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Virgile Tapiero

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THÈSE DE DOCTORAT DE

L'UNIVERSITE DE RENNES

ÉCOLE DOCTORALE N° 601

*Mathématiques, Télécommunications, Informatique, Signal, Systèmes,
Électronique*

Spécialité : *Mathématiques*

Par

Virgile TAPIERO

**Propriétés géométriques et ergodiques des
endomorphismes semi-extrémaux de $CP(2)$**

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À mon grand-père Jean Riss,

À mes parents dévoués,

À ma tante Katia

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Chapitre 1

Introduction

Soit f une application holomorphe de l'espace projectif complexe \mathbb{P}^k . Ce travail a pour objectif d'étendre des travaux en dimension $k = 1$ et $k \geq 2$ reliant la régularité du courant de Green T à la régularité de la masse de Monge-Ampère $\mu = T^{\wedge k}$ et aux valeurs des exposants de Lyapunov $\lambda_1 \geq \dots \geq \lambda_k$ de la mesure ergodique μ .

1.1 Dynamique complexe sur les espaces projectifs

1.1.1 Dynamique complexe en dimension 1

Les transformations holomorphes de \mathbb{P}^1 sont les fractions rationnelles θ . Gromov et Misiurewicz-Przytycki ont montré que l'entropie topologique d'une fraction θ de degré $d \geq 2$ est égale à $h_{\text{top}}(\theta) = \text{Log } d$; le principe variationnel stipule que cette quantité est la borne supérieure des entropies métriques des mesures de probabilité invariantes de \mathbb{P}^1 qui sont ergodiques pour θ . Cette borne supérieure est atteinte pour une unique mesure de probabilité μ_θ ergodique. Localement au voisinage de μ_θ —presque tout point z du support de μ_θ , le taux de croissance exponentiel des dérivées $(\theta^n)'(z)$ est mesuré par un coefficient λ_θ , dit de Lyapunov, et donné explicitement par :

$$\lambda_\theta = \int_{\mathbb{P}^1} \text{Log } |\theta'(z)| \, d\mu_\theta(z),$$

où $|\cdot|$ désigne n'importe quelle métrique hermitienne de \mathbb{P}^1 . Un fait remarquable est que $\lambda_\theta > 0$ lorsque $d \geq 2$. C'est une conséquence de l'inégalité de Margulis-Ruelle qui stipule que la somme des exposants de Lyapunov positifs du système ergodique $(\mathbb{P}^1, \theta, \mu_\theta)$ (comptés avec multiplicité) doit être plus grande que l'entropie métrique *i.e.* $2 \max\{\lambda_\theta, 0\} \geq \text{Log } d$. On a alors

$$\lambda_\theta \geq \frac{1}{2} \text{Log } d > 0. \tag{1.1}$$

Il est naturel de s'intéresser au cas d'égalité $\lambda_\theta = \frac{1}{2} \text{Log } d$.

Théorème 1.1.1 (Ledrappier [40], Zdunik [52]). *Soit θ une fraction rationnelle de degré $d \geq 2$. Les propriétés suivantes sont équivalentes :*

1. $\lambda_\theta = \frac{1}{2} \text{Log } d$.

2. $\mu_\theta \ll \text{Leb}_{\mathbb{P}^1} = \omega_{\mathbb{P}^1}$ la forme de Fubini-Study de \mathbb{P}^1 .
3. La dimension de Hausdorff $\text{HD}(\mu_\theta) = \inf_{\mu_\theta(A)=1, A \text{ borélien}} \text{HD}(A)$ est égale à 2.
4. θ est un exemple de Lattès.

Ici $\text{HD}(A)$ désigne la dimension de Hausdorff de $A \subset \mathbb{P}^1$ muni de sa métrique standard. Un exemple de Lattès est une fraction rationnelle θ de degré $d \geq 2$ qui se relève en une application affine D (de partie linéaire $z \mapsto \alpha z$, où $|\alpha| = \sqrt{d}$) sur un tore complexe \mathbb{C}/Λ , au moyen d'un revêtement holomorphe ramifié $\sigma : \mathbb{C}/\Lambda \rightarrow \mathbb{P}^1$ de degré fini, obtenu comme quotient du tore \mathbb{C}/Λ par un groupe fini d'automorphismes. Ces applications ont été très étudiées, voir par exemple l'article de Milnor [44].

1.1.2 Dynamique complexe en dimension ≥ 2

La volonté d'étendre le Théorème 1.1.1 en dimension supérieure $k \geq 2$, est venue de la généralisation de l'inégalité (1.1) sur les exposants de Lyapunov de la mesure d'entropie maximale des applications holomorphes de \mathbb{P}^k .

Nous renvoyons aux livres de Dinh-Sibony et Sibony [26, 48] pour une introduction à la dynamique des applications holomorphes de \mathbb{P}^k . Une telle application s'écrit en coordonnées homogènes $[P_0 : \dots : P_k]$, où les P_i sont des polynômes homogènes de même degré d sur \mathbb{C}^{k+1} , sans zéro commun (excepté l'origine). Le degré topologique de f (cardinal commun des fibres des points non valeurs critiques) est égal à d^k . Si $\omega_{\mathbb{P}^k}$ est la forme de Fubini-Study de \mathbb{P}^k , alors

$$T = \lim_{n \rightarrow +\infty} \frac{1}{d^n} (f^n)^* \omega_{\mathbb{P}^k}$$

définit un $(1, 1)$ -courant positif fermé sur \mathbb{P}^k , appelé le courant de Green de f . Ce courant admet des potentiels locaux plurisousharmoniques de régularité Hölder. En prenant k fois le produit extérieur,

$$\mu := T^{\wedge k} = \lim_{n \rightarrow +\infty} \frac{1}{d^{kn}} ((f^n)^* \omega_{\mathbb{P}^k})^{\wedge k}$$

définit une mesure positive et la normalisation $\int_{\mathbb{P}^k} \omega_{\mathbb{P}^k}^{\wedge k} = 1$ en fait une mesure de probabilité sur \mathbb{P}^k . Comme T admet des potentiels locaux continus, μ intègre les fonctions localement plurisousharmoniques, en particulier la fonction $\text{Log} |\det_{\mathbb{C}} df|$ qui possède une singularité logarithmique au niveau de l'ensemble critique de f . On observe que par construction T et μ vérifient des propriétés d'invariance :

$$f^* T = d \times T \text{ et } f_* \mu = d^k \times \mu.$$

En tant qu'opérateur sur $C^0(\mathbb{P}^k)$, on a $f_* f^* = d^k \times \text{Id}$, ce qui donne immédiatement

$$f_* \mu = \mu.$$

Autrement dit μ est une mesure de probabilité invariante par f . La mesure μ est mélangeante, donc ergodique. Comme en dimension 1, μ est l'unique mesure d'entropie maximale $h_{\text{top}}(f) = \text{Log } d^k$ (toujours calculée par Misiurewicz-Przytycki et Gromov). On l'appelle également la mesure d'équilibre de f . Ses exposants de Lyapunov sont bien définis car μ intègre la fonction $\text{Log} |\det_{\mathbb{C}} df|$.

Théorème 1.1.2 (Briend-Duval [13]). *Les exposants de Lyapunov $\lambda_1 \geq \dots \geq \lambda_k$ de μ vérifient :*

$$\lambda_1 \geq \dots \geq \lambda_k \geq \frac{1}{2} \text{Log } d. \quad (1.2)$$

Contrairement au cas de la dimension 1, la minoration des exposants de μ ne résulte pas de l'inégalité de Margulis-Ruelle. En effet, la minoration $2 \sum_{i=1}^k \max\{\lambda_i, 0\} \geq \text{Log } d^k$ n'implique pas que tous les exposants sont minorés par $\frac{1}{2} \text{Log } d$, elle donne cette information uniquement pour le plus grand exposant λ_1 . La démonstration de Briend-Duval consiste à montrer que le plus petit exposant λ_k est aussi minoré par $\frac{1}{2} \text{Log } d$, les arguments sont basés sur la théorie du pluripotential. On pourra consulter l'article de Briend-Duval [13] et celui de Buff [14]. Une autre preuve, due à de Thélin [21], utilise l'entropie et les variétés instables, sans faire appel à la théorie du pluripotential.

De façon analogue à la dimension 1, la question du cas d'égalité dans (1.2) s'est posée et plusieurs auteurs s'y sont attaqués. La différence principale avec la dimension 1 est l'apparition du courant de Green T (en dimension 1 il coïncide avec la mesure d'équilibre). Le Théorème 1.1.1 se généralise de la façon suivante :

Théorème 1.1.3 (Berteloot-Dupont-Loeb [6, 7, 3]). *Soit f une application holomorphe de \mathbb{P}^k de degré $d \geq 2$. Les propriétés suivantes sont équivalentes :*

1. $\lambda_1 = \dots = \lambda_k = \frac{1}{2} \text{Log } d$.
2. $\mu \ll \text{Leb}_{\mathbb{P}^k} := \omega_{\mathbb{P}^k}^{\wedge k}$
3. T coïncide sur un ouvert non vide de \mathbb{P}^k avec une $(1, 1)$ -forme différentielle lisse et > 0 sur cet ouvert.
4. f est un exemple de Lattès.

Une caractérisation en terme de la dimension de μ a également été démontrée [24]. Un exemple de Lattès de \mathbb{P}^k se définit de façon analogue à la dimension 1 : c'est une application de degré $d \geq 2$ qui se relève en une application affine D (de partie linéaire $\sqrt{d}U$, $U \in U_k(\mathbb{C})$) via un revêtement ramifié fini $\sigma : \mathbb{C}^k / \Lambda^k \rightarrow \mathbb{P}^2$, obtenu comme quotient d'un groupe fini d'automorphismes. Les applications de Lattès apparaissent dans d'autres situations, voir par exemple [25] où Dinh-Sibony classent les paires d'applications permutables de \mathbb{P}^k , et [33] où Favre-Pereira classent les applications de \mathbb{P}^k laissant un feuilletage invariant.

Dans cette thèse nous étudions les applications holomorphes de \mathbb{P}^2 qui possèdent un seul exposant minimal :

Définition 1.1.1 (Application semi-extrémale). *Soit f une application de \mathbb{P}^2 de degré $d \geq 2$, et soit $\lambda_1 \geq \lambda_2$ les exposants de Lyapunov de sa mesure d'équilibre μ . On dit que f est semi-extrémale si*

$$\lambda_1 > \lambda_2 = \frac{1}{2} \text{Log } d.$$

On peut étendre cette définition aux applications de \mathbb{P}^k , voir la Section 3.9. Une difficulté est d'obtenir des informations sur la mesure μ et le courant T des endomorphismes semi-extrémaux, de manière analogue au Théorème 1.1.3. Les premiers progrès sont dus à Dujardin [27] et Dupont-Taffin [32], voir plus bas. On peut aussi mentionner l'article [31], où Dupont-Rogue étudie la régularité du courant T des endomorphismes semi-extrémaux le long de disques holomorphes.

1.1.3 Premières conditions de semi-extrémalité

Le Théorème 1.1.3 montre que la mesure d'équilibre d'un endomorphisme semi-extrémal est singulière par rapport à la mesure de Lebesgue sur \mathbb{P}^2 . Le résultat suivant de Dujardin dégage une condition de régularité pour la mesure μ des applications semi-extrémales.

Théorème 1.1.4 (Dujardin [27]). *Soit f une application holomorphe de degré $d \geq 2$ sur \mathbb{P}^2 , de mesure d'équilibre μ et de courant de Green T .*

1. *Il existe un sous fibré \mathcal{T} de $T\mathbb{P}^2$ df -invariant tel que pour $T \wedge \omega_{\mathbb{P}^2}$ -presque tout x , $\dim(\mathcal{T}_x) \geq 1$ et pour tout $\vec{v} \in \mathcal{T}_x$:*

$$\limsup_{n \rightarrow +\infty} \frac{1}{n} \operatorname{Log} \|df^n \cdot \vec{v}\| \leq \frac{1}{2} \operatorname{Log} d.$$

2. *Si $\mu \ll T \wedge \omega_{\mathbb{P}^2}$ sur un ouvert $U \subset \mathbb{P}^2$ de mesure $\mu(U) > 0$, alors μ possède un exposant de Lyapunov minimal.*
3. *Sous l'hypothèse du point 2 et si de plus $\mu \not\ll \operatorname{Leb}_{\mathbb{P}^2}$ alors l'application f est semi-extrémale.*

Le troisième point est une conséquence du deuxième et du théorème d'Oseledec (voir Théorème 3.8.1). Comme $\lambda_1 > \lambda_2$, ce dernier fournit pour μ -presque tout x un vecteur non nul $\vec{v}_s(x)$ tel que $\limsup_n \frac{1}{n} \operatorname{Log} \|d_x f^n(\vec{v})\|$ est égal à λ_2 si \vec{v} est colinéaire à $\vec{v}_s(x)$, et à λ_1 sinon. On a donc $\lambda_2 \leq \frac{1}{2} \operatorname{Log} d$ et l'inégalité de Briend-Duval permet de conclure. La question suivante est l'analogie de $1. \implies 2.$ du Théorème 1.1.3 :

Question 1.1.2 (Dujardin [27]). *Un exposant minimal implique-t-il $\mu \ll T \wedge \omega_{\mathbb{P}^2}$?*

Dans la deuxième partie de la thèse nous donnons une réponse positive partielle à cette question en utilisant la théorie ergodique différentiable.

1.1.4 Pinceaux de droites et exemples semi-extrémaux

Dans l'article [23], Dinh démontre qu'une application polynomiale P de \mathbb{C}^k (qui se prolonge en une application holomorphe de \mathbb{P}^k) dont le support de sa mesure d'équilibre est laminé par des hypersurfaces réelles, est en fait homogène et induit sur $\mathbb{P}^k \setminus \mathbb{C}^k \simeq \mathbb{P}^{k-1}$ une application de Lattès. En particulier, si $k = 2$, le caractère homogène de P rend possible le calcul des exposants de μ qui sont $\lambda_1 = \operatorname{Log} d$ et $\lambda_2 = \frac{1}{2} \operatorname{Log} d$, P est donc semi-extrémale.

À notre connaissance, les seuls exemples d'applications semi-extrémales sont les applications préservant un pinceau de droites sur lequel l'action est Lattès. Quitte à effectuer un changement de coordonnées linéaire, on peut supposer que le pinceau est donné par les fibres de $\pi[z : w : t] = [z : w]$, $\pi : \mathbb{P}^2 \setminus \{[0 : 0 : 1]\} \longrightarrow \mathbb{P}^1$. Dans ce cas une application f préservant le pinceau est donnée par :

$$f[z : w : t] = [P(z, w) : Q(z, w) : R(z, w, t)]. \quad (1.3)$$

Le théorème 1.1.5 ci-dessous montre que f est semi-extrémale si et seulement si

$$\theta := [P(z, w) : Q(z, w)]$$

est une fraction de Lattès. On ne sait pas si toute application semi-extrémale laisse invariant un pinceau de droites. Dujardin pose dans [27] la question générale suivante.

Question 1.1.3 (Dujardin [27]). *Une application semi-extrémale contient-elle un Lattès 1-dimensionnel en un certain sens ?*

On peut poser la question plus précise suivante, voir [31]. Notons que Dabija-Jonsson [20] ont montré que si un endomorphisme semi-extremal de \mathbb{P}^2 préserve un pinceau de courbes, alors il s'agit d'un pinceau de droites.

Question 1.1.4. *Une application semi-extrémale de \mathbb{P}^2 préserve-t-elle un pinceau de droites ? ou une structure géométrique similaire, comme un feuilletage local ?*

Nous formulons plus bas une autre question (voir Question 1.2.1), où la condition de semi-extrémalité est remplacée par une condition plus forte d'absolue continuité de μ par rapport à T , motivée par la formule $\mu = T \wedge \pi^* \mu_\theta$ du théorème ci-dessous par une de ses conséquences (voir Théorème A.I). La première partie de la thèse donnera des éléments de réponse à cette question (voir Théorème B.I).

Théorème 1.1.5 (Dupont-Taffin [32, Theorem 1.1, Corollary 1.3]). *Soit f une application holomorphe de \mathbb{P}^2 de degré $d \geq 2$. Soit μ sa mesure d'équilibre, soit T son courant de Green T et soient $\lambda_1 \geq \lambda_2$ les exposants de Lyapunov de μ . On suppose que f préserve le pinceau de droites donné par π . Soient μ_θ la mesure d'équilibre de la fraction θ induite sur ce pinceau et soit λ_θ l'exposant de Lyapunov de μ_θ . Alors*

1. $\lambda_1 \geq \text{Log } d$ et $\lambda_\theta \in \{\lambda_1, \lambda_2\}$.
2. $\pi_* \mu = \mu_\theta$ et $\mu = T \wedge \pi^* \mu_\theta$.
3. Si θ est une fraction de Lattès alors $\mu \ll T \wedge \omega_{\mathbb{P}^2}$ et $\lambda_\theta = \lambda_2 = \frac{1}{2} \text{Log } d$.
4. Réciproquement si λ_2 est minimal alors θ est une fraction de Lattès.

L'article antérieur de Jonsson [37] fournit une étude analogue pour les applications produit semi-direct polynomiales dans \mathbb{C}^2 .

1.2 Applications semi-extrémales et feuilletages invariants

1.2.1 Coordonnées de Poincaré-Dulac

Soit f une application holomorphe de \mathbb{P}^2 de degré $d \geq 2$. On suppose que f préserve le pinceau de droites donné par les fibres de $\pi : [z : w : t] \mapsto [z : w]$. Soit θ la fraction rationnelle de degré d telle que $\theta \circ \pi = \pi \circ f$. Soit Ω_θ l'ensemble des points $p \in \mathbb{P}^1$ pour lesquels la mesure d'équilibre μ_θ est lisse et strictement positive sur un voisinage du point p . Lorsque θ est une application de Lattès, Berteloot-Loeb [6] ont montré que $A_\theta := \mathbb{P}^1 \setminus \Omega_\theta$ est fini (cf. Theorem 2.3.2). En gardant à l'esprit la question 1.1.4, nous commençons par donner une version précisée de la formule du théorème 1.1.5 :

$$\mu = T \wedge \pi^* \mu_\theta.$$

Soit $E_\theta := \pi^{-1}(A_\theta)$, et soit

$$R_{\mu,\theta} := \{a \in \text{Supp}(\mu) \setminus E_\theta, n - \text{périodique répulsif}, \frac{1}{n} \text{Log} |\det_{\mathbb{C}} d_a f^n| > \text{Log } d\}$$

La notion de coordonnées de Poincaré-Dulac est rappelée à la Section 2.2.

Théorème A.I. *Supposons que f préserve le pinceau de droites donné par π et que f induise sur ce pinceau une fraction de Lattès θ . Alors l'ensemble $R_{\mu,\theta}$ est dense dans $\text{Supp}(\mu) \setminus E_\theta$ et pour tout $a \in R_{\mu,\theta}$ de période $n \geq 1$, il existe (Z_a, W_a) des coordonnées de Poincaré-Dulac pour f^n au point a telles que :*

$$\mu = T \wedge dd^c |W_a|^2 \text{ sur } (\mathbb{P}^2, a). \quad (1.4)$$

1.2.2 Feuilletage invariant au voisinage de l'ensemble de Julia

Question 1.2.1. *La seule donnée de la formule (1.4), pour un seul point périodique répulsif $a \in \text{Supp}(\mu)$, implique-t-elle que f préserve un pinceau de droites ? ou une structure géométrique comme un feuilletage local ?*

Nous donnons une réponse partielle avec le Théorème B.I et les Corollaires D.I et E.I ci-dessous. L'ensemble exceptionnel $\mathcal{E}(f)$ de f est défini par :

$$\mathcal{E}(f) := \mathbb{P}^2 \setminus \{x, \mu_{x,n} \xrightarrow[n \rightarrow +\infty]{} \mu\}, \quad \mu_{x,n} := \frac{1}{d^{2n}} \sum_{y \in f^{-n}(x)} \delta_y,$$

où la limite est au sens de la topologie faible. $\mathcal{E}(f)$ est le plus grand ensemble algébrique propre, totalement invariant par f , voir [26]. Pour une application générique de \mathbb{P}^2 l'ensemble exceptionnel est vide. Pour une application préservant un pinceau de droites, $\mathcal{E}(f)$ contient le centre du pinceau.

Dans le théorème ci-dessous, un feuilletage \mathcal{F} sur un ouvert \mathcal{V} de \mathbb{P}^2 est dit invariant par f^N si $f^{-N}(\mathcal{V}) \cap \mathcal{V}$ n'est pas vide et si $(f^N)^*\mathcal{F} = \mathcal{F}$ sur $f^{-N}(\mathcal{V}) \cap \mathcal{V}$. Les sections 2.8.1 et 2.8.2 donnent des rappels sur les feuilletages holomorphes et la signification du tiré en arrière $(f^N)^*\mathcal{F}$.

Théorème B.I (Feuilletage invariant). *Soit f une application de \mathbb{P}^2 de degré $d \geq 2$ de mesure d'équilibre μ et de courant de Green T . Supposons que f vérifie :*

1. *Il existe un point N -périodique répulsif $a \in \text{Supp}(\mu)$ et des coordonnées de Poincaré-Dulac (Z_a, W_a) pour f^N au point a telles que :*

$$\mu = T \wedge dd^c |W_a|^2 \text{ sur } (\mathbb{P}^2, a).$$

2. *La mesure trace $\sigma_T = T \wedge \omega_{\mathbb{P}^2}$ n'est pas absolument continue par rapport à μ sur tout ouvert chargé par μ .*
3. *L'ensemble exceptionnel $\mathcal{E}(f)$ ne rencontre pas le support de la mesure μ .*

Alors il existe un feuilletage holomorphe \mathcal{F} sur un voisinage \mathcal{V} du support de la mesure μ tel que \mathcal{F} est invariant par f^N sur \mathcal{V} .

Ce théorème est le résultat principal de la première partie de ma thèse. Notons que pour obtenir l'hypothèse 2., il suffit de demander que σ_T ne soit pas absolument continue par rapport à μ sur un ouvert chargé par μ (voir Remarque 2.1.6), nous remercions T.-C. Dinh de nous avoir indiqué cette possibilité. La démonstration est résumée dans la Section 1.2.3. Un ingrédient majeur est le théorème de recollement (Patching Theorem) énoncé plus bas. Effectuons la décomposition de Radon-Nikodym de la mesure trace $\sigma_T = T \wedge \omega_{\mathbb{P}^2}$ par rapport à la mesure μ :

$$\sigma_T = \mu^a + \mu^s.$$

La mesure μ^a (resp. μ^s) est absolument continue (resp. singulière) par rapport à μ .

Théorème B.I.bis (De recollement). *Soit U une carte connexe de \mathbb{P}^2 munie de deux systèmes de coordonnées holomorphes (Z_1, W_1) et (Z_2, W_2) . On suppose que ces systèmes de coordonnées vérifient la contrainte :*

$$T \wedge dd^c |W_i|^2 \ll \mu \text{ sur } U, \quad i \in \{1, 2\}.$$

Si de plus $\mu(U) > 0$ et $\mu^s(U) > 0$, alors il existe un recollement modulo $\mathcal{O}^(U)$ des 1-formes holomorphes dW_1 et dW_2 , c'est-à-dire :*

$$\exists \beta \in \mathcal{O}^*(U) : dW_2 = \beta dW_1 \text{ sur } U.$$

La preuve repose sur le caractère diffus du courant T : il est plus gros qu'un courant d'intégration sur un ensemble analytique. T.-C. Dinh nous a signalé un autre argument, voir la Remarque 2.4.6, nous le remercions. Des exemples d'applications satisfaisant les hypothèses du Théorème B.I sont donnés par les applications semi-extrémales de la forme :

$$f = [P(z, w) : Q(z, w) : t^d] \text{ avec } \theta = [P(z, w) : Q(z, w)] \text{ Lattès.} \quad (1.5)$$

Le Théorème A.I implique le premier point. D'après [31], le point $[0 : 0 : 1]$ est un point fixe super attractif de f dont le bassin d'attraction est borné dans \mathbb{C}^2 et dont le bord coïncide avec le support de μ . Le support de T est le complémentaire dans \mathbb{P}^2 de ce bassin d'attraction, par conséquent le support de T et celui de μ ne coïncident sur aucun ouvert chargé par μ , cela assure le deuxième point. Enfin, il est connu que pour ce type d'application l'ensemble exceptionnel ne rencontre pas le support de μ , voir la Proposition 2.1.7.

1.2.3 Résumé de la preuve du Théorème B.I

Preuve du Théorème B.I.bis

Soient (Z_1, W_1) and (Z_2, W_2) des systèmes de coordonnées holomorphes définis sur un ouvert connexe U tel que $\mu(U) > 0$, $\mu^s(U) > 0$ et

$$(T \wedge dd^c |W_i|^2)|_U \ll \mu|_U, \quad i \in \{1, 2\}. \quad (1.6)$$

On veut montrer qu'il existe $\beta \in \mathcal{O}^*(U)$ telle que $dW_2 = \beta dW_1$ sur U .

Le principe de la preuve est de décomposer la mesure $T \wedge dd^c |Z_1|^2$ en une somme de deux mesures, une première absolument continue par rapport à μ et une deuxième absolument continue par rapport à μ^s . En supposant que $\mu^s(U) > 0$, cette deuxième mesure peut être renormalisée en une mesure de probabilité sur U , supposons pour simplifier que $\mu^s(U) = 1$ et que $T \wedge dd^c |Z_1|^2 = \mu^a + \mu^s$ (avec $\mu^a \ll \mu$) pour simplifier notre propos dans ce résumé. L'idée principale consiste ensuite à utiliser une borne inférieure sur la dimension ponctuelle de la trace (cf. Proposition 2.4.1) :

$$\forall x \in \mathbb{P}^2, \quad \underline{d}_{\sigma_T}(x) \geq 2 + \gamma_0,$$

ce qui donne une borne inférieure sur la dimension de Hausdorff des sous-ensembles de mesure positive pour μ^s (cf. Lemme 2.4.2) :

$$\inf_{\mu^s(A) > 0} \text{HD}(A) \geq 2 + \gamma_0 > 2.$$

Cette inégalité nous permet alors de démontrer un *lemme d'annulation*, cf. Lemme 2.4.3, pour toute fonction holomorphe h sur U :

$$(\mu^s \{x \in U : h(x) = 0\} = 1) \implies (h \equiv 0 \text{ on } U). \quad (1.7)$$

La suite de la démonstration consiste à décomposer la 1-forme dW_2 de la façon suivante :

$$dW_2 = \alpha dZ_1 + \beta dW_1, \quad (1.8)$$

et d'utiliser le lemme d'annulation pour montrer que la fonction α apparaissant dans cette décomposition est nulle sur U . On obtient alors le résultat recherché. Pour ce faire on utilise (1.8) pour calculer $dd^c|W_2|^2$ et on prend le produit extérieur avec le courant de Green :

$$T \wedge dd^c|W_2|^2 = |\alpha|^2 T \wedge dd^c|Z_1|^2 + 2\text{Re} \left[\alpha \bar{\beta} T \wedge \frac{i}{2} dZ_1 \wedge d\bar{W}_1 \right] + |\beta|^2 T \wedge dd^c|W_1|^2.$$

En utilisant l'inégalité de Cauchy-Schwarz, on déduit de cette égalité de mesures et de l'expression $\mu^s = T \wedge dd^c|Z_1|^2 - \mu^a$ que pour tout borélien $A \subset U$ sur lequel α et β sont bornées on a :

$$\int_A |\alpha|^2 d\mu^s \leq (T \wedge dd^c|W_2|^2)(A) + 2 \sup_A |\alpha\beta| \sqrt{(T \wedge dd^c|Z_1|^2)(A)} \sqrt{(T \wedge dd^c|W_1|^2)(A)}.$$

Enfin en utilisant l'hypothèse (1.6) on déduit de cette inégalité que $|\alpha|^2 \mu^s \ll \mu$ et donc nécessairement $\mu^s \{\alpha = 0\} = 1$. Par (1.7) on a donc $\alpha = 0$ sur U et le résultat recherché est démontré.

Preuve du Théorème B.I

Ce résultat de recollement de 1-formes coordonnées est utilisé dans la Section 2.5.4 pour étudier les fibres d'une application développante σ associée à des coordonnées de Poincaré-Dulac d'un cycle répulsif de l'application f .

Supposons désormais que f vérifie les hypothèses du Théorème B.I. Notons $J := \text{Supp}(\mu)$ le petit ensemble de Julia de f . Considérons des points x et y non critiques de σ et supposons que les images $p = \sigma(x)$ et $q = \sigma(y)$ appartiennent à J . Soient alors $U_x \ni x$ et $U_y \ni y$ des ouverts tels que $\sigma_p := \sigma|_{U_x}$ et $\sigma_q := \sigma|_{U_y}$ soient des biholomorphismes sur leurs images. On note $\Omega := \sigma(U_x) \cap \sigma(U_y)$, si on suppose que cet ensemble est non vide on peut alors considérer l'application suivante :

$$\phi := (\sigma_q^{-1} \circ \sigma_p) : \sigma_p^{-1}(\Omega) \longrightarrow \sigma_q^{-1}(\Omega).$$

Cette application a été étudiée avec succès par Berteloot-Loeb [7] qui ont prouvé que lorsque le courant de Green T est une $(1,1)$ -forme différentielle lisse et > 0 sur Ω , alors ϕ est la restriction sur $\sigma_p^{-1}(\Omega)$ d'une isométrie affine de \mathbb{C}^2 . Dans ce cas Berteloot-Loeb prouvèrent qu'un groupe d'isométries affines de \mathbb{C}^2 agit transitivement sur les fibres de σ , et ils en déduisirent que f est une application de Lattès de \mathbb{P}^2 au sens de la définition donnée plus haut dans cette introduction.

Dans le cadre de cette thèse on ne dispose pas de l'hypothèse de régularité de Berteloot-Loeb et on ne conclut pas que f est une application de Lattès. Dans notre

contexte nous utilisons le théorème de recollement (Théorème B.I.bis) pour montrer que l'application ϕ écrite sous la forme

$$\phi(z, w) = (A(z, w), B(z, w)),$$

doit vérifier que la seconde fonction coordonnée B est indépendante de la variable z :

$$B(z, w) = B(w). \quad (1.9)$$

Cette propriété implique que les images par σ_p et σ_q des lignes complexes horizontales $\{w = cste\}$ de $\sigma_p^{-1}(U_x)$ et de $\sigma_q^{-1}(U_y)$, se recollent sur Ω pour former un feuilletage (non singulier). Ce feuilletage est décrit par les 1-formes $(\sigma_p)_*dw$ et $(\sigma_q)_*dw$ qui sont égales modulo une fonction de $\mathcal{O}^*(\Omega)$. Précisément le théorème de recollement B.I.bis est utilisé pour montrer que le recollement modulo $\mathcal{O}^*(\Omega)$ des formes $(\sigma_p)_*dw$ et $(\sigma_q)_*dw$. En effet en exploitant l'hypothèse 1. du Théorème B.I nous prouvons (cf. Proposition 2.5.4) que

$$\mu = T \wedge dd^c|w \circ \sigma_p^{-1}|^2 \text{ et } \mu = T \wedge dd^c|w \circ \sigma_q^{-1}|^2 \text{ sur } \Omega,$$

et le théorème de recollement B.I.bis peut s'appliquer. On constate alors que la différentielle dB est égale à dw (modulo une fonction de $\mathcal{O}(\Omega)$) et on obtient (1.9).

Dans la suite de la démonstration nous fixons un bidisque $D_R := \mathbb{D}_R^2$ de rayon $R > 0$ suffisamment grand pour avoir $\sigma(D_R) \supset J$, un tel bidisque existe toujours lorsque l'on suppose que le point fixe a associé à σ appartient à J et que $J \cap \mathcal{E}(f) = \emptyset$ (hypothèse 3. du Théorème B.I), cf. Proposition 2.2.5. L'ensemble des valeurs critiques de $\sigma|_{D_R} : D_R \rightarrow \sigma(D_R)$ est noté E_R . En appliquant ce que nous venons de voir ci-dessus aux points p appartenant à l'ensemble $J \setminus E_R$, nous obtenons une famille $(V_p)_{p \in J \setminus E_R}$ de voisinages ouverts $V_p \ni p$ feuilletés par des feuilletages réguliers \mathcal{F}_p donnés par des 1-formes $(\sigma_p)_*dw$. Étant donnés deux points p et q de $J \setminus E_R$, la propriété (1.9) sur l'application de "changement de fibres" ϕ nous assure que les deux feuilletages \mathcal{F}_p et \mathcal{F}_q possèdent les mêmes droites tangentes sur les composantes connexes de $V_p \cap V_q$ chargées par μ . Par conséquent les deux feuilletages coïncident sur ces composantes connexes. Les feuilletages $(\mathcal{F}_p)_{p \in J \setminus E_R}$ se recollent donc en un feuilletage au voisinage de $J \setminus E_R$. Il s'agit ensuite de compléter ce feuilletage au niveau des points $q \in J \cap E_R$ qui sont des valeurs singulières de $\sigma|_{D_R}$.

Pour attribuer un voisinage feuilleté W_q à un point $q \in J \cap E_R$, l'idée consiste à utiliser le calcul des tangentes des feuilletages réguliers. En effet pour tout $p \in J \setminus E_R$, on a :

$$T_{\sigma(x)}\mathcal{F}_p = \mathbb{C} \cdot (d_x\sigma \cdot (1, 0)),$$

pour tout x appartenant à un ouvert spécial $IB_R(p) \subset D_R \setminus (\text{Crit } \sigma)$ vérifiant que $\sigma(IB_R(p)) = V_p$. Pour créer un feuilletage \mathcal{F}_q sur un ouvert $W_q \ni q$ qui va coïncider avec les feuilletages \mathcal{F}_p , il est naturel de créer un champ de vecteurs v_q sur un voisinage $W_q \ni q$ à partir de la différentielle $d\sigma \cdot (1, 0)$, car alors v_q sera tangent aux feuilletages \mathcal{F}_p et induira un feuilletage \mathcal{F}_q qui coïncidera avec le feuilletage \mathcal{F}_p (au voisinage des points de $J \cap V_p \cap W_q$). Pour construire v_q on moyenne sur les fibres de $\sigma|_{D_R}$ l'application $x \mapsto d_x\sigma \cdot (1, 0)$. Du fait que les fibres de $\sigma|_{D_R}$ sont finies on parvient à rendre rigoureuse cette construction. Le champ de vecteurs $v_q : W_q \rightarrow T\mathbb{P}^2$ engendre alors un feuilletage \mathcal{F}_q qui coïncide avec tout \mathcal{F}_p , $p \in J \setminus E_R$, sur les composantes

connexes de $V_p \cap W_q$ chargées par μ . Étant donné un autre point $q' \in J \cap E_R$ on peut vérifier que les feuilletages \mathcal{F}_q et $\mathcal{F}_{q'}$ coïncident sur les composantes connexes de $W_q \cap W_{q'}$ qui sont chargées par μ .

Une fois construit ces voisinages feuilletés $(V_p)_{p \in J \setminus E_R}$ et $(W_q)_{q \in J \cap E_R}$ comme ci-dessus, on recouvre par compacité l'ensemble J par un nombre fini d'entre eux :

$$J \subset \left(\bigcup_{i=1}^N V_{p_i} \right) \cup \left(\bigcup_{j=1}^M W_{q_j} \right).$$

Il est alors possible d'extraire un sous-recouvrement \mathcal{V} :

$$J \subset \mathcal{V} \subset \left(\bigcup_{i=1}^N V_{p_i} \right) \cup \left(\bigcup_{j=1}^M W_{q_j} \right),$$

de tel sorte que sur \mathcal{V} les feuilletages $(\mathcal{F}_{p_i})_{i=1, \dots, N}$ et $(\mathcal{F}_{q_j})_{j=1, \dots, M}$ se recollent en un feuilletage \mathcal{F} .

Pour terminer la démonstration du Théorème B.I il reste alors à montrer que \mathcal{F} est f^N -invariant sur \mathcal{V} . Quitte à remplacer f par f^N on peut supposer que $N = 1$ pour la fin de la démonstration. Pour cette dernière étape de la preuve on démontre que le feuilletage $\sigma^* \mathcal{F}$ coïncide avec le feuilletage en horizontales $\{w = cste\}_{cste \in \mathbb{C}}$ de \mathbb{C}^2 que l'on note \mathcal{F}_w , cf. Lemme 2.5.22. Pour justifier que $f^* \mathcal{F}$ coïncide avec \mathcal{F} (définition de l'invariance par f de \mathcal{F}), on se ramène à justifier que le feuilletage $\sigma^* \mathcal{F}$ est invariant par l'application polynomial $D(z, w) = (\chi_1 z + cw^k, \chi_2 w)$ qui vérifie $\sigma \circ D = f \circ \sigma$. Mais cette dernière propriété est automatique du fait que le feuilletage \mathcal{F}_w est invariant par l'application $D(z, w)$ puisque sa deuxième fonction coordonnée est une fonction linéaire de la variable w seule. La démonstration du Théorème B.I est alors terminée.

1.2.4 Applications du Théorème B.I

En utilisant la structure algébrique des feuilletages holomorphes de \mathbb{P}^2 , on observe :

Proposition C.I. *Soit f une application de \mathbb{P}^2 de degré $d \geq 2$ et soit \mathcal{F} un feuilletage f -invariant défini au voisinage d'un point fixe de f . Si \mathcal{F} admet une extension \mathcal{F}' à \mathbb{P}^2 , alors \mathcal{F}' est aussi f -invariant. Si de plus f est semi-extrémale alors dans des coordonnées homogènes adaptées, \mathcal{F}' est le pinceau de droite donné par $\pi[z : w : t] = [z : w]$. Dans ce cas l'application induite par f sur \mathbb{P}^1 est une application de Lattès.*

La preuve utilise le fait que les feuilletages de \mathbb{P}^2 qui sont caractérisés par la donnée d'une 1-forme polynomiale sur une carte affine, cf. Théorème 2.8.7. La seconde assertion est une conséquence de la classification de Favre-Pereira des couples (\mathcal{F}, f) , où \mathcal{F} est un feuilletage holomorphe singulier sur \mathbb{P}^2 et f une application de \mathbb{P}^2 qui préserve \mathcal{F} , voir le Théorème 2.8.5 et son Corollaire 2.8.6.

Avec les notations du Théorème B.I, en appliquant la Proposition C.I à f^N et au point fixe a on obtient :

Corollaire D.I. *Soit f une application satisfaisant les hypothèses du Théorème B.I. Soit \mathcal{F} un feuilletage holomorphe f^N -invariant défini sur un voisinage \mathcal{V} de $\text{Supp}(\mu)$ donné par ce Théorème. Si \mathcal{F} admet une extension \mathcal{F}' à \mathbb{P}^2 , alors*

1. \mathcal{F}' est un pinceau de droites.
2. Le support de μ n'est pas égal à \mathbb{P}^2 .

Pour le second point, si le support de μ était total, $\mathcal{F} = \mathcal{F}'$ serait un pinceau de droites préservé par (une itérée de) f . Le centre du pinceau serait alors un point fixe super attractif, nécessairement inclus dans le complémentaire du support de μ , contradiction.

Il serait intéressant d'étudier la question plus généralement :

Question 1.2.2. *La mesure d'équilibre d'une application semi-extrémale peut-elle avoir un support total dans \mathbb{P}^2 ?*

La Proposition C.I et le Corollaire D.I soulèvent la question de l'existence du prolongement du feuilletage \mathcal{F} . Une façon de faire est de passer par des propriétés Stein, voir la Section 2.8.4 :

Corollaire E.I. *Soit f une application de degré $d \geq 2$ sur \mathbb{P}^2 et soit μ sa mesure d'équilibre. Soit \mathcal{F} un feuilletage holomorphe défini sur un voisinage \mathcal{V} de $\text{Supp}(\mu)$. Si l'une des deux hypothèses suivantes est vérifiée*

4. *le support de μ est connexe et de complémentaire Stein dans \mathbb{P}^2 ,*
5. *le support de μ et le support de T coïncident,*

alors \mathcal{F} s'étend en un feuilletage holomorphe \mathcal{F}' sur \mathbb{P}^2 . En particulier si f vérifie les hypothèses du Théorème B.I et si l'hypothèse 4. (ou 5.) est vérifiée, alors f^N préserve un pinceau de droites, avec N la période du point périodique répulsif a du Théorème B.I.

Ce corollaire donne une réponse partielle à la question 1.2.1. Notons que l'hypothèse 5. est plus forte que l'hypothèse 4., du fait que le support de T est connexe et de complémentaire Stein dans \mathbb{P}^2 , voir la Section 2.8.4. Sous l'hypothèse 4. on utilise le théorème de prolongement suivant : un feuilletage sur un voisinage \mathcal{V} d'un compact connexe K de \mathbb{P}^2 de complémentaire Stein admet une unique extension à \mathbb{P}^2 , résultat dû à Lins Neto [42], voir Theorem 2.8.8. On pourra aussi consulter l'article de Canales [16].

Des exemples d'applications vérifiant les hypothèses 1., 3. du Théorème B.I et 5. du Corollaire E.I sont données par la *famille de Desboves élémentaire*, introduite et étudiée par Bonifant, Dabija et Milnor [10] (voir aussi [9]). C'est une famille $(f_\lambda)_{\lambda \in \mathbb{C}^*}$ de degré 4 préservant un pinceau de droites :

$$f_\lambda = [-z(z^3 + 2w^3) : w(2z^3 + w^3) : t(w^3 - z^3 + \lambda(z^3 + w^3 + t^3))], \quad \lambda \in \mathbb{C}^*. \quad (1.10)$$

Bianchi-Taffin [8] ont étudié cette famille du point de vue des bifurcations, ils ont montré que le lieu de bifurcation de cette famille est égal à \mathbb{C}^* . En fait $[-z(z^3 + 2w^3) : w(2z^3 + w^3)]$ est un exemple de Lattès, les f_λ sont donc semi-extrémales. Nous savons que les points 1., 3. et 5. sont vérifiés via le Théorème A.I et les résultats [8, Lemme A.1 & Proposition 6.16]. Pour le point 2. nous posons la question :

Question 1.2.3. *Que peut-on dire des applications f de courant de Green T et de mesure d'équilibre μ vérifiant $T \wedge \omega_{\mathbb{P}^2} \ll \mu$ sur un ouvert chargé par μ ? f est-elle une application de Lattès ?*

1.3 Absolue continuité des applications semi-extrémales

Dans la deuxième partie de la thèse nous revenons sur la première question de R. Dujardin, à savoir peut-on montrer que la mesure d'équilibre d'une application semi-extrémale est absolument continue par rapport à la trace du courant de Green ? Nous donnons une réponse positive partielle à cette question en adoptant un point de vue de dynamique différentiable.

1.3.1 Rappels de théorie ergodique

Extension naturelle. Fixons nous une application de \mathbb{P}^2 de degré $d \geq 2$ de mesure d'équilibre μ et de courant de Green T . On rappelle que (\mathbb{P}^2, f, μ) est un système dynamique ergodique et même mélangeant d'entropie topologique $\text{Log } d^2$. Un défaut néanmoins de ce système dynamique est qu'il n'est pas inversible, ce qui pose des difficultés pour utiliser les techniques de la dynamique différentiable, notamment la théorie de Pesin. Pour pallier à ce défaut on utilise l'extension naturelle de \mathbb{P}^2 . Il s'agit de l'espace des orbites

$$\widehat{\mathbb{P}^2} := \{\widehat{x} := (x_n)_{n \in \mathbb{Z}} : f(x_n) = x_{n+1}, \forall n \in \mathbb{Z}\}.$$

C'est un sous-ensemble compact de $(\mathbb{P}^2)^\mathbb{Z}$ sur lequel f agit naturellement comme un homéomorphisme via le décalage (shift) à gauche :

$$\widehat{f}(\widehat{x}) := (x_{n+1})_{n \in \mathbb{Z}}.$$

Par le théorème de Kolmogorov la mesure μ se relève à $\widehat{\mathbb{P}^2}$ de façon unique en une mesure de probabilité \widehat{f} -invariante vérifiant $(\pi_0)_* \widehat{\mu} = \mu$, où π_0 est la projection naturelle $\pi_0(\widehat{x}) = x_0$. Pour le théorème de Kolmogorov et les propriétés de l'extension naturelle nous renvoyons le lecteur à [19, Chapter 10]. Elle préserve la plupart des propriétés dynamiques du système de départ, ainsi $(\widehat{\mathbb{P}^2}, \widehat{f}, \widehat{\mu})$ est aussi un système ergodique et mélangeant d'entropie topologique $\text{Log } d^2$. L'entropie métrique $h_{\widehat{\mu}}(\widehat{f})$ vérifie :

$$h_{\widehat{\mu}}(\widehat{f}) = \text{Log } d^2. \quad (1.11)$$

Partitions mesurables. L'entropie peut se calculer à l'aide de *partitions mesurables*. Soit \mathcal{M} la tribu des boréliens de $\widehat{\mathbb{P}^2}$ complétée par rapport à la mesure $\widehat{\mu}$. Dans le contexte de la théorie ergodique, une partition ξ de $\widehat{\mathbb{P}^2}$ est une partition d'un ensemble $E \in \mathcal{M}$ de mesure totale. Les atomes de la partition ξ doivent également appartenir à \mathcal{M} . Pour le caractère mesurable d'une partition ξ , nous renvoyons à l'article de Rokhlin [46] pour une définition précise. Une propriété importante des partitions mesurables est que l'on peut désintégrer les mesures sur les atomes de ces partitions, c'est le théorème de Rokhlin [46, § 3.1]. Soit ξ une partition mesurable, le théorème de Rokhlin stipule que pour $\widehat{\mu}$ -presque tout \widehat{x} il existe une mesure de probabilité $\mu_{\xi, \widehat{x}}$ sur $\xi_{\widehat{x}}$, et que pour toute fonction continue φ sur $\widehat{\mathbb{P}^2}$:

$$\int_{\widehat{\mathbb{P}^2}} \varphi \, d\widehat{\mu} = \int_{\widehat{\mathbb{P}^2}} \left(\int_{\xi_{\widehat{x}}} \varphi(\widehat{y}) \, d\mu_{\xi, \widehat{x}}(\widehat{y}) \right) d\widehat{\mu}(\widehat{x}).$$

Le théorème de Kolmogorov-Sinaï. Étant données deux partitions ξ et ζ , en intersectant deux à deux leurs atomes respectifs on obtient une partition plus fine, notée $\xi \vee \zeta$, appelée le joint de ξ et ζ . Cette opération préserve le caractère mesurable des partitions : si $(\xi_n)_n$ est une suite de partitions mesurables alors $\bigvee_n \xi_n$ est encore une partition mesurable, voir [46]. Le décalage \widehat{f} préserve le caractère mesurable d'une partition par image directe et par image inverse. À une partition mesurable ξ on associe la partition mesurable $\xi_{\widehat{f}} := \bigvee_{p \in \mathbb{Z}} \widehat{f}^p(\xi)$. La partition ξ est *génératrice d'entropie* si les atomes de $\xi_{\widehat{f}}$ sont des points.

