta})_\alpha q_{2m} + (\bar{\theta}\bar{\theta})\theta_\alpha t_2 - (\bar{\theta}\bar{\theta})(\sigma^{mn}\theta)_\alpha t_{2mn} + (\theta\theta)(\bar{\theta}\bar{\theta})\mu_{2\alpha} \\ \bar{U}_2^{\dot{\alpha}} &= \bar{v}_2^{\dot{\alpha}} + \bar{\theta}^{\dot{\alpha}}\bar{s}_2 - (\bar{\sigma}^{mn}\bar{\theta})^{\dot{\alpha}}\bar{s}_{2mn} - (\bar{\sigma}^m\theta)^{\dot{\alpha}}\bar{w}_{2m} + (\bar{\theta}\bar{\theta})\bar{\eta}_2^{\dot{\alpha}} + (\theta\theta)\bar{\zeta}_2^{\dot{\alpha}} + (\theta\sigma^m\bar{\theta})\bar{r}_{2m}^{\dot{\alpha}} \\ &\quad - (\bar{\theta}\bar{\theta})(\bar{\sigma}^m\theta)^{\dot{\alpha}}\bar{q}_{2m} + (\theta\theta)\bar{\theta}^{\dot{\alpha}}\bar{t}_2 - (\theta\theta)(\bar{\sigma}^{mn}\bar{\theta})^{\dot{\alpha}}\bar{t}_{2mn} + (\theta\theta)(\bar{\theta}\bar{\theta})\bar{\mu}_2^{\dot{\alpha}} \end{aligned} \quad (3.9)$$

where the gauge parameter superfields $E_{1\alpha}, E_{2\alpha}$ are given by

$$\begin{aligned} E_{1\alpha} &= \Lambda_{1\alpha} + \theta_\alpha\Lambda_2 - (\sigma^{mn}\theta)_\alpha\Lambda_{2mn} + (\sigma^m\bar{\theta})_\alpha\Lambda_{3m} + (\theta\theta)\Lambda_{4\alpha} + (\bar{\theta}\bar{\theta})\Lambda_{5\alpha} + (\theta\sigma^m\bar{\theta})\Lambda_{6m\alpha} \\ &\quad + (\theta\theta)(\sigma^m\bar{\theta})_\alpha\Lambda_{7m} + (\bar{\theta}\bar{\theta})\theta_\alpha\Lambda_8 - (\bar{\theta}\bar{\theta})(\sigma^{mn}\theta)_\alpha\Lambda_{8mn} + (\theta\theta)(\bar{\theta}\bar{\theta})\Lambda_{9\alpha} \\ \bar{E}_1^{\dot{\alpha}} &= \bar{\Lambda}_1^{\dot{\alpha}} + \bar{\theta}^{\dot{\alpha}}\bar{\Lambda}_2 - (\bar{\sigma}^{mn}\bar{\theta})^{\dot{\alpha}}\bar{\Lambda}_{2mn} - (\bar{\sigma}^m\theta)^{\dot{\alpha}}\bar{\Lambda}_{3m} + (\bar{\theta}\bar{\theta})\bar{\Lambda}_4^{\dot{\alpha}} + (\theta\theta)\bar{\Lambda}_5^{\dot{\alpha}} + (\theta\sigma^m\bar{\theta})\bar{\Lambda}_{6m}^{\dot{\alpha}} \\ &\quad - (\bar{\theta}\bar{\theta})(\bar{\sigma}^m\theta)^{\dot{\alpha}}\bar{\Lambda}_{7m} + (\theta\theta)\bar{\theta}^{\dot{\alpha}}\bar{\Lambda}_8 - (\theta\theta)(\bar{\sigma}^{mn}\bar{\theta})^{\dot{\alpha}}\bar{\Lambda}_{8mn} + (\theta\theta)(\bar{\theta}\bar{\theta})\bar{\Lambda}_9^{\dot{\alpha}} \\ E_{2\alpha} &= \Upsilon_{1\alpha} + \theta_\alpha\Upsilon_2 - (\sigma^{mn}\theta)_\alpha\Upsilon_{2mn} + (\sigma^m\bar{\theta})_\alpha\Upsilon_{3m} + (\theta\theta)\Upsilon_{4\alpha} + (\bar{\theta}\bar{\theta})\Upsilon_{5\alpha} + (\theta\sigma^m\bar{\theta})\Upsilon_{6m\alpha} \\ &\quad + (\theta\theta)(\sigma^m\bar{\theta})_\alpha\Upsilon_{7m} + (\bar{\theta}\bar{\theta})\theta_\alpha\Upsilon_8 - (\bar{\theta}\bar{\theta})(\sigma^{mn}\theta)_\alpha\Upsilon_{8mn} + (\theta\theta)(\bar{\theta}\bar{\theta})\Upsilon_{9\alpha} \\ \bar{E}_2^{\dot{\alpha}} &= \bar{\Upsilon}_1^{\dot{\alpha}} + \bar{\theta}^{\dot{\alpha}}\bar{\Upsilon}_2 - (\bar{\sigma}^{mn}\bar{\theta})^{\dot{\alpha}}\bar{\Upsilon}_{2mn} - (\bar{\sigma}^m\theta)^{\dot{\alpha}}\bar{\Upsilon}_{3m} + (\bar{\theta}\bar{\theta})\bar{\Upsilon}_4^{\dot{\alpha}} + (\theta\theta)\bar{\Upsilon}_5^{\dot{\alpha}} + (\theta\sigma^m\bar{\theta})\bar{\Upsilon}_{6m}^{\dot{\alpha}} \\ &\quad - (\bar{\theta}\bar{\theta})(\bar{\sigma}^m\theta)^{\dot{\alpha}}\bar{\Upsilon}_{7m} + (\theta\theta)\bar{\theta}^{\dot{\alpha}}\bar{\Upsilon}_8 - (\theta\theta)(\bar{\sigma}^{mn}\bar{\theta})^{\dot{\alpha}}\bar{\Upsilon}_{8mn} + (\theta\theta)(\bar{\theta}\bar{\theta})\bar{\Upsilon}_9^{\dot{\alpha}} \end{aligned} \quad (3.10)$$

We also denote the dual field by $\tilde{\epsilon}^{mn}$ with

$$\tilde{\epsilon}^{mn} = \frac{1}{2}\epsilon^{mnpq}\epsilon_{pq}, \quad \epsilon^{mn} = -\frac{1}{2}\epsilon^{mnpq}\tilde{\epsilon}_{pq} \quad (3.11)$$

Obviously, the sum $(\epsilon_{mn} + i\tilde{\epsilon}_{mn})$ is self-dual. Some useful identities related to the field strengths are

$$\epsilon_{mn}\tilde{\epsilon}^{mk} = \frac{1}{4}\delta_n^k\epsilon_{ab}\tilde{\epsilon}^{ab}, \quad \epsilon_{mn}\epsilon^{mk} - \tilde{\epsilon}_{mn}\tilde{\epsilon}^{mk} = \frac{1}{2}\delta_n^k\epsilon_{ab}\epsilon^{ab} \quad (3.12)$$

3.2 Decoupled equations of motion in superspace

In [1], the equations of motion in superspace for the superfields present in (3.1) have been derived in the Lorenz gauge. The resulting equations couple the lower spin fields present in the \mathcal{C} superfield to the fields describing the massive spin-2 multiplet contained in V_m . Here we will go one step further and present a redefinition of the superfields that decouples the massive spin-2 multiplet from the other superfields in these equations.

Using the same notation as in [1], we start by rewriting the equations in the Lorenz gauge containing the massive spin-2 multiplet (equations (6.5) and (6.6) of [1])

$$\begin{aligned} 0 &= \bar{\sigma}^{m\dot{\alpha}\alpha} d_{\alpha 0} V_m - d_0^2 \bar{d}_0^{\dot{\alpha}} \mathcal{C}, \\ 0 &= \sigma_{\alpha\dot{\alpha}}^m \bar{d}_0^{\dot{\alpha}} V_m + \bar{d}_0^2 d_{\alpha 0} \mathcal{C}, \end{aligned} \quad (3.13)$$

and after the redefinition

$$V_m \rightarrow V_m + \frac{1}{3} \bar{\sigma}_m^{\dot{\alpha}\alpha} [d_{\alpha 0}, \bar{d}_{\dot{\alpha} 0}] \mathcal{C} + \frac{1}{6} \sigma_{m\alpha\dot{\alpha}} (\partial \theta_0^\alpha d_0^2 \bar{d}_0^{\dot{\alpha}} - \partial \bar{\theta}_0^{\dot{\alpha}} \bar{d}_0^2 d_0^\alpha + 16 \partial \theta_0^\alpha \partial \bar{\theta}_0^{\dot{\alpha}}) \mathcal{C}, \quad (3.14)$$

we obtain decoupled equations of motion for the superfield V_m

$$\begin{aligned} 0 &= \bar{\sigma}^{m\dot{\alpha}\alpha} d_{\alpha 0} V_m, \\ 0 &= \sigma_{\alpha\dot{\alpha}}^m \bar{d}_0^{\dot{\alpha}} V_m. \end{aligned} \quad (3.15)$$

Here we have used the fact that the superfield \mathcal{C} satisfies $d_0^2 \mathcal{C} = \bar{d}_0^2 \mathcal{C} = 0$, which come from the gauge-fixing conditions in [1], alongside with the relations

$$\begin{aligned} i \bar{d}_0^{\dot{\alpha}} \Pi_{\alpha\dot{\alpha} 0} d_0^\alpha \mathcal{C} &= 4 \mathcal{C}, \\ i d_0^\alpha \Pi_{\alpha\dot{\alpha} 0} \bar{d}_0^{\dot{\alpha}} \mathcal{C} &= 4 \mathcal{C}, \end{aligned} \quad (3.16)$$

that can be seen to follow from $d_0^2 \mathcal{C} = \bar{d}_0^2 \mathcal{C} = 0$, together with equation (A.10) of [1] and the mass-shell condition in superspace

$$(\Pi_0^m \Pi_{m 0} - 2) \mathcal{C} - 2 \partial \theta_0^\alpha d_{\alpha 0} \mathcal{C} - 2 \partial \bar{\theta}_{\dot{\alpha} 0} \bar{d}_0^{\dot{\alpha}} \mathcal{C} = 0. \quad (3.17)$$

The superfields V_m and \mathcal{C} are therefore decoupled. But the symmetric rank-2 tensor $h_{(mn)}$ and the spin-1 component C_m inside the superfield V_m remain coupled through the divergence constraint of $h_{(mn)}$:

$$\mathfrak{D}^m h_{(mn)} = -2i \tilde{\epsilon}_{nm} C^m. \quad (3.18)$$

Decoupling the fields in this constraint requires a further shift

$$h_{(mn)} \rightarrow h_{(mn)} + \left(i \tilde{\epsilon}_{mk} \mathfrak{D}^k C_n - i \tilde{\epsilon}_{mk} \mathfrak{D}_n C^k + \frac{i}{2} \eta_{mn} \tilde{\epsilon}_{pq} \mathfrak{D}^p C^q + (m \leftrightarrow n) \right). \quad (3.19)$$

By expanding the superfield equations in components, and taking into account the additional redefinition above, one can show that the equations (3.15) agree with those that we will derive from the Lagrangian in the unitary gauge, *i.e.* (4.99), (5.45) and (5.53) for the massive spin-2 and spin-3/2, respectively.

Note that, as it usually happens in manifestly space-time supersymmetric descriptions of the superstring in curved backgrounds, the gauge-fixed eqs. (3.15) have the same form as in the neutral case.⁵ In our description, the difference lies in the fact that the supersymmetric derivatives $d_{\alpha 0}$ and $\bar{d}_{\dot{\alpha} 0}$ receive contributions, proportional to the strength of the electromagnetic field ϵ , necessary for the preservation of the manifest $\mathcal{N} = 1$ $d = 4$ supersymmetry.

Let us comment on the equations of motion obtained for the lower-spin fields in Section 6 of ref. [1]. There, they were given for two spin-1/2 massive Dirac fermions, a massive complex scalar and a massive complex vector. The analysis done in Section 6 of [1] was incomplete for the lower-spin fields. We have checked explicitly that going further and using the gauge transformations of Section 5 in [1], one shows that the remaining lower-spin physical degrees of freedom reside in two complex scalar multiplets which, in components, describe two massive spin-1/2 Dirac fermions and 4 complex bosons. This conclusion agrees with the results presented in this work as well as with the analysis of [24] for the neutral case. Therefore, in the charged case, the action (3.1) describes a massive complex spin-2 multiplet coupled to two massive complex scalar multiplets.

In this section, we have shown how to decouple the equations of motion for the massive spin-2 multiplet using superfields. As outlined, this was done following the gauge-fixing conditions from Section 6 of [1], in which a Lorenz-type gauge is employed. However, in the analysis of the present work, we are adopting the unitary gauge to eliminate the unphysical degrees of freedom in the action (3.1) and, consequently, the field redefinitions which will be presented in the following sections to decouple the equations for the massive spin-2 multiplet do not need to have a direct relation to (3.14). Of course, this comment also applies for the redefinitions in the neutral case presented in Section 2.

4 Charged massive bosons

The notations we will use from here on are essentially a complex version of the real notations in Section 2.2. Gauge parameters topped with a hat or tilde are now combinations of the $E_{1\alpha}$ and \bar{E}_2^α components:

$$\hat{\Lambda}_{3m} \equiv \Lambda_{3m} + \bar{\Upsilon}_{3m}, \quad \tilde{\Lambda}_{3m} \equiv i(\Lambda_{3m} - \bar{\Upsilon}_{3m}) \quad (4.1)$$

likewise for $\hat{\Lambda}_8, \tilde{\Lambda}_8, \hat{\Lambda}_{8mn}, \tilde{\Lambda}_{8mn}$. The complex rank-2 tensor is decomposed in the same way as before:

$$h_{mn} = v_{mn} + f_{mn} + \frac{1}{4}\eta_{mn}h \quad (4.2)$$

⁵It is worth mentioning that these equations could have been obtained by considering only the superfield V_m , describing the massive spin-2 multiplet, in the vertex operator (effectively putting the lower-spin fields to zero) and performing the analysis in the Lorenz gauge as in [1]. Following similar steps, but now considering only the superfield in the vertex operator describing a massive spin- s multiplet, one can try to generalise (3.15) directly for massive spin- s fields. This would constitute of an extension of [21] to the supersymmetric case.

It is also more practical to use the following new (positively charged) fields

$$\begin{aligned}
\tau_1 &= t_1 + \bar{t}_2, & \tau_2 &= i(t_1 - \bar{t}_2) \\
\omega_{1m} &= w_{1m} + \bar{w}_{2m}, & \omega_{2m} &= i(w_{1m} - \bar{w}_{2m}) \\
\tau_{1mn} &= t_{1mn} + \bar{t}_{2mn}, & \tau_{2mn} &= i(t_{1mn} - \bar{t}_{2mn})
\end{aligned} \tag{4.3}$$

and, as $\{t_{1mn}, t_{2mn}\}$ remain self-dual, the relation $\tau_{1mn} = \frac{1}{2}\varepsilon_{mnrst}\tau_2^{rs}$ still holds.

4.1 Gauge transformations

Like in the neutral case, we start by listing here the gauge transformations under which the Lagrangian is invariant. We also indicate which fields, up to redefinitions, will be kept as physical at the end of the gauge fixing procedure. The gauge transformations of the different components of the superfields are as follows:

Fields in \mathcal{B} :

$$\begin{aligned}
\delta c_m &= -2\mathcal{D}_m\mathcal{D}_n\hat{\Lambda}_3^n + 2\mathcal{D}^2\hat{\Lambda}_{3m} - 8\mathcal{D}^n\tilde{\Lambda}_{8mn} + 2i\epsilon_{mn}\hat{\Lambda}_3^n - 2i\tilde{\epsilon}_{mn}\tilde{\Lambda}_3^n \\
\delta N_1 &= 0, & \delta N_2 &= 0 \\
\delta G &= \frac{1}{2}\mathcal{D}^2\mathcal{D}^m\tilde{\Lambda}_{3m} + \mathcal{D}^2\hat{\Lambda}_8 - i\epsilon^{mn}\mathcal{D}_m\tilde{\Lambda}_{3n} + i\tilde{\epsilon}^{mn}\mathcal{D}_m\hat{\Lambda}_{3n} + 2i\epsilon^{mn}\hat{\Lambda}_{8mn} \\
\delta\varphi &= -4\hat{\Lambda}_8 - 2\mathcal{D}^m\tilde{\Lambda}_{3m}
\end{aligned} \tag{4.4}$$

Among these fields, we will be able to eliminate G , φ . Also, only part of c_m , N_1 , N_2 will remain as physical degrees of freedom.

Fields in \mathcal{C} :

$$\begin{aligned}
\delta a_m &= -2\mathcal{D}_m\mathcal{D}_n\tilde{\Lambda}_3^n + 2\mathcal{D}^2\tilde{\Lambda}_{3m} + 8\mathcal{D}^n\hat{\Lambda}_{8mn} + 2i\epsilon_{mn}\tilde{\Lambda}_3^n + 2i\tilde{\epsilon}_{mn}\hat{\Lambda}_3^n \\
\delta M_1 &= 0, & \delta M_2 &= 0 \\
\delta D &= -\frac{1}{2}\mathcal{D}^2\mathcal{D}_m\hat{\Lambda}_3^m + \mathcal{D}^2\tilde{\Lambda}_8 + i\epsilon^{mn}\mathcal{D}_m\hat{\Lambda}_{3n} + i\tilde{\epsilon}^{mn}\mathcal{D}_m\tilde{\Lambda}_{3n} + 2i\epsilon^{mn}\tilde{\Lambda}_{8mn} \\
\delta\phi &= -4\tilde{\Lambda}_8 + 2\mathcal{D}^m\hat{\Lambda}_{3m}
\end{aligned} \tag{4.5}$$

Among these fields, we will be able to eliminate D , ϕ . Also, only part of a_m , M_1 , M_2 will remain as physical degrees of freedom.

Fields in V_m :

$$\begin{aligned}
\delta M_{1m} &= -4i\mathcal{D}_m\Lambda_2 - 8\Lambda_{7m} - 8i\mathcal{D}^n\Lambda_{2mn}, \\
\delta\bar{M}_{2m} &= 4i\mathcal{D}_m\bar{\Upsilon}_2 - 8\bar{\Upsilon}_{7m} + 8i\mathcal{D}^n\bar{\Upsilon}_{2mn} \\
\delta D_m &= 4\mathcal{D}_m\hat{\Lambda}_8 + 8\mathcal{D}^n\hat{\Lambda}_{8mn} + 2i\epsilon_{mn}\tilde{\Lambda}_3^n + 2i\tilde{\epsilon}_{mn}\hat{\Lambda}_3^n \\
\delta C_m &= -8\tilde{\Lambda}_{3m} \\
\delta h &= -8\mathcal{D}^m\hat{\Lambda}_{3m} + 32\tilde{\Lambda}_8 \\
\delta v_{mn} &= 4\left(\mathcal{D}_m\hat{\Lambda}_{3n} + \mathcal{D}_n\hat{\Lambda}_{3m}\right) - 2\eta_{mn}\mathcal{D}^k\hat{\Lambda}_{3k} \\
\delta f_{mn} &= 4\varepsilon_{mnpq}\mathcal{D}^p\tilde{\Lambda}_3^q + 16\tilde{\Lambda}_{8mn}
\end{aligned} \tag{4.6}$$

Among these fields, we will be able to eliminate M_{1m} , M_{2m} , D_m , f_{mn} . Also, only a combination of h and v_{mn} will remain as physical degrees of freedom.

Fields in $U_{1\alpha}, \bar{U}_2^{\dot{\alpha}}$:

$$\begin{aligned}
\delta\tau_1 &= 16\tilde{\Lambda}_8 + 8\mathcal{D}^2\tilde{\Lambda}_8 - 4\mathcal{D}^2\mathcal{D}^m\hat{\Lambda}_{3m} + 24i\epsilon^{mn}\tilde{\Lambda}_{8mn} + 4i\epsilon^{mn}\mathcal{D}_m\hat{\Lambda}_{3n} + 12i\tilde{\epsilon}^{mn}\mathcal{D}_m\tilde{\Lambda}_{3n} \\
\delta\tau_2 &= -16\hat{\Lambda}_8 - 8i\epsilon^{mn}\mathcal{D}_m\tilde{\Lambda}_{3n} \\
\delta\omega_{1m} &= 16\tilde{\Lambda}_{3m} + 8\mathcal{D}^2\tilde{\Lambda}_{3m} - 8\mathcal{D}_m\mathcal{D}_n\tilde{\Lambda}_3^n + 32\mathcal{D}^n\hat{\Lambda}_{8mn} - 8i\epsilon_{mn}\tilde{\Lambda}_3^n + 8i\tilde{\epsilon}_{mn}\hat{\Lambda}_3^n \\
\delta\omega_{2m} &= -16\hat{\Lambda}_{3m} - 8\mathcal{D}_m\mathcal{D}_n\hat{\Lambda}_3^n + 16\mathcal{D}_m\tilde{\Lambda}_8 + 16i\epsilon_{mn}\hat{\Lambda}_3^n \\
\delta\tau_{1mn} &= \frac{1}{2}\epsilon_{mnrst}\delta\tau_2^{rs} \\
\delta\tau_{2mn} &= -16\hat{\Lambda}_{8mn} - 8\left(\mathcal{D}_n\mathcal{D}^k\hat{\Lambda}_{8mk} - \mathcal{D}_m\mathcal{D}^k\hat{\Lambda}_{8nk}\right) + 2\mathcal{D}^2\left(\mathcal{D}_m\tilde{\Lambda}_{3n} - \mathcal{D}_n\tilde{\Lambda}_{3m}\right) \\
&\quad + 8i\left(\epsilon_{nk}\hat{\Lambda}_{8m}{}^k - \epsilon_{mk}\hat{\Lambda}_{8n}{}^k\right) + 8i\tilde{\epsilon}_{mn}\tilde{\Lambda}_8 + 4i\epsilon_{mn}\hat{\Lambda}_8 \\
&\quad + \left[2i\epsilon_{mk}\left(-2\mathcal{D}_n\tilde{\Lambda}_3^k + \mathcal{D}^k\tilde{\Lambda}_{3n}\right) - 2i\tilde{\epsilon}_{mk}\left(2\mathcal{D}_n\hat{\Lambda}_3^k + \mathcal{D}^k\hat{\Lambda}_{3n}\right) - (m \leftrightarrow n)\right] \\
\delta q_{1m} &= 16i\Lambda_{7m} + 4i(\epsilon_{mn} - i\tilde{\epsilon}_{mn})\mathcal{D}^n\Lambda_2 + 8i(\epsilon_{mn} - i\tilde{\epsilon}_{mn})\mathcal{D}_k\Lambda_2^{nk} + 16\epsilon_{mn}\Lambda_7^n \\
\delta\bar{q}_{2m} &= -16i\tilde{\Upsilon}_{7m} + 4i(\epsilon_{mn} + i\tilde{\epsilon}_{mn})\mathcal{D}^n\tilde{\Upsilon}_2 + 8i(\epsilon_{mn} + i\tilde{\epsilon}_{mn})\mathcal{D}_k\tilde{\Upsilon}_2^{nk} - 16\epsilon_{mn}\tilde{\Upsilon}_7^n \\
\delta s_1 &= 16i\Lambda_2 - 8\epsilon^{mn}\Lambda_{2mn} \\
\delta\bar{s}_2 &= -16i\tilde{\Upsilon}_2 + 8\epsilon^{mn}\tilde{\Upsilon}_{2mn} \\
\delta s_{1mn} &= 16i\Lambda_{2mn} + 4(\epsilon_{mn} + i\tilde{\epsilon}_{mn})\Lambda_2 - 8\left(\epsilon_{mk}\Lambda_{2n}{}^k - \epsilon_{nk}\Lambda_{2m}{}^k\right) \\
\delta\bar{s}_{2mn} &= -16i\tilde{\Upsilon}_{2mn} - 4(\epsilon_{mn} - i\tilde{\epsilon}_{mn})\tilde{\Upsilon}_2 + 8\left(\epsilon_{mk}\tilde{\Upsilon}_{2n}{}^k - \epsilon_{nk}\tilde{\Upsilon}_{2m}{}^k\right)
\end{aligned} \tag{4.7}$$

None of these fields will remain as a physical degree of freedom, though some will be part of the latter through the field redefinitions.

One can verify that the above gauge transformations reduce to (2.11)-(2.14) for $\epsilon = 0$. We point out here the gauge invariant combinations

$$\begin{aligned}
\delta\left[\bar{q}_{2m} - i\mathcal{D}^n\bar{s}_{2mn} - \frac{1}{2}i\mathcal{D}_m\bar{s}_2 - 2i(\eta_{mn} - i\epsilon_{mn})\bar{M}_2^m\right] &= 0 \\
\delta\left[-q_{1m} - i\mathcal{D}^n s_{1mn} - \frac{1}{2}i\mathcal{D}_m s_1 - 2i(\eta_{mn} - i\epsilon_{mn})M_1^n\right] &= 0
\end{aligned} \tag{4.8}$$

4.2 A compact bosonic Lagrangian

As in the neutral case, the bosonic Lagrangian \mathcal{L}_B can be separated into two parts, each of which is gauge invariant by itself:

$$\mathcal{L}_B = \mathcal{L}_1 + \mathcal{L}_2. \tag{4.9}$$

with \mathcal{L}_1 and \mathcal{L}_2 giving in the absence of electromagnetic fields the same Lagrangians as in the neutral case.

Here, \mathcal{L}_1 reads:

$$\begin{aligned}
\mathcal{L}_1 = & \frac{1}{2}\bar{q}_1^m q_{1m} + \frac{1}{2}\bar{q}_2^m q_{2m} - \frac{1}{8}\bar{s}_1 \mathcal{D}^2 s_1 - \frac{1}{8}\bar{s}_2 \mathcal{D}^2 s_2 \\
& + \left[i\bar{q}_1^m \left(M_{1m} - i\epsilon_{mn} M_1^n + \frac{1}{2}\mathcal{D}^n s_{1mn} + \frac{1}{4}\mathcal{D}_m s_1 - 3i\mathcal{D}_m M_1 + \mathcal{D}_m N_1 \right) + \text{h.c.} \right] \\
& + \left[i\bar{q}_2^m \left(M_{2m} + i\epsilon_{mn} M_2^n + \frac{1}{2}\mathcal{D}^n s_{2mn} + \frac{1}{4}\mathcal{D}_m s_2 - 3i\mathcal{D}_m M_2 + \mathcal{D}_m N_2 \right) + \text{h.c.} \right] \\
& + \left[\frac{1}{2}\mathcal{D}^m \bar{s}_1 (M_{1m} - i\epsilon_{mn} M_1^n - 3i\mathcal{D}_m M_1 + \mathcal{D}_m N_1) - \frac{1}{8}i\epsilon^{mn} \bar{s}_1 s_{1mn} + \text{h.c.} \right] \\
& + \left[\frac{1}{2}\mathcal{D}^m \bar{s}_2 (M_{2m} + i\epsilon_{mn} M_2^n - 3i\mathcal{D}_m M_2 + \mathcal{D}_m N_2) + \frac{1}{8}i\epsilon^{mn} \bar{s}_2 s_{2mn} + \text{h.c.} \right] \\
& - 3\bar{M}_1 (3\mathcal{D}^2 + 4) M_1 - 3\bar{M}_2 (3\mathcal{D}^2 + 4) M_2 \\
& + \left[6i\mathcal{D}^m \bar{M}_1 \left(M_{1m} - i\epsilon_{mn} M_1^n + \frac{1}{2}\mathcal{D}_m N_1 \right) + \frac{3}{2}\epsilon^{mn} \bar{M}_1 s_{1mn} + \text{h.c.} \right] \\
& + \left[6i\mathcal{D}^m \bar{M}_2 \left(M_{2m} + i\epsilon_{mn} M_2^n + \frac{1}{2}\mathcal{D}_m N_2 \right) - \frac{3}{2}\epsilon^{mn} \bar{M}_2 s_{2mn} + \text{h.c.} \right] \\
& - \bar{N}_1 (\mathcal{D}^2 - 4) N_1 + \left[2\mathcal{D}^m \bar{N}_1 (M_{1m} - i\epsilon_{mn} M_1^n) - \frac{1}{2}i\epsilon^{mn} \bar{N}_1 s_{1mn} + \text{h.c.} \right] \\
& - \bar{N}_2 (\mathcal{D}^2 - 4) N_2 + \left[2\mathcal{D}^m \bar{N}_2 (M_{2m} + i\epsilon_{mn} M_2^n) + \frac{1}{2}i\epsilon^{mn} \bar{N}_2 s_{2mn} + \text{h.c.} \right] \\
& + \frac{1}{2}\mathcal{D}^n s_{1mn} \mathcal{D}_k \bar{s}_1^{mk} + \left[\bar{s}_{1mn} \left(\mathcal{D}^m M_1^n - i\epsilon^{nk} \mathcal{D}^m M_{1k} \right) + \text{h.c.} \right] \\
& + \frac{1}{2}\mathcal{D}^n s_{2mn} \mathcal{D}_k \bar{s}_2^{mk} + \left[\bar{s}_{2mn} \left(\mathcal{D}^m M_2^n + i\epsilon^{nk} \mathcal{D}^m M_{2k} \right) + \text{h.c.} \right] \\
& + 2 \left[\left(\eta^{mk} - i\epsilon^{mk} \right) M_{1k} \right]^\dagger [(\eta_{mn} - i\epsilon_{mn}) M_1^n] \\
& + 2 \left[\left(\eta^{mk} + i\epsilon^{mk} \right) M_{2k} \right]^\dagger [(\eta_{mn} + i\epsilon_{mn}) M_2^n]
\end{aligned} \tag{4.10}$$

Remarkably, M_{1m} and \bar{M}_{2m} always appear with a factor $(\eta - i\epsilon)$, including in the gauge invariant expressions (4.8). Indeed, this feature can already be seen at the level of the action (3.1), where the superfield V_m is accompanied everywhere by one or two factors $(\eta - i\epsilon)$. These can be absorbed by the rescaling

$$\begin{aligned}
M_{1m} & \rightarrow (\eta_{mn} - i\epsilon_{mn})^{-1} M_1^n, \\
\bar{M}_{2m} & \rightarrow (\eta_{mn} - i\epsilon_{mn})^{-1} \bar{M}_2^n
\end{aligned} \tag{4.11}$$

Note that the terms of index 2 differ from those of index 1 only in an inversion of sign $\epsilon \leftrightarrow -\epsilon$. We can therefore limit ourselves in the following to writing explicitly only the terms with index 1.

We can then write the Lagrangian as:

$$\begin{aligned}
\mathcal{L}_1 = & \frac{1}{2} \bar{q}_1^m q_{1m} + \left[i \bar{q}_1^m \left(M_{1m} + \frac{1}{2} \mathfrak{D}^n s_{1mn} + \frac{1}{4} \mathfrak{D}_m s_1 - 3i \mathfrak{D}_m M_1 + \mathfrak{D}_m N_1 \right) + \text{h.c.} \right] \\
& - \frac{1}{8} \bar{s}_1 \mathfrak{D}^2 s_1 + \left[\frac{1}{2} \mathfrak{D}^m \bar{s}_1 \left(M_{1m} - 3i \mathfrak{D}_m M_1 + \mathfrak{D}_m N_1 \right) - \frac{1}{8} i \epsilon^{mn} \bar{s}_1 s_{1mn} + \text{h.c.} \right] \\
& - 3 \bar{M}_1 (3\mathfrak{D}^2 + 4) M_1 + \left[6i \mathfrak{D}^m \bar{M}_1 \left(M_{1m} + \frac{1}{2} \mathfrak{D}_m N_1 \right) + \frac{3}{2} \epsilon^{mn} \bar{M}_1 s_{1mn} + \text{h.c.} \right] \quad (4.12) \\
& - \bar{N}_1 (\mathfrak{D}^2 - 4) N_1 + \left[2 \mathfrak{D}^m \bar{N}_1 M_{1m} - \frac{1}{2} i \epsilon^{mn} \bar{N}_1 s_{1mn} + \text{h.c.} \right] \\
& + \frac{1}{2} \mathfrak{D}^n s_{1mn} \mathfrak{D}_k \bar{s}_1^{mk} + (\bar{s}_{1mn} \mathfrak{D}^m M_1^n + \text{h.c.}) + 2 \bar{M}_1^m M_{1m} + (1 \leftrightarrow 2, \epsilon \leftrightarrow -\epsilon)
\end{aligned}$$

$M_{\{1,2\}m}$ are auxiliary fields, and they have no explicit dependence on the electromagnetic background in their equations of motion,

$$M_{\{1,2\}m} = \frac{1}{2} i q_{\{1,2\}m} - \frac{1}{2} \mathfrak{D}^n s_{\{1,2\}mn} + 3i \mathfrak{D}_m M_{\{1,2\}} - \mathfrak{D}_m N_{\{1,2\}} - \frac{1}{4} \mathfrak{D}_m s_{\{1,2\}} \quad (4.13)$$

except of course in the covariant derivative. Integrating them out, $\{q_{\{1,2\}m}, s_{\{1,2\}mn}, s_{\{1,2\}}\}$ disappear, leaving only the gauge invariant fields $\{M_{\{1,2\}}, N_{\{1,2\}}\}$. This leads to the subsequent Lagrangian where the gauge is completely fixed:

$$\begin{aligned}
\mathcal{L}_1 = & 3 \bar{M}_1 (3\mathfrak{D}^2 - 4) M_1 + 3 (i \bar{M}_1 \mathfrak{D}^2 N_1 + \text{h.c.}) + \bar{N}_1 (\mathfrak{D}^2 + 4) N_1 \\
& + 3 \bar{M}_2 (3\mathfrak{D}^2 - 4) M_2 + 3 (i \bar{M}_2 \mathfrak{D}^2 N_2 + \text{h.c.}) + \bar{N}_2 (\mathfrak{D}^2 + 4) N_2 \quad (4.14)
\end{aligned}$$

As for the neutral case, it is convenient to introduce the following new complex scalar fields

$$\begin{aligned}
\mathcal{M}_1 &= M_1 + \bar{M}_2, & \mathcal{M}_2 &= i (M_1 - \bar{M}_2) \\
\mathcal{N}_1 &= N_1 + \bar{N}_2, & \mathcal{N}_2 &= i (N_1 - \bar{N}_2) \quad (4.15)
\end{aligned}$$

With the field redefinition

$$\mathcal{N}_1 \rightarrow \sqrt{2} \mathcal{N}_1 + 3 \mathcal{M}_2, \quad \mathcal{M}_1 \rightarrow \frac{\sqrt{2}}{3} \mathcal{M}_1 - \frac{1}{3} \mathcal{N}_2 \quad (4.16)$$

the complex scalars $\mathcal{M}_2, \mathcal{N}_2$ lose their kinetic terms. After integration, we get a Lagrangian of two decoupled massive complex scalars $\mathcal{M}_1, \mathcal{N}_1$ propagating in the electromagnetic background:

$$\mathcal{L}_1 = \bar{\mathcal{M}}_1 (-2 + \mathfrak{D}^2) \mathcal{M}_1 + \bar{\mathcal{N}}_1 (-2 + \mathfrak{D}^2) \mathcal{N}_1 \quad (4.17)$$

Obviously, (4.17) reduces to (2.21) in the neutral case.

We now turn our attention to the Lagrangian \mathcal{L}_2 :

$$\begin{aligned}
\mathcal{L}_2 = & \frac{1}{8} [(\eta^{mn} - i\epsilon^{mn}) C_n]^\dagger \mathfrak{D}^4 C_m - \frac{1}{2} \left\{ [(\eta^{mn} - i\epsilon^{mn}) C_n]^\dagger \mathfrak{D}^2 D_m + \text{h.c.} \right\} \\
& + 2 [(\eta^{mn} - i\epsilon^{mn}) D_n]^\dagger D_m + 2 \left\{ [(\eta^{mn} - i\epsilon^{mn}) C_n]^\dagger [(\eta_{mk} - i\epsilon_{mk}) D^k] + \text{h.c.} \right\} \\
& - [(\eta^{mn} - i\epsilon^{mn}) h_{nk}]^\dagger \left[(\eta_{ml} - i\epsilon_{ml}) h^{lk} + i\epsilon^{lk} h_{ml} - \frac{1}{2} \mathfrak{D}^2 h_m{}^k \right] \\
& + \frac{1}{2} [(\eta^{mn} - i\epsilon^{mn}) \mathfrak{D}^k h_{nk}]^\dagger \mathfrak{D}^l h_{ml} + 3 \left\{ [(\eta^{mn} - i\epsilon^{mn}) \mathfrak{D}^k h_{nk}]^\dagger \mathfrak{D}_m \phi + \text{h.c.} \right\} \\
& + \left\{ [(\eta^{mn} - i\epsilon^{mn}) C_n]^\dagger \left(\frac{3}{2} \mathfrak{D}^2 a_m - 3 \mathfrak{D}_m \mathfrak{D}_k a^k + 3i\epsilon_{mk} a^k + 3i\tilde{\epsilon}_{mk} \mathfrak{D}^k \phi \right. \right. \\
& \left. \left. + 2 \mathfrak{D}_m G + i\tilde{\epsilon}_{mk} c^k + \frac{1}{2} \mathfrak{D}_m \tau_2 + \mathfrak{D}^k \tau_{2mk} - \frac{1}{4} i\epsilon_{mk} \omega_1^k + \frac{1}{4} i\tilde{\epsilon}_{mk} \omega_2^k \right) + \text{h.c.} \right\} \\
& + \left\{ [(\eta^{mn} - i\epsilon^{mn}) h_n{}^k]^\dagger \varepsilon_{mkpq} \left(\frac{1}{4} \mathfrak{D}^p \omega_1^q - 3 \mathfrak{D}^p a^q + \frac{1}{2} \tau_2^{pq} \right) + \text{h.c.} \right\} \\
& + \left\{ [(\eta^{mn} - i\epsilon^{mn}) h_{nk}]^\dagger \left(-\mathfrak{D}_m c^k + i\tilde{\epsilon}_m{}^k \varphi - 3i\epsilon_m{}^k \phi - \frac{1}{4} \mathfrak{D}_m \omega_2^k - \frac{1}{4} \mathfrak{D}^k \omega_{2m} \right) + \text{h.c.} \right\} \\
& + \left\{ [(\eta^{mn} + i\epsilon^{mn}) h_{mn}]^\dagger \left(-6D + \frac{3}{2} \mathfrak{D}^2 \phi + \frac{1}{4} \mathfrak{D}^k \omega_{2k} + \frac{1}{2} \tau_1 \right) + \text{h.c.} \right\} \\
& + \left\{ [(\eta^{mn} - i\epsilon^{mn}) D_n]^\dagger (2 \mathfrak{D}_m \varphi - 6a_m + \omega_{1m}) + \text{h.c.} \right\} - \frac{1}{2} \mathfrak{D}^m \bar{c}_m \mathfrak{D}^n c_n - 2\bar{c}_m c^m \\
& + \left[\mathfrak{D}^m \bar{c}_m \left(-3D - \frac{3}{4} \mathfrak{D}^2 \phi + \frac{1}{2} \tau_1 - \frac{1}{4} \mathfrak{D}^n \omega_{2n} \right) + \text{h.c.} \right] - 6\bar{D} D - \frac{1}{8} \bar{\varphi} \mathfrak{D}^4 \varphi \\
& - \frac{33}{8} \bar{\phi} \mathfrak{D}^4 \phi - 12\bar{a}^m \mathfrak{D}^2 a_m - \frac{33}{2} \mathfrak{D}^m \bar{a}_m \mathfrak{D}^n a_n - 24i\epsilon^{mn} \bar{a}_m a_n + 6\bar{a}^m a_m \\
& + \left[\mathfrak{D}^2 \bar{\phi} \left(\frac{15}{2} D - \frac{9}{8} \mathfrak{D}^m \omega_{2m} - \frac{3}{4} \tau_1 \right) + \bar{D} \left(\frac{3}{2} \mathfrak{D}^m \omega_{2m} + 9\tau_1 - 12\phi \right) + \text{h.c.} \right] \\
& - \frac{1}{8} (\mathfrak{D}^m \bar{\omega}_{1m} \mathfrak{D}^n \omega_{1n} + \mathfrak{D}^m \bar{\omega}_{2m} \mathfrak{D}^n \omega_{2n}) + \tau_{2mn} \bar{\tau}_2^{mn} - \tau_1 \bar{\tau}_1 - 2\bar{G} G \\
& - \frac{1}{8} (\bar{\omega}_1^m \mathfrak{D}^2 \omega_{1m} - \bar{\omega}_2^m \mathfrak{D}^2 \omega_{2m}) + \left(\omega_1^m \mathfrak{D}^n \bar{\tau}_{2mn} - \frac{1}{4} \tau_1 \mathfrak{D}^m \bar{\omega}_{2m} - \frac{1}{4} \tau_2 \mathfrak{D}^m \bar{\omega}_{1m} + \text{h.c.} \right) \\
& - \frac{1}{8} i\epsilon^{mn} (3\bar{\omega}_{1m} \omega_{1n} + \bar{\omega}_{2m} \omega_{2n}) + \left[\mathfrak{D}^2 \bar{\varphi} \left(-\frac{1}{4} \tau_2 + \frac{3}{4} \mathfrak{D}^m a_m - \frac{1}{8} \mathfrak{D}^m \omega_{1m} - \frac{1}{2} G \right) + \text{h.c.} \right] \\
& + \left[\bar{G} \left(4\varphi + 3\mathfrak{D}^m a_m - \frac{1}{2} \mathfrak{D}^m \omega_{1m} - \tau_2 \right) + \mathfrak{D}^m \bar{a}_m \left(\frac{9}{4} \mathfrak{D}^n \omega_{1n} + \frac{3}{2} \tau_2 \right) + \text{h.c.} \right] \\
& + \left[\bar{\omega}_{1m} \left(\frac{3}{2} \mathfrak{D}^2 a^m + 3i\epsilon^{mn} a_n + \frac{3}{2} i\tilde{\epsilon}^{mn} \mathfrak{D}_n \phi \right) - 6\bar{a}_m \mathfrak{D}_n \tau_2^{mn} - i\bar{\varphi} \epsilon^{mn} \tau_{2mn} + \text{h.c.} \right] \\
& + \left[\frac{1}{2} i\bar{c}_m (\tilde{\epsilon}^{mn} \omega_{1n} - \epsilon^{mn} \omega_{2n}) - \frac{3}{2} i\epsilon^{mn} \bar{\omega}_{2m} \mathfrak{D}_n \phi - \frac{1}{8} i\tilde{\epsilon}^{mn} \omega_{2m} \bar{\omega}_{1n} + \text{h.c.} \right]
\end{aligned} \tag{4.18}$$

As above, the auxiliary fields in \mathcal{L}_2 are $\{D, D_m, G, \tau_1, \tau_2, \tau_{2mn}\}$, whose integration eliminates φ , and we are left with $\{f_{mn}, v_{mn}, \omega_{1m}, \omega_{2m}, a_m, C_m, c_m, h, \phi\}$. The following steps are essentially a straightforward generalisation of section 2.2.

We shall start with gauging away $\{f_{mn}, \omega_{1m}, \omega_{2m}, \phi\}$. Recall that in absence of the

background, $\partial_m \phi$ and ω_{2m} appear as Stückelberg fields

$$\begin{aligned} \delta \left(h + 8\phi + 2\partial^2 \phi + \frac{1}{2} \partial_m \omega_2^m \right) &= 0 \\ \delta \left(v_{mn} + \frac{1}{4} \partial_m \omega_{2n} + \frac{1}{4} \partial_n \omega_{2m} - \frac{1}{8} \eta_{mn} \partial_k \omega_2^k + 2\partial_m \partial_n \phi - \frac{1}{2} \eta_{mn} \partial^2 \phi \right) &= 0 \end{aligned} \quad (4.19)$$

But now, these expressions are not gauge invariant if one only promotes partial derivatives to covariant ones:

$$\begin{aligned} \delta \left(h + 8\phi + 2\mathfrak{D}^2 \phi + \frac{1}{2} \mathfrak{D}_m \omega_2^m \right) &= 8i\epsilon_{mn} \mathfrak{D}^m \hat{\Lambda}_3^n \\ \delta \left(v_{mn} + \frac{1}{4} \mathfrak{D}_m \omega_{2n} + \frac{1}{4} \mathfrak{D}_n \omega_{2m} - \frac{1}{8} \eta_{mn} \mathfrak{D}_k \omega_2^k + 2\mathfrak{D}_m \mathfrak{D}_n \phi - \frac{1}{2} \eta_{mn} \mathfrak{D}^2 \phi \right) &= 4i\epsilon_{mk} \mathfrak{D}_n \hat{\Lambda}_3^k + 4i\epsilon_{nk} \mathfrak{D}_m \hat{\Lambda}_3^k - 2i\eta_{mn} \epsilon_{kl} \mathfrak{D}^k \hat{\Lambda}_3^l \end{aligned} \quad (4.20)$$

Notice that

$$\delta (\omega_{2m} + 4\mathfrak{D}_m \phi) = -16 (\eta_{mn} - i\epsilon_{mn}) \hat{\Lambda}_3^n \quad (4.21)$$

where the ϵ term gives rise to the r.h.s. of (4.20). The $(1 - i\epsilon)$ factor can be absorbed by the rescaling

$$\omega'_{2m} \equiv (\eta_{mn} - i\epsilon_{mn})^{-1} (\omega_2^m + 4\mathfrak{D}^n \phi), \quad \delta \omega'_{2m} = -16 \hat{\Lambda}_{3m} \quad (4.22)$$

and ω'_{2m} appear in the new gauge invariant combinations,

$$\begin{aligned} \delta \left(h + 8\phi + \frac{1}{2} \mathfrak{D}_m \omega_2^m \right) &= 0 \\ \delta \left(v_{mn} + \frac{1}{4} \mathfrak{D}_m \omega'_{2n} + \frac{1}{4} \mathfrak{D}_n \omega'_{2m} - \frac{1}{8} \eta_{mn} \mathfrak{D}_k \omega_2^k \right) &= 0 \end{aligned} \quad (4.23)$$

This then is used to find a new gauge invariant spin-2 field, thereby going into a unitary gauge after a Stückelberg mechanism, as:

$$\begin{aligned} \tilde{h}_{mn} &\equiv v_{mn} + \frac{1}{4} \eta_{mn} (h + 8\phi) \\ &+ \frac{1}{4} (\eta_{nk} - i\epsilon_{nk})^{-1} \mathfrak{D}_m (\omega_2^k + 4\mathfrak{D}^k \phi) + \frac{1}{4} (\eta_{mk} - i\epsilon_{mk})^{-1} \mathfrak{D}_n (\omega_2^k + 4\mathfrak{D}^k \phi) \end{aligned} \quad (4.24)$$

which satisfies $\delta \tilde{h}_{mn} = 0$. Similarly, the last three lines of (2.25) are amended by a $(\eta - i\epsilon)^{-1}$ factor in front of ω_{1m} . The redefinition:

$$\omega'_{1m} \equiv (\eta_{mn} - i\epsilon_{mn})^{-1} (\omega_1^n - 4a^n), \quad \delta \omega'_{1m} = 16 \tilde{\Lambda}_{3m} \quad (4.25)$$

allows to write the gauge invariant combinations as

$$\begin{aligned} \delta \left(c_m + \frac{1}{2} \mathfrak{D}^n f_{mn} + \frac{1}{8} \mathfrak{D}^2 \omega'_{2m} - \frac{1}{8} \mathfrak{D}_m \mathfrak{D}_n \omega_2^{n'} + \frac{1}{8} i\epsilon_{mn} \omega_2^{n'} \right) &= 0 \\ \delta \left(a_m - \frac{1}{4} \Omega_m + \frac{1}{8} i\tilde{\epsilon}_{mn} \omega_2^{n'} \right) &= 0 \\ \delta \left(C_m + \frac{1}{2} \omega'_{1m} \right) &= 0 \end{aligned} \quad (4.26)$$

Thus, after integrating all the auxiliary fields in \mathcal{L}_2 , we can eliminate ω_{1m} , ω_{2m} , ϕ , f_{mn} by the two-step redefinition:

1. $\omega_{1m} \rightarrow (\eta_{mn} - i\epsilon_{mn})\omega_1^n + 4a_m, \quad \omega_{2m} \rightarrow (\eta_{mn} - i\epsilon_{mn})\omega_2^n - 4\mathfrak{D}_m\phi$
2. $c_m \rightarrow c_m - \frac{1}{2}\mathfrak{D}^n f_{mn} - \frac{1}{8}\mathfrak{D}^2\omega_{2m} + \frac{1}{8}\mathfrak{D}_m\mathfrak{D}_n\omega_2^n - \frac{1}{8}i\epsilon_{mn}\omega_2^n$
 $h \rightarrow h - 8\phi - \frac{1}{2}\mathfrak{D}^m\omega_{2m}, \quad v_{mn} \rightarrow v_{mn} + \frac{1}{8}\eta_{mn}\mathfrak{D}^k\omega_{2k} - \frac{1}{4}\mathfrak{D}_m\omega_{2n} - \frac{1}{4}\mathfrak{D}_n\omega_{2m}$
 $C_m \rightarrow C_m - \frac{1}{2}\omega_{1m}, \quad a_m \rightarrow a_m - \frac{1}{8}i\tilde{\epsilon}_{mn}\omega_2^n + \frac{1}{4}\epsilon_{mnpq}\mathfrak{D}^n f^{pq}$

Next, in order to decouple and normalise $\{a_m, C_m\}$ in the Lagrangian, we perform the redefinitions

$$C_m \rightarrow \sqrt{\frac{2}{3}}C_m, \quad a_m \rightarrow \frac{\sqrt{2}}{2}a_m - \frac{1}{\sqrt{6}}C_m \quad (4.27)$$

where

$$C_m \equiv (\eta_{mn} - i\epsilon_{mn})C^n \quad (4.28)$$

As a result, \mathcal{L}_2 is now

$$\begin{aligned} \mathcal{L}_2 = & \bar{C}^m\mathfrak{D}^2C_m + \mathfrak{D}^m\bar{C}_m\mathfrak{D}^n C_n - 2\bar{C}^m(\eta_{mn} - i\epsilon_{mn})C^n \\ & + 2\bar{a}^m a_m - i\epsilon_{mn}\bar{a}^m a^n + \mathfrak{D}^m\bar{a}_m\mathfrak{D}^n a_n - \left[\frac{i}{\sqrt{2}}\bar{a}^m \left(\epsilon^{nk}\epsilon_{mkpq}\mathfrak{D}^q h_n^p \right) + \text{h.c.} \right] \\ & - 2\bar{c}^m c_m + \left[\bar{c}^m \left(\sqrt{2}i\tilde{\epsilon}_{mn}a^n - \frac{2}{5}\mathfrak{D}_m h + \left(\eta^{nk} - i\epsilon^{nk} \right) \mathfrak{D}_n h_{mk} \right) + \text{h.c.} \right] \\ & - \frac{2}{5}\mathfrak{D}^m\bar{c}_m\mathfrak{D}^n c_n + \frac{1}{10}\bar{h}h - \bar{h}^{mn}h_{mn} + \frac{1}{2}\bar{h}_{mk}(\eta^{mn} - i\epsilon^{mn})\mathfrak{D}^2 h_n^k \\ & - \frac{1}{2}\bar{h}_{km}(\eta^{kl} - i\epsilon^{kl})\mathfrak{D}^m\mathfrak{D}^n h_{ln} - 3i\epsilon^{kl}\bar{h}_{lm}h_k^m + \frac{3}{2}\epsilon^{kl}\epsilon^{mn}\bar{h}_{ln}h_{km} - \frac{1}{2}\epsilon_k^m\epsilon^{kl}\bar{h}_{mn}h_l^n \end{aligned} \quad (4.29)$$

The vector field C_m is decoupled from all other fields and appears only as a rescaled \mathcal{C}_m . It has the Lagrangian of a charged vector boson, with the corresponding equation of motion

$$\mathfrak{D}^2\mathcal{C}_m - 2(\eta_{mn} - i\epsilon_{mn})\mathcal{C}^n - \mathfrak{D}_m\mathfrak{D}_n\mathcal{C}^n = 0 \quad (4.30)$$

The associated constraint is obtained by taking the divergence of (4.30):

$$[\mathfrak{D}^m, \mathfrak{D}^2]\mathcal{C}_m + 2i\epsilon^{mn}\mathfrak{D}_m\mathcal{C}_n - 2\mathfrak{D}^m\mathcal{C}_m = -2\mathfrak{D}^m\mathcal{C}_m = 0 \quad (4.31)$$

As for the remaining terms of the Lagrangian, a_m is now coupled to c_m and h_{mn} in the presence of the background, so dualisation with the (2.33)-(2.34) technique will inevitably couple the scalar A to other fields, making the Lagrangian more complicated. Such a form with $\{a_m, c_m\}$ replaced by $\{A, B\}$ will be presented later. First, we will study the equations of motion and constraints of (4.29), as well as the way the new spin-2 is modified by its couplings to the other fields.

4.3 The equations of motion

The Euler-Lagrange equations derived directly from the Lagrangian are complicated, not very illuminating and not in very useful forms. However, as we will see in this Section, a series of manipulations allows us to put them in the form of a Fierz-Pauli system, thus with a much more compact, and simple, expression for the equations of motion and associated constraints.

We start with the trace of the equations of motion of v_{mn} :

$$-\frac{1}{8}\mathfrak{D}^2 h - \frac{1}{2}\mathfrak{D}^m \mathfrak{D}^n v_{mn} + \frac{3}{16}\epsilon^{mn}\epsilon_{mn}h + \epsilon_{mk}\epsilon_n{}^k v^{mn} - \frac{1}{2}i\epsilon_{mn}\mathfrak{D}^n \mathfrak{D}_k v^{mk} - \mathfrak{D}^m c_m - i\epsilon_{mn}\mathfrak{D}^m c^n - \sqrt{2}i\tilde{\epsilon}_{mn}\mathfrak{D}^m a^n = 0 \quad (4.32)$$

and the equations of motion of h :

$$-\frac{3}{20}h + \frac{3}{32}\mathfrak{D}^2 h + \frac{3}{64}\epsilon^{mn}\epsilon_{mn}h - \frac{1}{8}\mathfrak{D}^m \mathfrak{D}^n v_{mn} - \frac{1}{8}i\epsilon_{mn}\mathfrak{D}^n \mathfrak{D}_k v^{mk} + \frac{1}{4}\epsilon_{nk}\epsilon_m{}^k v^{mn} + \frac{3}{20}\mathfrak{D}^m c_m - \frac{1}{4}i\epsilon^{mn}\mathfrak{D}_m c_n - \frac{1}{2\sqrt{2}}i\tilde{\epsilon}^{mn}\mathfrak{D}_m a_n = 0 \quad (4.33)$$

Taking the difference (4.32)−4×(4.33), we obtain

$$\mathfrak{D}^2 h = \frac{6}{5}h - \frac{16}{5}\mathfrak{D}^m c_m \quad (4.34)$$

We will need the explicit form of the equations of motion, and their divergence, of a_m

$$2a_m - i\epsilon_{mn}a^n - \mathfrak{D}_m \mathfrak{D}_n a^n - \frac{i}{\sqrt{2}}\epsilon^{nk}\epsilon_{mkpq}\mathfrak{D}^q h_n{}^p + \sqrt{2}i\tilde{\epsilon}_{mn}c^n = 0$$

$$2\mathfrak{D}^m a_m - i\epsilon_{mn}\mathfrak{D}^m a^n - \mathfrak{D}^2 \mathfrak{D}_m a^m - \frac{1}{4\sqrt{2}}\tilde{\epsilon}^{mn}\epsilon_{mn}h + \sqrt{2}i\tilde{\epsilon}_{mn}\mathfrak{D}^m c^n = 0 \quad (4.35)$$

and of c_m ,

$$-2c_m + \frac{2}{5}\mathfrak{D}_m \mathfrak{D}_n c^n + \sqrt{2}i\tilde{\epsilon}_{mn}a^n - \frac{2}{5}\mathfrak{D}_m h + (\eta^{nk} - i\epsilon^{nk})\mathfrak{D}_n h_{mk} = 0$$

$$-2\mathfrak{D}^m c_m + \frac{2}{5}\mathfrak{D}^2 \mathfrak{D}_m c^m + \sqrt{2}i\tilde{\epsilon}_{mn}\mathfrak{D}^m a^n - \frac{2}{5}\mathfrak{D}^2 h + (\eta^{nk} - i\epsilon^{nk})\mathfrak{D}^m \mathfrak{D}_n h_{mk} = 0 \quad (4.36)$$

At this stage, it is very complicated to decouple the equations of motion of the different fields, because those of h_{mn} contain many terms depending on ϵ . Indeed, being a component of the superfield V_m , h_{mn} always appears with one or two $(\eta - i\epsilon)$ factors. This suggests performing a rescaling that absorbs such factors:

$$\mathcal{H}_{mn} \equiv (\eta_{mk} - i\epsilon_{mk})h^k{}_n, \quad \mathcal{H} = h \quad (4.37)$$

Analogous rescalings were made for charged spin-2 states in the bosonic string case [18]. In fact, the rescaled spin-2 field in the Argyres-Nappi Lagrangian involves two factors, *i.e.* in the form $(\eta_{mk} - i\epsilon_{mk})(\eta_{nl} - i\epsilon_{nl})h^{kl}$. We will examine in a later subsection whether such rescaling is also useful for defining the field with spin-2 in our case.

The vectors $\{a_m, c_m\}$ also appear in the Lagrangian through their field strengths and duals, which we note:

$$\begin{aligned} F_{mn}(a) &\equiv \mathfrak{D}_m a_n - \mathfrak{D}_n a_m, & F_{mn}(c) &\equiv \mathfrak{D}_m c_n - \mathfrak{D}_n c_m \\ \tilde{F}_{mn}(a) &\equiv \frac{1}{2} \varepsilon_{mnpq} F^{pq}(a), & \tilde{F}_{mn}(c) &\equiv \frac{1}{2} \varepsilon_{mnpq} F^{pq}(c) \end{aligned} \quad (4.38)$$

The Lagrangian (4.29) can be written in a more compact form:

$$\begin{aligned} \mathcal{L}_2 &= \bar{\mathcal{C}}^m \mathfrak{D}^2 \mathcal{C}_m + \mathfrak{D}^m \bar{\mathcal{C}}_m \mathfrak{D}^n \mathcal{C}_n - 2\bar{\mathcal{C}}^m (\eta_{mn} - i\epsilon_{mn}) \mathcal{C}^n \\ &+ 2\bar{a}^m a_m - i\epsilon_{mn} \bar{a}^m a^n + \mathfrak{D}^m \bar{a}_m \mathfrak{D}^n a_n + \frac{1}{\sqrt{2}} \left[\bar{F}^{mn}(a) (F_{mn}(c) - \mathcal{H}_{[mn]}) + \text{h.c.} \right] \\ &- 2\bar{c}^m c_m - \frac{2}{5} \mathfrak{D}^m \bar{c}_m \mathfrak{D}^n c_n + \left[\bar{c}^m \left(-\frac{2}{5} \mathfrak{D}_m \mathcal{H} + \mathfrak{D}^n \mathcal{H}_{nm} \right) + \text{h.c.} \right] \\ &+ \frac{1}{2} \bar{\mathcal{H}}_{mn} \mathfrak{D}^2 h^{mn} + \frac{1}{2} \mathfrak{D}^n \bar{\mathcal{H}}_{mn} \mathfrak{D}_k h^{mk} - \bar{\mathcal{H}}^{(mn)} \mathcal{H}_{(mn)} + i\epsilon^{nk} \bar{\mathcal{H}}_{mn} h_k{}^m + \frac{1}{10} \bar{\mathcal{H}} \mathcal{H} \end{aligned} \quad (4.39)$$

Up to some trivial different choice of normalisation, (4.39) corresponds to (2.28) in the neutral case, so the equations of motion and constraints of the Lagrangian will reproduce (2.36) when ϵ is set to zero. The modification includes a mass term $-i\epsilon_{mn} \bar{a}^m a^n$, as well as a coupling of the dual field strength of a_m to the field strength of c_m and the anti-symmetric tensor $\mathcal{H}_{[mn]}$. This implies that a_m now appears in the constraint equations of h_{mn} .

The equation of motion of \mathcal{H}_{mn} , which we denote by \mathcal{R}_{mn} , takes a simpler form than that of h_{mn} . \mathcal{R}_{mn} reads:

$$\begin{aligned} \mathcal{R}_{mn} &\equiv -\frac{1}{\sqrt{2}} \varepsilon_{mnkl} \mathfrak{D}^k a^l + \frac{2}{5} \eta_{mn} \mathfrak{D}^k c_k - \mathfrak{D}_m c_n + \frac{1}{10} \eta_{mn} h + \frac{1}{2} \mathfrak{D}^2 h_{mn} - \frac{1}{2} \mathfrak{D}_n \mathfrak{D}^k h_{mk} \\ &- h_{mn} - \frac{3}{2} i\epsilon_{kn} h^k{}_m + \frac{1}{2} i\epsilon_m{}^k h_{kn} = 0 \end{aligned} \quad (4.40)$$

having the trace

$$\mathcal{R}^m{}_m = \frac{3}{5} \mathfrak{D}^m c_m - \frac{3}{5} h + \frac{1}{2} \mathfrak{D}^2 h - \frac{1}{2} \mathfrak{D}^m \mathfrak{D}^n h_{mn} = 0 \quad (4.41)$$

Plugging in (4.34), the above equation implies

$$\mathfrak{D}^m c_m + \frac{1}{2} \mathfrak{D}^m \mathfrak{D}^n h_{mn} = 0 \quad (4.42)$$

We can also compute the divergence of the equation of motion \mathcal{R}_{mn}

$$\begin{aligned} \mathfrak{D}^n \mathcal{R}_{mn} &= \frac{1}{10} \mathfrak{D}_m h - \mathfrak{D}^n h_{mn} + i\epsilon^{nk} \mathfrak{D}_k h_{mn} - \frac{3}{2} i\epsilon^{kn} \mathfrak{D}_n h_{mk} + \frac{1}{2} i\epsilon_{mk} \mathfrak{D}_n h^{kn} \\ &- \frac{1}{\sqrt{2}} i\tilde{c}_{mn} a^n - \frac{3}{5} \mathfrak{D}_m \mathfrak{D}_n c^n + i\epsilon_{mn} c^n \\ &= -c_m - \frac{2}{5} \mathfrak{D}_m \mathfrak{D}_n c^n + i\epsilon_{mn} c^n - \frac{1}{10} \mathfrak{D}_m h - \frac{1}{2} \mathfrak{D}^n h_{mn} + \frac{1}{2} i\epsilon_{mk} \mathfrak{D}_n h^{kn} = 0 \end{aligned} \quad (4.43)$$

where we used the equation of motion of c_m in the last line. Contracting \mathcal{R}_{mn} with the tensor $\tilde{\epsilon}$ and using the equation of motion of a_m , we obtain

$$\begin{aligned}\tilde{\epsilon}^{mn}\mathcal{R}_{mn} &= \sqrt{2}\epsilon^{mn}\mathcal{D}_m a_n - \tilde{\epsilon}^{mn}\mathcal{D}_m c_n + \frac{1}{2}\tilde{\epsilon}^{mn}\mathcal{D}_m\mathcal{D}^k h_{nk} - \frac{1}{4}\tilde{\epsilon}^{mn}\epsilon_{mn}h \\ &= \sqrt{2}i(\mathcal{D}^2 - 2)\mathcal{D}^m a_m + \tilde{\epsilon}^{mn}\mathcal{D}_m\left(c_n + \frac{1}{2}\mathcal{D}^k h_{nk}\right) = 0\end{aligned}\quad (4.44)$$

In order to derive a constraint on the trace of h_{mn} , we compute the double divergence of \mathcal{R}_{mn} and its contraction with ϵ :

$$\begin{aligned}\frac{1}{2}i\epsilon^{mn}\mathcal{R}_{mn} &= -\frac{34}{25}\mathcal{D}^m c_m + \frac{1}{5}\mathcal{D}^2\mathcal{D}^m c_m - \frac{1}{2}i\epsilon^{mn}\mathcal{D}_m c_n - \frac{6}{25}h - \frac{1}{4}i\epsilon^{nk}\mathcal{D}_n\mathcal{D}^m h_{mk} = 0 \\ \mathcal{D}^m\mathcal{D}^n\mathcal{R}_{mn} &= -\frac{2}{5}\mathcal{D}^2\mathcal{D}^m c_m + \frac{8}{25}\mathcal{D}^m c_m + i\epsilon_{mn}\mathcal{D}^m c^n - \frac{3}{25}h + \frac{1}{2}i\epsilon^{nk}\mathcal{D}_n\mathcal{D}^m h_{mk} = 0\end{aligned}\quad (4.45)$$

The sum ($i\epsilon^{mn}\mathcal{R}_{mn} + \mathcal{D}^m\mathcal{D}^n\mathcal{R}_{mn}$) gives then

$$h + 4\mathcal{D}^m c_m = 0\quad (4.46)$$

We retrieve that the trace is shifted by the divergence of c_m , as was the case for the neutral Lagrangian (2.28). Inserting this into (4.34) gives decoupled equations:

$$(\mathcal{D}^2 - 2)h = 0, \quad (\mathcal{D}^2 - 2)\mathcal{D}^m c_m = 0\quad (4.47)$$

The trace constraint (4.46) also allows to rewrite $\mathcal{D}^n\mathcal{R}_{mn}$ as

$$\mathcal{D}^n\mathcal{R}_{mn} = -(\eta_{mk} - i\epsilon_{mk})\left(c^k + \frac{1}{2}\mathcal{D}_n h^{kn}\right) = 0 \Rightarrow c_m + \frac{1}{2}\mathcal{D}^n h_{mn} = 0\quad (4.48)$$

The only difference with the neutral case is the covariant derivative.

The second equation of (4.36) can be simplified to

$$\sqrt{2}\tilde{\epsilon}^{mn}\mathcal{D}_m a_n = -2\epsilon^{mn}\mathcal{D}_m c_n + i\epsilon^{nk}\epsilon^m{}_n h_{mk}\quad (4.49)$$

whereas the first line of (4.44) gives

$$\sqrt{2}\epsilon^{mn}\mathcal{D}_m a_n - 2\tilde{\epsilon}^{mn}\mathcal{D}_m c_n - \frac{1}{4}i\tilde{\epsilon}^{mn}\epsilon_{mn}h = 0\quad (4.50)$$

Taking into account these results, the symmetric and anti-symmetric parts of \mathcal{R}_{mn} are respectively:

$$\begin{aligned}2\mathcal{R}_{(mn)} &= (\mathcal{D}^2 - 2)h_{mn} - 2i\left(\epsilon_{km}h^k{}_n + \epsilon_{kn}h^k{}_m\right) = 0 \\ 2\mathcal{R}_{[mn]} &= -\sqrt{2}\epsilon_{mnl}\mathcal{D}^k a^l - 2(\mathcal{D}_m c_n - \mathcal{D}_n c_m) + i\left(\epsilon_{km}h^k{}_n - \epsilon_{kn}h^k{}_m\right) = 0\end{aligned}\quad (4.51)$$

Here we recognise the first equation as an equation of motion, while the second establishes a duality relation between the field strength of a_m and the sum of the field strength of c_m and $\mathcal{H}_{[mn]} = \frac{1}{2}i(\epsilon_{km}h^k{}_n - \epsilon_{kn}h^k{}_m)$.

Replacing $\mathfrak{D}^n h_{mn}$ by $-2c_m$, the divergence $\mathfrak{D}^n \mathcal{R}_{(mn)}$ gives an equation for c_m

$$(\mathfrak{D}^2 - 2) c_m + 2i\epsilon_{mn} c^n = 0 \quad (4.52)$$

The divergence $\mathfrak{D}^n \mathcal{R}_{[mn]}$ combined with the equations of motion of c_m gives the same result. Different from the equation of motion of a massive charged vector boson with 3 propagating degrees of freedom, see for instance (4.30), the above equation lacks the divergence term $\mathfrak{D}_m \mathfrak{D}_n c^n$, because c_m has non-physical degrees of freedom.

The equations of motion of c_m (4.36) yields

$$4c_m + \frac{1}{2} \mathfrak{D}_m h = \sqrt{2} i \tilde{\epsilon}_{mn} a^n - i \epsilon^{nk} \mathfrak{D}_n h_{mk} \quad (4.53)$$

Finally, we can add $\epsilon_{mnpq} \mathfrak{D}^n \mathcal{R}^{pq}$ to the equation of motion of a_m to obtain

$$(\mathfrak{D}^2 - 2) a_m + 2i\epsilon_{mn} a^n = 0 \quad (4.54)$$

So far, we have derived decoupled equations of motion for $\{a_m, c_m, h_{mn}\}$ and a set of constraints that are not all independent of each other. Before determining the subset of independent constraints, we summarise the present results here:

- **Equations of motion**

$$\begin{aligned} (\mathfrak{D}^2 - 2) h &= 0, & (\mathfrak{D}^2 - 2) h_{mn} &= 2i \left(\epsilon_{km} h^k{}_n + \epsilon_{kn} h^k{}_m \right) \\ (\mathfrak{D}^2 - 2) \mathfrak{D}^m c_m &= 0, & (\mathfrak{D}^2 - 2) \mathfrak{D}^m a_m &= 0 \\ (\mathfrak{D}^2 - 2) a_m + 2i\epsilon_{mn} a^n &= 0, & (\mathfrak{D}^2 - 2) c_m + 2i\epsilon_{mn} c^n &= 0 \end{aligned} \quad (4.55)$$

- **Constraints**

$$\begin{aligned} i \left(\epsilon_{km} h^k{}_n - \epsilon_{kn} h^k{}_m \right) - 2 (\mathfrak{D}_m c_n - \mathfrak{D}_n c_m) &= \sqrt{2} \epsilon_{mnkl} \mathfrak{D}^k a^l \\ h + 4\mathfrak{D}^m c_m &= 0, & c_m + \frac{1}{2} \mathfrak{D}^n h_{mn} &= 0 \\ \sqrt{2} \tilde{\epsilon}^{mn} \mathfrak{D}_m a_n &= -2\epsilon^{mn} \mathfrak{D}_m c_n + i\epsilon^{nk} \epsilon^m{}_n h_{mk} \\ \sqrt{2} \epsilon^{mn} \mathfrak{D}_m a_n &= 2\tilde{\epsilon}^{mn} \mathfrak{D}_m c_n + \frac{1}{4} i \tilde{\epsilon}^{mn} \epsilon_{mn} h \\ 4c_m + \frac{1}{2} \mathfrak{D}_m h &= \sqrt{2} i \tilde{\epsilon}_{mn} a^n - i \epsilon^{nk} \mathfrak{D}_n h_{mk} \end{aligned} \quad (4.56)$$

Using the notations of (4.37)-(4.38), the first constraint can be rewritten as

$$\mathcal{H}_{[mn]} = \frac{1}{\sqrt{2}} \tilde{F}_{mn}(a) + F_{mn}(c) \quad (4.57)$$

from which we deduce the equation of motion and the divergence of the anti-symmetric tensor

$$\begin{aligned} (\mathfrak{D}^2 - 2) \mathcal{H}_{[mn]} + 2i \left(\epsilon_m{}^k \mathcal{H}_{[kn]} - \mathcal{H}_{[mk]} \epsilon^k{}_n \right) &= 0 \\ \mathfrak{D}^n \mathcal{H}_{[mn]} &= i\epsilon_{mn} c^n + \frac{i}{2} \epsilon^{nk} \mathfrak{D}_n h_{mk} \end{aligned} \quad (4.58)$$

As already stated, not all constraints in (4.56) are independent. The third and fourth lines can be obtained by contracting (4.57) with ϵ^{mn} and $\tilde{\epsilon}^{mn}$, respectively. Whereas the last line, formerly an independent constraint in the neutral case, can also be inferred from applying \mathfrak{D}^m on (4.57) in conjunction with the second line of (4.56):

$$\begin{aligned}\sqrt{2}\mathrm{i}\tilde{\epsilon}_{mn}a^n &= \mathrm{i}\epsilon_{km}\mathfrak{D}^n h^k{}_n + \mathrm{i}\epsilon^{nk}\mathfrak{D}_n h_{km} - 2\mathfrak{D}^n\mathfrak{D}_m c_n + 2\mathfrak{D}^2 c_m \\ &= 2\mathrm{i}\epsilon_{mn}c^n + \mathrm{i}\epsilon^{nk}\mathfrak{D}_n h_{km} - 2\mathfrak{D}_m(\mathfrak{D}_n c^n) + 2\mathrm{i}\epsilon_{mn}c^n + 4(c_m - \mathrm{i}\epsilon_{mn}c^n) \\ &= 4c_m + \mathrm{i}\epsilon^{nk}\mathfrak{D}_n h_{km} + \frac{1}{2}\mathfrak{D}_m h\end{aligned}\quad (4.59)$$

where we also used the equation of motion of c_m . As a side comment, given that the fourth line of (4.56) can be re-expressed as

$$\tilde{\epsilon}^{mn}\left(\mathcal{H}_{[mn]} - \frac{1}{\sqrt{2}}\tilde{F}_{mn}(a) - F_{mn}(c)\right) = 0 \quad (4.60)$$

which is valid for any non-vanishing ϵ , one may be tempted to conclude that the fourth line of (4.56) implies (4.57). In fact, the above equation holds for any $a'_m = a_m + \sqrt{2}x\tilde{\epsilon}_{mn}\mathfrak{D}^n\phi$, $c'_m = c_m + x\epsilon_{mn}\mathfrak{D}^n\phi$, with ϕ an arbitrary scalar field, and therefore does not have a unique solution.

In summary, only the first and second lines of (4.56) are independent, and hence are the only ones that will be considered for the counting of on-shell degrees of freedom. Here, the symmetric tensor h_{mn} along with two vector bosons $\{a_m, c_m\}$ counts $10 + 4 + 4$ complex degrees of freedom off-shell. The constraint $c_m = -\frac{1}{2}\mathfrak{D}^n h_{mn}$ removes 4 degrees of freedom, and $-\sqrt{2}\epsilon_{mnkl}\mathfrak{D}^k a^l - 2(\mathfrak{D}_m c_n - \mathfrak{D}_n c_m) + \mathrm{i}(\epsilon_{km}h^k{}_n - \epsilon_{kn}h^k{}_m) = 0$ removes 6 degrees of freedom, while $h = -4\mathfrak{D}^m c_m$ removes 1 degree of freedom. Therefore, we are left with 7 degrees of freedom on-shell in the Lagrangian (4.39). Together with the massive spin-1 \mathcal{C}_m and the physical scalars in \mathcal{L}_1 , the bosonic sector counts in total 12 complex degrees of freedom on-shell.

4.4 Decoupling the equations of motion

As vector fields dual to massive scalars, a_m and c_m satisfy the following equivalent sets of equations in the neutral case [32]:

$$\partial_m\partial_n V^n = 2V_m \iff \begin{cases} (\partial^2 - 2)V_m = 0 \\ \partial_m V_n - \partial_n V_m = 0 \end{cases} \quad (4.61)$$

Now consider a charged massive scalar Lagrangian $\mathcal{L} = \bar{A}(\mathfrak{D}^2 - 2)A$, we can introduce an auxiliary vector V_m endowed with a transformation that enables us to shift away the kinetic term of A , to integrate out the scalar and, at the end, obtain the equations of motion of the vector V_m dual to A . Effortlessly one generalises (4.61) to the charged case:

$$\mathfrak{D}_m\mathfrak{D}_n V^n = 2V_m \iff \begin{cases} (\mathfrak{D}^2 - 2)V_m + 2\mathrm{i}\epsilon_{mn}V^n = 0 \\ \mathfrak{D}_m V_n - \mathfrak{D}_n V_m = \frac{\mathrm{i}}{2}\epsilon_{mn}\mathfrak{D}_k V^k \end{cases} \quad (4.62)$$

Unsurprisingly, the first equation to the r.h.s. implies a gyromagnetic ratio $g = 2$ for V_m . Inspecting (4.35) and (4.36), we notice that a_m, c_m have coupled equations of motion. Alternatively, if one adopts the description at the r.h.s. above, the first equation $(\mathfrak{D}^2 - 2)V_m + 2i\epsilon_{mn}V^n = 0$ does decouple for $V_m = a_m, c_m$, as is written in (4.55). But the second one, which relates the field strength to the divergence of V_m , here given by (4.57) is coupled to the spin-2 field h_{mn} . In fact, there exists a redefinition of a_m and c_m that leads to decoupled equations of motion on shell:

$$\begin{aligned} a'_m &\equiv a_m - \frac{i}{2}\epsilon_{mn}a^n - \frac{i}{2\sqrt{2}}\tilde{\epsilon}^{nk}\mathfrak{D}_k h_{mn} + \frac{i}{2\sqrt{2}}\tilde{\epsilon}_{mn}\mathfrak{D}^n h + \sqrt{2}i\tilde{\epsilon}_{mn}c^n \\ c'_m &\equiv c_m - \frac{\sqrt{2}i}{4}\tilde{\epsilon}_{mn}a^n + \frac{i}{4}\epsilon^{nk}\mathfrak{D}_n h_{mk} \end{aligned} \quad (4.63)$$

in which case

$$\mathfrak{D}_m \mathfrak{D}_n a'^n = 2a'_m, \quad \mathfrak{D}_m \mathfrak{D}_n c'^n = 2c'_m \quad (4.64)$$

From the above equation we infer that a'_m, c'_m are equivalent to the gradient of a scalar and as such count one degree of freedom on shell. At first sight, it seems paradoxical because Eq. (4.57) may remove 6 degrees of freedom so we would be left with less degrees of freedom than before. In reality, if we rewrite the on-shell system (4.55)-(4.56) in terms of a'_m and c'_m , then Eq. (4.57) after this redefinition is no more an independent constraint: actually it can be obtained from the other constraints. We postpone the detailed discussion and the exact relation between the different constraints, to section 4.5 for the alternative form of the Lagrangian.

