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Constructive and asymptotic estimates for the solutions of some linear and nonlinear PDEs

Claudia Fonte Sánchez

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THÈSE DE DOCTORAT

DE L'UNIVERSITÉ PSL

Préparée à Université Paris Dauphine-PSL

**Constructive and asymptotic estimates for the solutions of
some linear and nonlinear PDEs**

Soutenue par

Claudia Fonte Sánchez

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Chapter 1

Introduction

This thesis is devoted to the study of asymptotic estimates for partial differential equations (PDE). These estimates can be in two directions:

1. The study of the limit when the number of individuals n included in a dynamic system becomes large.
2. The study of the asymptotic behavior when the time t becomes large.

In the first direction we go from the individual behavior of different agents at a microscopic level to the description of the collective behavior at a mesoscopic level through the probability density of the distribution of a population in certain states. In the second direction, we start in this work from the partial derivative model that describes the mesoscopic behavior and we study the existence of stationary or quasi-stationary solutions and their stability.

In this chapter, we introduce the different elements to be considered for our work, as well as the main results that will be presented in the following chapters.

1.1 Dynamical systems and mean-field limit

Dynamical system is the study of the particularity of time dependent evolving systems. The modern theory of it originated at the end of the 19th century with fundamental questions concerning the stability of the solar system. The questions raised in this context led to the emergence of a wide theoretical field with applications in physics, biology, astronomy, sociology and other areas.

We are interested in the evolution over time t of N variables

$$X^{(N)} = (X_1, \dots, X_N)$$

according to the rates of change

$$\frac{dX_1}{dt}, \dots, \frac{dX_N}{dt}.$$

Thus we obtain n differential equations

$$\frac{dX_i}{dt} = F_i(X^{(N)}). \tag{1.1.1}$$

State changes can be deterministic or random. When F_i referred to deterministic functions, it leads to a system of ordinary differential equations (ODE) as in the foundation work include on Birkhoff's 1927 seminal book [29]. On the other hand, the inclusion of random elements such as Gaussian noise or some stochastic jump process, leads us to consider stochastic differential equations (SDE) [126]. It is this second case in which we are interested here.

We talk then of N random variables X_i evolving in $E \subset \mathbb{R}^d$ (frequently $E = \mathbb{R}^d$ or $E = \mathbb{R}_+^d := \{x \in \mathbb{R}^d : x_i > 0\}$) and we define $P(E)$ the space of probability measures on E and $P(P(E))$ the space of probability measures on $P(E)$.

To go further, we consider that each individual is indistinguishable and follows a “typical behavior”. This is naturally described by exchangeable random variables (i.e. such that the probability laws are invariant under coordinates permutations) which are driven by a model equation with identical structure and parameters. This conditions are usually referred to as the “mean-field approximation”, we recommend F. Golse’s lecture [105].

We define the empirical measure $\mu^N \in P(E)$ of mean field approximation $X^{(N)}$ by

$$\mu^N = \frac{1}{N} \sum_{j=1}^N \delta_{X_j}.$$

We say that the system (1.1.1) has a mean-field limit if there exist $\mu \in P(E)$ such that μ^N converge to μ . The equation satisfied by μ is called then the mean-field limit equation.

The literature on mean-field approximation and mean-field limits is vast, see and references therein. One of the motivation of these works being the possibility of going from one measurement scale to another: from the microscopic level in which the trajectory of each individual can be separated to a mesoscopic level in which the behavior of a population is studied instead; this is from the trajectories of the evolution of different individuals to the density of the distribution of these in the states of interest. The applications of this type of result are varied, among them,

- The possibility of understanding how a change at the individual level affects the behavior of a population.
- Make estimates related to individual behavior from measurements on a larger scale (sometimes easier to obtain in practice).
- Model simplification: it is usually easier to carry out a qualitative study of the properties of a PDE with low dimensionality, than with a system of many particles.

On the other hand, we remark that the mean-field approximation assumptions do not imply the independence of the random variables X_i , however, under certain assumptions, asymptotic independence is obtained. This property is known as chaos propagation and is closed related to the mean-field limit. Indeed, let $G^N \in P(E)$ be the law of $X^{(n)}$, the mean-field approximation of the system, G_j^N the j th-marginal of G^N defined by

$$G_j^N = \int_{E^{N-j}} G^N dx_{j+1} \dots dx_N$$

then we say that there is chaos propagation if

$$\forall j \geq 1, \quad G_j^N \rightarrow \mu^{\otimes j} \text{ weakly in } P(E^j) \text{ as } N \rightarrow \infty.$$

It is know that the convergence of the empirical measure to the mean field limit is equivalent to the chaos propagation property above, see [121] and reference therein.

This brings us to another important application for statistics:

- The possibility of transferring properties of systems of independent variables to non-independent systems. For example, the law of large numbers or the central limit theorem.

Below we present two examples of dynamic systems describing the behavior of a neuronal population and discuss about their mean-field approximations.

1.1.1 Two dynamical systems on neuroscience

We present here two models, the FitzHugh-Nagumo (FhN) model and the time elapse model (TEM). Before describing each of them in details, we would like to comment on the common and divergent points between them.

In the first of these models (FhN) the activity of the neurons is modeled through the evolution of the voltage of each neuron, while in the second the variable of interest is the time elapsed between two spikes. To understand the biological interest of both of them, we would like to emphasize that information between one neuron and another is transmitted through discharges occurring at the synapse sites. Moreover, neuroscientists have come to the conclusion that most of the information can be encoded as a function of the time between spikes. When the variable being modeled is voltage, discharge times can be obtained by supposing that, if the voltage value exceeds a certain threshold, a discharge occurs. On the other hand, directly modeling the time between spikes is a relevant simplification.

In both cases when describing the behavior of each neuron, three elements are determinant

- i The behavior of a neuron when no discharge occurs.
- ii The interaction between neurons.
- iii Random noise (which can come from errors in transmission or small random external signals).

The FhN model is a simplification of the 4-dimensional Huxley-Hodgkin model (HH) that describes the physiological process of a discharge. In the HH model the one of the 4 variables correspond to the voltage and the other three to model the calcium, potassium and sodium channels involved in the discharge. We can say that the FhN model tries to make a more precise description of the way the spike is generated. Thus, in the absence of any other element, the model presents a description of the behavior of an isolated neuron that can (and has been) of interest on its own.

In the TEM model the spike generation process is not modeled.

About (ii) in the FhN model, electrical synaptic connections between neurons are considered, whereby one neuron has an inhibitory effect on another if its voltage is lower, and excitatory if the voltage is higher. We emphasize that this term influences during the whole time, the neurons are constantly interacting, which is less realistic since the neurons interact only during the discharge.

In the TEM model the behavior of the neuron is described through the firing rate at which the neurons spike, which is a function of all the spikes that have occurred up to the current time. In this sense, TEM offers a better description of the interaction, easily adaptable to the case of a population of inhibitory or excitatory neurons. In Raad *et. al.* work [213], they present a version that includes both inhibitory and excitatory populations. It also allows modeling effects such as delayed detection.

In the FhN model the randomness is introduced by the addition of a Gaussian noise modulated by a constant sigma, while in TEM it comes from considering a Hawkes process. Consequently, the first model is continuous in time while the second one has jumps every time a spike occurs. From the biological point of view, we should note that when a spike occurs the time between spikes is reset (to 0), which means that at the level of a neuron, the classical TEM model loses the memory of its previous activity, which makes it difficult to model phenomena such as adaptability and short term plasticity, which is the advantage of the version of the model presented in Chapter 7.

FitzHugh-Nagumo model

The FitzHugh-Nagumo (FhN) model was first introduced in the works of FitzHugh [85] and Nagumo [191] as a simplification of the Huxley-Hodgkin model (HH) that describes the evolution

of the membrane potential of a neuron [125]. The dynamics is based on two variables, a variable x which corresponds to the membrane potential and a recovery variable y , which satisfy the equations

$$\begin{aligned}\dot{x} &= F(x, y), \\ \dot{y} &= G(x, y),\end{aligned}$$

for $F(x, y) = x - x^3/3 - y + I$ and $G(x, y) = 1/c(x + a - by)$ with $a, I, c > 0$, $b \in \mathbb{R}$. On this model, I denotes the total membrane current and is a stimulus applied to the neuron, c determines the strength of damping while a and b govern two important characteristics of the oscillating solution, namely spike rate and spike duration [220]. With only these elements the system shows the most important properties of the 4-dimensional HH model such as refractoriness, insensitivity to further immediate stimulation after one discharge, and excitability, the ability to generate a large, rapid change of membrane voltage in response to a very small stimulus. Numerous works have been aimed at studying the ODE model and their properties, we reference the book of Rocsoreanu *et al.* [218] and the references therein.

More recently, specialists have been interested in the passage of the behavior of a neuron to neural networks, see for example Mischler *et al.* [180], Lucon and Poquet [160] and Baladron *et al.* [15]. When neurons interact through electrical synapses, it has been proposed that the evolution of N neurons satisfies

$$\begin{cases} dX_t^i = (F(X_t^i, Y_t^i) - \sum_{j=1}^N J_{ij}(X_t^i - X_t^j))dt + \sigma dB_t^i \\ dY_t^i = G(X_t^i, Y_t^i)dt \end{cases}, \quad (1.1.2)$$

where coefficients $J_{ij} > 0$ represent the effect of the interconnection between the neurons, and the term B_t^i refers, as usual, to independent Brownian motions.

We consider that the interactions are symmetric and identical for every pair of neurons in the network, which in particular implies that all neurons are connected. The strength of the interaction is measured by the parameter $J_{ij} = J$ that we re-parametrize as a function of a new parameter λ such that $J = \frac{\lambda}{N}$, where N is the number of neurons in the network. In this form, we obtain the mean field approximation,

$$\begin{cases} dX_t^i = (F(X_t^i, Y_t^i) - \frac{\lambda}{N} \sum_{j=1}^N (X_t^i - X_t^j))dt + \sigma dB_t^i \\ dY_t^i = G(X_t^i, Y_t^i)dt \end{cases}. \quad (1.1.3)$$

Note that this equation can be rewrite as

$$\begin{cases} dX_t^i = \left(F(X_t^i, Y_t^i) - \lambda(X_t^i - \int_{\mathbb{R}} x \mu_t^N(dx)) \right) dt + \sigma dB_t^i \\ dY_t^i = G(X_t^i, Y_t^i)dt, \end{cases}$$

which gives us an intuition about the form of the mean field limit equation

$$\begin{cases} dX_t = (F(X_t, Y_t) - \lambda(X_t - \mathbb{E}[X_t])) dt + \sigma dB_t \\ dY_t = G(X_t, Y_t)dt. \end{cases} \quad (1.1.4)$$

As a consequence the law μ_t of (X_t, Y_t) satisfied the mean-field limit equation,

$$\begin{cases} \partial_t \mu(x, y) = -\nabla((F, G)\mu(x, y)) - \nabla(\mu(x, y) f(x - \bar{x})\mu(d\bar{x}, d\bar{y})) + \frac{1}{2}\sigma^2 \partial_{xx} \mu(x, y) \\ \mu_{t=0} = \mu_0. \end{cases} \quad (1.1.5)$$

Indeed, in Baladron *et al.* work [15, theo. 4], the authors prove that the process $(X^{i,N}, Y^{i,N})$ the solution Eq. (1.1.3) for the neuron i converge in law to the process (X, Y) , the solution of the mean-field limit equation (1.1.4) when $N \rightarrow \infty$ for any $i \in \mathbb{Z}_+$. There they prove the mean-field limit convergence by a coupling argument. In chapter 10, we present a different approach to arrive at this result, that allows us to obtain in addition a control of the deviation of the probability measures through a concentration inequality in the spirit of Bernstein inequality.

Time elapse model

The second example we present here is the time elapse model (TEM) also known as “age dependent Hawkes process for neurons” [72] [60] [58] [212]. In effect, it is a Hawkes process described by the system of equations

$$\begin{cases} Z_t^i &= \int_0^t \int_0^\infty 1_{\{z \leq f(A_s^i, X_s)\}} \pi^i(ds, dz) \\ A_t^i &= A_0^i + t - \int_0^t A_s^i Z^i(ds) \\ X_t &= \frac{1}{N} \sum_{j=1}^n \int_0^t h(t-s, A_s^j) Z^j(ds), \end{cases}$$

with $(A_0, M_0) \sim u_0$ and where π is a Poisson random measure on $\mathbb{R}_+ \times \mathbb{R}_+$ with intensity $dsdz$ (see [223]). Here Z^i is a point process indicating the moment of the spike of the neuron i , A_t^i is the elapse time since the previous spikes of the neuron i at time t , and X_t gathers information on all network activity. We note that X_t is a function of all the spike until time t modulated by the kernel h that allows to consider some delay (e. g. exponential delay on t).

In Chevallier’s work [58, theo. 4.1], the author proves the mean-field convergence of the law of the stochastic process A^i toward ρ_t , the solution of the mean-field limit equation

$$\begin{cases} \partial_t \rho_t(a) + \partial_a \rho_t(a) = -f(a, \varepsilon x_t) \rho_t(a), \\ \rho_t(0) = \int_0^\infty f(a, \varepsilon x_t) \rho_t(a) da, \\ x_t = \int_0^t \int_0^\infty h(t-s) f(a, \varepsilon x_s) \rho_s(a) da ds, \\ \rho_0(a) = u_0(a), \end{cases} \quad (1.1.6)$$

also by a coupling argument.

The transport equation (1.1.6) has been widely used to study the properties of the behavior of neuronal population, in particular the convergence toward a stationary state when the interaction between neuron (here modulated by the influence of x_t in the firing rate f) is weak and the generation of oscillatory behavior when it is not. We can see for example the works of Pakdaman *et. al.* [201] for the study of this model and [48], [231], [232] for versions of it.

In Chapter 9, we present a different version of this model that deals with the lack of memory of the previous spike of a particular neuron in this model, to allow the description of effects of real neurons such that adaptability and short time plasticity. Then we make use of the mean field limit equation to study the long time behavior under weak interactions.

1.2 Partial differential equations models

As mentioned in the previous section, we refer to models at the mesoscopic level, when they describe the behavior of a given population by studying the probability density of finding an individual from that population in a given state. We have seen that such models can arrive from the mean field limit of a particle system when the number of individuals becomes large. In general, these models assume that we have a homogeneous population satisfying certain common rules and allow us to study the global behavior.

In the previous section we saw two first examples of these models. In the first case, from the FhN model with Brownian noise in only one of the variables, a kinetic Fokker-Planck type model is obtained for the probability density. In the second of a model based on a Poisson process, we arrive at a transport equation. Here are three more models that we will introduce in the following sections and to which we will dedicate chapters 6, 7 and 8 respectively.

- Diffusion models (which generally comes up considering a particle system with Gaussian noise in all variables i.e. considering a strictly positive sigma matrix affecting all states of particle i).

- A general version of the Kinetic Fokker-Planck model, considering in addition the case of a bounded domain with different boundary conditions.
- Mutation-Selection model, which is used in evolutionary theory.

1.2.1 Function spaces and weak solutions

In order to carry out a rigorous definition of our model (to ensure the existence and uniqueness of the PDE solution), we frequently work with the so-called “weak solutions” in a certain function space, even measures (distributions) spaces. Let us introduce these spaces and concepts first.

In general we will work in Banach lattice X , that is a Banach space $(X, \|\cdot\|)$ endowed with a compatible order relation \geq , which satisfies either $X = Y'$ or $X' = Y$ for another dual Banach lattice Y . On the applications, the Banach lattices X will be one the following examples (or small variation of it) :

- $X := C_0(E)$, the space of continuous functions which tend to 0 at infinity (when E is not a compact set) endowed with the uniform norm $\|\cdot\|_\infty$;
- $X := L^p(E) = L^p(E, \mathcal{E}, \mu)$, the Lebesgue space of functions associated to the Borel σ -algebra \mathcal{E} , a positive σ -finite measure μ and an exponent $p \in [1, \infty]$; in some cases, for a weight function m we will alternatively use the $L_m^p := \{f : fm \in L^p\}$ weighted space with the norm

$$\|f\|_{L_m^p} = \|fm\|_{L^p};$$

- $X := M^1(E) = (C_0(E))'$, the space of Radon measures defined as the dual space of $C_0(E)$.

In all the above examples, E denotes a σ -compact metric space, and we write $E = \cup E_R$, with $E_R \subset E_{R+1}$, E_R compact (generally $E \subset \mathbb{R}^d$). We always consider the test function space $X := C_c^\infty(E)$ of the infinity differentiable functions with compact support in E and occasional, the function space $X := H^1(E) = \{f \in L^2(E) : \partial f \in L^2(E)\}$, where ∂f is considered in the distribution sense i. e. $g = \partial f$ if

$$\int_E g\varphi = - \int_E f\partial\varphi; \quad \forall \varphi \in C_c^\infty(E).$$

For a bounded domain $\Omega \subset \mathbb{R}^d$ we could also consider $H_0^1(\Omega) = \{f \in H^1 : f|_{\partial\Omega} = 0\}$.

Let $\mathcal{L} : D(\mathcal{L}) \subset X \rightarrow X$ be a linear operator, we say that f is a weak solution of the equation

$$\partial_t f = \mathcal{L}f \tag{1.2.7}$$

if for all test function $\varphi \in C_c^\infty(E)$ we have

$$\langle f_T, \varphi_T \rangle = \langle f_0, \varphi_0 \rangle + \int_0^t \langle f_t, \mathcal{L}^* \varphi_t \rangle dt,$$

where $\langle \cdot, \cdot \rangle$ denote the dual application and \mathcal{L}^* the dual operator of \mathcal{L} (note that in any case the test function space is contained in the dual of X).

In what follows we present each model by defining the operator \mathcal{L} and in every case we consider the model represented by the evolution equation (1.2.7).

1.2.2 Diffusion models

A general elliptic operator in divergence form is given by

$$\mathcal{L}f := \partial_i(a_{ij}\partial_j f) + b_i\partial_i f + \partial_i(\beta_i f) + cf, \quad f \in H_0^1(\Omega), \tag{1.2.8}$$

where $\Omega \subset \mathbb{R}^d$ is a bounded domain (i.e. an open and connected set) or $\Omega = \mathbb{R}^d$, and for simplicity we always assume $d \geq 3$. We also always assume at least a boundedness and ellipticity condition on the (a_{ij}) matrix, namely

$$a_{ij} \in L^\infty(\Omega), \quad \exists \nu > 0, \quad \forall \xi \in \mathbb{R}^d, \quad a_{ij}\xi_i\xi_j \geq \nu|\xi|^2, \quad (1.2.9)$$

and some conditions on the coefficients b_i, β_j and c which will be described below depending on each particular case. We present four situations that we will consider in Chapter 6.

Diffusion with rough coefficients in a bounded domain

In the first case, we consider the general elliptic operator in divergence form (1.2.8) in the case of a bounded and smooth enough domain $\Omega \subset \mathbb{R}^d$ with general elliptic condition on a_{ij} as formulated above. We further assume

$$b_i, \beta_j \in L^r(\Omega), \quad c \in L^{r/2}(\Omega), \quad r > d, \quad (1.2.10)$$

as well as

$$c + \operatorname{div}\beta \leq c_0 \quad \text{and} \quad c - \operatorname{div}b \leq c_0, \quad (1.2.11)$$

for some $c_0 \in \mathbb{R}$.

Diffusion in \mathbb{R}^d with strong potential confinement

In the second case, we deal with an unbounded domain $\Omega = \mathbb{R}^d$ which brings us new difficulties. For a solution to be well defined, it must be ensured that the growth of the solutions in the region outside compact sets is controlled. One way to do this is to add a confinement condition. We consider then the elliptic operator in the form

$$\mathcal{L}f := \Delta f + b \cdot \nabla f + cf, \quad f \in H^1(\mathbb{R}^d), \quad (1.2.12)$$

with $b \in L^\infty_{\text{loc}}(\mathbb{R}^d)$, $c \in L^2_{\text{loc}}(\mathbb{R}^d)$ and a confinement condition that we impose through the properties of the *potential function* c , which is roughly speaking $c \rightarrow -\infty$ as $|x| \rightarrow \infty$. More precisely, we assume

$$\sigma_{i+} \in L^{d/2}, \quad \operatorname{meas}\{\sigma_i \geq K\} < \infty, \quad \forall K < 0, \quad (1.2.13)$$

with either $\sigma_1 := c + |b|^2/\kappa$ for some constant $\kappa \in (0, 4)$ or either $\sigma_2 := c + \operatorname{div}b/2$. For example, when we assume that

$$c \sim -|x|^\gamma \quad \text{and} \quad b \sim x|x|^{\beta-1} \quad \text{as} \quad |x| \rightarrow \infty,$$

the condition (1.2.13) for σ_1 is reached when $\gamma > \max(0, 2\beta)$ or when $\gamma = 2\beta > 0$ and some conditions on the constants involved in the behavior of the coefficients. In that context, the condition (1.2.13) for σ_2 is more general since it is reached when $\gamma > \max(0, \beta - 1)$ or when $\gamma = \beta - 1 > 0$ and some conditions on the constants involved in the behavior of the coefficients.

Diffusion in \mathbb{R}^d with weak potential confinement

In the third case, we return to the previous context of the elliptic operator (1.2.12) but now with a weaker confinement condition: we assume that c converges to a constant. With no loss of generality, we may assume $c \rightarrow 0$. More precisely, we consider the elliptic operator

$$\mathcal{L}f := \Delta f + b \cdot \nabla f + rcf, \quad (1.2.14)$$

now with $c \in C_0(\mathbb{R}^d)$, $b \in C_0(\mathbb{R}^d)$ and $r \in \mathbb{R}_+$ a parameter that we will make move.

Diffusion in \mathbb{R}^d with drift confinement

Finally, we consider the elliptic operator

$$\mathcal{L}f := \Delta f + b \cdot \nabla f + cf,$$

with a drift confinement as it is the case for the Fokker-Planck operator. More precisely, and for the sake of simplicity, we assume here

$$b = \nabla U, \quad U(x) = \frac{1}{\gamma} \langle x \rangle^\gamma, \quad \gamma > 0. \quad (1.2.15)$$

When $\gamma = 2$ and $c = x$, that operator corresponds to the classical harmonic Fokker-Planck operator which is known to be related to the standard Poincaré inequality and to the standard log-Sobolev inequality, see [11, 12, 233] or more recently [14, 138] and the references therein.

In the present situation, we impose that the contribution of c has lower influence at the infinity than the drift term b and we assume

$$c \in L_{loc}^\infty(\mathbb{R}^d), \quad \exists C_0, R_0 > 0, \quad \forall x \in B_{R_0}^c, \quad |c(x)| = o(|x|^{2(\gamma-1)}). \quad (1.2.16)$$

We further assume that

$$c \geq \operatorname{div} b \quad \text{when} \quad \gamma \in (0, 1]. \quad (1.2.17)$$

The action of the drift term will be revealed through the choice of a convenient "confining space". More precisely, for a weight function $m : \mathbb{R}^d \rightarrow [1, \infty)$, we will work in a weighted Lebesgue space.

1.2.3 Kinetic Fokker-Planck model

We present now a general version of the kinetic Fokker-Planck evolution equation, associated to the operator

$$\mathcal{L}f := -v \cdot \nabla_x f + \Delta_v f + b \cdot \nabla_v f + cf, \quad (1.2.18)$$

on functions $f : \mathcal{O} \rightarrow \mathbb{R}$, where $\mathcal{O} := \Omega \times \mathbb{R}^d$, $\Omega \subset \mathbb{R}^d$ is a domain, $b : \mathcal{O} \rightarrow \mathbb{R}^d$ is a given vector field and $c : \mathcal{O} \rightarrow \mathbb{R}$ is a given function. It can be interpreted as a simple model of particles dynamic which move and interacts with a thermal bath. Collisions are typically modeled by a Fokker-Planck operator $\Delta_v f + \operatorname{div}_v(vf)$ (when $b = v$ and $c = d$) which takes into account a thermal bath of (Gaussian) whitenoise.

We will consider the case when Ω is a bounded domain and the equation is complemented with a boundary condition. More precisely, we assume the classical balance between the values of the trace γf of f on the outgoing and incoming velocities subsets of the boundary

$$(\gamma_- f)(x, v) = \mathcal{R}_x(\gamma_+ f(x, \cdot))(v) \quad \text{on} \quad \Sigma_-, \quad (1.2.19)$$

where in this context we define $\Sigma_\pm^x := \{v \in \mathbb{R}^d; \pm v \cdot \nu_x > 0\}$ the sets of outgoing (Σ_+^x) and incoming (Σ_-^x) velocities at point $x \in \partial\Omega$, next the sets

$$\Sigma_\pm = \{(x, v) \in \Sigma; \pm \nu_x \cdot v > 0\} = \{(x, v); x \in \partial\Omega, v \in \Sigma_\pm^x\},$$

and finally the outgoing and incoming trace functions $\gamma_\pm f := \mathbf{1}_{\Sigma_\pm} \gamma f$. Here and in the sequel, ν_x denotes the unit normal outward vector field defined on the boundary set $\partial\Omega$. We similarly define the grazing velocity set

$$\Sigma_0 = \{(x, v) \in \Sigma; \nu_x \cdot v = 0\}.$$

The reflection operator \mathcal{R}_x is local in position, but can be local or nonlocal in the velocity variable, so that it writes

$$(\mathcal{R}_x g)(v) := \int_{\Sigma_+^x} r(x, v, v_*) g(v_*) v_* \cdot \nu_x dv_*,$$

for a reflection kernel $r : \partial\Omega \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$. Some classical general assumptions on r are

$$r \geq 0, \quad \mathcal{R}_x^* 1 = 1, \quad \mathcal{R}_x \mathcal{M} = \mathcal{M}, \quad (1.2.20)$$

for some positive function $\mathcal{M} = \mathcal{M}(v)$, see for instance [52, 53, 54]. The second (normalisation) condition corresponds to the fact that all the particles reaching the outgoing boundary are put back on the incoming boundary (no mass is lost) while the third (reciprocity) condition means (when \mathcal{M} is a Gaussian function) that the wall is in a local equilibrium state and is not influenced by the incoming particles. The normalization condition implies the local mass conservation

$$\int_{\Sigma_-^x} \mathcal{R}_x g |\nu \cdot v| dv = \int_{\Sigma_+^x} g \nu \cdot v dv, \quad (1.2.21)$$

while the three assumptions (1.2.20) together also imply

$$\int_{\Sigma_-^x} (\mathcal{R}_x g)^2 \mathcal{M}^{-1} |\nu \cdot v| dv \leq \int_{\Sigma_+^x} g^2 \mathcal{M}^{-1} \nu \cdot v dv, \quad (1.2.22)$$

In the sequel, we will rather consider the possibly position dependent Maxwell boundary condition operator

$$\mathcal{R}_x g = \alpha(x) \mathcal{D}_x g + \beta(x) \Gamma_x g, \quad (1.2.23)$$

where the accommodation coefficients $\alpha, \beta : \partial\Omega \rightarrow [0, 1]$ satisfy $\alpha(x) + \beta(x) =: \zeta(x) \leq 1$, Γ_x is the specular reflection operator

$$\Gamma_x(g(x, \cdot))(v) = g(x, \mathcal{V}_x v), \quad \mathcal{V}_x v = v - 2\nu(x)(\nu(x) \cdot v), \quad (1.2.24)$$

and \mathcal{D}_x is the diffusive operator

$$\mathcal{D}_x(g(x, \cdot))(v) = c_{\mathcal{M}} \mathcal{M}(v) \tilde{g}(x), \quad \tilde{g}(x) = \int_{\Sigma_+^x} g(x, w) \nu(x) \cdot w dw. \quad (1.2.25)$$

Here the constant $c_{\mathcal{M}} := (2\pi)^{1/2}$ is such that $c_{\mathcal{M}} \tilde{\mathcal{M}} = 1$ and \mathcal{M} stands for the standard Maxwellian

$$\mathcal{M}(v) := (2\pi)^{-d/2} \exp(-|v|^2/2), \quad (1.2.26)$$

or, more generally, $\mathcal{M} = \mathcal{M}(|v|) \geq 0$ is such that

$$\mathcal{D}_x^* 1 = 1, \quad \mathcal{D}_x \mathcal{M} = \mathcal{M}, \quad \langle v \rangle^\vartheta \mathcal{M} \in L^1(\mathbb{R}^d), \quad (1.2.27)$$

with $\vartheta \geq 1$ (that last condition is necessary in order that the second relation above makes sense). The boundary condition (1.2.23) corresponds to the *pure specular reflection* boundary condition when $\beta \equiv 1$ and it corresponds to the *pure diffusive* boundary condition when $\alpha \equiv 1$. When $\zeta \equiv 1$, the Maxwell boundary condition operator (1.2.23) satisfies (1.2.20). On the contrary, when $\zeta \neq 1$, the L^2 estimate (1.2.22) holds but not anymore the mass conservation (1.2.21). However, the following L^1 estimate

$$\int_{\Sigma_-^x} |\mathcal{R}_x g| |\nu \cdot v| dv \leq \zeta^* \int_{\Sigma_+^x} |g| \nu \cdot v dv \quad (1.2.28)$$

holds, with $0 \leq \sup \zeta \leq \zeta^* \leq 1$. Finally, the case $\zeta \equiv 0$ corresponds to the zero inflow problem.

1.2.4 Selection Mutation model

The last example that we present here is the mutation-selection evolution equation associated to the mutation-selection operator

$$\mathcal{L}f := J * f - W(x)f \quad (1.2.29)$$

defined on functions $f : \mathbb{R}^d \rightarrow \mathbb{R}$, $f = f(x)$, where J is a the mutation kernel, $*$ stands for the convolution operator and W is a confining potential. This equation appears in the modeling of genetic variability in evolutionary biology.

In this context, $f_t(x)$ represents the density of a population, at time t , of phenotypic trait x on the multi-dimensional phenotypic trait space \mathbb{R}^d . Here, the rate of change in f per generation is considered given by the following two components:

1. *Mutation.* Considering μ be the mutation rate per gene per generation, the effect of mutation on the present of a phenotypic trait x is usually modeling as $-\mu f(x) + \mu(\mathcal{J} * f)(x)$, where the first term gives the rate of loss of alleles having an average effect x by mutation to other alleles having a different effect and the second term gives the rate of gain from the mutation to alleles with average effect x from other alleles. Note that as we allow J to have a singular part, all this component can be include on the term $J * f$.
2. *Selection* The function $W(x)$ represents the fitness of the phenotype x and models the individual reproductive success, considering 0 as the optimum phenotypic value.

This model has been widely used in the literature; we refer, for example, to the works of M. Kimura [139], R. Lande [149], W.H. Fleming [86] and R. Bürger [42] as examples of biological applications.

1.3 Long time behavior

We now focus on the second asymptotic estimate that we address in this thesis, which will be the subject of most of the following chapters: the study of long time behavior for the evolution equations in the general form (1.2.7).

Roughly speaking, we are interested in understanding under what conditions there exists a profile f_∞ such that every solution f_t of Eq. (1.2.7) converges to f_∞ when t tends to infinity. When the equation (1.2.7) is mass conservative, this is

$$\langle f_t \rangle := \int_E f_t = \int_E f_0,$$

there is a wide range of literature that establish that under certain conditions, there is an unique positive stationary normalized solution f_∞ (with $\|f_\infty\|_X = 1$) such that we have the convergence

$$\|f_t - \langle f_0 \rangle f_\infty\|_X \leq \Theta(t) \|f_0 - f_\infty\|_X; \quad \forall f_0 \in X,$$

with a rate $\Theta(t)$.

If the problem is not mass conservative, we could not expect that f_t converge to a stationary solution anymore (as the mass growth or decrease depending on t). Yet, we first study the existence of a solution to the eigentriplet problem of finding $(\lambda_1, f_1, \phi_1) \in \mathbb{R} \times X \times Y$ such that

$$\mathcal{L}f_1 = \lambda_1 f_1, \quad f_1 \geq 0, \quad \|f_1\| = 1, \quad (1.3.30)$$

$$\mathcal{L}^* \phi_1 = \lambda_1 \phi_1, \quad \phi_1 \geq 0, \quad \|\phi_1\| = 1. \quad (1.3.31)$$

we can easily check that the quantity $\langle e^{-\lambda_1 t} f_t, \phi_1 \rangle$ is constant. Next, under a positive answer, one might expect that, granted the existence of such a triplet, a set of hypotheses could be arranged to obtain that

$$\|e^{-\lambda_1 t} f_t - \langle f, \phi_1 \rangle f_1\| \leq \Theta(t) \|f_0 - \langle f_0, \phi_1 \rangle f_1\|, \quad \forall t \geq 0, \quad \forall f \in X. \quad (1.3.32)$$

Such is the aim of the theoretical results presented in this thesis from chapter 2 to 5.

More in details, in this chapters, we revisit the Krein-Rutman theory for semigroups of positive operators in a Banach lattice framework and we provide some very general, efficient and handy results with constructive estimates about

1. the existence of a solution for the eigentriplet problem (1.3.30)-(1.3.31);
2. the geometry of the first eigenvalue;
3. the asymptotic stability of the first eigenvector with possible constructive rate of convergence.

This abstract theory is motivated and illustrated in chapters 6, 7 and 8 by the application of the general theory to the three models presented in the previous section.

The present work is motivated by new problems and ideas presented in the course on the Krein-Rutman theorem by P.-L. Lions at Collège de France [156] and by the recent contributions by Bansaye et al [17] and by Cañizo and Mischler [50] developing Harris techniques. Bringing and developing these ideas and techniques together with the more classical *spectral analysis* approach developed or synthesized in previous contributions by Krein and Rutman [142], by Arendt et al [7], by Mischler and Scher [177], by Bátkai et al [20] and many others, we are then able to significantly generalize and improve the Krein-Rutman theory for positive semigroups.

In what follows in this section, we present the results of the first four chapters and compare them with those existing in the literature.

1.3.1 The eigentriplet problem

We next consider a positive semigroup $S = S_{\mathcal{L}}$ on X , and we denote by \mathcal{L} its generator, by $D(\mathcal{L}) \subset X$ the domain of \mathcal{L} , by $\rho(\mathcal{L}) \subset \mathbb{C}$ the resolvent set of \mathcal{L} and by $\Sigma(\mathcal{L}) = \mathbb{C} \setminus \rho(\mathcal{L})$ the spectrum of \mathcal{L} . We also denote by S^* and \mathcal{L}^* the corresponding semigroup and generator on the dual space Y . We refer to the end of Section 2.1.1, for definition and related notations. We also remark that under the well-posedness of equation (1.2.7), the operator \mathcal{L} generate a semigroup associated with the solution of such equation, see for example the text book [82].

We now present some minimal condition for the existence and uniqueness of the triplet, in the spirit of what is established in the course of Lions [156].

First we ask,

(H1) $\exists \kappa_1 \in \mathbb{R}$ such that $\lambda - \mathcal{L}$ is invertible and $(\lambda - \mathcal{L})^{-1} : X_+ \rightarrow X_+$ for any $\lambda \geq \kappa_1$.

We then set

$$\mathcal{I} := \{\kappa \in \mathbb{R}; \lambda - \mathcal{L} \text{ is invertible, } (\lambda - \mathcal{L})^{-1} \geq 0 \text{ for any } \lambda \geq \kappa\}, \quad (1.3.33)$$

which is a non empty and non upper bounded interval due to **(H1)**. We finally set

$$\lambda_1 := \inf \mathcal{I} \in [-\infty, \kappa_1]. \quad (1.3.34)$$

(H2) Second, we ask $\exists \kappa_0 \in \mathbb{R}$ such that $\inf \mathcal{I} \geq \kappa_0$.

Under conditions **(H1)** and **(H2)**, there hold

$$\lambda_1 \in [\kappa_0, \kappa_2] \quad (1.3.35)$$

and

$$\exists \lambda_n \searrow \lambda_1, \exists \hat{f}_n \in D(\mathcal{L}) \cap X_+, \varepsilon_n := \lambda_n \hat{f}_n - \mathcal{L} \hat{f}_n \geq 0, \|\hat{f}_n\| = 1, \|\varepsilon_n\| \rightarrow 0. \quad (1.3.36)$$

Then we demand

(H3) for any sequence (\hat{f}_n) of X such that (2.1.20) holds, there exist $f_1 \in X_+ \setminus \{0\}$ and a subsequence $(\hat{f}_{n'})$ such that $\hat{f}_{n'} \rightharpoonup f_1$ for the weak convergence or the weak * convergence.

Under these condition for the primal problem, asking **(H3)** also for the dual, we can prove that the triplet exists. This is show in Theorem 2.1.20 in Chapter 2, where a prove is presented following the ideas of Lions' course.

The conditions **(H1)**-**(H2)**-**(H3)** are really weak, but somehow leaves room for the question of how to check it in applications. A possible more concrete conditions are

(H1*) \mathcal{L} is the generator of a positive semigroup $S = S_{\mathcal{L}}$;

(H2*) $\exists \kappa_0 \in \mathbb{R}, \exists \psi_0 \in Y_+ \setminus \{0\}$ such that $\mathcal{L}^* \psi_0 \geq \kappa_0 \psi_0$;

(H3*) there exist $\kappa_{\mathcal{B}} < \kappa_0$ and a splitting $\mathcal{L} = \mathcal{A} + \mathcal{B}$ such that $\mathcal{A} \in \mathcal{B}(X)$, $\mathcal{B} - \alpha$ is invertible and $\mathbb{R}_{\mathcal{B}}(\alpha) := (\alpha - \mathcal{B})^{-1}$ is uniformly bounded for any $\alpha \geq \kappa_{\mathcal{B}}$ and there is $N \geq 1$ such that

$$(\mathcal{R}_{\mathcal{B}}(\alpha)\mathcal{A})^N : \mathcal{C} \rightarrow \mathcal{C} \text{ is positive and uniformly bounded in } \alpha \geq \kappa_0 \quad (1.3.37)$$

$$(\mathcal{R}_{\mathcal{B}^*}(\alpha)\mathcal{A}^*)^N : \mathcal{C}^* \rightarrow \mathcal{C}^* \text{ is positive and uniformly bounded in } \alpha \geq \kappa_0 \quad (1.3.38)$$

for \mathcal{C} and \mathcal{C}^* are relatively sequentially compact for the corresponding weak topology in X and its dual Y and $0 \notin \bar{\mathcal{C}}$, where the closure is taken in the sense of the weak topology in both case.

Then we have the following result

Theorem 1.3.1. *Under **(H1*)**-**(H2*)**-**(H3*)** the first eigentriplet problem (1.3.30)-(1.3.31) admits a solution $(\lambda_1, f_1, \phi_1) \in \mathbb{R} \times X \times Y$.*

This result generalize the one of German school, see Bügner [43], Greiner *et. al.* [112], [111] [110] and Webb [244], that assume that \mathcal{C} have strong compactness and include the case of weak compactness in M^1 addressed by the probabilistic group [167, 168, 169, 170, 119, 50].

We also study the problem without spectral gap through a dynamic approach (see Theorem 2.2.4 in Chapter 2), which we believe is a novelty with respect to the literature.

For a more detailed discussion on the subject, please refer to the section 2.1.3 of chapter 2.

1.3.2 Geometry

A second issue is about the geometry of the spectrum and the principal eigenvalue problem. One way to characterize λ_1 , and that is related to the way we will solve the first eigentriplet problem, is

$$\lambda_1 := \sup\{\Re e \lambda; \lambda \in \Sigma(\mathcal{L})\} = \inf\{\kappa \in \mathbb{R}; \kappa' \in \rho(\mathcal{L}), \forall \kappa' \geq \kappa\}.$$

The boundary spectrum is then defined by

$$\Sigma_+(\mathcal{L}) := \Sigma(\mathcal{L}) \cap \bar{\Delta}_{\lambda_1} = \{\lambda \in \Sigma(\mathcal{L}); \Re e \lambda = \lambda_1\},$$

where Δ_α is the open half plan $\Delta_\alpha := \{z \in \mathbb{C}; \Re e z > \alpha\}$. We are next interested by describing $\Sigma_+(\mathcal{L})$ and to find conditions on \mathcal{L} so that at least we are in one of the following cases

1. λ_1 is algebraically and geometrically simple,
2. (better) $\Sigma_+(\mathcal{L})$ is trivial, namely

$$\Sigma_+(\mathcal{L}) \cap \Sigma_P(\mathcal{L}) = \{\lambda_1\}, \quad (1.3.39)$$

where Σ_P is the point spectrum set,

3. (even better) we have

$$\exists \kappa < \lambda_1; \quad \Sigma(\mathcal{L}) \cap \Delta_\kappa = \{\lambda_1\}. \quad (1.3.40)$$

In the second situation (1.3.40), we say that there is a spectral gap: a band separates the spectral value λ_1 to the remainder of the spectrum, while there is no spectral gap when (1.3.39) holds but (1.3.40) does not.

In Chapter 3 we will see that (1) is a consequence of the Strong Maximum Principle, see Theorem 3.3.11 while (2) is related with the “reverse Kato’s inequality condition”, see definition 4.3.10 and Theorem 4.4.14. The case (3) on the existence of spectral gap is addressed in Chapter 5 through some generalized Doeblin-Harris condition.

1.3.3 Stability

Three types of results are obtained closely linked to the geometry of the first eigenvalue. In the first case in which we have that λ_1 is algebraically and geometrically simple, we will see that under some weak compactness assumption we arrive to the conclusion

(CE1) for any $f \in X$, there holds

$$\frac{1}{T} \int_0^T \tilde{S}_t f dt \rightarrow \langle f, \phi_1 \rangle f_1, \quad \text{as } T \rightarrow \infty. \quad (1.3.41)$$

See Theorem 3.4.19.

If in addition we have the condition that assure that $\Sigma_+(\mathcal{L})$ is trivial, then we arrive to

(CE2) for any $f \in X$, there holds

$$\tilde{S}_t f \rightarrow \langle f, \phi_1 \rangle f_1, \quad \text{as } T \rightarrow \infty, \quad (1.3.42)$$

in a sense to be specified. See the Subsection 4.5 in Chapter 3.

Finally, when we can assure the existence of spectral gap, then under some condition to be specified later, we have

(CE3) there exist some constants $\kappa < 0$ and $C \geq 1$ such that for any $f \in X$, there holds

$$\|\tilde{S}(t)f - \langle f, \phi_1 \rangle f_1\| \leq \Theta(t) \|f - \langle f, \phi_1 \rangle f_1\|, \quad \forall t \geq 0, \forall f \in X, \quad (1.3.43)$$

with the decay rate function $\Theta(t) := C e^{\kappa t}$.

At the end of Chapter 3 we give some first result in this direction without quantitative estimation of κ , then we dedicate Chapter 5 to constructive ones.

The results in Chapter 3 are general well know. However, we rewrite them from different sources, placing them in the same framework and, in some cases, we provide new proofs that simplify the existing one. On the other hand, we consider the results in the Chapter 5 to be new.

1.4 Thesis plan and contributions

As mentioned above, from chapters 2 to 5 are dedicated to the theoretical development of the study of long time behavior. In succession, we deal with the existence of a solution to the eigenvalue triplet problem, then with the geometry and asymptotic behavior, without and with constructive quantification of convergence.

After that, we apply these results to the three examples introduced in section 1.2 in chapters 6, 7 and 8 respectively.

Contribution of chapter 6

Chapter 6 addressed the study of the diffusion models presented in section 1.2.2. For each of the introduced cases we prove the existence of the eigentriplet. In addition, we go through all the program until conclusion (CE3) with constructive estimates in the first case. In the case of the

second, third and fourth model, we prove all the condition until reaching conclusion **(CE2)**, of asymptotic convergence without spectral gap. Further to this, we consider that the method proposed in Chapter 5, which we exemplified in the first case, is adaptable to obtain estimates of constructive exponential convergence also in these cases.

We believe that these results are new and are an improvement over those previously reported in the literature.

Contribution of chapter 7

Chapter 7 is devoted to the Kinetic Fokker-Planck model, presented in section 1.2.3. In this case we will complete the whole program up to **(CE3)** without constructive rate. This result generalizes [152, Theorem 2.18] and [116, Theorem 6.8] for the zero inflow condition and [1, Theorems 1.6 & 1.7] for the torus case.

Contribution of chapter 8

We continue in Chapter 8 with the case of the Selection mutation model. Here we also complete the whole program up to **(CE3)** with constructive exponential rate. As far as we know, this result improves on the existing ones in the literature by weakening the conditions required of the kernel J . In particular, it generalizes the recent result obtained by Gabriel in [10], extending it to non-symmetric kernels.

Chapters 9 and 10 can be read independently from the previous 7 chapters, since they correspond to two separate works. Even if chapter 8 still deals with long time behavior.

Contribution of chapter 9

In chapter 9, we present a modified version of the TEM model introduced in section 1.1.1, which incorporates a memory variable. This new addition is significant from a biological point of view because it allows the model to describe phenomena such as adaptability and short term plasticity, as explained in Section 9.1.2. Next, we study the long time behavior of this model and prove that the conclusion **(CE3)** with constructive exponential rate is valid in the case of weak interaction between neurons. Two aspects should be commented in the context of the above results. On one hand, this model is mass-conservative, so the classical Doeblin-Harris theory as developed in [48] and [49] applies. This corresponds to the case where $(\lambda_1, \phi_1) = (0, 1)$. The proof will first establish that there is a stationary solution $f_1 = f_\infty$ and then an asymptotic result. On the other hand, we should emphasize that TEM model is nonlinear. To deal with this, we will divide the problem into two parts: first we will study a linearization of the problem, so we can apply the strategy described above and, in a second stage, we will use a perturbation argument to arrive at the conclusion for the nonlinear problem, when the nonlinear part is small.

Contribution of chapter 10

Finally, in Chapter 10, we study the particle system on the form (1.1.2) for general locally Lipschitz terms F and G and establish a Bernstein inequality-type concentration result that we use in two applications: First, to create an estimator of the parameters of the FhN model introduced in section 1.1.1; and then to extend the results of Della Maestra and Hoffmann in [73] to the non-globally Lipschitz coefficient case in position-velocity models; we create a nonparametric estimator of the measure μ_t . Finally, we conclude the chapter by performing an exploratory study of the feasibility of constructing a hypothesis test to check the existence of interactions between neurons through numerical simulations.

A more extensive explanation of the results of this part can be found in the introduction of chapter 10.

All chapters from 2 to 8 are the result of a work in collaboration with Stephane Mischler and Pierre Gabriel.

Chapter 2

Existence of solution of the eigentriplet problem

In this chapter, we address the existence of a solution to the first eigentriplet problem, as defined in the introduction. To do so, we take two approaches. In a first section we consider a stationary approach in which we look at the operator \mathcal{L} and its resolvent $\mathcal{R}_{\mathcal{L}}$, i.e. we look directly for a solution of the problem $\mathcal{L} - \lambda_1 = 0$. In a second section, we look at it from a dynamical point of view: we deal with the semigroup $S_{\mathcal{L}}(t)$ and the associated evolution problem proving the existence of the eigentriplet problem through a fixed point theorem.

2.1 Existence through a stationary problem approach

We start by presenting the basic material about the Banach lattice framework and conclude with a comparison with several previous works.

2.1.1 The Banach lattice framework

We start recalling the Banach lattice framework by stating (most of the time without proof) some well-known facts that one can find in reference monographs as [38, Chapitre II: Espaces de Riesz] or [222, 7, 16, 20].

Banach lattice. A real Banach lattice is a real Banach space $(X, \|\cdot\|)$ endowed with a partial order denoted by \geq (or \leq) such that the following holds:

(1) The set $X_+ := \{f \in X; f \geq 0\}$ is a nonempty convex cone (compatibility of the order with the vector space structure).

(2) For any $f \in X$, there exist some unique positive part $f_+ \in X_+$ and negative part $f_- \in X_+$ such that $f = f_+ - f_-$ which are minimal: $f = g - h$, $g, h \geq 0$ imply $g \geq f_+$ and $h \geq f_-$ (generation and properness of the positive cone). We set $|f| := f_+ + f_- \in X_+$ the absolute value of $f \in X$.

(3) For any $f, g \in X$, $|f| \leq |g|$ implies $\|f\| \leq \|g\|$ (compatibility of norm and order structures).

Under these assumptions, one can show that

- The convex cone X_+ is closed, pointed $X_+ \cap (-X_+) = \{0\}$ and generating $X = X_+ - X_+$.
- The lattice operations $f \mapsto f_+$, $f \mapsto f_-$ and $f \mapsto |f|$ are continuous (1-Lipschitz).
- The order intervals $\{h \in X; g \leq h \leq f\}$ are closed and bounded for any given $f, g \in X$, $f \geq g$.

It is worth emphasizing that one commonly defines the supremum and infimum operations by

$$f \vee g := g + (f - g)_+ \geq f, g, \quad f \wedge g := g - (g - f)_+ \leq f, g,$$

for any $f, g \in X$, and these operations can be used as an alternative way for defining a Banach lattice (the lattice structure referees indeed to these supremum and infimum operations). We may note the following elementary formulas

$$f_+ \wedge f_- = 0, \quad |||f||| = \|f\|, \quad \forall f \in X. \tag{2.1.1}$$

We write $f \perp g$ when $|f| \wedge |g| = 0$ or equivalently when $|f| + |g| = |f| \vee |g|$. In that case, we have

$$|f| + |g| = |f + g|.$$

Dual Banach lattice. On the dual space X' , we may naturally associate a dual order \geq (or \leq) by writing for $\varphi \in X'$

$$\varphi \geq 0 \text{ (or } \varphi \in X'_+) \text{ iff } \forall f \in X_+ \langle \varphi, f \rangle \geq 0.$$

For $\varphi \in X'$, there exist some unique $\varphi_{\pm} \in X'_+$ such that $\varphi = \varphi_+ - \varphi_-$ which also satisfy (and are defined by)

$$\forall f \in X_+, \quad \langle \varphi_{\pm}, f \rangle = \sup_{0 \leq g \leq f} \langle \pm \varphi, g \rangle.$$

One can show that the above conditions (1), (2) and (3) of a Banach lattice are fulfilled, and thus $X' = (X', \|\cdot\|_{X'}, \geq)$ is a Banach lattice. We observe that for any $f \in X_+$ there exists $f^* \in X'_+$ such that

$$\langle f^*, f \rangle = \|f\|^2 = \|f^*\|_{X'}^2, \tag{2.1.2}$$

as a classical corollary of the Hahn-Banach dominated extension theorem. Moreover, for any $f \in X$,

$$f \geq 0 \text{ iff } \langle \varphi, f \rangle \geq 0, \forall \varphi \in X'_+, \tag{2.1.3}$$

as an immediate application of the Hahn-Banach separation theorem. In other words, the restriction to X of the dual order in X'' associated to the order defined (by duality) on X' is nothing but the initial order, in particular the positive cone X'_+ is weakly $*$ closed.

The functional framework : The duality bracket. We consider X, Y such that $X = Y'$ with Y separable or $Y = X'$. We emphasize on the facts that

$$\text{for } f \in X : \quad f \in X_+ \text{ iff } \langle f, \varphi \rangle \geq 0, \forall \varphi \in Y_+, \tag{2.1.4}$$

$$\text{for } \varphi \in Y : \quad \varphi \in Y_+ \text{ iff } \langle f, \varphi \rangle \geq 0, \forall f \in X_+, \tag{2.1.5}$$

which are immediate consequences of (2.1.3) and of the definition of the dual order.

Examples. For the space $C_0(E)$, the order is defined by $f \geq 0$ iff $f(x) \geq 0$ for any $x \in E$. For a space $L^p(E, \mathcal{E}, \mu)$, $1 \leq p \leq \infty$, the order is defined by $f \geq 0$ iff $f(x) \geq 0$ for μ -a.e. $x \in E$. For the space $M^1(E)$, the order is defined by $f \geq 0$ iff in the Hahn decomposition $f = f_+ - f_-$, there holds $f_- = 0$, or equivalently, by duality: $f \geq 0$ iff $\langle f, \varphi \rangle \geq 0$ for any $\varphi \in C_0(E)$, $\varphi \geq 0$.

Because confinement will play a major role in our analysis, we will use some weighted version of the above space associated to a weight (continuous or Borel measurable) function $m : E \rightarrow (0, \infty)$ that we introduce now. We recall that E always denotes a σ -compact metric space, and we write $E = \cup E_R$, with $E_R \subset E_{R+1}$, E_R compact. In that context, we write $x_n \rightarrow \infty$ if for any $R \geq 1$ there exists n_R such that $x_n \notin E_R$ for any $n \geq n_R$.

- We denote by $C_{m,0}(E)$ the space

$$C_{m,0}(E) := \{ \varphi \in C(E); |\varphi(x)|/m(x) \rightarrow 0 \text{ as } x \rightarrow \infty \}$$

endowed with the norm $\|\varphi\|_{C_{m,0}} := \|\varphi/m\|_{C_0}$.

- We denote by $M_m^1(E) := (C_{m,0}(E))'$ the associated space of Radon measures.

- We denote by $L_m^p(E) = L_m^p(E, \mathcal{E}, \mu)$ the space

$$L_m^p(E) := \{f \in L_{loc}^1(E); \|f\|_{L_m^p} := \|fm\|_{L^p} < \infty\}.$$

It is worth emphasizing that $L_m^p(E, \mathcal{E}, \mu) = L^p(E, \mathcal{E}, m^p\mu)$ when $p \in [1, \infty)$.

Positive operator. We denote by $\mathcal{B}(X)$ the set of linear and bounded operators on X . We also denote by $\mathcal{K}(X)$ the subspace of compact operators. We say that a bounded operator $A \in \mathcal{B}(X)$ is positive, we write $A \geq 0$, if

$$Af \in X_+, \quad \forall f \in X_+.$$

We will also sometimes abuse notations by writing $A \in \mathcal{B}(X_+)$ for meaning that $A \geq 0$. For a positive operator $A \in \mathcal{B}(X)$, we have

$$|Af| \leq A|f|, \quad \forall f \in X, \quad \text{and} \quad \|A\| = \sup_{0 \leq f \in B_X} \|Af\|, \quad (2.1.6)$$

where B_X is the unit closed ball. More generally, we have

$$(Af) \vee (Ag) \leq A(f \vee g), \quad \forall f, g \in X. \quad (2.1.7)$$

For $A \in \mathcal{B}(X)$ and Y in duality with X , we uniquely define $A^* \in \mathcal{B}(Y)$ by

$$\langle Af, \phi \rangle = \langle f, A^*\phi \rangle, \quad \forall f \in X, \phi \in Y.$$

For $A \in \mathcal{B}(X)$ and $A^* \in \mathcal{B}(Y)$, there holds

$$A \geq 0 \quad \text{iff} \quad A^* \geq 0. \quad (2.1.8)$$

Let us present the elementary and classical but instructive proof of the direct implication, the reciprocal sense being similar. We assume thus $A \geq 0$. We take $\varphi \in Y_+$ and we define $\psi := A^*\varphi$. We then take $f \in X_+$ and we define $g := Af$, so that $g \geq 0$ by assumption. We compute

$$\langle \psi, f \rangle = \langle A^*\varphi, f \rangle = \langle \varphi, Af \rangle = \langle \varphi, g \rangle \geq 0.$$

Since $f \in X_+$ is arbitrary, we get $\psi \in Y_+$, and thus $A^* \geq 0$.

Semigroup, generator and spectrum. In this work, a semigroup $S = S(t) = (S_t)$ on X will always denote a semigroup of linear and bounded operators on a Banach lattice X which trajectories are

- either strongly continuous, namely, the mapping $t \mapsto S_t f$ is continuous for the norm of X for any fixed $f \in X$;

- either weakly * continuous, namely $X = Y'$ for some separable Banach lattice Y such that $\forall f \in X, \forall \phi \in Y, t \mapsto \langle S_t f, \phi \rangle_{X,Y}$ is continuous and $\forall t \geq 0, \forall \phi \in Y, f \mapsto \langle S_t f, \phi \rangle_{X,Y}$ is continuous. That is in particular the case when there exists a strongly continuous semigroup P on Y such that $S_t = P_t^*$ for any $t \geq 0$.

For a semigroup S , we denote by \mathcal{L} its generator and $D(\mathcal{L})$ the associated domain, and thus we sometimes write $S = S_{\mathcal{L}}$. We also denote the iterated domain defined recursively by $D(\mathcal{L}^k) := \{f \in D(\mathcal{L}^{k-1}), \mathcal{L}f \in D(\mathcal{L}^{k-1})\}$ for any $k \geq 2$ and $D(\mathcal{L}^\infty) := \bigcap_{k \geq 1} D(\mathcal{L}^k)$. We recall that $D(\mathcal{L})$ is dense in X and the graph of \mathcal{L} is closed in $X \times X$. We define the growth bound

$$\omega = \omega(S) := \limsup_{t \rightarrow \infty} \frac{1}{t} \log \|S(t)\| \in \mathbb{R} \cup \{-\infty\}, \quad (2.1.9)$$

so that

$$\forall \omega' > \omega, \quad \exists M \geq 1, \quad \|S(t)\|_{\mathcal{B}(X)} \leq M e^{\omega' t}, \quad \forall t \geq 0, \quad (2.1.10)$$

and ω is the infimum of $\omega' \in \mathbb{R}$ such that (2.1.10) holds. We say that S is a semigroup of contractions when S satisfies (2.1.10) with $M = 1$ and $\omega' = 0$.

The resolvent set $\rho(\mathcal{L})$ is the set of $z \in \mathbb{C}$ such that if $z - \mathcal{L} : D(\mathcal{L}) \rightarrow X$ is bijective and its inverse belongs to $\mathcal{B}(X)$. We define the resolvent operator by

$$\mathcal{R}(z) = \mathcal{R}_{\mathcal{L}}(z) := (z - \mathcal{L})^{-1}, \quad \forall z \in \rho(\mathcal{L}) \quad (2.1.11)$$

and the spectrum by $\Sigma(\mathcal{L}) := \mathbb{C} \setminus \rho(\mathcal{L})$. Denoting the half complex plane of abscissa $\alpha \in \mathbb{R}$ by

$$\Delta_{\alpha} := \{z \in \mathbb{C}; \Re(z) > \alpha\}, \quad (2.1.12)$$

we have $\rho(\mathcal{L}) \supset \Delta_{\omega}$ and for any $z \in \Delta_{\omega}$ there holds

$$\mathcal{R}(z) = \int_0^{\infty} S(t)e^{-zt} dt. \quad (2.1.13)$$

Positive semigroup. We say that a semigroup (S_t) on a Banach lattice X is positive if

$$S_t \geq 0, \quad \forall t \geq 0.$$

Lemma 2.1.1. *For a semigroup S on a Banach lattice X , there is equivalence between*

- (a) S is positive;
- (b) the associate resolvent operator \mathcal{R} is positive: $\mathcal{R}(\kappa) \geq 0$ for some (and thus any) $\kappa > \omega$.

It is immediate from Hille's identity (2.1.13) that (a) implies (b). The reciprocal implication comes from the relation $S(t) = \lim_{n \rightarrow \infty} [n/t \mathcal{R}(n/t)]^n$ at the foundation of the Hille-Yosida theory, see for instance [203, Theorem I.8.3].

2.1.2 Existence part of the Krein-Rutman theorem

From now-on in this section, we consider a Banach lattice X and an operator \mathcal{L} with dense domain and closed graph. Our goal is mainly to prove the existence part for the primal problem in the KR theorem, namely

$$\exists \lambda_1 \in \mathbb{R}, \exists f_1 \in X_+ \setminus \{0\}, \quad \mathcal{L}f_1 = \lambda_1 f_1. \quad (2.1.14)$$

We will also discuss the existence part for the dual problem at the end of the section.

We first assume

(H1) $\exists \kappa_1 \in \mathbb{R}$ such that $\lambda - \mathcal{L}$ is invertible and $(\lambda - \mathcal{L})^{-1} : X_+ \rightarrow X_+$ for any $\lambda \geq \kappa_1$.

We then set

$$\mathcal{I} := \{\kappa \in \mathbb{R}; \lambda - \mathcal{L} \text{ is invertible, } (\lambda - \mathcal{L})^{-1} \geq 0 \text{ for any } \lambda \geq \kappa\}, \quad (2.1.15)$$

which is a non empty and non upper bounded interval due to **(H1)**. We finally set

$$\lambda_1 := \inf \mathcal{I} \in [-\infty, \kappa_1]. \quad (2.1.16)$$

For the sake of completeness, we recall now some well known (which probably goes back to [111, 112]) and general facts about \mathcal{I} and λ_1 when \mathcal{L} is the generator of a positive semigroup. We also refer to [82, Section 1.b, Chapter VI] or [20, Chapter 12] and the references therein for more details.

Lemma 2.1.2. *When \mathcal{L} is the generator of a positive semigroup $S = S_{\mathcal{L}}$, then*

- (i) **(H1)** *automatically holds with any $\kappa_1 > \omega(S)$, so that $\lambda_1 \leq \omega(S)$;*
- (ii) $\Sigma(\mathcal{L}) \cap \Delta_{\lambda_1} = \emptyset$ *and the representation formula (2.1.13) holds true for any $z \in \Delta_{\lambda_1}$;*
- (iii) *it may happen that $\lambda_1 = -\infty$.*

The important property (b) is probably due to [112].

Proof of Lemma 2.1.2. The claim (i) is an immediate consequence of the representation formula (2.1.13) for any $\kappa_1 > \omega(S)$ and the positivity of $S(t)$ for any $t \geq 0$ (that is nothing but Lemma 2.1.1).

We prove (ii). Take $\lambda > \lambda_1$. From the classical identity

$$S(t)e^{-\lambda t} - I = (\mathcal{L} - \lambda) \int_0^t S(s)e^{-\lambda s} ds, \quad \forall t \geq 0,$$

and the positivity property of S , we have

$$0 \leq V(t) := \int_0^t S(s)e^{-\lambda s} ds = \mathcal{R}(\lambda) - \mathcal{R}(\lambda)S(t)e^{-\lambda t} \leq \mathcal{R}(\lambda),$$

for any $t \geq 0$. From that estimate, we get $\|V(t)\| \leq \|\mathcal{R}(\lambda)\|$. For any $z \in \Delta_{\lambda}$, an integration by part yields

$$\int_0^t e^{-zs} S(s) ds = e^{-(z-\lambda)t} V(t) + (z - \lambda) \int_0^t e^{-(z-\lambda)s} V(s) ds.$$

The estimate on V makes possible to pass to the limit $t \rightarrow \infty$ in the above identity, and we deduce

$$\mathcal{U}(z) := \int_0^{\infty} e^{-zs} S(s) ds = (z - \lambda) \int_0^{\infty} e^{-(z-\lambda)s} V(s) ds \in \mathcal{B}(X).$$

In that situation, one classically knows that $z \in \rho(\mathcal{L})$ and $(z - \mathcal{L})^{-1} = \mathcal{U}(z)$. We have thus established $\Sigma(\mathcal{L}) \cap \Delta_{\lambda} = \emptyset$ and we conclude the proof of (ii) by observing that (2.1.13) is then nothing but the above formula.

(iii) On $L^p(0, 1)$, $1 \leq p < \infty$, the translation semigroup defined for $a > 0$ by

$$S(t)f(x) := f(x + at)\mathbf{1}_{x+at \leq 1}, \quad \forall t \geq 0, x \in (0, 1),$$

is strongly continuous and positive. Since $S(t) \equiv 0$ for any $t \geq 1/a$, we have $\omega(S) = -\infty$, and thus $\lambda_1 = -\infty$ because of (i). \square

For further discussion, we give some probably classical results about the condition **(H1)** and some equivalent definitions of the set \mathcal{I} .

Lemma 2.1.3. *The operator \mathcal{L} satisfies **(H1)** if and only if the operator \mathcal{L}^* satisfies **(H1)**. Furthermore, under condition **(H1)** for \mathcal{L} (or \mathcal{L}^*), we have*

$$\mathcal{I} = \mathcal{I}_i, \quad \forall i = 2, 3, 4, \tag{2.1.17}$$

with

$$\begin{aligned} \mathcal{I}_2 &:= \{\kappa \in \mathbb{R}; \lambda - \mathcal{L} \text{ is invertible for any } \lambda \geq \kappa\}, \\ \mathcal{I}_3 &:= \{\kappa \in \mathbb{R}; \lambda - \mathcal{L}^* \text{ is invertible, } (\lambda - \mathcal{L}^*)^{-1} \geq 0 \text{ for any } \lambda \geq \kappa\}, \\ \mathcal{I}_4 &:= \{\kappa \in \mathbb{R}; \lambda - \mathcal{L}^* \text{ is invertible for any } \lambda \geq \kappa\}. \end{aligned}$$

Proof of Lemma 2.1.3. The equivalence of condition **(H1)** for the operators \mathcal{L} and \mathcal{L}^* is an immediate consequence of the identity $\rho(\mathcal{L}) = \rho(\mathcal{L}^*)$ (see for instance [136, Theorem III.6.22]) and the fact that $(\lambda - \mathcal{L})^{-1} : X_+ \rightarrow X_+$ iff $(\lambda - \mathcal{L}^*)^{-1} : Y_+ \rightarrow Y_+$ as recalled in (2.1.8). As a consequence, we have $\mathcal{I} = \mathcal{I}_3$ and $\mathcal{I}_2 = \mathcal{I}_4$.

We obviously have $\mathcal{I}_2 \subset \mathcal{I}$ and let us show the reverse inclusion. We denote $\mathcal{R} = \mathcal{R}_{\mathcal{L}}$. On the one hand, for any $z_0 \in \rho(\mathcal{L})$ and any $z \in \mathbb{C}$, $|z - z_0| < \|\mathcal{R}(z_0)\|^{-1}$, we know that

$$\mathcal{R}(z) = \mathcal{R}(z_0) \sum_{k=0}^{\infty} (z_0 - z)^k \mathcal{R}(z_0)^k, \quad (2.1.18)$$

which gives a proof of the fact that resolvent set $\rho(\mathcal{L})$ is open and that \mathcal{R} is an holomorphic function on $\rho(\mathcal{L})$. Formula (2.1.18) also ensures that for $\lambda_0, \lambda \in \mathbb{R}$, the condition $\mathcal{R}(\lambda_0) \geq 0$ implies that $\mathcal{R}(\lambda) \geq 0$ provided that $\lambda_0 - \lambda > 0$ is small enough and thus $\mathcal{R}(\lambda) \geq 0$ for any λ in the not upper bounded connected component of the set $\rho(\mathcal{L}) \cap \mathbb{R}$ thanks to a continuation argument. In particular, \mathcal{I} is an open set and $\mathcal{I} = \mathcal{I}_2$. \square

We next assume

(H2) $\exists \kappa_0 \in \mathbb{R}$ such that $\inf \mathcal{I} \geq \kappa_0$.

We point out several conditions under which **(H2)** is satisfied.

Lemma 2.1.4. *Condition **(H2)** holds under one of the four following conditions*

(i) $\exists \kappa_0 \in \mathbb{R}$, $\exists \psi_0 \in Y_+ \setminus \{0\}$ such that $\mathcal{L}^* \psi_0 \geq \kappa_0 \psi_0$, which means

$$\forall f \in D(\mathcal{L}) \cap X_+, \quad \langle \psi_0, (\kappa_0 - \mathcal{L})f \rangle \leq 0;$$

(ii) $\exists \kappa_0 \in \mathbb{R}$, $\exists f_0 \in X_+ \setminus \{0\}$ such that $\mathcal{L} f_0 \geq \kappa_0 f_0$, which means

$$\forall \psi \in D(\mathcal{L}^*) \cap Y_+, \quad \langle (\kappa_0 - \mathcal{L}^*)\psi, f_0 \rangle \leq 0;$$

(iii) \mathcal{L}^* is the generator of a positive semigroup $S^* = (S_t^*)$ and

$$\exists \kappa_0 \in \mathbb{R}, \exists \psi_0 \in Y_+ \setminus \{0\}, \exists T > 0 \text{ such that } S_T^* \psi_0 \geq e^{\kappa_0 T} \psi_0;$$

(iv) \mathcal{L} is the generator of a positive semigroup $S = (S_t)$ and

$$\exists \kappa_0 \in \mathbb{R}, \exists f_0 \in X_+ \setminus \{0\}, \exists T > 0 \text{ such that } S_T f_0 \geq e^{\kappa_0 T} f_0.$$

Proof of Lemma 2.1.4. In the three cases, we claim that $\kappa_0 \notin \mathcal{I}$ and thus $\inf \mathcal{I} \geq \kappa_0$. We argue by contradiction, assuming $\lambda_1 < \kappa_0$, so that $\kappa_0 \in \mathcal{I} = \mathcal{I}_i$ for any $i = 2, 3, 4$.

We assume **(i)**. For any $g \in X_+$, we define $f := (\kappa_0 - \mathcal{L})^{-1}g \in X_+$ and we compute

$$0 \leq \langle \psi_0, g \rangle = \langle \psi_0, (\kappa_0 - \mathcal{L})f \rangle \leq 0.$$

That implies $\langle \psi_0, g \rangle = 0$ for any $g \geq 0$, so that $\psi_0 = 0$ and a contradiction.

We assume **(ii)**. For any $\xi \in Y_+$, we define $\psi := (\kappa_0 - \mathcal{L}^*)^{-1}\xi \in Y_+$ and we compute

$$0 \leq \langle \xi, f_0 \rangle = \langle (\kappa_0 - \mathcal{L}^*)\psi, f_0 \rangle \leq 0.$$

That implies $\langle \xi, f_0 \rangle = 0$ for any $\xi \geq 0$, so that $f_0 = 0$ and a contradiction.

We assume first that **(iii)** holds for any $T > 0$. For any $f \in D(\mathcal{L}) \cap X_+ \setminus \{0\}$, we compute

$$\langle \psi_0, (\kappa_0 - \mathcal{L})f \rangle = -\frac{d}{dt} \langle \psi_0, e^{-\kappa_0 t} S_t f \rangle \leq 0,$$

which is precisely **(i)**. We assume now that **(iii)** holds. If $\kappa_0 \in \mathcal{I}$, for any $g \in X_+$, we may define $f = (\kappa_0 - \mathcal{L})^{-1}g \in X_+ \cap D(\mathcal{L})$ and from condition **(iii)**, we have

$$0 \leq \langle e^{-n\kappa_0 T} S_{nT} f - f, \psi_0 \rangle = \left\langle (\mathcal{L} - \kappa_0) \int_0^{nT} e^{-\kappa_0 t} S_t f dt, \psi_0 \right\rangle,$$

for any $n \in \mathbb{N}$. From the very definition of f , we also have

$$(\mathcal{L} - \kappa_0) \int_0^{nT} e^{-\kappa_0 t} S_t f dt = \int_0^{nT} e^{-\kappa_0 t} S_t (\mathcal{L} - \kappa_0) f dt = - \int_0^{nT} e^{-\kappa_0 t} S_t g dt \leq 0.$$

The two pieces of information together imply

$$\left\langle \int_0^{nT} e^{-\kappa_0 t} S_t g dt, \psi_0 \right\rangle = 0.$$

Passing to the limit $n \rightarrow \infty$ thanks to Lemma 2.1.2-**(ii)** and using (2.1.11)-(2.1.13), we obtain

$$0 = \left\langle \int_0^\infty e^{-\kappa_0 t} S_t g dt, \psi_0 \right\rangle = \langle f, \psi_0 \rangle = \langle g, (\kappa_0 - \mathcal{L}^*)^{-1} \psi_0 \rangle.$$

That implies $(\kappa_0 - \mathcal{L}^*)^{-1} \psi_0 = 0$ since g is arbitrary, what is not possible since $\psi_0 \neq 0$. The proof of **(H2)** under assumption **(iv)** is similar and thus skipped. \square

Remark 2.1.5. *In practice, we may build f_0 or ψ_0 through an explicit computation or use a barrier fonction and strong maximum principle techniques. We refer to Lemma 3.2.10 for a possible general result in that direction.*

Lemma 2.1.6. *Under conditions **(H1)** and **(H2)**, there hold*

$$\lambda_1 \in [\kappa_0, \kappa_1] \tag{2.1.19}$$

and

$$\exists \lambda_n \searrow \lambda_1, \exists \hat{f}_n \in D(\mathcal{L}) \cap X_+, \varepsilon_n := \lambda_n \hat{f}_n - \mathcal{L} \hat{f}_n \geq 0, \|\hat{f}_n\| = 1, \|\varepsilon_n\| \rightarrow 0. \tag{2.1.20}$$

Proof of Lemma 2.1.6. We obviously have $\lambda_1 \leq \kappa_1$ from assumption **(H1)** and $\lambda_1 \geq \kappa_0$ by assumption **(H2)**, so that (2.1.19) is proved.

Consider now a sequence $(\lambda_n)_{n \geq 2}$ such that $\lambda_n \searrow \lambda_1$ as $n \rightarrow \infty$. We eventually have $\|\mathcal{R}(\lambda_n)\| \rightarrow \infty$ as $n \rightarrow \infty$. On the contrary, we would have $\|\mathcal{R}(\lambda_{n'})\| \leq M$ for some subsequence $\lambda_{n'} \searrow \lambda_1$ and some constant $M > 0$. Because of (2.1.18) this implies that $(\lambda_{n'} - \varepsilon, \lambda_{n'}) \subset \mathcal{I}$ for any n' and some $\varepsilon > 0$, and this is in contradiction with the definition of λ_1 . The blow up $\|\mathcal{R}(\lambda_n)\| \rightarrow \infty$ means that

$$\exists f_n \in D(\mathcal{L}), \exists g_n \in X, \quad \mathcal{R}(\lambda_n)g_n = f_n, \|f_n\| \rightarrow \infty, \|g_n\| \leq 1.$$

By splitting $g_n = g_n^+ - g_n^-$, we get

$$f_n = \mathcal{R}(\lambda_n)g_n^+ - \mathcal{R}(\lambda_n)g_n^-$$

with

$$\|g_n^\pm\| \leq 1 \quad \text{and} \quad (\|\mathcal{R}(\lambda_n)g_n^+\| \rightarrow \infty \text{ or } \|\mathcal{R}(\lambda_n)g_n^-\| \rightarrow \infty).$$

Changing notations, we have thus

$$\exists f_n \geq 0, \exists g_n \geq 0, \quad \mathcal{R}(\lambda_n)g_n = f_n, \|f_n\| \rightarrow \infty, \|g_n\| \leq 1.$$

We get (2.1.20) by defining $\hat{f}_n := f_n/\|f_n\|$ and $\varepsilon_n := g_n/\|f_n\|$. \square

We learn a very similar proof in [156], from which our own proof is adapted. The same type of arguments can also be found in [20, proof of Theorem 12.15].

We finally assume that

(H3) for any sequence (\hat{f}_n) of X such that (2.1.20) holds, there exist $f_1 \in X_+ \setminus \{0\}$ and a subsequence $(\hat{f}_{n'})$ such that $\hat{f}_{n'} \rightharpoonup f_1$ for the weak convergence or the weak $*$ convergence.

We discuss several situations in which assumption **(H3)** holds. We start with a very classical framework due to Voigt [242], which is however quite restrictive since it is based on a strong compactness property assumed at the level of the associated semigroup of operators.

Lemma 2.1.7. *We assume that \mathcal{L} generates a positive semigroup S , that **(H2)** holds for a constant $\kappa_0 \in \mathbb{R}$ and that there exists $T > 0$ such that the splitting*

$$S_T = V_T + K_T, \quad (2.1.21)$$

holds with $\|V_T\|_{\mathcal{B}(X)} \leq e^{\kappa T}$, $\kappa < \kappa_0$, and $K_T \in \mathcal{K}(X)$. Then condition **(H3)** holds for the primal and the dual problems.

Proof of Lemma 2.1.7. The condition **(H1)** holds because of Lemma 2.1.2-(i). Let us then consider three sequences (λ_n) , (\hat{f}_n) and (ε_n) satisfying (2.1.20). Integrating along the rescaled flow, this yields

$$\begin{aligned} e^{-\lambda_n T} S_T \hat{f}_n - \hat{f}_n &= \int_0^T e^{-\lambda_n t} S_t (\mathcal{L} - \lambda_n) \hat{f}_n dt \\ &= - \int_0^T e^{-\lambda_n t} S_t \varepsilon_n dt =: \tilde{\varepsilon}_n, \end{aligned}$$

which also reads

$$V \hat{f}_n + K \hat{f}_n - e^{\lambda_n T} \hat{f}_n = e^{\lambda_n T} \tilde{\varepsilon}_n.$$

Since $e^{\lambda_n T} \geq e^{\kappa_0 T} > e^{\kappa T}$, the operator $e^{\lambda_n T} - V_T$ is invertible with inverse $Q(\lambda_n) := (e^{\lambda_n T} - V_T)^{-1}$ uniformly bounded and converging in $\mathcal{B}(X)$ to $Q(\lambda_1) = (e^{\lambda_1 T} - V_T)^{-1}$. We thus have

$$\hat{f}_n = w_n + v_n, \quad w_n := Q(\lambda_n) K_T \hat{f}_n, \quad v_n := -Q(\lambda_n) e^{\lambda_n T} \tilde{\varepsilon}_n,$$

with $\|v_n\|_X \rightarrow 0$ and (w_n) relatively compact in X , since there exist a subsequence (\hat{f}_{n_k}) and $g \in X$ such that $K_T \hat{f}_{n_k} \rightarrow g$ and next

$$w_{n_k} - Q(\lambda_1)g = (Q(\lambda_{n_k}) - Q(\lambda_1))K_T \hat{f}_{n_k} + Q(\lambda_1)(K_T \hat{f}_{n_k} - g) \rightarrow 0.$$

We deduce that $\hat{f}_{n_k} \rightarrow f_1$ strongly in X . Because of the positivity and normalized properties of \hat{f}_n , we get $f_1 \in X_+$, $\|f_1\|_X = 1$, and we conclude that **(H3)** holds for the primal problem

Observing that the dual semigroup S^* satisfies $S_T^* = V_T^* + K_T^*$ with $\|V_T^*\|_{\mathcal{B}(Y)} \leq e^{\kappa T}$ and $K_T^* \in \mathcal{K}(Y)$, the same proof implies that **(H3)** holds for the dual problem. \square

In the six next lemmas, we will assume that **(H1)**-**(H2)** holds associated to some constants $\kappa_i \in \mathbb{R}$, $\kappa_0 < \kappa_1$, and we always make the following splitting structure hypothesis

(HS1) there exists a splitting $\mathcal{L} = \mathcal{A} + \mathcal{B}$ such that $\mathcal{B} - \alpha$ is invertible for any $\alpha \geq \kappa_0$ and

$$\mathcal{V}(\alpha) := \sum_{i=0}^{N-1} (\mathcal{R}_{\mathcal{B}}(\alpha) \mathcal{A})^i \mathcal{R}_{\mathcal{B}}(\alpha), \quad \mathcal{W}(\alpha) := (\mathcal{R}_{\mathcal{B}}(\alpha) \mathcal{A})^N, \quad (2.1.22)$$

are bounded in $\mathcal{B}(X)$ uniformly with respect to $\alpha \geq \kappa_0$ and for some $N \geq 1$, where we recall that $\mathcal{R}_{\mathcal{B}}(\alpha) := (\alpha - \mathcal{B})^{-1}$.

We first present a result also based on a strong compactness property which is assumed to hold however at the level of the resolvent operator. We will be able to use that result in most of the applications.

Lemma 2.1.8. (1) We assume **(H1)**-**(H2)**-**(HS1)** and there exists $N \geq 1$ such that

$$\mathcal{W}(\alpha) \text{ is strongly compact locally uniformly on } \alpha \geq \kappa_0, \quad (2.1.23)$$

in the sense that if $\alpha_n \rightarrow \alpha$, $\alpha_n \geq \kappa_0$, and (g_n) is a bounded sequence in X , then there exist $f \in X$ and a subsequence (g_{n_k}) such that $\mathcal{W}(\alpha_{n_k})g_{n_k} \rightarrow f$ strongly in X . Then condition **(H3)** holds.

(2) We assume **(H1)**-**(H2)** and **(HS1)** where $\mathcal{R}_{\mathcal{B}}(\alpha)$ is bounded uniformly in $\alpha \geq \kappa_0$, $\mathcal{A} \in \mathcal{B}(X)$ and $\mathcal{W}(\alpha) \in \mathcal{K}(X)$ for any fixed $\alpha \geq \kappa_0$ and some $N \geq 1$. Then condition **(H3)** holds both for the primal and the dual problems.

Remark 2.1.9. (1) The property (2.1.23) holds if we assume $\mathcal{W}(\alpha) : X \rightarrow \mathcal{X}_1$ is bounded uniformly in $\alpha \geq \kappa_0$ and $\mathcal{X}_1 \subset X$ with strong compact embedding.

(2) The property (2.1.23) holds if we assume **(H1)**-**(H2)**-**(HS1)** together with the facts that $\mathcal{R}_{\mathcal{B}}(\alpha)$ and $\mathcal{R}_{\mathcal{B}}(\alpha)\mathcal{A}$ are bounded uniformly in $\alpha \geq \kappa_0$ and $\mathcal{W}(\alpha) \in \mathcal{K}(X)$ for any fixed $\alpha \geq \kappa_0$. Consider indeed $\alpha_n \rightarrow \alpha$, $\alpha_n \geq \kappa_0$, and (g_n) a bounded sequence in X . On the one hand, there exist $f \in X$ and a subsequence (g_{n_k}) such that $\mathcal{W}(\alpha)g_{n_k} \rightarrow f$ strongly in X , because $\mathcal{W}(\alpha) \in \mathcal{K}(X)$. On the other hand, using the resolvent identity $\mathcal{R}_{\mathcal{B}}(\lambda) - \mathcal{R}_{\mathcal{B}}(\mu) = (\mu - \lambda)\mathcal{R}_{\mathcal{B}}(\lambda)\mathcal{R}_{\mathcal{B}}(\mu)$, we have

$$\mathcal{W}(\alpha) - \mathcal{W}(\alpha_n) = (\alpha_n - \alpha) \sum_{j=1}^N (\mathcal{R}_{\mathcal{B}}(\alpha)\mathcal{A})^{N-j} \mathcal{R}_{\mathcal{B}}(\alpha) (\mathcal{R}_{\mathcal{B}}(\alpha_n)\mathcal{A})^j \rightarrow 0,$$

so that $\mathcal{W}(\alpha_{n_k})g_{n_k} \rightarrow f$ strongly in X , and (2.1.23) holds true.

Proof of Lemma 2.1.8. We first assume (1). Taking advantage of the splitting structure **(HS1)**, we write equation (2.1.20) as

$$(\lambda_n - \mathcal{B})\hat{f}_n = \mathcal{A}\hat{f}_n + \varepsilon_n,$$

or equivalently

$$\hat{f}_n = \mathcal{R}_{\mathcal{B}}(\lambda_n)\mathcal{A}\hat{f}_n + \mathcal{R}_{\mathcal{B}}(\lambda_n)\varepsilon_n.$$

Iterating that last identity and using the notations (2.1.22), we get

$$\hat{f}_n = w_n + v_n, \quad w_n := \mathcal{W}(\lambda_n)\hat{f}_n, \quad v_n := \mathcal{V}(\lambda_n)\varepsilon_n. \quad (2.1.24)$$

We observe that (w_n) is strongly relatively compact from (2.1.23) and $\|\hat{f}_n\|_X = 1$, so that there exist a subsequence (w_{n_k}) and $f_1 \in X$ such that $w_{n_k} \rightarrow f_1$ strongly in X . Since $v_n \rightarrow 0$ strongly in X , we deduce that $\hat{f}_{n_k} \rightarrow f_1$ strongly in X . We conclude that condition **(H3)** holds as in the proof of Lemma 2.1.7.

We next assume (2). As observed in Remark 2.1.9-(2), the property (2.1.23) holds and thus also the condition **(H3)** for the primal problem. We claim that the same locally uniform strong compactness property (2.1.23) holds for the dual problem at order $N + 1$ and thus condition **(H3)** holds for the dual problem. We may indeed use Remark 2.1.9-(2) since then $\mathcal{R}_{\mathcal{B}^*}(\alpha)$ and $\mathcal{A}^*\mathcal{R}_{\mathcal{B}^*}(\alpha)$ are bounded uniformly in $\alpha \geq \kappa_0$ and

$$(\mathcal{A}^*\mathcal{R}_{\mathcal{B}^*}(\alpha))^{N+1} = \mathcal{A}^*\mathcal{W}(\alpha)^*\mathcal{R}_{\mathcal{B}^*}(\alpha) \in \mathcal{K}(Y), \quad \forall \alpha \geq \kappa_0,$$

as a product of two bounded operator with a compact operator. \square

As we see now, strong compactness is not really necessary.

Lemma 2.1.10. We assume **(H1)**-**(H2)**-**(HS1)** and there exists $N \geq 1$ such that

$$\mathcal{W}(\alpha) : X \rightarrow \mathcal{X}_1 \subset X \text{ is positive and uniformly bounded in } \alpha \geq \kappa_0$$

and, denoting $\mathcal{X}_0 := X$, we assume that for any $R_1 \geq R_0 > 0$ the set

$$\mathcal{C} = \mathcal{C}_{R_0, R_1} := \{g \in X_+; \|g\|_{\mathcal{X}_0} \geq R_0, \|g\|_{\mathcal{X}_1} \leq R_1\} \quad (2.1.25)$$

is relatively sequentially compact for the weak topology $\sigma(X, Y)$ and $0 \notin \bar{\mathcal{C}}$, where the closure is taken in the sense of the weak topology $\sigma(X, Y)$. Then condition **(H3)** holds.

Remark 2.1.11. When $\mathcal{X}_1 \subset \mathcal{X}_0$ with strongly compact embedding the above set \mathcal{C} clearly satisfies the required conditions. In particular, Lemma 2.1.10 generalizes the result stated in Remark 2.1.9-(1).

Proof of Lemma 2.1.10. We go back to the proof of Lemma 2.1.8 and we start with (2.1.24). We recall that $\|\hat{f}_n\|_{\mathcal{X}_0} = 1$ and $\|v_n\|_{\mathcal{X}_0} \rightarrow 0$ from (2.1.20) and that $w_n \geq 0$ because $\mathcal{W}(\lambda_n)$ is a positive operator. We also observe that

$$\|w_n\|_{\mathcal{X}_1} \leq C_{\mathcal{W}} \|\hat{f}_n\|_{\mathcal{X}_0} = C_{\mathcal{W}}$$

and

$$\|w_n\|_{\mathcal{X}_0} \geq 1 - \|v_n\|_{\mathcal{X}_0} \geq 1/2$$

for any $n \geq n_*$, with $n_* \geq 1$ large enough, so that $w_n \in \mathcal{C} := \mathcal{C}_{1/2, C_{\mathcal{W}}}$ for any $n \geq n_*$. By the compactness properties of \mathcal{C} , there exist a subsequence (w_{n_k}) and $f_1 \in X_+ \setminus \{0\}$ such that $w_{n_k} \rightharpoonup f_1$ weakly $\sigma(X, Y)$. Since $v_n \rightarrow 0$ strongly in X , we deduce that $\hat{f}_{n_k} \rightharpoonup f_1$ weakly $\sigma(X, Y)$ and that ends the proof of **(H3)**. \square

We present a typical concrete application of the preceding result.

Lemma 2.1.12. We assume $X = L^p(E, \mathcal{E}, \mu)$, $p \in [1, \infty)$, **(H1)**-**(H2)**-**(HS1)** with $\mathcal{A} \geq 0$, $\mathcal{R}_{\mathcal{B}}(\alpha) \geq 0$ for $\alpha \geq \kappa_0$, and there exists $N \geq 1$ such that

$$\mathcal{W}(\alpha) : X \rightarrow \mathcal{X}_1 \text{ is uniformly bounded in } \alpha \geq \kappa_0, \quad (2.1.26)$$

for a subspace $\mathcal{X}_1 \subset X$ such that $\{g^p; g \geq 0, \|g\|_{\mathcal{X}_1} \leq R_1\}$ is a weakly compact subset of $L^1(E)$ for any $R_1 > 0$. Then condition **(H3)** holds.

Remark 2.1.13. (1) A typical example in the above statement is $\mathcal{X}_1 := L^q \cap L^p_m$ for some exponent $q > p$ and some weight function $m : E \rightarrow [1, \infty)$ such that $m(x) \rightarrow \infty$ as $x \rightarrow \infty$.

(2) The same result holds under the condition that if (u_n) is a nonnegative and bounded sequence in L^p then the nonnegative sequence $w_n := \mathcal{W}(\lambda_n)u_n$ is such that w_n^p is weakly compact in L^1 .

Proof of Lemma 2.1.12. For $0 < R_0 < R_1$, we define \mathcal{C} by (2.1.25) with $\mathcal{X}_0 := L^p$. From the weak compactness property made on \mathcal{X}_1 , we observe that

$$\alpha(R) := \sup_{g \in \mathcal{C}} \|g \mathbf{1}_{E_R^c}\|_{L^p} \rightarrow 0, \text{ as } R \rightarrow \infty,$$

and

$$\beta(M) := \sup_{g \in \mathcal{C}} \|g \mathbf{1}_{g \geq M}\|_{L^p} \rightarrow 0, \text{ as } M \rightarrow \infty.$$

For $g \in \mathcal{C}$, we may then write

$$R_0 \leq \|g\|_{L^p} \leq \|g \wedge M \mathbf{1}_{E_R}\|_{L^p} + \|g \mathbf{1}_{E_R^c}\|_{L^p} + \|g \mathbf{1}_{g \geq M}\|_{L^p}$$

and thus

$$M^{1-1/p} \|g \mathbf{1}_{E_R}\|_{L^1}^{1/p} \geq \|g \wedge M \mathbf{1}_{E_R}\|_{L^p} \geq R_0 - \alpha(R) - \beta(M) \geq R_0/2,$$

for some $R, M > 0$ large enough. On the one hand, from the reflexivity of L^p or the Dunford-Pettis theorem, the set \mathcal{C} is relatively sequentially compact for the weak topology $\sigma(L^p, L^{p'})$. On the other hand, because $\mathbf{1}_{E_R} \in L^{p'}$ the last estimate implies that any element $g^* \in \overline{\mathcal{C}}$, where the closure is taken in the sense of the weak topology $\sigma(L^p, L^{p'})$, satisfies

$$\langle g^*, \mathbf{1}_{E_R} \rangle = \|g^* \mathbf{1}_{E_R}\|_{L^1} \geq M^{p-1} (R_0/2)^p > 0,$$

and in particular $0 \notin \overline{\mathcal{C}}$. We deduce that (H3) holds as a consequence of Lemma 2.1.10. \square

We present a second kind of result where some weak compactness is required.

Lemma 2.1.14. *We assume (H1)-(H2)-(HS1) and there exists $N \geq 1$ such that*

$$\mathcal{W}(\alpha) : \mathcal{X}_0 \rightarrow X \subset \mathcal{X}_0 \text{ is uniformly bounded in } \alpha \geq \kappa_0 \quad (2.1.27)$$

and, denoting $\mathcal{X}_1 := X$, the set \mathcal{C} defined by (2.1.25) satisfies the same properties as the ones stated in Lemma 2.1.10. Then condition (H3) holds.

Remark 2.1.15. *If we replace the norm $\|\cdot\|_{\mathcal{X}_0}$ by a seminorm $\|f\|_{\mathcal{X}_0} := \langle |f|, \varphi_0 \rangle$, $\varphi_0 \in Y_+$, and we define \mathcal{C} accordingly by (2.1.25), and if we assume that $X = Y'$ with Y separable, then \mathcal{C} satisfies the same compactness properties as required in the statement of Lemma 2.1.10. If we further assume that (2.1.27) holds where \mathcal{X}_0 is endowed with the above seminorm, we may repeat the proof below in order to obtain that (H3) holds in that situation (see also Lemma 2.1.18 and its proof for a slightly generalized situation).*

Proof of Lemma 2.1.14. We start here again with (2.1.24). We have

$$1 = \|\hat{f}_n\|_{\mathcal{X}_1} \leq C_{\mathcal{W}} \|\hat{f}_n\|_{\mathcal{X}_0} + \|v_n\|_{\mathcal{X}_1},$$

and thus

$$\|\hat{f}_n\|_{\mathcal{X}_0} \geq C_{\mathcal{W}}^{-1} (1 - \|v_n\|_{\mathcal{X}_1}) \geq (2C_{\mathcal{W}})^{-1}$$

for any $n \geq n_*$, with $n_* \geq 1$ large enough, so that $\hat{f}_n \in \mathcal{C} := \mathcal{C}_{(2C_{\mathcal{W}})^{-1}, 1}$, for $n \geq n_*$. By the compactness properties of \mathcal{C} , there exist a subsequence (\hat{f}_{n_k}) and $f_1 \in X_+ \setminus \{0\}$ such that $\hat{f}_{n_k} \rightharpoonup f_1$ weakly $\sigma(X, Y)$. \square

We present a strong variant of Lemma 2.1.12 which is also a concrete consequence of Lemma 2.1.10 and Lemma 2.1.14.

Corollary 2.1.16. *We assume (H1)-(H2)-(HS1) in $X = L_{m_0}^{p_0}$, $1 \leq p_0 < \infty$, together with the facts that $\mathcal{R}_{\mathcal{B}}(\alpha)$ is positive and bounded in $\mathcal{B}(L_{m_0}^{p_0})$ and $\mathcal{B}(L_{m_1}^{p_1})$ uniformly in $\alpha \geq \kappa_0$, $0 \leq \mathcal{A} \in \mathcal{B}(L_{m_0}^{p_0})$ and $(\mathcal{R}_{\mathcal{B}}(\alpha)\mathcal{A})^N$ is bounded in $\mathcal{B}(L_{m_0}^{p_0}, L_{m_1}^{p_1})$ uniformly in $\alpha \geq \kappa_0$ for some $N \geq 1$, with $p_1 > p_0$ and m_1 such that $m_0/m_1 \in L^\vartheta$, $1/\vartheta := 1/p_0 - 1/p_1$. Then condition (H3) holds for both the primal and the dual problems.*

Proof of Corollary 2.1.16. On the one hand, we have

$$\begin{aligned} \mathcal{R}_{\mathcal{B}}(\alpha) + \cdots + (\mathcal{R}_{\mathcal{B}}(\alpha)\mathcal{A})^{N-1}\mathcal{R}_{\mathcal{B}}(\alpha) &\text{ is bounded in } \mathcal{B}(X) \text{ uniformly in } \alpha \geq \kappa_0, \\ \mathcal{W}(\alpha) := (\mathcal{R}_{\mathcal{B}}(\alpha)\mathcal{A})^N &\text{ is bounded in } \mathcal{B}(X, \mathcal{X}_1) \text{ uniformly in } \alpha \geq \kappa_0, \end{aligned}$$

with $\mathcal{X}_1 := L_{m_1}^{p_1} \subset X$ and thus $\{(gm_0)^{p_0}; g \geq 0, \|g\|_{\mathcal{X}_1} \leq R_1\}$ is a weakly compact subset of $L^1(E)$ for any $R_1 > 0$. Condition (H3) holds for the direct problem thanks to Lemma 2.1.12.

On the other hand, we set $Y := X' = L_{\nu_0}^{q_0}$, $q_0 := p'_0$, $\nu_0 := m_0^{-1}$, and we first observe that

$$\mathcal{R}_{\mathcal{B}^*}(\alpha) + \cdots + (\mathcal{R}_{\mathcal{B}^*}(\alpha)\mathcal{A}^*)^{N-1}\mathcal{R}_{\mathcal{B}^*}(\alpha) \text{ is bounded in } \mathcal{B}(Y) \text{ uniformly in } \alpha \geq \kappa_0.$$

We next observe that

$$(\mathcal{A}^* \mathcal{R}_{\mathcal{B}^*}(\alpha))^{N+1} = \mathcal{A}^* \mathcal{W}(\alpha) \mathcal{R}_{\mathcal{B}^*}(\alpha) \text{ is bounded in } \mathcal{B}(\mathcal{Y}_0, Y) \text{ uniformly in } \alpha \geq \kappa_0,$$

with $\mathcal{Y}_0 := L_{\nu_1}^{q_1}$, $q_1 := p_1'$, $\nu_1 := m_1^{-1}$. Because $\{(g\nu_1)^{q_1}; g \geq 0, \|g\|_Y \leq R_1\}$ is a weakly compact subset of $L^1(E)$ for any $R_1 > 0$, we have from the proof of Lemma 2.1.12 that the set \mathcal{C} defined by (2.1.25) for the norms of \mathcal{Y}_0 and $\mathcal{Y}_1 := Y$ satisfies the weak compactness property required in the statement of Lemma 2.1.10. We may thus apply Lemma 2.1.14 and we deduce that condition **(H3)** holds for the dual problem. \square

Another concrete consequence of Lemma 2.1.10 and Lemma 2.1.14 is the following.

Lemma 2.1.17. *We assume $X = M_{m_i}^1(E)$ for a continuous weight function m_i on E , $i = 0$ or $i = 1$, **(H1)**-**(H2)**-**(HS1)** and there exists $N \geq 1$ such that $(\mathcal{R}_{\mathcal{B}}(\alpha)\mathcal{A})^N : M_{m_0}^1(E) \rightarrow M_{m_1}^1(E)$ uniformly in $\alpha \geq \kappa_0$ for another continuous weight function m_{1-i} on E such that $m_1(x)/m_0(x) \rightarrow \infty$ as $x \rightarrow \infty$. We additionally assume that $\mathcal{A} \geq 0$ and $\mathcal{R}_{\mathcal{B}}(\alpha) \geq 0$ for $\alpha \geq \kappa_0$ when $i = 0$. Then condition **(H3)** holds.*

Proof of Lemma 2.1.17. We define $\mathcal{X}_i := M_{m_i}^1(E)$ and we consider the set \mathcal{C} defined by (2.1.25) which is clearly compact for the weak $*$ $\sigma(M_{m_1}^1, C_{m_1,0})$ topology. When $X = M_{m_0}^1$, the result follows from Lemma 2.1.10 while when $X = M_{m_1}^1$, the result is a consequence of Lemma 2.1.14. \square

We may slightly improve the preceding results by considering a more abstract framework and a somehow more general boundedness condition.

Lemma 2.1.18. *We assume $X = Y'$, Y separable, **(H1)**-**(H2)**-**(HS1)** and there exist $N \geq 1$, $\gamma \in [0, 1)$ and $\varphi \in Y_+ \setminus \{0\}$ such that for any $\alpha \geq \kappa_0$, there holds*

$$\|\mathcal{W}(\alpha)f\|_X \leq \gamma \|f\|_X + \langle f, \varphi \rangle_{X,Y}, \quad (2.1.28)$$

for all $f \in X_+$, or there holds

$$\|\mathcal{W}(\alpha)f\|_X \leq \gamma \|f\|_X + \langle \mathcal{W}(\alpha)f, \varphi \rangle_{X,Y}, \quad (2.1.29)$$

for all $f \in X_+$. Then condition **(H3)** holds true, and the limit f_1 satisfies $\langle f_1, \varphi \rangle_{X,Y} \geq 1 - \gamma > 0$.

The case $X = M_{m_1}^1(E)$ in Lemma 2.1.17 corresponds here to the first situation where (2.1.28) holds with $X := M_{m_1}^1(E)$, $\gamma := 0$, $Y := C_{m_0,0}(E)$ and $\varphi := m_0/m_1$.

Proof of Lemma 2.1.18. Starting with (2.1.24) and using (2.1.28), we have

$$\begin{aligned} \|\hat{f}_n\|_X &\leq \|\mathcal{W}(\lambda_n)\hat{f}_n\|_X + \|\mathcal{V}(\lambda_n)\varepsilon_n\|_X \\ &\leq \gamma \|\hat{f}_n\|_X + \langle \hat{f}_n, \varphi \rangle_{X,Y} + \|v_n\|_X, \end{aligned}$$

so that

$$\langle \hat{f}_n, \varphi \rangle_{X,Y} \geq 1 - \gamma - \|v_n\|_X.$$

By compactness, there are $f_1 \geq 0$ and a subsequence $(\hat{f}_{n'})$ such that $\hat{f}_{n'} \rightharpoonup f_1$ weak $*$ $\sigma(X, Y)$. Passing to the limit as $n' \rightarrow \infty$ in the above estimate, we find

$$\langle f_1, \varphi \rangle_{X,Y} = \lim_{n' \rightarrow \infty} \langle \hat{f}_{n'}, \varphi \rangle_{X,Y} \geq 1 - \gamma, \quad (2.1.30)$$

and in particular $f_1 \neq 0$.

Under the assumption (2.1.29), modifying slightly the previous argument, we have

$$\|\hat{f}_n\|_X \leq \gamma \|\hat{f}_n\|_X + \langle w_n, \varphi \rangle_{X,Y} + \|v_n\|_X,$$

which, together with

$$\langle \hat{f}_n, \varphi \rangle_{X,Y} = \langle w_n, \varphi \rangle_{X,Y} + \langle v_n, \varphi \rangle_{X,Y},$$

implies

$$\langle \hat{f}_n, \varphi \rangle_{X,Y} \geq 1 - \gamma - \|v_n\|_X + \langle v_n, \varphi \rangle_{X,Y}.$$

By compactness again, there are $f_1 \geq 0$ and a subsequence $(\hat{f}_{n'})$ such that $\hat{f}_{n'} \rightharpoonup f_1$ weak $*$ $\sigma(X, Y)$, and passing to the limit $n' \rightarrow \infty$ in the above estimate, we conclude again to (2.1.30). \square

Let us comment on Lemma 2.1.18 and in particular the condition (2.1.29).

In the case when $X = L^\infty(E, \mathcal{E}, \mu) = (L^1(E, \mathcal{E}, \mu))'$, we can relate condition (2.1.29) to the assumption that there exist $f_0 \in X_+$ and $\psi_0 \in Y_+ \setminus \{0\}$ such that

$$\|S_{\mathcal{L}}(t)f_0\|_X \leq \langle S_{\mathcal{L}}(t)f_0, \psi_0 \rangle, \quad \forall t \geq 0. \quad (2.1.31)$$

This last condition is reminiscent from conditions that appear in probabilistic inspired methods for the ergodicity of semigroups, see [55, Assumption (A2)] and [18, Assumption (A4)], which essentially build on [71, Condition \mathcal{Z}]. Assume indeed (2.1.31), let $\eta > \kappa_1 - \kappa_0 > 0$ and consider the trivial decomposition $\mathcal{L} = \mathcal{A} + \mathcal{B} = \eta + (\mathcal{L} - \eta)$. Then set $\kappa_{\mathcal{B}} := \kappa_1 - \eta < \kappa_0$, so that for any $\alpha > \kappa_{\mathcal{B}}$, $\mathcal{B} - \alpha = \mathcal{L} - (\eta + \alpha)$ is invertible since $\eta + \alpha > \eta + \kappa_{\mathcal{B}} = \kappa_1$. We thus have for any $\alpha > \kappa_{\mathcal{B}}$

$$\mathcal{W}(\alpha) := \eta(\alpha - \mathcal{B})^{-1} = \eta \int_0^\infty e^{-(\eta+\alpha)t} S_{\mathcal{L}}(t) dt$$

and (2.1.31) then ensures that

$$\|\mathcal{W}(\alpha)f_0\|_X \leq \langle \mathcal{W}(\alpha)f_0, \psi_0 \rangle.$$

We recover (2.1.29) with $\gamma = 0$ and the difference that f_0 is fixed here.

As a Corollary of Lemma 2.1.17 or Lemma 2.1.18 and anticipating on the material of part 2.2, we present now a situation very classical in stochastic processes theory.

Corollary 2.1.19. *We consider a positive semigroup $S = S_{\mathcal{L}}$ defined on a Radon space $X = M_\psi^1(E)$ for some weight functions ψ on E , in particular **(H1)** holds. We also assume that **(H2)** holds for some $\kappa_0 \in \mathbb{R}$. We finally assume the Lyapunov condition*

$$\mathcal{L}^*\psi \leq \kappa_{\mathcal{B}}\psi + M\chi, \quad (2.1.32)$$

with $\kappa_{\mathcal{B}} < \kappa_0$, $M \geq 0$ and $\chi \in C_{\psi,0}(E)$, $0 \leq \chi \leq \psi$. Then condition **(H3)** holds true.

