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École doctorale de sciences mathématiques de Paris centre

Thèse de doctorat

Discipline : Mathématiques

présentée par

Haowen Zhang

Arithmetic of homogeneneous spaces of algebraic groups over certain function fields

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dirigée par Cyril Demarche

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Abstract

This dissertation studies weak and strong approximation problems for homogeneous spaces of algebraic groups defined over some two-dimensional geometric global fields, or p-adic function fields.

We first consider a field K which is either (a) the function field of a curve over $\mathbb{C}((t))$ or (b) a finite extension of $\mathbb{C}((t_1, t_2))$. We show that the Brauer-Manin obstruction to weak approximation is the only one for connected reductive groups over K, generalizing the known results for K-tori obtained in by Colliot-Thélène, Harari and Izquierdo. Then we construct counterexamples showing that the Brauer-Manin obstruction to weak approximation is not enough for homogeneous spaces over K, and thus a more refined obstruction is needed. Following a construction by Izquierdo and Lucchini-Arteche in the study of local-global questions, we define an obstruction to weak approximation combining the Brauer-Manin obstruction and the descent obstruction with respect to torsors of quasi-trivial tori. We show that this obstruction to weak approximation is the only one for homogeneneous spaces of connected linear groups over K with connected or abelian stabilizers. For K of type (a), we also show that the Brauer-Manin obstruction to strong approximation is the only one for homogeneous spaces of quasi-split semisimple simply connected groups with toric stabilizers T such that $Ш^2(K, T)$ is finite.

Then we look at p-adic function fields K. Using the Rost invariant to which we relate Lichtenbaum's duality pairing for a p-adic curve, we construct homogeneous spaces of $\mathbf{SL}_n$ with semisimple simply connected stabilizers for which strong approximation away from a non-empty set of places fails. To account for such failure, we define a reciprocity obstruction to strong approximation using $H^3_{\mathrm{nr}}(Z, \mathbb{Q}/\mathbb{Z}(2))$, the subgroup of elements in $H^3(K(Z), \mathbb{Q}/\mathbb{Z}(2))$ that are unramified with respect to all the codimension 1 points of the K-variety Z. We show that for classifying varieties $Z = \mathbf{SL}_n / H$, this obstruction is related to the group $\mathrm{Inv}^3(H, \mathbb{Q}/\mathbb{Z}(2))$ of degree 3 cohomological invariants of H with coefficients in $\mathbb{Q}/\mathbb{Z}(2)$. Then we show that for certain classifying varieties Z including counterexamples we give above and $\mathbf{SL}_n / T$ for T a torus, the closure of $Z(K)$ in $Z(\mathbf{A}_K^S)$ equals the subset of elements in $Z(\mathbf{A}_K^S)$ that are orthogonal to $H^3_{\mathrm{nr},S}(Z, \mathbb{Q}/\mathbb{Z}(2))$, the subgroup of "trivial on S" elements in $H^3_{\mathrm{nr}}(Z, \mathbb{Q}/\mathbb{Z}(2))$. This means that the reciprocity obstruction to strong approximation away from S is the only one. Going beyond classifying varieties, we apply this reciprocity obstruction to K-tori, and we get results compatible to the known ones obtained by Harari and Izquierdo. At last, we explain that we can adapt a similar point of view for weak approximation problems for classifying varieties, linking the reciprocity obstruction to weak approximation with the unramified cohomological invariants. As a quick and direct application, combining Blinstein and Merkurjev's description of $\mathrm{Inv}^3_{\mathrm{nr}}(T, \mathbb{Q}/\mathbb{Z}(2))$ and the Poitou-Tate type exact sequence obtained by Harari, Scheiderer and Szamuely gives that the reciprocity obstruction to weak approximation is the only one for $\mathbf{SL}_n / T$ where T is a K-torus.

Résumé

Cette thèse étudie des problèmes d'approximation faible et forte pour les espaces homogènes de groupes algébriques définis sur certains corps globaux géométriques de dimension 2 ou des corps de fonctions p-adiques.

Nous considérons d'abord un corps K qui est soit (a) le corps de fonctions d'une courbe sur $\mathbb{C}(\!(t)\!)$, soit (b) une extension finie de $\mathbb{C}(\!(t_1, t_2)\!)$. Nous montrons que l'obstruction de Brauer-Manin à l'approximation faible est la seule pour les groupes réductifs connexes sur K, généralisant ainsi les résultats connus pour les K-tores obtenus par Colliot-Thélène, Harari et Izquierdo. Ensuite, nous construisons des contre-exemples montrant que l'obstruction de Brauer-Manin à l'approximation faible n'est pas suffisante pour les espaces homogènes sur K, et qu'une obstruction plus raffinée est nécessaire. En suivant une construction d'Izquierdo et Lucchini-Arteche dans l'étude de questions locales-globales, nous définissons une obstruction à l'approximation faible combinant l'obstruction de Brauer-Manin et l'obstruction de descente par rapport aux torseurs sous des K-tores quasi-triviaux. Nous montrons que cette obstruction à l'approximation faible est la seule pour les espaces homogènes de groupes linéaires connexes sur K avec des stabilisateurs connexes ou abéliens. Pour K de type (a), nous montrons également que l'obstruction de Brauer-Manin à l'approximation forte est la seule pour les espaces homogènes de groupes semi-simples simplement connexes quasi-déployés à stabilisateur torique T tel que $Ш^2(K, T)$ soit fini.

Ensuite, nous examinons les corps de fonctions p-adiques K. En utilisant l'invariant de Rost auquel nous relions l'accouplement de dualité de Lichtenbaum pour une courbe p-adique, nous construisons des espaces homogènes de $\mathbf{SL}_n$ avec des stabilisateurs semi-simples simplement connexes pour lesquels l'approximation forte en dehors d'un ensemble non vide de places ne vaut pas. Pour prendre en compte un tel défaut, nous définissons une obstruction de réciprocité à l'approximation forte en utilisant $H^3_{\mathrm{nr}}(Z, \mathbb{Q}/\mathbb{Z}(2))$, le sous-groupe des éléments de $H^3(K(Z), \mathbb{Q}/\mathbb{Z}(2))$ qui sont non-ramifiés par rapport à tous les points de codimension 1 de la K-variété Z. Nous montrons que pour les variétés classifiantes $Z = \mathbf{SL}_n / H$, cette obstruction est liée au groupe $\mathrm{Inv}^3(H, \mathbb{Q}/\mathbb{Z}(2))$ des invariants cohomologiques de degré 3 de H avec des coefficients dans $\mathbb{Q}/\mathbb{Z}(2)$. Ensuite, nous montrons que pour certaines variétés classifiantes Z incluant des contre-exemples à l'approximation forte que nous avons donnés ci-dessus et $\mathbf{SL}_n / T$ pour T un tore, l'adhérence de $Z(K)$ dans $Z(\mathbf{A}_K^S)$ est égale au sous-ensemble d'éléments dans $Z(\mathbf{A}_K^S)$ qui sont orthogonaux à $H^3_{\mathrm{nr},S}(Z, \mathbb{Q}/\mathbb{Z}(2))$, le sous-groupe des éléments « triviaux sur S » dans $H^3_{\mathrm{nr}}(Z, \mathbb{Q}/\mathbb{Z}(2))$. Cela signifie que l'obstruction de réciprocité à l'approximation forte en dehors de S est la seule. Allant au-delà des variétés classifiantes, nous appliquons cette obstruction de réciprocité aux K-tores et obtenons des résultats compatibles avec les résultats déjà connus par Harari et Izquierdo. Enfin, nous expliquons que nous pouvons adapter un point de vue similaire pour les problèmes d'approximation faible pour les variétés classifiantes, en reliant l'obstruction de réciprocité à l'approximation faible avec les invariants cohomologiques non-ramifiés. Comme application rapide et directe, la combinaison de la description de $\mathrm{Inv}^3_{\mathrm{nr}}(T, \mathbb{Q}/\mathbb{Z}(2))$ par Blinstein et Merkurjev et de la suite exacte à la Poitou-Tate obtenue

par Harari, Scheiderer et Szamuely montre que l'obstruction de réciprocité à l'approximation faible est la seule pour $\mathbf{SL}_n/T$ où T est un K-tore.

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Chapter 1

Introduction

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In this chapter, we first introduce arithmetic problems in classical situations over number fields, along with existing results of such problems for algebraic groups and their homogeneous spaces. Then we introduce and present known results about the types of fields for which we consider these problems in this dissertation, before stating the main findings in the last section of this chapter.

1.1 Classical situations

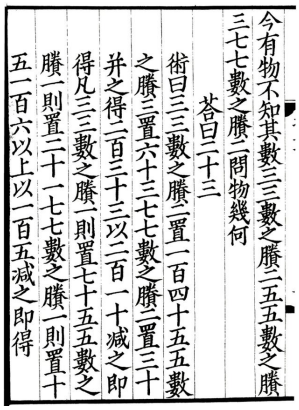
今有物不知其數三三數之賸二五五數之賸
三七七數之賸二問物幾何
荅曰二十三
術曰三三數之賸二置一百四十五五數
之賸三置六十三七七數之賸二置三十
并之得二百三十三以二百一十減之即
得凡三三數之賸一則置七十五五數之
賸一則置二十一七七數之賸一則置十
五一百六以上以一百五減之即得

Figure 1.1: Problem in Sunzi Suanjing:

Solve

$$\begin{cases} x \equiv 2 \bmod 3 \\ x \equiv 3 \bmod 5 \\ x \equiv 2 \bmod 7. \end{cases}$$

Questions about solving a system of linear congruence equations were raised as early as in the treatise Sunzi Suanjing written during 3rd to 5th centuries AD (Figure 1.1). Answers to such questions are now called the Chinese Remainder Theorem, affirming the surjectivity of the morphism $\mathbb{Z} \to \prod_{i=1}^{n} \mathbb{Z}/p_i^{k_i}\mathbb{Z}$ for finitely many prime numbers p_i and positive exponents $k_i \in \mathbb{N}$. This is equivalent to saying that the map $\mathbb{Z} \to \prod_{i=1}^{n} \mathbb{Z}_{p_i}$ has dense image in the topological product of p-adic integers for finitely many p_i, and hence we can think of the Chinese Remainder Theorem as an approximation theorem, allowing us to find an integer that simultaneously approximates a finite number of p-adic integers arbitrarily well. Going beyond the set of all integers, we can consider the set of integer solutions $Y(\mathbb{Z})$ of a system of polynomial equations (which defines a variety Y), and ask the surjectivity of $Y(\mathbb{Z}) \to \prod_{i=1}^{n} Y(\mathbb{Z}/p_i\mathbb{Z})$.

Let F be a number field. We denote by Ω_F the set of places of F, and F_v the completion of F at the place v which is a local field. In modern terms, the question of strong approximation for an F-variety Y asks the density of the F-rational points $Y(F)$ (supposed to be non-empty) in the adelic space $Y(\mathbf{A}_F^S) := \prod'_{v \notin S} Y(F_v)$ which is the topological restricted product with respect to integral points excluding places in a finite set S. If $Y(F)$ is dense in $Y(\mathbf{A}_F^S)$, we say that *strong approximation away from* S holds for Y: we can simultaneously approximate finitely many F_v-points with $v \notin S$ by a single F-rational point with the condition that this point is integral at all other places. With this terminology, the Chinese Remainder Theorem says that the affine line $\mathbb{A}^1_{\mathbb{Q}}$ satisfies strong approximation away from the place ∞.

We can generalize this question to the study of the closure $\overline{Y(F)}$ in the topological product $\prod_{v \in \Omega_F} Y(F_v)$ of local points. We say that *weak approximation* holds for Y if $\overline{Y(F)} = \prod_{v \in \Omega_F} Y(F_v)$, meaning that any collection of local points in finitely many places can be simultaneously approximated by a rational point. If the variety Y is proper, then $Y(\mathbf{A}_F^S) = \prod_{v \in \Omega_F \setminus S} Y(F_v)$. Thus if weak approximation holds for Y, then strong approximation holds for Y away from any finite set $S \subseteq \Omega_F$, in particular $S = \emptyset$. Apart from number fields, we can naturally ask these questions for function fields too.

Strong and weak approximation questions have been studied for algebraic groups, which are algebraic varieties Y for which the set of K-points $Y(K)$ is always non-empty for any field K and admits a group structure functorial in K. The following result was obtained by Kneser and Platonov in characteristic 0, and by Margulis and Prasad in positive characteristics.

Theorem 1.1.1 (cf. Theorem 7.12 of [PR93]). *Let G be a connected semisimple algebraic group over a global field F, and let S be a finite set of places of F. Then strong approximation away from S holds for G if and only if G is simply connected and for each almost simple factor G_i of G, there exists $v \in S$ such that $G_i(F_v)$ is noncompact.*

Unlike strong approximation, weak approximation is a stably birational property, hence satisfied by all stably rational varieties (e.g. a quasi-trivial torus). We have the following result due to Kneser, Harder and Platonov.

Theorem 1.1.2 (cf. Theorem 7.8 of [PR93]). *Let G be a semisimple group, either simply connected or adjoint, defined over a number field F. Then weak approximation holds for G.*

Before considering strong and weak approximation questions for more general varieties Y other than algebraic groups, one first asks naturally if our system of polynomial equations admits rational solutions at all, i.e. if the set of rational points $Y(F)$ is non-empty. A necessary condition for a positive answer is the existence of solutions over all of the local completions F_v. For certain types of equations, this turns out to be a sufficient condition too, for example, for quadrics in virtue of the Hasse-Minkowski Theorem. This gives rise to the definition of *the local-global principle* (also known as *the Hasse principle*): given a family $\mathcal{C}$ of algebraic varieties over a number field F, we say that it satisfies the *local-global principle* if $Y(\mathbf{A}_F) \neq \emptyset$ implies $Y(F) \neq \emptyset$ for every $Y \in \mathcal{C}$.

When the local-global principle does not hold, we hope to find certain obstructions explaining the failure. Manin introduced the Brauer set $Y(\mathbf{A}_F)^{\mathrm{Br}}$ in 1970, which is the left kernel of the pairing

$$\begin{aligned} Y(\mathbf{A}_F) \times \operatorname{Br} Y &\to \mathbb{Q}/\mathbb{Z} \\ ((x_v)_v, \alpha) &\mapsto \sum_v \operatorname{inv}_v(x_v^* \alpha) \end{aligned}$$

defined by using the functoriality of the Brauer group and the class field theory exact sequence

$$0 \to \operatorname{Br} F \to \bigoplus_v \operatorname{Br} F_v \xrightarrow{\Sigma \operatorname{inv}_v} \mathbb{Q}/\mathbb{Z} \to 0.$$

The Brauer set $Y(\mathbf{A}_F)^{\mathrm{Br}}$ contains $Y(F)$, giving rise to an obstruction to the existence of rational points. We say that there is a *Brauer-Manin obstruction to the local-global principle* for Y if $Y(\mathbf{A}_F) \neq \emptyset$ but $Y(\mathbf{A}_F)^{\mathrm{Br}} = \emptyset$. We say that *the Brauer-Manin obstruction to the local-global principle is the only one* if the implication $Y(\mathbf{A}_F)^{\mathrm{Br}} \neq \emptyset \implies Y(F) \neq \emptyset$ holds.

When Y has a F-rational point, we also get an obstruction to strong approximation by the continuity of this Brauer-Manin pairing. For a finite set S of places, we can consider

the projection $Y(\mathbf{A}_F) \to Y(\mathbf{A}_F^S)$ and say that the Brauer-Manin obstruction to strong approximation away from S is the only one if $Y(K)$ is dense in the image of $Y(\mathbf{A}_F)^{\mathrm{Br}}$ under this projection. With this definition, a series of work by Colliot-Thélène, Xu, Harari, Borovoi and Demarche shows that, under a necessary non-compactness assumption, the Brauer-Manin obstruction to strong approximation is the only one for homogeneous spaces of linear algebraic groups with connected stabilizers (or homogeneous spaces of connected algebraic groups with connected stabilizers, when assuming a suitable Tate-Shafarevich group to be finite). In [Dem22], Demeio improved these results by showing that it suffices to consider a less restrictive obstruction given by the modified Brauer group of "trivial on S" elements:

$$\mathrm{Br}_S Y := \mathrm{Ker}(\mathrm{Br}\, Y \to \prod_{v \in S} \mathrm{Br}\, Y_{F_v}). \tag{1.1}$$

The Brauer set $Y(\mathbf{A}_F^S)^{\mathrm{Br}_S Y}$ away from S is defined to be the elements in $Y(\mathbf{A}_F^S)$ which are orthogonal to $\mathrm{Br}_S Y$, and $Y(\mathbf{A}_F^S)^{\mathrm{Br}_S Y}$ always contains the closure $\overline{Y(K)}^S$ of K-points in $Y(\mathbf{A}_F^S)$. In our study of strong approximation problems in Chapter 4, we will adopt this point of view and also adapt it to define our analogous reciprocity obstruction.

Similarly, Colliot-Thélène and Sansuc (cf. [CTS87]) defined the Brauer-Manin obstruction to weak approximation by the pairing

$$\prod_{v \in \Omega_F} Y(F_v) \times \mathrm{Br}_{\mathrm{nr}}(F(Y)/F) \to \mathbb{Q}/\mathbb{Z}$$

using the unramified Brauer group, which is a birational invariant. The left kernel of the pairing contains the closure $\overline{Y(F)}$, and when they are equal we say that *the Brauer-Manin obstruction to weak approximation is the only one* for Y. With work by Sansuc and Borovoi, we have the following result:

Theorem 1.1.3 ([Bor96]). *For homogeneous spaces of connected linear algebraic groups over a number field, with connected or abelian stabilizers, the Brauer-Manin obstruction to the local-global principle and to weak approximation are the only ones.*

Afterwards, there has been growing interest in studying analogous problems over different fields other than number fields, for example, function fields of curves defined over various base fields. For such a field K, we can consider a natural set of places given by the set $X^{(1)}$ of codimension 1 points v with respect to which we can take completions K_v and formulate the strong and weak approximation problems. This can be illustrated by the following result of Colliot-Thélène:

Theorem 1.1.4 (Théorème 3.6 of [CT18]). *Let $K = \mathbb{C}(X)$ be the function field of smooth projective connected curve X over $\mathbb{C}$. Then strong approximation away from any non-empty set $S \subseteq X(\mathbb{C})$ holds for homogeneous spaces of semi-simple K-groups.*

However, strong approximation does not hold for tori of dimension > 0 over such a field. In fact, Harari and Izquierdo gave the following more precise and general result:

Theorem 1.1.5 (Théorème 3.6 of [HI19]). *Let $K = k(X)$ be the function field of a smooth projective geometrically integral curve X defined over a field k of characteristic 0. Let T be a K-torus. Then $T(K)$ is always discrete (hence closed) in $T(\mathbf{A}_K^S)$ for any finite $S \subseteq X^{(1)}$.*

In this dissertation, we will be mainly studying weak and strong approximation problems over the following types of fields: the p-adic function fields, which are function fields of smooth projective geometrically integral curves defined over a p-adic fields, and some two-dimensional geometric global fields, for example $\mathbb{C}((t_1, t_2))$, or the function field of a smooth projective geometrically integral curve defined over $\mathbb{C}((t))$. We give introductions and present existing results over these types of fields in the following sections, before stating the main findings of the dissertation in §1.4.

1.2 p-adic function fields

In addition to global function fields (i.e. function fields over finite fields), function fields over local fields are also of growing interest in arithmetic geometry. More generally, given k a complete discretely valued field, we call K a *semi-global field over* k if K is a one-variable function field over k, i.e. a finitely generated extension of k of transcendence degree 1 in which k is algebraically closed.

In recent years, there has been a series of work on local-global principles over semi-global fields in general, by Harbater, Hartmann, Krashen, Karemaker, Pop, Colliot-Thélène, Parimala, Suresh, Pirutka, Wu, Mishra etc. In fact there can be different choices of the set of places, for example, the set of all discrete valuations on K, or the subset of divisorial discrete valuations, or the even smaller subset of valuations coming from codimension 1 points of X when $K = k(X)$ for a smooth projective geometrically integral curve X over k. These different choices can give different results concerning local-global principles, cf. Example 7.10 of [CT+22].

Now we consider the prototypical case where k is a p-adic field, which gives a field K of cohomological dimension 3. Let X be a smooth projective geometrically integral curve over k with function field $K = k(X)$. We call K a p-adic function field. We take the set of places to be $X^{(1)}$, the set of codimension 1 points of X. For a given valuation $v \in X^{(1)}$, we denote by K_v the completion of K at v. The complex

$$H^3(K, \mathbb{Q}/\mathbb{Z}(2)) \to \bigoplus_{v \in X^{(1)}} H^3(K_v, \mathbb{Q}/\mathbb{Z}(2)) \xrightarrow{\Sigma \operatorname{inv}_v \circ \partial_v} \mathbb{Q}/\mathbb{Z} \tag{1.2}$$

of the generalized Weil reciprocity law enables us to define the so-called reciprocity obstruction, analogous to the Brauer-Manin obstruction over fields of cohomological dimension 2.

Harari and Szamuely studied the local-global principle with respect to the valuations from $X^{(1)}$ for torsors under tori and reductive groups over K. They first obtained a global duality theorem between the first Tate-Shafarevich group of the torus and the second Tate-Shafarevich group of the dual torus.

Theorem 1.2.1 (Theorem 0.1 of [HS16]). *There is a perfect pairing of finite groups*

$$Ш^1(T) \times Ш^2(T') \to \mathbb{Q}/\mathbb{Z}$$

where $Ш^i(T) := \mathrm{Ker}(H^i(K,T) \to \prod_{v \in X^{(1)}} H^i(K_v, T))$, *and* T' *denotes the dual torus of* T.

Building upon this duality theorem, they showed that the failure of local-global principle for torsors under tori are controlled by the subgroup of locally constant elements in the pairing of the reciprocity obstruction. Let

$$\begin{aligned} H^3_{\mathrm{lc}}(Z, \mathbb{Q}/\mathbb{Z}(2)) := &\,\mathrm{Ker}(H^3(Z, \mathbb{Q}/\mathbb{Z}(2))/\operatorname{Im}(H^3(K, \mathbb{Q}/\mathbb{Z}(2))) \\ &\to \prod_{v \in X^{(1)}} H^3(Z_{K_v}, \mathbb{Q}/\mathbb{Z}(2))/\operatorname{Im}(H^3(K_v, \mathbb{Q}/\mathbb{Z}(2)))) \end{aligned}$$

where Z is a K-variety. The pairing

$$Z(\mathbf{A}_K) \times H^3(Z, \mathbb{Q}/\mathbb{Z}(2)) \to \mathbb{Q}/\mathbb{Z}$$

induced by evaluating classes at adelic points is constant on the subgroup $H^3_{\mathrm{lc}}(Z, \mathbb{Q}/\mathbb{Z}(2))$, hence defining a morphism

$$\rho_Z : H^3_{\mathrm{lc}}(Z, \mathbb{Q}/\mathbb{Z}(2)) \to \mathbb{Q}/\mathbb{Z}$$

for Z such that $Z(\mathbf{A}_K) \neq \emptyset$. When Z is a torsor under a K-torus T, they showed that there is a map $Ш^2(T') \to H^3_{\mathrm{lc}}(Z, \mathbb{Q}/\mathbb{Z}(2))$ relating ρ to the duality pairing in Theorem 1.2.1, and in particular, the reciprocity obstruction to local-global principle is the only one for torsors under tori:

Theorem 1.2.2 (Theorem 0.2 of [HS16]). *Assume that* Z *is a torsor under a* K*-torus* T *such that* $Z(\mathbf{A}_K) \neq \emptyset$ *and* ρ_Z *is the zero map. Then* $Z(K) \neq \emptyset$.

To generalize this result to torsors under reductive groups, they combined the above theorem with the Rost invariant for simply connected linear algebraic groups, with the assumption of good reduction of the base curve.

Theorem 1.2.3 (Theorem 0.3 of [HS16]). *Let* Z *be a torsor under a reductive linear algebraic group* G *under* K. *Assume that the smooth projective curve* X *whose fraction field is* K *has good reduction and that the simply connected cover* G^{sc} *of the derived subgroup of* G *is quasi-split and has no* E_8 *factor. If* $Z(\mathbf{A}_K) \neq \emptyset$ *and* ρ_Z *is the zero map, then* $Z(K) \neq \emptyset$.

As for weak approximation, Harari, Scheiderer and Szamuely investigated such questions for tori over K. Let $A_\wedge$ denote the inverse limit of the quotients A/mA for an abelian group A. They first obtained a Poitou-Tate type exact sequence:

Theorem 1.2.4 (Theorem 0.1 of [HSS15]). *There is an exact sequence of topological groups*

$$\begin{tikzcd}
0 \arrow[r] & H^0(K,T)_\wedge \arrow[r] & \mathbf{P}^0(T)_\wedge \arrow[r] & H^2(K,T')^D \arrow[d] & \\
& H^1(K,T')^D \arrow[d] & \mathbf{P}^1(T) \arrow[l] & H^1(K,T) \arrow[l] & \\
& H^2(K,T) \arrow[r] & \mathbf{P}^2(T)_{\mathrm{tors}} \arrow[r] & (H^0(K,T')_\wedge)^D \arrow[r] & 0.
\end{tikzcd} \tag{1.3}$$

They used this sequence to analyze the defect of weak approximation for T and obtained the following description of the closure of K-points $T(K)$ in the topological product $\prod_{v\in X^{(1)}} T(K_v)$.

Theorem 1.2.5 (Theorem 0.2 of [HSS15]). *There is an exact sequence*

$$0 \to \overline{T(K)} \to \prod_{v\in X^{(1)}} T(K_v) \to Ш^2_\omega(T')^D \to Ш^1(T) \to 0, \tag{1.4}$$

where $Ш^2_\omega(T') \subseteq H^2(K,T')$ is the subgroup of classes whose image in $H^2(K_v,T')$ is trivial for all but finitely many v.

Then they showed that the above obstruction is compatible with the reciprocity obstruction to weak approximation first studied by Colliot-Thélène, and in particular, the reciprocity obstruction is the only one to weak approximation for a torus over K.

Theorem 1.2.6 (Theorem 0.3 of [HSS15]). *The left kernel of the pairing*

$$\prod_{v\in X^{(1)}} T(K_v) \times H^3_{\mathrm{nr}}(K(T)/K, \mathbb{Q}/\mathbb{Z}(2)) \to \mathbb{Q}/\mathbb{Z}$$

is the closure of the diagonal image of $T(K)$ for the product topology.

Tian generalized their results to connected reductive groups under certain assumptions which hold for quasi-split connected reductive groups and tori. He established a global duality for the Tate-Shafarevich groups of 2-term short complexes of K-tori.

Theorem 1.2.7 (Theorem 1.18 of [Tia21a], Theorem 3.5 of [Tia21b]). *Let $C = [T_1 \to T_2]$ be an arbitrary complex of K-tori concentrated in degrees -1 and 0. Let T_1' and T_2' be the respective dual torus of T_1 and T_2, and let $C' = [T_2' \to T_1']$. Let $Ш^i(C) := \mathrm{Ker}(H^i(K,C) \to \prod_{v\in X^{(1)}} H^i(K_v,C))$ be the Tate-Shafarevich group of the complex C. There is a perfect, functorial in C, pairing of finite groups for $0 \le i \le 2$:*

$$Ш^i(C) \times Ш^{2-i}(C') \to \mathbb{Q}/\mathbb{Z}.$$

Tian then considered the complex $C = [T^{\mathrm{sc}} \to T]$ where T is a maximal torus of G, and T^{sc} a maximal torus of G^{sc} which is the preimage of T under $G^{\mathrm{sc}} \to G^{\mathrm{ss}} \to G$ as in the diagrams (2.1) and (2.2). Under the assumption that G^{sc} satisfies weak approximation and it contains a quasi-trivial maximal torus, Tian gave an exact sequence similar to (1.4) measuring the defect of weak approximation for G, with $Ш^2_\omega(T')$ (resp. $Ш^1(T)$) replaced by $Ш^1_\omega(C')$ (resp. $Ш^1(C)$), and he showed that the reciprocity obstruction to weak approximation is the only one for such a G, with a morphism $Ш^1_\omega(C') \to H^3_{\mathrm{nr}}(K(G)/K, \mathbb{Q}/\mathbb{Z}(2))$ relating both obstructions (cf. Theorem 2.4 and Theorem 3.1 of [Tia21a]).

As for strong approximation, a classical and general result by Harder over Dedekind domains implies that strong approximation away from any non-empty set $S \subseteq X^{(1)}$ holds for a quasi-split semisimple simply connected K-group (for example $\mathbf{SL}_n$).

Harari and Izquierdo studied tori T and described the group $A(T) := T(\mathbf{A}_K)/\overline{T(K)} = T(\mathbf{A}_K)/T(K)$ (the last equality comes from Theorem 1.1.5) which measures the defect of strong approximation. Based on the exact sequence à la Poitou-Tate (1.3), they obtained the following results:

Theorem 1.2.8 (Corollaire 6.7 and 6.9 of [HI19]). *There is an injective morphism*

$$A(T)/\operatorname{Div} \hookrightarrow H^2(K, T')^D$$

where $A(T)/\operatorname{Div}$ denotes the quotient of $A(T)$ by its maximal divisible subgroup. The closure J of its image fits into an exact sequence

$$1 \to J \to H^2(K, T')^D \to Ш^2(T')^D \to 1.$$

There is a perfect pairing

$$A(T)_{\mathrm{tors}} \times \frac{H^2(K, T')_\wedge}{\operatorname{Im}(Ш^2(K, T'))} \to \mathbb{Q}/\mathbb{Z}.$$

1.3 Two-dimensional geometric global fields

We consider a field K of one of the following two types:

(a) *(Semi-global case).* The function field of a smooth projective curve X over $\mathbb{C}((t))$. This is a semi-global field as defined in §1.2. We let the set of places Ω be the set of valuations coming from $X^{(1)}$ the closed points of the curve X.

(b) *(Local case).* The field of fractions of a two-dimensional, excellent, henselian, local domain R with algebraically closed residue field of characteristic zero. An example of such a field is any finite extension of the field of fractions $\mathbb{C}((t_1, t_2))$ of the formal power series ring $\mathbb{C}[[t_1, t_2]]$. We let the set of places Ω be the set of valuations coming from prime ideals of height 1 in R.