The next step is to identify a new rank-2 symmetric tensor that yields a Fierz-Pauli system on-shell. A simple guess inspired by (2.38) is:

$$\mathcal{H}'_{mn} = h_{mn} + \eta_{mn}\mathfrak{D}^k c_k \quad (4.65)$$

which satisfies the deformed Fierz-Pauli equations of motion and the traceless constraint:

$$(\mathfrak{D}^2 - 2)\mathcal{H}'_{mn} + 2i[(\epsilon \cdot \mathcal{H}')_{mn} - (\mathcal{H}' \cdot \epsilon)_{mn}] = 0, \quad \mathcal{H}' = 0 \quad (4.66)$$

It has a non-vanishing divergence, but vanishing *double* divergence due to (4.46)

$$\mathfrak{D}^n \mathcal{H}'_{mn} = -2c_m - \frac{1}{4}\mathfrak{D}_m h = -\frac{1}{\sqrt{2}}i\tilde{\epsilon}_{mn}a^n + \frac{1}{2}i\epsilon^{nk}\mathfrak{D}_n h_{mk}, \quad \mathfrak{D}^m \mathfrak{D}^n \mathcal{H}'_{mn} = 0 \quad (4.67)$$

The divergence equation can be re-expressed as

$$\mathfrak{D}^n \left(\mathcal{H}'_{mn} + \frac{1}{\sqrt{2}}\tilde{F}_{mn}(a) - \frac{i}{2}\epsilon_n{}^k h_{mk} \right) = 0 \quad (4.68)$$

To obtain the divergence-free tensor, one might be tempted to redefine the expression in parenthesis (which is, by the way, traceless) as the new spin-2 field, but the extra terms are not symmetric. Instead, We can absorb in \mathcal{H}'_{mn} the symmetric term $\epsilon_{(m}{}^k h_{n)k}$, so that the r.h.s. of the divergence equation depends only on a_m and c_m . Thus, we introduce

$$\begin{aligned} \mathcal{H}''_{mn} &\equiv \mathcal{H}'_{mn} - \frac{i}{2}\epsilon_n{}^k h_{mk} - \frac{i}{2}\epsilon_m{}^k h_{nk} \\ &= \mathcal{H}_{(mn)} + \eta_{mn}\mathfrak{D}^k c_k \end{aligned} \quad (4.69)$$

Note that $\mathcal{H}_{(mn)}$ is the symmetric part of the rescaled h_{mn} , defined in (4.37). This new spin-2 field satisfies

$$\begin{aligned} (\mathfrak{D}^2 - 2) \mathcal{H}''_{mn} + 2i [(\epsilon \cdot \mathcal{H}'')_{mn} - (\mathcal{H}'' \cdot \epsilon)_{mn}] &= 0, \quad \mathcal{H}'' = 0 \\ \mathfrak{D}^n \mathcal{H}''_{mn} &= -\frac{i}{\sqrt{2}} \tilde{\epsilon}_{mn} a^n + i \epsilon_{mn} c^n \end{aligned} \quad (4.70)$$

The divergence is indeed vanishing when $\epsilon = 0$.

Another possibility is to consider the following double scaling

$$\tilde{\mathcal{H}}_{mn} = (\eta_{mk} - i\epsilon_{mk}) (\eta_{nl} - i\epsilon_{nl}) h^{kl} \quad (4.71)$$

which allows to obtain a Fierz-Pauli system in the bosonic case studied in [18]. To see if (4.71) is suitable for our case, let us first study the trace

$$\tilde{\mathcal{H}} = h - i\sqrt{2} \tilde{\epsilon}^{mn} \mathfrak{D}_m a_n - 2i\epsilon^{mn} \mathfrak{D}_m c_n \quad (4.72)$$

Compared to (4.65), the trace has extra contributions of order ϵ , so (4.71) could be modified to

$$\tilde{\mathcal{H}}'_{mn} = (\eta_{mk} - i\epsilon_{mk}) (\eta_{nl} - i\epsilon_{nl}) h^{kl} + \frac{1}{4} \eta_{mn} \left(4\mathfrak{D}^k c_k + i\sqrt{2} \tilde{\epsilon}^{kl} \mathfrak{D}_k a_l + 2i\epsilon^{kl} \mathfrak{D}_k c_l \right), \quad \tilde{\mathcal{H}}' = 0 \quad (4.73)$$

One can also check that (4.73) satisfies the equations of motion of deformed Fierz-Pauli:

$$(\mathfrak{D}^2 - 2) \tilde{\mathcal{H}}'_{mn} + 2i \left[(\epsilon \cdot \tilde{\mathcal{H}}')_{mn} + (\epsilon \cdot \tilde{\mathcal{H}}')_{nm} \right] = 0 \quad (4.74)$$

However, the divergence of (4.73) is relatively complicated and contains second derivatives:

$$\begin{aligned} \mathfrak{D}^n \tilde{\mathcal{H}}'_{mn} &= -\frac{1}{2} i \epsilon^{nk} \mathfrak{D}_n h_{mk} - \frac{i}{\sqrt{2}} \tilde{\epsilon}_{mn} a^n + 2i \epsilon_{mn} c^n \\ &\quad + \frac{\sqrt{2}}{4} i \tilde{\epsilon}^{kl} \mathfrak{D}_m \mathfrak{D}_k a_l + \frac{1}{2} i \epsilon^{kl} \mathfrak{D}_m \mathfrak{D}_k c_l - \epsilon_{mk} \epsilon_{nl} \mathfrak{D}^n h^{kl} \end{aligned} \quad (4.75)$$

thus the rescaling in (4.71) results in a cumbersome divergence constraint.

In fact, in order to absorb the divergence, *i.e.* to have a zero divergence tensor, while keeping the zero trace condition, higher derivatives and new $\mathcal{O}(\epsilon^2)$ terms have to be included in the $\tilde{\mathcal{H}}'_{mn}$ definition. Starting from a generic ansatz containing such terms, and imposing the divergence and trace constraints, we found the following new spin-2 field definition

$$\begin{aligned} \mathfrak{h}_{mn} &\equiv \frac{4}{3} h_{mn} - \frac{1}{3} \eta_{mn} h - \frac{i}{2} (\epsilon_m^k h_{kn} + \epsilon_{nk} h^k_m) + \frac{1}{3} (\mathfrak{D}_m c_n + \mathfrak{D}_n c_m) \\ &\quad - \frac{i}{2} (\epsilon_{mk} \mathfrak{D}^k c_n + \epsilon_{nk} \mathfrak{D}^k c_m - \epsilon_{mk} \mathfrak{D}_n c^k - \epsilon_{nk} \mathfrak{D}_m c^k + \eta_{mn} \epsilon^{kl} \mathfrak{D}_k c_l) \\ &\quad - \frac{1}{4} (\epsilon_{mk} \epsilon^{lk} h_{nl} + \epsilon_{nk} \epsilon^{lk} h_{ml} + 2\epsilon_{mk} \epsilon_{nl} h^{kl} - \eta_{mn} \epsilon^{kl} \epsilon^p_l h_{kp}) \\ &\quad + \frac{1}{2 - \epsilon\epsilon} \left[\frac{1}{12} \mathfrak{D}_m \mathfrak{D}_n h - \frac{1}{16} \epsilon_{mk} \epsilon^k_n h + \frac{i}{8} \epsilon_{mk} \mathfrak{D}^k \mathfrak{D}_n h - \frac{5\epsilon\epsilon}{96} \eta_{mn} h + (m \leftrightarrow n) \right] \\ &\quad - \frac{1}{\sqrt{2}} \frac{1}{2 + \epsilon\epsilon} \left[-\frac{i}{2} (\tilde{\epsilon}_{mk} \mathfrak{D}^k \mathfrak{D}_n + \tilde{\epsilon}_{nk} \mathfrak{D}^k \mathfrak{D}_m) \mathfrak{D}_l a^l + \frac{5}{8} (\epsilon\tilde{\epsilon}) \eta_{mn} \mathfrak{D}^k a_k \right. \\ &\quad \left. - \frac{1}{4} (\epsilon\tilde{\epsilon}) (\mathfrak{D}_m \mathfrak{D}_n + \mathfrak{D}_n \mathfrak{D}_m) \mathfrak{D}_k a^k + (\tilde{\epsilon}_{mk} \epsilon_{ln} \mathfrak{D}^k \mathfrak{D}^l + \tilde{\epsilon}_{nk} \epsilon_{lm} \mathfrak{D}^k \mathfrak{D}^l) \mathfrak{D}^p a_p \right] \end{aligned} \quad (4.76)$$

With the help of (4.55)-(4.56), one can check that \mathfrak{h}_{mn} yields a Fierz-Pauli system, *i.e.* it satisfies the same equation of motion, with now both $\mathfrak{D}^n \mathfrak{h}_{mn} = 0$ and $\mathfrak{h} = 0$. Note that if we set the electromagnetic field to zero, using the constraint $8c_m + \partial_m h = 0$, we find that \mathfrak{h}_{mn} reduces to $\mathfrak{h}_{mn} \rightarrow \frac{4}{3}(h_{mn} + \eta_{mn} \partial^k c_k)$.

We can now rewrite (4.55)-(4.56) as decoupled equations:

$$\begin{aligned} (\mathfrak{D}^2 - 2) \mathfrak{h}_{mn} &= 2i \left(\epsilon_{km} \mathfrak{h}^k{}_n + \epsilon_{kn} \mathfrak{h}^k{}_m \right) \\ \mathfrak{D}^n \mathfrak{h}_{mn} &= 0, \\ \mathfrak{h} &= 0 \\ \mathfrak{D}_m \mathfrak{D}_n a'^m &= 2a'_m, \quad \mathfrak{D}_m \mathfrak{D}_n c'^m = 2c'_m, \end{aligned} \tag{4.77}$$

4.5 A deformed Fierz-Pauli bosonic Lagrangian

The form of the Lagrangian (4.39) is compact, and the electromagnetic field dependent terms are relatively simple. After some redefinitions and manipulations, we have shown how the equations of motion and the constraints reduce to a simple system. However, for the spin-2 field one cannot immediately recognize a deformed Fierz-Pauli Lagrangian, *i.e.* visibly giving a Fierz-Pauli Lagrangian if the electromagnetic field is zero. We present here an alternative expression of \mathcal{L}_2 in the form of a Fierz-Pauli deformation, along with two complex scalars, hence containing only physical degrees of freedom.

The dualisation of $\{a_m, c_m\}$ to the physical scalars can be carried out in the same way as in the section 2.2. Notice that in the Lagrangian Eq. (4.39), the derivative terms of $\{a_m, c_m\}$ are $\mathfrak{D}^n \bar{a}_n \mathfrak{D}^m a_m - \frac{2}{5} \mathfrak{D}^m \bar{c}_m \mathfrak{D}^n c_n$, therefore, one can add auxiliary scalar terms $-\bar{A}A + \frac{2}{5} \bar{B}B$ and shift away the vector kinetic terms by

$$A \rightarrow \sqrt{2}A + \mathfrak{D}^m a_m, \quad B \rightarrow B + \mathfrak{D}^m c_m \tag{4.78}$$

Then, we use the equation of motion of c_m

$$c_m = -\frac{1}{5} \mathfrak{D}_m h - \frac{1}{5} \mathfrak{D}_m B + \frac{1}{2} \mathfrak{D}^n \mathcal{H}_{nm} + \frac{i}{\sqrt{2}} \tilde{\epsilon}_{mn} a^n \tag{4.79}$$

to integrate it out, and perform redefinitions analogous to (2.40)-(2.43):

$$B \rightarrow B + \frac{3}{2} h, \quad h \rightarrow h + 4B \tag{4.80}$$

After this step, the equation of motion of a_m can be written as:

$$\begin{aligned} a^m &= -\frac{1}{\sqrt{2}} \mathcal{A}^{mn} \left[\mathfrak{D}_n A + \frac{i}{2} \tilde{\epsilon}_{nl} \mathfrak{D}^l B + \frac{1}{8} (\epsilon \tilde{\epsilon}) \mathfrak{D}_n B - \frac{1}{2} \epsilon_{nlpq} \mathfrak{D}^l \mathcal{H}^{pq} \right. \\ &\quad \left. + \frac{i}{2} \tilde{\epsilon}_{nl} \mathfrak{D}_p \mathcal{H}^{pl} - \frac{i}{2} \tilde{\epsilon}_{nl} \mathfrak{D}^l \mathcal{H} \right] \\ &\equiv \mathbf{a}^m \end{aligned} \tag{4.81}$$

where

$$\mathcal{A}_{mn} \equiv \left(\eta_{mn} - \frac{i}{2} \epsilon_{mn} - \frac{1}{2} \tilde{\epsilon}_{mk} \tilde{\epsilon}^k{}_n \right)^{-1} \tag{4.82}$$

The above expression denoted by \mathbf{a}_m will be recurrent in a later calculation. The following notations are introduced for shorthand:

$$(\epsilon\epsilon) \equiv \epsilon^{mn}\epsilon_{mn}, \quad (\epsilon\tilde{\epsilon}) \equiv \epsilon^{mn}\tilde{\epsilon}_{mn} \quad (4.83)$$

Integrating out a_m , we obtain a deformed Fierz-Pauli Lagrangian:

$$\begin{aligned} \mathcal{L}_2 = & \bar{\mathcal{C}}^m \mathcal{D}^2 \mathcal{C}_m + \mathcal{D}^m \bar{\mathcal{C}}_m \mathcal{D}^n \mathcal{C}_n - 2\bar{\mathcal{C}}^m (\eta_{mn} - i\epsilon_{mn}) \mathcal{C}^n \\ & + \left[\bar{A} \mathcal{D}_m - \frac{i}{2} \tilde{\epsilon}_{mb} \bar{B} \mathcal{D}^b + \frac{1}{8} (\epsilon\tilde{\epsilon}) \bar{B} \mathcal{D}_m - \frac{1}{2} \epsilon_{mabc} \bar{\mathcal{H}}^{bc} \mathcal{D}^a - \frac{i}{2} \tilde{\epsilon}_{ma} \bar{\mathcal{H}}^{ba} \mathcal{D}_b + \frac{i}{2} \tilde{\epsilon}_{mb} \bar{\mathcal{H}} \mathcal{D}^b \right] \\ & \times \mathcal{A}^{mn} \left[\mathcal{D}_n A + \frac{i}{2} \tilde{\epsilon}_{nl} \mathcal{D}^l B + \frac{1}{8} (\epsilon\tilde{\epsilon}) \mathcal{D}_n B - \frac{1}{2} \epsilon_{nlpq} \mathcal{D}^l \mathcal{H}^{pq} + \frac{i}{2} \tilde{\epsilon}_{nl} \mathcal{D}_p \mathcal{H}^{pl} - \frac{i}{2} \tilde{\epsilon}_{nl} \mathcal{D}^l \mathcal{H} \right] \\ & - 2\bar{A}A + \bar{B} (\mathcal{D}^2 - 2) B - \frac{1}{2} \epsilon_{mn} \epsilon^{mk} \bar{B} \mathcal{D}^n \mathcal{D}_k B \\ & + \frac{1}{2} \left[i \left(\mathcal{D}^n \bar{\mathcal{H}}_{nm} \epsilon^{mk} \mathcal{D}_k B \right) - \frac{1}{2} (\epsilon\epsilon) \bar{\mathcal{H}} B + \text{h.c.} \right] \\ & + \frac{1}{2} \bar{\mathcal{H}}_{(mn)} \mathcal{D}^2 h^{mn} + \frac{1}{2} \mathcal{D}^n \bar{\mathcal{H}}_{mn} \mathcal{D}_k h^{mk} + \frac{1}{2} \mathcal{D}^n \bar{\mathcal{H}}_{nm} \mathcal{D}_k \mathcal{H}^{km} + \frac{1}{2} (\bar{\mathcal{H}}^{mn} \mathcal{D}_m \mathcal{D}_n h + \text{h.c.}) \\ & - 2\bar{\mathcal{H}}^{(mn)} \mathcal{H}_{(mn)} + \bar{\mathcal{H}}^{(mn)} h_{mn} - \frac{1}{2} \bar{\mathcal{H}} (\mathcal{D}^2 - 2) h \\ & + \left(\bar{\mathcal{H}}^{[mn]} + \frac{1}{2} i \epsilon^{mn} \bar{B} \right) \left(\mathcal{H}_{[mn]} - \frac{1}{2} i \epsilon_{mn} B \right) \end{aligned} \quad (4.84)$$

When $\epsilon = 0$, the above Lagrangian is decoupled, with two scalars A, B , one massive vector C_m and a spin-2 with the well-known Fierz-Pauli Lagrangian.

The inverse matrix in the third line of (4.84) is in general cumbersome to deal with, since its contraction with covariant derivatives has no particularly remarkable identity, and here this matrix is kept as it is. For small background, one can expand this inverse matrix as a series of ϵ^n . As an alternative, we will explain in Appendix C how it can be expressed concretely in terms of ϵ and its dual $\tilde{\epsilon}$. We would nevertheless expect the inverse matrix not to appear at the level of the equations of motion and constraints, for the reason that these equations can be obtained directly from (4.55) and (4.56) by appropriate redefinitions without dependence on \mathcal{A}_{mn} . The inverse matrix may only appear in the divergence constraint of the spin-2 where a_m is replaced by its equation of motion (4.81), but in this case we can eliminate \mathcal{A}_{mn} by contracting the constraint with its inverse $(\eta_{mn} - \frac{1}{2}\epsilon_{mn} - \frac{1}{2}\tilde{\epsilon}_{mk}\tilde{\epsilon}^k_n)$.

More precisely, the steps leading to the equations of motion and constraints are similar to those in Section 4.3. First, one obtains directly the trace constraint from:

$$-2 \frac{\delta \mathcal{L}_2}{\delta \bar{B}} + 4 \frac{\delta \mathcal{L}_2}{\delta h} + \eta_{mn} \frac{\delta \mathcal{L}_2}{\delta v_{mn}} = 2h = 0 \quad \Rightarrow \quad h = 0 \quad (4.85)$$

For simplicity, we introduce the notation:

$$\mathcal{P}_{mn} \equiv \frac{\delta \mathcal{L}_2}{\delta \bar{\mathcal{H}}^{mn}} = 0 \quad (4.86)$$

The divergence constraint, that we note \mathcal{V}_m , is derived from $\mathfrak{D}^n \mathcal{P}_{mn}$:

$$\begin{aligned} \mathfrak{D}^n \mathcal{P}_{mn} = 0 &\Rightarrow -(\eta_{mn} - i\epsilon_{mn}) \mathcal{V}^n = 0 \Rightarrow \mathcal{V}_m = 0 \\ \mathcal{V}_m &\equiv \frac{1}{2} \mathfrak{D}^n h_{mn} + \frac{1}{2} \mathfrak{D}^n \mathcal{H}_{nm} + \frac{i}{2} \epsilon_{mn} \mathfrak{D}^n B + \frac{i}{\sqrt{2}} \tilde{\epsilon}_{mn} \mathbf{a}^n \end{aligned} \quad (4.87)$$

When $\epsilon = 0$, the above equation implies $\partial^n h_{mn} = 0$.

On the other hand, it can be shown that the double divergence of h_{mn} vanishes due to

$$3\mathfrak{D}^m \mathcal{V}_m - \mathcal{P}^m{}_m = 2\mathfrak{D}^m \mathfrak{D}^n h_{mn} = 0 \Rightarrow \mathfrak{D}^m \mathfrak{D}^n h_{mn} = 0 \quad (4.88)$$

Taking into account the relations $h = 0 = \mathfrak{D}^m \mathfrak{D}^n h_{mn}$, we can infer the equation of motion of the scalar B from:

$$\frac{\delta \mathcal{L}_2}{\delta B} - 4 \frac{\delta \mathcal{L}_2}{\delta h} + \frac{2}{3} \mathcal{P}^m{}_m = (\mathfrak{D}^2 - 2) B = 0 \quad (4.89)$$

The equation of motion of the massive charged spin-2 field lies in the symmetric part of \mathcal{P}_{mn} , in conjunction with the divergence constraint:

$$\mathcal{P}_{mn} + \mathcal{P}_{nm} - \frac{2}{3} \eta_{mn} \mathcal{P}^k{}_k + \mathfrak{D}_m \mathcal{V}_n + \mathfrak{D}_n \mathcal{V}_m = (\mathfrak{D}^2 - 2) h_{mn} + 2i \left(\epsilon_m{}^k h_{nk} + \epsilon_n{}^k h_{mk} \right) = 0 \quad (4.90)$$

and the equation of motion of A arises from

$$\frac{\delta \mathcal{L}_2}{\delta A} + \frac{i}{2} \tilde{\epsilon}^{mn} \mathcal{P}_{mn} - \frac{i}{2} \tilde{\epsilon}^{mn} \mathfrak{D}_m \mathcal{V}_n = (\mathfrak{D}^2 - 2) A = 0 \quad (4.91)$$

So far, we have obtained the decoupled equations of motion of the physical fields $\{h_{mn}, A, B\}$, counting 12 degrees of freedom off shell, along with the trace and divergence constraints $h = 0$ and $\mathcal{V}_m = 0$, which remove 5 degrees of freedom on shell. One may wonder what happens to the anti-symmetric part of \mathcal{P}_{mn} , as previously it gave the constraint (4.57) which fixed the field strengths of $\{a_m, c_m\}$. However, for the Lagrangian (4.84), there must not be a new independent constraint coming from $\mathcal{P}_{[mn]}$, otherwise we will lose degrees of freedom. We will show below that $\mathcal{P}_{[mn]}$ is deduced from the divergence constraint.

As is noticed before, one can put the divergence constraint in a form that is independent of \mathcal{A}_{mn} :

$$\begin{aligned} \mathcal{V}'_m &\equiv \left(\eta_{mn} - \frac{i}{2} \epsilon_{mn} - \frac{1}{2} \tilde{\epsilon}_{mk} \tilde{\epsilon}^k{}_n \right) \mathcal{V}^n \\ &= \left[1 - \frac{1}{8} (\epsilon\epsilon) \right] \mathfrak{D}^n h_{mn} - \frac{i}{2} \epsilon_{mk} \mathfrak{D}_n h^{nk} - \frac{i}{2} \epsilon^{nk} \mathfrak{D}_n h_{mk} + \frac{1}{2} \epsilon_{mk} \epsilon_{nl} \mathfrak{D}^l h^{nk} - \frac{1}{4} \epsilon_{mk} \epsilon^{kn} \mathfrak{D}^l h_{nl} \\ &\quad + \frac{1}{4} \epsilon^{kn} \epsilon_k{}^l (\mathfrak{D}_m h_{nl} - \mathfrak{D}_l h_{mn}) + \frac{1}{8} (\epsilon\epsilon) \mathfrak{D}_m B + \frac{i}{2} \epsilon_{mn} \mathfrak{D}^n B \\ &\quad + \frac{1}{2} \epsilon_{mk} \epsilon^{kn} \mathfrak{D}_n B - \frac{1}{2} i \tilde{\epsilon}_{mn} \mathfrak{D}^n A \\ &= 0 \end{aligned} \quad (4.92)$$

In the meantime, we find it simpler to study the anti-symmetric part $\mathcal{P}_{[mn]}$ shifted by the divergence constraint:

$$\begin{aligned}\mathcal{P}'_{[mn]} &\equiv \mathcal{P}_{[mn]} + \frac{1}{2}\mathfrak{D}_m\mathcal{V}_n - \frac{1}{2}\mathfrak{D}_n\mathcal{V}_m \\ &= \frac{1}{2}\mathfrak{D}^k\mathfrak{D}_m h_{kn} - \frac{1}{2}\mathfrak{D}^k\mathfrak{D}_n h_{km} - \frac{i}{2}\epsilon_{mn}B - \frac{1}{\sqrt{2}}\epsilon_{mnkl}\mathfrak{D}^k\mathbf{a}^l\end{aligned}\quad (4.93)$$

The question is then whether or not $\mathcal{P}'_{[mn]}$ imposes an independent constraint.

Combining (4.87) and (4.93), one obtains a first relation where \mathbf{a}_m is absent

$$\begin{aligned}2(\epsilon\tilde{\epsilon})\mathcal{P}'_{[mn]} + 8i\epsilon_{mnkl}\epsilon^{lp}\mathfrak{D}^k\mathcal{V}_p \\ = 4i\epsilon_{mnkl}\epsilon^{qk}\mathfrak{D}^l\mathfrak{D}^p(\mathcal{H}_{pq} + h_{pq}) + (\epsilon\tilde{\epsilon})\left(\mathfrak{D}_k\mathfrak{D}_m h_n{}^k - \mathfrak{D}_k\mathfrak{D}_n h_m{}^k\right) \\ - 4\epsilon_{nk}\tilde{\epsilon}_{ml}\mathfrak{D}^l\mathfrak{D}^k B + 4\epsilon_{mk}\tilde{\epsilon}_{nl}\mathfrak{D}^l\mathfrak{D}^k B + 4i\tilde{\epsilon}_{mk}\tilde{\epsilon}^{kl}\tilde{\epsilon}_{ln}B\end{aligned}\quad (4.94)$$

Then, due to the following properties of the inverse matrix \mathcal{A}_{mn}

$$\begin{aligned}(\epsilon\tilde{\epsilon})\tilde{\epsilon}_{mk}\mathcal{A}^k{}_n &= 8\epsilon_{nk}\mathcal{A}_m{}^k - 4i\mathcal{A}^{kl}\epsilon_{mk}\epsilon_{nl} + 8\epsilon_{mn} \\ \tilde{\epsilon}_{mk}\tilde{\epsilon}_{nl}\mathcal{A}^{kl} &= \eta_{mn} - 2\mathcal{A}_{mn} - i\mathcal{A}_m{}^k\epsilon_{nk}\end{aligned}\quad (4.95)$$

we find

$$\begin{aligned}[(\epsilon\tilde{\epsilon})\eta_{mn} + 4i\tilde{\epsilon}_{mn}]\mathcal{V}^n \\ = [(\epsilon\tilde{\epsilon})\eta_{mk} + 4i\tilde{\epsilon}_{mk}]\left[\mathfrak{D}_n h^{kn} - \frac{i}{2}\epsilon_{ab}\mathfrak{D}^a h^{kb} + \frac{i}{2}\epsilon^{kn}\mathfrak{D}_n B\right] - \sqrt{2}[8 - (\epsilon\epsilon)]\mathbf{a}_m \\ - 4(2\eta_m{}^n + i\epsilon_m{}^n)\left[\mathfrak{D}_n A + \frac{i}{2}\tilde{\epsilon}_{nl}\mathfrak{D}^l B + \frac{1}{8}(\epsilon\tilde{\epsilon})\mathfrak{D}_n B - \frac{1}{2}\epsilon_{nlpq}\mathfrak{D}^l\mathcal{H}^{pq} + \frac{i}{2}\tilde{\epsilon}_{nl}\mathfrak{D}_p\mathcal{H}^{pl}\right]\end{aligned}\quad (4.96)$$

where \mathbf{a}_m is multiplied by a numerical factor, so the above relation is useful to cancel the $\epsilon_{mnkl}\mathfrak{D}^k\mathbf{a}^l$ term in (4.93). Having (4.92), (4.94) and (4.96) at hand, and after some tedious algebra, we get finally

$$\begin{aligned}\frac{1}{4}(\epsilon\tilde{\epsilon})\mathcal{P}'_{[mn]} + \left[\frac{1}{8}(\epsilon\epsilon) - 1\right]\epsilon_{mnkl}\mathcal{P}'^{[kl]} + i\epsilon_{mnkl}\epsilon^{lp}\mathfrak{D}^k\mathcal{V}_p + \epsilon_{mnkl}\mathfrak{D}^k\mathcal{V}^l \\ + \frac{1}{8}[(\epsilon\tilde{\epsilon})\eta_{mk} + 4i\tilde{\epsilon}_{mk}]\mathfrak{D}_n\mathcal{V}^k - \frac{1}{8}[(\epsilon\tilde{\epsilon})\eta_{nk} + 4i\tilde{\epsilon}_{nk}]\mathfrak{D}_m\mathcal{V}^k = 0\end{aligned}\quad (4.97)$$

where the traceless constraint and the equations of motion have been used as well. Simple manipulations of (4.97) lead to $\mathcal{P}'_{[mn]}$ in terms of \mathcal{V}_m , for small background. To conclude, the anti-symmetric part of the \mathcal{H}_{mn} equation of motion can be derived from the divergence constraint \mathcal{V}_m and therefore does not remove additional d.o.f. from the system. Besides, the vanishing double divergence condition $\mathfrak{D}^m\mathfrak{D}^n h_{mn} = 0$ results from $i\epsilon^{mn}\mathcal{P}'_{[mn]} + 2\mathfrak{D}^m\mathcal{V}_m$.

To decouple the spin-2 from the scalar fields on shell, we can write (4.92) as the divergence of a traceless symmetric tensor, which will be the modification to the new spin-2.

We find the following expression

$$\begin{aligned}
\mathfrak{h}_{mn} \equiv & \frac{4}{3}h_{mn} - \frac{i}{2} \left(\epsilon_m^k h_{kn} + \epsilon_{nk} h^k_m \right) - \frac{1}{6} \left(\mathfrak{D}_m \mathfrak{D}_k h^k_n + \mathfrak{D}_n \mathfrak{D}_k h^k_m \right) - \frac{i}{4} \eta_{mn} \left(\epsilon^{kl} \mathfrak{D}^p \mathfrak{D}_l h_{kp} \right) \\
& + \frac{i}{4} \left(\epsilon_{mk} \mathfrak{D}^l \mathfrak{D}^k h_{nl} + \epsilon_{nk} \mathfrak{D}^l \mathfrak{D}^k h_{ml} - \epsilon_{mk} \mathfrak{D}_l \mathfrak{D}_n h^{kl} - \epsilon_{nk} \mathfrak{D}_l \mathfrak{D}_m h^{kl} \right) \\
& + \frac{1}{2 - \epsilon\epsilon} \left[\frac{1}{6} (\epsilon\epsilon) (\mathfrak{D}_m \mathfrak{D}_n B + \mathfrak{D}_n \mathfrak{D}_m B) + \frac{i}{2} \left(\epsilon_{mk} \mathfrak{D}^k \mathfrak{D}_n B + \epsilon_{nk} \mathfrak{D}^k \mathfrak{D}_m B \right) \right. \\
& \quad \left. - \left(\frac{1}{6} + \frac{1}{8} \epsilon\epsilon \right) (\epsilon\epsilon) \eta_{mn} B + \frac{1}{2} (1 - \epsilon\epsilon) \epsilon_m^k \epsilon_{kn} B \right] \\
& + \frac{1}{2 + \epsilon\epsilon} \left[-\frac{i}{2} \left(\tilde{\epsilon}_{mk} \mathfrak{D}^k \mathfrak{D}_n A + \tilde{\epsilon}_{nk} \mathfrak{D}^k \mathfrak{D}_m A \right) + \frac{5}{8} (\epsilon\tilde{\epsilon}) \eta_{mn} A \right. \\
& \quad \left. - \frac{1}{4} (\epsilon\tilde{\epsilon}) (\mathfrak{D}_m \mathfrak{D}_n A + \mathfrak{D}_n \mathfrak{D}_m A) + \left(\tilde{\epsilon}_{mk} \epsilon_{ln} \mathfrak{D}^k \mathfrak{D}^l A + \tilde{\epsilon}_{nk} \epsilon_{lm} \mathfrak{D}^k \mathfrak{D}^l A \right) \right]
\end{aligned} \tag{4.98}$$

The trace of \mathfrak{h}_{mn} is zero and its divergence coincides with \mathcal{V}'_m , therefore vanishes as well.

In summary, we derived in this subsection a Lagrangian (4.84) for a charged massive spin-2 field which is in the form of a Fierz-Pauli Lagrangian deformed by the electromagnetic background. In addition, we managed to decouple completely the massive vector boson \mathcal{C}_m from all other fields. Setting $\epsilon = 0$, (4.84) reduces trivially to the free Fierz-Pauli Lagrangian with two decoupled charged scalars and one charged vector fields. A simple analysis of the equations of motion and constraints confirms causal propagation of the spin-2. Moreover, we found the expression for the spin-2 field (4.98) which leads to a completely decoupled system of equations of motion and constraints, the four-dimensional analogue of those of the Argyres-Nappi Lagrangian is:

$$\begin{aligned}
(\mathfrak{D}^2 - 2) \mathfrak{h}_{mn} + 2i \left(\epsilon_m^k \mathfrak{h}_{nk} + \epsilon_n^k \mathfrak{h}_{mk} \right) &= 0 \\
\mathfrak{D}^n \mathfrak{h}_{mn} &= 0, \\
\mathfrak{h} &= 0 \\
\mathfrak{D}^2 \mathcal{C}_m - 2(\eta_{mn} - i\epsilon_{mn}) \mathcal{C}^n - \mathfrak{D}_m \mathfrak{D}_n \mathcal{C}^n &= 0, \\
\mathfrak{D}^m \mathcal{C}_m &= 0 \\
(\mathfrak{D}^2 - 2) A &= 0, \\
(\mathfrak{D}^2 - 2) B &= 0
\end{aligned} \tag{4.99}$$

5 Charged massive fermions

Moving from real to complex superfields doubles the number of fermions, hence the introduction of the indices $\{1, 2\}$. The physical fermions will give a total of 12 complex degrees of freedom on shell. In this section, we will first fix the gauge of the fermionic Lagrangian after the expansion of the superfields into components, then integrate the non-physical degrees of freedom. Once a compact Lagrangian is derived, we proceed to the derivation of the equations of motion and identify the choice of physical fermions which produces decoupled equations. We can then present a fully explicit set of equations of motion and constraints for the spin-3/2 charged states in a constant electromagnetic background, in both the two and four component notations. Finally, an alternative form of the fermionic Lagrangian will be given and the equations of motion will be re-derived from this new form. Despite its length and complexity, this last Lagrangian reproduces the Rarita-Schwinger one in the neutral limit and leads directly to a system of decoupled on-shell equations.

In the derivation of the equations of motion, we prefer to drop the spinorial indices to lighten the expressions, as long as no ambiguity is present.

5.1 Gauge transformations

As we have done for the bosonic case, we begin by listing the gauge transformations of the different components:

Fields in \mathcal{B} :

$$\begin{aligned}
\delta\gamma_{1\alpha} &= -i\mathcal{D}^2\Lambda_{1\alpha} - 4\mathcal{D}_m(\sigma^m\bar{\Upsilon}_5)_\alpha + 2\mathcal{D}^m\Lambda_{6m\alpha} - 4i\Lambda_{9\alpha} \\
\delta\bar{\gamma}_2^{\dot{\alpha}} &= i\mathcal{D}^2\bar{\Upsilon}_1^{\dot{\alpha}} + 4\mathcal{D}_m(\bar{\sigma}^m\Lambda_5)^{\dot{\alpha}} + 2\mathcal{D}^m\bar{\Upsilon}_{6m}^{\dot{\alpha}} + 4i\bar{\Upsilon}_9^{\dot{\alpha}} \\
\delta\rho_{2\alpha} &= \frac{1}{2}\mathcal{D}_m\mathcal{D}^2(\sigma^m\bar{\Upsilon}_1)_\alpha + 2i\mathcal{D}^2\Lambda_{5\alpha} - i\mathcal{D}_m\mathcal{D}_n(\sigma^m\bar{\Upsilon}_6^n)_\alpha + 2\mathcal{D}_m(\sigma^m\bar{\Upsilon}_9)_\alpha - 2(\epsilon \cdot \sigma)_\alpha{}^\beta\Lambda_{5\beta} \\
\delta\bar{\rho}_1^{\dot{\alpha}} &= -\frac{1}{2}\mathcal{D}_m\mathcal{D}^2(\bar{\sigma}^m\Lambda_1)^{\dot{\alpha}} - 2i\mathcal{D}^2\bar{\Upsilon}_5^{\dot{\alpha}} - i\mathcal{D}_m\mathcal{D}_n(\bar{\sigma}^m\Lambda_6^n)^{\dot{\alpha}} - 2\mathcal{D}_m(\bar{\sigma}^m\Lambda_9)^{\dot{\alpha}} + 2(\epsilon \cdot \bar{\sigma})^{\dot{\alpha}}{}_{\dot{\beta}}\bar{\Upsilon}_5^{\dot{\beta}}
\end{aligned} \tag{5.1}$$

Fields in \mathcal{C} :

$$\begin{aligned}
\delta\xi_{1\alpha} &= \mathcal{D}^2\Lambda_{1\alpha} + 4i\mathcal{D}_m(\sigma^m\bar{\Upsilon}_5)_\alpha + 2i\mathcal{D}^m\Lambda_{6m\alpha} + 4\Lambda_{9\alpha} \\
\delta\bar{\xi}_2^{\dot{\alpha}} &= \mathcal{D}^2\bar{\Upsilon}_1^{\dot{\alpha}} + 4i\mathcal{D}_m(\bar{\sigma}^m\Lambda_5)^{\dot{\alpha}} - 2i\mathcal{D}^m\bar{\Upsilon}_{6m}^{\dot{\alpha}} + 4\bar{\Upsilon}_9^{\dot{\alpha}} \\
\delta\psi_{2\alpha} &= -\frac{1}{2}i\mathcal{D}_m\mathcal{D}^2(\sigma^m\bar{\Upsilon}_1)_\alpha - 2[\mathcal{D}^2 + i(\epsilon \cdot \sigma)]_\alpha{}^\beta\Lambda_{5\beta} - \mathcal{D}_m\mathcal{D}_n(\sigma^m\bar{\Upsilon}_6^n)_\alpha - 2i\mathcal{D}_m(\sigma^m\bar{\Upsilon}_9)_\alpha \\
\delta\bar{\psi}_1^{\dot{\alpha}} &= -\frac{1}{2}i\mathcal{D}_m\mathcal{D}^2(\bar{\sigma}^m\Lambda_1)^{\dot{\alpha}} - 2[\mathcal{D}^2 + i(\epsilon \cdot \bar{\sigma})]^{\dot{\alpha}}{}_{\dot{\beta}}\bar{\Upsilon}_5^{\dot{\beta}} + \mathcal{D}_m\mathcal{D}_n(\bar{\sigma}^m\Lambda_6^n)^{\dot{\alpha}} - 2i\mathcal{D}_m(\bar{\sigma}^m\Lambda_9)^{\dot{\alpha}}
\end{aligned} \tag{5.2}$$

Fields in V_m :

$$\begin{aligned}
\delta\chi_{1m\alpha} &= -4[2\sigma_m\bar{\Upsilon}_5 + \sigma^n\bar{\sigma}_m(\Lambda_{6n} - i\mathcal{D}_n\Lambda_1)]_\alpha \\
\delta\bar{\chi}_{2m}^{\dot{\alpha}} &= 4[2\bar{\sigma}_m\Lambda_5 - \bar{\sigma}^n\sigma_m(\bar{\Upsilon}_{6n} + i\mathcal{D}_n\bar{\Upsilon}_1)]^{\dot{\alpha}} \\
\delta\lambda_{2m\alpha} &= -4i(\sigma_n\bar{\sigma}_m\mathcal{D}^n\Lambda_5)_\alpha + 8(\sigma_m\bar{\Upsilon}_9)_\alpha + 2i\left(\sigma^n\bar{\sigma}^k\sigma_m\mathcal{D}_k\bar{\Upsilon}_{6n}\right)_\alpha + 2i(\epsilon \cdot \sigma)\sigma_m\bar{\Upsilon}_{1\alpha} \\
\delta\bar{\lambda}_{1m}^{\dot{\alpha}} &= 4i(\bar{\sigma}^n\sigma_m\mathcal{D}_n\bar{\Upsilon}_5)^{\dot{\alpha}} - 8(\bar{\sigma}_m\Lambda_9)^{\dot{\alpha}} + 2i\left(\bar{\sigma}^n\sigma^k\bar{\sigma}_m\mathcal{D}_k\Lambda_{6n}\right)^{\dot{\alpha}} - 2i(\epsilon \cdot \bar{\sigma})\bar{\sigma}_m\Lambda_1^{\dot{\alpha}}
\end{aligned} \tag{5.3}$$

Fields in $U_{1\alpha}, \bar{U}_2^{\dot{\alpha}}$:

$$\begin{aligned}
\delta v_{1\alpha} &= 2i(8 + \mathcal{D}^2) \Lambda_{1\alpha} - 8\mathcal{D}_m (\sigma^m \bar{\Upsilon}_5)_\alpha - 4\mathcal{D}^m \Lambda_{6m\alpha} + 8i\Lambda_{9\alpha} - 8(\epsilon \cdot \sigma)_\alpha{}^\beta \Lambda_{1\beta} \\
\delta \bar{v}_2^{\dot{\alpha}} &= -2i(8 + \mathcal{D}^2) \bar{\Upsilon}_1^{\dot{\alpha}} + 8\mathcal{D}^m (\bar{\sigma}_m \Lambda_5)^{\dot{\alpha}} - 4\mathcal{D}^m \bar{\Upsilon}_{6m}^{\dot{\alpha}} - 8i\bar{\Upsilon}_9^{\dot{\alpha}} + 8(\epsilon \cdot \bar{\sigma})^{\dot{\alpha}}{}_\beta \bar{\Upsilon}_1^{\dot{\beta}} \\
\delta \eta_{1\alpha} &= 16i\Lambda_{4\alpha} - 8(\epsilon \cdot \sigma)_\alpha{}^\beta \Lambda_{4\beta} \\
\delta \bar{\eta}_2^{\dot{\alpha}} &= -16i\bar{\Upsilon}_4^{\dot{\alpha}} + 8(\epsilon \cdot \bar{\sigma})^{\dot{\alpha}}{}_\beta \bar{\Upsilon}_4^{\dot{\beta}} \\
\delta \zeta_{1\alpha} &= -2\mathcal{D}^2 \mathcal{D}_m (\sigma^m \bar{\Upsilon}_1)_\alpha + 8i(2 + \mathcal{D}^2) \Lambda_{5\alpha} + 4i\mathcal{D}^m \mathcal{D}^n (\sigma_m \bar{\Upsilon}_{6n})_\alpha - 8\mathcal{D}^m (\sigma_m \bar{\Upsilon}_9)_\alpha \\
&\quad - 8(\epsilon \cdot \sigma)_\alpha{}^\beta \Lambda_{5\beta} + 4i\epsilon_{mn} \mathcal{D}^m (\sigma^n \bar{\Upsilon}_1)_\alpha \\
\delta \bar{\zeta}_2^{\dot{\alpha}} &= 2\mathcal{D}^2 \mathcal{D}_m (\bar{\sigma}^m \Lambda_1)^{\dot{\alpha}} - 8i(2 + \mathcal{D}^2) \bar{\Upsilon}_5^{\dot{\alpha}} + 4i\mathcal{D}^m \mathcal{D}^n (\bar{\sigma}_m \Lambda_{6n})^{\dot{\alpha}} + 8\mathcal{D}^m (\bar{\sigma}_m \Lambda_9)^{\dot{\alpha}} \\
&\quad + 8(\epsilon \cdot \bar{\sigma})^{\dot{\alpha}}{}_\beta \bar{\Upsilon}_5^{\dot{\beta}} - 4i\epsilon_{mn} \mathcal{D}^m (\bar{\sigma}^n \Lambda_1)^{\dot{\alpha}} \\
\delta r_{1m\alpha} &= 8i\mathcal{D}_m \mathcal{D}_n (\sigma^n \bar{\Upsilon}_5)_\alpha + 2\mathcal{D}_m (4\Lambda_{9\alpha} + 2i\mathcal{D}^n \Lambda_{6n\alpha} + \mathcal{D}^2 \Lambda_{1\alpha}) + 16i\Lambda_{6m\alpha} \\
&\quad - 8(\epsilon \cdot \sigma)_\alpha{}^\beta \Lambda_{6m\beta} + 8(\epsilon_{mn} - i\tilde{\epsilon}_{mn}) (\Lambda_{6\alpha}^n - i\mathcal{D}^n \Lambda_{1\alpha}) - 4(\epsilon_{kn} - i\tilde{\epsilon}_{kn}) (\sigma_m \bar{\sigma}^{kn} \bar{\Upsilon}_5)_\alpha \\
\delta \bar{r}_{2m}^{\dot{\alpha}} &= 8i\mathcal{D}_m \mathcal{D}_n (\bar{\sigma}^n \Lambda_5)^{\dot{\alpha}} + 2\mathcal{D}_m (4\bar{\Upsilon}_9^{\dot{\alpha}} - 2i\mathcal{D}^n \bar{\Upsilon}_{6n}^{\dot{\alpha}} + \mathcal{D}^2 \bar{\Upsilon}_1^{\dot{\alpha}}) - 16i\bar{\Upsilon}_{6m}^{\dot{\alpha}} \\
&\quad + 8(\epsilon \cdot \bar{\sigma})^{\dot{\alpha}}{}_\beta \bar{\Upsilon}_{6m}^{\dot{\beta}} - 8(\epsilon_{mn} + i\tilde{\epsilon}_{mn}) (\bar{\Upsilon}_6^{n\dot{\alpha}} + i\mathcal{D}^n \bar{\Upsilon}_1^{\dot{\alpha}}) - 4(\epsilon_{kn} + i\tilde{\epsilon}_{kn}) (\bar{\sigma}_m \sigma^{kn} \Lambda_5)^{\dot{\alpha}} \\
\delta \mu_{1\alpha} &= \frac{i}{2} \mathcal{D}^2 (\mathcal{D}^2 \Lambda_{1\alpha} + 2i\mathcal{D}^m \Lambda_{6m\alpha} + 4\Lambda_{9\alpha}) - 2\mathcal{D}^2 \mathcal{D}_m (\sigma^m \bar{\Upsilon}_5)_\alpha + 16i\Lambda_{9\alpha} - 8(\epsilon \cdot \sigma)_\alpha{}^\beta \Lambda_{9\beta} \\
&\quad - (\epsilon_{mn} - i\tilde{\epsilon}_{mn}) [2i\epsilon^{mn} \Lambda_{1\alpha} + 4i\mathcal{D}^n (\sigma^m \bar{\Upsilon}_5)_\alpha + 4i\mathcal{D}^m \Lambda_{6\alpha}^n] \\
\delta \bar{\mu}_2^{\dot{\alpha}} &= \frac{i}{2} \mathcal{D}^2 (-\mathcal{D}^2 \bar{\Upsilon}_1^{\dot{\alpha}} + 2i\mathcal{D}^n \bar{\Upsilon}_{6n}^{\dot{\alpha}} - 4\bar{\Upsilon}_9^{\dot{\alpha}}) + 2\mathcal{D}^2 \mathcal{D}_m (\bar{\sigma}^m \Lambda_5)^{\dot{\alpha}} - 16i\bar{\Upsilon}_9^{\dot{\alpha}} + 8(\epsilon \cdot \bar{\sigma})^{\dot{\alpha}}{}_\beta \bar{\Upsilon}_9^{\dot{\beta}} \\
&\quad + (\epsilon_{mn} + i\tilde{\epsilon}_{mn}) [2i\epsilon^{mn} \bar{\Upsilon}_1^{\dot{\alpha}} - 4i\mathcal{D}^m (\bar{\sigma}^n \Lambda_5)^{\dot{\alpha}} - 4i\mathcal{D}^m \bar{\Upsilon}_6^{n\dot{\alpha}}]
\end{aligned} \tag{5.4}$$

As in the neutral case, $\delta \eta_{1\alpha}$ and $\delta \bar{\eta}_2^{\dot{\alpha}}$ are algebraic in the parameters $\Lambda_{4\alpha}$ and $\bar{\Upsilon}_4^{\dot{\alpha}}$, which do not appear elsewhere, thus these fields should be absent in the Lagrangian. The algebraic dependence on different gauge parameters is unaffected by the presence of a background, thus we can still choose to gauge away algebraically $\{\xi_{j\alpha}, v_{j\alpha}, \zeta_{j\alpha}, r_{jm\alpha}\}$ and integrate out $\rho_{j\alpha}$, with $j = 1, 2$. The following gauge invariant combinations are useful to keep in mind:

$$\begin{aligned}
\delta \left(\rho_{2\alpha} + \frac{1}{2} i (\sigma^m \mathcal{D}_m \bar{\gamma}_2)_\alpha \right) &= -4(\epsilon \cdot \sigma)_\alpha{}^\beta \Lambda_{5\beta} \\
\delta \left(\bar{\rho}_1^{\dot{\alpha}} + \frac{1}{2} i (\bar{\sigma}^m \mathcal{D}_m \gamma_1)^{\dot{\alpha}} \right) &= 4(\epsilon \cdot \bar{\sigma})^{\dot{\alpha}}{}_\beta \bar{\Upsilon}_5^{\dot{\beta}} \\
\delta \left(\psi_{2\alpha} + \frac{1}{2} i (\sigma^m \mathcal{D}_m \bar{\xi}_2)_\alpha \right) &= -4i(\epsilon \cdot \sigma)_\alpha{}^\beta \Lambda_{5\beta} \\
\delta \left(\bar{\psi}_1^{\dot{\alpha}} + \frac{1}{2} i (\bar{\sigma}^m \mathcal{D}_m \xi_1)^{\dot{\alpha}} \right) &= -4i(\epsilon \cdot \bar{\sigma})^{\dot{\alpha}}{}_\beta \bar{\Upsilon}_5^{\dot{\beta}} \\
\delta (\zeta_{1\alpha} + 2(\sigma^m \mathcal{D}_m \bar{\xi}_2)_\alpha) &= 16i(\delta_\alpha{}^\beta + i(\epsilon \cdot \sigma)_\alpha{}^\beta) \Lambda_{5\beta} \\
\delta (\bar{\zeta}_2^{\dot{\alpha}} - 2(\bar{\sigma}^m \mathcal{D}_m \xi_1^{\dot{\alpha}})_\alpha) &= -16i(\delta^{\dot{\alpha}}{}_\beta + i(\epsilon \cdot \bar{\sigma})^{\dot{\alpha}}{}_\beta) \bar{\Upsilon}_5^{\dot{\beta}}
\end{aligned} \tag{5.5}$$

5.2 A compact fermionic Lagrangian

As with h_{mn} and C_m in the bosonic sector, fermions in the V_m superfield will also appear, in the Lagrangian, contracted with one or two $(1 - i\epsilon)$ factors. In order to make the formulas more concise, we introduce the *rescaled* spinors denoted with bold symbols:

$$\begin{aligned}\bar{\lambda}_{1m} &\equiv (\eta_{mn} - i\epsilon_{mn}) \bar{\lambda}_1^n, & \lambda_{2m} &\equiv (\eta_{mn} - i\epsilon_{mn}) \lambda_2^n \\ \chi_{1m} &\equiv (\eta_{mn} - i\epsilon_{mn}) \chi_1^n, & \bar{\chi}_{2m} &\equiv (\eta_{mn} - i\epsilon_{mn}) \bar{\chi}_2^n\end{aligned}\tag{5.6}$$

The Lagrangian of the fermionic fields extracted from (2.1) reads:

$$\begin{aligned}\mathcal{L}_F &= -\frac{i}{8} \left[4 (\lambda_1^m \sigma^n \mathcal{D}_n \bar{\lambda}_{1m}) - (\bar{\chi}_1^m \bar{\sigma}^n \sigma^k \bar{\sigma}^l \mathcal{D}_n \mathcal{D}_k \mathcal{D}_l \chi_{1m}) \right] \\ &\quad - \frac{1}{4} \left[(\bar{\chi}_1^m \bar{\sigma}^n \sigma^k \mathcal{D}_n \mathcal{D}_k \bar{\lambda}_{1m}) + (\lambda_1^m \sigma^n \bar{\sigma}^k \mathcal{D}_n \mathcal{D}_k \chi_{1m}) \right] \\ &\quad - \left[(\lambda_1^m \chi_{1m}) - 6 (\psi_1 \xi_1) - \frac{15}{4} (\psi_1 \sigma^m \bar{\sigma}^n \mathcal{D}_m \mathcal{D}_n \xi_1) + \text{h.c.} \right] \\ &\quad - \frac{33}{8} i \left[(\bar{\xi}_1 \bar{\sigma}^m \sigma^n \bar{\sigma}^k \mathcal{D}_m \mathcal{D}_n \mathcal{D}_k \xi_1) - 4 (\psi_1 \sigma^m \mathcal{D}_m \bar{\psi}_1) \right] \\ &\quad + \frac{3}{2} \left[i (\lambda_1^m \mathcal{D}_m \xi_1) - 2 (\lambda_1^m \sigma_m \bar{\psi}_1) + \frac{1}{2} (\bar{\chi}_1^m \bar{\sigma}_m \mathcal{D}^2 \xi_1) - i (\bar{\chi}_1^m \bar{\sigma}_m \sigma_n \mathcal{D}^n \bar{\psi}_1) \right. \\ &\quad \quad \left. - (\bar{\chi}_1^m \bar{\sigma}^n \mathcal{D}_m \mathcal{D}_n \xi_1) + 2i (\lambda_1^m \sigma_{mn} \mathcal{D}^n \xi_1) + \frac{1}{2} i \bar{\chi}_1^m (\epsilon \cdot \bar{\sigma}) \bar{\sigma}_m \xi_1 + \text{h.c.} \right] \\ &\quad - \frac{1}{4} \left[(\lambda_1^m \mathcal{D}_m v_1) + 2 (\lambda_1^m \sigma_{mn} \mathcal{D}^n v_1) - 2i (\bar{\chi}_1^m \bar{\sigma}_m \mu_1) + (\bar{\chi}_1^m \bar{\sigma}^n \sigma_m \mathcal{D}_n \bar{\zeta}_2) \right. \\ &\quad \quad + 2i (\lambda_1^m \sigma_m \bar{\zeta}_2) - i (\lambda_{1m} \sigma^n \bar{\sigma}^m r_{1n}) - \frac{1}{2} (\bar{\chi}_1^m \bar{\sigma}^n \mathcal{D}_n r_{1m}) + \frac{1}{2} (\bar{\chi}_1^m \bar{\sigma}^n \mathcal{D}_m r_{1n}) \\ &\quad \quad \left. + \frac{1}{2} (\bar{\chi}_1^m \bar{\sigma}_m \mathcal{D}_n r_1^n) - \frac{i}{2} \epsilon_{mnpq} (\bar{\chi}_1^m \bar{\sigma}^n \mathcal{D}^p r_1^q) + \frac{1}{2} \bar{\chi}_1^m (\epsilon \cdot \bar{\sigma}) \bar{\sigma}_m v_1 + \text{h.c.} \right] \\ &\quad + \frac{1}{4} \left[i (v_1 \sigma^m \mathcal{D}_m \bar{\mu}_1) - \frac{1}{4} (r_{1m} \sigma^m \mathcal{D}^2 \bar{v}_1) + \frac{1}{2} (r_{1m} \sigma^n \mathcal{D}^m \mathcal{D}_n \bar{v}_1) + (r_{1m} \sigma^m \bar{\mu}_1) + \text{h.c.} \right] \\ &\quad + \left[\frac{i}{2} (\zeta_1 \sigma^m \mathcal{D}_m \bar{\zeta}_1) - \frac{i}{8} (r_1^m \sigma^n \mathcal{D}_n \bar{r}_{1m}) - \frac{1}{8} \epsilon^{mnpq} (r_{1m} \sigma_n \mathcal{D}_k \bar{r}_{1l}) \right] \\ &\quad + \frac{i}{8} \tilde{\epsilon}_{mn} (v_1 \sigma^m \mathcal{D}^n \bar{v}_1) + \left[\frac{i}{8} r_{1m} (\epsilon \cdot \sigma) \sigma_m \bar{v}_1 + \text{h.c.} \right] \\ &\quad + \left[\frac{i}{2} (r_2^m \mathcal{D}_m \zeta_1) - (\mu_2 \zeta_1) - \frac{1}{4} (v_2 \mathcal{D}^2 \zeta_1) + \text{h.c.} \right] \\ &\quad + \frac{i}{2} \left(\rho_1 + \frac{1}{2} i \bar{\gamma}_1 \bar{\sigma}^m \mathcal{D}_m \right) \sigma^n \mathcal{D}_n \left(\bar{\rho}_1 + \frac{1}{2} i \bar{\sigma}^k \mathcal{D}_k \gamma_1 \right) - 2 [(\rho_1 \gamma_1) + \text{h.c.}] \\ &\quad + \frac{3}{4} \left[\left(\rho_1 + \frac{1}{2} i \bar{\gamma}_1 \bar{\sigma}^m \mathcal{D}_m \right) (i \sigma^m \bar{\sigma}^n \mathcal{D}_m \mathcal{D}_n \xi_1 + 2 \sigma^n \mathcal{D}_n \bar{\psi}_1) + \text{h.c.} \right]\end{aligned}\tag{5.7}$$

$$\begin{aligned}
& - \left[\bar{\chi}_1^m \left(\mathfrak{D}_m \bar{\rho}_1 - \frac{i}{2} \tilde{\epsilon}_{mn} \bar{\sigma}^n \gamma_1 \right) + \lambda_1^m \mathfrak{D}_m \gamma_1 + \text{h.c.} \right] \\
& + \frac{1}{8} \left[(\mathfrak{D}^2 \bar{v}_1 - 2i \bar{v}_1 (\epsilon \cdot \bar{\sigma}) - 4\bar{\mu}_1 + 4i \bar{r}_{1m} \bar{\sigma}^{mn} \mathfrak{D}_n) \left(\bar{\rho}_1 + \frac{1}{2} i \bar{\sigma}^k \mathfrak{D}_k \gamma_1 \right) + \text{h.c.} \right] \\
& + \left[\frac{i}{2} \bar{v}_1 (\epsilon \cdot \bar{\sigma}) \bar{\rho}_1 - \frac{i}{2} \bar{\zeta}_1 (\epsilon \cdot \bar{\sigma}) \bar{\gamma}_2 + \frac{i}{4} \bar{r}_1^m (\epsilon \cdot \bar{\sigma}) \bar{\sigma}_m \gamma_1 + \text{h.c.} \right] \\
& - \frac{9}{8} \left[(\mathfrak{D}^2 \bar{v}_1 - 2i \bar{v}_1 (\epsilon \cdot \bar{\sigma}) - 4\bar{\mu}_1 + 4i \bar{r}_{1m} \bar{\sigma}^{mn} \mathfrak{D}_n) \left(i \bar{\psi}_1 - \frac{1}{2} \bar{\sigma}^k \mathfrak{D}_k \xi_1 \right) + \text{h.c.} \right] \\
& + \left[3\bar{v}_1 (\epsilon \cdot \bar{\sigma}) \bar{\psi}_1 - \frac{3}{2} i \bar{\epsilon}^{mn} (\bar{r}_{1m} \bar{\sigma}_n \xi_1) + \frac{3i}{4} \bar{v}_1 (\epsilon \cdot \bar{\sigma}) \bar{\sigma}^m \mathfrak{D}_m \xi_1 \right. \\
& \quad + \frac{3}{2} (\bar{r}_1^m \bar{\sigma}^n \sigma_m \mathfrak{D}_n \bar{\psi}_1) - \frac{3i}{2} (\bar{\zeta}_1 \mathfrak{D}^2 \bar{\xi}_2) + \frac{3i}{4} (\bar{r}_1^m \bar{\sigma}_m \mathfrak{D}^2 \xi_1) \\
& \quad \left. + 3(\bar{\mu}_1 \bar{\sigma}^m \mathfrak{D}_m \xi_1) - 3(\bar{\zeta}_1 \bar{\sigma}^m \mathfrak{D}_m \psi_2) + \frac{3i}{2} (\bar{v}_1 \mathfrak{D}^2 \bar{\psi}_1) + \text{h.c.} \right] + (1 \leftrightarrow 2, \epsilon \leftrightarrow -\epsilon)
\end{aligned}$$

Setting $\epsilon = 0$ and $\mathfrak{D} \rightarrow \partial$, we retrieve (2.56). As we argued when analysing the gauge transformations, the fermions $\eta_{1\alpha}$, $\bar{\eta}_2^{\dot{\alpha}}$ are absent. Moreover, $\mu_{1\alpha}$, $\bar{\mu}_2^{\dot{\alpha}}$ are Lagrange multipliers.

For the gauge fixing procedure, we can directly generalise the one followed in the neutral case. We will illustrate it by listing its different steps:

1. We algebraically gauge away $v_{j\alpha}$, which is still the Stückelberg field of $r_{jm\alpha}$

$$\mu_{j\alpha} \rightarrow \mu_{j\alpha} + \frac{1}{4} \mathfrak{D}^2 v_{j\alpha}, \quad r_{jm\alpha} \rightarrow r_{jm\alpha} + i \mathfrak{D}_m v_{j\alpha}, \quad j = 1, 2 \quad (5.8)$$

2. We eliminate $\gamma_{j\alpha}$, $\psi_{j\alpha}$, $\chi_{jm\alpha}$ higher derivatives by the shifts

$$\begin{aligned}
\rho_{j\alpha} & \rightarrow \rho_{j\alpha} - \frac{1}{2} i (\sigma^m \mathfrak{D}_m \bar{\gamma}_j)_\alpha \\
\lambda_{jm\alpha} & \rightarrow \lambda_{jm\alpha} + \frac{i}{2} (\sigma^n \mathfrak{D}_n \bar{\chi}_{jm})_\alpha \\
\psi_{j\alpha} & \rightarrow \psi_{j\alpha} + \frac{i}{2} (\sigma^n \mathfrak{D}_n \bar{\xi}_j)_\alpha \\
r_{jm\alpha} & \rightarrow r_{jm\alpha} + 4 \mathfrak{D}_m \xi_{j\alpha}, \quad j = 1, 2 \\
\zeta_{1\alpha} & \rightarrow \zeta_{1\alpha} - 2 (\sigma^m \mathfrak{D}_m \bar{\xi}_2)_\alpha, \quad \bar{\zeta}_2^{\dot{\alpha}} \rightarrow \bar{\zeta}_2^{\dot{\alpha}} + 2 (\bar{\sigma}^m \mathfrak{D}_m \xi_1)^{\dot{\alpha}}
\end{aligned} \quad (5.9)$$

After this step, the spin-3/2 states $\chi_{jm\alpha}$ appear only in their rescaled form $\chi_{jm\alpha}$.

3. We algebraically gauge away $\xi_{j\alpha}$

$$\begin{aligned}
\gamma_{j\alpha} & \rightarrow \gamma_{j\alpha} - i \xi_{j\alpha}, \quad \bar{\lambda}_{mj}^{\dot{\alpha}} \rightarrow \bar{\lambda}_{mj}^{\dot{\alpha}} - 2 (\bar{\sigma}_m \xi_j)^{\dot{\alpha}} \\
\mu_{1\alpha} & \rightarrow \mu_{1\alpha} + 4i \xi_{1\alpha} - 2 (\epsilon \cdot \sigma)_\alpha^{\beta} \xi_{1\beta}, \quad \bar{\mu}_2^{\dot{\alpha}} \rightarrow \bar{\mu}_2^{\dot{\alpha}} - 4i \bar{\xi}_2^{\dot{\alpha}} + 2 (\epsilon \cdot \bar{\sigma})^{\dot{\alpha}}_{\beta} \bar{\xi}_2^{\dot{\beta}}
\end{aligned} \quad (5.10)$$

4. We rescale $\zeta_{j\alpha}$

$$\zeta_{1\alpha} \rightarrow \left(\delta_\alpha^\beta + i(\epsilon \cdot \sigma)_\alpha{}^\beta \right) \zeta_{1\beta}, \quad \bar{\zeta}_2^{\dot{\alpha}} \rightarrow \left(\delta_{\dot{\beta}}^{\dot{\alpha}} + i(\epsilon \cdot \bar{\sigma})_{\dot{\beta}}^{\dot{\alpha}} \right) \bar{\zeta}_2^{\dot{\beta}} \quad (5.11)$$

This extra step, compared to the neutral case, is due to the ϵ -dependent terms in the gauge transformations of $\zeta_{j\alpha}$. After the rescaling, $\zeta_{j\alpha}$ become pure gauge with: $\delta\zeta_{1\alpha} = 16i\Lambda_{5\alpha}$, $\delta\bar{\zeta}_2^{\dot{\alpha}} = -16i\bar{\Upsilon}_5^{\dot{\alpha}}$.

5. We algebraically gauge away $\zeta_{j\alpha}$

$$\begin{aligned} \gamma_{1\alpha} &\rightarrow \gamma_{1\alpha} - \frac{1}{2}i(\sigma^m \mathfrak{D}_m \bar{\zeta}_2)_\alpha, & \bar{\rho}_1^{\dot{\alpha}} &\rightarrow \bar{\rho}_1^{\dot{\alpha}} + \frac{1}{4}i[(\epsilon \cdot \bar{\sigma}) \bar{\zeta}_2]^{\dot{\alpha}}, \\ r_{1m\alpha} &\rightarrow r_{1m\alpha} - \frac{1}{2}i[\sigma_m(\epsilon \cdot \bar{\sigma}) \bar{\zeta}_2]_\alpha, & \bar{\psi}_1^{\dot{\alpha}} &\rightarrow \bar{\psi}_1^{\dot{\alpha}} + \frac{1}{4}[(\epsilon \cdot \bar{\sigma}) \bar{\zeta}_2]^{\dot{\alpha}}, \\ \bar{\lambda}_{1m}^{\dot{\alpha}} &\rightarrow \bar{\lambda}_{1m}^{\dot{\alpha}} + \mathfrak{D}_m \bar{\zeta}_2^{\dot{\alpha}}, & \chi_{1m\alpha} &\rightarrow \chi_{1m\alpha} - \frac{1}{2}i(\sigma_m \bar{\zeta}_2)_\alpha \\ \mu_{1\alpha} &\rightarrow \mu_{1\alpha} + \frac{1}{4}[\sigma^m(\epsilon \cdot \bar{\sigma}) \mathfrak{D}_m \bar{\zeta}_2]_\alpha + \frac{1}{2}[2i\delta_\alpha^\beta - (\epsilon \cdot \sigma)_\alpha{}^\beta](\sigma^m \mathfrak{D}_m \bar{\zeta}_2)_\beta \\ \gamma_{2\alpha} &\rightarrow \gamma_{2\alpha} - \frac{1}{2}i(\sigma^m \mathfrak{D}_m \bar{\zeta}_1)_\alpha, & \rho_{2\alpha} &\rightarrow \rho_{2\alpha} + \frac{1}{4}i[(\epsilon \cdot \sigma) \zeta_1]_\alpha, \\ \bar{r}_{2m}^{\dot{\alpha}} &\rightarrow \bar{r}_{2m}^{\dot{\alpha}} + \frac{1}{2}i[\bar{\sigma}_m(\epsilon \cdot \sigma) \zeta_1]^{\dot{\alpha}}, & \psi_{2\alpha} &\rightarrow \psi_{2\alpha} - \frac{1}{4}[(\epsilon \cdot \sigma) \zeta_1]_\alpha, \\ \lambda_{2m\alpha} &\rightarrow \lambda_{2m\alpha} + \mathfrak{D}_m \zeta_{1\alpha}, & \bar{\chi}_{2m}^{\dot{\alpha}} &\rightarrow \bar{\chi}_{2m}^{\dot{\alpha}} - \frac{1}{2}i(\bar{\sigma}_m \zeta_1)^{\dot{\alpha}} \\ \bar{\mu}_2^{\dot{\alpha}} &\rightarrow \bar{\mu}_2^{\dot{\alpha}} + \frac{1}{4}[\bar{\sigma}^m(\epsilon \cdot \sigma) \mathfrak{D}_m \zeta_1]^{\dot{\alpha}} + \frac{1}{2}[2i\delta_{\dot{\beta}}^{\dot{\alpha}} - (\epsilon \cdot \bar{\sigma})_{\dot{\beta}}^{\dot{\alpha}}](\bar{\sigma}^m \mathfrak{D}_m \zeta_1)^{\dot{\beta}} \end{aligned} \quad (5.12)$$

6. We algebraically gauge away $r_{jm\alpha}$

$$\begin{aligned} \mu_{j\alpha} &\rightarrow \mu_{j\alpha} - \frac{1}{2}i\mathfrak{D}^m r_{jm\alpha}, \quad j = 1, 2 \\ \chi_{1m\alpha} &\rightarrow \chi_{1m\alpha} + \frac{i}{4}(\eta_{mn} - i\epsilon_{mn})^{-1}(\sigma^k \bar{\sigma}^n r_{1k})_\alpha, \\ \bar{\chi}_{2m}^{\dot{\alpha}} &\rightarrow \bar{\chi}_{2m}^{\dot{\alpha}} - \frac{i}{4}(\eta_{mn} - i\epsilon_{mn})^{-1}(\bar{\sigma}^k \sigma^n \bar{r}_{2k})^{\dot{\alpha}} \end{aligned} \quad (5.13)$$

The result is a Lagrangian in the unitary gauge

$$\begin{aligned} \mathcal{L}_F &= -\frac{i}{2}[(\lambda_1^m \sigma^n \mathfrak{D}_n \bar{\lambda}_{1m}) + 2(\bar{\chi}_1^m \bar{\sigma}^n \mathfrak{D}_n \chi_{1m})] - [(\lambda_1^m \chi_{1m}) + \text{h.c.}] \\ &+ \frac{33}{2}i(\psi_1 \sigma^m \mathfrak{D}_m \bar{\psi}_1) + \frac{i}{2}(\rho_1 \sigma^m \mathfrak{D}_m \bar{\rho}_1) + 2i(\gamma_1 \sigma^m \mathfrak{D}_m \bar{\gamma}_1) \\ &+ \left[3i(\bar{\chi}_1^m \mathfrak{D}_m \bar{\psi}_1) - 3(\lambda_1^m \sigma_m \bar{\psi}_1) + \frac{1}{2}i(\bar{\chi}_1^m \bar{\sigma}_m \mu_1) - (\bar{\chi}_1^m \mathfrak{D}_m \bar{\rho}_1) - (\lambda_1^m \mathfrak{D}_m \gamma_1) + \text{h.c.} \right] \\ &+ \left[\frac{3}{2}(\rho_1 \sigma^m \mathfrak{D}_m \bar{\psi}_1) - 2(\rho_1 \gamma_1) - \frac{1}{2}(\mu_1 \rho_1) - \frac{9}{2}i(\mu_1 \psi_1) + \text{h.c.} \right] + (1 \leftrightarrow 2) \\ &- \left[\frac{1}{2}\bar{\chi}_1^m(\epsilon \cdot \bar{\sigma}) \bar{\sigma}_m \gamma_1 + \frac{1}{2}\chi_2^m(\epsilon \cdot \sigma) \sigma_m \bar{\gamma}_2 + \text{h.c.} \right] \end{aligned} \quad (5.14)$$

which gives (2.58) when the electromagnetic field strength is set to zero.

Here $\rho_{j\alpha}$ is a residual non-physical field. The fermion μ appears a Lagrange multiplier, leading to the same constraint as in the neutral case:

$$\rho_{j\alpha} = -9i\psi_{j\alpha} - i(\sigma_m \bar{\chi}_j^m)_\alpha, \quad j = 1, 2 \quad (5.15)$$

Also, using this constraint, ρ_j can be integrated to obtain a Lagrangian with only physical fermions (corresponding to (2.60) in the neutral case):

$$\begin{aligned} \mathcal{L}_F = & -\frac{i}{2} \left[(\lambda_1^m \sigma^n \mathcal{D}_n \bar{\lambda}_{1m}) + (\bar{\chi}_{1m} \bar{\sigma}^n \sigma^k \bar{\sigma}^m \mathcal{D}_k \chi_{1n}) \right] - [(\lambda_1^m \chi_{1m}) + \text{h.c.}] \\ & + 30i (\psi_1 \sigma^m \mathcal{D}_m \bar{\psi}_1) + 2i (\gamma_1 \sigma^m \mathcal{D}_m \bar{\gamma}_1) \\ & + [-3i (\chi_1^m \sigma_n \bar{\sigma}_m \mathcal{D}^n \psi_1) - 3 (\lambda_1^m \sigma_m \bar{\psi}_1) - 2i (\bar{\chi}_1^m \bar{\sigma}_m \gamma_1) - (\lambda_1^m \mathcal{D}_m \gamma_1) + \text{h.c.}] \\ & + [18i (\psi_1 \gamma_1) + \text{h.c.}] + (1 \leftrightarrow 2) - \left[\frac{1}{2} \bar{\chi}_1^m (\epsilon \cdot \bar{\sigma}) \bar{\sigma}_m \gamma_1 + \frac{1}{2} \chi_2^m (\epsilon \cdot \sigma) \sigma_m \bar{\gamma}_2 + \text{h.c.} \right] \end{aligned} \quad (5.16)$$

However, difficulties arise when one tries to generalise (2.61)-(2.69) to decouple spin-1/2 from spin-3/2 and to put the kinetic terms of the latter in a Rarita-Schwinger form. First, because the kinetic term of λ_m in (5.16) contains both the rescaled and the unscaled form, the generalisation of (2.61) will either generate multiple ϵ -dependent terms, making the new couplings and kinetic terms more complicated, or will introduce the inverse matrix $(\eta_{mn} - i\epsilon_{mn})^{-1}$ that is difficult to handle. Second, due to the additional ϵ -dependent couplings between χ_m and γ , (2.61) also creates ϵ -dependent couplings between χ_m and λ_m . Therefore, a direct generalisation of (2.61)-(2.69) does not allow to decouple the spin-1/2 from the spin-3/2 fields.

We are going to present in a later subsection the Lagrangian obtained by generalising (2.70), that has the advantages of being a deformation of the Rarita-Schwinger one, and of giving decoupled equations on shell. But we will start by presenting a more compact form of the fermionic Lagrangian, for which the derivation of the equations of motion and constraints is relatively simple.

To this end, the same field redefinition as (2.63) can be applied to simplify the Lagrangian

$$\chi_{jm\alpha} \rightarrow \chi_{jm\alpha} - 2(\sigma_m \bar{\psi}_j)_\alpha, \quad j = 1, 2 \quad (5.17)$$

In addition, we perform the following normalisations for convenience:

$$\bar{\psi}_j^{\dot{\alpha}} \rightarrow \frac{i}{2\sqrt{2}} \bar{\psi}_j^{\dot{\alpha}}, \quad \bar{\lambda}_{jm}^{\dot{\alpha}} \rightarrow \sqrt{2} \bar{\lambda}_{jm}^{\dot{\alpha}}, \quad j = 1, 2 \quad (5.18)$$

Since $\bar{\sigma}^m (\epsilon \cdot \sigma) \sigma_m = 0$, the terms between the last brackets of (5.16) are not shifted.

The resulting Lagrangian is:

$$\begin{aligned}
\mathcal{L}_F = & -\frac{i}{2} \left[2 (\lambda_1^m \sigma^n \mathcal{D}_n \bar{\lambda}_{1m}) + \left(\bar{\chi}_{1m} \bar{\sigma}^n \sigma^k \bar{\sigma}^m \mathcal{D}_k \chi_{1n} \right) \right] - \sqrt{2} [(\lambda_1^m \chi_{1m}) + \text{h.c.}] \\
& + \left[-\frac{i}{4} (\psi_1 \sigma^m \mathcal{D}_m \bar{\psi}_1) + 2i (\gamma_1 \sigma^m \mathcal{D}_m \bar{\gamma}_1) \right] \\
& + \left[\frac{3}{\sqrt{2}} (\chi_1^m \sigma_{mn} \mathcal{D}^n \psi_1) - \frac{1}{2\sqrt{2}} (\chi_1^m \mathcal{D}_m \psi_1) \right. \\
& \quad \left. - \frac{i}{2} (\lambda_1^m \sigma_m \bar{\psi}_1) - 2i (\bar{\chi}_1^m \bar{\sigma}_m \gamma_1) - \sqrt{2} (\lambda_1^m \mathcal{D}_m \gamma_1) + \text{h.c.} \right] \\
& + \left[\frac{1}{\sqrt{2}} (\psi_1 \gamma_1) + \text{h.c.} \right] + (1 \leftrightarrow 2) - \left[\frac{1}{2} \bar{\chi}_1^m (\epsilon \cdot \bar{\sigma}) \bar{\sigma}_m \gamma_1 + \frac{1}{2} \chi_2^m (\epsilon \cdot \sigma) \sigma_m \bar{\gamma}_2 + \text{h.c.} \right]
\end{aligned} \tag{5.19}$$

5.3 Equations of motion and new spin-3/2

We will only present the computational details for fermions of index 1, and we will provide at the end the analogous results for those of index 2, which are almost identical.

The equations of motion directly obtained from the Lagrangian (5.19) take the form:

$$\frac{i}{\sqrt{2}} (\sigma^n \mathcal{D}_n \bar{\lambda}_{1m})_\alpha = -(\eta_{mn} - i\epsilon_{mn}) \left(\chi_1^n + \frac{1}{2\sqrt{2}} i \sigma^n \bar{\psi}_1 + \mathcal{D}^n \gamma_1 \right)_\alpha \tag{5.20a}$$

$$\begin{aligned}
i (\bar{\sigma}^n \sigma^k \bar{\sigma}_m \mathcal{D}_k \chi_{1n})^{\dot{\alpha}} = & -2\sqrt{2} \bar{\lambda}_{1m}^{\dot{\alpha}} + 3\sqrt{2} (\bar{\sigma}_{mn} \mathcal{D}^n \bar{\psi}_1)^{\dot{\alpha}} - \frac{1}{\sqrt{2}} \mathcal{D}_m \bar{\psi}_1^{\dot{\alpha}} \\
& - 4i (\bar{\sigma}_m \gamma_1)^{\dot{\alpha}} - [(\epsilon \cdot \bar{\sigma}) \bar{\sigma}_m \gamma_1]^{\dot{\alpha}}
\end{aligned} \tag{5.20b}$$

$$\frac{i}{2\sqrt{2}} (\sigma^m \mathcal{D}_m \bar{\psi}_1)_\alpha = -3 (\sigma_{mn} \mathcal{D}^m \chi_1^n)_\alpha + \frac{1}{2} \mathcal{D}^m \chi_{1m\alpha} + \frac{\sqrt{2}}{2} i (\sigma^m \bar{\lambda}_{1m})_\alpha + \gamma_{1\alpha} \tag{5.20c}$$

$$i (\bar{\sigma}^m \mathcal{D}_m \gamma_1)^{\dot{\alpha}} = -i (\bar{\sigma}^m \chi_{1m})^{\dot{\alpha}} - \frac{\sqrt{2}}{2} \mathcal{D}^m \bar{\lambda}_{1m}^{\dot{\alpha}} - \frac{1}{4} [\bar{\sigma}^m (\epsilon \cdot \sigma) \chi_{1m}]^{\dot{\alpha}} - \frac{1}{2\sqrt{2}} \bar{\psi}_1^{\dot{\alpha}} \tag{5.20d}$$

where the first equation comes from the variation of \mathcal{L}_F with regard to $\bar{\lambda}_{1m}$ instead of the rescaled $\bar{\lambda}_{1m}$.

