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Adrien Morin

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THÈSE PRÉSENTÉE POUR OBTENIR LE GRADE DE  
**DOCTEUR DE L'UNIVERSITÉ DE  
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INFORMATIQUE

SPÉCIALITÉ MATHÉMATIQUES PURES

Par **Adrien MORIN**

Cohomologie Weil-étale en dimension 1 et valeurs spéciales de  
fonctions  $L$  en 0 et 1

Sous la direction de : **Baptiste MORIN**

Soutenue le 27 juin 2023

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# Chapitre 0

## Résumé étendu en français

### 0.1 Cohomologie Weil-étale

#### 0.1.1 La fonction zêta de Dedekind et la formule analytique du nombre de classes

La *fonction zêta de Riemann* est la somme infinie

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s},$$

qui converge pour  $\Re(s) > 1$ . Cette somme a une expression alternative, cette fois comme produit infini (une expression de ce genre est appelée *produit eulérien*) portant sur l'ensemble des nombres premiers

$$\zeta(s) = \prod_p \frac{1}{1 - p^{-s}}.$$

Cette seconde formule exprime un lien profond entre le comportement des nombres premiers et la fonction zêta de Riemann, et place cette dernière au centre de la théorie des nombres. Avec l'étude de la théorie algébrique des nombres, de nouvelles fonctions similaires sont apparues, les *fonctions zêta de Dedekind*. Soit  $K$  un corps de nombres dont on dénote l'anneau d'entiers  $\mathcal{O}_K$ . La fonction zêta de Dedekind associée à  $K$  est donnée par la série de Dirichlet

$$\zeta_K(s) = \sum_{I \text{ idéal de } \mathcal{O}_K} \text{card}(\mathcal{O}_K/I)^{-s},$$

qui converge pour  $\Re(s) > 1$ . Elle a, elle aussi, une expression en produit eulérien

$$\zeta_K(s) = \prod_{\mathfrak{p}} \frac{1}{1 - \text{card}(\mathcal{O}_K/\mathfrak{p})^{-s}}$$

portant sur les idéaux premiers non nuls  $\mathfrak{p}$  de  $\mathcal{O}_K$  (de manière équivalente, sur les points fermés du spectre de  $\mathcal{O}_K$ ). Comme pour la fonction zêta de Riemann, comprendre les fonctions zêta de Dedekind permet entre autres d'obtenir des informations sur les comportements des idéaux premiers dans les corps de nombres.

Une des premières occurrences du problème de la détermination des valeurs de fonctions zêta est le problème de Bâle, qui demande de déterminer une formule close pour la valeur



$\zeta(2)$  ; Euler a prouvé la fameuse formule  $\zeta(2) = \pi^2/6$ , et a plus généralement donné des formules pour  $\zeta(2n)$ , lorsque  $n \geq 1$ , en termes de nombres de Bernoulli. On sait, par le travail de Hecke, que la fonction zêta de Dedekind a un prolongement méromorphe au plan complexe tout entier, avec un pôle simple en  $s = 1$  ; on peut donc chercher à déterminer une formule close pour son résidu en  $s = 1$ . La *formule analytique du nombre de classes* est une telle formule :

$$\lim_{s \rightarrow 1} (s-1)\zeta_K(s) = \frac{2^{r_1}(2\pi)^{r_2}h_K R_K}{\omega_K \sqrt{|D_K|}}, \quad (0.1.1)$$

où  $r_1$  est le nombre de plongements réels de  $K$ ,  $2r_2$  est le nombre de plongements complexes,  $h_K = \text{card}(\text{Cl}(\mathcal{O}_K))$  est le nombre de classes,  $R_K$  est le régulateur,  $\omega_K$  est le nombre de racines de l'unité dans  $K$  et  $D_K$  est le discriminant (absolu) de  $K$ . Cette formule est surprenante : la fonction zêta est construite uniquement à partir de données locales en chaque premier (c'est-à-dire à partir de données venant des points fermés de  $\text{Spec}(\mathcal{O}_K)$ ) mais son résidu en 1 a une expression en termes de quantités définies globalement à partir de l'anneau des entiers  $\mathcal{O}_K$  ; toutes ces quantités sont importantes pour comprendre la structure de celui-ci.

Soit  $X$  une variété sur un corps fini. On peut alors imiter la définition de la fonction zêta de Dedekind, en utilisant les points fermés de  $X$  ; on définit ainsi la fonction zêta (arithmétique) de  $X$  par

$$\zeta_X(s) = \prod_{x \in X_0} \frac{1}{1 - N(x)^{-s}},$$

où  $N(x)$  est le cardinal du corps résiduel en un point fermé  $x \in X_0$ . La *cohomologie étale* a été développée en partie comme un outil pour comprendre cette fonction et ses valeurs. Plus précisément, la cohomologie étale est une théorie cohomologique : étant donné un objet « géométrique » (ici la variété  $X$ ) et des coefficients  $F$  (ici, les « faisceaux étales sur  $X$  »), elle donne des groupes abéliens  $H_{\text{et}}^i(X, F)$  pour tout entier  $i \geq 0$ . Ces groupes abéliens sont appelés groupes de cohomologie, et ils décrivent souvent des obstructions à certaines constructions : quand ils sont triviaux, la construction est possible, sinon ils donnent une certaine quantification de l'obstruction à la construction.

Revenons à la fonction zêta de Dedekind. En utilisant l'équation fonctionnelle, prouvée par Hecke, on peut reformuler la formule analytique du nombre de classes en une formule en  $s = 0$  :

$$\zeta_K^*(0) := \lim_{s \rightarrow 0} s^{1-r_1-r_2} \zeta_K(s) = -\frac{h_K R_K}{\omega_K}.$$

Dans cette formule, on peut observer que toutes les quantités sont en fait des invariants venant de la cohomologie étale :

$$h_K = \text{card}(H_{\text{et}}^1(\text{Spec}(\mathcal{O}_K), \mathbb{G}_m)),$$

le régulateur  $R_K$  est le discriminant de l'accouplement

$$H_{\text{et}}^0(X, \mathbb{G}_m) \times H_{\text{et},c}^1(X, \mathbb{Z}) \rightarrow \mathbb{R}$$

et

$$\omega_K = \text{card}(H_{\text{et}}^0(X, \mathbb{G}_m)_{\text{tor}}).$$

Mais cette interprétation, bien qu'intéressante, n'est pas complètement satisfaisante : la formule est en effet obtenue en prenant des morceaux par-ci, par-là de divers groupes de cohomologie étale, et en multipliant lesdits morceaux entre eux ; on voudrait une explication plus systématique. Avec le développement de la *cohomologie Weil-étale*, on peut affirmer qu'une telle explication est accessible.

## 0.1.2 Cohomologie Weil-étale et valeurs spéciales en $s = 0$ pour les variétés sur des corps finis

Dans le cas particulier du spectre  $\text{Spec}(k)$  d'un corps  $k$ , la cohomologie étale se calcule *via* la cohomologie du groupe de Galois de  $k$ , c'est-à-dire du groupe  $G_k = \text{Gal}(k^{\text{sep}}/k)$  des automorphismes  $k$ -linéaires d'une clôture séparable  $k^{\text{sep}}$  de  $k$ . Pour une variété  $X$  sur  $k$ , un faisceau étale sur  $X$  s'identifie à un faisceau  $G_k$ -équivariant sur la fibre géométrique  $X \times_k k^{\text{sep}}$ . A l'aide de la suite spectrale de Hochschild–Serre, on peut alors comprendre la cohomologie étale de  $X$  en fonction de la cohomologie du groupe de Galois de  $k$ , avec pour coefficients la cohomologie étale de la fibre géométrique.

Dans l'article [Lic05], Lichtenbaum a fait l'observation suivante : étant donnée une variété  $X$  sur un corps fini  $k$ , *remplacer le groupe de Galois de  $k$  par son groupe de Weil* mène à une théorie cohomologique avec des propriétés assez favorables, en tout cas plus favorables que la cohomologie étale. Plus précisément, soit  $W_k = \mathbb{Z} \subset \hat{\mathbb{Z}} = G_k$  le groupe de Weil de  $k$ , c'est-à-dire le sous-groupe de  $G_k$  engendré par le Frobenius. On définit alors la catégorie des faisceaux Weil-étale sur  $X$  comme la catégorie des faisceaux  $W_k$ -équivariants sur la fibre géométrique  $X \times_k k^{\text{sep}}$  ; c'est un topos. Comme on l'a dit, un faisceau étale sur  $X$  s'identifie à un faisceau  $G_k$ -équivariant sur la fibre géométrique, et obtient donc une structure canonique de faisceau Weil-étale *via* l'inclusion  $W_k \subset G_k$ . Cette opération définit un morphisme géométrique depuis le topos des faisceaux Weil-étale sur  $X$  vers le topos des faisceaux étales.

Lorsque  $X$  est projective et lisse, Lichtenbaum montre que les groupes de cohomologie correspondants  $H_W^i(X, \mathbb{Z})$  sont de type fini, et nuls pour des indices suffisamment grands. Un exemple facile qui met déjà en lumière des phénomènes intéressants est le cas  $X = \text{Spec}(k)$ . En effet, dans ce cas on a  $H_{\text{et}}^i(X, \mathbb{Z}) = H^i(\text{Gal}(k^{\text{sep}}/k), \mathbb{Z}) = \mathbb{Z}, 0, \mathbb{Q}/\mathbb{Z}, 0$  en degrés respectifs  $i = 0, i = 1, i = 2, i \geq 3$ , tandis que  $H_W^i(X, \mathbb{Z}) = H^i(W_k, \mathbb{Z}) = \mathbb{Z}, \mathbb{Z}, 0$  en degrés  $i = 0, i = 1, i \geq 2$ . On observe que  $\mathbb{Q}/\mathbb{Z}$  est remplacé par  $\mathbb{Z}$ , mais décalé un degré plus bas. C'est un phénomène qu'on observera de manière systématique en cohomologie Weil-étale : les termes de la forme  $\mathbb{Q}/\mathbb{Z}$  en cohomologie étale auront tendance à être remplacés en cohomologie Weil-étale par  $\mathbb{Z}$ , un degré plus bas. Le même phénomène survient usuellement lorsque l'on calcule une complétion  $\ell$ -adique dérivée : la complétion  $\ell$ -adique (dérivée) de  $\mathbb{Q}/\mathbb{Z}$  est

$$\mathbb{Q}/\mathbb{Z}_\ell^\wedge := R \lim_n (\mathbb{Q}/\mathbb{Z} \otimes^L \mathbb{Z}/\ell^n \mathbb{Z}) = R \lim_n \mathbb{Z}/\ell^n \mathbb{Z}[1] = \mathbb{Z}_\ell[1];$$

autrement dit  $\mathbb{Q}/\mathbb{Z}$  est remplacé par  $\mathbb{Z}_\ell$ , décalé un degré plus bas. Comme la cohomologie Weil-étale est souvent finiment engendrée, il s'ensuit qu'elle forme aussi souvent un modèle entier pour la cohomologie  $\ell$ -adique (voir par exemple [Gei04, 8.4]).

Notons  $r_X$  l'ordre d'annulation en 0 de  $\zeta_X$  et

$$\zeta_X^*(0) = \lim_{s \rightarrow 0} \zeta_X(s) s^{-r_X}$$

sa valeur spéciale en  $s = 0$ . De la cohomologie Weil-étale, Lichtenbaum extrait une caractéristique d'Euler  $\chi_X(\mathbb{Z})$  et prouve la formule

$$\zeta_X^*(0) = \pm \chi_X(\mathbb{Z})$$

en utilisant une autre formule, prouvée par Milne [Mil86], exprimant  $\zeta_X^*(0)$  en fonction de la cohomologie étale de  $X$ . C'est encore une fois surprenant : la fonction zêta est définie en utilisant des données locales uniquement, tandis que la caractéristique d'Euler Weil-étale est un *invariant global*. Elle contient des informations sur les comportements globaux sur  $X$ . Lichtenbaum conjecture aussi une formule similaire pour des schémas plus généraux sur  $k$  et une cohomologie Weil-étale à support compact.

### 0.1.3 Cohomologie Weil-étale pour les schémas arithmétiques et les coefficients motiviques

Dans des travaux ultérieurs, Flach, Geisser, Lichtenbaum, B. Morin, et d'autres auteurs se sont efforcés de généraliser les résultats précédents à des schémas arithmétiques quelconques (c'est-à-dire des schémas séparés de type fini sur  $\text{Spec}(\mathbb{Z})$ ) et à des coefficients motiviques  $\mathbb{Z}(n)$ ; voir par exemple [Lic09; Gei04; GS20] pour la situation au-dessus d'un corps fini et [Mor14; FM18] pour des schémas arithmétiques généraux. L'état de l'art est dû à B. Morin et Flach–B. Morin [FM18; Mor21]; dans ce qui suit,  $X$  est un schéma arithmétique de dimension  $d$  :

1. Supposons que  $X$  est propre et régulier. Pour  $n \geq 0$ , on dénote par  $\mathbb{Z}(n)$  le complexe de cycles de Bloch et pour  $n < 0$  on pose  $\mathbb{Z}(n) := \mathbb{Q}/\mathbb{Z}(n)[-1] = \bigoplus_p j_{p,!} \mu_p^{\otimes n}[-1]$ , où  $j_p : X[1/p] \rightarrow X$  est l'immersion ouverte. Flach–B. Morin introduisent dans [FM18] un complexe de cohomologie étale à support compact

$$R\Gamma_{\text{et},c}(X, \mathbb{Z}(n)) := \text{fib}(R\Gamma_{\text{et}}(X, \mathbb{Z}(n)) \rightarrow R\Gamma_{G_{\mathbb{R}}}(X(\mathbb{C}), (2i\pi)^n \mathbb{Z})),$$

où  $R\Gamma_{G_{\mathbb{R}}}(X(\mathbb{C}), -)$  dénote la cohomologie  $G_{\mathbb{R}}$ -équivariante sur l'espace topologique  $X(\mathbb{C})$  des points complexes de  $X$ , muni de sa topologie analytique. L'objectif de l'introduction de ce complexe est de corriger les phénomènes de 2-torsion qui se produisent en cohomologie motivique étale. Sous des hypothèses conjecturales dénotées  $\mathbf{AV}(X, n)$  (sur la dualité d'Artin–Verdier),  $\mathbf{L}(X, n)$  et  $\mathbf{L}(X, d - n)$  (sur la génération finie de la cohomologie motivique étale en certains degrés), ils construisent aussi un complexe de cohomologie Weil-étale à support compact  $R\Gamma_{W,c}(X, \mathbb{Z}(n))$ , défini par un triangle distingué (dans la 1-catégorie dérivée) :

$$R\text{Hom}(R\Gamma(X, \mathbb{Q}(n-d)), \mathbb{Q}[-2d-2]) \rightarrow R\Gamma_{\text{et},c}(X, \mathbb{Z}(n)) \rightarrow R\Gamma_{W,c}(X, \mathbb{Z}(n)) \rightarrow .$$

Plus généralement, pour tout schéma arithmétique  $X$  et pour chaque entier  $n \in \mathbb{Z}$ , il devrait exister un complexe parfait de groupes abéliens, le complexe Weil-étale à support compact  $R\Gamma_{W,c}(X, \mathbb{Z}(n))$ ; on parlera parfois du « complexe multiplicatif ».

2. Pour tout schéma arithmétique  $X$ , B. Morin définit dans [Mor21] un complexe  $L\Omega_{X/\mathbb{S}}^{<n}$ , appelé le complexe de de Rham dérivé relatif à la sphère en spectres modulo le  $n$ -ième terme de la filtration de Hodge; on parlera parfois du « complexe additif ».
3. Supposons que  $X$  est propre et régulier. La droite fondamentale est définie comme le groupe abélien libre de rang 1

$$\Delta_{X,n} = \det_{\mathbb{Z}}(R\Gamma_{W,c}(X, \mathbb{Z}(n))) \otimes_{\mathbb{Z}} \det_{\mathbb{Z}}(R\Gamma(X, L\Omega_{X/\mathbb{S}}^{<n})),$$

où  $\det_{\mathbb{Z}}$  est la construction du déterminant d'un complexe de [KM76], qui généralise l'application « puissance extérieure en degré maximal »  $M \mapsto \Lambda^{\text{rank} M} M$  aux complexes parfaits de groupes abéliens. La droite fondamentale a une partie « multiplicative »  $\det_{\mathbb{Z}}(R\Gamma_{W,c}(X, \mathbb{Z}(n)))$  et une partie « additive »  $\det_{\mathbb{Z}}(R\Gamma(X, L\Omega_{X/\mathbb{S}}^{<n}))$ .

Plus généralement, pour tout schéma arithmétique  $X$ , la droite fondamentale devrait être définie en fonction du complexe conjectural  $R\Gamma_{W,c}(X, \mathbb{Z}(n))$  et d'une cohomologie à support compact sur  $X$  à coefficients dans  $L\Omega_{X/\mathbb{S}}^{<n}$  adéquatement définie.

4. Supposons que  $X$  est propre et régulier et considérons la fibre

$$R\Gamma_c(X, \mathbb{R}(n)) := \text{fib}(R\Gamma(X, \mathbb{R}(n)) \rightarrow R\Gamma(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{R}(n)_{\mathcal{D}}))$$

de l'application qui va de la cohomologie étale à coefficients  $\mathbb{R}(n)$  vers la cohomologie  $G_{\mathbb{R}}$ -équivariante du complexe de Deligne  $\mathbb{R}(n)_{\mathcal{D}} := \text{fib}((2i\pi)^n \mathbb{R} \rightarrow \Omega_{X(\mathbb{C})}^{\leq n})$ . Flach–B. Morin introduisent une conjecture  $\mathbf{B}(X, n)$ , affirmant qu'un accouplement construit canoniquement

$$R\Gamma_c(X, \mathbb{R}(n)) \otimes^L R\Gamma(X, \mathbb{R}(d-n)) \rightarrow R\Gamma_c(X, \mathbb{R}(d)) \xrightarrow{\tau \geq 2d} H_c^{2d}(X, \mathbb{R}(d))[-2d] \\ \xrightarrow{tr} \mathbb{R}[-2d]$$

est un accouplement parfait de complexes parfaits de  $\mathbb{R}$ -espaces vectoriels. Sous la conjecture  $\mathbf{B}(X, n)$ , on peut construire une trivialisaton canonique

$$\lambda : \mathbb{R} \xrightarrow{\cong} \Delta_{X,n} \otimes \mathbb{R}$$

de la droite fondamentale.<sup>1</sup> En d'autres termes, le complexe additif et le complexe multiplicatif devraient être fortement liés après changement de base à  $\mathbb{R}$ .

Plus généralement, pour tout schéma arithmétique, il devrait exister une trivialisaton canonique de la droite fondamentale, venant d'un théorème de dualité à coefficients réels.

5. En utilisant la trivialisaton, on peut identifier  $\Delta_{X,n}$  à un réseau dans  $\mathbb{R}$ , et il est conjecturé que la valeur spéciale

$$\zeta_X^*(n) := \lim_{s \rightarrow n} \zeta_X(s)(s-n)^{-\text{ord}_{s=n} \zeta_X}$$

de  $\zeta_X$  en  $s = n$  est donnée au signe près par la formule suivante :<sup>2,3</sup>

$$\lambda(\zeta_X^*(n)^{-1}) \cdot \mathbb{Z} = \Delta_{X,n}. \quad (0.1.2)$$

En d'autres termes, le covolume du réseau  $\Delta_{X,n}$  à l'intérieur du  $\mathbb{R}$ -espace vectoriel 1-dimensionnel  $\Delta_{X,n} \otimes \mathbb{R}$  est  $\zeta_X^*(n)^{-1}$ . Si  $X$  est lisse et propre sur l'anneau d'entiers  $\mathcal{O}_K$  d'un corps de nombres, il a été montré par Flach–B. Morin qu'une variante de (0.1.2) (impliquant le complexe de de Rham dérivé relatif à  $\mathbb{Z}$  et un facteur correcteur venant de la théorie de Hodge  $p$ -adique, plutôt que le complexe  $L\Omega_{X/S}^{\leq n}$ ) est équivalente à la conjecture de valeurs spéciales de Bloch–Kato et Fontaine–Perrin-Riou. Cette variante est donc connue pour  $X = \text{Spec}(\mathcal{O}_K)$  lorsque  $n = 0, 1$ , ce qui donne une ré-interprétation Weil-étale de la formule analytique du nombre de classes équation (0.1.1), ou bien si  $K/\mathbb{Q}$  est une extension abélienne [FM18, 5.34]. Dans ces cas, par [Mor21, 1.6] et [FM18, 5.33], la conjecture (0.1.2) est aussi valide. Si  $X$  est lisse et propre sur un corps fini, la formule est valide si l'on suppose  $\mathbf{B}(X, n)$ ,  $\mathbf{L}(X, n)$  et  $\mathbf{L}(X, d-n)$  vraie, par réduction aux travaux de Lichtenbaum et Geisser ([Gei04], voir [FM18, 5.31] pour la réduction).

<sup>1</sup>Dans ce manuscrit, la trivialisaton sera plutôt écrite dans l'autre sens :  $\lambda : \Delta_{X,n} \otimes \mathbb{R} \xrightarrow{\cong} \mathbb{R}$ .

<sup>2</sup>Il y a aussi une conjecture sur l'ordre d'annulation

<sup>3</sup>Dans ce manuscrit, on écrira plutôt  $\lambda(\Delta_{X,n}) = \zeta_X^*(n)^{-1} \cdot \mathbb{Z}$ .

Une idée importante dans la construction de Flach–B. Morin est de construire le complexe Weil-étale en utilisant la dualité d’Artin–Verdier (la conjecture  $\mathbf{AV}(X, n)$ ), plutôt qu’en essayant de définir un topos Weil-étale et en prenant la théorie cohomologique associée. L’idée de cette définition un peu tortueuse, qui est apparue initialement dans [Mor14], provient du calcul, réalisé par Geisser [Gei04] sur les corps finis et par B. Morin [Mor11, §8, §9] pour le spectre d’un anneau d’entiers dans un corps de nombres, du foncteur dérivé du poussé en avant depuis le topos Weil-étale vers le topos étale. En souscrivant à la philosophie que « la cohomologie Weil-étale remplace la torsion de forme  $\mathbb{Q}/\mathbb{Z}$  par  $\mathbb{Z}[1]$  », essayons de motiver la définition : si l’on suppose les conjectures  $\mathbf{AV}(X, n)$ ,  $\mathbf{L}(X, n)$  et  $\mathbf{L}(X, d - n)$  valides, la torsion dans la cohomologie motivique  $R\Gamma_{\text{et},c}(X, \mathbb{Z}(n))$  en twist  $n$  s’identifie comme provenant du  $\mathbb{Q}/\mathbb{Z}$ -dual de la partie finiment engendrée de la cohomologie motivique  $R\Gamma_{\text{et}}(X, \mathbb{Z}(d - n))$  en twist  $d - n$ , et le reste du complexe  $R\Gamma_{\text{et},c}(X, \mathbb{Z}(n))$  est en fait de type fini. On veut obtenir un complexe de cohomologie Weil-étale parfait en remplaçant la torsion par quelque chose de type fini. En utilisant le triangle distingué  $\mathbb{Q} \rightarrow \mathbb{Q}/\mathbb{Z} \rightarrow \mathbb{Z}[1]$ , cet objectif peut être atteint si l’on construit un morphisme

$$R\text{Hom}(R\Gamma_{\text{et}}(X, \mathbb{Z}(d - n)), \mathbb{Q}[-2d - 2]) \rightarrow R\Gamma_{\text{et},c}(X, \mathbb{Z}(n))$$

qui « envoie le  $\mathbb{Q}$ -dual de la partie finiment engendrée de  $R\Gamma_{\text{et}}(X, \mathbb{Z}(d - n))$  sur son  $\mathbb{Q}/\mathbb{Z}$ -dual, *via* la projection canonique, dans  $R\Gamma_{\text{et},c}(X, \mathbb{Z}(n))$  », et que l’on considère son cône. La construction de Flach–B. Morin réalise exactement cela.<sup>4</sup>

Une autre observation fondamentale est qu’en caractéristique 0, contrairement à la caractéristique  $p$ , les droites  $\det_{\mathbb{Z}} R\Gamma_{W,c}(X, \mathbb{Z}(n))$  et  $\det_{\mathbb{Z}} R\Gamma(X, L\Omega_{X/S}^{\leq n})$  n’ont pas de trivialisations respectives, et on ne peut donc pas exprimer la valeur spéciale comme le produit d’une caractéristique d’Euler « multiplicative » et d’une « additive ». La droite fondamentale doit être considérée dans son ensemble pour construire une trivialisations.

*Remarque* ([FM18, §3.2]). Soit  $X$  un schéma arithmétique propre et régulier de dimension  $d$ . La conjecture  $\mathbf{AV}(X, n)$  est connue si :

- $X/\mathcal{O}_K$  est lisse ;
- $X/\mathbb{F}_q$  est lisse ;
- $X$  est une surface arithmétique (i.e  $X$  est connexe régulière de dimension 2, propre et plate sur  $\mathbb{Z}$ ) [FS22] ;
- $n \leq 0$  ou  $n \geq d$ .

La conjecture  $\mathbf{L}(X, n)$  est connue si :

- $d \leq 1$  ;
- $n = 1$  et  $X$  est une surface arithmétique ou un schéma de dimension 2, connexe, régulier et propre sur un corps fini, dans tous les cas avec un groupe de Brauer fini (de manière équivalente, la Jacobienne de la fibre générique a un groupe de Tate-Shafarevich fini [Gro68, §4]) ;
- $n \leq 0$  et  $X$  projectif lisse sur un corps fini, cf. [Gei04, 9.2] et [FM18, 3.17] ;
- $n \geq d - 1$  ou  $n \leq 1$ , lorsque  $X$  est projectif lisse sur un corps fini  $k$ , et est « construit à partir de produits de courbes projectives lisses sur  $k$  » (plus précisément, lorsque  $X$  appartient à la classe  $A(k)$  de [Gei04]), cf. [Gei04, 9.5] et [FM18, 3.17] ;
- $n \geq d$  ou  $n \leq 0$  lorsque  $X$  est régulier et « construit à partir de variétés sur les corps finis et de spectres d’anneaux d’entiers » (plus précisément, lorsque  $X$  appartient à la classe  $\mathcal{L}(\mathbb{Z})$  de [Mor14]), cf. [Mor14, §5.3].

<sup>4</sup>On peut observer que  $R\text{Hom}(R\Gamma_{\text{et}}(X, \mathbb{Z}(d - n)), \mathbb{Q}[-2d - 2]) = R\text{Hom}(R\Gamma_{\text{et}}(X, \mathbb{Z}(d - n)) \otimes \mathbb{Q}, \mathbb{Q}[-2d - 2]) = R\text{Hom}(R\Gamma(X, \mathbb{Q}(d - n)), \mathbb{Q}[-2d - 2])$ .

La conjecture  $\mathbf{B}(X, n)$  :

- se réduit, sous la conjecture de Parshin, à la conjecture de Beilinson sur la coïncidence des équivalences numérique et rationnelle, lorsque  $X$  est propre et lisse sur un corps fini [FM18, 2.7] ;
- est une version de la conjecture de Beilinson sur le comportement du régulateur, lorsque  $X$  est régulier, propre et plat sur  $\mathbb{Z}$  [FM18, §2.2] ;
- est connue si  $X$  est le spectre d'un anneau d'entiers dans un corps de nombres.

### 0.1.4 Faisceaux $\mathbb{Z}$ -constructibles en cohomologie Weil-étale

Soit  $X$  un schéma arithmétique irréductible de dimension 1. Un faisceau étale  $F$  sur  $X$  est dit  $\mathbb{Z}$ -constructible si sa restriction à un ouvert dense est localement constante, associée à un groupe abélien de type fini, et si ses germes en tout point sont des groupes abéliens de type fini. L'étude de la cohomologie Weil-étale avec des coefficients plus généraux que les coefficients motiviques a été initiée par Geisser dans l'article [Gei12], dans lequel il prouve un théorème de dualité pour la cohomologie Weil-étale à coefficients des faisceaux  $\mathbb{Z}$ -constructibles sur une courbe au-dessus d'un corps fini. Dans la thèse de Tran [Tra15], celui-ci propose une construction de cohomologie Weil-étale pour certains faisceaux  $\mathbb{Z}$ -constructibles (qu'il nomme faisceaux « fortement  $\mathbb{Z}$ -constructibles ») et leurs  $\mathbb{G}_m$ -duaux sur le spectre d'un anneau d'entiers dans un corps de nombres totalement imaginaire. Tran fait ensuite le lien, pour les faisceaux provenant du point générique par poussé en avant, avec les valeurs spéciales en  $s = 0, 1$  de fonctions  $L$  d'Artin associées à des représentations rationnelles (du groupe de Galois d'un corps de nombres totalement imaginaire). Il fait aussi une conjecture sur les 1-motifs et leurs fonctions  $L$ . Dans l'article [Tra16], Tran généralise sa construction (pour les faisceaux  $\mathbb{Z}$ -constructibles) au cas d'un corps de nombres quelconque (ce qui nécessite d'ailleurs de gérer correctement la 2-torsion) et prouve à nouveau une formule de valeurs spéciales en  $s = 0$  pour les fonctions  $L$  d'Artin de représentations rationnelles ; en utilisant l'équation fonctionnelle, il prouve aussi une formule de valeurs spéciales en  $s = 1$  pour des fonctions  $L$  de tores, en termes d'invariants de ceux-ci. Dans l'article [GS20], Geisser–Suzuki étudient la cohomologie Weil-étale d'une courbe lisse et propre sur un corps fini, à coefficients dans des modèles de Néron de 1-motifs sur le corps de fonctions, ou bien à coefficients des faisceaux  $\mathbb{Z}$ -constructibles. Ils prouvent une formule de valeur spéciale en  $s = 0$  pour les fonctions  $L$  de faisceaux  $\mathbb{Z}$ -constructibles, en  $s = 1$  pour les fonctions  $L$  de tores, et ils conjecturent une formule Weil-étale de valeur spéciale en  $s = 1$  pour la fonction  $L$  d'un 1-motif, qu'ils montrent être équivalente à la conjecture de Birch–Swinnerton-Dyer pour la partie abélienne de ce dernier.

## 0.2 Description des travaux

Ce travail consiste en l'extension de la formule sus-mentionnée de valeur spéciale, due à Flach–B. Morin, pour  $n = 0, 1$  aux faisceaux  $\mathbb{Z}$ -constructibles sur un schéma arithmétique intègre  $X$  de dimension 1. On s'inspirera du travail de Tran et Geisser–Suzuki mais on utilisera les méthodes de Flach–B. Morin qui donnent un cadre de travail plus robuste.

### 0.2.1 Valeurs spéciales en $s = 0$

Le chapitre II a sa propre introduction plus détaillée et contient l'article publié [Mor23], légèrement retravaillé. On définit la fonction  $L$  d'un faisceau  $\mathbb{Z}$ -constructible  $F$  sur  $X$  par

le produit eulérien

$$L_X(F, s) = \prod_{x \in X_{(0)}} \det(1 - N(x)^{-s} \text{Frob} \mid i_x^* F \otimes \mathbb{C})^{-1}$$

où  $i_x$  est l'inclusion du point fermé  $x \in X_0$  et  $\text{Frob}$  est le Frobenius géométrique, qui agit sur  $i_x^* F$ . On note  $L_X^*(F, 0) = \lim_{s \rightarrow 0} L_X(F, s) s^{-\text{ord}_{s=0} L_X(F, s)}$  la valeur spéciale de  $L_X(F, s)$  en  $s = 0$ . Un cas particulier de cette fonction  $L$  est le suivant : soit  $g : \text{Spec}(K) \rightarrow X$  l'inclusion du point générique ; si  $M$  est un  $\text{Gal}(K^{\text{sep}}/K)$ -module discret de type fini, alors  $g_* M$  est un faisceau  $\mathbb{Z}$ -constructible sur  $X$  et, lorsque  $X$  est propre et régulier,  $L_X(g_* M, s)$  est la fonction  $L$  d'Artin de  $M \otimes \mathbb{Q}$ , i.e.

$$L_X(g_* M, s) = L_K(M \otimes \mathbb{Q}, s) := \prod_{v \text{ place de } K} \det(1 - N(v)^{-s} \text{Frob} \mid (M \otimes \mathbb{Q})^{I_v})^{-1}.$$

En enlevant des points à  $X$ , on peut aussi naturellement considérer des fonctions  $L$  d'Artin partielles.

Lorsque  $X$  est régulier, en utilisant le cadre de travail de Flach–B. Morin, on définit dans la section II.3.1 un complexe Weil-étale à support compact  $R\Gamma_{W,c}(X, F)$  pour certains faisceaux  $\mathbb{Z}$ -constructibles, appelés faisceaux « gros » et « petits »<sup>5</sup>. On obtient une suite exacte longue en cohomologie pour les suites exactes courtes « gros-vers-petit »<sup>6</sup>. Comme  $n = 0$ , il n'y a pas de complexe de de Rham, donc la droite fondamentale est simplement

$$\Delta_X(F) := \det_{\mathbb{Z}} R\Gamma_{W,c}(X, F).$$

On prouve un théorème de dualité à coefficients réels, le théorème II.4.4, qui fait le lien entre la cohomologie à support compact à coefficients un faisceau étale  $F$  et les groupes de  $\text{Ext}$  de  $F$  vers le groupe multiplicatif  $\mathbb{G}_m$ . Ce théorème généralise la conjecture  $\mathbf{B}(X, 0)$  en un énoncé pour tous les faisceaux étales sur  $X$ . On déduit du théorème une trivialisaton de la droite fondamentale  $\Delta_X(F) \otimes \mathbb{R} \simeq \mathbb{R}$  lorsque  $F$  est gros ou petit. Cela permet de construire, pour tout faisceau gros ou petit  $F$ , une caractéristique d'Euler Weil-étale  $\chi_X(F)$  qui est multiplicative pour les suites exactes courtes de faisceaux gros ou petits. Tout faisceau  $\mathbb{Z}$ -constructible  $F$  peut se placer dans une suite exacte courte  $0 \rightarrow G \rightarrow F \rightarrow H \rightarrow 0$  avec  $G$  gros et  $H$  petit. En imposant la multiplicativité on obtient donc une caractéristique d'Euler Weil-étale définie pour tous les faisceaux  $\mathbb{Z}$ -constructibles et multiplicative pour les suites exactes courtes de faisceaux  $\mathbb{Z}$ -constructibles. Si le schéma  $X$  n'est pas régulier, il possède tout de même un morphisme quasi-fini  $f : X \rightarrow Y$  vers un schéma arithmétique  $Y$  intègre régulier et de dimension 1, et on peut définir  $\chi_X(F) := \chi_Y(f_! F)$  pour un faisceau  $\mathbb{Z}$ -constructible sur  $X$ , où  $f_!$  dénote le foncteur de poussé en avant propre le long de  $f$ . On prouve alors la formule de valeur spéciale (théorème II.6.24)

$$L_X^*(F, 0) = \pm \chi_X(F) \tag{0.2.1}$$

en se réduisant à la formule analytique du nombre de classes, qui avait déjà une interprétation Weil-étale par l'article [Mor14]. L'outil principal permettant cette réduction est l'induction d'Artin, qui énonce que les représentations rationnelles d'un groupe fini se comprennent essentiellement en termes des représentations induites de représentations triviales des sous-groupes.

<sup>5</sup>« big » et « tiny » dans la suite du texte.

<sup>6</sup>« big-to-tiny » dans la suite du texte

La caractéristique d’Euler Weil-étale a une expression explicite en termes de cohomologie étale à support compact à coefficients dans  $F$  et en termes de groupes d’Ext de  $F$  vers le groupe multiplicatif  $\mathbb{G}_m$  (ou bien plus généralement dans le cas singulier, vers le complexe de Deninger  $\mathbb{G}_X$ , voir [Den87a]). Le formalisme Weil-étale donne une manière d’assembler toutes ces données, qui étaient déjà disponibles, en un seul objet cohérent avec de bonnes propriétés de fonctorialité.

La formule (0.2.1) généralise aux schémas arithmétiques la formule similaire obtenue par Geisser–Suzuki dans l’article [GS20] pour les courbes propres et lisses sur un corps fini, et améliore tout en les généralisant les travaux de Tran [Tra16]. Si  $F$  est un faisceau constructible, la formule (0.2.1) est une généralisation de la formule de Tate pour la caractéristique d’Euler d’un corps global. Si  $M$  est un  $\text{Gal}(K^{\text{sep}}/K)$ -module discret de type fini et si  $X$  est propre, le théorème donne une formule de valeur spéciale pour la fonction  $L$  d’Artin de  $M \otimes \mathbb{Q}$  :

$$L_K^*(M \otimes \mathbb{Q}, 0) = \pm \chi_X(g_*M).$$

Un autre cas particulier intéressant du théorème est la généralisation suivante de la formule analytique du nombre de classes aux schémas singuliers : si  $X$  est affine, on a

$$\zeta_X^*(0) = \pm \frac{[\text{CH}_0(X)]R_X}{\omega},$$

où  $\text{CH}_0(X)$  est le 0-ième groupe de Chow,  $R_X$  est un régulateur et  $\omega$  est le nombre de racines de l’unité dans le corps de fonction de  $X$ . Si  $X$  est une courbe propre sur un corps fini, avec pour corps de constantes  $\mathbb{F}_q$ , on a d’autre part

$$\zeta_X^*(0) = \pm \frac{[\text{CH}_0(X)_{\text{tor}}]R_X}{\omega \log(q)}.$$

Le régulateur  $R_X$  a une définition similaire dans les deux cas, mais il est uniquement contrôlé par les singularités dans le cas d’une courbe propre ; en particulier, il s’annule si la courbe est régulière. En caractéristique 0, le régulateur est une modification du régulateur usuel (i.e. du régulateur de la normalisation), qui prend en compte les singularités.

Dans le chapitre III, on formalise la partie formelle de la preuve du théorème II.6.24, et on simplifie l’argument utilisé dans le chapitre II pour gérer les faisceaux constructibles, *via* un théorème de Swan. Soit  $\mathcal{A}$  une catégorie abélienne. Une caractéristique d’Euler  $\chi$  sur  $\mathcal{A}$  est une application  $\chi$  depuis les classes d’isomorphismes d’objets de  $\mathcal{A}$  vers un groupe abélien  $\Gamma$ , qui vérifie  $\chi(B) = \chi(A) + \chi(C)$  pour toute suite exacte courte  $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ . On montre les deux résultats suivants :

**Théorème.** *Soit  $\chi$  une caractéristique d’Euler sur la catégorie des faisceaux  $\mathbb{Z}$ -constructibles sur un schéma arithmétique  $X$ . Si  $\chi$  est à valeurs dans un groupe abélien sans torsion, alors elle s’annule sur les faisceaux constructibles.*

**Théorème.** *Soit  $\chi, \chi'$  deux caractéristiques d’Euler sur la catégorie des faisceaux  $\mathbb{Z}$ -constructibles sur un schéma arithmétique  $X$ , à valeurs dans un même groupe abélien sans torsion. Si  $\chi(\pi_*\mathbb{Z}) = \chi'(\pi_*\mathbb{Z})$  pour tout morphisme fini  $\pi : Y \rightarrow X$  de schémas arithmétiques avec  $Y$  normal et irréductible, alors  $\chi = \chi'$ .*

La stratégie de preuve est de généraliser le procédé de dévissage (§ II.2.0.1) déjà mis en place au chapitre II, tout en utilisant l’induction d’Artin et le théorème de Swan sus-mentionné.



## 0.2.2 Valeur spéciale en $s = 1$

Le chapitre IV est constitué de l'article [Mor22], et a sa propre introduction détaillée. Notons  $\mathbb{Z}_X^c$  le complexe de cycles de Bloch, avec l'indexation cohomologique (cf § IV.1.3.2). Lorsque  $X$  est régulier, on a  $\mathbb{Z}_X^c \simeq \mathbb{Z}(1)[2]$ . On travaille dorénavant en caractéristique 0, et donc  $X$  est le spectre d'un ordre dans un corps de nombres  $K$ . On note

$$F^D := R\mathcal{H}om_X(F, \mathbb{Z}_X^c);$$

que l'on considère comme une généralisation, contravariante en  $F$ , du twist motivique  $\mathbb{Z}(1)$  (tout du moins lorsque  $X$  est régulier, et a un décalage près). La dualité d'Artin–Verdier permet de considérer  $F^D$  comme le dual de  $F$ . Lorsque  $F$  est  $\mathbb{Z}$ -constructible, on définit la fonction  $L$  de  $F^D$  sur  $X$  par

$$L_X(F^D, s) := \prod_{x \in X_0} \det(1 - N(x)^{-s} \text{Frob} \mid (i_x^* F^D) \hat{\otimes} \mathbb{Q}_{\ell_x})^{-1}.$$

où  $\ell_x$  est un nombre premier, premier à la caractéristique résiduelle en  $x$ , et où pour un complexe  $M$  de faisceaux étales on a noté  $M \hat{\otimes} \mathbb{Q}_{\ell} := (R\lim_n (M \otimes^L \mathbb{Z}/\ell^n \mathbb{Z})) \otimes \mathbb{Q}$ , calculé sur le site pro-étale. Pour  $F = \mathbb{Z}$  et  $X$  régulier, on obtient  $L_X(\mathbb{Z}^D, s) = \zeta_X(s+1)$ . Si  $F = i_* M$  est le poussé en avant d'un  $\text{Gal}(k(x)^{\text{sep}}/k(x))$ -module discret de type fini  $M$  le long de l'inclusion  $i : x \rightarrow X$  d'un point fermé, on a  $L_X((i_* M)^D, s) = L_X(i_*(M^\vee), s)$  où le second terme est la fonction  $L$  d'un faisceau  $\mathbb{Z}$ -constructible, comme définie au chapitre II, et  $(-)^{\vee} := \mathcal{H}om(-, \mathbb{Z})$  est le dual  $\mathbb{Z}$ -linéaire. Lorsque  $X$  est régulier, si l'on note  $g : \text{Spec}(K) \rightarrow X$  l'inclusion du point générique de  $X$  et si  $M$  est un  $\text{Gal}(K^{\text{sep}}/K)$ -module discret de type fini, on a  $L_X(g_* M, s) = L_K(M \otimes \mathbb{Q}, s+1)$ , une fonction  $L$  d'Artin. Ainsi, génériquement la fonction  $L$  que l'on considère se comporte comme un fonction  $L$  d'Artin en la variable  $s+1$ , mais localement elle se comporte plutôt comme un facteur local en la variable  $s$ . Ceci est cohérent avec les diverses propriétés de functorialité de la construction  $(-)^D$  : on a  $\mathbb{Z}^D = \mathbb{Z}(1)[2]$  lorsque  $X$  est régulier, et l'on sait que  $\mathbb{Z}(1)$  est lié à la valeur spéciale en  $s=1$  de  $\zeta_X$ , tandis que pour l'inclusion  $i : x \rightarrow X$  d'un point fermé on a  $(i_* \mathbb{Z})^D = i_* \mathbb{Z}^{\vee} = i_* \mathbb{Z}$ , qui est lié à la valeur spéciale en 0 de  $\zeta_x$ .

On définit et étudie d'abord la cohomologie à support compact à coefficients  $F^D$ , en corrigeant la cohomologie étale du complexe  $F^D$  par la cohomologie de Betti aux places archimédiennes. En combinant la dualité d'Artin–Verdier usuelle avec la dualité pour la cohomologie de Tate du groupe de Galois aux places archimédiennes, on obtient une version « tordue » de la dualité d'Artin–Verdier faisant le lien entre la cohomologie à support compact de  $F^D$  et la cohomologie étale de  $F$ . A nouveau, suivant le formalisme établi par Flach–Morin, on définit alors la cohomologie Weil-étale à support compact à coefficients  $F^D$  pour certains faisceaux  $\mathbb{Z}$ -constructibles, les faisceaux « rouges » et « bleus »<sup>7</sup> (def. IV.3.1). Les résultats comme les méthodes sont semblables à ceux du chapitre II : en particulier, on obtient une suite exacte longue de cohomologie Weil-étale pour les suites exactes courtes « rouge-vers-bleu ». <sup>8</sup> Comme  $n=1$ , cette fois-ci la droite fondamentale a une partie additive : lorsque  $F$  est *modérément ramifié* (def. IV.4.1), on définit un complexe  $\text{Lie}_X(F^D)$ , le « complexe tangent de  $F^D$  », qui généralise d'une part le complexe de de Rham dérivé utilisé dans [FM18] et d'autre part dans certains cas l'algèbre de Lie du modèle de Néron d'un tore, comme utilisé dans [GS20]. Si  $F$  est modérément ramifié, on peut alors définir la droite fondamentale :

$$\Delta_X(F^D) = \det_{\mathbb{Z}} R\Gamma_{W,c}(X, F^D) \otimes \det_{\mathbb{Z}} \text{Lie}_X(F^D).$$

<sup>7</sup>« Red » et « blue » dans la suite du texte.

<sup>8</sup>« Red-to-blue » dans la suite du texte.

On prouve un théorème de dualité à coefficients réels faisant le lien entre la cohomologie de Deligne à support compact à coefficients  $F^D$  (c'est-à-dire la cohomologie étale, corrigée aux places archimédiennes par la cohomologie de Deligne, cf. def. IV.5.1) et la cohomologie étale de  $F$ . Ce théorème généralise la conjecture  $\mathbf{B}(X, 1)$ . On obtient ainsi une trivialisatoin  $\Delta_X(F^D) \otimes \mathbb{R} \simeq \mathbb{R}$  de la droite fondamentale. Cela donne une caractéristique d'Euler  $\chi_X(F^D)$  pour  $F$  modérément ramifié et rouge ou bleu ; cette dernière est multiplicative pour les suites exactes courtes rouge-vers-bleu de faisceaux modérément ramifiés. Tout faisceau  $\mathbb{Z}$ -constructible modérément ramifié peut être inséré dans une suite exacte courte entre deux faisceaux rouges et bleus modérément ramifiés, et donc en imposant la multiplicativité on obtient formellement une caractéristique d'Euler pour tous les faisceaux  $\mathbb{Z}$ -constructibles modérément ramifiés. Celle-ci a une expression explicite en fonction de puissances de 2 et de  $2\pi$ , de quantités « additives » définies à partir de  $F$  (en particulier une quantité du type « racine carré du discriminant »), de groupes de cohomologie étale à coefficients  $F$  et de groupes d'Ext de  $F$  vers le groupe multiplicatif  $\mathbb{G}_m$ . L'expression dans son ensemble ressemble beaucoup à la formule analytique du nombre de classes. On peut alors montrer la formule de valeurs spéciales pour les faisceaux  $\mathbb{Z}$ -constructibles modérément ramifiés :

$$L_X^*(F^D, 0) = \pm \chi_X(F^D).$$

En particulier, si  $M$  est un  $\text{Gal}(K^{\text{sep}}/K)$ -module discret de type fini, la fonction  $L$  de  $L_X((g_*M)^D, s)$  est la fonction  $L$  d'Artin  $L(M \otimes \mathbb{Q}, s + 1)$  et le théorème énonce alors la formule

$$L^*(M \otimes \mathbb{C}, 1) = \pm \chi_X((g_*M)^D).$$

Pour  $F = \mathbb{Z}$  lorsque  $X$  est régulier, on retrouve la formule analytique du nombre de classes en  $s = 1$ , comme dans [FM18]. Par les résultats du chapitre III, cela est essentiellement suffisant pour prouver la formule de valeurs spéciales. Celle-ci est de plus l'analogue dans le cas d'un corps de nombres du travail de Geisser–Suzuki dans le cas des corps de fonctions sur les valeurs spéciales de fonctions  $L$  de tores  $s = 1$  [GS20]. Enfin, si  $F$  est constructible, les deux complexes  $R\Gamma_{W,c}(X, F^D)$  et  $\text{Lie}_X(F^D)$  sont des complexes parfaits à groupes de cohomologie finis, et le théorème de valeurs spéciales énonce alors que leurs caractéristiques d'Euler multiplicatives, c'est-à-dire le produit alterné de l'ordre de leurs groupes de cohomologie, sont égales

$$\chi^\times(R\Gamma_{W,c}(X, F^D)) = \chi^\times(\text{Lie}_X(F^D)) = [F_\eta]^{[K:\mathbb{Q}]}.$$

Dans le chapitre V, on suit une remarque faite par T. Suzuki pour définir de manière inconditionnelle la partie additive de la droite fondamentale. L'observation de T. Suzuki est la suivante : même si le foncteur qui à un tore associe l'algèbre de Lie de son modèle de Néron n'est pas exact, son déterminant se comporte tout de même de manière multiplicative vis-à-vis des suites exactes courtes de tores. Notons  $g : \text{Spec}(K) \rightarrow X$  l'inclusion du point générique. Pour un faisceau étale  $F$  sur  $X$  modérément ramifié et sans torsion, le complexe  $\text{Lie}_X(F^D)$  coïncide (à un décalage près) avec l'algèbre de Lie du modèle de Néron du tore sur  $K$  dont le groupe de caractères est  $g^*F$ . Ainsi, dans le cas général d'un faisceau  $\mathbb{Z}$ -constructible, il est censé de définir la partie additive de la droite fondamentale en considérant une résolution à 2 termes de  $g^*F$  par des  $\text{Gal}(K^{\text{sep}}/K)$ -modules discrets de type fini sans torsion, et en prenant ensuite le déterminant du complexe formé par les algèbres de Lie des modèles de Néron des tores associés aux termes de la résolution. Ceci définit un  $\mathbb{Z}$ -fibré en droite gradué qui est multiplicatif vis-à-vis des suites exactes courtes, qui a le bon changement de base à  $\mathbb{Q}$  (c'est-à-dire l'algèbre de Lie du tore de groupe

de caractères  $g^*F$ ), et qui coïncide avec  $\det_{\mathbb{Z}} \text{Lie}_X(F^D)$  (à un décalage près) dans le cas modérément ramifié. Avec cette définition inconditionnelle de la droite fondamentale à notre disposition, on peut définir inconditionnellement la caractéristique d'Euler Weil-étale  $\chi_X(F^D)$  ce qui permet d'étendre la formule de valeurs spéciales à tous les faisceaux  $\mathbb{Z}$ -constructibles :

$$L_X^*(F^D, 0) = \pm \chi_X(F^D).$$

# Chapter I

## Introduction

### I.1 Weil-étale cohomology

#### I.1.1 The Dedekind zeta function and the analytic class number formula

The *Riemann zeta function* is the infinite sum

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s},$$

which converges for  $\Re(s) > 1$ . It has an alternate expression as an infinite product (which is called an *Euler product*) ranging over all prime numbers

$$\zeta(s) = \prod_p \frac{1}{1 - p^{-s}}.$$

This second formula expresses a deep link between the Riemann zeta function and the behaviour of prime numbers and makes the former a central object of study in number theory. With the study of algebraic number theory, new similar functions emerged, called *Dedekind zeta functions*. For a number field  $K$  with ring of integers  $\mathcal{O}_K$ , the Dedekind zeta function associated to  $K$  is given by the Dirichlet series

$$\zeta_K(s) = \sum_{I \text{ ideal of } \mathcal{O}_K} \text{card}(\mathcal{O}_K/I)^{-s},$$

converging for  $\Re(s) > 1$ , and has an Euler product expression

$$\zeta_K(s) = \prod_{\mathfrak{p}} \frac{1}{1 - \text{card}(\mathcal{O}_K/\mathfrak{p})^{-s}}$$

ranging over the non-zero prime ideals  $\mathfrak{p}$  of  $\mathcal{O}_K$  (that is, over the closed points of the spectrum of  $\mathcal{O}_K$ ). Similarly to the Riemann zeta function, understanding the Dedekind zeta functions gives an insight into the behavior of prime ideals in number fields.

