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Dynamic Epistemic Logic with Quantification And Normative Systems

THÈSE

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*Je dédie cette thèse
à ma famille.*

献给我的家人

Résumé

Nous vivons dans un monde où l'information est constamment mise à jour. Les logiques dotées d'opérateurs dynamiques sont capables de raisonner sur les changements d'information multi-agents. La logique épistémique dynamique est la logique du changement de connaissances. Par exemple, dans la logique de l'annonce publique, une annonce restreint le domaine aux états où elle est vraie pour représenter les conséquences du changement de connaissances. Une généralisation de la logique épistémique dynamique est l'extension avec quantification. Elle comble un fossé intéressant entre la logique modale propositionnelle et la logique du premier ordre en termes d'expressivité et de décidabilité. La logique des annonces publiques arbitraires étend la logique des annonces publiques en ajoutant la quantification sur toutes les annonces. Elle est plus expressive que les logiques modales et la logique des annonces publiques, mais indécidable.

Dans cette thèse, nous proposons de nouvelles versions de la logique d'annonce publique arbitraire. Tout d'abord, nous pouvons restreindre la quantification de toutes les annonces à des sous-ensembles de celles-ci. Plus précisément, les annonces concernées pourraient n'utiliser qu'un sous-ensemble de toutes les variables propositionnelles ou impliquer une formule donnée. Bien que ces variantes de la logique des annonces publiques arbitraires soient encore indécidables, nous montrons qu'il existe une hiérarchie intéressante de l'expressivité relative de ces variantes. Une autre approche est basée sur l'idée d'empaqueter des opérateurs ensemble. Les opérateurs groupés représentent des notions plus complexes et restreignent également l'expressivité des langages. La notion de "connaissabilité" peut être interprétée comme "il existe un changement d'information possible après lequel l'agent sait quelque chose". Il est naturel de la représenter par l'ensemble d'un opérateur dynamique quantifié et d'un opérateur épistémique. Dans cette thèse, nous proposons des logiques de connaissabilité utilisant de tels opérateurs. Nous montrons que s'il n'y a pas de restriction sur les annonces, alors la logique de connaissabilité est plus expressive que la logique d'annonce publique, mais indécidable. Si nous ne quantifions que les annonces booléennes, alors la logique a la même expressivité que la logique modale propositionnelle. En plus de l'extension avec la quantification, dans cette thèse, nous combinons également la logique épistémique dynamique avec un système normatif. La logique de mise à jour de flèches contient des modalités qui spécifient quelles relations doivent être préservées après la mise à jour par les conditions source et cible. Elle est conçue pour raisonner sur les changements de croyances multi-agents. Nous pouvons également l'utiliser pour représenter la notion de "norme" qui régule les comportements des agents de manière conséquente. Nous proposons une logique normative de mise à jour des flèches qui combine la logique de mise à jour de flèches et la logique temporelle normative. Elle conclut également la combinaison additive, multiplicative et séquentielle des normes. La logique normative de mise à jour de flèches permet de distinguer les façons statiques et dynamiques de considérer les normes comme une logique épistémique dynamique. Nous montrons son expressivité relative par rapport à d'autres logiques apparentées, sa décidabilité et sa complexité.

Mots-clés: logique dynamique, logique d'annonce publique arbitraire, connaissabilité, système normatif, expressivité, décidabilité

Abstract

We live in a world where information is constantly being updated. Logics with dynamic operators are capable of reasoning about multi-agent information change. Dynamic epistemic logic is the logic of knowledge change. For example, in public announcement logic, an announcement restricts the domain to states where it is true to represent the consequences of knowledge change. A generalization of dynamic epistemic logic is the extension with quantification. It bridges an interesting gap between propositional modal logic and first-order logic in expressivity and decidability. Arbitrary public announcement logic extends public announcement logic by adding quantification over all announcements. It is more expressive than modal logics and public announcement logic, but undecidable.

In this thesis, we propose some novel versions of arbitrary public announcement logic. Firstly, we can restrict the quantification from over all announcements to over subsets of them. To be specific, the announcements involved could use only a subset of all propositional variables or implied or implying a given formula. Even though these variants of arbitrary public announcement logic are still undecidable, we show there is an interesting hierarchy of relative expressivity of them. Another approach is based on the idea of packing operators together. Packed operators represent more complex notions and also restrict the expressivity of languages. The notion of “knowability” can be interpreted as “there is a possible information change after which the agent knows something”. It is natural to represent by the package of a quantified dynamic operator and an epistemic operator. In this thesis, we propose logics of knowability using such packed operators. We show that if there is no restriction on announcements, then the logic of knowability is more expressive than public announcement logic, but undecidable; if we only quantify over boolean announcements, then the logic has the same expressivity as propositional modal logic.

Besides the extension with quantification, in this thesis, we also combine dynamic epistemic logic with normative system. Arrow update logic contains modalities that specify which relations should be preserved after updating by source and target conditions. It is designed to reason about multi-agent belief change. We can also use it to represent the notion of “norm” which regulates behaviors of agents in a consequential way. We propose normative arrow update logic which combines arrow update logic and normative temporal logic. It also concludes additive, multiplicative and sequential combination of norms. Normative arrow update logic can distinguish between static and dynamic ways to consider norms as dynamic epistemic logic. We show its relative expressivity with other related logics, its decidability, and complexity.

Keywords: dynamic logic, arbitrary public announcement logic, knowability, normative system, expressivity, decidability

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Introduction

1.1 Background

Epistemic logic provides a formal language and set of rules for reasoning about what someone knows or believes. In epistemic logic, statements about knowledge and belief are represented using special modal operators called *epistemic operators*, such as “it is known that” or “it is believed that”. Modern epistemic logic originated in the 1950s and 1960s with a modal approach. In [108], von Wright introduced the idea of using modal logic to analyze knowledge and belief, and laid the groundwork for the development of epistemic logic. Hintikka’s seminal book *Knowledge and belief: An introduction to the logic of the two notions* [52] is widely viewed as the starting point of modern epistemic logic. Hintikka introduced *possible world semantics* (also known as *Kripke semantics*, see [60]) of epistemic logic under multi-agent scenarios. The *Kripke models* or *relational models* consist of a set of states, for each agent a binary relation called *accessibility relation* between states, and a *valuation* of propositional variables on each state. The properties of the accessibility relation, also called *frame properties*, define different classes of Kripke models. The logic of knowledge is typically interpreted in the class of *equivalence* (reflexive, transitive, and symmetric) models and the logic of belief is interpreted in the class of $S4$ (transitive and symmetric) models. For an agent a and an actual state, the accessible states are called *epistemic alternatives* of a . Intuitively, a considers all epistemic alternatives possible but not distinguishable for him. In this sense, what a knows is true for every epistemic alternative for him.

The possible world semantics of epistemic logic is the way we can determine whether statements about knowledge are true or not. There are some properties of knowledge which are always true on equivalence models and taken as axioms for reasoning about knowledge. Let a be an agent and p be a proposition:

- *Truthfulness*: If a knows p , then p is true;
- *Positive introspection*: If a knows p , then a knows that a knows p ;
- *Negative introspection*: If a does not know p , then a knows that a does not know p .

Those properties are widely accepted, but not undisputed. A classical problem is called *logical omniscience* (named by Hintikka [52]) which says the agents are supposed to be perfect reasoners who know every logical truth. There have been many attempts to solve this problem, like introducing the notion of *impossible worlds* [53] and *awareness* [36]. In the thesis, we still

adopt the possible world semantics as the framework of epistemic logic. The formal definition of epistemic logic is given in Chapter 2. We refer to the textbook [23] for more details on modal logic and textbooks [37, 100] for details on epistemic logic.

Epistemic logic gives a static perspective of interpreting epistemic scenarios at a certain point. However, information is continuously updated and communicated by agents. The basic epistemic logic is not able to model such changes and actions. One approach to modeling how knowledge evolves is to combine epistemic logic and temporal logic (the modal logic of time) in one logic (see [48, 37, 83]). Another approach we will survey is to extend epistemic logic with update modalities which are interpreted by updating a relational model. This type of logics is collectively known as *dynamic epistemic logic* (DEL) [104]. One way to distinguish between these logics is to see which components of the model they each update.

Public announcement logic (PAL) [76, 45] has an update modality called public announcement. After a truthful announcement is broadcast to all agents, the domain of the model is restricted to the states on which that announcement is true. Relations accessing removed states are also deleted. PAL is useful to formalize and solve epistemic problems involving public communications, such as the Muddy Children Puzzle [76].

Arrow update logic (AUL) [57], as its name indicates, has an update modality that makes a restriction on pairs in relations. Intuitively, when a pair in a relation of agent a is removed, it means a can distinguish such pair of states by information he is informed about. Meanwhile, the states are kept and there may be some relations with respect of other agents still linking that pair of states. Under epistemic context, it means this pair of states is still indistinguishable for those agents after a certain information-changing event. Arrow updates are more general than public announcements as the changing information could be private among a group of agents while the event is public. Unlike truthful public announcements, arrow updates could convey false information. Arrow updates may not preserve equivalence models. Therefore, we do not restrict the interpretation of AUL to equivalence models.

Action model logic (AML) [18] provides a generalization of information updates. *Action models* are Kripke-model-like structures in which the domain is a set of actions and a *precondition* is assigned to each action. The updated model in AML is the modal product of the relational model and the action model. The domain of the updated model is a set of pairs in form of (state, action) which means the action can be executed in the state. AML can model private announcements with limited observations by other agents. Similarly, *arrow update model logic* (AUML) [58] has *arrow update models* as a generalization of arrow updates.

1.2 Motivation and related work

From epistemic logic to dynamic epistemic logic, we have a dynamic perspective to reason about knowledge. From PAL to AML, we can model more complex epistemic actions. In this thesis, we focus more on another approach of generalizing DEL, namely by quantifying over epistemic actions. *Arbitrary public announcement logic* (APAL) [16] extends PAL by adding quantifiers over announcements. Quantifiers in APAL are still treated as modalities rather than quantifiers in first-order logic. When one is wondering if it is possible that after an announcement, a given statement is true, he can formalize it as there is a truthful announcement after which that statement is true.

Besides APAL, generalizing DEL with quantifiers over updates has been systematically investigated in recent years. *Arbitrary arrow update logic* (AAUL) [105] extends AUL with

quantifiers over information changing events involving arrow updates. *Arbitrary action model logic* (AAML) [47] adds quantifiers over action models to AML. In *Arbitrary arrow update model logic* (AAUML) [106] there are quantifiers over arrow update models.

Another approach is to investigate variants of APAL by restricting the range of quantifiers. Firstly, note that quantifiers in APAL do not quantify over all announcements in the language of APAL. To avoid circularity in the definition of the language of APAL, quantifiers can only quantify over quantifier-free formulas. Semantically, it has also been shown that the truth of a formula φ after an announcement with quantifiers does not imply that there is a quantifier-free announcement after which φ is true [62].

Group announcement logic (GAL) [1] quantifies over announcements made simultaneously by a group of agents. *Coalition announcement logic* (CAL) [5] quantifies over announcements made jointly by the counterpart of a group of agents. The constraints of quantifier in GAL and CAL are making to subsets of agents. *Positive announcement logic* (APAL⁺) [99] quantifies over positive formulas in which epistemic operators are never bound by negations. *Boolean announcement logic* (BAPAL) [98] quantifies over boolean (propositional) formulas.

Following a similar approach, we will propose several novel variants of APAL in Chapter 3. The first type of variant focuses on restricting propositional variables. Propositional variables are basic logical symbols for formal languages based on propositional logic. A set of propositional variables may be countably infinite. We may define a quantifier as only quantifying over announcements using a subset of propositional variables. To be specific, a subset of propositional variables could be finite (since an announcement could only contain finitely many variables), or only contain propositional variables that occur in the scope of the quantifier (since intuitively variables not occurring in the scope is irrelevant). The second type of variant contains quantifiers over formulas weaker or stronger than a given formula φ . That is to say, there are quantifiers over formulas implying φ or implied by φ . This type of variants focuses on how informative an announcement could be. Besides the intuition of these novel variants of APAL, we also have technical motivation to investigate them.

- Both APAL⁺ and BAPAL are incomparable in expressivity to APAL and more expressive than PAL [99, 98]. Thus constraint on quantifiers does not lead to weakening the expressivity. It should be interesting to compare expressivity between the variants we propose and APAL.
- The satisfiability problem of APAL is known to be undecidable [16]. However, BAPAL is shown to be decidable [98] and APAL⁺ is conjectured to be decidable. It is reasonable to expect some of the novel variants are also decidable.

APAL is also related to the notion of “knowability”. Roughly speaking, knowability is about what is knowable, or what can be known. Fitch’s paradox of knowability [39] is that if all truths are knowable, then all truths are known. Suppose p is an unknown truth, then “ p is an unknown truth” is true. It follows that “ p is an unknown truth” is knowable. When one knows it, one knows that p is true, which makes the sentence false. Therefore, the contradiction implies there is no unknown truth, that is to say, every truth is known. Fitch’s paradox indicates that under the framework of epistemic and modal logic, it is hard to formalize the notion of knowability. In [93], van Benthem suggested to use dynamic logic to interpret knowability. According to this view, “a statement φ is knowable” means φ is known after some announcement. It can be formalized by the language of APAL as “there is an announcement after which φ is known”. Another approach of knowability logic is based on relation restricting

update [109]. If we regard APAL as a logic of knowability, it may contain too many elements. For example, [97] proposed a knowability logic based on APAL without public announcements.

In chapter 4 we will propose some logics of knowability with an operator as “knowability operator”. Semantically, the truth condition of the knowability operator is identical to the combination of an existential quantifier and an epistemic operator. That is to say, in this logic of knowability, we also interpret “ φ is knowable” as there is an announcement after which φ is known. The knowability operator will be useful to investigate the properties of the knowability. Those logics of knowability are different by omitting quantifier, epistemic operator and public announcements respectively. They can be viewed as variants of APAL as well. The combined operator and distinction between these logics are not trivial, especially technically. The expressivity, axiomatization and decidability can change a lot.

For the third and last topic in this thesis, we explore a broader field of update logic than the epistemic context. Recall that in AUL, relations are updated by arrow updates. In general relational models, we can treat a pair in relations as a *transition* as in temporal logic. An arrow update (source, agent, target) can also be interpreted as an *action* by an agent. The agent does an action that changes such relation pairs. Thus arrow updates on transitions can be seen as restrictions on actions. In other words, arrow updates can be applied to characterize what agents should do, could do, or are allowed to do. Such notions are called *norms, rules, or laws*. The logics and systems reasoning about these notions are called *normative systems* or *deontic logic* (see [42, 70] for an overview). We will take *computation tree logic* (CTL) [31] as the framework of normative systems, in which sequential actions are taken on the branching timelines. CTL has temporal modalities that can range over arbitrary modal depth, thus statements like “ φ is *always* true” (φ is true on every state of every path) or “ φ will eventually be true” (there is a path on which φ is true at some point) can be formalized. As mentioned before, temporal logic is used to model information change as well. It is dynamic in the perspective of the flow of time, but static in terms of model updating. *Normative temporal logic* (NTL) [3] is a logic of normative systems based on CTL. Norms in NTL are interpreted as subsets of relations. Therefore, norms give constraints on actions.

In Chapter 5, we will investigate *normative arrow update logic* (NAUL) which was first proposed in [63]. In NAUL norms are interpreted as arrow updates. The source in an arrow update determines where norms should be applied; The target checks which transitions are compliant for the objective group of agents. Such settings allow interaction between formulas and norms (the source and target in an arrow update are formulas). More importantly, norms are explicit. What actions are allowed is formally defined, and we can even combine norms to compose more complex norms. For example, we can combine norms in an *additive* way so that agents can choose between norms; we can also combine norms in a *multiplicative* way so that agents must follow all of those norms; lastly, it is possible to combine norms in a *sequential* way so that agents follow given norms in order of priority.

Some technical results of NAUL have been shown in [63], such the relative expressivity of NAUL comparing to CTL and AUL and the complexity of the model checking problem. We are interested in whether the satisfiability problem of NAUL is decidable and its complexity.

1.3 Contribution and Overview

In this thesis, we propose several novel variants of existing dynamic epistemic logics and normative systems. In general, they can be viewed as extensions of DEL. Technically, we will

focus on logical properties, expressivity, axiomatization, and decidability of these logics. The structure and main contributions of the thesis are summarized as follows:

- **Chapter 2:** We introduce some formal definitions throughout this thesis. We also introduce formal definitions of some background logics mentioned above, such as epistemic logic, PAL, APAL, AUL, and CTL.
- **Chapter 3:** We propose several novel variants of APAL by restricting the range of quantifiers. We first introduce the syntax and semantics of those logics, then we show some valid formulas as logical properties of these logics. The main effort in this chapter is to show relative expressivity results between these novel variants, PAL and APAL. We give a map of the expressivity hierarchy of those logics. We also show axiomatization and decidability results which are very similar to the case of APAL.
- **Chapter 4:** We introduce three different logics for knowability. They all have the knowability operator, but quantifiers, public announcements, and epistemic operator are omitted respectively. We define the languages and semantics of those logics. We show some logical properties like validity. We show the relative expressivity of these of logic. To be specific, the logic of knowability without quantifiers is more expressive than PAL in the multi-agent case and equally expressive as PAL in the single-agent case. We give an axiomatization of that logic, and show the system is sound and complete. The logic of knowability without public announcements or epistemic operators is equally expressive as propositional logic.
- **Chapter 5:** We introduce normative arrow update logic. We give the formal definition of its language and semantics. The main contribution of this chapter is to show that the satisfiability problem of NAUL is decidable, by a tableau method.

The material that forms the main body of this dissertation is based on collaborations with various people: Chapter 3 is based on a joint paper with Hans van Ditmarsch, Louwe Kuijer and Igor Sedlar [102]. Chapter 4 is based on a joint paper with Jie Fan, Hans van Ditmarsch and Louwe Kuijer [65]. Chapter 5 is joint work with Hans van Ditmarsch and Louwe Kuijer.

2

Preliminaries

In this chapter, we will introduce some concepts and notions of modal logic. We will also introduce several existing logics formally.

2.1 Basic notions

2.1.1 Formal language

We study logic by formal language composed of symbols. For each specific logic, we set a finite set \mathbf{A} of agents and a countably infinite set \mathbf{P} of propositional variables as *signature*. A *well-formed formula* of a logic L is a string of symbols which can be identified by the syntactic definition of L . We often use p, q, r, \dots to denote propositional variables; a, b, c, \dots to denote agents; $\varphi, \psi, \chi, \dots$ to denote formulas; and \mathcal{L}_L to denote the set of all well-formed formulas. As our most basic logic, we use propositional modal logic.

Definition 1 (Language of propositional modal logic \mathcal{L}_{ML}). Given \mathbf{P} and \mathbf{A} , the language of propositional modal logic \mathcal{L}_{PL} is defined as:

$$\varphi ::= p \mid \neg\varphi \mid (\varphi \wedge \psi) \mid \Box_a\varphi$$

where $p \in \mathbf{P}, a \in \mathbf{A}$. We also use $\perp, \top, \varphi \rightarrow \psi, \varphi \vee \psi, \varphi \leftrightarrow \psi$ as the abbreviation of $\varphi \wedge \neg\varphi, \neg(\varphi \wedge \neg\varphi), \neg(\varphi \wedge \neg\psi), \neg(\neg\varphi \wedge \neg\psi), \neg(\varphi \wedge \neg\psi) \wedge \neg(\psi \wedge \neg\varphi)$ respectively, and $\Diamond_a\varphi$ as the abbreviation of $\neg\Box_a\neg\varphi$.

\mathcal{L}_{ML} is the most basic language in this thesis. Languages of the other logics can be viewed as variants or expansions of \mathcal{L}_{ML} . Generally speaking, a logic is a set of well-formed formulas. Semantically, it is the set of formulas that are always *true*; syntactically, it is the set of formulas that can be *proved*. We will use ML as an example to give the formal definition of a logic in both semantic and syntactical approaches.

2.1.2 Model

We use *relational models* also known as *Kripke models* [59] as the framework to determine the truth value of formulas in this thesis.

Definition 2. A model \mathcal{M} is a triple $\mathcal{M} = (S, R, V)$ where

- S is a domain of states;
- $R : \mathbf{A} \rightarrow S \times S$ is a binary relation on S ;
- $V : \mathbf{P} \rightarrow \mathcal{P}(S)$ is a valuation function where $\mathcal{P}(S)$ is the power set of S .

Given a model \mathcal{M} , we may refer to its domain relations and valuations as $S^{\mathcal{M}}, R_a^{\mathcal{M}}, V^{\mathcal{M}}$ respectively. The class of all Kripke models is denoted by \mathcal{K} . A model \mathcal{N} is a *submodel* of \mathcal{M} , notation $\mathcal{N} \subseteq \mathcal{M}$, if $S^{\mathcal{N}} \subseteq S^{\mathcal{M}}$, $R_a^{\mathcal{N}} = R_a^{\mathcal{M}} \cap (S^{\mathcal{N}} \times S^{\mathcal{N}})$ for all $a \in \mathbf{A}$, and $V^{\mathcal{N}}(p) = V^{\mathcal{M}}(p) \cap S^{\mathcal{N}}$. A *pointed model* (\mathcal{M}, s) is a model \mathcal{M} with a designated state s in S .

The class of all Kripke models is denoted by \mathcal{K} . The class of *equivalence* models in which R_a is an equivalence relation for each $a \in \mathbf{A}$ is denoted by $\mathcal{S5}$. We interpret formulas on pointed models. Given a model $\mathcal{M} = (S, R, V)$, the semantics of a logic L defines if a formula $\varphi \in \mathcal{L}$ is true on (\mathcal{M}, s) , written as $\mathcal{M}, s \models \varphi$.

Definition 3 (Semantics of propositional modal logic). Given $\mathcal{M} = (S, R, V)$, $s \in S$, $\varphi \in \mathcal{L}_{\text{ML}}$, $\mathcal{M}, s \models \varphi$ is defined as:

$$\begin{array}{lll} \mathcal{M}, s \models p & \text{iff} & s \in V(p) \\ \mathcal{M}, s \models \neg\varphi & \text{iff} & \text{not } \mathcal{M}, s \models \varphi \\ \mathcal{M}, s \models \varphi \wedge \psi & \text{iff} & \mathcal{M}, s \models \varphi \text{ and } \mathcal{M}, s \models \psi \\ \mathcal{M}, s \models \Box_a\varphi & \text{iff} & \text{for all } t \in S, sR_at \text{ implies } \mathcal{M}, t \models \varphi \end{array}$$

If φ is not true on (\mathcal{M}, s) , we write $\mathcal{M}, s \not\models \varphi$. If φ is true on (\mathcal{M}, s) , we also say (\mathcal{M}, s) *satisfies* φ . We say φ is *satisfiable* if there is some pointed model satisfying φ . If φ is true on all pointed models of \mathcal{M} , we say φ is true on \mathcal{M} or \mathcal{M} satisfies φ , written as $\mathcal{M} \models \varphi$. Given a class of models \mathcal{C} , we say φ is *valid* on \mathcal{C} if φ is true on every model in \mathcal{C} , written as $\models_{\mathcal{C}} \varphi$.

Given two pointed models (\mathcal{M}, s) and (\mathcal{N}, t) , and a logic L with language \mathcal{L}_L , $\mathcal{M}, s \equiv_L (\mathcal{N}, t)$ denotes: for all $\varphi \in \mathcal{L}_L$, $\mathcal{M}, s \models \varphi$ if and only if $\mathcal{N}, t \models \varphi$.

If there is some $\varphi \in \mathcal{L}_L$ such that $\mathcal{M}, s \models \varphi$ while $\mathcal{N}, t \not\models \varphi$, then we say φ or L can *distinguish* (\mathcal{M}, s) and (\mathcal{N}, t) . Apparently, if $(\mathcal{M}, s) \equiv_L (\mathcal{N}, t)$, then L cannot distinguish (\mathcal{M}, s) and (\mathcal{N}, t) . We determine the *expressivity* of a logic by how much it can distinguish between models. We can compare the expressivity between logics in terms of their distinguishability.

Definition 4. Let L and L' be two logics that are interpreted over relational models.

- L is *at least as expressive as* L' , notation $L' \preceq L$ if and only if for $\varphi \in \mathcal{L}'_{L'}$, there is a $\psi \in \mathcal{L}_L$ such that φ is equivalent to ψ ;
- L and L' are *equally expressive*, if and only if $L \preceq L'$ and $L' \preceq L$;
- L is *less expressive than* L' , notation $L \prec L'$ if and only if $L \preceq L'$ but $L' \not\preceq L$;
- L and L' are *incomparable*, notation $L \asymp L'$ if and only if $L \not\preceq L'$ and $L' \not\preceq L$.

Bisimulation [90] is a well-known notion of structural similarity. We will use this notion to show some expressivity results.

Definition 5 (Bisimulation). Let \mathcal{M} and \mathcal{N} be two models. A non-empty relation $Z \subseteq S^{\mathcal{M}} \times S^{\mathcal{N}}$ is a *bisimulation* between \mathcal{M} and \mathcal{N} if for all $Zst, a \in \mathbf{A}, p \in \mathbf{P}$,

atom : $s \in V^{\mathcal{M}}(p)$ if and only if $t \in V^{\mathcal{N}}(p)$;

forth : if $(s, s') \in R_a^{\mathcal{M}}$, then there is a $t' \in S^{\mathcal{N}}$ such that $(t, t') \in R_a^{\mathcal{N}}$ and $(s', t') \in Z$;

back : if $(t, t') \in R_a^{\mathcal{N}}$, then there is a $s' \in S^{\mathcal{M}}$ such that $(s, s') \in R_a^{\mathcal{M}}$ and $(s', t') \in Z$;

If there is a bisimulation Z between \mathcal{M} and \mathcal{N} , we write $\mathcal{M} \Leftrightarrow \mathcal{N}$. Two pointed models (\mathcal{M}, s) and (\mathcal{N}, t) are *bisimilar* if there is a bisimulation Z between \mathcal{M} and \mathcal{N} containing (s, t) , written as $(\mathcal{M}, s) \Leftrightarrow (\mathcal{N}, t)$. The following theorem shows if two pointed models are bisimilar, then they satisfies the same formulas of ML. The proof is known from the literature [23].

Theorem 1. *Let (\mathcal{M}, s) and (\mathcal{N}, t) be two pointed models. For all $\varphi \in \mathcal{L}_{ML}$, if $(\mathcal{M}, s) \Leftrightarrow (\mathcal{N}, t)$, then $(\mathcal{M}, s) \equiv_{ML} (\mathcal{N}, s)$.*

The converse of Theorem 1 does not hold in general, but it holds in restricted cases such as on image-finite models [23].

2.1.3 Axiomatization

An *axiomatization* is a syntactical characterization of a logic. It gives a set of formulas as axioms and a set of rules by which the other formulas in the logic can be derived from axioms. We also call an axiomatization a *Hilbert-system* or just *system*.

A *derivation* of a system \mathbf{X} is a sequence of formulas $\varphi_1, \varphi_2, \dots, \varphi_n$ where for any $i \in [1, n]$, φ_i is either an axiom of \mathbf{X} or the application of a rule of \mathbf{X} on several φ_j with $j < i$. If there is a derivation in \mathbf{X} that contains φ , then φ is a *theorem* or *provable* with respect to \mathbf{X} , written as $\vdash_{\mathbf{X}} \varphi$. We use $\mathbf{Th}(\mathbf{X})$ to denote the set of theorems of \mathbf{X} . The most basic system of modal logic is \mathbf{K} . It is the foundation of the other systems of modal logic.

Definition 6. The system \mathbf{K} consists the axioms of TAUT and K and is closed under the the rules of MP and NEC:

TAUT	all substitution instances of propositional tautologies
K	$\Box_a(\varphi \rightarrow \psi) \rightarrow (\Box_a\varphi \rightarrow \Box_a\psi)$
MP	from φ and $\varphi \rightarrow \psi$ infer ψ
NEC	from φ infer $\Box_a\varphi$

A system containing \mathbf{K} is called a *normal* modal system. \mathbf{K} is the weakest normal system. It means it has the least theorems. We may expand \mathbf{K} by adding new axioms, such as:

D	$\Box_a\varphi \rightarrow \Diamond_a\varphi$
T	$\Box_a\varphi \rightarrow \varphi$
B	$\varphi \rightarrow \Box_a\Diamond_a\varphi$
4	$\Box_a\varphi \rightarrow \Box_a\Box_a\varphi$
5	$\Diamond_a\varphi \rightarrow \Box_a\Diamond_a\varphi$

There are some common normal modal systems:

K	TAUT + MP + NEC + K
D	K + D
T	K + T
B	T + B
S4	T + 4
S5	T + B + 4 (or S4 + E)

A system \mathbf{X} is *sound* with respect to a class of models \mathcal{C} if for any theorem $\varphi \in \mathbf{Th}(\mathbf{X})$, φ is valid on \mathcal{C} , that is $\models_{\mathcal{C}} \varphi$. \mathbf{X} is *complete* with respect to a class of models \mathcal{C} if any valid formula φ on \mathcal{C} , $\vdash_{\mathbf{X}} \varphi$.

Theorem 2. [59, 60] \mathbf{K} is sound and complete with respect to all Kripke models. \mathbf{D} , \mathbf{T} , \mathbf{B} , $\mathbf{S4}$ and $\mathbf{S5}$ are sound and complete with respect to serial, reflexive, symmetric, transitive and equivalence class of models respectively.

2.1.4 Decision Problem

A problem is *decision problem* if it has a yes/no answer. Given a logic L , we are interested in the following decision problem:

Satisfiability problem:

Input an \mathcal{L}_L -formula φ .

Output YES if and only if there is a pointed model (\mathcal{M}, s) such that $\mathcal{M}, s \models \varphi$

A decision problem is *decidable* if it can be solved by computation. The concept of computation is interpreted by mathematical models, such as *Turing machine*[89]. This is known as *Church's Thesis* [29]. One can refer to [64] for an overview of the theory of computation. For decidable problems, the *complexity* determines how much *time* and *space* we need to solve them. We classify problems into different complexity classes.

Definition 7. Let f and g be functions from \mathbb{N} to \mathbb{N} . If there are positive constants c and k such that for all $n > k$, $f(n) \leq c \cdot g(n)$, then we say $f(n) = \mathcal{O}(g(n))$

- If a problem of input size n can be solved in $\mathcal{O}(f(n))$ time, then it is in the complexity class $\text{DTIME}(t(n))$.
- If a problem of input size n can be solved in $\mathcal{O}(f(n))$ time by a non-deterministic turing machine, then it is in the complexity class $\text{NTIME}(t(n))$.
- If a problem of input size n can be solved in $\mathcal{O}(f(n))$ space, then it is in the complexity class $\text{DSpace}(t(n))$.
- If a problem of input size n can be solved in $\mathcal{O}(f(n))$ space by a non-deterministic turing machine, then it is in the complexity class $\text{NSpace}(t(n))$.

•

$$\begin{aligned}
 \mathbf{P} &= \bigcup_{k>0} \text{DTIME}(n^k) \\
 \mathbf{NP} &= \bigcup_{k>0} \text{NTIME}(n^k) \\
 \text{EXPTIME} &= \bigcup_{k>0} \text{DTIME}(2^{\mathcal{O}(n^k)}) \\
 \text{NEXPTIME} &= \bigcup_{k>0} \text{NTIME}(2^{\mathcal{O}(n^k)}) \\
 \mathbf{PSPACE} &= \bigcup_{k>0} \text{DSPACE}(n^k) \\
 \mathbf{NPSPACE} &= \bigcup_{k>0} \text{NSPACE}(n^k) \\
 \text{EXPSpace} &= \bigcup_{k>0} \text{DSPACE}(2^{\mathcal{O}(n^k)}) \\
 \text{NEXPSpace} &= \bigcup_{k>0} \text{NSPACE}(2^{\mathcal{O}(n^k)})
 \end{aligned}$$

The relation of fundamental complexity classes is as follows (see [74]):

$$\mathbf{P} \subseteq \mathbf{NP} \subseteq \mathbf{PSPACE} \subseteq \text{EXPTIME} \subseteq \text{EXPSpace}$$

Let \mathbf{C} be a class of problems. A problem L is \mathbf{C} -complete if every problem in \mathbf{C} is polynomial time reducible to L and $L \in \mathbf{C}$.

2.2 Logics as background knowledge

2.2.1 Epistemic logic

Epistemic logic (EL) reasons about knowledge under multi-agent scenarios. It works on equivalence models. We use knowledge operators K_a which stands for “a knows”. Through this thesis, in the context of epistemic knowledge, we use K instead of \Box as the necessity modality.

Definition 8 (Language of epistemic logic \mathcal{L}_{EL}). Given \mathbf{P} and \mathbf{A} , the language of epistemic logic \mathcal{L}_{EL} is defined as:

$$\varphi ::= p \mid \neg\varphi \mid (\varphi \wedge \psi) \mid K_a\varphi$$

where $p \in \mathbf{P}, a \in \mathbf{A}$. We use $\hat{K}_a\varphi$ as the abbreviation of $\neg K_a\neg\varphi$.

For every agent a , $K_a\varphi$ means “agent a knows φ ”. We use \sim_a to denote equivalence relations of agent a . An *epistemic model* is a tuple $\mathcal{M} = (S, \sim, V)$. The semantics of EL is the same as Def. 3, except replacing \Box_a by K_a . For example, the clause of modality is defined as follows:

$$\mathcal{M}, s \models K_a\varphi \quad \text{iff} \quad \text{for all } t \in S, s \sim_a t \text{ implies } \mathcal{M}, t \models \varphi$$

Intuitively, states here are possible states of worlds. Given an agent a at a state s , the equivalence relation $s \sim_a t$ ensures all successors of s are indistinguishable for a . Thus, a knows φ if and only if φ is true on all states he considers possible.

S5 is the system of EL. $\mathbf{T} := K_a\varphi \rightarrow \varphi$ can be interpreted as “if a knows φ , then φ is true”; $\mathbf{4} := K_a\varphi \rightarrow K_aK_a\varphi$ means “if a knows φ , then a knows that he knows φ ”; $\mathbf{E} := \neg K_a\varphi \rightarrow K_a\neg K_a\varphi$ means “if a does not know φ , then he knows that he does not know φ ”. By Theorem 2, we know that **S5** is sound and complete with respect to the class of equivalence models.

2.2.2 Public announcement logic

Public announcement logic (PAL) was developed by [75] and independently by [45]. PAL adds public announcements to multi-agent epistemic logic. The information updating restricts epistemic models to states where the announcement is true.

Definition 9 (Language of PAL \mathcal{L}_{PAL}). Given \mathbf{A} and \mathbf{P} , the language of public announcement logic \mathcal{L}_{PAL} is defined as:

$$\varphi ::= p \mid \neg\varphi \mid (\varphi \wedge \psi) \mid K_a\varphi \mid [\varphi]\varphi$$

where $a \in \mathbf{A}$ and $p \in \mathbf{P}$. We use $\langle\psi\rangle\varphi$ to denote $\neg[\psi]\neg\varphi$.

Formula $[\psi]\varphi$ stands for “after announcing ψ , φ is true”. Formula $\langle\psi\rangle\varphi$ stands for “after some truthful announcement of ψ , φ is true”.

Definition 10 (Semantics of PAL). Given $\mathcal{M} = (S, \sim, V)$, $s \in S$, $\varphi \in \mathcal{L}_{\text{PAL}}$, $\mathcal{M}, s \models \varphi$ is defined as:

$$\begin{array}{lll} \mathcal{M}, s \models p & \text{iff} & s \in V(p) \\ \mathcal{M}, s \models \neg\varphi & \text{iff} & \mathcal{M}, s \not\models \varphi \\ \mathcal{M}, s \models \varphi \wedge \psi & \text{iff} & \mathcal{M}, s \models \varphi \text{ and } \mathcal{M}, s \models \psi \\ \mathcal{M}, s \models K_a\varphi & \text{iff} & \text{for all } t \in S, s \sim_a t \text{ implies } \mathcal{M}, t \models \varphi \\ \mathcal{M}, s \models [\psi]\varphi & \text{iff} & \mathcal{M}, s \models \varphi \text{ implies } \mathcal{M}|\psi, s \models \varphi \end{array}$$

where $\mathcal{M}|\varphi = (S', \sim', V')$ is such that $S' = \llbracket\varphi\rrbracket_{\mathcal{M}} = \{s \in S \mid \mathcal{M}, s \models \varphi\}$, $\sim'_a = \sim_a \cap (\llbracket\varphi\rrbracket_{\mathcal{M}} \times \llbracket\varphi\rrbracket_{\mathcal{M}})$, and $V'(p) = V(p) \cap \llbracket\varphi\rrbracket_{\mathcal{M}}$.

For the dual of $[\varphi]$, $\mathcal{M}, s \models \langle\varphi\rangle\psi$ if and only if $\mathcal{M}, s \models \varphi$ and $\mathcal{M}|\varphi, s \models \psi$.

Theorem 3 ([75]). *PAL is as expressive as EL.*

Definition 11. The axiomatization **PAL** is the extension of **S5** with the following axioms:

$$\begin{array}{ll} \mathbf{AP} & [\varphi]p \leftrightarrow (\varphi \rightarrow p) \\ \mathbf{AN} & [\varphi]\neg\psi \leftrightarrow (\varphi \rightarrow \neg[\varphi]\psi) \\ \mathbf{AC} & [\varphi](\psi \wedge \chi) \leftrightarrow ([\varphi]\psi \wedge [\varphi]\chi) \\ \mathbf{AK} & [\varphi]K_a\psi \leftrightarrow (\varphi \rightarrow K_a[\varphi]\psi) \\ \mathbf{AA} & [\varphi][\psi]\chi \leftrightarrow [\varphi \wedge [\varphi]\psi]\chi \end{array}$$

Theorem 4 ([75]). *The system **PAL** is sound and complete with respect to PAL.*

2.2.3 Arbitrary public announcement logic

Arbitrary public announcement logic (APAL), introduced by [16], quantifies over announcements to interpret the notion of “knowable”.

Definition 12 (Language of arbitrary public announcement logic $\mathcal{L}_{\text{APAL}}$). Given \mathbf{A} and \mathbf{P} , the language of arbitrary public announcement logic $\mathcal{L}_{\text{APAL}}$ is defined as:

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \psi \mid K_a\varphi \mid [\varphi]\varphi \mid [!]\varphi$$

where $a \in \mathbf{A}$ and $p \in \mathbf{P}$. We use $\langle ! \rangle \varphi$ to denote $\neg[!]\neg\varphi$.

Formula $[!]\varphi$ can be read as “after any truthful announcement, it holds that φ ”. The dual $\langle ! \rangle \varphi$ is read as “there is some truthful announcement after which it holds that φ ”.

The formal definition of semantics is given as:

Definition 13 (Semantics of APAL). Given $\mathcal{M} = (S, \sim, V)$, $s \in S$, $\varphi \in \mathcal{L}_{\text{PAL}}$, $\mathcal{M}, s \models \varphi$ is defined as:

$$\begin{array}{lll} \mathcal{M}, s \models p & \text{iff} & s \in V(p) \\ \mathcal{M}, s \models \neg\varphi & \text{iff} & \mathcal{M}, s \not\models \varphi \\ \mathcal{M}, s \models \varphi \wedge \psi & \text{iff} & \mathcal{M}, s \models \varphi \text{ and } \mathcal{M}, s \models \psi \\ \mathcal{M}, s \models K_a\varphi & \text{iff} & \text{for all } t \in S, s \sim_a t \text{ implies } \mathcal{M}, t \models \varphi \\ \mathcal{M}, s \models [\psi]\varphi & \text{iff} & \mathcal{M}, s \models \varphi \text{ implies } \mathcal{M}|\psi, s \models \varphi \\ \mathcal{M}, s \models [!]\varphi & \text{iff} & \text{for all } \psi \in \mathcal{L}_{\text{EL}}, \mathcal{M}|\psi, s \models \varphi \end{array}$$

where $\mathcal{M}|\varphi = (S', \sim', V')$ is such that $S' = \llbracket \varphi \rrbracket_{\mathcal{M}} = \{s \in S \mid \mathcal{M}, s \models \varphi\}$, $\sim'_a = \sim_a \cap (\llbracket \varphi \rrbracket_{\mathcal{M}} \times \llbracket \varphi \rrbracket_{\mathcal{M}})$, and $V'(p) = V(p) \cap \llbracket \varphi \rrbracket_{\mathcal{M}}$.

Theorem 5 ([16]). *Single-agent APAL and PAL are equally expressive. Multi-agent APAL is more expressive than PAL.*

The derivation rule involving the quantifier is formulated in terms of so-called *necessity forms* [46]. Consider a new symbol \sharp . The *necessity forms* are defined inductively as follows, where φ is a formula in some logical language \mathcal{L} and $a \in \mathbf{A}$.

$$\psi(\sharp) ::= \sharp \mid (\varphi \rightarrow \psi(\sharp)) \mid K_a\psi(\sharp) \mid [\varphi]\psi(\sharp)$$

A necessity form contains a unique occurrence of the symbol \sharp . If $\psi(\sharp)$ is a necessity form and $\varphi \in \mathcal{L}$, then $\psi(\varphi) \in \mathcal{L}$ is the substitution of \sharp by φ in $\psi(\sharp)$.

Definition 14. The axiomatization **APAL** is an expansion of **PAL** by adding the axiom **A!** and rule **R!**. In the rule **R!**, the expressions $\chi([\psi]\varphi)$ and $\chi([!]\varphi)$ are instantiations of a necessity form $\chi(\sharp)$.

$$\begin{array}{l} \mathbf{A!} \quad [!]\varphi \leftrightarrow [\psi]\varphi \quad \text{where } \psi \in \mathcal{L}_{\text{EL}} \\ \mathbf{R!} \quad \text{From } \chi([\psi]\varphi) \text{ for all } \psi \in \mathcal{L}_{\text{EL}} \text{ infer } \chi([!]\varphi) \end{array}$$

APAL is an infinitary axiomatization.

The soundness and completeness of **APAL** was shown in [16]. An error in that completeness proof was later corrected in [15]. Even later a simplified completeness proof was given in [17]. The system in [17] contains an additional derivation rule “From φ infer $[!]\varphi$ ” that is derivable in **APAL**.

2.2.4 Arrow update logic

Arrow update logic (AUL) was developed by [57]. AUL propose a different approach to update models other than PAL.

Definition 15 (Language of arrow update logic \mathcal{L}_{AUL}). Given \mathbf{A} and \mathbf{P} , the language of arbitrary public announcement logic \mathcal{L}_{AUL} is defined as:

$$\begin{aligned}\varphi &::= p \mid \neg\varphi \mid (\varphi \wedge \psi) \mid \Box_a\varphi \mid [U]\varphi \\ U &::= (\varphi, a, \psi) \mid (\varphi, a, \varphi), U\end{aligned}$$

where $a \in \mathbf{A}$ and $p \in \mathbf{P}$.

Triple (φ, a, ψ) is called an *arrow specification* where formula φ is called *source condition*, ψ is called *target condition*, and a is a label for an agent. U is called *arrow update*. Correspondingly, given an agent a and states s_1, s_2 , triple (s_1, a, s_2) is called a *transition*. (s_1, a, s_2) *starts* in s_1 and *ends* in s_2 , and is also denoted by $s_1 \xrightarrow{a} s_2$.

A *path* in \mathcal{M} is a (possibly finite) sequence $s_1 \xrightarrow{a_1} s_2, s_2 \xrightarrow{a_2} s_3, \dots$ of transitions in \mathcal{M} where each transition begins in the state where the previous transition ends. A single state s is considered a degenerate path that contains no transitions. A path P' *extends* a path P if P is an initial segment of P' .

Let $\mathcal{M} = (S, R, V)$ be a Kripke model and U is an arrow update. A transition $s_1 \xrightarrow{a} s_2$ satisfies U in \mathcal{M} if there is an arrow specification (φ, a, ψ) in U such that $\mathcal{M}, s_1 \models \varphi$ and $\mathcal{M}, s_2 \models \psi$. A path is a sequence $s_1 \xrightarrow{a_1} s_2 \xrightarrow{a_2} \dots$ of transition in \mathcal{M} . A path $s_1 \xrightarrow{a_1} s_2 \xrightarrow{a_2} \dots$ satisfies U if every transition $s_i \xrightarrow{a_i} s_{i+1}$ satisfies U . A U -path is *full* in \mathcal{M} if there is no U -path in \mathcal{M} that extends it.

Definition 16 (Semantics of AUL). Let $\mathcal{M} = (S, R, V)$ be a Kripke model, $s \in S$ and $\varphi \in \mathcal{L}_{\text{AUL}}$. $\mathcal{M}, s \models \varphi$ is defined as:

$$\begin{aligned}\mathcal{M}, s \models p &\quad \text{iff} \quad s \in V(p) \\ \mathcal{M}, s \models \neg\varphi &\quad \text{iff} \quad \mathcal{M}, s \not\models \varphi \\ \mathcal{M}, s \models \varphi \wedge \psi &\quad \text{iff} \quad \mathcal{M}, s \models \varphi \text{ and } \mathcal{M}, s \models \psi \\ \mathcal{M}, s \models \Box_a\varphi &\quad \text{iff} \quad \text{for all } t \in S, sR_at \text{ implies } \mathcal{M}, t \models \varphi \\ \mathcal{M}, s \models [U]\varphi &\quad \text{iff} \quad \mathcal{M} * U, s \models \varphi\end{aligned}$$

where $\mathcal{M} * U = (S, R * U, V)$ and for every $a \in \mathbf{A}$,

$$R * U = \{(s, s') \in R_a \mid s \xrightarrow{a} s' \text{ satisfies } U\}.$$

AUL* is a variant of AUL by adding an operator $\{U\}^*$ that looks through every full U -paths.

Definition 17 (Language of AUL*). Given \mathbf{A} and \mathbf{P} , the language of arbitrary public announcement logic $\mathcal{L}_{\text{AUL}^*}$ is defined as:

$$\begin{aligned}\varphi &::= p \mid \neg\varphi \mid (\varphi \wedge \psi) \mid K_a\varphi \mid [U]\varphi \mid \{U\}^*\varphi \\ U &::= (\varphi, a, \varphi) \mid (\varphi, a, \varphi), U\end{aligned}$$

where $a \in \mathbf{A}$ and $p \in \mathbf{P}$.

Definition 18 (Semantics of AUL*). We adopt the same definition and settings as for AUL, and

$$\mathcal{M}, s \models \{U\}^*\varphi \quad \text{iff} \quad \text{for every } U\text{-path } P \text{ starting in } s \text{ and every } s' \in P \text{ we have } \mathcal{M}, s' \models \varphi,$$

2.2.5 Computation tree logic

Computation tree logic (CTL) was first proposed by [31]. It reasons about time with possibly different paths in the future. CTL uses branching-time time structure and has quantifiers over paths.

Definition 19 (Language of CTL). Given \mathbf{P} , the language of computation tree logic \mathcal{L}_{CTL} is defined as:

$$\varphi ::= p \mid \neg\varphi \mid (\varphi \wedge \psi) \mid AX\varphi \mid AF\varphi \mid E(\varphi U\psi)$$

where $p \in \mathbf{P}$.

“A” and “E” are quantifiers over paths: “A” means “for all paths” and “E” means “there exists a path”. “X, F, U” are temporal modalities: “X” means “the next state”, “F” means “in the future”, and “U” means “until”. Therefore, $AX\varphi$ means “for the next state of all paths, φ is true”, and $AF\varphi$ means “for all paths, φ will be true at some state”. $E(\varphi U\psi)$ means “there is a path such that φ is true until a state where ψ is true”. The Def. 19 is based on a minimal set of operators. We may also use G as the dual of F . “G” means “always” or “globally”. $AG\varphi$ and $EG\varphi$ can be seen as an abbreviation of $\neg EF\neg\varphi$ and $\neg AF\neg\varphi$. The semantics of CTL is given as:

Definition 20 (Semantics of CTL). Let $\mathcal{M} = (S, R, V)$ be a Kripke model, $s \in S$ and $\varphi \in \mathcal{L}_{\text{CTL}}$. $\mathcal{M}, s \models \varphi$ is defined as:

$\mathcal{M}, s \models p$	iff	$s \in V(p)$
$\mathcal{M}, s \models \neg\varphi$	iff	$\mathcal{M}, s \not\models \varphi$
$\mathcal{M}, s \models \varphi \wedge \psi$	iff	$\mathcal{M}, s \models \varphi$ and $\mathcal{M}, s \models \psi$
$\mathcal{M}, s \models AX\varphi$	iff	for all $t \in S$ such that $s \mapsto t$, $\mathcal{M}, t \models \varphi$
$\mathcal{M}, s \models AF\varphi$	iff	for every full path P starting in s there is some $s_i \in P$ such that $\mathcal{M}, s_i \models \varphi$
$\mathcal{M}, s \models E(\varphi U\psi)$	iff	there is a -path P starting in s and $s_i \in P$ such that $\mathcal{M}, s_i \models \psi$ and for any $j < i$ we have $\mathcal{M}, s_j \models \varphi$

2.2.6 Normative temporal logic

Normative temporal logic (NTL) was developed by [3]. NTL reasons about normative systems which gives a set of *norms* to regulate behaviors of agents. Technically, it bases on CTL and labels paths with parameters as norms.

Let Σ_η be a set of normative systems, and $\eta \in \Sigma_\eta$ is a normative system. Given a Kripke model $\mathcal{M} = (S, R, V)$, let $I : \Sigma_\eta \rightarrow \mathcal{P}(R)$ be a function that assigns each η to a subset of R . If a pair $(s_1, s_2) \in I(\eta)$, we say $s_1 \mapsto s_2$ is a η -transition. A path $s_1 \mapsto s_2 \mapsto s_3 \dots$ is a η -path if every transition $s_i \mapsto s_{i+1}$ on it is in $I(\eta)$. A η -path is *full in \mathcal{M}* if there is no η -path in \mathcal{M} that extends it.