In absence of the electromagnetic background, the above equations give rise to

$$\begin{aligned}
i (\sigma^m \partial_m \bar{\psi})_\alpha = & -\sqrt{2} \gamma_\alpha, \quad i (\bar{\sigma}^m \partial_m \gamma)^{\dot{\alpha}} = -\sqrt{2} \bar{\psi}^{\dot{\alpha}} \\
(\bar{\sigma}^m \chi_m)^{\dot{\alpha}} = & 0, \quad (\sigma^m \bar{\lambda}_m)_\alpha = \frac{3}{\sqrt{2}} i \gamma_\alpha, \quad \partial^m \chi_{m\alpha} = 0, \quad \partial^m \bar{\lambda}_m^{\dot{\alpha}} = \frac{3}{2} \bar{\psi}^{\dot{\alpha}}
\end{aligned} \tag{5.21}$$

We also have coupled equations of motion for $\bar{\lambda}_m$, χ_m and the spin-1/2 fields. But, these equations can be put in a more convenient form by introducing

$$\bar{\lambda}'_m{}^{\dot{\alpha}} = \bar{\lambda}_m^{\dot{\alpha}} + \frac{i}{2\sqrt{2}} (\bar{\sigma}_m \gamma)^{\dot{\alpha}} - \frac{1}{2} \partial_m \bar{\psi}^{\dot{\alpha}} \tag{5.22}$$

which satisfy

$$\begin{aligned}
i (\bar{\sigma}^n \partial_n \chi_m)^{\dot{\alpha}} = & -\sqrt{2} \bar{\lambda}'_m{}^{\dot{\alpha}}, \quad i (\sigma^n \partial_n \bar{\lambda}'_m)_\alpha = -\sqrt{2} \chi_{m\alpha} \\
\partial^m \bar{\lambda}'_m{}^{\dot{\alpha}} = & 0, \quad (\sigma^m \bar{\lambda}'_m)_\alpha = 0
\end{aligned} \tag{5.23}$$

In the presence of an electromagnetic background, we will proceed in several steps. First, we compute $\bar{\sigma}^m$ (5.20a) (after substituting γ and ψ equations of motion):

$$\sqrt{2}\bar{\sigma}^{mn}\mathfrak{D}_m\bar{\lambda}_{1n} = -\frac{3}{2\sqrt{2}}\bar{\psi}_1 - \frac{i}{\sqrt{2}}(\epsilon \cdot \bar{\sigma})\bar{\psi}_1 - \epsilon_{mn}\bar{\sigma}^m\mathfrak{D}^n\gamma_1 - \frac{3}{4}\epsilon_{mn}\bar{\sigma}^m\chi_1^n + \frac{i}{4}\tilde{\epsilon}_{mn}\bar{\sigma}^m\chi_1^n \quad (5.24)$$

Taking the σ -trace of (5.20b) gives

$$5i\sigma^m\bar{\sigma}^n\mathfrak{D}_m\chi_{1n} + 4i\mathfrak{D}^m\chi_{1m} - 6i\gamma_1 + 2\sqrt{2}\sigma^m\bar{\lambda}_{1m} = 0 \quad (5.25)$$

which can be used to rewrite (5.20c) as

$$i(\sigma^m\mathfrak{D}_m\bar{\psi}_1)_\alpha = -\sqrt{2}\gamma_{1\alpha} - \frac{\sqrt{2}}{2}(\sigma^m\bar{\sigma}^n\mathfrak{D}_m\chi_{1n})_\alpha \quad (5.26)$$

The divergence of (5.20a) gives

$$\begin{aligned} 4\sigma^{mn}\mathfrak{D}_m\chi_{1n} &= -\sqrt{2}\epsilon_{mn}\sigma^m\bar{\lambda}_1^n + \frac{i}{2}\sigma^m\bar{\sigma}^n(\epsilon \cdot \sigma)\mathfrak{D}_m\chi_{1n} + 2i\epsilon_{mn}\mathfrak{D}^m\chi_1^n \\ &\quad - \frac{1}{\sqrt{2}}\epsilon_{mn}\sigma^n\mathfrak{D}^m\bar{\psi}_1 - \epsilon_{mn}\epsilon^{mn}\gamma_1 - 2i(\epsilon \cdot \sigma)\gamma_1 \\ &= -\sqrt{2}\epsilon_{mn}\sigma^m\bar{\lambda}_1^n - \frac{i}{2}(\epsilon \cdot \sigma)\sigma^m\bar{\sigma}^n\mathfrak{D}_m\chi_{1n} + i(\epsilon_{mn} - i\tilde{\epsilon}_{mn})\mathfrak{D}^m\chi_1^n \\ &\quad - i(\epsilon \cdot \sigma)\mathfrak{D}^m\chi_{1m} - \frac{1}{\sqrt{2}}\epsilon_{mn}\sigma^n\mathfrak{D}^m\bar{\psi}_1 - \epsilon_{mn}\epsilon^{mn}\gamma_1 - 2i(\epsilon \cdot \sigma)\gamma_1 \end{aligned} \quad (5.27)$$

where we replaced $\mathfrak{D}^2\gamma_1$ with (obtained via applying $\sigma^n\mathfrak{D}_n$ on (5.20d)):

$$\begin{aligned} \mathfrak{D}^2\gamma_1 &= -\frac{3}{2}\mathfrak{D}^m\chi_{1m} + 5\sigma^{mn}\mathfrak{D}_m\chi_{1n} - \frac{1}{4}i\sigma^m\bar{\sigma}^n(\epsilon \cdot \sigma)\mathfrak{D}_m\chi_{1n} \\ &\quad - \frac{\sqrt{2}}{2}i\sigma^m\mathfrak{D}_m\mathfrak{D}_n\bar{\lambda}_1^n - \frac{\sqrt{2}}{2}i\sigma^m\bar{\lambda}_{1m} - \gamma_1 + i(\epsilon \cdot \sigma)\gamma_1 \end{aligned} \quad (5.28)$$

Taking the divergence of the χ_{1m} equations of motion gives:

$$-i\bar{\sigma}^m\mathfrak{D}^2\chi_{1m} = -\frac{1}{\sqrt{2}}\mathfrak{D}^2\bar{\psi}_1 + \frac{3}{\sqrt{2}}i(\epsilon \cdot \bar{\sigma})\bar{\psi}_1 + \sqrt{2}\bar{\psi}_1 + 4i\bar{\sigma}^m\chi_{1m} - (\epsilon \cdot \bar{\sigma})\bar{\sigma}^m\mathfrak{D}_m\gamma_1 \quad (5.29)$$

Now, $\mathfrak{D}^2\bar{\psi}_1$ can be found by acting with $\bar{\sigma}^n\mathfrak{D}_n$ on (5.20c):

$$\begin{aligned} \frac{1}{\sqrt{2}}\mathfrak{D}^2\bar{\psi}_1 &= \frac{i}{\sqrt{2}}(\epsilon \cdot \bar{\sigma})\bar{\psi}_1 - \frac{1}{\sqrt{2}}\bar{\psi}_1 + 3i\bar{\sigma}^m\mathfrak{D}^2\chi_{1m} - 2i\bar{\sigma}^m\mathfrak{D}_m\mathfrak{D}_n\chi_1^n + 3(\epsilon \cdot \bar{\sigma})\bar{\sigma}^m\chi_{1m} \\ &\quad - \frac{1}{2}\bar{\sigma}^m(\epsilon \cdot \sigma)\chi_{1m} - 2i\bar{\sigma}^m\chi_{1m} - 2\sqrt{2}\bar{\sigma}^{mn}\mathfrak{D}_m\bar{\lambda}_{1n} \\ &= \frac{3i}{\sqrt{2}}(\epsilon \cdot \bar{\sigma})\bar{\psi}_1 + \sqrt{2}\bar{\psi}_1 + 3i\bar{\sigma}^m\mathfrak{D}^2\chi_{1m} - 2i\bar{\sigma}^m\mathfrak{D}_m\mathfrak{D}_n\chi_1^n + 2(\epsilon \cdot \bar{\sigma})\bar{\sigma}^m\chi_{1m} \\ &\quad - 2i\bar{\sigma}^m\chi_{1m} + \bar{\sigma}^m(\epsilon \cdot \sigma)\mathfrak{D}_m\gamma_1 - (\epsilon \cdot \bar{\sigma})\bar{\sigma}^m\mathfrak{D}_m\gamma_1 \end{aligned} \quad (5.30)$$

which inserted in equation (5.29) leads to:

$$3i\bar{\sigma}^m\chi_{1m} + 2i\bar{\sigma}^m\mathfrak{D}_m(\sigma^{nk}\mathfrak{D}_n\chi_{1k}) - \frac{1}{2}\bar{\sigma}^m(\epsilon \cdot \sigma)\mathfrak{D}_m\gamma_1 = 0 \quad (5.31)$$

We have computed the σ -trace and divergence of the spin-3/2 equations of motion, but they are not sufficient to determine the constraints due to the presence of higher (than first order) derivatives. Note that in the presence of an electromagnetic background, we have one more operation that is independent of $\sigma^m \times (\text{e.o.m.})_m$. This operation is $\sigma^m (\epsilon \cdot \bar{\sigma}) \times (\text{e.o.m.})_m$.⁶ While $\sigma^m \times (\text{e.o.m.})_m$ projects out σ -traceless components of the equation, $\sigma^m (\epsilon \cdot \bar{\sigma}) \times (\text{e.o.m.})_m$ recovers them, with an extra ϵ factor.

Acting with $\sigma^m (\epsilon \cdot \bar{\sigma})$ on (5.20b) gives:

$$\begin{aligned}
0 &= \sqrt{2} \sigma^m (\epsilon \cdot \bar{\sigma}) \bar{\chi}_{1m} + \epsilon_{mn} (\epsilon^{mn} - i\tilde{\epsilon}^{mn}) \gamma_1 + \sqrt{2} \epsilon_{mn} \sigma^n \mathfrak{D}^m \bar{\psi}_1 - \frac{1}{\sqrt{2}} (\epsilon \cdot \sigma) \sigma^m \mathfrak{D}_m \bar{\psi}_1 \\
&\quad - 2 (i\epsilon_{mn} + \tilde{\epsilon}_{mn}) \mathfrak{D}^m \chi_1^n \\
&= \sqrt{2} \sigma^m (\epsilon \cdot \bar{\sigma}) \bar{\chi}_{1m} + \epsilon_{mn} (\epsilon^{mn} - i\tilde{\epsilon}^{mn}) \gamma_1 + \sqrt{2} \epsilon_{mn} \sigma^n \mathfrak{D}^m \bar{\psi}_1 - i (\epsilon \cdot \sigma) \gamma_1 \\
&\quad - \frac{i}{2} (\epsilon \cdot \sigma) \sigma^m \bar{\sigma}^n \mathfrak{D}_m \chi_{1n} - 2 (i\epsilon_{mn} + \tilde{\epsilon}_{mn}) \mathfrak{D}^m \chi_1^n
\end{aligned} \tag{5.32}$$

where the second equality is due to (5.26). Then, adding the second line of the above equation to $-\frac{1}{2} (\epsilon \cdot \sigma) \times (5.25)$, we obtain

$$\begin{aligned}
-\sqrt{2} \epsilon_{mn} \sigma^m \bar{\chi}_1^n &= i (\epsilon \cdot \sigma) \gamma_1 + \frac{1}{2} \epsilon_{mn} (\epsilon^{mn} - i\tilde{\epsilon}^{mn}) \gamma_1 + \frac{\sqrt{2}}{2} \epsilon_{mn} \sigma^n \mathfrak{D}^m \bar{\psi}_1 - i (\epsilon \cdot \sigma) \mathfrak{D}^m \chi_{1m} \\
&\quad - (i\epsilon_{mn} + \tilde{\epsilon}_{mn}) \mathfrak{D}^m \chi_1^n - \frac{3i}{2} (\epsilon \cdot \sigma) \sigma^m \bar{\sigma}^n \mathfrak{D}_m \chi_{1n}
\end{aligned} \tag{5.33}$$

Inserting this into (5.27) leads to:

$$4 [1 + i (\epsilon \cdot \sigma)] \sigma^{mn} \mathfrak{D}_m \chi_{1n} = -i [1 + i (\epsilon \cdot \sigma)] (\epsilon \cdot \sigma) \gamma_1 \tag{5.34}$$

which multiplied on both sides by $[1 + i (\epsilon \cdot \sigma)]^{-1}$ gives:

$$\sigma^{mn} \mathfrak{D}_m \chi_{1n} = -\frac{i}{4} (\epsilon \cdot \sigma) \gamma_1 \tag{5.35}$$

This allows to simplify (5.31), which provides our first constraint:

$$\bar{\sigma}^m \chi_{1m} = 0 \tag{5.36}$$

Moreover, since $\sigma^m \bar{\sigma}^n \mathfrak{D}_m \chi_{1n} = 2\sigma^{mn} \mathfrak{D}_m \chi_{1n} - \mathfrak{D}^m \chi_{1m} = 0$, we can write our second constraint

$$\mathfrak{D}^m \chi_{1m} = -\frac{i}{2} (\epsilon \cdot \sigma) \gamma_1 \tag{5.37}$$

as well as the Dirac equation of $\bar{\psi}_1$:

$$i\sigma^m \mathfrak{D}_m \bar{\psi}_1 = -\sqrt{2} \gamma_1 \tag{5.38}$$

Also, (5.25) implies a trace constraint of $\bar{\chi}_{1m}$

$$\sigma^m \bar{\chi}_{1m} = \frac{3}{\sqrt{2}} i \gamma_1 - \frac{1}{\sqrt{2}} (\epsilon \cdot \sigma) \gamma_1 \tag{5.39}$$

⁶We assumed that the equation of motion has a global *dotted* index, (e.o.m.)^{dot}. In the opposite case, the corresponding operations should obviously be the conjugate ones.

which in conjunction with (5.24) yields:

$$-\frac{1}{\sqrt{2}}\mathfrak{D}^m\bar{\lambda}_{1m} = -\frac{3}{2\sqrt{2}}\bar{\psi}_1 - \sqrt{2}i(\epsilon \cdot \bar{\sigma})\bar{\psi}_1 + (\epsilon \cdot \bar{\sigma})\bar{\sigma}^m\mathfrak{D}_m\gamma_1 + \frac{1}{4}\bar{\sigma}^m(\epsilon \cdot \sigma)\chi_{1m} \quad (5.40)$$

Now add the above equation to (5.20d) gives:

$$i[1 + i(\epsilon \cdot \bar{\sigma})]\bar{\sigma}^m\mathfrak{D}_m\gamma_1 = -\sqrt{2}[1 + i(\epsilon \cdot \bar{\sigma})]\bar{\psi}_1 \quad (5.41)$$

so we get the Dirac equation for γ_1 :

$$i\bar{\sigma}^m\mathfrak{D}_m\gamma_1 = -\sqrt{2}\bar{\psi}_1 \quad (5.42)$$

Back to (5.40), the divergence constraint of $\bar{\lambda}_{1m}$ becomes

$$\mathfrak{D}^m\bar{\lambda}_{1m} = \frac{3}{2}\bar{\psi}_1 - \frac{\sqrt{2}}{4}\bar{\sigma}^m(\epsilon \cdot \sigma)\chi_{1m} \quad (5.43)$$

Finally, we need to find a generalisation of (5.22) for the case of propagation in an electromagnetic background, such that we arrive at new spin-3/2 fields that are decoupled from the spin-1/2 ones in the equations of motion and constraints. This is given by:

$$\begin{aligned} \bar{\lambda}'_{1m} &\equiv \bar{\lambda}_{1m} + \frac{i}{2\sqrt{2}}[1 - i(\epsilon \cdot \bar{\sigma})]\bar{\sigma}_m\gamma_1 - \frac{1}{2}[\eta_{mn} - i(\epsilon_{mn} + i\check{\epsilon}_{mn})]\mathfrak{D}^n\bar{\psi}_1 \\ \chi'_{1m} &\equiv \chi_{1m} + \frac{1}{2\sqrt{2}}(\epsilon \cdot \sigma)\sigma_m\bar{\psi}_1 \end{aligned} \quad (5.44)$$

and it leads to the equations of motion and constraints for the spin-3/2 fields:

$$\begin{aligned} i\sigma^n\mathfrak{D}_n\bar{\lambda}'_{1m} &= -\sqrt{2}(\eta_{mn} - i\epsilon_{mn})\chi_1{}^n \\ i\bar{\sigma}^n\mathfrak{D}_n\chi'_{1m} &= -\sqrt{2}\bar{\lambda}'_{1m}, \\ \mathfrak{D}^m\chi'_{1m} &= 0, \quad \mathfrak{D}^m\bar{\lambda}'_{1m} = -\frac{\sqrt{2}}{4}\bar{\sigma}^m(\epsilon \cdot \sigma)\chi'_{1m} \\ \bar{\sigma}^m\chi'_{1m} &= 0, \quad \sigma^m\bar{\lambda}'_{1m} = 0 \end{aligned} \quad (5.45)$$

as well as the Dirac equations for the spin-1/2 fields:

$$i\bar{\sigma}^m\mathfrak{D}_m\gamma_1 = -\sqrt{2}\bar{\psi}_1, \quad i\sigma^m\mathfrak{D}_m\bar{\psi}_1 = -\sqrt{2}\gamma_1 \quad (5.46)$$

The system (5.45) describes a spin-3/2 field with the correct number of degrees of freedom and a gyromagnetic ratio $g = 2$. Starting with 32 real degrees of freedom off-shell for $\{\chi'_{1m\alpha}, \bar{\lambda}'_{1m}{}^{\dot{\alpha}}\}$, the equations of motion remove 16 degrees of freedom and each divergence and σ -trace constraint removes 2, so we are left with 8 on-shell degrees of freedom for $\{\chi'_{1m\alpha}, \bar{\lambda}'_{1m}{}^{\dot{\alpha}}\}$. Note that the modification in the presence of an electromagnetic background is clearly manifest in the divergence constraint, and the equation of motion of $\bar{\lambda}'_{1m}$.

Next, let us express the equations for the spin-1/2 and spin-3/2 fields in four-component notations. We introduce the new Dirac spinors:

$$\Phi_1 \equiv \begin{pmatrix} \gamma_{1\alpha} \\ \bar{\psi}_1{}^{\dot{\alpha}} \end{pmatrix}, \quad \Psi_{1m} \equiv \begin{pmatrix} \chi'_{1m\alpha} \\ \bar{\lambda}'_{1m}{}^{\dot{\alpha}} \end{pmatrix} \quad (5.47)$$

We follow the notations in [35], where the γ -matrices are

$$\gamma^m = \begin{pmatrix} 0 & \sigma^m \\ \bar{\sigma}^m & 0 \end{pmatrix}, \quad \gamma^5 = \gamma^0 \gamma^1 \gamma^2 \gamma^3 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \quad (5.48)$$

and for shorthand, we note $\not{\mathcal{D}} \equiv \gamma^m \mathcal{D}_m$.

The spin-1/2 satisfies the Dirac equation in QED:

$$(i\not{\mathcal{D}} + \sqrt{2}) \Phi_1 = 0 \quad (5.49)$$

As for spin-3/2, the constraints can be written as

$$\left[\mathcal{D}^m - \frac{\sqrt{2}}{4} (\epsilon^{mn} + i\tilde{\epsilon}^{mn}) \gamma_n \right] \Psi_{1m} = 0, \quad \gamma^m \Psi_{1m} = 0 \quad (5.50)$$

With projection operators defined as $P_L = (1 + i\gamma^5)/2$, $P_R = (1 - i\gamma^5)/2$, we can write the equations of motion as

$$(i\not{\mathcal{D}} + \sqrt{2}) \Psi_{1m} = \sqrt{2}i\epsilon_{mn} \Psi_{1L}^n \quad (5.51)$$

The fermions of index 2 correspond to the conjugates of those of index 1 if the electromagnetic field intensity is set to zero. Their equations of motion and constraints can be worked out in an analogous way. The results are:

- **New spin-3/2 definitions**

$$\begin{aligned} \lambda'_{2m} &\equiv \lambda_{2m} + \frac{i}{2\sqrt{2}} [1 - i(\epsilon \cdot \sigma)] \sigma_m \bar{\gamma}_2 - \frac{1}{2} [\eta_{mn} - i(\epsilon_{mn} - i\tilde{\epsilon}_{mn})] \mathcal{D}^n \psi_2 \\ \bar{\chi}'_{2m} &\equiv \bar{\chi}_{2m} + \frac{1}{2\sqrt{2}} (\epsilon \cdot \bar{\sigma}) \bar{\sigma}_m \psi_2 \end{aligned} \quad (5.52)$$

- **Equations of motion and constraints:**

$$\begin{aligned} i\sigma^m \mathcal{D}_m \bar{\gamma}_2 &= -\sqrt{2} \psi_2, \quad i\bar{\sigma}^m \mathcal{D}_m \psi_2 = -\sqrt{2} \bar{\gamma}_2 \\ i\sigma^n \mathcal{D}_n \bar{\chi}'_{2m} &= -\sqrt{2} \lambda'_{2m}, \quad i\bar{\sigma}^n \mathcal{D}_n \lambda'_{2m} = -\sqrt{2} (\eta_{mn} - i\epsilon_{mn}) \bar{\chi}'_{2m} \\ \mathcal{D}^m \bar{\chi}'_{2m} &= 0, \quad \sigma^m \bar{\chi}'_{2m} = 0 \\ \mathcal{D}^m \lambda'_{2m} &= -\frac{\sqrt{2}}{4} \sigma^m (\epsilon \cdot \bar{\sigma}) \bar{\chi}'_{2m}, \quad \bar{\sigma}^m \lambda'_{2m} = 0 \end{aligned} \quad (5.53)$$

Likewise, we introduce the four-component fermions of opposite charge with regard to $\{\Phi_1, \Psi_{1m}\}$:

$$\Phi_2 \equiv \begin{pmatrix} \gamma_{2\alpha} \\ \bar{\psi}_2^{\dot{\alpha}} \end{pmatrix}, \quad \Psi_{2m} \equiv \begin{pmatrix} \chi'_{2m\alpha} \\ \bar{\lambda}'_{2m}{}^{\dot{\alpha}} \end{pmatrix} \quad (5.54)$$

They satisfy:

$$\begin{aligned} (i\not{\mathcal{D}} + \sqrt{2}) \Phi_2 &= 0 \\ \left[\mathcal{D}^m + \frac{\sqrt{2}}{4} (\epsilon^{mn} + i\tilde{\epsilon}^{mn}) \gamma_n \right] \Psi_{2m} &= 0, \quad \gamma^m \Psi_{2m} = 0 \\ (i\not{\mathcal{D}} + \sqrt{2}) \Psi_{2m} &= -\sqrt{2}i\epsilon_{mn} \Psi_{2L}^n \end{aligned} \quad (5.55)$$

One can observe that the γ -trace constraint does not change in the presence of the background, whereas the divergence constraint is modified by the electromagnetic field strength. The only difference compared to the equations of index 1 is the sign flipping $\epsilon \rightarrow -\epsilon$.

One may wonder what if the redefinitions (5.44) and (5.52) are performed at the level of the Lagrangian. To clarify this point, we apply the following field redefinition on (5.19):

$$\begin{aligned}
\bar{\lambda}_{1m} &\rightarrow \bar{\lambda}_{1m} - \frac{i}{2\sqrt{2}} [1 - i(\epsilon \cdot \bar{\sigma})] \bar{\sigma}_m \gamma_1 + \frac{1}{2} [\eta_{mn} - i(\epsilon_{mn} + i\tilde{\epsilon}_{mn})] \mathcal{D}^n \bar{\psi}_1 \\
\chi_{1m} &\rightarrow \chi_{1m} - \frac{1}{2\sqrt{2}} (\epsilon \cdot \sigma) \sigma_m \bar{\psi}_1 \\
\lambda_{2m} &\rightarrow \lambda_{2m} - \frac{i}{2\sqrt{2}} [1 - i(\epsilon \cdot \sigma)] \sigma_m \bar{\gamma}_2 + \frac{1}{2} [\eta_{mn} - i(\epsilon_{mn} - i\tilde{\epsilon}_{mn})] \mathcal{D}^n \psi_2 \\
\bar{\chi}_{2m} &\rightarrow \bar{\chi}_{2m} - \frac{1}{2\sqrt{2}} (\epsilon \cdot \bar{\sigma}) \bar{\sigma}_m \psi_2
\end{aligned} \tag{5.56}$$

and obtain

$$\begin{aligned}
\mathcal{L}_F &= -\frac{i}{2} \left[2(\lambda_1^m \sigma^n \mathcal{D}_n \bar{\lambda}_{1m}) + (\bar{\chi}_{1m} \bar{\sigma}^n \sigma^k \bar{\sigma}^m \mathcal{D}_k \chi_{1n}) \right] - \sqrt{2} [(\lambda_1^m \chi_{1m}) + \text{h.c.}] \\
&+ \left[\frac{3}{2\sqrt{2}} (\chi_1^m \sigma_m \sigma_n \mathcal{D}^n \psi_1) - \frac{i}{2} (\lambda_1^m \sigma_m \bar{\psi}_1) - \frac{3}{2} i (\bar{\chi}_1^m \bar{\sigma}_m \gamma_1) - \sqrt{2} (\lambda_1^m \mathcal{D}_m \gamma_1) + \text{h.c.} \right] \\
&+ \left[\frac{i}{4} (\psi_1 \sigma^m \mathcal{D}_m \bar{\psi}_1) + i(\gamma_1 \sigma^m \mathcal{D}_m \bar{\gamma}_1) \right] + \frac{1}{\sqrt{2}} [\bar{\gamma}_1 \mathcal{D}^2 \bar{\psi}_1 + \text{h.c.}] \\
&- \frac{i}{8} G^{mn} \bar{\gamma}_1 \bar{\sigma}_m [1 - i(\epsilon \cdot \sigma)] \sigma_k [1 - i(\epsilon \cdot \bar{\sigma})] \bar{\sigma}_n \mathcal{D}^k \gamma_1 \\
&+ \frac{i}{4} G^{mn} [\eta_{mp} + i(\epsilon_{mp} - i\tilde{\epsilon}_{mp})] [\eta_{nq} - i(\epsilon_{nq} + i\tilde{\epsilon}_{nq})] \psi_1 \sigma^k \mathcal{D}^p \mathcal{D}_k \mathcal{D}^q \bar{\psi}_1 \\
&- \frac{i}{4} (\epsilon_{mn} \epsilon^{mk} + \tilde{\epsilon}_{mn} \tilde{\epsilon}^{mk}) \psi_1 \sigma_k \mathcal{D}^n \bar{\psi}_1 - \frac{i}{2} \tilde{\epsilon}_{mn} (\psi_1 \sigma^n \mathcal{D}^m \bar{\psi}_1) + i\tilde{\epsilon}_{mn} (\bar{\gamma}_1 \bar{\sigma}^n \mathcal{D}^m \gamma_1) \\
&+ \frac{1}{4\sqrt{2}} \left\{ G^{mn} [\eta_{kp} - i(\epsilon_{kp} + i\tilde{\epsilon}_{kp})] [\eta_{nl} - i(\epsilon_{nl} + i\tilde{\epsilon}_{nl})] \bar{\gamma}_1 \bar{\sigma}_m \sigma^p \mathcal{D}^k \mathcal{D}^l \bar{\psi}_1 + \text{h.c.} \right\} \\
&- \left\{ \frac{i}{2} [\eta_{mk} - i(\epsilon_{mk} + i\tilde{\epsilon}_{mk})] \lambda_1^m \sigma_n \mathcal{D}^n \mathcal{D}^k \bar{\psi}_1 + \frac{1}{2\sqrt{2}} \lambda_1^m \sigma_n [1 - i(\epsilon \cdot \bar{\sigma})] \bar{\sigma}_m \mathcal{D}^n \gamma_1 + \text{h.c.} \right\} \\
&+ \frac{1}{2} \left[\lambda_1^m (\epsilon \cdot \sigma) \sigma_m \bar{\psi}_1 + \frac{1}{\sqrt{2}} (\epsilon \epsilon - i\epsilon \tilde{\epsilon}) \psi_1 \gamma_1 + \text{h.c.} \right] + (1 \leftrightarrow 2, \epsilon \leftrightarrow -\epsilon)
\end{aligned} \tag{5.57}$$

Where we denoted the inverse matrix by $G_{mn} \equiv (\eta_{mn} - i\epsilon_{mn})^{-1}$. It transforms as $G_{mn} \leftrightarrow G_{nm}$ under the sign flipping $\epsilon_{mn} \leftrightarrow -\epsilon_{mn}$.

We remark that the Lagrangian (5.57) obtained after redefinition contains higher derivatives even when $\epsilon = 0$. Nevertheless, this is not a cause for concern here as we find that it leads to the same system as in (5.45), and (5.53). To show this, we can start from the

corresponding equations of motion:⁷

$$\begin{aligned}
0 = & -i\sigma^n \mathcal{D}_n \bar{\lambda}_{1m} - \sqrt{2} \chi_{1m} + \sqrt{2i} \epsilon_{mn} \chi_1^n - \frac{i}{2} \sigma_m \bar{\psi}_1 - \frac{1}{2} \epsilon_{mn} \sigma^n \bar{\psi}_1 - \sqrt{2} \mathcal{D}_m \gamma_1 + \sqrt{2i} \epsilon_{mn} \mathcal{D}^n \gamma_1 \\
& - \frac{i}{2} [\eta_{mk} - i(\epsilon_{mk} + i\tilde{\epsilon}_{mk})] \sigma_n \mathcal{D}^n \mathcal{D}^k \bar{\psi}_1 - \frac{1}{2\sqrt{2}} \sigma_n [1 - i(\epsilon \cdot \bar{\sigma})] \bar{\sigma}_m \mathcal{D}^n \gamma_1 \\
& + \frac{1}{2} (\epsilon \cdot \sigma) \sigma_m \bar{\psi}_1 - \frac{i}{2} \epsilon_{mn} (\epsilon \cdot \sigma) \sigma^n \bar{\psi}_1
\end{aligned} \tag{5.58a}$$

$$0 = -\frac{i}{2} \bar{\sigma}^n \sigma^k \bar{\sigma}_m \mathcal{D}_k \chi_{1n} - \sqrt{2} \bar{\lambda}_{1m} + \frac{3}{2\sqrt{2}} \bar{\sigma}_m \sigma_n \mathcal{D}^n \bar{\psi}_1 - \frac{3i}{2} \bar{\sigma}_m \gamma_1 \tag{5.58b}$$

$$\begin{aligned}
0 = & i\bar{\sigma}^m \mathcal{D}_m \gamma_1 + \frac{1}{\sqrt{2}} \mathcal{D}^2 \bar{\psi}_1 + \sqrt{2} \mathcal{D}^m \bar{\lambda}_{1m} + \frac{3}{2} i\bar{\sigma}^m \chi_{1m} \\
& - \frac{i}{8} G^{mn} \bar{\sigma}_m [1 - i(\epsilon \cdot \sigma)] \sigma_k [1 - i(\epsilon \cdot \bar{\sigma})] \bar{\sigma}_n \mathcal{D}^k \gamma_1 \\
& + i\tilde{\epsilon}_{mn} \bar{\sigma}^n \mathcal{D}^m \gamma_1 + \frac{1}{4\sqrt{2}} G^{mn} [\eta_{kp} - i(\epsilon_{kp} + i\tilde{\epsilon}_{kp})] [\eta_{nl} - i(\epsilon_{nl} + i\tilde{\epsilon}_{nl})] \bar{\sigma}_m \sigma^p \mathcal{D}^k \mathcal{D}^l \bar{\psi}_1 \\
& + \frac{1}{2\sqrt{2}} \bar{\sigma}_m [1 - i(\epsilon \cdot \sigma)] \sigma_n \mathcal{D}^n \bar{\lambda}_1^m + \frac{1}{2\sqrt{2}} (\epsilon\epsilon + i\epsilon\tilde{\epsilon}) \bar{\psi}_1 = 0
\end{aligned} \tag{5.58c}$$

$$\begin{aligned}
0 = & \frac{i}{4} \sigma^m \mathcal{D}_m \bar{\psi}_1 + \frac{1}{\sqrt{2}} \mathcal{D}^2 \gamma_1 - \frac{3}{2\sqrt{2}} \sigma^m \bar{\sigma}^n \mathcal{D}_m \chi_{1n} + \frac{i}{2} \sigma^m \bar{\lambda}_{1m} - \frac{1}{2} \sigma^m (\epsilon \cdot \bar{\sigma}) \bar{\lambda}_{1m} \\
& - \frac{i}{2} \tilde{\epsilon}_{mn} \sigma^n \mathcal{D}^m \bar{\psi}_1 + \frac{i}{4} G^{mn} [\eta_{mp} + i(\epsilon_{mp} - i\tilde{\epsilon}_{mp})] [\eta_{np} - i(\epsilon_{np} + i\tilde{\epsilon}_{np})] \sigma^k \mathcal{D}^p \mathcal{D}_k \mathcal{D}^q \bar{\psi}_1 \\
& - \frac{i}{4} (\epsilon_{mn} \epsilon^{mk} + \tilde{\epsilon}_{mn} \tilde{\epsilon}^{mk}) \sigma_l \mathcal{D}^n \bar{\psi}_1 + \frac{i}{2} [\eta_{mk} + i(\epsilon_{mk} - i\tilde{\epsilon}_{mk})] \sigma_n \mathcal{D}^k \mathcal{D}^n \bar{\lambda}_1^m \\
& + \frac{1}{2\sqrt{2}} (\epsilon\epsilon - i\epsilon\tilde{\epsilon}) \gamma_1 + \frac{1}{4\sqrt{2}} G^{mn} [\eta_{kp} + i(\epsilon_{kp} - i\tilde{\epsilon}_{kp})] [\eta_{ml} + i(\epsilon_{ml} - i\tilde{\epsilon}_{ml})] \sigma^p \bar{\sigma}_n \mathcal{D}^l \mathcal{D}^k \gamma_1
\end{aligned} \tag{5.58d}$$

This system of equations can be transformed to an equivalent one, where the matrix G_{mn} doesn't appear, by a making suitable combination of the equations. G_{mn} appears in (5.58c) and (5.58d), thus these are the two equations that we will replace.

We start by simplifying the spin-1/2 equations of motion. We compute the combination (5.58c) $- \frac{i}{2\sqrt{2}} G^{pm} \bar{\sigma}_p [1 - i(\epsilon \cdot \sigma)] \times$ (5.58a):

$$\begin{aligned}
0 = & \frac{3}{2} i\bar{\sigma}^m \mathcal{D}_m \gamma_1 - \frac{1}{2} (\epsilon \cdot \bar{\sigma}) \bar{\sigma}^m \mathcal{D}_m \gamma_1 + \frac{1}{\sqrt{2}} \mathcal{D}^2 \bar{\psi}_1 + \frac{1}{\sqrt{2}} \bar{\psi}_1 + \sqrt{2} \mathcal{D}^m \bar{\lambda}_{1m} \\
& + 2i\bar{\sigma}^m \chi_{1m} + \frac{1}{2} \bar{\sigma}^m (\epsilon \cdot \sigma) \chi_{1m} = 0
\end{aligned} \tag{5.59}$$

⁷Only the calculation for fermions of index 1 is presented, and those of index 2 are completely analogous.

while, (5.58d)+ $\frac{1}{2}G^{pm}[\eta_{pq} + i(\epsilon_{pq} - i\tilde{\epsilon}_{pq})]\mathfrak{D}^q \times$ (5.58a)- $\frac{1}{2\sqrt{2}}\sigma^m(\epsilon \cdot \bar{\sigma}) \times$ (5.58b) gives:

$$\frac{i}{2}\sigma^m\bar{\lambda}_{1m} - \frac{3}{2\sqrt{2}}\sigma^m\bar{\sigma}^n\mathfrak{D}_m\chi_{1n} - \frac{1}{\sqrt{2}}\mathfrak{D}^m\chi_{1m} = 0 \quad (5.60)$$

Remarkably, (5.58a)-(5.58b) and (5.59)-(5.60) are equivalent to applying the field redefinition (5.56) on the former equations of motion (5.20a)-(5.20d). Therefore, the same steps and results, intermediate or final, obtained previously will be obtained here if we follow the same steps, only they take another form since they are connected by redefinitions of fields. In particular, the traceless condition (5.36) remains unchanged. More precisely, let us compute the divergence, σ -trace, as well as the $\sigma^m(\epsilon \cdot \bar{\sigma})$ projection of the χ_{1m} equations of motion:

$$\frac{i}{2}\bar{\sigma}^m\mathfrak{D}^2\chi_{1m} - \frac{1}{2}\bar{\sigma}^m(\epsilon \cdot \sigma)\chi_{1m} - \sqrt{2}\mathfrak{D}^m\bar{\lambda}_{1m} - \frac{3}{2\sqrt{2}}\mathfrak{D}^2\bar{\psi}_1 + \frac{3}{2\sqrt{2}}i(\epsilon \cdot \bar{\sigma})\bar{\psi}_1 - \frac{3}{2}i\bar{\sigma}^m\mathfrak{D}_m\gamma_1 = 0 \quad (5.61)$$

$$-2i\mathfrak{D}^m\chi_{1m} - \sqrt{2}\sigma^m\bar{\lambda}_{1m} - 3\sqrt{2}\sigma^m\mathfrak{D}_m\bar{\psi}_1 + 6i\gamma_1 = 0 \quad (5.62)$$

$$2i(\epsilon_{mn} - i\tilde{\epsilon}_{mn})\mathfrak{D}^m\chi_1^n - \sqrt{2}\sigma^m(\epsilon \cdot \bar{\sigma})\bar{\lambda}_{1m} = 0 \quad (5.63)$$

From (5.60)+ $\frac{i}{2\sqrt{2}}$ (5.62) we obtain:

$$\frac{i}{2}\sigma^m\mathfrak{D}_m\bar{\psi}_1 + \frac{1}{\sqrt{2}}\gamma_1 + \frac{1}{2\sqrt{2}}\sigma^m\bar{\sigma}^n\mathfrak{D}_m\chi_{1n} = 0 \quad (5.64)$$

which is exactly (5.26). The σ -trace of χ_{1m} can be inferred from the combination $\bar{\sigma}^m \times$ (5.58a)- $2\sqrt{2}i$ (5.59)- $\sqrt{2}i$ (5.61)- $2\bar{\sigma}^m\mathfrak{D}_m \times$ (5.60):

$$3\sqrt{2}\bar{\sigma}^m\chi_{1m} + \bar{\sigma}^m\mathfrak{D}_m \left[2\sqrt{2}\sigma^{nk}\mathfrak{D}_n\chi_{1k} - \frac{1}{2}(\epsilon \cdot \sigma)\sigma^n\mathfrak{D}_n\bar{\psi}_1 + \frac{i}{\sqrt{2}}(\epsilon \cdot \sigma)\gamma_1 \right] = 0 \quad (5.65)$$

One finds that the expression between brackets vanishes because the combination of equations $\frac{i}{\sqrt{2}}\sigma^n\mathfrak{D}_n \times$ (5.59)+ $\mathfrak{D}^m \times$ (5.58a)+ $i(\epsilon \cdot \sigma) \times$ (5.60)- $\frac{1}{2\sqrt{2}}$ (5.63) gives:

$$[1 + i(\epsilon \cdot \sigma)] \left[-2\sqrt{2}\sigma^{nk}\mathfrak{D}_n\chi_{1k} + \frac{1}{2}(\epsilon \cdot \sigma)\sigma^n\mathfrak{D}_n\bar{\psi}_1 - \frac{i}{\sqrt{2}}(\epsilon \cdot \sigma)\gamma_1 \right] = 0 \quad (5.66)$$

which implies then, as expected, $\bar{\sigma}^m\chi_{1m} = 0$. The Dirac equation for $\bar{\psi}_1$ then follows from (5.64). Back to (5.66) with $\bar{\psi}_1$ replaced by using its equations of motion, we deduce $\sigma^{mn}\mathfrak{D}_m\chi_{1n} = 0$ hence $\mathfrak{D}^m\chi_{1m} = 0$. As the for $\bar{\lambda}_{1m}$, one obtains $\sigma^m\bar{\lambda}_{1m} = 0$ using χ_{1m} constraints in (5.60). Furthermore, the condition (5.61) results in $\mathfrak{D}^m\bar{\lambda}_{1m} = -\frac{\sqrt{2}}{4}\bar{\sigma}^m(\epsilon \cdot \sigma)\chi_{1m}$. The Dirac equation of γ_1 comes from (5.59). Finally, plugging all the constraints and Dirac equations above, into (5.58a) and (5.58b), one obtains the same equations of motion for the spin-3/2 fields as before.

In summary, despite the presence of higher derivatives as well as couplings between spin-1/2 and spin-3/2, the Lagrangian (5.57) yields a *decoupled* system of equations of motion and constraints, given by (5.45).

5.4 A deformed Rarita-Schwinger Lagrangian

It can be assumed that the Lagrangian in the presence of a non-zero electromagnetic field would be a deformation of the one in the neutral case, (2.71). It should reduce to the latter, in fact to two copies, when the electromagnetic field vanishes. It would thus be a form of "Rarita-Schwinger plus Dirac" Lagrangians deformed by ϵ -dependent coefficients and additional terms. Although it contains higher derivatives, the Lagrangian of this type that we have found gives decoupled equations of motion and constraints. It also allows us to see explicitly how the "deformation" by the electromagnetic background can be written.

In the absence of an electromagnetic field, the Rarita-Schwinger Lagrangian has been obtained by redefining the fields according to (2.70). The generalisation of this redefinition to the charged case is as follows:

$$\begin{aligned}
\chi_{1m} &\rightarrow \sqrt{2}\sigma_{mn}\chi_1^n - i\sigma_m\bar{\psi}_1 - \frac{1}{2}(\epsilon \cdot \sigma)\sigma_m\bar{\psi}_1 \\
\bar{\lambda}_{1m} &\rightarrow \bar{\lambda}_{1m} - \frac{1}{\sqrt{2}}i[1 - i(\epsilon \cdot \bar{\sigma})]\bar{\sigma}_m\gamma_1 + [\eta_{mn} - i(\epsilon_{mn} + i\tilde{\epsilon}_{mn})]\mathcal{D}^n\bar{\psi}_1 \\
\gamma_1 &\rightarrow \frac{1}{\sqrt{2}}[1 - i(\epsilon \cdot \sigma)]\gamma_1 - \frac{i}{2}[1 + i(\epsilon \cdot \sigma)](\sigma^m\mathcal{D}_m\bar{\psi}_1) - \frac{1}{2}i(\sigma^m\bar{\lambda}_{1m}) + \frac{1}{2\sqrt{2}}(\sigma^m\bar{\sigma}^n\mathcal{D}_m\chi_{1n}) \\
\bar{\psi}_1 &\rightarrow \frac{i}{2}\bar{\psi}_1 - \frac{1}{2\sqrt{2}}(\bar{\sigma}^m\chi_{1m})
\end{aligned} \tag{5.67}$$

The corresponding formulas for index 2 are obtained by flipping a sign $\epsilon \leftrightarrow -\epsilon$.

The resulting expressions are very long. Therefore, to make them easier to read, we will separate the Lagrangian into three parts:

$$\mathcal{L}_F = \mathcal{L}_{\text{RSd}} + \mathcal{L}_{\text{km}} + \mathcal{L}_{\text{coupl}} \tag{5.68}$$

where:

- \mathcal{L}_{RSd} consists in a sum of Rarita-Schwinger and Dirac Lagrangians, thus of the same form as \mathcal{L}_F in the neutral case, where the partial derivatives are replaced by covariant ones.
- \mathcal{L}_{km} are the corrections of the kinetic and mass terms due to the electromagnetic background. They vanish when $\epsilon = 0$.
- $\mathcal{L}_{\text{coupl}}$ contains only new couplings between the spin-3/2 and spin-1/2 fields that are induced by the electromagnetic field. It vanishes when $\epsilon = 0$.

The first part takes the expected simple form

$$\begin{aligned}
\mathcal{L}_{\text{RSd}} &= -\frac{1}{2}\varepsilon^{mnkl}(\lambda_{1m}\sigma_n\mathcal{D}_k\bar{\lambda}_{1l}) + \frac{1}{2}\varepsilon_{mnkl}(\bar{\chi}_1^m\bar{\sigma}^n\mathcal{D}^k\chi_1^l) - \sqrt{2}[(\lambda_1^m\sigma_{mn}\chi_1^n) + \text{h.c.}] \\
&\quad - \frac{1}{2}i(\psi_1\sigma^m\mathcal{D}_m\bar{\psi}_1) - \frac{1}{2}i(\bar{\gamma}_1\bar{\sigma}^m\mathcal{D}_m\gamma_1) - \left[\frac{1}{\sqrt{2}}(\psi_1\gamma_1) + \text{h.c.}\right] + (1 \leftrightarrow 2)
\end{aligned} \tag{5.69}$$

and, it is the Lagrangian that was historically first considered and lead to the issues discussed in the introduction.

The new contribution to the kinetic and mass terms reads:

$$\begin{aligned}
\mathcal{L}_{\text{km}} = & -\frac{1}{2}i\tilde{\epsilon}^{mn}(\psi_1\sigma_m\mathcal{D}_n\bar{\psi}_1) - \frac{1}{2}\epsilon^{mk}G_{kn}\left(\psi_1\sigma_l\mathcal{D}_m\mathcal{D}^l\mathcal{D}^n\bar{\psi}_1\right) - \frac{i}{2}\psi_1(\epsilon\cdot\sigma)\sigma^m(\epsilon\cdot\bar{\sigma})\mathcal{D}_m\bar{\psi}_1 \\
& + \frac{1}{2}\epsilon^{mnkl}(\epsilon_{mp} - i\tilde{\epsilon}_{mp})(\epsilon_{lq} + i\tilde{\epsilon}_{lq})\psi_1\sigma_n\mathcal{D}^p\mathcal{D}_k\mathcal{D}^q\bar{\psi}_1 - i\epsilon^{mk}\epsilon_{kn}(\psi_1\sigma_m\mathcal{D}^n\bar{\psi}_1) \\
& - \frac{1}{2}\epsilon^{pm}G_{mk}\left(\epsilon^{kl} + i\tilde{\epsilon}^{kl}\right)(\epsilon_{pq} - i\tilde{\epsilon}_{pq})\psi_1\sigma_n\mathcal{D}^q\mathcal{D}^n\mathcal{D}_l\bar{\psi}_1 + i\epsilon_{mk}G^{kl}\epsilon_{ln}\psi_1\sigma^p\mathcal{D}^m\mathcal{D}_p\mathcal{D}^n\bar{\psi}_1 \\
& - \frac{i}{2}(\epsilon\epsilon)\tilde{\epsilon}^{mn}\psi_1\sigma_n\mathcal{D}_m\bar{\psi}_1 + \frac{i}{2}(\epsilon\tilde{\epsilon})\epsilon^{mn}\psi_1\sigma_n\mathcal{D}_m\bar{\psi}_1 + \frac{1}{4}\epsilon_{mk}G^{kn}\left(\bar{\gamma}_1\bar{\sigma}^m\sigma^l\bar{\sigma}_n\mathcal{D}_l\gamma_1\right) \\
& - \frac{1}{4}\left[i\epsilon_{mk}G^{kn}\bar{\gamma}_1\bar{\sigma}^m(\epsilon\cdot\sigma)\sigma^l\bar{\sigma}_n\mathcal{D}_l\gamma_1 + \text{h.c.}\right] - \frac{1}{4}\epsilon_{mk}G^{kn}\bar{\gamma}_1\bar{\sigma}^m(\epsilon\cdot\sigma)\sigma^l(\epsilon\cdot\bar{\sigma})\bar{\sigma}_n\mathcal{D}_l\gamma_1 \\
& - i\tilde{\epsilon}^{mn}\bar{\gamma}_1\bar{\sigma}_n\mathcal{D}_m\gamma_1 - \frac{i}{2}(\epsilon\epsilon)\bar{\gamma}_1\bar{\sigma}^m\mathcal{D}_m\gamma_1 - \frac{i}{2}\bar{\gamma}_1(\epsilon\cdot\bar{\sigma})\bar{\sigma}^m(\epsilon\cdot\sigma)\mathcal{D}_m\gamma_1 + \left[-\frac{i}{2}\psi_1(\epsilon\cdot\sigma)\gamma_1\right. \\
& \left. + \frac{1}{\sqrt{2}}\psi_1\sigma^m(\epsilon\cdot\bar{\sigma})\bar{\sigma}^n(\epsilon\cdot\sigma)\mathcal{D}_m\mathcal{D}_n\gamma_1 + \frac{i}{2\sqrt{2}}\epsilon_{mk}G^{kn}\bar{\gamma}_1\bar{\sigma}^m\sigma^l\mathcal{D}_l\mathcal{D}_n\bar{\psi}_1\right. \\
& \left. - \frac{1}{2\sqrt{2}}(\epsilon\epsilon + i\epsilon\tilde{\epsilon})(\psi_1\gamma_1) - \frac{1}{2\sqrt{2}}(\epsilon\epsilon - i\epsilon\tilde{\epsilon})(\psi_1\mathcal{D}^2\gamma_1) + \frac{1}{\sqrt{2}}\epsilon_{mk}G^{kn}\bar{\gamma}_1\bar{\sigma}^m(\epsilon\cdot\sigma)\sigma^l\mathcal{D}_l\mathcal{D}_n\bar{\psi}_1\right. \\
& \left. + \frac{i}{4\sqrt{2}}\epsilon_{mk}G^{kn}(\epsilon\epsilon - i\epsilon\tilde{\epsilon})\psi_1\sigma^l\bar{\sigma}_n\mathcal{D}^m\mathcal{D}_l\gamma_1 + \text{h.c.}\right] - \frac{1}{4}\epsilon_{mn}\left[\left(\bar{\chi}_1^m\bar{\sigma}^n\sigma^k\bar{\sigma}^l\mathcal{D}_k\chi_{1l}\right) + \text{h.c.}\right] \\
& - \frac{1}{4}\epsilon^{mk}G_{kn}\left(\bar{\chi}_1^l\bar{\sigma}_l\sigma_p\bar{\sigma}_q\mathcal{D}_m\mathcal{D}^p\mathcal{D}^n\chi_1^q\right) + \frac{1}{2}\epsilon_{mk}\left(\lambda_1^m\sigma^n\mathcal{D}_n\bar{\lambda}_1^k\right) \\
& + \frac{1}{4\sqrt{2}}\left[i(\lambda_1^m\Sigma_{mn}\chi_1^n) - 2i\epsilon_{mn}\left(\lambda_1^n\sigma^k\bar{\sigma}^l\mathcal{D}_k\mathcal{D}^m\chi_{1l}\right) + \text{h.c.}\right] + (1 \leftrightarrow 2, \epsilon \leftrightarrow -\epsilon)
\end{aligned} \tag{5.70}$$

while the new couplings between spin-1/2 and spin-3/2 fields are:

$$\begin{aligned}
\mathcal{L}_{\text{coupl}} = & \sqrt{2}i\lambda_1^m\sigma_{mn}(\epsilon\cdot\sigma)\mathcal{D}^n\gamma_1 - \frac{i}{\sqrt{2}}(\epsilon_{mn} - i\tilde{\epsilon}_{mn})(\lambda_1^m\mathcal{D}^n\gamma_1) \\
& + \frac{1}{2\sqrt{2}}i\epsilon_{mn}\left(\lambda_1^n\sigma^k\bar{\sigma}^m\mathcal{D}_k\gamma_1\right) + \frac{1}{2\sqrt{2}}\epsilon_{mn}\lambda_1^n\sigma^k(\epsilon\cdot\bar{\sigma})\bar{\sigma}^m\mathcal{D}_k\gamma_1 + \frac{1}{2}\epsilon^{mn}(\lambda_{1m}\sigma_n\bar{\psi}_1) \\
& - \frac{i}{4}(\epsilon\epsilon + i\epsilon\tilde{\epsilon})(\lambda_1^m\sigma_m\bar{\psi}_1) + \frac{i}{2}\epsilon^{mnkl}(\epsilon_{lq} + i\tilde{\epsilon}_{lq})\lambda_{1m}\sigma_n\mathcal{D}_k\mathcal{D}^q\bar{\psi}_1 \\
& + \frac{1}{2}\epsilon_{mk}\left(\eta^{kl} - i\epsilon^{kl} + \tilde{\epsilon}^{kl}\right)(\lambda_1^m\sigma^n\mathcal{D}_n\mathcal{D}_l\bar{\psi}_1) + \frac{1}{4}\bar{\chi}_1^m(\epsilon\cdot\bar{\sigma})\bar{\sigma}_m\gamma_1 \\
& - \frac{3}{2}\bar{\chi}_1^m\bar{\sigma}_m(\epsilon\cdot\sigma)\gamma_1 + \frac{1}{4}\epsilon^{mk}G_{kn}\left(\bar{\chi}_1^l\bar{\sigma}_l\sigma_p\bar{\sigma}^n\mathcal{D}_m\mathcal{D}^p\gamma_1\right) - \frac{i}{4\sqrt{2}}(\psi_1\Sigma^{mn}\mathcal{D}_m\chi_{1n}) \\
& - \frac{i}{\sqrt{2}}\bar{\chi}_1^m\bar{\sigma}_m(\epsilon\cdot\sigma)\sigma_n\mathcal{D}^n\bar{\psi}_1 - \frac{i}{4}\epsilon^{mk}G_{kn}\bar{\chi}_1^l\bar{\sigma}_l\sigma_p(\epsilon\cdot\bar{\sigma})\bar{\sigma}^n\mathcal{D}_m\mathcal{D}^p\gamma_1 \\
& + \frac{i}{4}\bar{\chi}_1^m(\epsilon\cdot\sigma)\bar{\sigma}_m(\epsilon\cdot\sigma)\gamma_1 + \frac{i}{2}(\epsilon\epsilon)(\bar{\chi}_1^m\bar{\sigma}_m\gamma_1) \\
& + \frac{1}{8\sqrt{2}}(\epsilon\epsilon + i\epsilon\tilde{\epsilon})(\bar{\chi}_1^m\bar{\sigma}_m\sigma_n\mathcal{D}^n\bar{\psi}_1) - \frac{1}{4\sqrt{2}}\bar{\chi}_1^m(\epsilon\cdot\bar{\sigma})\bar{\sigma}_m(\epsilon\cdot\sigma)\sigma_n\mathcal{D}^n\bar{\psi}_1 \\
& + \frac{1}{4\sqrt{2}}\bar{\chi}_1^m\bar{\sigma}_m(\epsilon\cdot\sigma)\sigma_n(\epsilon\cdot\bar{\sigma})\mathcal{D}^n\bar{\psi}_1 + \frac{i}{2\sqrt{2}}\epsilon^{mk}G_{kn}\left(\bar{\chi}_1^l\bar{\sigma}_l\sigma_p\mathcal{D}_m\mathcal{D}^p\mathcal{D}^n\bar{\psi}_1\right) \\
& - \frac{1}{2\sqrt{2}}(\epsilon_{mn} - i\tilde{\epsilon}_{mn})G^{mk}\epsilon_{kl}\psi_1\sigma^p\bar{\sigma}^q\mathcal{D}^m\mathcal{D}_p\mathcal{D}^l\chi_{1q} + \text{h.c.} + (1 \leftrightarrow 2, \epsilon \leftrightarrow -\epsilon)
\end{aligned} \tag{5.71}$$

In these expressions, we introduced the notation

$$\Sigma_{mn} \equiv \sigma_m \bar{\sigma}_n (\epsilon \cdot \sigma) - \sigma_m (\epsilon \cdot \bar{\sigma}) \bar{\sigma}_n - (\epsilon \cdot \sigma) \sigma_m \bar{\sigma}_n \quad (5.72)$$

The corresponding spin-3/2 fields equations of motion are

$$\begin{aligned} (\eta_{mp} - i\epsilon_{mp}) & \left[-\frac{1}{2} \epsilon^{pnkl} \sigma_n \mathcal{D}_k \bar{\lambda}_{1l} - \sqrt{2} \sigma^{pk} \chi_{1k} + \frac{i}{4\sqrt{2}} \Sigma^{pk} \chi_{1k} + \sqrt{2} i \sigma^{pn} (\epsilon \cdot \sigma) \mathcal{D}_n \gamma_1 \right. \\ & - \frac{i}{\sqrt{2}} (\epsilon^{pn} - i\tilde{\epsilon}^{pn}) \mathcal{D}_n \gamma_1 + \frac{1}{2} \epsilon^{pn} \sigma_n \bar{\psi}_1 - \frac{i}{4} (\epsilon\epsilon + i\epsilon\tilde{\epsilon}) \sigma^p \bar{\psi}_1 + \frac{i}{2} \epsilon^{pnkl} (\epsilon_{lq} + i\tilde{\epsilon}_{lq}) \sigma_n \mathcal{D}_k \mathcal{D}^q \bar{\psi}_1 \left. \right] \\ & + \frac{1}{2} \epsilon_{mk} \sigma^n \mathcal{D}_n \bar{\lambda}_1^k + \frac{i}{2\sqrt{2}} \epsilon_{mn} \sigma^k \bar{\sigma}^l \mathcal{D}_k \mathcal{D}^n \chi_{1l} - \frac{i}{2\sqrt{2}} \epsilon_{mn} \sigma^k \bar{\sigma}^n \mathcal{D}_k \gamma_1 \\ & - \frac{1}{2\sqrt{2}} \epsilon_{mn} \sigma^k (\epsilon \cdot \bar{\sigma}) \bar{\sigma}^n \mathcal{D}_k \gamma_1 + \frac{1}{2} \epsilon_{mk} \left(\eta^{kl} - i\epsilon^{kl} + \tilde{\epsilon}^{kl} \right) \sigma^n \mathcal{D}_n \mathcal{D}_l \bar{\psi}_1 = 0 \end{aligned} \quad (5.73)$$

$$\begin{aligned} & \frac{1}{2} \epsilon_{mnkl} \bar{\sigma}^n \mathcal{D}^k \chi_1^l - \frac{1}{4} \epsilon_{mn} \bar{\sigma}^n \sigma^k \bar{\sigma}^l \mathcal{D}_k \chi_{1l} + \frac{1}{4} \epsilon_{kl} \bar{\sigma}_m \sigma_n \bar{\sigma}^l \mathcal{D}^n \chi_1^k - \frac{1}{4} \epsilon_{lk} G^{kn} \bar{\sigma}_m \sigma_p \bar{\sigma}_q \mathcal{D}^l \mathcal{D}^p \mathcal{D}_n \chi_1^q \\ & - \sqrt{2} \bar{\sigma}_{mn} \bar{\lambda}_1^n - \frac{i}{4\sqrt{2}} \bar{\Sigma}_{nm} \bar{\lambda}_1^n + \frac{i}{2\sqrt{2}} \epsilon_{ln} \bar{\sigma}_m \sigma_k \mathcal{D}^l \mathcal{D}^k \bar{\lambda}_1^n + \frac{1}{4} (\epsilon \cdot \bar{\sigma}) \bar{\sigma}_m \gamma_1 - \frac{3}{2} \bar{\sigma}_m (\epsilon \cdot \sigma) \gamma_1 \\ & + \frac{1}{4} \epsilon_{lk} G^{kn} \bar{\sigma}_m \sigma_p \bar{\sigma}_n \mathcal{D}^l \mathcal{D}^p \gamma_1 - \frac{i}{4\sqrt{2}} \bar{\Sigma}_{nm} \mathcal{D}^n \bar{\psi}_1 - \frac{i}{\sqrt{2}} \bar{\sigma}_m (\epsilon \cdot \sigma) \sigma_n \mathcal{D}^n \bar{\psi}_1 \\ & + \frac{i}{2} (\epsilon\epsilon) \bar{\sigma}_m \gamma_1 - \frac{i}{4} \epsilon^{lk} G_{kn} \bar{\sigma}_m \sigma_p (\epsilon \cdot \bar{\sigma}) \bar{\sigma}^n \mathcal{D}_l \mathcal{D}^p \gamma_1 + \frac{1}{8\sqrt{2}} (\epsilon\epsilon + i\epsilon\tilde{\epsilon}) \bar{\sigma}_m \sigma_n \mathcal{D}^n \bar{\psi}_1 \\ & - \frac{1}{4\sqrt{2}} (\epsilon \cdot \bar{\sigma}) \bar{\sigma}_m (\epsilon \cdot \sigma) \sigma_n \mathcal{D}^n \bar{\psi}_1 + \frac{i}{2\sqrt{2}} \epsilon_{lk} G^{kn} \bar{\sigma}_m \sigma_p \mathcal{D}^l \mathcal{D}^p \mathcal{D}_n \bar{\psi}_1 + \frac{i}{4} (\epsilon \cdot \bar{\sigma}) \bar{\sigma}_m (\epsilon \cdot \sigma) \gamma_1 \\ & + \frac{1}{4\sqrt{2}} \bar{\sigma}_m (\epsilon \cdot \sigma) \sigma_n (\epsilon \cdot \bar{\sigma}) \mathcal{D}^n \bar{\psi}_1 - \frac{1}{2\sqrt{2}} (\epsilon_{qn} + i\tilde{\epsilon}_{qn}) G^{lk} \epsilon_{kn} \bar{\sigma}_m \sigma_p \mathcal{D}_l \mathcal{D}^p \mathcal{D}^q \bar{\psi}_1 = 0 \end{aligned} \quad (5.74)$$

Due to the length of the spin-1/2 equations of motion, only the first terms are written to make the adopted global factors explicit. They are:

$$-\frac{i}{2} \sigma^m \mathcal{D}_m \bar{\psi}_1 - \frac{i}{2} \tilde{\epsilon}_{mn} \sigma^m \mathcal{D}^n \bar{\psi}_1 - \frac{1}{\sqrt{2}} \gamma_1 + \dots = 0 \quad (5.75)$$

$$-\frac{i}{2} \bar{\sigma}^m \mathcal{D}_m \gamma_1 + \frac{1}{4} \epsilon_{mk} G^{kn} \bar{\sigma}^m \sigma^l \bar{\sigma}_n \mathcal{D}_l \gamma_1 - \frac{1}{\sqrt{2}} \bar{\psi}_1 + \dots = 0 \quad (5.76)$$

The calculation leading to the constraints is more tedious, but shares the same ideas as for the Lagrangian (5.57). Namely, one starts by eliminating the G -dependent terms appearing in the spin-1/2 equations of motion by the following manipulations.

We start by the combination: (5.76) $- \frac{i}{\sqrt{2}} G^{am} \bar{\sigma}_a \times$ (5.73) $- \frac{1}{\sqrt{2}} G^{am} \bar{\sigma}_a (\epsilon \cdot \sigma) \times$ (5.73):

$$\begin{aligned} [1 + i(\epsilon \cdot \bar{\sigma})] & \left[-\sqrt{2} \bar{\sigma}^{mn} \mathcal{D}_m \bar{\lambda}_{1n} - \frac{i}{\sqrt{2}} \bar{\sigma}^m (\epsilon \cdot \sigma) \sigma^n \mathcal{D}_m \mathcal{D}_n \bar{\psi}_1 - \frac{1}{\sqrt{2}} \bar{\psi}_1 - \frac{i}{\sqrt{2}} (\epsilon \cdot \bar{\sigma}) \bar{\psi}_1 \right. \\ & - \frac{i}{2} \bar{\sigma}^m \mathcal{D}_m \gamma_1 + \frac{1}{2} (\epsilon \cdot \bar{\sigma}) \bar{\sigma}^m \mathcal{D}_m \gamma_1 - \bar{\sigma}^m (\epsilon \cdot \sigma) \mathcal{D}_m \gamma_1 - \frac{3}{2} i \bar{\sigma}^m \chi_{1m} \\ & \left. + \frac{1}{2} (\epsilon \cdot \bar{\sigma}) \bar{\sigma}^m \chi_{1m} - \frac{1}{4} \bar{\sigma}^m (\epsilon \cdot \sigma) \chi_{1m} \right] = 0 \end{aligned} \quad (5.77)$$

then, (5.75)– $i(\epsilon_{ab} - i\tilde{\epsilon}_{ab})G^{bm}\mathcal{D}^a(5.73)+G^{am}\mathcal{D}_a\times(5.73)-\frac{1}{\sqrt{2}}\sigma^m(\epsilon\cdot\bar{\sigma})\times(5.74)$:

$$-\frac{i}{2}\sigma^m\mathcal{D}_m\bar{\psi}_1 - \frac{1}{\sqrt{2}}\gamma_1 + \frac{1}{2}(\epsilon\cdot\sigma)\sigma^m\mathcal{D}_m\bar{\psi}_1 - \frac{i}{\sqrt{2}}(\epsilon\cdot\sigma)\gamma_1 - \sqrt{2}\sigma^{mn}\mathcal{D}_m\chi_{1n} = 0 \quad (5.78)$$

Next, we compute and simplify the σ -trace, the divergence, and the projection on $\sigma^m(\epsilon\cdot\bar{\sigma})$ of the equations of motion of χ_{1m} . The combination $\sigma^m\times(5.74)+\sqrt{2}iG^{am}\mathcal{D}_a\times(5.73)$ gives:

$$-2i\sigma^{mn}\mathcal{D}_m\chi_{1n} + \frac{1}{\sqrt{2}}\sigma^m\bar{\lambda}_{1m} + 2(\epsilon\cdot\sigma)\gamma_1 + \sqrt{2}i(\epsilon\cdot\sigma)\sigma^m\mathcal{D}_m\bar{\psi}_1 = 0 \quad (5.79)$$

while $\mathcal{D}^m\times(5.74)-\frac{i}{\sqrt{2}}\bar{\sigma}^a\mathcal{D}_aG^{bm}\mathcal{D}_b\times(5.73)$ leads to:

$$\begin{aligned} & -\sqrt{2}\bar{\sigma}^{mn}\mathcal{D}_m\bar{\lambda}_{1n} + \frac{1}{4\sqrt{2}}i(\epsilon\cdot\bar{\sigma})\bar{\sigma}^m\sigma^n\mathcal{D}_m\bar{\lambda}_{1n} + \frac{1}{2}i\bar{\epsilon}^{mn}\bar{\sigma}_n\chi_{1m} + i\bar{\sigma}^m\mathcal{D}_m\sigma^{kn}\mathcal{D}_k\chi_{1n} \\ & -\frac{1}{8}(\epsilon\cdot\bar{\sigma})\bar{\sigma}^m\sigma^n\bar{\sigma}^k\mathcal{D}_m\mathcal{D}_n\chi_{1k} + \frac{1}{4}(\epsilon\cdot\bar{\sigma})\bar{\sigma}^m\mathcal{D}_m\gamma_1 - \frac{3}{2}\bar{\sigma}^m(\epsilon\cdot\sigma)\mathcal{D}_m\gamma_1 \\ & +\frac{i}{4}(\epsilon\cdot\bar{\sigma})\bar{\sigma}^m(\epsilon\cdot\sigma)\mathcal{D}_m\gamma_1 + \frac{1}{4\sqrt{2}}i(\epsilon\cdot\bar{\sigma})\bar{\sigma}^m\sigma^n\mathcal{D}_m\mathcal{D}_n\bar{\psi}_1 - \frac{3i}{2\sqrt{2}}\bar{\sigma}^m(\epsilon\cdot\sigma)\sigma^n\mathcal{D}_m\mathcal{D}_n\bar{\psi}_1 \\ & -\frac{1}{4\sqrt{2}}(\epsilon\cdot\bar{\sigma})\bar{\sigma}^m(\epsilon\cdot\sigma)\sigma^n\mathcal{D}_m\mathcal{D}_n\bar{\psi}_1 = 0 \end{aligned} \quad (5.80)$$

and $-\frac{1}{\sqrt{2}}\sigma^m(\epsilon\cdot\bar{\sigma})\times(5.74)-\frac{1}{2}(\epsilon\epsilon - i\epsilon\tilde{\epsilon})(5.78)$ to:

$$\begin{aligned} & \frac{1}{2}\sigma^m(\epsilon\cdot\bar{\sigma})\bar{\lambda}_{1m} - \frac{i}{\sqrt{2}}(\epsilon^{mn} - i\tilde{\epsilon}^{mn})\mathcal{D}_m\chi_{1n} \\ & +(\epsilon\epsilon - i\epsilon\tilde{\epsilon})\left(-\frac{i}{4}\sigma^m\bar{\lambda}_{1m} + \frac{1}{4\sqrt{2}}\sigma^m\bar{\sigma}^n\mathcal{D}_m\chi_{1n} + \frac{1}{\sqrt{2}}\sigma^{mn}\mathcal{D}_m\chi_{1n}\right) = 0 \end{aligned} \quad (5.81)$$

Using these relations, the σ -trace as well as the divergence for $\bar{\lambda}_{1m}$ equations of motion can be simplified by computing the combinations:

$$\bar{\sigma}^m\times(5.73)+i\epsilon_{ab}\bar{\sigma}^b\mathcal{D}^a\times(5.78)+\frac{\sqrt{2}(\epsilon\cdot\bar{\sigma})}{[1+i(\epsilon\cdot\bar{\sigma})]}(5.77)-\sqrt{2}i(5.80)-\frac{1}{2}i(\epsilon\cdot\bar{\sigma})\bar{\sigma}^a\mathcal{D}_a\times(5.78)$$

which gives:

$$\begin{aligned} & \frac{3}{\sqrt{2}}\bar{\sigma}^m\chi_{1m} + \bar{\sigma}^m\mathcal{D}_m\left[\frac{1}{4}(\epsilon\cdot\sigma)\sigma^n\bar{\lambda}_{1n} + \sqrt{2}\sigma^{kn}\mathcal{D}_k\chi_{1n} + \frac{i}{\sqrt{2}}(\epsilon\cdot\sigma)\sigma^{nk}\mathcal{D}_n\chi_{1k}\right. \\ & \left. + \frac{i}{4\sqrt{2}}(\epsilon\cdot\sigma)\sigma^n\bar{\sigma}^k\mathcal{D}_n\chi_{1k} + \frac{i}{\sqrt{2}}(\epsilon\cdot\sigma)\gamma_1 - \frac{1}{2}(\epsilon\cdot\sigma)\sigma^n\mathcal{D}_n\bar{\psi}_1\right] = 0 \end{aligned} \quad (5.82)$$

and $\mathcal{D}^m\times(5.73)+\frac{1}{\sqrt{2}}\epsilon_{ab}\sigma^b\mathcal{D}^a\times(5.77)+\frac{1}{2}(5.81)+\frac{1}{2}(\epsilon\epsilon)(5.78)+i(\epsilon\cdot\sigma)\times(5.78)$ which gives:

$$\begin{aligned} & [1+i(\epsilon\cdot\sigma)]\left[\frac{1}{4}(\epsilon\cdot\sigma)\sigma^n\bar{\lambda}_{1n} + \sqrt{2}\sigma^{kn}\mathcal{D}_k\chi_{1n} + \frac{i}{\sqrt{2}}(\epsilon\cdot\sigma)\sigma^{nk}\mathcal{D}_n\chi_{1k}\right. \\ & \left. + \frac{i}{4\sqrt{2}}(\epsilon\cdot\sigma)\sigma^n\bar{\sigma}^k\mathcal{D}_n\chi_{1k} + \frac{i}{\sqrt{2}}(\epsilon\cdot\sigma)\gamma_1 - \frac{1}{2}(\epsilon\cdot\sigma)\sigma^n\mathcal{D}_n\bar{\psi}_1\right] = 0 \end{aligned} \quad (5.83)$$

Thus, thanks to (5.83), the expression between brackets in (5.82) vanishes and we obtain the first constraint $\bar{\sigma}^m\chi_{1m} = 0$. Other constraints are found by inserting this traceless

condition into (5.77)-(5.83). Starting with the expression between brackets of (5.83), one replaces the $(\epsilon \cdot \sigma) \sigma^m \bar{\lambda}_{1m}$ term by $(\epsilon \cdot \sigma) \times$ (5.79), which implies

$$-\sqrt{2} \sigma^{mn} \mathfrak{D}_m \chi_{1n} + \frac{1}{2} (\epsilon \cdot \sigma) \sigma^m \mathfrak{D}_m \bar{\psi}_1 - \frac{i}{\sqrt{2}} (\epsilon \cdot \sigma) \gamma_1 = 0 \quad (5.84)$$

Combining it with (5.78), one gets the Dirac equation $i\sigma^m \mathfrak{D}_m \bar{\psi}_1 + \sqrt{2} \gamma_1 = 0$, which in turn yields $\sigma^{mn} \mathfrak{D}_m \chi_{1n} = 0$, hence $\mathfrak{D}^m \chi_{1m} = 0$. The constraints of $\bar{\lambda}_{1m}$ arise first from (5.79), which implies $\sigma^m \bar{\lambda}_{1m} = 0$. In addition, (5.80) gives rise to $\bar{\sigma}^{mn} \mathfrak{D}_m \bar{\lambda}_{1n} = -\frac{1}{4\sqrt{2}} \bar{\sigma}^m (\epsilon \cdot \sigma) \chi_{1m}$, therefore, $\mathfrak{D}^m \bar{\lambda}_{1m} = -\frac{\sqrt{2}}{4} \bar{\sigma}^m (\epsilon \cdot \sigma) \chi_{1m}$. The Dirac equation of γ_1 , $i\bar{\sigma}^m \mathfrak{D}_m \gamma_1 + \sqrt{2} \bar{\psi}_1 = 0$, then follows from (5.77). Finally, the equations of motion for spin-3/2 are obtained by applying all the previously mentioned equations of motion and constraints on (5.73) and (5.74).

6 Conclusions

We conclude by summarising the main results obtained here:

- We have shown explicitly how the Lagrangians of Fierz-Pauli and Rarita-Schwinger follow from the Open Superstring Field Theory action of [1, 24].
- We found the equations governing the four-dimensional propagation of a charged massive state with spin-2 in an electromagnetic background. Not surprisingly, we found the form obtained at the critical dimension of bosonic open strings in [18, 21].
- We found the explicit equations governing the four-dimensional propagation of a charged massive state with spin-3/2 in an electromagnetic background. These equations were not known before.
- We have written the effective Lagrangian, at the bilinear level, of the first massive level states of the four-dimensional open superstring in an electromagnetic background. The result was known in superspace thanks to [1], but here we obtained the result for the physical fields, without auxiliary ones. We have written the Lagrangian in several forms connected by redefinitions of the fields.

While we have solved here the problem raised in [2, 3] of writing the equations of motion for charged massive states with spin-3/2 or spin-2, we have not found the long sought Lagrangian from which to derive them. In the Lagrangians obtained the fields of interest are coupled to fields with lower spins.

Acknowledgements

We are grateful to Nathan Berkovits for useful discussions. CAD acknowledges FAPESP grant numbers 2020/10183-9 and 2022/14599-0 for financial support. The work of WK is supported by the Contrat Doctoral Spécifique Normalien (CDSN) of Ecole Normale Supérieure – PSL.

A Conventions

Throughout this work, we have followed the conventions in [35], with mostly positive metric $\eta_{mn} \sim (-1, 1, 1, 1)$, and Levi-Civita tensor $\epsilon_{0123} = -1$. The γ matrices are defined as

$$\gamma^m = \begin{pmatrix} 0 & \sigma^m \\ \bar{\sigma}^m & 0 \end{pmatrix}, \quad \gamma^5 = \gamma^0 \gamma^1 \gamma^2 \gamma^3 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \quad (\text{A.1})$$

The field redefinitions are written in the form

$$F \rightarrow aF + bG \quad (\text{A.2})$$

which means that the field F is replaced everywhere by $aF' + bG$, and the primes are dropped in the subsequent Lagrangian. The new gauge transformations are

$$\delta F' = \frac{1}{a}(\delta F - b\delta G), \quad \delta G' = \delta G \quad (\text{A.3})$$

A *self-dual* rank-2 tensor satisfies

$$\epsilon_{mnpq} S^{mn} = -2i S_{pq} \quad (\text{A.4})$$

Correspondingly, for an anti self-dual tensor, $\epsilon_{mnpq} S^{mn} = 2i S_{pq}$. Out of a generic rank-2 tensor, one is able to construct a self-dual combination

$$\mathcal{S}[A_{mn}] \equiv \frac{1}{4}(A_{mn} - A_{nm}) + \frac{i}{4}\epsilon_{mnpq} A^{pq}. \quad (\text{A.5})$$

This has been used to write gauge transformations of the self-dual fields t_{mn} , s_{mn} .