Théorème 1.3.1 (Kolmogorov-Sinaï [47, §9]). *Si la partition mesurable ξ est génératrice d'entropie et si son entropie est finie, alors $h_{\widehat{\mu}}(f) = h_{\widehat{\mu}}(\widehat{f}, \xi)$.*

L'entropie de ξ est la quantité $\int_{\widehat{\mathbb{P}^2}} -\text{Log } \widehat{\mu}(\xi_{\widehat{x}}) d\widehat{\mu}(\widehat{x})$. Le nombre $h_{\widehat{\mu}}(\widehat{f}, \xi)$ est l'entropie conditionnelle $H[\xi | \bigvee_{p \geq 1} \widehat{f}^{-p}(\xi)]$, où $H(\xi | \zeta) := \int_{\widehat{\mathbb{P}^2}} -\text{Log } \mu_{\zeta, \widehat{x}}(\xi_{\widehat{x}}) d\widehat{x}$.

Partitions décroissantes. Une partition mesurable η est décroissante si :

$$\forall \widehat{x} - \widehat{\mu} - p.p., \left(\widehat{f}^{-1} \eta \right)_{\widehat{x}} \subset \eta_{\widehat{x}}.$$

Une façon de créer une partition décroissante à partir d'une partition ξ est de prendre :

$$\eta := \bigvee_{p \geq 0} \widehat{f}^p(\xi).$$

Si la partition $\xi_n := \widehat{f}^{-1} \xi \vee \dots \vee \widehat{f}^{-n} \xi$ vérifie les propriétés suivantes :

$$\xi_n \text{ est génératrice d'entropie finie pour } \widehat{f}^{-n} \text{ et } H \left[\widehat{f}^{-n} \eta \mid \eta \right] = h_{\widehat{\mu}}(\widehat{f}^{-n}, \xi_n), \quad (1.12)$$

alors le théorème de Kolmogorov-Sinaï, la formule $h_{\widehat{\mu}}(\widehat{f}^{-n}) = n \times h_{\widehat{\mu}}(\widehat{f})$ et la formule (1.11) donnent :

$$\int_{\widehat{\mathbb{P}^2}} -\text{Log } \mu_{\eta, \widehat{x}} \left(\widehat{f}^{-n} \eta \right)_{\widehat{x}} d\widehat{\mu}(\widehat{x}) = \text{Log } d^{2n}. \quad (1.13)$$

Notons pour terminer cette section que la notion de partition mesurable a été utilisée par Bedford-Lyubich-Smillie [1] et par Cantat [17] pour établir l'unicité de la mesure d'entropie maximale pour les automorphismes polynomiaux de \mathbb{C}^2 et les automorphismes des surfaces K3.

1.3.2 La démonstration classique de l'absolue continuité

La preuve de Ledrappier en dimension 2. Expliquons brièvement la démonstration de $1. \implies 2.$ des Théorèmes 1.1.1 et 1.1.3. On trouve les détails dans les articles de Ledrappier [40] et de Dupont [28]. On donne ici les arguments pour une application f de \mathbb{P}^2 de degré $d \geq 2$. La première étape consiste à montrer l'existence d'une partition mesurable décroissante η vérifiant (1.12) et par suite vérifiant (1.13). Soient $\lambda_1 \geq \lambda_2$ les exposants de Lyapunov de la mesure d'entropie maximale μ . La formule de changement de variable pour la mesure de Lebesgue et des arguments classiques de théorie ergodique permettent de montrer la formule :

$$\int_{\widehat{\mathbb{P}^2}} -\text{Log } p_{\widehat{x}} \left(\widehat{f}^{-n} \eta \right)_{\widehat{x}} d\widehat{\mu}(\widehat{x}) = \int_{\widehat{\mathbb{P}^2}} \text{Log}(\det d_{x_0} f^n) d\widehat{\mu}(\widehat{x}) = 2n(\lambda_1 + \lambda_2), \quad (1.14)$$

où la mesure $p_{\hat{x}}$ est à densité par rapport à la mesure de Lebesgue :

$$p_{\hat{x}}(A) = \frac{1}{L(\hat{x})} \int_{\pi_0(A \cap \eta_{\hat{x}})} \prod_{i=1}^{+\infty} \frac{\det d_{x_{-i}} f^n}{\det d_{y_{-i}} f^n} d\text{Leb}_{\mathbb{P}^2}(y_0). \quad (1.15)$$

La suite $\hat{y} = (y_{-i})_i$ est donnée par l'injectivité de π_0 sur les atomes de η , qui est l'une des propriétés de la partition η , voir le Théorème 3.3.10. Le coefficient $L(\hat{x}) > 0$ est introduit pour avoir $p_{\hat{x}}(\eta_{\hat{x}}) = 1$.

Si les exposants de Lyapunov de μ sont minimaux, on a $2n(\lambda_1 + \lambda_2) = \text{Log } d^{2n}$. Les formules (1.13) et (1.14) sont alors égales, et l'inégalité de Jensen implique :

$$0 = \int_{\widehat{\mathbb{P}^2}} \text{Log} \frac{p_{\hat{x}}}{\mu_{\eta, \hat{x}}} \left(\widehat{f}^{-n} \eta \right)_{\hat{x}} d\widehat{\mu}(\hat{x}) \leq \text{Log} \int_{\widehat{\mathbb{P}^2}} \frac{p_{\hat{x}}}{\mu_{\eta, \hat{x}}} \left(\widehat{f}^{-n} \eta \right)_{\hat{x}} d\widehat{\mu}(\hat{x}). \quad (1.16)$$

On montre de plus que l'intégrale de droite dans (1.16) est égale à 1. On est ainsi dans le cas d'égalité de la formule de Jensen, donc $\mu_{\eta, \hat{x}} \left(\widehat{f}^{-n} \eta \right)_{\hat{x}} = p_{\hat{x}} \left(\widehat{f}^{-n} \eta \right)_{\hat{x}}$. Par suite on montre que $\mu_{\eta, \hat{x}} = p_{\hat{x}}$ en utilisant la propriété génératrice de la partition η . En utilisant le fait que $(\pi_0)_* p_{\hat{x}}$ est absolument continue par rapport à la mesure de Lebesgue sur \mathbb{P}^2 , on montre finalement que μ est elle-même absolument continue.

Convergence du produit infini. Un point crucial est de vérifier que la mesure $p_{\hat{x}}$ définie par (1.15) est bien définie sur $\eta_{\hat{x}}$ et que $(\pi_0)_* p_{\hat{x}} \ll \text{Leb}_{\mathbb{P}^2}$. Le produit infini est en effet bien défini, lisse, strictement positif et borné sur une boule $B(x_0, r(\hat{x}))$, voir [28]. On construit η pour que $\pi_0(\eta_{\hat{x}}) \subset B(x_0, r(\hat{x}))$.

Signalons que dans [28] l'inclusion $\pi_0(\eta_{\hat{x}}) \subset B(x_0, r(\hat{x}))$ n'est pas obtenue pour tout $\hat{x} \in \widehat{\mathbb{P}^2}$, cette inclusion est vraie lorsque \hat{x} appartient à borélien particulier \mathcal{A} . Comme $\widehat{\mu}(\mathcal{A}) > 0$, on peut utiliser le théorème de Birkhoff : pour $\widehat{\mu}$ -presque tout \hat{x} , il existe $n \geq 0$ tel que $\hat{x}_{-n} \in \mathcal{A}$. On définit alors la mesure $p_{\hat{x}}$ en s'appuyant sur la mesure $p_{\hat{x}_{-n}}$. Le borélien \mathcal{A} est appelé une boîte de Pesin : c'est une union de variétés instables disjointes centrées au même point dans \mathbb{P}^2 , et pour chaque $\hat{x} \in \mathcal{A}$, $\eta_{\hat{x}}$ est inclus dans la variété instable contenant \hat{x} . Sur une variété instable on peut faire converger le produit infini, voir [28, Lemme 3.3].

1.3.3 Branches inverses et variétés instables

Variétés instables. Soit \hat{x} une orbite $\widehat{\mu}$ -générique et soit $n \geq 0$. D'après le Théorème 3.8.2 il existe une branche inverse $f_{\hat{x}}^{-n}$ de f^n définie sur une boule $B(x_0, \eta_\varepsilon(\hat{x}))$ à valeurs dans une boule $B(x_{-n}, \eta_\varepsilon(\hat{x}_{-n}))$ telle que $f_{\hat{x}}^{-n}(x_0) = x_{-n}$ et telle que $\text{Lip}(f_{\hat{x}}^{-n}) \leq \beta_\varepsilon(\hat{x}) e^{-n(\lambda_2 - \varepsilon)}$, ici λ_2 désigne le plus petit exposant de μ .

On peut définir une variété instable associée à \hat{x} de la façon suivante :

Définition 1.3.1. *Pour tout $0 < r \leq 2\eta_\varepsilon(\hat{x})$ on définit*

$$W^u(\hat{x}, r) := \left\{ \widehat{z} \in \widehat{\mathbb{P}^2}, \exists t \in B(x_0, r) : z_{-n} = f_{\hat{x}}^{-n}(t), \forall n \in \mathbb{N} \right\}.$$

La terminologie de variété instable vient du fait que $\text{Lip}(f_{\hat{x}}^{-n}) \leq \beta_\varepsilon(\hat{x}) e^{-n(\lambda_2 - \varepsilon)}$.

Boîtes de Pesin. Le Théorème 3.3.5 dû à Briend [12] stipule pour $\widehat{\mu}$ -presque tout $c \in \mathbb{P}^2$, il existe $\rho_0 > 0, r_0 > 0$ tel que pour tout $\rho \geq \rho_0$ et tout arbre $\mathcal{T} \subset \pi_0^{-1}(c) \cap \{\eta_\varepsilon \geq 1/\rho, \beta_\varepsilon \leq \rho\}$, on dispose de la structure produit suivante pour $\widehat{\mu}$:

$$\forall 0 < r \leq r_0, \widehat{\mu}|_{\mathcal{A}} = \mu|_{B(c,r)} \otimes \mu_{\pi_0,c}, \quad \text{où } \mathcal{A} := \bigsqcup_{\widehat{z} \in \mathcal{T}} W^u(\widehat{z}, r). \quad (1.17)$$

L'ensemble \mathcal{A} est un borélien (boîte de Pesin) et $\mu_{\pi_0,c}$ est la mesure conditionnelle de $\widehat{\mu}$ sur la fibre $\pi_0^{-1}(c)$. En particulier $\widehat{\mu}(\mathcal{A}) = \mu(B(c, r)) \times \mu_{\pi_0,c}(\mathcal{T})$, qui est strictement positif dès que $\mu_{\pi_0,c}(\mathcal{T}) > 0$. Un arbre \mathcal{T} vérifiant la propriété (1.17) et vérifiant $\mu_{\pi_0,c}(\mathcal{T}) > 0$ est dit *régulier*. Notons que \mathcal{A} est utilisé pour faire converger le produit infini dans la formule (1.15).

1.3.4 Absolue continuité pour les endomorphismes semi-extrémaux

On rappelle que f est une application de \mathbb{P}^2 de degré $d \geq 2$, μ et T désignent la mesure d'équilibre et le courant de Green de f et $\lambda_1 \geq \lambda_2$ sont les exposants de Lyapunov de μ . On voudrait répondre à la question 1.1.2 de Dujardin à propos de l'absolue continuité de μ par rapport à la trace $\sigma_T = T \wedge \omega_{\mathbb{P}^2}$ lorsque μ possède un exposant minimal. Nous allons pouvoir adapter la méthode précédente lorsque les branches inverses et le cocycle stable d'Oseledec sont contrôlés sur la boîte de Pesin \mathcal{A} . Les hypothèses \mathcal{H}_1 et \mathcal{H}_2 sont résumées ci-dessous.

Formes normales pour les branches inverses. On utilise ici le Théorème 3.8.2 de Berteloot-Dupont-Molino (voir aussi l'article de Jonsson-Varolin [38]). Celui-ci affirme qu'il existe un borélien \mathcal{FN}_ε totalement invariant de $\widehat{\mu}$ -mesure 1 tel que pour tout \widehat{x} , le diagramme suivant commute pour tout $n \geq N(\widehat{x})$:

$$\begin{array}{ccc} B(x_{-n}, 2\eta_\varepsilon(\widehat{x}_{-n})) & \xleftarrow{f_{\widehat{x}}^{-n}} & B(x_0, 2\eta_\varepsilon(\widehat{x})) \\ \xi_{\widehat{x}_{-n}} = (Z_{\widehat{x}_{-n}}, W_{\widehat{x}_{-n}}) \downarrow & & \downarrow (Z_{\widehat{x}}, W_{\widehat{x}}) = \xi_{\widehat{x}} \\ \mathbb{D}^2(\rho_\varepsilon(\widehat{x}_{-n})) & \xleftarrow{R_{n,\widehat{x}}} & \mathbb{D}^2(\rho_\varepsilon(\widehat{x})) \end{array} \quad (1.18)$$

où $R_{n,\widehat{x}}$ est une application polynomiale dont la seconde composante est linéaire :

$$R_{n,\widehat{x}}^{(2)}(z, w) = \beta_{n,\widehat{x}} w, \quad e^{-n(\lambda_2+\varepsilon)} \leq |\beta_{n,\widehat{x}}| \leq e^{-n(\lambda_2-\varepsilon)}.$$

Le diagramme (1.18) fournit en particulier $W_{\widehat{x}_{-n}} \circ f_{\widehat{x}}^{-n} = \beta_{n,\widehat{x}} \times W_{\widehat{x}}$, avec $|\beta_{n,\widehat{x}}| \approx e^{-n\lambda_2}$.

Les hypothèses d'uniformité \mathcal{H}_1 et \mathcal{H}_2 . D'après le Théorème 3.8.1 d'Oseledec, il existe un borélien A_{os} totalement invariant par f et de μ -mesure 1 tel que le vecteur stable $\vec{v}_s : A_{os} \rightarrow T\mathbb{P}^2$ est bien défini. On note

$$\Delta(x, n) := \|d_x f^n \cdot \vec{v}_s(x)\|.$$

La notion d'arbre régulier a été introduite dans la section 1.3.3.

Définition 1.3.2.

\mathcal{H}_1 : il existe un arbre régulier \mathcal{T} et $\rho_0 > 0$ tels que pour tout $\widehat{z} \in \mathcal{T}$

$$\mathcal{R}_{\widehat{z}} := \inf \{ \eta_\varepsilon(\widehat{x}), \widehat{x} \in W^u(\widehat{z}, \eta_\varepsilon(\widehat{z})) \cap \mathcal{FN}_\varepsilon \} \geq \frac{1}{\rho_0}.$$

\mathcal{H}_2 : l'hypothèse \mathcal{H}_1 est vérifiée et pour tout $\widehat{z} \in \mathcal{T}$, il existe $\Delta_{\widehat{z}} > 1$ tel que :

$$\forall n \geq 0, \forall \widehat{x} \in W^u(\widehat{z}, \eta_\varepsilon(\widehat{z})) \cap \pi_0^{-1}(A_{os}), \frac{1}{\Delta_{\widehat{z}}} \leq \frac{\Delta(z_{-n}, n)}{\Delta(x_{-n}, n)} \leq \Delta_{\widehat{z}}.$$

Le réel $\beta_\varepsilon(\widehat{x})$ intervenant dans la majoration de $\text{Lip}(f_{\widehat{x}}^{-n})$ est en fait la distorsion du changement de coordonnées $\xi_{\widehat{x}}$ du diagramme (1.18). Cette distorsion et le rayon $\eta_\varepsilon(\widehat{x})$ vérifient la relation $\eta_\varepsilon(\widehat{x}) \leq \frac{1}{2\beta_\varepsilon(\widehat{x})}$ (voir la remarque 3.8.3). Si on utilise l'arbre régulier \mathcal{T} pour construire la boîte de Pesin \mathcal{A} , l'hypothèse \mathcal{H}_1 consiste à demander un contrôle uniforme de la distorsion sur \mathcal{A} :

$$\mathcal{H}_1 \iff \sup_{\widehat{x} \in \mathcal{A}} \beta_\varepsilon(\widehat{x}) \leq \rho_0 < +\infty.$$

Concernant l'hypothèse \mathcal{H}_2 , on dispose de la formule $\Delta(x_{-n}, n)|\beta_{n, \widehat{x}}| = \frac{M(\widehat{x})}{M(\widehat{x}_{-n})}$ (voir lemme 3.4.9) et des inégalités $\beta_\varepsilon^{-1}(\widehat{x}) \leq M(\widehat{x}) \leq 2$. Ainsi \mathcal{H}_2 est une hypothèse qui porte aussi sur les coefficients $\beta_{n, \widehat{x}}$. Elle permet de démontrer la Proposition 3.7.2 qui implique $q_{\widehat{x}} = q_{\widehat{y}}$ lorsque \widehat{y} parcourt l'atome $\eta_{\widehat{x}}$, pour tout $\widehat{x} \in \mathcal{A}$.

Le Théorème d'absolue continuité.

Théorème A.II. *Soit f une application de \mathbb{P}^2 de degré $d \geq 2$. Supposons que le plus petit exposant λ_2 de la mesure d'entropie maximale μ soit minimal égal à $\frac{1}{2} \text{Log } d$. Si l'hypothèse \mathcal{H}_2 est vérifiée, alors $\mu \ll T \wedge \omega_{\mathbb{P}^2}$.*

La démonstration repose sur le Théorème B.II qui nécessite l'hypothèse \mathcal{H}_1 , moins contraignante que \mathcal{H}_2 . Ce théorème fournit l'analogie des formules (1.13) et (1.14) lorsque les exposants de μ sont différents (la semi-extrémalité n'est pas nécessaire). La construction de la partition η reprend [28], elle utilise une boîte de Pesin \mathcal{A} , ainsi qu'une partition finie de \mathbb{P}^2 en ouverts sur lesquels f est injective, due à Buzzi [15].

Théorème B.II. *Soit f une application holomorphe de degré $d \geq 2$ de \mathbb{P}^2 . Supposons que les exposants de Lyapunov de la mesure d'entropie maximale μ satisfont $\lambda_1 > \lambda_2$, et supposons que l'hypothèse \mathcal{H}_1 est vérifiée. Alors il existe une partition mesurable η de $\widehat{\mathbb{P}^2}$ et une famille mesurable $\widehat{x} \mapsto q_{\widehat{x}}$ de mesures de probabilités sur $\widehat{\mathbb{P}^2}$ supportés par les atomes $\eta_{\widehat{x}}$ telle que :*

$$(\pi_0)_* q_{\widehat{x}} \ll T \wedge \omega_{\mathbb{P}^2},$$

$$\forall n \geq 0, \int_{\widehat{\mathbb{P}^2}} -\text{Log } q_{\widehat{x}} \left(\widehat{f}^{-n} \eta \right)_{\widehat{x}} d\widehat{\mu}(\widehat{x}) = \text{Log } d^n + 2n\lambda_2.$$

La partition mesurable η satisfait aussi la formule :

$$\forall n \geq 0, \int_{\widehat{\mathbb{P}^2}} -\text{Log } \mu_{\eta, \widehat{x}} \left(\widehat{f}^{-n} \eta \right)_{\widehat{x}} d\widehat{\mu}(\widehat{x}) = \text{Log } d^n + \text{Log } d^n.$$

Lorsque λ_2 est minimal égal à $\frac{1}{2} \text{Log } d$, les deux intégrales du Théorème B.II deviennent égales ($2n\lambda_2 = \text{Log } d^n$), on récupère alors l'inégalité de Jensen (1.16) en remplaçant la mesure $p_{\hat{x}}$ par la mesure $q_{\hat{x}}$. Nous montrons ensuite que les mesures conditionnelles $\mu_{\eta, \hat{x}}$ et $q_{\hat{x}}$ coïncident. Mais pour ce faire une étape préliminaire est de s'assurer que $q_{\hat{x}} = q_{\hat{y}}$ lorsque \hat{x} et \hat{y} sont dans le même atome de η , c'est ici que nous utilisons \mathcal{H}_2 comme expliquée plus haut. Cette difficulté n'apparaît pas dans la démonstration classique, car $p_{\hat{x}}$ (l'analogue de $q_{\hat{x}}$) est défini à l'aide d'un produit infini de jacobiens le long de \hat{x} . Finalement $(\pi_0)_* \mu_{\eta, \hat{x}} = (\pi_0)_* q_{\hat{x}} \ll T \wedge \omega_{\mathbb{P}^2}$, ce qui permet d'établir $\mu \ll T \wedge \omega_{\mathbb{P}^2}$, et donc le théorème A.II.

Pour démontrer le Théorème B.II, la nouveauté est l'utilisation des formes normales pour les branches inverses $f_{\hat{x}}^{-n}$, afin de définir les mesures $q_{\hat{x}}$. En effet l'invariance du courant de Green $f^*T = dT$ et le diagramme (1.18) permettent d'obtenir une formule de changement de variables

$$(f_{\hat{x}}^{-n})^* (T \wedge dd^c |W_{\hat{x}_{-n}}|^2) = d^{-n} |\beta_{n, \hat{x}}|^2 \times (T \wedge dd^c |W_{\hat{x}}|^2). \quad (1.19)$$

Il est alors pertinent de définir $q_{\hat{x}}$ sur $\eta_{\hat{x}}$ par la formule :

$$q_{\hat{x}} = \frac{(T \wedge dd^c |W_{\hat{x}}|^2) \circ \pi_0}{(T \wedge dd^c |W_{\hat{x}}|^2)(\pi_0(\eta_{\hat{x}}))}. \quad (1.20)$$

Un point crucial est de faire en sorte que l'exposant λ_2 apparaisse sans erreurs exponentielles $e^{\pm n\varepsilon}$ dans la première formule du Théorème B.II. La formule de changement de variable (1.19) fait apparaître $\beta_{n, \hat{x}}$ qui contient des erreurs exponentielles. Cette étape de nettoyage est assurée par intégration et utilisation des propriétés du cocycle d'Oseledec (voir l'intégrale du Théorème 3.8.1). Notons que les facteurs de normalisation $L(\hat{x}) := (T \wedge dd^c |W_{\hat{x}}|^2)(\pi_0(\eta_{\hat{x}}))$ pourraient aussi poser problème. Mais comme dans la méthode classique, le lemme 3.5.2 de Ledrappier-Strelcyn permet de les neutraliser par intégration.

Mentionnons enfin que $q_{\hat{x}}$ et $L(\hat{x})$ sont définis comme ci-dessus lorsque $\hat{x} \in \mathcal{A}$. Lorsque $\hat{x} \notin \mathcal{A}$, on utilise le théorème de Birkhoff pour revenir dans \mathcal{A} : il existe $n \geq 1$ tel que $\hat{x}_{-n} \in \mathcal{A}$. La mesure $q_{\hat{x}_{-n}}$ est alors définie par la formule (1.20) et on définit $q_{\hat{x}}$ à l'aide de $q_{\hat{x}_{-n}}$, voir la Définition 3.4.3. Notre méthode présente en effet une difficulté inhérente à l'utilisation des formes normales (qui n'apparaît pas dans le cas classique avec les mesures $p_{\hat{x}}$) : la mesure $q_{\hat{x}}$ est définie avec la submersion $W_{\hat{x}}$ qui est un objet local. Celle-ci est définie sur une boule $B(x_0, \eta_\varepsilon(\hat{x}))$ qui en général ne contient pas la projection $\pi_0(\eta_{\hat{x}})$. En revanche $\pi_0(\eta_{\hat{x}}) \subset B(x_0, \eta_\varepsilon(\hat{x}))$ lorsque $\hat{x} \in \mathcal{A}$.

Chapter 2

Invariant foliations for semi-extremal maps of $\mathbb{C}\mathbb{P}^2$

2.1 Introduction

This part concerns *semi-extremal* maps on \mathbb{P}^2 and holomorphic local foliations that they are likely to preserve. Let f be a holomorphic map of \mathbb{P}^2 of degree $d \geq 2$ and let μ and T be the equilibrium measure and the Green current of f . We recall that f is *semi-extremal* if $\lambda_1 > \lambda_2 = \frac{1}{2} \text{Log } d$, where $\lambda_1 \geq \lambda_2$ stands for the Lyapunov exponents of μ . Few is known on semi-extremal maps. Berteloot-Dupont [3] proved that in this case μ is totally singular with respect to the Lebesgue measure of \mathbb{P}^2 . Dujardin proved in [27, Theorem 3.6] :

Theorem 2.1.1 (Dujardin). *If $\mu \ll T \wedge \omega_{\mathbb{P}^2}$ and if $\mu \not\ll \text{Leb}_{\mathbb{P}^2}$ then f is semi-extremal.*

Dujardin also asked the question of the reverse implication :

Question 2.1.2 (Dujardin). *If f is semi-extremal, do we have $\mu \ll T \wedge \omega_{\mathbb{P}^2}$? Does f contain in some sense a one dimensional Lattès map ?*

We give a partial answer of the first question in Chapter 3. In this chapter we focus on the second question. We first observe in Theorem A.I that if f preserves a pencil of lines and induces a Lattès map on this pencil, then Formula (2.2) can be specified near n -periodic repelling points a by using a submersion W_a satisfying $|W_a \circ f^n| = \sqrt{d^n} |W_a|$. Precisely we obtain

$$\mu = T \wedge dd^c |W_a|^2 \text{ on } (\mathbb{P}^2, a). \quad (2.1)$$

Then our goal is to prove the reverse property : if f is a semi-extremal map satisfying (2.1) then, up to additional hypothesis on μ and T , we prove in Theorem B.I that f^n preserves a holomorphic foliation defined on a neighborhood of the Julia set $\text{Supp}(\mu)$. If the foliation can be extended to \mathbb{P}^2 , then it is a pencil of lines and f induces on it a Lattès map, see Corollaries D.I and E.I.

Semi-extremal maps preserving a pencil of lines. Let f be a holomorphic map of \mathbb{P}^2 of degree $d \geq 2$. Assume that f preserves the pencil of lines given by

$\pi[z : w : t] = [z : w]$. Let θ be the rational map induced by f on this pencil. Let Ω_θ be the set of points p for which μ_θ is smooth and strictly positive at the neighborhood of p . If θ is a Lattès map then the Lyapunov exponent λ_θ of μ_θ is minimal equal to $\frac{1}{2} \text{Log } d$ and $\mu_\theta \ll \omega_{\mathbb{P}^1}$. Moreover $A_\theta := \mathbb{P}^1 \setminus \Omega_\theta$ is a finite set (see Theorem 2.3.2) and thus

$$\mu_\theta = \varphi \omega_{\mathbb{P}^1}$$

where φ is a smooth positive function on $\mathbb{P}^1 \setminus A_\theta$. If $f = [P(z, w) : Q(z, w) : t^d]$ then $\theta = [P(z, w) : Q(z, w)]$ and explicit computations show

$$\mu \ll T \wedge \omega_{\mathbb{P}^2}$$

and

$$\lambda_1 = \text{Log } d \quad , \quad \lambda_2 = \lambda_\theta = \frac{1}{2} \text{Log } d.$$

More generally, we have :

Theorem 2.1.3 (Dupont-Taflin [32, Theorem 1.1, Corollary 1.3]). *Let f be a holomorphic map of \mathbb{P}^2 of degree $d \geq 2$. Let μ be its equilibrium measure, let T be its Green current and let $\lambda_1 \geq \lambda_2$ be the Lyapunov exponents of μ . Assume that f preserves the pencil of lines given by $\pi[z : w : t] = [z : w]$. Let μ_θ be the equilibrium measure of the rational map θ induced on the pencil and let λ_θ be the Lyapunov exponent of μ_θ . Then we have :*

1. $\lambda_1 \geq \text{Log } d$ and $\lambda_\theta \in \{\lambda_1, \lambda_2\}$.

2. $\pi_* \mu = \mu_\theta$ and

$$\mu = T \wedge \pi^* \mu_\theta. \tag{2.2}$$

3. If θ is Lattès then $\mu \ll T \wedge \omega_{\mathbb{P}^2}$ and $\lambda_\theta = \lambda_2 = \frac{1}{2} \text{Log } d$.

4. Conversely, if λ_2 is minimal then θ is Lattès.

This theorem shows that every map of the form

$$f[z : w : t] = [P(z, w) : Q(z, w) : R(z, w, t)], \tag{2.3}$$

where $\theta = [P(z, w) : Q(z, w)]$ is Lattès, provides a semi-extremal map. Up to our knowledge, these are the only known examples and the following question rises :

Question 2.1.4. *Does every semi-extremal map on \mathbb{P}^2 preserve a pencil of lines ?*

Keeping in mind this question, we specify the Dupont-Taflin Formula (2.2) when f is semi-extremal and preserves the pencil π with an induced Lattès map θ . Let $E_\theta := \pi^{-1}(A_\theta)$ (this is a finite union of projective lines) and let

$$R_{\mu, \theta} := \left\{ a \in \text{Supp}(\mu) \setminus E_\theta, \text{ } n\text{-periodic and repelling, } \frac{1}{n} \text{Log} |\det_{\mathbb{C}} d_a f^n| > \text{Log } d \right\}.$$

Theorem A.I. *Assume that f preserves a pencil of lines and induces a Lattès map θ on this pencil. Then $R_{\mu, \theta}$ is dense in $\text{Supp}(\mu) \setminus E_\theta$ and for any $a \in R_{\mu, \theta}$ of period $n \geq 1$, there exists (Z_a, W_a) Poincaré-Dulac coordinates for f^n near a such that :*

$$\mu = T \wedge dd^c |W_a|^2 \text{ on } (\mathbb{P}^2, a). \tag{2.4}$$

The notion of Poincaré-Dulac coordinates is recalled in Section 2.2. Given a repelling n -periodic point $a \in R_{\mu, \theta}$, the idea to prove (2.4) is to consider a normal form of the measure μ_θ at the neighborhood of $a_0 := \pi(a) \in \mathbb{P}^1$. A normal form of μ_θ is an equality $\mu_\theta = dd^c|W_0|^2$ where W_0 a holomorphic coordinate centered at a_0 which is invariant : $|W_0 \circ \theta^n| = \sqrt{d^n}|W_0|$. This normal form is provided by Theorem 2.3.3 due to Berteloot-Loeb. Pulling-back this normal form by π and using Dupont-Taflin formula (2.4), we obtain $\mu = T \wedge dd^c|W_0 \circ \pi|^2$ near a . The submersion $W_a := W_0 \circ \pi$ is invariant by f^n : $|W_a \circ f^n| = \sqrt{d^n}|W_a|$, and we prove in Lemma 2.2.3 that it can be completed into Poincaré-Dulac coordinates (Z_a, W_a) for the repelling fixed point a of f^n .

Invariant foliations near the Julia set. In the spirit of Question 2.1.4 :

Question 2.1.5. *Does Formula (2.4) imply that f preserves a pencil of lines ?*

We give a partial answer to this question in Theorem B.I, Corollaries D.I and E.I below. The exceptional set $\mathcal{E}(f)$ is defined by :

$$\mathcal{E}(f) := \mathbb{P}^2 \setminus \{x, \mu_{x,n} \xrightarrow{n \rightarrow +\infty} \mu\}, \quad (2.5)$$

where the limit is in the sense of the weak convergence and

$$\mu_{x,n} := \frac{1}{d^{2n}} \sum_{y \in f^{-n}(x)} \delta_y. \quad (2.6)$$

The set $\mathcal{E}(f)$ is the largest totally f -invariant proper algebraic subset of \mathbb{P}^2 [26]. For a generic map of \mathbb{P}^2 the exceptional set is empty. For a map preserving a pencil of lines, $\mathcal{E}(f)$ contains the center of the pencil.

In the statement of the theorem below, a holomorphic foliation \mathcal{F} on an open set $\mathcal{V} \subset \mathbb{P}^2$ is invariant by f^N if $f^{-N}(\mathcal{V}) \cap \mathcal{V}$ is not empty and if $(f^N)^*\mathcal{F} = \mathcal{F}$ on $f^{-N}(\mathcal{V}) \cap \mathcal{V}$. We refer to Sections 2.8.1 and 2.8.2 for reminders about holomorphic foliations and for the definition of $(f^N)^*\mathcal{F}$.

Theorem B.I (Invariant foliation). *Let f be a holomorphic map of \mathbb{P}^2 of degree $d \geq 2$, let $\mu = T \wedge T$ be its equilibrium measure. Assume that f satisfies :*

1. *There exist a N -periodic repelling point $a \in \text{Supp}(\mu)$ and Poincaré-Dulac coordinates (Z_a, W_a) for f^N at the point a such that :*

$$\mu = T \wedge dd^c|W_a|^2 \text{ on } (\mathbb{P}^2, a).$$

2. *The trace measure $\sigma_T = T \wedge \omega$ does not satisfy $\sigma_T \ll \mu$ on any neighborhood of every point of $\text{Supp}(\mu)$.*
3. *The exceptional set $\mathcal{E}(f)$ does not intersect $\text{Supp}(\mu)$.*

Then there exists a holomorphic foliation \mathcal{F} on an open neighborhood \mathcal{V} of $\text{Supp}(\mu)$ which is f^N -invariant.

Remark 2.1.6. *The second assumption is equivalent to the following one, we thank T.-C. Dinh for pointing us this fact : it suffices to assume that $\sigma_T \ll \mu$ on a single neighborhood Ω of a point in $\text{Supp}(\mu)$. Let us give briefly the argument.*

The Radon-Nikodym decomposition of σ_T with respect to μ is denoted $\sigma_T = \mu^a + \mu^s$, see (2.26). Assume that μ^s charges every open sub-set of Ω which is charged by μ . Let V be a neighborhood of a point in $\text{Supp}(\mu)$. Since μ is mixing there exists n_0 such that $\mu(f^{-n_0}\Omega \cap V) > 0$. Let $x_0 \in (\text{Supp}(\mu) \cap f^{-n_0}\Omega \cap V) \setminus \text{Crit}(f^{n_0})$ and let $B \subset V$ be a small ball centered at x_0 such that $f^{n_0}(B) \subset \Omega$ and f^{n_0} is injective on B . Observe that $\mu^s(f^{n_0}(B)) > 0$ by hypothesis. Since $(f^{n_0})^*T = d^{n_0}T$ and since $\omega_{\mathbb{P}^2}$ is quasi-invariant by the biholomorphism $f^{n_0} : B \rightarrow f^{n_0}(B)$, the measures $\sigma_T \circ f^{n_0}$ and σ_T are equivalent on B . Moreover the measures $\mu \circ f^{n_0}$ and μ are also equivalent on B since $(f^{n_0})^*\mu = d^{2n_0}\mu$. Hence $\mu^s(B) > 0$ as desired.

Mappings which satisfy the three hypothesis of the theorem are given by maps of the form :

$$f = [P(z, w) : Q(z, w) : t^d] \text{ with } \theta = [P(z, w) : Q(z, w)] \text{ Lattès.}$$

Theorem A.I provides the first item. According to [31], the point $[0 : 0 : 1]$ is a super-attractive fixed point of f , its attracting basin is bounded in \mathbb{C}^2 and the boundary of this basin is equal to $\text{Supp}(\mu)$. The support of T is the complement in \mathbb{P}^2 of this basin, thus the support of T and the support of μ never coincide on any open set charged by μ , it ensures the second item. Finally the third item is given by Proposition 2.1.7 below :

Proposition 2.1.7. *Assume that θ is not conjugated to a polynomial map neither to $z^{\pm d}$, then $\mathcal{E}(f) \cap \text{Supp}(\mu) = \emptyset$.*

Let us give the idea. Let $p \in \mathbb{P}^2 \setminus (\{[0 : 0 : 1]\} \cup \{t = 0\})$. Since $\mathcal{E}(\theta)$ is empty, the counting measure on the d^n preimages of $\pi(p)$ by θ^n converges to μ_θ . Further, restricting to each line L provided by those preimages by θ^n , the counting measure on the d^n preimages of p by f^n is close to the Lebesgue measure on the circle $\text{Supp}(\mu) \cap L$. This explains why $p \notin \mathcal{E}(f)$, see for instance Briend's thesis [11, Chapitre 3]. As a consequence $\mathcal{E}(f) = \{[0 : 0 : 1]\} \cup \{t = 0\}$, in particular $\mathcal{E}(f) \cap \text{Supp}(\mu) = \emptyset$.

Extension of foliations and applications of Theorem B.I. Using the algebraic structure of holomorphic foliations on \mathbb{P}^2 , we observe that :

Proposition C.I. *Let f be a holomorphic map of \mathbb{P}^2 of degree $d \geq 2$ and let \mathcal{F} be a f -invariant foliation defined on a neighborhood of a fixed point of f . If \mathcal{F} admits an extension \mathcal{F}' on \mathbb{P}^2 , then \mathcal{F}' is f -invariant by f . Moreover if f is semi-extremal then in appropriate coordinates, \mathcal{F}' is the pencil of lines given by $\pi[z : w : t] = [z : w]$. In this case the rational map induced by f on \mathbb{P}^1 is Lattès.*

The proof uses the fact that holomorphic foliations on \mathbb{P}^2 are characterized by polynomial 1-forms on affine charts, see Theorem 2.8.7. The second assertion comes from the classification, due to Favre-Pereira, of pairs (\mathcal{F}, f) where \mathcal{F} is a holomorphic foliation on \mathbb{P}^2 and f is an endomorphism of \mathbb{P}^2 preserving \mathcal{F} , see Theorem 2.8.5 and Corollary 2.8.6.

The foliation provided by Theorem B.I is f^N -invariant on a neighborhood of a fixed point of f^N , thus we can apply Proposition C.I to get :

Corollary D.I. *Let f be a mapping satisfying the assumptions of Theorem B.I. Let \mathcal{F} be a f^N -invariant holomorphic foliation on an open neighborhood \mathcal{V} of $\text{Supp}(\mu)$ given by this Theorem. If \mathcal{F} admits an extension \mathcal{F}' to \mathbb{P}^2 , then*

1. \mathcal{F}' is a pencil of lines.
2. The support of μ is not equal to \mathbb{P}^2 .

For the second item observe that if $\text{Supp}(\mu) = \mathbb{P}^2$, then $\mathcal{F} = \mathcal{F}'$ would be a pencil of lines preserved by an iterate f^N . The center of the pencil would be a super attractive fixed point for f^N and thus would belong to the Fatou set, contradiction.

Proposition C.I and Corollary D.I rise the question of extension of holomorphic foliations \mathcal{F} . A possible way is to use Steinness properties, see Section 2.8.4 :

Corollary E.I. *Let f be a degree $d \geq 2$ map on \mathbb{P}^2 and let μ its equilibrium measure. Let \mathcal{F} be a holomorphic foliation defined on an open neighborhood \mathcal{V} of $\text{Supp}(\mu)$. If one of the two following assumptions is satisfied*

4. *the support of μ is connected and its complementary set in \mathbb{P}^2 is Stein.*
5. *the support of μ and the support of T coincide.*

Then \mathcal{F} extends to a holomorphic foliation \mathcal{F}' on \mathbb{P}^2 . In particular if f satisfies the hypothesis of Theorem B.I and if 4. (or 5.) is satisfied, then f^N preserves a pencil of lines, where N is the period of the repelling periodic point a of Theorem B.I.

Hypothesis 5. actually implies hypothesis 4. since the support of T is connected and its complementary set in \mathbb{P}^2 is Stein, see Section 2.8.4. Under hypothesis 4. the proof uses the following extension theorem : a holomorphic foliation \mathcal{F} , defined on an open neighborhood \mathcal{V} of a compact connected subset K whose complementary set in \mathbb{P}^2 is Stein, extends to a holomorphic foliation on \mathbb{P}^2 . This result is due to Lins Neto [42], see Theorem 2.8.8. We also refer to Canales' article [16].

Examples of maps satisfying the hypothesis 1., 3. of Theorem B.I and 5. of Corollary E.I are given by the *elementary Desboves family*, introduced and studied by Bonifant, Dabija and Milnor [10] (see also [9]). This is a family $(f_\lambda)_{\lambda \in \mathbb{C}^*}$ of degree 4 maps preserving the pencil of lines defined by $\pi[z : w : t] = [z : w] :$

$$f_\lambda = [-z(z^3 + 2w^3) : w(2z^3 + w^3) : t(w^3 - z^3 + \lambda(z^3 + w^3 + t^3))], \lambda \in \mathbb{C}^*. \quad (2.7)$$

Bianchi-Taffin [8] studied this family from the bifurcation point of view, showing that its bifurcation locus is equal to \mathbb{C}^* . Actually $[-z(z^3 + 2w^3) : w(2z^3 + w^3)]$ is Lattès, therefore every f_λ is semi-extremal. We know that 1., 3. and 5. are satisfied by every f_λ thanks to Theorem A.I and the results [8]. For 2. we ask the following question :

Question 2.1.8. *What can we said about holomorphic maps f on \mathbb{P}^2 whose Green current T and equilibrium measure μ satisfy $T \wedge \omega \ll \mu$ on an open set $V \subset \mathbb{P}^2$ of measure $\mu(V) > 0$? Is f a Lattès map ?*

2.2 Poincaré-Dulac coordinates and repelling cycles

2.2.1 The Poincaré-Dulac theorem

We denote by \mathcal{G} the semi-group of germs of holomorphic maps fixing 0 in \mathbb{C}^2 . Let $\mathcal{G}^* \subset \mathcal{G}$ be the group of invertible elements of \mathcal{G} , it acts on \mathcal{G} by conjugation. Let $[g]$ denote the class of $g \in \mathcal{G}$ under the action of \mathcal{G}^* . Let $g \in \mathcal{G}$ and let χ_1 and χ_2 be the eigenvalues of d_0g , with $|\chi_1| \geq |\chi_2|$. We say that g is a *triangular germ* of multipliers (χ_1, χ_2) if $|\chi_2| > 0$ and if there exists $c \in \mathbb{C}$ and $k \geq 2$ such that :

$$g(z, w) = (\chi_1 z + cw^k, \chi_2 w) \text{ near } 0,$$

with $c = 0$ if $\chi_2^k \neq \chi_1$. If $\chi_2^k = \chi_1$ we say that the χ_i 's are resonant. A germ $g \in \mathcal{G}$ is *dilating* if the modulus of the eigenvalues of d_0g are strictly larger than 1.