Definition 21 (Language of NTL). Given \mathbf{P} and Σ_η , the language of normative temporal logic \mathcal{L}_{NTL} is defined as:

$$\varphi ::= p \mid \neg\varphi \mid (\varphi \wedge \psi) \mid \Box_\eta\varphi \mid F_\eta\varphi \mid E_\eta(\varphi U\psi)$$

where $\eta \in \Sigma_\eta$ and $p \in \mathbf{P}$. We use $\Diamond_\eta\varphi$, $\hat{F}_\eta\varphi$ and $A_\eta(\varphi U\psi)$ to denote the abbreviation of $\neg\Box_\eta\neg\varphi$, $\neg F_\eta\neg\varphi$ and $\neg E_\eta(\varphi U\psi)$

Definition 22 (Semantics of NTL). Let $\mathcal{M} = (S, R, V)$ be a Kripke model, $s \in S$ and $\varphi \in \mathcal{L}_{NTL}$. $\mathcal{M}, s \models \varphi$ is defined as:

$\mathcal{M}, s \models p$	iff	$s \in V(p)$
$\mathcal{M}, s \models \neg\varphi$	iff	$\mathcal{M}, s \not\models \varphi$
$\mathcal{M}, s \models \varphi \wedge \psi$	iff	$\mathcal{M}, s \models \varphi$ and $\mathcal{M}, s \models \psi$
$\mathcal{M}, s \models \Box_{\eta}\varphi$	iff	for all $t \in S$, $s \mapsto t \in I(\eta)$ implies $\mathcal{M}, t \models \varphi$
$\mathcal{M}, s \models F_{\eta}\varphi$	iff	for every full η -path P starting in s there is some $s_i \in P$ such that $\mathcal{M}, s_i \models \varphi$
$\mathcal{M}, s \models E_{\eta}(\varphi U \psi)$	iff	there is a η -path P starting in s and $s_i \in P$ such that $\mathcal{M}, s_i \models \psi$ and for any $j < i$ we have $\mathcal{M}, s_j \models \varphi$

3

Almost APAL

3.1 Introduction

The modal logic of knowledge was originally proposed to give a relational semantics for the perceived properties of knowledge, such as that what you know is true, and that you know what you know, and to contrast this with the properties of other epistemic notions such as belief [52]. Already in [52] the analysis of paradoxical phenomena that you cannot be informed of factual ignorance while ‘losing’ that ignorance, so-called Moorean phenomena [71], played an important role. On the heels of the logic of (single agent) knowledge came the multi-agent logics of knowledge, wherein similar phenomena are not so paradoxical: there is no issue with my knowledge of your ignorance. This led on the one hand to the development of group epistemic notions such as common knowledge [13, 68] and distributed knowledge [51], topics that we will bypass in this contribution. On the other hand this led to increased interest in the analysis of multiple agents informing each other of their ignorance and knowledge, often inspired by logic puzzles [72, 68]. This culminated in Plaza’s public announcement logic (PAL) [76], wherein such informative actions became full members of the logical language besides the knowledge modalities; parallel developments of dynamic but not epistemic logics of information change are [107, 91].

The logic PAL contains a dynamic operator representing the consequences of information change that is similarly observed by all agents, so-called public (and truthful) announcement. We let $[\psi]\varphi$ stand for ‘after truthful public announcement of ψ , φ (is true)’. Every PAL formula is equivalent to a formula without public announcements, so that PAL is as expressive as epistemic logic EL [75].

From PAL there were various directions for further generalization. One could consider public announcements in the presence of group epistemic operators such as common knowledge, or non-public information change such as private or secret announcements to some agents while other agents do not or only partially observe that. Both were simultaneously realized in action model logic [18]; parallel, now lesser known, developments are [45].

A different direction of generalizing PAL is to consider quantifying over announcements. Arbitrary public announcement logic APAL was proposed in [16] and contains a construct $[!]\varphi$ standing for ‘after any truthful public announcement, φ (is true)’, i.e., for all ψ , $[\psi]\varphi$. In order to avoid circularity, the APAL quantifier is only over announcements not containing

This chapter is based on the paper “Almost APAL”, H. van Ditmarsch, M. Liu, L.B. Kuijter, I. Sedlar. *Journal of Logic and Computation*, exac012, 2022. [102]

[!] modalities. There is an infinitary axiomatization for the logic [17], where an open question remains whether there is a finitary axiomatization. APAL is undecidable [41], and the complexity of model checking is PSPACE-complete [1]. There are versions of APAL with finitary axiomatizations or decidable satisfiability problems [27, 98, 19], or that model aspects of agency [1, 6, 44]. APAL is more expressive than PAL [16]. The relative expressivity of versions of APAL is rather intricate, and most relevant in view of potential applications. For example, group announcement logic GAL and APAL are incomparable in expressivity [44], and in GAL we can formalize goal reachability in finite two-principal security protocols [1].

In this chapter we investigate some novel versions of APAL. If we quantify over announcements only using atoms in *subsets* $Q \subseteq \mathbf{P}$ we obtain the logic SAPAL, and if these subsets are required to be *finite* we get FSAPAL. If we quantify over announcements only using atoms occurring in the formula under the *scope* of the quantifier, we obtain the logic SCAPAL. If we quantify over announcements *implying* a given formula ψ or *implied by* a given formula ψ and if such ψ may also contain quantifiers we obtain logic QIPAL and if they are not allowed to contain quantifiers we obtain IPAL.

Note that there is a strong, but not well-known, relation between quantification over public announcements and epistemic planning [24]. In the latter, we wish to satisfy some epistemic goal φ by finding a sequence of actions, that could be public announcements, successively transforming multi-agent models for the system until ultimately leading to a model satisfying goal φ . In the former, we wish to satisfy $\langle ! \rangle \varphi$ (for ‘there is an announcement, or a sequence of announcements, after which φ ’) by finding a sequence of announcements (successively transforming multi-agent models) after which φ . In both, undecidability can only be tamed by restricting what can be announced. The way to obtain decidability in epistemic planning is often to restrict the number of actions [12, 25]. This goes beyond merely restricting the number of atoms. Such an action may have a precondition of certain modal depth. But therefore, we no longer quantify over arbitrary modal depth as in the APAL versions considered here, but over bounded modal depth.

The common factor in each of the APAL variants considered in this chapter is that they restrict the domain of quantification to certain formulas that are considered relevant or permissible in some context. In APAL, $[!]\varphi$ means that φ holds after any truthful public announcement, including announcements that are completely irrelevant to the matter at hand. While it may be fun to read about Sherlock Holmes determining the identity of the killer based on, say, the weather in Berlin three days ago, such (seemingly) irrelevant announcements are not very useful in practice. Each of the variants under consideration here tries to solve this “irrelevant announcements” problem in some way.

For SAPAL and FSAPAL, whenever we use an arbitrary announcement operator, we need to specify the relevant domain of discourse $Q \subseteq P$ for that operator. We then consider only those announcements that are considered relevant by virtue of pertaining to this domain of discourse. Think of an expert witness who is only allowed to opine on matters within their area of expertise. For a more technical example, suppose that we want to allow users to query a database in limited ways, while keeping a certain fact p secret. One way to do this is to allow queries only about domain Q . We can then verify that p remains unknown after the answer to the query is given by checking that $[Q](\neg Kp \wedge \neg K\neg p)$ holds.

More generally, when modelling dynamics of a multi-agent system it is often the case that the vocabulary is finite. In particular, often only a finite number of atomic propositions are considered relevant for each given subtask of a problem to solve, where this vocabulary might vary between subtasks. In such cases, FSAPAL might be more suitable modelling tools than

‘generic’ APAL. For example, consider distributed systems wherein agents may communicate about their own local state value [61]. Similarly, in gossip protocols [56], the protocols wherein agents merely exchange their secrets are less powerful than those wherein they are permitted to exchange other information, such as who they previously communicated with. [50, 32]

In SCAPAL we also restrict announcements to some domain of discourse, but instead of adding this domain as a parameter to the operator, we consider a propositional variable p to be in the domain of discourse if it occurs inside the scope of the announcement operator. This is not always a good idea; in the database query example above, the secret p should definitely not be in the domain Q of queries. The assumption that variables are relevant because they appear in the scope does make sense in a conversational context, however: if we are debating the truth of φ , then announcements regarding the truth of any variable that occurs in φ are clearly relevant.

In IPAL, we do not restrict the domain of discourse, but instead limit (from below or above) how informative the announcement must be. An announcement $[\psi]$ eliminates all $\neg\psi$ states from consideration. As such, if χ is implied by ψ , and therefore holds at least on every state where ψ holds, then $[\chi]$ is at most as informative as $[\psi]$. Likewise, if ξ implies ψ then $[\xi]$ is at least as informative as $[\psi]$. Like SAPAL and FAPAL, this has applications in security protocols, where communications by principals need to satisfy information goals towards other principals as well as safety goals against eavesdroppers and other intruders [67, 77]. In such protocols, we may wonder whether it is possible to be at least as informative as ψ while not giving eavesdropper e knowledge of p , represented by the formula $\langle\psi^\dagger\rangle(\neg K_e p \wedge \neg K_e \neg p)$, or whether every communication at most as informative as ψ is safe, represented by the formula $[\psi^\uparrow](\neg K_e p \wedge \neg K_e \neg p)$.

IPAL can also be useful in situations where disclosing certain information is required (by law, by company policy or simply by social obligation), but disclosing more than the strict requirement is possible. Or, of course, in situations where disclosing certain information is forbidden.

For IPAL we were additionally motivated by the dynamic consequence relation based on PAL proposed in [92], and how the IPAL quantification (that like the PAL announcement is parametrized with a formula) can be seen as the condition for a substructural implication. See Section 3.7 for more details.

In addition to these applications, we were also originally motivated by the search for ‘tameable’ versions of APAL. Ideally, a ‘tame’ version of APAL would be decidable. Or, if not decidable, we could hope for a logic that is at least recursively enumerable (RE), and that therefore admits a finitary axiomatization. The reason that APAL is so poorly behaved is that its distinctive $[!]$ operator is extremely powerful. The corresponding operators $[Q]$ and $[\sqsubseteq]$ in FSAPAL and SCAPAL intuitively seem less powerful, suggesting that these logics might be tameable. Unfortunately, this turns out not to be the case. A pretty minor modification to the undecidability proof for APAL shows that FSAPAL and SCAPAL are undecidable, see Section 4.6.2. Even so, we hoped that the smaller domain of quantification would allow for a finitary axiomatization. While the domain of quantification for the arbitrary announcement operators in FSAPAL and SCAPAL is still infinite, and naive introduction rules for $[Q]$ and $[\sqsubseteq]$ are therefore infinitary, we had hoped to find introduction axioms for $[Q]$ and $[\sqsubseteq]$ using a finite (but unbounded) subset of the domain. There, too, we were frustrated, however; while we do present axiomatizations in Section 4.5 these use an infinitary introduction rule, similar to the corresponding rule in APAL.

In fact, even the intuition that $[Q]$ and $[\sqsubseteq]$ are less powerful than $[!]$ turns out to be only

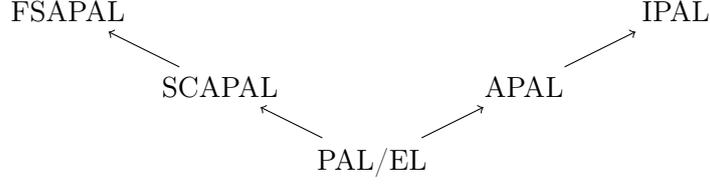


Figure 3.1: Expressivity hierarchy of logics presented in this work. An arrow means larger expressivity. Assume transitivity. Absence of an arrow means incomparability.

partially true. The domain of quantification of $[Q]$ and $[\sqsubseteq]$ is smaller than that of $[!]$, and, as a result, there are properties that can be expressed in APAL but not in FSAPAL or SCAPAL. But the smaller domain of quantification can also be used to express things in FSAPAL and SCAPAL that are unexpressible in APAL, see Section 3.4. So the expressive power of FSAPAL and SCAPAL is incomparable to, as opposed to strictly lower than, that of APAL.

With regard to SAPAL and IPAL, since it is possible to embed APAL in either of these logics, they are trivially at least as expressive as APAL, and their satisfiability problem is at least as hard as that of APAL. In Section 3.4 we show that both are in fact strictly more expressive than APAL.

So we did not strike gold in our search for tameable variants of APAL. Still, keeping in mind the applications discussed above, we argue that these logics are interesting in their own right. The expressivity results, which we consider the principal focus of our contribution, give a thorough overview of how these various attempts to limit the arbitrary announcement quantifier to some kind of relevant domain compare to each other. These results we consider of interest and non-trivial, so perhaps we did strike silver. Furthermore, that FSAPAL and SCAPAL are not tameable is a result in itself.

In Section 3.2 we introduce the syntax and semantics. In Section 3.3 we prove some modal properties of these quantifiers. Section 3.4 determines the expressivity hierarchy for the reported logics. It is shown in Fig. 3.1. Recall that \prec mean ‘strictly less expressive’ and \asymp ‘incomparable’, then the results are that PAL is strictly less expressive than any of the logics with quantifiers, and that SCAPAL \prec FSAPAL, APAL \asymp SCAPAL, APAL \asymp FSAPAL, IPAL \asymp SCAPAL, IPAL \asymp FSAPAL, and APAL \prec IPAL. Section 4.6.2 shows the undecidability of satisfiability of our APAL versions, and Section 4.5 provides complete axiomatizations for SAPAL and SCAPAL; these are similar to that for APAL. We conclude with Section 3.7 reinterpreting dynamic consequence in the IPAL setting.

3.2 Syntax and semantics: SAPAL, SCAPAL, QIPAL

Let a countably infinite set \mathbf{P} and a finite set of \mathbf{A} be given.

Definition 23 (Language). The logical language \mathcal{L} is defined inductively as:

$$\varphi ::= \top \mid p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid K_a\varphi \mid [\varphi]\varphi \mid [!]\varphi \mid [Q]\varphi \mid [\sqsubseteq]\varphi \mid [\varphi^\downarrow]\varphi \mid [\varphi^\uparrow]\varphi$$

where $p \in \mathbf{P}$, $a \in \mathbf{A}$, and $Q \subseteq \mathbf{P}$. By adding one of the *quantifiers* $[!]$, $[Q]$, $[\sqsubseteq]$, $[\varphi^\downarrow]\psi$ and $[\varphi^\uparrow]\psi$ to \mathcal{L}_{PAL} , we obtain, respectively, \mathcal{L}_{APAL} , \mathcal{L}_{SAPAL} and \mathcal{L}_{SCAPAL} , $\mathcal{L}_{QIPAL^\downarrow}$ and $\mathcal{L}_{QIPAL^\uparrow}$. Adding both $[\varphi^\downarrow]\psi$ and $[\varphi^\uparrow]\psi$ we obtain \mathcal{L}_{QIPAL} , and if the φ in $[\varphi^\downarrow]\psi$ and $[\varphi^\uparrow]\psi$ is restricted to \mathcal{L}_{PAL} , we get \mathcal{L}_{IPAL} . If the Q in $[Q]\varphi$ are (always) *finite* we get \mathcal{L}_{FSAPAL} .

The meaning of all constructs will be explained after defining the semantics. The dual modalities for $[Q]$, $[\subseteq]$, $[\varphi^\downarrow]$, and $[\varphi^\uparrow]$ are, respectively, $\langle Q \rangle$, $\langle \subseteq \rangle$, $\langle \varphi^\downarrow \rangle$, and $\langle \varphi^\uparrow \rangle$. Instead of $\varphi \in \mathcal{L}_X$ we also say that φ is an X formula. For any language \mathcal{L} , $\mathcal{L}|Q$ is the sublanguage only containing atoms in $Q \subseteq P$. Given $\varphi \in \mathcal{L}$, $P(\varphi)$ denotes the set of atoms occurring in φ . For $[\{p_1, \dots, p_n\}] \varphi$ we may write $[p_1 \dots p_n] \varphi$. The *modal depth* $d(\varphi)$ of a formula is the maximum stack of epistemic modalities; it is defined as: $d(\perp) = d(p) = 0$, $d(\varphi \wedge \psi) = \max\{d(\varphi), d(\psi)\}$, $d(K_a \varphi) = d(\varphi) + 1$, $d([\varphi] \psi) = d([\varphi^\downarrow] \psi) = d([\varphi^\uparrow] \psi) = d(\varphi) + d(\psi)$, and $d([\!] \varphi) = d([\subseteq] \varphi) = d([Q] \varphi) = d(\neg \varphi) = d(\varphi)$.

We let Γ , Σ and Δ denote finite sequences of formulas, where (Γ, Δ) denotes the concatenation of sequences (the parentheses are often omitted), and $|\Gamma|$ the length of a sequence. By induction on the length of Γ (and where φ, ψ are formulas) we define: $[\Gamma] \varphi := \varphi$ when $|\Gamma| = 0$, and $[\psi, \Gamma] \varphi := [\psi][\Gamma] \varphi$ when $|\psi, \Gamma| = n + 1$; similarly, $[\Gamma^\downarrow] \varphi := \varphi$ when $|\Gamma| = 0$, and $[(\psi, \Gamma)^\downarrow] \varphi := [\psi^\downarrow][\Gamma^\downarrow] \varphi$ when $|\psi, \Gamma| = n + 1$.

Definition 24 (Semantics). Given an epistemic model $\mathcal{M} = (S, \sim, V)$, $s \in S$ and $\varphi \in \mathcal{L}$ we inductively define $\mathcal{M}, s \models \varphi$ (φ is true in state s of model \mathcal{M}) as:

$\mathcal{M}, s \models p$	iff	$s \in V(p)$
$\mathcal{M}, s \models \neg \varphi$	iff	$\mathcal{M}, s \not\models \varphi$
$\mathcal{M}, s \models \varphi \wedge \psi$	iff	$\mathcal{M}, s \models \varphi$ and $\mathcal{M}, s \models \psi$
$\mathcal{M}, s \models K_a \varphi$	iff	for all $t \in S$, $s \sim_a t$ implies $\mathcal{M}, t \models \varphi$
$\mathcal{M}, s \models [\psi] \varphi$	iff	$\mathcal{M}, s \models \psi$ implies $\mathcal{M} \psi, s \models \varphi$
$\mathcal{M}, s \models [\!] \varphi$	iff	for any $\psi \in \mathcal{L}_{PAL}$: $\mathcal{M}, s \models [\psi] \varphi$
$\mathcal{M}, s \models [Q] \varphi$	iff	for any $\psi \in \mathcal{L}_{PAL} Q$: $\mathcal{M}, s \models [\psi] \varphi$
$\mathcal{M}, s \models [\subseteq] \varphi$	iff	for any $\psi \in \mathcal{L}_{PAL} P(\varphi)$: $\mathcal{M}, s \models [\psi] \varphi$
$\mathcal{M}, s \models [\chi^\downarrow] \varphi$	iff	for any $\psi \in \mathcal{L}_{PAL}$ implying χ : $\mathcal{M}, s \models [\psi] \varphi$
$\mathcal{M}, s \models [\chi^\uparrow] \varphi$	iff	for any $\psi \in \mathcal{L}_{PAL}$ implied by χ : $\mathcal{M}, s \models [\psi] \varphi$

where $\mathcal{M}| \varphi = (S', \sim', V')$ is such that $S' = \llbracket \varphi \rrbracket_{\mathcal{M}} = \{s \in S \mid \mathcal{M}, s \models \varphi\}$, $\sim'_a = \sim_a \cap (\llbracket \varphi \rrbracket_{\mathcal{M}} \times \llbracket \varphi \rrbracket_{\mathcal{M}})$, and $V'(p) = V(p) \cap \llbracket \varphi \rrbracket_{\mathcal{M}}$.

In the dual existential reading of the semantics of the quantifiers, the ψ in ‘there is a $\psi \in \mathcal{L}_{PAL}$ ’ is the *witness* of the quantifier. In the semantics of the last two, ‘ ψ implies χ ’ means $\models \psi \rightarrow \chi$ and ‘ ψ is implied by χ ’ means $\models \chi \rightarrow \psi$.

PAL and APAL *Public announcement logic* PAL and *arbitrary public announcement logic* APAL were already introduced.

SAPAL and FSAPAL The logic with construct $[Q] \varphi$, for ‘after any announcement only containing atoms in $Q \subseteq P$ ’, is called SAPAL, for APAL with quantification over formulas restricted to *subsets* of variables. If those subsets are required to be *finite* we get FSAPAL.

SCAPAL The logic with construct $[\subseteq] \varphi$, for ‘after any announcement only containing atoms occurring in φ ’, is called SCAPAL (where φ is the formula under the *scope* of the quantifier $[\subseteq]$).

QIPAL The logic with constructs $[\psi^\downarrow]\varphi$ and $[\psi^\uparrow]\varphi$ is called QIPAL; where $[\psi^\downarrow]\varphi$ stands for ‘after every announcement implying ψ , φ is true’, and $[\psi^\uparrow]\varphi$ stands for ‘after every announcement implied by ψ , φ is true’. In QIPAL we can reason over restrictions of a given model \mathcal{M} that are submodels of $\mathcal{M}|\psi$, or over restrictions that contain $\mathcal{M}|\psi$ as a submodel.

We define bounded bisimulation and obtain some elementary invariance results for our logics. They will be used much in the expressivity Section 3.4.

Definition 25 (Bounded bisimulation). Let \mathcal{M} and \mathcal{N} be epistemic models. For $n \in \mathbb{N}$ we define a sequence $Z^0 \supseteq \dots \supseteq Z^n$ of relations on $S^\mathcal{M} \times S^\mathcal{N}$.

A non-empty relation Z^0 is a 0-bisimulation if for all $Z^0 st$ and $p \in P$:

— **atoms**: $s \in V^\mathcal{M}(p)$ iff $t \in V^\mathcal{N}(p)$.

A non-empty relation Z^{n+1} is an $(n+1)$ -bisimulation if for all $Z^{n+1}st$, $a \in A$:

— $(n+1)$ -**forth**: if $s \sim_a^\mathcal{M} s'$, then there is a $t' \in S^\mathcal{N}$ s.t. $t \sim_a^\mathcal{N} t'$ and $Z^n s't'$.

— $(n+1)$ -**back**: if $t \sim_a^\mathcal{N} t'$, then there is a $s' \in S^\mathcal{M}$ s.t. $s \sim_a^\mathcal{M} s'$ and $Z^n s't'$.

If there exists a n -bisimulation Z^n between \mathcal{M} and \mathcal{N} we write $\mathcal{M} \Leftrightarrow^n \mathcal{N}$. (We also combine the notations \Leftrightarrow^Q and \Leftrightarrow^n in the obvious way, writing $\Leftrightarrow^{Q,n}$.)

Given pointed models (\mathcal{M}, s) and (\mathcal{N}, t) and a logic L with language \mathcal{L}_L , recall that $(\mathcal{M}, s) \equiv_L (\mathcal{N}, t)$ (for ‘ (\mathcal{M}, s) and (\mathcal{N}, t) are modally equivalent’) denotes: for all $\varphi \in \mathcal{L}_L$, $\mathcal{M}, s \models \varphi$ iff $\mathcal{N}, t \models \varphi$. Given $Q \subseteq P$ and $n \in \mathbb{N}$, annotations \equiv_L^n and \equiv_L^Q restrict the evaluated formulas $\varphi \in \mathcal{L}_L$ to those of modal depth $d(\varphi) \leq n$ and (resp.) to $\varphi \in \mathcal{L}_L|Q$. APAL is invariant for bisimilarity, but not for restricted bisimilarity or bounded bisimilarity: $(\mathcal{M}, s) \Leftrightarrow (\mathcal{N}, t)$ implies $(\mathcal{M}, s) \equiv_{APAL} (\mathcal{N}, t)$, whereas $(\mathcal{M}, s) \Leftrightarrow^n (\mathcal{N}, t)$ may not imply $(\mathcal{M}, s) \equiv_{APAL}^n (\mathcal{N}, t)$, and $(\mathcal{M}, s) \Leftrightarrow^Q (\mathcal{N}, t)$ may not imply $(\mathcal{M}, s) \equiv_{APAL}^Q (\mathcal{N}, t)$ [16, 99]. This is because the APAL modality $[\!]$ implicitly quantifies over formulas of arbitrarily large modal depth and over infinitely many atoms. All logics we consider in this paper are invariant for bisimilarity.

Lemma 1. For any L considered, $(\mathcal{M}, s) \Leftrightarrow (\mathcal{N}, t)$ implies $(\mathcal{M}, s) \equiv_L (\mathcal{N}, t)$.

Proof. For $L = \text{EL}, \text{PAL}$, this is known from the literature [23] for EL, and for PAL because EL and PAL are equally expressive [76]. For the other logics, let us for example consider SAPAL; the proof for all remaining logics is similar. By induction on the structure of φ we show that

For all $\varphi \in \mathcal{L}_{SAPAL}$ and for all pointed models $(\mathcal{M}, s), (\mathcal{N}, t)$:
 $(\mathcal{M}, s) \Leftrightarrow (\mathcal{N}, t)$ implies $\mathcal{M}, s \models \varphi$ iff $\mathcal{N}, t \models \varphi$.

All inductive cases are elementary except ‘public announcement’ and ‘quantifier’.

Case quantifier

$\mathcal{M}, s \models [Q]\psi$, iff $\mathcal{M}, s \models [\varphi]\psi$ for all $\varphi \in \mathcal{L}_{PAL}|Q$, iff $\mathcal{M}, s \models \varphi$ implies $\mathcal{M}|\varphi, s \models \psi$ for all $\varphi \in \mathcal{L}_{PAL}|Q$, iff (*) $\mathcal{N}, t \models \varphi$ implies $\mathcal{M}|\varphi, s \models \psi$ for all $\varphi \in \mathcal{L}_{PAL}|Q$, iff (**) $\mathcal{N}, t \models \varphi$ implies $\mathcal{N}|\varphi, t \models \psi$ for all $\varphi \in \mathcal{L}_{PAL}|Q$, iff $\mathcal{N}, t \models [\varphi]\psi$ for all $\varphi \in \mathcal{L}_{PAL}|Q$, iff $\mathcal{N}, t \models [Q]\psi$.

(*): By bisimulation invariance of PAL, we obtain $\mathcal{M}, s \models \varphi$ iff $\mathcal{N}, t \models \varphi$.

(**): Let $Z : (\mathcal{M}, s) \Leftrightarrow (\mathcal{N}, t)$. Define Z' between $\mathcal{M}|\varphi$ and $\mathcal{N}|\varphi$ as follows: $Z'uv$ iff $(Zuv$ and $\mathcal{M}, u \models \varphi)$. By bisimulation invariance for $\varphi \in \mathcal{L}_{PAL}$ it follows that also $\mathcal{N}, v \models \varphi$, so that Z' is indeed a relation between $\mathcal{M}|\varphi$ and $\mathcal{N}|\varphi$. We now show that $Z' : (\mathcal{M}|\varphi, s) \Leftrightarrow (\mathcal{N}|\varphi, t)$. The clause **atoms** is obviously satisfied. Concerning **forth** for some agent a , take any pair (v, v') such that $Z'vv'$ and let u in the domain of $\mathcal{M}|\varphi$ be such that $v \sim_a u$. As u is in the

domain of $\mathcal{M}|\varphi$, $\mathcal{M}, u \models \varphi$. From $Z'vv'$ follows Zvv' . As $v \sim_a u$ in $\mathcal{M}|\varphi$, also $v \sim_a u$ in \mathcal{M} . From Zvv' , $v \sim_a u$ in \mathcal{M} , and **forth** (for Z) it follows that there is u' in the domain of \mathcal{N} such that Zuu' and $v' \sim_a u'$. From Zuu' , $\mathcal{M}, u \models \varphi$, and bisimulation invariance for $\varphi \in \mathcal{L}_{PAL}$ it follows that $\mathcal{N}, u' \models \varphi$, i.e., u' is also in the domain of $\mathcal{N}|\varphi$. From Zuu' , $\mathcal{M}, u \models \varphi$, and the fact the u' is in the domain of $\mathcal{M}|\varphi$ it follows that $Z'uu'$, as required. This proves **forth**. The step **back** is shown similarly. Note that in particular $Z'st$. This therefore establishes that $Z' : (\mathcal{M}|\varphi, s) \Leftrightarrow (\mathcal{N}|\varphi, t)$, so that by definition $(\mathcal{M}|\varphi, s) \Leftrightarrow (\mathcal{N}|\varphi, t)$. By induction for ψ it now follows that $\mathcal{M}|\varphi, s \models \psi$ iff $\mathcal{N}|\varphi, t \models \psi$, as desired.

Case public announcement

The case public announcement, wherein we show that $\mathcal{M}, s \models [\varphi]\psi$ iff $\mathcal{N}, t \models [\varphi]\psi$, is shown fairly similarly to the case quantifier, except that in step (*) we do not use bisimulation invariance for $\varphi \in \mathcal{L}_{PAL}$ but we use the inductive hypothesis for $\varphi \in \mathcal{L}_{SAPAL}$, and similarly on two occasions in step (**). \square

Corollary 1. *Let $\varphi \in \mathcal{L}_L$ and $\mathcal{M}, s \models \varphi$. Then $(\mathcal{M}, s) \Leftrightarrow (\mathcal{N}, t)$ implies $(\mathcal{M}|\varphi, s) \Leftrightarrow (\mathcal{N}|\varphi, t)$.*

EL is also invariant under bounded bisimulation, with bound equal to the formula's modal depth. As every PAL formula is equivalent to an EL formula with equal modal depth (this is a special case of the translation introduced in [18]), it follows that PAL is similarly invariant. As we use a virtually identical result in subsequent proofs, we give a full proof here.

Lemma 2. *Let $n \in \mathbb{N}$ and $\varphi \in \mathcal{L}_{PAL}$ with $d(\varphi) = k \leq n$, models (M, s) and (N, t) , and $M, s \models \varphi$ be given. If $(M, s) \Leftrightarrow^n (N, t)$, then $(M|\varphi, s) \Leftrightarrow^{n-k} (N|\varphi, t)$.*

Proof. Let $Z^0 \supseteq \dots \supseteq Z^n$ be such that $Z^0 : (M, s) \Leftrightarrow^0 (N, t)$, \dots , $Z^n : (M, s) \Leftrightarrow^n (N, t)$. For all $i = 0, \dots, n-k$, let $Z_\varphi^i : \mathcal{D}(M) \rightarrow \mathcal{D}(N)$ be defined as: $Z_\varphi^i st$ iff $Z^{i+k}st$ and $M, s \models \varphi$. As $d(\varphi) \leq n$, from n -bisimulation invariance for PAL and $M, s \models \varphi$ also follows that $N, t \models \varphi$.

By natural induction on $n-k$ we show that $Z^n : (M, s) \Leftrightarrow^n (N, t)$ implies $Z_\varphi^{n-k} : (M|\varphi, s) \Leftrightarrow^{n-k} (N|\varphi, t)$, from which the required follows.

Case $n-k = 0$. We show **atoms**. We have that $Z_\varphi^0 st$ iff $Z^k st$, where the latter follows from $Z^k \supseteq Z^n$ and $Z^n st$. Therefore, $Z_\varphi^0 : (M|\varphi, s) \Leftrightarrow^0 (N|\varphi, t)$.

Case $n-k > 0$. We show $(n-k)$ -**forth**. Let $s \sim_a s'$ and $M, s' \models \varphi$, i.e., $s \sim_a s'$ in $M|\varphi$. From $Z^n : (M, s) \Leftrightarrow^n (N, t)$ and $s \sim_a s'$ follows that there is a $t' \sim_a t$ such that $Z^{n-1} : (M, s') \Leftrightarrow^{n-1} (N, t')$. As $n-k = n-d(\varphi) > 0$, $d(\varphi) < n$, so $d(\varphi) \leq n-1$. From $Z^{n-1} : (M, s') \Leftrightarrow^{n-1} (N, t')$, $M, s' \models \varphi$ and $d(\varphi) \leq n-1$ it follows by bisimulation invariance that $N, t' \models \varphi$. Therefore t' is in the domain of $N|\varphi$. By induction, from $Z^{n-1} : (M, s') \Leftrightarrow^{n-1} (N, t')$ it follows that $Z_\varphi^{n-k-1} : (M|\varphi, s') \Leftrightarrow^{n-k-1} (N|\varphi, t')$. Therefore, t' satisfies the requirement for $(n-k)$ -**forth** for relation Z_φ^{n-k} .

The clause $(n-k)$ -**back** is shown similarly. \square

Proposition 1. $(\mathcal{M}, s) \Leftrightarrow^Q (\mathcal{N}, t)$ implies $(\mathcal{M}, s) \equiv_{SAPAL}^Q (\mathcal{N}, t)$ and $(\mathcal{M}, s) \equiv_{SCAPAL}^Q (\mathcal{N}, t)$.

Proof. The proof is by induction on formulas true in (\mathcal{M}, s) . The crucial case quantifier is satisfied because (let $R \subseteq Q$): $\mathcal{M}, s \models [R]\varphi$, iff $\mathcal{M}, s \models [\psi]\varphi$ for all $\psi \in \mathcal{L}_{PAL}|R$, iff for all $\psi \in \mathcal{L}_{PAL}|R$, $\mathcal{M}, s \models \psi$ implies $\mathcal{M}|\psi, s \models \varphi$, iff (induction, Cor. 1) for all $\psi \in \mathcal{L}_{PAL}|R$, $\mathcal{N}, s \models \psi$ implies $\mathcal{N}|\psi, s \models \varphi$, iff (\dots) $\mathcal{N}, s \models [R]\varphi$.

The proof for SCAPAL is similar. \square

3.3 Modal properties of the quantifiers

We continue by discussing some peculiarities of the semantics, where we focus on modal properties of the quantifiers. We recall that APAL satisfies: $[!] \varphi \rightarrow \varphi$ (T), $[!] \varphi \rightarrow [!][!] \varphi$ (4), $\langle ! \rangle [!] \varphi \rightarrow [!] \langle ! \rangle \varphi$ (CR), and $[!] \langle ! \rangle \varphi \rightarrow \langle ! \rangle [!] \varphi$ (MK) [16, 99].

It may be useful to briefly consider the intuition behind these validities. The principle (T) is valid in APAL because if φ is true after every announcement, then in particular it is true after the uninformative announcement $[\top]$. So $[!] \varphi$ implies $[\top] \varphi$, which is in turn equivalent to φ . Validity of (4) is most easily seen in its dual form $\langle ! \rangle \langle ! \rangle \varphi \rightarrow \langle ! \rangle \varphi$; if $\langle ! \rangle \langle ! \rangle \varphi$ holds, then there are two announcements ψ and χ that, if announced after each other, will make φ true. The single announcement $\psi \wedge \langle \psi \rangle \chi$ (informally: “ ψ is true, and now χ is true as well”) has the same effect as announcing ψ and χ sequentially, so $\langle ! \rangle \varphi$ holds as well.

The properties (CR) and (MK) can be thought of as describing winning strategies when two players make one announcement each, with player one trying to make φ true and player two trying to make it false. Then (CR) states that if player one has a winning strategy when they make the first announcement, then they also have a winning strategy when they make the second announcement. Conversely, (MK) states that if player one can win when moving second, they can also win when they go first. The validity of these properties in APAL follows from the existence of a “most informative announcement” with respect to a given formula φ , that is available to either player. So player one has a winning strategy if and only if this specific announcement makes φ true.

3.3.1 SAPAL and FSAPAL

The logic SAPAL generalizes APAL, as $[P] \varphi$ is equivalent to $[!] \varphi$. We also considered FSAPAL where $Q \subseteq P$ in $[Q] \varphi$ is required to be finite.

Proposition 2. *SAPAL-valid are $[Q] \varphi \rightarrow \varphi$ (T) and $[Q \cup R] \varphi \rightarrow [Q][R] \varphi$ (4)*

Proof. The validity of $[Q] \varphi \rightarrow \varphi$ follows from the validity of $[\top] \varphi \leftrightarrow \varphi$. Just as for APAL, $[Q \cup R] \varphi \rightarrow [Q][R] \varphi$ is valid because two announcements can be made into one announcement, as in the PAL validity $[\psi] [\chi] \varphi \leftrightarrow [\psi \wedge [\psi] \chi] \varphi$, and because $P(\psi \wedge [\psi] \chi) \subseteq Q \cup R$ if $P(\psi) \subseteq Q$ and $P(\chi) \subseteq R$. \square

The SAPAL versions of CR and MK, $\langle Q \rangle [R] \varphi \rightarrow [Q] \langle R \rangle \varphi$ (CR) and $[Q] \langle R \rangle \varphi \rightarrow \langle Q \rangle [R] \varphi$ (MK) are not valid in SAPAL, however.

Proposition 3. *Neither $\langle Q \rangle [R] \varphi \rightarrow [Q] \langle R \rangle \varphi$ nor $[Q] \langle R \rangle \varphi \rightarrow \langle Q \rangle [R] \varphi$ is valid in SAPAL.*

Proof. Let $(M, 0)$ be the two state pointed model shown below.

$$\underline{0}(\overline{p}q) \xrightarrow{a} 1(p\overline{q})$$

Since q is false in both states, they are $\{q\}$ -bisimilar. As such, no informative $\{q\}$ -announcements are possible in this model or any of its submodels, in the sense that any such announcement holds either on all states or on no states.

As a result, we have $M, 0 \models \langle \neg p \rangle [\{q\}] K_a \neg p$ but $M, 0 \not\models [\top] \{q\} K_a \neg p$, and hence $M, 0 \not\models \langle \{p\} \rangle [\{q\}] K_a \neg p \rightarrow [\{p\}] \langle \{q\} \rangle K_a \neg p$. Similarly, we have $M, 0 \not\models [\{q\}] \langle \{p\} \rangle K_a \neg p \rightarrow \langle \{q\} \rangle [\{p\}] K_a \neg p$. \square

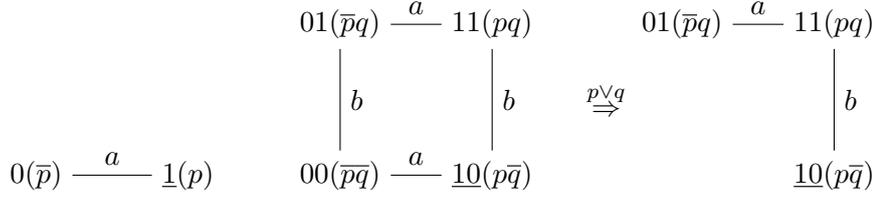


Figure 3.2: Model $(\mathcal{N}, 1)$ on the left, $(\mathcal{M}, 10)$ in the middle, $(\mathcal{M}|(p \vee q), 10)$ on the right.

Also note that all sets of variables in the above proof are finite, so CR and MK are not valid in FSAPAL either.

3.3.2 SCAPAL

The SCAPAL quantifier does not distribute over conjunction: $[\sqsubseteq]\varphi \wedge [\sqsubseteq]\psi$ is not equivalent to $[\sqsubseteq](\varphi \wedge \psi)$. This is easily demonstrated by an example.

Example 1. Consider model $(\mathcal{M}, 10)$ in Fig. 3.2 ($p\bar{q}$: p is true and q is false). Then:

$$\begin{aligned}
 \mathcal{M}, 10 &\not\models [\sqsubseteq]((K_ap \rightarrow K_bK_ap) \wedge \neg q) \\
 \mathcal{M}, 10 &\models [\sqsubseteq](K_ap \rightarrow K_bK_ap) \\
 \mathcal{M}, 10 &\models [\sqsubseteq]\neg q
 \end{aligned}$$

The first is false, because, as depicted:

$$\begin{aligned}
 \mathcal{M}, 10 &\models \langle p \vee q \rangle (K_ap \wedge \neg K_bK_ap), \text{ so} \\
 \mathcal{M}, 10 &\models \langle p \vee q \rangle ((K_ap \wedge \neg K_bK_ap) \vee q), \text{ and therefore} \\
 \mathcal{M}, 10 &\models \langle \sqsubseteq \rangle ((K_ap \wedge \neg K_bK_ap) \vee q), \text{ which is equivalent to} \\
 \mathcal{M}, 10 &\not\models [\sqsubseteq]((K_ap \rightarrow K_bK_ap) \wedge \neg q).
 \end{aligned}$$

The second is true because the only model restrictions containing 10 that we can obtain with formulas involving p are $\{10, 11\}$ and $\{10, 11, 00, 01\}$. The third is true because q is false in state 10.

Therefore, $[\sqsubseteq]\varphi \wedge [\sqsubseteq]\psi$ is not equivalent to $[\sqsubseteq](\varphi \wedge \psi)$.

Proposition 4. *Valid in SCAPAL are: $[\sqsubseteq]\varphi \rightarrow \varphi$ (T), $[\sqsubseteq]\varphi \rightarrow [\sqsubseteq][\sqsubseteq]\varphi$ (F), $[\sqsubseteq]\langle \sqsubseteq \rangle \varphi \rightarrow \langle \sqsubseteq \rangle [\sqsubseteq]\varphi$ (MK) and $\langle \sqsubseteq \rangle [\sqsubseteq]\varphi \rightarrow [\sqsubseteq]\langle \sqsubseteq \rangle \varphi$ (CR).*

Proof. T and F are valid for the same reason as in SAPAL. For CR and MK we can now (unlike for SAPAL) use the same method as in APAL, as in any state of a model we can announce the value of all variables occurring in φ . A proof of CR is found in [99, Prop. 3.10] (for the similar logic APAL⁺), which corrects the incorrect proof of CR for APAL in [16]. A proof of MK is found in [16]. \square

3.3.3 QIPAL and IPAL

We recall that in APAL the quantification is over $\varphi \in \mathcal{L}_{PAL}$. Fairly complex counterexamples demonstrate that $[\!] \varphi \rightarrow [\psi] \varphi$ is invalid for certain $\psi \in \mathcal{L}_{APAL}$ containing quantifiers [62]. Now in $[\psi^\dagger] \varphi$, $\psi \in \mathcal{L}_{QIPAL}$ may also contain quantifiers. This makes the relation to $[\!]$ unclear. In \mathcal{L}_{IPAL} , that ψ must be in \mathcal{L}_{PAL} and the relation is clearer.

Proposition 5. *Let $\psi \in \mathcal{L}_{PAL}$, $\chi \in \mathcal{L}_{IPAL}$ and pointed model (\mathcal{M}, s) be given. The following are equivalent:*

1. $\mathcal{M}, s \models \langle \psi^\downarrow \rangle \chi$
2. *there is a $\varphi \in \mathcal{L}_{PAL}$ such that $\models \varphi \rightarrow \psi$ and $\mathcal{M}, s \models \langle \varphi \rangle \chi$,*
3. *there is a $\varphi \in \mathcal{L}_{PAL}$ such that $\mathcal{M} \models \varphi \rightarrow \psi$ and $\mathcal{M}, s \models \langle \varphi \rangle \chi$,*
4. *there is a $\varphi \in \mathcal{L}_{PAL}$ such that $\mathcal{M}, s \models \langle \varphi \wedge \psi \rangle \chi$.*

Proof.

1 \Leftrightarrow 2 This is the semantics of the $\langle \psi^\downarrow \rangle$ quantifier (in dual form).

2 \Rightarrow 3 From $\models \varphi \rightarrow \psi$ it trivially follows that $\mathcal{M} \models \varphi \rightarrow \psi$.

3 \Rightarrow 4 Suppose that there is a $\varphi \in \mathcal{L}_{PAL}$ such that $\mathcal{M} \models \varphi \rightarrow \psi$ and $\mathcal{M}, s \models \langle \varphi \rangle \chi$. Because $\mathcal{M} \models \varphi \rightarrow \psi$, we have $\mathcal{M} \models \varphi \leftrightarrow (\varphi \wedge \psi)$, and therefore $\mathcal{M} \models \varphi = \mathcal{M} \models (\varphi \wedge \psi)$. From $\mathcal{M}, s \models \langle \varphi \rangle \chi$ then follows that $\mathcal{M}, s \models \langle \varphi \wedge \psi \rangle \chi$.

4 \Rightarrow 2 Suppose that there is a $\varphi \in \mathcal{L}_{PAL}$ such that $\mathcal{M}, s \models \langle \varphi \wedge \psi \rangle \chi$. Let $\varphi' = \varphi \wedge \psi$, and note that $\varphi' \in \mathcal{L}_{PAL}$. We have $\models \varphi' \rightarrow \psi$ and $\mathcal{M}, s \models \langle \varphi' \rangle \chi$. \square

So we can think of $\langle \psi^\downarrow \rangle$ as announcing $\varphi \wedge \psi$ for some φ . It is important, however, that the announcements of φ and ψ happen simultaneously. We cannot simply split $\langle \psi^\downarrow \rangle$ into an arbitrary announcement $\langle ! \rangle$ and the announcement $\langle \psi \rangle$, because the truth of ψ may be affected by the announcement of φ , and vice versa. Only under an additional constraint on ψ is such separation possible.

The *positive formulas* \mathcal{L}_{PAL}^+ are the PAL-fragment $p \mid \neg p \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid K_a \varphi \mid [\neg \varphi] \varphi$. The truth of positive formulas (corresponding to the universal fragment in first-order logic) is preserved after update [101].

Corollary 2. *Let $\psi \in \mathcal{L}_{PAL}^+$. Then $\langle \psi^\downarrow \rangle \chi$ implies $\langle ! \rangle \langle \psi \rangle \chi$.*

Proof. Let $\mathcal{M}, s \models \langle \psi^\downarrow \rangle \chi$. From Prop. 5.4 we obtain that there is $\varphi \in \mathcal{L}_{PAL}$ such that $\mathcal{M}, s \models \langle \varphi \wedge \psi \rangle \chi$. As ψ is positive, in any states where ψ is true it remains true after the update $\langle \varphi \wedge \psi \rangle$. An additional announcement of $\langle \psi \rangle$ therefore does not remove further states. So $\mathcal{M}, s \models \langle \varphi \wedge \psi \rangle \chi$ implies $\mathcal{M}, s \models \langle \varphi \wedge \psi \rangle \langle \psi \rangle \chi$.

By the definition of the APAL quantifier, it follows that $\mathcal{M}, s \models \langle ! \rangle \langle \psi \rangle \chi$. \square

Since every formula implies \top and is implied by \perp , both $[\top^\downarrow]$ and $[\perp^\uparrow]$ quantify over every formula in \mathcal{L}_{PAL} . We therefore have the following proposition.

Proposition 6. *Let $\varphi \in \mathcal{L}_{IPAL}$. Then $[\top^\downarrow] \varphi$ and $[\perp^\uparrow] \varphi$ are equivalent to $[!] \varphi$.*

Proof. Let model (\mathcal{M}, s) and $\varphi \in \mathcal{L}_{QIPAL}$ be given. Then: $\mathcal{M}, s \models [\top^\downarrow] \varphi$, iff $\mathcal{M}, s \models [\psi] \varphi$ for all $\psi \in \mathcal{L}_{PAL}$ with $\models \psi \rightarrow \top$, iff $\mathcal{M}, s \models [\psi] \varphi$ for all $\psi \in \mathcal{L}_{PAL}$, iff $\mathcal{M}, s \models [!] \varphi$.

Similarly, $\mathcal{M}, s \models [\perp^\uparrow] \varphi$, iff $\mathcal{M}, s \models [\psi] \varphi$ for all $\psi \in \mathcal{L}_{PAL}$ with $\models \perp \rightarrow \psi$, iff $\mathcal{M}, s \models [\psi] \varphi$ for all $\psi \in \mathcal{L}_{PAL}$, iff $\mathcal{M}, s \models [!] \varphi$. \square

Proposition 7. *Valid in QIPAL are $[\psi^\uparrow] \varphi \rightarrow \varphi$ (T) and also $[\psi^\uparrow] \varphi \rightarrow [\psi^\uparrow][\chi^\uparrow] \varphi$ and $[\psi^\downarrow] \varphi \rightarrow [\psi^\downarrow][\chi^\downarrow] \varphi$ (4)*

Proof. All proofs are as in Prop. 2 and 4. \square

However, $[\psi^\downarrow]\varphi \rightarrow \varphi$ (T) is invalid. This is because whenever $\mathcal{M}|\psi$ is a proper submodel of a given model \mathcal{M} , the trivial announcement is not allowed. For example, in any model where p is true but a does not know this, we have $[p^\downarrow]K_ap$ but not K_ap . Also, $[\psi^\uparrow]\varphi \rightarrow [\chi^\uparrow][\psi^\uparrow]\varphi$ and $[\psi^\downarrow]\varphi \rightarrow [\chi^\downarrow][\psi^\downarrow]\varphi$ are invalid, as the following example shows for the latter.

Example 2. Given is model M with two states s, t indistinguishable for a , and with p only true in s .

$$t(\bar{p}) \xrightarrow{a} s(p)$$

We have $M, s \models [(p \wedge K_ap)^\downarrow]\perp$, since $p \wedge K_ap$ holds on neither state, so any announcement implying $p \wedge K_ap$ cannot hold on any state either. Yet we also have $M, s_1 \not\models [\top^\downarrow][(p \wedge K_ap)^\downarrow]\perp$, with witnesses p for the first announcement and $p \wedge K_ap$ for the second.

3.4 Expressivity

We now address the relative expressivity of APAL, FSAPAL and SCAPAL and IPAL, where the proof that APAL is less expressive than IPAL is considerably more involved than the other proofs.

3.4.1 APAL $\not\leq$ FSAPAL and APAL $\not\leq$ SCAPAL

We show that there is an APAL-formula that can distinguish two pointed models that cannot be distinguished by any FSAPAL-formula. We use that APAL, unlike FSAPAL, quantifies over arbitrarily many atoms. The proof is similar to the proof that APAL $\not\leq$ PAL in [16].