Let us emphasize that we may assume some regularity on ψ by considering $\psi \in D(\mathcal{L}^*)$ so that (2.1.32) makes sense in X or just understand (2.1.32) in the weak sense:

$$\langle \mathcal{L}f, \psi \rangle \leq \kappa_{\mathcal{B}}\langle f, \psi \rangle + M\langle f, \chi \rangle, \quad \forall f \in D(\mathcal{L}) \cap X_+.$$

Proof of Corollary 2.1.19. We introduce the splitting $\mathcal{L} = \mathcal{A} + \mathcal{B}$ where \mathcal{A} is the multiplier (and thus bounded) operator $\mathcal{A} := M\chi/\psi$. As a bounded perturbation of \mathcal{L} , the operator \mathcal{B} is the generator of a semigroup $S_{\mathcal{B}}$. Defining $\tilde{S}_t := S_{\mathcal{L}}(t)e^{-Mt} \geq 0$ and $\mathcal{A}^c := M(1 - \chi/\psi) \geq 0$, we have the Duhamel formula

$$S_{\mathcal{B}} = \tilde{S} + \tilde{S}\mathcal{A}^c * S_{\mathcal{B}}$$

and iterating infinitely many times, we deduce the Dyson-Philips formula

$$S_{\mathcal{B}} = \sum_{k=0}^{\infty} (\tilde{S}\mathcal{A}^c)^{(*k)} * \tilde{S}.$$

That implies that $S_{\mathcal{B}} \geq 0$ as a combination of positive operators. Alternatively, from the very definition of \mathcal{B} , we have $\kappa - \mathcal{B} \leq (M + \kappa) - \mathcal{L}$ for any $\kappa \in \mathbb{R}$. Choosing $\kappa > \max(\omega(S_{\mathcal{L}}), \omega(S_{\mathcal{B}}))$ and using the direct implication in Lemma 2.1.1, we have $\mathcal{R}_{\mathcal{B}}(\kappa) \geq \mathcal{R}_{\mathcal{L}}(M + \kappa) \geq 0$. Using the reciprocal implication in Lemma 2.1.1, we obtain again that $S_{\mathcal{B}} \geq 0$.

Now, for $0 \leq f_0 \in D(\mathcal{B})$ and setting $f_t := S_{\mathcal{B}}(t)f_0$, we may compute

$$\frac{d}{dt} \langle f_t, \psi \rangle = \langle \mathcal{B}f_t, \psi \rangle \leq \kappa_{\mathcal{B}} \langle f_t, \psi \rangle,$$

so that

$$\|S_{\mathcal{B}}(t)f_0\|_{M_{\psi}^1} \leq e^{\kappa_{\mathcal{B}}t} \|f_0\|_{M_{\psi}^1}.$$

Using (2.1.13) we immediately and classically deduce

$$\|\mathcal{R}_{\mathcal{B}}(\alpha)\|_{\mathcal{B}(M_{\psi}^1)} \leq \frac{1}{\alpha - \kappa_{\mathcal{B}}}, \quad \forall \alpha > \kappa_{\mathcal{B}},$$

so that $\mathcal{R}_{\mathcal{B}}(\alpha)$ is bounded in $\mathcal{B}(M_{\psi}^1)$ and $\mathcal{R}_{\mathcal{B}}(\alpha)\mathcal{A}$ is bounded in $\mathcal{B}(M_{\chi}^1, M_{\psi}^1)$ uniformly for $\alpha \geq \kappa_0$. We apply Lemma 2.1.17 or Lemma 2.1.18 (with $N = 1$, $\gamma = 0$ and $\varphi = \frac{M}{\alpha - \kappa_{\mathcal{B}}}\chi$) in order to conclude. \square

In the proof of Corollary 2.1.19, we may alternatively use the trivial splitting $\mathcal{L} = \tilde{\mathcal{A}} + \tilde{\mathcal{B}} = r + (\mathcal{L} - r)$ for arbitrary $r > 0$, and reformulate the Lyapunov condition

$$(\alpha - \tilde{\mathcal{B}}^*)\psi \geq (\alpha + r - \kappa_{\mathcal{B}})\psi - M\chi,$$

for any $\alpha \geq \kappa_0$. Observing that $\tilde{\mathcal{W}}(\alpha) := \tilde{\mathcal{A}}\mathcal{R}_{\tilde{\mathcal{B}}}(\alpha) = r(\alpha - \tilde{\mathcal{B}})^{-1}$, we deduce

$$\tilde{\mathcal{W}}^*(\alpha)\psi \leq \frac{r}{r + \alpha - \kappa_{\mathcal{B}}}\psi + \frac{M}{r + \alpha - \kappa_{\mathcal{B}}}\tilde{\mathcal{W}}^*(\alpha)\chi.$$

We equivalently have

$$\|\tilde{\mathcal{W}}(\alpha)f\|_{M_{\psi}^1} \leq \gamma \|f\|_{M_{\psi}^1} + \langle \mathcal{W}(\alpha)f, \tilde{\chi} \rangle,$$

uniformly for any $\alpha \geq \kappa_0$, with $\gamma := \frac{r}{r + \kappa_0 - \kappa_{\mathcal{B}}} < 1$ and $\varphi := \frac{M}{r + \kappa_0 - \kappa_{\mathcal{B}}}\chi$, which is nothing but condition (2.1.29).

We finally come to the existence of a solution to the first eigenvalue problem and the first eigentriplet problem.

Theorem 2.1.20. *Under conditions (H1)-(H2)-(H3), the first eigenvalue problem (2.1.14) has a solution (λ_1, f_1) with λ_1 satisfying (2.1.19). When furthermore (H3) holds for the dual problem, then the first eigentriplet problem (1.3.30)-(1.3.31) admits a solution $(\lambda_1, f_1, \phi_1) \in \mathbb{R} \times X \times Y$.*

Theorem 2.1.20 generalizes some known versions of the existence part of the Krein-Rutman Theorem where either \mathcal{L} is assumed additionally to be the generator of a semigroup or to have strongly power compact resolvent or even where some additional conditions are made on the positive cone X_+ . As mentioned in the introduction, some possible references for these previous results are Krein-Rutman [142], Greiner in [109, Cor 1.2] and in [7, C-IV Thm 2.1] and Webb [244, Prop. 2.5], see also [43, Thm. 2], [177, Thm. 5.3], [156], [18, Thm 2.1], the textbook [20, Theorem 12.15] and the references therein.

Proof of Theorem 2.1.20. We first assume (H1)-(H2)-(H3). Because of Lemma 2.1.6, there exists a sequence (\hat{f}_n) of X such that (2.1.20) holds, and in particular

$$\langle \lambda_n \hat{f}_n, \phi \rangle - \langle \hat{f}_n, \mathcal{L}^* \phi \rangle = \langle \varepsilon_n, \phi \rangle, \tag{2.1.33}$$

for any $\phi \in D(\mathcal{L}^*)$. Because of condition (H3), we may pass to the limit $n' \rightarrow \infty$ in equation (2.1.33) and we deduce that (λ_1, f_1) satisfies (2.1.14) and (2.1.19).

We now additionally assume that **(H3)** holds for the dual problem. As recalled during the proof of Lemma 2.1.3 and by definition of λ_1 , we have $(\lambda_1, +\infty) \subset \rho(\mathcal{L}) = \rho(\mathcal{L}^*)$ and $\lambda_1 \in \Sigma(\mathcal{L}) = \Sigma(\mathcal{L}^*)$. Taking $\lambda_n \searrow \lambda_1$, we argue as in the proof of Lemma 2.1.6 and we get

$$\exists \widehat{\phi}_n \geq 0, \lambda_n \widehat{\phi}_n - \mathcal{L}^* \widehat{\phi}_n \rightarrow 0, \|\widehat{\phi}_n\| = 1.$$

Thanks to **(H3)** for the dual problem, we deduce that there exist a subsequence $(\widehat{\phi}_{n_k})$ and $\phi_1 \in X'$, $\|\phi_1\| = 1$ such that $\widehat{\phi}_{n_k} \rightarrow \phi_1$. We thus conclude that ϕ_1 is a solution to the dual problem (1.3.31) (for the same eigenvalue λ_1). \square

Let us conclude this section by some remarks.

Remark 2.1.21. (1) - As seen above, the condition **(H1)**-**(H2)** for the primal and the dual problems are equivalent, and thus one only has to check **(H1)**-**(H2)**-**(H3)** for the primal problem and **(H3)** for the dual problem in order to solve the first eigentriplet problem. It is worth emphasizing that condition **(H3)** on the dual problem is not a consequence of the condition **(H3)** on the primal problem. However, as presented in Lemma 2.1.7, Lemma 2.1.8 and Corollary 2.1.16, there exist several natural situations where both conditions **(H3)** for the primal and the dual problems hold together.

(2) - Alternatively, one may also assume **(H1)**-**(H2)**-**(H3)** for the dual problem, and then use a more classical fixed point theorem for proving the existence of a steady state for the rescaled semigroup by using for instance the MarkovKakutani fixed point theorem [134] as in [138, Theorem 5.1], the Tychonov fixed point theorem as in [95] or [83, Theorem 1.2] or a Birkhoff-Von Neumann type Theorem as in [50, Theorem 6.1]. For these last techniques, we also refer to Section 2.2, where such a dynamical approach is adapted to the present context. One may also use the usual Doblin-Harris theory, see for instance [119, 50] and the references therein.

2.1.3 Discussion

We discuss now the existence results presented in the preceding section.

For further references, let us first recall that when X is a Hilbert space and \mathcal{L} is self-adjoint, the first eigenvalue may be simply obtained thanks to the variational problem

$$\lambda_1 = \inf_{f \in X_+ \setminus \{0\}} \frac{\langle \mathcal{L}f, f \rangle}{\|f\|^2}. \quad (2.1.34)$$

We now explain how Theorem 2.1.20 is a generalization of the classical Krein-Rutman theorem stated in Theorem ???. We thus consider a Banach lattice X such that $X_{++} := \text{int} X_+ \neq \emptyset$ and an operator \mathcal{L} such that, for $\kappa_1 \in \mathbb{R}$ and any $\kappa > \kappa_1$, $\mathcal{R} := (\kappa - \mathcal{L})^{-1} : X \rightarrow X$ is compact and $\mathcal{R} : X_+ \setminus \{0\} \rightarrow X_{++}$, in particular **(H1)** holds true. As a first step, we recall the following very classical technical lemma of the KR theory.

Lemma 2.1.22. Assume $X_{++} := \text{int} X_+ \neq \emptyset$. For $g \in X_+$ and $f \in X_{++}$, there exists $C \geq 0$ such that $g \leq Cf$.

Proof of Lemma 2.1.22. We argue by contradiction. Otherwise, for any $n \geq 1$, we would have $f - g/n \in X_+^c \subset X_{++}^c$ and that last set is closed. Passing to the limit, we get $f \in X_{++}^c$, which is in contradiction with the assumption $f \in X_{++}$. \square

For a given $g_0 \in X_+ \setminus \{0\}$, we set $f_0 := \mathcal{R}g_0 \in X_{++}$. From Lemma 2.1.22, there exists $C_0 \geq 0$ such that $(\kappa - \mathcal{L})f_0 = g_0 \leq C_0 f_0$. That implies that **(ii)** holds with $\kappa_0 := \kappa - C_0$, and thus **(H2)** also holds. One may then define $\mu_1 := \kappa - \lambda_1$, with

$$\lambda_1 := \inf\{\lambda \in \mathbb{R}; (\lambda' - \mathcal{L})^{-1} \in \mathcal{B}(X), \forall \lambda' \in [\lambda, \kappa]\} \geq \kappa_0.$$

We recall that because of Lemma 2.1.6 (or its proof), there exist (λ_n) , (\hat{f}_n) and (ε_n) such that (2.1.20) holds, namely

$$\lambda_n \searrow \lambda_1, \hat{f}_n \geq 0, \varepsilon_n := \lambda_n \hat{f}_n - \mathcal{L}\hat{f}_n \geq 0, \|\hat{f}_n\| = 1, \|\varepsilon_n\| \rightarrow 0.$$

We may rewrite the equation as

$$\hat{f}_n = \mathcal{R}[\varepsilon_n + (\kappa - \lambda_n)\hat{f}_n],$$

so that (\hat{f}_n) belongs to a compact set of X because of the compactness assumption made on \mathcal{R} , so that **(H3)** holds true.

Because of Theorem 2.1.20, we deduce that there exists $f_1 \in X_+$ such that $\|f_1\| = 1$ and $\mathcal{L}f_1 = \lambda_1$. That implies $f_1 = \mu_1 \mathcal{R}f_1$, and thus that the existence part of Theorem ?? is a consequence of Theorem 2.1.20 for an operator \mathcal{R} which is the positive resolvent of an operator \mathcal{L} .

We would like to emphasize on the fact that our definition of the first eigenvalue by (2.1.15)-(2.1.16) bears some strong similarity with the definition of the first eigenvalue for elliptic operators in non divergence form as presented in [23]. Indeed, if $\lambda \in \mathcal{I}$, then

$$\exists f \in X_+ \setminus \{0\}, \quad \mathcal{L}f \leq \lambda f.$$

Assuming now that X is a space of functions (defined on a set E) and that $f(x) > 0$ for any $x \in E$, we deduce that

$$\lambda \geq \sup_E \frac{\mathcal{L}f}{f},$$

and thus λ_1 is characterized by

$$\lambda_1 = \inf_{f>0} \sup_E \frac{\mathcal{L}f}{f},$$

which is nothing but [23, (1.13)] (with a change of sign because of a different sign convention). We thus see that our formulation is a generalization at a more abstract level and for operators with positive inverse of that classical min-max approach for elliptic operators. Some more or less classical references on that subject are [78, 77], [195], [206], [25] and [24].

In particular in [25], two generalized principal eigenvalues

$$\lambda'_1 := \sup\{\kappa_0 \in \mathbb{R}; \exists g_0 \in \mathcal{C}_0 \mathcal{L}g_0 \leq \kappa_0 g_0\}$$

and

$$\lambda''_1 := \inf\{\kappa_1 \in \mathbb{R}; \exists g_1 \in \mathcal{C}_1 \mathcal{L}g_1 \geq \kappa_1 g_1\}$$

are defined for appropriate cones $\mathcal{C}_i \subset X_+ \setminus \{0\}$ for problems with lack of compactness. The links between the three quantities λ_1 , λ'_1 and λ''_1 are discussed as well as the possible not existence of a related generalized principal eigenfunction f_1 . The non existence of associated generalized principal eigenfunction should not be a surprise since it is the case when one considers $\mathcal{L} = \Delta$ in $X = L^2(\mathbb{R}^d)$ where $\mathcal{L}g_1 = \mathcal{L}^*g_1 = \lambda''_1 \psi$ with $0 < \psi = 1 \notin X = X'$ and $\lambda''_1 = 0$ but no associated generalized principal eigenfunction exists in X . We also refer to [156] where some examples of such a situation are discussed.

For its own interest and further discussions, we finally state and prove a slightly less general variant of Theorem ??.

Theorem 2.1.23. *Consider a Banach lattice with positive cone X_+ and a linear and bounded operator $\mathcal{R} : X \rightarrow X$ such that*

$$(i) \mathcal{R} : X_+ \rightarrow X_+;$$

(ii) $\exists g_2 \in X_+ \setminus \{0\}$, $\exists C_2 > 0$ such that $\mathcal{R}g_2 \leq C_2g_2$.

We define

$$K_2 := \{g \in X_+; \exists a > 0, g \leq ag_2\},$$

and next

$$A(g) := \inf\{a > 0; g \leq ag_2\}, \quad \text{if } g \in K_2,$$

as well as

$$\mathcal{J} := \{\mu \geq 0; \exists h \in K_2, h \geq \mu\mathcal{R}h + g_2\}.$$

We further assume

(iii) $\mu_1 := \sup \mathcal{J} < +\infty$.

(iv) Any upper bounded and increasing sequences (g^n) of X is convergent in the weak sense $\sigma(X, Y)$. More precisely, if $g_n \leq g_{n+1} \leq \bar{g} \in X$ for any $n \geq 1$, there exists $g \in X$, $g \leq \bar{g}$, such that $g_n \rightharpoonup g$.

(v) Any sequence (g^n) of normalized almost first eigenvectors is relatively compact. More precisely, for any sequence (g^n) of K_2 such that $A(g^n) = 1$, $g^n = \mu^n \mathcal{R}g^n + \varepsilon^n$, $\mu^n \nearrow \mu_1$ and $\varepsilon^n \rightarrow 0$, there exists $g \in K_2$ and a subsequence (g^{n_k}) such that $g^{n_k} \rightarrow g$ and $A(g) = 1$.

Then there exists $f_1 \in X_2$ such that $f_1 = \mu_1 \mathcal{R}f_1$ and $A(f_1) = 1$.

Proof of Theorem 2.1.23. We split the proof into three steps.

Step 1. We first establish that for any $\mu \in \mathcal{J}$, there exists $\tilde{g} = \tilde{g}_\mu \in K_2$ such that

$$\tilde{g} = \mu\mathcal{R}\tilde{g} + g_2. \quad (2.1.35)$$

We set $\tilde{g}_0 = 0$, $\tilde{g}_1 = g_2$, and we define (\tilde{g}_n) recursively by $\tilde{g}_{n+1} = \mu\mathcal{R}\tilde{g}_n + g_2$, for any $n \geq 1$. We claim that

$$0 \leq \tilde{g}_n \leq \tilde{g}_{n+1} \leq h, \quad \text{for any } n \geq 0,$$

where h enters in the definition of $\mu \in \mathcal{J}$. That is obviously true at order $n = 0$. Assuming that last inequality is proved at order $n - 1$ for $n \geq 1$, we compute

$$\tilde{g}_{n+1} = \mu\mathcal{R}\tilde{g}_n + g_2 \geq \mu\mathcal{R}\tilde{g}_{n-1} + g_2 = \tilde{g}_n$$

and

$$\tilde{g}_{n+1} = \mu\mathcal{R}\tilde{g}_n + g_2 \leq \mu\mathcal{R}h + g_2 \leq h,$$

which proves the same inequality at order n , and thus for any $n \geq 0$. Using the convergence property (iv) of upper bounded increasing sequences in X , we deduce that there exists $\tilde{g} \in X_2$ such that $\tilde{g}_n \rightarrow \tilde{g}$ and thus (2.1.35) holds.

Step 2. We obviously have $0 \in \mathcal{J}$ and \mathcal{J} is an interval because if (μ, h) satisfies the condition $\mu \in \mathcal{J}$ then so do (μ', h) for any $\mu' \in [0, \mu]$. We finally claim that \mathcal{J} is open. Take indeed $\mu \in \mathcal{J}$ and $\tilde{g} \in K_2$ such that (2.1.35) holds, what is possible due to Step 1. By definition, there would exist $A > 0$ such that $\tilde{g} \leq Ag_2$. Choosing $0 < \varepsilon \leq 1/(2AC_2)$ and $M \geq 2$, we compute

$$\begin{aligned} (M\tilde{g}) - (\mu + \varepsilon)\mathcal{R}(M\tilde{g}) &= Mg_2 - M\varepsilon\mathcal{R}\tilde{g} \\ &\geq Mg_2 - M\varepsilon A\mathcal{R}g_2 \geq M(1 - \varepsilon AC_2)g_2 \geq g_2, \end{aligned}$$

so that $\mu + \varepsilon \in \mathcal{J}$.

Step 3. We first establish by contradiction that $A(\tilde{g}_\mu) \nearrow \infty$ when $\mu \nearrow \mu_1$. If it was not the case, there exists $A \in (0, \infty)$ and a sequence (μ^n) such that $A(\tilde{g}_{\mu^n}) \leq A$ as $\mu^n \nearrow \mu_1$. Choosing $0 < \varepsilon \leq 1/(2AC_2)$ and $M \geq 2$ as in Step 2, the same computation gives

$$(M\tilde{g}_{\mu^n}) - (\mu + \varepsilon)\mathcal{R}(M\tilde{g}_{\mu^n}) \geq g_2,$$

so that $\mu^n + \varepsilon \in \mathcal{J}$. That means that $\mu^n + \varepsilon \leq \mu_1$, and a contradiction with the fact $\mu^n \nearrow \mu_1$. We next consider $\mu^n \nearrow \mu_1$ and we define

$$A^n := A(\tilde{g}_{\mu^n}), \quad \hat{g}^n := \frac{\tilde{g}_{\mu^n}}{A^n}, \quad \varepsilon_n := \frac{g_2}{A^n}, \quad \hat{g}^n = \mu^n T \hat{g}^n + \varepsilon_n.$$

We observe that $\varepsilon_n \rightarrow 0$ and $A(\hat{g}^n) = 1$. Because of the compactness assumption (v), we deduce that there exists $f_1 \in K_2$ and a subsequence (\hat{g}^{n_k}) such that $\hat{g}^{n_k} \rightarrow f_1$ and $A(f_1) = 1$. We conclude by passing to the limit in the above almost first eigenvalue equations. \square

We may compare Theorem 2.1.23 with the results presented in the previous section. When \mathcal{L} satisfies condition **(H1)**, we may set $\mathcal{R} := \mathcal{R}_{\mathcal{L}}(\kappa_1)$ so that $\mathcal{R} \in \mathcal{B}(X)$ and \mathcal{R} satisfies (i). In that case, Theorem 2.1.23 claims the existence of $f_1 \in K_2$ such that $\mathcal{L}f_1 = \lambda_1 f_1$, with $\lambda_1 := \kappa_1 - \mu_1$. The condition (ii) on \mathcal{R} translates as $\mathcal{L}g_2 \leq (\kappa_1 - 1/C_2)g_2$ which may be seen as an equivalent of condition **(H1)** (when working in the space $X_2 := K_2 - K_2$ with norm $\|g\|_2 := A_2(|g|)$ and \mathcal{L} generates a semigroup S . The hypothesis (iii) is nothing but **(H2)** and the hypothesis (iv) is very natural: it holds in the space $L^p(E)$ and $M^1(E)$ without additional condition on \mathcal{R} and it holds in a space of continuous functions when some additional uniform continuity assumption is made on the range of \mathcal{R} . Assumption (v) has to be compared with condition **(H3)**. It is worth emphasizing that when $X \subset L^p(E)$ and $g_2 > 0$ a.e., we simply have $A(g) = \|g/g_2\|_{L^\infty}$ for any $g \in X_+$. As a conclusion, although Theorem 2.1.20 and Theorem 2.1.23 bear some similarity none seems to be a consequence of the other. We believe that Theorem 2.1.20 is more flexible since it does not impose to work with the normalization associated to the seminorm $g \mapsto A(|g|)$ of L^∞ -type. It is also worth emphasizing that it is shown in [156] that Theorem ?? is also a particular case of Theorem 2.1.23 by essentially exploiting the fundamental Lemma 2.1.22.

2.2 Existence through a dynamical approach

In this part, we develop a dynamical approach for proving the existence part of the Krein-Rutman Theorem. We thus always consider a positive semigroup $S = S_{\mathcal{L}}$ on a Banach lattice X . We recover Theorem 2.1.20 under slightly reinforced assumptions. Above all, we are able to extend the existence part of the Krein-Rutman Theorem to a more general framework, namely to the case when \mathcal{L} only enjoys a suitable weakly dissipative condition.

2.2.1 About dissipativity

Let us start by recalling some classical definitions and results. We say that an operator \mathcal{L} defined in a Banach space X is dissipative if there is some number $\kappa \in \mathbb{R}$ such that

$$\forall f \in D(\mathcal{L}), \exists f^* \in J_f, \quad \Re \langle f^*, \mathcal{L}f \rangle \leq \kappa \|f\|^2,$$

where we define the associated dual set $J_f \subset X'$ of f by

$$J_f := \{\varphi \in X'; \langle \varphi, f \rangle = \|f\| = \|\varphi\|_{X'}\}. \tag{2.2.1}$$

In that situation and in order to be more precise, we should say that $\mathcal{L} - \kappa$ is dissipative. It is worth emphasizing that $J_f \neq \emptyset$ thanks to the corollary (2.1.2) of the Hahn-Banach dominated extension theorem. We say that an operator \mathcal{L} is hypodissipative in a Banach space X if there exist an equivalent norm $\|\cdot\|$ in X and a number $\kappa \in \mathbb{R}$ such that

$$\forall f \in D(\mathcal{L}), \exists f^* \in J_{f, \|\cdot\|}, \quad \Re \langle f^*, \mathcal{L}f \rangle \leq \kappa \|f\|^2, \tag{2.2.2}$$

where

$$J_{f, \|\cdot\|} := \{\varphi \in X'; \langle \varphi, f \rangle = \|f\| = \|\varphi\|_{X'}\}. \tag{2.2.3}$$

The only difference between the two definitions (2.2.1) and (2.2.3) comes from the norms in which the normalization is performed. When \mathcal{L} is the generator of a semigroup $S_{\mathcal{L}}$, one can show that the growth bound $\omega = \omega(S_{\mathcal{L}})$ defined in (2.1.9) also satisfies

$$\omega = \inf\{\kappa \in \mathbb{R}, (2.2.2) \text{ holds for some equivalent norm } \|\cdot\|\},$$

and $S_{\mathcal{L}}$ is a semigroup of contraction when \mathcal{L} is dissipative with $\kappa = 0$. At least formally, denoting $f_t := S(t)f$, for $f \in D(\mathcal{L})$, we deduce from (2.2.2) that

$$\frac{1}{2} \frac{d}{dt} \|f_t\|^2 = \Re e \langle (f_t)^*, \mathcal{L}f_t \rangle \leq \kappa \|f_t\|^2,$$

and together with the Grönwall lemma, we deduce

$$\|S(t)f\| \leq e^{\kappa t} \|f\|, \quad \forall t \geq 0,$$

which is nothing but (2.1.10). That last estimate is actually equivalent to the hypodissipativity estimate (2.2.2). Quite similarly, when

$$\exists \psi \in Y_+ \setminus \{0\}, \exists \kappa \in \mathbb{R}, \quad \pm \mathcal{L}^* \psi \leq \kappa \psi, \quad (2.2.4)$$

we may compute

$$\pm \frac{d}{dt} \langle f_t, \psi \rangle = \pm \langle \mathcal{L}f_t, \psi \rangle = \pm \langle f_t, \mathcal{L}^* \psi \rangle \leq \kappa \langle f_t, \psi \rangle,$$

and together with the Grönwall lemma, we get

$$\pm \langle S_t f, \psi \rangle \leq \pm e^{\pm \kappa t} \langle f, \psi \rangle, \quad \forall t \geq 0. \quad (2.2.5)$$

Two important more accurate versions of the previous ones are presented now. They will be on the main importance in the sequel. On the one hand, we may assume that \mathcal{L} satisfies a Lyapunov type condition, namely there exists $\psi_i \in Y_+$ and $\kappa \in \mathbb{R}$ such that

$$\mathcal{L}^* \psi_2 \leq \kappa \psi_2 + \psi_0, \quad (2.2.6)$$

with $\psi_2 > 0$ and $\psi_0/\psi_2 \rightarrow 0$ at infinity. For $f_t = S_{\mathcal{L}}(t)f$, $f \in D(\mathcal{L}) \cap X_+$, a similar computation as above gives

$$\frac{d}{dt} \langle f_t, \psi_2 \rangle = \langle f_t, \mathcal{L}^* \psi_2 \rangle \leq \kappa \langle f_t, \psi_2 \rangle + \langle f_t, \psi_0 \rangle.$$

Denoting $[f]_i := \langle |f|, \psi_i \rangle$ and using the Grönwall lemma, we classically deduce

$$[S(t)f]_2 \leq e^{\kappa t} [f]_2 + \int_0^t e^{\kappa(t-s)} [S(s)f]_0 ds, \quad \forall t \geq 0. \quad (2.2.7)$$

The Lyapunov condition (2.2.6) is particularly relevant and useful in a Radon measures space framework $X = M_{\psi_2}^1(E)$ for some weight function ψ_2 on E .

On the other hand, we may generalize the above Lyapunov condition by assuming the structure condition

(HS2) there exist a splitting $\mathcal{L} = \mathcal{A} + \mathcal{B}$ and $\kappa_{\mathcal{B}} \in \mathbb{R}$ such that \mathcal{A} is \mathcal{B} -bounded, that means

$$\exists C \geq 0, \forall f \in X, \quad \|\mathcal{A}f\| \leq C(\|f\| + \|\mathcal{B}f\|),$$

the operator \mathcal{B} generates a semigroup $S_{\mathcal{B}}$ and

$$\|(S_{\mathcal{B}}\mathcal{A})^{(*\ell)} * S_{\mathcal{B}}(t)\|_{\mathcal{B}(X)} = \mathcal{O}(e^{\alpha t}), \quad \forall t > 0, \quad (2.2.8)$$

for any $\ell \geq 0$ and $\alpha > \kappa_{\mathcal{B}}$. Here and below, for two functions $U : \mathbb{R}_+ \rightarrow \mathcal{B}(\mathcal{X}_0, \mathcal{X}_1)$ and $V : \mathbb{R}_+ \rightarrow \mathcal{B}(\mathcal{X}_1, \mathcal{X}_2)$, we define the convolution function

$$(V * U)(t) := \int_0^t V(t-s)U(s) ds,$$

when the integral is well-defined. For $U : \mathbb{R}_+ \rightarrow \mathcal{B}(\mathcal{X})$, we also recursively define $U^{(*0)} = I$ and $U^{*(\ell+1)} = U^{*(\ell)} * U$. Using this convolution notation, the Duhamel formula writes

$$S_{\mathcal{L}} = S_{\mathcal{B}} + S_{\mathcal{B}}\mathcal{A} * S_{\mathcal{L}},$$

and iterating this formula, for any $N \geq 1$, we get the following iterated Duhamel formula

$$S_{\mathcal{L}} = S_{\mathcal{B}} + \dots + (S_{\mathcal{B}}\mathcal{A})^{*(N-1)} * S_{\mathcal{B}} + (S_{\mathcal{B}}\mathcal{A})^{*(N)} * S_{\mathcal{L}}. \quad (2.2.9)$$

When $S_{\mathcal{L}}$ is well defined in another space $X_0 \supset X$ and the last iterated convolution term enjoys the regularity property $\|(S_{\mathcal{B}}\mathcal{A})^{*(N)}(t)\|_{\mathcal{B}(X_0, X)} = \mathcal{O}(e^{\alpha t})$ for all $t > 0$ and $\alpha > \kappa_{\mathcal{B}}$, we deduce from the above iterated Duhamel formula, the estimate

$$\|S(t)f\| \leq C_0 e^{\alpha t} \|f\| + C_1 \int_0^t e^{\alpha(t-s)} \|S(s)f\|_0 ds, \quad \forall t \geq 0, \alpha > \kappa_{\mathcal{B}}, \forall f \in X, \quad (2.2.10)$$

for some constants $C_i \geq 1$ and where $\|\cdot\|_0$ stands for the norm in X_0 . We may observe that the estimate (2.2.7) in the case of a Lyapunov condition is a particular case of (2.2.10) corresponding to the norms $\|\cdot\| = [\cdot]_2$ and $\|\cdot\|_0 = [\cdot]_0$. More specifically, in a Radon measures space framework, the splitting condition **(HS2)** is obtained by introducing the bounded operator $\mathcal{A}f := f\psi_0$ and the generator $\mathcal{B} := \mathcal{L} - \mathcal{A}$. Because of (2.2.6), we have $\mathcal{B}^*\psi_2 \leq \kappa\psi_2$, and arguing as for establishing (2.2.7), we have $[S_{\mathcal{B}}(t)f]_2 \leq e^{\kappa t}[f]_2$ for any $t \geq 0$ and $f \in X$. That last growth condition is equivalent to assuming that $\mathcal{B} - \kappa$ is dissipative for the norm $[\cdot]_2$, so that we have established that \mathcal{L} enjoys the splitting condition **(HS2)**.

2.2.2 Existence in the dissipative case

In this section, we give an existence result for a positive semigroup $S_{\mathcal{L}}$ on a Banach lattice X satisfying a kind of regularity/compactness assumption in the spirit of the structure condition **(HS2)** discussed above.

Theorem 2.2.1. *On a Banach lattice $X = Y'$, with Y separable Banach lattice, consider a positive semigroup $S = S_{\mathcal{L}}$ satisfying the growth bound (2.1.10), and set $\kappa_1 := \omega' + \log M$ for some $\omega' > \omega(S_{\mathcal{L}})$.*

We assume

- (1) $\exists \psi_0 \in Y_+ \setminus \{0\}$, $\exists \kappa_0 \in \mathbb{R}$ such that $[S(t)f]_0 \geq e^{\kappa_0 t}[f]_0$ for any $t \geq 0$ and $f \in X_+$, where we denote $[f]_0 := \langle |f|, \psi_0 \rangle$;
- (2) there exist $\kappa, C_0, C_1 \in \mathbb{R}$ with $\kappa < \kappa_0$, $C_0 \geq 1$ and $C_1 \geq 0$, such that

$$\|S(t)f\| \leq C_0 e^{\kappa t} \|f\| + C_1 \int_0^t e^{\kappa(t-s)} [S(s)f]_0 ds, \quad \forall t \geq 0, \forall f \in X. \quad (2.2.11)$$

Then there exist $\lambda_1 \in [\kappa_0, \kappa_1]$ and $f_1 \in X_+ \setminus \{0\}$ such that $\mathcal{L}f_1 = \lambda_1 f_1$.

Remark 2.2.2. (1) Assumption (2) in the statement of Theorem 2.2.1 holds when there exist V, W such that

$$S = V + W * S, \quad W \geq 0, \quad (2.2.12)$$

and there exist $\kappa, C_V, C_W \in \mathbb{R}$, $\kappa < \kappa_0$, $C_V \geq 1$, $C_W > 0$ such that

$$\|V(t)\|_{\mathcal{B}(X)} \leq C_V e^{\kappa t}, \quad \|W(t)\|_{\mathcal{B}(X_0, X)} \leq C_W e^{\kappa t}, \quad (2.2.13)$$

(2) Under the structural condition **(HS2)** together with some regularization effect on the semigroup of the type

$$\|(S_{\mathcal{B}}\mathcal{A})^{(*N)}(t)\|_{\mathcal{B}(\mathcal{X}_0, X)} = \mathcal{O}(e^{\kappa t}), \quad \forall t > 0, \kappa \in (\kappa_{\mathcal{B}}, \kappa_0),$$

we recover the above condition (2.2.12)-(2.2.13) with

$$V := S_{\mathcal{B}} + \cdots + (S_{\mathcal{B}}\mathcal{A})^{*(N-1)} * S_{\mathcal{B}}, \quad W := (S_{\mathcal{B}}\mathcal{A})^{(*N)} \quad (2.2.14)$$

because of the iterated Duhamel formula (2.2.9). In that case, the representation formula (2.1.13) holds true for any $z > \lambda_1$ from Lemma 2.1.2-(ii) and we easily compute

$$\mathcal{R}_L(z) = \mathcal{V}(z) + \mathcal{W}(z)\mathcal{R}_L(z), \quad \forall z > \lambda_1,$$

with

$$\mathcal{V}(z) := \int_0^\infty e^{-\lambda t} V(t) dt, \quad \mathcal{W}(z) := \int_0^\infty e^{-\lambda t} W(t) dt, \quad \forall z > \kappa.$$

We observe that \mathcal{W} satisfies (2.1.27) in Lemma 2.1.14 if W satisfies (2.2.13) and the set \mathcal{C} defined by (2.1.25) satisfies the same compactness properties as required in the statement of Lemma 2.1.10. We may thus apply Lemma 2.1.14 (see also Remark 2.1.15) and deduce that **(H3)** holds for the primal problem. We finally obtain the same conclusion as in Theorem 2.2.1 thanks to Theorem 2.1.20.

(3) Under the same structural condition **(HS2)** as above, but assuming now that

$$\|W(t)\|_{\mathcal{B}(X, \mathcal{X}_1)} = \mathcal{O}(e^{\kappa t}), \quad \forall t > 0, \kappa \in (\kappa_{\mathcal{B}}, \kappa_0),$$

with $W := (S_{\mathcal{B}}\mathcal{A})^{(*N)}$ and $\mathcal{X}_1 \subset X$ with strongly compact embedding, we observe that S does not necessary satisfies the assumptions of Theorem 2.2.1, but it rather satisfies the assumptions of Lemma 2.1.7 with $K_T := (W * S)(T)$ and $T > 0$ large enough. In that situation, we also obtain the same conclusion as in Theorem 2.2.1 thanks to Lemma 2.1.7 and Theorem 2.1.20.

Proof of Theorem 2.2.1. We split the proof into two steps.

Step 1. We define the set

$$\mathcal{C} := \{f \in X_+, [f]_0 = 1, \|f\| \leq R\},$$

for a convenient constant $R > 0$ to be fixed later. For any fixed $t > 0$, we next define the nonlinear weakly $\sigma(X, Y)$ continuous mapping

$$\Phi_t : \mathcal{C} \rightarrow X, \quad f \mapsto \frac{S_t f}{[S_t f]_0}.$$

Thanks to assumption **(1)**, we may observe that it is well defined because

$$[S_t f]_0 \geq e^{\kappa_0 t} [f]_0 = e^{\kappa_0 t} > 0. \quad (2.2.15)$$

For any $f \in \mathcal{C}$, we thus immediately have $\Phi_t f \geq 0$ and $[\Phi_t f]_0 = 1$. On the other hand, from assumption **(1)** again and the semigroup property, we have

$$[S(t)f]_0 \geq e^{\kappa_0(t-s)} [S(s)f]_0. \quad (2.2.16)$$

For $f \in \mathcal{C}$ and $t \geq 0$, we next compute

$$\begin{aligned} \|\Phi_t f\| &\leq C_0 e^{-\alpha t} \|f\| + C_1 \int_0^t e^{-\alpha(t-s)} ds \\ &\leq C_0 e^{-\alpha t} R + \frac{C_1}{\alpha}, \end{aligned}$$

where we have set $\alpha := \kappa_0 - \kappa_B > 0$. Fixing T_0 such that $C_0 e^{-\alpha T_0} \leq 1/2$ and next $R \geq 2C_1/\alpha$, we have thus $\Phi_{T_0} : \mathcal{C} \rightarrow \mathcal{C}$. Thanks to the Tykonov fixed point Theorem, there exists $f_{T_0} \in \mathcal{C}$ such that $\Phi_{T_0} f_{T_0} = f_{T_0}$. In other words, we have established the existence of $f_{T_0} \in X$ such that

$$f_{T_0} \geq 0, \quad [f_{T_0}]_0 = 1, \quad S_{T_0} f_{T_0} = e^{\lambda_1 T_0} f_{T_0}, \quad (2.2.17)$$

with $\lambda_1 := (1/T_0) \log[S_{T_0} f_{T_0}]_0 \in [\kappa_0, \kappa_1]$.

Step 2. Rewriting equation (2.2.17) as

$$0 = e^{-\lambda_1 T_0} S_{T_0} f_{T_0} - f_{T_0} = (\mathcal{L} - \lambda_1) \int_0^{T_0} e^{-\lambda_1 t} S_t f_{T_0} dt$$

and defining

$$f_1 := \int_0^{T_0} e^{-\lambda_1 t} S_t f_{T_0} dt,$$

we get that $f_1 \in X_+ \setminus \{0\}$ and satisfies $\mathcal{L} f_1 = \lambda_1 f_1$. \square

We present now a second proof based on a large times dynamical argument which is classical in the mean ergodicity theory of Von Neumann and Birkhoff introduced in [194, 30] and which will be adapted in the weak dissipativity case in Section 2.2.5 below.

Alternative Step 2. We define $\tilde{S}_t := S_t e^{-\lambda_1 t}$, so that f_{T_0} becomes a periodic state for \tilde{S}_t from (2.2.17), namely

$$\tilde{S}_t f_{T_0} = \tilde{S}_{t-kT_0} f_{T_0}, \quad k := [t/T_0], \quad \forall t > 0.$$

Using (2.2.15) and the above relation, we have

$$\begin{aligned} [\tilde{S}_t f_{T_0}]_0 &= [\tilde{S}_{t-kT_0} f_{T_0}]_0 \\ &\geq e^{(\kappa_0 - \lambda_1)(t-kT_0)} [f_{T_0}]_0 \geq e^{(\kappa_0 - \lambda_1)T_0} =: r_* > 0, \end{aligned}$$

for any $t \geq 0$. On the other hand, thanks to the growth bound (2.1.10), we have

$$\begin{aligned} \|\tilde{S}_t f_{T_0}\| &= \|\tilde{S}_{t-kT_0} f_{T_0}\| \\ &\leq M e^{(\kappa - \lambda_1)(t-kT_0)} \|f_{T_0}\| \leq M e^{(\kappa - \lambda_1)T_0} R =: R^* < \infty, \end{aligned}$$

for any $t \geq 0$. We finally define

$$u_T := \frac{1}{T} \int_0^T \tilde{S}_t f_{T_0} dt.$$

From the previous estimates, both sequences $(\tilde{S}_t f_{T_0})$ and (u_T) are bounded in

$$\mathbb{K} := \{f \in X; f \geq 0, [f]_0 \geq r_*, \|f\| \leq R^*\}.$$

By compactness, there exists a subsequence (u_{T_k}) and $f_1 \in \mathbb{K}$ such that $u_{T_k} \rightharpoonup f_1$ in a weak sense as $k \rightarrow \infty$. For any fixed $t > 0$, we observe that

$$\begin{aligned} \tilde{S}_t f_1 - f_1 &= \lim_{k \rightarrow \infty} \left\{ \frac{1}{T_k} \int_0^{T_k} \tilde{S}_t \tilde{S}_s f_{T_0} ds - \frac{1}{T_k} \int_0^{T_k} \tilde{S}_s f_{T_0} ds \right\} \\ &= \lim_{k \rightarrow \infty} \left\{ \frac{1}{T_k} \int_{T_k}^{T_k+t} \tilde{S}_s f_{T_0} ds - \frac{1}{T_k} \int_0^t \tilde{S}_s f_{T_0} ds \right\} = 0, \end{aligned}$$

where we have used that $(\tilde{S}_s f_{T_0})$ is uniformly bounded in the last line. As a consequence, f_1 is a stationary state for the rescaled semigroup \tilde{S}_t , and thus an eigenfunction associated to the eigenvalue λ_1 for the operator \mathcal{L} . \square

2.2.3 About weak dissipativity

In this section, we recall some definitions and results about the weak dissipativity. We say that the generator \mathcal{B} of a semigroup $S_{\mathcal{B}}$ is weakly dissipative in a Banach space X_i if there exist a second Banach space $X_{i-1} \supset X_i$ and some numbers $\kappa \in \mathbb{R}$ and $\sigma > 0$ such that

$$\forall f \in D(\mathcal{B}|_{X_i}), \exists f^* \in J_{f, X_i}, \quad \langle f^*, \mathcal{B}f \rangle \leq \kappa \|f\|_{X_i}^2 - \sigma \|f\|_{X_{i-1}}^2,$$

where we define the associated dual set $J_{f, X_i} \subset X_i'$ of f (for the norm $\|\cdot\|_{X_i}$) by

$$J_{f, X_i} := \{\varphi \in X_i'; \langle \varphi, f \rangle = \|f\|_{X_i}^2 = \|\varphi\|_{X_i'}^2\}. \quad (2.2.18)$$

By translation, we may assume that $\kappa = 0$, an hypothesis we will always make in the sequel of this section. We will furthermore assume the splitting structure $\mathcal{L} = \mathcal{A} + \mathcal{B}$ with \mathcal{A} \mathcal{B} -bounded and \mathcal{B} weakly dissipative.

More precisely, we assume that there exists one more Banach lattice $X_0 \supset X_1 \supset X_2 := X$, with norm denoted by $\|\cdot\|_k := \|\cdot\|_{X_k}$, such that \mathcal{B} generates a semigroup and is weakly dissipative in each X_k : for any $k = 1, 2$

$$\forall f \in D(\mathcal{B}|_{X_k}), \exists f^* \in J_{f, X_k}, \quad \langle f^*, \mathcal{B}f \rangle_{X_k', X_k} \leq -\sigma \|f\|_{k-1}^2. \quad (2.2.19)$$

This classically implies (or we can take the next inequality as a definition of the weak dissipativity condition) that

$$\frac{d}{dt} \|S_{\mathcal{B}}(t)f\|_k + \sigma \|S_{\mathcal{B}}(t)f\|_{k-1} \leq 0, \quad \forall t \geq 0, \forall f \in X_k, \forall k = 1, 2. \quad (2.2.20)$$

We assume that X_k is dense into X_{k-1} for $k = 1, 2$ and that X_1 is an interpolated space between X_0 and X_2 in the sense that there exists a continuous and strictly decreasing function $\eta : (0, 1] \rightarrow [0, \infty)$, $\eta(\varepsilon) \rightarrow \infty$ when $\varepsilon \rightarrow 0$, $\eta(1) = 0$, such that

$$\|f\|_1 \leq \varepsilon \|f\|_2 + \eta(\varepsilon) \|f\|_0, \quad \forall \varepsilon \in (0, 1], \forall f \in X_2. \quad (2.2.21)$$

From (2.2.20) with $k = 2$, we deduce

$$\|S_{\mathcal{B}}(t)f\|_2 \leq \|f\|_2, \quad \forall t \geq 0, \forall f \in X_2. \quad (2.2.22)$$

Next, for $k = 1$, gathering the weak dissipativity condition (2.2.20), the interpolation condition (2.2.21) and the non expansion property (2.2.22) in the space X_2 , we get

$$\begin{aligned} \frac{d}{dt} \|S_{\mathcal{B}}(t)f\|_1 + \frac{\sigma}{\eta(\varepsilon)} \|S_{\mathcal{B}}(t)f\|_1 &\leq \frac{\sigma\varepsilon}{\eta(\varepsilon)} \|S_{\mathcal{B}}(t)f\|_2 \\ &\leq \frac{\sigma\varepsilon}{\eta(\varepsilon)} \|f\|_2, \end{aligned}$$

for any $t \geq 0$, $\varepsilon \in (0, 1)$ and $f \in X_2$. We deduce

$$\frac{d}{dt} \left(\|S_{\mathcal{B}}(t)f\|_1 e^{\frac{\sigma}{\eta(\varepsilon)} t} \right) \leq \frac{\sigma\varepsilon}{\eta(\varepsilon)} e^{\frac{\sigma}{\eta(\varepsilon)} t} \|f\|_2,$$

and thanks to the Grönwall lemma, we obtain

$$\|S_{\mathcal{B}}(t)f\|_1 \leq \Theta(t) \|f\|_2, \quad (2.2.23)$$

for any $t \geq 0$ and $f \in X_2$, with

$$\Theta(t) := \inf_{\varepsilon \in (0, 1)} (e^{-\frac{\sigma}{\eta(\varepsilon)} t} + \varepsilon) \rightarrow 0 \quad \text{as } t \rightarrow +\infty. \quad (2.2.24)$$

On the other hand, using the representation formula

$$\mathcal{R}_{\mathcal{B}}(z)f = \int_0^\infty e^{-zt} S_{\mathcal{B}}(t)f dt, \quad \forall z \in \Delta_0, \forall f \in X_2,$$

together with (2.2.20), we get

$$\sigma \|\mathcal{R}_{\mathcal{B}}(z)f\|_1 \leq \int_0^\infty \sigma \|S_{\mathcal{B}}(t)f\|_1 dt \leq \|f\|_2,$$

for any $z \in \bar{\Delta}_0$ and $f \in X_2$. We next assume that

$$\Theta(t)^{-1} \|\mathcal{A}S_{\mathcal{B}}(t)f\|_1 + \int_0^\infty \|\mathcal{A}S_{\mathcal{B}}(t)f\|_1 dt \lesssim \|f\|_1, \quad (2.2.25)$$

that there exist $\alpha > 1$, $N \geq 1$, $C \geq 1$ such that

$$\sup_{x+iy \in \Delta_0} \|\mathcal{A}\mathcal{R}_{\mathcal{B}}^{1+\varepsilon_1}(x+iy) \dots \mathcal{A}\mathcal{R}_{\mathcal{B}}^{1+\varepsilon_N}(x+iy)f\|_2 \leq \frac{C}{\langle y \rangle^\alpha} \|f\|_2, \quad (2.2.26)$$

for any $\varepsilon \in \{0, 1\}^N$, $\varepsilon_1 + \dots + \varepsilon_N \leq 1$, and that

$$\sup_{z \in \Delta_0} \|(\mathcal{R}_{\mathcal{B}}(z)\mathcal{A})^N f\|_{\mathcal{X}_1} \leq \|f\|_1, \quad (2.2.27)$$

with \mathcal{X}_1 compactly imbedded in X_1 . The necessity to add (ε_i) in (2.2.26) is probably purely technical and not restrictive for applications. In examples, we can take $N = 2N'$, when

$$\sup_{x+iy \in \Delta_0} \|(\mathcal{A}\mathcal{R}_{\mathcal{B}})^{N'}(x+iy)f\|_3 \leq \frac{C}{\langle y \rangle^\alpha} \|f\|_2, \quad (2.2.28)$$

for some convenient space X_3 such that $\mathcal{A} : X_1 \rightarrow X_3$ and $\sup_{z \in \Delta_0} \|\mathcal{R}_{\mathcal{B}}(z)\|_{\mathcal{B}(X_3, X_2)} < \infty$. At the level of the semigroup, (2.2.28) is typically a consequence of

$$\|(\mathcal{A}S_{\mathcal{B}})^{(*N'')}(t)\|_{\mathcal{B}(X_2, X_3^\zeta)} \in L^1(\mathbb{R}_+),$$

with $\zeta > 0$, where $X_3^\zeta := \{f \in X_3, \mathcal{L}^\zeta f \in X_3\}$ stands for the (possibly fractional) domain for the operator defined in X_3 . However, (2.2.26) is a bit more general than that last estimate. We refer to [177, 179, 178, 176] for precise definition, examples and discussion. For further references, we observe that (2.2.23) and (2.2.25) together imply

$$\begin{aligned} \frac{1}{T} \int_0^T \|(S_{\mathcal{B}} * \mathcal{A}S_{\mathcal{B}})(t)f\|_1 dt &\leq \frac{1}{T} \int_0^T \int_0^t \|S_{\mathcal{B}}(t-s)\mathcal{A}S_{\mathcal{B}}(s)f\|_1 ds dt \\ &\leq \frac{1}{T} \int_0^T \int_0^T \|S_{\mathcal{B}}(u)\|_{\mathcal{B}(X_2, X_1)} \|\mathcal{A}S_{\mathcal{B}}(s)f\|_1 du ds \\ &\lesssim \frac{1}{T} \int_0^T \Theta(u) du \|f\|_2. \end{aligned}$$

Arguing in a similar way for any $\ell \geq 1$, we establish

$$\frac{1}{T} \int_0^T \|(S_{\mathcal{B}} * (\mathcal{A}S_{\mathcal{B}})^{(*\ell)})(t)f\|_1 dt \lesssim \frac{1}{T} \int_0^T \Theta du \|f\|_2 \rightarrow 0 \text{ as } T \rightarrow \infty. \quad (2.2.29)$$

For synthesizing and for further references, let us now bring out some possible general framework for semigroup enjoying weak dissipativity. We introduce the following structure condition on a semigroup $S_{\mathcal{L}}$ and its generator \mathcal{L} by assuming

(HS3) there exist a splitting $\mathcal{L} = \mathcal{A} + \mathcal{B}$, some Banach lattices $X_2 \subset X_1$, an integer $N \geq 1$ and some decaying functions $\Theta_i : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ with $\Theta_1(t) \rightarrow 0$ as $t \rightarrow \infty$, $\Theta_2 \in L^1(\mathbb{R}_+)$ such that \mathcal{A} is positive, \mathcal{B} generates a positive semigroup $S_{\mathcal{B}}$ and the following estimates hold

$$\|(S_{\mathcal{B}}\mathcal{A})^{(*\ell)} * S_{\mathcal{B}}\|_{\mathcal{B}(X_2, X_1)} = \mathcal{O}(\Theta_1), \quad \forall \ell \in \{0, \dots, N-1\}, \quad (2.2.30)$$

$$\|(S_{\mathcal{B}}\mathcal{A})^{(*N)}\|_{\mathcal{B}(X_1, X_2)} = \mathcal{O}(\Theta_2). \quad (2.2.31)$$

We now particularize our discussion to a Radon measures framework. We assume that there exist some weight functions ψ_i on E , $\psi_0 \lesssim \psi_1 \leq \psi_2$, with $\psi_2(x)/\psi_1(x) \rightarrow \infty$ as $x \rightarrow \infty$ so that $M_{\psi_2}^1 \subset\subset M_{\psi_1}^1$ (compact imbedding for the weak convergence), a function $\chi \in C_c(E)$, $0 \leq \chi \leq 1$, and a constant $M \geq 0$ such that

$$(i) \quad \mathcal{L}^*\psi_1 \leq -\psi_0 + M\chi;$$

$$(ii) \quad \mathcal{L}^*\psi_2 \leq M\chi;$$

$$(iii) \quad \psi_1 \leq \varepsilon\psi_2 + \eta(\varepsilon)\psi_0 \text{ for any } \varepsilon > 0,$$

for a function $\eta : (0, 1] \rightarrow (0, \infty)$ such that $\eta(1) = 0$, $\eta(\varepsilon) \rightarrow \infty$ when $\varepsilon \rightarrow 0$, and

$$t \mapsto \Theta(t) := \inf_{\varepsilon \in (0, 1)} (e^{-\frac{t}{\eta(\varepsilon)}} + \varepsilon) \in L^1(0, \infty). \quad (2.2.32)$$

It is worth emphasizing that from the very definition, we have automatically that Θ is positive and decreasing, $\Theta(0) = 1$ and $\Theta(t) \rightarrow 0$ as $t \rightarrow \infty$. Arguing similarly as we did during the proof of Corollary 2.1.19 and the end of Section 2.2.1, we introduce the splitting

$$\mathcal{A} := M\chi, \quad \mathcal{B} := \mathcal{L} - \mathcal{A},$$

and we establish that $S_{\mathcal{B}}$ is a positive semigroup on $X = M_{\psi_2}^1(E)$. More precisely, for $0 \leq f_0 \in D(\mathcal{B})$ in the domain of $S_{\mathcal{B}}$ and denoting $f_t := S_{\mathcal{B}}(t)f_0$, we may compute

$$\frac{d}{dt} \int f_t \psi_2 \leq \int f_t \mathcal{B}^* \psi_2 \leq 0$$

and similarly

$$\frac{d}{dt} \int f_t \psi_1 \leq \int f_t \mathcal{B}^* \psi_1 \leq - \int f_t \psi_0.$$

Integrating both differential inequalities, we deduce $S_{\mathcal{B}} \in L_t^\infty(\mathcal{B}(M_{\psi_i}^1))$, $i = 1, 2$ and

$$\int_0^\infty \|S_{\mathcal{B}}(t)f_0\|_{M_{\psi_0}^1} dt \leq \|f_0\|_{M_{\psi_1}^1}, \quad \forall f_0 \in M_{\psi_1}^1.$$

We may make a slight (but important) improvement of the previous estimate by proceeding similarly as we did for proving (2.2.23). Using the same notations as in the above computation, we indeed have

$$\frac{d}{dt} \int f_t \psi_1 + \frac{1}{\eta(\varepsilon)} \int f_t \psi_1 \leq \frac{\varepsilon}{\eta(\varepsilon)} \int f_t \psi_2 \leq \frac{\varepsilon}{\eta(\varepsilon)} \int f_0 \psi_2,$$

where we have used (i) and (iii) in the first inequality and the previous $L_t^\infty(\mathcal{B}(M_{\psi_2}^1))$ bound in the second inequality. Integrating in time, we deduce

$$\|S_{\mathcal{B}}(t)f\|_{M_{\psi_1}^1} \leq \Theta(t)\|f\|_{M_{\psi_2}^1}, \quad \forall t > 0.$$

Taking $X_i := M_{\psi_i}^1$ and $N = 1$, we see that \mathcal{L} then satisfies **(HS3)** with $\Theta_i = \Theta$.

2.2.4 First existence result in the weakly dissipative case

We first come back to the proof of Theorem 2.1.20 and explain what goes wrong when we try to adapt it to a weak dissipativity context. More precisely, we assume that $S_{\mathcal{L}}$ is a positive semigroup (so that **(H1)** holds) satisfying $\mathcal{L}^*\psi_0 \geq 0$ for some $\psi_0 \in X' \setminus \{0\}$ (so that **(H2)** holds) and the splitting structure **(HS3)** for some bounded operator \mathcal{A} and some weakly dissipative operator \mathcal{B} , in the sense that (2.2.19) holds. In such a situation, we may define

$$\lambda_1 := \inf\{\lambda \in \mathbb{R}; \mathcal{R}_{\mathcal{L}}(\kappa) \in \mathcal{B}(X), \forall \kappa \geq \lambda\} \geq 0,$$

and there exist sequences (λ_n) of \mathbb{R} and (\hat{f}_n) of X_+ such that

$$\lambda_n \searrow \lambda_1 \geq 0, \quad \|\hat{f}_n\| = 1, \quad \varepsilon_n := \lambda_n \hat{f}_n - \mathcal{L}\hat{f}_n \rightarrow 0 \text{ in } X,$$

thanks to Lemma 2.1.6. In the simplest situation, we may further assume that $\mathcal{R}_{\mathcal{B}}(\kappa) : X_1 \rightarrow X_0$ is uniformly bounded in $\kappa \geq \lambda_1$ and $\mathcal{A} : X_0 \rightarrow X_1$ with $X = X_1 \subset X_0$. The issue is that even in that case, we may write

$$\hat{f}_n = \mathcal{R}_{\mathcal{B}}(\lambda_n)\mathcal{A}\hat{f}_n + \mathcal{R}_{\mathcal{B}}(\lambda_n)\varepsilon_n,$$

but it is not clear how to conclude that (\hat{f}_n) belongs to a compact set in X because it is not clear that $\mathcal{R}_{\mathcal{B}}(\lambda_n)\varepsilon_n \rightarrow 0$ in X .

The next result aim precisely to establish that last convergence under suitable quite strong (although natural and true in some examples) assumptions on the operator \mathcal{L} . The proof is adapted from [138, Section 6.3] and mixes some dynamical argument together with the stationary approach developed in Section 2.1.2.

Theorem 2.2.3. *Consider a positive semigroup $S_{\mathcal{L}}$ in a Banach lattice $X = X_2 \subset X_1 \subset X_0$ such that its generator \mathcal{L} satisfies*

(1) *there exists $\psi_0 \in D(\mathcal{L}^*)$, $\psi_0 \geq 0$, $\psi_0 \neq 0$, such that $\mathcal{L}^*\psi_0 \geq 0$.*

(2) *$\mathcal{L} = \mathcal{A} + \mathcal{B}$ with \mathcal{A} and \mathcal{B} satisfying (2.2.23), (2.2.25), (2.2.26) and (2.2.27);*

Then, there exist $\lambda_1 \geq 0$ and $f_1 \in X_1$ such that

$$\|f_1\|_{X_1} = 1, \quad f_1 \geq 0, \quad \mathcal{L}f_1 = \lambda_1 f_1. \quad (2.2.33)$$

Proof of Theorem 2.2.3. We split the proof into four steps.

Step 1. We know from Lemma 2.1.2 and Lemma 2.1.4-(i) that **(H1)** and **(H2)** hold. We may then define $\lambda_1 \geq 0$ with the help of (2.1.16). If $\lambda_1 > 0$, we see that $\mathcal{V}(\alpha)$ defined in (2.1.22) is bounded in $\mathcal{B}(X)$ uniformly on $\alpha \geq \kappa_0 := \lambda_1/2$ because of (2.2.23) and (2.2.25), and that $\mathcal{W}(\alpha)$ also defined in (2.1.22) satisfies (2.1.23) because of (2.2.25) and Remark 2.1.9-(1). Using Lemma 2.1.8, we get that **(H3)** holds, and we conclude thanks to Theorem 2.1.20 in that case.

In the sequel, we always assume $\lambda_1 = 0$.

Step 2. Let us fix $f_0 \in D(\mathcal{L})$ such that $f_0 \geq 0$ and $C_0 := \langle f_0, \psi_0 \rangle > 0$, which exists by definition of ψ_0 . Denoting $f(t) := S_{\mathcal{L}}(t)f_0$, we have

$$\frac{d}{dt}\langle f(t), \psi_0 \rangle = \langle \mathcal{L}f(t), \psi_0 \rangle = \langle f(t), \mathcal{L}^*\psi_0 \rangle \geq 0,$$

which in turns implies

$$\langle f(t), \psi_0 \rangle \geq C_0, \quad \forall t \geq 0.$$

Step 3. We claim that $\|\mathcal{R}_{\mathcal{L}}(0)\|_{\mathcal{B}(X_2, X_1)} = +\infty$. That in particular implies $\|\mathcal{R}_{\mathcal{L}}(0)\|_{\mathcal{B}(X)} = +\infty$ and thus $0 \in \Sigma(\mathcal{L})$. We assume by contradiction that $\mathcal{K}_{2,1} := \|\mathcal{R}_{\mathcal{L}}(0)\|_{\mathcal{B}(X_2, X_1)} < +\infty$. First, because $S_{\mathcal{L}}$ is positive, we have

$$|\mathcal{R}_{\mathcal{L}}(z)f| \leq \int_0^{\infty} e^{-t\Re z} S_{\mathcal{L}}(t)|f| dt = |\mathcal{R}_{\mathcal{L}}(\Re z)|f|,$$

from which we deduce

$$\|\mathcal{R}_{\mathcal{L}}(z)\|_{\mathcal{B}(X_2, X_1)} \leq \|\mathcal{R}_{\mathcal{L}}(\Re z)\|_{\mathcal{B}(X_2, X_1)}, \quad \forall z \in \Delta_0.$$

As a consequence, we have

$$\sup_{y \in \mathbb{R}} \|\mathcal{R}_{\mathcal{L}}(iy)\|_{\mathcal{B}(X_2, X_1)} \leq \mathcal{K}_{2,1}. \quad (2.2.34)$$

We write the representation formulas (taken from [177, (2.21)])

$$S_{\mathcal{L}}(t)f = \mathcal{T}_0(t) + \lim_{M \rightarrow \infty} \mathcal{T}_{1,M}(t)$$

with

$$\mathcal{T}_0(t) := \sum_{\ell=0}^{N-1} S_{\mathcal{B}} * (\mathcal{A}S_{\mathcal{B}})^{(*\ell)}(t)f$$

and

$$\mathcal{T}_{1,M}(t) := \frac{i}{2\pi} \int_{a-iM}^{a+iM} e^{zt} \mathcal{R}_{\mathcal{L}}(z) (\mathcal{A}\mathcal{R}_{\mathcal{B}}(z))^N f dz,$$

for any $f \in D(\mathcal{L})$, $t \geq 0$ and $a > 0$. On the one hand, from (2.2.29), we have

$$\frac{1}{T} \int_0^T \mathcal{T}_0(t) dt \rightarrow 0 \text{ in } X_1, \text{ as } T \rightarrow \infty. \quad (2.2.35)$$

On the other hand, we estimate the contribution of the Cesàro mean of $\mathcal{T}_{1,M}$. Integrating by part, we have

$$\mathcal{T}_{1,M}(t) = \frac{1}{t} \frac{i}{2\pi} \int_{a-iM}^{a+iM} e^{zt} \frac{d}{dz} [\mathcal{R}_{\mathcal{L}}(z) (\mathcal{A}\mathcal{R}_{\mathcal{B}}(z))^N] f dz,$$

with

$$\frac{d}{dz} [\mathcal{R}_{\mathcal{L}}(z) (\mathcal{A}\mathcal{R}_{\mathcal{B}}(z))^N] = \sum_{\varepsilon \in \mathbb{N}^{N+1}, |\varepsilon|=1} \mathcal{R}_{\mathcal{L}}(z)^{1+\varepsilon_0} \mathcal{A}\mathcal{R}_{\mathcal{B}}^{1+\varepsilon_1}(z) \dots \mathcal{A}\mathcal{R}_{\mathcal{B}}^{1+\varepsilon_N}(z).$$

Together with condition (2.2.26) and estimate (2.2.34), we get

$$\begin{aligned} & \left\| \frac{d}{dz} [\mathcal{R}_{\mathcal{L}}(z) (\mathcal{A}\mathcal{R}_{\mathcal{B}}(z))^N] f \right\|_1 \\ & \leq (\mathcal{K}_{2,1} + \mathcal{K}_{2,1}^2)^N \sup_{\varepsilon \in \mathbb{N}^N, |\varepsilon| \leq 1} \|\mathcal{A}\mathcal{R}_{\mathcal{B}}^{1+\varepsilon_1}(z) \dots \mathcal{A}\mathcal{R}_{\mathcal{B}}^{1+\varepsilon_N}(z) f\|_2 \\ & \leq \frac{C_1}{\langle y \rangle^\alpha} \|f\|_2, \end{aligned}$$

uniformly for any $z = x + iy \in \Delta_0$, for some constant $C_1 > 0$. We deduce

$$\left\| \lim_{M \rightarrow \infty} \mathcal{T}_{1,M}(t) \right\|_1 \leq \frac{1}{t} \frac{1}{2\pi} \int_{\mathbb{R}} \frac{C_1}{\langle y \rangle^\alpha} dy \|f\|_2 \rightarrow 0, \quad (2.2.36)$$

as $t \rightarrow \infty$. Gathering (2.2.35) and (2.2.36), we conclude in particular that

$$\frac{1}{T} \int_0^T S_{\mathcal{L}}(t)f_0 dt \rightarrow 0 \text{ in } X_1, \text{ as } T \rightarrow \infty,$$

which is in contradiction with the estimate of Step 2.

Step 4. Conclusion. Taking advantage of the convenient blow up of $\mathcal{R}_{\mathcal{L}}(\lambda)$ as $\lambda \searrow 0$ established in the previous Step 2, we may now argue similarly as in the proof of Theorem 2.1.20. From Step 2, there exists a sequence (λ_n) such that $\lambda_n \rightarrow 0$ and

$$\|\mathcal{R}_{\mathcal{L}}(\lambda_n)\|_{\mathcal{B}(X_2, X_1)} \rightarrow \infty.$$

That means that there exist (f_n) and (g_n) such that

$$\|f_n\|_{X_1} \rightarrow \infty, \quad \|g_n\|_{X_2} = 1, \quad f_n = \mathcal{R}_{\mathcal{L}}(\lambda_n) g_n,$$

or equivalently that there exist (\hat{f}_n) and (ε_n) (by defining $\hat{f}_n := f_{n\pm}/\|f_{n\pm}\|_{X_1}$, $\varepsilon_n := g_{n\pm}/\|f_{n\pm}\|_{X_1}$) satisfying

$$\|\hat{f}_n\|_{X_1} = 1, \quad \hat{f}_n \geq 0, \quad \|\varepsilon_n\|_{X_2} \rightarrow 0, \quad \varepsilon_n = (\lambda_n - \mathcal{L}) \hat{f}_n. \quad (2.2.37)$$

As in the proof of Lemma 2.1.8, we deduce that (2.1.24) holds, that is

$$\hat{f}_n = \sum_{\ell=0}^{N-1} (\mathcal{R}_{\mathcal{B}}(\lambda_n)\mathcal{A})^\ell \mathcal{R}_{\mathcal{B}}(\lambda_n)\varepsilon_n + (\mathcal{R}_{\mathcal{B}}(\lambda_n)\mathcal{A})^N \hat{f}_n. \quad (2.2.38)$$

Using the uniform boundedness

$$(\mathcal{R}_{\mathcal{B}}(\lambda_n)\mathcal{A})^\ell \mathcal{R}_{\mathcal{B}}(\lambda_n) \in \mathcal{B}(X_2, X_1), \quad (\mathcal{R}_{\mathcal{B}}(\lambda_n)\mathcal{A})^N \in \mathcal{B}(X_1, \mathcal{X}_1), \quad \mathcal{X}_1 \subset\subset X_1,$$

we deduce that (\hat{f}_n) belongs to a compact set of X_1 , or in other words, that there exist a subsequence of (\hat{f}_n) (not relabeled) and $f_1 \in X_1$ such that $\hat{f}_n \rightarrow f_1$ in X_1 . We may pass to the limit in (2.2.37), and we get (2.2.33). \square

2.2.5 Second existence result in the weakly dissipative case

Using a pure dynamical approach adapted from the second proof of Theorem 2.2.1 and from [50, Theorem 6.1], we establish a second existence result which is less demanding in terms of conditions on the semigroup $S_{\mathcal{L}}$.

Theorem 2.2.4. *Consider a positive semigroup $S = S_{\mathcal{L}}$ on a Banach lattice $X = Y'$ for a separable Banach lattice Y . We assume*

(i) *there exists $\psi_0 \in Y_+ \setminus \{0\}$ such that $[S_t f]_0 \geq [f]_0$ for any $f \in X_+$ and $f \mapsto [f]_0 := \langle |f|, \psi_0 \rangle$ is a norm on X . We then denotes \mathcal{X} the vector space X endowed with this norm $[\cdot]_0$;*

(ii) *there exist $v \in L^\infty(\mathbb{R}_+; \mathcal{B}(X))$ and $0 \leq w \in L^1(\mathbb{R}_+; \mathcal{B}(\mathcal{X}, X))$ such that*

$$S = V + W * S, \quad (2.2.39)$$

and we set

$$M := \sup_{t \geq 0} \|v(t)\|_{\mathcal{B}(X)} < \infty, \quad \Theta(t) := \|w(t)\|_{\mathcal{B}(\mathcal{X}, X)} \in L^1(\mathbb{R}_+). \quad (2.2.40)$$

Then there exists a pair $(\lambda_1, f_1) \in \mathbb{R}_+ \times X_+ \setminus \{0\}$ such that $\mathcal{L}f_1 = \lambda_1 f_1$.

Remark 2.2.5. (1) When $S_{\mathcal{L}}$ satisfies **(HS3)** then (2.2.39) holds with

$$V := \sum_{\ell=0}^{N-1} S_{\mathcal{B}} * (\mathcal{A}S_{\mathcal{B}})^{(*\ell)}, \quad W := (S_{\mathcal{B}}\mathcal{A})^{(*N)}. \quad (2.2.41)$$

(2) By definition of the norm $[\cdot]_0$ of \mathcal{X} , we see that \mathcal{X} is a weighted L^1 space or a weighted Radon measures space. In many applications, when both \mathcal{X} and X are Radon measures spaces, one can choose $N = 1$. On the other hand, when X is for instance a (possibly weighted) L^p space, one must take $N \geq 2$ in most of the applications. In condition (ii), the first bound is not really demanding and almost automatic in view of the estimates exhibited in Section 2.2.3. The second bound is a kind of regularity estimate reminiscent of the enlarging and shrinkage technique developed in [188, 113, 174].

Proof of Theorem 2.2.4. We split the proof into three steps.

Step 1. We define

$$R := \max(2\|\Theta\|_{L^1}, \|g_0\|),$$

for some $g_0 \in X_+$ such that $[g_0]_0 = 1$, and next the nonempty convex and compact (in the sense of \mathcal{X}) set

$$\mathcal{C} := \{f \in X_+; [f]_0 = 1, \|f\| \leq R\},$$

as well as the increasing function

$$\lambda(t) := \inf_{f \in \mathcal{C}} [S(t)f]_0, \quad \forall t \geq 0.$$

We have the alternative

- **(1)** $\sup \lambda > 2M$,
- **(2)** $\sup \lambda \leq 2M$.

Step 2. We assume that the first term **(1)** of the alternative holds true, or in other words, there exists $T_0 > 0$ such that

$$\forall f \in \mathcal{C}, \quad [S_{T_0}f]_0 \geq 2M. \quad (2.2.42)$$

We define as before

$$\Phi_{T_0}f := \frac{S_{T_0}f}{[S_{T_0}f]_0}, \quad \forall f \in \mathcal{C}.$$

By construction, for any $f \in \mathcal{C}$, we have $\Phi_{T_0}f \geq 0$ and $[\Phi_{T_0}f]_0 = 1$. On the other hand, using the splitting structure (2.2.39) and the estimates (2.2.40), we have

$$\|S(t)f\| \leq M\|f\| + \int_0^t \Theta(t-s)[S(s)f]_0 ds.$$

From hypothesis (i) and the semigroup property, we also have

$$[S_t f]_0 \geq [S_s f]_0, \quad \forall t \geq s \geq 0.$$

The two above estimates together imply

$$\begin{aligned} \|\Phi_{T_0}f\| &\leq \frac{M\|f\|}{[S_{T_0}f]_0} + \int_0^{T_0} \Theta(T_0-s) \frac{[S_s f]_0}{[S_{T_0}f]_0} ds \\ &\leq \frac{1}{2}\|f\| + \|\Theta\|_{L^1} \leq R, \end{aligned}$$

for any $f \in \mathcal{C}$. We have thus proved $\Phi_{T_0} : \mathcal{C} \rightarrow \mathcal{C}$. Thanks to the Tykonov fixed point Theorem, there exists $f_{T_0} \in \mathcal{C}$ such that $\Phi_{T_0}f_{T_0} = f_{T_0}$. In other words, we have built a pair of “almost eigenvalue and eigenfunction”

$$f_{T_0} \geq 0, \quad [f_{T_0}]_0 = 1, \quad S_{T_0}f_{T_0} = e^{\lambda_1 T_0} f_{T_0},$$

with $e^{\lambda_1 T_0} = [S_{T_0}f]_0$ and thus $\lambda_1 \in [0, \kappa_1]$. We conclude to the existence of $f_1 \in \mathcal{C}$ such that $\mathcal{L}f_1 = \lambda_1 f_1$ really similarly as in Step 2 of the Second proof of Theorem 2.2.1.

Step 3. We assume that the second term **(2)** of the alternative holds true. In that case, for any $n \geq 1$, there exists $f_n \in \mathcal{C}$ such that $[S(n)f_n]_0 \leq 2M$. By compactness, there exists $f_0 \in \mathcal{C}$ and a subsequence (f_{n_k}) such that $f_{n_k} \rightharpoonup f_0 \in \mathcal{C}$ and

$$\forall t \geq 0, \quad \forall k (n_k \geq t), \quad [S(t)f_{n_k}]_0 \leq [S(n_k)f_{n_k}]_0 \leq 2M,$$

so that

$$\forall t \geq 0, \quad [S(t)f_0]_0 \leq 2M. \quad (2.2.43)$$

Using this particular initial datum, we argue similarly as in [50, proof of Theorem 6.1], and we conclude to the existence of a stationary state. More precisely, we come back to the splitting structure (2.2.39) of the semigroup S and we introduce the associated Cesàro means

$$U_T := \frac{1}{T} \int_0^T S(t) dt, \quad V_T := \frac{1}{T} \int_0^T v(t) dt, \quad K_T := \frac{1}{T} \int_0^T (w * S)(t) dt, \quad (2.2.44)$$

for any $T > 0$. We obviously have

$$\|V_T\|_{\mathcal{B}(X)} \leq \frac{1}{T} \int_0^T \|v(t)\|_{\mathcal{B}(X)} dt \leq M.$$

On the other hand, we have

$$\int_0^T (w * S)(t) dt = \int_0^T \int_s^T w(t-s) dt S(s) ds \leq \int_0^T w(\tau) d\tau \int_0^T S(s) ds,$$

thanks to the Fubini theorem and the positivity of the two operators involved in this integral formula. We deduce

$$\begin{aligned} \|K_T f_0\| &\leq \left\| \int_0^T w(\tau) d\tau \frac{1}{T} \int_0^T S(s) f_0 ds \right\| \\ &\leq \int_0^\infty \|w(\tau)\|_{\mathcal{B}(X,X)} d\tau \left[\frac{1}{T} \int_0^T S(s) f_0 ds \right]_0 = \|\Theta\|_{L^1} [U_T f_0]_0, \end{aligned}$$

thanks to assumption (ii), so that $K_T f_0$ is uniformly bounded in X thanks to (2.2.43) and the elementary estimate $[U_T f_0]_0 \leq [S_T f_0]_0$. We then deduce that $U_T = V_T + K_T$ satisfies

$$\|U_T f_0\| \leq M \|f_0\| + 2M \|\Theta\|_{L^1} \quad \text{and} \quad 1 \leq [S_T f_0]_0 \leq 2M,$$

for any $T > 0$. By compactness, there exists $T_k \rightarrow +\infty$ and $f_1 \in X_+$ such that $U_{T_k} f \rightharpoonup f_1$ weakly in X . Thanks to the second inequality, we have $[f_1]_0 \geq 1$. We then argue thanks to the usual mean ergodic theorem trick. For any fixed $s > 0$, we observe that

$$\begin{aligned} S(s) f_1 - f_1 &= \lim_{k \rightarrow \infty} \left\{ \frac{1}{T_k} \int_0^{T_k} S(s) S(t) f_0 dt - \frac{1}{T_k} \int_0^{T_k} S(t) f_0 dt \right\} \\ &= \lim_{k \rightarrow \infty} \left\{ \frac{1}{T_k} \int_{T_k}^{T_k+s} S(t) f_0 dt - \frac{1}{T_k} \int_0^s S(t) f_0 dt \right\} \end{aligned}$$

weakly in X . By the lower semicontinuous property of the norm $[\cdot]_0$, we deduce

$$[S(s) f_1 - f_1]_0 \leq \liminf_{k \rightarrow \infty} \left\{ \frac{1}{T_k} \int_{T_k}^{T_k+s} [S(t) f_0]_0 dt + \frac{1}{T_k} \int_0^s [S(t) f_0]_0 dt \right\} = 0,$$

so that f_1 is a stationary solution, and thus f_1 is an eigenfunction associated to the eigenvalue $\lambda_1 = 0$. \square

Chapter 3

Irreducibility and the geometry of the first eigenvalue

In this chapter, we are concerned with the geometric part of the Krein-Rutman theorem for an unbounded operator \mathcal{L} on a Banach lattice X . We assume that the conclusions of the existence part are achieved, namely

(C1) the first primal and dual eigenvalue problem has a solution (λ_1, f_1, ϕ_1) : there exist $\lambda_1 \in \mathbb{R}$, $f_1 \in X_+ \cap D(\mathcal{L})$, $\phi_1 \in Y_+ \cap D(\mathcal{L}^*)$ such that

$$\|f_1\| = 1, \quad \mathcal{L}f_1 = \lambda_1 f_1, \quad \|\phi_1\| = 1, \quad \mathcal{L}^*\phi_1 = \lambda_1 \phi_1. \quad (3.0.1)$$

By construction, we also have $\Sigma(\mathcal{L}) \subset \{z \in \mathbb{C}, \Re(z) \leq \lambda_1\}$.

Assuming that S is positive as for the existence part and an additional strong maximum principle property, we analyze the first eigenvalue problem.

3.1 More about positivity

For further references, we introduce several notions which are strongly related to the positivity property for semigroups.

The signum operator sign . In a real Banach lattice X , we say that $\text{sign } f \in \mathcal{B}(X, X'')$ is a signum operator for $f \in X$, if it satisfies the following properties

$$\begin{aligned} (\text{sign } f) f &= |f|, \\ (\text{sign } f) g &\leq |g|, \quad \forall g \in X. \end{aligned}$$

In the sequel, we will always assume that such an operator exists. We refer to [7, Sections C.I & C.II] for a general introduction to the topic. In practice, we will only need a weak formulation of the sign operator (see below) which may be defined only in some subspace $\mathcal{X} \subset X$. We always additionally assume that the signum operator satisfies

$$\begin{aligned} (\text{sign } (-f)) (-g) &= (\text{sign } f)g, \quad \forall g \in X, \\ (\text{sign } f) g &= g, \quad \forall g \in X, \text{ if } f \in X_+, \end{aligned}$$

We also define

$$\text{sign}_+ f := \frac{1}{2}(I + \text{sign } f).$$

• When X is a space of functions, the sign operator $\text{sign } f$ associated to $f \in X$ corresponds to the multiplication by the function $\text{sign } f := \mathbf{1}_{f>0} - \mathbf{1}_{f<0}$. When $X := L^p(E)$, we obviously see that $\text{sign } f \in \mathcal{B}(L^p(E))$ for any $f \in L^p(E)$. On the other hand, when $X := C_0(E)$, we only have

$\text{sign } f \in \mathcal{B}(C_0(E); \mathcal{M}^\infty(E))$, where $\mathcal{M}^\infty(E)$ denotes the space of uniformly bounded measurable functions, so that $\mathcal{M}^\infty(E) \subset (C_0(E))''$. In a space of bounded measures $X = M^1(E)$, we may define the sign operator by means of the Radon-Nikodym theorem. For a given $f \in M^1(E)$, using Hahn decomposition, there exists indeed a measurable function $\alpha : E \rightarrow \{-1, 1\}$ such that $f = \alpha|f|$, and we then define $(\text{sign } f)g = \alpha g$ for any $g \in M^1(E)$.

• When X is σ -order complete, in the sense that any increasing and upper bounded sequence has a supremum (a common least upper bound), the operator sign exists and is more regular, namely $\text{sign } f \in \mathcal{B}(X)$ for any $f \in X$, see [190] and also [7, Section C.I.8]. We recover in particular that $\text{sign } f \in \mathcal{B}(L^p(E))$ for any $f \in L^p(E)$.

Weak principle maximum and Kato's inequality. We introduce now two definitions formulated on an operator \mathcal{L} which are almost equivalent to the positivity property of the semigroup S when \mathcal{L} is the generator of S .

- We say that the operator \mathcal{L} satisfies the *weak maximum principle* when

$$\kappa \in \mathbb{R}, f \in D(\mathcal{L}) \text{ and } (\kappa - \mathcal{L})f \geq 0 \quad \text{imply} \quad f \geq 0; \quad (3.1.2)$$

- We say that the operator \mathcal{L} satisfies *Kato's inequality* when

$$\forall f \in D(\mathcal{L}), \quad \mathcal{L}|f| \geq (\text{sign } f)\mathcal{L}f. \quad (3.1.3)$$

Since $|f|$ does not necessarily belong to $D(\mathcal{L})$, the correct way to understand Kato's inequality is

$$\forall f \in D(\mathcal{L}), \quad \forall \psi \in D(\mathcal{L}^*) \cap Y_+, \quad \langle |f|, \mathcal{L}^*\psi \rangle \geq \langle (\text{sign } f)\mathcal{L}f, \psi \rangle. \quad (3.1.4)$$

We immediately see from the definitions that (3.1.3) is equivalent to assuming

$$\forall f \in D(\mathcal{L}), \quad \mathcal{L}f_+ \geq (\text{sign}_+ f)\mathcal{L}f. \quad (3.1.5)$$

Remark 3.1.1. We complement Lemma 2.1.1, by claiming that for a semigroup $S = S_{\mathcal{L}}$ on a Banach lattice X , there is equivalence between the fact that S is positive and $\kappa - \mathcal{L}$ satisfies the weak maximum principle for any $\kappa > \omega(\mathcal{L})$, what is straightforward using that these properties are equivalent to the fact that $\mathcal{R}_{\mathcal{L}}(\kappa) \geq 0$ for any $\kappa > \omega(\mathcal{L})$. These properties also imply that Kato's inequality holds true, see [190, 5], [8, Proposition 1.1], [6, Remark 3.10] and the textbook [7, Theorems C.II.2.4, C.II.2.6 and Remark C-II.3.12].

We end this chapter by introducing notions of strict positivity which strengthen the previous defined positivity condition.

Strict order. We may define a first strict order $>$ (or $<$) on X by writing for $f \in X$

$$f > 0 \quad \text{if} \quad f \in X_+ \setminus \{0\}$$

and similarly a strict order $>$ (or $<$) on X' by writing for $\psi \in X'$

$$\psi > 0 \quad \text{if} \quad \psi \in X'_+ \setminus \{0\}.$$

We may next define a second (and stronger) strict order \gg (or \ll) on X by writing for $f \in X$

$$f \gg 0 \text{ or } f \in X_{++} \quad \text{iff} \quad \forall \psi \in X'_+ \setminus \{0\}, \quad \langle \psi, f \rangle > 0,$$

and similarly a strict order \gg (or \ll) on X' by writing for $\psi \in X'$

$$\psi \gg 0 \text{ or } \psi \in X'_{++} \quad \text{iff} \quad \forall g \in X_+ \setminus \{0\}, \quad \langle \psi, g \rangle > 0.$$

On the two Banach lattices X and Y , we have thus three positivity notions with \gg (associated to X_{++} and Y_{++}) stronger than $>$ (associated to $X_+ \setminus \{0\}$ and $Y_+ \setminus \{0\}$) which itself is stronger than \geq (associated to X_+ and Y_+).

Let us comment on the notion of strict positivity.

Examples 3.1.2. In the space $C_0(E)$, the strict order is defined by $f \gg 0$ iff $f(x) > 0$ for any $x \in E$. In a space $L^p(E, \mathcal{E}, \mu)$, $1 \leq p \leq \infty$, the strict order is defined by $f \gg 0$ iff $f(x) > 0$ for μ -a.e. $x \in E$. In the space $M^1(E)$, the strict order is defined by duality by $f \gg 0$ iff $\langle f, \varphi \rangle > 0$ for any $\varphi \in C_0(E)$, $\varphi \geq 0$, $\varphi \neq 0$.

Remark 3.1.3. In a Banach lattice X such that $\text{int}X_+ \neq \emptyset$, the common definition of the strict order is $X_{++} := \text{int}X_+$. In particular, in the case when E is compact and $X = C_0(E) = C(E)$, we have $\text{int}X_+ \neq \emptyset$ and the definition of X_{++} introduced in Examples 3.1.2 coincides with $\text{int}X_+$. In all the other examples considered, we have $\text{int}(X_+) = \emptyset$, and thus our definition of the strict order does not coincide with the one defined through the set $\text{int}(X_+)$.

Remark 3.1.4. Another notion of strict order can be defined through the notions of ideals and quasi-interior points as briefly explained now, see [7] or [20, Chapter 10] and the references therein for details. Defining the segment $[g_1, g_2]$ and the set I_f for $g_1, g_2 \in X$ and $f \in X_+ \setminus \{0\}$ by

$$[g_1, g_2] := \{g \in X; g_1 \leq g \leq g_2\}, \quad I_f := \bigcup_{k \geq 0} [-kf, kf],$$

one shows that I_f is an ideal in the sense that $g \in I_f$ implies $|g| \in I_f$ and $0 \leq g \leq f$ implies $g \in I_f$. We say that f is an order unit if $I_f = X$. When $\text{int}X_+ \neq \emptyset$, we find that f is an order unit iff $f \in \text{int}X_+$ from Lemma 2.1.22, so that we recover the notion of strict positivity defined above. On the other hand, we say that f is a quasi-interior point if $\bar{I}_f = X$. When $X = L^p(E, \mathcal{E}, \mu)$, $1 \leq p < \infty$, μ is a σ -finite diffuse (or atomless) measure, one shows that f is a quasi-interior point iff $f > 0$ a.e., see [20, Examples 10.16], so that we also recover the notion of strict positivity defined above by defining $f \in X_{++}$ iff $\bar{I}_f = X$.

We finally point out the following result. For a semigroup $S = S_{\mathcal{L}}$ in a Banach lattice, under the mild assumption that there exists a strictly positive subeigenvector for the dual problem, namely

$$\exists \phi \in X'_{++}, \exists b \in \mathbb{R}, \quad \mathcal{L}^* \phi \leq b \phi,$$

Kato's inequality implies that S is positive, see [8, Theorem 1.6].

3.2 Irreducibility and strong maximum principle

We present some material involving the strict positivity.

In the sequel, for both spaces we will always assume $X_{++} \neq \emptyset$, $Y_{++} \neq \emptyset$ and

$$f_+ \in X_{++} \text{ implies } f \in X_{++}, \quad \phi_+ \in Y_{++} \text{ implies } \phi \in Y_{++}. \quad (3.2.6)$$

Lemma 3.2.5. *The property (3.2.6) holds true when X is a reflexive space, $X = L^p$, $p \in [1, \infty]$, $X = C_0$ or $X = M^1$.*

Proof of Lemma 3.2.5. Assume first that X is reflexive and fix $f \in X$ such that $f_+ \in X_{++}$. There exists $0 \leq \varphi^* \in X'$, $\|\varphi^*\| = 1$ such that

$$\|f_-\| = \langle \varphi^*, f_- \rangle = \sup_{0 \leq \psi \leq \varphi^*} \langle \psi, -f \rangle,$$

where we have used the corollary (2.1.2) of the Hahn-Banach dominated extension theorem in the first equality and the definition f_- as an element of X'' in the second equality. There next exists $0 \leq \psi^* \leq \varphi^*$, such that

$$\|f_-\| = \langle \psi^*, -f \rangle = \langle \psi^*, f_- - f_+ \rangle,$$

where we have used that $B_{X'}$ is compact for the weakly $*$ topology $\sigma(X', X)$ in the first equality. We deduce

$$0 \leq \langle \varphi^* - \psi^*, f_- \rangle = -\langle \psi^*, f_+ \rangle \leq 0,$$

with $\langle \psi^*, f_+ \rangle < 0$ if $\psi^* \neq 0$. That implies $\psi^* = 0$ and $f_- = 0$.

We next assume $X = L^p(E, \mathcal{E}, \mu)$ and we take $f \in X$ such that $f_+ \in X_{++}$. From the definition of X_{++} made explicit in Examples 3.1.2, we have $f_+(x) = \max(f(x), 0) > 0$ a.e., so that $f(x) > 0$ a.e. and finally $f_-(x) = 0$ a.e.. We last assume $X = M^1(E, \mathcal{E})$ and we take again $f \in X$ such that $f_+ \in X_{++}$. Writing $f = \alpha|f|$, where $\alpha : E \rightarrow \{-1, 1\}$ is the measurable function related to Hahn decomposition as introduced in Section 3.1, the condition $f_+ = \alpha_+|f| > 0$ means that $\alpha_+(x) > 0$ for $|f|$ -a.e. $x \in E$. We deduce again $\alpha(x) > 0$ for $|f|$ -a.e. $x \in E$ and thus $f_- = \alpha_-|f| = 0$. \square

For an operator $A \in \mathcal{B}(X)$, we have yet formalized a positivity condition in section 2.1.1, by (P1) $A \geq 0$ if $A : X_+ \rightarrow X_+$.

Other possible definition of positivity may be

$$(P2) \ A : X_+ \setminus \{0\} \rightarrow X_+ \setminus \{0\};$$

$$(P3) \ A : X_{++} \rightarrow X_{++}.$$

We now define a stronger notion of positivity, named as strong positivity condition, by

$$(P4) \ A > 0 \text{ if } A : X_+ \setminus \{0\} \rightarrow X_{++}.$$

We list without proof some elementary properties about these different notions and also refer to Section 5.1 for further discussion. We have (P2) \Rightarrow (P1), (P3) \Rightarrow (P1) as well as (P4) \Rightarrow ((P3), (P2)). We also have $A : X_+ \rightarrow X_+$ iff $A^* : Y_+ \rightarrow Y_+$; $A : X_{++} \rightarrow X_{++}$ iff $A^* : Y_{++} \rightarrow Y_{++}$; $A : X_+ \setminus \{0\} \rightarrow X_{++}$ iff $A^* : Y_+ \setminus \{0\} \rightarrow Y_{++}$.

We say that $\lambda - \mathcal{L}$ satisfies the strong maximum principle if

$$f \in X_+ \cap D(\mathcal{L}), (\lambda - \mathcal{L})f \geq 0 \quad \text{imply} \quad f \gg 0 \text{ or } f = 0. \quad (3.2.7)$$

We say that a positive semigroup S is irreducible if

$$\forall f \in X_+ \setminus \{0\}, \forall \phi \in Y_+ \setminus \{0\}, \exists \tau > 0 \quad \langle S_\tau f, \phi \rangle > 0. \quad (3.2.8)$$

A semigroup S is classically said to be irreducible and aperiodic if the above positivity condition holds for all sufficiently large times, namely

$$\forall f \in X_+ \setminus \{0\}, \forall \phi \in Y_+ \setminus \{0\}, \exists T > 0, \forall \tau \geq T \quad \langle S_\tau f, \phi \rangle > 0. \quad (3.2.9)$$

Other notions of strong positivity for the semigroup S are

$$\exists T > 0, \quad S_T : X_+ \setminus \{0\} \rightarrow X_{++}, \quad (3.2.10)$$

$$\exists T > 0, \quad \int_0^T S(t)dt : X_+ \setminus \{0\} \rightarrow X_{++}. \quad (3.2.11)$$

We summarize some possible implications between the previous positivity notions.

Lemma 3.2.6. *For a positive semigroup S , the following hold:*

- (1) *The pointwise strong positivity condition (3.2.10) implies the condition (3.2.11);*
- (2) *The integral strong positivity condition (3.2.11) implies the irreducibility condition (3.2.8), but the reverse implication is false. Similarly, the irreducibility and aperiodicity condition (3.2.9) implies the irreducibility condition (3.2.8), but the reverse implication is false;*
- (3) *The irreducibility condition (3.2.8) is equivalent to the fact that $\mathcal{R}_{\mathcal{L}}(\lambda) : X_+ \setminus \{0\} \rightarrow X_{++}$, for any $\lambda > \lambda_1$ as well as to the fact that $\lambda - \mathcal{L}$ satisfies the strong maximum principle (3.2.7) for any $\lambda > \lambda_1$.*

The result is very classic, at least for strongly positive semigroup, see e.g. [7, Definition C.3.1] or [20, Proposition 14.10]. For the sake of completeness, we however present a short proof.

Proof of Lemma 3.2.6. We prove (1). We assume (3.2.10) and we fix $g \in X_+ \setminus \{0\}$, $\phi \in Y_+ \setminus \{0\}$, so that $\langle S(T)g, \phi \rangle > 0$. Observing that the function $t \mapsto \langle S(t)g, \phi \rangle$ is continuous, there exists $\varepsilon > 0$ such that $\langle S(t)g, \phi \rangle > 0$ for any $t \in [T - \varepsilon, T]$, so that

$$\left\langle \int_0^T S(t)dtg, \phi \right\rangle = \int_0^T \langle S(t)g, \phi \rangle dt > 0.$$

Because $\phi \in Y_+ \setminus \{0\}$ may be chosen arbitrary, we deduce (3.2.11).

We prove (2). We assume now (3.2.11) and we fix $g \in X_+ \setminus \{0\}$, $\phi \in Y_+ \setminus \{0\}$, so that

$$\int_0^T \langle S(t)g, \phi \rangle dt = \left\langle \int_0^T S(t)dtg, \phi \right\rangle > 0,$$

by assumption. We get (3.2.8) by observing that the function $t \mapsto \langle S(t)g, \phi \rangle$ must be positive somewhere on $[0, T]$. For the reverse implication we refer to [27, 93], where is studied an example of growth-fragmentation operator associated to mitosis satisfying the irreducibility condition (3.2.8) but not the integral strong positivity condition (3.2.11) nor the irreducibility and aperiodicity condition (3.2.9), see also Section ??.

We prove (3). We finally assume (3.2.8). From the above continuity argument, for any $g \in X_+ \setminus \{0\}$, $\phi \in Y_+ \setminus \{0\}$ there exist $\tau > \varepsilon > 0$ such that $\langle S(t)g, \phi \rangle > 0$ for any $t \in [\tau - \varepsilon, \tau + \varepsilon]$. As a consequence and thanks to the representation formula (2.1.13) for some fixed $\lambda > \lambda_1$ which holds thanks to Lemma 2.1.2-(ii), we have

$$\langle \mathcal{R}_{\mathcal{L}}(\lambda)g, \phi \rangle = \left\langle \int_0^\infty e^{-\lambda t} S(t)dtg, \phi \right\rangle > 0.$$

Because $\phi \in Y_+ \setminus \{0\}$ is arbitrary, we have established that $\mathcal{R}_{\mathcal{L}}(\lambda)g \in X_{++}$ for any $g \in X_+ \setminus \{0\}$. In other words, when $\lambda > \lambda_1$ and $f \in X_+ \cap D(\mathcal{L})$ satisfy $g := (\lambda - \mathcal{L})f \geq 0$, we deduce that $f = \mathcal{R}_{\mathcal{L}}(\lambda)g \in X_{++}$, what is the strong maximum principle. This one is obviously equivalent to the strong positivity property $\mathcal{R}_{\mathcal{L}}(\lambda) : X_+ \setminus \{0\} \rightarrow X_{++}$. On the other way round, writing the above identity as

$$\int_0^\infty e^{-\lambda t} \langle S(t)g, \phi \rangle dt = \langle \mathcal{R}_{\mathcal{L}}(\lambda)g, \phi \rangle,$$

we see that the strong maximum principle implies that the RHS term is positive for any $g \in X_+ \setminus \{0\}$, $\phi \in Y_+ \setminus \{0\}$. As a consequence, the LHS term is positive and there exists $\tau > 0$ such that $\langle S(\tau)g, \phi \rangle > 0$, which is nothing but the irreducibility condition (3.2.8). \square

We present two other elementary results about the strong maximum principle.

Lemma 3.2.7. *Consider \mathcal{L} satisfying (H1) and $\lambda \in \mathbb{R}$. Then the following assertions are equivalent*

- (1) $\lambda - \mathcal{L}$ satisfies the strong maximum principle for any $f \in D(\mathcal{L}) \cap X_+$;
- (2) $\lambda - \mathcal{L}$ satisfies the strong maximum principle for any $f \in D(\mathcal{L}^k) \cap X_+$ for some $k \geq 1$;
- (3) $\lambda - \mathcal{L}^*$ satisfies the strong maximum principle for any $\phi \in D(\mathcal{L}^*) \cap Y_+$;
- (4) $\lambda - \mathcal{L}$ satisfies the strong maximum principle for any $\phi \in D((\mathcal{L}^*)^\ell) \cap Y_+$ for some $\ell \geq 1$.

Proof of Lemma 3.2.7. Assume that $\lambda - \mathcal{L}$ satisfies the strong maximum principle for some $\lambda \in \mathbb{R}$ and $k \geq 1$ and consider $\phi \in D(\mathcal{L}^*) \cap Y_+ \setminus \{0\}$ such that $(\lambda - \mathcal{L}^*)\phi \geq 0$. For any $\kappa > \max(\lambda, \lambda_1)$ and $g \in D(\mathcal{L}^{k-1}) \cap X_+ \setminus \{0\}$, thanks to (H1) and the strong maximum principle, there exists $f \in D(\mathcal{L}^k) \cap X_{++}$ such that $(\kappa - \mathcal{L})f = g$. As a consequence, we have

$$\begin{aligned} \langle \phi, g \rangle &= \langle \phi, (\kappa - \mathcal{L})f \rangle \\ &= \langle (\kappa - \mathcal{L}^*)\phi, f \rangle \geq (\kappa - \lambda) \langle \phi, f \rangle > 0. \end{aligned}$$

Since $g \in D(\mathcal{L}^{k-1}) \cap X_+$ is arbitrary and $D(\mathcal{L}^{k-1}) \cap X_+$ is dense in X_+ , we deduce that $\phi \gg 0$. We have proved that $\lambda - \mathcal{L}^*$ satisfies the strong maximum principle. The other implications can be proved similarly. \square

Remark 3.2.8. *We may replace the condition (1) by assuming that $\lambda - \mathcal{L}$ satisfies the strong maximum principle for $f \in \mathcal{C} \cap X_+$ for a subspace $\mathcal{C} \subset D(\mathcal{L})$ such that $(\lambda - \mathcal{L})^{-1} \in \mathcal{B}(\mathcal{C})$ and \mathcal{C} is dense in X .*

The strong maximum principle can be seen as a consequence of the weak maximum principle together with the existence of a family of strictly positive barrier functions. We give now a typical result which can be applied (or modified in order to be applied) in many situations.

Lemma 3.2.9. *We assume that*

(i) *the operator $\lambda - \mathcal{L}$ satisfies the weak maximum principle;*

(ii) *there exists a subset $\mathcal{G} \subset X_{++} \cap D(\mathcal{L})$ such that $\forall f \in D(\mathcal{L}) \cap X_+ \setminus \{0\}, \exists g \in \mathcal{G}$ such that $(g - f)_+ \in D(\mathcal{L})$ and $(\mathcal{L} - \lambda)g \geq 0$.*

Then $\lambda - \mathcal{L}$ satisfies the strong maximum principle.

Proof of Lemma 3.2.9. We consider $f \in D(\mathcal{L}) \cap X_+ \setminus \{0\}$ such that $(\lambda - \mathcal{L})f \geq 0$ and choose $g \in \mathcal{G}$ such that $h := (g - f)_+ \in D(\mathcal{L})$. We remark that from Kato's inequality

$$(\mathcal{L} - \lambda)h \geq \text{sign}_+(g - f)(\mathcal{L} - \lambda)(g - f) \geq 0.$$

As a consequence of the weak maximum principle, we have $h \leq 0$. That implies $h = 0$, so that $g - f \leq 0$ and finally $f \gg 0$. \square

The above barrier functions technique is also useful for obtaining the condition **(H2)** (possibly in a constructive way).

Lemma 3.2.10. *For an operator \mathcal{L} , we assume that*

(i) *the condition **(H1)** holds with a constant $\kappa_1 \in \mathbb{R}$;*

(ii) *the hypothesis (ii) in Lemma 3.2.9 holds;*

(iii) *there exists $h_0 \in X_+ \setminus \{0\}$ such that for any $g \in \mathcal{G}$ there exists $\varepsilon > 0$ such that $g \geq \varepsilon h_0$.*

*Then the property **(H2)** holds true.*

Proof of Lemma 3.2.10. Thanks to assumption (i), we may define $f_0 \in D(\mathcal{L}) \cap X_+ \setminus \{0\}$ as the solution to the equation $(\kappa_1 - \mathcal{L})f_0 = h_0$. From the proof of Lemma 3.2.9 and condition (iii), there exists $g \in \mathcal{G}$ and next $\varepsilon > 0$ such that $f_0 \geq g \geq \varepsilon h_0$. Coming back to the equation, we have

$$\mathcal{L}f_0 = \kappa_1 f_0 - h_0 \geq (\kappa_1 - \varepsilon^{-1})f_0,$$

so that condition **(H2)** holds true with $\kappa_0 := \kappa_1 - \varepsilon^{-1}$ thanks to Lemma 2.1.4-(ii). \square

3.3 The geometry of the first eigenvalue problem

We come back on and state a result about the geometry of the first eigenvalue.

On the one hand, we assume that X is a Banach lattice such that

(X1) the signum operator is well define in X , $X_{++} \neq \emptyset$ and the property (3.2.6) holds true.

On the other hand, we consider an operator \mathcal{L} on X which satisfies the conclusion **(C1)** about the existence of a solution (λ_1, f_1, ϕ_1) to the first eigentriplet problem. We also assume

(H1') the weak maximum principle

$$\lambda > \lambda_1, f \in D(\mathcal{L}), (\lambda - \mathcal{L})f \geq 0 \quad \text{imply} \quad f \geq 0 \quad (3.3.12)$$

and its *Kato's inequalities* counterpart

$$(\text{sign}f)\mathcal{L}f \leq \mathcal{L}|f|, \quad (\text{sign}_+f)\mathcal{L}f \leq \mathcal{L}f_+, \quad (3.3.13)$$

as well as

(H4) the strong maximum principle

$$\lambda \geq \lambda_1, f \in X_+ \cap D(\mathcal{L}), (\lambda - \mathcal{L})f \geq 0 \quad \text{imply} \quad f \gg 0 \text{ or } f = 0. \quad (3.3.14)$$

We may then state our main result in this section, where we recall that $N(A)$ denotes the null space associated to the operator A .