Theorem 2.2.1 (Poincaré-Dulac). *Every dilating germ g belongs to the class of a triangular germ whose multipliers (χ_1, χ_2) satisfy $|\chi_1| \geq |\chi_2| > 1$.*

Let $f : (\mathbb{P}^2, a) \rightarrow (\mathbb{P}^2, a)$ a holomorphic germ fixing a point $a \in \mathbb{P}^2$. Let $\xi : (\mathbb{P}^2, a) \rightarrow (\mathbb{C}^2, 0)$ a centered chart. The eigenvalues of the germ $g := (\xi \circ f \circ \xi^{-1}) \in \mathcal{G}$ do not depend on the chart ξ and we say that f is a *dilating germ at a* if g is a dilating germ. In this case by Poincaré-Dulac theorem $g \in [D]$ with $D(z, w) = (\chi_1 z + cw^k, \chi_2 w)$, where c, k, χ_1 and χ_2 satisfy the properties given above. If $\varphi \in \mathcal{G}^*$ satisfies in the group \mathcal{G}^* the relation $D = \varphi \circ g \circ \varphi^{-1}$, then the germ of biholomorphism $\sigma_0 := (\varphi \circ \xi)^{-1}$ provides a commutative diagram :

$$\begin{array}{ccc} (\mathbb{C}^2, 0) & \xrightarrow{D} & (\mathbb{C}^2, 0) \\ \sigma_0 \downarrow & & \downarrow \sigma_0 \\ (\mathbb{P}^2, a) & \xrightarrow{f} & (\mathbb{P}^2, a) \end{array} \quad (2.8)$$

Definition 2.2.2 (Poincaré-Dulac coordinates). *The map $\sigma_0^{-1} : (\mathbb{P}^2, a) \rightarrow (\mathbb{C}^2, 0)$ induces coordinates $\sigma_0^{-1} = (Z_a, W_a)$, we call them Poincaré-Dulac coordinates for the germ f of (\mathbb{P}^2, a) .*

2.2.2 Completion into Poincaré-Dulac coordinates

Let us consider a dilating germ $g \in [D]$, where

$$D(z, w) = (h(z, w), \chi_2 w).$$

We denote $\chi_1 := \partial_z h(0, 0)$ and we assume

$$|\chi_1| > |\chi_2| > 1. \quad (2.9)$$

The germs g and D are conjugated by $\xi \in \mathcal{G}^* : \xi \circ g \circ \xi^{-1} = D$. The second component of $\xi = (Z, W)$ gives a coordinate W satisfying $W \circ g = \chi_2 W$. Our purpose is to complete W into a system of coordinates (\tilde{Z}, W) which provides Poincaré-Dulac coordinates for g . This technical trick is used to prove Theorem A.I.

Lemma 2.2.3. *There exists $\tilde{\xi} = (\tilde{Z}, W) \in \mathcal{G}^*$ and a triangular germ $\tilde{D}(z, w) = (\chi_1 z + \tilde{c}w^k, \chi_2 w)$ such that*

$$\tilde{\xi} \circ g \circ \tilde{\xi}^{-1} = \tilde{D}.$$

Proof : Since $|\chi_1| > |\chi_2| > 1$, Poincaré-Dulac theorem yields the following commutative diagram with $\xi' \in \mathcal{G}^*$:

$$\begin{array}{ccc} (\mathbb{C}^2, 0) & \xrightarrow{D'} & (\mathbb{C}^2, 0) \\ \xi' \uparrow & & \uparrow \xi' \\ (\mathbb{C}^2, 0) & \xrightarrow{D} & (\mathbb{C}^2, 0) \end{array} \quad (2.10)$$

where

$$D'(z, w) = (\chi_1 z + cw^k, \chi_2 w) \text{ and } c = 0 \text{ if } \chi_2^k \neq \chi_1.$$

Let W' be the second component of ξ' , our goal is to show that W' is given by $W' : (z, w) \mapsto \gamma \times w$ on $(\mathbb{C}^2, 0)$ for some constant $\gamma \in \mathbb{C}^*$. Taking $\tilde{\xi} := \xi'' \circ \xi' \circ \xi$ with $\xi''(z, w) := (z, \frac{1}{\gamma}w)$ and $\tilde{D} := \xi'' \circ D' \circ (\xi'')^{-1}$, the desired conclusion follows.

As a holomorphic function, W' is a power series near 0 of the form :

$$W'(z, w) = \sum_{r \geq 1} (\gamma_{r,0} z^r + \gamma_{r-1,1} z^{r-1} w + \cdots + \gamma_{0,r} w^r) = \sum_{r \geq 1} \mathcal{O}'(r).$$

The function h also expands into a power series near 0 of the form :

$$h(z, w) = \sum_{r \geq 1} (H_{r,0} z^r + H_{r-1,1} z^{r-1} w + \cdots + H_{0,r} w^r) = [H_{1,0} z + H_{0,1} w] + \mathcal{O}_{\geq 2}.$$

Of course $H_{1,0}$ is equal to χ_1 (we recall that χ_1, χ_2 are the eigenvalues of $d_0 D$). The commutative diagram (2.10) provides

$$\chi_2 W'(z, w) = W'(h(z, w), \chi_2 w) \text{ on } (\mathbb{C}^2, 0). \quad (2.11)$$

Observe that (2.11) implies (looking at terms of order 1 in the power series) :

$$\chi_2 \gamma_{1,0} z + \chi_2 \gamma_{0,1} w = (\gamma_{1,0} H_{1,0} z + \gamma_{1,0} H_{0,1} w) + \gamma_{0,1} \chi_2 w.$$

Since $H_{1,0} = \chi_1$ we deduce, looking at coefficients of the variable z , that $\chi_2 \gamma_{1,0} = \gamma_{1,0} \chi_1$. Since $\chi_1 \neq \chi_2$, we get

$$\gamma_{1,0} = 0 \text{ i.e. } \mathcal{O}'(1) = \gamma_{0,1} w. \quad (2.12)$$

We can now prove by induction on $r \geq 2$ that :

$$\forall k \in \{2, \dots, r\}, \mathcal{O}'(k) = 0 \text{ i.e. } W'(z, w) = \gamma_{0,1} w + \sum_{k \geq r+1} \mathcal{O}'(k) \text{ on } (\mathbb{C}^2, 0). \quad (2.13)$$

• We proceed as before for $r = 2$. Using (2.11) and (2.12), we get :

$$\chi_2 \gamma_{2,0} z^2 + \chi_2 \gamma_{1,1} z w + \chi_2 \gamma_{0,2} w^2 = \gamma_{2,0} (H_{1,0} z + H_{0,1} w)^2 + \gamma_{1,1} (H_{1,0} z + H_{0,1} w) (\chi_2 w) + \gamma_{0,2} (\chi_2 w)^2.$$

Since $\chi_1 = H_{1,0}$ we deduce, looking at the coefficients of z^2 , that $\chi_2\gamma_{2,0} = \gamma_{2,0}\chi_1^2$, it yields $\gamma_{2,0} = 0$ by (2.9). Using again $\chi_1 = H_{1,0}$ and looking at the coefficient of zw we obtain $\chi_2\gamma_{1,1} = \gamma_{1,1}\chi_1\chi_2$, thus $\gamma_{1,1} = 0$ by (2.9). At last looking at the coefficients of w^2 we obtain $\chi_2\gamma_{0,2} = \gamma_{0,2}\chi_2^2$, and thus $\gamma_{0,2} = 0$. We have proved $\mathcal{O}'(2) = 0$.

•Let $r \geq 2$ and assume that (2.13) holds, our aim is to prove $\mathcal{O}'(r+1) = 0$. Observe that the equations (2.11) and (2.12) injected in (2.13) imply :

$$\chi_2\gamma_{r+1,0}z^{r+1} + \chi_2\gamma_{r,1}z^r w + \cdots + \chi_2\gamma_{0,r+1}w^{r+1} = \sum_{j=0}^{r+1} \gamma_{r+1-j,j} [H_{1,0}z + H_{0,1}w]^{r+1-j} (\chi_2 w)^j.$$

Looking at the terms z^{r+1} and using $\chi_1 = H_{1,0}$ we have $\chi_2\gamma_{r+1,0} = \gamma_{r+1,0}\chi_1^{r+1}$, thus $\gamma_{r+1,0} = 0$ by (2.9). Then, as the case $r = 2$, the equation coming from the terms $z^r w$ becomes simpler and one can deduce that $\gamma_{r,1} = 0$ using (2.9). By induction, as in the case $r = 2$, we deduce by similar arguments that successively $\gamma_{r-1,2} = \gamma_{r-2,3} = \cdots = \gamma_{0,r+1} = 0$. So we have $\mathcal{O}'(r+1) = 0$ as desired. \square

2.2.3 The developing map associated to a repelling fixed point

Let us fix a dilating germ f acting on (\mathbb{P}^2, a) , a triangular germ $D(z, w) = (\chi_1 z + cw^k, \chi_2 w)$ with $k \geq 2$ such that $f \in [D]$, and Poincaré-Dulac coordinates $\sigma_0^{-1} = (Z_a, W_a)$. We assume that f is induced by a holomorphic map of \mathbb{P}^2 of degree $d \geq 2$.

From the local diagram (2.8) we shall construct a global commutative diagram semi-conjugating f to D via a holomorphic map $\sigma : \mathbb{C}^2 \rightarrow \mathbb{P}^2$. The construction of σ is classical, see for instance Berteloot-Loeb [6] when D is linear.

Let U_a be a connected open neighborhood of a on which σ_0^{-1} is defined. We denote $U_0 := \sigma^{-1}(U_a)$. Let $\mathbb{D}_\varepsilon^2 \subset U_a \cap D^{-1}(U_a)$ be a small bidisc such that $D^{-1}(\mathbb{D}_\varepsilon^2) \subset \mathbb{D}_\varepsilon^2$. In particular the following diagram commutes

$$\begin{array}{ccc} \mathbb{D}_\varepsilon^2 & \xrightarrow{D} & D(\mathbb{D}_\varepsilon^2) \\ \sigma_0 \downarrow & & \downarrow \sigma_0 \\ \sigma_0(\mathbb{D}_\varepsilon^2) & \xrightarrow{f} & f(\sigma_0(\mathbb{D}_\varepsilon^2)) \end{array} \quad (2.14)$$

Proposition 2.2.4 (The developing map).

1. *There exists an increasing sequence of integers $(n_k)_k$ such that $D^{-n_k}(\mathbb{D}_k^2) \subset \mathbb{D}_\varepsilon^2$. In particular the restriction σ_k of $f^{n_k} \circ \sigma_0 \circ D^{-n_k}$ to \mathbb{D}_k^2 is well defined. It satisfies $f \circ \sigma_k = \sigma_k \circ D$ on $\mathbb{D}_k^2 \cap D^{-1}(\mathbb{D}_k^2)$.*
2. *σ_k is an open mapping and $\#\sigma_k^{-1}(p) \leq d^{2n_k}$ for every $p \in \sigma_k(\mathbb{D}_k^2)$.*
3. *The critical values $E_k := \sigma_k(\text{Crit } \sigma_k)$ are included in $f^{n_k}(\text{Crit } f^{n_k})$, which is an algebraic subset of codimension 1 of \mathbb{P}^2 .*
4. *For every positive integers $k \leq l$, $\sigma_k = \sigma_{k+l}$ on \mathbb{D}_k^2 . Hence one can define the holomorphic map $\sigma : \mathbb{C}^2 \rightarrow \mathbb{P}^2$ by*

$$\sigma = \lim_{k \rightarrow +\infty} \sigma_k.$$

5. The following diagram commutes

$$\begin{array}{ccc} \mathbb{C}^2 & \xrightarrow{D} & \mathbb{C}^2 \\ \sigma \downarrow & & \downarrow \sigma \\ \mathbb{P}^2 & \xrightarrow{f} & \mathbb{P}^2 \end{array}$$

6. The map σ is open and have discrete fibers on \mathbb{C}^2 .

7. If $E := \sigma(\text{Crit } \sigma)$ then $E = \bigcup_{k \geq 0} E_k$ and $\mu(E) = 0$.

Proof :

1. We have $D^n = (\chi_1^n z + n c \chi_1^{n-1} w^k, \chi_2^n w)$ and $D^{-n} = (\chi_1^{-n} z - n c \chi_1^{-(n-1)} w^k, \chi_2^{-n} w)$. Hence, for every $k \geq 1$, there exists n_k such that $D^{-n_k}(\mathbb{D}_k^2) \subset \mathbb{D}_\varepsilon^2$. Let σ_k be the restriction of $f^{n_k} \circ \sigma_0 \circ D^{-n_k}$ to \mathbb{D}_k^2 . One can assume that n_k is increasing. We have $f \circ \sigma_k = \sigma_k \circ D$ on $\mathbb{D}_k^2 \cap D^{-1}(\mathbb{D}_k^2)$ thanks to the commutative diagram (2.14).

2. The holomorphic map f^{n_k} is a ramified covering of \mathbb{P}^2 of degree d^{2n_k} , thus it is an open mapping according to [45, Proposition 4, p. 132]. Since $\sigma_0 \circ D^{-n_k} : \mathbb{D}_k^2 \rightarrow \sigma_0(D^{-n_k}(\mathbb{D}_k^2))$ is a biholomorphism, the map σ_k is also open. For every $p \in \sigma_k(\mathbb{D}_k^2)$, we have $\sigma_k^{-1}(p) = (D^{n_k} \circ \sigma_0^{-1})(f^{-n_k}(p)) \cap \mathbb{D}_k^2$, hence $\#\sigma_k^{-1}(p) \leq d^{2n_k}$.

3. By definition, $\text{Crit}(\sigma_k) \subset D^{n_k} \circ \sigma_0^{-1}(\text{Crit } f^{n_k})$ and thus $E_k := \sigma_k(\text{Crit } \sigma_k) \subset f^{n_k}(\text{Crit } f^{n_k})$. Note that $\text{Crit}(f^{n_k})$ is an algebraic subset of \mathbb{P}^2 of codimension 1. Since f^k is a proper holomorphic map with finite fibers, Remmert's theorem (see [45, Theorem 2 p. 129]) ensures that $f^{n_k}(\text{Crit } f^{n_k})$ is an algebraic subset of codimension 1.

4. Let $y \in \mathbb{C}^2$ and assume that $D^m(y) \in \mathbb{D}_\varepsilon^2$ for some $m \geq 1$. From the inclusion $D^{-1}(\mathbb{D}_\varepsilon^2) \subset \mathbb{D}_\varepsilon^2$, we get $\{y, D(y), \dots, D^m(y)\} \subset \mathbb{D}_\varepsilon^2$. Now if $p \in D^n(\mathbb{D}_\varepsilon^2) \cap D^{n+m}(\mathbb{D}_\varepsilon^2)$ with $p = D^n(x) = D^{n+m}(y)$, then $D^m(y) = x \in \mathbb{D}_\varepsilon^2$ and thus, as explained above, $D^j(y) \in \mathbb{D}_\varepsilon^2$ for any $j \in \{0, \dots, m\}$. By the commutative diagram (2.14) we get

$$f^m \circ \sigma_0(y) = f^{m-1} \circ \sigma_0 \circ D(y) = \dots = \sigma_0(D^m(y)) = \sigma_0(x).$$

Composing by f^n we obtain $f^{n+m} \circ \sigma_0 \circ D^{-(n+m)}(D^{n+m}(y)) = f^n \circ \sigma_0 \circ D^{-n}(D^n(x))$. If $k \leq l$ then taking $n = n_k$ and $m = n_l - n_k \geq 0$, one has $\sigma_{k+l} = \sigma_l$ on $D^{n_k}(\mathbb{D}_\varepsilon^2) \cap D^{n_l}(\mathbb{D}_\varepsilon^2) \supset \mathbb{D}_k^2$. It allows to define $\sigma : \mathbb{C}^2 \rightarrow \mathbb{P}^2$ by $\sigma|_{\mathbb{D}_k^2} := \sigma_k$ for any $k \geq 0$. It also satisfies $\sigma = \lim_k \sigma_k$.

5. Let us fix $k \geq 0$. According to the preceding item $\sigma|_{\mathbb{D}_k^2} = \sigma_k$. Thus we deduce using the first item that $f \circ \sigma = \sigma \circ D$ on $\mathbb{D}_k^2 \cap D^{-1}(\mathbb{D}_k^2)$. For each $p \in \mathbb{C}^2$ there exists $k \geq 0$ large enough such that $p \in \mathbb{D}_k^2 \cap D^{-1}(\mathbb{D}_k^2)$, thus $f \circ \sigma(p) = \sigma \circ D(p)$. Hence $f \circ \sigma = \sigma \circ D$ on \mathbb{C}^2 .

6. The map σ is open since every $\sigma_k : \mathbb{D}_k^2 \rightarrow \mathbb{P}^2$ is open, by the first item. It has discrete fibers since $\sigma|_{\mathbb{D}_k^2} = \sigma_k$ and σ_k has finite fibers.

7. We know that $\sigma|_{\mathbb{D}_k^2} = \sigma_k$, hence $E_k \subset E$. If $p \in E$ then there exist $x \in \text{Crit}(\sigma)$ such that $\sigma(x) = p$ and $k \geq 1$ such that $x \in \mathbb{D}_k^2$. Then $x \in \text{Crit}(\sigma_k)$ and $p \in E_k$.

The equilibrium measure μ satisfies $\mu = T \wedge T$ where T is the Green current, and T admits *p.s.h.* local continuous potentials, see [26, Proposition 1.18]. Thus μ does not charge algebraic subsets of \mathbb{P}^2 , see [26, Corollary A.32]. We deduce that $\mu(E_k) \leq \mu(f^k(\text{Crit } f^k)) = 0$ for any $k \geq 0$, and thus $\mu(E) = 0$. \square

An important point is to know that $\sigma(\mathbb{C}^2)$ is large enough to contain $\text{Supp}(\mu)$. The exceptional set $\mathcal{E}(f)$ and the measures $\mu_{x,n}$ are defined by (2.5) and (2.6).

Proposition 2.2.5. *If the fixed point a belongs to the support of μ , then*

$$\text{Supp}(\mu) \setminus \mathcal{E}(f) \subset \mathbb{P}^2 \setminus \mathcal{E}(f) \subset \sigma(\mathbb{C}^2).$$

Proof : Let $q \in \mathbb{P}^2 \setminus \mathcal{E}(f)$ and let $\rho \in C^0(\mathbb{P}^2, \mathbb{R}^+)$ be a continuous function such that $\rho \equiv 1$ on a ball B_a centered at a and such that $\text{Supp}(\rho) \subset 2B_a$, we can assume that σ_0^{-1} is defined on $2B_a$. By definition of $\mathcal{E}(f)$ one has $\lim_n \langle \mu_{q,n}, \rho \rangle = \langle \mu, \rho \rangle \geq \mu(B_a) > 0$ because a belongs to the support of μ . Thus there exists $n_k \geq 1$ such that $\langle \mu_{q,n_k}, \rho \rangle > 0$ and then, by definition of μ_{q,n_k} , there exists a point $q_{n_k} \in f^{-n_k}(q)$ such that $\rho(q_{n_k}) > 0$. Since the support of ρ is included in $2B_a$ we have $q_{n_k} \in 2B_a \subset \text{Dom}(\sigma_0^{-1})$. This allows us to consider $p_{n_k} := D^{n_k} \circ \sigma_0^{-1}(q_{n_k})$, it satisfies $\sigma(p_{n_k}) = f^{n_k}(q_{n_k}) = q$. \square

2.3 Proof of Theorem A.I

Let f be a holomorphic map of degree $d \geq 2$ on \mathbb{P}^2 of equilibrium measure $\mu = T \wedge T$ whose Lyapunov exponents are denoted $\lambda_1 \geq \lambda_2$. Let us suppose that f preserves the pencil of lines given by the standard linear fibration $\pi[z : w : t] = [z : w]$. Let θ be the rational map on \mathbb{P}^1 satisfying $\pi \circ f = \theta \circ \pi$. We thus have

$$f[z : w : t] = [P(z, w) : Q(z, w) : R(z, w, t)] , \theta[z : w] = [P(z, w) : Q(z, w)].$$

Note that the relation $\pi_*\mu = \mu_\theta$ (provided by Theorem 2.1.3) immediately implies

$$\pi(\text{Supp}(\mu)) = \text{Supp}(\mu_\theta).$$

2.3.1 Projections of repelling cycles by π

We denote $R(f) := \bigcup_{N \in \mathbb{N}^*} R_N(f)$ where $R_N(f)$ is the set of repelling periodic points $a \in \text{Supp}(\mu)$ whose period is equal to N . We define in the same way $R(\theta)$. We also recall that $R(f)$ (resp. $R(\theta)$) is dense in $\text{Supp}(\mu)$ (resp. $\text{Supp}(\mu_\theta)$), see Briend-Duval [13]. The center of the pencil $[0 : 0 : 1]$ is an attractive fixed point and thus belongs to the Fatou set $\mathbb{P}^2 \setminus \text{Supp}(T)$. We shall not need the property of density stated in the next lemma, we mention it for sake of completeness.

Lemma 2.3.1. *The map π maps repelling cycles of f to repelling cycles of θ . Moreover $\pi(R(f))$ is dense in $\text{Supp}(\mu_\theta)$.*

Proof : Let us verify $\pi(R(f)) \subset R(\theta)$. Let $a = [a_1 : a_2 : a_3] \neq [0 : 0 : 1]$. Assume that $a \in \text{Supp}(\mu)$ and that a is a repelling N -periodic point. In particular $\theta^N(\pi(a)) = \pi(a)$. Let us assume for instance that $a_3 \neq 0$ and $a_2 \neq 0$. The other cases can be treated similarly. Let φ_0 and ψ_0 be the charts of \mathbb{P}^2 and \mathbb{P}^1 given by $\varphi_0([z : w : 1]) = (z, w)$

and $\psi_0([z : w]) = z/w$, and let f_0^N and θ_0^N be the local expressions of f^N and θ^N in these charts. The following diagram is commutative :

$$\begin{array}{ccc} (\mathbb{C}^2, (a_1, a_2)) & \xrightarrow{f_0^N} & (\mathbb{C}^2, (a_1, a_2)) \\ \psi_0 \circ \pi \circ \varphi_0^{-1} \downarrow & & \downarrow \psi_0 \circ \pi \circ \varphi_0^{-1} \\ (\mathbb{C}, a_1/a_2) & \xrightarrow{\theta_0^N} & (\mathbb{C}, a_1/a_2) \end{array} \quad (2.15)$$

The map $\psi_0 \circ \pi \circ \varphi_0^{-1}$ being a submersion at (a_1, a_2) , the diagram implies that $(\theta_0^N)'(a_1/a_2)$ is an eigenvalue of $d_a f^N$. Since a is repelling, $|(\theta_0^N)'(a_1/a_2)| > 1$ and $[a_1 : a_2]$ is a repelling fixed point of θ^N as desired.

Let us prove now that $\pi(R(f))$ is dense in $\text{Supp}(\mu_\theta)$. If W is an open set in \mathbb{P}^1 intersecting $\text{Supp}(\mu_\theta)$, then $V := \pi^{-1}(W)$ satisfies $\mu(V) = \pi_*\mu(W) = \mu_\theta(W) > 0$ and thus $V \cap \text{Supp}(\mu) \neq \emptyset$. This implies that $R(f) \cap V \neq \emptyset$ since $R(f)$ is dense in $\text{Supp}(\mu)$, and one concludes by using the projection π . \square

2.3.2 Lattès mappings on \mathbb{P}^1 and multipliers on \mathbb{P}^2

We begin by recalling two results due to Berteloot-Loeb. For a rational map θ of degree $d \geq 2$ on \mathbb{P}^1 , we denote

- A *regular point* $p \in \mathbb{P}^1$ is a point such that μ_θ is a strictly positive smooth $(1, 1)$ -form on an open neighborhood of p in \mathbb{P}^1 .
- Ω_θ is the set of regular points and $A_\theta := \mathbb{P}^1 \setminus \Omega_\theta$.
- $R_1(\theta)$ is the set of repelling fixed points of θ , it is contained in $\text{Supp}(\mu_\theta)$.

Theorem 2.3.2 (Berteloot-Loeb, [6, Theorem 1.1 & Proposition 2.5]). *If θ is Lattès map then A_θ is a finite set. If θ is not a Lattès map then $A_\theta = \mathbb{P}^1$.*

Theorem 2.3.3 (Berteloot-Loeb, [7, Proposition 4.1]). *Let $a \in R_1(\theta)$, $D(z) := \theta'(a)z$ and suppose that $a \in \Omega_\theta$. There exists an invertible holomorphic germ $\sigma_0 : (\mathbb{C}, 0) \rightarrow (\mathbb{P}^1, a)$ such that the following diagram commutes*

$$\begin{array}{ccc} (\mathbb{C}, 0) & \xrightarrow{D} & (\mathbb{C}, 0) \\ \sigma_0 \downarrow & & \downarrow \sigma_0 \\ (\mathbb{P}^1, a) & \xrightarrow{\theta} & (\mathbb{P}^1, a) \end{array} \quad (2.16)$$

and such that $\sigma_0^*\mu_\theta = dd^c|z|^2$ (which implies $|\theta'(a)| = \sqrt{d}$).

Now we deal with the multipliers of repelling cycles for mappings of \mathbb{P}^2 . Briend-Duval [13] proved the equidistribution of n -periodic repelling points inside the support of μ . The link between the multipliers of these periodic points and the Lyapunov exponents $\lambda_1 \geq \lambda_2$ of μ has been given by Berteloot-Dupont-Molino [5]. Let $\varepsilon > 0$ and $R_\mu^{n,\varepsilon}$ be the set of n -periodic repelling points $a \in \text{Supp}(\mu)$ satisfying

$$\frac{1}{n} \text{Log} |\det_{\mathbb{C}} d_a f^n| \geq \lambda_1 + \lambda_2 - 2\varepsilon.$$

Proposition 2.3.4 (Berteloot-Dupont-Molino [5, Lemma 4.5]). *Let $V \subset \mathbb{P}^2$ be an open set satisfying $\mu(V) > 0$. There exists $n_\varepsilon \geq 1$ such that for any $n \geq n_\varepsilon$:*

$$\text{Card}(R_\mu^{n,\varepsilon} \cap V) \geq d^{2n}(1-\varepsilon)^3 \mu(V). \quad (2.17)$$

In particular $R_\mu^\varepsilon := \bigcup_{n \geq 1} R_\mu^{n,\varepsilon}$ is dense in $\text{Supp}(\mu)$.

We note that (2.17) is actually proved in [5, lemma 4.5] for $V = \mathbb{P}^2$, but the same proof works for a general open set V charged by μ . Observe that if $\lambda_1 > \lambda_2$, then $\lambda_1 + \lambda_2 > \log d$, we thus get :

Corollary 2.3.5. *Let R_μ^n denote the set of n -periodic repelling points satisfying*

$$\frac{1}{n} \text{Log} |\det_{\mathbb{C}} d_a f^n| > \text{Log } d.$$

If $\lambda_1 > \lambda_2$ then $R_\mu := \bigcup_{n \geq 1} R_\mu^n$ is dense in $\text{Supp}(\mu)$.

2.3.3 Proof of Theorem A.I

Let us recall the statement. We mention that $E_\theta = \pi^{-1}(A_\theta)$ and that A_θ is defined in Section 2.3.2.

Theorem A.I. *Assume that f preserves a pencil of lines and induces a Lattès map θ on this pencil. Then $R_{\mu,\theta} = R_\mu \setminus E_\theta$ is dense in $\text{Supp}(\mu) \setminus E_\theta$ and for any $a \in R_{\mu,\theta}$ of period N , there exists (Z_a, W_a) Poincaré-Dulac coordinates for f^N near a such that*

$$\mu = T \wedge dd^c |W_a|^2 \text{ on } (\mathbb{P}^2, a).$$

Corollary 2.3.5 implies the first assertion, it remains to prove the second one. Let us fix $a \in \text{Supp}(\mu)$ a repelling N -periodic point satisfying $\frac{1}{N} \text{Log} |\chi_1 \chi_2| > \text{Log } d$, where χ_1, χ_2 are the eigenvalues of $d_a f^N$ (with $|\chi_1| \geq |\chi_2|$). Let $a_0 := \pi(a)$, it is a repelling fixed point of θ^N by Lemma 2.3.1. We recall that $\Omega_\theta = \mathbb{P}^1 \setminus A_\theta$.

Lemma 2.3.6. *With the preceding notations, if $a_0 \in \Omega_\theta$ then*

1. $|\chi_1| > |\chi_2| = \sqrt{d^N}$,
2. *There exists Poincaré-Dulac coordinates (Z_a, W_a) for f^N such that :*

$$\mu = T \wedge dd^c |W_a|^2 \text{ and } W_a \circ f^N = \chi_2 W_a \text{ on } (\mathbb{P}^2, a).$$

Proof : Since f and f^N have same equilibrium measure μ and Green current T (similarly for θ and θ^N), we can assume that $N = 1$. By assumption the fixed point a_0 belongs to Ω_θ , hence by Theorem 2.3.3 there exists a Poincaré-Dulac coordinate W_0 near a_0 such that

$$\mu_\theta = dd^c |W_0|^2 \text{ on } (\mathbb{P}^1, a_0). \quad (2.18)$$

Let U_0 be an open neighborhood of a_0 in \mathbb{P}^1 on which (2.18) holds, and denote

$$W_a := W_0 \circ \pi \text{ on } \pi^{-1}(U_0) \ni a.$$

Then injecting the normal form (2.18) into Dupont-Taffin formula (2.2) one gets :

$$\mu = T \wedge dd^c |W_a|^2 \text{ on } \pi^{-1}(U_0). \quad (2.19)$$

At this stage, by the commutative diagram (2.16), we have :

$$W_a \circ f = W_0 \circ \pi \circ f = W_0 \circ \theta \circ \pi = \lambda \times W_0 \circ \pi = \lambda W_a, \quad (2.20)$$

where $\lambda = \theta'(a_0)$. According to Theorem 2.3.3, it satisfies $|\lambda| = \sqrt{d}$.

Let \tilde{Z}_a be a submersion on an open neighborhood U_a of a such that $U_a \subset \pi^{-1}(U_0)$ and such that $\tilde{\xi}_a := (\tilde{Z}_a, W_a)$ define holomorphic coordinates centered at a . If $\tilde{D}_f := \tilde{\xi}_a \circ f \circ (\tilde{\xi}_a)^{-1}$ and $p_w : (z, w) \mapsto w$, then

$$p_w \circ \tilde{D}_f = p_w \circ \tilde{\xi}_a \circ f \circ (\tilde{\xi}_a)^{-1} = W_a \circ f \circ (\tilde{\xi}_a)^{-1} = \lambda W_a \circ (\tilde{\xi}_a)^{-1} = \lambda p_w,$$

where the third equality is given by (2.20). This shows that \tilde{D}_f has the form

$$\tilde{D}_f(z, w) = (h(z, w), \lambda w), \quad (2.21)$$

where $h \in \mathcal{O}(\mathbb{C}^2, 0)$. By construction $d_0 \tilde{D}_f$ has the same eigenvalues χ_1 and χ_2 than $d_a f$ which by assumption satisfy $|\chi_1 \chi_2| > d$. From (2.21) we deduce that λ is also an eigenvalue of $d_0 \tilde{D}_f : \lambda \in \{\chi_1, \chi_2\}$. But $|\lambda| = \sqrt{d}$, thus we deduce that $\chi_2 = \lambda$ and that $|\chi_1| > |\chi_2|$. By Lemma 2.2.3 there exist a chart ξ_a such that the diagram

$$\begin{array}{ccc} (\mathbb{C}^2, 0) & \xrightarrow{D_f} & (\mathbb{C}^2, 0) \\ \xi_a \uparrow & & \uparrow \xi_a \\ (\mathbb{C}^2, 0) & \xrightarrow{f} & (\mathbb{C}^2, 0) \end{array}$$

commutes, where $D_f(z, w) = (\chi_1 z + cw^k, \lambda w)$ and $p_w \circ \xi_a = W_a$. Finally $\xi_a = (Z_a, W_a)$ are Poincaré-Dulac coordinates for f at a and $\mu = T \wedge dd^c |W_a|^2$ near a . \square

2.4 The Patching Theorem

2.4.1 Pointwise lower dimension of measures

Let ν be a finite borel measure on \mathbb{P}^2 . The *pointwise lower dimension* of ν at $x \in \mathbb{P}^2$ is defined by

$$\underline{d}_\nu(x) := \liminf_{r \rightarrow 0^+} \frac{\text{Log } \nu B(x, r)}{\text{Log } r}.$$

If $\tilde{\nu}$ is another finite borel measure we have

$$\underline{d}_{(\nu + \tilde{\nu})}(x) = \inf \{ \underline{d}_\nu(x), \underline{d}_{\tilde{\nu}}(x) \}. \quad (2.22)$$

Let $\sigma_T = T \wedge \omega_{\mathbb{P}^2}$ be the trace measure of T . If $\nu \leq C \sigma_T$ for some $C > 0$, it follows from (2.22) with $\tilde{\nu} = C \sigma_T - \nu$ that

$$\forall x \in \mathbb{P}^2, \underline{d}_\nu(x) \geq \underline{d}_{\sigma_T}(x). \quad (2.23)$$

Since T has continuous Hölder potentials, we can get a lower estimate for the lower pointwise dimension of σ_T :

Proposition 2.4.1 (Dinh-Sibony [26, Proposition 1.18]). *For every $x \in \mathbb{P}^2$ we have $T = dd^c(u_x)$ on a small ball B_x , where u_x is a plurisubharmonic function on B_x which is γ -Hölder for any $\gamma \in]0, \gamma_0[$, where*

$$\gamma_0 = \min \left\{ 1, \frac{\text{Log}(d)}{\text{Log}(d_\infty)} \right\} > 0 \quad \text{and} \quad d_\infty = \lim_{n \rightarrow +\infty} \left(\sup_{p \in \mathbb{P}^2} \|d_p f^n\| \right)^{1/n}.$$

In particular, $\underline{d}_{\sigma_T}(x) \geq 2 + \gamma_0$ for every $x \in \mathbb{P}^2$.

2.4.2 Hausdorff dimension of measures and a vanishing lemma

Let $\text{HD}(A)$ denote the Hausdorff dimension of $A \subset \mathbb{P}^2$. We shall need the following classical result proved by Young. Let ν be a finite borel measure on \mathbb{P}^2 .

Proposition 2.4.2 (Young [50, Proposition 2.1]). *Assume that*

$$\forall x \in \mathbb{P}^2 - \nu - a.e., \quad \underline{d}_\nu(x) \geq \gamma.$$

Then $\text{HD}(A) \geq \gamma$ for every borel set A satisfying $\nu(A) > 0$.

This proposition implies the following one.

Proposition 2.4.3 (Vanishing lemma). *Let U be a connected open set of \mathbb{P}^2 such that $\nu(U) > 0$. Assume that there exists $\gamma > 2$ such that*

$$\forall x \in \mathbb{P}^2 - \nu - a.e., \quad \underline{d}_\nu(x) \geq \gamma.$$

If $h \in \mathcal{O}(U)$ is a holomorphic function satisfying :

$$\nu\{x \in U : h(x) = 0\} > 0, \tag{2.24}$$

then $h \equiv 0$ on U .

Proof : We get from (2.24) and Proposition 2.4.2 that

$$\text{HD}(\{x \in U : h(x) = 0\}) \geq \gamma > 2. \tag{2.25}$$

If h were not null on U , then the analytic set $\{x \in U : h(x) = 0\}$ would have Hausdorff dimension 2, see for instance Chirka's book [18, Corollary 1 p.23]. But this is not compatible with (2.25). \square

By applying this Proposition with $\gamma := 2 + \gamma_0$ and then using (2.23) and Proposition 2.4.1, we get :

Corollary 2.4.4. *Let $U \subset \mathbb{P}^2$ be a connected open set charged by ν and assume that $\nu \leq C\sigma_T$ for some constant $C > 0$. Let $h \in \mathcal{O}(U)$ such that $h(p) = 0$ for ν -almost every point $p \in U$. Then $h \equiv 0$ on U .*

2.4.3 The Patching Theorem

We introduce the Radon-Nikodym decomposition of the trace measure $\sigma_T = T \wedge \omega_{\mathbb{P}^2}$ with respect to μ :

$$\sigma_T = \mu^a + \mu^s. \quad (2.26)$$

Here μ^a is absolutely continuous with respect to μ and $\mu^s \perp \mu$. Our aim is to prove :

Theorem 2.4.5 (Patching Theorem). *Let U be a connected chart of \mathbb{P}^2 with two systems of holomorphic coordinates (Z_1, W_1) and (Z_2, W_2) . Let us suppose that these systems of coordinates satisfy :*

$$T \wedge dd^c |W_i|^2 \ll \mu \text{ on } U, \quad i \in \{1, 2\}. \quad (2.27)$$

If moreover $\mu(U) > 0$ and $\mu^s(U) > 0$, then there is a patching of the holomorphic 1-forms dW_1 and dW_2 :

$$\exists \beta \in \mathcal{O}^*(U) : dW_2 = \beta dW_1 \text{ on } U.$$

Proof : For any differentiable function h on U , we use the following notations :

$$\frac{\partial h}{\partial Z_1} := \frac{\partial}{\partial z} [h \circ \xi_1^{-1}] \circ \xi_1 \quad \text{and} \quad \frac{\partial h}{\partial W_1} := \frac{\partial}{\partial w} [h \circ \xi_1^{-1}] \circ \xi_1, \quad (2.28)$$

where $\xi_1 = (Z_1, W_1)$ is the chart associated to the coordinates (Z_1, W_1) . We are going to prove that $dW_2 = \frac{\partial W_2}{\partial W_1} dW_1$, the function $\frac{\partial W_2}{\partial W_1}$ being defined by (2.28) with $W_2 = p_w \circ \xi_2$. It is a local problem, it is sufficient to prove the equality on each U' open with $\overline{U'} \subset U$. So we can assume without loss of generality that the holomorphic functions Z_1, W_1, Z_2 and W_2 are holomorphic on a neighborhood of \overline{U} and thus the partial derivatives $\partial W_2 / \partial Z_1$ and $\partial W_2 / \partial W_1$ are bounded on U .

The Green current T on U can be written in the coordinates (Z_1, W_1) as a $(1, 1)$ -differential form with complex measures coefficients :

$$T|_U = \sigma_1 \frac{i}{2} dZ_1 \wedge d\overline{Z}_1 + \left(\Lambda \frac{i}{2} dZ_1 \wedge d\overline{W}_1 \right) + \overline{\left(\Lambda \frac{i}{2} dZ_1 \wedge d\overline{W}_1 \right)} + \sigma_2 \frac{i}{2} dW_1 \wedge d\overline{W}_1,$$

with σ_1 and σ_2 the positive measures given by :

$$\sigma_1 = T \wedge \frac{i}{2} dW_1 \wedge d\overline{W}_1 \quad \text{and} \quad \sigma_2 = T \wedge \frac{i}{2} dZ_1 \wedge d\overline{Z}_1,$$

and Λ is the complex measure given by :

$$\Lambda = -T \wedge \frac{i}{2} dW_1 \wedge d\overline{Z}_1.$$

More synthetically we use a matrix to write the current T :

$$T|_U = \begin{bmatrix} \sigma_1 & \overline{\Lambda} \\ \Lambda & \sigma_2 \end{bmatrix}_{(Z_1, W_1)}$$

This matrix of measures evaluated on a borel set $A \subset U$ gives a positive hermitian matrix. In particular a Cauchy-Schwarz inequality holds, for every borel set A in U :

$$|\Lambda(A)| \leq \sqrt{\sigma_1(A)} \sqrt{\sigma_2(A)}. \quad (2.29)$$

The trace of the matrix

$$\sigma_U := \sigma_1 + \sigma_2$$

gives a positive measure which is equivalent to the trace measure σ_T on U :

$$\exists C > 0 : \frac{1}{C} \sigma_T|_U \leq \sigma_U \leq C \sigma_T|_U \quad (2.30)$$

The measure $\mu = T \wedge T$ charges the open set U , thus the trace $\sigma_1 + \sigma_2$ is not null on U . Actually each term σ_1 and σ_2 of the trace measure is not null on U . This is a non trivial result : the idea is that if σ_1 or σ_2 were null, then the potentiel of T would be harmonic on every (say) horizontal disc contained in U , which implies that μ is null on U by a Theorem of Briend, details can be found in Dupont-Rogue's article [30, §3.3]. Let us denote for convenience :

$$\lambda := \sigma_2 \text{ and } \mu_1^a := \sigma_1,$$

we recall that by hypothesis (2.27) $\mu_1^a = \psi_1 \mu|_U \ll \mu$, with $\psi_1 \in L^1(\mu|_U)$. Let us now write the Radon-Nikodym decomposition of the measure λ with respect to $\mu|_U$:

$$\lambda = h_1 \mu|_U + \mu_1^s,$$

with $h_1 \in L^1(\mu|_U)$ a non negative function, and μ_1^s a positive measure on U singular with respect to μ . There exists \mathcal{A}_1 a borel set of measure $\mu(\mathcal{A}_1) = 0$ such that $\mu_1^s(\cdot \cap \mathcal{A}_1) = \mu_1^s(\cdot)$. Let us verify that the measures μ^s and μ_1^s are equivalent, μ^s being defined in (2.26) :

$$\frac{1}{C} \mu^s|_U \leq \mu_1^s \leq C \mu^s|_U \quad (2.31)$$

with the same $C > 0$ than in (2.30). Indeed by construction

$$\mu_1^s \leq \lambda \leq \sigma_U \leq C \sigma_T|_U \leq C(\mu^a + \mu^s),$$

and restricting to \mathcal{A}_1 it gives $\mu_1^s \leq C(0 + \mu^s(\cdot \cap \mathcal{A}_1)) \leq C \mu^s$. Similarly we have

$$\mu^s|_U \leq \sigma_T|_U \leq C(\lambda + \mu_1^a) = C((h_1 + \psi_1)\mu|_U + \mu_1^s),$$

and it implies $\mu^s|_U \leq C \mu_1^s$ if we restrict to a set of full measure for μ^s and of measure 0 for μ . In particular since by hypothesis $\mu^s(U) > 0$, Equation (2.31) implies that :

$$\mu_1^s(U) > 0. \quad (2.32)$$

The idea now is to decompose dW_2 in terms of the 1-forms dZ_1 and dW_1 :

$$dW_2 = \frac{\partial W_2}{\partial Z_1} dZ_1 + \frac{\partial W_2}{\partial W_1} dW_1.$$

We use the following notations :

$$\alpha := \frac{\partial W_2}{\partial Z_1}, \quad \beta := \frac{\partial W_2}{\partial \bar{W}_1} \quad \text{and} \quad \gamma := \alpha \bar{\beta}.$$

With these notations we have :

$$dd^c |W_2|^2 = |\alpha|^2 dd^c |Z_1|^2 + 2\operatorname{Re} \left[\gamma \frac{i}{2} dZ_1 \wedge d\bar{W}_1 \right] + |\beta|^2 dd^c |W_1|^2.$$

Wedging by T this equality, we get :

$$\underbrace{T \wedge dd^c |W_2|^2}_{=: \mu_2^a} = |\alpha|^2 \underbrace{(T \wedge dd^c |Z_1|^2)}_{=: \lambda} + 2\operatorname{Re} \left[\underbrace{\gamma \left(T \wedge \frac{i}{2} dZ_1 \wedge d\bar{W}_1 \right)}_{=: \Lambda'} \right] + |\beta|^2 \underbrace{(T \wedge dd^c |W_1|^2)}_{=: \mu_1^a},$$

and thus we have :

$$\mu_2^a = |\alpha|^2 \lambda + \Lambda' + |\beta|^2 \mu_1^a, \quad (2.33)$$

with Λ' a signed measure which satisfies for any borel set $A \subset U$:

$$|\Lambda'(A)| \leq 2 \left(\max_U |\gamma| \right) |\Lambda(A)| \leq 2 \left(\max_U |\gamma| \right) \sqrt{\mu_1^a(A)} \sqrt{\lambda(A)}, \quad (2.34)$$

where the second inequality comes from (2.29). Let $A \subset U$ be any borel set, then using (2.33) and using the fact that $\lambda = h_1 \mu|_U + \mu_1^s$ we get :

$$\begin{aligned} 0 &\leq \int_A |\alpha|^2 d\mu_1^s = \mu_2^a(A) - \int_A |\beta|^2 d\mu_1^a - \Lambda'(A) - \int_A h_1 |\alpha|^2 d\mu \\ &\leq \mu_2^a(A) + |\Lambda'(A)| \\ &\leq \mu_2^a(A) + 2 \left(\max_U |\gamma| \right) \sqrt{\mu_1^a(A)} \sqrt{\lambda(A)} \text{ by (2.34)}. \end{aligned}$$

Since μ_1^a and μ_2^a are absolutely continuous with respect to μ and since $\mu_1^s \perp \mu$ on U , there exists $\mathcal{A} \subset U$ a borel set of measure 0 for μ_1^a and μ_2^a such that $\mu_1^s = \mu_1^s(\cdot \cap \mathcal{A})$. So using the preceding inequalities we have for any $A \subset U$:

$$0 \leq \int_A |\alpha|^2 d\mu_1^s = \int_{A \cap \mathcal{A}} |\alpha|^2 d\mu_1^s \leq 0 + 2 \left(\max_U |\gamma| \right) \sqrt{0} \sqrt{\lambda(A \cap \mathcal{A})} = 0,$$

and thus for any $A \subset U$:

$$\int_A |\alpha|^2 d\mu_1^s = 0.$$

The measure μ_1^s charges the open set U by (2.32), hence $\alpha(p) = 0$ for μ_1^s -almost every point $p \in U$. By construction $\mu_1^s \leq C\sigma_T$ for some $C > 0$, so we can apply Corollary 2.4.4 with $\nu := \mu_1^s$ to conclude that $\alpha \equiv 0$ on U . Thus we obtain $dW_2 = \beta dW_1$ on U , and finally $\beta \in \mathcal{O}^*(U)$ since W_1 and W_2 are submersions on U . \square

Remark 2.4.6. We thank T.-C. Dinh for the following observations.