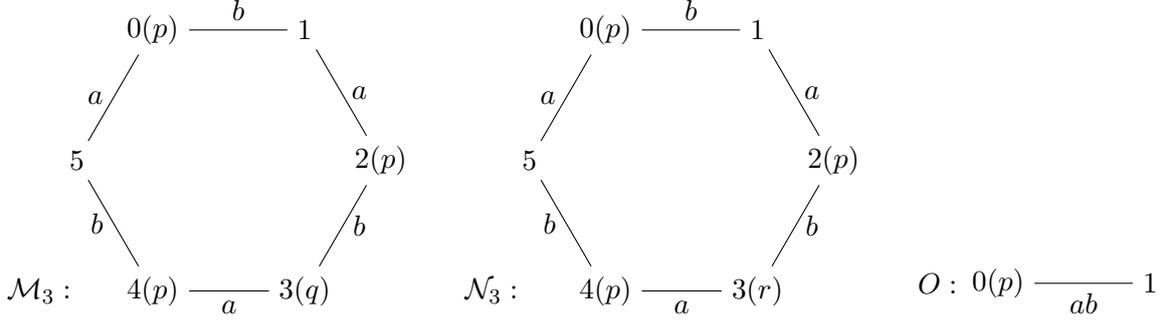
Proposition 8. *APAL $\not\leq$ FSAPAL and APAL $\not\leq$ SCAPAL.*

Proof. Consider APAL formula $\langle ! \rangle(K_ap \wedge \neg K_b K_ap)$, and assume towards a contradiction that ψ is an equivalent FSAPAL formula. Let $q \notin P(\psi)$. Now consider models $(\mathcal{M}, 10)$ and $(\mathcal{N}, 1)$ in Fig. 3.2, where the value of q in states 0 and 1 of \mathcal{N} is irrelevant. These models are $P(\psi)$ -bisimilar. We now have that:

1. $\mathcal{M}, 10 \models \langle ! \rangle(K_ap \wedge \neg K_b K_ap)$
Observe that $\mathcal{M}|\langle p \vee q \rangle \models K_ap \wedge \neg K_b K_ap$. This model is shown in Fig. 3.2.
2. $\mathcal{N}, 1 \not\models \langle ! \rangle(K_ap \wedge \neg K_b K_ap)$
3. $\mathcal{M}, 10 \models \psi$ iff $\mathcal{N}, 1 \models \psi$
By Prop. 1, $(\mathcal{M}, 10) \Leftrightarrow^{P(\psi)} (\mathcal{N}, 1)$ implies $(\mathcal{M}, 10) \equiv_{FSAPAL}^{P(\psi)} (\mathcal{N}, 1)$.

The third item contradicts the first two items. Therefore APAL $\not\leq$ FSAPAL.

As Prop. 1 also applies to SCAPAL, this also proves that APAL $\not\leq$ SCAPAL. \square


 Figure 3.3: The models \mathcal{M}_3 and \mathcal{N}_3

3.4.2 SCAPAL $\not\equiv$ APAL and FSAPAL $\not\equiv$ APAL

The proof is similar to that of the previous section, but more involved. We now show that the assumption that there is an APAL formula ψ equivalent to SCAPAL formula $\langle \subseteq \rangle (\neg q \wedge K_a p \wedge \neg K_b K_a p)$ leads to a contradiction. Prior to that we present models and lemmas used in the proof.

Consider models \mathcal{M}_n and \mathcal{N}_n as follows, where $n \in \mathbb{N}$ is odd. Model $\mathcal{M}_n = (S, \sim, V)$ is such that (i) $S = [0, 2n - 1]$, (ii) for any $i < n$, $2i \sim_b (2i + 1)$ and, except for $i = 0$, $(2i - 1) \sim_a 2i$ and also $(2n - 1) \sim_a 0$, and (iii) for any $i < n$, variable p is true in states $2i$, variable q is only true in state n and variable r is always false. Model \mathcal{N}_n is like model \mathcal{M}_n except that variable r is only true in n and variable q is always false. Fig. 3.3 depicts \mathcal{M}_3 and \mathcal{N}_3 .

Lemma 3. *Let $\mathcal{M} \subseteq \mathcal{M}_n$, $\mathcal{N} \subseteq \mathcal{N}_n$, $i, j, k \in \mathbb{N}$, with $i \in \mathcal{M}$ and $j \in \mathcal{N}$ be such that $(\mathcal{M}, i) \simeq^k (\mathcal{N}, j)$. Then for all $\chi \in \mathcal{L}_{PAL}$ such that $\mathcal{M}, i \models \chi$ there is a $\chi' \in \mathcal{L}_{PAL}$ such that $\mathcal{N}, j \models \chi'$ and $(\mathcal{M}|\chi, i) \simeq^k (\mathcal{N}|\chi', j)$. Furthermore, for all $\chi' \in \mathcal{L}_{PAL}$ such that $\mathcal{N}, j \models \chi'$ there is a $\chi \in \mathcal{L}_{PAL}$ such that $\mathcal{M}, i \models \chi$ and $(\mathcal{M}|\chi, i) \simeq^k (\mathcal{N}|\chi', j)$.*

Proof. Without loss of generality, we can assume that \mathcal{M} and \mathcal{N} are connected. We begin by showing that every state s of \mathcal{M} is uniquely identifiable by some formula $\varphi_s \in \mathcal{L}_{PAL}$. If the q -state is reachable from s , then the identifying formula is based on the shortest path to the q -state, and the agents along that path. For example, in \mathcal{M}_3 state 5 is the only state from which the q -state, state 3, is reachable by taking a b -edge followed by an a -edge, but not by only following an a -edge or only a b -edge. Hence, state 5 in \mathcal{M}_3 is uniquely identified by the formula $\mathcal{M}_b \mathcal{M}_a q \wedge \neg \mathcal{M}_a q \wedge \neg \mathcal{M}_b q$. If the q -state is not reachable from s and \mathcal{M} contains at least two states, then there is a “leftmost” state in \mathcal{M} , which can be uniquely identified by the formula $\varphi_{left} = K_a p \vee K_b \neg p$. The state s can then be uniquely identified by its distance to this leftmost state. If \mathcal{M} contains only one state, it can be identified trivially by \top .

Because \mathcal{M} is a finite model and each state can be uniquely identified by a formula, each submodel of \mathcal{M} is the extension of a disjunction of such formulas. Every state of \mathcal{N} is similarly uniquely identifiable, so each submodel of \mathcal{N} is also the extension of some formula.

In order for (\mathcal{M}, i) and (\mathcal{N}, j) to be k -bisimilar it is necessary and sufficient that (i) the q and r state are not reachable in k steps from (\mathcal{M}, i) and (\mathcal{N}, j) , respectively, (ii) there is a leftmost (resp. rightmost) state reachable from (\mathcal{M}, i) in less than k steps if and only if there is a leftmost (resp. rightmost) state reachable from (\mathcal{N}, j) in less than k steps. Condition (i) is always preserved in submodels. In order to guarantee that $(\mathcal{M}|\chi, i) \simeq^k (\mathcal{N}|\chi', j)$ it therefore

suffices to preserve (ii), which can be done by taking χ or χ' to be the formula such that χ holds on a state $l \leq k$ steps to the left/right of (\mathcal{M}, i) if and only if χ' holds l steps to the left/right of (\mathcal{N}, j) . □

In general, two k -bisimilar states need not be k -indistinguishable in APAL. This is because the $[!]$ operator quantifies over formulas of arbitrary depth. For submodels of \mathcal{M}_n and \mathcal{N}_n , however, k -bisimilarity does imply k -indistinguishability.

Lemma 4. *Let $\mathcal{M} \subseteq \mathcal{M}_n$, $\mathcal{N} \subseteq \mathcal{N}_n$ and $i, j, k \in \mathbb{N}$, with $i \in \mathcal{M}$ and $j \in \mathcal{N}$. If $(\mathcal{M}, i) \simeq^k (\mathcal{N}, j)$, then $(\mathcal{M}, i) \equiv_{APAL}^k (\mathcal{N}, j)$.*

Proof. We show the equivalent formulation:

For all $\varphi \in \mathcal{L}_{APAL}$, $\mathcal{M} \subseteq \mathcal{M}_n$, $\mathcal{N} \subseteq \mathcal{N}_n$ and $i, j, k \in \mathbb{N}$ with $i \in \mathcal{M}$ and $j \in \mathcal{N}$: if $(\mathcal{M}, i) \simeq^k (\mathcal{N}, j)$ and $d(\varphi) \leq k$, then $\mathcal{M}, i \models \varphi$ iff $\mathcal{N}, j \models \varphi$.

The proof is by induction on the structure of φ . The cases of interest are $K_b\varphi$, $[\psi]\varphi$, and $[!]\varphi$. As k -bisimilarity is a symmetric relation, it suffices to show only one direction of the equivalence.

Case $K_a\varphi$: Suppose $d(K_a\varphi) \leq k$. We have $\mathcal{M}, i \models K_a\varphi$ iff for all $i' \sim_a i$, $\mathcal{M}, i' \models \varphi$. As $(\mathcal{M}, i) \simeq^k (\mathcal{N}, j)$, for all $j' \sim_a j$ there is some $i' \sim_a i$ such that $(\mathcal{M}, i') \simeq^{k-1} (\mathcal{N}, j')$. As $d(K_a\varphi) \leq k$, $d(\varphi) \leq k-1$. Therefore, by induction, $\mathcal{N}, j' \models \varphi$. And therefore $\mathcal{N}, j \models K_a\varphi$.

Case $[\psi]\varphi$: Suppose $d([\psi]\varphi) \leq k$, and $\mathcal{M}, i \models [\psi]\varphi$. Let $d(\psi) = x$ and $d(\varphi) = y$, then $x + y = d(\psi) + d(\varphi) = d([\psi]\varphi) \leq k$. By definition, $\mathcal{M}, i \models [\psi]\varphi$ iff $\mathcal{M}, i \models \psi$ implies $\mathcal{M}|_{\psi}, i \models \varphi$. From $\mathcal{M}, i \models \psi$, $(\mathcal{M}, i) \simeq^k (\mathcal{N}, j)$ and $d(\psi) = x \leq k$ and induction we obtain $\mathcal{N}, j \models \psi$. From $(\mathcal{M}, i) \simeq^k (\mathcal{N}, j)$, $\mathcal{M}, i \models \psi$, $d(\psi) = x \leq k - y$, a part identical to that of Lemma 2 except that where bisimulation invariance for PAL is used on $\psi \in \mathcal{L}_{PAL}$ we now use induction on $\psi \in \mathcal{L}_{APAL}$, we obtain that $(\mathcal{M}|_{\psi}, i) \simeq^y (\mathcal{N}|_{\psi}, j)$. From that, $\mathcal{M}|_{\psi}, i \models \varphi$, $d(\varphi) = y$ and induction we obtain $\mathcal{N}|_{\psi}, j \models \varphi$. Then, $\mathcal{N}, j \models \psi$ implies $\mathcal{N}|_{\psi}, j \models \varphi$ is by definition $\mathcal{N}, j \models [\psi]\varphi$.

Case $[!]\varphi$: Suppose towards a contradiction that $\mathcal{N}, j \not\models [!]\varphi$. Then there is some $\chi' \in \mathcal{L}_{PAL}$ such that $\mathcal{N}, j \models \chi'$ and $\mathcal{N}|_{\chi'}, j \not\models \varphi$. By assumption $(\mathcal{M}, i) \simeq^k (\mathcal{N}, j)$, so the conditions of Lemma 3 are satisfied. So there is a $\chi \in \mathcal{L}_{PAL}$ such that $\mathcal{M}, i \models \chi$ and $(\mathcal{M}|_{\chi}, i) \simeq^k (\mathcal{N}|_{\chi}, j)$. The induction hypothesis and the fact that $\mathcal{N}|_{\chi'}, j \not\models \varphi$ then imply that $\mathcal{M}|_{\chi}, i \not\models \varphi$. We therefore have $\mathcal{M}, i \not\models [\chi]\varphi$, contradicting $\mathcal{M}, i \models [!]\varphi$. From this contradiction, we conclude that $\mathcal{N}, j \models [!]\varphi$. □

Proposition 9. *SCAPAL $\not\equiv$ APAL.*

Proof. Consider \mathcal{L}_{SCAPAL} formula $\varphi = \langle \subseteq \rangle (\neg q \wedge K_{ap} \wedge \neg K_b K_{ap})$. Let ψ be the supposedly equivalent \mathcal{L}_{APAL} formula. Take $n > d(\psi)$. We now show that:

1. $\mathcal{M}_n, 0 \models \langle \subseteq \rangle (\neg q \wedge K_{ap} \wedge \neg K_b K_{ap})$
2. $\mathcal{N}_n, 0 \not\models \langle \subseteq \rangle (\neg q \wedge K_{ap} \wedge \neg K_b K_{ap})$
3. $\mathcal{M}_n, 0 \models \psi$ iff $\mathcal{N}_n, 0 \models \psi$

These items are proved by the following arguments:

1. The state n is distinguished by formula q . This allows us to distinguish each finite subset of the domain, in the usual way, in \mathcal{L}_{EL} (note that there is no mirror symmetry along the $0-n$ ‘diameter’ of the circular models \mathcal{M}_n and \mathcal{N}_n). Thus there is a formula $\eta \in \mathcal{L}_{EL}|q$ that distinguishes the set of states $\{0, 1\}$. We now have that:

$$\begin{aligned} \mathcal{M}_n, 0 &\models \eta \\ \mathcal{M}_n|_{\eta}, 0 &\models \neg q \wedge K_a p \wedge \neg K_b K_a p \\ \mathcal{M}_n, 0 &\models \langle \eta \rangle (\neg q \wedge K_a p \wedge \neg K_b K_a p) \\ \mathcal{M}_n, 0 &\models \langle \subseteq \rangle (\neg q \wedge K_a p \wedge \neg K_b K_a p) \end{aligned}$$

2. On the other hand, $\mathcal{N}_n, 0 \not\models \langle \subseteq \rangle (\neg q \wedge K_a p \wedge \neg K_b K_a p)$. This is because we cannot use that r is only true in n , as $r \notin P(\neg q \wedge K_a p \wedge \neg K_b K_a p)$, and because $(\mathcal{N}_n, 0) \simeq^{pq} (O, 0)$. Clearly $O, 0 \not\models \langle \subseteq \rangle (\neg q \wedge K_a p \wedge \neg K_b K_a p)$.
3. However, $\mathcal{M}_n, 0 \models \psi$ iff $\mathcal{N}_n, 0 \models \psi$. This follows from Lemma 4, as $n > d(\psi)$ and $(\mathcal{M}_n, 0) \stackrel{d(\psi)}{\leftrightarrow} (\mathcal{N}_n, 0)$. \square

Proposition 10. $FSAPAL \not\leq APAL$.

Proof. As Prop. 9, but we now take FSAPAL formula $\langle q \rangle (\neg q \wedge K_a p \wedge \neg K_b K_a p)$ instead of SCAPAL formula $\langle \subseteq \rangle (\neg q \wedge K_a p \wedge \neg K_b K_a p)$. \square

As $[!]\varphi$ is equivalent to $[P]\varphi$ we rather trivially have that $APAL \preceq SAPAL$, so that with Prop. 10 and its consequence $SAPAL \not\leq APAL$ we immediately obtain:

Corollary 3. $APAL \prec SAPAL$.

3.4.3 SCAPAL \prec FSAPAL

We first show that $SCAPAL \preceq FSAPAL$, and then show that $SCAPAL \prec FSAPAL$.

Proposition 11. $SCAPAL \preceq FSAPAL$.

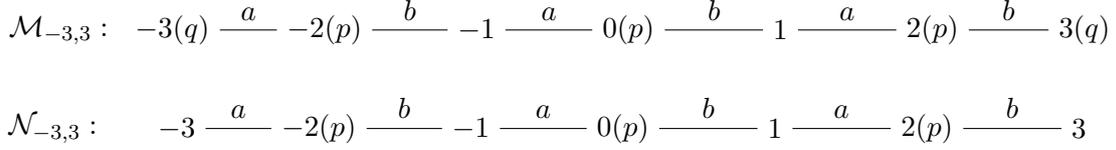
Proof. It is trivial that $SCAPAL \preceq FSAPAL$, since $\models [\subseteq]\varphi \leftrightarrow [P(\varphi)]\varphi$. Formally, we inductively define a translation function f from SCAPAL to FSAPAL by

$$\begin{aligned} f(p) &= p & f(\varphi \vee \psi) &= f(\varphi) \vee f(\psi) & f([\varphi]\psi) &= [f(\varphi)]f(\psi) \\ f(\neg\varphi) &= \neg f(\varphi) & f(K_a\varphi) &= K_a f(\varphi) & f([\subseteq]\varphi) &= [P(\varphi)]f(\varphi) \end{aligned}$$

In the final line we could equivalently have written $f([\subseteq]\varphi) = [P(f(\varphi))]f(\varphi)$, as f does not affect the set of atoms that occur in a formula. We then have $\models \varphi \leftrightarrow f(\varphi)$ (which is shown by induction), and therefore $SCAPAL \preceq FSAPAL$. \square

We now show $SCAPAL \prec FSAPAL$. In the proof we use models $\mathcal{M}_{-n,n}$ and $\mathcal{N}_{-n,n}$ similar to \mathcal{M}_n and \mathcal{N}_n used in the previous subsection. They are depicted in Fig. 3.4 for $n = 3$, compare to Fig. 3.3. (Imagine ‘cutting open’ \mathcal{M}_3 and \mathcal{N}_3 at the q resp. r state, and remove r as we can now use the distinguishing power of p on the edges of the chain.) Similarly to Lemma 4, we first show a Lemma 5.

Lemma 5. *Let $\mathcal{M} \subseteq \mathcal{M}_{-n,n}$, $\mathcal{N} \subseteq \mathcal{N}_{-n,n}$ and $i, j, k \in \mathbb{N}$, with $i \in \mathcal{M}$ and $j \in \mathcal{N}$. If $(\mathcal{M}, i) \simeq^k (\mathcal{N}, j)$, then $(\mathcal{M}, i) \equiv_{SCAPAL}^k (\mathcal{N}, j)$.*


 Figure 3.4: The models $\mathcal{M}_{-3,3}$ and $\mathcal{N}_{-3,3}$

Proof. We show by formula induction that $\mathcal{M}, i \models \varphi$ iff $\mathcal{N}, j \models \varphi$ for any $\varphi \in \mathcal{L}_{SCAPAL}$ with $d(\varphi) \leq k$. Cases $K_a\psi$ and $[\chi]\psi$ are the same. The case quantifier $[\subseteq]\psi$ is different and shown as follows.

First, suppose that $q \notin P(\psi)$. Then from $(\mathcal{M}, i) \leftrightarrow^{P(\psi)} (\mathcal{N}, j)$ and Lemma 1 it directly follows that $\mathcal{M}, i \models [\subseteq]\psi$ iff $\mathcal{N}, j \models [\subseteq]\psi$.

Next, suppose that $q \in P(\psi)$; w.l.o.g. we may also assume that $p \in P(\psi)$. By assumption, $(\mathcal{M}, i) \leftrightarrow^k (\mathcal{N}, j)$. Just as for Lemma 3, every $\mathcal{M}' \subseteq \mathcal{M}$ is definable in \mathcal{M} by a formula in $\mathcal{L}_{PAL}|pq$, and every $\mathcal{N}' \subseteq \mathcal{N}$ is definable in \mathcal{N} by a formula in $\mathcal{L}_{PAL}|pq$. It follows that for every $\chi \in \mathcal{L}_{PAL}|pq$ with $\mathcal{M}, i \models \chi$ there is a $\xi \in \mathcal{L}_{PAL}|pq$ such that $(\mathcal{M}|\chi, i) \leftrightarrow^k (\mathcal{N}|\xi, j)$, and vice versa. Therefore, $\mathcal{M}, i \models [\subseteq]\psi$ iff $\mathcal{N}, j \models [\subseteq]\psi$. \square

Proposition 12. $SCAPAL \prec FSAPAL$.

Proof. We proceed as usual, however, with distinguishing FSAPAL formula $\langle q \rangle (K_ap \wedge \neg K_b K_ap)$. Let ψ be the supposedly equivalent \mathcal{L}_{SCAPAL} formula. Take $n > d(\psi)$. Then:

1. $\mathcal{M}_{-n,n}, 0 \models \langle q \rangle (K_ap \wedge \neg K_b K_ap)$
2. $\mathcal{N}_{-n,n}, 0 \not\models \langle q \rangle (K_ap \wedge \neg K_b K_ap)$ (obvious)
3. $\mathcal{M}_{-n,n}, 0 \models \psi$ iff $\mathcal{N}_{-n,n}, 0 \models \psi$ (use $(\mathcal{M}_{-n,n}, 0) \leftrightarrow^{d(\psi)} (\mathcal{N}_{-n,n}, 0)$ & Lemma 5) \square

3.4.4 Results for IPAL

Let us first present all our results for IPAL in relation to the other logics in the contribution, with the exception of the proof that $APAL \prec IPAL$, our main result.

Proposition 13. $APAL \preceq IPAL$.

Proof. This follows from Prop. 6 that $[\top^\downarrow]\varphi$ is equivalent to $[\!]\varphi$. \square

We can also obtain strictness.

Proposition 14. $APAL \prec IPAL$.

Proof. The proof of this result is rather involved and presented in the next subsection. \square

The relative expressivity between IPAL and FSAPAL/SCAPAL mirrors the results already obtained between APAL and FSAPAL/SCAPAL.

Proposition 15. $IPAL \asymp FSAPAL$ and $IPAL \asymp SCAPAL$.

Proof. FSAPAL $\not\leq$ IPAL and SCAPAL $\not\leq$ IPAL are shown as FSAPAL $\not\leq$ APAL (Prop. 10) and SCAPAL $\not\leq$ APAL (Prop. 9), except that in the inductive case for the quantifier of the proof of Lemma 4 we do not consider all witnesses ψ for the quantifier $\langle ! \rangle$ but only those that imply the given χ in $\langle \chi^\downarrow \rangle$ or that are implied by the given χ in $\langle \chi^\downarrow \rangle$.

From APAL \preceq IPAL, APAL $\not\leq$ FSAPAL and APAL $\not\leq$ SCAPAL (Prop. 8), we immediately obtain IPAL $\not\leq$ FSAPAL and IPAL $\not\leq$ SCAPAL. \square

3.4.5 APAL \prec IPAL

This section contains the proof of Proposition 14.

Let us start this proof by defining the sets of models that we will use. These models consist of a base part

$$\begin{array}{ccc} s_1(p) & \xrightarrow{b} & s_2(p, q) \\ | & & \\ a & & \\ | & & \\ t_1(p) & \xrightarrow{b} & t_2(p, q) \end{array}$$

plus a number of *branches* of the form, for some $l \in \mathbb{N}$

$$u_0 \xrightarrow{b} u_1(p) \xrightarrow{a} u_2(p) \quad \dots \quad u_{l-2}(p) \xrightarrow{a/b} u_{l-1}(p)$$

The atom p holds in every state except u_1 , and the accessibility alternates between a and b , so the final agent may be a or b depending on whether l is even. We refer to the state u_0 as the *root* of the branch.

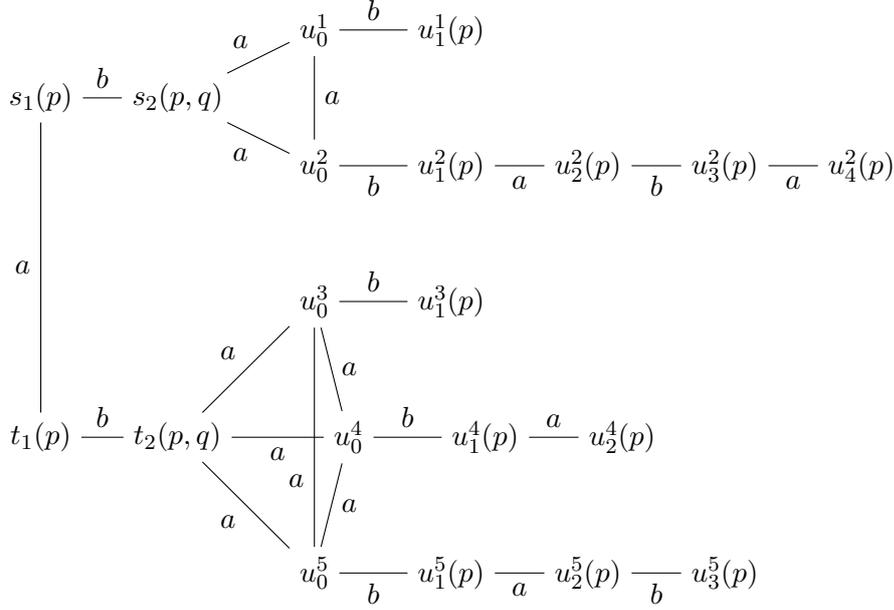
The models that we will consider consist of a base part, where both s_2 and t_2 are a -attached to any finite number of branches, possibly of different lengths. An example of such a model, where s_2 is attached to two branches of length 2 and 5, while t_2 is attached to three branches of length 2, 3 and 4, is shown in Figure 3.5. We say that two states in the model are *on the same side* if one can be reached from the other without using the a -edge between s_1 and t_1 , and *on the other side* otherwise.

We then divide these types of models into two sets: a set \mathfrak{N} where there is at least one length l such that both s_2 and t_2 are attached to at least one branch of length l , and a set \mathfrak{M} where there is no such shared length l . We will first show that IPAL can uniformly distinguish between these sets.

Lemma 6. *Let $\mathcal{M} \in \mathfrak{M}$ and $\mathcal{N} \in \mathfrak{N}$. Furthermore, let $\varphi = [p^\uparrow](\psi_1 \rightarrow \langle \top^\downarrow \rangle \psi_2)$, where $\psi_1 = \hat{K}_b \hat{K}_a \neg p$ and $\psi_2 = K_b \neg q \wedge \hat{K}_a \hat{K}_b q$. Then $\mathcal{M}, s_1 \models \varphi$ and $\mathcal{N}, s_1 \not\models \varphi$.*

Proof. The key observation is that announcing any epistemic formula χ implied by p can remove access to a branch by removing the state u_0^i for that branch, but it cannot change the length of a branch, or remove any of the states s_1, s_2, t_1 and t_2 , since all other states satisfy p , and therefore also χ .

Let χ then be any epistemic formula implied by p such that $\mathcal{M}|_\chi, s_1 \models \hat{K}_b \hat{K}_a \neg p$. Then at least one branch on the top side of the model is retained. Because $\mathcal{M} \in \mathfrak{M}$, there is no branch on the bottom side of the model that has the same length. This implies that $\mathcal{M}|_\chi, s_2$ and


 Figure 3.5: Typical model used in the proof that APAL \prec IPAL

$\mathcal{M}|_\chi$, t_2 are distinguishable by a modal formula. As such, there is an epistemic announcement ξ that removes s_2 while retaining s_1, t_1 and t_2 , so we have $(\mathcal{M}|_\chi)|_\xi, s_1 \models \psi_2$.

This suffices to show that $\mathcal{M}, s_1 \models [p^\uparrow](\psi_1 \rightarrow \langle \top^\downarrow \rangle \psi_2)$.

Regarding \mathcal{N} , since it is a member of \mathfrak{N} there is some l such that the top and bottom sides of \mathcal{N} both have a branch of length l . Let χ be the epistemic formula that retains a $\neg p$ state only if it is the root of a branch of length exactly l . Then we have $\mathcal{N}|_\chi, s_1 \models \psi_1$. Yet in $\mathcal{N}|_\chi$, the top and bottom side of the model are bisimilar, so there is no announcement that would retain t_2 while removing s_2 . Hence $\mathcal{N}|_\chi, s_1 \not\models \langle \top^\downarrow \rangle \psi_2$.

This suffices to show that $\mathcal{N}, s_1 \not\models [p^\uparrow](\psi_1 \rightarrow \langle \top^\downarrow \rangle \psi_2)$. \square

Left to show is that there is no APAL formula that similarly distinguishes between \mathfrak{M} and \mathfrak{N} . Unfortunately, this proof is significantly more complex than the other expressivity proofs in this paper. It is therefore useful to first introduce a few auxiliary definitions and lemmas.

Definition 26. Let (X, x) be a submodel of a model of type \mathfrak{M} or \mathfrak{N} . We classify (X, x) based on which worlds are retained, in the following way:

- If $x = u_i^j$ and u_0^j is not reachable from x in X , then (X, x) is a *dead branch*.
- If at least one state u_0^j is reachable from x in X and on the same side, then this side of (X, x) is a *bouquet*. If furthermore s_2 or t_2 is reachable on the same side, then the bouquet has a *stem* of length 1. If s_1 or s_2 is also reachable on the same side, then the bouquet has a *stem* of length 2.
- If s_1, s_2, t_1 or t_2 is same-side reachable from (X, x) but no state u_0^j is, then this side of (X, x) is a *dead stem*. Two dead stems have the same form if they both retain their q world or both remove it, and both retain their $\neg q$ world or both remove it.

Definition 27. Let (X, x) and (Y, y) be pointed submodels of models of type \mathfrak{M} or \mathfrak{N} and let $k \in \mathbb{N}$. We say that (X, x) and (Y, y) are k -akin if one of the following three conditions holds for both this side of the models and, if reachable, the other side:

- they are both dead branches,
- they are both bouquets with the same stem length and
 - for every $l \leq k$, if the bouquet in X has a branch of length l then so does the one in Y , and vice versa,
 - if the bouquet in X has exactly $m \leq (k + 1)^2$ branches of different lengths greater than k then so does the one in Y , and vice versa,
 - if the top and bottom sides of X and Y are each bouquets with stem 2, then the top and bottom side of X are bisimilar if and only if the top and bottom side of Y are bisimilar,
- (X, x) and (Y, y) are both dead stems of the same form.

Definition 28. Let $k \in \mathbb{N}$, and let (X, x) and (Y, y) be k -akin. The relation \approx^k is the restriction of the following relation to the connected parts of the two models:

- $(X, s_i) \approx^k (Y, s_i)$ and $(X, t_i) \approx^k (Y, t_i)$ for $i \in \{1, 2\}$,
- for every $0 \leq i \leq k - 1$, if (X, u_i^j) lies in a branch of length at most k , then $(X, u_i^j) \approx^k (Y, u_i^{j'})$, where $(Y, u_i^{j'})$ lies in the branch of Y at the same length at the same side¹, and vice versa,
- for every $0 \leq i \leq k - 1$, if (X, u_i^j) lies in a branch of length greater than k , then $(X, u_i^j) \approx^k (Y, u_i^{j'})$ for every $u_i^{j'}$ that lies in a branch of the same side of length greater than k , and vice versa,
- for every $i, i' \geq k$, if (X, u_i^j) and $(Y, u_{i'}^{j'})$ are on the same side, then $(X, u_i^j) \approx^k (Y, u_{i'}^{j'})$.

We will use \approx^k as the invariant in our inductive proof. One important property of \approx^k is that it is a k -bisimulation.

Lemma 7. *If $(X, x) \approx^k (Y, y)$ then for every a - or b -successor x' of x there is an a - or b -successor y' of y such that $(X, x') \approx^{k-1} (Y, y')$, and vice versa.*

Lemma 8. *The relation \approx^k is a k -bisimulation.*

The proofs are conceptually very simple, but still requires a lot of notation and different cases, so we omit them.

We have now completed all the preliminary work, and can prove the result that we are after.

Lemma 9. *Let $k \in \mathbb{N}$, and let $(X, x) \approx^k (Y, y)$. Then for every φ of depth at most k , we have $X, x \models \varphi$ iff $Y, y \models \varphi$.*

¹Which must exist because (X, x) and (Y, y) are k -akin.

Proof. By induction on formula construction. If φ is Boolean, then the lemma follows immediately from the fact that \approx^k is a k -bisimulation.

Suppose then as induction hypothesis that the lemma holds for all φ' that are strict subformulas of φ . Assume towards a contradiction that φ distinguishes between (X, x) and (Y, y) . Since the conditions of the lemma are symmetric we can assume without loss of generality that $X, x \models \varphi$ and $Y, y \not\models \varphi$.

A Boolean combination of formulas distinguishes between two states only if one of the combined formulas does. If the main connective of φ is Boolean it therefore follows immediately from the induction hypothesis that φ does not distinguish between (X, x) and (Y, y) . This leaves three cases for the main connective of φ : K_a , $[\psi]$ and $[!]$.

Suppose that $\varphi = K_a\psi$. Then $Y, y \not\models K_a\psi$, so there is an a -successor y' of y such that $Y, y' \not\models \psi$. By $(X, x) \approx^k (Y, y)$ there is an a -successor x' of x such that $(X, x') \approx^{k-1} (Y, y')$. By the induction hypothesis, together with the fact that $d(\psi) \leq k - 1$, we then have $X, x' \not\models \psi$, and therefore $X, x \not\models K_a\psi$, contradicting our assumption that $K_a\psi$ distinguishes between (X, x) and (Y, y) .

Suppose that $\varphi = [\psi]\chi$. Then $X, x \models [\psi]\chi$, and therefore either $X, x \not\models \psi$ or $X|\psi, x \models \chi$. In the first case, by the induction hypothesis we also have $Y, y \not\models \psi$, which implies that $Y, y \models [\psi]\chi$ contradicting the assumption that $[\psi]\chi$ distinguishes between the two pointed models.

In the second case, compare the models $X|\psi$ and $Y|\psi$. Because ψ is, by the induction hypothesis, invariant under \approx^k , for every branch in $X|\psi$ that is cut off at a length at most k , its counterpart in $Y|\psi$ is cut off at the same length. The states s_1, s_2, t_1 and t_2 are similarly retained in one model if and only if they are retained in the other. It follows that $(X|\psi, x) \approx^k (Y|\psi, y)$. By the induction hypothesis the two models are therefore indistinguishable by χ . This contradicts the assumption that $X, x \models [\psi]\chi$ and $Y, y \not\models [\psi]\chi$.

Finally, suppose that $\varphi = [!]\chi$. We assumed $X, x \models \varphi$ and $Y, y \not\models \varphi$, so there is some epistemic formula ψ such that $Y, y \not\models [\psi]\chi$. We will create an epistemic formula ψ' such that $(X|\psi', x) \approx^k (Y|\psi, y)$.

First, note that the top and bottom side of X are bisimilar if and only if the top and bottom side of Y are. If top and bottom are bisimilar, we can perform a bisimilarity contraction on both models, obtaining a model where one side has been removed entirely. As such, we can assume without loss of generality that if both sides of the model still exist, then they are non-bisimilar. Because all models under consideration are finite this also implies that the top and bottom sides are distinguishable by a modal formula.

As a result, every state can be uniquely identified by (1) whether it is on the top or bottom side, (2) if it is in a branch, the length of that branch and (3) its position in the branch or in the stem. By using a disjunction of characterizing formulas we can create a formula that retains an arbitrary set of identifiable states.

We now create ψ' as follows:

1. For each side, ψ' holds on the stem states iff ψ does.
2. For each side, if ψ removes a branch of length $l \leq k$, or trims it to a length $l' < l$, then so does ψ' .
3. For each side, if ψ cuts down a branch of length $l > k$ to $l' \leq k - 1$, then ψ' cuts down a branch of length $l'' > k$ to l' .

4. For each side, if ψ retains $m \leq k^2$ branches of length at least $k - 1$, then ψ' retains exactly m such branches as well.
5. For each side, if ψ retains $m > k^2$ branches of length at least $k - 1$ then ψ' retains $m' > k^2$ such branches.
6. If y ends up in a short branch in Y then x 's branch in X is cut to the same length.

Note that items 3–5 can be done because of the condition that X and Y contained either the same number of (different length) long branches, or more than $(k + 1)^2$ of them. Some long branches may be “consumed” to provide the branches of length $l \leq k - 1$. But ψ and ψ' consume the same number of branches, and at most $k - 1$ of them. So X has enough long branches to provide either the same number of long branches in $X|\psi'$ and $Y|\psi$ or at least k^2 of them.

By construction, the models $X|\psi'$ and $Y|\psi$ satisfy the conditions for being $(k - 1)$ -akin. Furthermore, x and y are in the same relative position, so $(X|\psi', x) \approx^{k-1} (Y|\psi, y)$. By the induction hypothesis we therefore have $X|\psi', x \models \chi$ iff $Y|\psi, y \models \chi$. This contradicts our assumption that $X, x \models [!]\chi$ and $Y, y \not\models [!]\chi$.

In each case, we arrived at a contradiction. So φ does not distinguish between (X, x) and (Y, y) , completing the induction step and thereby the proof. \square

We have now shown that there is an IPAL formula that distinguishes between \mathfrak{M} and \mathfrak{N} (Lemma 6) and that there is no APAL formula that similarly distinguishes the two sets (Lemma 9). This implies that there is no APAL formula that is equivalent to the distinguishing IPAL formula. So we have $\text{IPAL} \not\preceq \text{APAL}$. Together with $\text{APAL} \preceq \text{IPAL}$ (Proposition 13), this yields the result that that we were after, namely that $\text{APAL} \prec \text{IPAL}$ (Proposition 14).

3.5 Decidability and undecidability of satisfiability

The satisfiability problem of APAL is decidable when there is only one agent, whereas it is undecidable when there are at least two agents [41, 6]. The approach is by encoding/formalizing an undecidable tiling problem into APAL [22]. There are some decidable logics with quantification over information change, for example Boolean arbitrary public announcement logic [98]. It is therefore a relevant question whether the APAL versions considered in this paper are decidable. It turns out that they are all *undecidable* (for more than one agent). We prove this by referring to the undecidability proof in [6] and listing, for each of SAPAL, SCAPAL and IPAL, the exact changes needed in that proof in order to show undecidability. For all proof details and proof structure we refer to [6].

Proposition 16. *The satisfiability problem for SAPAL, FSAPAL, SCAPAL and IPAL is undecidable.*

Proof. In [6] it is shown that, given a finite set C of colours, there is an APAL formula φ that formalizes an undecidable tiling problem of tiles coloured with C . We cannot determine whether φ is satisfiable as this would solve the tiling problem. Therefore the satisfiability problem of APAL is undecidable. This formula φ has many constituents that describe properties that need to be satisfied by the tiling, and APAL quantifiers occur in the formulas describing such properties (see Example 3 below). For each of SAPAL, SCAPAL and IPAL there is a very simple way to translate these $\mathcal{L}_{\text{APAL}}$ formulas into equivalent $\mathcal{L}_{\text{SAPAL}}$, $\mathcal{L}_{\text{SCAPAL}}$, respectively

\mathcal{L}_{IPAL} formulas. Furthermore, for SAPAL the translation only uses finite sets of variables, so it is a translation to FSAPAL as well.

First, we note that the APAL undecidability proof in [6] only uses two agents and a *finite set* $C \cup \Lambda$ of atoms that is the union of a finite set C of colours plus a set $\Lambda = \{u, d, l, r, \heartsuit, \clubsuit, \diamond, \spadesuit\}$. Let us at least explain the intuitive meaning of these different atoms. The properties formalized in the proof describe the requirements to tile an infinite grid where a square in the grid has four sides u, d, l, r (for ‘up’, ‘down’, ‘left’ and ‘right’) and where each square is labeled with one of $\heartsuit, \clubsuit, \diamond, \spadesuit$. The four sides of the tiles have colours from C and the colours of adjoining tiles positioned on the grid have to match. No other atoms are required.

The required truth (value) preserving translations from \mathcal{L}_{APAL} to \mathcal{L}_X , where X is one of SAPAL, SCAPAL and IPAL, are now as follows. We recall the above $\varphi \in \mathcal{L}_{APAL}$ encoding the tiling. Then:

- For SAPAL, replace each occurrence of $[!]$ in φ by $[C \cup \Lambda]$.
- For SCAPAL, let $\top_{C \cup \Lambda} := \bigwedge_{p \in C \cup \Lambda} (p \vee \neg p)$. Now replace each subformula of φ of shape $[!]\psi$ by $[\subseteq](\psi \wedge \top_{C \cup \Lambda})$.²
- For IPAL, replace each occurrence of $[!]$ in φ by $[\top^\downarrow]$.

These translations are indeed adequate. For SAPAL it is sufficient to observe that the set of atoms P considered is $C \cup \Lambda$ and that $[!]\varphi$ is equivalent to $[P]\varphi$ for the entire (finite) set of atoms. The case SCAPAL is slightly more complex, as the witnesses of a constituent of shape $[!]\psi$ of the tiling formula φ may need more atoms than are occurring in the formula ψ (as demonstrated below in Example 3). The translation simply forces any formula bound by a quantifier to employ all atoms in the language by adding another conjunct that does not affect the truth value as it is always true. Finally, Prop. 6 showed that $[\top^\downarrow]\varphi$ is equivalent to $[!]\varphi$.

Apart from these translations, no other adjustments to the proof in [6] are needed. \square

Example 3. A constituent of the formula φ encoding the tiling of the plain is as follows, where \mathfrak{s} and \mathfrak{e} are the two agents used in the proof. It says that for any square of the infinite grid labelled with a \heartsuit , there is some square below some square to the left of some square above some square to the right of that square, that is n -bisimilar (that is, a square that is also labelled with \heartsuit , but now the occurrence on the right-hand side of the formula below). See [6, page 617].

$$c_{apal}(\heartsuit) := \heartsuit \rightarrow [!](K_{\mathfrak{s}}(r \rightarrow (K_{\mathfrak{e}}(l \rightarrow (K_{\mathfrak{s}}(u \rightarrow K_{\mathfrak{e}}(d \rightarrow K_{\mathfrak{s}}(l \rightarrow K_{\mathfrak{e}}(r \rightarrow K_{\mathfrak{s}}(d \rightarrow K_{\mathfrak{e}}(u \rightarrow \hat{K}_{\mathfrak{s}}\heartsuit))))))))))$$

For SAPAL this $c_{apal}(\heartsuit)$ is translated into:

$$c_{fsapal}(\heartsuit) := \heartsuit \rightarrow [\Lambda_C](K_{\mathfrak{s}}(r \rightarrow (K_{\mathfrak{e}}(l \rightarrow (K_{\mathfrak{s}}(u \rightarrow K_{\mathfrak{e}}(d \rightarrow K_{\mathfrak{s}}(l \rightarrow K_{\mathfrak{e}}(r \rightarrow K_{\mathfrak{s}}(d \rightarrow K_{\mathfrak{e}}(u \rightarrow \hat{K}_{\mathfrak{s}}\heartsuit))))))))))$$

For SCAPAL, this $c_{apal}(\heartsuit)$ is translated into the following. Here, it is relevant to observe that in the proof in [6] the witness formula for this occurrence of $[!]$ contains the atoms \clubsuit, \diamond and

²More properly, we should see this as an inductively defined translation $t : \mathcal{L}_{APAL} \rightarrow \mathcal{L}_{SCAPAL}$ with only non-trivial clause $t([!]\varphi) := [\subseteq](t(\varphi) \wedge \top_{C \cup \Lambda})$.

♠ that do not occur in $c_{apal}(\heartsuit)$. Therefore, without the trivially true conjunct $\top_{C\cup A}$ used in the translation, this witness would not have been available. Merely replacing $[!]$ by $[\sqsubseteq]$ in $c_{apal}(\heartsuit)$ would have resulted a formula with a different meaning.

$$c_{scapal}(\heartsuit) := \heartsuit \rightarrow [\sqsubseteq](K_s(r \rightarrow (K_c(l \rightarrow (K_s(u \rightarrow K_c(d \rightarrow K_s(l \rightarrow K_c(r \rightarrow K_s(d \rightarrow K_c(u \rightarrow \hat{K}_s\heartsuit)))))))))) \wedge \top_{C\cup A})$$

For IPAL, $c_{apal}(\heartsuit)$ is translated into:

$$c_{ipal}(\heartsuit) = \heartsuit \rightarrow [\top^\perp](K_s(r \rightarrow (K_c(l \rightarrow (K_s(u \rightarrow K_c(d \rightarrow K_s(l \rightarrow K_c(r \rightarrow K_s(d \rightarrow K_c(u \rightarrow \hat{K}_s\heartsuit))))))))))$$

3.6 Axiomatization

In this section we report on axiomatizations of the logics under consideration. The known axiomatization of APAL is infinitary (non-RE) [16, 17]. The infinitary axiomatizations of SAPAL (and also of FSAPAL, as a special case) and SCAPAL are straightforward variations of the axiomatization of APAL, and can similarly be proved sound and complete. It requires merely checking very few and very local changes of the completeness proof, as we will see. These axiomatizations we can confidently present as results. It seems the axiomatization of QIPAL (and of IPAL, as a special case) is similarly a variation of that of APAL, but the adjustments there are larger and require checking details in various parts of the completeness proof. It seems then advisable to redo the entire proof, so that the concerned reader can check the correctness of the argument. This is beyond the scope of our current investigation and therefore, as they say, referred to further research. However, the value of such further research may be limited, if the conjectured axiomatization is the only outcome. More adventurous pursuits, such as the reported search for finitary (RE) axiomatizations for APAL variations, may then be worthier.

Definition 29 (Axiomatizations SAPAL and SCAPAL). The axiomatization **SAPAL** of SAPAL is as **APAL** (Def. 14) but where the axiom and rule involving the quantifier are replaced by (where $Q \subseteq \mathcal{P}$):

$$\begin{array}{ll} \mathbf{A}[Q] & [Q]\varphi \rightarrow [\psi]\varphi \text{ where } \psi \in \mathcal{L}_{EL}|Q \\ \mathbf{R}[Q] & \text{From } \chi([\psi]\varphi) \text{ for all } \psi \in \mathcal{L}_{EL}|Q \text{ infer } \chi([Q]\varphi) \end{array}$$

The axiomatization **SCAPAL** of SCAPAL is as **APAL** but where the axiom and rule involving the quantifier are replaced by:

$$\begin{array}{ll} \mathbf{A}[\sqsubseteq] & [\sqsubseteq]\varphi \rightarrow [\psi]\varphi \text{ where } \psi \in \mathcal{L}_{EL}|P(\varphi) \\ \mathbf{R}[\sqsubseteq] & \text{From } \chi([\psi]\varphi) \text{ for all } \psi \in \mathcal{L}_{EL}|P(\varphi) \text{ infer } \chi([\sqsubseteq]\varphi) \end{array}$$

Proposition 17. *The axiomatizations **SAPAL** and **SCAPAL** are sound and complete.*

Proof. It suffices to sketch the proof. The soundness of the axiomatizations **SAPAL** and **SCAPAL** is evident as the axiom and the rule follow the semantics of, respectively, the $[Q]$ and $[\sqsubseteq]$ quantifier. All remaining axioms and rules are standard from PAL. The completeness proof proceeds exactly as in [17], with very minimal changes: the quantifier $[!]$ only features in the subinductive case $[\psi][!]\chi$ and in the inductive case $[!]\psi$ of the proof of the Truth Lemma

[17, pages 75–76]. Apart from changing the notation of the quantifier, it suffices to replace four occurrences of the word ‘epistemic formulas’, i.e., $\psi \in \mathcal{L}_{EL}$, by ‘epistemic formulas in $\mathcal{L}_{EL}|Q$ ’ respectively ‘epistemic formulas in $\mathcal{L}_{EL}|P(\varphi)$.’ This minimal change is sufficient because the Truth Lemma for **APAL** is proved by a lexicographic complexity measure wherein $[\psi]\varphi$ is less complex than $[!]\varphi$ for any $\psi \in \mathcal{L}_{EL}$, for the simple reason that $[\psi]\varphi$ contains one less quantifier than $[!]\varphi$. Similarly, $[\psi]\varphi$ is less complex than $[Q]\varphi$ and than $[P(\varphi)]\varphi$ for any $\psi \in \mathcal{L}_{EL}$. No other changes are required in the completeness proof. \square

Let us now consider IPAL^\downarrow . Given the semantics of the quantifier and Proposition 5.4 the candidate axiom and rule are as follows:

$$\begin{array}{ll} \mathbf{A}[\downarrow] & [\eta^\downarrow]\varphi \rightarrow [\psi \wedge \eta]\varphi \text{ where } \psi \in \mathcal{L}_{EL} \\ \mathbf{R}[\downarrow] & \text{From } \chi([\psi \wedge \eta]\varphi) \text{ for all } \psi \in \mathcal{L}_{EL} \text{ infer } \chi([\eta^\downarrow]\varphi) \end{array}$$

This still seems to be sufficient to demonstrate completeness, with, given the presence of an additional formula η , minor further adjustments of the proof for APAL.

We have not considered the case IPAL^\uparrow .

Now consider QIPAL. Instead of the changed axiom and rule above we would now need two rules (and two similar rules for the other quantifier):

$$\begin{array}{ll} \mathbf{RA}[\downarrow] & \text{From } \psi \rightarrow \eta \text{ infer } [\eta^\downarrow]\varphi \rightarrow [\psi]\varphi \text{ where } \psi \in \mathcal{L}_{EL} \\ \mathbf{RR}[\downarrow] & \text{From } \chi([\psi]\varphi) \text{ for all } \psi \in \mathcal{L}_{EL} \text{ such that } \psi \rightarrow \eta, \text{ infer } \chi([\eta^\downarrow]\varphi) \end{array}$$

It may be that completeness can still be obtained for this system, but this would require more checks, for example we appear to need a slightly changed complexity measure in the completeness proof, such that $\psi \rightarrow \eta <_{d_\square}^{Size} \chi([\eta^\downarrow]\varphi)$ [17, page 68]. At this stage it therefore seems best to relegate all this to conjectures.

3.7 IPAL, substructural logics and dynamic consequence

3.7.1 Introduction

In this section we discuss the motivation for the (Q)IPAL \downarrow -quantifier, connecting it with the implication connective of *substructural logics* [43, 73, 78, 85]. This connection is explored also via a brief study of a *dynamic consequence relation* [92, 95] arising from the notion of IPAL validity.

In a nutshell, our semantics of $[\varphi^\downarrow]\psi$ is loosely inspired by the satisfaction clause for implication in the relational semantics for substructural logic, according to which “ φ implies ψ ” is satisfied in a state iff combining that state with any state satisfying φ will result in a state satisfying ψ . Information update is one natural reading of “combining states” and “any state satisfying φ ” translates in the information update setting into looking at updates with any formula implying φ . The dynamic consequence relation arising from the notion of IPAL validity is not closed under most of the usual structural rules, nor under substitution, although it satisfies a form of weakening even stronger than that satisfied by van Benthem’s dynamic consequence arising from PAL. Details follow.