Theorem 3.3.11. *We assume that X is a Banach lattice satisfying (X1). We consider an unbounded operator \mathcal{L} on X which satisfies the conclusion (C1) about the existence of a solution (λ_1, f_1, ϕ_1) to the first eigentriplet eigenvalue problem, the weak maximum principles and Kato's inequality (H1'), as well as the strong maximum principle (H4).*

Then the following hold

i) $f_1 \gg 0$, $\phi_1 \gg 0$ and λ_1 is the unique eigenvalue associated to a positive eigenvector.

ii) λ_1 is algebraically simple:

$$\begin{aligned} N((\mathcal{L} - \lambda_1)^k) &= \text{Span}(f_1), \quad \forall k \geq 1, \\ N((\mathcal{L}^* - \lambda_1)^k) &= \text{Span}(\phi_1), \quad \forall k \geq 1, \end{aligned}$$

in particular f_1 (resp. ϕ_1) is the unique positive and normalized eigenvector of \mathcal{L} (resp. \mathcal{L}^) associated to λ_1 . Finally, the projection on the first eigenspace (associated to λ_1) writes*

$$\Pi f := \langle f, \phi_1 \rangle f_1.$$

Remark 3.3.12. (1) *It is worth emphasizing again that (3.3.12) is true when \mathcal{L} is the generator of a positive semigroup, and that (3.3.14) is true when $S_{\mathcal{L}}$ enjoys additional strong positivity (or irreducibility) condition as formulated in (3.2.8), (3.2.9), (3.2.10) or (3.2.11). As a consequence, the conclusion of Theorem 3.3.11 holds true when \mathcal{L} is the generator of a positive semigroup which satisfies the hypotheses of the existence part of the Krein-Rutman Theorem 2.1.20 and one of the additional above strict positivity conditions.*

(2) *Theorem 3.3.11 has to be compared with the seminal Krein and Rutman Theorem ?? ([142]), to the many results gathered in [7, Part C-III] (see in particular [7, Proposition C.3.5], [7, Theorem C.3.8] and the original paper [111]) and to the more recent contributions [177, Theorem 5.3], [20, Theorem 14.15] and [138, Theorem 5.1]. Probably many of the conclusions of Theorem 3.3.11 are very similar (or even included) in the material of [7, Part C-III]. However, our assumptions slightly different since we do not make explicit reference to a positive semigroup but rather refer to the weak and strong maximum principle.*

(3) *Our proof is quite direct and elementary and uses similar arguments as those used during the proof of [177, Theorem 4.3] and [138, Theorem 5.1]. We learnt this kind of technique in the (less abstract and general) proof of the uniqueness part of [204, Lemma 2.1].*

(4) *From ii), we deduce that \mathcal{L} decomposes according to $X = X_0 \oplus X_1$ with $X_1 := \text{Span } f_1$ and $X_0 := (\text{Span } \phi_1)^\perp = \{f \in X; \langle f, \phi_1 \rangle = 0\}$ in the sense of [136, §III.5.6]. More precisely, $X = X_0 \oplus X_1$ is a topological direct sum, $\mathcal{L} : X_0 \cap D(\mathcal{L}) \rightarrow X_0$ and $\mathcal{L} : X_1 \rightarrow X_1$.*

The proof of Theorem 3.3.11 is split into the following Lemma 3.3.13, Lemma 3.3.15, Lemma 3.3.16 and Lemma 3.3.18.

Lemma 3.3.13. *Under assumptions (C1) and (H4), the solution (λ_1, f_1, ϕ_1) to the first eigentriplet problem satisfies*

$$f_1 \gg 0 \quad \text{and} \quad \phi_1 \gg 0. \quad (3.3.15)$$

Proof of Lemma 3.3.13. By definition of f_1 , we have

$$f_1 \in X_+ \setminus \{0\}, \quad (\lambda_1 - \mathcal{L})f_1 \geq 0,$$

and thus $f_1 \gg 0$ from **(H4)**. Thanks to Lemma 3.2.7, the strong maximum principle holds for $\lambda_1 - \mathcal{L}^*$, and the same proof leads to the same conclusion, namely $\phi_1 \gg 0$. \square

Remark 3.3.14. *It is worth emphasizing that the same conclusion holds when we only assume that $f_1 \in X_+$ and $\phi_1 \in Y_+$ satisfy*

$$\|f_1\| = 1, \quad \mathcal{L}f_1 = \lambda_1 f_1, \quad \|\phi_1\| = 1, \quad \mathcal{L}^*\phi_1 = \lambda_1^* \phi_1, \quad (3.3.16)$$

what we see by just repeating the argument. In that case, we deduce that $\lambda_1^ = \lambda_1$ by writing*

$$\lambda_1 \langle f_1, \phi_1 \rangle = \langle \mathcal{L}f_1, \phi_1 \rangle = \langle f_1, \mathcal{L}^*\phi_1 \rangle = \lambda_1^* \langle f_1, \phi_1 \rangle,$$

and observing that $\langle f_1, \phi_1 \rangle \neq 0$.

Lemma 3.3.15. *Under assumptions **(C1)** and **(H4)**, λ_1 is the unique eigenvalue associated to a positive eigenvector for \mathcal{L} (resp. for \mathcal{L}^*).*

Proof of Lemma 3.3.15. Consider $\lambda \in \mathbb{C}$ and $f \in X_+ \setminus \{0\}$ such that $\mathcal{L}f = \lambda f$. We compute

$$0 = \langle (\lambda - \mathcal{L})f, \phi_1 \rangle = \langle f, (\lambda - \mathcal{L}^*)\phi_1 \rangle = (\lambda - \lambda_1) \langle f, \phi_1 \rangle,$$

and thus $\lambda = \lambda_1$ since $\langle f, \phi_1 \rangle > 0$. The same proof applies to the dual problem. \square

Lemma 3.3.16. *Under assumptions **(C1)**, **(H1')** and **(H4)**, we have $N(\mathcal{L} - \lambda_1) = \text{Span}(f_1)$ (resp. $N(\mathcal{L}^* - \lambda_1) = \text{Span}(\phi_1)$). In particular, f_1 (resp. ϕ_1) is unique (because of the positivity and normalization condition).*

Proof of Lemma 3.3.16. Consider a eigenfunction $f \in X \setminus \{0\}$ associated to the eigenvalue λ_1 . First, we observe from Kato's inequality that

$$\lambda_1 |f| = \lambda_1 \text{sign}(f) f = \text{sign}(f) \mathcal{L}f \leq \mathcal{L}|f|.$$

That inequality is in fact an equality, otherwise we would have

$$\lambda_1 \langle |f|, \phi_1 \rangle \neq \langle \mathcal{L}|f|, \phi_1 \rangle = \langle |f|, \mathcal{L}^*\phi_1 \rangle = \lambda_1 \langle |f|, \phi_1 \rangle,$$

and a contradiction. As a consequence, $|f|$ is a solution to the eigenvalue problem $\lambda_1 |f| = \mathcal{L}|f|$, so that

$$\lambda_1 f_{\pm} = \mathcal{L}f_{\pm},$$

by writing $f_{\pm} = (|f| \pm f)/2$. The strong maximum principle assumption implies $f_{\pm} \gg 0$ or $f_{\pm} = 0$, and thus $f_+ \gg 0$ or $f_- \gg 0$ thanks to Lemma 3.3.13. Without loss of generality we may assume $f_+ \gg 0$. From (3.2.6), we then deduce $f \gg 0$. We introduce the normalized eigenfunctions $\tilde{f} := rf$ and $\tilde{f}_1 = r_1 f_1$ with

$$\langle \tilde{f}, \phi_1 \rangle = \langle \tilde{f}_1, \phi_1 \rangle = 1. \quad (3.3.17)$$

Now, thanks to Kato's inequality again, we write

$$\lambda_1 (\tilde{f} - \tilde{f}_1)_+ = \text{sign}_+(\tilde{f} - \tilde{f}_1) \mathcal{L}(\tilde{f} - \tilde{f}_1) \leq \mathcal{L}(\tilde{f} - \tilde{f}_1)_+,$$

and for the same reason as above that last inequality is in fact an inequality. The strong maximum principle implies that either $(\tilde{f} - \tilde{f}_1)_+ = 0$, or in other words $\tilde{f} \leq \tilde{f}_1$, either $(\tilde{f} - \tilde{f}_1)_+ > 0$ or in other words $\tilde{f} > \tilde{f}_1$. Because of the identity (3.3.17) and the fact that $\phi_1 \in X'_+ \setminus \{0\}$, the second case in the above alternative is not possible. Repeating the same argument with $(\tilde{f}_1 - \tilde{f})_+$ we get that $\tilde{f}_1 \leq \tilde{f}$ and we conclude with $\tilde{f} = \tilde{f}_1$. The same proof applies to the dual problem. \square

Remark 3.3.17. Under the same hypotheses as in Lemma 3.3.16, we have $\psi \in \text{span}(\phi_1)$ if $\psi \in Y_+$ satisfies $\mathcal{L}^*\psi \geq \lambda_1\psi$ and $g \in \text{span}(f_1)$ if $g \in X_+$ satisfies $\mathcal{L}g \geq \lambda_1g$. In the second case, we indeed cannot have $\mathcal{L}^*g - \lambda_1g \in X_+ \setminus \{0\}$, since this would imply

$$\langle \mathcal{L}g - \lambda_1g, \phi_1 \rangle > 0,$$

and this would be in contradiction with the fact that

$$\langle \mathcal{L}g - \lambda_1g, \phi_1 \rangle = \langle g, \mathcal{L}^*\phi_1 - \lambda_1\phi_1 \rangle = 0.$$

We thus must have $\mathcal{L}g - \lambda_1g = 0$ and we conclude thanks to Lemma 3.3.16. The same proof applies to the dual problem.

Lemma 3.3.18. Under assumptions **(C1)**, **(H1')** and **(H4)**, λ_1 is algebraically simple.

Proof of Lemma 3.3.18. We use an induction argument. We have already proved that $N((\mathcal{L} - \lambda_1)^k) = \text{Span}(f_1)$ for $k = 1$. Assume then the result proved for any ℓ , $1 \leq \ell \leq k$, and consider $f \in N((\mathcal{L} - \lambda_1)^{k+1})$. That means that $(\mathcal{L} - \lambda_1)f \in N((\mathcal{L} - \lambda_1)^k)$, and thus $(\mathcal{L} - \lambda_1)f = rf_1$, with $r \in \mathbb{R}$, thanks to the induction hypothesis. If $r = 0$, then $f \in N(\mathcal{L} - \lambda_1) = \text{Span}(f_1)$. Otherwise, $r \neq 0$, and then

$$\lambda_1 \langle f, \phi_1 \rangle = \langle f, \mathcal{L}^*\phi_1 \rangle = \langle \mathcal{L}f, \phi_1 \rangle = \langle \lambda_1f + rf_1, \phi_1 \rangle,$$

which in turn implies $r\langle f_1, \phi_1 \rangle = 0$ and a contradiction. That concludes the proof. \square

3.4 Mean ergodicity

We deduce from the above analysis a first classical and general but rough information about the long-time behaviour of the trajectories associated to a semigroup. More precisely, assuming the existence and uniqueness of the first eigentriplet (λ_1, f_1, ϕ_1) for the generator \mathcal{L} of a semigroup S and introducing the rescaled semigroup $\tilde{S}_t := e^{-\lambda_1 t} S(t)$, we wish to establish the following mean ergodic property

(CE1) for any $f \in X$, there holds

$$\frac{1}{T} \int_0^T \tilde{S}_t f dt \rightarrow \langle f, \phi_1 \rangle f_1, \quad \text{as } T \rightarrow \infty, \quad (3.4.18)$$

in a sense to be specified.

Theorem 3.4.19. Consider a positive semigroup S on a Banach lattice X and assume that its generator \mathcal{L} satisfies the conclusions of Theorem 3.3.11 about the existence and uniqueness of the first eigentriplet (λ_1, f_1, ϕ_1) . With the above notations, we assume furthermore that

- (1) (\tilde{S}_t) is bounded;
- (2) B_X is weakly compact for a topology which makes $f \mapsto \langle f, \phi_1 \rangle$ continuous.

Then, the above mean ergodic property **(CE1)** holds for the topology introduced in (2).

The result is not new, see for instance [82, Corollary V.4.6]. We present however the very short proof for completeness and further discussion.

Proof of Theorem 3.4.19. Fix $f \in X$ and define

$$u_T := \frac{1}{T} \int_0^T \tilde{S}_t f dt.$$

From (1), we have

$$\|u_T\| \leq \frac{1}{T} \int_0^T \|\tilde{S}_t f\| dt \leq M \|f\|, \quad \forall T > 0.$$

We also compute

$$\langle u_T, \phi_1 \rangle = \frac{1}{T} \int_0^T \langle \tilde{S}_t f, \phi_1 \rangle dt = \langle f, \phi_1 \rangle, \quad \forall T > 0.$$

Thanks to assumption **(2)**, we deduce that there exists $f^* \in X$ and a sequence (T_k) such that

$$u_{T_k} \rightarrow f^* \quad \text{and} \quad \langle f^*, \phi_1 \rangle = \langle f, \phi_1 \rangle.$$

Because $(\tilde{S}_t f)$ is bounded, we may use the usual ergodicity trick as in the second proof of Theorem 2.2.1 and for any $t > 0$, we have

$$\tilde{S}_t f^* - f^* = \lim_{k \rightarrow \infty} \frac{1}{T_k} \left\{ \int_{T_k}^{T_k+t} \tilde{S}_s f ds - \int_0^t \tilde{S}_s f ds \right\} = 0.$$

We have established $(\mathcal{L} - \lambda_1)f^* = 0$, so that $f^* \in \text{Span}(f_1)$ and more precisely $f^* = \langle f, \phi_1 \rangle f_1$. By uniqueness of the limit, it is the whole family (u_T) which converges to f^* . \square

We present a variant of the previous result in which we see that in a very general framework (including all the applications we present in the second part of this work) the above hypothesizes **(1)** and **(2)** are not needed (or more precisely are automatically satisfied).

Theorem 3.4.20. (1) Consider a Banach lattice $X \subset L_{\text{loc}}^1(E, \mathcal{E}, \mu)$ and a positive semigroup S on X such that its generator \mathcal{L} satisfies the conclusions of Theorem 3.3.11 about the existence, positivity and uniqueness of the first eigentriplet (λ_1, f_1, ϕ_1) , then the mean ergodic convergence **(CE1)** holds for the weak topology of $L_{\phi_1}^1$.

(2) If we additionally assume that

$$\mathcal{X}^k := \{f \in \mathcal{X}; \mathcal{L}^j f \in \mathcal{X}, \forall j \leq k\} \subset L_{\text{loc}}^1, \quad (3.4.19)$$

with strong compact embedding for some $k \geq 1$, where we denote by \mathcal{X} the space X endowed with the norm $L_{\phi_1}^1$, then the mean ergodic convergence **(CE1)** holds for the strong topology of $L_{\phi_1}^1$ for any f belonging to the closure of $D(\mathcal{L}^k)$ (for the strong topology of X).

Proof of Theorem 3.4.20. Step 1. We first recall a very classical result about conservative semigroups. Denoting $\tilde{S}_t := e^{-\lambda_1 t} S(t)$, we observe that this rescaled semigroup satisfies

- (i) $\tilde{S}_t \geq 0$;
- (ii) $\tilde{S}_t f_1 = f_1$ for any $t \geq 0$;
- (iii) $\langle \tilde{S}_t g, \phi_1 \rangle = \langle g, \phi_1 \rangle$ for any $g \in X$ and $t \geq 0$.

We denote $[f]_1 := \langle |f|, \phi_1 \rangle$ which is a norm on X (we use here that $\phi_1 \gg 0$) and \tilde{S}_t is obviously a contraction for this one since for any $f \in X$ there holds

$$|\tilde{S}_t f| = |\tilde{S}_t f_+ - \tilde{S}_t f_-| \leq \tilde{S}_t f_+ + \tilde{S}_t f_- = \tilde{S}_t |f|$$

using **(i)** in the inequality, and next

$$[\tilde{S}_t f]_1 = \langle |\tilde{S}_t f|, \phi_1 \rangle \leq \langle \tilde{S}_t |f|, \phi_1 \rangle = [f]_1, \quad (3.4.20)$$

using **(iii)** in the last equality. Denoting by \mathcal{X} the completion of X for this norm (so that we may identify \mathcal{X} to a closed subspace of $L_{\phi_1}^1$), we may extend \tilde{S}_t to \mathcal{X} by uniform continuity and this extension still satisfies the properties **(i)**-**(ii)**-**(iii)** on \mathcal{X} . Consider now $f \in X$ such that $H(f/f_1)f_1 \in X$ for some convex function $H : \mathbb{R} \rightarrow \mathbb{R}$, where we use here that $X \subset L_{\text{loc}}^1$, and thus in particular $f_1 > 0$ a.e. on E , in order to give a sense to the term $H(f/f_1)f_1$. From **(ii)**, we have

$$\ell[(\tilde{S}_t f)/f_1]f_1 = \tilde{S}_t[\ell(f/f_1)f_1],$$

for any real affine function ℓ . Next from (i) and (2.1.7), we have

$$H[(\tilde{S}_t f)/f_1]f_1 \leq \tilde{S}_t[H(f/f_1)f_1],$$

because of $H = \sup_{\ell \leq H} \ell$ and the supremum can be taken on a numerable set of affine functions. Thanks to (iii), we conclude that

$$\langle H[(\tilde{S}_t f)/f_1]f_1, \phi_1 \rangle \leq \langle H[f/f_1]f_1, \phi_1 \rangle, \quad \forall t \geq 0. \quad (3.4.21)$$

Step 2. We normalize $\langle f_1, \phi_1 \rangle = 1$. For $f \in \mathcal{X} \subset L^1_{\phi_1}$ so that $f\phi_1 = (f/f_1)f_1\phi_1 \in L^1$, the de la Vallée Poussin theorem tells us that there exists an even and convex function $H : \mathbb{R} \rightarrow \mathbb{R}_+$ such that $H(s)/s \rightarrow +\infty$ as $s \rightarrow \infty$ and $H(f/f_1)f_1\phi_1 \in L^1$. Using the notations of the proof of Theorem 3.4.19, the Jensen inequality and the above estimate (3.4.21), we deduce

$$\int_E H(u_T/f_1)f_1\phi_1 d\mu \leq \frac{1}{T} \int_0^T \int_E H(\tilde{S}_t f)/f_1 f_1\phi_1 d\mu dt \leq \int_E H(f/f_1)f_1\phi_1 d\mu,$$

for any $T > 0$. Now, for any $A \in \mathcal{E}$ and $T, K > 0$, we have

$$\begin{aligned} \int_E u_T \mathbf{1}_A \phi_1 d\mu &= \int_E \frac{u_T}{f_1} \mathbf{1}_{\frac{|u_T|}{f_1} > K} \mathbf{1}_A f_1 \phi_1 d\mu + \int_E \frac{u_T}{f_1} \mathbf{1}_{\frac{|u_T|}{f_1} \leq K} \mathbf{1}_A f_1 \phi_1 d\mu \\ &\leq \frac{K}{H(K)} \int_E H(u_T/f_1) f_1 \phi_1 d\mu + K \int_E \mathbf{1}_A f_1 \phi_1 d\mu \\ &\leq \frac{K}{H(K)} \int_E H(f/f_1) f_1 \phi_1 d\mu + K \int_E \mathbf{1}_A f_1 \phi_1 d\mu, \end{aligned}$$

from what we immediately deduce that (u_T) belongs to a weak compact set of $L^1_{\phi_1}$. We conclude that (3.4.18) holds for the weak convergence in $L^1_{\phi_1}$ as in the proof of Theorem 3.4.19.

Step 3. We now additionally assume that (3.4.19) holds with strong compact embedding for some $k \geq 1$. Taking $f \in D(\mathcal{L}^k)$, we compute

$$\langle |\mathcal{L}^j(\tilde{S}_t f)|, \phi_1 \rangle = \langle |\tilde{S}_t(\mathcal{L}^j f)|, \phi_1 \rangle \leq \langle |\mathcal{L}^j f|, \phi_1 \rangle,$$

for any $j \leq k$ and any $t \geq 0$, and thus the same bound holds for (u_T) . From (3.4.19), we deduce that up to the extraction of a subsequence, (u_T) converges a.e. on E . Together with the weak convergence in $L^1_{\phi_1}$ yet established, we classically deduce that the whole family (u_T) converges for the strong topology in $L^1_{\phi_1}$. We conclude that the same holds for any $f \in X$ by taking advantage of the fact that $D(\mathcal{L}^k)$ is dense in X for the strong topology of X and thus for the strong topology of \mathcal{X} . \square

Remark 3.4.21. (1) A similar conclusion holds as in Theorem 3.4.20 when we assume $X \subset M^1_{\text{loc}}$, $D(\mathcal{L}^k) \subset L^1_{\text{loc}}$ and $D(\mathcal{L}^{*k}) \subset L^1_{\text{loc}}$ for some $k \geq 1$ instead of $X \subset L^1_{\text{loc}}$. For $f \in D(\mathcal{L}^k) \subset L^1_{\text{loc}}$, we may indeed repeat the proof of Theorem 3.4.20 and we obtain the same conclusion. We next define \mathcal{X} as the closure of $D(\mathcal{L}^k)$ for the norm $[\cdot]_1$. We conclude that (3.4.18) holds weakly in $L^1_{\phi_1}$ for any $f \in \mathcal{X}$ by a density argument.

(2) The proof of Theorem 3.4.20 is based on so-called General Relative Entropy (GRE) techniques as developed for instance in [158], [171] and [26]. These ones are known to be useful for some classes of PDEs and for stochastic semigroups in order to establish uniform in time estimates and longtime convergence results.

The main interest of the two previous results is that they do not ask any new information on the semigroup but they are just based on the eigentriplet stationary problem. The shortcoming is that they are formulated in terms of the norm $[\cdot]_1$ instead of the norm of X . We present a second variant of Theorem 3.4.19 which is well adapted to the splitting framework developed in Sections 2.1 and 2.2 and is precisely formulated in a weak or strong topology of a space $X_0 \supset X$.

Theorem 3.4.22. *Consider a positive semigroup $S = S_{\mathcal{L}}$ such that \mathcal{L} satisfies the conclusions of Theorem 3.3.11 about the existence and uniqueness of the first eigentriplet (λ_1, f_1, ϕ_1) . Assume furthermore that S satisfies the splitting structure introduced in **(HS2)** in section 2.2.2 or **(HS3)** in Section 2.2.2, or more precisely, there exist two families of operators $(V(t))$ and $(W(t))$ such that*

$$S = V + W * S,$$

a real number $\kappa \leq \lambda_1$ and some decaying functions $\Theta_i : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ with $\Theta_1(t) \rightarrow 0$ as $t \rightarrow \infty$, $\Theta_2 \in L^1(\mathbb{R}_+)$ such that the following estimates hold

$$\|V(t)e^{-\kappa t}\|_{\mathcal{B}(X)} = \mathcal{O}(1), \quad \|V(t)e^{-\kappa t}\|_{\mathcal{B}(X, X_0)} = \mathcal{O}(\Theta_1), \quad (3.4.22)$$

$$\|W(t)e^{-\kappa t}\|_{\mathcal{B}(X_0, X_1)} = \mathcal{O}(\Theta_2), \quad (3.4.23)$$

with $X_1 \subset X_0 \subset X_0$, where X_0 is the space X endowed with the norm $[g]_1 := \langle |g|, \phi_1 \rangle$.

(1) Assuming furthermore that $X_1 \subset X_0$ with compact embedding for the weak or the strong topology in X_0 and this topology makes $f \mapsto \langle f, \phi_1 \rangle$ continuous. Then the conclusion of Theorem 3.4.19 holds true for the above strong or weak topology.

(2) Assuming furthermore that $X \subset L^1_{\text{loc}}$ and that the space \mathcal{X}^k defined by (3.4.19) is strongly compact embedded in L^1_{loc} for some $k \geq 1$. Then the conclusion of Theorem 3.4.20-(2) holds true for the strong topology of X_0 .

Proof of Theorem 3.4.22. We define

$$\tilde{V}(t) := V(t)e^{-\lambda_1 t}, \quad \tilde{W}(t) := W(t)e^{-\lambda_1 t},$$

so that

$$\tilde{S} = \tilde{V} + \tilde{W} * \tilde{S},$$

and

$$M := \sup_{t \geq 0} \|\tilde{V}(t)\|_{\mathcal{B}(X)} < \infty, \quad \|\tilde{V}\|_{\mathcal{B}(X, X_0)} \lesssim \tilde{\Theta}_1 \in C_0(\mathbb{R}_+), \quad (3.4.24)$$

$$\tilde{\Theta}_2(t) := \|\tilde{W}(t)\|_{\mathcal{B}(X_0, X_1)} \in L^1(\mathbb{R}_+).$$

Step 1. We furthermore assume (1) and that the weak topology of X_0 makes $f \mapsto \langle f, \phi_1 \rangle$ continuous. We denote by \mathcal{T} the weak or the strong topology X_0 (depending of the assumption made on the embedding $X_1 \subset X_0$). For $f_0 \in X$, we split

$$f(t) := \tilde{S}_t f_0 = v(t) + k(t), \quad v(t) := \tilde{V}(t)f_0, \quad k(t) := (\tilde{W} * \tilde{S})(t)f_0,$$

and we observe that $\|v(t)\|_{X_0} \rightarrow 0$ as $t \rightarrow \infty$ from the second estimate in (3.4.24). On the other hand, we have

$$\sup_{t \geq 0} \|k(t)\|_{X_1} \leq \|\tilde{W}\|_{L^1} \sup_{t \geq 0} \|\tilde{S}_t f_0\|_{X_0} \leq \|\tilde{W}\|_{L^1} \|f_0\|_{X_0},$$

from (3.4.20). In particular, $k(t)$ belongs to a compact set of \mathcal{T} , so that $(f(t))_{t \geq 0}$ also belongs to a compact set for the same topology \mathcal{T} . The same argument used on the Cesàro function (u_T) defined during the proof of Theorem 3.4.19 implies that there exist $f^* \in X$ and a sequence (T_k) such that

$$u_{T_k} \rightarrow f^* \text{ in the sense of } \mathcal{T} \quad \text{and} \quad \langle f^*, \phi_1 \rangle = \langle f, \phi_1 \rangle,$$

the last identity following from the assumption that $f \mapsto \langle f, \phi_1 \rangle$ continuous for \mathcal{T} . We may then conclude as in the proof of Theorem 3.4.19.

Step 2. We furthermore assume (2), and by linearity we may assume $f_0 \in X$, $\langle f_0, \phi_1 \rangle = 0$. We recall that (3.4.18) holds for the strong topology of $L^1_{\phi_1}$ from Theorem 3.4.20 and that $\|v(t)\|_{X_0} \rightarrow 0$ as $t \rightarrow \infty$ from Step 1. Arguing as in Step 3 of the proof of Theorem 2.2.4, we have

$$\begin{aligned} K(T) &:= \frac{1}{T} \int_0^T (W * \tilde{S})(t) dt = \frac{1}{T} \int_0^T W(s) \int_0^{T-s} \tilde{S}(u) du ds \\ &= \int_0^T W(s) \frac{T-s}{T} \tilde{U}(T-s) ds, \end{aligned}$$

where $\tilde{U}_T := U_T e^{-\lambda_1 T}$, U_t is defined by (2.2.44), so that $u_T = \tilde{U}_T f_0$ and $[u_T]_1 \rightarrow 0$ as $T \rightarrow \infty$ from Theorem 3.4.20. As a consequence, we have

$$\begin{aligned} \|K(T)f_0\|_{\mathcal{X}_1} &\leq \int_0^{T/2} \Theta_2(s) [\tilde{U}(T-s)]_0 ds + \int_{T/2}^T \Theta(s) [\tilde{U}(T-s)]_0 ds \\ &\leq \|\tilde{\Theta}_2\|_{L^1} \sup_{t \geq T/2} [\tilde{U}(t)]_0 + \int_{T/2}^{\infty} \tilde{\Theta}_2(s) ds \sup_{t \geq 0} [\tilde{U}(t)]_0 \rightarrow 0, \end{aligned}$$

as $T \rightarrow \infty$. All together, we have established that $\|u_T\|_{X_0} \rightarrow 0$ as $T \rightarrow \infty$. \square

Chapter 4

The geometry of the boundary point spectrum

We summarize the results established up to now by assuming that the main conclusions in the previous sections are achieved, namely

(C2) the first eigentriplet problem (3.0.1) has a unique solution (λ_1, f_1, ϕ_1) , and furthermore, $f_1 \gg 0$ and $\phi_1 \gg 0$. In that situation, we usually normalize the eigenvectors by $\langle f_1, \phi_1 \rangle = 1$ and either $\|f_1\| = 1$ or $\|\phi_1\| = 1$.

In this chapter, we aim to describe one step further the geometry of the spectrum and more precisely to get some information on the boundary point spectrum

$$\Sigma_P^+(\mathcal{L}) := \Sigma_P(\mathcal{L}) \cap \overline{\Delta}_{\lambda_1}.$$

That will be possible by introducing first a suitable and usual complexification framework and next by assuming a stronger positivity property on \mathcal{L} or on the associated semigroup. Here and for further references below, we recall that we define the sets

$$\Sigma_d(\mathcal{L}) \subset \Sigma_P(\mathcal{L}) \subset \Sigma(\mathcal{L}),$$

where the point spectrum set $\Sigma_P(\mathcal{L})$ is the set of eigenvalues, namely $\lambda \in \Sigma_P(\mathcal{L})$ if $N(\mathcal{L} - \lambda) \neq \{0\}$, and the discrete spectrum set $\Sigma_d(\mathcal{L})$ is the set of eigenvalues which are isolate and have finite algebraic multiplicity.

4.1 Complexification and the sign operator

We present some materials, most of them being very classical, about the sign operator in a complex Banach lattice and we refer to [222, 7] for more details.

Complexification. The complexification space $X_{\mathbb{C}}$ associated to a real Banach lattice X is defined by $X_{\mathbb{C}} := X + iX$ so that $f \in X_{\mathbb{C}}$ if $f = g + ih$, $g, h \in X$. In general, we just write X without mentioning the field, although when we need to specify it we write $X_{\mathbb{C}}$ or $X_{\mathbb{R}}$. We extend on $X_{\mathbb{C}}$ the order defined on $X_{\mathbb{R}}$ by writing

$$f = g + ih \geq 0 \quad \text{if} \quad g \geq 0 \text{ and } h = 0.$$

The conjugate \bar{f} of a complex vector $f = g + ih$ is classically defined by $\bar{f} = g - ih$. We then define the modulus

$$|f| := \sup_{\theta \in [0, 2\pi]} (g \cos \theta + h \sin \theta), \tag{4.1.1}$$

which indeed exists for such a family of vectors. One checks the usual modulus properties:

$$|f| \geq 0, \quad |f| = 0 \text{ iff } f = 0, \quad |\lambda f| = |\lambda| |f|, \quad |f + g| \leq |f| + |g|,$$

for any $f, g \in X$ and $\lambda \in \mathbb{C}$. We finally define the norm on $X_{\mathbb{C}}$ by

$$\|f\| := \| |g + ih| \|_{X_{\mathbb{R}}},$$

and we observe that $X_{\mathbb{C}}$ has then a complex Banach lattice structure. We extend the definition of $A \in \mathcal{B}(X_{\mathbb{R}})$ to $X_{\mathbb{C}}$ by setting

$$A(g + ih) = Ag + iAh, \quad \forall g + ih \in X_{\mathbb{C}}.$$

The operator sign. We may classically extend the sign operator defined in Section 3.1 to the present complex Banach lattice framework. Instead of dealing with the most general case, we will use some regularity assumption on the Banach lattice X which is suitable for our purpose and that we present below. Similarly as in Remark 3.1.4, for $f \in X$, we define

$$X_f := \bigcup_n \{g \in X; |g| \leq n|f|\},$$

and next, similarly as in Theorem 2.1.23, we define

$$A_f[g] := \inf\{C > 0; |g| \leq C|f|\}, \quad \forall g \in X_f.$$

We summarize the regularity assumption on the Banach lattice X we need by assuming :

(X2) For any $f \in X$ such that $|f| \in X_{++}$, there exists a sign operator $\text{sign } f \in \mathcal{B}(X)$, with the following properties

$$\text{sign } f \circ \text{sign } \bar{f} = I, \quad (\text{sign } f)f = |f|, \quad (4.1.2)$$

$$(\text{sign } f)g = (\text{sign}(uf))(ug), \quad |(\text{sign } f)g| \leq |g|, \quad \forall g \in X, \forall u \in \mathbb{S}^1. \quad (4.1.3)$$

Furthermore, $X = \overline{X}_f$, the closure of X_f , and for all $f \in X$ and $g \in X_f$

$$(g \in X_{\mathbb{R}} \text{ and } |g| \leq C|f|) \Leftrightarrow A_f[g - ir|f|] \leq \sqrt{C^2 + r^2}, \forall r \in \mathbb{R}. \quad (4.1.4)$$

For a space of functions, the sign operator is defined as the multiplication by (abusing notations)

$$\text{sign } f := \bar{f}/|f|, \quad \forall f \in X, |f| \in X_{++}. \quad (4.1.5)$$

Lemma 4.1.1. With (4.1.5), the property **(X2)** holds when $X = L^p(E, \mathcal{E}, \mu)$ or $X = C_0(E)$.

Proof of Lemma 4.1.1. For $f \in X$, $|f| \in X_{++}$, we just indicate the proof of $\overline{X}_f = X$, the other algebraic properties being clear from the definition (4.1.5). When $f \in L^p$ such that $|f| > 0$ μ -a.e. and $0 \leq g \in L^p$, we set $g_n := g \wedge (n|f|)$. We have $0 \leq g_n \leq g$ and $g_n \rightarrow g$ strongly L^p if $p < \infty$ and weakly $*$ L^∞ if $p = \infty$. The general case $g \in L^p$ is dealt in the usual way by introducing positive and negative parts and next real and imaginary part. That concludes the proof of $\overline{X}_f = L^p$. The proof of $\overline{X}_f = C_0(E)$ is similar. \square

A sign operator satisfying **(X2)** can actually be built by using Kakutani's theorem in general Banach lattices whenever $|f|$ is a quasi-interior point, see for instance [20, Chapter 14.3]. In a space of measures $X = M^1(E)$, there is no quasi-interior point and no sign operator can be defined in that way. Yet, we can associate to f such that $|f| \gg 0$ a sign operator by means of the Radon-Nikodym theorem. Denoting $\alpha : E \rightarrow \mathbb{S}^1$ the measurable function such that $f = \alpha|f|$, the multiplication by $\bar{\alpha}/|\alpha|$ defines a sign operator $\text{sign } f \in \mathcal{B}(X)$ which satisfies the first properties in **(X2)**. However, it does not satisfy $X = \overline{X}_f$, since there is no quasi-interior point in $M^1(E)$. Measure spaces thus not enjoy **(X2)**.

Lemma 4.1.2. *Assume (X2) and $f \in X_{++}$. Consider a linear operator $\mathcal{Q} : X_f \rightarrow X_f$ such that $\mathcal{Q}f = f$ and $A_f(\mathcal{Q}g) \leq A_f(g)$ for any $g \in X_f$. Then $\mathcal{Q} \geq 0$.*

Proof of Lemma 4.1.2. Take $0 \leq g \in X_f$ such that $g \leq 2Cf$, $C > 0$, and observe that

$$-Cf \leq g - Cf \leq Cf.$$

For any $r \in \mathbb{R}$, we compute

$$\begin{aligned} A_f[(\mathcal{Q}g) - Cf - irf] &= A_f[\mathcal{Q}(g - Cf - irf)] \\ &\leq A_f[g - Cf - irf] \\ &\leq \sqrt{C^2 + r^2}, \end{aligned}$$

by using the non expansion property of \mathcal{Q} and the claim (4.1.4). Using again (4.1.4), we deduce $-Cf \leq (\mathcal{Q}g) - Cf \leq Cf$ and the conclusion. \square

We generalize Kato's inequality (3.1.3) to the present complex framework by saying that an operator \mathcal{L} on X satisfies (the complex) Kato's inequality if

$$\forall f \in D(\mathcal{L}), \quad \Re e(\text{sign}f)\mathcal{L}f \leq \mathcal{L}|f|, \quad (4.1.6)$$

possibly in a dual sense as in (3.1.4). As for the real Kato's inequality, when \mathcal{L} is the generator of a semigroup, Kato's inequality (4.1.6) is a consequence of the positivity of the semigroup, and we refer to Remark 3.1.1 for references about this claim.

4.2 On the subgroup and discrete structure of the point boundary spectrum

In this section, we establish that the point boundary spectrum enjoys a subgroup structure under the same kind of hypothesis as considered in the previous sections.

Lemma 4.2.3. *Under assumptions (X2), (C1), (H1') and (H4), for any $\lambda \in \Sigma_P^+(\mathcal{L}) \setminus \{0\}$ the associated normalized eigenfunction f satisfies $|f| = f_1$.*

Proof of Lemma 4.2.3. By definition $\mathcal{L}f = \lambda f$ and $f \in D(\mathcal{L})$. By linearity of the operator sign and thanks to (4.1.2) and Kato's inequality (4.1.6), we have

$$\lambda_1 |f| = \Re e[\lambda(\text{sign}f)f] = \Re e(\text{sign}f)(\lambda f) = \Re e(\text{sign}f)\mathcal{L}f \leq \mathcal{L}|f|.$$

By the duality argument introduced during the proof of Lemma 3.3.16, we must have $\lambda_1 |f| = \mathcal{L}|f|$ and the conclusion. \square

Theorem 4.2.4. *Assume (X2) and (C2), and denote $\tilde{\mathcal{L}} = \mathcal{L} - \lambda_1$. Under assumptions (H1') and (H4), the set $\mathcal{S} := \Sigma_P(\tilde{\mathcal{L}}) \cap i\mathbb{R}$ is an additive subgroup and $\dim N(\tilde{\mathcal{L}} - i\alpha)^k = 1$ for any $i\alpha \in \mathcal{S}$ and $k \geq 1$.*

Theorem 4.2.4 is similar to [7, C-III, Cor. 2.12] and [20, Prop. 14.15]. Our proof is very similar to the proof of [20, Proposition 14.15]. However, it is more direct and avoid the use of the $C(K)$ algebra and Kakutani Theorem [133] (see also [165, Theorem 2.1.3]).

Proof of Theorem 4.2.4. We split the proof into three steps.

Step 1. We consider f associated to an eigenvalue $i\alpha \in \Sigma_P(\tilde{\mathcal{L}}) \setminus \{0\}$, and we define

$$T(t) := (\text{sign}f)e^{-i\alpha t}\tilde{\mathcal{S}}(t)(\text{sign}\bar{f}).$$

Observing that $\tilde{S}(t)f = e^{i\alpha t}f$, we have

$$T(t)|f| = (\text{sign } f)e^{-i\alpha t}\tilde{S}(t)f = (\text{sign } f)f = |f| = \tilde{S}(t)|f|.$$

On the other hand we have

$$|T(t)g| \leq |\tilde{S}(t)(\text{sign } \bar{f})g| \leq \tilde{S}(t)|g|, \quad \forall g \in X,$$

which, by positivity of $\tilde{S}(t)$, yields

$$|T(t)g| \leq A_f(g)\tilde{S}(t)|f| = A_f(g)|f|, \quad \forall g \in X_f.$$

Because $|f| = f_1 \gg 0$ from Lemma 4.2.3 and **(C2)**, we can apply Lemma 4.1.2 to $|f|$ and $\mathcal{Q} := T(t)$. We deduce that $T(t) \geq 0$ on $X_{|f|} = X_f$, and then on $X = \bar{X}_f$. As a consequence, $0 \leq T(t)g = |T(t)g| \leq \tilde{S}(t)g$ for any $g \geq 0$. In other words, we have $0 \leq \tilde{S}(t) - T(t)$ and then $0 \leq \tilde{S}(t)^* - T(t)^*$. We must have $\tilde{S}(t)^* - T(t)^* = 0$. Otherwise, there would exist $\psi \in X'_+ \setminus \{0\}$ such that $(\tilde{S}(t)^* - T(t)^*)\psi \in X'_+ \setminus \{0\}$, and we find a contradiction by computing

$$0 < \langle (\tilde{S}(t)^* - T(t)^*)\psi, f_1 \rangle = \langle \psi, (\tilde{S}(t) - T(t))f_1 \rangle = 0.$$

We have thus established that $\tilde{S}(t) = T(t)$.

Step 2. Consider $\alpha, \beta \in \mathbb{R}$ and $f, g \in X \setminus \{0\}$ such that $\tilde{\mathcal{L}}f = i\alpha f$ and $\tilde{\mathcal{L}}g = i\beta g$, and suppose first that $(\text{sign } \bar{f}) : D(\mathcal{L}) \rightarrow D(\mathcal{L})$. From Step 1 and the fact that $(\text{sign } \bar{f}) \circ \text{sign } f = I$, for any $h \in D(\mathcal{L})$, we may compute

$$\begin{aligned} \tilde{\mathcal{L}}h &= \lim_{t \rightarrow 0} \frac{1}{t}(\tilde{S}(t)h - h) \\ &= (\text{sign } f) \lim_{t \rightarrow 0} \frac{1}{t}(e^{-i\alpha t}\tilde{S}(t)(\text{sign } \bar{f})h - (\text{sign } \bar{f})h) \\ &= (\text{sign } f)(\tilde{\mathcal{L}} - i\alpha)(\text{sign } \bar{f})h, \end{aligned}$$

or in other words $\tilde{\mathcal{L}} - i\alpha = (\text{sign } \bar{f})\tilde{\mathcal{L}}(\text{sign } f)$. We have similarly $\tilde{\mathcal{L}} - i\beta = (\text{sign } \bar{g})\tilde{\mathcal{L}}(\text{sign } g)$. Both equations together imply

$$\tilde{\mathcal{L}} - i(\alpha + \beta) = (\text{sign } \bar{f})(\text{sign } \bar{g})\tilde{\mathcal{L}}(\text{sign } g)(\text{sign } f).$$

Defining $h := (\text{sign } \bar{f})(\text{sign } \bar{g})f_1$, so that $(\text{sign } g)(\text{sign } f)h = f_1$, we get $\tilde{\mathcal{L}}h = i(\alpha + \beta)h$, and finally $i(\alpha + \beta) \in \mathcal{S}$, so that the additive subgroup structure is established.

When the condition $(\text{sign } \bar{f}) : D(\mathcal{L}) \rightarrow D(\mathcal{L})$ is not granted, we modify the above argument by using a resolvent approach. For some $\lambda > 0$, we compute

$$\begin{aligned} (\lambda - \tilde{\mathcal{L}})^{-1} &= \int_0^\infty e^{-\lambda t}\tilde{S}(t) dt \\ &= (\text{sign } f) \int_0^\infty e^{-(\lambda + i\alpha)t}\tilde{S}(t) dt (\text{sign } \bar{f}) \\ &= (\text{sign } f)(\lambda + i\alpha - \tilde{\mathcal{L}})^{-1}(\text{sign } \bar{f}). \end{aligned}$$

Repeating the argument, we obtain

$$(\lambda + i(\alpha + \beta) - \tilde{\mathcal{L}})^{-1} = (\text{sign } \bar{f})(\text{sign } \bar{g})(\lambda - \tilde{\mathcal{L}})^{-1}(\text{sign } g)(\text{sign } f).$$

Applying that last identity to the vector $h = (\text{sign } \bar{f})(\text{sign } \bar{g})f_1$ and using that $(\lambda - \tilde{\mathcal{L}})^{-1}f_1 = \lambda^{-1}f_1$, we deduce $(\lambda + i(\alpha + \beta) - \tilde{\mathcal{L}})^{-1}h = \lambda^{-1}h$. In other words, we have again $\tilde{\mathcal{L}}h = i(\alpha + \beta)h$, and we conclude as above.

Step 3. From the fact that $(\text{sign } f)$ is an invertible operator and the equation

$$(\tilde{\mathcal{L}} - i\alpha)^k = (\text{sign } f)^{-1}(\tilde{\mathcal{L}})^k(\text{sign } f),$$

we see from Theorem 3.3.11-(ii) that $N(\tilde{\mathcal{L}} - i\alpha)^k = (\text{sign } f)^{-1}N(\tilde{\mathcal{L}})^k = (\text{sign } f)^{-1}\text{Span}f_1$ for any $k \geq 1$, so that its dimension is one. \square

Making an additional splitting structure hypothesis as yet introduced in Section 2.1.2, we may significantly improve the conclusion. We first recall a classical result on the spectrum of an operator which holds under some (power) compactness assumption on the resolvent.

Theorem 4.2.5. *We assume that \mathcal{L} satisfies the splitting structure **(HS1)** with $(\mathcal{A}\mathcal{R}_{\mathcal{B}}(z))^N \in \mathcal{K}(X)$ for some $N \geq 1$ and any $z \in \Delta_{\kappa_0}$, where we recall that $\mathcal{K}(X)$ denote the space of compact operators. Then $\Sigma(\mathcal{L}) \cap \Delta_{\kappa_0} \subset \Sigma_d(\mathcal{L})$.*

Theorem 4.2.5 is a consequence of [242, Corollary 1.1]. For the sake of completeness, we present the idea of the proof. We also refer to [177, proof of Theorem 3.1] for a possible elementary proof.

A sketch of the proof of Theorem 4.2.5. Iterating the formula $\mathcal{R}_{\mathcal{L}} = \mathcal{R}_{\mathcal{B}} + \mathcal{R}_{\mathcal{B}}\mathcal{A}\mathcal{R}_{\mathcal{L}}$, we deduce

$$\mathcal{K}(z)\mathcal{R}_{\mathcal{L}}(z) = \mathcal{V}(z)$$

with

$$\mathcal{K} := I - (\mathcal{A}\mathcal{R}_{\mathcal{B}})^N, \quad \mathcal{V} := \mathcal{R}_{\mathcal{B}} + \dots + \mathcal{R}_{\mathcal{B}}(\mathcal{A}\mathcal{R}_{\mathcal{B}})^{N-1}.$$

Because \mathcal{K} is holomorphic on Δ_{κ_0} , it is a compact perturbation of the identity and $\mathcal{K}(z) \rightarrow I$ when $\Re z \rightarrow \infty$, one may use the theory of *degenerate-meromorphic functions* of Ribarič and Vidav [215], and conclude that $\mathcal{K}(z)$ is invertible outside of a discrete set \mathcal{D} of Δ_{κ_0} . That implies that $\Sigma(\mathcal{L}) \cap \Delta_{\kappa_0} = \mathcal{D}$ is a discrete set of Δ_{κ_0} . On the other hand, thanks to the Fredholm alternative [89], one deduces that the eigenspace associated to each spectral value $\lambda \in \mathcal{D}$ is non zero and finite dimensional, so that $\lambda \in \Sigma_d(\mathcal{L})$. \square

We end this section by a result which gives a more accurate description of the geometry of the boundary spectrum, and is a variant of the classical results [7, C–III, Thm. 3.12], [82, Thm. VI.1.12], [20, Thm. 14.17].

Theorem 4.2.6. *Assume **(X2)** for both the spaces X and Y , **(C2)**, **(H1')**, **(H4)** and additionally that the splitting structure **(HS1)** holds with $(\mathcal{A}\mathcal{R}_{\mathcal{B}}(z))^N \in \mathcal{K}(X)$ for some $N \geq 1$ and any $z \in \Delta_{\kappa_0}$. Then the set $\Sigma_P(\tilde{\mathcal{L}}) \cap i\mathbb{R}$ is a **discrete additive subgroup** of $i\mathbb{R}$ and any of its elements is an **algebraically simple eigenvalue**. More precisely,*

- either $\Sigma_P(\tilde{\mathcal{L}}) \cap i\mathbb{R} = \{0\}$ and the projection on the first eigenspace (associated to λ_1) writes

$$\Pi f := \langle f, \phi_1 \rangle f_1;$$

- or $\Sigma_P(\tilde{\mathcal{L}}) \cap i\mathbb{R} = i\alpha\mathbb{Z}$ for some $\alpha > 0$ and there exists a sequence $(g_k, \psi_k)_{k \in \mathbb{Z}}$ such that $\mathcal{L}g_k = (\lambda_1 + ik\alpha)g_k$, $\mathcal{L}^*\psi_k = (\lambda_1 + ik\alpha)\psi_k$, and $\langle g_k, \psi_\ell \rangle = \delta_{k\ell}$.

Proof of Theorem 4.2.6. Combining Theorem 4.2.4 and Theorem 4.2.5, we immediately get that the subgroup $\mathcal{S} := \Sigma_P(\tilde{\mathcal{L}}) \cap i\mathbb{R}$ satisfies $\mathcal{S} \subset \Sigma_d(\mathcal{L})$, it is thus discrete and made of algebraically simple eigenvalues. The first case $\Sigma_P(\tilde{\mathcal{L}}) \cap i\mathbb{R} = \{0\}$ falls yet in the conclusions of Theorem 3.3.11. \square

In the second case, where the boundary spectrum is not trivial, the existence of a projection on the boundary eigenspace $\overline{\text{Span}}(g_k)_{k \in \mathbb{Z}}$ is ensured by the Jacobs–de Leeuw–Glicksberg theorem provided that \mathcal{L} is the generator of a relatively compact semigroup, see for instance [20,

Theorem A.39 and Proposition A.40] and the references therein. We also refer to [136, paragraphs III.6.4 and III.6.5] for very classical results on the projector on the direct sum of eigenspaces associated to eigenvalues belonging to a subset of the spectrum. We can even give an explicit expression of this projection in terms of (g_k) and (ψ_k) under the form of a Fejér type sum, see Theorem 4.5.19.

4.3 Stronger positivity

In order to go one step further and establish the triviality of the boundary point spectrum, we need to reinforce the positivity of the semigroup or its generator. One possible condition will be the following.

The reverse strong positivity condition

For $A \geq 0$, we recall that from (2.1.6), we have

$$|Af| \leq A|f|, \quad \forall f \in X, \quad (4.3.7)$$

and we observe that the above inequality is an equality when $Af = uA|f|$ for some $u \in \mathbb{S}^1$. We focus now on the case of equality in (4.3.7).

Definition 4.3.7. We say that A satisfies the “reverse strong positivity condition” for a subclass of vectors $\mathcal{C} \subset X$ if for any $f \in \mathcal{C}$

$$|Af| = A|f| \quad \text{implies} \quad \exists u \in \mathbb{S}^1, Af = uA|f|. \quad (4.3.8)$$

We observe that $A \gg 0$ implies the strict positivity for non-signed vectors in $X_{\mathbb{R}}$.

Lemma 4.3.8. Consider an operator $A \gg 0$ and assume X is reflexive. For $f \in X_{\mathbb{R}}$ such that $\pm f \notin X_+$, there holds

$$|Af| \ll A|f|.$$

Proof of Lemma 4.3.8. We consider $f \in X_{\mathbb{R}}$ such that $f_{\pm} \neq 0$. We claim that $|Af| \ll A|f|$. We first recall that

$$Af_+ = Af + Af_- \geq Af,$$

so that $Af_+ \geq (Af)_+$, and similarly $Af_- \geq (Af)_-$. Consider the case $(Af)_+ > 0$. We argue similarly as in the proof of Lemma 3.2.5. By definition, for any $\phi \gg 0$, we have

$$\begin{aligned} 0 < \langle (Af)_+, \phi \rangle &= \sup_{0 \leq \psi \leq \phi} \langle Af, \psi \rangle = \langle Af, \psi^* \rangle = \langle f, A^* \psi^* \rangle \\ &< \langle f_+, A^* \psi^* \rangle = \langle Af_+, \psi^* \rangle \leq \langle Af_+, \phi \rangle, \end{aligned}$$

where the strict inequality comes from the fact that $\psi^* > 0$ because $\langle (Af)_+, \phi \rangle > 0$, so that $A^* \psi^* \gg 0$ and finally $\langle f_-, A^* \psi^* \rangle > 0$ because $f_- > 0$. We have thus $(Af)_+ < Af_+$. Similarly, we establish $(Af)_- < Af_-$ when $(Af)_- > 0$.

As a conclusion, in the three cases $Af = 0$, $(Af)_+ \neq 0$ and $(Af)_- \neq 0$, we have

$$|Af| = (Af)_+ + (Af)_- \ll Af_+ + Af_- = A|f|,$$

which is the desired strict inequality. \square

We believe that a similar result also holds true for complex vectors in a general Banach lattice framework. We do not try to prove such a statement but we rather establish the corresponding complex version for at least one of our standard examples of concrete Banach spaces in which the definition of the absolute value $|f|$ of a vector $f \in X$ is more tractable. We start with the example of continuous functions.

Lemma 4.3.9. *Consider an operator $A \gg 0$ on $X \subset C_b(E)$ endowed with the uniform norm, for some locally and σ -compact metric space E . For $f \in X$ such that $|f| \gg 0$, we have*

$$|Af| = A|f| \quad \text{implies} \quad \exists u \in \mathbb{S}^1, \quad Af = uA|f|.$$

Proof of Lemma 4.3.9. We assume by contradiction that $f = g + ih \in X$, $A|f| \equiv |Af|$ on E and $\forall v \in \mathbb{S}^1$, $|f| > \Re(fv)$ (in the sense ≥ 0 and $\neq 0$). Pick up $x \in E$ and denote by $v = e^{i\theta}$ the element of \mathbb{S}^1 such that $|(Af)(x)| = v(Af)(x)$. We thus have

$$A|f|(x) = |(Af)(x)| = \Re(v(Af)(x)) = \cos\theta(Ag)(x) - \sin\theta(Ah)(x).$$

On the other hand, because of $A \gg 0$ and A is linear, we have

$$\begin{aligned} A|f|(x) &> (A\Re(fv))(x) \\ &= (A(\cos\theta g - \sin\theta h))(x) \\ &= \cos\theta(Ag)(x) - \sin\theta(Ah)(x), \end{aligned}$$

and a contradiction. We have established that there exists $v \in \mathbb{S}^1$ such that $|f| \equiv \Re(fv)$. Now, we have

$$\sqrt{(\Re(fv))^2 + (\Im(fv))^2} = |fv| = |f| = \Re(fv),$$

which in turn implies $\Im(fv) = 0$, since $\Re(fv) \gg 0$. That is here that we use the assumption $|f| \gg 0$ and not only $f \in X_+ \setminus \{0\}$. We conclude that $|f| = fv$ and thus that $f = u|f|$, with $u := v^{-1} \in \mathbb{S}^1$. \square

The reverse Kato's inequality condition

We recall that it has been stated in section 3.1 that the generator \mathcal{L} of a positive semigroup $S(t)$ satisfies Kato's inequality which in a complex framework writes

$$\forall f \in X, \quad \Re(\text{sign}f)\mathcal{L}f \leq \mathcal{L}|f|. \quad (4.3.9)$$

We observe that if $f = u|f|$ for some $u \in \mathbb{S}^1$, we have

$$\Re(\text{sign}f)\mathcal{L}f = \text{sign}(u^{-1}f)\mathcal{L}(u^{-1}f) = \mathcal{L}|f|,$$

which is the case of equality in Kato's inequality.

Definition 4.3.10. *We say that \mathcal{L} satisfies a "reverse Kato's inequality condition" for a class of vectors $\mathcal{C} \subset D(\mathcal{L})$ if for any $f \in \mathcal{C}$ the case of equality in Kato's inequality*

$$\mathcal{L}|f| = \Re(\text{sign}f)\mathcal{L}f$$

implies

$$\exists u \in \mathbb{C}, \quad f = u|f|.$$

In some situation, we may prove directly that the "reverse Kato's inequality condition" holds by reasoning at the level of the operator \mathcal{L} , see for instance [138, Proof of Theorem 5.1]. That is also a consequence of the strong positivity of the semigroup as we see below.

Lemma 4.3.11. *Consider a semigroup S and its generator \mathcal{L} . On the set \mathcal{C} of vectors $f \in X \setminus \{0\}$ such that*

$$\exists \lambda \in \mathbb{C}, \quad \mathcal{L}f = \lambda f, \quad \mathcal{L}|f| = (\Re\lambda)|f|, \quad (4.3.10)$$

there is equivalence between:

- (i) $S(t)$ satisfies the "reverse strong positivity condition" for some (and thus any) $t > 0$;
- (ii) \mathcal{L} satisfies the "reverse Kato's inequality condition".

Remark 4.3.12. When $X \subset L_{\text{loc}}^1$, the “reverse Kato’s inequality condition” (ii) implies the “reverse strong positivity condition” (i) on the class \mathcal{C} of vectors such that $f \in D(\mathcal{L})$, $0 \ll |f| \in D(\mathcal{L})$. Assume indeed that \mathcal{L} satisfies (ii) and consider $f \in \mathcal{C}$ such that $S_t|f| = |S_t f|$ for any $t \geq 0$. By differenciating, we get

$$(\text{sign } f)\mathcal{L}f = \mathcal{L}|f|. \quad (4.3.11)$$

From the “reverse Kato’s inequality condition”, we deduce that $f = u|f|$ for some $u \in \mathbb{S}^1$, so that (i) holds.

Proof of Lemma 4.3.11. In what follows, we fix $f \in X \setminus \{0\}$ such that (4.3.10) holds, and we compute

$$\Re e(\text{sign } f)\mathcal{L}f = \Re e(\text{sign } f)(\lambda f) = (\Re e \lambda)|f| = \mathcal{L}|f|. \quad (4.3.12)$$

For any $t > 0$, we also have $S_t f = e^{\lambda t} f$, $S_t|f| = e^{\Re e \lambda t}|f|$, and thus

$$|S_t f| = S_t|f|. \quad (4.3.13)$$

Assuming the “reverse Kato’s inequality condition”, we deduce from (4.3.12) that $f = u|f|$ for some $u \in \mathbb{S}^1$, thus $S_t f = u S_t|f|$ for some $u \in \mathbb{S}^1$, which is the conclusion of the “reverse strong positivity condition” when (4.3.13) holds.

On the other way round, assuming the “reverse strong positivity condition” for some $T > 0$, we deduce from (4.3.13) for $T > 0$ that there exists $v \in \mathbb{S}^1$ such that

$$e^{\lambda T} f = S_T f = v S_T|f| = v e^{\Re e \lambda T}|f|.$$

That implies that $f = u|f|$ with $u = v e^{-i(\Im m \lambda)T}$, which is nothing but the conclusion of the “reverse Kato’s inequality condition” when (4.3.12) holds. \square

We summarize the material developed above in the following main result of the section.

Theorem 4.3.13. Assume that S is a positive semigroup on X with $X = L^p(E, \mathcal{E}, \mu)$, $X = M^1(E, \mathcal{E})$ or $X = C_0(E)$, for some locally and σ -compact metric space E and denote by (E_k) a sequence of increasing compact sets such that $E = \lim E_k$. We furthermore assume that for any $k \geq 1$ there exists $T > 0$ such that S_T is strictly positive on E_k , in the sense that

$$\forall f \in X_+ \setminus \{0\}, f|_{E_k} \neq 0, \forall \phi \in X'_+ \setminus \{0\}, \text{supp } \phi \subset E_k, \quad \langle S_T f, \phi \rangle > 0. \quad (4.3.14)$$

Then \mathcal{L} satisfies the “reverse Kato’s inequality condition” on the set \mathcal{C} of eigenvectors introduced in Lemma 4.3.11.

Proof of Theorem 4.3.13. Let us consider $f \in X \setminus \{0\}$ such that (4.3.10) holds, so that $S_t|f| = e^{(\Re e \lambda)t}|f|$ for any $t \geq 0$. On the one hand, we may fix $k \geq 1$ such that $|f| \neq 0$ on E_k . Then for any $\ell \geq k$, there exists $T_\ell > 0$ such that (4.3.14) holds, so that

$$e^{(\Re e \lambda)T_\ell} \langle |f|, \phi \rangle = \langle S_{T_\ell}|f|, \phi \rangle > 0,$$

for any $\phi \in X'_+ \setminus \{0\}$, $\text{supp } \phi \subset E_\ell$. That implies $|f| > 0$ on E_ℓ for any $\ell \geq k$, and thus $|f| \gg 0$. Next, as in the proof of Lemma 4.3.11, we observe that

$$|S_{T_\ell} f| = S_{T_\ell}|f|, \quad \forall \ell \geq k.$$

Repeating the proof of Lemma 4.3.9, Lemma ?? and Lemma ??, we deduce that there exists $u_\ell \in \mathbb{S}^1$ such that $S_{T_\ell} f = u_\ell S_{T_\ell}|f|$ on E_ℓ , or equivalently there exists $v_\ell \in \mathbb{S}^1$ such that $f = v_\ell|f|$ on E_ℓ , with $v_\ell := u_\ell e^{-i(\Im m \lambda)T_\ell}$. Because $E_\ell \supset E_1$, we have established that $f = v_1|f|$ on E which is the conclusion of the “reverse Kato’s inequality condition” when (4.3.10) holds. \square

4.4 On the triviality of the boundary spectrum

As in section 3.3, we still assume the existence **(C1)** of a solution $(\lambda_1, f_1, \phi_1) \in \mathbb{R} \times X_+ \times Y_+$ to the first eigenvalue problem (3.0.1), that the condition **(H1')** holds, so that \mathcal{L} enjoys the weak maximum principle (3.3.12) and its **complex Kato's inequalities** counterpart (4.3.9) as well as the strong maximum principle **(H4)**.

We introduce the additional assumption:

(H5) the "reverse Kato's inequality condition" (as defined in Definition 4.3.10) holds true for the class \mathcal{C} defined in Lemma 4.3.11: for $f \in X \setminus \{0\}$ such that

$$\exists \lambda \in \mathbb{C}, \quad \mathcal{L}f = \lambda f, \quad \mathcal{L}|f| = (\Re \lambda)|f| = \Re(\text{sign} f)\mathcal{L}f, \quad (4.4.15)$$

we have

$$\exists u \in \mathbb{C}, \quad f = u|f|.$$

On the other hand, we do not need the structure assumption **(X2)**.

We are then able to make a more accurate analyse of the geometry of the spectrum.

Theorem 4.4.14. *Consider an unbounded operator \mathcal{L} on a Banach lattice X which satisfies **(C1)**, **(H1')**, **(H4)** and **(H5)**. Then λ_1 is the unique eigenvalue with largest real part:*

$$\Sigma_P^+(\mathcal{L}) = \{\lambda_1\}.$$

Remark 4.4.15. (1) *It is worth emphasizing again that (4.3.9) is true when \mathcal{L} is the generator of a positive semigroup and that (4.4.15) is true when $S_{\mathcal{L}}(T)$ satisfies the "reverse strong positivity condition" for some $T > 0$ as a consequence of Lemma 4.3.11, see also Theorem 4.3.13.*

(2) *During the proof we use similar arguments as in [138, Theorem 5.1].*

(3) *When instead of **(H4)** and **(H5)** we only assume that \mathcal{L} is the generator of a positive semigroup, we find $\Sigma(\mathcal{L}) \cap \Delta_{\lambda_1} = \emptyset$ as established in Lemma 2.1.2-(ii).*

Proof of Theorem 4.4.14. Consider an eigenvalue $\lambda \in \mathbb{C}$ with normalized eigenvector $f \in X \setminus \{0\}$, so that $\|f\| = 1$, $\mathcal{L}f = \lambda f$. Thanks to the complex Kato's inequality, we have

$$(\Re \lambda)|f| = \Re \text{sign}(f)(\lambda f) = \Re \text{sign}(f)(\mathcal{L}f) \leq \mathcal{L}|f|.$$

We consider two cases:

When the above inequality is not an equality, we have

$$(\Re \lambda)\langle |f|, \phi_1 \rangle < \langle \mathcal{L}|f|, \phi_1 \rangle = \langle |f|, \mathcal{L}^* \phi_1 \rangle = \lambda_1 \langle |f|, \phi_1 \rangle,$$

and thus $\Re \lambda < \lambda_1$.

When on the contrary the above inequality is an equality, then $|f|$ is a positive eigenvector associated to the eigenvalue $\Re \lambda$. Lemma 3.3.15 implies $\Re \lambda = \lambda_1$ and Lemma 3.3.16 together with (2.1.1) imply $|f| = f_1$. In other words, f satisfies (4.4.15) and thus $f \in \text{Span}(f_1)$ from assumption **(H5)**, in particular $\lambda = \lambda_1$. \square

An alternative result is the following, where we make the alternative assumption:

(H5') a positive and irreducible semigroup S is aperiodic if it enjoys (3.2.9), namely

$$\forall f \in X_+ \setminus \{0\}, \forall \phi \in Y_+ \setminus \{0\}, \exists T > 0, \forall \tau \geq T \quad \langle S_\tau f, \phi \rangle > 0.$$

Theorem 4.4.16. *On a Banach lattice X , consider a positive semigroup S which satisfies the irreducibility and aperiodicity condition **(H5')**, and such that its generator \mathcal{L} satisfies **(C2)**. Then λ_1 is the unique eigenvalue with largest real part.*

Proof of Theorem 4.4.16. We introduce the notations $\tilde{S}_t := S_t e^{-\lambda_1 t}$ and $\tilde{\mathcal{L}} := \mathcal{L} - \lambda_1$. Assume that $f = g + ih \in X$, $g, h \in X_{\mathbb{R}}$, is an eigenfunction associated to the eigenvalue $\lambda = \lambda_1 + i\alpha \in \mathbb{C}$, $\alpha > 0$, so that

$$\tilde{\mathcal{L}}(g + ih) = i\alpha(g + ih) = \frac{2\pi i}{t_0}(g + ih),$$

for some $t_0 > 0$. On the one hand, because $\alpha \neq 0$, we must have $g \neq 0$ and $h \neq 0$, and because of

$$\alpha \langle g, \phi_1 \rangle = \langle \tilde{\mathcal{L}}h, \phi_1 \rangle = \langle h, \tilde{\mathcal{L}}^* \phi_1 \rangle = 0,$$

and $\phi_1 \gg 0$, we have $g_+ \neq 0$ and $g_- \neq 0$. As a consequence and because of (3.2.6), there exists $\psi \in X_+ \setminus \{0\}$ such that $\langle g_+, \psi \rangle = 0$. On the other hand, we compute

$$\tilde{S}_{t_0}(g + ih) = e^{i\alpha t_0}(g + ih) = g + ih,$$

from what we deduce $\tilde{S}_{t_0}g = g$, because S_t is real. On the other hand, because S_t is positive, we have

$$g_+ = (\tilde{S}_{t_0}g)_+ \leq \tilde{S}_{t_0}g_+,$$

and next

$$\langle \phi_1, g_+ \rangle \leq \langle \phi_1, \tilde{S}_{t_0}g_+ \rangle = \langle \tilde{S}_{t_0}^* \phi_1, g_+ \rangle = \langle \phi_1, g_+ \rangle,$$

so that the inequality is an equality (remind that $\phi_1 \gg 0$), and thus

$$\tilde{S}_{t_0}g_+ = g_+.$$

We conclude that

$$\langle \tilde{S}_{kt_0}g_+, \psi \rangle = \langle g_+, \psi \rangle = 0, \quad \forall k \geq 0,$$

what is in contradiction with **(H5')**. We have established that $\Sigma_+(\mathcal{L}) = \{\lambda_1\}$. \square

We end this section with a third situation where the triviality of the boundary spectrum can be shown as a immediate consequence of Theorem 4.2.6.

Theorem 4.4.17. *We assume **(X2)** and there exists $M > 0$ large enough such that $\lambda - \mathcal{L}$ is invertible in $\mathcal{B}(X)$ for any $\lambda \in \mathbb{C}$, $\Re \lambda = \lambda_1$, $|\lambda| \geq M$. Then λ_1 is the unique eigenvalue with largest real part. When furthermore the same holds for any $\lambda \in \mathbb{C}$, $\Re \lambda \geq \lambda_1 - \varepsilon$, $|\lambda| \geq M$ and $\mathcal{W} \in \mathcal{K}(X)$ then \mathcal{L} enjoys a spectral gap.*

We summarize the main results established in this section as follows.

(C3) the first eigentriplet problem (3.0.1) has a solution (λ_1, f_1, ϕ_1) , furthermore this one is unique, $f_1 \gg 0$, $\phi_1 \gg 0$, λ_1 is algebraically simples (for both \mathcal{L} and \mathcal{L}^*) and $\Sigma_P^+(\mathcal{L}) = \{\lambda_1\}$.

4.5 Ergodicity

Thanks to the above analyze, we are able to formulate some convergence results on the trajectories associated to a semigroup.

More precisely, assuming the existence and uniqueness of the first eigentriplet (λ_1, f_1, ϕ_1) for the generator \mathcal{L} of a semigroup S and still denoting the rescaled semigroup $\tilde{S}_t := e^{-\lambda_1 t} S(t)$, we wish in particular to establish the following ergodic property

(CE2) for any $f \in X$, there holds

$$\tilde{S}_t f \rightarrow \langle f, \phi_1 \rangle f_1, \quad \text{as } T \rightarrow \infty, \quad (4.5.16)$$

in a sense to be specified.

We start with a simple result which take advantage of some dissipativity property of the semigroup formulated by a "reverse positivity condition". We next present some more involved results which use directly the spectral information. It is worth emphasizing that our results in this section do not use any spectral gap property what contrasts with the results we will present in the next section.

Theorem 4.5.18. *Consider a positive semigroup S on a Banach lattice X such that its generator \mathcal{L} enjoys the conclusions **(C2)** of existence, uniqueness and strict positivity of the first eigentriplet (λ_1, f_1, ϕ_1) and let us set $\tilde{S}_t := e^{-\lambda_1 t} S_t$. We denote \mathcal{X}_0 the space X endowed with the norm $[\cdot]$, with $[f] := \langle |f|, \phi_1 \rangle$. Assume furthermore that*

(1) *for any $f \in X$, the trajectory $(\tilde{S}_t f)_{t \geq 0}$ belongs to a compact set of a normed space \mathcal{X} , with $\mathcal{X} \subset \mathcal{X}_0$;*

(2) *(S_t) satisfies the reverse positivity condition for semigroups*

$$|S_t f| = S_t |f|, \quad \forall t > 0, \quad \text{implies} \quad \exists T > 0, \quad \exists u_T \in \mathbb{S}^1, \quad S_T f = u_T S_T |f|. \quad (4.5.17)$$

*Then, the ergodicity property **(CE2)** holds in the sense of the norm of \mathcal{X} .*

Let us comment on hypotheses made in the statement of Theorem 4.5.18. Hypothesis **(1)** can be obtained as a consequence of a Lyapunov (or growth) condition reminiscent of the structure condition **(HS3)** introduced in Section 2.2.3 and an irreducibility condition. Typically, we assume first

$$\|\tilde{S}(t)f\| \leq M\|f\| + K[\tilde{S}(t)f]_0,$$

with $[g]_0 := \langle |g|, \psi_0 \rangle$, $\psi_0 \in Y_+ \setminus \{0\}$, what can be established under the very general condition (ii) of Theorem 2.2.4. Next we need to be able to prove that $\psi_0 \leq r\phi_1$ for some $r > 0$. In concrete situations, we may take ψ_0 with compact support and then the above inequality is a consequence of the standard strong maximum principle. We deduce

$$\begin{aligned} \|\tilde{S}(t)f\| &\leq M\|f\| + Kr\langle |\tilde{S}(t)f|, \phi_1 \rangle \\ &\leq M\|f\| + Kr\langle \tilde{S}(t)|f|, \phi_1 \rangle \\ &= M\|f\| + Kr\langle |f|, \phi_1 \rangle, \end{aligned}$$

so that (\tilde{S}_t) is bounded. The hypothesis **(1)** is in fact a bit more demanding, but also quite natural. Assume that $S_{\mathcal{L}}$ enjoys the splitting structure introduced in section 2.2.1 and section 2.2.3, so that

$$\tilde{S} = V + K, \quad (4.5.18)$$

with

$$V := \tilde{S}_{\mathcal{B}} + \cdots + (\tilde{S}_{\mathcal{B}}\mathcal{A})^{*(N-1)} * \tilde{S}_{\mathcal{B}}, \quad K := (\tilde{S}_{\mathcal{B}}\mathcal{A})^{*(N)} * \tilde{S}, \quad \tilde{S}_{\mathcal{B}}(t) := e^{-\lambda_1 t} S_{\mathcal{B}}(t).$$

In some applications, we typically have

$$\|V(t)f_0\| \leq \Theta(t)\|f_0\|, \quad \|(\tilde{S}_{\mathcal{B}}\mathcal{A})^{*(N)}\|_{\mathcal{B}(\mathcal{X}_0, \mathcal{X}_1)} \leq \Theta$$

with $\Theta \in L^1(\mathbb{R}_+) \cap C_0(\mathbb{R}_+)$, $\mathcal{X}_1 \subset \mathcal{X}_0$ compact. In that situation, we deduce **(1)**.

Proof of Theorem 4.5.18. We fix $f \in X$ and without loss of generality, we may assume that $\langle f, \phi_1 \rangle = 0$. We observe that

$$\langle |\tilde{S}_t f|, \phi_1 \rangle = \langle |\tilde{S}_{t-s} \tilde{S}_s f|, \phi_1 \rangle \leq \langle \tilde{S}_{t-s} |\tilde{S}_s f|, \phi_1 \rangle = \langle |\tilde{S}_s f|, \phi_1 \rangle, \quad (4.5.19)$$

for any $t \geq s$. We deduce that (\tilde{S}_t) is a dynamical system with compact trajectoires in \mathcal{X} and $\mathcal{H}(f) := \langle |f|, \phi_1 \rangle$ is a Lyapunov functional. As a consequence, from the La Salle invariance principle, we have

$$\inf_{g \in \omega_{\mathcal{H}}} \langle |\tilde{S}_t f - g|, \phi_1 \rangle \rightarrow 0 \quad \text{as } t \rightarrow \infty, \quad (4.5.20)$$

with

$$\omega_{\mathcal{H}} := \{g \in X; \langle g, \phi_1 \rangle = 0, \forall t \in \mathbb{R}, \mathcal{H}(\tilde{S}_t g) = \inf_{s > 0} \mathcal{H}(\tilde{S}_s f)\}. \quad (4.5.21)$$

We characterize $\omega_{\mathcal{H}}$. Picking up $g \in \omega_{\mathcal{H}}$, we observe that

$$\langle |\tilde{S}_t g|, \phi_1 \rangle = \langle |g|, \phi_1 \rangle = \langle |g|, \tilde{S}_t^* \phi_1 \rangle = \langle \tilde{S}_t |g|, \phi_1 \rangle, \quad \forall t \geq 0,$$

so that

$$\langle \tilde{S}_t |g| - |\tilde{S}_t g|, \phi_1 \rangle = 0, \quad \forall t \geq 0.$$

In particular, using that $|\tilde{S}_t g| \leq \tilde{S}_t |g|$, we have

$$\tilde{S}_t |g| = |\tilde{S}_t g|, \quad \forall t \geq 0. \quad (4.5.22)$$

Because of the reverse positivity condition for semigroups (4.5.17), there exist $T > 0$ and $u_T \in \mathbb{S}^1$ such that

$$\tilde{S}_t g = u_T \tilde{S}_t |g|.$$

As a consequence, by definition of the set $\omega_{\mathcal{H}}$, we have

$$0 = \langle g, \phi_1 \rangle = \langle \tilde{S}_t g, \phi_1 \rangle = u_T \langle \tilde{S}_t |g|, \phi_1 \rangle = u_T \langle |g|, \phi_1 \rangle.$$

Because $u_T \neq 0$, we conclude that $g = 0$. In other words, we have established that $\omega_{\mathcal{H}} = \{0\}$ and together with (4.5.20), we obtain (4.5.16). \square

In the next variant, we formulate the aperiodicity condition in term of inverse Kato's inequality stated at the level of the generator instead of the above reverse positivity condition stated at the level of the semigroup.

We present now a variant of the previous result which holds under the sole assumption **(C2)**.

Theorem 4.5.19. *Consider a strongly continuous semigroup S on a Banach lattice X such that its generator \mathcal{L} enjoys the conclusions **(C2)** on the existence and uniqueness of the first eigentriplet problem (λ_1, f_1, ϕ_1) . We set $\tilde{S}_t := e^{-\lambda_1 t} S_t$ and we assume that the trajectories $(\tilde{S}_t f)_{t \geq 0}$ are relatively compact for all $f \in X$. Then we have the following convergence results.*

1. *If the (point) boundary spectrum is trivial (in the sense of conclusion **(C3)**), then the ergodicity property **(CE2)** holds in the sense of the norm of X .*
2. *If the boundary spectrum is discrete, non-trivial, and made of algebraically simple eigenvalues, then for all $f \in X$ the projection*

$$\Pi f = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{\ell=0}^n \sum_{k=-\ell}^{\ell} \langle f, \psi_k \rangle g_k$$

is well defined and

$$\|\tilde{S}_t f - \tilde{S}_t \Pi f\| \rightarrow 0 \quad \text{as } t \rightarrow +\infty.$$

The previous result can be compared for instance to [20, Theorems 14.19], although our hypotheses are slightly more general. Our proof is also more direct than in [20] and it additionally provides an explicit expression of the projection on the boundary eigenspace $\overline{\text{Span}(g_k)_{k \in \mathbb{Z}}}$. The proof of this result relies on the theory of almost periodic functions. The concept of almost periodicity dates back to H. Bohr and there is a large literature on the subject. We refer for instance to the book of Corduneanu [61] for a comprehensive introduction. There are several equivalent definitions of almost periodic functions and we will use the following one. A function $f \in C_b(\mathbb{R}, X)$, *i.e.* a bounded continuous function from \mathbb{R} to X , is said to be almost periodic if the set $\{f(\cdot + \tau), \tau \in \mathbb{R}\}$ is relatively compact in $C_b(\mathbb{R}, X)$. The set of almost periodic functions is a sub-algebra of $C_b(\mathbb{R}, X)$, and also the closure of the space of periodic functions in $C_b(\mathbb{R}, X)$.

Proof of Theorem 4.5.19. Step 1. Let $f \in X$. Since the trajectory $(\tilde{S}_t f)_{t \geq 0}$ is relatively compact, we infer from [120, Theorem 8] (with $U(\tau, t) = S_t$ and thus no periodicity condition on U) the existence of an almost periodic eternal solution g of the rescaled semigroup \tilde{S} , *i.e.* a function $g : \mathbb{R} \rightarrow X$ such that $g(t + \tau) = \tilde{S}_\tau g(t)$ for all $t \in \mathbb{R}$ and $\tau \geq 0$, such that

$$\lim_{t \rightarrow +\infty} \|\tilde{S}_t f - g(t)\| = 0.$$

The end of the proof consists in characterizing the function g in the two cases (1) and (2). For $\lambda \in \mathbb{R}$, we define the Bohr transformation of the almost-periodic function g by

$$c_\lambda(g) = \lim_{T \rightarrow +\infty} \frac{1}{T} \int_0^T e^{-i\lambda t} g(t) dt,$$

which is known to exist, see [61, Theorem 3.4], since $e^{-i\lambda t} g(t)$ is also almost periodic. Since $e^{-i\lambda t} g(t)$ is besides an eternal solution of the semigroup $e^{-i\lambda t} \tilde{S}_t$ with infinitesimal generator $\mathcal{L}_\lambda = \mathcal{L} - \lambda_1 - i\lambda$, we have that

$$\mathcal{L}_\lambda \int_0^T e^{-i\lambda t} g(t) dt = g(T) - g(0).$$

Dividing by T the above expression, and passing to the limit $T \rightarrow +\infty$ and using that \mathcal{L}_λ is a closed operator, we get

$$\mathcal{L}_\lambda c_\lambda(g) = 0.$$

In other words, we have established

$$\mathcal{L} c_\lambda(g) = (\lambda_1 + i\lambda) c_\lambda(g)$$

and $\lambda_1 + i\lambda$ is an eigenvalue of \mathcal{L} if $c_\lambda(g) \neq 0$.

Step 2. We deduce that if the boundary spectrum is trivial, then necessarily $c_\lambda(g) = 0$ for all $\lambda \neq 0$. By the uniqueness theorem, see for instance [61, Theorem 4.7], we get that g is constant. Due to the conservation law $\langle \tilde{S}_t f, \phi_1 \rangle = \langle f, \phi_1 \rangle$ and the simplicity of the eigenvalue 0, we get that $g = \langle f, \phi_1 \rangle f_1$ and the result of case (1) is proved.

Step 3. In the case (2), the boundary spectrum is discrete and given by $\{\lambda_1 + i\alpha k, k \in \mathbb{Z}\}$ for some $\alpha > 0$. As a consequence, any λ such that $c_\lambda(g) \neq 0$ is necessarily of the form $\lambda = \alpha k$ for some $k \in \mathbb{Z}$. By the uniqueness theorem, g is then a α -periodic function which is given, by virtue of Fejér's theorem, by

$$g(t) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{\ell=0}^n \sum_{k=-\ell}^{\ell} c_{\alpha k}(g) e^{i\alpha k t}.$$

Consider (g_k, ψ_k) two positive direct and dual eigenvectors of \mathcal{L} associated to the eigenvalue $i\alpha k$ such that $\langle g_k, \psi_k \rangle = 1$. From the conservation laws $\langle \tilde{S}_t f, \psi_k \rangle = \langle f, \psi_k \rangle e^{i\alpha k t}$ and the algebraic simplicity of the eigenvalues $i\alpha k$, we get that $c_{\alpha k}(g) = \langle f, \phi_k \rangle g_k$, and the result is proved. \square

We will many time apply the following variant and immediate consequence of Theorem 4.5.19.

Theorem 4.5.20. *Consider a positive semigroup S on a Banach lattice X such that its generator \mathcal{L} enjoys the conclusions **(C2)** of existence, uniqueness and strict positivity of the first eigentriplet (λ_1, f_1, ϕ_1) and let us set $\tilde{S}_t := e^{-\lambda_1 t} S_t$. We assume that $X \subset L_{\text{loc}}^1$ and that for some $k \geq 1$ the space \mathcal{X}^k defined in (3.4.19) satisfies $\mathcal{X}^k \subset L_{\text{loc}}^1$ with strongly compact embedding. We also assume that \mathcal{L} satisfies the reverse Kato's inequality, in the sense that the conditions*

$$f \in D(\mathcal{L}^\infty), \quad 0 \ll |f| \in D(\mathcal{L}), \quad \mathcal{L}|f| = \Re(\text{sign} f) \mathcal{L}f, \quad (4.5.23)$$

implies $f = u|f|$ for some $u \in \mathbb{S}^1$. Then, for any $f \in X$, there holds

$$\tilde{S}_t f \rightarrow f_1 \langle f, \phi_1 \rangle \text{ strongly in } L_{\phi_1}^1 \text{ as } t \rightarrow \infty. \quad (4.5.24)$$

Remark 4.5.21. *Theorem 4.5.20 enjoys some strong similarities with the main consequences of the General Relative Entropy technique developed in [171], see in particular [171, Theorem 3.2], [171, Theorem 4.3] and [171, Theorem 5.2]. In particular, the aperiodicity condition (4.5.23) may be compared with the fact that the first eigenvector f_1 is the unique (normalized and nonnegative) vector $f \in X$ with vanishing dissipation of entropy $\mathcal{D}(f) = 0$ as defined in [171] or more generally that $\text{span}f_1$ is the unique invariant space on which the functional \mathcal{D} vanishes. The present formulation is more abstract and probably more general. The drawback is the condition $\mathcal{X}^k \subset L_{\text{loc}}^1$ with strongly compact embedding which can be avoided in [171], by using some weak compactness argument and the lower semicontinuity property of \mathcal{D} . That is explained by the fact that our proof uses rather the La Salle invariance principle (similarly as in the proof of [83, Theorem 3.2]) instead of a entropy dissipation argument.*

Although Theorem 4.5.19 deals with strong compactness assumption, it also allows handling weak compactness conditions, as attested by the following variant.

Theorem 4.5.22. *Consider a positive semigroup S on a Banach lattice X such that its generator \mathcal{L} satisfies the conclusions **(C3)** about the existence, positivity and uniqueness of the first eigentriplet (λ_1, f_1, ϕ_1) and the triviality of the boundary spectrum. We set $\tilde{S}_t := e^{-\lambda_1 t} S_t$. We assume furthermore that*

(1) *either $X = Y'$, Y separable, and the trajectory $(\tilde{S}_t f)_{t \geq 0}$ is bounded for any $f \in X$. We denote by \mathcal{T} the weak $\ast\sigma(Y', Y)$ topology;*

(2) *or $X \subset L_{\text{loc}}^1(E, \mathcal{E}, \mu)$ and we denote by \mathcal{T} the weak topology of $L_{\phi_1}^1$.*

*In both case, the ergodicity property **(CE2)** holds in the sense of the weak convergence in \mathcal{T} .*

Remark 4.5.23. *When instead of assumption **(C3)** in the statement of Theorem 4.5.22, we assume **(C2)** and $\Sigma_P^+(\mathcal{L})$ is discrete and made of algebraically simple eigenvalues, then the conclusion (2) of Theorem 4.5.19 holds in the sense of the weak convergence in \mathcal{T} .*

Proof of Theorem 4.5.22. We split the proof into two steps.

Step 1. We first assume (1). Since Y is separable, we can find a sequence $(\varphi_n)_{n \geq 1} \subset Y$ which satisfies $\|\varphi_n\| = 1$ and $\text{span}(\varphi_n)$ is dense in Y . Then we can define on X the norm $\|\cdot\|_*$ by setting

$$\|f\|_* = \sum_{n=1}^{\infty} 2^{-n} |\langle f, \varphi_n \rangle|. \quad (4.5.25)$$

On bounded subsets of X the topology of this norm is the same as the weak- \ast topology, it is worth emphasizing

$$f_n \rightharpoonup f \ast \sigma(Y', Y) \Leftrightarrow (\sup \|f_n\| < \infty \quad \text{and} \quad \|f_n - f\|_* \rightarrow 0).$$

Since by assumption the trajectory $(\tilde{S}_t f)$ is bounded, it is weakly- \ast relatively compact, and so relatively compact in $(X, \|\cdot\|_*)$. It is also clear that the semigroup S is continuous for the weak norm $\|\cdot\|_*$. The normed space $(X, \|\cdot\|_*)$ is not a Banach space, but the proof of Theorem 4.5.19 actually only requires, for applying [120, Theorem 8], that the closed balls of X are complete metric spaces, which is the case for the distance induced by $\|\cdot\|_*$. Applying Theorem 4.5.19-(1) then yields the result.

Step 2. We next assume (2). We consider $f \in X$ and repeating the proof of Step 2 in Theorem 3.4.20, we get that $(S_t f)_{t \geq 0}$ belongs to a weak compact set of $L_{\phi_1}^1$. We define the norm $\|\cdot\|_*$ by (4.5.25) for a sequence $(\varphi_n)_{n \geq 1} \subset C_c(E)$ which satisfies $\|\varphi_n\|_{L^\infty} = 1$ and $\text{span}(\varphi_n)$ is dense in $C_c(E)$. The trajectory $(\tilde{S}_t f)$ is then relatively compact in $(X, \|\cdot\|_*)$ and the semigroup S is continuous for the weak norm $\|\cdot\|_*$. We conclude as in the previous step. \square

4.6 A word about spectral analysis argument

The aim of this section is to recall some more or less classical results which makes possible to slightly improve the conclusions of the results presented in the previous section by additionally assume some spectral gap at the level of the operator or the semigroup. More precisely, we are interested by some accurate versions of a partial, but principal *spectral mapping theorem* which asserts that

$$\Sigma(e^{t\mathcal{L}}) \cap \Delta_{e^{\kappa t}} = e^{t\Sigma(\mathcal{L}) \cap \Delta_{\kappa}}, \quad \forall t \geq 0, \quad (4.6.26)$$

for some $\kappa < \lambda_1$, and even more precisely, we wish to establish the following geometric (or exponential) asymptotic stability

(CE3) there exist some constants $\kappa < 0$ and $C \geq 1$ such that for any $f \in X$, there holds

$$\|\tilde{S}(t)f - \langle f, \phi_1 \rangle f_1\| \leq \Theta(t) \|f - \langle f, \phi_1 \rangle f_1\|, \quad \forall t \geq 0, \forall f \in X, \quad (4.6.27)$$

with the decay rate function $\Theta(t) := C e^{\kappa t}$.

In order to discuss the several results we present, we recall the splitting framework

$$S = V + W * S, \quad \|V(t)\|_{\mathcal{B}(X)} + \|W(t)\|_{\mathcal{B}(X)} \lesssim e^{\kappa t}, \quad (4.6.28)$$

for the same $\kappa \in \mathbb{R}$ as above. We start by recalling the spectral mapping theorem for the point spectrum, and its proof, which is instructive.

Lemma 4.6.24 (Spectral mapping theorem for point spectrum). *For a semigroup $(S_t)_{t \geq 0}$ with infinitesimal generator \mathcal{L} we have*

$$\Sigma_P(S_t) \setminus \{0\} = e^{t\Sigma_P(\mathcal{L})}, \quad \forall t \geq 0.$$

Proof of Lemma 4.6.24. The inclusion $e^{t\Sigma_P(\mathcal{L})} \subset \Sigma_P(S_t) \setminus \{0\}$ is clear. Now let $\xi \in \Sigma_P(S_t) \setminus \{0\}$, that is $\xi \in \mathbb{C} \setminus \{0\}$ such that $S_t f = \xi f$ for some $f \in X_{\mathbb{C}} \setminus \{0\}$, and let $\lambda \in \mathbb{C}$ such that $\xi = e^{\lambda t}$ and $\phi \in X'$ such that $\langle \phi, f \rangle \neq 0$. For any $k \in \mathbb{Z}$ we have $\xi = e^{\lambda t + 2ik\pi}$ and so

$$0 = e^{-(\lambda + \frac{2ik\pi}{t})t} S_t f - f = \left(\mathcal{L} - \lambda - \frac{2ik\pi}{t} \right) \int_0^t e^{-(\lambda + \frac{2ik\pi}{t})s} S_s f ds.$$

If the last integral is non-zero for some $k \in \mathbb{Z}$, we deduce that $\lambda + \frac{2ik\pi}{t}$ is an eigenvalue of \mathcal{L} and the result is proved. Assume by contradiction that $\int_0^t e^{-(\lambda + \frac{2ik\pi}{t})s} S_s f ds = 0$ for all $k \in \mathbb{Z}$. This means that the continuous t -periodic complex-valued function $s \mapsto e^{-\lambda s} \langle \phi, S_s f \rangle$ has all its Fourier coefficients equal to zero, which is not possible since this function is not equally zero (its value at $s = 0$ is not zero). \square

We next present a very classical result about the exponential stability of f_1 which is based on the quasi-compact semigroup framework of Voigt [242] (see also [7, B-IV-2] and [82, Section V.3]) and which is a more accurate version of Lemma 2.1.7 and Theorem 4.2.6.

Theorem 4.6.25. *Let $(S_t)_{t \geq 0}$ be a positive irreducible semigroup on a Banach lattice X satisfying the hypotheses of Lemma 2.1.7, namely: **(H2)** holds for a constant $\kappa_0 \in \mathbb{R}$ and there exists $T > 0$ such that the splitting*

$$S_T = V_T + K_T, \quad (4.6.29)$$

holds with $\|V_T\|_{\mathcal{B}(X)} \leq e^{\kappa T}$, $\kappa < \kappa_0$, and $K_T \in \mathcal{K}(X)$. Then there exists a unique solution (λ_1, f_1, ϕ_1) to the eigentriplet and the exponential stability (4.6.27) holds (without constructive estimate).

Remark 4.6.26. *In the splitting framework (4.6.28) the critical hypothesis $K_T \in \mathcal{K}(X)$ may be obtained by assuming (and proved the applications) that*

$$\|W(t)\|_{\mathcal{B}(X, \mathcal{X}_1)} \lesssim e^{\kappa t}, \quad \forall t \geq 0, \quad \mathcal{X}_1 \subset X \text{ compact.}$$

In fact, in many applications, we are also able to establish $\mathcal{X}_1 \subset D(\mathcal{L}^\beta)$, for some $\beta > 0$, without too much more work.

Theorem 4.6.25 is in fact nothing but [20, Theorems 14.18] (see also [244, Section 2], [82, Theorem V.3.7] or [7, C-IV, Thm. 2.1 & Rk. 2.2]). We give however a short proof of Theorem 4.6.25 since it is simpler and more direct than the ones we usual find in the literature and in particular does not refer to subtil results about the spectrum and its essential part.

Proof of Theorem 4.6.25. First step. From Lemma 2.1.7, we already know that (H1), (H2) and (H3) hold. Together with the irreducibility which is nothing but **(H4)** from Lemma 3.2.6, we may apply Theorem 3.3.11 and conclude to the existence, uniqueness and strict positivity result about the eigentriplet solution (λ_1, f_1, ϕ_1) .

Second step. We claim that $\Sigma(\mathcal{L}) \cap \{z \in \mathbb{C}, \Re(z) \geq \kappa_0\}$ is also made of a finite number of isolated eigenvalues with finite geometric multiplicity. We indeed set $\beta_0 := e^{\kappa_0 T}$. Since for any $\lambda \in B_{\beta_0}^c := \{z \in \mathbb{C}, |z| \geq \beta_0\}$ the operator $\lambda - V_T$ is invertible, we easily see that $\lambda \in B_{\beta_0}^c$ is in the spectrum of S_T if and only if 0 is in the spectrum of $I - (\lambda - V_T)^{-1}K_T$, or in the spectrum of $I - K_T(\lambda - V_T)^{-1}$. Indeed, solving $(\lambda - S)f = g$ is equivalent to, on the one hand,

$$(I - (\lambda - V_T)^{-1}K_T)f = (\lambda - V_T)^{-1}g,$$

and in the other hand,

$$(I - K_T(\lambda - V_T)^{-1})(\lambda - V_T)f = g.$$

So if $\lambda \in \Sigma(S_T) \cap B_{\beta_0}^c$ then $1 \in \Sigma((\lambda - V_T)^{-1}K_T)$. Since $(\lambda - V_T)^{-1}K_T$ is a compact operator, the classical Fredholm alternative (see for instance [40, Theorem 6.6]) asserts that its spectrum is made of eigenvalues with finite geometric multiplicity, and then so does for $\Sigma(S_T) \cap B_{\beta_0}^c$. We can also prove, by adapting the proof of [40, Lemma VI.2], that these eigenvalues are isolated, and thus $\Sigma(S_T) \cap B_{\beta_0}^c$ is made of a finite number of isolated eigenvalues with finite geometric multiplicity. Since $e^{T\Sigma(\mathcal{L})} \subset \Sigma(S_T)$, we deduce that $\Sigma(\mathcal{L}) \cap \{z \in \mathbb{C}, \Re(z) \geq \kappa_0\}$ is also made of a finite number of isolated eigenvalues with finite geometric multiplicity.

Third step. We prove the existence of a spectral gap and we conclude.

Since $\Sigma(\mathcal{L}) \cap \{z \in \mathbb{C}, \Re(z) \geq \kappa_0\}$ is finite, λ_1 is simple, and the boudary spectrum of \mathcal{L} is a group, we deduce the existence of $\varepsilon > 0$ such that $\Sigma(\mathcal{L}) \cap \{z \in \mathbb{C}, \Re(z) \geq \lambda_1 - \varepsilon\} = \{\lambda_1\}$. The spectral mapping theorem in Lemma 4.6.24 then ensures that $\Sigma(S_T) \cap \{z \in \mathbb{C}, |z| \geq e^{(\lambda_1 - \varepsilon)T}\} = \{e^{\lambda_1 T}\}$ and that $e^{\lambda_1 T}$ is simple with eigenspace spanned by f_1 . The restriction S_T^\perp of S_T to the invariant subspace $X_\perp := \{f \in X, \langle \phi_1, f \rangle = 0\}$ thus has a spectral radius smaller than $e^{(\lambda_1 - \varepsilon)T}$. The spectral radius formula (see [221, Thm 10.13] for instance) then ensures that

$$\lim_{n \rightarrow \infty} \|S_{nT}^\perp\|^{1/n} = r(S_T^\perp) \leq e^{(\lambda_1 - \varepsilon)T}.$$

This guarantees, for any $\eta \in (0, \varepsilon)$, the existence of a constant $C_\eta > 0$ such that for all $f \in X_\perp$ and all $t \geq 0$

$$\|e^{-\lambda_1 t} S_t f\| \leq C_\eta e^{-\eta t} \|f\|,$$

and the proof is complete. \square

Let us now present a variant of another classical result known as the Gearhart-Prüss Theorem in [98, 210], see also the contributions of Herbst [123] and Greiner [7, A-III.7] as well as the more constructive proof [82, Theorem V.1.11] based on techniques developed in or related to [247, 31].

Theorem 4.6.27. *Consider a positive semigroup S on a Banach lattice X such that its generator \mathcal{L} satisfies the conclusions **(C2)** about the existence, positivity and uniqueness of the first eigentriplet (λ_1, f_1, ϕ_1) . We assume furthermore that X is an Hilbert space and that there exist $\kappa < \lambda_1$ and $R > 0$ such that*

- (i) $\sup_{z \in \Delta_\kappa \setminus B_R} \|\mathcal{R}_{\mathcal{L}}(z)\|_{\mathcal{B}(X)} < \infty$;
- (ii) $\Sigma(\mathcal{L}) \cap \Delta_\kappa \subset \Sigma_d(\mathcal{L}) \cap B_R$.

Then the exponential stability (4.6.27) holds (without constructive estimate).

Proof of Theorem 4.6.27. The spectral information **(C2)** and (ii) together imply **(C3)** (because of Theorem 4.2.4) and that there exists $\kappa^* \in (\kappa, \lambda_1)$, such that $\Sigma(\mathcal{L}) \cap \Delta_{\kappa^*} = \{\lambda_1\}$. The operator \mathcal{L} on $X_0 := (\text{vect}\{f_1\})^\perp$ thus satisfies $\sup_{z \in \Delta_{\kappa^*}} \|\mathcal{R}_{\mathcal{L}}(z)\|_{\mathcal{B}(X_0)} < \infty$, and we conclude thanks to [82, Theorem V.1.11]. The lack of constructivity here only comes from the fact that our assumptions do not provide any information on the spectral gap $\lambda_1 - \kappa > 0$. \square

Remark 4.6.28. *Except of the Hilbert space framework, the assumptions made in Theorem 4.6.27 are slightly weaker than those of Theorem 4.6.25, and are indeed established during the proof of Theorem 4.6.25: such an information at the level of the resolvent is a bit easier to establish than a similar estimate at the level of the semigroup. In the splitting framework (4.6.28) and its resolvent counterpart (2.1.22), we typically only have to show*

$$\sup_{\kappa \leq \Re z \leq \kappa_1} \|\mathcal{V}(z)\|_{\mathcal{B}(X)} < \infty, \quad \lim_{r \rightarrow \infty} \sup_{\kappa \leq \Re z \leq \kappa_1, |\Im z| \geq r} \|\mathcal{W}(z)\|_{\mathcal{B}(X)} = 0, \quad (4.6.30)$$

for some $\kappa < \lambda_1$, and $\mathcal{W}(z) \in \mathcal{X}(X)$ for any $z \in \Delta_\kappa$. That last claim is classical (see for instance [113]) and we only briefly sketch the proof. On the one hand, from the first and the last estimates, we deduce that $\Sigma(\mathcal{L}) \cap \Delta_\kappa \subset \Sigma_d(\mathcal{L})$ thanks to Theorem 4.2.5. As in the proof of Theorem 4.2.5 and with the usual notations, we also have

$$(I - \mathcal{W}(z))\mathcal{R}_{\mathcal{L}}(z) = \mathcal{V}(z), \quad \forall z \in \Delta_\kappa,$$

where $I - \mathcal{W}(z)$ is invertible and $\|(I - \mathcal{W}(z))^{-1}\|_{\mathcal{B}(X)} \leq 2$ for any $z \in \mathbb{C}$ such that $\kappa \leq \Re z \leq \kappa_1$, $|\Im z| \geq R$ and R is large enough. We immediately deduce that the condition (i) in Theorem 4.6.27 holds.

We end this section by a more recent result which is similar to the Gearhart-Prüss Theorem but is not restricted to an Hilbert space.

Theorem 4.6.29. *Consider a positive semigroup S on a Banach lattice X such that its generator \mathcal{L} satisfies the conclusions **(C2)** about the existence, positivity and uniqueness of the first eigentriplet (λ_1, f_1, ϕ_1) . We further assume that $\mathcal{L} = \mathcal{A} + \mathcal{B}$ with $0 \leq \mathcal{A} \in \mathcal{B}(X)$, $S_{\mathcal{B}} \geq 0$ and the associated operators V and W defined by (2.2.14) satisfy (4.6.28) for some $\kappa < \lambda_1$ and that the resolvent counterpart \mathcal{W} defined by (2.1.22) satisfies (4.6.30) and more precisely*

$$\sup_{\kappa \leq \Re z \leq \kappa_1} \|\langle z \rangle^\alpha \mathcal{W}(z)\|_{\mathcal{B}(X)} < \infty,$$

with $\alpha > 1$. Then the exponential stability (4.6.27) holds (without constructive estimate).

The proof of Theorem 4.6.29 is a mere adaptation of [177, Theorem 3.1] (see also [179]) and it is thus skipped. The needed estimates are a bit stronger than those of Remark 4.6.28, but in the applications, they are not really more demanding. They also holds at the level of the resolvent instead of what is assumed in the statement of Theorem 4.6.25.

We conclude by emphasizing again on the fact that all the above results are not constructive. We propose in the next part an alternative approach which is constructive.

Chapter 5

Quantitative stability

In this chapter we establish some quantitative stability results in the spirit of the Doblin, Harris, Meyn-Tweedie theory for Markov semigroup.

5.1 About quantified positivity conditions

We briefly discuss some positivity conditions related to the strong maximum principle and barriers techniques. The issue is about how quantify the strong maximum principle

$$f \in X_+ \setminus \{0\}, (\kappa_1 - \mathcal{L})f \geq 0 \text{ imply } f > 0 \text{ or } f \gg 0$$

or the related strong positivity of the associated semigroup. A possible way can be achieved with the help of a barrier functions family $\mathcal{G} \subset X_+$ and a second weaker (semi)norm $[\cdot]$ used for normalization. Let us then introduce the two conditions

$$\forall R > 0, \exists g_i \in \mathcal{G}, \forall f \in X_+, [f] = 1, \|f\| \leq R, \quad (5.1.1)$$

we have

$$(i) S_T f \geq g_1 \text{ (for some } T > 0)$$

or

$$(ii) f \geq g_2 \text{ if } (\kappa_1 - \mathcal{L})f \geq 0.$$

Point (ii) is a quantified version of the strong maximum principle when $\mathcal{G} \subset X_{++}$ and it is always a consequence of the positivity condition (i). Assume indeed that (i) holds (for some $T > 0$) and that f satisfies the requirements (5.1.1) and $(\kappa_1 - \mathcal{L})f \geq 0$. We then write

$$\frac{d}{dt}(e^{(\mathcal{L}-\kappa_1)t}f) = e^{(\mathcal{L}-\kappa_1)t}(\mathcal{L} - \kappa_1)f \leq 0,$$

so that

$$f \geq e^{(\mathcal{L}-\kappa_1)T}f = e^{-\kappa_1 T} S_T f \geq e^{-\kappa_1 T} g_2 =: g_1,$$

with g_2 given by condition (i). The reciprocal implication is not clear, see however Lemma 3.2.6-(3).

Let us now make a list of possible quantified positivity conditions for a linear (and continuous) operator $A : X \rightarrow X$:

$$(P1') \exists g_0 \in X_+ \setminus \{0\}, \exists \psi_0 \in X_+ \setminus \{0\}, \forall f \in X_+, Af \geq g_0 \langle f, \psi_0 \rangle \text{ (Harris condition);}$$

$$(P2') \exists g_0 \in X_+ \setminus \{0\}, \exists \psi_0 \in X'_{++}, \forall f \in X_+, Af \geq g_0 \langle f, \psi_0 \rangle \text{ (Doblin condition);}$$

$$(P3') \exists g_0 \in X_{++}, \exists \psi_0 \in X'_+ \setminus \{0\}, \forall f \in X_+, Af \geq g_0 \langle f, \psi_0 \rangle \text{ (dual Doblin condition);}$$

$$(P4') \exists g_0 \in X_{++}, \exists \psi_0 \in X'_{++}, \forall f \in X_+, Af \geq g_0 \langle f, \psi_0 \rangle \text{ (strong Doblin condition).}$$

We summarize some elementary relations between these conditions and those listed in Section 3.2.