A summary of some shorthand notations used in this work:

$$\begin{aligned} (\epsilon \cdot \sigma) &\equiv \epsilon^{mn} \sigma_{mn} = i \tilde{\epsilon}^{mn} \sigma_{mn}, & (\epsilon \cdot \bar{\sigma}) &\equiv \epsilon^{mn} \bar{\sigma}_{mn} = -i \tilde{\epsilon}^{mn} \bar{\sigma}_{mn} \\ \epsilon\epsilon &\equiv \epsilon^{mn} \epsilon_{mn}, & \tilde{\epsilon}\tilde{\epsilon} &\equiv \tilde{\epsilon}^{mn} \tilde{\epsilon}_{mn} \\ G_{mn} &\equiv (\eta_{mn} - i\epsilon_{mn})^{-1}, & \mathcal{A}_{mn} &\equiv \left(\eta_{mn} - \frac{i}{2}\epsilon_{mn} - \frac{1}{2}\tilde{\epsilon}_{mk}\tilde{\epsilon}^k{}_n \right)^{-1} \\ \Sigma_{mn} &\equiv \sigma_m \bar{\sigma}_n (\epsilon \cdot \sigma) - \sigma_m (\epsilon \cdot \bar{\sigma}) \bar{\sigma}_n - (\epsilon \cdot \sigma) \sigma_m \bar{\sigma}_n \end{aligned} \quad (\text{A.6})$$

B Spinor algebra results

Without background

Some helpful σ -matrix identities are

$$\begin{aligned}
\sigma_{\alpha\dot{\alpha}}^m \bar{\sigma}^{n\dot{\alpha}\beta} &= 2\sigma^{mn} \alpha^\beta - \eta^{mn} \delta_{\alpha}^\beta, & \bar{\sigma}^{m\dot{\alpha}\alpha} \sigma_{\alpha\dot{\beta}}^n &= 2\bar{\sigma}^{mn\dot{\alpha}}_{\dot{\beta}} - \eta^{mn} \delta_{\dot{\beta}}^{\dot{\alpha}} \\
(\sigma^m \bar{\sigma}_m)_{\alpha}^{\beta} &= -4\delta_{\alpha}^{\beta}, & \sigma_{\alpha\dot{\alpha}}^n \bar{\sigma}^{m\dot{\alpha}\beta} \sigma_{n\beta\dot{\gamma}} &= 2\sigma_{\alpha\dot{\gamma}}^m, & \bar{\sigma}^{n\dot{\alpha}\alpha} \sigma_{\alpha\dot{\beta}}^m \bar{\sigma}_n^{\dot{\beta}\gamma} &= 2\bar{\sigma}^{m\dot{\alpha}\gamma} \\
\sigma^{mn} \alpha^\beta \sigma_{n\beta\dot{\alpha}} &= \sigma_{n\alpha\dot{\beta}} \bar{\sigma}^{nm\dot{\beta}}_{\dot{\alpha}} = -\frac{3}{2}\sigma_{\alpha\dot{\alpha}}^m, & \bar{\sigma}^{mn\dot{\beta}}_{\dot{\alpha}} \bar{\sigma}_n^{\dot{\alpha}\alpha} &= \bar{\sigma}_n^{\dot{\beta}\beta} \sigma^{nm} \beta^\alpha = -\frac{3}{2}\bar{\sigma}^{m\dot{\beta}\alpha} \\
\sigma_{m\alpha\dot{\alpha}} \bar{\sigma}^{nk\dot{\alpha}}_{\dot{\beta}} \bar{\sigma}^{m\dot{\beta}\beta} &= 0, & \bar{\sigma}_m^{\dot{\alpha}\alpha} \sigma^{nk} \alpha^\beta \sigma_{\beta\dot{\beta}}^m &= 0 \\
\sigma_{m\beta\dot{\alpha}} \bar{\sigma}^{n\dot{\alpha}\alpha} \sigma_{\alpha\dot{\gamma}}^k \bar{\sigma}^{m\dot{\gamma}\gamma} &= 4\eta^{nk} \delta_{\beta}^{\gamma}, & \bar{\sigma}_m^{\dot{\alpha}\alpha} \sigma_{\alpha\dot{\beta}}^n \bar{\sigma}^{k\dot{\beta}\gamma} \sigma_{\gamma\dot{\gamma}}^m &= 4\eta^{nk} \delta_{\dot{\beta}}^{\dot{\alpha}} \\
\varepsilon_{mnkl} (\sigma^k \bar{\sigma}^l)_{\alpha}^{\beta} &= -4i\sigma_{mn\alpha}^{\beta}, & \varepsilon_{mnkl} (\bar{\sigma}^k \sigma^l)^{\dot{\alpha}}_{\dot{\beta}} &= 4i\bar{\sigma}_{mn}^{\dot{\alpha}}_{\dot{\beta}} \\
\sigma^m \bar{\sigma}^n \sigma^k &= \eta^{mk} \sigma^n - \eta^{nk} \sigma^m - \eta^{mn} \sigma^k + i\varepsilon^{mnkl} \sigma_l \\
\bar{\sigma}^m \sigma^n \bar{\sigma}^k &= \eta^{mk} \bar{\sigma}^n - \eta^{nk} \bar{\sigma}^m - \eta^{mn} \bar{\sigma}^k - i\varepsilon^{mnkl} \bar{\sigma}_l
\end{aligned} \tag{B.1}$$

The two-spinors satisfy the following properties

$$\begin{aligned}
\gamma\psi &\equiv \gamma^\alpha \psi_\alpha, & \bar{\psi}\bar{\gamma} &\equiv \bar{\psi}_{\dot{\alpha}} \bar{\gamma}^{\dot{\beta}} \\
\gamma\psi &= \psi\gamma, & \bar{\gamma}\bar{\psi} &= \bar{\psi}\bar{\gamma} \\
\gamma\sigma^m \bar{\psi} &= -\bar{\psi}\bar{\sigma}^m \gamma, & \gamma\sigma^m \bar{\sigma}^n \psi &= \psi\sigma^n \bar{\sigma}^m \gamma \\
\gamma\sigma^{mn} \psi &= -\psi\sigma^{mn} \gamma, & \bar{\gamma}\bar{\sigma}^{mn} \bar{\psi} &= -\bar{\psi}\bar{\sigma}^{mn} \bar{\gamma}
\end{aligned} \tag{B.2}$$

See also the appendices of [35] for other relations and details.

With background

$$\begin{aligned}
\bar{\sigma}^{m\dot{\alpha}\alpha} (\epsilon \cdot \sigma)_{\alpha}^{\beta} &= -\epsilon^{mn} \bar{\sigma}_n^{\dot{\alpha}\beta} - i\tilde{\epsilon}^{mn} \bar{\sigma}_n^{\dot{\alpha}\beta}, & (\epsilon \cdot \sigma)_{\alpha}^{\beta} \sigma_{\beta\dot{\beta}}^m &= \epsilon^{mn} \sigma_{n\alpha\dot{\beta}} + i\tilde{\epsilon}^{mn} \sigma_{n\alpha\dot{\beta}} \\
\sigma_{\alpha\dot{\alpha}}^m (\epsilon \cdot \bar{\sigma})^{\dot{\alpha}}_{\dot{\beta}} &= -\epsilon^{mn} \sigma_{n\alpha\dot{\beta}} + i\tilde{\epsilon}^{mn} \sigma_{n\alpha\dot{\beta}}, & (\epsilon \cdot \bar{\sigma})^{\dot{\alpha}}_{\dot{\beta}} \bar{\sigma}^{m\dot{\beta}\alpha} &= \epsilon^{mn} \bar{\sigma}_n^{\dot{\alpha}\alpha} - i\tilde{\epsilon}^{mn} \bar{\sigma}_n^{\dot{\alpha}\alpha} \\
(\epsilon \cdot \sigma)_{\alpha}^{\beta} (\epsilon \cdot \sigma)_{\beta}^{\gamma} &= -\frac{1}{2}\epsilon^{mn} (\epsilon_{mn} + i\tilde{\epsilon}_{mn}) \delta_{\alpha}^{\gamma} \\
(\epsilon \cdot \bar{\sigma})^{\dot{\alpha}}_{\dot{\beta}} (\epsilon \cdot \bar{\sigma})^{\dot{\beta}}_{\dot{\gamma}} &= -\frac{1}{2}\epsilon^{mn} (\epsilon_{mn} - i\tilde{\epsilon}_{mn}) \delta_{\dot{\beta}}^{\dot{\alpha}} \\
\tilde{\epsilon}_m^k \sigma_{kn}^{\dot{\alpha}} \dot{\gamma} - i\epsilon_n^k \sigma_{km}^{\dot{\alpha}} \dot{\gamma} &= \frac{i}{2} (\epsilon \cdot \sigma)^{\dot{\alpha}}_{\dot{\gamma}} \eta_{mn}, & \tilde{\epsilon}_m^k \bar{\sigma}_{kn}^{\dot{\alpha}} \dot{\beta} + i\epsilon_n^k \bar{\sigma}_{km}^{\dot{\alpha}} \dot{\beta} &= -\frac{i}{2} (\epsilon \cdot \bar{\sigma})^{\dot{\alpha}}_{\dot{\beta}} \eta_{mn} \\
\sigma_{\alpha\dot{\alpha}}^m (\epsilon \cdot \bar{\sigma})^{\dot{\alpha}}_{\dot{\beta}} \bar{\sigma}^{n\dot{\beta}\beta} - \sigma_{\alpha\dot{\alpha}}^n (\epsilon \cdot \bar{\sigma})^{\dot{\alpha}}_{\dot{\beta}} \bar{\sigma}^{m\dot{\beta}\beta} &= 2(\epsilon^{mn} - i\tilde{\epsilon}^{mn}) \delta_{\alpha}^{\beta} \\
\bar{\sigma}^{m\dot{\alpha}\alpha} (\epsilon \cdot \sigma)_{\alpha}^{\beta} \sigma_{\beta\dot{\beta}}^n - \bar{\sigma}^{n\dot{\alpha}\alpha} (\epsilon \cdot \sigma)_{\alpha}^{\beta} \sigma_{\beta\dot{\beta}}^m &= 2(\epsilon^{mn} + i\tilde{\epsilon}^{mn}) \delta_{\dot{\beta}}^{\dot{\alpha}} \\
\sigma_{\alpha\dot{\alpha}}^m \bar{\sigma}^{n\dot{\alpha}\gamma} (\epsilon \cdot \sigma)_{\gamma}^{\beta} + (\epsilon \cdot \sigma)_{\alpha}^{\gamma} \sigma_{\gamma\dot{\gamma}}^m \bar{\sigma}^{n\dot{\gamma}\beta} &= -2(\epsilon^{mn} + i\tilde{\epsilon}^{mn}) \delta_{\alpha}^{\beta} - 2\eta^{mn} (\epsilon \cdot \sigma)_{\alpha}^{\beta} \\
\bar{\sigma}^{m\dot{\alpha}\alpha} \sigma_{\alpha\dot{\gamma}}^n (\epsilon \cdot \bar{\sigma})^{\dot{\gamma}}_{\dot{\beta}} + (\epsilon \cdot \bar{\sigma})^{\dot{\alpha}}_{\dot{\beta}} \bar{\sigma}^{m\dot{\gamma}\beta} \sigma_{\beta\dot{\beta}}^n &= -2(\epsilon^{mn} - i\tilde{\epsilon}^{mn}) \delta_{\dot{\beta}}^{\dot{\alpha}} - 2\eta^{mn} (\epsilon \cdot \bar{\sigma})^{\dot{\alpha}}_{\dot{\beta}}
\end{aligned} \tag{B.3}$$

C Inverse matrices with the field strength ϵ

Due to the presence of the $(\eta - i\epsilon)$ factors in the superspace action, the inverse matrix $(\eta - i\epsilon)^{-1}$ is recurrent in the Lagrangian after redefinition and in the equations of motion. In this work, for the sake of simplicity, we have not developed the inverse matrices in the Lagrangian, such as (5.57). Nevertheless, they can be written explicitly in terms of ϵ and its dual $\tilde{\epsilon}$. For a constant $a \sim \mathcal{O}(1)$:

$$(a\eta_{mn} - i\epsilon_{mn})^{-1} = \frac{1}{F} \left[a^3 \eta_{mn} + a^2 i\epsilon_{mn} - a\tilde{\epsilon}_{mk}\tilde{\epsilon}^k{}_n + \frac{i}{4}(\epsilon\tilde{\epsilon})\tilde{\epsilon}_{mn} \right] \quad (\text{C.1})$$

with $F \equiv a^4 - \frac{1}{16}(\epsilon\tilde{\epsilon})^2 - \frac{a^2}{2}(\epsilon\epsilon)$.

The denominator of the r.h.s. is a non-vanishing number for any small electromagnetic field strength. When $a = 1$, we have for instance

$$\begin{aligned} \text{Tr}(\eta - i\epsilon)^{-1} &= \eta^{mn}(\eta_{mn} - i\epsilon_{mn})^{-1} = \frac{4 - (\epsilon\epsilon)}{1 - \frac{1}{16}(\epsilon\tilde{\epsilon})^2 - \frac{1}{2}(\epsilon\epsilon)} \\ \epsilon^{mn}(\eta_{mn} - i\epsilon_{mn})^{-1} &= \frac{i(\epsilon\epsilon) + \frac{i}{4}(\epsilon\tilde{\epsilon})^2}{1 - \frac{1}{16}(\epsilon\tilde{\epsilon})^2 - \frac{1}{2}(\epsilon\epsilon)} \\ (\eta_{mn} - i\epsilon_{mn})^{-1} \bar{\sigma}^m \sigma^k \bar{\sigma}^n &= \frac{1}{1 - \frac{1}{16}(\epsilon\tilde{\epsilon})^2 - \frac{1}{2}(\epsilon\epsilon)} \left[2\eta^{mk} + 2\tilde{\epsilon}^{mk} + 2\epsilon^m{}_n \epsilon^{nk} - \frac{1}{2}(\epsilon\tilde{\epsilon})\epsilon^{mk} \right] \bar{\sigma}_m \end{aligned} \quad (\text{C.2})$$

It is possible to work out other forms of inverse matrices thanks to (C.1). In particular, we encountered in (4.84) the factor $(\eta_{mn} - \frac{i}{2}\epsilon_{mn} - \frac{1}{2}\tilde{\epsilon}_{mk}\tilde{\epsilon}^k{}_n)^{-1}$, which can be found by decomposing $(\eta_{mn} - \frac{i}{2}\epsilon_{mn} - \frac{1}{2}\tilde{\epsilon}_{mk}\tilde{\epsilon}^k{}_n)$ into a product:

$$\left(\eta_{mn} - \frac{i}{2}\epsilon_{mn} - \frac{1}{2}\tilde{\epsilon}_{mk}\tilde{\epsilon}^k{}_n \right) = \frac{1}{2} \left[\frac{1}{2}(1+x)\eta_{mk} - i\epsilon_{mk} \right] \left[\frac{1}{2}(1-x)\delta^k{}_n - i\epsilon^k{}_n \right] \quad (\text{C.3})$$

where $x = \sqrt{2(\epsilon\epsilon) - 7}$. A helpful property of these inverses matrices containing $\eta, \epsilon, \tilde{\epsilon}$, that we denote generically by K_{mn} , is

$$K_{mn}\epsilon^{nk} = \epsilon_{mn}K^{nk}, \quad K_{mn}\tilde{\epsilon}^{nk} = \tilde{\epsilon}_{mn}K^{nk} \quad (\text{C.4})$$

Besides, while deriving the constraints of the fermionic Lagrangians, we used the inverse of $[1 + i(\epsilon \cdot \sigma)]$. Due to the third line of (B.3), we know that

$$[1 + i(\epsilon \cdot \sigma)]_\alpha{}^\gamma [1 - i(\epsilon \cdot \sigma)]_\gamma{}^\beta = [1 - i(\epsilon \cdot \sigma)]_\alpha{}^\gamma [1 + i(\epsilon \cdot \sigma)]_\gamma{}^\beta = \left[1 - \frac{1}{2}(\epsilon\epsilon + i\epsilon\tilde{\epsilon}) \right] \delta_\alpha{}^\beta \quad (\text{C.5})$$

therefore,

$$[1 + i(\epsilon \cdot \sigma)]^{-1}{}_\alpha{}^\beta = \frac{1}{1 - \frac{1}{2}(\epsilon\epsilon + i\epsilon\tilde{\epsilon})} [1 - i(\epsilon \cdot \sigma)]_\alpha{}^\beta \quad (\text{C.6})$$

The inverse of $[1 \pm i(\epsilon \cdot \bar{\sigma})]$ is worked out analogously.

D Physical and non-physical fields

It is insightful to have a qualitative understanding about the role of each component of the superfields.

- The $(\theta\theta)(\bar{\theta}\bar{\theta})$ components of $\{V_m, \mathcal{B}, \mathcal{C}\}$ are auxiliary fields.
- All the gauge parameters appear at least once algebraically in the transformations, hence correspondingly certain fields turn out to be pure gauges.
- In going to the unitary gauge adopted in our work, we identify the Stückelberg fields by looking at the field redefinition.

The roles of different components are summarised in the following table:

	V_m	\mathcal{B}	\mathcal{C}	$U_{1\alpha}, \bar{U}_2^{\dot{\alpha}}$
Physical	$v_{mn}, h, C_m,$ $\chi_{jm}, \bar{\lambda}_{jm}$	$\mathcal{N}_1, B, \gamma_{j\alpha}$	$\mathcal{M}_1, A, \bar{\psi}_{j\alpha}$	
Auxiliary	M_{jm}, D_m	$\rho_{j\alpha}, G, \mathcal{N}_2$	D, \mathcal{M}_2	τ_j, τ_{jmn}
Stückelberg			ϕ	$v_{j\alpha}, \omega_{2m}, s_j$
Pure gauge (except Stückelberg)	f_{mn}	φ	$\xi_{j\alpha}$	$r_{jm\alpha}, \zeta_{j\alpha},$ $\omega_{1m}, q_{jm},$ s_{jmn}
Others				$\mu_{j\alpha}, \eta_{j\alpha}$

where in the last line, $\mu_{j\alpha}$ appear as Lagrange multipliers. As for the components $\eta_{j\alpha}$, being also pure gauges, they do not contribute to the Lagrangian.

References

- [1] K. Benakli, N. Berkovits, C. A. Daniel and M. Lize, “Higher-spin states of the superstring in an electromagnetic background,” *JHEP* **12** (2021), 112 doi:10.1007/JHEP12(2021)112 [arXiv:2110.07623 [hep-th]].
- [2] P. A. M. Dirac, *Proc. Roy. Soc. Lond. A* **155** (1936), 447-459 doi:10.1098/rspa.1936.0111
- [3] M. Fierz and W. Pauli, “On relativistic wave equations for particles of arbitrary spin in an electromagnetic field,” *Proc. Roy. Soc. Lond. A* **173** (1939), 211-232 doi:10.1098/rspa.1939.0140
- [4] K. Johnson and E. C. G. Sudarshan, “Inconsistency of the local field theory of charged spin-3/2 particles,” *Annals Phys.* **13** (1961), 126-145 doi:10.1016/0003-4916(61)90030-6
- [5] G. Velo and D. Zwanziger, “Propagation and quantization of Rarita-Schwinger waves in an external electromagnetic potential,” *Phys. Rev.* **186** (1969), 1337-1341 doi:10.1103/PhysRev.186.1337
- [6] G. Velo and D. Zwanziger, “Noncausality and other defects of interaction lagrangians for particles with spin one and higher,” *Phys. Rev.* **188** (1969), 2218-2222 doi:10.1103/PhysRev.188.2218
- [7] G. Velo, “Anomalous behaviour of a massive spin two charged particle in an external electromagnetic field,” *Nucl. Phys. B* **43** (1972), 389-401 doi:10.1016/0550-3213(72)90027-2
- [8] M. Kobayashi and Y. Takahashi, “The Rarita-schwinger Paradoxes,” *J. Phys. A* **20** (1987), 6581 doi:10.1088/0305-4470/20/18/053
- [9] A. Hasumi, R. Endo and T. Kimura, “Dirac Quantization of Massive spin-3/2 Particle Coupled With Magnetic Field,” *J. Phys. A* **12** (1979), L217 doi:10.1088/0305-4470/12/8/006
- [10] W. Cox, “On the Lagrangian and Hamiltonian Constraint Algorithms for the Rarita-schwinger Field Coupled to an External Electromagnetic Field,” *J. Phys. A* **22** (1989), 1599-1608 doi:10.1088/0305-4470/22/10/015
- [11] M. Kobayashi and A. Shamaly, “Minimal Electromagnetic Coupling for Massive Spin-2 Fields,” *Phys. Rev. D* **17** (1978), 2179 doi:10.1103/PhysRevD.17.2179
- [12] S. Deser, V. Pascalutsa and A. Waldron, “Massive spin-3/2 electrodynamics,” *Phys. Rev. D* **62** (2000), 105031 doi:10.1103/PhysRevD.62.105031 [arXiv:hep-th/0003011 [hep-th]].
- [13] P. Federbush, “Minimal electromagnetic coupling for spin two particles” *Il Nuovo Cimento* **19** (1961), 572–573 doi:10.1007/bf02733252.
- [14] S. Ferrara, M. Porrati and V. L. Telegdi, “ $g = 2$ as the natural value of the tree-level gyromagnetic ratio of elementary particles,” *Phys. Rev. D* **46** (1992), 3529-3537 doi:10.1103/PhysRevD.46.3529
- [15] A. Abouelsaood, C. G. Callan, Jr., C. R. Nappi and S. A. Yost, “Open Strings in Background Gauge Fields,” *Nucl. Phys. B* **280** (1987), 599-624 doi:10.1016/0550-3213(87)90164-7
- [16] C. P. Burgess, “Open String Instability in Background Electric Fields,” *Nucl. Phys. B* **294** (1987), 427-444 doi:10.1016/0550-3213(87)90590-6
- [17] V. V. Nesterenko, “The Dynamics of Open Strings in a Background Electromagnetic Field,” *Int. J. Mod. Phys. A* **4** (1989), 2627-2652 doi:10.1142/S0217751X89001023

- [18] P. C. Argyres and C. R. Nappi, “Massive Spin-2 Bosonic String States in an Electromagnetic Background,” *Phys. Lett. B* **224** (1989), 89-96 doi:10.1016/0370-2693(89)91055-1
- [19] P. C. Argyres and C. R. Nappi, “Spin 1 Effective Actions From Open Strings,” *Nucl. Phys. B* **330** (1990), 151-173 doi:10.1016/0550-3213(90)90305-W
- [20] M. Porrati and R. Rahman, “Notes on a Cure for Higher-Spin Acausality,” *Phys. Rev. D* **84** (2011), 045013 doi:10.1103/PhysRevD.84.045013 [arXiv:1103.6027 [hep-th]].
- [21] M. Porrati, R. Rahman and A. Sagnotti, “String Theory and The Velo-Zwanziger Problem,” *Nucl. Phys. B* **846** (2011), 250-282 doi:10.1016/j.nuclphysb.2011.01.007 [arXiv:1011.6411 [hep-th]].
- [22] M. Porrati and R. Rahman, “Causal Propagation of a Charged spin-3/2 Field in an External Electromagnetic Background,” *Phys. Rev. D* **80** (2009), 025009 doi:10.1103/PhysRevD.80.025009 [arXiv:0906.1432 [hep-th]].
- [23] S. Deser and A. Waldron, “Inconsistencies of massive charged gravitating higher spins,” *Nucl. Phys. B* **631** (2002), 369-387 doi:10.1016/S0550-3213(02)00199-2 [arXiv:hep-th/0112182 [hep-th]].
- [24] N. Berkovits and M. M. Leite, “Superspace action for the first massive states of the superstring,” *Phys. Lett. B* **454** (1999), 38-42 doi:10.1016/S0370-2693(99)00334-2 [arXiv:hep-th/9812153 [hep-th]].
- [25] T. Gregoire, M. D. Schwartz and Y. Shadmi, “Massive supergravity and deconstruction,” *JHEP* **07** (2004), 029 doi:10.1088/1126-6708/2004/07/029 [arXiv:hep-th/0403224 [hep-th]].
- [26] Martín-García, Jose M., “xAct: Efficient tensor computer algebra for the Wolfram Language,” <http://xact.es/>.
- [27] K. Benakli, C. A. Daniel and W. Ke, “Open Superstring First Mass Level Effective Lagrangian: Massive Spin-3/2 Fields in an Electromagnetic Background,” [arXiv:2211.13691 [hep-th]].
- [28] K. Benakli, C. A. Daniel and W. Ke, “Open superstring first mass level effective Lagrangian: Massive spin-2 in an electromagnetic background,” *Phys. Lett. B* **838** (2023), 137680 doi:10.1016/j.physletb.2023.137680 [arXiv:2211.13689 [hep-th]].
- [29] N. Berkovits, “SuperPoincare invariant superstring field theory,” *Nucl. Phys. B* **450** (1995), 90-102 [erratum: *Nucl. Phys. B* **459** (1996), 439-451] doi:10.1016/0550-3213(95)00259-U [arXiv:hep-th/9503099 [hep-th]].
- [30] N. Berkovits and M. M. Leite, “First massive state of the superstring in superspace,” *Phys. Lett. B* **415** (1997), 144-148 doi:10.1016/S0370-2693(97)01269-0 [arXiv:hep-th/9709148 [hep-th]].
- [31] N. Berkovits, “A New description of the superstring,” [arXiv:hep-th/9604123 [hep-th]].
- [32] T. L. Curtright and P. G. O. Freund, “MASSIVE DUAL FIELDS,” *Nucl. Phys. B* **172** (1980), 413-424 doi:10.1016/0550-3213(80)90174-1
- [33] S. Deser, P. K. Townsend and W. Siegel, “Higher Rank Representations of Lower Spin,” *Nucl. Phys. B* **184** (1981), 333-350 doi:10.1016/0550-3213(81)90222-4
- [34] E. S. Fradkin and A. A. Tseytlin, “Nonlinear Electrodynamics from Quantized Strings,” *Phys. Lett. B* **163** (1985), 123-130 doi:10.1016/0370-2693(85)90205-9

- [35] J. Wess and J. Bagger, “Supersymmetry and supergravity,” Princeton University Press, 1992, ISBN 978-0-691-02530-8

Chapter 3

(Too) slow gravitinos

The theory of elementary spin-3/2 particles was initiated in the 1930s. In a 1936 paper [5], Dirac has pointed out the importance of writing out the equations of motion for such states, which not only are of mathematical interest, but they also allow for a direct application to elementary particles of spin-3/2 if they are observed in the future, or at least to the composite states in the point particle approximation. The Lagrangian of an uncharged spin-3/2 field in a flat space was established by Rarita and Schwinger in 1941 [6], employing a vector-spinor subject to transverse and γ -traceless constraints. As we have seen in the previous chapter, the massive *charged* spin-3/2, or more generally massive charged higher spin ($s > 1$) states minimally coupled to electromagnetism, suffer from inconsistencies known as the Velo-Zwanziger problem. Indeed, the problematic propagation of spin-3/2 often occurs in a non-trivial background. In addition to the aforementioned electromagnetic background, serious phenomenological issues can arise also in a curved spacetime.

Gravitational particle production in a curved background is an important source of the observed dark matter density in modern cosmological models, ranging from scalar candidates to higher spin fields. The spin-3/2 case has drawn a particular attention partly due to the fact that, the gravitinos, *i.e.* the spin-3/2 partners of the graviton, are an indispensable building block of supergravity theories. For a brief understanding of this point, recall that gauging the ordinary Yang-Mill symmetry requires the introduction of a gauge field A_μ^a transforming as

$$\delta A_\mu^a = \partial_\mu \Lambda^a \tag{3.1}$$

with Λ^a the gauge parameter. In supergravity where supersymmetry is gauged, the symmetry parameter is not a scalar or vector, but a spinor ϵ_α , with $\alpha = 1, 2$ a spinor index, and therefore the corresponding gauge field should be a vector-spinor χ_μ^α (*viz.* of spin-3/2). One can further show that this vector-spinor sits in the same multiplet as the graviton, hence the name “gravitino”. In the super-Higgs mechanism, the gravitino acquires a mass term by eating up the spin-1/2 goldstino, and obtains subsequently the helicity $\pm 1/2$ states. This mechanism explains why no goldstino has been observed so far.

The equations of motion as well as the production of the gravitinos, in a curved background, are studied by various authors two decades ago [16–19, 67]. The gravitinos can be produced in the early universe, through both thermal scatterings after inflation during reheating, and non-thermal generation by inflaton oscillations. Depending on the specific inflationary model, the latter may turn out to be much more efficient than the former [16,

17], resulting in a dangerously abundant generation of helicity-1/2 gravitinos, and eventually jeopardizing the big bang nucleosynthesis.

Another problematic issue of gravitino production has been put forward in a more recent work by Kolb, Long and McDonough [20] concerning the “too slow gravitinos”. In fact, it was noticed earlier [68, 69] that in a fluid background breaking at the same time supersymmetry and Lorentz invariance, the *sound speed* extracted from the dispersion relation, of the helicity-1/2 gravitino, is lower than the speed of light. Gravitinos with such behavior are named therein “slow gravitinos”. The situation is similar for the curved background, which may also give rise to slow gravitinos. Intriguingly, while investigating the sound speed of the gravitinos in a Friedmann-Lemaître-Robertson-Walker (FLRW) spacetime, the authors of [20] have discovered that for masses lighter than the Hubble expansion rate $m_{3/2} \lesssim H$, the sound speed may vanish, in which case the gravitino production is divergent, and this phenomenon is called “catastrophic gravitino production”. Note that such scenario differs from the previous problem of over-abundant gravitino production. The divergence here signifies that the cancellation of the sound speed removes the cutoff on the particle momentum, therefore, modes of arbitrarily large momentum can be produced, which signals the breakdown of the effective field theory. Subsequently, a Swampland Conjecture is formulated in [70], stating that gravitinos in consistent effective theories of quantum gravity must not have vanishing sound speed.

Does supergravity survive this requirement? In the presence of one chiral multiplet, it has been shown already in [18, 19], that the sound speed is always the speed of light. Meanwhile, in models with multiple chiral multiplets, subject to constraints, the sound speed may vanish [20], and may even become superluminal [71]. The latter situation implies acausality and appears in non-linear supersymmetric models, which will not be our focus thereafter. The question that we will address here is rather: what is the gravitino sound speed in supergravity with *linear* supersymmetry? To this end, we study the simplest case of $\mathcal{N} = 1$, $d = 4$ supergravity with two multiplets, and point out the subtlety in the definition of a physical speed, due to the mixing of two fermions. We will show, through explicit examples, that the physical sound speed never vanishes, and in certain cases, it is equal to the speed of light.

The following sections are structured as follow: in Sections 3.1 and 3.2, we summarize the massive spin-3/2 action and equations in a flat and curved spacetime respectively, introducing the notion of sound speed. We also review the catastrophic gravitino production in [20], related to the vanishing of the gravitino sound speed. The following section presents detailed calculation and diagonalisation of the coupled equations of motion, both in the case with two chiral multiplets and with one chiral plus one vector multiplet. In the end, we draw our conclusions. Conventions and notations are listed in Appendix B.

3.1 Spin-3/2 in a flat spacetime

Let us recall some basic properties of the spin-3/2 field and its decomposition in the momentum space. The treatment here follows closely that in [20] but with conventions of [19]. In the Rarita-Schwinger construction [6], the spin-3/2 is described by a vector spinor ψ_μ , originating from the direct product of the $\left(\frac{1}{2}, \frac{1}{2}\right)$ vector representation, and the $\left(\frac{1}{2}, 0\right) \oplus \left(0, \frac{1}{2}\right)$ spinor representation of the Lorentz group. The result is a reducible representation:

$$\left(\frac{1}{2}, \frac{1}{2}\right) \otimes \left[\left(\frac{1}{2}, 0\right) \oplus \left(0, \frac{1}{2}\right)\right] = \left(\frac{1}{2}, 1\right) \oplus \left(\frac{1}{2}, 0\right) \oplus \left(1, \frac{1}{2}\right) \oplus \left(0, \frac{1}{2}\right) \quad (3.2)$$

which corresponds to one spin-3/2 component, including helicities $\pm 3/2$ and $\pm 1/2$, and two spin-1/2 components, where the latter can be projected out by a constraint. The Rarita-Schwinger action is conventionally written as

$$S_{\text{RS}} = \int d^4x \left[-\left(\bar{\psi}_\mu \gamma^{\mu\rho\nu} \partial_\rho \psi_\nu\right) + m \bar{\psi}_\mu \gamma^{\mu\nu} \psi_\nu \right] \quad (3.3)$$

where $\gamma^{\mu\nu} = (\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu)/2$, $\gamma^{\mu\nu\rho} = (\gamma^\mu \gamma^\rho \gamma^\nu - \gamma^\nu \gamma^\rho \gamma^\mu)/2$. The Euler-Lagrange equation gives

$$-\gamma^{\mu\rho\nu} \partial_\rho \psi_\nu + m \gamma^{\mu\nu} \psi_\nu = 0 \quad (3.4)$$

Acting with ∂_μ and γ_μ on the above equation, we obtain

$$\gamma^{\mu\nu} \partial_\mu \psi_\nu = 0, \quad -2\gamma^{\mu\nu} \partial_\mu \psi_\nu + 3m \gamma^\mu \psi_\mu = 0 \quad (3.5)$$

which then implies the divergence and γ -trace constraints

$$\partial^\mu \psi_\mu = 0, \quad \gamma^\mu \psi_\mu = 0 \quad (3.6)$$

in addition to the Dirac equation $(i\gamma^\nu \partial_\nu - m) \psi_\mu = 0$. We are then left with four physical degrees of freedom corresponding to the four helicity states of spin-3/2. As a side comment, the massless case can be obtained by directly fixing $m = 0$ in the Rarita-Schwinger Lagrangian, which introduces a gauge invariance $\delta\psi_\mu = \partial_\mu \Lambda$ and the on-shell degrees of freedom is reduced to two. We will focus thereafter on the massive spin-3/2 only. To see how each helicity state behaves, let us move to the momentum space by the Fourier decomposition

$$\psi_\mu(t, \mathbf{x}) = \int \frac{d^3\mathbf{k}}{(2\pi)^3} \psi_{\mu, \mathbf{k}}(t) e^{i\mathbf{k}\cdot\mathbf{x}} \quad (3.7)$$

and for simplicity we choose the direction of the momentum to be $\mathbf{k} = (0, 0, k_z)$. The four components of the vector-spinor, ψ_0, \dots, ψ_3 , are mixed through the constraint equations, though the Dirac equation is satisfied by each of them. The helicity eigenstates, that are orthogonal to each other, are found by spinor-vector projectors constructed from

- The spin-1 helicity projectors ($s = \pm 1, 0$):

$$(M_s)^\mu{}_\nu \equiv -(\epsilon_s^*)^\mu (\epsilon_s)_\nu \quad (3.8)$$

where ϵ_s are the polarization vectors.

- The spinor projectors ($s = \pm 1/2$):

$$S_{\pm} = \frac{1}{2} (1 \pm i\gamma^1\gamma^2) \quad (3.9)$$

Orthogonal combinations of them yield the vector-spinor projectors for the four helicity states $\pm 3/2, \pm 1/2$:

$$\begin{aligned} (P_{\pm 3/2})^{\mu}{}_{\nu} &= S_{\pm} (M_{\pm})^{\mu}{}_{\nu} \\ (P_{\pm 1/2})^{\mu}{}_{\nu} &= S_{\pm} (M_0)^{\mu}{}_{\nu} + S_{\mp} (M_{\pm})^{\mu}{}_{\nu} \end{aligned} \quad (3.10)$$

The helicity eigenstates are obtained from the above projectors. In terms of the four space-time components ψ_0, \dots, ψ_3 , their expressions are

$$\begin{aligned} \psi_{1/2, \mathbf{k}} &= \frac{\sqrt{6}}{2} (\psi_{1, \mathbf{k}} - \gamma^1\gamma^2\psi_{2, \mathbf{k}}) \\ \psi_{3/2, \mathbf{k}} &= \frac{1}{\sqrt{2}} (\psi_{1, \mathbf{k}} + \gamma^1\gamma^2\psi_{2, \mathbf{k}}) \end{aligned} \quad (3.11)$$

They obey separately the Dirac equation:

$$\begin{aligned} (\gamma^0\partial_0 + ik_z\gamma^3 + m) \psi_{1/2, \mathbf{k}} &= 0 \\ (\gamma^0\partial_0 + ik_z\gamma^3 + m) \psi_{3/2, \mathbf{k}} &= 0 \end{aligned} \quad (3.12)$$

The sound speed c_s is defined as the coefficient in front of the momentum squared k^2 in the dispersion relation, namely

$$\omega_{\pm} = \pm\sqrt{c_s^2 k^2 + \dots} \quad (3.13)$$

Here, we have the usual relativistic dispersion relation for the two eigenstates: $\omega_{\pm} = \pm\sqrt{k^2 + m^2}$, and the sound speed is therefore trivially the speed of light. We will see in the next section that in a curved background, due to the different treatments of space and time, the dispersion relations of the helicity $\pm 1/2, \pm 3/2$ modes are modified by the scale factor.

3.2 Spin-3/2 in a curved spacetime: catastrophic production?

3.2.1 Rarita-Schwinger action and equations

We follow the references [19, 20] to review the curved-space Rarita-Schwinger action and the particular case of FLRW metric. For more detailed introductions of curved-space quantum field theory, see for example [28, 30].

Moving from the Minkowski metric $\eta_{\alpha\beta}$ to the metric of a curved spacetime $g_{\mu\nu}$, it is convenient to introduce the frame field (or *vierbein*) $e^{\alpha}{}_{\mu}$ that allows to properly describe spinors. At every spacetime point, the two metrics are related by

$$g_{\mu\nu} = e^{\alpha}{}_{\mu}(x)e^{\beta}{}_{\nu}(x)\eta_{\alpha\beta} \quad (3.14)$$

Note that the metric is defined by $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$. The first index of $e^a{}_\mu$ is raised and lowered by the Minkowski metric η_{ab} while the second one is raised and lowered by $g_{\mu\nu}$. The torsion-free *spin connection* is

$$\omega_\mu^{cd}(e) = 2e^{\nu[c}\partial_{[\mu}e_{\nu]}^{d]} - e_{a\mu}e^{\nu[c}e^{\sigma d]}\partial_\nu e_\sigma^a \quad (3.15)$$

which, when contracted with the generator of local Lorentz transformations in the spinor representation $\bar{\gamma}_{cd}$, is used to define the covariant derivative acting on spinors. The γ -matrices in curved spacetime are obtained via contraction of the constant γ -matrices with the frame field:

$$\gamma_\mu \equiv e^a{}_\mu \bar{\gamma}_a \quad (3.16)$$

In our notation, the constant γ -matrices have an overbar. The covariant derivatives on different fields are given in Appendix B.

The Rarita-Schwinger action generalized to the curved spacetime is:

$$S_{\text{RS}} = \int d^4x \sqrt{-g} \left[-\bar{\Psi}_\mu \gamma^{\mu\rho\nu} (\mathcal{D}_\rho \Psi_\nu) + m \bar{\Psi}_\mu \gamma^{\mu\nu} \Psi_\nu \right] \quad (3.17)$$

Let us consider the particular case of FLRW metric $ds^2 = a(\eta)^2(-d\eta^2 + dx^2)$ where a is the scale factor and η is the conformal time. The factor $\sqrt{-g}$ is equal to a^4 , and the covariant derivative acts as $\mathcal{D}_\mu \Psi_\nu = \left(\partial_\mu + \frac{1}{2} a H \eta_{\mu\rho} \bar{\gamma}^{\rho 0} \right) \Psi_\nu$. The Hubble rate is given by $H \equiv \partial_\eta a / a^2$. The γ -matrices are related by $\gamma_\mu = a^{-1} \bar{\gamma}_\mu$. We furthermore introduce the rescaling

$$\psi_\mu(\eta, \mathbf{x}) = a^{1/2}(\eta) \Psi_\mu(\eta, \mathbf{x}) \quad (3.18)$$

The equation of motion for ψ_μ , derived from (3.17), reads

$$\bar{\gamma}^{\mu\rho\nu} \partial_\rho \psi_\nu + aH \left(\bar{\gamma}^\mu \eta^{\nu 0} - \bar{\gamma}^\nu \eta^{\mu 0} \right) \psi_\nu - am \bar{\gamma}^{\mu\nu} \psi_\nu = 0 \quad (3.19)$$

The Dirac equation is now modified by the time-dependent scale factor $a(\eta)$, with a rescaled mass $a(\eta)m$ and an extra Hubble-dependent term. We recover the equation in Minkowski spacetime by setting $a = 1$. The constraint equations can be obtained in a similar way, applying on the equation of motion respectively $\bar{\gamma}_\mu$ and the covariant derivative \mathcal{D}_μ , then combining the two results. The detailed calculation, which is not fundamental for our discussion, will not be carried out here.

3.2.2 The sound speed

Now, we go straightforwardly to the gravitino sound speed in curved space. We first Fourier-decompose the rescaled field ψ_μ as

$$\psi_\mu(\eta, \mathbf{x}) = \int \frac{d^3\mathbf{k}}{(2\pi)^3} \psi_{\mu,\mathbf{k}}(\eta) e^{i\mathbf{k}\cdot\mathbf{x}} \quad (3.20)$$

and same as before, the two orthogonal helicity eigenstates are obtained by helicity projectors (always choosing $\mathbf{k} = (0, 0, k_z)$), with

$$\begin{aligned} \psi_{1/2,\mathbf{k}} &= \sqrt{\frac{3}{2}} \left(\psi_{1,\mathbf{k}} - \bar{\gamma}^1 \bar{\gamma}^2 \psi_{2,\mathbf{k}} \right) \\ \psi_{3/2,\mathbf{k}} &= \sqrt{\frac{1}{2}} \left(\psi_{1,\mathbf{k}} + \bar{\gamma}^1 \bar{\gamma}^2 \psi_{2,\mathbf{k}} \right) \end{aligned} \quad (3.21)$$

The equation of motion (3.19) implies in the momentum space for the two eigenstates:

$$\begin{aligned} \left[\bar{\gamma}^0 \partial_\eta + ik_z \bar{\gamma}^3 + am \right] \psi_{3/2,k} &= 0 \\ \left[\bar{\gamma}^0 \partial_\eta + ik_z \left(C_A + iC_B \bar{\gamma}^0 \right) \bar{\gamma}^3 + am \right] \psi_{1/2,k} &= 0 \end{aligned} \quad (3.22)$$

with

$$\begin{aligned} C_A &= \frac{1}{3(H^2 + m^2)^2} \left[(m^2 - H^2) \left(-\frac{1}{3}R - H^2 + 3m^2 \right) - 4Hm \frac{\partial_\eta m}{a} \right] \\ C_B &= \frac{2m}{3(H^2 + m^2)^2} \left[H \left(-\frac{1}{3}R - H^2 + 3m^2 \right) + (m^2 - H^2) \frac{\partial_\eta m}{ma} \right]. \end{aligned} \quad (3.23)$$

Comparing the curved space mode equations with (3.12), one can notice that the helicity $\pm 3/2$ state still obeys the same Dirac equation as before but with a time-dependent mass $a(\eta)m$, whereas the other helicity $\pm 1/2$ mode is modified by the Hubble rate and an additional $\bar{\gamma}^0$ term in front of the momentum. In order to extract the dispersion relations, we write out explicitly the γ -matrices and put the equations (3.22) in the form

$$\begin{aligned} i\partial_\eta \psi_{3/2,k} &= \mathcal{A}_{3/2} \psi_{3/2,k} \\ i\partial_\eta \psi_{1/2,k} &= \mathcal{A}_{1/2} \psi_{1/2,k} \end{aligned} \quad (3.24)$$

where

$$\mathcal{A}_{3/2}(\eta) = \begin{pmatrix} am & k \\ k & -am \end{pmatrix}, \quad \mathcal{A}_{1/2}(\eta) = \begin{pmatrix} am & (C_A + iC_B)k \\ (C_A - iC_B)k & -am \end{pmatrix} \quad (3.25)$$

The dispersion relations correspond to the eigenvalues of the above matrices:

$$\omega_{3/2,\pm} = \pm \sqrt{k^2 + a^2 m^2}, \quad \omega_{1/2,\pm} = \pm \sqrt{(C_A^2 + C_B^2)k^2 + a^2 m^2} \quad (3.26)$$

We can immediately remark that the sound speed of the helicity $\pm 3/2$ state is the speed of light, same as in the flat spacetime, meanwhile, the sound speed of helicity $\pm 1/2$ turns out to be

$$c_s^2 \equiv C_A^2 + C_B^2 = \frac{1}{9(H^2 + m^2)^2} \left[\left(-\frac{1}{3}R - H^2 + 3m^2 \right)^2 + 4 \frac{(\partial_\eta m)^2}{a^2} \right] \quad (3.27)$$

Consider the gravitino production from inflaton oscillations at the end of inflation, with inflaton potential $V(\phi)$. The Freedman equations imply

$$c_s^2 = \frac{(p - 3m_{3/2}^2 M_P^2)^2}{(\rho + 3m_{3/2}^2 M_P^2)^2} + \frac{4M_P^4 (\partial m_{3/2}/\partial t)^2}{(\rho + 3m_{3/2}^2 M_P^2)^2} \quad (3.28)$$

where ρ and p are the energy density and the pressure. M_P and $m_{3/2}$ are the Planck mass and gravitino mass. t is the cosmic time. It is claimed in [20] that, whenever this speed vanishes (possible when $H/m > 1$), the gravitino production will be divergent. One of the arguments was based on the dimensionless measure of nonadiabaticity $A_k(\eta)$, given by

$$A_k(\eta) \equiv \frac{\partial_\eta \omega_k}{\omega_k^2} \quad (3.29)$$

which for helicity $\pm 1/2$ takes the form

$$A_k = \frac{a^3 H m^2 + c_s (\partial_\eta c_s) k^2}{(a^2 m^2 + c_s^2 k^2)^{3/2}} \quad (3.30)$$

The particle production rate is related to the variation speed of the dispersion relation, and becomes efficient when $A_k \gg 1$. When c_s vanishes, occurring when $H/m > 1$, we have then $A_k = H/m$, which not only exceeds one, but also is independent of momentum. The latter point is crucial for the catastrophic production, because it means that particles of arbitrarily high momentum can be produced and there is no momentum cutoff.

The main question in this chapter is whether this sound speed can vanish for the gravitino in supergravity, in the simplest setup of $\mathcal{N} = 1$, $d = 4$. In the supergravity context, the gravitino propagation in the case of one chiral multiplet has been studied by various authors [16–19], and the gravitino equation of motion in the presence of arbitrary number of chiral multiplets is discussed in [19]. To enter into the subject, we provide here a quick glimpse of some general features of gravitino propagation in supergravity, with FLRW background and only one chiral multiplet, while a more elaborated presentation will be given in the next sections.

The gravitino equation of motion derived from the supergravity Lagrangian takes the form

$$(\not{D} + m) \psi_\mu = \left(\mathcal{D}_\mu - \frac{m}{2} \gamma_\mu \right) \gamma_\nu \psi^\nu \quad (3.31)$$

The gravitino mass is $m = e^{\mathcal{K}/2} \frac{W}{M_P^2}$ where \mathcal{K} and W are the Kähler potential and superpotential, respectively. The divergence constraint is

$$\mathcal{D}^\mu \psi_\mu - \not{D} \gamma^\mu \psi_\mu + \frac{3}{2} m \gamma^\mu \psi_\mu = 0 \quad (3.32)$$

and after some tedious algebra, the trace constraint can be put in the form

$$\gamma^0 \psi_0 = \hat{A} \gamma^i \psi_i \quad (3.33)$$

\hat{A} is a matrix whose general expression for any number of chiral multiplets is provided in [19]. The idea is again to find the equation of motion for each helicity state in the momentum space. It is convenient in this case to decompose the Fourier mode¹ into its transverse part ψ_i^T , the γ -trace $\gamma^i \psi_i$ and the trace $\mathbf{k} \cdot \psi$:

$$\psi_i = \psi_i^T + \left(\frac{1}{2} \gamma_i - \frac{1}{2} \hat{k}_i (\hat{\mathbf{k}} \cdot \gamma) \right) \gamma^j \psi_j + \left(\frac{3}{2} \hat{k}_i - \frac{1}{2} \gamma_i (\hat{\mathbf{k}} \cdot \gamma) \right) \hat{\mathbf{k}} \cdot \psi \quad (3.34)$$

By the constraint equation (3.32), $\mathbf{k} \cdot \psi$ can be expressed in terms of $\gamma^i \psi_i$, so the decomposition (3.34) involves only the γ -trace corresponding to helicity $\pm 1/2$, and the transverse part, representing helicity $\pm 3/2$. Their equations of motion are

$$\left(\bar{\gamma}^0 \partial_0 + i \bar{\gamma}^i k_i + am \right) \bar{\Psi}^T = 0, \quad \bar{\Psi}^T \equiv a^{1/2} \bar{\psi}^T \quad (3.35)$$

and

$$\left(\partial_0 - i \gamma^j k_j \gamma_0 \hat{A} + \hat{B} \right) \left(\bar{\gamma}^i \psi_i \right) = 0 \quad (3.36)$$

¹By an abuse of language and for the sake of simplicity, the Fourier mode is also written as ψ .

The mass matrix \hat{B} depends on \hat{A} and the scale factor a . The following steps are clear and we can extract from the above equations the dispersion relations, then the sound speeds. For the transverse component (helicity $\pm 3/2$) we have $c_s = 1$, whereas the longitudinal one (helicity $\pm 1/2$) has $c_s = |\hat{A}|^2$. The latter in the FLRW background is shown to be equal to 1 and therefore both helicity $\pm 3/2$ and $\pm 1/2$ propagate at the speed of light.

$\mathcal{N} = 1$, $d = 4$ supergravity with one chiral multiplet is exempt from catastrophic gravitino production. Is this still the case if we have more than one multiplet? When there are two spin-1/2 fermions, sitting either in two chiral or one chiral plus one vector multiplet, the equations of motion generalized from Eq (3.35)-(3.36) are a coupled system, and a diagonalization procedure is necessary to find the equation of the physical fermion. The sound speed is then recovered from the dispersion relation of the physical fermion. In the following, we present the work [72] where we compute the sound speed in the case of two multiplets, and show that it never vanishes, hence no catastrophic production would occur.

3.3 Publication

Salvage of Too Slow Gravitinos

I. Antoniadis,^{a,b,1} K. Benakli,^{a,2} and W. Ke^{a,3}

^a*Laboratoire de Physique Théorique et Hautes Energies (LPTHE),
UMR 7589, Sorbonne Université et CNRS,
4 place Jussieu, 75252 Paris Cedex 05, France*

^b*Institute for Theoretical Physics, KU Leuven, Celestijnenlaan 200D, B-3001 Leuven, Belgium*
E-mail: antoniad@lpthe.jussieu.fr, kbenakli@lpthe.jussieu.fr,
wke@lpthe.jussieu.fr

ABSTRACT: Gravitinos can inherit a non-relativistic dispersion relation while propagating in a background breaking both supersymmetry and Lorentz symmetry spontaneously. This is because the longitudinal mode velocity is controlled by the sound speed in the background. It was pointed out recently by Kolb, Long and McDonough that the production of gravitinos might diverge when this sound speed vanishes. We argue that in the framework of cosmological models with linearly spontaneously broken realised supersymmetry, where the physical fermions are combinations of the vacuum goldstino and the inflatino, the gravitino longitudinal mode has a relativistic dispersion relation and therefore avoids the catastrophic production. We illustrate this in some explicit examples.

Contents

1	Introduction	1
2	The General case with two fermions	2
2.1	The mixing matrices	5
2.2	Dispersion relations for the physical fermions	7
3	Examples	9
3.1	Two chiral multiplets	9
3.2	One chiral multiplet and one vector multiplet	11
4	Conclusions	13
A	Notations	13

1 Introduction

In the super-Higgs mechanism, a spin 1/2 fermion, the goldstino, combines with the gravitino and provides it with the appropriate number of degrees of freedom for a massive spin 3/2 [1–5]. When the spin 1/2 propagates in a generic background with a non-relativistic dispersion relation, for instance at the sound speed $c_s < 1$ in fluids, the result of the super-Higgs mechanism has been denoted "slow gravitinos" [6–10]. Such situations might occur in cosmological backgrounds as cosmological solutions treat time and space differently [11–15].

At the end of inflation, during the period of reheating, the inflaton dissipates its energy while oscillating around the minimum of its potential. This energy is in part converted into a non-thermal production of gravitinos. This process was studied in [12–15] where the equations of propagation of the different gravitino modes were established. In particular, there is a copious production of the helicities $\pm 1/2$ components of the gravitinos, the goldstino fermion at the moment, which was also numerically evaluated in those papers and in [16, 17].

Recently, [18, 19] have reconsidered this process. The equation of motion of the fermion θ describing the longitudinal mode in the vacuum after reheating can be written as

$$[\bar{\gamma}^0 \partial_0 + i\bar{\gamma}^i k_i c_s] \theta + \dots = 0, \quad (1.1)$$

where the \dots stand for mass and mixing terms with other fermions. The sound speed is identified then as given by:

$$c_s^2 = \frac{\left(p - 3m_{3/2}^2 M_P^2\right)^2}{\left(\rho + 3m_{3/2}^2 M_P^2\right)^2} + \frac{4M_P^4 \left(\partial m_{3/2}/\partial t\right)^2}{\left(\rho + 3m_{3/2}^2 M_P^2\right)^2}, \quad (1.2)$$

where M_P is the reduced Planck mass, $m_{3/2}$ the gravitino mass, while ρ, p denote the energy density and pressure of the matter system. It was then noticed in [18, 19] that a catastrophic gravitino production occurs whenever the above sound speed vanishes.

In the supergravity cosmological models we discuss here where supersymmetry is linearly realised (but spontaneously broken), the slow goldstino is not an eigenstate of the Hamiltonian. It mixes with the fermion whose scalar partner has a non-vanishing kinetic energy, for instance the inflatino in the works mentioned above. These give rise at each moment, after diagonalisation, to combinations of the two fermions that are eigenstates of the Hamiltonian and c_s is not a physical quantity. This has motivated [10] to resort to non-linear realisation of supersymmetry in order to project out the second fermion and construct models of slow gravitinos. If one keeps both states, it was pointed out in [20] that $c_s^2 = 0$ does not lead to catastrophic gravitino production. More precisely, as the coefficients of the momentum vector \vec{k} in the kinetic term are now described by a mixing matrix that is known not to be singular as $c_s^2 = 0$, no catastrophic production is expected.

The aim of this work is to derive explicitly the general dispersion relations of the physical fermions in standard linear supergravity and show that not only as expected the mixing matrix in the fermion kinetic terms is not singular, but, it can be diagonalised to the identity matrix. Using the non-adiabaticity coefficient, we conclude that in this background catastrophic production does not occur. Furthermore, we show in particular cases that the physical sound speed is equal to one independently of the value of c_s^2 ; namely, for degenerate fermion masses, the dispersion relation is relativistic despite the time dependent gravitational background.

In section 2, we consider the general case with a goldstino combination of two fermions. We exhibit through diagonalisation of the kinetic terms the generic form of the dispersion relations of the physical fermions. We argue that these do not lead to catastrophic gravitino production. One of the novelties of our work is to consider also the case where one of the fermions arises from a vector multiplet. Section 3 presents some examples where the whole diagonalisation can be carried out explicitly and where the dispersion relation takes a relativistic form.

2 The General case with two fermions

We consider an $N = 1$ supergravity model where in addition to the graviton and gravitino ψ_ν , one has two possible sources of supersymmetry breaking *in the vacuum*. The first is a potentially non-vanishing D -term \mathcal{P} for a $U(1)$ vector multiplet. The second possibility uses the non-vanishing F -term of a chiral multiplet. In addition, during the cosmological evolution, there is an extra source of supersymmetry breaking given by the non-vanishing kinetic energy of a rolling scalar, the inflaton. To describe this system we consider a vector supermultiplet with field strength and gaugino denoted as $F_{\mu\nu}$ and λ , respectively, as well as one or two chiral multiplets consisting in scalars ϕ_i and fermions χ_i , $i = 1, 2$. The

corresponding Lagrangian is given by: ¹

$$\begin{aligned}
e^{-1}\mathcal{L} = & -\frac{1}{2}M_{\text{P}}^2 R - g_i^j \left(\hat{\partial}_\mu \phi^i \right) \left(\hat{\partial}^\mu \phi_j \right) - V \\
& - \frac{1}{2}M_{\text{P}}^2 \bar{\psi}_\mu R^\mu + \frac{1}{2}m \bar{\psi}_{\mu R} \gamma^{\mu\nu} \psi_{\nu R} + \frac{1}{2}m^* \bar{\psi}_{\mu L} \gamma^{\mu\nu} \psi_{\nu L} \\
& + (\text{Re } f) \left[-\frac{1}{4}F_{\mu\nu} F^{\mu\nu} - \frac{1}{2}\bar{\lambda} \mathcal{D}\lambda \right] + \frac{1}{4}i (\text{Im } f) \left[F_{\mu\nu} \tilde{F}^{\mu\nu} - \hat{\partial}_\mu (\bar{\lambda} \gamma_5 \gamma^\mu \lambda) \right] \\
& + \frac{1}{4} \{ (\text{Re } f) \bar{\psi}_\mu \gamma^{\nu\rho} F_{\nu\rho} \gamma^\mu \lambda - [f^i \bar{\chi}_i \gamma^{\mu\nu} F_{\mu\nu}^- \lambda_L + \text{h.c.}] \} \\
& - g_i^j [\bar{\chi}_j \mathcal{D}\chi^i + \bar{\chi}^i \mathcal{D}\chi_j] - m^{ij} \bar{\chi}_i \chi_j - m_{ij} \bar{\chi}^i \chi^j \\
& - 2m_{i\alpha} \bar{\chi}^i \lambda - 2m^{i\alpha} \bar{\chi}_i \lambda - m_{R,\alpha\beta} \bar{\lambda}_R \lambda_R - m_{L,\alpha\beta} \bar{\lambda}_L \lambda_L \\
& + \left(2g_j^i \bar{\psi}_{\mu R} \gamma^{\nu\mu} \chi^j \hat{\partial}_\nu \phi_i + \bar{\psi}_R \cdot \gamma v_L + \text{h.c.} \right)
\end{aligned} \tag{2.1}$$

where L, R subscripts refer to the left and right chiralities, respectively. Moreover, χ_i is a left-handed field while χ^i is right-handed, and ϕ^i denotes the complex conjugate of ϕ_i . The kinetic term of the gravitino is defined as $R^\mu = \gamma^{\mu\rho\sigma} \mathcal{D}_\rho \psi_\sigma$. The covariant derivatives as well as the mass terms in this Lagrangian can be found in Appendix A. The Greek index α in the gaugino mass terms $m_{i\alpha}$ is set to 1, since there is at most one vector multiplet, in which case there is also only one chiral multiplet and thus the latin index i is also set to 1. The field strengths $F_{\mu\nu}, \tilde{F}^{\mu\nu}, F_{\mu\nu}^-$ are irrelevant for our discussion, and are defined in [15].

The Kähler metric g_i^j is given by the Kähler potential K with

$$g_i^j = \frac{\partial}{\partial \phi^i} \frac{\partial}{\partial \phi_j} K \tag{2.2}$$

and the gravitino mass $m_{3/2}$ is determined by the superpotential W as well as the Kähler potential

$$m_{3/2} = |m| M_{\text{P}}^{-2}, \quad m \equiv e^{\frac{K}{2M_{\text{P}}^2}} W \tag{2.3}$$

The scalar potential is a sum of the F -term and D -term contributions, with m_i the Kähler covariant derivative of m and $f \equiv 1/g^2$ denoting the gauge kinetic function, assumed to be constant:

$$V = V_F + V_D, \quad V_F = -3M_{\text{P}}^{-2} |m|^2 + m_i (g_j^i)^{-1} m^j, \quad V_D = \frac{1}{2} g^2 \mathcal{P}^2 \tag{2.4}$$

where \mathcal{P} is the Killing potential.

In the following, we will consider a flat universe described by the Friedmann-Lemaître-Robertson-Walker (FLRW) metric $ds^2 = a^2(\eta) (-d\eta^2 + d\mathbf{x}^2)$ where a is the scale factor and $\eta \equiv x^0$ is the conformal time. The determinant of the vierbein is then $e = a^4$. We further introduce the dot derivative with respect to the physical time t , with $\dot{} \equiv a^{-1} \partial_0 $. The Hubble rate is defined as $H \equiv \dot{a}/a$.

¹The Lagrangian with a generic number of chiral and vector multiplets can be found in [15].

Throughout this work, we will assume real backgrounds, and use the plane wave expansion for the fermions $\Psi(\eta, \vec{k}) = \exp(i\vec{k} \cdot \vec{x})\Psi(\eta)$. Useful notations are:

$$\begin{aligned}
\alpha &\equiv \rho + 3M_P^{-2}m^2, & \alpha_1 &\equiv p - 3M_P^{-2}m^2, & \alpha_2 &\equiv 2\dot{m}, \\
\hat{A} &= \hat{A}_1 + \bar{\gamma}^0 \hat{A}_2 \equiv \frac{1}{\alpha} (\alpha_1 + \bar{\gamma}^0 \alpha_2), \\
\hat{B} &= \hat{B}_1 + \bar{\gamma}^0 \hat{B}_2 \equiv -\frac{3}{2}\dot{a}\hat{A} - \frac{1}{2}M_P^{-2}ma\bar{\gamma}^0(1 + 3\hat{A}) \\
n_i &\equiv g_i^j \dot{\phi}_j, & n^i &\equiv g_j^i \dot{\phi}^j, & |\dot{\phi}|^2 &\equiv g_i^j \dot{\phi}_j \dot{\phi}^i \\
\xi^i &\equiv m^i + \bar{\gamma}^0 n^i, & \xi_i &\equiv m_i + \bar{\gamma}^0 n_i \\
\Delta^2 &\equiv 1 - \frac{\alpha_1^2}{\alpha^2} - \frac{\alpha_2^2}{\alpha^2} = \frac{4}{\alpha^2} \left[\dot{\phi}^i \dot{\phi}_j m_k m^\ell \left(g_\ell^{k-1} g_i^j - \delta_i^k \delta_\ell^j \right) + \frac{1}{2} |\dot{\phi}|^2 g^2 \mathcal{P}^2 \right]
\end{aligned} \tag{2.5}$$

In the FLRW background, the energy density and pressure are given in terms of the Hubble parameter as

$$\rho = 3M_P^2 H^2, \quad p = -M_P^2 \left(3H^2 + 2\dot{H} \right) \tag{2.6}$$

Before choosing a gauge, the goldstino v in the last line of (2.1) takes the form

$$v = \xi^{\dagger i} \chi_i + \xi_i^{\dagger} \chi^i + \frac{i}{2} \gamma_5 \mathcal{P} \lambda \tag{2.7}$$

To describe the theory in the supersymmetry broken phase, we follow [15] and introduce the combination of spin-1/2 fermions

$$\begin{aligned}
\theta &\equiv \bar{\gamma}^i \psi_i, & \Upsilon &\equiv a (n_i \chi^i + n^i \chi_i) \\
\Xi_R &= -m^k g_k^{-1j} m_{ji} \chi^i + \bar{\gamma}^0 \dot{\phi}_j (m^{ji} \chi_i + m^j \lambda_L) + iM_P^{-2} m \mathcal{P} \lambda_R - i g^2 \mathcal{P} m_i \chi^i
\end{aligned} \tag{2.8}$$

In the unitary gauge $v = 0$, (2.7) then relates the gaugino to the chiral fermions. The spinors in (2.8) are *a priori* independent. However, as we consider the case of two fermions, θ will be associated with the longitudinal component in the vacuum of the gravitino in the unitary gauge. The fermion Υ describes the correction to this mode from supersymmetry breaking by the rolling scalar kinetic energy. In this case, Υ and Ξ are proportional to each other, with

$$\Xi = -a^{-1} \hat{F} \Upsilon \tag{2.9}$$

For two chiral multiplets, the matrix \hat{F} is provided in [15]. In the presence of a D -term, the remaining chiral multiplet is written as (χ_1, ϕ_1) , and the kinetic energy becomes $|\dot{\phi}|^2 = g_1^1 \dot{\phi}_1^2$. We find (for non-vanishing $|\dot{\phi}|^2$ and \mathcal{P}):

$$\hat{F} = \frac{\dot{V}}{2|\dot{\phi}|^2} - \frac{\dot{\mathcal{P}}}{\mathcal{P}} + \bar{\gamma}^0 \left((g_1^1)^{-1} m_{11} - \frac{\dot{\mathcal{P}}}{\mathcal{P}} \frac{m^1}{n^1} + \frac{2m}{M_P^2} \right) \tag{2.10}$$

As emphasised by [15–17], the equations of motions for θ and Υ are coupled together, thus spin-1/2 particles produced are not necessarily the longitudinal component of the gravitino, but the fermions that diagonalise the Hamiltonian. We call them the *physical fermions* thereafter. Moreover, the spin-1/2 fermions (θ, Υ) have non-canonical kinetic

terms and thus will be rescaled before diagonalising their equations of motion. It was noticed in [16, 17], in the two chiral multiplets case, that the kinetic terms can be written as

$$\mathcal{L} \supset -\frac{4a}{\alpha\Delta^2}\bar{\Upsilon}\bar{\gamma}^0\partial_0\Upsilon - \frac{\alpha}{4k^2}a^3\bar{\theta}\bar{\gamma}^0\partial_0\theta \quad (2.11)$$

Here we are going to generalise the above expression in the presence of a vector multiplet with a non-vanishing D -term. Υ can be projected onto the left-handed gaugino by

$$P_L\xi^{\dagger 1}\Upsilon = -\frac{1}{2}aP_L\xi^{\dagger 1}\bar{\gamma}^0\left(\xi_1\chi^1 + \xi^1\chi_1 + \frac{i}{2}\gamma_5\mathcal{P}\lambda\right) \quad (2.12)$$

where in the first line we used the unitary gauge condition $v = 0$ from (2.7). The right-handed gaugino is obtained by charge conjugation:

$$P_R\xi_1^\dagger\Upsilon = \frac{i}{2}an_1\mathcal{P}P_R\lambda \quad (2.13)$$

On the other hand, it is easier to project Υ onto the chiral fermions given its definition:

$$\chi_1 = \frac{P_L\Upsilon}{an^1}, \quad \chi^1 = \frac{P_R\Upsilon}{an_1} \quad (2.14)$$

Expressing the gaugino and chiral fermion kinetic terms in terms of Υ , we find

$$\mathcal{L} \supset -\frac{4aV_D}{\alpha^2\Delta^2}(\bar{\Upsilon}\bar{\gamma}^0\partial_0\Upsilon) - \frac{4a(\alpha - V_D)}{\alpha^2\Delta^2}\bar{\Upsilon}\bar{\gamma}^0\partial_0\Upsilon = -\frac{4a}{\alpha\Delta^2}\bar{\Upsilon}\bar{\gamma}^0\partial_0\Upsilon \quad (2.15)$$

As for θ , one uses the fact that the spatial component of the gravitino can be decomposed into

$$\vec{\psi} = \vec{\psi}^T + \frac{1}{k^2}\left[\vec{k}(\bar{\gamma}^i k_i) + \frac{1}{2}i(3\vec{k} - \bar{\gamma}(\bar{\gamma}^i k_i))(\dot{a}\bar{\gamma}^0 + M_P^{-2}am)\right]\theta \quad (2.16)$$

where ψ^T corresponds to the transverse mode. Inserting the above equation into the gravitino kinetic term in the Lagrangian, we recover the same form as in (2.11). Consequently, in presence of one chiral multiplet and one vector multiplet, the kinetic terms of θ and Υ are the same as for two chiral multiplets, up to a redefinition of α and Δ . We thus can use the same rescaling in the two cases, allowing to have canonical fields $\{\Psi_1, \Psi_2\}$:

$$\theta = \frac{2i\bar{\gamma}^i k_i}{(\alpha a^3)^{1/2}}\Psi_1, \quad \Upsilon = \frac{\Delta}{2}\left(\frac{\alpha}{a}\right)^{1/2}\Psi_2 \quad (2.17)$$

2.1 The mixing matrices

In the basis $\{\Psi_1, \Psi_2\}$, the spin- $\frac{1}{2}$ part of the Lagrangian takes the form:

$$\begin{aligned} \mathcal{L}_{\Psi_1\Psi_2} = & -\bar{\Psi}_1\left[\bar{\gamma}^0\partial_0\Psi_1 - \frac{1}{2}\frac{\partial_0(\alpha a^3)}{\alpha a^3}\bar{\gamma}^0\Psi_1 + \bar{\gamma}^0\hat{B}\Psi_1 + i\bar{\gamma}^i k_i\hat{A}^\dagger\Psi_1 - i\bar{\gamma}^i k_i\Delta\bar{\gamma}^0\Psi_2\right] \\ & -\bar{\Psi}_2\left[\bar{\gamma}^0\partial_0\Psi_2 + \frac{\partial_0(\Delta\sqrt{\frac{\alpha}{a}})}{\Delta\sqrt{\frac{\alpha}{a}}}\bar{\gamma}^0\Psi_2 + \bar{\gamma}^0\hat{B}^\dagger\Psi_2 + i\bar{\gamma}^i k_i\hat{A}\Psi_2 + 2\dot{a}\bar{\gamma}^0\Psi_2\right. \\ & \left. + \frac{am}{M_P^2}\Psi_2 + a\bar{\gamma}^0\hat{F}\Psi_2 + i\bar{\gamma}^0\bar{\gamma}^i k_i\Delta\Psi_1\right] \end{aligned} \quad (2.18)$$

where the different parameters are given in (2.5) and $\hat{F} = \hat{F}_1 + \bar{\gamma}^0 \hat{F}_2$ is defined in (2.9). The explicit form of (2.18) depends on the specific model. We will illustrate some examples in Section 3.

With Ψ designating the vector $(\Psi_1, \Psi_2)^T$, the above Lagrangian can be put in a simple form

$$\mathcal{L}_{\Psi_1 \Psi_2} = -\bar{\Psi} [\bar{\gamma}^0 \partial_0 + i\bar{\gamma}^i k_i N + M] \Psi \quad (2.19)$$

with equations of motion

$$[\bar{\gamma}^0 \partial_0 + i\bar{\gamma}^i k_i N + M]_{mn} \Psi_n = 0, \quad m, n \in \{1, 2\}. \quad (2.20)$$

The N and M mixing matrices are given by²

$$\begin{aligned} M &= \mathbb{1}_4 \begin{pmatrix} -\hat{B}_2 & 0 \\ 0 & \hat{B}_2 + \frac{am}{M_P^2} - a\hat{F}_2 \end{pmatrix} + \bar{\gamma}^0 \begin{pmatrix} -\frac{1}{2} \frac{\partial_0(\alpha a^3)}{\alpha a^3} + \hat{B}_1 & 0 \\ 0 & \frac{\partial_0(\Delta \sqrt{\frac{\alpha}{a}})}{\Delta \sqrt{\frac{\alpha}{a}}} + \hat{B}_1 + 2\dot{a} + a\hat{F}_1 \end{pmatrix} \\ &= \mathbb{1}_4 \begin{pmatrix} -\hat{B}_2 & 0 \\ 0 & \hat{B}_2 + \frac{am}{M_P^2} - a\hat{F}_2 \end{pmatrix} \end{aligned} \quad (2.21)$$

and

$$N = N_1 + \bar{\gamma}^0 N_2 = \mathbb{1}_4 \begin{pmatrix} \hat{A}_1 & 0 \\ 0 & \hat{A}_1 \end{pmatrix} + \bar{\gamma}^0 \begin{pmatrix} -\hat{A}_2 & -\Delta \\ -\Delta & \hat{A}_2 \end{pmatrix} \quad (2.22)$$

where $\mathbb{1}_4$ is a 4×4 unit matrix. We should stress that in the decompositions of M and N , the 2×2 matrices act on the basis $\{\Psi_1, \Psi_2\}$, while $\bar{\gamma}^0$ is a 4×4 matrix acting on the spinor indices. In the first line of (2.21), the $\bar{\gamma}^0$ -dependent part vanishes both for two chiral multiplets and for one chiral multiplet with a D -term due to the property

$$\hat{B}_1 = \frac{1}{2} (\hat{B} + \hat{B}^\dagger) = \frac{a\dot{\alpha}}{2\alpha} + \frac{3\dot{a}}{2} \quad (2.23)$$

Note also that for Δ appearing in the denominator of Υ kinetic term (2.15), there seems to be a singularity at $\Delta \rightarrow 0$, but this singularity cancels out in the mixing matrices M, N , because there is another Δ in the denominator of \hat{F}_1 compensating the one in front of the Υ kinetic term.

In the $\{\Psi_1, \Psi_2\}$ basis, only N contributes to the mixing. One might be tempted to diagonalize N so as to decouple the two fermions, but in general the mixing matrices depend on time, thus a unitary transformation to the basis diagonalising N would also be time-dependent, which by time derivative gives a contribution to the mass matrix, rendering M non-diagonal. Though on general grounds, we will not provide an analytical expression of the physical fermions in terms of $\{\Psi_1, \Psi_2\}$, as long as we consider the catastrophic production in [18, 19] it is not necessary to carry out the entire diagonalization of (2.18).

²The difference of some signs compared to [16, 17, 20] is due to the $\bar{\gamma}^0$ convention.

2.2 Dispersion relations for the physical fermions

Expressing (1.2) in terms of the parameters of (2.5), we observe that the sound speed (1.2) amounts to the norm of N_{11} , namely

$$c_s^2 = \hat{A}_1^2 + \hat{A}_2^2 = 1 - \Delta^2 \quad (2.24)$$

In the case of a single chiral multiplet, the fields Υ and Ξ vanish in the unitary gauge, and we are left with θ . The norm of $N \equiv N_{11}$ enters into the dispersion relation as the gravitino velocity. It is a well-known result [12–14] that $\hat{A}_1^2 + \hat{A}_2^2 = 1$ for one chiral multiplet, thus the gravitino sound speed is the speed of light. However, when two fermions are present, Υ cannot be omitted and the physical fermions are combinations of θ and Υ . The question is then raised whether (2.24) is the sound speed of a *physical* propagating state.

One can check that the mixing matrix N in (2.22) is unitary, thus it can be written as an exponential of a phase

$$N = \exp(2\Phi\bar{\gamma}^0) = \cos(2\Phi) + \bar{\gamma}^0 \sin(2\Phi), \quad \Phi^\dagger = \Phi \quad (2.25)$$

Furthermore, notice that N_1 and N_2 are real, then Φ is a real, thus symmetric matrix. By a unitary transformation $\hat{\Psi} = \exp(\bar{\gamma}^0\Phi)\Psi$, the exponent in (2.25) is taken away, making N equal to the identity, and the Lagrangian (2.18) in the new basis $\hat{\Psi} = (\hat{\Psi}_1, \hat{\Psi}_2)^T$ becomes

$$\mathcal{L}_{\hat{\Psi}_1\hat{\Psi}_2} = -\bar{\hat{\Psi}} \left[\bar{\gamma}^0 \partial_0 + i\bar{\gamma}^i k_i + \hat{M} \right] \hat{\Psi}. \quad (2.26)$$

The new mass matrix

$$\begin{aligned} \hat{M} &= \hat{M}_1 + \bar{\gamma}^0 \hat{M}_2 = \exp(\bar{\gamma}^0\Phi)M \exp(-\bar{\gamma}^0\Phi) + \partial_0\Phi \\ \hat{M}_1 &= \cos(\Phi)M \cos(\Phi) + \sin(\Phi)M \sin(\Phi) + \partial_0\Phi, \\ \hat{M}_2 &= \sin(\Phi)M \cos(\Phi) - \cos(\Phi)M \sin(\Phi) \end{aligned} \quad (2.27)$$

is in general non-diagonal, due to the off-diagonal elements of Φ . We obtain therefore a system of two propagating fermions subject to oscillations due to the time-dependent mixing in their mass matrix.

As a side remark, the matrix Φ of (2.25) is a phase and defined up to a constant, as long as $N_1 = \cos(2\Phi)$, $N_2 = \sin(2\Phi)$ are satisfied. On the other hand, the transformation matrix $\exp(\bar{\gamma}^0\Phi) = \cos(\Phi) + \bar{\gamma}^0 \sin(\Phi)$ may take a minus sign according to the choice of Φ , but the Lagrangian (2.26) is independent of this choice. Moreover, the constant ambiguity does not change $\partial_0\Phi$ and the minus signs in $\cos(\Phi)$, $\sin(\Phi)$ are compensated in the expressions of (2.27). As a result, this ambiguity has no effect on the Lagrangian or on the mass matrix.

Since \hat{M}_2 is antisymmetric, we can further perform an orthogonal transformation [16, 17] in order to eliminate this matrix

$$\hat{\Psi} = L\tilde{\Psi}, \quad \text{with} \quad \left(\partial_0 + \hat{M}_2 \right) L = 0. \quad (2.28)$$

Thus, we arrive at a Lagrangian where the mixing comes only from the mass matrix, that is $\bar{\gamma}^0$ -independent

$$\mathcal{L}_{\tilde{\Psi}_1\tilde{\Psi}_2} = -\bar{\tilde{\Psi}} \left[\bar{\gamma}^0 \partial_0 + i\bar{\gamma}^i k_i + L^T \hat{M}_1 L \right] \tilde{\Psi}. \quad (2.29)$$

$\tilde{M} \equiv L^T \hat{M}_1 L$ is real and symmetric, hence it can be diagonalised by an orthogonal matrix C , with

$$\mu = \text{diag}(\mu_1, \mu_2) = C^T \tilde{M} C. \quad (2.30)$$

The energy squared eigenvalues for the fermions are then of the form

$$E_i^2 = k^2 + \mu_i^2. \quad (2.31)$$

Although the momentum squared is multiplied by 1, yet one cannot conclude from (2.31) that the sound speed of the physical fermions is the speed of light.

To have a closer look at the propagation of physical degrees of freedom, we follow the approach in [16, 17, 21] and expand $\tilde{\Psi}_i$ into creation and annihilation operators:

$$\tilde{\Psi}_i(x) = C_{ij} \int \frac{d^3 \mathbf{x}}{(2\pi)^{3/2}} e^{i\mathbf{k}\cdot\mathbf{x}} \left[U_r^{j\ell}(k, \eta) a_r^\ell(k) + V_r^{j\ell}(k, \eta) b_r^{\dagger\ell}(-k) \right], \quad (2.32)$$

where $r = \pm$ denotes the helicity components and a summation over repeated indices is understood. The spinorial Fourier coefficients are written in terms of the helicity eigenfunctions ψ_\pm and mode functions (matrices) U_\pm, V_\pm :

$$U_r^{ij} \equiv \left[\frac{U_\pm^{ij}}{\sqrt{2}} \psi_{r,+}, r \frac{U_\pm^{ij}}{\sqrt{2}} \psi_{r,-} \right]^T, \quad V_r^{ij} \equiv \left[\frac{V_\pm^{ij}}{\sqrt{2}} \psi_{r,+}, r \frac{V_\pm^{ij}}{\sqrt{2}} \psi_{r,-} \right]^T. \quad (2.33)$$

Since U_\pm and V_\pm are related by charge conjugation invariance of $\tilde{\Psi}_i$, we can restrict ourselves to the mode equations of U_\pm . Taking the momentum along the x^3 -axis and defining the antisymmetric matrix

$$\Gamma \equiv C^T \partial_0 C, \quad (2.34)$$

the equations of motion of $\tilde{\Psi}_i$ result in

$$i\partial_0 \begin{pmatrix} U_+ \\ U_- \end{pmatrix} = D \begin{pmatrix} U_+ \\ U_- \end{pmatrix}, \quad D = \begin{pmatrix} -i\Gamma - \mu & -k\mathbb{1}_2 \\ -k\mathbb{1}_2 & -i\Gamma + \mu \end{pmatrix} \quad (2.35)$$

where D is a 4×4 hermitian matrix, whose diagonal blocks encode the time dependence. Its real eigenvalues are

$$\begin{aligned} \omega_{1,\pm} &= \pm \left[\Gamma_{12}^2 + k^2 + \frac{1}{2} (\mu_1^2 + \mu_2^2) + \left(\frac{1}{4} (\mu_1^2 - \mu_2^2)^2 + \Gamma_{12}^2 (4k^2 + (\mu_1 + \mu_2)^2) \right)^{\frac{1}{2}} \right]^{\frac{1}{2}} \\ \omega_{2,\pm} &= \pm \left[\Gamma_{12}^2 + k^2 + \frac{1}{2} (\mu_1^2 + \mu_2^2) - \left(\frac{1}{4} (\mu_1^2 - \mu_2^2)^2 + \Gamma_{12}^2 (4k^2 + (\mu_1 + \mu_2)^2) \right)^{\frac{1}{2}} \right]^{\frac{1}{2}} \end{aligned} \quad (2.36)$$

where we denoted the (1, 2) element of Γ by Γ_{12} .