The first instance of the problem of understanding the values of zeta functions is the Basel problem, which asks for a closed form for  $\zeta(2)$ ; Euler proved the celebrated formula  $\zeta(2) = \pi^2/6$ , and gave more generally formulas for  $\zeta(2n)$  when  $n \geq 1$  in terms of Bernoulli numbers. By work of Hecke, it is known that the Dedekind zeta functions have

a meromorphic continuation on the whole complex plane with a simple pole at  $s = 1$  so one can ask about their residue at  $s = 1$ . The *analytic class number formula* gives the answer:

$$\lim_{s \rightarrow 1} (s - 1) \zeta_K(s) = \frac{2^{r_1} (2\pi)^{r_2} h_K R_K}{\omega_K \sqrt{|D_K|}}, \quad (\text{I.1.1})$$

where  $r_1$  is the number of real embeddings of  $K$ ,  $2r_2$  is the number of complex embeddings,  $h_K = \text{card}(\text{Cl}(\mathcal{O}_K))$  is the class number,  $R_K$  is the regulator,  $\omega_K$  is the number of roots of unity in  $K$  and  $D_K$  is the (absolute) discriminant of  $K$ . This formula is striking: the zeta function is assembled only from local data at each prime (i.e. from knowledge on the closed points of  $\text{Spec}(\mathcal{O}_K)$ ) but its residue at 1 has an expression in terms of quantities defined from the whole ring of integers  $\mathcal{O}_K$ ; all those quantities are important to understand the structure of the ring of integers.

When  $X$  is a variety over a finite field, one can mimic the definition of the Dedekind zeta function using the (closed) points of  $X$ ; thus the zeta function of  $X$  is defined by

$$\zeta_X(s) = \prod_{x \in X_0} \frac{1}{1 - N(x)^{-s}},$$

where  $N(x)$  is the cardinality of the residue field at a closed point  $x \in X_0$ . *Étale cohomology* was developed in part as a tool to understand this function and its values. Étale cohomology is a cohomology theory: given a geometric object (here the variety  $X$ ) and some coefficients  $F$  (here called “étale sheaves on  $X$ ”), it outputs abelian groups  $H_{\text{ét}}^i(X, F)$  for all integers  $i \geq 0$ . These abelian groups are called cohomology groups and they often describe the obstruction to some constructions: when they are 0, the construction is possible, otherwise they give some sort of quantification of what goes wrong.

Coming back to the Dedekind zeta function, using its functional equation (proved by Hecke) we can reformulate the analytic class number formula as a formula at  $s = 0$ :

$$\zeta_K^*(0) := \lim_{s \rightarrow 0} s^{1-r_1-r_2} \zeta_K(s) = -\frac{h_K R_K}{\omega_K}.$$

In this formula, one can observe that all the invariants come from étale cohomology:

$$h_K = \text{card}(H_{\text{ét}}^1(\text{Spec}(\mathcal{O}_K), \mathbb{G}_m)),$$

the regulator  $R_K$  is the discriminant of a pairing

$$H_{\text{ét}}^0(X, \mathbb{G}_m) \times H_{\text{ét},c}^1(X, \mathbb{Z}) \rightarrow \mathbb{R}$$

and

$$\omega_K = \text{card}(H_{\text{ét}}^0(X, \mathbb{G}_m)_{\text{tor}}).$$

But this interpretation is not completely satisfying: the formula is obtained by taking bits and pieces from various étale cohomology groups and taking their products; we would like a systematic explanation. This was arguably achieved with the development of *Weil-étale cohomology*.

## I.1.2 Weil-étale cohomology and special value at $s = 0$ for varieties over finite fields

In the particular case of the spectrum  $\text{Spec}(k)$  of a field  $k$ , étale cohomology is computed in terms of the cohomology of the Galois group of  $k$ , that is of the group  $G_k = \text{Gal}(k^{\text{sep}}/k)$  of  $k$ -linear automorphisms of a separable closure  $k^{\text{sep}}$  of  $k$ . For a variety  $X$  over  $k$ , an étale sheaf on  $X$  identifies with a  $G_k$ -equivariant sheaf on the base change  $X \times_k k^{\text{sep}}$ . Using the Hochschild-Serre spectral sequence, the étale cohomology of  $X$  can then be understood in terms of the cohomology of the Galois group of  $k$ , with coefficients the étale cohomology of  $X \times_k k^{\text{sep}}$ .

In [Lic05], Lichtenbaum made the following observation: for a variety  $X$  over a finite field  $k$ , replacing the Galois group of  $k$  by the Weil group of  $k$  leads to a cohomology theory with favourable properties, better behaved than étale cohomology. Namely, let  $W_k = \mathbb{Z} \subset \hat{\mathbb{Z}} = G_k$  be the Weil group of  $k$ , that is the subgroup generated by the Frobenius. Then the category of Weil-étale sheaves on  $X$  is defined as the category of  $W_k$ -equivariant étale sheaves on  $X \times_k k^{\text{sep}}$ ; this is a topos. As we said, a  $G_k$ -equivariant sheaf on  $X \times_k k^{\text{sep}}$  identifies with a sheaf on  $X$ , and has thus a canonical structure of  $W_k$ -equivariant sheaf through the inclusion  $W_k \subset G_k$ . This defines a geometric morphism from the topos of Weil-étale sheaves to the topos of étale sheaves.

For  $X$  smooth projective, the corresponding cohomology groups  $H_W^i(X, \mathbb{Z})$  are shown to be finitely generated, and zero for large indexes. An easy example that already showcases some interesting phenomenon is the case  $X = \text{Spec}(k)$ . In this case one has  $H_{\text{ét}}^i(X, \mathbb{Z}) = H^i(\text{Gal}(k^{\text{sep}}/k), \mathbb{Z}) = \mathbb{Z}, 0, \mathbb{Q}/\mathbb{Z}, 0$  in degree  $i = 0, i = 1, i = 2, i \geq 3$ , while  $H_W^i(X, \mathbb{Z}) = H^i(W_k, \mathbb{Z}) = \mathbb{Z}, \mathbb{Z}, 0$  in degree  $i = 0, i = 1, i \geq 2$ . We see that  $\mathbb{Q}/\mathbb{Z}$  gets replaced by  $\mathbb{Z}$ , shifted down by one degree. This will be a general phenomenon in Weil-étale cohomology: the terms  $\mathbb{Q}/\mathbb{Z}$  in étale cohomology will tend to get replaced by  $\mathbb{Z}$ , shifted one degree down, in Weil-étale cohomology. The same phenomenon happens usually when taking a derived  $\ell$ -adic completion: the (derived)  $\ell$ -adic completion of  $\mathbb{Q}/\mathbb{Z}$  is

$$\mathbb{Q}/\mathbb{Z}_\ell^\wedge := R\lim_n (\mathbb{Q}/\mathbb{Z} \otimes^L \mathbb{Z}/\ell^n \mathbb{Z}) = R\lim_n \mathbb{Z}/\ell^n \mathbb{Z}[1] = \mathbb{Z}_\ell[1],$$

so  $\mathbb{Q}/\mathbb{Z}$  gets replaced by  $\mathbb{Z}_\ell$ , shifted one degree down. As Weil-étale cohomology is often finitely generated, it turns out that it is thus also often an integral model for  $\ell$ -adic cohomology<sup>1</sup>.

Denote by  $r_X$  the vanishing order at 0 of  $\zeta_X$  and

$$\zeta_X^*(0) = \lim_{s \rightarrow 0} \zeta_X(s) s^{-r_X}$$

its special value at  $s = 0$ . Lichtenbaum extracts from the Weil-étale cohomology groups a Weil-étale Euler characteristic  $\chi_X(\mathbb{Z})$  and shows the formula

$$\zeta_X^*(0) = \pm \chi_X(\mathbb{Z})$$

by using a formula, proven by Milne [Mil86], expressing  $\zeta_X^*(0)$  in terms of étale cohomology. Again this is striking: the zeta function is defined using local data only and the Weil-étale Euler characteristic is a *global invariant*, it contains informations about global phenomena on  $X$ . Lichtenbaum also conjectures a similar formula for more general schemes over  $k$  and Weil-étale cohomology with compact support.

<sup>1</sup>See for instance [Gei04, 8.4]

### I.1.3 Weil-étale cohomology for arithmetic schemes and motivic coefficients

In subsequent work, Flach, Geisser, Lichtenbaum, B. Morin, and others have strived to generalize these results to arbitrary arithmetic schemes  $X$  (finite type separated schemes over  $\text{Spec}(\mathbb{Z})$ ) and motivic coefficients  $\mathbb{Z}(n)$ ; see e.g. [Lic09; Gei04; GS20] for the situation over finite fields and [Mor14; FM18] for general arithmetic schemes. The state of the art is due to B. Morin and Flach–B. Morin [FM18; Mor21]; in what follows,  $X$  is an arithmetic scheme of dimension  $d$ :

1. Suppose that  $X$  is proper and regular. For  $n \geq 0$ , denote by  $\mathbb{Z}(n)$  Bloch’s cycle complex and for  $n < 0$  put  $\mathbb{Z}(n) := \mathbb{Q}/\mathbb{Z}(n)[-1] = \bigoplus_p j_{p,!} \mu_{p^\infty}^{\otimes n}[-1]$  where  $j_p : X[1/p] \rightarrow X$  is the open immersion. Flach–B. Morin introduce in [FM18] a complex of étale cohomology with compact support

$$R\Gamma_{\text{et},c}(X, \mathbb{Z}(n)) := \text{fib}(R\Gamma_{\text{et}}(X, \mathbb{Z}(n)) \rightarrow R\Gamma_{G_{\mathbb{R}}}(X(\mathbb{C}), (2i\pi)^n \mathbb{Z}))$$

where  $R\Gamma_{G_{\mathbb{R}}}(X(\mathbb{C}), -)$  denotes  $G_{\mathbb{R}}$ -equivariant cohomology on the topological space  $X(\mathbb{C})$  of complex points of  $X$ , endowed with its analytic topology. The aim of this complex is to correct the 2-torsion phenomena happening in étale motivic cohomology. Under some conjectural hypotheses  $\mathbf{AV}(X, n)$  (on Artin–Verdier duality),  $\mathbf{L}(X, n)$  and  $\mathbf{L}(X, d - n)$  (on finite generation of étale motivic cohomology in certain degrees) they also construct a Weil-étale complex with compact support  $R\Gamma_{W,c}(X, \mathbb{Z}(n))$ , defined by a distinguished triangle (in the derived 1-category):

$$R\text{Hom}(R\Gamma(X, \mathbb{Q}(n - d)), \mathbb{Q}[-2d - 2]) \rightarrow R\Gamma_{\text{et},c}(X, \mathbb{Z}(n)) \rightarrow R\Gamma_{W,c}(X, \mathbb{Z}(n)) \rightarrow .$$

More generally, for every arithmetic scheme  $X$  and for each  $n \in \mathbb{Z}$ , there should exist a perfect complex of abelian groups, the Weil-étale complex with compact support  $R\Gamma_{W,c}(X, \mathbb{Z}(n))$ ; we also refer to it as the “multiplicative complex”.

2. B. Morin defines in [Mor21] for every arithmetic scheme  $X$  a complex  $L\Omega_{X/\mathbb{S}}^{\leq n}$  called the derived de Rham complex relative to the sphere spectrum modulo the  $n$ -th step of the Hodge filtration; we refer to it as the “additive complex”.
3. When  $X$  is proper regular, the fundamental line is then defined as the free abelian group of rank 1

$$\Delta_{X,n} = \det_{\mathbb{Z}}(R\Gamma_{W,c}(X, \mathbb{Z}(n))) \otimes \det_{\mathbb{Z}}(R\Gamma(X, L\Omega_{X/\mathbb{S}}^{\leq n})),$$

where  $\det_{\mathbb{Z}}$  is the determinant construction of [KM76], which generalizes the top exterior power map  $M \mapsto \Lambda^{\text{rank} M} M$  to perfect complexes of abelian groups. It has a “multiplicative” part  $\det_{\mathbb{Z}}(R\Gamma_{W,c}(X, \mathbb{Z}(n)))$  and an “additive” part  $\det_{\mathbb{Z}}(R\Gamma(X, L\Omega_{X/\mathbb{S}}^{\leq n}))$ .

More generally, for every arithmetic scheme  $X$ , the fundamental line should be defined in terms of the conjectural complex  $R\Gamma_{W,c}(X, \mathbb{Z}(n))$  and an appropriately-defined compactly supported cohomology on  $X$  of  $L\Omega_{X/\mathbb{S}}^{\leq n}$ .

4. Suppose that  $X$  is regular and proper and consider the fiber

$$R\Gamma_c(X, \mathbb{R}(n)) := \text{fib}(R\Gamma(X, \mathbb{R}(n)) \rightarrow R\Gamma(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{R}(n)_{\mathcal{D}}))$$

of the map from étale cohomology with  $\mathbb{R}(n)$ -coefficients to  $G_{\mathbb{R}}$ -equivariant cohomology of the Deligne complex  $\mathbb{R}(n)_{\mathcal{D}} := \text{fib}((2i\pi)^n \mathbb{R} \rightarrow \Omega_{X(\mathbb{C})}^{\leq n})$ . Flach–B. Morin introduce a conjecture  $\mathbf{B}(X, n)$ , stating that a canonically constructed pairing

$$R\Gamma_c(X, \mathbb{R}(n)) \otimes^L R\Gamma(X, \mathbb{R}(d-n)) \rightarrow R\Gamma_c(X, \mathbb{R}(d)) \xrightarrow{\tau \geq 2d} H_c^{2d}(X, \mathbb{R}(d))[-2d] \\ \xrightarrow{tr} \mathbb{R}[-2d]$$

is a perfect pairing of perfect complexes of  $\mathbb{R}$ -vector spaces. Under  $\mathbf{B}(X, n)$ , a canonical trivialization

$$\lambda : \mathbb{R} \xrightarrow{\simeq} \Delta_{X,n} \otimes \mathbb{R}$$

of the fundamental line can be defined.<sup>2</sup> In other words, the additive complex and the multiplicative complex should be strongly related after tensoring with  $\mathbb{R}$ .

More generally, for every arithmetic scheme there should exist a canonical trivialization of the fundamental line, coming from some duality with  $\mathbb{R}$ -coefficients.

5. Using the trivialization,  $\Delta_{X,n}$  identifies with a lattice inside  $\mathbb{R}$ , and the special value

$$\zeta_X^*(n) := \lim_{s \rightarrow n} \zeta_X(s) (s-n)^{-\text{ord}_{s=n} \zeta_X}$$

of  $\zeta_X$  at  $s = n$  is conjectured to be given up to sign by the following formula:<sup>34</sup>

$$\lambda(\zeta_X^*(n)^{-1}) \cdot \mathbb{Z} = \Delta_{X,n}. \quad (\text{I.1.2})$$

In other words, the covolume of the lattice  $\Delta_{X,n}$  inside the 1-dimensional  $\mathbb{R}$ -vector space  $\Delta_{X,n} \otimes \mathbb{R}$  is  $\zeta_X^*(n)^{-1}$ . For  $X$  smooth proper over the ring of integers  $\mathcal{O}_K$  of a number field, a variant of (I.1.2) (involving the derived de Rham complex  $L\Omega_{X/\mathbb{Z}}^{\leq n}$  relative to  $\mathbb{Z}$  and a correction factor coming from  $p$ -adic Hodge theory, instead of  $L\Omega_{X/\mathbb{S}}^{\leq n}$ ) is shown by Flach–B. Morin to be compatible with the special value conjecture of Bloch–Kato and Fontaine–Perrin–Riou. This variant is thus known for  $X = \text{Spec}(\mathcal{O}_K)$  if  $n = 0, 1$ , giving a Weil–étale interpretation of the analytic class number formula Equation (I.1.1), or if  $K/\mathbb{Q}$  is abelian [FM18, 5.34]. By [Mor21, 1.6] and [FM18, 5.33], (I.1.2) also holds in this case. For  $X$  smooth proper over a finite field, the formula holds under  $\mathbf{B}(X, n)$ ,  $\mathbf{L}(X, n)$  and  $\mathbf{L}(X, d-n)$  by reduction to work of Lichtenbaum and Geisser ([Gei04], see [FM18, 5.31] for the reduction).

One important idea of the construction of Flach–B. Morin is that the Weil–étale complex is constructed using Artin–Verdier duality (conjecture  $\mathbf{AV}(X, n)$ ), instead of trying to define a Weil–étale topos and taking the corresponding cohomology theory. The idea of this definition, which first appeared in [Mor14], comes from the computation of the derived pushforward functor from the Weil–étale topos to the étale topos, by Geisser [Gei04] over finite fields and B. Morin [Mor11, §8, §9] for the spectrum of a ring of integers in a number field. With the Weil–étale philosophy that “ $\mathbb{Q}/\mathbb{Z}$ -shaped torsion gets replaced by  $\mathbb{Z}[1]$ ”, let us try to motivate how one could re-discover this definition: under conjectures  $\mathbf{AV}(X, n)$ ,  $\mathbf{L}(X, n)$  and  $\mathbf{L}(X, d-n)$ , the torsion in motivic cohomology  $R\Gamma_{\text{et},c}(X, \mathbb{Z}(n))$  is identified as coming from the  $\mathbb{Q}/\mathbb{Z}$ -dual of the finite type part of motivic cohomology  $R\Gamma_{\text{et}}(X, \mathbb{Z}(d-n))$ , and the rest of  $R\Gamma_{\text{et},c}(X, \mathbb{Z}(n))$  is actually of finite type. We want to obtain a perfect

<sup>2</sup>In this manuscript, the trivialization will instead be written  $\lambda : \Delta_{X,n} \otimes \mathbb{R} \xrightarrow{\simeq} \mathbb{R}$ .

<sup>3</sup>There is also a conjecture for the vanishing order

<sup>4</sup>In this manuscript  $\lambda(\Delta_{X,n}) = \zeta_X^*(n)^{-1} \cdot \mathbb{Z}$ .



complex of Weil-étale cohomology by replacing the torsion by something of finite type; using the fiber sequence  $\mathbb{Q} \rightarrow \mathbb{Q}/\mathbb{Z} \rightarrow \mathbb{Z}[1]$ , this can be done if we can construct a map

$$R\mathrm{Hom}(R\Gamma_{\mathrm{et}}(X, \mathbb{Z}(d-n)), \mathbb{Q}[-2d-2]) \rightarrow R\Gamma_{\mathrm{et,c}}(X, \mathbb{Z}(n))$$

that “sends the  $\mathbb{Q}$ -dual of the finite type part of  $R\Gamma_{\mathrm{et}}(X, \mathbb{Z}(d-n))$  to its  $\mathbb{Q}/\mathbb{Z}$ -dual, via the canonical projection, inside  $R\Gamma_{\mathrm{et,c}}(X, \mathbb{Z}(n))$ ”, and consider its mapping cone. The construction of Flach–B. Morin achieves this.<sup>5</sup>

Another fundamental observation is that in characteristic 0, contrarily to the characteristic  $p$  case,  $\det_{\mathbb{Z}} R\Gamma_{W,c}(X, \mathbb{Z}(n))$  and  $\det_{\mathbb{Z}} R\Gamma(X, L\Omega_{X/\mathbb{S}}^{\leq n})$  do not have trivializations on their own, so we cannot express the special value as the product of a “multiplicative” Euler characteristic and an “additive” Euler characteristic. The fundamental line must be considered as a whole.

*Remark* ([FM18, §3.2]). Let  $X$  be a regular proper arithmetic scheme of dimension  $d$ . Conjecture **AV**( $X, n$ ) is known if:

- $X/\mathcal{O}_K$  is smooth;
- $X/\mathbb{F}_q$  is smooth;
- $X$  is an arithmetic surface (i.e  $X$  is connected regular of dimension 2 and flat and proper over  $\mathbb{Z}$ ) [FS22];
- $n \leq 0$  or  $n \geq d$ .

Conjecture **L**( $X, n$ ) is known if:

- $d \leq 1$ ;
- $n = 1$  and  $X$  is an arithmetic surface or connected regular of dimension 2 and proper over a finite field, with finite Brauer group (equivalently the Jacobian of the generic fiber has finite Tate-Shafarevich group by [Gro68, §4]);
- $n \leq 0$  and  $X$  smooth projective over a finite field, see [Gei04, 9.2] and [FM18, 3.17];
- $n \geq d - 1$  or  $n \leq 1$  where  $X$  is smooth projective over a finite field  $k$ , and is “built from products of smooth projective curves over  $k$ ” (more precisely in the class  $A(k)$  of [Gei04]), see [Gei04, 9.5] and [FM18, 3.17];
- $n \geq d$  or  $n \leq 0$  and  $X$  regular “built from varieties over finite fields and spectra of number rings” (more precisely,  $X$  belongs to the the class  $\mathcal{L}(\mathbb{Z})$  of [Mor14]), see [Mor14, §5.3].

Conjecture **B**( $X, n$ ):

- reduces under Parshin’s conjecture to Beilinson’s conjecture on the coincidence of rational and numerical equivalence when  $X$  is smooth proper over a finite field [FM18, 2.7];
- is a version of Beilinson’s conjectures on the behaviour of the regulator for  $X$  regular flat and proper over  $\mathbb{Z}$  [FM18, §2.2];
- is known if  $X$  is the spectrum of a number ring.

### I.1.4 $\mathbb{Z}$ -constructible sheaves in Weil-étale cohomology

If  $X$  is an irreducible arithmetic scheme of dimension 1, an étale sheaf  $F$  on  $X$  is  $\mathbb{Z}$ -constructible if on a dense open subscheme it is locally constant associated to a finite type

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<sup>5</sup>Observe that  $R\mathrm{Hom}(R\Gamma_{\mathrm{et}}(X, \mathbb{Z}(d-n)), \mathbb{Q}[-2d-2]) = R\mathrm{Hom}(R\Gamma_{\mathrm{et}}(X, \mathbb{Z}(d-n)) \otimes \mathbb{Q}, \mathbb{Q}[-2d-2]) = R\mathrm{Hom}(R\Gamma(X, \mathbb{Q}(d-n)), \mathbb{Q}[-2d-2])$ .

abelian group, and if its stalks at all points are finite type abelian groups. The study of Weil-étale cohomology with coefficients more general than motivic coefficients was initiated by Geisser in [Gei12], where a duality theorem for Weil-étale cohomology with coefficients in  $\mathbb{Z}$ -constructible sheaves on a curve over a finite field is proven. In his thesis [Tra15], Tran proposes a construction of Weil-étale cohomology for some  $\mathbb{Z}$ -constructible sheaves (which he calls “strongly  $\mathbb{Z}$ -constructible sheaves”) and their  $\mathbb{G}_m$ -duals on the spectrum of the ring of integers in a totally imaginary number field. He then links it, for those sheaves coming by pushforward from the generic point, to special values at  $s = 0, 1$  of Artin  $L$ -functions of rational representations. He also makes a conjecture for 1-motives and their  $L$ -function. In [Tra16], Tran refines his construction for  $\mathbb{Z}$ -constructible sheaves to arbitrary number fields (which necessitates to handle correctly the 2-torsion) and proves again a special value formula for Artin  $L$ -functions of rational representations; using the functional equation he also proves a special value formula at  $s = 1$  for  $L$ -functions of tori. In [GS20], Geisser–Suzuki study the Weil-étale cohomology on the smooth proper curve over a finite field with coefficients in Néron models of 1-motives over the function field and in  $\mathbb{Z}$ -constructible sheaves. They prove special value formulas at  $s = 0$  for  $L$ -functions of  $\mathbb{Z}$ -constructible sheaves, at  $s = 1$  for  $L$ -functions of tori, and they conjecture a Weil-étale special value formula at  $s = 1$  for the  $L$ -function of a 1-motive, which they prove to be equivalent to the Birch–Swinnerton-Dyer conjecture for its abelian part.

## I.2 Outline

This work consists in extending the above-mentioned special value formula of Flach–B. Morin (I.1.2) for  $n = 0, 1$  to  $\mathbb{Z}$ -constructible coefficients on an integral arithmetic scheme  $X$  of dimension 1, by taking inspiration from the work of Tran and Geisser–Suzuki but using the methods of Flach–B. Morin which give a more robust framework to work in.

### I.2.1 Special value at $s = 0$

Chapter II consists of the published article [Mor23], slightly reworked, and has its own more detailed introduction. The  $L$ -function of a  $\mathbb{Z}$ -constructible sheaf  $F$  on  $X$  is defined by the Euler product

$$L_X(F, s) = \prod_{x \in X(0)} \det(1 - N(x)^{-s} \text{Frob} \mid i_x^* F \otimes \mathbb{C})^{-1}$$

where  $i_x$  is the inclusion of the closed point  $x$  and  $\text{Frob}$  is the geometric Frobenius, which acts on  $i_x^* F$ . Let  $L_X^*(F, 0) = \lim_{s \rightarrow 0} L_X(F, s) s^{-\text{ord}_{s=0} L_X(F, s)}$  denote the special value at  $s = 0$ . A particular case of this  $L$ -function is the following: let  $g : \text{Spec}(K) \rightarrow X$  be the inclusion of the generic point; if  $M$  is a discrete  $\text{Gal}(K^{\text{sep}}/K)$ -module of finite type, then  $g_* M$  is a  $\mathbb{Z}$ -constructible sheaf on  $X$  and, if  $X$  is regular and proper,  $L_X(g_* M, s)$  is the Artin  $L$ -function of  $M \otimes \mathbb{C}$ , i.e.

$$L_X(g_* M, s) = L_K(M \otimes \mathbb{C}, s) := \prod_{v \text{ place of } K} \det(1 - N(v)^{-s} \text{Frob} \mid (M \otimes \mathbb{C})^{I_v})^{-1}.$$

By removing points from  $X$  we can also naturally consider partial Artin  $L$ -functions.

When  $X$  is regular, using the framework of Flach–B. Morin we define in Subsection II.3.1 a Weil-étale complex with compact support  $R\Gamma_{W,c}(X, F)$  for certain  $\mathbb{Z}$ -constructible sheaves,

the big sheaves and the tiny sheaves (Definition II.3.1), that yields a long exact cohomology sequence for “big-to-tiny” short exact sequences. Since  $n = 0$ , there is no de Rham complex, so the fundamental line is

$$\Delta_X(F) := \det_{\mathbb{Z}} R\Gamma_{W,c}(X, F).$$

We prove a duality theorem for  $\mathbb{R}$ -coefficients, Theorem II.4.4, relating compactly supported cohomology with coefficients in a  $\mathbb{Z}$ -constructible sheaf  $F$  to Ext groups from  $F$  to the multiplicative group  $\mathbb{G}_m$ , and generalizing  $\mathbf{B}(X, 0)$  to a statement for all étale sheaves on  $X$ . We deduce a trivialization of the fundamental line  $\Delta_X(F) \otimes \mathbb{R} \simeq \mathbb{R}$ . From this we obtain for any big or tiny sheaf  $F$  a Weil-étale Euler characteristic  $\chi_X(F)$ , multiplicative with respect to short exact sequences of big or tiny sheaves. As any  $\mathbb{Z}$ -constructible sheaf  $F$  fits in a short exact sequence  $0 \rightarrow G \rightarrow F \rightarrow H \rightarrow 0$  with  $G$  big and  $H$  tiny, by enforcing multiplicativity we get a Weil-étale Euler characteristic for any  $\mathbb{Z}$ -constructible sheaf, multiplicative with respect to short exact sequences. If  $X$  is not necessarily regular, it has nonetheless a quasi-finite morphism  $f : X \rightarrow Y$  to a regular integral arithmetic scheme of dimension 1 and we can put  $\chi_X(F) := \chi_Y(f_! F)$ , where  $f_!$  denotes the proper pushforward functor along  $f$ . We then show the special value formula (Theorem II.6.24)

$$L_X^*(F, 0) = \pm \chi_X(F) \tag{I.2.1}$$

by reducing to the analytic class number formula, which already had a Weil-étale interpretation by [Mor14]. The main tool for the reduction is Artin induction, which states that one essentially understands a rational representation of a finite group from the induced representations of the trivial representation along all subgroups. The Weil-étale Euler characteristic has an explicit expression in terms of étale cohomology with compact support with coefficients in  $F$  and étale Ext groups from  $F$  to the multiplicative group  $\mathbb{G}_m$  (or Deninger’s complex  $\mathbb{G}_X$  in the singular case, see [Den87a]). The Weil-étale formalism gives a way to package this already available information into *one* object with good functoriality properties. Formula (I.2.1) generalizes to arithmetic schemes the similar formula obtained by Geisser and Suzuki in [GS20] for smooth proper curves over finite fields, and improves and generalizes also the work of Tran [Tra16]. If  $F$  is constructible, (I.2.1) is a generalization of Tate’s formula for the Euler characteristic of a global field. If  $M$  is a discrete  $\text{Gal}(K^{\text{sep}}/K)$ -module of finite type and  $X$  is proper, the theorem gives a special value formula for the Artin  $L$ -function

$$L_K^*(M \otimes \mathbb{C}, 0) = \pm \chi_X(g_* M).$$

Another special case of the theorem is the following generalization to singular schemes of the analytic class number formula: if  $X$  is affine one has

$$\zeta_X^*(0) = \pm \frac{[\text{CH}_0(X)]R_X}{\omega},$$

where  $\text{CH}_0(X)$  is the 0-th Chow group,  $R_X$  is a regulator and  $\omega$  is the number of roots of unity in the function field of  $X$ . If  $X$  is a proper curve over a finite field with field of constants  $\mathbb{F}_q$ , one has

$$\zeta_X^*(0) = \pm \frac{[\text{CH}_0(X)_{\text{tor}}]R_X}{\omega \log(q)}.$$

The regulator  $R_X$  has a similar definition in both cases but in the proper curve case it is only controlled by the singularities; in particular it vanishes if the curve is regular. In

characteristic 0 it is a modification of the classical regulator that takes into account the singularities.

In Chapter III, we abstract away the formal part of the proof of Theorem II.6.24, and simplify the argument used to handle constructible sheaves using a theorem of Swan. Let  $\mathcal{A}$  be an abelian category. An Euler characteristic  $\chi$  on  $\mathcal{A}$  is a map  $\chi$  from the isomorphism classes of  $\mathcal{A}$  to an abelian group  $\Gamma$ , such that  $\chi(B) = \chi(A) + \chi(C)$  for every short exact sequence  $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ . We show the two following results:

**Theorem.** *Let  $\chi$  be an Euler characteristic on the category of  $\mathbb{Z}$ -constructible sheaves on an arithmetic scheme  $X$ . If  $\chi$  takes its values in a torsion-free abelian group then it vanishes on constructible sheaves.*

**Theorem.** *Let  $\chi, \chi'$  be two Euler characteristics on the category of  $\mathbb{Z}$ -constructible sheaves on an arithmetic scheme  $X$ , with values in a torsion-free abelian group. If  $\chi(\pi_*\mathbb{Z}) = \chi'(\pi_*\mathbb{Z})$  for  $\pi : Y \rightarrow X$  any finite morphism of arithmetic schemes with  $Y$  normal and irreducible, then  $\chi = \chi'$ .*

The method is to generalize the dévissage process already used in Chapter II, together with Artin induction and the aforementioned theorem of Swan.

## I.2.2 Special value at $s = 1$

Chapter IV consists of the article [Mor22] and has its own more detailed introduction. Denote by  $\mathbb{Z}_X^c$  Bloch's cycle complex with cohomological indexing (see § IV.1.3.2); then  $\mathbb{Z}_X^c \simeq \mathbb{Z}(1)[2]$  when  $X$  is regular. We work in characteristic 0; thus  $X = \text{Spec}(\mathcal{O})$  is the spectrum of an order in a number field  $K$ . Put

$$F^D := R\mathcal{H}om_X(F, \mathbb{Z}_X^c);$$

we see it as a generalization, contravariant in  $F$ , of  $\mathbb{Z}(1)$  (at least when  $X$  is regular, and up to a shift). Moreover, Artin–Verdier duality tells us that we can rightfully consider  $F^D$  as the dual of  $F$ . When  $F$  is  $\mathbb{Z}$ -constructible, we define the  $L$ -function of  $F^D$  on  $X$  by

$$L_X(F^D, s) := \prod_{x \in X_0} \det(1 - N(x)^{-s} \text{Frob} \mid (i_x^* F^D) \hat{\otimes} \mathbb{Q}_{\ell_x})^{-1}.$$

where  $\ell_x$  is a prime number coprime to  $\text{char}(\kappa(x))$  and for a complex of étale sheaves  $M$ , we put  $M \hat{\otimes} \mathbb{Q}_\ell := (R\lim_n (M \otimes^L \mathbb{Z}/\ell^n \mathbb{Z})) \otimes \mathbb{Q}$  computed on the proétale site. For  $F = \mathbb{Z}$  and  $X$  regular we have  $L_X(\mathbb{Z}^D, s) = \zeta_X(s + 1)$ . If  $F = i_* M$ , where  $i : x \rightarrow X$  is the inclusion of a closed point and  $M$  a finite type discrete  $\text{Gal}(k(x)^{\text{sep}}/k(x))$ -module, we have  $L_X((i_* M)^D, s) = L_X(i_*(M^\vee), s)$  where the latter is the  $L$ -function of a  $\mathbb{Z}$ -constructible sheaf as defined above and  $(-)^\vee := \mathcal{H}om(-, \mathbb{Z})$ . If  $X$  is regular,  $g : \text{Spec}(K) \rightarrow X$  denotes the inclusion of the generic point and  $M$  is a finite type discrete  $\text{Gal}(K^{\text{sep}}/K)$ -module, we have  $L_X(g_* M, s) = L_K(M \otimes \mathbb{Q}, s + 1)$ , an Artin  $L$ -function. Thus the  $L$ -function we introduce behaves generically like an Artin  $L$ -function in the variable  $s + 1$  but locally it behaves more like a local factor in the variable  $s$ . This is coherent with the behaviour of the construction  $(-)^D$ : we have  $\mathbb{Z}^D = \mathbb{Z}(1)[2]$  when  $X$  is regular, which must be related to the special value at  $s = 1$  of  $\zeta_X$ , while for the inclusion of a closed point  $i : x \rightarrow X$  we have  $(i_* \mathbb{Z})^D = i_*(\mathbb{Z}^\vee) = i_* \mathbb{Z}$ , which must be related to the special value at 0 of  $\zeta_x$ .

We first define and study cohomology with compact support with coefficients in  $F^D$  by correcting the étale cohomology of  $F^D$  by Betti cohomology at the archimedean places. The

combination of usual Artin–Verdier duality and duality for Tate cohomology of the Galois group at archimedean places gives a “twisted” Artin–Verdier duality relating compactly supported étale cohomology for  $F^D$  with the étale cohomology for  $F$ . Again following the framework of Flach–B. Morin we then define Weil–étale cohomology with compact support with coefficients in  $F^D$  for some  $\mathbb{Z}$ -constructible sheaves, the red sheaves and the blue sheaves (Definition IV.3.1). The results and methods are similar to Chapter II: in particular, we obtain a long exact Weil–étale cohomology sequence for “red-to-blue” short exact sequences. As  $n = 1$ , this time the fundamental line should have an additive part: when  $F$  is *tamely ramified* (Definition IV.4.1), we define a complex  $\mathrm{Lie}_X(F^D)$ , the “tangent complex of  $F^D$ ” which generalizes both the derived de Rham complex used in [FM18] for  $n = 1$  and in some cases the Lie algebra for the Néron model of a torus used in [GS20]; in that case, if  $F$  is moreover red or blue the fundamental line is

$$\Delta_X(F^D) = \det_{\mathbb{Z}} R\Gamma_{W,c}(X, F^D) \otimes \det_{\mathbb{Z}} \mathrm{Lie}_X(F^D).$$

We prove a duality theorem for  $\mathbb{R}$ -coefficients, relating Deligne compactly supported cohomology of  $F^D$  (meaning étale cohomology corrected at the archimedean places by Deligne cohomology, see Definition IV.5.1) to étale cohomology of  $F$  and generalizing  $\mathbf{B}(X, 1)$ . We thus obtain a trivialization  $\Delta_X(F^D) \otimes \mathbb{R} \simeq \mathbb{R}$  of the fundamental line. This yields an Euler characteristic  $\chi_X(F^D)$  for  $F$  tamely ramified and red or blue, multiplicative in red-to-blue short exact sequence. Any tamely ramified  $\mathbb{Z}$ -constructible sheaf fits in a short exact sequence between two red and blue tamely ramified sheaves, so by enforcing multiplicativity we obtain an Euler characteristic for all tamely ramified  $\mathbb{Z}$ -constructible sheaves. It has an explicit expression in terms of powers of 2,  $2\pi$ , “additive” quantities derived from  $F$  including a square-root-of-discriminant type quantity, étale cohomology groups with coefficients in  $F$  and Ext groups from  $F$  into the multiplicative group  $\mathbb{G}_m$ ; the whole expression is reminiscent of the analytic class number formula. We can then show the special value formula for tamely ramified  $\mathbb{Z}$ -constructible sheaves:

$$L_X^*(F^D, 0) = \pm \chi_X(F^D).$$

In particular, when  $M$  is a discrete  $\mathrm{Gal}(K^{\mathrm{sep}}/K)$ -module of finite type, the  $L$ -function  $L_X((g_*M)^D, s)$  equals the Artin  $L$ -function  $L(M \otimes \mathbb{C}, s + 1)$  and the theorem is

$$L^*(M \otimes \mathbb{C}, 1) = \pm \chi_X((g_*M)^D).$$

When  $F = \mathbb{Z}$  and  $X$  is regular, one recovers the analytic class number formula at  $s = 1$ , as in [FM18]. By the results of Chapter III, this is essentially sufficient to prove the result. This formula is also the analogue in the number field case of the work of Geisser–Suzuki on special value formulas for tori over function fields at  $s = 1$  [GS20]. If  $F$  is constructible tamely ramified, both  $R\Gamma_{W,c}(X, F^D)$  and  $\mathrm{Lie}_X(F^D)$  are perfect complexes with finite cohomology groups, and the theorem says that their multiplicative Euler characteristic, that is the alternating product of the order of their cohomology groups, are equal:

$$\chi^\times(R\Gamma_{W,c}(X, F^D)) = \chi^\times(\mathrm{Lie}_X(F^D)) = [F_\eta]^{[K:\mathbb{Q}]}.$$

In Chapter V, we follow a remark made by T. Suzuki to define unconditionally the additive part of the fundamental line. The observation of T. Suzuki is the following: although the Lie algebra of the Néron model of a torus is not an exact functor, its determinant still behaves multiplicatively with respect to short exact sequences of tori.

Denote by  $g : \text{Spec}(K) \rightarrow X$  the inclusion of the generic point. For  $F$  tamely ramified and torsion-free,  $\text{Lie}_X(F^D)$  coincides<sup>6</sup> with the Lie algebra of the Néron model of the torus with character group  $g^*F$  on  $\text{Spec}(K)$ ; thus in the general case we can define the additive part of the fundamental line by taking a 2-step resolution of  $g^*F$  by torsion-free discrete  $\text{Gal}(K^{\text{sep}}/K)$ -modules and taking the determinant of the corresponding complex of Lie algebras for the Néron models of the associated tori. This defines a graded  $\mathbb{Z}$ -line that is multiplicative with respect to short exact sequences, has the correct base-change to  $\mathbb{Q}$  (namely the Lie algebra of the torus with character group  $g^*F$ ), and which coincides<sup>7</sup> with  $\det_{\mathbb{Z}} \text{Lie}_X(F^D)$  in the tamely ramified case. Having an unconditional construction of the fundamental line, we deduce an unconditional definition of the Weil-étale Euler characteristic  $\chi_X(F^D)$  which allows us to extend the special value formula to all  $\mathbb{Z}$ -constructible sheaves:

$$L_X^*(F^D, 0) = \pm \chi_X(F^D).$$

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<sup>6</sup>up to a shift

<sup>7</sup>up to a shift

# Chapter II

## Special values of $L$ -functions on regular arithmetic schemes of dimension 1

### II.1 Introduction

#### II.1.1 Context

In [Lic09], Lichtenbaum conjectured the existence of a Weil-étale topology for arithmetic schemes and that the Euler characteristic of the corresponding cohomology with compact support, constructed *via* a determinant, should give special values of zeta functions at  $s = 0$ . Lichtenbaum then gave a tentative definition of the cohomology groups in the case of the spectrum  $\mathrm{Spec}(\mathcal{O}_K)$  of the ring of integers in a number field and showed that the conjecture holds if the cohomology groups vanish in degrees strictly greater than 3; unfortunately, they were shown to be non-vanishing in [Fla08].

In [Mor14] and [FM18], Flach and B. Morin used a new approach by defining a Weil-étale complex in the derived category of abelian groups, as the fiber of a morphism constructed *via* Artin-Verdier duality; they also introduced some higher regulator pairings. On the other hand, in [Tra16], Tran constructed a Weil-étale cohomology with compact support for a certain class of  $\mathbb{Z}$ -constructible sheaves on the spectrum  $U = \mathrm{Spec}(\mathcal{O}_{K,S})$  of the ring of  $S$ -integers in a number field  $K$  and used it to prove a formula for the special values at  $s = 0$  of Artin  $L$ -functions associated to rational representations of the absolute Galois group of  $K$ . Other relevant works include Chiu's thesis [Chi11], Beshenov's thesis [Bes18] and Geisser and Suzuki's recent paper [GS20].

#### II.1.2 Results

In this chapter, using the approach of Flach-B. Morin we both improve and generalize all results of [Tra16] and treat also smooth curves over finite fields as in [GS20]: we construct a Weil-étale complex for a larger class of  $\mathbb{Z}$ -constructible sheaves, and deduce an Euler characteristic defined for all  $\mathbb{Z}$ -constructible sheaves and with good functoriality properties. This enables us to prove a formula for the special value of the  $L$ -function of a  $\mathbb{Z}$ -constructible sheaf at  $s = 0$ . We deduce a special value formula for Artin  $L$ -functions of rational representations of the Galois group of a global field  $K$ ; as a special case, we obtain a special value formula for the arithmetic zeta function of  $X$ , which generalizes to the singular case the analytic class formula and its function field variant.

Let  $U$  be a 1-dimensional regular irreducible scheme of finite type over  $\mathbb{Z}$ , with function field  $K$ , and denote by  $S$  the set of missing places of  $U$  (including archimedean ones). We

recall in Section II.2 the compactly supported cohomology  $R\Gamma_c(U, -)$  (resp.  $R\hat{\Gamma}_c(U, -)$ ), defined as in [Mil06] by correcting étale cohomology with Galois cohomology (resp. Tate cohomology) at the places in  $S$ ; we write  $H_c^i(U, -)$  (resp.  $\hat{H}_c^i(U, -)$ ) for the corresponding cohomology groups. A  $\mathbb{Z}$ -constructible sheaf  $F$  on  $U$  will be called big if  $\text{Ext}_U^1(F, \mathbb{G}_m)$  is finite and tiny if  $H_c^1(U, F)$  is finite; a morphism  $F \rightarrow G$  is big-to-tiny if  $F$  and  $G$  are big,  $F$  is big and  $G$  is tiny or both  $F$  and  $G$  are tiny. A short exact sequence  $0 \rightarrow F \rightarrow G \rightarrow H \rightarrow 0$  will be called big-to-tiny if all morphisms are big-to-tiny. We prove in Subsections II.3.3 and II.6.3:

**Theorem II.A.** *For every big or tiny sheaf  $F$  on  $U$ , there exists a Weil-étale complex with compact support  $R\Gamma_{W,c}(U, F)$ , well-defined up to unique isomorphism. It sits in a distinguished triangle*

$$R\text{Hom}(R\text{Hom}_U(F, \mathbb{G}_m), \mathbb{Q}[-3]) \rightarrow R\Gamma_c(U, F) \rightarrow R\Gamma_{W,c}(U, F) \rightarrow .$$

*It is a perfect complex, functorial in big-to-tiny morphisms, and it yields a long exact cohomology sequence for big-to-tiny short exact sequences. If  $\pi : U' \rightarrow U$  is the normalization in a finite Galois extension of  $K$  and  $j : V \subset U$  is an open immersion, we have canonical isomorphisms  $R\Gamma_{W,c}(U, \pi_*F) \simeq R\Gamma_{W,c}(U', F)$  and  $R\Gamma_{W,c}(U, j_!F) \simeq R\Gamma_{W,c}(V, F)$ .*

This distinguished triangle was inspired by [Mor14]: for  $U$  proper and  $F = \mathbb{Z}$  this is exactly the distinguished triangle in B. Morin's article. Unfortunately, the defects of the derived 1-category prevented us from obtaining a well-defined complex for every  $\mathbb{Z}$ -constructible sheaf, and we only got a long exact cohomology sequence instead of a distinguished triangle.

Next, following [FM18], we construct in Section II.4 a regulator pairing, for a complex  $F$  in the left bounded derived category of étale sheaves  $D^+(U_{\text{ét}})$ , that generalizes the logarithmic embedding of the Dirichlet  $S$ -units theorem. We show that this pairing is perfect after base change to  $\mathbb{R}$ :

**Theorem II.B.** *Let  $F$  be a  $\mathbb{Z}$ -constructible sheaf on  $U$ . There is a regulator pairing*

$$R\Gamma_c(U, F) \otimes R\text{Hom}_U(F, \mathbb{G}_m) \rightarrow \mathbb{R}[-1],$$

*functorial in  $F$ . The induced map after base change to  $\mathbb{R}$*

$$R\text{Hom}_U(F, \mathbb{G}_m)_{\mathbb{R}} \rightarrow R\text{Hom}(R\Gamma_c(U, F), \mathbb{R}[-1])$$

*is a natural isomorphism for  $F \in D^+(U_{\text{ét}})$ .*

The method we use to prove the theorem was inspired by [Tra16]; it relies on dévissage of  $\mathbb{Z}$ -constructible sheaves *via* Artin induction at the generic point. This method was also used in [GS20] to treat the case of curves over a finite field.

We show in Section II.5 that the perfectness of the regulator pairing gives a canonical trivialization of the  $\mathbb{R}$ -determinant of the Weil-étale complex

$$\lambda : \det_{\mathbb{R}}(R\Gamma_{W,c}(U, F) \otimes \mathbb{R}) \xrightarrow{\simeq} \mathbb{R}.$$

The  $\mathbb{Z}$ -determinant  $\det_{\mathbb{Z}}(R\Gamma_{W,c}(U, F))$  is a lattice in the 1-dimensional  $\mathbb{R}$ -vector space  $\det_{\mathbb{R}}(R\Gamma_{W,c}(U, F) \otimes \mathbb{R})$  and we define in Section II.6:



**Definition.** • Let  $F$  be a big or tiny sheaf on  $U$ . The (Weil-étale) Euler characteristic of  $F$  is the positive real number  $\chi_U(F)$  such that

$$\lambda(\det_{\mathbb{Z}}(R\Gamma_{W,c}(U, F))) = \frac{1}{\chi_U(F)} \mathbb{Z} \hookrightarrow \mathbb{R}.$$

We denote by  $E_U(F)$  the secondary Euler characteristic

$$E_U(F) := \sum_i (-1)^i \cdot i \cdot \dim_{\mathbb{R}}(H_{W,c}^i(U, F) \otimes \mathbb{R}),$$

where  $H_{W,c}^i(U, F)$  denotes hypercohomology of  $R\Gamma_{W,c}(U, F)$ .

- Let  $F$  be a  $\mathbb{Z}$ -constructible sheaf on  $U$ . There exists a short exact sequence  $0 \rightarrow F' \rightarrow F \rightarrow F'' \rightarrow 0$  with  $F'$  big and  $F''$  tiny; define

$$\chi_U(F) := \chi_U(F')\chi_U(F'').$$

It does not depend on the chosen sequence. Proceed similarly to define  $E_U(F)$ .

The point of this construction, and its main advantage over the construction of Tran, is the good functoriality properties of the Euler characteristic which are inherited from the functoriality properties of the Weil-étale complex:

**Proposition** (II.6.19, II.6.20, II.6.21). *If  $0 \rightarrow F \rightarrow G \rightarrow H \rightarrow 0$  is a short exact sequence of  $\mathbb{Z}$ -constructible sheaves on  $U$ , then*

$$\chi_U(G) = \chi_U(F)\chi_U(H).$$

Moreover, if  $\pi : U' \rightarrow U$  is the normalization in a finite Galois extension of  $K$  (resp.  $j : V \rightarrow U$  is an open subscheme) and  $F$  is a  $\mathbb{Z}$ -constructible sheaf on  $U'$  (resp.  $V$ ), we have  $\chi_U(\pi_* F) = \chi_{U'}(F)$  (resp.  $\chi_U(j_! F) = \chi_V(F)$ ).