3.7.2 Substructural logics and implication

Substructural logics are logics weaker than classical Boolean logic. The name reflects the fact that their Gentzen-style formulations are obtained, roughly speaking, by omitting some (or

all) *structural rules* of Gentzen’s sequent calculus for intuitionistic logic, most prominently *weakening* (i), *contraction* (c) and *commutativity* or “exchange” (e)

$$\frac{\Gamma, \Delta \Rightarrow \psi}{\Gamma, \varphi, \Delta \Rightarrow \psi} \text{ (i)} \quad \frac{\Gamma, \varphi, \varphi, \Delta \Rightarrow \psi}{\Gamma, \varphi, \Delta \Rightarrow \psi} \text{ (c)} \quad \frac{\Gamma, \varphi, \chi, \Delta \Rightarrow \psi}{\Gamma, \chi, \varphi, \Delta \Rightarrow \psi} \text{ (e)}.$$

Other structural rules featuring in this contribution are *strong weakening* (si), *left monotonicity* (lm), *cautious monotonicity* (cm) and *reflexivity* (r).

$$\frac{\Gamma, \Delta \Rightarrow \varphi}{\Gamma, \Sigma, \Delta \Rightarrow \varphi} \text{ (si)} \quad \frac{\Gamma \Rightarrow \varphi}{\psi, \Gamma \Rightarrow \varphi} \text{ (lm)} \quad \frac{\Gamma \Rightarrow \varphi \quad \Gamma, \Delta \Rightarrow \psi}{\Gamma, \varphi, \Delta \Rightarrow \psi} \text{ (cm)} \quad \frac{}{\Gamma, \varphi \Rightarrow \varphi} \text{ (r)}$$

Substructural logics have general algebraic semantics [43] but — similarly as in modal logic — models of a more concrete kind are better at facilitating fruitful interpretations. Substructural logics also have general *relational semantics* [81, 82], directly inspired by Kripke semantics for modal logic. In order to understand the key aspects of this semantics, one needs to take into account the role played by *implication*, namely, that implication *internalizes consequence* in the sense that

$$\Gamma, \varphi \Rightarrow \psi \quad \text{iff} \quad \Gamma \Rightarrow \varphi \rightarrow \psi.$$

Relational semantics for substructural logics treat implication as a *binary modal operator*, that is, the relational models contain a *ternary accessibility relation* R between states (pieces of information) x, y, z that is referred to in the satisfaction (denoted \models) condition for formulas of the form $\varphi \rightarrow \psi$:

$$x \models \varphi \rightarrow \psi \quad \text{iff} \quad \text{for all } y \text{ and } z, Rxyz \text{ and } y \models \varphi \text{ imply } z \models \psi.$$

This is an obvious generalization of the standard Kripke satisfaction condition for formulas of the form $\Box\psi$. General readings indicating the relation of the ternary semantics to various notions of conditionality have been proposed in [21]. Another approach (that is, however, not completely orthogonal to the former one) is to read R in terms of *combining pieces of information*. Dunn and Restall point out that:

“perhaps the best reading [of $Rxyz$] is to say that the combination of the pieces of information x and y (not necessarily the union) is a piece of information in z ” [35, p. 67].

Restall adds that:

“a body of information warrants $\varphi \rightarrow \psi$ if and only if whenever you *update* that information with new information which warrants φ , the resulting (perhaps new) body of information warrants ψ ” [79, p. 362] (notation adjusted).

On the informational reading, substructural implication clearly resembles an information update operator; see also [10, 11] where it is observed that dynamic epistemic logic can be seen as a two-sorted substructural logic, and that the product update is a special case of the ternary accessibility relation. The question is, what kind of update operator does substructural implication represent? Our semantics of $[\varphi^\dagger]\psi$ modify PAL announcements so that the result reflects the “non-determinism” of substructural implication — in evaluating $\varphi \rightarrow \psi$ at a given state, there is no one “canonical” piece of information representing φ that is combined with the

given state (think of the truth set of φ in the PAL satisfaction clause), but usually a number of them is considered. In the semantics of $[\varphi^\downarrow]\psi$ the role of these various pieces of information is played by formulas implying φ (or, rather, by truth sets of formulas implying φ).

The question is, how does this notion compare to substructural implication on the one hand and to PAL announcements on the other. A study of dynamic consequence relations is a particularly useful way of comparison.

3.7.3 Dynamic consequence

We now define a novel dynamic consequence relation \Rightarrow^\downarrow .

Definition 30 (Dynamic consequence). Let Σ be a finite (possibly empty) sequence of EL-formulas and φ a EL-formula.

$$\begin{array}{ll} \text{IPAL dynamic consequence} & \Sigma \Rightarrow^\downarrow \varphi \quad \text{iff} \quad \models [\Sigma^\downarrow]\varphi \\ \text{PAL dynamic consequence} & \Sigma \Rightarrow^! \varphi \quad \text{iff} \quad \models [\Sigma]\varphi \end{array}$$

Relation \Rightarrow^\downarrow can be seen as a variant of $\Rightarrow^!$ which is van Benthem's dynamic consequence relation in its "local" version [92, 95].

We can see that, trivially, the (Q)IPAL \downarrow -quantifier internalizes \Rightarrow^\downarrow similarly as substructural implication internalizes \Rightarrow : we have $\Gamma, \varphi \Rightarrow^\downarrow \psi$ iff $\Gamma \Rightarrow^\downarrow [\varphi^\downarrow]\psi$. As shown below, \Rightarrow^\downarrow differs from $\Rightarrow^!$ by satisfying a stronger version of weakening, and it shares with $\Rightarrow^!$ a number of other properties usually not present in substructural consequence relations (such as not being closed under substitution). This shows that, despite certain resemblances, one would need to further modify the (Q)IPAL \downarrow -quantifier to mimic substructural implication in a PAL-like setting, and vice versa.

Lemma 10. $\models [\psi^\downarrow]\varphi$ implies $\models [\psi^\downarrow][\chi^\downarrow]\varphi$.

Proof. This follows from Proposition 7 that $[\psi^\downarrow]\varphi \rightarrow [\psi^\downarrow][\chi^\downarrow]\varphi$. □

We recall from Example 2 that $\not\models [\psi^\downarrow]\varphi \rightarrow [\chi^\downarrow][\psi^\downarrow]\varphi$. Despite that, we still have that:

Lemma 11. $\models [\psi^\downarrow]\varphi$ implies $\models [\chi^\downarrow][\psi^\downarrow]\varphi$.

Proof. Assume $\models [\psi^\downarrow]\varphi$. Now suppose towards a contradiction that $\not\models [\chi^\downarrow][\psi^\downarrow]\varphi$. Let (M, s) be such that $M, s \not\models [\chi^\downarrow][\psi^\downarrow]\varphi$. Then there is η implying χ such that $M|\eta, s \not\models [\psi^\downarrow]\varphi$. This contradicts assumption $\models [\psi^\downarrow]\varphi$. □

Proposition 18. IPAL dynamic consequence is closed under strong weakening (si).

Proof. Lemma 10 says in other words that $\psi \Rightarrow^\downarrow \varphi$ implies $\psi\chi \Rightarrow^\downarrow \varphi$, whereas Lemma 11 says in other words that $\psi \Rightarrow^\downarrow \varphi$ implies $\chi\psi \Rightarrow^\downarrow \varphi$. We can show that, for arbitrary sequences, $\Gamma, \Delta \Rightarrow^\downarrow \varphi$ implies $\Gamma, \Sigma, \Delta \Rightarrow^\downarrow \varphi$, by an induction on the length of the sequences involved, using the above sequent representations of Lemma 10 and Lemma 11. □

As observed by van Benthem, $\Rightarrow^!$ does not satisfy (si). For example, $[\neg K_{ap}]\neg K_{ap}$ is valid, but $[\neg K_{ap}][p]\neg K_{ap}$ is not valid. This is therefore a difference between \Rightarrow^\downarrow and $\Rightarrow^!$.

As a corollary to Proposition 18, \Rightarrow^\downarrow also satisfies the structural rules *left monotonicity* (lm) and *cautious monotonicity* (cm). These are also satisfied by $\Rightarrow^!$ [95].

Finally note that \Rightarrow^\downarrow (as well as \Rightarrow^\uparrow or any other conceivable dynamic consequence relation involving public announcements) does not satisfy reflexivity (r). For example, it is elementary that $\models [p^\downarrow]p$ (that is, $p \Rightarrow^\downarrow p$), whereas on the other hand, just as elementary, $\not\models [(p \wedge \neg K_a p)^\downarrow](p \wedge \neg K_a p)$. Just as PAL is not closed under substitution, also IPAL is not closed under substitution. Hence, \Rightarrow^\downarrow is not a consequence relation in the Tarskian sense.

3.8 Conclusions

In this chapter, we investigated some logics that are almost APAL but not quite: the logics FSAPAL, SCAPAL and IPAL. They distinguish themselves by their widely varying relative expressivity. On the other hand, their axiomatizations are very similar to that of APAL, and they also have undecidable satisfiability problems. We have shown that the IPAL quantifier, motivated by the satisfaction clause for substructural implication, yields a substructural dynamic consequence relation differing from van Benthem's dynamic consequence based on PAL.

Logics for knowability

4.1 Introduction

Intuitively, a proposition is known to you, if you know it; in contrast, a proposition is knowable for you, if you can get to know it. The knowability paradox is that if all truths are knowable, then all truths are actually known. The standard references for the knowability paradox are [39] and [26]. However, following Salerno’s archival efforts the obligatory precursor to that Church’s ‘anonymous’ referee report of what (much) later became [39]:

(...) there is always a true proposition which it is empirically impossible for a to know at time t . For let k be a true proposition which is unknown to a at time t , and let k' be the proposition that k is true but unknown to a at time t . Then k' is true. But it would seem that if a knows k' at time t , then a must know k at time t , and must also know that he does not know k at time t . ([30], reprinted in [84])

Fitch finally writes:

If there is some true proposition which nobody knows (or has known, or will know) to be true, then there is some true proposition that nobody can know to be true.
[39, p.139]

Formally, ‘proposition φ is knowable’ later became $\diamond K\varphi$ [26], where \diamond is some modal diamond, representing a process, or time, or some alethic modality of truth. This modal diamond does not yet occur in [39]. Let us sketch the paradox. The existence of unknown truths is semi-formalized as “there is a proposition φ such that $\varphi \wedge \neg K\varphi$ ”. That all truths are knowable is semi-formalized as “for all propositions ψ , $\psi \rightarrow \diamond K\psi$ ”. Fitch’s paradox is that the existence of unknown truths is inconsistent with the requirement that all truths are knowable. This can now be easily shown: let ψ be $\varphi \wedge \neg K\varphi$, then we get $(\varphi \wedge \neg K\varphi) \rightarrow \diamond K(\varphi \wedge \neg K\varphi)$. On the assumption of $\varphi \wedge \neg K\varphi$, we therefore obtain $\diamond K(\varphi \wedge \neg K\varphi)$. Whatever the interpretation of \diamond , this will result in having to evaluate $K(\varphi \wedge \neg K\varphi)$. But this is inconsistent for knowledge, as can be shown by very simple means: since knowing a conjunction entails knowing each of the conjuncts, we obtain $K\varphi$ and $K\neg K\varphi$ from this, and from the latter and that knowledge

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entails truth, $\neg K\varphi$, and $K\varphi \wedge \neg K\varphi$ is inconsistent.³ This is of course Church’s argument cited above. It is also inconsistent for belief, as was already observed by Hintikka [52].

Knowability is a subjective concept; it is possible that a proposition is knowable for an agent but not for another. Take the proposition “it is raining but Alice does not know it” as an example. This proposition is not knowable for Alice, as above. But the proposition is knowable for another agent Bob, who may be aware of Alice’s ignorance. We are moving from $\diamond K\varphi$ to $\diamond K_a\varphi$ and $\diamond K_b\varphi$.

Since Fitch’s 1963 publication, the topic of *knowability* has done the rounds of philosophical communities, see e.g. [84, 88, 34]. The knowability paradox is relevant in verificationism and anti-realism. The verification principle requires a non-analytic, meaningful true sentence to be empirically verifiable [14]. Replace ‘empirically verifiable’ for ‘knowable’ (or recall ‘empirically impossible for a to know’, cited above) and we are there. Anti-realism or non-realism is the philosophy that denies the existence of an objective reality of entities. In other words, there are no true unknowable propositions: a true proposition about the objective reality that has no counterpart in a knowing subject would be such an unknowable proposition [33].

A dynamic view for knowability was subsequently proposed by van Benthem [93]. According to this dynamic view, knowable means ‘known after some announcement’, where ‘announcement’ is the truthfully public announcement of what is indeed known as PAL [76]. A logic extending public announcement logic with this notion of knowability was proposed in the logic APAL [16].

Unlike PAL, APAL is undecidable, has an infinitary axiomatization, and even model checking is already highly complex (PSPACE complete [1]). In [103] it was subsequently shown that after all *everything is knowable* in the sense that in this logic, $\langle ! \rangle K\varphi \vee \langle ! \rangle K\neg\varphi$ is valid; in other words, everything is knowable to be true or false. But some kind of cheating is involved: for example, $p \wedge \neg K_a p$ is ‘knowable’ in this sense, because after Bob announcing this to Alice it has become false: Alice now knows p , $K_a p$, which entails $\neg(p \wedge \neg K_a p)$.

In this investigation we will consider the combination $\langle ! \rangle K$ as a primitive modality in the logical language, and investigate the properties of various logics with this modality. Instead of $\langle ! \rangle K$, or rather $\langle ! \rangle K_i$ for an agent i , we will then write $\langle ! \rangle_i$, but this is mere syntactic sugar: the point is that we are not allowed to use the $\langle ! \rangle$ modality independently, but only followed by K_i . This technique of packing or bundling a knowledge modality with another modality (or a quantifier) was pioneered in works by [66]. As one may see, this packing can help us see the logical properties of knowability, such as McKinsey and Church-Rosser, more clearly. As can be expected, this may affect the properties of the logic, for example its expressivity, or complexity, or even the existence of an axiomatization. Such logics with a primitive ‘knowability’ modality $\langle ! \rangle_i$ will be called *logics for knowability*.⁴ We will focus on matters involving expressivity, axiomatization and decidability of such knowability logics. In particular, we show that the logic that is like APAL but instead of the \diamond modality has the packed $\langle ! \rangle_i$ modality also has a complete axiomatization, and we demonstrate various logical properties of $\langle ! \rangle_i$. Moreover, as we will show, although the full knowability logic is undecidable for at least three agents, two of its fragments *are* decidable, since both of them are equally expressive as the classical propositional logic.

³Instead of using the two properties of knowledge in question, one can show in the monotone logic of unknown truths [38] that the unknown truths $\varphi \wedge \neg K\varphi$ is not known.

⁴Although the method to pack two modalities into one is different from the usual modelling of the knowability paradox, the formalization of the paradox still requires two modalities, namely the novel knowability modality as well as the knowledge modality (see Coro. 6 below).

The remainder is organized as follows. After introducing the syntax and semantics of knowability logics and other related logics (Section 4.2), we investigate the logical properties of knowability and also a fragment of positive formulas in Section 4.3. Section 4.4 introduces the bisimulation for a knowability logic LK and compares the relative expressivity of LK and some related logics. Section 4.5 proposes an axiomatization of LK and shows its soundness. Section 4.6 shows its completeness of **LK**, and explore the decidability of LK, which turns out to be undecidable when there are at least three agents. We then propose two decidable knowability logics, which are both equally expressive as the classical propositional logic PL, and axiomatize them in Section 4.7. Finally we conclude with some future work in Section 4.8.

4.2 Syntax and Semantics

In what follows, we let \mathbf{P} denote a denumerable set of propositional variables, and \mathbf{A} a finite set of agents.

Definition 31 (Languages). We consider various fragments of the following recursively defined language \mathcal{L} :

$$\varphi ::= p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid K_i\varphi \mid \langle\varphi\rangle\varphi \mid \langle!\rangle_i\varphi \mid \langle!\rangle\varphi$$

where $p \in \mathbf{P}$ and $i \in \mathbf{A}$.

Without the construct $\langle!\rangle\varphi$, we obtain the language LK of *knowability logic*; without the construct $K_i\varphi$ as well, we obtain the language LK⁻; without the construct $\langle\varphi\rangle\varphi$ further, we obtain the language LK⁼. Without the construct $\langle!\rangle_i\varphi$, we obtain the language APAL of arbitrary public announcement logic; without additionally the construct $\diamond\varphi$, we obtain the language PAL of public announcement logic; without additionally the construct $\langle\varphi\rangle\varphi$, we obtain the language EL; without even the construct $K_i\varphi$ additionally, we obtain the language PL of classical propositional logic.

Although we have different primitives $\langle!\rangle_i$ and $\langle!\rangle$, we could alternatively have defined $\langle!\rangle_i$ by abbreviation as the ‘packing’ or ‘bundling’ of K_i and $\langle!\rangle$, namely as $\langle!\rangle_i\varphi := \langle!\rangle K_i\varphi$, such that the inductive definition of LK could have been given as $\varphi ::= p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid K_i\varphi \mid \langle\varphi\rangle\varphi \mid \langle!\rangle K_i\varphi$. Instead, we will now after the presentation of the semantics have this as a property of the complete language \mathcal{L} . The main focus of our investigations is the logic LK.

Intuitively, $K_i\varphi$, $\langle\psi\rangle\varphi$, $\langle!\rangle_i\varphi$, and $\langle!\rangle\varphi$ are read, respectively, “agent i knows that φ ”, “after some truthful announcement of ψ , it holds that φ ”, “ φ is knowable for agent i ”, “after some truthful announcement, it holds that φ ”. Other connectives are defined as usual. In particular, we abbreviate $\hat{K}_i\varphi$, $[\psi]\varphi$, $[\!]_i\varphi$, and $[\!]\varphi$ as, respectively, $\neg K_i\neg\varphi$, $\neg\langle\psi\rangle\neg\varphi$, $\neg\langle!\rangle_i\neg\varphi$, and $\neg\langle!\rangle\neg\varphi$. Moreover, $\text{var}(\varphi)$ is the set of propositional variables occurring in φ .

Definition 32 (Semantics). Given an epistemic model $\mathcal{M} = \langle S, \{R_i \mid i \in \mathbf{A}\}, V \rangle$ and a state $s \in S$, the formulas of \mathcal{L} are interpreted recursively as follows:

$$\begin{aligned} \mathcal{M}, s \models p & \iff s \in V(p) \\ \mathcal{M}, s \models \neg\varphi & \iff \mathcal{M}, s \not\models \varphi \\ \mathcal{M}, s \models \varphi \wedge \psi & \iff \mathcal{M}, s \models \varphi \text{ and } \mathcal{M}, s \models \psi \\ \mathcal{M}, s \models K_i\varphi & \iff \mathcal{M}, t \models \varphi \text{ for all } t \in R_i(s) \\ \mathcal{M}, s \models \langle\psi\rangle\varphi & \iff \mathcal{M}, s \models \psi \text{ and } \mathcal{M}|_\psi, s \models \varphi \\ \mathcal{M}, s \models \langle!\rangle_i\varphi & \iff \text{for some formula } \psi \in \text{EL} : \mathcal{M}, s \models \langle\psi\rangle K_i\varphi \\ \mathcal{M}, s \models \langle!\rangle\varphi & \iff \text{for some formula } \psi \in \text{EL} : \mathcal{M}, s \models \langle\psi\rangle\varphi \end{aligned}$$

where $\mathcal{M}|_\psi = \langle S', \{R'_i \mid i \in \mathbf{A}\}, V' \rangle$ is such that $S' = \llbracket \varphi \rrbracket_{\mathcal{M}} = \{s \in S \mid \mathcal{M}, s \models \varphi\}$, $R'_i = R_i \cap (\llbracket \varphi \rrbracket_{\mathcal{M}} \times \llbracket \varphi \rrbracket_{\mathcal{M}})$, and $V'(p) = V(p) \cap \llbracket \varphi \rrbracket_{\mathcal{M}}$.

A formula φ is *valid*, notation: $\models \varphi$, if for all models \mathcal{M} and all states s in \mathcal{M} , we have $\mathcal{M}, s \models \varphi$. Given any two states s, t in \mathcal{M} and any formula φ , we say that s and t *agree on* φ , if $\mathcal{M}, s \models \varphi$ iff $\mathcal{M}, t \models \varphi$.

Note that in the semantic definition of $\langle ! \rangle_i \varphi$, the quantification is restricted to EL-formulas. This is to avoid circularity of the definition. As EL is expressively equivalent to PAL, we can also define the semantics of $\langle ! \rangle_i$ as follows:

$$\mathcal{M}, s \models \langle ! \rangle_i \varphi \iff \text{for some formula } \psi \in \text{PAL} : \mathcal{M}, s \models \langle \psi \rangle K_i \varphi$$

For convenience, we also give the semantics of $\llbracket ! \rrbracket_i$ as follows.

$$\mathcal{M}, s \models \llbracket ! \rrbracket_i \varphi \iff \text{for all formulas } \psi \in \text{EL} : \mathcal{M}, s \models [\psi] \hat{K}_i \varphi$$

From Definition 32 it follows that $\models \langle \psi \rangle K_i \varphi \rightarrow \langle ! \rangle_i \varphi$, where $\psi \in \text{EL}$. We can also use its equivalent version $\llbracket ! \rrbracket_i \varphi \rightarrow [\psi] \hat{K}_i \varphi$ (where $\psi \in \text{EL}$), which means intuitively that if $\neg \varphi$ is unknowable ($\neg \langle ! \rangle_i \neg \varphi$), then after any announcement $\neg \varphi$ is unknown ($[\psi] \neg K_i \neg \varphi$).

Proposition 19. *For all $\varphi \in \mathcal{L}$, $\models \langle ! \rangle_i \varphi \leftrightarrow \langle ! \rangle K_i \varphi$.*

Proof. By definition of the semantics. □

Due to the presence of the knowability operators, in the completeness proof, we need to use a method of induction with, on one hand, the size of formulas (as usual), and on the other hand, the depth of knowability operators. These two notions are combined into the notion of complexity. This notion and the next proposition will be also used in proving the proof theoretical results in Prop. 27 and Sec. 4.5.2.

Definition 33 (Complexity). The *complexity of a formula* consists of two aspects: size and $\langle ! \rangle$ -depth, which are defined as follows.

The *size* of a formula φ , notation: $\text{Size}(\varphi)$, is a positive natural number, defined recursively as follows:

$$\begin{aligned} \text{Size}(p) &= 1 \\ \text{Size}(\neg \varphi) &= 1 + \text{Size}(\varphi) \\ \text{Size}(\varphi \wedge \psi) &= 1 + \max\{\text{Size}(\varphi), \text{Size}(\psi)\} \\ \text{Size}(K_i \varphi) &= 3 + \text{Size}(\varphi) \\ \text{Size}(\langle \psi \rangle \varphi) &= \text{Size}(\psi) + 3 \cdot \text{Size}(\varphi) \\ \text{Size}(\langle ! \rangle_i \varphi) &= 1 + \text{Size}(\varphi) \end{aligned}$$

The *knowable-depth* of a formula φ , notation $dk(\varphi)$, is a natural number, defined recursively as follows:

$$\begin{aligned} dk(p) &= 0 \\ dk(\neg \varphi) &= dk(\varphi) \\ dk(\varphi \wedge \psi) &= \max\{dk(\varphi), dk(\psi)\} \\ dk(K_i \varphi) &= dk(\varphi) \\ dk(\langle \psi \rangle \varphi) &= dk(\psi) + dk(\varphi) \\ dk(\langle ! \rangle_i \varphi) &= 1 + dk(\varphi) \end{aligned}$$

With the definitions of size and knowable-depth in hand, we define $<_{\diamond}^S$ as a binary relation between formulas such that

$$\varphi <_{\diamond}^S \psi \iff \text{either } dk(\varphi) < dk(\psi), \text{ or } dk(\varphi) = dk(\psi) \text{ and } Size(\varphi) < Size(\psi).$$

If $\varphi <_{\diamond}^S \psi$, then we say that φ is less complex than ψ .

One may easily show by induction that $dk(\varphi) = 0$ for all $\varphi \in \text{EL}$. And also, it is easily computed that $Size([\psi]\varphi) = 4 + Size(\psi) + 3 \cdot Size(\varphi)$.

Proposition 20. In (\circ) and $(\circ\circ)$, $\psi \in \text{EL}$.

$$\begin{array}{l}
 \varphi <_{\diamond}^S \neg\varphi & \left| \right. & \psi <_{\diamond}^S \langle\psi\rangle p & \left| \right. & \langle\psi\rangle\chi <_{\diamond}^S \langle\psi\rangle(\varphi \wedge \chi) \\
 \varphi <_{\diamond}^S \varphi \wedge \psi & \left| \right. & p <_{\diamond}^S \langle\psi\rangle p & \left| \right. & \psi <_{\diamond}^S \langle\psi\rangle K_i \varphi \\
 \psi <_{\diamond}^S \varphi \wedge \psi & \left| \right. & \psi <_{\diamond}^S \langle\psi\rangle \neg\varphi & \left| \right. & K_i[\psi]\varphi <_{\diamond}^S \langle\psi\rangle K_i \varphi \\
 \varphi <_{\diamond}^S K_i \varphi & \left| \right. & \langle\psi\rangle\varphi <_{\diamond}^S \langle\psi\rangle \neg\varphi & \left| \right. & \langle\langle\psi\rangle\chi\rangle\varphi <_{\diamond}^S \langle\psi\rangle\langle\chi\rangle\varphi \\
 (\circ) \langle\psi\rangle K_i \varphi <_{\diamond}^S \langle!\rangle_i \varphi & \left| \right. & \langle\psi\rangle\varphi <_{\diamond}^S \langle\psi\rangle(\varphi \wedge \chi) & \left| \right. & \langle\chi\rangle\langle\psi\rangle K_i \varphi <_{\diamond}^S \langle\chi\rangle\langle!\rangle_i \varphi & (\circ\circ) \\
 \psi <_{\diamond}^S \langle\psi\rangle\varphi & \left| \right. & \varphi <_{\diamond}^S \langle\psi\rangle\varphi & \left| \right. & &
 \end{array}$$

Proof. We take some of them as examples.

- $\langle\psi\rangle K_i \varphi <_{\diamond}^S \langle!\rangle_i \varphi$: this is because $dk(\langle\psi\rangle K_i \varphi) = dk(\varphi) < 1 + dk(\varphi) = dk(\langle!\rangle_i \varphi)$.
- $p <_{\diamond}^S \langle\psi\rangle p$: this is because $dk(p) \leq dk(\psi) + dk(p) = dk(\langle\psi\rangle p)$ and $Size(p) = 1 < Size(\psi) + 3 \cdot Size(p) = Size(\langle\psi\rangle p)$.
- $K_i[\psi]\varphi <_{\diamond}^S \langle\psi\rangle K_i \varphi$: this is because $dk(K_i[\psi]\varphi) = dk(\psi) + dk(\varphi) = dk(\langle\psi\rangle K_i \varphi)$, and $Size(K_i[\psi]\varphi) = 3 + 4 + Size(\psi) + 3 \cdot Size(\varphi) = 7 + Size(\psi) + 3 \cdot Size(\varphi) < 9 + Size(\psi) + 3 \cdot Size(\varphi) = Size(\langle\psi\rangle K_i \varphi)$.
- $\langle\langle\psi\rangle\chi\rangle\varphi <_{\diamond}^S \langle\psi\rangle\langle\chi\rangle\varphi$: this is because $dk(\langle\langle\psi\rangle\chi\rangle\varphi) = dk(\psi) + dk(\chi) + dk(\varphi) = dk(\langle\psi\rangle\langle\chi\rangle\varphi)$, and $Size(\langle\langle\psi\rangle\chi\rangle\varphi) = Size(\psi) + 3 \cdot Size(\chi) + 3 \cdot Size(\varphi) < Size(\psi) + 3 \cdot Size(\chi) + 9 \cdot Size(\varphi) = Size(\langle\psi\rangle\langle\chi\rangle\varphi)$.

□

Note that in the definition of $Size(K_i \varphi)$, the number 3 is the least natural number to provide $K_i[\psi]\varphi <_{\diamond}^S \langle\psi\rangle K_i \varphi$. In contrast, in [17], $Size(K_i \varphi)$ is defined to be $1 + Size(\varphi)$, in other words, plus 1 rather than plus 3.

4.3 Logical properties of knowability

This section explores the logical properties of the knowability operator in the logic LK.

It has been shown in [103] that *everything is knowable*, in the sense that $\diamond K_i \varphi \vee \diamond K_i \neg\varphi$ is valid. In LK this becomes $\langle!\rangle_i \varphi \vee \langle!\rangle_i \neg\varphi$ and indeed this is also valid, by a very similar proof (only the case quantifier is occasionally different). For clarity we give the entire proof.

Given a model \mathcal{M} , the valuation of propositional variable p is *constant* on its domain S if $V(p) = S$ or $V(p) = \emptyset$, i.e., if any two states in S agree on the value of p .

Proposition 21 ([16, Lemma 3.2]). *Let $\varphi \in \text{LK}$, and let \mathcal{M} be a model with constant values for all variables occurring in φ . Then $\mathcal{M} \models \varphi$ or $\mathcal{M} \models \neg\varphi$.*

Proof. Suppose that each propositional variable occurring in φ has constant value on \mathcal{M} . If $V(p) = S$, that is, $\mathcal{M} \models p \leftrightarrow \top$, then $\mathcal{M} \models \varphi \leftrightarrow \varphi(\top/p)$; if $V(p) = \emptyset$, that is, $\mathcal{M} \models p \leftrightarrow \perp$, then $\mathcal{M} \models \varphi \leftrightarrow \varphi(\perp/p)$. We denote the result obtained by substituting \top or \perp for all propositional variables in φ in that way as φ^\emptyset . Obviously, $\mathcal{M} \models \varphi \leftrightarrow \varphi^\emptyset$. Note that φ^\emptyset contains no propositional variables.

We now show by induction on the structure of φ that $\models \varphi^\emptyset \leftrightarrow \top$ or $\models \varphi^\emptyset \leftrightarrow \perp$. Cases atom, conjunction and negation are trivial. Further:

- $\models K_i \top \leftrightarrow \top$ and $\models K_i \perp \leftrightarrow \perp$;
- $\models \langle \top \rangle \top \leftrightarrow \top$, $\models \langle \top \rangle \perp \leftrightarrow \perp$, $\models \langle \perp \rangle \top \leftrightarrow \perp$, and $\models \langle \perp \rangle \perp \leftrightarrow \perp$;
- $\models \langle ! \rangle_i \top \leftrightarrow \top$ and $\models \langle ! \rangle_i \perp \leftrightarrow \perp$ (in particular, $\models \top \rightarrow \langle ! \rangle_i \top$ follows from the correctness of knowledge after the trivial announcement of \top).

Therefore $\models \varphi^\emptyset \leftrightarrow \top$ or $\models \varphi^\emptyset \leftrightarrow \perp$. Combining this with $\mathcal{M} \models \varphi \leftrightarrow \varphi^\emptyset$, we derive that $\mathcal{M} \models \varphi \leftrightarrow \top$ or $\mathcal{M} \models \varphi \leftrightarrow \perp$, that is, $\mathcal{M} \models \varphi$ or $\mathcal{M} \models \neg\varphi$, respectively. \square

Theorem 6 ([103, Thm. 1]). *For all $\varphi \in LK$, we have*

$$\models \langle ! \rangle_i \varphi \vee \langle ! \rangle_i \neg\varphi.$$

Proof. Given any model \mathcal{M} and s in \mathcal{M} , define δ_s^φ as the characteristic formula of the restriction of the valuation in s to $\text{var}(\varphi)$:

$$\delta_s^\varphi = \bigwedge \{p \mid p \in \text{var}(\varphi) \text{ and } \mathcal{M}, s \models p\} \wedge \bigwedge \{\neg p \mid p \in \text{var}(\varphi) \text{ and } \mathcal{M}, s \not\models p\}.$$

For all $p \in \text{var}(\varphi)$, we obviously have

$$\mathcal{M}, s \models p \text{ or } \mathcal{M}, s \models \neg p,$$

and therefore

$$\mathcal{M}|_{\delta_s^\varphi}, s \models p \text{ or } \mathcal{M}|_{\delta_s^\varphi}, s \models \neg p$$

and even

$$\mathcal{M}|_{\delta_s^\varphi} \models p \text{ or } \mathcal{M}|_{\delta_s^\varphi} \models \neg p.$$

Then by Prop. 21, we have

$$\mathcal{M}|_{\delta_s^\varphi} \models \varphi \text{ or } \mathcal{M}|_{\delta_s^\varphi} \models \neg\varphi.$$

Thus

$$\mathcal{M}|_{\delta_s^\varphi} \models K_i \varphi \text{ or } \mathcal{M}|_{\delta_s^\varphi} \models K_i \neg\varphi.$$

Since $s \in \mathcal{M}|_{\delta_s^\varphi}$, we have

$$\mathcal{M}|_{\delta_s^\varphi}, s \models K_i \varphi \text{ or } \mathcal{M}|_{\delta_s^\varphi}, s \models K_i \neg\varphi.$$

Therefore,

$$\mathcal{M}, s \models \langle \delta_s^\varphi \rangle K_i \varphi \text{ or } \mathcal{M}, s \models \langle \delta_s^\varphi \rangle K_i \neg\varphi,$$

that is,

$$\mathcal{M}, s \models \langle ! \rangle_i \varphi \vee \langle ! \rangle_i \neg\varphi.$$

As \mathcal{M} and s are arbitrary, we now conclude that

$$\models \langle ! \rangle_i \varphi \vee \langle ! \rangle_i \neg\varphi.$$

\square

Since $\langle ! \rangle_i \varphi \vee \langle ! \rangle_i \neg \varphi$ is equivalent to $\neg \langle ! \rangle_i \neg \varphi \rightarrow \langle ! \rangle_i \varphi$, and since $[!]_i$ is the dual of $\langle ! \rangle_i$, we immediately have

Corollary 4. *For all $\varphi \in LK$, $\models [!]_i \varphi \rightarrow \langle ! \rangle_i \varphi$.*

However, we recall that although every formula is knowable in the sense of Thm. 6, this does not mean that every true formula is knowable (to be true), as the announcement may ‘flip’ the value of the formula in question. Fitch [39] showed that there is an unknowable truth, for example $\not\models (p \wedge \neg K_i p) \rightarrow \langle ! \rangle_i (p \wedge \neg K_i p)$. In fact, we have a stronger result: every unknown truth is unknowable; in Salerno’s term in [84, p. 32], this says that “Fitch-conjunctions are unknowable.”

Proposition 22. $\models \neg \langle ! \rangle_i (\varphi \wedge \neg K_i \varphi)$.

Proof. Suppose not, that is, there is a pointed model (\mathcal{M}, s) such that $\mathcal{M}, s \not\models \neg \langle ! \rangle_i (\varphi \wedge \neg K_i \varphi)$, then $\mathcal{M}, s \models \langle ! \rangle_i (\varphi \wedge \neg K_i \varphi)$. This means that for some formula $\psi \in \text{EL}$ such that $\mathcal{M}, s \models \psi$ and $\mathcal{M}|_\psi, s \models K_i (\varphi \wedge \neg K_i \varphi)$. The latter entails that $\mathcal{M}|_\psi, s \models K_i \varphi \wedge K_i \neg K_i \varphi$. Since $\models K_i \varphi \rightarrow \varphi$, we have $\mathcal{M}|_\psi, s \models K_i \varphi \wedge \neg K_i \varphi$: a contradiction. \square

Consequently, we have $\models (\varphi \wedge \neg K_i \varphi) \rightarrow \neg \langle ! \rangle_i (\varphi \wedge \neg K_i \varphi) \wedge (\varphi \wedge \neg K_i \varphi)$, which says that if it is an unknown truth that φ , it is an unknowable truth that it is an unknown truth that φ ; in short, every unknown truth is itself unknowable, see [39, Thm. 2] and [110, p. 154].

Corollary 5. $\langle ! \rangle_i (\varphi \wedge \neg K_i \varphi)$ is unsatisfiable. That is, there is no pointed model satisfying $\langle ! \rangle_i (\varphi \wedge \neg K_i \varphi)$.

In comparison, $\langle ! \rangle_j (p \wedge \neg K_i p)$ is satisfiable, as one may easily check. This tells us that the notion of knowability is a subjective concept: the proposition $p \wedge \neg K_i p$ is unknowable for the agent i but knowable for another agent j , as mentioned in the introduction.

Also, as we mentioned in the introduction, the knowability paradox says that if all truths are knowable, then all truths are actually known. This can be shown semantically as follows.

Corollary 6. *If $\models \varphi \rightarrow \langle ! \rangle_i \varphi$ for all φ , then $\models \varphi \rightarrow K_i \varphi$ for all φ .*

Proof. Suppose that $\models \varphi \rightarrow \langle ! \rangle_i \varphi$ for all φ . Then of course, $\models \varphi \wedge \neg K_i \varphi \rightarrow \langle ! \rangle_i (\varphi \wedge \neg K_i \varphi)$ for all φ . By Prop. 22, we have $\models \neg (\varphi \wedge \neg K_i \varphi)$ for all φ , and therefore $\models \varphi \rightarrow K_i \varphi$ for all φ . \square

Proposition 23. $\models K_i \varphi \rightarrow \langle ! \rangle_i \varphi$

Proof. This is because $\models K_i \varphi \rightarrow \langle \top \rangle K_i \varphi$ and $\models \langle \top \rangle K_i \varphi \rightarrow \langle ! \rangle_i \varphi$. \square

We continue our survey of the properties of the knowability operator with a number of validities only involving that operator.

Theorem 7. $\models \langle ! \rangle_i \langle ! \rangle_i \varphi \rightarrow \langle ! \rangle_i \varphi$

Proof. Let $\mathcal{M} = \langle S, \{R_i \mid i \in \mathbf{A}\}, V \rangle$ and $s \in S$. First, suppose that $\mathcal{M}, s \models \langle ! \rangle_i \langle ! \rangle_i \varphi$, then for some $\psi \in \text{EL}$: $\mathcal{M}, s \models \langle \psi \rangle K_i \langle ! \rangle_i \varphi$. This means that $\mathcal{M}, s \models \psi$ and $\mathcal{M}|_\psi, s \models K_i \langle ! \rangle_i \varphi$. Since R_i is an equivalence relation and equivalence relations are closed under public announcements, $R_i|_\psi$ is an equivalence relation as well. Thus $\mathcal{M}|_\psi, s \models \langle ! \rangle_i \varphi$, which entails that for some $\chi \in \text{EL}$: $\mathcal{M}|_\psi, s \models \langle \chi \rangle K_i \varphi$, which amounts to saying that $\mathcal{M}|_\psi, s \models \chi$ and $(\mathcal{M}|_\psi)|_\chi, s \models K_i \varphi$.

Summarizing the above results, we have that for some $\psi, \chi \in \text{EL}$: $\mathcal{M}, s \models \psi$ and $\mathcal{M}|_\psi, s \models \chi$ and $(\mathcal{M}|_\psi)|_\chi, s \models K_i \varphi$. As a sequence of two announcements is an announcement [104, Prop. 4.17], it directly follows that $\mathcal{M}|_{\langle \psi \rangle \chi}, s \models K_i \varphi$. From $\mathcal{M}, s \models \langle \psi \rangle \chi$ and $\mathcal{M}|_{\langle \psi \rangle \chi}, s \models K_i \varphi$ it now follows that $\mathcal{M}, s \models \langle ! \rangle_i \varphi$. \square

The following result indicates that $\langle ! \rangle_i$ (and thus $[!]_i$) are monotone.

Proposition 24. *If $\models \varphi \rightarrow \psi$, then $\models \langle ! \rangle_i \varphi \rightarrow \langle ! \rangle_i \psi$ and $\models [!]_i \varphi \rightarrow [!]_i \psi$.*

Proof. Straightforward from the semantics. \square

Note that $\langle ! \rangle_i$ is not *regular*. In other words, $\not\models \langle ! \rangle_i \varphi \wedge \langle ! \rangle_i \psi \rightarrow \langle ! \rangle_i (\varphi \wedge \psi)$: one may easily construct a pointed model (\mathcal{M}, s) such that $\mathcal{M}, s \models \langle ! \rangle_i p$ and $\mathcal{M}, s \models \langle ! \rangle_i \neg p$ but $\mathcal{M}, s \not\models \langle ! \rangle_i (p \wedge \neg p)$.

The next result states that unknowable truths are themselves unknowable.

Corollary 7. $\models \neg \langle ! \rangle_i (\varphi \wedge \neg \langle ! \rangle_i \varphi)$.

Proof. By Prop. 23, $\models K_i \varphi \rightarrow \langle ! \rangle_i \varphi$, thus $\models \varphi \wedge \neg \langle ! \rangle_i \varphi \rightarrow \varphi \wedge \neg K_i \varphi$. Then from Prop. 24, it follows that $\models \langle ! \rangle_i (\varphi \wedge \neg \langle ! \rangle_i \varphi) \rightarrow \langle ! \rangle_i (\varphi \wedge \neg K_i \varphi)$. Finally, using Prop. 22, we conclude that $\models \neg \langle ! \rangle_i (\varphi \wedge \neg \langle ! \rangle_i \varphi)$. \square

Proposition 25. $\models \langle ! \rangle_i \varphi \rightarrow \langle ! \rangle_i K_i \varphi$.

Proof. Let $\mathcal{M} = \langle S, R, V \rangle$ and $s \in S$. Suppose that $\mathcal{M}, s \models \langle ! \rangle_i \varphi$, then for some $\psi \in \text{EL}$, $\mathcal{M}, s \models \langle \psi \rangle K_i \varphi$. Since R_i is an equivalence relation, $\models K_i \varphi \rightarrow K_i K_i \varphi$, and thus $\mathcal{M}, s \models \langle \psi \rangle K_i K_i \varphi$. Therefore $\mathcal{M}, s \models \langle ! \rangle_i K_i \varphi$. \square

Theorem 8. $\models \langle ! \rangle_i \varphi \rightarrow \langle ! \rangle_i \langle ! \rangle_i \varphi$.

Proof. By Prop. 23, $\models K_i \varphi \rightarrow \langle ! \rangle_i \varphi$. Then by Prop. 24, $\models \langle ! \rangle_i K_i \varphi \rightarrow \langle ! \rangle_i \langle ! \rangle_i \varphi$. Now due to Prop. 25, $\models \langle ! \rangle_i \varphi \rightarrow \langle ! \rangle_i \langle ! \rangle_i \varphi$. \square

Corollary 8. $\models \langle ! \rangle_i \varphi \leftrightarrow \langle ! \rangle_i \langle ! \rangle_i \varphi$, and thus $\models \langle ! \rangle_i \langle ! \rangle_i \varphi \leftrightarrow \langle ! \rangle_i K_i \varphi$, $\models \langle ! \rangle_i K_i \varphi \leftrightarrow \langle ! \rangle_i \varphi$, and $\models K_i \langle ! \rangle_i \varphi \rightarrow \langle ! \rangle_i K_i \varphi$.

Corollary 9. $\models [!]_i \varphi \rightarrow \langle ! \rangle_i [!]_i \varphi$ and $\models [!]_i \varphi \rightarrow [!]_i \langle ! \rangle_i \varphi$. As a consequence, $\models [!]_i \langle ! \rangle_i \varphi \rightarrow \langle ! \rangle_i \varphi$ and $\models \langle ! \rangle_i [!]_i \varphi \rightarrow \langle ! \rangle_i \varphi$.

Proof. By Coro. 8, we have $\models [!]_i \varphi \leftrightarrow [!]_i [!]_i \varphi$. By Coro. 4, we infer that $\models [!]_i [!]_i \varphi \rightarrow \langle ! \rangle_i [!]_i \varphi$, and therefore $\models [!]_i \varphi \rightarrow \langle ! \rangle_i [!]_i \varphi$; by Coro. 4 and Prop. 24, $\models [!]_i [!]_i \varphi \rightarrow [!]_i \langle ! \rangle_i \varphi$, and therefore $\models [!]_i \varphi \rightarrow [!]_i \langle ! \rangle_i \varphi$. \square

We have shown that $\models [!]_i \langle ! \rangle_i \varphi \rightarrow \langle ! \rangle_i \varphi$. However, $[!]_i \varphi \rightarrow \varphi$ is not valid, since its equivalent $\varphi \rightarrow \langle ! \rangle_i \varphi$ is not valid. Prop. 22 demonstrated that some true propositions are not knowable, for example $\varphi = p \wedge \neg K_i p$. This also shows that $\models \langle ! \rangle_i \varphi \leftrightarrow \varphi$ does *not* hold for all $\varphi \in \mathcal{L}_{\mathbf{LK}}$, though it does hold for all $\varphi \in \mathcal{L}_{\mathbf{PL}}$ [16, Prop. 3.11.2].

Lemma 12. *Let $\varphi \in \text{LK}$, and let \mathcal{M} be a model where all states agree on each propositional variable occurring in φ . Then $\mathcal{M} \models \varphi \rightarrow [!]_i \varphi$.*

Proof. Let s be any state in \mathcal{M} , and $\mathcal{M}, s \models \varphi$. Now consider any EL-formula ψ such that $\mathcal{M}, s \models \psi$. Let \mathcal{M}' be the disjoint union of \mathcal{M} and $\mathcal{M}|_\psi$. The valuation of atoms in $\text{var}(\varphi)$ is also constant on \mathcal{M}' . By Prop. 21, it follows that $\mathcal{M}' \models \varphi$ or $\mathcal{M}' \models \neg \varphi$. If $\mathcal{M}' \models \neg \varphi$, then it contradicts $\mathcal{M}, s \models \varphi$. Thus $\mathcal{M}' \models \varphi$, and therefore $\mathcal{M}|_\psi \models \varphi$. That is to say, for any state t such that $s R_i t$ in $\mathcal{M}|_\psi$, we have $\mathcal{M}|_\psi, t \models \varphi$. By semantics, it follows that $\mathcal{M}|_\psi, s \models K_i \varphi$, and thus $\mathcal{M}|_\psi, s \models \hat{K}_i \varphi$. As ψ is arbitrary, by semantics we know that $\mathcal{M}, s \models [!]_i \varphi$. So far we have shown that $\mathcal{M}, s \models \varphi \rightarrow [!]_i \varphi$. As s is arbitrary in \mathcal{M} , $\mathcal{M} \models \varphi \rightarrow [!]_i \varphi$. \square

In what follows, we show that the McKinsey property (MK) and the Church–Rosser property (CR) hold for LK.

Theorem 9 (MK). $\models [!]_i \langle ! \rangle_i \varphi \rightarrow \langle ! \rangle_i [!]_i \varphi$

Proof. Let a model $\mathcal{M} = \langle S, \{R_i \mid i \in \mathbf{A}\}, V \rangle$ and a state $s \in S$ be given. Suppose that $\mathcal{M}, s \models [!]_i \langle ! \rangle_i \varphi$. Then by the semantics, for all $\psi \in \text{EL}$, we have $\mathcal{M}, s \models [\psi] \hat{K}_i \langle ! \rangle_i \varphi$. Consider δ_s^φ in the proof of Thm. 6. It is obvious that $\mathcal{M}, s \models \delta_s^\varphi$ and $\delta_s^\varphi \in \text{EL}$, thus $\mathcal{M}|_{\delta_s^\varphi}, s \models \hat{K}_i \langle ! \rangle_i \varphi$. Since all states in $\mathcal{M}|_{\delta_s^\varphi}$ have constant values for variables in φ , by Lemma 12 we have $\mathcal{M}|_{\delta_s^\varphi} \models \varphi \rightarrow [!]_i \varphi$ and its dual $\mathcal{M}|_{\delta_s^\varphi} \models \langle ! \rangle_i \varphi \rightarrow \varphi$, therefore $\mathcal{M}|_{\delta_s^\varphi} \models \langle ! \rangle_i \varphi \rightarrow [!]_i \varphi$. Note that all states in $\mathcal{M}|_{\delta_s^\varphi}$ also have constant values for variables in $\langle ! \rangle_i \varphi$. Then by Prop. 21 we have $\mathcal{M}|_{\delta_s^\varphi} \models \langle ! \rangle_i \varphi$ or $\mathcal{M}|_{\delta_s^\varphi} \models \neg \langle ! \rangle_i \varphi$. As $\mathcal{M}|_{\delta_s^\varphi}, s \models \hat{K}_i \langle ! \rangle_i \varphi$, there is a state t such that $\mathcal{M}|_{\delta_s^\varphi}, t \models \langle ! \rangle_i \varphi$, contradicting $\mathcal{M}|_{\delta_s^\varphi} \models \neg \langle ! \rangle_i \varphi$. Thus $\mathcal{M}|_{\delta_s^\varphi} \models \langle ! \rangle_i \varphi$. From that and $\mathcal{M}|_{\delta_s^\varphi} \models \langle ! \rangle_i \varphi \rightarrow [!]_i \varphi$ already obtained above, it follows that $\mathcal{M}|_{\delta_s^\varphi} \models [!]_i \varphi$. Therefore, for any state s' such that $sR_i s'$ in $\mathcal{M}|_{\delta_s^\varphi}$ we have $\mathcal{M}|_{\delta_s^\varphi}, s' \models [!]_i \varphi$. By semantics, $\mathcal{M}|_{\delta_s^\varphi}, s \models K_i [!]_i \varphi$, and therefore $\mathcal{M}, s \models \langle ! \rangle_i [!]_i \varphi$. \square

Theorem 10 (CR). $\models \langle ! \rangle_i [!]_i \varphi \rightarrow [!]_i \langle ! \rangle_i \varphi$

Proof. Let a model $\mathcal{M} = \langle S, \{R_i \mid i \in \mathbf{A}\}, V \rangle$ and a state $s \in S$ be given. Suppose that $\mathcal{M}, s \models \langle ! \rangle_i [!]_i \varphi$. By semantics, for some $\psi \in \text{EL}$: $\mathcal{M}, s \models \langle \psi \rangle K_i [!]_i \varphi$. Then $\mathcal{M}, s \models \psi$ and for any t in $\mathcal{M}|_\psi$ such that $sR_i t$, $\mathcal{M}|_\psi, t \models [!]_i \varphi$. Consider δ_s^φ in the proof of Thm. 6, it is an EL-formula and thus $(\mathcal{M}|_\psi)|_{\delta_s^\varphi}, t \models \hat{K}_i \varphi$.