The field K has similar properties comparable to that of the usual global fields, making it natural to consider arithmetic problems for varieties over K. We list the following common behavior shared by K and totally imaginary number fields (cf. Theorem 5.5 of [HHK09], Theorem 1.2, 1.4, 4.7 of [CTGP04], §10.1 of [CTH15]):

(i) the cohomological dimension is two;

(ii) index and exponent of central simple algebras coincide;

(iii) Serre's conjecture II holds: for any semisimple simply connected group G over K, we have $H^1(K, G) = 1$;

(iv) weak approximation is satisfied for semisimple simply connected groups over such fields.

By Proposition 2.1.(v) of [CTH15] and Théorème 1.6 of [Izq19], we also have the following exact sequence for K analogous to the class field theory exact sequence for number fields, allowing us to define the Brauer-Manin obstruction:

$$\operatorname{Br} K \to \bigoplus_{v \in \Omega} \operatorname{Br} K_v \to \mathbb{Q}/\mathbb{Z} \to 0.$$

For K of type (a), Colliot-Thélène and Harari studied tori over K. Adapting the arguments in [HS16] which treat the case of p-adic function fields and give Theorem 1.2.1, they obtained the following analogous dualities:

Theorem 1.3.1 ([CTH15]). *Let T be a K-torus with character group $\widehat{T}$. There are pairings of torsion groups*

$$Ш^1(K, T) \times Ш^2(K, \widehat{T}) \to \mathbb{Q}/\mathbb{Z}(-1)$$
$$Ш^1(K, \widehat{T}) \times Ш^2(K, T) \to \mathbb{Q}/\mathbb{Z}(-1)$$

where $\mathbb{Q}/\mathbb{Z}(-1)$ denotes the direct limit of $\mathbb{Z}/m(-1) := \operatorname{Hom}(\mu_m, \mathbb{Z}/m)$, which is (non-canonically) isomorphic to $\mathbb{Q}/\mathbb{Z}$. Quotienting by the maximal divisible subgroups, we get perfect pairing of finite groups

$$Ш^1(K, T) \times Ш^2(K, \widehat{T})/\operatorname{Div} \to \mathbb{Q}/\mathbb{Z}(-1)$$

$$Ш^1(K, \widehat{T}) \times Ш^2(K, T)/\operatorname{Div} \to \mathbb{Q}/\mathbb{Z}(-1).$$

Similarly to Theorem 1.2.2 which is a consequence of Theorem 1.2.1, the above Theorem 1.3.1 yields that the obstruction to the local-global principle for a homogeneous spaces Z of a K-torus is controlled by the subgroup of locally constant elements in the algebraic Brauer group $\operatorname{Br}_1 Z$, as in the case of number fields.

However, they showed examples where weak approximation for a torus T is not controlled by the unramified Brauer group $\operatorname{Br}_{\mathrm{nr}} T$, different from the case of global fields. Another consequence of such counter-examples is the following result:

Corollary 1.3.2 (Corollaire 9.16 of [CTH15]). *There exists a flasque K-torus Q such that the following complex is not exact:*

$$H^1(K,Q) \to \bigoplus_{v\in\Omega} H^1(K_v,Q) \to Ш^1_\omega(\widehat{Q})^D \tag{1.5}$$

where A^D denotes the group of continuous homomorphisms from A to $\mathbb{Q}/\mathbb{Z}(-1)$.

Therefore, there is no Poitou-Tate type exact sequence in general as in the case of p-adic function fields (1.3). But if we suppose that $Ш^2(K,T)$ is finite for the K-torus T, we do have an exact sequence (cf. Proposition 3.16(ii) of [Izq16])

$$H^1(K,T) \to \mathbf{P}^1(T) \to H^1(K,\widehat{T})^D.$$

With the crucial hypothesis that there exists a field extension L of K splitting the torus T such that $Ш^2(L,\mathbb{G}_m)=0$, Izquierdo established the following results:

Theorem 1.3.3 (Théorème 3.20 and 3.22 of [Izq16]). *For K of type (a) and a K-torus T, let L/K be a finite extension splitting T. Suppose that $Ш^2(L,\mathbb{G}_m)=0$. Then there is an exact sequence*

$$\begin{tikzcd}
0 \arrow[r] & Ш^3(\widehat{T})^D \arrow[r] & H^0(K,T)_\wedge \arrow[r] & \mathbf{P}^0(T)_\wedge \arrow[r] & H^2(K,\widehat{T})^D \arrow[d] \\
 & & H^1(K,\widehat{T})^D & \mathbf{P}^1(T) \arrow[l] & H^1(K,T) \arrow[l]
\end{tikzcd}$$

and an exact sequence

$$\begin{tikzcd}
 & H^1(K,T)^D \arrow[d] & \mathbf{P}^1(\widehat{T}) \arrow[l] & & \\
 & H^2(K,\widehat{T}) \arrow[r] & \mathbf{P}^2(\widehat{T})_{\mathrm{tors}} \arrow[r] & (H^0(K,T)_\wedge)^D \arrow[d] & \\
0 & (\varprojlim_m {}_mT(\overline{K}))^D \arrow[l] & \mathbf{P}^3(\widehat{T}) \arrow[l] & H^3(K,\widehat{T}) \arrow[l] &
\end{tikzcd}$$

where $\widehat{T}$ denotes the character group of T.

Colliot-Thélène and Harari described the defect of weak approximation for a K-torus T using the modified version $Б_\omega$ of a Brauer group:

Theorem 1.3.4 (Théorème 9.11 of [CTH15]). *Let T be a K-torus. Then there is an exact sequence*

$$1 \to \overline{T(K)} \to \prod_{v\in\Omega} T(K_v) \to Б_\omega(T)^D \to Б(T)^D \to 1 \tag{1.6}$$

where $Б(T)$ (resp. $Б_\omega(T)$) is defined to be the subgroup of $\mathrm{Br}_1(T)/\mathrm{Br}(K)$ containing elements vanishing in $\mathrm{Br}_1(T_{K_v})/\mathrm{Br}(K_v)$ for all places (resp. almost all places) $v\in\Omega$.

For K of type (b), Izquierdo obtained results of the same form as Theorem 1.3.1 and Theorem 1.3.4 (cf. Théorème 3.8 and Théorème 4.9 of [Izq19]).

Then Izquierdo and Lucchini-Arteche studied obstructions to the local-global principle for homogeneous spaces with connected or abelian stabilizers over K of both types (a) and (b). They gave examples showing that the usual Brauer-Manin obstruction is not enough to explain the failure of the local-global principle:

Theorem 1.3.5 (Theorem 1.1 of [IA21])**.** *Let K be a field of type (a) or (b) as above. There exists a finite extension L of K and a homogeneous space Z/L under $\mathbf{SL}_{n,L}$ for some $n \geq 1$ with toric stabilizers for which the Brauer set $Z(\mathbf{A}_L)^{\mathrm{Br}}$ is non-empty, that has points in all completions of L, but has no L-rational points.*

To account for such failures of the local-global principle for homogeneous spaces over K, they combined the Brauer-Manin obstruction given by the modified Brauer group Б, and the descent obstruction which is another very usual obstruction:

Theorem 1.3.6 (Theorem 5.3 of [IA21])**.** *Let K be a field of type (a) or (b) as above. Let Z be a homogeneous space of a connected linear group G with connected or abelian geometric stabilizers. Assume that Z has points in every completion of K. Then there exists a torsor $W \to Z$ of a quasi-trivial torus T such that, if the Brauer set $W(\mathbf{A}_K)^{\text{Б}(W)}$ is empty, then Z has a K-rational point.*

To get a more conceptual result, they introduced the following obstruction to the local-global principle, combining the Brauer-Manin obstruction and the descent obstruction with respect to torsors under quasi-trivial tori:

Definition 1.3.1 (Definition 5.5 of [IA21])**.** For Z any varieties, we define

$$Z(\mathbf{A}_K)^{\mathrm{qt},\text{Б}} := \bigcap_{\substack{f:W \xrightarrow{T} Z \\ T \text{ quasi-trivial}}} f(W(\mathbf{A}_K)^{\text{Б}}). \tag{1.7}$$

We have $Z(K) \subseteq Z(\mathbf{A}_K)^{\mathrm{qt},\text{Б}} \subseteq Z(\mathbf{A}_K)$, and the above Theorem 1.3.5 gives

Corollary 1.3.7 (Corollary 5.6 of [IA21])**.** *Let K be a field of type (a) or (b) as above. Let Z be a homogeneous space of a connected linear group G with connected or abelian geometric stabilizers. Assume that Z has points in every completion of K and $Z(\mathbf{A}_K)^{qt,\text{Б}} \neq \emptyset$. Then Z has a K-rational point.*

They also compared this obstruction to the descent obstruction with respect to torsors under general tori (which cuts out the subset $Z(\mathbf{A}_K)^{\mathrm{tor}}$), and they showed that $Z(\mathbf{A}_K)^{\mathrm{tor}} \subseteq Z(\mathbf{A}_K)^{\mathrm{qt},\text{Б}}$ (cf. Theorem 6.4 of [IA21]).

As for strong approximation, Izquierdo and Harari studied tori over fields of type (a), and they obtained results similar to Theorem 1.2.8. In this case, we see that strong approximation is to some extent controlled by the Brauer group $\mathrm{Br}_{\mathrm{a}}(T)$ which is isomorphic to $H^2(K, \widehat{T})$:

Theorem 1.3.8 (Théorème 5.3 and 5.4 of [HI19]). *Let K be a field of type (a) and let T be a K-torus. There is an injective morphism*

$$A(T)/\operatorname{Div} \hookrightarrow H^2(K, \widehat{T})^D$$

where $A(T)/\operatorname{Div}$ denotes the quotient of $A(T) := T(\mathbf{A}_K)/\overline{T(K)} = T(\mathbf{A}_K)/T(K)$ by its maximal divisible subgroup. The closure J of its image fits into an exact sequence

$$1 \to J \to H^2(K, \widehat{T})^D \to Ш^2(\widehat{T})^D \to 1.$$

There is a perfect pairing

$$A(T)_{\mathrm{tors}} \times \frac{\varprojlim_m H^2(K, \widehat{T}/m)}{\operatorname{Im}(Ш^2(K, \widehat{T})/\operatorname{Div})} \to \mathbb{Q}/\mathbb{Z}.$$

1.4 Main results of the dissertation

Two-dimensional geometric function fields

Let K be a field of type (a) or (b) as in §1.3.

Using dévissage arguments, we first generalize Theorem 1.3.4 for tori to all connected linear groups G over K:

Theorem 1.4.1 (Theorem 3.2.1). *Let G be a connected linear group over K of type (a) or (b) as above. Then there is an exact sequence*

$$1 \to \overline{G(K)} \to \prod_{v \in \Omega} G(K_v) \to Б_\omega(G)^D \to Б(G)^D \to 1$$

where $Б(G)$ (resp. $Б_\omega(G)$) is defined to be the subgroup of $\operatorname{Br}_1(G)/\operatorname{Br}(K)$ containing elements vanishing in $\operatorname{Br}_1(G_{K_v})/\operatorname{Br}(K_v)$ for all places (resp. almost all places) $v \in \Omega$, and A^D denotes the group of continuous homomorphisms from A to $\mathbb{Q}/\mathbb{Z}(-1)$.

In other words, the defect of weak approximation is controlled by the modified Brauer group $Б_\omega$. However, this does not hold anymore for homogeneous spaces, shown by our example:

Theorem 1.4.2 (Theorem 3.1.2). *Let Q be a flasque torus such that*

$$H^1(K, Q) \to \bigoplus_{v \in \Omega} H^1(K_v, Q) \to Ш^1_\omega(\widehat{Q})^D$$

is not exact (for example, the one in (1.5)). Embed Q into some $\mathbf{SL}_n$ and consider the homogeneous space $Z := \mathbf{SL}_n/Q$. Then the closure $\overline{Z(K)}$ in $\prod_{v \in \Omega} Z(K_v)$ is strictly smaller than the subset of elements that are orthogonal to the modified Brauer group $Б_\omega(Z)$.

Therefore, we need to go beyond the Brauer-Manin obstruction to explain the defect of weak approximation. We use a similar construction to the one (1.7) used by Izquierdo and Lucchini-Arteche, combining the Brauer-Manin obstruction and the descent obstruction. With arguments involving the fibration method, We show that this obstruction is the only one to weak approximation for homogeneous spaces over K:

Theorem 1.4.3 (Theorem 3.1.3). *For G a connected linear group over K, let Z be a homogeneous space of G with connected or abelian geometric stabilizers. Define*

$$Z(K_\Omega)^{qt,Ш_\omega} := \bigcap_{\substack{f:W \xrightarrow{T} Z \\ T \text{ quasi-trivial}}} f(W(K_\Omega)^{Ш_\omega}).$$

where f runs over torsors $W \to Z$ under quasi-trivial tori T. Then $Z(K_\Omega)^{qt,Ш_\omega} = \overline{Z(K)}$.

For K of type (a), we get the following result on strong approximation for homogeneneous spaces of quasi-split semisimple simply connected groups with toric stabilizers T, based on parts of the Poitou-Tate type exact sequence given by Izquierdo in [Izq16] under the hypothesis of the finiteness of $Ш^2(K,T)$.

Theorem 1.4.4 (Theorem 3.4.1). *Let $K = \mathbb{C}((t))(X)$ be the function field of a smooth projective geometrically integral curve X over $\mathbb{C}((t))$. Let E be a quasi-split semisimple simply connected algebraic group over K. Let T be a K-torus such that $Ш^2(K,T)$ is finite. Consider the homogeneous space $Z = E/T$. For a non-empty finite set of places $S \subseteq \Omega$, there is an exact sequence of pointed sets*

$$1 \to \overline{Z(K)}^S \to Z(\mathbf{A}_K^S) \to (\mathrm{Br}_S Z / \mathrm{Br}_S K)^D \to Ш(Z)^D \to 1$$

where $\mathrm{Br}_S Z := \mathrm{Ker}(\mathrm{Br}\, Z \to \prod_{v \in S} \mathrm{Br}\, Z_{K_v})$. In particular, the Brauer-Manin obstruction to strong approximation away from S is the only one.

p-adic function fields

Let K be the function field of a smooth projective geometrically integral curve X over a p-adic field k.

We first show that we have very different behavior of strong approximation over K compared to number fields:

Theorem 1.4.5. *There are homogeneous spaces of the form $Z = \mathbf{SL}_n / H$ with H semisimple simple connected such that strong approximation away from a non-empty set $S \subseteq X^{(1)}$ does not hold for Z.*

Our constructions include H of inner type A (Theorem 4.2.6, Corollary 4.2.7 and Example 4.2.8), and outer type A (Corollary 4.2.11 and Example 4.2.13).

For H of inner type A, we have $H = \mathbf{SL}_1(A)$ for a simple central K-algebra A and $Z = \mathbf{SL}_n/H$ is a stably rational K-variety. We show that the maps $H^1(K_v, H) \to H^3(K_v, \mathbb{Q}/\mathbb{Z}(2))$ given by the Rost invariant composed with the sum in the complex (1.2) of the generalized Weil reciprocity law can be calculated in terms of the Lichtenbaum duality pairing $\operatorname{Br} X \times \operatorname{Pic} X \to \mathbb{Q}/\mathbb{Z}$ induced by evaluation on closed points. As a consequence, we give explicit descriptions of $H^1(K_v, H)$, which are finite cyclic groups, and conditions under which the map $H^1(K, H) \to \bigoplus_{v \in X^{(1)} \setminus S} H^1(K_v, H)$ is not surjective, yielding counter-examples to strong approximation for $\mathbf{SL}_n/H$.

For H of outer type A, we have $H = \mathbf{SU}(A, \tau)$ for a central simple L-algebra A with L/K-unitary involution τ, where L/K is a separable quadratic extension. We describe the images of $H^1(K_v, H)$ under the Rost invariant with respect to different ramification types, and we give counter-examples to strong approximation for $\mathbf{SL}_n/H$ based on calculations in the case of inner type A.

To account for such failure, we define a reciprocity obstruction. We use the group $H^3_{\mathrm{nr}}(Z, \mathbb{Q}/\mathbb{Z}(2))$, which is the subgroup of elements in $H^3(K(Z), \mathbb{Q}/\mathbb{Z}(2))$ that are unramified with respect to all codimension 1 points of Z. Analogous to the definition (1.1) of the subgroup of "trivial on S" elements in a Brauer group used by Demeio, we define

$$H^3_{\mathrm{nr},S}(Z, \mathbb{Q}/\mathbb{Z}(2)) := \operatorname{Ker}(H^3_{\mathrm{nr}}(Z, \mathbb{Q}/\mathbb{Z}(2)) \to \prod_{v \in S} H^3_{\mathrm{nr}}(Z_{K_v}, \mathbb{Q}/\mathbb{Z}(2)))$$

and we denote by $\overline{H}^3_{\mathrm{nr},S}(Z, \mathbb{Q}/\mathbb{Z}(2))$ its quotient by constant elements. We define a pairing

$$Z(\mathbf{A}_K^S) \times \overline{H}^3_{\mathrm{nr}}(Z, \mathbb{Q}/\mathbb{Z}(2)) \to \mathbb{Q}/\mathbb{Z}, \quad ((x_v)_{v \notin S}, \alpha) \mapsto \sum_{v \notin S} \alpha(x_v).$$

The subset of elements in $Z(\mathbf{A}_K^S)$ that are orthogonal to $\overline{H}^3_{\mathrm{nr},S}(Z, \mathbb{Q}/\mathbb{Z}(2))$ contains $Z(K)$ in virtue of the generalized Weil reciprocity law (1.2), and also its closure $\overline{Z(K)}^S$ by continuity of the pairing, giving rise to a reciprocity obstruction to strong approximation away from S.

When Z is of the form $\mathbf{SL}_n/H$, we show that we can also define a pairing using $\operatorname{Inv}^3(H, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{norm}}$, the normalized degree 3 cohomological invariants of H with coefficients in $\mathbb{Q}/\mathbb{Z}(2)$, and we have the following compatibility given be the commutative diagram

$$\begin{tikzcd}
Z(\mathbf{A}_K^S) \arrow[d, equal] & \times & \operatorname{Inv}^3(H, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{norm}} \arrow[r] \arrow[d, "\simeq"] & \mathbb{Q}/\mathbb{Z} \arrow[d, equal] \\
Z(\mathbf{A}_K^S) & \times & \overline{H}^3_{\mathrm{nr}}(Z, \mathbb{Q}/\mathbb{Z}(2)) \arrow[r] & \mathbb{Q}/\mathbb{Z}.
\end{tikzcd}$$

As a consequence, there is a reciprocity obstruction to strong approximation for the examples we constructed. Then we show that for certain Z (e.g. the one in Example 4.2.8, or $\mathbf{SL}_n/T$ for T a torus), this obstruction is the only one.

Theorem 1.4.6 (Theorem 4.3.10)**.** *Let $Z = E/\mathbf{SL}_1(A)$ with $[A] \in \operatorname{Br} X$ of exponent m, and the special rational group E is split semisimple simply connected. For the p-adic curve*

X, suppose $\mathrm{Pic}^0(X)/m = 0$, or equivalently ${}_mH^1(k, \mathrm{Pic}\,\overline{X}) = 0$ (for example $X = \mathbb{P}^1_k$ the projective line satisfies this condition). Then there is an exact sequence of pointed sets

$$1 \to \overline{Z(K)}^S \to Z(\mathbf{A}_K^S) \to \overline{H}^3_{\mathrm{nr},S}(Z, \mathbb{Q}/\mathbb{Z}(2))^D \to 1$$

for $S \subseteq X^{(1)}$ a non-empty finite set of places. In particular, the reciprocity obstruction to strong approximation away from S is the only one for Z.

The group $\overline{H}^3_{\mathrm{nr},S}(Z, \mathbb{Q}/\mathbb{Z}(2))$ measures the defect of strong approximation away from S for Z, and is finite cyclic of order $\gcd(\frac{I(S)}{I(X)}, m)$ where $I(X)$ (resp. $I(S)$) is the index of X (resp. S). In particular, strong approximation away from S holds for Z if and only of $I(S)/I(X)$ is coprime to m, and such an S always exists, for example S such that $I(S) = I(X)$.

The following result is obtained by exploiting the Poitou-Tate type exact sequence (1.3) along with the description of $\mathrm{Inv}^3(T, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{norm}}$ given by Blinstein and Merkurjev in [BM13].

Theorem 1.4.7 (Theorem 4.3.11)**.** *Let $Z = E/T$ be a classifying variety of a torus T over K a p-adic function field, where the special rational group E is split semisimple simply connected. Then there is an exact sequence of pointed sets*

$$1 \to \overline{Z(K)}^S \to Z(\mathbf{A}_K^S) \to \overline{H}^3_{\mathrm{nr},S}(Z, \mathbb{Q}/\mathbb{Z}(2))^D$$

for $S \subseteq X^{(1)}$ a non-empty finite set of places. In other words, the reciprocity obstruction to strong approximation away from S is the only one for Z. If we suppose furthermore that $\mathrm{CH}^2(Z) \to H^0(K, \mathrm{CH}^2(\overline{Z}))$ is surjective, then there is an exact sequence of pointed sets

$$1 \to \overline{Z(K)}^S \to Z(\mathbf{A}_K^S) \to \overline{H}^3_{\mathrm{nr},S}(Z, \mathbb{Q}/\mathbb{Z}(2))^D \to \overline{H}^3_{\mathrm{nr},X^{(1)}}(Z, \mathbb{Q}/\mathbb{Z}(2))^D \to 1.$$

Then we show that our reciprocity obstruction to strong approximation not only works for classifying varieties, but also applies to tori, compatible with the results in Theorem 1.2.8 obtained by Harari and Izquierdo:

Theorem 1.4.8 (Corollary 4.3.16)**.** *The morphism $T(\mathbf{A}_K) \to H^3_{\mathrm{nr}}(T, \mathbb{Q}/\mathbb{Z}(2))^D$ induces injective morphisms*

$$A(T)/\operatorname{Div} \hookrightarrow H^3_{\mathrm{nr}}(T, \mathbb{Q}/\mathbb{Z}(2))^D,$$

$$A(T)_{\mathrm{tors}} \hookrightarrow (H^3_{\mathrm{nr}}(T, \mathbb{Q}/\mathbb{Z}(2))_\wedge)^D$$

where $H^3_{\mathrm{nr}}(T, \mathbb{Q}/\mathbb{Z}(2))_\wedge := \varprojlim_m H^3_{\mathrm{nr}}(T, \mathbb{Q}/\mathbb{Z}(2))/m$, the group $A(T) := T(\mathbf{A}_K)/\overline{T(K)} = T(\mathbf{A}_K)/T(K)$ measures the defect of strong approximation, and $A(T)/\operatorname{Div}$ denotes the quotient of $A(T)$ by its maximal divisible subgroup.

Finally, we explain that for classifying varieties, our comparison between the two pairings of reciprocity obstruction and cohomological invariants has a parallel form for weak

approximation problems too, given by the commutative diagram

$$\begin{tikzcd}
\prod_{v\in X^{(1)}} Z(K_v) & \times & \mathrm{Inv}^3_{\mathrm{nr}}(H,\mathbb{Q}/\mathbb{Z}(2))_{\mathrm{norm}} \arrow[r] \arrow[d, "\simeq"'] & \mathbb{Q}/\mathbb{Z} \arrow[d, equal] \\
\prod_{v\in X^{(1)}} Z(K_v) \arrow[u, equal] & \times & H^3_{\mathrm{nr}}(K(Z)/K,\mathbb{Q}/\mathbb{Z}(2))/\operatorname{Im}(H^3(K,\mathbb{Q}/\mathbb{Z}(2))) \arrow[r] & \mathbb{Q}/\mathbb{Z}
\end{tikzcd} \tag{1.8}$$

where $\mathrm{Inv}^3_{\mathrm{nr}}(H,\mathbb{Q}/\mathbb{Z}(2))_{\mathrm{norm}}$ denotes the subgroup of $\mathrm{Inv}^3(H,\mathbb{Q}/\mathbb{Z}(2))_{\mathrm{norm}}$ consisting of the unramified invariants.

As a quick and direct application, we obtain the following theorem giving answers to weak approximation problems for classifying varieties of tori over K, combining the Poitou-Tate type exact sequence (1.3) and the description of $\mathrm{Inv}^3_{\mathrm{nr}}(T,\mathbb{Q}/\mathbb{Z}(2))_{\mathrm{norm}}$ in the work [BM13] of Blinstein and Merkurjev. This result was also obtained by Linh in a different way in his very recent work (cf. Theorem B of [Lin22]).

Theorem 1.4.9 (Theorem 4.4.2)**.** *Let $Z = E/T$ be a classifying variety of a torus T over K. Then the reciprocity obstruction to weak approximation is the only one for Z. In fact, there is a morphism*

$$Ш^1_S(K,T') \to H^3_{\mathrm{nr}}(K(Z)/K,\mathbb{Q}/\mathbb{Z}(2))/\operatorname{Im}(H^3(K,\mathbb{Q}/\mathbb{Z}(2)))$$

such that the subset of elements in $\prod_{v\in S} Z(K_v)$ orthogonal to the image of $Ш^1_S(K,T')$ in $H^3_{\mathrm{nr}}(K(Z)/K,\mathbb{Q}/\mathbb{Z}(2))/\operatorname{Im}(H^3(K,\mathbb{Q}/\mathbb{Z}(2)))$ with respect to the pairing (1.8) already equals the closure $\overline{Z(K)}$ in the topological product $\prod_{v\in S} Z(K_v)$, where $Ш^1_S(T') := \operatorname{Ker}(H^1(K,T') \to \prod_{v\in X^{(1)}\setminus S} H^1(K_v,T'))$ for a finite set S.

Chapter 2

Preliminaries

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This chapter of preliminaries presents useful tools and notions that will be employed in the rest of the dissertation.

2.1 Dévissage

Dévissage techniques consist of finding a nice resolution for an algebraic group G, i.e. an exact sequence where G fits in and the other terms in this sequence satisfy good properties which we hope to transfer to G or we can use to simplify our computations. For example, we can sometimes realize G as a quotient variety of simpler algebraic groups to better understand G. Now we collect a number of such useful dévissage resolutions.

Let G be a linear algebraic F-group. First, we can consider the exact sequence

$$1 \to G^\circ \to G \to G^{\mathrm{f}} \to 1$$

where G° is the neutral connected component of G, and the group G^{f} of connected components of G is finite.

Now we consider G a connected linear group defined over a field F of characteristic 0. Let G^{u} be the unipotent radical of G. It is the largest unipotent F-group which is normal in G. There is a canonical exact sequence

$$1 \to G^{\mathrm{u}} \to G \to G^{\mathrm{red}} \to 1$$

where G^{red} is the largest reductive quotient group of G.

For G a connected reductive group over F (of arbitrary characteristic), we denote by G^{ss} its derived subgroup, which is a semi-simple F-group normal in G. Let $G^{\mathrm{tor}} := G/G^{\mathrm{ss}}$ which is a F-torus and the largest toric quotient of G. We have the canonical exact sequence of reductive F-groups

$$1 \to G^{\mathrm{ss}} \to G \to G^{\mathrm{tor}} \to 1.$$

Let G^{sc} be the simply connected cover of G^{ss}. There is a canonical exact sequence

$$1 \to \mu \to G^{\mathrm{sc}} \to G^{\mathrm{ss}} \to 1$$

where μ is a finite F-group of multiplicative type, central in G^{sc}.

Finally, we consider algebraic tori over F. Recall that a torus T is called *flasque* (resp. *coflasque*) if $H^1(L, \check{T}) = 0$ (resp. $H^1(L, \widehat{T}) = 0$) for every finite extension L/F. By [CTS87], there always exist the following exact sequences of tori for a given F-torus T (the last two sequences are dual to the first two sequences by taking the dual tori):

- $1 \to S_1 \to P_1 \to T \to 1$ where S_1 is flasque and P_1 is quasi-trivial (this is called a *flasque resolution* of T);
- $1 \to P_2 \to Q_2 \to T \to 1$ where Q_2 is coflasque and P_2 is quasi-trivial (this is called a *coflasque resolution* of T);
- $1 \to T \to P_3 \to Q_3 \to 1$ where Q_3 is coflasque and P_3 is quasi-trivial;
- $1 \to T \to S_4 \to P_4 \to 1$ where S_4 is flasque and P_4 is quasi-trivial.

Putting all this information together, we get a flasque resolution of a connected reductive group G, i.e. an exact sequence $1 \to S \to H \to G \to 1$ fitting into the following commutative diagram with exact rows and columns where S is a flasque torus and P is a quasi-trivial torus (cf. [CT08]):

$$\begin{tikzcd}
 & 1 \arrow[d] & 1 \arrow[d] & 1 \arrow[d] & \\
1 \arrow[r] & \mu \arrow[r] \arrow[d] & G^{\mathrm{sc}} \arrow[r] \arrow[d] & G^{\mathrm{ss}} \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & S \arrow[r] \arrow[d] & H \arrow[r] \arrow[d] & G \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & N \arrow[r] \arrow[d] & P \arrow[r] \arrow[d] & G^{\mathrm{tor}} \arrow[r] \arrow[d] & 1 \\
 & 1 & 1 & 1. &
\end{tikzcd} \tag{2.1}$$

We have $P = H^{\mathrm{tor}}$, and the kernel N of $P \to G^{\mathrm{tor}}$ is a group of multiplicative type. The finite F-group μ is also the kernel of the morphism $S \to P$. It follows that N is a torus, as a quotient of the torus S by the finite group μ.

Let $\rho : G^{\mathrm{sc}} \to G^{\mathrm{ss}} \to G$ be the composite map and let $T \subset G$ be a maximal torus. Then $T^{\mathrm{sc}} := \rho^{-1}(T)$ is a maximal torus of G^{sc}. Applying Appendice A of [CT08] to the maximal torus T, we obtain a commutative diagram

$$\begin{tikzcd}
 & 1 \arrow[d] & 1 \arrow[d] & 1 \arrow[d] & \\
1 \arrow[r] & \mu \arrow[r] \arrow[d] & T^{\mathrm{sc}} \arrow[r] \arrow[d] & T \cap G^{\mathrm{ss}} \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & S \arrow[r] \arrow[d] & T_H \arrow[r] \arrow[d] & T \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & N \arrow[r] \arrow[d] & P \arrow[r] \arrow[d] & G^{\mathrm{tor}} \arrow[r] \arrow[d] & 1 \\
 & 1 & 1 & 1 &
\end{tikzcd} \tag{2.2}$$

with exact rows and columns, where $T_H \subset H$ is a maximal torus of H.