Note that when the fermions have degenerate mass $\mu_1 = \mu_2 = \bar{\mu}$, then from (2.30), $\hat{M}_1 = \bar{\mu}\mathbb{1}_2$ and $\Gamma_{12} = 0$. Thus, in this case we recover the relativistic dispersion relation with time-dependent mass:

$$\omega_{1,\pm} = \omega_{2,\pm} = \pm \sqrt{k^2 + \bar{\mu}^2} \quad (2.37)$$

In general, the dispersion relations of the physical fermions in (2.36) are very different from that in [19], describing the vacuum helicity-1/2 mode of the gravitino. We recall the latter for clarity:

$$\omega_k \equiv \sqrt{c_s^2 k^2 + a^2 m_{3/2}^2} \quad (2.38)$$

One of the arguments for catastrophic gravitino production at $c_s = 0$ is based on the adiabaticity violation. The dimensionless coefficient of non-adiabaticity is defined as [22]

$$\mathcal{A}_k \equiv \frac{\partial_0 \omega_k}{\omega_k^2} \quad (2.39)$$

Indeed, in (2.38), the momentum is multiplied by c_s , so when $c_s = 0$, the coefficient of non-adiabaticity is independent of k , and can even exceed one under some circumstances implying particle production with an arbitrarily large momentum. This is, however, not the case here: the sound speed of (2.24) does not enter explicitly³ the physical dispersion relations, and cannot suppress the momentum dependence when it vanishes. Therefore, c_s is not, at least not directly, responsible for the divergent particle production.

To see if particles of arbitrarily large momenta can actually be produced, we assume that the only source of non-adiabaticity is due to the variations of frequencies of the two physical fermions, with the coefficient \mathcal{A}_k being a sum of them. We then consider the limit of high momenta $k \gg \mu_i$ and $k \gg \Gamma_{12}$, where Γ_{12} is roughly the time derivative of the logarithm of masses (see its definition (2.34)), defining a scale related to the variation of masses. In this limit, the non-adiabaticity coefficient at leading order becomes

$$\mathcal{A}_k \equiv \frac{\partial_0 \omega_{1,+}}{\omega_{1,+}^2} + \frac{\partial_0 \omega_{2,+}}{\omega_{2,+}^2} \approx -6\Gamma_{12} \partial_0 \Gamma_{12} \frac{1}{k^3} \quad (2.40)$$

implying that \mathcal{A}_k falls as k^{-3} , and thus particles with arbitrarily large k cannot be produced.

3 Examples

3.1 Two chiral multiplets

Given two chiral multiplets (χ_1, ϕ_1) , (χ_2, ϕ_2) , we investigate the case with $\Delta = 1$ *at all times*, so that the sound speed defined in (2.24) vanishes, and according to [18, 19], the gravitino production is expected to diverge. In this toy example, the physical fermions as well as their equations of motion can be explicitly obtained. For simplicity, the Kähler potential is taken to be canonical, so that the expression of Δ becomes:

$$\Delta = \frac{2}{\alpha} \left(m_1 \dot{\phi}_2 - m_2 \dot{\phi}_1 \right) = 1 \quad (3.1)$$

On the other hand, from (2.5) one sees that $\Delta = 1$ is equivalent to the conditions $\alpha_1 = \alpha_2 = 0$. The latter implies that the gravitino mass is constant which is equivalent to the condition $m_1 \dot{\phi}_1 + m_2 \dot{\phi}_2 = 0$. The former implies $\dot{\phi}_1^2 + \dot{\phi}_2^2 = m_1^2 + m_2^2$, where we used

³ Γ_{12} can potentially depend on c_s , because Γ is related to the diagonalization of the mixing matrices, which contain Δ .

the expression (2.4) for the potential in the absence of D -term contribution. One possible setup for a solution is therefore $m_2 = \dot{\phi}_1 = 0$ and $m_1 = \dot{\phi}_2 \neq 0$, in other words, ϕ_1 breaks supersymmetry via its F -term and ϕ_2 via its kinetic term, by the same amount.

The mixing matrix N then takes the form

$$N = \bar{\gamma}^0 \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \quad (3.2)$$

while, for convenience, we write M as

$$M = \mathbb{1}_4 \begin{pmatrix} M_{11} & 0 \\ 0 & M_{22} \end{pmatrix}, \quad M_{11}, M_{22} \neq 0. \quad (3.3)$$

The angle Φ in (2.25) is constant because N does not depend on time; we then choose

$$\Phi = \frac{3\pi}{4} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \exp(\bar{\gamma}^0 \Phi) = -\frac{1}{\sqrt{2}} \mathbb{1}_2 + \frac{1}{\sqrt{2}} \bar{\gamma}^0 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (3.4)$$

Upon the unitary transformation $\hat{\Psi} = \exp(\bar{\gamma}^0 \Phi) \Psi$, we obtain the new mass matrix \hat{M} with

$$\hat{M}_1 = \frac{M_{11} + M_{22}}{2} \mathbb{1}_2, \quad \hat{M}_2 = \frac{M_{11} - M_{22}}{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (3.5)$$

We now look for the orthogonal matrix L cancelling the $\bar{\gamma}^0$ component of \hat{M} . Parametrising L by an angle τ , from (2.28) we get

$$L = \begin{pmatrix} \cos \tau(\eta) & -\sin \tau(\eta) \\ \sin \tau(\eta) & \cos \tau(\eta) \end{pmatrix}, \quad \tau'(\eta) = \frac{M_{11} - M_{22}}{2} \quad (3.6)$$

Once such an orthogonal transformation is found, the Lagrangian in the $\tilde{\Psi} = L^T \exp(\bar{\gamma}^0 \Phi) \Psi$ basis becomes

$$\mathcal{L}_{\tilde{\Psi}_1 \tilde{\Psi}_2} = -\tilde{\Psi} \left[\bar{\gamma}^0 \partial_0 + i \bar{\gamma}^i k_i + \frac{M_{11} + M_{22}}{2} \right] \tilde{\Psi}. \quad (3.7)$$

Thus, the mass matrix in this particular case is diagonal and one concludes that $\tilde{\Psi} = \{\tilde{\Psi}_1, \tilde{\Psi}_2\}$ are the physical fermions. They have degenerate mass and their equations of motion are decoupled:

$$\left[\bar{\gamma}^0 \partial_0 + i \bar{\gamma}^i k_i + \frac{M_{11} + M_{22}}{2} \right] \tilde{\Psi}_j = 0, \quad j \in \{1, 2\} \quad (3.8)$$

These are just standard Dirac equations with time-dependent mass, similar to the transverse mode of the gravitino. As a result, the physical fermions, which are linear combinations of θ and Υ , have the dispersion relation

$$\omega^2 = k^2 + \left(\frac{M_{11} + M_{22}}{2} \right)^2. \quad (3.9)$$

The coefficient of non-adiabaticity is suppressed by large momenta, and particle production is not expected to be divergent in this case, despite the fact that the speed of sound (1.2) associated to θ vanishes.

3.2 One chiral multiplet and one vector multiplet

Another example with two fermions is one chiral multiplet (χ_1, ϕ_1) accompanied by a vector multiplet with non-vanishing D -term. The simplest model one may consider is that of a constant (Fayet-Iliopoulos) D -term when the vector multiplet gauges the R-symmetry [23]. One can then consider two possibilities. The first consists of a neutral chiral multiplet, in which case the superpotential vanishes. Again, we take the Kähler potential to be canonical. The scalar potential is then

$$V = V_D = \frac{1}{2}g^2\mathcal{P}^2 \quad (3.10)$$

with \mathcal{P} constant. Since the gravitino is massless in this model, all mass terms appearing in (2.10) vanish, and we have $\hat{F} = 0$. Let

$$w \equiv \frac{p}{\rho}, \quad \text{with } p = |\dot{\phi}|^2 - V_D, \quad \rho = |\dot{\phi}|^2 + V_D. \quad (3.11)$$

In this model, the scalar equation of motion can be easily solved. We have

$$\ddot{\phi} + 3H\dot{\phi} = 0 \quad (3.12)$$

leading to $\dot{\phi} = e^{-3Ht}$, where the constant of integration is absorbed by a redefinition of the origin of time. The expression of w is then

$$w(t) = \frac{2e^{-6Ht} - g^2\mathcal{P}^2}{2e^{-6Ht} + g^2\mathcal{P}^2}. \quad (3.13)$$

Moreover, the parameters in (2.5) and Δ can be written as

$$\hat{A} = \hat{A}_1 = w, \quad \hat{B} = \hat{B}_1 = -\frac{3\dot{a}}{2}(1 - w^2), \quad \Delta^2 = 1 - w^2. \quad (3.14)$$

The sound speed defined in (2.24) is therefore simply given by w , and the mixing matrices in this case are

$$M = 0, \quad N = \begin{pmatrix} w & 0 \\ 0 & w \end{pmatrix} + \bar{\gamma}^0 \begin{pmatrix} 0 & -\sqrt{1-w^2} \\ -\sqrt{1-w^2} & 0 \end{pmatrix} \quad (3.15)$$

It follows that the physical fermions production and their equations of motion are determined only by w . Here, we study some particular limits:

- When $w \rightarrow 1$, the pressure and the energy density are equal, implying that the D -term is vanishing. This limit amounts to a theory with a single chiral multiplet. N is the identity matrix whereas $M = 0$, so Ψ_i are the physical fermions described by the massless Dirac equation and propagating at the speed of light.⁴
- On the other hand, the same situation occurs for $w \rightarrow -1$, which means that $|\dot{\phi}|^2 \rightarrow 0$, or equivalently $t \rightarrow +\infty$ in (3.13), so that maximal symmetry is unbroken. The equations of motion of the physical fermions differ from the massless Dirac equation by a minus sign in front of $\bar{\gamma}^i k_i$, but the dispersion relation $\omega^2 = k^2$ is unchanged compared to the previous case.

⁴This agrees also with the literature for the one chiral multiplet case; see the discussion in Section 2.2.

- Another special value is $w = 0$, corresponding to zero pressure and $e^{-6Ht} = V_D$, which is always satisfied for a certain time t . The mixing matrix N is identical to (3.2) for two chiral multiplets. The diagonalisation can be carried out in exactly the same way and we obtain two physical fermions with degenerate mass (massless here). Their equations of motion are the massless Dirac equation and we do not expect a divergent particle production, even though $c_s = 0$.

More generally, the matrix Φ in (2.25) can be chosen as

$$\Phi = -\frac{1}{2} \arccos(w) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (3.16)$$

while for $w \neq \pm 1$

$$\hat{M} = \partial_0 \Phi = \frac{\partial_0 w}{2\sqrt{1-w^2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (3.17)$$

The matrix \hat{M} has no $\bar{\gamma}^0$ component and can be diagonalised by a constant orthogonal matrix C . Thus $\Gamma = 0$, leading again to relativistic dispersion relations. The details of particle production can be worked out by doing the expansion (2.32), and solving numerically differential equations for the Bogolyubov coefficients, which will not be discussed here.

Finally, we comment briefly on the second possibility where the chiral field has a non-vanishing R-charge and the Kähler potential is non-canonical. Consider for instance a realistic model of inflation driven by supersymmetry breaking [24], where the Kähler potential and the superpotential are

$$K = \phi^1 \phi_1 + A(\phi^1 \phi_1)^2, \quad W = f \phi_1 \quad (3.18)$$

with f a constant and $|\phi_1|$ playing the role of the inflaton, while its phase is absorbed in the gauge field to make it massive. A is a small positive constant, so that the potential has a maximum at the origin allowing hilltop inflation with the slow-roll parameter η controlled by A . In this case, the D -term part of the potential is given by

$$V_D = \frac{q^2}{2} (1 + \phi^1 \phi_1 + 2A(\phi^1 \phi_1)^2)^2, \quad (3.19)$$

with q a constant parameter corresponding to the R-charge of ϕ_1 , that must be small compared to the F -term so that V_D is subdominant during inflation. It follows that Δ has the form:

$$\alpha^2 \Delta^2 = 4V_D(1 + 4A\phi^1 \phi_1)\dot{\phi}^1 \dot{\phi}_1, \quad \alpha^2 = (\rho + 3M_P^{-2}|m|^2)^2. \quad (3.20)$$

Note that it appears possible to have $\Delta^2 < 0$ for some negative values of A , leading to $c_s > 1$ according to equation (2.24). However, this region is unphysical since the Kähler metric becomes negative. This is actually similar to the situation that can be obtained in pathological models where Υ is dropped out by constraints, leading to $c_s > 1$ [20].

We will now investigate again the case of $\Delta = 1$ at all times. From (2.5), one sees that the condition $\alpha_2 = 0$ implies that the gravitino mass is constant with $m_1 = 0$, which is equivalent that ϕ_1 has vanishing F -term and breaks supersymmetry only by its kinetic

energy. This may indeed be satisfied around the vacuum at the minimum of the potential after the end of inflation. There, ϕ_1 is in general far from the maximum at the origin where corrections to the Kähler potential (3.18) become important and change its form. On the other hand, the condition $\alpha_1 = 0$ implies $|\dot{\phi}_1|^2 = V_D$ where the latter is now given by $V_D = (q^2/2)(1 + K^1\phi_1)^2$ with $K^1 \equiv \partial K/\partial\phi_1$. The analysis of the two fermions θ and Υ can now proceed as in the previous subsection of two chiral multiplets, giving rise to two decoupled equations of motion with a relativistic dispersion relation.

4 Conclusions

We have studied the equations of motion for the longitudinal modes of gravitinos in supergravity models where supersymmetry is linearly realised but spontaneously broken. We have considered the general case of two supermultiplets. One contains a scalar field ϕ that has non-vanishing kinetic energy, $\partial_\mu\phi \neq 0$. In a cosmological background, this scalar can be identified with the inflaton which is time dependent. The other multiplet is at the origin of the gravitino mass in the vacuum at late times. We have found that, after diagonalisation of the Hamiltonian, in all cases the dispersion relations of the propagating fermions take relativistic forms with in general a time-dependent mixing mass matrix. While this might be expected, it is shown here explicitly. Such cases are not expected to show a catastrophic production of gravitinos.

We did not discuss here the non-linear models as those considered in [10, 20, 25] as it is not clear to us which microscopic supergravity Lagrangian is at the origin of the constraint imposed there on the inflaton superfield. We note that in these cases one ends up with one fermion propagating in peculiar backgrounds. The result of [18, 19] constrains the background on which Rarita-Schwinger fields are allowed to propagate.

Acknowledgements

Work partially performed by I.A. as International Professor of the Francqui Foundation, Belgium. The work of K.B. is supported by the Agence Nationale de Recherche under grant ANR-15-CE31-0002 ‘‘HiggsAutomator’’. I.A. would like to thank Toine Van Proeyen for enlightening discussions.

A Notations

Our notations follow mostly that in [15], where the flat space γ -matrices are

$$\bar{\gamma}^0 = \begin{pmatrix} i\mathbb{1}_2 & 0 \\ 0 & -i\mathbb{1}_2 \end{pmatrix}, \quad \bar{\gamma}^i = \begin{pmatrix} 0 & -i\sigma_i \\ i\sigma_i & 0 \end{pmatrix}, \quad \gamma_5 = \begin{pmatrix} 0 & -\mathbb{1}_2 \\ -\mathbb{1}_2 & 0 \end{pmatrix} \quad (\text{A.1})$$

The Minkowski metric has signature $(-, +, +, +)$, and for cosmological applications, we used the FLRW metric. The curved space γ -matrices, noted γ^μ , are then related to the flat space γ -matrices by $\gamma^\mu = a^{-1}\bar{\gamma}^\mu$. The left and right projections are defined as

$$P_L = \frac{1}{2}(1 + \gamma_5) \quad , \quad P_R = \frac{1}{2}(1 - \gamma_5) \quad (\text{A.2})$$

Note that for chiral fermions, $P_L \chi_i = \chi_i$ and $P_R \chi^i = \chi^i$. The charge conjugation matrix in this convention is given by $C = \bar{\gamma}^0 \bar{\gamma}^2$. Some useful charge conjugates are

$$\begin{aligned} \chi_i^C &= \chi^i, & \phi_i^C &= \phi^i, & \mathcal{P}^C &= \mathcal{P}, & \lambda^C &= \lambda \\ \bar{\gamma}_\mu^C &= \bar{\gamma}_\mu, & \gamma_5^C &= -\gamma_5, & P_L^C &= P_R \end{aligned} \quad (\text{A.3})$$

In the Lagrangian (2.1), the covariant derivative of the scalar is $\hat{\partial}_0 = \partial_0 - \frac{i}{2} A_0^B \gamma_5$. In the cosmological context considered, the spatial derivatives of the scalar vanish, and for real backgrounds we have $A_0^B = 0$ and thus $\hat{\partial}_0 = \partial_0$. Keeping the above simplifications, the covariant derivatives acting on the chiral fermions, the gaugino and the gravitino are respectively

$$\begin{aligned} D_\mu \chi_i &\equiv \left(\partial_\mu + \frac{1}{4} \omega_\mu^{ab} \bar{\gamma}_{ab} \right) \chi_i + \Gamma_i^{jk} \chi_j \partial_\mu \phi_k, & \mathcal{D}_\mu \lambda &= \left(\partial_\mu + \frac{1}{4} \omega_\mu^{ab} \bar{\gamma}_{ab} \right) \lambda \\ D_\mu \psi_\nu &= \left(\left(\partial_\mu + \frac{1}{4} \omega_\mu^{ab} \bar{\gamma}_{ab} + \frac{1}{2} i \gamma_5 A_\mu \right) \delta_\nu^\lambda - \Gamma_{\mu\nu}^\lambda \right) \psi_\lambda \end{aligned} \quad (\text{A.4})$$

where ω_μ^{ab} stands for the spin connection and A_μ is the $U(1)$ gauge field. The Christoffel connection $\Gamma_{\mu\nu}^\lambda$ differs from the Kähler connection, where the latter corresponds to $\Gamma_i^{jk} \equiv g^{-1l} \partial^j g_l^k$. We use the notation $\bar{\gamma}_{ab} \equiv [\bar{\gamma}_a, \bar{\gamma}_b] / 2$.

Having introduced the Kähler covariant derivative \mathcal{D}^i , the mass terms are

$$m^i \equiv \mathcal{D}^i m = \partial^i m + \frac{\partial^i K}{2M_{\text{P}}^2} m, \quad m^{ij} \equiv \mathcal{D}^i \mathcal{D}^j m = \left(\partial^i + \frac{\partial^j K}{2M_{\text{P}}^2} \right) m^j - \Gamma_k^{ij} m^k \quad (\text{A.5})$$

$$m_{i\alpha} = -i \left[\partial_i \mathcal{P} - \frac{1}{4} (\text{Re } f)^{-1} \mathcal{P} f_i \right], \quad m_{R,\alpha\beta} = -\frac{1}{4} f_i g_j^{-1i} m^j \quad (\text{A.6})$$

where the subscript i in f denotes derivative with respect to ϕ^i .

References

- [1] P. Fayet and J. Iliopoulos, *Spontaneously Broken Supergauge Symmetries and Goldstone Spinors*, Phys. Lett. B **51** (1974), 461-464
- [2] D. V. Volkov and V. A. Soroka, *Higgs Effect for Goldstone Particles with Spin 1/2*, JETP Lett. **18** (1973), 312-314
- [3] P. Fayet, *Mixing Between Gravitational and Weak Interactions Through the Massive Gravitino*, Phys. Lett. B **70** (1977), 461
- [4] S. Deser and B. Zumino, *Broken Supersymmetry and Supergravity*, Phys. Rev. Lett. **38** (1977), 1433-1436
- [5] E. Cremmer, B. Julia, J. Scherk, S. Ferrara, L. Girardello and P. van Nieuwenhuizen, *Spontaneous Symmetry Breaking and Higgs Effect in Supergravity Without Cosmological Constant*, Nucl. Phys. B **147** (1979), 105
- [6] K. Benakli, Y. Oz and G. Policastro, *The Super-Higgs Mechanism in Fluids*, JHEP **02** (2014), 015 [arXiv:1310.5002 [hep-th]].

- [7] K. Benakli, L. Darmé and Y. Oz, *The Slow Gravitino*, JHEP **10** (2014), 121 [arXiv:1407.8321 [hep-ph]].
- [8] K. Benakli and L. Darmé, *Off-trail SUSY*, PoS **PLANCK2015** (2015), 019 [arXiv:1511.02044 [hep-ph]].
- [9] Y. Kahn, D. A. Roberts and J. Thaler, *The goldstone and goldstino of supersymmetric inflation*, JHEP **10** (2015), 001 [arXiv:1504.05958 [hep-th]].
- [10] S. Ferrara, R. Kallosh and J. Thaler, *Cosmology with orthogonal nilpotent superfields*, Phys. Rev. D **93** (2016) no.4, 043516 [arXiv:1512.00545 [hep-th]].
- [11] A. Schenkel and C. F. Uhlemann, *Quantization of the massive gravitino on FLRW spacetimes*, Phys. Rev. D **85** (2012), 024011 [arXiv:1109.2951 [hep-th]].
- [12] R. Kallosh, L. Kofman, A. D. Linde and A. Van Proeyen, *Gravitino production after inflation*, Phys. Rev. D **61** (2000), 103503 [arXiv:hep-th/9907124 [hep-th]].
- [13] G. F. Giudice, I. Tkachev and A. Riotto, *Nonthermal production of dangerous relics in the early universe*, JHEP **08** (1999), 009 [arXiv:hep-ph/9907510 [hep-ph]].
- [14] G. F. Giudice, A. Riotto and I. Tkachev, *Thermal and nonthermal production of gravitinos in the early universe*, JHEP **11** (1999), 036 [arXiv:hep-ph/9911302 [hep-ph]].
- [15] R. Kallosh, L. Kofman, A. D. Linde and A. Van Proeyen, *Superconformal symmetry, supergravity and cosmology*, Class. Quant. Grav. **17** (2000), 4269-4338 [erratum: Class. Quant. Grav. **21** (2004), 5017] [arXiv:hep-th/0006179 [hep-th]].
- [16] H. P. Nilles, M. Peloso and L. Sorbo, *Coupled fields in external background with application to nonthermal production of gravitinos*, JHEP **04** (2001), 004 [arXiv:hep-th/0103202].
- [17] H. P. Nilles, M. Peloso and L. Sorbo, *Nonthermal production of gravitinos and inflatinos*, Phys. Rev. Lett. **87** (2001), 051302 [arXiv:hep-ph/0102264 [hep-ph]].
- [18] E. W. Kolb, A. J. Long and E. McDonough, *Catastrophic Production of Slow Gravitinos*, [arXiv:2102.10113 [hep-th]].
- [19] E. W. Kolb, A. J. Long and E. McDonough, *The Gravitino Swampland Conjecture*, [arXiv:2103.10437 [hep-th]].
- [20] E. Dudas, M. A. G. Garcia, Y. Mambrini, K. A. Olive, M. Peloso and S. Verner, *Slow and Safe Gravitinos*, [arXiv:2104.03749 [hep-th]].
- [21] Y. Ema, K. Mukaida, K. Nakayama and T. Terada, *Nonthermal Gravitino Production after Large Field Inflation*, JHEP **11** (2016), 184 [arXiv:1609.04716 [hep-ph]].
- [22] L. Kofman, A. D. Linde and A. A. Starobinsky, *Towards the theory of reheating after inflation*, Phys. Rev. D **56** (1997), 3258-3295 [arXiv:hep-ph/9704452 [hep-ph]].
- [23] D. Z. Freedman, *Supergravity with axial gauge invariance*, Phys. Rev. D **15** (1977) 1173
- [24] I. Antoniadis, A. Chatrabhuti, H. Isono and R. Knoops, *Inflation from Supersymmetry Breaking*, Eur. Phys. J. C **77** (2017) no.11, 724 [arXiv:1706.04133 [hep-th]].
- [25] T. Terada, *Minimal Supergravity Inflation without Slow Gravitino*, [arXiv:2104.05731 [hep-th]].

Chapter 4

Higgs mass constraints on a supersymmetric solution of the muon $g-2$ anomaly

We now move to the second part of this manuscript, concerning beyond-the-SM solutions to the experimental anomalies, with a focus on supersymmetric candidates. In this Chapter, we will assume high-scale supersymmetry, namely, the new particles are too heavy to be detected by the current collider. In this scenario, constraining the parameter space becomes a difficult problem.

Contrary to the Standard Model, in supersymmetry, the Higgs quartic coupling is not a free parameter, but is in general determined by the D -term and F -term at tree level, and through radiative corrections, it contains information about all new particles in the theory, including their mass and couplings. Therefore, it can be used to constrain the parameters. On the other hand, experimental discrepancies also serve as an important input, and in this Chapter we choose the $(g - 2)_\mu$ anomaly. Joining these two inputs, we obtain various constraints on the parameters and their dependence on each other.

In Section 4.1, we recall some basic notions of $(g - 2)_\mu$ and supersymmetric models that accommodate this anomaly. In Section 4.2, we turn to the main object of this study – the Higgs mass, explaining the dependence of Higgs mass on other parameters of the theory, and outlining our strategies to constrain supersymmetric models. After a summary of results in 4.3, we apply these strategies and present the case study on a specific supersymmetric model in Section 4.4.

4.1 Muon $g-2$ and supersymmetry

The discovery of a Higgs boson – the last missing piece of the Standard Model – by ATLAS and CMS, as well as the measurement of its properties points to a very good agreement with the SM predictions [21, 22, 73, 74]. Despite the success of the SM in a majority of the experiments, it fails to deal with, *e.g.*, the hierarchy problem or the unification with the gravitational interaction. In addition, several significant experimental anomalies have been reported in recent years, that can hardly be explained by the SM only. One example is the muon anomalous magnetic moment $a_\mu \equiv (g - 2)/2$. Reported by Muon $g-2$ Collaboration at Fermilab in 2021 [75], its value turns out to be consistent with the previous measurement by the E281 experiment at BNL [76], and the combination of the two measurements gives

for the muon anomalous magnetic dipole moment $a_\mu^{\text{exp}} = 116592061(41) \times 10^{-11}$. In view of the corresponding SM prediction [77] $a_\mu^{\text{SM}} = 116591810(43) \times 10^{-11}$, the measured muon anomalous magnetic dipole moment is to date one of the largest deviations, approximately 4.2σ , with regard to the SM.

The unsatisfactory theoretical properties of the SM, along with the experimental discrepancies, call for beyond-the-SM (BSM) physics in order to accommodate the aforementioned issues. One well-motivated BSM candidate is supersymmetry, featuring an extended Higgs sector and superpartners for all SM particles. SUSY models can provide an explanation for the observed a_μ , and also for other anomalies such as the W boson mass, as we will discuss in the next chapter. Though a variety of non-SUSY candidates exists, here we will be focusing on SUSY theories.

Starting with the basic notion, in practice, the contribution to a_μ is computed from the following operator

$$\mathcal{L}_{\text{MDM}} = \frac{ie}{4m_\mu} a_\mu \bar{\mu} \gamma_{\alpha\beta} \mu F^{\alpha\beta} \quad (4.1)$$

where m_μ corresponds to the muon mass, and $F_{\alpha\beta}$ the photon field strength. We define $\gamma_{\alpha\beta} = [\gamma_\alpha, \gamma_\beta]/2$. The one-loop contribution to a_μ in SUSY consists of two types: i) neutralino-smuon loop diagrams, ii) chargino-sneutrino loop diagrams.

In the simplest setup, the Minimal Supersymmetric Standard Model (MSSM), these diagrams lead to a shift of the magnetic dipole moment with regard to the SM prediction, which takes the form [78]:

$$\Delta a_\mu^{\text{MSSM}} \propto \frac{g^2 (g')^2}{192\pi^2} \frac{m_\mu^2}{M_{\tilde{\mu}}^2} \frac{M_{1,2}^2 \mu}{M_{\tilde{\mu}}^2} \frac{\tan \beta}{1 + \epsilon_l \tan \beta} \times F \left(\frac{M_{1,2}^2}{M_{\tilde{\mu}}^2}, \frac{\mu^2}{M_{\tilde{\mu}}^2} \right) \quad (4.2)$$

where g, g' are the EW gauge couplings. $M_{\tilde{\mu}}$ is the smuon mass, whereas $M_{1,2}$ are the masses of the EW gauginos, for the gauge group $U(1), SU(2)$ respectively. μ is a parameter appearing in the MSSM superpotential and is related to the higgsino mass. ϵ_l is a function of different model parameters, whose complete form is irrelevant for our discussion here. $\tan \beta \equiv v_u/v_d$ is the ratio between the vacuum expectation values (vevs) of the Higgs doublets H_u, H_d which give mass to the up- and down- type fermions, respectively. A representative diagram involved in the MSSM is the following, with a smuon and neutralinos circulating in the loop:

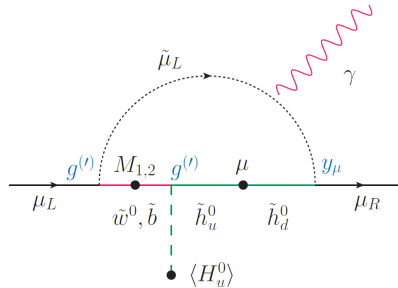


Figure 4.1: Example of one-loop contribution to a_μ in the MSSM

Here, the muon Yukawa coupling that determines the smuon-muon-higgsino vertex is given in the MSSM by $y_\mu^{\text{MSSM}} = g_\mu^{\text{SM}} / \cos \beta$, with g_μ^{SM} its SM counterpart. For large $\tan \beta$

values, this relation can be approximated by $y_\mu^{\text{MSSM}} \approx g_\mu^{\text{SM}} \tan \beta$, which gives rise to an enhancement of a_μ for large $\tan \beta$. In the meantime, we can see from Eq. (4.2) that a_μ is also suppressed by m_μ/M_S , where M_S represents the mass scale of the SUSY particles. These features are more transparent if one adopts the simplification that all SUSY particles have the same mass (M_S), so one can infer from (4.2) that $\Delta a_\mu^{\text{MSSM}} \propto m_\mu^2 \tan \beta / M_S^2$. With an appropriate choice of parameters, the MSSM is able to yield the observed a_μ .

On the experiment side, however, no strong evidence of BSM particles has shown up so far in direct collider searches, and stringent constraints are imposed on the mass of SUSY particles. Note that, in the MSSM, $\tan \beta$ cannot be arbitrarily large, because the bottom and tau Yukawa couplings follow the same relation as for the muon one: $y_{b,\tau}^{\text{MSSM}} = g_{b,\tau}^{\text{SM}} / \cos \beta = m_{b,\tau} / (v \cos \beta) \approx m_{b,\tau} \tan \beta / v$. As the bottom, tau masses are much larger than the muon mass, a too large $\tan \beta$ value will result in non-perturbative bottom and tau Yukawa couplings when evolved to the GUT scale. The requirement of perturbativity puts an upper bound on $\tan \beta$ [79]. Meanwhile, in order to achieve the experimental muon anomalous magnetic dipole moment, the mass ratio m_μ/M_S should not be too small, either. The interplay of the two bounds yields a prediction of the SUSY mass in the MSSM, restricted to the few-hundred-GeV range, which is in some tension with the direct searches for SUSY particles at the LHC. Though specific regions of the MSSM parameter space are still viable, typically those with a ‘‘compressed’’ SUSY mass spectrum, such scenarios will not be considered here.

In view of the absence of new particles in collider searches, a plausible possibility would be that the scale of New Physics is much higher than the EW scale, thus is out of reach with the current experimental facilities. In this case, which we consider thereafter, the new particles involved in the SUSY model must be much heavier than the SM ones. While the a_μ constraint on the MSSM leads to an upper bounded SUSY mass scale M_S , some extensions of the MSSM, with a larger parameter space and particle content, are able to accommodate the measured a_μ with a sufficiently high SUSY scale.

A possible candidate is proposed by Almannshofer *et al.* in [80], consisting in a supersymmetric four-Higgs-doublet model, which allows to achieve the measured a_μ value with multi-TeV smuon, higgsino and gaugino masses. In this ‘‘Flavorful Supersymmetric Standard Model’’ (FSSM), two of the four Higgs doublets, H_u and H_d couple only to fermions of the third generation, whereas the other two, H'_u and H'_d , provide masses to fermions of the first and second generations, and they have much smaller vevs than those of H_u and H_d . One of the advantages of the FSSM, compared to the MSSM, is that now the muon and bottom/tau Yukawa couplings are determined by different relations, because their masses are given by different Higgs doublets. Consequently, the muon Yukawa coupling in this model can be of $\mathcal{O}(1)$, yielding a sufficient enhancement to a_μ , without implying non-perturbative bottom and tau Yukawa couplings.

An important question in these BSM candidates is how the parameter space of the model is constrained. Given that multi-TeV range SUSY particles cannot be probed by the current colliders, alternative signatures may be used to infer the parameter ranges. In the specific case of the FSSM, apart from the muon anomalous magnetic dipole moment, some constraints on the flavor structure of the Yukawa couplings are also imposed due to the extended Higgs and higgsino sector, as is discussed in [80]. In this chapter, we investigate one particularly interesting quantity: the Higgs mass. We will show how, through a case study of the FSSM, Higgs mass in conjunction with the muon ($g - 2$) anomaly restricts the behavior of various parameters, such as the SUSY mass scale, the vev ratios (*i.e.* $\tan \beta$'s), and the muon Yukawa

coupling. We will start by reviewing some general properties of the Higgs mass in SUSY, as well as the methodology used to constrain a given high scale SUSY model.

4.2 Higgs mass in supersymmetry

The (squared-)physical mass of a scalar field is extracted from its inverse propagator. At tree level, this mass is generally determined by the bilinear, trilinear and quartic couplings in the scalar potential, in addition to the scalar vev. The equation satisfied by the vev, *i.e.* the *minimum condition*, relates the various couplings. In the SM, the gauge symmetry along with renormalizability results in a scalar potential of the Higgs boson up to quartic order, and by the minimum condition of the scalar potential, one can express the tree-level Higgs mass in terms of the quartic coupling and the vev only. For a quick reminder, the scalar potential of the Higgs doublet in the SM is

$$V(\Phi) = -\mu^2 \Phi^\dagger \Phi + \frac{\lambda}{2} (\Phi^\dagger \Phi)^2 \quad (4.3)$$

The negative mass term allows for spontaneous symmetry breaking (SSB). In the unitary gauge, the Higgs doublet and its vev take the form

$$\Phi = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ h + v \end{pmatrix} \xrightarrow{\text{SSB}} \langle \Phi \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix} \quad (4.4)$$

where h is the Higgs boson whose vev is zero. v conventionally sets the electroweak scale with $v \approx 246\text{GeV}$. The minimum condition is simply $-\mu^2 + \frac{1}{2}\lambda v^2 = 0$, leading to the tree-level Higgs mass $m_h^2 = \lambda v^2$.

In contrast to the SM, the quartic coupling of the Higgs boson is not a free parameter in SUSY, but is determined at tree level by the D -term, involving electroweak gauge couplings and the F -term from the superpotential, and therefore it fixes the leading order prediction for the SM-like Higgs mass. To illustrate this point, recall that the supersymmetric scalar potential is the sum of F - and D - terms. Ignoring for the moment the soft SUSY-breaking potential, the scalar potential is obtained by

$$\begin{aligned} V(z^i, z_j^\dagger) &= \sum_i \left| \frac{dW}{dz^i} \right|^2 + \frac{1}{2} \sum_a g_a^2 (z_i^\dagger T_j^{ai} z^j)^2 \\ &= \sum_i |F^i|^2 + \frac{1}{2} \sum_a (D^a)^2 \end{aligned} \quad (4.5)$$

where W is the superpotential and g_a the gauge couplings. For the MSSM, the superpotential is

$$W = -Y_d \hat{q} \cdot \hat{H}_d \hat{d} - Y_e \hat{l} \cdot \hat{H}_d \hat{e} + Y_u \hat{q} \cdot \hat{H}_u \hat{u} + \mu \hat{H}_u \cdot \hat{H}_d \quad (4.6)$$

We denote the left-handed lepton and quark $SU(2)$ doublets by \hat{l} , \hat{q} , whereas the corresponding right-handed singlets are \hat{u} , \hat{d} , \hat{e} . The fermion superfields are coupled to Higgs *via* Yukawa couplings Y_u , Y_d , Y_e . The two Higgs doublets are given by $H_u = (H_u^+, H_u^0)$, $H_d = (H_d^0, H_d^-)$, which consist of the scalar components of the superfields \hat{H}_u , \hat{H}_d . The dot product is defined as $H_u \cdot H_d = H_u^+ H_d^- - H_u^0 H_d^0$. The only non-vanishing vevs are

$$\langle H_u^0 \rangle = v_u / \sqrt{2} = v \sin \beta / \sqrt{2}, \quad \langle H_d^0 \rangle = v_d / \sqrt{2} = v \cos \beta / \sqrt{2} \quad (4.7)$$

satisfying $v_u^2 + v_d^2 = v^2$. The F -term gives for H_u^0, H_d^0 :

$$V_F \supset |\mu|^2 \left(|H_u^0|^2 + |H_d^0|^2 \right) \quad (4.8)$$

whereas the quartic terms of H_u^0, H_d^0 arise from the D -term:

$$V_D \supset \frac{1}{8} (g^2 + g'^2) \left(|H_u^0|^2 - |H_d^0|^2 \right)^2 \quad (4.9)$$

It is then obvious that the quartic coupling is determined by the gauge couplings only. The μ parameter can be furthermore replaced using the minimum condition, so that the tree-level SM-like Higgs mass is expressed in terms of the gauge couplings and the vevs. We omit here the detailed calculation and give directly the final result

$$\left(M_h^2 \right)^{\text{tree}} = \frac{g^2 + g'^2}{4} v^2 \cos^2 2\beta = M_Z^2 \cos^2 2\beta, \quad (4.10)$$

which is bounded from above by the Z boson mass, $M_Z \approx 91 \text{ GeV}$, manifestly lower than what is measured. As we will see later, the same tree-level upper bound also applies to the SM-like Higgs mass in the FSSM. In fact, the Higgs quartic coupling receives significant contributions from radiative corrections that allow the Higgs mass to reach the measured value. The higher order corrections in a generic SUSY model comprise loops with superparticles and BSM Higgs bosons, thus all SUSY mass parameters as well as the couplings affect the Higgs mass prediction, where the most relevant corrections generally arise from loops mediated by the superpartner of the top quark, stop. At one loop, the stop correction has the following form

$$\Delta\lambda^{\tilde{t}} \approx \frac{3y_t^4}{8\pi^2} \left(\ln \frac{M_{\tilde{t}}^2}{Q^2} + \frac{X_t^2}{M_{\tilde{t}}^2} - \frac{X_t^4}{12M_{\tilde{t}}^4} \right) \quad (4.11)$$

where $M_{\tilde{t}}$ is a common mass for the left and right stops, y_t is the top Yukawa coupling, X_t is the left-right mixing parameter in the stop mass matrix, and Q is the renormalization scale. The smuon contribution to the quartic coupling, though small compared to the stop contribution, is also of interest here. The latter is dominated by a one-loop diagram with four external Higgs legs and Higgs-smuon-smuon vertices, which is approximately

$$\Delta\lambda^{\tilde{\mu}} \approx -\frac{y_\mu^4}{96\pi^2} \left(\frac{\mu}{M_{\tilde{\mu}}} \right)^4 \quad (4.12)$$

Coming back to the muon anomalous magnetic dipole moment, we recall that certain SUSY parameters, such as the smuon mass, enter the expression of Δa_μ , as can be seen in Eq. (4.2) for the MSSM. It is therefore well-justified to associate the Higgs mass with the a_μ measurement to jointly constrain the SUSY parameter space. To provide a qualitative example of this interplay, notice that a larger enhancement of a_μ can be obtained for a larger muon Yukawa coupling, which according to (4.12) results in a more important *negative* contribution to the Higgs quartic coupling from smuons, and in turn, this requires a larger *positive* stop correction in (4.11) to compensate the smuon one. We will explicitly illustrate this behavior in the next section.

Experiments have measured the value of the SM-like Higgs boson mass to a very high precision, with $M_h = 125.10 \pm 0.14 \text{ GeV}$ from a combined ATLAS and CMS result [81]. It

is then of particular importance to calculate as precisely as possible the theoretical Higgs mass prediction in BSM models. Considerable progress has been achieved for the MSSM and beyond, and various methods, including fixed-order as well as Effective Field Theory (EFT) calculations, have been improved in the past years. The state of the art of Higgs mass prediction in SUSY models is reported in [82]. In the hierarchical scenario we are considering, the fixed order calculation is inadequate, since it gives rise to powers of $\ln(M_S^2/M_t^2)$ where M_t is the top quark mass, and in the case of large mass hierarchies, these logarithmic terms will obstruct the convergence of the perturbative expansion. We will instead adopt the EFT approach, namely, we assume that all SUSY masses are above M_S , much higher than the EW scale, so that the heavy SUSY particle can be integrated out at M_S , leaving as an imprint *threshold corrections* to the Higgs quartic coupling. The MSSM threshold corrections are well studied in the literature [83, 84], where the full one-loop and dominant two-loop expressions are computed. For a given generic model, supersymmetric or not, the formulae for one-loop threshold corrections are available in [85], and are readily applicable to the FSSM. On the other hand, the quartic coupling is extracted at the EW scale (chosen to be the top mass) from the measured Higgs mass then evolved up to M_S using the two-loop renormalization group equations (RGEs) of the SM. Finally, the matching of the quartic coupling at the scale M_S yields a constraint on the model parameters. While a brief summary of the FSSM and the main results are given in Sec. 4.3, details of this approach will be illustrated in Sec. 4.4.

4.3 FSSM model content and constraints from the Higgs mass

The FSSM comprises four Higgs doublets, with the superpotential

$$\begin{aligned}
W = & \mu_{ud} \hat{H}_u \hat{H}_d + \mu_{u'd'} \hat{H}'_u \hat{H}'_d + \mu_{u'd} \hat{H}'_u \hat{H}_d + \mu_{ud'} \hat{H}_u \hat{H}'_d \\
& - \left(Y_u \hat{H}_u + Y'_u \hat{H}'_u \right) \hat{Q} \hat{U}^c + \left(Y_d \hat{H}_d + Y'_d \hat{H}'_d \right) \hat{Q} \hat{D}^c + \left(Y_\ell \hat{H}_d + Y'_\ell \hat{H}'_d \right) \hat{L} \hat{E}^c
\end{aligned} \tag{4.13}$$

The first line generalizes the μ -term in the MSSM, and the second line gives new contributions to the Yukawa Lagrangian. We also add the following two simplifications:

- We will not consider flavor-violating processes and therefore the Yukawa couplings $Y_u^{(l)}$, $Y_d^{(l)}$, $Y_l^{(l)}$ are diagonal.
- As we mentioned earlier, the Higgs doublets H_u , H_d provide masses to the third generation fermions, whereas both the first and second generation fermion masses are provided by H'_u , H'_d . The Yukawa couplings of the first generation are suppressed with respect to the second generation ones, hence we can reasonably ignore the former.

We denote the vev's of the neutral components of H_u , H_d , H'_u , H'_d by v_u , v_d , v'_u , v'_d . It is convenient to define

$$\tan \tilde{\beta} \equiv \left(\frac{v_u^2 + v_u'^2}{v_d^2 + v_d'^2} \right)^{1/2}, \quad \tan \beta_u \equiv \frac{v_u}{v'_u}, \quad \tan \beta_d \equiv \frac{v_d}{v'_d} \tag{4.14}$$

so the tree-level Higgs mass, generated by the D -term, is $(M_h^2)^{\text{tree}} = M_Z^2 \cos^2 2\tilde{\beta}$. The first constraint on the parameters arises from the matching of the Higgs quartic coupling. At the EW scale, it is extracted from the Higgs mass using the public code `mr` [86], then we evolve

this quartic coupling up to the SUSY scale (M_S) using SM RGEs. The result is matched to the FSSM prediction, which at tree-level is analogous to the MSSM prediction up to replacing $\beta \rightarrow \tilde{\beta}$: $\lambda^{\text{tree}}(Q) = \frac{1}{4} [g^2(Q) + g'^2(Q)] \cos^2 2\tilde{\beta}$, where Q denotes the renormalization scale. The one-loop threshold corrections are composed of:

$$\Delta\lambda^{\text{reg}} + \Delta\lambda^{\tilde{f}} + \Delta\lambda^H + \Delta\lambda^\chi \quad (4.15)$$

where $\Delta\lambda^{\text{reg}}$ accounts for the difference of the renormalization schemes, and $\Delta\lambda^{\tilde{f}}$, $\Delta\lambda^H$, $\Delta\lambda^\chi$ are respectively the contributions from sfermions, heavy Higgses, higgsinos as well as EW gauginos. In practice, the threshold corrections are obtained using the general formulae in [85]. Next, we employ the FSSM prediction of a_μ in [80], but with the simplification of identical masses $M_{\tilde{\mu}_L} = M_{\tilde{\mu}_R} = M_{\tilde{\mu}}$, $\mu = M_1 = M_2$, so the expression reduces to

$$\Delta a_\mu^{\text{FSSM}} = \frac{1}{192\pi^2} \frac{M_\mu^2}{M_{\tilde{\mu}}^2} \frac{\tan\beta \tan\beta_d}{1 + \epsilon_\ell \tan\beta \tan\beta_d} \left[g'^2 f_1 \left(M_\chi^2 / M_{\tilde{\mu}}^2 \right) + 5g^2 f_2 \left(M_\chi^2 / M_{\tilde{\mu}}^2 \right) \right] \quad (4.16)$$

The precise form of f_1 , f_2 , ϵ_ℓ will be given in the next section. We can see that compared to the MSSM prediction (4.2), the FSSM has an additional enhancement $\tan\beta_d$, assuming that $\tan\beta_d \gg 1$. The second constraint corresponds to $\Delta a_\mu^{\text{FSSM}} = 251 \times 10^{-11}$ – that the FSSM reproduces the experimental value of a_μ .

Combining the two constraints, we are able to determine qualitatively the parameter space of the FSSM, for example, the stop mass $M_{\tilde{t}}$, smuon mass $M_{\tilde{\mu}}$, muon Yukawa coupling and the $\tan\beta$ factors. In the particular case of the FSSM, and a chosen smuon mass range [1 TeV, 5 TeV], we notice that the smuon Yukawa coupling increases with $M_{\tilde{\mu}}$ but stays $\mathcal{O}(1)$. On the other hand, the stop mass ranges from several TeV to approximately 100 TeV depending on the benchmark, exhibiting in certain cases a strongly hierarchical scenario. The plots are shown in Figure 1 and Figure 2 of the next section. The goal of this study is not simply constraining the FSSM but rather the application of a more general concept, that one can extract useful information from the Higgs mass in the case of high-scale SUSY models, and experimental anomalies, such as a_μ , help to constrain the parameter space in conjunction with the Higgs mass.

4.4 Publication

Higgs-mass constraints on a supersymmetric solution of the muon $g - 2$ anomaly

Wenqi Ke^a and Pietro Slavich^a

^a *Sorbonne Université, CNRS, Laboratoire de Physique Théorique et Hautes Energies (LPTHE),
UMR 7589, 4 place Jussieu, 75005 Paris, France.*

Abstract

The prediction for the quartic coupling of the SM-like Higgs boson constrains the parameter space of SUSY models, even in scenarios where all of the new-particle masses are above the scale probed so far by the LHC. We study the implications of the Higgs-mass prediction on a recently-proposed SUSY model that features two pairs of Higgs doublets, and provides a solution to the $(g - 2)_\mu$ anomaly thanks to a suitable enhancement of the muon Yukawa coupling.

1 Introduction

The discovery of a Higgs boson with mass around 125 GeV and properties compatible with the predictions of the Standard Model (SM) [1–4], combined with the negative (so far) results of the searches for additional new particles at the LHC, point to scenarios with at least a mild hierarchy between the electroweak (EW) scale and the scale of beyond-the-SM (BSM) physics. In this case, the SM plays the role of an effective field theory (EFT) valid between the two scales. The requirement that a given BSM model include a state that can be identified with the observed Higgs boson can translate into important constraints on the model’s parameter space.

One of the prime candidates for BSM physics is supersymmetry (SUSY), which predicts scalar partners for all SM fermions, as well as fermionic partners for all bosons. A remarkable feature of SUSY extensions of the SM is the requirement of an extended Higgs sector, with additional neutral and charged bosons. In contrast to the case of the SM, the masses of the Higgs bosons are not free parameters, as SUSY requires all quartic scalar couplings to be related to the gauge and Yukawa couplings. Moreover, radiative corrections to the tree-level predictions for the quartic scalar couplings introduce a dependence on all of the SUSY-particle masses and couplings.¹ In a hierarchical scenario such as the one described above, the prediction of the SUSY model for the quartic self-coupling of its lightest Higgs scalar, which plays the role of the SM Higgs boson, must coincide with the SM coupling λ_{SM} extracted at the EW scale from the measured value of the Higgs mass and evolved up to the SUSY scale with appropriate renormalization group equations (RGEs). This condition can be used to constrain some yet-unmeasured parameters of the SUSY model, such as, e.g., the masses of the scalar partners of the top quarks, the stops.

While the new particles predicted by SUSY models – or, for that matter, those predicted by any other BSM model – have yet to show up at the LHC, precision experiments have seen tantalizing deviations from the predictions of the SM, particularly in measurements involving muons. Over the past few years the LHCb collaboration reported hints of lepton flavor violation in rare B decays [6–9], and earlier in 2021 the Muon $g-2$ Collaboration at Fermilab reported a new measurement [10] of the muon anomalous magnetic moment $a_\mu \equiv (g-2)_\mu/2$, consistent with the previous measurement by the E821 experiment at BNL [11]. In what might be considered currently the most striking deviation from the predictions of the SM, the combination of the two experimental results for the muon anomalous magnetic moment, $a_\mu^{\text{exp}} = 116\,592\,061(41) \times 10^{-11}$, differs by 4.2σ from the state-of-the-art SM prediction given in ref. [12], $a_\mu^{\text{SM}} = 116\,591\,810(43) \times 10^{-11}$, which is based on refs. [13–32].

Supersymmetric extensions of the SM can accommodate an explanation for the observed discrepancy $\Delta a_\mu \equiv a_\mu^{\text{exp}} - a_\mu^{\text{SM}} = (251 \pm 59) \times 10^{-11}$. In the minimal of such extensions, the MSSM, a suitable contribution to a_μ can arise from one-loop diagrams involving smuons, higgsinos and EW gauginos (namely, the SUSY partners of muons, Higgs bosons and EW gauge bosons). This contribution is suppressed by the ratio M_μ^2/M_S^2 – where M_μ is the muon mass and M_S represents the mass scale of

¹We point the reader to ref. [5] for a recent review of Higgs-mass predictions in SUSY models.

the relevant SUSY particles – but it can be enhanced by a large value of the parameter $\tan \beta \equiv v_u/v_d$, i.e. the ratio of the vacuum expectation values (vevs) of the Higgs doublets H_u and H_d , which give mass to the up-type and down-type fermions, respectively. However, since the Yukawa couplings of the down-type fermions f_d in the MSSM are related to their SM counterparts by $y_{f_d}^{\text{MSSM}} = g_{f_d}^{\text{SM}}/\cos \beta$, the requirement that the bottom and tau couplings remain perturbative up to the GUT scale sets an upper limit on the acceptable values of $\tan \beta$, see e.g. ref. [33]. When such limit is taken into account, the masses of the SUSY particles entering the diagrams that provide the required contribution to a_μ are typically restricted to the few-hundred-GeV range. This results in some tension with the direct searches for SUSY particles at the LHC, although specific regions of the MSSM parameter space – typically, those with a “compressed” SUSY mass spectrum – remain still viable.²

Recently, a new SUSY model in which a suitable contribution to a_μ can be obtained even with smuon, higgsino and gaugino masses in the multi-TeV range was proposed in ref. [37]. The Higgs sector of the “Flavorful Supersymmetric Standard Model” (FSSM)³ consists of four doublets, two of which, H_u and H_d , couple only to quarks and leptons of the third generation, whereas the other two, H'_u and H'_d , have much smaller vevs and provide masses to the fermions of the first and second generation. In this model the muon Yukawa coupling y_μ^{FSSM} , which determines the higgsino–muon–smuon and Higgs–smuon–smuon couplings entering the diagrams that contribute to a_μ , can be of $\mathcal{O}(1)$ without implying non-perturbative values for y_τ^{FSSM} and y_b^{FSSM} . Indeed, the SUSY contribution to a_μ in the FSSM is enhanced by v_u/v'_d , which can greatly exceed the enhancement achievable in the MSSM when $v'_d \ll v_d$, in turn allowing for a stronger suppression by M_μ^2/M_S^2 .

Scenarios where all of the SUSY particles have masses in the multi-TeV range will be probed directly only at future colliders. However, as discussed in ref. [37], the extended Higgs/higgsino sector of the FSSM can accommodate interesting flavor-changing effects both in the lepton sector and in the quark sector, leading to constraints on the flavor structure of the Yukawa couplings. As mentioned earlier, a further constraint stems from the requirement that the lightest scalar in the Higgs sector be identified with the SM-like Higgs boson discovered at the LHC. Compared with the case of the MSSM, the presence of additional particles in the Higgs/higgsino sector and of additional $\mathcal{O}(1)$ couplings in the superpotential can affect the FSSM prediction for the SM-like Higgs mass, leading to different constraints on the parameter space of the model.

In this paper we study the Higgs-mass prediction of the FSSM and its interplay with the solution of the $(g - 2)_\mu$ anomaly. In the calculation of the Higgs mass we rely on the EFT approach, as appropriate to a hierarchical scenario where the BSM physics is somewhat removed from the EW scale. In section 2 we introduce the Higgs sector of the FSSM. In section 3 we obtain the one-loop threshold correction to the quartic Higgs coupling, adapting to the model under consideration the general formulas given in ref. [39]. Combined with two-loop RGEs for the SM couplings, this allows

²For recent surveys of explanations of the $(g - 2)_\mu$ anomaly in the MSSM see e.g. refs. [34,35]. For an earlier study of $(g - 2)_\mu$ in MSSM scenarios with TeV-scale SUSY masses and very large $\tan \beta$ see ref. [36].

³We remark that this acronym had already been used in ref. [38] to denote a model with “Fake” Split SUSY.

for the next-to-leading-logarithmic (NLL) resummation of the corrections to the Higgs mass enhanced by powers of $\ln(M_S/M_t)$ (where, as usual, we take M_t as a proxy for the EW scale). We also point out a potential issue stemming from large threshold corrections to the strange Yukawa coupling in case the four-doublet construction of the FSSM is extended to the quark sector. In section 4 we discuss the constraints on the parameter space of the FSSM that arise from the combined requirements of an appropriate prediction for the Higgs mass and a solution to the $(g-2)_\mu$ anomaly. Section 5 contains our conclusions. Finally, in the appendix we provide explicit formulas for the tree-level Higgs mass matrices in the FSSM.

2 The Higgs sector of the FSSM

In this section we describe the Higgs and higgsino sectors of the FSSM, focusing on the hierarchical scenario in which the lightest scalar plays the role of the SM Higgs boson, while the remaining physical Higgs states are heavier.

The FSSM includes two $SU(2)$ doublets of chiral superfields with positive hypercharge, \hat{H}_u and \hat{H}'_u , and two doublets with negative hypercharge, \hat{H}_d and \hat{H}'_d . The superpotential can be decomposed as $W = W_\mu + W_Y$, where W_μ generalizes the “ μ term” of the MSSM:

$$W_\mu = \mu_{ud} \hat{H}_u \hat{H}_d + \mu_{u'd'} \hat{H}'_u \hat{H}'_d + \mu_{ud} \hat{H}'_u \hat{H}_d + \mu_{u'd'} \hat{H}_u \hat{H}'_d, \quad (1)$$

whereas W_Y contains the interactions of the Higgs doublets with the quark and lepton superfields:

$$W_Y = - (Y_u \hat{H}_u + Y'_u \hat{H}'_u) \hat{Q} \hat{U}^c + (Y_d \hat{H}_d + Y'_d \hat{H}'_d) \hat{Q} \hat{D}^c + (Y_\ell \hat{H}_d + Y'_\ell \hat{H}'_d) \hat{L} \hat{E}^c, \quad (2)$$

where all gauge and generation indices are understood. In ref. [37], where the focus is on the leptonic sector, the coupling Y_ℓ is defined as a rank-1 matrix whose only non-zero element is $(3, 3)$, providing a tree-level mass to the tau lepton proportional to v_d . The coupling Y'_ℓ is instead defined as a rank-3 matrix which provides mass and mixing terms proportional to v'_d to all of the charged leptons. In this setup the muon Yukawa coupling can in principle be larger than the bottom and tau ones, as long as $v'_d \ll v_d$. As discussed in ref. [37], the current bounds on lepton-flavor violating processes give rise to constraints on the off-diagonal elements of Y'_ℓ , which anyway are not relevant to the prediction for a_μ at the considered level of accuracy. Finally, ref. [37] mentions that a similar construction can be implemented in the quark sector.

In this work we do not consider flavor-violating processes in either the lepton or the quark sector, but we rather focus on the interplay of the effects of $\mathcal{O}(1)$ flavor-diagonal couplings on the predictions for the SM-like Higgs mass and for a_μ . We therefore adopt for simplicity a pared-down version of W_Y , in which we include only flavor-diagonal couplings for the second and third generations:

$$\begin{aligned} W_Y = & -y'_c \hat{H}'_u \hat{Q}_2 \hat{U}_2^c + y'_s \hat{H}'_d \hat{Q}_2 \hat{D}_2^c + y'_\mu \hat{H}'_d \hat{L}_2 \hat{E}_2^c \\ & -y'_t \hat{H}'_u \hat{Q}_3 \hat{U}_3^c + y'_b \hat{H}'_d \hat{Q}_3 \hat{D}_3^c + y'_\tau \hat{H}'_d \hat{L}_3 \hat{E}_3^c \\ & -y_t \hat{H}_u \hat{Q}_3 \hat{U}_3^c + y_b \hat{H}_d \hat{Q}_3 \hat{D}_3^c + y_\tau \hat{H}_d \hat{L}_3 \hat{E}_3^c. \end{aligned} \quad (3)$$

As to the first-generation couplings, they are necessarily suppressed with respect to those of the second generation, because in the FSSM both generations receive their masses from v'_u and v'_d .

In addition to mass terms for gauginos and sfermions, which are the same as in the MSSM, the soft SUSY-breaking Lagrangian of the FSSM contains mass terms and B -terms for all of the Higgs doublets

$$\begin{aligned}
-\mathcal{L}_{\text{soft}} \supset & m_{uu}^2 H_u^\dagger H_u + m_{dd}^2 H_d^\dagger H_d + m_{u'u'}^2 H_u'^\dagger H_u' + m_{d'd'}^2 H_d'^\dagger H_d' \\
& + \left(m_{uu'}^2 H_u^\dagger H_u' + m_{d'd'}^2 H_d^\dagger H_d' + \text{h.c.} \right) \\
& + \left(B_{ud} H_u H_d + B_{u'd'} H_u' H_d' + B_{u'd} H_u' H_d + B_{ud'} H_u H_d' + \text{h.c.} \right), \quad (4)
\end{aligned}$$

as well as trilinear interaction terms analogous to those in the superpotential

$$\begin{aligned}
-\mathcal{L}_{\text{soft}} \supset & -y'_c A'_c H_u Q_2 U_2^c + y'_s A'_s H_d' Q_2 D_2^c + y'_\mu A'_\mu H_d' L_2 E_2^c \\
& -y'_t A'_t H_u' Q_3 U_3^c + y'_b A'_b H_d' Q_3 D_3^c + y'_\tau A'_\tau H_d' L_3 E_3^c \\
& -y_t A_t H_u Q_3 U_3^c + y_b A_b H_d Q_3 D_3^c + y_\tau A_\tau H_d L_3 E_3^c. \quad (5)
\end{aligned}$$

The tree-level Higgs mass spectrum of a model with three pairs of doublets has been discussed in ref. [40], whose approach can be easily adapted to the case of two pairs of doublets. In order to identify the state that plays the role of the SM-like Higgs boson, we rotate the four doublets to the so-called ‘‘Higgs basis’’, in which only one of the doublets acquires a non-zero vev defined by $v^2 \equiv v_u^2 + v_u'^2 + v_d^2 + v_d'^2$. To this purpose, we first rotate the doublets with the same hypercharge:

$$\begin{pmatrix} \Phi_u \\ \Phi_u' \end{pmatrix} = \begin{pmatrix} \sin \beta_u & \cos \beta_u \\ \cos \beta_u & -\sin \beta_u \end{pmatrix} \begin{pmatrix} H_u \\ H_u' \end{pmatrix}, \quad \begin{pmatrix} \Phi_d \\ \Phi_d' \end{pmatrix} = \begin{pmatrix} \sin \beta_d & \cos \beta_d \\ \cos \beta_d & -\sin \beta_d \end{pmatrix} \begin{pmatrix} -\epsilon H_d^* \\ -\epsilon H_d'^* \end{pmatrix}, \quad (6)$$

where the rotation angles are defined by $\tan \beta_u \equiv v_u/v_u'$ and $\tan \beta_d \equiv v_d/v_d'$. The antisymmetric tensor ϵ , with $\epsilon_{12} = 1$, acts on the complex conjugates of H_d and H_d' so that all doublets in the new basis have the same hypercharge. In this basis, the vevs of the neutral components of the four doublets become $\langle \Phi_u^0 \rangle = (v_u^2 + v_u'^2)^{1/2}$, $\langle \Phi_d^0 \rangle = (v_d^2 + v_d'^2)^{1/2}$, and $\langle \Phi_u'^0 \rangle = \langle \Phi_d'^0 \rangle = 0$. The two doublets that acquire vevs are further rotated as

$$\begin{pmatrix} \Phi_h \\ \Phi_H \end{pmatrix} = \begin{pmatrix} \cos \tilde{\beta} & \sin \tilde{\beta} \\ -\sin \tilde{\beta} & \cos \tilde{\beta} \end{pmatrix} \begin{pmatrix} \Phi_d \\ \Phi_u \end{pmatrix}, \quad \tan \tilde{\beta} \equiv \left(\frac{v_u^2 + v_u'^2}{v_d^2 + v_d'^2} \right)^{1/2}, \quad (7)$$

so that $\langle \Phi_h^0 \rangle = v$ and $\langle \Phi_H^0 \rangle = 0$, i.e., in the Higgs basis the doublet Φ_h is entirely responsible for the breaking of the EW symmetry (EWSB).

The mass matrices for the scalar, pseudoscalar and charged components of the four doublets in the Higgs basis are given in the appendix. They depend on the μ parameters defined in eq. (1) and on the soft SUSY-breaking mass and B parameters defined in eq. (4), plus the EW gauge couplings, the vev v and the angles β_u , β_d and $\tilde{\beta}$. The minimum conditions of the scalar potential are used to

remove the dependence of the mass matrices on four combinations of the original parameters. Most importantly, the terms that mix the components of Φ_h with the components of the remaining doublets are either zero or proportional to v^2 (more specifically, to M_Z^2). In a hierarchical scenario in which the masses of the BSM Higgs bosons are significantly higher than the EW scale, we can thus neglect their mixing with Φ_h , and identify the latter directly with the Higgs boson of the SM. In contrast, the scalar, pseudoscalar and charged components of the three remaining doublets Φ_H , Φ'_u and Φ'_d do mix with each other.⁴ However, under the approximation of neglecting terms proportional to v^2 , the respective 3×3 mass matrices are all the same. We can then combine the eigenstates of the scalar, pseudoscalar and charged mass matrices into three heavy doublets H_i (with $i = 1, 2, 3$), whose masses we denote as M_{H_i} . The condition for their decoupling from the lightest doublet is then $M_{H_i} \gg M_Z$.

We now focus on the properties of the SM-like doublet Φ_h . The tree-level mass of its scalar component is

$$(M_h^2)^{\text{tree}} = M_Z^2 \cos^2 2\tilde{\beta} , \quad (8)$$

which differs from the analogous result in the decoupling limit of the MSSM only via the replacement of β with $\tilde{\beta}$. In the scenarios of interest for the solution to the $(g-2)_\mu$ anomaly, one has $v'_d \ll v_d$. If the condition $v'_u \ll v_u$ also holds, $\tan \beta$ and $\tan \tilde{\beta}$ are numerically very close to each other, hence the tree-level prediction for the SM-like Higgs mass in the FSSM is essentially the same as in the MSSM.

The SM-like couplings of Φ_h to second-generation quarks and leptons are related at the tree level to the superpotential couplings in eq. (3) by

$$g_c = y'_c \sin \tilde{\beta} \cos \beta_u , \quad g_{s,\mu} = y'_{s,\mu} \cos \tilde{\beta} \cos \beta_d , \quad (9)$$

while the couplings to third-generation fermions read

$$g_t = y_t \sin \tilde{\beta} \sin \beta_u + y'_t \sin \tilde{\beta} \cos \beta_u , \quad g_{b,\tau} = y_{b,\tau} \cos \tilde{\beta} \sin \beta_d + y'_{b,\tau} \cos \tilde{\beta} \cos \beta_d . \quad (10)$$

The relevant difference with the MSSM, in the context of the solution of the $(g-2)_\mu$ anomaly, is the additional suppression by $\cos \beta_d$ in the couplings of the SM-like Higgs to down-type fermions of the second generation. Consequently, in the FSSM superpotential of eq. (3), the muon Yukawa coupling can in principle be even larger than the bottom and tau ones, as long as $\tan \beta_d \gg 1$.

For what concerns the couplings of Φ_h to sfermions, the quartic couplings are proportional to the squared Yukawa couplings g_f^2 defined as in eqs. (9) and (10). The main difference with respect to the MSSM stems from the left-right mixing parameters entering the trilinear Higgs-sfermion couplings in the combination $g_f X_f$. Those for the second-generation sfermions read

$$X_c = A'_c - \cot \beta \tan \beta_u \left(\mu_{u'd} + \mu_{u'd'} \cot \beta_d \right) , \quad X_{s,\mu} = A'_{s,\mu} - \tan \beta \tan \beta_d \left(\mu_{ud'} + \mu_{ud'} \cot \beta_u \right) , \quad (11)$$

⁴Note that in this study we do not consider the possibility of CP violation in the Higgs sector, hence the scalar and pseudoscalar components of the three heavy doublets mix separately.

while those for the third-generation sfermions read

$$\begin{aligned}
X_t &= \frac{A_t - \cot \beta \left(\mu_{ud} + \mu_{ud'} \cot \beta_d \right)}{1 + \frac{y'_t}{y_t} \cot \beta_u} + \frac{A'_t - \cot \beta \tan \beta_u \left(\mu_{u'd} + \mu_{u'd'} \cot \beta_d \right)}{1 + \frac{y_t}{y'_t} \tan \beta_u}, \\
X_{b,\tau} &= \frac{A_{b,\tau} - \tan \beta \left(\mu_{ud} + \mu_{u'd} \cot \beta_u \right)}{1 + \frac{y'_{b,\tau}}{y_{b,\tau}} \cot \beta_d} + \frac{A'_{b,\tau} - \tan \beta \tan \beta_d \left(\mu_{ud'} + \mu_{u'd'} \cot \beta_u \right)}{1 + \frac{y_{b,\tau}}{y'_{b,\tau}} \tan \beta_d}. \quad (12)
\end{aligned}$$

Again, the relevant aspect of eqs. (11) and (12) in the context of the solution of the $(g-2)_\mu$ anomaly is the enhancement of the Higgs-smuon trilinear coupling by a factor $\tan \beta_d$ with respect to the MSSM case (note that, to facilitate the comparison, we expressed the trilinear couplings in terms of $\tan \beta = \tan \tilde{\beta} \sin \beta_u / \sin \beta_d$). If we also assume $\tan \beta_u \gg 1$, the enhanced part of X_μ involves only the superpotential parameter $\mu_{ud'}$. We note, on the other hand, that for $\tan \beta_{u,d} \gg 1$ there are no further enhancements with respect to the MSSM in the trilinear Higgs couplings to third-generation sfermions, as long as the ‘‘primed’’ top, bottom and tau Yukawa couplings remain at most of $\mathcal{O}(1)$.

We finally comment on the higgsino masses. In the hierarchical scenario considered in our study, we assume that both gaugino and higgsino masses are somewhat removed from the EW scale. In this case, the mixing between EW gauginos and higgsinos induced by EWSB can be neglected, and the four two-component fermions $\tilde{h}_u, \tilde{h}_d, \tilde{h}'_u,$ and \tilde{h}'_d combine into two Dirac fermions. Following ref. [37], we define the angles θ_u and θ_d that diagonalize the higgsino mass matrix as

$$\begin{pmatrix} \cos \theta_d & \sin \theta_d \\ -\sin \theta_d & \cos \theta_d \end{pmatrix} \begin{pmatrix} \mu_{ud} & \mu_{u'd} \\ \mu_{ud'} & \mu_{u'd'} \end{pmatrix} \begin{pmatrix} \cos \theta_u & \sin \theta_u \\ -\sin \theta_u & \cos \theta_u \end{pmatrix} = \begin{pmatrix} \mu & 0 \\ 0 & \tilde{\mu} \end{pmatrix}, \quad (13)$$

and we use the Dirac masses μ and $\tilde{\mu}$ and the two rotation angles as input parameters in our analysis. We note that the numerical results in ref. [37] are obtained for the parameter choices $\theta_u = \theta_d = \pi/4$ and $\tilde{\mu} = \mu$, which in terms of the original superpotential parameters correspond to $\mu_{ud'} = -\mu_{u'd} = \mu$ and $\mu_{ud} = \mu_{u'd'} = 0$. While these choices might look *ad hoc*, they are in fact quite appropriate, because they make the dependence of the numerical results on $\mu_{ud'}$ – the parameter that determines the leading contributions from smuon loops to both a_μ and the Higgs-mass correction – more transparent.

3 Higgs-mass calculation in the EFT approach

For our calculation of the radiative corrections to the Higgs mass in the FSSM we adopt an EFT approach in which the effective theory valid below the scale M_S that characterizes the SUSY-particle masses is just the SM. Rather than computing the prediction for the Higgs mass from a full set of high-energy FSSM parameters, and then comparing it with the value measured at the LHC, we follow a more convenient procedure that uses the measured Higgs mass directly as an input parameter. From

the Higgs mass we extract the quartic Higgs coupling λ_{SM} at the EW scale, evolve it up to the SUSY scale with the RGEs of the SM, and then require that $\lambda_{\text{SM}}(M_S)$ coincide with the FSSM prediction for the quartic coupling of the lightest Higgs scalar. This procedure allows us to determine one of the FSSM parameters, such as, e.g., a common mass term for the stops.

We obtain a full one-loop prediction for the quartic coupling of the SM-like Higgs doublet Φ_h in the FSSM. Combined with the one-loop determination of the $\overline{\text{MS}}$ -renormalized parameters of the SM Lagrangian at the EW scale, and with the two-loop RGEs of the SM for the evolution up to the SUSY scale, this allows for the NLL resummation of the corrections to the SM-like Higgs mass. However, when they are available we use two-loop results for the determination of the SM parameters and three-loop RGEs for their evolution. While in the absence of a full two-loop calculation of the quartic coupling this cannot be claimed to improve the overall accuracy of the calculation, it does not degrade it either. Indeed, in the EFT approach the EW-scale and SUSY-scale sides of the calculation are separately free of large logarithmic corrections, and the inclusion of additional pieces in only one side does not entail the risk of spoiling crucial cancellations between large corrections.

We use the public code `mr` [41], based on the two-loop calculation of ref. [42], to determine the parameters of the SM Lagrangian – neglecting all Yukawa couplings except the top and bottom ones – in the $\overline{\text{MS}}$ renormalization scheme at the scale $Q_{\text{EW}} = M_t$. We take as input for the code a set of seven physical observables that we fix to their current PDG values [43], namely $G_F = 1.1663787 \times 10^{-5} \text{ GeV}^{-2}$, $M_h = 125.25 \text{ GeV}$, $M_Z = 91.1876 \text{ GeV}$, $M_W = 80.379 \text{ GeV}$, $M_t = 172.76 \text{ GeV}$, $M_b = 4.78 \text{ GeV}$ and $\alpha_s(M_Z) = 0.1179$. The remaining SM parameters that we need to determine are the tau Yukawa coupling and the Yukawa couplings of the second generation. For the leptons we take as input the physical masses $M_\tau = 1.776 \text{ GeV}$ and $M_\mu = 105.66 \text{ MeV}$, and obtain the $\overline{\text{MS}}$ Yukawa couplings directly at the scale $Q_{\text{EW}} = M_t$ via the one-loop relation [44]

$$g_\ell(Q_{\text{EW}}) = \sqrt{2\sqrt{2}G_F} M_\ell \left[1 + \frac{\alpha}{4\pi} \left(3 \ln \frac{M_\ell^2}{Q_{\text{EW}}^2} - 4 \right) + \delta^{\text{EW}}(Q_{\text{EW}}) \right], \quad (\ell = \tau, \mu), \quad (14)$$

where the EW correction stemming from the renormalization of G_F reads⁵

$$\delta^{\text{EW}}(Q_{\text{EW}}) = \frac{G_F}{8\pi^2\sqrt{2}} \left[3M_t^2 \left(\frac{1}{2} - \ln \frac{M_t^2}{Q_{\text{EW}}^2} \right) + \frac{M_h^2}{4} \right]. \quad (15)$$

For the second-generation quarks we take as input the $\overline{\text{MS}}$ -renormalized masses $m_c(m_c) = 1.27 \text{ GeV}$ and $m_s(2 \text{ GeV}) = 93 \text{ MeV}$ [43], which we evolve up to the scale Q_{EW} at the NLL level in QCD by means of eqs. (D4) and (D5) of ref. [45]. We then include the one-loop QED and EW corrections according to:

$$g_q(Q_{\text{EW}}) = \sqrt{2\sqrt{2}G_F} m_q(Q_{\text{EW}}) \left[1 + \frac{3\alpha}{4\pi} Q_q^2 \ln \frac{m_q^2}{Q_{\text{EW}}^2} + \delta^{\text{EW}}(Q_{\text{EW}}) \right], \quad (q = c, s), \quad (16)$$

⁵We use here an approximate formula from ref. [44] which includes only the contributions from the top Yukawa coupling and the quartic Higgs coupling. Anyway, the overall effect of this correction is only about 0.5%.

where Q_q is the electric charge of the quark q , and $\delta^{\text{EW}}(Q_{\text{EW}})$ is given in eq. (15).

For the evolution of the SM couplings from the EW scale to the SUSY scale we use the set of three-loop RGEs provided in refs. [46–48], which however include only the third-generation Yukawa couplings. For the couplings of the second generation we use 2-loop RGEs from ref. [49], which are sufficient to our aim of a NLL resummation of the large logarithmic effects. Following refs. [46–48], we neglect the tiny contributions of the Yukawa couplings of the first two generations within the beta functions, apart from the overall multiplicative factors.

Once the SM couplings are evolved up to the SUSY scale, they are matched to the corresponding FSSM couplings, which enter the prediction for the quartic Higgs coupling. Since the Yukawa couplings enter only from one loop onwards, the tree-level relations in eqs. (9) and (10) are in principle sufficient for the NLL calculation of the Higgs-mass prediction. It is nevertheless convenient to take into account the one-loop “SUSY-QCD” corrections controlled by the strong gauge coupling g_3 to the relation between the quark Yukawa couplings of the SM and those of the FSSM. This amounts to redefining the quark Yukawa couplings as

$$\hat{g}_q(Q) = \frac{g_q(Q)}{1 - \Delta g_q}, \quad (q = t, b, c, s), \quad (17)$$

where $g_q(Q)$ are given in eqs. (9) and (10), and the correction Δg_q reads

$$\Delta g_q = -\frac{g_3^2}{12\pi^2} \left[1 + \ln \frac{M_3^2}{Q^2} + \tilde{F}_6 \left(\frac{M_{\tilde{q}_L}}{M_3} \right) + \tilde{F}_6 \left(\frac{M_{\tilde{q}_R}}{M_3} \right) - \frac{X_q}{M_3} \tilde{F}_9 \left(\frac{M_{\tilde{q}_L}}{M_3}, \frac{M_{\tilde{q}_R}}{M_3} \right) \right], \quad (18)$$

where: M_3 is the gluino mass; $M_{\tilde{q}_L}$ and $M_{\tilde{q}_R}$ are the soft SUSY-breaking mass parameters for the scalar partners of the left- and right-handed quarks, respectively; X_q are the left-right mixing parameters given in eqs. (11) and (12); the functions $\tilde{F}_6(x)$ and $\tilde{F}_9(x, y)$ are defined in the appendix A of ref. [50]. As was recently discussed in a systematic way in refs. [51, 52] for the case of the MSSM, the use of the corrected Yukawa couplings \hat{g}_q absorbs (“resums”) in the one-loop contribution to the quartic Higgs coupling a tower of higher-order corrections involving powers of $g_3^2 X_q/M_S$, where M_S denotes the scale of the squark and gluino masses. In case X_q/M_S contains terms that are numerically enhanced (e.g., by a large ratio of vevs), such “resummation” ensures a better convergence of the perturbative expansion. We note that contributions to Δg_q controlled by the Yukawa couplings and by the EW gauge couplings also exist, but we do not consider them in our study as they are generally subdominant to those controlled by the strong gauge coupling.⁶

For what concerns the Yukawa couplings of the leptons, the only one-loop corrections that can be enhanced by a large ratio of vevs are those controlled by the EW gauge couplings. In particular, in the FSSM the muon Yukawa coupling is subject to corrections enhanced by $v_u/v'_d = \tan \beta \tan \beta_d$,

⁶This is not necessarily the case for the $\tan \beta$ -enhanced $\mathcal{O}(g_t^2)$ contribution to Δg_b , but in the FSSM that contribution depends on μ_{ud} , and vanishes in the scenarios considered in this paper.

which, following ref. [37], we absorb in the coupling via the redefinition

$$\hat{g}_\mu(Q) = \frac{g_\mu(Q)}{1 + \epsilon_\ell \tan \beta \tan \beta_d}, \quad (19)$$

where the explicit formula for ϵ_ℓ is given in ref. [37]. Being controlled by the EW gauge couplings, the term ϵ_ℓ is itself of $\mathcal{O}(10^{-3})$ only, but the overall correction in eq. (19) is not negligible for the values of $\tan \beta \tan \beta_d$ in the few-hundred range that – as will be seen in section 4 – are relevant to the solution of the $(g - 2)_\mu$ anomaly. For the tau Yukawa coupling, on the other hand, the analogous EW correction is enhanced at most by $\tan \beta$, still with an $\mathcal{O}(10^{-3})$ prefactor. We can thus neglect this correction in our analysis and define $\hat{g}_\tau(Q) = g_\tau(Q)$.