Let $\eta \in \text{EL}$ be arbitrary such that $\mathcal{M}, s \models \eta$. The valuation of atoms in $\text{var}(\varphi)$ is constant on $(\mathcal{M}|_\eta)|_{\delta_s^\varphi}$. By Prop. 21, we have $(\mathcal{M}|_\eta)|_{\delta_s^\varphi} \models \varphi$ or $(\mathcal{M}|_\eta)|_{\delta_s^\varphi} \models \neg \varphi$. Since $\psi \in \text{EL}$ and $\mathcal{M}, s \models \psi$, we have also $(\mathcal{M}|_\psi)|_{\delta_s^\varphi} \models \varphi$ or $(\mathcal{M}|_\psi)|_{\delta_s^\varphi} \models \neg \varphi$. As $(\mathcal{M}|_\psi)|_{\delta_s^\varphi}, t \models \hat{K}_i \varphi$, there must be a t' such that $(\mathcal{M}|_\psi)|_{\delta_s^\varphi}, t' \models \varphi$ which contradicts $(\mathcal{M}|_\psi)|_{\delta_s^\varphi} \models \neg \varphi$. Thus we obtain that $(\mathcal{M}|_\psi)|_{\delta_s^\varphi} \models \varphi$. Consider the disjoint union \mathcal{M}' of $(\mathcal{M}|_\psi)|_{\delta_s^\varphi}$ and $(\mathcal{M}|_\eta)|_{\delta_s^\varphi}$. Since \mathcal{M}' has constant values for variables in φ as well, we conclude that $\mathcal{M}' \models \varphi$, and therefore $(\mathcal{M}|_\eta)|_{\delta_s^\varphi} \models \varphi$. Let s' be any state such that $sR_i s'$ in $(\mathcal{M}|_\eta)|_{\delta_s^\varphi}$. Now we know that $(\mathcal{M}|_\eta)|_{\delta_s^\varphi}, s' \models \varphi$. Then $(\mathcal{M}|_\eta)|_{\delta_s^\varphi}, s \models K_i \varphi$, and thus $\mathcal{M}|_\eta, s \models \langle ! \rangle_i \varphi$. This follows that $\mathcal{M}|_\eta, s \models \hat{K}_i \langle ! \rangle_i \varphi$. As $\eta \in \text{EL}$ is arbitrary, we conclude that $\mathcal{M}, s \models [!]_i \langle ! \rangle_i \varphi$. \square

As we have seen above, not every true formula is knowable. In contrast, every valid formula is knowable, in symbol: $\models \varphi$ implies $\models \langle ! \rangle_i \varphi$, as easily shown. This then follows that $\models \langle ! \rangle_i \top$. Besides, it may be worth noting that the knowability operators are *not* normal.

Proposition 26. $\not\models \langle ! \rangle_i (\varphi \rightarrow \psi) \rightarrow (\langle ! \rangle_i \varphi \rightarrow \langle ! \rangle_i \psi)$

Proof. Consider the following model \mathcal{M} :

$$\underline{s} : p \text{ --- } i \text{ --- } t : \neg p$$

- $\mathcal{M}, s \models \langle ! \rangle_i (p \rightarrow p \wedge \neg K_i p)$: firstly, note that $\mathcal{M}, s \models p \rightarrow p \wedge \neg K_i p$ and $\mathcal{M}, t \models p \rightarrow p \wedge \neg K_i p$, thus $\mathcal{M}, s \models K_i (p \rightarrow p \wedge \neg K_i p)$. By Prop. 23, $\mathcal{M}, s \models \langle ! \rangle_i (p \rightarrow p \wedge \neg K_i p)$.
- $\mathcal{M}, s \models \langle ! \rangle_i p$: clearly, $\mathcal{M}, s \models \langle p \rangle K_i p$, thus $\mathcal{M}, s \models \langle ! \rangle_i p$.
- $\mathcal{M}, s \not\models \langle ! \rangle_i (p \wedge \neg K_i p)$: this follows directly from Prop. 22.

\square

This refutes the claim that “knowable-in-principle, knowability, is closed under consequence” in [9].

We conclude this section with the fragment of the *positive formulas* in LK. The fragment, denoted LK^+ , is inductively defined as follows:

$$\varphi ::= p \mid \neg p \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid K_i \varphi \mid [\neg \varphi] \varphi \mid [!]_i \varphi$$

In modal logic, the fragment of the language where negations do not bind (box-type) epistemic modalities is known as the *positive* fragment [94, 101, 16]. It corresponds to the universal fragment in first-order logic. It has the property that it preserves truth under submodels. Intuitively, this is because a box modality says that something is true in all accessible worlds, so if you go to a submodel it is still true in all remaining accessible worlds, whatever remains. The result we present here is a generalization of a similar result in [16]. We should point out the surprising negation in the inductive clause $[\neg \varphi] \varphi$. This has to do with the semantics of public announcement. Note that we have that $\mathcal{M}, s \models [\neg \varphi] \psi$, iff (by the semantics of public announcement) $\mathcal{M}, s \models \neg \varphi$ implies $\mathcal{M}|_{\neg \varphi}, s \models \psi$, iff (propositionally) $\mathcal{M}, s \models \varphi$ or $\mathcal{M}|_{\neg \varphi}, s \models \psi$. In the last formulation the negation has disappeared! This aspect will also play a role in the proof of the subsequent proposition.

We say that φ is *successful*, if after being announced, φ still holds; in symbol, $\models [\varphi] \varphi$. The following result states that positive formulas are successful.

Proposition 27. *For all $\varphi \in LK^+$, we have $\models [\varphi] \varphi$.*

Proof. We show the following claim: For any \mathcal{M}' and \mathcal{M}'' with $\mathcal{M}'' \subseteq \mathcal{M}'$, $s \in S^{\mathcal{M}''}$ and $\varphi \in LK^+$: If $\mathcal{M}', s \models \varphi$, then $\mathcal{M}'', s \models \varphi$.

The proof is by induction on the complexity of φ . Recall that the notion of complexity is given in Def. 33.

- φ is atomic: Since the valuation of atoms is local, it is trivial.
- Boolean cases: It is straightforward by induction hypothesis.
- φ is $K_i \psi$: Suppose $\mathcal{M}', s \models K_i \psi$, by semantics $\mathcal{M}', s' \models \psi$ for any s' such that $sR_i^{\mathcal{M}'} s'$. Consider any t such that $sR_i^{\mathcal{M}''} t$. Since $\mathcal{M}'' \subseteq \mathcal{M}'$, we have $sR_i^{\mathcal{M}'} t$. Thus $\mathcal{M}', t \models \psi$, and then by inductive hypothesis $\mathcal{M}'', t \models \psi$. By semantics again, it follows that $\mathcal{M}'', s \models K_i \psi$.
- φ is $[\neg \psi_1] \psi_2$. Suppose $\mathcal{M}', s \models [\neg \psi_1] \psi_2$ and $\mathcal{M}'', s \models \neg \psi_1$. By induction hypothesis, $\mathcal{M}', s \models \neg \psi_1$. By semantics, $\mathcal{M}'|_{\neg \psi_1}, s \models \psi_2$. Note that $\mathcal{M}''|_{\neg \psi_1} \subseteq \mathcal{M}'|_{\neg \psi_1}$, then by induction hypothesis $\mathcal{M}''|_{\neg \psi_1}, s \models \psi_2$. By semantics $\mathcal{M}'', s \models [\neg \psi_1] \psi_2$.
- φ is $[!]_i \psi$. Suppose $\mathcal{M}', s \models [!]_i \psi$. Assume, for reductio, that $\mathcal{M}'', s \not\models [!]_i \psi$. By semantics, there is a $\chi \in EL$ such that $\mathcal{M}'', s \models \chi$ and $\mathcal{M}''|_{\chi}, s \not\models \hat{K}_i \psi$. As $\mathcal{M}''|_{\chi} \subseteq \mathcal{M}'' \subseteq \mathcal{M}'$, by induction hypothesis, we infer that $\mathcal{M}', s \not\models \hat{K}_i \psi$, that is, $\mathcal{M}', s \not\models [\top] \hat{K}_i \psi$. Then $\mathcal{M}', s \not\models [!]_i \psi$, contrary to the supposition.

Given any $\varphi \in LK^+$, for any model \mathcal{M} and $s \in S^{\mathcal{M}}$: If $\mathcal{M}, s \models \varphi$, then no matter what submodel of \mathcal{M} that φ defines, it follow that $\mathcal{M}|_{\varphi}, s \models \varphi$ by the above claim. By semantics it means $\mathcal{M}, s \models [\varphi] \varphi$. Since (\mathcal{M}, s) is arbitrary, we conclude $\models [\varphi] \varphi$. \square

4.4 Bisimulation and Expressivity

4.4.1 Bisimulation

In this part, we show that the notion of bisimilarity is tailored for the logic of knowability LK. That is, LK is invariant under bisimulation.

Given pointed models (\mathcal{M}, s) and (\mathcal{N}, t) and a language L , $(\mathcal{M}, s) \equiv_L (\mathcal{N}, t)$ denotes: for all $\varphi \in \mathcal{L}_L$, $\mathcal{M}, s \models \varphi$ iff $\mathcal{N}, t \models \varphi$.

Proposition 28. *For all pointed models (\mathcal{M}, s) and (\mathcal{M}', s') , if $(\mathcal{M}, s) \Leftrightarrow (\mathcal{M}', s')$, then $(\mathcal{M}, s) \equiv_{LK} (\mathcal{M}', s')$.*

Proof. Suppose that $(\mathcal{M}, s) \Leftrightarrow (\mathcal{M}', s')$, we show for all $\varphi \in \text{LK}$: $\mathcal{M}, s \models \varphi$ if and only if $\mathcal{M}', s' \models \varphi$. The proof proceeds with induction on the structure of φ . As it is known that PAL is invariant for bisimulation, we need only present the case $\langle ! \rangle_i \psi$.

Assume that $\mathcal{M}, s \models \langle ! \rangle_i \psi$. Then there is an EL-formula χ such that $\mathcal{M}, s \models \chi$ and $\mathcal{M}|_\chi, s \models K_i \psi$. As $(\mathcal{M}, s) \Leftrightarrow (\mathcal{M}', s')$ and χ is an EL-formula, $\mathcal{M}', s' \models \chi$. Consider a relation $Z|_\chi$ as the bisimulation Z between \mathcal{M} and \mathcal{M}' restricted to $\mathcal{M}|_\chi$ and $\mathcal{M}'|_\chi$. We can check $Z|_\chi$ is also a bisimulation and $(s, s') \in Z|_\chi$. Therefore, for any $t' \in \mathcal{M}'|_\chi$ such that $s' R_i t'$, there is a $t \in \mathcal{M}|_\chi$ such that $s R_i t$ and $(\mathcal{M}|_\chi, t) \Leftrightarrow (\mathcal{M}'|_\chi, t')$, which by induction hypothesis implies that $\mathcal{M}|_\chi, t \models \psi$ if and only if $\mathcal{M}'|_\chi, t' \models \psi$. Since $\mathcal{M}|_\chi, s \models K_i \psi$, for any $t \in \mathcal{M}|_\chi$ such that $s R_i t$: $\mathcal{M}|_\chi, t \models \psi$. Then by induction hypothesis, $\mathcal{M}|_\chi, t' \models \psi$, and hence $\mathcal{M}'|_\chi, s' \models K_i \psi$. We have now shown $\mathcal{M}', s' \models \chi$ and $\mathcal{M}'|_\chi, s' \models K_i \psi$. It then follows that $\mathcal{M}', s' \models \langle ! \rangle_i \psi$. The other direction is similar. \square

Proposition 29. *For all image-finite models \mathcal{M} and \mathcal{N} , for all s in \mathcal{M} and t in \mathcal{N} , if $(\mathcal{M}, s) \equiv_{LK} (\mathcal{N}, t)$, then $(\mathcal{M}, s) \Leftrightarrow (\mathcal{N}, t)$.*

Proof. Let \mathcal{M} and \mathcal{N} be image-finite. Suppose that $(\mathcal{M}, s) \equiv_{LK} (\mathcal{N}, t)$. Since LK is an extension of EL, it follows that $(\mathcal{M}, s) \equiv_{EL} (\mathcal{N}, t)$. By the Hennessy-Milner theorem of EL (see e.g. [23]), we have $(\mathcal{M}, s) \Leftrightarrow (\mathcal{N}, t)$, as desired. \square

4.4.2 Expressivity

In this part, we shall compare the expressive powers of our logic LK, *PAL*, and *APAL*. It turns out that in the case of single-agent, the three logics are equally expressive; however, in the case of multi-agent, *LK* is more expressive than *PAL*. First, we introduce the definition of related concepts.

Proposition 30. *In the single-agent case, LK and APAL are equally expressive. As a corollary, LK and PAL are equally expressive on the single-agent case.*

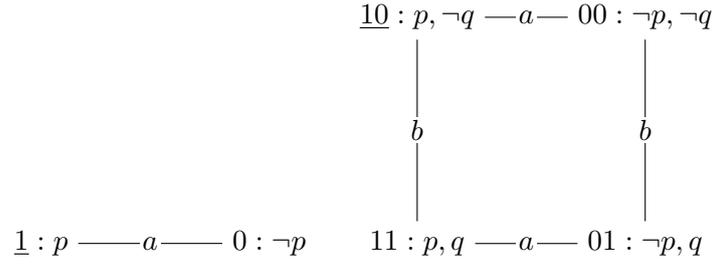
Proof. Recall that in the single-agent case, APAL is equally expressive as EL (thus *PAL*) [16, Prop. 3.12]. Moreover, LK is an extension of EL. This entails that LK is at least as expressive as APAL in single-agent case. Besides, as LK is a fragment of APAL due to the definability of $\langle ! \rangle_i$ in terms of $\langle ! \rangle$ and K_i , APAL is at least as expressive as LK. Therefore, in the single-agent case, LK and APAL are equally expressive. \square

The following result is shown as in the proof of [16, Prop. 3.13] via slight revisions. To make the exposition self-contained, we prove it in the following.

Proposition 31. *LK is more expressive than PAL.*

Proof. First, as LK is an extension of PAL with the knowability operators, $\text{PAL} \preceq \text{LK}$. It suffices to show that $\text{LK} \not\preceq \text{PAL}$. We show that $\langle ! \rangle_a(p \wedge \neg K_b K_a p)$ is *not* equivalent to any PAL-formula.

Suppose not, then as PAL is equally expressive as EL, the given knowability formula is equivalent to an EL-formula, say ψ . Because ψ is finite, it contains only finite many propositional variables. Let q be a propositional variable not occurring in ψ . Consider the following models, where the left-hand side is \mathcal{M} and the right-hand side is \mathcal{M}' :



Since $(\mathcal{M}, 1)$ and $(\mathcal{M}', 10)$ are bisimilar for atoms other than q , we have that $\mathcal{M}, 1 \models \psi$ iff $\mathcal{M}', 10 \models \psi$. However, $\mathcal{M}, 1 \not\models \langle ! \rangle_a(p \wedge \neg K_b K_a p)$ but $\mathcal{M}', 10 \models \langle ! \rangle_a(p \wedge \neg K_b K_a p)$. The argument for the former is as follows: every announcement that makes a know that p at 1 (that is, $\mathcal{M}, 1 \models K_a p$) must delete the state 0, and therefore $K_a \neg K_b K_a p$ is false at 1. To see the latter, just notice that $\mathcal{M}', 10 \models \langle p \vee q \rangle (K_a p \wedge K_a \neg K_b K_a p)$, which is equivalent to $\mathcal{M}', 10 \models \langle p \vee q \rangle K_a (p \wedge \neg K_b K_a p)$, and therefore $\mathcal{M}', 10 \models \langle ! \rangle_a(p \wedge \neg K_b K_a p)$. \square

We conjecture that LK is less expressive than APAL. In the concluding Section 4.8 we will explain in some detail why this is a difficult problem.

4.5 Axiomatization

4.5.1 Proof system and soundness

Definition 34. The system **LK** consists of the following axioms and is closed under the following rules.

TAUT	all instances of propositional tautologies
K	$K_i(\varphi \rightarrow \psi) \rightarrow (K_i\varphi \rightarrow K_i\psi)$
T	$K_i\varphi \rightarrow \varphi$
4	$K_i\varphi \rightarrow K_iK_i\varphi$
5	$\neg K_i\varphi \rightarrow K_i\neg K_i\varphi$
!ATOM	$\langle \psi \rangle p \leftrightarrow (\psi \wedge p)$
!NEG	$\langle \psi \rangle \neg \varphi \leftrightarrow (\psi \wedge \neg \langle \psi \rangle \varphi)$
!CON	$\langle \psi \rangle (\varphi \wedge \chi) \leftrightarrow (\langle \psi \rangle \varphi \wedge \langle \psi \rangle \chi)$
!K	$\langle \psi \rangle K_i\varphi \leftrightarrow (\psi \wedge K_i[\psi]\varphi)$
!!	$\langle \psi \rangle \langle \chi \rangle \varphi \leftrightarrow \langle \langle \psi \rangle \chi \rangle \varphi$
Dual	$\langle ! \rangle_i \varphi \leftrightarrow \neg [!]_i \neg \varphi$
AKK	$[!]_i \varphi \rightarrow [\psi] \hat{K}_i \varphi$, where $\psi \in \text{EL}$
MP	$\frac{\varphi \quad \varphi \rightarrow \psi}{\psi}$
NECK	$\frac{\varphi}{K_i\varphi}$
RM $\langle \cdot \rangle$	$\frac{\varphi \rightarrow \psi}{\langle \chi \rangle \varphi \rightarrow \langle \chi \rangle \psi}$
RKb	$\frac{\eta([\psi] \hat{K}_i \varphi) \text{ for all } \psi \in \text{EL}}{\eta([!]_i \varphi)}$

A formula φ is a *theorem* of **LK**, or φ is *provable* in **LK**, notation $\vdash \varphi$, if φ is either an instantiation of an axiom, or obtained by applying inferences to axioms. We use **Thm** for the set of all theorems of **LK**.

Note that although our reduction axioms are different from the more familiar ones from e.g. [104, 17], we will show that they are provable from ours (see Prop. 35).

Also note that we include Dual as an axiom. This is because we are now using $\langle ! \rangle_i$ rather than $[!]_i$ as modal primitives. This is similar to some case in the minimal normal modal logic, e.g. [23, Sec. 1.6], where the possibility operator \diamond instead of the necessity operator \square is used as a modal primitive and $\diamond\varphi \leftrightarrow \neg\square\neg\varphi$ is used as an axiom. The axiom Dual will be used later, namely in the proofs of RE (Prop. 34), Prop. 39 and Prop. 40.

To see the intuition of AKK, we can use its dual form (also denoted AKK):

$$\langle \psi \rangle K_i \varphi \rightarrow \langle ! \rangle_i \varphi, \text{ where } \psi \in \text{EL}.$$

Intuitively, this formula says that if φ is known after some announcement, then φ is knowable.

Proposition 32. **LK** is sound with respect to the class of all frames.

Proof. By the soundness of public announcement logic, it remains only to show the soundness of Dual, AKK and RKb. The soundness of Dual is obtained from the semantics of $\langle ! \rangle_i$ and $[!]_i$. The soundness of AKK is straightforward by semantics of $[!]_i$. To show the soundness of RKb, we show a stronger result:

$$(*) \quad \text{For all } (\mathcal{M}, s), \text{ if } \mathcal{M}, s \models \eta([\psi] \hat{K}_i \varphi) \text{ for all } \psi \in \text{EL}, \text{ then } \mathcal{M}, s \models \eta([!]_i \varphi).$$

The proof proceeds by induction on the structure of admissible forms.

- Base case \sharp . Since $\sharp([\psi]\hat{K}_i\varphi) = [\psi]\hat{K}_i\varphi$ and $\sharp([\!]_i\varphi) = [\!]_i\varphi$, $(*)$ follows directly from the semantics of $[\!]_i\varphi$.
- Inductive cases. We assume by induction hypothesis (IH) that $(*)$ holds for $\eta(\sharp)$, we show that $(*)$ also holds for the cases $\chi \rightarrow \eta(\sharp)$, $K_i\eta(\sharp)$ and $[\chi]\eta(\sharp)$, as follows.
 - Case $\chi \rightarrow \eta(\sharp)$. Note that $(\chi \rightarrow \eta(\sharp))([\psi]\hat{K}_i\varphi) = \chi \rightarrow \eta([\psi]\hat{K}_i\varphi)$ and $(\chi \rightarrow \eta(\sharp))([\!]_i\varphi) = \chi \rightarrow \eta([\!]_i\varphi)$. Our goal is to show that for all (\mathcal{M}, s) , if $\mathcal{M}, s \models \chi \rightarrow \eta([\psi]\hat{K}_i\varphi)$ for all $\psi \in \text{EL}$, then $\mathcal{M}, s \models \chi \rightarrow \eta([\!]_i\varphi)$. For this, suppose that $\mathcal{M}, s \models \chi \rightarrow \eta([\psi]\hat{K}_i\varphi)$ for all $\psi \in \text{EL}$ and $\mathcal{M}, s \models \chi$, then $\mathcal{M}, s \models \eta([\psi]\hat{K}_i\varphi)$ for all $\psi \in \text{EL}$. By (IH), we infer that $\mathcal{M}, s \models \eta([\!]_i\varphi)$, as desired.
 - Case $K_i\eta(\sharp)$. Note that $(K_i\eta(\sharp))([\psi]\hat{K}_i\varphi) = K_i\eta([\psi]\hat{K}_i\varphi)$ and $(K_i\eta(\sharp))([\!]_i\varphi) = K_i\eta([\!]_i\varphi)$. Our goal is to show that for all (\mathcal{M}, s) , if $\mathcal{M}, s \models K_i\eta([\psi]\hat{K}_i\varphi)$ for all $\psi \in \text{EL}$, then $\mathcal{M}, s \models K_i\eta([\!]_i\varphi)$. For this, suppose that $\mathcal{M}, s \models K_i\eta([\psi]\hat{K}_i\varphi)$ for all $\psi \in \text{EL}$, and for any t in \mathcal{M} such that sR_it , then $\mathcal{M}, t \models \eta([\psi]\hat{K}_i\varphi)$ for all $\psi \in \text{EL}$. By (IH), we derive that $\mathcal{M}, t \models \eta([\!]_i\varphi)$. Therefore, $\mathcal{M}, s \models K_i\eta([\!]_i\varphi)$, as desired.
 - Case $[\chi]\eta(\sharp)$. Note that $([\chi]\eta(\sharp))([\psi]\hat{K}_i\varphi) = [\chi]\eta([\psi]\hat{K}_i\varphi)$ and $([\chi]\eta(\sharp))([\!]_i\varphi) = [\chi]\eta([\!]_i\varphi)$. Our goal is to show that for all (\mathcal{M}, s) , if $\mathcal{M}, s \models [\chi]\eta([\psi]\hat{K}_i\varphi)$ for all $\psi \in \text{EL}$, then $\mathcal{M}, s \models [\chi]\eta([\!]_i\varphi)$. For this, suppose that $\mathcal{M}, s \models [\chi]\eta([\psi]\hat{K}_i\varphi)$ for all $\psi \in \text{EL}$ and $\mathcal{M}, s \models \chi$, then $\mathcal{M}|_\chi, s \models \eta([\psi]\hat{K}_i\varphi)$ for all $\psi \in \text{EL}$. By (IH), we obtain that $\mathcal{M}|_\chi, s \models \eta([\!]_i\varphi)$. Therefore, $\mathcal{M}, s \models [\chi]\eta([\!]_i\varphi)$, as desired.

□

4.5.2 Proof theoretical results

In this subsection we present some proof theoretical results for LK. Almost all proofs are in the Appendix, as they are rather lengthy.

In the first place, some common alternative derivation rules are derivable in the system **LK** (where Lemma 13 is essential in showing Prop 34).

Proposition 33. *The following rule is derivable in SLK:*

$$\text{RM}[\cdot] \quad \frac{\varphi \rightarrow \psi}{[\chi]\varphi \rightarrow [\chi]\psi}$$

Proof. We have the following derivation in *SLK*.

$$\begin{array}{lll} (i) & \varphi \rightarrow \psi & \text{assumption} \\ (ii) & \neg\psi \rightarrow \neg\varphi & (i) \\ (iii) & \langle\chi\rangle\neg\psi \rightarrow \langle\chi\rangle\neg\varphi & (ii), \text{RM}\langle\cdot\rangle \\ (iv) & \neg\langle\chi\rangle\neg\varphi \rightarrow \neg\langle\chi\rangle\neg\psi & (iii) \\ (v) & [\chi]\varphi \rightarrow [\chi]\psi & (iv), \text{Def. } [\cdot] \end{array}$$

□

Lemma 13. *For all φ, ψ and χ , if $\vdash \psi \leftrightarrow \chi$, then $\vdash \langle\psi\rangle\varphi \leftrightarrow \langle\chi\rangle\varphi$.*

Proof. Assume that $\vdash \psi \leftrightarrow \chi$, to show that $\vdash \langle \psi \rangle \varphi \leftrightarrow \langle \chi \rangle \varphi$. The proof goes by induction on the complexity of φ (recall that the notion of the complexity of a formula is given in Def. 33).

- Case p . We have the following derivation in **LK**.

$$\begin{array}{lll}
(i) & \langle \psi \rangle p \leftrightarrow (\psi \wedge p) & \text{!ATOM} \\
(ii) & \langle \chi \rangle p \leftrightarrow (\chi \wedge p) & \text{!ATOM} \\
(iii) & (\psi \wedge p) \leftrightarrow (\chi \wedge p) & \text{assumption} \\
(iv) & \langle \psi \rangle p \leftrightarrow \langle \chi \rangle p & (i)\text{--}(iii)
\end{array}$$

- Case $\neg\varphi$. Recall that φ is less complex than $\neg\varphi$ (Prop. 20). By induction hypothesis (IH), $\vdash \langle \psi \rangle \varphi \leftrightarrow \langle \chi \rangle \varphi$. We have the following derivation in **LK**.

$$\begin{array}{lll}
(i) & \langle \psi \rangle \neg\varphi \leftrightarrow (\psi \wedge \neg\langle \psi \rangle \varphi) & \text{!NEG} \\
(ii) & \langle \chi \rangle \neg\varphi \leftrightarrow (\chi \wedge \neg\langle \chi \rangle \varphi) & \text{!NEG} \\
(iii) & (\psi \wedge \neg\langle \psi \rangle \varphi) \leftrightarrow (\chi \wedge \neg\langle \chi \rangle \varphi) & \text{assumption, IH} \\
(iv) & \langle \psi \rangle \neg\varphi \leftrightarrow \langle \chi \rangle \neg\varphi & (i)\text{--}(iii)
\end{array}$$

- Case $\varphi_1 \wedge \varphi_2$. Recall that both φ_1 and φ_2 are less complex than $\varphi_1 \wedge \varphi_2$ (Prop. 20). By induction hypothesis (IH), $\vdash \langle \psi \rangle \varphi_1 \leftrightarrow \langle \chi \rangle \varphi_1$ and $\vdash \langle \psi \rangle \varphi_2 \leftrightarrow \langle \chi \rangle \varphi_2$. We have the following derivation in **LK**.

$$\begin{array}{lll}
(i) & \langle \psi \rangle (\varphi_1 \wedge \varphi_2) \leftrightarrow (\langle \psi \rangle \varphi_1 \wedge \langle \psi \rangle \varphi_2) & \text{!CON} \\
(ii) & \langle \chi \rangle (\varphi_1 \wedge \varphi_2) \leftrightarrow (\langle \chi \rangle \varphi_1 \wedge \langle \chi \rangle \varphi_2) & \text{!CON} \\
(iii) & \langle \psi \rangle \varphi_1 \leftrightarrow \langle \chi \rangle \varphi_1 & \text{IH} \\
(iv) & \langle \psi \rangle \varphi_2 \leftrightarrow \langle \chi \rangle \varphi_2 & \text{IH} \\
(v) & (\langle \psi \rangle \varphi_1 \wedge \langle \psi \rangle \varphi_2) \leftrightarrow (\langle \chi \rangle \varphi_1 \wedge \langle \chi \rangle \varphi_2) & (iii), (iv) \\
(vi) & \langle \psi \rangle (\varphi_1 \wedge \varphi_2) \leftrightarrow \langle \chi \rangle (\varphi_1 \wedge \varphi_2) & (i), (ii), (v)
\end{array}$$

- Case $K_i\varphi$. Recall that φ is less complex than $K_i\varphi$ (Prop. 20). By induction hypothesis (IH), $\vdash \langle \psi \rangle \varphi \leftrightarrow \langle \chi \rangle \varphi$. We have the following derivation in **LK**.

$$\begin{array}{lll}
(i) & \langle \psi \rangle K_i\varphi \leftrightarrow (\psi \wedge K_i[\psi]\varphi) & \text{!K} \\
(ii) & \langle \chi \rangle K_i\varphi \leftrightarrow (\chi \wedge K_i[\chi]\varphi) & \text{!K} \\
(iii) & \langle \psi \rangle \varphi \leftrightarrow \langle \chi \rangle \varphi & \text{IH} \\
(iv) & \langle \psi \rangle \neg\varphi \leftrightarrow \langle \chi \rangle \neg\varphi & (iii), \text{ similar to the case } \neg\varphi \\
(v) & [\psi]\varphi \leftrightarrow [\chi]\varphi & (iv), \text{ Def. } [\cdot] \\
(vi) & K_i[\psi]\varphi \leftrightarrow K_i[\chi]\varphi & (v), \text{ NECK, K, MP} \\
(vii) & (\psi \wedge K_i[\psi]\varphi) \leftrightarrow (\chi \wedge K_i[\chi]\varphi) & (vi), \text{ assumption} \\
(viii) & \langle \psi \rangle K_i\varphi \leftrightarrow \langle \chi \rangle K_i\varphi & (i), (ii), (vii)
\end{array}$$

- Case $\langle \varphi_1 \rangle \varphi_2$. Recall that φ_1 is less complex than $\langle \varphi_1 \rangle \varphi_2$ (Prop. 20). By induction hypothesis (IH), $\vdash \langle \psi \rangle \varphi_1 \leftrightarrow \langle \chi \rangle \varphi_1$. We have the following derivation in **LK**.

$$\begin{array}{lll}
(i) & \langle \psi \rangle \langle \varphi_1 \rangle \varphi_2 \leftrightarrow \langle \langle \psi \rangle \varphi_1 \rangle \varphi_2 & \text{!!} \\
(ii) & \langle \chi \rangle \langle \varphi_1 \rangle \varphi_2 \leftrightarrow \langle \langle \chi \rangle \varphi_1 \rangle \varphi_2 & \text{!!} \\
(iii) & \langle \psi \rangle \varphi_1 \leftrightarrow \langle \chi \rangle \varphi_1 & \text{IH} \\
(iv) & \langle \langle \psi \rangle \varphi_1 \rangle \varphi_2 \leftrightarrow \langle \langle \chi \rangle \varphi_1 \rangle \varphi_2 & \text{IH by (iii)} \\
(v) & \langle \psi \rangle \langle \varphi_1 \rangle \varphi_2 \leftrightarrow \langle \chi \rangle \langle \varphi_1 \rangle \varphi_2 & (i), (ii), (iv)
\end{array}$$

- Case $\langle ! \rangle_i \varphi$. Let ϑ be any EL-formula. Recall that $\langle \vartheta \rangle K_i \varphi$ is less complex than $\langle ! \rangle_i \varphi$ (Prop. 20), and thus $\neg[\vartheta] \hat{K}_i \neg \varphi$ is less expressive than $\langle ! \rangle_i \varphi$. By induction hypothesis (IH), $\vdash \langle \psi \rangle \neg[\vartheta] \hat{K}_i \neg \varphi \leftrightarrow \langle \chi \rangle \neg[\vartheta] \hat{K}_i \neg \varphi$. Then $\vdash \neg \langle \psi \rangle \neg[\vartheta] \hat{K}_i \neg \varphi \leftrightarrow \neg \langle \chi \rangle \neg[\vartheta] \hat{K}_i \neg \varphi$. By Def. $[\cdot]$, $\vdash [\psi][\vartheta] \hat{K}_i \neg \varphi \leftrightarrow [\chi][\vartheta] \hat{K}_i \neg \varphi$. We denote this by $(*)$. Then we have the following derivation in **LK**.

(i)	$[\!]_i \neg \varphi \rightarrow [\vartheta] \hat{K}_i \neg \varphi$	AKK
(ii)	$[\psi][\!]_i \neg \varphi \rightarrow [\psi][\vartheta] \hat{K}_i \neg \varphi$	(i), RM $[\cdot]$ (Prop. 33)
(iii)	$[\psi][\!]_i \neg \varphi \rightarrow [\chi][\vartheta] \hat{K}_i \neg \varphi$	(ii), $(*)$
(iv)	$[\psi][\!]_i \neg \varphi \rightarrow [\chi][\!]_i \neg \varphi$	(iii), RKb
(v)	$\langle \chi \rangle \langle ! \rangle_i \varphi \rightarrow \langle \psi \rangle \langle ! \rangle_i \varphi$	(iv)
(vi)	$\langle \psi \rangle \langle ! \rangle_i \varphi \rightarrow \langle \chi \rangle \langle ! \rangle_i \varphi$	similar to the proof of (v)
(vii)	$\langle \psi \rangle \langle ! \rangle_i \varphi \leftrightarrow \langle \chi \rangle \langle ! \rangle_i \varphi$	(v), (vi)

□

Proposition 34. *The following rule called **RE** (for ‘replacement of equivalents’) is derivable in **LK**:*

$$\frac{\psi \leftrightarrow \chi}{\varphi(p/\psi) \leftrightarrow \varphi(p/\chi)}$$

Proof. Assume that $\vdash \psi \leftrightarrow \chi$, we show $\vdash \varphi(p/\psi) \leftrightarrow \varphi(p/\chi)$ by induction on the complex of φ . Recall that the notion of complexity is given in Def. 33.

- $\varphi = p$. Then $\varphi(p/\psi) = \psi$ and $\varphi(p/\chi) = \chi$. By assumption, we have immediately that $\vdash \varphi(p/\psi) \leftrightarrow \varphi(p/\chi)$.
- $\varphi = q \neq p$. Then $\varphi(p/\psi) = \varphi(p/\chi) = q$. It is then clear that $\vdash \varphi(p/\psi) \leftrightarrow \varphi(p/\chi)$.
- $\varphi = \neg \vartheta$. Then $\varphi(p/\psi) = \neg \vartheta(p/\psi)$ and $\varphi(p/\chi) = \neg \vartheta(p/\chi)$. Since ϑ is less complex than φ (Prop. 20), by induction hypothesis (IH), $\vdash \vartheta(p/\psi) \leftrightarrow \vartheta(p/\chi)$. Then $\vdash \neg \vartheta(p/\psi) \leftrightarrow \neg \vartheta(p/\chi)$.
- $\varphi = \varphi_1 \wedge \varphi_2$. Then $\varphi(p/\psi) = \varphi_1(p/\psi) \wedge \varphi_2(p/\psi)$ and $\varphi(p/\chi) = \varphi_1(p/\chi) \wedge \varphi_2(p/\chi)$. Since both φ_1 and φ_2 are less complex than φ (Prop. 20), by induction hypothesis (IH), $\vdash \varphi_1(p/\psi) \leftrightarrow \varphi_1(p/\chi)$ and $\vdash \varphi_2(p/\psi) \leftrightarrow \varphi_2(p/\chi)$. Then $\vdash \varphi(p/\psi) \leftrightarrow \varphi(p/\chi)$.
- $\varphi = K_i \vartheta$. Then $\varphi(p/\psi) = K_i \vartheta(p/\psi)$ and $\varphi(p/\chi) = K_i \vartheta(p/\chi)$. Since ϑ is less complex than φ , by induction hypothesis (IH), $\vdash \vartheta(p/\psi) \leftrightarrow \vartheta(p/\chi)$. Then using **NECK**, **K** and **MP**, we obtain that $\vdash \varphi(p/\psi) \leftrightarrow \varphi(p/\chi)$.
- $\varphi = \langle \varphi_1 \rangle \varphi_2$. Then $\varphi(p/\psi) = \langle \varphi_1(p/\psi) \rangle \varphi_2(p/\psi)$ and $\varphi(p/\chi) = \langle \varphi_1(p/\chi) \rangle \varphi_2(p/\chi)$. Since both φ_1 and φ_2 are less complex than φ (Prop. 20), by induction hypothesis (IH), $\vdash \varphi_1(p/\psi) \leftrightarrow \varphi_1(p/\chi)$ and $\vdash \varphi_2(p/\psi) \leftrightarrow \varphi_2(p/\chi)$. From the former and Lemma 13, it follows that $\vdash \langle \varphi_1(p/\psi) \rangle \varphi_2(p/\psi) \leftrightarrow \langle \varphi_1(p/\chi) \rangle \varphi_2(p/\psi)$; from the latter and **RM** $\langle \cdot \rangle$, it follows that $\vdash \langle \varphi_1(p/\chi) \rangle \varphi_2(p/\psi) \leftrightarrow \langle \varphi_1(p/\chi) \rangle \varphi_2(p/\chi)$. Then $\vdash \varphi(p/\psi) \leftrightarrow \varphi(p/\chi)$.
- $\varphi = \langle ! \rangle_i \vartheta$. Then $\varphi(p/\psi) = \langle ! \rangle_i \vartheta(p/\psi)$ and $\varphi(p/\chi) = \langle ! \rangle_i \vartheta(p/\chi)$. Let η be any EL-formula. By Prop. 20, $\langle \eta \rangle K_i \vartheta$ is less complex than φ , so is $[\eta] \hat{K}_i \neg \vartheta$. Then by induction

hypothesis (IH), $\vdash [\eta]\hat{K}_i\neg\vartheta(p/\psi) \leftrightarrow [\eta]\hat{K}_i\neg\vartheta(p/\chi)$. We then have the following derivation in **LK**.

(i)	$[\!]_i\neg\vartheta(p/\chi) \rightarrow [\eta]\hat{K}_i\neg\vartheta(p/\chi)$	AKK
(ii)	$[\!]_i\neg\vartheta(p/\chi) \rightarrow [\eta]\hat{K}_i\neg\vartheta(p/\psi)$	(i), IH
(iii)	$[\!]_i\neg\vartheta(p/\chi) \rightarrow [\!]_i\neg\vartheta(p/\psi)$	(ii), RKb
(iv)	$\neg[\!]_i\neg\vartheta(p/\psi) \rightarrow \neg[\!]_i\neg\vartheta(p/\chi)$	(iii)
(v)	$\langle\!\rangle_i\vartheta(p/\psi) \rightarrow \langle\!\rangle_i\vartheta(p/\chi)$	(iv), Dual
(vi)	$\langle\!\rangle_i\vartheta(p/\chi) \rightarrow \langle\!\rangle_i\vartheta(p/\psi)$	similar to the proof of (v)
(vii)	$\langle\!\rangle_i\vartheta(p/\psi) \leftrightarrow \langle\!\rangle_i\vartheta(p/\chi)$	(v), (vi)

□

Recall that the axiomatization of public announcement logic, denoted **PAL**, is given in e.g. [104, Sec. 4.8].

Proposition 35. $\mathbf{PAL} \subseteq \mathbf{LK}$.

Proof. We need only show the reduction axioms of **PAL** are derivable in **LK**.

$[\varphi]p$		
$\leftrightarrow \neg\langle\!\rangle\varphi\neg p$		Def. $[\cdot]$
$\leftrightarrow \neg(\varphi \wedge \neg\langle\!\rangle\varphi p)$!NEG
$\leftrightarrow \neg(\varphi \wedge \neg(\varphi \wedge p))$!ATOM
$\leftrightarrow (\varphi \rightarrow p)$		TAUT
$[\varphi]\neg\psi$		
$\leftrightarrow \neg\langle\!\rangle\varphi\neg\neg\psi$		Def. $[\cdot]$
$\leftrightarrow \neg(\varphi \wedge \neg\langle\!\rangle\varphi\neg\psi)$!NEG
$\leftrightarrow \neg(\varphi \wedge [\varphi]\psi)$		Def. $[\cdot]$
$\leftrightarrow (\varphi \rightarrow \neg[\varphi]\psi)$		TAUT
$[\varphi](\psi \wedge \chi)$		
$\leftrightarrow \neg\langle\!\rangle\varphi\neg(\psi \wedge \chi)$		Def. $[\cdot]$
$\leftrightarrow \neg(\varphi \wedge \neg\langle\!\rangle\varphi(\psi \wedge \chi))$!NEG
$\leftrightarrow \neg(\varphi \wedge \neg(\langle\!\rangle\varphi\psi \wedge \langle\!\rangle\varphi\chi))$!CON
$\leftrightarrow \neg((\varphi \wedge \neg\langle\!\rangle\varphi\psi) \vee (\varphi \wedge \neg\langle\!\rangle\varphi\chi))$		TAUT
$\leftrightarrow \neg(\langle\!\rangle\varphi\neg\psi \vee \langle\!\rangle\varphi\neg\chi)$!NEG
$\leftrightarrow ([\varphi]\psi \wedge [\varphi]\chi)$		TAUT, Def. $[\cdot]$
$[\varphi]K_i\psi$		
$\leftrightarrow \neg\langle\!\rangle\varphi\neg K_i\psi$		Def. $[\cdot]$
$\leftrightarrow \neg(\varphi \wedge \neg\langle\!\rangle\varphi K_i\psi)$!NEG
$\leftrightarrow \neg(\varphi \wedge \neg(\varphi \wedge K_i[\varphi]\psi))$!K
$\leftrightarrow (\varphi \rightarrow \varphi \wedge K_i[\varphi]\psi)$		TAUT
$\leftrightarrow (\varphi \rightarrow K_i[\varphi]\psi)$		TAUT

$$\begin{aligned}
 & [\varphi][\psi]\chi \\
 \Leftrightarrow & \neg\langle\varphi\rangle\neg\neg\langle\psi\rangle\neg\chi && \text{Def. } [\cdot] \\
 \Leftrightarrow & \neg\langle\varphi\rangle\langle\psi\rangle\neg\chi && \text{RM}\langle\cdot\rangle \\
 \Leftrightarrow & \neg\langle\langle\varphi\rangle\psi\rangle\neg\chi && \text{!!} \\
 \Leftrightarrow & \neg\langle\varphi \wedge [\varphi]\psi\rangle\neg\chi \\
 \Leftrightarrow & [\varphi \wedge [\varphi]\psi]\chi && \text{Def. } [\cdot]
 \end{aligned}$$

where the penultimate ‘ \Leftrightarrow ’ follows from $\vdash \langle\varphi\rangle\psi \leftrightarrow (\varphi \wedge [\varphi]\psi)$ and Lemma 13. The proof for $\vdash \langle\varphi\rangle\psi \leftrightarrow (\varphi \wedge [\varphi]\psi)$ is as follows.

$$\begin{aligned}
 & \langle\varphi\rangle\psi \\
 \Leftrightarrow & \langle\varphi\rangle\neg\neg\psi && \text{TAUT, RM}\langle\cdot\rangle \\
 \Leftrightarrow & (\varphi \wedge \neg\langle\varphi\rangle\neg\psi) && \text{!NEG} \\
 \Leftrightarrow & (\varphi \wedge [\varphi]\psi) && \text{Def. } [\cdot]
 \end{aligned}$$

□

Proposition 36. *The following axiom is provable:*

$$\text{AKK}^* \quad \langle\psi\rangle K_i\varphi \rightarrow \diamond_i^K \varphi, \text{ where } \psi \in \mathcal{L}_{PAL}$$

Proof. It is known that for any PAL -formula ψ , there is an EL-formula ψ' such that $\models \psi \leftrightarrow \psi'$. By the completeness of **PAL**, we have $\vdash_{\text{PAL}} \psi \leftrightarrow \psi'$. By Prop. 35, $\text{PA} \subseteq \text{LK}$, thus $\vdash_{\text{LK}} \psi \leftrightarrow \psi'$. Then by **AKK** and **RE**, **AKK*** is derivable. □

Proposition 37. *Let $\varphi \in \text{LK}$. $\vdash \varphi \leftrightarrow \langle\top\rangle\varphi$*

Proof. By induction on the complexity of LK-formulas φ (recall the notion of complexity of a formula is given in Def. 33).

- Case p .

$$\begin{aligned}
 (i) \quad & \langle\top\rangle p \leftrightarrow (\top \wedge p) && \text{!ATOM} \\
 (ii) \quad & (\top \wedge p) \leftrightarrow p && \text{TAUT} \\
 (iii) \quad & \langle\top\rangle p \leftrightarrow p && (i), (ii)
 \end{aligned}$$

- Case $\neg\varphi$. Recall that φ is less complex than $\neg\varphi$, that is, $\varphi <_{\diamond}^S \neg\varphi$ (Prop. 20). By induction hypothesis (IH), $\vdash \langle\top\rangle\varphi \leftrightarrow \varphi$.

$$\begin{aligned}
 (i) \quad & \langle\top\rangle\neg\varphi \leftrightarrow (\top \wedge \neg\langle\top\rangle\varphi) && \text{!NEG} \\
 (ii) \quad & (\top \wedge \neg\langle\top\rangle\varphi) \leftrightarrow \neg\langle\top\rangle\varphi && \text{TAUT} \\
 (iii) \quad & \langle\top\rangle\neg\varphi \leftrightarrow \neg\langle\top\rangle\varphi && (i), (ii) \\
 (iv) \quad & \langle\top\rangle\varphi \leftrightarrow \varphi && \text{IH} \\
 (v) \quad & \langle\top\rangle\neg\varphi \leftrightarrow \neg\varphi && (iii), (iv)
 \end{aligned}$$

- Case $\varphi \wedge \psi$. Recall that both φ and ψ are less complex than $\varphi \wedge \psi$ (Prop. 20). By induction hypothesis (IH), $\vdash \langle\top\rangle\varphi \leftrightarrow \varphi$ and $\vdash \langle\top\rangle\psi \leftrightarrow \psi$.

$$\begin{aligned}
 (i) \quad & \langle\top\rangle(\varphi \wedge \psi) \leftrightarrow (\langle\top\rangle\varphi \wedge \langle\top\rangle\psi) && \text{!CON} \\
 (ii) \quad & \langle\top\rangle\varphi \leftrightarrow \varphi && \text{IH} \\
 (iii) \quad & \langle\top\rangle\psi \leftrightarrow \psi && \text{IH} \\
 (iv) \quad & \langle\top\rangle(\varphi \wedge \psi) \leftrightarrow (\varphi \wedge \psi) && (i)-(iii)
 \end{aligned}$$

- Case $K_i\varphi$. Recall that φ is less complex than $K_i\varphi$ (Prop. 20). By induction hypothesis (IH), $\vdash \langle \top \rangle \varphi \leftrightarrow \varphi$.

$$\begin{array}{lll}
(i) & \langle \top \rangle K_i\varphi \leftrightarrow (\top \wedge K_i[\top]\varphi) & \text{!K} \\
(ii) & \top \wedge K_i[\top]\varphi \leftrightarrow K_i\neg\langle \top \rangle\neg\varphi & \text{TAUT, Def. } [\cdot] \\
(iii) & K_i\neg\langle \top \rangle\neg\varphi \leftrightarrow K_i\neg(\top \wedge \neg\langle \top \rangle\varphi) & \text{!NEG, RE} \\
(iv) & K_i\neg(\top \wedge \neg\langle \top \rangle\varphi) \leftrightarrow K_i\langle \top \rangle\varphi & \text{TAUT, RE} \\
(v) & \langle \top \rangle\varphi \leftrightarrow \varphi & \text{IH} \\
(vi) & K_i\langle \top \rangle\varphi \leftrightarrow K_i\varphi & (v), \text{RE} \\
(vii) & \langle \top \rangle K_i\varphi \leftrightarrow K_i\varphi & (i)\text{-}(iv), (vi)
\end{array}$$

- Case $\langle \psi \rangle \varphi$. Recall that ψ is less complex than $\langle \psi \rangle \varphi$ (Prop. 20). By induction hypothesis (IH), $\vdash \langle \top \rangle \psi \leftrightarrow \psi$.

$$\begin{array}{lll}
(i) & \langle \top \rangle \langle \psi \rangle \varphi \leftrightarrow \langle \langle \top \rangle \psi \rangle \varphi & \text{!!} \\
(ii) & \langle \top \rangle \psi \leftrightarrow \psi & \text{IH} \\
(iii) & \langle \top \rangle \langle \psi \rangle \varphi \leftrightarrow \langle \psi \rangle \varphi & (i), (ii), \text{RE}
\end{array}$$

- Case $\langle ! \rangle_i \varphi$. Let ψ be any EL-formula. Recall that $\langle \psi \rangle K_i\varphi$ is less complex than $\langle ! \rangle_i \varphi$ (Prop. 20). By induction hypothesis (IH), $\vdash \langle \top \rangle \langle \psi \rangle K_i\varphi \leftrightarrow \langle \psi \rangle K_i\varphi$.

$$\begin{array}{lll}
(i) & [!]_i\neg\varphi \rightarrow [\psi]\hat{K}_i\neg\varphi & \text{AKK} \\
(ii) & [\psi]\hat{K}_i\neg\varphi \leftrightarrow [\top][\psi]\hat{K}_i\neg\varphi & \text{IH} \\
(iii) & [!]_i\neg\varphi \rightarrow [\top][\psi]\hat{K}_i\neg\varphi & (i), (ii) \\
(iv) & [!]_i\neg\varphi \rightarrow [\top][!]_i\neg\varphi & (iii), \text{RKb} \\
(v) & [\top][!]_i\neg\varphi \rightarrow [\top][\psi]\hat{K}_i\neg\varphi & (i), \text{RM}[\cdot] \\
(vi) & [\top][!]_i\neg\varphi \rightarrow [\psi]\hat{K}_i\neg\varphi & (ii), (v) \\
(vii) & [\top][!]_i\neg\varphi \rightarrow [!]_i\neg\varphi & (vi), \text{RKb} \\
(viii) & [\top][!]_i\neg\varphi \leftrightarrow [!]_i\neg\varphi & (iv), (vii) \\
(ix) & \langle \top \rangle \langle ! \rangle_i \varphi \leftrightarrow \langle ! \rangle_i \varphi & (viii), \text{RE}
\end{array}$$

□

Corollary 10. $\vdash [\top]\varphi \leftrightarrow \varphi$ for all $\varphi \in LK$.