Similarly, for a connected reductive F-group G, we have the following commutative

diagram of coflasque resolution:

$$\begin{tikzcd}
 & 1 \arrow[d] & 1 \arrow[d] & 1 \arrow[d] & \\
1 \arrow[r] & \mu \arrow[r] \arrow[d] & G^{\mathrm{sc}} \arrow[r] \arrow[d] & G^{\mathrm{ss}} \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & P_0 \arrow[r] \arrow[d] & H_0 \arrow[r] \arrow[d] & G \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & N_0 \arrow[r] \arrow[d] & Q \arrow[r] \arrow[d] & G^{\mathrm{tor}} \arrow[r] \arrow[d] & 1 \\
 & 1 & 1 & 1. &
\end{tikzcd} \tag{2.3}$$

where Q is a coflasque torus and P_0 is a quasi-trivial torus, and the rows and columns are exact sequences.

For a connected reductive group G defined over a field F of characteristic 0, an exact sequence

$$1 \to \mu_1 \to R \to G \to 1$$

is called a *special covering* of G if μ_1 is finite central and R is the product of a semisimple connected F-group by a quasi-trivial F-torus. Every semisimple connected F-group admits a special covering. By Lemme 1.10 of [San81], for every connected reductive F-group G, there exists an integer $n > 0$ and a quasi-trivial F-torus P_G such that the product $G^n \times_F P_G$ admits a special covering.

2.2 Cohomological invariants

2.2.1 Definitions

Let G be a smooth algebraic group defined over a field K. For F a field extension of K, we can identify $H^1(F, G)$ with the set of isomorphism classes of torsors of G_F over F. Then $H^1(*, G) : F \mapsto H^1(F, G)$ is a functor from the category of field extensions over K to the category of sets. Let M be another functor from the category field extensions over K to the category of abelian groups. An *invariant of* G *with values in* M (or an M*-invariant of* G is a morphism of functors

$$I : H^1(*, G) \to M,$$

or equivalently, a collection of maps $I_F : H^1(F, G) \to M(F)$ for every field extensions F/K such that for every morphism $F' \to F$, we have a commutative diagram

$$\begin{tikzcd}
H^1(F', G) \arrow[r, "I_{F'}"] \arrow[d] & M(F') \arrow[d] \\
H^1(F, G) \arrow[r, "I_F"] & M(F)
\end{tikzcd}$$

where the vertical maps are induced by $F' \to F$ by functoriality. The set $\mathrm{Inv}(G, M)$ of all M-invariants of G has the structure of an abelian group. A morphism $G' \to G$ of algebraic groups induce a morphism

$$\mathrm{Inv}(G, M) \to \mathrm{Inv}(G', M).$$

Therefore, we have a contravariant functor $\mathrm{Inv}(*, M)$ from the category of algebraic groups over K to the category of abelian groups.

An element $m \in M(K)$ defines a *constant* M-invariant of G by sending an element in $H^1(F, G)$ to the image of m under the morphism $M(K) \to M(F)$ induced by the inclusion $K \to F$. The subgroup of constant invariants in $\mathrm{Inv}(G, M)$ is isomorphic to $M(K)$.

An M-invariant of G is called normalized if it takes the distinguished element of $H^1(K, G)$ to zero, and we denote by $\mathrm{Inv}(G, M)_{\mathrm{norm}}$ the subgroup of normalized elements in $\mathrm{Inv}(G, M)$. Then we have an isomorphism

$$\mathrm{Inv}(G, M) \simeq \mathrm{Inv}(G, M)_{\mathrm{norm}} \oplus M(K).$$

2.2.2 Cohomological invariants with values in $H^d(*, \mathbb{Q}/\mathbb{Z}(d-1))$

For every integer $j \geq 0$, we define $\mathbb{Q}/\mathbb{Z}(j)$ to be the direct sum over all prime numbers p of $\mathbb{Q}_p/\mathbb{Z}_p(j)$ in the derived category of sheaves of abelian groups on the big étale site of $\mathrm{Spec}\, F$, where $\mathbb{Q}_p/\mathbb{Z}_p(j) = \mathrm{colim}_n\, \mu_{p^n}^{\otimes j}$ if $p \neq \mathrm{char}\, K$, with $\mu_{p^n}^{\otimes j}$ being the sheaf of p^nth roots of unity, and $\mathbb{Q}_p/\mathbb{Z}_p(j) = \mathrm{colim}_n\, W_n\Omega^j_{\log}[-j]$ if $p = \mathrm{char}\, K$, with $W_n\Omega^j_{\log}$ being the sheaf of logarithmic de Rham–Witt differentials (cf. [Kah96]).

Now we look at the functor M defined by

$$M(F) = H^d(F, \mathbb{Q}/\mathbb{Z}(d-1)).$$

We denote by $\mathrm{Inv}^d(G, \mathbb{Q}/\mathbb{Z}(d-1))$ the group of M-invariants of G for such an M, and d is called the degree.

When the degree $d = 1$, we have $\mathrm{Inv}^1(G, \mathbb{Q}/\mathbb{Z})_{\mathrm{norm}} = 0$ for G connected (cf. Proposition 31.15 of [Knu98]).

When the degree $d = 2$, the group $H^2(F, \mathbb{Q}/\mathbb{Z}(1))$ is canonically isomorphic to the Brauer group $\mathrm{Br}\, F$. The invariants with values in the Brauer group have been studied by Blinstein and Merkurjev:

Theorem 2.2.1 (Theorem 2.4 of [BM13]). *Let G be a smooth connected linear algebraic group over a field K. Assume that G is reductive if* $\mathrm{char}\, K > 0$. *Then there is an isomorphism*

$$\mathrm{Inv}^2(G, \mathbb{Q}/\mathbb{Z}(1))_{\mathrm{norm}} \simeq \mathrm{Pic}\, G.$$

When the degree $d = 3$, Rost determined $\mathrm{Inv}^3(G, \mathbb{Q}/\mathbb{Z}(2))$ when G is simply connected quasi-simple, and in this case $\mathrm{Inv}^3(G, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{norm}}$ is finite cyclic with a canonical

generator R_G called the *Rost invariant* (cf. Part II of [GMS03]). In [Mer16a], Merkurjev studied $\mathrm{Inv}^3(G, \mathbb{Q}/\mathbb{Z}(2))$ when G is an arbitrary semisimple group. In [BM13], Blinstein and Merkurjev studied the case when G is a torus. In [Mer16b], Merkurjev studied the case when G is a split reductive group.

2.2.3 Classifying varieties

A connected algebraic group E over K is called *special* if $H^1(F, G) = 1$ for every field extensions F/K. Examples of special groups include $\mathbf{GL}_n, \mathbf{SL}_n, \mathbf{Sp}_{2n}$ and their products. We have $\mathrm{Inv}(E, M)_{\mathrm{norm}} = 0$ for every M if E is special. Choose an injective morphism ρ of G into a special group E and the factor variety $Z = E/\rho(G)$ is called a *classifying variety* of G, because $E(F)$-orbits in the set $Z(F)$ classify torsors of G over F: we have

$$H^1(F, G) = Z(F)/E(F)$$

identifying $H^1(F, G)$ with the orbit space of the action of $E(F)$ on $Z(F)$ (cf. Corollary 28.4 of [Knu98]). Different choices of ρ give stably birational classifying varieties. For a point $x \in Z(F)$, we write x^*E for the fiber of the H-torsor $E \to Z$ above the point x and it is an H-torsor over F, whose class in $H^1(F, H)$ is the image of $[E]$ under the map $H^1(Z, H) \to H^1(F, H)$ induced by x. In particular, we have the generic H-torsor ξ^*E over $K(Z)$ corresponding to the generic point $\xi \in Z$.

Theorem 2.2.2 (Rost, cf. Theorem 3.3. of [GMS03]). *Suppose that the morphism $M(F) \to M(F((t)))$ is injective for every field extension F/K. Then the morphism*

$$\mathrm{Inv}(G, M) \to M(K(Z)), \quad I \mapsto I_{K(Z)}(\xi^*E) \tag{2.4}$$

is injective.

The assumption in this theorem is satisfied by the functor $F \mapsto H^d(F, \mathbb{Q}/\mathbb{Z}(d-1))$ that we defined in §2.2.2:

Proposition 2.2.3 (Proposition A.9 of [GMS03]). *The natural morphism*

$$H^d(F, \mathbb{Q}/\mathbb{Z}(d-1)) \to H^d(F((t)), \mathbb{Q}/\mathbb{Z}(d-1))$$

is injective.

When M satisfies certain conditions (the notion of *cycle module* defined in [Ros96]), Rost showed that we can identify $\mathrm{Inv}(G, M)$ with a more precise subgroup of $M(K(Z))$. Examples of cycle modules include M such that $M(F) = H^d(F, \mathbb{Q}_p/\mathbb{Z}_p(d-1))$ when p is coprime to $\mathrm{char}\, K$, but when $p = \mathrm{char}\, K$ this is not a cycle module anymore.

Now we consider functor M defined by

$$F \mapsto H^d(F, \mathbb{Q}/\mathbb{Z}(d-1)) = \varinjlim H^d(F, \mu_n^{\otimes(d-1)})$$

over a field K of characteristic 0. Such an M is a cycle module.

First we need the following notion before stating Rost's result. As a consequence of Gersten's conjecture for étale cohomology proved by Bloch and Ogus (cf. [BO74]), we have:

Theorem-Definition 2.2.4 (cf. Theorem 4.1.1 of [CT95], see also (1.1) of [CT15])**.** Let Z be a smooth geometrically integral K-variety. The following subgroups of $H^d(K(Z), \mathbb{Q}/\mathbb{Z}(d-1))$ coincide, that we define to be $H^d_{\mathrm{nr}}(Z, \mathbb{Q}/\mathbb{Z}(d-1))$:

(1) the group $H^0_{\mathrm{Zar}}(Z, \mathcal{H}^d(\mathbb{Q}/\mathbb{Z}(d-1)))$ of global sections of the Zariski sheaf $\mathcal{H}^d(Z, \mathbb{Q}/\mathbb{Z}(d-1))$ which is the sheaf associated to the Zariski presheaf $U \mapsto H^d_{\text{ét}}(U, \mathbb{Q}/\mathbb{Z}(d-1))$;

(2) the group of elements $\alpha \in H^d(K(Z), \mathbb{Q}/\mathbb{Z}(d-1))$ that are unramified with respect to any codimension 1 point P of Z, i.e. we have $\partial_{\mathcal{O}_{Z,P}}(\alpha) = 0 \in H^{d-1}(\kappa_P, \mathbb{Q}/\mathbb{Z}(d-2))$ or equivalently α comes from a class in $H^d(\mathcal{O}_{Z,P}, \mathbb{Q}/\mathbb{Z}(d-1))$;

(3) the group of elements in $H^d(K(Z), \mathbb{Q}/\mathbb{Z}(d-1))$ which at any point $P \in Z$ come from a class in $H^d(\mathcal{O}_{Z,P}, \mathbb{Q}/\mathbb{Z}(d-1))$.

Then we can state the theorem due to Rost describing precisely the image of the morphism in (2.4).

Theorem 2.2.5 (Rost, cf. Part 1, Appendix C of [GMS03])**.** *There is an invective morphism*

$$\begin{aligned} \theta : \mathrm{Inv}^d(G, \mathbb{Q}/\mathbb{Z}(d-1)) &\to H^d_{\mathrm{nr}}(Z, \mathbb{Q}/\mathbb{Z}(d-1)) \\ I &\mapsto I_{K(Z)}(\xi^* E). \end{aligned} \tag{2.5}$$

If the special group E is split semisimple simply connected, then θ is an isomorphism.

To illustrate furthermore how classifying varieties Z of G interact with the group of cohomological invariants of G, we give the following theorem describing the degree 3 cohomological invariants of a torus, where the Chow groups of Z come into play:

Theorem 2.2.6 (Theorem B of [BM13])**.** *Let T be an algebraic torus over a field K. Then there is an exact sequence*

$$\begin{aligned} 1 \to \mathrm{CH}^2(Z)_{\mathrm{tors}} \to H^1(K, T') \to \mathrm{Inv}^3(T, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{norm}} \\ \to H^0(K, \mathrm{CH}(\overline{Z}))/\operatorname{Im}(\mathrm{CH}^2(Z)) \to H^2(K, T'). \end{aligned}$$

Here T' denotes the dual torus of T, i.e. the torus whose character group is the cocharacter group of T.

Remark 2.2.7. We can also consider $M : G \mapsto H^d(G, \mathbb{Q}/\mathbb{Z}(j))$ for j not necessarily equal to $d-1$, and we denote by $\mathrm{Inv}^d(G, \mathbb{Q}/\mathbb{Z}(j))$ the group of such M-invariants of G.

Remark 2.2.8. When $\operatorname{char} K = p$, we cannot use the theory of cycle modules of Rost. However, Blinstein and Merkurjev defined in [BM13] a subgroup of *balanced* elements in $H^0_{\mathrm{Zar}}(Z, \mathcal{H}^d(\mathbb{Q}/\mathbb{Z}(j)))$, and they showed that $\mathrm{Inv}^d(G, \mathbb{Q}/\mathbb{Z}(j))$ is isomorphic to such a subgroup under certain assumptions.

Using flasque resolutions of a tori, Blinstein and Merkurjev turned a torus into a classifying variety and obtained:

Theorem 2.2.9 (Theorem C of [BM13])**.** *let T be a torus over K and let $1 \to S \to P \to T \to 1$ be a flasque resolution of T, i.e. S is a flasque torus and P is a quasi-trivial torus. Then there is a natural isomorphism*

$$H^d_{\mathrm{nr}}(K(T)/K, \mathbb{Q}/\mathbb{Z}(d-1)) \simeq \mathrm{Inv}^d(S, \mathbb{Q}/\mathbb{Z}(d-1)).$$

2.2.4 Unramified cohomological invariants

We still suppose $\operatorname{char} K = 0$. An invariant $I \in \mathrm{Inv}^d(G, \mathbb{Q}/\mathbb{Z}(d-1))$ is called *unramified* if for every field extension F/K and every element $a \in H^1(F, G)$, we have $I(a)$ inside the subgroup $H^3_{\mathrm{nr}}(F/K, \mathbb{Q}/\mathbb{Z}(d-1))$ of elements unramified with respect to all discrete valuations of F trivial on K. Under the identification (2.5) when the special group E is split semisimple simply connected, the subgroup $H^d_{\mathrm{nr}}(K(Z)/K, \mathbb{Q}/\mathbb{Z}(d-1))$ of $H^d_{\mathrm{nr}}(Z, \mathbb{Q}/\mathbb{Z}(d-1))$ is identified with the group $\mathrm{Inv}^d_{\mathrm{nr}}(G, \mathbb{Q}/\mathbb{Z}(d-1))$ of unramified invariants (cf. Proposition 4.1 of [Mer16b]). We denote by $\mathrm{Inv}^d_{\mathrm{nr}}(G, \mathbb{Q}/\mathbb{Z}(d-1))_{\mathrm{norm}}$ the group of unramified normalized invariants.

There have been many studies on unramified degree 3 cohomological invariants have been widely studied. Merkurjev studied $\mathrm{Inv}^3_{\mathrm{nr}}(G, \mathbb{Q}/\mathbb{Z}(2))$ for all classical semisimple simply connected groups G, and he showed in [Mer02] that $\mathrm{Inv}^3_{\mathrm{nr}}(G, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{norm}}$ can be nontrivial only when G is of type ${}^2\mathsf{A}_{n-1}$, ${}^2\mathsf{D}_3$ or ${}^1\mathsf{D}_4$ under certain conditions.

Theorem 2.2.10 (Theorem 5.3 of [Mer02])**.** *Let G be a simply connected group of type ${}^2\mathsf{A}_{n-1}$, i.e. $G = \mathbf{SU}(A, \tau)$ where A is a central simple L-algebra of degree $n \geq 3$ with L/K-unitary involution τ over a quadratic separable field extension L/K. The group $\mathrm{Inv}^3_{\mathrm{nr}}(G, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{norm}}$ is cyclic of order 2 generated by $\exp(A)R_G$, except for the following cases (when this group is trivial):*

1. *$\exp(A)$ is odd;*
2. *n is a 2-power and $\exp(A) = n$;*
3. *n is a 2-power, $\exp(A) = n/2$ and the discriminant algebra is split.*

Theorem 2.2.11 (Theorem 8.5 of [Mer02])**.** *Let G be a simply connected group of type D_n (we exclude groups of trialitarian type in D_4), i.e. $G = \mathbf{Spin}(A, \sigma)$ for a central simple algebra A of degree $2n$ over K with an orthogonal involution σ. Let $C(A, \sigma)$ be the Clifford algebra.*

Then $\mathrm{Inv}^3_{\mathrm{nr}}(G, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{norm}}$ *is trivial, except for the following cases (when this group is cyclic of order* 2 *generated by* $2R_G$*):*

1. $n = 3$, $\mathrm{disc}(\sigma)$ *is not trivial,* A *is not split and* $\exp(C) = 2$*;*
2. $n = 4$, $\mathrm{disc}(\sigma)$ *is trivial,* A *is not split and neither component* C^+ *nor* C^- *of* C *splits.*

Garibaldi computed the remaining cases when G is *exceptional*, i.e. when G is of type G_2, ${}^3\mathsf{D}_4$, ${}^6\mathsf{D}_4$, F_4, E_6, E_7 or E_8, completing the computations of $\mathrm{Inv}^3_{\mathrm{nr}}(G, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{norm}}$ for G semisimple simply connected. He showed in [Gar06] that for G a simple simply connected exceptional algebraic group, the group $\mathrm{Inv}^3_{\mathrm{nr}}(G, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{norm}}$ is $\mathbb{Z}/2\mathbb{Z}$ if G is of type ${}^3\mathsf{D}_4$ with a nontrivial Tits algebra; otherwise $\mathrm{Inv}^3_{\mathrm{nr}}(G, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{norm}}$ is trivial.

Blinstein and Merkurjev described unramified cohomological invariants for a torus with an auxiliary flasque torus. For T a torus over K, by Lemma 0.6 of [CTS87]), we can find an exact sequence $1 \to T \to T_1 \to P \to 1$ with T_1 a flasque torus and P a quasi-trivial torus.

Theorem 2.2.12 (Theorem 5.5 of [BM13])**.** *There is a natural isomorphism*

$$\mathrm{Inv}^3(T_1, \mathbb{Q}/\mathbb{Z}(2)) \simeq \mathrm{Inv}^3_{\mathrm{nr}}(T, \mathbb{Q}/\mathbb{Z}(2)).$$

Chapter 3

Homogeneous spaces over some two-dimensional geometric global fields

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The results of the first three sections of this chapter appeared in my article [Zha23]. In this article, we study obstructions to weak approximation for connected linear groups and homogeneous spaces with connected or abelian stabilizers over finite extensions of $\mathbb{C}(\!(x,y)\!)$ or function fields of curves over $\mathbb{C}(\!(t)\!)$. We show that for connected linear groups, the usual Brauer-Manin obstruction works as in the case of tori. However, this Brauer-Manin obstruction is not enough for homogeneous spaces, as shown by the examples we give. We then construct an obstruction using torsors under quasi-trivial tori that explains the failure of weak approximation.

In the last section, we show that the Brauer-Manin obstruction to strong approximation is the only one for homogeneous spaces of quasi-split semisimple simply connected groups with toric stabilizers T such that $Ш^2(K,T)$ is finite, where K is the function field of a curve over $\mathbb{C}(\!(t)\!)$.

3.1 Introduction

Given an algebraic variety Z over a number field K, we say that Z satisfies weak approximation if $Z(K)$ is dense in $\prod_{v\in\Omega} Z(K_v)$ with respect to the product of the v-adic topologies, where K_v denotes the v-adic completion and Ω denotes the set of places of K. Weak approximation is satisfied for simply connected groups, and when it is not satisfied, we want to characterize the closure $\overline{Z(K)}$ of the set of rational points in $\prod_{v\in\Omega} Z(K_v)$ as some subset of elements satisfying certain compatibility conditions, for example, as the Brauer-Manin set $Z(K_\Omega)^{\mathrm{Br}}$ (cf. Section 8.2 of [Poo17]), and this gives an obstruction to weak approximation.

Going beyond number fields, recently there has been an increasing interest in studying analogous questions over some two-dimensional geometric global fields. As in [IA21], we consider a field K of one of the following two types:

(a) the function field of a smooth projective curve C over $\mathbb{C}(\!(t)\!)$;

(b) the fraction field of a local, henselian, two-dimensional, excellent domain A with algebraically closed residue field of characteristic 0 (e.g. any finite extension of $\mathbb{C}(\!(x,y)\!)$, the field of Laurent series in two variables over the field of complex numbers).

In case (a), one can consider the set Ω of valuations coming from the closed points of the curve C. Colliot-Thélène/Harari (cf. [CTH15]) proved the following exact sequence describing the obstruction to weak approximation for the case of tori T:

$$1 \to \overline{T(K)} \to \prod_{v\in\Omega} T(K_v) \to Ш_\omega(T)^D \to Ш(T)^D \to 1 \tag{3.1}$$

where $Ш(Z)$ (resp. $Ш_\omega(Z)$) is defined to be the subgroup of $\mathrm{Br}_1(Z)/\mathrm{Br}(K)$ containing elements vanishing in $\mathrm{Br}_1(Z_v)/\mathrm{Br}(K_v)$ for all places (resp. almost all places) $v \in C^{(1)}$, and A^D denotes the group of continuous homomorphisms from A to $\mathbb{Q}/\mathbb{Z}(-1)$.

In case (b), one can take Ω to be the set of valuations coming from prime ideals of height one in A. Izquierdo (cf. [Izq19]) proved the exact sequence 3.1 in such situations for tori T.

Using some dévissage arguments as in [CTH15], we can first generalize this result to a connected linear group G over K.

Theorem 3.1.1. *(Theorem 3.2.1) We keep the notation as above. For a connected linear group G over K of the type (a) or (b), there is an exact sequence*

$$1 \to \overline{G(K)} \to G(K_\Omega) \to Ш_\omega(G)^D \to Ш(G)^D \to 1. \tag{3.2}$$

In particular, the obstruction to weak approximation is controlled by the Brauer set $G(K_\Omega)^{Ш_\omega} := \mathrm{Ker}(G(K_\Omega) \to Ш_\omega(G)^D)$. However, such an exact sequence does not generalize to homogeneous spaces, and we do find counter-examples:

Theorem 3.1.2 (Proposition 3.3.1)**.** *Let Q be a flasque torus such that*

$$H^1(K,Q) \to \oplus_{v\in\Omega} H^1(K_v,Q) \to Ш^1_\omega(\widehat{Q})^D$$

is not exact (for example the one constructed in Corollary 9.16 of [CTH15]). Embed Q into some $\mathbf{SL}_n$ and let $Z := \mathbf{SL}_n/Q$. Then $\overline{Z(K)} \subsetneq Z(K_\Omega)^{Ш_\omega}$.

In order to better understand the obstruction to weak approximation for homogeneous spaces over fields of the type (a) or (b), we should somehow combine the Brauer-Manin obstruction with the descent obstruction, another natural tool used in the study of such questions, as done by Izquierdo and Lucchini-Arteche in [IA21] for the study of obstruction to rational points. We present the following result which applies in particular to homogeneous spaces under connected stabilizers:

Theorem 3.1.3 (Proposition 3.3.4)**.** *For G a connected linear group, we consider a homogeneous space Z under G with geometric stabilizer $\bar{H}$ such that G^{ss} is simply connected and $\bar{H}^{torf}$ is abelian. Define*

$$Z(K_\Omega)^{qt,Ш_\omega} := \bigcap_{\substack{f:W\xrightarrow{T}Z \\ T \text{ quasi-trivial}}} f(W(K_\Omega)^{Ш_\omega}).$$

where f runs over torsors $W \to Z$ under quasi-trivial tori T. Then $Z(K_\Omega)^{qt,Ш_\omega} = \overline{Z(K)}$.

Notation and preliminaries

The notation will be fixed in this section and used throughout this article. The setting is pretty much the same as in [IA21] where they treated the problem of Hasse principle for such varieties.

Cohomology The cohomology groups we consider are always in terms of Galois cohomology or étale cohomology.

Fields Throughout this article, we consider a field K of one of the following two types:

(a) the function field of a smooth projective curve C over $\mathbb{C}((t))$;

(b) the field of fractions of a two-dimensional, excellent, henselian, local domain A with algebraically closed residue field of characteristic zero. An example of such a field is any finite extension of the field of fractions $\mathbb{C}((X,Y))$ of the formal power series ring $\mathbb{C}[[X,Y]]$.

Both these two types of fields share a number of properties that hold also for totally imaginary number fields: (cf. Thm. 5.5 of [HHK09], Thm. 1.2 and Thm. 1.4 of [CTGP04])

(i) the cohomological dimension is two,

(ii) index and exponent of central simple algebras coincide,

(iii) for any semisimple simply connected group G over K, we have $H^1(K, G) = 1$.

(iv) there is a natural set Ω of rank one discrete valuations (i.e. with values in $\mathbb{Z}$), with respect to which one can take completions. For type (a), let Ω be the set of valuations coming from $C^{(1)}$ the set of closed points of the curve C. For type (b), let Ω be the set of valuations coming from prime ideals of height one in A.

Weak approximation is satisfied for semisimple simply connected groups over such fields (cf. Theorem 4.7 of [CTGP04] for type (b), and §10.1 of [CTH15] for type (a)).

Sheaves and abelian groups For $i > 0$, we denote by $\mu_n^{\otimes i}$ the i-fold tensor product of the étale sheaf μ_n of n-th roots of unity with itself. We set $\mu_n^0 = \mathbb{Z}/n\mathbb{Z}$ and for $i < 0$, and we define $\mu_n^{\otimes i} = \mathrm{Hom}(\mu_n^{\otimes(-i)}, \mathbb{Z}/n\mathbb{Z})$. Over the fields K we consider, since K contains an algebraically closed field and thus all the roots of unity, we have a (non-canonical) isomorphism $\mu_n \simeq \mathbb{Z}/n\mathbb{Z}$. Denote by $\mathbb{Q}/\mathbb{Z}(i)$ the direct limit of the sheaves $\mu_m^{\otimes i}$ for all $m > 0$. By choosing a compatible system of primitive n-th roots of unity for every n (for example $\xi_n = \exp(2\pi i/n)$), we can identify $\mathbb{Q}/\mathbb{Z}(i)$ with $\mathbb{Q}/\mathbb{Z}$.

For an abelian group A (that we always suppose to be equipped with the discrete topology if there isn't any other topology defined on it), we denote by A^D the group of continuous homomorphisms from A to $\mathbb{Q}/\mathbb{Z}(-1)$. The functor $A \mapsto A^D$ is an anti-equivalence of categories between torsion abelian groups and profinite groups.

Weak approximation and Brauer-Manin obstruction Given a set Ω of places of K, let $Z(K_\Omega) := \prod_{v \in \Omega_K} Z(K_v)$ be the topological product where each $Z(K_v)$ is equipped with the v-adic topology, and K_v denotes the completion at v. We say that a K-variety Z satisfies weak approximation with respect to Ω if the set of rational points $Z(K)$, considered through the diagonal map as a subset of $Z(K_\Omega)$, is a dense subset. Weak approximation is a birational invariant of smooth geometrically integral varieties, which is a consequence of the implicit function theorem for K_v (cf. Theorem 9.5.1 of [CTS21] and the argument in the Proposition 12.2.3 of [CTS21]).

There are varieties for which weak approximation fails, thus we try to introduce obstructions that give a more precise description of the closure of $Z(K)$ inside $Z(K_\Omega)$, thus explaining such failures.

For a variety Z, the cohomological Brauer group $\mathrm{Br}(Z)$ is defined to be $H^2(Z, \mathbb{G}_m)$. Define also

- $\mathrm{Br}_0(Z) := \mathrm{Im}(\mathrm{Br}(K) \to \mathrm{Br}(Z))$,
- $\mathrm{Br}_1(Z) := \mathrm{Ker}(\mathrm{Br}(Z) \to \mathrm{Br}(Z_{\bar{K}}))$,

- $\mathrm{Br}_{\mathrm{a}}(Z) := \mathrm{Br}_1(Z)/\mathrm{Br}_0(Z)$,
- Let $w \in Z(K)$ be a K-point of Z, we set $\mathrm{Br}_w(Z) := \mathrm{Ker}(w^* : \mathrm{Br}(Z) \to \mathrm{Br}(K))$ and $\mathrm{Br}_{1,w} := \mathrm{Ker}(w^* : \mathrm{Br}_1(Z) \to \mathrm{Br}(K))$. We have a canonical isomorphism $\mathrm{Br}_{1,w}(Z) \simeq \mathrm{Br}_{\mathrm{a}}(Z)$.
- $Б(Z) := \mathrm{Ker}(\mathrm{Br}_{\mathrm{a}}(Z) \to \prod_{v\in\Omega} \mathrm{Br}_{\mathrm{a}}(Z_{K_v}))$,
- $Б_S(Z) := \mathrm{Ker}(\mathrm{Br}_{\mathrm{a}}(Z) \to \prod_{v\in\Omega\setminus S} \mathrm{Br}_{\mathrm{a}}(Z_{K_v}))$ where S is a finite subset of Ω,
- $Б_\omega(Z)$: subgroup of $\mathrm{Br}_{\mathrm{a}}(Z)$ containing the elements vanishing in $\mathrm{Br}_{\mathrm{a}}(Z_{K_v})$ for almost all places $v \in \Omega$.