At this stage, an issue with the SUSY-QCD correction to the strange Yukawa coupling might be worth mentioning. Inspection of eq. (11) shows that the trilinear Higgs-squark coupling X_s entering the correction Δg_s in eq. (18) includes the term $\mu_{ud'} \tan \beta \tan \beta_d$, which is the same combination of parameters entering the dominant higgsino-gaugino-smuon contribution to a_μ . Being controlled by the strong gauge coupling, Δg_s is of the order of $10^{-2} \times \tan \beta \tan \beta_d$, and can easily reach and even exceed unity for the values of $\tan \beta \tan \beta_d$ relevant to the solution of the $(g - 2)_\mu$ anomaly. A particularly obnoxious situation occurs when $\Delta g_s \simeq 1$, in which case the corrected coupling \hat{g}_s in eq. (17) blows up, leading to unphysically large corrections and numerical instabilities. Since the higgsino-gaugino-smuon contribution to a_μ takes the sign required to account for the observed anomaly when $(\mu_{ud'} M_2) > 0$, the condition $\Delta g_s \simeq 1$ requires $(M_2 M_3) < 0$, as is the case in scenarios with anomaly mediated SUSY breaking (AMSB).⁷ Even in scenarios where the SUSY-QCD correction suppresses \hat{g}_s rather than enhancing it, the condition $|\Delta g_s| > 1$ means that the radiative correction to the strange quark mass arising from squark-gluino diagrams exceeds, and possibly by far, the tree-level contribution. This would complicate any attempt (which we do not make in this paper anyway) to obtain a realistic flavor structure for the quark sector of the FSSM. We remark that a trivial way out from this complication would consist in applying the four-doublet construction only to the lepton sector, and have all of the quarks receive their masses from the doublets H_u and H_d .

We now describe the one-loop matching condition for the quartic Higgs coupling in the FSSM. At a renormalization scale Q of the order of the SUSY particle masses, it takes the form

$$\lambda_{\text{SM}}(Q) = \frac{1}{4} [g^2(Q) + g'^2(Q)] \cos^2 2\tilde{\beta} + \Delta\lambda^{\text{reg}} + \Delta\lambda^{\tilde{f}} + \Delta\lambda^H + \Delta\lambda^X, \quad (20)$$

where g and g' are the EW gauge couplings. Again, we see that the tree-level matching condition differs from the analogous result in the MSSM only via the replacement of β with $\tilde{\beta}$. We assume that the EW gauge couplings are SM parameters renormalized in the $\overline{\text{MS}}$ scheme, i.e. we use directly the values obtained via RG evolution from the EW scale. Following ref. [50], we also assume that the

⁷In the case of the MSSM with AMSB, the interplay between contributions to $(g - 2)_\mu$ and SUSY-QCD corrections to the quark couplings was discussed earlier in ref. [53].

angle $\tilde{\beta}$ is renormalized in such a way as to remove entirely the wave-function-renormalization (WFR) contributions that mix the SM-like Higgs doublet with the heavy doublets.

The one-loop correction $\Delta\lambda^{\text{reg}}$ accounts for the fact that SUSY determines the quartic Higgs coupling in the $\overline{\text{DR}}$ scheme, whereas λ_{SM} and the EW gauge couplings in eq. (20) are defined in the $\overline{\text{MS}}$ scheme. It reads [50]

$$(4\pi)^2\Delta\lambda^{\text{reg}} = -\frac{g'^4}{4} - \frac{g^2g'^2}{2} - \left(\frac{3}{4} - \frac{\cos^2 2\tilde{\beta}}{6}\right)g^4. \quad (21)$$

Concerning the remaining one-loop threshold corrections in eq. (20), $\Delta\lambda^{\tilde{f}}$ arises from diagrams that involve the sfermions, $\Delta\lambda^H$ from diagrams that involve the heavy Higgs doublets, and $\Delta\lambda^\chi$ from diagrams that involve higgsinos and EW gauginos. Each of these three corrections can in turn be decomposed as a sum of three terms:

$$\Delta\lambda^p = \Delta\lambda^{p, \text{1PI}} + \Delta\lambda^{p, \text{WFR}} + \Delta\lambda^{p, \text{gauge}}, \quad (p = \tilde{f}, H, \chi). \quad (22)$$

The first term on the r.h.s. of the equation above denotes the contribution of one-particle-irreducible (1PI) diagrams with particles of type p in the loop and four external Higgs fields; the second term involves the contributions of particles of type p to the WFR of the Higgs field, which multiply the tree-level quartic coupling; the third term contains additional corrections stemming from the fact that the SUSY prediction for the quartic Higgs coupling involves the gauge couplings of the MSSM, whereas we interpret the gauge couplings in the tree-level part of eq. (20) as SM parameters.

To obtain the four-Higgs diagrams entering $\Delta\lambda^{p, \text{1PI}}$ and the self-energy diagrams entering $\Delta\lambda^{p, \text{WFR}}$ we use the general results from ref. [39] (see sections B.3 and B.1.1, respectively, of that paper). This saves us the trouble of actually calculating one-loop Feynman diagrams, but requires that we adapt to the case of the FSSM the notation of ref. [39] for masses and interactions of scalars and fermions in a general renormalizable theory. The additional corrections in $\Delta\lambda^{p, \text{gauge}}$ can instead be obtained by adapting the MSSM shifts of the gauge couplings, see eqs. (19) and (20) of ref. [50], to the FSSM case of two Dirac higgsinos with masses μ and $\tilde{\mu}$, and three heavy Higgs doublets with masses M_{H_i} .

We find that the sfermion contribution to the quartic Higgs coupling, $\Delta\lambda^{\tilde{f}}$, has the same form as the corresponding contribution in the MSSM, see eq. (A1) of ref. [54], trivially extended to the case of non-zero Yukawa couplings for the second generation. However, in the FSSM case the angle β is replaced by $\tilde{\beta}$, and the trilinear Higgs-sfermion couplings X_f are those given in our eqs. (11) and (12). In contrast, the heavy-Higgs and higgsino-gaugino contributions differ from the corresponding MSSM contributions, due to the extended Higgs/higgsino sector of the FSSM. The full formulas for $\Delta\lambda^H$ and, especially, $\Delta\lambda^\chi$ for generic values of all relevant parameters are lengthy and not particularly illuminating, therefore we make them available on request in electronic form. In the following we provide instead explicit results for all three contributions in the simplified FSSM scenario that we will use in section 4 to explore the interplay between the prediction for the Higgs mass and the solution of the $(g-2)_\mu$ anomaly.

In the sfermion sector, we assume degenerate soft SUSY-breaking mass parameters $M_{\tilde{f}_{12}}$ for all first- and second-generation sfermions and $M_{\tilde{f}_3}$ for all third-generation sfermions. The sfermion contribution to the quartic Higgs coupling then becomes:

$$\begin{aligned}
(4\pi)^2 \Delta\lambda^{\tilde{f}} &= \left[2(3\hat{g}_c^4 + 3\hat{g}_s^4 + \hat{g}_\mu^4) + \frac{\bar{g}^2}{2}(3\hat{g}_c^2 - 3\hat{g}_s^2 - \hat{g}_\mu^2) \cos 2\tilde{\beta} + \frac{2}{3} \left(g^4 + \frac{5}{3} g'^4 \right) \cos^2 2\tilde{\beta} \right] \ln \frac{M_{\tilde{f}_{12}}^2}{Q^2} \\
&+ \left[2(3\hat{g}_t^4 + 3\hat{g}_b^4 + \hat{g}_\tau^4) + \frac{\bar{g}^2}{2}(3\hat{g}_t^2 - 3\hat{g}_b^2 - \hat{g}_\tau^2) \cos 2\tilde{\beta} + \frac{1}{3} \left(g^4 + \frac{5}{3} g'^4 \right) \cos^2 2\tilde{\beta} \right] \ln \frac{M_{\tilde{f}_3}^2}{Q^2} \\
&+ \sum_{f=c,s,\mu} \hat{g}_f^2 N_c \frac{X_f^2}{M_{\tilde{f}_{12}}^2} \left[2\hat{g}_f^2 \left(1 - \frac{X_f^2}{12 M_{\tilde{f}_{12}}^2} \right) + \frac{\bar{g}^2}{12} \cos 2\tilde{\beta} (3c_f - \cos 2\tilde{\beta}) \right] \\
&+ \sum_{f=t,b,\tau} \hat{g}_f^2 N_c \frac{X_f^2}{M_{\tilde{f}_3}^2} \left[2\hat{g}_f^2 \left(1 - \frac{X_f^2}{12 M_{\tilde{f}_3}^2} \right) + \frac{\bar{g}^2}{12} \cos 2\tilde{\beta} (3c_f - \cos 2\tilde{\beta}) \right], \quad (23)
\end{aligned}$$

where \hat{g}_f are the loop-corrected Yukawa couplings defined in eqs. (17)–(19), the trilinear Higgs-sfermion couplings X_f are given in eqs. (11) and (12), and we defined: $\bar{g}^2 \equiv g^2 + g'^2$; $N_c = 3$ for quarks and $N_c = 1$ for leptons; $c_f = 1$ for $f = c, t$ and $c_f = -1$ for $f = s, b, \mu, \tau$.

In the Higgs sector, we assume that there is no mixing between the three doublets Φ_H , Φ'_u and Φ'_d , i.e. we take the 3×3 matrix R_H that rotates the heavy doublets from the Higgs basis to the basis of mass eigenstates to be the identity. We also assume a common mass M_H for all three of the doublets. The heavy-Higgs contribution to the quartic Higgs coupling then becomes:

$$(4\pi)^2 \Delta\lambda^H = \frac{1}{64} \left[16g^4 + 8g'^4 + 7\bar{g}^4 - 4(\bar{g}^4 - 2g'^4) \cos 4\tilde{\beta} - 3\bar{g}^4 \cos 8\tilde{\beta} \right] \ln \frac{M_H^2}{Q^2} - \frac{3\bar{g}^4}{16} \sin^2 4\tilde{\beta}. \quad (24)$$

Finally, for the higgsino sector we consider the same scenario as in ref. [37], namely $\theta_u = \theta_d = \pi/4$ and $\tilde{\mu} = \mu$, so that our choices for μ determine directly the relevant parameter $\mu_{ud'}$. We also assume a common mass M_χ for the higgsinos and the EW gauginos, i.e. $M_\chi \equiv M_1 = M_2 = \mu = \tilde{\mu}$. The higgsino-gaugino contribution to the quartic Higgs coupling then becomes:

$$\begin{aligned}
(4\pi)^2 \Delta\lambda^\chi &= -\frac{1}{24} \left[47g^4 + 12g^2g'^2 + 13g'^4 + (11g^4 - 12g^2g'^2 + g'^4) \cos 4\tilde{\beta} \right] \ln \frac{M_\chi^2}{Q^2} \\
&- \frac{1}{48} \left[93g^4 + 50g^2g'^2 + 27g'^4 + (3g^4 - 10g^2g'^2 - 3g'^4) \cos 4\tilde{\beta} \right. \\
&\quad + 2(3g^4 + 4g^2g'^2 + g'^4) \cos 4\tilde{\beta} \sin 2\tilde{\beta} \sin(\beta_d - \beta_u) \\
&\quad - 2(45g^4 + 28g^2g'^2 + 15g'^4) \sin 2\tilde{\beta} \sin(\beta_d - \beta_u) \\
&\quad \left. - 2(3g^4 + 2g^2g'^2 + g'^4) \sin^2 2\tilde{\beta} \cos 2(\beta_d - \beta_u) \right]. \quad (25)
\end{aligned}$$

The inspection of eqs. (24) and (25) shows that the heavy-Higgs and higgsino-gaugino contributions to the quartic Higgs coupling all involve four powers of the EW gauge couplings. Their numerical impact is thus going to be modest, unless there is a significant hierarchy between the matching scale Q and the mass scales M_H and M_χ . In contrast, the sfermion contributions include terms depending on the top Yukawa coupling \hat{g}_t , which is of $\mathcal{O}(1)$, as well as terms involving other Yukawa couplings in which the smallness of \hat{g}_f can be compensated by a large ratio $X_{\tilde{f}}/M_{\tilde{f}}$. In particular, the contribution that will be relevant to our discussion in section 4 is the one involving the muon Yukawa coupling, which for $\tan\tilde{\beta} \gg 1$ reads

$$(4\pi)^2 \Delta\lambda^{\tilde{\mu}} \approx \hat{g}_\mu^2 \frac{X_\mu^2}{M_{\tilde{\mu}}^2} \left[2\hat{g}_\mu^2 \left(1 - \frac{X_\mu^2}{12 M_{\tilde{\mu}}^2} \right) + \frac{\bar{g}^2}{6} \right], \quad (26)$$

where \hat{g}_μ and X_μ are defined in eqs. (19) and (11), respectively, and by $M_{\tilde{\mu}}$ we denote a common mass parameter for the scalar partners of the left- and right-handed muons (note that $M_{\tilde{\mu}} = M_{\tilde{f}_{12}}$ in our simplified scenario). We recall that X_μ contains a term enhanced by $\tan\beta \tan\beta_d$, and indeed when the combination $(\mu_{ud'}/M_{\tilde{\mu}}) \tan\beta \tan\beta_d$ is large enough to overcome the smallness of \hat{g}_μ the smuon contribution to the quartic Higgs coupling becomes large and negative. As will be discussed in the next section, an increased positive contribution from a different SUSY sector is then necessary to maintain the correct prediction for the SM-like Higgs mass. In particular, the stop contribution to the quartic Higgs coupling is dominated by the terms involving four powers of \hat{g}_t , which read

$$(4\pi)^2 \Delta\lambda^{\tilde{t}} \approx 6\hat{g}_t^4 \left(\ln \frac{M_t^2}{Q^2} + \frac{X_t^2}{M_t^2} - \frac{X_t^4}{12 M_t^4} \right), \quad (27)$$

where by $M_{\tilde{t}}$ we denote a common mass parameter for the scalar partners of the left- and right-handed top (with $M_{\tilde{t}} = M_{\tilde{f}_3}$ in our simplified scenario) and X_t is defined in eq. (12). The non-logarithmic terms in eq. (27) are maximized for $X_t = \sqrt{6} M_{\tilde{t}}$, and a further increase in $\Delta\lambda^{\tilde{t}}$ can arise from the logarithmic term when the stop mass is pushed to higher values.

Finally, we note that in the FSSM the strange-squark contribution to the quartic Higgs coupling is subject to the same enhancement by $\tan\beta \tan\beta_d$ as the smuon contribution. However, the strange-squark contribution is generally subdominant, because the strange Yukawa coupling is smaller than the muon one at the matching scale. A possible exception is the pathological case discussed earlier, in which a SUSY correction $\Delta g_s \simeq 1$ in eq. (17) causes the strange coupling \hat{g}_s to blow up.

4 Higgs-mass constraints and $(g-2)_\mu$

We now investigate the interplay between the constraints on the FSSM parameter space arising from the solution to the $(g-2)_\mu$ anomaly and those arising from the prediction for the quartic Higgs coupling. To keep the number of independent parameters manageable, we employ the simplifying assumptions for the SUSY mass spectrum described in the previous section. Namely, we adopt common mass scales $M_{\tilde{f}_{12}}$, $M_{\tilde{f}_3}$, M_H and M_χ for first/second-generation sfermions, third-generation sfermions, heavy Higgs

bosons and higgsinos/EW-gauginos, respectively. Note that we will henceforth refer to the common mass parameters for the sfermions as $M_{\tilde{\mu}}$ and $M_{\tilde{t}}$, because those are the masses of the first/second and third generation, respectively, that are most relevant to our discussion of $(g-2)_\mu$ and of the Higgs mass constraint. We will also keep referring to the collective scale of the SUSY particle masses as M_S . A further simplifying assumption consists in neglecting all contributions from the ‘‘primed’’ Yukawa couplings for the third generation, namely y'_t , y'_b , and y'_τ in eq. (3), as they do not give rise to contributions enhanced by large ratios of vevs. Finally, we take directly as input the stop mixing parameter X_t , which enters the stop contribution to $\Delta\lambda^{\tilde{f}}$, thereby fixing via eq. (12) the soft SUSY-breaking trilinear coupling A_t as a function of the other parameters.⁸ For the remaining trilinear couplings in eq. (5), which are not involved in any enhanced contributions to $\Delta\lambda^{\tilde{f}}$, we assume $A'_c = A'_s = A'_\mu = 0$ and $A_b = A_\tau = A_t$.

In the FSSM, $(g-2)_\mu$ receives contributions from one-loop diagrams involving smuons, higgsinos and EW gauginos that are enhanced by $v_u/v'_d = \tan\beta \tan\beta_d$. Explicit formulas for these contributions with full dependence on the relevant FSSM parameters – but under the assumption $M_S \gg v$, i.e. neglecting the effects of EWSB on the SUSY-particle masses and mixing – are given ref. [37]. In the simplified scenario considered here, where in particular $M_{\tilde{\mu}_L} = M_{\tilde{\mu}_R} = M_{\tilde{\mu}}$, $\mu = \tilde{\mu} = M_1 = M_2 = M_\chi$, and $\theta_u = \theta_d = \pi/4$, they reduce to

$$\Delta a_\mu^{\text{FSSM}} = \frac{1}{192\pi^2} \frac{M_\mu^2}{M_{\tilde{\mu}}^2} \frac{\tan\beta \tan\beta_d}{1 + \epsilon_\ell \tan\beta \tan\beta_d} \left[g'^2 f_1(M_\chi^2/M_{\tilde{\mu}}^2) + 5g^2 f_2(M_\chi^2/M_{\tilde{\mu}}^2) \right], \quad (28)$$

where

$$f_1(x) = \frac{6x}{(1-x)^4} \left[7 + 4x - 11x^2 + 2(1 + 6x + 2x^2) \ln x \right], \quad f_1(1) = 1, \quad (29)$$

$$f_2(x) = \frac{6}{5(1-x)^4} \left[4 + 11x - 16x^2 + x^3 + 2x(7 + 2x) \ln x \right], \quad f_2(1) = 1, \quad (30)$$

and ϵ_ℓ represents the correction to the muon Yukawa coupling introduced in eq. (19). In our simplified scenario, the formula given in ref. [37] for this correction reduces to

$$\epsilon_\ell = \frac{g'^2}{64\pi^2} g_1(M_\chi^2/M_{\tilde{\mu}}^2) - \frac{3g^2}{64\pi^2} g_2(M_\chi^2/M_{\tilde{\mu}}^2), \quad (31)$$

where

$$g_1(x) = \frac{2x}{(1-x)^2} \left[3 - 3x + (1+2x) \ln x \right], \quad g_2(x) = \frac{2x}{(1-x)^2} \left[-1 + x - \ln x \right], \quad g_1(1) = g_2(1) = 1. \quad (32)$$

The requirement that the smuon-higgsino-gaugino contribution in eq. (28) provide the solution of the $(g-2)_\mu$ anomaly corresponds to $\Delta a_\mu^{\text{FSSM}} = 251 \times 10^{-11}$. For a given choice of values of $M_{\tilde{\mu}}$ and M_χ , this can be solved for the product $\tan\beta \tan\beta_d$, which in turn determines the enhancement of the mixing

⁸Our assumption $y'_t = y'_b = y'_\tau = 0$ implies that the second term on the r.h.s. of each line of eq. (12) vanishes.

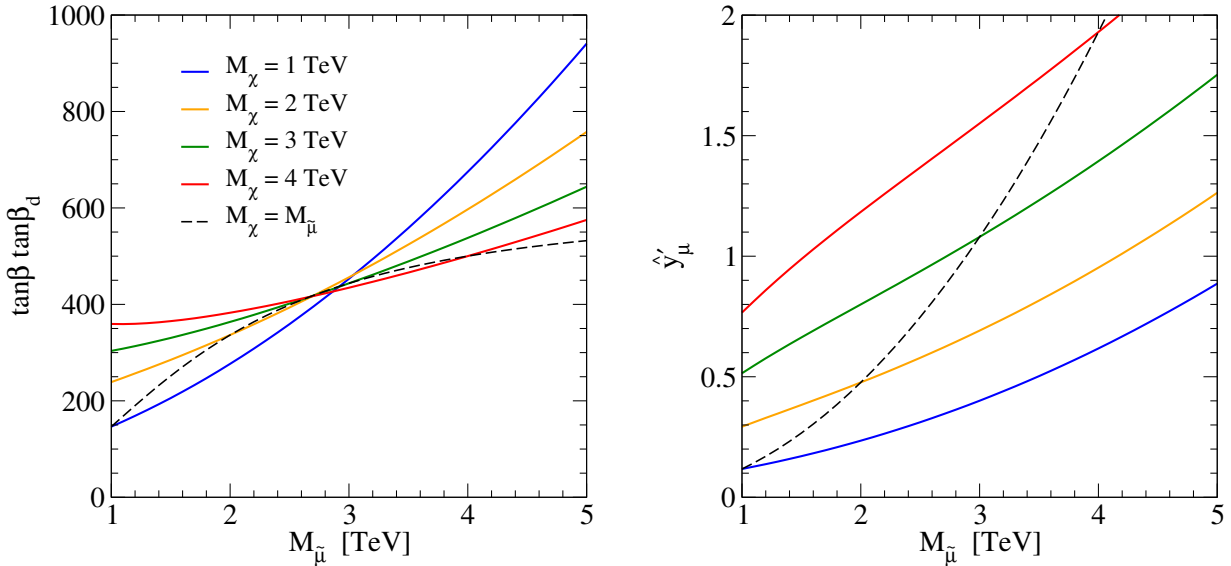


Figure 1: *Left:* Values of the product $\tan\beta \tan\beta_d$ that result in $\Delta a_\mu^{\text{FSSM}} = 251 \times 10^{-11}$, as a function of the smuon mass and for different values of a common mass for higgsinos and EW gauginos. *Right:* Values of the loop-corrected muon Yukawa coupling of the FSSM that correspond to the solution for $\tan\beta \tan\beta_d$ shown in the left plot. The meaning of the lines is the same as in the left plot.

parameter X_μ entering the smuon contribution to $\Delta\lambda^{\tilde{f}}$. The requirement that the FSSM prediction for the quartic Higgs coupling agree with the value obtained by evolving λ_{SM} from the EW scale up to the SUSY scale can then be used to determine one of the remaining FSSM parameters. In particular, it seems reasonable to determine one of the parameters that enter the dominant contribution to $\Delta\lambda^{\tilde{f}}$, i.e. the one involving the stops.

To set the stage for our discussion, in the left plot of figure 1 we show the values of $\tan\beta \tan\beta_d = v_u/v_d$ that result in the desired smuon-higgsino-gaugino contribution to a_μ , as a function of the common smuon mass $M_{\tilde{\mu}}$ and for different values of M_χ . In particular, the blue, yellow, green and red solid lines correspond to gaugino and higgsino masses of 1, 2, 3 and 4 TeV, respectively, while the black dashed line corresponds to the choice $M_\chi = M_{\tilde{\mu}}$. The EW gauge couplings entering eqs. (28) and (31) are evaluated at the scale $Q = M_{\tilde{\mu}}$, but we found qualitatively similar results for any choice of scale in the few-TeV range. The plot shows that the values of $\tan\beta \tan\beta_d$ necessary to obtain $\Delta a_\mu^{\text{FSSM}} = 251 \times 10^{-11}$ are typically in the few-hundreds range, and can reach as much as 940 for the largest considered value of $M_{\tilde{\mu}}$.

In the right plot of figure 1 we show instead the values of the loop-corrected muon Yukawa coupling of the FSSM at the scale $Q = M_{\tilde{\mu}}$,

$$\hat{y}'_\mu(M_{\tilde{\mu}}) = \frac{g_\mu(M_{\tilde{\mu}})/(\cos\tilde{\beta}\cos\beta_d)}{1 + \epsilon_\ell \tan\beta \tan\beta_d}, \quad (33)$$

that correspond to the solutions for $\tan\beta \tan\beta_d$ shown in the left plot. The FSSM coupling \hat{y}'_μ controls the higgsino-muon-smuon and Higgs-smuon-smuon interactions involved in the smuon-higgsino-

gaugino contributions to a_μ , as well as the Higgs-smuon-smuon interaction involved in the smuon contribution to the quartic Higgs coupling. The solid and dashed lines have the same meaning as in the left plot. We set $\tan\beta_u = 100$ in order to determine the angle $\tilde{\beta}$ entering eq. (33), but the results are essentially independent of this choice as long as $\tan\beta_u \gg 1$. The plot shows that larger values of \hat{y}'_μ are required to obtain the desired contribution to a_μ when the smuon mass increases, due to the $M_{\tilde{\mu}}^{-2}$ suppression in eq. (28). Also, for a fixed value of the smuon mass, larger values of \hat{y}'_μ are required when M_χ increases. We remark that, while the considered values of $\hat{y}'_\mu(M_{\tilde{\mu}})$ are all perturbative, further constraints on this scenario could arise if we required that \hat{y}'_μ remain perturbative all the way up to the GUT scale.

In the presence of a large muon Yukawa coupling, the smuon contribution to the quartic Higgs coupling in eq. (26) is dominated by a negative term

$$(4\pi)^2 \Delta\lambda^{\tilde{\mu}} \approx -\frac{\hat{y}'_\mu{}^4}{6} \left(\frac{M_\chi}{M_{\tilde{\mu}}}\right)^4, \quad (34)$$

and can become substantial. An increased positive contribution from loops involving the remaining SUSY particles is then needed to satisfy the constraint arising from the measured value of the Higgs mass. As mentioned in the previous section, such positive contribution can most easily come from the stops, which themselves have generally large couplings to the SM-like Higgs boson.

In figure 2 we plot the values of the common stop mass $M_{\tilde{t}}$ that are required to obtain the correct prediction for the quartic Higgs coupling in FSSM scenarios where the smuon-higgsino-gaugino loops provide the desired contribution to $(g-2)_\mu$. We set the matching scale Q in eqs. (20) and (23)–(25) equal to the stop mass, as this ensures a full NNLL resummation of the large logarithmic corrections controlled by the top Yukawa coupling. All of the running couplings entering our calculation are then computed at the scale $Q = M_{\tilde{t}}$. The common mass parameters for the smuons, $M_{\tilde{\mu}}$, and for higgsinos and EW gauginos, M_χ , are varied as in figure 1 (note that we add a purple solid line for $M_\chi = 3.5$ TeV). We fix $\tan\tilde{\beta} = 20$, thus ensuring that the tree-level prediction for λ_{SM} in eq. (20) is essentially saturated, and $\tan\beta_u = 100$ (the latter choice has little impact on our results as long as $\tan\beta_u \gg 1$). In contrast, $\tan\beta_d$ is computed in each point from the requirement that $\Delta a_\mu^{\text{FSSM}} = 251 \times 10^{-11}$. The stop mixing parameter is fixed to the value $X_t = \sqrt{6} M_{\tilde{t}}$ that maximizes the one-loop stop contribution to the quartic Higgs coupling, see eq. (27). Our choices for $\tan\tilde{\beta}$ and X_t ensure that the values of the stop mass shown in figure 2 are about the minimal ones that provide the correct prediction for the Higgs mass (in other words, different choices for these parameters would result in an overall upward shift of all lines in the figure). For the remaining free parameters of our simplified FSSM scenario we choose $M_H = M_3 = M_\chi$. The choice of the common mass parameter for the heavy Higgs doublets has only a small impact on our results, because the corresponding contributions to the quartic Higgs coupling, see eq. (24), all involve four powers of the EW gauge couplings. The choice of the gluino mass affects our calculation only through the corrections to the quark Yukawa couplings in eqs. (17) and (18), and its qualitative impact on our results is generally not substantial. Our choice of a positive value

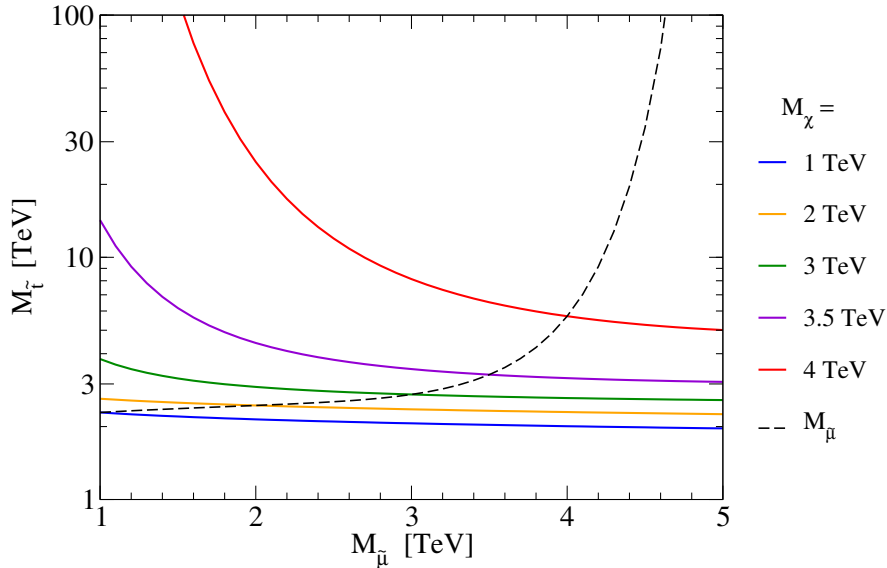


Figure 2: Values of the stop mass $M_{\tilde{t}}$ that result in the correct prediction for the Higgs mass when $\tan \beta_d$ is fixed by the requirement that the smuon-higgsino-gaugino contribution solve the $(g-2)_\mu$ anomaly, as a function of the smuon mass $M_{\tilde{\mu}}$ and for different values of the common higgsino/gaugino mass M_χ . The remaining free parameters of our simplified FSSM scenario are fixed as $\tan \tilde{\beta} = 20$, $\tan \beta_u = 100$, $X_t = \sqrt{6} M_{\tilde{t}}$ and $M_H = M_3 = M_\chi$.

for the ratio X_t/M_3 enhances the loop-corrected top Yukawa coupling \hat{y}_t , whereas a negative value would suppress it and require somewhat heavier stops in order to satisfy the Higgs-mass constraint. However, we recall that, for positive values of X_s/M_3 (i.e., negative values of $\mu_{ud'}/M_3$), a fine-tuned choice of parameters such that $\Delta g_s \simeq 1$ might lead the strange Yukawa coupling to blow up, resulting in a large negative strange-squark contribution to the quartic Higgs coupling.

Figure 2 shows that there are regions of the FSSM parameter space in which the interplay between the requirements of a suitable contribution to a_μ and of a correct prediction for the quartic Higgs coupling implies a strongly hierarchical SUSY spectrum, with the stops being significantly heavier than smuons, higgsinos and EW gauginos. Unsurprisingly, in the scenario with $M_\chi = M_{\tilde{\mu}}$ (the black dashed line) this happens for the largest values of $M_{\tilde{\mu}}$, when a large muon Yukawa coupling \hat{y}'_μ is needed to counteract the $M_{\tilde{\mu}}^{-2}$ suppression of the smuon-higgsino-gaugino contribution to a_μ , see eq. (28), and results in a large negative contribution to the quartic Higgs coupling, see eq. (34). Moreover, the left ends of the red, purple and (to a lesser extent) green lines show that very heavy stops may be needed also at lower values of $M_{\tilde{\mu}}$ – where figure 1 shows that $\hat{y}'_\mu \lesssim 1$ is sufficient for the solution of the $(g-2)_\mu$ anomaly – if the smuon contribution to the quartic Higgs coupling in eq. (34) is enhanced by a large ratio $M_\chi/M_{\tilde{\mu}}$. Even in regions of the parameter space where the SUSY spectrum is not strongly hierarchical, such as the right end of the red line in figure 2, the minimal value of $M_{\tilde{t}}$ required to obtain the correct Higgs-mass prediction can be significantly higher than the 2–3 TeV typically found in the MSSM. Finally, we find that for $M_\chi \lesssim 2$ TeV (the yellow and blue lines) the smuon contribution to the quartic Higgs coupling never becomes substantial in our simplified scenario: at

low $M_{\tilde{\mu}}$ – where the desired contribution to $(g - 2)_{\mu}$ is obtained with $\hat{y}'_{\mu} \lesssim 0.5$ – there is little or no enhancement from the $(M_{\chi}/M_{\tilde{\mu}})^4$ term in eq. (34), whereas at high $M_{\tilde{\mu}}$ the corresponding suppression wins over the enhancement of \hat{y}'_{μ} . The mild residual dependence of the yellow and blue lines on $M_{\tilde{\mu}}$ and M_{χ} stems from the terms controlled by the EW gauge couplings in eqs. (23)–(25).

We remark that in our simplified FSSM scenario, where $M_{\chi} \equiv M_1 = M_2 = \mu = \tilde{\mu}$, the condition $M_{\chi}/M_{\tilde{\mu}} > 1$ implies that the LSP is a heavy sneutrino, which is generally disfavored by Dark Matter considerations [55, 56]. While a detailed study of Dark Matter constraints on the FSSM parameter space is beyond the scope of our paper, using the general formula for $\Delta\lambda^x$ we verified that very heavy stops may be required even in scenarios where the LSP is always an EW gaugino. In particular, if we fix $M_1 = 1$ TeV and $M_{\chi} \equiv M_2 = \mu = \tilde{\mu}$ we still find a rise in the stop mass similar to the black dashed line in figure 2 for $M_{\tilde{\mu}} = M_{\chi} \gtrsim 4$ TeV. However, we no longer see the rise at low $M_{\tilde{\mu}}$ in the lines corresponding to larger M_{χ} , because in this region the desired contribution to $(g - 2)_{\mu}$ is obtained with lower values of \hat{y}'_{μ} than in the case of degenerate gaugino masses.

It is interesting to note that, in contrast to what we find for the FSSM, in the MSSM the combination of the constraints from the Higgs-mass prediction and from $(g - 2)_{\mu}$ can yield *upper* bounds on the stop masses, see e.g. ref. [57]. Indeed, in the absence of large and negative contributions from other sectors, a large and positive contribution from heavy-stop loops to the prediction for λ_{SM} must be compensated for by a suppression of the tree-level prediction via a lower value of $\tan\beta$. However, in the MSSM this would also suppress the smuon-higgsino-gaugino contribution to a_{μ} , jeopardizing the solution of the $(g - 2)_{\mu}$ anomaly. The interplay of the two constraints is different in the FSSM, because in this model the ratio of vevs that determines the tree-level prediction for λ_{SM} can be varied independently of the ratio of vevs that enhances the smuon-higgsino-gaugino contribution to a_{μ} .

Finally, we remark that, when the stops are an order of magnitude (or more) heavier than the other SUSY particles, our calculation of the Higgs mass loses accuracy, and we would need to build a two-step EFT setup in which the stops are separately integrated out of the FSSM at a scale comparable with their mass. However, the aim of our study is not a precise determination of masses that, for the time being, are beyond experimental reach, but rather a qualitative insight on the structure of this heavy-SUSY model.⁹ The possibility of a hierarchical mass spectrum has obvious implications for the prospects of probing the FSSM at future colliders, and from the model-building point of view it might also complicate any attempt to devise a suitable mechanism of SUSY breaking.

⁹For the same reason we do not take into account the theoretical uncertainties of our predictions for λ_{SM} and a_{μ} .

5 Conclusions

A scenario for particle physics that is now looking increasingly plausible is the one where new physics manifests itself in one or more deviations from the SM predictions for rare processes or precision observables, but the BSM particles responsible for those deviations are too heavy to be discovered at the LHC. In this case, all possible clues should be exploited to unravel the structure of the heavy BSM sector, also to guide the searches for the new particles at future colliders. If a model that aims to explain the observed anomalies is supersymmetric, it will generally involve a prediction for the quartic coupling λ_{SM} of the SM-like Higgs boson. Since all of the new particles in the SUSY model affect λ_{SM} through radiative corrections, its prediction can reveal correlations between the sectors of the model that are involved in the observed anomalies and those that are not.

In this paper we studied the Higgs-mass constraints on the parameter space of a supersymmetric four-Higgs-doublet model, the FSSM [37], which was recently proposed as a solution of the $(g - 2)_\mu$ anomaly [10, 11] with SUSY particle masses beyond the current reach of the LHC. We followed the modern approach of taking M_h as an input rather than an output of our calculation, and we relied on an EFT setup to account at the NLL order for the large logarithmic corrections to the relation between the measured value of M_h and the value of λ_{SM} at the SUSY scale. In our one-loop calculation of the prediction for λ_{SM} we adapted to the case of the FSSM the results derived in ref. [39] for a general renormalizable theory. We provided explicit formulas for the one-loop correction to the quartic Higgs coupling in a simplified FSSM scenario, but we make the result for the general FSSM available on request in electronic form.

We found that the prediction for λ_{SM} establishes interesting relations between the parameters that contribute to $(g - 2)_\mu$, namely the masses of smuons, higgsinos and EW gauginos, and the parameters in the stop sector. In particular, there are scenarios with a suitable SUSY contribution to $(g - 2)_\mu$ in which the stops need to be considerably heavier than smuons, higgsinos and EW gauginos, in order to compensate for a large and negative smuon contribution to the prediction for λ_{SM} . The possibility of a hierarchical SUSY mass spectrum should be taken into account when assessing the prospects of probing the FSSM at future colliders.

As mentioned in ref. [37], further investigations of the FSSM could address the flavor structure of the quark sector, in which case the large corrections to the strange Yukawa coupling discussed in section 3 of our paper would need to be taken into account. Other directions of investigation could address the extended Higgs sector of the FSSM, exploring e.g. the collider phenomenology of the heavy Higgs bosons, or the stability of the scalar potential. With the present study, we aimed to provide a proof of concept – applicable also to other models and other anomalies – of how the Higgs-mass prediction can be used, in combination with other observables, to shed some light on the hidden structure of a SUSY model with heavy superparticles.

Acknowledgments

We thank Karim Benakli and Dominik Stöckinger for useful discussions. The work of W. K. is supported by the Contrat Doctoral Spécifique Normalien (CDSN) of Ecole Normale Supérieure – PSL. The work of P. S. is supported in part by French state funds managed by the Agence Nationale de la Recherche (ANR), in the context of the grant ‘‘HiggsAutomator’’ (ANR-15-CE31-0002).

Appendix

We provide here explicit formulas for the tree-level mass matrices of the Higgs bosons in the FSSM. In the Higgs basis, we decompose the four SU(2) doublets (all with positive hypercharge) as

$$\Phi_h = \begin{pmatrix} G^+ \\ v + \frac{1}{\sqrt{2}}(h + iG^0) \end{pmatrix}, \quad \Phi_H = \begin{pmatrix} H^+ \\ \frac{1}{\sqrt{2}}(H + iA) \end{pmatrix}, \quad \Phi'_{d,u} = \begin{pmatrix} \phi'_{d,u}{}^+ \\ \frac{1}{\sqrt{2}}(\phi'_{d,u} + i a'_{d,u}) \end{pmatrix}. \quad (\text{A1})$$

In this basis the mass matrices for the scalar (h, H, ϕ'_d, ϕ'_u), pseudoscalar (G^0, A, a'_d, a'_u) and charged ($G^+, H^+, \phi'_d{}^+, \phi'_u{}^+$) components of the four doublets read

$$\mathcal{M}_S^2 = \begin{pmatrix} M_Z^2 \cos^2 2\tilde{\beta} & -\frac{1}{2} M_Z^2 \sin 4\tilde{\beta} & 0 & 0 \\ -\frac{1}{2} M_Z^2 \sin 4\tilde{\beta} & M_Z^2 \sin^2 2\tilde{\beta} + 2b_{12}/\sin 2\tilde{\beta} & -b_{32}/\cos \tilde{\beta} & b_{14}/\sin \tilde{\beta} \\ 0 & -b_{32}/\cos \tilde{\beta} & M_{\Phi'_d}^2 + \frac{1}{2} M_Z^2 \cos 2\tilde{\beta} & -b_{34} \\ 0 & b_{14}/\sin \tilde{\beta} & -b_{34} & M_{\Phi'_u}^2 - \frac{1}{2} M_Z^2 \cos 2\tilde{\beta} \end{pmatrix}, \quad (\text{A2})$$

$$\mathcal{M}_P^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2b_{12}/\sin 2\tilde{\beta} & -b_{32}/\cos \tilde{\beta} & b_{14}/\sin \tilde{\beta} \\ 0 & -b_{32}/\cos \tilde{\beta} & M_{\Phi'_d}^2 + \frac{1}{2} M_Z^2 \cos 2\tilde{\beta} & -b_{34} \\ 0 & b_{14}/\sin \tilde{\beta} & -b_{34} & M_{\Phi'_u}^2 - \frac{1}{2} M_Z^2 \cos 2\tilde{\beta} \end{pmatrix}, \quad (\text{A3})$$

$$\mathcal{M}_\pm^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & M_W^2 + 2b_{12}/\sin 2\tilde{\beta} & -b_{32}/\cos \tilde{\beta} & b_{14}/\sin \tilde{\beta} \\ 0 & -b_{32}/\cos \tilde{\beta} & M_{\Phi'_d}^2 - (M_W^2 - \frac{1}{2} M_Z^2) \cos 2\tilde{\beta} & -b_{34} \\ 0 & b_{14}/\sin \tilde{\beta} & -b_{34} & M_{\Phi'_u}^2 + (M_W^2 - \frac{1}{2} M_Z^2) \cos 2\tilde{\beta} \end{pmatrix}, \quad (\text{A4})$$

where the mass parameters $M_{\Phi'_{d,u}}^2$ and the mixing parameters¹⁰ b_{ij} are combinations of the original mass parameters and B -terms as defined in eqs. (1) and (4):

¹⁰Our notation for the mixing parameters b_{ij} follows ref. [40]. Note however that the upper-left 4×4 blocks of the mass matrices shown in eqs. (31), (34) and (35) of that paper correspond to a different basis, namely $(\epsilon\Phi_d^*, \Phi_u, \epsilon\Phi'_d{}^*, \Phi'_u)$.

$$M_{\Phi'_d}^2 = (m_{dd}^2 + \mu_{ud}^2 + \mu_{u'd}^2) \cos^2 \beta_d + (m_{d'd'}^2 + \mu_{u'd'}^2 + \mu_{ud'}^2) \sin^2 \beta_d - (m_{dd'}^2 + \mu_{ud}\mu_{ud'} + \mu_{u'd'}\mu_{u'd}) \sin 2\beta_d , \quad (\text{A5})$$

$$M_{\Phi'_u}^2 = (m_{uu}^2 + \mu_{ud}^2 + \mu_{u'd}^2) \cos^2 \beta_u + (m_{u'u'}^2 + \mu_{u'd'}^2 + \mu_{ud'}^2) \sin^2 \beta_u - (m_{uu'}^2 + \mu_{ud}\mu_{u'd} + \mu_{u'd'}\mu_{ud'}) \sin 2\beta_u , \quad (\text{A6})$$

$$b_{12} = \sin \beta_d (B_{ud} \sin \beta_u + B_{u'd} \cos \beta_u) + \cos \beta_d (B_{ud'} \sin \beta_u + B_{u'd'} \cos \beta_u) , \quad (\text{A7})$$

$$b_{32} = \cos \beta_d (B_{ud} \sin \beta_u + B_{u'd} \cos \beta_u) - \sin \beta_d (B_{ud'} \sin \beta_u + B_{u'd'} \cos \beta_u) , \quad (\text{A8})$$

$$b_{14} = \sin \beta_d (B_{ud} \cos \beta_u - B_{u'd} \sin \beta_u) + \cos \beta_d (B_{ud'} \cos \beta_u - B_{u'd'} \sin \beta_u) , \quad (\text{A9})$$

$$b_{34} = \cos \beta_d (B_{ud} \cos \beta_u - B_{u'd} \sin \beta_u) - \sin \beta_d (B_{ud'} \cos \beta_u - B_{u'd'} \sin \beta_u) . \quad (\text{A10})$$

The minimum conditions of the Higgs potential have been used to remove four combinations of the original parameters from eqs. (A2)–(A4). In the limit of unbroken EW symmetry (i.e., $v \rightarrow 0$), which we adopt in the calculation of the matching condition for the quartic coupling of the SM-like Higgs, the mixing between the SM-like scalar h and the three heavy scalars vanishes, and the 3×3 sub-matrices for the masses of the scalar, pseudoscalar and charged components of the heavy doublets Φ_H , Φ'_d and Φ'_u all reduce to:

$$\mathcal{M}_H^2 = \begin{pmatrix} 2b_{12}/\sin 2\tilde{\beta} & -b_{32}/\cos \tilde{\beta} & b_{14}/\sin \tilde{\beta} \\ -b_{32}/\cos \tilde{\beta} & M_{\Phi'_d}^2 & -b_{34} \\ b_{14}/\sin \tilde{\beta} & -b_{34} & M_{\Phi'_u}^2 \end{pmatrix} . \quad (\text{A11})$$

We can then introduce a 3×3 orthogonal matrix R_H that rotates the three heavy doublets of the Higgs basis into a basis of mass eigenstates:

$$\begin{pmatrix} H_1 \\ H_2 \\ H_3 \end{pmatrix} = R_H \begin{pmatrix} \Phi_H \\ \Phi'_d \\ \Phi'_u \end{pmatrix} , \quad \text{diag} (M_{H_1}^2, M_{H_2}^2, M_{H_3}^2) = R_H \mathcal{M}_H^2 R_H^T . \quad (\text{A12})$$

References

- [1] **CMS** Collaboration, S. Chatrchyan *et al.*, *Observation of a New Boson at a Mass of 125 GeV with the CMS Experiment at the LHC*. Phys. Lett. B **716** (2012) 30–61, [arXiv:1207.7235](#) [[hep-ex](#)].
- [2] **ATLAS** Collaboration, G. Aad *et al.*, *Observation of a new particle in the search for the Standard Model Higgs boson with the ATLAS detector at the LHC*. Phys. Lett. B **716** (2012) 1–29, [arXiv:1207.7214](#) [[hep-ex](#)].
- [3] **ATLAS, CMS** Collaboration, G. Aad *et al.*, *Combined Measurement of the Higgs Boson Mass in pp Collisions at $\sqrt{s} = 7$ and 8 TeV with the ATLAS and CMS Experiments*. Phys. Rev. Lett. **114** (2015) 191803, [arXiv:1503.07589](#) [[hep-ex](#)].
- [4] **ATLAS, CMS** Collaboration, G. Aad *et al.*, *Measurements of the Higgs boson production and decay rates and constraints on its couplings from a combined ATLAS and CMS analysis of the LHC pp collision data at $\sqrt{s} = 7$ and 8 TeV*. JHEP **08** (2016) 045, [arXiv:1606.02266](#) [[hep-ex](#)].
- [5] P. Slavich *et al.*, *Higgs-mass predictions in the MSSM and beyond*. Eur. Phys. J. C **81** (2021) no. 5, 450, [arXiv:2012.15629](#) [[hep-ph](#)].
- [6] **LHCb** Collaboration, R. Aaij *et al.*, *Test of lepton universality using $B^+ \rightarrow K^+ \ell^+ \ell^-$ decays*. Phys. Rev. Lett. **113** (2014) 151601, [arXiv:1406.6482](#) [[hep-ex](#)].
- [7] **LHCb** Collaboration, R. Aaij *et al.*, *Test of lepton universality with $B^0 \rightarrow K^{*0} \ell^+ \ell^-$ decays*. JHEP **08** (2017) 055, [arXiv:1705.05802](#) [[hep-ex](#)].
- [8] **LHCb** Collaboration, R. Aaij *et al.*, *Search for lepton-universality violation in $B^+ \rightarrow K^+ \ell^+ \ell^-$ decays*. Phys. Rev. Lett. **122** (2019) no. 19, 191801, [arXiv:1903.09252](#) [[hep-ex](#)].
- [9] **LHCb** Collaboration, R. Aaij *et al.*, *Test of lepton universality in beauty-quark decays*. [arXiv:2103.11769](#) [[hep-ex](#)].
- [10] **Muon g-2** Collaboration, B. Abi *et al.*, *Measurement of the Positive Muon Anomalous Magnetic Moment to 0.46 ppm*. Phys. Rev. Lett. **126** (2021) no. 14, 141801, [arXiv:2104.03281](#) [[hep-ex](#)].
- [11] **Muon g-2** Collaboration, G. W. Bennett *et al.*, *Final Report of the Muon E821 Anomalous Magnetic Moment Measurement at BNL*. Phys. Rev. D **73** (2006) 072003, [arXiv:hep-ex/0602035](#).
- [12] T. Aoyama *et al.*, *The anomalous magnetic moment of the muon in the Standard Model*. Phys. Rept. **887** (2020) 1–166, [arXiv:2006.04822](#) [[hep-ph](#)].
- [13] T. Aoyama, M. Hayakawa, T. Kinoshita, and M. Nio, *Complete Tenth-Order QED Contribution to the Muon $g-2$* . Phys. Rev. Lett. **109** (2012) 111808, [arXiv:1205.5370](#) [[hep-ph](#)].

- [14] T. Aoyama, T. Kinoshita, and M. Nio, *Theory of the Anomalous Magnetic Moment of the Electron*. *Atoms* **7** (2019) no. 1, 28.
- [15] A. Czarnecki, W. J. Marciano, and A. Vainshtein, *Refinements in electroweak contributions to the muon anomalous magnetic moment*. *Phys. Rev. D* **67** (2003) 073006, [arXiv:hep-ph/0212229](#). [Erratum: *Phys.Rev.D* 73, 119901 (2006)].
- [16] C. Gnendiger, D. Stöckinger, and H. Stöckinger-Kim, *The electroweak contributions to $(g - 2)_\mu$ after the Higgs boson mass measurement*. *Phys. Rev. D* **88** (2013) 053005, [arXiv:1306.5546](#) [hep-ph].
- [17] M. Davier, A. Hoecker, B. Malaescu, and Z. Zhang, *Reevaluation of the hadronic vacuum polarisation contributions to the Standard Model predictions of the muon $g - 2$ and $\alpha(m_Z^2)$ using newest hadronic cross-section data*. *Eur. Phys. J. C* **77** (2017) no. 12, 827, [arXiv:1706.09436](#) [hep-ph].
- [18] A. Keshavarzi, D. Nomura, and T. Teubner, *Muon $g - 2$ and $\alpha(M_Z^2)$: a new data-based analysis*. *Phys. Rev. D* **97** (2018) no. 11, 114025, [arXiv:1802.02995](#) [hep-ph].
- [19] G. Colangelo, M. Hoferichter, and P. Stoffer, *Two-pion contribution to hadronic vacuum polarization*. *JHEP* **02** (2019) 006, [arXiv:1810.00007](#) [hep-ph].
- [20] M. Hoferichter, B.-L. Hoid, and B. Kubis, *Three-pion contribution to hadronic vacuum polarization*. *JHEP* **08** (2019) 137, [arXiv:1907.01556](#) [hep-ph].
- [21] M. Davier, A. Hoecker, B. Malaescu, and Z. Zhang, *A new evaluation of the hadronic vacuum polarisation contributions to the muon anomalous magnetic moment and to $\alpha(m_Z^2)$* . *Eur. Phys. J. C* **80** (2020) no. 3, 241, [arXiv:1908.00921](#) [hep-ph]. [Erratum: *Eur.Phys.J.C* 80, 410 (2020)].
- [22] A. Keshavarzi, D. Nomura, and T. Teubner, *$g - 2$ of charged leptons, $\alpha(M_Z^2)$, and the hyperfine splitting of muonium*. *Phys. Rev. D* **101** (2020) no. 1, 014029, [arXiv:1911.00367](#) [hep-ph].
- [23] A. Kurz, T. Liu, P. Marquard, and M. Steinhauser, *Hadronic contribution to the muon anomalous magnetic moment to next-to-next-to-leading order*. *Phys. Lett. B* **734** (2014) 144–147, [arXiv:1403.6400](#) [hep-ph].
- [24] K. Melnikov and A. Vainshtein, *Hadronic light-by-light scattering contribution to the muon anomalous magnetic moment revisited*. *Phys. Rev. D* **70** (2004) 113006, [arXiv:hep-ph/0312226](#).
- [25] P. Masjuan and P. Sanchez-Puertas, *Pseudoscalar-pole contribution to the $(g_\mu - 2)$: a rational approach*. *Phys. Rev. D* **95** (2017) no. 5, 054026, [arXiv:1701.05829](#) [hep-ph].
- [26] G. Colangelo, M. Hoferichter, M. Procura, and P. Stoffer, *Dispersion relation for hadronic light-by-light scattering: two-pion contributions*. *JHEP* **04** (2017) 161, [arXiv:1702.07347](#) [hep-ph].
- [27] M. Hoferichter, B.-L. Hoid, B. Kubis, S. Leupold, and S. P. Schneider, *Dispersion relation for hadronic light-by-light scattering: pion pole*. *JHEP* **10** (2018) 141, [arXiv:1808.04823](#) [hep-ph].

- [28] A. Gérardin, H. B. Meyer, and A. Nyffeler, *Lattice calculation of the pion transition form factor with $N_f = 2 + 1$ Wilson quarks*. Phys. Rev. D **100** (2019) no. 3, 034520, [arXiv:1903.09471](#) [hep-lat].
- [29] J. Bijnens, N. Hermansson-Truedsson, and A. Rodríguez-Sánchez, *Short-distance constraints for the $HLbL$ contribution to the muon anomalous magnetic moment*. Phys. Lett. B **798** (2019) 134994, [arXiv:1908.03331](#) [hep-ph].
- [30] G. Colangelo, F. Hagelstein, M. Hoferichter, L. Laub, and P. Stoffer, *Longitudinal short-distance constraints for the hadronic light-by-light contribution to $(g - 2)_\mu$ with large- N_c Regge models*. JHEP **03** (2020) 101, [arXiv:1910.13432](#) [hep-ph].
- [31] T. Blum, N. Christ, M. Hayakawa, T. Izubuchi, L. Jin, C. Jung, and C. Lehner, *Hadronic Light-by-Light Scattering Contribution to the Muon Anomalous Magnetic Moment from Lattice QCD*. Phys. Rev. Lett. **124** (2020) no. 13, 132002, [arXiv:1911.08123](#) [hep-lat].
- [32] G. Colangelo, M. Hoferichter, A. Nyffeler, M. Passera, and P. Stoffer, *Remarks on higher-order hadronic corrections to the muon $g-2$* . Phys. Lett. B **735** (2014) 90–91, [arXiv:1403.7512](#) [hep-ph].
- [33] W. Altmannshofer and D. M. Straub, *Viability of MSSM scenarios at very large $\tan \beta$* . JHEP **09** (2010) 078, [arXiv:1004.1993](#) [hep-ph].
- [34] M. Chakraborti, S. Heinemeyer, and I. Saha, *The new “MUON $G-2$ ” result and supersymmetry*. Eur. Phys. J. C **81** (2021) no. 12, 1114, [arXiv:2104.03287](#) [hep-ph].
- [35] P. Athron, C. Balázs, D. H. Jacob, W. Kotlarski, D. Stöckinger, and H. Stöckinger-Kim, *New physics explanations of a_μ in light of the FNAL muon $g - 2$ measurement*. JHEP **09** (2021) 080, [arXiv:2104.03691](#) [hep-ph].
- [36] M. Bach, J.-h. Park, D. Stöckinger, and H. Stöckinger-Kim, *Large muon $(g - 2)$ with TeV-scale SUSY masses for $\tan \beta \rightarrow \infty$* . JHEP **10** (2015) 026, [arXiv:1504.05500](#) [hep-ph].
- [37] W. Altmannshofer, S. A. Gadam, S. Gori, and N. Hamer, *Explaining $(g - 2)_\mu$ with Multi-TeV Sleptons*. JHEP **07** (2021) 118, [arXiv:2104.08293](#) [hep-ph].
- [38] K. Benakli, L. Darmé, M. D. Goodsell, and P. Slavich, *A Fake Split Supersymmetry Model for the 126 GeV Higgs*. JHEP **05** (2014) 113, [arXiv:1312.5220](#) [hep-ph].
- [39] J. Braathen, M. D. Goodsell, and P. Slavich, *Matching renormalisable couplings: simple schemes and a plot*. Eur. Phys. J. C **79** (2019) no. 8, 669, [arXiv:1810.09388](#) [hep-ph].
- [40] N. Escudero, C. Munoz, and A. M. Teixeira, *FCNCs in supersymmetric multi-Higgs doublet models*. Phys. Rev. D **73** (2006) 055015, [arXiv:hep-ph/0512046](#).
- [41] B. A. Kniehl, A. F. Pikelner, and O. L. Veretin, *mr: a C++ library for the matching and running of the Standard Model parameters*. Comput. Phys. Commun. **206** (2016) 84–96, [arXiv:1601.08143](#) [hep-ph].

- [42] B. A. Kniehl, A. F. Pikelner, and O. L. Veretin, *Two-loop electroweak threshold corrections in the Standard Model*. Nucl. Phys. B **896** (2015) 19–51, [arXiv:1503.02138 \[hep-ph\]](#).
- [43] **Particle Data Group** Collaboration, P. A. Zyla *et al.*, *Review of Particle Physics*. PTEP **2020** (2020) no. 8, 083C01.
- [44] R. Hempfling and B. A. Kniehl, *On the relation between the fermion pole mass and M_S Yukawa coupling in the standard model*. Phys. Rev. D **51** (1995) 1386–1394, [arXiv:hep-ph/9408313](#).
- [45] G. Degrossi, P. Gambino, and P. Slavich, *SusyBSG: A Fortran code for $BR[B \rightarrow X(s) \gamma]$ in the MSSM with Minimal Flavor Violation*. Comput. Phys. Commun. **179** (2008) 759–771, [arXiv:0712.3265 \[hep-ph\]](#).
- [46] A. V. Bednyakov, A. F. Pikelner, and V. N. Velizhanin, *Anomalous dimensions of gauge fields and gauge coupling beta-functions in the Standard Model at three loops*. JHEP **01** (2013) 017, [arXiv:1210.6873 \[hep-ph\]](#).
- [47] A. V. Bednyakov, A. F. Pikelner, and V. N. Velizhanin, *Yukawa coupling beta-functions in the Standard Model at three loops*. Phys. Lett. B **722** (2013) 336–340, [arXiv:1212.6829 \[hep-ph\]](#).
- [48] A. V. Bednyakov, A. F. Pikelner, and V. N. Velizhanin, *Higgs self-coupling beta-function in the Standard Model at three loops*. Nucl. Phys. B **875** (2013) 552–565, [arXiv:1303.4364 \[hep-ph\]](#).
- [49] M.-x. Luo and Y. Xiao, *Two loop renormalization group equations in the standard model*. Phys. Rev. Lett. **90** (2003) 011601, [arXiv:hep-ph/0207271](#).
- [50] E. Bagnaschi, G. F. Giudice, P. Slavich, and A. Strumia, *Higgs Mass and Unnatural Supersymmetry*. JHEP **09** (2014) 092, [arXiv:1407.4081 \[hep-ph\]](#).
- [51] T. Kwasnitza, D. Stöckinger, and A. Voigt, *Improved MSSM Higgs mass calculation using the 3-loop FlexibleEFTHiggs approach including x_t -resummation*. JHEP **07** (2020) no. 07, 197, [arXiv:2003.04639 \[hep-ph\]](#).
- [52] T. Kwasnitza and D. Stöckinger, *Resummation of terms enhanced by trilinear squark-Higgs couplings in the MSSM*. JHEP **08** (2021) 070, [arXiv:2103.08616 \[hep-ph\]](#).
- [53] B. C. Allanach, G. Hiller, D. R. T. Jones, and P. Slavich, *Flavour Violation in Anomaly Mediated Supersymmetry Breaking*. JHEP **04** (2009) 088, [arXiv:0902.4880 \[hep-ph\]](#).
- [54] E. Bagnaschi, J. Pardo Vega, and P. Slavich, *Improved determination of the Higgs mass in the MSSM with heavy superpartners*. Eur. Phys. J. C **77** (2017) no. 5, 334, [arXiv:1703.08166 \[hep-ph\]](#).
- [55] T. Falk, K. A. Olive, and M. Srednicki, *Heavy sneutrinos as dark matter*. Phys. Lett. B **339** (1994) 248–251, [arXiv:hep-ph/9409270](#).
- [56] C. Arina and N. Fornengo, *Sneutrino cold dark matter, a new analysis: Relic abundance and detection rates*. JHEP **11** (2007) 029, [arXiv:0709.4477 \[hep-ph\]](#).

- [57] M. Badziak, Z. Lalak, M. Lewicki, M. Olechowski, and S. Pokorski, *Upper bounds on sparticle masses from muon $g - 2$ and the Higgs mass and the complementarity of future colliders*. JHEP **03** (2015) 003, [arXiv:1411.1450](#) [hep-ph].

Chapter 5

Higgs alignment and W boson mass in Dirac gaugino models

Extensions of the SM often involve an enlarged scalar sector, and examples are the MSSM which contains two Higgs doublets and the FSSM including four Higgs doublets. While only one fundamental scalar has been discovered, one must address an important question: how do these models incorporate the SM-like Higgs boson? This question finds one trivial answer in the previous Chapter, where we considered high-scale SUSY models, in other words, the non SM-like Higgs bosons are much heavier than the SM-like one, so they decouple from the latter in the low-energy limit. A different, and phenomenologically interesting scenario is that the BSM particles are within reach of the current (or at least the near future) collider, but their mixing with the SM Higgs are suppressed by some underlying symmetries of the model. This is conventionally called *alignment without decoupling*, see for example [87–91].

Such possibility is realized, for example, in *Dirac gaugino models* [24, 92–99] – supersymmetric models where gauginos have Dirac masses. At tree level, alignment without decoupling is automatic [100], in addition, it is relatively stable under quantum corrections [101]. Beyond their favorable properties regarding alignment without decoupling, Dirac gaugino models also account for the recently measured W boson mass thanks to the presence of a triplet scalar. In this Chapter, we examine several Dirac gaugino models, beginning with their alignment properties and subsequently assessing their ability to enhance the W boson mass. In Section 5.1, we recall relevant notions and properties of Higgs alignment. In Section 5.2, we present the basic construction of Dirac gaugino models, as well as its implications in Higgs alignment and the W boson mass.

5.1 Higgs alignment

The main idea of Higgs alignment is that, in the presence of multiple scalars, the eigenstate of the scalar mixing matrix that acquires a non-zero vev is aligned with the observed Higgs boson and its properties turn out to be SM-like. To give a basic illustration, let us consider one of the simplest extensions of the scalar sector: the two-Higgs doublet model

(2HDM). We put the two doublets in the form

$$H_d = \begin{pmatrix} H_d^0 \\ H_d^- \end{pmatrix}, \quad H_u = \begin{pmatrix} H_u^+ \\ H_u^0 \end{pmatrix} \quad (5.1)$$

where

$$H_d^0 \equiv (v_d + h_d^0 + ia_d^0)/\sqrt{2}, \quad H_u^0 \equiv (v_u + h_u^0 + ia_u^0)/\sqrt{2} \quad (5.2)$$

We assume that $U(1)_{\text{em}}$ and CP symmetries are not spontaneously broken by the vacuum, therefore only neutral components acquire non-zero vev and all the vevs (here denoted by v_u, v_d) are real. In addition, they obey the relation $v_u^2 + v_d^2 = v^2$, with $v \approx 246$ GeV. The most general renormalizable and $SU(2)$ -invariant scalar potential of this model is given by

$$V = m_{H_u}^2 |H_u|^2 + m_{H_d}^2 |H_d|^2 + (m_{12}^2 H_u \cdot H_d + h.c.) + \frac{\lambda_1}{2} |H_d|^4 + \frac{\lambda_2}{2} |H_u|^4 + \lambda_3 |H_u|^2 |H_d|^2 + \lambda_4 |H_u \cdot H_d|^2 + \left[\frac{\lambda_5}{2} (H_u \cdot H_d)^2 - (\lambda_6 |H_d|^2 + \lambda_7 |H_u|^2) H_u \cdot H_d + h.c. \right] \quad (5.3)$$

where the dot product is defined as:

$$H_u \cdot H_d \equiv \epsilon_{ij} H_u^i H_d^j = H_u^+ H_d^- - H_u^0 H_d^0 \quad (5.4)$$

H_u and H_d have opposite hypercharges. One can also redefine the basis in order to have two hypercharge +1 doublets:

$$\Phi_{dj} \equiv \epsilon_{ij} H_d^{*i}, \quad \Phi_u^j \equiv H_u^j \quad (5.5)$$

A rotation of Φ_u^j and Φ_d^j yields the so-called *Higgs basis*, where only one of the doublets acquires a non-zero vev:

$$H_1 \equiv \frac{v_d \Phi_d + v_u \Phi_u}{v}, \quad H_2 \equiv \frac{-v_u \Phi_d + v_d \Phi_u}{v} \quad (5.6)$$

with

$$H_1 = \begin{pmatrix} \varphi^+ \\ (v + H^{\text{SM}} + i\chi)/\sqrt{2} \end{pmatrix}, \quad H_2 = \begin{pmatrix} H^+ \\ (H^{\text{NSM}} + iA)/\sqrt{2} \end{pmatrix} \quad (5.7)$$

The neutral CP-even component of H_1 is identified with the SM-like Higgs boson H^{SM} , and the Goldstone bosons φ^\pm, χ all lie in H_1 . The non-SM particles sit in the other doublet H_2 . This procedure can be readily applied to an arbitrary number of doublets, as we have already seen in the case of the FSSM, Eq. (6)-(7) of Section 4.4. In the Higgs basis, the scalar potential (5.3) is rewritten as

$$V = Y_1 H_1^\dagger H_1 + Y_2 H_2^\dagger H_2 + [Y_3 H_1^\dagger H_2 + h.c.] + \frac{1}{2} Z_1 (H_1^\dagger H_1)^2 + \frac{1}{2} Z_2 (H_2^\dagger H_2)^2 + Z_3 (H_1^\dagger H_1) (H_2^\dagger H_2) + Z_4 (H_1^\dagger H_2) (H_2^\dagger H_1) + \left\{ \frac{1}{2} Z_5 (H_1^\dagger H_2)^2 + [Z_6 (H_1^\dagger H_1) + Z_7 (H_2^\dagger H_2)] H_1^\dagger H_2 + h.c. \right\} \quad (5.8)$$

The new coefficients are related to the old ones by the rotation angle β , with $\tan\beta \equiv v_u/v_d$. Finally, we infer from (5.8) the CP-even squared mass matrix in the basis $\{H^{SM}, H^{NSM}\}$:

$$\mathcal{M}^2 = \begin{pmatrix} Z_1 v^2 & Z_6 v^2 \\ Z_6 v^2 & m_A^2 + Z_5 v^2 \end{pmatrix} \quad (5.9)$$

where $m_A^2 = -\frac{m_{12}^2}{s_\beta c_\beta} - \lambda_5 v^2$ is the squared mass of the CP-odd neutral Higgs field. We can now rephrase the requirement for Higgs alignment as: one of the mass eigenstates of \mathcal{M}^2 – conventionally considered to be the lightest one – is the SM-like Higgs boson H^{SM} . In terms of the matrix elements, it amounts to the conditions

$$Z_6 = 0 \quad \text{and} \quad Z_1 v^2 = m_h^2 \approx (125 \text{ GeV})^2 \quad (5.10)$$

which correspond to the scenario of *exact* alignment. Nevertheless, a moderate amount of deviations from (5.10), typically due to quantum corrections, is tolerated as long as they are not in conflict with precision measurements.

When the non-SM scalars contain an electroweak singlet, with decomposition $S = v_s + (H^S + ia_s^0)/\sqrt{2}$, the alignment conditions are very similar. Let us extend the 2HDM to include S , then same as before, one must rotate the two doublets $\{H_u, H_d\}$ to the Higgs basis, while the singlet vev is unaffected by this change of basis. In the scalar potential, there are new singlet self-interactions and singlet-doublet couplings. We write the subsequent CP-even squared-mass matrix in the Higgs basis $\{H^{SM}, H^{NSM}, H^S\}$ as:

$$\mathcal{M}^2 = \begin{pmatrix} \mathcal{M}_{11}^2 & \mathcal{M}_{12}^2 & \mathcal{M}_{13}^2 \\ & \mathcal{M}_{22}^2 & \mathcal{M}_{23}^2 \\ & & \mathcal{M}_{33}^2 \end{pmatrix} \quad (5.11)$$

Again, Higgs alignment requires that the SM-like scalar be decoupled from the others:

$$\mathcal{M}_{11}^2 = m_h^2, \quad \mathcal{M}_{12}^2 = 0, \quad \mathcal{M}_{13}^2 = 0 \quad (5.12)$$

While the above discussions involve the general 2HDM and 2HDM plus a singlet, what is the situation in some simple supersymmetric models? As we have presented in the last Chapter, the MSSM is a supersymmetric 2HDM, whose scalar potential is determined by the F - and D - terms at tree level. Higgs alignment in the MSSM is extensively studied in the literature [91, 102–106], which points to a possible alignment without decoupling, but this requires a specific choice of parameter regions where the tree-level and loop contributions to Z_6 approximately cancel. *A priori*, no symmetry of the model implies this cancellation, so alignment without decoupling in this case necessitates some amount of tuning. Similar analyses have also been performed on the NMSSM [107, 108] which is the singlet extension of the MSSM, yielding various constraints on the scalar potential, but exact alignment is again not automatically satisfied.

5.2 Dirac gaugino models

Gauginos are the superpartners of the gauge bosons that can acquire mass through SUSY breaking. They are the fermionic component of the vector multiplet, which in the

four-component notation is represented by a Majorana fermion. Whether gauginos have a Majorana ($\tilde{M}\lambda\lambda + \text{h.c.}$) or Dirac ($M\lambda\psi + \text{h.c.}$) mass is an important question because a Majorana mass term explicitly breaks R-symmetry, while a Dirac mass term preserves it.

In the MSSM, the gauginos have Majorana masses that reside in the soft SUSY-breaking Lagrangian, hence R-symmetry is broken. There are also viable models where gauginos have Dirac masses, and this class of models is called *Dirac gaugino models*. First introduced in [24] in order to provide mass to gluinos while preserving R-symmetry, Dirac gauginos have been, since then, substantially studied in non-minimal extensions of the SM. The simplest content comprises three chiral superfields that supplement the MSSM. Each chiral multiplet is associated with a gauge group, and transforms under the adjoint representation. We denote the singlet, $SU(2)$ triplet and $SU(3)$ color octet by \mathbf{S} , \mathbf{T} , \mathbf{O}_g respectively. The chiral multiplets enlarge the field content, with new complex scalars:

$$S = \frac{S_R + iS_I}{\sqrt{2}}, \quad T = \frac{1}{2} \begin{pmatrix} T_0 & \sqrt{2}T_+ \\ \sqrt{2}T_- & -T_0 \end{pmatrix}, \quad O_g^{(a)} = \frac{O_{gR}^{(a)} + iO_{gI}^{(a)}}{\sqrt{2}} \quad (5.13)$$

where $T_0 = \frac{1}{\sqrt{2}}(T_R + iT_I)$, $T_{\pm} = \frac{1}{\sqrt{2}}(T_{\pm R} + iT_{\pm I})$. The fermionic components pair with gauginos to form a Dirac mass term. Details about the superpotential and the Lagrangian will be presented in Section 5.4. For the following discussions, we restrict to the couplings between the adjoint superfields and the MSSM Higgs superfields in the superpotential:

$$W \supset \lambda_S \mathbf{S} \mathbf{H}_u \cdot \mathbf{H}_d + 2\lambda_T \mathbf{H}_d \cdot \mathbf{T} \mathbf{H}_u \quad (5.14)$$

In the first studies of Higgs alignment in Dirac gaugino models, one often assumed that the adjoint scalars S and T are very heavy, and consequently can be integrated out (see for example [97]). Since the color octet is not coupled to the scalars S and T , the resulting effective theory is a 2HDM. In the superpotential, λ_S , λ_T are not completely arbitrary. If one considers an underlying $\mathcal{N} = 2$ SUSY in the gauge sector at a high energy scale, which contains an $SU(2)_R$ symmetry [96, 100, 109], these coupling constants are actually fixed by the gauge couplings:

$$\lambda_S = \frac{g'}{\sqrt{2}}, \quad \lambda_T = \frac{g}{\sqrt{2}} \quad (5.15)$$

Interestingly, (5.15) will imply that alignment without decoupling at tree level is *automatic*, *i.e.* the off-diagonal element of the mass matrix is vanishing ($Z_6 = 0$). In addition, including quantum corrections, the alignment condition is shown to be radiatively stable [101].

The above discussion is based on the effective theory where S and T are integrated out. Revisiting this assumption, one may be wondering how light can the singlet and triplet be. Note that the triplet vev in this model is constrained by electroweak precision measurements of the “ ρ parameter”, to which the triplet vev gives a tree-level contribution. Recall that ρ is related to the W boson mass through:

$$M_W^2 \equiv \rho \cos^2 \theta_W M_Z^2 \quad (5.16)$$

with θ_W the Weinberg angle. According to [97], this vev is bounded by $v_T \lesssim 3$ GeV. Small triplet vevs are conventionally ensured by a large triplet mass¹, so in this case the triplet mass is bounded from below, $m_T \gtrsim 1$ TeV.

¹As we will see in Section 5.4, the squared mass of the triplet is, approximately, proportional to the inverse of its vev.

On the other hand, it has not been sufficiently investigated to what extent the singlet can be light. The constraint mainly comes from the mixing between the singlet and the light Higgs, because the larger the mixing is, the stronger the direct search bound will be. One may choose to keep, or integrate out the triplet scalar, then investigate how the resulting model is restricted by approximate alignment. These scenarios will be examined in Section 5.4.

Recently, hints of new physics appeared in the direct measurement of the W boson mass by the CDF Collaboration [25], with a better precision than the previous LEP, Tevatron and LHC results. The reported value is:

$$M_W^{\text{CDF}} = 80.4335 \pm 0.0094 \text{ GeV} \quad (5.17)$$

and from the SM prediction [110, 111]:

$$M_W^{\text{SM}} = 80.357 \pm 0.006 \text{ GeV} \quad (5.18)$$

the deviation is as large as 7σ . While waiting for confirmation from other experiments, it is interesting to scrutinize how Dirac gaugino models accommodate this discrepancy. As we mentioned before, the triplet vev has a tree-level contribution to the ρ parameter, in the form:

$$\rho \equiv 1 + \Delta\rho_{\text{tree}} + \Delta\rho = 1 + 4\frac{v_T^2}{v^2} + \Delta\rho \quad (5.19)$$

where $\Delta\rho$ stands for higher order corrections. The tree-level enhancement of m_W from a triplet is well-known. It arises from the kinetic term of the triplet that includes a quartic coupling between two W bosons and two triplet scalars that acquire a vev v_T . Therefore, Dirac gaugino models provide an intriguing explanation to the new CDF result, thanks to the adjoint triplet naturally present in these models. As a first approximation, we take the observed values of M_Z and $\cos\theta_W$, so the tree-level deviation of the ρ parameter is $\Delta^{\text{CDF}}\rho_{\text{tree}} = 0.0012$, implying $v_T \approx 4 \text{ GeV}$, which is around the upper bound given previously.

For a more detailed analysis, the tree-level estimation is far from being sufficient. In fact, Dirac gaugino models also give rise to important quantum corrections to m_W , and the latter may even compete with tree-level contributions for some specific parameter regions. To take into account these corrections, we will not restrict to Dirac gaugino models with an underlying $SU(2)_R$ symmetry, so the couplings λ_S and λ_T are arbitrary, and we will start from a general set of parameters that also allows for some R-symmetry breaking. In the meantime, we will compare different variants of Dirac gaugino models (with specific choices of parameters), involving both alignment properties and the predictions of the W boson mass. This work will be presented in Section 5.4.