For a  $\mathbb{Z}$ -constructible sheaf  $F$  on  $U$ , define the  $L$ -function associated to  $F$  by the usual Euler product with local factors given by the action of the geometric<sup>1</sup> Frobenius on  $F_v \otimes \mathbb{C}$ , where  $F_v$  is the pullback of  $F$  at a closed point  $v \in U$ . Denote by  $n = \text{ord}_{s=0} L_U(F, s)$  the vanishing order of  $L_U(F, s)$  at  $s = 0$  and

$$L_U^*(F, 0) = \lim_{s \rightarrow 0} L_U(F, s) s^{-n}$$

the special value of  $L_U(F, s)$  at  $s = 0$ . Our main theorem then is:

**Theorem II.C** (Special value formula). *Let  $F$  be a  $\mathbb{Z}$ -constructible sheaf on a 1-dimensional regular irreducible scheme  $U$  of finite type over  $\text{Spec}(\mathbb{Z})$ . Then*

$$\begin{aligned} \text{ord}_{s=0} L_U(F, s) &= \sum_i (-1)^i \cdot i \cdot \dim_{\mathbb{R}}(H_{W,c}^i(U, F) \otimes \mathbb{R}), \\ L_U^*(F, 0) &= \pm \chi_U(F). \end{aligned}$$

---

<sup>1</sup>We explain the choice of the geometric Frobenius and its impact on our results in the remark after Theorem II.6.24

We prove the theorem in Section II.6. Again the proof is by dévissage, using the functoriality properties of the Euler characteristic. This theorem shows in particular that the Weil-étale Euler characteristic of a constructible sheaf is 1. Suppose that  $U$  is either  $\mathrm{Spec}(\mathcal{O}_{K,S})$  or an open subscheme of the smooth proper curve with function field  $K$ , with closed complement  $S$ . Then for  $x \in U_0$  we have  $(g_*M)_x = M^{I_x}$ , where  $I_x$  is an inertia subgroup for the place  $x$ , and thus  $L_U(g_*M, s)$  is the Artin  $L$ -function of the rational  $G_K$ -representation  $M \otimes \mathbb{Q}$ , modulo the local factors at  $S$ .

Let  $X$  be a 1-dimensional irreducible scheme of finite type over  $\mathbb{Z}$ , with function field  $K$ . Theorem II.C immediately implies:

**Corollary II.D.** *Let  $f : X \rightarrow T$  denote either a quasi-finite morphism  $X \rightarrow T := \mathbb{P}_{\mathbb{F}_p}^1$  (in the function field case), which exists by the projective Noether normalization lemma, or the quasi-finite structural morphism  $X \rightarrow T := \mathrm{Spec}(\mathbb{Z})$  (in the number field case); then  $f_!$  is well-defined. Let  $F$  be a  $\mathbb{Z}$ -constructible sheaf on  $X$ . We have  $\mathrm{ord}_{s=0} L_X(F, s) = E_T(f_!F)$  and*

$$L_X^*(F, 0) = \pm \chi_T(f_!F).$$

The Weil-étale Euler characteristic is computed explicitly and doesn't depend on  $f$ ; this is done through the use of Deninger's complex  $\mathbb{G}_X$  from [Den87a], which identifies with a shift of Bloch's cycle complex  $\mathbb{Z}_X^c(0)$ .

We deduce from this a special value formula for the arithmetic zeta function of  $X$ :

**Corollary II.E.** *Suppose  $X$  is affine. We have  $\mathrm{ord}_{s=0} \zeta_X = \mathrm{rank}_{\mathbb{Z}}(\mathrm{CH}_0(X, 1))$  and*

$$\zeta_X^*(0) = -\frac{[\mathrm{CH}_0(X)]R_X}{\omega}.$$

**Corollary II.F.** *Suppose  $X$  is a proper curve, and denote by  $\mathbb{F}_q$  the field of constants of  $K$ . We have  $\mathrm{ord}_{s=0} \zeta_X = \mathrm{rank}_{\mathbb{Z}}(\mathrm{CH}_0(X, 1)) - 1$  and*

$$\zeta_X^*(0) = -\frac{[\mathrm{CH}_0(X)_{\mathrm{tor}}]R_X}{\omega \log(q)}.$$

Here  $\omega$  is the number of roots of unity in the field of functions,  $R_X$  is a regulator and  $\mathrm{CH}_0(X, i)$  denotes higher Chow groups. Those two corollaries give a generalization of the classical special value formula of the zeta function of a global field at  $s = 0$ . The result seems to be new; we note that a similar formula at  $s = 1$  for  $X$  furthermore affine and reduced was established by Jordan–Poonen in [JP20].

### II.1.3 Review of the article of Tran and comparison with his results

In [Tra16], Tran shows that there should be a generalization of Weil-étale cohomology to  $\mathbb{Z}$ -constructible sheaves on an open subscheme  $U$  of the spectrum of the ring of integers in a number field  $K$ . Tran defines a Weil-étale complex for every  $\mathbb{Z}$ -constructible sheaf  $F$  on  $U$ :

$$R\Gamma_W(U, F) = \tau^{\leq 1} R\hat{\Gamma}_c(U, F) \oplus \tau^{\geq 2} R\mathrm{Hom}(R\mathrm{Hom}_U(F, \mathbb{G}_m), \mathbb{Z}[-2]).$$

For a place  $v$  of  $K$ , let  $K_v$  denote the completion of  $K$  at  $v$  if  $v$  is archimedean and the henselization at  $v$  otherwise, and denote by  $F_{\eta_v}$  the pullback of  $F$  to  $\mathrm{Spec}(K_v)$ . Via the

canonical map  $R\Gamma_W(U, F) \rightarrow R\hat{\Gamma}_c(U, F) \rightarrow \prod_{v \in S} R\hat{\Gamma}(K_v, F_{\eta_v})$ , Tran then constructs the complex

$$D_F = \text{Cone}\left(R\Gamma_W(U, F) \rightarrow \prod_{v \in S} R\hat{\Gamma}(K_v, F_{\eta_v})\right)[-1].$$

Tran defines the concept of strongly  $\mathbb{Z}$ -constructible sheaf as a  $\mathbb{Z}$ -constructible sheaf for which  $H_c^0(U, F)$ ,  $H^1(U, F)$  and  $\text{Ext}_U^1(F, \mathbb{G}_m)$  are finite and the pairing  $\langle \cdot, \cdot \rangle_F: \prod_{v \in S} H^0(K_v, F_{\eta_v})/H^0(U, F) \times \text{Hom}_U(F, \mathbb{G}_m) \rightarrow \mathbb{R}$  is perfect modulo torsion. Everything is then done under the hypothesis that the considered sheaf is strongly  $\mathbb{Z}$ -constructible.

For such a sheaf  $F$ , Tran shows that there is an isomorphism  $\Theta(F) : H^1(D_F) \otimes \mathbb{R} \rightarrow H^2(D_F) \otimes \mathbb{R}$  and defines

$$\chi_{\text{Tran}, U}(F) = \prod_{i=0}^5 [H^i(D_F)_{\text{tor}}] |\det(\Theta(F))| \prod_{v \text{ real}} h(K_v, F_{\eta_v})^3,$$

where  $|\det(\Theta(F))|$  is computed in bases modulo torsion of  $H^1(D_F)$  and  $H^2(D_F)$ , and for an archimedean place  $v$  the Herbrand quotient of the  $\text{Gal}(\mathbb{C}/K_v)$ -module  $F_{\eta_v}$  is denoted by  $h(K_v, F_{\eta_v})$ . Tran then gives an explicit formula ([Tra16, Theorem 4.5]) and this Euler characteristic is proven to be multiplicative for short exact sequences  $0 \rightarrow F \rightarrow G \rightarrow H \rightarrow 0$  if either  $K$  is totally imaginary or  $H^1(U, G) = 0$  and  $H^1(K_v, G_{\eta_v}) = 0$  for all  $v \in S$  ([Tra16, Prop. 4.17, 4.18]). After some explicit computations of the Euler characteristic, Tran's main result is

**Theorem** ([Tra16, Prop. 3.11, Theorem 5.4]). *Let  $M$  be a torsion-free discrete  $G_K$ -module of finite type and  $g : \text{Spec}(K) \rightarrow U$  the inclusion of the generic point. Then  $g_*M$  is strongly  $\mathbb{Z}$ -constructible,  $\text{ord}_{s=0} L_S(M, s) = \text{rank}_{\mathbb{Z}} \text{Hom}_U(F, \mathbb{G}_m)$  and*

$$L_S^*(M, 0) = \pm \chi_{\text{Tran}, U}(g_*M).$$

This is shown using Artin induction on the generic point to reduce to the case of zeta functions. The idea of using Artin induction is crucial in this and also in our work, and to our knowledge Tran was the first to use it. Finally, such a  $G_K$ -module  $M$  is the character group of a torus  $T$  over  $K$ , and using the functional equation of Artin  $L$ -functions tran gives formulas for both the special value at  $s = 0$  and  $s = 1$  of  $L_S(M \otimes \mathbb{Q}, s)$  in terms of invariants of  $T^2$

Let us now describe our improvements:

- The class of strongly  $\mathbb{Z}$ -constructible sheaves is a proper subset of the class of big sheaves. Sheaves of the form  $f_*\mathbb{Z}$ , where  $f : X \rightarrow U$  is finite and  $X$  is singular, irreducible and not unibranch, are big but never strongly  $\mathbb{Z}$ -constructible: in this case,  $H^1(U, f_*\mathbb{Z})$  is infinite and Tran's pairing  $\langle \cdot, \cdot \rangle_{f_*\mathbb{Z}}$  is not perfect modulo torsion.
- We are able to handle sheaves of the form  $i_*M$  where  $i : v \rightarrow U$  is the inclusion of a closed point and  $M$  is a finite type, not necessarily finite, discrete  $\text{Gal}(\kappa(v)^{\text{sep}}/\kappa(v))$ -module, as those are tiny.
- Our Weil-étale complex with compact support gives a long exact cohomology sequence for big-to-tiny short exact sequences. Tran proves that there is a long exact cohomology sequence for his Weil-étale complex only for short exact sequences of big sheaves.

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<sup>2</sup>Since  $M \otimes \mathbb{Q}$  is a rational representation, it is self-dual so the functional equation relates  $L_K(M \otimes \mathbb{Q}, s)$  with  $L_K(M \otimes \mathbb{Q}, 1 - s)$ .

- The pairing  $\langle \cdot, \cdot \rangle_F$  of Tran is corrected and generalized to our regulator pairing, and we show in theorem II.B that the latter is always perfect and defined unconditionally on the left bounded derived category. This is what enables us to do away with some of the restrictions imposed to strongly  $\mathbb{Z}$ -constructible sheaves and to generalize to big sheaves.
- Our Weil-étale complex with compact support is a perfect complex cohomologically concentrated in degree  $[0, 2d + 1]$  (where  $d = 1$  is the dimension) and its tensor product with  $\mathbb{R}$  has the correct secondary Euler characteristic: it gives the order of vanishing at  $s = 0$  of the  $L$ -function. Those properties are in adequation with the conjectural framework of Flach–B. Morin.<sup>3</sup> On the other hand, the complex of Tran is unbounded; in particular, Tran’s complex is not quasi-isomorphic to our Weil-étale complex.
- Our Euler characteristic is given *on the nose* by the Weil-étale complex with compact support and fits in the conjectural framework of Flach–B. Morin, as the covolume of the determinant of Weil-étale cohomology inside its tensor product with  $\mathbb{R}$ , *via* the trivialization induced by the regulator pairing. There is no need for correcting factors.
- We obtain a multiplicative Euler characteristic defined for all  $\mathbb{Z}$ -constructible sheaves. This enables us to have a general special value formula. Tran proves the multiplicativity only under some restrictions and thus only has a special value formula for  $L$ -functions of integral  $G_K$ -representations. In this situation our characteristics obviously coincide since they both give the special value.
- Our results were initially in characteristic zero only, but the generalization to characteristic  $p$  was straightforward once the formalism was in place. However, all of our results in characteristic  $p$  have already been shown in [GS20], although the formalism there is slightly different; in particular, their trivialization of the determinant of the Weil-étale complex tensored with  $\mathbb{R}$  is done with respect to the base field  $\mathbb{F}_q$  and comes from the generator  $e \in H_{\mathbb{W}}^1(\mathbb{F}_q, \mathbb{Z}) = \text{Hom}(W_{\mathbb{F}_q}, \mathbb{Z}) = \mathbb{Z}$  corresponding to the Frobenius. This explains the presence of a factor  $\log(q)$  in their formulas. Our trivialization is done *via* the regulator pairing, so the factor  $\log(q)$  is integrated in the regulator part of the Euler characteristic. See the second remark after Definition II.6.1. We note that the framework of Geisser and Suzuki only works in characteristic  $p$ , while our framework works in both zero and non-zero characteristic.

In summary, our Euler characteristic is defined as the determinant of the full Weil-étale complex, with no correction factor involved. It follows that our method works in complete generality.

## II.2 Étale cohomology with compact support

We will be working with the derived  $\infty$ -category of abelian groups  $\mathcal{D}(\mathbb{Z})$ ; this is a stable  $\infty$ -category whose homotopy category is the ordinary derived category  $D(\mathbb{Z})$ .<sup>4</sup> Moreover, the projective model structure on unbounded chain complexes of abelian groups is a symmetric monoidal model structure, hence by [Lur17, 4.1.7.6] the underlying  $\infty$ -category

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<sup>3</sup>For an explanation of the presence of this secondary characteristic using Deninger’s conjectural cohomology  $H_{\text{dyn},c}^*(-)$ , see [FM20, Proposition 4.7]

<sup>4</sup>For the theory of derived and stable  $\infty$ -categories, see [Lur17, chap. 1]

$\mathcal{D}(\mathbb{Z})$  is closed symmetric monoidal (see also [NS18, A.7]), with inner hom  $R\mathrm{Hom}(-, -)$  and tensor product  $- \otimes^L -$ , computed as usual. If  $\mathcal{C}$  is an  $\infty$ -category, then the  $\infty$ -category of functors  $\mathrm{Fun}(\mathcal{C}, \mathcal{D}(\mathbb{Z}))$  is again a stable  $\infty$ -category. We denote by  $\mathrm{fib}$ ,  $\mathrm{cofib}$  and  $[1]$  the fiber, cofiber and shift functors.

Let  $U$  be a regular irreducible scheme of finite type over  $\mathbb{Z}$  and of dimension 1. We denote by  $K$  its function field. It is a global field; either  $K$  is a number field and  $U = \mathrm{Spec}(\mathcal{O}_{K,S})$  is the spectrum of the ring of  $S$ -integers for  $S$  a finite set of places of  $K$  including the set  $S_\infty$  of all archimedean places, or  $K$  is a function field of characteristic  $p > 0$  and  $U$  is an open subscheme of the unique connected smooth proper curve  $C$  with function field  $K$ , in which case we denote by  $S$  the set of missing points<sup>5</sup>. If  $S \neq \emptyset$  then  $U$  is affine [Goo69].

We recall the Milne étale cohomology with compact support from [Mil06]. For each place  $v$  of  $K$ , let  $K_v$  be the completion of  $K$  at  $v$  if  $v$  is archimedean and the field of fraction of the henselization  $\mathcal{O}_v^h$  otherwise. We will denote pullback to  $\eta_v := \mathrm{Spec}(K_v)$  (via the composite  $\eta_v \rightarrow \mathrm{Spec}(K) \rightarrow U$ ) by  $(-)_\eta_v$ . Let  $R\Gamma(K_v, -)$  denote the étale cohomology on  $\mathrm{Spec}(K_v)$ . We have a natural morphism

$$R\Gamma(U, -) \rightarrow R\Gamma(K_v, (-)_\eta_v).$$

Now let  $R\hat{\Gamma}(K_v, -)$  denote the Tate group cohomology of  $\mathrm{Gal}(K_v^{\mathrm{sep}}/K_v)$  when  $v$  is archimedean, and ordinary cohomology of profinite groups otherwise. We have a natural morphism

$$R\Gamma(K_v, -) \rightarrow R\hat{\Gamma}(K_v, -)$$

for any place  $v$ . We then define

- The (ordinary) étale complex with compact support

$$R\Gamma_c(U, -) := \mathrm{fib}(R\Gamma(U, -) \rightarrow \prod_{v \in S} R\Gamma(K_v, (-)_\eta_v)).$$

- The (Tate) étale complex with compact support

$$R\hat{\Gamma}_c(U, -) := \mathrm{fib}(R\Gamma(U, -) \rightarrow \prod_{v \in S} R\hat{\Gamma}(K_v, (-)_\eta_v)).$$

where we take the fiber in the  $\infty$ -category  $\mathrm{Fun}(\mathrm{Sh}(U_{\mathrm{et}}), \mathcal{D}(\mathbb{Z}))$  of functors from the category of abelian sheaves on  $U_{\mathrm{et}}$  to the derived  $\infty$ -category of abelian groups  $\mathcal{D}(\mathbb{Z})$ . The corresponding cohomology groups are denoted by  $H_c^i(U, -)$  and  $\hat{H}_c^i(U, -)$ .

*Remark.*

- Only the Tate étale complex with compact support is introduced in [Mil06] but we will use both crucially.
- If  $j : U \rightarrow C$  is an open subscheme of the proper curve  $C$ ,  $R\Gamma_c(U, -)$  is the usual cohomology with compact support  $R\Gamma(C, j_!(-))$  (see Proposition II.2.1.d) and  $R\Gamma_c(U, -) = R\hat{\Gamma}_c(U, -)$ .

The various functoriality properties of the étale complex with compact support are investigated in [Mil06]. Let us recall them (for instance for the ordinary complex with compact support) :

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<sup>5</sup>see [JP20, 3.1]

**Proposition II.2.1** ([Mil06, II.2.3]).

a. For any sheaf  $F$  on  $U$ , there is a long exact sequence

$$\cdots \rightarrow H_c^i(U, F) \rightarrow H^i(U, F) \rightarrow \prod_{v \in S} H^i(K_v, F_{\eta_v}) \rightarrow H_c^{i+1}(U, F) \rightarrow \cdots \quad (\text{II.2.1})$$

b. A short exact sequence  $0 \rightarrow F \rightarrow G \rightarrow H$  on  $U$  gives a fiber sequence

$$R\Gamma_c(U, F) \rightarrow R\Gamma_c(U, G) \rightarrow R\Gamma_c(U, H).$$

c. For any closed immersion  $i : Z \hookrightarrow U$  and sheaf  $F$  on  $Z$ , we have a natural isomorphism

$$R\Gamma_c(U, i_*(-)) \simeq R\Gamma(Z, -).$$

d. For any open immersion  $j : V \rightarrow U$ , we have a natural isomorphism

$$R\Gamma_c(U, j_!(-)) \simeq R\Gamma_c(V, -).$$

Thus for any sheaf  $F$  on  $U$ , there is a fiber sequence

$$R\Gamma_c(V, F|_V) \rightarrow R\Gamma_c(U, F) \rightarrow \bigoplus_{v \in U \setminus V} R\Gamma(v, i_v^* F),$$

where  $i_v : v \rightarrow U$  is the inclusion of a closed point.

e. For any finite map  $\pi : U' \rightarrow U$  with  $U'$  regular irreducible, we have a natural isomorphism

$$R\Gamma_c(U, \pi_*(-)) \simeq R\Gamma_c(U', -).$$

*Proof.* See [Mil06]. We have to say something about b. ; it follows from the following lemma.  $\square$

**Lemma II.2.2** ( $3 \times 3$  Lemma). Let  $\mathcal{C}$  be a stable  $\infty$ -category and let  $A \rightarrow B \rightarrow C$ ,  $A'' \rightarrow B'' \rightarrow C''$  be two fiber sequences in  $\mathcal{C}$ . Consider a commutative diagram

$$\begin{array}{ccccc} A & \longrightarrow & B & \longrightarrow & C \\ \downarrow & & \downarrow & & \downarrow \\ A'' & \longrightarrow & B'' & \longrightarrow & C'' \end{array}$$

Let  $A', B', C'$  denote the respective fibers of  $A \rightarrow A''$ , etc. Then there is a commutative diagram

$$\begin{array}{ccccc} A' & \longrightarrow & B' & \longrightarrow & C' \\ \downarrow & & \downarrow & & \downarrow \\ A & \longrightarrow & B & \longrightarrow & C \\ \downarrow & & \downarrow & & \downarrow \\ A'' & \longrightarrow & B'' & \longrightarrow & C'' \end{array}$$

and  $A' \rightarrow B' \rightarrow C'$  is a fiber sequence.

*Proof.* By [Lur17, 1.1.1.8], the fiber functor  $\text{fib}$  preserves all limits, hence preserves fiber sequences.  $\square$

Recall that a  $\mathbb{Z}$ -constructible sheaf on  $U$  is an abelian sheaf  $F$  on  $U_{\text{et}}$  such that  $U$  can be written as the union of locally closed subschemes  $Z_i$ , with  $F|_{Z_i}$  locally constant associated to an abelian group of finite type. Equivalently, because  $U$  is of dimension 1, this means that there is an open subscheme  $V \subset U$  with closed complement  $Z$ , such that  $F|_V$  is locally constant associated to an abelian group of finite type, and on the other hand for every closed point  $v \in Z$ , the pullback  $F_v$  is a discrete  $G_v := \text{Gal}(\kappa(v)^{\text{sep}}/\kappa(v))$ -module of finite type, where  $\kappa(v)$  is the residue field at  $v$ .

**II.2.0.1 Dévissage for  $\mathbb{Z}$ -constructible sheaves** Let us discuss the dévissage argument we will often use ; this methodology comes from [Tra16] and was used in [GS20]. Let  $F$  be a  $\mathbb{Z}$ -constructible sheaf. There is a short exact sequence

$$0 \rightarrow F_{\text{tor}} \rightarrow F \rightarrow F/F_{\text{tor}} \rightarrow 0$$

where  $F_{\text{tor}}$  is torsion hence constructible, and  $F/F_{\text{tor}}$  is torsion free. We will often be able to handle constructible sheaves ; let us thus suppose that  $F$  is torsion-free. By Artin induction ([Swa60, 4.1,4.4]), there exists for the pullback  $F_\eta$  to the generic point a short exact sequence of the form

$$0 \rightarrow (F_\eta)^n \oplus \bigoplus \text{ind}_{G_K}^{G_{L_i}} \mathbb{Z} \rightarrow \bigoplus \text{ind}_{G_K}^{G_{L'_j}} \mathbb{Z} \rightarrow N \rightarrow 0$$

where  $n$  is an integer,  $N$  is a finite  $G_K$ -module and  $L_i, L'_j$  are a finite number of finite Galois extensions of  $K$ . By spreading out, there exists a dense open subscheme  $V \subset U$ , finite normalization morphisms of  $V$  in the above finite Galois extensions of  $K$  of the form  $\pi_i : W_i \rightarrow V$ ,  $\pi'_j : W'_j \rightarrow V$  and a constructible sheaf  $G$  on  $V$  such that there is a short exact sequence

$$0 \rightarrow (F|_V)^n \oplus \bigoplus (\pi_i)_* \mathbb{Z} \rightarrow \bigoplus (\pi'_j)_* \mathbb{Z} \rightarrow G \rightarrow 0.$$

Finally, for  $Z$  the closed complement of  $V$ , there is a short exact sequence

$$0 \rightarrow j_!(F|_V) \rightarrow F \rightarrow i_* i^* F \rightarrow 0$$

where  $j : V \hookrightarrow U$  is the open inclusion) and  $i_* i^* F = \bigoplus_{v \in Z} (i_v)_* i_v^* F$  (with  $i : Z \hookrightarrow U$  resp.  $i_v : v \hookrightarrow U$  the closed inclusion). We can again use Artin induction on each point of  $Z$ .

With this dévissage, general arguments about  $\mathbb{Z}$ -constructible sheaves that are functorial with respect to the operations  $j_!$ ,  $i_*$ ,  $\pi_*$  mentioned above reduce to statements about constructible sheaves and the constant sheaf  $\mathbb{Z}$ , on  $U$  and on a closed point. Once we have handled the case of a point, the short exact sequence  $0 \rightarrow j_! \mathbb{Z} \rightarrow \mathbb{Z} \rightarrow i_* \mathbb{Z} \rightarrow 0$  will enable us to further suppose that  $U$  is either affine or proper, depending on which situation we prefer.

**II.2.0.2 The comparison complex** Let  $v$  be an archimedean place of  $K$ . The complex  $R\Gamma(K_v, -)$  is represented by the complex of homogeneous cochains  $C^*(K_v, -)$ , obtained by taking Hom from the standard free resolution of  $\mathbb{Z}$  into a given module. Moreover, the complex  $R\hat{\Gamma}(K_v, -)$  is represented by the complex  $\hat{C}^*(K_v, -)$  obtained by taking Hom from the complete standard resolution of  $\mathbb{Z}$ . There is a natural arrow  $C^*(K_v, -) \rightarrow \hat{C}^*(K_v, -)$ . Let us introduce the ‘‘homology complex’’  $C_*(K_v, -)$ , a complex in negative degree obtained by tensoring a module with the standard free resolution of  $\mathbb{Z}$ . The norm map  $N : M \rightarrow M$  for any discrete  $G_{K_v}$ -module  $M$  induces an arrow  $C_*(K_v, -) \rightarrow C^*(K_v, -)$ , and the sequence

$$C_*(K_v, -) \rightarrow C^*(K_v, -) \rightarrow \hat{C}^*(K_v, -)$$

is a fiber sequence. Let us introduce a “comparison complex”

$$T := \text{fib} \left( \prod_{v \in S_\infty} R\Gamma(K_v, (-)_{\eta_v}) \rightarrow \prod_{v \in S_\infty} R\hat{\Gamma}(K_v, (-)_{\eta_v}) \right) = \prod_{v \in S_\infty} C_*(K_v, (-)_{\eta_v}).$$

Thus  $T$  computes homology at the archimedean places.

Applying the  $3 \times 3$  lemma (Lemma II.2.2), we find a commutative diagram of fiber sequences

$$\begin{array}{ccccc} \text{fib}(R\Gamma_c(U, -) \rightarrow R\hat{\Gamma}_c(U, -)) & \longrightarrow & 0 & \longrightarrow & T \\ \downarrow & & \downarrow & & \downarrow \\ R\Gamma_c(U, -) & \longrightarrow & R\Gamma(U, -) & \longrightarrow & \prod_{v \in S} R\Gamma(K_v, (-)_{\eta_v}) \\ \downarrow & & \downarrow = & & \downarrow \\ R\hat{\Gamma}_c(U, -) & \longrightarrow & R\Gamma(U, -) & \longrightarrow & \prod_{v \in S} R\hat{\Gamma}(K_v, (-)_{\eta_v}) \end{array}$$

We deduce an isomorphism  $\text{fib} \left( R\Gamma_c(U, -) \rightarrow R\hat{\Gamma}_c(U, -) \right) = T[-1]$ , that is

$$\text{cofib} \left( R\Gamma_c(U, -) \rightarrow R\hat{\Gamma}_c(U, -) \right) = T.$$

We have

$$H^i(T(F)) = \begin{cases} \prod_{v \in S_\infty} \hat{H}^{i-1}(K_v, F_{\eta_v}), & i < 0, \\ \prod_{v \in S_\infty} H_0(K_v, F_{\eta_v}), & i = 0, \\ 0, & i > 0. \end{cases} \quad (\text{II.2.2})$$

**Proposition II.2.3.** *Let  $F$  be a  $\mathbb{Z}$ -constructible sheaf on  $U$ . Then  $H_c^i(U, F) = \hat{H}_c^i(U, F)$  for  $i \geq 2$  and  $H_c^1(U, F) \rightarrow \hat{H}_c^1(U, F)$  is surjective.*

*Proof.* The long exact sequence of cohomology associated to the fiber sequence  $R\Gamma_c(U, F) \rightarrow R\hat{\Gamma}_c(U, F) \rightarrow T(F)$  gives the claim.  $\square$

**Proposition II.2.4.** *Let  $F$  be a  $\mathbb{Z}$ -constructible sheaf on  $U$ . Then  $H_c^i(U, F)$  and  $\hat{H}_c^i(U, F)$  are of finite type for  $i = 0, 1$ .*

*Proof.* The cofiber  $T(F)$  of  $R\Gamma_c(U, F) \rightarrow R\hat{\Gamma}_c(U, F)$  computes homology at the archimedean places, thus its cohomology groups are finite type. We reduce to showing the statement only for  $\hat{H}_c^i(U, F)$ . By the dévissage argument (§ II.2.0.1) and the functoriality properties from Proposition II.2.1, we have to treat three cases: the constructible case, the case of the constant sheaf  $\mathbb{Z}$ , and the case of Galois cohomology of a discrete  $G_v$ -module of finite type for  $v$  a closed point.

- The first is part of the statement of Artin–Verdier duality (see Theorem II.3.2),
- The second is a computation using the long exact sequence Equation (II.2.1): we have  $H^0(U, \mathbb{Z}) = \mathbb{Z}$  and  $H^1(U, \mathbb{Z}) = \text{Hom}_{\text{cont}}(\pi_1^{\text{SGA}3}(U), \mathbb{Z}) = 0$  because  $U$  is normal and noetherian hence the étale fundamental group  $\pi_1^{\text{SGA}3}(U) = \pi_1^{\text{SGA}1}(U)$  is profinite. Then Equation (II.2.1) gives an exact sequence

$$0 \rightarrow \hat{H}_c^0(U, \mathbb{Z}) \rightarrow \mathbb{Z} \rightarrow \bigoplus_{v \in S \setminus S_\infty} \mathbb{Z} \oplus \bigoplus_{v \text{ real}} \mathbb{Z}/2\mathbb{Z} \rightarrow \hat{H}_c^1(U, \mathbb{Z}) \rightarrow 0.$$



- Let  $G$  be a profinite group and  $M$  a discrete  $G$ -module of finite type (as an abelian group). It is clear that  $H^0(G, M) = M^G$  is also of finite type. There exists an open subgroup  $H$  such that  $H$  acts trivially on  $G$ ; then the inflation-restriction exact sequence reduces us to verifying that  $H^1(G, M)$  is of finite type for  $M$  a discrete  $G$ -module of finite type with trivial action. Here we have  $G = G_v = \hat{\mathbb{Z}}$ ; but  $H^1(\hat{\mathbb{Z}}, \mathbb{Z}) = \text{Hom}_{\text{cont}}(\hat{\mathbb{Z}}, \mathbb{Z}) = 0$  and  $H^1(\hat{\mathbb{Z}}, \mathbb{Z}/n\mathbb{Z}) = \text{Hom}_{\text{cont}}(\hat{\mathbb{Z}}, \mathbb{Z}/n\mathbb{Z}) \simeq \mathbb{Z}/n\mathbb{Z}$  so we are done.

□

## II.3 The Weil-étale complex with compact support

### II.3.1 Construction of the Weil-étale complex with compact support

**Definition II.3.1.** *Let  $F$  be a  $\mathbb{Z}$ -constructible sheaf on  $U$ . We say that*

- $F$  is a big sheaf if  $\text{Ext}_U^1(F, \mathbb{G}_m)$  is torsion (hence finite) <sup>6</sup>
- $F$  is a tiny sheaf if  $H_c^1(U, F)$  is torsion (hence finite)
- A big-to-tiny morphism is a morphism of sheaves  $F \rightarrow G$  where either  $F$  and  $G$  are both big, or both tiny, or  $F$  is big and  $G$  is tiny; a big-to-tiny short exact sequence is a short exact sequence with big-to-tiny morphisms.

*Remark.*

- A constructible sheaf is both big and tiny (see Theorem II.3.2)
- A constant sheaf defined by a finite type abelian group is big, except when  $U = C$  is a proper curve. Indeed, this reduces to the case of  $\mathbb{Z}$ , and  $\text{Ext}_U^1(\mathbb{Z}, \mathbb{G}_m) = H^1(U, \mathbb{G}_m) = \text{Pic}(U)$  is finite when  $U$  is affine<sup>7</sup>
- Let  $F$  be a locally constant sheaf defined by a finite type abelian group. Let  $\pi : V \rightarrow U$  be a finite Galois cover with group  $G$  trivializing  $F$ ; the low degree exact sequence of the spectral sequence

$$H^p(G, \text{Ext}_V^q(F|_V, \mathbb{G}_m)) \Rightarrow \text{Ext}_U^{p+q}(F, \mathbb{G}_m)$$

shows from the constant case that  $F$  is big, except when  $U = C$ .

- A  $\mathbb{Z}$ -constructible sheaf supported on a finite closed subscheme is tiny; indeed, if  $i : v \hookrightarrow U$  is the inclusion of a closed point, we have that  $H_c^1(U, i_*M) = H^1(G_v, M)$  is finite by the argument given in the proof of Proposition II.2.4.
- On  $U = C$ , a locally constant sheaf defined by a finite type abelian group is tiny. This reduces to the case of  $\mathbb{Z}$ , and we have  $H_c^1(C, \mathbb{Z}) = H^1(C, \mathbb{Z}) = \text{Hom}_{\text{cont}}(\pi_1(U), \mathbb{Z}) = 0$ .
- By Theorem II.4.4,  $\text{Ext}_U^1(F, \mathbb{G}_m)$  is finite if and only if  $H_c^0(U, F)$  is finite, and similarly  $H_c^1(U, F)$  is finite if and only if  $\text{Hom}_U(F, \mathbb{G}_m)$  is finite.

<sup>6</sup>As noted in the introduction, these are a generalization of the strongly  $\mathbb{Z}$ -constructible sheaves of [Tra16] in the number field case

<sup>7</sup>This is well-known : see for instance [Mor89, 2.3]

Let  $F$  be a big or tiny sheaf. We construct in this section a complex  $R\Gamma_{W,c}(U, F)$  in the derived category of the integers  $D(\mathbb{Z})$ , exact and functorial in big-to-tiny morphisms. We encountered some difficulty and failed to define it for all  $\mathbb{Z}$ -constructible sheaves, but this will be sufficient to obtain a well-defined Euler characteristic. Following the approach of [Mor14] and [FM18], the complex fits in a distinguished triangle

$$R\mathrm{Hom}(R\mathrm{Hom}_U(F, \mathbb{G}_m), \mathbb{Q}[-3]) \rightarrow R\Gamma_c(U, F) \rightarrow R\Gamma_{W,c}(U, F) \rightarrow$$

where the map  $R\mathrm{Hom}(R\mathrm{Hom}_U(F, \mathbb{G}_m), \mathbb{Q}[-3]) \rightarrow R\Gamma_c(U, F)$  is lifted from the canonical map  $R\mathrm{Hom}(R\mathrm{Hom}_U(F, \mathbb{G}_m), \mathbb{Q}[-3]) \rightarrow R\mathrm{Hom}(R\mathrm{Hom}_U(F, \mathbb{G}_m), \mathbb{Q}/\mathbb{Z}[-3])$  via Artin–Verdier duality.

*Remark.* The point is to use the short exact sequence  $0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Q} \rightarrow \mathbb{Q}/\mathbb{Z} \rightarrow 0$  to replace the factors  $\mathbb{Q}/\mathbb{Z}$  appearing in the cohomology groups of  $R\Gamma_c(U, F)$  by factors  $\mathbb{Z}$ : Artin–Verdier duality (which we’ll state below) says exactly that the factors  $\mathbb{Q}/\mathbb{Z}$  are of the form  $\mathrm{Hom}_{\mathrm{Ab}}(A, \mathbb{Q}/\mathbb{Z})$  for  $A$  going through the factors  $\mathbb{Z}$  of the cohomology groups of  $R\mathrm{Hom}_U(F, \mathbb{G}_m)$ . Thus the kernel and cokernel of a well-constructed map

$$\mathrm{Hom}_{\mathrm{Ab}}(\mathrm{Ext}_U^{3-i}(F, \mathbb{G}_m), \mathbb{Q}) \rightarrow \mathrm{Hom}_{\mathrm{Ab}}(\mathrm{Ext}_U^{3-i}(F, \mathbb{G}_m), \mathbb{Q}/\mathbb{Z}) \rightarrow H_c^i(U, F)$$

should give finite type abelian groups. Our construction will be a derived version of this observation.

Put

$$\begin{aligned} D &:= R\mathrm{Hom}(R\mathrm{Hom}_U(-, \mathbb{G}_m), \mathbb{Q}[-3]), \\ E &:= R\mathrm{Hom}(R\mathrm{Hom}_U(-, \mathbb{G}_m), \mathbb{Q}/\mathbb{Z}[-3]). \end{aligned}$$

Artin–Verdier duality gives a pairing  $R\hat{\Gamma}_c(U, -) \otimes R\mathrm{Hom}_U(-, \mathbb{G}_m) \rightarrow \mathbb{Q}/\mathbb{Z}[-3]$  that induces an “Artin–Verdier” arrow

$$R\hat{\Gamma}_c(U, -) \rightarrow E.$$

We have  $H^i(D) = \mathrm{Ext}_U^{3-i}(-, \mathbb{G}_m)^\dagger$  and  $H^i(E) = \mathrm{Ext}_U^{3-i}(-, \mathbb{G}_m)^D$ , where  $(-)^\dagger := \mathrm{Hom}(-, \mathbb{Q})$  and  $(-)^D := \mathrm{Hom}(-, \mathbb{Q}/\mathbb{Z})$ . The cohomology groups of the Artin–Verdier arrow are described as following:

**Theorem II.3.2** (Artin–Verdier duality, [Mil06, II.3.1]). *Let  $F$  be a  $\mathbb{Z}$ -constructible sheaf on  $U$ . The groups  $\mathrm{Ext}_U^i(F, \mathbb{G}_m)$  and  $\hat{H}_c^i(U, F)$  are finite for  $i < 0$ ,  $i > 4$ , of finite type for  $i = 0, 1$  and torsion of cofinite type for  $i = 2, 3$ . Moreover  $\hat{H}_c^i(U, F) = 0$  for  $i \geq 4$ . The Artin–Verdier arrow induces an isomorphism for  $i = 0, 1$*

$$\hat{H}_c^i(U, F)^\wedge \xrightarrow{\cong} \mathrm{Ext}_U^{3-i}(F, \mathbb{G}_m)^D$$

where  $(-)^\wedge$  is the profinite completion with respect to subgroups of finite index, and for  $i \neq 0, 1$  an isomorphism

$$\hat{H}_c^i(U, F) \xrightarrow{\cong} \mathrm{Ext}_U^{3-i}(F, \mathbb{G}_m)^D.$$

If  $F$  is constructible, all groups are finite and the Artin–Verdier arrow is an isomorphism.

We recall that an abelian group is torsion of cofinite type if it is of the form  $(\mathbb{Q}/\mathbb{Z})^n \oplus A$  where  $A$  is finite and  $n$  an integer.

The theorem implies in particular that  $D$  is cohomologically concentrated in degree 2 and 3.

*Remark.* In [Mil06], the theorem is expressed in its dual form ; we obtain the one above by applying  $(-)^* := \text{Hom}_{\text{cont}}(-, \mathbb{R}/\mathbb{Z})$ , which coincides with  $\text{Hom}_{\text{cont}}(-, \mathbb{Q}/\mathbb{Z})$  for profinite groups (where  $\mathbb{Q}/\mathbb{Z}$  has the discrete topology) and with  $(-)^D$  for discrete torsion groups. We use furthermore that for  $M$  discrete of finite type,  $(M^\wedge)^* = M^*$  and the Pontryagin duality  $M = M^{**}$  for any abelian locally compact  $M$ .

The fact that  $\hat{H}_c^4(U, F) = 0$  does not appear in [Mil06] but is easily deduced by the dévissage argument (§ II.2.0.1) from the functoriality properties of  $\hat{H}_c^i(U, -)$ , the case of  $\mathbb{Z}$ , which is [Mil06, II.2.11(d)], the case of constructible sheaves, which is [Mil06, II.3.12(a)], and the case of Galois cohomology of  $\hat{\mathbb{Z}}$ , which follows from the strict cohomological dimension of  $\hat{\mathbb{Z}}$  being 2 (see for instance [Ser68, XIII.1]).

We want to construct an arrow  $\beta_F$  in  $D(\mathbb{Z})$  making the following diagram commute

$$\begin{array}{ccc} D_F & \xrightarrow{\beta_F} & R\Gamma_c(U, F) \longrightarrow R\hat{\Gamma}_c(U, F) \\ & \searrow \pi & \downarrow \text{Artin-Verdier} \\ & & E_F \end{array}$$

Let us analyze  $\text{Hom}_{D(\mathbb{Z})}(D_F, R\Gamma_c(U, G))$  for  $F$  big or  $G$  tiny. There is a spectral sequence for  $\text{Hom}_{D(\mathbb{Z})}(K, L)$  ([Ver96, III.4.6.10]) of the form

$$E_2^{p,q} = \prod_{i \in \mathbb{Z}} \text{Ext}^p(H^i(K), H^{i+q}(L)) \Rightarrow H^{p+q}(R \text{Hom}(K, L)) = \text{Hom}(K, L[p+q]) = E^{p+q}$$

For abelian groups, we have  $\text{Ext}_{\mathbb{Z}}^i = 0$  for  $i \neq 0, 1$  so the sequence degenerates and gives a short exact sequence

$$0 \rightarrow \prod_i \text{Ext}^1(H^i(K), H^{i-1}(L)) \rightarrow \text{Hom}_{D(\mathbb{Z})}(K, L) \rightarrow \prod_i \text{Hom}(H^i(K), H^i(L)) \rightarrow 0. \quad (\text{II.3.1})$$

In our case, this gives

$$\begin{aligned} 0 &\rightarrow \text{Ext}^1(\text{Ext}_U^1(F, \mathbb{G}_m)^\dagger, H_c^1(U, G)) \times \text{Ext}^1(\text{Hom}_U(F, \mathbb{G}_m)^\dagger, H_c^2(U, G)) \\ &\rightarrow \text{Hom}_{D(\mathbb{Z})}(D_F, R\Gamma_c(U, G)) \rightarrow \prod_i \text{Hom}(H^i(D_F), H^i(R\Gamma_c(U, G))) \rightarrow 0 \end{aligned}$$

where  $(-)^\dagger := \text{Hom}(-, \mathbb{Q})$ . Since  $H_c^2(U, -)$  is torsion of cofinite type and either  $\text{Ext}_U^1(F, \mathbb{G}_m)$  or  $H_c^1(U, G)$  are finite, the two terms in the product on the left vanish and we have an isomorphism

$$\text{Hom}_{D(\mathbb{Z})}(D_F, R\Gamma_c(U, G)) \xrightarrow{\cong} \text{Hom}(H^2(D_F), H_c^2(U, G)) \times \text{Hom}(H^3(D_F), H_c^3(U, G))$$

because  $D_F$  is cohomologically concentrated in degree 2 and 3. The Artin-Verdier arrow is an isomorphism in degree 2 and 3, and  $\hat{H}_c^i(U, F) = H_c^i(U, F)$  for  $i \geq 2$ , so the arrow  $\beta_F$  exists and is unique: it is given by the composition

$$\text{Ext}_U^{3-i}(F, \mathbb{G}_m)^\dagger \rightarrow \text{Ext}_U^{3-i}(F, \mathbb{G}_m)^D \xleftarrow{\cong} \hat{H}_c^i(U, F) \xleftarrow{\cong} H_c^i(U, F)$$

for  $i = 2, 3$ . Moreover a big-to-tiny morphism  $f : F \rightarrow G$  gives rise to a commutative square

$$\begin{array}{ccc} D_F & \xrightarrow{\beta_F} & R\Gamma_c(U, F) \\ \downarrow f_* & & \downarrow f_* \\ D_G & \xrightarrow{\beta_G} & R\Gamma_c(U, G) \end{array}$$

Indeed by the above isomorphism it suffices to check commutativity on cohomology groups in degree 2 and 3.

**Definition II.3.3.** *Let  $F$  be a big or tiny sheaf on  $U$ . We define*

$$R\Gamma_{W,c}(U, F) := \text{Cone}(\beta_F),$$

*the Weil-étale cohomology with compact support with coefficients in  $F$ .*

### II.3.2 Computing the Weil-étale cohomology with compact support

**Definition II.3.4.** *Let  $F$  be a big or tiny sheaf on  $U$ . We denote by  $H_{W,c}^i(U, F) := H^i(R\Gamma_{W,c}(U, F))$  the Weil-étale cohomology groups with compact support.*

The complex  $D_F$  is cohomologically concentrated in degree 2 and 3, so the long exact cohomology sequence gives  $H_{W,c}^i(U, F) = H_c^i(U, F)$  for  $i \leq 0$ ,  $i \geq 4$ . In particular  $H_{W,c}^i(U, F) = 0$  for  $i \notin [0, 3]$ . The remaining long exact sequence reads

$$\begin{aligned} 0 &\longrightarrow H_c^1(U, F) \longrightarrow H_{W,c}^1(U, F) \\ &\longrightarrow \text{Ext}^1(F, \mathbb{G}_m)^\dagger \xrightarrow{\text{nat}} H_c^2(U, F) \longrightarrow H_{W,c}^2(U, F) \\ &\longrightarrow \text{Hom}(F, \mathbb{G}_m)^\dagger \xrightarrow{\text{nat}} H_c^3(U, F) \longrightarrow H_{W,c}^3(U, F) \longrightarrow 0. \end{aligned}$$

From the construction, we see that the natural arrows  $\text{Ext}_U^{3-i}(F, \mathbb{G}_m)^\dagger \rightarrow H_c^i(U, F) \simeq \hat{H}_c^i(U, F) \simeq \text{Ext}_U^{3-i}(F, \mathbb{G}_m)^D$  for  $i = 2, 3$  identify with the post-composition with the projection  $\mathbb{Q} \rightarrow \mathbb{Q}/\mathbb{Z}$ . We thus get short exact sequences and an isomorphism

$$\begin{aligned} 0 &\rightarrow H_c^1(U, F) \rightarrow H_{W,c}^1(U, F) \rightarrow \text{Hom}(\text{Ext}_U^1(F, \mathbb{G}_m), \mathbb{Z}) \rightarrow 0, \\ 0 &\rightarrow (\text{Ext}_U^1(F, \mathbb{G}_m)_{\text{tor}})^D \rightarrow H_{W,c}^2(U, F) \rightarrow \text{Hom}(\text{Hom}_U(F, \mathbb{G}_m), \mathbb{Z}) \rightarrow 0, \\ &H_{W,c}^3(U, F) \simeq (\text{Hom}_U(F, \mathbb{G}_m)_{\text{tor}})^D. \end{aligned}$$

In particular, all cohomology groups are of finite type, hence  $R\Gamma_{W,c}(U, F)$  is a perfect complex.

### II.3.3 Functoriality of the Weil-étale complex

**Theorem II.3.5.** *Let  $F$  be either a big or tiny sheaf on  $U$ . The complex  $R\Gamma_{W,c}(U, F)$  is well-defined up to unique isomorphism, is functorial in big-to-tiny morphisms, and gives a long exact cohomology sequence for big-to-tiny short exact sequences.*

*Proof.* Let us fix a choice of a cone of  $\beta_F$  for every big or tiny sheaf  $F$ . Let  $f : F \rightarrow G$  be a big-to-tiny morphism. We consider the beginning of a morphism distinguished triangle

$$\begin{array}{ccccccc} D_F & \xrightarrow{\beta_F} & R\Gamma_c(U, F) & \longrightarrow & R\Gamma_{W,c}(U, F) & \longrightarrow & D_F[1] \\ \downarrow f_* & & \downarrow f_* & & & & \downarrow f_* \\ D_G & \xrightarrow{\beta_G} & R\Gamma_c(U, G) & \longrightarrow & R\Gamma_{W,c}(U, G) & \longrightarrow & D_G[1] \end{array}$$

The groups  $H^i(D_F)$  are  $\mathbb{Q}$ -vector spaces and zero for  $i \neq 2, 3$  and the Weil-étale cohomology groups with compact support are of finite type and zero for  $i \neq 0, 1, 2, 3$ . The exact sequence Equation (II.3.1) for  $\mathrm{Hom}_{D(\mathbb{Z})}$  gives

$$\begin{aligned} 0 \rightarrow \prod_i \mathrm{Ext}^1(H^i(D_F), H_{W,c}^i(U, F)) &\rightarrow \mathrm{Hom}_{D(\mathbb{Z})}(D_F[1], R\Gamma_{W,c}(U, F)) \\ &\rightarrow \prod_i \mathrm{Hom}(H^{i+1}(D_F), H_{W,c}^i(U, F)) \rightarrow 0, \end{aligned}$$

$$\begin{aligned} 0 \rightarrow \prod_i \mathrm{Ext}^1(H_{W,c}^i(U, F), H_{W,c}^{i-1}(U, G)) &\rightarrow \mathrm{Hom}_{D(\mathbb{Z})}(R\Gamma_{W,c}(U, F), R\Gamma_{W,c}(U, G)) \\ &\rightarrow \prod_i \mathrm{Hom}(H_{W,c}^i(U, F), H_{W,c}^i(U, G)) \rightarrow 0, \end{aligned}$$

so that the abelian group  $\mathrm{Hom}_{D(\mathbb{Z})}(D_F[1], R\Gamma_{W,c}(U, G))$  is uniquely divisible and the abelian group  $\mathrm{Hom}_{D(\mathbb{Z})}(R\Gamma_{W,c}(U, F), R\Gamma_{W,c}(U, G))$  is of finite type. In the long exact sequence obtained by applying  $\mathrm{Hom}_{D(\mathbb{Z})}(-, R\Gamma_{W,c}(U, G))$  to the upper triangle, the image of  $\mathrm{Hom}_{D(\mathbb{Z})}(D_F[1], R\Gamma_{W,c}(U, G))$  in  $\mathrm{Hom}_{D(\mathbb{Z})}(R\Gamma_{W,c}(U, F), R\Gamma_{W,c}(U, G))$  must be uniquely divisible and of finite type, hence is trivial and we get an injection

$$0 \rightarrow \mathrm{Hom}_{D(\mathbb{Z})}(R\Gamma_{W,c}(U, F), R\Gamma_{W,c}(U, G)) \rightarrow \mathrm{Hom}_{D(\mathbb{Z})}(R\Gamma_c(U, F), R\Gamma_{W,c}(U, G))$$

Therefore a morphism completing the morphism of triangles, which exists by axiom, is uniquely determined by the requirement that it makes the square

$$\begin{array}{ccc} R\Gamma_c(U, F) & \longrightarrow & R\Gamma_{W,c}(U, F) \\ \downarrow f_* & & \downarrow \\ R\Gamma_c(U, G) & \longrightarrow & R\Gamma_{W,c}(U, G) \end{array}$$

commute. Thus there is a unique morphism  $f_* : R\Gamma_{W,c}(U, F) \rightarrow R\Gamma_{W,c}(U, G)$  completing the diagram of distinguished triangles above, which gives at the same time the uniqueness and desired functoriality of  $R\Gamma_{W,c}(U, -)$  (for the uniqueness, consider the case of two different choices of cone of  $\beta_F$  and  $\mathrm{id} : F \rightarrow F$ ).