Proof. By Prop. 37, $\vdash \langle \top \rangle\neg\varphi \leftrightarrow \neg\varphi$. Thus $\vdash \neg\langle \top \rangle\neg\varphi \leftrightarrow \neg\neg\varphi$. By Def. $[\cdot]$, we obtain $\vdash [\top]\varphi \leftrightarrow \varphi$. □

Proposition 38. If $\vdash \varphi \rightarrow \psi$, then $\vdash [!]_i\varphi \rightarrow [!]_i\psi$.

Proof. Assume that $\vdash \varphi \rightarrow \psi$, we have the following derivation in \mathbb{LK} , where χ is any EL-formula:

$$\begin{array}{lll}
(i) & \neg\psi \rightarrow \neg\varphi & \text{assumption, TAUT} \\
(ii) & K_i\neg\psi \rightarrow K_i\neg\varphi & (i), \text{NECK, K, MP} \\
(iii) & \langle \chi \rangle K_i\neg\psi \rightarrow \langle \chi \rangle K_i\neg\varphi & (ii), \text{RM}(\cdot) \\
(iv) & \langle \chi \rangle K_i\neg\varphi \rightarrow \langle ! \rangle_i\neg\varphi & \text{AKK} \\
(v) & \langle \chi \rangle K_i\neg\psi \rightarrow \langle ! \rangle_i\neg\varphi & (iii), (iv) \\
(vi) & \neg\langle ! \rangle_i\neg\varphi \rightarrow \neg\langle \chi \rangle K_i\neg\psi & (v) \\
(vii) & \neg\langle \chi \rangle K_i\neg\psi \leftrightarrow \neg\langle \chi \rangle\neg\neg K_i\neg\psi & \text{TAUT, RM}(\cdot) \\
(viii) & \neg\langle \chi \rangle K_i\neg\psi \leftrightarrow [\chi]\hat{K}_i\psi & (vii), \text{Def. } [\cdot], \text{Def. } \hat{K}_i \\
(ix) & [!]_i\varphi \rightarrow [\chi]\hat{K}_i\psi & (vi), (viii), \text{Def. } [!]_i \\
(x) & [!]_i\varphi \rightarrow [!]_i\psi & (ix), \text{RKb}
\end{array}$$

□

Recall that in Prop. 7 and Prop. 8 we show that $\models \langle ! \rangle_i \langle ! \rangle_i \varphi \rightarrow \langle ! \rangle_i \varphi$ and $\models \langle ! \rangle_i \varphi \rightarrow \langle ! \rangle_i \langle ! \rangle_i \varphi$, respectively. We can also give a syntactic proof of them.

Proposition 39. $\vdash \langle ! \rangle_i \varphi \rightarrow \langle ! \rangle_i \langle ! \rangle_i \varphi$

Proof. We have the following derivation in **LK**, where χ is any EL-formula:

- | | | |
|--------|---|------------------|
| (i) | $[\!]_i \hat{K}_i \neg \varphi \rightarrow [\chi] \hat{K}_i \hat{K}_i \neg \varphi$ | AKK |
| (ii) | $\hat{K}_i \hat{K}_i \neg \varphi \rightarrow \hat{K}_i \neg \varphi$ | 4 |
| (iii) | $[\chi] \hat{K}_i \hat{K}_i \neg \varphi \rightarrow [\chi] \hat{K}_i \neg \varphi$ | (ii), Prop. 33 |
| (iv) | $[\!]_i \hat{K}_i \neg \varphi \rightarrow [\chi] \hat{K}_i \neg \varphi$ | (i), (iii) |
| (v) | $[\!]_i \hat{K}_i \neg \varphi \rightarrow [\!]_i \neg \varphi$ | (iv), RKb |
| (vi) | $[\!]_i \neg \varphi \rightarrow [\top] \hat{K}_i \neg \varphi$ | AKK |
| (vii) | $[\top] \hat{K}_i \neg \varphi \leftrightarrow \hat{K}_i \neg \varphi$ | Coro. 10 |
| (viii) | $[\!]_i \neg \varphi \rightarrow \hat{K}_i \neg \varphi$ | (vi), (vii) |
| (ix) | $[\!]_i [\!]_i \neg \varphi \rightarrow [\!]_i \hat{K}_i \neg \varphi$ | (viii), Prop. 38 |
| (x) | $[\!]_i [\!]_i \neg \varphi \rightarrow [\!]_i \neg \varphi$ | (ix), (v) |
| (xi) | $\langle ! \rangle_i \varphi \rightarrow \langle ! \rangle_i \langle ! \rangle_i \varphi$ | (x), RE, Dual |

□

Proposition 40. $\vdash \langle ! \rangle_i \langle ! \rangle_i \varphi \rightarrow \langle ! \rangle_i \varphi$

Proof. We have the following derivation in **LK**, where ψ, χ are any EL-formulas (thus $\langle \psi \rangle \chi \in \mathcal{L}_{PAL}$).

- | | | |
|--------|---|------------------|
| (i) | $[\!]_i \neg \varphi \rightarrow \hat{K}_i [\!]_i \neg \varphi$ | T |
| (ii) | $[\psi] [\!]_i \neg \varphi \rightarrow [\psi] \hat{K}_i [\!]_i \neg \varphi$ | (i), Prop. 33 |
| (iii) | $[\!]_i \neg \varphi \rightarrow [\psi \wedge [\psi] \chi] \hat{K}_i \neg \varphi$ | AKK* |
| (iv) | $[\psi \wedge [\psi] \chi] \hat{K}_i \neg \varphi \leftrightarrow [\psi] [\chi] \hat{K}_i \neg \varphi$ | Prop. 35 |
| (v) | $[\!]_i \neg \varphi \rightarrow [\psi] [\chi] \hat{K}_i \neg \varphi$ | (iii), (iv) |
| (vi) | $[\!]_i \neg \varphi \rightarrow [\psi] [\!]_i \neg \varphi$ | (v), RKb |
| (vii) | $[\!]_i \neg \varphi \rightarrow [\psi] \hat{K}_i [\!]_i \neg \varphi$ | (vi), (ii) |
| (viii) | $[\!]_i \neg \varphi \rightarrow [\!]_i [\!]_i \neg \varphi$ | (vii), RKb |
| (ix) | $\langle ! \rangle_i \langle ! \rangle_i \varphi \rightarrow \langle ! \rangle_i \varphi$ | (viii), RE, Dual |

□

We conclude this section with a derivable rule.

Proposition 41. *The following rule is derivable in LK:*

$$\frac{\varphi}{\langle ! \rangle_i \varphi}$$

Proof. We have the following derivation in **LK**.

- | | | |
|-------|------------------------------------|----------------|
| (i) | φ | assumption |
| (ii) | $K_i \varphi$ | (i), NECK |
| (iii) | $\langle \top \rangle K_i \varphi$ | (ii), Prop. 37 |
| (iv) | $\langle ! \rangle_i \varphi$ | (iii), AKK |

□

For contrast, note that $\varphi \rightarrow \langle ! \rangle_i \varphi$ is *not* derivable (see the remarks before Prop. 22).

4.6 Completeness and Decidability

4.6.1 Completeness

This section deals with a demonstration of the completeness of **LK**. The canonical model will be based on a notion of *maximal consistent theory*, rather than the more familiar notion of *maximal consistent set*. The reason of defining consistency for a theory rather than any set of formulas, is because we need the closure condition under **RKb**, which is indispensable in the completeness proof.

Definition 35 (MCT). A set Γ of formulas is said to be a *theory*, if besides containing **Thm**, it is also closed under the rules **MP** and **RKb**. A theory Γ is said to be *consistent*, if $\perp \notin \Gamma$; Γ is said to be *maximal*, if for all φ , $\varphi \in \Gamma$ or $\neg\varphi \in \Gamma$. Γ is a *maximal consistent theory* (MCT), if it is a theory which is consistent and maximal.

One may easily check that **Thm** is the smallest theory.

Define $s + \varphi$ as $\{\psi \mid \varphi \rightarrow \psi \in s\}$. We omit the proof details of the following result.

Proposition 42. *Let $\varphi \in \mathbf{LK}$ and s be a theory. Then*

1. $s + \varphi$ is a theory, and $s \cup \{\varphi\} \subseteq s + \varphi$.
2. $s + \varphi$ is consistent iff $\neg\varphi \notin s$.

Lindenbaum's Lemma can be proven as [16, Lemma 4.12], with only corresponding changes of the rule **RKb**. Thus we omit the proof details.

Lemma 14 (Lindenbaum's Lemma). *Every consistent theory can be extended to a MCT.*

Definition 36 (Canonical Model). The canonical model for **LK** is $\mathcal{M}^c = \langle S^c, \{R_i^c \mid i \in \mathbf{A}\}, V^c \rangle$, where

- S^c is the set of all MCTs;
- For all $i \in \mathbf{A}$, $sR_i^c t$ iff $\{\varphi \mid K_i\varphi \in s\} \subseteq t$;
- $V^c(p) = \{s \in S^c \mid p \in s\}$.

Using axioms **T**, **4** and **5**, we can show that each R_i^c is an equivalence relation. Thus \mathcal{M}^c is indeed a model.

The following proposition can be shown as in [17, Lemma 7]. Thus again, we omit the proof details.

Proposition 43. *Let $s \in S^c$, $\psi \in \mathbf{LK}$, and $i \in \mathbf{A}$ such that $K_i\psi \notin s$. Then there exists $t \in S^c$ such that $sR_i^c t$ and $\psi \notin t$.*

Lemma 15 (Truth Lemma). *For all formulas $\varphi \in \mathbf{LK}$ and all $s \in S^c$, we have*

$$\mathcal{M}^c, s \models \varphi \iff \varphi \in s.$$

Proof. It is straightforward to show that $<_{\diamond}^S$ is a well-founded strict partial order between formulas. Let $\varphi \in \text{LK}$ and $s \in S^c$, we proceed with $<_{\diamond}^S$ -induction on φ , that is, with induction on the complexity of φ .

- $\varphi = p$. We have $\mathcal{M}^c, s \models p \iff s \in V^c(p) \stackrel{\text{Def. } V^c}{\iff} p \in s$.
- $\varphi = \neg\psi$. Recall that $\psi <_{\diamond}^S \neg\psi$ (Prop. 20). We have

$$\begin{aligned} \mathcal{M}^c, s \models \neg\psi &\iff \mathcal{M}^c, s \not\models \psi \\ &\stackrel{\text{IH}}{\iff} \psi \notin s \\ &\iff \neg\psi \in s. \end{aligned}$$

- $\varphi = \psi \wedge \chi$. Recall that $\psi <_{\diamond}^S \psi \wedge \chi$ and $\chi <_{\diamond}^S \psi \wedge \chi$ (Prop. 20). We have

$$\begin{aligned} \mathcal{M}^c, s \models \psi \wedge \chi &\iff \mathcal{M}^c, s \models \psi \text{ and } \mathcal{M}^c, s \models \chi \\ &\stackrel{\text{IH}}{\iff} \psi \in s \text{ and } \chi \in s \\ &\iff \psi \wedge \chi \in s. \end{aligned}$$

- $\varphi = K_i\psi$. Recall that $\psi <_{\diamond}^S K_i\psi$ (Prop. 20). We have

$$\begin{aligned} \mathcal{M}^c, s \models K_i\psi &\iff \mathcal{M}^c, t \models \psi \text{ for all } t \in R_i^c(s) \\ &\stackrel{\text{IH}}{\iff} \psi \in t \text{ for all } t \in R_i^c(s) \\ &\stackrel{(*)}{\iff} K_i\psi \in s. \end{aligned}$$

The equivalence $(*)$ follows from the definition of R_i^c and Proposition 43.

- $\varphi = \langle\psi\rangle p$. Recall that $\psi <_{\diamond}^S \langle\psi\rangle p$ and $p <_{\diamond}^S \langle\psi\rangle p$ (Prop. 20). We have

$$\begin{aligned} \mathcal{M}^c, s \models \langle\psi\rangle p &\iff \mathcal{M}^c, s \models \psi \text{ and } \mathcal{M}^c, s \models p \\ &\stackrel{\text{IH}}{\iff} \psi \in s \text{ and } p \in s \\ &\iff \psi \wedge p \in s \\ &\stackrel{\text{Ax. } \text{!ATOM}}{\iff} \langle\psi\rangle p \in s. \end{aligned}$$

- $\varphi = \langle\psi\rangle\neg\chi$. Recall that $\psi <_{\diamond}^S \langle\psi\rangle\neg\chi$ and $\langle\psi\rangle\chi <_{\diamond}^S \langle\psi\rangle\neg\chi$ (Prop. 20). We have

$$\begin{aligned} \mathcal{M}^c, s \models \langle\psi\rangle\neg\chi &\iff \mathcal{M}^c, s \models \psi \text{ and } \mathcal{M}^c, s \not\models \langle\psi\rangle\chi \\ &\stackrel{\text{IH}}{\iff} \psi \in s \text{ and } \langle\psi\rangle\chi \notin s \\ &\iff \psi \in s \text{ and } \neg\langle\psi\rangle\chi \in s \\ &\stackrel{\text{Ax. } \text{!NEG}}{\iff} \langle\psi\rangle\neg\chi \in s. \end{aligned}$$

- $\varphi = \langle\psi\rangle(\chi_1 \wedge \chi_2)$. Recall that $\langle\psi\rangle\chi_1 <_{\diamond}^S \langle\psi\rangle(\chi_1 \wedge \chi_2)$ and $\langle\psi\rangle\chi_2 <_{\diamond}^S \langle\psi\rangle(\chi_1 \wedge \chi_2)$ (Prop. 20). We have

$$\begin{aligned} \mathcal{M}^c, s \models \langle\psi\rangle(\chi_1 \wedge \chi_2) &\iff \mathcal{M}^c, s \models \langle\psi\rangle\chi_1 \text{ and } \mathcal{M}^c, s \models \langle\psi\rangle\chi_2 \\ &\stackrel{\text{IH}}{\iff} \langle\psi\rangle\chi_1 \in s \text{ and } \langle\psi\rangle\chi_2 \in s \\ &\iff \langle\psi\rangle\chi_1 \wedge \langle\psi\rangle\chi_2 \in s \\ &\stackrel{\text{Ax. } \text{!CON}}{\iff} \langle\psi\rangle(\chi_1 \wedge \chi_2) \in s. \end{aligned}$$

- $\varphi = \langle \psi \rangle K_i \chi$. Recall that $\psi <_{\diamond}^S \langle \psi \rangle K_i \chi$ and $K_i[\psi]\varphi <_{\diamond}^S \langle \psi \rangle K_i \chi$ (Prop. 20). We have

$$\begin{aligned}
 \mathcal{M}^c, s \models \langle \psi \rangle K_i \chi &\iff \mathcal{M}^c, s \models \psi \text{ and } \mathcal{M}^c, s \models K_i[\psi]\chi \\
 &\stackrel{\text{IH}}{\iff} \psi \in s \text{ and } K_i[\psi]\chi \in s \\
 &\iff \psi \wedge K_i[\psi]\chi \in s \\
 &\stackrel{\text{Ax. !CON}}{\iff} \langle \psi \rangle K_i \chi \in s.
 \end{aligned}$$

- $\varphi = \langle \psi \rangle \langle \chi \rangle \delta$. Recall that $\langle \langle \psi \rangle \chi \rangle \delta <_{\diamond}^S \langle \psi \rangle \langle \chi \rangle \delta$ (Prop. 20). We have

$$\begin{aligned}
 \mathcal{M}^c, s \models \langle \psi \rangle \langle \chi \rangle \delta &\iff \mathcal{M}^c, s \models \langle \langle \psi \rangle \chi \rangle \delta \\
 &\stackrel{\text{IH}}{\iff} \langle \langle \psi \rangle \chi \rangle \delta \in s \\
 &\stackrel{\text{Ax. !!}}{\iff} \langle \psi \rangle \langle \chi \rangle \delta \in s.
 \end{aligned}$$

- $\varphi = \langle \psi \rangle \langle ! \rangle_i \chi$. We have

$$\begin{aligned}
 \mathcal{M}^c, s \models \langle \psi \rangle \langle ! \rangle_i \chi &\iff \mathcal{M}^c, s \models \psi \text{ and } \mathcal{M}^c|_{\psi}, s \models \langle ! \rangle_i \chi \\
 &\iff \mathcal{M}^c, s \models \psi \text{ and } \mathcal{M}^c|_{\psi}, s \models \langle \delta \rangle K_i \chi \text{ for some } \delta \in \text{EL} \\
 &\iff \mathcal{M}^c, s \models \langle \psi \rangle \langle \delta \rangle K_i \chi \text{ for some } \delta \in \text{EL} \\
 &\stackrel{\text{IH}}{\iff} \langle \psi \rangle \langle \delta \rangle K_i \chi \in s \text{ for some } \delta \in \text{EL} \\
 &\stackrel{(1)}{\iff} [\psi][\delta] \hat{K}_i \neg \chi \notin s \text{ for some } \delta \in \text{EL} \\
 &\stackrel{(**)}{\iff} [\psi][!]_i \neg \chi \notin s \\
 &\stackrel{(2)}{\iff} \langle \psi \rangle \langle ! \rangle_i \chi \in s.
 \end{aligned}$$

Recall that $\langle \psi \rangle \langle \delta \rangle K_i \chi <_{\diamond}^S \langle \psi \rangle \langle ! \rangle_i \chi$ for any $\delta \in \text{EL}$ (Prop. 20), thus we can use the induction hypothesis (IH) in the fourth step. In (**), the left-to-right direction follows from Axiom AKK and rule RM[·], and the other direction is because s is closed under the rule RKb for the admissible form $[\psi]\sharp$. (1) and (2) hold due to the maximal consistency of s .

- $\varphi = \langle ! \rangle_i \psi$. We have

$$\begin{aligned}
 \mathcal{M}^c, s \models \langle ! \rangle_i \psi &\iff \mathcal{M}^c, s \models \langle \chi \rangle K_i \psi \text{ for some } \chi \in \text{EL} \\
 &\stackrel{\text{IH}}{\iff} \langle \chi \rangle K_i \psi \in s \text{ for some } \chi \in \text{EL} \\
 &\stackrel{(a)}{\iff} [\chi] \hat{K}_i \neg \psi \notin s \text{ for some } \chi \in \text{EL} \\
 &\stackrel{(***)}{\iff} [!]_i \neg \psi \notin s \\
 &\stackrel{(b)}{\iff} \langle ! \rangle_i \psi \in s.
 \end{aligned}$$

Recall that $\langle \chi \rangle K_i \psi <_{\diamond}^S \langle ! \rangle_i \psi$ for any $\chi \in \text{EL}$ (Prop. 20), thus we can use the induction hypothesis (IH) in the second step. The equivalence (***) is due to Axiom AKK and the fact that s is closed under the rule RKb for the possible form \sharp . (a) and (b) hold because of the maximal consistency of s .

□

With the Truth Lemma in mind, we obtain the completeness theorem as usual.

Theorem 11 (Completeness Theorem). *LK is sound and complete with respect to the class of frames. That is, if $\models \varphi$, then $\vdash \varphi$.*

Proof. The soundness is immediate. For the completeness, suppose $\not\models \varphi$, i.e. $\varphi \notin \mathbf{Thm}$. Since \mathbf{Thm} is a theory, it is closed under MP, thus $\neg\neg\varphi \notin \mathbf{Thm}$. By Prop. 42, $\mathbf{Thm} + \{\neg\varphi\}$ is a consistent theory and $\neg\varphi \in \mathbf{Thm} + \{\neg\varphi\}$. By Lindenbaum's Lemma (Lemma 14), there exists $t \in S^c$ with $\mathbf{Thm} + \{\neg\varphi\} \subseteq t$, and thus $\neg\varphi \in t$, that is, $\varphi \notin t$. Due to the Truth Lemma (Lemma 15), we obtain $\mathcal{M}^c, t \not\models \varphi$. Moreover, as remarked before, \mathcal{M}^c is a model. Therefore $\not\models \varphi$. \square

4.6.2 Decidability

Recall that the satisfiability problem of *APAL* is shown to be undecidable when there are at least two agents [6, 41]. The approach is by reducing an undecidable tiling problem into *APAL* [6]. Following the same approach, we may infer that LK is also undecidable when there are at least three agents. We will sketch the main idea of the proof.

In [6] an *APAL*-formula φ is defined such that a certain finite set of tiles Γ tiles the infinite plain $\mathbb{N} \times \mathbb{N}$, if and only if φ is satisfiable on a certain model \mathcal{M} defined for two agents a and b . We can transform φ into an LK-formula ψ by substituting all quantifiers \square in φ for knowability operators $[\!]_i$, and we can change the model \mathcal{M} into a model \mathcal{M}^{LK} that is the same as \mathcal{M} except that we add another agent i that has the identity relation on the domain. Since for any state t in the model, t has itself as the only i -successor, it follows for any subformula ϑ of φ :

$$\mathcal{M}^{LK} \models \vartheta \leftrightarrow \hat{K}_i \vartheta$$

For example, a constituent of the formula φ is:

$$c_{apal}(\heartsuit) := \heartsuit \rightarrow \square(K_s(r \rightarrow (K_c(l \rightarrow (K_s(u \rightarrow K_c(d \rightarrow K_s(l \rightarrow K_c(r \rightarrow K_s(d \rightarrow K_c(u \rightarrow \hat{K}_s \heartsuit))))))))))$$

It is transformed into:

$$c_{lk}(\heartsuit) := \heartsuit \rightarrow [\!]_i(K_s(r \rightarrow (K_c(l \rightarrow (K_s(u \rightarrow K_c(d \rightarrow K_s(l \rightarrow K_c(r \rightarrow K_s(d \rightarrow K_c(u \rightarrow \hat{K}_s \heartsuit))))))))))$$

and $\mathcal{M}^{LK}, t \models c_{lk}(\heartsuit)$ if and only if $\mathcal{M}, t \models c_{apal}(\heartsuit)$.

This may sufficiently demonstrate that a detailed proof of the undecidability of the satisfiability of LK would be nearly identical to the proof in [6]. Therefore, LK is undecidable for at least three agents. Whether LK is decidable for only two agents needs further investigation.

In what follows, we will give two decidable knowability logics.

4.7 Decidable knowability logics

4.7.1 Logic $LK^=$

We recall that the language of the logic $LK^=$ was defined as the fragment

$$\varphi ::= p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \langle \! \! \rangle_i \varphi$$

In this fragment we can no longer quantify over all epistemic formulas, but, for a similar treatment of the quantifier, over all Booleans only. Its semantics are:

$$\mathcal{M}, s \models \langle ! \rangle_i \varphi \iff \text{there is a } \psi \in \text{PL such that } \mathcal{M}, s \models \psi \text{ and for all } t \in R_i(s), \mathcal{M}|_{\psi}, t \models \varphi$$

This quantification is therefore like the one in so-called Boolean arbitrary public announcement logic BAPAL [98]

$$\mathcal{M}, s \models \langle ! \rangle \varphi \iff \text{there is } \psi \in \mathcal{L}_{PL} \text{ such that } \mathcal{M}, s \models \psi \text{ and } \mathcal{M}|_{\psi}, s \models \varphi$$

As the semantics of the quantifier in $\text{LK}^=$ are different, the properties of the quantifier $\langle ! \rangle_i$ that were observed in Section 4.3 now have to be shown again. It is straightforward that $\langle ! \rangle_i \varphi$ implies $\langle ! \rangle \varphi$.

It may be interesting and surprising to see that the knowability operators are dispensable in classical propositional logic. That is to say, the addition of knowability operators does not increase the expressive power of classical propositional logic.

Proposition 44. *$\text{LK}^=$ is equally expressive as PL.*

Proof. As $\text{LK}^=$ extends PL, $\text{LK}^=$ is at least as expressive as PL. It suffices to prove that PL is at least as expressive as $\text{LK}^=$.

For this, let φ be a formula in the language of $\text{LK}^=$. We prove that φ is equivalent to a formula in PL. The proof is by induction on the number of $\langle ! \rangle_i$ modalities in φ .

If φ contains no $\langle ! \rangle_i$ modality, then φ is already in PL, and we are done. Otherwise, consider a subformula $\langle ! \rangle_i \psi$ of φ such that $\psi \in \text{PL}$.

We first show that $\models \langle ! \rangle_i \psi \leftrightarrow \psi$.

Let $\mathcal{M} = \langle S, R, V \rangle$ and $s \in S$ be given.

Assume that $\mathcal{M}, s \models \langle ! \rangle_i \psi$. By definition, there is a $\chi \in \text{PL}$ such that $\mathcal{M}, s \models \chi$ and for all $t \in R_i(s)$, $\mathcal{M}|_{\chi}, t \models \psi$. In particular, $\mathcal{M}|_{\chi}, s \models \psi$. Therefore, as ψ is Boolean and as the valuation does not change after model restriction, we have $\mathcal{M}, s \models \psi$.

Conversely, assume that $\mathcal{M}, s \models \psi$. Consider the characteristic formula δ_s^ψ defined as in the proof of Thm. 6. Then $\mathcal{M}, s \models \delta_s^\psi$, and also $\mathcal{M}|_{\delta_s^\psi}, s \models \psi$. As the valuation of the variables in ψ is constant on $\mathcal{M}|_{\delta_s^\psi}$, it follows from Prop. 21 that $\mathcal{M}|_{\delta_s^\psi} \models \psi$, and therefore $\mathcal{M}|_{\delta_s^\psi}, t \models \psi$ for all $t \in R_i(s)$. From that and $\mathcal{M}, s \models \delta_s^\psi$ it follows by semantics that $\mathcal{M}, s \models \langle ! \rangle_i \psi$.

This proves $\models \langle ! \rangle_i \psi \leftrightarrow \psi$. Now replace $\langle ! \rangle_i \psi$ by ψ in φ . Let the result be φ' . Note that $\models \varphi \leftrightarrow \varphi'$. As φ' contains one less knowability modality than φ , by induction hypothesis we can conclude that φ' is equivalent to a Boolean formula φ'' . From $\models \varphi \leftrightarrow \varphi'$ and $\models \varphi' \leftrightarrow \varphi''$ it follows that $\models \varphi \leftrightarrow \varphi''$. \square

It may be instructive to present an example.

Example 4. We will show that the formula $\langle ! \rangle_i \langle ! \rangle_j ([!]_k (p \rightarrow q) \vee [!]_k \neg r)$, read “it is knowable for i that it is knowable for j that either it is unknowable for k that p does not imply q or it is unknowable for k that r ”, is equivalent to a Boolean formula. The proof is as follows.

$$\begin{aligned} & \langle ! \rangle_i \langle ! \rangle_j ([!]_k (p \rightarrow q) \vee [!]_k \neg r) \\ \leftrightarrow & \langle ! \rangle_i \langle ! \rangle_j (\neg [!]_k \neg (p \rightarrow q) \vee \neg [!]_k \neg r) \\ \leftrightarrow & \langle ! \rangle_i \langle ! \rangle_j (\neg \neg (p \rightarrow q) \vee \neg \neg r) \\ \leftrightarrow & \langle ! \rangle_i \langle ! \rangle_j ((p \rightarrow q) \vee \neg r) \\ \leftrightarrow & \langle ! \rangle_i ((p \rightarrow q) \vee \neg r) \\ \leftrightarrow & (p \rightarrow q) \vee \neg r \end{aligned}$$

In what follows, we show the properties of Church-Rosser and McKinsey hold for LK^- . For this, we define a translation from $\mathcal{L}_{\text{LK}^-}$ to \mathcal{L}_{PL} .

Definition 37. Define $t : \mathcal{L}_{\text{LK}^-} \rightarrow \mathcal{L}_{\text{PL}}$ as follows.

$$\begin{aligned} t(p) &= p \\ t(\neg\varphi) &= \neg t(\varphi) \\ t(\varphi \wedge \psi) &= t(\varphi) \wedge t(\psi) \\ t(\langle ! \rangle_i \varphi) &= t(\varphi). \end{aligned}$$

Intuitively, t removes every occurrence of $\langle ! \rangle_i$ in the formulas of LK^- .

It is straightforward to compute that $t([\!]_i \varphi) = \neg \neg t(\varphi)$. This translation helps us show the properties of Church-Rosser and McKinsey holds for LK^- , namely, $\langle ! \rangle_i [\!]_i \varphi \rightarrow [\!]_i \langle ! \rangle_i \varphi$ and $[\!]_i \langle ! \rangle_i \varphi \rightarrow \langle ! \rangle_i [\!]_i \varphi$, respectively, are valid on the semantics of LK^- . To see this, we first show the following result.

Lemma 16. For all $\varphi \in \mathcal{L}_{\text{LK}^-}$, we have

$$\models \varphi \leftrightarrow t(\varphi).$$

Proof. By induction on $\varphi \in \mathcal{L}_{\text{LK}^-}$.

- $\varphi = p \in \mathbf{P}$. Since $t(p) = p$, we obviously have $\models p \leftrightarrow t(p)$.
- $\varphi = \neg\psi$. By induction hypothesis, $\models \psi \leftrightarrow t(\psi)$. Then $\models \neg\psi \leftrightarrow t(\neg\psi)$.
- $\varphi = \psi \wedge \chi$. By induction hypothesis, $\models \psi \leftrightarrow t(\psi)$ and $\models \chi \leftrightarrow t(\chi)$. Then $\models (\psi \wedge \chi) \leftrightarrow t(\psi \wedge \chi)$.
- $\varphi = \langle ! \rangle_i \psi$. By induction hypothesis, $\models \psi \leftrightarrow t(\psi)$. Then $\models \langle ! \rangle_i \psi \leftrightarrow \langle ! \rangle_i t(\psi)$. Since $t(\psi) \in \mathcal{L}_{\text{PL}}$, by the proof of Prop. 44, $\models \langle ! \rangle_i t(\psi) \leftrightarrow t(\psi)$. This follows that $\models \langle ! \rangle_i \psi \leftrightarrow t(\psi)$.⁵ As $t(\langle ! \rangle_i \psi) = t(\psi)$, we conclude that $\models \langle ! \rangle_i \psi \leftrightarrow t(\langle ! \rangle_i \psi)$.

□

Theorem 12 (CR and MK). $\models \langle ! \rangle_i [\!]_i \varphi \leftrightarrow [\!]_i \langle ! \rangle_i \varphi$.

Proof. Note that $t(\langle ! \rangle_i [\!]_i \varphi) = t([\!]_i \varphi) = \neg \neg t(\varphi)$ and $t([\!]_i \langle ! \rangle_i \varphi) = \neg \neg t(\langle ! \rangle_i \varphi) = \neg \neg t(\varphi)$. Thus $t(\langle ! \rangle_i [\!]_i \varphi) = t([\!]_i \langle ! \rangle_i \varphi)$. By Lemma 16, we have $\models \langle ! \rangle_i [\!]_i \varphi \leftrightarrow t(\langle ! \rangle_i [\!]_i \varphi)$ and $\models [\!]_i \langle ! \rangle_i \varphi \leftrightarrow t([\!]_i \langle ! \rangle_i \varphi)$. Therefore, $\models \langle ! \rangle_i [\!]_i \varphi \leftrightarrow [\!]_i \langle ! \rangle_i \varphi$. □

Now we add an axiomatization for LK^- . In retrospect, Lemma 16 essentially gives us the following reduction-like axiom (denoted **Red**):

$$\langle ! \rangle_i \varphi \leftrightarrow \varphi.$$

Intuitively, **Red** removes all $\langle ! \rangle_i$ operators from formulas in LK^- within finitely many steps.

We use \mathbf{LK}^- to denote $\mathbf{PL} + \text{Red}$, in which \mathbf{PL} is the classical propositional calculus. In what follows, we will show that \mathbf{LK}^- is determined by the class of epistemic frames. For this, we first need an important result.

⁵Note that Prop. 44 only shows that $\models \langle ! \rangle_i \chi \leftrightarrow \chi$ holds for every $\chi \in \text{PL}$, but it does not show this statement holds for any LK^- -formula. This is what we are doing here.

Lemma 17. For all $\varphi \in \mathcal{L}_{LK^=}$, we have

$$\vdash \varphi \leftrightarrow t(\varphi).$$

Proof. By induction on $\varphi \in \mathcal{L}_{LK^=}$.

- $\varphi = p \in \mathbf{P}$. As $t(p) = p$, we have $\vdash p \leftrightarrow t(p)$.
- $\varphi = \neg\psi$. By induction hypothesis, $\vdash \psi \leftrightarrow t(\psi)$, and thus $\vdash \neg\psi \leftrightarrow \neg t(\psi)$, that is, $\vdash \neg\psi \leftrightarrow t(\neg\psi)$.
- $\varphi = \psi \wedge \chi$. By induction hypothesis, we have $\vdash \psi \leftrightarrow t(\psi)$ and $\vdash \chi \leftrightarrow t(\chi)$. Therefore, $\vdash (\psi \wedge \chi) \leftrightarrow t(\psi \wedge \chi)$.
- $\varphi = \langle ! \rangle_i \psi$. By induction hypothesis, $\vdash \psi \leftrightarrow t(\psi)$. By axiom **Red**, $\vdash \langle ! \rangle_i \psi \leftrightarrow \psi$. Moreover, $t(\langle ! \rangle_i \psi) = t(\psi)$. Then we conclude that $\vdash \langle ! \rangle_i \psi \leftrightarrow t(\langle ! \rangle_i \psi)$.

□

Theorem 13. $\mathbf{LK}^=$ is sound and complete with respect to the class of all frames.

Proof. For the soundness, it remains only to show the validity of axiom **Red**. By Lemma 16, $\models \langle ! \rangle_i \varphi \leftrightarrow t(\langle ! \rangle_i \varphi)$ and $\models \varphi \leftrightarrow t(\varphi)$. As $t(\langle ! \rangle_i \varphi) = t(\varphi)$, we therefore obtain $\models \langle ! \rangle_i \varphi \leftrightarrow \varphi$.

As for the completeness, suppose $\models \varphi$, then by Lemma 16, $\models t(\varphi)$. Since $t(\varphi) \in \mathcal{L}_{PL}$, by the completeness of PL, $\vdash_{\mathbf{PL}} t(\varphi)$. Since $\mathbf{PL} \subseteq \mathbf{LK}^=$, then $\vdash t(\varphi)$. Now using Lemma 17, we conclude that $\vdash \varphi$, as desired. □

Remark 1. With axiom **Red** in hand, we can even give a *syntactic* proof of CR and MK in $\mathbf{LK}^=$ (without use of completeness), because we can derive that $\vdash \langle ! \rangle_i \varphi \leftrightarrow \varphi$ and $\vdash [!]_i \varphi \leftrightarrow \varphi$. Therefore, both $\langle ! \rangle_i [!]_i \varphi$ and $[!]_i \langle ! \rangle_i \varphi$ are provably equivalent to φ . Therefore, $\vdash \langle ! \rangle_i [!]_i \varphi \leftrightarrow [!]_i \langle ! \rangle_i \varphi$.

4.7.2 Logic \mathbf{LK}^-

One may naturally ask whether the announcement operators increase the expressivity in $\mathbf{LK}^=$. Again, the answer is negative. Recall that when the announcement operators are added to $\mathbf{LK}^=$, we obtain the language \mathbf{LK}^- . In other words, \mathbf{LK}^- is defined recursively as follows.

$$\varphi ::= p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \langle \varphi \rangle \varphi \mid \langle ! \rangle_i \varphi$$

Proposition 45. \mathbf{LK}^- is equally expressive as PL.

Proof. As \mathbf{LK}^- extends PL, \mathbf{LK}^- is at least as expressive as PL. It suffices to show that PL is at least as expressive as \mathbf{LK}^- .

For this, let φ be a formula in the language of \mathbf{LK}^- . We show that φ is equivalent to a formula in PL. The proof is by induction on the number of $\langle \cdot \rangle$ modalities in φ .

If φ contains no $\langle \cdot \rangle$ modality, then φ is a formula in the language of $\mathbf{LK}^=$. As we shown in Prop. 44, φ is equivalent to a PL-formula. Otherwise, consider a subformula $\langle \chi \rangle \psi$ of φ such that $\psi, \chi \in \mathbf{LK}^=$. By Prop. 44 again, each of ψ and χ is equivalent to some PL-formula. Then by using the reduction axioms concerning announcements and Boolean formulas, we can infer that $\langle \chi \rangle \psi$ is equivalent to a PL-formula, namely $\chi \wedge \psi$. Now replace $\langle \chi \rangle \psi$ by $\chi \wedge \psi$ in φ . Let the result be φ' . Note that $\models \varphi \leftrightarrow \varphi'$. As φ' contains one less $\langle \cdot \rangle$ modality than φ , by induction hypothesis we conclude that φ' is equivalent to a formula φ'' in PL. From $\models \varphi \leftrightarrow \varphi'$ and $\models \varphi' \leftrightarrow \varphi''$, it follows that $\models \varphi \leftrightarrow \varphi''$. □

Also, we give a concrete example to illustrate the result.

Example 5. We will show that the formula $\langle ! \rangle_i \langle p \rangle \langle ! \rangle_j \langle \langle ! \rangle_i (q \wedge r) \rangle (p \rightarrow q)$, read “it is knowable for i that after a truthful announcement of p , it is knowable for j that after a truthful announcement of the fact that the conjunction of q and r is knowable for i , p implies q ”, is equivalent to a Boolean formula, as follows.

$$\begin{aligned}
 & \langle ! \rangle_i \langle p \rangle \langle ! \rangle_j \langle \langle ! \rangle_i (q \wedge r) \rangle (p \rightarrow q) \\
 \Leftrightarrow & \langle ! \rangle_i \langle p \rangle \langle ! \rangle_j \langle q \wedge r \rangle (p \rightarrow q) \\
 \Leftrightarrow & \langle ! \rangle_i \langle p \rangle \langle ! \rangle_j ((q \wedge r) \wedge (p \rightarrow q)) \\
 \Leftrightarrow & \langle ! \rangle_i \langle p \rangle ((q \wedge r) \wedge (p \rightarrow q)) \\
 \Leftrightarrow & \langle ! \rangle_i (p \wedge (q \wedge r) \wedge (p \rightarrow q)) \\
 \Leftrightarrow & p \wedge (q \wedge r) \wedge (p \rightarrow q) \\
 \Leftrightarrow & p \wedge q \wedge r
 \end{aligned}$$

Also, we can axiomatize \mathbf{LK}^- over the class of all frames. Define \mathbf{LK}^- as the smallest extension of $\mathbf{LK}^=$ plus the following axiom $\text{Red}^?$:

$$\langle \varphi \rangle \psi \leftrightarrow (\varphi \wedge \psi).$$

In what follows, we show the properties of Church-Rosser and McKinsey also hold for \mathbf{LK}^- . For this, we define a translation from $\mathcal{L}_{\mathbf{LK}^-}$ to $\mathcal{L}_{\mathbf{PL}}$.

Definition 38. Define $t' : \mathcal{L}_{\mathbf{LK}^-} \rightarrow \mathcal{L}_{\mathbf{PL}}$ as follows.

$$\begin{aligned}
 t'(p) &= p \\
 t'(\neg \varphi) &= \neg t'(\varphi) \\
 t'(\varphi \wedge \psi) &= t'(\varphi) \wedge t'(\psi) \\
 t'(\langle \varphi \rangle \psi) &= t'(\varphi) \wedge t'(\psi) \\
 t'(\langle ! \rangle_i \varphi) &= t'(\varphi).
 \end{aligned}$$

That is, t' extends t for the fragment $\mathbf{LK}^=$ in Def. 37 with the extra case $\langle \varphi \rangle \psi$.

Lemma 18. For all $\varphi \in \mathcal{L}_{\mathbf{LK}^-}$, we have

$$\vDash \varphi \leftrightarrow t'(\varphi).$$

Proof. By induction on $\varphi \in \mathcal{L}_{\mathbf{LK}^-}$. By Lemma 16, it suffices to show the case that $\varphi = \langle \psi \rangle \chi$.

By induction hypothesis, $\vDash \psi \leftrightarrow t'(\psi)$ and $\vDash \chi \leftrightarrow t'(\chi)$. Thus $\vDash \langle \psi \rangle \chi \leftrightarrow \langle t'(\psi) \rangle t'(\chi)$. Since $t'(\chi) \in \mathbf{PL}$, $\vDash \langle t'(\psi) \rangle t'(\chi) \leftrightarrow (t'(\psi) \wedge t'(\chi))$. As $t'(\langle \psi \rangle \chi) = t'(\psi) \wedge t'(\chi)$, we conclude that $\vDash \langle \psi \rangle \chi \leftrightarrow t'(\langle \psi \rangle \chi)$. \square

Then as in Thm. 12, we can show that the properties of Church-Rosser and McKinsey hold for \mathbf{LK}^- .

Theorem 14 (CR and MK). $\vDash \langle ! \rangle_i [!]_i \varphi \leftrightarrow [!]_i \langle ! \rangle_i \varphi$.

In what follows, we will also show that \mathbf{LK}^- is determined by the class of all frames. For this, we show

Lemma 19. For all $\varphi \in \mathbf{LK}^-$, we have

$$\vdash \varphi \leftrightarrow t'(\varphi).$$

Proof. By induction on $\varphi \in \mathbf{LK}^-$. The cases for $\varphi \in \mathbf{LK}^=$ formulas is similar as in Lemma 17. It remains only to prove the case that $\varphi = \langle \psi \rangle \chi$.

By induction hypothesis, $\vdash \psi \leftrightarrow t'(\psi)$ and $\vdash \chi \leftrightarrow t'(\chi)$. Thus $\vdash (\psi \wedge \chi) \leftrightarrow (t'(\psi) \wedge t'(\chi))$. By axiom **Red'** and definition of t' , we derive that $\vdash \langle \psi \rangle \chi \leftrightarrow t'(\langle \psi \rangle \chi)$. \square

Theorem 15. \mathbf{LK}^- is sound and complete with respect to the class of all frames.

Proof. For the soundness, by Thm. 13, it suffices to show the validity of axiom **Red'**. By Lemma 18, $\models \langle \varphi \rangle \psi \leftrightarrow t'(\langle \varphi \rangle \psi)$, $\models \varphi \leftrightarrow t'(\varphi)$, and $\models \psi \leftrightarrow t'(\psi)$. By definition of t' , $t'(\langle \varphi \rangle \psi) = t'(\varphi) \wedge t'(\psi)$. Therefore, $\models \langle \varphi \rangle \psi \leftrightarrow (\varphi \wedge \psi)$.

As for the completeness, suppose $\models \varphi$, then by Lemma 18, $\models t(\varphi)$. Since $t(\varphi) \in \mathcal{L}_{PL}$, by the completeness of PL, $\vdash_{\mathbf{PL}} t(\varphi)$. Since $\mathbf{PL} \subseteq \mathbf{LK}^-$, we have $\vdash t(\varphi)$. Now using Lemma 19, we conclude that $\vdash \varphi$, as desired. \square

Similar to Remark 1, we can also give a *syntactic* proof of CR and MK in \mathbf{LK}^- without use of completeness.

As both $\mathbf{LK}^=$ and \mathbf{LK}^- are equally expressive as PL, and PL is decidable, we have the following decidability result.

Theorem 16. $\mathbf{LK}^=$ and \mathbf{LK}^- are both decidable.

4.8 Conclusion and future work

We have proposed three knowability logics, namely LK, \mathbf{LK}^- and $\mathbf{LK}^=$. We compared the relative expressivity of the three logics and other related logics. It turns out that in the single-agent case, LK is equally expressive as arbitrary public announcement logic APAL and public announcement logic PAL, whereas in the multi-agent case, LK is more expressive than PAL. In contrast, both \mathbf{LK}^- and $\mathbf{LK}^=$ are equally expressive as classical propositional logic PL. We axiomatized the three knowability logics and showed their soundness and completeness. We showed that the properties of Church-Rosser (CR) and McKinsey (MK) holds for all three knowability logics, both syntactically and semantically. LK is undecidable for at least three agents; in contrast, \mathbf{LK}^- and $\mathbf{LK}^=$ are both decidable for any number of agents.

We currently see three topics for future research.

Firstly, one may investigate whether LK is already undecidable for only two agents.

Secondly, we would wish to determine whether LK is less expressive than APAL. We have a proof that $\mathbf{LK} \prec \mathbf{APAL}$ on the class of reflexive models, but we have not yet managed to modify this proof to work with $S5$ models. The issue with $S5$ models is that they provide far less freedom to make certain states distinguishable while others are indistinguishable. For example, if s_1 and s_2 in an $S5$ model are distinguishable and t_1 and t_2 are a -successors of s_1 and s_2 , respectively, and only of those states, then t_1 and t_2 cannot be indistinguishable. As a consequence, potential $S5$ counterexamples to LK being as expressive as APAL need to be for more complex than the counterexamples for reflexive models, and are therefore harder to find. We do still conjecture that such counterexamples exist, and therefore that $\mathbf{LK} \prec \mathbf{APAL}$ on $S5$ models, but so far we have not managed to find them.

Finally, an remaining important open question is what the axiomatization is of the logic with the language of LK but without public announcements, so that the semantics of the quantifier is given directly (and equivalently). A similar open question remains for the logic APAL but without the public announcement in the language (see also [99]) where this is

discussed at some length). In such cases, we can no longer resort to the public announcement in the axiom and in the derivation rule for the quantifier, and it is very unclear how to proceed alternatively.

5

Normative arrow update logic

5.1 Introduction

Deontic logic is the study of rules, norms, obligations and permissions, through logical means [80, 8, 28, 42, 55], and this has also been extensively investigated in dynamic modal logics [69, 20, 96, 54, 49]. In the field of deontic logic, there is a sub-field that studies rules or norms by comparing the situation where a rule is not in effect, or not being followed, to the situation where the rule/norm is obeyed. There is no universally accepted name for this sub-field, but “social laws” [86, 87, 40] and “normative systems” [2, 4] are often used. We will use the term *normative systems*, and refer to the behavioural restrictions under consideration as *norms*.

A logic of normative systems is concerned with what things agents are capable of doing, and what they are allowed to do if a norm is enacted. It therefore requires a model of agency at its core. Any model of agency will do, but the most commonly used choices are labeled transition systems with a CTL-like logic of agency [31] and outcome function transition systems with ATL-like logic [7]. Here, we will follow the CTL-style approach. This means that a model is a labeled transition system, i.e., it contains a set S of states and a set $\{R(a) \mid a \in \mathcal{A}\}$ of accessibility relations, where $R(a) \subseteq S \times S$. A transition $(s_1, s_2) \in R(a)$, is an action or an agent that changes the state of the world from s_1 to s_2 .

In order to choose a course of action, we need to decide whether we should adopt a norm and then check if an action is allowed by the norm. Whether an action a is allowed may depend on a logical condition φ before the action takes place, so on the situation in s_1 , and also may depend on a logical condition ψ after the action took place, so on a condition satisfied in s_2 . We refer to s_1 as the *source* of the action, to φ as a *source condition*, to s_2 as the *target*, and to ψ as a *target condition*. A source condition φ determines whether we should adopt a norm and a target condition ψ checks which actions are allowed. For norms with both source and target conditions one cannot reduce multiple source conditions to one (for example by taking the disjunction), nor multiple target condition to one. A norm in our formalism will be therefore represented by a list of clauses, each with a source condition and a target condition. This is as in *arrow update logic* [57, 105]. The arrow eliminating updates in arrow update logic now correspond to adherence to norms.

We will also introduce more complex ways to describe norms, so we will refer to such a list of clauses as an *atomic norm*. We distinguish four ways to combine norms. If N_1 and N_2 are norms, then

- $\neg N_1$ is the negation of N_1 , and allows exactly those actions that are disallowed by N_1 ,

- $N_1 + N_2$ is the additive combination of N_1 and N_2 , and allows exactly those actions that are allowed by N_1 or N_2 ,
- $N_1 \times N_2$ is the multiplicative combination of N_1 and N_2 , and allows exactly those actions that are allowed by both N_1 and N_2 .
- $N_1 \circ N_2$ is the sequential composition of N_1 and N_2 , and allows exactly those actions that are allowed by N_2 in the transition system restricted to those actions that are allowed by N_1 .

We further distinguish *static* from *dynamic* applications of norms. A liveness condition such as “if the norm N is obeyed, then φ is guaranteed to be true at every time in the future” can be formalized in two ways, which we denote $[N]G\varphi$ (dynamic) and $G_N\varphi$ (static). The difference lies in whether the norm N is assumed to hold during the evaluation of φ : when evaluating $[N]G\varphi$, everything inside the scope of $[N]$ is considered in the transition system restricted to the actions allowed by N . When evaluating $G_N\varphi$, on the other hand, the “forever in the future” operator G is evaluated in the system restricted to N -allowed actions, but φ is evaluated in the non-restricted system.

The dynamic operator $[N]$ can be expressed using only the static operators, and the combined norms can be expressed using only atomic norms. The combined and dynamic norms affect the succinctness of the language, and thus the complexity of decision problems, they do not affect its expressivity. The logic will be called NAUL, Normative Arrow Update Logic. We will now formally define its syntax and semantics and then investigate the complexity of satisfiability with a tableau method.

5.2 Language and Semantics

Let \mathcal{A} be a finite set of agents and \mathcal{P} a countably infinite set of propositional variables.

Recall that we have defined relational models, transitions and paths in Chapter 2. Now we introduce the language of NAUL.

Definition 39. The formulas of \mathcal{L}_{NAUL} are given by

$$\begin{aligned} \varphi &::= p \mid \neg\varphi \mid \varphi \vee \varphi \mid [N]\varphi \mid \Box_N\varphi \mid G_N\varphi \mid F_N\varphi \\ \mathcal{N} &::= (\varphi, B, \varphi) \mid \overline{(\varphi, B, \varphi)} \mid \mathcal{N}, (\varphi, B, \varphi) \mid \mathcal{N}, \overline{(\varphi, B, \varphi)} \\ N &::= \mathcal{N} \mid -N \mid N + N \mid N \times N \mid N \circ N \end{aligned}$$

where $p, \in \mathcal{P}$ and $B \subseteq \mathcal{A}$.