Similar to the exact sequence from Class Field Theory for number fields, there is also an exact sequence (cf. Prop. 2.1.(v) of [CTH15] and Thm. 1.6 of [Izq19])

$$\mathrm{Br}(K) \to \bigoplus_{v\in\Omega} \mathrm{Br}(K_v) \to \mathbb{Q}/\mathbb{Z} \to 0$$

and a Brauer-Manin pairing

$$\begin{aligned} Z(K_\Omega) \times Б_\omega(Z) &\to \mathbb{Q}/\mathbb{Z} \\ ((P_v), \alpha) &\mapsto \sum_{v\in\Omega} < P_v, \alpha > \end{aligned}$$

with the following property: $Z(K)$ is contained in the subset $Z(K_\Omega)^{Б_\omega}$ of $Z(K_\Omega)$ defined as $\{(P_v) \in Z(K_\Omega) : ((P_v), \alpha) = 0 \text{ for all } \alpha \in Б_\omega(Z)\}$: the set of points that are orthogonal to $Б_\omega(Z)$. When $Z(K)$ is non-empty and $\overline{Z(K)} = Z(K_\Omega)^{Б_\omega}$, we say that the Brauer-Manin obstruction with respect to $Б_\omega$ is the only one to weak approximation.

Tate-Shafarevich groups For a Galois module M over the field K and an integer $i \geq 0$, we define the following Tate-Shafarevich groups:

- $Ш^i(K, M) := \mathrm{Ker}(H^i(K, M) \to \prod_{v\in\Omega} H^i(K_v, M))$,
- $Ш^i_S(K, M) := \mathrm{Ker}(H^i(K, M) \to \prod_{v\in\Omega\setminus S} H^i(K_v, M))$ where S is a finite subset of Ω,
- $Ш^i_\omega(K, M)$: subgroup of $H^i(K, M)$ containing the elements vanishing in $H^i(K_v, M)$ for almost all places $v \in \Omega$.

Algebraic groups and homogeneous spaces For a linear algebraic K-group G, the following notation will be used:

- $D(G)$: the derived subgroup of G,

- G°: the neutral connected component of G,
- $G^{\mathrm{f}} := G/G^\circ$ the group of connected components of G, which is a finite group,
- G^{u} : the unipotent radical of G°,
- $G^{\mathrm{red}} := G^\circ/G^{\mathrm{u}}$ which is a reductive group,
- $G^{\mathrm{ss}} := D(G^{\mathrm{red}})$ which is a semisimple group,
- $G^{\mathrm{tor}} := G^{\mathrm{red}}/G^{\mathrm{ss}}$ which is a torus,
- $G^{\mathrm{ssu}} := \mathrm{Ker}(G^\circ \to G^{\mathrm{tor}})$ which is an extension of G^{ss} by G^{u},
- $G^{\mathrm{torf}} := G/G^{\mathrm{ssu}}$ which is an extension of G^{f} by G^{tor},
- $\widehat{G}$: the Galois module of the geometric characters of G.

A unipotent group over K a field of characteristic 0 is isomorphic to an affine space $\mathbb{A}^n_K$, thus K-rational and satisfies weak approximation.

A torus T is said to be quasi-trivial if $\widehat{T}$ is an induced $\mathrm{Gal}(\bar{K}/K)$-module.

3.2 Weak approximation for connected linear groups

The aim of this section is to prove the following theorem which concerns the Brauer-Manin obstruction to weak approximation for connected linear groups:

Theorem 3.2.1. *Let K be a field of the type (a) or (b) and let G be a connected linear group over K. Then there is an exact sequence*

$$1 \to \overline{G(K)} \to G(K_\Omega) \to Б_\omega(G)^D \to Б(G)^D \to 1.$$

We complete the proof by a series of lemmas.

Lemma 3.2.2. *Suppose that Theorem 4.1 holds for G^n where n is a positive integer, then it also holds for G.*

Proof. It follows from the fact that all of the operations in the exact sequence above take products to products. (For example, the closure of a product is the product of closures.) □

An exact sequence

$$1 \to F \to H \times_K P \to G \to 1$$

where H is semi-simple simply connected, P is a quasitrivial K-torus and F is finite and central is called a special covering of the reductive K-group G. For any reductive K-group G, there exists an integer $n > 0$ such that G^n admit a special covering. By the lemma above, we can suppose without loss of generality that G admits a special covering itself.

Lemma 3.2.3. *Let S be a finite set of places in Ω. The following sequence is exact*

$$1 \to \overline{G(K)} \to \prod_{v \in S} G(K_v) \to Б_S(G)^D$$

where G is a reductive connected linear group.

Proof. The proof is exactly the same as in Lemma 9.6 of [CTH15], using the special covering and the fact that H and P both satisfy weak approximation (so does their product) and $H^1(K, H) = H^1(K, P) = H^1(K, H \times_K P) = 0$. We have a commutative diagram where the rows are exact and the columns are complexes.

$$\begin{tikzcd}
H \times_K P(K) \arrow[r] \arrow[d] & G(K) \arrow[r] \arrow[d] & H^1(K, F) \arrow[r] \arrow[d] & 1 \\
\prod_{v \in S} H \times_K P(K_v) \arrow[r] \arrow[d] & \prod_{v \in S} G(K_v) \arrow[r] \arrow[d] & \prod_{v \in S} H^1(K_v, F) \arrow[r] \arrow[d] & 1 \\
Б_S(H \times P)^D \arrow[r] & Б_S(G)^D \arrow[r] & Ш^1_S(\widehat{F})^D \arrow[r] & 1
\end{tikzcd}$$

The exactness of the last row comes from the fact that $H^1(K, \widehat{F}) \simeq \mathrm{Ker}(\mathrm{Br_a}(G) \to \mathrm{Br_a}(H \times_K P))$ and $\mathrm{Br_a}(H \times_K P) \simeq H^2(K, \widehat{P})$ (Cor 7.4 and Lemme 6.9 of [San81]), with the latter isomorphism inducing an isomorphism $Б_S(H \times P) \simeq Ш^2_S(\widehat{P})$ by the same argument as in Section 8.1 of [CTH15]. The commutativity of the right-bottom square can be deduced from Lemme 8.11 of [San81].

□

Lemma 3.2.4. *Let S be a finite set of places in Ω. The following sequence is exact*

$$1 \to \overline{G(K)} \to \prod_{v \in S} G(K_v) \to Б_S(G)^D$$

where G is a connected linear group over K.

Proof. We use the resolution $1 \to G^{\mathrm{u}} \to G \to G^{\mathrm{red}} \to 1$ and the induced diagram

$$\begin{tikzcd}
G^{\mathrm{u}}(K) \arrow[r] \arrow[d] & G(K) \arrow[r] \arrow[d] & G^{\mathrm{red}}(K) \arrow[r] \arrow[d] & 1 \\
\prod_{v \in S} G^{\mathrm{u}}(K_v) \arrow[r] & \prod_{v \in S} G(K_v) \arrow[r] \arrow[d] & \prod_{v \in S} G^{\mathrm{red}}(K_v) \arrow[d] & \\
& Б_S(G)^D \arrow[r] & Б_S(G^{\mathrm{red}})^D. &
\end{tikzcd}$$

taking into account the vanishing of $H^1(K, G^{\mathrm{u}})$. The unipotent radical G^{u} satisfies weak approximation (see Section 3.1). The exactness of the rightmost column is known from the previous lemma, then a diagram chasing gives the result. □

Lemma 3.2.5. *Let S be a finite set of places in Ω. The sequence*

$$\prod_{v\in S} G(K_v) \to Ш_S(G)^D \to Ш(G)^D \tag{3.3}$$

is exact, where G is a connected linear group over K.

Proof. As in the previous lemmas, we can first treat the case where G is reductive. Suppose now that G is reductive, we use a coflasque resolution

$$1 \to P \to G' \to G \to 1$$

where P is a quasitrivial torus and G' fits into the exact sequence

$$1 \to G^{\mathrm{sc}} \to G' \to T \to 1$$

where T is a coflasque torus and G^{sc} is a semisimple simply connected group. Suppose that the exactness of the sequence in question is known for G' (replacing G). Then we can prove the exactness for G by chasing in the following diagram

$$\begin{tikzcd}
 & \prod_{v\in S} G'(K_v) \arrow[r] \arrow[d] & \prod_{v\in S} G(K_v) \arrow[r] \arrow[d] & 0 \\
Ш_S(P)^D \arrow[r] \arrow[d, "\simeq"] & Ш_S(G')^D \arrow[r] \arrow[d] & Ш_S(G)^D \arrow[r] \arrow[d] & 0 \\
Ш(P)^D \arrow[r] & Ш(G')^D \arrow[r] & Ш(G)^D \arrow[r] & 0.
\end{tikzcd}$$

The leftmost arrow is an isomorphism by Proposition 2.6 of [CTH15]. The last two rows are exact, and this can be proved by chasing the following diagram, as done in the proof of Lemma 4.4 of [Bor96].

$$\begin{tikzcd}
 & & \mathrm{Br}(K) \arrow[dl, dashed, "f_1"'] \arrow[d, dashed, "f_2"] & \\
0 \arrow[r] & \mathrm{Br}_1(G) \arrow[r, "f_4"] \arrow[d, "f_6"] & \mathrm{Br}_1(G') \arrow[r, "f_5"] \arrow[d, "f_7"] & \mathrm{Br}_{\mathrm{a}}(P) \arrow[d, "f_3"] \\
0 \arrow[r] & \mathrm{Br}_1(G_v) \arrow[r, "f_9"] & \mathrm{Br}_1(G'_v) \arrow[r, "f_{10}"] & \mathrm{Br}_{\mathrm{a}}(P_v) \\
 & & \mathrm{Br}(K_v) \arrow[ul, dashed, "f_{11}"] \arrow[u, dashed, "f_8"] &
\end{tikzcd}$$

(The exactness of the rows can be deduced from Proposition 6.10 of [San81] and the fact that $\mathrm{Pic}(P) = H^1(K, \widehat{P}) = 0$ since $\widehat{P}$ is a permutation module.)

The exactness of the sequence 3.3 is known for T (Corollary 9.9 of [CTH15] for K of type (a), and Theorem 4.9 of [Izq19] for K of type (b)), by a similar diagram chasing, we can prove the result for G', thus completing the proof. The essential condition we need in this diagram chasing is that we should have an injection $Ш_S(G')^D \hookrightarrow Ш_S(T)^D$, or equivalently a surjection $Ш_S(T) \twoheadrightarrow Ш_S(G')$. This is indeed true because we have an exact sequence $\mathrm{Br}_{\mathrm{a}}(T) \to \mathrm{Br}_{\mathrm{a}}(G') \to \mathrm{Br}_{\mathrm{a}}(G^{\mathrm{sc}})$ and $\mathrm{Br}(G^{\mathrm{sc}}) = \mathrm{Br}(K)$ since G^{sc} a simply connected group over a field of characteristic 0 (cf. Corollary in §0 of [Gil09b]).

Now for a connected linear group G (not necessarily reductive), we use again the resolution $1 \to G^{\mathrm{u}} \to G \to G^{\mathrm{red}} \to 1$ and the induced commutative diagram

$$\begin{tikzcd}
 & \prod_{v\in S} G(K_v) \arrow[r] \arrow[d] & \prod_{v\in S} G^{\mathrm{red}}(K_v) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & Ш_S(G)^D \arrow[r] \arrow[d] & Ш_S(G^{\mathrm{red}})^D \arrow[d] & \\
 & Ш(G)^D \arrow[r] & Ш(G^{\mathrm{red}})^D &
\end{tikzcd}$$

taking into account the vanishing of $H^1(K_v, G^{\mathrm{u}})$. The injectivity of the middle line follows from the fact that $\mathrm{Br}_{\mathrm{a}}(G^{\mathrm{u}}) = \mathrm{Br}_{\mathrm{a}}(\mathbb{A}^n_K) = (\mathrm{Br}_{\mathrm{a}}(\mathbb{A}^1_K))^n = 0$ (cf. Lemme 6.6 of [San81], Theorem 4.5.1(viii) of [CTS21]). Then a diagram chasing gives the desired result. □

Corollary 3.2.1. *(Theorem 3.2.1) With the same notation as above, there is an exact sequence*

$$1 \to \overline{G(K)} \to G(K_\Omega) \to Ш_\omega(G)^D \to Ш(G)^D \to 1.$$

The defect of weak approximation $A(G) := \mathrm{coker}(\overline{G(K)} \to G(K_\Omega))$ *is finite, and* G *satisfies* weak weak approximation, i.e. there exists a finite set of places S_0 such that $G(K)$ is dense in $\prod_{v\notin S_0} G(K_v)$.

Proof. We extend the result with respect to a finite set S of places to all the Ω by following the same proof used in Cor. 9.9 of [CTH15]. Let $A_S(G) := \mathrm{coker}(\overline{G(K)} \to \prod_{v\in S}(G(K_v)))$ measuring the failure of weak approximation. Let $1 \to Q \to R \to G \to 1$ be a flasque resolution of G (cf. Proposition-Definition 3.1 of [CT08]). By Prop. 9.1 of [CTH15] (for K of type (a)) and Theorem 3.7 with its remark in [CTGP04] (for K of type (b)), we have $A_S(G) = \mathrm{coker}(H^1(K, Q) \to \prod_{v\in S} H^1(K_v, Q))$ which is finite and $H^1(K_v, Q) = 0$ for all v outside the finite set S_0 of places of bad reduction of Q. Therefore, when S does not meet S_0, we have that $A_S(G)$ is trivial, proving weak weak approximation. On the other hand, for $S_0 \subseteq S$, we have $A_S(G) = A(G)$. Let $S_0 \subseteq S \subseteq S'$, and then we have a commutative

diagram

$$\begin{tikzcd}
1 \arrow[r] & A_{S'}(G) \arrow[r] \arrow[d] & Ш_{S'}(G)^D \arrow[r] \arrow[d] & Ш(G)^D \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & A_S(G) \arrow[r] & Ш_S(G)^D \arrow[r] & Ш(G)^D \arrow[r] & 1
\end{tikzcd}$$

where the left and the right vertical maps are isomorphisms. Thus the middle map is also an isomorphism and $Ш_S(G) = Ш_\omega(G)$. □

3.3 Weak approximation for homogeneous spaces

Now we look at weak approximation for homogeneous spaces over such fields K. The usual Brauer-Manin obstruction we used in last section is not enough in this case, as shown by the following construction of examples:

Proposition 3.3.1. *Let Q be a flasque torus such that*

$$H^1(K, Q) \to \oplus_{v\in\Omega} H^1(K_v, Q) \to Ш^1_\omega(\widehat{Q})^D$$

is not exact. Such a torus exists (see Corollary 9.16 of [CTH15] for a construction.) Embed Q into some $\mathbf{SL}_n$ and let $Z := \mathbf{SL}_n / Q$. Then $\overline{Z(K)} \subsetneq Z(K_\Omega)^{Ш_\omega}$.

Proof. We consider the following commutative diagram with exact rows:

$$\begin{tikzcd}
 & Z(K) \arrow[r, "f_1"] \arrow[d, "f_2"] & H^1(K, Q) \arrow[r] \arrow[d, "f_3"] & 1 \\
 & Z(K_\Omega) \arrow[r, "f_4"] \arrow[d, "f_5"] & \prod_{v\in\Omega} H^1(K_v, Q) \arrow[r] \arrow[d, "f_6"] & 1 \\
1 \arrow[r] & Ш_\omega(Z)^D \arrow[r, "f_7"] & Ш^1_\omega(\widehat{Q})^D \arrow[r] & 1.
\end{tikzcd}$$

The vanishing of $\mathrm{Br_a}(\mathbf{SL}_n)$ on the bottom-left corner comes from the Corollary of [Gil09b] (section 0). We prove the result by diagram chasing. Since the right column is not exact, there exists $a \in \prod_{v\in\Omega} H^1(K_v, Q)$ such that $f_6(a)$ vanishes but $a \notin f_3(H^1(K, Q))$. Since f_4 is surjective, we can find $b \in Z(K_\Omega)$ such that $f_4(b) = a$. By the commutativity of the bottom square, $f_5(b)$ vanishes. We prove that $b \notin \overline{Z(K)}$ by contradiction. We suppose $b \in \overline{Z(K)}$. The torus Q being flasque implies that $\prod_{v\in\Omega} H^1(K_v, Q)$ is finite (cf. Prop. 9.1 of [CTH15] for K of type (a); Thm. 3.7 with its remark in [CTGP04] for K of type (b)), thus the preimage $f_4^{-1}(a)$ is open (f_4 is continuous, cf. [Čes15]), containing $b \in \overline{Z(K)}$, so we should be able to find $c \in Z(K)$ lying in $f_4^{-1}(a)$. Then $f_3(f_1(c)) = a$ by the commutativity of the top square, contradicting $a \notin \operatorname{Im} f_3$. Therefore, we found $b \in Z(K_\Omega)^{Ш_\omega} \backslash \overline{Z(K)}$. □

Therefore, we need to find other obstructions. In the rest of this section, we consider homogeneous spaces Z under a connected linear group G with geometric stabilizer $\bar{H}$ such that G^{ss} is simply connected and $\bar{H}^{\mathrm{torf}}$ is abelian. This is the assumption already used in [IA21] and [Bor96], and is satisfied by every homogeneous space under a connected linear K-group with connected stabilizers (cf. Lemma 5.2 in [Bor96]).

We can find $Z \leftarrow W \rightarrow W'$ such that

- W is a K-homogeneous space under $G \times T$,
- T is a quasitrivial torus into which H^{torf} injects, where H^{torf} is the canonical K-form of $\bar{H}^{\mathrm{torf}}$ associated to Z,
- $W \rightarrow Z$ is a T-torsor,
- W' is the quotient variety Z/G^{ss}, which is also a homogeneous space of $G^{\mathrm{tor}} \times T$ with geometric stabilizer $\bar{H}^{\mathrm{torf}}$ and the fibers of $W \rightarrow W'$ are homogeneous spaces of G^{ssu} with geometric stabilizers $\bar{H}^{\mathrm{ssu}}$.

Indeed, we can embed H^{tor} in a quasi-trivial torus T and consider the diagonal morphism $H \rightarrow G \times T$ induced by the inclusion $H \hookrightarrow G$ and the composition $H \rightarrow H^{\mathrm{tor}} \rightarrow T$. Then we define $W = (G \times T)/H$, and $W \rightarrow Z$ is induced by the projection to the first coordinate. Define W' to be the quotient variety W/G^{ssu} and we get what we want.

Proposition 3.3.2. *The fiber W_P above a K-point $P \in W'(K)$ satisfies weak approximation.*

Proof. Since H^{ss} is semi-simple, we consider its simply connected covering $1 \rightarrow F \rightarrow H^{\mathrm{sc}} \rightarrow H^{\mathrm{ss}} \rightarrow 1$ where F is finite and H^{sc} is simply connected.

We first prove that this covering induces isomorphisms $H^1(K, H^{\mathrm{ss}}) \simeq H^2(K, F)$ and $H^1(K_v, H^{\mathrm{ss}}) \simeq H^2(K_v, F)$ for all $v \in \Omega$. For K and K_v, the two conditions in Theorem 2.1 of [CTGP04] are satisfied, and thus we have a bijection $H^1(K, H^{\mathrm{ad}}) \rightarrow H^2(K, \mu)$ coming from the central isogeny

$$1 \rightarrow \mu \rightarrow H^{\mathrm{sc}} \rightarrow H^{\mathrm{ad}} \rightarrow 1$$

associated to the center μ of H^{sc}. Since F is contained in the center μ, we have the exact sequence

$$1 \rightarrow \mu/F \rightarrow H^{\mathrm{ss}} \rightarrow H^{\mathrm{ad}} \rightarrow 1$$

and the commutative diagram with exact rows

$$\begin{array}{ccccccc}
H^1(K, \mu/F) & \longrightarrow & H^1(K, H^{\mathrm{ss}}) & \longrightarrow & H^1(K, H^{\mathrm{ad}}) & \longrightarrow & H^2(K, \mu/F) \\
\| & & \downarrow & & \| & & \| \\
H^1(K, \mu/F) & \longrightarrow & H^2(K, F) & \longrightarrow & H^2(K, \mu) & \longrightarrow & H^2(K, \mu/F).
\end{array}$$

Then the four-lemma gives the surjectivity of $H^1(K, H^{\mathrm{ss}}) \to H^2(K, F)$. The injectivity comes from the vanishing of $H^1(K, H^{\mathrm{sc}})$. The proof is the same when K is replaced by K_v.

Then, we prove that

$$H^2(K, F) \to \oplus_{v \in S} H^2(K_v, F)$$

is surjective, where S is a finite set of places. This is proved by chasing the following commutative diagram

$$\begin{tikzcd}
 & \oplus_{v \notin S} H^2(K_v, F) \arrow[d] \arrow[dr, two heads] & & \\
H^2(K, F) \arrow[r] \arrow[d] & \oplus_{v \in \Omega} H^2(K_v, F) \arrow[r] \arrow[d] & H^0(K, \widehat{F})^D \arrow[r] & 1. \\
H^2(K, F) \arrow[r] & \oplus_{v \in S} H^2(K_v, F) \arrow[d] & & \\
 & 1 & &
\end{tikzcd}$$

The row in the middle is exact. (See [Izq16] Theorem 2.7 applying $d = 0$ for K of type (a). For K of type (b), use Cor. 2.3 of [Izq19] and the exact sequence

$$H^2(U, F) \to \bigoplus_{v \in X \setminus U} H^2(K_v, F) \to H^3_c(U, F)$$

with X and U defined in [Izq19], and then take direct limit over U.) The surjectivity of $\oplus_{v \notin S} H^2(K_v, F) \to H^0(K, \widehat{F})^D$ comes from the injectivity taking the duals $H^0(K, \widehat{F}) \hookrightarrow \oplus_{v \notin S} H^0(K_v, \widehat{F})$. This yields the surjectivity of $H^2(K, F) \to \prod_{v \in S} H^2(K_v, F)$, which is $H^1(K, H^{\mathrm{ss}}) \to \oplus_{v \in S} H^1(K_v, H^{\mathrm{ss}})$.

Finally, by Lemme 1.13 of [San81], we have $H^1(K, H^{\mathrm{ss}}) = H^1(K, H^{\mathrm{ssu}})$ and $H^1(K_v, H^{\mathrm{ss}}) = H^1(K_v, H^{\mathrm{ssu}})$. With the same argument as in Proposition 3.2 of [San81], we can prove that G^{ssu} satisfies weak approximation (cf. Proposition 4.1 of [Bor66] for the vanishing of $H^1(K, H^u)$ and $H^1(K_v, H^u)$. The unipotent radical H^{u} satisfies weak approximation). Therefore, we consider the commutative diagram with exact rows (the set $W_P(K)$ is non-empty by Proposition 3.1 of [IA21]):

$$\begin{tikzcd}
G^{\mathrm{ssu}}(K) \arrow[r] \arrow[d] & W_P(K) \arrow[r] \arrow[d] & H^1(K, H^{\mathrm{ssu}}) \arrow[r] \arrow[d, two heads] & 1 \\
\prod_{v \in S} G^{\mathrm{ssu}}(K_v) \arrow[r] & \prod_{v \in S} W_P(K_v) \arrow[r] & \prod_{v \in S} H^1(K_v, H^{\mathrm{ssu}}) \arrow[r] & 1.
\end{tikzcd}$$

A diagram chasing then gives the desired result. □

Proposition 3.3.3. *We have* $W(K_\Omega)^{\mathrm{Б}_\omega(W)} = \overline{W(K)}$.

Proof. Consider $(P_v) \in W(K_\Omega)^{Ш_\omega(W)}$, and any open subset U containing (P_v), we want to find $Q \in U \cap W(K)$ using fibration method.

Denote by (P'_v) the image of (P_v) under the induced map $g : W(K_\Omega) \to W'(K_\Omega)$. By the functorality of the Brauer-Manin pairing, (P'_v) lies in $W'(K_\Omega)^{Ш_\omega(W')}$. Since $W \to W'$ is smooth, and the K_v are henselian, we have open maps $W(K_v) \to W'(K_v)$, and thus g is open. In particular, $V = g(U) \subseteq W'(K_\Omega)$ is also open. Since W' is a torus (cf. Theorem 3.2 of [IA21]), we have $W'(K_\Omega)^{Ш_\omega(W')} = \overline{W'(K)}$. Therefore, there exists $P' \in V \cap W'(K)$. Let $(Q_v) \in g^{-1}(P') \cap U \subseteq W(K_\Omega)$, and it can be seen as in $W_{P'}(K_\Omega)$ too, as shown in the following diagram.

$$\begin{tikzcd}
 & W \arrow[r] & W' \\
 & W_{P'} \arrow[u] \arrow[r] \arrow[ur, phantom, "\square"] & \operatorname{Spec} K \arrow[u, "P'"'] \\
\coprod_{v\in\Omega} \operatorname{Spec} K_v \arrow[uur] \arrow[ur, dotted, "(Q_v)"] \arrow[urr] & &
\end{tikzcd}$$

This diagram also shows that $W_{P'}(K_\Omega) \subseteq W(K_\Omega)$, and $U_{|_{W_{P'}(K_\Omega)}} := U \bigcap W_{P'}(K_\Omega)$ is an open neighborhood of (Q_v) in $W_{P'}(K_\Omega)$. Since $W_{P'}$ satisfies weak approximation by proposition 3.3.2, there exists $Q \in W_{P'}(K) \cap U_{|_{W_{P'}(K_\Omega)}}$, and thus in $W(K) \cap U$, which proves that $W(K_\Omega)^{Ш_\omega(W)} \subseteq \overline{W(K)}$. $\square$

As in [IA21], we are naturally led to consider the following definition.

Definition 3.3.1. For an arbitrary K-variety Z, we define

$$Z(K_\Omega)^{\mathrm{qt},Ш_\omega} := \bigcap_{f:W \xrightarrow{T} Z, T \text{ quasi-trivial}} f(W(K_\Omega)^{Ш_\omega}).$$

Using the torsor $W \to Z$ defined above, we have $Z(K_\Omega)^{\mathrm{qt},Ш_\omega} \subseteq \overline{Z(K)}$. In fact, this is an equality.

Proposition 3.3.4. *Let the notation be as above. Then* $Z(K_\Omega)^{qt,Ш_\omega} = \overline{Z(K)}$.

Proof. Since for all f, we have

$$Z(K) \subseteq f(W(K)) \subseteq f(W(K_\Omega)^{Ш_\omega}),$$

so $Z(K) \subseteq Z(K_\Omega)^{\mathrm{qt},Ш_\omega}$. To prove Proposition 3.3.4, it is equivalent to prove that $Z(K_\Omega)^{\mathrm{qt},Ш_\omega}$ is closed. It suffices to prove that $f(W(K_\Omega)^{Ш_\omega})$ is closed for every torus $f : W \xrightarrow{T} Z$ with T quasitrivial. Since f is smooth and the K_v are henselian, the induced map $f : W(K_\Omega) \to Z(K_\Omega)$ is open. Now we'll prove that $f(W(K_\Omega)^{Ш_\omega})$ does not meet $f(W(K_\Omega) \backslash W(K_\Omega)^{Ш_\omega})$, and combined with the surjectivity of f, we'll get $f(W(K_\Omega)^{Ш_\omega})$ closed.

Fixing $w \in W(K)$ which is a K-point of W, we have the canonical isomorphism $\mathrm{Br}_{\mathrm{a}}(W) \simeq \mathrm{Br}_{1,w}(W)$. Define the map $i_w : T \to W$ as $t \mapsto t.w$, and we have an induced map $i_w^* : \mathrm{Br}_{1,w}(W) \to \mathrm{Br}_{1,e}(T)$. Let α be an element in $Ш_\omega(W)$ and we still denote by α its image in $\mathrm{Br}_{1,w}(W)$. Let $(x_v), (y_v) \in W(K_\Omega)$ above (P_v). We'll prove that $\sum_{v\in\Omega} < x_v, \alpha >= \sum_{v\in\Omega} < y_v, \alpha >$. Let $m : T \times W \to W$ denotes the action of T on W. Since W_{P_v} is a K_v-torsor under T, there exists $t_v \in T(K_v)$ such that $x_v = t_v.y_v$. By the functorality of the Brauer-Manin paring, we have $< x_v, \alpha >=< (t_v, y_v), m^*\alpha >$. By Lemme 6.6 of [San81], we have $\mathrm{Br}_{1,(w,e)}(W \times T) = \mathrm{Br}_{1,w}(W) \times \mathrm{Br}_{1,e}(T)$, and the formula (24) in [BD13] gives $m^*\alpha = p_T^* i_w^*(\alpha) + p_W^*\alpha$ where p_T and p_W are the two natural projections. Then $< x_v, \alpha >=< t_v, i_w^*(\alpha) > + < y_v, \alpha >$. But P is a quasi-trivial torus, we have $Ш_\omega(P) \simeq Ш(P)$ (cf. Prop. 2.6 and §8.1 of [CTH15]). Therefore, by the exact sequence in Theorem 3.2.1, the map $P(K_\Omega) \to Ш_\omega(P)^D$ is 0, i.e. $\sum_{v\in\Omega} < t_v, i_w^*(\alpha) >= 0$ and we get what we want. □

Remark 3.3.2. Actually the above proof also shows that

$$Z(K_\Omega)^{\mathrm{qt},Ш_\omega} = \bigcap_{f:W \xrightarrow{T} Z, T \text{ quasi-trivial}} \bigcap_{\alpha \in Ш_\omega(W)} f(W(K_\Omega)^\alpha).$$

With this description, we can prove as done in Theorem 6.4 of [IA21] that

$$Z(K_\Omega)^{\mathrm{tor}} \subseteq Z(K_\Omega)^{\mathrm{qt},Ш_\omega}. \tag{3.4}$$

But this does not necessarily give an obstruction to weak approximation since we do not know if $Z(K_\Omega)^{\mathrm{tor}}$ is closed. We do not know if (3.4) is an equality. One could wonder whether a "purely descent" description of $Z(K_\Omega)^{\mathrm{qt},Ш_\omega}$ exists.

3.4 Strong approximation for homogeneous spaces with toric stabilizers

We consider K of type (a). For a smooth geometrically integral K-variety Z having local points everywhere, there is a Brauer-Manin pairing

$$Z(\mathbf{A}_K) \times \mathrm{Br}\, Z \to \mathbb{Q}/\mathbb{Z}(-1)$$

whose left kernel contains $Z(K)$ (cf. §8.2 of [CTH15]). By continuity of the pairing, the closure $\overline{Z(K)}^S$ of $Z(K)$ in $Z(\mathbf{A}_K^S)$ is orthogonal to $\mathrm{Br}_S\, Z := \mathrm{Ker}(\mathrm{Br}\, Z \to \prod_{v\in S} \mathrm{Br}\, Z_{K_v})$ with respect to the pairing

$$\begin{aligned} Z(\mathbf{A}_K^S) \times \mathrm{Br}_S\, Z &\to \mathbb{Q}/\mathbb{Z}(-1) \\ ((x_v)_{v\notin S}, \alpha) &\mapsto \sum_{v\notin S} \alpha(x_v) \end{aligned}$$

where S is a finite set of places. This gives rise to a Brauer-Manin obstruction to strong approximation away from S. It is also equivalent to consider $\mathrm{Br}_S\, Z/\mathrm{Br}\, K$ instead of $\mathrm{Br}_S\, Z$.