- We can drop $\mu(U) > 0$ in the statement of Theorem 2.4.5.
- Moreover, he proposed a simpler argument for the proof : if we assume that $dW_1 \wedge dW_2$ is not null on U , then $dW_1 \wedge dW_2 \neq 0$ on a Zariski open set V of U . In V , the coordinates (W_1, W_2) create a chart and thus we have :

$$\mu^s \leq \sigma_T \ll T \wedge (dd^c |W_1|^2 + dd^c |W_2|^2) \ll \mu.$$

This property extends on U because T has no mass on $U \setminus V$ which is an analytic set. Hence $\mu^s(U) = 0$ and the result follows. This argument only needs that T is a closed positive current with no mass on analytic sub-sets.

2.5 Proof of Theorem B.I

Let f be a degree $d \geq 2$ holomorphic map of equilibrium measure $\mu = T \wedge T$, we denote $J = \text{Supp}(\mu)$. We want to prove Theorem B.I. Let a be a repelling N -periodic point of f with Poincaré-Dulac coordinates $\sigma_0^{-1} = (Z_a, W_a)$. We assume that the following formula holds

$$\mu = T \wedge dd^c |W_a|^2 \text{ on } (\mathbb{P}^2, a). \quad (2.35)$$

We also assume that σ_T does not satisfy $\sigma_T \ll \mu$ on any neighborhood of every point of J and that the exceptional set of f does not intersect J . Let $D(z, w) = (\chi_1 z + cw^k, \chi_2 w)$ be the polynomial map such that $f^N \circ \sigma = \sigma \circ D$, with σ the developing map of σ_0 given by Proposition 2.2.4.

2.5.1 Pull back by the developing map

Our purpose in this section is to see how (2.35) is lifted by σ on \mathbb{C}^2 . Let us specify how the $(1, 1)$ closed positive current σ^*T and the borel measure $\sigma^*\mu$ are defined on \mathbb{C}^2 . We refer to [2, 34, 48, 26] for general accounts on currents.

For every $p \in \mathbb{C}^2$, let us write $T = dd^c u$ on an open neighborhood of $\sigma(p)$, u being a continuous psh function. Then σ^*T is defined near p by $dd^c(u \circ \sigma)$. Since u is bounded, $\sigma^*\mu := \sigma^*T \wedge \sigma^*T$ is well defined in the sense of Bedford-Taylor. These currents are representable by integration (they have order 0), hence the trace of σ^*T and $\sigma^*\mu$ are borel measures on \mathbb{C}^2 . These two measures are finite on bounded open subsets by Chern-Levine-Nirenberg inequality, thus they are Borel regular measures (Radon measures) on \mathbb{C}^2 .

Proposition 2.5.1. *Let f be a degree $d \geq 2$ map on \mathbb{P}^2 of equilibrium measure $\mu = T \wedge T$. Assume the first hypothesis of Theorem B.I so that the formula (2.35) holds for a repelling N -periodic point $a \in \text{Supp}(\mu)$. With the same notations than above :*

1. $|\chi_2| = \sqrt{d^N}$.
2. The pull back $\sigma^*\mu$ and the pull back σ^*T are related on \mathbb{C}^2 by

$$\sigma^*\mu = \sigma^*T \wedge dd^c |w|^2 \text{ on } \mathbb{C}^2,$$

where w is the second standard coordinate on \mathbb{C}^2 .

Proof : Without loss of generality we can assume that $N = 1$. Using $f^*\mu = d^2\mu$, $f^*T = dT$ and $W_a \circ f = \chi_2 W_a$, we get by pulling back (2.35) by f :

$$\mu = \frac{|\chi_2|^2}{d} (T \wedge dd^c |W_a|^2) \text{ on } (\mathbb{P}^2, a).$$

Using again (2.35) one gets $\mu = \frac{|\chi_2|^2}{d} \mu$ on (\mathbb{P}^2, a) . Finally $|\chi_2|^2 = d$ since $a \in \text{Supp}(\mu)$.

Let us prove the second item. We want to show that $\nu := \sigma^*T \wedge dd^c |w|^2$ and $\sigma^*\mu$ are equal. Since they are borel regular measures, it suffices to prove that they coincide on compact sets K of \mathbb{C}^2 . Let us fix a compact set $K \subset \mathbb{C}^2$. Let k be large enough

such that $K \subset \mathbb{D}_k^2$, in particular $\sigma = \sigma_k = f^{n_k} \circ \sigma_0 \circ D^{-n_k}$ on K . Since $f^{n_k^*} \mu = d^{2n_k} \mu$, we get

$$(\sigma^* \mu)(K) = [(D^{-n_k})^* \sigma_0^* (f^{n_k^*} \mu)](K) = d^{2n_k} \times [(D^{-n_k})^* \sigma_0^* \mu](K).$$

We obtain by the formula (2.35) :

$$(\sigma^* \mu)(K) = d^{2n_k} \times [(D^{-n_k})^* \sigma_0^* T] \wedge (dd^c |W_a \circ \sigma_0 \circ D^{-n_k}|^2)(K).$$

Now using

$$W_a \circ \sigma_0 \circ D^{-n_k} = w \circ \sigma_0^{-1} \circ \sigma_0 \circ D^{-n_k} = w \circ D^{-n_k} = \chi_2^{-n_k} \times w,$$

we deduce that :

$$(\sigma^* \mu)(K) = d^{2n_k} |\chi_2|^{-2n_k} \times [(D^{-n_k})^* \sigma_0^* T] \wedge dd^c |w|^2(K).$$

But $|\chi_2| = \sqrt{d}$ by the first item, hence we obtain :

$$(\sigma^* \mu)(K) = [(D^{-n_k})^* \sigma_0^* (d^{n_k} T)] \wedge dd^c |w|^2(K).$$

Using the invariant relation $d^{n_k} T = f^{n_k^*} T$ one deduces :

$$(\sigma^* \mu)(K) = [(D^{-n_k})^* \sigma_0^* f^{n_k^*} T] \wedge dd^c |w|^2(K).$$

Finally, by recalling that $\sigma|_K = f^{n_k} \circ \sigma_0 \circ D^{-n_k}|_K$, we get

$$(\sigma^* \mu)(K) = [(\sigma^* T) \wedge dd^c |w|^2](K) = \nu(K).$$

This proves that the measures $\sigma^* \mu$ and ν are equal on \mathbb{C}^2 . \square

2.5.2 Outline of the construction of the foliation

We recall that our aim is to construct an f^N -invariant foliation on a neighborhood of $J = \text{Supp}(\mu)$. We proceed in several steps. The notion of holomorphic foliation and the formalism of 1-forms and vector fields are recalled in Section 2.8.1.

Step 1. A first step is to construct a bidisc $D_R := \mathbb{D}_R^2$ with $R > 0$ large enough to have $\sigma(D_R) \supset J$. The existence of this bidisc is a consequence of the fact that the repelling periodic point a belongs to J and that $J \cap \mathcal{E}(f) = \emptyset$. This step is essential because it allows us to work with the open mapping $\sigma_R := \sigma|_{D_R} : D_R \rightarrow \sigma(D_R)$ which has finite fibers and whose critical values E_R are included in a codimension 1 analytic sub-set of \mathbb{P}^2 .

Step 2. The map σ_R may not be proper and thus may not be a (ramified) covering map. To avoid technical issues, we introduce for each $p \in J \setminus E_R$, some special inverse branches of σ_R defined on a neighborhood of p . Their images are denoted $\text{IB}_R(p)$.

Step 3. We begin the construction near the regular values of σ_R . We associate to each point $p \in J \setminus E_R$ a 1-form ω_p , without zeroes, on $V_p = \sigma(\text{IB}_R(p))$. It provides a foliation \mathcal{F}_p on V_p . Then we prove that for every non trivial intersection $V_p \cap V_{p'}$, we have $\mathcal{F}_p = \mathcal{F}_{p'}$ on any connected component $\mathcal{C} \subset V_p \cap V_{p'}$ such that $\mu(\mathcal{C}) > 0$.

Step 4. We construct in Section 2.5.5 a (possibly singular) foliation \mathcal{F}_q on a neighborhood W_q of q for each critical values $q \in J \cap E_R$. We check in Section 2.5.5 that on each non trivial intersection $V_p \cap W_q$, the two foliations \mathcal{F}_p and \mathcal{F}_q coincide on each connected component of $V_p \cap W_q$ which is charged by μ . Similarly $\mathcal{F}_q = \mathcal{F}_{q'}$ on each connected component of $W_q \cap W_{q'}$ charged by μ , see Proposition 2.5.15.

Step 5. In Section 2.5.6 we use the compactness of J to cover it by adapted open sets on which all the local foliations \mathcal{F}_p and \mathcal{F}_q patch together. We obtain thus a foliation \mathcal{F} on a neighborhood \mathcal{V} of J . We also prove that there is an open set $\mathcal{W} \subset \mathbb{C}^2$ such that $\sigma(\mathcal{W}) = \mathcal{V}$ and such that $\sigma^*\mathcal{F}$ coincide with the horizontal foliation of \mathbb{C}^2 on \mathcal{W} .

Step 6. In Section 2.5.7 we check that the foliation \mathcal{F} is invariant by (a power of) f . In order to do that, we improve the preceding computation of $\sigma^*\mathcal{F}$ by proving that $\sigma^*\mathcal{F}$ coincide with the horizontal foliation of \mathbb{C}^2 on $\sigma^{-1}(\mathcal{V})$. Then using the commutative relation $\sigma \circ f = D \circ \sigma$ and the fact that the horizontal foliation of \mathbb{C}^2 is D -invariant, we conclude that \mathcal{F} is invariant by (a power) of f .

2.5.3 Inverse branches near regular values in the Julia set

Since the repelling fixed point a of f^N belongs to J and since $J \cap \mathcal{E}(f) = \emptyset$, we have that $\text{Supp}(\mu) \subset \sigma(\mathbb{C}^2)$ by Proposition 2.2.5.

By Proposition 2.2.4, the map σ is equal to $\sigma_k = f^{n_k} \circ \sigma_0 \circ D^{-n_k}$ on \mathbb{D}_k^2 . Let now $p \in J \subset \sigma(\mathbb{C}^2)$ and let $k \geq 1$ be such that $p \in \sigma(\mathbb{D}_k^2)$. There exists $q_k \in \mathbb{D}_k^2$ such that $p = \sigma(q_k) = \sigma_k(q_k)$. Let B_{q_k} be a ball centered at q_k such that $B_{q_k} \subset \mathbb{D}_k^2$. Since σ_k is open, $\sigma(B_{q_k}) = \sigma_k(B_{q_k})$ is an open neighborhood of p .

By compactness we can cover J by a finite number of such neighborhoods

$$J \subset \mathcal{U} := \sigma(B_{q_{k_1}}) \cup \cdots \cup \sigma(B_{q_{k_N}}).$$

Let $R := \max\{k_1, \dots, k_N\}$ so that $B_{q_{k_1}} \cup \cdots \cup B_{q_{k_N}} \subset D_R := \mathbb{D}_R^2$. We have :

$$\sigma_R := (f^{n_R} \circ \sigma_0 \circ D^{-n_R})|_{D_R} : D_R \longrightarrow \sigma_R(D_R) \supset \mathcal{U} \supset J. \quad (2.36)$$

According to Proposition 2.2.4, the critical values of σ_R satisfy $E_R \subset f^{n_R}(\text{Crit } f^{n_R})$. In particular E_R is included in a closed proper analytic subset of $\mathbb{C}\mathbb{P}^2$ and has empty interior. Let $\text{Reg}_{\sigma_R}(J) := J \setminus E_R$, it has full μ -measure.

Any point $p \in \mathcal{U}$ satisfies $1 \leq \#(\sigma_R^{-1}(p)) \leq d^{2n_R}$. So for every $p \in \text{Reg}_{\sigma_R}(J)$ there exists $n_p \in \{1, \dots, d^{2n_R}\}$ such that $\sigma_R^{-1}(p) = \{x_{1,p}, \dots, x_{n_p,p}\}$. For every $j \in \{1, \dots, n_p\}$ there exists an open connected neighborhood $U_{j,p} \subset D_R$ of $x_{j,p}$ such that

$$\sigma_{j,p} := \sigma_R|_{U_{j,p}} : U_{j,p} \longrightarrow \sigma(U_{j,p}) =: V_{j,p} \subset \mathcal{U}$$

is a biholomorphism. We can assume that every $V_{j,p}$ does not intersect E_R . We denote

$$V_p := \bigcap_{j=1}^{n_p} V_{j,p}. \quad (2.37)$$

Remark 2.5.2 (to fix possible problems near the boundary of D_R). *Let $p \in \text{Reg}_{\sigma_R}(J)$ and let $p' \in V_p$. Even up to a reduction of V_p , the fiber of p' by σ_R may not be given by the inverse branches $\sigma_{1,p}^{-1}, \dots, \sigma_{n_p,p}^{-1}$. Indeed, if p admits a preimage by σ on the boundary of D_R , the inclusion $\{\sigma_{1,p}^{-1}(p'), \dots, \sigma_{n_p,p}^{-1}(p')\} \subset \sigma_R^{-1}(p')$ could be strict.*

To avoid this difficulty, we introduce special sets of inverse branches :

Definition 2.5.3. For every $p \in \text{Reg}_{\sigma_R}(J)$, we define

$$\text{IB}_R(p) := \{x' \in D_R, \exists j \in \{1, \dots, n_p\}, \exists p' \in V_p : x' = \sigma_{j,p}^{-1}(p')\}. \quad (2.38)$$

Here the abbreviation "IB" stands for inverse branches.

2.5.4 Construction of the foliation near regular values

In this section we introduce for every $p \in \text{Reg}_{\sigma_R}(J)$ a foliation \mathcal{F}_p on V_p . We shall patch together the foliations $(\mathcal{F}_p)_{p \in \text{Reg}_{\sigma_R}(J)}$ (and also the foliations on neighborhoods of singular values $q \in J \cap E_R$ constructed in Section 2.5.5). We shall use the following distribution of tangent complex lines, for $p \in \text{Reg}_{\sigma_R}(J)$:

$$\mathcal{D}_p : \begin{cases} V_p & \longrightarrow \mathbb{P}(T\mathbb{P}^2) \\ p' & \longmapsto [d_{x'}\sigma_R \cdot (1, 0)] \text{ for any } x' \in \text{IB}_R(p) \text{ such that } \sigma_R(x') = p' \end{cases} \quad (2.39)$$

It will describe the directions followed by the different local foliations we will construct. The notation $[\vec{v}]$ stands for the complex line $\mathbb{C} \cdot \vec{v}$ of $T_p\mathbb{P}^2$. We verify in the next proposition that the maps \mathcal{D}_p are well defined.

Proposition 2.5.4. Assume that f and the N -periodic repelling point a satisfy the hypothesis of Theorem B.I with the same notations.

1. Let $x, y \in \mathbb{C}^2 \setminus (\text{Crit } \sigma)$ and let us assume that $p := \sigma(x)$ and that $q := \sigma(y)$ belong to J . Let $U_{x,p}$ be an open connected neighborhood of x such that

$$\sigma_{x,p} := \sigma|_{U_{x,p}} : U_{x,p} \longrightarrow \sigma(U_{x,p}) =: V_{x,p}$$

is a biholomorphism. Similarly let $\sigma_{y,q} := \sigma|_{U_{y,q}} : U_{y,q} \longrightarrow \sigma(U_{y,q}) =: V_{y,q}$ be a biholomorphism. Assume that $\Omega := V_{x,p} \cap V_{y,q}$ is connected and that $\mu(\Omega) > 0$. In particular $\mu^s(U) > 0$ by the hypothesis 2. of Theorem B.I, where μ^s is defined by (2.26). Let ϕ be the map

$$\phi := (\sigma_{y,q}^{-1} \circ \sigma_{x,p}) : \sigma_{x,p}^{-1}(\Omega) \longrightarrow \sigma_{y,q}^{-1}(\Omega).$$

Then ϕ satisfies on $\sigma_{x,p}^{-1}(\Omega)$:

$$\phi(z, w) = (A(z, w), B(w)),$$

where $A(z, w)$ and $B(w)$ are two holomorphic functions.

2. Let $p \in \text{Reg}_{\sigma_R}(J)$ and $p' \in V_p$. Then for every $x', y' \in \sigma^{-1}(p') \cap \text{IB}_R(p)$, the complex lines $[d_{x'}\sigma \cdot (1, 0)]$ and $[d_{y'}\sigma \cdot (1, 0)]$ are equal. In particular the map $\mathcal{D}_p : V_p \longrightarrow \mathbb{P}(T\mathbb{P}^2)$ is well defined.
3. Let $p, q \in \text{Reg}_{\sigma_R}(J)$. If \mathcal{C} is a connected component of $V_p \cap V_q$ of measure $\mu(\mathcal{C}) > 0$, then $\mathcal{D}_p = \mathcal{D}_q$ on \mathcal{C} .

Proof :

1. Let us write $\phi = (A(z, w), B(z, w))$ and let us prove that $B(z, w)$ does not depend on z . To do so we use the Proposition 2.5.1 :

$$\sigma^* \mu = \sigma^* T \wedge dd^c |w|^2 \text{ on } \mathbb{C}^2.$$

Restricting this equation on $U_{x,p}$ and on $U_{y,q}$ we get on $\Omega = V_{x,p} \cap V_{y,q}$:

$$\begin{cases} \sigma_{x,p}^* \mu = \sigma_{x,p}^* T \wedge dd^c |w|^2 \text{ on } U_{x,p} \\ \sigma_{y,q}^* \mu = \sigma_{y,q}^* T \wedge dd^c |w|^2 \text{ on } U_{y,q} \end{cases} \implies \begin{cases} \mu = T \wedge dd^c |W_p|^2 \text{ on } \Omega \\ \mu = T \wedge dd^c |W_q|^2 \text{ on } \Omega \end{cases}$$

with the notations

$$\sigma_{x,p}^{-1} = (Z_p, W_p) \text{ and } \sigma_{y,q}^{-1} = (Z_q, W_q).$$

By hypothesis Ω is connected, $\mu(\Omega) > 0$ and $\mu^s(U) > 0$, thus we can apply the Patching Theorem 2.4.5 (see (2.28) for a definition of the partial derivatives) :

$$dW_q = \frac{\partial W_q}{\partial W_p} dW_p \text{ and } \frac{\partial W_q}{\partial Z_p} = 0 \text{ on } \Omega.$$

So we have on $\sigma_{x,p}^{-1}(\Omega)$:

$$dB = \sigma_{x,p}^* dW_q = 0 \times dz + \left(\frac{\partial W_q}{\partial W_p} \circ \sigma_{x,p} \right) \times dw,$$

in particular $\partial_z B \equiv 0$ on $\sigma_{x,p}^{-1}(\Omega)$.

2. Since $x', y' \in \text{IB}_R(p)$ and $\sigma(x') = \sigma(y') = p'$, there exist i, j such that $x' = \sigma_{i,p}^{-1}(p')$ and $y' = \sigma_{j,p}^{-1}(p')$. Let us assume $i = 1$ and $j = 2$ for simplicity. Observe that $\sigma_{1,p}^{-1}$ and $\sigma_{2,p}^{-1}$ are defined on the connected set V_p by construction, and $\mu(V_p) > 0$ since V_p is an open neighborhood of $p \in J$. So we can apply the point 1. with $p = q$, to conclude that the map $\phi := (\sigma_{2,p}^{-1} \circ \sigma_{1,p})$ satisfies $\phi(z, w) = (A(z, w), B(w))$ on $\sigma_{1,p}^{-1}(V_p)$. In particular we have :

$$d_{x'} \phi = \begin{bmatrix} \partial_z A(x') & \partial_w A(x') \\ 0 & B'(x') \end{bmatrix}.$$

We deduce :

$$\begin{aligned} d_{x'} \sigma \cdot (1, 0) &= d_{x'} \sigma_{1,p} \cdot (1, 0) = d_{x'} (\sigma_{2,p} \circ \phi) \cdot (1, 0) \\ &= d_{y'} \sigma_{2,p} \cdot d_{x'} \phi \cdot (1, 0) = \partial_z A(x') \times d_{y'} \sigma \cdot (1, 0). \end{aligned}$$

The coefficient $\partial_z A(x')$ is not equal to 0 because ϕ is a biholomorphism. Finally

$$[d_{x'} \sigma \cdot (1, 0)] = [d_{y'} \sigma \cdot (1, 0)] \in \mathbb{P}(T_{p'} \mathbb{P}^2),$$

which proves that \mathcal{D}_p is well defined on V_p .

3. Let \mathcal{C} be a connected component of $V_p \cap V_q$, and let us assume that $\mu(\mathcal{C}) > 0$. We mention that \mathcal{C} is itself an open set of \mathbb{P}^2 . Let us fix p' an element of \mathcal{C} , and let

$x \in \sigma^{-1}(p') \cap \text{IB}_R(p)$ and $y \in \sigma^{-1}(p') \cap \text{IB}_R(q)$. By definition of $\text{IB}_R(p)$ there exists $i \in \{1, \dots, n_p\}$ such that $x = \sigma_{i,p}^{-1}(p')$. Similarly there exists $j \in \{1, \dots, n_q\}$ such that $y = \sigma_{j,q}^{-1}(p')$. Let now $U_x := \sigma_{i,p}^{-1}(\mathcal{C}) \subset U_{i,p}$ be a neighborhood of x , $\mathcal{C} \subset V_p$ is indeed contained in the domain of definition of $\sigma_{i,p}^{-1}$. Similarly let us define $U_y := \sigma_{j,q}^{-1}(\mathcal{C}) \subset U_{j,q}$ which contains y . We have $\sigma(U_x) = \sigma(U_y) = \mathcal{C}$, and by hypothesis \mathcal{C} is a connected neighborhood of p' such that $\mu(\mathcal{C}) > 0$. The map $\phi := (\sigma_{j,q}^{-1} \circ \sigma_{i,p}) : U_x \rightarrow U_y$ is then well defined, and according to the first item (applied with $\Omega = \mathcal{C}$), ϕ has the form $\phi(z, w) = (A(z, w), B(w))$ on U_x . As in the proof of the preceding item, we can compute the matrix $d_x \phi$ and check that $d_x \sigma \cdot (1, 0) = \partial_z A(x) \times d_y \sigma \cdot (1, 0)$. We also have $\partial_z A(x) \neq 0$ and thus using again the second item we have :

$$\mathcal{D}_p(p') = [d_x \sigma \cdot (1, 0)] = [d_y \sigma \cdot (1, 0)] = \mathcal{D}_q(p').$$

The proof is then complete. \square

Lemma 2.5.5. *Let $p \in \text{Reg}_{\sigma_R}(J)$. Then the holomorphic 1-forms*

$$\omega_{j,p} := d(w \circ \sigma_{j,p}^{-1}), \quad j \in \{1, \dots, n_p\},$$

are equal modulo a multiplicative function in $\mathcal{O}^(V_p)$.*

Proof : By the point 1. of Proposition 2.5.4, for each i, j we have on V_p that $\sigma_{j,p}^{-1} \circ \sigma_{i,p} = (A_{ij}(z, w), B_{ij}(w))$ and thus $\sigma_{i,p}^* \omega_{j,p} = d(w \circ \sigma_{j,p}^{-1} \circ \sigma_{i,p}) = B'_{ij}(w) dw$. Then $\omega_{j,p} = (B'_{ij} \circ (w \circ \sigma_{i,p}^{-1})) \times \omega_{i,p}$. Because the function $w \circ \sigma_{i,p}^{-1}$ is a submersion, we have $(B'_{ij} \circ (w \circ \sigma_{i,p}^{-1})) \in \mathcal{O}^*(V_p)$. \square

Thanks to Lemma 2.5.5 we can put the following definition. We refer to Section 2.8.1 for the definition of foliations using 1-forms. We also denote \mathcal{F}_w the horizontal foliation on \mathbb{C}^2 given by the 1-form dw .

Definition 2.5.6. *For every $p \in \text{Reg}_{\sigma_R}(J)$ we denote \mathcal{F}_p the non singular foliation on V_p defined by one of the holomorphic 1-forms $\omega_{j,p}$. We note that $\sigma|_{\text{IB}_R(p)}^* \mathcal{F}_p = \mathcal{F}_w$.*

Finally one can reformulate the results of the present section as follows.

Proposition 2.5.7.

1. *If $p \in \text{Reg}_{\sigma_R}(J)$, then $T_{p'} \mathcal{F}_p = \mathcal{D}_p(p')$ for every $p' \in V_p$.*
2. *For every $p, q \in \text{Reg}_{\sigma_R}(J)$, \mathcal{F}_p and \mathcal{F}_q coincide on every connected component $\mathcal{C} \subset V_p \cap V_q$ such that $\mu(\mathcal{C}) > 0$.*

2.5.5 Construction of the foliation near singular values

For every singular value q of σ_R which belongs to the Julia set J , we construct an open neighborhood $W_q \ni q$ and a foliation \mathcal{F}_q (possibly with singularities) on W_q which is tangent to directions $(\mathcal{D}_p)_{p \in \text{Reg}_{\sigma_R}(J)}$ on neighborhoods of points of $\text{Reg}_{\sigma_R}(J) \cap W_q$, see Proposition 2.5.13. Thanks to this property \mathcal{F}_q will coincide with the regular foliations $(\mathcal{F}_p)_{p \in \text{Reg}_{\sigma_R}(J)}$ on connected components of $\bigcup_{p \in \text{Reg}_{\sigma_R}(J)} V_p \cap W_q$ which have μ -positive measure, see Proposition 2.5.15. We finish the construction of the desired foliation in Section 2.5.6 using the compactness of J .

To create \mathcal{F}_q the idea is to construct, using $d\sigma \cdot (1, 0)$, a vector field on a neighborhood of q . Let us fix a singular value $q \in J \cap E_R$. We recall that $\sigma(D_R) \supset J$ and according to Proposition 2.2.4, the fiber $\sigma^{-1}(q)$ is discrete in \mathbb{C}^2 . Thus we can consider a preimage $y \in D_R$ and an open set $U_y \subset D_R$ containing y such that $\overline{U}_y \cap \sigma^{-1}(q) = \{y\}$. Let B_y be a centered ball at y such that $\overline{B}_y \subset U_y$.

Lemma 2.5.8.

1. *There exists a connected neighborhood $U_{y,q} \subset B_y$ of y , such that $W_q := \sigma(U_{y,q})$ is a connected neighborhood of q satisfying :*

$$\sigma^{-1}(p) \cap U_y \subset B_y, \quad \forall p \in W_q. \quad (2.40)$$

2. *Let $p \in W_q \setminus E_R$ and let $n_{p,q}$ be the cardinality of $\sigma^{-1}(p) \cap U_y$, it satisfies $1 \leq n_{p,q} < +\infty$. There exist a ball $B_p \subset W_q \setminus E_R$ centered at p and a family of inverse branches $\sigma_{1,p,q}^{-1}, \dots, \sigma_{n_{p,q},p,q}^{-1}$ such that for all $p' \in B_p$:*

$$\sigma^{-1}(p') \cap U_y = \left\{ \sigma_{1,p,q}^{-1}(p'), \dots, \sigma_{n_{p,q},p,q}^{-1}(p') \right\}. \quad (2.41)$$

Proof :

1. Let us assume to the contrary that for any neighborhood of q there exists a preimage by σ of a point in this neighborhood which belongs to $U_y \setminus B_y$. Then looking at smaller and smaller neighborhoods of q we obtain a sequence $(p_n)_n$ of points converging to q , and a sequence $(x_n)_n$ of $U_y \setminus B_y$ such that $\sigma(x_n) = p_n$. Taking $x \in \overline{U}_y \setminus B_y$ a cluster value of $(x_n)_n$, we have by continuity of σ that, up to a sub-sequence, $\sigma(x) = \lim_n \sigma(x_n) = \lim_n p_n = q$. Thus we have $x \in \sigma^{-1}(q) \cap \overline{U}_y \setminus B_y$, but this is a contradiction since $\sigma^{-1}(q) \cap \overline{U}_y = \{y\} \subset B_y$.

So we have proved the existence of a neighborhood W'_q of q such that for all $p \in W'_q$, $\sigma^{-1}(p) \cap U_y \subset B_y$. Taking a small connected neighborhood $U_{y,q}$ of y included in B_y , one has $\sigma(U_{y,q}) \subset W'_q$. We complete the proof by setting $W_q := \sigma(U_{y,q})$.

2. Observe first that because $\sigma(B_y) \supset W_q$, the fiber $\sigma^{-1}(p) \cap U_y$ is not empty. As explained above the fibers of σ are discrete in \mathbb{C}^2 , thus the fiber $\sigma^{-1}(p) \cap U_y$ is finite. In particular we have $1 \leq n_{p,q} < +\infty$. Let $(\sigma_{j,p,q})_{j=1, \dots, n_{p,q}}$ be a family of inverse branches of σ such that $\sigma^{-1}(p) \cap U_y = \{\sigma_{j,p,q}^{-1}(p), 1 \leq j \leq n_{p,q}\}$. Let us fix a ball $B_p \subset W_q \setminus E_R$ centered at p such that all these inverse branches are defined on B_p . We denote

$$\text{IB}(p, q) := \bigcup_{j=1}^{n_{p,q}} \sigma_{j,p,q}^{-1}(B_p).$$

Since $\sigma^{-1}(p) \cap U_y \subset B_y$ by (2.40), up to a reduction of the radius of the ball B_p , we can assume that $\text{IB}(p, q) \subset B_y$. Let us verify that, up to a reduction of the ball B_p , we have for every $p' \in B_p$:

$$\sigma^{-1}(p') \cap U_y \subset \text{IB}(p, q), \quad (2.42)$$

in particular it gives (2.41). Indeed, to the contrary let us assume that there exists a strictly decreasing sequence $B_{n,p} \subset B_p$ of balls centered at p such that for any $n \geq 1$ there exist $p'_n \in B_{n,p}$ and $y'_n \in (\sigma^{-1}(p'_n) \cap U_y) \setminus \text{IB}(p, q)$. Then using (2.40) we have

$y'_n \in B_y$. Up to consider a sub-sequence, we deduce that $y'_n \rightarrow y' \in \overline{B_y} \setminus \text{IB}(p, q)$. We have by construction $\lim_n p'_n = p$, thus by continuity $\sigma(y') = \lim_n \sigma(y'_n) = \lim_n p'_n = p$. We have thus constructed $y' \in (\sigma^{-1}(p) \cap U_y) \setminus \text{IB}(p, q)$, but it is a contradiction since $\sigma^{-1}(p) \cap U_y \subset \text{IB}(p, q)$. The ball $B_p \subset W_q \setminus E_R$ satisfying (2.41) thus exists. \square

Definition 2.5.9. *Let us fix $q \in J \cap E_R$. Let W_q and U_y be the open sets given by Lemma 2.5.8. We define a vector field $v_q : W_q \setminus E_R \rightarrow T\mathbb{P}^2$ by :*

$$\forall p \in W_q \setminus E_R, \quad v_q(p) := \sum_{x \in \sigma^{-1}(p) \cap U_y} h(x) d_x \sigma \cdot (1, 0) \in T_p \mathbb{P}^2.$$

The function $h \in \mathcal{O}(\mathbb{C}^2)$ is chosen to have $v_q \neq 0$ on $W_q \setminus E_R$.

Lemma 2.5.10.

1. The vector field $v_q : W_q \setminus E_R \rightarrow T\mathbb{P}^2$ is holomorphic.
2. The function $h \in \mathcal{O}(\mathbb{C}^2)$ chosen such that $v_q \neq 0$ exists. Moreover one can choose h equal to a polynomial function on \mathbb{C}^2 .

Proof :

1. Let $p \in W_q \setminus E_R$ fixed. Let $(\sigma_{j,p,q}^{-1})_{j \in \{1, \dots, n_{p,q}\}}$ be the family of inverse branches defined on a ball B_p given by the second item of Lemma 2.5.8. According to (2.41) we have for any $p' \in B_p$:

$$v_q(p') = \sum_{j=1}^{n_{p,q}} (h \circ \sigma_{j,p,q}^{-1})(p') d_{\sigma_{j,p,q}^{-1}(p')} \sigma \cdot (1, 0). \quad (2.43)$$

This formula (2.43) shows that v_q is holomorphic on B_p . Since these arguments are valid for any $p \in W_q \setminus E_R$, the vector field $v_q : W_q \setminus E_R \rightarrow T\mathbb{P}^2$ is holomorphic.

2. Let us explain why the function h exists. Let us fix $p \in W_q \setminus E_R$ and let $\Phi : B_p \rightarrow (\mathbb{C}^2)^{n_{p,q}}$ be the map defined for any $p' \in B_p$ by :

$$\Phi(p') := (\sigma_{1,p,q}^{-1}(p'), \dots, \sigma_{n_{p,q},p,q}^{-1}(p')).$$

The map Φ is injective and holomorphic, thus $\Phi(B_p)$ is an open sub-set of $(\mathbb{C}^2)^{n_{p,q}}$. Let $Z_{p,q}$ be the set defined by :

$$Z_{p,q} := \{((Z_1, W_1), \dots, (Z_{n_{p,q}}, W_{n_{p,q}})) \in (\mathbb{C}^2)^{n_{p,q}}, \exists i < j : Z_i = Z_j\}.$$

$Z_{p,q}$ is a finite union of codimension 1 algebraic sub-sets of $(\mathbb{C}^2)^{n_{p,q}}$. Thus the set $\Phi(B_p) \setminus Z_{p,q}$ is a non empty open sub-set of $(\mathbb{C}^2)^{n_{p,q}}$, so let us fix an arbitrary element $p' \in B_p$ such that $\Phi(p') \notin Z_{p,q}$. Let us denote $\Phi(p') = (x'_1, \dots, x'_{n_{p,q}}) = ((Z_1, W_1), \dots, (Z_{n_{p,q}}, W_{n_{p,q}}))$, and for any $j \in \{1, \dots, n_{p,q}\}$, $\vec{V}_j := d_{x'_j} \sigma \cdot (1, 0)$. Since $x'_1, \dots, x'_{n_{p,q}}$ are not critical points of σ , the vectors $\vec{V}_1, \dots, \vec{V}_{n_{p,q}}$ are not equal to $\vec{0}$. Thus taking $\lambda_1 = 1$ and $\lambda_2 = \dots = \lambda_{n_{p,q}} = 0$, we have a non trivial linear combination $\sum_{j=1}^{n_{p,q}} \lambda_j \vec{V}_j = \vec{V}_1 \neq \vec{0}$. Since $\Phi(p') \notin Z_{p,q}$, the complex numbers $Z_1, \dots, Z_{n_{p,q}}$ are pairwise distinct, thus there exists a polynomial function $P(Z)$ of one complex variable

such that $P(Z_j) = \lambda_j$ for any $j \in \{1, \dots, n_{p,q}\}$. Now if we define $h(Z, W) := P(Z)$ on \mathbb{C}^2 , we deduce from (2.43) :

$$v_q(p') = \sum_{j=1}^{n_{p,q}} P(Z_j) \times \vec{V}_j = \sum_{j=1}^{n_{p,q}} \lambda_j \vec{V}_j = \vec{V}_1 \neq \vec{0}.$$

Finally we have proved the existence of a polynomial function h on \mathbb{C}^2 such that v_q is not identically null on $W_q \setminus E_R$. \square

We prove now that v_q extends holomorphically through E_R . We recall that E_R is included in an analytic sub-set of codimension 1 of $\sigma(D_R)$:

$$E_R \subset E'_R, \text{ with } E'_R := f^{n_R}(\text{Crit } f^{n_R}) \cap \sigma(D_R). \quad (2.44)$$

We shall use the following theorem of extension (see [36, Proposition 1.1.7]) :

Theorem 2.5.11 (Riemann extension theorem). *Let $V \subset \mathbb{C}^2$ be a connected open set and let $F \subset V$ be an analytic sub-set of $\text{codim}_{\mathbb{C}}(F) \geq 1$. Let $\varphi : V \setminus F \rightarrow \mathbb{C}$ be a holomorphic function such that there exists a constant $M < +\infty$ satisfying $|\varphi(z, w)| \leq M$, $\forall (z, w) \in V \setminus F$. Then φ admits a unique extension $\tilde{\varphi} \in \mathcal{O}(V)$.*

Proposition 2.5.12. *Up to a reduction of the neighborhood W_q of q , the vector field v_q admits a unique holomorphic extension on W_q , still denoted v_q .*

Proof : In this proof we extend v_q through the analytic sub-set E'_R defined by (2.44), it extends in particular v_q through E_R .

Up to a reduction of the neighborhood W_q of q , we can assume that there exists $\xi = (Z, W) : W_q \rightarrow \mathbb{C}^2$ a chart, holomorphic on a neighborhood of \overline{W}_q . We can define $v_* := \xi_* v_q$ the push-forward of v on $\xi(W_q \setminus E'_R) = \xi(W_q) \setminus F'$, with $F' := \xi(E'_R \cap W_q)$. The vector field v_* can be written under the form

$$v_* = (\alpha, \beta),$$

with $\alpha, \beta \in \mathcal{O}(V \setminus F')$.

Let us fix a point $p \in W_q \setminus E'_R$ and let us consider the ball $B_p \subset W_q \setminus E_R$ centered at p we have constructed in Lemma 2.5.8. Up to a reduction of this ball one can assume that $B_p \subset W_q \setminus E'_R$. Let us now give an upper bound for $|\alpha|$ and $|\beta|$ on $\xi(B_p)$ which does not depend on p . To do so we estimate $\|v_*\|$ on $\xi(B_p)$. Let $p' \in B_p$ and let us use the expression (2.43) of v_q on B_p . We get that the vector $v_*(\xi(p'))$ is given by (writing $s_j := \sigma_{j,p,q}^{-1}$ for sake of simplicity) :

$$v_*(\xi(p')) = \sum_{j=1}^{n_{p,q}} (h \circ s_j(p')) \times (d_{p'} \xi \circ d_{s_j(p')} \sigma) \cdot (1, 0).$$

So we have $\|v_*(\xi(p'))\| \leq n_{p,q} \times \sup_{D_R} |h| \times \|d_{p'} \xi\| \times \sup_{D_R} \|d\sigma \cdot (1, 0)\|$, for any $p' \in B_p$.

Since h is a holomorphic function bounded on the bidisc D_R and since $n_{p,q} \leq d^{2n_R}$ (the maximal cardinality of a fiber of σ inside the bidisc D_R), we deduce that :

$$\forall (z, w) \in \xi(B_p) \setminus F', \|v_*(z, w)\| \leq d^{2n_R} M_q,$$

where M_q is a constant independent of p :

$$M_q := \sup_{D_R} |h| \times \sup_{W_q} \|d\xi\| \times \sup_{D_R} \|d\sigma \cdot (1, 0)\| < +\infty.$$

So we have a uniform bound $d^{2n_R} M_q$ for α and β on $V \setminus F'$. By the Riemann extension theorem 2.5.11, α and β admit unique extensions $\tilde{\alpha}$ and $\tilde{\beta}$ on V . Then the vector field $\xi^*(\tilde{\alpha}, \tilde{\beta})$ extends uniquely v_q on W_q . \square

The interest of this vector field v_q is that for each $p \in \text{Reg}_{\sigma_R}(J) \cap W_q$, it is tangent to the distribution of directions \mathcal{D}_p on a neighborhood of p included in $V_p \cap W_q$:

Lemma 2.5.13. *Let $p \in \text{Reg}_{\sigma_R}(J) \cap W_q$. Let B_p be the ball constructed in Lemma 2.5.8. There exists a ball $B_{p,q} \subset B_p \cap V_p \cap W_q$ centered at p such that :*

$$\forall t \in B_{p,q}, v_q(t) \in \mathcal{D}_p(t).$$

Proof : Let B_p be the ball constructed in Lemma 2.5.8. According to (2.43), we have the following formula for any $p' \in B_p$:

$$v_q(p') = \sum_{j=1}^{n_{p,q}} (h \circ \sigma_{j,p,q}^{-1})(p') d_{\sigma_{j,p,q}^{-1}(p')} \sigma \cdot (1, 0).$$

Observe that, because $p \in \text{Reg}_{\sigma_R}(J)$, the inverse branches $\sigma_{j,p,q}^{-1}$ which appear are some of those we have constructed in Section 2.5.3 for the regular value p of σ :

$$\forall j \in \{1, \dots, n_{p,q}\}, \forall p' \in B_p, \sigma_{j,p,q}^{-1}(p') \in \text{IB}_R(p),$$

we refer to Definition 2.5.3. Hence by definition of \mathcal{D}_p in (2.39), we have for any $p' \in B_p$ and for any $j \in \{1, \dots, n_{p,q}\}$, $\left[d_{\sigma_{j,p,q}^{-1}(p')} \sigma \cdot (1, 0) \right] \in \mathcal{D}_p(p')$ and thus $v_q(p') \in \mathcal{D}_p(p')$. If $B_{p,q}$ is a ball centered at p and included in $B_p \cap V_p \cap W_q$, the conclusion follows. \square

Up to a reduction of the neighborhood $U_{y,q}$ of y , we can assume that $W_q = \sigma(U_{y,q})$ is equipped with holomorphic coordinates (z_q, w_q) . The vector field v_q can be expressed in this coordinates using two holomorphic functions α_q and β_q on W_q :

$$v_q = \alpha_q \frac{\partial}{\partial z_q} + \beta_q \frac{\partial}{\partial w_q}.$$

Let ω_q be the holomorphic 1-form defined by :

$$\omega_q := \beta_q dz_q - \alpha_q dw_q,$$

this form satisfies $\omega_q(v_q) = 0$. Since $v_q \neq 0$ on W_q , we get $\omega_q \neq 0$ on W_q .

Definition 2.5.14. *The (possibly singular) foliation induced by the 1-form ω_q on W_q is denoted \mathcal{F}_q .*

Observe that the equation $\omega_q(v_q) = 0$ ensures that the leaves of the foliation \mathcal{F}_q are tangent to the vector field v_q , at least where $v_q \neq 0$. We refer to Section 2.8.1 for more details on foliations.

Proposition 2.5.15.

1. For any $q \in J \cap E_R$ and for any $p \in \text{Reg}_{\sigma_R}(J)$, the foliations \mathcal{F}_p and \mathcal{F}_q coincide on each connected component $\mathcal{C} \subset V_p \cap W_q$ such that $\mu(\mathcal{C}) > 0$.
2. For any $(q, q') \in (J \cap E_R)^2$, the foliations \mathcal{F}_q and $\mathcal{F}_{q'}$ coincide on each connected component $\mathcal{C} \subset W_q \cap W_{q'}$ such that $\mu(\mathcal{C}) > 0$.
3. For $q \in J \cap E_R$, $\sigma^* \mathcal{F}_q = \mathcal{F}_w$ on $U_{y,q}$, where $U_{y,q}$ is defined in Lemma 2.5.8.

Proof :

1. Let \mathcal{C} be a connected component of $V_p \cap W_q$ and let us assume that $\mu(\mathcal{C}) > 0$. We have $\mu(\text{Reg}_{\sigma_R}(J)) = 1$ so there exists $\tilde{p} \in \mathcal{C} \cap \text{Reg}_{\sigma_R}(J)$. In particular $\tilde{p} \in W_q$ and according to Lemma 2.5.13, one has :

$$\forall t \in B_{\tilde{p},q} \setminus \{v_q = 0\}, \quad T_t \mathcal{F}_q = [v_q(t)] = \mathcal{D}_p(t). \quad (2.45)$$

Since the ball $B_{\tilde{p},q}$ is included in $V_{\tilde{p}}$, we have also from Propositions 2.5.7 and 2.5.4 :

$$\forall t \in B_{\tilde{p},q}, \quad T_t \mathcal{F}_{\tilde{p}} = \mathcal{D}_{\tilde{p}}(t) = \mathcal{D}_p(t). \quad (2.46)$$

We deduce from (2.45) and (2.46) that :

$$T_t \mathcal{F}_{\tilde{p}} = T_t \mathcal{F}_q, \quad \forall t \in B_{\tilde{p},q} \setminus \{v_q = 0\}.$$

Using the equivalence (2.60) in appendix, it implies $\mathcal{F}_q = \mathcal{F}_{\tilde{p}}$ on $B_{\tilde{p},q}$. By Proposition 2.5.7 the two foliations \mathcal{F}_p and $\mathcal{F}_{\tilde{p}}$ coincide on the connected component of $V_p \cap V_{\tilde{p}}$ charged by μ . Thus we deduce that \mathcal{F}_p and $\mathcal{F}_{\tilde{p}}$ coincide on the connected component $\tilde{\mathcal{C}}$ of $B_{\tilde{p},q} \cap \mathcal{C}$ containing \tilde{p} , since $\mu(\tilde{\mathcal{C}}) > 0$ because $\tilde{p} \in \text{Supp}(\mu)$. At last we have $\mathcal{F}_q = \mathcal{F}_{\tilde{p}} = \mathcal{F}_p$ on $\tilde{\mathcal{C}}$. But $\tilde{\mathcal{C}}$ is a non trivial open sub-set of \mathcal{C} , so we can use analytic continuation (Lemma 2.8.3) to conclude that finally $\mathcal{F}_q = \mathcal{F}_p$ on \mathcal{C} .