Lemma 5.1.1. *We have $(P2') \Rightarrow (P2) \Rightarrow (P1)$, $(P3') \Rightarrow (P3) \Rightarrow (P1)$, $(P4') \Rightarrow ((P4), (P3'), (P2'))$ as well as $(P4) \Rightarrow ((P3), (P2))$.*

We also have: A satisfies $(P2')$ iff A^ satisfies $(P3')$; A satisfies $(P3')$ iff A^* satisfies $(P2')$; A satisfies $(P4')$ iff A^* satisfies $(P4')$.*

We finally have: A satisfies $(P2')$ implies $\exists g_0 \in X_+ \setminus \{0\}$, $\exists \kappa > 0$, $Ag_0 \geq \kappa g_0$

Proof of Lemma 5.1.1. We assume $Af \geq g_0 \langle f, \psi_0 \rangle$ for any $f \in X_+$ and some $g_0 \in X_+$, $\psi_0 \in X'_+$. For any $\phi \in X' \setminus \{0\}$ and $f \in X_+$, we have

$$\langle A^* \phi, f \rangle = \langle \phi, Af \rangle \geq \langle \phi, g_0 \langle f, \psi_0 \rangle \rangle,$$

which implies $A^* \phi \geq \psi_0 \langle \phi, g_0 \rangle$. We thus deduce that A satisfies $(P2')$ (resp. $(P3')$, $(P4)$) implies that A^* satisfies $(P3')$ (resp. $(P2')$, $(P4)$). \square

We conclude this introductory section by emphasizing on the fact (as already mentioned above) that $S_{\mathcal{L}}$ satisfies (Pi') implies $\mathcal{R}_{\mathcal{L}}(\lambda)$ satisfies (Pi') for any $\lambda \geq \lambda_1$ and $i = 1, \dots, 4$.

5.2 Asymptotic stability under Doblin condition

We start with a simple situation. We assume the Doblin condition, namely

$$\exists T > 0, \exists \psi_0 \gg 0, \exists g_0 > 0, \quad \forall f \geq 0, S_T f \geq g_0 \langle \psi_0, f \rangle, \quad (5.2.2)$$

together with the companion positivity condition

$$\exists r_0 > 0, \quad \langle \phi_1, g_0 \rangle \geq r_0, \quad (5.2.3)$$

as well as the strong additional boundedness assumption

$$\exists R_0 > 0, \quad \phi_1 \leq R_0 \psi_0. \quad (5.2.4)$$

When $\psi_0 := 1 \in X' \subset L^\infty$, the condition in (5.2.4) is automatically satisfied with $R_0 := \|\phi_1\| = 1$. In the case when $S_{\mathcal{L}}^*$ enjoys a splitting structure similar to (4.5.18) and more precisely

$$\|\tilde{S}^*(t)\phi\| \leq \Theta(t)\|\phi\| + \int_0^t \Theta(t-s)[\tilde{S}^*(s)\phi]_{g_0} ds,$$

with $\Theta \in L^1(\mathbb{R}_+) \cap C_0(\mathbb{R}_+)$, we deduce that

$$1 = \|\phi_1\| = \|\tilde{S}^*(t)\phi_1\| \leq \Theta(t) + \int_0^t \Theta(t-s)[\phi_1]_{g_0} ds, \quad \forall t > 0.$$

Passing to the limit $t \rightarrow \infty$, we get (5.2.3) with $r_0 := \|\Theta\|_{L^1}^{-1}$. Another condition can be formulated as follows. We assume that $D(\mathcal{L}^\infty) \subset L_{\text{loc}}^1$ and the splitting property $\mathcal{L} = \mathcal{A} + \mathcal{B}$ with $\mathcal{A} \in \mathcal{B}(X)$, $\mathcal{R}_{\mathcal{B}}(\lambda) \in \mathcal{B}(X) \cap \mathcal{B}(X_+)$ for any $\lambda \geq \kappa$, with $\kappa < \kappa_0 \leq \lambda_1$, and the additional regularity condition

$$(\mathcal{R}_{\mathcal{B}^*}(\lambda)\mathcal{A}^*)^N : L_{g_0}^1 \rightarrow L_{\psi_0}^{\infty-1}, \quad \forall \lambda > \kappa. \quad (5.2.5)$$

The dual eigenvector ϕ_1 satisfies

$$(\lambda - \mathcal{B}^*)\phi_1 = \mathcal{A}^*\phi_1, \quad \lambda_1 > \kappa,$$

and then $\phi_1 = (\mathcal{R}_{\mathcal{B}^*}(\lambda_1)\mathcal{A}^*)^N \phi_1$, so that (5.2.3)-(5.2.4) holds with the normalization condition $r_0 := 1$ and $R_0 := \|(\mathcal{R}_{\mathcal{B}^*}(\lambda)\mathcal{A}^*)^N\|_{\mathcal{B}(L_{g_0}^1, L_{\psi_0}^{\infty-1})}$.

We are then able to formulate a first quantified stability result.

Theorem 5.2.2. *Consider a semigroup S on a Banach lattice X such that its generator \mathcal{L} enjoys the conclusion **(C1)** on the existence of the first eigentriplet (λ_1, f_1, ϕ_1) . We assume furthermore the Doblin condition (5.2.2)–(5.2.4)–(5.2.3). Then the exponential stability (4.6.27) in the norm $[\cdot]_{\psi_0}$ holds true, with constructive constants.*

The proof closely follows the usual contraction argument in the Doblin result, see for instance [170], [92, Theorem 11] or [50, Theorem 2.1]. We do not explicitly assume the irreducibility of the semigroup, but the Doblin condition (5.2.2)–(5.2.4)–(5.2.3) is in many aspects a strong positivity condition. In particular, our result implies the uniqueness of the first eigentriplet (λ_1, f_1, ϕ_1) and the triviality of the boundary spectrum.

Proof of Theorem 5.2.2. The two conditions (5.2.2) and (5.2.4) together imply the modified Doblin condition

$$\exists T > 0, \exists g_1 > 0, \quad \forall f \geq 0, S_T f \geq g_1 \langle \phi_1, f \rangle,$$

with $g_1 := g_0/R_0$. Take f such that $\langle \phi_1, f \rangle = 0$, so that $\langle \phi_1, f_{\pm} \rangle = r := \langle \phi_1, |f| \rangle / 2 \geq 0$ and thus

$$S_T f_{\pm} \geq g_1 \langle \phi_1, f_{\pm} \rangle = r g_1.$$

We write

$$|S_T f| \leq |S_T f_+ - r g_1| + |S_T f_- - r g_1| = S_T |f| - 2r g_1.$$

We deduce

$$\langle \phi_1, |S_T f| \rangle \leq \langle S_T^* \phi_1, |f| \rangle - 2r \langle \phi_1, g_1 \rangle = \left(e^{\lambda_1 T} - \langle \phi_1, g_1 \rangle \right) \langle \phi_1, |f| \rangle.$$

In other words, setting $\tilde{S}_t := e^{-\lambda_1 t} S_t$, we have

$$[\tilde{S}_T f]_{\phi_1} \leq \gamma [f]_{\phi_1},$$

with $\gamma < 1$ which depends explicitly of r_0, R_0, T and the estimates on λ_1 . We then classically deduce the exponential convergence in the $[\cdot]_{\phi_1}$ norm. Now, the dual condition associated to the Doblin hypothesis (5.2.2) is

$$\forall \psi \in X'_+, \quad S_T^* \psi \geq \psi_0 \langle \psi, g_0 \rangle.$$

In particular, the first dual eigenvector ϕ_1 satisfies

$$\phi_1 = e^{-\lambda_1 T} S_T^* \phi_1 \geq e^{-\lambda_1 T} \psi_0 \langle \phi_1, g_0 \rangle = e^{-\lambda_1 T} r_0 \psi_0. \quad (5.2.6)$$

Together with condition in (5.2.4), we see that $[\cdot]_{\phi_1}$ and $[\cdot]_{\psi_0}$ are equivalent norm, and we immediately obtain the exponential convergence in the $[\cdot]_{\psi_0}$ norm (with constructive constants). \square

5.3 Asymptotic stability under Harris condition

The Doblin condition (5.2.2)–(5.2.4)–(5.2.3) is too much demanding for many applications. In this section, we make the following somehow more general Harris type condition

$$\begin{cases} \exists T > 0, \forall R > 0, \exists g_R > 0 \text{ such that} \\ \forall f \geq 0, \|f\| \leq R [f]_{\phi_1} \text{ there holds } S_T f \geq g_R [f]_{\phi_1}, \end{cases} \quad (5.3.7)$$

that we complement with the following Lyapunov condition

$$\|\tilde{S} f\| \leq \gamma_L \|f\| + K [f]_{\phi_1}, \quad (5.3.8)$$

with $\tilde{S} = S_T e^{-\lambda_1 T}$, $\gamma_L \in (0, 1)$, $K \geq 0$. We replace the positivity condition (5.2.3) by

$$\exists r_R > 0, \quad \langle \phi_1, g_R \rangle \geq r_R. \quad (5.3.9)$$

As we have seen several times, condition (5.3.8) is some kind of regularity hypothesis which is natural under a splitting structure on the semigroup $S_{\mathcal{L}}$.

Theorem 5.3.3. *Consider a semigroup S on a Banach lattice X such that its generator \mathcal{L} enjoys the conclusions (C1) on the existence of the first eigentriplet (λ_1, f_1, ϕ_1) . We assume furthermore the Harris condition (5.3.7) together with the Lyapunov condition (5.3.8) and the positivity condition (5.3.9). Then the exponential stability (4.6.27) in the norm of X holds true, with constructive constants.*

Of course, in order that Theorem 5.3.3 really gives a constructive convergence result, we have to establish (5.3.7), (5.3.8) and (5.3.9) in a constructive way.

Proof of Theorem 5.3.3. On the one hand, we have

$$[\tilde{S}f]_{\phi_1} \leq \langle \tilde{S}|f|, \phi_1 \rangle = \langle |f|, \tilde{S}^* \phi_1 \rangle = [f]_{\phi_1}. \quad (5.3.10)$$

On the other hand, we wish to establish the coupling property

$$[\tilde{S}f]_{\phi_1} \leq \gamma_H [f]_{\phi_1} \quad \text{if} \quad \|f\| \leq A[f]_{\phi_1} \quad \text{and} \quad \langle f, \phi_1 \rangle = 0, \quad (5.3.11)$$

for any $A > K/(1 - \gamma_L)$ and some $\gamma_H \in (0, 1)$. In order to do so we fix $A > K/(1 - \gamma_L)$ and we set $R := 2A$. We next consider $f \in X$, such that $\langle f, \phi_1 \rangle = 0$ and $\|f\| \leq A[f]_{\phi_1}$, so that

$$\|f_{\pm}\| \leq \|f\| \leq A[f]_{\phi_1} = R[f_{\pm}]_{\phi_1}.$$

Using the Harris condition (5.3.7), we deduce

$$\tilde{S}f_{\pm} \geq \vartheta g_R, \quad \vartheta := \frac{1}{2} e^{-\lambda_1 T} [f]_{\phi_1}.$$

Similarly as in the proof of f of Theorem 5.2.2, we next compute

$$|\tilde{S}f| \leq |\tilde{S}f_+ - \vartheta g_R| + |\tilde{S}f_- - \vartheta g_R| \leq \tilde{S}|f| - 2\vartheta g_R$$

and then

$$\begin{aligned} [\tilde{S}f]_{\phi_1} &\leq \langle \tilde{S}|f| - 2\vartheta g_R, \phi_1 \rangle \\ &= \langle |f|, \tilde{S}^* \phi_1 \rangle - 2\vartheta \langle g_R, \phi_1 \rangle \\ &= (1 - e^{-\lambda_1 T} \langle g_R, \phi_1 \rangle) [f]_{\phi_1}, \end{aligned}$$

which in turn implies (5.3.11) with $\gamma_H := 1 - e^{-\lambda_1 T} \langle g_R, \phi_1 \rangle$.

Now, the two estimates (5.3.10) and (5.3.11) together give

$$[Sf]_{\phi_1} \leq \gamma_H [f]_{\phi_1} + \frac{1 - \gamma_H}{A} \|f\|. \quad (5.3.12)$$

From (5.3.12) and the Lyapunov condition (5.3.8), we deduce that

$$U^{n+1} = MU^n$$

with

$$U^n := \begin{pmatrix} \|S^n f\| \\ [S^n f]_{\phi_1} \end{pmatrix} \quad \text{and} \quad M := \begin{pmatrix} \gamma_L & K \\ \frac{1 - \gamma_H}{A} & \gamma_H \end{pmatrix}.$$

The eigenvalues of M are

$$\mu_{\pm} := \frac{1}{2} (T \pm \sqrt{T^2 - 4D}),$$

with

$$T := \text{tr} M = \gamma_L + \gamma_H, \quad D := \det M = \gamma_L \gamma_H - (1 - \gamma_H) \frac{K}{A}.$$

We observe that

$$\gamma_L \gamma_H > D > \gamma_L \gamma_H - (1 - \gamma_H)(1 - \gamma_L) = T - 1,$$

so that

$$(\gamma_H - \gamma_L)^2 = T^2 - 4\gamma_L \gamma_H < T^2 - 4D < T^2 - 4(T - 1) = (T - 2)^2$$

and finally

$$\alpha := \max(|\mu_+|, |\mu_-|) < \max(\gamma_H, \gamma_L, |T - 1|, 1) = 1.$$

We conclude that $\|M^n\| \lesssim \alpha^n$, from what we immediately conclude. \square

5.4 The weak dissipativity case

In this section, we consider a weak dissipative semigroup (S_t) as considered in Section 2.2.3 and in a sense we make precise now. We consider four Banach lattices $X_3 \subset X_2 \subset X_1 \subset X_0 = X$.

We first make the same kind of Harris type condition as in the previous section, namely

Hypothesis (H) (Doblin-Harris) condition (5.3.7) holds for the same time $T > 0$ and for both norms $\|\cdot\| = \|\cdot\|_{X_0}$ and $\|\cdot\| = \|\cdot\|_{X_2}$ as well as the companion positivity condition (5.3.9) holds;

Instead of the strong Lyapunov condition (5.3.8), we assume

Hypothesis (L) (weak Lyapunov) there exist a constant $K \geq 0$ such that

$$\begin{aligned} \|\tilde{S}f\|_1 + \|\tilde{S}f\|_0 &\leq \|f\|_1 + K[f]_{\phi_1}, \quad \forall f \in X_1, \\ \|\tilde{S}f\|_3 + \|\tilde{S}f\|_2 &\leq \|f\|_3 + K[f]_{\phi_1}, \quad \forall f \in X_3, \end{aligned}$$

with $\tilde{S} = S_T e^{-\lambda_1 T}$;

Hypothesis (I) (interpolation) there exists an increasing function $\xi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, $\lambda \mapsto \xi_\lambda$, such that

$$\lambda \|f\|_1 \leq \|f\|_0 + \xi_\lambda \|f\|_3, \quad \forall \lambda > 0, \quad \xi_\lambda / \lambda \rightarrow 0 \quad \text{as } \lambda \rightarrow 0.$$

Theorem 5.4.4. *Consider a semigroup S on a Banach lattice X such that its generator \mathcal{L} enjoys the conclusions (C1) on the existence of the first eigentriplet (λ_1, f_1, ϕ_1) . We assume furthermore the three above conditions of weak confinement (L), Doblin-Harris strong irreducibility (H) and interpolation (I). Then, there exist some constructive decay rate functions Θ and $\tilde{\Theta}$ such that*

$$\|S^n f\|_{X_1} \lesssim \Theta(n) \|f\|_{X_3}, \quad \forall n \geq 1, \tag{5.4.13}$$

and

$$\|S^n f\| \lesssim \tilde{\Theta}(n) \|f\|_{X_3}, \quad \forall n \geq 1, \tag{5.4.14}$$

for any $f \in X_3$, $\langle f, \phi_1 \rangle = 0$. More precisely, the decay rate functions Θ and $\tilde{\Theta}$ are defined by

$$\Theta(t) := \inf_{\lambda} \Theta_{\zeta\lambda}(t), \quad \tilde{\Theta}(t) := t^{-1} \Theta([t/2]), \tag{5.4.15}$$

for a constructive constant $\zeta \in (0, 1)$, the infimum being taken over all the decreasing function $\lambda : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, $t \mapsto \lambda_t$, and Θ_λ is defined by

$$\Theta(t) := \inf_{\lambda > 0} (e^{-\lambda t} + \frac{\xi_\lambda}{\lambda}). \tag{5.4.16}$$

The proof follows closely the proof of [50, Thm 4.8]. We start with the following key argument of non expansive mapping result on a well chosen norm.

Proposition 5.4.5. *Consider a positive semigroup (S_t) which satisfies both above conditions of weak confinement **(L)** and Doblin-Harris strong irreducibility **(H)**. There exist some equivalent norms $\|\cdot\|_1$ to $\|\cdot\|_1$ and $\|\cdot\|_3$ to $\|\cdot\|_3$ such that \tilde{S}_t is a non expansive mapping for the two new norms $\|\cdot\|_1$ and $\|\cdot\|_3$. More precisely, there exists $\alpha > 0$ such that*

$$\|\tilde{S}f\|_1 + \alpha \|\tilde{S}f\|_0 \leq \|f\|_1, \quad \forall f \in X_1, \langle f, \phi_1 \rangle = 0, \quad (5.4.17)$$

$$\|\tilde{S}f\|_3 + \alpha \|\tilde{S}f\|_2 \leq \|f\|_3, \quad \forall f \in X_3, \langle f, \phi_1 \rangle = 0. \quad (5.4.18)$$

Proof of Proposition 5.4.5. We define

$$\|f\|_1 := [f]_{\phi_1} + \delta \|f\|_0 + \beta \|f\|_1, \quad (5.4.19)$$

with $\beta > \delta > 0$ conveniently chosen. We take $\beta := (1 - \gamma_H)/K$, $\delta := (1 - \gamma_H)/A$. We define $\|\cdot\|_3$ in the same way. In what follows, we then only establish (5.4.17), the proof of (5.4.18) being exactly the same.

We fix $f \in X_1$, $\langle f, \phi_1 \rangle = 0$, and we recall

$$[\tilde{S}f]_{\phi_1} \leq [f]_{\phi_1}. \quad (5.4.20)$$

We also recall that from (5.3.11), for any $A > 0$, there exists $\gamma_H = \gamma_H(A) \in (0, 1)$ such that the following coupling property holds

$$[\tilde{S}f]_{\phi_1} \leq \gamma_H [f]_{\phi_1} \quad \text{if} \quad \|f\|_0 \leq A [f]_{\phi_1}. \quad (5.4.21)$$

We fix $A > K$ and we observe that the following alternative holds

$$\|f\|_0 \leq A [f]_{\phi_1} \quad (5.4.22)$$

or

$$\|f\|_0 > A [f]_{\phi_1}. \quad (5.4.23)$$

Case 1. Under condition (5.4.22), we use (5.4.21) and the first estimate in **(L)**, and we deduce

$$\begin{aligned} \|\tilde{S}f\|_1 &= [\tilde{S}f]_{\phi_1} + \delta \|\tilde{S}f\|_0 + \beta \|\tilde{S}f\|_1 \\ &\leq \gamma_H [f]_{\phi_1} + \beta \|f\|_1 + \beta K [f]_{\phi_1} - (\beta - \delta) \|\tilde{S}f\|_0. \end{aligned}$$

From our choice of $\beta > 0$ we have $\gamma_H + \beta K = 1$, and we conclude that (5.4.17) holds with $\alpha := \beta - \delta > 0$.

Case 2. Under condition (5.4.23), the first Lyapunov condition in **(L)** implies

$$\|\tilde{S}f\|_1 + \|\tilde{S}f\|_0 \leq \|f\|_1 + \frac{K}{A} \|f\|_0.$$

Together with the non expansivity estimate (5.4.20), we get

$$[\tilde{S}f]_{\phi_1} + \beta \|\tilde{S}f\|_1 + \beta \|\tilde{S}f\|_0 \leq [f]_{\phi_1} + \beta \|f\|_1 + \delta \|f\|_0,$$

and we conclude to (5.4.17) again. \square

The subgeometric convergence result is a straightforward consequence of Proposition 5.4.5 and an interpolation argument.

Proposition 5.4.6. *Assume that S satisfies the hypotheses of Theorem 5.4.4. Then (5.4.13) and (5.4.14) hold true with the same decay rate functions Θ and $\tilde{\Theta}$ given by (5.4.15) (up to a modification of the constant ζ).*

Proof of Proposition 5.4.6. We recall that we have already proven (5.4.17) and (5.4.18). From (5.4.17) and the interpolation condition **(I)**, we deduce

$$\|\tilde{S}f\|_1 + \lambda\alpha\|\tilde{S}f\|_1 \leq \|f\|_1 + \xi_\lambda\alpha\|\tilde{S}f\|_3.$$

We observe next that from the very definition of the $\|\cdot\|_1$ norm

$$\|\tilde{S}f\|_1 + \frac{\alpha}{\lambda}\|\tilde{S}f\|_1 \geq Z_\lambda\|\tilde{S}f\|_1, \quad Z_\lambda = 1 + \kappa\lambda \in (1, 2],$$

for some $\kappa > 0$ and any $\lambda \in (0, \lambda_0)$, $\lambda_0 > 0$, and that from the very definition of the $\|\cdot\|_3$ norm

$$\alpha\xi_\lambda\|\tilde{S}f\|_3 \leq B\xi_\lambda\|\tilde{S}f\|_3,$$

for some $B > 0$. The three above estimates together imply

$$Z_\lambda\|\tilde{S}f\|_1 \leq \|f\|_1 + B\xi_\lambda\|\tilde{S}f\|_3.$$

Using the second estimate (5.4.18) and repeating the same proof, we have

$$Z_{\lambda_{n+1}}\|\tilde{S}^{n+1}f\|_1 \leq \|\tilde{S}^n f\|_1 + B\xi_{\lambda_{n+1}}\|f\|_3,$$

for any $n \geq 0$ and for any $\lambda_{n+1} > 0$. The discrete Grönwall lemma implies

$$\|\tilde{S}^n f\|_1 \leq A_n\|f\|_1 + \sum_{k=1}^n A_{k,n}\xi_{\lambda_k}B\|f\|_3, \quad \forall n \geq 0, \quad (5.4.24)$$

where we have defined

$$A_n := \prod_{k=1}^n a_k, \quad A_{k,n} = A_n/A_k = \prod_{i=k+1}^n a_i, \quad a_i := Z_{\lambda_i}^{-1}.$$

Observing that

$$A_{k,n} \lesssim e^{-\kappa\sum_{i=k}^n \lambda_i} \lesssim e^{\kappa(\Lambda(k) - \Lambda(n))}, \quad \text{with } \Lambda(t) := \int_0^t \lambda_s ds$$

and $\lambda_s := \lambda_i$ if $s \in (i-1, i]$, we immediately conclude that the first estimate (5.4.13) holds true. We come back to the first inequality in (5.4.17) that we iterate and sum up in order to obtain

$$\|\tilde{S}^n f\|_1 + \alpha \sum_{k=[n/2]+1}^n \|\tilde{S}^k f\|_0 \leq \|\tilde{S}^{[n/2]} f\|_1,$$

for any $n \geq 1$. Together with the non expansion inequality

$$[\tilde{S}^n f]_{\phi_1} \leq [\tilde{S}^k f]_{\phi_1} \lesssim \|\tilde{S}^k f\|_0, \quad \forall n \geq k,$$

and the first estimate (5.4.13), we deduce

$$(n - [n/2] - 1)\alpha[\tilde{S}^n f]_{\phi_1} \lesssim \Theta([n/2])\|f\|_3,$$

which is nothing but (5.4.14). □

Chapter 6

Application to Diffusion models

In this part, we consider a general elliptic operator in divergence form

$$\mathcal{L}f := \partial_i(a_{ij}\partial_j f) + b_i\partial_i f + \partial_i(\beta_i f) + cf, \quad f \in H_0^1(\Omega), \quad (6.0.1)$$

where $\Omega \subset \mathbb{R}^d$ is a bounded domain (i.e. an open and connected set) or $\Omega = \mathbb{R}^d$, and for simplicity we always assume $d \geq 3$. We also always assume at least a boundedness and ellipticity condition on the (a_{ij}) matrix, namely

$$a_{ij} \in L^\infty(\Omega), \quad \exists \nu > 0, \quad \forall \xi \in \mathbb{R}^d, \quad a_{ij}\xi_i\xi_j \geq \nu|\xi|^2, \quad (6.0.2)$$

and some conditions on the coefficients b_i , β_j and c which will be described below.

We aim to establish the existence of (λ_1, f_1, ϕ_1) solution to the first eigentriplet problem

$$\lambda_1 \in \mathbb{R}, \quad 0 < f_1 \in H_0^1, \quad \mathcal{L}f_1 = \lambda_1 f_1, \quad 0 < \phi_1 \in H_0^1, \quad \mathcal{L}^*\phi_1 = \lambda_1 \phi_1, \quad (6.0.3)$$

and the existence of some (constructive) reate function Θ such that the rescaled semigroup $\tilde{\mathcal{S}}$ associated to the generator $\tilde{\mathcal{L}} = \mathcal{L} - \lambda_1$ satisfies

$$\|\tilde{\mathcal{S}}(t)f - \langle f, \phi_1 \rangle f_1\|_{H_0} \leq \Theta(t)\|f - \langle f, \phi_1 \rangle f_1\|_H, \quad (6.0.4)$$

for any $t \geq 0$ and any $f \in H$, with $H \subset H_0 \subset L^2$.

6.1 Diffusion with rough coefficients in a bounded domain

In this section, we consider the general elliptic operator in divergence form (1.2.8) in the case of a bounded and smooth enough domain $\Omega \subset \mathbb{R}^d$ with general elliptic condition on a_{ij} as formulated above. We further assume

$$b_i, \beta_j \in L^r(\Omega), \quad c \in L^{r/2}(\Omega), \quad r > d, \quad (6.1.5)$$

as well as

$$c + \operatorname{div}\beta \leq c_0 \quad \text{and} \quad c - \operatorname{div}b \leq c_0, \quad (6.1.6)$$

for some $c_0 \in \mathbb{R}$.

In that situation, the first eigentriplet problem (6.0.3) in a slightly less general framework (all the coefficients belong to L^∞) is considered in [156]. We explain with all details the existence proof by following more or less the arguments presented in [156] and next we present a proof of the geometric part and the stability part by taking advantage of the abstract material developed in the previous sections. It is worth emphasizing that our proof of the uniqueness of the first eigenfunction significantly differs from the one presented in [156] which is based on a dissipativity argument, probably related to the reverse Kato's inequality condition. The framework considered

here is the usual generalized solutions or weak solutions framework which goes back at least to Stampacchia [226, 225], but it is reminiscent of previous contributions by Friedrichs [91, 90], Gårding [96], De Giorgi [68], Nash [192], Morrey [184], Moser [186, 187, 185], Ladyzhenskaya, Solonnikov, Ural'ceva [148, 146], Oleinik, Kruzhkov [196] and many others. Lot of the functional arguments are picked up from the book of Gilbarg and Trudinger, and more specifically from [104, Chapter 8], and also in recent notes by Kavian [137] and Vasseur [239]. It is worth emphasizing that the present analysis does not apply directly to elliptic operator in non divergence form, although this framework is considered in [156]. We expect that all the results developed below can be generalized to a non divergence form framework, for example the one developed in [23], but we do not follow this line of research in the present work.

The proof of (6.0.3) and (6.0.4) are straightforward consequences of the abstract results developed in the previous sections once we have been able to check that the corresponding hypotheses are fulfilled. In the sequel, we will then show how these hypotheses are met in the present context.

Condition (H1). We recall that a weak (or variational) solution to the elliptic equation

$$\mathcal{L}f = g \in H^{-1}(\Omega), \quad f \in H_0^1(\Omega),$$

is a function $f \in H_0^1(\Omega)$ such that

$$D_{\mathcal{L}}(f, w) = \langle g, w \rangle, \quad \forall w \in H_0^1(\Omega), \quad (6.1.7)$$

where the (negative) Dirichlet form $D_{\mathcal{L}}$ is defined by

$$D_{\mathcal{L}}(f, w) := - \int_{\Omega} (a_{ij} \partial_j f + \beta_i f) \partial_i w + \int_{\Omega} (b_i \partial_i f w + c f w),$$

for any $f, w \in H_0^1(\Omega)$. Most of the time, we will simply write

$$\langle \mathcal{L}f, w \rangle = \langle g, w \rangle, \quad \forall w \in H_0^1(\Omega), \quad (6.1.8)$$

instead of (6.1.7). For the reader convenience, we repeat here some estimates picked up in [225]. For $\lambda \in \mathbb{R}$ and $f \in H_0^1(\Omega)$, we start with

$$\begin{aligned} \langle (\lambda - \mathcal{L})f, f \rangle &= \int_{\Omega} a_{ij} \partial_i f \partial_j f + \int_{\Omega} (\beta_i - b_i) \partial_i f f + \int_{\Omega} (\lambda - c) f^2 \\ &\geq \|f \sqrt{c_-}\|_{L^2}^2 + \nu \|\nabla f\|_{L^2}^2 - \|\beta - b\|_{L^2} \|\nabla f\|_{L^2} - \|\sqrt{c_+} f\|_{L^2}^2 + \lambda \|f\|_{L^2}^2 \\ &\geq \|f \sqrt{c_-}\|_{L^2}^2 + \frac{\nu}{2} \|\nabla f\|_{L^2}^2 - \frac{1}{2\nu} \|\beta - b\|_{L^2}^2 - \|\sqrt{c_+} f\|_{L^2}^2 + \lambda \|f\|_{L^2}^2, \end{aligned}$$

using the Cauchy-Schwarz inequality and the Young inequality, and next

$$\begin{aligned} \langle (\lambda - \mathcal{L})f, f \rangle &\geq \|f \sqrt{c_-}\|_{L^2}^2 + \frac{\nu}{4} \|\nabla f\|_{L^2}^2 + \left(\lambda - \frac{M}{2\nu} - M^{1/2}\right) \|f\|_{L^2}^2 \\ &\quad + \frac{\nu}{4} C_{\Omega} \|f\|_{L^{2^*}}^2 - \frac{1}{2\nu} \|\beta - b\|_{L^2} \|\mathbf{1}_{|\beta-b| \geq M} f\|_{L^2} - \|\sqrt{c_+} \mathbf{1}_{c_+ \geq M} f\|_{L^2}^2 \\ &\geq \|f \sqrt{c_-}\|_{L^2}^2 + \frac{\nu}{4} \|\nabla f\|_{L^2}^2 + \left(\lambda - \frac{M}{2\nu} - M^{1/2}\right) \|f\|_{L^2}^2 \\ &\quad + \left(\frac{\nu}{4} C_{\Omega} - \frac{1}{2\nu} \|\beta - b\|_{L^d} \|\mathbf{1}_{|\beta-b| \geq M}\|_{L^d} - \|c_+ \mathbf{1}_{c_+ \geq M}\|_{L^{d/2}}\right) \|f\|_{L^{2^*}}^2, \end{aligned}$$

using the Sobolev inequality (with associated constant C_{Ω}) and the Holder inequality. Choosing $M > 0$ large enough in such a way that the last term is positive, and next $\kappa_1 > 0$ large enough, we deduce for instance that

$$\langle (\lambda - \mathcal{L})f, f \rangle \geq \|f \sqrt{c_-}\|_{L^2}^2 + \frac{\nu}{4} \|\nabla f\|_{L^2}^2 + \|f\|_{L^2}^2, \quad \forall \lambda \geq \kappa_1. \quad (6.1.9)$$

Thanks to the Lax-Milgram theorem and the above coercivity estimate, we deduce that $\lambda - \mathcal{L}$ is invertible, and more precisely the mapping $(\lambda - \mathcal{L})^{-1} : H^{-1} \rightarrow H_0^1(\Omega)$ is well defined. We also claim that $\lambda - \mathcal{L}$ enjoys a weak principle maximum, and more precisely

$$f \in H_0^1(\Omega), \quad (\lambda - \mathcal{L})f \geq 0 \quad \text{imply} \quad f \geq 0. \quad (6.1.10)$$

Indeed, for such a function $f \in H_0^1(\Omega)$, we take $w = f_- \in H_0^1(\Omega)$, as a test function, and elementary Sobolev space calculus together with the previous estimate yields

$$\begin{aligned} 0 &\leq \langle (\lambda - \mathcal{L})f, f_- \rangle = -\langle (\lambda - \mathcal{L})f_-, f_- \rangle \\ &\leq -\|f_- \sqrt{c_-}\|_{L^2}^2 - \frac{\nu}{4} \|\nabla f_-\|_{L^2}^2 - \|f_-\|_{L^2}^2 \leq 0, \end{aligned}$$

so that $f_- = 0$ and $f \geq 0$. We thus deduce $(\lambda - \mathcal{L})^{-1} : L_+^2 \rightarrow L_+^2$, and from J.-L. Lions theory on parabolic equation (see for instance [155, Chapter 3]), we next deduce that \mathcal{L} is the generator in L^2 of a positive semigroup $S_{\mathcal{L}}$, so that **(H1)** holds. It is worth emphasizing at this point that the semigroup S built thanks to Lions' theory is defined by $S(t)f_0 = f$ for any $f_0 \in L^2$, where $f \in \mathcal{E} := C([0, \infty); L^2) \cap L_{\text{loc}}^2([0, \infty); H_0^1) \cap H_{\text{loc}}^1([0, \infty); H^{-1})$ is the unique (variational) solution to the equation

$$(f(T), g(T))_{L^2} - (f_0, g(0))_{L^2} = \int_0^T \{ \langle \partial_t g, f \rangle_{H^{-1}, H_0^1} + D_{\mathcal{L}}(f, g) \} ds, \quad (6.1.11)$$

for any $T > 0$ and $g \in \mathcal{E}$. Choosing $g = f$ in the above equation, we classically compute

$$\frac{1}{2} \|f(t)\|_{L^2}^2 - \frac{1}{2} \|f_0\|_{L^2}^2 - \int_0^t D_{\mathcal{L}}(f, f) ds = 0, \quad \forall t > 0,$$

which together with (6.1.9) imply

$$\frac{1}{t} \int_0^t \frac{\nu}{4} \|\nabla f\|_{L^2}^2 ds \leq -\left(\frac{f(t) - f_0}{t}, \frac{f(t) + f_0}{2} \right)_{L^2} + \frac{\kappa_1}{t} \int_0^t \|f\|_{L^2}^2 ds, \quad \forall t > 0.$$

When $f_0 \in D(\mathcal{L})$, the RHS is bounded and there thus exists a sequence $t_n \rightarrow 0$ such that $\|\nabla f(t_n)\|_{L^2}$ is bounded. That implies $f_0 \in H_0^1(\Omega)$ and thus $D(\mathcal{L}) \subset H_0^1(\Omega)$. Similarly, we may consider the dual Dirichlet form $D^*(f, g) := D_{\mathcal{L}}(g, f)$ and build an associated positive semigroup S^* thanks to Lions' theory described above. More precisely $S^*(t)g_0 = g$ for any $t \geq 0$ and $g_0 \in L^2$, where $g \in \mathcal{E}$ is the unique (variational) solution to the equation

$$(g(t), f(t))_{L^2} - (g_0, f(0))_{L^2} = \int_0^t \{ \langle \partial_t f, g \rangle_{H^{-1}, H_0^1} + D^*(g, f) \} ds,$$

for any $t > 0$ and $f \in \mathcal{E}$. Now, we fix $T > 0$, $g_T \in L^2$ and we set $g(t) := S^*(T - t)g_T$, so that g is a solution to the backward evolution equation

$$-\partial_t g = \mathcal{L}^* g, \quad g(T) = g_T,$$

with

$$\mathcal{L}^* g := \partial_j (a_{ij} \partial_i g) - \partial_i (b_i g) - \beta_i \partial_i g + c g.$$

The variational formulation of this last problem is

$$(g_T, f(T))_{L^2} - (g(0), f(0))_{L^2} = \int_0^T \{ \langle \partial_t f, g \rangle_{H^{-1}, H_0^1} - D^*(g, f) \} ds, \quad (6.1.12)$$

for any $f \in \mathcal{E}$. Summing up (6.1.11) and (6.1.12) with $f(t) := S(t)f_0$ for $f_0 \in L^2$ and $g(t) := S^*(T - t)g_T$ for $g_T \in L^2$, we deduce

$$(S(T)f_0, g_T)_{L^2} = (S^*(T)g_T, f_0)_{L^2}.$$

In other words, we have established that $S^* = (S_{\mathcal{L}})^*$ and thus that \mathcal{L}^* is the generator of the semigroup S^* .

Condition (H2). Let us consider a ball B_R , $R > 0$, such that $B_{4R} \subset \Omega$ and next the solution

$$f_0 \in H_0^1(\Omega), \quad (\kappa_1 - \mathcal{L})f_0 = \mathbf{1}_{B_R}, \quad (6.1.13)$$

which exists from the above discussion. We next recall some classical results. On the one hand, from [226, Section 3 & Section 4] or [104, Theorem 8.15] (see also the original papers [68, 192, 186]), the following global L^∞ De Giorgi-Nash-Moser type estimate

$$\|f_+\|_{L^\infty(\Omega)} \lesssim \|f_+\|_{L^2(\Omega)} + \|g\|_{L^{r/2}(\Omega)} \quad (6.1.14)$$

holds for any subsolution

$$f \in H_0^1(\Omega), \quad (\lambda - \mathcal{L})f \leq g \in L^{r/2}(\Omega).$$

The local estimate variant [104, Theorem 8.18] (or *weak Harnack inequality*)

$$\|f\|_{L^p(B_{2R})} \lesssim \inf_{B_R} f + \|g\|_{L^{r/2}(\Omega)}, \quad \forall p \in [1, 2^*/2), \quad (6.1.15)$$

also holds for a nonnegative supersolution

$$f \in H^1(\Omega), \quad f \geq 0 \text{ on } B_{4R} \subset \Omega, \quad (\lambda - \mathcal{L})f \geq g \in L^{r/2}(\Omega),$$

from what one deduces that a strong maximum principle [104, Theorem 8.19] holds, and more precisely, for any $f \in H_0^1(\Omega)$, we have

$$\mathcal{L}f \leq 0 \text{ in } \Omega, \quad f \geq 0 \text{ in } \Omega \quad \text{imply} \quad f \equiv 0 \text{ or } f > 0 \text{ a.e. in } \Omega. \quad (6.1.16)$$

When indeed $f \not\equiv 0$, we may choose $B_{4R} \subset \Omega$ such that $\|f\|_{L^1(B_{2R})} > 0$ and thus $\inf_{B_R} f > 0$ from (6.1.15) (with $g = 0$). We conclude that f is positive thanks to a connexity argument. Although the statements in [104, Chapter 8] are written assuming L^∞ bound on the coefficients b, β, c , it is observed at the end of [104, section 8.5] that the theory may be extended under less restrictive conditions on the coefficients as those assumed here. This theory is developed in [226] although the above strong maximum principle is not explicitly written. On the other hand, the following Hölder regularity estimate [226, Théorème 7.1] and [104, Theorem 8.29] (see also the original papers [68, 192, 186]) of De Giorgi-Nash-Moser type

$$\|f\|_{C^\alpha(\Omega)} \leq C \|(\lambda - \mathcal{L})f\|_{L^\infty(\Omega)} \quad (6.1.17)$$

holds true for some $\alpha = \alpha(a_{ij}) \in (0, 1)$ and $C > 0$. These last two pieces of information together and the fact that $f_0 \not\equiv 0$ imply that there exists a constant $\theta > 0$ such that $f_0 \geq \theta \mathbf{1}_{B_R}$, and thus

$$\mathcal{L}f_0 \geq (\kappa_1 - \theta^{-1})f_0.$$

That is condition **(i)** in Lemma 2.1.4, so that condition **(H2)** holds thanks to Lemma 2.1.4. Presented in that way, the above estimate is not really constructive, but the constant $\theta := \inf_{B_R} (\kappa_1 - \mathcal{L})^{-1} \mathbf{1}_{B_R}$ can also be considered as a geometric quantity associated to geometric properties of the operator and the domain.

First constructive argument for (H2). In the case when \mathcal{L} is self-adjoint, that corresponds to the case $a_{ij} = a_{ji}$ and $b_i + \beta_i = 0$, we classically know (that has been recalled in Section 2.1.3, see (2.1.34)) that

$$\lambda_1 = \inf_{f \in X_+ \setminus \{0\}} \frac{\langle \mathcal{L}f, f \rangle}{\|f\|^2} = \inf_{f \in H_0^1, \|f\|_{L^2} = 1} \int_{\mathcal{O}} \{a \nabla f \cdot \nabla f + c f^2\},$$

from what and the Sobolev imbedding, we get

$$\lambda_1 \geq \inf_{f \in H_0^1, \|f\|_{L^2}=1} \{(\nu C_\Omega - \|c_- \mathbf{1}_{c_- \geq M}\|_{L^{d/2}}) \|f\|_{L^{2^*}}^2 - M\} \geq -M,$$

by choosing M large enough. That gives an explicit lower bound on λ_1 .

Second constructive argument for (H2). We give another constructive argument without assuming any self-adjointness property. We rather assume

$$(\partial_i b_i - c)_+ \in M^1(\Omega), \quad b_i + \beta_i - \partial_j a_{ij} \in M^1(\Omega). \quad (6.1.18)$$

We fix $h_0 \in C_c^2(\Omega)$ such that $c_0 \mathbf{1}_{B_\rho} \leq h_0 \leq c_0 \mathbf{1}_{B_{3\rho/2}}$ with $B_{8\rho} \subset \Omega$ and $\|h_0\|_{L^2} = 1$. We next define f_0 as the (positive) solution to

$$f_0 \in H_0^1(\Omega), \quad (\kappa_1 - \mathcal{L})f_0 = h_0, \quad (6.1.19)$$

so that $f_0 \in C^\alpha(\Omega)$ from (6.1.14) and (6.1.17), and similarly

$$\tilde{f}_0 \in H_0^1(B_{2\rho}), \quad (\kappa_1 - \mathcal{L})\tilde{f}_0 = h_0, \quad (6.1.20)$$

so that $\tilde{f}_0 \in C^\alpha(B_{2\rho})$ from (6.1.14) and (6.1.17). We observe that $0 \leq \tilde{f}_0 \leq f_0$ thanks to the weak maximum principle. We then compute

$$1 = \|h_0\|_{L^2}^2 = \int_{B_{2\rho}} h_0 (\kappa_1 - \mathcal{L})\tilde{f}_0 = \int_{B_{2\rho}} \tilde{f}_0 (\kappa_1 - \mathcal{L}^*)h_0 \leq \|\tilde{f}_0\|_{L^\infty} \|(\kappa_1 - \mathcal{L}^*)h_0\|_{M^1},$$

where the last term is finite because of the additional hypothesis (6.1.18). We conclude to a first constructive lower bound $\|\tilde{f}_0\|_{L^\infty(B_{2\rho})} \geq c_1 > 0$. Because of the Holder continuity, we also have $\|\tilde{f}_0\|_{L^1(B_{2\rho})} \geq c_2$ with constructive constant $c_2 = c_2(c_1, \alpha, d) > 0$. Thanks to (6.1.15) (with $g = 0$), we obtain

$$\begin{aligned} f_0 &\geq \mathbf{1}_{B_{2\rho}} \inf_{B_{2\rho}} f_0 \geq \mathbf{1}_{B_{2\rho}} C_{wH} \|f_0\|_{L^1(B_{2\rho})} \\ &\geq \mathbf{1}_{B_{2\rho}} C_{wH} \|\tilde{f}_0\|_{L^1(B_{2\rho})} \geq C_{wH} c_2 c_0^{-1} h_0. \end{aligned}$$

Because all the inequalities are constructive and proceeding as above, we deduce that condition **(ii)** in Lemma 2.1.4 holds and thus also **(H2)** with constructive constant $\kappa_0 := \kappa_1 - C_{wH}^{-1} c_2^{-1} c_0$. Finally, because of $(\kappa_1 - \mathcal{L})f_0 = 0$ on $\Omega \setminus B_{2\rho}$, we may apply the Harnack inequality [104, Corollary 8.21], and we classically deduce there exist constructive constants $C > 0$ and $C_\varrho > 0$ for any $\varrho > 0$ such that

$$C_\varrho \mathbf{1}_{\omega_\varrho} \leq f_0 \leq C, \quad (6.1.21)$$

with $\omega_\varrho := \{x \in \Omega; \delta(x) > \varrho\}$ and $\delta(x) := d(x, \partial\Omega)$ is the distance to the boundary function.

We can also get a constructive argument for **(H2)** by asking that condition **(i)** in Lemma 2.1.4 holds. We may for instance verify that the dual counterpart of the above constructive argument holds when $(c + \partial_i \beta_i)_- \in M^1$ and $b_i + \beta_i + \partial_j a_{ji} \in M^1$. More precisely, we establish in a similar way as above that the solution to the problem

$$\psi_0 \in H_0^1(\Omega), \quad (\kappa_1 - \mathcal{L}^*)\psi_0 = h_0, \quad (6.1.22)$$

satisfies

$$\kappa_0 \psi_0 \leq \mathcal{L}^* \psi_0 \leq \kappa_1 \psi_0, \quad (6.1.23)$$

for some constructive constants $\kappa_0 \leq \kappa_1$. Similarly as above again, there exist constructive constants $C > 0$ and $C_\varrho > 0$ for any $\varrho > 0$ such that

$$C_\varrho \mathbf{1}_{\omega_\varrho} \leq \psi_0 \leq C. \quad (6.1.24)$$

Third constructive argument for (H2). We write

$$\mathcal{L}f = a_{ij}\partial_{ij}^2 f + \tilde{b}_i\partial_i f + \tilde{c}f, \quad (6.1.25)$$

with $\tilde{b}_i := b_i + \partial_j a_{ji} + \beta_i$ and $\tilde{c} := c + \partial_i \beta_i$. We further assume $\tilde{b}_i, \tilde{c} \in L^\infty$. In that case, we may also obtain an explicit lower bound on λ_1 by proceeding in the following way. We define $f_0(x) := \chi(|x|)$ with $\chi \in C_c^1(\mathbb{R}_+) \cap W^{2,\infty}(\mathbb{R}_+)$, $\mathbf{1}_{[0,1/3]} \leq \chi \leq \mathbf{1}_{[0,1]}$, $\chi' \leq 0$ on $[0,1]$, $\chi(s) := n^2(1-s)^2/2$ on $[\iota_n, 1]$, $\iota_n := 1 - 1/(2n)$, for some $n \geq 1$ to be chosen. As a consequence, $\chi'' = n^2$ on $[\iota_n, 1]$, $|\chi'| \leq n$ on $[\iota_n, 1]$ and $\chi \geq 1/2$ on $[0, \iota_n]$. Denoting $s := |x|$, we compute

$$\mathcal{L}f_0 = a_{ij}\{\chi''(s)\hat{x}_i\hat{x}_j + \chi'(s)\frac{\delta_{ij} - \hat{x}_i\hat{x}_j}{s}\} + \tilde{b}(x) \cdot \hat{x}\chi'(s) + \tilde{c}(x)\chi(s).$$

For n large enough, we get

$$\begin{aligned} \mathcal{L}f_0 &\geq n^2\nu - n2A - nB - C \geq 0 \quad \text{on } B_1 \setminus B_{\iota_n}, \\ \mathcal{L}f_0 &\geq -A\{\|\chi''\|_{L^\infty} + \|\chi'(s)/s\|_{L^\infty}\} - B\|\chi'\|_{L^\infty} - C \geq \kappa_0\chi \quad \text{on } B_{\iota_n}, \end{aligned}$$

with $A := \|a\|_{L^\infty(B_1)}$, $B := \|\tilde{b}\|_{L^\infty(B_1)}$, $C := \|\tilde{c}\|_{L^\infty(B_1)}$ and $\kappa_0 \in \mathbb{R}_-$. As a conclusion, we have again established condition **(ii)** in Lemma 2.1.4, so that condition **(H2)** holds.

Fourth constructive argument for (H2). We present a last situation when we are able to prove a quantitative version of condition **(H2)**. We assume that $a \in C^0(\bar{\Omega})$, $\operatorname{div}\beta \in L^{r/2}$, as well as $\tilde{b}_i \in L^r$ and $\tilde{c} \in L^{r/2}$ in the definition of (6.1.25). We define h_0 and f_0 as in the second constructive argument for (H2), so that (6.1.18) holds. Choosing $p \in (1, 2)$ defined by $1/p := 1/r + 1/2 > 2/r + 1/2^*$, we observe that

$$\begin{aligned} \|\kappa_1 f_0 - \tilde{b}_i\partial_i f_0 - \tilde{c}f_0 - h_0\|_{L^p} &\lesssim \kappa_1\|f_0\|_{L^2} + \|\tilde{b}_i\|_{L^r}\|\partial_i f_0\|_{L^2} + \|\tilde{c}\|_{L^{r/2}}\|f_0\|_{L^{2^*}} + \|h_0\|_{L^2} \\ &\lesssim \|h_0\|_{L^2}, \end{aligned}$$

from equation (6.1.19) and the coercivity estimate (6.1.9). From the Calderon-Zygmund regularity theory [47] or [104, Theorem 9.14], we also know that

$$\|f_0\|_{W^{2,p}(\Omega)} \lesssim \|a_{ij}\partial_{ij}^2 f_0\|_{L^p(\Omega)}. \quad (6.1.26)$$

Writing $a_{ij}\partial_{ij}^2 f_0 = \kappa_1 f_0 - \tilde{b}_i\partial_i f_0 - \tilde{c}f_0 - h_0$ and using the two above estimates, we deduce

$$\|f_0\|_{W^{2,1}(\Omega)} \lesssim \|h_0\|_{L^2(\Omega)}. \quad (6.1.27)$$

On the other hand, from (6.1.8) and the Poincaré inequality, we have

$$1 = \|h_0\|_{L^2}^2 = \langle (\kappa_1 - \mathcal{L})f_0, h_0 \rangle \lesssim \|\nabla f_0\|_{L^2} \|\nabla h_0\|_{L^2}.$$

Together with the estimate (6.1.27) and the Gagliardo-Nirenberg inequality

$$\|\nabla f\|_{L^2} \lesssim \|D^2 f\|_{L^1}^{1/2} \|f\|_{L^\infty}^{1/2},$$

we obtain a lower bound $\|f_0\|_{L^\infty} \geq C_0 > 0$. We then conclude as in the second constructive argument for (H2).

Condition (H3). Because of Rellich-Kondrachov theorem on the compact embedding $H_0^1 \subset L^2$, the mapping $(\lambda - \mathcal{L})^{-1} : L^2 \rightarrow L^2$ is compact for any $\lambda \geq \kappa_1$. As a consequence, introducing the splitting $\mathcal{L} = \mathcal{A} + \mathcal{B}$ with $\mathcal{A} := \kappa_1 - \kappa_{\mathcal{B}}$, $\kappa_{\mathcal{B}} \in \mathbb{R}$ arbitrary, the operator $\mathcal{R}_{\mathcal{B}}(\lambda) = (\lambda + \kappa_1 - \kappa_{\mathcal{B}} - \mathcal{L})^{-1}$ is bounded uniformly on $\lambda \geq \kappa_{\mathcal{B}}$ and it is compact for any $\lambda \geq \kappa_{\mathcal{B}}$. We deduce from Lemma 2.1.8-(2) that **(H3)** holds for both the primal and the dual problems.

We may thus apply Theorem 2.1.20 and deduce the existence of a solution (λ_1, f_1, ϕ_1) to the first eigentriplet problem

$$\lambda_1 \in \mathbb{R}, \quad 0 \leq f_1 \in H_0^1, \quad \mathcal{L}f_1 = \lambda_1 f_1, \quad 0 \leq \phi_1 \in H_0^1, \quad \mathcal{L}^* \phi_1 = \lambda_1 \phi_1, \quad (6.1.28)$$

where both equations must be understood in the variational sense as a consequence of the discussion at the end of the proof of condition **(H1)**.

Condition (H4). The strong maximum principle holds as already mentioned in the paragraph dedicated to condition **(H2)**. As a consequence and thanks to Theorem 3.3.11, we know that the first eigentriplet problem (6.0.3) has a unique solution (λ_1, f_1, ϕ_1) which satisfies $f_1 > 0$, $\phi_1 > 0$, $N(\mathcal{L} - \lambda_1)^k = \text{Span}(f_1)$ and $N(\mathcal{L}^* - \lambda_1)^k = \text{Span}(\phi_1)$ for any $k \geq 1$.

Condition (H5). Consider $f \in D(\mathcal{L}^\infty)$ such that $0 < |f| \in D(\mathcal{L}^\infty)$ and

$$\mathcal{L}|f| = \Re e(\text{sign } f) \mathcal{L}f,$$

so that multiplying both term of the equation by $|f|$ and integrating, we have

$$\Re e \langle \mathcal{L}f, \bar{f} \rangle = \langle \mathcal{L}|f|, |f| \rangle.$$

We next compute

$$\Re e \langle \mathcal{L}f, \bar{f} \rangle = - \int_{\Omega} a_{kj} \Re e(\partial_j f \partial_k \bar{f}) + \int_{\Omega} (b_k - \beta_k) \Re e(\bar{f} \partial_k f) + \int_{\Omega} c |f|^2,$$

and

$$\langle \mathcal{L}|f|, |f| \rangle = - \int_{\Omega} a_{kj} \partial_j |f| \partial_k |f| + \int_{\Omega} (b_k - \beta_k) \Re e(\bar{f} \partial_k f) + \int_{\Omega} c |f|^2,$$

where in the last equality, we have used that $\partial_k |f| = \frac{1}{|f|} \Re e(\bar{f} \partial_k f)$. From the three above equations, we deduce

$$\int_{\Omega} a_{kj} [\partial_j |f| \partial_k |f| - \Re e(\partial_j f \partial_k \bar{f})] = 0.$$

Introducing the real and complex part decomposition $f = u + iv$, and similarly as in [138, Proof of Theorem 5.1], we next compute

$$\begin{aligned} & \partial_j |f| \partial_k |f| - \Re e(\partial_j f \partial_k \bar{f}) \\ &= \frac{1}{|f|^2} [uv(\partial_k u \partial_j v + \partial_k v \partial_j u) - u^2 \partial_j v \partial_k v - v^2 \partial_j u \partial_k u] \\ &= \frac{1}{|f|^2} (u \partial_j v - v \partial_j u)(u \partial_k v - v \partial_k u), \end{aligned}$$

so that from the ellipticity condition on a , we have $u \partial_k v - v \partial_k u = 0$ a.e. on Ω . On the other hand, from De Giorgi-Nash-Moser regularity estimates (6.1.14) and (6.1.17), f has Hölder regularity. In particular both functions u and v are continuous. Because $|f| \not\equiv 0$, one of the two function is not identically vanishing, say for instance $v \not\equiv 0$. There exists some points $x_0 \in \Omega$ such that $v(x_0) \neq 0$, say for instance $v(x_0) > 0$. Denoting by ω the connected component of the set $\{x \in \Omega; v(x) > 0\}$ containing x_0 , we have $\nabla(u/v) = 0$ on ω . Hence $u = \alpha v$ on ω for some $\alpha \in \mathbb{R}$, which implies that there exists $\sigma \in \mathbb{S}^1$ such that $f = \sigma |f|$ on ω . If $\omega \neq \Omega$, we would have $|f| = 0$ on $\partial\omega \cap \Omega \neq \emptyset$, which would be a contradiction with the fact that $|f| > 0$. We conclude that $\omega = \Omega$ and thus that $f = \sigma |f|$, which is nothing but the reverse Kato's inequality condition **(H5)**.

At this stage, we may use Theorem 4.4.14, in order to get the conclusion **(C3)** on the triviality of the boundary punctual spetrum.

In order to go one step further and establish the asymptotic stability of f_1 , we may use the two following approaches which are consequences respectively of Lemma 6.1.1 and Lemma 6.1.2.

Lemma 6.1.1. *For any $R > 0$, the set*

$$\mathcal{K} := \{f \in D(\mathcal{L}); [f] \leq R, [\mathcal{L}f] \leq R\}$$

is strongly compact in $L^1_{\text{loc}}(\Omega)$, where $[g] := \|g\|_{L^1_{\phi_1}}$.

Proof of Lemma 6.1.1. Consider $f \in \mathcal{K}$ so that $f \in H^1_0(\Omega)$ and

$$\partial_i(a_{ij}\partial_j f) + b_i\partial_i f + \partial_i(\beta_i f) + cf = g \in L^2(\Omega).$$

From the renormalization theory of elliptic equations and the GRE trick (see for instance [171] and the references therein) for any renormalizing function $H \in C^2(\mathbb{R})$, there holds

$$\begin{aligned} H''(u)f_1\phi_1 a \nabla u \cdot \nabla u &= \operatorname{div}(a\phi_1 \nabla(H(u)f_1)) - \operatorname{div}(f_1 H(u)a \nabla \phi_1) \\ &\quad + \operatorname{div}((b + \beta)H(u)f_1\phi_1) + gH'(u)f_1\phi_1, \end{aligned}$$

with $u := f/f_1$. Considering $H \in W^{2,\infty}$ the even (and convex) function such that $H(0) = 0$ and $H'' := \mathbf{1}_{[n,n+1]}$, so that in particular $|H'(s)| \leq 1$, and integrating the previous equation, we deduce

$$\nu \int_{|u| \in [n,n+1]} |\nabla u|^2 f_1 \phi_1 \leq \int |g| f_1 \phi_1 \leq \|f_1\|_{L^\infty} R.$$

We proceed along the line of the proof of [32, Theorem 1]. For a fixed $\omega \subset\subset \Omega$, we define $B_n := \{x \in \omega; |u(x)| \in [n, n+1]\}$. Using that $f_1 > 0$ and $\phi_1 > 0$, there exists a constructive constant $C_{\omega,R} > 0$ such that

$$\int_{B_n} |\nabla u|^2 \leq C_\omega^2, \quad \forall n \geq 0.$$

From the Cauchy-Schwarz inequality, we have

$$\int_{B_n} |\nabla u| \leq C_\omega \operatorname{meas}(B_n)^{1/2}, \quad \forall n \geq 0. \quad (6.1.29)$$

On the other hand, denoting by $1^* := d/(d-1)$ the Sobolev exponent, we have

$$\int_{B_n} |\nabla u| \leq C_{\omega,R} \left(n^{-1^*} \int_{B_n} |u|^{1^*} \right)^{1/2}.$$

Summing up and using the Cauchy-Schwarz inequality again, we have

$$\begin{aligned} \sum_{n \geq 1} \int_{B_n} |\nabla u| &\leq C_{\omega,R} \left(\sum_{n \geq 1} n^{-1^*} \right)^{1/2} \left(\sum_{n \geq 1} \int_{B_n} |u|^{1^*} \right)^{1/2} \\ &\leq C_{\omega,R} \left(\sum_{n \geq 1} n^{-1^*} \right)^{1/2} \|u\|_{L^{1^*}}^{1^*/2}. \end{aligned}$$

Together with (6.1.29) for $n = 0$, we deduce

$$\|\nabla u\|_{L^1(\omega)} \leq C'_{\omega,R} (1 + \|\nabla u\|_{L^1(\omega)}^{1^*/2}).$$

Because $1^*/2 \leq 3/4 < 1$ (recall that $d \geq 3$), we can kill the last term, and we obtain the estimate

$$\|\nabla(f/f_1)\|_{L^1(\omega)} \leq C'', \quad \forall f \in \mathcal{K},$$

for some constant $C'' := C''_{\omega,R} > 0$. We classically conclude thanks to the Rellich-Kondrachov theorem. \square

From the above lemma and Theorem 4.5.20, we deduce that $\tilde{S}(t)f \rightarrow \langle f, \phi_1 \rangle f_1$ in the $L^1_{\phi_1}$ norm sense as $t \rightarrow \infty$ for any $f \in L^2(\Omega)$. The alternative approach is based on the following result.

Lemma 6.1.2. *Setting $\kappa := \kappa_0 - 1$, there exist $A, \alpha, R > 0$ such that*

- (i) $\sup_{z \in \Delta_\kappa} \langle y \rangle^\alpha \|\mathcal{R}_B(z)\|_{\mathcal{B}(L^2; H_0^1)} + \sup_{z \in \Delta_\kappa \setminus B_R} \|\mathcal{R}_L(z)\|_{\mathcal{B}(L^2; H_0^1)} < \infty$,
- (ii) $\Sigma(\mathcal{L}) \cap \Delta_\kappa \subset \Sigma_d(\mathcal{L}) \cap B_R$,

where $B := \mathcal{L} - A$ and $z = x + iy$, $x, y \in \mathbb{R}$.

Proof of Lemma 6.1.2. Let us consider an a priori solution to the stationary problem

$$f \in H_0^1, \quad z = x + iy \in \Delta_\kappa, \quad (\mathcal{L} + z)f = g \in L^2.$$

This one satisfies

$$\left| -\int (a\nabla f + \beta f) \cdot \nabla \bar{f} + \int b \cdot \nabla f \bar{f} + (c + z)|f|^2 \right| = \left| \int g \bar{f} \right|.$$

Using the elliptic condition, the Cauchy-Schwarz inequality and triangular inequalities, we get

$$\begin{aligned} \left| \int g \bar{f} \right| &\geq \left| \int a\nabla f \nabla \bar{f} + ((c + x)_+ + iy)|f|^2 \right| - \left| \int b \cdot \nabla f \bar{f} - \beta f \cdot \nabla \bar{f} + (c + x)_- |f|^2 \right| \\ &\geq \frac{\nu}{2} \|\nabla f\|_{L^2}^2 + \left(\frac{|y|}{2} - x_-\right) \|f\|_{L^2}^2 - \|(b + |\beta|)f\|_{L^2} \|\nabla f\|_{L^2} - \|\sqrt{c_-} f\|_{L^2}^2. \end{aligned}$$

Using next similar arguments and those introduced in the paragraph dedicated to condition **(H1)** and with similar definition for the constant $M := M(b, \beta, c) > 0$, we deduce

$$\left| \int g \bar{f} \right| \geq \left(\frac{|y|}{2} - x_- - M\right) \|f\|_{L^2}^2 + \frac{\nu}{4} \|\nabla f\|_{L^2}^2.$$

Defining the sectorial set

$$\mathcal{S} := \{z = x + iy \in \mathbb{C}; |y| > 2x_- + M\},$$

we have established the a priori estimates

$$\begin{aligned} \|f\|_{L^2} &\leq \left(\frac{|y|}{2} - x_- - M\right)^{-1/2} \|g\|_{L^2}, \\ \|\nabla f\|_{L^2} &\leq 2\nu^{-1/2} \left(\frac{|y|}{2} - x_- - M\right)^{-1/4} \|g\|_{L^2}, \end{aligned}$$

for any $z \in \mathcal{S}$. We classically and immediately deduce that $\rho(\mathcal{L}) \supset \mathcal{S}$ and the resolvent estimate $\|\mathcal{R}_L(z)\|_{\mathcal{B}(L^2; H_0^1)} \lesssim \left(\frac{|y|}{2} - x_- - M\right)^{-1/2} + \left(\frac{|y|}{2} - x_- - M\right)^{-1/4}$ for any $z \in \mathcal{S}$, and in particular the estimate **(i)** holds true.

On the other hand, because \mathcal{L} has compact resolvent as established just above or during the proof of **(H3)** and using the Fredholm alternative, we have $\Sigma(\mathcal{L}) = \Sigma_d(\mathcal{L})$ and $\Sigma(\mathcal{L}) \cap \Delta_\kappa$ is finite for any $\kappa \in \mathbb{R}$, what is nothing but the property **(ii)**. \square

From the above lemma and Theorem 4.6.27 or Theorem 4.6.29, we deduce that $\tilde{S}(t)f \rightarrow \langle f, \phi_1 \rangle \phi_1$ in the L^2 norm sense as $t \rightarrow \infty$ for any $f \in L^2(\Omega)$ with exponential rate.

We may summarize our analysis in the following result.

Theorem 6.1.3. *Consider the elliptic operator (1.2.8) in a bounded domain and assume that the coefficients satisfy (6.0.2), (6.1.5), (6.1.6) and (6.1.18). Then the conclusions **(C3)** holds as well as **(CE2)** in $L^1_{\phi_1}$ norm and **(CE3)** in L^2 with non constructive rate.*

It is however worth emphasizing again that the above approach is definitively not constructive. We propose now an alternative approach which is constructive.

Quantitative estimate of stability.

Using the Doblin-Harris type approach presented in Section 5, we are able to establish a rate of convergence to the principal dynamic, at least in a regular framework. We thus make some regularity assumptions on Ω and additional regularity assumptions on the coefficients.

- For the domain, we assume that there exist constant $r_\Omega > 0$ such that for any $x \in \Omega$ there is $y \in \Omega$ such that $x \in B(y, r_\Omega) \subset \Omega$, in particular, for any $x \in \partial\Omega$ there is $y \in \Omega$ such that $x \in \partial B(y, r_\Omega)$, $B(y, r_\Omega) \subset \Omega$. We also assume that Ω is $C^{1,1}$.

- For the coefficients, we assume $a_{ij} \in C(\bar{\Omega})$, $\tilde{b}_i, \tilde{c} \in L^\infty(\Omega)$, where \tilde{b}_i and \tilde{c} are defined in (6.1.25).

Theorem 6.1.4. *Consider the elliptic operator (1.2.8) in a bounded domain and assume that the assumptions of Theorem 6.1.3 hold together with the additional regularity assumptions on the coefficients and the boundary. Then the conclusion (CE3) holds with constructive exponential rate.*

The proof of Theorem 6.1.4 follows from Theorem 5.3.3. We split the proof into several steps.

- **Step 1. Regularity estimates.** Thanks to De Giorgi-Nash-Moser regularity technique for parabolic equations developed for instance in [147] (in Russian), [237, Theorem 1.3, Theorem 2.2] as well as more recently in [137, Lemma 2.7] and [114, Theorem 1.1], there exists $\alpha = \alpha(a_{ij}) \in (0, 1)$ and for any $T_1 > T_0 > 0$ and any $\varrho \in (0, 1)$, there exist constructive constants $C_i = C_i(\|f\|_{L_t^\infty L_x^2}, T, \tau, r)$ such that any solution $f \in L^\infty(0, \infty; L^2(\Omega))$ to the parabolic equation $\partial_t f = \mathcal{L}f$ satisfies

$$\|f\|_{L^\infty([T_0, T_1] \times \Omega)} \leq C_1, \quad \|f\|_{C^\alpha([T_0, T_1] \times \omega_\varrho)} \leq C_2, \quad (6.1.30)$$

with $\omega_r := \{x \in \Omega; d(x, \partial\Omega) > r\}$. More precisely, in order to establish the second estimate in (6.1.30) with constructive constant, one may observe that the proof of [114, Proposition 2.4] may be repeated in order to get that solutions to the parabolic equation considered in the present framework fall into De Giorgi classes as defined in [114, Definition 2.3], and thus [114, Theorem 1.1] applies.

On the other hand, in this context and because of the regularity assumptions, we may establish a more accurate regularity estimate. More precisely, by gathering the Sobolev inequality and the Calderon-Zygmund estimate (6.1.26), we obtain the classical constructive regularity estimate

$$\|u\|_{C^{0,1}(\Omega)} \lesssim \|u\|_{W^{2,d+1}(\Omega)} \lesssim \|(\kappa_1 - \mathcal{L})u\|_{L^{d+1}(\Omega)}, \quad (6.1.31)$$

see for instance Theorem 7.10, Theorem 7.25 and Lemma 9.17 in [104]. Iterating the same kind of arguments, we get

$$\|u\|_{C^{0,1}(\Omega)} \leq C \|(\kappa_1 - \mathcal{L})^k u\|_{L^2(\Omega)}, \quad (6.1.32)$$

with constructive constants C and k .

- **Step 2. Harnack estimate.** We claim that for any $0 \leq t_0 < T$ and $\varrho > 0$, there exist a constant $C_H > 0$ such that, for any $f_0 \in L^2$, the associated solution $f := S_{\mathcal{L}} f_0$ satisfies

$$\sup_{\omega_\varrho} f_{t_0} \leq C_H \inf_{\omega_\varrho} f_T. \quad (6.1.33)$$

The proof mainly follows from Aronson-Serrin [10] (see also [185, 128, 130, 143, 129, 236, 237, 144] for similar results). First, we know from [10, Theorem 3] that

$$\max_{Q^*(\rho)} f \leq C \min_{Q(\rho)} f, \quad (6.1.34)$$

for any $\rho > 0$, $t > 0$ such that $Q^*(3\rho) \subset (0, \infty) \times \Omega$, where $Q(\rho) := [t - \rho^2, t] \times \mathcal{C}(\rho)$, $Q^*(\rho) := [t - 8\rho^2, t - 7\rho^2] \times \mathcal{C}(\rho)$ and $\mathcal{C}(\rho)$ is a cube with length ρ . To avoid technical issues we assume that w_ϱ is convex. In other case, the geometrical condition given above implies that there is $N \in \mathbb{Z}_+$ such that any two points $x, y \in \Omega$ can be connected by a polygonal path of at most N segments, and we can argue as follows for any segment. We define $D := \sup_{a, b \in \Omega} d(a, b)$ the diameter of Ω and we choose $r' < \varrho/7$ such that

$$7(\lfloor \frac{D}{2r'} \rfloor + 1)(r')^2 < T - t_0.$$

For any $x, y \in \omega_\varrho$, we also define $N_c = \lfloor \frac{|x-y|}{r'} \rfloor$. Since ω_ϱ is convex, $r' < \varrho/7$, we have that the family of cubes $\{C(x_i, 2r')\}_{i=0, N_c}$ of center x_i and length $2r'$ for $x_i = x + \frac{(x-y)i}{N_c}$ satisfy that $C(x_i, 6r') \subset \Omega$ and $C(x_i, 2r') \cap C(x_{i+1}, 2r') \neq \emptyset$ for any $i = 0, \dots, N_c$. As a consequence, we can apply Aronson-Serrin estimate (6.1.34) for each cube to obtain

$$\max_{C(x_i, 2r')} f_{t_i} \leq C_{2r'} \min_{C(x_i, 2r')} f_{t_{i+1}},$$

with $t_i = t_0 + 7i(2r')^2$. Taking $y_i \in C(x_i, 2r') \cap C(x_{i+1}, 2r')$, we deduce

$$\max_{C(x_i, 2r')} f_{t_i} \leq C_{2r'} \min_{C(x_i, 2r')} f_{t_{i+1}} \leq C_{2r'} f_{t_{i+1}}(y_i) \leq C_{2r'} \max_{C(x_{i+1}, 2r')} f_{t_{i+1}} \leq C_{2r'}^2 \min_{C(x_{i+1}, 2r')} f_{t_{i+2}}.$$

By induction, we obtain

$$f_{t_0}(x) \leq \max_{C(x_1, 2r')} f_{t_1} \leq C_{2r'}^{N_c} \min_{C(x_{N_c}, 2r')} f_{t_{N_c}} \leq C_{2r'}^{N_c} f_{t_{N_c}}(y),$$

with $t_{N_c} = t_0 + 7N_c r'^2 \leq T$. Note that in any case the constant $C_{2r'}$ is the same since it only depends on the length $2r'$ and the coefficient of the equation. We have thus established (6.1.33) with $C_H := C_{2r'}^{\lfloor \frac{D}{2r'} \rfloor + 1}$.

On the other hand, we state an improved version of the already mentioned stationary Harnack inequality. Because of the interior ball condition the Hopf Lemma (see for instance the proof of [104, Lemma 3.4]) claims that for any $\varrho \in (0, r_\Omega/2]$ there exists a constructive constant $\alpha > 0$ such that if $u \in W^{2,p}(\Omega)$, $p > d$, is such that

$$u \geq \mathbf{1}_{\omega_\varrho}, \quad (\kappa_1 - \mathcal{L}^*)u \geq 0,$$

then u satisfies

$$u \geq \chi(x) := e^{-\alpha(2\varrho - \delta(x))^2} - e^{-\alpha(2\varrho)^2} \text{ on } \omega_\varrho^c. \quad (6.1.35)$$

Let us give two applications of the above sharp regularity and positivity estimates. First, recalling (6.1.24) and using (6.1.31) and (6.1.35), we deduce that there exist two constructive constants $c_i \in (0, \infty)$ such that

$$c_0 \delta \leq \psi_0 \leq c_1 \delta \text{ on } \Omega. \quad (6.1.36)$$

Consider now $f_1 \in H_0^1(\Omega)$ the positive first eigenfunction with normalization $\|f_1\|_{L^2} = 1$. Using the estimate of regularity (6.1.32) on the iterated equation $(\kappa_1 - \mathcal{L})^k f_1 = (\kappa_1 - \lambda_1)^k f_1$, we have

$$\|f_1\|_{L^\infty(\Omega)} \leq \|f_1\|_{C^{0,1}(\Omega)} \leq C_1,$$

for some constructive constant $C_1 \in (0, \infty)$. Next using the elementary inequality

$$1 = \int_\Omega f_1^2 \leq \|f_1\|_{L^\infty} \|f_1\|_{L^1} \leq C_1 \|f_1\|_{L^1},$$

we deduce

$$\begin{aligned} |\Omega| \sup_{\omega_\varrho} f_1 &\geq \int_{\omega_\varrho} f_1 = \|f_1\|_{L^1} - \int_{\omega_\varrho^c} f_1 \\ &\geq 1/C_1 - C_1 |\omega_\varrho^c| \geq 1/(2C_1), \end{aligned}$$

by choosing $\varrho \in (0, r_\Omega/2)$ small enough. Then, from the Harnack inequality [104, Corollary 8.21], we deduce

$$\inf_{\omega_\varrho} f_1 \geq C_H \sup_{\omega_\varrho} f_1 \geq C_H (2C_1 |\Omega|)^{-1}.$$

Finally, from the above Hopf lemma and the above Lipschitz continuity, we have established

$$c_0 \delta \leq f_1 \leq c_1 \delta \quad \text{on } \Omega, \quad (6.1.37)$$

for two constructive constants $c_i \in (0, \infty)$. The same arguments on the normalized and positive first dual eigenfunction ϕ_1 lead to the same estimate

$$c_0 \delta \leq \phi_1 \leq c_1 \delta \quad \text{on } \Omega. \quad (6.1.38)$$

In particular, for any such $\varrho \in (0, r_\Omega/2)$, we have

$$\langle \phi_1, \mathbf{1}_{\omega_\varrho} \rangle \geq r_\varrho, \quad (6.1.39)$$

with constructive constant r_ϱ , what is nothing but condition (5.3.9) in the Harris theorem that we will use below.

- **Step 3. Splitting of \mathcal{L} .** We introduce the splitting $\mathcal{L} = \mathcal{A} + \mathcal{B}$, with $\mathcal{A}f = \mathcal{M} \mathbf{1}_{\omega_\varrho} f$, $\mathcal{M} \geq 0$ large enough and $\varrho > 0$ small enough that we fix just below. Using (6.1.9), we observe that

$$\begin{aligned} (\mathcal{B}f, f)_{L^2} &= (\mathcal{L}f, f)_{L^2} - \mathcal{M} \|f\|_{L^2}^2 + \mathcal{M} \int_{\omega_\varrho^c} f^2 \\ &\leq -\frac{\nu}{4} \|\nabla f\|_{L^2}^2 + (\kappa_1 - \mathcal{M}) \|f\|_{L^2}^2 + \mathcal{M} |\omega_\varrho^c|^{4/d} \|f\|_{L^{2^*}}^2 \leq \kappa_0 \|f\|_{L^2}^2, \end{aligned}$$

by choosing first $\mathcal{M} \geq \kappa_1 - \kappa_0$ and next $\varrho > 0$ small enough in order to be able to throw away the last term using the negative first term and the Sobolev inequality. We deduce

$$S_{\mathcal{B}}(t) : L^2 \rightarrow L^2 \quad \text{with bound } \mathcal{O}(e^{\kappa_0 t}). \quad (6.1.40)$$

On the other hand, denoting $f_t := S_{\mathcal{L}}(t)f$ for $f \in L^2(\Omega)$ and recalling that ψ_0 defined by (6.1.22) satisfies (6.1.23), we have

$$\frac{d}{dt} \int |f_t| \psi_0 \leq \int \mathcal{L} |f_t| \psi_0 \leq \int |f_t| \mathcal{L}^* \psi_0 \leq \kappa_1 \int |f_t| \psi_0,$$

so that

$$\int |f_t| \psi_0 \leq e^{\kappa_1 t} \int |f_0| \psi_0. \quad (6.1.41)$$

Arguing in the same way for $S_{\mathcal{B}}$ and using (6.1.36), we have established

$$S_{\mathcal{L}}(t), S_{\mathcal{B}}(t) : L_\delta^1 \rightarrow L_\delta^1 \quad \text{with bound } \mathcal{O}(e^{\kappa_1 t}). \quad (6.1.42)$$

For a solution to the evolution equation $\partial_t f = \mathcal{C}f$, $\mathcal{C} = \mathcal{L}$ or $\mathcal{C} = \mathcal{B}$, we also classically compute

$$\begin{aligned} \frac{d}{dt} \int f^2 \psi_0 &= 2 \int (\mathcal{C}f) f \psi_0 \\ &= -2 \int (\nabla f \cdot a \nabla f) \psi_0 + \int f^2 \mathcal{C}^* \psi_0. \end{aligned}$$

Thanks to (6.1.23) again, we have

$$\frac{d}{dt} \int f^2 \psi_0 \leq -2\nu \int |\nabla f|^2 \psi_0 + \kappa_1 \int f^2 \psi_0, \quad (6.1.43)$$

from what we deduce

$$S_{\mathcal{L}}(t), S_{\mathcal{B}}(t) : L^2(\delta) \rightarrow L^2(\delta) \quad \text{with bound } \mathcal{O}(e^{\kappa_1 t/2}).$$

In the sequel, we will need the following version of Nash inequality.

Lemma 6.1.5 (weighted Nash inequality). *There exists a constructive constant C_N such that*

$$\|f\|_{L^2(\delta)} \leq C_N \|\nabla f\|_{L^2(\delta)}^{\frac{d+1}{d+2}} \|f\|_{L^1_\delta}^{\frac{1}{d+2}}, \quad \forall f \in H^1(\delta). \quad (6.1.44)$$

Proof of Lemma 6.1.5. For $\varepsilon > 0$, we define

$$f_\varepsilon(x) := \frac{1}{\delta_\varepsilon(x)} \int_{B(x,\varepsilon)} f(y) \delta(y) dy, \quad \delta_\varepsilon(x) = \delta(B(x,\varepsilon)) := \int_{B(x,\varepsilon)} \delta(y) dy,$$

and $B(x,\varepsilon) := \{y \in \Omega; |x-y| < \varepsilon\}$. It is worth emphasizing that

$$\varepsilon^{d+1} \lesssim \delta_\varepsilon(x) \lesssim \varepsilon^d, \quad \forall \varepsilon > 0. \quad (6.1.45)$$

For $f \in H^1(\delta)$, we compute

$$\begin{aligned} \|f - f_\varepsilon\|_{L^2(\delta)}^2 &= \int_\Omega \left| \frac{1}{\delta_\varepsilon(x)} \int_{B(x,\varepsilon)} (f(y) - f(x)) \delta(y) dy \right|^2 \delta(x) dx \\ &\leq \int_\Omega \int_\Omega \mathbf{1}_{|y-x| \leq \varepsilon} |f(y) - f(x)|^2 \frac{\delta(y)}{\delta_\varepsilon(x)} \delta(x) dx dy \\ &\leq \varepsilon^2 \int_0^{1/2} \int_\Omega \int_\Omega |\nabla f((1-t)x + ty)|^2 \frac{\delta(y)}{\delta_\varepsilon(x)} \delta(x) dx dy dt \\ &\quad + \varepsilon^2 \int_{1/2}^1 \int_\Omega \int_\Omega |\nabla f((1-t)x + ty)|^2 \frac{\delta(y)}{\delta_\varepsilon(x)} \delta(x) dx dy dt \\ &\lesssim \varepsilon^2 \int_0^{1/2} \int_\Omega \int_\Omega |\nabla f(z)|^2 \frac{\delta(y)}{\varepsilon^{d+1}} dy 2\delta(z) \frac{dz}{(1-t)^d} dt \\ &\quad + \varepsilon^2 \int_{1/2}^1 \int_\Omega \int_\Omega |\nabla f(z)|^2 \frac{2\delta(z)}{\varepsilon^{d+1}} \delta(x) \frac{dz}{t^d} dt dx \end{aligned}$$

where for the last inequality we have used the first inequality in (6.1.45), the fact that $\delta(x) \leq 2\delta(z)$ when $0 < t < 1/2$ and the fact that $\delta(y) \leq 2\delta(z)$ when $1/2 < t < 1$. Using the second inequality in (6.1.45), we straightforwardly obtain

$$\|f - f_\varepsilon\|_{L^2(\delta)}^2 \leq C_1 \varepsilon \|\nabla f\|_{L^2(\delta)}^2, \quad \forall \varepsilon > 0,$$

for a constant $C_1 > 0$. On the other hand, we also observe that

$$\|f_\varepsilon\|_{L^\infty} \leq \frac{C_2}{\varepsilon^{d+1}} \|f\|_{L^1_\delta}.$$

Writing now

$$f^2 = f(f - f_\varepsilon) + f f_\varepsilon$$

and using the above two estimates, we deduce

$$\begin{aligned} \|f\|_{L^2_\delta}^2 &\leq \|f\|_{L^2_\delta} \|f - f_\varepsilon\|_{L^2_\delta} + \|f\|_{L^1_\delta} \|f_\varepsilon\|_{L^\infty} \\ &\leq \|f\|_{L^2_\delta} C_1 \varepsilon^{1/2} \|\nabla f\|_{L^2_\delta} + C_2 \varepsilon^{-d-1} \|f\|_{L^1_\delta}^2 \\ &\leq \frac{1}{2} \|f\|_{L^2_\delta}^2 + \frac{C_1}{2} \varepsilon \|\nabla f\|_{L^2_\delta}^2 + C_2 \varepsilon^{-d-1} \|f\|_{L^1_\delta}^2, \end{aligned}$$

and we obtain the weighted Nash inequality (6.1.44) by choosing $\varepsilon := (\|f\|_{L^1_\delta}^2 / \|\nabla f\|_{L^2_\delta}^2)^{1/(d+2)}$. \square

Defining

$$u := \int |f_t| \psi_0 dx e^{-2\kappa t}, \quad v := \int f_t^2 \psi_0 dx e^{-2\kappa t},$$

with $\kappa := \kappa_{1+}$, coming back to (6.1.43) and using (6.1.36), the Nash inequality (6.1.44) and the estimate (6.1.41), we get

$$\begin{aligned} v'(t) &\leq -2\nu c_0 \int |\nabla f_t|^2 \delta e^{-2\kappa t} \\ &\leq -2\nu c_0 C_N^{-2\frac{d+2}{d+1}} \frac{\left(\|f_t\|_{L^2(\delta)}^2 e^{-2\kappa t}\right)^{\frac{d+2}{d+1}}}{\left(\|f_t\|_{L^1(\delta)}^2 e^{-2\kappa t}\right)^{\frac{1}{d+1}}} \\ &\leq -C \frac{v(t)^{1+\alpha}}{u(0)^{2\alpha}}, \end{aligned}$$

with $C := 2\nu C_N^{-2\frac{d+2}{d+1}} c_0^{1+2\frac{d+2}{d+1}} c_1^{-\frac{2}{d+1}}$ and $\alpha := 1/(d+1)$. Integrating in time, we deduce

$$v(t) \leq \frac{\alpha^{1/\alpha} u(0)^2}{C^{1/\alpha} t^{1/\alpha}}, \quad \forall t > 0.$$

We have thus established that there exist constructive constants $K > 0$ and $\kappa \geq 0$ such that

$$\|S_{\mathcal{L}}(t)f\|_{L^2(\psi_0)} \leq K \frac{e^{\kappa t}}{t^{(d+1)/2}} \|f\|_{L^1(\psi_0)}, \quad \forall f \in L^1(\psi_0). \quad (6.1.46)$$

From that last result, the estimates (6.1.36) and the properties of \mathcal{A} , we deduce that for $N \geq 1$ large enough

$$(S_{\mathcal{B}}\mathcal{A})^{(*N)} : L^1(\delta) \rightarrow L^2(\delta) \quad \text{with bound } \mathcal{O}(e^{\kappa t}). \quad (6.1.47)$$

We refer to [113, Proposition 3.9], [175, Proposition 2.5] and [138, Lemma 2.4] for details.

- **Step 4. Lyapunov condition.** We may next write

$$\tilde{S}_{\mathcal{L}} = V + W * \tilde{S}_{\mathcal{L}},$$

with

$$V := \tilde{S}_{\mathcal{B}} + \dots + (\tilde{S}_{\mathcal{B}}\mathcal{A})^{*(N-1)}, \quad W := (\tilde{S}_{\mathcal{B}}\mathcal{A})^{(*N)}.$$

On the one hand, using that $\mathcal{A} : L^2 \rightarrow L^2$ is bounded and (6.1.40), we deduce that

$$V : L^2 \rightarrow L^2, \quad \text{with bound } \mathcal{O}(e^{\kappa t}),$$

for any $\kappa \in (\kappa_0 - \kappa_1, 0)$. On the other hand, using that $\mathcal{A} : L^2_\delta \rightarrow L^2$ is bounded as well as (6.1.42) for $S_{\mathcal{L}}$, (6.1.47), (6.1.38), (6.1.36) and (6.1.40), we deduce that

$$W * \tilde{S}_{\mathcal{L}} : L^1_{\phi_1} \rightarrow L^2, \quad \text{with bound } \mathcal{O}(e^{\kappa' t}),$$

for any $\kappa' > \kappa_1 - \kappa_0$. We may thus fix $t = T$ large enough such that the following Lyapunov inequality holds

$$\|\tilde{S}_{\mathcal{L}}(T)f\|_{L^2} \leq \frac{1}{2} \|f\|_{L^2} + M_T \|f\|_{L^1_{\phi_1}}, \quad (6.1.48)$$

which is nothing but (5.3.8) in the hypothesis of the Harris theorem.

- **Step 5. Harris condition** Let $R > 0$ and consider $0 \leq f_0 \in L^2$ such that $\|f_0\|_2 \leq R\langle f_0, \psi_0 \rangle$. We set $\tilde{f}_t := e^{-\lambda_1 t} S_{\mathcal{L}}(t)f_0$. From the first inequality in (6.1.23), we have

$$\frac{d}{dt} \langle \tilde{f}_t, \psi_0 \rangle = \langle \tilde{f}_t, (\mathcal{L}^* - \lambda_1)\psi_0 \rangle \geq -(\lambda_1 - \kappa_0) \langle \tilde{f}_t, \psi_0 \rangle,$$

and then, thanks to Gronwall lemma again, we obtain,

$$\langle \tilde{f}_t, \psi_0 \rangle \geq e^{-(\lambda_1 - \kappa_0)t} \langle f_0, \psi_0 \rangle.$$

This estimate, together with the previous step, shows that

$$\begin{aligned} \int_{\omega_\varrho} \tilde{f}_{t_0}(x) \psi_0 dx &= \int_{\Omega} \tilde{f}_{t_0}(x) \psi_0 dx - \int_{\omega_\varrho^c} \tilde{f}_{t_0}(x) \psi_0 dx \\ &\geq e^{-(\lambda_1 - \kappa_0)t_0} \langle f_0, \psi_0 \rangle - \|\tilde{f}_{t_0}\|_2 \|\psi_0\|_\infty |\omega_\varrho^c|^{1/2} \\ &\geq e^{-(\kappa_1 - \kappa_0)t_0} \langle f_0, \psi_0 \rangle - e^{(\kappa_1 - \kappa_0)t_0} \|f_0\|_2 \|\psi_0\|_\infty |\omega_\varrho^c|^{1/2} \\ &\geq \left(e^{-(\kappa_1 - \kappa_0)t_0} - R e^{(\kappa_1 - \kappa_0)t_0} \|\psi_0\|_\infty |\omega_\varrho^c|^{1/2} \right) \langle f_0, \psi_0 \rangle. \end{aligned}$$

Choosing $\varrho > 0$ small enough, we get

$$\int_{\omega_\varrho} \tilde{f}_{t_0}(x) \psi_0 dx \geq \gamma \langle f_0, \psi_0 \rangle, \quad \gamma := \frac{1}{2} e^{-(\lambda_1 - \kappa_0)t_0}.$$

As a consequence, there is $x_{t_0}^f \in \omega_\varrho$ such that

$$\tilde{f}_{t_0}(x_{t_0}^f) \geq \frac{1}{|\omega_\varrho|} \int_{\omega_\varrho} \tilde{f}_{t_0}(x) dx \geq \frac{1}{|\Omega|c_1\varrho} \int_{\omega_\varrho} \tilde{f}_{t_0}(x) \psi_0 dx \geq \frac{\gamma}{|\Omega|c_1\varrho} \langle f_0, \psi_0 \rangle.$$

On the other hand, from the Harnack inequality (6.1.33) established in Step 2, we know that for any $T > t_0$ exists C_H such that

$$\tilde{f}_{t_0}(x_{t_0}^f) \leq \sup_{\omega_\varrho} \tilde{f}_{t_0} \leq C_H \inf_{\omega_\varrho} \tilde{f}_T.$$

The two last estimates together with (6.1.38) and (6.1.36) imply the Harris type estimate

$$\tilde{f}_T = \tilde{S}(T)f_0 \geq g_R \langle f_0, \phi_1 \rangle, \quad (6.1.49)$$

with $g_R := \frac{c_0\gamma}{C_H|\Omega|c_1^2\varrho} \mathbf{1}_{\omega_\varrho}$, which is nothing but (5.3.7) in Harris theorem.

- **Step 6. Conclusion.** Because of the constructive estimates (6.1.39), (6.1.48) and (6.1.49), we may apply the Harris type Theorem 5.3.3, and we conclude to the exponential stability (4.6.27) in the norm of $L^2(\Omega)$ with constructive constants.

6.2 Diffusion in \mathbb{R}^d with strong potential confinement

We consider in this section the elliptic operator

$$\mathcal{L}f := \Delta f + b \cdot \nabla f + cf, \quad f \in H^1(\mathbb{R}^d), \quad (6.2.50)$$

with $b \in L_{\text{loc}}^\infty(\mathbb{R}^d)$, $c \in L_{\text{loc}}^2(\mathbb{R}^d)$ and a confinement condition that we impose through the properties of the *potential function* c , which is roughly speaking $c \rightarrow -\infty$ as $|x| \rightarrow \infty$. More precisely, we assume

$$\sigma_{i+} \in L^{d/2}, \quad \text{meas}\{\sigma_i \geq K\} < \infty, \quad \forall K < 0, \quad (6.2.51)$$

with either $\sigma_1 := c + |b|^2/\kappa$ for some constant $\kappa \in (0, 4)$ or either $\sigma_2 := c + \operatorname{div}b/2$. When we assume that

$$c \sim -|x|^\gamma \quad \text{and} \quad b \sim x|x|^{\beta-1} \quad \text{as} \quad |x| \rightarrow \infty,$$

the condition (6.2.51) for σ_1 is reached when $\gamma > \max(0, 2\beta)$ or $\gamma = 2\beta > 0$ and some conditions on the constants involved in the behavior of the coefficients. In that context, the condition (6.2.51) for σ_2 is more general since it is reached when $\gamma > \max(0, \beta - 1)$ or $\gamma = \beta - 1 > 0$ and some conditions on the constants involved in the behavior of the coefficients.

A similar framework is considered in [156] and for the reader convenience we just briefly check that it falls in the framework developed before by slightly modifying the arguments presented in the previous section. The integrability conditions on b and c may be probably weakened. For the sake of clarity we do not follow this line of research but rather focus on the new arguments which are necessary in order to deal with the unbounded domain $\Omega = \mathbb{R}^d$.

Condition (H1). The definition of the operator is still made through the formula (6.1.8). Under assumption (6.2.51) on σ_1 , denoting $\theta_1 := 1 - \kappa/4$ and proceeding exactly as in the previous section during the proof of (6.1.9), for any $f \in H^1(\mathbb{R}^d)$ and $\lambda \in \mathbb{R}$, we have

$$\begin{aligned} \langle (\lambda - \mathcal{L})f, f \rangle &= \int_{\mathbb{R}^d} |\nabla f|^2 + \int_{\mathbb{R}^d} f b \cdot \nabla f + \int_{\mathbb{R}^d} (\lambda - c)f^2 \\ &\geq \theta_1 \int_{\mathbb{R}^d} |\nabla f|^2 + \int_{\mathbb{R}^d} (\lambda - \sigma_1)f^2, \end{aligned}$$

by using successively the Cauchy-Schwarz inequality and the Young inequality. On the other hand, under assumption (6.2.51) on σ_2 , denoting $\theta_2 := 1$, for any $f \in H^1(\mathbb{R}^d)$ and $\lambda \in \mathbb{R}$, we write

$$\langle (\lambda - \mathcal{L})f, f \rangle = \theta_2 \int_{\mathbb{R}^d} |\nabla f|^2 + \int_{\mathbb{R}^d} (\lambda - \sigma_2)f^2,$$

by performing one integration by part in the previous equation. In both cases, for and any $M > 0$, proceeding again as in the previous section during the proof of (6.1.9), and denoting from now on $\sigma = \sigma_i$, $\theta = \theta_i$ we have

$$\langle (\lambda - \mathcal{L})f, f \rangle \geq \frac{\theta}{2} \|\nabla f\|_{L^2}^2 + \|\sqrt{\sigma_-}f\|_{L^2}^2 + (\lambda - M)\|f\|_{L^2}^2 + \left(\frac{\theta C_S}{2} - \|\sigma \mathbf{1}_{\sigma \geq M}\|_{L^{d/2}}\right) \|f\|_{L^{2^*}}^2,$$

by using the Sobolev inequality (with associated constant C_S) and the Holder inequality. Taking $M > 0$ large enough, and next $\kappa_1 > 0$ large enough, we finally obtain

$$\langle (\lambda - \mathcal{L})f, f \rangle \geq \frac{\theta}{2} \|\nabla f\|_{L^2}^2 + \|\sqrt{\sigma_-}f\|_{L^2}^2 + \|f\|_{L^2}^2, \quad \forall \lambda \geq \kappa_1. \quad (6.2.52)$$

With the same arguments as in the previous section, we conclude that \mathcal{L} is the generator in L^2 of a positive semigroup $S_{\mathcal{L}}$, so that **(H1)** holds.

Condition (H2). We may for instabce use the third constructive argument presented in section 6.1 and we establish

$$\exists f_0 \in H_0^1 \setminus \{0\}, \quad f_0 \geq 0, \quad \exists \kappa_0 \in \mathbb{R}, \quad \mathcal{L}f_0 \geq \kappa_0 f_0.$$

That is condition **(ii)** in Lemma 2.1.4, so that condition **(H2)** holds.

Condition (H3). We introduce again the splitting $\mathcal{L} = \mathcal{A} + \mathcal{B}$ with $\mathcal{A} := \kappa_1 - \kappa_0 + 1$, so that from (6.2.52), the operator $\lambda - \mathcal{B} = (\lambda - \kappa_0 + 1) + (\kappa_1 - \mathcal{L})$ is invertible for any $\lambda \geq \kappa_{\mathcal{B}} := \kappa_0 - 1$. We claim that the operator $(\lambda - \mathcal{B})^{-1}$ is compact for any $\lambda \geq \kappa_{\mathcal{B}}$. For that purpose, let us consider a sequence (f_n) such that $(\lambda - \mathcal{B})f_n$ is bounded in L^2 and we have to prove that (f_n) is

relatively strongly compact. When condition (6.2.51) holds and because of the estimate (6.2.52) and the very definition of \mathcal{B} , we have

$$\frac{\theta}{2} \|\nabla f_n\|_{L^2}^2 + \|\sqrt{\sigma_-} f_n\|_{L^2}^2 + \|f_n\|_{L^2}^2 \leq C, \quad (6.2.53)$$

for some constant $C \in \mathbb{R}_+$. Because of the Rellich-Kondrachov theorem, we just have to show that

$$\lim_{R \rightarrow \infty} \sup_n \int_{B_R^c} f_n^2 = 0.$$

But that last convergence may be established using the assumption (6.2.51) in the following way. We write

$$\begin{aligned} \int_{B_R^c} f_n^2 &= \int_{B_R^c \cap \{\sigma \geq K\}} f_n^2 + \int_{B_R^c \cap \{\sigma < K\}} f_n^2 \\ &\leq \|f_n\|_{L^{2^*}}^{\frac{d-2}{d}} [\text{meas}(B_R^c \cap \{\sigma \geq K\})]^{\frac{2}{d}} + \frac{1}{|K|} \int \sigma_- f_n^2, \end{aligned}$$

for any $K < 0$, by using the Holder inequality. Using next the Sobolev inequality, the estimate (6.2.53) and the assumption (6.2.51), we deduce

$$\limsup_{R \rightarrow \infty} \int_{B_R^c} f_n^2 \lesssim \limsup_{R \rightarrow \infty} \inf_{K < 0} \left\{ [\text{meas}(B_R^c \cap \{\sigma \geq K\})]^{\frac{2}{d}} + \frac{1}{|K|} \right\} = 0,$$

and the claim is proved. As a consequence, we may apply Lemma 2.1.8-(2) and we deduce that **(H3)** holds for both the primal and the dual problems.

Condition (H4). As in [138, Proposition 5.4], we establish the strong maximum principle by exhibiting a barrier function and using Lemma 3.2.9. An alternative argument should be to adapt the proof based on the Harnack inequality as presented in the previous section. Let us then consider $f \in D(\mathcal{L}^k) \cap X_+ \setminus \{0\}$ such that $(\lambda - \mathcal{L})f \geq 0$ with k large enough ($k > d/2$ must be suitable) and $\lambda \geq \lambda_1$ large enough but fixed ($\lambda \geq \kappa_1$ is suitable). Using a very classically bootstrap argument based on iterated application of the Calderon-Zygmund elliptic regularity theorem and the Morrey estimate, we have $f \in C(\mathbb{R}^d)$. By assumption, there thus exist $x_0 \in \mathbb{R}^d$, and two constants $\tau, r > 0$ such that $f \geq \tau$ on $B(x_0, r)$ and we take choose $x_0 = 0$ in order to simplify the notations. We next fix $R > r$ and we observe that the function

$$g(x) := \tau^*(g_0(|x|) - g_0(R)), \quad g_0(s) := \exp(\sigma r^2/2 - \sigma s^2/2),$$

satisfies

$$\begin{aligned} (\tau^*)^{-1}(\lambda - \mathcal{L})g &= (\lambda - c)(g_0 - g_0(R)) + (d\sigma - \sigma b \cdot x - \sigma^2 r^2) g_0 \\ &\leq [2(|\lambda| + \|c\|_{L^\infty(B_R)}) + \sigma(d + \|b \cdot x\|_{L^\infty(B_R)}) - \sigma^2 r^2] g_0 \leq 0 \end{aligned}$$

on $\mathcal{O} := B(0, R) \setminus B(0, r)$ for $\sigma > 0$ large enough. We next fix τ^* such that $g = \tau$ on $\partial B(x_0, r)$. We also observe that from (6.2.52), $\lambda - \mathcal{L}$ is coercive on \mathcal{O} , in the sense that

$$\forall h \in H_0^1(\mathcal{O}) \quad ((\lambda - \mathcal{L})h, h)_{L^2(\mathcal{O})} \geq \|h\|_{L^2(\mathcal{O})}.$$

In particular, $\lambda - \mathcal{L}$ satisfies the weak maximum principle as explained in the proof of (6.1.10). Arguing as in the proof of Lemma 3.2.9, we deduce that $f \geq g > 0$ on \mathcal{O} , what we also see directly by observing that $h := (g - f)_+ \in H_0^1(\mathcal{O})$, $(\lambda - \mathcal{L})h \leq 0$ and using the weak maximum principle implies $h \leq 0$, thus $h \equiv 0$ and finally $f \geq g$. Because $R > r$ can be chosen arbitrarily large, we conclude with $f > 0$ on \mathbb{R}^d .

Condition (H5). The reverse Kato's inequality condition is proved by using local arguments, so that it holds for the same reasons as in the previous section. Similarly, because the argument are local, the conclusion of Lemma 6.1.1 holds here.

As a consequence, using Theorem 2.1.20, Theorem 3.3.11, Theorem 4.4.14 and Theorem 4.5.20, we may summarize our analysis in the following result.

Theorem 6.2.6. *Consider the elliptic operator (6.2.50) in the whole space and assume that the coefficients satisfy (6.2.51). Then the conclusions **(C3)** holds as well as **(CE2)** in $L^1_{\phi_1}$.*

We do not present an exponential constructive estimate, which we believe is possible to prove, but would require significantly more development.

6.3 Diffusion in \mathbb{R}^d with weak potential confinement

We consider in this section the same elliptic operator (6.2.50) with now a weak confinement condition assuming that c converges to a constant. With no loss of generality, we may assume $c \rightarrow 0$. More precisely, we consider the elliptic operator

$$\mathcal{L}f := \Delta f + b \cdot \nabla f + rcf, \quad (6.3.54)$$

with $c \in C_0(\mathbb{R}^d)$, $b \in C_0(\mathbb{R}^d)$ and $r \in \mathbb{R}_+$ a parameter. When not necessary in the discussion we will take $r = 1$. The associated first eigenvalue problem in such a situation has been studied in [156, 8th and 9th courses] to which we refer for more details. We define

$$\lambda_1 = \lambda_1(r) := \inf\{\kappa \in \mathbb{R}; (\lambda - \mathcal{L})^{-1} \text{ well defined and positive for any } \lambda \geq \kappa\}.$$

Proceeding exactly as in the proof of **(H1)** in the preceding section, we see that the operator $\lambda - \mathcal{L}$ is invertible for any $\lambda > \|c_+\|_{L^\infty}$, and then its inverse is positive. Because the proof of **(H2)** in the preceding section also applies here, we deduce that the infimum λ_1 of the set \mathcal{I} of real resolvent values is well defined with $\lambda_1 \in (\kappa_0, \kappa_1)$, for some constructive constants $\kappa_i \in \mathbb{R}$.

We split now the discussion into two cases.

Case 1. We start considering **the case** $b = 0$. In that case, \mathcal{L} is self-adjoint so that λ_1 is also characterized by

$$\lambda_1 = \sup_{\|f\|_{L^2}=1} \mathcal{E}(f),$$

with

$$\mathcal{E}(f) := (\mathcal{L}f, f) = r \int cf^2 - \int |\nabla f|^2.$$

We make the following elementary observations :

- We claim that $\lambda_1 \geq 0$. Taking $f_n(x) := n^{-d/2}u(x/n)$ for some function $u \in H^1(\mathbb{R}^d)$, $\|u\|_{L^2}=1$, we compute

$$\begin{aligned} -\mathcal{E}(f_n) &= \int |\nabla f_n|^2 - \int_{B_R} rcf_n^2 - \int_{B_R^c} rcf_n^2 \\ &\leq \frac{1}{n^2} \int |\nabla u|^2 + \|rc\|_{L^\infty(B_R)} \int_{B_{R/n}} u^2 + \|rc\|_{L^\infty(B_R^c)}, \end{aligned}$$

for any $R > 0$, so that

$$-\lambda_1 \leq \limsup(-\mathcal{E}(f_n)) \leq 0.$$

- We claim that $\lambda_1 = 0$ when $c \leq 0$. In that case, we have $\mathcal{E}(f) \leq 0$ for any $f \in H^1(\mathbb{R}^d)$, and we deduce the reverse inequality $\lambda_1 \leq 0$. In particular, as a function $\lambda_1 = \lambda_1(r)$ of $r \geq 0$, we have $\lambda_1(0) = 0$. We also claim that $\lambda_1(r) \rightarrow \infty$ as $r \rightarrow \infty$ when $c_+ \not\equiv 0$. We may indeed fix $f \in H^1(\mathbb{R}^d)$, $\|f\|_{L^2}=1$, $\text{supp } f \subset \text{supp } c_+$, and we compute

$$\mathcal{E}(f) = r \int_{\mathbb{R}^d} c_+ f^2 - \int |\nabla f|^2 \rightarrow \infty, \quad \text{as } r \rightarrow \infty.$$

- We finally observe that $\lambda_1 : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is convex since it is defined as the supremum of linear functions $r \rightarrow \mathcal{E}(f)$ for any fixed $f \in H^1(\mathbb{R}^d)$. As a consequence, we have the following alternative:

- $\lambda_1 \equiv 0$;
- $\exists r_0 \in [0, \infty)$ such that $\lambda_1(r) = 0$ for $r \leq r_0$ and $\lambda_1(r) > 0$ for $r > r_0$.