Remark 2. In NAUL we use only three temporal operators: \Box_N , G_N and F_N . These temporal operators include an implicit universal quantification over all paths, so we could have denoted them in a more CTL-like fashion as AX_N , AG_N and AF_N . Operators corresponding to the other temporal operators from CTL can be defined in NAUL. For example, $E(\varphi_1 U_N \varphi_2)$ can be defined as $\neg G_{(\varphi_1, \mathcal{A}, \top) \times N} \neg \varphi_2$.

In NAUL, the set of subformulas (*SubF*) or subnorms (*SubN*) of a formula φ (or a norm N) includes all formulas or norms *occur* in φ (or N).

Strictly speaking a norm of type \mathcal{N} is a list of clauses, but we abuse notation by identifying it with the set of its clauses. Additionally, we use a number of abbreviations. We refer to

norms of type \mathcal{N} as *atomic norms* and norms of type N simply as *norms*. Note that every atomic norm is also a norm.

Definition 40. We use $\wedge, \rightarrow, \leftrightarrow, \bigwedge, \bigvee$ and \diamond_N in the usual way as abbreviations. Furthermore, we use \hat{G}_N and \hat{F}_N as abbreviations for $\neg G_N \neg$ and $\neg F_N \neg$. We write \square_B for $\square_{(\top, B, \top)}$, G_B for $G_{(\top, B, \top)}$ and F_B for $F_{(\top, B, \top)}$. Finally, we use \square, G and F for $\square_{\mathcal{A}}, G_{\mathcal{A}}$ and $F_{\mathcal{A}}$.

The semantics of \mathcal{L}_{NAUL} are given by the following two interdependent definitions.

Definition 41. Let $\mathcal{M} = (S, R, v)$ be a relational model and N a norm. A transition $s_1 \xrightarrow{a} s_2$ satisfies N in \mathcal{M} if one of the following is holds:

1. N is an atomic norm, there is a positive clause $(\varphi, B, \psi) \in N$ such that $\mathcal{M}, s_1 \models \varphi$, $a \in B$ and $\mathcal{M}, s_2 \models \psi$. Furthermore, there is no negative clause $(\overline{\varphi}, B, \overline{\psi}) \in N$ such that $\mathcal{M}, s_1 \models \varphi$, $a \in B$ and $\mathcal{M}, s_2 \models \psi$,
2. N is of the form $\neg N_1$ and $s_1 \xrightarrow{a} s_2$ does not satisfy N_1 ,
3. N is of the form $N_1 + N_2$ and $s_1 \xrightarrow{a} s_2$ satisfies N_1 or N_2 in \mathcal{M} ,
4. N is of the form $N_1 \times N_2$ and $s_1 \xrightarrow{a} s_2$ satisfies N_1 and N_2 in \mathcal{M} ,
5. N is of the form $N_1 \circ N_2$, $s_1 \xrightarrow{a} s_2$ satisfies N_1 in \mathcal{M} and the transition $s_1 \xrightarrow{a} s_2$ satisfies N_2 in $\mathcal{M} * N_1$.

A path $s_1 \xrightarrow{a_1} s_2 \xrightarrow{a_2} s_3 \cdots$ is an N -path in \mathcal{M} if every transition $s_i \xrightarrow{a_i} s_{i+1}$ in the path satisfies N in \mathcal{M} . An N -path is *full* in \mathcal{M} if there is no N -path in \mathcal{M} that extends it.

When the model \mathcal{M} is clear from context, we say simply that a transition satisfies N or that a path is an N -path.

Definition 42. Let $\mathcal{M} = (S, R, v)$ be a transition system and $s \in S$. The relation \models is given as follows.

$$\begin{aligned}
 \mathcal{M}, s \models p & \Leftrightarrow s \in v(p) \text{ for } p \in \mathcal{P}, \\
 \mathcal{M}, s \models \neg \varphi & \Leftrightarrow \mathcal{M}, s \not\models \varphi, \\
 \mathcal{M}, s \models \varphi_1 \vee \varphi_2 & \Leftrightarrow \mathcal{M}, s \models \varphi_1 \text{ or } \mathcal{M}, s \models \varphi_2, \\
 \mathcal{M}, s \models \square_N \varphi & \Leftrightarrow \text{for every transition } s \xrightarrow{a} s' \text{ that} \\
 & \text{satisfies } N, \text{ we have } \mathcal{M}, s' \models \varphi, \\
 \mathcal{M}, s \models G_N \varphi & \Leftrightarrow \text{for every } N\text{-path } P \text{ starting in } s \text{ and} \\
 & \text{every } s' \in P \text{ we have } \mathcal{M}, s' \models \varphi, \\
 \mathcal{M}, s \models F_N \varphi & \Leftrightarrow \text{for every full } N\text{-path } P \text{ starting in } s \text{ there is} \\
 & \text{some } s' \in P \text{ such that } \mathcal{M}, s' \models \varphi, \\
 \mathcal{M}, s \models [N] \varphi & \Leftrightarrow \mathcal{M} * N, s \models \varphi
 \end{aligned}$$

where $\mathcal{M} * N = (S, R * N, v)$ and, for every $a \in \mathcal{A}$,

$$R * N(a) = \{(s, s') \in R(a) \mid s \xrightarrow{a} s' \text{ satisfies } N\}.$$

Recall that the single state s is a degenerate path with no transitions. So every transition in s satisfies every norm N , so it is an N -path. As a result, $\mathcal{M}, s \models G_N \varphi$ implies $\mathcal{M}, s \models \varphi$.

5.3 Example: Self-driving Cars

We will give a simple example of NAUL. Suppose we have a racetrack where a number of self-driving cars operate. We want to equip cars with norms that will guarantee that they avoid

- (a) collisions with each other and stationary objects;
- (b) “deadlock” situations where no one can act.

Let $coll$ be the proposition variable that represents “a collision happens”. Note that situations where no one can act are represented by $\Box\perp$.

For (a), we create a norm N_c such that if no collision has occurred then it should prevent collisions for every point in the future. N_c is therefore successful if we have $\neg coll \rightarrow [N_c]G\neg coll$. The simplest way is to disallow any action, then N_c is $(\perp, \mathcal{A}, \perp)$. However, we would like to let N_c allow at least one action to avoid deadlock. Thus we take $N_c := (\top, \mathcal{A}, \neg F coll)$. It is indeed successful as we have $\models \neg coll \rightarrow [N_c]G\neg coll$.

For (b), we interpret it as “there must be some available action that is not only possible but also allowed”, and then we construct a N_d such that $[N_d]G\Diamond\top$ holds. We should take $N_d := (\top, \mathcal{A}, \neg F\Box\perp)$. This gives us $\models \neg F\Box\perp \rightarrow [N_d]G\Diamond\top$. In other words, as long as there is an infinite path the norm N_d forces agents to follow such a path.

For combining N_c and N_d , $N_c \times N_d$ allows agents to perform actions that result in a situation where movement, while possible, is disallowed because it will lead to a collision. The sequential combination solves this problem: the norm $N_c \circ N_d$ allows exactly those actions that lead to neither collisions nor situations where agents cannot or are not allowed to act. In other words, we have $\models \neg F(coll \vee \Box\perp) \rightarrow [N_c \circ N_d]G(\neg coll \wedge \Diamond\top)$.

The self-driving cars example is also useful for illustrating the difference between the static operators \Box_N , G_N , and F_N on the one hand, and the dynamic operator $[N]$ on the other. We have $\mathcal{M}, s \models G_N\varphi$ if φ holds after every sequence of action that starts in s and is allowed by N . Importantly, during the evaluation of φ it is not assumed that everyone follows N . We have $\mathcal{M}, s \models [N]G\varphi$ if, under the assumption that all agents follow N permanently from now on, every sequence of actions leads to a φ state. In this case, during the evaluation of φ , we do assume that all agents follow N .

Sometimes we may require that N_c not only avoids collisions, but also situations where a single mistake could cause a collision. We cannot phrase this stronger success condition as $[N_c]\varphi$ for any φ . After all, the φ in $[N_c]\varphi$ is evaluated under the assumption that all agents follow the norm N_c —so no mistakes are made. This is where the static operator G_{N_c} is useful. Consider the formula $G_{N_c}(\neg coll \wedge \Box\neg coll)$. The \Box in that formula is not evaluated under the assumption that the agents follow N_c , so $G_{N_c}(\neg coll \wedge \Box\neg coll)$ holds exactly if every sequence of actions allowed by N_c leads to a state where there is no collision and no single action can cause a collision.

5.4 Expressivity

We will compare the expressivity of NAUL with other related logics. Firstly, NAUL and NTL are obviously incomparable. The different ways to define norms make it impossible to compare NTL and NAUL with each other. Therefore, we will focus on comparing NAUL with CTL and AUL*. It has been shown NAUL is strictly more expressive than CTL and AUL* [63].

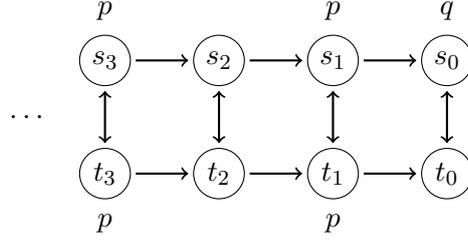


Figure 5.1: The model \mathcal{M}_{CTL} . The states s_i and t_i satisfy p iff i is odd.

Remark 3. CTL is usually interpreted over different models than AUL* and NAUL. In particular, CTL tends to use single-agent serial models. Strictly speaking, this makes it impossible to compare the expressivity of NAUL to that of CTL. This problem can be solved by either extending CTL to multi-agent non-serial models—which can be done in a straightforward way—or by restricting AUL* and NAUL to single-agent serial models. The results presented here hold regardless of which of these solutions we use.

5.4.1 CTL \prec NAUL

Recall that we have introduced the basic definition of CTL in Sec. 2.2.5. First, we show that NAUL is at least as expressive as CTL. The operators \neg, \vee, AX and AF are also NAUL operators, although AX and AF are denoted \Box and F in NAUL. As such, it suffices to show that EU can be defined in NAUL.

Lemma 20. *NAUL is at least as expressive as CTL.*

Proof. The operators \neg, \vee, AX and AF in CTL are also NAUL operators, although AX and AF are denoted \Box and F in NAUL. As such, it suffices to show that EU can be defined in NAUL. We prove the following:

$$\models E(\varphi U \psi) \leftrightarrow \neg G_{(\varphi, \mathcal{A}, \top)} \neg \psi$$

By semantics of NAUL $\mathcal{M}, s \models \neg G_{(\varphi, \mathcal{A}, \top)} \neg \psi$ if and only if there is a $(\varphi, \mathcal{A}, \top)$ path from s that contains a ψ state. Because such a path is a $(\varphi, \mathcal{A}, \top)$ path, φ is true on every state before the ψ state. As such, by semantics of CTL, $\mathcal{M}, s \models E(\varphi U \psi) \leftrightarrow \neg G_{(\varphi, \mathcal{A}, \top)} \neg \psi$. This is true for any \mathcal{M}, s , so $\models E(\varphi U \psi) \leftrightarrow \neg G_{(\varphi, \mathcal{A}, \top)} \neg \psi$. \square

Left to show is that CTL is not at least as expressive as NAUL. Consider the model \mathcal{M}_{CTL} shown in Figure 5.1, and note that the NAUL formula $[(p, \mathcal{A}, \neg p), (\neg p, \mathcal{A}, p)]G\neg q$ distinguishes between \mathcal{M}_{CTL}, s_i and \mathcal{M}_{CTL}, t_i for all $i \in \mathbb{N}$. We show that there is no CTL formula that similarly distinguishes s_i from t_i .

Lemma 21. *Let φ be any CTL formula, and let n be the modal depth of φ . Then φ does not distinguish between \mathcal{M}_{CTL}, s_i and \mathcal{M}_{CTL}, t_i for $i > n$.*

Proof. By induction. As base case, suppose $n = 0$. Then φ is a Boolean formula, so it cannot distinguish between s_i and t_i for $i \neq 0$. Assume as induction hypothesis that the lemma holds for all $n' < n$.

If a Boolean combination distinguishes between two states then so does at least one of the combined formulas, so we can assume without loss of generality that the main connective of φ is AX, AF or EU .

- Suppose $\varphi = AX\psi$. In order for φ to distinguish between s_i and t_i , ψ must distinguish between t_i and s_i or s_{i-1} and t_{i-1} . This contradicts the induction hypothesis, since ψ is of modal depth $n - 1$.
- Suppose $\varphi = AF\psi$. There are two possibilities. Firstly, ψ may hold on either s_i or t_i . Then, by the induction hypothesis it holds on both s_i and t_i . As such, φ holds on both s_i and t_i , and therefore does not distinguish between them.

The second possibility is that ψ holds on neither s_i nor t_i . Suppose φ does not hold on s_i , so there is some path $s_i \mapsto x_1 \mapsto x_2 \mapsto \dots$ that does not contain a ψ state. Then the path $t_i \mapsto s_i \mapsto x_1 \mapsto x_2 \mapsto \dots$ also does not contain a ψ state. So φ does not hold on t_i . Analogously, if φ does not hold on t_i then it does not hold on s_i . This shows that φ does not distinguish between s_i and t_i .

- Suppose $\varphi = E(\psi_1 U \psi_2)$. There are three possibilities. The first possibility is that ψ_2 holds on either of s_i and t_i and therefore—by the induction hypothesis—on both. Then φ holds on both states, and therefore does not distinguish between them.

The second possibility is that both ψ_1 and ψ_2 hold on neither state. Then φ holds on neither state, and therefore does not distinguish between them.

The final possibility is that ψ_1 holds on neither state, but ψ_2 holds on either and therefore—by the induction hypothesis—both states. Suppose φ holds on s_i . Then there is some path $s_i \mapsto x_1 \mapsto x_2 \mapsto \dots$ that satisfies ψ_1 until ψ_2 . This implies that the path $t_i \mapsto s_i \mapsto x_1 \mapsto x_2 \mapsto \dots$ also satisfies ψ_1 until ψ_2 , so φ holds on t_i as well. Analogously, this reasoning shows that if φ holds on t_i then it also holds on s_i . This shows that φ does not distinguish between s_i and t_i .

In all cases, φ doesn't distinguish s_i from t_i . This completes the induction step and thereby the proof. \square

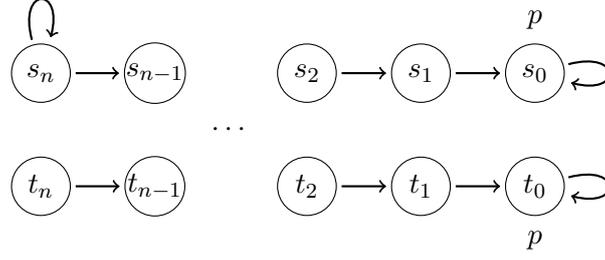
Theorem 17. *NAUL is strictly more expressive than CTL.*

Proof. Lemma 20 shows that NAUL is at least as expressive as CTL. Lemma 21 shows that there is no CTL formula equivalent to $[(p, \mathcal{A}, \neg p), (\neg p, \mathcal{A}, p)]G\neg q$, so CTL is not at least as expressive as NAUL. \square

5.4.2 AUL* \prec NAUL

The only difference between NAUL and AUL* is that NAUL has an F_N operator while AUL* does not. NAUL is therefore trivially at least as expressive as AUL*. Left to show is that AUL* is not at least as expressive as NAUL. In order to do so, we will use a sequence of models \mathcal{M}_{AUL}^n , which is shown in Figure 5.2. For reasons of brevity we will assume that AUL* does not contain the $[N]$ operator; we can safely do this because $[N]$ can be seen as an abbreviation in both NAUL and AUL*.

Lemma 22. *Let φ be any AUL* formula, and let m be the modal depth of φ . Then, for every $n > m$ and every $n \geq i, j > m$, φ does not distinguish between \mathcal{M}_{AUL}^n, s_i and \mathcal{M}_{AUL}^n, t_j . Furthermore, for every $n > i \geq 0$, φ does not distinguish between \mathcal{M}_{AUL}^n, s_i and \mathcal{M}_{AUL}^n, t_i .*

Figure 5.2: The model \mathcal{M}_{AUL}^n .

Proof. The second claim in the lemma is trivial: for every $i < n$, the states s_i and t_i are bisimilar and these logics respect bisimilarity. It remains to show that φ cannot distinguish between s_i and t_j for $i, j > m$. We do this by induction. As base case, suppose $m = 0$. For every $i, j > 0$, the states s_i and t_j agree on all propositional variables, so φ does not distinguish between them.

Suppose then as induction hypothesis that $m > 0$ and that the lemma holds for all $m' < m$. If a Boolean combination of formulas distinguishes between two states then so does at least one of the combined formulas, so we can assume without loss of generality that the main connective of φ is \Box_N or G_N .

- Suppose $\varphi = \Box_N \psi$. In order for φ to distinguish between s_i and t_j it is necessary for either ψ or one of the formulas in N to distinguish between s_i and t_j , or between s_{i-1} and t_{j-1} . Each of the formulas in N as well as ψ are of modal depth $\leq m - 1$, so by the induction hypothesis they cannot distinguish between these states. This implies that φ does not distinguish between s_i and t_j .
- Suppose $\varphi = G_N \psi$. In order to distinguish between s_i and t_j , exactly one of the states must have a path containing a $\neg\psi$ state. There are two ways this could happen: either there is some k such that exactly one of s_k and t_k satisfies ψ , or there is a k such that s_k and t_k both satisfy $\neg\psi$, but only one of them is reachable from s_i or t_j by an N -path.

The first option cannot occur; the induction hypothesis implies that ψ cannot distinguish between s_k and t_k for any k . The second option also cannot occur. Such a reachability difference would require some formula in N to distinguish between s_k and t_k with $k < n$ or between s_k and t_l with $k, l > m - 1$. The induction hypothesis implies that neither distinction is possible.

In both cases, φ doesn't distinguish s_i from t_j . This completes the induction step and thereby the proof. \square

Theorem 18. *NAUL is strictly more expressive than AUL*.*

Proof. NAUL is trivially at least as expressive as AUL*. From Lemma 22 it follows that there is no AUL* formula equivalent to the NAUL formula Fp . \square

5.5 Satisfiability Problem

Definition 43. The satisfiability problem for NAUL is defined as follows:

- **Input** an NNF formula φ .
- **Output** YES if and only if there is a pointed model (\mathcal{M}, s) such that $\mathcal{M}, s \models \varphi$

In this section, we present a tableau method to show the satisfiability problem of NAUL is decidable. We will use **negation normal form** (NNF) of formulas or norms. An NNF formula only has negation on literals. An NNF norm only has negations on atomic norms instead of clauses.

Definition 44 (Negation normal form (NNF)). Given a set of variables \mathbf{P} and a finite set of agents \mathcal{A} .

$$\varphi ::= p \mid \neg p \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \Box_N \varphi \mid \Diamond_N \varphi \mid G_N \varphi \mid \hat{G}_N \varphi \mid F_N \varphi \mid \hat{F}_N \varphi \mid [N]\varphi \mid \langle N \rangle \varphi$$

$$\mathcal{N} ::= (\varphi, a, \varphi) \mid \mathcal{N}, (\varphi, a, \varphi)$$

$$N ::= \mathcal{N} \mid \overline{\mathcal{N}} \mid N + N \mid N \times N \mid N \circ N$$

where $p \in \mathbf{P}$, $a \in \mathcal{A}$.

Theorem 19. *Every NAUL-formula or norm can be transformed to an equivalent formula or norm in NNF.*

Proof. For NAUL-formulas, it can be shown easily by an induction. As for atomic norms, since the order of clauses in an atomic norm does not matter, given an atomic NAUL-norm \mathcal{N} , and \mathcal{N}^+ as all positive clauses, \mathcal{N}^- as all negative clauses of \mathcal{N} , clearly \mathcal{N} is equivalent to $\mathcal{N}^+ + \mathcal{N}^-$ which is an NNF norm. As for the negations of combined norms, we have the following transformations:

- $\overline{\overline{N}} = N$
- $\overline{N_1 + N_2} = \overline{N_1} \times \overline{N_2}$
- $\overline{N_1 \times N_2} = \overline{N_1} + \overline{N_2}$
- $\overline{N_1 \circ N_2} = \overline{N_1} + N_1 \circ \overline{N_2}$

□

Given an NAUL-formula φ or NAUL-norm N , the time of transforming it into an NNF formula φ' or NNF norm N' and the size of φ' or N' is polynomial in the size of φ or N .

5.5.1 Tableau method

We introduce some concepts related to tableau method.

Definition 45 (Term). There are two types of terms:

F-term $\langle s; \lambda; \varphi \rangle$ where $s \in S$, λ is a sequence of norms, φ is a formula. It means the model has been updated by λ and φ is true on s .

N-term $\langle s_1 \xrightarrow{a} s_2; \lambda; \eta \rangle$ where $s_1, s_2 \in S$, λ, η are sequences of norms. It means the transition $s_1 \xrightarrow{a} s_2$ satisfies η successively after the model is updated by λ .

Definition 46 (Tableau). A tableau T is a structure $T = (W, V, E, \pi)$ where W is an infinite set, and V is a finite set, E is a binary relation on V . Given a set of terms L , $\pi : V \rightarrow \mathbf{P}(L)$ is a labelling map.

Let A, C_1, \dots, C_n be sets of terms. A *tableau rule* is represented as

$$\frac{A}{C_1 \mid \dots \mid C_n}$$

Above the line, A is the *antecedent*; below the line, there are *consequents*. A tableau rule is applicable on a node if the node has terms as an instance of the antecedent. If there are multiple consequents after applying a rule, one need to choose one of them.

Definition 47 (Interpretability). Given a model $\mathcal{M} = (S, R, v)$, it *interpret* (noted as \models_T) a set of terms T if any term in T satisfies:

- $\mathcal{M} \models_T \langle s; \lambda; \varphi \rangle$ if and only if $\mathcal{M} * \lambda, s \models \varphi$.
- $\mathcal{M} \models_T \langle s_1 \xrightarrow{a} s_2; \lambda; \eta \rangle$ if and only if $s_1 \xrightarrow{a} s_2$ satisfies η on $\mathcal{M} * \lambda$.

A set of terms T is *interpretable* if there exists a model \mathcal{M} such that \mathcal{M} interprets all terms in T .

Definition 48. Given a tableau T , we define an order \prec on all terms of T as

- $\langle s; \lambda; \varphi \rangle \prec \langle s; \lambda'; \psi \rangle$ if φ is a subformula of ψ .
- $\langle s \xrightarrow{a} s'; \lambda; \eta \rangle \prec r \langle s; \lambda'; \varphi \rangle$ if η is a parameter of some operator in φ .
- $\langle s; \lambda; \varphi \rangle \prec \langle s \xrightarrow{a} s'; \lambda'; \eta \rangle$ if φ is in some clause of η .
- $\langle s \xrightarrow{a} s'; \lambda; \eta \rangle \prec \langle s \xrightarrow{a} s'; \lambda; \eta' \rangle$ if η is a sub-norm of η' ;
- $\langle s \xrightarrow{a} s'; \lambda; N' \rangle \prec \langle s \xrightarrow{a} s'; \lambda'; N \rangle$ if λ is an initial segment of λ' .

Now we define the tableau rules for NAUL. We omit terms which remain the same after applying a certain rule. Let ϵ be the norm $(\top, \mathcal{A}, \top)$ after which nothing is updated.

Definition 49 (Tableau rules). ϵ is the norm $(\top, \mathcal{A}, \top)$ after which nothing is updated.

$$\begin{aligned}
 (lit) \quad & \frac{\langle s; \lambda; p \rangle}{\langle s; \epsilon; p \rangle} \quad \frac{\langle s; \lambda; \neg p \rangle}{\langle s; \epsilon; \neg p \rangle} \\
 (\wedge) \quad & \frac{\langle s; \lambda; \varphi \wedge \psi \rangle}{\langle s; \lambda; \varphi \rangle, \langle s; \lambda; \psi \rangle} \\
 (\vee) \quad & \frac{\langle s; \lambda; \varphi \vee \psi \rangle}{\langle s; \lambda; \varphi \rangle \mid \langle s; \lambda; \psi \rangle} \\
 (G) \quad & \frac{\langle s; \lambda; G_N \varphi \rangle}{\langle s; \lambda; \varphi \rangle, \langle s; \lambda; \Box_N G_N \varphi \rangle} \\
 (F) \quad & \frac{\langle s; \lambda; F_N \varphi \rangle}{\langle s; \lambda; \varphi \rangle \mid \langle s; \lambda; \Diamond_N \top \rangle, \langle s; \lambda; \Box_N F_N \varphi \rangle}
 \end{aligned}$$

$$\begin{aligned}
 (\hat{G}) & \frac{\langle s; \lambda; \hat{G}_N \varphi \rangle}{\langle s; \lambda; \varphi \rangle \mid \langle s; \lambda; \diamond_N \hat{G}_N \varphi \rangle} \\
 (\hat{F}) & \frac{\langle s; \lambda; \hat{F}_N \varphi \rangle}{\langle s; \lambda; \varphi \rangle, \langle s; \lambda; \square_N \perp \rangle \mid \langle s; \lambda; \varphi \rangle, \langle s; \lambda; \diamond_N \hat{F}_N \varphi \rangle} \\
 (\text{Dynamic}) & \frac{\langle s; \lambda; [N] \varphi \rangle}{\langle s; \lambda, N; \varphi \rangle} \quad \frac{\langle s; \lambda; \langle N \rangle \varphi \rangle}{\langle s; \lambda, N; \varphi \rangle} \\
 (\diamond) & \frac{\langle s; \lambda; \diamond_N \varphi \rangle}{\langle s'; \lambda; \varphi \rangle, \langle s \xrightarrow{a_1} s'; \lambda; N \rangle \mid \cdots \mid \langle s'; \lambda; \varphi \rangle, \langle s \xrightarrow{a_n} s'; \lambda; N \rangle} \\
 (\square) & \frac{\langle s; \lambda; \square_N \varphi \rangle, \langle s \xrightarrow{a} s'; \epsilon; \lambda \rangle}{\langle s'; \lambda; \varphi \rangle, \langle s \xrightarrow{a} s'; \lambda; N \rangle \mid \langle s \xrightarrow{a} s'; \lambda; -N' \rangle} \quad (-N' \text{ is the NNF of } -N)
 \end{aligned}$$

Suppose an atomic norm $\mathcal{N} = (\varphi_1, a_1, \psi_1), \dots, (\varphi_n, a_n, \psi_n)$, for any $i \in [1, n]$

$$\begin{aligned}
 (\text{Atomic}) & \frac{\langle s \xrightarrow{a_i} s'; \lambda; \mathcal{N} \rangle}{\langle s; \lambda; \varphi_i \rangle, \langle s'; \lambda; \psi_i \rangle} \\
 (\text{Neg}) & \frac{\langle s \xrightarrow{a} s'; \lambda; \overline{\mathcal{N}} \rangle}{\langle s; \lambda; \bigwedge_{j \in K_1} \neg \varphi_j \rangle, \langle s'; \lambda; \bigwedge_{j \in K_2} \neg \psi_j \rangle \mid \cdots} \quad (\neg \varphi_j, \neg \psi_j \text{ are transformed to NNF formulas}) \\
 & \text{For any disjoint } K_1, K_2 \subseteq [1, n] \text{ such that } K_1 \cup K_2 = \{i \mid (\varphi_i, a, \psi_i) \in \mathcal{N}\} \\
 (\text{Add}) & \frac{\langle s \xrightarrow{a} s'; \lambda; N_1 + N_2 \rangle}{\langle s \xrightarrow{a} s'; \lambda; N_1 \rangle \mid \langle s \xrightarrow{a} s'; \lambda; N_2 \rangle} \\
 (\text{Multi}) & \frac{\langle s \xrightarrow{a} s'; \lambda; N_1 \times N_2 \rangle}{\langle s \xrightarrow{a} s'; \lambda; N_1 \rangle, \langle s \xrightarrow{a} s'; \lambda; N_2 \rangle} \\
 (\text{Seq}) & \frac{\langle s \xrightarrow{a} s'; \lambda; N_1 \circ N_2 \rangle}{\langle s \xrightarrow{a} s'; \lambda, N_1; N_2 \rangle} \\
 (\text{DN}) & \frac{\langle s \xrightarrow{a} s'; \lambda, N_1; N_2 \rangle}{\langle s \xrightarrow{a} s'; \lambda; N_1 \rangle}
 \end{aligned}$$

(lit), (\wedge) and (\vee) are Boolean rules. (G), (F), (\hat{G}), (\hat{F}) handle temporal modalities. (G) says if we have $G_N \varphi$ at a word s , then we have φ as well as $\square_N G_N \varphi$ at s . (F) says if we have $F_N \varphi$ at s , then whether we have φ , or s has some N -successor ($\diamond_N \top$ is true) and $\square_N F_N \varphi$. (\hat{G}) says if we have $\hat{G}_N \varphi$ at s , then whether we have φ or $\diamond_N \hat{G}_N \varphi$ at s . (\hat{F}) says if we have $\hat{F}_N \varphi$ at s , then we have φ at s and whether s has no N -successor or it has $\diamond_N \hat{F}_N \varphi$.

(\diamond) says if we have $\diamond_N \varphi$ at s , then we can choose an agent $a \in \mathcal{A}$ to “assume” that there is a transition $s \xrightarrow{a} s'$ satisfying N and we have φ at s' . Note that (\diamond) is the only rule that generates new states and whether a state can be actually generated will be examined later. (\square) says if we have $\square_N \varphi$ at s and transition $s \xrightarrow{a} s'$ exists, then whether we have φ at s' or $s \xrightarrow{a} s'$ does not satisfy N . (Dynamic) handle dynamic operators. It says if we have $[N] \varphi$ (or $\langle N \rangle \varphi$) at s updated by λ , then we have φ at s updated by λ then by N .

The other rules handle norms. (Atomic) says if we have atomic norm \mathcal{N} for $s \xrightarrow{a_i} s'$ where a_i occurs in some clause $(\varphi_i, a_i, \psi_i) \in \mathcal{N}$, then we have φ_i at s and ψ_i at s' . (Neg) says if we have $\bar{\mathcal{N}}$ for $s \xrightarrow{a} s'$, then given $\{i \mid (\varphi_i, a, \psi_i) \in \mathcal{N}\}$ we choose some $K_1, K_2 \subseteq [1, n]$ such that $K_1 \cup K_2 = \{i \mid (\varphi_i, a, \psi_i) \in \mathcal{N}\}$ and $\bigwedge_{j \in K_1} \neg\varphi_j$ is at s and $\bigwedge_{j \in K_2} \neg\psi_j$ is at s' . As a result, none of clause in \mathcal{N} will be satisfied by $s \xrightarrow{a} s'$. (Add), (Multi) and (Seq) are standard with respect to Def 41. (DN) says if $s \xrightarrow{a} s'$ satisfies some norm N_2 after updating by λ , N_1 , then it satisfies N_1 after updating by λ . A special case of (DN) is

$$(DN^*) \frac{\langle s \xrightarrow{a} s'; \lambda; N \rangle}{\langle s \xrightarrow{a} s'; \epsilon; \lambda \rangle}$$

(DN*) says if transition $s \xrightarrow{a} s'$ is updated by λ , then it satisfies λ .

Besides above tableau rules, we also need principles to delete inconsistent states, set an order of applying rules, and avoid infinite consequents

Definition 50 (Tableau principles). Given an NNF formula φ , we start from the root with label $\langle s_0; \epsilon; \varphi \rangle$. We have the following the principles of generating a tableau of φ :

- (Inc) If a node has inconsistent literals (as well as \perp), then mark it as “deleted”. If all consequents are marked deleted, then mark the antecedent as deleted. In particular, if one node have no consequent then mark it as deleted directly.
- (Exh) We should apply rules to terms with respect to one state until no rule is applicable on that state. When no rule is applicable on a state s , we mark s as “exhausted”. After that, we can apply rules to terms with respect to its successors.
- (Cyc) When a state s are marked as “exhausted”, one needs to check if there some exhausted ancestor s^* of s which has the same F-terms with s on some node t^* . If so, we should add $(t, t^*) \in E$ and mark s as “exhausted” as well. If a state s is merged with some ancestor, then all successors of s are also marked as “exhausted”, and we stop to explore any term with respect to these successors further. In addition, let $\sim \subseteq S \times S$ be an equivalent relation, and use $s^* \sim s$ to “merge” these two state to a reflexive state.
- (Eve \hat{G}) If all consequences of an antecedent t are marked as deleted, then mark t as “deleted”. If $\hat{G}_N\varphi$ is in some term of a node t with respect to a state s , and there is no reachable state from s such that φ occurs in some term, then mark t as “deleted”.
- (EveF) If $F_N\varphi$ is in some term of a node t with respect to a state s , and there exists a full branch from s on which φ does not occur in any term of state on that branch, then mark t as “deleted”.

If there is no rule applicable any more, the procedure of generating the tableau terminate, and the tableau is *complete*. If the root of a complete tableau T is not marked as “deleted”, then we call a path from the root to a leaf node on T an *open branch*. If a complete tableau has at least one open branch, then we call it an *open tableau*.

Proposition 46. *For any NNF-formula φ , the procedure of generate a tableau for φ will terminate.*

Example 6. Consider NAUL-formula

$$F\Diamond\neg p \wedge p \wedge G_{(\top, \mathcal{A}, p)}(\Diamond p \wedge \Diamond\neg p)$$

It says p is true, and for every full-path there is some state that has $\neg p$ successor, and for every state on a full (\top, \mathcal{A}, p) -path it has both a p -successor and a $\neg p$ -successor. Now we check if there is an open tableau for it.

We start from t_r with the label $\langle s_0; \epsilon; F\Diamond\neg p \wedge p \wedge G_{(\top, \mathcal{A}, p)}(\Diamond p \wedge \Diamond\neg p) \rangle$. We use (\wedge) rule 3 times

$$t_1 : \langle s_0; \epsilon; F\Diamond\neg p \rangle, \langle s_0; \epsilon; p \rangle, \langle s_0; \epsilon; G_{(\top, \mathcal{A}, p)}(\Diamond p \wedge \Diamond\neg p) \rangle$$

Then apply (F) to the first term we have two branches of t_1 , we choose one of them as:

$$t_2 : \langle s_0; \epsilon; \Diamond\neg p \rangle, \langle s_0; \epsilon; p \rangle, \langle s_0; \epsilon; G_{(\top, \mathcal{A}, p)}(\Diamond p \wedge \Diamond\neg p) \rangle$$

We explore the branch from t_2 to check whether it is open. For t_2 , we should apply (G) to get a consequence

$$t_3 : \langle s_0; \epsilon; \Diamond\neg p \rangle, \langle s_0; \epsilon; p \rangle, \langle s_0; \epsilon; \Diamond p \rangle, \langle s_0; \epsilon; \Box_{(\top, \mathcal{A}, p)}G_{(\top, \mathcal{A}, p)}(\Diamond p \wedge \Diamond\neg p) \rangle$$

Now we apply (\Diamond) twice by choosing an arbitrary agent a and b ,

$$t_4 : \langle s_0; \epsilon; p \rangle, \langle s_0; \epsilon; \Box_{(\top, \mathcal{A}, p)}G_{(\top, \mathcal{A}, p)}(\Diamond p \wedge \Diamond\neg p) \rangle \\ \langle s_0 \xrightarrow{a} s_1; \epsilon; (\top, \mathcal{A}, \top) \rangle, \langle s_1; \epsilon; p \rangle, \langle s_0 \xrightarrow{b} s_2; \epsilon; (\top, \mathcal{A}, \top) \rangle, \langle s_2; \epsilon; \neg p \rangle$$

We apply (Atomic) , the consequent has trivial terms like $\langle s_0; \epsilon; \top \rangle$ which we omit. Thus

$$t_5 : \langle s_0; \epsilon; p \rangle, \langle s_0; \epsilon; \Box_{(\top, \mathcal{A}, p)}G_{(\top, \mathcal{A}, p)}(\Diamond p \wedge \Diamond\neg p) \rangle \\ \langle s_0 \xrightarrow{a} s_1; \epsilon; \epsilon \rangle, \langle s_1; \epsilon; p \rangle, \langle s_0 \xrightarrow{b} s_2; \epsilon; \epsilon \rangle, \langle s_2; \epsilon; \neg p \rangle$$

Now (\Box) is applicable for t_5 ,

$$t_6 : \langle s_0; \epsilon; p \rangle, \langle s_1; \epsilon; p \rangle, \langle s_2; \epsilon; \neg p \rangle, \langle s_0 \xrightarrow{a} s_1; \epsilon; \epsilon \rangle, \langle s_0 \xrightarrow{b} s_2; \epsilon; \epsilon \rangle \\ \langle s_1; \epsilon; G_{(\top, \mathcal{A}, p)}\Diamond p \wedge \Diamond\neg p \rangle, \langle s_0 \xrightarrow{a} s_1; \epsilon; (\top, \mathcal{A}, p) \rangle, \langle s_0 \xrightarrow{b} s_2; \epsilon; \overline{(\top, \mathcal{A}, p)} \rangle$$

We can apply (Atomic) on $\langle s_0 \xrightarrow{a} s_1; \epsilon; (\top, \mathcal{A}, p) \rangle$, and the result is trivial. We also can apply (Neg) on $\langle s_0 \xrightarrow{a} s_1; \epsilon; \overline{(\top, \mathcal{A}, p)} \rangle$, and result only has duplicate terms as well.

$$t_7 : \langle s_0; \epsilon; p \rangle, \langle s_1; \epsilon; p \rangle, \langle s_2; \epsilon; \neg p \rangle, \langle s_0 \xrightarrow{a} s_1; \epsilon; \epsilon \rangle, \langle s_0 \xrightarrow{b} s_2; \epsilon; \epsilon \rangle, \langle s_1; \epsilon; G_{(\top, \mathcal{A}, p)}\Diamond p \wedge \Diamond\neg p \rangle$$

Now s_0 should be marked as “exhausted”. We continue to explore s_1 and s_2 . Obviously, s_2 should be marked as “exhausted” immediately. As for s_1 , we apply (G)

$$t_8 : \langle s_0; \epsilon; p \rangle, \langle s_1; \epsilon; p \rangle, \langle s_2; \epsilon; \neg p \rangle, \langle s_0 \xrightarrow{a} s_1; \epsilon; \epsilon \rangle, \langle s_0 \xrightarrow{b} s_2; \epsilon; \epsilon \rangle \\ \langle s_1; \epsilon; \Diamond p \rangle, \langle s_1; \epsilon; \Diamond\neg p \rangle, \langle s_1; \epsilon; \Box_{(\top, \mathcal{A}, p)}G_{(\top, \mathcal{A}, p)}\Diamond p \wedge \Diamond\neg p \rangle$$

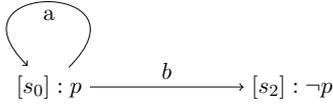
Notice s_1 has the same F-terms on t_8 as s_0 on t_3 , so we will have a similar procedure as t_3 to t_8 with respect to s_0 . We omit details to save space. Let c, d be arbitrary agents, s_3, s_4 be two states in S , the following node t_l is accessible from t_8 :

$$\begin{aligned} t_l : & \langle s_0; \epsilon; p \rangle, \langle s_1; \epsilon; p \rangle, \langle s_2; \epsilon; \neg p \rangle, \langle s_0 \xrightarrow{a} s_1; \epsilon; \epsilon \rangle, \langle s_0 \xrightarrow{b} s_2; \epsilon; \epsilon \rangle \\ & \langle s_3; \epsilon; p \rangle, \langle s_4; \epsilon; \neg p \rangle, \langle s_1 \xrightarrow{c} s_3; \epsilon; \epsilon \rangle, \langle s_1 \xrightarrow{d} s_4; \epsilon; \epsilon \rangle \\ & \langle s_3; \epsilon; \diamond p \rangle, \langle s_3; \epsilon; \diamond \neg p \rangle, \langle s_3; \epsilon; \Box_{(\top, \mathcal{A}, p)} G_{(\top, \mathcal{A}, p)} \diamond p \wedge \diamond \neg p \rangle \end{aligned}$$

Now s_1 should be marked as exhausted. By (Cyc), s_1 has the same F-terms on t_l as s_0 on t_8 . Thus we add the pair (t_l, t_8) to E and mark s_1 as “exhausted”, and mark its successors s_3 and s_4 as exhausted too. Now the procedure of generating the tableau of φ terminates. There is $F\diamond\neg p$ on s_0 , and $\diamond\neg p$ is also on s_0 , thus this branch meets (EveF). Therefore, it is an open branch. The corresponding model $\mathcal{M} = (S, R, V)$ is

- $S = \{[s_0], [s_2]\}$
- $R = \{[s_0] \xrightarrow{a} [s_0], [s_0] \xrightarrow{b} [s_2]\}$
- $V(p) = \{[s_0]\}, V(\neg p) = \{[s_2]\}$

where $[s] = \{s' \in S \mid s \sim s'\}$.



Example 7. Consider NAUL-formula

$$p \wedge F\neg p \wedge G_{(\top, \mathcal{A}, p)}(\diamond p \wedge \diamond \neg p)$$

It says p is true, every full path has a state $\neg p$ is true, and for every state on a full (\top, \mathcal{A}, p) -path it has both a p -successor and a $\neg p$ -successor. It is unsatisfiable because it requires a full path where p is true on each state, which contradicts $F\neg p$. Therefore, there should be no open tableau for it. We check it by tableau method.

We start from t_r with the label

$$t_r : \langle s_0; \epsilon; p \wedge F\neg p \wedge G_{(\top, \mathcal{A}, p)}(\diamond p \wedge \diamond \neg p) \rangle$$

We apply (\wedge) rule on t_r multi times to a consequence t_1 as:

$$t_1 : \langle s_0; \epsilon; p \rangle, \langle s_0; \epsilon; F\neg p \rangle, \langle s_0; \epsilon; G_{(\top, \mathcal{A}, p)}(\diamond p \wedge \diamond \neg p) \rangle$$

Then we should apply (F) rule on $\langle s_0; \epsilon; F\neg p \rangle$. We will have two branches t_2 and t_3 as:

$$t_2 : \langle s_0; \epsilon; p \rangle, \langle s_0; \epsilon; \neg p \rangle, \langle s_0; \epsilon; G_{(\top, \mathcal{A}, p)}(\diamond p \wedge \diamond \neg p) \rangle$$

and

$$t_3 : \langle s_0; \epsilon; p \rangle, \langle s_0; \epsilon; \Box F\neg p \rangle, \langle s_0; \epsilon; \diamond \top \rangle, \langle s_0; \epsilon; G_{(\top, \mathcal{A}, p)}(\diamond p \wedge \diamond \neg p) \rangle$$

As we can observe, t_2 will be marked as deleted by have both $\langle s_0; \epsilon; p \rangle$ and $\langle s_0; \epsilon; \neg p \rangle$. Thus we can just explore t_3 . We should apply (G), then get a consequent t_4 of t_3 as:

$$t_4 : \langle s_0; \epsilon; p \rangle, \langle s_0; \epsilon; \Box F \neg p \rangle, \langle s_0; \epsilon; \Diamond \top \rangle, \langle s_0; \epsilon; \Diamond p \wedge \Diamond \neg p \rangle, \langle s_0; \epsilon; \Box_{(\top, \mathcal{A}, p)} G_{(\top, \mathcal{A}, p)}(\Diamond p \wedge \Diamond \neg p) \rangle$$

Now we should apply (\Diamond) for $\Diamond p$ and $\Diamond \neg p$. There are branches for each agent in \mathcal{A} . We choose an arbitrary $a, b \in \mathcal{A}$ as an example:

$$t_5 : \langle s_0; \epsilon; p \rangle, \langle s_0; \epsilon; \Box F \neg p \rangle, \langle s_0; \epsilon; \Box_{(\top, \mathcal{A}, p)} G_{(\top, \mathcal{A}, p)}(\Diamond p \wedge \Diamond \neg p) \rangle \\ \langle s_1; \epsilon; p \rangle, \langle s_0 \xrightarrow{a} s_1; \epsilon; (\top, \mathcal{A}, \top) \rangle, \langle s_2; \epsilon; \neg p \rangle, \langle s_0 \xrightarrow{b} s_2; \epsilon; (\top, \mathcal{A}, \top) \rangle$$

By omitting trivial and duplicate terms, we should apply (\Box). For $\langle s_0; \epsilon; \Box F \neg p \rangle$, it is obvious that the only possible branch is:

$$t_6 : \langle s_0; \epsilon; p \rangle, \langle s_0; \epsilon; \Box_{(\top, \mathcal{A}, p)} G_{(\top, \mathcal{A}, p)}(\Diamond p \wedge \Diamond \neg p) \rangle, \langle s_1; \epsilon; p \rangle, \langle s_1; \epsilon; F \neg p \rangle, \\ \langle s_2; \epsilon; \neg p \rangle, \langle s_2; \epsilon; F \neg p \rangle, \langle s_0 \xrightarrow{a} s_1; \epsilon; \epsilon \rangle, \langle s_0 \xrightarrow{b} s_2; \epsilon; \epsilon \rangle$$

Now (\Box) rule is still applicable. We omit some steps to give the only possible branch:

$$t_7 : \langle s_0; \epsilon; p \rangle, \langle s_1; \epsilon; p \rangle, \langle s_1; \epsilon; F \neg p \rangle, \langle s_1; \epsilon; G_{(\top, \mathcal{A}, p)}(\Diamond p \wedge \Diamond \neg p) \rangle, \\ \langle s_2; \epsilon; \neg p \rangle, \langle s_2; \epsilon; F \neg p \rangle, \langle s_0 \xrightarrow{a} s_1; \epsilon; \epsilon \rangle, \langle s_0 \xrightarrow{b} s_2; \epsilon; \epsilon \rangle$$

Now we can mark s_0 as exhausted. For s_2 , by applying (F), it will be marked as exhausted with the only F-term $\langle s_2; \epsilon; \neg p \rangle$. For s_1 , there is a similar procedure as t_1 to t_7 with respect to s_0 . Like the above example, we also omit detail. Let c, d be arbitrary agents, s_3, s_4 be two states. t_l is an accessible consequent of t_7 :

$$t_l : \langle s_0; \epsilon; p \rangle, \langle s_1; \epsilon; p \rangle, \langle s_1; \epsilon; F \neg p \rangle, \langle s_2; \epsilon; \neg p \rangle, \langle s_0 \xrightarrow{a} s_1; \epsilon; \epsilon \rangle, \langle s_0 \xrightarrow{b} s_2; \epsilon; \epsilon \rangle \\ \langle s_3; \epsilon; p \rangle, \langle s_4; \epsilon; \neg p \rangle, \langle s_4; \epsilon; F \neg p \rangle, \langle s_1 \xrightarrow{c} s_3; \epsilon; \epsilon \rangle, \langle s_1 \xrightarrow{d} s_4; \epsilon; \epsilon \rangle \\ \langle s_3; \epsilon; F \neg p \rangle, \langle s_3; \epsilon; G_{(\top, \mathcal{A}, p)}(\Diamond p \wedge \Diamond \neg p) \rangle$$

Now s_1 is exhausted. We notice that s_1 has the same F-terms on t_l with s_0 on t_7 . By (Cyc), we add (t_l, t_7) to E and mark s_3, s_4 as exhausted. The procedure of generating this branch ends. Since $F \neg p$ is on a F-term of s_0 , and $\neg p$ does not occur on the full branch $s_0 \xrightarrow{a} s_0$, therefore by (EveF) t_7 should be marked deleted. Since a, b is arbitrary, all branches of t_4 will be marked as deleted similarly. As a result t_r will be marked as deleted by (Inc). It follows that there is no open branch for this formula.

5.5.2 Soundness and Completeness

Proposition 47 (Soundness). *Given an NNF-formula φ , if φ is satisfiable then there is an open tableau rooted at $(s_0; \epsilon; \varphi)$.*

Proof. We show all tableau rules preserve interpretability. By Def 47, it is obvious that interpretability implies satisfiability.

- (*lit*) and (\wedge) preserve interpretability obviously. For (\vee), if one of its consequences is interpretable, then so is the antecedent.