Let us show the exactness. Let  $0 \rightarrow F \rightarrow G \rightarrow H \rightarrow 0$  be a big-to-tiny short exact sequence. We obtain a commutative diagram, where the two left columns and all rows are distinguished triangles.

$$\begin{array}{ccccccc} D_F & \xrightarrow{\beta_F} & R\Gamma_c(U, F) & \longrightarrow & R\Gamma_{W,c}(U, F) & \longrightarrow & D_F[1] \\ \downarrow & & \downarrow & & \downarrow \exists! & & \downarrow \\ D_G & \xrightarrow{\beta_G} & R\Gamma_c(U, G) & \longrightarrow & R\Gamma_{W,c}(U, G) & \longrightarrow & D_G[1] \\ \downarrow & & \downarrow & & \downarrow \exists! & & \downarrow \\ D_H & \xrightarrow{\beta_H} & R\Gamma_c(U, H) & \longrightarrow & R\Gamma_{W,c}(U, H) & \longrightarrow & D_H[1] \\ \downarrow & & \downarrow & & \downarrow \exists! & & \downarrow \\ D_F[1] & \xrightarrow{\beta_F[1]} & R\Gamma_c(U, F)[1] & \longrightarrow & R\Gamma_{W,c}(U, F)[1] & \longrightarrow & D_F[2] \end{array} \tag{II.3.2}$$

Actually we still have to construct the expected arrow  $R\Gamma_{W,c}(U, H) \rightarrow R\Gamma_{W,c}(U, F)[1]$ . We reason as before: the exact sequence Equation (II.3.1) gives

$$\begin{aligned} 0 \rightarrow \mathrm{Ext}^1(\mathrm{Ext}_U^1(H, \mathbb{G}_m)^\dagger, H_c^2(U, F)) \times \mathrm{Ext}^1(\mathrm{Hom}_U(H, \mathbb{G}_m)^\dagger, H_c^3(U, F)) \\ \rightarrow \mathrm{Hom}_{D(\mathbb{Z})}(D_H, R\Gamma_c(U, F)[1]) \rightarrow \prod_i \mathrm{Hom}(H^i(D_H), H_c^{i+1}(U, F)) \rightarrow 0. \end{aligned}$$

Since  $H_c^i(U, F) = \hat{H}_c^i(U, F)$  is torsion of cofinite type for  $i \geq 2$  and zero for  $i \geq 4$ , we readily see that the left term is 0, hence a canonical isomorphism

$$\mathrm{Hom}_{D(\mathbb{Z})}(D_H, R\Gamma_c(U, F)[1]) \xrightarrow[\cong]{H^2} \mathrm{Hom}(H^2(D_H), H_c^3(U, F)).$$

Thus in diagram Equation (II.3.2), we can check that the bottom left square commutes on cohomology groups in degree 2 ; but then the corresponding square is

$$\begin{array}{ccc} \mathrm{Ext}_U^1(H, \mathbb{G}_m)^\dagger & \longrightarrow & H_c^2(U, H) = \mathrm{Ext}_U^1(H, \mathbb{G}_m)^D \\ \downarrow & & \downarrow \\ \mathrm{Hom}_U(F, \mathbb{G}_m)^\dagger & \longrightarrow & H_c^3(U, F) = \mathrm{Hom}_U(H, \mathbb{G}_m)^D \end{array}$$

which clearly commutes. Now the existence and uniqueness of the wanted arrow follows by the same methods as for the others.

Let us now take the cohomology groups in the third column; this gives a long sequence

$$\cdots \rightarrow H_{W,c}^i(U, F) \rightarrow H_{W,c}^i(U, G) \rightarrow H_{W,c}^i(U, H) \rightarrow H_{W,c}^{i+1}(U, F) \rightarrow \cdots$$

and we have to check that it is exact. We will use results from [Stacks, Tag 0BKH] regarding derived completions. We have  $D_F \otimes^L \mathbb{Z}/n\mathbb{Z} = 0$ . This gives an isomorphism

$$R\Gamma_c(U, F) \otimes^L \mathbb{Z}/n\mathbb{Z} \simeq R\Gamma_{W,c}(U, F) \otimes^L \mathbb{Z}/n\mathbb{Z}$$

hence by taking derived limits ([Stacks, Tag 0922]) an isomorphism

$$R\Gamma_c(U, F)_p^\wedge \simeq R\Gamma_{W,c}(U, F)_p^\wedge \simeq R\Gamma_{W,c}(U, F) \otimes \mathbb{Z}_p$$

between the derived  $p$ -completions ; the last isomorphism holds because  $R\Gamma_{W,c}(U, F)$  is a perfect complex and  $\mathbb{Z}_p$  is flat ([Stacks, Tag 0EEV]). Derived  $p$ -completion is an exact functor ([Stacks, Tag 091V]) so the functor  $R\Gamma_c(U, -)_p^\wedge$  is exact. Now we can just check that our long sequence is exact after tensoring with  $\mathbb{Z}_p$  for each prime  $p$ , which follows from the exactness of  $R\Gamma_{W,c}(U, -) \otimes \mathbb{Z}_p$ .  $\square$

## II.4 The regulator pairing

### II.4.1 Construction

We denote by  $\otimes$  the derived tensor product in  $\mathcal{D}(\mathbb{Z})$  and  $A_{\mathbb{R}} := A \otimes \mathbb{R}$  for any abelian group or complex of abelian groups.

Taking inspiration from [FM18, 2.2], we define in this section a pairing on the left bounded derived  $\infty$ -category  $\mathcal{D}^+(U_{\mathrm{et}})$  of étale sheaves on  $U$

$$R\Gamma_c(U, -) \otimes R\mathrm{Hom}_U(-, \mathbb{G}_m) \rightarrow \mathbb{R}[-1]$$

with values in the derived  $\infty$ -category  $\mathcal{D}(\mathbb{Z})$ , and we show that it is perfect after base change to  $\mathbb{R}$ . The pairing is functorial, meaning that the induced morphism

$$R\mathrm{Hom}_U(-, \mathbb{G}_m) \rightarrow R\mathrm{Hom}(R\Gamma_c(U, -), \mathbb{R}[-1])$$

is a natural transformation. When  $U$  is a curve over  $\mathbb{F}_q$ , we will be able to define the pairing with value in  $\mathbb{Q}[-1]^8$ , and its base change to  $\mathbb{Q}$  will be perfect.

The derived  $\infty$ -category  $\mathcal{D}(U_{\mathrm{et}})$  is naturally enriched in the symmetric monoidal  $\infty$ -category  $\mathcal{D}(\mathbb{Z})$  via the complexes  $R\mathrm{Hom}_U(-, -)$ ; hence for  $F, G \in \mathcal{D}(U_{\mathrm{et}})$  we have the Ext pairings<sup>9</sup>

$$R\Gamma(U, F) \otimes R\mathrm{Hom}_U(F, G) \rightarrow R\Gamma(U, G),$$

$$\prod_{v \in S} R\Gamma(K_v, F_{\eta_v}) \otimes R\mathrm{Hom}_U(F, G) \rightarrow \prod_{v \in S} R\Gamma(K_v, F_{\eta_v}) \otimes \prod_{v \in S} R\mathrm{Hom}_{K_v}(F_{\eta_v}, G_{\eta_v}) \rightarrow \prod_{v \in S} R\Gamma(K_v, G_{\eta_v}).$$

Since the derived tensor product is exact, we have a commutative diagram

$$\begin{array}{ccccc} R\Gamma_c(U, F) \otimes R\mathrm{Hom}_U(F, G) & \rightarrow & R\Gamma(U, F) \otimes R\mathrm{Hom}_U(F, G) & \rightarrow & \left( \prod_{v \in S} R\Gamma(K_v, F_{\eta_v}) \right) \otimes R\mathrm{Hom}_U(F, G) \\ & & \downarrow & & \downarrow \\ R\Gamma_c(U, G) & \longrightarrow & R\Gamma(U, G) & \longrightarrow & \prod_{v \in S} R\Gamma(K_v, G_{\eta_v}) \end{array}$$

where both rows are fiber sequences, hence an induced morphism

$$R\Gamma_c(U, F) \otimes R\mathrm{Hom}_U(F, G) \rightarrow R\Gamma_c(U, G).$$

Specialise for  $G = \mathbb{G}_m[0]$ ; we now wish to construct a map  $R\Gamma_c(U, \mathbb{G}_m) \rightarrow \mathbb{R}[-1]$ , resp. a factorization through  $\mathbb{Q}[-1]$  of that map when  $U$  is a curve. We begin by computing the cohomology groups  $H_c^i(U, \mathbb{G}_m)$ .

**Definition II.4.1.** *Define*

$$I'_K = \{(\alpha_v) \in \prod_{v \text{ place de } K} K_v^\times, \alpha_v \in \mathcal{O}_{K_v}^\times \text{ for almost all } v\},$$

*the henselian idele group. Define the henselian idele class group and henselian  $S$ -idele class group*

$$\begin{aligned} C'_K &= I'_K / K^\times, \\ C'_{K,S} &= \mathrm{coker}\left(\prod_{v \in U_0} \mathcal{O}_{K_v}^\times \rightarrow I'_K \rightarrow C'_K\right), \end{aligned}$$

where  $U_0$  denotes the closed points of  $U$ .

*Remark.*

- These differ from the usual definitions because we use for  $K_v$  the fraction field of the henselization at  $v$  when  $v$  is non-archimedean.
- By the snake Lemma we have  $C'_{K,S} \simeq (\bigoplus_{v \in U_0} \mathbb{Z} \oplus \prod_{v \in S} K_v^\times) / K^\times$ . It follows that  $C'_{K,S} = \mathrm{Pic}(C)$  when  $U = C$  is a proper curve.

<sup>8</sup>Though it will depend on  $q$ ; the  $\mathbb{R}$ -valued regulator doesn't depend on  $q$

<sup>9</sup>coming from the Yoneda pairing, noting that  $R\Gamma(U, F) = R\mathrm{Hom}_U(\mathbb{Z}, F)$

**Proposition II.4.2** (“Hilbert 90”). *We have  $H_c^0(U, \mathbb{G}_m) = 0$ , resp.  $H_c^0(U, \mathbb{G}_m)$  is finite when  $U$  is affine resp.  $U = C$  is a proper curve, and  $H_c^1(U, \mathbb{G}_m) = C'_{K,S}$ . For  $i \geq 2$ ,*

$$H_c^i(U, \mathbb{G}_m) = \hat{H}_c^i(U, \mathbb{G}_m) = \begin{cases} \mathbb{Q}/\mathbb{Z}, & i = 3, \\ 0, & i > 3. \end{cases}$$

*Proof.* For  $i = 0$  this comes from the defining long exact sequence

$$0 \rightarrow H_c^0(U, \mathbb{G}_m) \rightarrow \mathcal{O}_U(U)^\times \rightarrow \prod_{v \in S} K_v^\times \rightarrow \dots$$

and for  $U = C$  a proper regular curve of characteristic  $p > 0$ ,  $\mathcal{O}_C(C)$  is a finite extension of  $\mathbb{F}_p$ . We already saw in Proposition II.2.3 that  $H_c^i(U, -) = \hat{H}_c^i(U, -)$  for  $i > 1$ , thus for  $i > 1$  see [Mil06, III.2.6].

Let us treat the case  $i = 1$ . Define a complex  $R\Gamma_{\text{Zar},c}$  similarly to  $R\Gamma_c$ , but using Zariski cohomology on  $U$  and  $\text{Spec}(K_v)$  instead of étale cohomology. Let  $\varepsilon$  be the morphism of sites  $U_{\text{Zar}} \rightarrow U_{\text{et}}$ . We denote by  $\varepsilon_* : F \mapsto F \circ \varepsilon$  the direct image functor and  $\varepsilon^*$  its left adjoint. We have  $\varepsilon_*(\mathbb{G}_m)_{U_{\text{et}}} = (\mathbb{G}_m)_{U_{\text{Zar}}}$ , thus  $\tau^{\leq 0} R\varepsilon_* \mathbb{G}_m = \mathbb{G}_m$  and we get a morphism

$$R\Gamma_{\text{Zar}}(U, \mathbb{G}_m) = R\Gamma_{\text{Zar}}(U, \tau^{\leq 0} R\varepsilon_* \mathbb{G}_m) \rightarrow R\Gamma_{\text{Zar}}(U, R\varepsilon_* \mathbb{G}_m) = R\Gamma(U, \mathbb{G}_m).$$

This combined with the similar result on  $\text{Spec}(K_v)$  induce a canonical morphism

$$R\Gamma_{\text{Zar},c}(U, \mathbb{G}_m) \rightarrow R\Gamma_c(U, \mathbb{G}_m).$$

This induces a commutative diagram in cohomology

$$\begin{array}{ccccccc} 0 & \rightarrow & \mathcal{O}_U(U)^\times & \rightarrow & \prod_{v \in S} K_v^\times & \rightarrow & H_{\text{Zar},c}^1(U, \mathbb{G}_m) \rightarrow H_{\text{Zar}}^1(U, \mathbb{G}_m) = \text{Pic}(U) \rightarrow 0 \\ & & \parallel & & \parallel & & \downarrow & & \parallel \\ 0 & \rightarrow & \mathcal{O}_U(U)^\times & \rightarrow & \prod_{v \in S} K_v^\times & \rightarrow & H_c^1(U, \mathbb{G}_m) \rightarrow H^1(U, \mathbb{G}_m) = \text{Pic}(U) \rightarrow 0 \end{array}$$

where the identification on the right follows from Hilbert’s theorem 90. Thus by the five lemma

$$H_{\text{Zar},c}^1(U, \mathbb{G}_m) \xrightarrow{\sim} H_c^1(U, \mathbb{G}_m).$$

On the Zariski site of  $U$ , we have a short exact sequence

$$0 \rightarrow \mathbb{G}_m \rightarrow g_* \mathbb{G}_m \rightarrow \bigoplus_{v \in U_0} (i_v)_* \mathbb{Z} \rightarrow 0,$$

where  $g : \text{Spec}(K) \hookrightarrow U$  is the inclusion of the generic point and  $i_v$  is the inclusion of a closed point  $v \in U$ . This gives a distinguished triangle

$$R\Gamma_{\text{Zar},c}(U, \mathbb{G}_m) \rightarrow R\Gamma_{\text{Zar},c}(U, g_* \mathbb{G}_m) \rightarrow \bigoplus_{v \in U_0} \mathbb{Z} \rightarrow$$

For  $q > 0$ , the sheaf  $R^q g_* F$  is the sheaf associated to

$$V \mapsto H_{\text{Zar}}^q(V \times_U \text{Spec}(K), F) = H_{\text{Zar}}^q(\text{Spec}(K), F) = 0,$$

hence  $g_* = Rg_*$  is exact and

$$R\Gamma_{\text{Zar}}(U, g_* \mathbb{G}_m) = R\Gamma_{\text{Zar}}(\text{Spec}(K), \mathbb{G}_m) = K^\times[0].$$



Thus there is an identification

$$R\Gamma_{\text{Zar},c}(U, g_*\mathbb{G}_m) \simeq \text{fib} \left( K^\times[0] \rightarrow \prod_{v \in S} K_v^\times[0] \right) = \left[ K^\times \rightarrow \prod_{v \in S} K_v^\times \right],$$

where  $K^\times$  is in degree zero. We find

$$R\Gamma_{\text{Zar},c}(U, \mathbb{G}_m) \simeq \text{fib} \left( \left[ K^\times \rightarrow \prod_{v \in S} K_v^\times \right] \rightarrow \bigoplus_{v \in U_0} \mathbb{Z}[0] \right) = \left[ K^\times \rightarrow \bigoplus_{v \in U_0} \mathbb{Z} \oplus \prod_{v \in S} K_v^\times \right],$$

hence the result.  $\square$

The map  $\sum_v \log |\cdot|_v$  is well-defined on the henselian idele group  $I'_K$  and by the product formula induces a map on the henselian  $S$ -idele class group  $tr : C'_{K,S} \rightarrow \mathbb{R}$ . For  $v \notin S$ , we have in particular on the factor  $\mathbb{Z}$  associated to  $v$ :

$$tr(1_v) = -\log(N(v)), \quad (\text{II.4.1})$$

where  $N(v) = [\kappa(v)]$  is the cardinality of the residue field at  $v$ ; this follows from the conventions chosen for the product formula. If  $U$  is a curve over  $\mathbb{F}_q$ , then  $tr$  factors through  $\mathbb{Z}$  as  $C'_{K,S} \xrightarrow{tr_q} \mathbb{Z} \xrightarrow{\log(q)} \mathbb{R}$  with  $tr_q$  the map induced by  $\sum_v \log_q |\cdot|_v$ . Then for  $v \notin S$ , we have  $tr_q(1_v) = -[\kappa(v) : \mathbb{F}_q]$ . In particular, if  $U = C$  is a proper curve, then  $tr_q = \text{deg}$  is the degree map  $\text{Pic}(C) \rightarrow \mathbb{Z}$ .

We can consider the composite<sup>10</sup>

$$R\Gamma_c(U, \mathbb{G}_m) \rightarrow \tau^{\geq 1} R\Gamma_c(U, \mathbb{G}_m)_{\mathbb{R}} = H_c^1(U, \mathbb{G}_m)_{\mathbb{R}}[-1] = (C'_{K,S})_{\mathbb{R}}[-1] \xrightarrow{tr} \mathbb{R}[-1].$$

When  $U$  is a curve over  $\mathbb{F}_q$ , we can also consider the composite

$$R\Gamma_c(U, \mathbb{G}_m) \rightarrow \tau^{\geq 1} R\Gamma_c(U, \mathbb{G}_m)_{\mathbb{Q}} = H_c^1(U, \mathbb{G}_m)_{\mathbb{Q}}[-1] = (C'_{K,S})_{\mathbb{Q}}[-1] \xrightarrow{tr_q \otimes \mathbb{Q}} \mathbb{Q}[-1],$$

whose composite with  $\mathbb{Q} \xrightarrow{1 \rightarrow \log(q)} \mathbb{R}$  is the previous map<sup>11</sup>

**Definition II.4.3.** *The regulator pairing is the pairing for étale sheaves on  $U$*

$$R\Gamma_c(U, F) \otimes R\text{Hom}_U(F, \mathbb{G}_m) \rightarrow R\Gamma_c(U, \mathbb{G}_m) \rightarrow \mathbb{R}[-1]$$

induced by  $tr$ .

If  $U$  is a curve over  $\mathbb{F}_q$ , the rational regulator pairing is the pairing

$$R\Gamma_c(U, F) \otimes R\text{Hom}_U(F, \mathbb{G}_m) \rightarrow R\Gamma_c(U, \mathbb{G}_m) \rightarrow \mathbb{Q}[-1]$$

induced by  $tr_q$ .

Note that the rational regulator pairing depends on the choice of the base field.

<sup>10</sup>The  $\mathbb{R}$ -vector space  $(C'_{K,S})_{\mathbb{R}}$  is quite big, which can be surprising, but it doesn't matter since we take the trace map to  $\mathbb{R}$ .

<sup>11</sup>Note that we have to tensor with  $\mathbb{Q}$  to suppress the torsion coming from  $H_c^3(U, \mathbb{G}_m) = \mathbb{Q}/\mathbb{Z}$ .

## II.4.2 Perfectness of the regulator pairing after base change to $\mathbb{R}$

We call a pairing of perfect complexes  $A \otimes B \rightarrow C$  perfect if the natural morphism  $A \rightarrow R\mathrm{Hom}(B, C)$  is an isomorphism.

**Theorem II.4.4.** *Let  $U$  be a 1-dimensional regular irreducible scheme of finite type over  $\mathrm{Spec}(\mathbb{Z})$  and let  $F$  be a  $\mathbb{Z}$ -constructible sheaf on  $U$ . The regulator pairing*

$$R\Gamma_c(U, F) \otimes R\mathrm{Hom}_U(F, \mathbb{G}_m) \rightarrow R\Gamma_c(U, \mathbb{G}_m) \rightarrow \mathbb{R}[-1]$$

*is perfect after base change to  $\mathbb{R}$  for every  $\mathbb{Z}$ -constructible sheaf  $F$ . More generally, the map*

$$R\mathrm{Hom}_U(F, \mathbb{G}_m)_{\mathbb{R}} \rightarrow R\mathrm{Hom}(R\Gamma_c(U, F)_{\mathbb{R}}, \mathbb{R}[-1])$$

*is an isomorphism for every  $F \in \mathcal{D}^+(U_{\mathrm{et}})$ . When  $U$  is a curve over  $\mathbb{F}_q$ , the theorem holds over  $\mathbb{Q}$ .*

*Remark.*

- For  $U = \mathrm{Spec}(\mathcal{O}_K)$  and  $F = \mathbb{Z}$  this is conjecture  $\mathbf{B}(U, 0)$  of [FM18] (note that  $d = 1$ ), which is shown in *ibid.*, section 2.2 to be equivalent to Beilinson's conjecture. The latter is known for  $U = \mathrm{Spec}(\mathcal{O}_K)$ .
- For  $F \in \mathcal{D}^+(U_{\mathrm{et}})$  not  $\mathbb{Z}$ -constructible, the vector spaces involved may not be finite dimensional in which case the dual map  $R\Gamma_c(U, F)_{\mathbb{R}} \rightarrow R\mathrm{Hom}(R\mathrm{Hom}_U(F, \mathbb{G}_m)_{\mathbb{R}}, \mathbb{R}[-1])$  will not be an isomorphism.

*Proof.* If  $U$  is a curve over  $\mathbb{F}_q$ , the rational regulator pairing induces the regulator pairing up to a non-zero factor  $\log(q)$ ; hence for curves we will only prove the perfectness of the rational regulator pairing after base change to  $\mathbb{Q}$ .

We have to show that  $R\mathrm{Hom}_U(F, \mathbb{G}_m)_{\mathbb{R}} \rightarrow R\mathrm{Hom}(R\Gamma_c(U, F), \mathbb{R}[-1])$  is an isomorphism (resp.  $R\mathrm{Hom}_U(F, \mathbb{G}_m)_{\mathbb{Q}} \rightarrow R\mathrm{Hom}(R\Gamma_c(U, F), \mathbb{Q}[-1])$  if  $U$  is a curve). This is a natural transformation that behaves well with fiber sequences and filtered colimits, hence we first reduce to the case of bounded complexes, then by filtering with the truncations to sheaves, then to  $\mathbb{Z}$ -constructible sheaves. By the dévissage argument (§ II.2.0.1), we now have to check that

1. the regulator pairing behaves well with the pushforward  $\pi_*$  coming from a finite generically Galois morphism  $\pi : V \rightarrow U$ ;
2. the regulator pairing behaves well with the extension by zero  $j_!$  coming from an open subscheme  $j : V \hookrightarrow U$ ;
3. the regulator pairing for a  $\mathbb{Z}$ -constructible sheaf supported on a closed point identifies with a pairing in Galois cohomology that is perfect after base change to  $\mathbb{Q}$ ;
4. the theorem is true for a constructible sheaf;
5. the theorem is true for the constant sheaf  $\mathbb{Z}$ .

**II.4.2.1 Case of the pushforward by a finite generically Galois cover.** Let  $\pi : V \rightarrow U$  be the normalization of  $U$  in a finite Galois extension  $L/K$  and let  $F$  be a  $\mathbb{Z}$ -constructible sheaf on  $V$ . Then  $V = \text{Spec}(\mathcal{O}_{L,S'})$  or  $V = C' \setminus S'$  where  $S'$  are the places of  $L$  above those of  $S$  and  $C'$  is the connected smooth proper curve with function field  $L$  in the function field case. Denote by  $g_v : \text{Spec}(K_v) \rightarrow U$  (resp.  $g_w : \text{Spec}(L_w) \rightarrow V$ ) and  $\pi_w : \text{Spec}(L_w) \rightarrow \text{Spec}(K_v)$  the natural morphisms for  $v \in S$ ,  $w \in S'$ . We remark that ([Tra16, 3.6])  $g_v^* \pi_* = \prod_{w|v} (\pi_w)_* g_w^*$ . On the one hand, Proposition II.2.1.e gives a natural isomorphism  $R\Gamma_c(V, -) = R\Gamma_c(U, \pi_* -)$ . On the other hand, the norm induces a map  $Nm : \pi_* \mathbb{G}_m \rightarrow \mathbb{G}_m$  such that the composite

$$N : R\text{Hom}_V(F, \mathbb{G}_m) \xrightarrow{\pi_*} R\text{Hom}_U(\pi_* F, \pi_* \mathbb{G}_m) \xrightarrow{Nm_*} R\text{Hom}_U(\pi_* F, \mathbb{G}_m)$$

is an isomorphism ([Mil06, II.3.9]). The following diagrams are commutative by functoriality of the Yoneda pairing:

$$\begin{array}{ccc} R\text{Hom}_V(\mathbb{Z}, F) \otimes R\text{Hom}_V(F, \mathbb{G}_m) & \longrightarrow & R\Gamma(V, \mathbb{G}_m) = R\text{Hom}_V(\mathbb{Z}, \mathbb{G}_m) \\ \downarrow & & \downarrow \\ R\text{Hom}_U(\mathbb{Z}, \pi_* F) \otimes R\text{Hom}_U(\pi_* F, \pi_* \mathbb{G}_m) & \longrightarrow & R\Gamma(U, \pi_* \mathbb{G}_m) \\ \downarrow \text{id} \otimes Nm_* & & \downarrow Nm_* \\ R\Gamma(U, \pi_* F) \otimes R\text{Hom}_U(\pi_* F, \mathbb{G}_m) & \longrightarrow & R\Gamma(U, \mathbb{G}_m) \end{array}$$

$$\begin{array}{ccc} R\Gamma(L_w, F_{\eta_w}) \otimes R\text{Hom}_{L_w}(F_{\eta_w}, \mathbb{G}_m) & \longrightarrow & R\Gamma(L_w, \mathbb{G}_m) \\ \downarrow & & \downarrow \\ R\Gamma(K_v, (\pi_w)_* F_{\eta_w}) \otimes R\text{Hom}_{K_v}((\pi_w)_* F_{\eta_w}, (\pi_w)_* \mathbb{G}_m) & \longrightarrow & R\Gamma(K_v, (\pi_w)_* \mathbb{G}_m) \\ \downarrow \text{id} \otimes (N_{L_w/K_v})_* & & \downarrow (N_{L_w/K_v})_* \\ R\Gamma(K_v, (N_{L_w/K_v})_* F_{\eta_w}) \otimes R\text{Hom}_{K_v}((N_{L_w/K_v})_* F_{\eta_w}, \mathbb{G}_m) & \longrightarrow & R\Gamma(K_v, \mathbb{G}_m) \end{array}$$

$$\begin{array}{ccc} R\text{Hom}_V(F, \mathbb{G}_m) & \xrightarrow{\prod_{w|v} j_w^*} & \prod_{w|v} R\text{Hom}_{L_w}(F_{\eta_w}, \mathbb{G}_m) \\ \downarrow Nm_* \circ \pi_* & & \downarrow \prod_{w|v} (N_{L_w/K_v})_* \circ (\pi_w)_* \\ R\text{Hom}_U(\pi_* F, \mathbb{G}_m) & \xrightarrow{j_v^*} & \prod_{w|v} R\text{Hom}_{K_v}((\pi_w)_* F_{\eta_w}, \mathbb{G}_m) \end{array}$$

We obtain a diagram:

$$\begin{array}{ccccccc} R\Gamma_c(V, F) \otimes R\text{Hom}_V(F, \mathbb{G}_m) & \longrightarrow & R\Gamma_c(V, \mathbb{G}_m) & \longrightarrow & (C_{L,S'})_{\mathbb{R}}[-1] & \xrightarrow{tr} & \mathbb{R}[-1] \\ \downarrow \simeq & & \downarrow & & \downarrow Nm_* & & \parallel \\ R\Gamma_c(U, \pi_* F) \otimes R\text{Hom}_U(\pi_* F, \mathbb{G}_m) & \longrightarrow & R\Gamma_c(U, \mathbb{G}_m) & \longrightarrow & (C_{K,S})_{\mathbb{R}}[-1] & \xrightarrow{tr} & \mathbb{R}[-1] \end{array}$$

The left hand side is commutative by the previous diagrams. The long exact sequence Equation (II.2.1) gives

$$0 \rightarrow \mathcal{O}_U(U)^\times \rightarrow \prod_{v \in S} K_v^\times \rightarrow C'_{K,S} \rightarrow \text{Pic}(U).$$

Thus the map  $tr$  factors through

$$C'_{K,S} \otimes \mathbb{R} = \left( \left( \prod_{v \in S} K_v^\times \right) / \mathcal{O}_U(U)^\times \right) \otimes \mathbb{R}$$

on which it is given by

$$(\alpha_v) \mapsto \sum_{v \in S} \log(|\alpha_v|_v),$$

so the right hand side is commutative by the following computation:

$$\sum_{v \in S} \log \left| \prod_{w|v} N_{L_w/K_v}(\alpha_w) \right|_v = \sum_{w \in S'} \log |N_{L_w/K_v}(\alpha_w)|_v = \sum_{w \in S'} \log |\alpha_w|_w.$$

We have proved that the theorem is true for  $F$  if and only if it is true for  $\pi_* F$ ; the reasoning is exactly the same for the rational regulator pairing when  $U$  is a curve, except in the case where  $U = C$  is a proper curve in which case this follows from the compatibility of the degree maps of  $U$  and  $V$  with the canonical map  $\text{Pic}(V) \rightarrow \text{Pic}(U)$ .

**II.4.2.2 Case of the extension by zero along an open inclusion.** Let  $j : V \hookrightarrow U$  be an open subscheme and let  $F$  be a  $\mathbb{Z}$ -constructible sheaf on  $V$ . Write  $V = \text{Spec } \mathcal{O}_{K,S'}$  or  $V = C \setminus S'$ . Using Proposition II.2.1.d and the identification  $j^* \mathbb{G}_m = \mathbb{G}_m$ , we obtain a commutative diagram

$$\begin{array}{ccccccc} R\Gamma_c(V, F) \otimes R\text{Hom}_V(F, \mathbb{G}_m) & \longrightarrow & R\Gamma_c(V, \mathbb{G}_m) & \longrightarrow & (C_{K,S'})_{\mathbb{R}}[-1] & \xrightarrow{tr} & \mathbb{R}[-1] \\ \downarrow \simeq & & \downarrow & & \downarrow & & \parallel \\ R\Gamma_c(U, j_! F) \otimes R\text{Hom}_U(j_! F, \mathbb{G}_m) & \longrightarrow & R\Gamma_c(U, \mathbb{G}_m) & \longrightarrow & (C_{K,S})_{\mathbb{R}}[-1] & \xrightarrow{tr} & \mathbb{R}[-1] \end{array}$$

which shows that the theorem is true for  $F$  if and only if it is true for  $j_! F$ . The reasoning is the same for the rational regulator pairing when  $U$  is a curve.

**II.4.2.3 Case of a sheaf supported on a closed point.** Consider a  $\mathbb{Z}$ -constructible sheaf supported on a closed point  $v$ . Let  $G_v = \text{Gal}(\kappa(v)^{\text{sep}}/\kappa(v))$ . The sheaf is of the form  $i_* M$  where  $M$  is a discrete  $G_v$ -module of finite type and  $i : v \hookrightarrow U$  is the closed inclusion. The functor  $i_*$  has a right adjoint in the derived category  $Ri^!$  such that  $i_* Ri^! \simeq \text{id}$ , and one has an identification ([Maz73, section 1, eq. 1.2])

$$Ri^! \mathbb{G}_m \simeq \mathbb{Z}[-1]. \tag{II.4.2}$$

Proposition II.2.1.c gives a natural isomorphism  $R\Gamma_c(U, i_* M) = R\Gamma(v, M)$  and we obtain morphisms

$$\begin{aligned} R\Gamma(v, \mathbb{Z}[-1]) &= R\Gamma_c(U, i_* \mathbb{Z}[-1]) = R\Gamma_c(U, i_* Ri^! \mathbb{G}_m) \xrightarrow{\eta_*} R\Gamma_c(U, \mathbb{G}_m), \\ R\text{Hom}_v(M, \mathbb{Z}[-1]) &= R\text{Hom}_v(M, Ri^! \mathbb{G}_m) \xrightarrow[\simeq]{\eta^* \circ i^*} R\text{Hom}_U(i_* M, \mathbb{G}_m), \end{aligned}$$

where  $\eta : i_* Ri^! \mathbb{G}_m \rightarrow \mathbb{G}_m$  is the counit of the adjunction. The functoriality of the Yoneda pairing then implies that the following diagram is commutative:

$$\begin{array}{ccc} R\Gamma(v, M) \otimes R\text{Hom}_v(M, \mathbb{Z}[-1]) & \longrightarrow & R\Gamma(v, \mathbb{Z}[-1]) \\ \downarrow \simeq & & \downarrow \\ R\Gamma_c(U, i_* M) \otimes R\text{Hom}_U(i_* M, \mathbb{G}_m) & \longrightarrow & R\Gamma_c(U, \mathbb{G}_m) \end{array}$$

The Galois cohomology groups of  $G_v \simeq \hat{\mathbb{Z}}$  with coefficients in  $\mathbb{Z}$  are given by

$$H^i(\hat{\mathbb{Z}}, \mathbb{Z}) = \begin{cases} \mathbb{Z}, & i = 0, \\ \mathrm{Hom}_{\mathrm{cont}}(\hat{\mathbb{Z}}, \mathbb{Z}) = 0, & i = 1, \\ \mathrm{Hom}_{\mathrm{cont}}(\hat{\mathbb{Z}}, \mathbb{Q}/\mathbb{Z}) = \mathbb{Q}/\mathbb{Z}, & i = 2, \\ 0, & i > 2, \end{cases}$$

hence

$$\tau^{\geq 1} R\Gamma(G_v, \mathbb{Z}[-1])_{\mathbb{Q}} \simeq H^0(G_v, \mathbb{Z})_{\mathbb{Q}}[-1] = \mathbb{Q}[-1].$$

The map  $\eta_* : \mathbb{Z} = H^0(G_v, \mathbb{Z}) = H_c^0(U, i_*\mathbb{Z}) \rightarrow H_c^1(U, \mathbb{G}_m) = C'_{K,S}$  sends  $\mathbb{Z}$  to the factor indexed by  $v$  in  $C'_{K,S}$ . Following remark II.4.1, we get the commutative diagram Diagram II.1.

**II.4.2.4 Case of the Galois cohomology of a finite field.** Since  $-\log(N(v)) \neq 0$ , we are reduced by the previous section (and in particular by Diagram II.1) to showing that

$$R\Gamma(v, M) \otimes R\mathrm{Hom}_v(M, \mathbb{Z}) \rightarrow R\Gamma(v, \mathbb{Z}) \rightarrow \mathbb{Q}[0]$$

is perfect after base change to  $\mathbb{Q}$  for any discrete  $G_v$ -module  $M$  of finite type as an abelian group; this will imply that it is also perfect after base change to  $\mathbb{R}$ . We show it on cohomology groups. Galois cohomology is torsion and Artin–Verdier duality gives that  $\mathrm{Ext}_v^i(M, \mathbb{Z}) = \mathrm{Ext}_U^{i+1}(i_*M, \mathbb{G}_m)$  (Equation (II.4.2)) is finite for  $i \geq 1$ , so the pairing is trivially perfect in degree  $i \neq 0$  after base change to  $\mathbb{Q}$ . Thus it suffices to show the result in degree 0, i.e. to show that the pairing

$$M^{G_v} \times \mathrm{Hom}_{G_v}(M, \mathbb{Z}) \rightarrow \mathbb{Q}$$

is perfect after base change to  $\mathbb{Q}$ . Let  $H$  be an open subgroup acting trivially on  $M$ ; then  $M^{G_v} = (M)^{G_v/H}$  and  $\mathrm{Hom}_{G_v}(M, \mathbb{Z}) = \mathrm{Hom}_{G_v/H}(M, \mathbb{Z})$ , i.e. we reduce to the case of a cyclic group  $G := G_v/H$  acting on the finite type abelian group  $M$ . We have furthermore  $\mathrm{Hom}_G(M, \mathbb{Z}) = \mathrm{Hom}(M_G, \mathbb{Z})$  (where  $M_G$  denotes coinvariants). Let  $F$  denote a generator of  $G$ . The exact sequence

$$0 \rightarrow M^G \rightarrow M \xrightarrow{F-1} M \rightarrow M_G \rightarrow 0$$

gives identifications  $(M^G)_{\mathbb{Q}} = (M_{\mathbb{Q}})^G$  and  $(M_G)_{\mathbb{Q}} = (M_{\mathbb{Q}})_G$  by tensoring with  $\mathbb{Q}$ , and also gives that those vector spaces have the same dimension. Let  $N : M_G \rightarrow M^G$  denote the norm and  $\pi : M^G \rightarrow M \rightarrow M_G$  the restriction of the canonical projection to  $M^G$ . We notice that  $N \circ \pi = |G| \cdot \mathrm{Id}$ , so  $\pi_{\mathbb{Q}}$  is an isomorphism. In the pairing

$$M_{\mathbb{Q}}^G \times \mathrm{Hom}(M_{G, \mathbb{Q}}, \mathbb{Q}) \rightarrow \mathbb{Q},$$

the left hand terms have same dimension, so it suffices to prove that the left kernel is trivial, which follows immediately for surjectivity of  $\pi_{\mathbb{Q}}$ .

**II.4.2.5 Case of a constructible sheaf.** Let  $F$  be a constructible sheaf on  $U$ . Then cohomology with compact support differs from Tate cohomology with compact support by finite groups. Artin–Verdier duality thus shows that all cohomology groups  $H_c^i(U, F)$  and  $\mathrm{Ext}_U^i(F, \mathbb{G}_m)$  are finite, hence the pairing is trivial after base change to  $\mathbb{R}$  (or  $\mathbb{Q}$  when  $U$  is a curve), so it is perfect.

$$\begin{array}{ccccccc}
R\Gamma(v, M) \otimes R\mathrm{Hom}_v(M, \mathbb{Z}[-1]) & \longrightarrow & R\Gamma(v, \mathbb{Z}[-1]) & \longrightarrow & \tau^{\geq 1} R\Gamma(v, \mathbb{Z}[-1])_{\mathbb{Q}} & \xrightarrow{=} & \mathbb{Q}[-1] \\
\downarrow \simeq & & \downarrow & & \downarrow & & \downarrow \scriptstyle{-\log(N^{(v)})} \\
R\Gamma_c(U, i_* M) \otimes R\mathrm{Hom}_U(i_* M, \mathbb{G}_m) & \longrightarrow & R\Gamma_c(U, \mathbb{G}_m) & \longrightarrow & \tau^{\geq 1} R\Gamma_c(U, \mathbb{G}_m)_{\mathbb{R}} = (C'_{K,S})_{\mathbb{R}}[-1] & \xrightarrow{tr} & \mathbb{R}[-1]
\end{array}$$

Diagram II.1

**II.4.2.6 Case of the constant sheaf  $\mathbb{Z}$ .** Consider the constant sheaf  $\mathbb{Z}$  on  $U$ . If  $U$  is a curve over  $\mathbb{F}_q$ , the previous discussion along with the dévissage argument (§ II.2.0.1) allows us to suppose that  $U = C$  is a proper curve. We will show perfectness of the regulator pairing on cohomology groups. Since  $U$  is noetherian and normal,  $H^q(U, \mathbb{Z})$  is torsion for  $q > 0$  ([Mil06, II.2.10]) and  $\pi_1^{SGA3}(U) = \pi_1^{SGA1}(U)$  is profinite, hence  $H^1(U, \mathbb{Z}) = \text{Hom}_{\text{cont}}(\pi_1^{\text{ét}}(U), \mathbb{Z}) = 0$ . The same argument holds for  $\mathbb{Z}$  on  $\text{Spec}(K_v)$  for  $v \in S$ . Thus:

- If  $U = \text{Spec}(\mathcal{O}_{K,S})$ , we have the exact sequence Equation (II.2.1)

$$0 \rightarrow H_c^0(U, \mathbb{Z}) \rightarrow \mathbb{Z} \rightarrow \prod_{v \in S} \mathbb{Z} \rightarrow H_c^1(U, \mathbb{Z}) \rightarrow 0$$

and we obtain

$$H_c^0(U, \mathbb{Z}) = 0, \quad H_c^1(U, \mathbb{Z}) = \left( \prod_{v \in S} \mathbb{Z} \right) / \mathbb{Z}, \quad H_c^i(U, \mathbb{Z})_{\mathbb{R}} = 0 \text{ for } i > 1. \quad (\text{II.4.3})$$

On the other hand

$$\text{Ext}_U^i(\mathbb{Z}, \mathbb{G}_m) = H^i(U, \mathbb{G}_m) = \begin{cases} \mathcal{O}_{K,S}^{\times}, & i = 0, \\ \text{Pic}(U), & i = 1, \\ \text{torsion of cofinite type}, & i \geq 2. \end{cases}$$

The pairing  $H_c^i(U, \mathbb{Z})_{\mathbb{R}} \times \text{Ext}^{1-i}(\mathbb{Z}, \mathbb{G}_m)_{\mathbb{R}} \rightarrow \mathbb{R}$  is thus perfect for  $i = 0$  by finiteness of the  $S$ -ideal class group and trivially perfect for  $i \neq 0, 1$ . In degree  $i = 1$ , the regulator pairing identifies with the map

$$\begin{cases} \left( \prod_{v \in S} \mathbb{Z} \right) / \mathbb{Z} \times \mathcal{O}_{K,S}^{\times} & \longrightarrow & \mathbb{R} \\ ((\alpha_v), u) & \longmapsto & \sum \alpha_v \log |u|_v, \end{cases}$$

which is non-degenerate modulo torsion by the theorem of  $S$ -units of Dirichlet (see for instance [Nar04, 3.3.12]). Indeed, under the identification

$$\text{Hom}\left(\left(\prod_{v \in S} \mathbb{R}\right) / \mathbb{R}, \mathbb{R}\right) \simeq \{(x_i) \in \mathbb{R}^s, \sum x_i = 0\},$$

the map

$$\mathcal{O}_{K,S}^{\times} \rightarrow \text{Hom}\left(\left(\prod_{v \in S} \mathbb{R}\right) / \mathbb{R}, \mathbb{R}\right) \subset \mathbb{R}^s$$

induced by the pairing is the usual logarithmic embedding.

- If  $U = C$  is a proper curve over  $\mathbb{F}_q$ , we have  $H^0(C, \mathbb{Z}) = \mathbb{Z}$ ,  $H^i(C, \mathbb{Z})$  torsion for  $i > 0$  and

$$\text{Ext}_C^i(\mathbb{Z}, \mathbb{G}_m) = H^i(C, \mathbb{G}_m) = \begin{cases} \mathcal{O}_C(C)^{\times} & i = 0 \\ \text{Pic}(C) & i = 1 \\ \text{torsion of cofinite type} & i \geq 2 \end{cases}$$

Since  $C$  is proper, normal and connected,  $\mathcal{O}_C(C) = K \cap \bar{\mathbb{F}}_q$  is the field of constants of  $K$ , and a finite extension of  $\mathbb{F}_q$ ; in particular  $H^0(C, \mathbb{G}_m)$  is finite. The pairing  $H^i(C, \mathbb{Z})_{\mathbb{Q}} \times \text{Ext}^{1-i}(\mathbb{Z}, \mathbb{G}_m)_{\mathbb{Q}} \rightarrow \mathbb{Q}$  is thus trivially perfect for  $i \neq 0$ . In degree  $i = 0$ ,

the rational regulator pairing identifies with the top row in the following commutative diagram :

$$\begin{array}{ccc} \mathbb{Q} \times \mathrm{Pic}(C)_{\mathbb{Q}} & \longrightarrow & \mathbb{Q} \\ \simeq \downarrow \mathrm{id} \times \mathrm{deg}_{\mathbb{Q}} & & \parallel \\ \mathbb{Q} \times \mathbb{Q} & \xrightarrow{r, s \mapsto rs} & \mathbb{Q} \end{array}$$

The bottom pairing is perfect, so it remains only to show that the degree map is an isomorphism rationally, which follows from the exact sequence  $0 \rightarrow \mathrm{Pic}^0(C) \rightarrow \mathrm{Pic}(C) \xrightarrow{\mathrm{deg}} f\mathbb{Z} \rightarrow 0$ ,<sup>12</sup> and the finiteness of the class group of degree 0 divisors  $\mathrm{Pic}^0(C)$ .<sup>13</sup>

□

## II.5 Splitting of the Weil-étale complex after base change to $\mathbb{R}$

We denote by  $(-)_{\mathbb{Q}} := - \otimes^L \mathbb{Q}$  (resp.  $(-)_{\mathbb{R}}$ ) the base change by  $\mathbb{Q}$  (resp.  $\mathbb{R}$ ).