2. Let \mathcal{C} be a connected component of $W_q \cap W_{q'}$ and let us assume that $\mu(\mathcal{C}) > 0$. As in the preceding point, there exists $p \in \mathcal{C} \cap \text{Reg}_{\sigma_R}(J)$. The connected component \mathcal{C}' of $\mathcal{C} \cap V_p$ containing $p \in \text{Supp}(\mu)$ is open in \mathbb{P}^2 , and thus satisfies $\mu(\mathcal{C}') > 0$. According to the previous item (applied first to q and p and then secondly to q' and p) we have $\mathcal{F}_q = \mathcal{F}_p = \mathcal{F}_{q'}$ on \mathcal{C}' . Using again Lemma 2.8.3 we have at last $\mathcal{F}_q = \mathcal{F}_{q'}$ on \mathcal{C} .

3. Let $p \in \text{Reg}_{\sigma_R}(J) \cap W_q$ and let $B_{p,q}$ be the ball constructed in Lemma 2.5.13. Let $\sigma_{j,p,q}^{-1}$ be an inverse branch of σ which appears in the formula (2.43) and such that $x_j := \sigma_{j,p,q}^{-1}(p)$ belongs to $\text{IB}_R(p) \cap U_{y,q}$. Observe that $\sigma_{j,p,q}^{-1}(p) \in \text{IB}_R(p)$ by definition of $\text{IB}_R(p)$. The fact that we can choose j such that $x_j \in U_{y,q}$ comes from the fact that $\sigma(U_{y,q}) = W_q$, see Lemma 2.5.8.

According to the first item, the foliations \mathcal{F}_p and \mathcal{F}_q coincide on the ball $B_{p,q} \subset V_p \cap W_q$. Thus we have :

$$\sigma^* \mathcal{F}_q = \sigma^* \mathcal{F}_p \text{ on } \sigma^{-1}(B_{p,q}).$$

But we have also $\sigma^* \mathcal{F}_p = \mathcal{F}_w$ on $\text{IB}_R(p)$, see Definition 2.5.6. So we deduce that :

$$\sigma^* \mathcal{F}_q = \mathcal{F}_w \text{ on } \sigma^{-1}(B_{p,q}) \cap \text{IB}_R(p),$$

and the point x_j belongs to $\sigma^{-1}(B_{p,q}) \cap \text{IB}_R(p) \cap U_{y,q}$. So we have found a non trivial open sub-set of $U_{y,q}$ on which $\sigma^* \mathcal{F}_q$ and \mathcal{F}_w coincide, thus we have $\sigma^* \mathcal{F}_q|_{U_{y,q}} = \mathcal{F}_w|_{U_{y,q}}$ by analytic continuation (see Lemma 2.8.3). \square

2.5.6 Finite covering of the Julia set by foliated neighborhoods

We have constructed for each point $p \in J$ a neighborhood \mathcal{O}_p of p ($\mathcal{O}_p := V_p$ if $p \in \text{Reg}_{\sigma_R}(J)$ or $\mathcal{O}_p = W_p$ if $p \in J \cap E_R$), equipped with a foliation \mathcal{F}_p on \mathcal{O}_p . Propositions 2.5.7 and 2.5.15 ensure that two foliations \mathcal{F}_p and \mathcal{F}_q of this collection coincide on $\mathcal{O}_p \cap \mathcal{O}_q$ when the connected components of $\mathcal{O}_p \cap \mathcal{O}_q$ are charged by μ . But if there is a connected component $\mathcal{C} \subset \mathcal{O}_p \cap \mathcal{O}_q$ which is not charged by μ , we can not conclude that \mathcal{F}_p and \mathcal{F}_q coincide on \mathcal{C} .

To avoid this problem we use the compactness of J to extract an adapted sub-covering of the covering $\bigcup_{p \in J} \mathcal{O}_p \supset J$. Before extracting, we modify the open sets \mathcal{O}_p as follows :

- We can reduce the neighborhoods $(\mathcal{O}_p)_{p \in J}$ to have $\mathcal{O}_p = B(p, 2r_p)$ a ball of radius $2r_p > 0$, and we denote $\mathcal{O}'_p := B(p, r_p)$ the half ball.
- We can extract from $(\mathcal{O}'_p)_{p \in J}$ a finite covering of J :

$$J \subset \bigcup_{k=1}^L \mathcal{O}'_{p_k} = \left(\bigcup_{i=1}^N V'_{p_i} \right) \cup \left(\bigcup_{j=1}^M W'_{q_j} \right). \quad (2.47)$$

- We modify, for any $k \in \{1, \dots, L\}$, the open sets \mathcal{O}'_{p_k} to deal with problems of positiveness with respect to μ :

$$\tilde{\mathcal{O}}_{p_k} := \mathcal{O}'_{p_k} \setminus \left(\bigcup_l \overline{\mathcal{O}'_{p_k} \cap \mathcal{O}'_{p_l}} \right), \quad (2.48)$$

where the union runs on the integers $l \in \{1, \dots, L\}$ such that $\overline{\mathcal{O}'_{p_k} \cap \mathcal{O}'_{p_l}} \cap J \neq \emptyset$.

- Observe that because we do not deprive from \mathcal{O}'_{p_k} elements of J , we still have an open covering of J :

$$J \subset \bigcup_{k=1}^L \tilde{\mathcal{O}}_{p_k} = \left(\bigcup_{i=1}^N \tilde{V}_{p_i} \right) \cup \left(\bigcup_{j=1}^M \tilde{W}_{q_j} \right) =: \mathcal{V}. \quad (2.49)$$

With this covering we can prove the following lemma :

Lemma 2.5.16. *For any integers $k, l \in \{1, \dots, L\}$, if the intersection $\tilde{\mathcal{O}}_{p_k} \cap \tilde{\mathcal{O}}_{p_l}$ is not empty then the foliations \mathcal{F}_{p_k} and \mathcal{F}_{p_l} are equal on $\mathcal{O}_{p_k} \cap \mathcal{O}_{p_l}$.*

Proof : If $\tilde{\mathcal{O}}_{p_k} \cap \tilde{\mathcal{O}}_{p_l} \neq \emptyset$, by construction (2.48) it means that $\overline{\mathcal{O}'_{p_k} \cap \mathcal{O}'_{p_l}} \cap J \neq \emptyset$ and thus $\mu(\mathcal{O}_{p_k} \cap \mathcal{O}_{p_l}) > 0$. Moreover the intersection $\mathcal{O}_{p_k} \cap \mathcal{O}_{p_l}$ is connected since \mathcal{O}_{p_k} and \mathcal{O}_{p_l} are balls. Then according to Propositions 2.5.7 and 2.5.15, the two foliations \mathcal{F}_{p_k} and \mathcal{F}_{p_l} are equal on $\mathcal{O}_{p_k} \cap \mathcal{O}_{p_l}$. \square

In consequence of this lemma, the foliations $(\mathcal{F}_{p_i})_{i \in \{1, \dots, N\}}$ and $(\mathcal{F}_{q_j})_{j \in \{1, \dots, M\}}$ patch all together on \mathcal{V} to form a (possibly singular) foliation :

Definition 2.5.17. *We denote*

$$\mathcal{F} := \left(\bigcup_{i=1}^N \mathcal{F}_{p_i} |_{\tilde{V}_{p_i}} \right) \cup \left(\bigcup_{j=1}^M \mathcal{F}_{q_j} |_{\tilde{W}_{q_j}} \right) \quad (2.50)$$

the foliation on the covering \mathcal{V} (2.49) given by this patching.

To finish the proof of Theorem B.I it remains to show that \mathcal{F} is f^N -invariant on \mathcal{V} . It is done in the next section and use the Proposition 2.5.18 below that we prove in this section.

To do so we introduce an open set $\mathcal{W} \subset \mathbb{C}^2$ such that $\sigma(\mathcal{W}) = \mathcal{V}$ and on which we are able to compute $\sigma^*\mathcal{F}$. The set \mathcal{W} is defined as follows :

— We denote for each p_i :

$$\widetilde{\text{IB}}_R(p_i) := \text{IB}_R(p_i) \cap \sigma^{-1}(\widetilde{V}_{p_i}).$$

— We denote for each q_j :

$$\widetilde{U}_{y_j, q_j} := U_{y_j, q_j} \cap \sigma^{-1}(\widetilde{W}_{q_j}),$$

where U_{y_j, q_j} is defined in Lemma 2.5.8. The open set \mathcal{W} is then given by :

$$\mathcal{W} := \left(\bigcup_{i=1}^N \widetilde{\text{IB}}_R(p_i) \right) \cup \left(\bigcup_{j=1}^M \widetilde{U}_{y_j, q_j} \right). \quad (2.51)$$

For each i and j one has $\sigma(\widetilde{\text{IB}}_R(p_i)) = \widetilde{V}_{p_i}$ and $\sigma(\widetilde{U}_{y_j, q_j}) = \widetilde{W}_{q_j}$, and thus

$$\sigma(\mathcal{W}) = \mathcal{V}. \quad (2.52)$$

Proposition 2.5.18. *The foliation $\sigma^*\mathcal{F}$ and \mathcal{F}_w satisfy :*

$$\sigma^*\mathcal{F} = \mathcal{F}_w \text{ on } \mathcal{W}.$$

Proof : By construction (2.51) of \mathcal{W} and by definition (2.50) of \mathcal{F} , it is sufficient to prove that for each i the foliation $\sigma^*\mathcal{F}_{p_i}$ coincide with \mathcal{F}_w on $\widetilde{\text{IB}}_R(p_i)$, and that for each j the foliation $\sigma^*\mathcal{F}_{q_j}$ coincide with \mathcal{F}_w on \widetilde{U}_{y_j, q_j} . But it is a direct consequence of Definition 2.5.6 and of the third item of Proposition 2.5.15. \square

2.5.7 Invariance of the foliation near the Julia set

We want to prove that \mathcal{F} is invariant by f^N in the sense of Definition 2.8.4. We recall that the integer $N \geq 1$ is the period of the repelling cycle a . Without loss of generality, we can assume that $N = 1$. The idea is to extend the result of Proposition 2.5.18 on $\sigma^{-1}(\mathcal{V})$. To do so we need the following lemma which allows us to lift continuous path by σ .

Lemma 2.5.19. *Let $r > 0$ and $D_r := \mathbb{D}_r^2$. Let us denote C_r the critical set of $\sigma|_{D_r} : D_r \rightarrow \sigma(D_r)$, and E_r the critical values of this map. Let $x_0 \in D_r \setminus C_r$ and let $\gamma : [0, 1] \rightarrow \sigma(D_r) \setminus E_r$ be a Lipschitz path such that $\gamma(0) = \sigma(x_0)$. Then there exists an unique continuous path $\tilde{\gamma} : [0, 1] \rightarrow D_r \setminus C_r$ such that $\tilde{\gamma}(0) = x_0$ and $\sigma \circ \tilde{\gamma}(t) = \gamma(t)$, for any $t \in [0, 1]$.*

Proof : Since σ is locally injective on $D_r \setminus C_r$, one can check that if $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ are two desired lifts of γ by σ , then the set $\{t \in [0, 1] : \tilde{\gamma}_1(t) = \tilde{\gamma}_2(t)\}$ is not empty, open and closed, thus equal to $[0, 1]$. It gives the uniqueness of $\tilde{\gamma}$ if it exists.

It remains now to prove the existence of $\tilde{\gamma}$. To do so we proceed in several steps. First we need to control the norm of the differential of inverse branches of σ defined on a neighborhood of $\gamma([0, 1])$. For each $(x, p) \in (D_r \setminus C_r) \times (\sigma(D_r) \setminus E_r)$ such that $\sigma(x) = p$, let $\sigma_{x,p}^{-1}$ be an inverse branch of σ such that $\sigma_{x,p}^{-1}(p) = x$. Let $V_{x,p}$ be a (convex) neighborhood of p such that $\bar{V}_{x,p} \subset \sigma(D_r) \setminus E_r$ and such that $\sigma_{x,p}^{-1}$ is defined on $\bar{V}_{x,p}$, we also denote $U_{x,p} := \sigma_{x,p}^{-1}(V_{x,p})$. We can assume that $\bar{U}_{x,p} \subset D_r \setminus E_r$. Since $\gamma([0, 1])$ is compact and since $\gamma([0, 1]) \subset \sigma(D_r) \setminus E_r$, there exists a finitely many inverse branches $\sigma_{x_1, p_1}^{-1}, \dots, \sigma_{x_N, p_N}^{-1}$ such that :

$$\gamma([0, 1]) \subset \bigcup_{j=1}^N V_{x_j, p_j}. \quad (2.53)$$

We denote $U_{x_0} := \bigcup_{j=1}^N U_{x_j, p_j}$ which satisfies $\bar{U}_{x_0} \subset D_r \setminus E_r$. We also denote :

$$C_0 := \max_{j=1, \dots, N} \max_{\bar{V}_{x_j, p_j}} \|d\sigma_{x_j, p_j}^{-1}\| < +\infty.$$

Let κ be the lipschitz constant of γ . We introduce now the set \mathcal{X} defined by :

$$\begin{aligned} \mathcal{X} := \{t \in [0, 1], \exists \tilde{\gamma} \in C^0([0, t], U_{x_0}) : \tilde{\gamma}(0) = x_0, \sigma \circ \tilde{\gamma}(s) = \gamma(s), \forall s \in [0, t], \\ \|\tilde{\gamma}(s_1) - \tilde{\gamma}(s_2)\| \leq C_0 \kappa |s_1 - s_2|, \forall s_1, s_2 \in [0, t]\}. \end{aligned}$$

We denote $p_0 := \gamma(0)$. There exists $j_0 \in \{1, \dots, N\}$ such that $p_0 \in V_{x_{j_0}, p_{j_0}}$ and $x_0 = \sigma_{x_{j_0}, p_{j_0}}^{-1}(p_0)$. Indeed, we can add the set V_{x_0, p_0} associated to (x_0, p_0) in the covering (2.53) to ensure that j_0 exists. Let $t_0 \in]0, 1]$ such that $\gamma([0, t_0]) \subset V_{x_{j_0}, p_{j_0}}$, and let $\tilde{\gamma}_0 : [0, t_0] \rightarrow U_{x_0}$ defined for any $s \in [0, t_0]$ by :

$$\tilde{\gamma}_0(s) := \sigma_{x_{j_0}, p_{j_0}}^{-1} \circ \gamma(s).$$

By definition of the constants κ and C_0 , one can observe that $t_0 \in \mathcal{X}$ by using the map $\tilde{\gamma}_0 : [0, t_0] \rightarrow U_{x_0}$. Actually one can observe that $[0, t_0] \subset \mathcal{X}$. Let now t be any element of \mathcal{X} . By definition of \mathcal{X} there exists a map $\tilde{\gamma} : [0, t] \rightarrow U_{x_0}$ such that $\tilde{\gamma}(0) = x_0$, $\sigma \circ \tilde{\gamma} = \gamma|_{[0, t]}$ and

$$\|\tilde{\gamma}(s_1) - \tilde{\gamma}(s_2)\| \leq C_0 \kappa |s_1 - s_2|, \forall s_1, s_2 \in [0, t]. \quad (2.54)$$

If $t = 1$ then $\tilde{\gamma}$ is the desired lift of σ and the proof is complete. So let us assume that $t < 1$. Let $j \in \{1, \dots, N\}$ such that $\gamma(t) \in V_{x_j, p_j}$ and such that $\tilde{\gamma}(t) = \sigma_{x_j, p_j}^{-1}(\gamma(t))$. Observe that the existence of j is given by the fact that $\tilde{\gamma}(t) \in U_{x_0}$. Let $\varepsilon > 0$ be such that $t < t + \varepsilon < 1$ and $\gamma([t, t + \varepsilon]) \subset V_{x_j, p_j}$. Then one can extend $\tilde{\gamma}$ to $\tilde{\tilde{\gamma}} : [0, t + \varepsilon] \rightarrow U_{x_0}$ by defining $\tilde{\tilde{\gamma}}|_{[0, t]} := \tilde{\gamma}$ and

$$\forall s \in [t, t + \varepsilon], \tilde{\tilde{\gamma}}(s) := \sigma_{x_j, p_j}^{-1} \circ \gamma(s).$$

Observe that $\tilde{\tilde{\gamma}}(t) = \sigma_{x_j, p_j}^{-1} \circ \gamma(t)$, thus $\tilde{\tilde{\gamma}}$ is well defined and is continuous on $[0, t + \varepsilon]$. Let $s_1 \leq s_2$ be two elements of $[0, t + \varepsilon]$. If $s_2 \leq t$ then $\|\tilde{\tilde{\gamma}}(s_1) - \tilde{\tilde{\gamma}}(s_2)\| \leq C_0 \kappa |s_2 - s_1|$

by using (2.54). If $t \leq s_1$ then $\|\tilde{\gamma}(s_1) - \tilde{\gamma}(s_2)\| \leq C_0\kappa|s_2 - s_1|$ by using the fact that $\tilde{\gamma}(s_i) = \sigma_{x_j, p_j}^{-1} \circ \gamma(s_i)$, $i = 1, 2$. If $s_1 \leq t \leq s_2$ then by the same arguments :

$$\begin{aligned} \|\tilde{\gamma}(s_1) - \tilde{\gamma}(s_2)\| &\leq \|\tilde{\gamma}(s_1) - \tilde{\gamma}(t)\| + \|\tilde{\gamma}(t) - \tilde{\gamma}(s_2)\| \\ &\leq C_0\kappa(t - s_1) + C_0\kappa(s_2 - t) = C_0\kappa(s_2 - s_1). \end{aligned}$$

Thus we have :

$$\|\tilde{\gamma}(s_1) - \tilde{\gamma}(s_2)\| \leq C_0\kappa|s_2 - s_1|, \quad \forall s_1, s_2 \in [0, t + \varepsilon].$$

So the existence of the map $\tilde{\gamma} : [0, t + \varepsilon] \rightarrow U_{x_0}$ shows that $[0, t + \varepsilon] \subset \mathcal{X}$.

We deduce that either $\mathcal{X} = [0, t_{\max}[$ with $t_{\max} < 1$, or $\mathcal{X} = [0, 1[$ or $\mathcal{X} = [0, 1]$. To finish the proof it remains to show that the case $\mathcal{X} = [0, t_{\max}[$ and the case $\mathcal{X} = [0, 1[$ are not possible. To do so it is sufficient to show that if $\mathcal{X} \supset [0, s_0[$ for some $s_0 \in]0, 1]$, then $s_0 \in \mathcal{X}$. So let $s_0 \in]0, 1]$ be such that $\mathcal{X} \supset [0, s_0[$ and let us prove that $s_0 \in \mathcal{X}$. Let now (t_n) be a strictly increasing sequence of $[0, s_0[$ which tends to s_0 , and for each $n \geq 0$ let $\tilde{\gamma}_n : [0, t_n] \rightarrow U_{x_0}$ be a continuous map which is given by definition of $t_n \in \mathcal{X}$. We need the following Fact, which follows from the local injectivity of σ on $D_r \setminus C_r$, as explained at the beginning of the proof.

Fact 2.5.20. *For every $t \in [0, 1]$, if it exists, the lift of $\gamma|_{[0, t]}$ by σ inside $D_r \setminus C_r$ taking x_0 as value at $t = 0$, is unique.*

According to this fact we have $\tilde{\gamma}_{n+1}|_{[0, t_n]} = \tilde{\gamma}_n$ for any $n \geq 0$. In other words, the map $\tilde{\gamma} : [0, s_0[\rightarrow U_{x_0}$ defined for any $t \in [0, s_0[$ by :

$$\tilde{\gamma}(t) := \tilde{\gamma}_n(t) \text{ for any } n \text{ such that } t \in [0, t_n],$$

is well defined and is continuous on $[0, s_0[$. By definition of \mathcal{X} observe that the map $\tilde{\gamma}$ satisfies the properties : $\tilde{\gamma}(0) = x_0$, $\sigma \circ \tilde{\gamma}(t) = \gamma(t)$ for any $t \in [0, s_0[$, and :

$$\|\tilde{\gamma}(s_1) - \tilde{\gamma}(s_2)\| \leq C_0\kappa|s_1 - s_2| \text{ for any } s_1, s_2 \in [0, s_0[. \quad (2.55)$$

It remains to prove now that $\tilde{\gamma}$ admits an extension by continuity at s_0 with $\tilde{\gamma}(s_0) \in U_{x_0}$. Indeed in this case, by continuity of σ we would have the existence of a lift $\tilde{\gamma} : [0, s_0] \rightarrow U_{x_0}$ of γ by σ and we would have $s_0 \in \mathcal{X}$ by continuity arguments. So let us prove now that $\tilde{\gamma}$ extends by continuity at s_0 . To do so let $(s_n)_n$ be a Cauchy sequence of $[0, s_0[$. By (2.55) we have for any $n, m \in \mathbb{N}$:

$$\|\tilde{\gamma}(s_n) - \tilde{\gamma}(s_m)\| \leq C_0\kappa|s_n - s_m|,$$

thus $(\tilde{\gamma}(s_n))_n$ is a Cauchy sequence of U_{x_0} . We deduce that the sequence $(\tilde{\gamma}(s_n))_n$ converges in $\overline{U_{x_0}}$ to some limit l . Similarly let $(s'_n)_n$ be another Cauchy sequence, and let l' be the limit of $(\tilde{\gamma}(s'_n))$. By using again (2.55) we have for any $n \geq 0$:

$$\begin{aligned} |l - l'| &\leq |l - \tilde{\gamma}(s_n)| + |\tilde{\gamma}(s_n) - \tilde{\gamma}(s'_n)| + |\tilde{\gamma}(s'_n) - l'| \\ &\leq |l - \tilde{\gamma}(s_n)| + C_0\kappa|s_n - s'_n| + |\tilde{\gamma}(s'_n) - l'|. \end{aligned}$$

Thus if the two Cauchy sequences (s_n) and (s'_n) tends to s_0 , then the right-hand side of the above inequality tends to 0 and thus $l = l'$. Observe that $l \in \overline{U_{x_0}}$. But

$\sigma(l) = \gamma(s_0) \in \bigcup_{j=1}^N V_{x_j, p_j}$, hence l actually belongs to U_{x_0} . So $\tilde{\gamma}$ can be defined at s_0 by $\tilde{\gamma}(s_0) := l$, and $\tilde{\gamma} : [0, 1] \rightarrow U_{x_0}$ is a continuous lift of γ by σ . Finally $s_0 \in \mathcal{X}$ as desired. We conclude that $\mathcal{X} = [0, 1]$ and the proof is complete. \square

Applying this lemma to a continuous path with end point in the Julia set, we get :

Proposition 2.5.21. *Let \mathcal{C} be a connected component of $\sigma^{-1}(\mathcal{V})$. Then there exists $x \in \mathcal{C} \setminus (\text{Crit } \sigma)$ such that $\sigma(x) \in J \setminus E_R$.*

Proof : Let $\Omega \subset \mathcal{V}$ be the connected component of \mathcal{V} which contains the connected set $\sigma(\mathcal{C}) \subset \mathcal{V}$. By construction (2.49) of \mathcal{V} , one can observe that the open set Ω satisfies $\Omega \cap J \neq \emptyset$. Let now $r > 0$ be large enough such that $\sigma(D_r) \supset \mathcal{V}$ and such that $D_r \cap \mathcal{C} \neq \emptyset$, where D_r denote the bidisc of radius r . Let E_r be the critical values of the map $\sigma|_{D_r} : D_r \rightarrow \sigma(D_r)$. Then $\sigma(D_r \cap \mathcal{C})$ is an open set included in Ω , and since $E_r \cup E_R$ is included in a analytic sub-set of \mathbb{P}^2 of codimension 1 (see Proposition 2.2.4), we have that $\emptyset \neq \sigma(D_r \cap \mathcal{C}) \setminus (E_r \cup E_R) \subset \Omega \setminus (E_r \cup E_R)$. So let us fix $x_0 \in D_r \cap \mathcal{C}$ such that $p_0 := \sigma(x_0) \in \Omega \setminus (E_r \cup E_R)$. Actually the connected set Ω is pathwise connected, since $E_r \cup E_R$ is included in a codimension 1 sub-set of \mathbb{P}^2 , the set $\Omega \setminus (E_r \cup E_R)$ is also pathwise connected. Since Ω is an open set such that $J \cap \Omega \neq \emptyset$ and since $\mu(E_r \cup E_R) = 0$, we have that $J \cap \Omega \setminus (E_r \cup E_R) \neq \emptyset$. So there exist $p_1 \in J \cap \Omega \setminus (E_r \cup E_R)$ and a continuous path $\gamma : [0, 1] \rightarrow \Omega \setminus (E_r \cup E_R)$ such that $\gamma(0) = p_0$ and $\gamma(1) = p_1$. We can assume that this path is Lipschitz on $[0, 1]$. According to Lemma 2.5.19 there exists $\tilde{\gamma} : [0, 1] \rightarrow D_r \setminus (\text{Crit } \sigma)$ a continuous path such that $\tilde{\gamma}(0) = x_0$ and such that $\sigma \circ \tilde{\gamma} = \gamma$ on $[0, 1]$. Let $x_1 := \tilde{\gamma}(1)$, we have $\sigma(x_1) = p_1 \in J \setminus E_R$. Moreover $\tilde{\gamma}([0, 1]) \subset \sigma^{-1}(\mathcal{V})$ since $\sigma(\tilde{\gamma}([0, 1])) = \gamma([0, 1]) \subset \Omega \subset \mathcal{V}$. Since $\mathcal{C} \ni x_0 = \tilde{\gamma}(0)$ and since \mathcal{C} is a connected component of $\sigma^{-1}(\mathcal{V})$, we deduce that $\tilde{\gamma}([0, 1]) \subset \mathcal{C}$ (because $\tilde{\gamma}([0, 1])$ is connected). This shows that $x_1 \in \mathcal{C}$ and the proof is complete. \square

We can prove now that the foliation $\sigma^*\mathcal{F}$ is horizontal on $\sigma^{-1}(\mathcal{V})$:

Lemma 2.5.22. *The foliation $\sigma^*\mathcal{F}$ defined on $\sigma^{-1}(\mathcal{V})$ coincide with the horizontal foliation \mathcal{F}_w of \mathbb{C}^2 .*

Proof : We have to show that on each connected component \mathcal{C} of $\sigma^{-1}(\mathcal{V})$, the foliations $\sigma^*\mathcal{F}$ and \mathcal{F}_w coincide on \mathcal{C} . Let now \mathcal{C} be such a connected component. According to Proposition 2.5.21 there exists $x_0 \in \mathcal{C} \setminus (\text{Crit } \sigma)$ such that $p := \sigma(x_0) \in J \setminus E_R$. Thus there exists $U_{x_0} \subset \mathcal{C}$ a connected open neighborhood of x_0 such that $\sigma_{x_0} := \sigma|_{U_{x_0}} : U_{x_0} \rightarrow \sigma(U_{x_0}) =: V_{x_0}$ is a biholomorphism. By (2.52) there exists $y \in \mathcal{W}$ such that $\sigma(y) = p$. Since $y \in D_R$ and since p is not a critical value of $\sigma|_{D_R}$, y is not a critical value of σ . So let $U_y \subset \mathcal{W}$ be a open neighborhood of y such that $\sigma_y := \sigma|_{U_y} : U_y \rightarrow \sigma(U_y) =: V_y$ is a biholomorphism. Up to a reduction of U_{x_0} , we can assume that $V_{x_0} \subset V_y$.

Let us define $\phi := \sigma_y^{-1} \circ \sigma_{x_0} : U_{x_0} \rightarrow \sigma_y^{-1}(V_{x_0})$, this map is well defined because $V_{x_0} \subset V_y$. Observe that V_{x_0} is a neighborhood of $p \in J$, thus this set has positive μ -measure. By construction V_{x_0} is also connected. Thus according to the first item of Proposition 2.5.4, the map ϕ has the form $\phi(z, w) = (A(z, w), B(w))$. According to Proposition 2.5.18 we have $\sigma_y^*(\mathcal{F}|_{V_y}) = \mathcal{F}_w|_{U_y}$, thus $\mathcal{F}|_{V_y} = (\sigma_y)_* [\sigma_y^*(\mathcal{F}|_{V_y})] = (\sigma_y)_*(\mathcal{F}_w|_{U_y})$ is a foliation which is described by the 1-form $(\sigma_y)_* dw$ (because \mathcal{F}_w is

described by the 1–form dw). Then the pull-back $\sigma_{x_0}^*(\mathcal{F}|_{V_{x_0}})$ is described on U_{x_0} by the 1–form :

$$\sigma_{x_0}^*(\sigma_y)_*dw = \phi^*dw = B'(w)dw.$$

We deduce that $\sigma_{x_0}^*\mathcal{F}|_{U_{x_0}} = \mathcal{F}_w|_{U_{x_0}}$, since the function $B'(w)$ is not trivial (ϕ is a biholomorphism) and since that \mathcal{F}_w is represented by the 1–form dw . This shows that $\sigma^*\mathcal{F} = \mathcal{F}_w$ on $\emptyset \neq U_{x_0} \subset \mathcal{C}$. Thus $\sigma^*\mathcal{F} = \mathcal{F}_w$ on \mathcal{C} by analytic continuation (Lemma 2.8.3). The conclusion follows. \square

Proposition 2.5.23. *The foliation \mathcal{F} is f –invariant on \mathcal{V} , i.e. $f^*\mathcal{F} = \mathcal{F}$ on $f^{-1}(\mathcal{V}) \cap \mathcal{V} \supset J$ which is not empty.*

Proof : It is sufficient to prove that for any $p, q \in \mathcal{V}$ such that $f(p) = q$, there exists $W \subset \mathcal{V}$ an open neighborhood of p and there exists $V \subset \mathcal{V}$ an open neighborhood of q satisfying $f(W) \subset V$ and such that $(f^*\mathcal{F}|_V)|_W = \mathcal{F}|_W$.

Let x be a preimage by σ of p and let $y := D(x)$, we have that $\sigma(y) = f(\sigma(x)) = f(p) = q$: y is a σ –preimage of q . We can consider an open set $U_x \subset \sigma^{-1}(\mathcal{V})$ containing x such that $W := \sigma(U_x)$ is a open neighborhood of p in \mathcal{V} . Similarly we can consider a neighborhood $U_y \subset \sigma^{-1}(\mathcal{V})$ of y such that $V := \sigma(U_y)$ is a neighborhood of q in \mathcal{V} . Since $D(x) = y$ and $f(p) = q$, we can reduce the neighborhood U_x of x such that $D(U_x) \subset U_y$ and such that $f(W) \subset V$.

We denote $\sigma_p := \sigma|_{U_x} : U_x \longrightarrow W$ and $\sigma_q := \sigma|_{U_y} : U_y \longrightarrow V$. According to Lemma 2.5.22, the foliations $\sigma_p^*\mathcal{F}$ and $\sigma_q^*\mathcal{F}$ satisfy :

$$\sigma_p^*\mathcal{F} = \mathcal{F}_w|_{U_x} \text{ and } \sigma_q^*\mathcal{F} = \mathcal{F}_w|_{U_y}. \quad (2.56)$$

Let $\mathcal{G} := (f^*\mathcal{F}|_V)|_W$, to finish the proof it remains to prove that $\mathcal{G} = \mathcal{F}|_W$. To do so we compute the pull-back $\sigma_p^*\mathcal{G}$:

$$\begin{aligned} \sigma_p^*\mathcal{G} &= ((f \circ \sigma_p)^*(\mathcal{F}|_V))|_{U_x} = ((\sigma \circ D)^*\mathcal{F}|_V)|_{U_x} \\ &= ((\sigma_q \circ D)^*\mathcal{F}|_V)|_{U_x} \\ &= (D^*(\sigma_q^*\mathcal{F}|_V))|_{U_x}, \end{aligned}$$

because $\sigma \circ D = \sigma_q \circ D$ on U_x since $D(U_x) \subset U_y$. Using then (2.56) we deduce that :

$$\sigma_p^*\mathcal{G} = (D^*(\sigma_q^*\mathcal{F}|_V))|_{U_x} = (D^*\mathcal{F}_w)|_{U_x}.$$

Knowing that \mathcal{F}_w is described by the 1–form dw and that $D^*dw = \chi_2 dw$ with $\chi_2 \in \mathbb{C}^*$, we have that $D^*\mathcal{F}_w = \mathcal{F}_w$ on \mathbb{C}^2 and so we have :

$$\sigma_p^*\mathcal{G} = \mathcal{F}_w|_{U_x}.$$

Using again (2.56) we have $\sigma_p^*(\mathcal{F}|_W) = \mathcal{F}_w|_{U_x}$ and thus :

$$\sigma_p^*\mathcal{G} = \sigma_p^*(\mathcal{F}|_W).$$

If σ_p is injective on U_x one can push forward by $\sigma_p : U_x \longrightarrow W$ this last equality and one gets $\mathcal{G} = \mathcal{F}|_W$. If σ_p is not injective on U_x , we can consider a non trivial open set $U'_x \subset U_x$ on which σ_p is injective (the critical values of σ have empty interior) and push forward by $\sigma_p|_{U'_x} : U'_x \longrightarrow \sigma(U'_x) =: W'$ to get $\mathcal{G}|_{W'} = \mathcal{F}|_{W'}$. Up to a reduction of the neighborhood U_x of x , we can assume that $W = \sigma(U_x)$ is connected and thus $\mathcal{G} = \mathcal{F}|_W$ by analytic continuation (see Lemma 2.8.3). \square

2.6 Proof of Proposition C.I

2.6.1 From local to global invariance

We prove the first part of Proposition C.I. Let f be a holomorphic map of \mathbb{P}^2 of degree $d \geq 2$, and let $V \subset \mathbb{P}^2$ be an open set such that f admits a fixed point a in V . We can suppose that $a \in V_0 := V \cap U_0$, where U_0 is the affine chart $\{[z_0 : z_1 : z_2] : z_2 \neq 0\}$. Let (x, y) be the affine coordinates of U_0 . Let \mathcal{F} be a foliation defined on V_0 and let us assume that \mathcal{F} is f -invariant.

Lemma 2.6.1. *If \mathcal{F} admits an extension \mathcal{F}' on \mathbb{P}^2 , then \mathcal{F}' is also f -invariant.*

Proof : By Theorem 2.8.7 the foliation \mathcal{F}' is defined on U_0 by a polynomial 1-form $\omega' = P(x, y)dx + Q(x, y)dy$, where P and Q are polynomials without any common factor. We shall compute the form $f^*\omega'$. Let us write $f = [F_0 : F_1 : F_2]$. We denote $\tilde{F}_i(x, y) := F_i(x, y, 1)$ and $\tilde{P} := P \circ H$, $\tilde{Q} := Q \circ H$ with $H := (\tilde{F}_0/\tilde{F}_2, \tilde{F}_1/\tilde{F}_2)$. Then

$$f^*\omega' = \tilde{P} \times d(\tilde{F}_0/\tilde{F}_2) + \tilde{Q} \times d(\tilde{F}_1/\tilde{F}_2) \text{ on } U_0 \cap f^{-1}(U_0),$$

and one can compute explicitly $d(\tilde{F}_0/\tilde{F}_2)$ and $d(\tilde{F}_1/\tilde{F}_2)$ to get

$$f^*\omega' = \frac{R}{h}dx + \frac{S}{h}dy \text{ on } U_0 \cap f^{-1}(U_0),$$

where R, S are two polynomials on \mathbb{C}^2 and $h := \tilde{F}_2^2 \in \mathcal{O}^*(U_0 \cap f^{-1}(U_0))$. Let $k = \text{GCD}(R, S)$ and let \tilde{R}, \tilde{S} be polynomials such that $R = k\tilde{R}$ and $S = k\tilde{S}$. Then :

$$f^*\omega' = \frac{k}{h} \times (\tilde{R}dx + \tilde{S}dy) \text{ on } f^{-1}(U_0) \cap U_0, \quad (2.57)$$

where $\text{GCD}(\tilde{R}, \tilde{S}) = 1$. According to Theorem 2.8.7 the polynomial 1-form $\omega'' := \tilde{R}dx + \tilde{S}dy$ on U_0 defines a foliation \mathcal{F}'' on \mathbb{P}^2 . Moreover on $f^{-1}(U_0) \cap U_0$ we have by construction $\omega'' \wedge f^*\omega' = 0$, it means that the two foliations \mathcal{F}'' and $f^*\mathcal{F}$ are equal on $f^{-1}(U_0) \cap U_0$. By analytic continuation (Lemma 2.8.3) we deduce that $\mathcal{F}'' = f^*\mathcal{F}$ on U_0 . Every connected non empty open set of \mathbb{P}^2 intersect the affine chart U_0 , thus by applying again Lemma 2.8.3 we deduce that $\mathcal{F}'' = f^*\mathcal{F}$ on \mathbb{P}^2 .

Now let us verify that $\mathcal{F}'' = \mathcal{F}'$. According to Section 2.8.3, it suffices to prove that ω' and ω'' define the same foliation on U_0 . Let $U_a \subset V_0$ be a connected open set containing a and let ω be a 1-form defining \mathcal{F} on U_a . Since a is fixed by f , it belongs to $f^{-1}(U_a) \cap U_a =: V_a$. But $\mathcal{F}'|_{V_0} = \mathcal{F}$ and \mathcal{F} is f -invariant by assumption, so the foliation $\mathcal{F}'|_{V_0}$ is also f -invariant and thus $f^*\omega' \wedge \omega' = 0$ on V_a . Since we have on $V_a \setminus \{k = 0\}$ that $\frac{h}{k}f^*\omega' = \omega''$ (by (2.57)), we deduce that :

$$\omega'' \wedge \omega' = \frac{h}{k} (f^*\omega' \wedge \omega') = 0, \text{ on } V_a \setminus \{k = 0\}.$$

It implies that $\mathcal{F}'' = \mathcal{F}'$ on $V_a \setminus \{k = 0\}$. The open set $V_a \setminus \{k = 0\}$ is not empty otherwise we would have $k = 0$ on U_0 (k is a polynomial map on the affine chart U_0) and thus by (2.57) we would have $f^*\omega' = 0$ on $f^{-1}(U_0) \cap U_0$, which can not happen since ω' is not identically equal to 0 on U_0 . By analytic continuation (Lemma 2.8.3) we deduce that $\mathcal{F}'' = \mathcal{F}'$ on \mathbb{P}^2 . The proof is complete. \square

2.6.2 Semi-extremality implies an invariant pencil of lines

We prove in this section the second part of Proposition C.I. Let f be a semi-extremal map of \mathbb{P}^2 which preserves a foliation \mathcal{F} defined on a neighborhood of a fixed point. Assume that \mathcal{F} extends to a global foliation \mathcal{F}' on \mathbb{P}^2 . By Lemma 2.6.1, \mathcal{F}' is invariant by f on \mathbb{P}^2 . Since f is semi-extremal, Corollary 2.8.6 (provided by Favre-Pereira classification) asserts that \mathcal{F}' is a pencil of lines, as desired.

2.7 Proof of Corollary E.I

By Theorem B.I, there exists a holomorphic foliation \mathcal{F} on an open neighborhood \mathcal{V} of $J = \text{Supp}(\mu)$. It satisfies $(f^N)^*\mathcal{F} = \mathcal{F}$ on $f^{-N}(\mathcal{V}) \cap \mathcal{V}$, where N is the period of the repelling periodic point a . If $\text{Supp}(\mu)$ is connected and $\mathbb{P}^2 \setminus \text{Supp}(\mu)$ is Stein, then \mathcal{F} extends to a foliation \mathcal{F}' on \mathbb{P}^2 by Theorem 2.8.8. According to Corollary D.I, \mathcal{F}' is a pencil of lines, given by $\pi[x : y : z] = [x : y]$ up to a linear change of coordinates.

* * * Appendix of Chapter 2 * * *

2.8 Foliations and Stein property

2.8.1 Holomorphic foliations on complex surfaces

Definition of foliations. Let X be a complex surface, for any open sub-set $U \subset X$ we denote $\Omega^1(U)$ the set of holomorphic 1-forms on U . Let us also denote $\text{Cov}(X)$ the set of all open, connected, and locally finite coverings $(U_i)_{i \in I}$ of X . We denote $\text{FOL}(X)$ the set of collections $(U_i, \omega_i)_{i \in I}$, where $(U_i)_{i \in I} \in \text{Cov}(X)$ and $\omega_i \in \Omega^1(U_i) \setminus \{0\}$ for every $i \in I$, such that the following compatibility condition is satisfied :

$$\omega_i \wedge \omega_j \equiv 0 \text{ on } U_i \cap U_j. \quad (2.58)$$

Two elements $F := (U_i, \alpha_i)_i$ and $G := (V_j, \beta_j)_j$ of $\text{FOL}(X)$ are *equivalent*, denoted $F \sim G$, if for each $i \in I$ and $j \in J$ such that $U_i \cap V_j \neq \emptyset$, one has :

$$\alpha_i \wedge \beta_j \equiv 0 \text{ on } U_i \cap V_j.$$

We denote $\mathcal{Fol}(X) := \text{FOL}(X) / \sim$ the quotient space by the equivalence relation \sim .

Definition 2.8.1. A *singular holomorphic foliation* \mathcal{F} on X is an element of $\mathcal{Fol}(X)$.

Tangent distributions and vector fields. Let $\mathcal{F} = [(U_i, \omega_i)_{i \in I}] \in \mathcal{Fol}(X)$ be a foliation on X , and let us set $X_0 := X \setminus (\cup_{i \in I} \{\omega_i = 0\})$. Since $\{\omega_i = 0\}$ have codimension ≥ 1 , and the covering is locally finite, the set X_0 is open and $\overline{X_0} = X$.

For any point $p \in X_0$, the compatibility condition (2.58) implies that the kernel $\text{Ker}(\omega_i)_p$ is a complex line which does not depend of the choice of the form ω_i such that $p \in U_i$. Moreover, because we work in complex dimension 2, the distribution of complex lines $p \in X_0 \mapsto \text{Ker}(\omega_i)_p$ is integrable in the sense of Frobenius. We have thus the existence of a sub-bundle $T\mathcal{F} \subset TX_0$ such that $T_p\mathcal{F} = \text{Ker}(\omega_i)_p$ for any $p \in X_0$ and for any $i \in I$ such that $p \in U_i$.

There exists an other construction of $T\mathcal{F}$ using vector fields. Let us assume that each open set U_i is equipped with holomorphic coordinates (Z_i, W_i) , we can write $\omega_i = f_i dZ_i + g_i dW_i$ where $f_i, g_i \in \mathcal{O}(U_i)$. The vector field $v_i := g_i \frac{\partial}{\partial Z_i} - f_i \frac{\partial}{\partial W_i}$ satisfies :

$$\omega_i(v_i) = 0 \text{ on } U_i.$$

In particular one has for any $p \in U_i \cap X_0$:

$$T_p\mathcal{F} = \text{Ker}(\omega_i)_p = \mathbb{C} \cdot v_i(p).$$

Let now $\mathcal{G} = [(V_j, \omega_j)_{j \in J}] \in \mathcal{Fol}(X)$ be another foliation. Up to consider a common sub-covering of $(U_i)_{i \in I}$ and $(V_j)_{j \in J}$, we can assume that $(U_i)_{i \in I} = (V_j)_{j \in J}$. Let us write $\eta_i = h_i dZ_i + k_i dW_i$ on U_i and let $w_i := k_i \frac{\partial}{\partial Z_i} - h_i \frac{\partial}{\partial W_i}$ be the associated vector field. We denote as before $X_1 := X \setminus (\cup_{i \in I} \{\eta_i = 0\})$. One has for any $p \in U_i \cap X_1$:

$$T_p\mathcal{G} = \text{Ker}(\eta_i)_p = \mathbb{C} \cdot w_i(p).$$

Lemma 2.8.2. *Let $\mathcal{F} = [(U_i, \omega_i)_{i \in I}]$ and $\mathcal{G} = [(U_i, \eta_i)_{i \in I}]$ be two elements of $\mathcal{Fol}(X)$. Assume that each open set U_i is equipped with holomorphic coordinates (Z_i, W_i) . With the notations introduced above one has for every $i \in I$:*

$$1. \quad \mathcal{F}|_{U_i} = \mathcal{G}|_{U_i} \iff (\omega_i \wedge \eta_i = 0 \text{ on } U_i) \iff (\omega_i(w_i) = 0 \text{ on } U_i). \quad (2.59)$$

$$2. \quad \mathcal{F}|_{U_i} = \mathcal{G}|_{U_i} \iff (T_p\mathcal{F} = T_p\mathcal{G}, \forall p \in U_i \cap X_0 \cap X_1). \quad (2.60)$$

Proof :

1. The first equivalence is our definition of foliations. For the second equivalence, just observe that $\omega_i \wedge \eta_i = \begin{vmatrix} f_i & h_i \\ g_i & k_i \end{vmatrix} dZ_i \wedge dW_i$ and that $\omega_i(w_i) = f_i k_i - g_i h_i = \begin{vmatrix} f_i & h_i \\ g_i & k_i \end{vmatrix}$.

2. We have to prove the reverse implication. Let us assume that $T_p\mathcal{F} = T_p\mathcal{G}$ for any $p \in U_i \cap X_0 \cap X_1$. Since $T_p\mathcal{F} = \text{Ker}(\omega_i)_p$ and $T_p\mathcal{G} = \mathbb{C} \cdot w_i(p)$, we get $\omega_i(w_i)(p) = 0$ for any $p \in U_i \cap X_0 \cap X_1$. Since U_i is connected, $\omega_i(w_i)(p) = 0$ for every $p \in U_i$ by analytic continuation. Hence $\mathcal{F}|_{U_i} = \mathcal{G}|_{U_i}$ according to (2.59). \square

Let us provide one more criterion to check that two foliations are equal.