Concerning the value of r_0 , it may happen that $r_0 > 0$, and that is the case when $c \in L^{d/2}$ because of the Sobolev inequality, or that $r_0 = 0$, and that is the case for instance when $c \geq 0$, $c(x) = |x|^{-m}$ for $x \in B_R^c$, $m \in (0, 2)$, $R > 0$. To prove that last claim, we may take the same sequence (f_n) as above, and we compute

$$\begin{aligned} \mathcal{E}(f_n) &\geq \int_{B_R^c} r|x|^{-m} f_n^2 - \int |\nabla f_n|^2 dx \\ &= \frac{r}{n^m} \int_{B_{R/n}^c} |x|^{-m} u^2 - \frac{1}{n^2} \int |\nabla u|^2 dx > 0, \end{aligned}$$

for n large enough (whatever is the value of $r > 0$).

About condition (H3). It is established in [156] that when $\lambda_1 = 0$, the condition **(H3)** is not satisfied and there does not exist a first eigenfunction $f_1 \in L^2(\mathbb{R}^d)$ to the operator \mathcal{L} defined by (6.3.54). We refer to [156, 8th course] for a proof of that result. On the other hand, we claim that the condition **(H3)** is satisfied when $\lambda_1 > 0$. Consider indeed three sequences (λ_n) of \mathbb{R} , (f_n) of $H^1(\mathbb{R}^d)$ and (ε_n) of $L^2(\mathbb{R}^d)$ such that $(\lambda_n - \mathcal{L})f_n = \varepsilon_n$, $\varepsilon_n, f_n \geq 0$, $\|f_n\|_{L^2} = 1$, for any $n \geq 1$, $\lambda_n \rightarrow \lambda_1$ and $\varepsilon_n \rightarrow 0$ in L^2 as $n \rightarrow \infty$. We then have

$$\lambda_n - \mathcal{E}(f_n) = \langle (\lambda_n - \mathcal{L})f_n, f_n \rangle = \langle \varepsilon_n, f_n \rangle \rightarrow 0,$$

as $n \rightarrow \infty$. By definition of \mathcal{E} and boundedness of c , we see that (f_n) is bounded in H^1 . As a consequence, up to the extraction of a subsequence, we have $f_n \rightarrow f_1 \geq 0$ in L^2_{loc} and thus next $(\lambda_1 - \mathcal{L})f_1 = 0$ in the variational sense and

$$\int c f_n^2 \rightarrow \int c f_1^2, \quad \|\nabla f_1\|_{L^2} \leq \liminf \|\nabla f_n\|_{L^2},$$

where we have used the dominated convergence theorem of Lebesgue and the fact that $c \rightarrow 0$ at infinity in order to get the first convergence. We finally deduce

$$\mathcal{E}(f_1) \geq \limsup \mathcal{E}(f_n) = \lambda_1 > 0,$$

so that $f_1 \not\equiv 0$, and the condition **(H3)** is verified.

As a conclusion, for a self-adjoint operator, condition **(H3)** is automatically fulfilled by its adjoint and the conditions **(H4)** and **(H5)** have been proved in a general situation, including the present framework. The same conclusions of existence and uniqueness of a first eigentriplet solution (λ_1, f_1, ϕ_1) as in section 6.2 hold true when $\lambda_1 > 0$.

Case 2. We consider the **general case** $b \in C_0(\mathbb{R}^d)$.

• We claim that $\lambda_1 \geq 0$. Adapting the second constructive argument in the proof of **(H2)** in Section 6.1, we consider $\chi \in C_c^1(\mathbb{R}_+) \cap W^{2,\infty}(\mathbb{R}_+)$ such that $\mathbf{1}_{[0,1/2]} \leq \chi \leq \mathbf{1}_{[0,1]}$, $\chi' \leq 0$ on $[0, 1]$, $\chi(s) := (1-s)^2/2$ on $[\eta, 1]$ with $\eta \in (1/2, 1)$ large enough in such a way that

$$\chi''(s) + (d-1)\chi'(s)/s \geq 1/2, \quad \forall s \in (\eta, 1), \quad (6.3.55)$$

and define $f_0(x) := \chi(|x - x_0|/n)$ for $|x_0|$ large enough to be chosen later. We have $\text{supp } f_0 \subset B_n(x_0)$ for any $n \geq 1$ and we compute

$$\mathcal{L}f_0(x) = \frac{1}{n^2} \left\{ \chi''(r/n) + \frac{d-1}{r/n} \chi'(r/n) \right\} + \frac{1}{n} b(x) \cdot \hat{y} \chi'(r/n) + c(x) \chi(r/n)$$

where $y = x - x_0$ and $r = |y|$. On $B_{n\eta}(x_0)$, we have

$$\mathcal{L}f_0(x) \geq -\frac{\|\chi''\|_\infty}{n^2} - \frac{d-1}{n^2} \left\| \frac{\chi'(r)}{r} \right\|_\infty - \frac{\|\chi'\|_\infty}{n} \sup_{B_n(x_0)} |b| - \|\chi\|_\infty \sup_{B_n(x_0)} |c|.$$

On $B_n(x_0) \setminus B_{\eta n}(x_0)$, thanks to (6.3.55), we have

$$\mathcal{L}f_0(x) \geq \frac{1}{2n^2} - \frac{\|\chi'\|_\infty}{n} \sup_{B_n(x_0)} |b| - \|\chi\|_\infty \sup_{B_n(x_0)} |c|.$$

Let now fix $\epsilon > 0$ and choose first n large enough so that

$$-\frac{\|\chi''\|_\infty}{n^2} - \frac{d-1}{n^2} \left\| \frac{\chi'(r)}{r} \right\|_\infty \geq -\frac{\epsilon}{2} \inf_{(0,\eta)} \chi.$$

Then, using that $b, c \in C_0(\mathbb{R}^d)$, we can take $|x_0|$ large enough so that

$$-\frac{\|\chi'\|_\infty}{n} \sup_{B_n(x_0)} |b| - \|\chi\|_\infty \sup_{B_n(x_0)} |c| \geq -\frac{\epsilon}{2} \inf_{(0,\eta)} \chi$$

and

$$\frac{\|\chi'\|_\infty}{n} \sup_{B_n(x_0)} |b| + \sup_{B_n(x_0)} |c| \leq \frac{1}{2n^2}.$$

Gathering the above inequalities, we obtain

$$\mathcal{L}f_0 \geq -\epsilon f_0,$$

and the condition **(H2)** is verified with $\kappa_0 = -\epsilon$. Because $\epsilon > 0$ can be choose arbitrarily small, we conclude with $\lambda_1 \geq 0$.

- We claim that $\lambda_1 = 0$ when $\sigma_2 \leq 0$. Indeed, we have already seen that

$$\langle \mathcal{L}f, f \rangle = - \int_{\mathbb{R}^d} |\nabla f|^2 + \int_{\mathbb{R}^d} \sigma_2 f^2,$$

from which we deduce that

$$\frac{d}{dt} \|S_t f\|^2 = 2 \langle \mathcal{L}f, f \rangle \leq 0.$$

This ensures that **(H1)** is verified with $\kappa_1 = 0$ and so $\lambda_1 \leq 0$.

- We claim that $\lambda_1 > 0$ when $c_+ \neq 0$ and $r > 0$ is large enough. For simplifying notations and up to translation and dilatation, we may reduce to the case $c \geq c_0 \mathbf{1}_{B(0,1)}$ with $c_0 > 0$. Adapting the second constructive argument in the proof of **(H2)** in Section 6.1, we consider $\chi \in C_c^1(\mathbb{R}_+) \cap W^{2,\infty}(\mathbb{R}_+)$, $\mathbf{1}_{[0,1/2]} \leq \chi \leq \mathbf{1}_{[0,1]}$, $\text{supp} \chi = [0, 1]$, $\chi''(1) = 1$, $\chi' \leq 0$ on $[0, 1]$ and we set $f_0(x) := \chi(|x|)$. We compute

$$\mathcal{L}f_0 = \chi''(|x|) + \chi'(|x|)((d-1)/|x| + b \cdot \hat{x}) + rc(x)\chi(|x|).$$

On the one hand, we fix $\eta \in (1/2, 1)$, $1 - \eta$ small enough, in such a way that

$$\|\chi'\|_{L^\infty(\eta,1)}(2(d-1) + \|b\|_{L^\infty}) \leq 1/4, \quad 1/2 \leq \|\chi''\|_{L^\infty(\eta,1)},$$

and thus

$$\mathcal{L}f_0 \geq \frac{1}{4} \geq \frac{1}{4} f_0 \quad \text{on } B(0,\eta)^c.$$

On the other hand, we fix $r > 0$, large enough, in such a way that

$$\|\chi''\|_{L^\infty} + \|\chi'\|_{L^\infty}(2(d-1) + \|b\|_{L^\infty}) \leq \kappa(r) := \frac{1}{2} r c_0 \inf_{[0,\eta]} \chi,$$

and thus

$$\mathcal{L}f_0 \geq \kappa(r) \geq \kappa(r) f_0 \quad \text{on } B(0,\eta).$$

As a conclusion, we have established that condition **(ii)** in the statement Lemma 2.1.4 holds with $\kappa_0 := \min(1/4, \kappa(r))$, and that ends the constructive proof of condition **(H2)** by using Lemma 2.1.4. That implies in particular the claim since then $\lambda_1 \geq \kappa_0 > 0$.

• We finally claim again that **(H3)** holds when $\lambda_1 > 0$. To see that, we consider again three sequences (λ_n) of \mathbb{R} , (f_n) of $H^1(\mathbb{R}^d)$ and (ε_n) of $L^2(\mathbb{R}^d)$ such that $(\lambda_n - \mathcal{L})f_n = \varepsilon_n$, $\varepsilon_n, f_n \geq 0$, $\|f_n\|_{L^2} = 1$, for any $n \geq 1$, $\lambda_n \searrow \lambda_1$ and $\varepsilon_n \rightarrow 0$ in L^2 as $n \rightarrow \infty$. As a consequence, we have

$$\lambda_n + \int |\nabla f_n|^2 - \int f_n b \cdot \nabla f_n - \int c f_n^2 = ((\lambda_n - \mathcal{L})f_n, f_n) = \langle \varepsilon_n, f_n \rangle \rightarrow 0,$$

as $n \rightarrow \infty$. Using the boundedness of c , b and λ_n , we see that (f_n) is bounded in H^1 . As a consequence, up to the extraction of a subsequence, we have $f_n \rightarrow f_1 \geq 0$ in L^2_{loc} . We assume by contradiction that $f_1 \equiv 0$. We deduce that

$$\int c f_n^2 \rightarrow 0, \quad \int f_n b \cdot \nabla f_n \rightarrow 0,$$

where we have used the dominated convergence theorem of Lebesgue and the fact that $b, c \rightarrow 0$ at infinity. We thus obtain

$$0 < \lambda_1 \leq \lambda_n + \int |\nabla f_n|^2 - \int f_n b \cdot \nabla f_n + \int c f_n^2 + \langle \varepsilon_n, f_n \rangle \rightarrow 0,$$

and our contradiction. So that $f_1 \not\equiv 0$, and the condition **(H3)** is verified.

For the dual problem, from the above analysis, we know that there exist two sequences (ϕ_n) of $H^1(\mathbb{R}^d)$, (ε_n) of $L^2(\mathbb{R}^d)$ such that $(\lambda_n - \mathcal{L}^*)\phi_n = \varepsilon_n$, $\varepsilon_n, \phi_n \geq 0$ and $\|\phi_n\|_{L^2} = 1$, for any $n \geq 1$, and $\varepsilon_n \rightarrow 0$ in L^2 as $n \rightarrow \infty$. But we face the same situation as previously, since again

$$\lambda_n + \int |\nabla \phi_n|^2 - \int \phi_n b \cdot \nabla \phi_n - \int c \phi_n^2 = ((\lambda_n - \mathcal{L}^*)\phi_n, \phi_n) = \langle \varepsilon_n, \phi_n \rangle \rightarrow 0,$$

and thus the same conclusion, namely $\phi_n \rightarrow \phi_1$, with $\phi_1 \in H^1(\mathbb{R}^d)$, $\phi_1 \geq 0$, $\phi_1 \not\equiv 0$.

Conclusion. The conditions **(H4)** and **(H5)** have been proved in a general situation, including the present framework. The same conclusions as in section 6.2 hold true when $r > 0$ is large enough (and thus $\lambda_1 > 0$).

6.4 Diffusion in \mathbb{R}^d with drift confinement

We now consider the elliptic operator

$$\mathcal{L}f := \Delta f + b \cdot \nabla f + cf,$$

with a drift confinement as it is the case for the Fokker-Planck operator. More precisely, and for the sake of simplicity, we assume here

$$b = \nabla U, \quad U(x) = \frac{1}{\gamma} \langle x \rangle^\gamma, \quad \gamma > 0. \quad (6.4.56)$$

When $\gamma = 2$ and $c = x$, that operator corresponds to the classical harmonic Fokker-Planck operator which is known to be related to the standard Poincaré inequality and to the standard log-Sobolev inequality, see [11, 12, 233] or more recently [14, 138] and the references therein. When $c = \text{div}b$, the operator \mathcal{L} is on divergence form and $\mathcal{L}^*1 = 0$, so that $(0, 1) \in \mathbb{R} \times L^\infty(\mathbb{R}^d)$ is a solution to the dual first eigenvalue problem. Existence of stationary solution f_1 (which is also the first eigenfunction) and its stability have been widely studied. We refer for instance to [234, 217, 94, 13] as well as to [138, 174, 113] which techniques will be adapted here.

In the present situation, we impose that the contribution of c has lower influence at the infinity than the drift term b and we assume

$$c \in L_{loc}^\infty(\mathbb{R}^d), \quad \exists C_0, R_0 > 0, \quad \forall x \in B_{R_0}^c, \quad |c(x)| = o(|x|^{2(\gamma-1)}). \quad (6.4.57)$$

We further assume that

$$c \geq \operatorname{div} b \quad \text{when} \quad \gamma \in (0, 1]. \quad (6.4.58)$$

The action of the drift term will be revealed through the choice of a convenient ‘‘confining space’’. More precisely, for a weight function $m : \mathbb{R}^d \rightarrow [1, \infty)$, we will work in a weighted Lebesgue space. Our analysis is based on the following elementary computation which can be readily adapted from [138, Lemma 3.8], [174, Lemma 3.8] and [113, Lemma 2.1].

Lemma 6.4.7. *For any $p \in [1, \infty)$, any weight function m and any smooth, rapidly decaying function f , we have*

$$\int (\mathcal{L}f) f |f|^{p-2} m^p = -(p-1) \int |\nabla f|^2 |f|^{p-2} m^p + \int |f|^p m^p \varphi_1, \quad (6.4.59)$$

with

$$\varphi_1 := (p-1) \frac{|\nabla m|^2}{m^2} + \frac{\Delta m}{m} + \left(c - \frac{1}{p} \operatorname{div} b \right) - b \cdot \frac{\nabla m}{m} \quad (6.4.60)$$

as well as

$$\int (\mathcal{L}f) f |f|^{p-2} m^p = -(p-1) \int |\nabla(fm)|^2 |fm|^{p-2} + \int |f|^p m^p \varphi_2, \quad (6.4.61)$$

with

$$\varphi_2 := 2\left(1 - \frac{1}{p}\right) \frac{|\nabla m|^2}{m^2} + \left(\frac{2}{p} - 1\right) \frac{\Delta m}{m} + \left(c - \frac{1}{p} \operatorname{div} b \right) - b \cdot \frac{\nabla m}{m}. \quad (6.4.62)$$

In order to simplify the discussion, we restrict ourself to the exponent $p = 2$ and to the exponential weight function $m = e^{a\langle x \rangle^s}$, $s \in (0, \gamma]$, $a > 0$. We thus work in the Banach lattice $X := L_m^2$. We observe that

$$\begin{aligned} \frac{\nabla m}{m} &= sax\langle x \rangle^{s-2} \sim sa|x|^{s-1}, \\ \frac{\Delta m}{m} &= sad\langle x \rangle^{s-2} + s(s-2)a|x|^2\langle x \rangle^{s-4} + (sa)^2|x|^2\langle x \rangle^{2s-4} \sim (sa)^2|x|^{2s-2}, \\ \operatorname{div} b &= d\langle x \rangle^{\gamma-2} + (\gamma-2)|x|^2\langle x \rangle^{\gamma-4} \sim (d+\gamma-2)|x|^{\gamma-2}, \\ b \cdot \frac{\nabla m}{m} &= sax\langle x \rangle^{s-2} \cdot x\langle x \rangle^{\gamma-2} \sim sa|x|^{s+\gamma-2}, \end{aligned}$$

so that the contribution of $(c - \operatorname{div} b/2)$ is always negligible at infinity, and we get

$$\varphi_i \sim 2(sa)^2|x|^{2s-2} - sa|x|^{s+\gamma-2}. \quad (6.4.63)$$

We denote

$$\begin{aligned} a' &:= sa > 0 \quad \text{if} \quad s \in (0, \gamma), \\ a' &:= a\gamma - 2(a\gamma)^2 > 0 \quad \text{if} \quad s = \gamma \text{ and } a \in (0, 1/(\sqrt{2}\gamma)). \end{aligned}$$

We then face to three cases :

- (i) $\gamma > 1$: taking $s \in ((2-\gamma)_+, \gamma)$, we have $\varphi_i \sim -a'|x|^{s+\gamma-2} \rightarrow -\infty$ with $s+\gamma-2 > 0$;
- (ii) $\gamma = 1$: taking $s = \gamma$, $a < 1/(\sqrt{2}\gamma)$, we have $\varphi_i \rightarrow -a'$;
- (iii) $\gamma \in (0, 1)$: taking $s = \gamma$, $a < 1/(\sqrt{2}\gamma)$, we have $\varphi_i \sim -a'|x|^{2\gamma-2} \rightarrow 0$ with $2\gamma-2 < 0$.

Condition (H1). In any of the above cases, we have from (6.4.59)

$$((\lambda - \mathcal{L})f, f) = \int |\nabla f|^2 m^2 + \int (\lambda - \varphi_1) f^2 m^2,$$

for $\lambda \in \mathbb{R}$, with $\inf(\lambda - \varphi_1) > 0$ for $\lambda \geq \kappa_1$ and $\kappa_1 > 0$ large enough. We deduce that $\lambda - \mathcal{L}$ is coercive for $\lambda \geq \kappa_1$. With the same arguments as in section 6.1, we conclude that \mathcal{L} is the generator in L_m^2 of a positive semigroup $\mathcal{S}_{\mathcal{L}}$, so that **(H1)** holds.

Condition (H2). When $\gamma > 1$, the same arguments as in Section 6.2 imply that condition **(H2)** holds for some $\kappa_0 \in \mathbb{R}$. When $\gamma \in (0, 1]$, we have $\mathcal{L}^*1 = c - \operatorname{div}b \geq 0$ from (6.4.58) and **(H2)** holds with $\kappa_0 = 0$.

Conditions (H4) and (H5). The strong maximum principle holds here because for instance we may apply the same barrier function argument as presented in Section 6.2. The reverse Kato's inequality condition is proved by using local arguments, so that it holds for the same reasons as in the previous section.

Condition (H3). We define the multiplication operator \mathcal{A} and the elliptic operator \mathcal{B} by

$$\mathcal{A} := M\chi_R, \quad \mathcal{B} := \mathcal{L} - \mathcal{A},$$

for $M, R > 0$ and $\chi_R(x) := \chi(x/R)$ with $\chi \in \mathcal{D}(\mathbb{R}^d)$, $\mathbf{1}_{B_1} \leq \chi \leq \mathbf{1}_{B_2}$. We fix $\kappa_{\mathcal{B}} < \kappa_0$ in case (i), $\kappa_{\mathcal{B}} := -a'/4$ in case (ii) and $\kappa_{\mathcal{B}} := 0$ in case (iii), and we set $a'' := a'/2$. Choosing $M, R > 0$ large enough, from Lemma 6.4.7 and the discussion which follows, we deduce that

$$((\mathcal{B} - \alpha)f, f) \leq - \int |\nabla f|^2 m^2 - a'' \int f^2 (\mathbf{1}_{B_1} + \mathbf{1}_{B_1^c} |x|^{s+\gamma-2}) m^2, \quad (6.4.64)$$

for any $\alpha \geq \kappa_{\mathcal{B}}$ and any nice function f . We classically deduce that $\alpha - \mathcal{B}$ is coercive and thus invertible. We discuss the three different cases.

- In the first case $\gamma > 1$, so that $s + \gamma - 2 > 0$, we see that the operator $\mathcal{R}_{\mathcal{B}}(\alpha)$ is compact from Rellich-Kondrachov theorem, so that also $\mathcal{W}(\alpha) := \mathcal{R}_{\mathcal{B}}(\alpha)\mathcal{A}$ for any $\alpha \geq \kappa_{\mathcal{B}}$. We may thus apply Lemma 2.1.8-(2) and we deduce that **(H3)** holds for both the primal and the dual problems.

- In the case $\gamma = 1$, so that $2\gamma - 2 \leq 0$, the operator $\mathcal{R}_{\mathcal{B}}(\alpha)$ is not compact anymore. However, for any sequence (f_n) which is bounded in L_m^2 , we define the sequence (g_n) by $g_n := \mathcal{A}f_n$, and (g_n) is bounded in $L_{\tilde{m}}^2$ with $\tilde{m} := e^{\tilde{a}\langle x \rangle^\gamma}$, $\tilde{a} \in (a, 1/\sqrt{2\gamma})$. Using the dissipativity estimate (6.4.64) in $L_{\tilde{m}}^2$, we see that $\mathcal{B} - \alpha$ is dissipative in $L_{\tilde{m}}^2$ for any $\alpha \geq \kappa_{\mathcal{B}}$, and more precisely the sequence (h_n) defined by $h_n := \mathcal{R}_{\mathcal{B}}(\alpha)g_n$ satisfies

$$\int |\nabla h_n|^2 m^2 + \tilde{a}'' \int h_n^2 (\mathbf{1}_{B_1} + \mathbf{1}_{B_1^c} |x|^{2\gamma-2}) \tilde{m}^2 \leq \int g_n^2 \tilde{m}^2.$$

Using that $|x|^{2\gamma-2} \tilde{m}^2 / m^2 \rightarrow \infty$ as $|x| \rightarrow \infty$, that implies that (h_n) is relatively compact in L_m^2 .

More precisely, the above estimates show that $\mathcal{W}(\alpha) := \mathcal{R}_{\mathcal{B}}(\alpha)\mathcal{A} : L_m^2 \rightarrow H_m^1 \cap L_{m^\sharp}^2$ with $m^\sharp := m^{1/2} \tilde{m}^{1/2}$ and in particular we have established that $\mathcal{W}(\alpha) := \mathcal{R}_{\mathcal{B}}(\alpha)\mathcal{A}$ is a compact operator in L_m^2 uniformly on $\alpha \geq \kappa_{\mathcal{B}}$ because of the Rellich-Kondrachov theorem and the fact that $m = o(m^\sharp)$. Since $\mathcal{R}_{\mathcal{B}}(\alpha)$ is bounded in $\mathcal{B}(L_m^2)$ uniformly for any $\alpha > \kappa_{\mathcal{B}}$, the operator \mathcal{L} satisfies the splitting structure **(HS1)** and, applying Lemma 2.1.8-(2), we deduce that **(H3)** holds for both the primal and the dual problems.

At this stage, when $\gamma \geq 1$, we obtain a solution (λ_1, f_1, ϕ_1) to the first eigentriplet problem (6.0.3) by using Theorem 2.1.20.

Condition (HS3). In the case $\gamma \in (0, 1)$, the same as in the case $\gamma = 1$ holds except that $\mathcal{R}_{\mathcal{B}}(\alpha)$ is not uniformly bound in $\mathcal{B}(L_m^2)$ for $\alpha \geq \kappa_{\mathcal{B}}$ because we are in the critical case $\kappa_{\mathcal{B}} = \kappa_0$. We do not know how to adapt the stationary approach in that situation and we thus aim to use a dynamical approach through the use of Theorem 2.2.4 with the above splitting $\mathcal{L} = \mathcal{A} + \mathcal{B}$ and $N := [d/4] + 1$. We set $X = X_1 := L_m^2$ and $X_0 := L^1$. The proof of condition **(HS3)** is an immediate consequence of the following estimate.

Proposition 6.4.8. *We define $\Theta_\zeta(t) := e^{-\zeta t^{\gamma/(2-\gamma)}}$. For $N := [d/4] + 1$, there hold*

- (i) $S_{\mathcal{B}} \in L_t^\infty(\mathcal{B}(X_1))$;
- (ii) $S_{\mathcal{B}} \mathcal{A} \Theta_\zeta^{-1} \in L_t^\infty(\mathcal{B}(X_i))$ for $i = 0, 1$ and any $\zeta \in (0, \zeta^*)$;
- (iii) $(S_{\mathcal{B}} \mathcal{A})^{(*N)} \Theta_\zeta^{-1} \in L_t^\infty(\mathcal{B}(X_0, X_1))$ for any $\zeta \in (0, \zeta^*/2)$.

The proof of Proposition 6.4.8 is similar to the proofs of [138, Lemma 2.1], [138, Lemma 2.2], [138, Lemma 2.3] and [138, Lemma 2.4]. For the sake of completeness we however present the main lines of the proof. We start with a technical result that we will use during the proof of Proposition 6.4.8.

Lemma 6.4.9. *Consider two Banach spaces X_0, X_1 and a function $u : \mathbb{R}_+ \rightarrow \mathcal{B}(X_0) + \mathcal{B}(X_1)$ which satisfies*

- (a) $u \Theta^{-1} \in L^\infty(0, \infty; \mathcal{B}(X_0) \cap \mathcal{B}(X_1))$;
- (b) $u \wp \in L^\infty(0, \infty; \mathcal{B}(X_0, X_1))$;

for any exponentially decaying function $\Theta = \Theta_\zeta = e^{-\zeta t^\varsigma}$, $\forall \zeta \in (0, \zeta^*)$, and for the power function $\wp := t^{-\alpha}$, with $\zeta^* > 0$, $\varsigma \in (0, 1]$ and $\alpha \geq 0$ fixed. Then

- (c) *there exists N such that $u^{(*N)} \tilde{\Theta} \in L^\infty(0, \infty; \mathcal{B}(X_0, X_1))$,*

for any $\tilde{\Theta} = \Theta_{\tilde{\zeta}}$, $\tilde{\zeta} \in (0, \zeta^*/2)$.

Proof of Lemma 6.4.9. A similar argument is developed in [113, Lemma 2.17], [174, Lemma 2.4], [175, Proposition 2.5] and [138, Lemma 2.4].

Step 1. Consider two functions v and w which satisfy the estimate (a). For $\mathcal{X} = X_0$ or $\mathcal{X} = X_1$, we compute

$$\begin{aligned} \|v * w(t)\|_{\mathcal{X} \rightarrow \mathcal{X}} &\leq \int_0^t \|v(t-s)w(s)\|_{\mathcal{X} \rightarrow \mathcal{X}} ds \\ &\leq \int_0^t C_{\mathcal{X}}^v \Theta(t-s) C_{\mathcal{X}}^w \Theta(s) ds \leq C_{\mathcal{X}}^v C_{\mathcal{X}}^w t \Theta(t), \end{aligned}$$

with obvious notation and where we have used that $\Theta(t-s)\Theta(s) \leq \Theta(t)$ for any $0 < s < t$. Since for any $\zeta' \in (0, \zeta)$, there exists a constant C such that $t\Theta_\zeta(t) \leq C\Theta_{\zeta'}(t)$ for any $t \geq 0$, we see that the function $v * w$ satisfies the same estimate (a) for any $\Theta = \Theta_\zeta$, $\zeta \in (0, \zeta^*)$.

Step 2. Consider two functions v and w which satisfy the estimates (a) and (b) with $\alpha \geq 1$. We compute

$$\begin{aligned} \|v * w(t)\|_{X_0 \rightarrow X_1} &\leq \int_0^{t/2} \|v(t-s)w(s)\|_{X_0 \rightarrow X_1} ds + \int_{t/2}^t \|v(t-s)w(s)\|_{X_0 \rightarrow X_1} ds \\ &\leq \int_0^{t/2} C_{01}^v (t-s)^{-\alpha} C_0^w \Theta(s) ds + \int_{t/2}^t C_1^v \Theta(t-s) C_{01}^w s^{-\alpha} ds \\ &= [C_1^v C_{01}^w + C_{01}^v C_0^w] \Theta(0) t^{-\alpha+1} \int_0^{1/2} (1-\tau)^{-\alpha} d\tau, \end{aligned}$$

with obvious notation and we have used that Θ is a decaying function. As a consequence, the function $v * w$ satisfies estimate (b) with an exponent $\alpha - 1$ instead of α .

Step 3. Consider two functions v and w which satisfy the estimates **(a)** and **(b)** with $\alpha \in [0, 1)$. We compute

$$\begin{aligned} \|v * w(t)\|_{X_0 \rightarrow X_1} &\leq \int_0^{t/2} \|v(t-s)w(s)\|_{X_0 \rightarrow X_1} ds + \int_{t/2}^t \|v(t-s)w(s)\|_{X_0 \rightarrow X_1} ds \\ &\leq \int_0^{t/2} C_1^v \Theta(t-s) C_{01}^w s^{-\alpha} ds + \int_{t/2}^t C_{01}^v (t-s)^{-\alpha} C_0^w \Theta(s) ds \\ &\leq C_1^v C_{01}^w \Theta(t/2) \int_0^{t/2} s^{-\alpha} ds + C_{01}^v C_0^w \Theta(t/2) \int_{t/2}^t (t-s)^{-\alpha} ds \\ &= [C_1^v C_{01}^w + C_{01}^v C_0^w] \Theta(t/2) \frac{t^{1-\alpha}}{1-\alpha}, \end{aligned}$$

with the same obvious notation and we have used again that Θ is a decaying function.

Step 4. Iterating $n := [\alpha]$ times steps 1 and 2, we get that $u^{(*n)}$ still satisfies estimate **(a)** and satisfies the estimate **(b)** for the exponent $\alpha - [\alpha] \in (0, 1)$. We then conclude that **(c)** holds with $N := n + 1$ and any $\tilde{\zeta} \in (0, \zeta^*/2)$ by using the third step. \square

Proof of Proposition 6.4.8. We classically establish that \mathcal{B} generates a positive semigroup $S_{\mathcal{B}}$ in both spaces X_i and we thus concentrate on the announced estimates. On the one hand, proceeding as for the proof of (6.4.64), we have

$$\int (\mathcal{B}f)(\text{sign } f)m \leq -a'' \int |f|(\mathbf{1}_{B_1} + \mathbf{1}_{B_1^c}|x|^{s+\gamma-2})m, \quad (6.4.65)$$

for any nice function f and any weight function $m = m_a$, with $m_a(x) := e^{a\langle x \rangle^\gamma}$, $a \in (a_1, a_2)$, $0 < a_1 < a_2 < 1/(\sqrt{2}\gamma)$, where we define $a'' := a\gamma/2 - (a\gamma)^2$. That exactly means that \mathcal{B} is weakly dissipative in L_m^1 as defined in (2.2.19). From the discussion in Section 2.2.3, we deduce that $S_{\mathcal{B}}$ is a semigroup of contractions and satisfies the associated decay estimate (2.2.23), (2.2.24), and more precisely

$$\|S_{\mathcal{B}}(t)f\|_{L_{m_a}^1} \leq \|f\|_{L_{m_a}^1}, \quad \|S_{\mathcal{B}}(t)f\|_{L_{m_a}^1} \leq \Theta_\zeta(t)\|f\|_{L_{m_{a'}}^1}, \quad (6.4.66)$$

for any $a, a' \in (a_1, a_2)$, $a < a'$, $\zeta \in (0, \zeta_*)$, $\zeta_* := (a' - a)^{(2-2\gamma)/(2-\gamma)}(a'\gamma(1 - a'\gamma))^{\gamma/(2-\gamma)}$. We refer to [138, Lemma 2.1] for details. Using that $\mathcal{A} : L^1 \rightarrow L_m^1$ is bounded, that establishes (ii) in X_0 .

Similarly, starting from (6.4.61) and proceeding as in the proof of (6.4.64), we get

$$(\mathcal{B}f, f)_{L_m^2} \leq - \int |\nabla(fm)|^2 - a'' \int f^2(\mathbf{1}_{B_1} + \mathbf{1}_{B_1^c}|x|^{s+\gamma-2})m^2, \quad (6.4.67)$$

for any nice function f and any weight function $m = m_a$ as above. Throwing away the first term at the RHS and arguing as we did in L_m^1 , we obtain that $S_{\mathcal{B}}$ satisfies

$$\|S_{\mathcal{B}}(t)f\|_{L_{m_a}^2} \leq \|f\|_{L_{m_a}^2}, \quad \|S_{\mathcal{B}}(t)f\|_{L_{m_a}^2} \leq \Theta_\zeta(t)\|f\|_{L_{m_{a'}}^2}, \quad (6.4.68)$$

for any $a, a' \in (a_1, a_2)$. Using that $\mathcal{A} : L_{m_a}^2 \rightarrow L_{m_{a'}}^2$ is bounded, that establishes (i) and (ii) in X_1 .

On the other hand, throwing away the second term at the RHS in (6.4.67), for any trajectory $f_t = S_{\mathcal{B}}(t)f_0$, f_0 in the domain of \mathcal{B} in L_m^2 , we have

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^d} f_t^2 m^2 dx \leq - \int_{\mathbb{R}^d} |\nabla(f_t m)|^2 dx.$$

Using Nash's inequality which for some constant $C_N \in (0, \infty)$ stipulates that

$$\int_{\mathbb{R}^d} g^2 dx \leq C_N \left(\int_{\mathbb{R}^d} |\nabla g|^2 dx \right)^{\frac{d}{d+2}} \left(\int_{\mathbb{R}^d} |g| dx \right)^{\frac{4}{d+2}}, \quad \forall g,$$

with $g := f_t m$ and the first estimate in (6.4.66), we deduce

$$F'(t) \leq -2C'_N F(t)^{-4/d} G(t)^{1+\frac{2}{d}} \leq -2C'_N F(t)^{-4/d} G(0)^{1+\frac{2}{d}}, \quad (6.4.69)$$

with $C'_N := C_N^{-1-2/d}$ and where for brevity of notations we have set

$$F(t) := \|f_t\|_{L^2(m)}^2, \quad G(t) := \|f_t\|_{L^1(m)}.$$

Integrate the differential inequality (6.4.69), we find

$$\|S_{\mathcal{B}}(t)f_0\|_{L^2_m}^2 \lesssim t^{-d/4} \|f_0\|_{L^1_m}, \quad \forall t > 0,$$

and using that $\mathcal{A} : L^1 \rightarrow L^1_m$, we next obtain

$$S_{\mathcal{B}}(t)\mathcal{A}t^{d/4} \in L^\infty(0, \infty; \mathcal{B}(X_0, X_1)). \quad (6.4.70)$$

Setting with $u(t) := S_{\mathcal{B}}(t)\mathcal{A}$, we see that u satisfies **(a)** in Lemma 6.4.9 thanks to (ii) in X_0 and X_1 . Furthermore, u satisfies **(b)** in Lemma 6.4.9 thanks to (6.4.70). Using Lemma 6.4.9, we conclude that condition (iii) holds. \square

We come back to the proof of **(HS3)**. Gathering (i) and (ii) in X_1 in Proposition 6.4.8, we get that $(S_{\mathcal{B}}\mathcal{A})^{(*\ell)} * S_{\mathcal{B}} \in L_t^\infty(\mathcal{B}(X_1))$ for any $\ell \in \{0, \cdot, N-1\}$, $N := [d/4] + 1$. Using that $\Theta \in L^1(0, \infty)$ and (iii) in Proposition 6.4.8, we deduce that $(S_{\mathcal{B}}\mathcal{A})^{(*N)} \in L^1(0, \infty; \mathcal{B}(X_0, X_1))$.

We may now handle the existence part of the first eigenvalue problem. On the one hand, recalling **(H2)**, we have $\mathcal{L}^*\psi_0 \geq 0$ with $\psi_0 = 1$ so that the condition (i) in Theorem 2.2.4 holds. On the other hand, the condition (ii) in Theorem 2.2.4 is an immediate consequence of **(HS3)** as emphasized in Remark 2.2.5-(1). As a conclusion, the hypotheses of Theorem 2.2.4 are thus met, and we deduce that there exists $(\lambda_1, f_1) \in \mathbb{R}_+ \times L^2_{m^+}$ solution to the first eigenvalue problem. Because the strong maximum principle **(H4)** holds, we have $f_1 > 0$ on \mathbb{R}^d .

In order to prove the existence of a first positive eigenfunction for the dual problem, we argue in the following way. We start observing that we have the alternative: $\lambda_1 = 0$ or $\lambda_1 > 0$.

- In the first case, we may argue as in Remark 3.3.17. We indeed have in the same time $\mathcal{L}^*\psi_0 \geq 0$ and $\langle \mathcal{L}^*\psi_0, f_1 \rangle = \langle \psi_0, \mathcal{L}f_1 \rangle = 0$, so that $\mathcal{L}^*\psi_0 = 0$ because $f_1 > 0$. The function $\phi_1 := \psi_0$ is thus a solution to the first dual eigenvalue problem.

- In the second case $\lambda_1 > 0$, we may argue as in the case $\gamma = 1$ above. On the one hand, the operator $\mathcal{R}_{\mathcal{B}}(\alpha)$ is uniformly bounded in L^2_m for any $\alpha \geq \kappa_{\mathcal{B}} := \lambda_1/2 > 0$ and on the other hand the operator $\mathcal{W}(\alpha) := \mathcal{R}_{\mathcal{B}}(\alpha)\mathcal{A} : L^2_m \rightarrow H^1_m \cap L^2_{m^\sharp}$ is uniformly bounded for any $\alpha \geq \kappa_{\mathcal{B}}$ with $m = o(m^\sharp)$, so that $H^1_m \cap L^2_{m^\sharp} \subset L^2_m$ is compact. We may thus apply Theorem 2.1.20 and we conclude to the existence of a solution $(\lambda'_1, f'_1, \phi'_1)$ to the eigentriplet problem.

The conditions **(H4)** and **(H5)** being true in a general situation as well as the conclusions of Lemma 6.1.1, as an intermediate conclusion, we have established under the general condition $\gamma > 0$ in (6.4.56) that yet the same conclusions as in section 6.2 hold true.

Chapter 7

The kinetic Fokker-Planck equation

In this chapter, we consider the kinetic Fokker-Planck evolution equation associated to the operator

$$\mathcal{L}f := -v \cdot \nabla_x f + \Delta_v f + b \cdot \nabla_v f + cf, \quad (7.0.1)$$

on functions $f : \mathcal{O} \rightarrow \mathbb{R}$, where $\mathcal{O} := \Omega \times \mathbb{R}^d$, $\Omega \subset \mathbb{R}^d$ is a domain, $b : \mathcal{O} \rightarrow \mathbb{R}^d$ is a given vector field and $c : \mathcal{O} \rightarrow \mathbb{R}$ is a given function.

As we said in the introduction, we will consider the case when Ω is a bounded domain and the equation is complemented with a boundary condition. For ease of reading, we will reintroduce the key notation and elements of this chapter.

We assume the classical balance between the values of the trace γf of f on the outgoing and incoming velocities subsets of the boundary

$$(\gamma_- f)(x, v) = \mathcal{R}_x(\gamma_+ f(x, \cdot))(v) \quad \text{on } \Sigma_-, \quad (7.0.2)$$

where in this context we define $\Sigma_\pm^x := \{v \in \mathbb{R}^3; \pm v \cdot \nu_x > 0\}$ the sets of outgoing (Σ_+^x) and incoming (Σ_-^x) velocities at point $x \in \partial\Omega$, next the sets

$$\Sigma_\pm = \{(x, v) \in \Sigma; \pm \nu_x \cdot v > 0\} = \{(x, v); x \in \partial\Omega, v \in \Sigma_\pm^x\},$$

and finally the outgoing and incoming trace functions $\gamma_\pm f := \mathbf{1}_{\Sigma_\pm} \gamma f$. Here and in the sequel, ν_x denotes the unit normal outward vector field defined on the boundary set $\partial\Omega$. We similarly define the grazing velocity set

$$\Sigma_0 = \{(x, v) \in \Sigma; \nu_x \cdot v = 0\}.$$

The reflection operator \mathcal{R}_x is local in position, but can be local or nonlocal in the velocity variable, so that it writes

$$(\mathcal{R}_x g)(v) := \int_{\Sigma_+^x} r(x, v, v_*) g(v_*) v_* \cdot \nu_x dv_*,$$

for a reflection kernel $r : \partial\Omega \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$. Some classical general assumptions on r are

$$r \geq 0, \quad \mathcal{R}_x^* \mathbf{1} = \mathbf{1}, \quad \mathcal{R}_x \mathcal{M} = \mathcal{M}, \quad (7.0.3)$$

for some positive function $\mathcal{M} = \mathcal{M}(v)$, see for instance [52, 53, 54]. The second (normalisation) condition corresponds to the fact that all the particles reaching the outgoing boundary are put back on the incoming boundary (no mass is lost) while the third (reciprocity) condition means (when \mathcal{M} is a Gaussian function) that the wall is in a local equilibrium state and is not influenced by the incoming particles. The normalization condition implies the local mass conservation

$$\int_{\Sigma_-^x} \mathcal{R}_x g |\nu \cdot v| dv = \int_{\Sigma_+^x} g \nu \cdot v dv, \quad (7.0.4)$$

while the three assumptions (7.0.3) on r together also imply

$$\begin{aligned} \int_{\Sigma_x^-} (\mathcal{R}_x g)^2 \mathcal{M}^{-1} |\nu \cdot v| dv &\leq \int_{\Sigma_x^-} (\mathcal{R}_x (g^2 / \mathcal{M})) (\mathcal{R}_x \mathcal{M}) \mathcal{M}^{-1} |\nu \cdot v| dv \\ &= \int_{\Sigma_x^+} g_*^2 \mathcal{M}_*^{-1} (\mathcal{R}^* 1) \nu \cdot v_* dv_*, \end{aligned}$$

where we have used the Cauchy-Schwarz inequality (and the fact that $r \geq 0$) in the first line and the reciprocity condition in the second line. As a consequence, we have

$$\int_{\Sigma_x^-} (\mathcal{R}_x g)^2 \mathcal{M}^{-1} |\nu \cdot v| dv \leq \int_{\Sigma_x^+} g^2 \mathcal{M}^{-1} \nu \cdot v dv, \quad (7.0.5)$$

where we have used the normalization condition in that last step. In the sequel, we will rather consider the possibly position dependent Maxwell boundary condition operator

$$\mathcal{R}_x g = \alpha(x) \mathcal{D}_x g + \beta(x) \Gamma_x g, \quad (7.0.6)$$

where the accommodation coefficients $\alpha, \beta : \partial\Omega \rightarrow [0, 1]$ satisfy $\alpha(x) + \beta(x) =: \zeta(x) \leq 1$, Γ_x is the specular reflection operator

$$\Gamma_x(g(x, \cdot))(v) = g(x, \mathcal{V}_x v), \quad \mathcal{V}_x v = v - 2\nu(x)(\nu(x) \cdot v), \quad (7.0.7)$$

and \mathcal{D}_x is the diffusive operator

$$\mathcal{D}_x(g(x, \cdot))(v) = c_{\mathcal{M}} \mathcal{M}(v) \tilde{g}(x), \quad \tilde{g}(x) = \int_{\Sigma_x^+} g(x, w) \nu(x) \cdot w dw. \quad (7.0.8)$$

Here the constant $c_{\mathcal{M}} := (2\pi)^{1/2}$ is such that $c_{\mathcal{M}} \tilde{\mathcal{M}} = 1$ and \mathcal{M} stands for the standard Maxwellian

$$\mathcal{M}(v) := (2\pi)^{-d/2} \exp(-|v|^2/2), \quad (7.0.9)$$

or, more generally, $\mathcal{M} = \mathcal{M}(|v|) \geq 0$ is such that

$$\mathcal{D}_x^* 1 = 1, \quad \mathcal{D}_x \mathcal{M} = \mathcal{M}, \quad \langle v \rangle^\vartheta \mathcal{M} \in L^1(\mathbb{R}^d), \quad (7.0.10)$$

with $\vartheta \geq 1$ (that last condition is necessary in order that the second relation above makes sense). The boundary condition (7.0.6) corresponds to the *pure specular reflection* boundary condition when $\beta \equiv 1$ and it corresponds to the *pure diffusive* boundary condition when $\alpha \equiv 1$. When $\zeta \equiv 1$, the Maxwell boundary condition operator (7.0.6) satisfies (7.0.3). On the contrary, when $\zeta \neq 1$, the L^2 estimate (7.0.5) holds but not anymore the mass conservation (7.0.4). However, the following L^1 estimate

$$\int_{\Sigma_x^-} |\mathcal{R}_x g| |\nu \cdot v| dv \leq \zeta^* \int_{\Sigma_x^+} |g| \nu \cdot v dv \quad (7.0.11)$$

holds, with $0 \leq \sup \zeta \leq \zeta^* \leq 1$. Finally, the case $\zeta \equiv 0$ corresponds to the zero inflow problem.

Let us finally mention that the regularity needed on the domain Ω may be formulated in the following way: we assume that Ω is locally on one side of $\partial\Omega$ and there exists a function $\delta = \delta_\Omega \in W^{2,\infty}(\mathbb{R}^d)$ such that for all x in an interior neighborhood of $\partial\Omega$ one has $\delta(x) = \text{dist}(x, \partial\Omega)$ and the vector field ν defined on \mathbb{R}^d by $x \mapsto \nu(x) = \nu_x := -\nabla_x \delta(x)$ coincides with the previously defined unit outward normal vector field on $\partial\Omega$ and satisfies $\|\nu\|_{L^\infty} = 1$. We also assume that the Lebesgue measure on $\partial\Omega$ is well defined and it is denoted by $d\sigma_x$.

7.1 The trace problem

We consider in this section the trace problem for a solution $g = g(x, v)$ to the stationary Vlasov-Fokker-Planck equation

$$\mathcal{M}g := v \cdot \nabla_x g - b \cdot \nabla_v g - \Delta_v g = G \quad \text{in } \mathcal{O}, \quad (7.1.12)$$

for a given a vector field $b = b(x, v)$, a source term $G = G(x, v)$ and for a solution $g = g(t, x, v)$ to the evolution Vlasov-Fokker-Planck equation

$$\partial_t g + v \cdot \nabla_x g - b \cdot \nabla_v g - \Delta_v g = G \quad \text{in } (0, T) \times \mathcal{O}, \quad (7.1.13)$$

for a given a vector field $b = b(t, x, v)$, a source term $G = G(t, x, v)$. The second problem has been considered first in [51] and next in [172, Section 4], where a strong (renormalized) trace function is proved to exist. In the sequel, we recall these results and slightly extending them by considering a possible $L^2 H^{-1}$ source term. We introduce some notations. We denote

$$d\xi := |\nu(x) \cdot v| dv d\sigma_x \quad \text{and} \quad d\xi^2 := (\nu(x) \cdot \hat{v})^2 dv d\sigma_x$$

the measures on the boundary set Σ . We denote by \mathcal{B}_1 the class of renormalized functions $\beta \in W_{loc}^{2,\infty}(\mathbb{R})$ such that β'' has a compact support, by \mathcal{B}_2 the class of functions $\beta \in W_{loc}^{2,\infty}(\mathbb{R})$ such that $\beta'' \in L^\infty(\mathbb{R})$ and by $\mathcal{D}_0(\bar{\mathcal{O}})$ the space of test functions $\varphi \in \mathcal{D}(\bar{\mathcal{O}})$ such that $\varphi = 0$ on Σ_0 . We finally define the dual operator

$$\mathcal{M}^* \varphi := -v \cdot \nabla_x \varphi + \operatorname{div}_v(b\varphi) - \Delta_v \varphi.$$

Theorem 7.1.1. *We consider $g, b \in L_{loc,x}^2 H_{loc,v}^1$, $G \in L_{loc,x}^2 H_{loc,v}^{-1}$ and we assume that g is a solution to the stationary Vlasov-Fokker-Planck equation (7.1.12). Then there exists $\gamma g \in L_{loc}^2(\Sigma, d\xi^2)$ such that the following Green renormalized formula*

$$\begin{aligned} \iint_{\mathcal{O}} (\beta(g) \mathcal{M}^* \varphi - \beta''(g) |\nabla_v g|^2 \varphi) dv dx + \langle G, \beta'(g) \varphi \rangle &= \\ = \iint_{\Sigma} \beta(\gamma g) \varphi \nu(x) \cdot v dv d\sigma_x & \end{aligned} \quad (7.1.14)$$

holds for any renormalized function $\beta \in \mathcal{B}_1$ and any test functions $\varphi \in \mathcal{D}(\bar{\mathcal{O}})$, as well as for any renormalized function $\beta \in \mathcal{B}_2$ and any test functions $\varphi \in \mathcal{D}_0(\bar{\mathcal{O}})$. It is worth emphasizing that $\beta'(g) \varphi \in L_x^2 H_v^1$ so that the duality product $\langle G, \beta'(g) \varphi \rangle$ is well defined.

If furthermore $\gamma_{\mp} g \in L_{loc}^2(\Sigma; d\xi)$ then $\gamma_{\pm} g \in L_{loc}^2(\Sigma; d\xi)$ and (7.1.14) holds for any renormalized function $\beta \in \mathcal{B}_2$ and any test functions $\varphi \in \mathcal{D}(\bar{\mathcal{O}})$.

Proof of Theorem 7.1.1. We only allude the proof which uses very similar arguments as those presented in [75, 172]. In the one hand, considering the mollifier $(\rho_\varepsilon)_{\varepsilon>0}$ defined by

$$\rho_\varepsilon(z) = \frac{1}{\varepsilon^d} \rho(z/\varepsilon), \quad 0 \leq \rho \in \mathcal{D}(\mathbb{R}^d), \quad \operatorname{supp} \rho \subset B_1, \quad \int_{\mathbb{R}^N} \rho(z) dz = 1,$$

with $z := (x, y)$, we get that g_ε is smooth and satisfies

$$g_\varepsilon \rightarrow g \quad \text{in } L_{loc,x}^2 H_{loc,v}^1, \quad \mathcal{M}g_\varepsilon = G_\varepsilon \rightarrow G \quad \text{in } L_{loc,x}^2 H_{loc,v}^{-1},$$

which is nothing but a variant of [76, Lemma II.1]. The function g_ε being smooth, for any $\beta \in C^2$ such that $\beta' \in C_b^1$, we may differentiate $\beta(g_\varepsilon)$ and we get

$$\mathcal{M}\beta(g_\varepsilon) + \beta''(g_\varepsilon) |\nabla_v g_\varepsilon|^2 = \beta'(g_\varepsilon) G_\varepsilon \quad \text{in } \mathcal{O},$$

with. We may thus pass to the limit as $\varepsilon \rightarrow 0$ and we obtain (7.1.14). \square

We also write without proof a stability result that we will use several times in the sequel.

Proposition 7.1.2. *Let us consider three sequences (g_k) , (b_k) of $L^2_{\text{loc},x}H^1_{\text{loc},v}$ and (G_k) of $L^2_{\text{loc},x}H^{-1}_{\text{loc},v}$ such that*

$$v \cdot \nabla_x g_k - b_k \cdot \nabla_v g_k - \Delta_v g_k = G_k \quad \text{in } \mathcal{D}'(\mathcal{O})$$

for any $k \geq 1$ and three functions $g, b \in L^2_{\text{loc},x}H^1_{\text{loc},v}$ and $G \in L^2_{\text{loc},x}H^{-1}_{\text{loc},v}$ such that $g_k \rightarrow g$ strongly in $L^2_{\text{loc},x}H^1_{\text{loc},v}$, $b_k \rightharpoonup b$ weakly in $L^2_{\text{loc}}(\bar{\mathcal{O}})$ and $G_k \rightarrow G$ strongly in $L^2_{\text{loc},x}H^{-1}_{\text{loc},v}$. Then g satisfies (7.1.12) and, up to the extraction of a subsequence, $\gamma g_k \rightarrow \gamma g$ a.e. on $\Sigma \setminus \Sigma_0$.

(2) If $g_k \rightharpoonup g$ weakly in $L^1_{\text{loc}}(\bar{\mathcal{O}})$ then g satisfies (7.1.14) and, up to the extraction of a subsequence, $\gamma g_k \xrightarrow{r} \gamma g$ on $\Sigma \setminus \Sigma_0$.

Where $g_n \xrightarrow{r} g$ denote the renormalized convergence to be defined as follows: a sequence (g_n) of $L(E)$ converges in the renormalized sense to g if for any $\delta \in (0, 1]$ (or for any $\delta \in D \subset (0, 1]$, $D := \{\delta_m\}$ for a sequence $\delta_m \searrow 0$) there exists $\beta_\delta \in L^\infty(E)$ such that

$$\beta_\delta(g_n) \rightharpoonup \bar{\beta}_\delta * \sigma(L^\infty, L^1) \text{ as } n \rightarrow \infty \quad \text{and} \quad \beta_1(\bar{\beta}_\delta) \rightarrow \beta_1(g) \text{ in } L^1_{\text{loc}}(\bar{\mathcal{O}}) \text{ as } \delta \rightarrow 0.$$

We may observe that $g_n \rightharpoonup g$ weakly $L^1(E)$ implies $g_n \xrightarrow{r} g$ up to the extraction of a subsequence and that $g_n \rightarrow g$ a.e. in E implies $g_n \xrightarrow{r} g$. We refer to [172] for more material about renormalized convergence.

7.2 Well-posedness problem with inflow term at the boundary

We consider the kinetic Fokker-Planck operator \mathcal{L} defined in (7.0.1) and we start revisiting the well posedness problem

$$(\lambda - \mathcal{L})f = \mathfrak{F} \quad \text{in } \mathcal{O}, \quad \gamma_- f = \mathfrak{g} \quad \text{on } \Sigma_-, \quad (7.2.15)$$

for given data $\mathfrak{F} : \mathcal{O} \rightarrow \mathbb{R}$ and $\mathfrak{g} : \Sigma_- \rightarrow \mathbb{R}$.

For a given weight function $m : \mathbb{R}^d \rightarrow [1, \infty)$, we define the measure $d\xi_m := m^2 |\nu(x) \cdot v| dv d\sigma_x$ on the boundary Σ . We next define $L^2 H^1_m = L^2 H^1_m(\mathcal{O})$ the space associated to the Hilbert norm defined by

$$\|f\|_{L^2 H^1_m}^2 := \|f\|_{L^2_m}^2 + \|\nabla_v f\|_{L^2_m}^2,$$

and we assume that m satisfies the Poincaré type inequality

$$\|f \frac{\nabla m}{m}\|_{L^2_m} \lesssim \|f\|_{L^2 H^1_m}, \quad \forall f \in L^2 H^1_m. \quad (7.2.16)$$

Such a Poincaré inequality is classically known to be true when $m := \mathcal{M}^{-\vartheta}$, \mathcal{M} is the Maxwellian (7.0.9) and $\vartheta > 0$. We also define

$$L^2 H_m^{-1} := \{\mathfrak{F} = g + \text{div}_v G; \quad g, G_i \in L^2_m(\mathcal{O})\},$$

so that when $m = 1$ the space $L^2 H_m^{-1}$ is nothing but the space of continuous and linear mappings on $L^2 H^1$. For $\mathfrak{F} \in L^2 H_m^{-1}$ and $f \in L^2 H^1_m$, we may thus write

$$\langle \mathfrak{F}, f m^2 \rangle \leq \|\mathfrak{F}\|_{L^2 H_m^{-1}} \|f\|_{L^2 H^1_m}.$$

We finally define in this context

$$W_2 := \{f \in L^2 H^1_m; \quad \hat{v} \cdot \nabla_x f \in L^2 H_m^{-1}\},$$

and

$$W_{2,\Sigma} := \{g \in W_2; \quad \gamma g \in L^2(\Sigma; d\xi_m)\},$$

with $W_{2,\Sigma} \neq W_2$ in general.

Theorem 7.2.3. *Let us fix a vector field $b \in H_{\text{loc}}^1(\bar{\mathcal{O}})$, a function $c \in L^\infty(\mathcal{O})$, a weight function $m : \mathbb{R}^d \rightarrow [1, \infty)$ and let us assume that $b/\langle v \rangle \in L^\infty(\mathcal{O})$, that (7.2.16) holds and that*

$$\lambda^* := \text{ess sup } \varpi < \infty, \quad \varpi := c + \frac{\Delta m^2}{2m^2} - \frac{1}{2} \text{div} b - b \cdot \frac{\nabla m}{m}. \quad (7.2.17)$$

For any $\mathfrak{F} \in L^2 H_m^{-1}$, $\mathfrak{g} \in L^2(\Sigma_-; d\xi_m)$ and $\lambda > \lambda^$, there exists a unique solution $f \in W_{2,\Sigma}$ to the Dirichlet problem (7.2.15). We have furthermore $f \geq 0$ if $\mathfrak{F} \geq 0$ and $\mathfrak{g} \geq 0$.*

A similar result is proved in [70, Appendix A] in the case $\Omega = \mathbb{R}^d$. Also observe that (7.2.17) holds with $m := \mathcal{M}^{-1/2}$ when \mathcal{M} is the standard Maxwellian (7.0.9) and $b(v) = \vartheta v$, with $\vartheta > 1/2$.

Proof of Theorem 7.2.3. We split the proof into five steps.

Step 1. A priori estimates. Multiplying the first equation in (7.2.15) by fm^2 , performing several integrations by part in the velocity variable and using the Green formula, we have

$$\int_{\mathcal{O}} (\lambda - \varpi) f^2 m^2 + \frac{1}{2} \int_{\Sigma} (\gamma f)^2 m^2 \nu \cdot v + \int_{\mathcal{O}} |\nabla_v f|^2 m^2 = \langle \mathfrak{F}, fm^2 \rangle.$$

Fixing $\lambda > \lambda^*$, using the Young inequality

$$\|\mathfrak{F}\|_{L^2 H_m^{-1}} \|f\|_{L^2 H_m^1} \leq \left(\frac{1}{2(\lambda - \lambda^*)} + \frac{1}{2} \right) \|\mathfrak{F}\|_{L^2 H_m^{-1}}^2 + \frac{\lambda - \lambda^*}{2} \|f\|_{L_m^2}^2 + \frac{1}{2} \|\nabla_v f\|_{L_m^2}^2$$

and the boundary condition on the incoming set Σ_- in (7.2.15), we deduce

$$(\lambda - \lambda^*) \int_{\mathcal{O}} f^2 m^2 + \int_{\Sigma_+} (\gamma_+ f)^2 d\xi_m + \int_{\mathcal{O}} |\nabla_v f|^2 m^2 \leq \frac{1 + \lambda - \lambda^*}{\lambda - \lambda^*} \|\mathfrak{F}\|_{L^2 H_m^{-1}}^2 + \int_{\Sigma_-} \mathfrak{g}^2 d\xi_m. \quad (7.2.18)$$

• Because of the first equation in (7.2.15) and the above estimate, we find

$$\hat{v} \cdot \nabla_x f = \frac{1}{\langle v \rangle} (\mathfrak{F} - \lambda f + \Delta_v f + b \cdot \nabla_v f + cf) \in L^2 H_m^{-1}, \quad (7.2.19)$$

so that $f \in W_2$.

• Multiplying the first equation in (7.2.15) by $f\psi$, $\psi := \nu(x) \cdot \tilde{v} m^2$ where here and below we use the notations $\hat{v} := v/\langle v \rangle$, $\tilde{v} := v/\langle v \rangle^2$, $\langle v \rangle^2 := 1 + |v|^2$, and using the Green formula and one integration by part in the velocity variable, we get

$$\begin{aligned} \frac{1}{2} \int_{\Sigma} (\gamma f)^2 (\nu \cdot \hat{v})^2 m^2 &= \frac{1}{2} \int_{\mathcal{O}} f^2 \hat{v} \cdot D_x \nu_x \hat{v} m^2 - \int_{\mathcal{O}} |\nabla_v f|^2 \psi \\ &+ \int_{\mathcal{O}} f \nabla_v f (b\psi - \nabla_v \psi) + \int_{\mathcal{O}} f^2 \psi (c - \lambda) + \langle \mathfrak{F}, f\psi \rangle. \end{aligned}$$

Observing that

$$|\langle \mathfrak{F}, f\psi \rangle| \leq \|\mathfrak{F}\|_{L^2 H_m^{-1}} \|f\nu(x) \cdot \tilde{v}\|_{L^2 H_m^1} \lesssim \|\mathfrak{F}\|_{L^2 H_m^{-1}} \|f\|_{L^2 H_m^1}$$

and

$$\|f \nabla_v \psi\|_{L^2} \lesssim \|f\|_{L^2 H_m^1}^2,$$

recalling that $b/\langle v \rangle \in L^\infty(\mathcal{O})$ and using the Cauchy-Schwarz inequality, we deduce

$$\|\gamma f\|_{L^2(\Sigma; d\xi_m^2)}^2 \leq C(1 + |\lambda|) \|f\|_{L^2 H_m^1}^2 + C \|\mathfrak{F}\|_{L^2 H_m^{-1}} \|f\|_{L^2 H_m^1}, \quad (7.2.20)$$

for some constant $C = C(b, c, m, \nu)$, with $d\xi_m^2 := (\nu \cdot \hat{v})^2 m^2 dv d\sigma_x$.

• For latter reference, we establish an estimate about the behaviour of the solution near the boundary. We now introduce the following Lions-Perthame [157] type weight function

$$\psi := 2\delta(x)^{1/2}\nu(x) \cdot \tilde{v}, \quad (7.2.21)$$

and we observe that $\psi = 0$ on Σ , $\langle v \rangle \psi \in L^\infty(\mathcal{O})$, $\nabla_v \psi \in L^\infty(\mathcal{O})$ and

$$v \cdot \nabla_x \psi = \frac{1}{\delta(x)^{1/2}}(\hat{v} \cdot \nu(x))^2 + 2\delta(x)^{1/2}\hat{v} \cdot D_x \nu(x)\hat{v}.$$

Multiplying the first equation in (7.2.15) by $f\psi$, we have

$$\frac{1}{2} \int_{\mathcal{O}} v \cdot \nabla_x f^2 \psi - \int_{\mathcal{O}} f \frac{b}{\langle v \rangle} \cdot \nabla_v f \langle v \rangle \psi + \int_{\mathcal{O}} \nabla_v (f\psi) \cdot \nabla_v f + \int_{\mathcal{O}} (\lambda - c) f^2 \psi = \langle F, f\psi \rangle.$$

Using Cauchy-Schwarz and Young inequalities, we deduce

$$\int_{\mathcal{O}} f^2 \frac{(\hat{v} \cdot \nu(x))^2}{\delta(x)^{1/2}} dv dx \leq C(1 + |\lambda|)(\|f\|_{L^2 H^1}^2 + \|F\|_{L^2 H^{-1}}^2), \quad (7.2.22)$$

for some constant $C = C(b, c, n)$.

• We finally state a somehow classical regularity estimate when $\mathcal{F} \in L_m^2(\mathcal{O})$. Taking advantage of the fact that $\mathcal{F} \in L_m^2$ and $f \in L^2 H_m^1$ and localizing the problem by introducing the function $g := f\chi_\varepsilon \in L_x^2 H_v^1(\mathbb{R}^d \times \mathbb{R}^d)$, $\chi_\varepsilon \in C_c^2(\mathcal{O})$, $\mathbf{1}_{\mathcal{O}_\varepsilon} \leq \chi_\varepsilon \leq 1$, $\mathcal{O}_\varepsilon := \{(x, v) \in \mathcal{O}; \delta(x) > \varepsilon, |v| \leq 1/\varepsilon\}$, we have

$$v \cdot \nabla_x g - \Delta_v g + \langle v \rangle^2 g = \mathcal{G} \text{ in } \mathcal{D}'(\mathbb{R}^d \times \mathbb{R}^d),$$

with

$$\mathcal{G} := (\mathcal{F} - \lambda f - cf - b \cdot \nabla_v f)\chi_\varepsilon - 2\nabla_v f \cdot \nabla_v \chi_\varepsilon + \langle v \rangle^2 f\chi_\varepsilon \in L^2(\mathbb{R}^d \times \mathbb{R}^d).$$

From the quantitative Hormander's hypoellipticity estimate of Hérau & Pravda-Starov [122, Proposition 2.1], we then have

$$\|D_x^{2/3} g\|_{L^2} + \|D_v^2 g\|_{L^2} \lesssim \|\mathcal{G}\|_{L^2} + \|g\|_{L^2}.$$

Coming back to the function f and using the previous estimates, we deduce

$$\|D_x^{2/3} f\|_{L^2(\mathcal{O}_\varepsilon)} + \|D_v^2 f\|_{L^2(\mathcal{O}_\varepsilon)} \leq C(\|\mathcal{L}f\|_{L^2(\mathcal{O})} + \|f\|_{L^2(\mathcal{O})}), \quad (7.2.23)$$

for a constant $C = C(\lambda, \varepsilon) > 0$.

Step 2. Existence. We assume $\mathfrak{g} = 0$. A possible way for proving the existence is to use Lions' variant of the Lax-Milgram theorem [154, Chap III, §1] as in [19, 70] and as we proceed now. Defining the bilinear form $\mathcal{E} : L^2 H_m^1(\mathcal{O}) \times C_c^1(\mathcal{O} \cup \Sigma_-) \rightarrow \mathbb{R}$, by

$$\begin{aligned} \mathcal{E}(f, \varphi) &= \int_{\mathcal{O}} (\lambda - \mathcal{L})f\varphi m^2 \\ &:= \int_{\mathcal{O}} (\lambda f - b \cdot \nabla_v f - cf)\varphi m^2 + \nabla_v f \cdot \nabla_v(\varphi m^2) - f(v \cdot \nabla_x \varphi)m^2, \end{aligned}$$

for any $f \in L^2 H_m^1(\mathcal{O})$ and $\varphi \in C_c^1(\mathcal{O})$, we observe that this one is coercive, namely

$$\begin{aligned} \mathcal{E}(\varphi, \varphi) &= \int_{\mathcal{O}} (\lambda - \varpi)\varphi^2 m^2 + \int_{\mathcal{O}} |\nabla_v \varphi|^2 m^2 + \frac{1}{2} \int_{\Sigma_-} (\gamma_- \varphi)^2 d\xi_m \\ &\geq \kappa \|\varphi\|_{L^2 H_m^1}^2, \end{aligned}$$

for any $\varphi \in C_c^1(\mathcal{O} \cup \Sigma_-)$, with $\kappa := \min(\lambda - \lambda^*, 1) > 0$. From the above mentioned Lions' theorem, for any $\mathfrak{F} \in L^2 H_m^{-1}$, there exists $f \in L^2 H_m^1$ such that

$$\mathcal{E}(f, \varphi) = \langle \mathfrak{F}, \varphi m^2 \rangle, \quad \forall \varphi \in C_c^1(\mathcal{O} \cup \Sigma_-). \quad (7.2.24)$$

In particular, f satisfies the first equation in (7.2.15) in the distributional sense $\mathcal{D}'(\mathcal{O})$, and thus from (7.2.19), we deduce that $f \in W_2$. Thanks to the trace Theorem 7.1.1 and the estimate (7.2.20), the function f admits a trace $\gamma f \in L^2(\Sigma; d\xi_m^2)$. Using the Green formula (7.1.14) with $\beta = \text{id} \in \mathcal{B}_1$, we have

$$\iint_{\mathcal{O}} (f(\mathcal{L}^* - \lambda)\varphi + \mathfrak{F}\varphi) \, dv dx = \iint_{\Sigma} \gamma f \varphi \nu(x) \cdot v \, dv d\sigma_x, \quad (7.2.25)$$

for any $\varphi \in \mathcal{D}_0(\bar{\mathcal{O}})$. Particularizing to $\varphi \in \mathcal{D}(\mathcal{O} \cup \Sigma_-)$ and comparing with (7.2.24), we deduce that $\gamma_- f = 0$.

Step 3. Existence. The general case $\mathbf{g} \in L^2(\Sigma_-; d\xi_m)$. When $\mathbf{g} \in C_c^2(\Sigma_-)$, there exists a function $\mathfrak{h} \in C_c^2(\mathcal{O} \cup \Sigma_-)$ such that $\mathfrak{h}|_{\Sigma_-} = \mathbf{g}$ and we consider the source term $G := \mathfrak{F} + (\mathcal{L} - \lambda)\mathfrak{h} \in L^2 H_m^{-1}$ as well as the problem

$$(\lambda - \mathcal{L})g = G \text{ in } \mathcal{O}, \quad \gamma_- g = 0 \text{ on } \Sigma_-.$$

From Step 2, there exists a solution $g \in W_{2,\Sigma}$ to this problem and we set $f := g + \mathfrak{h}$. In such a way that $f \in W_{2,\Sigma}$ and satisfies

$$\begin{aligned} \int_{\mathcal{O}} f(\lambda - \mathcal{L}^*)\varphi &= \int_{\mathcal{O}} g(\lambda - \mathcal{L}^*)\varphi + \int_{\mathcal{O}} \mathfrak{h}(\lambda - \mathcal{L}^*)\varphi \\ &= \int_{\mathcal{O}} G\varphi + \int_{\mathcal{O}} (\lambda - \mathcal{L})\mathfrak{h}\varphi - \int_{\Sigma} \mathfrak{h}|_{\Sigma}\varphi \nu \cdot v, \end{aligned}$$

and thus

$$\int_{\mathcal{O}} f(\lambda - \mathcal{L}^*)\varphi = \int_{\mathcal{O}} \mathfrak{F}\varphi - \int_{\Sigma_-} \mathbf{g}\varphi \nu \cdot v, \quad (7.2.26)$$

for any $\varphi \in C_c^2(\mathcal{O} \cup \Sigma_-)$. Together with (7.2.25), we get that $\gamma_- f = \mathbf{g}$ on Σ_- . In order to deal with the general case $\mathbf{g} \in L^2(\Sigma_-; d\xi_m)$, we introduce a sequence (\mathbf{g}^n) of $C_c^2(\Sigma_-)$ such that $\mathbf{g}^n \rightarrow \mathbf{g}$ in $L^2(\Sigma_-, d\xi_m)$ and we next consider the associated sequence of solutions (f^n) of $W_{2,\Sigma}$ just built above. Using the estimates exhibited in Step 1, we get that (f^n) is a Cauchy sequence in W_2 , so that it converges to a limit $f \in W_{2,\Sigma}$. We may pass to the limit in (7.2.26) written for the sequence (f^n) and deduce that the same equation holds at the limit for f .

Step 4. Uniqueness. Consider two weak solutions $f_i \in W_2$ to the equation (7.2.15) in the sense that

$$\mathcal{E}(f_i, \varphi) = \langle \mathfrak{F}, \varphi m^2 \rangle, \quad \forall \varphi \in C_c^1(\mathcal{O} \cup \Sigma_-).$$

In particular, the difference $f := f_2 - f_1 \in W_2$ satisfies

$$\mathcal{E}(f, \varphi) = 0, \quad \forall \varphi \in C_c^1(\mathcal{O} \cup \Sigma_-),$$

and from the above discussion $\gamma_- f = 0 \in L^2(\Sigma_-; d\xi_m)$. Thanks to the trace Theorem 7.1.1, we deduce that $\gamma f \in L_{\text{loc}}^2(\Sigma; d\xi_m)$ and we may choose $\beta(s) = s^2$ in the Green formula (7.1.14): we get

$$\int_{\mathcal{O}} f^2 \{v \cdot \nabla_x \varphi - \text{div}_v(b\varphi) + \Delta_v \varphi + 2f(c - \lambda)\varphi\} - 2|\nabla_v f|^2 \varphi = \int_{\Sigma_+} (\gamma f)^2 \nu \cdot v \varphi,$$

for any test function $\varphi \in C_c^2(\bar{\mathcal{O}})$. Choosing $\varphi = m^2 \chi_\varrho$, with $\chi_\varrho(v) := \chi(v/\varrho)$, $\chi \in C_c^2(\mathbb{R}^d)$, $\mathbf{1}_{B_1} \leq \chi \leq \mathbf{1}_{B_2}$, we deduce

$$\int_{\mathcal{O}} f^2 m^2 \left\{ (\lambda - \varpi) \chi_\varrho + \frac{1}{2} b \cdot \nabla \chi_\varrho - \frac{\nabla m}{m} \cdot \nabla \chi_\varrho - \Delta \chi_\varrho \right\} \leq 0.$$

Because $f \in L^2 H_m^1$, we may pass to the limit $\varrho \rightarrow \infty$ thanks to the dominated convergence theorem and we obtain

$$\int_{\mathcal{O}} f^2 m^2 (\lambda - \varpi) \leq 0,$$

and thus $f = 0$.

Step 5. Positivity. We assume now that $\mathfrak{F} \geq 0$ and $\mathfrak{g} \geq 0$. We proceed similarly as in the previous step by considering $\beta(s) = s_-^2$, $\varphi = m^2 \chi_M$. Letting $M \rightarrow \infty$, we deduce

$$\int_{\mathcal{O}} f_-^2 m^2 (\lambda - \varpi) \leq 0,$$

and thus $f_- = 0$. □

Summing up, gathering the above estimates (7.2.18), (7.2.19), (7.2.20), (7.2.22), (7.2.23), we see that there exists a constant $C > 0$ such that any function $f \in D(\mathcal{L})$ satisfies

$$\begin{aligned} \|f\|_{L^2 H_m^1} + \|\hat{v} \cdot \nabla_x f\|_{L^2 H_m^{-1}} + \|f \frac{\hat{v} \cdot \nu}{\delta^{1/4}}\|_{L^2} \\ + \|\gamma f\|_{L^2(\Sigma; d\xi_m^2)} + \|\gamma_+ f\|_{L^2(\Sigma; d\xi_m)} \leq C(\|f\|_{L^2} + \|\mathcal{L}f\|_{L^2}) \end{aligned} \quad (7.2.27)$$

and for any $\varepsilon > 0$ there exists a constant C_ε such that any function $f \in D(\mathcal{L})$ satisfies

$$\|D_x^{2/3} f\|_{L^2(\mathcal{O}_\varepsilon)} + \|D_v^2 f\|_{L^2(\mathcal{O}_\varepsilon)} \leq C_\varepsilon(\|f\|_{L^2} + \|\mathcal{L}f\|_{L^2}).$$

7.3 Well-posedness problem with reflection condition at the boundary

We consider now the well posedness problem associated to the stationary equation

$$(\lambda - \mathcal{L})f = \mathfrak{F} \quad \text{in } \mathcal{O}, \quad \gamma_- f = \mathcal{R}\gamma_+ f \quad \text{on } \Sigma_-, \quad (7.3.28)$$

for a given datum $\mathfrak{F} : \mathcal{O} \rightarrow \mathbb{R}$, where the kinetic Fokker-Planck operator \mathcal{L} is still defined by (7.0.1) and the reflexion operator \mathcal{R} is described in (7.0.6), (7.0.7), (7.0.8).

Theorem 7.3.4. *Let us fix a vector field $b \in H_{\text{loc}}^1(\bar{\mathcal{O}})$, a function $c \in L^\infty(\mathcal{O})$ which satisfy the assumptions of Theorem 7.2.3 with a given weight function $m : \mathbb{R}^d \rightarrow [1, \infty)$ for the pure specular reflection case $\alpha \equiv 0$ and with the weight function $m := \mathcal{M}^{-1/2}$ when $\alpha \neq 0$, where \mathcal{M} is the Gaussian function (7.0.9) or a more general equilibrium satisfying (7.0.10). In that last case, we furthermore assume one of the two following conditions*

(i) $1 - \zeta + \alpha^2/2 \geq \delta_* > 0$, and we observe that $L^2(\Sigma; d\xi_m) \subset L^1(\Sigma; d\xi)$,

(ii) $\langle v \rangle^2 \mathcal{M} \in L^1$, and we observe that $L^2(\Sigma; d\xi_m^2) \subset L^1(\Sigma; d\xi)$,

where we recall that we have defined $d\xi_m := m^2 |\nu(x) \cdot v| dv d\sigma_x$ and $d\xi_m^2 := m^2 (\nu(x) \cdot \hat{v})^2 dv d\sigma_x$.

For any $\mathfrak{F} \in L^2 H_m^{-1}$ and $\lambda > \lambda^*$, there exists at least one solution $f \in W_2$ to the Dirichlet problem (7.3.28). Assuming furthermore that

$$\lambda^{**} := \text{ess sup}(c - \text{div}b) < \infty, \quad (7.3.29)$$

and $\lambda > \lambda^{**}$, the solution f is unique and $f \geq 0$ if $\mathfrak{F} \geq 0$.

It is worth emphasizing that the assumptions of Theorem 7.3.4 hold when $b = v$ and $m := \mathcal{M}^{-1/2}$. We also emphasize on the fact that the additional assumptions (i) or (ii) are made in order to prove the uniqueness of the solution during the proof.

Proof of Theorem 7.3.4. We split the proof into four steps.

Step 1. A priori estimates. We multiply the first equation in (7.3.28) by fm^2 . As in Step 1 of the proof of Theorem 7.2.3, we get

$$\int_{\mathcal{O}} (\lambda - \varpi) f^2 m^2 + \frac{1}{2} \int_{\Sigma} (\gamma f)^2 m^2 \nu \cdot v + \int_{\mathcal{O}} |\nabla_v f|^2 m^2 = \langle \mathfrak{F}, fm^2 \rangle.$$

Using for instance [28, Lemma 3.1], we have

$$\int_{\Sigma} (\gamma f)^2 m^2 \nu \cdot v \geq \int_{\Sigma_+} [(1 - \zeta)(\gamma_+ f)^2 + \alpha(\mathcal{D}^\perp \gamma_+ f)^2] d\xi_m =: \mathcal{E}_{\zeta, \alpha}(\gamma_+ f) \geq 0, \quad (7.3.30)$$

with $\mathcal{D}^\perp g := g - \mathcal{D}g$. Using that the contribution of the boundary is nonnegative in the first estimate, we first deduce

$$(\lambda - \lambda^*) \|f\|_{L_m^2}^2 + \|\nabla f\|_{L_m^2}^2 \leq \|\mathfrak{F}\|_{L^2 H_m^{-1}} \|f\|_{L^2 H_m^1},$$

for $\lambda > \lambda^*$, so that

$$\min(\lambda - \lambda^*, 1) \|f\|_{L^2 H_m^1} \leq \|\mathfrak{F}\|_{L^2 H_m^{-1}}.$$

From the three above estimates together, for $\lambda > \lambda^*$, we obtain

$$\int_{\mathcal{O}} (\lambda - \varpi)_+ f^2 m^2 + \int_{\mathcal{O}} |\nabla_v f|^2 m^2 + \frac{1}{2} \mathcal{E}_{\zeta, \alpha}(\gamma_+ f) \leq \frac{1}{\min(\lambda - \lambda^*, 1)} \|\mathfrak{F}\|_{L^2 H_m^{-1}}^2. \quad (7.3.31)$$

There is no difficulty for also getting the pieces of information (7.2.19), (7.2.20), (7.2.22) and (7.2.23), so that in particular $f \in W_2$. It is worth emphasizing here that when $\langle v \rangle^2 \mathcal{M} \in L^1$, we have $L^2(d\xi_m^2) \subset L^1(\Sigma; d\xi)$ by using the Cauchy-Schwarz and (7.2.20), so that in particular the boundary condition is well defined.

Let us show now how the last conclusion also holds under condition (i) in the statement of the Theorem. We then assume $\vartheta = 1$ in (7.0.10) and we show how to establish an additional a priori estimate. We indeed know from (7.2.20) that

$$\int_{\Sigma_-} (\alpha \mathcal{D}(\gamma_+ f))^2 (\nu \cdot \hat{v})^2 m^2 dv d\sigma_x \leq \int_{\Sigma} (\gamma f)^2 (\nu \cdot \hat{v})^2 m^2 dv d\sigma_x \leq C_\lambda \|\mathfrak{F}\|_{L^2 H_m^{-1}}^2,$$

and similarly as in [9] or [172, proof of Lemma 2.2] that

$$1 = \int_{\Sigma_-^x} |\nu(x) \cdot v| \mathcal{M} dv = C \int_{\Sigma_-^x} (\nu(x) \cdot \hat{v})^2 \mathcal{M} dv, \quad \forall x \in \partial\Omega,$$

for some constant $C \in (0, \infty)$, so that

$$\int_{\Sigma_-} (\alpha \mathcal{D}(\gamma_+ f))^2 d\xi_m = C \int_{\Sigma_-} (\alpha \mathcal{D}(\gamma_+ f))^2 (\nu \cdot \hat{v})^2 m^2 \leq C C_\lambda \|\mathfrak{F}\|_{L^2 H_m^{-1}}^2. \quad (7.3.32)$$

Summing up (7.3.31) and (7.3.32), and using that

$$(\gamma_+ f)^2 \leq 2(\mathcal{D}^\perp \gamma_+ f)^2 + 2(\mathcal{D} \gamma_+ f)^2,$$

we deduce that

$$\int_{\Sigma_+} [1 - \zeta + \alpha^2/2] (\gamma_+ f)^2 d\xi_m \leq C_\lambda \|\mathfrak{F}\|_{L^2 H_m^{-1}}^2. \quad (7.3.33)$$

Defining

$$f \in W_{2, \mathcal{R}} := \{g \in W_2; \gamma_- g = \mathcal{R} \gamma_+ g\},$$

we see that $W_{2, \mathcal{R}} = W_{2, \Sigma}$ if $1 - \zeta + \alpha^2/2 \geq \delta_* > 0$, but it is worth emphasizing that we may have $W_{2, \mathcal{R}} \neq W_{2, \Sigma}$ in the general case.

Step 2. Existence when $\mathfrak{F} \geq 0$. With the help of Theorem 7.2.3, we define $f_0 = 0$ and, recursively for any $n \geq 1$, we define $f_n \in W_{2,\Sigma}$ as the solution of

$$(\lambda - \mathcal{L})f_n = \mathfrak{F} \text{ in } \mathcal{O}, \quad \gamma_- f_n = \mathcal{R}\gamma_+ f_{n-1} \text{ on } \Sigma_-. \quad (7.3.34)$$

It is worth emphasizing here that $\gamma_+ f_{n-1} \in L^2(\Sigma_+; d\xi_m)$ implies $\mathcal{R}(\gamma_+ f_{n-1}) \in L^2(\Sigma_-; d\xi_m)$ because of (7.0.5). We also notice that $f_n \geq 0$ because $\mathfrak{F} \geq 0$. By linearity

$$(\lambda - \mathcal{L})(f_{n+1} - f_n) = 0 \text{ in } \mathcal{O}, \quad \gamma_-(f_{n+1} - f_n) = \mathcal{R}\gamma_+(f_n - f_{n-1}) \text{ on } \Sigma_-,$$

and we thus show recursively that $f_{n+1} - f_n \geq 0$. In other words, (f_n) is an increasing sequence and thus also is (γf_n) . From (7.3.30), we have

$$\begin{aligned} \int_{\Sigma} (\gamma f_n)^2 d\xi_m &= \int_{\Sigma_+} (\gamma_+ f_n)^2 d\xi_m - \int_{\Sigma_-} (\mathcal{R}\gamma_- f_{n-1})^2 d\xi_m \\ &\geq \int_{\Sigma_+} (\gamma_+ f_n)^2 d\xi_m - \int_{\Sigma_-} (\mathcal{R}\gamma_- f_n)^2 d\xi_m \geq \mathcal{E}_{\zeta, \alpha}(\gamma_+ f_n), \end{aligned}$$

so that the estimate (7.3.31) holds true for f_n (instead of f) uniformly in $n \geq 1$. From the monotonous convergence theorem, there exists $f \in L^2 H_m^1$ satisfying (7.3.31), (7.3.33), (7.2.20) and such that $f_n \nearrow f$ a.e. Thanks to Proposition 7.1.2, we have $\gamma f_n \nearrow \gamma f$ a.e. on Σ , from what we deduce that $\mathcal{R}\gamma_+ f_n \rightarrow \mathcal{R}\gamma_+ f$ in $L^2(\Sigma_-; d\xi_m)$ thanks to the monotonous convergence theorem. As a consequence, we may pass to the limit in the weak formulation of (7.3.34), and we get that f is a solution of (7.3.28). We may also pass to the liminf in the estimate (7.3.31) written for f_n , and we thus deduce that the same estimate holds for f .

Step 3. Existence when $\mathfrak{F} \in L^2 H_m^{-1}$. When $\mathfrak{F} \in L_m^2$, we may introduce the splitting $\mathfrak{F} = \mathfrak{F}_+ - \mathfrak{F}_-$, just use the previous step for \mathfrak{F}_{\pm} and conclude by linearity of the equation. When $\mathfrak{F} \notin L_m^2$, we proceed similarly as in [172] and in the following way. We first assume $\zeta \leq \zeta^* \in [0, 1)$ and we consider the mapping $\Psi : W_{2,\Sigma} \rightarrow W_{2,\Sigma}$, $g \mapsto f = \Psi(g)$, where f is the solution to the stationary problem

$$\begin{cases} (\lambda - \mathcal{L})f = \mathfrak{F} & \text{in } \mathcal{O} \\ \gamma_- f = \mathcal{R}\gamma_+ g & \text{on } \Sigma_-. \end{cases} \quad (7.3.35)$$

The space $W_{2,\Sigma}$ is endowed with the norm $\|\cdot\|_{W_{2,\Sigma}}$ defined by

$$\|g\|_{W_{2,\Sigma}}^2 = \|g\|_{L_m^2}^2 + \|\nabla_v g\|_{L_m^2}^2 + \|\gamma_+ g\|_{L_m^2(d\xi_1)}^2.$$

From (7.2.18) and the estimate $\|\mathcal{R}g\|_{L^2(\Sigma_-; d\xi_m)} \leq \zeta^* \|g\|_{L^2(\Sigma_+; d\xi_m)}$ what we obtain by repeating the proof of (7.0.5), we deduce

$$\begin{aligned} \frac{1}{C_\lambda} \|f\|_{L^2 H_m^1}^2 + \|\gamma_+ f\|_{L^2(\Sigma_+; d\xi_m)}^2 &\leq C_\lambda \|\mathfrak{F}\|_{L^2 H_m^{-1}}^2 + \|\mathcal{R}\gamma_+ g\|_{L^2(\Sigma_-; d\xi_m)}^2 \\ &\leq C_\lambda \|\mathfrak{F}\|_{L^2 H_m^{-1}}^2 + \zeta^* \|\gamma_+ g\|_{L^2(\Sigma_+; d\xi_m)}^2, \end{aligned}$$

for some constant $C_\lambda > 0$. By linearity of (7.3.35), we deduce that for two functions $g_1, g_2 \in W_{2,\Sigma}$, and denoting $f_i := \Psi(g_i)$, we have

$$\frac{1}{C_\lambda} \|f_2 - f_1\|_{L^2 H_m^1}^2 + \|\gamma_+ f_2 - \gamma_+ f_1\|_{L^2(\Sigma_+; d\xi_m)}^2 \leq \zeta^* \|\gamma_+ g_2 - \gamma_+ g_1\|_{L^2(\Sigma_+; d\xi_m)}^2,$$

so that Ψ is a contraction in $W_{2,\Sigma}$. By the Banach fixed point theorem, we deduce that there exists a solution $f \in W_{2,\Sigma}$ to the equation (7.3.28) in that case. Finally, in order to deal with the case $\zeta^* = 1$, we consider a sequence (ζ_n^*) of $[0, 1)$ such that $\zeta_n^* \nearrow 1$ and the associated sequence

(f_n) of solutions in $W_{2,\Sigma}$ associated to the equation (7.3.28) with the modified reflection kernel $\mathcal{R}_n g := \zeta_n^* \mathcal{R}g$. From (7.3.31) and (7.2.20), that sequence satisfies

$$\|f_n\|_{L^2 H_m^1}^2 + \|\gamma f_n\|_{L^2(\Sigma; d\xi_m^2)}^2 + \mathcal{E}_{1,\alpha}(\gamma_+ f_n) \leq C_\lambda \|\mathfrak{F}\|_{L^2 H_m^{-1}}^2.$$

When $\alpha \neq 0$, the above estimate or (7.3.33) also implies that $(\gamma_+ f_n)$ belongs to a weakly compact set of $L^1(\Sigma_+; d\xi)$. As a consequence, there exist $f \in W_2$ and $\bar{\gamma}_\pm$ two functions defined on Σ_\pm such that, up to the extraction of a subsequence,

$$\begin{aligned} f_n &\rightharpoonup f \quad L^2 H_m^1, & \gamma_\pm f_n &\rightharpoonup \bar{\gamma}_\pm \quad L^2(\Sigma_\pm; d\xi_m^2), \\ \gamma_+ f_n &\rightharpoonup \bar{\gamma}_+ \quad L^1(\Sigma_+; d\xi), & \mathcal{R}\gamma_+ f_n &\rightharpoonup \mathcal{R}\bar{\gamma}_+ \quad L^1(\Sigma_-; d\xi), \end{aligned}$$

where we have used (7.0.11) for the last convergence. Using Proposition 7.1.2, we may thus pass to the limit in the equation (7.3.28) satisfied by f_n with modified reflection kernel \mathcal{R}_n and we get that f is a solution of (7.3.28). In the pure specular reflection case $\alpha \equiv 0$, only the first line of convergences holds, but that it is enough in order to pass to the limit in the equations (we refer to [173, 172] for similar arguments).

Step 4. Other properties. We further assume $\lambda > \lambda^{**}$. We proceed similarly as in [164]. Consider two weak solutions $f_i \in W_2$ to the equation (7.3.28). In particular, the difference $f := f_2 - f_1 \in W_2$ satisfies

$$(\lambda - \mathcal{L})f = 0 \quad \text{in } \mathcal{O}, \quad \gamma_- f = \mathcal{R}\gamma_+ f \quad \text{on } \Sigma_-.$$

Using the Green renormalized formula (7.1.14), we have

$$0 = \int_{\mathcal{O}} \beta'(f)(\lambda - c)f\varphi + \beta''(f)|\nabla f|^2\varphi + \beta(f)(\operatorname{div}_v(b\varphi) - v \cdot \nabla_x \varphi - \Delta_v \varphi) + \int_{\Sigma} \beta(\gamma f)\nu \cdot v\varphi.$$

for any $\beta \in C^2(\mathbb{R})$, $\beta' \in C_b^1(\mathbb{R})$ and any test function $\varphi \in C_c^2(\bar{\mathcal{O}})$. We choose $\varphi = \varphi(v) \geq 0$, $\beta \geq 0$ and $\beta'' \geq 0$, so that

$$0 \geq \int_{\mathcal{O}} \beta'(f)(\lambda - c)f\varphi + \beta(f)(\operatorname{div}_v(b\varphi) - \Delta_v \varphi) + \int_{\Sigma} \beta(\gamma f)\nu \cdot v\varphi.$$

By an approximation argument, we may now take $\beta(s) = |s|$, and we get

$$0 \geq \int_{\mathcal{O}} |f| \{(\lambda - c)\varphi + (\operatorname{div}_v(b\varphi) - \Delta_v \varphi)\} + \int_{\Sigma} |\gamma f| \nu \cdot v\varphi.$$

We observe that in any cases we have $f \in L_m^2(\mathcal{O}) \subset L^1(\mathcal{O})$ and $\gamma f \in L^1(\Sigma; d\xi)$. By an approximation argument, we may now take $\varphi = 1$ and using the L^1 estimate (7.0.11) on \mathcal{R} (with $\zeta^* = 1$), we get

$$\begin{aligned} 0 &\geq \int_{\Sigma_-} |\mathcal{R}\gamma_+ f| |\nu \cdot v| - \int_{\Sigma_+} |\gamma_+ f| |\nu \cdot v| \\ &\geq \int_{\mathcal{O}} |f| \{\lambda - c + \operatorname{div}_v b\} \geq (\lambda - \lambda_{**}) \int_{\mathcal{O}} |f|. \end{aligned}$$

We deduce that $f = 0$. The proof of the positivity property follows the same arguments but choosing $\beta(s) = s_-$. \square

For latter reference, we state the counterpart of the preceding result for the kinetic Fokker-Planck evolution equation.

Theorem 7.3.5. *Let us make the same assumptions as in Theorem 7.3.4. For any $f_0 \in L_m^2$, there exists a unique solution $f \in C([0, T]; L_m^2) \cap L^2(0, T; H_m^1)$ for any $T > 0$ to the kinetic Fokker-Planck evolution equation*

$$\begin{cases} \partial_t f = \mathcal{L}f & \text{in } (0, \infty) \times \mathcal{O} \\ \gamma_- f = \mathcal{R}\gamma_+ f & \text{on } (0, \infty) \times \Sigma_-, \end{cases} \quad (7.3.36)$$

with \mathcal{L} defined in (7.0.1) and \mathcal{R} defined in (7.0.6).

The proof of Theorem 7.3.5 is skipped since it is a mere adaptation of the proof of Theorem 7.3.4. We refer to [248, Cor. 2.7, Lem. 2.8 and Cor. 2.8] where similar well-posedness results are established (see also [172] for the existence part).

7.4 The first eigenvalue problem in a domain with reflection at the boundary

We consider now the first eigenvalue problem for the kinetic Fokker-Planck operator (7.0.1) in a domain with reflection at the boundary, namely

$$\begin{cases} \lambda f + v \cdot \nabla_x f - \Delta_v f - b \cdot \nabla_v f - cf = 0 & \text{in } \mathcal{O} \\ \gamma_- f = \mathcal{R}\gamma_+ f & \text{on } \Sigma_-, \end{cases} \quad (7.4.37)$$

and the associated dual problem. In this section, we assume that b and c satisfy the assumptions of Theorem 7.2.3 with the weight function $m := \mathcal{M}^{-1/2}$ when $\alpha \neq 0$ and for a given weight function $m : \mathbb{R}^d \rightarrow [1, \infty)$ when $\alpha \equiv 0$ and \mathcal{R} is given by (7.0.6). We additionally assume that

$$\liminf_{|(x,v)| \rightarrow \infty} \varpi(x, v) = -\infty, \quad (7.4.38)$$

where we recall that ϖ is defined in (7.2.17). When \mathcal{M} is the Gaussian function, we find

$$\varpi = c + \frac{|v|^2 + d}{2} - \frac{1}{2} \operatorname{div} b - b \cdot v,$$

so that (7.4.38) holds when b is typically a bounded perturbation of the vector field $b_0(v) = \vartheta_0 v$, $\vartheta_0 > 1/2$, and more precisely

$$\operatorname{div}_v b \in L^\infty(\mathcal{O}) \quad \text{and} \quad \inf_{x \in \Omega} \liminf_{|v| \rightarrow \infty} (b \cdot v \langle v \rangle^{-2}) \geq \vartheta_0 > 1/2.$$

The above condition is quite technical but can be seen as a compatibility condition between the thermalization due to the boundary and to the Fokker-Planck collisional operator. We are then able to work in the functional space $X := L_m^2(\mathcal{O})$.

Theorem 7.4.6. *Under the above conditions, the first eigentriplet problem associated to (7.0.1) has a unique solution $(\lambda_1, f_1, \phi_1) \in \mathbb{R} \times X \times X'$ with $f_1 > 0$ and $\phi_1 > 0$.*

The proof of Theorem 7.4.6 follows from Theorem 2.1.20, Theorem 3.3.11 and Theorem 4.4.14 as a consequence of conditions **(H1)**–**(H5)**. We prove now that each of these conditions is satisfied. Theorem 7.4.6 generalizes [152, Theorem 2.12] where the same problem is tackled for the zero inflow condition ($\alpha = \beta = 0$) with $b = v$ and $c = 0$ by using the classical Krein-Rutman theorem [142] in the space $X = C_b(\bar{\mathcal{O}})$. We also refer to [116, Theorem 6.8] for a variant and somehow generalisation of [152].

Condition (H1). From Theorem 7.3.4, the operator \mathcal{L} satisfies **(H1)** with

$$\kappa_1 := \max(\lambda^*, \lambda^{**}),$$

with λ^* defined by (7.2.17) and λ^{**} defined by (7.3.29). For later reference, let us state more precisely, the available estimates for f . On the one hand, repeating the proof of Step 1 in the proof of Theorem 7.3.4, we establish that for any $\lambda > \kappa_1$ and $\mathfrak{F} \in L_m^2$, the solution $f \in W_2$ to the Dirichlet problem (7.3.28) satisfies

$$\int_{\mathcal{O}} (\lambda - \varpi)_+ f^2 m^2 + \int_{\mathcal{O}} |\nabla_v f|^2 m^2 + \frac{1}{2} \mathcal{E}_{\zeta, \alpha}(\gamma + f) \leq \frac{1}{\lambda - \lambda^*} \|\mathfrak{F}\|_{L_m^2}^2. \quad (7.4.39)$$

On the other hand, adapting the proof of (7.2.22), we straightforwardly obtain

$$\int_{\mathcal{O}} f^2 \frac{(\hat{v} \cdot \nu(x))^2}{\delta(x)^{1/2}} dv dx \leq C \int \mathfrak{F}^2 m^2, \quad (7.4.40)$$

for some constant $C = C(b, c, \nu, \lambda)$. For $\varepsilon_x, \varepsilon_v, \varrho > 0$, let us now define

$$\mathcal{U} := \{(x, v) \in \mathcal{O}; d(x, \partial\Omega) > \varepsilon_x, |v| < \varrho\}, \quad (7.4.41)$$

and compute

$$\int_{\mathcal{U}^c} f^2 m^2 \leq \int f^2 m^2 \mathbf{1}_{|v| \geq \varrho} + \int f^2 m^2 \mathbf{1}_{A_x} + \int f^2 m^2 \mathbf{1}_B,$$

with

$$A_x := \{v \in B_{\varrho}, (\hat{v} \cdot \nu(x))^2 \leq \varepsilon_v^2\}, \quad B := \{(x, v); |v| \leq \varrho, (\hat{v} \cdot n)^2 \geq \varepsilon_v^2, d(x, \partial\Omega) \leq \varepsilon_x\}.$$

For the second term, we have

$$\begin{aligned} \int f^2 m^2 \mathbf{1}_{A_x} &\leq \int |A_x|^{2/r'} \|f(x, \cdot)\|_{L_v^r}^2 dx \\ &\lesssim (\varrho^{d-1} \varepsilon_v)^{2/r'} \|f\|_{L^2 H_m^1}^2, \end{aligned}$$

where we have used the Holder inequality with $r \in (1, 2^*/2)$ in the first line and the Sobolev inequality in the second line. For the third term, we have

$$\int f^2 m^2 \mathbf{1}_B \leq m^2(\varrho) \frac{\varepsilon_x^{1/2}}{\varepsilon_v^2} \int_{\mathcal{O}} f_n^2 \frac{(\hat{v} \cdot \nu(x))^2}{\delta(x)^{1/2}}.$$

Gathering these last estimates with (7.4.39) and (7.4.40), we have established that the solution f to equation (7.3.28) furthermore satisfies

$$\int_{\mathcal{U}^c} f^2 m^2 \leq C \left(\frac{1}{\langle \varrho \rangle^2} + \varrho^{d-1} \varepsilon_v + m^2(\varrho) \frac{\varepsilon_x^{1/2}}{\varepsilon_v^2} \right) \int \mathfrak{F}^2 m^2, \quad (7.4.42)$$

for a constant $C = C(b, c, \Omega, \lambda)$ and for any $\varepsilon_x, \varepsilon_v, \varrho > 0$.

The strong maximum principle. Let us now consider a function $0 \leq f \in W_2 \setminus \{0\}$ which satisfies the Dirichlet problem (7.3.28) associated to $\lambda > \kappa_1$ and a source term $0 \leq \mathfrak{F} \in L_m^2 \cap L^\infty$. In order to simplify the discussion, we assume that the normalization $\|f\|_{L_m^2} = 1$ holds. For proving the strong maximum principle, we briefly explain how we may adapt the arguments we have presented for the diffusive equation in Chapter 6 by taking in particular advantage of the above established estimates, the regularity results established in [106, 115] and some spreading positivity results we learnt in [241, Corollary A.20]. We proceed in three steps.

Step 1. On the one hand, from (7.4.42), we may choose conveniently $\varrho^{-1}, \varepsilon_v, \varepsilon_x > 0$ small enough in such a way that

$$\int_{\mathcal{U}^c} f^2 m^2 \leq \frac{1}{2} \|f\|_{L_m^2}^2,$$

where \mathcal{U} is defined by (7.4.41). Because of the normalization condition, we have

$$\int_{\mathcal{U}} f^2 m^2 = \|f\|_{L_m^2}^2 \quad (7.4.43)$$

and consequently $f(x_0, v_0)^2 \geq \delta_0^2 := \|f\|_{L_m^2}^2 (2\|\mathbf{1}_{\mathcal{U}}\|_{L_m^2}^2)^{-1}$ for at least one point $(x_0, v_0) \in \mathcal{U}$.

Step 2. On the other hand, let us recall some integrability and regularity results established in [106] for a solution g to the kinetic Fokker-Planck evolution equation

$$\partial_t g + v \cdot \nabla_x g = \Delta_v g + B \cdot \nabla_v g + s \quad \text{in } \mathcal{V},$$

or a sub-solution

$$\partial_t g + v \cdot \nabla_x g \leq \Delta_v g + B \cdot \nabla_v g + s \quad \text{in } \mathcal{V},$$

for some bounded set $\mathcal{V} \subset (0, T) \times \mathcal{O}$, $s \in L^2(\mathcal{V})$ and $B \in L^\infty(\mathcal{V})$. For that purpose, given some (t^*, x^*, v^*) , we define

$$Q_r := \{(t, x, v); t \in (t^* - r^2, t^*], |x - x^* - (t - t^*)v^*| < r^3, |v - v^*| < r\}.$$

We claim then that there exist $2 < p < q < \infty$, $\alpha \in (0, 1)$ and for any $0 < r_1 < r_0$ there exists C such that

$$\|g\|_{L^p(Q_{r_1})} \leq C (\|g\|_{L^2(Q_{r_0})} + \|s\|_{L^2(Q_{r_0})}) \quad (7.4.44)$$

for any nonnegative subsolution g on Q_{r_0} from [106, Theorem 6],

$$\|g\|_{L^\infty(Q_{r_1})} \leq C (\|g\|_{L^2(Q_{r_0})} + \|s\|_{L^q(Q_{r_0})}) \quad (7.4.45)$$

for any nonnegative subsolution g on Q_{r_0} from [106, Theorem 12] and

$$\|g\|_{C^\alpha(Q_{r_1})} \leq C (\|g\|_{L^2(Q_{r_0})} + \|s\|_{L^\infty(Q_{r_0})}) \quad (7.4.46)$$

for any solution g on Q_{r_0} from [106, Theorem 3]. As a consequence of (7.4.44) and a classical covering argument, for any bounded set $\mathcal{U} \subset \bar{\mathcal{U}} \subset \mathcal{O}$, there exist $C_0 = C_0(\mathcal{U})$ and $C_1 = C_1(\mathcal{U}, \lambda)$ such that

$$\|f\|_{L^p(\mathcal{U})} \leq C_0 (\|f\|_{L^2(\mathcal{O})} + \|\mathfrak{F} + cf - \lambda f\|_{L^2(\mathcal{O})}) \leq C_1 (\|f\|_{L^2(\mathcal{O})} + \|\mathfrak{F}\|_{L^2(\mathcal{O})}).$$

Observing that for $\varrho = p/2 > 1$, we have

$$v \cdot \nabla_x f^\varrho - \Delta_v f^\varrho - b \cdot \nabla_v f^\varrho + \varrho f^{\varrho-1} (\lambda f - cf - \mathfrak{F}) = -4 \frac{(\varrho-1)}{\varrho} |\nabla(f^{\varrho/2})|^2 \leq 0,$$

so that f^ϱ is a weak sub-solution to the kinetic Fokker-Planck equation, we may repeat the argument and obtain in that way that $f \in L^{p_k}(\mathcal{U})$ for any $k \geq 1$, with $p_k := \varrho^k 2$. Now, choosing k such that $p_k \geq q$ and using (7.4.45) (as well as again a classical covering argument), we get

$$\|f\|_{L^\infty(\mathcal{U})} \lesssim \|f\|_{L^2(\mathcal{O})} + \|\mathfrak{F} + cf - \lambda f\|_{L^q(\mathcal{O})} \lesssim \|f\|_{L^2(\mathcal{O})} + \|\mathfrak{F}\|_{L^q(\mathcal{O})}.$$

Using finally (7.4.46), we deduce that there exists a constant $C = C(\mathcal{U}, \lambda)$ such that

$$\|f\|_{C^\alpha(\mathcal{U})} \lesssim \|f\|_{L^2(\mathcal{O})} + \|\mathfrak{F}\|_{L^\infty(\mathcal{O})}.$$

Together with the conclusion of the first step, we deduce that there exists a constructive constant $r_0 > 0$ such that $f \geq \delta_0 \mathbf{1}_{B((x_0, v_0), r_0)}$.