**Proposition II.5.1.** *Let  $F$  be a big or tiny sheaf on  $U$ . The Weil-étale complex with compact support splits rationally, naturally in big-to-tiny morphisms and big-to-tiny short exact sequences:*

$$R\Gamma_{W,c}(U, F)_{\mathbb{Q}} = R\Gamma_c(U, F)_{\mathbb{Q}} \oplus D_{F,\mathbb{Q}}[1].$$

Moreover, by the perfectness of the regulator pairing after base change to  $\mathbb{R}$ , there is an isomorphism, natural in big-to-tiny morphisms and big-to-tiny short exact sequences:

$$\tau_F : R\Gamma_{W,c}(U, F)_{\mathbb{R}} \xrightarrow{\simeq} D_{F,\mathbb{R}}[1] \oplus D_{F,\mathbb{R}}[2]. \quad (\text{II.5.1})$$

*Proof.* By tensoring with  $\mathbb{Q}$  the defining triangle of  $R\Gamma_{W,c}(U, F)$  and rotating once, we obtain a distinguished triangle

$$R\Gamma_c(U, F)_{\mathbb{Q}} \xrightarrow{i_F} R\Gamma_{W,c}(U, F)_{\mathbb{Q}} \xrightarrow{p_F} D_{F,\mathbb{Q}}[1] \rightarrow R\Gamma_c(U, F)_{\mathbb{Q}}[1].$$

Let us show that this triangle splits canonically. We will prove the existence of a section of  $p$ :

$$s : D_{F,\mathbb{Q}}[1] \rightarrow R\Gamma_{W,c}(U, F)_{\mathbb{Q}}.$$

Let  $G$  be another big or tiny sheaf. Consider the exact sequence obtained by applying  $\mathrm{Hom}(D_{F,\mathbb{Q}}[1], -)$  to the above triangle for  $G$  :

$$\begin{array}{ccc} \mathrm{Hom}(D_{F,\mathbb{Q}}[1], R\Gamma_c(U, G)_{\mathbb{Q}}) & \longrightarrow & \mathrm{Hom}(D_{F,\mathbb{Q}}[1], R\Gamma_{W,c}(U, G)_{\mathbb{Q}}) \\ & \searrow & \nearrow \\ & \mathrm{Hom}(D_{F,\mathbb{Q}}[1], D_{G,\mathbb{Q}}[1]) & \longrightarrow \mathrm{Hom}(D_{F,\mathbb{Q}}[1], R\Gamma_c(U, G)_{\mathbb{Q}}[1]). \end{array}$$

Since  $D_{F,\mathbb{Q}}[1]$  is cohomologically concentrated in degrees 3 and 4 and  $R\Gamma_c(U, G)_{\mathbb{Q}}$  is concentrated in degrees 0 and 1, the usual argument with the exact sequence Equation (II.3.1) shows that both left and right terms are zero, hence a canonical isomorphism:

$$\mathrm{Hom}(D_{F,\mathbb{Q}}[1], R\Gamma_{W,c}(U, G)_{\mathbb{Q}}) \xrightarrow{\simeq} \mathrm{Hom}(D_{F,\mathbb{Q}}[1], D_{G,\mathbb{Q}}[1]).$$

<sup>12</sup>We have  $f = [\mathcal{O}_C(C) : \mathbb{F}_q]$ , see for instance [Wei95, VII.5, cor. 5]

<sup>13</sup>See for instance [Wei95, IV.4 Thm 7]; the group  $k_{\mathbb{A}}^1/k^\times \Omega(\emptyset)$  of *ibid.* identifies with  $\mathrm{Pic}^0(C)$ , as discussed at the beginning of *ibid.*, Ch. VI



The same argument also gives

$$\mathrm{Hom}(D_{F,\mathbb{Q}}[1], R\Gamma_{W,c}(U, G)_{\mathbb{Q}}[1]) \xrightarrow{\cong} \mathrm{Hom}(D_{F,\mathbb{Q}}[1], D_{G,\mathbb{Q}}[2]).$$

Let  $s_F$  be the inverse image of the identity under

$$\mathrm{Hom}(D_{F,\mathbb{Q}}[1], R\Gamma_{W,c}(U, F)_{\mathbb{Q}}) \xrightarrow{\cong} \mathrm{Hom}(D_{F,\mathbb{Q}}[1], D_{F,\mathbb{Q}}[1]).$$

The following diagram gives a morphism of distinguished triangles (note that the right square does commute, since  $\mathrm{Hom}(D_{F,\mathbb{Q}}[1], R\Gamma_c(U, F)_{\mathbb{Q}}[1]) = 0$ ):

$$\begin{array}{ccccccc} R\Gamma_c(U, F)_{\mathbb{Q}} & \longrightarrow & R\Gamma_c(U, F)_{\mathbb{Q}} \oplus D_{F,\mathbb{Q}}[1] & \longrightarrow & D_{F,\mathbb{Q}}[1] & \longrightarrow & R\Gamma_c(U, F)_{\mathbb{Q}}[1] \\ \parallel & & \downarrow (i_F, s_F) & & \parallel & & \parallel \\ R\Gamma_c(U, F)_{\mathbb{Q}} & \xrightarrow{i} & R\Gamma_{W,c}(U, F)_{\mathbb{Q}} & \longrightarrow & D_{F,\mathbb{Q}}[1] & \xrightarrow{p} & R\Gamma_c(U, F)_{\mathbb{Q}}[1] \end{array}$$

hence by the five lemma an isomorphism

$$(i_F, s_F) : R\Gamma_c(U, F)_{\mathbb{Q}} \oplus D_{F,\mathbb{Q}}[1] \xrightarrow{\cong} R\Gamma_{W,c}(U, F)_{\mathbb{Q}}.$$

Let us see the naturality of the above isomorphism. Let  $F \rightarrow G$  be a big-to-tiny morphism. We have already proved that  $i$  is natural (see the proof of Theorem II.3.5), so it suffices to show that  $s$  is a natural transformation, i.e. that the following diagram commutes:

$$\begin{array}{ccc} D_{F,\mathbb{Q}}[1] & \xrightarrow{s_H} & R\Gamma_{W,c}(U, F) \\ \downarrow & & \downarrow \\ D_{G,\mathbb{Q}}[1] & \xrightarrow{s_G} & R\Gamma_{W,c}(U, G) \end{array}$$

The horizontal arrows in the following diagram are isomorphisms, and the diagram is commutative by commutativity of Equation (II.3.2), whence we conclude with a simple diagram chasing:

$$\begin{array}{ccc} \mathrm{Hom}(D_{F,\mathbb{Q}}[1], R\Gamma_{W,c}(U, F)_{\mathbb{Q}}) & \xrightarrow{\cong} & \mathrm{Hom}(D_{F,\mathbb{Q}}[1], D_{F,\mathbb{Q}}[1]) \\ \downarrow & & \downarrow \\ \mathrm{Hom}(D_{F,\mathbb{Q}}[1], R\Gamma_{W,c}(U, G)_{\mathbb{Q}}) & \xrightarrow{\cong} & \mathrm{Hom}(D_{F,\mathbb{Q}}[1], D_{G,\mathbb{Q}}[1]) \\ \uparrow & & \uparrow \\ \mathrm{Hom}(D_{G,\mathbb{Q}}[1], R\Gamma_{W,c}(U, G)_{\mathbb{Q}}) & \xrightarrow{\cong} & \mathrm{Hom}(D_{G,\mathbb{Q}}[1], D_{G,\mathbb{Q}}[1]) \end{array}$$

Let  $0 \rightarrow F \rightarrow G \rightarrow H \rightarrow 0$  be a big-to-tiny short exact sequence. Similarly to the previous step,  $i$  is natural, and it remains to see that the diagram

$$\begin{array}{ccc} D_{H,\mathbb{Q}}[1] & \xrightarrow{s_H} & R\Gamma_{W,c}(U, H) \\ \downarrow & & \downarrow \\ D_{F,\mathbb{Q}}[2] & \xrightarrow{s_F[1]} & R\Gamma_{W,c}(U, F)[1] \end{array}$$

commutes. We conclude by a similar diagram chasing in the diagram

$$\begin{array}{ccc}
\mathrm{Hom}(D_{H,\mathbb{Q}}[1], R\Gamma_{W,c}(U, H)_{\mathbb{Q}}) & \xrightarrow{\cong} & \mathrm{Hom}(D_{H,\mathbb{Q}}[1], D_{H,\mathbb{Q}}[1]) \\
\downarrow & & \downarrow \\
\mathrm{Hom}(D_{H,\mathbb{Q}}[1], R\Gamma_{W,c}(U, F)_{\mathbb{Q}}[1]) & \xrightarrow{\cong} & \mathrm{Hom}(D_{H,\mathbb{Q}}[1], D_{F,\mathbb{Q}}[2]) \\
\uparrow & & \uparrow \\
\mathrm{Hom}(D_{F,\mathbb{Q}}[2], R\Gamma_{W,c}(U, F)_{\mathbb{Q}}[1]) & \xrightarrow{\cong} & \mathrm{Hom}(D_{F,\mathbb{Q}}[2], D_{F,\mathbb{Q}}[2])
\end{array}$$

The regulator pairing gives an isomorphism, natural in complexes of étale sheaves  $F \in \mathcal{D}^+(U)$

$$\begin{aligned}
R\Gamma_c(U, F)_{\mathbb{R}} &\xrightarrow{\cong} R\mathrm{Hom}((R\mathrm{Hom}_U(F, \mathbb{G}_m), \mathbb{R}[-1])) = R\mathrm{Hom}(R\mathrm{Hom}_U(F, \mathbb{G}_m), \mathbb{Q}[-1]) \otimes \mathbb{R} \\
&= D_{F,\mathbb{R}}[2],
\end{aligned}$$

hence the claimed isomorphism natural in big-to-tiny morphisms and big-to-tiny short exact sequences

$$\tau_F : R\Gamma_{W,c}(U, F)_{\mathbb{R}} \xrightarrow{\cong} D_{F,\mathbb{R}}[1] \oplus D_{F,\mathbb{R}}[2].$$

□

*Remark.*

- If  $U$  is a curve the same holds over  $\mathbb{Q}$ , using the rational regulator pairing.
- The above isomorphism implies that the functor  $R\Gamma_{W,c}(U, -)_{\mathbb{R}}$  is exact in big-to-tiny short exact sequences.

## II.6 The Weil-étale Euler characteristic

### II.6.1 Construction

In this section we will use the determinant construction of Knudsen-Mumford [KM76], and the subsequent work of Breuning, Burns and Knudsen, in particular [Bre08]. Let  $R$  be a Noetherian ring; denote by  $\mathrm{Proj}_R$  the exact category of projective finite type  $R$ -modules,  $\mathrm{Gr}^b(\mathrm{Mod}_R^{ft})$  the bounded graded abelian category of finite type  $R$ -modules and  $D_{perf}(R)$  the derived category of perfect complexes. For  $\mathcal{P}$  a Picard groupoid, the Picard groupoid of determinants  $\mathrm{det}(\mathcal{C}, \mathcal{P})$  for  $\mathcal{C}$  an exact or triangulated category is defined in [Knu02] and [Bre11] respectively. Their objects are determinants, that is pairs  $f = (f_1, f_2)$  where  $f_1$  is a functor  $\mathcal{C}^{iso} \rightarrow \mathcal{P}$  and  $f_2$  maps each exact sequence/distinguished triangle  $\Delta : X \rightarrow Y \rightarrow Z$  to an isomorphism  $f_2(\Delta) : f_1(Y) \rightarrow f_1(X) \otimes f_1(Z)$ , satisfying certain axioms. When the context is clear we'll abuse notation and write  $f$  for both  $f_1$  and  $f_2$ . There is a canonical inclusion  $I : \mathrm{Proj}_R \hookrightarrow \mathrm{Gr}^b(\mathrm{Mod}_R^{ft})$  where  $I(A)$  is  $A$  in degree 0 and 0 elsewhere. We extend  $I$  naturally to exact sequences. Thus  $I$  defines a restriction functor

$$I^* : \mathrm{det}(\mathrm{Gr}^b(\mathrm{Mod}_R^{ft}), \mathcal{P}) \rightarrow \mathrm{det}(\mathrm{Proj}_R, \mathcal{P})$$

for any Picard groupoid  $\mathcal{P}$ , given by  $f = (f_1, f_2) \mapsto (f_1 \circ I, f_2 \circ I)$ . On the other hand, there is a functor  $H : D_{perf}(R) \rightarrow \mathrm{Gr}^b(\mathrm{Mod}_R^{ft})$  given by  $X \mapsto (H^i(X))_i$ , and for a distinguished triangle

$$\Delta : X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w}$$

we define  $H(\Delta)$  to be the following exact sequence, induced by the long exact cohomology sequence:

$$0 \rightarrow \ker(H(u)) \rightarrow H(X) \rightarrow H(Y) \rightarrow H(Z) \xrightarrow{\partial} \ker(H(u))[1] \rightarrow 0.$$

Thus  $H$  defines an extension functor  $H^* : \det(\mathrm{Gr}^b(\mathrm{Mod}_R^{ft}), \mathcal{P}) \rightarrow \det(D_{perf}(R), \mathcal{P})$ , mapping a determinant  $(g_1, g_2)$  to  $(g_1 \circ H, f_2)$  where  $f_2$  is deduced from applying  $g_2$  to the exact sequences of the form  $H(\Delta)$  for  $\Delta$  a distinguished triangle [Bre11, Prop. 5.8].

The functor  $I$  factors as  $\mathrm{Proj}_R \rightarrow \mathrm{Mod}_R^{ft} \rightarrow \mathrm{Gr}^b(\mathrm{Mod}_R^{ft})$  hence for  $R$  a regular ring,  $I^*$  is an equivalence of categories for any Picard category  $\mathcal{P}$  by [Bre11, Prop. 5.5] and Quillen's resolution Theorem [Qui73, Cor. 2 in §4] (see also the proof of [Bre08, Prop. 3.4]). Note that by [Bre11, Lemma 5.10, Prop. 5.11],  $H^*$  is also an equivalence of categories since there is an identification  $D_{perf}(R) \simeq D^b(\mathrm{Mod}_R^{ft})$ .

Consider the usual determinant functor

$$\det_R \in \det(\mathrm{Proj}_R, \mathcal{P}_R), \quad M \mapsto (\Lambda^{top} M, \mathrm{rank} M)$$

of [KM76] with values in the Picard groupoid of graded  $R$ -lines, that is of pairs  $(L, n)$  where  $L$  is an invertible  $R$ -module and  $n : \mathrm{Spec}(R) \rightarrow \mathbb{Z}$  a locally constant function, with morphisms  $(L, n) \rightarrow (L', m)$  the isomorphisms  $L \rightarrow L'$  if  $n = m$  and none otherwise.

*Remark* ([BF03, Section 2.5]). We have  $\pi_0 \mathcal{P}_R = H^0(\mathrm{Spec}(R), \mathbb{Z}) \oplus \mathrm{Pic}(R)$  and  $\pi_1 \mathcal{P}_R = R^\times$ . Hence by  $K$ -theoretic computations,  $\det_R$  is the universal determinant for  $R$  local, or semi-simple, or the ring of integers in a number field.

For basic properties of determinant functors on triangulated categories, see [Bre11, Section 3]. Let  $g_R \in \det(\mathrm{Gr}^b(\mathrm{Mod}_R^{ft}), \mathcal{P}_R)$  be a determinant functor extending  $\det_R$ , i.e. such that  $I^*(g_R) = \det_R$ . Define then  $f_R := H^*(g_R) \in \det(D_{perf}(R), \mathcal{P}_R)$ .

We specialize now to  $R = \mathbb{Z}, \mathbb{R}$ . Whenever it makes sense, denote by  $B$  the base change functor  $X \mapsto X \otimes_{\mathbb{Z}} \mathbb{R}$ . The following diagram is commutative

$$\begin{array}{ccccc} \mathrm{Proj}_{\mathbb{Z}} & \xleftarrow{I} & \mathrm{Gr}^b(\mathrm{Mod}_{\mathbb{Z}}^{ft}) & \xleftarrow{H} & D_{perf}(\mathbb{Z}) \\ \downarrow B & & \downarrow B & & \downarrow B \\ \mathrm{Proj}_{\mathbb{R}} & \xleftarrow{I} & \mathrm{Gr}^b(\mathrm{Mod}_{\mathbb{R}}^{ft}) & \xleftarrow{H} & D_{perf}(\mathbb{R}) \end{array}$$

and there is a natural symmetric monoidal functor  $B : \mathcal{P}_{\mathbb{Z}} \rightarrow \mathcal{P}_{\mathbb{R}}$ , hence we obtain a commutative diagram

$$\begin{array}{ccccc} \det(\mathrm{Proj}_{\mathbb{Z}}, \mathcal{P}_{\mathbb{Z}}) & \xleftarrow{I^*} & \det(\mathrm{Gr}^b(\mathrm{Mod}_{\mathbb{Z}}^{ft}), \mathcal{P}_{\mathbb{Z}}) & \xrightarrow{H^*} & \det(D_{perf}(\mathbb{Z}), \mathcal{P}_{\mathbb{Z}}) \\ \downarrow B_* & & \downarrow B_* & & \downarrow B_* \\ \det(\mathrm{Proj}_{\mathbb{Z}}, \mathcal{P}_{\mathbb{R}}) & \xleftarrow{I^*} & \det(\mathrm{Gr}^b(\mathrm{Mod}_{\mathbb{Z}}^{ft}), \mathcal{P}_{\mathbb{R}}) & \xrightarrow{H^*} & \det(D_{perf}(\mathbb{Z}), \mathcal{P}_{\mathbb{R}}) \\ B^* \uparrow & & B^* \uparrow & & B^* \uparrow \\ \det(\mathrm{Proj}_{\mathbb{R}}, \mathcal{P}_{\mathbb{R}}) & \xleftarrow{I^*} & \det(\mathrm{Gr}^b(\mathrm{Mod}_{\mathbb{R}}^{ft}), \mathcal{P}_{\mathbb{R}}) & \xrightarrow{H^*} & \det(D_{perf}(\mathbb{R}), \mathcal{P}_{\mathbb{R}}) \end{array}$$

where upper stars denote precomposition and lower stars denote postcomposition. From this diagram, we see that there is an isomorphism of determinants  $\gamma : B \circ g_{\mathbb{Z}} \simeq B^*(g_{\mathbb{R}})$  inducing an isomorphism

$$H^*(\gamma) : B \circ f_{\mathbb{Z}} \simeq B^*(f_{\mathbb{R}}). \quad (\text{II.6.1})$$

Applying the determinant construction to the isomorphism Equation (II.5.1), we get a trivialization

$$\lambda_F : f_{\mathbb{R}}(R\Gamma_{W,c}(U, F)_{\mathbb{R}}) \simeq f_{\mathbb{R}}(D_{F,\mathbb{R}}[1] \oplus D_{F,\mathbb{R}}[2]) \xleftarrow{\simeq} f_{\mathbb{R}}(D_{F,\mathbb{R}}[1]) \otimes f_{\mathbb{R}}(D_{F,\mathbb{R}}[2]) \xrightarrow{\simeq} \mathbb{R},$$

where the last isomorphism holds because for any perfect complex  $X$ , the distinguished triangle  $X \rightarrow 0 \rightarrow X[1] \rightarrow$  gives an isomorphism

$$f_{\mathbb{R}}(X) \otimes f_{\mathbb{R}}(X[1]) \xrightarrow{\simeq} f_{\mathbb{R}}(0) = \mathbb{R}.$$

The canonical isomorphism

$$H^*(\gamma)_{R\Gamma_{W,c}(U,F)} : f_{\mathbb{Z}}(R\Gamma_{W,c}(U, F)) \otimes_{\mathbb{Z}} \mathbb{R} \xrightarrow{\simeq} f_{\mathbb{R}}(R\Gamma_{W,c}(U, F)_{\mathbb{R}})$$

gives a natural embedding

$$f_{\mathbb{Z}}(R\Gamma_{W,c}(U, F)) \hookrightarrow f_{\mathbb{R}}(R\Gamma_{W,c}(U, F)_{\mathbb{R}})$$

of the underlying (ungraded) lines, i.e. of abelian groups.

*Remark.* Define the Picard groupoid of embedded graded lines  $\mathcal{P}_{\mathbb{Z} \rightarrow \mathbb{R}}$ , whose objects are pairs  $(f : L \rightarrow V, n)$  where  $L$  is a free  $\mathbb{Z}$ -modules of rank 1,  $V$  an  $\mathbb{R}$ -vector space of dimension 1 and  $n \in \mathbb{Z}$ , with a map  $f : L \rightarrow V$  of abelian groups such that the induced map  $L \otimes \mathbb{R} \rightarrow V$  is an isomorphism. The projection  $\mathcal{P}_{\mathbb{Z} \rightarrow \mathbb{R}} \rightarrow \mathcal{P}_{\mathbb{Z}}$  and the functor  $\mathcal{P}_{\mathbb{Z}} \rightarrow \mathcal{P}_{\mathbb{Z} \rightarrow \mathbb{R}}$ ,  $(L, n) \mapsto (L \rightarrow L \otimes \mathbb{R}, n)$  are inverse equivalences of Picard groupoids. The above embedding can be seen as an object in  $\mathcal{P}_{\mathbb{Z} \rightarrow \mathbb{R}}$ .

**Definition II.6.1.** Let  $F$  be a big or tiny sheaf on  $U$ . The (Weil-étale) Euler characteristic of  $F$  is the positive real number  $\chi_U(F)$  such that

$$\lambda(f_{\mathbb{Z}}(R\Gamma_{W,c}(U, F))) = (\chi_U(F))^{-1} \mathbb{Z} \hookrightarrow \mathbb{R}.$$

**Definition II.6.2.** Let  $F$  be a big or tiny sheaf on  $U$ . We define the secondary Euler characteristic

$$E_U(F) := \sum_i (-1)^i \cdot i \cdot \dim_{\mathbb{R}}(H_{W,c}^i(U, F)_{\mathbb{R}}).$$

*Remark.* By Theorem II.4.4 and the computations of subSubsection II.3.2 we have

$$\begin{aligned} E_U(F) &= -1(\text{rank}_{\mathbb{Z}} H_c^1(U, F) + \text{rank}_{\mathbb{Z}} \text{Ext}_U^1(F, \mathbb{G}_m)) + 2(\text{rank}_{\mathbb{Z}} \text{Hom}_U(F, \mathbb{G}_m)) \\ &= \text{rank}_{\mathbb{Z}}(\text{Hom}_U(F, \mathbb{G}_m)) - \text{rank}_{\mathbb{Z}}(\text{Ext}_U^1(F, \mathbb{G}_m)) \\ &= \text{rank}_{\mathbb{Z}}(H_c^1(U, F)) - \text{rank}_{\mathbb{Z}}(H_c^0(U, F)). \end{aligned}$$

*Remark.* Denote by  $\cup\theta$  the map

$$R\Gamma_{W,c}(U, F)_{\mathbb{R}} \simeq D_{F,\mathbb{R}}[1] \oplus D_{F,\mathbb{R}}[2] \rightarrow D_{F,\mathbb{R}}[2] \rightarrow D_{F,\mathbb{R}}[2] \oplus D_{F,\mathbb{R}}[3] \simeq R\Gamma_{W,c}(U, F)_{\mathbb{R}}[1].$$

Then  $\cup\theta$  induces an exact sequence

$$\dots \rightarrow H_{W,c}^{i-1}(U, F)_{\mathbb{R}} \xrightarrow{\cup\theta} H_{W,c}^i(U, F)_{\mathbb{R}} \xrightarrow{\cup\theta} H_{W,c}^{i+1}(U, F)_{\mathbb{R}} \rightarrow \dots$$

which gives a trivialization

$$\lambda : \bigotimes_{\mathbb{R}}_{i \in \mathbb{Z}} \det(H_{W,c}^i(U, F)_{\mathbb{R}})^{(-1)^i} \xrightarrow{\simeq} \mathbb{R},$$

and we can define alternatively  $\chi_U(F)$  such that

$$\lambda\left(\bigotimes_{\mathbb{Z}} \det(H_{W,c}^i(U, F))^{(-1)^i}\right) = (\chi_U(F))^{-1} \mathbb{Z} \hookrightarrow \mathbb{R}.$$

When  $X$  is a proper smooth curve over  $\mathbb{F}_q$ , Flach and B. Morin have shown that  $R\Gamma_{W,c}(X, \mathbb{Z}) = R\Gamma_c(X_W, \mathbb{Z})$  ([FM18, Theorem 3.20]), in which case they observed furthermore that  $\cup\theta = \log(q) \cup e$  where  $e \in H^1(W_{\mathbb{F}_q}, \mathbb{Z}) = \text{Hom}(W_{\mathbb{F}_q}, \mathbb{Z})$  sends the Frobenius to 1 ([FM20, The map  $\cup\theta$  versus  $\cup e$ ], see also [Mor16]).

**Theorem II.6.3.** *The secondary Euler characteristic and the Weil-étale Euler characteristic are respectively additive and multiplicative with respect to big-to-tiny short exact sequences. That is, given a big-to-tiny short exact sequence*

$$0 \rightarrow F \rightarrow G \rightarrow H \rightarrow 0,$$

we have

$$\begin{aligned} E_U(G) &= E_U(F) + E_U(H), \\ \chi_U(G) &= \chi_U(F)\chi_U(H). \end{aligned}$$

*Proof.* The result for  $E_U$  is immediate given the remark above, the long exact sequence of Ext groups and the fact that  $\text{Ext}^i(F, \mathbb{G}_m)$  is zero for  $i < 0$  and torsion for  $i > 1$ .

Let  $\Delta$  denote the (non-distinguished) triangle

$$R\Gamma_{W,c}(U, F) \rightarrow R\Gamma_{W,c}(U, G) \rightarrow R\Gamma_{W,c}(U, H) \rightarrow .$$

By Theorem II.3.5,  $H(\Delta)$  is an exact sequence in  $\text{Gr}^b(\text{Mod}_{\mathbb{Z}}^{ft})$ , and we can consider the following diagram:

$$\begin{array}{ccc} f_{\mathbb{Z}}(R\Gamma_{W,c}(U, F)) \otimes f_{\mathbb{Z}}(R\Gamma_{W,c}(U, H)) & \xrightarrow{g_{\mathbb{Z}}(H\Delta)} & f_{\mathbb{Z}}(R\Gamma_{W,c}(U, G)) \\ \downarrow & & \downarrow \\ (f_{\mathbb{Z}}(R\Gamma_{W,c}(U, F)))_{\mathbb{R}} \otimes (f_{\mathbb{Z}}(R\Gamma_{W,c}(U, H)))_{\mathbb{R}} & \xrightarrow{(g_{\mathbb{Z}}(H\Delta))_{\mathbb{R}}} & (f_{\mathbb{Z}}(R\Gamma_{W,c}(U, G)))_{\mathbb{R}} \\ \downarrow H^*(\gamma)_{R\Gamma_{W,c}(U, F)} \otimes H^*(\gamma)_{R\Gamma_{W,c}(U, H)} & & \downarrow H^*(\gamma)_{R\Gamma_{W,c}(U, G)} \\ f_{\mathbb{R}}(R\Gamma_{W,c}(U, F)_{\mathbb{R}}) \otimes_{\mathbb{R}} f_{\mathbb{R}}(R\Gamma_{W,c}(U, H)_{\mathbb{R}}) & \xrightarrow{g_{\mathbb{R}}(H(\Delta_{\mathbb{R}})) = f_{\mathbb{R}}(\Delta_{\mathbb{R}})} & f_{\mathbb{R}}(R\Gamma_{W,c}(U, G)_{\mathbb{R}}) \\ \downarrow f_{\mathbb{R}}(\tau_F) \otimes f_{\mathbb{R}}(\tau_H) & & \downarrow f_{\mathbb{R}}(\tau_G) \\ f_{\mathbb{R}}(D_{F,\mathbb{R}}[1] \oplus D_{F,\mathbb{R}}[2]) \otimes_{\mathbb{R}} f_{\mathbb{R}}(D_{H,\mathbb{R}}[1] \oplus D_{H,\mathbb{R}}[2]) & \longrightarrow & f_{\mathbb{R}}(D_{G,\mathbb{R}}[1] \oplus D_{G,\mathbb{R}}[2]) \\ \uparrow & & \uparrow \\ f_{\mathbb{R}}(D_{F,\mathbb{R}}[1]) \otimes_{\mathbb{R}} f_{\mathbb{R}}(D_{F,\mathbb{R}}[2]) \otimes_{\mathbb{R}} f_{\mathbb{R}}(D_{H,\mathbb{R}}[1]) \otimes_{\mathbb{R}} f_{\mathbb{R}}(D_{H,\mathbb{R}}[2]) & \longrightarrow & f_{\mathbb{R}}(D_{G,\mathbb{R}}[1]) \otimes_{\mathbb{R}} f_{\mathbb{R}}(D_{G,\mathbb{R}}[2]) \\ \downarrow & & \downarrow \\ \mathbb{R} \otimes_{\mathbb{R}} \mathbb{R} & \xrightarrow{\text{mult}} & \mathbb{R} \end{array} \tag{II.6.2}$$

- Recall that we have set  $f_{\mathbb{Z}} := H^*(g_{\mathbb{Z}})$ , hence the top arrow would be  $f_{\mathbb{Z}}(\Delta)$  if  $\Delta$  was actually distinguished. Note that  $\Delta \otimes \mathbb{R}$  is distinguished. The top square is a well-defined map in  $\mathcal{P}_{\mathbb{Z} \rightarrow \mathbb{R}}$ , in particular it is a commutative square of abelian groups.
- Put  $H_{W,c}^*(U, -) := H^*(R\Gamma_{W,c}(U, -))$ . Since  $\gamma$  is a morphism of determinants, there is a commutative diagram associated to the exact sequence  $H\Delta$ :

$$\begin{array}{ccc} B \circ g_{\mathbb{Z}}(H_{W,c}^*(U, F)) \otimes B \circ g_{\mathbb{Z}}(H_{W,c}^*(U, H)) & \xrightarrow{B \circ g_{\mathbb{Z}}(H\Delta)} & B \circ g_{\mathbb{Z}}(H_{W,c}^*(U, G)) \\ \downarrow \gamma_{H_{W,c}^*(U, F)} \otimes \gamma_{H_{W,c}^*(U, H)} & & \downarrow \gamma_{H_{W,c}^*(U, G)} \\ g_{\mathbb{R}}(H_{W,c}^*(U, F)_{\mathbb{R}}) \otimes_{\mathbb{R}} g_{\mathbb{R}}(H_{W,c}^*(U, H)_{\mathbb{R}}) & \xrightarrow{g_{\mathbb{R}}(H\Delta)_{\mathbb{R}}} & g_{\mathbb{R}}(H_{W,c}^*(U, G)_{\mathbb{R}}) \end{array}$$

Unwinding the definitions, we see that the above diagram is exactly the second square.

- The third square commutes because  $\tau$  defines an isomorphism of distinguished triangles.
- The commutativity of the fourth square is a formal consequence of the associativity and commutativity axioms of the determinant, as we prove in the next Lemma II.6.4.
- The last square commutes by [Bre11, Lemma 3.6(ii) and its proof], and the fact that the unit structure on  $\mathbb{R} = f_{\mathbb{R}}(0)$  coming from the distinguished triangle  $0 \rightarrow 0 \rightarrow 0 \rightarrow$  is given by the multiplication map. Since the image by the multiplication map of  $x\mathbb{Z} \otimes y\mathbb{Z} \subset \mathbb{R} \otimes \mathbb{R}$  is  $xy\mathbb{Z} \subset \mathbb{R}$ , we are done.

□

**Lemma II.6.4.** *Let  $[-] : \mathcal{T} \rightarrow \mathcal{P}$  be a determinant functor on a triangulated category. Let  $\Delta : A \rightarrow B \rightarrow C \rightarrow$  and  $\Delta' : A' \rightarrow B' \rightarrow C' \rightarrow$  be distinguished triangles. The following diagram, coming from the  $3 \times 3$  diagram with columns  $\Delta, \Delta \oplus \Delta', \Delta'$ , commutes:*

$$\begin{array}{ccc}
 [B \oplus B'] & \longrightarrow & [B][B'] \\
 \downarrow & & \downarrow \\
 [A \oplus A'][C \oplus C'] & & \\
 \downarrow & & \downarrow \\
 [A][A'][C][C'] & \xrightarrow{\text{flip}} & [A][C][A'][C']
 \end{array}$$

We will follow below the convention that unmarked arrows will be obtained by functoriality of the determinant, applied to a distinguished triangle, and will go from the middle term to the product of the outer terms. We write the products as concatenations for compactness.

*Proof.* Consider diagram Diagram II.2: The outer diagram is exactly the one we want to prove to be commutative. Square 1 commutes by associativity, applied to the octahedron diagram

$$\begin{array}{ccccc}
 A \oplus A' & \longrightarrow & B \oplus A' & \longrightarrow & C \\
 \parallel & & \downarrow & & \downarrow \\
 A \oplus A' & \longrightarrow & B \oplus B' & \longrightarrow & C \oplus C' \\
 & & \downarrow & & \downarrow \\
 & & C' & \xlongequal{\quad} & C'
 \end{array}$$

Square 2 similarly commutes by the octahedron diagram

$$\begin{array}{ccccc}
 B & \longrightarrow & B \oplus A' & \longrightarrow & A' \\
 \parallel & & \downarrow & & \downarrow \\
 B & \longrightarrow & B \oplus B' & \longrightarrow & B' \\
 & & \downarrow & & \downarrow \\
 & & C' & \xlongequal{\quad} & C'
 \end{array}$$

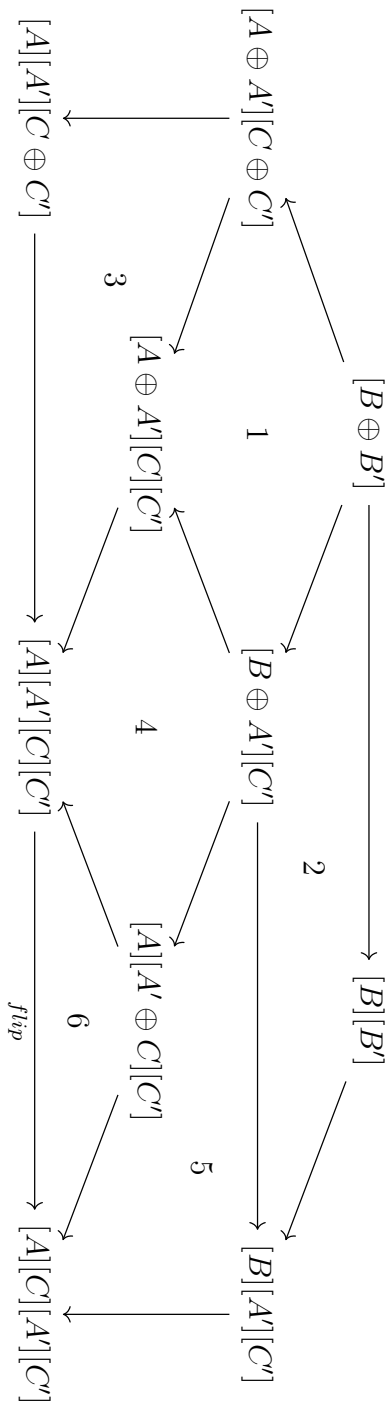


Diagram II.2

Square 3 is trivially commutative. Square 4 is obtained by tensoring on the right with  $[C']$  the associativity diagram coming from the following octahedron:

$$\begin{array}{ccccc}
A & \longrightarrow & A \oplus A' & \longrightarrow & A' \\
\parallel & & \downarrow & & \downarrow \\
A & \longrightarrow & B \oplus A' & \longrightarrow & A' \oplus C \\
& & \downarrow & & \downarrow \\
& & C & \xlongequal{\quad\quad\quad} & C
\end{array}$$

Square 5 is obtained by tensoring on the right with  $[C']$  the associativity diagram coming from the following octahedron:

$$\begin{array}{ccccc}
A & \longrightarrow & B & \longrightarrow & C \\
\parallel & & \downarrow & & \downarrow \\
A & \longrightarrow & B \oplus A' & \longrightarrow & A' \oplus C \\
& & \downarrow & & \downarrow \\
& & A' & \xlongequal{\quad\quad\quad} & A'
\end{array}$$

Finally, square 6 is commutative by the commutativity axiom. □

## II.6.2 Computations

We now use the results from Subsection II.3.2 and Appendix A to compute the Weil-étale characteristic.

**Definition II.6.5** (The regulator). *Let  $F$  be a  $\mathbb{Z}$ -constructible sheaf. Fix bases modulo torsion of  $H_c^0(U, F)$ ,  $H_c^1(U, F)$ ,  $\text{Ext}_U^1(F, \mathbb{G}_m)$  and  $\text{Hom}_U(F, \mathbb{G}_m)$ . Let  $R_0(F)$  be the absolute value of the determinant of the pairing*

$$H_c^0(U, F)_{\mathbb{R}} \times \text{Ext}_U^1(F, \mathbb{G}_m)_{\mathbb{R}} \rightarrow \mathbb{R}$$

*and  $R_1(F)$  the absolute value of the determinant of the pairing*

$$H_c^1(U, F)_{\mathbb{R}} \times \text{Hom}_U(F, \mathbb{G}_m)_{\mathbb{R}} \rightarrow \mathbb{R}$$

*in those bases. Those quantities do not depend on the choices, and we define the regulator  $R(F)$  of  $F$ :*

$$R(F) := \frac{R_1(F)}{R_0(F)}.$$

**Proposition II.6.6.** *Let  $F$  be a big sheaf on  $U$ . Then*

$$\chi_U(F) = \frac{R(F)[\text{Ext}_U^1(F, \mathbb{G}_m)][H_c^0(U, F)]}{[\text{Hom}_U(F, \mathbb{G}_m)_{\text{tor}}][H_c^1(U, F)_{\text{tor}}]}.$$

**Proposition II.6.7.** *Let  $F$  be a tiny sheaf on  $U$ . Then*

$$\chi_U(F) = \frac{R(F)[\text{Ext}_U^1(F, \mathbb{G}_m)_{\text{tor}}][H_c^0(U, F)_{\text{tor}}]}{[\text{Hom}_U(F, \mathbb{G}_m)][H_c^1(U, F)]}.$$



**Corollary II.6.8.** *Let  $i : v \hookrightarrow U$  be the inclusion of a closed point of  $U$  and  $M$  be a finitely generated discrete  $G_v$ -module. Then  $i_*M$  is tiny and*

$$\chi_U(i_*M) = \frac{[H^0(G_v, M)_{\text{tor}}]}{[H^1(G_v, M)]R(M)(\log N(v))^{\text{rank}_{\mathbb{Z}}(H^0(G_v, M))}},$$

where  $N(v) = |\kappa(v)|$  and  $R(M)$  is the determinant of the pairing

$$M_{\mathbb{Q}}^{G_v} \times \text{Hom}_{G_v}(M, \mathbb{Q}) \rightarrow \mathbb{Q}$$

after a choice of bases modulo torsion of  $M^{G_v}$  and  $\text{Hom}_{G_v}(M, \mathbb{Z})$ .

*Remark.*

- Note the factor  $(\log N(v))^{\text{rank}_{\mathbb{Z}}(H^0(G_v, M))}$  which comes from diagram Diagram II.1; there is no sign here because we take absolute values.
- The value  $\chi_v(M) := \chi_U(i_*M)$  doesn't depend on the embedding  $i : v \rightarrow U$  hence  $\chi_v$  is intrinsic to the scheme  $v$ .

**Proposition II.6.9.** *Suppose  $U = \text{Spec } \mathcal{O}_{K,S}$  and denote by  $R_S$ ,  $h_S$  and  $\omega$  the  $S$ -regulator,  $S$ -class number and number of roots of unity of  $K$  respectively. We have*

$$\begin{aligned} \chi_U(\mathbb{Z}) &= \frac{h_S R_S}{\omega}, \\ E_U(\mathbb{Z}) &= \text{rank}_{\mathbb{Z}} \mathcal{O}_{K,S}^{\times} = [S] - 1. \end{aligned}$$

We obtain from the analytic class number formula for the  $S$ -zeta function of  $K$  (the arithmetic zeta function of  $U$ ) that  $E_U(\mathbb{Z}) = \text{ord}_{s=0} \zeta_U$  and the formula for the special value  $\lim_{s \rightarrow 0} s^{-E_U(\mathbb{Z})} \zeta_U(s) = -\chi_U(\mathbb{Z})$ .<sup>14</sup>

*Proof.* In the construction of the regulator pairing, we noticed that for  $F = \mathbb{Z}$  it simply identifies with the classical regulator ; hence  $R(\mathbb{Z})$  is the  $S$ -regulator  $R_S$  of  $\mathcal{O}_{K,S}$ . Moreover, we have

$$\begin{aligned} \text{Hom}_U(\mathbb{Z}, \mathbb{G}_m)_{\text{tor}} &= (\mathcal{O}_{K,S}^{\times})_{\text{tor}} = \mu(K), \\ \text{Ext}_U^1(\mathbb{Z}, \mathbb{G}_m) &= H_{\text{et}}^1(U, \mathbb{G}_m) = \text{Pic}(U). \end{aligned}$$

Finally, the long exact sequence Equation (II.2.1) gives

$$0 \rightarrow H_c^0(U, \mathbb{Z}) \rightarrow \mathbb{Z} \xrightarrow{\Delta} \prod_{v \in S} \mathbb{Z} \rightarrow H_c^1(U, \mathbb{Z}) \rightarrow H_{\text{et}}^1(U, \mathbb{Z}),$$

where  $\Delta$  is the diagonal inclusion. We already saw that  $H_{\text{et}}^1(U, \mathbb{Z}) = 0$ . Thus  $H_c^1(U, \mathbb{Z}) \simeq \prod_{v \in S} \mathbb{Z}/\mathbb{Z}$  is free and  $H_c^0(U, \mathbb{Z}) = 0$ . We obtain

$$\begin{aligned} E_U(F) &= \text{rank}_{\mathbb{Z}}(\text{Hom}_U(\mathbb{Z}, \mathbb{G}_m)) - \text{rank}_{\mathbb{Z}}(\text{Ext}_U^1(\mathbb{Z}, \mathbb{G}_m)) = \text{rank}_{\mathbb{Z}}(\mathcal{O}_{K,S}^{\times}), \\ \chi_U(F) &= \frac{R_S[\text{Pic}(U)]}{[\mu(K)]} = \frac{h_S R_S}{\omega}. \end{aligned}$$

□

---

<sup>14</sup>see for instance [Tat84, I.2.2]

**Proposition II.6.10.** *Let  $U = C$  be the smooth proper curve associated to the function field  $K$ , and let  $\mathbb{F}_q$  be its field of constants. We have*

$$\chi_C(\mathbb{Z}) = \frac{h}{\omega \log(q)},$$

$$E_C(\mathbb{Z}) = -\text{rank}_{\mathbb{Z}} \text{Pic}(C) = -1.$$

where  $h = [\text{Pic}^0(C)]$  is the cardinality of the group of classes of degree 0 divisors and  $\omega$  the number of roots of unity of  $K$ .<sup>15</sup> We verify again that  $\text{ord}_{s=0} \zeta_C = E_C(\mathbb{Z})$  and  $\zeta_C^*(0) = -\chi_C(\mathbb{Z})$  for  $\zeta_C$  the arithmetic zeta function of  $C$ .<sup>16</sup>

*Proof.* The constant sheaf  $\mathbb{Z}$  is tiny on  $C$  because  $H^1(C, \mathbb{Z}) = 0$ , so we use Proposition II.6.7. We consider  $C$  as a curve over the field of constants  $\mathbb{F}_q$  of its function field, hence the degree map is surjective [Wei95, VII.5, cor. 5] and we have an exact sequence

$$0 \rightarrow \text{Pic}^0(C) \rightarrow \text{Pic}(C) \xrightarrow{\text{deg}} \mathbb{Z} \rightarrow 0$$

with  $\text{Pic}^0(C)$  the class group of degree zero divisors, which is finite by [Wei95, IV.4 Thm 7]. We have  $H^0(C, \mathbb{Z}) = \mathbb{Z}$  and  $H^0(C, \mathbb{G}_m) = \mathbb{F}_q^\times = \mu(K)$ . We obtain

$$\chi_U(\mathbb{Z}) = \frac{R(\mathbb{Z})[\text{Pic}^0(C)]}{\omega}.$$

Finally, by the discussion in paragraph II.4.2.6, the rational regulator pairing for  $C$  identifies via the degree map to the trivial pairing  $\mathbb{Q} \times \mathbb{Q} \rightarrow \mathbb{Q}$  with the natural integral bases<sup>17</sup>, hence the comparison between the regulator pairing and the rational regulator pairing gives  $R(\mathbb{Z}) = 1/\log(q)$ .  $\square$

### II.6.3 Functoriality of the Euler characteristic

**Proposition II.6.11.** *Let  $L/K$  be a finite Galois extension and  $\pi : V \rightarrow U$  the normalization of  $U$  in  $L$ . If  $F$  is a big or tiny sheaf on  $V$ , so is  $\pi_* F$  and*

$$\chi_U(\pi_* F) = \chi_V(F),$$

$$E_U(\pi_* F) = E_V(F).$$

*Proof.* As seen in paragraph II.4.2.1, we have isomorphisms

$$R \text{Hom}_U(\pi_* F, \mathbb{G}_m) \simeq R \text{Hom}_V(F, \mathbb{G}_m),$$

$$R\Gamma_c(U, \pi_* F) \simeq R\Gamma_c(V, F),$$

so the first assertion is immediate, as well as the equality  $E_U(\pi_* F) = E_V(F)$ . Similarly, there is also an isomorphism

$$R\hat{\Gamma}_c(U, \pi_* F) \simeq R\hat{\Gamma}_c(V, F).$$

Write

$$D_F^U := R \text{Hom}(R \text{Hom}_U(F, \mathbb{G}_m), \mathbb{Q}[-3]),$$

$$E_F^U := R \text{Hom}(R \text{Hom}_U(F, \mathbb{G}_m), \mathbb{Q}/\mathbb{Z}[-3]),$$

$$\beta_F^U \text{ for the arrow constructed in Subsection II.3.1.}$$

<sup>15</sup>Note that  $\omega = q - 1$

<sup>16</sup>see for instance [Wei95, VII.6, Proposition 4]

<sup>17</sup>Here we use that the degree map is surjective

We have seen in paragraph II.4.2.1 that the above isomorphisms induce isomorphisms of the regulator pairing

$$\begin{array}{ccc} R\Gamma_c(V, F) \otimes R\mathrm{Hom}_V(F, \mathbb{G}_m) & \longrightarrow & \mathbb{R}[-1] \\ \downarrow \simeq & & \parallel \\ R\Gamma_c(U, \pi_*F) \otimes R\mathrm{Hom}_U(\pi_*F, \mathbb{G}_m) & \longrightarrow & \mathbb{R}[-1] \end{array}$$

and Artin–Verdier pairing

$$\begin{array}{ccc} R\hat{\Gamma}_c(V, F) \otimes R\mathrm{Hom}_V(F, \mathbb{G}_m) & \longrightarrow & \mathbb{Q}/\mathbb{Z}[-3] \\ \downarrow \simeq & & \parallel \\ R\hat{\Gamma}_c(U, \pi_*F) \otimes R\mathrm{Hom}_U(\pi_*F, \mathbb{G}_m) & \longrightarrow & \mathbb{Q}/\mathbb{Z}[-3] \end{array}$$

Thus there is a commutative diagram

$$\begin{array}{ccccccc} D_F^V & \longrightarrow & E_F^V & \longleftarrow & R\hat{\Gamma}_c(V, F) & \longleftarrow & R\Gamma_c(V, F) \\ \downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq \\ D_{\pi_*F}^U & \longrightarrow & E_{\pi_*F}^U & \longleftarrow & R\hat{\Gamma}_c(U, \pi_*F) & \longleftarrow & R\Gamma_c(U, \pi_*F) \end{array}$$

Since  $\beta_{\pi_*F}^U$  and  $\beta_F^V$  are defined uniquely by their cohomology in degree 2 and 3 (and more generally morphisms from  $D_F^V$  to  $R\Gamma_c(U, \pi_*F)$  verify this property, by the same argument as for  $\beta$ ), it follows that the following diagram is commutative

$$\begin{array}{ccc} D_F^V & \xrightarrow{\beta_F^V} & R\Gamma_c(V, F) \\ \downarrow \simeq & & \downarrow \simeq \\ D_{\pi_*F}^U & \xrightarrow{\beta_{\pi_*F}^U} & R\Gamma_c(U, \pi_*F) \end{array}$$

Thus there exists an induced isomorphism on the cones, giving an isomorphism of distinguished triangles:

$$\begin{array}{ccccc} D_F^V & \longrightarrow & R\Gamma_c(V, F) & \longrightarrow & R\Gamma_{W,c}(V, F) \\ \downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq \\ D_{\pi_*F}^U & \xrightarrow{\beta_{\pi_*F}^U} & R\Gamma_c(U, \pi_*F) & \longrightarrow & R\Gamma_{W,c}(U, \pi_*F) \end{array}$$

The induced isomorphism is moreover unique, similarly to how we argued for the uniqueness of the complex  $R\Gamma_{W,c}$ . All the identifications we have made are compatible and we obtain  $\chi_U(\pi_*F) = \chi_V(F)$  by taking determinants.  $\square$

**Proposition II.6.12.** *Let  $j : V \subset U$  be an open immersion. If  $F$  is a big or tiny sheaf on  $V$ , then so is  $j_!F$ , and we have*

$$\begin{aligned} \chi_U(j_!F) &= \chi_V(F), \\ E_U(j_!F) &= E_V(F). \end{aligned}$$

*Proof.* The proof proceeds as the previous one, using Proposition II.2.1, the adjunction  $j_! \dashv j^*$  and the results from paragraph II.4.2.2.  $\square$

## II.6.4 The $L$ -function of a $\mathbb{Z}$ -constructible sheaf

Let  $X$  be an irreducible scheme of finite type over  $\mathbb{Z}$  and of dimension 1, and let  $F$  be a  $\mathbb{Z}$ -constructible sheaf on  $X$ . We define the  $L$ -function associated to  $F$  and go over its functorial properties.

**Definition II.6.13.** *Let  $i : v \rightarrow X$  be a closed point of  $X$ . The local factor with respect to  $F$  and  $v$  is*

$$L_v(F, s) = \det(I - N(v)^{-s} \text{Frob}_v | F_v \otimes \mathbb{C})^{-1},$$

where  $F_v$  is the discrete  $G_v := \text{Gal}(\kappa(v)^{\text{sep}}/\kappa(v))$ -module corresponding to  $i^*F$  and  $\text{Frob}_v \in G_v$  is the (geometric) Frobenius.

The  $L$ -function associated to  $F$  is

$$L_X(F, s) = \prod_{v \in X_0} L_v(F, s),$$

where  $X_0$  is the set of closed points of  $X$ . It is well-defined and holomorphic for  $\Re(s) > 1$ .

*Remark.*

- For  $F = \mathbb{Z}$  we obtain the arithmetic zeta function of  $X$ :

$$L_X(\mathbb{Z}) = \zeta_X.$$

- If  $i : v \rightarrow X$  is the inclusion of a closed point and  $M$  is a discrete  $G_v$ -module of finite type, we have

$$L_X(i_*M, s) = \det(I - N(v)^{-s} \text{Frob}_v | M \otimes \mathbb{C})^{-1}.$$

Let us investigate the functorial properties of the  $L$ -functions of  $\mathbb{Z}$ -constructible sheaves.

**Proposition II.6.14.**

1. If  $0 \rightarrow F \rightarrow G \rightarrow H \rightarrow 0$  is an exact sequence of  $\mathbb{Z}$ -constructible sheaves on  $X$ , then

$$L_X(G) = L_X(F)L_X(H).$$

2. Let  $\pi : Y \rightarrow X$  be a finite morphism, with  $Y$  irreducible. If  $F$  is a  $\mathbb{Z}$ -constructible sheaf on  $Y$ , then  $\pi_*F$  is  $\mathbb{Z}$ -constructible and

$$L_X(\pi_*F) = L_Y(F). \tag{II.6.3}$$

3. Let  $j : V \subset X$  an open subscheme and let  $F$  be a  $\mathbb{Z}$ -constructible sheaf on  $V$ . Then

$$L_X(j_!F) = L_V(F).$$

*Proof.* Only 2. is non-trivial. The pushforward  $\pi_*$  commutes with base change so we have

$$(\pi_*F)_v = \prod_{w|v} \text{ind}_{G_w}^{G_v} (F_w),$$

where the product is on finite places  $w$  above  $v$ , hence

$$L_v(\pi_*F, s) = \prod_{w|v} L_w(F, s)$$

since local factors behave well with respect to induced modules [Neu99b, prop. VII.10.4.(iv)].  $\square$

**Proposition II.6.15.** *Denote by  $g : \text{Spec}(K) \rightarrow X$  the generic point of  $X$  and let  $M$  be a discrete  $G_K$ -module of finite type. If  $v$  is a non-singular point of  $X$ , thus corresponding to a place of  $K$ , then*

$$(g_*M)_v = M^{I_v}$$

*with its natural  $G_v$ -action, where  $I_v$  is the inertia subgroup of a place  $\bar{v}$  of  $K^{\text{sep}}$  above  $v$ .*

*Proof.* This is local so we can suppose  $X$  is affine and regular. Fix an embedding  $K^{\text{sep}} \hookrightarrow \bar{K}^{\text{sep}}$ ; this determines an extension  $\bar{v}$  of  $v$  on  $K^{\text{sep}}$ . Let  $\mathcal{O}_v^{sh}$  be the strict henselization of  $X$  at  $v$ ; then  $\mathcal{O}_v^{sh}$  is the integral closure of  $\mathcal{O}_X(X)$  in the ring of integers of the maximal unramified extension  $K_v^{un}$  of the completion of  $K$  at  $v$ . Let  $K_v^{sh}$  denote the field of fractions. The above embedding identifies  $K_v^{sh}$  with the subfield of  $K^{\text{sep}}$  fixed by the inertia subgroup  $I_v$ . The formula for stalks of pushforwards then gives

$$(g_*M)_v = H^0(K_v^{sh}, M) = M^{I_v}$$

endowed with its natural  $G_v \simeq D_v/I_v$ -action, where  $D_v = \{\sigma, \sigma\bar{v} = \bar{v}\} \subset G_K$  is the decomposition group.  $\square$

In particular, for a discrete  $G_K$ -module of finite type  $M$  and  $X$  regular proper, we have

$$L_X(g_*M, s) = L_K(M \otimes \mathbb{Q}, s),$$

where the latter is the Artin  $L$ -function of the rational representation  $M \otimes \mathbb{Q}$  of  $G_K$ . We now want to link the  $L$ -function of  $F$  to a classical Artin  $L$ -function of a representation of  $G_K$ . By restricting  $F$  to the regular locus of  $X$ , we modify its  $L$ -function by a finite number of local factors. We can thus suppose that  $X$  is regular. The kernel and cokernel of the canonical map  $F \rightarrow g_*g^*F$  are  $\mathbb{Z}$ -constructible and supported on closed subschemes. From the proposition above, we deduce that the  $L$ -function of  $F$  on  $X$  differs from the Artin  $L$ -function of the rational  $G_K$ -representation  $g^*F \otimes \mathbb{Q}$  by a finite number of local factors.