5.3 The setup and main results

The superpotential of the Dirac gaugino model under consideration is, for the Higgs sector:

$$W = \mu\mathbf{H}_u \cdot \mathbf{H}_d + \lambda_S\mathbf{S}\mathbf{H}_u \cdot \mathbf{H}_d + 2\lambda_T\mathbf{H}_d \cdot \mathbf{T}\mathbf{H}_u \quad (5.20)$$

where λ_S and λ_T will be taken as independent parameters. The Yukawa superpotential is same as in the MSSM, given by

$$W_{\text{Yukawa}} = Y_u^{ij} \mathbf{U}^c_i \mathbf{Q}_j \cdot \mathbf{H}_u - Y_d^{ij} \mathbf{D}^c_i \mathbf{Q}_j \cdot \mathbf{H}_d - Y_e^{ij} \mathbf{E}^c_i \mathbf{L}_j \cdot \mathbf{H}_d \quad (5.21)$$

The effective superpotential for the Dirac gaugino masses is:

$$W_{\text{DG}} = \sqrt{2}\theta^\alpha [m_{\text{DY}} \mathbf{S} \mathcal{W}_{1\alpha} + 2m_{\text{D2}} \text{tr}(\mathbf{T} \mathcal{W}_{2\alpha}) + 2m_{\text{D3}} \text{tr}(\mathbf{O} \mathcal{W}_{3\alpha})] \quad (5.22)$$

with $\mathbf{O} \equiv \frac{1}{2} \lambda^a \mathbf{O}^a$ and $\mathcal{W}_{i\alpha}$ denote the chiral gauge-strength superfields. We also take into account $U(1)_R$ R-symmetry breaking effects, encoded in the superpotential:

$$\begin{aligned} W_{NR} = & \xi_S \mathbf{S} + \frac{M_S}{2} \mathbf{S}^2 + \frac{\kappa}{3} \mathbf{S}^3 + \lambda_{ST} \mathbf{S} \text{tr}(\mathbf{T} \mathbf{T}) + \lambda_{SO} \mathbf{S} \text{tr}(\mathbf{O} \mathbf{O}) \\ & + M_T \text{tr}(\mathbf{T} \mathbf{T}) + M_O \text{tr}(\mathbf{O} \mathbf{O}) + \frac{\lambda_O}{3} \text{tr}(\mathbf{O} \mathbf{O} \mathbf{O}) \end{aligned} \quad (5.23)$$

The soft SUSY-breaking Lagrangian contains R-symmetry preserving and breaking terms, that can be found in the next section. The above information enables us to derive the tree-level mass matrix $(\mathcal{M}^2)_{\text{tree}}$ in the basis $\{h, H, S_R, T_R^0\}$. In particular, if we insert the conditions (5.15), then $(\mathcal{M}^2_{11})_{\text{tree}} = M_Z^2$. For a more complete analysis, we must also include radiative corrections to the entries of the mass matrix, as well as to Eq. (5.15), which will partially spoil the alignment. Some specific choices of parameters – depending for example on whether R-symmetry is broken – will result in different alignment properties, and these cases will be listed in Sec. 2.4 in the next section. One of them which is of particular interest is referred to as the *aligned DGNMSSM*², where the triplet is decoupled, and the light Higgs does not mix with the heavy Higgs or with the singlet, while allowing for a possible R-symmetry violation. Tree-level alignment can be achieved in this model through a specific set of parameters.

The second aspect we consider is the W boson mass prediction in Dirac gaugino models. In view of the recent CDF result, an enhancement of M_W can be achieved in two ways: the vev of the triplet scalar and the quantum corrections. The latter receive a significant contribution from the triplet scalar interactions. As is mentioned before, the tree-level triplet vev is estimated to be $v_T \approx 4$ GeV in order to match $\Delta M_W^2 = 11$ (GeV)². On the other hand, an evaluation of quantum corrections is carried out numerically. The different models are then compared in terms of their compatibility with collider constraints, and the aforementioned aligned DGNMSSM, for example, agrees with the experimental bounds. In particular, we observe the importance of quantum corrections to the W boson mass, which are at least as large as the tree-level contribution from the triplet expectation value v_T and therefore, a sensible analysis of W boson mass must go beyond tree level.

5.4 Publication

²This acronym originates from the fact that in this model, we have two Higgs doublets and one singlet, same as in the NMSSM.

W boson mass in minimal Dirac gaugino scenarios

Karim Benakli^a, Mark Goodsell^b, Wenqi Ke^c and Pietro Slavich^d

Sorbonne Université, CNRS, Laboratoire de Physique Théorique et Hautes Energies (LPTHE), F-75005 Paris, France.

Abstract

We investigate the conditions for alignment in Dirac Gaugino models with minimal matter content. This leads to several scenarios, including an aligned Dirac Gaugino NMSSM that allows a light singlet scalar. We then investigate the compatibility of minimal Dirac Gaugino models with an enhanced W boson mass, using a new precise computation of the quantum corrections included in the code SARAH 4.15.0.

^akbenakli@lpthe.jussieu.fr

^bgoodsell@lpthe.jussieu.fr

^cwke@lpthe.jussieu.fr

^dslavich@lpthe.jussieu.fr

Contents

1	Introduction	2
2	Dirac Gaugino Models with Automatic Tree-level Alignment	4
2.1	Field content and interactions	4
2.2	The electroweak scalar sector and alignment	7
2.3	The electroweak fermion sector	13
2.4	Scenarios	13
3	W mass in Dirac Gaugino models	14
3.1	Numerical setup	16
3.2	MSSM without μ term	17
3.3	W mass in the MDGSSM	18
3.4	W mass in the aligned MDGSSM	21
3.5	W mass in the aligned DGNMSSM	22
3.6	W mass in the general DGNMSSM	24
4	Conclusions	25

1 Introduction

While constraints on heavy Higgs bosons in supersymmetric models are rather stringent at large $\tan\beta$, excluding masses above a TeV, at small to moderate $\tan\beta$ direct searches do not place significant limits; only indirect constraints from $B \rightarrow s\gamma$ limit a heavy charged Higgs boson to be above 568 GeV, roughly independent of $\tan\beta$ [1]. On the other hand, in the Minimal Supersymmetric Standard Model (MSSM), this region is likely excluded for an additional neutral Higgs boson below a few hundred GeV due to modifications to the SM-like Higgs boson couplings. This has led to a lot of interest in extensions of the MSSM (or variants of the Two Higgs Doublet Model) where *alignment without decoupling* is possible [2–9], that is where the mixing between the SM-like Higgs boson and other scalars is minimised so that it aligns with the expectation values and has SM-like couplings.

Dirac Gaugino models [10–18], (see also, for example, [19–65]) accommodate scenarios where alignment without decoupling is *automatic* at tree-level [15,66–70]. Under the assumption of an $N = 2$ supersymmetry in the gauge sector at some scale, these models contain an approximate $SU(2)_R$ R-symmetry which guarantees the tree-level alignment [67]. An investigation of the effects of quantum corrections showed that it is even radiatively stable [66], with competing effects partially cancelling.

These models have many interesting phenomenological properties, and have been extensively studied in the literature. They involve, at a minimum, an extension of the MSSM by three adjoint chiral superfields, one for each gauge group; the fermions from these pair with the gauginos to give them a Dirac mass. This means the presence of new scalar fields, in singlet, triplet and octet representations.

Actually, it has not adequately been investigated to what extent the *singlet* could be light in such models. One condition for this to be the case is that it should not

disturb the couplings of the light Higgs – in other words, we should have some amount of alignments without decoupling. ATLAS and CMS both give constraints on the overall signal strength of the Higgs boson μ to be [71, 72]:

$$\mu = 1.06 \pm 0.07 \text{ (ATLAS)}, \quad \mu = 1.02^{+0.07}_{-0.06} \text{ (CMS)}. \quad (1.1)$$

If the Higgs boson h mixes with an inert singlet s , then we can write the mass eigenstates \tilde{h}, S as

$$\begin{pmatrix} h \\ s \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} \\ -S_{12} & S_{11} \end{pmatrix} \begin{pmatrix} \tilde{h} \\ S \end{pmatrix} \quad (1.2)$$

then we will find that

$$\mu = |S_{11}|^2 \leq 1. \quad (1.3)$$

Hence if we allow a 3σ deviation from the ATLAS result, we require

$$1 - |S_{11}|^2 = |S_{12}|^2 \leq 0.15 \longrightarrow |S_{12}| < 0.39. \quad (1.4)$$

While this still allows a moderate amount of mixing, the larger the mixing between the flavour eigenstates, the stronger the direct search bounds on the singlet will be. Therefore in this work we will consider the conditions for an approximate alignment in which the light Higgs mixes neither with the Heavy Higgs nor with the singlet (the triplet being decoupled). This will lead to a scenario that we refer to as the *aligned DGNMSSM*.

Recently, the CDF experiment reported a new measurement of the mass of the W boson [73]. Compared to the SM prediction [73–75], this gives as averages (combined Tevatron+LEP [73, 76–83]):

$$M_W^{\text{Tevatron+LEP}} = 80424 \pm 9 \text{ MeV}, \quad M_W^{\text{SM}} = 80356 \pm 6 \text{ MeV}. \quad (1.5)$$

If we take the central value of the top quark mass to be 172.89 GeV then the central SM prediction becomes 80352 MeV [84]. These differ by 7 standard deviations, although measurements at the LHC [85, 86] also differ from the combination of Tevatron+LEP by 4 standard deviations, so at this stage confirmation is required by other experiments. Nevertheless, a modification to the W boson mass is one of the most generic effects of new light particles coupling to the electroweak sector, so such a hint is tantalising.

It has generally been assumed in Dirac Gaugino models that the adjoint scalars should be heavy; indeed, the requirement that the triplet scalar vacuum expectation value (vev) must be very small compared to the Standard Model Higgs one – otherwise it would generate a large ρ parameter - is usually ensured by giving the triplet a heavy mass. Amusingly, following the new measurement, the simplest explanation for the enhanced W boson mass is exactly an expectation value for the neutral component of such a triplet. In this work we shall investigate that possibility in minimal Dirac Gaugino models.

Such a triplet scalar also comes along with electroweak fermions, which can modify the quantum corrections too. Therefore a *precise* computation is required. While a

preliminary such computation was performed for the MRSSM [87] using an update to FlexibleSUSY [88, 89], and a related computation was performed for the same model in [90], that model lacks a natural enhancement to the W boson quantum corrections. In this work we introduce a similarly precise computation in the package SARAH-4.15.0 and use it to examine the compatibility of our aligned DGNMSSM, along with four other scenarios – the “MSSM without μ term,” the MDGSSM, the aligned MDGSSM and the general DGNMSSM – with the new measurement of the W boson mass, or a naive world average value of $M_W^{\text{world average}} = 80411 \pm 15$ MeV.

This work is organised as follows. In section 2 we summarise the essential details of the class of Dirac Gaugino models, including the vacuum minimisation conditions and mass matrices. The conditions for alignment are reviewed and a comparison is made with the cases of the MSSM and NMSSM. We also introduce the different variants we shall consider: the MSSM without μ -term; the MDGSSM, the DGNMSSM and the aligned DGNMSSM. In Section 3 we will study the predictions for all of these classes of models for the W boson mass, examining in particular the effects of a precise computation of the quantum effects. We present our conclusions in section 4.

2 Dirac Gaugino Models with Automatic Tree-level Alignment

2.1 Field content and interactions

We shall consider in this work the extension of the MSSM by a minimal matter content to allow Dirac Gaugino masses, as in [18, 35]. The additional superfields consist of three chiral multiplets, in adjoint representations of the SM gauge group factors (DG-adjoints): a singlet \mathbf{S} , an $SU(2)_W$ triplet \mathbf{T}^a , and an $SU(3)_C$ octet \mathbf{O}^a . If we require gauge-coupling unification, even more states should be added to the model. For instance, for an $(SU(3))^3$ Grand Unification, the minimal set of chiral multiplets includes also extra Higgs-like doublets $\mathbf{R}_{u,d}$ as well as two pairs of vector-like right-handed electron $\mathbf{E}'_{1,2}$ in $(\mathbf{1}, \mathbf{1})_1$ and $\tilde{\mathbf{E}}'_{1,2}$ in $(\mathbf{1}, \mathbf{1})_{-1}$. We will not consider these states here.

In order to develop an intuition for the different interactions involved, it is helpful to consider a simple picture where the model descends from a supersymmetric theory in D dimensions. The different states can appear in different sectors: some live in the whole D -dimensional bulk, others are localised on four-dimensional hyper-surfaces (branes) at points of the extra dimensions of coordinates $x_i = \{x_i^a\}$, $a = 5, \dots, D$. The corresponding Lagrangian can be written as

$$\int d^D x \mathcal{L} = \int d^D x \{ \mathcal{L}_{bulk} + \sum_i \delta^{(D-4)}(x - x_i) \mathcal{L}_{boundaries}^{(i)} \}, \quad (2.1)$$

where we have not explicitly written the metric factors. The four-dimensional theory arises after a truncation keeping only the compactification zero modes:

$$\mathcal{L}^{4d} = \mathcal{L}_{bulk}^{4d} + \mathcal{L}_{boundaries} , \quad (2.2)$$

$$\mathcal{L}_{bulk}^{4d} = \int d^{D-4}x \mathcal{L}_{bulk} , \quad (2.3)$$

$$\mathcal{L}_{boundaries} = \int d^{D-4}x \sum_i \delta^{(D-4)}(x - x_i) \mathcal{L}_{boundaries}^{(i)} . \quad (2.4)$$

A tree-level alignment in the Higgs sector appears in a class of models where the bulk theory leads to a four-dimensional Lagrangian with interactions governed by an $N = 2$ extended SUSY. In particular, the SM gauge fields and the DG-adjoint fields arise as $N = 2$ vector supermultiplets, and the two Higgs chiral superfields \mathbf{H}_d and \mathbf{H}_u form an $N = 2$ hypermultiplet, interacting through the superpotential

$$\mathcal{L}_{bulk}^{4d} \supset \int d^2\theta \{ \mu \mathbf{H}_u \cdot \mathbf{H}_d + \lambda_S \mathbf{S} \mathbf{H}_u \cdot \mathbf{H}_d + 2\lambda_T \mathbf{H}_d \cdot \mathbf{T} \mathbf{H}_u \} , \quad (2.5)$$

where $\mathbf{T} \equiv \frac{1}{2}\sigma^a \mathbf{T}^a$, and the dot product is defined as

$$\mathbf{H}_u \cdot \mathbf{H}_d \equiv \epsilon_{ij} \mathbf{H}_u^i \mathbf{H}_d^j = \mathbf{H}_u^+ \mathbf{H}_d^- - \mathbf{H}_u^0 \mathbf{H}_d^0 . \quad (2.6)$$

The $N = 2$ SUSY has a global $SU(2)_R$ R-symmetry that rotates between the generators of the two $N = 1$ supercharges. The scalar components S , T^a of \mathbf{S} and \mathbf{T}^a , respectively, are singlets of $SU(2)_R$. This R-symmetry rotates then between the auxiliary fields F_Σ^a of the adjoint superfields $\Sigma^a \in \{\mathbf{S}, \mathbf{T}^a\}$ and the auxiliary component D^a of the corresponding chiral gauge superfields $\mathcal{W}_{i\alpha}^a$ for $U(1)_Y$ and $SU(2)_W$. This implies that

$$(\text{Re}(F_\Sigma^a) , D^a , \text{Im}(F_\Sigma^a)) \quad (2.7)$$

form a triplet of $SU(2)_R$. As a consequence, in order that the interactions (2.5) of \mathbf{S} and \mathbf{T}^a with the two Higgs doublets preserve $SU(2)_R$, the couplings λ_S and λ_T must be related through¹

$$\lambda_S = \frac{g_Y}{\sqrt{2}}, \quad \text{and} \quad \lambda_T = \frac{g_2}{\sqrt{2}} \quad (2.8)$$

to the couplings g_Y and g_2 of the $U(1)_Y$ and $SU(2)_W$ gauge groups, respectively. Below the scale where the $N = 2$ SUSY is broken to $N = 1$, these relationships are spoiled by a small amount through renormalisation group running, so in numerical evaluations we must treat the couplings λ_S and λ_T as independent parameters.

In addition to the $SU(2)_R$ R-symmetry which is broken in $N = 1$ (chiral) sectors, there is a global $U(1)_R$ R-symmetry under which the superspace coordinates θ^α carry a -1 charge. The $U(1)_R$ charges of the \mathbf{H}_u and \mathbf{H}_d superfields are R_{H_u} and R_{H_d} ,

¹In the discussion of the W boson mass, we shall relax this condition and study also generic Minimal Dirac Gaugino models with arbitrary values for λ_S and λ_T . All of the description of the models presented in this section holds for these models except for the $N = 2$ SUSY and $SU(2)_R$ global R-symmetry that are broken.

respectively. They are arbitrary but subject to the constraint $R_{H_u} + R_{H_d} = 2$. The DG-adjoint superfields \mathbf{S} , \mathbf{T}^a , and \mathbf{O}^a are R-neutral. Below, we shall classify the different $N = 1$ interactions following whether they preserve or break the $U(1)_R$ symmetry.

The boundary Lagrangian can be split into different contributions:

$$\mathcal{L}_{boundaries}^{4d} = \mathcal{L}_{localised}^{bulk} + \int d^2\theta \{W_{Yukawa} + W_{DG} + W_{NR}\} + \Delta\mathcal{L}^{soft}. \quad (2.9)$$

Here, we denote by $\mathcal{L}_{localised}^{bulk}$ kinetic and interaction terms already present in the bulk theory \mathcal{L}_{bulk}^{4d} but appearing with relative coefficients that violate $N = 2$ supersymmetry. Such terms can a priori be present because the boundary theory preserves only $N = 1$ SUSY, thus the coefficients of these terms are less constrained. Here, for simplicity, we assume such terms to vanish at tree level, to be only generated by quantum loops after supersymmetry breaking, and will therefore be accounted for in our analysis, at least in part, through the radiative corrections.

Also in (2.9), we have the usual MSSM Yukawa superpotential W_{Yukawa} with the couplings responsible for the quark and lepton masses:

$$W_{Yukawa} = Y_u^{ij} \mathbf{U}^c_i \mathbf{Q}_j \cdot \mathbf{H}_u - Y_d^{ij} \mathbf{D}^c_i \mathbf{Q}_j \cdot \mathbf{H}_d - Y_e^{ij} \mathbf{E}^c_i \mathbf{L}_j \cdot \mathbf{H}_d, \quad (2.10)$$

which arises on the brane where the matter field supermultiplets are localised.

In this work, we consider a typical scale for the soft terms, for example squarks or gaugino masses, to be in the phenomenologically interesting range $m_{soft} \sim \text{TeV}$. If we denote by Λ a higher scale, for instance related to supersymmetry breaking messenger mass scale or to the Planck scale, then we can consider the relative strength of the diverse SUSY-breaking terms as an expansion in powers of $\frac{m_{soft}}{\Lambda}$. We will assume that SUSY-breaking terms in the gauge sector preserve the $U(1)_R$ R-symmetry, giving rise to Dirac gaugino masses, while Majorana masses might be generated only by higher-order interaction terms, therefore suppressed by additional powers of hidden-sector couplings and/or $\frac{m_{soft}}{\Lambda}$ where Λ could be the Planck scale (for gravity-induced effects). The effective superpotential for the Dirac gaugino masses reads

$$W_{DG} = \sqrt{2}\theta^\alpha \left[m_{DY} \mathbf{S} \mathcal{W}_{1\alpha} + 2m_{D2} \text{tr}(\mathbf{T} \mathcal{W}_{2\alpha}) + 2m_{D3} \text{tr}(\mathbf{O} \mathcal{W}_{3\alpha}) \right], \quad (2.11)$$

where $\mathbf{O} \equiv \frac{1}{2} \lambda^a \mathbf{O}^a$, and $\mathcal{W}_{i\alpha}$ are the chiral gauge-strength superfields. Finally, the superpotential W_{NR} contains terms that break explicitly the $U(1)_R$ R-symmetry:

$$\begin{aligned} W_{NR} = & \xi_S \mathbf{S} + \frac{M_S}{2} \mathbf{S}^2 + \frac{\kappa}{3} \mathbf{S}^3 + \lambda_{ST} \mathbf{S} \text{tr}(\mathbf{T}\mathbf{T}) + \lambda_{SO} \mathbf{S} \text{tr}(\mathbf{O}\mathbf{O}) \\ & + M_T \text{tr}(\mathbf{T}\mathbf{T}) + M_O \text{tr}(\mathbf{O}\mathbf{O}) + \frac{\lambda_O}{3} \text{tr}(\mathbf{O}\mathbf{O}\mathbf{O}). \end{aligned} \quad (2.12)$$

The soft SUSY-breaking Lagrangian $\Delta\mathcal{L}^{soft}$ can in turn be split in two parts. The first contains the scalar mass and interaction terms that preserve the $U(1)_R$ R-

symmetry:

$$\begin{aligned}
-\Delta\mathcal{L}_R^{soft} &= m_{H_u}^2 |H_u|^2 + m_{H_d}^2 |H_d|^2 + m_S^2 |S|^2 + 2m_T^2 \text{tr}(T^\dagger T) + 2m_O^2 \text{tr}(O^\dagger O) \\
&+ (m_Q^2)^{ij} Q_i^\dagger Q_j + (m_U^2)^{ij} U_i^{c\dagger} U_j^c + (m_D^2)^{ij} D_i^{c\dagger} D_j^c + (m_L^2)^{ij} L_i^\dagger L_j + (m_E^2)^{ij} E_i^{c\dagger} E_j^c \\
&+ \left(t_S S + \frac{1}{2} B_S S^2 + \frac{1}{3} T_\kappa S^3 + T_{ST} S \text{tr}(TT) + T_{SO} S \text{tr}(OO) \right. \\
&\quad \left. + B_T \text{tr}(TT) + B_O \text{tr}(OO) + \frac{1}{3} T_O \text{tr}(OOO) + \text{h.c.} \right). \tag{2.13}
\end{aligned}$$

The second part of $\Delta\mathcal{L}^{soft}$ contains the scalar mass and interaction terms that break the $U(1)_R$ R-symmetry:

$$\begin{aligned}
-\Delta\mathcal{L}_{NR}^{soft} &\supset B_\mu H_u \cdot H_d + T_S S H_u \cdot H_d + 2T_T H_d \cdot T H_u \\
&+ T_u^{ij} U_i^c Q_j \cdot H_u - T_d^{ij} D_i^c Q_j \cdot H_d - T_e^{ij} E_i^c L_j \cdot H_d + \text{h.c.}, \tag{2.14}
\end{aligned}$$

as well as the Majorana mass terms M_i (with $i = 1, 2, 3$) for the gauginos. In general, the mechanisms that break R-symmetry and SUSY could be independent of each other, hence in eqs. (2.13) and (2.14) we refrained from defining the soft SUSY-breaking trilinear couplings as proportional to the corresponding superpotential couplings. In this work we assume that the soft SUSY-breaking Higgs-sfermion-sfermion interactions in the second line of eq. (2.14) are suppressed with respect to the R-conserving sfermion mass terms in eq. (2.13). This can be realised in our D -dimensional picture if the quark and lepton superfields are localised on a brane that differs from the one where the breaking of the R-symmetry takes place.

Since our study focuses on the electroweak sector of Dirac Gaugino models, we assume for simplicity that the scalar octet O^a is heavy and can be integrated out of the theory. To insulate the singlet sector from threshold corrections involving the heavy octet, we also neglect the singlet-octet interaction term proportional to λ_{SO} in the R-violating part of the superpotential, eq. (2.12), as well as the analogous term proportional to T_{SO} in the R-conserving part of the soft SUSY-breaking Lagrangian, eq. (2.13). Similarly, since they cannot appear in some of our scenarios, for simplicity in the following we shall also neglect λ_{ST}, T_{ST} and the tadpole terms t_S, ξ_S .

2.2 The electroweak scalar sector and alignment

We can now discuss the neutral scalar sector of this class of models. The vacuum expectation values² (vevs) of the neutral components of the doublets H_d and H_u are related by $v_u^2 + v_d^2 = v^2$, where $v \simeq 246$ GeV is the electroweak scale, and we define $\tan\beta = v_u/v_d$. The neutral singlet and triplet scalars S and T^0 obtain vevs $\langle S \rangle = v_S/\sqrt{2}$ and $\langle T^0 \rangle = v_T/\sqrt{2}$, respectively. These lead to effective μ and B_μ

²It should be emphasised that, throughout this work, we assume that CP symmetry is not spontaneously broken by the vacuum. Therefore, all vevs are real.

parameters:

$$\mu_{\text{eff}} \equiv \mu + \frac{1}{\sqrt{2}}(\lambda_S v_S + \lambda_T v_T), \quad (2.15)$$

$$B_{\mu, \text{eff}} \equiv B_\mu + \frac{1}{\sqrt{2}}(\lambda_S M_S + T_S)v_S + \frac{1}{\sqrt{2}}(\lambda_T M_T + T_T)v_T + \frac{1}{2}\lambda_S \kappa v_S^2. \quad (2.16)$$

The vevs v_S and v_T are then determined as a solution for the coupled cubic equations:

$$\begin{aligned} \kappa^2 v_S^3 + \frac{1}{\sqrt{2}}(T_\kappa + 3\kappa M_S)v_S^2 + \tilde{m}_{S_R}^2 v_S \\ + \frac{v^2}{2} \left[\sqrt{2}\lambda_S \mu_{\text{eff}} - g_Y m_{D_Y} c_{2\beta} - \left(\frac{1}{\sqrt{2}}T_S + \frac{1}{\sqrt{2}}\lambda_S M_S + \lambda_S v_S \kappa \right) s_{2\beta} \right] = 0, \end{aligned} \quad (2.17)$$

$$\tilde{m}_{T_R}^2 v_T + \frac{v^2}{2} \left[\sqrt{2}\lambda_T \mu_{\text{eff}} + g_2 m_{D_2} c_{2\beta} - \frac{1}{\sqrt{2}}(T_T + \lambda_T M_T) s_{2\beta} \right] = 0, \quad (2.18)$$

where

$$\begin{aligned} \tilde{m}_{S_R}^2 &\equiv M_S^2 + m_S^2 + B_S + 4m_{D_Y}^2, \\ \tilde{m}_{T_R}^2 &\equiv M_T^2 + m_T^2 + B_T + 4m_{D_2}^2, \end{aligned} \quad (2.19)$$

are effective mass-squared parameters (at zero expectation value) for the real components S_R and T_R^0 of the neutral singlet and triplet scalars (the analogous masses for the imaginary components S_I and T_I^0 are $\tilde{m}_{S_I}^2 = M_S^2 + m_S^2 - B_S$ and $\tilde{m}_{T_I}^2 = M_T^2 + m_T^2 - B_T$, respectively). We know that v_T must be small – namely, less than a few GeV – to avoid an overlarge tree-level $\Delta\rho$, so to a good approximation we can set $v_T = 0$ in the vacuum minimisation equation for v_S , eq. (2.17), and decouple it from the one for v_T , eq. (2.18); this would allow the cubic equation for v_S to be solved using standard techniques. However, the current state of technology for the computation of loop corrections assumes that we take expectation values as being valid for the true minimum of the full quantum-corrected potential, so in our numerical studies we must take them as *inputs*. Especially for v_T this can lead to complications; see [91] for a recent discussion of this issue.

To discuss the alignment in the Higgs sector, it is now convenient to introduce the so-called Higgs basis for the two doublets,

$$\Phi_1 \equiv \frac{v_d \Phi_d + v_u \Phi_u}{v}, \quad \Phi_2 \equiv \frac{-v_u \Phi_d + v_d \Phi_u}{v}, \quad (2.20)$$

where we defined for convenience two doublets with positive hypercharge, $\Phi_u^j \equiv H_u^j$ and $\Phi_d^j \equiv -\epsilon_{ij} H_d^{*j}$. In the Higgs basis the two doublets can be decomposed as

$$\Phi_1 = \begin{pmatrix} G^+ \\ (v + h + iG^0)/\sqrt{2} \end{pmatrix}, \quad \Phi_2 = \begin{pmatrix} H^+ \\ (H + iA)/\sqrt{2} \end{pmatrix}, \quad (2.21)$$

i.e., only the neutral component of Φ_1 has a non-zero vev, and the would-be-Goldstone bosons, G^\pm and G^0 , all lie in Φ_1 . In general, the neutral CP-even fields h and H

mix with the neutral CP-even components of the singlet and the triplet. In the basis $\{h, H, S_R, T_R^0\}$, the tree-level mass matrix reads

$$(\mathcal{M}^2)_{\text{tree}} = \begin{pmatrix} M_Z^2 + \Delta_h s_{2\beta}^2 & \Delta_h s_{2\beta} c_{2\beta} & \Delta_{hs} & \Delta_{ht} \\ \Delta_h s_{2\beta} c_{2\beta} & M_A^2 - \Delta_h s_{2\beta}^2 & \Delta_{Hs} & \Delta_{Ht} \\ \Delta_{hs} & \Delta_{Hs} & \tilde{m}_S^2 & \lambda_S \lambda_T \frac{v^2}{2} \\ \Delta_{ht} & \Delta_{Ht} & \lambda_S \lambda_T \frac{v^2}{2} & \tilde{m}_T^2 \end{pmatrix}, \quad (2.22)$$

where

$$\Delta_h = \frac{v^2}{2}(\lambda_S^2 + \lambda_T^2) - M_Z^2, \quad (2.23)$$

$$\Delta_{hs} = v \left[\sqrt{2} \lambda_S \mu_{\text{eff}} - g_Y m_{DY} c_{2\beta} - \left(\frac{1}{\sqrt{2}} (T_S + \lambda_S M_S) + v_S \kappa \lambda_S \right) s_{2\beta} \right] \quad (2.24)$$

$$\Delta_{Hs} = v \left[- \left(\frac{1}{\sqrt{2}} (T_S + \lambda_S M_S) + v_S \kappa \lambda_S \right) c_{2\beta} + g_Y m_{DY} s_{2\beta} \right], \quad (2.25)$$

$$\Delta_{ht} = v \left[\sqrt{2} \lambda_T \mu_{\text{eff}} + g_2 m_{D2} c_{2\beta} - \frac{1}{\sqrt{2}} (T_T + \lambda_T M_T) s_{2\beta} \right], \quad (2.26)$$

$$\Delta_{Ht} = -v \left[\frac{1}{\sqrt{2}} (T_T + \lambda_T M_T) c_{2\beta} + g_2 m_{D2} s_{2\beta} \right], \quad (2.27)$$

$$M_A^2 = \frac{2B_{\mu, \text{eff}}}{s_{2\beta}}, \quad (2.28)$$

$$\tilde{m}_S^2 = \tilde{m}_{SR}^2 + \lambda_S^2 \frac{v^2}{2} - \kappa \lambda_S \frac{v^2}{2} s_{2\beta} + 3\kappa^2 v_S^2 + \sqrt{2} v_S (T_\kappa + 3\kappa M_S), \quad (2.29)$$

$$\tilde{m}_T^2 = \tilde{m}_{TR}^2 + \lambda_T^2 \frac{v^2}{2}. \quad (2.30)$$

Exact alignment in the Higgs sector is obtained when one of the eigenstates of the CP-even mass matrix – in this work, we take it to be the lightest one – is aligned in field space with the direction of the SM Higgs vev, and thus has SM-like couplings to gauge bosons and matter fermions. This is equivalent to requiring that h itself be an eigenstate of \mathcal{M}^2 , or in other words that $\mathcal{M}_{1j}^2 = 0$ with $j = 2, 3, 4$. If we make the reasonable assumption that the triplet is heavy, then this can be relaxed to just $j = 2, 3$. In addition, we will also refer in this work to cases where the singlet can be light without being potentially ruled out by direct searches. In this case we will require the supplementary condition that $\mathcal{M}_{23}^2 = 0$.

We start our discussion by focusing on the alignment between the two doublets. The use in eq. (2.23) of the $N = 2$ condition for the singlet and triplet superpotential couplings, see eq. (2.8), implies $\Delta_h = 0$ and $M_h = M_Z$, i.e. alignment is automatically realised at the tree level in this class of models, and the tree-level mass of the SM-like Higgs boson is independent of $\tan \beta$ but well below the value observed at the LHC. It is however well known that, in SUSY models, the radiative corrections to the Higgs mass matrix play a crucial role in lifting the prediction for the mass of the SM-like Higgs boson up to the observed value. Moreover, the radiative corrections to the condition in eq. (2.8) for the superpotential couplings of the adjoint superfields can

become relevant if the scale $M_{N=2}$ where the $N = 2$ SUSY is broken to $N = 1$ is much larger than the scale where the Higgs mass matrix is computed. All of these corrections inevitably affect also the condition for alignment in the Higgs sector. As was discussed in ref. [66], in DG models the element that mixes the two doublets in the loop-corrected mass matrix can be recast as

$$\mathcal{M}_{12}^2 = \frac{1}{\tan\beta} [\mathcal{M}_{11}^2 - M_Z^2] - \frac{v^2 \tan\beta}{1 + \tan^2\beta} \left[\left(\lambda_S^2 - \frac{g'^2}{2} \right) + \left(\lambda_T^2 - \frac{g^2}{2} \right) \right] + (\dots), \quad (2.31)$$

where \mathcal{M}_{11}^2 contains the dominant one-loop contribution from top and stops. The latter consists in a term enhanced by $y_t^4 \ln(M_{\text{SUSY}}^2/m_t^2)$, where y_t is the top Yukawa coupling and M_{SUSY} denotes for simplicity a common soft SUSY-breaking mass parameter for the stops. The second term in eq. (2.31) accounts for the deviation of the superpotential couplings from the $SU(2)_R$ condition in eq. (2.8), and the ellipses denote one-loop top/stop contributions that are suppressed by small ratios of parameters, one-loop contributions that involve couplings other than y_t , and higher-loop contributions. Close to alignment, the loop-corrected mass-matrix element \mathcal{M}_{11}^2 can be empirically identified with the observed mass of the SM-like Higgs boson, $M_h^2 \approx 2 M_Z^2$. Therefore, eq. (2.31) shows that the radiative corrections included in \mathcal{M}_{11}^2 tend to destroy the tree-level alignment in the Higgs sector of DG models. However, when $M_{N=2}$ is large the evolution of λ_S and λ_T down to the scale where the Higgs mass matrix is computed makes the second term in eq. (2.31) negative, and partially compensates for the misalignment induced by the top/stop contributions.

It is instructive to compare the condition for doublet alignment in DG models with the analogous conditions in the MSSM and in the NMSSM. In the case of the MSSM, discussed e.g. in refs. [6, 92], one finds

$$\mathcal{M}_{12}^2 = \frac{1}{\tan\beta} [\mathcal{M}_{11}^2 - M_Z^2 c_{2\beta}] + \frac{6 y_t^2 m_t^2 \mu A_t}{16\pi^2 M_{\text{SUSY}}^2} \left(1 - \frac{A_t^2}{6 M_{\text{SUSY}}^2} \right) + (\dots), \quad (2.32)$$

where $A_t \equiv T_u^{33}/y_t$ is the soft SUSY-breaking Higgs-stop-stop interaction parameter, and again the ellipses denote sub-dominant terms. It appears that, in the MSSM, the alignment condition $\mathcal{M}_{12}^2 = 0$ can be realised radiatively when a large value of $\tan\beta$ suppresses the first term in eq. (2.32), while the parameters M_{SUSY} , A_t and μ combine in such a way that the second term is large and negative. In contrast, in DG models the contributions to \mathcal{M}_{12}^2 analogous to the second term in eq. (2.32) are suppressed by the assumption that $A_t \ll M_{\text{SUSY}}$, see the comments after eq. (2.14), thus doublet alignment cannot be realised in this way. We remark however that, even with the $N = 2$ condition for the superpotential couplings, in DG models \mathcal{M}_{12}^2 is smaller by a factor between 2 and 3 – depending on $\tan\beta$, which we assume to be greater than 1 – with respect to the case of the MSSM with small A_t .

In the case of the NMSSM, discussed e.g. in refs. [7, 9], the mixing between h and H is given by

$$\mathcal{M}_{12}^2 = \frac{1}{\tan\beta} [\mathcal{M}_{11}^2 - M_Z^2 c_{2\beta} - \lambda_S^2 v^2 s_\beta^2] + \frac{6 y_t^2 m_t^2 \tilde{\mu} A_t}{16\pi^2 M_{\text{SUSY}}^2} \left(1 - \frac{A_t^2}{6 M_{\text{SUSY}}^2} \right) + (\dots), \quad (2.33)$$

where $\tilde{\mu} \equiv \mu + \lambda_S v_S / \sqrt{2}$. Comparing with the case of the MSSM, eq. (2.32), we see that the condition $\mathcal{M}_{12}^2 = 0$ can be realised even in the absence of a large contribution from the terms proportional to $\mu_{\text{eff}} A_t / M_{\text{SUSY}}^2$, as long as the singlet-doublet superpotential coupling takes values in the range $\lambda_S^2 \approx (3-4) M_Z^2 / v^2$, where the exact numerical coefficient depends on the value of $\tan \beta$. As first pointed out in ref. [7], this condition singles out the region of the NMSSM parameter space where $\lambda_S \approx 0.7 \pm 0.05$, a much larger value than would be implied by the $SU(2)_R$ condition in DG models.

To summarise, the $SU(2)_R$ R-symmetry implies exact alignment at the tree level in the Higgs-doublet sector of the DG models, but the alignment is partially spoiled by the radiative corrections that are necessary to obtain a realistic value for the SM-like mass. Alignment in the MSSM can be realised only through radiative corrections, for large $\tan \beta$ and for specific choices of the parameters in the stop sector. Finally, doublet alignment in the NMSSM can be realised even without the help of radiative corrections for an appropriate choice of λ_S , which – differently from the DG case with $SU(2)_R$ R-symmetry – is treated as a free parameter.

The second condition for Higgs alignment in DG models is $\mathcal{M}_{13}^2 = 0$, i.e. vanishing mixing between h and S_R . Including the dominant contributions from stop loops, we find:

$$\mathcal{M}_{13}^2 = \Delta_{hs} - \frac{6y_t \lambda_S c\beta}{16\pi^2} m_t (A_t - \mu_{\text{eff}} \cot \beta) \ln \frac{M_{\text{SUSY}}^2}{Q^2} + (\dots), \quad (2.34)$$

where the tree-level mixing term Δ_{hs} is given in eq. (2.24), μ_{eff} is given in eq. (2.15), and Q is the renormalisation scale at which the parameters entering Δ_{hs} are expressed. We assumed again a common soft SUSY-breaking mass term M_{SUSY} for the stops, and we neglected terms suppressed by powers of $m_t^2 / M_{\text{SUSY}}^2$. The various terms that contribute to Δ_{hs} arise from different sectors of the D -dimensional picture discussed earlier in this section: namely, μ and λ_S enter the bulk superpotential in eq. (2.5); m_{1D} enters the R-conserving boundary superpotential in eq. (2.11); M_S and κ enter the R-violating boundary superpotential in eq. (2.12); T_S enters the R-violating SUSY-breaking Lagrangian in eq. (2.14). Therefore, even if we assume the $N = 2$ SUSY relation of eq. (2.8) between λ_S and g_Y , a vanishing \mathcal{M}_{13}^2 can only result from an accidental cancellation between unrelated terms. We also note that, in contrast to the case of \mathcal{M}_{12}^2 , the radiative correction to \mathcal{M}_{13}^2 is not enhanced by $\tan \beta$ with respect to the tree-level part. Thus, its qualitative impact on our discussion of the alignment conditions is limited, as long as the scale Q is not too far from M_{SUSY} .

The minimum conditions of the scalar potential can be exploited to express the mass parameters for the doublets and the singlet in terms of the other Lagrangian parameters and of the vevs v_d , v_u and v_S . In particular, we obtain a relation between the mass parameter \tilde{m}_{SR}^2 for the real component of the singlet, see eq. (2.19), and the matrix element \mathcal{M}_{13}^2 given in eq. (2.34):

$$\tilde{m}_{SR}^2 = -\frac{v}{2v_S} \mathcal{M}_{13}^2 - \frac{v_S}{\sqrt{2}} \left(T_\kappa + \sqrt{2} \kappa^2 v_S + 3\kappa M_S \right). \quad (2.35)$$

Eq. (2.35) above shows that the condition of vanishing mixing between h and S_R , however realised, carries implications for the mass of the singlet. The diagonal element

for the singlet in the scalar mass matrix is

$$\begin{aligned} \mathcal{M}_{33}^2 &= \tilde{m}_{S_R}^2 + \frac{1}{2}\lambda_S^2 v^2 + \sqrt{2}T_\kappa v_S + \kappa \left(3\kappa v_S^2 + 3\sqrt{2}M_S v_S - \lambda_S s_\beta c_\beta v^2 \right) \\ &\quad + \frac{3y_t^2 \lambda_S^2 c_\beta^2}{32\pi^2} v^2 \ln \frac{M_{\text{SUSY}}^2}{Q^2} + (\dots), \end{aligned} \quad (2.36)$$

where we applied the same approximations as in eq. (2.34) for the one-loop correction in the second line.

We now discuss the simplest case in which the global $U(1)_R$ R-symmetry is preserved in the superpotential but broken by soft SUSY-breaking terms, in which case we can set κ to zero. Since this implies a vanishing quartic self-coupling for the singlet, the stability of the scalar potential requires that we also assume $T_\kappa = 0$, even if the trilinear self-coupling of the singlet resides in the R-conserving part of the soft SUSY-breaking Lagrangian. In this scenario, which we shall refer to as the *aligned MDGSSM*, eq. (2.35) shows that the alignment condition $\mathcal{M}_{13}^2 \approx 0$ requires $\tilde{m}_{S_R}^2 \approx 0$ or $v_S \ll v$. The first of these two options implies that the CP-even mass eigenstate that is mostly singlet is relatively light: setting $\tilde{m}_{S_R}^2 = 0$ and $\kappa = T_\kappa = 0$ in eq. (2.36), and neglecting the small effect of the one-loop correction, we find that the value $\lambda_S \approx 0.7$ favored by the alignment condition for the Higgs doublets in the NMSSM, see eq. (2.34), leads to $\mathcal{M}_{33}^2 \approx (122 \text{ GeV})^2$, whereas the value $\lambda_S \approx 0.25$ implied in our Dirac-gaugino model by the $N = 2$ SUSY relation of eq. (2.8) leads to $\mathcal{M}_{33}^2 \approx (44 \text{ GeV})^2$. We remark that the mixing between S_R and the heavier, non-SM-like scalar H , which is controlled by \mathcal{M}_{23}^2 , is suppressed when $M_A^2 \gg \mathcal{M}_{33}^2$, and would in any case lower the mass of the singlet-like eigenstate.

The definition in eq. (2.19) shows that even the vanishing of $\tilde{m}_{S_R}^2$ requires a cancellation between terms that arise from different sectors of our D -dimensional construction: m_{1D} from the R-conserving boundary superpotential in eq. (2.11), M_S from the R-violating boundary superpotential in eq. (2.12), m_S^2 and B_S from the R-conserving soft SUSY-breaking Lagrangian in eq. (2.13). If such cancellation is not realised, the alternative requirement for alignment implied by eq. (2.35) when $\kappa = 0$ is that $v_S \ll v$. This is not problematic as long as a suitable higgsino mass is provided by the μ term in the bulk superpotential, see eq. (2.5).

Finally, the third condition for Higgs alignment in DG models is $\mathcal{M}_{14}^2 = 0$, i.e. vanishing mixing between h and T_R^0 . The formulas for the relevant mass-matrix elements and for the minimum condition, including the dominant one-loop corrections from top and stop loops, are similar to eqs. (2.34)–(2.36), with the obvious singlet-to-triplet replacements but without terms analogous to those controlled by κ in the singlet case:

$$\mathcal{M}_{14}^2 = \Delta_{ht} - \frac{6y_t \lambda_T c_\beta}{16\pi^2} m_t (A_t - \mu_{\text{eff}} \cot \beta) \ln \frac{M_{\text{SUSY}}^2}{Q^2} + (\dots), \quad (2.37)$$

$$\mathcal{M}_{44}^2 = \tilde{m}_{T_R}^2 + \frac{1}{2}\lambda_T^2 v^2 + \frac{3y_t^2 \lambda_T^2 c_\beta^2}{32\pi^2} v^2 \ln \frac{M_{\text{SUSY}}^2}{Q^2} + (\dots), \quad (2.38)$$

$$\tilde{m}_{T_R}^2 = -\frac{v}{2v_T} \mathcal{M}_{14}^2. \quad (2.39)$$

The discussion of the constraints on the triplet mass induced by the condition of doublet-triplet alignment follows the lines of the discussion of singlet-triplet alignment for $\kappa = 0$, with the important difference that the condition $v_T \ll v$ must in any case be satisfied to avoid an excessive contribution to $\Delta\rho$. As a consequence, it is not necessary to require $\tilde{m}_{TR}^2 \approx 0$ to obtain approximate alignment. Nevertheless, we remark that the condition of exact alignment $\mathcal{M}_{14}^2 = 0$ would imply $\mathcal{M}_{44}^2 \approx M_W^2$ through eqs. (2.37)–(2.39).

2.3 The electroweak fermion sector

We now outline the mass spectrum of the electroweak fermions, which will be relevant for our discussion examining the W boson mass. The neutralino mass matrix, in the basis $\tilde{S}, \tilde{B}, \tilde{T}^0, \tilde{W}^0, \tilde{H}_d^0, \tilde{H}_u^0$ reads:

$$\begin{pmatrix} M_S + \sqrt{2}\kappa v_S & m_{DY} & 0 & 0 & -\frac{\sqrt{2}\lambda_S}{g_Y} M_{ZSW} s_\beta & -\frac{\sqrt{2}\lambda_S}{g_Y} M_{ZSW} c_\beta \\ m_{DY} & M_1 & 0 & 0 & -M_{ZSW} c_\beta & M_{ZSW} s_\beta \\ 0 & 0 & M_T & m_{D2} & -\frac{\sqrt{2}\lambda_T}{g_2} M_{ZCW} s_\beta & -\frac{\sqrt{2}\lambda_T}{g_2} M_{ZCW} c_\beta \\ 0 & 0 & m_{D2} & M_2 & M_{ZCW} c_\beta & -M_{ZCW} s_\beta \\ -\frac{\sqrt{2}\lambda_S}{g_Y} M_{ZSW} s_\beta & -M_{ZSW} c_\beta & -\frac{\sqrt{2}\lambda_T}{g_2} M_{ZCW} s_\beta & M_{ZCW} c_\beta & 0 & -\mu_{\text{eff}} \\ -\frac{\sqrt{2}\lambda_S}{g_Y} M_{ZSW} c_\beta & M_{ZSW} s_\beta & -\frac{\sqrt{2}\lambda_T}{g_2} M_{ZCW} c_\beta & -M_{ZCW} s_\beta & -\mu_{\text{eff}} & 0 \end{pmatrix} \quad (2.40)$$

The chargino masses, $-\frac{1}{2}((v^-)^T \mathcal{M}_{\chi^\pm} v^+ + h.c.)$ in the basis $v^+ = (\tilde{T}^+, \tilde{W}^+, \tilde{H}_u^+)$, $v^- = (\tilde{T}^-, \tilde{W}^-, \tilde{H}_d^-)$, are given by

$$\mathcal{M}_{\chi^\pm} = \begin{pmatrix} M_T & m_{D2} + g_2 v_T & \lambda_T v c_\beta \\ m_{D2} - g_2 v_T & M_2 & \frac{g_2 v}{\sqrt{2}} s_\beta \\ -\lambda_T v s_\beta & \frac{g_2 v}{\sqrt{2}} c_\beta & \mu_{\text{eff}} - \sqrt{2}\lambda_T v_T \end{pmatrix} \quad (2.41)$$

and do not depend on κ , but we have written for completeness the Majorana gaugino masses $M_{1,2}$ for the bino and wino respectively.

2.4 Scenarios

We have described the general features of minimal Dirac Gaugino models, and explained how some values of the couplings allow Higgs alignment to automatically occur at tree-level. Below, we shall consider the following different scenarios corresponding to specific choices of the model parameters:

- **General MDGSSM**

In the general MDGSSM, the only source of R -symmetry violation comes from a small B_μ term, which is a radiatively stable condition (the renormalisation group running will not generate other R -symmetry violating terms from a B_μ term). This excludes all of W_{NR} ; the only supersymmetric parameters beyond those of the MSSM retained are λ_S, λ_T , but those are allowed to have any value. Supersoft masses [13] are allowed, and soft supersymmetry-breaking masses are allowed, but squark/sfermion trilinears and T_S, T_T are not. On the other hand, trilinears involving the adjoint scalars may be allowed (so T_κ, T_{ST}, T_{SO}) but may

be argued to be small in typical models. The couplings λ_S, λ_T enhance the Higgs mass and W mass at the same time; while large values of λ_T were previously deemed problematic for the ρ parameter, they are now a virtue.

- **MSSM without μ term (RIP)**

This model, described in [14], is identical to the general MDGSSM except that the μ -term is set to zero (although a small v_S generates a tiny effective μ). In the MSSM this would yield massless higgsinos, but here the higgsinos obtain a mass through λ_T which causes them to mix with the triplet fermion. Thus the μ problem of the MSSM is solved, at the expense of an upper bound on the chargino masses, which, as we shall see, leads to the model being ruled out.

- **Aligned MDGSSM**

By taking λ_S, λ_T to their their $N = 2$ values given in Eq. (2.8) in the general MDGSSM we guarantee alignment with the heavy Higgs at tree level. If we further impose that $\Delta_{hs} = 0$, as described above, then the singlet also has negligible mixing with the SM-like Higgs. We shall refer to this scenario as the *aligned MDGSSM*.

- **DGNMSSM**

If we instead allow R-symmetry violation in the superpotential and the associated soft-breaking trilinears – in particular for κ, T_κ and T_S – we can generate μ and B_μ terms through a substantial expectation value for the singlet. The model thus resembles the NMSSM, especially if we set the μ, B_μ terms (and M_S, M_T) to zero; this was proposed in [18]. We shall therefore refer to this scenario as the DGNMSSM.

- **Aligned DGNMSSM**

We can achieve alignment in the DGNMSSM by setting λ_S, λ_T to their $N = 2$ values and taking $\Delta_{hs} = 0$. We shall refer to this scenario as the aligned DGNMSSM; in this work, when we consider the W boson mass, we shall also enforce $\Delta_{Hs} = 0$ which guarantees that the singlet couplings to SM fields are small, rendering light singlets safe from collider searches.

3 W mass in Dirac Gaugino models

Dirac gaugino models offer two methods of explaining an enhancement of the W boson mass with respect to the SM: either quantum corrections or a tree-level expectation value for the triplet scalar. In the case of quantum corrections, there are new contributions to the W mass compared to the MSSM, again coming from interactions related to the adjoint triplet; in particular the coupling λ_T .

Recall the definition of the ρ parameter is $M_W^2 \equiv \rho c_W^2 M_Z^2$; through the presence of the adjoint triplet, DG models contain a tree-level modification to this relation compared to the SM:

$$\rho \equiv 1 + \Delta\rho_{\text{tree}} + \Delta\rho = 1 + 4\frac{v_T^2}{v^2} + \Delta\rho. \quad (3.1)$$

One could consider a modification to c_W instead of ρ as an explanation of an enhanced W mass, but this is discounted based on electroweak precision tests (see e.g. [93]); on the other hand, a triplet expectation value is one of the most generic and acceptable ways of enhancing the W mass at tree level. Naively we could then take the observed value of M_Z and the standard value of c_W and infer the value of $\Delta\rho$ to obtain a given value of M_W . However, in the SM and in any BSM theory it is necessary to take certain electroweak observables as input, and in our setup we will take the conventional choice of the Z mass, G_F and α . When we modify ρ this gives both a modification to M_W and a small modification to $\sin^2\theta_W$. At tree-level this is

$$\begin{aligned}\Delta_{\text{tree}}M_W^2 &= \frac{c_W^2}{c_W^2 - s_W^2} (M_W^2)_{\text{SM}} \Delta\rho_{\text{tree}}, \\ \Delta_{\text{tree}}s_W^2 &= -\frac{s_W^2 c_W^2}{c_W^2 - s_W^2} \Delta\rho_{\text{tree}},\end{aligned}\tag{3.2}$$

so if we want to obtain $M_W = 80.424$ MeV we need $\Delta M_W^2 = 11$ (GeV)². Taking $s_W^2 = 0.23121$ this gives

$$\begin{aligned}\Delta^{\text{CDF}}\rho_{\text{tree}} &= 0.0012 \\ \Delta_{\text{tree}}^{\text{CDF}}s_W^2 &= -4 \times 10^{-4}.\end{aligned}\tag{3.3}$$

Interpreted as a tree-level expectation value for the triplet, this yields

$$v_T \simeq 4 \text{ GeV},\tag{3.4}$$

which was previously at the upper bound of what was acceptable.

In the minimal Dirac gaugino model, we have from equation (2.39)

$$\tilde{m}_{TR}^2 = -\frac{v}{2v_T} \Delta_{ht} = -\frac{\Delta_{ht}}{\sqrt{\Delta\rho_{\text{tree}}}}.\tag{3.5}$$

For a small v_T , if we do not tune $\Delta_{ht} \approx 0$, then triplets must be heavy. However, we also need heavy winos to evade collider bounds and this implies large m_{D2} : the connection between electroweakino masses and the expectation value of the triplet is a novel feature of this class of models. In [64] the conclusion was that for winos above 700 GeV there were essentially no constraints on the higgsinos beyond LEP. If the contribution from m_{D2} dominates Δ_{ht} this implies

$$m_T^2 \sim -\frac{g_2 v m_{D2} c_{2\beta}}{\sqrt{\Delta\rho_{\text{tree}}}} \sim (1.8 \text{ TeV})^2 \times \left(\frac{700 \text{ GeV}}{m_{D2}}\right) \times \left(\frac{c_{2\beta}}{-1}\right) \times \sqrt{\frac{0.0012}{\Delta\rho}},\tag{3.6}$$

which is a natural scale for supersymmetric scalars. Of course, we can have lighter winos provided that the neutralino is not too light, above around 200 to 300 GeV [64].

However, the model also contains ample room for quantum corrections to also enhance the W mass. In the following, we shall investigate this for the different scenarios described in section 2.4.

3.1 Numerical setup

In order to accurately compute the quantum corrections to the W mass, we use a new EFT approach, closely related to that of [87], implemented in the spectrum-generator-generator **SARAH**. We use the expression:

$$M_W^2 = (M_W^2)_{SM} \left(1 + \frac{s_W^2}{c_W^2 - s_W^2} \left[\frac{c_W^2}{s_W^2} (\Delta\rho_{\text{tree}} + \Delta\rho) - \Delta r_W - \Delta\alpha \right] \right) \quad (3.7)$$

where $(M_W^2)_{SM}$ is the full two-loop W mass in the SM, as computed in [75], depending on the pole masses of the Higgs boson, top quark, α_s and $\Delta\alpha_{\text{had}}^{(5)}$. We use the interpolating function from that paper. When we use the average values of the Higgs mass of 125.09 GeV and the top quark mass of 172.89 GeV this function gives us $M_W = 80.354$ GeV, just 2 MeV higher than the current world average for the W boson mass in the SM. The expressions computed in the square brackets are the *differences* between the high-energy theory (HET) and the SM:

$$\begin{aligned} \Delta\rho &\equiv \text{Re} \left[\frac{\Pi_{ZZ}^{\text{HET}}(M_Z^2)}{M_Z^2} - \frac{\Pi_{WW}^{\text{HET}}(M_W^2)}{M_W^2} \right] - \text{Re} \left[\frac{\Pi_{ZZ}^{\text{SM}}(M_Z^2)}{M_Z^2} - \frac{\Pi_{WW}^{\text{SM}}(M_W^2)}{M_W^2} \right] \\ \Delta r_W &\equiv \left[\frac{\Pi_{WW}^{\text{HET}}(0)}{M_W^2} - \frac{\Pi_{WW}^{\text{HET}}(M_W^2)}{M_W^2} + \delta_{VB}^{\text{HET}} \right] - \left[\frac{\Pi_{WW}^{\text{SM}}(0)}{M_W^2} - \frac{\Pi_{WW}^{\text{SM}}(M_W^2)}{M_W^2} + \delta_{VB}^{\text{SM}} \right]. \end{aligned} \quad (3.8)$$

$\Delta\alpha$ are now the gauge threshold corrections between the HET and the SM for the electromagnetic gauge coupling divided by α (so they do not now depend on $\Delta\alpha_{\text{had}}^{(5)}$).

In **SARAH**, we compute the expression in square brackets at the matching scale, which is the mass of the heavy particles. We are therefore ignoring the running from that scale down to the electroweak scale, which is of controllable size, but will nevertheless be included in a future development of the code. One could argue that we should instead perform the matching at the electroweak scale, but then all of the loop functions will contain large logarithms and there can be larger, spurious, running of the couplings of the high-energy theory which can spoil the results.

In order to be a strict one-loop matching between the HET and the SM, the weak mixing angle in the above must be the \overline{MS} or \overline{DR} value at the matching scale Q :

$$s_W^2 = \frac{g_Y^2(Q)}{g_2^2(Q) + g_Y^2(Q)}. \quad (3.9)$$

To extract this, we match the calculations of the Z-boson mass, $\alpha(Q)$ – and we also compute the decay of the muon at the matching scale. In practice, this means that we extract the couplings in the SM at the top mass scale without including the effects of new physics, then run them up to the matching scale. The threshold corrections to α between the two theories are simple to compute since it is unbroken and yield $\Delta\alpha$; the couplings $g_Y(Q), g_2(Q)$ in the high-energy theory are chosen to solve the equation

$$c_W^2 s_W^2 = (c_W^2 s_W^2)_{SM} \times \frac{(1 + \Delta\alpha + \Delta\hat{r})}{1 + \Delta\rho_{\text{tree}}} \quad (3.10)$$

where

$$\Delta\hat{r} \equiv \left[(1 + \Delta\rho_{\text{tree}}) \frac{\Pi_{WW}^{HET}(0)}{M_W^2} - \frac{\Pi_{ZZ}^{HET}(M_Z^2)}{M_Z^2} + \delta_{VB}^{HET} \right] - \left[\frac{\Pi_{WW}^{SM}(0)}{M_W^2} - \frac{\Pi_{ZZ}^{SM}(M_Z^2)}{M_Z^2} + \delta_{VB}^{SM} \right], \quad (3.11)$$

which is done iteratively by progressively running up and down and updating at each step, along with all the other quantities in the high-energy theory. The value for $(c_W^2 s_W^2)_{\text{SM}}$ includes a compensatory term for corrections from $\overline{\text{MS}}$ to $\overline{\text{DR}}$ if needed.

In the **SARAH** model file, the couplings $\lambda_S, \lambda_T, T_S, T_T$ are defined differently to the above: we have $\mathbf{1am} = -\lambda_S, \text{LT} = \sqrt{2}\lambda_T, \text{T}[\mathbf{1am}] = -T_S, \text{T}[\text{LT}] = \sqrt{2}T_T$. Hence in our plots and benchmark points we list the values in terms of $-\lambda_S, \sqrt{2}\lambda_T, -T_S$, which makes the correspondence with the numerical codes exact.

3.2 MSSM without μ term

The ‘‘MSSM without μ term’’ proposed in [14] was an intriguing solution to the μ problem. It was however challenged by the requirement of having a high enough Higgs mass, chargino mass and not too large ρ ; indeed the lack of intersection of points satisfying the latter two was demonstrated in [35]. It might therefore be tempting to revisit this model in light of the new data about the W mass. However, we shall demonstrate here that it is conclusively ruled out.

Putting aside the Higgs mass constraint, the see-saw effect on the charginos is a problem. LEP put a lower limit on the mass of the lightest chargino of 94 GeV [94–96]. In that model the chargino mass is

$$\mathcal{M}_{\chi^\pm} \xrightarrow{v_T \simeq 0} \begin{pmatrix} 0 & m_{D2} & \frac{2\lambda_T}{g^2} M_Z c_W c_\beta \\ m_{D2} & 0 & \sqrt{2} M_Z c_W s_\beta \\ -\frac{2\lambda_T}{g} M_Z c_W s_\beta & \sqrt{2} M_Z c_W c_\beta & 0 \end{pmatrix} \quad (3.12)$$

It is known that it is possible to fulfil the LEP bound by a careful choice of λ_T and m_{D2} : a large value of λ_T as well as m_{D2} around 107 GeV is needed to maximize the mass of the lightest chargino. It would then be made of a higgsino-wino mixture with *two* charginos that are light and one (wino-like) somewhat heavier. Unfortunately, subsequent LHC searches are especially sensitive to winos up to about 800 GeV, see [64]. It might be possible to evade this constraint if the light wino and neutralino are close enough in mass so that decays such as $\tilde{\chi}^\pm \rightarrow \tilde{\chi}^0 + W^\pm$ are not possible. Most likely this is difficult or impossible to achieve, but without a detailed investigation we cannot exclude the possibility that some region of parameter space might evade direct LHC searches; we can only apply the LEP constraint as a hard lower bound on the chargino mass.

In [35] it was demonstrated that that the corrections to $\Delta\rho$ were correlated with the lightest chargino mass; in order to evade the LEP bound, λ_T has to be large and this drives large $\Delta\rho$. Here we can give a striking confirmation of this observation by plotting the W mass against the lightest chargino mass for a sample of $\mathcal{O}(70000)$ spectra generated using **SARAH**. We fix the octet scalar, and all squark and slepton soft masses via $m_O^2 = m_q^2 = m_l^2 = 10 \text{ TeV}^2$, and fix the gluino mass to 3 TeV; this

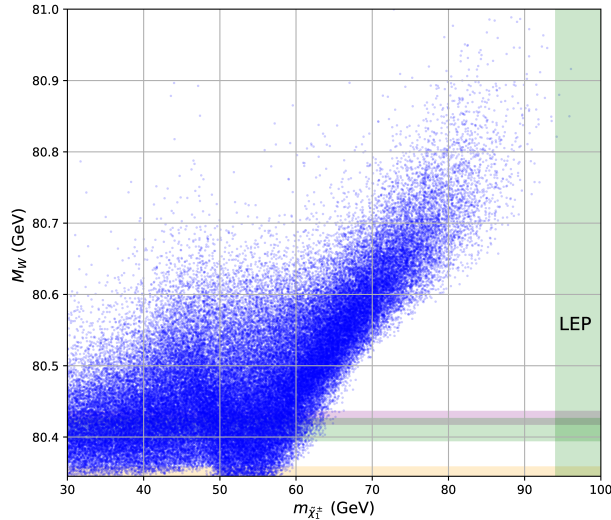


Figure 1: W boson mass vs lightest chargino mass for points generated in the ‘MSSM without μ -term.’ The horizontal bands show the SM, Tevatron and world average masses for the W boson mass; the vertical green band shows the LEP constraint on the lightest chargino.

ensures that they are beyond all current and near-future bounds. Then we scan over the ranges:

$$m_{D_Y} \in [100, 500] \text{ GeV}, \quad m_{D_2} \in [100, 250] \text{ GeV}, \quad v_S \in [-50, 50] \text{ GeV}, \quad v_T \in [-5, 5] \text{ GeV} \\ B_\mu \in [10^4, 10^6] (\text{GeV})^2, \quad \lambda_T \in [-1, 1], \quad \lambda_S \in [-1.5, 1.5], \quad \tan \beta \in [2, 50]. \quad (3.13)$$

We use a Markov Chain Monte-Carlo (MCMC) algorithm to generate points (this helps to obtain points with non-tachyonic spectra with Higgs mass close to the observed value compared to a random scan). There is not intended to be a genuine statistical interpretation of the distribution of the points, but the overall envelope should show where valid sets of parameters exist. The results are shown in figure 1, where the LEP constraint is shown as a vertical green band. It can be clearly seen that it is not possible to both satisfy the LEP constraint and have an acceptable value for the W boson mass; from the W boson mass alone we would predict a chargino of mass below 65 GeV.

3.3 W mass in the MDGSSM

In the minimal Dirac Gaugino extension of the Standard Model (MDGSSM) the most typical scenario is to assume that R-symmetry is broken only via a B_μ term, i.e. the Higgs sector is special. It is identical to the previous model except that we allow a μ -term. This, however, makes all the difference: now the higgsino mass is not bounded from below, not requiring a large mixing with the winos; and further the enhancement to the Higgs mass is under control.

We perform a new MCMC scan with parameters allowed to vary within the ranges:

$$\begin{aligned}
m_{DY} &\in [100, 700] \text{ GeV}, & m_{D2} &\in [100, 1200] \text{ GeV}, & v_S &\in [-50, 50] \text{ GeV}, & v_T &\in [-5, 5] \text{ GeV} \\
\mu &\in [0, 1000] \text{ GeV}, & B_\mu &\in [10^4, 10^6] (\text{ GeV})^2, \\
\sqrt{2}\lambda_T &\in [-1.5, 1.5], & \lambda_S &\in [-1.5, 1.5], & \tan\beta &\in [2, 50].
\end{aligned}
\tag{3.14}$$

We fix the octet scalar, and all squark and slepton soft masses via $m_O^2 = m_{\tilde{q}}^2 = m_{\tilde{l}}^2 = 10 \text{ TeV}^2$, and fix the gluino mass to 3 TeV. We choose a likelihood function to be a product of a gaussian in the Higgs mass with mean 125 GeV and standard deviation 3 GeV, a gaussian in the W mass with mean 80.413 GeV and standard deviation 20 MeV, and a sigmoid on the constraints (given as the maximum ratio of predicted cross-section to observed, across all channels) from `HiggsBounds5` [97–100], which strongly suppresses the likelihood when the observed cross-section ratio is greater than one, but is otherwise close to unity. This choice of likelihood function is merely a device to select desirable points, and the distribution of the points is not meant to have a statistical interpretation in terms of their Bayesian likelihood; in particular, the theory uncertainty on the Higgs mass does not have a statistical interpretation, and a window of 3 GeV is a conservative estimate of the average error, since we use the latest two-loop corrections in the generalised effective potential and gaugeless limit [60, 101–103] with pole-mass matching onto the SM [104] (see [105] for a recent review). Such a conservative window of 3 GeV is employed because only a relatively small proportion of points actually generate a spectrum, and in principle a two-stage procedure pre-filtering points along those suggested in [64, 106] would probably be more efficient – or in addition the ability to invert the vacuum minimisation relations and compute v_S, v_T instead of treating them as inputs, but this is not yet automatically possible in the code in a way that would correctly incorporate the loop corrections to the Higgs masses [91].

We show plots in figure 2 for M_W against v_T (left plot), and then for M_W against λ_T (right plot), to show the points that benefit from large quantum corrections as the means of enhancing the W mass: the tree-level expectation just from modifying v_T is shown as a solid red curve on the left plot. We also show as a dashed red curve the value of M_W that would be obtained by insisting that the shift in $\Delta\rho$ only modifies M_W without changing $\sin\theta_W$ (if a different method of matching onto the SM parameters were used, for example).

All points shown satisfy all Higgs bounds; have a charged Higgs heavier than 600 GeV (so are safe from $B \rightarrow s\gamma$ constraints [1]); have charginos heavier than the LEP limit and winos heavier than 600 GeV. These baseline selections are marked as blue points in the plots; there are about 36000, of which about 10000 have $|v_T| < 1 \text{ GeV}$. Points shown in yellow further have charginos heavier than 200 GeV, $m_{D2} > 700 \text{ GeV}$; while those marked in green have charginos heavier than 250 GeV and $m_{D2} > 800 \text{ GeV}$ (about 1600 points in our sample survive these cuts). The green points are thus almost certainly guaranteed to be safe from current collider bounds (although they may yet be probed in future). The requirement of heavy charged Higgs scalars sets the MSSM-like neutral and pseudoscalar masses to be heavy, and essentially guarantees the safety of all selected points from constraints on the couplings of the SM-like Higgs. Nevertheless, we also filtered the green points with constraints from `HiggsSignals` [107, 108]. The

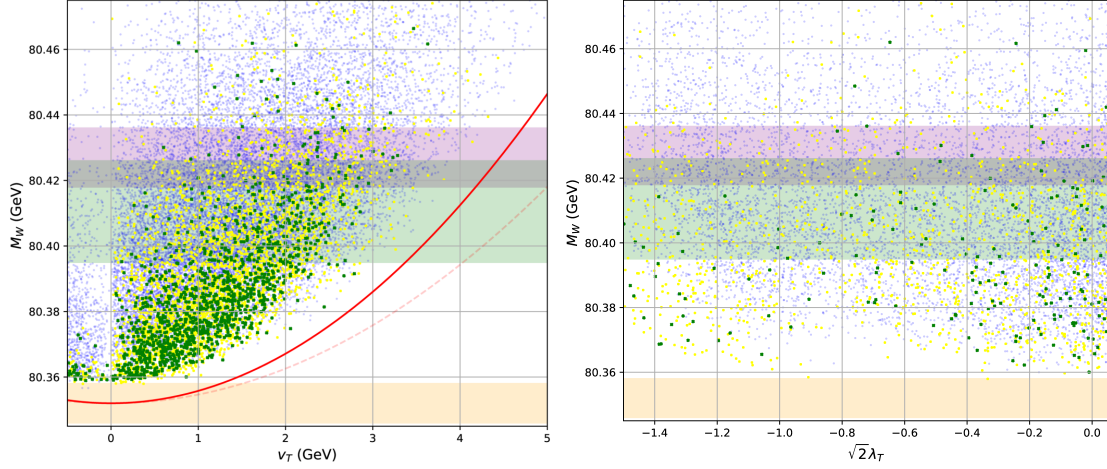


Figure 2: Left: W boson mass vs v_T in the MDGSSM, with the red curve showing the tree-level prediction. Right: W boson mass vs $\sqrt{2}\lambda_T$ in the MDGSSM. Colours of the points are described in the text, with those obeying the strictest cuts shown in green. The colourful horizontal bands show the SM range in light orange; the Tevatron+LEP average in purple, and a conservative world average in green.

	MDG1	MDG2	MDG3
m_{DY} (GeV)	280	285	245
m_{D2} (GeV)	983	941	940
μ (GeV)	276	255	353
$\tan\beta$	48	46	47
$-\lambda_S$	1.179	1.074	1.112
$\sqrt{2}\lambda_T$	-0.487	0.502	0.099
B_μ (GeV) ²	828838	794477	938787
v_S (GeV)	5.0	4.3	1.4
v_T (GeV)	2.3	2.7	2.8
m_{h_1} (GeV)	125.4	124.7	124.7
m_{h_2} (GeV)	3120.9	2274.3	2405.8
m_{A_1} (GeV)	2394.2	1221.4	1456.5
$m_{H_1^\pm}$ (GeV)	2400.2	1213.0	1455.3
$m_{\tilde{\chi}_1^0}$ (GeV)	217.9	231.5	233.0
$m_{\tilde{\chi}_1^\pm}$ (GeV)	255.0	275.9	362.0
m_W (GeV)	80.425	80.420	80.421

Table 1: Benchmark points for the MDGSSM. Input parameters are given above the double line, and masses of the most important particles below.

different categories of points show the expected wider range of enhancements to the W mass as the charginos become lighter. We provide a selection of benchmark points, with the input parameters and crucial data, in table 1.

The first clear observation is that the quantum corrections to the W mass are at least as important as the tree-level contribution from the expectation value v_T , and the generic contribution to the W mass is positive with no points below the red curve. The asymmetry of the plot with v_T is due to the fact that we only take positive Dirac gaugino masses. It is important to note that in the red curves we take the SM value of the W boson mass to be 80.352 GeV, whereas the fitting function of [75] as employed in SARAH gives a value of 80.354 GeV for the central values; and gives 80.356 for a Higgs mass at the lower bound of our permitted range of 122 GeV. However, it is clear that the quantum corrections in our sample are generally more important than $\Delta\rho_{\text{tree}}$. Indeed, with the parameter ranges we have chosen, we are not within the range of masses required for decoupling of the quantum corrections (we have checked that the quantum corrections to M_W smoothly drop to near zero as the masses of all particles are raised to about 2 TeV or higher).

The second observation is that there is no clear correlation between the W mass and the parameter λ_T . Due to our requirement of a large wino mass, the selected points have light neutralinos/charginos of mixed bino/higgsino type. The mass splittings among the higgsinos – and thus the contribution to the W mass – can be driven large by λ_S and λ_T . These couplings also enhance the Higgs mass at tree-level, so large values are favoured in the scans because we fixed the stop masses at $\sqrt{10}$ TeV. In this class of model there is therefore no particular preference for one or the other coupling.

Since the selected points generally have a mixed bino/higgsino LSP, they have good dark matter candidates, but it is expected that the relic density should be underdense. A detailed investigation of the dark matter-collider complementarity for scenarios satisfying the latest W mass data along the lines of [64] would be an interesting subject for future work, provided that the latest LHC analyses can be recast. As mentioned above, the possibility of a large λ_T (and the presence of the singlet scalar/fermion) distinguishes higgsinos in this scenario from those in the MSSM. It is clear that this class of models provides a very natural explanation for an enhancement to the W mass compared to the Standard Model.

3.4 W mass in the aligned MDGSSM

In the aligned MDGSSM (where the only source of R-symmetry breaking is the B_μ term, and we take $T_T = 0$), we choose the parameters to induce alignment at tree-level in the MDGSSM (so $\lambda_S = g_Y/\sqrt{2}$, $\lambda_T = g_2/\sqrt{2}$ and $m_{DY} = c_{2\beta}\mu_{\text{eff}}$) but taking $T_S = 0$ (which means the mixing of the singlet with the heavy Higgs cannot vanish unless it is heavy). To study this scenario we perform a scan with the same strategy as before except that now, since λ_S, λ_T are fixed, it is necessary to vary the masses of the stops/sbottoms to allow us to find the observed value of the Higgs mass; there is also therefore a preference for models with larger $\tan\beta$ since the tree-level contributions to the Higgs mass at low $\tan\beta$ are not sufficient. We therefore use a common mass for the third generation squarks M_{SUSY} ; we set $m_{Q,33}^2 = m_{U,33}^2 = m_{D,33}^2 = M_{\text{SUSY}}^2$ (the other squarks and sleptons we retain fixed at $\sqrt{10}$ TeV). The remaining parameters

	AMDG1	AMDG2	AMDG3	AMDG4	AMDG5	AMDG6
M_{SUSY} (GeV)	8636.2	4550.9	5181.9	8436.1	7357.2	5454.9
m_{D2} (GeV)	1037	1249	1149	1107	1219	994
$\tan \beta$	2	2	3	10	3	2
μ (GeV)	158.2	160.4	165.8	174.1	159.5	201.6
v_S (GeV)	-19.0	-12.7	-8.4	-0.3	-5.0	-13.7
v_T (GeV)	1.5	1.5	1.1	0.3	1.9	1.1
m_{h_1} (GeV)	126.6	122.8	123.2	123.6	127.4	122.2
m_{h_2} (GeV)	457.7	312.3	308.5	421.8	465.8	487.5
m_{h_3} (GeV)	1282.2	799.7	935.2	1251.6	1176.5	762.3
m_{h_4} (GeV)	2667.0	2965.0	3350.8	7104.7	2847.1	2933.0
$m_{H_1^\pm}$ (GeV)	1284.0	793.5	933.4	1257.7	1174.3	765.4
$m_{\tilde{\chi}_1^0}$ (GeV)	106.3	110.4	120.5	172.6	128.0	134.5
$m_{\tilde{\chi}_1^\pm}$ (GeV)	168.4	169.6	177.3	193.1	173.1	211.5
m_W (GeV)	80.363	80.365	80.362	80.361	80.369	80.362

Table 2: Benchmark points with a light singlet in the Aligned MDGSSM

are scanned via the same MCMC algorithm in the ranges:

$$\begin{aligned}
m_{D2} &\in [400, 1500] \text{ GeV}, & v_S &\in [-250, 250] \text{ GeV}, & v_T &\in [-5, 5] \text{ GeV} \\
\mu &\in [-1000, 1000] \text{ GeV}, & B_\mu &\in [10^4, 10^6] (\text{GeV})^2, \\
M_{\text{SUSY}} &\in [2, 10] \text{ TeV}, & \tan \beta &\in [2, 50].
\end{aligned} \tag{3.15}$$

We mostly find points with small μ/m_{DY} and very little enhancement to the W mass because the electroweakinos tend to be light. We give benchmark points in table 2 and plots in figure 3 which demonstrate the lack of enhancement and scarcity of points (790 survived from a scan for one million).

3.5 W mass in the aligned DGNMSSM

We turn now to the case of the aligned DGNMSSM described in section 2.4. We set the couplings λ_S, λ_T to their $N = 2$ values and then choose m_{DY} and T_S to make Δ_{hs} and Δ_{Hs} vanish. This leads to

$$\begin{aligned}
m_{DY} &= c_{2\beta} \mu_{\text{eff}} \\
T_S &= -g_Y (\kappa v_S - \sqrt{2} s_{2\beta} \mu_{\text{eff}}).
\end{aligned} \tag{3.16}$$

This has an interesting consequence because in this model

$$B_{\mu, \text{eff}} = \frac{1}{\sqrt{2}} T_S v_S + \frac{1}{2} \lambda_S \kappa v_S^2 = -\frac{1}{2\sqrt{2}} g_Y \kappa v_S^2 + g_Y s_{2\beta} v_S \mu_{\text{eff}}. \tag{3.17}$$

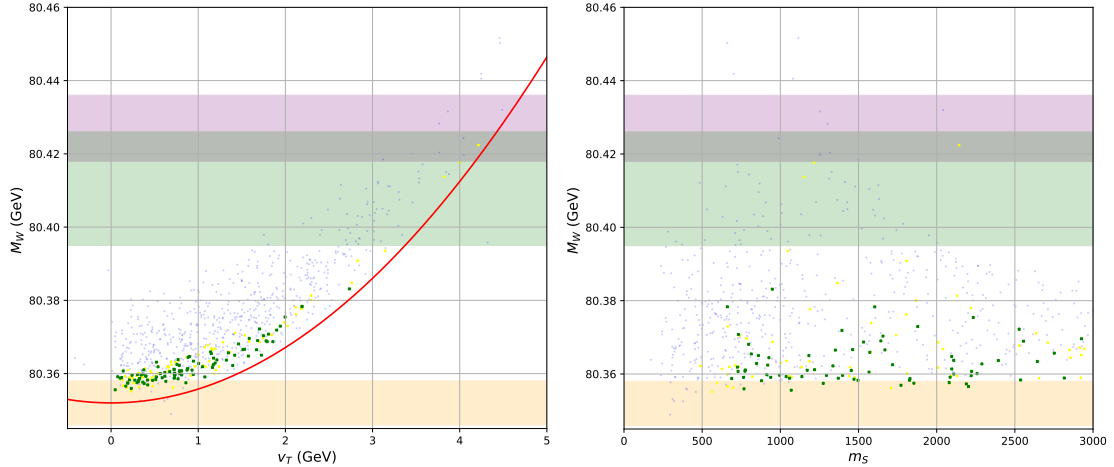


Figure 3: Points in the aligned MDGSSM. Left: M_W against triplet expectation value. Right: M_W against singlet-like scalar mass. Colour coding as for previous plots.