Lemma 2.8.3. *Let \mathcal{F} and \mathcal{G} be two foliations on a connected open set $U \subset X$ equipped with holomorphic coordinates (Z, W) . Assume that there exist ω and η two non trivial holomorphic 1-forms on U which represent \mathcal{F} and \mathcal{G} . If $\mathcal{F}|_V = \mathcal{G}|_V$, where V is a non empty open sub-set of U , then $\mathcal{F} = \mathcal{G}$ on U .*

Proof : Let us write $\omega = fdZ + gdW$ and $\eta = hdZ + kdW$, where f, g, h and k belong to $\mathcal{O}(U)$. If $\mathcal{F}|_V = \mathcal{G}|_V$ on $V \subset U$, then $\omega \wedge \eta = 0$ on V . Since $\omega \wedge \eta = \begin{vmatrix} f & h \\ g & k \end{vmatrix} dZ \wedge dW$, we deduce that $\begin{vmatrix} f & h \\ g & k \end{vmatrix} = 0$ on V and thus on U by analytic continuation. It means that $\omega \wedge \eta = 0$ on U and $\mathcal{F} = \mathcal{G}$ on U as desired. \square

Remark on the size of singularities. Let X be a complex surface and let $U \subset X$ be a non empty connected open sub-set equipped with holomorphic coordinates (Z, W) . For any holomorphic 1-form $\omega \in \Omega^1(U) \setminus \{0\}$, there exists a function $\phi \in \mathcal{O}(U) \setminus \{0\}$ and a 1-form ω' on U , such that $\omega = \phi \times \omega'$ and $\text{codim}(\{\omega' = 0\}) = 2$. We refer for

instance to [43, p. 33] for a proof. Moreover according to our definition of foliations, ω and ω' represent the same class in $\mathcal{F}ol(U)$. In other words, we can assume that a foliation on X is defined by holomorphic 1-forms which have isolated singularities. If $\mathcal{F} = [(U_i, \omega_i)_{i \in I}] \in \mathcal{F}ol(X)$, its singular set $\text{Sing}(\mathcal{F})$ is defined as the set of points $p \in X$ such that $\omega_i(p) = 0$ for every $i \in I$, it is a discrete sub-space of X .

2.8.2 Pull-back of foliations and invariance

Pull-back. Let again X and Y be two complex surfaces and let $f : Y \rightarrow X$ be a holomorphic surjective map which is not constant on each connected component of Y . Let $\mathcal{F} = [(U_i, \omega_i)_{i \in I}] \in \mathcal{F}ol(X)$ be a foliation on X . For each $i \in I$, the 1-form $f^*\omega_i$ is not null on each connected component of $f^{-1}(U_i)$ (f is not constant on this connected component). Observe also that for each non trivial intersections $U_i \cap U_j \neq \emptyset$, one has :

$$f^*\omega_i \wedge f^*\omega_j = f^*(\omega_i \wedge \omega_j) = 0 \text{ on } f^{-1}(U_i) \cap f^{-1}(U_j).$$

Let $f^*\mathcal{F}$ be the foliation on Y given by (the class of) the family $(f^*\omega_i)_{i \in I}$. One immediately checks that $f^*\mathcal{F}$ only depends on the foliation \mathcal{F} .

Invariance. We define the invariance of a foliation by a holomorphic map by :

Definition 2.8.4. Let $f : X \rightarrow X$ be a holomorphic map on a complex surface. Let $\mathcal{V} \subset X$ be a non empty open sub-set of X such that $f^{-1}(\mathcal{V}) \cap \mathcal{V} \neq \emptyset$, and such that $f : f^{-1}(\mathcal{V}) \rightarrow \mathcal{V}$ is non constant on each connected component of $f^{-1}(\mathcal{V})$. Let \mathcal{F} be a foliation on \mathcal{V} . We say that \mathcal{F} is invariant by f on \mathcal{V} if the pull-back of \mathcal{F} by $f : f^{-1}(\mathcal{V}) \rightarrow \mathcal{V}$, denoted $f^*\mathcal{F}$, satisfies $f^*\mathcal{F}|_{f^{-1}(\mathcal{V}) \cap \mathcal{V}} = \mathcal{F}|_{f^{-1}(\mathcal{V}) \cap \mathcal{V}}$.

Classification on \mathbb{P}^2 . On $X = \mathbb{P}^2$ the pairs (\mathcal{F}, f) where $f^*\mathcal{F} = \mathcal{F}$, have been classified by Favre-Pereira. We shall need the following simplified version of their classification :

Theorem 2.8.5 (Favre-Pereira [33]). Let $f : \mathbb{P}^2 \rightarrow \mathbb{P}^2$ be a holomorphic map of degree $d \geq 2$. Let \mathcal{F} be a foliation on \mathbb{P}^2 invariant by f . Then in appropriate homogeneous coordinates $[x : y : z]$ on \mathbb{P}^2 , one of the following cases holds :

1. \mathcal{F} is the pencil on lines $\pi[z : w : t] = [z : w]$. In this case $f = [P(x, y) : Q(x, y) : R(x, y, z)]$ with P, Q, R homogeneous polynomials of degree d .
2. \mathcal{F} is not a pencil of lines through a point in \mathbb{P}^2 and f has the form :
 - i. $f = [x^d : y^d : z^d]$ or
 - ii. $f = [z^d : x^d : y^d]$ or
 - iii. $f = [x^d : y^d : R(x, y, z)]$, $R(x, y, z)$ depends on the three variables (x, y, z) , or
 - iv. $f = [y^d : x^d : R(x, y, z)]$, $R(x, y, z)$ depends on the three variables (x, y, z) .

From Favre-Pereira's theorem we deduce the following corollary :

Corollary 2.8.6. Let f be a degree $d \geq 2$ map on \mathbb{P}^2 . If f preserves a foliation \mathcal{F} on \mathbb{P}^2 and if f is semi-extremal, then in appropriate homogeneous coordinates \mathcal{F} is a pencil of lines given by the fibers of $\pi[z : w : t] = [z : w]$.

2.8.3 Structure of foliations on \mathbb{P}^2

Let $U_0 = \{[z_0 : z_1 : z_2] : z_2 \neq 0\}$ be an affine chart, let $x := z_0/z_2$ and $y := z_1/z_2$ be the associated affine coordinates. Given two polynomials $P(x, y)$ and $Q(x, y)$ without any common factor, then the 1-form

$$\omega = P(x, y)dx + Q(x, y)dy \quad (2.61)$$

defines a foliation on $U_0 = \mathbb{P}^2 \setminus l_0$, where $l_0 = \{z_2 = 0\}$. It turns out that this foliation extends uniquely to \mathbb{P}^2 , see for instance Zakeri's article [51, §1.9]. A fundamental point is that the converse is true, see [51, Corollary 1.21].

Theorem 2.8.7. *Let \mathcal{F} be a foliation on \mathbb{P}^2 . There is a unique (up to a multiplication by a constant) polynomial 1-form of the form (2.61) on U_0 which defines \mathcal{F} .*

This provides a criterion to prove that two foliations on \mathbb{P}^2 are equal : if there exists a 1-form on U_0 of the form (2.61) that characterizes them, then they must be equal on \mathbb{P}^2 . We can also say that two foliations on \mathbb{P}^2 are equal if they coincide on the affine chart U_0 .

2.8.4 Stein manifolds and extension of foliations

Stein manifolds. (see Demailly [22]) A Stein manifold V is a complex manifold which satisfies the following points :

1. V is holomorphically convex.
2. $\mathcal{O}(V)$ separates locally the points : for any $x \in V$ there exists a neighborhood W of x such that for any $y \in W$ there exists $f \in \mathcal{O}(V)$ such that $f(x) \neq f(y)$.

The first condition implies that there exists a *psh* function $\psi \in C^\infty(V, \mathbb{R})$ which is an exhaustion of V . The second condition implies that there exists a strictly *psh* function $u \in C^\infty(V, \mathbb{R})$. Thus $\rho = \psi + u : V \rightarrow \mathbb{R}$ is a strictly *psh* exhaustion of V , we say that V is strictly pseudoconvex. Conversely the existence of such a strictly *psh* exhaustion on a complex manifold always implies that it is a Stein manifold, this is characterization of Stein manifolds is due to Grauert [35].

Extension of foliations. With the preceding notations, the level sets $\{\rho < t\}_{t \in \mathbb{R}}$ are strictly pseudo-convex and exhaust the Stein manifold V . By using Hartogs extension, one can extend a holomorphic function u defined on a level set $\{\rho > t\}$ to a slightly bigger one $\{\rho > t - \varepsilon\}$, with $\varepsilon > 0$. This procedure allows to extend u on V .

Lins Neto used this idea to extend holomorphic foliations, by considering the slope of the tangent lines, see also Canales [16, Theorem 5.1].

Theorem 2.8.8 (Lins Neto [42, §2.2]). *Let K be a connected compact in \mathbb{P}^2 and let \mathcal{V} be an open neighborhood of K on which a holomorphic foliation \mathcal{F} is defined. If $V = \mathbb{P}^2 \setminus K$ is Stein, then \mathcal{F} extends to a unique foliation \mathcal{F}' on \mathbb{P}^2 .*

Fatou sets are Stein. We will apply Lins Neto's theorem in the following context. Let f be a holomorphic map of \mathbb{P}^2 of degree $d \geq 2$ and let T be its Green current. Forneaess-Sibony proved in [34] that $\text{Supp}(T)$ is connected, and Ueda proved in [49] that $\mathbb{P}^2 \setminus \text{Supp}(T)$ is Stein. In particular we can apply Lins Neto's theorem to the compact set $K = \text{Supp}(\mu)$ when $\text{Supp}(\mu) = \text{Supp}(T)$.

Chapter 3

Absolute continuity for semi-extremal maps of $\mathbb{C}\mathbb{P}^2$

3.1 Introduction

This part concerns the ergodic theory of holomorphic dynamical systems, it deals with Pesin's formula and its generalizations. Let θ be a rational map on \mathbb{P}^1 of degree $d \geq 2$ and let $\omega_{\mathbb{P}^1}$ be the spherical $(1, 1)$ -form on \mathbb{P}^1 . We recall that the probability measure $\mu_\theta = \lim_{n \rightarrow +\infty} \frac{1}{d^n} \theta^{n*} \omega_{\mathbb{P}^1}$ is the unique measure of maximal entropy, equal to $\text{Log } d$. It satisfies $\theta^* \mu = d\mu$ and is mixing. The Lyapunov exponent of μ_θ satisfies $\lambda_\theta \geq \frac{1}{2} \text{Log } d$ by Margulis-Ruelle inequality, and moreover

$$\lambda_\theta = \frac{1}{2} \text{Log } d \iff \mu \ll \text{Leb}_{\mathbb{P}^1} = \omega_{\mathbb{P}^1}. \quad (3.1)$$

The direct implication was proved by Ledrappier [40], the reverse one by Ledrappier-Strelcyn [41], the arguments rely on measurable partitions. In our context the reverse implication can be obtained by inserting the density of μ in the relation $\theta^* \mu = d\mu$.

For a holomorphic map f of degree $d \geq 2$ on \mathbb{P}^2 , as explained in the introduction of the manuscript, the measure μ of maximal entropy of f is constructed similarly as the 1-dimensional case, satisfy $f^* \mu = d^2 \times \mu$ and is mixing. We recall that Margulis-Ruelle inequality fails to show that the Lyapunov exponents $\lambda_1 \geq \lambda_2$ of μ are greater than or equal to $\frac{1}{2} \text{Log } d$. This lower bound has been proved by Briend-Duval, we refer to 1.1.2. The counterpart of (3.1) takes the form

$$\lambda_1 = \lambda_2 = \frac{1}{2} \text{Log } d \iff \mu \ll \text{Leb}_{\mathbb{P}^2} = \omega_{\mathbb{P}^2} \wedge \omega_{\mathbb{P}^2}. \quad (3.2)$$

The direct implication was obtained by Dupont in [28] following the classical arguments of [40] and in [29, Theorem D] using a central limit theorem for the observable $\text{Log det } df$. One can prove the reverse implication by inserting the density of μ in $f^* \mu = d^2 \mu$ and using the lower bound on the Lyapunov exponents. The (adapted) equivalence (3.2) is valid on every \mathbb{P}^k , $k \geq 1$.

The measure μ is actually equal to $T \wedge T$, where T is the Green current $\lim_{n \rightarrow +\infty} \frac{1}{d^n} f^{n*} \omega_{\mathbb{P}^2}$. It satisfies $f^*T = dT$ and can be interpreted as a singular invariant metric on \mathbb{P}^2 . Let us recall some facts given in Theorem 1.1.3. Berteloot-Loeb [7] proved that if T is smooth and positive on a non empty open set of \mathbb{P}^2 , then f is a Lattès map : f can be lifted to an affine map on a complex torus via a finite ramified covering $\sigma : \mathbb{C}^2/\Lambda \rightarrow \mathbb{P}^2$. Moreover σ^*T is equal to the standard hermitian form $\frac{i}{2}dz \wedge d\bar{z} + \frac{i}{2}dw \wedge d\bar{w}$. In particular one gets $T \ll \omega_{\mathbb{P}^2}$ and $\omega_{\mathbb{P}^2} \ll T$ since T does not charge analytic sets, and one gets $\lambda_1 = \lambda_2 = \frac{1}{2} \text{Log } d$ by $f^*T = dT$. Berteloot-Dupont [3] established later that if $\mu = T \wedge T \ll \omega_{\mathbb{P}^2} \wedge \omega_{\mathbb{P}^2}$, then T satisfies Berteloot-Loeb smoothness condition, the arguments use normal forms, the invariance of T and pluripotential theory.

More recently, Dujardin studied the Fatou directions associated to endomorphisms of \mathbb{P}^k and proved $(\mu \ll T \wedge \omega_{\mathbb{P}^2} \implies \lambda_2 = \frac{1}{2} \text{Log } d)$, as mentioned in introduction. We recall that he also asked the question of the reverse implication : "*Does $\lambda_2 = \frac{1}{2} \text{Log } d$ imply $\mu \ll T \wedge \omega_{\mathbb{P}^2}$?*", see Question 1.1.2.

Theorem A below gives a positive answer to Dujardin's question assuming hypothesis \mathcal{H}_2 , defined in Definition 3.2.1. Roughly speaking, \mathcal{H}_2 asserts that there exists a collection \mathcal{A} of disjoint unstable manifolds centered at some $c_0 \in \mathbb{P}^2$ on which the inverse branches of f and the Oseledec stable cocycle are uniformly controlled.

Theorem A.II. *Let f be a holomorphic map of degree $d \geq 2$ on \mathbb{P}^2 . Assume that the smallest Lyapunov exponent λ_2 of the maximal entropy measure μ is minimal equal to $\frac{1}{2} \text{Log } d$. If hypothesis \mathcal{H}_2 holds, then $\mu \ll T \wedge \omega_{\mathbb{P}^2}$.*

Let us note that the equivalence (3.2) gives a positive answer to Dujardin's question when $\lambda_1 = \lambda_2 = \frac{1}{2} \text{Log } d$. Indeed, in that case, f is a Lattès map and $\omega_{\mathbb{P}^2} \approx T$ as explained above. So we have to focus on the case $\lambda_1 > \lambda_2 = \frac{1}{2} \text{Log } d$ of semi-extremal endomorphisms. For the present the only examples of semi-extremal mappings that we know preserve a pencil of lines $\pi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$. Such mappings have been studied in general by Dupont-Taflin, see 1.1.5. In particular if θ is a Lattès map on \mathbb{P}^1 induced by f on \mathbb{P}^1 , $\mu_\theta = \varphi \text{Leb}_{\mathbb{P}^1}$ and the following precise description of μ holds :

$$\mu = (\varphi \circ \pi)T \wedge \pi^* \omega_{\mathbb{P}^1} \ll T \wedge \omega_{\mathbb{P}^2}.$$

Our result does not give such a product structure for μ with respect to T . Nonetheless, our hypothesis \mathcal{H}_2 does not rely on the existence of an invariant pencil of lines but on less rigid dynamical aspects, that we hope should be less restrictive. Another open question is to find, if it exists, a geometric characterization of semi-extremal mappings.

The proof of Theorem A relies partly on the classical partition method employed in [40] and [28]. The novelty, in order to deal with semi-extremality, is to introduce normal forms for the iterated inverse branches of f . We outline our arguments in Section 3.2.

Remark 3.1.1. *Our present result on \mathbb{P}^2 extends to higher dimensions with the same strategy, the details are given in dimension 2 for sake of simplicity. The statements on \mathbb{P}^k , $k \geq 2$, are given in Section 3.9.*

3.2 Outline of proofs

3.2.1 The classical proof for two minimal exponents

We recall briefly the classical proof of

$$\lambda_1 = \lambda_2 = \frac{1}{2} \text{Log } d \implies \mu \ll \text{Leb}_{\mathbb{P}^2}. \quad (3.3)$$

We refer the reader to [39, 40] for the one dimensional case (on $[0, 1]$ and on \mathbb{P}^1) and to [28] for its extension to \mathbb{P}^k . The proof relies on the construction of a decreasing measurable partition η , generator of entropy, such that :

$$\forall n \geq 0, \quad \text{Log } d^{2n} = \int_{\widehat{\mathbb{P}^2}} -\text{Log } \mu_{\widehat{x}} \left(\widehat{f}^{-n} \eta \right)_{\widehat{x}} d\widehat{\mu}(\widehat{x}). \quad (3.4)$$

We use here the natural extension $(\widehat{\mathbb{P}^2}, \widehat{f}, \widehat{\mu}) \xrightarrow{\pi_0} (\mathbb{P}^2, f, \mu)$ in order to work with an invertible dynamical system, see Section 3.8.2 for details. We denote $\mathcal{B}(\widehat{\mathbb{P}^2})$ the σ -algebra of borel sets of $\widehat{\mathbb{P}^2}$. Given a measurable partition ζ of $\widehat{\mathbb{P}^2}$, $\zeta_{\widehat{x}}$ is the atom of ζ containing $\widehat{x} \in \widehat{\mathbb{P}^2}$. Let $\mu_{\widehat{x}}$ denote the conditional measure of μ on the atom $\eta_{\widehat{x}}$. The idea to prove (3.3) is to introduce for $\widehat{\mu}$ -almost every $\widehat{x} \in \widehat{\mathbb{P}^2}$ a probability measure $p_{\widehat{x}}$ on $\eta_{\widehat{x}}$ which is absolutely continuous with respect to $\text{Leb}_{\mathbb{P}^2}$ and such that (3.4) is satisfied when $\mu_{\widehat{x}}$ is replaced by $p_{\widehat{x}}$. Indeed in this case, for any $n \geq 0$, we have the equality :

$$\int_{\widehat{\mathbb{P}^2}} -\text{Log } p_{\widehat{x}} \left(\widehat{f}^{-n} \eta \right)_{\widehat{x}} d\widehat{\mu}(\widehat{x}) = \text{Log } d^{2n} = \int_{\widehat{\mathbb{P}^2}} -\text{Log } \mu_{\widehat{x}} \left(\widehat{f}^{-n} \eta \right)_{\widehat{x}} d\widehat{\mu}(\widehat{x}). \quad (3.5)$$

It implies by Jensen inequality :

$$0 = \int_{\widehat{\mathbb{P}^2}} \text{Log } \frac{p_{\widehat{x}}}{\mu_{\widehat{x}}} \left(\widehat{f}^{-n} \eta \right)_{\widehat{x}} d\widehat{\mu}(\widehat{x}) \leq \text{Log } \int_{\widehat{\mathbb{P}^2}} \frac{p_{\widehat{x}}}{\mu_{\widehat{x}}} \left(\widehat{f}^{-n} \eta \right)_{\widehat{x}} d\widehat{\mu}(\widehat{x}). \quad (3.6)$$

Other properties of η (see for instance Theorem 3.3.10) allow to prove $\int \frac{p_{\widehat{x}}}{\mu_{\widehat{x}}} \left(\widehat{f}^{-n} \eta \right)_{\widehat{x}} d\widehat{\mu}(\widehat{x}) = 1$, hence (3.6) is an equality. The strict concavity of Log then implies $p_{\widehat{x}} \left(\widehat{f}^{-n} \eta \right)_{\widehat{x}} = \mu_{\widehat{x}} \left(\widehat{f}^{-n} \eta \right)_{\widehat{x}}$. Finally, the generating property of η yields $p_{\widehat{x}} = \mu_{\widehat{x}}$, hence $\widehat{\mu} = \int p_{\widehat{x}} d\widehat{\mu}(\widehat{x}) =: \widehat{p}$ on $\widehat{\mathbb{P}^2}$. Using the fact that $p_{\widehat{x}}$ is absolutely continuous (which has not be used so far), we get as desired

$$\mu = (\pi_0)_* \widehat{p} \ll \text{Leb}_{\mathbb{P}^2}.$$

Practically, the measures $p_{\widehat{x}}$ are defined by :

$$\forall A \in \mathcal{B}(\widehat{\mathbb{P}^2}), \quad p_{\widehat{x}}(A) = \frac{1}{L(\widehat{x})} \int_{\pi_0(A \cap \eta_{\widehat{x}})} \prod_{i=1}^{+\infty} \frac{\det d_{x-i} f}{\det d_{y-i} f} d\text{Leb}(y_0), \quad (3.7)$$

where $\widehat{y} = (y_{-i})_i$ is given by the injectivity of π_0 on the atoms of η , and $L(\widehat{x}) > 0$ ensures $p_{\widehat{x}}(\eta_{\widehat{x}}) = 1$. Observe that if $\widehat{z} \in \eta_{\widehat{x}}$ then $p_{\widehat{x}} = p_{\widehat{z}}$. Indeed, for every $A \subset \eta_{\widehat{x}}$ one has :

$$p_{\widehat{z}}(A) = \frac{L(\widehat{x})}{L(\widehat{z})} \prod_{i=1}^{+\infty} \frac{\det d_{z-i} f}{\det d_{x-i} f} \times p_{\widehat{x}}(A),$$

so we have $p_{\widehat{z}} = p_{\widehat{x}}$ since these are probability measures $\eta_{\widehat{x}}$. We thank R. Dujardin for asking us to specify this important fact. Now, using the change of variable formula for the Lebesgue measure, the decreasing property of η and the injectivity of f^n on $\pi_0((\widehat{f}^{-n}\eta)_{\widehat{x}})$ (see Theorem 3.3.10) one obtains the fundamental formula

$$\int_{\widehat{\mathbb{P}}^2} -\text{Log } p_{\widehat{x}} \left(\widehat{f}^{-n}\eta \right)_{\widehat{x}} d\widehat{\mu}(\widehat{x}) = \int_{\widehat{\mathbb{P}}^2} \text{Log } (\det d_{x_0} f^n) d\widehat{\mu}(\widehat{x}). \quad (3.8)$$

The right hand side is equal to $2n\lambda_1 + 2n\lambda_2$ by classical ergodic theory. It is equal to $\text{Log } d^{2n}$ when λ_1 and λ_2 are minimal, in this case (3.5) is satisfied and absolute continuity follows.

An important step is to establish the convergence of the integral (3.7). The problem is settled in [28, 40] by using a Pesin box \mathcal{A} . Roughly speaking \mathcal{A} is a disjoint union $\bigsqcup_{\widehat{z} \in \mathcal{T}} W^u(\widehat{z}, \mathcal{R})$ of unstable manifolds, where \mathcal{T} is a subset of a fiber $\pi_0^{-1}(c)$. The unstable manifolds are defined using inverse branches $f_{\widehat{x}}^{-n}$. A crucial property of \mathcal{A} is that $\eta_{\widehat{x}} \subset W^u(\widehat{z}, \mathcal{R})$ for any $\widehat{x} \in W^u(\widehat{z}, \mathcal{R})$ and $\widehat{z} \in \mathcal{T}$. Indeed, this inclusion implies the convergence of the infinite product $y_0 \in \pi_0(\eta_{\widehat{x}}) \mapsto \prod_{i=1}^{+\infty} \frac{\det d_{x_{-i}} f}{\det d_{y_{-i}} f}$, see for instance [28, Lemme 3.3]. When \widehat{x} does not belong to \mathcal{A} , Birkhoff ergodic theorem allows us to get back to \mathcal{A} . In our work we shall use Pesin boxes, see Section 3.3, but we will not have to prove the convergence of an infinite product, that step will indeed be contained in a normal form theorem for inverse branches.

3.2.2 Outline of the proof of Theorem A.II

We resume our proof of $\lambda_1 > \lambda_2 = \frac{1}{2} \text{Log } d \implies \mu \ll T \wedge \omega_{\mathbb{P}^2}$ under \mathcal{H}_2 . We start by proving the following theorem, whose first formula replaces Equation (3.8). The hypothesis \mathcal{H}_1 is given in Definition 3.2.1, it is less restrictive than \mathcal{H}_2 .

Theorem B.II. *Let f be a holomorphic map of degree $d \geq 2$ on \mathbb{P}^2 . Assume that the Lyapunov exponents of the maximal entropy measure μ satisfy $\lambda_1 > \lambda_2$ and that \mathcal{H}_1 holds. Then there exist a measurable partition η of $\widehat{\mathbb{P}}^2$ and a measurable family $\widehat{x} \mapsto q_{\widehat{x}}$ of probability measures on $\widehat{\mathbb{P}}^2$ supported on $\eta_{\widehat{x}}$ such that*

$$(\pi_0)_* q_{\widehat{x}} \ll T \wedge \omega_{\mathbb{P}^2},$$

$$\forall n \geq 0, \int_{\widehat{\mathbb{P}}^2} -\text{Log } q_{\widehat{x}} \left(\widehat{f}^{-n}\eta \right)_{\widehat{x}} d\widehat{\mu}(\widehat{x}) = \text{Log } d^n + 2n\lambda_2.$$

The measurable partition η also satisfies the formula :

$$\forall n \geq 0, \int_{\widehat{\mathbb{P}}^2} -\text{Log } \mu_{\widehat{x}} \left(\widehat{f}^{-n}\eta \right)_{\widehat{x}} d\widehat{\mu}(\widehat{x}) = \text{Log } d^n + \text{Log } d^n.$$

Theorem B implies Theorem A when $\lambda_2 = \frac{1}{2} \text{Log } d$. Indeed, in this case, Formula (3.5) is satisfied replacing $p_{\widehat{x}}$ by $q_{\widehat{x}}$. The same arguments than before (involving Jensen inequality and the properties of η) then imply $q_{\widehat{x}} = \mu_{\widehat{x}}$. In Section 3.6.2 we deduce from this equality and from $(\pi_0)_* q_{\widehat{x}} \ll T \wedge \omega_{\mathbb{P}^2}$ that $\mu \ll T \wedge \omega_{\mathbb{P}^2}$.

Compared with the classical method, our novelty is to use normal forms for inverse branches $f_{\widehat{x}}^{-n}$ to construct $q_{\widehat{x}}$. More precisely, by Theorem 3.8.2, there exist

controlled holomorphic change of coordinates $(Z_{\hat{x}}, W_{\hat{x}})$ such that the following diagram commutes

$$\begin{array}{ccc} B(x_{-n}, 2\eta_\varepsilon(\hat{x}_{-n})) & \xleftarrow{f_{\hat{x}}^{-n}} & B(x_0, 2\eta_\varepsilon(\hat{x})) \\ \xi_{\hat{x}_{-n}} = (Z_{\hat{x}_{-n}}, W_{\hat{x}_{-n}}) \downarrow & & \downarrow (Z_{\hat{x}}, W_{\hat{x}}) = \xi_{\hat{x}} \\ \mathbb{D}^2(\rho_\varepsilon(\hat{x}_{-n})) & \xleftarrow{R_{n, \hat{x}}} & \mathbb{D}^2(\rho_\varepsilon(\hat{x})) \end{array} \quad (3.9)$$

where the second coordinate of $R_{n, \hat{x}}$ is linear and satisfies

$$R_{n, \hat{x}}^{(2)}(z, w) = \beta_{n, \hat{x}} w \quad , \quad e^{-n(\lambda_2 + \varepsilon)} \leq |\beta_{n, \hat{x}}| \leq e^{-n(\lambda_2 - \varepsilon)}.$$

Using $f^*T = dT$ and the diagram above, we obtain the change of variable formula

$$(f_{\hat{x}}^{-n})^* (T \wedge dd^c |W_{\hat{x}_{-n}}|^2) = d^{-n} |\beta_{n, \hat{x}}|^2 \times (T \wedge dd^c |W_{\hat{x}}|^2).$$

We define the probability measures $q_{\hat{x}}$ on $\eta_{\hat{x}}$ by

$$q_{\hat{x}} = \frac{(T \wedge dd^c |W_{\hat{x}}|^2) \circ \pi_0}{(T \wedge dd^c |W_{\hat{x}}|^2)(\pi_0(\eta_{\hat{x}}))}. \quad (3.10)$$

The crucial part of our work is the proof of the first formula of Theorem B. It is important that λ_2 comes in this formula without any $e^{\pm n\varepsilon}$ error term (such terms could appear in $\beta_{n, \hat{x}}$), this is settled in Section 3.5 by using properties of the stable Oseledec cocycle (the integral formula provided by Theorem 3.8.1) and the classical Lemma 3.5.2 of Ledrappier-Strelcyn. Actually $q_{\hat{x}}$ is defined by (3.10) only when $\hat{x} \in \mathcal{A}$. If $\hat{x} \notin \mathcal{A}$ we have to introduce an adapted formula, see Section 3.4.

3.2.3 Hypothesis \mathcal{H}_1 and \mathcal{H}_2

We fix once for all ε small with respect to the Lyapunov exponents $\lambda_1 > \lambda_2$. By Theorem 3.8.1, there exists of borel set A_{os} of full μ -measure on which the unitary stable vector $\vec{v}_s : A_{os} \rightarrow T\mathbb{P}^2$ is defined. We denote

$$\Delta(x, n) := \|d_x f^n \cdot \vec{v}_s(x)\|.$$

Let \mathcal{FN}_ε be the set of full $\hat{\mu}$ -measure provided by the normal form Theorem 3.8.2. In particular, the inverse branches

$$f_{\hat{x}}^{-n} : B(x_0, 2\eta_\varepsilon(\hat{x})) \rightarrow B(x_{-n}, 2\eta_\varepsilon(\hat{x}_{-n}))$$

are defined for $\hat{x} \in \mathcal{FN}_\varepsilon$. Let $\beta_\varepsilon(\hat{x})$ be the distortion of the change of coordinates $\xi_{\hat{x}}$ specified in the diagram (3.9). As in Briend's article [12], for every $0 < r \leq 2\eta_\varepsilon(\hat{x})$, we define the unstable manifold by

$$W^u(\hat{x}, r) := \left\{ \hat{z} \in \widehat{\mathbb{P}}^2, \exists t \in B(x_0, r) : z_{-n} = f_{\hat{x}}^{-n}(t), \forall n \in \mathbb{N} \right\}.$$

Theorem 3.3.5 of Briend gives a borel set Br_ε of full μ -measure such that for any $c \in Br_\varepsilon$, there exist $r_c > 0$ and $\rho_c > 0$ satisfying $\hat{\mu} = \mu|_{B(c, r)} \otimes \mu_{\pi_0, c}$ on $\bigsqcup_{\hat{z} \in \mathcal{T}} W^u(\hat{z}, r)$, for any $0 < r \leq r_c$, any $\rho \geq \rho_c$ and any borel set $\mathcal{T} \subset \pi_0^{-1}(c) \cap \{\eta_\varepsilon \geq 1/\rho, \beta_\varepsilon \leq \rho\}$. For any choice of $\rho \geq \rho_c$, we shall say that a borel set $\mathcal{T} \subset \pi_0^{-1}(c) \cap \{\eta_\varepsilon \geq 1/\rho, \beta_\varepsilon \leq \rho\}$ is a *regular tree* if $\mu_{\pi_0, c}(\mathcal{T}) > 0$. We refer to Section 3.3.2 for more details.

Definition 3.2.1.

\mathcal{H}_1 : there exist $c_0 \in Br_\varepsilon$ and \mathcal{T}_0 an associated regular tree, and there exists $\rho_0 > 0$ such that for every $\widehat{z} \in \mathcal{T}_0$:

$$\mathcal{R}_{\widehat{z}} := \inf\{\eta_\varepsilon(\widehat{x}), \widehat{x} \in W^u(\widehat{z}, \eta_\varepsilon(\widehat{z})) \cap \mathcal{FN}_\varepsilon\} \geq \frac{1}{\rho_0}.$$

\mathcal{H}_2 : \mathcal{H}_1 is satisfied and for every $\widehat{z} \in \mathcal{T}_0$, there exists $\Delta_{\widehat{z}} > 1$ such that :

$$\forall n \geq 0, \forall \widehat{x} \in W^u(\widehat{z}, \eta_\varepsilon(\widehat{z})) \cap \pi_0^{-1}(A_{os}), \frac{1}{\Delta_{\widehat{z}}} \leq \frac{\Delta(z_{-n}, n)}{\Delta(x_{-n}, n)} \leq \Delta_{\widehat{z}}.$$

Let us explain why we introduce \mathcal{H}_1 and \mathcal{H}_2 . The measure $q_{\widehat{x}}$ is built using the submersion $W_{\widehat{x}}$ defined on $B(x_0, 2\eta_\varepsilon(\widehat{x}))$, but this ball does not contain $\pi_0(\eta_{\widehat{x}})$ in general. This induces difficulties to implement the classical partition method. This did not occur for the construction of $p_{\widehat{x}}$, which used the function $x \mapsto \det(d_x f)$ defined on \mathbb{P}^2 . The hypothesis \mathcal{H}_1 thus sets an uniform control for the function $\widehat{x} \mapsto \eta_\varepsilon(\widehat{x})$ on the Pesin box \mathcal{A} .

Another difficulty, which did not appear for the family $(p_{\widehat{x}})_{\widehat{x}}$, is to check that $q_{\widehat{x}} = q_{\widehat{y}}$ when \widehat{x} and \widehat{y} belong to the same atom of η , see Proposition 3.4.5 (proved in Section 3.7). That property is fundamental to show Theorem 3.6.1. To get it we show in Section 3.7 that it suffices to control $(n, \widehat{x}) \mapsto \Delta(z_{-n}, n)/\Delta(x_{-n}, n)$ when \widehat{x} runs over the unstable manifolds $W^u(\widehat{z}, \mathcal{R})$ of the Pesin box \mathcal{A} , as specified in \mathcal{H}_2 .

3.3 Construction of Pesin boxes and partitions

3.3.1 Unstable manifolds and Buzzi's partition

The following proposition leads to the definition of \mathcal{P} -address.

Proposition 3.3.1 (Buzzi [15, Section 4], see also Dupont [28, Section 2.4]). *There exists a partition $\mathcal{P} = (\mathcal{P}_j)_{j \in \{1, \dots, N\}}$ of a full μ -measure set of \mathbb{P}^2 , whose atoms are open sets of \mathbb{P}^2 and such that :*

1. *The map f is injective on each atom \mathcal{P}_j .*
2. *If \mathcal{P}_x denotes the atom of \mathcal{P} which contains x , then $f_{\widehat{x}}^{-n}(B(x_0, 2\eta_\varepsilon(\widehat{x}))) \subset \mathcal{P}_{x_{-n}}$ for every $n \geq 0$ and for $\widehat{\mu}$ -almost every \widehat{x} .*

Definition 3.3.2.

1. *The \widehat{f} -invariant set $\widehat{\mathcal{P}} := \bigcap_{n \in \mathbb{Z}} \widehat{f}^n(\pi_0^{-1}(\bigcup_{j=1}^N \mathcal{P}_j))$ has full $\widehat{\mu}$ -measure. We say that $\widehat{x}, \widehat{y} \in \widehat{\mathcal{P}}$ have the same \mathcal{P} -address if $\mathcal{P}_{x_{-n}} = \mathcal{P}_{y_{-n}}$ for every $n \geq 0$. By Proposition 3.3.1, the \mathcal{P} -address is constant on the unstable manifolds.*
2. *Let*

$$\Lambda_\varepsilon := \pi_0^{-1}(A_{os}) \cap \mathcal{FN}_\varepsilon \cap \widehat{\mathcal{P}},$$

it is a totally invariant set of full $\widehat{\mu}$ -measure on which $\vec{v}_s \circ \pi_0$, the \mathcal{P} -address and the inverse branches $f_{\widehat{x}}^{-n}$ are defined.

Lemma 3.3.3. *For any $\widehat{z} \in \Lambda_\varepsilon$ we have :*

1. Let $\hat{x} \in W^u(\hat{z}, 2\eta_\varepsilon(\hat{z})) \cap \Lambda_\varepsilon$ and $n \geq 0$. Then the maps $f_{\hat{x}}^{-n}$ and $f_{\hat{z}}^{-n}$ coincide on $B(x_0, 2\eta_\varepsilon(\hat{x})) \cap B(z_0, 2\eta_\varepsilon(\hat{z}))$.
2. Let $0 < r \leq 2\eta_\varepsilon(\hat{z})$ and $\hat{x} \in W^u(\hat{z}, r)$. If $\hat{w} \in \widehat{\mathcal{P}}$ satisfies
 - (a) \hat{x} and \hat{w} have the same \mathcal{P} -address,
 - (b) $w_0 \in B(z_0, r)$,
 then $\hat{w} \in W^u(\hat{z}, r)$. If moreover $w_0 = z_0$ then $\hat{w} = \hat{z}$.

Proof : Let $t_0 \in B(x_0, 2\eta_\varepsilon(\hat{x})) \cap B(z_0, 2\eta_\varepsilon(\hat{z}))$. By Proposition 3.3.1 and Definition 3.3.2 we have for every $k \geq 0$:

$$\mathcal{P}_{f_{\hat{x}}^{-k}(t_0)} = \mathcal{P}_{x_{-k}} = \mathcal{P}_{z_{-k}} = \mathcal{P}_{f_{\hat{z}}^{-k}(t_0)}, \quad (3.11)$$

where $\mathcal{P}_{x_{-k}} = \mathcal{P}_{z_{-k}}$ comes from $\hat{x} \in W^u(\hat{z}, 2\eta_\varepsilon(\hat{x}))$. If $f_{\hat{x}}^{-j}(t_0) = f_{\hat{z}}^{-j}(t_0)$ then $f(f_{\hat{x}}^{-(j+1)}(t_0)) = f(f_{\hat{z}}^{-(j+1)}(t_0))$. By using (3.11) and the fact that f is injective on the atoms of \mathcal{P} (see Proposition 3.3.1), we get $f_{\hat{x}}^{-(j+1)}(t_0) = f_{\hat{z}}^{-(j+1)}(t_0)$. An induction on $j \geq 0$ thus implies $f_{\hat{x}}^{-j}(t_0) = f_{\hat{z}}^{-j}(t_0)$ for every $j \geq 0$, since it is true for any t_0 the conclusion follows. The second item can be proved similarly. \square

3.3.2 Pesin boxes and Briend's theorem

Definition 3.3.4. A Pesin box is a quintuplet $(P, r, \rho, \mathcal{T}, c)$ where $c \in \mathbb{P}^2$ and

1. \mathcal{T} is a borel subset of $\Lambda_\varepsilon \cap \{\pi_0 = c\}$ such that $2\eta_\varepsilon(\hat{z}) \geq r > 0$ for every $\hat{z} \in \mathcal{T}$,
2. $\beta_\varepsilon(\hat{z}) \leq \rho$ for every $\hat{z} \in \mathcal{T}$,
3. $P = \bigsqcup_{\hat{z} \in \mathcal{T}} W^u(\hat{z}, r)$.

In the third item the unstable manifolds are pairwise disjoint, it is a consequence of the first item of Lemma 3.3.3 and the fact that the unstable manifolds are centered at the same point c . For any $\hat{z} \in \mathcal{T}$, let us define

$$\varphi_{\hat{z}} : y_0 \in B(c, r) \mapsto (f_{\hat{z}}^{-n}(y_0))_{n \in \mathbb{Z}} \in \widehat{\mathbb{P}^2},$$

where $f_{\hat{z}}^n = f^n$ if $n \geq 0$. For any $\hat{y} \in P$, let $\tilde{\pi}(\hat{y})$ be the unique $\hat{z} \in \mathcal{T}$ such that $\varphi_{\hat{z}}(y_0) = \hat{y}$. Then P is homeomorphic to $B(c, r) \times \mathcal{T}$ via the following continuous bijective mappings :

$$\Psi : \begin{cases} B(c, r) \times \mathcal{T} & \longrightarrow P \\ (y_0, \hat{z}) & \longmapsto \varphi_{\hat{z}}(y_0) \end{cases}, \quad \Theta : \begin{cases} P & \longrightarrow B(c, r) \times \mathcal{T} \\ \hat{y} & \longmapsto (y_0, \tilde{\pi}(\hat{y})) \end{cases}$$

In particular the Pesin box P is a borel set of $\widehat{\mathbb{P}^2}$. The continuity of Ψ and Θ can be checked by hands using the product topology of $\widehat{\mathbb{P}^2}$, they are implicit in [12].

By Rokhlin's theorem [46, §3.1], $\hat{\mu}$ admits a decomposition on the fibers of π_0 :

$$\forall A \in \mathcal{B}(\widehat{\mathbb{P}^2}), \quad \hat{\mu}(A) = \int_{\mathbb{P}^2} \mu_{\pi_0, c}(A \cap T_c) \, d\mu(c), \quad (3.12)$$

where $\mu_{\pi_0, c}$ is the conditional measure of $\hat{\mu}$ on the fiber $T_c := \{\pi_0 = c\}$. The measure $\hat{\mu}$ is a product on Pesin boxes :

Theorem 3.3.5 (Briend [12, Theorem 4.1]). *There exists a borel set Br_ε of full μ -measure such that for every $c \in Br_\varepsilon$, there exists $r_c > 0$ and $\rho_c > 0$ such that :*

$$\Theta_*(\widehat{\mu}|_P) = \mu|_{B(c,r)} \otimes \mu_{\pi_0,c}|_{\mathcal{T}},$$

for every Pesin box $(P, r, \rho, \mathcal{T}, c)$ satisfying $r \leq r_c$ and $\rho \geq \rho_c$.

3.3.3 A special Pesin box \mathcal{A}

In this section we construct a special Pesin box \mathcal{A} of $\widehat{\mu}$ -positive measure on which the dynamics has good properties. We use it to construct the decreasing partition η in Section 3.3.4 and the family of conditional measures $q_{\widehat{x}}$ under the hypothesis \mathcal{H}_1 and \mathcal{H}_2 in Section 3.4. We proceed in several steps. First let Br_ε be the set of full μ -measure of points $c \in \text{Supp}(\mu)$ that satisfy Theorem 3.3.5. Then for $\rho > 0$ we define :

$$V := \{\widehat{z} \in \Lambda_\varepsilon : \eta_\varepsilon(\widehat{z}) \geq 1/\rho, \beta_\varepsilon(\widehat{z}) \leq \rho\}$$

which is of positive $\widehat{\mu}$ -measure for $\rho > 0$ large enough. Thus applying (3.12) with $A = V$, there exists a set of positive μ -measure $E \subset Br_\varepsilon$ such that for every $c \in E$, the conditional measure $\mu_{\pi_0,c}$ charges the set $V \cap T_c$. Then for every $c \in E$ and $r \in]0, \frac{1}{\rho}]$ we define

$$P_{c,r} := \bigsqcup_{\widehat{z} \in V \cap T_c} W^u(\widehat{z}, r).$$

We deduce that $(P_{c,r}, r, \rho, V \cap T_c, c)$ is a Pesin box with respect to the Definition 3.3.4. Moreover by Theorem 3.3.5, we get for every $\rho \geq \rho_c$ and $0 < r \leq r_c$:

$$\widehat{\mu}(P_{c,r}) = \mu(B(c, r)) \times \mu_{\pi_0,c}(V \cap T_c) > 0. \quad (3.13)$$

Lemma 3.3.6. [28, Lemma 2.2] *There exists $S \subset [1/4\rho, 1/2\rho]$ a borel set of full Lebesgue measure such that for every $s \in S$, there exists $\gamma : \widehat{\mathbb{P}}^2 \rightarrow]0, 1]$ a measurable function satisfying*

$$\forall \widehat{x} - \widehat{\mu} - a.e., \forall n \in \mathbb{N}, |d(x_{-n}, c) - s| \geq \gamma(\widehat{x})e^{-n\varepsilon}.$$

Let us fix $\mathcal{R} \in S$, set $\mathcal{T} := V \cap T_c$ and define

$$\mathcal{A} := P_{c,\mathcal{R}} = \bigsqcup_{\widehat{z} \in \mathcal{T}} W^u(\widehat{z}, \mathcal{R}) \subset \pi_0^{-1}(B(c, \mathcal{R})). \quad (3.14)$$

We still have $\widehat{\mu}(\mathcal{A}) > 0$ by (3.13). This choice of $\mathcal{R} \leq 1/2\rho$ is used in the proof of Proposition 3.3.7, and the choice of $\mathcal{R} \in S$ given by Lemma 3.3.6 is important for the proof of Lemma 3.3.9 below.