From this discussion, we deduce:

**Proposition II.6.16.** *The  $L$ -function of a  $\mathbb{Z}$ -constructible sheaf is meromorphic on  $\mathbb{C}$ .*

*Proof.* If  $v \in X$  is a closed point, the local factor at  $v$  is obviously meromorphic; thus we are reduced to the meromorphicity of Artin  $L$ -functions, which is known.  $\square$

**Definition II.6.17.** *Let  $F$  be a  $\mathbb{Z}$ -constructible sheaf on  $X$ . We let  $n = \text{ord}_{s=0} L_X(F, s)$  the order of  $L_X(F, s)$  at  $s = 0$  and*

$$L_X^*(F, 0) = \lim_{s \rightarrow 0} L_X(F, s) s^{-n}$$

*the special value of  $L_X(F, s)$  at  $s = 0$ .*

We will relate in Subsection II.6.7 and Section II.7 the order and special value at  $s = 0$  to the previously constructed Euler characteristic, by reducing via the dévissage argument (§ II.2.0.1) to special cases.

## II.6.5 Extending the definition of the Weil-étale Euler characteristic

In this section we construct a well-defined functorial Euler characteristic for any  $\mathbb{Z}$ -constructible sheaf, bypassing a full construction of the Weil-étale complex, which will enable us to obtain a special values theorem.

**Definition II.6.18.** *Let  $F$  be a  $\mathbb{Z}$ -constructible sheaf on  $U$ . Let  $j : V \subset U$  be an open affine non-empty subscheme such that  $F|_V$  is locally constant and let  $i : Z := U \setminus V \hookrightarrow U$  be the inclusion of the closed complement. Put  $F_V := j_!(F|_V)$  and  $F_Z := i_*i^*F$ . By the remark below Definition II.3.1 and by Proposition II.6.12,  $F_V$  is big and  $F_Z$  is tiny. We thus define  $E_U(F)$  and the Weil-étale Euler characteristic of  $F$  as:*

$$\begin{aligned} E_U(F) &= E_U(F_V) + E_U(F_Z), \\ \chi_U(F) &= \chi_U(F_V)\chi_U(F_Z). \end{aligned}$$

*Remark.* If  $F$  is big or tiny, the above exact sequence is big-to-tiny, hence this definition is coherent with the previous one for  $F$ .

We first show that those quantities do not depend on the choice of  $V$ :

*Proof.* Let  $j' : V' \subset U$  be another open affine subscheme with  $F|_{V'}$  locally constant and let  $i' : Z' \hookrightarrow U$  be its closed complement. Without loss of generality we can suppose  $V' \subset V$ . Let  $t : T := V \setminus V' \hookrightarrow U$  be the closed inclusion. The open-closed decomposition lemma gives big-to-tiny short exact sequences

$$\begin{aligned} 0 \rightarrow (t_*t^*F) \rightarrow i'_*i'^*F \rightarrow i_*i^*F \rightarrow 0, \\ 0 \rightarrow j'_!F|_{V'} \rightarrow j_!F|_V \rightarrow t_*t^*F \rightarrow 0. \end{aligned}$$

The proof follows formally. □

We can now investigate functoriality properties:

**Proposition II.6.19.** *Let  $0 \rightarrow F \rightarrow G \rightarrow H \rightarrow 0$  be a short exact sequence of  $\mathbb{Z}$ -constructible sheaves on  $U$ . Then*

$$\chi_U(G) = \chi_U(F)\chi_U(H).$$

*Proof.* Choose a common open subscheme  $V$  on which the restriction of each sheaf is locally constant, then apply the open-closed decomposition lemma, giving a commutative diagram with exact rows and columns (using notation from Definition II.6.18):

$$\begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & F_V & \longrightarrow & F & \longrightarrow & F_Z \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & G_V & \longrightarrow & G & \longrightarrow & G_Z \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & H_V & \longrightarrow & H & \longrightarrow & H_Z \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & & 0 & & 0 \end{array}$$

The left column (resp. right) is a short exact sequence of big (resp. tiny) sheaves, so the proof follows formally.  $\square$

**Proposition II.6.20.** *Let  $L/K$  be a finite Galois extension and  $\pi : V \rightarrow U$  the normalization of  $U$  in  $L$ . If  $F$  is a  $\mathbb{Z}$ -constructible sheaf on  $V$ , then*

$$\begin{aligned}\chi_U(\pi_*F) &= \chi_V(F), \\ E_U(\pi_*F) &= E_V(F).\end{aligned}$$

*Proof.* We can fix open affine subschemes  $j : U' \subset U$ ,  $j' : V' \rightarrow V$  such that  $F|_{V'}$  is locally constant and  $\pi' := \pi|_{V'} : V' \rightarrow U'$  is the normalization of  $U'$  in  $L$ . Then  $(\pi_*F)|_{U'} = \pi'_*(F|_{V'})$  is locally constant. Denote by  $i : Z = U \setminus U' \hookrightarrow U$  and  $i' : Z' = V \setminus V' \hookrightarrow V$  the closed inclusions. The open-closed decomposition lemma gives short exact sequences

$$\begin{aligned}0 \rightarrow (\pi_*F)_{U'} \rightarrow \pi_*F \rightarrow (\pi_*F)_Z \rightarrow 0, \\ 0 \rightarrow F_{V'} \rightarrow F \rightarrow F_{Z'} \rightarrow 0.\end{aligned}$$

Denote by  $\pi_Z$  the base change of  $\pi$  to  $Z$ . Then  $\pi$  is finite so  $i_*i^*\pi_* = i_*\pi_{Z,*}i'^* = \pi_*i'_*i'^*F$  thus  $(\pi_*F)_Z = \pi_*F_{Z'}$  and we get

$$\chi_U(\pi_*F) := \chi_{U'}(\pi'_*F)\chi_U((\pi_*F)_Z) = \chi'_{V'}(F)\chi_U(\pi_*F_{Z'}) = \chi'_{V'}(F)\chi_V(F_{Z'}) =: \chi_V(F).$$

$\square$

**Proposition II.6.21.** *Let  $j : V \subset U$  be an open immersion. Let  $F$  be a  $\mathbb{Z}$ -constructible sheaf on  $V$ . We have*

$$\begin{aligned}\chi_U(j_!F) &= \chi_V(F), \\ E_U(j_!F) &= E_V(F).\end{aligned}$$

*Proof.* Let  $j' : V' \subset V$  an open affine subscheme with  $F|_{V'}$  locally constant, and let  $i' : Z' := V \setminus V' \hookrightarrow V$ ,  $i : Z = U \setminus V' \hookrightarrow U$  denote the closed inclusions. Then  $(j_!F)|_{V'} = F|_{V'}$ ,  $i^*j_!F = t_*i'^*F$  (with  $t$  the clopen immersion  $Z' \subset Z$ ) and  $i_*t_* = j_!i'_*$  so the short exact sequences given by the open-closed decomposition lemma gives

$$\begin{aligned}\chi_U(j_!F) &:= \chi_U((jj')_!(j_!F)|_{V'})\chi_U(i_*i^*j_!F) = \chi_U((jj')_!F|_{V'})\chi_U(j_!i'_*i'^*F) \\ &= \chi_V(j'_!F|_{V'})\chi_V(i'_*i'^*F) \\ &=: \chi_V(F).\end{aligned}$$

$\square$

## II.6.6 Explicit formula for the Weil-étale Euler characteristic

**Proposition II.6.22.** *Let  $F$  be a  $\mathbb{Z}$ -constructible sheaf on  $U$ . Then*

$$\chi_U(F) = \frac{[H_c^0(U, F)_{\text{tor}}][\text{Ext}_U^1(F, \mathbb{G}_m)_{\text{tor}}]R(F)}{[H_c^1(U, F)_{\text{tor}}][\text{Hom}_U(F, \mathbb{G}_m)_{\text{tor}}]}.$$

*Proof.* Let  $F_V, F_Z$  be as in Definition II.6.18, so that  $\chi_U(F) := \chi_U(F_V)\chi_U(F_Z)$ . Consider the two following exact sequences, viewed as acyclic cochain complexes :

$$\begin{aligned} A^\bullet : 0 &\rightarrow H_c^0(U, F_V) \rightarrow H_c^0(U, F) \rightarrow H_c^0(U, F_Z) \\ &\rightarrow H_c^1(U, F_V) \rightarrow H_c^1(U, F) \rightarrow H_c^1(U, F_Z) \rightarrow I \rightarrow 0, \\ B^\bullet : 0 &\rightarrow \mathrm{Hom}_U(F_Z, \mathbb{G}_m) \rightarrow \mathrm{Hom}_U(F, \mathbb{G}_m) \rightarrow \mathrm{Hom}_U(F_V, \mathbb{G}_m) \\ &\rightarrow \mathrm{Ext}_U^1(F_Z, \mathbb{G}_m) \rightarrow \mathrm{Ext}_U^1(F, \mathbb{G}_m) \rightarrow \mathrm{Ext}_U^1(F_V, \mathbb{G}_m) \rightarrow J \rightarrow 0, \end{aligned}$$

where  $I$  is the image of  $H_c^1(U, F_Z)$  in  $H_c^2(U, F_V)$  and  $J$  is the image of  $\mathrm{Ext}_U^1(F_V, \mathbb{G}_m)$  in  $\mathrm{Ext}_U^2(F_Z, \mathbb{G}_m)$ . We choose the convention of putting the first non-zero term of  $A$  in degree 0 and the last non-zero term of  $B$  in degree 2. Since  $F_V$  is big and  $F_Z$  is tiny, using Artin–Verdier duality we have isomorphisms between finite groups  $H_c^1(U, F_Z) \simeq \mathrm{Ext}_U^2(F_Z, \mathbb{G}_m)^D$  and  $H_c^2(U, F_V) \simeq \mathrm{Ext}_U^1(F_V, \mathbb{G}_m)^D$ , thus Pontryagin duality gives  $I \simeq J^D$ .

The regulator pairing gives an isomorphism  $\phi : B_{\mathbb{R}} \rightarrow \mathrm{Hom}(A_{\mathbb{R}}, \mathbb{R})[-1]$ ,<sup>18</sup> so Lemma A.0.4 together with Propositions II.6.6 and II.6.7 give:

$$\begin{aligned} 1 &= \frac{[H_c^0(U, F)_{\mathrm{tor}}][\mathrm{Ext}_U^1(F, \mathbb{G}_m)_{\mathrm{tor}}]R(F)}{[H_c^1(U, F)_{\mathrm{tor}}][\mathrm{Hom}_U(F, \mathbb{G}_m)_{\mathrm{tor}}]} \cdot \frac{[J]}{[I]} \cdot \frac{1}{\chi_U(F_V)\chi_U(F_Z)} \\ &= \frac{[H_c^0(U, F)_{\mathrm{tor}}][\mathrm{Ext}_U^1(F, \mathbb{G}_m)_{\mathrm{tor}}]R(F)}{[H_c^1(U, F)_{\mathrm{tor}}][\mathrm{Hom}_U(F, \mathbb{G}_m)_{\mathrm{tor}}]} \cdot \frac{1}{\chi_U(F)}, \end{aligned}$$

where  $R(F)$  is the regulator of  $F$  of Definition II.6.5; the result follows.  $\square$

## II.6.7 Computations, part II, and the special value theorem

Our aim is to prove theorem II.C by using the dévissage argument (§ II.2.0.1) to reduce to special cases. The case of  $F = \mathbb{Z}$  was already shown; let us handle now the case of a constructible sheaf.

**Proposition II.6.23.** *Let  $F$  be a constructible sheaf on  $U$ . Then  $\chi_U(F) = 1$  and  $E_U(F) = 0$ . As the  $L$ -function of a constructible sheaf is 1, we have  $\mathrm{ord}_{s=0} L_U(F, s) = E_U(F)$  and  $L_U^*(F, 0) = \chi_U(F)$ .*

*Proof.* Let us first treat the case where  $F$  is supported on a finite set ; by functoriality, this reduces to the case where  $F$  is of the form  $i_*M$ , with  $i : v \rightarrow U$  the inclusion of a closed point and  $M$  a finite discrete  $G_v$ -module. By Corollary II.6.8, we have

$$\chi_U(i_*M) = \frac{[H^0(G_v, M)]}{[H^1(G_v, M)]}.$$

Since  $M$  is finite, we have [Ser68, XIII.1 prop. 1] that  $H^1(G_v, M) = M/(\varphi - 1)M$ , where  $\varphi$  is the Frobenius. Hence the exact sequence

$$0 \rightarrow H^0(G_v, M) \rightarrow M \xrightarrow{\varphi - 1} M \rightarrow H^1(G_v, M) \rightarrow 0 \quad (\text{II.6.4})$$

gives the result by taking cardinals.

Let us now treat the general case. We have seen in paragraph II.4.2.5 that the regulator pairing of  $F$  is trivial after base change to  $\mathbb{R}$ , thus  $R(F) = 1$  and  $E_U(F) = 0$ . By Proposition II.6.22 and Artin–Verdier duality we have

$$\chi_U(F) = \frac{[\hat{H}_c^2(U, F)][H_c^0(U, F)]}{[\hat{H}_c^3(U, F)][H_c^1(U, F)]}.$$

<sup>18</sup>hence the choice of indexing convention



Suppose we are in the number field case. We want to apply [Mil06, II.2.13]; by functoriality we can throw away a finite number of points, and thus we can assume that there is an integer  $m$  invertible on  $U$  such that  $mF = 0$ . Thus using [Mil06, II.2.13], we obtain further

$$\chi_U(F) = \frac{[\hat{H}_c^1(U, F)][H_c^0(U, F)]}{[\hat{H}_c^0(U, F)][H_c^1(U, F)]} \prod_{v \in S_\infty} [H^0(K_v, F_{\eta_v})].$$

Recall that the cofiber  $T$  of  $R\Gamma_c(U, -) \rightarrow R\hat{\Gamma}_c(U, -)$  computes homology at the archimedean places (equation Equation (II.2.2)). For a cyclic group  $G$  with generator  $s$  and a  $G$ -module  $M$ , the exact sequence

$$0 \rightarrow H^0(G, M) \rightarrow M \xrightarrow{s-1} M \rightarrow H_0(G, M) \rightarrow 0$$

shows that  $[H_0(G, M)] = [H^0(G, M)]$ , thus

$$[H^0(T_F)] = \prod_{v \in S_\infty} [H^0(K_v, F_{\eta_v})].$$

Moreover, the Herbrand quotient of a finite  $G$ -module is 1 so we have, using the exact sequence Equation (II.2.1):

$$[\hat{H}_c^{-1}(U, F)] = \prod_{v \in S_\infty} [\hat{H}^{-1}(K_v, F_{\eta_v})] = \prod_{v \in S_\infty} [\hat{H}^{-2}(K_v, F_{\eta_v})] = [H^{-1}(T_F)].$$

The proposition now follows from the long exact cohomology sequence

$$\begin{aligned} 0 \rightarrow \hat{H}_c^{-1}(U, F) \rightarrow H^{-1}(T_F) \rightarrow H_c^0(U, F) \\ \rightarrow \hat{H}_c^0(U, F) \rightarrow H^0(T_F) \rightarrow H_c^1(U, F) \rightarrow \hat{H}_c^1(U, F) \rightarrow 0 \end{aligned}$$

by taking cardinals.

In the function field case, by functoriality we can suppose that  $U = C$  is a proper smooth curve over  $k = \mathbb{F}_p$ . Put  $\bar{X} := X \times_k \bar{k}$  where  $\bar{k}$  is an algebraic closure of  $k$ , and let  $\bar{F}$  denote the pullback of  $F$  to  $\bar{X}$ . Then the cohomology groups  $H^i(\bar{X}, \bar{F})$  are finite [Mil80, VI.2.1]. For  $M$  a finite  $G_k := \text{Gal}(\bar{k}/k)$ -module, we have  $H^i(G_k, M) = 0$  for  $i \geq 2$  thus the spectral sequence

$$E_2^{p,q} = H^p(G_k, H^q(\bar{X}, \bar{F})) \Rightarrow H^{p+q}(X, F)$$

degenerates at the  $E_2$  page to give short exact sequences

$$0 \rightarrow H^1(G_k, H^{i-1}(\bar{X}, \bar{F})) \rightarrow H^i(X, F) \rightarrow H^i(\bar{X}, \bar{F})^{G_k} \rightarrow 0$$

for all  $i \in \mathbb{Z}$ . From Equation (II.6.4) we deduce moreover that  $[H^1(G_k, H^i(\bar{X}, \bar{F}))] = [H^i(\bar{X}, \bar{F})^{G_k}]$ , which concludes the proof.  $\square$

We now prove theorem II.C:

**Theorem II.6.24** (Special values formula for  $\mathbb{Z}$ -constructible sheaves). *Let  $F$  be a  $\mathbb{Z}$ -constructible sheaf on a 1-dimensional regular irreducible scheme  $U$  of finite type over  $\text{Spec}(\mathbb{Z})$  and let  $F$  be a  $\mathbb{Z}$ -constructible sheaf on  $U$ . Then*

$$\begin{aligned} \text{ord}_{s=0} L_U(F, s) &= E_U(F), \\ L_U^*(F, 0) &= \pm \chi_U(F). \end{aligned}$$

*Proof.* Given the functorial properties of the quantities involved, using the dévissage argument (§ II.2.0.1) we reduce to the special cases of a constructible sheaf, which is the previous proposition, of the constant sheaf  $\mathbb{Z}$  on  $\mathrm{Spec}(\mathcal{O}_{K,S})$  or on a regular proper curve, which are Proposition II.6.9 and Proposition II.6.10, and of a pushforward  $i_*M$  of a finite type torsion-free discrete  $G_v$ -module for  $i : v \hookrightarrow U$  a closed point.

Let  $v = \mathrm{Spec}(\mathbb{F}_q)$  be the spectrum of a finite field and let  $M$  be a finite type  $G_v$ -module. Define the Weil-étale Euler characteristic of  $M$  by  $\chi_v(M) := \chi_{\mathrm{Spec}(\mathbb{Z})}(f_*M)$  where  $f : v \rightarrow \mathrm{Spec}(\mathbb{Z})$  is the structural morphism. Then  $f$  decomposes as  $v \xrightarrow{\pi} v_0 \xrightarrow{i} \mathrm{Spec}(\mathbb{Z})$  with  $v_0 = \mathrm{Spec}(\mathbb{F}_p)$ . Since  $\pi$  is finite étale, we have

$$Rf^! \mathbb{G}_m = \pi^* Ri^! \mathbb{G}_m = \pi^* \mathbb{Z}[-1] = \mathbb{Z}[-1].$$

Moreover  $R\Gamma_c(\mathrm{Spec}(\mathbb{Z}), f_*M) = R\Gamma(v, M)$  and the canonical map

$$\mathbb{Z} = H^0(G_v, \mathbb{Z}) \rightarrow H^0(G_{v_0}, \mathbb{Z}) = \mathbb{Z}$$

is multiplication by  $[\mathbb{F}_q : \mathbb{F}_p]$ . It follows using Proposition II.6.22 and II.6.8, and an analysis similar to the one in § II.4.2.3 (to compute  $R(f_*M)$ ) that  $\chi_v(M) = \chi_U(i_*M)$  as soon as  $i : v \hookrightarrow U$  realizes  $v$  as a closed point of  $U$ . If  $\pi : w = \mathrm{Spec}(\mathbb{F}_{q^n}) \rightarrow v$  is a finite morphism and  $M$  is a finite type  $G_w$ -module, we immediately find  $\chi_v(\mathrm{ind}_{G_v}^{G_w} M) = \chi_w(M)$ .

We have to show that  $\mathrm{ord}_{s=0} L_v(M, s) = E_U(i_*M)$  and

$$L_v^*(M, 0) = \chi_v(M)$$

for  $M$  a finite type torsion-free discrete  $G_v$ -module; since  $\chi_v$  and  $E_U(i_*M) = \mathrm{rank}_{\mathbb{Z}} H^0(G_v, M)$  behave well with induction, and the case of  $M$  finite was already treated, by Artin induction we reduce to  $M = \mathbb{Z}$ . Then  $L_v(\mathbb{Z}, s) = 1/(1 - q^{-s})$ , so  $L_v^*(\mathbb{Z}, 0) = 1/\log q$  with order 1; on the other hand,  $\mathrm{rank}_{\mathbb{Z}} H^0(G_v, \mathbb{Z}) = 1$  and Corollary II.6.8 gives  $\chi_v(\mathbb{Z}) = 1/\log(q)$ . This concludes the proof.  $\square$

*Remark.*

- In [GS20], the sign in Theorem II.6.24 is identified.
- The formula for proper smooth curves was already shown in [GS20, 3.1]; note that the  $\log(q)$  factor in *ibid.* is integrated in our Weil-étale Euler characteristic, coming from the regulator pairing.
- We chose to define the  $L$ -functions of  $\mathbb{Z}$ -constructible sheaves with geometric Frobenii to fit with the modern definitions;  $\mathbb{Z}$ -constructible sheaves are geometric data. But, we could have defined the  $L$ -functions with arithmetic Frobenii instead and the formulas for the order and special value would still hold. Indeed, it is easy to see that a local factor has the same special value *up to sign* at  $s = 0$  whether it be defined with arithmetic or geometric Frobenius. By the devissage argument, the  $L$ -functions of  $\mathbb{Z}$ -constructible sheaves are essentially, up to changing some local factors and taking powers, products of arithmetic zeta functions and their inverses. Since zeta functions are insensitive to the difference between arithmetic and geometric Frobenius, it follows that up to sign the special value at  $s = 0$  is the same whether we define the  $L$ -function with arithmetic or geometric Frobenii.

## II.7 Special values for $L$ -functions of $\mathbb{Z}$ -constructible sheaves in the non-regular case

We obtain as a corollary a result generalizing and precisizing [Tra16]; we allow singular points, we work also on curves, and there is no need for a correcting factor for the 2-torsion in the definition of the Weil-étale Euler characteristic in the presence of real places.

First, in the regular case, write  $U = \text{Spec}(\mathcal{O}_{K,S})$  in the number field case or  $U = C \setminus S$  for  $C$  the smooth complete curve with function  $K$  in the function field case. The following definition is standard:

**Definition.** *Let  $V$  be a finite-dimensional discrete complex representation of  $G_K$ . The  $S$ -partial Artin  $L$ -function of  $V$  is*

$$L_{K,S}(V, s) := \prod_{v \notin S} \det(I - N(v)^{-s} \text{Frob} | V^{I_v} )^{-1}.$$

If  $g : \text{Spec}(K) \rightarrow U$  is the inclusion of the generic point, we have seen that for a discrete  $G_K$ -module  $M$  of finite type, the sheaf  $g_*M$  is  $\mathbb{Z}$ -constructible and  $L_U(g_*M, s) = L_{K,S}(M \otimes \mathbb{Q}, s)$ . Hence the following corollary:

**Corollary II.7.1.** *Let  $M$  be a discrete  $G_K$ -module of finite type. Then*

$$\begin{aligned} \text{ord}_{s=0} L_{K,S}(M \otimes \mathbb{Q}, s) &= E_U(g_*M), \\ L_{K,S}^*(M \otimes \mathbb{Q}, 0) &= \pm \chi_U(g_*M). \end{aligned}$$

*Explicitly, we have:*

$$\begin{aligned} E_U(g_*M) &= \text{rank}_{\mathbb{Z}} H_c^1(U, g_*M) - \text{rank}_{\mathbb{Z}} H_c^0(U, g_*M) \\ &= \text{rank}_{\mathbb{Z}} \text{Hom}_U(g_*M, \mathbb{G}_m) - \text{rank}_{\mathbb{Z}} \text{Ext}_U^1(g_*M, \mathbb{G}_m), \\ \chi_U(g_*M) &= \frac{R(g_*M)[\text{Hom}_U(g_*M, \mathbb{G}_m)_{\text{tor}}][H_c^0(U, g_*M)_{\text{tor}}]}{[\text{Ext}_U^1(g_*M, \mathbb{G}_m)_{\text{tor}}][H_c^1(U, g_*M)_{\text{tor}}]}, \end{aligned}$$

where  $R(g_*M) := \frac{R_1(g_*M)}{R_0(g_*M)}$  with  $R_0(g_*M)$  the absolute value of the determinant of the pairing

$$H_c^0(X, g_*M)_{\mathbb{R}} \times \text{Ext}_U^1(g_*M, \mathbb{G}_m)_{\mathbb{R}} \rightarrow \mathbb{R}$$

and  $R_1(g_*M)$  the absolute value of the determinant of the pairing

$$H_c^1(U, g_*M)_{\mathbb{R}} \times \text{Hom}_U(g_*M, \mathbb{G}_m)_{\mathbb{R}} \rightarrow \mathbb{R}$$

*in bases modulo torsion.*

Let  $K$  be a global field and  $X$  be an irreducible scheme, finite type over  $\mathbb{Z}$ , of dimension 1 and with function field  $K$ . If the morphism  $X \rightarrow \text{Spec}(\mathbb{Z})$  factors through  $\text{Spec}(\mathbb{F}_p)$ ,  $X$  is a curve over  $\mathbb{F}_p$ . Otherwise morphism  $X \rightarrow \text{Spec}(\mathbb{Z})$  is dominant so it is quasi-finite, hence by Zariski's main theorem  $X$  is the open subscheme of the spectrum of a finite  $\mathbb{Z}$ -algebra, thus if  $X$  is reduced it is an open subscheme of the spectrum of an order in  $K$ . We let  $Z$  denote the singular locus of  $X$  and  $U$  the regular locus.

Following [Den87a], define the complex of sheaves

$$\mathbb{G}_X = \left[ g_* \mathbb{G}_{m,\eta} \rightarrow \bigoplus_{v \in X_0} (i_v)_* \mathbb{Z} \right]$$

where  $X_0$  denotes the closed points,  $\eta = \text{Spec}(K)$  is the generic point and  $g$  (resp.  $i_v$ ) is the canonical morphism  $\eta \rightarrow X$  (resp.  $v \rightarrow X$ ). If  $X$  is regular, we have  $\mathbb{G}_X \simeq \mathbb{G}_{m,X}$  [Gro67, 21.6.9].

Let  $\pi : Y \rightarrow X_{red} \rightarrow X$  denote the normalization of  $X_{red}$ . Define cohomology with compact support as in [Mil06, II.6], through the fiber sequence

$$R\Gamma_c(X, -) \rightarrow R\Gamma(X, -) \rightarrow \prod_{v \in S} R\Gamma(K_v, (-)_{\eta_v}),$$

where  $S$  is the set of places of  $K$  not corresponding to a point of  $Y$ . We let again  $H_c^i(X, -)$  denote the corresponding cohomology groups; Propositions II.2.1 and II.2.4, as well as Artin–Verdier duality (Theorem II.3.2) still hold; this is easily seen by comparing  $X$  to its regular locus  $U$ .

**Corollary II.7.2.** *Let  $f : X \rightarrow T$  denote either a quasi-finite morphism  $X \rightarrow T := \mathbb{P}_{\mathbb{F}_p}^1$  (in the function field case), which exists by the projective Noether normalization lemma, or the quasi-finite structural morphism  $X \rightarrow T := \text{Spec}(\mathbb{Z})$  (in the number field case); then  $f_!$  is well-defined. Let  $F$  be a  $\mathbb{Z}$ -constructible sheaf on  $X$ . We have the order and special value formula at  $s = 0$ :*

$$\begin{aligned} \text{ord}_{s=0} L_X(F, s) &= E_T(f_!F), \\ L_X^*(F, 0) &= \pm \chi_T(f_!F). \end{aligned}$$

Explicitly, we have:

$$\begin{aligned} E_T(f_!F) &= \text{rank}_{\mathbb{Z}} H_c^1(X, F) - \text{rank}_{\mathbb{Z}} H_c^0(X, F) = \text{rank}_{\mathbb{Z}} \text{Hom}_X(F, \mathbb{G}_X) - \text{rank}_{\mathbb{Z}} \text{Ext}^1(F, \mathbb{G}_X), \\ \chi_T(f_!F) &= \frac{R(F)[\text{Hom}_X(F, \mathbb{G}_X)_{\text{tor}}][H_c^0(X, F)_{\text{tor}}]}{[\text{Ext}_X^1(F, \mathbb{G}_X)_{\text{tor}}][H_c^1(X, F)_{\text{tor}}]}, \end{aligned}$$

where  $R(F) := \frac{R_1(F)}{R_0(F)}$  with  $R_0(F)$  the absolute value of the determinant of the pairing

$$H_c^0(X, F)_{\mathbb{R}} \times \text{Ext}_X^1(F, \mathbb{G}_X)_{\mathbb{R}} \rightarrow \mathbb{R}$$

and  $R_1(F)$  the absolute value of the determinant of the pairing

$$H_c^1(X, F)_{\mathbb{R}} \times \text{Hom}_X(F, \mathbb{G}_X)_{\mathbb{R}} \rightarrow \mathbb{R}$$

in bases modulo torsion.

*Proof.* We have  $L_T(f_!F, s) = L_X(F, s)$  and  $R\Gamma_c(T, f_!F) = R\Gamma_c(X, F)$ . The finite base change theorem implies that

$$Rf^! \mathbb{G}_{m,T} = Rf^! \mathbb{G}_T = \mathbb{G}_X,$$

which shows that  $R\text{Hom}_T(f_!g_*M, \mathbb{G}_m) = R\text{Hom}_X(g_*M, \mathbb{G}_X)$ . Theorem II.6.24 then gives the first result, and for the second it suffices to apply Proposition II.6.22.  $\square$

**Corollary II.7.3.** *Let  $f : X \rightarrow T$  denote either a quasi-finite morphism  $X \rightarrow T := \mathbb{P}_{\mathbb{F}_p}^1$  (in the function field case), which exists by the projective Noether normalization lemma, or the quasi-finite structural morphism  $X \rightarrow T := \text{Spec}(\mathbb{Z})$  (in the number field case); then  $f_!$  is well-defined. We have the order and special value formula at  $s = 0$ :*

$$\begin{aligned} \text{ord}_{s=0} L_{K,X}(M, s) &= E_T(f_!g_*M), \\ L_{K,X}^*(M, 0) &= \pm \chi_T(f_!g_*M). \end{aligned}$$

*Proof.* The sheaf  $g_*M$  is  $\mathbb{Z}$ -constructible. □

**Corollary II.7.4.** *Suppose  $X$  is affine. We have the order and special value formula at  $s = 0$  for the arithmetic zeta function of  $X$ :*

$$\begin{aligned} \text{ord}_{s=0}\zeta_X &= \text{rank}_{\mathbb{Z}}(\text{CH}_0(X, 1)), \\ \zeta_X^*(0) &= -\frac{[\text{CH}_0(X)]R_X}{\omega}, \end{aligned}$$

where  $R_X$  is the absolute value of the determinant of the regulator pairing

$$H_c^1(X, \mathbb{Z})_{\mathbb{R}} \times \text{CH}_0(X, 1)_{\mathbb{R}} \rightarrow \mathbb{R}$$

after a choice of bases modulo torsion and  $\omega$  is the number of roots of unity in  $K$ .

*Proof.* We apply cocorollary II.7.2 to  $F = \mathbb{Z}$ . Using  $R^q g_* \mathbb{G}_m = 0$  for  $q > 0$  (see [Mil06, II.1.4]), hypercohomology computations give

$$\begin{aligned} H^0(X, \mathbb{G}_X) &= \ker(K^\times \xrightarrow{\sum \text{mult}_v} \bigoplus_{v \in X_0} \mathbb{Z}) = \text{CH}_0(X, 1), \\ H^1(X, \mathbb{G}_X) &= \text{coker}(K^\times \xrightarrow{\sum \text{mult}_v} \bigoplus_{v \in X_0} \mathbb{Z}) = \text{CH}_0(X). \end{aligned}$$

Moreover it is clear from the above that  $\text{CH}_0(X, 1)_{\text{tor}} = \mu(K)$ .

From the short exact sequence

$$0 \rightarrow \mathbb{Z} \rightarrow \pi_* \mathbb{Z} \rightarrow \bigoplus_{v \in Z} i_{v,*} (\bigoplus_{\pi(w)=v} \text{ind}_{G_v}^{G_w} \mathbb{Z}) / \mathbb{Z} \rightarrow 0$$

with  $\pi : Y \rightarrow X$  the normalization,  $G_v = \text{Gal}(\kappa(v)^{\text{sep}}/\kappa(v))$  and  $G_w$  the Galois groups of the residue fields, we find by taking cohomology with compact support that  $H_c^0(X, \mathbb{Z}) = 0$  and  $H_c^1(X, \mathbb{Z})$  is torsion-free<sup>19</sup>. The regulator  $R(\mathbb{Z})$  contains no contribution from the perfect pairing of trivial vector spaces

$$H_c^0(X, \mathbb{Z})_{\mathbb{R}} \times \text{CH}_0(X)_{\mathbb{R}} \rightarrow \mathbb{R},$$

hence we have  $R(\mathbb{Z}) = R_X$ .

If  $i : v \rightarrow X$  is a closed point and  $M$  is a finite type discrete  $G_v$ -module, the eigenvalues of the Frobenius must be either 1,  $-1$  or pairs of complex conjugates. It follows that the  $L$ -function of  $i_*M$  has a positive special value at  $s = 0$ . Let  $U$  be the regular locus of  $X$ ; we deduces from this that the sign of  $\zeta_X^*(0)$  is the same as that of  $\zeta_U^*(0)$ , which is known to be  $-1$ . □

*Remark.*

- We can compute  $\text{rank}_{\mathbb{Z}} \text{CH}_0(X, 1) = s + t - 1$  where  $s = [S]$  and  $t = \sum_{v \in Z} ([\pi^{-1}(v)] - 1)$  by using the localization sequences associated to the open-closed decompositions  $U \hookrightarrow X \hookleftarrow Z$  and  $U \hookrightarrow Y \hookleftarrow \pi^{-1}Z$ .
- Note that the proof above shows that  $\text{CH}_0(X) = \text{Ext}_X^1(\mathbb{Z}, \mathbb{G}_X)$  is torsion, and Artin–Verdier duality states that it is finite type, hence it is finite.

<sup>19</sup>See the computations in the proof of Proposition II.6.9.

**Corollary II.7.5.** *Suppose  $X$  is a proper curve, and denote by  $\mathbb{F}_q$  the field of constants of  $K$ . We have the order and special value formula at  $s = 0$  for the arithmetic zeta function of  $X$ :*

$$\begin{aligned} \text{ord}_{s=0} \zeta_X &= \text{rank}_{\mathbb{Z}}(\text{CH}_0(X, 1)) - 1, \\ \zeta_X^*(0) &= -\frac{[\text{CH}_0(X)_{\text{tor}}]R_X}{\omega \log(q)}, \end{aligned}$$

where  $R_X$  is the absolute value of the determinant of the regulator pairing

$$H^1(X, \mathbb{Z})_{\mathbb{R}} \times \text{CH}_0(X, 1)_{\mathbb{R}} \rightarrow \mathbb{R}$$

after a choice of bases modulo torsion and  $\omega$  is the number of roots of unity in  $K^{20}$ .

*Proof.* As in the previous proof we have  $H^0(X, \mathbb{G}_X) = \text{CH}_0(X, 1)$ ,  $\text{CH}_0(X, 1)_{\text{tor}} = \mu(K)$  and  $H^1(X, \mathbb{G}_X) = \text{CH}_0(X)$ , but this time we have  $H^0(X, \mathbb{Z}) = \mathbb{Z}$ . We now want to identify the pairing  $H^0(X, \mathbb{Z})_{\mathbb{R}} \times \text{CH}_0(X)_{\mathbb{R}} \rightarrow \mathbb{R}$  coming from the regulator. For a closed point  $w \in Y$  above  $v \in X$ , denote by  $f_w := [\kappa_w : \kappa_v]$  the residual degree. Considering  $Y$  as a curve over  $\mathbb{F}_p$ , the degree map  $\text{deg}_Y : \text{Pic}(Y) \rightarrow \mathbb{Z}$  has image  $f\mathbb{Z}$ , where  $p^f = q$ . This easily implies that there is a similarly-defined degree map  $\text{deg}_X : \text{CH}_0(X) \rightarrow \mathbb{Z}$  with image  $f\mathbb{Z}$ . The snake lemma applied to

$$\begin{array}{ccccccc} 0 & \longrightarrow & K^\times / \mathcal{O}_Y(Y)^\times & \longrightarrow & \bigoplus_{w \in Y_0} \mathbb{Z} & \longrightarrow & \text{Pic}(Y) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & K^\times / \text{CH}_0(X, 1) & \longrightarrow & \bigoplus_{v \in X_0} \mathbb{Z} & \longrightarrow & \text{CH}_0(X) \longrightarrow 0 \end{array}$$

shows that  $\text{coker}(\text{Pic}(Y) \rightarrow \text{CH}_0(X)) = \bigoplus_{v \in Z} \mathbb{Z}/m_v\mathbb{Z}$ , where  $m_v = \text{pgcd}\{f_w, \pi(w) = v\}$ . Since  $\text{coker}(\text{Pic}(Y) \rightarrow \text{CH}_0(X)) = \text{coker}(\text{Pic}^0(Y) \rightarrow \ker(\text{deg}_X))$ , it follows from the finiteness of  $\text{Pic}^0(Y)$  that  $\ker(\text{deg}_X)$  is finite and hence equals  $\text{CH}_0(X)_{\text{tor}}$ . It is clear from the constructions that the pairing  $H^0(X, \mathbb{Z})_{\mathbb{R}} \times \text{CH}_0(X)_{\mathbb{R}} \rightarrow \mathbb{R}$  comes from the pairing  $H^0(X, \mathbb{Z}) \times \text{CH}_0(X) \rightarrow \text{CH}_0(X) \xrightarrow{\text{deg}} f\mathbb{Z}$ , hence similarly to the proof of II.6.10 we find that its determinant is  $f \log(p) = \log(q)$ ; it follows that  $R(\mathbb{Z}) = R_X / \log(q)$ .

The sign of  $\zeta_X^*(0)$  is proven to be  $-1$  similarly to the previous proof.  $\square$

*Remark.*

- We can compute  $\text{rank}_{\mathbb{Z}} \text{CH}_0(X, 1) = t$  where  $t = \sum_{v \in Z} ([\pi^{-1}(v)] - 1)$  by using the localization sequences associated to the open-closed decompositions  $U \hookrightarrow X \hookrightarrow Z$  and  $U \hookrightarrow Y \hookrightarrow \pi^{-1}Z$ . Thus if  $X$  is unibranch we have  $R_X = 1$ .
- We showed in the proof above that  $\text{CH}_0(X)_{\text{tor}}$  is the group of classes of 0-cycles of degree 0.

*Remark.* In [JP20], Jordan and Poonen prove an analytic class number formula for the zeta function of a reduced affine finite type  $\mathbb{Z}$ -scheme of pure dimension 1. They note that even for non-maximal orders in number fields, the formula seems new. Let us compare their results with ours. Define the completed zeta function

$$\widehat{\zeta}_K(s) = |d_K|^{s/2} \zeta_Y(s) \prod_{v \in S} L_v(\mathbb{Z}, s),$$

---

<sup>20</sup>Thus  $\omega = q - 1$ .

where  $\pi : Y \rightarrow X$  is as before the normalization of  $X$  in  $K$ ,  $d_K$  is the discriminant of  $K$ ,  $L_v(\mathbb{Z}, s)$  is the usual local factor if  $v$  is nonarchimedean and

$$\begin{aligned} L_v(\mathbb{Z}, s) &= 2(2\pi)^{-s}\Gamma(s), \quad v \text{ complex;} \\ L_v(\mathbb{Z}, s) &= \pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right), \quad v \text{ real.} \end{aligned}$$

By comparing  $\zeta_X$  and  $\zeta_Y$ , we get the equality

$$\zeta_X(s) = \widehat{\zeta}_K(s)|d_K|^{-s/2} \prod_{w \in S} L_w(\mathbb{Z}, s)^{-1} \prod_{v \in Z} \left( \frac{L_v(\mathbb{Z}, s)}{\prod_{\pi(w)=v} L_w(\mathbb{Z}, s)} \right).$$

Using the functional equation  $\widehat{\zeta}_K(s) = \widehat{\zeta}_K(1-s)$ , we find the order and special value at  $s = 0$ :

$$\begin{aligned} \text{ord}_{s=0} \zeta_X &= s + t - 1, \\ \zeta_X^*(0) &= -\frac{h(\mathcal{O})R(\mathcal{O})}{\omega(\mathcal{O})[\widetilde{\mathcal{O}} : \mathcal{O}]} \prod_{v \in Z} \frac{\gamma_v}{\prod_{\pi(w)=v} \gamma_w}, \end{aligned}$$

where  $Y = \text{Spec}(\widetilde{\mathcal{O}})$ ,  $\gamma_v = \frac{1-N(v)^{-1}}{\log(N(v))}$ ,  $h(\mathcal{O}) = [\text{Pic}(X)]$ ,  $R(\mathcal{O})$  is the covolume of the image in  $\mathbb{R}^s$  of  $\mathcal{O}^\times$  via the classical regulator of  $\widetilde{\mathcal{O}}$ , and  $\omega(\mathcal{O}) = \mathcal{O}_{\text{tor}}^\times$ . Our formula at  $s = 0$  seems to be better in several aspects: it does not suppose  $X$  to be reduced nor affine and it is defined in terms of global invariants of  $X$ .

# Chapter III

## Interlude: Euler characteristics for $\mathbb{Z}$ -constructible sheaves

In this chapter, we study Euler characteristics for categories of  $\mathbb{Z}$ -constructible sheaves on arithmetic schemes, and how one can do a dévissage for such sheaves to understand the Euler characteristics. The aim is to abstract away the part of the proof of Theorem II.6.24 that is completely formal. Recall that an Euler characteristic  $\chi$  on an abelian category  $\mathcal{A}$  is a map  $\chi$  from the isomorphism classes of  $\mathcal{A}$  to an abelian group  $\Gamma$ , such that  $\chi(B) = \chi(A) + \chi(C)$  for every short exact sequence  $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ . Equivalently, it is a group homomorphism from the Grothendieck group  $G(\mathcal{A})$  of the monoid of isomorphism classes of  $\mathcal{A}$  to  $\Gamma$ .

### III.1 Vanishing on constructible sheaves

Let  $X$  be an arithmetic scheme. We denote  $\mathrm{Sh}_{\mathbb{Z}}(X)^{\mathrm{constr}}$  for the category of  $\mathbb{Z}$ -constructible sheaves on  $X$ . The first main theorem we prove is the following

**Theorem III.1.1.** *Let  $\chi$  be an Euler characteristic on  $\mathrm{Sh}_{\mathbb{Z}}(X)^{\mathrm{constr}}$ . If  $\chi$  takes its values in a torsion-free abelian group then it vanishes on constructible sheaves.*

The above theorem essentially appeared for  $X$  the spectrum of a number ring in Chen's thesis [Che17]. As any object in the bounded derived category of an abelian category has a finite filtration with graded pieces its cohomology groups, the result generalizes to Euler characteristics on  $D^b(\mathrm{Sh}_{\mathbb{Z}}(X)^{\mathrm{constr}})$ . Thus a particular case of Theorem III.1.1 is the following: given an Euler characteristic  $\chi$  for the category of  $\mathbb{Z}$ -constructible sheaves on  $\mathrm{Spec}(\mathbb{Z})$  with values in a torsion-free abelian group, then for all bounded complexes  $C$  of étale sheaves with constructible cohomology sheaves we have  $\chi(C) = 0$ . Moreover, if  $f : X \rightarrow \mathrm{Spec}(\mathbb{Z})$  is an arithmetic scheme, we can define an Euler characteristic for constructible sheaves on  $X$  by putting  $\chi_X(F) := \chi_{\mathbb{Z}}(Rf_!F)$ , which is well-defined, and then  $\chi_X(F) = 0$  for all constructible sheaves on  $X$ . This can be interesting if  $\chi_X$  has an intrinsic description in terms of  $X$ ; for instance if  $\chi_{\mathbb{Z}}$  is the Weil-étale Euler characteristic for  $\mathrm{Spec}(\mathbb{Z})$  constructed in Chapter II, then for a constructible sheaf  $F$  on  $X$  we have

$$\chi_X(F) = \prod_{i \in \mathbb{Z}} [H_c^i(X, F)]^{(-1)^i},$$

where  $H_c^i(X, -)$  is a compactly supported cohomology taking into account the cohomology above the archimedean place of  $\mathbb{Q}$ .



Let  $\pi$  be a finite group, let  $R$  be a ring and denote  $G(R[\pi])$  the Grothendieck group of the category of finitely generated  $R[\pi]$ -modules. Up to the dévissage which we will explain later, the theorem follows from a theorem of Swan:

**Theorem III.1.2** ([Swa63, Corollary 1]). *Let  $\mathcal{O}_K$  be the ring of integers in a number field  $K$ . The kernel of the natural base change map  $G(\mathcal{O}_K[\pi]) \rightarrow G(K[\pi])$  is finite.*

We will mostly use the following corollary:

**Corollary III.1.3.** *Let  $G$  be a profinite group and let  $\chi$  be an Euler characteristic on the category of finite type discrete  $G$ -modules. If  $\chi$  takes its values in a torsion-free abelian group then it vanishes on finite  $G$ -modules.*

*Proof.* Let  $H$  be an open normal subgroup of  $G$ . The full abelian subcategory  $\mathcal{C}_H$  of finite type discrete  $G_K$ -modules on which  $H$  acts trivially is equivalent to the category of finite type  $\mathbb{Z}[G/H]$ -modules, so the restriction of  $\chi$  to it is an Euler characteristic, and thus factors through  $G(\mathbb{Z}[G/H])$ . If  $M$  is a finite  $G$ -module, then  $M$  belongs to  $\mathcal{C}_H$  for some open normal subgroup  $H$ . Moreover, we have  $M \otimes \mathbb{Q} = 0$  so the class of  $M$  in  $G(\mathbb{Q}[G/H])$  is 0. By Theorem III.1.2 we deduce that the class of  $M$  is of finite order  $n$  in  $G(\mathbb{Z}[G/H])$ ; then  $n\chi([M]) = \chi([M]^n) = 0$  so  $\chi(M) = 0$  because the value group of  $\chi$  is torsion-free.  $\square$

As another corollary to Theorem III.1.1, we reprove Tate's formula for the Euler characteristic of a global field  $K$  by reduction<sup>1</sup> to our formula for the Weil-étale Euler characteristic on the spectrum of the ring of  $S$ -integers in  $K$ :

**Corollary III.1.4** (Tate, [Mil06, I.5.1]). *Let  $K$  be a number field and let  $S$  be a finite set of places including the set  $S_\infty$  of all archimedean places. Denote by  $K_S$  the maximal extension of  $K$  unramified outside  $S$  and  $G_{K,S} := \text{Gal}(K_S/K)$ . Let  $M$  be a finite  $G_{K,S}$ -module whose order is invertible on  $X := \text{Spec}(\mathcal{O}_{K,S})$ . Define*

$$\chi(G_{K,S}, M) := \frac{[H^0(G_{K,S}, M)][H^2(G_{K,S}, M)]}{[H^1(G_{K,S}, M)]}.$$

Then we have

$$\chi(G_{K,S}, M) = \frac{1}{[M]^d} \prod_{v \in S_\infty} [H^0(G_v, M)],$$

where  $G_v := \text{Gal}(K_v^{\text{sep}}/K_v)$  and  $d = [K : \mathbb{Q}]$ .