Based on parts of a Poitou-Tate type exact sequence given by Izquierdo in [Izq16] under the hypothesis of the finiteness of $Ш^2(K,T)$, we get the following result on strong approximation for homogeneneous spaces of quasi-split semisimple simply connected groups with toric stabilizers T.

Theorem 3.4.1. *Let $K = \mathbb{C}((t))(X)$ be the function field of a smooth projective geometrically integral curve X over $\mathbb{C}((t))$. Let E be a quasi-split semisimple simply connected algebraic group over K. Let T be a K-torus such that $Ш^2(K,T)$ is finite. Consider the homogeneous space $Z = E/T$. For a finite non-empty set $S \subseteq X^{(1)}$, there is an exact sequence of pointed sets*

$$1 \to \overline{Z(K)}^S \to Z(\mathbf{A}_K^S) \to (\operatorname{Br}_S Z/\operatorname{Br}_S K)^D \to Ш(Z)^D \to 1.$$

In particular, the Brauer-Manin obstruction to strong approximation away from S is the only one.

Proof. There is a commutative diagram with exact rows

$$\begin{tikzcd}
H^1(K,T) \arrow[r, equal] \arrow[d] & H^1(K,T) \arrow[d] & \\
\mathbf{P}^1_S(T) \arrow[d] & \mathbf{P}^1(T) \arrow[l, two heads] \arrow[d] & \prod_{v\in S} H^1(K_v,T) \arrow[l, hook'] \arrow[d, "\simeq"] \\
(\operatorname{Ker}(H^1(K,\widehat{T}) \to \prod_{v\in S} H^1(K_v,\widehat{T})))^D \arrow[d] & H^1(K,\widehat{T})^D \arrow[l, two heads] & (\prod_{v\in S} H^1(K_v,\widehat{T}))^D \arrow[l, hook'] \\
Ш^1(K,\widehat{T})^D \arrow[d] & & \\
1 & &
\end{tikzcd}$$

where the middle column is an exact sequence by Proposition 3.16(ii) of [Izq16], and the right-most isomorphism comes from the local duality in Proposition 3.4 of [CTH15]. A diagram chasing shows that the first three terms in the left-most column form an exact sequence. The exactness of the rest of this column follows by dualization, noting also that $\mathbf{P}^1_S(T)^D \simeq \mathbf{P}^1_S(\widehat{T})$ (cf. Proposition 3.12 and Corollaire 3.15 of [Izq16]).

We have a morphism

$$H^1(K,\widehat{T}) \to \operatorname{Br} Z, \quad a \mapsto a \cup [E]$$

by taking cup-product with the class $[E] \in H^1(Z,T)$, and this induces an isomorphism $H^1(K,\widehat{T}) \oplus \operatorname{Br} K \simeq \operatorname{Br} Z$ (cf. Proposition 2.9 and 2.10 of [CTX09]). Therefore, we get isomorphisms

$$\operatorname{Ker}(H^1(K,\widehat{T}) \to \prod_{v\in S} H^1(K_v,\widehat{T})) \simeq \operatorname{Br}_S Z/\operatorname{Br}_S K$$

$$Ш^1(K,\widehat{T}) \simeq \mathrm{Ker}(\mathrm{Br}\, Z/\mathrm{Br}\, K \to \prod_{v\in\Omega} \mathrm{Br}\, Z_{K_v}/\mathrm{Br}\, K_v) \simeq Б(Z)$$

where the last isomorphism follows from the definition of $Б(Z)$ and the fact that $\mathrm{Br}(Z_{\bar{K}}) \simeq \mathrm{Pic}\, T_{\bar{K}} \oplus \mathrm{Br}\, \bar{K} = 0$ (cf. Proposition 2.10 of [CTX09]). These maps fit into the following commutative diagram

$$\begin{tikzcd}
E(K) \arrow[r] \arrow[d] & Z(K) \arrow[r] \arrow[d] & H^1(K,T) \arrow[r] \arrow[d] & 1 \\
E(\mathbf{A}_K^S) \arrow[r] & Z(\mathbf{A}_K^S) \arrow[r] \arrow[d] & \mathbf{P}^1_S(T) \arrow[r] \arrow[d] & 1 \\
 & (\mathrm{Br}_S\, Z/\,\mathrm{Br}_S\, K)^D \arrow[r, "\simeq"] \arrow[d] & (\mathrm{Ker}(H^1(K,\widehat{T}) \to \prod_{v\in S} H^1(K_v,\widehat{T})))^D \arrow[d] & \\
 & Б(Z)^D \arrow[r, "\simeq"] \arrow[d] & Ш^1(K,\widehat{T})^D \arrow[d] & \\
 & 1 & 1 &
\end{tikzcd}$$

whose commutativity follows from Proposition 2.9 of [CTX09]. The open subset $X\backslash S$ is affine whose ring of regular functions is a Dedekind domain, so strong approximation away from S holds for the quasi-split semisimple simply connected group E (cf. Satz 2.2.1 of [Har67]). Then a diagram chasing gives the desired result. $\square$

Chapter 4

Homogeneous spaces over p-adic function fields

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Over function fields of p-adic curves, we construct stably rational varieties in the form of homogeneous spaces of $\mathbf{SL}_n$ with semisimple simply connected stabilizers and we show that strong approximation away from a non-empty set of places fails for such varieties. The construction combines the Lichtenbaum duality and the degree 3 cohomological invariants of the stabilizers. We then establish a reciprocity obstruction which accounts for this failure of strong approximation. We show that this reciprocity obstruction to strong approximation is the only one for counterexamples we constructed, and also for classifying varieties of tori. We also show that this reciprocity obstruction to strong approximation is compatible with known results for tori. At the end, we explain how a similar point of view shows that the reciprocity obstruction to weak approximation is the only one for classifying varieties of tori over p-adic function fields.

4.1 Introduction

In recent years, there has been growing interest in studying the arithmetic of algebraic groups and their homogeneous spaces defined over fields of cohomological dimension strictly greater than 2, and in particular, function fields K of p-adic curves. For example, in [HS16] and [HSS15], Harari, Szamuely and Scheiderer studied local-global principle and weak approximation questions for tori over such fields K, with respect to the valuations coming from the codimension 1 points of the p-adic curve. For both questions, they showed that the reciprocity obstruction is the only one. Tian generalized their results on weak approximation to quasi-split reductive groups in his work [Tia21a]. Very recently in [Lin22], Linh studied homogeneous spaces of $\mathbf{SL}_n$ with toric stabilizers, and more generally, stabilizers in the form of an extension of a group of multiplicative type by a unipotent group. He showed that the reciprocity obstruction is the only one to both the local-global principle and weak approximation.

Being a generalized form of the Chinese Remainder Theorem, strong approximation is a natural question in the study of the arithmetic of algebraic groups and their homogeneous spaces. For tori over p-adic function fields, Harari and Izquierdo described the defect of strong approximation in [HI19], based on a Poitou-Tate type exact sequence obtained in [HSS15].

We study strong and weak approximation questions for homogeneous spaces over a p-adic function fields. Let K be the function field of a smooth proper geometrically integral curve X defined over p-adic field k. When studying a homogeneous space $Z := G/H$ for G a connected K-group G, it is natural to use the construction

$$\begin{tikzcd}
G \arrow[d, "H"] & G \times T \arrow[l] \arrow[r] \arrow[d, "H"] & G^{\mathrm{tor}} \times T \arrow[d, "H^{\mathrm{tor}}"] \\
Z := G/H & W := G \times T/H \arrow[l, "T"'] \arrow[r, "G^{\mathrm{ss}}"] & W' := Z/G^{\mathrm{ss}}
\end{tikzcd}$$

as in §3.3, where T is a quasi-trivial K-torus into which we embed H^{tor}, and the quotient variety W' is a torus. The fibers of $W \to W'$ are homogeneous spaces of G^{ss} (that we may always suppose to be simply connected up to replacing the ambient group conveniently) with semisimple stabilizers. Using fibration methods, we hence reduce to studying the arithmetic of tori and homogeneous spaces of semisimple simply connected groups with semisimple stabilizers. Since there are already works on tori, we are motivated to study the latter case.

First, we consider homogeneous spaces Z of $\mathbf{SL}_n$ over K with stabilizers that are semisimple simply connected of type A. Using the Rost invariant, to which we relate the Lichtenbaum's duality pairing $\operatorname{Br} X \times \operatorname{Pic} X \to \mathbb{Q}/\mathbb{Z}$, we obtain examples where strong approximation away from a non-empty set S of places does not hold for Z, showing very different behavior compared to number fields.

In order to explain such failure, we define a reciprocity obstruction to strong approxi-

mation away from S for a K-variety Z using the group

$$H^3_{\mathrm{nr},S}(Z,\mathbb{Q}/\mathbb{Z}(2)) := \mathrm{Ker}(H^3_{\mathrm{nr}}(Z,\mathbb{Q}/\mathbb{Z}(2))) \to \prod_{v\in S} H^3_{\mathrm{nr}}(Z_{K_v},\mathbb{Q}/\mathbb{Z}(2))$$

where $H^3_{\mathrm{nr}}(Z,\mathbb{Q}/\mathbb{Z}(2))$ denotes the subgroup the elements in $H^3(K(Z),\mathbb{Q}/\mathbb{Z}(2))$ that are unramified with respect to all the codimension 1 points of Z. For classifying varieties $Z = \mathbf{SL}_n/H$, we relate this obstruction to the group $\mathrm{Inv}^3(H,\mathbb{Q}/\mathbb{Z}(2))$ of degree 3 cohomological invariants of H with coefficients in $\mathbb{Q}/\mathbb{Z}(2)$. Then we show in Theorem 4.3.10 that this obstruction is the only one for counterexamples we constructed, meaning that the closure of $Z(K)$ in $Z(\mathbf{A}_K^S)$ equals the subset of elements in $Z(\mathbf{A}_K^S)$ that are orthogonal to $H^3_{\mathrm{nr},S}(Z,\mathbb{Q}/\mathbb{Z}(2))$.

We also apply this obstruction to $\mathbf{SL}_n/T$ for T a torus. Using Blinstein and Merkurjev's description of $\mathrm{Inv}^3(T,\mathbb{Q}/\mathbb{Z}(2))$ in [BM13] along with the Poitou-Tate type exact sequence in [HSS15], we show in Theorem 4.3.11 that, also in this case, the reciprocity obstruction to strong approximation away from S is the only one for $Z = \mathbf{SL}_n/T$ over K a p-adic function field.

Going beyond classifying varieties, we show in Corollary 4.3.16 that for $Z = T$ a torus, this reciprocity obstruction to strong approximation is compatible with the known results in [HI19].

At last, we explain that we can adapt a similar point of view for weak approximation problems for classifying varieties, linking the reciprocity obstruction to weak approximation with the unramified cohomological invariants. As a quick and direct application, we combine Blinstein and Merkurjev's description of $\mathrm{Inv}^3_{\mathrm{nr}}(T,\mathbb{Q}/\mathbb{Z}(2))$ in [BM13] with the Poitou-Tate type exact sequence in [HSS15] and we get Theorem 4.4.2 saying that the reciprocity obstruction to weak approximation is the only one for $\mathbf{SL}_n/T$ over K a p-adic function field (this method is different from [Lin22] where Linh obtained the same results for $\mathbf{SL}_n/T$). Weak approximation for $\mathbf{SL}_n/H$ with H semisimple simply connected is still an open question, and information from $\mathrm{Inv}^3_{\mathrm{nr}}(H,\mathbb{Q}/\mathbb{Z}(2))$ might shed light on possible directions for such problems.

Preliminaries

Motivic complexes

Let X be a smooth scheme. Lichtenbaum (cf. [Lic87][Lic07]) defined the motivic complexes $\mathbb{Z}(i)$ for $i = 0,1,2$, of étale sheaves on X. We write $H^*(X,\mathbb{Z}(i))$ for the étale (hyper)cohomology groups of X with values in $\mathbb{Z}(i)$. The complex $\mathbb{Z}(0)$ equals the constant sheaf $\mathbb{Z}$ and $\mathbb{Z}(1) = \mathbb{G}_{m,X}[-1]$, hence $H^n(X,\mathbb{Z}(1)) = H^{n-1}(X,\mathbb{G}_{m,X})$. In particular, $H^3(X,\mathbb{Z}(1)) = \mathrm{Br}\,X$, the cohomological Brauer group of X. The complex $\mathbb{Z}(2)$ is concentrated in degrees 1 and 2 and there is a product map $\mathbb{Z}(1)\otimes^{\mathbf{L}}\mathbb{Z}(1)\to\mathbb{Z}(2)$ (cf. Proposition 2.5 of [Lic87]). Over the small Zariski site X_{Zar}, we have the complex $\mathbb{Z}(2)_{\mathrm{Zar}}$ concentrated

in degree ≤ 2 defined in a similar way to $\mathbb{Z}(2)$.

We denote by $A(i)$ the complex $A \otimes \mathbb{Z}(i)$ for an abelian group A (similarly for $\mathbb{Z}(i)_{\mathrm{Zar}}$). If X is defined over a field whose characteristic does not divide m, we have a quasi-isomorphism of complexes of étale sheaves $\mathbb{Z}/m\mathbb{Z} \otimes^{\mathbf{L}} \mathbb{Z}(i) \simeq \mu_m^{\otimes i}$, where μ_m is the étale sheaf of mth roots of unity, and we set $\mu_m^0 = \mathbb{Z}/m\mathbb{Z}$. Thus we shall also use the notation $\mathbb{Q}/\mathbb{Z}(i)$ for the direct limit of the sheaves $\mu_m^{\otimes i}$ for all $m > 0$ when the base field has characteristic 0.

The exact triangle in the derived category of étale sheaves

$$\mathbb{Z}(i) \to \mathbb{Q}(i) \to \mathbb{Q}/\mathbb{Z}(i) \to \mathbb{Z}(i)[1]$$

yields the connecting morphism

$$H^n(X, \mathbb{Q}/\mathbb{Z}(i)) \to H^{n+1}(X, \mathbb{Z}(i)),$$

which is an isomorphism if $X = \operatorname{Spec} F$ for a field F and $n > i$ (cf. Lemma 1.1 of [Kah93]).

Classifying varieties and cohomological invariants

A connected algebraic group E defined over a field K is called *special* if $H^1(F, E) = 1$ for all field extensions F/K. Let H be a smooth algebraic group over K. Choose an embedding $\rho : H \hookrightarrow E$ into a special rational group E. Examples of special rational groups E include $\mathbf{SL}_n, \mathbf{Sp}_{2n}, \mathbf{GL}_1(A)$ for a central simple K-algebra A. The variety $Z = E/\rho(H)$ is called a *classifying variety* of H. Different choices of ρ give stably birational classifying varieties. (cf. §2 of [Mer02].)

For every field extension F/K, the set $H^1(F, H)$ classifying H-torsors over F can be identified with $Z(F)/E(F)$, the orbit space of the action of $E(F)$ on $Z(F)$. For a point $x \in Z(F)$, we write x^*E for the fiber of the H-torsor $E \to Z$ above the point x and it is an H-torsor over F, and its class in $H^1(F, H)$ is the image of $[E]$ under the map $H^1(Z, H) \to H^1(F, H)$ induced by x. In particular, we have the generic H-torsor ξ^*E over $K(Z)$ corresponding to the generic point $\xi \in Z$.

Let H be an algebraic group over a field K. The group $\mathrm{Inv}^d(H, \mathbb{Q}/\mathbb{Z}(d-1))$ of degree d cohomological invariants of H consists of morphisms of functors

$$H^1(*, H) \to H^d(*, \mathbb{Q}/\mathbb{Z}(d-1))$$

from the category of field extensions of K to the category of sets. An invariant is called *normalized* if it takes the trivial H-torsor to zero. The normalized invariants form a subgroup $\mathrm{Inv}^d(H, \mathbb{Q}/\mathbb{Z}(d-1))_{\mathrm{norm}}$ of $\mathrm{Inv}^d(H, \mathbb{Q}/\mathbb{Z}(d-1))$ and there is a natural isomorphism

$$\mathrm{Inv}^d(H, \mathbb{Q}/\mathbb{Z}(d-1)) \simeq H^d(K, \mathbb{Q}/\mathbb{Z}(d-1) \oplus \mathrm{Inv}^d(H, \mathbb{Q}/\mathbb{Z}(d-1))_{\mathrm{norm}}.$$

Rost proved that if H is absolutely simple simply connected, then $\mathrm{Inv}^3(H, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{norm}}$ is a finite cyclic group generated by the *Rost invariant* R_H (cf. Theorem 9.11 of [GMS03]).

p-adic function fields

Let k be a finite extension of $\mathbb{Q}_p$, and let X be a smooth proper geometrically integral curve over k. Let K be the function field of X. There is a natural set of places: the set of all closed points of X that we denote by $X^{(1)}$. For a closed point $v \in X^{(1)}$, we denote by $\kappa(v)$ its residue field, and K_v the completion of K for the discrete valuation induced by v. Then $\kappa(v)$ is also the residue field of the ring of integers $\mathcal{O}_v$ and is a finite extension of k. Therefore K_v is a *2-dimensional local fields*, i.e. a field complete with respect to a discrete valuation whose residue field is a classical local field.

We have a complex in virtue of the generalized Weil reciprocity law (cf. Theorem 6.4.3 of [GS17]):

$$H^3(K, \mathbb{Q}/\mathbb{Z}(2)) \to \bigoplus_{v \in X^{(1)}} H^3(K_v, \mathbb{Q}/\mathbb{Z}(2)) \xrightarrow{\Sigma} \mathbb{Q}/\mathbb{Z} \tag{4.1}$$

where the map Σ is given by the composition

$$\bigoplus_{v \in X^{(1)}} H^3(K_v, \mathbb{Q}/\mathbb{Z}(2)) \xrightarrow[\simeq]{\partial_v} \bigoplus_{v \in X^{(1)}} H^2(\kappa(v), \mathbb{Q}/\mathbb{Z}(1)) \xrightarrow{\Sigma \operatorname{Cor}_{\kappa(v)/k}} H^2(k, \mathbb{Q}/\mathbb{Z}(1)) = \mathbb{Q}/\mathbb{Z}.$$

The above notation of our fields will be fixed for the rest of the article.

Strong approximation

Given a smooth geometrically integral K-scheme Z, for a non-empty open subscheme $U \subseteq X$ sufficiently small, we can find a smooth geometrically integral U-scheme $\mathcal{Z}$ such that $\mathcal{Z} \times_U K \simeq Z$. We can thus define the adelic space (which is actually independent of the choice of the model $\mathcal{Z}$):

$$Z(\mathbf{A}_K) := \varinjlim_{\substack{U' \subseteq U \\ U' \neq \emptyset}} \prod_{v \in X^{(1)} \setminus U'} Z(K_v) \times \prod_{v \in U'^{(1)}} \mathcal{Z}(\mathcal{O}_v).$$

The problem of strong approximation studies the closure $\overline{Z(K)}$ inside the adelic space $Z(\mathbf{A}_K)$ with the restricted product topology, possibly away from a finite set S of places (in this case, we study $\overline{Z(K)}^S$ the closure of $Z(K)$ in $Z(\mathbf{A}_K^S) := \prod'_{v \notin S} Z(K_v)$ the restricted product with respect to integral points excluding places in S). If $Z(K)$ is dense in $Z(\mathbf{A}_K^S)$, we say that Z satisfies *strong approximation away from* S: we can simultaneously approximate finitely many K_v-points with $v \notin S$ by a single K-rational point with the condition that this point is integral at all other places. For $S \subseteq S'$, strong approximation away from S implies strong approximation away from S'.

Strong approximation for semisimple groups has been studied in the general setting. A K-group G is said to be *quasi-split* if it has a Borel subgroup defined over K. A K-group G is called *absolutely almost simple* if over the algebraic closure it becomes an extension of a simple group by a finite normal (hence central) subgroup.

Theorem 4.1.1 (Gille, Corollaire 5.1 of [Gil09a]). *Let R be a Dedekind domain with function field K and let G be a semisimple simply connected K-group which is absolutely almost simple and isotropic. Suppose the K-variety G is retract rational. Then strong approximation holds for G over* $\operatorname{Spec} R$.

The above theorem implies an earlier result:

Theorem 4.1.2 (Harder, Satz 2.2.1 of [Har67]). *Let R be a Dedekind domain with function field K. Let G be a quasi-split semisimple simply connected K-group. Then strong approximation holds for G over* $\operatorname{Spec} R$.

Corollary 4.1.3. *Let X be a smooth proper geometrically integral curve over a p-adic field k, with function field $K = k(X)$. Let G be a quasi-split semisimple simply connected K-group. Then strong approximation away from a non-empty finite set of places $S \subseteq X^{(1)}$ holds for G.*

Proof. The open subset $V := X \backslash S$ is affine and the ring of regular functions $k[V]$ is a Dedekind domain. □

When studying strong approximation for classifying varieties $Z = E/H$, we usually make use of the following commutative diagram

$$\begin{tikzcd} E(K) \arrow[r] \arrow[d] & Z(K) \arrow[r] \arrow[d] & H^1(K,H) \arrow[r] \arrow[d] & 1 \\ E(\mathbf{A}_K^S) \arrow[r] & Z(\mathbf{A}_K^S) \arrow[r] & \mathbf{P}_S^1(H) \arrow[r] & 1 \end{tikzcd}$$

where

$$\mathbf{P}_S^i(H) := \prod_{v \notin S}{}' H^i(K_v, H) \tag{4.2}$$

denotes the restricted topological product of the $H^i(K_v, H)$ for all $v \in X^{(1)} \backslash S$ with respect to the images of the maps $H^i(\mathcal{O}_v, \mathcal{H})$ for $v \in U$.

Notation

Unless otherwise stated, all (hyper)cohomology groups are taken with respect to the étale cohomology.

Given an abelian group A, we denote by ${}_mA$ the m-torsion subgroup of A. The notation $A_\wedge$ is the inverse limit of the quotients A/nA for all $n > 0$. For A a topological abelian group, denote by $A^D := \operatorname{Hom}_{\text{cont}}(A, \mathbb{Q}/\mathbb{Z})$ the group of continuous homomorphisms from A to $\mathbb{Q}/\mathbb{Z}$. When there is no other topology defined on A, we equip A with the discrete topology. The functor $A \mapsto A^D$ is an anti-equivalence of categories between torsion abelian groups and profinite groups.

4.2 Failure of strong approximation

Over p-adic function fields, we construct varieties of the form $\mathbf{SL}_n/H$ with H semisimple simply connected and we show that strong approximation away from a finite set of places fails for such varieties.

4.2.1 Inner type

For a smooth proper geometrically integral curve X over a p-adic field k, we have the following dualities due to Lichtenbaum:

Theorem 4.2.1 (Lichtenbaum, cf. [Lic69] Theorem 4)**.** *There are pairings*

$$\psi : \operatorname{Br} X \times \operatorname{Pic} X \to \mathbb{Q}/\mathbb{Z} \tag{4.3}$$

$$\rho : H^1(k, \operatorname{Pic} \overline{X}) \times \operatorname{Pic}^0(X) \to \mathbb{Q}/\mathbb{Z}$$

where ψ is induced by evaluation on closed points. These pairings induce dualities

$$\psi^* : \operatorname{Br} X \simeq \operatorname{Pic} X^D \tag{4.4}$$

$$\rho^* : H^1(k, \operatorname{Pic}(\overline{X})) \simeq \operatorname{Pic}^0(\overline{X})^D. \tag{4.5}$$

Remark 4.2.2. If $[A]$ comes from $\operatorname{Br} k$ via the restriction map, then $\psi([A], z) = (\deg z)[A]$.

Construction 4.2.3. Let H be an absolutely simple simply connected group over K of inner type A. Then $H = \mathbf{SL}_1(A)$ for A a simple central K-algebra. Suppose the class $[A] \in \operatorname{Br} K$ lies in $\operatorname{Br}_{\mathrm{nr}}(K/k) = \operatorname{Br} X$. Embed H into a special rational group E which is split semisimple simply connected (for example, $E = \mathbf{SL}_n$ or $\mathbf{Sp}_{2n}$). Consider the classifying variety $Z := E/H$. Since H also embeds into the special rational group $\mathbf{GL}_1(A)$, we have that Z is stably birational to the classifying variety $\mathbf{GL}_1(A)/\mathbf{SL}_1(A) = \mathbb{G}_m$ which is rational. Therefore, the variety Z is stably rational. In particular, weak approximation holds for Z.

The Rost invariant R_H has an explicit formula in this situation (cf. [GQM11]): Let $[x] \in H^1(F, H) = F^\times / \operatorname{Nrd}((A \otimes F)^\times)$, then we have

$$R_H([x]) = (x) \cup [A] \in H^3(F, \mathbb{Q}/\mathbb{Z}(2))$$

where (x) denotes the class in $H^1(K, \mu_m) = F^\times/F^{\times m}$ of any representative of $[x]$, and m is the exponent of A. Moreover, the order of $R_H \in \operatorname{Inv}^3(H, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{norm}}$ equals m (cf. Theorem 11.5 of [GMS03]).

We have the following commutative diagram where the first row is the complex (4.1) by the generalized Weil reciprocity:

$$\begin{tikzcd} H^3(K, \mathbb{Q}/\mathbb{Z}(2)) \arrow[r] & \bigoplus_{v\in X^{(1)}} H^3(K_v, \mathbb{Q}/\mathbb{Z}(2)) \arrow[r, "\Sigma"] & \mathbb{Q}/\mathbb{Z}. \\ H^1(K, H) \arrow[u, "R_H"] \arrow[r] & \bigoplus_{v\in X^{(1)}} H^1(K_v, H) \arrow[u, "\oplus R_H"] \arrow[ur, "f_H"'] & \end{tikzcd} \tag{4.6}$$

Now we show that the composed map f_H defined by this diagram is related to the pairing ψ in (4.3):

Proposition 4.2.4. *Let $a = ([a_v])_v \in \bigoplus_{v\in X^{(1)}} H^1(K_v, H) = \bigoplus_{v\in X^{(1)}} K_v^\times / \operatorname{Nrd}((A \otimes K_v)^\times)$ represented by $a_v \in K_v^\times$. Denote by n_v the valuation of a_v with respect to v and let $z := \sum_{v\in X^{(1)}} n_v \cdot v \in \operatorname{Pic} X$. Then*

$$\psi([A], z) = f_H(a).$$

Proof. We have $f_H(a) = \sum_{v\in X^{(1)}} \operatorname{Cor}_{\kappa(v)/k}(\partial_v(R_H([a_v])))$. By linearity, it suffices to show

$$\psi([A], n_v \cdot v) = \operatorname{Cor}_{\kappa(v)/k}(\partial_v(R_H([a_v]))).$$

Choose a uniformizer π_v of the place v, and write $a_v = u\pi_v^{n_v}$ with u a unit in $\mathcal{O}_v$. Then

$$\partial_v(R_H([a_v])) = \partial_v((u\pi_v^{n_v}) \cup [A]) = \partial_v((u) \cup [A]) + n_v\partial_v((\pi_v) \cup [A]).$$

In fact $\partial_v((\pi_v) \cup [A])$ is the image of $[A]$ under the specialization map

$$s^2_{-\pi_v} : H^2(K_v, \mu_m) \to H^2(\kappa(v), \mu_m)$$

with respect to the uniformizer $-\pi_v$ (cf. Construction 6.8.6 of [GS17]). Actually $s^2_{-\pi_v}([A])$ is independent of the choice of the uniformizer π_v, since $\partial_v([A]) = 0$, implying that $[A]$ lies in the image of $H^2(\kappa(v), \mu_m) \xrightarrow{\text{Inf}} H^2(K_v, \mu_m)$ (cf. Proposition 6.8.7 and Corollary 6.8.8 of [GS17]). The fact that $[A]$ is unramified at v also gives $\partial_v((u) \cup [A]) = 0$ (cf. Lemma 6.8.4 of [GS17]). But $\psi([A], v)$ is just the evaluation of $[A]$ at the closed point v followed by $\operatorname{Cor}_{\kappa(v)/k}$, so we have

$$\psi([A], n_v \cdot v) = \operatorname{Cor}_{\kappa(v)/k}(n_v\partial_v((\pi_v) \cup [A])) = \operatorname{Cor}_{\kappa(v)/k}(\partial_v(R_H([a_v])))$$

giving the desired result. □

The above calculations also give the following result, which will be used later.

Proposition 4.2.5. *There is a bijection*

$$\begin{aligned} H^1(K_v, H) &\simeq \mathbb{Z}/m_v\mathbb{Z} \\ [a_v] &\mapsto \textit{valuation of } a_v \bmod m_v \end{aligned}$$

where m_v denotes the exponent of $A \otimes K_v \in \operatorname{Br} K_v$ and $a_v \in K_v^\times / K_v^{\times m_v}$ represents the class $[a_v] \in H^1(K_v, H)$.