The hypothesis \mathcal{H}_1 and \mathcal{H}_2 allow to get much properties that will be crucial to define the measures $q_{\widehat{x}}$.

Proposition 3.3.7 (Use of \mathcal{H}_1 and \mathcal{H}_2).

1. *Under hypothesis \mathcal{H}_1 , we can replaced in the definition (3.14) the point c and the tree \mathcal{T} by a point $c_0 \in Br_\varepsilon$ and a regular tree \mathcal{T}_0 to ensure that for every $\widehat{x} \in \mathcal{A} \cap \Lambda_\varepsilon$:*

$$2\mathcal{R} \leq \eta_\varepsilon(\widehat{x}) \text{ and } \beta_\varepsilon(\widehat{x}) \leq \frac{1}{2\mathcal{R}}. \quad (3.15)$$

Moreover we still have $\widehat{\mu}(\mathcal{A}) > 0$.

2. Under hypothesis \mathcal{H}_2 , we can replaced in the definition (3.14) the point c and the tree \mathcal{T} by a point $c_0 \in Br_\varepsilon$ and a regular tree \mathcal{T}_0 to ensure that (3.15) is satisfied for any $\hat{x} \in \mathcal{A} \cap \Lambda_\varepsilon$; and moreover that for every $\hat{z} \in \mathcal{T}_0$ there exists $\Delta_{\hat{z}} > 1$ such that :

$$\forall n \geq 0, \quad \forall \hat{x} \in W^u(\hat{z}, \mathcal{R}) \cap \Lambda_\varepsilon, \quad \frac{1}{\Delta_{\hat{z}}} \leq \frac{\Delta(z_{-n}, n)}{\Delta(x_{-n}, n)} \leq \Delta_{\hat{z}}.$$

Moreover we still have $\widehat{\mu}(\mathcal{A}) > 0$.

Remark 3.3.8. In practice in the rest of the text, when we use \mathcal{H}_1 or \mathcal{H}_2 we keep the notation c and \mathcal{T} instead of c_0 and \mathcal{T}_0 for sake of simplicity.

Proof : We focus on the first item, the second one is obtained similarly. By Definition 3.2.1 of \mathcal{H}_1 , there exists a point $c_0 \in Br_\varepsilon$ and a tree \mathcal{T}_0 included in $\pi_0^{-1}(c_0) \cap \{\eta_\varepsilon \geq 1/\rho_{c_0}, \beta_\varepsilon \leq \rho_{c_0}\}$ such that $\mu_{\pi_0, c_0}(\mathcal{T}) > 0$. According to the preceding construction, we can assume $\rho \geq \rho_{c_0}$ and we can replace c and \mathcal{T} by c_0 and \mathcal{T}_0 to define \mathcal{A} . One can also assume $\rho \geq \rho_0$ in the preceding discussion, where ρ_0 is provided by \mathcal{H}_1 in Definition 3.2.1. Then by definition of \mathcal{H}_1 and $\mathcal{R} \in S \subset [1/4\rho, 1/2\rho]$, one has for every $\hat{z} \in \mathcal{T}_0$ and $\hat{x} \in W^u(\hat{z}, \mathcal{R}) \cap \mathcal{FN}_\varepsilon$:

$$\eta_\varepsilon(\hat{x}) \geq \mathcal{R}_{\hat{z}} \geq \frac{1}{\rho_0} \geq \frac{1}{\rho} \geq 2\mathcal{R}.$$

From Remark 3.8.3 we get $\beta_\varepsilon(\hat{x}) \leq \frac{1}{\eta_\varepsilon(\hat{x})} \leq \frac{1}{2\mathcal{R}}$. At last we still have $\widehat{\mu}(\mathcal{A}) > 0$ because the tree \mathcal{T}_0 is supposed to be regular : $\mu_{\pi_0, c_0}(\mathcal{T}_0) > 0$. \square

3.3.4 A decreasing partition η of the natural extension

We construct the measurable partition η . We refer to [1, 17, 28, 46, 47] for accounts on the theory on measurable partitions and entropy. We mention that in [1] (resp. in [17]) the theory of measurable partitions is used to prove the uniqueness of the measure of maximal entropy for polynomial automorphisms of \mathbb{C}^2 (resp. for automorphisms of K3 surfaces).

For sake of simplicity, we denote by g the left shift \widehat{f} . The arguments are borrowed from [28]. We use the partition \mathcal{P} and the Pesin box \mathcal{A} respectively defined in Proposition 3.3.1 and in Section 3.3.3. We define the measurable partitions

$$\xi := \pi_0^{-1}(\mathcal{P}) \bigvee \{\mathcal{A}, \mathcal{A}^c\} \quad \text{and} \quad \eta := \bigvee_{p \in \mathbb{N}} g^p(\xi).$$

The partition η satisfies the following properties.

Lemma 3.3.9.

1. For $\widehat{\mu}$ -almost every \hat{x} , the elements of $\eta_{\hat{x}}$ and \hat{x} have the same \mathcal{P} -address.
2. $\forall \hat{z} \in \mathcal{T}, \forall \hat{x} \in W^u(\hat{z}, \mathcal{R}), \eta_{\hat{x}} \subset W^u(\hat{z}, \mathcal{R}) \subset \mathcal{A}$.
3. There exists a measurable function $\eta'_\varepsilon : \widehat{\mathbb{P}^2} \rightarrow]0, 1]$ such that for $\widehat{\mu}$ -almost every \hat{x} :

- (a) $0 < \eta'_\varepsilon(\hat{x}) \leq \eta_\varepsilon(\hat{x}) \leq 1$.
- (b) $\forall n \geq 0, f_{\hat{x}}^{-n}(B(x_0, \eta'_\varepsilon(\hat{x}))) \subset B(c, \mathcal{R})$ or $f_{\hat{x}}^{-n}(B(x_0, \eta'_\varepsilon(\hat{x}))) \subset B(c, \mathcal{R})^c$.
- (c) $W^u(\hat{x}, \eta'_\varepsilon(\hat{x})) \subset \eta_{\hat{x}}$.

Proof :

1. The \mathcal{P} -address is constant on the atoms of $\widehat{\mathcal{P}}$, hence on the atoms of η .
2. Let $\hat{x}' \in \eta_{\hat{x}}$, by the previous point \hat{x}' and \hat{x} have the same \mathcal{P} -address. One has $\hat{x} \in \mathcal{A}$ by definition of \mathcal{A} , hence $\eta_{\hat{x}} \subset \mathcal{A}$ because η is thinner than $\{\mathcal{A}, \mathcal{A}^c\}$. Thus \hat{x}' belongs to \mathcal{A} and there is $\hat{z}' \in \mathcal{T}$ such that $\hat{x}' \in W^u(\hat{z}', \mathcal{R})$. Since the \mathcal{P} -address is constant on unstable manifolds, we deduce that the \mathcal{P} -address of \hat{z}' is the one of \hat{x}' , that is the one of \hat{x} . Then, since $z'_0 = c = z_0$, we can apply the point 2. of Lemma 3.3.3 (applied with $\widehat{w} := \hat{z}'$) to conclude that $\hat{z}' = \hat{z}$ and thus $\hat{x}' \in W^u(\hat{z}, \mathcal{R})$.
3. Let $\eta'_\varepsilon(\hat{x}) := \min \left\{ \mathcal{R}, \eta_\varepsilon(\hat{x}), \frac{\gamma(\hat{x})}{2\beta_\varepsilon(\hat{x})} \right\}$, where $\gamma(\hat{x})$ is the function given by Lemma 3.3.6 associated to the element \mathcal{R} of S . This choice of \mathcal{R} ensures that the diameter of $f_{\hat{x}}^{-n}(B(x_0, \eta_\varepsilon(\hat{x})))$ decreases exponentially faster as $n \rightarrow +\infty$ than the distance between x_{-n} and the boundary of the ball $B(c, \mathcal{R})$. It gives the point (b). We refer to [28, Lemma 4.2] for details and for a proof of (b) \implies (c). \square

The other important properties of the partition η are gathered in the following theorem. Given ζ a measurable partition, we denote $\mathcal{M}(\zeta)$ the completion with respect to $\widehat{\mu}$ of the σ -algebra generated by the atoms of ζ . Let \mathcal{M} be the completion with respect to $\widehat{\mu}$ of the σ -algebra of borel sets of $\widehat{\mathbb{P}^2}$.

Theorem 3.3.10. *The following properties hold for the partition η :*

1. η is decreasing : $\forall n \geq 0$, for $\widehat{\mu}$ -almost every \hat{x} , $(g^{-n}\eta)_{\hat{x}} \subset \eta_{\hat{x}}$.
2. π_0 is injective on $\eta_{\hat{x}}$ for $\widehat{\mu}$ -almost every \hat{x} .
3. For $\widehat{\mu}$ -almost every \hat{x} , f^n is injective on $\pi_0((g^{-n}\eta)_{\hat{x}})$ for every $n \geq 0$.
4. for every $n \geq 0$ and for $\widehat{\mu}$ -almost every \hat{x} , $\eta_{\hat{x}}$ is a countable union of atoms of $g^{-n}\eta$.
5. $\bigvee_{n \geq 0} \mathcal{M}(g^{-n}\eta)$ coincide with \mathcal{M} .
6. for every $n \geq 0$, $\int_{\widehat{\mathbb{P}^2}} -\text{Log } \mu_{\hat{x}}(g^{-n}\eta)_{\hat{x}} d\widehat{\mu}(x) = \text{Log } d^{2n}$.

The proof follows classical arguments, developed for instance in [28]. Let us explain the last item. By definition $\int_{\widehat{\mathbb{P}^2}} -\text{Log } \mu_{\hat{x}}((g^{-n}\eta)_{\hat{x}}) d\widehat{\mu}(\hat{x})$ is equal to the conditional entropy $H(g^{-n}\eta|\eta)$. Moreover, following [28, Section 4.3] one can prove that $H(g^{-n}\eta|\eta)$ is equal to the relative entropy $h_{\widehat{\mu}}(g^{-n}, \xi_n)$, where $\xi_n := g^{-1}\xi \vee \dots \vee g^{-n}\xi$. But ξ_n is a generator of finite entropy for g^{-n} [28, Proposition 4.1], hence $h_{\widehat{\mu}}(g^{-n}, \xi_n) = h_{\widehat{\mu}}(g^{-n}) = nh_{\widehat{\mu}}(g)$ by Kolmogorov-Sinai theorem [47, §9]. One finally obtains the last item from $h_{\widehat{\mu}}(g) = h_{\mu}(f) = \text{Log } d^2$.

From the point 3. we deduce the following elementary lemma. The purpose of the lemma is to prove that for any $\hat{x} \in \mathcal{A} \cap \Lambda_\varepsilon$, the identity $f_{\hat{x}}^{-n} \circ f^n = \text{Id}$ on $\pi_0[(g^{-n}\eta)_{\hat{x}_{-n}}]$ is true if we assume \mathcal{H}_1 . It is used in the proof of Proposition 3.4.7.

Lemma 3.3.11. *For any $\hat{x} \in \mathcal{A} \cap \Lambda_\varepsilon$ and for every $n \in \mathbb{N}$ one has :*

1. $f_{\hat{z}}^{-n}[\pi_0(\eta_{\hat{x}})] = \pi_0[(g^{-n}\eta)_{\hat{x}_{-n}}]$ with $\hat{z} \in \mathcal{T}$ such that $\hat{x} \in W^u(\hat{z}, \mathcal{R})$.
2. If we assume \mathcal{H}_1 then $f_{\hat{x}}^{-n} \circ f^n|_{\pi_0[(g^{-n}\eta)_{\hat{x}_{-n}}]} = \text{Id}|_{\pi_0[(g^{-n}\eta)_{\hat{x}_{-n}}]}$.

Proof : Let $\hat{y} \in \eta_{\hat{x}}$. Since $\hat{x} \in \mathcal{A}$ there exists $\hat{z} \in \mathcal{T}$ such that $\eta_{\hat{x}} \subset W^u(\hat{z}, \mathcal{R})$ (by Lemma 3.3.9). Then $\hat{y} \in W^u(\hat{z}, \mathcal{R})$ and so, by definition of the unstable manifold, we have $f_{\hat{z}}^{-n}(y_0) = y_{-n} \in \pi_0((g^{-n}\eta)_{\hat{x}_{-n}})$ since $g^{-n}\hat{y} \in g^{-n}(\eta_{\hat{x}}) = (g^{-n}\eta)_{\hat{x}_{-n}}$. Thus we have the inclusion " \subset ". The reverse inclusion " \supset " is obtained with the same arguments.

For the second item, observe that by Proposition 3.4.1 (given below) we have using \mathcal{H}_1

$$\pi_0(\eta_{\hat{x}}) \subset B(x_0, \eta_\varepsilon(\hat{x})) \subset \text{Dom}(f_{\hat{x}}^{-n}),$$

and thus $f^n(\pi_0[(g^{-n}\eta)_{\hat{x}_{-n}}]) = \pi_0[g^n g^{-n}(\eta_{\hat{x}})] = \pi_0(\eta_{\hat{x}})$ belongs to the domain of definition of the holomorphic map $f_{\hat{x}}^{-n}$. At this stage the map $h := f_{\hat{x}}^{-n} \circ f^n|_{\pi_0[(g^{-n}\eta)_{\hat{x}_{-n}}]}$ is well defined. Let us verify that $h = \text{Id}$. By definition one has :

$$\forall p \in \pi_0[(g^{-n}\eta)_{\hat{x}_{-n}}], \quad f^n(h(p)) = f^n(p). \quad (3.16)$$

Since $\pi_0(\eta_{\hat{x}}) \subset B(x_0, \eta_\varepsilon(\hat{x}))$ we can use the first point of Lemma 3.3.3 which asserts that $f_{\hat{x}}^{-n}$ and $f_{\hat{z}}^{-n}$ coincide on $\pi_0(\eta_{\hat{x}})$. Thus by using the point 1. we have :

$$h(\pi_0[(g^{-n}\eta)_{\hat{x}_{-n}}]) = f_{\hat{x}}^{-n} \circ \pi_0[\eta_{\hat{x}}] = f_{\hat{z}}^{-n}(\pi_0[\eta_{\hat{x}}]) = \pi_0[(g^{-n}\eta)_{\hat{x}_{-n}}].$$

Then (3.16) and injectivity of f^n on $\pi_0[(g^{-n}\eta)_{\hat{x}_{-n}}]$ (given by Theorem 3.3.10) imply $h(p) = p$ on $\pi_0[(g^{-n}\eta)_{\hat{x}_{-n}}]$. \square

3.4 The measures $q_{\hat{x}}$

3.4.1 Introduction of normal forms

For every $\hat{x} \in \Lambda_\varepsilon$, Theorem 3.8.2 provides holomorphic coordinates

$$\xi_{\hat{x}} : B(x_0, 2\eta_\varepsilon(\hat{x})) \longrightarrow \mathbb{D}^2(\rho_\varepsilon(\hat{x})) \quad , \quad \xi_{\hat{x}} = (Z_{\hat{x}}, W_{\hat{x}}). \quad (3.17)$$

We will define $q_{\hat{x}}$ on $\eta_{\hat{x}}$. The following lemma ensures that $\pi_0(\eta_{\hat{x}})$ is contained in the domain of definition of $W_{\hat{x}}$ for every $\hat{x} \in \mathcal{A} \cap \Lambda_\varepsilon$. The definition of \mathcal{A} is given in Equation (3.14).

Proposition 3.4.1. *Assume \mathcal{H}_1 and let $\hat{x} \in \mathcal{A} \cap \Lambda_\varepsilon$. Let \hat{z} be the unique element of \mathcal{T} such that $\hat{x} \in W^u(\hat{z}, \mathcal{R})$. Then*

$$\pi_0(\eta_{\hat{x}}) \subset B(z_0, \mathcal{R}) \subset B(x_0, \eta_\varepsilon(\hat{x})) \subset \text{Dom}(W_{\hat{x}}).$$

In particular, $\pi_0(\eta_{\hat{x}}) \subset B(z_0, \mathcal{R}) \subset B(x_0, \eta_\varepsilon(\hat{x})) \cap B(y_0, \eta_\varepsilon(\hat{y}))$ for every $\hat{y} \in \eta_{\hat{x}} \cap \Lambda_\varepsilon$.

Proof : By Lemma 3.3.9 one has $\eta_{\hat{x}} \subset W^u(\hat{z}, \mathcal{R})$, which proves the first inclusion. Since $x_0 \in B(z_0, \mathcal{R})$, we have $B(z_0, \mathcal{R}) \subset B(x_0, 2\mathcal{R})$. Proposition 3.3.7 finally gives $2\mathcal{R} \leq \eta_\varepsilon(\hat{x})$. \square

Proposition 3.4.2. *Let $\hat{x} \in \Lambda_\varepsilon$.*

1. *The measure $T \wedge dd^c |W_{\hat{x}}|^2$ is not null on every ball $B(x_0, r)$ with $0 < r \leq \eta_\varepsilon(\hat{x})$.*
2. *For every $n \geq 0$, $(T \wedge dd^c |W_{\hat{x}_{-n}}|^2) \circ \pi_0 \circ g^{-n}(\eta_{\hat{x}}) > 0$.*

Proof : The first assertion is a consequence of [30, § 3.3], which asserts that if h is a holomorphic submersion defined on a neighborhood of a point $x \in \text{Supp}(\mu)$ then $(T \wedge dd^c |h|^2)(B(x, r)) > 0$ for every small $r > 0$. For the second assertion, we use the inclusion

$$\eta_{\hat{x}} \supset W^u(\hat{x}, \eta'_\varepsilon(\hat{x}))$$

given by the point 3.(c) of Lemma 3.3.9. We deduce by definition of $W^u(\hat{x}, \eta'_\varepsilon(\hat{x}))$ that :

$$\pi_0[g^{-n}(\eta_{\hat{x}})] \supset \pi_0(\{\hat{y}_{-n}, \hat{y} \in W^u(\hat{x}, \eta'_\varepsilon(\hat{x}))\}) = f_{\hat{x}}^{-n}(B(x_0, \eta'_\varepsilon(\hat{x}))) \supset B(x_{-n}, r'),$$

where $r' > 0$ is small enough to get the last inclusion. The first item applied to \hat{x}_{-n} then allows to conclude. \square

Let us now define the measures $q_{\hat{x}}$. We recall that $\xi_{\hat{x}}$ is defined in (3.17). Let

$$M(\hat{x}) := \|d_0 \xi_{\hat{x}}^{-1} \cdot (0, 1)\| \quad , \quad \Delta(x_{-p}, p) = \|d_{x_{-p}} f^p \cdot \vec{v}_s(x_{-p})\|.$$

Definition 3.4.3. *Assume \mathcal{H}_1 . For every $\hat{x} \in \Lambda_\varepsilon$ we define*

$$\tilde{q}_{\hat{x}} := d^p (\Delta(x_{-p}, p))^2 M(\hat{x}_{-p})^2 \times (T \wedge dd^c |W_{\hat{x}_{-p}}|^2) \circ \pi_0 \circ g^{-p} \text{ on } \eta_{\hat{x}},$$

where $p \geq 0$ is any integer such that \hat{x}_{-p} belongs to \mathcal{A} . The probability measure $q_{\hat{x}}$ is then

$$\forall A \in \mathcal{B}(\widehat{\mathbb{P}^2}), \quad q_{\hat{x}}(A) := \frac{\tilde{q}_{\hat{x}}(A \cap \eta_{\hat{x}})}{L(\hat{x})},$$

the normalization $L(\hat{x}) := \tilde{q}_{\hat{x}}(\eta_{\hat{x}})$ being > 0 by Proposition 3.4.2.

Remark 3.4.4.

1. *Up to a modification of Λ_ε using Birkhoff ergodic theorem, the set of integers $\text{Rec}_{\hat{x}}(\mathcal{A}) := \{p \geq 0, \hat{x}_{-p} \in \mathcal{A}\}$ is infinite for every $\hat{x} \in \Lambda_\varepsilon$.*
2. *The fact that $\tilde{q}_{\hat{x}}$ does not depend on p is established in Proposition 3.4.7. We need to introduce p so that $T \wedge dd^c |W_{\hat{x}_{-p}}|^2$ is well defined on $\pi_0(\eta_{\hat{x}_{-p}})$ (see Proposition 3.4.1), which contains $\pi_0(g^{-p}(\eta_{\hat{x}}))$ by the decreasing property of η .*
3. *By Theorem 3.3.10, π_0 is injective on the atoms of $g^{-p}\eta$, thus $\tilde{q}_{\hat{x}}$ is a measure.*
4. *If $\hat{x} \in \mathcal{A}$, then $\tilde{q}_{\hat{x}} = M(\hat{x})^2 (T \wedge dd^c |W_{\hat{x}}|^2) \circ \pi_0$ on $\eta_{\hat{x}}$.*

Proposition 3.4.5. *Assume \mathcal{H}_2 , then for $\hat{\mu}$ -almost every $\hat{x} \in \widehat{\mathbb{P}^2}$:*

$$\hat{y} \in \eta_{\hat{x}} \cap \Lambda_\varepsilon \implies q_{\hat{x}} = q_{\hat{y}}.$$

In particular we can define a measure \hat{q} on $\mathcal{B}(\widehat{\mathbb{P}^2})$ whose conditionals on η are given by the measures $q_{\hat{x}}$:

$$\forall A \in \mathcal{B}(\widehat{\mathbb{P}^2}), \quad \hat{q}(A) = \int_{\widehat{\mathbb{P}^2}} q_{\hat{x}}(A \cap \eta_{\hat{x}}) d\hat{\mu}(\hat{x}).$$

Remark 3.4.6. From the proof of Theorem 3.8.2, one can construct $\widehat{x} \mapsto \xi_{\widehat{x}}$ as limits of holomorphic maps hence the dependence on \widehat{x} is measurable. Thus for every borel set $A \subset \widehat{\mathbb{P}^2}$, the function $\widehat{x} \mapsto (T \wedge dd^c |W_{\widehat{x}}|^2)(\pi_0(A \cap \eta_{\widehat{x}}))$ is measurable and the integrals above $A \mapsto \int q_{\widehat{x}}(A \cap \eta_{\widehat{x}}) d\widehat{\mu}(\widehat{x})$ make sense.

The proof of Proposition 3.4.5 is provided in Section 3.7.

3.4.2 The definition of $q_{\widehat{x}}$ does not depend on $p \in \text{Rec}_{\widehat{x}}(\mathcal{A})$

The set of integers $\text{Rec}_{\widehat{x}}(\mathcal{A})$ is defined in Remark 3.4.4.

Proposition 3.4.7. Assume \mathcal{H}_1 . Let $\widehat{x} \in \Lambda_\varepsilon$, $(p, q) \in \text{Rec}_{\widehat{x}}(\mathcal{A})^2$ and $A \subset \eta_{\widehat{x}}$ be a borel set. Let

$$E^p := d^p \Delta(x_{-p}, p)^2 M(\widehat{x}_{-p})^2 (T \wedge dd^c |W_{\widehat{x}_{-p}}|^2) \circ \pi_0 \circ g^{-p}(A),$$

and let E^q be defined similarly replacing p by q . Then $E^p = E^q$.

We need two lemmas.

Lemma 3.4.8. For every $\widehat{x} \in \Lambda_\varepsilon$ the following formula holds on $B(x_0, 2\eta_\varepsilon(\widehat{x}))$:

$$W_{\widehat{x}_{-n}} \circ f_{\widehat{x}}^{-n} = \beta_{n, \widehat{x}} \times W_{\widehat{x}}, \quad \forall n \geq N(\widehat{x}).$$

In particular, for every $n \geq N(\widehat{x})$, we get on $B(x_0, 2\eta_\varepsilon(\widehat{x}))$:

$$(f_{\widehat{x}}^{-n})^*(T \wedge dd^c |W_{\widehat{x}_{-n}}|^2) = d^{-n} |\beta_{n, \widehat{x}}|^2 \times (T \wedge dd^c |W_{\widehat{x}}|^2).$$

Proof : The commutative diagram (3.9) given by Theorem 3.8.2 provides for $n \geq N(\widehat{x})$:

$$W_{\widehat{x}_{-n}} \circ f_{\widehat{x}}^{-n} = W \circ (\xi_{\widehat{x}_{-n}} \circ f_{\widehat{x}}^{-n}) = (W \circ R_{n, \widehat{x}}) \circ \xi_{\widehat{x}} = \beta_{n, \widehat{x}} \times W_{\widehat{x}},$$

where $W : \mathbb{C}^2 \rightarrow \mathbb{C}$ is the projection on the second coordinate. □

Lemma 3.4.9.

1. For every $\widehat{x} \in \Lambda_\varepsilon$ and $n, k \geq 0$, $\Delta(x, n)\Delta(x, k) = \Delta(x, n+k)$.
2. For every $\widehat{x} \in \Lambda_\varepsilon$ and $n \geq N(\widehat{x})$, $\Delta(x_{-n}, n)|\beta_{n, \widehat{x}}| = \frac{M(\widehat{x})}{M(\widehat{x}_{-n})}$.

Proof : The first item comes from the chain rule formula. For the second item, the commutative diagram (3.9) of Theorem 3.8.2 gives for $n \geq N(\widehat{x})$:

$$|\beta_{n, \widehat{x}}| = \left\| d_{x_{-n}} \xi_{\widehat{x}_{-n}} \circ d_{x_0} f_{\widehat{x}}^{-n} \circ d_0 (\xi_{\widehat{x}})^{-1} \cdot (0, 1) \right\|.$$

One concludes by Theorems 3.8.1 and 3.8.2 which yield $d_0 (\xi_{\widehat{x}})^{-1} \cdot (0, 1) = M(\widehat{x}) \vec{v}_s(x_0)$,

$$d_{x_0} f_{\widehat{x}}^{-n} \cdot \vec{v}_s(x_0) = \Delta(x_{-n}, n)^{-1} \vec{v}_s(x_{-n}),$$

and $\|d_{x_{-n}} \xi_{\widehat{x}_{-n}} \cdot \vec{v}_s(x_{-n})\| = M(\widehat{x}_{-n})^{-1}$. □

Proof of Proposition 3.4.7 : Let $\hat{x} \in \Lambda_\varepsilon$ and let $A \subset \eta_{\hat{x}}$ be a borel set. We have to prove $E^p = E^q$ for any $p, q \in \text{Rec}_{\hat{x}}(\mathcal{A})$. It suffices to show the following property :

$$\forall p, q \in \text{Rec}_{\hat{x}}(\mathcal{A}), (p - q \geq N(\hat{x}) + N(\hat{x}_{-q})) \implies E^p = E^q \quad : (P)$$

Indeed since $\text{Rec}_{\hat{x}}(\mathcal{A})$ is infinite, for any $p, q \in \text{Rec}_{\hat{x}}(\mathcal{A})$ we can find $r \in \text{Rec}_{\hat{x}}(\mathcal{A})$ such that $r - p \geq N(\hat{x}) + N(\hat{x}_{-p})$ and $r - q \geq N(\hat{x}) + N(\hat{x}_{-q})$. Thus if (P) is true, $E^r = E^p$ and $E^r = E^q$, which implies $E^p = E^q$. Let us prove now the property (P). Let $p, q \in \text{Rec}_{\hat{x}}(\mathcal{A})$ such that $p - q \geq N(\hat{x}) + N(\hat{x}_{-q})$. We define :

$$F^p := (T \wedge dd^c |W_{\hat{x}_{-p}}|^2) \circ \pi_0 \circ g^{-p}(A)$$

and similarly F^q by replacing p by q . Proving $E^p = E^q$ amounts to show the equality :

$$d^p \Delta(x_{-p}, p)^2 M(\hat{x}_{-p})^2 F^p = d^q \Delta(x_{-q}, q)^2 M(\hat{x}_{-q})^2 F^q. \quad (3.18)$$

Since $\hat{x}_{-q} \in \mathcal{A}$ and $g^{-p}(\eta_{\hat{x}}) \subset (g^{-n}\eta)_{\hat{x}_{-q-n}}$, one gets by Lemma 3.3.11 (applied to \hat{x}_{-q} with $n = p - q$) :

$$f_{\hat{x}_{-q}}^{-(p-q)} \circ f_U^{p-q}|_{\pi_0(g^{-p}(\eta_{\hat{x}}))} = \text{Id}|_{\pi_0(g^{-p}(\eta_{\hat{x}}))}. \quad (3.19)$$

Let us denote $U := \pi_0(g^{-p}(\eta_{\hat{x}}))$ and $f_U^{p-q} := f_U^{p-q}|_U$. Using hypothesis \mathcal{H}_1 , Proposition 3.4.1 tells us that $\pi_0(\eta_{\hat{x}_{-p}}) \subset B(x_{-p}, \eta_\varepsilon(\hat{x}_{-p})) \subset \text{Dom}(W_{\hat{x}_{-p}})$, and thus $\pi_0(g^{-p}(A)) \subset \pi_0(g^{-p}(\eta_{\hat{x}})) \subset \pi_0(\eta_{\hat{x}_{-p}}) \subset \text{Dom}(W_{\hat{x}_{-p}})$. By (3.19) and by $(f_U^{p-q})^*T = d^{p-q}T$, we obtain

$$\begin{aligned} F^p &= \left[T \wedge (f_{\hat{x}_{-q}}^{-(p-q)} \circ f_U^{p-q})^* dd^c |W_{\hat{x}_{-p}}|^2 \right] \circ \pi_0 \circ g^{-p}(A) \\ &= d^{q-p} \left[(f_U^{p-q})^* T \wedge (f_U^{p-q})^* dd^c |W_{\hat{x}_{-p}} \circ f_{\hat{x}_{-q}}^{-(p-q)}|^2 \right] \circ \pi_0 \circ g^{-p}(A). \end{aligned}$$

Then by injectivity of f_U^{p-q} we have :

$$F^p = d^{q-p} \left[T \wedge dd^c |W_{\hat{x}_{-p}} \circ f_{\hat{x}_{-q}}^{-(p-q)}|^2 \right] \circ (f_U^{p-q}) \circ \pi_0 \circ g^{-p}(A).$$

Using the relation $f \circ \pi_0 = \pi_0 \circ g$, we have $(f_U^{p-q}) \circ \pi_0 \circ g^{-p}(A) = \pi_0 \circ g^{-q}(A)$. Moreover, by Lemma 3.4.8 with \hat{x} replaced by \hat{x}_{-q} and $n = p - q \geq N(\hat{x}_{-q})$, we get :

$$W_{\hat{x}_{-p}} \circ f_{\hat{x}_{-q}}^{-(p-q)} = \beta_{p-q, \hat{x}_{-q}} \times W_{\hat{x}_{-q}} \text{ on } B(x_{-q}, 2\eta_\varepsilon(\hat{x}_{-q})).$$

We deduce :

$$F^p = d^{q-p} |\beta_{p-q, \hat{x}_{-q}}|^2 \times F^q.$$

Now we replace $|\beta_{p-q, \hat{x}_{-q}}|^2$ by applying the second item of Lemma 3.4.9 to \hat{x}_{-q} with $n = p - q \geq N(\hat{x}_{-q})$. We get

$$d^p M(\hat{x}_{-p})^2 F^p = d^q M(\hat{x}_{-q})^2 F^q \times \frac{1}{\Delta(x_{-p}, p - q)^2}.$$

To conclude, we use $\Delta(x_{-p}, p - q)\Delta(x_{-q}, q) = \Delta(x_{-p}, p)$ which comes from Lemma 3.4.9 applied to \hat{x}_{-p} , $n = p - q$ and $k = q$. \square

3.5 Proof of Theorem B.II

The second formula of Theorem B is contained in Theorem 3.3.10. Let us now show for every $n \geq 0$ the first formula of Theorem B :

$$\text{Log } d^n + 2n\lambda_2 = \int_{\widehat{\mathbb{P}^2}} -\text{Log } q_{\widehat{x}}(g^{-n}\eta)_{\widehat{x}} d\widehat{\mu}(\widehat{x}). \quad (3.20)$$

Proposition 3.5.1. *For $\widehat{\mu}$ -almost every \widehat{x} and for every $n \geq 0$:*

$$L(\widehat{x})q_{\widehat{x}}((g^{-n}\eta)_{\widehat{x}}) = \frac{1}{d^n} \frac{L(\widehat{x}_n)}{\Delta(x_0, n)^2}. \quad (3.21)$$

Proof : Let $\widehat{x} \in \Lambda_\varepsilon$ and $p \in \mathbb{N}$ such that $\widehat{x}_{-p} \in \mathcal{A}$. Let us fix $n \in \mathbb{N}$ and let $q := n + p$ so that $\widehat{x}_{n-q} = \widehat{x}_{-p} \in \mathcal{A}$. By definition of $q_{\widehat{x}}$ and the inclusion $(g^{-n}\eta)_{\widehat{x}} \subset \eta_{\widehat{x}}$:

$$L(\widehat{x})q_{\widehat{x}}((g^{-n}\eta)_{\widehat{x}}) = d^p \Delta(x_{-p}, p)^2 M(\widehat{x}_{-p})^2 \times [T \wedge dd^c |W_{\widehat{x}_{-p}}|^2] \circ \pi_0 \circ g^{-p}((g^{-n}\eta)_{\widehat{x}}).$$

Using $p = q - n$ and $g^{-p}((g^{-n}\eta)_{\widehat{x}}) = g^{-q}(\eta_{\widehat{x}_n})$ we have :

$$L(\widehat{x})q_{\widehat{x}}((g^{-n}\eta)_{\widehat{x}}) = \frac{1}{d^n} \Delta(x_{n-q}, p)^2 d^q M(\widehat{x}_{n-q})^2 [T \wedge dd^c |W_{\widehat{x}_{n-q}}|^2] \circ \pi_0 \circ g^{-q}(\eta_{\widehat{x}_n}).$$

By definition of $L(\widehat{x}_n)$ (use Definition 3.4.3 replacing in it \widehat{x} by \widehat{x}_n and p by q) we have

$$L(\widehat{x})q_{\widehat{x}}((g^{-n}\eta)_{\widehat{x}}) = \frac{\Delta(x_{n-q}, p)^2}{d^n \Delta(x_{n-q}, q)^2} \times L(\widehat{x}_n).$$

The first item of Lemma 3.4.9 applied to \widehat{x}_{n-q} , p and $q - p$ completes the proof. \square

By taking Log in Equation (3.21), we obtain :

$$\text{Log } d^n + 2 \text{Log } \Delta(x_0, n) = -\text{Log } q_{\widehat{x}}(g^{-n}\eta)_{\widehat{x}} + \text{Log } \frac{L(\widehat{x}_n)}{L(\widehat{x})}. \quad (3.22)$$

The third item of Theorem 3.8.1 asserts that $\int_{\widehat{\mathbb{P}^2}} \text{Log } \Delta(x_0, n) d\widehat{\mu}(\widehat{x}) = n\lambda_2$. To finish the proof of Formula (3.20), it remains to show that $h_n := \text{Log} \left(\frac{L \circ g^n}{L} \right) \in L^1(\widehat{\mu})$ and satisfies $\int h_n d\widehat{\mu} = 0$. We use the following classical lemma, the original statement is stated with Log^- instead of Log^+ , but the proof also works with Log^+ .

Lemma 3.5.2 (Ledrappier-Strelcyn [41, Proposition 2.2]). *Let $n \geq 0$ and let φ be a positive measurable function on $\widehat{\mathbb{P}^2}$. If $\text{Log}^+ \left(\frac{\varphi \circ g^n}{\varphi} \right) \in L^1(\widehat{\mu})$ then :*

$$\text{Log} \left(\frac{\varphi \circ g^n}{\varphi} \right) \in L^1(\widehat{\mu}) \text{ and } \int \text{Log} \left(\frac{\varphi \circ g^n}{\varphi} \right) d\widehat{\mu} = 0.$$

The assumptions of this lemma are satisfied. Indeed, Equation (3.22) implies for $\widehat{\mu}$ -almost every \widehat{x} in $\{h_n \geq 0\}$:

$$0 \leq h_n(\widehat{x}) = \text{Log}(d^n \Delta(x_0, n)^2) + \text{Log } q_{\widehat{x}}((g^{-n}\eta)_{\widehat{x}}) \leq \text{Log}(d^n \Delta(x_0, n)^2).$$

Moreover, $\widehat{x} \mapsto \text{Log } \Delta(x_0, n)$ belongs to $L^1(\widehat{\mu})$ by Theorem 3.8.1. Therefore we can apply Lemma 3.5.2 to get $h_n \in L^1(\widehat{\mu})$ and $\int h_n d\widehat{\mu} = 0$ as desired.

To complete the proof of Theorem B we prove now that $(\pi_0)_* q_{\widehat{x}} \ll (T \wedge \omega_{\mathbb{P}^2}) = \sigma_T$ for $\widehat{\mu}$ -almost every \widehat{x} . By Birkhoff ergodic theorem

$$\widehat{\mu} \left(\bigcup_{N \geq 0} g^N(\mathcal{A}) \right) = 1, \quad (3.23)$$

since $\widehat{\mu}(\mathcal{A}) > 0$. So it is sufficient to prove :

Lemma 3.5.3. *Assume \mathcal{H}_1 , let $N \geq 0$ and $\widehat{x} \in g^N(\mathcal{A}) \cap \Lambda_\varepsilon$. Then $(\pi_0)_* q_{\widehat{x}} \ll \sigma_T$.*

For every $\widehat{x} \in \Lambda_\varepsilon$, the measure $T \wedge dd^c |W_{\widehat{x}}|^2$ is absolutely continuous with respect to the trace measure $\sigma_T = T \wedge \omega_{\mathbb{P}^2}$.

Definition 3.5.4. *We can define the Radon-Nikodym derivative :*

$$D_{\widehat{x}} := \frac{d(T \wedge dd^c |W_{\widehat{x}}|^2)}{d\sigma_T} \text{ on } B(x_0, 2\eta_\varepsilon(\widehat{x})).$$

Now we prove the lemma using the fact that the measures σ_T and $(f^N)^*(T \wedge \omega_{\mathbb{P}^2})$ have continuous local potentials so that they do not charge analytic sub-sets of \mathbb{P}^2 of codimension ≥ 1 .

Proof of Lemma 3.5.3 : Let us fix a borel set B of \mathbb{P}^2 , and let us define the constant $C(N, \widehat{x}) := d^N \Delta(x_{-N}, N)^2 M(\widehat{x}_{-N})^2 / L(\widehat{x})$ so that

$$q_{\widehat{x}}(\pi_0^{-1}(B)) = C(N, \widehat{x}) \times (T \wedge dd^c |W_{\widehat{x}_{-N}}|^2) \circ \pi_0 \circ g^{-N}[\eta_{\widehat{x}} \cap \pi_0^{-1}(B)].$$

Now observe that $g^{-N}[\eta_{\widehat{x}} \cap \pi_0^{-1}B] = (g^{-N}\eta)_{\widehat{x}_{-N}} \cap (g^{-N}\pi_0^{-1}B) = (g^{-N}\eta)_{\widehat{x}_{-N}} \cap \pi_0^{-1}(f^{-N}B)$. Recall that f^N is injective on $\pi_0[(g^{-N}\eta)_{\widehat{x}_{-N}}]$ by Theorem 3.3.10. Let h_N denote the inverse of the restriction of f^N to $\pi_0[(g^{-N}\eta)_{\widehat{x}_{-N}}]$, and let $B' := B \cap \pi_0(\eta_{\widehat{x}})$. We obtain

$$q_{\widehat{x}}(\pi_0^{-1}(B)) / C(N, \widehat{x}) \leq (T \wedge dd^c |W_{\widehat{x}_{-N}}|^2)(h_N(B')) =: l_{N, \widehat{x}}(B').$$

Using Definition 3.5.4, we have

$$l_{N, \widehat{x}}(B') = \int_{h_N(B')} D_{\widehat{x}_{-N}} d\sigma_T.$$

Now it suffices to show that $\sigma_T(B) = 0 \implies l_{N, \widehat{x}}(B') = 0$. Let E_N be the continuous function on \mathbb{P}^2 defined on $\mathbb{P}^2 \setminus \text{Crit}(f^N)$ by $E_N(p) := \|(d_p f^N)^{-1}\|^{-2}$ and extended by 0 on $\text{Crit}(f^N)$. It satisfies $E_N \times \omega_{\mathbb{P}^2} \leq (f^N)^* \omega_{\mathbb{P}^2}$ and therefore

$$\sigma_T \leq E_N^{-1} \times T \wedge (f^N)^* \omega_{\mathbb{P}^2} = E_N^{-1} \times d^{-N} (f^N)^* (T \wedge \omega_{\mathbb{P}^2}) \text{ on } \mathbb{P}^2 \setminus \text{Crit}(f^N),$$

where we used $(f^N)^* T = d^N T$ for the equality. The preceding estimate holds on \mathbb{P}^2 since σ_T and $(f^N)^*(T \wedge \omega_{\mathbb{P}^2})$ do not charge $\text{Crit}(f^N)$ as explained earlier. Hence

$$l_{N, \widehat{x}}(B') \leq d^{-N} \int_{h_N(B')} E_N^{-1} D_{\widehat{x}_{-N}} d(f^N)^* [T \wedge \omega_{\mathbb{P}^2}].$$

Since $f^N : h_N(B') \rightarrow B'$ is invertible with inverse map $h_N|_{B'}$, we get

$$l_{N,\hat{x}}(B') \leq d^{-N} \int_{B'} (E_N^{-1} D_{\hat{x}_{-N}}) \circ h_N \, d[T \wedge \omega_{\mathbb{P}^2}].$$

That implies $\sigma_T(B) = 0 \implies l_{N,\hat{x}}(B') = 0$, as desired. \square

3.6 Absolute continuity of μ with respect to T

3.6.1 Proof of $\hat{q} = \hat{\mu}$

The probability measure \hat{q} is defined in Proposition 3.4.5.

Theorem 3.6.1. *Assume \mathcal{H}_2 and $\lambda_1 > \lambda_2 = \frac{1}{2} \text{Log } d$.*

1. *For $\hat{\mu}$ -almost every \hat{x} and for every $n \geq 0$, $q_{\hat{x}}(g^{-n}\eta)_{\hat{x}} = \mu_{\hat{x}}(g^{-n}\eta)_{\hat{x}}$.*
2. *For $\hat{\mu}$ -almost every \hat{x} , $q_{\hat{x}} = \mu_{\hat{x}}$ and so $\hat{q} = \hat{\mu}$.*

Proof : The second item is a classical consequence of the first one by using the generating property of η , see for instance [28, Lemme 5.6]. Let us prove the first item. By Theorem 3.3.10 we know that for any $n \geq 0$:

$$\int_{\widehat{\mathbb{P}^2}} -\text{Log } \mu_{\hat{x}}(g^{-n}\eta)_{\hat{x}} \, d\hat{\mu}(\hat{x}) = \text{Log } d^n + \text{Log } d^n.$$

But we have by Theorem B (using \mathcal{H}_1) for every $n \geq 0$:

$$\int_{\widehat{\mathbb{P}^2}} -\text{Log } q_{\hat{x}}(g^{-n}\eta)_{\hat{x}} \, d\hat{\mu}(\hat{x}) = \text{Log } d^n + 2n\lambda_2.$$

If λ_2 is minimal equal to $\frac{1}{2} \text{Log } d$, we get $\text{Log } d^n = 2n\lambda_2$ and thus :

$$\int_{\widehat{\mathbb{P}^2}} -\text{Log } q_{\hat{x}}(g^{-n}\eta)_{\hat{x}} \, d\hat{\mu}(\hat{x}) = - \int_{\widehat{\mathbb{P}^2}} \text{Log } \mu_{\hat{x}}(g^{-n}\eta)_{\hat{x}} \, d\hat{\mu}(\hat{x}).$$

By Jensen inequality we deduce :

$$0 = \int_{\widehat{\mathbb{P}^2}} \text{Log } \frac{q_{\hat{x}}}{\mu_{\hat{x}}}(g^{-n}\eta)_{\hat{x}} \, d\hat{\mu}(\hat{x}) \leq \text{Log } \int_{\widehat{\mathbb{P}^2}} \frac{q_{\hat{x}}}{\mu_{\hat{x}}}(g^{-n}\eta)_{\hat{x}} \, d\hat{\mu}(\hat{x}). \quad (3.24)$$

By definition of the conditional measures of $\hat{\mu}$ with respect to η , we get

$$\int_{\widehat{\mathbb{P}^2}} \frac{q_{\hat{x}}}{\mu_{\hat{x}}}(g^{-n}\eta)_{\hat{x}} \, d\hat{\mu}(\hat{x}) = \int_{\widehat{\mathbb{P}^2}} \int_{\eta_{\hat{x}}} \frac{q_{\hat{y}}}{\mu_{\hat{y}}}(g^{-n}\eta)_{\hat{y}} \, d\mu_{\hat{x}}(\hat{y}) \, d\hat{\mu}(\hat{x}).$$

But by Proposition 3.4.5 we have $q_{\hat{x}} = q_{\hat{y}}$ for every $\hat{y} \in \eta_{\hat{x}} \cap \Lambda_\varepsilon$. Thus we have

$$\int_{\eta_{\hat{x}}} \frac{q_{\hat{y}}}{\mu_{\hat{y}}}(g^{-n}\eta)_{\hat{y}} \, d\mu_{\hat{x}}(\hat{y}) = \int_{\eta_{\hat{x}}} \frac{q_{\hat{x}}}{\mu_{\hat{x}}}(g^{-n}\eta)_{\hat{y}} \, d\mu_{\hat{x}}(\hat{y}).$$

Now by Theorem 3.3.10 $\eta_{\hat{x}}$ is a countable union $\bigsqcup_{j \in \mathbb{N}} A_{j,n}^{\hat{x}}$ of atoms of $g^{-n}\eta$. We infer

$$\int_{\eta_{\hat{x}}} \frac{q_{\hat{x}}}{\mu_{\hat{x}}}(g^{-n}\eta)_{\hat{y}} \, d\mu_{\hat{x}}(\hat{y}) = \sum_{j=0}^{+\infty} \int_{A_{j,n}^{\hat{x}}} \frac{q_{\hat{x}}}{\mu_{\hat{x}}}(A_{j,n}^{\hat{x}}) \, d\mu_{\hat{x}}(\hat{y}) = \sum_{j=0}^{+\infty} \frac{q_{\hat{x}}}{\mu_{\hat{x}}}(A_{j,n}^{\hat{x}}) \times \mu_{\hat{x}}(A_{j,n}^{\hat{x}})$$

which is equal to $q_{\hat{x}}(\bigsqcup_j A_{j,n}^{\hat{x}}) = 1$. Finally, $\int_{\widehat{\mathbb{P}^2}} \frac{q_{\hat{x}}}{\mu_{\hat{x}}}(g^{-n}\eta)_{\hat{x}} \, d\hat{\mu}(\hat{x}) = 1$, which implies the equality in Equation (3.24). The strict concavity of Log completes the proof. \square

3.6.2 Proof of Theorem A.II

Let us assume in this section \mathcal{H}_2 and $\lambda_1 > \lambda_2 = \frac{1}{2} \text{Log } d$. Our aim is to prove

$$\mu \ll \sigma_T \text{ on } \mathbb{P}^2.$$

Observe that for a borel set B :

$$\mu(B) = \widehat{\mu}(\pi_0^{-1}(B)) = \widehat{q}(\pi_0^{-1}(B)) \leq \sum_{N=0}^{+\infty} \int_{g^N(\mathcal{A})} q_{\widehat{x}}(\pi_0^{-1}(B)) \, d\widehat{\mu}(\widehat{x}),$$

where the second equality comes from Theorem 3.6.1 and the inequality comes from (3.23). In particular we have

$$\mu(B) \leq \sum_{N=0}^{+\infty} \int_{g^N(\mathcal{A})} (\pi_0)_* q_{\widehat{x}}(B) \, d\widehat{\mu}(\widehat{x}),$$

and since the measures $(\pi_0)_* q_{\widehat{x}}$ are absolutely continuous with respect to σ_T by Lemma 3.5.3, the conclusion follows.