Step 3. From [241, Corollary A.20], we deduce that for any bounded set $\mathcal{U} \subset \bar{\mathcal{U}} \subset \mathcal{O}$, there exists a constructive constant $\delta = \delta(\delta_0, r_0, \mathcal{U}) > 0$ such that

$$f(x, v) \geq \delta \quad \text{for any } (x, v) \in \mathcal{U},$$

where it is worth emphasizing that the hypothesis $b, c \in C(\mathcal{O})$ made in [241, Corollary A.20] is not really necessary and can be replaced by $b, c \in L^\infty(\mathcal{U})$. Because \mathcal{U} may be chosen arbitrary, we have established that $f > 0$ on \mathcal{O} and the strong maximum principle.

Condition (H2). For a given function $0 \leq h_0 \in C_c^2(\mathcal{O})$ normalized by $\|h_0\|_{L_m^2} = 1$, we define $f_0 \in D(\mathcal{L})$ as the solution to

$$(\kappa_1 - \mathcal{L})f_0 = h_0 \quad \text{in } \mathcal{O}, \quad \gamma_- f_0 = \mathcal{R}\gamma_+ f_0 \quad \text{on } \Sigma_-.$$

Taking advantage of the fact that h_0 has compact support, we compute

$$1 = \int_{\mathcal{O}} h_0^2 m^2 = \int_{\mathcal{O}} (\kappa_1 - \mathcal{L})f_0 h_0 m^2 = \int_{\mathcal{O}} f_0 (\kappa_1 - \mathcal{L}^*)(h_0 m^2) \leq C_1 \|f_0\|_{L_m^2},$$

with $C_1 := \|m^{-1}(\kappa_1 - \mathcal{L}^*)(h_0 m^2)\|_{L^2}$. On the other hand, from (7.2.27), we have

$$\|f_0\|_{L^2 H_m^1} + \|f_0\|_{\delta^{1/4}} \leq C_2, \quad (7.4.47)$$

for a constant C_2 only depending on $\|h_0\|_{L_m^2}$, κ_1 and the constant C which appears in (7.2.27). Arguing as in (7.4.43), we deduce that

$$\int_{\mathcal{U}} f_0^2 m^2 \geq (2C_1)^{-1}, \quad \text{supp } h_0 \subset \mathcal{U}, \quad (7.4.48)$$

with $\mathcal{U} = \mathcal{U}_\varrho$ defined in (7.4.41) and $\varrho > 0$ small enough (chosen constructively from C_2 and C_1). From the above constructive strong maximum principle, we deduce that $f_0 \geq \varepsilon \mathbf{1}_{\mathcal{U}} \geq 1/C_0 h_0$ for some $\varepsilon, C_0 > 0$. We conclude as in the Second constructive argument for (H2) in Section 6.1. Coming back indeed to the equation, we have

$$\mathcal{L}f_0 = \kappa_1 f_0 - h_0 \geq \kappa_1 f_0 - \|h_0\|_{L^\infty} \mathbf{1}_{\mathcal{U}} \geq (\kappa_1 - \|h_0\|_{L^\infty} C_0) f_0,$$

so that (H2) holds with $\kappa_0 := \kappa_1 - \|h_0\|_{L^\infty} C_0$ from Lemma 2.1.4-(ii).

Condition (H3). Let us fix $\kappa < \kappa_0$ arbitrary. We define $\mathcal{B}f := \mathcal{L}f - n\chi_R(v)f$ for any $f \in W_{2, \mathcal{R}}$, with $\chi_R \in \mathcal{D}(\mathbb{R}^d)$ such that $\mathbf{1}_{B_R} \leq \chi_R \leq \mathbf{1}_{B_{2R}}$ and for some given $n, R \geq 0$ to be specified below. We observe that, at least formally,

$$\int f m^2 (\mathcal{B} - \kappa) f = \int_{\mathcal{O}} (\varpi - \kappa - n\chi_R) f^2 m^2 - \frac{1}{2} \int_{\Sigma} (\gamma f)^2 m^2 \nu \cdot v - \int_{\mathcal{O}} |\nabla_v f|^2 m^2.$$

Thanks to (7.4.38), there exists a constant $R > 0$ such that

$$\sup_{v \in \mathbb{R}^d \setminus B_R} \varpi \leq \kappa.$$

Choosing $n := \sup \varpi_+ - \kappa$, we deduce that $\varpi - \kappa - n\chi_R \leq 0$. On the other hand, because of (7.3.30), the contribution of the boundary term in the above identity is non positive. We thus deduce that $(\mathcal{B} - \kappa)$ is dissipative in L_m^2 . We now establish that the associated operator \mathcal{B} has compact resolvent. For $\mathfrak{F} \in L_m^2$, we consider $f \in L_m^2$ the solution to

$$-\mathcal{B}f = \mathfrak{F} \quad \text{in } \mathcal{O}, \quad \gamma_- f = \mathcal{R}\gamma_+ f \quad \text{on } \Sigma_-, \quad (7.4.49)$$

which existence follows from Theorem 7.3.4. From the above discussion (with $\kappa = -1$) and the same arguments as in Step 1 of the proof of Theorem 7.2.3, we have

$$\int f^2 \langle \varpi \rangle_- m^2 + 2 \int |\nabla_v f|^2 m^2 \leq \int \mathfrak{F}^2 m^2. \quad (7.4.50)$$

Together with the regularity estimate (7.2.23) and the compact imbedding $H^{2/3}(\mathcal{U}) \subset L^2(\mathcal{U})$, we conclude that \mathcal{B} has compact resolvent. The operator \mathcal{A} on L_m^2 defined by $\mathcal{A}f := n\chi_R(v)f$ being bounded, we may apply Lemma 2.1.8-(2) and we deduce that **(H3)** holds for both the primal and the dual problems.

Condition (H4) is nothing but the yet established strong maximum principle.

A variant of condition (H5). Consider (f, λ) a pair of eigenfunction and eigenvalue such that $\lambda \in \Sigma_{P^+}(\mathcal{L})$. Arguing similarly as in the proof of condition **(H5)** in Section 6.1, we know that

$$\tilde{\mathcal{L}}f = i\vartheta f, \quad \vartheta \in \mathbb{R}, \quad \tilde{\mathcal{L}}|f| = 0$$

and introducing the real and complex part decomposition $f = g + ih$, we have

$$\int_{\mathcal{O}} \frac{1}{|f|^2} |g\nabla_v h - h\nabla_v g|^2 = 0,$$

and finally $g\nabla_v h - h\nabla_v g = 0$ a.e. on \mathcal{O} . Because of the regularity estimate presented during the above proof of the strong maximum principle, the functions f has Hölder regularity, and thus g and h are continuous on \mathcal{O} . Because $|f| \not\equiv 0$, we may claim that there exists a point $(x_0, v_0) \in \mathcal{O}$ such that $h(x_0, v_0) > 0$ for instance. Denoting by ω the connected component of $\{(x, v) \in \mathcal{O}; h(x, v) > 0\}$ containing (x_0, v_0) , we have $\nabla(g/h) = 0$ on ω , and thus $g = \alpha(x)h$ on ω for some continuous function $\alpha : \Omega \rightarrow \mathbb{R}$. Coming back to the eigenvalue equation that we may write in the following system form

$$\tilde{\mathcal{L}}g = -\vartheta h, \quad \tilde{\mathcal{L}}h = \vartheta g,$$

we compute

$$-\vartheta h = \tilde{\mathcal{L}}(\alpha h) = \alpha \tilde{\mathcal{L}}h - hv \cdot \nabla_x \alpha = \alpha \vartheta g - hv \cdot \nabla_x \alpha \quad \text{on } \omega,$$

so that

$$-\vartheta = \alpha^2 \vartheta - v \cdot \nabla_x \alpha \quad \text{on } \omega.$$

We deduce that α is a constant on ω and finally $\vartheta = 0$. We have thus established that $\lambda = \lambda_1$.

At this stage, we may use Theorem 2.1.20, Theorem 3.3.11 and Theorem 4.4.14, in order to get the conclusions **(C1)**, **(C2)** and **(C3)** about the existence and uniqueness of the eigentriplet (λ_1, f_1, ϕ_1) which satisfies $f_1 > 0$, $\phi_1 > 0$, λ_1 is algebraically simple and on the triviality of the boundary punctual spetrum.

We briefly explain how we may deduce the stability of f_1 by adapting some arguments developed in [171] and already mentioned. On the one hand, we know from [171, Lemma 1.1] that any solution f to the rescaled evolution equation (7.3.36) with \mathcal{L} replaced by $\tilde{\mathcal{L}} = \mathcal{L} - \lambda_1$ satisfies

$$\partial_t(H(X)f_1\phi_1) + \operatorname{div}_x(vH(X)f_1\phi_1) - \operatorname{div}_v(\phi_1^2 \nabla_v(H(X)f_1/\phi_1)) = -H''(X)f_1\phi_1|\nabla_v X|^2,$$

for any convex function $H : \mathbb{R} \rightarrow \mathbb{R}$ and with $X := f/f_1$. After integration, we get

$$\frac{d}{dt} \int_{\mathcal{O}} H(X)f_1\phi_1 + \int_{\Sigma} \nu \cdot v H(X)f_1\phi_1 = - \int_{\mathcal{O}} H''(X)f_1\phi_1|\nabla_v X|^2, \quad (7.4.51)$$

When $H(s) := |s|$, the boundary term is

$$\begin{aligned} \int_{\Sigma} |\gamma f| \gamma \phi_1 \nu \cdot v &= \int_{\Sigma_+} |\gamma_+ f| \mathcal{R}^* \gamma_- \phi_1 \nu \cdot v - \int_{\Sigma_-} |\mathcal{R} \gamma_+ f| \gamma_- \phi_1 \nu \cdot v \\ &\geq \int_{\Sigma_+} |\gamma_+ f| \mathcal{R}^* \gamma_- \phi_1 |\nu \cdot v| - \int_{\Sigma_-} \mathcal{R} |\gamma_+ f| \gamma_- \phi_1 |\nu \cdot v| = 0, \end{aligned}$$

from what we deduce the non expansive property

$$\int_{\mathcal{O}} |f_{t_1}| \phi_1 \leq \int_{\mathcal{O}} |f_{t_0}| \phi_1, \quad \forall t_1 \geq t_0 \geq 0. \quad (7.4.52)$$

On the other hand, from the Cauchy-Schwarz inequality, we have

$$(\mathcal{R} \gamma_+ f)^2 \leq (\mathcal{R} \gamma_+ f_1) \mathcal{R}(\gamma_+ f^2 / \gamma_+ f_1),$$

so that

$$\int_{\Sigma_-} \frac{(\mathcal{R} \gamma_+ f)^2}{\mathcal{R} \gamma_+ f_1} \gamma_- \phi_1 |\nu \cdot v| \leq \int_{\Sigma_-} \mathcal{R}(\gamma_+ f^2 / \gamma_+ f_1) \gamma_- \phi_1 |\nu \cdot v|$$

and finally

$$\int_{\Sigma} (\gamma f)^2 (\gamma f_1)^{-1} \gamma \phi_1 \nu \cdot v \leq 0.$$

When $H(s) = s^2$, the equation (7.4.51) and the last inequality imply

$$\frac{dt}{dt} \int_{\mathcal{O}} f_1 \phi_1 (f/f_1)^2 + 2 \int_{\mathcal{O}} f_1 \phi_1 |\nabla_v (f/f_1)|^2 \leq 0. \quad (7.4.53)$$

We next recall a classical compactness result.

Lemma 7.4.7. *Let (g_n) be a sequence of functions such that*

$$(g_n) \text{ is bounded in } L^\infty(0, T; L^2_{xv, \text{loc}}) \cap L^2(0, T; L^2_{x, \text{loc}} H^1_{v, \text{loc}})$$

and

$$\partial_t g_n + v \cdot \nabla_x g - \Delta_v g_n = G_n \text{ bounded in } L^2_{\text{loc}},$$

then (g_n) belongs to a strong compact set of L^2_{loc} .

Proof of Lemma 7.4.7. We just sketch it. Because

$$\partial_t g_n + v \cdot \nabla_x g = \Delta_v g_n + G_n \text{ bounded in } L^2_{tx} H_v^{-1},$$

the usual averaging lemma in [107, 74] implies that

$$(g_n * \rho) \text{ belongs to a strong compact set of } L^2_{\text{loc}},$$

for any $\rho \in \mathcal{D}(\mathbb{R}^d)$. On the other hand, introducing a mollifiers sequence (ρ_ε) and writing then

$$g_n = (g_n - g_n * \rho_\varepsilon) + g_n * \rho_\varepsilon,$$

we see that the first term is small uniformly in n as $\varepsilon \rightarrow 0$ and the second term is relatively compact thanks to the first step, from what we immediately conclude. \square

Now, for $0 \leq f_0 \in L^1_{\phi_1}$, we introduce the sequence $f_{0,k} := (f_0 \wedge k) \mathbf{1}_{\mathcal{U}_k} \in L^2(f_1^{-1} \phi_1) \cap L^2$, with $\mathcal{U}_k := \{(x, v) \in \mathcal{O}; \delta(x) > 1/k, |v| \leq k\}$, and the associated solution $f_k \in L^\infty(0, T; L^2) \cap L^2(0, \infty; L^2_x H_v^1)$. Because of (7.4.53), for any increasing sequence (t_n) which converges to ∞ and for any function $\varphi_m \in \mathcal{D}(\mathcal{O})$, $\mathbf{1}_{\mathcal{U}_m} \leq \varphi_m \leq 1$, the rescaled and truncated function $g_n(t) := f_k(t + t_n) f_1^{-1} e^{-\lambda_1(t+t_n)} \varphi_m$ meet the hypothesis of Lemma 7.4.7, from what we classically deduce that the sequence of $\tilde{f}_n(t) := f_k(t + t_n) f_1^{-1} e^{-\lambda_1(t+t_n)}$ is relatively strongly compact in L^2_{loc} . Repeating the proof of Theorem 3.4.20 and Theorem 4.5.20 (see also [171, Thm. 3.2]), we deduce that $\tilde{f}_n(t) \rightarrow \langle f_{0,k}, \phi_1 \rangle f_1$ as $t \rightarrow \infty$. Together with the above non expansive property (7.4.53), we deduce that $f_t \rightarrow \langle f_0, \phi_1 \rangle f_1$ in $L^1_{\phi_1}$ as $t \rightarrow \infty$.

We summarize our convergence result in the following theorem.

Theorem 7.4.8. *For any $f_0 \in L^2$, it holds $f_t \rightarrow \langle f_0, \phi_1 \rangle f_1$ in $L^1_{\phi_1}$ as $t \rightarrow \infty$.*

Theorem 7.4.8 generalizes [152, Theorem 2.18] and [116, Theorem 6.8] for the zero inflow condition and [1, Theorems 1.6 & 1.7] for the torus case. It is worth emphasizing that in these papers the longtime convergence is established with exponential rate (with constructive estimate in [1]). In [152] the proof is based on a representation formula for the associated semigroup S which is proved to have a kernel $p_t \in (L^1 \cap L^\infty \cap C^\infty)(\mathcal{O})$ for any $t > 0$ (see [152, Thm. 2.4 & 2.6] as well as [214, 124, 151]). One then classically deduces that $S_t \in \mathcal{K}(X)$ for any $t > 0$ and $X = L^p$, $p \in [1, \infty]$, or $X = C_0$ (see [152, Thm. 2.18]), and next one may apply Theorem 4.6.25.

We follow now a similar approach as in [152, 116]. We start with a series of technical results. Here, we make the additional assumption

$$\varpi^\sharp(x, v) := \sup_{1 \leq p \leq \infty} w_p(x, v) \leq \kappa_2 < \infty, \quad (7.4.54)$$

with

$$\varpi_p := \frac{(2-p)\Delta m}{p} + \frac{2|\nabla m|^2}{p' m^2} + c - \frac{1}{p} \frac{\operatorname{div}(bm^p)}{m^p}.$$

Lemma 7.4.9. *For any fixed $\kappa < \kappa_0$ there exists $\varrho_x > 0$, $\varrho_v > 0$ and $\kappa_2 \in \mathbb{R}$ such that defining $\mathcal{A}f := n\xi_{\varrho_v}(v)\zeta_{\varrho_x}(x)f$ with $n \geq 0$, $\xi_{\rho_v} \in \mathcal{D}(\mathbb{R}^d)$, $\mathbf{1}_{|v| \leq \rho_v} \leq \xi_{\rho_v} \leq \mathbf{1}_{|v| \leq 2\rho_v}$, $\zeta_{\rho_x} \in \mathcal{D}(\Omega)$, $\mathbf{1}_{\delta(x) \geq \rho_x/2} \leq \zeta_{\rho_x} \leq \mathbf{1}_{\delta(x) \geq \varrho_x}$, and next $\mathcal{B} := \mathcal{L} - \mathcal{A}$, there hold*

$$\|S_{\mathcal{B}}(t)\|_{\mathcal{B}(L^2_m)} \lesssim e^{\kappa t}, \quad \forall t \geq 0, \quad (7.4.55)$$

$$\|S_{\mathcal{B}}(t)\|_{\mathcal{B}(L^p_{m_p})} \lesssim e^{\kappa_2 t}, \quad \forall t \geq 0, \quad \forall p \in (2, \infty], \quad (7.4.56)$$

with $m_p := \mathcal{M}^{1-1/p}$.

Proof of Lemma 7.4.9. We first recall from Step 1 of the proof of Theorem 7.2.3 and (7.3.30) that

$$\begin{aligned} (\mathcal{L}f, f)_{L^2_m} &= -\int |\nabla f|^2 m^2 - \frac{1}{2} \int_{\Sigma} (\gamma f)^2 m^2 \nu \cdot v + \int f^2 \varpi m^2 \\ &\leq -\int |\nabla f|^2 m^2 + \int f^2 \varpi m^2 \end{aligned}$$

and, with ψ defined in (7.2.21),

$$\begin{aligned} (-\mathcal{L}f, f)_{L^2_\psi} &= -\frac{1}{2} \int f^2 (v \cdot \nabla_x \psi) - \int f \frac{b}{\langle v \rangle} \cdot \nabla_v f \langle v \rangle \psi + \int \nabla_v (f\psi) \cdot \nabla_v f - \int cf^2 \psi \\ &\leq -\int f^2 \frac{(\hat{v} \cdot \nu(x))^2}{\delta(x)^{1/2}} dv dx + C \int (f^2 + |\nabla f|^2). \end{aligned}$$

Defining then $\tilde{m} := m - \beta\psi$, with $\beta > 0$ small enough, and summing up the two previous estimates, we get

$$(\mathcal{L}f, f)_{L^2_{\tilde{m}}} \leq -\beta \int f^2 \frac{(\hat{v} \cdot \nu(x))^2}{\delta(x)^{1/2}} - \frac{1}{2} \int |\nabla f|^2 m^2 + \int f^2 (\varpi m^2 + 1).$$

Similarly as in (7.4.41), we define

$$\mathcal{U} := \{(x, v) \in \mathcal{O}; \delta(x) > \varrho_x, |v| < \varrho_v\},$$

and we observe that

$$\mathcal{U}^c \subset A \cup B \cup C,$$

with

$$A := \{v \in B_{\varrho_v}, |\hat{v} \cdot \nu(x)| \leq \varepsilon_v\}, \quad B := \{v \in B_{\varrho_v}, |\hat{v} \cdot n| \geq \varepsilon_v, \delta(x) \leq \varrho_x\},$$

for some $\varepsilon_x > 0$, and $C := B_{\rho_v}^c$. We next repeat the proof of (7.4.42), and we get

$$\int_{\mathcal{U}^c} f^2 m^2 \lesssim (\varrho_v^{d-1} \varepsilon_v)^{2/r'} \int |\nabla_v f|^2 + m(\varrho_v)^2 \frac{\varrho_x^{1/2}}{\varepsilon_v^2} \int f^2 \frac{(\hat{v} \cdot \nu(x))^2}{\delta(x)^{1/2}} + \frac{1}{\varpi_-(\rho_v)} \int f^2 \varpi_- m^2.$$

Observing that

$$\int f^2 (\varpi m^2 + 1) \leq \kappa \int f^2 \tilde{m}^2 + C_\kappa \int_{\mathcal{U}} f^2 m^2 + C_\kappa \int_{\mathcal{U}^c} f^2 m^2$$

with $C_\kappa := \sup(\varpi + 2 - \kappa)_+ < \infty$, and $\mathcal{A} \geq C_\kappa \mathbf{1}_{\mathcal{U}}$ for $n := C_\kappa$, altogether, we conclude with

$$(\mathcal{B}f, f)_{L_m^2} \leq \kappa \|f\|_{L_m^2}.$$

We then classically deduce that (7.4.55) holds.

Similarly as for the first estimate and in the proof of [174, Lem. 3.8], for any smooth, rapidly decaying and positive function f , we have

$$\int (\mathcal{L}f) f^{p-1} m^p = - \int_{\Sigma} \frac{(m\gamma f)^p}{p} \nu \cdot v - (p-1) \int |\nabla(mf)|^2 (mf)^{p-2} dx + \int f^p \varpi_p m^p.$$

From Darozès-Guiraud (or Jensen) inequality, we know that the first (boundary) term is nonpositive (see [67] or [172, Rem. 6.4]) and we then classically conclude to (7.4.56). \square

Lemma 7.4.10. *There exists a finite family $2 = p_0 < p_1 < \dots < p_k < \infty$ and $\alpha \in (0, 1)$ such that such that both $\mathcal{C} = \mathcal{B}, \mathcal{L}$, for any $T > \tau > 0$ and $\mathcal{V} \subset\subset \mathcal{O}$,*

$$\int_{\tau}^T \|\mathcal{A}S_{\mathcal{C}}(t)f_0\|_{L_m^{p_j}} dt \leq C_{p_{j-1}}^{p_j} \|f_0\|_{L_m^{p_{j-1}}}, \quad j = 1 \dots, k, \quad (7.4.57)$$

$$\sup_{t \in [\tau, T]} \|\mathcal{A}S_{\mathcal{B}}(t)f_0\|_{L^\infty} \leq C_{p_k}^\infty \|f_0\|_{L^{p_k}}, \quad (7.4.58)$$

$$\sup_{t \in [\tau, T]} \|S_{\mathcal{B}}(t)f_0\|_{C^\alpha(\mathcal{V})} \leq C_\infty^\alpha \|f_0\|_{L^\infty}. \quad (7.4.59)$$

Proof of Lemma 7.4.10. For $0 \leq f_0 \in L_m^2$, let us denote $f := S_{\mathcal{B}}f_0$ which thus satisfies the PDE

$$\partial_t f - \mathcal{B}f = s := cf \quad \text{in } \mathcal{D}'((0, T) \times \mathcal{O}).$$

Let us fix two open sets U_i such that $[\tau, T] \times \text{supp}\xi \times \text{supp}\zeta \subset U_0 \subset\subset U_1 \subset\subset (0, T) \times \mathcal{O}$. From [106, Thm 6] and a covering lemma, there exists a constant $\bar{C}_0 > 0$ and $p_1 > 2$ such that

$$\|f\|_{L^{p_1}(U_0)} \leq \bar{C}_0 (\|f\|_{L^2(U_1)} + \|s\|_{L^2(U_1)}).$$

The estimate (7.4.57) for $j = 1$ then follows from Theorem 7.3.5 (and the classical underlying energy estimate). On the other hand, [106, Thm 12] similarly implies that there exists a constant $\bar{C}_k > 0$ and $p_k \in (p_1, \infty)$ such that

$$\|f\|_{L^\infty(U_0)} \leq \bar{C}_k (\|f\|_{L^2(U_1)} + \|s\|_{L^{p_k}(U_1)}),$$

and interpolating with the previous estimate, we get

$$\|f\|_{L^{p_j}(U_0)} \leq \bar{C}_{j-1} (\|f\|_{L^2(U_1)} + \|s\|_{L^{p_{j-1}}(U_1)}), \quad \forall j, 2 \leq j \leq k-1.$$

The growth bound (7.4.56) and the two last estimates imply (7.4.58) and (7.4.57) for any $2 \leq j \leq k-1$. Finally, [106, Thm 3] similarly implies that there exists a constant $\bar{C}_{k+1} > 0$ and $\alpha \in (0, 1)$ such that

$$\|f\|_{C^\alpha(U_0)} \leq \bar{C}_{k+1} (\|f\|_{L^2(U_1)} + \|s\|_{L^\infty(U_1)}),$$

from what we deduce (7.4.59) in the same way. \square

Theorem 7.4.11. *The conclusion (CE3) holds in L_m^2 with exponential rate but non constructive rate.*

Proof of Theorem 7.4.11. We introduce the splitting

$$S_{\mathcal{L}} = V + K_1^c + K_2^c + K,$$

where we set

$$K := \Upsilon S_{\mathcal{B}} *_{\tau} K_{k+2}, \quad K_1^c := W * S_{\mathcal{L}} - S_{\mathcal{B}} *_{\tau} K_{k+2}, \quad K_2^c := (1 - \Upsilon) S_{\mathcal{B}} *_{\tau} K_{k+2}.$$

Here we define V and W with the usual notations (2.2.41) where we take $N = k + 2$. Next for two functions (of operators) a and b and $0 < \tau < T$, we define

$$(a *_{\tau} b)(t) := \int_{\tau}^{t-\tau} a(t-s)b(s) ds \text{ if } t \in [\tau, T - \tau], \quad (a *_{\tau} b)(t) := 0 \text{ if } t \in [\tau, T - \tau]^c,$$

and then recursively $K_0 = \text{Id}$, $K_1 := \mathcal{A}S_{\mathcal{L}}$, $K_{\ell} := (\mathcal{A}S_{\mathcal{B}}) *_{\tau} K_{\ell-1}$ for $\ell \geq 2$. We finally define $\Upsilon g := \chi_{\varepsilon} g$ with $\chi_{\varepsilon} \in \mathcal{D}(\mathcal{O})$, $\mathbf{1}_{\mathcal{U}_{2\varepsilon}} \leq \chi_{\varepsilon} \leq \mathbf{1}_{\mathcal{U}_{\varepsilon}}$, with $\mathcal{U}_{\varepsilon} := \{|v| \leq 1/\varepsilon, \delta(x) > \varepsilon\}$. The sequel of the proof is split into two steps.

Step 1. On the one hand, we compute

$$\begin{aligned} \| (S_{\mathcal{B}} *_{\tau} K_{k+2}) f_0 \|_{L_m^{p_1}} &\leq \| S_{\mathcal{B}} \|_{L^{\infty}(\mathcal{B}(L_m^{p_1}))} \int_{\tau}^{T-\tau} \left\| \int_{\tau}^{t-\tau} \mathcal{A}S_{\mathcal{B}}(t-s) K_{k+1}(s) ds f_0 \right\|_{L_m^{p_1}} dt \\ &\leq C_T \int_{\tau}^T \int_{\tau}^T \| \mathcal{A}S_{\mathcal{B}}(t) K_{k+1}(s) f_0 \|_{L_m^{p_1}} dt ds \\ &\leq C_T C_2^{p_1} \int_{\tau}^T \| K_{k+1}(s) f_0 \|_{L_m^2} ds, \end{aligned}$$

and thus

$$\| (S_{\mathcal{B}} *_{\tau} K_{k+2}) f_0 \|_{L_m^{p_1}} \leq C_T \| f_0 \|_{L_m^2}, \quad (7.4.60)$$

where we have used (7.4.56) in the first line, the Fubini theorem in the second line, (7.4.57) with $j = 1$ in the third line and several times (7.4.55) in the last line.

From the very definition of \mathcal{A} and $S_{\mathcal{B}}$ as well as (7.4.55), we may fix $\kappa_{\mathcal{B}} \in (\kappa, \kappa_0)$ and next $T > 0$ such that $\|V(T)\|_{\mathcal{B}(L_m^2)} \leq \frac{1}{3} e^{\kappa_{\mathcal{B}} T}$. We may next use (7.4.55) and fix $\tau > 0$ small enough such that

$$\|K_1^c\|_{\mathcal{B}(L_m^2)} \leq \tau C_T \leq \frac{1}{3} e^{\kappa_{\mathcal{B}} T}.$$

Last, because of (7.4.60), we may fix $\varepsilon > 0$ small enough, in such a way that

$$\|K_2^c f_0\|_{L_m^2} \leq \eta(\varepsilon) \| (S_{\mathcal{B}} *_{\tau} K_{k+2}) f_0 \|_{L_m^{p_1}} \leq \frac{1}{3} e^{\kappa_{\mathcal{B}} T} \| f_0 \|_{L_m^2}.$$

The three last estimates together, we have established

$$\|V(T) + K_1^c + K_2^c\|_{\mathcal{B}(L_m^2)} \leq e^{\kappa_{\mathcal{B}} T}. \quad (7.4.61)$$

Step 2. Performing the same kind of computations as for proving (7.4.60), we get

$$\begin{aligned} \int_0^T \| (\mathcal{A}S_{\mathcal{B}} *_{\tau} K_{j-1})(s) f_0 \|_{L_m^{p_j}} ds &\leq \int_0^T \int_{\tau}^{T-\tau} \| \mathcal{A}S_{\mathcal{B}}(t) K_{j-1}(s) f_0 \|_{L_m^{p_j}} dt ds \\ &\leq C_{p_{j-1}}^{p_j} \int_0^T \| K_{j-1}(s) f_0 \|_{L_m^{p_{j-1}}} ds, \end{aligned}$$

for $j = 1, \dots, k+1$, $p_{k+1} := \infty$. Iterating, and using (7.4.57) with $j = 1$, we get

$$\int_0^T \|K_{k+1}(s)f_0\|_{L_m^\infty} ds \lesssim \|f_0\|_{L_m^2}.$$

Similarly, we may write

$$\begin{aligned} \sup_{[\tau, T]} \|K_{k+2}f_0\|_{L_m^\infty} &\leq \sup_{t \in [\tau, T]} \int_\tau^t \|\mathcal{A}S_{\mathcal{B}}(t-s)K_{k+1}(s)f_0\|_{L_m^\infty} ds \\ &\leq \sup_{t \in [\tau, T]} \|\mathcal{A}S_{\mathcal{B}}(s)\|_{\mathcal{B}(L_m^\infty)} \int_\tau^T \|K_{k+1}(s)f_0\|_{L_m^\infty} ds \end{aligned}$$

and

$$\begin{aligned} \|Kf_0\|_{C^\alpha(\mathcal{O})} &\leq \int_\tau^{T-\tau} \|S_{\mathcal{B}}(t-s)K_{k+2}(s)f_0\|_{C^\alpha(\mathcal{U}_\varepsilon)} ds \\ &\leq C_\infty^\alpha T \sup_{[\tau, T]} \|K_{k+2}f_0\|_{L_m^\infty}. \end{aligned}$$

The three last estimates together and the compact support condition $\text{supp}\chi_\varepsilon \subset\subset \mathcal{O}$ imply

$$\|Kf_0\|_{C^\alpha \cap L_m^2} \lesssim \|f_0\|_{L_m^2}, \quad \forall f_0 \in L_m^2,$$

from what we deduce that $K \in \mathcal{K}(L_m^2)$. We may apply Theorem 4.6.25 in order to conclude. \square

Chapter 8

A mutation-selection model

In this chapter, we consider the mutation-selection evolution equation associated to the mutation-selection operator

$$\mathcal{L}f := J * f - W(x)f \quad (8.0.1)$$

defined on functions $f : \mathbb{R}^d \rightarrow \mathbb{R}$, $f = f(x)$, where J is a the mutation kernel, $*$ stands for the convolution operator and W is a confining potential. This equation appears in the modeling of genetic variability in evolutionary biology.

As we say in the introduction, $f_t(x)$ represents the density of a population, at time t , of phenotypic trait x on the multi-dimensional phenotypic trait space \mathbb{R}^d . the kernel J is related to mutations and the potential W to the selection process.

Different selection potentials have been considered. The case of linear fitness $W(x) = -x$ is completely studied by the works of M. Alfaro and R. Carles [2](Laplacian case, Equation (1.4)) and R. Burger and M.E. Gil [42] whereas the quadratic case, that is $W(x) := x^2$ (say $N = 1$), was considered in M. Kimura and analyzed by R. Bürger [44] for Equation (1.1) and by M. Alfaro and R. Carles [2]. We also can find more realistic cases for which fitness functions are defined by a linear combination of two components (e.g. birth and death rates), each maximized by different optimal values of the underlying trait, a typical case being $W(x) := x^4 - x^2$ with is studied in Alfaro and M. Veruete [4].

The study of the eigenvalue problem associated with this model has been of interest to the scientific community for some time. In particular the works of R. Bürger and I.M. Bomze [45] and also O. Bonnefon, J. Coville, and G. Legendre [36] for bounded domains, show that the lack of regularizing effect of the integral operator compared to the Laplace operator can arrive at a concentration phenomenon which leads to the formation of singular measure for the first eigenfunction. A first fundamental question is to identify sharp conditions on J and W that ensure the existence of a principal eigenfunction. Some of those conditions have been presented in [3]. In this chapter, we present a relaxation of these conditions, mainly by removing the J symmetry assumption, see remark 12.2 (3) below.

Under these relaxed assumptions, we present a quantification of the spectral gap, and its stability result that generalizes the one shown in [3].

We assume that the mutation kernel J is a positive finite measure of \mathbb{R}^d which is lower bounded on a neighborhood of the origin, or in other words

$$0 \leq J \in M^1(\mathbb{R}^d), \quad J \geq J_* \mathbf{1}_{B_r}, \quad (8.0.2)$$

for some constants $J_*, r > 0$. We also assume that the selection potential $W : \mathbb{R}^d \rightarrow \mathbb{R}$ is continuous and satisfies

$$W(x) > W(0) = 0, \quad \forall x \neq 0, \quad W(x) \rightarrow +\infty \text{ as } |x| \rightarrow \infty. \quad (8.0.3)$$

We finally assume the following compatibility condition between mutation and selection: there exist $\beta > 0$ and a bounded Borel set $A \subset \mathbb{R}^d$ such that

$$a := \operatorname{ess\,inf}_{x \in A_\beta} \int_{x-A_\beta} \frac{J(dz)}{W(x-z)} > 1, \quad (8.0.4)$$

$$J = J_1 + J_2, \quad J_1 \in C_c^1(\mathbb{R}^d), \quad \kappa_* := \|J_2\|_1 := \int_{\mathbb{R}^d} dJ_2 < \kappa_0 := (a-1)\beta, \quad (8.0.5)$$

where we use the notation $A_\beta = A \cap \{W \geq \beta\}$. In the sequel, we work in the Banach lattice $X := L^1(\mathbb{R}^d)$.

Theorem 8.0.1. *Under the above assumptions, we can arrive to the following conclusions,*

1. *The first eigentriplet problem (1.3.30)-(1.3.31) admits a unique solution $(\lambda_1, f_1, \phi_1) \in \mathbb{R} \times X_+ \times X'_+$, with $\lambda_1 \geq \kappa_0$, $0 < f_1 \in L^1_{(W)}(\mathbb{R}^d) \cap L^\infty_{(W)}(\mathbb{R}^d)$ and $0 < \phi_1 \in L^1_{(W)}(\mathbb{R}^d) \cap L^\infty_{(W)}(\mathbb{R}^d)$.*
2. *Moreover, \mathcal{L} generates a semigroup $S_{\mathcal{L}}$ on $X = L^1(\mathbb{R}^d)$ and for any $f_0 \in X$, there holds*

$$\|e^{-\lambda_1 t} S_{\mathcal{L}}(t) f_0 - \langle \phi_1, f_0 \rangle f_1\|_{L^1} \leq C e^{-\alpha t} \|f_0 - \langle \phi_1, f_0 \rangle f_1\|_{L^1}, \quad (8.0.6)$$

for any $t \geq 0$ and for some constructive constants $C, \alpha > 0$.

Let us comment on the above result which is our main result in this section.

Remark 8.0.2.

(1) *Assumption (8.0.4) is satisfied when W is small enough in a neighborhood of the origin. It is for instance satisfied if $W^{-1} \notin L^1(B_1)$. That is in particular the case in dimension $d = 1$ when W is Lipschitz, because of the condition $W(0) = 0$.*

(2) *Assume $J(x) = \varepsilon^{-d} \rho(\varepsilon^{-1}x)$ with $\rho \in C_c^1(\mathbb{R}^d) \cap \mathcal{P}(\mathbb{R}^d)$ and $\rho \geq \rho_* \mathbf{1}_{B_1}$, $\rho_* > 0$, so that $J = J_1$ and $J_2 = 0$, and $W = W(|x|)$. We may observe that for $\beta > 0$ and $\varepsilon > 0$ small enough*

$$\inf_{\beta \leq W(x) < 2\beta} \int_{\beta \leq W(y) < 2\beta} \frac{J(x-y)}{W(y)} dy =: a \geq \frac{\rho_*}{2\beta} \operatorname{meas}\{\mathbb{R}_+^d \cap B_1\} > 1,$$

so that (8.0.4) holds with $A := \{W(x) < 2\beta\}$.

(3) *Assumption (8.0.4) is similar to [153, Condition (2.3)], see also [3, Assumption 2.6] and the comparison with [3, Assumption 2.4], as well as [43, Condition (3.7)-(3.8)] and [46, p. 250, Note added in proof.]. On the other hand, the conditions on J are relaxed since J may have singular part, and more importantly, it is not assumed to be symmetric as in [3, Assumption 2.2].*

(4) *Optimal conditions linking J and W for the existence of a spectral gap are still unknown. In the recent paper [3], using variational methods in a L^2 framework, the authors obtain a quantified spectral gap and the associated exponential stability when the mutation kernel J is additionally assumed to be symmetric. Up to our knowledge, Theorem 8.0.1 is the very first result providing a quantified spectral gap for a non-symmetric mutation kernel J .*

(5) *Condition (8.0.4) can be compared to the condition*

$$\bar{a} := \operatorname{ess\,sup}_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} \frac{J(x-y)}{W(y)} dy < 1,$$

under which no first eigenfunction may exist in X . First, we claim that $\lambda_1 \geq 0$. Indeed, considering $\varepsilon > 0$ and $f_\varepsilon = \mathbf{1}_{B_\varepsilon}$, we have

$$\mathcal{L}f_\varepsilon \geq -\left(\inf_{B_\varepsilon} W\right)f_\varepsilon,$$

so that the condition **(H2)** holds for $\kappa_0 = -\inf_{B_\epsilon} W$ for any $\epsilon > 0$. Since W is continuous and $W(0) = 0$, we deduce that $\lambda_1 \geq 0$ by passing to the limit $\epsilon \rightarrow 0$. Assume now by contradiction that there exists $f_1 \in X_+ \setminus \{0\}$ such that

$$\lambda_1 f_1(x) = \mathcal{L}f_1(x) = \int_{\mathbb{R}^d} J(x-y)f_1(y) dy - W(x)f_1(x) \quad (8.0.7)$$

and define, for any $\epsilon > 0$, the function $\varphi_\epsilon(x) = \frac{1}{\epsilon+W(x)} \in L^\infty(\mathbb{R}^d)$. Testing (8.0.7) against φ_ϵ we get for any $\epsilon \in (0, 1)$

$$\begin{aligned} 0 \leq \lambda_1 \langle f_1, \varphi_1 \rangle &\leq \lambda_1 \langle f_1, \varphi_\epsilon \rangle = \iint \frac{J(x-y)}{\epsilon+W(x)} f_1(y) dx dy - \int \frac{W(x)}{\epsilon+W(x)} f_1(x) dx \\ &\leq \bar{a} \int f_1 - \int \frac{W(x)}{\epsilon+W(x)} f_1(x) dx \end{aligned}$$

and passing to the limit $\epsilon \rightarrow 0$ we obtain the contradiction $0 \leq \lambda_1 \langle f_1, \varphi_1 \rangle \leq (\bar{a} - 1) \int f_1 < 0$. However, there always exists a principal eigenvector f_1 in $M^1(\mathbb{R}^d)$.

The proof of Theorem 8.0.1 follows from Theorem 2.1.20, Theorem 3.3.11 and Theorem 4.4.14 as a consequence of conditions **(H1)**–**(H5)** that we establish now. Setting $D(\mathcal{L}) := L^1_{\langle W \rangle}(\mathbb{R}^d)$, we observe that \mathcal{L} is an unbounded closed operator with dense domain $D(\mathcal{L})$.

Condition (H1) and (H1'). We define the semigroup

$$S_W(t)f(x) := e^{-W(x)t}f(x), \quad \forall f \in L^p, p \in [1, \infty],$$

which is clearly a positive semigroup of contractions. We next define $S_{\mathcal{L}}$ as a bounded perturbation of S_W . It is also positive and it satisfies the growth estimate $\|S_{\mathcal{L}}(t)\|_{\mathcal{B}(L^p)} \leq e^{\|J\|_1 t}$, where we recall that $\|J\|_1$ stands for the L^1 norm or the total variation norm of J . We deduce that **(H1)** holds true with $\kappa_1 := 1 = \|J\|_1$ thanks to Lemma 2.1.2-(i).

Multiplying $\mathcal{L}f$ by $\text{sign } f$, for $f \in D(\mathcal{L})$, we immediately get Kato's inequality

$$(\text{sign } f)\mathcal{L}f = (\text{sign } f)J * f - W|f| \leq J * |f| - W|f| = \mathcal{L}|f|.$$

Condition (H2). Let us define $f_0 := \frac{1}{W(x)}\mathbf{1}_{A_\beta}$, where A_β is introduced in condition (8.0.4). We compute

$$\begin{aligned} \mathcal{L}f_0 &= J * \left(\mathbf{1}_{A_\beta} \frac{1}{W} \right) - \mathbf{1}_{A_\beta} \geq \left(J * \left(\mathbf{1}_{A_\beta} \frac{1}{W} \right) - 1 \right) \mathbf{1}_{A_\beta} \\ &\geq \left(\text{ess inf}_{x \in A_\beta} \left[J * \left(\mathbf{1}_{A_\beta} \frac{1}{W} \right) \right] - 1 \right) \mathbf{1}_{A_\beta} \\ &= (a-1)\mathbf{1}_{A_\beta} \geq (a-1) \frac{\beta}{W} \mathbf{1}_{A_\beta} = \kappa_0 f_0, \end{aligned}$$

where in the second equality we have used the very definition of a in assumption (8.0.4). We conclude that **(H2)** holds thanks to Lemma 2.1.4-(ii).

Condition (H3). We introduce the splitting

$$\mathcal{L} = \mathcal{A} + \mathcal{B}, \quad \mathcal{A}f := J_1 * f, \quad \mathcal{B}f := J_2 * f - W(x)f. \quad (8.0.8)$$

Arguing as in the proof of condition **(H1)**, we see that \mathcal{B} is the generator a positive semigroup in $L^p(\mathbb{R}^d)$, $1 \leq p \leq \infty$, with growth bound $\omega(S_{\mathcal{B}}) \leq \kappa_*$ and thus $(\alpha - \mathcal{B})$ is invertible for any $\alpha \geq \kappa_0 > \kappa_*$, with

$$\|(\alpha - \mathcal{B})^{-1}\|_{\mathcal{B}(L^p)} \leq \frac{1}{\alpha - \kappa_*}. \quad (8.0.9)$$

Next, observing that

$$(W + \alpha)h = (\alpha - \mathcal{B})h + J_2 * h,$$

for any $h \in \mathcal{D}(\mathcal{L})$ and $\alpha \geq \kappa_0$, we deduce that

$$(W + \alpha)(\alpha - \mathcal{B})^{-1}g = g + J_2 * ((\alpha - \mathcal{B})^{-1}g), \quad (8.0.10)$$

for any $g \in X$ and $\alpha \geq \kappa_0$. Together with (8.0.9), we deduce

$$\|(\alpha - \mathcal{B})^{-1}g\|_{L_W^p} \leq \|g\|_{L^p} + \|J_2 * ((\alpha - \mathcal{B})^{-1}g)\|_{L^p} \leq \frac{\alpha}{\alpha - \kappa_*} \|g\|_{L^p}, \quad (8.0.11)$$

for any $g \in L^p$ and $\alpha \geq \kappa_0$. Defining $\mathcal{W}(\alpha) := (\alpha - \mathcal{B})^{-1}\mathcal{A}$, we finally deduce from (8.0.10) the identity

$$\mathcal{W}(\alpha)f = \frac{1}{W + \alpha}\mathcal{A}f + \frac{1}{W + \alpha}J_2 * ((\alpha - \mathcal{B})^{-1}\mathcal{A}f),$$

for any $f \in X$ and $\alpha \geq \kappa_0$. We may then compute

$$\|\mathcal{W}(\alpha)f\|_{L^\infty} \leq \frac{1}{\alpha}\|\mathcal{A}f\|_{L^\infty} + \frac{1}{\alpha}\|J_2\|_1\|(\alpha - \mathcal{B})^{-1}\mathcal{A}f\|_{L^\infty},$$

and together with (8.0.9) for $p = \infty$, we deduce

$$\|\mathcal{W}(\alpha)f\|_{L^\infty} \leq \|J_1\|_\infty \frac{1}{\alpha - \kappa_*} \|f\|_{L^1}, \quad (8.0.12)$$

for any $f \in X$ and $\alpha \geq \kappa_0$. Starting from the same identity, we prove in a similar way

$$\|\mathcal{W}(\alpha)f\|_{L_W^\infty} \leq \|J_1\|_\infty \frac{\alpha}{\alpha - \kappa_*} \|f\|_{L^1}, \quad (8.0.13)$$

for any $f \in X$ and $\alpha \geq \kappa_0$. As a conclusion and gathering (8.0.9), (8.0.11), (8.0.12) and (8.0.13), we have established that

$$\mathcal{W}(\alpha) : L^1 \rightarrow L_{\langle W \rangle}^1 \cap L_{\langle W \rangle}^\infty, \quad (8.0.14)$$

with uniform bound for any $\alpha \geq \kappa_0$. Observing that $L_{\langle W \rangle}^1 \cap L_{\langle W \rangle}^\infty \subset L^1$ is weakly compact and using Lemma 2.1.12 with $p = 1$, we deduce that **(H3)** holds. We can actually strengthen the compactness by noticing that $\mathcal{A} : L^1 \rightarrow L_W^1 \cap W^{1,1}$ is bounded because of assumption (8.0.5). This ensures that $\mathcal{A} : L^1 \rightarrow L^1$ is compact, from what we deduce that $\mathcal{W}(\alpha) : L^1 \rightarrow L^1$ is strongly compact for all $\alpha \geq \kappa_0$. We may thus apply Lemma 2.1.8-(2) to infer that condition **(H3)** holds for both the primal and the dual problems.

Condition (H4). Assume that $\lambda \geq \lambda_1$ and $f \in D(\mathcal{L}) = L_{\langle W \rangle}^1$ satisfy

$$\|f\|_{L^1} = 1, \quad f \geq 0, \quad (\lambda - \mathcal{L})f \geq 0. \quad (8.0.15)$$

Denoting $W_R := \inf_{B_R^c} W$, we compute

$$\int_{B_R} f \geq \int_{\mathbb{R}^d} f - \frac{1}{W_R} \int_{B_R^c} fW \geq 1 - \frac{1}{W_R} \|f\|_{L_{\langle W \rangle}^1} \geq 1/2,$$

for $R > 0$ large enough by taking advantage of the fact that $W(x)$ tend to infinity when $|x| \rightarrow \infty$. In particular, there exists $x_0^f \in B_R$ such that

$$\int_{B_{r/2}(x_0^f)} f \geq \delta := \frac{1}{2} \left(\frac{r}{2R} \right)^d > 0,$$

where we recall that r is defined in (8.0.2). We deduce that

$$(J * f)(x) \geq J_* \int_{B_{r/2}(x_0^f)} f(y) dy \mathbf{1}_{B_{r/2}(x_0^f)}(x) \geq J_* \delta \mathbf{1}_{B_{r/2}(x_0^f)}(x).$$

Using the equation (8.0.15), we obtain

$$f(x) \geq \frac{(J * f)(x)}{W(x) + \lambda} \geq \frac{J_* \delta}{W[R] + \lambda} \mathbf{1}_{B_{r/2}(x_0^f)}(x),$$

for $W[R] = \sup_{B_R} W(x)$. With that last information and (8.0.2) again, we have now

$$J * f \geq \frac{J_*}{2^d} \frac{J_* \delta}{W[R] + \lambda} \mathbf{1}_{B_r(x_0^f)},$$

and, iterating the argument, we deduce

$$f \geq \frac{J_*^m}{2^{(m-1)d} (W[R] + \lambda)^{m-1}} \delta \mathbf{1}_{B_{mr/2}(x_0^f)} \geq \bar{\gamma} \mathbf{1}_{B_R},$$

with $\bar{\gamma} = \bar{\gamma}(R) > 0$ for $m = m(R)$ large enough. Choosing R an integer, we have proved that

$$f \geq h_0 := \bar{\gamma}(R) \mathbf{1}_{B_R} + \sum_{n \geq R} \bar{\gamma}(n+1) \mathbf{1}_{B_{n+1} \setminus B_n} > 0. \quad (8.0.16)$$

That means that the **(H4)** holds, with constructive lower bound.

Condition (H5). Let us consider $f \in L^1_{\langle W \rangle} \setminus \{0\}$ and $\lambda \in \mathbb{C}$ such that (4.4.15) holds, in particular

$$\mathcal{L}|f| = (\Re \lambda)|f| \quad \text{and} \quad \mathcal{L}|f| = \Re(\text{sign} f) \mathcal{L}f. \quad (8.0.17)$$

The first equality means that $\Re \lambda$ is an eigenvalue associated to a positive eigenfunction, and Lemma 3.3.15 then enforces $\Re \lambda = \lambda_1$. Lemma 3.3.16 subsequently ensures that $|f| \in (\text{Span } f_1)_+ \setminus \{0\}$, and in particular $|f| \gg 0$. Throwing away the term $W|f|$ in each side of the second identity in (8.0.17), we have

$$\Re \frac{\bar{f}}{|f|} (J * f) = J * |f|.$$

Integrating this equation, we get

$$\int_{\mathbb{R}^{2d}} J(x-y) \Re \left[|f(y)| - \frac{\bar{f}(x)}{|f(x)|} f(y) \right] dy = 0.$$

From the positivity condition (8.0.2) on J , we deduce

$$|f(y)| - \frac{\bar{f}(x)}{|f(x)|} f(y) = \Re \left[|f(y)| - \frac{\bar{f}(x)}{|f(x)|} f(y) \right] = 0, \quad \forall x, y \in \mathbb{R}^d, |x-y| < r,$$

and thus $\bar{f}(x)/|f(x)| = \bar{u}$ for any $x \in \mathbb{R}^d$ for a constant $u \in \mathbb{C}$. That ends the proof of the reverse Kato's inequality **(H5)**.

Proof of theorem 8.0.1 part (1). We may use Theorem 2.1.20 in order to establish the existence of a solution $(\lambda_1, f_1, \phi_1) \in (0, +\infty) \times L^1 \times L^\infty$ to the first eigen triplet problem (1.3.30)-(1.3.31). From Theorem 3.3.11 and Theorem 4.4.14, this solution is unique, $f_1 \gg 0$, $\phi_1 \gg 0$, λ_1 is algebraically simple for both \mathcal{L} and \mathcal{L}^* and it is the unique eigenvalue in $\Sigma_+(\mathcal{L})$.

Due to (8.0.14), we actually have $f_1 \in L^1_{\langle W \rangle} \cap L^\infty_{\langle W \rangle}$. Observing that \mathcal{L}^* is of the same type as \mathcal{L} ,

$$\mathcal{L}^* \phi = \check{J} * \phi - W \phi, \quad \check{J}(x) := J(-x),$$

and considering the dual problem as a primal problem in L^1 , Theorem 2.1.20 also provides the existence of $\lambda_1^* > 0$ and $0 \ll \phi_1^* \in L^1_{\langle W \rangle} \cap L^\infty_{\langle W \rangle}$ such that

$$\mathcal{L}^* \phi_1^* = \lambda_1^* \phi_1^*.$$

Because of the f_1 and ϕ_1 are strictly positive, we have in fact $\lambda_1^* = \lambda_1$ and the simplicity of λ_1 then yields that $\text{Span } \phi_1^* = \text{Span } \phi_1$. This ensures that $\phi_1 \in L^1_{\langle W \rangle} \cap L^\infty_{\langle W \rangle}$ and also that ϕ_1 enjoys the explicit lower bound (8.0.16). Besides, we can prove

$$\|\phi_1\|_{L^\infty_{\langle W \rangle}} \leq \|J_1\|_{L^1} \frac{\lambda_1}{\lambda_1 - \kappa_*} \|\phi_1\|_{L^\infty} \leq \|J_1\|_{L^1} \frac{\kappa_1}{\kappa_0 - \kappa_*} \|\phi_1\|_{L^\infty}$$

by arguing similarly as for (8.0.13). \square

In order to prove Theorem 8.0.1 part (2) with constructive constants we use a Doeblin-Harris type argument

Lemma 8.0.3 (Lyapunov Condition). *Under the above assumptions, there are $\gamma_L \in (0, 1)$, $T > 0$ and $K > 0$ such that*

$$\|\tilde{S}_T f\|_{L^1} \leq \gamma_L \|f\|_{L^1} + K \|f\|_{\phi_1}.$$

Proof. Writing $f_t = S(t)f$, we have

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^d} |f_t| &\leq \int_{\mathbb{R}^d} J * |f_t| - \int_{\mathbb{R}^d} (W + \lambda_1) |f_t| \\ &\leq \|J\|_1 \int_{\mathbb{R}^d} |f_t| - \int_{\mathbb{R}^d} (W + \lambda_1) |f_t| \\ &\leq \int_{\mathbb{R}^d \setminus B_R} (\|J\|_1 - W - \lambda_1) |f_t| + \frac{\|J\|_1 - \lambda_1}{\alpha_R} \int_{B_R} |f_t| \phi_1 \\ &\leq - \int_{\mathbb{R}^d \setminus B_R} |f_t| + \frac{\|J\|_1 - \lambda_1}{\alpha_R} \int_{\mathbb{R}^d} |f_t| \phi_1, \end{aligned}$$

for some α_R the bound by below of ϕ_1 in B_R . Since

$$\int_{\mathbb{R}^d} |f_t| \phi_1 \leq \int_{\mathbb{R}^d} S|f_0| \phi_1 = \int_{\mathbb{R}^d} |f_0| \phi_1,$$

we get

$$\|S(t)f\| \leq e^{-t} \|f\| + \frac{1}{\alpha_R} (1 - e^{-t}) \|f\|_{\phi_1},$$

by Grönwall's lemma. \square

Lemma 8.0.4 (Harris' condition). *Under the assumption above, there exists $\psi_0 \in X'_{++}$, $g_0 \in X_+$ and $T > 0$ such that*

$$S_T f \geq g_0 \langle f, \psi_0 \rangle, \quad \forall f \in X_+. \quad (8.0.18)$$

Proof. Step 1. proof of (8.0.18). From Duhamel's formula (2.2.9) we have

$$S_{\mathcal{L}} = S_{\mathcal{B}} + \cdots + (S_{\mathcal{B}} \mathcal{A})^{*(N-1)} * S_{\mathcal{B}} + (S_{\mathcal{B}} \mathcal{A})^{*(N)} * S_{\mathcal{L}}.$$

We note that

$$(S_{\mathcal{B}} \mathcal{A} * S_{\mathcal{B}}) f(x) = \int_0^t S_{\mathcal{B}}(t-s) \mathcal{A} S_{\mathcal{B}}(s) f ds = \int_0^t [\mathcal{A}(f e^{W(x)s})] e^{-W(x)(t-s)} ds.$$

For any $R > r$, $x \in B_R$, it is satisfied that

$$\mathcal{A}(f e^{W(x)s})(x) = \int_{\mathbb{R}^d} J(x-y) f(y) e^{-W(y)s} dy \geq J_* e^{-W[2R]s} \int_{B_r(x)} f(y) dy$$

with $W[R]$ defined as in the proof of (8.0.16). Then we get

$$(S_{\mathcal{B}} \mathcal{A} * S_{\mathcal{B}}) f(x) \geq \mathbf{1}_{B_R}(x) J_* t e^{-W[2R]t} \int_{B_r(x)} f(y) dy.$$

Subsequently, we obtain that

$$S_{\mathcal{B}}\mathcal{A} * (S_{\mathcal{B}}\mathcal{A} * S_{\mathcal{B}})f(x) \geq \mathbf{1}_{B_{R-r}}(x) \int_0^t J_* s e^{-W[2R]t} \mathcal{A} \left(\mathbf{1}_{B_R}(x) \int_{B_r(x)} f(y) dy \right) ds,$$

with

$$\mathcal{A} \left(\mathbf{1}_{B_R}(x) \int_{B_r(x)} f(y) dy \right) = \int_{\mathbb{R}^d} J(x-y) \mathbf{1}_{B_R}(y) \int_{B_r(y)} f(z) dz dy \geq J_* \int_{B_r(x)} \int_{B_r(y)} f(z) dz dy.$$

We claim that for all $a \geq r$,

$$\int_{B_r(x)} \int_{B_a(y)} f(z) dz dy \geq |B_{r/4}| \int_{B_{a+r/2}(x)} f(z) dz.$$

Indeed, we deduce

$$\int_{B_r(x)} \int_{B_a(y)} f(z) dz dy = \int_{B_r(x)} \int_{\mathbb{R}^d} \mathbf{1}_{B_a(y)}(z) f(z) dz dy = \int_{\mathbb{R}^d} f(z) \int_{B_r(x)} \mathbf{1}_{B_a(z)}(y) dy dz$$

and, since for all $z \in B_{a+r/2}(x)$,

$$B_{\frac{r}{4}} \left(\frac{z-x}{|z-x|} \frac{3r}{4} + x \right) \subset B_r(x) \cap B_a(z),$$

we have

$$\int_{B_r(x)} \mathbf{1}_{B_a(z)}(y) dy \geq |B_{r/4}| \mathbf{1}_{B_{a+r/2}(x)}(z),$$

and consequently,

$$\int_{B_r(x)} \int_{B_a(y)} f(z) dz dy \geq |B_{r/4}| \int_{B_{a+r/2}(x)} f(z) dz.$$

We have obtained that

$$S_{\mathcal{B}}\mathcal{A} * (S_{\mathcal{B}}\mathcal{A} * S_{\mathcal{B}})f(x) \geq \mathbf{1}_{B_{R-r}}(x) J_*^2 t^2 / 2 e^{-W[2R]t} \int_{B_{r+r/2}(x)} f(y) dy.$$

Iterating the same argument we arrive to

$$(S_{\mathcal{B}}\mathcal{A})^{(*n)} S_{\mathcal{B}}f(x) \geq \mathbf{1}_{B_{R-nr}}(x) J_*^n \frac{t^n}{n!} e^{-W[2R]t} \int_{B_{r+(n-1)r/2}(x)} f(y) dy.$$

In consequence, for $R = (n+1)r$, we get

$$(S_{\mathcal{B}}\mathcal{A})^{(*n)} S_{\mathcal{B}}f(x) \geq \mathbf{1}_{B_r}(x) J_*^n \frac{t^n}{n!} e^{-W[2(n+1)r]t} \int_{B_{(n-1)r/2}(0)} f(y) dy.$$

Coming back to the Duhamel formula (2.2.9), we deduce

$$S_{\mathcal{L}}f(x) \geq g_0(x) \sum_{n=2}^{\infty} \frac{(J_*t)^n}{n!} e^{-W[2(n+1)r]t} \int_{B_{(n-1)r/2}} f(y) dy,$$

from where (8.0.18) follows with

$$\psi_0 := \sum_{n=2}^{\infty} \frac{(J_*t)^n}{n!} e^{-W[2(n+1)r]t} \mathbf{1}_{B_{(n-1)r/2}}$$

□

Proof of Theorem 8.0.1 part (2). Let us consider $A > 0$ and $f \in X_+$ such that $\|f\| \leq A[f]_{\phi_1}$. For any integer $n \geq 1$, we have

$$\begin{aligned} [f]_{\phi_1} &= \int_{B_n} f \phi_1 + \int_{B_n^c} f \phi_1 \leq \alpha_n \langle f, \psi_0 \rangle + \beta_n \|f\| \\ &\leq \alpha_n \langle f, \psi_0 \rangle + \beta_n A [f]_{\phi_1}, \end{aligned}$$

with $\alpha_n = \|\phi_1\|_{L^\infty_{\langle W \rangle}} / \inf_{B_n} \psi_0$ and $\beta_n = \|\phi_1\|_{L^\infty_{\langle W \rangle}} / \inf_{B_n^c} W$. Choosing n_A such that $\beta_{n_A} A \leq 1/2$, we deduce the constructive estimate

$$[f]_{\phi_1} \leq 2\alpha_{n_A} \langle f, \psi_0 \rangle,$$

and thus that (5.3.7) holds with $g_A := (2\alpha_{n_A})^{-1}g_0$. Because of the constructive lower bound (8.0.16) on ϕ_1 , we have

$$\langle \phi_1, g_R \rangle \geq (2\alpha_{n_A})^{-1} \langle h_0, g_0 \rangle =: r_A,$$

which provides (5.3.9) in a quantified way. The two above estimates and the Lyapunov condition established in Lemma 8.0.3 ensure that we may apply the Harris-Doblin Theorem 5.3.3 and thus conclude to (8.0.6) with constructive rate. \square

Chapter 9

Time elapse model with memory

This chapter is the result of a collaboration with Valentin Schmutz.

9.1 Introduction

Multidimensional mean-field models in theoretical neuroscience are challenging to analyse [216, 240, 21, 189] but their study is a necessary step towards understanding how multiple timescales present at the single-neuron level [208, 230] affect the dynamics of large networks of neurons.

One-dimensional mean-field equations for populations of spiking neurons with deterministic drift with stochastic jumps have been a subject of mathematical studies since the works of Pakdaman, Perthame and Salort [199, 202, 198], providing rigorous foundations to earlier works in theoretical neuroscience [245, 101, 100, 99]. These population equations correspond to the mean-field limit of large networks of interacting neurons [69, 88, 59]. However, they are derived from spiking neuron models that are of the ‘renewal’ type (with the exception of [198]), which means that, while they capture the effect of neuronal refractoriness, they neglect slower neuronal timescales, like those of spike frequency adaptation and short-term synaptic plasticity.

To take into account slow neuronal timescales, state-of-the-art phenomenological spiking neuron models are multidimensional [140, 230] or kernel-based [235, 205, 207] (and see [102, Ch. 6.4]). In the following, we consider a class of neuron models that characterize neuronal refractoriness by an ‘age’ variable (the time elapsed since last spike) and effects of spike frequency adaptation or short-term synaptic plasticity by a ‘leaky memory’ variables. For this class of neuron models, the mean-field limit is characterized by a multidimensional transport equation with a nonlocal boundary condition [223]. In this work, we study the long time behavior of the solutions to the equation proposed in [223], in the two-dimensional case.

9.1.1 The age- and leaky memory-structured model

The population model we consider describes the evolution of a density ρ_t over the state-space $(a, m) \in \mathbb{R}_+ \times \mathbb{R}_+^*$, where a and m are the ‘age’ and ‘leaky memory’ variables of the neuron, and $\rho_t(a, m)$ represents the density of neurons in state (a, m) at time t .

The nonlinear evolution problem for the density ρ_t , for the initial datum u_0 , writes

$$\partial_t \rho_t + \nabla \cdot (b \rho_t) = -f(a, m, \varepsilon x_t) \rho_t, \quad (9.1.1a)$$

$$\rho_t(0, m) = \mathbb{1}_{m > \gamma(0)} \left| (\gamma^{-1})'(m) \right| \int_0^\infty f(a, \gamma^{-1}(m), \varepsilon x_t) \rho_t(a, \gamma^{-1}(m)) da, \quad (9.1.1b)$$

$$x_t = \int_0^t \int_0^\infty \int_0^\infty h(t-s, a, m) f(a, m, \varepsilon x_s) \rho_s(a, m) da dm ds, \quad (9.1.1c)$$

$$\rho_0 = u_0. \quad (9.1.1d)$$

The dynamics of the model can be decomposed in three elements: (i) the behavior of neurons between spikes, (ii) the spike-triggered jumps and (iii) the interaction between neurons, which we discuss in turn.

(i) Between spikes, neurons are transported along the vector field $b(a, m) = (1, -\lambda m)$, with $\lambda > 0$ ($\nabla \cdot$ denotes the divergence operator over the state-space).

(ii) Neurons spike at a rate $f(a, m, \varepsilon x_t)$, where $f : \mathbb{R}_+ \times \mathbb{R}_+^* \times \mathbb{R} \rightarrow \mathbb{R}_+$ is the ‘firing rate function’ corresponding to the stochastic intensity of the spike generation process and $\varepsilon \in \mathbb{R}$ is the connection strength. When a neuron spikes, its age a is reset to 0 and its leaky memory variable m jumps to $\gamma(m)$, where $\gamma : \mathbb{R}_+ \rightarrow \mathbb{R}_+^*$ is the ‘jump mapping’ and is assumed to be a strictly increasing \mathcal{C}^1 -diffeomorphism. As a consequence, the border condition (9.1.1b) has a simple interpretation: the density of neurons in state $(0, m)$ at time t is equal to the marginal density of those neurons that have their leaky memory variable in state $\gamma^{-1}(m)$ and spike at time t . The indicator function $\mathbb{1}_{m > \gamma(0)}$ reflects the fact that m is always strictly positive and the term $|(\gamma^{-1})'(m)|$ is necessary to guarantee the conservation of the total mass of neurons. Indeed, formally,

$$\begin{aligned} \partial_t \int \rho_t &= \int \mathbb{1}_{m > \gamma(0)} |(\gamma^{-1})'(m)| \int_0^\infty f(a, \gamma^{-1}(m), \varepsilon x_t) \rho_t(a, \gamma^{-1}(m)) da dm - \int f(a, m, \varepsilon x_t) \rho_t \\ &= 0, \end{aligned}$$

by a change of variable.

(iii) Neurons interact through the ‘total postsynaptic potential’ x_t , which integrates the past spiking activity of the population, filtered by the ‘interaction function’ $h : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+^* \rightarrow \mathbb{R}$. x_t , weighted by the connection strength $\varepsilon \in \mathbb{R}$, influences the firing rate f . If we write $N(t)$ for the mean firing rate

$$N(t) := \int_0^\infty \int_0^\infty f(a, m, \varepsilon x_t) \rho_t(a, m) da dm,$$

and if we take h independent of a and m , then x_t takes the form

$$x_t = \int_0^t h(t-s) N(s) ds,$$

where now h is simply a delay kernel, as in [101, 100, 99, 199]. In our formulation, h in Eq. (9.1.1c) allows to model more general interactions. For example, in Sec. 9.1.2, we show that by choosing $h(t, a, m) = \hat{h}(t)(1-m)$, we can include the effects of a classical short-term synaptic plasticity model [238].

9.1.2 Motivation

The model (9.1.1) extends the time elapsed neuron network model [199] (see also [100, 99]) by the addition of a leaky memory variable which can accumulate over spikes (as opposed to the age variable which is reset to 0 at each spike) and hence introduces a slow timescale in the population dynamics. Such a slow timescale is typically used to account for some form of fatigue mechanism, which can act on the spiking activity (spike frequency adaptation) or on synaptic transmission (short-term synaptic depression). Slow fatigue at the single neuron level can lead to nontrivial emergent behaviors at the population level, like population bursts [243, 103, 97] (see Fig. 1), which have not been observed in the age- or voltage-structured models of [199] and [69] (but see [198]). Even though some population equations have been successfully used in the computational neuroscience literature to study emergent behaviors in networks of neurons with fatigue, these population equations were obtained at the cost of a timescale separation approximation [103, 97] or a ‘mixing’ assumption [193, 224], making them inexact. In contrast, the model (9.1.1) is the exact mean-field limit [223] for spiking neuron models with spike-frequency adaptation or short-term synaptic depression, as we discuss now.

Spike frequency adaptation

The recent spike history of a neuron can modulate its firing rate f , leading to spike frequency adaptation [22]. If h is independent of a and m and if $\gamma(m) = m + \hat{\Gamma}$, for a fixed $\hat{\Gamma} > 0$, (9.1.1) becomes

$$\partial_t \rho_t + \nabla \cdot (b \rho_t) = -f(a, m, \varepsilon x_t) \rho_t, \quad (9.1.2a)$$

$$\rho_t(0, m) = \mathbb{1}_{m > \hat{\Gamma}} \int_0^\infty f(a, m - \hat{\Gamma}, \varepsilon x_t) \rho_t(a, m - \hat{\Gamma}) da, \quad (9.1.2b)$$

$$x_t = \int_0^t h(t-s) \int_0^\infty \int_0^\infty f(a, m, \varepsilon x_s) \rho_s(a, m) da dm ds, \quad (9.1.2c)$$

$$\rho_0 = u_0. \quad (9.1.2d)$$

If $\eta : \mathbb{R}_+ \rightarrow \mathbb{R}$ is a bounded function such that $\lim_{a \rightarrow +\infty} \eta(a) = 0$ (η is the ‘refractory kernel’ [102, Sec. 9.3]), we can define f more explicitly:

$$f(a, m, \varepsilon x_t) := \hat{f}(\eta(a) - m + \varepsilon x_t), \quad (9.1.2e)$$

where $\hat{f} : \mathbb{R} \rightarrow \mathbb{R}_+$ is typically a non-decreasing function. Since m makes jumps of size $\hat{\Gamma} > 0$ at each spike and decays exponentially at rate λ between spikes, m accumulates over spikes, which decreases the firing rate f (Eq. (9.1.2e)), leading to spike frequency adaptation [22]. More specifically, Eq. (9.1.2) is a population equation for adaptive SRM₀ (Spike Response Model) neurons [132, 102].

Populations of spiking neurons with spike frequency adaptation exhibit self-sustained population bursts when the connectivity strength is sufficiently strong [243, 103, 97]. In Fig. 1, we show simulations of (9.1.2) for two different connectivity strengths ε . For large ε , we observe self-sustained bursts, whereas for small ε , we observe relaxation to a stationary state. For comparison, in the Appendix, we show similar simulations for the time elapsed neuron network model [199], where, as expected, we only observe self-sustained oscillations or relaxation to a stationary state.

Short-term synaptic depression

The recent spike history of a presynaptic neuron can modulate the synaptic transmission, leading to short-term synaptic plasticity [249]. We will consider here the case of depressive synapses and use the model of [238] (with a change of variable for convenience). In this case, the state-space is $(a, m) \in \mathbb{R}_+ \times]0, 1[$. Taking f independent of m , and choosing h and γ of the form $h(t, a, m) := \hat{h}(t)(1 - m)$ and $\gamma(m) := 1 - v + vm$ for a fixed $v \in]0, 1[$, (9.1.1) becomes

$$\partial_t \rho_t + \nabla \cdot (b \rho_t) = -f(a, m, \varepsilon x_t) \rho_t, \quad (9.1.3a)$$

$$\rho_t(0, m) = \mathbb{1}_{m > \gamma(0)} \frac{1}{v} \int_0^\infty f(a, \varepsilon x_t) \rho_t(a, \gamma^{-1}(m)) da, \quad (9.1.3b)$$

$$x_t = \int_0^t \hat{h}(t-s) \int_0^1 \int_0^\infty (1-m) f(a, \varepsilon x_s) \rho_s(a, m) da dm ds, \quad (9.1.3c)$$

$$\rho_0 = u_0. \quad (9.1.3d)$$

Note that the term $\frac{1}{v}$ on the RHS of Eq. (9.1.3b) simply comes from the fact that $|(\gamma^{-1})'(m)| = \frac{1}{v}$, for all $m \in]0, 1[$. Here, at each spike, m makes strictly positive jumps which size tends to 0 as m tends to 1 (since $\gamma(1) = 1$) and decays exponentially at rate λ between spikes. If m is close to 1, synaptic transmission is weak because of the factor $(1 - m)$ in Eq. (9.1.3c).

As observed in [219], the stationary state of populations of neurons with short-term synaptic plasticity can be described by a simple formula, which we prove in Sec. 9.4.3.

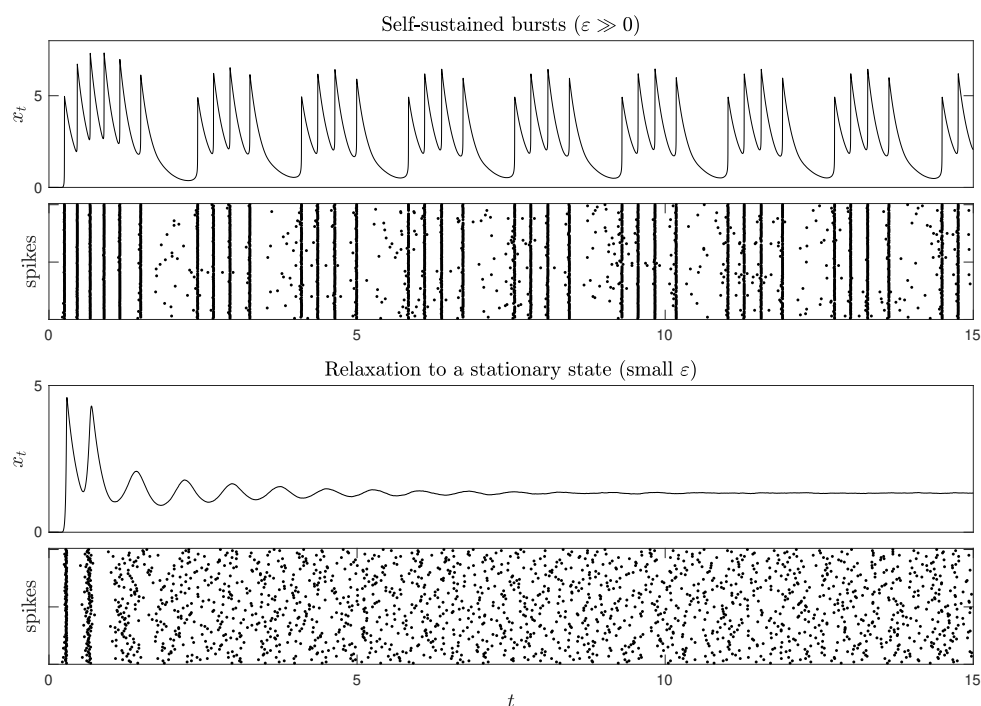


Figure 9.1: Depending on the connectivity strength ε , a population of adaptive SRM₀ can exhibit self-sustained bursts ($\varepsilon \gg 0$) or relaxation to a stationary state (small ε). We show simulations of a network of $5 \cdot 10^5$ adaptive SRM₀ neurons, approximating the mean-field limit Eq. (9.1.2), with identical parameters (except for ε) and identical initial conditions. The raster plots below the plots for the time-evolution of the total postsynaptic potential x_t represent the spikes of 100 randomly selected neurons.

9.1.3 Assumptions and main results

The main result of this work is the exponential stability of (9.1.1) in the weak connectivity regime (**Theorem 3**) – or, more explicitly, there exists $\varepsilon^{**} > 0$ such that (9.1.1) is exponentially stable for all connectivity strength $\varepsilon \in]-\varepsilon^{**}, +\varepsilon^{**}[$. Before proving the exponential stability, we first establish the well-posedness of (9.1.1) in the appropriate function space (**Theorem 1**) and show that stationary solutions exist and are unique for sufficiently weak connectivity (**Theorem 2**).

Here, we study the weak solutions to (9.1.1) for an initial datum in $L_+^1 := L^1(\mathbb{R}_+ \times \mathbb{R}_+^*, \mathbb{R}_+)$ and write $L_+^1(\mathbb{R}_+^*) := L^1(\mathbb{R}_+^*, \mathbb{R}_+)$.

Definition (Solutions). $(\rho, x) \in \mathbb{C}(\mathbb{R}_+, L_+^1) \times \mathbb{C}(\mathbb{R}_+)$ is a solution to (9.1.1), for the initial datum $u_0 \in L_+^1$, if

$$x_t = \int_0^t \int_0^\infty \int_0^\infty h(t-s, a, m) f(a, m, \varepsilon x_s) \rho_s(a, m) da dm ds, \quad \forall t \geq 0, \quad (9.1.4a)$$

and if for all $\varphi \in \mathbb{C}_c^\infty(\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+^*)$,

$$\begin{aligned} 0 &= \int_0^\infty \int_0^\infty u_0(a, m) \varphi(0, a, m) da dm \\ &+ \int_0^\infty \int_0^\infty \int_0^\infty \rho_t(a, m) \left\{ [\partial_t + \partial_a - \lambda m \partial_m] \varphi + (\varphi(t, 0, \gamma(m)) - \varphi(t, a, m)) f(a, m, \varepsilon x_t) \right\} da dm dt. \end{aligned} \quad (9.1.4b)$$

To prove the well-posedness of (9.1.1), we need some simple assumptions of the firing rate function f and the interaction function h :

Assumptions 1. f is bounded and L_f -Lipschitz, i.e.

$$|f(a, m, x) - f(a^*, m^*, x^*)| \leq L_f (|a - a^*| + |m - m^*| + |x - x^*|),$$

and h is bounded and continuous.

Since we want to apply Harris' theorem, the well-posedness in L^1 (which is treated in [223]) is not enough and we need the well-posedness in a weighted L^1 space (where the weight satisfies a Lyapunov condition [166]) with a global-in-time estimate in the weighted L^1 norm.

Using the weight function

$$w : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow [1, \infty), \quad (a, m) \mapsto 1 + m,$$

we define the function space

$$L_+^1(w) := \left\{ g \in L^1(\mathbb{R}_+ \times \mathbb{R}_+^*, \mathbb{R}_+) \mid \|g\|_{L^1(w)} := \int_0^\infty \int_0^\infty g(a, m) w(a, m) da dm < \infty \right\}.$$

To obtain a global-in-time estimate in the $L_+^1(w)$ norm, we further need that the jump sizes of γ are bounded:

Assumptions 2. There exists a bounded function $\Gamma : \mathbb{R}_+^* \rightarrow \mathbb{R}_+^*$ such that for all $m \in \mathbb{R}_+^*$, $\gamma(m) = m + \Gamma(m)$.

Theorem 1 (Well-posedness). Grant Assumption 1. For any initial datum $u_0 \in L_+^1$, there exists a unique weak solution (ρ, x) to (9.1.1). This solution satisfies

$$I \text{ (} L^1\text{-stability)} \quad \|\rho_t\|_{L^1} = \|u_0\|_{L^1}, \quad \forall t > 0,$$

II (Global bound in $L^1_+(w)$) if, in addition, Assumption 2 holds and $u_0 \in L^1_+(w)$, then

$$\forall t > 0, \quad \|\rho_t\|_{L^1(w)} \leq \|u_0\|_{L^1(w)} e^{-\alpha t} + \frac{b}{\alpha}(1 - e^{-\alpha t}), \quad (9.1.5)$$

for some constants $\alpha > 0$ and $b \in \mathbb{R}$.

In contrast to [223], the well-posedness proof presented here does not involve any probabilistic argument. The proof consists of two consecutive applications of Banach's fixed-point theorem, where a first fixed-point gives the unique solution to a linearized version of (9.1.1) which is then used in a second fixed-point treating the nonlinearity of (9.1.1).

The second step towards the exponential stability proof is the study of the existence and uniqueness of the stationary solutions to (9.1.1). For this step, we require:

Assumptions 3.

I There exists $\Delta_{abs} > 0$ and $\sigma > 0$ such that

$$f(a, m, x) \geq \sigma, \quad \forall (a, m, x) \in [\Delta_{abs}, +\infty[\times \mathbb{R}_+^* \times \mathbb{R}.$$

II There exists $C_\gamma \in]0, 1]$ such that $C_\gamma \leq \gamma' \leq 1$.

III $\bar{h}(a, m) = \int_0^\infty h(t, a, m) dt$ is bounded.

The first point of Assumption 3 sets a lower bound on the firing rate function f for any $a \geq \Delta_{abs}$ and hence allows for an absolute refractory period $\Delta_{abs} > 0$, i.e. a period of time following a spike during which $f = 0$ (which is an important neurodynamical feature [102, Sec. 1.1]). This assumption is also used in [48].

In the second point of Assumption 3, the lower bound $0 < C_\gamma \leq \gamma'$ guarantees that γ is strictly increasing, which reflects the idea that m is a 'leaky memory' variable of the past neuronal activity. On the other hand, the upper bound $\gamma' \leq 1$, which can be rewritten in terms of the jump size function Γ as $\Gamma' \leq 0$, prevents the variable m from growing too fast and allows for a potential saturation of the memory, as in the example with short-term synaptic plasticity (9.1.3). The third point of Assumption 3 reflects the fact that a single spike has a finite impact on the neuron that receives it.

We emphasize that the two examples shown above, spike frequency adaptation (9.1.2) and short-term synaptic depression (9.1.3), satisfy Assumption 3.

Theorem 2 (Stationary solutions). *Grant Assumptions 1 – 3.*

I There exists a stationary solution to (9.1.1).

II There exists $\varepsilon^* > 0$ such that for all $\varepsilon \in]-\varepsilon^*, +\varepsilon^*[$, the stationary solution to (9.1.1) is unique.

Over the course of this work, we obtained the existence of the stationary solution by two different approaches. The first approach is based on the Doeblin-Harris method [118] and is similar to that of [48]. First, we show that when x_t is fixed and time-invariant in (9.1.1) (neurons are non-interacting), the system satisfies a Harris condition – this constitutes a key result of this work –, and we can use Harris' theorem to get the stationary solution. Then, we use the Lipschitz continuity the stationary solutions with respect to the fixed x to prove the existence of a stationary solution for arbitrary connectivity strengths ε . Finally, for ε small enough, we also get the uniqueness of the stationary solution, by Banach's fixed-point theorem.

The second approach relies on the fact that the stationary solutions solve an integral equation, for which we can show that a solution exists by Schauder's fixed-point theorem. In the process,

we get several estimates on the stationary solutions, namely that they are continuous, bounded, and exponentially decaying in m . However, this approach does not give uniqueness.

As mentioned above, the application of Harris' theorem requires us to consider solutions in the weighted space $L^1(w)$. However, in the case where the state-space of the leaky memory variable m is bounded, the situation is simpler: we can use Doeblin's theorem in L^1 . The following assumption guarantees that m stays in a bounded state-space:

Assumptions 4. *There exists $G > 0$ such that for all $m \in \mathbb{R}_+^*$, $\gamma(m) < G$.*

Note that this assumption is satisfied in the example with short-term synaptic plasticity (9.1.3), with $G = 1$.

Finally, to study the exponential stability of (9.1.1), we need an exponential decay on h :

Assumptions 5. *There exists $\mathfrak{h}, C_h > 0$ such that $h(t, a, m) \leq C_h e^{-\mathfrak{h}t}$, $\forall (t, a, m)$.*

By a perturbation argument similar to that of [181], we obtain our main result:

Theorem 3 (Exponential stability in the weak connectivity regime). *Grant Assumptions 1 – 3 and 5. For any $W > 0$, there exists $\varepsilon_W^{**} > 0$ such that for $\varepsilon \in]-\varepsilon_W^{**}, +\varepsilon_W^{**}[$, there exists $C \geq 1$ and $c_W > 0$ such that for all initial data $u_0 \in L_+^1(w)$ with $\|u_0\|_{L^1} = 1$ and $\|u_0\|_{L^1(w)} \leq W$, the solution (ρ, x) to (9.1.1) satisfies*

$$\|\rho_t - \rho_\infty\|_{L^1(w)} + |x_t - x_\infty| \leq C e^{-c_W t} (\|u_0 - \rho_\infty\|_{L^1(w)} + 1), \quad \forall t \geq 0, \quad (9.1.6)$$

where (ρ_∞, x_∞) is the unique stationary solution given by Theorem 2 (ii).

If, in addition, we grant Assumption 4, then there exists $\varepsilon^{**} > 0$ such that for all $\varepsilon \in]-\varepsilon^{**}, +\varepsilon^{**}[$, there exists $C' \geq 1$ and $c > 0$ such that for all initial data $u_0 \in L_+^1$ with $\|u_0\|_{L^1} = 1$,

$$\|\rho_t - \rho_\infty\|_{L^1} + |x_t - x_\infty| \leq C' e^{-ct} (\|u_0 - \rho_\infty\|_{L^1} + 1), \quad \forall t \geq 0. \quad (9.1.7)$$

From the neuronal modeling point of view, this result is not surprising: when the connection strength is weak enough, neurons do not synchronize and the population activity converges to a stationary state. This was already proved for simpler one-dimensional models (see below) and the addition of a leaky memory variable carrying the effect of spike frequency adaptation or short-term synaptic plasticity does not change this behavior.

9.1.4 Discussion of the methods

The asymptotic stability of the age-structured model of [199] in the weak connectivity regime has been studied using entropy methods (assuming that f is a step-function) [199, 202], spectral analysis of semigroups in Banach spaces [182, 181] or Doeblin's theorem [48]. For the treatment of the strong connectivity regime, we refer to [199, 181].

On the closely related voltage-structured model [69], [65] also proved asymptotic stability in the weak connectivity regime using Laplace transform techniques. For this model, the nonlinear stability of the stationary solutions has been recently studied in [62] (see also [79]) and can identify Hopf bifurcations [64].

Doeblin's theorem has also been used in [81] in the case of the 'threshold crossing' neuronal population equation of [197]. Note that closely related methods have been used by probabilists to study the ergodicity of single neuron models [127, 80].

Our approach combines strategies from [182] and [48], even though [48] uses Doeblin's instead of Harris' theorem. On the one hand, our proof is based on the application of Harris' theorem for the linear problem, which simplifies the proof of [182]. On the other hand, we use an argument from in [182] to deal with delay effects, which are not considered in [48]. Note that our model is two-dimensional (by the addition of the leaky memory variable), whereas the aforementioned works only considered one-dimensional models.

9.1.5 Plan of the chapter

The proof of Theorem 1 (Well-posedness) is presented in **Section 9.2**. In **Section 9.3**, we prove the exponential stability of (9.1.1) in the non-interacting case $\varepsilon = 0$ using Harris' or Doeblin's theorem. The proof of Theorem 2 (Stationary solutions) is presented in **Section 9.4** which is divided in three parts: in the first part, we present a proof which uses the exponential stability of the non-interacting case; in the second part, we present an alternative proof for the existence of stationary solutions which does not involve the Doeblin-Harris method; and in the last part, we present a proof for the formula of [219] in the case of short-term synaptic plasticity (9.1.3). Finally, **Section 9.5** is dedicated to the proof of Theorem 3 (Exponential stability in the weak connectivity regime).

9.2 Well-posedness

This section is dedicated to the proof of Theorem 1, which we decompose in several lemmas. First, we verify the *a priori* L^1 -stability of the solutions to (9.1.1), a technical result we use later in the proof. Then, we introduce a linearized version of (9.1.1) and show that it is well-posed by an application of Banach's fixed-point theorem. Another Banach's fixed-point argument is used to treat the nonlinearity of (9.1.1) and concludes the proof of the well-posedness in L^1 . Finally, we prove the global bound in $L^1_+(w)$ (point (ii) of Theorem 1), which we will use to apply Harris' theorem in the next sections.

Lemma 1 (*A priori L^1 -stability*). *Grant Assumption 1. If (ρ, x) is a weak solution to (9.1.1) for the initial datum $u_0 \in L^1_+$, then*

$$\|\rho_t\|_{L^1} = \|u_0\|_{L^1}, \quad \forall t > 0.$$

Proof. By a standard cut-off in time argument, we have that for all $T > 0$ and for all $\varphi \in C_c^\infty(\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+^*)$,

$$\begin{aligned} & \int_0^\infty \int_0^\infty \rho_T(a, m) \varphi(T, a, m) da dm - \int_0^\infty \int_0^\infty u_0(a, m) \varphi(0, a, m) da dm = \\ & \int_0^T \int_0^\infty \int_0^\infty \rho_t(a, m) \left\{ [\partial_t + \partial_a - \lambda m \partial_m] \varphi + (\varphi(t, 0, \gamma(m)) - \varphi(t, a, m)) f(a, m, \varepsilon x_t) \right\} da dm dt. \end{aligned}$$

Let χ be a function in $C_c^\infty(\mathbb{R}_+ \times \mathbb{R}_+^*, \mathbb{R}_+)$ such that

$$\chi(a, m) = 1, \quad \text{for all } a^2 + m^2 \leq 1.$$

For all $n \in \mathbb{N}^*$, we write $\tilde{\varphi}^n \in C^\infty(\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+^*)$ the classical solution to the transport equation

$$\partial_t \tilde{\varphi}^n(t, a, m) + \partial_a \tilde{\varphi}^n(t, a, m) - \lambda m \partial_m \tilde{\varphi}^n(t, a, m) = 0, \quad (9.2.8a)$$

$$\tilde{\varphi}^n(0, a, m) = \chi(a/n, m/n). \quad (9.2.8b)$$

Because of the finite speed of propagation of the transport equation, for all n , there exists a function $\varphi^n \in C_c^\infty(\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+^*)$ such that $\varphi^n(t, a, m) = \tilde{\varphi}^n(t, a, m)$, for all $(t, a, m) \in [0, T] \times \mathbb{R}_+ \times \mathbb{R}^*$. Hence, for all $n \in \mathbb{N}^*$,

$$\begin{aligned} & \int_0^\infty \int_0^\infty \varphi^n(T, a, m) \rho_T(a, m) da dm - \int_0^\infty \int_0^\infty \varphi^n(0, a, m) u_0(a, m) da dm = \\ & \int_0^T \int_0^\infty \int_0^\infty \left\{ \partial_t \varphi^n + \partial_a \varphi^n - \lambda m \partial_m \varphi^n + (\varphi^n(t, 0, \gamma(m)) - \varphi^n(t, a, m)) f(a, m, \varepsilon x_t) \right\} \rho_t(a, m) da dm dt. \end{aligned}$$

As φ_n is a solution to Eq. (9.2.8a) on time $[0, T]$, we get

$$\begin{aligned} \int_0^\infty \int_0^\infty \varphi^n(T, a, m) \rho_T(a, m) da dm - \int_0^\infty \int_0^\infty \varphi^n(0, a, m) u_0(a, m) da dm = \\ \int_0^T \int_0^\infty \int_0^\infty \left\{ (\varphi^n(t, 0, \gamma(m)) - \varphi^n(t, a, m)) f(a, m, \varepsilon x_t) \right\} \rho_t(a, m) da dm dt. \end{aligned}$$

For all $(t, a, m) \in [0, T] \times \mathbb{R}_+ \times \mathbb{R}_+^*$, $\varphi^n(t, a, m) \xrightarrow{n \rightarrow \infty} 1$, since the initial datum tends to 1 as $n \rightarrow \infty$ (Eq. (9.2.8b)) and by finite speed of propagation. Thus, by dominated convergence, we get

$$\int_0^\infty \int_0^\infty \rho_T(a, m) da dm - \int_0^\infty \int_0^\infty u_0(a, m) da dm = 0. \quad (9.2.9)$$

Since ρ is nonnegative, this concludes the proof. \square

Lemma 1 will allow us to prove the well-posedness of (9.1.1) by the means of fixed-point arguments. Let us first introduce a linearized version of Eq. (9.1.1): for all $x \in \mathcal{C}(\mathbb{R}_+)$, we consider the linear evolution problem

$$\partial_t \rho_t + \nabla \cdot (b \rho_t) = -f(a, m, \varepsilon x_t) \rho_t, \quad (9.2.10a)$$

$$\rho_t(0, m) = \mathbb{1}_{m > \gamma(0)} \left| (\gamma^{-1})'(m) \right| \int_0^\infty f(a, \gamma^{-1}(m), \varepsilon x_t) \rho_t(a, \gamma^{-1}(m)) da, \quad (9.2.10b)$$

$$\rho_0 = u_0. \quad (9.2.10c)$$

We can see Eq. (9.2.10) as the Kolmogorov forward equation of a time-dependent Markov process. Indeed, we can rewrite Eqs. (9.2.10a) and (9.2.10b) as

$$\partial_t \rho_t = \mathcal{L}_t \rho_t \quad (9.2.11)$$

where, for all suitable test function $\phi : \mathbb{R}_+ \times \mathbb{R}_+^* \rightarrow \mathbb{R}$,

$$\mathcal{L}_t^* \phi(a, m) = b(a, m) \cdot \nabla \phi(a, m) + [\phi(0, \gamma(m)) - \phi(a, m)] f(a, m, \varepsilon x_t). \quad (9.2.12)$$

\mathcal{L}_t^* is the time-dependent generator of a piecewise deterministic Markov process with degenerate jumps.

The linearized equation (9.2.10) will play a special role in the following sections and it therefore deserves its own proposition:

Proposition 1 (Well-posedness of the linearized equation (9.2.10)). *Grant Assumption 1. For any initial datum $u_0 \in L_+^1$ and any $x \in \mathcal{C}(\mathbb{R}_+)$, there exists a unique weak solution $\rho^x \in \mathcal{C}(\mathbb{R}_+, L_+^1)$ to Eq. (9.2.10). Furthermore, ρ^x satisfies*

I For all $t > 0$ and for all $m \in \mathbb{R}_+^*$,

$$\begin{aligned} \rho_t^x(0, m) &= \mathbb{1}_{m > \gamma(0)} \left| (\gamma^{-1})'(m) \right| \int_0^\infty f(a, \gamma^{-1}(m), \varepsilon x) \rho_t^x(a, \gamma^{-1}(m)) da, \\ \rho_t^x(a, m) &= \begin{cases} u_0(a - t, e^{\lambda t} m) \exp \left(\lambda t - \int_0^t f(a - t + s, e^{\lambda(t-s)} m, \varepsilon x) ds \right) & \text{if } a \geq t, \\ \rho_{t-a}^x(0, e^{\lambda a} m) \exp \left(\lambda a - \int_{t-a}^t f(a - t + s, e^{\lambda(t-s)} m, \varepsilon x) ds \right) & \text{if } 0 < a < t. \end{cases} \end{aligned}$$

II For all $t > 0$ and for all $\phi \in \mathcal{C}_c^\infty(\mathbb{R}_+ \times \mathbb{R}_+^*)$,

$$\langle \rho_t^x, \phi \rangle = \langle u_0, \phi \rangle + \int_0^t \langle \rho_s^x, \mathcal{L}_s^* \phi \rangle ds. \quad (9.2.14)$$

Proof. Fix $x \in \mathcal{C}(\mathbb{R}_+)$. For all $p \in \mathbb{C}(\mathbb{R}_+, L_+^1(\mathbb{R}_+^*))$ and $u_0 \in L_+^1$, we know, from the standard theory of transport equations, that there is a unique weak solution to

$$\begin{aligned}\partial_t \rho_t + \nabla \cdot (b \rho_t) &= -f(a, m, \varepsilon x_t) \rho_t, \\ \rho_t(0, m) &= p_t(m), \\ \rho_0 &= u_0,\end{aligned}$$

which we denote $\rho^{x,p}$ and is given by the representation formula,

$$\rho_t^{x,p}(a, m) := \begin{cases} u_0(a-t, e^{\lambda t} m) \exp\left(\lambda t - \int_0^t f(a-t+s, e^{\lambda(t-s)} m, \varepsilon x_s) ds\right) & \text{if } a \geq t, \\ p_{t-a}(e^{\lambda a} m) \exp\left(\lambda a - \int_{t-a}^t f(a-t+s, e^{\lambda(t-s)} m, \varepsilon x_s) ds\right) & \text{if } 0 < a < t. \end{cases}$$

$\rho^{x,p}$ is in $\mathbb{C}(\mathbb{R}_+, L^1)$, since

$$\forall t \in \mathbb{R}_+, \quad \left\| \rho_t^{x,p} \right\|_{L^1} \leq \|u_0\|_{L^1} + \int_0^t \|p_s\|_{L^1} ds.$$

We have

$$\left(\mathbb{1}_{m > \gamma(0)} \left| (\gamma^{-1})'(m) \right| \int_0^\infty f(a, \gamma^{-1}(m), x) \rho_t^{x,p}(a, \gamma^{-1}(m)) da \right)_{(t,m) \in \mathbb{R}_+ \times \mathbb{R}_+^*} \in \mathbb{C}(\mathbb{R}_+, L_+^1(\mathbb{R}_+^*))$$

since

$$\begin{aligned}\forall t \in \mathbb{R}_+, \quad \int_{\gamma(0)}^\infty \left| (\gamma^{-1})'(m) \right| \int_0^\infty f(a, \gamma^{-1}(m), x) \rho_t^{x,p}(a, \gamma^{-1}(m)) dadm \\ \leq \|f\|_\infty \left\| \rho_t^{x,p} \right\|_{L^1} \leq \|f\|_\infty \left(\|u_0\|_{L^1} + \int_0^t \|p_s\|_{L^1} ds \right).\end{aligned}$$

Hence, we can define, for any $T > 0$, the operator Φ_T^x :

$$\begin{aligned}\mathbb{C}([0, T], L_+^1(\mathbb{R}_+^*)) &\rightarrow \mathbb{C}([0, T], L_+^1(\mathbb{R}_+^*)) \\ p &\mapsto \left(\mathbb{1}_{m > \gamma(0)} \left| (\gamma^{-1})'(m) \right| \int_0^\infty f(a, \gamma^{-1}(m), x) \rho_t^{x,p}(a, \gamma^{-1}(m)) da \right)_{(t,m) \in [0, T] \times \mathbb{R}_+^*}.\end{aligned}$$

For any $p, q \in \mathbb{C}([0, T], L_+^1(\mathbb{R}_+^*))$,

$$\begin{aligned}\left\| \Phi_T^x(p) - \Phi_T^x(q) \right\|_{\mathbb{C}([0, T], L^1)} &\leq \|f\|_\infty \sup_{t \in [0, T]} \left\| \rho_t^{x,p} - \rho_t^{x,q} \right\|_{L^1} \\ &\leq \|f\|_\infty \int_0^T \|p_s - q_s\|_{L^1} ds \\ &\leq T \|f\|_\infty \|p - q\|_{\mathbb{C}([0, T], L^1)}.\end{aligned}$$

Therefore, if $0 < T < \|f\|_\infty^{-1}$, Φ_T^x is a contraction. By Banach's fixed-point theorem, there exists a unique $\rho^x \in \mathbb{C}([0, T], L_+^1)$ solving Eq. (9.2.10). Since the choice of the contracting T does not depend on the initial datum, we can iterate the above argument on successive time intervals of length T and conclude that there exists a unique $\rho^x \in \mathbb{C}(\mathbb{R}_+, L_+^1)$ solving Eq. (9.2.10) for which the formula (i) is satisfied. Then, (ii) follows from a standard cut-off in time argument. \square

Now, we can prove the existence and uniqueness of a solution to the nonlinear problem (9.1.1) by the means of a second application of Banach's fixed-point theorem.

Proof of the well-posedness of (9.1.1) in L^1 . For any $x \in \mathcal{C}(\mathbb{R}_+)$, we take the ρ^x given by Proposition 1. We have

$$\left(\int_0^t \int_{\mathbb{R}_+ \times \mathbb{R}_+^*} h(t-s) f(\varepsilon x_s) \rho_s^x \, dadmds \right)_{t \in \mathbb{R}_+} \in \mathbb{C}(\mathbb{R}_+)$$

since

$$\forall t \in \mathbb{R}_+, \quad \left| \int_0^t \int_{\mathbb{R}_+ \times \mathbb{R}_+^*} h(t-s) f(\varepsilon x_s) \rho_s^x \, dadmds \right| \leq \|h\|_\infty \|f\|_\infty \int_0^t \|\rho_s^x\|_{L^1} \, ds.$$

Hence, for any $T > 0$, we can define the operator

$$\begin{aligned} \Psi_T : \mathbb{C}([0, T]) &\rightarrow \mathbb{C}([0, T]) \\ x &\mapsto \left(\int_0^t \int_{\mathbb{R}_+ \times \mathbb{R}_+^*} h(t-s) f(\varepsilon x_s) \rho_s^x \, dadmds \right)_{t \in [0, T]}. \end{aligned}$$

For any $x, y \in \mathbb{C}([0, T])$, we have

$$\begin{aligned} \|\Psi_T(x) - \Psi_T(y)\|_{\mathbb{C}([0, T])} &\leq T \|h\|_\infty \sup_{t \in [0, T]} \int_{\mathbb{R}_+ \times \mathbb{R}_+^*} |f(\varepsilon x_t) \rho_t^x - f(\varepsilon y_t) \rho_t^y| \, dadm \\ &\leq T \|h\|_\infty \sup_{t \in [0, T]} \left(\varepsilon L_f |x_t - y_t| \|\rho_t^x\|_{L^1} + \|f\|_\infty \|\rho_t^x - \rho_t^y\|_{L^1} \right). \end{aligned}$$

By Grönwall's lemma, $\|\rho_t^x\|_{L^1} \leq \|u_0\|_{L^1} \exp(\|f\|_\infty t)$, since

$$\forall t \in [0, T], \quad \|\rho_t^x\|_{L^1} \leq \|u_0\|_{L^1} + \|f\|_\infty \int_0^t \|\rho_s^x\|_{L^1} \, ds.$$

On the other hand, we have, for all $t \in [0, T]$,

$$\begin{aligned} \|\rho_t^x - \rho_t^y\|_{L^1} &\leq \int_0^t \int_0^\infty \left| \rho_s^x(0, m) \exp\left(-\int_s^t f(u-s, e^{-\lambda(u-s)} \gamma(m), \varepsilon x_u) du\right) \right. \\ &\quad \left. - \rho_s^y(0, m) \exp\left(-\int_s^t f(u-s, e^{-\lambda(u-s)} \gamma(m), \varepsilon y_u) du\right) \right| \, dmds \\ &\leq \|f\|_\infty \int_0^t \|\rho_s^x - \rho_s^y\|_{L^1} \, ds + t \varepsilon \|f\|_\infty L_f \|x - y\|_{\mathbb{C}([0, T])} \int_0^t \|\rho_s^x\|_{L^1} \, ds. \end{aligned}$$

Hence, by Grönwall's lemma, for all $t \in [0, T]$,

$$\|\rho_t^x - \rho_t^y\|_{L^1} \leq \varepsilon L_f \|u_0\|_{L^1} \frac{(\exp(\|f\|_\infty t) - 1)^2}{\|f\|_\infty} \|x - y\|_{\mathbb{C}([0, T])}.$$

Gathering the bounds, we get

$$\|\Psi_T(x) - \Psi_T(y)\|_{\mathbb{C}([0, T])} \leq T \varepsilon \|h\|_\infty L_f \|u_0\|_{L^1} \exp(\|f\|_\infty T) [1 + \exp(\|f\|_\infty T)] \|x - y\|_{\mathbb{C}([0, T])}.$$

For T small enough, Ψ_T is a contraction and, by Banach's fixed-point theorem, has a unique fixed-point. Thus, there exists a unique solution $(\rho, x) \in \mathcal{C}([0, T], L_+^1)$. Since, by Lemma 1, $\|\rho_T\|_{L^1} = \|u_0\|_{L^1}$, we can iterate this argument on successive time intervals of length T and conclude that there exists a unique solution in $\mathcal{C}(\mathbb{R}_+, L_+^1)$. \square

To conclude the proof of Theorem 1, it remains to show the estimate Eq. (9.1.5). Under Assumption 2, the weight function

$$w : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow [1, \infty), \quad (a, m) \mapsto 1 + m$$

satisfies $w(a, m) \rightarrow \infty$ when $m \rightarrow \infty$ and the Lyapunov condition on m :

$$\exists \alpha > 0, b \geq 0 \quad \text{such that} \quad \mathcal{L}_t^* w \leq -\alpha w + b. \quad (9.2.16)$$

Indeed, for all $(t, a, m) \in \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+^*$,

$$\mathcal{L}_t^* w(a, m) = -\lambda m + \Gamma(m)f(a, m, \varepsilon x_t) \leq -\lambda w(a, m) + \lambda + \|\Gamma\|_\infty \|f\|_\infty.$$

Importantly, the constants α and b do not depend on x .