- (G): Suppose $\mathcal{M} \models_T \langle s; \lambda; G_N \varphi \rangle$, then we have $\mathcal{M} * \lambda, s \models G_N \varphi$. By semantics, for every N -path P starting from s and every $s' \in P$, $\mathcal{M} * \lambda, s' \models \varphi$. Since s is on any N -path starting from s , we have $\mathcal{M} * \lambda, s \models \varphi$. For every s' such that $s \xrightarrow{a} s'$ satisfying N , we also have $\mathcal{M} * \lambda, s' \models G_N \varphi$. Thus by semantics, $\mathcal{M} * \lambda, s \models \Box_N G_N \varphi$.
- (F): Suppose $\mathcal{M} \models_T \langle s; \lambda; F_N \varphi \rangle$, then we have $\mathcal{M} * \lambda, s \models F_N \varphi$. By semantics, for every full N -path P starting from s , there is some $s' \in P$ such that $\mathcal{M} * \lambda, s' \models \varphi$. Since s is in every N -path starting from s , it is sufficient if $\mathcal{M} * \lambda, s \models \varphi$. Otherwise, we have there is some N -successor s' of s such that $\mathcal{M} * \lambda, s' \models F_N \varphi$. In this case, $\mathcal{M} * \lambda, s \models \Diamond_N \top$ and $\mathcal{M} * \lambda, s \models \Diamond_N F_N \varphi$.
- (\hat{G}): Suppose $\mathcal{M} \models_T \langle s; \lambda; \hat{G}_N \varphi \rangle$, then we have $\mathcal{M} * \lambda, s \models \hat{G}_N \varphi$. By semantics, there exists some N -path P starting from s on which there is some s' such that $\mathcal{M} * \lambda, s' \models \varphi$. If $\mathcal{M} * \lambda, s \models \varphi$, then $\mathcal{M} * \lambda, s \models \hat{G}_N \varphi$ holds. Otherwise, there is some N -successor s' of s such that $\mathcal{M} * \lambda, s' \models \hat{G}_N \varphi$. Thus we have $\mathcal{M} * \lambda, s \models \Diamond_N \top$ and $\mathcal{M} * \lambda, s \models \Diamond_N \hat{G}_N \varphi$.
- (\hat{F}): Suppose $\mathcal{M} \models_T \langle s; \lambda; \hat{F}_N \varphi \rangle$, then we have $\mathcal{M} * \lambda, s \models \hat{F}_N \varphi$. By semantics, there is some N -path P on which for every s' we have $\mathcal{M} * \lambda, s' \models \varphi$. If $P = \{s\}$, then s has no N -successor, that is, $\mathcal{M} * \lambda, s \models \Box_N \perp$ and $\mathcal{M} * \lambda, s \models \varphi$; If P some state other than s , then s has some N -successors, that is, $\mathcal{M} * \lambda, s \models \Diamond_N \top$ and there is N -successor s' such that $\mathcal{M} * \lambda, s' \models \hat{F}_N \varphi$, thus $\mathcal{M} * \lambda, s \models \Diamond_N \hat{F}_N \varphi$.
- (\Diamond_N): Suppose $\mathcal{M} \models_T \langle s; \lambda; \Diamond_N \varphi \rangle$, then we have $\mathcal{M} * \lambda, s \models \Diamond_N \varphi$. By semantics, there is a transition $s \xrightarrow{a} s'$ satisfying N and $\mathcal{M} * \lambda, s' \models \varphi$. Then we have $\mathcal{M} \models_T \langle s'; \lambda; \varphi \rangle$ and $\mathcal{M} \models_T \langle s \xrightarrow{a} s'; \lambda; N \rangle$ for some $a \in \mathcal{A}$.
- (\Box_N): Suppose $\mathcal{M} \models_T \langle s; \lambda; \Box_N \varphi \rangle$ and $\mathcal{M} \models_T \langle s \xrightarrow{a} s'; \epsilon; \lambda \rangle$, then we have $\mathcal{M} * \lambda, s \models \Box_N \varphi$ and $s \xrightarrow{a} s'$ satisfies λ on \mathcal{M} . If $s \xrightarrow{a} s'$ satisfies N on $\mathcal{M} * \lambda$, then by semantics we have $\mathcal{M} * \lambda, s' \models \varphi$. Thus $\mathcal{M} \models_T \langle s'; \lambda; \varphi \rangle$. If $s \xrightarrow{a} s'$ does not satisfy N on $\mathcal{M} * \lambda$, then it satisfies $\neg N$ on $\mathcal{M} * \lambda$. Thus $\mathcal{M} \models_T \langle s \xrightarrow{a} s'; \lambda; \neg N \rangle$.
- (Dynamic): Suppose $\mathcal{M} \models_T \langle s; \lambda; [N] \varphi \rangle$, then we have $\mathcal{M} * \lambda, s \models [N] \varphi$. By semantics, $\mathcal{M} * \lambda * N, s \models \varphi$. Thus $\mathcal{M} \models_T \langle s; \lambda, N; \varphi \rangle$. The case $\mathcal{M} \models_T \langle N \rangle \varphi$ is similar.
- (Atomic): Suppose $\mathcal{M} \models_T \langle s \xrightarrow{a_i} s'; \lambda; \mathcal{N} \rangle$, then $s \xrightarrow{a_i} s'$ satisfies \mathcal{N} on $\mathcal{M} * \lambda$. It follows that $\mathcal{M} * \lambda, s \models \varphi_i$ and $\mathcal{M} * \lambda, s' \models \psi_i$. Thus $\mathcal{M} \models_T \langle s; \lambda; \varphi_i \rangle, \langle s'; \lambda; \psi_i \rangle$.
- (Neg): Suppose $\mathcal{M} \models_T \langle s \xrightarrow{a} v; \lambda; \overline{\mathcal{N}} \rangle$, then $s \xrightarrow{a} v$ satisfies $\overline{\mathcal{N}}$ on $\mathcal{M} * \lambda$. It follows that $s \xrightarrow{a} v$ satisfies no clause with respect to a in \mathcal{N} . Thus let $K_1 = \{i \mid \mathcal{M} * \lambda, v \models \neg \varphi_i \text{ for any } (\varphi_i, a, \psi_i) \in \mathcal{N}\}$ and $K_2 = \{i \mid \mathcal{M} * \lambda, v \models \neg \psi_i \text{ for any } (\varphi_i, a, \psi_i) \in \mathcal{N}\}$. Therefore, we have $K_1 \cup K_2 = \{i \mid (\varphi_i, a_i, \psi_i) \in \mathcal{N}\}$, and $\mathcal{M} \models_T \langle v; \lambda; \bigwedge_{i \in K_1} \neg \varphi_i \rangle$ and $\mathcal{M} \models_T \langle v; \lambda; \bigwedge_{i \in K_2} \neg \psi_i \rangle$.
- (Add), (Multi), (Seq) is straightforward by the definition.

- (DN): Suppose $\mathcal{M} \models_T \langle s \xrightarrow{a} s'; \lambda, N_1; N_2 \rangle$, then $s \xrightarrow{a} s'$ satisfies N_2 on $\mathcal{M} * \lambda * N_1$. By definition, if $s \xrightarrow{a} s'$ is on $\mathcal{M} * \lambda * N_1$, then it satisfies N_1 on $\mathcal{M} * \lambda$ as well. Thus $\mathcal{M} \models_T \langle s \mapsto s'; \lambda; N_1 \rangle$.
- (Cyc) If there is a trace-back link between two nodes, then those two nodes have the same terms with respect to some state. Thus interpretability is preserved by this trace-back link as well. The other principles do not add consequences.

Suppose φ is satisfiable, then there is a pointed model $\mathcal{M}, s \models \varphi$. Thus $\mathcal{M} \models_T \langle s; \epsilon; \varphi \rangle$. Since every tableau rule preserves interpretability, there is no inconsistent of literals. Let s be s_0 , then there is an open tableau rooted at $\langle s_0; \epsilon; \varphi \rangle$. \square

Proposition 48 (Completeness). *Given an NNF-formula φ , if there is an open tableau rooted at $\langle s_0; \epsilon; \varphi \rangle$, then φ is satisfiable.*

Proof. Suppose there is an open tableau T rooted at $\langle s_0; \lambda; \varphi \rangle$. Let T^* be a full branch on T . We construct a model $\mathcal{M} = (S, R, v)$ where

- $S = \{[s] \mid \langle s; \lambda; \psi \rangle \text{ is in } T^*\}$
- $R = \{s \xrightarrow{a} s' \mid \langle s \xrightarrow{a} s'; \epsilon; \epsilon \rangle \in T^*\} \cup \{s \xrightarrow{a} s \mid a \in \mathcal{A}, |[s]| > 1\}$
- $v(s) = \{p \mid \langle s; \epsilon; p \rangle \in T^*\}$

where $[s] = \{s' \in S \mid s \sim s'\}$.

We show the following claims:

1. if $\langle s; \lambda; \psi \rangle$ is in T^* , then $\mathcal{M} * \lambda, s \models \psi$.
2. if $\langle s_1 \xrightarrow{a} s_2; \lambda; N \rangle$ is in T^* , then $s_1 \xrightarrow{a} s_2$ is in $\mathcal{M} * \lambda$ and satisfies η on $\mathcal{M} * \lambda$ (Note that any sequence of norms η can be transformed into a norm N).

Make an induction on all terms by the order \prec in Def. 48 to show the above claims. For Claim 2,

- If $\langle s_1 \xrightarrow{a_i} s_2; \lambda; \mathcal{N} \rangle \in T^*$ where $\mathcal{N} = (\varphi_1, a_1, \psi_1), \dots, (\varphi_n, a_n, \psi_n)$, $i \in [1, n]$, then by (Atomic) rule $\langle s_1; \lambda; \varphi_i \rangle, \langle s; \lambda; \psi_i \rangle \in T^*$. Then by IH, we have $\mathcal{M} * \lambda, s_1 \models \varphi$, $\mathcal{M} * \lambda, s_2 \models \psi$. Thus $s_1 \xrightarrow{a_i} s_2$ satisfies \mathcal{N} on $\mathcal{M} * \lambda$.
- If $\langle s_1 \xrightarrow{a} s_2; \lambda; \overline{\mathcal{N}} \rangle \in T^*$ where $\mathcal{N} = (\varphi_1, a_1, \psi_1), \dots, (\varphi_n, a_n, \psi_n)$, $i \in [1, n]$, then by (Neg) rule, $\langle s_1; \lambda; \bigwedge_{i \in K_1} \neg \varphi_i \rangle \in T^*$ and $\langle s_2; \lambda; \bigwedge_{i \in K_2} \neg \psi_i \rangle \in T^*$ for some disjoint $K_1 \cup K_2 = \{i \mid (\varphi_i, a, \psi_i) \in \mathcal{N}\}$. Thus, no φ_i and ψ_i are satisfied simultaneously so that no clause in \mathcal{N} with respect to a is satisfied. Therefore, $s_1 \xrightarrow{a} s_2$ satisfies $\overline{\mathcal{N}}$.
- The cases of $\langle s_1 \mapsto s_2; \lambda; N_1 + N_2 \rangle \in T^*$ and $\langle s_1 \mapsto s_2; \lambda; N_1 \times N_2 \rangle \in T^*$ is straightforward by IH.

- If $\langle s_1 \xrightarrow{a} s_2; \lambda; N_1 \circ N_2 \rangle \in T^*$, then by (Seq) rule, $\langle s_1 \xrightarrow{a} s_2; \lambda; N_1; N_2 \rangle \in T^*$. By (DN) rule, we also have $\langle s_1 \xrightarrow{a} s_2; \lambda; N_1 \rangle \in T^*$. Thus by IH, $s_1 \xrightarrow{a} s_2$ is in $\mathcal{M} * \lambda$ and satisfies N_1 on $\mathcal{M} * \lambda$. Since $\langle s_1 \xrightarrow{a} s_2; \lambda; N_1; N_2 \rangle \prec \langle s_1 \xrightarrow{a} s_2; \lambda; N_1; N_1 \circ N_2 \rangle \prec \langle s_1 \xrightarrow{a} s_2; \lambda; N_1 \circ N_2 \rangle$, by IH, $s_1 \xrightarrow{a} s_2$ is in $(\mathcal{M} * \lambda) * N_1$ and it satisfies N_2 on $(\mathcal{M} * \lambda) * N_1$. Therefore, $s_1 \xrightarrow{a} s_2$ is in $\mathcal{M} * \lambda$ and satisfies $N_1 \circ N_2$ on $\mathcal{M} * \lambda$.

For Claim 1,

- If $\langle s; \lambda; p \rangle \in T^*$, then by the rule (lit), $\langle s; \epsilon; p \rangle \in T^*$. Thus $\mathcal{M}, s \models p$, and then $\mathcal{M} * \lambda, s \models p$. Similarly, if $\langle s; \lambda; \neg p \rangle \in T^*$, then $\mathcal{M} * \lambda, s \models \neg p$.
- If $\langle s; \lambda; \varphi_1 \wedge \varphi_2 \rangle \in T^*$, by rule (\wedge) $\langle s; \lambda; \varphi_1 \rangle \in T^*$ and $\langle s; \lambda; \varphi_2 \rangle \in T^*$. By IH, $\mathcal{M} * \lambda, s \models \varphi_1$ and $\mathcal{M} * \lambda, s \models \varphi_2$. Thus $\mathcal{M} * \lambda, s \models \varphi_1 \wedge \varphi_2$. Similarly, if $\langle s; \lambda; \varphi_1 \vee \varphi_2 \rangle \in T^*$, then by IH $\mathcal{M} * \lambda, s \models \varphi_1 \vee \varphi_2$.
- If $\langle s; \lambda; \diamond_N \psi \rangle \in T^*$, then by rule (\diamond_N), we have $\langle s'; \lambda; \psi \rangle \in T^*$ and $\langle s \xrightarrow{a} s'; \lambda; N \rangle \in T^*$ for some $a \in \mathcal{A}$. By IH, $s \xrightarrow{a} s'$ satisfies N in $\mathcal{M} * \lambda$ and $\mathcal{M} * \lambda, s' \models \psi$. Then by semantics, we have $\mathcal{M} * \lambda, s \models \diamond_N \psi$.
- Suppose $\langle s; \lambda; \square_N \psi \rangle \in T^*$. For any N -successor s' of s on $\mathcal{M} * \lambda$, if there is some $\langle s \xrightarrow{a} s'; \epsilon; \lambda \rangle \in T^*$, then for any such term with respect to any s' , by (\square_N) rule, $\langle s'; \lambda; \psi \rangle \in T^*$ or $\langle s \xrightarrow{a} s'; \lambda; -N \rangle \in T^*$. $\langle s'; \lambda; \psi \rangle \in T^*$, then by IH $\mathcal{M} * \lambda, s' \models \psi$. If $\langle s \xrightarrow{a} s'; \lambda; -N \rangle \in T^*$, then by Claim 1, $s \xrightarrow{a} s'$ satisfies $-N$, that is to say, $s \xrightarrow{a} s'$ does not satisfy N . Thus by semantics, $\mathcal{M}, s \models \square_N \psi$.
- Suppose $\langle s; \lambda; G_N \psi \rangle \in T^*$. Let $P = s \xrightarrow{a_1} s_1 \xrightarrow{a_2} s_2 \cdots \xrightarrow{a_n} s_{n+1}$ be any N -path starting from s . We show that $\langle s'; \lambda; \square_N G_N \psi \rangle \in T^*$ and $\mathcal{M} * \lambda, s' \models \psi$ for any $s' \in P$ by induction on $n + 1$. By (G) rule, $\langle s; \lambda; \psi \rangle \in T^*$ and $\langle s; \lambda; \square_N G_N \psi \rangle \in T^*$. Since $\langle s; \lambda; \psi \rangle \in T^*$, by IH we have $\mathcal{M} * \lambda, s \models \psi$. Assume $\langle s_n; \lambda; \square_N G_N \psi \rangle \in T^*$ and $\mathcal{M} * \lambda, s_n \models \psi$. By (G) rule again, we have $\langle s_{n+1}; \lambda; \square_N G_N \psi \rangle \in T^*$. Since $s_n \xrightarrow{a_n} s_{n+1} \in R$, we have $\langle s_n \xrightarrow{a_n} s_{n+1}; \lambda; N \rangle \in T^*$. Then by (\square_N) rule, we have $\langle s_{n+1}; \lambda; \psi \rangle \in T^*$. By IH, we have $\mathcal{M} * \lambda, s_{n+1} \models \psi$. Thus for every $s' \in P$, we have $\mathcal{M} * \lambda, s' \models \psi$. As P is arbitrary, by semantics $\mathcal{M} * \lambda, s \models G_N \psi$.
- Suppose $\langle s; \lambda; \hat{G}_N \psi \rangle \in T^*$. Towards contradiction, we assume for any s' in any N -path starting from s , $\mathcal{M} * \lambda, s' \not\models \psi$. By IH, it means $\langle s'; \lambda; \psi \rangle \notin T^*$. Hence, $\langle s; \lambda; \hat{G}_N \psi \rangle$ must be marked as deleted. It is a contradiction. Thus there is some s' in some N -path starting from s such that $\mathcal{M} * \lambda, s' \models \psi$. By semantics, we have $\mathcal{M} * \lambda, s \models \hat{G}_N \psi$.
- Suppose $\langle s; \lambda; F_N \psi \rangle \in T^*$. Towards contradiction, we assume there is an N -path P on which for any s' , $\mathcal{M} * \lambda, s' \not\models \psi$. By IH, it means $\langle s'; \lambda; \psi \rangle \notin T^*$. Therefore, there is a full branch on which ψ is not in any label. It follows $\langle s; \lambda; F_N \psi \rangle$ must be marked as deleted. It is a contradiction. Hence, for any N -path starting from s there is some s' such that $\mathcal{M} * \lambda, s' \models \psi$. By semantics, $\mathcal{M} * \lambda, s \models F_N \psi$.

- If $\langle s; \lambda; \hat{F}_N \psi \rangle \in T^*$, then $\langle s; \lambda; \psi \rangle, \langle s; \lambda; \Box_N \perp \rangle \in T$ or $\langle s; \lambda; \psi \rangle, \langle s; \lambda; \Diamond_N \hat{F}_N \psi \rangle \in T$.
 - If we choose the first branch, then assume there is an N -transition $s \xrightarrow{\alpha} s' \in R$, and $\langle s \xrightarrow{\alpha} s'; \lambda; N \rangle \in T^*$. By (\Box_N) rule, $\langle s'; \lambda; \perp \rangle \in T$. However, this term must be marked as deleted. It is a contradiction. Thus there is no N -transition of s , that is to say, the singleton s is a full N -path. By semantics, we have $\mathcal{M} * \lambda, s \models \hat{F}_N \psi$.
 - If we choose the second branch, then towards contradiction we suppose for any full N -path $P = s \xrightarrow{\alpha_1} w_1 \xrightarrow{\alpha_2} w_2 \cdots \xrightarrow{\alpha_n} w_{n+1}$ there is some $w_i \in P (i \in [1, n+1])$ such that $\mathcal{M} * \lambda, s_i \not\models \psi$. We show that $\langle s; \lambda; \hat{F}_N \psi \rangle \notin T^*$ by induction from s_i to s . By IH we have $\langle s_i; \lambda; \psi \rangle \notin T^*$. It follows that $\langle s_i; \lambda; \hat{F}_N \psi \rangle \notin T^*$ by (\hat{F}) rule. Furthermore, it follows that $\langle s_{i-1}; \lambda; \Diamond_N \hat{F}_N \psi \rangle \notin T^*$. Since s_{i-1} has N -successor s_i , it follows that $\langle s_{i-1}; \lambda; \hat{F}_N \psi \rangle \notin T^*$. Similarly, if $\langle s_1; \lambda; \hat{F}_N \psi \rangle \notin T^*$, then $\langle s; \lambda; \hat{F}_N \psi \rangle \notin T^*$. It is a contradiction. Thus there is a full N -path on which for every s' we have $\mathcal{M} * \lambda, s' \models \psi$. By semantics, $\mathcal{M} * \lambda, s \models \hat{F}_N \psi$.
- If $\langle s; \lambda; [N]\psi \rangle \in T^*$, then by (Dynamic) rule, $\langle s; \lambda; N; \psi \rangle \in T^*$. By IH, $\mathcal{M} * (\lambda; N), s \models \psi$. It is equal to $(\mathcal{M} * \lambda) * N, s \models \psi$. If $\langle s; \lambda; \langle N \rangle \psi \rangle \in T^*$ then similarly we have $(\mathcal{M} * \lambda) * N, s \models \psi$.

□

Theorem 20. For any NNF formula φ , φ is satisfiable if and only if there is an open tableau rooted at (s_0, ϵ, φ) .

5.5.3 EXPSPACE-membership

Theorem 21. The satisfiability problem of NAUL is in EXPSPACE.

Proof. Let φ be an NNF formula, and T be an open tableau for φ . We show the following claims:

1. The depth of T is at most exponential.
2. The width of T is at most double exponential.
3. The procedure can be done in double exponential amount of time.

Proof of 1:

Note that tableau rules does not decompose formulas strictly, thus the sizes of formulas in the consequences may be larger than the sizes of formulas in the antecedents. However we can give an upper bound of how many terms a single open branch in T has.

The *agenda* $Ag(\varphi)$ of a formula φ is the smallest set containing ϵ , $SubF(\varphi)$ as well as $SubN(\varphi)$ and satisfying the following conditions:

- If $\psi \in Ag(\varphi)$, then $\neg\psi^* \in Ag(\varphi)$;
- If $G_N \psi \in Ag(\varphi)$, then $\Box_N G_N \psi \in Ag(\varphi)$;
- If $\hat{G}_N \psi \in Ag(\varphi)$, then $\Diamond_N \hat{G}_N \psi \in Ag(\varphi)$;

- If $F_N\psi \in Ag(\varphi)$, then $\diamond_N\top, \square_N F_N\psi \in Ag(\varphi)$;
- If $\hat{F}_N\psi \in Ag(\varphi)$, then $\square_N\perp, \diamond_N\hat{F}_N\psi \in Ag(\varphi)$;
- If $N \in Ag(\varphi)$, then $\overline{N}^* \in Ag(\varphi)$.

Clearly, the cardinality of $Ag(\varphi)$ is polynomial in $|\varphi|$.

For any F-term $\langle s; \lambda; \psi \rangle$ or N-term $\langle s \xrightarrow{\alpha} s'; \lambda; N \rangle$ occurring in T when s is marked as exhausted, it can be shown that $\psi, N \in Ag(\varphi)$ and all elements of λ are in $Ag(\varphi)$ by examining every rule. For (G), (F), (\hat{G}) and (\hat{F}), the formulas in consequences are all concluded in $Ag(\varphi)$ by the conditions above. For the (\square_N) rule, if $\square_N\psi \in Ag(\varphi)$, then $N \in Ag(\varphi)$ and $\overline{N}^* \in Ag(\varphi)$ as well. For the (Neg) rule, if $\overline{N} \in Ag(\varphi)$, then for any formula ψ occurring in some clause of \mathcal{N} we have $\psi \in Ag(\varphi)$. Thus $\neg\psi^* \in Ag(\varphi)$ as well. The other rules are trivial as formulas and norms in the consequences are all in $Ag(\varphi)$.

Firstly, we could give an upper bound of how many states there are in one open branch. We have shown the formulas of all F-terms are within $Ag(\varphi)$. The norms in λ of a term come from dynamic operators $[N]$ or $\langle N \rangle$ in φ . However the order of λ is not arbitrary, it follows the modal depth of dynamic operators in φ . Thus the possibility of λ is up to subsets of $Ag(\varphi)$ rather than higher power of $Ag(\varphi)$. Since two exhausted states get merged if they have the same F-terms, we can get at most $O(2^{|\varphi|}) \times O(2^{|\varphi|}) = O(2^{|\varphi|})$ many different exhausted states, so at most exponential in the size of φ .

Secondly, we could give an upper bound of how many transitions are generated from one state. Note that the (\diamond_N) rule is the only rule that generates new transitions. The frequency that (\diamond_N) rule is applied depends on, on one hand, how many \diamond modality as well as F, \hat{G}, \hat{F} are in φ . On the other hand, if we have to choose the second branch when applying (\square_N) rule, then there will be some new \diamond, F, \hat{F} or \hat{G} modalities which do not occur in φ , but occurs in negations of subformulas in N . In this case, the frequency of applying (\diamond_N) rule with respect to one \square_N modality is up to the size of N which is bounded by the size of φ . As the amount of modalities is bounded by the size of φ , one state has at most $O(|\varphi|) \times O(|\varphi|) = O(|\varphi|^2)$ arrows, so at most polynomial in the size of φ .

Therefore, the upper bound of the amount of F-terms is in $O(2^{|\varphi|}) \times O(2^{|\varphi|}) \times O(2^{|\varphi|}) = O(2^{|\varphi|})$, and the upper bound of the amount of N-terms is in $O(2^{|\varphi|}) \times O(|\varphi|^2) \times O(2^{|\varphi|}) \times O(2^{|\varphi|}) = O(2^{|\varphi|})$, so the amount of terms is at most exponential in the size of φ . One open branch has at most exponential depth as well, as there are at most exponentially many exhausted states with the same F-terms. This is because if a state is merged with some ancestor, then we will stop exploring terms of it. Therefore, the frequency that each state can be merged is no more than the number of paths starting from it. Since each exhausted state has polynomial many arrows to other states, it can be merged at most exponentially many times. In short, the depth of one open branch is in $O(2^{|\varphi|}) \times O(2^{|\varphi|}) \times O(2^{|\varphi|}) \times O(2^{|\varphi|}) = O(2^{|\varphi|})$.

Proof of 2:

The rule leads to exponentially many branches is (Neg). Given an atomic norm \mathcal{N} , $|\overline{\mathcal{N}}|$ is bounded by $|\varphi|$. The cardinality of branches is in $O(2^{|\varphi|})$. As there are at most exponentially many terms in one branch, the width of T is in $O(2^{|\varphi|^2})$, so at most double exponential in the size of φ .

Proof of 3:

The algorithm contains:

- applying tableau rules,
- checking, marking and pruning the tableau by principles,
- transforming formulas with negation into NNF.

For each branch, as there are at most exponentially many terms in the size of φ , all of three procedures above can be done in an exponential amount of time. To be specific, applying rules contains searching suitable premises and executing. The input of searching is the power set of labels on some node, which is exponential in the size of φ and the executions of applying rules are no more than the amount of terms; the input of checking inconsistency and states with the same terms is exponential in the size of φ and can be done in exponential time; the frequency of transforming NNF formulas is at most exponential and each transformation can be done in polynomial time.

To sum up, as we can reuse the space for each open branch, the procedure is in EXPSPACE. \square

5.6 Conclusion

We have presented a logic named *normative arrow update logic* (NAUL). NAUL is based on techniques and ideas from *arrow update logic* (AUL*) and *normative temporal logic* NTL. In NAUL, we can combine norms in three ways: additive, multiplicative and sequential. We can also distinguish static and dynamic ways to consider norms. We compare the expressivity of NAUL with other relative logics. We have shown that NAUL is strictly more expressive than CTL and AUL*. Furthermore, we have shown that the satisfiability problem of NAUL is decidable via a tableau method and the complexity of this problem is in EXPSPACE.

For the further research, firstly, we conjecture the satisfiability problem of NAUL is EXPSPACE-hard but have no proof yet. We also need to find an axiomatization of NAUL. Secondly, we are interested in finding tractable fragments of NAUL. Finally, it may be interesting to develop a variant of arbitrary arrow update logic (AAUL) [105] like NAUL. It would have quantifier over norms and express “there is some norm that guarantees φ ”.

6

Conclusion

In this thesis, we studied the extension of dynamic epistemic logic with quantification and normative systems.

Dynamic epistemic logic is an umbrella term for a class of logics that reason about knowledge and belief with information changing. The extension of dynamic epistemic logic with quantification has been extensively studied since arbitrary public announcement logic (APAL). In Chapter 3, we proposed several new variants by restricting the domain of quantifiers in APAL:

- The logic with construct $[Q]\varphi$ as a variant of APAL with the quantification over formulas restricted to subsets of variables (SAPAL). A special case of SAPAL is the subsets of variables are finite (FSAPAL).
- The logic with construct $[\sqsubseteq]\varphi$ as a variant of APAL with the quantification over formulas that only contain variables occurring in the scope of the quantifier $[\sqsubseteq]$ (SCAPAL).
- The logic with construct $[\psi^\uparrow]\varphi$ and $[\psi^\downarrow]\varphi$ as a variant of APAL with the quantification over formulas implying or implied by ψ (QIPAL). A special case of QIPAL is that quantified formulas are quantifier-free (IPAL).

In Section 3.3 we investigated the modal properties of these logics. We have shown that (CR) and (MK) are valid in SCAPAL but invalid in SAPAL and FSAPAL (Prop. 2 and Prop. 4). In Section 3.4, we compare the expressivity of these logics with APAL, epistemic logic, and each other. We found an interesting expressivity hierarchy of these variants, APAL, and epistemic logic (Fig. 3.1). To sum up, all these logics are more expressive than epistemic logic. FSAPAL are more expressive than SCAPAL and both of them are incomparable with APAL. IPAL is more expressive than APAL. We also showed that the satisfiability problem for SAPAL, FSAPAL, SCAPAL and IPAL is undecidable by modifying the proof of the APAL case slightly (Prop. 16). Lastly, we showed that the IPAL quantifier yields a substructural dynamic consequence relation differing from the version in [92, 95] based on public announcement logic.

In Chapter 4, we focus on the notion of “knowability”. In APAL, “ φ is knowable (to an agent a)” means “there is an announcement after which φ is known (by a)”. However, APAL does not have an operator representing knowability. Thus, we proposed a logic for knowability (LK) by packing the quantifier and modality together as the operator of knowability $\langle ! \rangle_i$. We investigated some logical properties of knowability by LK. For example, we showed that $\langle ! \rangle_i \varphi \leftrightarrow \langle ! \rangle_i \langle ! \rangle_i \varphi$ is valid. We also showed that (MK) and (CR) hold for LK (Theorem 14). The

expressivity of LK is equal to PAL under the single-agent system, but LK is more expressive than PAL under the multi-agent system. We also showed that APAL is at least as expressive as LK and conjectured APAL is strictly more expressive than LK. We gave the axiomatization of LK and showed its completeness and soundness (Theorem 11). For the satisfiability problem of LK, by a modification of the proof in [6], we knew that it is undecidable for at least three agents. We also proposed another two logics for knowability, namely without the construct $K_i\varphi$ (LK^-) and without $\langle\psi\rangle\varphi$ ($LK^=$). We found that LK^- and $LK^=$ are equally expressive as PL. Therefore, these two logics are decidable but lack expressivity.

Both novel variants of APAL in Chapter 3 and logics of knowability in Chapter 4 are the extension of dynamic epistemic logic with quantification and under the framework of public announcement logic. In Chapter 5, we investigated normative arrow update logic (NAUL) which uses arrow update logic as the framework of normative systems. NAUL follows a CTL-like approach in which a model is a labeled transition system. A norm is interpreted as an arrow update that has a source and a target condition. Intuitively, the source condition determines whether the norm should be adopted and the target condition tells which actions are allowed. Here action is depicted as a transition from the situation before to the situation after the action. With these settings, we can combine norms in different ways. The complex norms enable better formalization of multiple orders and orders for different agents. In NAUL, we can distinguish the static and dynamic applications of norms by using dynamic and temporal operators. Even though combined norms and dynamic operators can be expressed by atomic norms and static operators, they affect the succinctness of the language and the complexity of decision problems. It has been shown that NAUL is more expressive than CTL and AUL^* . The main contribution of Chapter 5 is to show NAUL is decidable by a tableau method. We further showed the satisfiability problem for NAUL is in EXPSpace.

The three themes in this thesis are a few case studies of the extension of dynamic epistemic logic. In the future, we would like to continue to develop our research. Various questions and directions could be further investigated. Firstly, there are some open questions in this thesis:

- It is unclear whether LK is less expressive than APAL. We have a proof on reflexive models, but have not found a proof on S5 models. It is also worth investigating that whether LK is already undecidable for only two agents.
- We conjecture the satisfiability problem of NAUL is EXPSpace-hard but have no proof yet. We also need to find an axiomatization of NAUL.

Secondly, we point out several related researches that could be done in the future:

- In Chapter 3, logics with restricted quantification are based on the framework of APAL. Similarly, group announcement logic [1], coalition announcement logic [5], positive announcement logic [99], and boolean announcement logic [98] are all variants of APAL. We may extend this approach to restricting quantifiers over other epistemic actions, such as arbitrary action model logic [47], and arbitrary arrow update logic [106]. It would be interesting to investigate the relative expressivity and decidability.
- Most results for dynamic epistemic logics with quantification over information change are for individual knowledge or belief modalities, but not for group modalities such as common knowledge and distributed knowledge.
- What is the axiomatization of the logic with the language of LK but without public announcements? We need to find alternatives for axioms involving public announcement.

Packing quantifiers with subsequent modalities may bring decidable variants of logics with quantification over information change.

- The framework of NAUL could be further improved. NAUL is a logic for multi-agent systems, but it lacks interaction between agents and cannot be specific about actions by different agents. For example, it is hard to model “agent a does nothing” since other agents may still change situations before and after a ’s inaction.
- It may be interesting to develop a variant of arbitrary arrow update logic (AAUL) [105] like NAUL. It would have quantifiers over norms and express “there is some norm that guarantees φ ” but in a setting of more private information change.

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Résumé étendu

Contexte

La *logique épistémique* fournit un langage formel et un ensemble de règles pour raisonner sur ce que quelqu'un sait ou croit. Dans la logique épistémique, les déclarations relatives à la connaissance et à la croyance sont représentées à l'aide d'opérateurs modaux spéciaux appelés *opérateurs épistémiques*, tels que "on sait que" or "on croit que". La logique épistémique moderne a vu le jour dans les années 1950 et 1960 avec une approche modale. Dans [108], von Wright a introduit l'idée d'utiliser la logique modale pour analyser la connaissance et la croyance, et a fondé du développement de la logique épistémique. L'ouvrage fondateur de Hintikka *Knowledge and belief: An introduction to the logic of the two notions* [52] est largement considéré comme le point de départ de la logique épistémique moderne. Hintikka a introduit la *sémantique des mondes possibles* (également connue sous le nom de *sémantique de Kripke*, voir [60]) de la logique épistémique dans le cadre de scénarios multi-agents. Les *modèles de Kripke* ou *modèles relationnels* consistent en un ensemble d'états, pour chaque agent une relation binaire appelée *relation d'accessibilité* entre les états, et une *évaluation* des variables propositionnelles sur chaque état. Les propriétés de la relation d'accessibilité, également appelées *propriétés de cadre*, définissent différentes classes de modèles de Kripke. La logique de la connaissance est typiquement interprétée dans la classe des modèles *d'équivalence* (réflexifs, transitifs et symétriques) et la logique de la croyance est interprétée dans la classe des modèles S_4 (transitifs et symétriques). Pour un agent a et un état réel, les états accessibles sont appelés alternatives épistémiques de a . Intuitivement, a considère toutes les alternatives épistémiques possibles mais non distinguables pour lui. En ce sens, ce que a sait est vrai pour chaque alternative épistémique pour lui.

La sémantique des mondes possibles de la logique épistémique est la façon dont nous pouvons déterminer si les déclarations sur la connaissance sont vraies ou non. Certaines propriétés de la connaissance sont toujours vraies dans les modèles d'équivalence et sont considérées comme des axiomes pour le raisonnement sur la connaissance. Soit a un agent et p une proposition:

- *Véracité*: Si a sait p , alors p est vrai;
- *Introspection positive*: Si a sait p , alors a sait que a sait p ;
- *Introspection négative*: Si a ne sait pas p , alors a sait que a ne sait pas p .

Ces propriétés sont largement acceptées, mais pas incontestées. Un problème classique est celui de *l'omniscience logique* (nommé par Hintikka [52]), selon lequel les agents sont censés être des raisonneurs parfaits qui connaissent toutes les vérités logiques. De nombreuses tentatives ont

été faites pour résoudre ce problème, comme l'introduction de la notion de *mondes impossibles* [53] et la *prise de conscience* [36]. Dans la thèse, nous adoptons toujours la sémantique des mondes possibles comme cadre de la logique épistémique. La définition formelle de la logique épistémique est donnée au chapitre 2. Nous renvoyons au manuel [23] pour plus de détails sur la logique modale et aux manuels [37, 100] pour plus de détails sur la logique épistémique.

La logique épistémique donne une perspective statique de l'interprétation des scénarios épistémiques à un moment donné. Cependant, les informations sont continuellement mises à jour et communiquées par les agents. La logique épistémique de base n'est pas en mesure de modéliser de tels changements et actions. Une approche de la modélisation de l'évolution des connaissances consiste à combiner la logique épistémique et la logique temporelle (la logique modale du temps) en une seule logique [48, 37, 83]. Une autre approche que nous étudierons consiste à étendre la logique épistémique avec des modalités de mise à jour qui sont interprétées par la mise à jour d'un modèle relationnel. Ce type de logiques est collectivement connu sous le nom de *dynamic epistemic logic* (DEL, *logique épistémique dynamique*) [104]. Une façon de distinguer ces logiques est de voir quels composants du modèle elles mettent à jour.

Public announcement logic (PAL, *logique d'annonce publique*) [76, 45] possède une modalité de mise à jour appelée annonce publique. Après la diffusion d'une annonce véridique à tous les agents, le domaine du modèle est restreint aux états pour lesquels cette annonce est vraie. Les relations accédant aux états supprimés sont également supprimées. PAL est utile pour formaliser et résoudre des problèmes épistémiques impliquant des communications publiques, tels que l'énigme des enfants sales (Muddy Children Puzzle) [76].

Arrow update logic (AUL, *logique de mise à jour des flèches*) [57], comme son nom l'indique, possède une modalité de mise à jour qui impose une restriction sur les paires dans les relations. Intuitivement, lorsqu'une paire dans une relation de l'agent a est supprimée, cela signifie que a peut distinguer cette paire d'états par des informations dont il est informé. Entre-temps, les états sont conservés et il peut y avoir des relations avec d'autres agents qui relient encore cette paire d'états. Dans un contexte épistémique, cela signifie que cette paire d'états est toujours indiscernable pour ces agents après un certain événement modifiant l'information. Les mises à jour fléchées sont plus générales que les annonces publiques, car l'information changeante peut être privée au sein d'un groupe d'agents alors que l'événement est public. Contrairement aux annonces publiques véridiques, les mises à jour de flèches peuvent transmettre de fausses informations. Les mises à jour de flèches peuvent ne pas préserver les modèles d'équivalence. Par conséquent, nous ne limitons pas l'interprétation de AUL aux modèles d'équivalence.

Action model logic (AML, *logique des modèles d'action*) [18] fournit une généralisation des mises à jour de l'information. *Modèles d'action* sont des structures de type modèle de Kripke dans lesquelles le domaine est un ensemble d'actions et une *précondition* est attribuée à chaque action. Le modèle actualisé en AML est le produit modal du modèle relationnel et du modèle d'action. Le domaine du modèle actualisé est un ensemble de paires sous la forme de (état, action), ce qui signifie que l'action peut être exécutée dans l'état. L'AML peut modéliser des annonces privées avec des observations limitées par d'autres agents. De même, *arrow update model logic* (AUML, *logique des modèles de mise à jour des flèches*) [58] a *modèles de mise à jour des flèches* comme généralisation des mises à jour des flèches.

Motivation et travaux connexes

De la logique épistémique à la logique épistémique dynamique, nous disposons d'une perspective dynamique pour raisonner sur la connaissance. De PAL à AML, nous pouvons modéliser des actions épistémiques plus complexes. Dans cette thèse, nous nous concentrons davantage sur une autre approche de la généralisation de la DEL, à savoir la quantification des actions épistémiques. *Arbitrary public announcement logic* (APAL, *logique des annonces publiques arbitraires*) [16] étend PAL en ajoutant des quantificateurs sur les annonces. Les quantificateurs dans APAL sont toujours traités comme des modalités plutôt que comme des quantificateurs dans la logique du premier ordre. Lorsque l'on se demande s'il est possible qu'après une annonce, un énoncé donné soit vrai, on peut le formaliser en disant qu'il existe une annonce véridique après laquelle cet énoncé est vrai.

Outre APAL, la généralisation de la DEL avec des quantificateurs sur les mises à jour a été systématiquement étudiée ces dernières années. *Arbitrary arrow update logic* (AAUL, *logique de mise à jour arbitraire des flèches*) [105] étend AUL avec des quantificateurs sur les événements de changement d'information impliquant des mises à jour de flèches. *Arbitrary action model logic* (AAML, *logique du modèle d'action arbitraire*) [47] ajoute à AML des quantificateurs sur les modèles d'action. Dans *Arbitrary arrow update model logic* (AAUML, *logique arbitraire des modèles de mise à jour des flèches*) [106], il y a des quantificateurs sur les modèles de mise à jour des flèches.

Une autre approche consiste à étudier les variantes d'APAL en restreignant la gamme des quantificateurs. Tout d'abord, il convient de noter que les quantificateurs dans APAL ne quantifient pas toutes les annonces dans le langage d'APAL. Pour éviter toute circularité dans la définition du langage APAL, les quantificateurs ne peuvent quantifier que des formules sans quantificateur. Sémantiquement, il a également été montré que la vérité d'une formule φ après une annonce avec quantificateurs n'implique pas qu'il existe une annonce sans quantificateurs après laquelle φ est vraie [62].

Group announcement logic (GAL, *logique d'annonce de groupe*) [1] quantifie les annonces faites simultanément par un groupe d'agents. *Coalition announcement logic* (CAL, *logique d'annonce de la coalition*) [5] quantifie les annonces faites conjointement par la contrepartie d'un groupe d'agents. Les contraintes du quantificateur dans GAL et CAL s'appliquent à des sous-ensembles d'agents. *Positive announcement logic* (APAL⁺, *logique des annonces positives*) [99] quantifie les formules positives dans lesquelles les opérateurs épistémiques ne sont jamais liés par des négations. *Boolean announcement logic* (BAPAL, *logique d'annonce booléenne*) [98] quantifie les formules booléennes (propositionnelles).

En suivant une approche similaire, nous proposerons plusieurs nouvelles variantes d'APAL dans le chapitre 3. Le premier type de variante se concentre sur la restriction des variables propositionnelles. Les variables propositionnelles sont des symboles logiques de base pour les langages formels basés sur la logique propositionnelle. Un ensemble de variables propositionnelles peut être infini. Nous pouvons définir un quantificateur comme quantifiant uniquement les annonces à l'aide d'un sous-ensemble de variables propositionnelles. Plus précisément, un sous-ensemble de variables propositionnelles peut être fini (puisque une annonce ne peut contenir qu'un nombre fini de variables), ou ne contenir que des variables propositionnelles qui apparaissent dans la portée du quantificateur (puisque, intuitivement, les variables qui n'apparaissent pas dans la portée n'ont pas d'importance). Le deuxième type de variante contient des quantificateurs sur des formules plus faibles ou plus fortes qu'une formule donnée φ . En d'autres termes, il existe des quantificateurs sur des formules impliquant φ ou impliquées

par φ . Ce type de variantes se concentre sur le degré d'information d'une annonce. Outre l'intuition de ces nouvelles variantes d'APAL, nous avons également une motivation technique pour les étudier.

- APAL⁺ et BAPAL sont incomparables en termes d'expressivité à APAL et plus expressifs que PAL [99, 98]. Ainsi, la contrainte sur les quantificateurs ne conduit pas à un affaiblissement de l'expressivité. Il serait intéressant de comparer l'expressivité entre les variantes que nous proposons et APAL.
- Le problème de satisfiabilité de APAL est connu pour être indécidable [16]. Cependant, il a été démontré que BAPAL est décidable [98] et APAL⁺ est conjecturé comme étant décidable. On peut raisonnablement s'attendre à ce que certaines des nouvelles variantes soient également décidables.

L'APAL est également lié à la notion de "connaissabilité". En gros, la connaissabilité concerne ce qui est connaissable, ou ce qui peut être connu. Le paradoxe de Fitch sur la connaissabilité est que si toutes les vérités sont connaissables, alors toutes les vérités sont connues. Supposons que p est une vérité inconnue, alors " p est une vérité inconnue" est vrai. Il s'ensuit que " p est une vérité inconnue" est connaissable. Lorsqu'on le sait, on sait que p est vrai, ce qui rend la phrase fausse. Par conséquent, la contradiction implique qu'il n'y a pas de vérité inconnue, c'est-à-dire que toute vérité est connue. Le paradoxe de Fitch indique que dans le cadre de la logique épistémique et modale, il est difficile de formaliser la notion de connaissabilité. Dans [93], van Benthem a suggéré d'utiliser la logique dynamique pour interpréter la connaissabilité. Selon ce point de vue, "un énoncé φ est connaissable" signifie que φ est connu après une certaine annonce. Elle peut être formalisée par le langage APAL comme "il y a une annonce après laquelle φ est connu". Une autre approche de la logique de la connaissabilité est basée sur la mise à jour restrictive des relations. Si nous considérons APAL comme une logique de connaissabilité, elle peut contenir trop d'éléments. Par exemple, [97] a proposé une logique de connaissabilité basée sur APAL sans annonces publiques.

Dans le chapitre 4, nous proposerons quelques logiques de connaissabilité avec un opérateur appelé "opérateur de connaissabilité". Sémantiquement, la condition de vérité de l'opérateur de connaissabilité est identique à la combinaison d'un quantificateur existentiel et d'un opérateur épistémique. En d'autres termes, dans cette logique de la connaissabilité, nous interprétons également " φ est connaissable" comme il existe une annonce après laquelle φ est connu. L'opérateur de connaissabilité sera utile pour étudier les propriétés de la connaissabilité. Ces logiques de connaissabilité sont différentes en omettant respectivement le quantificateur, l'opérateur épistémique et les annonces publiques. Elles peuvent également être considérées comme des variantes de APAL. L'opérateur combiné et la distinction entre ces logiques ne sont pas triviaux, en particulier sur le plan technique. L'expressivité, l'axiomatisation et la décidabilité peuvent changer considérablement.

Pour le troisième et dernier sujet de cette thèse, nous explorons un champ plus large de la logique de mise à jour que le contexte épistémique. Rappelons que dans AUL, les relations sont mises à jour par des mises à jour de flèches. Dans les modèles relationnels généraux, nous pouvons traiter une paire de relations comme une *transition* comme dans la logique temporelle. Une mise à jour de flèche (source, agent, cible) peut également être interprétée comme une *action* par un agent. L'agent effectue une action qui modifie ces paires de relations. Ainsi, les mises à jour de flèches sur les transitions peuvent être considérées comme des restrictions sur les actions. En d'autres termes, les mises à jour des flèches peuvent être appliquées pour

caractériser ce que les agents devraient faire, pourraient faire ou sont autorisés à faire. Ces notions sont appelées *normes, règles ou lois*. Les logiques et les systèmes qui raisonnent sur ces notions sont appelés *systèmes normatifs* ou *logique déontique* (voir [42, 70] pour une vue d’ensemble). Nous prendrons la *computation tree-like logic* (CTL, *logique des arbres de calcul*) [31] comme cadre des systèmes normatifs, dans lesquels des actions séquentielles sont entreprises sur les lignes temporelles ramifiées. La CTL possède des modalités temporelles qui peuvent s’étendre sur une profondeur modale arbitraire, ce qui permet de formaliser des énoncés tels que “ φ est *toujours* vrai” (φ est vrai à chaque état de chaque chemin) ou “ φ sera éventuellement vrai” (il existe un chemin sur lequel φ est vrai à un moment donné). Comme indiqué précédemment, la logique temporelle est également utilisée pour modéliser le changement d’information. Elle est dynamique dans la perspective de l’écoulement du temps, mais statique en termes de mise à jour du modèle. *Normative temporal logic* (NTL, *logique temporelle normative*) [3] est une logique de systèmes normatifs basée sur la CTL. Dans NTL, les normes sont interprétées comme des sous-ensembles de relations. Par conséquent, les normes donnent des contraintes sur les actions.

Dans le chapitre 5, nous étudierons *normative arrow update logic* (NAUL, *logique normative de mise à jour des flèches*) qui a été proposée pour la première fois dans [63]. Dans NAUL, les normes sont interprétées comme des mises à jour de flèches. La source d’une mise à jour de flèche détermine où les normes doivent être appliquées ; la cible vérifie quelles transitions sont conformes pour le groupe objectif d’agents. Ces paramètres permettent une interaction entre les formules et les normes (la source et la cible d’une mise à jour de flèche sont des formules). Plus important encore, les normes sont explicites. Les actions autorisées sont formellement définies, et nous pouvons même combiner des normes pour composer des normes plus complexes. Par exemple, nous pouvons combiner des normes de manière *additive* afin que les agents puissent choisir entre les normes ; nous pouvons également combiner des normes de manière *multiplicative* afin que les agents doivent suivre toutes ces normes ; enfin, il est possible de combiner des normes de manière *séquentielle* afin que les agents suivent des normes données dans l’ordre de priorité.

Certains résultats techniques de NAUL ont été montrés dans [63], comme l’expressivité relative de NAUL par rapport à CTL et AUL et la complexité du problème de vérification de modèle. Nous nous intéressons à la question de savoir si le problème de satisfiabilité de NAUL est décidable et à sa complexité.

Contributions

Dans cette thèse, nous proposons plusieurs nouvelles variantes de logiques épistémiques dynamiques et de systèmes normatifs existants. En général, ils peuvent être considérés comme des extensions de DEL. Techniquement, nous nous concentrerons sur les propriétés logiques, l’expressivité, l’axiomatisation et la décidabilité de ces logiques. La structure et les principales contributions de la thèse sont résumées ci-dessous :

- **Chapitre 2:** Nous introduisons certaines définitions formelles tout au long de cette thèse. Nous introduisons également des définitions formelles de certaines logiques de base mentionnées ci-dessus, telles que la logique épistémique, PAL, APAL, AUL et CTL.
- **Chapitre 3:** Nous proposons plusieurs nouvelles variantes d’APAL en restreignant l’éventail des quantificateurs. Nous introduisons d’abord la syntaxe et la sémantique

de ces logiques, puis nous montrons quelques formules valides en tant que propriétés logiques de ces logiques. L'effort principal de ce chapitre est de montrer les résultats d'expressivité relative entre ces nouvelles variantes, PAL et APAL. Nous donnons la hiérarchie d'expressivité de ces logiques. Nous montrons également des résultats d'axiomatisation et de décidabilité qui sont très similaires au cas d'APAL.

- **Chapitre 4:** Nous introduisons trois logiques différentes pour la connaissabilité. Elles possèdent toutes l'opérateur de connaissabilité, mais les quantificateurs, les annonces publiques et l'opérateur épistémique sont respectivement omis. Nous définissons les langages et la sémantique de ces logiques. Nous montrons certaines propriétés logiques comme la validité. Nous montrons l'expressivité relative de ces logiques. En particulier, la logique de la connaissabilité sans quantificateurs est plus expressive que PAL dans le cas multi-agent et tout aussi expressive que PAL dans le cas mono-agent. Nous proposons une axiomatisation de cette logique et montrons que le système est sain et complet. La logique de la connaissabilité sans annonces publiques ni opérateurs épistémiques est aussi expressive que la logique propositionnelle.
- **Chapitre 5:** Nous introduisons la logique normative de mise à jour des flèches. Nous proposons la définition formelle de son langage et de sa sémantique. La principale contribution de ce chapitre est de montrer que le problème de satisfiabilité de NAUL est décidable, par une méthode de tableau.

Le matériel qui constitue le corps principal de cette thèse est basé sur des collaborations avec différentes personnes : Le chapitre 3 est basé sur un travail joint avec Hans van Ditmarsch, Louwe Kuijer et Igor Sedlar [102]. Le chapitre 4 est basé sur un article travail joint Jie Fan, Hans van Ditmarsch et Louwe Kuijer [65]. Le chapitre 5 est un travail joint avec Hans van Ditmarsch et Louwe Kuijer.