3.7 Proof of Proposition 3.4.5

In this section we assume \mathcal{H}_2 and prove for $\widehat{\mu}$ -almost every $\widehat{x} \in \widehat{\mathbb{P}}^2$:

$$\widehat{y} \in \eta_{\widehat{x}} \cap \Lambda_\varepsilon \implies q_{\widehat{x}} = q_{\widehat{y}} \text{ on } \eta_{\widehat{x}}. \quad (3.25)$$

Section 3.7.1 only needs \mathcal{H}_1 to consider the measures $q_{\widehat{x}}$. We will need \mathcal{H}_2 in Section 3.7.2.

3.7.1 Reduction to the case $\widehat{x} \in \mathcal{A}$

Proposition 3.7.1. *If Equation (3.25) holds for $\widehat{\mu}$ -almost every \widehat{x} in \mathcal{A} , then it holds for $\widehat{\mu}$ -almost every \widehat{x} in $\widehat{\mathbb{P}}^2$.*

Proof : The assumption implies that there exists $E \subset \Lambda_\varepsilon \cap \mathcal{A}$ of measure $\widehat{\mu}(E) = \widehat{\mu}(\mathcal{A})$ such that for every $\widehat{x} \in E$, (3.25) is true. Then by Birkhoff ergodic theorem, there exists $F \subset \Lambda_\varepsilon$ of measure $\widehat{\mu}(F) = 1$ such that for every $\widehat{x} \in F$, there exists $n \geq 0$ satisfying $\widehat{x}_{-n} \in E$. Now let us fix $\widehat{x} \in F$ and let $n \geq 0$ such that $\widehat{x}_{-n} \in E$. Let $\widehat{y} \in \Lambda_\varepsilon \cap \eta_{\widehat{x}}$. Then because η is decreasing and is thinner than $\{\mathcal{A}, \mathcal{A}^c\}$, one has $\widehat{y}_{-n} \in \eta_{\widehat{x}_{-n}} \subset \{\mathcal{A}, \mathcal{A}^c\}_{\widehat{x}_{-n}} = \mathcal{A}$. So we have $q_{\widehat{y}_{-n}} = q_{\widehat{x}_{-n}}$ by definition of E , which can be written as

$$\frac{1}{L(\widehat{y}_{-n})} \widetilde{q}_{\widehat{y}_{-n}} = \frac{1}{L(\widehat{x}_{-n})} \widetilde{q}_{\widehat{x}_{-n}}. \quad (3.26)$$

Now using the definition of $q_{\widehat{y}}$ we have :

$$L(\widehat{y})q_{\widehat{y}} = d^n \Delta(y_{-n}, n)^2 \times M(\widehat{y}_{-n})^2 (T \wedge dd^c |W_{\widehat{y}_{-n}}|^2) \circ \pi_0 \circ g^{-n} \lfloor_{\eta_{\widehat{y}}}.$$

Since $g^{-n}(\eta_{\widehat{y}}) \subset \eta_{\widehat{y}_{-n}}$, one can replace $\pi_0 \circ g^{-n} \lfloor_{\eta_{\widehat{y}}}$ by $\pi_0 \lfloor_{\eta_{\widehat{y}_{-n}}} \circ g^{-n} \lfloor_{\eta_{\widehat{y}}}$. Moreover, because $\widehat{y}_{-n} \in \mathcal{A}$, we get by Remark 3.4.4 :

$$\widetilde{q}_{\widehat{y}_{-n}} = M(\widehat{y}_{-n})^2 (T \wedge dd^c |W_{\widehat{y}_{-n}}|^2) \circ \pi_0 \lfloor_{\eta_{\widehat{y}_{-n}}}.$$

We finally deduce

$$L(\hat{y})q_{\hat{y}} = d^n \Delta(y_{-n}, n)^2 \times \tilde{q}_{\hat{y}_{-n}} \circ g^{-n}|_{\eta_{\hat{y}}},$$

and similarly $L(\hat{x})q_{\hat{x}} = d^n \Delta(x_{-n}, n)^2 \times \tilde{q}_{\hat{x}_{-n}} \circ g^{-n}|_{\eta_{\hat{x}}}$. Equation (3.26) yields

$$q_{\hat{y}} = \frac{\Delta(y_{-n}, n)^2 L(\hat{y}_{-n}) L(\hat{x})}{\Delta(x_{-n}, n)^2 L(\hat{x}_{-n}) L(\hat{y})} \times q_{\hat{x}}.$$

The ratio is equal to 1 and $q_{\hat{y}} = q_{\hat{x}}$ since these are probability measures on $\eta_{\hat{x}}$. \square

3.7.2 The case $\hat{x} \in \mathcal{A}$

In this section we assume \mathcal{H}_2 and $\lambda_1 > \lambda_2$, and we prove Equation (3.25) for $\hat{x} \in \mathcal{A} \cap \Lambda_\varepsilon$. Let us begin with some remarks.

Let \hat{z} be the unique element of \mathcal{T} such that $\hat{x} \in W^u(\hat{z}, \mathcal{R})$. Lemma 3.3.9 yields $\eta_{\hat{x}} \subset W^u(\hat{z}, \mathcal{R}) \subset \mathcal{A}$, and Proposition 3.4.1 gives

$$\forall \hat{y} \in \eta_{\hat{x}} \cap \Lambda_\varepsilon, \quad \pi_0(\eta_{\hat{x}}) \subset B(z_0, \mathcal{R}) \subset B(x_0, \eta_\varepsilon(\hat{x})) \cap B(y_0, \eta_\varepsilon(\hat{y})). \quad (3.27)$$

By Lemma 3.3.3, the maps $f_{\hat{x}}^{-n}, f_{\hat{y}}^{-n}$ and $f_{\hat{z}}^{-n}$ coincide on $B(z_0, \mathcal{R})$, thus we get

$$f_{\hat{z}}^{-n} \circ \pi_0(\eta_{\hat{x}}) \subset f_{\hat{z}}^{-n}(B(z_0, \mathcal{R})) \subset B(x_{-n}, \eta_\varepsilon(\hat{x}_{-n})) \cap B(y_{-n}, \eta_\varepsilon(\hat{y}_{-n})) \quad (3.28)$$

for any $n \geq \max\{N(\hat{x}), N(\hat{y})\}$, see Theorem 3.8.2.

We will denote for each $n \geq 0$, $B_n := B(x_n, 2\eta_\varepsilon(\hat{x}_{-n})) \cap B(y_n, 2\eta_\varepsilon(\hat{y}_{-n}))$. Observe that $\pi_0(\eta_{\hat{x}}) \subset \frac{1}{2}B_0$.

Proposition 3.7.2. *Assume \mathcal{H}_2 and $\lambda_1 > \lambda_2$. Let $\hat{x} \in \mathcal{A} \cap \Lambda_\varepsilon$, $\hat{z} \in \mathcal{T}$ and $\hat{y} \in \eta_{\hat{x}} \cap \Lambda_\varepsilon$ as before. Assume that $\hat{x}_{-n} \in \mathcal{A} \cap \{\beta_\varepsilon \leq \tau\}$, where $n \geq \max\{N(\hat{x}), N(\hat{y})\}$ and $\tau > 0$. Then*

$$q_{\hat{y}} = u_n |C_n \circ f_{\hat{z}}^{-n} \circ \pi_0|^2 \times q_{\hat{x}} + J_n \quad \text{on } \eta_{\hat{x}},$$

where

1. $0 \leq u_n \leq \frac{L(\hat{x})}{L(\hat{y})} 4\tau^2 \Delta_{\hat{z}}^4$,
2. $C_n : f_{\hat{z}}^{-n}(B_0) \rightarrow \mathbb{C}$ is a holomorphic function bounded by $1/\mathcal{R}$,
3. J_n is a real valued measure on $\eta_{\hat{x}}$ satisfying

$$|J_n| \leq c_0 e^{-n(\lambda_1 - \lambda_2 - 4\varepsilon)} \sigma_T \circ \pi_0$$

for some $c_0 = c_0(\hat{x}, \hat{y}, \mathcal{R})$.

We give the proof in Section 3.7.3 and Section 3.7.4. Let us see how this Proposition implies Equation (3.25), completing the proof of Proposition 3.4.5. Taking $\tau > 0$ large enough, by Poincaré recurrence theorem, for $\hat{\mu}$ -almost every $\hat{x} \in \mathcal{A} \cap \{\beta_\varepsilon \leq \tau\} \cap \Lambda_\varepsilon$, for any $\hat{y} \in \eta_{\hat{x}} \cap \Lambda_\varepsilon$, there exists $(n_k)_k$ satisfying

$$\hat{x}_{-n_k} \in \mathcal{A} \cap \{\beta_\varepsilon \leq \tau\} \text{ and } n_k \geq \max\{N(\hat{x}), N(\hat{y})\}.$$

Proposition 3.7.2 and the fact that $\text{Lip}(f_{\hat{z}}^{-n}) \rightarrow 0$ (see Theorem 3.8.2) imply that, up to a subsequence, $(C_n \circ f_{\hat{z}}^{-n})_n$ converges uniformly on $\pi_0(\eta_{\hat{x}}) \subset \frac{1}{2}B_0$ to a constant function F . Proposition 3.7.2 also implies that $(u_n)_n$ converges to some $u \geq 0$ and that $\lim_n J_n = 0$ (since $\lambda_1 > \lambda_2$). Therefore

$$q_{\hat{y}} = \lim_{n \rightarrow +\infty} u_n |C_n \circ f_{\hat{z}}^{-n} \circ \pi_0|^2 \times q_{\hat{x}} = uF \times q_{\hat{x}} \text{ on } \eta_{\hat{x}}.$$

Hence $uF = 1$ and $q_{\hat{y}} = q_{\hat{x}}$, since they are probability measures.

3.7.3 Proof of Proposition 3.4.5 in the non resonant case

Let \hat{x} and \hat{y} be as in Proposition 3.7.2. Recall that $B_n := B(x_{-n}, 2\eta_\varepsilon(\hat{x}_{-n})) \cap B(y_{-n}, 2\eta_\varepsilon(\hat{y}_{-n}))$, it is not empty when $n \geq \max\{N(\hat{x}), N(\hat{y})\}$ by Equation (3.28). We consider in the following integers n larger than $\max\{N(\hat{x}), N(\hat{y})\}$. We define on B_n the functions

$$C_n^Z := \frac{\partial W_{\hat{y}_{-n}}}{\partial Z_{\hat{x}_{-n}}} := \frac{\partial}{\partial z} \left[W \circ \xi_{\hat{y}_{-n}} \circ \xi_{\hat{x}_{-n}}^{-1} \right] \circ \xi_{\hat{x}_{-n}}$$

$$C_n^W := \frac{\partial W_{\hat{y}_{-n}}}{\partial W_{\hat{x}_{-n}}} = \frac{\partial}{\partial w} \left[W \circ \xi_{\hat{y}_{-n}} \circ \xi_{\hat{x}_{-n}}^{-1} \right] \circ \xi_{\hat{x}_{-n}}.$$

We have

$$\frac{i}{2} dW_{\hat{y}_{-n}} \wedge d\overline{W}_{\hat{y}_{-n}} = \sum_{A,B \in \{Z,W\}} C_n^A \overline{C_n^B} \times \frac{i}{2} dA_{\hat{x}_{-n}} \wedge d\overline{B}_{\hat{x}_{-n}} \text{ on } B_n. \quad (3.29)$$

We now pull back Equation (3.29) by $f_{\hat{x}}^{-n} : B_0 \rightarrow B_n$. We assume in this section that the Lyapunov exponents are not resonant, see Section 3.7.4 for the resonant case. By Theorem 3.8.2, the map $(z, w) \mapsto R_{n,\hat{x}}(z, w)$ is linear and for every $n \geq N(\hat{x})$:

$$Z_{\hat{x}_{-n}} \circ f_{\hat{x}}^{-n} = \alpha_{n,\hat{x}} \times Z_{\hat{x}} \text{ and } W_{\hat{x}_{-n}} \circ f_{\hat{x}}^{-n} = \beta_{n,\hat{x}} \times W_{\hat{x}}.$$

As explained before, by Equation (3.27) $\pi_0(\eta_{\hat{x}})$ is included in the intersection of the domains of definition of the maps $f_{\hat{x}}^{-n}$, $f_{\hat{y}}^{-n}$ and $f_{\hat{z}}^{-n}$, and by Lemma 3.3.3 these maps coincide on this intersection, thus on $\pi_0(\eta_{\hat{x}})$. The pull back of Equation (3.29) by $f_{\hat{z}}^{-n} : \pi_0(\eta_{\hat{x}}) \rightarrow f_{\hat{z}}^{-n}(\pi_0(\eta_{\hat{x}}))$ then gives :

$$|\beta_{n,\hat{y}}|^2 \times \frac{i}{2} dW_{\hat{y}} \wedge d\overline{W}_{\hat{y}} = \sum_{A,B \in \{Z,W\}} F_n^{AB} \times \alpha_{n,\hat{x}}^A \overline{\alpha_{n,\hat{x}}^B} \times \frac{i}{2} dA_{\hat{x}} \wedge d\overline{B}_{\hat{x}} \text{ on } \pi_0(\eta_{\hat{x}}), \quad (3.30)$$

where

$$F_n^{AB} := C_n^A \circ f_{\hat{z}}^{-n} \times \overline{C_n^B \circ f_{\hat{z}}^{-n}}, \quad \alpha_{n,\hat{x}}^Z := \alpha_{n,\hat{x}}, \quad \alpha_{n,\hat{x}}^W := \beta_{n,\hat{x}}.$$

Let us denote $\lambda_{AB} := T \wedge \frac{i}{2} dA_{\hat{x}} \wedge d\overline{B}_{\hat{x}}$. Since $\hat{y} \in \mathcal{A}$ the definition of $q_{\hat{y}}$ yields

$$(T \wedge \frac{i}{2} dW_{\hat{y}} \wedge d\overline{W}_{\hat{y}}) \circ \pi_0|_{\eta_{\hat{x}}} = \frac{L(\hat{y})}{M(\hat{y})^2} \times q_{\hat{y}},$$

with a similar formula for $\hat{x} \in \mathcal{A}$. Wedging Equation (3.30) by T , we deduce on $\eta_{\hat{x}}$:

$$q_{\hat{y}} = u_n |C_n \circ f_{\hat{z}}^{-n} \circ \pi_0|^2 \times q_{\hat{x}} + J_n, \quad (3.31)$$

where

$$u_n := \frac{L(\hat{x})M(\hat{y})^2 |\beta_{n,\hat{x}}|^2}{L(\hat{y})M(\hat{x})^2 |\beta_{n,\hat{y}}|^2}, \quad C_n := C_n^W,$$

$$J_n := \frac{M(\hat{y})^2}{L(\hat{y})|\beta_{n,\hat{y}}|^2} \sum_{(A,B) \neq (W,W)} (F_n^{AB} \circ \pi_0) \times \alpha_{n,\hat{x}}^A \overline{\alpha_{n,\hat{x}}^B} \cdot \lambda_{AB} \circ \pi_0|_{\eta_{\hat{x}}}. \quad (3.32)$$

— Let us now prove that u_n is bounded by a constant independent of n . Lemma 3.4.9 implies $|\beta_{n,\hat{x}}|^2/M(\hat{x})^2 = M(\hat{x}_{-n})^{-2}\Delta(x_{-n},n)^{-2}$ (with a similar formula for \hat{y}). We deduce

$$u_n = \frac{L(\hat{x})M(\hat{y}_{-n})^2\Delta(y_{-n},n)^2}{L(\hat{y})M(\hat{x}_{-n})^2\Delta(x_{-n},n)^2} = \frac{L(\hat{x})}{L(\hat{y})} \times \frac{M(\hat{y}_{-n})^2}{M(\hat{x}_{-n})^2} \times \frac{\Delta(y_{-n},n)^2}{\Delta(z_{-n},n)^2} \frac{\Delta(z_{-n},n)^2}{\Delta(x_{-n},n)^2}.$$

We then get $0 \leq u_n \leq \frac{L(\hat{x})}{L(\hat{y})}4\tau^2\Delta_{\hat{z}}^4$ by using the inequalities $\beta_\varepsilon^{-1}(\hat{w}) \leq M(\hat{w}) \leq 2$ (see Theorem 3.8.2), $\hat{x}_{-n} \in \{\beta_\varepsilon \leq \tau\}$ and Proposition 3.3.7 recalling that $\hat{x}, \hat{y} \in W^u(\hat{z}, \mathcal{R}) \cap \Lambda_\varepsilon$.

— Let us prove that the function $C_n : f_{\hat{z}}^{-n}(B_0) \rightarrow \mathbb{C}$ is bounded by $1/\mathcal{R}$. We verify more generally that

$$|C_n^Z|, |C_n^W| \leq 1/\mathcal{R} \text{ on } B_n. \quad (3.33)$$

By Theorem 3.8.2, the change of coordinates $H := \xi_{\hat{y}_{-n}} \circ \xi_{\hat{x}_{-n}}^{-1}$ satisfies $\|dH\| \leq 2\beta_\varepsilon(\hat{y}_{-n})$ on B_n . Since $\hat{y}_{-n} \in \mathcal{A} \cap \Lambda_\varepsilon$, Proposition 3.3.7 yields $\|dH\| \leq 1/\mathcal{R}$, which implies Equation (3.33).

— It remains to prove the upper estimate on $|J_n|$. Since $f_{\hat{z}}^{-n}(B_0) \subset B_n$, by definition of F_n^{AB} and from Equation (3.33), we have $|F_n^{AB}| \leq 1/\mathcal{R}^2$ on B_0 . Now let us observe (a similar inequality holds for λ_{WW})

$$\lambda_{ZZ} = T \wedge dd^c|Z_{\hat{x}}|^2 \leq \|d\xi_{\hat{x}}\|^2(T \wedge \omega_{\mathbb{P}^2}) \leq \beta_\varepsilon(\hat{x})^2\sigma_T \text{ on } B(x_0, 2\eta_\varepsilon(\hat{x})).$$

The same estimate holds for $|\lambda_{ZW}|$ and $|\lambda_{WZ}|$, since $|\lambda_{ZW}| \leq \sqrt{\lambda_{ZZ}}\sqrt{\lambda_{WW}}$ by Cauchy-Schwarz inequality. Finally, Theorem 3.8.2 provides $|\alpha_{n,\hat{x}}| \cdot |\beta_{n,\hat{x}}|/|\beta_{n,\hat{y}}|^2 \leq e^{-n(\lambda_1 - \lambda_2 - 4\varepsilon)}$ and one has (since $\pi_0(\eta_{\hat{x}}) \subset B_0$)

$$|J_n| \leq c_0 e^{-n(\lambda_1 - \lambda_2 - 4\varepsilon)} \sigma_T \circ \pi_0|_{\eta_{\hat{x}}}$$

by setting $c_0 := \frac{M(\hat{y})^2\beta_\varepsilon(\hat{x})^2}{L(\hat{y})\mathcal{R}^2}$. This completes the proof of Proposition 3.7.2. \square

3.7.4 Proof of Proposition 3.4.5 in the resonant case

When the Lyapunov exponents satisfy $\lambda_1 = k\lambda_2$ for some $k \geq 2$, the first component of the map $R_{n,\hat{x}}$ may be not linear (see Theorem 3.8.2) :

$$R_{n,\hat{x}}(z, w) = (\alpha_{n,\hat{x}}z + \gamma_{n,\hat{x}}w^k, \beta_{n,\hat{x}}w).$$

The proof of Proposition 3.7.2 can be modified as follows. Equation (3.31) becomes :

$$q_{\hat{y}} = u_n |C_n \circ f_{\hat{z}}^{-n} \circ \pi_0|^2 \times q_{\hat{x}} + \tilde{J}_n \text{ on } \eta_{\hat{x}},$$

where \tilde{J}_n is a new sequence satisfying $|\tilde{J}_n| \leq \tilde{c}_0 e^{-n(\lambda_1 - \lambda_2 - 4\varepsilon)} \sigma_T \circ \pi_0|_{\eta_{\hat{x}}}$ for some $\tilde{c}_0(\hat{x}, \hat{y}, \mathcal{R}) > 0$. Indeed, Equation (3.30) is replaced by :

$$\begin{aligned} |\beta_{n,\hat{y}}|^2 \frac{i}{2} (dW_{\hat{y}} \wedge d\bar{W}_{\hat{y}}) &= F_n^{ZZ} |\alpha_{n,\hat{x}}|^2 \frac{i}{2} dZ_{\hat{x}} \wedge d\bar{Z}_{\hat{x}} \\ &+ 2\text{Re} \left[\left(kF_n^{ZZ} \alpha_{n,\hat{x}} \bar{\gamma}_{n,\hat{x}} \bar{W}_{\hat{x}}^{k-1} + F_n^{ZW} \alpha_{n,\hat{x}} \bar{\beta}_{n,\hat{x}} \right) \frac{i}{2} dZ_{\hat{x}} \wedge d\bar{W}_{\hat{x}} \right] \\ &+ (2\text{Re} [kF_n^{ZW} \gamma_{n,\hat{x}} \bar{\beta}_{n,\hat{x}} W_{\hat{x}}^{k-1}] + k^2 F_n^{ZZ} |\gamma_{n,\hat{x}}|^2 |W_{\hat{x}}|^{2(k-1)}) \frac{i}{2} dW_{\hat{x}} \wedge d\bar{W}_{\hat{x}} \\ &+ F_n^{WW} |\beta_{n,\hat{x}}|^2 \frac{i}{2} dW_{\hat{x}} \wedge d\bar{W}_{\hat{x}}. \end{aligned}$$

From this new expression one can compute \tilde{J}_n as we did for Equation (3.32), and get on $\eta_{\hat{x}}$:

$$\begin{aligned} \frac{L(\hat{y})|\beta_{n,\hat{y}}|^2}{M(\hat{y})^2} \tilde{J}_n &= (F_n^{ZZ} \circ \pi_0) |\alpha_{n,\hat{x}}|^2 \lambda_{ZZ} \circ \pi_0 \\ &+ 2\operatorname{Re} \left[\left(k (F_n^{ZZ} \circ \pi_0) \alpha_{n,\hat{x}} \bar{\gamma}_{n,\hat{x}} \left(\overline{W_{\hat{x}}^{k-1}} \circ \pi_0 \right) + (F_n^{ZW} \circ \pi_0) \alpha_{n,\hat{x}} \bar{\beta}_{n,\hat{x}} \right) \lambda_{ZW} \circ \pi_0 \right] \\ &+ 2\operatorname{Re} \left[k (F_n^{ZW} \circ \pi_0) \gamma_{n,\hat{x}} \bar{\beta}_{n,\hat{x}} \left(W_{\hat{x}}^{k-1} \circ \pi_0 \right) \right] \lambda_{WW} \circ \pi_0 \\ &+ k^2 (F_n^{ZZ} \circ \pi_0) |\gamma_{n,\hat{x}}|^2 |W_{\hat{x}} \circ \pi_0|^{2(k-1)} \lambda_{WW} \circ \pi_0. \end{aligned}$$

We get a constant \tilde{c}_0 by using the upper bounds $|\beta_{n,\hat{y}}|^{-1} \leq e^{n(\lambda_2+\varepsilon)}$, $|\alpha_{n,\hat{x}}| \leq e^{-n(\lambda_1-\varepsilon)}$, $|\beta_{n,\hat{x}}| \leq e^{-n(\lambda_2-\varepsilon)}$, $|\gamma_{n,\hat{x}}| \leq M_\varepsilon(\hat{x})e^{-n(\lambda_1-\varepsilon)}$ (Theorem 3.8.2) and the fact that $W_{\hat{x}}$ is bounded on $B(x_0, \eta_\varepsilon(\hat{x})) \supset \pi_0(\eta_{\hat{x}})$.

* * * Appendix of Chapter 3 * * *

3.8 Classical results

Let f be an endomorphism of \mathbb{P}^2 of degree $d \geq 2$. Let μ be its equilibrium measure and let us denote $\lambda_1 \geq \lambda_2$ the Lyapunov exponents of μ . Let $\operatorname{Crit}(f)$ denote the critical set of f and $\mathcal{C} := \bigcup_{n \in \mathbb{Z}} f^n(\operatorname{Crit}(f))$. We know that $\mu(\mathcal{C}) = 0$, see [26, 48].

3.8.1 Oseledec Theorem

Theorem 3.8.1. *Assume that $\lambda_1 > \lambda_2$. There exists a totally invariant borel set A_{os} of full μ -measure disjoint from \mathcal{C} and a measurable map $v_s : A_{os} \rightarrow \mathbb{P}(T\mathbb{P}^2)$ such that*

$$\forall x \in A_{os}, \forall \vec{v} \in v_s(x) \setminus \{0\}, \lim_{n \rightarrow \infty} \frac{1}{n} \operatorname{Log} \|d_x f^n \cdot \vec{v}\| = \lambda_2.$$

Moreover $d_x f^n(v_s(x)) = v_s(f^n(x))$. If $\vec{v} \in T_x \mathbb{P}^2 \setminus v_s(x)$ then the limit above is equal to λ_1 . We denote

$$\chi_f(x) := \operatorname{Log} \|d_x f \cdot \vec{v}_s(x)\|,$$

where $\vec{v}_s(x)$ is any vector of norm 1 in the complex line $v_s(x)$. The following points hold :

1. $\chi_f \in L^1(\mu)$,
2. for every $x \in A_{os}$, $\frac{1}{n} \sum_{i=0}^{n-1} \chi_f(f^i(x)) = \frac{1}{n} \chi_{f^n}(x)$. In particular, $\int \chi_f d\mu = \lambda_2$.
3. $\chi_{f^n} \circ \pi_0 \in L^1(\hat{\mu})$ and

$$\int_{\hat{\mathbb{P}}^2} \chi_{f^n}(x_0) d\hat{\mu}(\hat{x}) = n\lambda_2.$$

We denote $\Delta(x, n) := \|d_x f^n \cdot \vec{v}_s(x)\|$.

The proof of the first item can be found in [48, Section 3.7]. The second one comes from the chain rule formula and Birkhoff ergodic theorem. The third item relies on $(\pi_0)_* \hat{\mu} = \mu$.

3.8.2 Normal forms for inverse branches

Let $\widehat{\mathbb{P}^2}$ be the set of orbits $\{\widehat{x} = (x_n)_{n \in \mathbb{Z}}, f(x_n) = x_{n+1}\}$. We denote $\pi_0 : \widehat{\mathbb{P}^2} \rightarrow \mathbb{P}^2$ the projection $\pi_0 : \widehat{x} \mapsto x_0$ and \widehat{f} the left shift. By Kolmogorov extension theorem, there exists a unique \widehat{f} -invariant probability measure on $\widehat{\mathbb{P}^2}$ such that $(\pi_0)_* \widehat{\mu} = \mu$. Let $X := \mathbb{P}^2 \setminus \mathcal{C}$ and $\widehat{X} := \{\widehat{x} \in \widehat{\mathbb{P}^2} : x_n \in X, \forall n \in \mathbb{Z}\}$, this is a \widehat{f} -invariant subset of $\widehat{\mathbb{P}^2}$ of full $\widehat{\mu}$ -measure.

The following theorem was proved by Berteloot-Dupont-Molino [5], it gives normal forms for the iterated inverse branches of the dynamical system. We refer to the articles [38] by Jonsson-Varolin and [4] by Berteloot-Dupont for related results. A real-valued function φ on $\widehat{\mathbb{P}^2}$ is ε -tempered if $e^{-|n|\varepsilon} \varphi \leq \varphi \circ \widehat{f}^n \leq e^{|n|\varepsilon} \varphi$ for every $n \in \mathbb{Z}$.

Theorem 3.8.2. *Let f be an endomorphism of \mathbb{P}^2 of degree $d \geq 2$. Let ε be small with respect to the Lyapunov exponents $\lambda_1 > \lambda_2$ of μ . There exist a totally \widehat{f} -invariant borel set $\mathcal{FN}_\varepsilon \subset \widehat{X}$ of full $\widehat{\mu}$ -measure, ε -tempered functions $\rho_\varepsilon, \eta_\varepsilon : \mathcal{FN}_\varepsilon \rightarrow]0, 1]$, $\beta_\varepsilon, M_\varepsilon : \mathcal{FN}_\varepsilon \rightarrow [1, +\infty[$ and a function $N : \mathcal{FN}_\varepsilon \rightarrow \mathbb{N}$ satisfying the following properties for every $\widehat{x} \in \mathcal{FN}_\varepsilon$:*

1. *there is an injective holomorphic map $\xi_{\widehat{x}} : B(x_0, 2\eta_\varepsilon(\widehat{x})) \rightarrow \mathbb{D}^2(\rho_\varepsilon(\widehat{x}))$ such that :*
 - (a) $\xi_{\widehat{x}}(x_0) = 0$ and $v_s(x_0) = (d_0 \xi_{\widehat{x}}^{-1} \cdot (0, 1)) \mathbb{C}$,
 - (b) $\forall p, q \in B(x_0, 2\eta_\varepsilon(\widehat{x}))$, $\frac{1}{2} \text{dist}(p, q) \leq \|\xi_{\widehat{x}}(p) - \xi_{\widehat{x}}(q)\| \leq \beta_\varepsilon(\widehat{x}) \text{dist}(p, q)$.
2. *there is a sequence of holomorphic maps $(f_{\widehat{x}}^{-n})_n$ such that $f^n \circ f_{\widehat{x}}^{-n} = \text{Id}$ on $B(x_0, 2\eta_\varepsilon(\widehat{x}))$, $f_{\widehat{x}}^{-n}(x_0) = x_{-n}$ and $\text{Lip}(f_{\widehat{x}}^{-n}) \leq \beta_\varepsilon(\widehat{x}) e^{-n(\lambda_2 - \varepsilon)}$,*
3. *the diagram (3.9) in Section 3.2.2 commutes for any $n \geq N(\widehat{x})$.*

The map $R_{n, \widehat{x}}$ is equal to

$$R_{n, \widehat{x}}(z, w) = (\alpha_{n, \widehat{x}} z + \gamma_{n, \widehat{x}} w^k, \beta_{n, \widehat{x}} w),$$

where $k = \lambda_1 / \lambda_2$. Moreover $\gamma_{n, \widehat{x}} = 0$ if $k \notin \{2, 3, \dots\}$, and

1. $e^{-n(\lambda_1 + \varepsilon)} \leq |\alpha_{n, \widehat{x}}| \leq e^{-n(\lambda_1 - \varepsilon)}$ and $|\gamma_{n, \widehat{x}}| \leq M_\varepsilon(\widehat{x}) e^{-n(\lambda_1 - \varepsilon)}$.
2. $e^{-n(\lambda_2 + \varepsilon)} \leq |\beta_{n, \widehat{x}}| \leq e^{-n(\lambda_2 - \varepsilon)}$.

Remark 3.8.3. *The functions $\eta_\varepsilon, \rho_\varepsilon, \beta_\varepsilon$ are related by $2\eta_\varepsilon = \rho_\varepsilon / \beta_\varepsilon \leq 1 / \beta_\varepsilon$, see for instance [4].*

3.9 Results in higher dimensions

Let f be a holomorphic mapping of degree $d \geq 2$ on \mathbb{P}^k , let $\mu = T^{\wedge k}$ be the measure of maximal entropy and $\lambda_1 \geq \dots \geq \lambda_k$ be its Lyapunov exponents. Assume that $k \geq 2$ and that there exists $r \in \{1, \dots, k-1\}$ such that :

$$\lambda_1 \geq \dots \geq \lambda_r > \lambda_{r+1} \geq \dots \geq \lambda_k. \quad (3.34)$$

The case $\lambda_{r+1} = \dots = \lambda_r = \frac{1}{2} \text{Log } d$ generalizes the notion of semi-extremal maps on \mathbb{P}^2 . To extend our method on \mathbb{P}^k , we have to consider a new stable cocycle $\Delta(x, n)$ and to consider several submersions $W_{\widehat{x}}$.

By classical ergodic theory, Theorem 3.8.1 extends as follows. Assuming (3.34), there exists $A_{os} \subset \mathbb{P}^k$ a totally invariant borel set of full μ -measure, such that for every $x \in A_{os}$ there exists a complex sub-vector space $V_s(x) \subset T_x \mathbb{P}^k$ of dimension $k - r$ satisfying :

- $d_x f^n : V_s(x) \longrightarrow V_s(f^n x)$ is a \mathbb{C} -linear isomorphism.
- $\forall \vec{v} \in V_s(x) \setminus \{0\}$, $\lim_{n \rightarrow +\infty} \frac{1}{n} \text{Log} \|d_x f^n \cdot \vec{v}\| \in \{\lambda_{r+1}, \dots, \lambda_k\}$.
- $\forall \vec{v} \in T_x \mathbb{P}^k \setminus V_s(x)$, $\lim_{n \rightarrow +\infty} \frac{1}{n} \text{Log} \|d_x f^n \cdot \vec{v}\| \in \{\lambda_1, \dots, \lambda_r\}$.

By setting

$$\Delta(x, n) := |\det_{\mathbb{C}}(d_x f^n : V_s(x) \longrightarrow V_s(f^n x))|, \quad (3.35)$$

we also have :

$$\int_{\mathbb{P}^k} \text{Log} \Delta(x, n) \, d\mu(x) = n(\lambda_{r+1} + \dots + \lambda_k).$$

Theorem 3.8.2 extends on \mathbb{P}^k with the commutative diagram

$$\begin{array}{ccc} B(x_{-n}, 2\eta_\varepsilon(\widehat{x}_{-n})) & \xleftarrow{f_{\widehat{x}}^{-n}} & B(x_0, 2\eta_\varepsilon(\widehat{x})) \\ \xi_{\widehat{x}_{-n}} \downarrow & & \downarrow \xi_{\widehat{x}} \\ \mathbb{D}^k(\rho_\varepsilon(\widehat{x}_{-n})) & \xleftarrow{R_{n, \widehat{x}}} & \mathbb{D}^k(\rho_\varepsilon(\widehat{x})) \end{array}$$

Assuming that there is no resonance between $\lambda_{r+1} \geq \dots \geq \lambda_k$, the map $R_{n, \widehat{x}}$ has the form

$$\left(\alpha_{n, \widehat{x}}^1 z_1 + P_1(z_2, \dots, w_k), \dots, \alpha_{n, \widehat{x}}^r z_r + P_r(w_{r+1}, \dots, w_k), \beta_{n, \widehat{x}}^{r+1} w_{r+1}, \dots, \beta_{n, \widehat{x}}^k w_k \right), \quad (3.36)$$

where $e^{-n(\lambda_j + \varepsilon)} \leq |\alpha_{n, \widehat{x}}^j|, |\beta_{n, \widehat{x}}^j| \leq e^{-n(\lambda_j - \varepsilon)}$. The polynomials P_j are sums of

$$\gamma z_{j+1}^{\alpha_{j+1}} \dots z_r^{\alpha_r} w_{r+1}^{\beta_{r+1}} \dots w_k^{\beta_k},$$

where the length of $(\alpha_{j+1}, \dots, \beta_k) \in \mathbb{N}^{k-j}$ is ≥ 2 , and γ is bounded as $\gamma_{n, \widehat{x}}$ in Theorem 3.8.2. If $\lambda_{j+1}^{\alpha_{j+1}} \dots \lambda_k^{\beta_k} \neq \lambda_j$, then $\gamma = 0$. If we write $\xi_{\widehat{x}} : B(x_0, \eta_\varepsilon(\widehat{x})) \longrightarrow \mathbb{D}^k(\rho_\varepsilon(\widehat{x}))$ as

$$\xi_{\widehat{x}} = (Z_{\widehat{x}}^1, \dots, Z_{\widehat{x}}^r, W_{\widehat{x}}^{r+1}, \dots, W_{\widehat{x}}^k).$$

then it satisfies

$$d_0 \xi_{\widehat{x}}^{-1} (\{0\}^r \times \mathbb{C}^{k-r}) = V_s(x).$$

Now we set

$$M(\widehat{x}) := |\det_{\mathbb{C}}(d_0 \xi_{\widehat{x}}^{-1} : \{0\}^r \times \mathbb{C}^{k-r} \longrightarrow V_s(x))|$$

and we recall that the cocycle $\Delta(x, n)$ is defined in (3.35).

Definition 3.9.1. Assume \mathcal{H}_1 . For every $\widehat{x} \in \Lambda_\varepsilon$, we define

$$\tilde{q}_{\widehat{x}} := (d^r)^p (\Delta(x_{-p}, p))^2 M(\widehat{x}_{-p})^2 \times \left(T^r \wedge dd^c |W_{\widehat{x}_{-p}}^{r+1}|^2 \wedge \dots \wedge dd^c |W_{\widehat{x}_{-p}}^k|^2 \right) \circ \pi_0 \circ \widehat{f}^{-p}|_{\eta_{\widehat{x}}},$$

where $p \geq 0$ is such that \widehat{x}_{-p} belongs to \mathcal{A} . Then $q_{\widehat{x}} := \frac{1}{L(\widehat{x})} \tilde{q}_{\widehat{x}}$, where $L(\widehat{x}) := \tilde{q}_{\widehat{x}}(\eta_{\widehat{x}})$.

Since there is no resonance between the $k - r$ smallest exponents, the $k - r$ last components of $R_{n,\widehat{x}}$ are linear (see (3.36)), and one can verify as in Proposition 3.4.7 that the definition of $\tilde{q}_{\widehat{x}}$ does not depend on p . The following formula, which extends Lemma 3.4.9, holds true

$$\Delta(x_{-n}, n) \prod_{i=r+1}^k |\beta_{n,\widehat{x}}^i| = \frac{M(\widehat{x})}{M(\widehat{x}_{-n})},$$

(note that it does not require the linearity for the last components of $R_{n,\widehat{x}}$). Using similar arguments as for $k = 2$, Theorems A and B extend to \mathbb{P}^k as follows.

Theorem 3.9.2. *Let f be an endomorphism of degree $d \geq 2$ on \mathbb{P}^k with $k \geq 2$. Assume that the Lyapunov exponents of the measure of maximal entropy μ satisfy $\lambda_1 \geq \dots \geq \lambda_r > \lambda_{r+1} \geq \dots \geq \lambda_k$ for some $r \in \{1, \dots, k-1\}$. Assume that there is no resonance between the $k - r$ smallest exponents.*

1. *If \mathcal{H}_1 holds, then there exist a measurable partition η and a measurable family $\widehat{x} \mapsto q_{\widehat{x}}$ of probability measures on $\widehat{\mathbb{P}}^2$ supported on $\eta_{\widehat{x}}$ which satisfy*

$$(\pi_0)_* q_{\widehat{x}} \ll T^r \wedge \omega_{\mathbb{P}^k}^{k-r},$$

$$\forall n \geq 0, \int_{\widehat{\mathbb{P}}^k} -\text{Log } q_{\widehat{x}}(\widehat{f}^{-n}\eta)_{\widehat{x}} d\widehat{\mu}(x) = \text{Log } (d^r)^n + 2n(\lambda_{r+1} + \dots + \lambda_k).$$

2. *If \mathcal{H}_2 holds and if $\lambda_{r+1} = \dots = \lambda_k = \frac{1}{2} \text{Log } d$, then $\mu \ll T^r \wedge \omega_{\mathbb{P}^k}^{k-r}$.*

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Titre : Propriétés géométriques et ergodiques des endomorphismes semi-extrémaux de $CP(2)$
Mot clés : Dynamique holomorphe, courant de Green, mesure d'équilibre, exposants de Lyapunov

Résumé : La thèse étudie les propriétés géométriques et dynamiques des endomorphismes semi-extrémaux du plan projectif complexe : ce sont les applications holomorphes possédant un seul exposant de Lyapunov minimal.

Le premier chapitre traite des aspects géométriques de ces applications. Les exemples connus préservent un pinceau de droites, nous observons tout d'abord que cela impose une formule pluri-potentialiste reliant le courant de Green et la mesure d'équilibre près des points périodiques répulsifs du petit ensemble de Julia. Nous montrons ensuite que, réciproquement, lorsque cette formule est vérifiée, et sous des hypothèses portant sur le courant de Green et la mesure d'équilibre, l'application laisse un feuilletage holomorphe invariant au voisinage du petit ensemble de Julia. Il s'agit là d'un phénomène de rigidité, dont la preuve passe par l'étude d'applications de Poincaré associées aux points péri-

diques répulsifs. Lorsque le feuilletage se prolonge au plan projectif (par exemple lorsque le complémentaire du petit ensemble de Julia est de Stein), l'application laisse invariant un pinceau de droites.

Le deuxième chapitre est consacré aux propriétés dynamiques des endomorphismes semi-extrémaux. Ceux préservant un pinceau de droites possèdent une mesure d'équilibre absolument continue par rapport au courant de Green. R. Dujardin a montré que cette condition garantit la minimalité d'un exposant de Lyapunov. Nous montrons que la propriété réciproque est vraie, sous une certaine condition d'uniformité sur les variétés instables, ce qui répond partiellement à une question posée par R. Dujardin. La preuve associe une méthode classique de théorie ergodique (basée sur des partitions mesurables et sur l'entropie) à un théorème de linéarisation des branches inverses le long d'orbites génériques dans le passé.

Title: Geometric and ergodic properties of semi-extremal endomorphisms of $CP(2)$
Keywords: Holomorphic dynamics, Green current, equilibrium measure, Lyapunov exponents

Abstract: In this manuscript we study geometric and dynamical aspects of semi-extremal maps on $CP(2)$: they are maps with a unique minimal Lyapunov exponent.

Chapter one is devoted to geometric properties. The known examples preserve a pencil of lines on the projective plane. We observe that it imposes a pluri-potential equation involving the Green current and the equilibrium measure near periodic repelling points of the small Julia set. Conversely, assuming that this formula holds near one repelling periodic point and additional hypothesis on the Green current and the equilibrium measure, we prove that the map preserves a holomorphic foliation defined on a neighborhood of the small Julia set. It is a rigidity phenomenon whose proof uses a Poincaré map given by the repelling cycle. If the fo-

liation extends to a global foliation on the projective plane (for example if the small Julia set has a Stein complementary), then it is an invariant pencil of lines. Chapter two concerns dynamical aspects of semi-extremal maps. Those which preserve a pencil of lines have an equilibrium measure absolutely continuous with respect to the Green current. R. Dujardin have showed that this condition imposes a minimal Lyapunov exponent. We prove that the converse is true under some uniform conditions at the level of the unstable manifolds of the dynamical system, it is a partial answer to a question asked by R. Dujardin. The proof uses jointly a classic method of ergodic theory (based on measurable partitions and on entropy) and a linearization theorem of inverse branches of the map along generic backward orbits.