Lemma 2 (Global bound in $L_+^1(w)$). *Grant Assumptions 1 and 2. If the initial datum u_0 is in $L_+^1(w)$, then $\rho_t \in L_+^1(w)$ for all $t \geq 0$. Moreover,*

$$\forall t > 0, \quad \|\rho_t\|_{L^1(w)} \leq \|u_0\|_{L^1(w)} e^{-\alpha t} + \frac{b}{\alpha}(1 - e^{-\alpha t}), \quad (9.2.17)$$

where the constants α and b are taken from the Lyapunov condition (9.2.16).

Proof. We divide the proof in two steps: first, we prove that the solution is stable in $L_+^1(w)$ with a weaker and time dependent bound; then, we use this first bound to apply the dominated convergence theorem and obtain Eq. (9.2.17) by Grönwall's lemma.

Step 1. Fix any $T > 0$. Let $\chi \in \mathbb{C}_c^\infty(\mathbb{R}_+, \mathbb{R}_+)$ be a non-increasing function such that $\chi(x) = 1$ if $0 \leq x \leq 1$ and $\chi(x) = 0$ if $x > 2$. For all $n \in \mathbb{N}^*$, let us write $\varphi_k(a)\chi_n(m) := \chi(a/k)\chi(m/n)$. We also consider $g_M(w)$ a smooth approximation of $w \wedge M$, such that $\|g'\|_\infty \leq 1$ and $M\mathbb{1}_{w \geq M} \leq g(w) \leq M$. For all n, k and M , $g_M(w)\chi_n\varphi_k \in \mathbb{C}_c^\infty(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+)$. Hence, by Proposition 1 (ii),

$$\forall n \in \mathbb{N}^*, \quad \langle \rho_T, g_M(w)\chi_n\varphi_k \rangle = \langle u_0, g_M(w)\chi_n\varphi_k \rangle + \int_0^T \langle \rho_t, \mathcal{L}_x^*(g_M(w)\chi_n\varphi_k) \rangle dt,$$

where

$$\begin{aligned} \mathcal{L}_x^*(g_M(w)\chi_n\varphi_k) &= \partial_a(g_M(w)\chi_n\varphi_k) - \lambda m \partial_m(g_M(w)\chi_n\varphi_k) \\ &\quad + (g_M(w(\gamma(m)))\chi_n(\gamma(m))\varphi_k(0) - g_M(w)\chi_n\varphi_k) f \\ &= g_M(w)\chi_n \frac{1}{k} \chi'(a/k) - \lambda m g_M(w)\varphi_k \frac{1}{n} \chi'(m/n) - \lambda m g'_M(w)\chi_n\varphi_k \\ &\quad + (g_M(w(\gamma(m)))\chi_n(\gamma(m))\varphi_k(0) - g_M(w)\chi_n\varphi_k) f. \end{aligned}$$

From the L^1 -stability and the fact that both $g_M(w)\partial_m\chi_n$ and $g_M(w)\chi_n$ are bounded and have compact support, we can go to the limit in k using the dominated convergence theorem:

$$\begin{aligned} \langle \rho_T, g_M(w)\chi_n \rangle &= \langle u_0, g_M(w)\chi_n \rangle \\ &\quad + \int_0^T \left\langle \rho_t, -\lambda m g_M(w) \frac{1}{n} \chi'(m/n) - \lambda m g'_M(w)\chi_n \right\rangle dt \\ &\quad + \int_0^T \langle \rho_t, (g_M(w(\gamma(m)))\chi_n(\gamma(m)) - g_M(w)\chi_n) f \rangle dt, \end{aligned} \quad (9.2.18)$$

On the other hand, from the properties of χ and g_M , we have

$$\left| \lambda m g_M(w) \frac{1}{n} \chi'(m/n) \right| \leq \lambda g_M(w) \frac{2n}{n} \|\chi'\|_\infty \leq 2\lambda M \|\chi'\|_\infty$$

and

$$|\lambda m g'_M(w) \chi_n| \leq \lambda g_M(w) \leq \lambda M,$$

whence

$$\begin{aligned} \langle \rho_T, g_M(w) \chi_n \rangle &\leq \langle u_0, g_M(w) \chi_n \rangle \\ &+ \int_0^T \langle \rho_t, \lambda g_M(w) \|\chi'\|_\infty + \lambda g_M(w) \rangle dt \\ &+ \int_0^T \langle \rho_t, (g_M(w(\gamma(m))) \chi_n(\gamma(m)) - g_M(w) \chi_n) f \rangle dt \end{aligned}$$

and we can take the limit in n using the dominated convergence theorem:

$$\begin{aligned} \langle \rho_T, g_M(w) \rangle &\leq \langle u_0, g_M(w) \rangle \\ &+ \int_0^T \langle \rho_t, \lambda g_M(w) \|\chi'\|_\infty + \lambda g_M(w) \rangle dt \\ &+ \int_0^T \langle \rho_t, (g_M(w(\gamma(m))) - g_M(w)) f \rangle dt. \end{aligned}$$

From the properties of γ , we get

$$w(0, \gamma(m)) \leq w(0, m + \|\Gamma\|_\infty) \leq (1 + \|\Gamma\|_\infty) w(a, m)$$

and

$$g_M(w(0, \gamma(m))) \leq (1 + \|\Gamma\|_\infty) g_M(w(m)).$$

This, together with the fact that f is bounded, shows that there exists a constant C , which does not depend on M , such that

$$\langle \rho_T, g_M(w) \rangle \leq \langle u_0, g_M(w) \rangle + C \int_0^T \langle \rho_t, g_M(w) \rangle dt,$$

and we can apply Grönwall's lemma to obtain

$$\langle \rho_T, g_M(w) \rangle \leq \langle u_0, g_M(w) \rangle e^{Ct}.$$

Then, it follows from Fatou lemma that $\rho_T \in L^1_+(w)$.

Step 2. To improve the previous estimate, we come back to (9.2.18) and use dominated convergence in n and M (**Step 1.** guarantees domination) to show

$$\langle \rho_T, w \rangle = \langle u_0, w \rangle + \int_0^T \langle \rho_t, \mathcal{L}_x^* w \rangle dt.$$

By the Lyapunov condition (9.2.16),

$$\|\rho_T\|_{L^1(w)} \leq \|u_0\|_{L^1(w)} - \alpha \int_0^T \|\rho_t\|_{L^1(w)} dt + Tb.$$

and by Grönwall's lemma, we have, for all $t \in [0, T]$,

$$\|\rho_T\|_{L^1(w)} \leq \|u_0\|_{L^1(w)} e^{-\alpha t} + \frac{b}{\alpha} (1 - e^{-\alpha t}).$$

Since T can be chosen arbitrarily large, this achieves the proof. \square

Remark. Following the same steps as in the proof above, we can show that the bound Eq. (9.2.17) also holds for the linearized equation (9.2.10) and does not depend on x nor the constants α and b .

9.3 Exponential stability in the non-interacting case

If $x \in \mathcal{C}(\mathbb{R}_+)$ in the linearized equation (9.2.10) is time-invariant, i.e. $x \equiv \tilde{x}$ for some $\tilde{x} \in \mathbb{R}$, then Eq. (9.2.10) can be seen as the dynamics of a non-interacting population of neurons. In this section, we prove the exponential stability in the non-interacting case using Harris' or Doeblin's theorem. This is the key result of this work and will allow us to prove the existence and uniqueness of the stationary solution to (9.1.1) (Section 9.4) and the exponential convergence to it (Section 9.5).

For $\tilde{x} \in \mathbb{R}$, $u_0 \in L^1$, we denote $\rho_t^{\tilde{x}}$ the unique solution to Eq. (9.2.10) for the initial datum u_0 and $x \equiv \tilde{x}$, given by Proposition 1. We write, using the semigroup notation,

$$S_t^{\tilde{x}} u_0 := \rho_t^{\tilde{x}}, \quad \forall t \geq 0. \quad (9.3.19)$$

To show that the Eq. (9.3.19) is exponentially stable we will use Harris' theorem in the general case or Doeblin's theorem if Assumption 4 is granted. In both cases, the main technical difficulty is to verify the Doeblin minoration condition (Lemma 3) as the jumps of the process described by Eq. (9.2.12) are degenerate and the model is two-dimensional.

Lemma 3 (Doeblin minoration condition). *Grant Assumptions 1 and 3. Fix any $x \in \mathbb{R}$. For all $R > 0$, there exists $T > 0$ and a positive non-zero measure ν such that*

$$\forall u_0 \in L_+^1, \quad S_T^{\tilde{x}} u_0 \geq \nu \int_{\mathbb{R}_+ \times]0, R]} u_0 \, dadm. \quad (9.3.20)$$

Proof. We proceed in two steps. First (**Step 1**), we choose a time $T > 0$ and a rectangle $[0, \bar{a}] \times [\underline{m}, \bar{m}] \subset \mathbb{R}_+ \times \mathbb{R}_+^*$ (with nonzero Lebesgue measure) and show that the density $S_T^{\tilde{x}} u_0 \in L^1$ has a lower bound on $[0, \bar{a}] \times [\underline{m}, \bar{m}]$ which depends on a Lebesgue integral in \mathbb{R}_+^2 involving u_0 . Then (**Step 2**), we perform a change of variable to express this lower bound in terms of $\int_{\mathbb{R}_+ \times]0, R]} u_0 \, dadm$. The proof only relies on the expression of $S_t^{\tilde{x}} u_0$ given by the method of characteristics (see Proposition 1) and this allows treating a typically probabilistic question – the Doeblin minoration condition – from a transport point of view. This is possible because $S_t^{\tilde{x}}$ is the stochastic (mass-conservative) semigroup of a piecewise deterministic Markov process.

The constants Δ_{abs} , σ and C_γ are taken from Assumption 3.

Step 1:

Fix $R > 0$. Since $\gamma(e^{-\lambda \Delta_{\text{abs}}} \gamma(0)) > \gamma(0)$ and $\gamma(e^{-\lambda t} \gamma(e^{-\lambda \Delta_{\text{abs}}} R)) \rightarrow \gamma(0)$ as $t \rightarrow \infty$, there exists $\bar{a} > 0$ and $T > \bar{a} + \Delta_{\text{abs}}$ such that

$$\underline{m} =: \gamma(e^{-\lambda(T-\bar{a}-\Delta_{\text{abs}})} \gamma(e^{-\lambda \Delta_{\text{abs}}} R)) < e^{-\lambda \bar{a}} \gamma(e^{-\lambda \Delta_{\text{abs}}} \gamma(0)) =: \bar{m}. \quad (9.3.21)$$

Eq. (9.3.21) has the following heuristic interpretation: if we see $S_t^{\tilde{x}}$ as the stochastic semigroup of the piecewise deterministic Markov process defined by the generator Eq. (9.2.12), for any initial point $(a_0, m_0) \in \mathbb{R}_+ \times]0, R]$ and any landing point $(a, m) \in [0, \bar{a}] \times [\underline{m}, \bar{m}]$ at time T , there is a 'possible' trajectory going from (a_0, m_0) to (a, m) , with exactly two jumps (spikes). Since the trajectories of the process are determined by the jump times, we will exploit the fact that these 'possible' trajectories correspond to jump times with strictly positive probability density. Below, we take a transport point of view on this probabilistic argument.

For all $(a, m) \in [0, \bar{a}] \times [\underline{m}, \bar{m}]$,

$$\begin{aligned}
(S_T^{\tilde{x}} u_0)(a, m) &\geq \mathbb{1}_{\{a < T\}} (S_{T-a}^{\tilde{x}} u_0)(0, e^{\lambda a} m) \exp \left(\lambda a - \int_{T-a}^T f(a - T + s, e^{\lambda(T-s)} m, \tilde{x}) ds \right) \\
&\geq \mathbb{1}_{\{a < T\}} e^{-\|f\|_\infty T} e^{\lambda a} (S_{T-a}^{\tilde{x}} u_0)(0, e^{\lambda a} m) \\
&\geq \mathbb{1}_{\{a < T\}} e^{-\|f\|_\infty T} \sigma e^{\lambda a} \left| (\gamma^{-1})'(e^{\lambda a} m) \right| \int_{\Delta_{\text{abs}}}^\infty (S_{T-a}^{\tilde{x}} u_0)(a', \gamma^{-1}(e^{\lambda a} m)) da' \\
&= \mathbb{1}_{\{a < T\}} e^{-\|f\|_\infty T} \sigma \left| \frac{d}{dm} \gamma^{-1}(e^{\lambda a} m) \right| \int_{\Delta_{\text{abs}}}^\infty \underbrace{(S_{T-a}^{\tilde{x}} u_0)(a', \gamma^{-1}(e^{\lambda a} m))}_{(\star)} da'.
\end{aligned}$$

Above, we went back in time to the last jump time $T - a$. Let us notice that $\gamma^{-1}(e^{\lambda a} m) \geq \gamma^{-1}(e^{\lambda a} \underline{m}) > 0$. We can therefore define

$$a_{a,m}^* := \frac{1}{\lambda} \left(\log \gamma(0) - \log \gamma^{-1}(e^{\lambda a} m) \right).$$

Note that $a_{a,m}^*$ satisfies $\gamma^{-1}(e^{\lambda a_{a,m}^*} \gamma^{-1}(e^{\lambda a} m)) = 0$. In other words, $a_{a,m}^*$ is the minimal time between the last and second last jumps for a trajectory landing at (a, m) at time T . We can easily verify that, by our choice of $\{T, \bar{a}, \underline{m}, \bar{m}\}$, $\Delta_{\text{abs}} \leq a_{a,m}^* < T - a - \Delta_{\text{abs}}$. This guarantees that it is possible to make two jumps in $[0, T]$ and land at (a, m) at time T while respecting the absolute refractoriness of the neuron (i.e. there needs to be a time interval $\geq \Delta_{\text{abs}}$ between jumps). This allows us to go further back in time to the second last jump:

For all $a' \in [a_{a,m}^*, T - a - \Delta_{\text{abs}}]$,

$$\begin{aligned}
(\star) &\geq \mathbb{1}_{\{a' < T-a\}} e^{-\|f\|_\infty T} \sigma \left| (\gamma^{-1})'(e^{\lambda a'} \gamma^{-1}(e^{\lambda a} m)) \right| e^{\lambda a'} \\
&\quad \int_{\Delta_{\text{abs}}}^\infty \underbrace{(S_{T-a-a'}^{\tilde{x}} u_0)(a'', \gamma^{-1}(e^{\lambda a'} \gamma^{-1}(e^{\lambda a} m)))}_{(\star\star)} da''.
\end{aligned}$$

Then, we can go further back to time 0 to get u_0 :

$$(\star\star) \geq \mathbb{1}_{\{a'' \geq T-a-a'\}} e^{-\|f\|_\infty T} e^{\lambda(T-a-a')} u_0(a'' - (T-a-a'), e^{\lambda(T-a-a')} \gamma^{-1}(e^{\lambda a'} \gamma^{-1}(e^{\lambda a} m))).$$

Putting all the lower bounds together, we get

$$\begin{aligned}
(S_T^{\tilde{x}} u_0)(a, m) &\geq \mathbb{1}_{\{a < T\}} e^{-3\|f\|_\infty T} \sigma^2 \\
&\quad \int_{a_{a,m}^*}^{T-a-\Delta_{\text{abs}}} \int_{T-a-a'}^\infty \left| \frac{d}{dm} e^{\lambda(T-a-a')} \gamma^{-1}(e^{\lambda a'} \gamma^{-1}(e^{\lambda a} m)) \right| \\
&\quad u_0(a'' - (T-a-a'), e^{\lambda(T-a-a')} \gamma^{-1}(e^{\lambda a'} \gamma^{-1}(e^{\lambda a} m))) da'' da'.
\end{aligned}$$

Since $\gamma' \leq 1$ (Assumption 3),

$$\left| \frac{d}{dm} e^{\lambda(T-a-a')} \gamma^{-1}(e^{\lambda a'} \gamma^{-1}(e^{\lambda a} m)) \right| \geq e^{\lambda T}.$$

Thus,

$$(S_T^{\tilde{x}} u_0)(a, m) \geq \mathbb{1}_{\{a < T\}} e^{(\lambda - 3\|f\|_\infty)T} \sigma^2 \int_{a_{a,m}^*}^{T-a-\Delta_{\text{abs}}} \int_0^\infty u_0(a_0, e^{\lambda(T-a-a')} \gamma^{-1}(e^{\lambda a'} \gamma^{-1}(e^{\lambda a} m))) da_0 da'. \quad (9.3.22)$$

We have obtained that on $[0, \bar{a}] \times [\underline{m}, \bar{m}]$, the density $(S_T^{\tilde{x}}u_0)$ is lower bounded by a constant depending on a Lebesgue integral on \mathbb{R}_+^2 involving u_0 .

Step 2:

Now, we want express the lower bound Eq. (9.3.22) in terms of $\int_{\mathbb{R}_+ \times]0, R]} u_0 \, da \, dm$ by a change of variable. Let us define the function $\psi_{a,m}^T$:

$$\psi_{a,m}^T : [a_{a,m}^*, T - a - \Delta_{\text{abs}}] \rightarrow \mathbb{R}_+, \quad a' \mapsto e^{\lambda(T-a-a')} \gamma^{-1}(e^{\lambda a'} \gamma^{-1}(e^{\lambda a} m)).$$

We verify that $(\psi_{a,m}^T)' > 0$:

For all $a' \in [a_{a,m}^*, T - a]$,

$$(\psi_{a,m}^T)'(a') = \lambda e^{\lambda(T-a-a')} \left\{ (\gamma^{-1})'(e^{\lambda a'} \gamma^{-1}(e^{\lambda a} m)) e^{\lambda a'} \gamma^{-1}(e^{\lambda a} m) - \gamma^{-1}(e^{\lambda a'} \gamma^{-1}(e^{\lambda a} m)) \right\}. \quad (9.3.23)$$

As $\Gamma > 0$ and $\gamma' \leq 1$ (Assumption 3), we have

$$\begin{aligned} (\psi_{a,m}^T)'(a') &> \lambda e^{\lambda(T-a-a')} \left\{ (\gamma^{-1})'(e^{\lambda a'} \gamma^{-1}(e^{\lambda a} m)) e^{\lambda a'} \gamma^{-1}(e^{\lambda a} m) - e^{\lambda a'} \gamma^{-1}(e^{\lambda a} m) \right\} \\ &= \lambda e^{\lambda(T-a)} \gamma^{-1}(e^{\lambda a} m) \underbrace{\left\{ (\gamma^{-1})'(e^{\lambda a'} \gamma^{-1}(e^{\lambda a} m)) - 1 \right\}}_{\geq 1} \geq 0. \end{aligned}$$

Therefore, $\psi_{a,m}^T$ is a strictly increasing C^1 -diffeomorphism from $[a_{a,m}^*, T - a - \Delta_{\text{abs}}]$ to $[\psi_{a,m}^T(a_{a,m}^*), \psi_{a,m}^T(T - a - \Delta_{\text{abs}})]$. We can now rewrite Eq. (9.3.22):

$$\begin{aligned} (S_T^{\tilde{x}}u_0)(a, m) &\geq e^{(\lambda - 3\|f\|_\infty)T} \sigma^2 \int_{a_{a,m}^*}^{T-a-\Delta_{\text{abs}}} \int_0^\infty u_0(a_0, \psi_{a,m}^T(a')) \, da_0 \, da' \\ &= e^{(\lambda - 3\|f\|_\infty)T} \sigma^2 \int_{\psi_{a,m}^T(a_{a,m}^*)}^{\psi_{a,m}^T(T-a-\Delta_{\text{abs}})} \int_0^\infty u_0(a_0, m_0) \left| ((\psi_{a,m}^T)^{-1})'(m_0) \right| \, da_0 \, dm_0. \end{aligned}$$

Going back to Eq. (9.3.23), and using the fact that there exists C_γ such that $C_\gamma \leq \gamma' \leq 1$ (Assumption 3), we have, for all $a' \in [a_{a,m}^*, T - a - \Delta_{\text{abs}}]$,

$$(\psi_{a,m}^T)'(a') \leq \lambda e^{\lambda(T-a-a')} C_\gamma^{-1} e^{\lambda a'} \gamma^{-1}(e^{\lambda a} m) \leq \lambda e^{\lambda T} C_\gamma^{-1} \bar{m}.$$

Hence,

$$(S_T^{\tilde{x}}u_0)(a, m) \geq \frac{e^{-3\|f\|_\infty T} \sigma^2 C_\gamma}{\lambda \bar{m}} \int_{\psi_{a,m}^T(a_{a,m}^*)}^{\psi_{a,m}^T(T-a-\Delta_{\text{abs}})} \int_0^\infty u_0(a_0, m_0) \, da_0 \, dm_0.$$

In addition, by our choice of $\{T, \bar{a}, \underline{m}, \bar{m}\}$, we have

$$\begin{aligned} \psi_{a,m}^T(a_{a,m}^*) &= 0, \\ \psi_{a,m}^T(T - a - \Delta_{\text{abs}}) &= e^{\lambda \Delta_{\text{abs}}} \gamma^{-1}(e^{\lambda(T-a-\Delta_{\text{abs}})} \gamma^{-1}(e^{\lambda a} m)) > R. \end{aligned}$$

Therefore,

$$(S_T^{\tilde{x}}u_0)(a, m) \geq \frac{e^{-3\|f\|_\infty T} \sigma^2 C_\gamma}{\lambda \bar{m}} \int_0^R \int_0^\infty u_0(a_0, m_0) \, da_0 \, dm_0.$$

Since we have supposed that $(a, m) \in [0, \bar{a}] \times [\underline{m}, \bar{m}]$, this concludes the proof. \square

With the Lyapunov condition (9.2.16) and the Doeblin minoration condition (9.3.20), we can apply a version of Harris' theorem:

Theorem 4 (Harris). *Grant Assumptions 1 – 3. For all $\tilde{x} \in \mathbb{R}$, there exists a unique $\rho_\infty^{\tilde{x}} \in L_+^1(w)$ with $\|\rho_\infty^{\tilde{x}}\|_{L^1} = 1$ such that $S_t^{\tilde{x}} \rho_\infty^{\tilde{x}} = \rho_\infty^{\tilde{x}}$, for all $t \geq 0$, and there exists $K \geq 1$ and $\mathfrak{a} > 0$ such that for all initial data $u_0 \in L_+^1(w)$ with $\|u_0\|_{L^1} = 1$,*

$$\|S_t^{\tilde{x}} u_0 - \rho_\infty^{\tilde{x}}\|_{L^1(w)} \leq K e^{-\mathfrak{a}t} \|u_0 - \rho_\infty^{\tilde{x}}\|_{L^1(w)}, \quad \forall t \geq 0. \quad (9.3.24)$$

Furthermore, by Lemma 2, we have that $\|\rho_\infty^{\tilde{x}}\|_{L^1(w)} \leq \frac{b}{\alpha}$, where the constants α and b are taken from the Lyapunov condition (9.2.16).

Proof. This is a classic result which proof can be found in the work of Hairer and Mattingly [118]. \square

If, in addition, Assumption 4 holds, we can simply apply Doeblin's theorem:

Theorem 5 (Doeblin). *Grant Assumptions 1, 3 and 4. For all $\tilde{x} \in \mathbb{R}$, there exists a unique $\rho_\infty^{\tilde{x}} \in L_+^1$ with $\|\rho_\infty^{\tilde{x}}\|_{L^1} = 1$ such that $S_t^{\tilde{x}} \rho_\infty^{\tilde{x}} = \rho_\infty^{\tilde{x}}$, for all $t \geq 0$, and there exists $K \geq 1$ and $\mathfrak{a} > 0$ such that for all initial data $u_0 \in L_+^1$ with $\|u_0\|_{L^1} = 1$,*

$$\|S_t^{\tilde{x}} u_0 - \rho_\infty^{\tilde{x}}\|_{L^1} \leq K e^{-\mathfrak{a}t} \|u_0 - \rho_\infty^{\tilde{x}}\|_{L^1}, \quad \forall t \geq 0. \quad (9.3.25)$$

Proof. See, for example, Theorem 2.3 in [48]. \square

We say that $\rho_\infty^{\tilde{x}}$ is the *invariant probability measure* of the semigroup $(S_t^{\tilde{x}})_{t \in \mathbb{R}_+}$. Note that both theorems imply the next corollary.

Corollary 1. *Grant Assumptions 1 – 3. For all $\tilde{x} \in \mathbb{R}$, there exists a unique $\rho_\infty^{\tilde{x}} \in L_+^1(w)$ with $\|\rho_\infty^{\tilde{x}}\|_{L^1} = 1$ solving*

$$\partial_a \rho_\infty^{\tilde{x}}(a, m) - \lambda \partial_m (m \rho_\infty^{\tilde{x}}(a, m)) = -f(a, m, \tilde{x}) \rho_\infty^{\tilde{x}}(a, m), \quad (9.3.26a)$$

$$\rho_\infty^{\tilde{x}}(0, m) = \mathbb{1}_{m > \gamma(0)} \left| (\gamma^{-1})'(m) \right| \int_0^\infty f(a, \gamma^{-1}(m), \tilde{x}) \rho_\infty^{\tilde{x}}(a, \gamma^{-1}(m)) da, \quad (9.3.26b)$$

in the weak sense. Furthermore, we have that $\rho_\infty^{\tilde{x}} \in \mathcal{C}(\mathbb{R}_+, L_+^1(\mathbb{R}_+^*)) \cap L^\infty(\mathbb{R}_+, L_+^1(\mathbb{R}_+^*))$.

9.4 Stationary solutions for arbitrary connectivity strength

In this section, we study the stationary solutions to (9.1.1), namely the solution to

$$\partial_a \rho_\infty(a, m) - \lambda \partial_m (m \rho_\infty(a, m)) = -f(a, m, \varepsilon x_\infty) \rho_\infty(a, m), \quad (9.4.27a)$$

$$\rho_\infty(0, m) = \mathbb{1}_{m > \gamma(0)} \left| (\gamma^{-1})'(m) \right| \int_0^\infty f(a, \gamma^{-1}(m), \varepsilon x_\infty) \rho_\infty(a, \gamma^{-1}(m)) da, \quad (9.4.27b)$$

$$x_\infty = \int_0^\infty \int_0^\infty \bar{h}(a, m) f(a, m, \varepsilon x_\infty) \rho_\infty(a, m) da dm. \quad (9.4.27c)$$

Definition. $(\rho_\infty, x_\infty) \in L_+^1(w) \cap \mathcal{C}(\mathbb{R}_+, L_+^1(\mathbb{R}_+^*)) \cap L^\infty(\mathbb{R}_+, L_+^1(\mathbb{R}_+^*)) \times \mathbb{R}_+$ is a stationary solution to (9.1.1) if $\|\rho_\infty\|_{L^1} = 1$ and if it solves Eq. (9.4.27) in the weak sense.

9.4.1 Existence and uniqueness using the Doeblin-Harris method.

We present two Lipschitz continuity results, which will allow us to prove the existence (and the uniqueness when ε is small) of stationary solutions. The following lemma plays the same role as Theorem 4.5 in [48]:

Lemma 4 (Lipschitz continuity at finite T). *Grant Assumptions 1 – 3. For all initial data $u_0 \in L^1_+(w)$ and for all $T > 0$, there exists a constant $C_{T\|u_0\|_{L^1(w)}} > 0$ such that*

$$\forall \tilde{x}_1, \tilde{x}_2 \in \mathbb{R}, \quad \left\| S_T^{\tilde{x}_1} u_0 - S_T^{\tilde{x}_2} u_0 \right\|_{L^1(w)} \leq C_{T\|u_0\|_{L^1(w)}} |\tilde{x}_1 - \tilde{x}_2|. \quad (9.4.28)$$

Proof. For all $t > 0$,

$$\begin{aligned} & \left\| S_t^{\tilde{x}_1} u_0 - S_t^{\tilde{x}_2} u_0 \right\|_{L^1(w)} \\ &= \int_0^\infty \int_t^\infty \left| u_0(a-t, e^{\lambda t} m) \exp\left(\lambda t - \int_0^t f(a-t+s, e^{\lambda(t-s)} m, \tilde{x}_1) ds\right) \right. \\ & \quad \left. - u_0(a-t, e^{\lambda t} m) \exp\left(\lambda t - \int_0^t f(a-t+s, e^{\lambda(t-s)} m, \tilde{x}_2) ds\right) \right| w(a, m) dadm \\ & \quad + \int_0^\infty \int_0^t \left| \rho_{t-a}^{\tilde{x}_1}(0, e^{\lambda a} m) \exp\left(\lambda a - \int_{t-a}^t f(a-t+s, e^{\lambda(t-s)} m, \tilde{x}_1) ds\right) \right. \\ & \quad \left. - \rho_{t-a}^{\tilde{x}_2}(0, e^{\lambda a} m) \exp\left(\lambda a - \int_{t-a}^t f(a-t+s, e^{\lambda(t-s)} m, \tilde{x}_2) ds\right) \right| w(a, m) dadm \\ &=: Q_1 + Q_2. \end{aligned}$$

$$\begin{aligned} Q_1 &= \int_0^\infty \int_0^\infty u_0(a, m) \left| \exp\left(-\int_0^t f(a+s, e^{-\lambda s} m, \tilde{x}_1) ds\right) - \exp\left(-\int_0^t f(a+s, e^{-\lambda s} m, \tilde{x}_2) ds\right) \right| \\ & \quad w(a+t, e^{-\lambda t} m) dadm \\ &\leq \int_0^\infty \int_0^\infty u_0(a, m) \left(\int_0^t |f(a+s, e^{-\lambda s} m, \tilde{x}_1) - f(a+s, e^{-\lambda s} m, \tilde{x}_2)| ds \right) w(a+t, e^{-\lambda t} m) dadm \\ &\leq t L_f |\tilde{x}_1 - \tilde{x}_2| \int_0^\infty \int_0^\infty u_0(a, m) w(a+t, e^{-\lambda t} m) dadm \\ &\leq t L_f \|u_0\|_{L^1(w)} |\tilde{x}_1 - \tilde{x}_2|, \end{aligned}$$

where in the last inequality we used

$$w(a+t, e^{-\lambda t} m) \leq w(a, m), \quad \forall a \geq 0, m \geq 0. \quad (9.4.29)$$

$$\begin{aligned} Q_2 &= \int_0^\infty \int_0^t \left| \rho_{t-a}^{\tilde{x}_1}(0, m) \exp\left(-\int_{t-a}^t f(a-t+s, e^{\lambda(t-s-a)} m, \tilde{x}_1) ds\right) \right. \\ & \quad \left. - \rho_{t-a}^{\tilde{x}_2}(0, m) \exp\left(-\int_{t-a}^t f(a-t+s, e^{\lambda(t-s-a)} m, \tilde{x}_2) ds\right) \right| w(a, e^{-\lambda a} m) dadm. \end{aligned}$$

By changes of variables,

$$\begin{aligned}
 Q_2 &= \int_0^\infty \int_0^t \left| \rho_s^{\tilde{x}_1}(0, m) \exp\left(-\int_0^{t-s} f(u, e^{-\lambda u} m, \tilde{x}_1) du\right) \right. \\
 &\quad \left. - \rho_s^{\tilde{x}_2}(0, m) \exp\left(-\int_0^{t-s} f(u, e^{-\lambda u} m, \tilde{x}_2) du\right) \right| w(t-s, e^{-\lambda(t-s)} m) ds dm \\
 &\leq \int_0^\infty \int_0^t \left| \rho_s^{\tilde{x}_1}(0, m) \right| \exp\left(-\int_0^{t-s} f(u, e^{-\lambda u} m, \tilde{x}_1) du\right) - \exp\left(-\int_0^{t-s} f(u, e^{-\lambda u} m, \tilde{x}_2) du\right) \Big| \\
 &\quad \left. w(t-s, e^{-\lambda(t-s)} m) ds dm \right. \\
 &\quad \left. + \int_0^\infty \int_0^t \left| \rho_s^{\tilde{x}_1}(0, m) - \rho_s^{\tilde{x}_2}(0, m) \right| w(t-s, e^{-\lambda(t-s)} m) ds dm \right. \\
 &=: Q_{2,1} + Q_{2,2}
 \end{aligned}$$

$$\begin{aligned}
 Q_{2,1} &\leq t \|f\|_\infty L_f |\tilde{x}_1 - \tilde{x}_2| \int_0^t \int_0^\infty \int_0^\infty |(\gamma^{-1})'(m)| \rho_s^{\tilde{x}_1}(a, \gamma^{-1}(m)) w(t, m) da dm ds \\
 &\leq t \|f\|_\infty L_f |\tilde{x}_1 - \tilde{x}_2| \int_0^t \int_0^\infty \int_0^\infty \rho_s^{\tilde{x}_1}(a, m) w(t, m + \|\Gamma\|_\infty) da dm ds \\
 &\leq t(1 + \|\Gamma\|_\infty) \|f\|_\infty L_f |\tilde{x}_1 - \tilde{x}_2| \int_0^t \left\| \rho_s^{\tilde{x}_1} \right\|_{L^1(w)} ds,
 \end{aligned}$$

where in the last inequality we used

$$w(t, m + \|\Gamma\|_\infty) = 1 + m + \|\Gamma\|_\infty \leq (1 + \|\Gamma\|_\infty) w(a, m), \quad \forall a \geq 0, m \geq 0. \quad (9.4.30)$$

By Lemma 2,

$$Q_{2,1} \leq t^2 (1 + \|\Gamma\|_\infty) \|f\|_\infty L_f \left(\|u_0\|_{L^1(w)} + \frac{b}{\alpha} \right) |\tilde{x}_1 - \tilde{x}_2|.$$

$$\begin{aligned}
 Q_{2,2} &\leq \|f\|_\infty \int_0^t \int_0^\infty \int_0^\infty |(\gamma^{-1})'(m)| \left| \rho_s^{\tilde{x}_1}(a, \gamma^{-1}(m)) - \rho_s^{\tilde{x}_2}(a, \gamma^{-1}(m)) \right| w(t, m) da dm ds \\
 &\leq \|f\|_\infty \int_0^t \int_0^\infty \int_0^\infty \left| \rho_s^{\tilde{x}_1}(a, m) - \rho_s^{\tilde{x}_2}(a, m) \right| w(t, m + \|\Gamma\|_\infty) da dm ds \\
 &\leq (1 + \|\Gamma\|_\infty) \|f\|_\infty \int_0^t \left\| S_s^{\tilde{x}_1} u_0 - S_s^{\tilde{x}_2} u_0 \right\|_{L^1(w)} ds,
 \end{aligned}$$

where again, in the last inequality, we used Eq. (9.4.30). Fix $T > 0$. Gathering the bounds for Q_1 , $Q_{2,1}$ and $Q_{2,2}$ we see that there exists constants $C > 0$ and $C'_{T, \|u_0\|_{L^1(w)}} > 0$ such that, for all $t \in [0, T]$,

$$\left\| S_t^{\tilde{x}_1} u_0 - S_t^{\tilde{x}_2} u_0 \right\|_{L^1(w)} \leq C \int_0^t \left\| S_s^{\tilde{x}_1} u_0 - S_s^{\tilde{x}_2} u_0 \right\|_{L^1(w)} ds + t C'_{T, \|u_0\|_{L^1(w)}} |\tilde{x}_1 - \tilde{x}_2|.$$

By Grönwall's lemma, for all $t \in [0, T]$,

$$\left\| S_t^{\tilde{x}_1} u_0 - S_t^{\tilde{x}_2} u_0 \right\|_{L^1(w)} \leq \frac{C'_{T, \|u_0\|_{L^1(w)}} |\tilde{x}_1 - \tilde{x}_2|}{C} (\exp(Ct) - 1). \quad (9.4.31)$$

Since Eq. (9.4.31) holds for all $t \in [0, T]$, this achieves the proof. \square

Lemma 5 (Lipschitz continuity at $T = \infty$). *Grant Assumptions 1 – 3. Writing $\rho_\infty^{\tilde{x}} \in L_+^1(w)$ the invariant probability measure given by Theorem 4 for any $\tilde{x} \in \mathbb{R}$, the function*

$$\Upsilon : \mathbb{R}_+ \rightarrow \mathbb{R}_+, \quad \Upsilon(x) = \int_0^\infty \int_0^\infty \bar{h}(a, m) f(a, m, \varepsilon x) \rho_\infty^{\varepsilon x}(a, m) da dm$$

is Lipschitz and there exists $C > 0$ such that

$$\forall x_1, x_2 \in \mathbb{R}_+, \quad |\Upsilon(x_1) - \Upsilon(x_2)| \leq |\varepsilon| C |x_1 - x_2|.$$

Proof. Since f is Lipschitz in x , we have, for any $x_1, x_2 \in \mathbb{R}_+$,

$$\begin{aligned} |\Upsilon(x_1) - \Upsilon(x_2)| &\leq \left\| \bar{h} \right\|_\infty \left\{ \|f\|_\infty \|\rho_\infty^{\varepsilon x_1} - \rho_\infty^{\varepsilon x_2}\|_{L^1} + L_f |\varepsilon| |x_1 - x_2| \right\} \\ &\leq \left\| \bar{h} \right\|_\infty \left\{ \|f\|_\infty \|\rho_\infty^{\varepsilon x_1} - \rho_\infty^{\varepsilon x_2}\|_{L^1(w)} + L_f |\varepsilon| |x_1 - x_2| \right\}, \end{aligned}$$

from where we only need to bound the first term on the right hand side. We can use Theorem 4 and Lemma 4: for any $T \in \mathbb{R}_+$,

$$\begin{aligned} \|\rho_\infty^{\varepsilon x_1} - \rho_\infty^{\varepsilon x_2}\|_{L^1(w)} &= \|S_T^{\varepsilon x_1} \rho_\infty^{\varepsilon x_1} - S_T^{\varepsilon x_1} \rho_\infty^{\varepsilon x_2} + S_T^{\varepsilon x_1} \rho_\infty^{\varepsilon x_2} - S_T^{\varepsilon x_2} \rho_\infty^{\varepsilon x_2}\|_{L^1(w)} \\ &\leq K e^{-\mathbf{a}T} \|\rho_\infty^{\varepsilon x_1} - \rho_\infty^{\varepsilon x_2}\|_{L^1(w)} + C_T |\varepsilon| |x_1 - x_2|, \end{aligned}$$

where K and \mathbf{a} are the exponential stability constants of Theorem 4. Choosing T such that $K e^{-\mathbf{a}T} = 1/2$, we get

$$\|\rho_\infty^{\varepsilon x_1} - \rho_\infty^{\varepsilon x_2}\|_{L^1(w)} \leq 2C_T |\varepsilon| |x_1 - x_2|.$$

Gathering the bounds concludes the proof. \square

Theorem 6 (Stationary solutions). *Grant Assumptions 1 – 3. We have*

I *There exists a stationary solution to (9.1.1).*

II *There exists $\varepsilon^* > 0$ such that for all $\varepsilon \in]-\varepsilon^*, +\varepsilon^*[$, the stationary solution to (9.1.1) is unique.*

Proof. For all $\tilde{x} \in \mathbb{R}$, let us write $\rho_\infty^{\tilde{x}} \in L_+^1(w)$ the unique invariant measure given by Theorem 4 and let us also take the function Υ from Lemma 5. By Corollary 1, $(\rho_\infty, x_\infty) \in L_+^1(w) \cap \mathcal{C}(\mathbb{R}_+, L_+^1(\mathbb{R}_+^*)) \cap L^\infty(\mathbb{R}_+, L_+^1(\mathbb{R}_+^*)) \times \mathbb{R}_+$ is a weak solution to Eq. (9.4.27) if and only if $\rho_\infty = \rho_\infty^{\varepsilon x_\infty}$ and x_∞ is a fixed-point of Υ . Hence, the study of the existence and the uniqueness of stationary solutions is reduced to the study of the existence and the uniqueness of the fixed-point of Υ .

Since for all $x \in \mathbb{R}_+$, $\|\rho_\infty^{\varepsilon x}\|_{L^1} = 1$, we have that for all $x \in \mathbb{R}_+$, $\Upsilon(x) \leq \left\| \bar{h} \right\|_\infty \|f\|_\infty$. Therefore, the set $[0, \left\| \bar{h} \right\|_\infty \|f\|_\infty]$ (which is compact and convex) is stable by Υ . Then, the continuity of Υ guarantees the existence of a fixed-point, which proves (i).

To obtain (ii), we observe that the Lipschitz constant of Υ is $|\varepsilon|C$: if we take $|\varepsilon| < \varepsilon^* := C^{-1}$, Υ is a contraction and we can apply Banach's fixed-point theorem to conclude. \square

9.4.2 Alternative proof for the existence using Schauder's fixed-point theorem

We include here an alternative proof for the existence of a stationary solution, which is interesting for two reasons: on the one hand, it does not rely on the Harris-Doebelin method, and on the other hand, it provides some estimates on the stationary solutions.

For any $(\tilde{u}, \tilde{x}) \in L_+^1(] \gamma(0), +\infty[) \times \mathbb{R}$, consider the transport equation

$$\begin{aligned} \partial_a \varrho(a, m) - \lambda \partial_m (m \varrho(a, m)) &= -f(a, m, \tilde{x}) \varrho(a, m), \\ \varrho(0, m) &= \tilde{u}(m). \end{aligned}$$

It has a unique weak solution $\rho_\infty^{\tilde{u}, \tilde{x}} \in \mathcal{C}(\mathbb{R}_+, L_+^1(\mathbb{R}_+^*)) \cap L^\infty(\mathbb{R}_+, L_+^1(\mathbb{R}_+^*))$ given by the method of characteristics: for all $(a, m) \in \mathbb{R}_+ \times \mathbb{R}_+^*$,

$$\rho_\infty^{\tilde{u}, \tilde{x}}(a, m) = \tilde{u}(e^{\lambda a} m) \exp\left(\lambda a - \int_0^a f(s, e^{\lambda(a-s)} m, \tilde{x}) ds\right). \quad (9.4.32)$$

We can now define the operator $\Phi := (\Phi_1, \Phi_2)$ on $L_+^1(] \gamma(0), +\infty[) \times \mathbb{R}$ where, for all $(\tilde{u}, \tilde{x}) \in L_+^1(] \gamma(0), +\infty[) \times \mathbb{R}$,

$$\Phi_1(\tilde{u}, \tilde{x})(m) := \mathbb{1}_{m > \gamma(0)} \left| (\gamma^{-1})'(m) \right| \int_0^\infty f(a, \gamma^{-1}(m), \tilde{x}) \rho_\infty^{\tilde{u}, \tilde{x}}(a, \gamma^{-1}(m)) da, \quad (9.4.33a)$$

$$\Phi_2(\tilde{u}, \tilde{x}) := \int_0^\infty \int_0^\infty \bar{h}(a, m) f(a, m, \tilde{x}) \rho_\infty^{\tilde{u}, \tilde{x}}(a, m) da dm. \quad (9.4.33b)$$

(ρ_∞, x_∞) is a stationary solution if and only if it is a fixed-point of Φ . Whence, we get the *a priori* estimates:

Lemma 6. *Grant Assumptions 1 and 3. There exists $\theta \in]0, 1[$ such that for all $(\tilde{u}, \tilde{x}) \in L_+^1(] \gamma(0), +\infty[) \times \mathbb{R}$,*

I $\|\Phi_1(\tilde{u}, \tilde{x})\|_{L^1} = \|\tilde{u}\|_{L^1}.$

II For all $m \in \mathbb{R}_+^*$, $|\Phi_1(\tilde{u}, \tilde{x})(m)| \leq \mathbb{1}_{m > \gamma(0)} \frac{\|f\|_\infty}{\lambda \gamma^{-1}(m)} \|\tilde{u}\|_{L^1}.$

III

$$\int_0^\infty \Phi_1(\tilde{u}, \tilde{x})(m) m dm \leq \max\left(\int_0^\infty \tilde{u}(m) m dm, \frac{\gamma(0)}{1-\theta} \|\tilde{u}\|_{L^1}\right).$$

IV For all $\beta \in]0, \frac{\min(f)}{\lambda}[$,

$$\int_{\gamma(0)}^\infty \frac{\Phi_1(\tilde{u}, \tilde{x})(m)}{\gamma^{-1}(m)^\beta} dm \leq \frac{\|f\|_\infty}{\lambda \gamma(0)^\beta} \left(\frac{\min(f)}{\lambda} - \beta\right) \|\tilde{u}\|_{L^1}.$$

V $\Phi_2(\tilde{u}, \tilde{x}) \leq \|\bar{h}\|_\infty \|\tilde{u}\|_{L^1}.$

Proof. (i) By changes of variables on m ,

$$\begin{aligned} \|\Phi_1(\tilde{u}, \tilde{x})\|_{L^1} &= \int_0^\infty \int_0^\infty f(a, m, \tilde{x}) \tilde{u}(e^{\lambda a} m) \exp\left(\lambda a - \int_0^a f(s, e^{\lambda(a-s)} m, \tilde{x}) ds\right) da dm \\ &= \int_0^\infty \tilde{u}(m) \underbrace{\int_0^\infty f(a, e^{-\lambda a} m, \tilde{x}) \exp\left(-\int_0^a f(s, e^{-\lambda s} m, \tilde{x}) ds\right) da}_{=1 \quad (\text{by Assumption 3 (i)})} dm. \end{aligned}$$

(ii)

$$\begin{aligned} |\Phi_1(\tilde{u}, \tilde{x})(m)| &\leq \mathbb{1}_{m > \gamma(0)} \|f\|_\infty \int_0^\infty \tilde{u}(e^{\lambda a} \gamma^{-1}(m)) \exp(\lambda a) da \\ &= \mathbb{1}_{m > \gamma(0)} \frac{\|f\|_\infty}{\lambda \gamma^{-1}(m)} \int_0^\infty \tilde{u}(e^{\lambda a} \gamma^{-1}(m)) \gamma^{-1}(m) \lambda \exp(\lambda a) da \\ &= \mathbb{1}_{m > \gamma(0)} \frac{\|f\|_\infty}{\lambda \gamma^{-1}(m)} \underbrace{\int_{\gamma^{-1}(m)}^\infty \tilde{u}(y) dy}_{\leq \|\tilde{u}\|_{L^1}} \end{aligned}$$

where for the last equality we used the change of variable $y = e^{\lambda a} \gamma^{-1}(m)$.

(iii) Performing the same change of variable as for (i) and using the fact that $\gamma(m) \leq \gamma(0) + m$, $\forall m \in \mathbb{R}_+$ (since $\gamma' \leq 1$), we have

$$\begin{aligned} & \int_0^\infty \Phi_1(\tilde{u}, \tilde{x})(m) m dm \\ &= \int_0^\infty \tilde{u}(m) \int_0^\infty \gamma(e^{-\lambda a} m) f(a, e^{-\lambda a} m, \tilde{x}) \exp\left(-\int_0^a f(s, e^{-\lambda s} m, \tilde{x}) ds\right) da dm \\ &\leq \int_0^\infty \tilde{u}(m) m \underbrace{\int_0^\infty e^{-\lambda a} f(a, e^{-\lambda a} m, \tilde{x}) \exp\left(-\int_0^a f(s, e^{-\lambda s} m, \tilde{x}) ds\right) da}_{=: \vartheta(m)} dm + \gamma(0) \|\tilde{u}\|_{L^1}. \end{aligned}$$

There exists $\theta \in]0, 1[$ such that for all $m \in \mathbb{R}_+^*$, $\vartheta(m) < 1$:

Fix $\epsilon > 0$.

$$\begin{aligned} \vartheta(m) &\leq \int_0^\epsilon f(a, e^{-\lambda a} m, \tilde{x}) \exp\left(-\int_0^a f(s, e^{-\lambda s} m, \tilde{x}) ds\right) da \\ &\quad + \int_\epsilon^\infty e^{-\lambda \epsilon} f(a, e^{-\lambda a} m, \tilde{x}) \exp\left(-\int_0^a f(s, e^{-\lambda s} m, \tilde{x}) ds\right) da \\ &= 1 - (1 - e^{-\lambda \epsilon}) \int_\epsilon^\infty f(a, e^{-\lambda a} m, \tilde{x}) \exp\left(-\int_0^a f(s, e^{-\lambda s} m, \tilde{x}) ds\right) da \\ &= 1 - (1 - e^{-\lambda \epsilon}) \exp\left(-\int_0^\epsilon f(s, e^{-\lambda s} m, \tilde{x}) ds\right) \\ &\leq 1 - (1 - e^{-\lambda \epsilon}) \exp(-\|f\|_\infty \epsilon) =: \theta < 1. \end{aligned}$$

Whence,

$$\int_0^\infty \Phi_1(\tilde{u}, \tilde{x})(m) m dm \leq \theta \int_0^\infty \tilde{u}(m) m dm + \gamma(0) \|\tilde{u}\|_{L^1}.$$

To see that

$$\int_0^\infty \Phi_1(\tilde{u}, \tilde{x})(m) m dm \leq \max\left(\int_0^\infty \tilde{u}(m) m dm, \frac{\gamma(0)}{1-\theta} \|\tilde{u}\|_{L^1}\right),$$

we can distinguish three cases: if $\int_0^\infty \tilde{u}(m) m dm = \infty$, the inequality is trivial; if $\frac{\gamma(0)}{1-\theta} \|\tilde{u}\|_{L^1} \leq \int_0^\infty \tilde{u}(m) m dm < +\infty$, then

$$\begin{aligned} \int_0^\infty \Phi_1(\tilde{u}, \tilde{x})(m) m dm &\leq \int_0^\infty \tilde{u}(m) m dm - (1-\theta) \int_0^\infty \tilde{u}(m) m dm + \gamma(0) \|\tilde{u}\|_{L^1} \\ &\leq \int_0^\infty \tilde{u}(m) m dm; \end{aligned}$$

and finally if $\int_0^\infty \tilde{u}(m) m dm < \frac{\gamma(0)}{1-\theta} \|\tilde{u}\|_{L^1}$, then

$$\int_0^\infty \Phi_1(\tilde{u}, \tilde{x})(m) m dm \leq \theta \frac{\gamma(0)}{1-\theta} \|\tilde{u}\|_{L^1} + \gamma(0) \|\tilde{u}\|_{L^1} = \frac{\gamma(0)}{1-\theta} \|\tilde{u}\|_{L^1}.$$

(iv)

$$\begin{aligned} \int_{\gamma(0)}^\infty \frac{\Phi_1(\tilde{u}, \tilde{x})(m)}{\gamma^{-1}(m)^\beta} dm &= \int_0^\infty \int_0^\infty \frac{1}{m} f(a, m, \tilde{x}) \tilde{u}(e^{\lambda a} m) \exp\left(\lambda a - \int_0^a f(s, e^{\lambda(a-s)} m, \tilde{x})\right) da dm \\ &\leq \|f\|_\infty \int_0^\infty \int_0^\infty \frac{1}{m^\beta} \tilde{u}(e^{\lambda a} m) \exp(\lambda a - \min(f)a) da dm, \end{aligned}$$

making the change of variable $y = e^{\lambda a} m$:

$$\begin{aligned} &= \|f\|_\infty \int_0^\infty \int_m^\infty \frac{1}{\lambda m^{1+\beta}} \tilde{u}(y) \exp\left(-\min(f) \frac{1}{\lambda} \ln\left(\frac{y}{m}\right)\right) dy dm \\ &= \frac{\|f\|_\infty}{\lambda} \int_0^\infty \int_m^\infty m^{\min(f)/\lambda - 1 - \beta} \tilde{u}(y) y^{-\min(f)/\lambda} dy dm, \end{aligned}$$

using Fubini's theorem and the fact that $\min(f)/\lambda - \beta > 0$:

$$\begin{aligned} &= \frac{\|f\|_\infty}{\lambda} \int_0^\infty \tilde{u}(y) y^{-\min(f)/\lambda} \underbrace{\int_0^y m^{\min(f)/\lambda - 1 - \beta} dm}_{= \frac{y^{\min(f)/\lambda - \beta}}{\min(f)/\lambda - \beta}} dy \\ &= \frac{\|f\|_\infty}{\lambda} \left(\frac{\min(f)}{\lambda} - \beta \right) \int_0^\infty \tilde{u}(y) y^{-\beta} dy. \end{aligned}$$

Finally, it is easy to check that $\int_0^\infty \tilde{u}(y) y^{-\beta} dy \leq \gamma(0)^{-\beta} \|\tilde{u}\|_{L^1}$.

(v) Use Eq. (9.4.33b) and see the proof of (i). \square

By these estimates, we see that there exists $\beta, C_1, C_2, C_3, C_4 > 0$ such that the set $\mathcal{C} \times B \in L^1(\cdot, +\infty) \times \mathbb{R}$, where

$$\mathcal{C} := \left\{ u \in L^1_+(\cdot, +\infty) \mid \|u\|_{L^1} \leq 1; u \leq \frac{C_1}{\gamma^{-1}(\cdot)} \text{ a.e.}; \int_0^\infty u(m) m dm \leq C_2; \int_{\gamma(0)}^\infty \frac{u(m)}{\gamma^{-1}(m)^\beta} dm \leq C_3 \right\}$$

and $B := [-C_4, +C_4]$, is stable by the operator Φ .

In order to apply Schauder's fixed-point theorem, we will need

Lemma 7. *Grant Assumptions 1 and 3. \mathcal{C} is convex, closed and compact for the weak topology $\sigma(L^1, L^\infty)$.*

Proof. It is easy to verify that \mathcal{C} is convex. Since \mathcal{C} is convex, it suffices to show that it is strongly closed to show that it is weakly closed. Let u_n be a sequence of elements of \mathcal{C} which converge strongly to $u \in L^1(\cdot, +\infty)$. By the strong convergence, $\|u\|_{L^1} \leq 1$. We can extract a subsequence u_{n_k} such that u_{n_k} converges to u a.e. Taking the pointwise limit, we have that $u \leq \frac{C_1}{\gamma^{-1}(\cdot)}$ a.e. Furthermore, by Fatou's lemma,

$$\int_{\gamma(0)}^\infty u(m) m dm \leq \liminf_{k \rightarrow +\infty} \int_{\gamma(0)}^\infty u_{n_k}(m) m dm \leq C_2$$

and

$$\int_{\gamma(0)}^\infty \frac{u(m)}{\gamma^{-1}(m)^\beta} dm \leq \liminf_{k \rightarrow +\infty} \int_{\gamma(0)}^\infty \frac{u_{n_k}(m)}{\gamma^{-1}(m)^\beta} dm \leq C_3.$$

Hence, \mathcal{C} is strongly closed.

To show that \mathcal{C} is weakly compact, we will show that

- $\sup_{u \in \mathcal{C}} \|u\|_{L^1} < \infty$,
- $\forall \epsilon > 0, \exists R > 0$ such that $\int_R^\infty u(m) dm < \epsilon$ for all $u \in \mathcal{C}$,
- \mathcal{C} is equi-integrable, i.e. $\forall \epsilon > 0, \exists \delta > 0$ such that for all Borel set $A \subset \mathbb{R}_+$ with $|A| \leq \delta$ and for all $u \in \mathcal{C}$, $\int_A u(m) dm \leq \epsilon$,

and use Dunford-Pettis theorem. (a.) is clearly verified. (b.) is also verified since for all $R > 0$, $\int_R^\infty u(m) dm \leq \frac{1}{R} \int_0^\infty u(m) m dm \leq \frac{C_2}{R}$. To show (c.), let us first observe that for all $\delta_1 > 0$,

$$\int_{\gamma(0)}^{\gamma(0) + \delta_1} u(m) dm \leq \gamma^{-1}(\gamma(0) + \delta_1)^\beta \int_{\gamma(0)}^\infty \frac{u(m)}{\gamma^{-1}(m)^\beta} dm \leq \gamma^{-1}(\gamma(0) + \delta_1)^\beta C_3.$$

For any $\epsilon > 0$, let us choose $\delta_1 > 0$ such that $\gamma^{-1}(\gamma(0) + \delta_1)^\beta C_3 \leq \frac{\epsilon}{2}$. Then, for all Borel set $A \subset \mathbb{R}_+$ with $|A| \leq \delta$,

$$\int_A u(m)dm \leq \int_{\gamma(0)}^{\gamma(0)+\delta_1} u(m)dm + \int_{A \setminus [0, \gamma(0)+\delta_1]} u(m)dm \leq \frac{\epsilon}{2} + \delta \frac{C_1}{\gamma^{-1}(\gamma(0) + \delta_1)}.$$

Hence, we can choose $\delta = \min\left(\delta_1, \frac{\epsilon \gamma^{-1}(\gamma(0)+\delta_1)}{2C_1}\right)$ and (c.) is verified. By the Dunford-Pettis theorem, \mathcal{C} is weakly relatively compact. Finally, since \mathcal{C} is weakly closed, \mathcal{C} is weakly compact. \square

We can now give an alternative proof of the existence of stationary solutions to (9.1.1) for arbitrary connectivity strength ϵ :

Proof of Theorem 2 (i). We verify that the operator Φ is weakly continuous: For any sequence $(u_n, x_n) \rightarrow (u, x)$ in $\mathcal{C} \times \mathbb{R}$ and for any $\varphi \in L^\infty(\mathbb{R}_+)$,

$$\left| \int (\Phi_1(u_n, x_n) - \Phi_1(u, x))\varphi(m)dm \right| \leq Q_1^n + Q_2^n + Q_3^n,$$

where

$$\begin{aligned} Q_1^n &:= \left| \int_0^\infty \int_0^\infty (u_n(e^{\lambda a} m) - u(e^{\lambda a} m))\varphi(\gamma(m))e^{\lambda a} f(a, m, x) e^{-\int_0^a f(\tau, e^{\lambda(a-\tau)}m, x)d\tau} dadm \right|, \\ Q_2^n &:= \|\varphi\|_\infty \int_0^\infty \int_0^\infty u_n(e^{\lambda a} m)e^{\lambda a} |f(a, m, x) - f(a, m, x_n)| e^{-\int_0^a f(\tau, e^{\lambda(a-\tau)}m, x)d\tau} dadm, \\ Q_3^n &:= \|\varphi\|_\infty \int_0^\infty \int_0^\infty u_n(e^{\lambda a} m)e^{\lambda a} f(a, m, x_n) \left| e^{-\int_0^a f(\tau, e^{\lambda(a-\tau)}m, x)d\tau} - e^{-\int_0^a f(\tau, e^{\lambda(a-\tau)}m, x_n)d\tau} \right| dadm. \end{aligned}$$

Making the change of variable $ydy = e^{\lambda a}mdm$ in Q_1 we get

$$Q_1^n = \left| \int_0^\infty (u_n(y) - u(y)) \int_0^\infty \varphi(\gamma(ye^{-\lambda a}))f(a, ye^{-\lambda a}, x) e^{-\int_0^a f(\tau, e^{-\lambda\tau}y, x)d\tau} dadm \right|.$$

Since u_n converges to u in $\sigma(L^1, L^\infty)$ and

$$\begin{aligned} \int_0^\infty \varphi(\gamma(ye^{-\lambda a}))f(a, ye^{-\lambda a}, x) e^{-\int_0^a f(\tau, e^{-\lambda\tau}y, x)d\tau} da \\ \leq \|\varphi\|_\infty \int_0^\infty f(a, ye^{-\lambda a}, x) e^{-\int_0^a f(\tau, e^{-\lambda\tau}y, x)d\tau} da = \|\varphi\|_\infty, \end{aligned}$$

Q_1^n converges to 0. On the other hand, since f is bounded and Lipschitz, $Q_2^n, Q_3^n \leq \|u_n\|_{L^1} C|x_n - x| \leq C|x_n - x|$. Whence, Φ_1 is a continuous operator with respect to the weak topology $\sigma(L^1, L^\infty)$.

The continuity of Φ_2 is shown analogously, taking $\varphi = h$ (h is a bounded).

Since \mathcal{C} is stable by Φ , convex and weakly compact (Lemma 7), we can apply Schauder's fixed-point theorem to obtain the existence of a fixed-point, which gives the existence of a stationary solution. \square

Corollary 2. *Grant Assumptions 1 and 3. If f is of class C^k , then $u(m)$ is a function of class C^k for all $m > \gamma(0)$. Consequently, the stationary solutions of (9.1.1) are of class C^k .*

Proof. If (u, \tilde{x}) is a fixed-point of Φ , then

$$\begin{aligned} u(m) = \mathbb{1}_{m > \gamma(0)} \left| (\gamma^{-1})'(m) \right| \int_0^\infty f(a, \gamma^{-1}(m), \tilde{x}) u(e^{\lambda a} \gamma^{-1}(m)) \\ \exp\left(\lambda a - \int_0^a f(s, e^{\lambda(a-s)} \gamma^{-1}(m), \tilde{x}) ds\right) da. \quad (9.4.34) \end{aligned}$$

Making the change of variable $y = e^{\lambda a \gamma^{-1}(m)}$ in a , as in the estimate (ii) of Lemma 6, we obtain

$$u(m) = \mathbb{1}_{m > \gamma(0)} \frac{|(\gamma^{-1})'(m)|}{\lambda \gamma^{-1}(m)} \int_{\gamma^{-1}(m)}^{\infty} f(g(y, m), y, \tilde{x}) u(y) \exp\left(-\int_0^{g(y, m)} f(s, e^s y, \tilde{x}) ds\right) dy, \quad (9.4.35)$$

where $g(y, m) = \ln \frac{y}{\lambda(\gamma^{-1}(m))}$. We conclude with a bootstrap argument: if u is L^1 , then the right hand side of Eq. (9.4.35) is a continuous function of m , meaning that u is continuous. But if u is continuous, then the right hand side is of class C^1 , etc. \square

Corollary 3. *Grant Assumptions 1 and 3. There exists a constant $C > 0$, such that the stationary solution ρ_∞ satisfies,*

$$\rho_\infty(a, m) \leq \frac{C e^{-\sigma(a - \Delta_{abs})}}{m}. \quad (9.4.36)$$

Proof. From the previous theorem it follows that there is C such that $u(m) \leq C/m$, which, together with (9.4.32), implies

$$\rho_\infty(a, m) \leq \bar{C} \frac{e^{-\int_0^a f(s, e^{\lambda(a-s)\gamma^{-1}(m)}, \tilde{x}) ds}}{m}. \quad (9.4.37)$$

The estimate follows from Assumption 3 (i). \square

9.4.3 Formula in the case of short-term synaptic depression

In general, there is no explicit formula for the invariant probability measure solving Eq. (9.3.26). However, in the case of short-term synaptic depression Eq. (9.1.3), we can derive an explicit expression for the total postsynaptic potential

$$X(\tilde{x}) := \int_0^\infty \hat{h}(t) \int_0^1 \int_0^\infty (1-m) f(a, \tilde{x}) \rho_\infty^{\tilde{x}}(a, m) da dm dt, \quad (9.4.38)$$

for any $\tilde{x} \in \mathbb{R}$. This fact has been reported in the theoretical neuroscience literature [219]; we provide here a rigorous and analytic justification for it.

For all $\tilde{x} \in \mathbb{R}$, let us introduce the quantities

$$\begin{aligned} I^{\tilde{x}} &:= \int_0^\infty a f(a, \tilde{x}) \exp\left(-\int_0^a f(s, \tilde{x}) ds\right) da = \int_0^\infty \exp\left(-\int_0^a f(s, \tilde{x}) ds\right) da, \\ P^{\tilde{x}}(\lambda) &:= \int_0^\infty e^{-\lambda a} f(a, \tilde{x}) \exp\left(-\int_0^a f(s, \tilde{x}) ds\right) da. \end{aligned}$$

$I^{\tilde{x}}$ can be interpreted as the mean inter-spike interval of a neuron receiving a constant input \tilde{x} . $P^{\tilde{x}}(\lambda)$ can be seen as the Laplace transform of the inter-spike interval distribution of that neuron, evaluated in λ .

Proposition 2. *Grant Assumptions 1 and 3. For all $\tilde{x} \in \mathbb{R}$,*

$$X(\tilde{x}) = \int_0^\infty \hat{h}(t) dt \frac{1}{I^{\tilde{x}}} \left\{ \frac{1 - P^{\tilde{x}}(\lambda)}{1 - v P^{\tilde{x}}(\lambda)} \right\}.$$

Proof. Using the method of characteristics (i.e. combining Eqs. (9.3.26b) and (9.4.32)), we have

$$\begin{aligned} 1 &= \int_0^1 \int_0^\infty \rho_\infty^{\tilde{x}}(a, m) da dm = \int_0^1 \int_0^\infty \mathbb{1}_{e^{\lambda a} m < 1} \rho_\infty^{\tilde{x}}(0, e^{\lambda a} m) \exp\left(\lambda a - \int_0^a f(s, \tilde{x}) ds\right) da dm \\ &= \int_0^1 \int_0^\infty \rho_\infty^{\tilde{x}}(0, m) \exp\left(-\int_0^a f(s, \tilde{x}) ds\right) da dm = I^{\tilde{x}} \int_0^1 \rho_\infty^{\tilde{x}}(0, m) dm. \end{aligned}$$

Whence,

$$\int_0^1 \int_0^\infty f(a, \tilde{x}) \rho_\infty^{\tilde{x}}(a, m) dadm = \int_0^1 \rho_\infty^{\tilde{x}}(0, m) dm = \frac{1}{I^{\tilde{x}}}.$$

On the other hand,

$$\begin{aligned} & \int_0^1 \int_0^\infty mf(a, \tilde{x}) \rho_\infty^{\tilde{x}}(a, m) dadm \\ &= \int_0^1 \int_0^\infty \mathbb{1}_{e^{\lambda a} m < 1} mf(a, \tilde{x}) \rho_\infty^{\tilde{x}}(0, e^{\lambda a} m) \exp\left(\lambda a - \int_0^a f(s, \tilde{x}) ds\right) dadm \\ &= \int_0^1 \int_0^\infty e^{-\lambda a} mf(a, \tilde{x}) \rho_\infty^{\tilde{x}}(0, m) \exp\left(-\int_0^a f(s, \tilde{x}) ds\right) dadm \\ &= P^{\tilde{x}}(\lambda) \int_0^1 m \rho_\infty^{\tilde{x}}(0, m) dm \end{aligned}$$

and

$$\begin{aligned} \int_0^1 m \rho_\infty^{\tilde{x}}(0, m) dm &= \int_0^1 m \mathbb{1}_{m > 1-v} \frac{1}{v} \int_0^\infty f(a, \tilde{x}) \rho_\infty^{\tilde{x}}\left(a, 1 - \frac{1-m}{v}\right) dadm \\ &= \int_0^1 (1-v+vm) \int_0^\infty f(a, \tilde{x}) \rho_\infty^{\tilde{x}}(a, m) dadm \\ &= \frac{1-v}{I^{\tilde{x}}} + v P^{\tilde{x}}(\lambda) \int_0^1 m \rho_\infty^{\tilde{x}}(0, m) dm. \end{aligned}$$

Whence,

$$\int_0^1 m \rho_\infty^{\tilde{x}}(0, m) dm = \frac{1-v}{I^{\tilde{x}}(1-vP^{\tilde{x}}(\lambda))}$$

and

$$\int_0^1 \int_0^\infty mf(a, \tilde{x}) \rho_\infty^{\tilde{x}}(a, m) dadm = \frac{P^{\tilde{x}}(\lambda)(1-v)}{I^{\tilde{x}}(1-vP^{\tilde{x}}(\lambda))}.$$

Finally, we have

$$\begin{aligned} X(\tilde{x}) &= \int_0^\infty \hat{h}(t) dt \left\{ \int_0^1 \int_0^\infty f(a, \tilde{x}) \rho_\infty^{\tilde{x}}(a, m) dadm - \int_0^1 \int_0^\infty mf(a, \tilde{x}) \rho_\infty^{\tilde{x}}(a, m) dadm \right\} \\ &= \int_0^\infty \hat{h}(t) dt \frac{1}{I^{\tilde{x}}} \left\{ 1 - \frac{P^{\tilde{x}}(\lambda)(1-v)}{(1-vP^{\tilde{x}}(\lambda))} \right\} = \int_0^\infty \hat{h}(t) dt \frac{1}{I^{\tilde{x}}} \left\{ \frac{1-P^{\tilde{x}}(\lambda)}{1-vP^{\tilde{x}}(\lambda)} \right\}. \end{aligned}$$

□

9.5 Exponential stability in the weak connectivity regime

To study the long time behavior (9.1.1) in the weak connectivity regime, we perturb the non-interacting case (9.3.19), taking $\tilde{x} = \varepsilon x_\infty$, where x_∞ is given by the unique stationary solution to (9.1.1) when $\varepsilon \in]-\varepsilon^*, +\varepsilon^*[$ (ε^* is taken from Theorem 2 (ii)). In this section, we keep the small ε fixed and we work under Assumptions 1 – 3 and 5. We roughly follow the same line of argument as [181, Sec. 5].

For convenience, we first rewrite (9.1.1) in a more formal and compact form:

$$\partial_t \rho_t = -\partial_a \rho_t + \lambda \partial_m (m \rho_t) - f(\varepsilon x_t) \rho_t + \delta_0^a (\gamma \circ \Pi)_* (f(\varepsilon x_t) \rho_t), \quad (9.5.39a)$$

$$x_t = \int_0^t \int f(t-s) f(\varepsilon x_s) \rho_s dadm ds, \quad (9.5.39b)$$

$$\rho_0 = u_0, \quad (9.5.39c)$$

where δ_0^a indicates that (singular) mass enters in $a = 0^1$, $\Pi : (a, m) \mapsto m$ is the projection on m and $*$ denotes the pushforward measure. To write Eq. (9.5.39) as an evolution equation, we introduce an auxiliary transport equation on $\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+^*$

$$\begin{aligned}\partial_t \zeta_t &= -\partial_s \zeta_t + \delta_0^s f(\varepsilon x_t) \rho_t, \\ \zeta_0 &= 0,\end{aligned}$$

which solution is given by the method of characteristics:

$$\zeta_t(s) = \mathbb{1}_{s \leq t} f(\varepsilon x_{t-s}) \rho_{t-s}, \quad \forall (t, s) \in \mathbb{R}_+^* \times \mathbb{R}_+.$$

Using the auxiliary equation, Eq. (9.5.39) is equivalent to

$$\partial_t(\rho_t, \zeta_t) = \left(-\partial_a \rho_t + \lambda \partial_m(m \rho_t) - f(\varepsilon x_t) \rho_t + \delta_0^a (\gamma \circ \Pi)_*(f(\varepsilon x_t) \rho_t), -\partial_s \zeta_t + \delta_0^s f(\varepsilon x_t) \rho_t \right), \quad (9.5.40a)$$

$$(\rho_0, \zeta_0) = (u_0, 0), \quad (9.5.40b)$$

where $x_t := \int_0^\infty \int h(s) \zeta_t(s) da dm ds$.

By Theorem 2, for all $\varepsilon \in]-\varepsilon^*, +\varepsilon^*[$, there exists a unique stationary solution (ρ_∞, x_∞) and we have

$$-\partial_a \rho_\infty + \lambda \partial_m(m \rho_\infty) - f(\varepsilon x_\infty) \rho_\infty + \delta_0^a (\gamma \circ \Pi)_*(f(\varepsilon x_\infty) \rho_\infty) = 0. \quad (9.5.41)$$

Now, we write Eq. (9.5.40) as the sum of a linear equation and a perturbation:

$$\partial_t(\rho_t, \zeta_t) = \Lambda(\rho_t, \zeta_t) + (Z_t^{(1)}, Z_t^{(2)}), \quad (9.5.42a)$$

$$(\rho_0, \zeta_0) = (u_0, 0), \quad (9.5.42b)$$

where

$$\Lambda(\rho_t, \zeta_t) := \left(-\partial_a \rho_t + \lambda \partial_m(m \rho_t) - f(\varepsilon x_\infty) \rho_t + \delta_0^a (\gamma \circ \Pi)_*(f(\varepsilon x_\infty) \rho_t), -\partial_s \zeta_t + \delta_0^s f(\varepsilon x_\infty) \rho_t \right),$$

$$Z_t^{(1)} := [f(\varepsilon x_\infty) - f(\varepsilon x_t)] \rho_t + \delta_0^a (\gamma \circ \Pi)_*([f(\varepsilon x_t) - f(\varepsilon x_\infty)] \rho_t),$$

$$Z_t^{(2)} := \delta_0^s [f(\varepsilon x_t) - f(\varepsilon x_\infty)] \rho_t.$$

Let us put $\zeta_\infty(s) := f(\varepsilon x_\infty) \rho_\infty$, $\forall s \in \mathbb{R}_+$. Then, using Eq. (9.5.41), by the linearity of the operator Λ and writing $\bar{\rho}_t := \rho_t - \rho_\infty$ and $\bar{\zeta}_t := \zeta_t - \zeta_\infty$, we get

$$\partial_t(\bar{\rho}_t, \bar{\zeta}_t) = \Lambda(\bar{\rho}_t, \bar{\zeta}_t) + (Z_t^{(1)}, Z_t^{(2)}), \quad (9.5.43a)$$

$$(\bar{\rho}_0, \bar{\zeta}_0) = (u_0 - \rho_\infty, -\zeta_\infty). \quad (9.5.43b)$$

Writing $(S_t^\Lambda)_{t \in \mathbb{R}_+}$ the semigroup associated with the operator Λ , we have, by Duhamel's formula,

$$(\bar{\rho}_t, \bar{\zeta}_t) = S_t^\Lambda(\bar{\rho}_0, \bar{\zeta}_0) + \int_0^t S_{t-s}^\Lambda(Z_s^{(1)}, Z_s^{(2)}) ds, \quad \forall t \geq 0. \quad (9.5.44)$$

Let us define the weighted space

$$L_+^1(\mu) := \left\{ \zeta \in L^1(\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+^*, \mathbb{R}_+) \mid \int_0^\infty \|\zeta(s)\|_{L^1} \|h\|_\infty e^{-hs} ds < \infty \right\}.$$

¹ δ_0^a should not be confused with the Dirac distribution $\delta_{0=a}$. Using $\delta_{0=a}$, by integration by parts of weak solutions, Eq. (9.5.39a) should write

$$\partial_t \rho_t = -\partial_a \rho_t + \lambda \partial_m(m \rho_t) - f(\varepsilon x_t) \rho_t + \delta_{0=a} \left\{ (\gamma \circ \Pi)_*(f(\varepsilon x_t) \rho_t) - \rho_t(0, \cdot) \right\}.$$

Note that, for all $t \geq 0$,

$$\begin{aligned} |x_t - x_\infty| &= \left| \int_0^\infty \int h(s) \zeta_t(s) \, dadm \, ds - \int_0^\infty \int h(s) \zeta_\infty(s) \, dadm \, ds \right| \\ &\leq \int_0^\infty \|h\|_\infty e^{-\mathfrak{h}s} \|\zeta_t(s) - \zeta_\infty(s)\|_{L^1} \, ds = \|\bar{\zeta}_t\|_{L^1(\mu)}. \end{aligned}$$

Also, we have, for all $t \geq 0$,

$$\left\| Z_t^{(1)} \right\|_{L^1} \leq |\varepsilon| 2L_f \|\rho_t\|_{L^1} |x_t - x_\infty| \leq |\varepsilon| 2L_f \|\bar{\zeta}_t\|_{L^1(\mu)}, \quad (9.5.45a)$$

$$\left\| Z_t^{(1)} \right\|_{L^1(w)} \leq |\varepsilon| 2L_f \|\rho_t\|_{L^1(w)} |x_t - x_\infty| \leq |\varepsilon| 2L_f \left(\|u_0\|_{L^1(w) + \frac{b}{a}} \right) \|\bar{\zeta}_t\|_{L^1(\mu)}, \quad (9.5.45b)$$

$$\left\| Z_t^{(2)} \right\|_{L^1(\mu)} \leq |\varepsilon| \|h\|_\infty L_f \|\rho_t\|_{L^1} |x_t - x_\infty| \leq |\varepsilon| \|h\|_\infty L_f \|\bar{\zeta}_t\|_{L^1(\mu)}, \quad (9.5.45c)$$

where we have used Theorem 1 (ii) in the first line and Lemma 1 in the second.

Lemma 8. *Grant Assumptions 1 – 3 and 5 and take $(\bar{\rho}_0, \bar{\zeta}_0)$ as in Eq. (9.5.43). There exists $K_1 \geq 1$ and $\mathfrak{a}_1 > 0$ such that, for all initial data $u_0 \in L^1_+(w)$ with $\|u_0\|_{L^1} = 1$,*

$$\left\| S_t^\Lambda(\bar{\rho}_0, \bar{\zeta}_0) \right\|_{L^1(w) \times L^1(\mu)} \leq K_1 e^{-\mathfrak{a}_1 t} \left\| (\bar{\rho}_0, \bar{\zeta}_0) \right\|_{L^1(w) \times L^1(\mu)}, \quad \forall t \geq 0. \quad (9.5.46)$$

If in addition, we grant Assumption 4, then there exists $K_2 \geq 1$ and $\mathfrak{a}_2 > 0$ such that, for all initial data $u_0 \in L^1_+$ with $\|u_0\|_{L^1} = 1$,

$$\left\| S_t^\Lambda(\bar{\rho}_0, \bar{\zeta}_0) \right\|_{L^1 \times L^1(\mu)} \leq K_2 e^{-\mathfrak{a}_2 t} \left\| (\bar{\rho}_0, \bar{\zeta}_0) \right\|_{L^1 \times L^1(\mu)}, \quad \forall t \geq 0. \quad (9.5.47)$$

Proof. We write $(S_t^\Lambda(\bar{\rho}_0, \bar{\zeta}_0)^{(1)}, S_t^\Lambda(\bar{\rho}_0, \bar{\zeta}_0)^{(2)}) := S_t^\Lambda(\bar{\rho}_0, \bar{\zeta}_0)$ the first and second component of $S_t^\Lambda(\bar{\rho}_0, \bar{\zeta}_0)$.

By Theorem 4, there exists $K \geq 0$ and $\mathfrak{a} > 0$ such that,

$$\left\| S_t^\Lambda(\bar{\rho}_0, \bar{\zeta}_0)^{(1)} \right\|_{L^1(w)} \leq K e^{-\mathfrak{a}t} \|\bar{\rho}_0\|_{L^1(w)}, \quad \forall t \geq 0.$$

Then,

$$\begin{aligned} \left\| S_t^\Lambda(\bar{\rho}_0, \bar{\zeta}_0)^{(2)} \right\|_{L^1(\mu)} &= \int_0^t \left\| f(\varepsilon x_\infty) S_{t-s}^\Lambda(\bar{\rho}_0, \bar{\zeta}_0)^{(1)} \right\|_{L^1} C_h e^{-\mathfrak{h}s} \, ds \\ &\quad + \int_t^\infty \left\| \bar{\zeta}_0(s) \right\|_{L^1} C_h e^{-\mathfrak{h}s} \, ds \\ &\leq C_h \left\{ \|f\|_\infty K \int_0^t e^{-\mathfrak{a}(t-s)} e^{-\mathfrak{h}s} \, ds \|\bar{\rho}_0\|_{L^1(w)} + e^{-\mathfrak{h}t} \|\bar{\zeta}_0\|_{L^1(\mu)} \right\}. \end{aligned}$$

Gathering the bounds on the two components and observing that the function $t \mapsto \int_0^t e^{-\mathfrak{a}(t-s)} e^{-\mathfrak{h}s} \, ds$ decays exponentially, we conclude that there exists $K_1 \geq 1$ and $\mathfrak{a}_1 > 0$ such that Eq. (9.5.46) holds.

For Eq. (9.5.47), we use Theorem 5 and follow the same argument. \square

We can now prove our main result:

Proof of Theorem 3. By Duhamel's formula (9.5.44), Eq. (9.5.46) in Lemma 8 and the bounds Eqs. (9.5.45), for all $t \geq 0$,

$$\begin{aligned} \|(\bar{\rho}_t, \bar{\zeta}_t)\|_{L^1(w) \times L^1(\mu)} &\leq \|S_t^\Lambda(\bar{\rho}_0, \bar{\zeta}_0)\|_{L^1(w) \times L^1(\mu)} + \int_0^t \|S_{t-s}^\Lambda(Z_s^{(1)}, Z_s^{(2)})\|_{L^1(w) \times L^1(\mu)} ds \\ &\leq K_1 e^{-\mathbf{a}_1 t} \|(\bar{\rho}_0, \bar{\zeta}_0)\|_{L^1(w) \times L^1(\mu)} + K_1 \int_0^t e^{-\mathbf{a}_1(t-s)} \|(Z_s^{(1)}, Z_s^{(2)})\|_{L^1(w) \times L^1(\mu)} ds \\ &\leq K_1 e^{-\mathbf{a}_1 t} \|(\bar{\rho}_0, \bar{\zeta}_0)\|_{L^1(w) \times L^1(\mu)} + |\varepsilon| \tilde{C}_W \int_0^t e^{-\mathbf{a}_1(t-s)} \|(\bar{\rho}_s, \bar{\zeta}_s)\|_{L^1(w) \times L^1(\mu)} ds \\ &=: \mathcal{Q}(t), \end{aligned}$$

where \tilde{C}_K is a constant depending on W . We have, for all $t \geq 0$,

$$\begin{aligned} \frac{d}{dt} \mathcal{Q}(t) &= -\mathbf{a}_1 \mathcal{Q}(t) + |\varepsilon| \tilde{C}_W \|(\bar{\rho}_t, \bar{\zeta}_t)\|_{L^1(w) \times L^1(\mu)} \\ &\leq \left(-\mathbf{a}_1 + |\varepsilon| \tilde{C}_W\right) \mathcal{Q}(t). \end{aligned}$$

Whence, by Grönwall's lemma,

$$\forall t \geq 0, \quad \mathcal{Q}(t) \leq K_1 \|(\bar{\rho}_0, \bar{\zeta}_0)\|_{L^1(w) \times L^1(\mu)} \exp\left(\left(-\mathbf{a}_1 + |\varepsilon| \tilde{C}_W\right) t\right).$$

For all $t \geq 0$, we have

$$\|\rho_t - \rho_\infty\|_{L^1(w)} + |x_t - x_\infty| \leq \|(\bar{\rho}_t, \bar{\zeta}_t)\|_{L^1(w) \times L^1(\mu)} \leq \mathcal{Q}(t)$$

and

$$\|\bar{\zeta}_0\|_{L^1(\mu)} \leq \int_0^\infty \|f(\varepsilon x_\infty) \rho_\infty\|_{L^1} C_h e^{-\mathfrak{h}s} ds \leq \frac{\|f\|_\infty C_h}{\mathfrak{h}}.$$

Whence, choosing $\varepsilon_W^{**} := \frac{\mathfrak{a}_1}{\tilde{C}_W} \wedge \varepsilon^*$, we easily see that there exists $C \geq 1$ and $c_W > 0$ such that Eq. (9.1.6) holds.

For Eq. (9.1.7), we use Eq. (9.5.47) instead of Eq. (9.5.46) and follow the same argument. \square

Appendix

Here, we compare simulations of Eq. (9.1.2) with simulations of the time elapsed neuron network model [199].

If, the firing rate function f does not depend on m and if we put

$$f(a, \varepsilon x_t) := \hat{f}(\eta(a) + \varepsilon x_t), \quad (9.5.48)$$

then Eq. (9.1.2) reduces to the time elapsed neuron network model

$$\partial_t \rho_t(a) + \partial_a \rho_t(a) = -f(a, \varepsilon x_t) \rho_t(a), \quad (9.5.49a)$$

$$\rho_t(0) = \int_0^\infty f(a, \varepsilon x_t) \rho_t(a) da, \quad (9.5.49b)$$

$$x_t = \int_0^t h(t-s) \int_0^\infty \int_0^\infty f(a, \varepsilon x_s) \rho_s(a) da ds, \quad (9.5.49c)$$

$$\rho_0(a) = u_0(a). \quad (9.5.49d)$$

Eq. (9.5.49) is the population equation for non-adaptive SRM₀ neurons (or age-dependent nonlinear Hawkes processes) [59]. As reported previously, Eq. (9.5.49) exhibits self-sustained oscillations for large ε or relaxation to a stationary state for small ε (see Fig. 2).

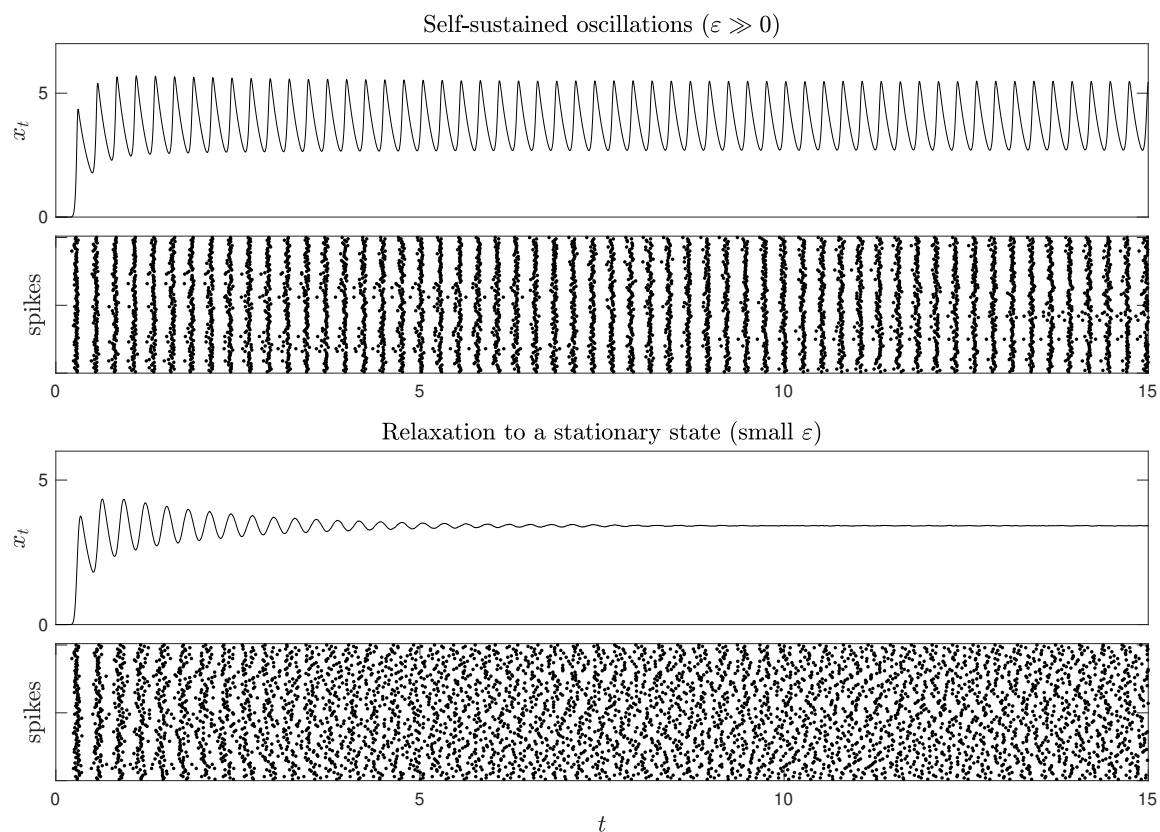


Figure 9.2: Same as Fig. 1 but for the time elapsed neuron network model Eq. (9.5.49). Simulations of a network of $5 \cdot 10^5$ non-adaptive SRM_0 neurons, approximating Eq. (9.5.49), with identical parameters (except for ε) and identical initial conditions. Neuronal parameters are the same as in Fig. 1, except that f is replaced by Eq. (9.5.48). The ε have also been adapted.

Chapter 10

Concentration inequality for dynamical systems and applications

This chapter is the result of a work in progress in collaboration with Sylvain Delattre and Marc Hoffmann.

10.1 Introduction

We are interested in studying the stochastic system of N interacting agents

$$Z_t = (Z_t^1, \dots, Z_t^N), \quad t \in [0, T], \quad (10.1.1)$$

where each agent $Z_t^i = (X_t^i, Y_t^i)$ is described by two states X_t^i, Y_t^i evolving in the Euclidean space \mathbb{R}^d and solves

$$\begin{cases} dX_t^i = F(X_t^i, Y_t^i)dt + \frac{1}{N} \sum_{j=1}^N H(X_t^i - X_t^j, Y_t^i - Y_t^j) + \sigma dB_t^i \\ dY_t^i = G(X_t^i, Y_t^i)dt \\ \mathcal{L}(X_0, Y_0) = \mu_0^{\otimes N}, \end{cases} \quad (10.1.2)$$

where B^i are independent \mathbb{R}^d -valued Brownian motions, a constant $\sigma > 0$, and the coefficient F , G and H satisfy regularity and the growth conditions to be specified below.

This type of model was introduced back in the 1960s [163], originally in work in statistical physics to describe the interaction of particles in plasma physics, the state (X, Y) corresponding to the position and velocity of electrons or ions. This remains one of the main current applications and continues to be of interest to the mathematical community as illustrated by the works of Guillin *et al.*[117], Bresch *et al.*[39] and Boley *et al.* [34]. Beyond that, the 2010s brought an expansion of the field of application, spreading its use to collective animal behavior and population dynamics Bolley *et al.* [33], Mogilner *et al.* [183]; opinion dynamics Chazelle *et al.* [56], finance Fouque and Sun[87] and neuroscience Baladron *et al.* [15]. The latter is the main motivation for this work, in particular, the FitzHugh-Nagumo model that we describe below in details and has already been presented in the work of Mischler *et al.* [180], Lucon and Poquet [159],[160] and Quinao and Touboul[211]. The word agents, in this case, corresponds to neurons interacting in the same network interacting through electrical synapses. The coefficients F and G describe the part of the behavior of each agent that depends only on its own state, we call it "single agent behavior", while H describes the interaction between the agents and the last term introduces some randomness, whose intensity is modulated by the σ parameter.

The objective of this work goes in three related directions. First of all, we are interested in studying the behavior of such a system when the number of agents N is large. More in details,

we prove that the empirical measure of the system

$$\mu_t^N = \frac{1}{N} \sum_{j=1}^N \delta_{X^j} \quad (10.1.3)$$

is close to μ_t the solution of the associated Fokker Planck equation

$$\partial_t \mu(x, y) = -\nabla \cdot ((F, G)\mu(x, y)) - \nabla \cdot (\mu(x, y) \int H(x - \bar{x}, y - \bar{y}) \mu(d\bar{x}, d\bar{y})) + 1/2\sigma^2 \partial_{xx} \mu(x, y), \quad (10.1.4)$$

In order to do so, we analyze the mean-field limit of the system and establish a concentration inequality. Extensive methodology has been developed to address these problems, see for example Tanaka and Hitsuda [229], Sznitman [227], [228], Fernandez *et al.* [84], Bolley [35], Massart [162] among myriad references. Our result follows in spirit that of Della Maestra and Hoffmann [73] comparable to the Bernstein deviation inequality for independent data as it appears in Boucheron *et al.* [37].

The model (10.1.2) considered here is similar to the one studied by Bolley *et al.* in [33]. In that work, the authors prove the existence of a limit equation, and then use classical coupling techniques to establish a mean field limit. Here we present a different strategy to arrive at the limit equation and we obtain a new concentration inequality that relates the empirical measure of the system and the resulting law of this limit equation in a stronger result. Our perspective follows the strategy proposed by the recent work of Lacker [145] and Della Maestra and Hoffmann [73], which we extend here to accommodate to continuous coefficients not necessarily bounded (as in Lacker) or globally Lipschitz (as in Della Maestra), and we deal with the degeneracy that arises when considering a two-state model like the one described above.

As a first application of the concentration result, we estimate the parameters of the FitzHugh-Nagumo model.

As a second application, we take advantage of it to arrive at a nonparametric estimate of μ_t , following the arguments of Della Maestra and Hoffmann [73] and extending them to our conditions.

Finally, on the three and last stage of this chapter, we move towards the construction of a statistical test to answer the question of the existence or not of interactions between the agents. For different models in the field of neuroscience, it is widely accepted that the behavior of the voltage and the time between discharges of a population of neurons differs significantly when the connectivity between their is low or high. In chapter 1, we see that when the interaction is weak, the distribution of the state of the neurons, the elapse time between discharges, converges to a steady state, while when the connectivity increases the distribution depends on time and some patterns can appear, describing, for example, oscillations. The same behavior is observed in the time elapse model proposed by Pakdaman *et al.* [200], [201], in the integrate and fire model, see for example Cormier *et al.* [66], [63] and for the FhN model, Mischler *et al.* [180].

This leads us to the question, how connected are neurons in practice? when one sees the appearance of patterns in the measurements, is this because the population of neurons is receiving different stimulation every short time, so that although the connectivity is weak, it never manages to reach the steady state or is it a stable pattern without external intervention caused by the connectivity between neurons? Our interest is to create a test that, based on the experience in a short window of time, can tell us if the interaction between the agents is weak or strong. In that sense, we use our first result to anticipate a suitable candidate of an estimator and a rejection zone and we test it in different simulations, testing its validity at least numerically.

10.1.1 The FitzHugh-Nagumo model

The main motivation for this work is the Fitzhugh-Nagumo model (FhN) for populations of neurons. The FhN model was first introduced in the works of FitzHugh [85] and Nagumo

[191] as a simplification of the Huxley-Hodgkin model (HH) that describes the evolution of the membrane potential of a neuron [125]. The dynamics is based on two variables, a variable x which corresponds to the membrane potential and a recovery variable y , which satisfy the equations

$$\begin{aligned}\dot{x} &= F(x, y), \\ \dot{y} &= G(x, y),\end{aligned}$$

for $F(x, y) = x - x^3/3 - y + I$ and $G(x, y) = 1/c(x + a - by)$ with $a, I, c > 0$, $b \in \mathbb{R}$. On this model, I denotes the total membrane current and is a stimulus applied to the neuron, c determines the strength of damping while a and b govern two important characteristics of the oscillating solution, namely spike rate and spike duration [220]. With only these elements the system shows the most important properties of the 4-dimensional HH model such as refractoriness, insensitivity to further immediate stimulation after one discharge, and excitability, the ability to generate a large, rapid change of membrane voltage in response to a very small stimulus. Numerous works have been aimed at studying the ODE model and their properties, we reference the book of Rocsoreanu *et al.* [218] and the references therein.

More recently, specialists have been interested in the passage of the behavior of a neuron to neural networks, see for example Mischler *et al.* [180], Lucon and Poquet [160] and Baladron *et al.* [15]. When neurons interact through electrical synapses, it has been proposed that the evolution of N neurons satisfies

$$\begin{cases} dX_t^i = (F(X_t^i, Y_t^i) - \sum_{j=1}^N J_{ij}(X_t^i - X_t^j))dt + \sigma dB_t^i \\ dY_t^i = G(X_t^i, Y_t^i)dt \end{cases} \quad (10.1.5)$$

where coefficients $J_{ij} > 0$ represent the effect of the interconnection between the neurons, and the term B_t^i refers, as usual, to independent Brownian motions.

We consider that the interactions are symmetric and identical for every pair of neurons in the network, which in particular implies that all neurons are connected. The strength of the interaction is measured by the parameter $J_{ij} = J$ that we re-parametrize as a function of a new parameter λ such that $J = \frac{\lambda}{N}$, where N is the number of neurons in the network.

In section 10.5 we propose a method to estimate the parameters $\theta = (I, a, b, c, \lambda, \sigma^2)$ based on the observation of the moments of the activity of a neuronal population. The estimation of the parameters of the FhN model has been approached through different methods, see Che *et al.* [57] for the least squares method, Rudi *et al.* [220] for Neural Networks and Jensen and Ditlevsen [131] for Markov chain Monte Carlo approach. Yet in most cases the estimations target a selection of the parameters of the FhH equation for a single neuron from measurements of the voltage of neurons whose activity is assumed to be independent. Here we present an alternative method that includes the interaction between neurons.

10.1.2 Notation

Let us introduce some notation and definitions. We set an integer $d \geq 1$ and a time horizon $T > 0$. We denote by \mathcal{C} the space of the continuous paths from $[0, T]$ to \mathbb{R}^{2d} . For the rest of this work, we use the notation $Z = (Z_t)_{t \in [0, T]} = (X_t, Y_t)_{t \in [0, T]}$ for the canonical process in \mathcal{C} . The existence of a solution for any of the stochastic equation that appear in this work, can be understood equivalently as the existence of a probability such that the canonical process is a solution of the equation. We work in the probability space $\mathcal{P}(\mathcal{C})$, the space of all the probabilities over \mathcal{C} , endowed with the Wasserstein 1-metric

$$W_1(\mu, \nu) = \inf_{m \in \Gamma(\mu, \nu)} \int_{\mathcal{C} \times \mathcal{C}} |z_1 - z_2| m(dz_1, dz_2) = \sup_{|\varphi|_{Lip} \leq 1} \int \varphi d(\mu - \nu), \quad (10.1.6)$$

where $\Gamma(\mu_t, \nu_t)$ denote the set of probability measures on $\mathcal{C} \times \mathcal{C}$ with marginals μ and ν . Also, we will use the notation \mathcal{C}_X to refer to the space of continuous functions of the continuous path from $[0, T]$ to \mathbb{R}^d and $\mathcal{P}(\mathcal{C}_X)$ for the corresponding probability space. Finally we denote \mathcal{C}^N the space of the continuous path from $[0, T]$ to $(\mathbb{R}^{2d})^N$ and $\mathcal{P}(\mathcal{C}^N)$ the associated probabilities space.

We call P the probability such that $W_t = \sigma^{-1}X_t$ is a standard Brownian motion and consider $(\mathcal{F}_t)_{t \geq 0}$ the natural filtration of W_t .

10.1.3 Main results

Let μ_t^N be the empirical measure as defined in (10.1.3), we write the previous system as

$$\begin{cases} dX_t^i = (F(t, X_t^i, Y_t^i) + \int H(X_t^i - x, Y_t^i - y) \mu_t^N(dx, dy))dt + \sigma dB_t^i \\ dY_t^i = G(t, X_t^i, Y_t^i)dt \\ \mathcal{L}(X_0, Y_0) = \mu_0^{\otimes N}. \end{cases} \quad (10.1.7)$$

We are interested in the relation when N is going to infinity between μ_t^N and μ_t , the unique weak solution of

$$\begin{cases} \partial_t \mu(x, y) = -\nabla((F, G)\mu(x, y)) - \nabla(\mu(x, y) \int H(x - \bar{x}, y - \bar{y}) \mu(d\bar{x}, d\bar{y})) + \frac{1}{2} \sigma^2 \partial_{xx} \mu(x, y) \\ \mu_{t=0} = \mu_0 \end{cases}, \quad (10.1.8)$$

which correspond to the law of the stochastic process that solves

$$\begin{cases} d\bar{X}_t = (F(\bar{X}_t, \bar{Y}_t) + \int H(\bar{X}_t - x, \bar{Y}_t - y) \mu_t(dx, dy))dt + \sigma d\bar{B}_t \\ d\bar{Y}_t = G(\bar{X}_t, \bar{Y}_t)dt \\ \mathcal{L}(\bar{X}_0, \bar{Y}_0) = \mu_0, \end{cases} \quad (10.1.9)$$

for \bar{B}_t an \mathbb{R}^d -value Brownian motion and $\mu_t = \mathcal{L}(\bar{X}_t, \bar{Y}_t)$ i.e. the kinetic McKean-Vlasov equation. We begin by proving the well-posedness of equation (10.1.9). Next, we prove a Bernstein inequality which establishes the concentration of the empirical measure close to the measure μ_t .

The limit equation

In fact, our first result deals with a more general framework, we prove the existence and uniqueness of the solution to the equation

$$\begin{cases} dX_t = b_1(t, X_t, Y_t, \mu_t)dt + \sigma dB_t \\ dY_t = b_2(t, X_t, Y_t)dt \\ \mathcal{L}(X_0, Y_0) = \mu_0, \end{cases} \quad (10.1.10)$$

where μ_t denote the law of X_t , under the following set of assumptions

Assumptions 6. For some $k_4 > 0$ and all $p \geq 1$ the initial condition μ_0 satisfies

$$\int_{\mathbb{R}^{2d}} |x|^{2p} \mu_0(dz) \leq k_4 p!.$$

Assumptions 7. (A) b_1 is locally Lipschitz and can be written as

$$b_1(t, x, y, \mu) = \int_{\mathbb{R}^d} \tilde{b}_1(t, x, y, u, v) \mu(du, dv)$$

for some measurable $\tilde{b}_1(t, x, y, u, v) : [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \rightarrow \mathbb{R}^d$ which is uniformly Lipschitz on the variables (u, v) . More precisely, there exists $k_1 > 0$ such that, for all (t, x, y) ,

$$|\tilde{b}_1(t, x, y, u_1, v_1) - \tilde{b}_1(t, x, y, u_2, v_2)| \leq k_1(|u_1 - u_2| + |v_1 - v_2|).$$

(B) $b_2(t, x, y)$ is a Lipschitz continuous function, this is, there exist $k_2 > 0$ such that

$$|b_2(t_1, x_1, y_1) - b_2(t_2, x_2, y_2)| \leq k_2(|t_1 - t_2| + |x_1 - x_2| + |y_1 - y_2|).$$

(C) There is a constant $k_3 > 0$ that such that

$$x\tilde{b}_1(t, x, y, u, v) \leq k_3(1 + |x|^2 + |y|^2 + |u|^2 + |v|^2),$$

and

$$yb_2(t, x, y) \leq k_3(1 + |x|^2 + |y|^2).$$

Let us note that equation (10.1.9) is contained in the framework of equation (10.1.10) with the correspondence

$$\begin{aligned} \tilde{b}_1(t, x, y, u, v) &= F(t, x, y) + H(x - u, y - v), \\ b_2 &= G. \end{aligned}$$

Then, Assumption 7(B) requires a locally Lipschitz condition for the transport coefficient F and globally Lipschitz condition for the interaction coefficient H . In addition, Assumption 7(C) asks a growth condition to be satisfied on all the coefficients.

Under such assumptions we are able to prove the following theorem,

Theorem 7. *Grant Assumptions 6 and 7, the Eq. (10.1.10) has a unique strong solution.*

Bernstein inequality

In a similar spirit that in Eq. (10.1.10) we consider a more compact formulation of Eq. (10.1.2), given by

$$\begin{cases} dX_t^i = b_1(t, X_t^i, Y_t^i, \mu_t^N)dt + \sigma(t, X_t^i)dB_t^i \\ dY_t^i = b_2(t, X_t^i, Y_t^i)dt \\ \mathcal{L}(X_0^i, Y_0^i) = \mu_0^{\otimes N} \end{cases}, \quad i = 1, \dots, N, \quad (10.1.11)$$

for initial variable $Z^i = (X^i, Y^i)$ which satisfies $\mathbb{E}|Z_0^i|^p \leq k_0^p$. The existence and uniqueness of a solution for this system under Assumptions 6 and 7 is a classical result, see for example [209] and [161]. Equivalently, there is a probability, that we denote \mathbb{P}^N , such that $Z = (Z_1, \dots, Z_N)$ the canonical process in \mathcal{C}^N is a solution of this equation.

Let $\rho(dt)$ denote a probability measure in $[0, T]$; define,

$$\nu^N(dt, dz) = \mu_t^N(dz) \otimes \rho(dt), \quad \nu(dt, dz) = \mu_t(dz) \otimes \rho(dt).$$

We argue as in [73] in order to proof the next result.

Theorem 8. *Grant Assumptions 6 and 7, there exist $\alpha_1, \alpha_2, R > 0$ such that*

$$\mathbb{P}^N \left(\int_{[0, T] \times \mathbb{R}^{2d}} \phi(t, z) (\nu^N(dt, dz) - \nu(dt, dz)) \geq \gamma \right) \leq \alpha_1 \exp \left(-\alpha_2 \frac{N\gamma^2}{|\phi|_{L^2(\nu)}^2 + CR\gamma} \right) \quad (10.1.12)$$

for all $\gamma \geq 0$ and every $\phi : [0, T] \times \mathbb{R}^{2d} \rightarrow \mathbb{R}$ such that there is $k > 0$ which satisfies $|\phi(t, z)| \leq C|z|^k$ for some $C > 0$.

Remark. *Theorem 8 extend [73, theo. 18] to accommodate locally Lipschitz coefficient and the position-velocity scheme of (10.4.41). We also remark that the conclusion is slightly stronger as we allow the function ϕ to be unbounded (with polynomial growth).*

FithHugh-Nagumo model

As we mention above, we apply the previous result to estimate the parameters of the FhN model described by equation (10.1.5).