The main ingredient is the following proposition which relates étale cohomology of locally constant constructible sheaves on  $X$  and Galois cohomology of finite  $G_S$ -modules:

**Proposition III.1.5** ([Mil06, Proposition II.2.9]). *Put  $\eta = \text{Spec}(K)$  and let  $F$  be a locally constant constructible sheaf on  $X$ . Let  $M$  denote  $F_{\overline{\eta}}$ . If the order of  $M$  is invertible on  $X$  then*

$$H^i(G_{K,S}, M) = H^i(X, F), \forall i \geq 0.$$

*Proof of Corollary III.1.4.* Denote by  $S_f$  the set of finite places in  $S$ ,  $G_v$  the Galois group of the completion (resp. henselization)  $K_v$  of  $K$  at an archimedean (resp. finite) place  $v$ ,  $F_v$  the pullback of  $F$  to  $\text{Spec}(K_v)$ , and  $\chi_X$  the Weil-étale Euler characteristic constructed

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<sup>1</sup>We still use Tate's formula for the Euler characteristic of a *local* field.

in Chapter II. As we noted above, since  $F$  is constructible we have by Proposition II.6.22, Artin–Verdier duality [Mil06, II.3.1(b)] and Proposition II.2.3 that

$$\chi_X(F) = \prod_{i=0}^3 [H_c^i(X, F)]^{(-1)^i}.$$

Using the above proposition, the defining long exact sequence of compactly supported cohomology gives a long exact sequence

$$\cdots \rightarrow H_c^i(X, F) \rightarrow H^i(G_{K,S}, M) \rightarrow \prod_{v \in S} H^i(G_v, F_v) \rightarrow \cdots.$$

As  $F$  is constructible, we have  $H_c^i(X, F) = 0$  for  $i > 3$  [Mil06, II.3.12(a)] and  $H^i(G_v, F_v) = 0$  for a finite place  $v$  and  $i > 2$ , so we can end the sequence at  $i = 3$  as

$$\cdots \rightarrow H_c^3(X, F) \rightarrow H^3(G_{K,S}, M) \rightarrow \prod_{v \in S_\infty} H^3(G_v, F_v) \rightarrow 0.$$

Denote by  $\chi_v(N) := \prod [H^i(G_v, N)]^{(-1)^i}$  the Euler characteristic of the local field  $K_v$  when  $v$  is finite. Then

$$\chi(G_{K,S}, M) = [H^3(G_{K,S}, M)] \chi_X(F) \prod_{v \in S_f} \chi_v(F_v) \prod_{v \in S_\infty} \prod_{i=0}^3 [H^i(G_v, F_v)]^{(-1)^i}.$$

We have an isomorphism  $H^3(G_{K,S}, M) \simeq \prod_{v \in S_\infty} H^3(G_v, F_v)$  by [Mil06, I.4.10(c)], the Herbrand quotient of a finite module is 1 and  $\chi_X(F) = 1$  by Theorem III.1.1, thus the above reduces to

$$\chi(G_{K,S}, M) = \prod_{v \in S_f} \chi_v(F_v) \prod_{v \in S_\infty} [H^0(G_v, F_v)].$$

Let  $m$  denote the order of  $M$ ; this is also the order of  $F_v$  when the latter is seen as a  $G_v$ -module. The computation of the Euler characteristic of a local field [Mil06, I.2.8] gives for any finite place  $v$  of  $K$  that

$$\chi_v(F_v) = \frac{1}{(\mathcal{O}_v^h : m\mathcal{O}_v^h)}.$$

By hypothesis  $m$  is invertible in  $\mathcal{O}_v^h$  for any finite place  $v$  of  $K$  not in  $S$ . The product formula thus gives

$$\prod_{v \in S_f} \chi_v(F_v) = \frac{1}{\prod_{v \in S_\infty} |m|_v} = \frac{1}{m^d}.$$

□

*Remark.*

- A similar reduction shows the analogous formula for a global function field.
- The two non-trivial steps in the above are the vanishing of  $\chi_X(F)$  which follows from Theorem III.1.1 and hence from Swan’s theorem, and the computation of  $\chi_v(F_v)$  for a finite place  $v$ . We wonder whether there exists a proof of the latter computation also relying on Swan’s theorem. Let us explain what we mean by this: we hope

that there would exist an "arithmetic Euler characteristic" for  $\mathbb{Z}$ -constructible sheaves on the spectrum of a  $p$ -adic field  $K$ , which would be defined through a trivialization of a fundamental line formed with the determinants of an additive and a multiplicative complex. For constructible coefficients, the multiplicative complex should be given by Galois cohomology, and the additive complex should have finite cohomology groups that are "easily" computable, and with alternating product equal to  $1/(\mathcal{O}_K : [M]\mathcal{O}_K)$ . Then Swan's theorem would imply immediately Tate's formula. This picture is a bit naive however as cohomology groups with  $\mathbb{Z}$  coefficients tend to have  $\mathbb{Z}_p$  or  $\mathbb{Q}_p/\mathbb{Z}_p$  terms in them, so there is no hope of modifying them to obtain a finitely generated cohomology.

- Deninger has proposed an alternative proof and a generalization of Tate's formula for proper arithmetic schemes in [Den87b].

In the following we abuse notations and make no distinction between sheaves on  $\text{Spec}(K)$  and discrete  $G_K$ -modules.

**Lemma III.1.6.** *Let  $X$  be a normal integral arithmetic scheme and denote by  $g : \text{Spec}(K) \rightarrow X$  the inclusion of its generic point. Let  $L/K$  be a finite Galois extension. There exists a dense open  $U$  of  $X$  such that for any discrete  $G_K$ -module  $M$  on which  $G_L$  acts trivially, the sheaf  $(g_*M)|_U$  is locally constant with stalk  $M$  (in particular, it is  $\mathbb{Z}$ -constructible if  $M$  is finite type). Moreover, if  $M$  is of finite type the stalks of  $g_*M$  at codimension 1 points are of finite type. In particular, if  $X$  is of dimension 1 then  $g_*M$  is  $\mathbb{Z}$ -constructible for any finite type discrete  $G_K$ -module  $M$ .*

*Proof.* Denote by  $\pi' : \text{Spec}(L) \rightarrow \text{Spec}(K)$  the morphism corresponding to  $L/K$  and  $\pi : Y \rightarrow X$  the normalization of  $X$  in  $L$ . We let  $U$  be the étale locus of  $\pi$ ,  $g_U : \text{Spec}(K) \rightarrow U$  the inclusion of the generic point,  $\pi_U : V := \pi^{-1}(U) \rightarrow U$  the corestriction of  $\pi$  to  $U$  and  $g' : \text{Spec}(L) \rightarrow V$  the inclusion of the generic point. Then  $\pi$  is finite and  $\pi_U$  is finite étale. Let  $M$  be a sheaf on  $\text{Spec}(K)$  on which  $G_L$  acts trivially, that is such that  $\pi'^*M$  is constant. As  $V$  is normal integral,  $g'_*\pi'^*M$  is constant. Let us show that  $\pi_U^*((g_*M)|_U) = g'_*\pi'^*M$ . We have  $(g_*M)|_U = g_{U,*}M$ . As  $\pi_U$  is étale, if  $W$  is an étale  $V$ -scheme we have  $\pi_U^*(g_{U,*}M)(W) = (g_{U,*}M)(W) = M(W \times_U \text{Spec}(K))$ . On the other hand, we have  $\text{Spec}(L) = V \times_U \text{Spec}(K)$  and  $\pi'$  is étale so we obtain  $g'_*\pi'^*M(W) = \pi'^*M(W \times_V \text{Spec}(L)) = M(W \times_V V \times_U \text{Spec}(K)) = M(W \times_U \text{Spec}(K))$ .

We thus have shown that  $g_*M$  is locally constant on the dense open subset  $U$ ; its stalk at the generic point is  $M$ . Moreover, for  $x$  a point of codimension 1 let  $K_x^{sh}$  denote the fraction field of the strictly henselian local ring at  $x$ , and  $I_x$  its Galois group. Then  $(g_*M)_{\bar{x}} = H^0(\text{Spec}(K_x^{sh}), M) = M^{I_x}$  is of finite type when  $M$  is of finite type.  $\square$

The following definition<sup>2</sup> will be used in the proof of Theorem III.1.1:

**Definition III.1.7.** *Let  $X$  be an arithmetic scheme. Let  $F$  be a constructible sheaf on  $X$ . If the support of  $F$  does not contain any generic point of  $X$ , we say that  $F$  is negligible.*

Note that if a sheaf  $F$  is negligible, it is supported on a closed subscheme  $i : Z \rightarrow X$  containing no generic point hence  $\dim Z < \dim X$ , and  $F$  can be written  $F = i_*G$  for the constructible sheaf  $G = i^*F$  on  $Z$ . We now prove the main theorem:

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<sup>2</sup>suggested by Tran in his thesis [Tra15]

*Proof of Theorem III.1.1.* The proof proceeds by induction on the dimension of  $X$ . Without loss of generality, we can suppose that  $X$  is reduced. If  $\dim X = 0$ , then  $X$  is a disjoint union of spectra of finite fields, so we reduce to the case  $X = \text{Spec}(k)$  where  $k$  is a finite field. The result is now given by Corollary III.1.3.

If  $\dim X = d > 0$ , we first treat negligible sheaves. If  $F$  is negligible, there is a closed subscheme  $i : Z \rightarrow X$  with  $\dim Z < \dim X$  and a constructible sheaf  $G$  on  $Z$  such that  $F = i_*G$ . As  $i_*$  is exact and preserves  $\mathbb{Z}$ -constructibles, the function  $\chi \circ i_*$  is an Euler characteristic on  $Z$  and thus by the induction hypothesis  $\chi(F) = \chi \circ i_*(G) = 0$ .

We now reduce to the normal case: denote by  $\pi : Y \rightarrow X$  the normalization of  $X$ ; it is a finite morphism, and  $Y$  is the disjoint union of the normalizations of the (finitely many) irreducible components of  $X$ , which are each of dimension  $\leq d$ . Then  $\pi^*F$  is a constructible sheaf on  $Y$ ,  $\pi_*$  is exact and respects  $\mathbb{Z}$ -constructibles and the cokernel  $Q$  in the exact sequence

$$0 \rightarrow F \rightarrow \pi_*\pi^*F \rightarrow Q \rightarrow 0$$

is negligible. We thus find on the one hand that  $\chi(F) = (\chi \circ \pi_*)(\pi^*F)$  and on the other hand that  $\chi \circ \pi_*$  is an Euler characteristic on  $Y$ , so we can reduce to the case of a normal arithmetic scheme of dimension  $d$ .

Suppose now that  $X$  is integral normal and denote by  $g : \eta = \text{Spec}(K) \rightarrow X$  the inclusion of its generic point. Let  $L/K$  be a finite Galois extension such that  $G_L$  acts trivially on  $M := g^*F$ , and put  $G := G_K/G_L$ . Take a dense open  $j : U \rightarrow X$  as in Lemma III.1.6 and let  $g_U : \eta \rightarrow U$  denote the canonical map. We claim that  $\chi \circ j_!g_{U,*}$  is an Euler characteristic on finite type  $G$ -modules. Let

$$0 \rightarrow N' \rightarrow N \rightarrow N'' \rightarrow 0$$

be an exact sequence of  $G$ -modules of finite type. Then the sequence

$$0 \rightarrow g_{U,*}N' \rightarrow g_{U,*}N \rightarrow g_{U,*}N'' \rightarrow 0$$

is an exact sequence of locally constant  $\mathbb{Z}$ -constructible sheaves by construction of  $U$  so we obtain an exact sequence of  $\mathbb{Z}$ -constructible sheaves on  $X$  after applying  $j_!$ . This shows the claim. Therefore  $\chi \circ j_!g_{U,*}$  vanishes on finite  $G$ -modules by Corollary III.1.3.

Denote by  $i : Z \rightarrow X$  the closed complement of  $U$ . As  $U$  is dense and  $F$  is constructible, the sheaf  $i_*i^*F$  is negligible so the localization sequence

$$0 \rightarrow j_!F|_U \rightarrow F \rightarrow i_*i^*F \rightarrow 0$$

implies that  $\chi(F) = \chi(j_!F|_U)$ . On the other hand, the kernel and cokernel of the canonical map  $j_!F|_U \rightarrow j_!g_{*,U}M$  are also negligible so we find  $\chi(j_!F|_U) = \chi(j_!g_{*,U}M) = 0$ ; this implies that  $\chi(F) = 0$ .  $\square$

*Remark.* We stated our theorems for arithmetic schemes but this will work more generally for any family of schemes such that

1. in dimension 0 we obtain, in the reduced case, a finite disjoint union of spectra of fields (for instance Noetherian schemes) and
2. the normalization of the reduction is always a finite morphism (for instance Nagata schemes).

In particular the above applies to all Noetherian Nagata schemes, as the induction argument reasons on the schemes that have an immersion to the considered scheme, and those are also Noetherian and Nagata.

## III.2 Vanishing on $\mathbb{Z}$ -constructible sheaves

Using a similar dévissage, but relying on Artin induction now, we can prove our second main theorem:

**Theorem III.2.1.** *Let  $\chi, \chi'$  be two Euler characteristics on the category of  $\mathbb{Z}$ -constructible sheaves on an arithmetic scheme  $X$ , with values in a torsion-free abelian group. If  $\chi(\pi_*\mathbb{Z}) = \chi'(\pi_*\mathbb{Z})$  for  $\pi : Y \rightarrow X$  any finite morphism of arithmetic schemes with  $Y$  normal irreducible, then  $\chi = \chi'$ .*

Let us recall the statement of Artin induction, and its integral version:

**Theorem III.2.2** ([Swa60, 4.1,4.4]). *Let  $\pi$  be a finite group. Then the subgroup generated by classes of representations induced from the trivial representation of subgroups of  $\pi$  is of finite index in  $G(\mathbb{Q}[\pi])$ .*

**Corollary III.2.3.** *Let  $G$  be a profinite group and let  $M$  be a finite type discrete  $G$ -module. There exists an integer  $n$ , finitely many open subgroups  $(H_i)_{i \in I}, (H_j)_{j \in J}$  of  $G$  and a finite  $G$ -module  $N$  together with an exact sequence*

$$0 \rightarrow M^n \oplus \bigoplus_{i \in I} \text{ind}_G^{H_i} \mathbb{Z} \rightarrow \bigoplus_{j \in J} \text{ind}_G^{H_j} \mathbb{Z} \rightarrow N \rightarrow 0.$$

We deduce a preliminary version of Theorem III.2.1:

**Corollary III.2.4.** *Let  $G$  be a profinite group and let  $\chi, \chi'$  be two Euler characteristics on the category of finite type discrete  $G$ -modules, with values in a torsion-free abelian group. If  $\chi(\text{ind}_G^H \mathbb{Z}) = \chi'(\text{ind}_G^H \mathbb{Z})$  for every open subgroup  $H$  of  $G$  then  $\chi = \chi'$ .*

*Proof.* It suffices to prove that if  $\chi(\text{ind}_G^H \mathbb{Z}) = 0$  for all open subgroups then  $\chi = 0$ . By Theorem III.1.1 we have  $\chi(M) = 0$  for finite  $M$ . If  $M$  is an arbitrary finite type discrete  $G$ -module, using the short exact sequence

$$0 \rightarrow M_{\text{tor}} \rightarrow M \rightarrow M/\text{tor} \rightarrow 0$$

we reduce to the case where  $M$  is torsion-free. Then using an exact sequence as in the above corollary, we find

$$n\chi(M) = \sum_{j \in J} \chi(\text{ind}_G^{H_j} \mathbb{Z}) - \sum_{i \in I} \chi(\text{ind}_G^{H_i} \mathbb{Z}) - \chi(N) = 0.$$

As the value group of  $\chi$  is torsion-free, we find  $\chi(M) = 0$ . □

We now prove Theorem III.2.1 by a dévissage similar to the proof of Theorem III.1.1.

*Proof of Theorem III.2.1.* It suffices to show that if  $\chi(\pi_*\mathbb{Z}) = 0$  for all finite morphisms  $\pi : Y \rightarrow X$  with  $Y$  normal irreducible then  $\chi = 0$ . We reason by induction on  $\dim X$ , and we can suppose that  $X$  is reduced. In the dimension 0 case, the result is Corollary III.2.4. If  $\dim X > 0$ , we already know by Theorem III.2.1 that  $\chi$  vanishes on constructible sheaves. Let  $F$  be a  $\mathbb{Z}$ -constructible sheaf on  $X$ . If the support of  $F$  does not contain any generic point, then  $F$  is supported on a closed subscheme  $i : Z \rightarrow X$  with  $\dim Z < \dim X$  so the result applied to  $\chi \circ i_*$  gives  $\chi(F) = 0$ . Let  $\pi : Y \rightarrow X$  denote the normalization of  $X$ ; then the supports of the kernel and cokernel of  $F \rightarrow \pi_*\pi^*F$  do not contain any

generic point, so we can reduce to the case where  $X$  is integral and normal. Denote by  $g : \text{Spec}(K) \rightarrow X$  the inclusion of the generic point, let  $L/K$  be a finite Galois extension such that  $G_L$  acts trivially on  $M := g^*F$ , and put  $G = G_K/G_L$ . Take a dense open  $j : U \rightarrow X$  as in Lemma III.1.6 and let  $g_U : \text{Spec}(K) \rightarrow U$  denote the canonical morphism and  $i : Z \rightarrow X$  the inclusion of the closed complement of  $U$ ; we have  $\dim Z < \dim X$ . We have seen that  $\chi \circ j_!g_{U,*}$  is an Euler characteristic on finite type  $G$ -modules. If  $H$  is a subgroup of  $G$  corresponding to an intermediate extension  $L/E/K$ , denote by  $\pi : Y \rightarrow X$  the normalization of  $X$  in  $E$  and  $\pi_U : V \rightarrow U$  its corestriction to  $U$ . Let  $g'_V$  denote the morphism  $\text{Spec}(E) \rightarrow V$ ; since  $V$  is normal, we have

$$g_{U,*}\text{ind}_G^H \mathbb{Z} = \pi_{U,*}g_{V,*}\mathbb{Z} = \pi_{U,*}\mathbb{Z}$$

hence

$$\chi(j_!g_{U,*}\text{ind}_G^H \mathbb{Z}) = \chi(j_!\pi_{U,*}\mathbb{Z}) = \chi(\pi_*\mathbb{Z}) = 0$$

since the cokernel of  $j_!\pi_{U,*}\mathbb{Z} \rightarrow \pi_*\mathbb{Z}$  is supported on  $Z$ . We can then apply Corollary III.2.4 which yields  $\chi \circ j_!g_{U,*}(M) = 0$ . Finally, the kernel and cokernel of  $j_!F|_U \rightarrow j_!g_{U,*}M$  do not contain the generic point and the cokernel of the injection  $j_!F|_U \rightarrow F$  is supported on  $Z$ , so we find

$$\chi(F) = \chi(j_!F|_U) = \chi(j_!g_{U,*}M) = 0.$$

□

# Chapter IV

## Tori over number fields and special values at $s = 1$

### IV.1 Introduction

#### IV.1.1 Results

**IV.1.1.1 A new  $L$ -function.** Let  $\mathcal{O}$  be an order in a number field  $K$  and  $X = \text{Spec}(\mathcal{O})$ . Denote by  $\mathbb{Z}_X^c := \mathbb{Z}_X^c(0)$  Bloch's cycle complex (with cohomological indexing, see § IV.1.3.2), defined on the étale site of  $X$ ; in particular if  $X$  is regular we have  $\mathbb{Z}_X^c = \mathbb{G}_m[1]$ . A sheaf of abelian groups  $F$  on the étale site of  $X$  is  $\mathbb{Z}$ -constructible if on a dense open it is locally constant associated to a finite type abelian group, and moreover the stalks at all geometric points are finite type abelian groups. If  $F$  is a  $\mathbb{Z}$ -constructible sheaf, put  $F^D := R\mathcal{H}om_X(F, \mathbb{Z}_X^c)$ . The cohomology of  $F^D$  is related to the compactly supported cohomology of  $F$  by *Artin–Verdier duality*; thus we think of it as computing homology with coefficients in  $F$ . To the complex  $F^D$ , we associate an  $L$ -function:

**Definition.** For a complex of étale sheaves  $M$ , we put

$$M \hat{\otimes} \mathbb{Q}_\ell := (R\lim_n (M \otimes^L \mathbb{Z}/\ell^n \mathbb{Z})) \otimes \mathbb{Q},$$

computed on the proétale site. For each closed point  $x$  of  $X$ , let  $\ell = \ell_x$  be a prime number such that  $\ell \neq \text{char}(\kappa(x))$  and  $L_x$  the usual local factor defined using the geometric Frobenius  $\varphi$ : for a finite dimensional  $\mathbb{Q}_\ell$ -representation  $V$ , it is given by

$$L_x(V, s) := \det(1 - \varphi N(x)^{-s} | V)^{-1}$$

with  $N(x) = \text{card}(\kappa(x))$ . We define the  $L$ -function of  $F^D$  by

$$L_X(F^D, s) := \prod_{x \in X_0} L_x((i_x^* F^D) \hat{\otimes} \mathbb{Q}_{\ell_x}, s).$$

We compute explicitly the local factors, show that they are well-defined and that the Euler product converges for  $s > 1$ . In fact, denote by  $g : \text{Spec}(K) \rightarrow X$  the inclusion of the generic point and  $G_K = \text{Gal}(K^{\text{sep}}/K)$ . If  $V$  is the rational representation of  $G_K$  corresponding to  $g^* F \otimes \mathbb{Q}$ , then the  $L$ -function of  $F^D$  equals up to a finite number of factors the Artin  $L$ -function  $L_K(V, s+1)$ .<sup>1</sup> In particular, the  $L$ -function extends to a meromorphic

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<sup>1</sup>Rational representations are self-dual.

function on  $\mathbb{C}$ . If  $F$  is a  $\mathbb{Z}$ -constructible sheaf on  $\mathrm{Spec}(K)$  associated to a finite type integral  $G_K$ -representation  $M$ , we have moreover  $L_X(g_*F)^D, s) = L_K(M \otimes \mathbb{Q}, s + 1)$ . We have as a special case  $L_X(\mathbb{Z}^D, s) = \zeta_X(s + 1)$  if  $X$  is regular. On the other hand, let  $i : x \rightarrow X$  be the inclusion of a closed point and let  $F$  be a  $\mathbb{Z}$ -constructible sheaf on  $x$ . In Subsection II.6.4 we introduced  $L$ -functions for  $\mathbb{Z}$ -constructible sheaves on  $X$ ; then  $L_X((i_*F)^D, s)$  is the  $L$ -function of the  $\mathbb{Z}$ -constructible sheaf  $i_*\mathcal{H}om_x(F, \mathbb{Z})$ . Finally, if  $j : U \hookrightarrow X$  is the inclusion of a dense open subscheme, we have  $L_X((j_*\mathbb{Z})^D, s) = \zeta_U(s + 1) \times \prod_{x \in X \setminus U} \frac{\zeta_x(s+1)}{\zeta_x(s)}$ .

**IV.1.1.2 Weil-étale cohomology and special values.** We apply the Weil-étale formalism of Flach–B. Morin [FM18] to give a special value formula at  $s = 0$  of this new  $L$ -function. The idea of Weil-étale cohomology originates in [Lic05], where Lichtenbaum constructs a Weil-étale topos for varieties over a finite field and links it to the special value at  $s = 0$  of zeta functions. Further work was done by Geisser over a finite field [Gei04]. For schemes over  $\mathrm{Spec}(\mathbb{Z})$ , attempts at the definition of a Weil-étale topos were made by Lichtenbaum [Lic09] and Flach–B. Morin [FM10] but its cohomology doesn’t behave well in high degree. Another approach was instigated by B. Morin in [Mor14] and refined by Flach–B. Morin in [FM18]: instead of constructing the Weil-étale topos, one only constructs Weil-étale cohomology complexes in the derived category of abelian groups which fit into a certain distinguished triangle. This distinguished triangle comes heuristically from the pushforward from the Weil-étale topos to the étale topos<sup>2</sup>.

Other relevant works on Weil-étale cohomology include Chiu’s thesis [Chi11], Beshenov’s thesis [Bes21a; Bes21b] and Tran’s article [Tra16]. Work related to the study of the Weil-étale cohomology of  $F^D$  are Geisser–Suzuki’s article [GM21] and Tran’s thesis [Tra15]. As far as the author knows, Tran was the first to observe that for an integral representation  $M$  of the Galois group of a number field  $K$ , the special value of the Artin  $L$ -function of  $M \otimes \mathbb{Q}$  at  $s = 1$  should be related to Weil-étale cohomology of the dual of the pushforward of  $M$  to  $\mathrm{Spec}(\mathcal{O}_K)$ .

Following the formalism of Flach–B. Morin, we should construct for each  $F$  a “multiplicative” complex<sup>3</sup>, the Weil-étale complex (with compact support) which we think of as “Weil-étale homology” with coefficients in  $F$ , and an “additive” complex<sup>4</sup>, an analogue of Milne’s correcting factor in special value formulas for zeta functions of varieties over finite fields. The right object to consider is then the fundamental line  $\Delta_X(F^D)$ , a free abelian group of rank 1 which is defined as the product of the determinants of the additive and multiplicative complexes. In the general situation of an arithmetic scheme, contrary to the case over a finite field, the additive and multiplicative complexes are linked to each other through phenomena happening on complex points. They cannot be studied independently to get a special value formula; instead one of the fundamental insights of [FM18] is that one has to study them together through the fundamental line<sup>5</sup>. The fundamental line should have a canonical trivialization  $\Delta_X(F^D) \otimes \mathbb{R} \xrightarrow{\sim} \mathbb{R}$  after base change to  $\mathbb{R}$ , which enables one to construct a multiplicative Euler characteristic as the covolume of  $\Delta_X(F^D)$  inside  $\Delta_X(F^D) \otimes \mathbb{R}$ . This Euler characteristic should give the special value of the  $L$ -function up to sign.

<sup>2</sup>The pushforward had been computed by Geisser [Gei04] in the case over a finite field and by B. Morin in the case of the spectrum of a ring of integers in a number field [Mor11, §8, §9].

<sup>3</sup>Here “multiplicative” suggests that the complex is linked to motivic cohomology: the latter involves the units, the Picard group, etc.

<sup>4</sup>Here “additive” suggests that the complex is linked to coherent phenomena/ de Rham cohomology

<sup>5</sup>This idea has its origin in the formulation of Fontaine–Perrin-Riou of the Bloch–Kato conjecture on special values of  $L$ -functions



**IV.1.1.3 The Weil-étale complex** We introduce compactly supported cohomology and “Tate” compactly supported cohomology complexes<sup>6</sup>

$$R\Gamma_{c,B}(X, F^D), \quad R\hat{\Gamma}_{c,B}(X, F^D).$$

The two bear to each other the same kind of relationship as ordinary and Tate cohomology: the fiber of the canonical morphism (induced by the norm map)  $R\Gamma_{c,B}(X, F^D) \rightarrow R\hat{\Gamma}_{c,B}(X, F^D)$  computes some homology at the archimedean places. Moreover, we prove an Artin–Verdier duality statement for  $R\hat{\Gamma}_{c,B}(X, F^D)$ , which suggests that compactly supported cohomology of  $F^D$  should be thought of as “étale homology” of  $F$ :

**Theorem IV.A** (Artin–Verdier duality for  $F^D$ , see Subsection IV.2.2). *Let  $F$  be an étale sheaf on the spectrum  $X$  of an order in a number field. There is a natural pairing*

$$R\Gamma(X, F) \otimes^L R\hat{\Gamma}_{c,B}(X, F^D) \rightarrow \mathbb{Q}/\mathbb{Z}[-2].$$

*It induces a map*

$$R\hat{\Gamma}_{c,B}(X, F^D) \rightarrow R\mathrm{Hom}(R\Gamma(X, F), \mathbb{Q}/\mathbb{Z}[-2]).$$

*If  $F$  is  $\mathbb{Z}$ -constructible, the above map is an isomorphism in degree  $\neq -1, 0$ , and an isomorphism after profinite completion of the left hand side in degree  $-1, 0$ .*

The proof proceeds by twisting the usual Artin–Verdier theorem [Mil06, II.3.1] by a duality at archimedean places. With the Artin–Verdier-like duality theorem in our hands, we can construct a Weil-étale complex using the methodology of [Mor14]: the cohomology groups with compact support of  $F^D$  have a finite type part and a torsion of cofinite type part (i.e. the  $\mathbb{Q}/\mathbb{Z}$ -dual of a finite type abelian group) and Artin–Verdier duality says that the torsion cofinite type part is the  $\mathbb{Q}/\mathbb{Z}$ -dual of some étale cohomology of  $F$ . Taking inspiration from Geisser’s and B. Morin’s computation of the derived pushforward from the Weil-étale topos to the étale topos [Gei04; Mor11], there should exist a fundamental distinguished triangle saying that Weil-étale cohomology with compact support of  $F^D$  is obtained by replacing the torsion cofinite type part of cohomology with compact support of  $F^D$  by a finite type part, using the short exact sequence  $0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Q} \rightarrow \mathbb{Q}/\mathbb{Z} \rightarrow 0$ . This suggests the existence of a map  $\mathrm{Hom}(H^{2-i}(X, F), \mathbb{Q}) \rightarrow H_{c,B}^i(X, F^D)$  making the diagram

$$\begin{array}{ccc} \mathrm{Hom}(H^{2-i}(X, F), \mathbb{Q}) & \dashrightarrow & H_{c,B}^i(X, F^D) \\ \downarrow & & \downarrow \\ \mathrm{Hom}(H^{2-i}(X, F), \mathbb{Q}/\mathbb{Z}) & \xleftarrow{\text{Artin-Verdier}} & \hat{H}_{c,B}^i(X, F^D) \end{array}$$

commute; taking kernels and cokernels will then give something of finite type. We are thus led to consider the existence of a map  $R\mathrm{Hom}(R\Gamma(X, F), \mathbb{Q}[-2]) \xrightarrow{?} R\Gamma_{c,B}(X, F^D)$ , the cone of which we want to name Weil-étale cohomology with compact support of  $F^D$ . We achieve the construction of such a map with good functoriality properties in  $D(\mathbb{Z})$  for a large class of  $\mathbb{Z}$ -constructible sheaves, which we name red sheaves and blue sheaves.

**Definition.** *Let  $F$  be a  $\mathbb{Z}$ -constructible sheaf on  $X$ . We say that*

<sup>6</sup>The subscript  $B$  refers to the way we correct the cohomology at infinity, which involves the Tate twist  $\mathbb{Z}(1) = 2i\pi\mathbb{Z}$

- $F$  is red if  $H_{c,B}^0(X, F^D)$  is torsion, hence finite.
- $F$  is blue if  $H^1(X, F)$  is torsion, hence finite.
- A red-to-blue morphism is a morphism of sheaves  $F \rightarrow G$  where either  $F$  and  $G$  are both blue, or are both red, or  $F$  is red and  $G$  is blue; a red-to-blue short exact sequence is a short exact sequence with red-to-blue morphisms.

Examples of red sheaves are extensions by zero of locally constant  $\mathbb{Z}$ -constructible sheaves on a regular open subscheme. Examples of blue sheaves are  $\mathbb{Z}$ -constructible sheaves supported on a finite closed subscheme. The important point is that there are “enough red and blue sheaves”, meaning that any  $\mathbb{Z}$ -constructible sheaf can be put in a short exact sequence where the first term is red and the last is blue.<sup>7</sup>

**Theorem IV.B** (Existence of the Weil-étale complex, see Section IV.3). *For every red or blue sheaf  $F$  on the spectrum  $X$  of an order in a number field, there exists a Weil-étale complex with compact support  $R\Gamma_{W,c}(X, F^D) \in D(\mathbb{Z})$ , well-defined up to unique isomorphism. It sits in a distinguished triangle*

$$R\mathrm{Hom}(R\Gamma(X, F), \mathbb{Q}[-2]) \rightarrow R\Gamma_{c,B}(X, F^D) \rightarrow R\Gamma_{W,c}(X, F^D) \rightarrow .$$

*It is a perfect complex, functorial in red-to-blue morphisms, and it yields a long exact cohomology sequence for red-to-blue short exact sequences. If  $Y = \mathrm{Spec}(\mathcal{O}')$  is the spectrum of an order in a number field with a finite dominant morphism  $\pi : Y \rightarrow X$ , we have a canonical isomorphism  $R\Gamma_{W,c}(X, (\pi_*F)^D) \simeq R\Gamma_{W,c}(Y, F^D)$ .*

The Weil-étale cohomology with compact support of  $F^D$  is constructed from the complex  $R\Gamma_{c,B}(X, F^D)$ , so it can be thought of as some “Weil-étale homology” of  $F$ . The idea of such a homology theory is not new: Geisser had defined “arithmetic homology” for curves over finite fields [Gei12], and there is a tentative construction of a Weil-étale complex of  $F^D$  over the spectrum of a ring of integers in a totally imaginary number field in Tran’s thesis [Tra15]. Finally, let  $K$  be a function field associated to a smooth proper curve  $C$  over a finite field. Geisser–Suzuki showed in [GS20] that for a given torus over  $K$ , its connected Néron model  $\mathcal{T}^\circ$  over  $C$  is of the form  $F^D$  for  $F$  a complex defined in terms of the character group of  $T$ . They then linked the special value of the  $L$ -function of the torus at  $s = 1$  to Weil-étale cohomology of  $\mathcal{T}^\circ$ ; this suggested that our approach might be fruitful (see § IV.1.2.2 for a precise comparison).

Having an  $\infty$ -categorical construction of the map

$$R\mathrm{Hom}(R\Gamma(X, F), \mathbb{Q}[-2]) \rightarrow R\Gamma_{c,B}(X, F^D)$$

would have been better, but this construction has eluded us. Instead, we use a kind of miraculous vanishing of  $\mathrm{Ext}^1$  groups, for red or blue sheaves, which enables to define the map by only specifying the maps it induces in cohomology. This vanishing is not true for arbitrary  $\mathbb{Z}$ -constructible sheaves, which explains our restriction. It will turn out that to define the Weil-étale Euler characteristic this is not a problem, because of the existence of “enough red and blue sheaves”.

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<sup>7</sup>Namely, take the short exact localization sequence associated to a regular dense open subscheme on which the sheaf is locally constant.

**IV.1.1.4 The additive complex** We now have the multiplicative part of the fundamental line, so we turn to the additive part, which we dub the tangent space of  $F^D$ :

**Definition** (The tangent space of  $F^D$ ). *Let  $F$  be a  $\mathbb{Z}$ -constructible sheaf on  $X$  and denote by  $g : \text{Spec}(K) \rightarrow X$  the inclusion of the generic point. We identify  $g^*F$  with a discrete  $G_K$ -module  $M$ . Let also  $\mathcal{O}_K$  denote the ring of integers in  $K$ .*

- We say that  $F$  is tamely ramified if for each  $x \in X_0$ , the wild ramification group at  $x$  acts trivially on  $M$ . Let  $K^t$  be the maximal tamely ramified extension of  $K$ ,<sup>8</sup> and  $G_K^t := \text{Gal}(K^t/K)$ ; if  $F$  is tamely ramified,  $M$  carries a natural action of  $G_K^t$ .
- Suppose that  $F$  is tamely ramified. The tangent space of  $F^D$  is the complex

$$\text{Lie}_X(F^D) := R\text{Hom}_{G_K^t}(M, \mathcal{O}_{K^t}[1]).$$

- Suppose that  $F$  is tamely ramified and red or blue. The fundamental line is

$$\Delta_X(F^D) := \det_{\mathbb{Z}} R\Gamma_{W,c}(X, F^D) \otimes \det_{\mathbb{Z}} \text{Lie}_X(F^D)^{-1}.$$

Our definition of the tangent space is made so that in particular, when  $X = \text{Spec}(\mathcal{O}_K)$  is regular, the tangent space of  $\mathbb{Z}^D = \mathbb{Z}_X^c = \mathbb{G}_m[1]$  is  $R\Gamma(X, \mathbb{G}_a[1]) = \mathcal{O}_K[1]$ , and the tangent space of the dual of a sheaf supported on a finite closed subscheme is 0. This is as expected from the  $L$ -function we introduce: indeed for  $\mathbb{Z}^D$  it gives the zeta function at  $s+1$  so its special value at 0 is the special value of the Dedekind zeta function at 1 and should involve contributions from  $\mathcal{O}_K$  (for instance, the discriminant); this was shown already in the Weil-étale formalism by Flach–B. Morin in the case of  $\mathbb{Z}(1) = (\mathbb{Z}^D)[-2]$  for  $X$  regular [FM18]. On the other hand, for the dual of a sheaf supported on a finite closed subscheme we find an  $L$ -function as in Subsection II.6.4, for which there is no “additive” contribution for the special value at 0, as we saw. The restriction to tamely ramified sheaves is justified by a theorem of Noether [Noe32], which implies that  $\mathcal{O}_{K^t}$  is cohomologically trivial. In the next chapter, we will show how to remove the tamely ramified hypothesis, at least at the level of the fundamental line. The definition was inspired by the article [GS20], which uses the Lie algebra of the Néron model of a torus  $T$  over  $K$  to form the additive part of the fundamental line giving the special value at  $s=1$  of the  $L$ -function of  $T$  (see § IV.1.2.2). Our definition is a generalization of this construction (in the tamely ramified case) that includes finite groups of multiplicative type, as shows the following:

**Proposition** (see Corollary IV.4.7). *Denote by  $g : \text{Spec}(K) \rightarrow X$  the canonical map, and let  $F$  be a  $\mathbb{Z}$ -constructible sheaf on  $X$ . Suppose that  $F$  is a tamely ramified and that  $g^*F$  is torsion-free. Let  $T$  denote the torus over  $K$  with character group  $g^*F$  and let  $\mathcal{T}$  denote its Néron model over  $\text{Spec}(\mathcal{O}_K)$ . Then there is a canonical isomorphism*

$$\text{Lie}_X(F^D) \simeq \text{Lie}(\mathcal{T})[1].$$

**IV.1.1.5 Trivialization of the fundamental line** Now that we have our fundamental line, to obtain the Euler characteristic we should seek a canonical trivialization. We propose a contravariant generalization of the complex  $R\Gamma_c(X, \mathbb{R}(1))$  of [FM18]<sup>9</sup>:

<sup>8</sup>See Appendix C for a reminder.

<sup>9</sup>The complex of Flach–B. Morin is the mapping fiber of the Beilinson regulator between motivic cohomology (tensored with  $\mathbb{R}$ ) and (real) Deligne cohomology on the complex points, in weight 1; see [FM18, § 2.1]

**Definition.** Let  $F$  be an étale sheaf on  $X$  and let  $\alpha^*F$  denote the  $G_{\mathbb{R}}$ -equivariant sheaf obtained by pulling back  $F$  to  $X(\mathbb{C})$ . The map  $\log | - | : \mathbb{C}^\times \rightarrow \mathbb{R}$  induces a natural map

$$\text{Log} : R\Gamma(X, F^D)_{\mathbb{R}} \rightarrow R\text{Hom}_{G_{\mathbb{R}}, X(\mathbb{C})}(\alpha^*F, \mathbb{R}[1]).$$

For  $F$  a  $\mathbb{Z}$ -constructible sheaf on  $X$ , we define the Deligne compactly supported complex with coefficients in  $F^D$  by

$$R\Gamma_{c, \mathcal{D}}(X, F_{\mathbb{R}}^D) := \text{fib}(R\Gamma(X, F^D)_{\mathbb{R}} \xrightarrow{\text{Log}} R\text{Hom}_{G_{\mathbb{R}}, X(\mathbb{C})}(\alpha^*F, \mathbb{R}[1])).$$

Then, again following [FM18], we introduce Weil–Arakelov complexes:

**Definition.** Let  $F$  be a  $\mathbb{Z}$ -constructible sheaf on  $X$ . We define the Weil–Arakelov complex of  $F^D$  as the complex:

$$R\Gamma_{\text{ar}, c}(X, F_{\mathbb{R}}^D) := R\Gamma_{c, \mathcal{D}}(X, F_{\mathbb{R}}^D)[-1] \oplus R\Gamma_{c, \mathcal{D}}(X, F_{\mathbb{R}}^D).$$

The determinant of the Weil–Arakelov complex has a canonical trivialization; we will obtain the trivialization of the fundamental line by relating the fundamental line with the determinant of the Weil–Arakelov complex through a duality theorem and the rational splitting of Weil–étale cohomology:

**Theorem IV.C** (Duality theorem for  $\mathbb{R}$ -coefficients, see Section IV.5). *Let  $F$  be an étale sheaf on the spectrum  $X$  of an order in a number field. There is a natural pairing*

$$(R\Gamma(X, F) \otimes \mathbb{R}) \otimes_{\mathbb{R}}^L R\Gamma_{c, \mathcal{D}}(X, F_{\mathbb{R}}^D) \rightarrow \mathbb{R}[0].$$

It induces a map

$$R\Gamma_{c, \mathcal{D}}(X, F_{\mathbb{R}}^D) \rightarrow R\text{Hom}(R\Gamma(X, F), \mathbb{R}),$$

which is an isomorphism for  $F \in D^+(X_{\text{et}})$ . If moreover  $F$  is a bounded complex with  $\mathbb{Z}$ -constructible cohomology groups, both sides are perfect complexes of  $\mathbb{R}$ -vector spaces.

**Proposition** (Rational splitting of Weil–étale cohomology, see Proposition IV.3.12). *Let  $F$  be a red or blue sheaf on  $X$ . The defining distinguished triangle of Weil–étale cohomology splits rationally to give an isomorphism*

$$R\Gamma_{W, c}(X, F^D) \otimes \mathbb{Q} \xrightarrow{\cong} R\text{Hom}(R\Gamma(X, F), \mathbb{Q}[-1]) \oplus R\Gamma_{c, B}(X, F^D) \otimes \mathbb{Q},$$

natural in red-to-blue morphisms and red-to-blue short exact sequences and compatible with finite dominant morphisms between spectra of orders in number fields.

The duality theorem and the rational splitting imply that there is a distinguished triangle

$$R\Gamma_{\text{ar}, c}(X, F_{\mathbb{R}}^D) \rightarrow R\Gamma_{W, c}(X, F^D) \otimes \mathbb{R} \rightarrow \text{Lie}_X(F^D) \otimes \mathbb{R},$$

which gives the natural trivialization

$$\begin{aligned} \lambda : \Delta_X(F^D)_{\mathbb{R}} &= \det_{\mathbb{R}}(R\Gamma_{W, c}(X, F^D) \otimes \mathbb{R}) \otimes \det_{\mathbb{R}}(\text{Lie}_X(F^D) \otimes \mathbb{R})^{-1} \\ &\xrightarrow{\cong} \det_{\mathbb{R}}(R\Gamma_{\text{ar}, c}(X, F_{\mathbb{R}}^D)) \\ &\xrightarrow{\cong} \mathbb{R}. \end{aligned}$$

### IV.1.1.6 The Euler characteristic and the special value theorem

**Definition.**

- Let  $F$  be a tamely ramified red or blue sheaf on  $X$ . The Weil-étale Euler characteristic of  $F^D$  is the positive real number  $\chi_X(F^D)$  such that

$$\lambda(\Delta_X(F^D)) = \chi_X(F^D)^{-1} \cdot \mathbb{Z} \hookrightarrow \mathbb{R}.$$

- Let  $F$  be a tamely ramified  $\mathbb{Z}$ -constructible sheaf on  $X$ . There exists a short exact sequence  $0 \rightarrow F' \rightarrow F \rightarrow F'' \rightarrow 0$  with  $F'$  red and tamely ramified and  $F''$  blue and tamely ramified; define

$$\chi_X(F^D) = \chi_X((F')^D) \chi_X((F'')^D).$$

*It does not depend on the chosen sequence.*

The constructed Euler characteristic is multiplicative thanks to the functoriality properties of our constructions. We have an explicit computation:

**Proposition** (see Proposition IV.7.9). *Let  $F$  be a tamely ramified  $\mathbb{Z}$ -constructible sheaf on  $X$  and suppose that  $X$  is regular. We have*

$$\chi_X(F^D) = \frac{(2\pi)^{r_2(F)} 2^{r_1(F)} [H^0(X, F)_{\text{tor}}] [\text{Ext}_X^1(F, \mathbb{G}_m)_{\text{tor}}] R(F^D)}{[H^1(X, F)_{\text{tor}}] [\text{Hom}_X(F, \mathbb{G}_m)_{\text{tor}}] [\text{Ext}_{G_K^t}^1(F_\eta, \mathcal{O}_{K^t})] [N_2] \text{Disc}(F)},$$

where  $r_1(F)$  and  $r_2(F)$  are some positive integers,  $N_2$  is a certain finite 2-torsion abelian group,  $R(F^D)$  is a regulator-type real number and  $\text{Disc}(F)$  is a square-root-of-discriminant-type real number.

Now that we have an Euler characteristic at our disposition, the general method following work of Tran [Tra16] (also used in [GM21] and Chapter II) is to reduce the special value formula using the results of Chapter III to computations in specific cases. We obtain:

**Theorem IV.D** (Special value formula, see Subsection IV.7.2). *Let  $F$  be a  $\mathbb{Z}$ -constructible sheaf on the spectrum  $X$  of an order in a number field. We have the vanishing order formula*

$$\text{ord}_{s=0} L_X(F^D, s) = \sum (-1)^i \cdot \dim_{\mathbb{R}} H_{\text{ar},c}^i(X, F_{\mathbb{R}}^D).$$

*If  $F$  is tamely ramified and red or blue, we have the special value formula*

$$\lambda^{-1}(L_X^*(F^D, 0)^{-1} \cdot \mathbb{Z}) = \Delta_X(F^D).$$

*In general, if  $F$  is a tamely ramified  $\mathbb{Z}$ -constructible sheaf, we have the special value formula*

$$L_X^*(F^D, 0) = \pm \chi_X(F^D).$$

For  $F = \mathbb{Z}$  we find the analytic class formula for the Dedekind zeta function, so this can be seen as a wide generalization of the analytic class number formula. In the singular case, it does not give a special value formula at  $s = 1$  of the zeta function: indeed if  $Z$  is the singular locus and  $\pi : Y \rightarrow X$  is the normalization,  $L_X(\mathbb{Z}^D, s) = \zeta_{X \setminus Z}(s+1) \times \prod_{z \in Z} \prod_{\pi(y)=z} \frac{\zeta_y(s+1)}{\zeta_y(s)}$ .

## IV.1.2 Comparison with other works

**IV.1.2.1 [Tra15]** Our work represents a significant improvement on Tran’s thesis. The construction of Tran, which takes place over  $X = \text{Spec}(\mathcal{O}_K)$  in the totally imaginary case, involves a complex  $D_{FD}$  for so-called “strongly  $\mathbb{Z}$ -constructible sheaves” with a naturally attached real number  $\chi_{\text{Tran}}(F^D)$ . In the previous chapters of his thesis, Tran had constructed a complex  $D_F$  with a naturally attached real number<sup>10</sup>  $\chi_{\text{Tran}}(F)$  such that, for  $M$  a finite type torsion-free discrete  $G_K$ -module corresponding to an étale sheaf  $Y$  on  $\text{Spec}(K)$  and  $g : \text{Spec}(K) \rightarrow X$  the canonical morphism,  $L_K^*(M, 0) = \pm \chi_{\text{Tran}}(g_*Y)$ . Let  $d = \text{rank}_{\mathbb{Z}} Y$ ; Tran computes

$$\chi_{\text{Tran}}((g_*Y)^D) = \frac{\chi_{\text{Tran}}(F)(2\pi)^{nd/2}}{\sqrt{|\Delta_K|}^d}$$

and essentially deduces from the functional equation the formula

$$L_K^*(M, 1) = \pm \frac{\chi_{\text{Tran}}((g_*Y)^D)}{N_{K/\mathbb{Q}}(\mathfrak{f}(M))},$$

with  $\mathfrak{f}(M)$  the Artin conductor. The quantity  $N_{K/\mathbb{Q}}(\mathfrak{f}(M))$  is shown to be related to two natural integral structures on the Lie algebra  $\text{Lie}(D(M))$  of the torus  $D(M)$  with character group  $M$ .<sup>11</sup>

Tran’s thesis only works out the case of a totally imaginary number field, which avoids the handling of factors of 2 linked to real places. Our construction takes places over an order in an arbitrary number field; in particular we have to be careful with the 2-torsion, which is achieved through our compactly supported cohomology, and with singularities, which are handled by using the dualizing complex  $\mathbb{Z}^c$  instead of  $\mathbb{G}_m[1]$ . Though we restrict ourselves to tamely ramified  $\mathbb{Z}$ -constructible coefficients (and this condition is lifted in Chapter V), we obtain a multiplicative Euler characteristic, given *on the nose* by the fundamental line, which describes the special value at  $s = 0$  of the  $L$ -function  $L_X(F^D, s)$  (which includes as a special case Artin  $L$ -functions at  $s = 1$  for tamely ramified rational representations of  $G_K$ ). We also have to place some restrictions on the  $\mathbb{Z}$ -constructible sheaves to obtain a construction of Weil-étale complexes, but these restrictions are significantly weaker than the “strongly  $\mathbb{Z}$ -constructible” condition of Tran and give a well-behaved complex that has good functorial properties.