Proof. The map given by the Rost invariant

$$R_H : H^1(K_v, H) \to H^3(K_v, \mathbb{Q}/\mathbb{Z}(2))$$

is injective in this case (cf. Theorem 5.3 of [CTPS12]). Therefore, we can identify $H^1(K_v, H)$ with its image $R_H(H^1(K_v, H))$. Applying the isomorphism $\partial_v : H^3(K_v, \mathbb{Q}/\mathbb{Z}(2)) \simeq \mathrm{Br}(\kappa(v))$, we see that $H^1(K_v, H) \simeq \partial_v(R_H(H^1(K_v, H)))$ is in fact a cyclic group generated by $\partial_v(R_H([\pi_v])) = v^*([A]) \in \mathrm{Br}(\kappa(v))$ with the calculations in Proposition 4.2.4, where $v^*([A])$ denotes the evaluation of $[A]$ at v. By Proposition 6.8.7 and Corollary 6.8.8 of [GS17], the class $A \otimes K_v \in \mathrm{Br}\, K_v$ comes from $v^*([A])$ under the injection $\mathrm{Br}(\kappa(v)) \hookrightarrow \mathrm{Br}\, K_v$, and thus they have the same order. □

In virtue of the Lichtenbaum duality (4.4), the non-vanishing of the pairing ψ gives non-vanishing f_H, which can then be used to fabricate examples of non-surjective $H^1(K, H) \to \bigoplus_{v \in X^{(1)}} H^1(K_v, H_v)$. Moreover, this map can be non-surjective even away from a finite set S of places:

Theorem 4.2.6. *If the set of values taken by $\psi([A], -)$ at the divisors supported in $S \subseteq X^{(1)}$ do not cover all the values taken at the divisors supported in $X^{(1)} \setminus S$, i.e.*

$$\{\psi([A], z) | \operatorname{Supp}(z) \subseteq X^{(1)} \backslash S\} \not\subseteq \{\psi([A], z) | \operatorname{Supp}(z) \subseteq S\},$$

then

$$H^1(K, H) \to \bigoplus_{v \in X^{(1)} \backslash S} H^1(K_v, H)$$

is not surjective. As a result, strong approximation away from S does not hold for Z.

Proof. Take a divisor $z_0 = n_v \cdot v$ supported in $X^{(1)} \backslash S$ such that

$$\psi([A], z_0) \notin \{\psi([A], z) | \operatorname{Supp}(z) \subseteq S\}.$$

Let $a := ([\pi_v^{n_v}])_v \in \bigoplus_{v \in X^{(1)} \backslash S} H^1(K_v, H)$. By Proposition 4.2.4, the images of elements in $\bigoplus_{v \in S} H^1(K_v, H)$ under f_H lie in $\{\psi([A], z) | \operatorname{Supp}(z) \subseteq S\}$, and thus cannot cancel out $f_H(a) = \psi([A], z_0)$. Then chasing the diagram (4.6) shows that a is not in the image of $H^1(K, H)$.

Now consider the following commutative diagram

$$\begin{tikzcd}
Z(K) \arrow[r] \arrow[d] & H^1(K, H) \arrow[r] \arrow[d] & 1 \\
Z(\mathbf{A}_K^S) \arrow[r] & \bigoplus_{v \in X^{(1)} \backslash S} H^1(K_v, H) \arrow[r] & 1
\end{tikzcd}$$

with exact rows. We identify $\oplus_{v \in X^{(1)} \backslash S} H^1(K_v, H)$ with the topological restricted product $\mathbf{P}^1_S(H)$ because $H^1(\mathcal{O}_v, \mathcal{H}) = H^1(\kappa(v), H) = 0$ by Serre's Conjecture II proved for p-adic fields (cf. Theorem 1 of §4.1, [KJ69]). The map $Z(\mathbf{A}_K^S) \to \bigoplus_{v \in X^{(1)} \backslash S} H^1(K_v, H)$ is

locally constant. Take one element $a \in \bigoplus_{v \in X^{(1)} \setminus S} H^1(K_v, H)$ which is not in the image of $H^1(K, H)$, for example the a we just constructed. There exists an open subset V of $z \in Z(\mathbf{A}_K^S)$ which maps to a. A diagram chasing shows that V doesn't meet $Z(K)$. □

For $S \subseteq X^{(1)}$, the index $I(S)$ of S is defined to be the greatest common divisor of the degrees of all $v \in S$. The index $I(X)$ of X is $I(S)$ for $S = X^{(1)}$.

Corollary 4.2.7. *Let A be a simple central k-algebra of exponent m and consider $H := \mathbf{SL}_1(A_K)$. Let $z_0 = \sum_{v \in X^{(1)}} n_v \cdot v$ be a divisor such that $\deg z_0 \not\equiv 0 \bmod m$. Let S be a set of places in $X^{(1)}$ not meeting the support of z_0 such that $\deg(z_0)$ is not contained in the subgroup generated by $I(S)$ in $\mathbb{Z}/m\mathbb{Z}$. Then strong approximation away from S does not hold for Z.*

Proof. By Remark 4.2.2, we have $\psi([A], z_0) = \deg z_0/m \in {}_m\mathrm{Br}k \simeq \frac{1}{m}\mathbb{Z}/\mathbb{Z} \subseteq \mathbb{Q}/\mathbb{Z}$. When z is supported in S, the value $\psi([A], z)$ is always a multiple of $I(S)/m$, which cannot be equal to $\deg(z_0)/m$. The condition in Proposition 4.2.6 is thus satisfied. □

Example 4.2.8. Let $X = \mathbb{P}^1_k$ be the projective line over $k = \mathbb{Q}_p$. Then the function field $K = \mathbb{Q}_p(t)$. Consider the quaternion algebra $A = (p, u)$ over K with u a non-square unit in $\mathbb{Z}_p^\times$. Then $[A] \in {}_2\mathrm{Br}K$ comes from the generator of ${}_2\mathrm{Br}X = {}_2\mathrm{Br}k$. Let H be the semisimple simply connected group $\mathbf{SL}_1(A)$, and consider the stably rational homogeneous space $Z = \mathbf{SL}_n/H$. If $S \subseteq X^{(1)}$ is a subset containing only places of even degrees, for example $(t^2 - p), (t^4 - p), (t^6 - p)$ etc, then strong approximation away from S does not hold for Z. In fact, we will see later that strong approximation away from S holds for Z if and if only the index of S is odd, and the reciprocity obstruction is the only one (cf. Theorem 4.3.10).

Remark 4.2.9. Our example shows that we have a different behavior over p-adic function fields compared to the classical situations over number fields. For H a semisimple and connected algebraic group defined over a number field F, the canonical map $H^1(F, H) \to \prod_{v \in \Omega_F} H^1(F_v, H)$ is always surjective (cf. §5.3 of [KJ69]). The group $\mathbf{SL}_n$ satisfies strong approximation away from a non-empty set $S \subseteq \Omega_F$ of places. A diagram chasing argument shows that strong approximation away from a non-empty $S \subseteq \Omega_F$ always holds for $\mathbf{SL}_n/H$. In our case over a p-adic function field K, we still have strong approximation away from a non-empty $S \subseteq X^{(1)}$ for $\mathbf{SL}_n$ (cf. Corollary 4.1.3), but not for $\mathbf{SL}_n/H$ where H is an absolutely simple simply connected algebraic group. We will give another example with H of outer type A (cf. Example 4.2.13).

4.2.2 Outer type

Now consider H a simply connected group of outer type A_{n-1}, i.e. $H = \mathbf{SU}(A, \tau)$, where A is a central simple algebra of degree $n \geq 3$ over a (separable) quadratic extension L of K with a unitary involution τ which leaves K element-wise invariant. The existence of such

a L/K-unitary involution on A is equivalent to $\mathrm{Cor}_{L/K}([A]) = 0 \in \mathrm{Br}\, K$ (cf. Theorem 3.1 of [Knu98]). Let

$$\mathrm{SSym}(A,\tau)^\times = \{(s,z) \in A^\times \times L^\times | \tau(s) = s, \mathrm{Nrd}_A(s) = N_{L/K}(z)\}$$

which is a homogeneous space for $\mathbf{GL}_1(A)$ under the operation given by $x : (s,z) \mapsto (xs\tau(x), z\,\mathrm{Nrd}_A(x))$. There is an exact sequence $1 \to H \to \mathbf{GL}_1(A) \to \mathrm{SSym}(A,\tau)^\times \to 1$ which then induces the exact sequence

$$\mathbf{GL}_1(A_{F\otimes L}) \to \mathrm{SSym}(A_{F\otimes L},\tau)^\times \to H^1(F,H) \to H^1(F,\mathbf{GL}_1(A)) = \{1\}$$

for any field extensions F/K. In other words, there is a canonical bijection between $H^1(F,H)$ and $\mathrm{SSym}(A_{F\otimes L},\tau)^\times/\approx$ where the equivalence relation $\approx$ is defined by $(s,z) \approx (s',z')$ if and only of $s' = xs\tau(x)$ and $z' = z\,\mathrm{Nrd}_{A_{F\otimes L}}(x)$ for some $x \in A^\times_{F\otimes L}$ (cf. (29.18) of [Knu98] and §5.5 of [KJ69]).

Over the quadratic extension L/K, the group H is isomorphic to $\mathbf{SL}_1(A)$. Under the field extension map, the Rost invariant R_H maps to the Rost invariant R_{H_L}. The corestriction map for the field extension L/K takes R_{H_L} to $2R_H$. Using the formula of the Rost invariant for $\mathbf{SL}_1(A)$, we have

$$2R_H(s,z) = \mathrm{Cor}_{F\otimes L/F}((z) \cup [A_{F\otimes L}]) \in H^3(F,\mathbb{Q}/\mathbb{Z}(2))$$

over a field extension F/K, where $(s,z) \in \mathrm{SSym}(A_{F\otimes L},\tau)^\times/\approx$.

We investigate different places $v \in X^{(1)}$ in terms of their ramification types. If $L_v := K_v \otimes L$ is a field, then the valuation of v on K_v extends uniquely to L_v (which we denote by $\tilde{v}$); moreover, if the residue field $\kappa(\tilde{v})$ of L_v is a field extension of degree 2 (resp. 1) of the residue field $\kappa(v)$ of K_v, we call such a place *inert* (resp. *ramified*). For an inert place v, the field extension L_v/K_v is unramified. If L_v is not a field, then we have $L_v \simeq K_v \times K_v$, and such a place is called *totally split*.

If v is totally split, then L embeds into K_v. The group H_{K_v} is of inner type, and equals $\mathbf{SL}_1(A_{K_v})$. We can thus apply our results in §4.2.1. In particular, the order of the Rost invariant $R_{H_{K_v}}$ equals the exponent m_v of A_{K_v}. If $[A_{K_v}]$ is unramified at v, we can show that there is a bijection $H^1(K_v,H) \simeq \mathbb{Z}/m_v\mathbb{Z}$ as in Proposition 4.2.5.

Proposition 4.2.10. *For an inert place v such that $[A_{L_v}]$ is unramified at $\tilde{v}$, the image of $2R_H : H^1(K_v,H) \to H^3(K_v,\mathbb{Q}/\mathbb{Z}(2))$ is 0.*

Proof. We use the diagram in Proposition 8.6 of [GMS03] which is commutative with exact rows:

$$\begin{tikzcd}
1 \arrow[r] & H^i(\kappa(\tilde{v}),\mu_m^{\otimes j}) \arrow[r] \arrow[d, "e\cdot\mathrm{Cor}_{\kappa(\tilde{v})/\kappa(v)}"] & H^i(L_v,\mu_m^{\otimes j}) \arrow[r, "\partial_{\tilde{v}}"] \arrow[d, "\mathrm{Cor}_{L_v/K_v}"] & H^{i-1}(\kappa(\tilde{v}),\mu_m^{\otimes(j-1)}) \arrow[r] \arrow[d, "\mathrm{Cor}_{\kappa(\tilde{v})/\kappa(v)}"] & 1 \\
1 \arrow[r] & H^i(\kappa(v),\mu_m^{\otimes j}) \arrow[r] & H^i(K_v,\mu_m^{\otimes j}) \arrow[r, "\partial_v"] & H^{i-1}(\kappa(v),\mu_m^{\otimes(j-1)}) \arrow[r] & 1
\end{tikzcd} \tag{4.7}$$

where the ramification index $e = 1$ in our case. Since $[A_{L_v}]$ is unramified at $\tilde{v}$, it comes from a certain $\bar{\alpha} \in \mathrm{Br}(\kappa(\tilde{v}))$ in the top row of the diagram. The condition $\mathrm{Cor}_{L_v/K_v}([A_{L_v}]) = 0$ then gives $\mathrm{Cor}_{\kappa(\tilde{v})/\kappa(v)}(\bar{\alpha}) = 0$ by taking $i = 2$ and $j = 1$ in the left square of the diagram. Now we compose $2R_H$ with the isomorphism $\partial_v : H^3(K_v, \mathbb{Q}/\mathbb{Z}(2)) \xrightarrow{\simeq} \mathrm{Br}\,\kappa(v)$ and we get

$$\partial_v(\mathrm{Cor}_{L_v/K_v}((z) \cup [A_{L_v}])) = \mathrm{Cor}_{\kappa(\tilde{v})/\kappa(v)}(\partial_{\tilde{v}}((z) \cup [A_{L_v}])) = \mathrm{Cor}_{\kappa(\tilde{v})/\kappa(v)}(\bar{\alpha}^{\tilde{v}(z)}) = 0$$

by taking $i = 3$ and $j = 2$ in the right square of the diagram, where we denote by $\tilde{v}(z)$ the valuation of z with respect to $\tilde{v}$. □

Corollary 4.2.11. *Suppose there is a totally split place v_0 such that $[A_{K_{v_0}}]$ is unramified at v_0 with exponent ≥ 3. Let $S \subseteq X^{(1)}$ be a finite set containing inert places v such that $[A_{L_v}]$ is unramified at $\tilde{v}$. Then*

$$H^1(K, H) \to \bigoplus_{v \in X^{(1)} \setminus S} H^1(K_v, H)$$

is not surjective. As a result, strong approximation away from S does not hold for $Z = E/H$ which is a classifying variety of H.

Proof. Proposition 4.2.5 applies to $H^1(K_{v_0}, H) = H^1(K_{v_0}, \mathbf{SL}_1(A))$ and we know that it is in fact a cyclic group of order equal to the exponent of $[A_{K_{v_0}}]$. Construct

$$(a_v)_v \in \oplus_{v \in X^{(1)} \setminus S} H^1(K_v, H)$$

such that

$$a_{v_0} = [\pi_{v_0}] \in K_{v_0}^\times / \mathrm{Nrd}(A_{K_{v_0}}) \simeq H^1(K_{v_0}, H)$$

for a uniformizer π_{v_0} of K_{v_0} and $a_v = 0$ elsewhere. Then apply Proposition 4.2.4 and we get $f_H(a)$ is of order ≥ 3. For $v \in S$, the images of $H^1(K_v, H)$ under f_H only give elements of order at most 2, and thus cannot cancel out $f_H(a)$. The same diagram chasing argument as in Theorem 4.2.6 then gives the result. □

There are only finitely many places left: those v such that the field extension L_v/K_v is ramified, or $[A_{L_v}]$ is ramified at $\tilde{v}$. In fact, these two conditions cannot hold at the same time.

If L_v/K_v is ramified, then by Lemma 6.3 of [PS22], we have $[A_{L_v}] = \alpha_0 \otimes L_v$ for some unramified $\alpha_0 \in \mathrm{Br}_{\mathrm{nr}}(K_v/K)$. The compatibility between residue maps and restriction maps (cf. Proposition 1.4.6 of [CTS21]) shows that $[A_{L_v}]$ is unramified at $\tilde{v}$. Moreover, the exponent of $[A_{L_v}]$ is at most 2 because

$$0 = \mathrm{Cor}_{L_v/K_v}([A_{L_v}]) = \mathrm{Cor}_{L_v/K_v}(\mathrm{Res}_{L_v/K_v}(\alpha_0)) = 2\alpha_0.$$

Lemma 4.2.12. *For $v \in X^{(1)}$, we have*

$$H^1(K_v, \mu_m) \simeq K_v^\times / K_v^{\times m} = \{\pi^r \delta^s u^t | (r, s, t) \in (\mathbb{Z}/m\mathbb{Z})^3\} \simeq (\mathbb{Z}/m\mathbb{Z})^3$$

where π is a uniformizer for K_v, the element $\delta \in K_v$ is a unit whose image $\bar{\delta} \in \kappa(v)$ is a uniformizer for $\kappa(v)$, and u is a unit in K_v whose image $\bar{u} \in \kappa(v)$ is a unit of order m in $\kappa(v^\times)/\kappa(v)^{\times m}$.

If K_v contains a primitive mth root of unity ω (e.g. when $m|(p-1)$), then we have

$$H^2(K_v, \mu_m) \simeq {}_m\mathrm{Br}(K_v) = \{r(\delta, u)_\omega + s(\pi, \delta)_\omega + t(\pi, u)_\omega | (r, s, t) \in (\mathbb{Z}/m\mathbb{Z})^3\} \simeq (\mathbb{Z}/m\mathbb{Z})^3$$

where $(a, b)_\omega$ is a cyclic algebra of degree m.

Proof. The exact rows in (4.7) are in fact split (cf. Corollary 6.8.8 of [GS17]). Taking $i = 1, j = 1$ gives the first statement, noting that

$$H^1(\kappa(v), \mu_m) \simeq \kappa(v)^\times/\kappa(v)^{\times m} = \{\bar{\delta}^s \bar{u}^t | (s, t) \in (\mathbb{Z}/m\mathbb{Z})^2\} \simeq (\mathbb{Z}/m\mathbb{Z})^2$$

which is obtained by applying the same exact sequence to the complete discretely valued field $\kappa(v)$.

A primitive mth root of unity ω induces an isomorphism $\mu_m \simeq \mathbb{Z}/m\mathbb{Z}$. The group $H^2(\kappa(v), \mu_m) \simeq {}_m\mathrm{Br}(\kappa(v)) \simeq \mathbb{Z}/m\mathbb{Z}$ is then generated by the cyclic algebra $(\bar{\delta}, \bar{u})_{\bar{\omega}}$, where $\bar{\omega}$ denotes the image of ω in the residue field $\kappa(v)$. Take $i = 2, j = 1$ in the split exact sequence (4.7). A retract of the injection ${}_m\mathrm{Br}\,\kappa(v) \hookrightarrow {}_m\mathrm{Br}\,K_v$ is given by the specialization map $s^2_\pi : a \mapsto \partial_v((-\pi) \cup a)$. The calculations

$$s^2_\pi(r(\delta, u)_\omega + s(\pi, \delta)_\omega + t(\pi, u)_\omega) = r(\bar{\delta}, \bar{u})_{\bar{\omega}},$$
$$\partial_v(r(\delta, u)_\omega + s(\pi, \delta)_\omega + t(\pi, u)_\omega) = \bar{\delta}^s \bar{u}^t$$

give the second statement. □

If the field extension L_v/K_v is unramified and $[A_{L_v}]$ has index $n \geq 3$, then by Proposition 6.6 of [PS22], there is a primitive nth root of unity ρ in L_v such that $N_{L_v/K_v}(\rho) = 1$ and $[A_{L_v}] = (\pi, \delta)_\rho$, where π and δ are as in Lemma 4.2.12. In particular, $[A_{L_v}]$ is ramified at $\tilde{v}$. There exists an L_v/K_v-unitary involution τ on $(\pi, \delta)_\rho$ such that τ leaves i and j invariant, where i and j are generators of $(\pi, \delta)_\rho$ such that $i^n = \pi, j^n = \delta, ij = \rho ji$.

Example 4.2.13. Let $X = \mathbf{P}^1_k$ be the projective line over k with function field $K = k(t)$. Let $k' = k[T]/g(T)$ be an unramified quadratic extension of k, such that there is a primitive mth root of unity ρ with $N_{k'/k}(\rho) = 1$ and $m \geq 3$. Examples include $k' = k[T]/(T^2+T+1)$ over $k = \mathbb{Q}_5$ with ρ a primitive 3rd root of unity, and $k' = k[T]/(T^2 - \frac{1+\sqrt{5}}{2}T + 1)$ over $k = \mathbb{Q}_3(\sqrt{-7})$ with ρ a primitive 5th root of unity, etc. Let $L = k'(X_{k'}) = k'(t)$ which is a quadratic extension of K. Let δ be a uniformizer of k, which stays a uniformizer of k' since the extension k'/k is supposed to be unramified. Let $u \in k'$ be a unit with order m in $k'^\times/k'^{\times m}$. Consider the cyclic L-algebra $A = (\delta, u)_\rho$. The non-trivial L/K-automorphism extends to an L/K-unitary involution τ on A. Let $H = \mathbf{SU}(A, \tau)$ and $Z = E/H$ be a classifying variety. The place $v_0 = (g(t))$ is totally split, and $H^1(K_{v_0}, H)$ is cyclic of order m. There are no ramified places or places v where $[A_{L_k}]$ is ramified. Let S be a subset of places v where L_v/K_v are (unramified) field extensions. Then strong approximation away from S does not hold for Z.

4.3 Reciprocity obstruction to strong approximation

We establish a reciprocity obstruction to strong approximation. This obstruction for classifying varieties is related to the degree 3 cohomological invariants. As a result, the failure of strong approximation constructed in Section 4.2 is explained by this reciprocity obstruction. We prove that this reciprocity obstruction to strong approximation is the only one in certain situations, including the case of Example 4.2.8, and classifying varieties of tori. We also explain that for K-tori, this reciprocity obstruction to strong approximation is compatible with the known results in [HI19].

4.3.1 Defining the reciprocity obstruction

Let Z be a smooth geometrically integral K-variety. As a consequence of Gersten's conjecture for étale cohomology proved by Bloch and Ogus (cf. [BO74]), we have:

Theorem-Definition 4.3.1 (cf. Theorem 4.1.1 of [CT95], see also (1.1) of [CT15])**.** The following subgroups of $H^3(K(Z), \mathbb{Q}/\mathbb{Z}(2))$ coincide, that we define to be $H^3_{\mathrm{nr}}(Z, \mathbb{Q}/\mathbb{Z}(2))$:

(1) the group $H^0_{\mathrm{Zar}}(Z, \mathcal{H}^3(\mathbb{Q}/\mathbb{Z}(2)))$ of global sections of the Zariski sheaf $\mathcal{H}^3(Z, \mathbb{Q}/\mathbb{Z}(2))$ which is the sheaf associated to the Zariski presheaf $U \mapsto H^3_{\text{ét}}(U, \mathbb{Q}/\mathbb{Z}(2))$;

(2) the group of elements $\alpha \in H^3(K(Z), \mathbb{Q}/\mathbb{Z}(2))$ that are unramified with respect to any codimension 1 point P of Z, i.e. we have $\partial_{\mathcal{O}_{Z,P}}(\alpha) = 0 \in H^2(\kappa(P), \mathbb{Q}/\mathbb{Z}(1))$ or equivalently α comes from a class in $H^3(\mathcal{O}_{Z,P}, \mathbb{Q}/\mathbb{Z}(2))$;

(3) the group of elements in $H^3(K(Z), \mathbb{Q}/\mathbb{Z}(2))$ which at any point $P \in Z$ come from a class in $H^3(\mathcal{O}_{Z,P}, \mathbb{Q}/\mathbb{Z}(2))$.

Remark 4.3.2. Since we do not suppose Z to be proper, the group $H^3_{\mathrm{nr}}(Z, \mathbb{Q}/\mathbb{Z}(2))$ can be strictly bigger than its subgroup $H^3_{\mathrm{nr}}(K(Z)/K, \mathbb{Q}/\mathbb{Z}(2))$, which is the group of elements in $H^3(K(Z), \mathbb{Q}/\mathbb{Z}(2))$ that are unramified with respect to all discrete valuations of $K(Z)$ trivial on K. Let Z^c be a smooth compactification of Z, then we have $H^3_{\mathrm{nr}}(Z^c, \mathbb{Q}/\mathbb{Z}(2)) = H^3_{\mathrm{nr}}(K(Z)/K, \mathbb{Q}/\mathbb{Z}(2))$. We will see later in Theorem 4.3.10 that for Z in our Construction 4.2.3, the group $H^3_{\mathrm{nr}}(Z, \mathbb{Q}/\mathbb{Z}(2))/\operatorname{Im}(H^3(K, \mathbb{Q}/\mathbb{Z}(2)))$ is of order m which accounts for the defect of strong approximation for Z, but

$$H^3_{\mathrm{nr}}(K(Z)/K, \mathbb{Q}/\mathbb{Z}(2))/\operatorname{Im}(H^3(K, \mathbb{Q}/\mathbb{Z}(2))) = 0$$

since Z is stably birational.

Now we define a pairing

$$Z(\mathbf{A}_K) \times H^3_{\mathrm{nr}}(Z, \mathbb{Q}/\mathbb{Z}(2)) \to \mathbb{Q}/\mathbb{Z} \tag{4.8}$$

in a similar way as in §2.2 of [CTPS16], where their original definition applies to $H^3_{\mathrm{nr}}(K(Z)/Z, \mathbb{Q}/\mathbb{Z}(2))$ instead of $H^3_{\mathrm{nr}}(Z, \mathbb{Q}/\mathbb{Z}(2))$ (see also Lemma 4.1 of [HSS15]). Let $\alpha \in H^3_{\mathrm{nr}}(Z, \mathbb{Q}/\mathbb{Z}(2))$. For any field extension F/K and any point $x \in Z(F)$ with image $P \in Z$, we can lift α to a (unique) element of $H^3(\mathcal{O}_{Z,P}, \mathbb{Q}/\mathbb{Z}(2))$ in virtue of 4.3.1(3), and define $\alpha(x)$ to be its image under the pullback $H^3(\mathcal{O}_{Z,P}, \mathbb{Q}/\mathbb{Z}(2)) \to H^3(F, \mathbb{Q}/\mathbb{Z}(2))$. This defines a pairing

$$\begin{aligned} Z(K_v) \times H^3_{\mathrm{nr}}(Z, \mathbb{Q}/\mathbb{Z}(2)) &\to H^3(K_v, \mathbb{Q}/\mathbb{Z}(2)) = \mathbb{Q}/\mathbb{Z} \\ (x, \alpha) &\mapsto \alpha(x). \end{aligned} \tag{4.9}$$

By shrinking the open subset $U \subseteq X$ over which we have the model $\mathcal{Z} \to U$ of Z, the element α lies in $H^3(\mathcal{O}_{\mathcal{Z},Q}, \mathbb{Q}/\mathbb{Z}(2))$ for all but finitely many codimension 1 points Q of the k-variety $\mathcal{Z}$. By the assumption on α, the exceptional Q lie in finitely many closed fibers $\mathcal{Z}_{v_1} \cdots, \mathcal{Z}_{v_r}$ of $\mathcal{Z} \to U$. Hence for $v \neq v_i$, we have $\alpha \in H^3(\mathcal{O}_{\mathcal{Z},P}, \mathbb{Q}/\mathbb{Z}(2))$ for all points $P \in \mathcal{Z}_v$. Therefore, for x coming from $\mathcal{Z}(\mathcal{O}_v)$, we have $\alpha(x) \in H^3(\mathcal{O}_v, \mathbb{Q}/\mathbb{Z}(2)) = H^3(\kappa(v), \mathbb{Q}/\mathbb{Z}(2)) = 0$, noting that the residue field $\kappa(v)$ has cohomological dimension 2. We can thus sum up these maps to get a well-defined pairing (4.8) for the adelic space.

Remark 4.3.3. In the particular case when Z is proper (so strong approximation is equivalent to weak approximation), the above pairing (4.8) becomes

$$\prod_{v \in X^{(1)}} Z(K_v) \times H^3_{\mathrm{nr}}(K(Z)/K, \mathbb{Q}/\mathbb{Z}(2)) \to \mathbb{Q}/\mathbb{Z}$$

which is exactly the pairing in §4 of [HSS15] giving the reciprocity obstruction to weak approximation.

For a subset $S \subseteq X^{(1)}$, we define the subgroup of "trivial on S" elements:

$$H^3_{\mathrm{nr},S}(Z, \mathbb{Q}/\mathbb{Z}(2)) := \mathrm{Ker}(H^3_{\mathrm{nr}}(Z, \mathbb{Q}/\mathbb{Z}(2)) \to \prod_{v \in S} H^3_{\mathrm{nr}}(Z_{K_v}, \mathbb{Q}/\mathbb{Z}(2))).$$

In virtue of the complex (4.1) of the generalized Weil reciprocity law, the subset $Z(K) \subseteq Z(\mathbf{A}_K^S)$ is orthogonal to $H^3_{\mathrm{nr},S}(Z, \mathbb{Q}/\mathbb{Z}(2))$ with respect to the pairing

$$Z(\mathbf{A}_K^S) \times H^3_{\mathrm{nr}}(Z, \mathbb{Q}/\mathbb{Z}(2)) \to \mathbb{Q}/\mathbb{Z}, \quad ((x_v)_{v \notin S}, \alpha) \mapsto \sum_{v \notin S} \alpha(x_v).$$

The same holds for the closure $\overline{Z(K)}^S$ of $Z(K)$ in $Z(\mathbf{A}_K^S)$ by continuity of the pairing, giving rise to a reciprocity obstruction to strong approximation away from S. Modulo constant elements, we can also replace $H^3_{\mathrm{nr},S}(Z, \mathbb{Q}/\mathbb{Z}(2))$ by

$$\overline{H}^3_{\mathrm{nr},S}(Z, \mathbb{Q}/\mathbb{Z}(2)) := H^3_{\mathrm{nr},S}(Z, \mathbb{Q}/\mathbb{Z}(2)) / \mathrm{Im}(\mathrm{Ker}(H^3(K, \mathbb{Q}/\mathbb{Z}(2)) \to \prod_{v \in S} H^3(K_v, \mathbb{Q}/\mathbb{Z}(2))))$$

which is a subgroup of $\overline{H}^3_{\mathrm{nr}}(Z, \mathbb{Q}/\mathbb{Z}(2)) := H^3_{\mathrm{nr}}(Z, \mathbb{Q}/\mathbb{Z}(2)) / \mathrm{Im}(H^3(K, \mathbb{Q}/\mathbb{Z}(2)))$. We say that *the reciprocity obstruction to strong approximation away from S is the only one* if the subset of elements in $Z(\mathbf{A}_K^S)$ orthogonal to $H^3_{\mathrm{nr},S}(Z, \mathbb{Q}/\mathbb{Z}(2))$ (or equivalently $\overline{H}^3_{\mathrm{nr},S}(Z, \mathbb{Q}/\mathbb{Z}(2))$) equals $\overline{Z(K)}^S$.