Then

$$M_A^2 = g_Y v_S \left(2\mu_{\text{eff}} - \frac{\kappa v_S}{\sqrt{2}s_{2\beta}} \right) = g_Y v_S^2 \left(g_Y - \frac{\kappa}{\sqrt{2}s_{2\beta}} \right). \quad (3.18)$$

We perform a scan using the same strategy as the previous sections; we use a common mass for the third generation squarks M_{SUSY} ; we set $m_{Q,33}^2 = m_{U,33}^2 = m_{D,33}^2 = M_{\text{SUSY}}^2$ (the other squarks and sleptons we retain fixed at $\sqrt{10}$ TeV). Then we scan with the parameter ranges, using the same likelihood function as before:

$$\begin{aligned} M_{\text{SUSY}} &\in [2000, 10000] \text{ GeV}, \\ m_{D2} &\in [400, 1500] \text{ GeV}, \quad v_S \in [-1500, 1500] \text{ GeV}, \quad v_T \in [-5, 5] \text{ GeV} \\ \kappa &\in [-1.5, 1.5], \quad T_\kappa \in [-2000, 2000] \text{ GeV}, \quad \tan \beta \in [2, 50]. \end{aligned} \quad (3.19)$$

In our scans we impose that all Higgs searches are satisfied using `HiggsBounds` and `HiggsSignals`. We show the results for the W boson mass in figure 4 where the points have the same colour coding as in the previous sections. It is apparent that in this model it is complicated to enhance the W boson mass. This is because we have only a small quantum effect from λ_S, λ_T , but also because the lightest neutralinos are typically rather light: since $\mu_{\text{eff}} = g_Y v_S / 2$ we need a large $v_S \gtrsim 500$ GeV (or 1 TeV for our more stringent points) to have heavy enough higgsinos. Then we need κ negative and not too small to avoid a too-small pseudoscalar/charged Higgs mass (if we neglect κ then M_A is bounded by $g_Y v_S$, so $M_A > 600$ GeV requires $v_S \gtrsim 1700$ GeV, at the limit of our search range). So this implies that the singlino is generally heavy compared to m_{DY} :

$$\frac{m_{DY}}{\sqrt{2}\kappa v_S} = \frac{g_Y c_{2\beta}}{2\sqrt{2}\kappa}. \quad (3.20)$$

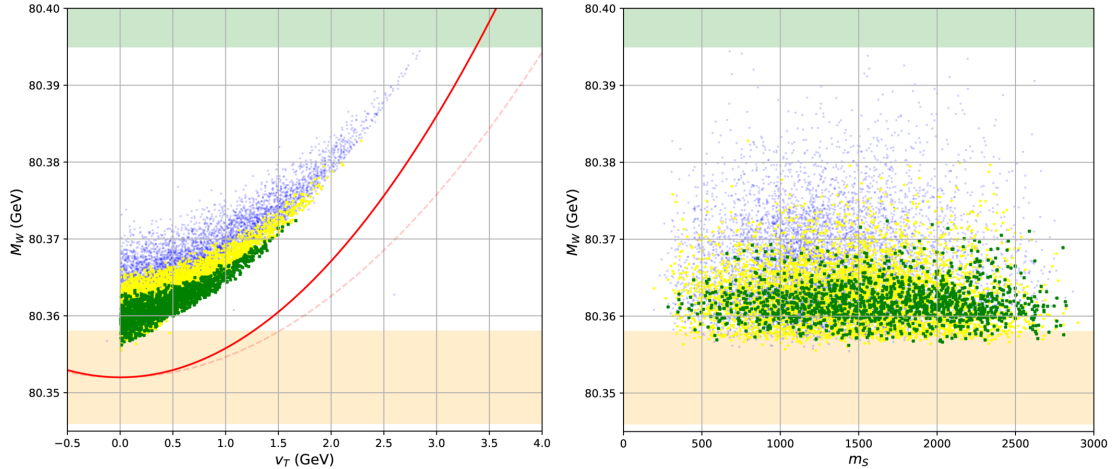


Figure 4: W boson mass in the aligned DGNMSSM. Left: W boson mass against triplet expectation value. Right: W boson mass against singlet-like Higgs mass. Colour coding of points is described in the text.

In figure 4 we show the scan results with the same colour coding as in previous sections. It is clear that in this scenario, models which can explain a large W mass are driven by a larger v_T with some modest quantum corrections enhancing the mass by $\mathcal{O}(10)$ MeV; there is very little spread due to the lack of variation in λ_T . However, it is difficult to find points with large enough v_T that satisfy other bounds.

Another feature of the selected points is that in almost all cases $v_T > 0$; considering $T_T = M_T = 0$ in equation (2.18) means that in order for $\tilde{m}_{TR}^2 > 0$ (so that, at least, the pseudoscalar triplet should be non-tachyonic, since we take $B_T = 0$) and $v_T < 0$ we would need $\mu_{\text{eff}} > |m_{D2}c_{2\beta}|$. But we need large $\tan\beta$ to obtain the correct Higgs mass, and therefore $g_Y v_S/2 \gtrsim m_{D2}$; for our selected points we require a minimum of $m_{D2} > 600$ GeV, and so v_S would again be beyond our search range.

Since we are interested here in alignment, we may have a light singlet scalar without falling foul of either light Higgs or heavy Higgs searches. In this limit we have

$$\tilde{m}_S^2 \rightarrow \frac{1}{4} \left[2v_S(\sqrt{2}T_\kappa + 3\sqrt{2}\kappa M_S + 4v_S\kappa^2) + v^2 g_Y (g_Y - \sqrt{2}\kappa s_{2\beta}) \right]. \quad (3.21)$$

T_κ of opposite sign to v_S allows the singlet to be made light while making M_A arbitrarily heavy. In the scan, we do not impose any likelihood bias to search for points with a light singlet, but we show benchmark points passing all constraints which have light singlet masses in table 3. They show a W mass consistent with the SM prediction.

3.6 W mass in the general DGNMSSM

Finally we consider the general DGNMSSM, where we allow the values of λ_S, λ_T to vary and do not fix the values of m_{DY} or T_S to require alignment but scan over them. This means that we will not focus on light singlet (or doublet) scalars. Similar to the aligned case, there is still a see-saw effect on the lightest neutralino mass due to the

	A-DGN1	A-DGN2	A-DGN3	A-DGN4
M_{SUSY} (GeV)	6368.4	5186.5	8219.5	8702.9
m_{D2} (GeV)	802	991	932	923
$\tan \beta$	26	11	29	22
κ	-0.418	-0.332	-0.446	-0.418
v_S (GeV)	1417.4	1402.4	1337.0	1488.8
v_T (GeV)	0.2	0.1	0.1	0.8
T_κ (GeV)	-595.6	-267.5	-648.0	-639.4
m_{h_1} (GeV)	124.1	122.9	125.2	123.4
m_{h_2} (GeV)	353.5	417.6	347.2	353.9
m_{h_3} (GeV)	1838.5	1284.5	1843.0	1803.0
m_{h_4} (GeV)	6710.7	12226.0	10817.9	4056.2
$m_{H_1^\pm}$ (GeV)	1841.4	1288.5	1846.1	1806.2
$m_{\tilde{\chi}_1^0}$ (GeV)	72.2	84.2	64.5	75.2
$m_{\tilde{\chi}_1^\pm}$ (GeV)	280.1	274.6	265.1	291.9
m_W (GeV)	80.362	80.361	80.361	80.363

Table 3: Benchmark points for the “aligned DGNMSSM”. Input parameters are given above the double line, and masses of the most important particles below.

non-zero singlino mass, which can drive down the quantum corrections to the W boson, but a large $|\lambda_T|$ can compensate for this and also help enhance the SM-like Higgs mass.

We perform a scan using the strategy as in sections 3.2 and 3.3, with the parameter ranges:

$$\begin{aligned}
m_{DY} \in [100, 700] \text{ GeV}, \quad m_{D2} \in [150, 1200] \text{ GeV}, \quad v_S \in [-700, 700] \text{ GeV}, \quad v_T \in [-5, 5] \text{ GeV} \\
\kappa \in [-1.5, 1.5], \quad T_\kappa \in [-2000, 2000] \text{ GeV}, \quad T_S \in [-4000, 4000] \text{ GeV}, \\
\lambda_S \in [-1.5, 1.5], \quad \sqrt{2}\lambda_T \in [-1.5, 1.5], \quad \tan \beta \in [2, 50].
\end{aligned} \tag{3.22}$$

We give plots in figure 5 with the same colour coding as in the previous sections; the difference in the distribution to the previous examples is rather striking. It is clear that in this scenario a large negative λ_T and a positive v_T is favoured; this gives a tree-level enhancement to the Higgs mass and a loop-level enhancement to the W -boson mass. The singlino component will mix less with the higgsinos than in the MDGSSM because of the $\sqrt{2}\kappa v_S$ singlino mass, and thus the effect of λ_S on the W mass is diminished. The asymmetry in the signs of λ_T and v_T can be explained by the fact that we only take positive Dirac gaugino masses in the scans.

4 Conclusions

We have shown that an aligned Dirac Gaugino NMSSM is possible and compatible with current collider constraints; it can even lead to relatively light singlet scalars that may be of interest to future searches (although would be rather difficult to find directly

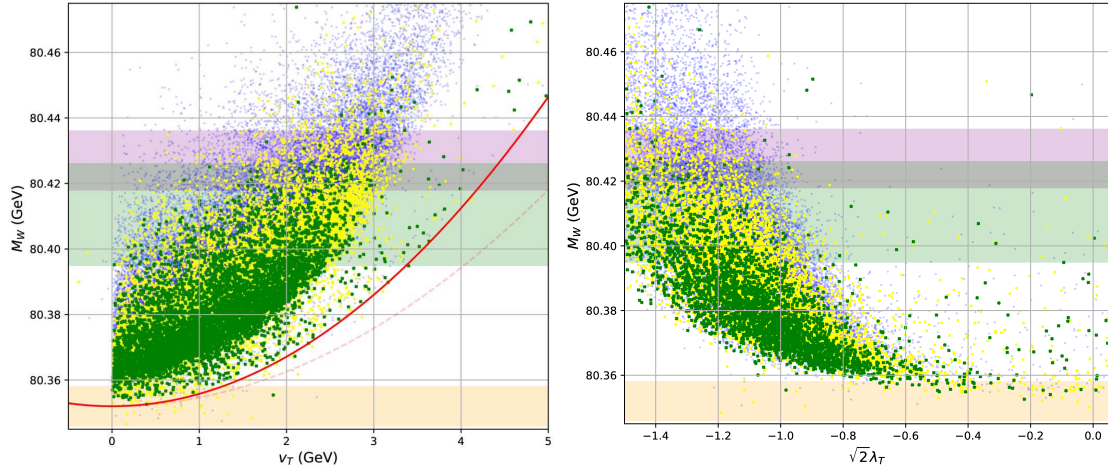


Figure 5: W boson mass in the general DGNMSSM. Left: W boson mass against triplet expectation value. Right: W boson mass against λ_T . Colour coding of points as in previous figures.

	DGN1	DGN2	DGN3	DGN4	DGN5
m_{DY} (GeV)	392	298	410	380	292
m_{D2} (GeV)	927	971	841	1003	805
κ	1.391	-1.369	-1.266	-1.309	-1.361
$\tan \beta$	9	23	21	30	30
$-\lambda_S$	0.727	-0.893	-0.544	0.554	-0.677
$\sqrt{2}\lambda_T$	-1.426	1.496	1.463	-1.303	-1.296
$-T_S$ (GeV)	3077	3747	-2496	-3002	1183
T_κ (GeV)	1139	350	-1292	-571	728
v_S (GeV)	-658.1	-539.1	574.5	524.7	-482.8
v_T (GeV)	2.7	2.3	1.7	2.5	2.2
m_{h_1} (GeV)	125.1	125.3	125.8	124.6	124.7
m_{h_2} (GeV)	1017.6	937.1	665.0	831.8	740.1
m_{A_1} (GeV)	757.6	93.7	778.1	115.1	502.2
$m_{H_1^\pm}$ (GeV)	2793.1	3195.1	1281.7	778.5	806.5
$m_{\tilde{\chi}_1^0}$ (GeV)	115.0	87.2	123.6	110.9	90.7
$m_{\tilde{\chi}_1^\pm}$ (GeV)	278.8	273.6	265.5	254.8	268.8
m_W (GeV)	80.421	80.421	80.424	80.420	80.422

Table 4: Benchmark points for the general DGNSSM. Input parameters are given above the double line, and masses of the most important particles below.

as they are difficult to produce). Such a model favours a W boson mass compatible with or just above the SM prediction. We also showed how two different Dirac Gaugino scenarios can easily be compatible with an enhanced W boson mass, including a precise computation of the quantum corrections for the first time, which are now incorporated automatically in the package **SARAH**. We also used this computation to add more nails to the coffin of the “MSSM without μ term.”

We have been conservative in our application of collider constraints and concluded that the MDGSSM models would typically contain underdense dark matter densities. However, it would be interesting to examine the issue of dark matter and collider constraints again in all of these classes of models when all the latest searches for electroweakinos have been recast; we have provided ample benchmark points for this purpose. In the DGNMSSM or its aligned version, if we impose strict R-parity or have a heavy gravitino (by no means entirely obvious assumptions), it may be that we require a Higgs funnel to obtain the correct relic density, which would require a sophisticated search strategy to find allowed parameter ranges, along e.g. the lines of [106]. However, it is also likely that a light singlino in the aligned DGNMSSM could fulfil the role of the Higgs funnel. We leave these questions to future work.

Acknowledgments

The work of W. K. is supported by the Contrat Doctoral Spécifique Normalien (CDSN) of Ecole Normale Supérieure – PSL. K. B., P. S. and M. D. G. acknowledge support from the grant “HiggsAutomator” of the Agence Nationale de la Recherche (ANR) (ANR-15-CE31-0002). M. D. G. also acknowledges support from the ANR via grant “DMwithLLPatLHC”, (ANR-21-CE31-0013). M. D. G. thanks Werner Porod and Hannah Day for interesting discussions. W. K. thanks Pierre Fayet for illuminating discussions during the early stages of this work.

References

- [1] M. Misiak and M. Steinhauser, *Weak radiative decays of the B meson and bounds on M_{H^\pm} in the Two-Higgs-Doublet Model*. *Eur. Phys. J.* **C77** (2017) no. 3, 201, [arXiv:1702.04571 \[hep-ph\]](#).
- [2] S. Davidson and H. E. Haber, *Basis-independent methods for the two-Higgs-doublet model*. *Phys. Rev. D* **72** (2005) 035004, [arXiv:hep-ph/0504050](#). [Erratum: *Phys.Rev.D* 72, 099902 (2005)].
- [3] P. S. Bhupal Dev and A. Pilaftsis, *Maximally Symmetric Two Higgs Doublet Model with Natural Standard Model Alignment*. *JHEP* **12** (2014) 024, [arXiv:1408.3405 \[hep-ph\]](#). [Erratum: *JHEP* 11, 147 (2015)].
- [4] J. Bernon, J. F. Gunion, H. E. Haber, Y. Jiang, and S. Kraml, *Scrutinizing the alignment limit in two-Higgs-doublet models: $m_h=125$ GeV*. *Phys. Rev. D* **92** (2015) no. 7, 075004, [arXiv:1507.00933 \[hep-ph\]](#).

- [5] J. Bernon, J. F. Gunion, H. E. Haber, Y. Jiang, and S. Kraml, *Scrutinizing the alignment limit in two-Higgs-doublet models. II. $m_H=125$ GeV*. *Phys. Rev. D* **93** (2016) no. 3, 035027, [arXiv:1511.03682 \[hep-ph\]](#).
- [6] M. Carena, I. Low, N. R. Shah, and C. E. M. Wagner, *Impersonating the Standard Model Higgs Boson: Alignment without Decoupling*. *JHEP* **04** (2014) 015, [arXiv:1310.2248 \[hep-ph\]](#).
- [7] M. Carena, H. E. Haber, I. Low, N. R. Shah, and C. E. M. Wagner, *Alignment limit of the NMSSM Higgs sector*. *Phys. Rev. D* **93** (2016) no. 3, 035013, [arXiv:1510.09137 \[hep-ph\]](#).
- [8] H. E. Haber, S. Heinemeyer, and T. Stefaniak, *The Impact of Two-Loop Effects on the Scenario of MSSM Higgs Alignment without Decoupling*. *Eur. Phys. J. C* **77** (2017) no. 11, 742, [arXiv:1708.04416 \[hep-ph\]](#).
- [9] N. M. Coyle and C. E. M. Wagner, *Dynamical Higgs field alignment in the NMSSM*. *Phys. Rev. D* **101** (2020) no. 5, 055037, [arXiv:1912.01036 \[hep-ph\]](#).
- [10] P. Fayet, *MASSIVE GLUINOS*. *Phys. Lett. B* **78** (1978) 417–420.
- [11] J. Polchinski and L. Susskind, *Breaking of Supersymmetry at Intermediate-Energy*. *Phys. Rev. D* **26** (1982) 3661.
- [12] L. J. Hall and L. Randall, *$U(1)$ -R symmetric supersymmetry*. *Nucl. Phys. B* **352** (1991) 289–308.
- [13] P. J. Fox, A. E. Nelson, and N. Weiner, *Dirac gaugino masses and supersoft supersymmetry breaking*. *JHEP* **08** (2002) 035, [arXiv:hep-ph/0206096](#).
- [14] A. E. Nelson, N. Rius, V. Sanz, and M. Unsal, *The Minimal supersymmetric model without a μ term*. *JHEP* **08** (2002) 039, [arXiv:hep-ph/0206102](#).
- [15] I. Antoniadis, K. Benakli, A. Delgado, and M. Quiros, *A New gauge mediation theory*. *Adv. Stud. Theor. Phys.* **2** (2008) 645–672, [arXiv:hep-ph/0610265](#).
- [16] G. Belanger, K. Benakli, M. Goodsell, C. Moura, and A. Pukhov, *Dark Matter with Dirac and Majorana Gaugino Masses*. *JCAP* **08** (2009) 027, [arXiv:0905.1043 \[hep-ph\]](#).
- [17] K. Benakli and M. D. Goodsell, *Dirac Gauginos in General Gauge Mediation*. *Nucl. Phys. B* **816** (2009) 185–203, [arXiv:0811.4409 \[hep-ph\]](#).
- [18] K. Benakli, M. D. Goodsell, and A.-K. Maier, *Generating μ and $B\mu$ in models with Dirac Gauginos*. *Nucl. Phys. B* **851** (2011) 445–461, [arXiv:1104.2695 \[hep-ph\]](#).
- [19] G. D. Kribs, E. Poppitz, and N. Weiner, *Flavor in supersymmetry with an extended R -symmetry*. *Phys. Rev. D* **78** (2008) 055010, [arXiv:0712.2039 \[hep-ph\]](#).
- [20] S. D. L. Amigo, A. E. Blechman, P. J. Fox, and E. Poppitz, *R -symmetric gauge mediation*. *JHEP* **01** (2009) 018, [arXiv:0809.1112 \[hep-ph\]](#).
- [21] K. Benakli and M. D. Goodsell, *Dirac Gauginos and Kinetic Mixing*. *Nucl. Phys. B* **830** (2010) 315–329, [arXiv:0909.0017 \[hep-ph\]](#).

- [22] S. Y. Choi, J. Kalinowski, J. M. Kim, and E. Popenza, *Scalar gluons and Dirac gluinos at the LHC*. *Acta Phys. Polon.* **B40** (2009) 2913–2922, [arXiv:0911.1951 \[hep-ph\]](#).
- [23] K. Benakli and M. D. Goodsell, *Dirac Gauginos, Gauge Mediation and Unification*. *Nucl. Phys. B* **840** (2010) 1–28, [arXiv:1003.4957 \[hep-ph\]](#).
- [24] S. Y. Choi, D. Choudhury, A. Freitas, J. Kalinowski, J. M. Kim, and P. M. Zerwas, *Dirac Neutralinos and Electroweak Scalar Bosons of $N=1/N=2$ Hybrid Supersymmetry at Colliders*. *JHEP* **08** (2010) 025, [arXiv:1005.0818 \[hep-ph\]](#).
- [25] L. M. Carpenter, *Dirac Gauginos, Negative Supertraces and Gauge Mediation*. *JHEP* **09** (2012) 102, [arXiv:1007.0017 \[hep-th\]](#).
- [26] S. Abel and M. Goodsell, *Easy Dirac Gauginos*. *JHEP* **06** (2011) 064, [arXiv:1102.0014 \[hep-th\]](#).
- [27] R. Davies, J. March-Russell, and M. McCullough, *A Supersymmetric One Higgs Doublet Model*. *JHEP* **04** (2011) 108, [arXiv:1103.1647 \[hep-ph\]](#).
- [28] K. Benakli, *Dirac Gauginos: A User Manual*. *Fortsch. Phys.* **59** (2011) 1079–1082, [arXiv:1106.1649 \[hep-ph\]](#).
- [29] C. Frugiuele and T. Gregoire, *Making the Sneutrino a Higgs with a $U(1)_R$ Lepton Number*. *Phys. Rev.* **D85** (2012) 015016, [arXiv:1107.4634 \[hep-ph\]](#).
- [30] H. Itoyama and N. Maru, *D-term Dynamical Supersymmetry Breaking Generating Split $N=2$ Gaugino Masses of Mixed Majorana-Dirac Type*. *Int. J. Mod. Phys.* **A27** (2012) 1250159, [arXiv:1109.2276 \[hep-ph\]](#).
- [31] E. Bertuzzo and C. Frugiuele, *Fitting Neutrino Physics with a $U(1)_R$ Lepton Number*. *JHEP* **05** (2012) 100, [arXiv:1203.5340 \[hep-ph\]](#).
- [32] R. Davies, *Dirac gauginos and unification in F-theory*. *JHEP* **10** (2012) 010, [arXiv:1205.1942 \[hep-th\]](#).
- [33] C. Frugiuele, T. Gregoire, P. Kumar, and E. Ponton, *' $L=R$ ' - $U(1)_R$ as the Origin of Leptonic 'RPV'*. *JHEP* **03** (2013) 156, [arXiv:1210.0541 \[hep-ph\]](#).
- [34] C. Frugiuele, T. Gregoire, P. Kumar, and E. Ponton, *' $L=R$ ' - $U(1)_R$ Lepton Number at the LHC*. *JHEP* **05** (2013) 012, [arXiv:1210.5257 \[hep-ph\]](#).
- [35] K. Benakli, M. D. Goodsell, and F. Staub, *Dirac Gauginos and the 125 GeV Higgs*. *JHEP* **06** (2013) 073, [arXiv:1211.0552 \[hep-ph\]](#).
- [36] E. Dudas, M. Goodsell, L. Heurtier, and P. Tziveloglou, *Flavour models with Dirac and fake gluinos*. *Nucl. Phys.* **B884** (2014) 632–671, [arXiv:1312.2011 \[hep-ph\]](#).
- [37] K. Benakli, L. Darmé, M. D. Goodsell, and P. Slavich, *A Fake Split Supersymmetry Model for the 126 GeV Higgs*. *JHEP* **05** (2014) 113, [arXiv:1312.5220 \[hep-ph\]](#).
- [38] K. Benakli, L. Darmé, and M. D. Goodsell, *(O)Mega Split*. *JHEP* **11** (2015) 100, [arXiv:1508.02534 \[hep-ph\]](#).

- [39] A. Arvanitaki, M. Baryakhtar, X. Huang, K. van Tilburg, and G. Villadoro, *The Last Vestiges of Naturalness*. *JHEP* **03** (2014) 022, [arXiv:1309.3568 \[hep-ph\]](#).
- [40] H. Itoyama and N. Maru, *D-term Triggered Dynamical Supersymmetry Breaking*. *Phys. Rev.* **D88** (2013) no. 2, 025012, [arXiv:1301.7548 \[hep-ph\]](#).
- [41] S. Chakraborty and S. Roy, *Higgs boson mass, neutrino masses and mixing and keV dark matter in an $U(1)_R$ -lepton number model*. *JHEP* **01** (2014) 101, [arXiv:1309.6538 \[hep-ph\]](#).
- [42] C. Csaki, J. Goodman, R. Pavesi, and Y. Shirman, *The $m_D - b_M$ problem of Dirac gauginos and its solutions*. *Phys. Rev.* **D89** (2014) no. 5, 055005, [arXiv:1310.4504 \[hep-ph\]](#).
- [43] H. Itoyama and N. Maru, *126 GeV Higgs Boson Associated with D-term Triggered Dynamical Supersymmetry Breaking*. *Symmetry* **7** (2015) no. 1, 193–205, [arXiv:1312.4157 \[hep-ph\]](#).
- [44] H. Beauchesne and T. Gregoire, *Electroweak precision measurements in supersymmetric models with a $U(1)_R$ lepton number*. *JHEP* **05** (2014) 051, [arXiv:1402.5403 \[hep-ph\]](#).
- [45] K. Benakli, *(Pseudo)goldstinos, SUSY fluids, Dirac gravitino and gauginos*. *EPJ Web Conf.* **71** (2014) 00012, [arXiv:1402.4286 \[hep-ph\]](#).
- [46] E. Bertuzzo, C. Frugiuele, T. Gregoire, and E. Ponton, *Dirac gauginos, R symmetry and the 125 GeV Higgs*. *JHEP* **04** (2015) 089, [arXiv:1402.5432 \[hep-ph\]](#).
- [47] P. Dießner, J. Kalinowski, W. Kotlarski, and D. Stöckinger, *Higgs boson mass and electroweak observables in the MRSSM*. *JHEP* **12** (2014) 124, [arXiv:1410.4791 \[hep-ph\]](#).
- [48] K. Benakli, M. Goodsell, F. Staub, and W. Porod, *Constrained minimal Dirac gaugino supersymmetric standard model*. *Phys. Rev. D* **90** (2014) no. 4, 045017, [arXiv:1403.5122 \[hep-ph\]](#).
- [49] M. D. Goodsell and P. Tziveloglou, *Dirac Gauginos in Low Scale Supersymmetry Breaking*. *Nucl. Phys.* **B889** (2014) 650–675, [arXiv:1407.5076 \[hep-ph\]](#).
- [50] D. Busbridge, *Constrained Dirac gluino mediation*. [arXiv:1408.4605 \[hep-ph\]](#).
- [51] S. Chakraborty, A. Datta, and S. Roy, *$h \rightarrow \gamma\gamma$ in $U(1)_R$ -lepton number model with a right-handed neutrino*. *JHEP* **02** (2015) 124, [arXiv:1411.1525 \[hep-ph\]](#). [Erratum: *JHEP*09,077(2015)].
- [52] R. Ding, T. Li, F. Staub, C. Tian, and B. Zhu, *Supersymmetric standard models with a pseudo-Dirac gluino from hybrid F - and D -term supersymmetry breaking*. *Phys. Rev.* **D92** (2015) no. 1, 015008, [arXiv:1502.03614 \[hep-ph\]](#).
- [53] D. S. M. Alves, J. Galloway, M. McCullough, and N. Weiner, *Goldstone Gauginos*. *Phys. Rev. Lett.* **115** (2015) no. 16, 161801, [arXiv:1502.03819 \[hep-ph\]](#).

- [54] D. S. M. Alves, J. Galloway, M. McCullough, and N. Weiner, *Models of Goldstone Gauginos*. *Phys. Rev.* **D93** (2016) no. 7, 075021, [arXiv:1502.05055 \[hep-ph\]](#).
- [55] L. M. Carpenter and J. Goodman, *New Calculations in Dirac Gaugino Models: Operators, Expansions, and Effects*. *JHEP* **07** (2015) 107, [arXiv:1501.05653 \[hep-ph\]](#).
- [56] S. P. Martin, *Nonstandard supersymmetry breaking and Dirac gaugino masses without supersoftness*. *Phys. Rev.* **D92** (2015) no. 3, 035004, [arXiv:1506.02105 \[hep-ph\]](#).
- [57] M. D. Goodsell, M. E. Krauss, T. Müller, W. Porod, and F. Staub, *Dark matter scenarios in a constrained model with Dirac gauginos*. *JHEP* **10** (2015) 132, [arXiv:1507.01010 \[hep-ph\]](#).
- [58] W. Kotlarski, *Sgluons in the same-sign lepton searches*. *JHEP* **02** (2017) 027, [arXiv:1608.00915 \[hep-ph\]](#).
- [59] K. Benakli, L. Darmé, M. D. Goodsell, and J. Harz, *The Di-Photon Excess in a Perturbative SUSY Model*. *Nucl. Phys.* **B911** (2016) 127–162, [arXiv:1605.05313 \[hep-ph\]](#).
- [60] J. Braathen, M. D. Goodsell, and P. Slavich, *Leading two-loop corrections to the Higgs boson masses in SUSY models with Dirac gauginos*. *JHEP* **09** (2016) 045, [arXiv:1606.09213 \[hep-ph\]](#).
- [61] S. Chakraborty and J. Chakraborty, *Natural emergence of neutrino masses and dark matter from R-symmetry*. *JHEP* **10** (2017) 012, [arXiv:1701.04566 \[hep-ph\]](#).
- [62] G. Chalons, M. D. Goodsell, S. Kraml, H. Reyes-González, and S. L. Williamson, *LHC limits on gluinos and squarks in the minimal Dirac gaugino model*. *JHEP* **04** (2019) 113, [arXiv:1812.09293 \[hep-ph\]](#).
- [63] L. M. Carpenter, T. Murphy, and M. J. Smylie, *Exploring color-octet scalar parameter space in minimal R-symmetric models*. *JHEP* **11** (2020) 024, [arXiv:2006.15217 \[hep-ph\]](#).
- [64] M. D. Goodsell, S. Kraml, H. Reyes-González, and S. L. Williamson, *Constraining Electroweakinos in the Minimal Dirac Gaugino Model*. *SciPost Phys.* **9** (2020) no. 4, 047, [arXiv:2007.08498 \[hep-ph\]](#).
- [65] L. M. Carpenter and T. Murphy, *Color-octet scalars in Dirac gaugino models with broken R symmetry*. *JHEP* **05** (2021) 079, [arXiv:2012.15771 \[hep-ph\]](#).
- [66] K. Benakli, M. D. Goodsell, and S. L. Williamson, *Higgs alignment from extended supersymmetry*. *Eur. Phys. J. C* **78** (2018) no. 8, 658, [arXiv:1801.08849 \[hep-ph\]](#).
- [67] K. Benakli, Y. Chen, and G. Lafforgue-Marmet, *R-symmetry for Higgs alignment without decoupling*. *Eur. Phys. J. C* **79** (2019) no. 2, 172, [arXiv:1811.08435 \[hep-ph\]](#).

- [68] K. Benakli, Y. Chen, and G. Lafforgue-Marmet, *Predicting Alignment in a Two Higgs Doublet Model*. *MDPI Proc.* **13** (2019) no. 1, 2, [arXiv:1812.02208 \[hep-ph\]](#).
- [69] K. Benakli, Y. Chen, and G. Lafforgue-Marmet, *An Emergent Higgs Alignment*. *PoS CORFU2018* (2019) 014, [arXiv:1904.04248 \[hep-ph\]](#).
- [70] J. Ellis, J. Quevillon, and V. Sanz, *Doubling Up on Supersymmetry in the Higgs Sector*. *JHEP* **10** (2016) 086, [arXiv:1607.05541 \[hep-ph\]](#).
- [71] **CMS** Collaboration, *Combined Higgs boson production and decay measurements with up to 137 fb^{-1} of proton-proton collision data at $\sqrt{s} = 13 \text{ TeV}$* .
- [72] **ATLAS** Collaboration, *A combination of measurements of Higgs boson production and decay using up to 139 fb^{-1} of proton-proton collision data at $\sqrt{s} = 13 \text{ TeV}$ collected with the ATLAS experiment*.
- [73] **CDF** Collaboration, T. Aaltonen *et al.*, *High-precision measurement of the W boson mass with the CDF II detector*. *Science* **376** (2022) no. 6589, 170–176.
- [74] M. Awramik, M. Czakon, A. Freitas, and G. Weiglein, *Precise prediction for the W boson mass in the standard model*. *Phys. Rev. D* **69** (2004) 053006, [arXiv:hep-ph/0311148](#).
- [75] G. Degrossi, P. Gambino, and P. P. Giardino, *The $m_W - m_Z$ interdependence in the Standard Model: a new scrutiny*. *JHEP* **05** (2015) 154, [arXiv:1411.7040 \[hep-ph\]](#).
- [76] **OPAL** Collaboration, G. Abbiendi *et al.*, *Measurement of the mass and width of the W boson*. *Eur. Phys. J. C* **45** (2006) 307–335, [arXiv:hep-ex/0508060](#).
- [77] **L3** Collaboration, P. Achard *et al.*, *Measurement of the mass and the width of the W boson at LEP*. *Eur. Phys. J. C* **45** (2006) 569–587, [arXiv:hep-ex/0511049](#).
- [78] **ALEPH** Collaboration, S. Schael *et al.*, *Measurement of the W boson mass and width in e^+e^- collisions at LEP*. *Eur. Phys. J. C* **47** (2006) 309–335, [arXiv:hep-ex/0605011](#).
- [79] **DELPHI** Collaboration, J. Abdallah *et al.*, *Measurement of the Mass and Width of the W Boson in e^+e^- Collisions at $\sqrt{s} = 161\text{-GeV} - 209\text{-GeV}$* . *Eur. Phys. J. C* **55** (2008) 1–38, [arXiv:0803.2534 \[hep-ex\]](#).
- [80] **ALEPH, DELPHI, L3, OPAL, LEP Electroweak** Collaboration, S. Schael *et al.*, *Electroweak Measurements in Electron-Positron Collisions at W -Boson-Pair Energies at LEP*. *Phys. Rept.* **532** (2013) 119–244, [arXiv:1302.3415 \[hep-ex\]](#).
- [81] **D0** Collaboration, V. M. Abazov *et al.*, *Measurement of the W Boson Mass with the D0 Detector*. *Phys. Rev. Lett.* **108** (2012) 151804, [arXiv:1203.0293 \[hep-ex\]](#).
- [82] **CDF** Collaboration, T. Aaltonen *et al.*, *Precise measurement of the W -boson mass with the CDF II detector*. *Phys. Rev. Lett.* **108** (2012) 151803, [arXiv:1203.0275 \[hep-ex\]](#).

- [83] **CDF, D0** Collaboration, T. A. Aaltonen *et al.*, *Combination of CDF and D0 W-Boson Mass Measurements*. *Phys. Rev. D* **88** (2013) no. 5, 052018, [arXiv:1307.7627 \[hep-ex\]](#).
- [84] S. Heinemeyer, W. Hollik, G. Weiglein, and L. Zeune, *Implications of LHC search results on the W boson mass prediction in the MSSM*. *JHEP* **12** (2013) 084, [arXiv:1311.1663 \[hep-ph\]](#).
- [85] **ATLAS** Collaboration, M. Aaboud *et al.*, *Measurement of the W-boson mass in pp collisions at $\sqrt{s} = 7$ TeV with the ATLAS detector*. *Eur. Phys. J. C* **78** (2018) no. 2, 110, [arXiv:1701.07240 \[hep-ex\]](#). [Erratum: *Eur.Phys.J.C* 78, 898 (2018)].
- [86] **LHCb** Collaboration, R. Aaij *et al.*, *Measurement of the W boson mass*. *JHEP* **01** (2022) 036, [arXiv:2109.01113 \[hep-ex\]](#).
- [87] P. Athron, M. Bach, D. H. J. Jacob, W. Kotlarski, D. Stöckinger, and A. Voigt, *Precise calculation of the W boson pole mass beyond the Standard Model with FlexibleSUSY*. [arXiv:2204.05285 \[hep-ph\]](#).
- [88] P. Athron, J.-h. Park, D. Stöckinger, and A. Voigt, *FlexibleSUSY – a meta spectrum generator for supersymmetric models*. *Nucl. Part. Phys. Proc.* **273-275** (2016) 2424–2426, [arXiv:1410.7385 \[hep-ph\]](#).
- [89] P. Athron, M. Bach, D. Harries, T. Kwasnitza, J.-h. Park, D. Stöckinger, A. Voigt, and J. Ziebell, *FlexibleSUSY 2.0: Extensions to investigate the phenomenology of SUSY and non-SUSY models*. *Comput. Phys. Commun.* **230** (2018) 145–217, [arXiv:1710.03760 \[hep-ph\]](#).
- [90] P. Diessner and G. Weiglein, *Precise prediction for the W boson mass in the MRSSM*. *JHEP* **07** (2019) 011, [arXiv:1904.03634 \[hep-ph\]](#).
- [91] J. Braathen, M. D. Goodsell, S. Paßehr, and E. Pinsard, *Expectation management*. *Eur. Phys. J. C* **81** (2021) no. 6, 498, [arXiv:2103.06773 \[hep-ph\]](#).
- [92] M. Carena, H. E. Haber, I. Low, N. R. Shah, and C. E. M. Wagner, *Complementarity between Nonstandard Higgs Boson Searches and Precision Higgs Boson Measurements in the MSSM*. *Phys. Rev.* **D91** (2015) no. 3, 035003, [arXiv:1410.4969 \[hep-ph\]](#).
- [93] M. Du, Z. Liu, and P. Nath, *CDF W mass anomaly from a dark sector with a Stueckelberg-Higgs portal*. [arXiv:2204.09024 \[hep-ph\]](#).
- [94] **ALEPH** Collaboration, A. Heister *et al.*, *Search for charginos nearly mass degenerate with the lightest neutralino in e^+e^- collisions at center-of-mass energies up to 209-GeV*. *Phys. Lett. B* **533** (2002) 223–236, [arXiv:hep-ex/0203020](#).
- [95] J. Abdallah *et al.*, *Searches for supersymmetric particles in e^+e^- collisions up to 208 GeV, and interpretation of the results within the MSSM*.
- [96] LEP2 SUSY Working Group, *ALEPH, DELPHI, L3 and OPAL experiments, note LEPSUSYWG/02-04.1*. <http://lepsusy.web.cern.ch/lepsusy/Welcome.html>.

- [97] P. Bechtle, O. Brein, S. Heinemeyer, G. Weiglein, and K. E. Williams, *HiggsBounds: Confronting Arbitrary Higgs Sectors with Exclusion Bounds from LEP and the Tevatron*. *Comput. Phys. Commun.* **181** (2010) 138–167, [arXiv:0811.4169 \[hep-ph\]](#).
- [98] P. Bechtle, O. Brein, S. Heinemeyer, G. Weiglein, and K. E. Williams, *HiggsBounds 2.0.0: Confronting Neutral and Charged Higgs Sector Predictions with Exclusion Bounds from LEP and the Tevatron*. *Comput. Phys. Commun.* **182** (2011) 2605–2631, [arXiv:1102.1898 \[hep-ph\]](#).
- [99] P. Bechtle, O. Brein, S. Heinemeyer, O. Stål, T. Stefaniak, G. Weiglein, and K. E. Williams, *HiggsBounds – 4: Improved Tests of Extended Higgs Sectors against Exclusion Bounds from LEP, the Tevatron and the LHC*. *Eur. Phys. J. C* **74** (2014) no. 3, 2693, [arXiv:1311.0055 \[hep-ph\]](#).
- [100] P. Bechtle, D. Dercks, S. Heinemeyer, T. Klingl, T. Stefaniak, G. Weiglein, and J. Wittbrodt, *HiggsBounds-5: Testing Higgs Sectors in the LHC 13 TeV Era*. *Eur. Phys. J. C* **80** (2020) no. 12, 1211, [arXiv:2006.06007 \[hep-ph\]](#).
- [101] M. Goodsell, K. Nickel, and F. Staub, *Generic two-loop Higgs mass calculation from a diagrammatic approach*. *Eur. Phys. J.* **C75** (2015) no. 6, 290, [arXiv:1503.03098 \[hep-ph\]](#).
- [102] J. Braathen and M. D. Goodsell, *Avoiding the Goldstone Boson Catastrophe in general renormalisable field theories at two loops*. *JHEP* **12** (2016) 056, [arXiv:1609.06977 \[hep-ph\]](#).
- [103] J. Braathen, M. D. Goodsell, and F. Staub, *Supersymmetric and non-supersymmetric models without catastrophic Goldstone bosons*. *Eur. Phys. J.* **C77** (2017) no. 11, 757, [arXiv:1706.05372 \[hep-ph\]](#).
- [104] F. Staub and W. Porod, *Improved predictions for intermediate and heavy Supersymmetry in the MSSM and beyond*. *Eur. Phys. J. C* **77** (2017) no. 5, 338, [arXiv:1703.03267 \[hep-ph\]](#).
- [105] P. Slavich *et al.*, *Higgs-mass predictions in the MSSM and beyond*. *Eur. Phys. J. C* **81** (2021) no. 5, 450, [arXiv:2012.15629 \[hep-ph\]](#).
- [106] M. D. Goodsell and A. Joury, *Active learning BSM parameter spaces*. [arXiv:2204.13950 \[hep-ph\]](#).
- [107] P. Bechtle, S. Heinemeyer, O. Stål, T. Stefaniak, and G. Weiglein, *HiggsSignals: Confronting arbitrary Higgs sectors with measurements at the Tevatron and the LHC*. *Eur. Phys. J. C* **74** (2014) no. 2, 2711, [arXiv:1305.1933 \[hep-ph\]](#).
- [108] P. Bechtle, S. Heinemeyer, T. Klingl, T. Stefaniak, G. Weiglein, and J. Wittbrodt, *HiggsSignals-2: Probing new physics with precision Higgs measurements in the LHC 13 TeV era*. *Eur. Phys. J. C* **81** (2021) no. 2, 145, [arXiv:2012.09197 \[hep-ph\]](#).

Conclusions

The theory of spins $s \leq 1$ has been established for a long time. Meanwhile, the construction of a consistent action for higher spin particles, both in the massive and massless, charged and uncharged cases, continues to be studied to this day. In particular, the propagation of charged massive higher spins suffers from severe pathologies, historically known as the Velo-Zwanziger problem. In Chapter 2, we addressed this issue and provided an answer in the case of charged massive spin-3/2 and spin-2, in a constant electromagnetic background. In the Lagrangians, the spin-3/2 and spin-2 are coupled to lower spins. Though we have not found a way to decouple different states off shell, we were able to decouple them at the level of equations of motion and constraints, through field redefinitions.

For spin-2, we recovered on shell the Fierz-Pauli system, same as that derived from the Argyres-Nappi Lagrangian, but our Lagrangian is more complicated due to the presence of more states on the mass level. For spin-3/2, we obtained equations of motion and constraints that, as far as we know, do not exist in the literature. This on-shell system guarantees causality, correct degrees of freedom and gyromagnetic ratio $g = 2$. It is interesting to note that in [11], the γ -trace constraint $\gamma^\mu \psi_\mu = 0$ was only enforced up to a certain order, and it turned out that none of the proposed modifications to the Lagrangian escaped from acausality. In [59], it was shown that causality is ensured if this trace constraint is exact to all orders. In our case, we precisely find the constraint $\gamma^\mu \psi_\mu = 0$, whereas the divergence constraint itself is shifted by electromagnetic coupling.

The higher spin studies are not only of field-theoretic interest, but also important phenomenologically. Many authors have investigated the production of higher spin particles in early universe, and particular attention has been paid to gravitinos in supergravity theories. In Chapter 3, we studied a pathological gravitino propagation pointed out in [20]: the gravitino production becomes divergent when its sound speed vanishes. Our question was whether such behavior exists in linear $\mathcal{N} = 1$, $d = 4$ supergravity. When the matter content is one chiral multiplet, the sound speed is always equal to the speed of light [16–18]. In the presence of more than one fermion, the definition of a physical sound speed needs more care, which involves diagonalizing a coupled system of equations of motion. We have shown that, both for two chiral multiplets and one chiral plus one vector multiplet, the sound speed never vanishes and therefore no divergent production will happen.

In Chapter 4 and 5, we moved to SUSY candidates for experimental anomalies. First, we have investigated in Chapter 4 the role of Higgs mass in constraining high-scale SUSY models. Our starting points are that, on one hand, the Higgs quartic coupling receives radiative corrections from all SUSY particles, and on the other, the Higgs mass is measured with a high precision, and therefore can be used as an input. When the SUSY scale is much higher than the EW scale, it is reasonable to adopt an EFT approach. For a proof of concept,

we have taken the example of the FSSM, a supersymmetric four-Higgs-doublet model, which was proposed in [80] as a solution to the muon $(g - 2)$ anomaly with multi-TeV sleptons. We have shown that, the Higgs mass in conjunction with the muon $(g - 2)$ yields various constraints on the parameters of the FSSM. The principle of our method is general, and can be applied to other (high-scale SUSY) models, or other anomalies.

It is also phenomenologically interesting to consider SUSY models that are within the reach of current colliders. As SUSY extensions of the SM feature an extended scalar sector, the crucial point is that one of the scalars must be the SM-like Higgs, while its coupling to other scalars should be sufficiently suppressed to have escaped detection. In Chapter 5, we discussed the Dirac gaugino models that have good alignment properties compared to the MSSM and the NMSSM. And intriguingly, thanks to the triplet scalar in the model, one naturally obtains a tree-level enhancement to the W boson mass, so as to accommodate the recent CDF measurement. A numerical analysis is performed, pointing out the importance of quantum corrections to m_W , that may become as large as the tree-level contribution.

Appendix A

Conventions in Chapter 2

Spinor conventions

We have followed the conventions in [26], with mostly positive metric $\eta_{mn} \sim (-1, 1, 1, 1)$, and Levi-Civita tensor $\varepsilon_{0123} = -1$. The hermitian σ -matrices are

$$\begin{aligned} \sigma^0 &= \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} & \sigma^1 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ \sigma^2 &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} & \sigma^3 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \end{aligned} \quad (\text{A.1})$$

The γ matrices are defined as

$$\gamma^m = \begin{pmatrix} 0 & \sigma^m \\ \bar{\sigma}^m & 0 \end{pmatrix}, \quad \gamma^5 = \gamma^0 \gamma^1 \gamma^2 \gamma^3 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \quad (\text{A.2})$$

The higher rank γ -matrices are defined as

$$\gamma^{mn} = \frac{1}{2} [\gamma^m, \gamma^n], \quad \gamma^{mnk} = \frac{1}{2} \{ \gamma^m, \gamma^{nk} \}, \quad \dots \quad (\text{A.3})$$

The spinor indices are raised and lowered by the antisymmetric ε -symbol, whose non-vanishing components are

$$\varepsilon_{21} = \varepsilon^{12} = 1, \quad \varepsilon_{12} = \varepsilon^{21} = -1 \quad (\text{A.4})$$

Denoting $\sigma^\mu = (\sigma^0, \vec{\sigma})$, σ^μ and $\bar{\sigma}^\mu$ are related by

$$\begin{aligned} \bar{\sigma}^{\mu\dot{\alpha}\alpha} &= \varepsilon^{\dot{\alpha}\beta} \varepsilon^{\alpha\gamma} \sigma_{\beta\gamma}^\mu \\ \bar{\sigma}^0 &= \sigma^0 \\ \bar{\sigma}^{1,2,3} &= -\sigma^{1,2,3}. \end{aligned} \quad (\text{A.5})$$

The conjugate of the spinor has dotted index and an overbar $(\chi_\alpha)^\dagger = \bar{\chi}_{\dot{\alpha}}$. The spinor indices are contracted in the ‘‘northwest-southeast’’ (NW-SE) manner, namely, $(\lambda\chi) = \lambda^\alpha \chi_\alpha = -\chi_\alpha \lambda^\alpha = (\chi\lambda)$, $(\bar{\lambda}\bar{\chi}) = \bar{\lambda}_{\dot{\alpha}} \bar{\chi}^{\dot{\alpha}} = -\bar{\chi}^{\dot{\alpha}} \bar{\lambda}_{\dot{\alpha}} = (\bar{\chi}\bar{\lambda})$.

From two components to four components

The four-component notation can be restored by introducing the γ -matrices (A.2) and the four-component Dirac spinors are written as

$$\Psi_D = \begin{pmatrix} \chi_\alpha \\ \bar{\lambda}^{\dot{\alpha}} \end{pmatrix} \quad (\text{A.6})$$

with the Dirac adjoint defined as

$$\bar{\Psi}_D \equiv \Psi_D^\dagger i\gamma^0 = -i \begin{pmatrix} \lambda^\alpha & \bar{\chi}_{\dot{\alpha}} \end{pmatrix} \quad (\text{A.7})$$

Conversely, the Weyl spinors in Ψ_D are selected by the chiral projectors $P_L = (1 + i\gamma^5)/2$, $P_R = (1 - i\gamma^5)/2$.

Field redefinitions

The field redefinitions are written in the form

$$F \rightarrow aF + bG \quad (\text{A.8})$$

which means that the field F is replaced everywhere by $aF' + bG$, and the primes are dropped in the subsequent Lagrangian. The new gauge transformations are

$$\delta F' = \frac{1}{a}(\delta F - b\delta G), \quad \delta G' = \delta G \quad (\text{A.9})$$

Dual fields

A *self-dual* rank-2 tensor satisfies

$$\varepsilon_{mnpq} S^{mn} = -2i S_{pq} \quad (\text{A.10})$$

Correspondingly, for an anti self-dual tensor, $\varepsilon_{mnpq} S^{mn} = 2i S_{pq}$. Out of a generic rank-2 tensor, one is able to construct a self-dual combination

$$\mathcal{S}[A_{mn}] \equiv \frac{1}{4}(A_{mn} - A_{nm}) + \frac{i}{4}\varepsilon_{mnpq} A^{pq}. \quad (\text{A.11})$$

This has been used to write gauge transformations of the self-dual fields t_{mn} , s_{mn} .

Shorthand notations

A summary of some shorthand notations used in this work:

$$\begin{aligned} (\epsilon \cdot \sigma) &\equiv \epsilon^{mn} \sigma_{mn} = i\tilde{\epsilon}^{mn} \sigma_{mn}, & (\epsilon \cdot \bar{\sigma}) &\equiv \epsilon^{mn} \bar{\sigma}_{mn} = -i\tilde{\epsilon}^{mn} \bar{\sigma}_{mn} \\ \epsilon\epsilon &\equiv \epsilon^{mn} \epsilon_{mn}, & \tilde{\epsilon}\tilde{\epsilon} &\equiv \tilde{\epsilon}^{mn} \tilde{\epsilon}_{mn} \\ G_{mn} &\equiv (\eta_{mn} - i\epsilon_{mn})^{-1}, & \mathcal{A}_{mn} &\equiv \left(\eta_{mn} - \frac{i}{2}\epsilon_{mn} - \frac{1}{2}\tilde{\epsilon}_{mk}\tilde{\epsilon}^k{}_n \right)^{-1} \\ \Sigma_{mn} &\equiv \sigma_m \bar{\sigma}_n (\epsilon \cdot \sigma) - \sigma_m (\epsilon \cdot \bar{\sigma}) \bar{\sigma}_n - (\epsilon \cdot \sigma) \sigma_m \bar{\sigma}_n \end{aligned} \quad (\text{A.12})$$

Appendix B

Conventions in Chapter 3

Our notations follow mostly that in [19], where the flat space γ -matrices are

$$\bar{\gamma}^0 = \begin{pmatrix} i\mathbb{1}_2 & 0 \\ 0 & -i\mathbb{1}_2 \end{pmatrix}, \quad \bar{\gamma}^i = \begin{pmatrix} 0 & -i\sigma_i \\ i\sigma_i & 0 \end{pmatrix}, \quad \gamma_5 = \begin{pmatrix} 0 & -\mathbb{1}_2 \\ -\mathbb{1}_2 & 0 \end{pmatrix} \quad (\text{B.1})$$

The Minkowski metric has the signature $(-, +, +, +)$, and for cosmological applications, we used the FLRW metric. The curved space γ -matrices, noted γ^μ , are then related to the flat space γ -matrices by $\gamma^\mu = a^{-1}\bar{\gamma}^\mu$.

The left and right projections are defined as

$$P_L = \frac{1}{2}(1 + \gamma_5) \quad , \quad P_R = \frac{1}{2}(1 - \gamma_5) \quad (\text{B.2})$$

Note that for chiral fermions, $P_L\chi_i = \chi_i$ and $P_R\chi^i = \chi^i$. The charge conjugation matrix in this convention is given by $C = \bar{\gamma}^0\bar{\gamma}^2$. Some useful charge conjugates are

$$\begin{aligned} \chi_i^C &= \chi^i, & \phi_i^C &= \phi^i, & \mathcal{P}^C &= \mathcal{P}, & \lambda^C &= \lambda \\ \bar{\gamma}_\mu^C &= \bar{\gamma}_\mu, & \gamma_5^C &= -\gamma_5, & P_L^C &= P_R \end{aligned} \quad (\text{B.3})$$

The covariant derivative of the scalar is $\hat{\partial}_0 = \partial_0 - \frac{i}{2}A_0^B\gamma_5$. In the cosmological context considered, the spatial derivatives of the scalar vanish, and for real backgrounds we have $A_0^B = 0$, so $\hat{\partial}_0 = \partial_0$.

Keeping the above simplifications, the covariant derivatives acting on the chiral fermions, the gaugino and the gravitino are respectively

$$\begin{aligned} \mathcal{D}_\mu\chi_i &\equiv \left(\partial_\mu + \frac{1}{4}\omega_\mu^{ab}\bar{\gamma}_{ab} \right) \chi_i + \Gamma_i^{jk}\chi_j\partial_\mu\phi_k, & \mathcal{D}_\mu\lambda &= \left(\partial_\mu + \frac{1}{4}\omega_\mu^{ab}\bar{\gamma}_{ab} \right) \lambda \\ \mathcal{D}_\mu\psi_\nu &= \left(\left(\partial_\mu + \frac{1}{4}\omega_\mu^{ab}\bar{\gamma}_{ab} + \frac{1}{2}i\gamma_5 A_\mu \right) \delta_\nu^\lambda - \Gamma_{\mu\nu}^\lambda \right) \psi_\lambda \end{aligned} \quad (\text{B.4})$$

ω_μ^{ab} stands for the spin connection and A_μ is the $U(1)$ gauge field. The Christoffel connection $\Gamma_{\mu\nu}^\lambda$ differs from the Kähler connection, where latter corresponds to $\Gamma_i^{jk} \equiv g^{-1l}_i\partial^j g_l^k$. We used the notation $\bar{\gamma}_{ab} \equiv [\bar{\gamma}_a, \bar{\gamma}_b]/2$.

Introducing also the Kähler covariant derivative \mathcal{D}^i , the mass terms are

$$\begin{aligned}
m^i &\equiv \mathcal{D}^i m = \partial^i m + \frac{\partial^i K}{2M_{\text{P}}^2} m, & m^{ij} &\equiv \mathcal{D}^i \mathcal{D}^j m = \left(\partial^i + \frac{\partial^j K}{2M_{\text{P}}^2} \right) m^j - \Gamma_k^{ij} m^k \\
m_{i\alpha} &= -i \left[\partial_i \mathcal{P} - \frac{1}{4} (\text{Re } f)^{-1} \mathcal{P} f_i \right], & m_{R,\alpha\beta} &= -\frac{1}{4} f_i g_j^{-1i} m^j
\end{aligned} \tag{B.5}$$

The subscript i in f denotes the derivative with respect to ϕ^i .

Bibliography

- [1] E. P. Wigner, “On Unitary Representations of the Inhomogeneous Lorentz Group”, *Annals Math.* **40**, edited by Y. S. Kim and W. W. Zachary, 149 (1939).
- [2] P. A. M. Dirac, “The quantum theory of the electron”, *Proc. Roy. Soc. Lond. A* **117**, 610 (1928).
- [3] V. E. Barnes et al., “Observation of a Hyperon with Strangeness Minus Three”, *Phys. Rev. Lett.* **12**, 204 (1964).
- [4] B. Aubert et al. (BaBar), “Measurement of the spin of the Omega- hyperon at BABAR”, *Phys. Rev. Lett.* **97**, edited by A. Sissakian, G. Kozlov, and E. Kolganova, 112001 (2006), [arXiv:hep-ex/0606039](#).
- [5] P. A. M. Dirac, “Relativistic wave equations”, *Proc. Roy. Soc. Lond. A* **155**, 447 (1936).
- [6] W. Rarita and J. Schwinger, “On a theory of particles with half integral spin”, *Phys. Rev.* **60**, 61 (1941).
- [7] M. Fierz and W. Pauli, “On relativistic wave equations for particles of arbitrary spin in an electromagnetic field”, *Proc. Roy. Soc. Lond. A* **173**, 211 (1939).
- [8] L. P. S. Singh and C. R. Hagen, “Lagrangian formulation for arbitrary spin. 1. The boson case”, *Phys. Rev. D* **9**, 898 (1974).
- [9] L. P. S. Singh and C. R. Hagen, “Lagrangian formulation for arbitrary spin. 2. The fermion case”, *Phys. Rev. D* **9**, 910 (1974).
- [10] P. Federbush, “Minimal electromagnetic coupling for spin two particles”, *Il Nuovo Cimento* **19**, 572 (1961).
- [11] S. Deser, V. Pascalutsa, and A. Waldron, “Massive spin 3/2 electrodynamics”, *Phys. Rev. D* **62**, 105031 (2000), [arXiv:hep-th/0003011](#).
- [12] A. Abouelsaood, C. G. Callan Jr., C. R. Nappi, and S. A. Yost, “Open Strings in Background Gauge Fields”, *Nucl. Phys. B* **280**, 599 (1987).
- [13] K. Benakli, N. Berkovits, C. A. Daniel, and M. Lize, “Higher-spin states of the superstring in an electromagnetic background”, *JHEP* **12**, 112 (2021), [arXiv:2110.07623 \[hep-th\]](#).
- [14] P. C. Argyres and C. R. Nappi, “Massive Spin-2 Bosonic String States in an Electromagnetic Background”, *Phys. Lett. B* **224**, 89 (1989).
- [15] P. C. Argyres and C. R. Nappi, “Spin 1 Effective Actions From Open Strings”, *Nucl. Phys. B* **330**, 151 (1990).

- [16] G. F. Giudice, A. Riotto, and I. Tkachev, “Thermal and nonthermal production of gravitinos in the early universe”, *JHEP* **11**, 036 (1999), [arXiv:hep-ph/9911302](#).
- [17] G. F. Giudice, I. Tkachev, and A. Riotto, “Nonthermal production of dangerous relics in the early universe”, *JHEP* **08**, 009 (1999), [arXiv:hep-ph/9907510](#).
- [18] R. Kallosh, L. Kofman, A. D. Linde, and A. Van Proeyen, “Gravitino production after inflation”, *Phys. Rev. D* **61**, 103503 (2000), [arXiv:hep-th/9907124](#).
- [19] R. Kallosh, L. Kofman, A. D. Linde, and A. Van Proeyen, “Superconformal symmetry, supergravity and cosmology”, *Class. Quant. Grav.* **17**, [Erratum: *Class.Quant.Grav.* **21**, 5017 (2004)], 4269 (2000), [arXiv:hep-th/0006179](#).
- [20] E. W. Kolb, A. J. Long, and E. McDonough, “Catastrophic Production of Slow Gravitinos”, (2021), [arXiv:2102.10113 \[hep-th\]](#).
- [21] G. Aad et al. (ATLAS), “Observation of a new particle in the search for the Standard Model Higgs boson with the ATLAS detector at the LHC”, *Phys. Lett. B* **716**, 1 (2012), [arXiv:1207.7214 \[hep-ex\]](#).
- [22] S. Chatrchyan et al. (CMS), “Observation of a New Boson at a Mass of 125 GeV with the CMS Experiment at the LHC”, *Phys. Lett. B* **716**, 30 (2012), [arXiv:1207.7235 \[hep-ex\]](#).
- [23] D. Buttazzo, G. Degrassi, P. P. Giardino, G. F. Giudice, F. Sala, A. Salvio, and A. Strumia, “Investigating the near-criticality of the Higgs boson”, *JHEP* **12**, 089 (2013), [arXiv:1307.3536 \[hep-ph\]](#).
- [24] P. Fayet, “MASSIVE GLUINOS”, *Phys. Lett. B* **78**, 417 (1978).
- [25] T. Aaltonen et al. (CDF), “High-precision measurement of the W boson mass with the CDF II detector”, *Science* **376**, 170 (2022).
- [26] J. Wess and J. Bagger, *Supersymmetry and supergravity* (Princeton University Press, Princeton, NJ, USA, 1992).
- [27] S. P. Martin, “A Supersymmetry primer”, *Adv. Ser. Direct. High Energy Phys.* **18**, edited by G. L. Kane, 1 (1998), [arXiv:hep-ph/9709356](#).
- [28] G. Dall’Agata and M. Zagermann, *Supergravity: From First Principles to Modern Applications*, Vol. 991, Lecture Notes in Physics (July 2021).
- [29] G. Dall’Agata and M. Zagermann, “Simple Supergravity”, (2022), [arXiv:2212.10044 \[hep-th\]](#).
- [30] D. Z. Freedman and A. Van Proeyen, *Supergravity* (Cambridge Univ. Press, Cambridge, UK, May 2012).
- [31] P. Fayet and S. Ferrara, “Supersymmetry”, *Phys. Rept.* **32**, 249 (1977).
- [32] S. J. Gates, M. T. Grisaru, M. Rocek, and W. Siegel, *Superspace Or One Thousand and One Lessons in Supersymmetry*, Vol. 58, Frontiers in Physics (1983), [arXiv:hep-th/0108200](#).
- [33] S. Coleman and J. Mandula, “All possible symmetries of the S matrix”, *Phys. Rev.* **159**, 1251 (1967).
- [34] R. Haag, J. T. Łopuszański, and M. Sohnius, “All possible generators of supersymmetries of the s-matrix”, *Nuclear Physics B* **88**, 257 (1975).

- [35] A. Salam and J. A. Strathdee, “Supergauge Transformations”, *Nucl. Phys. B* **76**, 477 (1974).
- [36] L. O’Raifeartaigh, “Spontaneous Symmetry Breaking for Chiral Scalar Superfields”, *Nucl. Phys. B* **96**, 331 (1975).
- [37] P. Fayet and J. Iliopoulos, “Spontaneously Broken Supergauge Symmetries and Goldstone Spinors”, *Phys. Lett. B* **51**, 461 (1974).
- [38] P. Fayet, “Supergauge Invariant Extension of the Higgs Mechanism and a Model for the electron and Its Neutrino”, *Nucl. Phys. B* **90**, 104 (1975).
- [39] A. E. Nelson and N. Seiberg, “R symmetry breaking versus supersymmetry breaking”, *Nucl. Phys. B* **416**, 46 (1994), [arXiv:hep-ph/9309299](#).
- [40] S. Deser and B. Zumino, “Consistent Supergravity”, *Phys. Lett. B* **62**, edited by A. Salam and E. Sezgin, 335 (1976).
- [41] D. Z. Freedman, P. van Nieuwenhuizen, and S. Ferrara, “Progress Toward a Theory of Supergravity”, *Phys. Rev. D* **13**, 3214 (1976).
- [42] D. V. Volkov and V. A. Soroka, “Higgs Effect for Goldstone Particles with Spin 1/2”, *JETP Lett.* **18**, [*Pisma Zh. Eksp. Teor. Fiz.*18,529(1973)], 312 (1973).
- [43] S. Deser and B. Zumino, “Broken Supersymmetry and Supergravity”, *Phys. Rev. Lett.* **38**, 1433 (1977).
- [44] E. Cremmer, B. Julia, J. Scherk, P. van Nieuwenhuizen, S. Ferrara, and L. Girardello, “Super-higgs effect in supergravity with general scalar interactions”, *Phys. Lett. B* **79**, 231 (1978).
- [45] S. Ferrara and A. Van Proeyen, “Mass Formulae for Broken Supersymmetry in Curved Space-Time”, *Fortsch. Phys.* **64**, 896 (2016), [arXiv:1609.08480 \[hep-th\]](#).
- [46] K. Benakli, C. A. Daniel, and W. Ke, “Open superstring first mass level effective Lagrangian: Massive spin-3/2 fields in an electromagnetic background”, *Phys. Lett. B* **839**, 137788 (2023), [arXiv:2211.13691 \[hep-th\]](#).
- [47] K. Benakli, C. A. Daniel, and W. Ke, “Open superstring first mass level effective Lagrangian: Massive spin-2 in an electromagnetic background”, *Phys. Lett. B* **838**, 137680 (2023), [arXiv:2211.13689 \[hep-th\]](#).
- [48] K. Benakli, C. A. Daniel, and W. Ke, “Spin-3/2 and spin-2 charged massive states in a constant electromagnetic background”, in (Feb. 2023), [arXiv:2302.06630 \[hep-th\]](#).
- [49] V. Bargmann and E. P. Wigner, “Group Theoretical Discussion of Relativistic Wave Equations”, *Proc. Nat. Acad. Sci.* **34**, 211 (1948).
- [50] C. Fronsdal, “On the theory of higher spin fields”, *Nuovo Cimento* **9**, 10 .1007 / BF02747684 (1958).
- [51] S.-J. Chang, “Lagrange formulation for systems with higher spin”, *Phys. Rev.* **161**, 1308 (1967).
- [52] G. Velo and D. Zwanziger, “Propagation and quantization of Rarita-Schwinger waves in an external electromagnetic potential”, *Phys. Rev.* **186**, 1337 (1969).
- [53] G. Velo and D. Zwanziger, “Noncausality and other defects of interaction lagrangians for particles with spin one and higher”, *Phys. Rev.* **188**, 2218 (1969).

- [54] G. Velo, “Anomalous behaviour of a massive spin two charged particle in an external electromagnetic field”, *Nucl. Phys. B* **43**, 389 (1972).
- [55] M. Hortacsu, “Demonstration of noncausality for the rarita-schwinger equation”, *Phys. Rev. D* **9**, 928 (1974).
- [56] R. Courant and D. Hilbert, *Methods of mathematical physics*, vol. 2 (Interscience Publ., 1965).
- [57] S. Ferrara, M. Porrati, and V. L. Telegdi, “ $g = 2$ as the natural value of the tree-level gyromagnetic ratio of elementary particles”, *Phys. Rev. D* **46**, 3529 (1992).
- [58] M. Porrati, R. Rahman, and A. Sagnotti, “String Theory and The Velo-Zwanziger Problem”, *Nucl. Phys. B* **846**, 250 (2011), [arXiv:1011.6411 \[hep-th\]](#).
- [59] M. Porrati and R. Rahman, “Causal Propagation of a Charged Spin 3/2 Field in an External Electromagnetic Background”, *Phys. Rev. D* **80**, 025009 (2009), [arXiv:0906.1432 \[hep-th\]](#).
- [60] S. Deser and A. Waldron, “Inconsistencies of massive charged gravitating higher spins”, *Nucl. Phys. B* **631**, 369 (2002), [arXiv:hep-th/0112182](#).
- [61] M. Porrati and R. Rahman, “Notes on a Cure for Higher-Spin Acausality”, *Phys. Rev. D* **84**, 045013 (2011), [arXiv:1103.6027 \[hep-th\]](#).
- [62] S. M. Klishevich, “Electromagnetic interaction of massive spin 3 state from string theory”, *Int. J. Mod. Phys. A* **15**, 395 (2000), [arXiv:hep-th/9805174](#).
- [63] N. Berkovits and M. M. Leite, “First massive state of the superstring in superspace”, *Phys. Lett. B* **415**, 144 (1997), [arXiv:hep-th/9709148](#).
- [64] N. Berkovits and M. M. Leite, “Superspace action for the first massive states of the superstring”, *Phys. Lett. B* **454**, 38 (1999), [arXiv:hep-th/9812153](#).
- [65] T. Gregoire, M. D. Schwartz, and Y. Shadmi, “Massive supergravity and deconstruction”, *JHEP* **07**, 029 (2004), [arXiv:hep-th/0403224](#).
- [66] T. L. Curtright and P. G. O. Freund, “MASSIVE DUAL FIELDS”, *Nucl. Phys. B* **172**, 413 (1980).
- [67] M. Bastero-Gil and A. Mazumdar, “Gravitino production in hybrid inflationary models”, *Phys. Rev. D* **62**, 083510 (2000), [arXiv:hep-ph/0002004](#).
- [68] K. Benakli, Y. Oz, and G. Policastro, “The Super-Higgs Mechanism in Fluids”, *JHEP* **02**, 015 (2014), [arXiv:1310.5002 \[hep-th\]](#).
- [69] K. Benakli, L. Darmé, and Y. Oz, “The Slow Gravitino”, *JHEP* **10**, 121 (2014), [arXiv:1407.8321 \[hep-ph\]](#).
- [70] E. W. Kolb, A. J. Long, and E. Mcdonough, “The Gravitino Swampland Conjecture”, (2021), [arXiv:2103.10437 \[hep-th\]](#).
- [71] E. Dudas, M. A. G. Garcia, Y. Mambrini, K. A. Olive, M. Peloso, and S. Verner, “Slow and Safe Gravitinos”, (2021), [arXiv:2104.03749 \[hep-th\]](#).
- [72] I. Antoniadis, K. Benakli, and W. Ke, “Salvage of too slow gravitinos”, *JHEP* **11**, 063 (2021), [arXiv:2105.03784 \[hep-th\]](#).

- [73] G. Aad et al. (ATLAS, CMS), “Combined Measurement of the Higgs Boson Mass in pp Collisions at $\sqrt{s} = 7$ and 8 TeV with the ATLAS and CMS Experiments”, *Phys. Rev. Lett.* **114**, 191803 (2015), [arXiv:1503.07589 \[hep-ex\]](#).
- [74] G. Aad et al. (ATLAS, CMS), “Measurements of the Higgs boson production and decay rates and constraints on its couplings from a combined ATLAS and CMS analysis of the LHC pp collision data at $\sqrt{s} = 7$ and 8 TeV”, *JHEP* **08**, 045 (2016), [arXiv:1606.02266 \[hep-ex\]](#).
- [75] B. Abi et al. (Muon $g-2$), “Measurement of the Positive Muon Anomalous Magnetic Moment to 0.46 ppm”, *Phys. Rev. Lett.* **126**, 141801 (2021), [arXiv:2104.03281 \[hep-ex\]](#).
- [76] G. W. Bennett et al. (Muon $g-2$), “Final Report of the Muon E821 Anomalous Magnetic Moment Measurement at BNL”, *Phys. Rev. D* **73**, 072003 (2006), [arXiv:hep-ex/0602035](#).
- [77] T. Aoyama et al., “The anomalous magnetic moment of the muon in the Standard Model”, *Phys. Rept.* **887**, 1 (2020), [arXiv:2006.04822 \[hep-ph\]](#).
- [78] T. Moroi, “The Muon anomalous magnetic dipole moment in the minimal supersymmetric standard model”, *Phys. Rev. D* **53**, [Erratum: *Phys.Rev.D* 56, 4424 (1997)], 6565 (1996), [arXiv:hep-ph/9512396](#).
- [79] W. Altmannshofer and D. M. Straub, “Viability of MSSM scenarios at very large $\tan\beta$ ”, *JHEP* **09**, 078 (2010), [arXiv:1004.1993 \[hep-ph\]](#).
- [80] W. Altmannshofer, S. A. Gadam, S. Gori, and N. Hamer, “Explaining $(g - 2)$ with multi-TeV sleptons”, *JHEP* **07**, 118 (2021), [arXiv:2104.08293 \[hep-ph\]](#).
- [81] P. A. Zyla et al. (Particle Data Group), “Review of Particle Physics”, *PTEP* **2020**, 083C01 (2020).
- [82] P. Slavich, S. Heinemeyer, et al., “Higgs-mass predictions in the MSSM and beyond”, *Eur. Phys. J. C* **81**, edited by P. Slavich and S. Heinemeyer, 450 (2021), [arXiv:2012.15629 \[hep-ph\]](#).
- [83] E. Bagnaschi, G. F. Giudice, P. Slavich, and A. Strumia, “Higgs Mass and Unnatural Supersymmetry”, *JHEP* **09**, 092 (2014), [arXiv:1407.4081 \[hep-ph\]](#).
- [84] E. Bagnaschi, J. Pardo Vega, and P. Slavich, “Improved determination of the Higgs mass in the MSSM with heavy superpartners”, *Eur. Phys. J. C* **77**, 334 (2017), [arXiv:1703.08166 \[hep-ph\]](#).
- [85] J. Braathen, M. D. Goodsell, and P. Slavich, “Matching renormalisable couplings: simple schemes and a plot”, *Eur. Phys. J. C* **79**, 669 (2019), [arXiv:1810.09388 \[hep-ph\]](#).
- [86] B. A. Kniehl, A. F. Pikelner, and O. L. Veretin, “mr: a C++ library for the matching and running of the Standard Model parameters”, *Comput. Phys. Commun.* **206**, 84 (2016), [arXiv:1601.08143 \[hep-ph\]](#).
- [87] J. F. Gunion, H. E. Haber, G. L. Kane, and S. Dawson, *The Higgs Hunter’s Guide*, Vol. 80 (2000).
- [88] S. Davidson and H. E. Haber, “Basis-independent methods for the two-Higgs-doublet model”, *Phys. Rev. D* **72**, [Erratum: *Phys.Rev.D* 72, 099902 (2005)], 035004 (2005), [arXiv:hep-ph/0504050](#).

- [89] J. Bernon, J. F. Gunion, H. E. Haber, Y. Jiang, and S. Kraml, “Scrutinizing the alignment limit in two-Higgs-doublet models: $m_h=125$ GeV”, *Phys. Rev. D* **92**, 075004 (2015), [arXiv:1507.00933 \[hep-ph\]](#).
- [90] J. Bernon, J. F. Gunion, H. E. Haber, Y. Jiang, and S. Kraml, “Scrutinizing the alignment limit in two-Higgs-doublet models. II. $m_H=125$ GeV”, *Phys. Rev. D* **93**, 035027 (2016), [arXiv:1511.03682 \[hep-ph\]](#).
- [91] M. Carena, H. E. Haber, I. Low, N. R. Shah, and C. E. M. Wagner, “Complementarity between Nonstandard Higgs Boson Searches and Precision Higgs Boson Measurements in the MSSM”, *Phys. Rev. D* **91**, 035003 (2015), [arXiv:1410.4969 \[hep-ph\]](#).
- [92] J. Polchinski and L. Susskind, “Breaking of Supersymmetry at Intermediate-Energy”, *Phys. Rev. D* **26**, 3661 (1982).
- [93] L. J. Hall and L. Randall, “U(1)-R symmetric supersymmetry”, *Nucl. Phys. B* **352**, 289 (1991).
- [94] P. J. Fox, A. E. Nelson, and N. Weiner, “Dirac gaugino masses and supersoft supersymmetry breaking”, *JHEP* **08**, 035 (2002), [arXiv:hep-ph/0206096](#).
- [95] A. E. Nelson, N. Rius, V. Sanz, and M. Unsal, “The Minimal supersymmetric model without a mu term”, *JHEP* **08**, 039 (2002), [arXiv:hep-ph/0206102](#).
- [96] I. Antoniadis, K. Benakli, A. Delgado, and M. Quiros, “A New gauge mediation theory”, *Adv. Stud. Theor. Phys.* **2**, 645 (2008), [arXiv:hep-ph/0610265](#).
- [97] G. Belanger, K. Benakli, M. Goodsell, C. Moura, and A. Pukhov, “Dark Matter with Dirac and Majorana Gaugino Masses”, *JCAP* **08**, 027 (2009), [arXiv:0905.1043 \[hep-ph\]](#).
- [98] K. Benakli and M. D. Goodsell, “Dirac Gauginos in General Gauge Mediation”, *Nucl. Phys. B* **816**, 185 (2009), [arXiv:0811.4409 \[hep-ph\]](#).
- [99] K. Benakli, M. D. Goodsell, and A.-K. Maier, “Generating mu and Bmu in models with Dirac Gauginos”, *Nucl. Phys. B* **851**, 445 (2011), [arXiv:1104.2695 \[hep-ph\]](#).
- [100] K. Benakli, Y. Chen, and G. Lafforgue-Marmet, “R-symmetry for Higgs alignment without decoupling”, *Eur. Phys. J. C* **79**, 172 (2019), [arXiv:1811.08435 \[hep-ph\]](#).
- [101] K. Benakli, M. D. Goodsell, and S. L. Williamson, “Higgs alignment from extended supersymmetry”, *Eur. Phys. J. C* **78**, 658 (2018), [arXiv:1801.08849 \[hep-ph\]](#).
- [102] M. Carena, H. E. Haber, S. Heinemeyer, W. Hollik, C. E. M. Wagner, and G. Weiglein, “Reconciling the two loop diagrammatic and effective field theory computations of the mass of the lightest CP - even Higgs boson in the MSSM”, *Nucl. Phys. B* **580**, 29 (2000), [arXiv:hep-ph/0001002](#).
- [103] H. E. Haber and D. O’Neil, “Basis-independent methods for the two-Higgs-doublet model. II. The Significance of $\tan\beta$ ”, *Phys. Rev. D* **74**, [Erratum: *Phys.Rev.D* **74**, 059905 (2006)], 015018 (2006), [arXiv:hep-ph/0602242](#).
- [104] H. E. Haber, “The Higgs data and the Decoupling Limit”, in 1st Toyama International Workshop on Higgs as a Probe of New Physics 2013 (Dec. 2013), [arXiv:1401.0152 \[hep-ph\]](#).

- [105] P. Bechtle, H. E. Haber, S. Heinemeyer, O. Stål, T. Stefaniak, G. Weiglein, and L. Zeune, “The Light and Heavy Higgs Interpretation of the MSSM”, *Eur. Phys. J. C* **77**, 67 (2017), [arXiv:1608.00638 \[hep-ph\]](#).
- [106] H. E. Haber, S. Heinemeyer, and T. Stefaniak, “The Impact of Two-Loop Effects on the Scenario of MSSM Higgs Alignment without Decoupling”, *Eur. Phys. J. C* **77**, 742 (2017), [arXiv:1708.04416 \[hep-ph\]](#).
- [107] M. Carena, H. E. Haber, I. Low, N. R. Shah, and C. E. M. Wagner, “Alignment limit of the NMSSM Higgs sector”, *Phys. Rev. D* **93**, 035013 (2016), [arXiv:1510.09137 \[hep-ph\]](#).
- [108] N. M. Coyle and C. E. M. Wagner, “Dynamical Higgs field alignment in the NMSSM”, *Phys. Rev. D* **101**, 055037 (2020), [arXiv:1912.01036 \[hep-ph\]](#).
- [109] K. Benakli, Y. Chen, and G. Lafforgue-Marmet, “An Emergent Higgs Alignment”, *PoS CORFU2018*, edited by K. Anagnostopoulos et al., 014 (2019), [arXiv:1904.04248 \[hep-ph\]](#).
- [110] M. Awramik, M. Czakon, A. Freitas, and G. Weiglein, “Precise prediction for the W boson mass in the standard model”, *Phys. Rev. D* **69**, 053006 (2004), [arXiv:hep-ph/0311148](#).
- [111] G. Degrandi, P. Gambino, and P. P. Giardino, “The $m_W - m_Z$ interdependence in the Standard Model: a new scrutiny”, *JHEP* **05**, 154 (2015), [arXiv:1411.7040 \[hep-ph\]](#).