For simplicity, we re-parameterize the model in order to obtain linear dependence on the parameters, we make $\bar{c} = \frac{1}{c}$, $\bar{a} = \frac{1}{c}a$ and $\bar{b} = \frac{1}{c}b$, for which G take the form $G(x, y) = \bar{c}x + \bar{a} - \bar{b}y$. The goal of this section is to find an estimator of the parameter vector $\theta = (I, \bar{a}, \bar{b}, \bar{c}, \lambda, \sigma^2)$ considering as known data the evolution of the network up to a time T filtered by a smooth function in the sense: we know the elements of the set

$$K = \{(\mu_t^N, \varphi) : \varphi \in C^\infty(\mathbb{R}^{2d}), \forall t \in [0, T]\}$$

where $\langle \mu, \varphi \rangle = \int_{\mathbb{R}^{2d}} \varphi \mu(dz)$.

We introduce the notation

$$O_t^i = \int_{\mathbb{R}^2} x^i + y^i \mu_t(dx, dy); \quad m_t^{i,j} = \int_{\mathbb{R}^2} x^i y^j \mu_t(dx, dy), \quad (10.1.13)$$

for the momentum of μ_t , \widehat{O}_N and $\widehat{m}^{i,j}_N$ by integrating with respect to the empirical measure instead of μ_t , we define the matrix M such that each row is given by

$$M_i = (i \int_0^T m_s^{i-1,0}, i \int_0^T m_s^{1,i-1}, i \int_0^T m_s^{0,i-1}, -i \int_0^T m_s^{0,i}, -i \int_0^T (m_s^{i,0} + m_s^{1,0} m_s^{i-1,0}), \\ \frac{i(i-1)}{2} \int_0^T m_s^{i-2,i}),$$

the independent term $\Lambda = (\Lambda_1, \dots, \Lambda_6)$, with $\Lambda_i = O_0^i + i \int_0^T (m_s^{i,0} - \frac{m_s^{i+2,0}}{3} + m_s^{i-1,1})$ and the corresponding \widehat{M}_N , $\widehat{\Lambda}_N$, replacing μ_t by μ_t^N .

The vector

$$\widehat{\theta}_N = \widehat{M}_N^{-1}(\widehat{O}_N - \widehat{\Lambda}_N) \mathbf{1}_{\|\widehat{M}_N^{-1}\|_{\mathcal{F}} > \delta_N}.$$

is an estimator of the parameter vector θ . Indeed, the following theorem is satisfied.

Theorem 9. *Under Assumption 6, let θ be the parameter vector corresponding to the FitzHugh-Nagumo model defined by equation (10.1.5) and $\widehat{\theta}_N$ defined as above, it is satisfied that,*

$$\mathbb{P}_N(\|\widehat{\theta}_N - \theta\|_\infty \geq \gamma) \leq c_1 e^{-c_2 N \gamma^{1/2}} + c'_1 e^{-c'_2 N} + c''_1 e^{-c''_2 N \gamma}. \quad (10.1.14)$$

Nonparametric oracle estimation of μ_t

As a consequence of Theorem 8 and [141, Theo. 2.1] we can follow step by step the proof of the oracle estimate [73, Theo. 7] to obtain the same oracle inequality that is obtained in such result. For the sake of completeness we include this theorem, slightly adapted when needed.

Let $K_1 : (0, T) \rightarrow \mathbb{R}$ and $K_2 : \mathbb{R}^d \rightarrow \mathbb{R}$ be two bounded and compactly supported kernel functions such that

$$\int_0^T K_1(t) dt = \int_{\mathbb{R}^{2d}} K_2(z) dz = 1.$$

For $h > 0$ we denote,

$$K_1^{(h)}(t) = h^{-1} K_1(h^{-1}t); \quad K_2^{(h)}(z) = h^{-d} K_2(h^{-1}z).$$

We construct a family of estimators of $\mu_{t_0}(z_0)$ depending on h by setting

$$\hat{\mu}_h^N(t_0, z_0) = \int_{\mathbb{R}^{2d}} K_2^{(h)}(z_0 - z) \mu_{t_0}^N(dx) \quad (10.1.15)$$

We fix $(t_0, x_0) \in (0, T] \times \mathbb{R}^d$ and a discrete set

$$\mathcal{H}_1^N \subset \left[N^{-1/d} (\log N)^{2/d}, 1 \right],$$

of admissible bandwidths such that $\text{Card}(\mathcal{H}_1^N) \lesssim N$. The algorithm, based on Lepski's principle, requires the family of estimators

$$\left(\hat{\mu}_h^N(t_0, x_0), h \in \mathcal{H}_1^N \right)$$

defined in 10.1.15 and selects an appropriate bandwidth \hat{h}^N from data $\mu_{t_0}^N(dx)$. Writing $\{x\}_+ = \max(x, 0)$, define

$$A_h^N = \max_{h' \leq h, h' \in \mathcal{H}_1^N} \left\{ \left(\hat{\mu}_h^N(t_0, x_0) - \hat{\mu}_{h'}^N(t_0, x_0) \right)^2 - \left(V_h^N + V_{h'}^N \right) \right\}_+,$$

where

$$V_h^N = \varpi_1 |K|_2^2 (\log N) N^{-1} h^{-d}, \varpi_1 > 0 \quad (10.1.16)$$

Let

$$\hat{h}^N \in \operatorname{argmin}_{h \in \mathcal{H}_1^N} \left(A_h^N + V_h^N \right). \quad (10.1.17)$$

The data driven Goldenshluger-Lepski estimator of $\mu_{t_0}(x_0)$ defined by

$$\hat{\mu}_{\text{GL}}^N(t_0, x_0) = \hat{\mu}_{\hat{h}^N}^N(t_0, x_0)$$

is specified by K_2 and ϖ_1 .

We define

$$\mathcal{B}_h^N(\mu)(t_0, x_0) = \sup_{h' \leq h, h' \in \mathcal{H}_1^N} \left| \int_{\mathbb{R}^d} K_{h'}(x_0 - x) \mu_{t_0}(x) dx - \mu_{t_0}(x_0) \right|. \quad (10.1.18)$$

Theorem 10 (Oracle estimate). *Grant Assumptions 6 and 7. Let $(t_0, x_0) \in (0, T] \times \mathbb{R}^{2d}$, the following oracle inequality holds true:*

$$\mathbb{E}_{\mathbb{P}^N} \left[\left(\hat{\mu}_{\text{GL}}^N(t_0, x_0) - \mu_{t_0}(x_0) \right)^2 \right] \lesssim \min_{h \in \mathcal{H}_1^N} \left(\mathcal{B}_h^N(\mu)(t_0, x_0)^2 + V_h^N \right),$$

for large enough N , up to a constant depending on (t_0, x_0) , $|K|_\infty$ and \mathfrak{b} , provided $\hat{\mu}_{\text{GL}}^N(t_0, x_0)$ is calibrated with $\varpi_1 \geq 16\kappa_2^{-1}\kappa_3$, where κ_2 is specified in Theorem 18 and κ_3 is a (local) upper bound of μ_{t_0} , see Lemma 23 below.

10.1.4 Plan of the Chapter

The sections of this chapter are organized as follows: in Section 10.2 we recall some classical result that will be useful for the next section. In continuation, in Section 10.3 we will deal with proving Theorem 7 in the general context of formulation (10.1.10) under Assumptions 6 and 7. Then in section 10.4 we will prove the Bernstein's inequality stated in theorem 8. Section 10.5 is dedicated to the proof of Theorem 9 and Section 10.6 of this of Theorem 10. Finally in section 10.7 we explore numerically the possibility of testing the interaction bases on the previous section.

10.2 Brief summary of classical results

Two classical results are essential for the development of this chapter, Girsanov's theorem and Bernstein's inequality. For a smoother reading, we briefly introduce both results in this section.

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$ be a filtered probability space, W a centered d -dimensional Brownian motion with respect to the filtration \mathcal{F} and L_t an adapted measurable process satisfying

$$P \left(\int_0^T |L_t|^2 < \infty \right) = 1; \quad \forall T \geq 0. \quad (10.2.19)$$

We introduce the process

$$\mathcal{E}(L_t) = \exp(L_t - \frac{1}{2}\langle L \rangle_t), \quad (10.2.20)$$

where $\langle L \rangle_t$ denote the quadratic variation of L . This process is well defined under condition (10.2.19), see for example [135, Sec. 3.5].

The Girsanov theorem can be stated as follows

Theorem 11 (Girsanov (1960)). *Assume that $\mathcal{E}(L_t)$ defined above is a martingale, then the process defined by*

$$\tilde{W}_t^{(i)} = W_t^{(i)} - \int_0^t L_s^{(i)} ds; \quad 1 \leq i \leq d, \quad 0 \leq t < \infty \quad (10.2.21)$$

is a Brownian motion under the probability Q given by

$$Q|_{\mathcal{F}_t} = \mathcal{E}(L_t)P|_{\mathcal{F}_t}.$$

Proof. A proof of such a result can be found in Karatzas' book [[135] section 3.5] as well as in Le Gall's book [[150] section 5.5]. \square

In practice, we would like to start from a convenient process L such that the theorem provides us with a probability such that the canonical process solve our stochastic equation. That leaves us with the need to prove that $\mathcal{E}(L_t)$ is a martingale for our choice of L . We add here a sufficient condition for that.

Let L be a continuous local martingale such that $L_0 = 0$ then $\mathcal{E}(L_t)$ is also a local martingale, hence a supermartingale, and is a martingale if and only if

$$\mathbb{E}[\mathcal{E}(L_t)] = 1; \quad \forall t \geq 0. \quad (10.2.22)$$

One condition to assure this is the following,

Lemma 9 (Novikov Condition). *Let L be a continuous martingale such that $L_0 = 0$ and*

$$\mathbb{E}[\exp \frac{1}{2}\langle L \rangle_t] < \infty, \quad \forall t \geq 0.$$

then $\mathbb{E}[\mathcal{E}(L_t)] = 1$, and thus $\mathcal{E}(L_t)$ is a martingale.

Proof. The proof of this lemma can be found in Karatzas's book [[135] prop 5.12]. \square

The second result that is important to us is the Bernstein's inequality. We conclude this section by stating the result as presented in Boucheron et al. [37].

Theorem 12 (Bernstein's inequality). *Let X_1, \dots, X_n be independent real-valued random variables. Assume that there exist positive numbers v and c such that $\sum_{i=1}^n \mathbb{E}[X_i^2] \leq v$ and*

$$\sum_{i=1}^n \mathbb{E}[(X_i)_+^q] \leq \frac{q!}{2} v c^{q-2}, \quad \forall q \geq 3,$$

then for all $t \geq 0$,

$$P \left(\sum_{i=1}^n (X_i - \mathbb{E}X_i) \geq t \right) \leq e^{-\frac{t^2}{2(v+ct)}}. \quad (10.2.23)$$

For one proof of this result we reference [[37] theorem 2.10 and corollary 2.11].

We finish this section by recalling a classical definition of a sub-Gaussian random variable, see for example [41].

Definition 10.2.5. *A real-valued random variable Z such that $\mathbb{E}[Z] = 0$ is λ^2 sub-Gaussian if one of the following conditions is satisfied, each statement implying the next:*

(i) *Laplace transform condition*

$$\mathbb{E}[\exp(zZ)] \leq \exp\left(\frac{1}{2}\lambda^2 z^2\right) \text{ for every } z \in \mathbb{R}.$$

(ii) *Moment condition*

$$\mathbb{E}\left[Z^{2p}\right] \leq p! \left(4\lambda^2\right)^p \text{ for every integer } p \geq 1.$$

(iii) *Orlicz condition*

$$\mathbb{E}\left[\exp\left(\frac{1}{8\lambda^2}Z^2\right)\right] \leq 2.$$

(iv) *Laplace transform condition (bis)*

$$\mathbb{E}[\exp(zZ)] \leq \exp\left(\frac{24}{2}\lambda^2 z^2\right) \text{ for every } z \in \mathbb{R}.$$

We will also use the following additive property of sub-Gaussian random variables: if the random variables Z_i are independent and λ_i^2 sub-Gaussian, then $\rho(Z_1 + Z_2)$ is $|\rho|^2(\lambda_1^2 + \lambda_2^2)$ subGaussian for every $\rho \in \mathbb{R}$.

10.3 Existence and uniqueness proof

This section is dedicated to the proof of theorem 7. It is based on an argument derived from Girsanov’s theorem in a similar way to how it is done in Lacker [145]. However, as we mentioned before, our work extends to equations with unbounded coefficients, locally Lipschitz. We emphasize that, for us, the proof of the existence of a solution is understood as finding a probability such that the canonical process is the solution of equation (10.1.10).

The proof is divided into two main stages. In a first step, we start from a fixed measure $\bar{\mu}$, which belongs to a suitable closed subset Ξ of $\mathcal{P}(\mathcal{C})$ with respect to the metric W_1 defined in (10.1.6). We consider the equation

$$\begin{cases} dX_t = b_1(t, X_t, Y_t, \bar{\mu}_t)dt + \sigma dB_t^i \\ dY_t = b_2(t, X_t, Y_t)dt \\ \mathcal{L}(X_0, Y_0) = \mu_0 \end{cases}, \tag{10.3.24}$$

for this fixed $\bar{\mu}$. According to the classical theory of stochastic equations, which can be reviewed for example in Protter’s textbook [209], a solution of such equation exists up to a stopping time \mathfrak{T} with respect to the filtration $(\mathcal{F}_t)_{t \geq 0}$ generated by the Brownian B_t . We prove that once Assumptions 6 and 7 are satisfied, the probability of the event $\{\mathfrak{T} = \infty\}$ is 1, and, furthermore, the probability $P^{\bar{\mu}}$ that solves the equation (10.3.24) has a representation from the probability P , for which $\sigma^{-1}X_t$ is a Brownian motion.

In a second stage, the problem is posed as the existence of a fixed point of the operator $\Phi(\mu)$ that associates the corresponding solution of the equation (10.3.24) to each $\mu \in \Xi$. The existence of the fixed point is proved thanks to Banach’s theorem using the representation found in the first stage.

Preparation of the proof

Before we begin the proof of the theorem, let us make a series of observations, some of them written as lemmas.

One first remark is that, given $X_t : [0, T] \rightarrow \mathbb{R}^d$ a continuous function, since b_2 is a globally Lipschitz function, the equation

$$dY_t = b_2(t, X_t, Y_t)dt \quad (10.3.25)$$

has a unique solution associated to the initial data $Y_0 = y_0 \in \mathbb{R}^d$, which does not explode in finite time. In other words, the random variable Y_t can be written as a function of the path of X until the time t ; we use the notation $Y_t(X_{0 \leq s \leq t})$ to make it clear when necessary. Consequently, the equations (10.3.24) and (10.1.10) can be written just in terms of X . Moreover we have the following lemma.

Lemma 10. *Let $\|X\|_{L^\infty[0, T]} \leq R$, Y_t defined from $X_{0 \leq s \leq t}$ by Eq. (10.3.25); then there is a constant C_R depending on T and the Lipschitz constant k_2 given by Assumption 7-(2), such that $\|Y\|_{L^\infty[0, T]} \leq C_R(T)$.*

Proof. By definition

$$|Y_t| \leq |Y_0| + \int_0^t |b_2(s, X_s, Y_s)| ds \leq |Y_0| + \int_0^t (k_2(s + 2\|X\|_\infty + |Y_s| + |Y_0|) + |b_2(0, X_0, Y_0)|) ds$$

we conclude by Gronwall's lemma. \square

In what follows, we will use the notation $\mu_Z = \mu(dx, dy)$ to emphasize that that we are referring to the joint law of X and Y while the notations μ_X and μ_Y stand for the marginal distribution with respect to X and Y respectively. In term of probability, the previous key observation shows that for the solution μ_Z of Eq. (10.1.9) we can find a probability $\mu_X \in \mathcal{P}(\mathcal{C}_X)$ such that

$$\int_{\mathcal{C}} g(z) \mu_Z(dz) = \int_{\mathcal{C}} g(x, y) \delta_{y=y(x)}(dy) \mu_X(dx) = \int_{\mathcal{C}_X} g(x, y(x)) \mu_X(dx),$$

where $y(x)$ is the solution of Eq. (10.3.25) corresponding to $x = (x_t)_{0 \leq t \leq T} \in \mathcal{C}_X$. To refer to the relation between μ_Z and μ_X we will use the notation

$$\mu_Z = \mu_X \otimes \delta_{Y(X)}.$$

Next, we present an important consequence of Assumptions 6 and 7-(3) concerning the second moment of the solution of Eq. (10.3.24).

Lemma 11. *There exist a function $C(t) : [0, T] \rightarrow \mathbb{R}_+$, depending only on k_3, σ, μ_0 , such that if*

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} |z|^2 \bar{\mu}_t(dz) \leq C(t),$$

for all $0 \leq t \leq T$, then the canonical process Z_t has bounded second moment under $P^{\bar{\mu}}$. More precisely, let $\mu = \mathcal{L}_{P^{\bar{\mu}}}(Z_t)$ be the law of the solution of Eq. (10.3.24) then

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} |z|^2 \mu_t(dz) \leq C(t).$$

Proof. Let $\mu = \mathcal{L}_{P^{\bar{\mu}}}(Z_t)$, by Ito's formula, μ satisfies the equation

$$\partial_t \mu = -\nabla(b\mu) + \frac{1}{2} \sigma^2 \partial_{xx} \mu, \quad (10.3.26)$$

from where we obtain, after integrating by part, that

$$\begin{aligned}
\frac{d}{dt} \int_{\mathbb{R}^d \times \mathbb{R}^d} |z|^2 \mu_t(dz) &= \int_{\mathbb{R}^d \times \mathbb{R}^d} 2zb(t, z, \bar{\mu}_t) \mu_t(dz) + 1/2 \int_{\mathbb{R}^d \times \mathbb{R}^d} 2\sigma^2 \mu_t(dz), \\
&= \int_{\mathbb{R}^d \times \mathbb{R}^d} 2x \int_{\mathbb{R}^d \times \mathbb{R}^d} b_1(t, z, w) \bar{\mu}_t(dw) \mu_t(dz) + 2k_3 + \sigma^2, \\
&\leq 2k_3 \int_{\mathbb{R}^d \times \mathbb{R}^d} \int_{\mathbb{R}^d \times \mathbb{R}^d} (|z|^2 + |w|^2) \bar{\mu}_t(dw) \mu_t(dz) + \sigma^2, \\
&= 2k_3 \int_{\mathbb{R}^d \times \mathbb{R}^d} |z|^2 \mu_t(dz) + 2k_3 C(t) + 2k_3 + \sigma^2,
\end{aligned}$$

where we have made use of Assumptions 7-(1),(3)

We get a problem of the form

$$u'(t) \leq C_1(u(t) + C(t) + 1), \quad (10.3.27)$$

with $u(t) = \int_{\mathbb{R}^d \times \mathbb{R}^d} |z|^2 \mu_t(dz)$, $C_1 = 2k_3 + \sigma^2$. We would like to find $C(t)$ such that the equation (10.3.27) implies that $u(t) \leq C(t)$. This is satisfied if

$$C(t) = (M + \frac{1}{2})e^{2Mt} - \frac{1}{2},$$

with $M = \max\{C_1, u(0)\}$. □

We consider $\Xi \subset \mathcal{P}(\mathcal{C})$ the space of all the probablyly measures such that $\int_{\mathbb{R}^d \times \mathbb{R}^d} |z|^2 \mu_t(dz) \leq C(t)$ for all $t \in [0, T]$,

$$\Xi = \{\mu \in \mathcal{P}(\mathcal{C}) : \int_{\mathbb{R}^d \times \mathbb{R}^d} |z|^2 \mu_t(dz) \leq C(t), \forall t \in [0, T]\}.$$

This is a closed subspace of $\mathcal{P}(\mathcal{C})$. Indeed, let $\mu^n \in \Xi$ be converging to μ for the distance W_1 , then for all $t \in [0, T]$, μ_t^n converge weakly to μ_t , this is

$$\int \varphi \mu_t^n(dz) \rightarrow \int \varphi \mu_t(dz)$$

for all test function $\varphi \in C_c(\mathbb{R}^{2d})$. Taking $\varphi_R = \chi(z/R) \geq 0$ with $\chi(z) = 1$ for all $|z| \leq 1$, $\chi(z) = 0$ for all $|z| > 2$ we get

$$\int m \varphi_R \mu_t^n(dz) \rightarrow \int m \varphi_R \mu_t(dz); \quad z \in \mathbb{R}^{2d}$$

which implies that $\int m \varphi_R \mu(dz) \leq C_1$, and then, applying Fatou lemma, we obtain

$$\int m \mu_t(dz) \leq \liminf \int m \varphi_R \mu_t(dz) \leq C(t).$$

One of the consequence of the previous lemma is the following observation.

Corollary 4. *For all $\mu \in \Xi$, it is satisfied that*

$$zb(t, z, \mu) \leq k_3(1 + |z|^2 + C(t)) \leq k_3(1 + C(T) + |z|^2). \quad (10.3.28)$$

Lemma 12. *Grant Assumptions 6 and 7, for all $T > 0$, there is a constant $C_2(k_3) > 0$ such that for all $p \geq 2$, we have*

$$\mathbb{E}|\bar{Z}_t|^p \leq \sigma^2(p/2)! C_2^{p/2} \quad (10.3.29)$$

for all $t \in [0, T]$, $\bar{Z}_t = (\bar{X}_t, \bar{Y}_t)$ solution of 10.3.24.

The proof is classical, yet we present here a version adapted from the one of Mao [161, Chap. 2 Theo 4.1] to our condition.

Proof. Applying Ito's Formula

$$\begin{aligned}
(1 + |Z_t|^2)^{\frac{p}{2}} &= (1 + |Z_0|^2)^{\frac{p}{2}} + p \int_0^t (1 + |Z_s|^2)^{\frac{p-2}{2}} z^T b(s, z, \bar{\mu}) \\
&\quad + \sigma^2 \frac{p}{2} \int_0^t (1 + |Z_s|^2)^{\frac{p-2}{2}} + \sigma^2 \frac{p(p-2)}{2} \int_0^t (1 + |Z_s|^2)^{\frac{p-4}{2}} |Z_s|^2 \\
&\quad + p \int_0^t (1 + |Z_s|^2)^{\frac{p-2}{2}} Z_s^T \sigma dB_s \\
&\leq 2^{\frac{p-2}{2}} (1 + |Z_0|^p) + p \int_0^t (1 + |Z_s|^2)^{\frac{p-2}{2}} (Z_s^T b(s, Z_s, \bar{\mu}) + \sigma^2 \frac{p-1}{2}) \\
&\quad + p \int_0^t (1 + |Z_s|^2)^{\frac{p-2}{2}} Z_s^T \sigma dB_s
\end{aligned}$$

From Young's inequality we have that $\frac{p-1}{2}(1 + |Z_s|^2)^{\frac{p-2}{2}} \leq (1 + |Z_s|^2)^{\frac{p}{2}} + 2\frac{(p/2)^{\frac{p}{2}}}{p}$, plugging this and the result in corollary 4 to the previous estimate, we obtain

$$\begin{aligned}
(1 + |Z_t|^2)^{\frac{p}{2}} &\leq 2^{\frac{p-2}{2}} (1 + |Z_0|^p) + p \int_0^t (1 + |Z_s|^2)^{\frac{p-2}{2}} k_3 (1 + \|C\| + |Z_s|^2) \\
&\quad + \int_0^t (p(1 + |Z_s|^2)^{\frac{p}{2}} + 2\sigma^2 (p/2)^{\frac{p}{2}}) + p \int_0^t (1 + |Z_s|^2)^{\frac{p-2}{2}} Z_s^T \sigma dB_s
\end{aligned}$$

From where, it is some constant α such that,

$$(1 + |Z_t|^2)^{\frac{p}{2}} \leq 2^{\frac{p-2}{2}} (1 + |Z_0|^p) + p\alpha \int_0^t (1 + |Z_s|^2)^{\frac{p}{2}} \quad (10.3.30)$$

$$+ 2\sigma^2 (p/2)^{\frac{p}{2}} T + p \int_0^t (1 + |Z_s|^2)^{\frac{p-2}{2}} Z_s^T \sigma dB_s \quad (10.3.31)$$

Let us define for $n \geq 0$ the stopping time

$$\tau_n = T \wedge \inf_{t \in [0, T]} \{|Z(t)| \geq n\}.$$

clearly, from the non-explosion of the solution, $\tau_n \rightarrow T$ a.s., moreover from (10.3.30) and the properties of Ito's integral, we get

$$\begin{aligned}
\mathbb{E}(1 + |Z_{t \wedge \tau_n}|^2)^{\frac{p}{2}} &\leq 2^{\frac{p-2}{2}} (1 + \mathbb{E}|Z_0|^p) + p\alpha \mathbb{E} \int_0^{t \wedge \tau_n} (1 + |Z_s|^2)^{\frac{p}{2}} ds + 2\sigma^2 (p/2)^{\frac{p}{2}} t \\
&\leq 2^{\frac{p-2}{2}} (1 + \mathbb{E}|Z_0|^p) + p\alpha \mathbb{E} \int_0^t (1 + |Z_{s \wedge \tau_n}|^2)^{\frac{p}{2}} ds + 2\sigma^2 (p/2)^{\frac{p}{2}} t.
\end{aligned}$$

Applying Gronwall's lemma we obtain

$$\mathbb{E}(1 + |Z_{t \wedge \tau_n}|^2)^{\frac{p}{2}} \leq 2^{\frac{p-2}{2}} e^{p\alpha t} (1 + \mathbb{E}|Z_0|^p) + (p/2)^{\frac{p}{2}} \sigma^2 e^{p\alpha t} \leq 2^{\frac{p-2}{2}} (1 + \mathbb{E}|Z_0|^p) e^{p\alpha t} + (p/2)! \sigma^2 e^{p(\alpha t + 1/2)}.$$

Letting $n \rightarrow \infty$ yields

$$\mathbb{E}(1 + |Z_t|^2)^{\frac{p}{2}} \leq \sigma^2 (p/2)! C_2^{p/2}.$$

for some constant C_2 which come from $\mathbb{E}|Z_0|^p \leq C(\frac{p}{2})!$. □

In consequence, we have the next corollary.

Corollary 5. *Let the initial condition Z_0 be such that $\mathbb{E}|Z_0|^p \leq k_0^p$ for some constant k_0 . Z_t satisfies*

$$\sup_{[0, T]} \mathbb{E}[e^{\frac{1}{2C_2}} |Z_t|^2] \leq 1 + \sigma^2.$$

Proof. From lemma 12, for $p \geq 1$, we have

$$\sup_{[0, T]} \mathbb{E}[e^{\frac{1}{2C_2}|Z_t|^2}] = 1 + \sum_{p \geq 1} \frac{2^{-p}}{p! C_2^p} \mathbb{E}|Z_t|^{2p} \leq 1 + \sigma^2 \sum_{p \geq 1} 2^{-p} = 1 + \sigma^2. \quad (10.3.32)$$

□

From the last corollary and proposition 6.3 of [108], it follows that there is a constant $k_5 > 0$ that only depends on C_2 such that for any measures μ and ν in Ξ , it is satisfied that

$$W_1(\mu_t, \nu_t) \leq k_5 \sqrt{\mathcal{H}_t(\mu|\nu)}; \quad \forall t \in [0, T].$$

where $\mu_t = \mu_{\cdot \wedge t}$, $\nu_t = \nu_{\cdot \wedge t} \in P(\mathcal{C})$ and $\mathcal{H}_t(\mu|\nu) = \mathcal{H}(\mu_t|\nu_t)$ denote the relative entropy

$$\mathcal{H}(\mu|\nu) = \int_{\mathcal{C}} \frac{d\mu}{d\nu} \log \frac{d\mu}{d\nu} d\nu.$$

Proof of theorem 7

Proof. As we mention above, we divide the proof in two steps: first, we fix $\bar{\mu} \in \Xi$ and prove the existence and uniqueness of a solution of (10.3.24) for this fixed $\bar{\mu}$. Then we prove through a fixed point argument that there exists μ such that it is the law of the component X_t of the corresponding solution of (10.3.24) for this μ .

Step 1. Let $\bar{\mu} \in \Xi$ be fixed, we use a Girsanov's argument to prove that we have a solution $P^{\bar{\mu}} \otimes \delta_{Y(X_{0 \leq s \leq T})}$ with $P^{\bar{\mu}}$ that satisfies

$$P^{\bar{\mu}}|_{\mathcal{F}_t} = M_t P|_{\mathcal{F}_t}$$

for

$$M_t = \mathcal{E}_t \left(\int_0^t \sigma^{-1} b_1(s, X_s, Y_s, \bar{\mu}) \cdot dW_s \right), \quad (10.3.33)$$

where we define $\mathcal{E}_t(M) = \exp(M_t - \frac{1}{2} \langle M \rangle_t)$. We recall that P is the measure such that $W_t = \sigma^{-1} X_t$ is a Brownian motion, with X_t the canonical process in \mathcal{C}_X , and $(\mathcal{F}_t)_{t \geq 0}$ the natural filtration of W_t .

Grant Assumptions 6 and 7, there is a probability such that Z is a solution of (10.1.10) until a time of explosion \mathfrak{T} . Since Y_t is unequivocally determined by $X_{0 \leq s \leq t}$, we can write this probability as $P^{\bar{\mu}} \otimes \delta_{Y(X_{0 \leq s \leq t})}$, with $P^{\bar{\mu}} \in \mathcal{P}(\mathcal{C}_X)$. We define, for $R > 0$,

$$\tau_R = \inf\{t \geq 0, |X_t| \geq R\}.$$

By lemma 10, there is C_R such that $|Y_t| \leq C_R$ in $[0, \tau_R]$, which implies that $\tau = \sup_{R > 0} \tau_R \leq \mathfrak{T}$. We note that for any R , τ_R is a stopping time with respect to the filtration $(\mathcal{F}_t)_{0 \leq t \leq T}$.

Let us define

$$f_R(x) = \begin{cases} x & \text{if } |x| \leq R \\ Rx/|x| & \text{if } |x| > R \end{cases}$$

and $b_{1R}(t, X_t, Y_t, \bar{\mu}) = b_1(t, f_R(X), Y_t(f_R(X)), \bar{\mu})$. The process

$$M_t^R = \mathcal{E}_t \left(\int_0^t \sigma^{-1} b_{1R}(s, X_s, Y_s(X_{\cdot \wedge t}), \bar{\mu}) \cdot dW_s \right). \quad (10.3.34)$$

is a martingale under P . Indeed, by the definition of b_{1R} , M_t^R satisfies Novikov criterion, lemma 9, i.e.

$$\mathbb{E} \left[\exp \frac{1}{2} \int_0^T |\sigma^{-1} b_{1R}(t, X_t, Y_t(X_{\cdot \wedge t}), \bar{\mu})|^2 dt \right] < \infty \quad (10.3.35)$$

for all $0 \leq T < \infty$. It follows then that $M_{t \wedge \tau_R}^R$ is also a martingale and, as they coincide in $[0, \tau_R]$, $M_{t \wedge \tau_R}$ is also a martingale.

Girsanov theorem shows then that there is the probability measure P_R

$$P_R|_{\mathcal{F}_{t \wedge \tau_R}} = M_{t \wedge \tau_R} P \text{ on } (\mathcal{C}, \mathcal{F}_T)$$

such that $W_{t \wedge \tau_R}^\mu = W_{t \wedge \tau_R} - \int_0^{t \wedge \tau_R} \sigma^{-1} b_1(s, X_s, Y_s, \bar{\mu}) ds$ is a standard Brownian motion. By uniqueness of the solution in $[0, \mathfrak{T}]$ we get $P^\mu|_{\mathcal{F}_{t \wedge \tau_R}} = P_R|_{\mathcal{F}_{t \wedge \tau_R}} = M_{t \wedge \tau_R} P$.

For all $h \in C^2(\mathbb{R}^2)$ we define,

$$\mathcal{L}_t h(x, y) := \frac{1}{2} \sigma \partial_{xx} h(x, y) + b(t, x, y, \mu) \cdot \nabla h(x, y).$$

The key part of the argument is to find f such that

- $\mathcal{L}_t f \leq C f$,
- $f(Z_{\tau_R}) \geq g(R)$ for some g such that $g(R) \rightarrow \infty$ when $R \rightarrow \infty$.

Then we can proceed in a similar way as it is done in [246] to conclude that $\lim_{R \rightarrow \infty} P^{\bar{\mu}}(\tau_R > t) = 1$ for all $t \geq 0$.

Let $f(z) = \frac{|z|^2 + 1}{2}$; by corollary 4, we have $\mathcal{L}_t f \leq C f$ for $C = 2(\|\sigma^2\|_\infty + k_3)$. Applying Ito's formula to $e^{-Ct} f(x, y)$, we get for $0 \leq s \leq t$

$$\begin{aligned} e^{-Ct} f(X_t, Y_t) &= e^{-Cs} f(X_s, Y_s) + \int_s^t e^{-Cu} \mathcal{L}_u f du - C \int_s^t e^{-Cu} f du + \int_s^t \nabla f \sigma dW^\mu \\ &\leq e^{-Cs} f(X_s, Y_s) + \int_s^t \nabla f \sigma dW^\mu \end{aligned}$$

Evaluating in $s \wedge \tau_R$, $t \wedge \tau_R$ and taking conditional expectation with respect to the σ -field \mathcal{F}_s , we obtain

$$\mathbb{E}_{\mathbb{P}^{\bar{\mu}}} [e^{-Ct \wedge \tau_R} f(X_t, Y_t) | \mathcal{F}_s] \leq e^{-Cs \wedge \tau_R} f(X_s, Y_s)$$

since the first term is \mathcal{F}_s -measurable and the second is a centred Gaussian with respect to $P^{\bar{\mu}}$. From the fact that $e^{-Ct \wedge \tau_R} f(x, y)$ is a supermartingale, we infer

$$\mathbb{E}_{\mathbb{P}^{\bar{\mu}}} [e^{-Ct \wedge \tau_R} f(X_t, Y_t)] \leq \mathbb{E}_{\mathbb{P}^{\bar{\mu}}} [f(X_0, Y_0)].$$

Since $f(Z_{\tau_R}) \geq g(R) = R^2/2$ in $[\tau_R < +\infty]$, we get

$$\begin{aligned} P^{\bar{\mu}}(\tau_R \leq t) &= e^{Ct}/g(R) \mathbb{E}_{P^{\bar{\mu}}} [1_{\tau_R \leq t} e^{-Ct} g(R)] \leq e^{Ct}/g(R) \mathbb{E}_{P^{\bar{\mu}}} [1_{\tau_R \leq t} e^{-Ct \wedge \tau_R} f(Z_{t \wedge \tau_R})] \\ &\leq e^{Ct}/g(R) \mathbb{E}_{P^{\bar{\mu}}} [e^{-Ct \wedge \tau_R} f(Z_{t \wedge \tau_R})] \\ &\leq e^{Ct}/g(R) f(Z_0) \end{aligned}$$

which goes to 0 when R goes to infinity. From $\lim P^{\bar{\mu}}(\tau_R > t) = 1$ we conclude,

$$\int M_t dP \geq \int 1_{\tau_R > t} M_{t \wedge \tau_R} dP = P^{\bar{\mu}}(\tau_R > t) \rightarrow 1$$

which implies that M_t is a martingale. We can then apply Girsanov's theorem directly to M_t , to conclude that $P^\mu|_{\mathcal{F}_t} = M_t P$ and $W_t^\mu = W_t - \int_0^t \sigma^{-1} b_1(s, X_s, Y_s, \bar{\mu}) ds$ is a Brownian Motion.

Step 2.

We define $\Phi(\mu) = P^\mu \otimes \delta_{Y(X)}$. Let A be a subset of the σ -field generated by X_t , $\mathcal{C}_Y = C([0, T]; \mathbb{R}^d)$, we have

$$\int_A \frac{d\Phi(\nu)_t}{d\Phi(\mu)_t}(x, y(x)) dP^\mu(x) = \int_{A \otimes \mathcal{C}_Y} \frac{d\Phi(\nu)_t}{d\Phi(\mu)_t}(x, y) d\Phi(\nu)_t = \Phi(\mu)_t(A \otimes \mathcal{C}_Y) = P^\mu(A).$$

It follows,

$$\frac{dP_t^\nu}{dP_t^\mu}(x) = \frac{d\Phi(\nu)_t}{d\Phi(\mu)_t}(x, y(x)).$$

In consequence, we have that

$$\begin{aligned} H_t(\Phi(\mu)|\Phi(\nu)) &= -\int_{\mathcal{C}} \log \frac{d\Phi(\nu)_t}{d\Phi(\mu)_t} d\Phi(\mu)_t = -\int_{\mathcal{C}_X} \log \frac{d\Phi(\nu)_t}{d\Phi(\mu)_t}(x_{\cdot \wedge t}, y(x_{\cdot \wedge t})) P^\mu(dx), \\ &= -\mathbb{E}^{P^\mu} \left[\log \frac{dP_t^\nu}{dP_t^\mu}(X_{\cdot \wedge t}) \right], \\ &= -\mathbb{E}^{P^\mu} \left[\log \mathbb{E}^{P^\mu} \left[\frac{dP_t^\nu}{dP_t^\mu} | X_{\cdot \wedge t} \right] \right]. \end{aligned}$$

Since X_t and W_t generate the same filtration we get

$$\begin{aligned} H_t(\Phi(\mu)|\Phi(\nu)) &= -\mathbb{E}^{P^\mu} \left[\log \mathbb{E}^{P^\mu} \left[\frac{dP_t^\nu}{dP_t^\mu} | \mathcal{F}_{\cdot \wedge t} \right] \right] \\ &= \frac{1}{2} \mathbb{E}_{P^\mu} \left[\int_0^t |\sigma^{-1} b_1(s, Z, \nu) - \sigma^{-1} b_1(s, Z, \mu)|^2 ds \right] \end{aligned}$$

where we use the result from the previous step for the second equality.

On the other hand, we have that if b_1 satisfies Assumption 7-(1), then

$$|b_1(s, z, \nu) - b_1(s, z, \mu)| = \left| \int_{\mathbb{R}^d} \tilde{b}_1(t, z, u) d(\mu - \nu) \right| \leq k_1 \sup_{|\varphi|_{Lip} \leq 1} \int \varphi d(\mu - \nu) = k_1 W_1(\mu, \nu).$$

Gathering everything, we get,

$$W_1(\Phi(\mu), \Phi(\nu)) \leq k_1 k_5 \sqrt{H_t(\Phi(\mu)|\Phi(\nu))} \leq k_1 k_5 \sqrt{1/2 |\sigma|^2 t} W_1(\mu, \nu).$$

We use Banach's fixed point theorem to conclude. □

10.4 Bernstein's Inequality

This section is dedicated to the proof of theorem 2. This is based on a change of measures argument which allows us to use a Bernstein inequality. Here is a short sketch of the proof to make it easier to read.

1. First we rewrite equation (10.1.12) in terms of Bernstein inequality. Let us define

$$\mathcal{A}^N = \left\{ \int_{[0, T] \times \mathbb{R}^{2d}} \phi(t, z) (\nu^N(dt, dz) - \nu(dt, dz)) \geq \gamma \right\} \quad (10.4.36)$$

$$= \left\{ \sum_{i=1}^N \left(\int_{[0, T]} \phi(t, Z^i) \rho(dt) - \int_{[0, T] \times \mathbb{R}^{2d}} \phi(t, z) \rho(dt) \mu(dz) \right) \geq N\gamma \right\} \quad (10.4.37)$$

$$= \left\{ \sum_{i=1}^N \beta_i \geq N\gamma \right\}, \quad (10.4.38)$$

for $\beta_i = \int_{[0, T]} \phi(t, Z^i) \rho(dt) - \int_{[0, T] \times \mathbb{R}^{2d}} \phi(t, z) \rho(dt) \mu(dz)$.

We use one generalization of Bernstein's Inequality which can be found in Boucheron et al. [37]. This establishes that if the β_i are independent (in our case this is equivalent to says that the Z^i are independent), centred ($\mathbb{E}(\beta_i) = 0$), with uniformly bounded second moment, and satisfy that for all integers $p \geq 0$

$$\mathbb{E}|\beta_i|^p \leq p! R^{p-2} V(\beta_i) \quad (10.4.39)$$

for some constant R , V denoting the variance, then

$$\mathbb{P}\left(\sum_{i=1}^N \beta_i \geq N\gamma\right) \leq 2 \exp\left(-\frac{(N\gamma)^2}{2(\sum_{i=1}^N \mathbb{E}[\beta_i^2] + R\gamma)}\right)$$

what is the result.

The difficulty here is that the variable Z^i , solutions of (10.1.11), are not independent. In order to solve this, the strategy is based on two more steps:

2. We consider a new probability measure $\bar{\mathbb{P}}^N$ for which the variables Z^i are independent, $\mathbb{E}_{\bar{\mathbb{P}}^N}[\beta_i] = 0$ and the random variables β_i satisfy the moment estimates, so that the inequality is valid for this new probability.

3. We prove that

$$\mathbb{P}^N(\mathcal{A}) \leq C \bar{\mathbb{P}}^N(\mathcal{A})^\delta, \quad (10.4.40)$$

for some $C, \delta > 0$, in order to obtain the desired inequality for the original probability measure.

Definition of the probability measure $\bar{\mathbb{P}}^N$

Let μ be the law of the unique solution of Eq. (10.1.10). Define $\bar{\mathbb{P}}^N = \mu_t^{\otimes N}$ as the unique probability measure for which the canonical process solve

$$\begin{cases} dX_t^i = b_1(t, X_t^i, Y_t^i, \mu_t)dt + \sigma(t, X_t^i)d\bar{B}_t^i \\ dY_t^i = b_2(t, X_t^i, Y_t^i)dt \\ \mathcal{L}(X_0^i, Y_0^i) = \mu_0^{\otimes N} \end{cases}, \quad (10.4.41)$$

where

$$\bar{B}_t^i = \int_0^t \sigma^{-1}(s, Z_s^i)(dZ_s^i - b_1(s, Z_s^i, \mu_s)ds), \quad 1 \leq i \leq N, \quad (10.4.42)$$

are independent Brownian motions. That is, the variables $Z^i = (X^i, Y^i)$ correspond to N independent samples of the solution of equation 10.1.10, whose well-posedness is given by theorem 7. Since $\mathcal{L}(Z^i) = \mu_t$, is clear from its definition that $\mathbb{E}_{\bar{\mathbb{P}}^N}[\beta^i] = 0$. Next lemma conclude with the second step establish above.

Lemma 13. *Under the probability $\bar{\mathbb{P}}^N$, the second moment of β_i is bounded by a constant V and all the remaining moments satisfy the estimate (10.4.39) for a proper constant R .*

Proof. Let $\xi = \int_{[0,T] \times \mathbb{R}^{2d}} \phi(t, z) \rho(dt) \mu(dz)$, p a positive integer, then

$$\mathbb{E}_{\bar{\mathbb{P}}^N} |\beta_i|^p = \int_{\mathbb{R}^{2d}} \left| \int_{[0,T]} \phi(t, z) \rho(dt) - \xi \right|^p \mu(dz).$$

Since $|\phi(t, z)| \leq C|z|^k$ and $|\xi|^p \leq \int C^p |z|^{p+k} \mu(dz)$, using the inequality $|a + b|^p \leq 2^{p-1}(|a|^p + |b|^p)$, we get

$$\mathbb{E}_{\bar{\mathbb{P}}^N} |\beta_i|^p \leq 2^p C^p \int_{\mathbb{R}^{2d}} |z|^{p+k} \mu(dz).$$

From where, considering the estimates of the moment of Z_t given by lemma 12, we obtain,

$$\begin{aligned}\mathbb{E}_{\bar{\mathbb{P}}^N} |\beta_i|^2 &\leq 4 \int_{\mathbb{R}^{2d}} |z|^{2+k} \mu(dz) \\ &\leq C^2 2^{2+k} (1 + k_0^{2+k}) e^{\alpha(2+k)T} = V \\ \mathbb{E}_{\bar{\mathbb{P}}^N} |\beta_i|^p &\leq 2^{p+k} (1 + \mathbb{E}|Z_0|^{p+k}) e^{\alpha(p+k)T} \\ &\leq V(2e^{\alpha T} k_0)^{p-2} = V(CR)^{p-2},\end{aligned}$$

for $R = 2e^{\alpha T} k_0$. □

Relation between the probability measures: proof of inequality (10.4.40)

Let us define

$$\bar{M}_t^N = \sum_{i=1}^N \int_0^t ((\sigma^{-1}b_1)(s, X_s, Y_s, \mu_s^N) - (\sigma^{-1}b_1)(s, X_s, Y_s, \mu_s)) d\bar{B}_s^i,$$

which is a local martingale with respect to $\bar{\mathbb{P}}^N$. Moreover, as one consequence of the next proposition, Norikov's criterion, $\mathcal{E}_t(\bar{M}^N) = \exp(\bar{M}_t^N - \frac{1}{2}\langle \bar{M}^N \rangle_t)$ is a martingale under $\bar{\mathbb{P}}^N$. Then, applying Girsanov's theorem, since the solution of (10.1.11) is unique, we get

$$\frac{d\mathbb{P}^N}{d\bar{\mathbb{P}}^N} = \mathcal{E}_t(\bar{M}^N).$$

Proposition 3. *For every $\tau > 0$ there exist $\delta_0 > 0$ such that*

$$\sup_{N \geq 1} \sup_{[0, T]} \mathbb{E}[\exp(\tau(\langle \bar{M}^N \rangle_{t+\delta} - \langle \bar{M}^N \rangle_t))] \leq \bar{C},$$

for every $0 \leq \delta \leq \delta_0$ and some constant \bar{C} .

Proof. Let $\mathcal{E}_s^N(x, y) = b_1(s, x, y, \mu_s^N) - b_1(s, x, y, \mu_s)$, thanks to Assumption 7-(2),

$$|\mathcal{E}_s^N(X_s, Y_s)|^2 \leq \int |\bar{b}_1(s, X_s, Y_s, u)(\mu_s^N - \mu_s)(du)|^2.$$

From where

$$\begin{aligned}\tau(\langle \bar{M}^N \rangle_{t+\delta} - \langle \bar{M}^N \rangle_t) &= \sum_{i=1}^N \int_t^{t+\delta} ((\sigma^{-1}b_1)(s, X_s^i, Y_s^i, \mu_s^N) - (\sigma^{-1}b_1)(s, X_s^i, Y_s^i, \mu_s)) ds \\ &\leq \tau |\sigma^{-1}|_\infty \sum_{i=1}^N \int_t^{t+\delta} |\mathcal{E}_s^N(X_s^i, Y_s^i)|^2 \\ &\leq \kappa \int_t^{t+\delta} \sum_{i=1}^N \int |\bar{b}_1(s, X_s^i, Y_s^i, u)(\mu_s^N - \mu_s)(du)|^2\end{aligned}$$

with $\kappa = \tau |\sigma^{-1}|_\infty$. By Jensen's inequality together with the exchangeability of Z_t we have

$$\begin{aligned}\mathbb{E}[\exp(\tau(\langle \bar{M}^N \rangle_{t+\delta} - \langle \bar{M}^N \rangle_t))] &\leq \frac{1}{\delta} \int_t^{t+\delta} \mathbb{E}[\exp(\kappa \delta \sum_{i=1}^N \int |\bar{b}_1(s, X_s^i, Y_s^i, u)(\mu_s^N - \mu_s)(du)|^2)] \\ &\leq \sup_{[0, T]} \mathbb{E}_{\bar{\mathbb{P}}}^N[\exp(\kappa \delta N \int |\bar{b}_1(s, X_s^1, Y_s^1, u)(\mu_s^N - \mu_s)(du)|^2)] \\ &\leq \sup_{[0, T]} \mathbb{E}_{\bar{\mathbb{P}}}^N[\exp(\kappa \delta \frac{1}{N} \sum_{j=1}^N |A^{1,j}|^2)] \\ &\leq \sup_{[0, T]} \frac{1}{2} \left(\mathbb{E}_{\bar{\mathbb{P}}}^N[\exp(\kappa \delta \frac{1}{N} |A^{1,1}|^2)] + \mathbb{E}_{\bar{\mathbb{P}}}^N[\exp(\kappa \delta \frac{1}{N} \sum_{j=2}^N |A^{1,j}|^2)] \right),\end{aligned}$$

where,

$$A_s^{i,j} = \bar{b}_1(s, X_s^i, Y_s^i, X_s^j) - \int_{\mathbb{R}^d} \bar{b}_1(s, X_s^i, Y_s^i, u) \mu_s(du).$$

As a consequence of lemma 12 and the Assumption 7(1) on b_1 we have,

$$\mathbb{E}[\bar{A}_t^{i,j}|^{2p}] \leq \sigma^2 p! C_2^p, \quad (10.4.43)$$

which is the moment condition on the definition of a sub-Gaussian random variable. In consequences, as the variables $A_s^{1,j}$ for $j = 2, \dots, N$ are independent under \mathbb{P}^N , $\sum_{j=2}^N A_s^{1,j}$ is a $(N-1)c'$ sub-Gaussian for some constant C' depending on C_2 and by the moment condition again

$$\mathbb{E}[|\sum_{j=2}^N A_s^{1,j}(X, Y)|^{2p}] \leq p! 4^p (N-1)^p C'^p.$$

Hence,

$$\mathbb{E}_{\mathbb{P}^N}[\exp(\kappa\delta \frac{1}{N} \sum_{j=2}^N |A^{1,j}|^2)] = 1 \sum_{p \geq 1} \frac{(\kappa\delta)^p}{p!} \frac{1}{N^p} p! 4^p (N-1)^p C'^p < \infty,$$

and since the first term,

$$\mathbb{E}_{\mathbb{P}^N}[\exp(\kappa\delta \frac{1}{N} |A^{1,1}|^2)],$$

is clearly bounded as a consequence of (10.4.43) the result follows. \square

Let now $\mathcal{A} \in \mathcal{F}_T$, since $\bar{\mathbb{P}}^N$ and \mathbb{P}^N coincide in \mathcal{F}_0 , we have

$$\mathbb{P}^N(\mathcal{A}) = \mathbb{E}_{\mathbb{P}^N}[\mathbb{P}^N(\mathcal{A}|\mathcal{F}_0)] = \mathbb{E}_{\bar{\mathbb{P}}^N}[\mathbb{P}^N(\mathcal{A}|\mathcal{F}_0)].$$

The next result is the key to arrive at the relation 10.4.40,

Lemma 14. *For any division $0 = t_0 < \dots < t_K \leq T$ and $\mathcal{A} \in \mathcal{F}_T$, it is satisfied that*

$$\mathbb{E}_{\bar{\mathbb{P}}^N}[\mathbb{P}(\mathcal{A}|\mathcal{F}_0)] = \mathbb{E}_{\bar{\mathbb{P}}^N}[\mathbb{P}(\mathcal{A}|\mathcal{F}_{t_K})]^{1/4K} \prod_{j=1}^K \mathbb{E}_{\bar{\mathbb{P}}^N}[\exp(2(\langle \bar{M}^N \rangle_{t_j} - \langle \bar{M}^N \rangle_{t_{j-1}}))]^{j/4}.$$

Indeed, from this result, and proposition 3 with $\tau = 2$, $t_j = jT/K$ and K large enough so $t_j - t_{j-1} \leq \delta_0$, we can conclude that

$$\begin{aligned} \mathbb{P}^N(\mathcal{A}) &\leq \mathbb{E}_{\bar{\mathbb{P}}^N}[\mathbb{P}(\mathcal{A}|\mathcal{F}_{t_K})]^{1/4K} \prod_{j=1}^K \mathbb{E}_{\bar{\mathbb{P}}^N}[\exp(2(\langle \bar{M}^N \rangle_{t_j} - \langle \bar{M}^N \rangle_{t_{j-1}}))]^{j/4} \\ &\leq \bar{\mathbb{P}}^N(\mathcal{A})^{1/4K} \sup_{N \geq 1} \sup_{t \in [0, T - \delta_0]} \left(\mathbb{E}_{\bar{\mathbb{P}}^N}[\exp(2(\langle \bar{M}^N \rangle_{t+\delta_0} - \langle \bar{M}^N \rangle_t))] \right)^{K(K+1)/8} \\ &\leq \bar{C}^{K(K+1)/8} \bar{\mathbb{P}}^N(\mathcal{A})^{1/4K}. \end{aligned}$$

The proof of lemma 14 can be found in Lacker [145] and Della Maestra [73]. For completeness, we include a version of this in the appendix.

Conclusion, proof of theorem 8

We now have all the ingredients to complete the proof of Theorem 8. Let \mathcal{A}^N be defined as in 10.4.36, we have from step 3 that, $\mathbb{P}^N(\mathcal{A}^N) \leq \bar{C}^{K(K+1)/8} \bar{\mathbb{P}}^N(\mathcal{A}^N)^{1/4K}$. Moreover, from step 2, the random variables β_i satisfy the conditions to apply Bernstein Inequality under $\bar{\mathbb{P}}^N$, from where,

$$\mathbb{P}^N(\mathcal{A}^N) \leq \bar{C}^{K(K+1)/8} \left(2 \exp \left(- \frac{(N\gamma)^2}{2(\sum_{i=1}^N \mathbb{E}[\beta_i^2] + R\gamma)} \right) \right)^{1/4K},$$

which conclude the result for $\alpha_1 = \bar{C}^{K(K+1)/8} 2^{1/4K}$ and $\alpha_2 = \frac{1}{2} \frac{1}{4^K}$.

10.5 Application to the parameter estimation of FitzHugh-Nagumo model

Here, we present an application of the concentration inequality (10.1.12) to the estimation of the parameters of the FitzHugh-Nagumo model presented in the introduction.

Given μ_t the solution of equation (10.1.5), let us compute the moments of order i . We define O_t^i and $m_t^{i,j}$ as in (10.1.13). Since μ_t is a solution of the equation, it is satisfied that

$$\begin{aligned} O_T^i &= O_0^i + \int_0^T \int_{\mathbb{R}^2} \{i(x - x^3/3 - y + I)x^{i-1} \\ &\quad + i(\bar{c}x + \bar{a} - \bar{b}y)y^{i-1} + \frac{1}{2}\sigma^2 i(i-1)x^{i-2}\} \mu_t(dx, dy) \\ &= O_0^i + i \int_0^T (m^{i,0} - \frac{m^{i+2,0}}{3} - m^{i-1,1} + Im^{i-1,0}) \\ &\quad + \bar{c} \int_0^T m^{1,i-1} + \bar{a} \int_0^t m^{0,i-1} - \bar{b} \int_0^T m^{0,i} - \lambda \int_0^T (m^{i,0} + m^{1,0}m^{i-1,0}) \\ &\quad + \sigma^2 \int_0^T \frac{i(i-1)}{2} m^{i-2,0}. \end{aligned}$$

Let us consider the six first moment O_T^i with $i = 1, 6$, we obtain a linear system of six equation, which we write with the matrix formulation $O = M\theta + \Lambda$, with $O = (O_T^1, \dots, O_T^6)^T$, the rows of M given by

$$M_{i,\dots} = (i \int_0^T m_s^{i-1,0}, i \int_0^T m_s^{1,i-1}, i \int_0^T m^{0,i-1}, -i \int_0^T m^{0,i}, -i \int_0^T (m^{i,0} + m^{1,0}m^{i-1,0}), \frac{i(i-1)}{2} \int_0^T m^{i-2,i})$$

and the independent term $\Lambda = (\Lambda_1, \dots, \Lambda_6)$, with $\Lambda_i = O_0^i + i \int_0^T (m^{i,0} - \frac{m_s^{i+2,0}}{3} + m_s^{i-1,1})$. Note that all moments are well defined, according to lemma 12.

In consequence, if O, M and Λ are known, we can estimate the parameter vector as $\theta = M^{-1}(O - \Lambda)$. Let μ_t^N the empirical measure, \hat{O}_N, \hat{M}_N and $\hat{\Lambda}_N$ the associated approximations from replacing μ for the empirical measure, then, we obtain an estimator for the parameter vector given by

$$\hat{\theta}_N = \hat{M}_N^{-1}(\hat{O}_N - \hat{\Lambda}_N) \mathbf{1}_{\|\hat{M}_N^{-1}\| > \delta_N}.$$

Proof of theorem 3

Proof. Since

$$\begin{aligned} \|\hat{\theta}_N - \theta\|_\infty &= \|M^{-1}(O - \Lambda) - \hat{M}_N^{-1}(\hat{O}_N - \hat{\Lambda}_N)\|_\infty \\ &\leq \|(M^{-1} - \hat{M}_N^{-1})(\hat{O}_N - \hat{\Lambda}_N)\|_\infty + \|M^{-1}(\hat{O}_N - O)\|_\infty + \|M^{-1}(\hat{\Lambda}_N - \Lambda)\|_\infty, \end{aligned}$$

it is satisfied that,

$$\begin{aligned} P(\|\hat{\theta}_N - \theta\|_\infty \geq \gamma) &\leq P(\|(M^{-1} - \hat{M}_N^{-1})(\hat{O}_N - \hat{\Lambda}_N)\|_\infty \geq \frac{\gamma}{3}) \\ &\quad + P(\|M^{-1}(\hat{O}_N - O)\|_\infty \geq \frac{\gamma}{3}) + P(\|M^{-1}(\hat{\Lambda}_N - \Lambda)\|_\infty \geq \frac{\gamma}{3}) \\ &= I + II + III. \end{aligned}$$

We now proceed to bound each of these terms. For the last two, let us observe that

$$\|\hat{O}_N - O\|_\infty = \max_{i=1,6} \langle \mu_T^N - \mu_T, x^i + y^i \rangle; \quad (10.5.44)$$

$$\|\hat{\Lambda}_N - \Lambda\|_\infty = \max_{i=1,6} i \int_0^T \langle \mu_t^N - \mu_t, x^i - \frac{x^{i+2}}{3} + x^{i-1}y \rangle dt. \quad (10.5.45)$$

Then we get the estimate by applying the Bernstein Inequality shown in theorem 8. Indeed, taking $\phi(x, y) = x^i + y^i$, $\rho = \delta_T$ in theorem 8 we obtain

$$P\left(\langle \mu_T^N - \mu_T, x^i + y^i \rangle \geq \frac{\gamma}{3\|M^{-1}\|_\infty}\right) \leq \alpha_1 \exp\left\{-\alpha_2 \frac{N(\gamma/(3\|M^{-1}\|_\infty))^2}{C_1 + RC\gamma/(3\|M^{-1}\|_\infty)}\right\},$$

where C_1 is an upper bound for all moments of μ less than 12. Note that $|\phi| \leq C|z|^i$ for some constant C (uniformly for all i).

Analogously, for $\phi(x, y) = x^i - \frac{x^{i+2}}{3} + x^{i-1}y$, ρ the Lebesgue's measure in $[0, T]$, C' such that $|\phi| \leq C'|z|^2$ we get

$$P\left(i \int_0^T \langle \mu_t^N - \mu_t, x^i - \frac{x^{i+2}}{3} + x^{i-1}y \rangle dt \geq \frac{\gamma}{3\|M^{-1}\|_\infty}\right) \leq \alpha_1 \exp\left\{-\alpha_2 \frac{N(\gamma/(18\|M^{-1}\|_\infty))^2}{C'_1 + RC'\gamma/(18\|M^{-1}\|_\infty)}\right\},$$

for C'_1 upper bound of 16th-moment of μ .

$$I \leq P(\|\widehat{O}_N - O\|_\infty \geq \frac{\gamma}{3\|M^{-1}\|_\infty}) \leq 6\alpha_1 \exp\left\{-\alpha_2 \frac{N(\gamma/(3\|M^{-1}\|_\infty))^2}{C_1 + RC\gamma/(3\|M^{-1}\|_\infty)}\right\}, \quad (10.5.46)$$

$$II \leq P(\|\widehat{\Lambda}_N - \Lambda\|_\infty \geq \frac{\gamma}{3\|M^{-1}\|_\infty}) \leq 6\alpha_1 \exp\left\{-\alpha_2 \frac{N(\gamma/(18\|M^{-1}\|_\infty))^2}{C'_1 + RC'\gamma/(18\|M^{-1}\|_\infty)}\right\}, \quad (10.5.47)$$

It only remains then to bound I . Let us denote $A = \{\|(M^{-1} - \widehat{M}_N^{-1})(\widehat{O}_N - \widehat{\Lambda}_N)\|_\infty \geq \frac{\gamma}{3}\}$, for all $L > 0$, we can write A as

$$A = \left(A \cap \{\|\widehat{O}_N - \widehat{\Lambda}_N\| \geq L\}\right) \cup \left(\{\|\widehat{O}_N - \widehat{\Lambda}_N\| < L\} \cap \{\|(M^{-1} - \widehat{M}_N^{-1})\| \geq \frac{\gamma}{3L}\}\right).$$

This implies that

$$\begin{aligned} P(A) &\leq P(\{\|\widehat{O}_N - \widehat{\Lambda}_N\| \geq L\}) + P(\{\|(M^{-1} - \widehat{M}_N^{-1})\| \geq \frac{\gamma}{3L}\}) \\ &\leq P(\{\|\widehat{O}_N\| \geq L\}) + P(\{\|\widehat{\Lambda}_N\| \geq L\}) + P(\{\|(M^{-1} - \widehat{M}_N^{-1})\| \geq \frac{\gamma}{3L}\}) \end{aligned}$$

Let's set $L = \gamma^{1/2} + \|O\| + \|\Lambda\|$ then

$$P(A) \leq P(\{\|\widehat{O}_N - O\| \geq \gamma^{1/2}\}) + P(\{\|\widehat{\Lambda}_N - \Lambda\| \geq \gamma^{1/2}\}) + P(\{\|(M^{-1} - \widehat{M}_N^{-1})\| \geq \frac{\gamma^{1/2}}{3}\})$$

The first two terms can be bounded using Bernstein's inequality as above. To proceed with the third one, we must first perform the following algebraic trick.

Let us denote $\xi = M - \widehat{M}_N$, then

$$\widehat{M}_N^{-1} - M^{-1} = (M - \xi)^{-1} - M^{-1} = (1 - M^{-1}\xi)^{-1}M^{-1} - M^{-1}.$$

There are two possibilities, or $\|M^{-1}\xi\|_F < \frac{1}{2}$ or $\|M^{-1}\xi\|_F \geq \frac{1}{2}$. In the first case, we have that $(1 - M^{-1}\xi) = \sum_{j \geq 0} (M^{-1}\xi)^j$, from where

$$\begin{aligned} \widehat{M}_N^{-1} - M^{-1} &= \left(\sum_{j \geq 0} (M^{-1}\xi)^j\right)M^{-1} - M^{-1} = \left(\sum_{j \geq 1} (M^{-1}\xi)^j\right)M^{-1}, \\ \|\widehat{M}_N^{-1} - M^{-1}\|_F &\leq \|\xi\| \|M^{-1}\|^2 \sum_{j \geq 0} \|\xi M^{-1}\|^j = \|\xi\| \frac{\|M^{-1}\|^2}{1 - \|\xi M^{-1}\|} \leq 2\|\xi\| \|M^{-1}\|^2. \end{aligned}$$

In conclusion,

$$P(\|(M^{-1} - \widehat{M}_N^{-1})\| \geq \frac{\gamma^{1/2}}{3}) \leq P(\|\xi\|_F \geq \frac{\gamma^{1/2}\|M\|_F^2}{6}) + P(\|\xi\|_F \geq 1/2\|M\|_F)$$

since $\|\xi\|_F \leq 6 \max_{1 \leq i, j \leq 6} |m_{ij}|$, applying again Bernstein inequality for each coefficient of the matrix, we obtain,

$$P(\|(M^{-1} - \widehat{M}_N^{-1})\| \geq \frac{\gamma^{1/2}}{3}) \leq \alpha_1 \exp \left\{ -\alpha_2 \frac{N\gamma(\|M\|_F/6)^2}{C_1'' + RC''\gamma^{1/2}(\|M\|_F/6)} \right\} \\ + \alpha_1 \exp \left\{ -\alpha_2 \frac{N(1/12\|M\|_F)^2}{C_2'' + RC''1/12\|M\|_F} \right\}$$

Gathering all together, we conclude

$$\mathbb{P}_N(\|\widehat{\theta}_N - \theta\|_\infty \geq \gamma) \leq \alpha_1' \exp \left\{ -\alpha_2' \frac{N\gamma}{1 + RC\gamma^{1/2}} \right\} + \alpha_1 \exp \left\{ -\alpha_2 \frac{N(1/12\|M\|_F)^2}{C_2'' + RC''1/12\|M\|_F} \right\}$$

□

10.6 Nonparametric estimation of μ_t

Let's start by stating the following lemma, which is a consequence of [141, Theo. 2.1] and take the role of [73, lemm. 23].

Lemma 15. *Grant Assumptions 6, 7 and $(t_0, z_0) \in (0, T] \times \mathbb{R}^{2d}$. Let $r > 0$ and $[r_1, r_2] \subset (0, T)$, there exist κ_3, κ_4 depending on z_0, r_1, r_2, r and \mathbf{b} such that*

$$0 < \kappa_5 \leq \inf_{t \in [r_1, r_2], |z - z_0| \leq r} \mu_t(z) \leq \sup_{t \in [r_1, r_2], |z - z_0| \leq r} \mu_t(z) \leq \kappa_6.$$

In turn, for a compactly supported kernel K , this implies the existence of $r = r(K)$ such that the estimate

$$|K_h(z_0 - \cdot)|_{L^2(\mu_{t_0})}^2 = \int_{\mathbb{R}^{2d}} h^{-4d} K(h^{-1}z)^2 \mu_{t_0}(z_0 - z) dz \leq \kappa_3(r) h^{-2d} |K|_2^2 \quad (10.6.48)$$

holds true.

Lemma 16. *In the setting of Theorem 10. if K is a bounded and compactly supported kernel and $h \in \mathcal{H}_1^N$, we have*

$$\mathbb{E}_{\mathbb{P}^N} \left[\left(\widehat{\mu}_h^N(t_0, z_0) - \mu_{t_0}(z_0) \right)^2 \right] \lesssim \mathcal{B}_h^N(\mu)(t_0, z_0)^2 + V_h^N,$$

up to a constant that depends (continuously) on (t_0, z_0) , $|K|_\infty$ and \mathbf{b} , and where $\mathcal{B}_h^N(\mu)(t_0, z_0)$ is defined in (10.1.18) and V_h^N in (10.1.16).

Proof. Write $\widehat{\mu}_h^N(t_0, z_0) - \mu_{t_0}(z_0) = I + II$, with

$$I = \int_{\mathbb{R}^d} K_h(z_0 - z) \mu_{t_0}(z) dz - \mu_{t_0}(z_0)$$

and

$$II = \int_{\mathbb{R}^d} K_h(z_0 - z) \left(\mu_{t_0}^N(dz) - \mu_{t_0}(z) dz \right).$$

We have $I^2 \leq \mathcal{B}_h^N(\mu)(t_0, z_0)^2$ for the squared bias term. For the variance term, using successively Theorem 8 and the estimate (10.6.48), we have

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}^N} [II^2] &= \int_0^\infty \mathbb{P}^N (|II| \geq u^{1/2}) du \\
&\leq 2\kappa_1 \int_0^\infty \exp \left(-\frac{\kappa_2 N u}{|K_h(z_0 - \cdot)|_{L^2(\mu_{t_0})}^2 + |K_h(z_0 - \cdot)|_\infty u^{1/2}} \right) du \\
&\leq 2\kappa_1 \int_0^\infty \exp \left(-\frac{\kappa_2 N h^{2d} u}{\kappa_3 |K|_2^2 + |K|_\infty u^{1/2}} \right) du \\
&\lesssim (N h^{2d})^{-1} \left(1 + (N h^{2d})^{-1} \right) \\
&\lesssim V_h^N
\end{aligned}$$

where we used (48) and the fact that $\max_{h \in \mathcal{H}_1^N} (N h^{2d})^{-1} \lesssim 1$. \square

Completion of proof of Theorem 10 Recall that \hat{h}^N denotes the data-driven bandwidth defined in 10.1.17.

Step 1: For $h \in \mathcal{H}_1^N$, we successively have

$$\begin{aligned}
&\mathbb{E}_{\mathbb{P}^N} \left[\left(\hat{\mu}_{GL}^N(t_0, z_0) - \mu_{t_0}(z_0) \right)^2 \right] \\
&\lesssim \mathbb{E}_{\mathbb{P}^N} \left[\left(\hat{\mu}_{GL}^N(t_0, z_0) - \hat{\mu}_h^N(t_0, z_0) \right)^2 \right] + \mathbb{E}_{\mathbb{P}^N} \left[\left(\hat{\mu}_h^N(t_0, z_0) - \mu_{t_0}(z_0) \right)^2 \right] \\
&\lesssim \mathbb{E}_{\mathbb{P}^N} \left[\left\{ \left(\hat{\mu}_{\hat{h}^N}^N(t_0, z_0) - \hat{\mu}_h^N(t_0, z_0) \right)^2 - V_h^N - V_{\hat{h}^N}^N \right\}_+ + V_h^N + V_{\hat{h}^N}^N \right] + \mathbb{E}_{\mathbb{P}^N} \left[\left(\hat{\mu}_h^N(t_0, z_0) - \mu_{t_0}(z_0) \right)^2 \right] \\
&\lesssim \mathbb{E}_{\mathbb{P}^N} \left[A_{\max(\hat{h}^N, h)}^N + V_h^N + V_{\hat{h}^N}^N \right] + \mathbb{E}_{\mathbb{P}^N} \left[\left(\hat{\mu}_h^N(t_0, z_0) - \mu_{t_0}(z_0) \right)^2 \right] \\
&\lesssim \mathbb{E}_{\mathbb{P}^N} [A_h^N] + V_h^N + \mathbb{E}_{\mathbb{P}^N} [A_{\hat{h}^N}^N + V_{\hat{h}^N}^N] + \mathbb{E}_{\mathbb{P}^N} \left[\left(\hat{\mu}_h^N(t_0, z_0) - \mu_{t_0}(z_0) \right)^2 \right] \\
&\lesssim \mathbb{E}_{\mathbb{P}^N} [A_h^N] + V_h^N + \mathcal{B}_h^N(\mu)(t_0, z_0)^2,
\end{aligned}$$

where we applied Lemma 16 in order to obtain the last line.

Step 2: We first estimate A_h^N . Write $\mu_h(t_0, z_0)$ for $\int_{\mathbb{R}^{2d}} K_h(z_0 - z) \mu_{t_0}(z) dz$. For $h, h' \in \mathcal{H}_1^N$ with $h' \leq h$, since

$$\begin{aligned}
&\left(\hat{\mu}_h^N(t_0, z_0) - \hat{\mu}_{h'}^N(t_0, z_0) \right)^2 \\
&\leq 4 \left(\hat{\mu}_h^N(t_0, z_0) - \mu_h(t_0, z_0) \right)^2 + 4 \left(\mu_h(t_0, z_0) - \mu_{t_0}(z_0) \right)^2 + 4 \left(\mu_{h'}(t_0, z_0) - \mu_{t_0}(z_0) \right)^2 \\
&\quad + 4 \left(\hat{\mu}_{h'}^N(t_0, z_0) - \mu_{h'}(t_0, z_0) \right)^2
\end{aligned}$$

we have

$$\begin{aligned}
\left(\hat{\mu}_h^N(t_0, z_0) - \hat{\mu}_{h'}^N(t_0, z_0) \right)^2 - V_h^N - V_{h'}^N &\leq 8\mathcal{B}_h^N(\mu)(t_0, z_0)^2 + \left(4 \left(\hat{\mu}_h^N(t_0, z_0) - \mu_h(t_0, z_0) \right)^2 - V_h^N \right) \\
&\quad + \left(4 \left(\hat{\mu}_{h'}^N(t_0, z_0) - \mu_{h'}(t_0, z_0) \right)^2 - V_{h'}^N \right)
\end{aligned}$$

using $h' \leq h$ in order to bound $\left(\hat{\mu}_{h'}^N(t_0, z_0) - \mu_{h'}(t_0, z_0) \right)^2$ by the bias at scale h . Taking maximum

over $h' \leq h$, we obtain

$$\begin{aligned} & \max_{h' \leq h} \left\{ \left(\widehat{\mu}_h^N(t_0, z_0) - \widehat{\mu}_{h'}^N(t_0, z_0) \right)^2 - V_h^N - V_{h'}^N \right\}_+ \\ & \leq 8\mathcal{B}_h^N(\mu)(t_0, z_0)^2 + \left\{ 4 \left(\widehat{\mu}_h^N(t_0, z_0) - \mu_h(t_0, z_0) \right)^2 - V_h^N \right\}_+ \\ & \quad + \max_{h' \leq h} \left\{ 4 \left(\widehat{\mu}_{h'}^N(t_0, z_0) - \mu_{h'}(t_0, z_0) \right)^2 - V_{h'}^N \right\}_+ \end{aligned} \quad (10.6.49)$$

Step 3: We estimate the expectation of the first stochastic term in the right-hand side of (10.6.49). We refine the computation of the term II in the proof of Lemma 16. By Theorem 8 and using estimates

$$\int_{\nu}^{\infty} \exp(-u^r) du \leq 2r^{-1} \nu^{1-r} \exp(-\nu^r), \quad \nu, r > 0, \quad \nu \geq (2/r)^{1/r},$$

and

$$\int_0^{\infty} \exp\left(-\frac{au^p}{b+cu^{p/2}}\right) du \leq C_p \max\left(\left(\frac{a}{b}\right)^{-1/p}, \left(\frac{a}{c}\right)^{-2/p}\right), \quad a, b, c, p > 0,$$

with $C_p = 2 \int_0^{\infty} \exp\left(-\frac{1}{2}(\min(\sqrt{u}, u))^p\right) du$, stemming from the rough bound

$$\exp\left(-\frac{au^p}{b+cu^{p/2}}\right) \leq \exp\left(-\frac{au^p}{2b}\right) + \exp\left(-\frac{au^{p/2}}{2c}\right), \quad u > 0.$$

we have

$$\begin{aligned} & \mathbb{E}_{\mathbb{P}^N} \left[\left\{ 4 \left(\widehat{\mu}_h^N(t_0, z_0) - \mu_h(t_0, z_0) \right)^2 - V_h^N \right\}_+ \right] \\ & = \int_0^{\infty} \mathbb{P}^N \left(4 \left(\widehat{\mu}_h^N(t_0, z_0) - \mu_h(t_0, z_0) \right)^2 - V_h^N \geq u \right) du \\ & = \int_0^{\infty} \mathbb{P}^N \left(\left| \widehat{\mu}_h^N(t_0, z_0) - \mu_h(t_0, z_0) \right| \geq \frac{1}{2} \left(V_h^N + u \right)^{1/2} \right) du \\ & \leq 2\kappa_1 \int_{V_h^N}^{\infty} \exp\left(-\frac{\kappa_2 N h^{2d} \frac{1}{4} u}{\kappa_3 |K|_2^2 + |K|_{\infty} \frac{1}{2} u^{1/2}}\right) du \\ & \lesssim \int_{V_h^N}^{\infty} \exp\left(-\frac{\kappa_2 N h^{2d} u}{8\kappa_3 |K|_2^2}\right) du + \int_{V_h^N}^{\infty} \exp\left(-\frac{\kappa_2 N h^{2d} u^{1/2}}{4|K|_{\infty}}\right) du \\ & \lesssim (N h^{2d})^{-1} \exp\left(-\frac{\kappa_2 N h^{2d} V_h^N}{8\kappa_3 |K|_2^2}\right) + (N h^{2d})^{-2} N h^{2d} (V_h^N)^{1/2} \exp\left(-\frac{\kappa_2 N h^{2d} (V_h^N)^{1/2}}{4|K|_{\infty}}\right) \\ & \lesssim (N h^{2d})^{-1} N^{-\varpi_1 \kappa_2 / (8\kappa_3)} + (N h^{2d})^{-3/2} (\log N)^{1/2} \exp\left(-\frac{\kappa_2 |K|_2 \varpi_1^{1/2} (\log N)^{5/2}}{4|K|_{\infty}}\right), \\ & \lesssim N^{-2} \end{aligned}$$

as soon as $\varpi_1 \geq 16\kappa_2^{-1}\kappa_3$, thanks to $\max_{h \in \mathcal{H}_1^N} (N h^{2d})^{-1} \lesssim 1$, and using $\min_{h \in \mathcal{H}_1^N} h \geq (N^{-1}(\log N)^2)^{1/d}$ to show that the second term is negligible in front of N^{-2} .

Step 4: For the second stochastic term, we have the rough estimate

$$\mathbb{E}_{\mathbb{P}^N} \left[\max_{h' \leq h} \left\{ 4 \left(\widehat{\mu}_{h'}^N(t_0, z_0) - \mu_{h'}(t_0, z_0) \right)^2 - V_{h'}^N \right\}_+ \right]$$

$$\leq \sum_{h' \leq h} \mathbb{E}_{\mathbb{P}^N} \left[\left\{ 4 \left(\widehat{\mu}_{h'}^N(t_0, z_0) - \mu_{h'}(t_0, z_0) \right)^2 - V_{h'}^N \right\}_+ \right] \lesssim \text{Card}(\mathcal{H}_1^N) N^{-2} \lesssim N^{-1}$$

where we used Step 3 to bound each term $\mathbb{E}_{\mathbb{P}^N} \left[\left\{ 4 \left(\widehat{\mu}_{h'}^N(t_0, z_0) - \mu_{h'}(t_0, z_0) \right)^2 - V_{h'}^N \right\}_+ \right]$ independently of h together with $\text{Card}(\mathcal{H}_1^N) \lesssim N$. In conclusion, we have through Steps 2-4 that $\mathbb{E}_{\mathbb{P}^N} \left[A_h^N \right] \lesssim N^{-1} + \mathcal{B}_h^N(\mu)(t_0, z_0)^2$. Therefore, from Step 1, we conclude

$$\mathbb{E}_{\mathbb{P}^N} \left[\left(\widehat{\mu}_{GL}^N(t_0, z_0) - \mu_{t_0}(z_0) \right)^2 \right] \lesssim \mathcal{B}_h^N(\mu)(t_0, z_0)^2 + V_h^N + N^{-1}$$

for any $h \in \mathcal{H}_1^N$. Since $N^{-1} \lesssim V_h^N$ always, the proof of Theorem 10 is complete.

10.7 Testing of the interaction between agents

the last part of this chapter is exploratory only, and is a joint ongoing work with S. Delattre and M. Hoffmann. We propose the construction of a statistical test to decide whether we have existence of connectivity between agents in a dynamic system such as the one presented in model (10.1.2), assuming that the coefficients F , G and H are unknown. To understand the difficulties that this objective presents and to introduce our strategy, let us begin with an important observation related to propagation of chaos for mean field problems.

One might think that a possible strategy for testing connectivity is to test the independence between agents, since it is intuitive to assume that if such interaction exists, the state of an agent will depend on the state of those connected to it and vice versa. However, the phenomenon of propagation of chaos indicates that when the number of agents increases, it is impossible to distinguish between a system of N interrelated particles and that of N independent particles, each of them following the law of limit distribution.

More precisely, let us call P^N , the joint distribution interrelated according to the system of equations (10.1.2) and $P^{\otimes N}$ the distribution of N independent particles each of them following the law (10.1.9), in the article of Della Maestra and Hoffmann, it has been proved that, for not large coefficients or short time windows,

$$\limsup_{N \rightarrow \infty} \|P^N - P^{\otimes N}\|_{TV} < 1 \quad (10.7.50)$$

where $\|\cdot\|_{TV}$ denote the total variation distance.

In particular, this implies that there exist no test of the null hypothesis $H_0 : P^N = P^{\otimes N}$ against $H_1 : P^N \neq P^{\otimes N}$ such that the first error and the second error converge both asymptotically to zero. More precisely, for all test T and rejection zone Ω , the first error $\alpha = P(T \in \Omega | H_0) = P^{\otimes N}(\Omega)$ and second error $\beta = P(T \notin \Omega | H_1) = P^N(\Omega^c)$ satisfy

$$P^{\otimes N}(\Omega) + P^N(\Omega^c) \geq 1 - \|P^N - P^{\otimes N}\|_{TV} > 0,$$

for all N large enough.

Consequently, another type of strategy has to be considered. We suggest to take advantage of the mean field convergence and redefine our problem in a new approach, related to the closeness of the system to equilibrium.

Indeed, we would like to use the result on theorem 8 and transfer properties from μ_t , the limit measure given by the solution of equation (10.1.9), to the empirical measure μ_t^N . More in details, if we take a look at equations (10.1.9) and (10.4.41), we notice that under the hypothesis of no interactions $H_0 : H = 0$, the coefficient b does not depend on time, while in the case

of interactions, b depends on t through μ_t . This motivates the construction of a test that is interesting on the hypothesis " b no depending on t ", against the alternative of the dependence. This approach, however, presents a new difficulty: in case the system is in an equilibrium state, $\mu_t = \mu_\infty$, the null hypothesis is again indistinguishable from the alternative.

We conjecture that in the case where the system is in equilibrium it is not possible to create a test that distinguishes whether there are interactions, but we lack of a proof at the present time. We nevertheless can propose a strategy based on a hierarchical test, where we first test that whether the system is at equilibrium or not, and then, based on a rejection of this hypothesis, we can check whether or not there are interactions, by testing whether the drift coefficient depends on time or not.

In the following we perform some exploratory work based on numerical simulations to underline the idea of such a test construction. After which we present a first theoretical result in a simplified model.

It has been proved in Mischler for the FhN model presented in the introduction [citation], that if the interaction between neurons is weak or null, the system converges to an equilibrium state, while when the interaction grows different patterns may appear such as synchronous behavior of neurons.

In Figure 10.7, we present the typical behavior of the voltage of a neuronal population (result of one simulation), when there is not interaction. We show three cases, in top we observe the trajectories and a density plot of a population of 1000 neurons without interaction and with an initial distribution close to equilibrium (that we have obtained by letting the system evolve for a long time; in the middle row, we show the same plots for a similar population, also without considering interaction between neurons but with an uniform initial distribution in $[-1, 1]$; then in the last row, we see a case when we start far from equilibrium all neurons start in the same state $Z_0 = (2, 2)$. We see how when we start from equilibrium the distribution is maintained over time, when in the other two cases, the population is moving towards the steady state.

In the other hand, in Figure 10.7 we show, first the typical evolution in long time without interaction and with increasing interaction ($J = 1$ and $J = 10$). We see that, when the interactions are weaker, we still have an equilibrium state while when the interactions are stronger a time-dependent pattern appears.

We will explore our test idea on several simulations of these cases.

Equilibrium test

To start with it, we pick two bounded and compactly supported kernel functions $K_1 : (0, T) \rightarrow \mathbb{R}$ and $K_2 : \mathbb{R}^d \rightarrow \mathbb{R}$ such that

$$\int_0^T K_1(t) dt = \int_{\mathbb{R}^{2d}} K_2(z) dz = 1.$$

In our simulation we will consider the Epanechnikov kernel,

$$K(u) = \frac{3}{4}(1 - u^2); \quad \text{supp } K = \{|u| \leq 1\}.$$

for t and each component of z .

For $h > 0$ we denote,

$$K_1^{(h)}(t) = h^{-1} K_1(h^{-1}t); \quad K_2^{(h)}(z) = h^{-d} K_2(h^{-1}z).$$

We construct a family of estimators of $\mu_{t_0}(z_0)$ depending on h by setting

$$\hat{\mu}_h^N(t_0, z_0) = \int_{\mathbb{R}^{2d}} K_2^{(h)}(z_0 - z) \mu_{t_0}^N(dx) \tag{10.7.51}$$

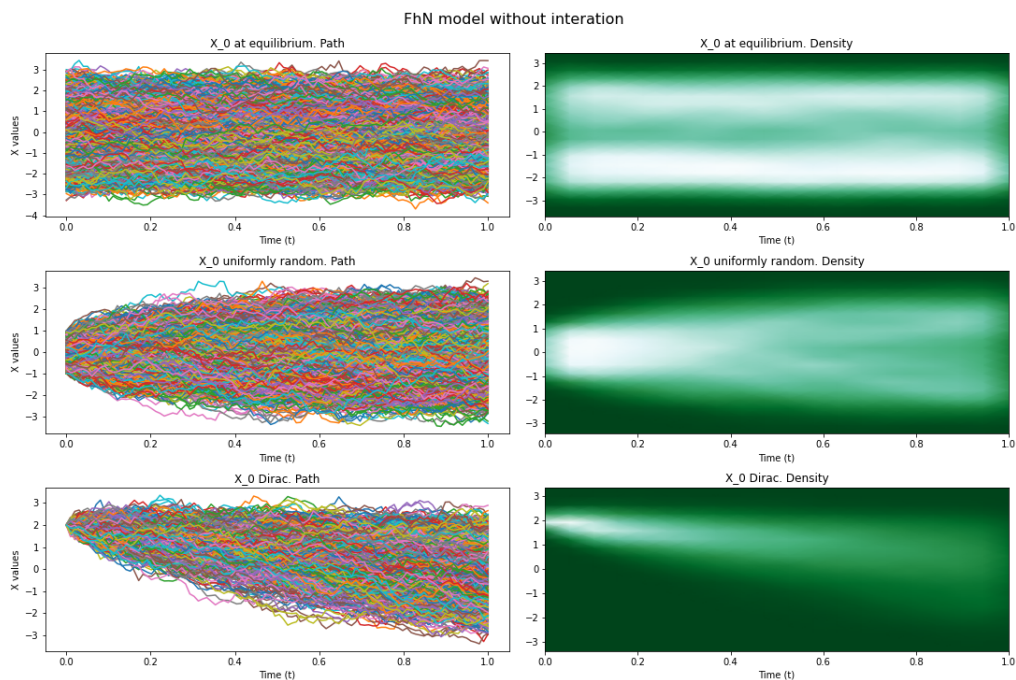


Figure 10.1: We show the voltage of three typical simulations of the model without interaction with three different initial conditions. In the top row, from the steady state; in the middle row, from a uniform distribution in $[-1, 1]^2$ and finally in the last row, when all neurons initialize in the same state (corresponding to a Dirac distribution in $(2, 2)$).

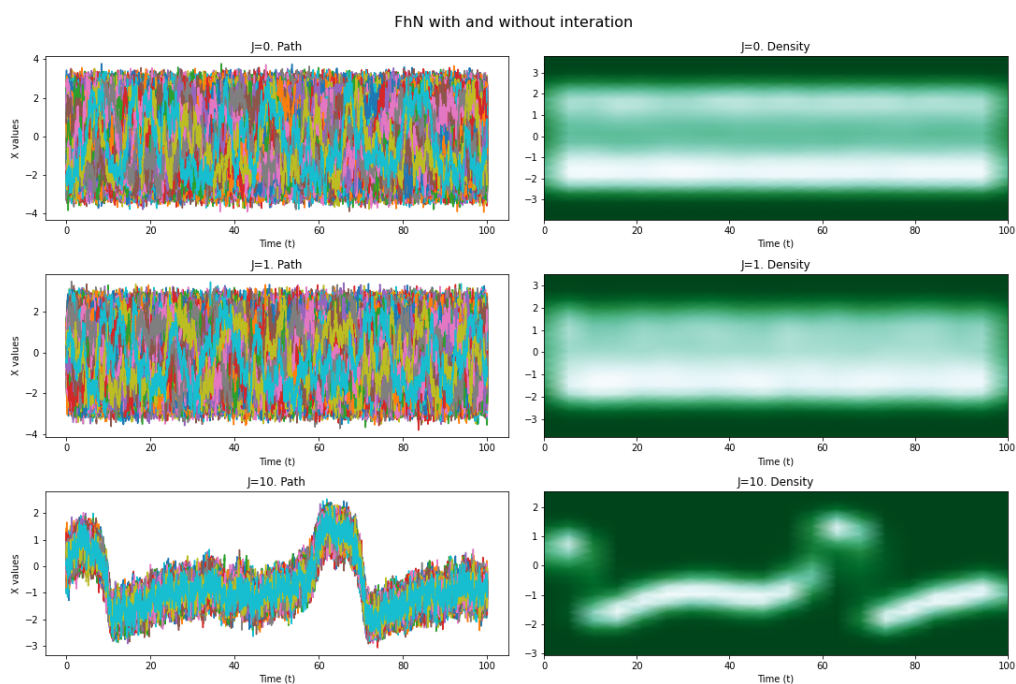


Figure 10.2: We show the voltage of three typical simulations evolving in a long time (in $[0, 100]$) without interaction, with weak interaction and with strong interaction.

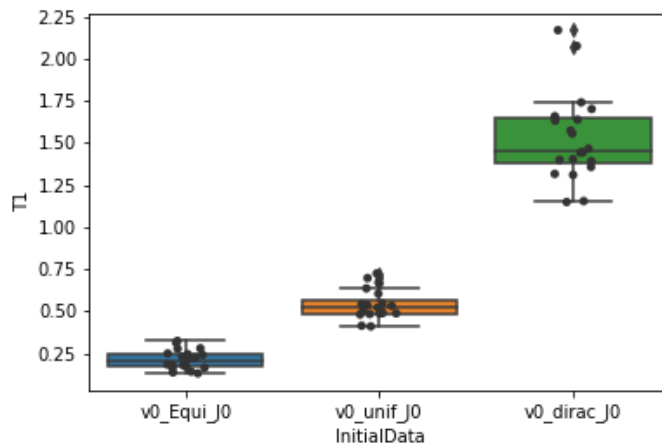


Figure 10.3: Box plot showing the distribution of T_1^N test values starting at equilibrium or not. The leftmost plot corresponding to the case when we start at the equilibrium, in the middle, when we start from the uniform distribution, and the last one when the system is far from the equilibrium (corresponding to the third case presented).

how is suggested in section 10.6. The performance benefits of such an estimator are shown by the oracle inequality given in Lemma 16. We regularise it on t by taking the convolution with the kernel K_1 ,

$$\tilde{\mu}_{\bar{h}}^N(t_0, z_0) = \int_0^T \int_{\mathbb{R}^{2d}} (K_1 \otimes K_2)_{\bar{h}}(t_0 - t, z_0 - z) \mu_{t_0}^N(dz) dt, \quad (10.7.52)$$

where for $\bar{h} = (h_1, h_2)$, $h_i > 0$, we define

$$(K_1 \otimes K_2)_{\bar{h}}(t, z) = K_1^{(h_1)}(t) K_2^{(h_2)}(z).$$

Under the hypothesis of being on the equilibrium, $\mu_t = \mu_\infty$, it is expected that $\partial_t \mu_t = 0$. We rely our construction on this argument by considering that the equilibrium test will be a function of $T_1 = \int |\partial_t \mu(z)|^2 dz$ which can be estimated by

$$T_1^N(t) = \int_{\mathbb{R}^{2d}} |\partial_t \tilde{\mu}_{\bar{h}}^N(t, z)|^2 dz. \quad (10.7.53)$$

In Figure 3, we can compare the values of T_1^N , over 20 simulations of the three cases without interaction. From the box plot we can see that there is significant statistical different of the indicator T_1^N between the case when the system is close to the equilibrium with respect to the case in where is not from, increasing as we move further away from equilibrium. We see this behavior again when we include interactions (Figure 4) although in this case with more variability.

In conclusion, these explorations suggest the validity of such an indicator as the basis for the creation of a test for the equilibrium hypothesis.

Interaction test

We continue now considering that we are not close to equilibrium and exploring the possibility of testing the connectivity between the agents.

Let $(t_0, z_0) \in (0, T) \times \mathbb{R}^{2d}$ we denote

$$\pi^N(dt, dz) = N^{-1} \sum_{i=1}^N Z^i(dt) \delta_{Z^i}(dz),$$

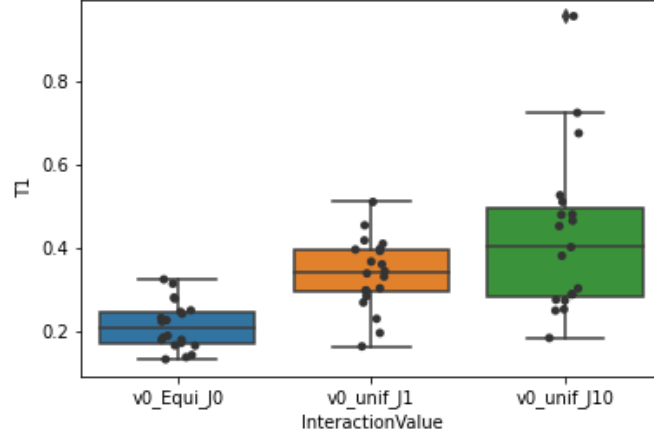


Figure 10.4: Box plot showing the distribution of T_1^N test values starting at equilibrium in the case without interaction or when there is not equilibrium (cases with interaction). The leftmost plot corresponding to the case when we start at the equilibrium with $J = 0$, and the middle and right, to increase the interaction in correspondence with the models presented in Figure 2.

the \mathbb{R}^d -value random measure defined by

$$\int_{[0,T] \times \mathbb{R}^{2d}} \phi(t, z) \pi^N(dt, dz) = \frac{1}{N} \sum_{i=1}^N \int_0^T \phi(t, Z_t^i) dZ_t^i$$

for a test function $\phi(t, z) : [0, T] \times \mathbb{R}^{2d} \rightarrow \mathbb{R}$. We recall that $Z^i = (X_i, Y_i)$ and rewriting equation 10.1.7 in a more synthetic form we have

$$dZ^i = b(Z_t^i, \mu_t^N) dt + \bar{\sigma} d\bar{B}_t^i$$

where $\bar{\sigma}$ is a diagonal $2d$ -matrix with σ in the diagonal elements corresponding to X and 0 in the diagonal elements corresponding to Y , and \bar{B}_t^i being a $2d$ -dimensional Brownian motion. We have that, at least formally,

$$\frac{1}{N} \sum_{i=1}^N \int_0^T \phi(t, Z_t^i) dZ_t^i = \frac{1}{N} \sum_{i=1}^N \int_0^T \phi(t, z) b(z, \mu_t^N) \mu_t^N dt + \frac{1}{N} \sum_{i=1}^N \int_0^T \bar{\sigma} \phi(t, Z_t^i) d\bar{B}_t^i$$

with

$$\mathbb{E} \left[\frac{1}{N} \sum_{i=1}^N \int_0^T \bar{\sigma} \phi(t, Z_t^i) d\bar{B}_t^i \right] = 0,$$

since ϕ is bounded. This suggests the convergence in expected value of $\pi^N(t, z)$ to $\pi(t, z) = b(z, \mu_t(z)) \mu_t(z)$. Such kind of result has been proven in Della Maestra and Hoffmann in [73, Theorem 9] for elliptic case (non-degenerated matrix $\bar{\sigma}$) and Lipschitz coefficient b , as it is the case of the linear model (??). More in details, they propose for b the estimator

$$\hat{b}_h^N(t_0, x_0)_w = \frac{\hat{\pi}_h^N(t_0, x_0)}{\hat{\mu}_h^N(t_0, x_0) \vee w},$$

where

$$\pi_h^N(t_0, x_0) = \int_{[0,T] \times \mathbb{R}^d} (K_1 \otimes K_2)_h(t_0 - t, x_0 - x) \pi^N(dt, dx).$$

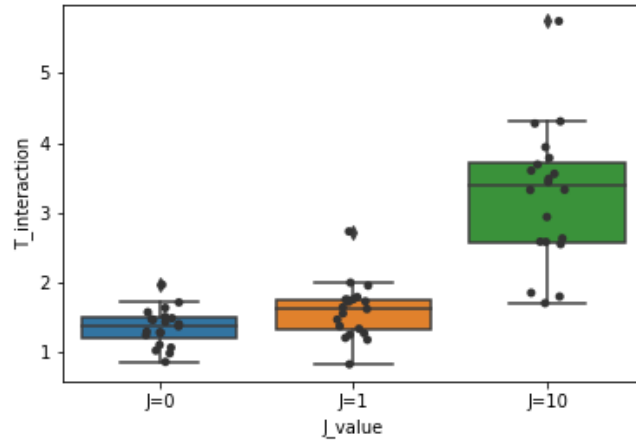


Figure 10.5: We compare the values of T_2^N when there is and there is not interaction. We see significant difference between the simulation with large interaction and without interaction but this difference is less clear when interaction are small.

Following this approach, we have

$$\partial_t b(z, \mu_t(z)) = \partial_t \left(\frac{\pi_t(z)}{\mu_t(z)} \right) = \frac{\partial_t \pi_t(z) \mu_t(z) - \pi_t(z) \partial_t \mu_t(z)}{\mu_t^2(z)},$$

and the hypothesis b is not depending on t coincide with

$$T_2(t, z) = \partial_t \pi_t(z) \mu_t(z) - \pi_t(z) \partial_t \mu_t(z) = 0,$$

that could be approximate by

$$\hat{T}_2^N(t, z) = \partial_t \hat{\pi}_t^N(z) \hat{\mu}_t^N(z) - \hat{\pi}_t^N(z) \partial_t \hat{\mu}_t^N(z).$$

In Figure 5 we explore the viability of such approach on simulation of the FhN model. We realize several simulations for each of the case, with and without interaction and for both models, and we compute the value of the indicator \hat{T}_2^N for each of them. We see on the boxplot graphics that we have significant difference between the simulation with large interaction and without interaction but this difference is less clear when interaction are small. This could be explained by the fact that when interactions are weak the system behaves similarly to when there are no interactions. This suggests that this approach may be successful in detecting when we are in the presence of strong interactions but perhaps less effective in detecting weak interactions.

10.8 Appendix

10.8.1 Proof of lemma 14

We now turn to the proof of lemma 5. The proof follows the arguments of Della Maestra [73] and Lacker [145].

Proof. Let it be the partition $0 = t_0 < \dots < t_K \leq T$, for $j - 1$ we deduce,

$$\begin{aligned} \mathbb{E}_{\bar{\mathbb{P}}^N}[\mathbb{P}^N(\mathcal{A}^N | \mathcal{F}_{t_{j-1}})] &= \mathbb{E}_{\bar{\mathbb{P}}^N}[\mathbb{E}_{\mathbb{P}^N}[\mathbb{P}^N(\mathcal{A}^N | \mathcal{F}_{t_j}) | \mathcal{F}_{t_{j-1}}]] \\ &= \mathbb{E}_{\bar{\mathbb{P}}^N}[\mathbb{E}_{\bar{\mathbb{P}}^N}[\frac{\mathcal{E}_{t_j}(\bar{M}^N)}{\mathcal{E}_{t_{j-1}}(\bar{M}^N)} \mathbb{P}^N(\mathcal{A}^N | \mathcal{F}_{t_j}) | \mathcal{F}_{t_{j-1}}]] \\ &= \mathbb{E}_{\bar{\mathbb{P}}^N}[\frac{\mathcal{E}_{t_j}(\bar{M}^N)}{\mathcal{E}_{t_{j-1}}(\bar{M}^N)} \mathbb{P}^N(\mathcal{A}^N | \mathcal{F}_{t_j})], \end{aligned}$$

where the second equality comes from applying the lemma 3.5.3 of [135]. Following its definition we can write that

$$\frac{\mathcal{E}_{t_j}(\bar{M}^N)}{\mathcal{E}_{t_{j-1}}(\bar{M}^N)} = \mathcal{E}_{t_j}(2(\bar{M}^N - \bar{M}_{t_{j-1}}^N))^{1/2} (\exp(\langle \bar{M}^N \rangle_{t_j} - \langle \bar{M}^N \rangle_{t_{j-1}}))^{1/2}.$$

As we have proven that $\mathcal{E}(\bar{M}^N)$ is a martingale under $\bar{\mathbb{P}}^N$, then, for $t \geq t_{j-1}$ we have that $\mathcal{E}_{t_j}(2(\bar{M}^N - \bar{M}_{t_{j-1}}^N))$ is a martingale and

$$\mathbb{E}_{\bar{\mathbb{P}}^N}[\mathcal{E}_{t_j}(2(\bar{M}^N - \bar{M}_{t_{j-1}}^N))^{1/2}] = 1.$$

Gathering all the estimates and applying the Cauchy-Schwarz's inequality twice, we obtain

$$\begin{aligned} \mathbb{E}_{\bar{\mathbb{P}}^N}[\frac{\mathcal{E}_{t_j}(\bar{M}^N)}{\mathcal{E}_{t_{j-1}}(\bar{M}^N)} \mathbb{P}^N(\mathcal{A}^N | \mathcal{F}_{t_j})] &\leq \mathbb{E}_{\bar{\mathbb{P}}^N}[\mathbb{P}^N(\mathcal{A}^N | \mathcal{F}_{t_j})^2 \exp(\langle \bar{M}^N \rangle_{t_j} - \langle \bar{M}^N \rangle_{t_{j-1}})]^{1/2} \\ &\leq \mathbb{E}_{\bar{\mathbb{P}}^N}[\mathbb{P}^N(\mathcal{A}^N | \mathcal{F}_{t_j})^4]^{1/4} \mathbb{E}_{\bar{\mathbb{P}}^N}[\exp 2(\langle \bar{M}^N \rangle_{t_j} - \langle \bar{M}^N \rangle_{t_{j-1}})]^{1/4} \end{aligned}$$

Then, since $\mathbb{P}^N(\mathcal{A}^N | \mathcal{F}_{t_j}) \leq 1$, we get

$$\mathbb{E}_{\bar{\mathbb{P}}^N}[\mathbb{P}^N(\mathcal{A}^N | \mathcal{F}_{t_{j-1}})] \leq \mathbb{E}_{\bar{\mathbb{P}}^N}[\mathbb{P}^N(\mathcal{A}^N | \mathcal{F}_{t_j})]^{1/4} \mathbb{E}_{\bar{\mathbb{P}}^N}[\exp 2(\langle \bar{M}^N \rangle_{t_j} - \langle \bar{M}^N \rangle_{t_{j-1}})]^{1/4}$$

We conclude by repeating the argument over the elements of the subdivision. □

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RÉSUMÉ

Cette thèse est consacrée à l'étude des estimations asymptotiques pour les équations aux dérivées partielles (EDP) dans deux directions : l'analyse du comportement à long terme des solutions des EDP et l'étude de la limite lorsque le nombre d'individus inclus dans un système dynamique devient grand. Dans une première partie de la thèse, nous abordons la première de ces estimations. Pour cela, nous étudions successivement le problème des éléments propres en étendant le théorème de Krein-Rutman, puis la géométrie de la première valeur propre et enfin, ses implications pour le comportement asymptotique en temps long. En particulier, nous étendons la théorie de Doeblin-Harris pour les semigroupes non-conservateurs, en obtenant des résultats de couverture exponentielle avec un taux constructif. Nous appliquons ces résultats à quatre exemples : des modèles de diffusion, un modèle cinétique de Fokker-Planck, un modèle de sélection-mutation et un modèle de temps écoulé pour une population neuronale. Dans une seconde partie, nous étudions l'existence d'un résultat de concentration de type inégalité de Bernstein pour les modèles position-vitesse que nous appliquons ensuite à trois problèmes : à l'estimation des paramètres du modèle de FitzHugh-Nagumo, à l'estimation non paramétrique de coefficients de modèles généraux et, enfin, un travail exploratoire pour la construction d'un test statistique de la connectivité entre particules dans un système dynamique.

MOTS CLÉS

PDE, neurosciences, comportement asymptotique, comportement à long terme, inégalité de Bernstein, modèle de diffusion, cinétique de Fokker-Planck

ABSTRACT

This thesis is devoted to the study of asymptotic estimates for partial differential equations (PDEs) in two directions, the analysis of the long time behavior of PDEs solutions and the study of the limit when the number of individuals included in a dynamical system becomes large. In a first part of the thesis, we address the first of these estimates. For this, we study successively the eigenvalue problem by extending the Krein-Rutman theorem, then the geometry of the first eigenvalue and finally, its implications for the asymptotic behavior in long time. In particular, we extend the Doeblin-Harris theory for nonconservative semigroups, obtaining exponential convergence results with constructive rate. We apply these results to four examples: diffusion models, a kinetic Fokker-Planck model, a selection-mutation model and an elapsed time model for neuronal population. In a second part, we study the existence of a concentration result of the Bernstein inequality type for position-velocity models which we then apply to three problems: to the estimation of the parameters of the FitzHugh-Nagumo model, to the nonparametric estimation of general model coefficients and, finally, an exploratory work for the construction of a statistical test of the connectivity between particles in a dynamical system.

KEYWORDS

PDE, neuroscience, asymptotic behavior, Long-term behavior, diffusion model, Bernstein inequality Fokker-Planck kinetics