Remark 4.3.4. In the study of local-global principal for torsors under tori over K, Harari and Szamuely (cf. [HS16]) used the reciprocity obstruction given by a different pairing

$$Z(\mathbf{A}_K) \times H^3(Z, \mathbb{Q}/\mathbb{Z}(2)) \to \mathbb{Q}/\mathbb{Z}.$$

This pairing is compatible with our pairing (4.8) via a map $H^3(Z, \mathbb{Q}/\mathbb{Z}(2)) \to H^3_{\mathrm{nr}}(Z, \mathbb{Q}/\mathbb{Z}(2))$ that we describe now. By results of Bloch and Ogus ([BO74]), the spectral sequence

$$E_2^{p,q} = H^p_{\mathrm{Zar}}(Z, \mathcal{H}^q(\mu_m^{\otimes j})) \implies H^{p+q}_{\text{ét}}(Z, \mu_m^{\otimes j})$$

gives rise to an exact sequence

$$H^3(Z, \mu_m^{\otimes j}) \to H^0(Z, \mathcal{H}^3(\mu_m^{\otimes j})) \to H^2(Z, \mathcal{H}^2(\mu_m^{\otimes j})) \to H^4(Z, \mu_m^{\otimes 2})$$

which for $j = 2$ can be rewritten as

$$H^3(Z, \mu_m^{\otimes 2}) \to H^3_{\mathrm{nr}}(Z, \mu_m^{\otimes 2}) \to \mathrm{CH}^2(Z)/m \to H^4(Z, \mu_m^{\otimes 2})$$

where $\mathrm{CH}^2(Z)$ denotes the second Chow group of codimension 2 cycles modulo rational equivalence. When $\mathrm{CH}^2(Z)/m$ vanishes, the obstructions given by $H^3(Z, \mu_m^{\otimes 2})$ and $H^3_{\mathrm{nr}}(Z, \mu_m^{\otimes 2})$ are the same. In general, the obstruction given by $H^3_{\mathrm{nr}}(Z, \mathbb{Q}/\mathbb{Z}(2))$ is potentially more restrictive than the obstruction given by $H^3_{\mathrm{nr}}(Z, \mathbb{Q}/\mathbb{Z}(2))$.

Remark 4.3.5. In the literature of strong approximation problems away from S over a number field F, the Brauer-Manin obstruction used to be defined using the projection $\pi_S : Z(\mathbf{A}_F) \to Z(\mathbf{A}_F^S)$, saying that the Brauer-Manin obstruction to strong approximation away from S is the only one if $Z(F)$ is dense in $\pi_S(Z(\mathbf{A})^{\mathrm{Br}})$. Demeio introduced the modified Brauer group of "trivial on S" elements (cf. §6 of [Dem22])

$$\mathrm{Br}_S Z := \mathrm{Ker}(\mathrm{Br}\, Z \to \prod_{v \in S} \mathrm{Br}\, Z_{F_v})$$

and the corresponding Brauer set $Z(\mathbf{A}_F^S)^{\mathrm{Br}_S}$ (for which we can also replace $\mathrm{Br}_S Z$ by $\mathrm{Br}_S Z/\operatorname{Im}(\mathrm{Br}_S F)$). This gives a less restrictive obstruction compared to the one defined using projection, since $\pi_S(Z(\mathbf{A}_F)^{\mathrm{Br}}) \subseteq Z(\mathbf{A}_F^S)^{\mathrm{Br}_S}$; moreover, we have $\overline{Z(F)}^S \subseteq Z(\mathbf{A}_F^S)^{\mathrm{Br}_S}$ while $\pi_S(Z(\mathbf{A}_F)^{\mathrm{Br}})$ can be strictly smaller than $\overline{Z(F)}^S$ (cf. Proposition 4.8 of [Dem22]). We adopt this point of view and we will show later (cf. Theorem 4.3.10 and 4.3.11) that in our case, the group $\overline{H}^3_{\mathrm{nr},S}(Z, \mathbb{Q}/\mathbb{Z}(2))$ of "trivial on S" elements cuts out exactly the closure $\overline{Z(K)}^S$ with respect to the reciprocity obstruction.

4.3.2 Application to classifying varieties

Now we calculate this reciprocity obstruction we just defined for classifying varieties. Let H be an algebraic group over K. Let $Z = E/H$ be a classifying variety where the special rational group E is split semisimple simply connected (for example, $E = \mathbf{SL}_n$ or $\mathbf{Sp}_{2n}$).

The following morphism is defined by evaluating an invariant $I \in \mathrm{Inv}^3(H, \mathbb{Q}/\mathbb{Z}(2))$ at the generic H-torsor $\xi^* E$ (the fiber of the H-torsor $E \to Z$ above the generic point $\xi \in Z$):

$$\mathrm{Inv}^3(H, \mathbb{Q}/\mathbb{Z}(2)) \to H^3(K(Z), \mathbb{Q}/\mathbb{Z}(2)).$$

Rost proved that this map is injective, and more precisely, there is an isomorphism

$$\theta : \mathrm{Inv}^3(H, \mathbb{Q}/\mathbb{Z}(2)) \simeq H^3_{\mathrm{nr}}(Z, \mathbb{Q}/\mathbb{Z}(2)) \tag{4.10}$$

(cf. Part 2, Theorem 3.3 and Part 1, Appendix C of [GMS03]). The decomposition

$$\mathrm{Inv}^3(H, \mathbb{Q}/\mathbb{Z}(2)) = \mathrm{Inv}^3(H, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{norm}} \oplus H^3(K, \mathbb{Q}/\mathbb{Z}(2))$$

then induces an isomorphism

$$\begin{aligned} H^3_{\mathrm{nr},S}(Z, \mathbb{Q}/\mathbb{Z}(2)) \simeq \mathrm{Ker}(\mathrm{Inv}^3(H, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{norm}} \to \prod_{v \in S} \mathrm{Inv}^3(H_{K_v}, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{norm}}) \\ \oplus \mathrm{Ker}(H^3(K, \mathbb{Q}/\mathbb{Z}(2)) \to \prod_{v \in S} H^3(K_v, \mathbb{Q}/\mathbb{Z}(2))) \end{aligned}$$

and modulo the constant elements, we get

$$\overline{H}^3_{\mathrm{nr},S}(Z, \mathbb{Q}/\mathbb{Z}(2)) \simeq \mathrm{Ker}(\mathrm{Inv}^3(H, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{norm}} \to \prod_{v \in S} \mathrm{Inv}^3(H_{K_v}, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{norm}}). \tag{4.11}$$

Proposition 4.3.6. *There is a well-defined pairing*

$$\begin{aligned} Z(\mathbf{A}_K^S) \times \mathrm{Inv}^3(H, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{norm}} &\to \mathbb{Q}/\mathbb{Z} \\ ((x_v)_v, I) &\mapsto \sum_{v \in X^{(1)} \setminus S} \mathrm{Cor}_{\kappa(v)/k}(\partial_v(I_{K_v}(x_v^* E))) \end{aligned} \tag{4.12}$$

induced by evaluating an invariant I at the H-torsors $x_v^* E$ over K_v, where $x_v^* E$ denotes the fiber of the H-torsor $E \to Z$ above the point $x_v \in Z(K_v)$. This pairing (4.12) is compatible with (4.8) in the sense of the commutative diagram:

$$\begin{tikzcd} Z(\mathbf{A}_K^S) & \times & \mathrm{Inv}^3(H, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{norm}} \arrow[r] \arrow[d, "\theta", "\simeq"'] & \mathbb{Q}/\mathbb{Z} \arrow[d, equal] \\ Z(\mathbf{A}_K^S) \arrow[u, equal] & \times & \overline{H}^3_{\mathrm{nr}}(Z, \mathbb{Q}/\mathbb{Z}(2)) \arrow[r] & \mathbb{Q}/\mathbb{Z}. \end{tikzcd}$$

Proof. Let $b \in H^3_{\mathrm{nr}}(Z, \mathbb{Q}/\mathbb{Z}(2))$ and $x \in Z(F)$ be a point over any field extension F/K. We use the description of the isomorphism θ in Appendix A.2, p.461 of [Mer02]. We define the class $b(x) \in H^3(F, \mathbb{Q}/\mathbb{Z}(2))$ as the image of b under the pull-back morphism

$$x^* : H^0_{\mathrm{Zar}}(Z, \mathcal{H}^3(\mathbb{Q}/\mathbb{Z}(2))) \to H^0_{\mathrm{Zar}}(\mathrm{Spec}\, F, \mathcal{H}^3(\mathbb{Q}/\mathbb{Z}(2))) = H^3(F, \mathbb{Q}/\mathbb{Z}(2))$$

with respect to $x : \mathrm{Spec}\, F \to Z$. Thus we get a map

$$\hat{I}_K : Z(K) \to H^3(L, \mathbb{Q}/\mathbb{Z}(2)), \quad x \mapsto b(x)$$

which is exactly how we defined our pairing (4.9). This $\hat{I}_K$ defines an invariant $I \in \mathrm{Inv}^3(H, \mathbb{Q}/\mathbb{Z}(2))$ with $\theta(I) = b$ since the map b is constant on the orbits of the $E(K)$-action on $Z(E)$. Therefore, we have a commutative diagram

$$\begin{array}{ccccc} Z(K_v) & \times & \mathrm{Inv}^3(H, \mathbb{Q}/\mathbb{Z}(2)) & \longrightarrow & \mathbb{Q}/\mathbb{Z} \\ \| & & \simeq \downarrow \theta & & \| \\ Z(K_v) & \times & H^3_{\mathrm{nr}}(Z, \mathbb{Q}/\mathbb{Z}(2)) & \longrightarrow & \mathbb{Q}/\mathbb{Z}. \end{array}$$

Now we sum up these maps and quotient by $H^3(K, \mathbb{Q}/\mathbb{Z}(2))$ and we get the desired result. □

As a result, there is an reciprocity approximation to strong approximation which accounts for our constructions in Section 4.2. Now we study more precise questions of how this reciprocity obstruction controls strong approximation, for example, is this obstruction the only one? For this end, we first need to take a closer took at our p-adic function field K.

Proposition 4.3.7. *Let X be a smooth proper geometrically integral curve over a p-adic field k, with index $I(X)$ and function field K. Let $[A_0] \in \mathrm{Br}\, k$ of exponent m_0 which maps to $[A] \in \mathrm{Br}\, K$ of exponent m. Then*

$$m = \frac{m_0}{\gcd(I(X), m_0)},$$

and we can always choose $[A_1] \in \mathrm{Br}\, k$ of exponent $mI(x)$ which maps to $[A] \in \mathrm{Br}\, K$.

Proof. Since $[A_{K_v}] \in \mathrm{Br}\, K_v$ is unramified, it comes from its evaluation $v^*[A_{K_v}] \in \mathrm{Br}\, \kappa(v)$ at the place v, which is also the image of $[A_0]$ under the restriction map $\mathrm{Br}\, k \to \mathrm{Br}(\kappa(v))$, given by multiplication by $[\kappa(v) : k] = \deg v$ if we identify both $\mathrm{Br}\, k$ and $\mathrm{Br}(\kappa(v))$ with $\mathbb{Q}/\mathbb{Z}$. Therefore, the exponent m_v of $v^*[A_{K_v}]$ equals $\frac{m_0}{\gcd(\deg v, m_0)}$, which is also the exponent of A_{K_v} because of the injection $\mathrm{Br}\, \kappa(v) \hookrightarrow \mathrm{Br}\, K_v$. Each exponent m_v of A_{K_v} divides the exponent m of A, and thus their least common multiple $\mathrm{lcm}(m_v)_{v \in X^{(1)}}$ divides m. On the other hand $\mathrm{lcm}(m_v)_{v \in X^{(1)}}$ annihilates all the $v^*[A_{K_v}]$, and thus $\mathrm{lcm}(m_v)_{v \in X^{(1)}}[A] \in \mathrm{Br}\, X$ vanishes at all the closed points $v \in X^{(1)}$. By Lichtenbaum's duality (4.4), this implies that $\mathrm{lcm}(m_v)_{v \in X^{(1)}}[A] = 0 \in \mathrm{Br}\, X$ and hence $m | \mathrm{lcm}(m_v)_{v \in X^{(1)}}$. Therefore, we have $m = \mathrm{lcm}(m_v)_{v \in X^{(1)}}$. But

$$\mathrm{lcm}(m_v)_{v \in X^{(1)}} = \mathrm{lcm}(\frac{m_0}{\gcd(\deg v, m_0)})_{v \in X^{(1)}} = \frac{m_0}{\gcd(I(X), m_0)},$$

giving the first statement.

In particular, taking $m = 1$ shows that $\mathrm{Ker}(\mathrm{Br}\, k \to \mathrm{Br}\, K)$ is the cyclic subgroup of order $I(X)$ in $\mathrm{Br}\, k \simeq \mathbb{Q}/\mathbb{Z}$. Choose $\alpha \in \mathrm{Ker}(\mathrm{Br}\, k \to \mathrm{Br}\, K)$ of order exactly $I(X)$. Then $[A_1] = [A_0] + \alpha \in \mathrm{Br}\, k$ has exponent

$$\mathrm{lcm}(m_0, I(X)) = \frac{m_0 I(X)}{\gcd(I(X), m_0)} = mI(X)$$

and $[A_1]$ maps to $[A] \in \operatorname{Br} K$ as desired. $\square$

Remark 4.3.8. In fact, the particular case $m = 1$ showing that $\operatorname{Ker}(\operatorname{Br} k \to \operatorname{Br} K)$ has order $I(X)$ gives exactly Theorem 1 of [Roq66] and Theorem 3 of [Lic69], where in the latter paper Lichtenbaum reduced the calculations to the period of X and used the duality (4.5).

Remark 4.3.9. In the proof of Proposition 4.3.7, we see that Lichtenbaum's duality (4.4) can actually give a local-global principle for $\operatorname{Br} K$, i.e. the natural map

$$\operatorname{Br} K \to \prod_{v \in X^{(1)}} \operatorname{Br} K_v$$

is injective. If $\alpha \in \operatorname{Br} K$ vanishes in $\operatorname{Br} K_v$, then α is unramified because the residue map factors through the completion. Hence α lies in $\operatorname{Br} X$ with evaluation $v^*(\alpha) = 0 \in \operatorname{Br} \kappa(v)$ for all places v, and the duality (4.4) shows that $\alpha = 0 \in \operatorname{Br} K$. See also Corollary 10.5.5 and Remark 10.5.7(2) of [CTS21].

Theorem 4.3.10. *Let $Z = E/\mathbf{SL}_1(A)$ as in Construction 4.2.3, with A of exponent m. For the p-adic curve X, suppose $\operatorname{Pic}^0(X)/m = 0$, or equivalently ${}_mH^1(k, \operatorname{Pic} \overline{X}) = 0$ by the Lichtenbaum duality (4.5) (for example $X = \mathbb{P}^1_k$ the projective line satisfies this condition). Then there is an exact sequence of pointed sets*

$$1 \to \overline{Z(K)}^S \to Z(\mathbf{A}_K^S) \to \overline{H}^3_{\mathrm{nr},S}(Z, \mathbb{Q}/\mathbb{Z}(2))^D \to 1$$

for $S \subseteq X^{(1)}$ a non-empty finite set of places. In particular, the reciprocity obstruction to strong approximation away from S is the only one for Z.

The group $\overline{H}^3_{\mathrm{nr},S}(Z, \mathbb{Q}/\mathbb{Z}(2))$ measures the defect of strong approximation away from S for Z, and is finite cyclic of order $\gcd(\frac{I(S)}{I(X)}, m)$ where $I(X)$ (resp. $I(S)$) is the index of X (resp. S). In particular, strong approximation away from S holds for Z if and only of $I(S)/I(X)$ is coprime to m, and such an S always exists, for example S such that $I(S) = I(X)$.

Proof. We claim that there is a commutative diagram as follows, and the right column is an exact sequence:

$$\begin{tikzcd}
E(K) \arrow[r] \arrow[d] & Z(K) \arrow[r] \arrow[d] & H^1(K, H) \arrow[r] \arrow[d] & 1 \\
E(\mathbf{A}_K^S) \arrow[r] & Z(\mathbf{A}_K^S) \arrow[r] \arrow[d] & \bigoplus_{v \in X^{(1)} \setminus S} H^1(K_v, H) \simeq \bigoplus_{v \in X^{(1)} \setminus S} \mathbb{Z}/m_v\mathbb{Z} \arrow[r] \arrow[d, "f_H"', "(n_v)_v \mapsto \Sigma n_v \cdot \deg v / I(X)"] & 1 \\
& \overline{H}^3_{\mathrm{nr},S}(Z, \mathbb{Q}/\mathbb{Z}(2))^D \arrow[r, "h \mapsto h(\frac{m}{\gcd(\frac{I(S)}{I(X)}, m)} R_H)", "\simeq"'] & \mathbb{Z}/\gcd(\frac{I(S)}{I(X)}, m)\mathbb{Z} \arrow[d] & \\
& & 1 &
\end{tikzcd}$$

where $m_v = mI(X)/\gcd(\deg v, mI(X))$, and we identify $\bigoplus_{v\in X^{(1)}\setminus S} H^1(K_v, H)$ again with $\mathbf{P}^1_S(H)$ as explained in Theorem 4.2.6.

First we prove that it holds when we take $S = \emptyset$ and the corresponding $\gcd(\frac{I(S)}{I(X)}, m) = m$.

Definition and exactness of the right column. The Hochschild-Serre spectral sequence

$$H^p(k, H^q(\overline{X}, \mathbb{G}_m)) \implies H^{p+q}(X, \mathbb{G}_m)$$

along with $\mathrm{Br}(\overline{X}) = 0$ gives the exact sequence

$$0 \to \mathrm{Pic}\, X \to H^0(k, \mathrm{Pic}\, \overline{X}) \to \mathrm{Br}\, k \to \mathrm{Br}\, X \to H^1(k, \mathrm{Pic}\, \overline{X}).$$

Since ${}_mH^1(k, \mathrm{Pic}\, \overline{X}) = 0$, the class $[A] \in {}_m\mathrm{Br}\, X$ comes from $\mathrm{Br}\, k$. By Proposition 4.3.7, we can choose $[A_0] \in \mathrm{Br}\, k$ of exponent $mI(X)$ which maps to $[A] \in \mathrm{Br}\, K$. The exponent of A_{K_v} equals $m_v = mI(X)/\gcd(\deg v, mI(X))$ as explained in the proof of Proposition 4.3.7.

Let $([a_v])_v \in \bigoplus_{v\in X^{(1)}} H^1(K_v, H)$ represented by $a_v \in K_v^\times$, and let $z_0 = \sum_{v\in X^{(1)}} n_v \cdot v \in \mathrm{Pic}\, X$ where n_v is the valuation of a_v at v. Then $(n_v)_v \in \bigoplus_{v\in X^{(1)}} \mathbb{Z}/m_v\mathbb{Z}$ corresponds to $([a_v])_v \in \bigoplus_{v\in X^{(1)}} H^1(K_v, H)$ under the isomorphism of Proposition 4.2.5. By Proposition 4.2.4 and Remark 4.2.2, the image of f_H (the map $\Sigma \circ \oplus R_H$ in the diagram (4.6)) is contained in the subgroup ${}_{mI(X)}\mathrm{Br}\, k = \mathbb{Z}/mI(X)\mathbb{Z}$ generated by $[A_0]$, and under this identification $[A_0] = 1 \in \mathbb{Z}/mI(X)\mathbb{Z}$, we have

$$f_H(([a_v])_v) = \deg z_0 \bmod mI(X) = \sum_{v\in X^{(1)}} (n_v \bmod m_v) \cdot \deg v \mod mI(X). \tag{4.13}$$

Note that the sum on the right is indeed well-defined because $mI(X)|(m_v \cdot \deg v)$. The image of the degree map in $\mathbb{Z}/mI(X)\mathbb{Z}$ is the cyclic subgroup of order m generated by $I(X)$. Hence dividing (4.13) by $I(X)$, we can identify the image of f_H with $\mathbb{Z}/m\mathbb{Z}$, taking $I(X)[A_0] \in \mathrm{Br}\, k$ to $1 \in \mathbb{Z}/m\mathbb{Z}$ as indicated in the diagram.

Now suppose $f_H(([a_v])_v) = 0 = \deg z_0/I(X) \bmod m$, or equivalently $\deg z_0[A_0] = 0 \in {}_{mI(X)}\mathrm{Br}\, k$. Therefore $m|(\deg z_0/I(X))$, and $z_0 \in \mathrm{Pic}\, X/m$ has trivial image in $\mathbb{Z}/m\mathbb{Z}$ in the following exact sequence

$$\mathrm{Pic}^0(X)/m \to \mathrm{Pic}\, X/m \xrightarrow{\deg/I(X)} \mathbb{Z}/m\mathbb{Z} \to 0 \tag{4.14}$$

obtained by $\otimes\mathbb{Z}/m\mathbb{Z}$ the exact sequence

$$0 \to \mathrm{Pic}^0(X) \to \mathrm{Pic}\, X \xrightarrow{\deg/I(X)} \mathbb{Z} \to 0.$$

But $\mathrm{Pic}^0(X)/m = 0$ by our assumption, we can thus find a global function $a \in K^\times/K^{\times m}$ such that the valuation of a at v is congruent to n_v modulo m, and a fortiori modulo m_v because $m_v|m$. So the class $[a] \in H^1(K, H)$ has image $([a_v])_v$ under the map $H^1(K, H) \to \oplus_{v\in X^{(1)}} H^1(K_v, H)$. This proves the exactness at the second term.

Given any $l \in \mathbb{Z}/m\mathbb{Z}$, there exists $\sum_{v\in X^{(1)}} l_v \cdot v \in \operatorname{Pic} X/m$ such that

$$\deg(\sum_{v\in X^{(1)}} l_v \cdot v)/I(X) = l \in \mathbb{Z}/m\mathbb{Z}$$

by the exact sequence (4.14). Then $(l_v)_v \in \bigoplus_{v\in X^{(1)}} \mathbb{Z}/m_v\mathbb{Z}$ has image $l = \mathbb{Z}/m\mathbb{Z}$ under the map f_H, proving the exactness at the third term.

Definition and commutativity of the bottom square. By Proposition 4.2.4, we can identify $\overline{H}^3_{\mathrm{nr}}(Z, \mathbb{Q}/\mathbb{Z}(2))^D$ with $(\mathrm{Inv}^3(H, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{norm}})^D$ in the sequence. Since $\mathrm{Inv}^3(H, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{norm}}$ is a finite cyclic group of order m generated by the Rost invariant R_H, we have an isomorphism $(\mathrm{Inv}^3(H, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{norm}})^D \to \mathbb{Z}/m\mathbb{Z}$ by evaluating an element at R_H. Then the formula (4.12), the definition of f_H, and the identification of $\mathbb{Z}/m\mathbb{Z}$ as the cyclic subgroup of $\mathbb{Z}/mI(X)\mathbb{Z}$ generated by $I(X)$ altogether give the commutativity.

Now consider $S \neq \emptyset$. We use the identification (4.11). Under the field extension map, the Rost invariant R_H maps to the Rost invariant $R_{H_{K_v}}$ of order m_v. Therefore, an element $n_0 R_H \in \mathrm{Inv}^3(H, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{norm}}$ is in $\overline{H}^3_{\mathrm{nr},S}(Z, \mathbb{Q}/\mathbb{Z}(2))$ if and only of $m_v | n_0$ for all $v \in S$, or equivalently n_0 is a multiple of

$$\mathrm{lcm}(m_v)_{v\in S} = \mathrm{lcm}(\frac{mI(X)}{\gcd(\deg v, mI(X))})_{v\in S} = \frac{mI(X)}{\gcd(I(S), mI(X))} = \frac{m}{\gcd(\frac{I(S)}{I(X)}, m)},$$

noting that $I(X)|I(S)$. Then

$$\frac{m}{\gcd(\frac{I(S)}{I(X)}, m)} R_H \in \overline{H}^3_{\mathrm{nr},S}(Z, \mathbb{Q}/\mathbb{Z}(2))$$

is a generator of this cyclic group of order $\gcd(\frac{I(S)}{I(X)}, m)$, and evaluation at this generator gives an isomorphism $\overline{H}^3_{\mathrm{nr},S}(Z, \mathbb{Q}/\mathbb{Z}(2))^D \to \mathbb{Z}/\gcd(\frac{I(S)}{I(X)}, m)\mathbb{Z}$.

Any $(n_v)_v \in \bigoplus_{v\in X^{(1)}\setminus S} \mathbb{Z}/m_v\mathbb{Z}$ can be completed by elements from $\bigoplus_{v\in S} \mathbb{Z}/m_v\mathbb{Z}$, whose images under f_H give exactly the subgroup generated by $I(S)/I(X)$ in $\mathbb{Z}/m\mathbb{Z}$. Therefore, the exactness of the right column and also the commutativity follow from the previous case $S = \emptyset$.

Since the special rational group E is split semisimple simply connected, it satisfies strong approximation away from S (cf. Corollary 4.1.3). The rows in the diagram are exact sequences, then a diagram chasing gives the desired result. □

As another application of this reciprocity obstruction to strong approximation, we study classifying varieties of tori.

Theorem 4.3.11. *Let $Z = E/T$ be a classifying variety of a torus T over K a p-adic function field, where the special rational group E is split semisimple simply connected. Then there is an exact sequence of pointed sets*

$$1 \to \overline{Z(K)}^S \to Z(\mathbf{A}_K^S) \to \overline{H}^3_{\mathrm{nr},S}(Z, \mathbb{Q}/\mathbb{Z}(2))^D \tag{4.15}$$

for $S \subseteq X^{(1)}$ a non-empty finite set of places. In other words, the reciprocity obstruction to strong approximation away from S is the only one for Z. If we suppose furthermore that $\mathrm{CH}^2(Z) \to H^0(K, \mathrm{CH}^2(\overline{Z}))$ is surjective, then there is an exact sequence of pointed sets

$$1 \to \overline{Z(K)}^S \to Z(\mathbf{A}_K^S) \to \overline{H}^3_{\mathrm{nr},S}(Z, \mathbb{Q}/\mathbb{Z}(2))^D \to \overline{H}^3_{\mathrm{nr},X^{(1)}}(Z, \mathbb{Q}/\mathbb{Z}(2))^D \to 1. \tag{4.16}$$

Proof. We claim that there is a commutative diagram as follows, and the right column is an exact sequence:

$$\begin{tikzcd}
E(K) \arrow[r] \arrow[d] & Z(K) \arrow[r] \arrow[d] & H^1(K,T) \arrow[r] \arrow[d] & 1 \\
E(\mathbf{A}_K^S) \arrow[r] & Z(\mathbf{A}_K^S) \arrow[r] \arrow[d] & \mathbf{P}^1_S(T) \arrow[r] \arrow[d] & 1 \\
 & \overline{H}^3_{\mathrm{nr},S}(Z, \mathbb{Q}/\mathbb{Z}(2))^D \arrow[r] \arrow[d] & Ш^1_{X^{(1)}\setminus S}(K,T')^D \arrow[d] & \\
 & \overline{H}^3_{\mathrm{nr},X^{(1)}}(Z, \mathbb{Q}/\mathbb{Z}(2))^D \arrow[r] \arrow[d] & Ш^1(K,T')^D \arrow[d] & \\
 & 1 & 1 &
\end{tikzcd} \tag{4.17}$$

where T' denotes the dual torus of T (i.e. the torus whose character group is the cocharacter group of T), and

$$Ш^1_{X^{(1)}\setminus S}(K,T) := \mathrm{Ker}(H^1(K,T) \to \prod_{v \in S} H^1(K_v, T)),$$

$$Ш^1(K,T) := \mathrm{Ker}(H^1(K,T) \to \prod_{v \in X^{(1)}} H^1(K_v, T)).$$

Blinstein and Merkurjev gave a description of $\mathrm{Inv}^3(T, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{norm}}$ by the exact sequence (cf. Theorem B and Lemma 4.2 of [BM13])

$$\begin{aligned} 1 \to \mathrm{CH}^2(Z)_{\mathrm{tors}} \to H^1(K,T') \xrightarrow{\alpha} \mathrm{Inv}^3(T, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{norm}} \\ \to H^0(K, \mathrm{CH}^2(\overline{Z}))/\operatorname{Im}(\mathrm{CH}^2(Z)) \to H^2(K,T'). \end{aligned} \tag{4.18}$$

The morphism α is constructed as follows: for every $a \in H^1(K,T')$, $b \in H^1(F,T)$ and every field extension F/K, the invariant $\alpha(a)$ sends b to $a_F \cup b$ under the cup-product pairing

$$H^1(F,T') \otimes H^1(F,T) \to H^4(F, \mathbb{Z}(2)) \simeq H^3(F, \mathbb{Q}/\mathbb{Z}(2)) \tag{4.19}$$

which is defined using the quasi-isomorphisms

$$T \simeq \widehat{T'} \otimes^{\mathbf{L}} \mathbb{Z}(1)[1], \quad T' \simeq \widehat{T} \otimes^{\mathbf{L}} \mathbb{Z}(1)[1] \tag{4.20}$$

and pairings

$$\widehat{T}' \otimes \widehat{T} \to \mathbb{Z}, \quad \mathbb{Z}(1)[1] \otimes^{\mathbf{L}} \mathbb{Z}(1)[1] \to \mathbb{Z}(2)[2]. \tag{4.21}$$

On the other hand, there is a Poitou-Tate type exact sequence of topological groups established by Harari, Scheiderer and Szamuely (cf. Theorem 0.1 of [HSS15]):

$$\cdots \to H^1(K,T) \to \mathbf{P}^1(T) \to H^1(K,T')^D \to \cdots \tag{4.22}$$

More precisely, the image of the group $H^1(K,T)$ in $\mathbf{P}^1(T)$ is the right kernel of the pairing

$$H^1(K,T') \times \mathbf{P}^1(T) \xrightarrow{\Sigma} \mathbb{Q}/\mathbb{Z}$$

induced by

$$H^1(K_v,T') \times H^1(K_v,T) \to H^3(K_v,\mathbb{Q}/\mathbb{Z}(2)) \tag{4.23}$$

which are defined exactly by using (4.19). Taking T to be T' in (4.22), and noting that any elements of $\mathbf{P}^1_S(T')$ can be completed by 0's into an element of $\mathbf{P}^1(T')$, we get an exact sequence

$$Ш^1_{X^{(1)}\setminus S}(K,T') \to \mathbf{P}^1_S(T') \to H^1(K,T)^D$$

which can be completed into the exact sequence

$$1 \to Ш^1(K,T') \to Ш^1_{X^{(1)}\setminus S}(K,T') \to \mathbf{P}^1_S(T') \to H^1(K,T)^D \tag{4.24}$$

by definition. The pairing (4.23) is actually a perfect duality, and the subgroups $H^1(\mathcal{O}_v,\mathcal{T}) \subseteq H^1(K_v,T)$ and $H^1(\mathcal{O}_v,\mathcal{T}') \subseteq H^1(K_v,T')$ are exact annihilators of each other (Proposition 1.1 and 1.3 of [HSS15]). Therefore, by dualizing (4.24), we get the exactness of the right column in (4.17).

The commutative diagram

$$\begin{array}{ccc} H^1(K,T') & \xrightarrow{\quad\alpha\quad} & \mathrm{Inv}^3(T,\mathbb{Q}/\mathbb{Z}(2)) \\ \downarrow & & \downarrow \\ \prod_{v\in S} H^1(K_v,T') & \xrightarrow{\prod\alpha} & \prod_{v\in S}\mathrm{Inv}^3(T_{K_v},\mathbb{Q}/\mathbb{Z}(2)) \end{array}$$

induces a morphism

$$\begin{aligned} Ш^1_{X^{(1)}\setminus S}(K,T') \to \mathrm{Ker}(\mathrm{Inv}^3(T,\mathbb{Q}/\mathbb{Z}(2))_{\mathrm{norm}} \to \prod_{v\in S}\mathrm{Inv}^3(T_{K_v},\mathbb{Q}/\mathbb{Z}(2))_{\mathrm{norm}}) \\ \simeq \overline{H}^3_{\mathrm{nr},S}(Z,\mathbb{Q}/\mathbb{Z}(2)). \end{aligned} \tag{4.25}$$

Now applying Proposition 4.3.6 and the compatibility between the morphism α (4.19) and the pairing (4.23), we get the commutativity of the two squares at the bottom of (4.17).

Since E satisfies strong approximation away from S (cf. Corollary 4.1.3), chasing the diagram (4.17) gives the desired exactness of (4.15).

Under the further assumption of the surjectivity of $\mathrm{CH}^2(Z) \to H^0(K,\mathrm{CH}^2(\overline{Z}))$, the morphism α is thus surjective by (4.18), and so is the induced morphism (4.25). Hence the two bottom rows in (4.17) are injective. Now a diagram chasing gives the exact sequence (4.16). □

Example 4.3.12. We consider the torus $T' = R_{L/K}(\mathbb{G}_{m,L})/\mathbb{G}_m$ in Example 4.14 of [BM13], where L/K is a degree n field extension induced by a continuous surjective group morphism from the absolute Galois group of K to the symmetric group S_n. For example, the generic maximal torus of the group $\mathbf{PGL}_n$ is of this form. The dual torus T of T' is the norm one torus $R^{(1)}_{L/K}(\mathbb{G}_{m,L})$. Let $Z = E/T$ be a classifying variety. Then $H^0(K, \mathrm{CH}^2(\overline{Z}))/\operatorname{Im}(\mathrm{CH}(Z))$ is trivial, and we have $\overline{H}^3_{\mathrm{nr}}(Z, \mathbb{Q}/\mathbb{Z}(2)) \simeq \mathrm{Br}(L/K)$. Since local-global principle holds for the Brauer group of K (cf. Remark 4.3.9), the group $\overline{H}^3_{\mathrm{nr},X^{(1)}}(Z, \mathbb{Q}/\mathbb{Z}(2))$ is trivial. We apply (4.16) and get the exact sequence of pointed sets

$$1 \to \overline{Z(K)}^S \to Z(\mathbf{A}_K^S) \to \mathrm{Ker}(\mathrm{Br}(L/K) \to \prod_{v\in S} \mathrm{Br}(K_v \otimes L/K_v))^D \to 1.$$

Remark 4.3.13. The exact sequence in Theorem 4.3.10 actually also takes the form (4.16), because

$$\overline{H}^3_{\mathrm{nr},X^{(1)}}(Z, \mathbb{Q}/\mathbb{Z}(2)) \subseteq \overline{H}^3_{\mathrm{nr},S'}(Z, \mathbb{Q}/\mathbb{Z}(2)) = 1$$

for some finite set S' such that $I(S') = I(X)$. Therefore, we may hope this exact sequence to hold in other more general situations.

In fact, the commutative diagram in the proof of Proposition 4.3.6 showing local compatibility is generally true for $\mathrm{Inv}^d(H, \mathbb{Q}/\mathbb{Z}(d-1))$ and $H^d_{\mathrm{nr}}(Z, \mathbb{Q}/\mathbb{Z}(d-1))$ over any field of characteristic 0 (while in characteristic p, we can identify $\mathrm{Inv}^d(H, \mathbb{Q}/\mathbb{Z}(d-1))$ with the subgroup of "balanced elements" in $H^0_{\mathrm{Zar}}(Z, \mathcal{H}^d(\mathbb{Q}/\mathbb{Z}(d-1)))$ under certain conditions, cf. Theorem A of [BM13]). In particular, taking $d = 2$, we get the group $\mathrm{Inv}(H, \mathrm{Br})$ of invariants with values in the Brauer group, and

$$H^0_{\mathrm{Zar}}(Z, \mathcal{H}^2(\mathbb{Q}/\mathbb{Z}(1))) = H^2(Z, \mathbb{Q}/\mathbb{Z}(1)) = \mathrm{Br}\, Z.$$

This enables us to relate $\mathrm{Inv}(H, \mathrm{Br})$ with the Brauer-Manin pairing, which is used for studying the arithmetic of varieties over a number field F, for which we get the following commutative diagram

$$\begin{array}{ccccc} Z(\mathbf{A}_K) & \times & \mathrm{Inv}(H, \mathrm{Br}) & \longrightarrow & \mathbb{Q}/\mathbb{Z} \\ \| & & \downarrow\simeq & & \| \\ Z(\mathbf{A}_K) & \times & \mathrm{Br}\, Z & \longrightarrow & \mathbb{Q}/\mathbb{Z}. \end{array} \tag{4.26}$$

Now we explain how this is compatible with known results of strong approximation over F a number field.

For H a connected linear group over a field K of characteristic 0, Blinstein and Merkurjev (cf. Theorem 2.4 of [BM13]) showed that

$$\mathrm{Pic}\, H \oplus \mathrm{Br}\, K \xrightarrow[\simeq]{\nu_1} \mathrm{Inv}(H, \mathrm{Br}) \simeq \mathrm{Br}\, Z$$

whose construction and proof are based on the exact sequence

$$\mathrm{Pic}\, W \to \mathrm{Pic}\, H \to \mathrm{Br}\, Y \to \mathrm{Br}\, W \tag{4.27}$$

associated to a smooth K-variety Y and a Y-torsor W under H, as defined in Proposition 6.10 of [San81]. In fact, if we use the exact sequence in Theorem 2.8 of [BD13] which is of the same form as (4.27) but with $\operatorname{Pic} H \to \operatorname{Br} Y$ constructed in a different way via the abelian group $\operatorname{Ext}^c_F(H, \mathbb{G}_m)$ of isomorphism classes of central extensions of K-algebraic groups of H by $\mathbb{G}_m$, we can still get an isomorphism

$$\operatorname{Pic} H \oplus \operatorname{Br} K \xrightarrow[\simeq]{\nu_2} \operatorname{Inv}(H, \operatorname{Br}) \simeq \operatorname{Br} Z \tag{4.28}$$

with the same proof as in Theorem A of [BM13].

Now for F a number field, the above diagram (4.26) along with the identification (4.28) gives rise to the following commutative diagram, where the right column is an exact sequence of pointed sets (cf. Corollary 2.5 and Proposition 2.6 of [Kot86]):

$$\begin{tikzcd}
E(F) \arrow[r] \arrow[d] & Z(F) \arrow[r] \arrow[d] & H^1(F, H) \arrow[r] \arrow[d] & 1 \\
E(\mathbf{A}_F) \arrow[r] & Z(\mathbf{A}_F) \arrow[r] \arrow[d] & \mathbf{P}^1(H) \arrow[r] \arrow[d] & 1 \\
 & (\operatorname{Br} Z / \operatorname{Im}(\operatorname{Br} F))^D \arrow[r, "\simeq"] & (\operatorname{Pic} H)^D. &
\end{tikzcd}$$

For $S \subseteq \Omega_F$ such that E satisfies strong approximation away from S (e.g. $E = \mathbf{SL}_n$ with any non-empty finite set S), a diagram chasing argument shows that the Brauer-Manin obstruction to strong approximation away from S (defined using projection and the whole Brauer group, cf. Remark 4.3.5) is the only one for the classifying variety Z. See Theorem 3.7 of [CTX09] where these results were obtained for the first time.

Now we apply Demeio's version of obstruction given by $\operatorname{Br}_S Z$ (see Remark 4.3.5). Let S be a set containing only non-archimedean places and denote by $Z(\mathbf{A}_F)_\bullet$ the adelic points where each archimedean component is collapsed to the (discrete) topological space of its connected components. We have the following commutative diagram

$$\begin{tikzcd}
E(F) \arrow[r] \arrow[d] & Z(F) \arrow[r] \arrow[d] & H^1(F, H) \arrow[r] \arrow[d] & 1 \\
E(\mathbf{A}_F^S)_\bullet \arrow[r] & Z(\mathbf{A}_F^S)_\bullet \arrow[r] \arrow[d] & \mathbf{P}^1_S(H) \arrow[r] \arrow[d] & 1 \\
 & (\operatorname{Br}_S Z / \operatorname{Im}(\operatorname{Br}_S F))^D \arrow[r, "\simeq"] & (\operatorname{Ker}(\operatorname{Pic} H \to \prod_{v \in S} \operatorname{Pic} H_{F_v}))^D &
\end{tikzcd} \tag{4.29}$$

where the bottom row is induced by (4.28).

The right column is an exact sequence of pointed sets, which can be seen by chasing the

following commutative diagram (cf. Theorem 1.2 of [Kot86] for the right-most bijection):

$$\begin{tikzcd}
H^1(F,H) \arrow[r, equal] \arrow[d] & H^1(F,H) \arrow[d] & \\
\mathbf{P}^1_S(H) \arrow[d] & \mathbf{P}^1(H) \arrow[l, two heads] \arrow[r, hook] \arrow[d] & \prod_{v\in S} H^1(F_v,H) \arrow[d, "\simeq"] \\
(\mathrm{Ker}(\mathrm{Pic}\, H \to \prod_{v\in S} \mathrm{Pic}\, H_{F_v}))^D & (\mathrm{Pic}\, H)^D \arrow[l, two heads] \arrow[r, hook] & (\prod_{v\in S} \mathrm{Pic}\, H_{F_v})^D.
\end{tikzcd} \tag{4.30}$$

If $E(F)$ is dense in $E(\mathbf{A}_F^S)_\bullet$ (for example $E = \mathbf{SL}_n$), then chasing the diagram (4.29) gives $\overline{Z(F)}^S = Z(\mathbf{A}_F^S)_\bullet^{\mathrm{Br}_S Z}$, meaning that the Brauer-Manin obstruction to strong approximation away from S is the only one for Z. See Theorem 6.1 and Remark 6.3 of Demeio's work [Dem22] for the general result.

4.3.3 Application to tori

For a torus over a p-adic function field, strong approximation is far from being satisfied. In fact, Harari and Izquierdo (cf. [HI19]) showed that for K the function field of a smooth projective X over any field k of characteristic 0, any finite set $S \subseteq X^{(1)}$, and any K-torus T, the image of $T(K)$ in $T(\mathbf{A}_K^S)$ is a discrete (hence closed) subgroup. In particular, strong approximation never holds for a torus of dimension > 0. The group $A(T) := T(\mathbf{A}_K)/\overline{T(K)}$ measuring the defect of strong approximation is thus the same as $T(\mathbf{A}_K)/T(K)$. With the exact sequence

$$1 \to H^0(K,T)_\wedge \to \mathbf{P}^0(T)_\wedge \to H^2(K,T')^D \to Ш^2(K,T')^D \to 1 \tag{4.31}$$

from the Poitou-Tate type exact sequence in [HSS15], where the pairing

$$\mathbf{P}^0(T) \times H^2(K,T') \to \mathbb{Q}/\mathbb{Z}$$

comes from the cup-products induced by $T \otimes T' \to \mathbb{Z}(2)[2]$ (cf. (4.20) and (4.21)), Harari and Izquierdo described $A(T)$ when K is a p-adic function field:

Theorem 4.3.14 (Corollaire 6.7 and Corollaire 6.9 of [HI19]). *There is an injective morphism*

$$A(T)/\operatorname{Div} \hookrightarrow H^2(K,T')$$

where $A(T)/\operatorname{Div}$ denotes the quotient of $A(T)$ by its maximal divisible subgroup. The closure J of its image fits into an exact sequence

$$1 \to J \to H^2(K,T')^D \to Ш^2(T')^D \to 1.$$

There is a perfect pairing

$$A(T)_{\mathrm{tors}} \times \frac{H^2(K,T')_\wedge}{\mathrm{Im}(Ш^2(K,T'))} \to \mathbb{Q}/\mathbb{Z}.$$

Now we explain that this is compatible with our reciprocity obstruction given by $H^3_{\mathrm{nr}}(T, \mathbb{Q}/\mathbb{Z}(2))$.

Proposition 4.3.15. *There is a morphism $H^2(K, T') \to H^3_{\mathrm{nr}}(T, \mathbb{Q}/\mathbb{Z}(2))$ which fits into the following commutative diagram*

$$\begin{tikzcd}
H^0(K_v, T) \arrow[d, equal] & \times & H^2(K, T') \arrow[r] \arrow[d] & H^3(K_v, \mathbb{Q}/\mathbb{Z}(2)) \arrow[d, equal] \\
T(K_v) & \times & H^3_{\mathrm{nr}}(T, \mathbb{Q}/\mathbb{Z}(2)) \arrow[r, "(4.9)"] & H^3(K_v, \mathbb{Q}/\mathbb{Z}(2)).
\end{tikzcd}$$

Proof. Given an element $a : \operatorname{Spec} K_v \to T$ in $H^0(K_v, T)$, the functoriality of the cup-products induced by $T \otimes T' \to \mathbb{Z}(2)[2]$ gives the following commutative diagram:

$$\begin{tikzcd}
H^0(K_v, T) & \times & H^2(K, T') \arrow[r] \arrow[d, equal] & H^4(K_v, \mathbb{Z}(2)) \\
H^0(T, T) \arrow[u, "a^*"] & \times & H^2(K, T') \arrow[r] & H^4(T, \mathbb{Z}(2)), \arrow[u, "a^*"']
\end{tikzcd}$$

where a^* denotes the pullback maps induced by a. The identity $\mathrm{Id} \in H^0(T, T)$ is mapped to a under a^*, and thus we get the commutativity of the upper part of the following diagram:

$$\begin{tikzcd}
H^0(K_v, T) \arrow[d, equal] & \times & H^2(K, T') \arrow[r] \arrow[d, "\mathrm{Id} \cup -"] & H^4(K_v, \mathbb{Z}(2)) \arrow[d, equal] \\
T(K_v) \arrow[d, equal] & \times & H^4(T, \mathbb{Z}(2)) \arrow[r, "\text{evaluation}"] \arrow[d, "(4.33)"] & H^4(K_v, \mathbb{Z}(2)) \\
T(K_v) & \times & H^3_{\mathrm{nr}}(T, \mathbb{Q}/\mathbb{Z}(2)) \arrow[r, "(4.9)"] & H^3(K_v, \mathbb{Q}/\mathbb{Z}(2)) \arrow[u, "\simeq"']
\end{tikzcd} \tag{4.32}$$

where the morphism $H^2(K, T') \to H^4(T, \mathbb{Z}(2))$ is given by taking the cup-product with $\mathrm{Id} \in H^0(T, T)$.

Now we construct and prove the commutativity of the lower part of this diagram. Recall that (cf. Proposition 2.9 of [Kah12]) for a smooth K-variety Z, we have a natural map $H^4(Z, \mathbb{Z}(2)) \to H^3_{\mathrm{nr}}(Z, \mathbb{Q}/\mathbb{Z}(2))$ defined as follows. Let $\alpha : Z_{\text{ét}} \to Z_{\mathrm{Zar}}$ be the change-of-sites map. The natural map $\mathbb{Q}(2)_{\mathrm{Zar}} \to R\alpha_*\mathbb{Q}(2)$ is an isomorphism in the derived category of Zariski sheaves (cf. Lemma 2.5 and Theorem 2.6 of [Kah12]). Since $\mathbb{Q}(2)_{\mathrm{Zar}}$ is concentrated in degree ≤ 2, we have $R^4\alpha_*\mathbb{Q}(2) = R^3\alpha_*\mathbb{Q}(2) = 0$, hence $R^4\alpha_*\mathbb{Z}(2) \simeq R^3\alpha_*\mathbb{Q}/\mathbb{Z}(2)$. Then the Leray spectral sequence for α yields an edge map

$$H^4(Z, \mathbb{Z}(2) \to H^0_{\mathrm{Zar}}(Z, R^4\alpha_*\mathbb{Z}(2)) \simeq H^0_{\mathrm{Zar}}(Z, R^3\alpha_*\mathbb{Q}/\mathbb{Z}(2)) \overset{4.3.1}{=} H^3_{\mathrm{nr}}(Z, \mathbb{Q}/\mathbb{Z}(2)). \tag{4.33}$$

The construction of the Gersten resolution implies we have the commutative diagram

$$\begin{tikzcd}
H^4(Z, \mathbb{Z}(2)) \arrow[r] \arrow[d, "(4.33)"] & H^4(K(Z), \mathbb{Z}(2)) \\
H^3_{\mathrm{nr}}(Z, \mathbb{Q}/\mathbb{Z}(2)) \arrow[r, hook] & H^3(K(Z), \mathbb{Q}/\mathbb{Z}(2)) \arrow[u, "\simeq"']
\end{tikzcd}$$

where $H^4(Z, \mathbb{Z}(2)) \to H^4(K(Z), \mathbb{Z}(2))$ is the restriction map. This gives the commutativity of the lower part of (4.32). □

Corollary 4.3.16. *The morphism $T(\mathbf{A}_K) \to H^3_{\mathrm{nr}}(T, \mathbb{Q}/\mathbb{Z}(2))^D$ induces injective morphisms*

$$A(T)/\operatorname{Div} \hookrightarrow H^3_{\mathrm{nr}}(T, \mathbb{Q}/\mathbb{Z}(2))^D,$$

$$A(T)_{\mathrm{tors}} \hookrightarrow (H^3_{\mathrm{nr}}(T, \mathbb{Q}/\mathbb{Z}(2))_\wedge)^D.$$

Proof. Proposition 4.3.15 along with (4.31) gives the exact sequence

$$1 \to T(K) \to T(\mathbf{A}_K) \to H^3_{\mathrm{nr}}(T, \mathbb{Q}/\mathbb{Z}(2))^D.$$

Now apply Lemme 2.7 and Théorème 6.6 of [HI19], and we get the first injection.

The morphism constructed in Proposition 4.3.15 gives $H^2(K, T')/m \to H^3_{\mathrm{nr}}(T, \mathbb{Q}/\mathbb{Z}(2))/m$ for all $m > 0$, and thus a morphism $H^2(K, T')_\wedge \to H^3_{\mathrm{nr}}(T, \mathbb{Q}/\mathbb{Z}(2))_\wedge$, compatible with the pairing in Theorem 4.3.14, yielding the second injection. □

4.4 Weak approximation for classifying varieties

In the study of weak approximation problems over K a p-adic function field, we can define a pairing (cf. [HSS15])

$$\prod_{v \in X^{(1)}} Z(K_v) \times H^3_{\mathrm{nr}}(K(Z)/K, \mathbb{Q}/\mathbb{Z}(2)) \to \mathbb{Q}/\mathbb{Z} \tag{4.34}$$

whose left kernel contains the closure $\overline{Z(K)}$ inside $\prod_{v \in X^{(1)}} Z(K_v)$ with respect to the product topology, defining the reciprocity obstruction to weak approximation.

For $Z = E/H$ a classifying variety where the special rational group E is split semisimple simply connected, we can establish the same compatibility between the reciprocity obstruction and the cohomological invariants.

An invariant $I \in \operatorname{Inv}^3(H, \mathbb{Q}/\mathbb{Z}(2))$ is called *unramified* if for every field extension F/K and every element $H^1(F, H)$, we have $I(a)$ in $H^3_{\mathrm{nr}}(F/K, \mathbb{Q}/\mathbb{Z}(2))$ the group of elements unramified with respect to all discrete valuations of F trivial on K. Under the identification (4.10), the subgroup $H^3_{\mathrm{nr}}(K(Z)/K, \mathbb{Q}/\mathbb{Z}(2))$ of $H^3_{\mathrm{nr}}(Z, \mathbb{Q}/\mathbb{Z}(2))$ is identified with the group $\operatorname{Inv}^3_{\mathrm{nr}}(H, \mathbb{Q}/\mathbb{Z}(2))$ of unramified invariants (cf. Proposition 4.1 of [Mer16b]). We denote by $\operatorname{Inv}^3_{\mathrm{nr}}(H, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{norm}}$ the group of unramified normalized invariants.

Proposition 4.4.1. *There is a well-defined pairing*

$$\prod_{v \in X^{(1)}} Z(K_v) \times \operatorname{Inv}^3_{\mathrm{nr}}(H, \mathbb{Q}/\mathbb{Z}(2)) \to \mathbb{Q}/\mathbb{Z}$$

$$((x_v)_v, I) \mapsto \sum_{v \in X^{(1)}} \operatorname{Cor}_{\kappa(v)/k}(\partial_v(I_{K_v}(x_v^* E)))$$

induced by evaluating an invariant I at the H-torsors x_v^*E over K_v, where x_v^*E denotes the fiber of the H-torsor $E \to Z$ above the point $x_v \in Z(K_v)$. This pairing is compatible with (4.34) in the sense of the following commutative diagram

$$\begin{array}{ccccc} \prod_{v\in X^{(1)}} Z(K_v) & \times & \mathrm{Inv}^3_{\mathrm{nr}}(H,\mathbb{Q}/\mathbb{Z}(2))_{\mathrm{norm}} & \longrightarrow & \mathbb{Q}/\mathbb{Z} \\ \| & & \simeq\downarrow\theta & & \| \\ \prod_{v\in X^{(1)}} Z(K_v) & \times & H^3_{\mathrm{nr}}(K(Z)/K,\mathbb{Q}/\mathbb{Z}(2))/\operatorname{Im}(H^3(K,\mathbb{Q}/\mathbb{Z}(2))) & \longrightarrow & \mathbb{Q}/\mathbb{Z}. \end{array} \tag{4.35}$$

Proof. Same as that of Proposition 4.3.6. □

As an application, we show how this method gives answers to weak approximation problems for classifying varieties of tori over K. The following result was also obtained by Linh in a different way in his very recent work (cf. Theorem B of [Lin22]). Now we give our proof using cohomological invariants.

Theorem 4.4.2. *Let $Z = E/T$ be a classifying variety of a torus T over K. Then the reciprocity obstruction to weak approximation is the only one for Z. In fact, there is a morphism*

$$Ш^1_S(T') \to H^3_{\mathrm{nr}}(K(Z)/K,\mathbb{Q}/\mathbb{Z}(2))/\operatorname{Im}(H^3(K,\mathbb{Q}/\mathbb{Z}(2)))$$

such that the subset of elements in $\prod_{v\in S} Z(K_v)$ orthogonal to the image of $Ш^1_S(T')$ in $\frac{H^3_{\mathrm{nr}}(K(Z)/K,\mathbb{Q}/\mathbb{Z}(2))}{\operatorname{Im}(H^3(K,\mathbb{Q}/\mathbb{Z}(2)))}$ with respect to the pairing (4.34) already equals the closure $\overline{Z(K)}$ in the topological product $\prod_{v\in S} Z(K_v)$, where $Ш^1_S(T') := \operatorname{Ker}(H^1(K,T') \to \prod_{v\in X^{(1)}\setminus S} H^1(K_v,T'))$ for a finite set S.

Proof. Blinstein and Merkurjev (cf. Theorem 5.5 of [BM13]) gave a natural isomorphism

$$\begin{aligned} \mathrm{Inv}^3(T_1,\mathbb{Q}/\mathbb{Z}(2)) &\simeq \mathrm{Inv}^3_{\mathrm{nr}}(T,\mathbb{Q}/\mathbb{Z}(2)) \\ I_1 &\mapsto (b \mapsto I_1(g_*(b))) \end{aligned}$$

where T_1 is a flasque torus, fitting into the exact sequence $1 \to T \xrightarrow{g} T_1 \to P \to 1$ with P a quasi-trivial torus (such a resolution for a given T always exists by Lemma 0.6 of [CTS87]). We have its dual exact sequence $1 \to P' \to T_1' \to T' \to 1$ where P' being the dual of a quasi-trivial torus is still quasi-trivial. The induced exact sequence

$$1 \to H^1(K,T_1') \to H^1(K,T') \to H^2(K,P')$$

gives an isomorphism $Ш^1_S(T_1') \simeq Ш^1_S(T')$ because $Ш^2_S(P') = 0$ by Lemma 3.2 of [HSS15]. Consider the composite map

$$\begin{aligned} Ш^1_S(T') \simeq Ш^1_S(T_1') \hookrightarrow H^1(K,T_1') &\xrightarrow[(4.19)]{\alpha} \mathrm{Inv}^3(T_1,\mathbb{Q}/\mathbb{Z}(2))_{\mathrm{norm}} \\ \simeq \mathrm{Inv}^3_{\mathrm{nr}}(T,\mathbb{Q}/\mathbb{Z}(2))_{\mathrm{norm}} &\simeq H^3_{\mathrm{nr}}(K(Z)/K,\mathbb{Q}/\mathbb{Z}(2))/\operatorname{Im}(H^3(K,\mathbb{Q}/\mathbb{Z}(2))) \end{aligned} \tag{4.36}$$

which gives rise to the map at the bottom of the following diagram

$$\begin{tikzcd}
E(K) \arrow[r] \arrow[d] & Z(K) \arrow[r] \arrow[d] & H^1(K,T) \arrow[r] \arrow[d] & 1 \\
\prod_{v\in S} E(K_v) \arrow[r] & \prod_{v\in S} Z(K_v) \arrow[r] \arrow[d] & \prod_{v\in S} H^1(K_v,T) \arrow[r] \arrow[d] & 1 \\
 & \left(\frac{H^3_{\mathrm{nr}}(K(Z)/K,\mathbb{Q}/\mathbb{Z}(2))}{\mathrm{Im}(H^3(K,\mathbb{Q}/\mathbb{Z}(2)))}\right)^D \arrow[r] & Ш^1_S(T')^D. &
\end{tikzcd} \tag{4.37}$$

The right column of the diagram comes from the Poitou-Tate type exact sequence (4.22) and is exact by the same argument as in the proof in Proposition 4.3.11. Now we check that the bottom square of (4.37) is commutative. In virtue of Proposition 4.4.1, it suffices to show the commutativity of the following diagram

$$\begin{tikzcd}
H^1(K_v,T) \arrow[d, equal] & \times & Ш^1_S(T') \arrow[r, "(4.19)"] \arrow[d, "(4.36)"] & \mathbb{Q}/\mathbb{Z} \arrow[d, equal] \\
H^1(K_v,T) & \times & \mathrm{Inv}^3_{\mathrm{nr}}(T,\mathbb{Q}/\mathbb{Z}(2)) \arrow[r] & \mathbb{Q}/\mathbb{Z}.
\end{tikzcd}$$

Let $a \in Ш^1_S(T')$. Let $a_1 \in H^1(K,T'_1)$, $I_1 \in \mathrm{Inv}^3(T_1,\mathbb{Q}/\mathbb{Z}(2))_{\mathrm{norm}}$, $I \in \mathrm{Inv}^3_{\mathrm{nr}}(T,\mathbb{Q}/\mathbb{Z}(2))_{\mathrm{norm}}$ be the images of a through the consecutive maps (4.36). For all $b \in H^1(K_v,T)$, we have

$$I(b) = I_1(g_*(b)) = a_1 \cup g_*(b) = a \cup b$$

where $I(b)$ (resp. $a \cup b$) gives exactly the pairing between b and I (resp. a) with respect to the lower (resp. upper) row, yielding the commutativity.

The special rational group E satisfies weak approximation, then chasing the diagram (4.37) gives the desired result. □

It is still an open question whether weak approximation holds for $Z = \mathbf{SL}_n/H$ where H is a semisimple simply connected group over K a p-adic function field. In case of a negative answer, we might hope to find a reciprocity obstruction using $H^3_{\mathrm{nr}}(K(Z)/K,\mathbb{Q}/\mathbb{Z}(2)) \simeq \mathrm{Inv}^3_{\mathrm{nr}}(H,\mathbb{Q}/\mathbb{Z}(2))$. Merkurjev studied $\mathrm{Inv}^3_{\mathrm{nr}}(H,\mathbb{Q}/\mathbb{Z}(2))$ for all classical semisimple simply connected groups G, and he showed in [Mer02] that $\mathrm{Inv}^3_{\mathrm{nr}}(H,\mathbb{Q}/\mathbb{Z}(2))_{\mathrm{norm}}$ is trivial except for H of type ${}^2\mathsf{A}_{n-1}$, ${}^2\mathsf{D}_3$ or ${}^1\mathsf{D}_4$ under certain conditions, where $\mathrm{Inv}^3_{\mathrm{nr}}(H,\mathbb{Q}/\mathbb{Z}(2))_{\mathrm{norm}} \simeq \mathbb{Z}/2\mathbb{Z}$. Garibaldi computed the remaining cases when H is *exceptional*, i.e. when H is of type G_2, ${}^3\mathsf{D}_4$, ${}^6\mathsf{D}_4$, F_4, E_6, E_7 or E_8, and he showed in [Gar06] that for H a simple simply connected exceptional algebraic group, the group $\mathrm{Inv}^3_{\mathrm{nr}}(G,\mathbb{Q}/\mathbb{Z}(2))_{\mathrm{norm}}$ is $\mathbb{Z}/2\mathbb{Z}$ if H is of type ${}^3\mathsf{D}_4$ with a nontrivial Tits algebra; otherwise $\mathrm{Inv}^3_{\mathrm{nr}}(H,\mathbb{Q}/\mathbb{Z}(2))_{\mathrm{norm}}$ is trivial. It would be interesting to see if weak approximation could fail for $\mathbf{SL}_n/H$ in the case of a non-trivial $\mathrm{Inv}^3_{\mathrm{nr}}(H,\mathbb{Q}/\mathbb{Z}(2))_{\mathrm{norm}}$.

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