**IV.1.2.2 [GS20]** Our work can be seen as adapting and generalizing to the number field case a part of Geisser–Suzuki’s work on special values of 1-motives over a function field  $K$ , specifically the part about tori. The analogy between our work and theirs is as follows: let  $K$  be a global field, let  $T$  be a torus over  $K$  with character group  $M$ , let  $X$  be either  $\text{Spec}(\mathcal{O}_K)$  or the smooth complete curve with function field  $K$  and let  $g : \text{Spec}(K) \rightarrow X$  be the canonical morphism. In the function field case, Geisser–Suzuki consider the connected Néron model  $\mathcal{T}^\circ$  of  $T$  over  $X$ . They prove that  $\mathcal{T}^\circ \simeq R\mathcal{H}om_X(\tau^{\leq 1} Rg_*M, \mathbb{G}_m)$  as an étale sheaf. The cup product with a generator  $e \in H_W^1(X, \mathbb{Z}) \simeq \mathbb{Z}$  induces a trivialization on  $\det_{\mathbb{R}}(R\Gamma_W(X, \mathcal{T}^\circ) \otimes \mathbb{R})$  while the complex  $R\Gamma_{\text{Zar}}(X, \mathcal{L}ie(\mathcal{T}^\circ)) \otimes \mathbb{R}$  is trivial; this induces a

<sup>10</sup>We do not say Euler characteristic because the multiplicativity is not proven

<sup>11</sup>It seems to us that there is a mistake there, as Tran claims that  $\text{Lie}(D(M)) = \text{Hom}_{\mathbb{Z}}(M, K)$  while the correct formula should be  $\text{Lie}(D(M)) = \text{Hom}_{G_K}(M, K^{\text{sep}})$ . It is not clear to us what impact this potential mistake has on his results.

trivialization  $\lambda_e : ((\det_{\mathbb{Z}} R\Gamma_W(X, \mathcal{T}^\circ))^{-1} \otimes \det_{\mathbb{Z}} R\Gamma_{\text{Zar}}(X, \mathcal{L}ie(\mathcal{T}^\circ)) \otimes \mathbb{R} \xrightarrow{\cong} \mathbb{R}$ . After some reformulation, the special value formula for the  $L$ -function of  $T$  is<sup>12</sup>

**Theorem** ([GS20, thm. 4.6]).

$$\lambda_e^{-1}(L^*(T, 0) \cdot \mathbb{Z}) = (\det_{\mathbb{Z}} R\Gamma_W(X, \mathcal{T}^\circ))^{-1} \otimes \det_{\mathbb{Z}} R\Gamma_{\text{Zar}}(X, \mathcal{L}ie(\mathcal{T}^\circ)).$$

In the number field case, put  $F := \tau^{\leq 1} Rg_* M$  so that

$$F^D = R\mathcal{H}om_X(\tau^{\leq 1} Rg_* M, \mathbb{G}_m)[1] = \mathcal{T}^\circ[1].$$

The sheaf  $g_* M$  is red and moreover  $R^1 g_* M$  is constructible, so the conclusions of both thm. A and thm. B hold without modification for  $F$  and we have a Weil-étale complex with compact support  $R\Gamma_{W,c}(X, \mathcal{T}^\circ) := R\Gamma_{W,c}(X, F^D)[-1]$ . We have  $L_X(F^D, s) = L(T, s)^{-1}$ . Finally when  $M$  is tamely ramified, since  $R^1 g_* M$  is supported on a finite closed subscheme we have

$$\text{Lie}_X(F^D) = \text{Lie}_X((g_* M)^D) = \text{Lie}(\mathcal{T})[1] = \text{Lie}(\mathcal{T}^\circ)[1] = R\Gamma_{\text{Zar}}(X, \mathcal{L}ie(\mathcal{T}^\circ))[1]$$

so our formula is the direct analogue of the previous one:

**Theorem.**

$$\lambda^{-1}(L^*(T, 0) \cdot \mathbb{Z}) = (\det_{\mathbb{Z}} R\Gamma_{W,c}(X, \mathcal{T}^\circ))^{-1} \otimes \det_{\mathbb{Z}} R\Gamma_{\text{Zar}}(X, \mathcal{L}ie(\mathcal{T}^\circ)).$$

Since we consider all coefficients  $F^D$  for  $F$  a  $\mathbb{Z}$ -constructible sheaf, we gain some flexibility which allows us to consider the case where  $X$  is singular. The main technical difficulties in our situation seem to be the definition of Weil-étale cohomology and of the tangent space of  $F^D$ ,<sup>13</sup> while in the function field case Weil-étale cohomology is a well-established construction by work of Lichtenbaum and Geisser [Lic05; Gei04].

**IV.1.2.3 [JP20]** Jordan–Poonen proved an analytic class number formula for the zeta function of a 1-dimensional affine reduced arithmetic scheme. Our work is related but different in nature, as the functions we consider are not the same: for  $j : U \rightarrow X$  an open immersion, we have  $L_X((j_! \mathbb{Z})^D, s) = \zeta_U(s+1)$  only when  $U = X$  and  $X$  is regular. Let  $X = \text{Spec}(\mathcal{O})$  be the spectrum of an order in a number field  $K$ , let  $j : U \rightarrow X$  be an open subscheme and let  $S_f$  be the finite places missing from the normalization of  $U$ . Put  $A := \mathcal{O}_U(U)$ . Their special value formula for  $U$  is

$$\zeta_U^*(1) = \frac{2^{r_1} (2\pi)^{r_2} h(A) R(A) \prod_{x \in S_f} ((1 - N(x)^{-1}) / \log N(x))}{\omega(A) \sqrt{|\Delta_A|}},$$

with  $r_1, r_2$  the number of real and complex places of  $K$ ,  $h(A) := [\text{Pic}(U)]$ ,  $R(A)$  is the covolume of  $A^\times \subset \mathcal{O}_{K,S}^\times$  under the usual logarithmic embedding,  $\omega(A) = [(A^\times)_{\text{tor}}]$  and  $\Delta_A$  is the discriminant of  $A$ , that is  $\det(\text{Tr}(e_i e_j))$  for a  $\mathbb{Z}$ -basis  $e_i$  of  $A$ .

<sup>12</sup>We use the  $L$ -function  $L(T, s)$  of the 1-motive of  $[0 \rightarrow T]$ , which is related to the Hasse-Weil  $L$ -function  $L_{\text{HW}}(T, s)$  by  $L(T, s) = L_{\text{HW}}(T, s+1)$ . This is why our formula takes place at  $s = 0$  while the formula in [GS20] is for  $s = 1$ .

<sup>13</sup>Although we construct in the next chapter the additive part of the fundamental line, which we interpret as the determinant of the tangent space of  $F^D$ , we do not introduce a tangent space in the tamely ramified case; the additive part of the fundamental line is constructed as the determinant of a complex which is not exact in  $F$ , so we are not sure if the latter deserves the name of “tangent space” of  $F^D$ : since all our constructions are derived, we would very much like the tangent space to be exact in  $F!$

Meanwhile our formula is computed explicitly as

$$L_X^*((j!\mathbb{Z})^D, 0) = \frac{2^{r_1}(2\pi)^{r_2} h_U R_U}{\omega \sqrt{|\Delta_K|}},$$

with  $h_U = [\text{CH}_0(U)]$ ,  $R_U$  the regulator from Corollary II.7.4 and  $\omega$  the number of roots of unity in  $K$ .

### IV.1.3 Notations

**IV.1.3.1 Schemes** Let  $Y_{\text{et}}$  denote the étale site of a scheme  $Y$  and  $Y_{\text{proet}}$  the pro-étale site [BS15]. There is morphism of topoi  $\nu : \text{Sh}(Y_{\text{proet}}) \rightarrow \text{Sh}(Y_{\text{et}})$  such that  $\nu^*$  is fully faithful. A sheaf of abelian groups on  $Y_{\text{et}}$  will be called an étale sheaf for short.

An arithmetic scheme is a scheme separated of finite type over  $\text{Spec}(\mathbb{Z})$ ; an arithmetic curve is a dimension 1 arithmetic scheme. In this chapter, we will consider proper integral arithmetic curves that are flat over  $\text{Spec}(\mathbb{Z})$ . Such a scheme is the spectrum of an order  $\mathcal{O}$  in a number field  $K$ . For the rest of the chapter, we set  $X = \text{Spec}(\mathcal{O})$ . The generic point will be denoted by  $g : \eta = \text{Spec}(K) \rightarrow X$ . We write  $G_K := \text{Gal}(K^{\text{sep}}/K)$  for the Galois group of  $K$ . If  $v$  is a finite place (resp. an archimedean place) of  $K$ , let  $K_v$  denote the henselian local field (resp. complete local field) at  $v$  and  $g_v : \eta_v = \text{Spec}(K_v) \rightarrow X$  the canonical morphism. If  $x$  is a closed point of  $X$ , denote by  $i_x : x \rightarrow X$  the inclusion (or  $i$  when the context is clear),  $G_x = \text{Gal}(\kappa(x)^{\text{sep}}/\kappa(x))$  the Galois group of the residue field at  $x$  and  $N(x) = [\kappa(x)]$  the cardinality of the residue field at  $x$ . When the context allows, we will abuse notation and write  $v$  for a regular closed point of  $X$  corresponding to a finite place  $v$  of  $K$ . If  $U$  is an open subscheme of  $X$ , let  $j : U \rightarrow X$  denote the corresponding morphism. We denote by  $X_0$  the set of closed points of  $X$ .

For  $F$  an étale sheaf on  $X$ , we put  $F_\eta := g^*F$ , resp.  $F_x = i_x^*F$  for  $x$  a closed point, and we identify it with a discrete  $G_K$ -module, resp. a discrete  $G_x$ -module. In general, we will make no distinction between an étale sheaf on the spectrum of a field and the associated Galois module.

**IV.1.3.2 Dualizing complex** Denote by  $z_0(X, i)$  the free abelian group generated by closed integral subschemes of  $X \times \Delta^i$ , of relative dimension  $i$ , which intersect all faces properly. The dualizing complex  $\mathbb{Z}_X^c := \mathbb{Z}_X^c(0)$  (also called Bloch's cycle complex) is the complex of étale sheaves with  $z_0(-, -i)$  in degree  $i$  and differentials the sum of face maps [Gei10]. Let

$$\mathbb{G}_X := \left[ g_*\mathbb{G}_m \rightarrow \bigoplus_{x \in X_0} i_{x,*}\mathbb{Z} \right]$$

denote Deninger's dualizing complex [Den87a]. In our case we have  $\mathbb{Z}_X^c \simeq \mathbb{G}_X[1]$  [Nar89].

For  $F$  an étale sheaf on  $X$ , we denote by  $F^D := R\mathcal{H}om_X(F, \mathbb{Z}_X^c)$  the derived internal hom in the derived category of étale sheaves on  $X$ .

**IV.1.3.3 Group cohomology** If  $G$  is a finite group, we denote by  $R\Gamma(G, -)$  the derived functor of  $G$ -invariants. Let  $P^\bullet$  be the standard complete resolution of  $\mathbb{Z}$ . Put furthermore  $R\hat{\Gamma}(G, -) := R\text{Hom}_G(P^\bullet, -)$ . This latter functor computes Tate cohomology.

Let  $G$  be a profinite group and  $H \subset G$  an open subgroup. If  $M$  is a discrete  $H$ -module, then the induction of  $M$  to  $G$  is the  $G$ -module  $\text{ind}_H^G M := \text{Cont}_H(G, M)$  of  $H$ -equivariant continuous maps  $G \rightarrow M$ . Let  $L/K$  be a finite extension of fields and  $F$  an



étale sheaf on  $\text{Spec}(L)$  corresponding to a discrete  $G_L$ -module  $M$ . If  $\pi$  denotes the map  $\text{Spec}(L) \rightarrow \text{Spec}(K)$  then  $\pi_* F$  corresponds to the discrete  $G_K$ -module  $\text{ind}_{G_L}^{G_K} M$ .

If  $M$  is an abelian group, denote by  $M^\vee := R\text{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$  its (derived) linear dual,  $M^* := \text{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$  its Pontryagin dual, and  $M^\dagger := \text{Hom}_{\mathbb{Z}}(M, \mathbb{Q})$ <sup>14</sup>. If  $V$  is a vector space over a field  $E$ , let  $V^\vee$  denote its linear dual.

**IV.1.3.4 Complex points** We endow the complex points  $X(\mathbb{C})$  with the analytic topology; in our case the complex points correspond to the embeddings  $K \rightarrow \mathbb{C}$  and the topology is discrete. Denote by  $\text{Sh}(G_{\mathbb{R}}, X(\mathbb{C}))$  the topos of  $G_{\mathbb{R}}$ -equivariant sheaves on  $X(\mathbb{C})$ . There is a morphism of topoi  $\alpha : \text{Sh}(G_{\mathbb{R}}, X(\mathbb{C})) \rightarrow \text{Sh}(X_{\text{ét}})$ . Put  $\mathbb{Z}(1) := 2i\pi\mathbb{Z} \in \text{Sh}(G_{\mathbb{R}}, X(\mathbb{C}))$ , and for  $M$  a  $G_{\mathbb{R}}$ -equivariant sheaf of abelian groups on  $X(\mathbb{C})$ , and

$$\begin{aligned} M(1) &:= M \otimes \mathbb{Z}(1) \text{ with diagonal action,} \\ M^\vee &:= R\mathcal{H}om_{G_{\mathbb{R}}, X(\mathbb{C})}(M, \mathbb{Z}), \\ M^\vee(1) &:= R\mathcal{H}om_{G_{\mathbb{R}}, X(\mathbb{C})}(M, \mathbb{Z}(1)). \end{aligned}$$

The  $G_{\mathbb{R}}$ -equivariant cohomology of a complex  $C \in D(\text{Sh}(G_{\mathbb{R}}, X(\mathbb{C})))$  is defined as

$$R\Gamma_{G_{\mathbb{R}}}(X(\mathbb{C}), C) := R\Gamma(G_{\mathbb{R}}, R\Gamma(X(\mathbb{C}), C)).$$

We also denote by

$$R\hat{\Gamma}_{G_{\mathbb{R}}}(X(\mathbb{C}), C) := R\hat{\Gamma}(G_{\mathbb{R}}, R\Gamma(X(\mathbb{C}), C))$$

the Tate  $G_{\mathbb{R}}$ -equivariant cohomology. The norm map  $N$  induces a fiber sequence

$$\mathbb{Z} \otimes_{\mathbb{Z}[G]}^L R\Gamma(X(\mathbb{C}), C) \xrightarrow{N} R\Gamma_{G_{\mathbb{R}}}(X(\mathbb{C}), C) \rightarrow R\hat{\Gamma}_{G_{\mathbb{R}}}(X(\mathbb{C}), C).$$

*Remark.* Since  $X(\mathbb{C})$  is discrete, a  $G_{\mathbb{R}}$ -equivariant sheaf on  $X(\mathbb{C})$  is the data of

- for each embedding  $\sigma : K \rightarrow \mathbb{C}$ , an abelian group  $F_\sigma$ , and
- an isomorphism  $F_\sigma \rightarrow F_{\bar{\sigma}}$  for  $\sigma$  complex and an action of  $G_{\mathbb{R}}$  on  $F_\sigma$  for  $\sigma$  real.

Choose an embedding  $\sigma_v$  for each archimedean place and put  $F_v = F_{\sigma_v}$ . We then have

$$R\Gamma(X(\mathbb{C}), F) = \prod_{v \text{ real}} F_v \times \prod_{v \text{ complex}} \text{ind}_{\{1\}}^{G_{\mathbb{R}}} F_v$$

as a  $G_{\mathbb{R}}$ -module, and thus

$$R\Gamma_{G_{\mathbb{R}}}(X(\mathbb{C}), F) = \prod_{v \text{ archimedean}} R\Gamma(G_{K_v}, F_v).$$

This breaks down if  $X(\mathbb{C})$  is not discrete.

If  $F$  is an étale sheaf on  $X$  and  $v$  is a place of  $K$ , denote by  $F_v$  its pullback to  $\text{Spec}(K_v)$ ; thus  $(\alpha^* F)_v = F_v$ .

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<sup>14</sup>“The dagger kills torsion.”

**IV.1.3.5 Determinants** Let  $A = \mathbb{Z}$  or  $\mathbb{R}$ . A complex  $C$  in the derived category  $D(A)$  is perfect if it is bounded with finite type cohomology groups. We will use the determinant construction of Knudsen-Mumford [KM76], and the subsequent work of Breuning, Burns and Knudsen, in particular [Bre08; Bre11]. Denote by  $\text{Proj}_A$  the exact category of projective finite type  $A$ -modules,  $\text{Gr}^b(\text{Mod}_A^{\text{ft}})$  the bounded graded abelian category of finite type  $A$ -modules,  $D_{\text{perf}}(A)$  the derived category of perfect complexes and  $\mathcal{P}_A$  the Picard groupoid of graded  $A$ -lines. The usual determinant functor

$$\det_A \in \det(\text{Proj}_A, \mathcal{P}_A), \quad M \mapsto (\Lambda^{\text{rank}_A M} M, \text{rank}_A M)$$

extends to a determinant functor  $g_A \in \det(\text{Gr}^b(\text{Modft}_A), \mathcal{P}_A)$ . Moreover, the graded cohomology functor  $H : D_{\text{perf}}(A) \rightarrow \text{Gr}^b(\text{Modft}_A)$  induces a functor

$$H^* : \det(\text{Gr}^b(\text{Modft}_A), \mathcal{P}_A) \rightarrow \det(D_{\text{perf}}(A), \mathcal{P}_A),$$

and we put  $\det_A := H^* g_A$ .<sup>15</sup> There is a canonical isomorphism  $(\det_{\mathbb{Z}} C) \otimes \mathbb{R} \simeq \det_{\mathbb{R}}(C \otimes \mathbb{R})$ .

**IV.1.3.6 Derived  $\infty$ -categories** We will use the theory of stable  $\infty$ -categories, see [Lur17]. If  $\mathcal{X}$  is a topos, denote by  $\mathcal{D}(\mathcal{X})$  the derived  $\infty$ -category associated to abelian group objects in  $\mathcal{X}$ . It is a stable  $\infty$ -category whose homotopy category is the usual derived 1-category  $D(\mathcal{X})$ . In particular  $\mathcal{D}(X)$  will denote the derived  $\infty$ -category of étale sheaves on  $X$ ,  $\mathcal{D}(\mathbb{Z})$  the derived  $\infty$ -category of abelian groups and  $\mathcal{D}(G_{\mathbb{R}}, X(\mathbb{C}))$  the derived  $\infty$ -category of  $G_{\mathbb{R}}$ -equivariant sheaves on  $X(\mathbb{C})$ .

A stable  $\infty$ -category has all finite limits and colimits. Moreover, pushouts are pullbacks and reciprocally. The homotopy category of a stable  $\infty$ -category has a canonical structure of triangulated category. If  $\mathcal{C}$  is a stable  $\infty$ -category and  $A \in \mathcal{C}$ , denote by  $A[1]$  the shift of  $A$ ,<sup>16</sup> which is given by the following pushout:

$$\begin{array}{ccc} A & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & A[1] \end{array}$$

If  $f : A \rightarrow B$  is a morphism in  $\mathcal{C}$ , we define  $\text{fib}(f)$  and  $\text{cofib}(f)$  by the following pullback and pushout diagrams:

$$\begin{array}{ccc} \text{fib}(f) & \longrightarrow & A \\ \downarrow & & \downarrow f \\ 0 & \longrightarrow & B \end{array} \quad \begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \text{cofib}(f) \end{array}$$

A sequence  $A \rightarrow B \rightarrow C$  is called a fiber sequence if  $A = \text{fib}(B \rightarrow C)$  (or equivalently  $C = \text{cofib}(A \rightarrow B)$ ). A fiber sequence induces a distinguished triangle in the homotopy category.

<sup>15</sup>This convention is chosen to simplify proofs later on, but of course all determinant functors extending  $\det_A \in \det(\text{Proj}_A, \mathcal{P}_A)$  are isomorphic

<sup>16</sup>It is usually called the suspension and denoted by  $\Sigma$  in the general setting of stable  $\infty$ -categories, but we will only be working with derived categories so this choice of slightly non-canonical notation is not really harmful.

## IV.2 Cohomology with compact support of $F^D$

### IV.2.1 Definitions

The  $G_{\mathbb{R}}$ -equivariant sheaf  $\alpha^*\mathbb{Z}_X^c = \alpha^*\mathbb{G}_X[1]$  is  $\overline{\mathbb{Q}}^\times[1]$ ,<sup>17</sup> with action of  $G_{\mathbb{R}}$  via decomposition groups at archimedean places. We have a canonical morphism  $\alpha^*\mathbb{Z}_X^c \rightarrow \mathbb{C}^\times[1]$  and the short exact sequence  $0 \rightarrow 2i\pi\mathbb{Z} \rightarrow \mathbb{C} \rightarrow \mathbb{C}^\times \rightarrow 0$  gives a map  $\mathbb{C}^\times[1] \rightarrow 2i\pi\mathbb{Z}[2]$ , whence a composite arrow  $\alpha^*\mathbb{Z}_X^c \rightarrow \mathbb{Z}(1)[2]$ . Let  $F, G$  be abelian sheaves on  $X_{\text{et}}$ . The functor  $\alpha^*$  is strict monoidal, hence from the arrow

$$\alpha^*(R\mathcal{H}om_X(F, G) \otimes^L F) \rightarrow \alpha^*G$$

obtained by applying  $\alpha^*$  to the natural map, we obtain by adjunction a canonical map

$$\alpha^*R\mathcal{H}om_X(F, G) \rightarrow R\mathcal{H}om_{G_{\mathbb{R}}, X(\mathbb{C})}(\alpha^*F, \alpha^*G).$$

In particular, there is a natural transformation

$$\alpha^*(F^D) \rightarrow R\mathcal{H}om_{G_{\mathbb{R}}, X(\mathbb{C})}(\alpha^*F, \mathbb{C}^\times[1]) \rightarrow (\alpha^*F)^\vee(1)[2],$$

hence maps

$$R\Gamma(X, F^D) \rightarrow R\Gamma_{G_{\mathbb{R}}}(X(\mathbb{C}), (\alpha^*F)^\vee(1)[2]) \rightarrow R\hat{\Gamma}_{G_{\mathbb{R}}}(X(\mathbb{C}), (\alpha^*F)^\vee(1)[2]).$$

**Definition IV.2.1.** *The corrected cohomology with compact support of  $F^D$  is the fiber of the first morphism:*

$$R\Gamma_{c,B}(X, F^D) := \text{fib}(R\Gamma(X, F^D) \rightarrow R\Gamma_{G_{\mathbb{R}}}(X(\mathbb{C}), (\alpha^*F)^\vee(1)[2])).$$

*The Tate corrected cohomology with compact support of  $F^D$  is the fiber of the composite morphism:*

$$R\hat{\Gamma}_{c,B}(X, F^D) := \text{fib}(R\Gamma(X, F^D) \rightarrow R\hat{\Gamma}_{G_{\mathbb{R}}}(X(\mathbb{C}), (\alpha^*F)^\vee(1)[2])).$$

*We also recall the definition of Tate cohomology with compact support<sup>18</sup> of  $F$  from Section II.2 as the fiber:*

$$R\hat{\Gamma}_c(X, F) := \text{fib}(R\Gamma(X, F) \rightarrow R\hat{\Gamma}_{G_{\mathbb{R}}}(X(\mathbb{C}), \alpha^*F)).$$

*Remark.* The above is only defined for sheaves of the specific form  $F^D$ . It is not a functor on the whole category of sheaves. We think of it as a contravariant functor in  $F$ . As we will see later, Tate corrected cohomology with compact support of  $F^D$  is in an ‘‘Artin–Verdier-like’’ duality with étale cohomology of  $F$ , so we can think of it as an étale homology of  $F$ .

**Proposition IV.2.2.** *Let  $F$  be an étale sheaf on  $X$ . The canonical map  $\alpha^*\mathbb{Z}_X^c \rightarrow \mathbb{Z}(1)[2]$  induces an isomorphism*

$$R\hat{\Gamma}_c(X, F^D) \xrightarrow{\cong} R\hat{\Gamma}_{c,B}(X, F^D).$$

<sup>17</sup>This is an abuse of notation, coming either from the choice of an embedding of  $K$  in  $\overline{\mathbb{Q}}$  (and the choice doesn’t matter since all we do after is compute cohomology), or by defining number fields as subfields of  $\overline{\mathbb{Q}}$  that are finite over  $\mathbb{Q}$ .

<sup>18</sup>The definition is not original but the terminology is, see [Mil06, II.2, Cohomology with compact support]

*Proof.* We have to show that

$$R\hat{\Gamma}(G_{\mathbb{R}}, \alpha^*(F^D)) \xrightarrow{\simeq} R\hat{\Gamma}(G_{\mathbb{R}}, (\alpha^*F)^\vee(1)[2])$$

is an isomorphism. It is enough to show it at real places. For  $v$  a real place, denote by  $g_v : \text{Spec}(K_v) \rightarrow X$  the canonical map. As  $g_v$  is proétale, we have  $g_v^*(F^D) = R\mathcal{H}om_{G_{\mathbb{R}}}(g_v^*F, \overline{\mathbb{Q}}^\times[1])$ , and we thus want to show

$$R\hat{\Gamma}(G_{\mathbb{R}}, R\mathcal{H}om_{G_{\mathbb{R}}}(g_v^*F, \overline{\mathbb{Q}}^\times[1])) \xrightarrow{\simeq} R\hat{\Gamma}(G_{\mathbb{R}}, R\mathcal{H}om_{G_{\mathbb{R}}}(g_v^*F, 2i\pi\mathbb{Z}[2]))$$

is an isomorphism. All terms now depend only on  $M := g_v^*F$  so we can reason by Artin induction on any finite type  $G_{\mathbb{R}}$ -module  $M$ .

We first treat the case  $M = \mathbb{Z}$ : from the exact sequence  $0 \rightarrow 2i\pi\mathbb{Z} \rightarrow \mathbb{C} \rightarrow \mathbb{C}^\times \rightarrow 0$  we find an isomorphism

$$R\hat{\Gamma}(G_{\mathbb{R}}, \mathbb{C}^\times[1]) \xrightarrow{\simeq} R\hat{\Gamma}(G_{\mathbb{R}}, \mathbb{Z}(1)[2])$$

Moreover,  $\overline{\mathbb{Q}}^\times$  and  $\mathbb{C}^\times$  have the same Tate cohomology: both are zero in odd degree and in even degree, both equal  $\mathbb{Z}/2\mathbb{Z}$  via the sign map because a real algebraic number which is a norm is the square of a real algebraic number, hence the norm of an algebraic number.

If  $M = \text{ind}_{\{0\}}^{G_{\mathbb{R}}}\mathbb{Z}$  is induced, we have that  $R\mathcal{H}om_{G_{\mathbb{R}}}(M, N) \simeq \text{ind}_{\{0\}}^{G_{\mathbb{R}}}N$  is induced for any  $G_{\mathbb{R}}$ -module  $N$ , so it has trivial Tate cohomology.

Finally, if  $M$  is finite  $R\mathcal{H}om_{G_{\mathbb{R}}}(M, \overline{\mathbb{Q}}^\times)$  and  $R\mathcal{H}om_{G_{\mathbb{R}}}(M, \mathbb{C}^\times)$  are canonically isomorphic because  $\overline{\mathbb{Q}}^\times$  and  $\mathbb{C}^\times$  have the same torsion. Since  $\mathbb{C}$  is uniquely divisible and has no torsion, we have moreover  $R\mathcal{H}om_{G_{\mathbb{R}}}(M, \mathbb{C}^\times) \xrightarrow{\simeq} R\mathcal{H}om_{G_{\mathbb{R}}}(M, 2i\pi\mathbb{Z}[1])$ . We thus obtain the required isomorphism on Tate cohomology.

By Artin induction we obtain the required isomorphism for any finite type  $G_{\mathbb{R}}$ -module  $M$ ; by taking filtered colimits, we obtain the statement for any  $G_{\mathbb{R}}$ -module  $M$  since Tate cohomology, being a derived Hom, commutes with derived limits.  $\square$

## IV.2.2 Artin–Verdier duality for $F^D$

Let  $X$  be the spectrum of an order in a number field. For a complex of abelian groups, denote  $R\text{Hom}(-, \mathbb{Q}/\mathbb{Z}) = \text{Hom}^\bullet(-, \mathbb{Q}/\mathbb{Z})$  by  $(-)^*$ . Artin–Verdier duality gives a map  $R\Gamma(X, F^D) = R\text{Hom}_X(F, \mathbb{G}_X[1]) \xrightarrow{AV} R\hat{\Gamma}_c(X, F)^*[-2]$  which is “almost” an isomorphism. We want to modify this duality at the complex points to obtain an Artin–Verdier duality relating  $R\hat{\Gamma}_{c,B}(X, F^D)$  and  $R\Gamma(X, F)$ . We have fiber sequences

$$\begin{aligned} R\hat{\Gamma}_{G_{\mathbb{R}}}(X(\mathbb{C}), (\alpha^*F)^\vee(1)[1]) &\rightarrow R\hat{\Gamma}_{c,B}(X, F^D) \rightarrow R\Gamma(X, F^D), \\ R\hat{\Gamma}_c(X, F) &\rightarrow R\Gamma(X, F) \rightarrow R\hat{\Gamma}_{G_{\mathbb{R}}}(X(\mathbb{C}), \alpha^*F). \end{aligned}$$

To obtain an Artin–Verdier duality statement, we will construct in the next three paragraphs pairings

$$\begin{aligned} R\hat{\Gamma}_{G_{\mathbb{R}}}(X(\mathbb{C}), (\alpha^*F)^\vee(1)) \otimes^L R\hat{\Gamma}_{G_{\mathbb{R}}}(X(\mathbb{C}), \alpha^*F) &\rightarrow \mathbb{Q}/\mathbb{Z}[-3], \\ R\Gamma(X, F) \otimes^L R\hat{\Gamma}_{c,B}(X, F^D) &\rightarrow \mathbb{Q}/\mathbb{Z}[-2], \end{aligned}$$

such that we have a morphism of fiber sequences given by the adjoint maps:

$$\begin{array}{ccccc} R\hat{\Gamma}_{G_{\mathbb{R}}}(X(\mathbb{C}), (\alpha^*F)^\vee(1)[1]) &\longrightarrow & R\hat{\Gamma}_{c,B}(X, F^D) &\longrightarrow & R\Gamma(X, F^D) \\ \downarrow & & \downarrow & & \downarrow AV \\ R\hat{\Gamma}_{G_{\mathbb{R}}}(X(\mathbb{C}), \alpha^*F)^*[-2] &\longrightarrow & R\Gamma(X, F)^*[-2] &\longrightarrow & R\hat{\Gamma}_c(X, F)^*[-2] \end{array} \quad (\text{IV.2.1})$$

**IV.2.2.1 Construction of the pairing for Tate cohomology on  $X(\mathbb{C})$ .** Let us construct the first pairing: Tate cohomology  $R\hat{\Gamma}_{G_{\mathbb{R}}}(X(\mathbb{C}), -) = R\hat{\Gamma}_{G_{\mathbb{R}}}(-) \circ R\Gamma(X(\mathbb{C}), -)$  is lax-monoidal [NS18, I.3.1], hence the natural evaluation map  $(\alpha^*F)^\vee(1)[1] \otimes^L \alpha^*F \rightarrow 2i\pi\mathbb{Z}[1]$  gives a map

$$R\hat{\Gamma}_{G_{\mathbb{R}}}(X(\mathbb{C}), (\alpha^*F)^\vee(1)) \otimes^L R\hat{\Gamma}_{G_{\mathbb{R}}}(X(\mathbb{C}), \alpha^*F) \rightarrow R\hat{\Gamma}_{G_{\mathbb{R}}}(X(\mathbb{C}), 2i\pi\mathbb{Z}).$$

Since  $\hat{H}_{G_{\mathbb{R}}}^i(X(\mathbb{C}), 2i\pi\mathbb{Z}) = 0$  for  $i$  even, there is a canonical isomorphism<sup>19</sup>

$$R\hat{\Gamma}_{G_{\mathbb{R}}}(X(\mathbb{C}), 2i\pi\mathbb{Z}) \simeq \bigoplus_i \hat{H}_{G_{\mathbb{R}}}^i(X(\mathbb{C}), 2i\pi\mathbb{Z})[-i].$$

Let  $r_1$  be the number of real places of the function field of  $K$ . The natural map  $\mathbb{C}^\times \rightarrow 2i\pi\mathbb{Z}[1]$  induces an identification  $\hat{H}^3(G_{\mathbb{R}}, X(\mathbb{C}), 2i\pi\mathbb{Z}) = (\mathbb{R}^\times/\mathbb{R}_{>0}^\times)^{r_1}$ . We compose the previous pairing with a projection and a sum map to get the pairing

$$\begin{aligned} R\hat{\Gamma}_{G_{\mathbb{R}}}(X(\mathbb{C}), (\alpha^*F)^\vee(1)) \otimes^L R\hat{\Gamma}_{G_{\mathbb{R}}}(X(\mathbb{C}), \alpha^*F) &\rightarrow R\hat{\Gamma}_{G_{\mathbb{R}}}(X(\mathbb{C}), 2i\pi\mathbb{Z}) \\ &\xrightarrow{H^3} \hat{H}_{G_{\mathbb{R}}}^3(X(\mathbb{C}), 2i\pi\mathbb{Z})[-3] \\ &= (\mathbb{R}^\times/\mathbb{R}_{>0}^\times)^{r_1}[-3] \\ &\xrightarrow{\Sigma} \mathbb{Q}/\mathbb{Z}[-3]. \end{aligned} \quad (\text{IV.2.2})$$

**IV.2.2.2 Construction of the pairing for  $R\hat{\Gamma}_{c,B}(X, F^D)$ .** In the following diagrams, we will write  $R\Gamma, R\Gamma_{G_{\mathbb{R}}}, R\hat{\Gamma}_{G_{\mathbb{R}}}, R\hat{\Gamma}_c, R\hat{\Gamma}_{c,B}$  for  $R\Gamma(X, -), R\Gamma_{G_{\mathbb{R}}}(X(\mathbb{C}), -), R\hat{\Gamma}_{G_{\mathbb{R}}}(X(\mathbb{C}), -), R\hat{\Gamma}_c(X, -)$  and  $R\hat{\Gamma}_{c,B}(X, -)$  respectively, and we will also write  $\otimes$  for the derived tensor product. The natural transformations

$$R\Gamma(X, -) \rightarrow R\Gamma_{G_{\mathbb{R}}}(X(\mathbb{C}), -) \rightarrow R\hat{\Gamma}_{G_{\mathbb{R}}}(X(\mathbb{C}), -) \rightarrow$$

are lax monoidal [NS18, I.3.1].<sup>20</sup> so the following diagram commutes:

$$\begin{array}{ccccc} R\Gamma(F) \otimes R\Gamma(F^D) & \xrightarrow{\hspace{10em}} & R\Gamma(\mathbb{Z}_X^c) & & \\ \downarrow & & \downarrow & & \\ R\Gamma(F) \otimes R\Gamma_{G_{\mathbb{R}}}(F_{\mathbb{C}}^\vee(1)[2]) & \rightarrow & R\Gamma_{G_{\mathbb{R}}}(\alpha^*F) \otimes R\Gamma_{G_{\mathbb{R}}}(F_{\mathbb{C}}^\vee(1)[2]) & \rightarrow & R\Gamma_{G_{\mathbb{R}}}(2i\pi\mathbb{Z}[2]) \\ \downarrow & & \downarrow & & \downarrow \\ R\Gamma(F) \otimes R\hat{\Gamma}_{G_{\mathbb{R}}}(F_{\mathbb{C}}^\vee(1)[2]) & \rightarrow & R\hat{\Gamma}_{G_{\mathbb{R}}}(\alpha^*F) \otimes R\hat{\Gamma}_{G_{\mathbb{R}}}(F_{\mathbb{C}}^\vee(1)[2]) & \rightarrow & R\hat{\Gamma}_{G_{\mathbb{R}}}(2i\pi\mathbb{Z}[2]) \end{array}$$

<sup>19</sup>Indeed, there is always such an isomorphism (non-canonical) in the homotopy category  $D(\mathbb{Z})$ , and there is a canonical one here because there are no  $\text{Ext}^1$  between two consecutive cohomology groups

<sup>20</sup>Since the lax monoidal structure for  $R\Gamma(X, -)$  and  $R\Gamma_{G_{\mathbb{R}}}(X(\mathbb{C}), -)$  come from the strict monoidal structure on their left adjoint, this is a formal consequence of the commutative triangle of right adjoints with strict monoidal left adjoints

$$\begin{array}{ccc} \mathcal{D}(G_{\mathbb{R}}, X(\mathbb{C})) & \xrightarrow{\alpha_*} & \mathcal{D}(X_{\text{et}}) \\ & \searrow R\Gamma_{G_{\mathbb{R}}}(X(\mathbb{C}), -) & \swarrow R\Gamma(X, -) \\ & \mathcal{D}(\mathbb{Z}) & \end{array}$$

We thus obtain the left dotted map in the following commutative diagram where the top row is a fiber sequence:

$$\begin{array}{ccccc}
R\Gamma(X, F) \otimes^L R\hat{\Gamma}_{c,B}(X, F^D) & \longrightarrow & R\Gamma(X, F) \otimes^L R\Gamma(X, F^D) & \longrightarrow & R\Gamma(X, F) \otimes^L R\hat{\Gamma}_{G_{\mathbb{R}}}(X(\mathbb{C}), F_{\mathbb{C}}^{\vee}(1)[2]) \\
\downarrow \text{dotted} & & \downarrow & & \downarrow \\
R\hat{\Gamma}_{c,B}(X, \mathbb{Z}^D) & \longrightarrow & R\Gamma(X, \mathbb{Z}_X^c) & \longrightarrow & R\hat{\Gamma}_{G_{\mathbb{R}}}(X(\mathbb{C}), 2i\pi\mathbb{Z}[2])
\end{array}$$

We have

$$\tau^{\geq 2} R\hat{\Gamma}_{c,B}(X, \mathbb{Z}^D) = \tau^{\geq 2} R\hat{\Gamma}_c(X, \mathbb{Z}_X^c) = \mathbb{Q}/\mathbb{Z}[-2]$$

by Proposition IV.2.2 and [Mil06, II.2.6, II.6.1], hence we get a pairing

$$R\Gamma(X, F) \otimes^L R\hat{\Gamma}_{c,B}(X, F^D) \rightarrow R\hat{\Gamma}_{c,B}(X, \mathbb{Z}^D) \xrightarrow{\tau^{\geq 2}} \mathbb{Q}/\mathbb{Z}[-2]. \quad (\text{IV.2.3})$$

**IV.2.2.3 The morphism of fiber sequences.** The following cube is commutative by compatibility of the involved pairings:

$$\begin{array}{ccccc}
R\Gamma(F) \otimes R\Gamma(F^D) & \longrightarrow & R\Gamma(F) \otimes R\hat{\Gamma}_{G_{\mathbb{R}}}((\alpha^*F)^{\vee}(1)[2]) & & \\
\downarrow & \searrow & \downarrow & \searrow & \\
& & R\Gamma(\mathbb{Z}_X^c) & \longrightarrow & R\hat{\Gamma}_{G_{\mathbb{R}}}(2i\pi\mathbb{Z}[2]) \\
& & \downarrow & & \downarrow \\
R\hat{\Gamma}_{G_{\mathbb{R}}}(\alpha^*F) \otimes R\Gamma(F^D) & \longrightarrow & R\hat{\Gamma}_{G_{\mathbb{R}}}(\alpha^*F) \otimes R\hat{\Gamma}_{G_{\mathbb{R}}}((\alpha^*F)^{\vee}(1)[2]) & & \\
& \searrow & \downarrow & \searrow & \\
& & R\hat{\Gamma}_{G_{\mathbb{R}}}(\alpha^*\mathbb{Z}_X^c) & \longrightarrow & R\hat{\Gamma}_{G_{\mathbb{R}}}(2i\pi\mathbb{Z}[2])
\end{array}$$

hence by adding the fibers of horizontal and vertical maps we deduce a morphism of  $3 \times 3$  diagrams depicted in Diagram IV.1. Doing so, we recover in the middle layer the pairing (IV.2.3).

Let  $A, A', B, B', C, C'$  be objects of  $D(\text{Ab})$ , with maps  $A \rightarrow A', B' \rightarrow B, C \rightarrow C'$ . Given two pairings  $A \otimes^L B \rightarrow C$  and  $A' \otimes^L B' \rightarrow C'$ , the commutativity of the two following induced diagrams is equivalent:

$$\begin{array}{ccc}
A \longrightarrow A' & & A \otimes^L B' \longrightarrow A \otimes^L B \longrightarrow C \\
\downarrow & & \searrow & & \downarrow \\
R\mathcal{H}om(B, C) \longrightarrow R\mathcal{H}om(B', C') & & & & A' \otimes^L B' \longrightarrow C'
\end{array}$$

We now prove that the natural diagram (IV.2.1) commutes. By the above, this reformulates to the commutativity of a subdiagram of (IV.1), together with the following commutative diagram:

$$\begin{array}{ccccc}
R\hat{\Gamma}(2i\pi\mathbb{Z}[1]) = R\hat{\Gamma}(\mathbb{C}^{\times}) & \longrightarrow & R\hat{\Gamma}_{c,B}(\mathbb{Z}^D) & \xleftarrow{\simeq} & R\hat{\Gamma}_c(\mathbb{Z}_X^c) \\
& \searrow \Sigma \circ H^2 & \downarrow \tau^{\geq 2} & \swarrow \tau^{\geq 2} & \\
& & \mathbb{Q}/\mathbb{Z}[-2] & & 
\end{array}$$

which follows from the proof of [Mil06, II.2.6].

$$\begin{array}{ccccccc}
R\hat{\Gamma}_c(F) \otimes R\hat{\Gamma}_{c,B}(F^D) & \longrightarrow & R\hat{\Gamma}_c(F) \otimes R\Gamma(F^D) & \longrightarrow & R\hat{\Gamma}_c(F) \otimes R\hat{\Gamma}_{G_{\mathbb{R}}}((\alpha^*F)^{\vee}(1)[2]) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
R\Gamma(F) \otimes R\hat{\Gamma}_{c,B}(F^D) & \longrightarrow & R\Gamma(F) \otimes R\Gamma(F^D) & \longrightarrow & R\Gamma(F) \otimes R\hat{\Gamma}_{G_{\mathbb{R}}}((\alpha^*F)^{\vee}(1)[2]) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
R\hat{\Gamma}_{c,B}(\mathbb{Z}^c) & \xrightarrow{\cong} & R\hat{\Gamma}_c(\mathbb{Z}^c) & \longrightarrow & R\hat{\Gamma}_c(\mathbb{Z}^c) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
R\hat{\Gamma}_{c,B}(\mathbb{Z}^D) & \longrightarrow & R\Gamma(\mathbb{Z}^c) & \longrightarrow & R\hat{\Gamma}_{G_{\mathbb{R}}}(2i\pi\mathbb{Z}[2]) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
R\hat{\Gamma}_{G_{\mathbb{R}}}(\alpha^*F) \otimes R\hat{\Gamma}_{c,B}(F^D) & \longrightarrow & R\hat{\Gamma}_{G_{\mathbb{R}}}(\alpha^*F) \otimes R\Gamma(F^D) & \longrightarrow & R\hat{\Gamma}_{G_{\mathbb{R}}}(\alpha^*F) \otimes R\hat{\Gamma}_{G_{\mathbb{R}}}((\alpha^*F)^{\vee}(1)[2]) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & R\hat{\Gamma}_{G_{\mathbb{R}}}(\alpha^*\mathbb{Z}^c) & \xrightarrow{\cong} & R\hat{\Gamma}_{G_{\mathbb{R}}}(2i\pi\mathbb{Z}[2]) & \longrightarrow & 0
\end{array}$$

Diagram IV.1

**Theorem IV.2.3** (Artin–Verdier duality for  $F^D$ ). *Let  $F$  be a  $\mathbb{Z}$ -constructible sheaf on the spectrum  $X$  of an order in a number field. The pairing*

$$R\Gamma(X, F) \otimes^L R\hat{\Gamma}_{c,B}(X, F^D) \rightarrow \mathbb{Q}/\mathbb{Z}[-2]$$

*induces a map*

$$R\hat{\Gamma}_{c,B}(X, F^D) \rightarrow R\Gamma(X, F)^*[-2],$$

*which is an isomorphism in degree  $\neq -1, 0$ , and an isomorphism after profinite completion of the left hand side in degree  $-1, 0$ . In particular if  $F$  is constructible then the map is an isomorphism.*

*Remark.* The statement also holds more generally for a bounded complex  $F \in \mathcal{D}^b(X)$  such that  $H^0(F)$  is  $\mathbb{Z}$ -constructible and  $H^i(F)$  is constructible for  $i \neq 0$  (by filtering with the truncations). If we reformulate the theorem as the map  $R\hat{\Gamma}_{c,B}(X, F^D) \otimes \hat{\mathbb{Z}} \rightarrow R\Gamma(X, F)^*[-2]$  being an isomorphism, this generalizes to bounded complexes with  $\mathbb{Z}$ -constructible cohomology sheaves.

*Proof.* By the diagram (IV.2.1) and Artin–Verdier duality for singular schemes [Mil06, II.6.2], it suffices to show that the pairing

$$R\hat{\Gamma}_{G_{\mathbb{R}}}(X(\mathbb{C}), (\alpha^*F)^\vee(1)[1]) \otimes^L R\hat{\Gamma}_{G_{\mathbb{R}}}(X(\mathbb{C}), \alpha^*F) \rightarrow \mathbb{Q}/\mathbb{Z}[-2]$$

is a perfect pairing between complexes of abelian groups with finite cohomology groups. To prove that the pairing is perfect, it suffices to do it at every real place, that is for the Galois cohomology of  $G_{\mathbb{R}}$ ; thus it suffices to prove the next proposition.  $\square$

**Proposition IV.2.4.** *The canonical pairing*

$$R\hat{\Gamma}(G_{\mathbb{R}}, M^\vee(1)) \otimes^L R\hat{\Gamma}(G_{\mathbb{R}}, M) \rightarrow \mathbb{Q}/\mathbb{Z}[-3]$$

*is perfect for  $M$  any discrete  $G_{\mathbb{R}}$ -module of finite type.*

*Proof.* We will reduce to Theorem B.0.1 for the finite group  $G_{\mathbb{R}} = \mathbb{Z}/2\mathbb{Z}$ . Denote by  $\varepsilon$  the augmentation  $\mathbb{Z}[G_{\mathbb{R}}] \rightarrow \mathbb{Z}$  and  $I_{G_{\mathbb{R}}} = (s-1)\mathbb{Z}$  the augmentation ideal. We have  $H^1(G_{\mathbb{R}}, 2i\pi\mathbb{Z}) = \text{Ext}_{G_{\mathbb{R}}}^1(\mathbb{Z}, 2i\pi\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$ , and the non-zero class  $t$  corresponds to a morphism  $\mathbb{Z} \rightarrow 2i\pi\mathbb{Z}[1]$  in the derived category coming from the equivalence class of the non-split exact sequence

$$0 \rightarrow 2i\pi\mathbb{Z} = I_{G_{\mathbb{R}}} \rightarrow \mathbb{Z}[G_{\mathbb{R}}] \xrightarrow{\varepsilon} \mathbb{Z} \rightarrow 0.$$

Since  $I_{G_{\mathbb{R}}} = 2i\pi\mathbb{Z}$  as a  $G_{\mathbb{R}}$ -module, we obtain an isomorphism

$$R\hat{\Gamma}(G_{\mathbb{R}}, M) \xrightarrow[\text{(id}_M \otimes t)_*]{\simeq} R\hat{\Gamma}(G_{\mathbb{R}}, M(1))[1].$$

Note that  $M^\vee(1) = (M^\vee)(1)$ . From the above isomorphism we deduce an isomorphism of pairings

$$\begin{array}{ccccccc} R\hat{\Gamma}(G_{\mathbb{R}}, M^\vee) \otimes^L R\hat{\Gamma}(G_{\mathbb{R}}, M) & \longrightarrow & R\hat{\Gamma}(G_{\mathbb{R}}, \mathbb{Z}) & \longrightarrow & \hat{H}^2(G_{\mathbb{R}}, \mathbb{Z})[-2] & \longrightarrow & \mathbb{Q}/\mathbb{Z}[-2] \\ \downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq & & \parallel \\ R\hat{\Gamma}(G_{\mathbb{R}}, M^\vee(1))[1] \otimes^L R\hat{\Gamma}(G_{\mathbb{R}}, M) & \longrightarrow & R\hat{\Gamma}(G_{\mathbb{R}}, 2i\pi\mathbb{Z})[1] & \longrightarrow & \hat{H}^3(G_{\mathbb{R}}, 2i\pi\mathbb{Z})[-2] & \longrightarrow & \mathbb{Q}/\mathbb{Z}[-2] \end{array}$$



which shows that it suffices to check that the natural pairing

$$R\hat{\Gamma}(G_{\mathbb{R}}, M^{\vee}) \otimes^L R\hat{\Gamma}(G_{\mathbb{R}}, M) \rightarrow R\hat{\Gamma}(G_{\mathbb{R}}, \mathbb{Z}) \rightarrow \hat{H}^2(G_{\mathbb{R}}, \mathbb{Z})[-2] \rightarrow \mathbb{Q}/\mathbb{Z}[-2]$$

is perfect. In a similar way, the non-zero class  $u \in H^2(G_{\mathbb{R}}, \mathbb{Z})$  corresponds to a map  $\mathbb{Z} \rightarrow \mathbb{Z}[2]$  whose fiber comes from induced modules<sup>21</sup>, and we also get an isomorphism of pairings

$$\begin{array}{ccccccc} R\hat{\Gamma}(G_{\mathbb{R}}, M^{\vee}) \otimes^L R\hat{\Gamma}(G_{\mathbb{R}}, M) & \longrightarrow & R\hat{\Gamma}(G_{\mathbb{R}}, \mathbb{Z}) & \longrightarrow & \hat{H}^0(G_{\mathbb{R}}, \mathbb{Z})[0] & \longrightarrow & \mathbb{Q}/\mathbb{Z}[0] \\ \downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq & & \parallel \\ R\hat{\Gamma}(G_{\mathbb{R}}, M^{\vee}) \otimes^L R\hat{\Gamma}(G_{\mathbb{R}}, M)[2] & \longrightarrow & R\hat{\Gamma}(G_{\mathbb{R}}, \mathbb{Z})[2] & \longrightarrow & \hat{H}^2(G_{\mathbb{R}}, \mathbb{Z})[0] & \longrightarrow & \mathbb{Q}/\mathbb{Z}[0] \end{array}$$

so equivalently it suffices to check that the natural pairing

$$R\hat{\Gamma}(G_{\mathbb{R}}, M^{\vee}) \otimes^L R\hat{\Gamma}(G_{\mathbb{R}}, M) \rightarrow R\hat{\Gamma}(G_{\mathbb{R}}, \mathbb{Z}) \rightarrow \hat{H}^0(G_{\mathbb{R}}, \mathbb{Z})[0] \rightarrow \mathbb{Q}/\mathbb{Z}[0]$$

is perfect; this is Theorem B.0.1. □

We finish with a study of the behaviour of our corrected compactly supported cohomology with respect to finite dominant morphisms.

**Proposition IV.2.5.** *Let  $Y = \text{Spec}(\mathcal{O}')$  be the spectrum of an order in a number field with a finite dominant morphism  $\pi : Y \rightarrow X$  and let  $F$  be a  $\mathbb{Z}$ -constructible sheaf on  $Y$ . We have canonical isomorphisms*

$$\begin{aligned} R\Gamma_{c,B}(X, (\pi_* F)^D) &\xrightarrow{\simeq} R\Gamma_{c,B}(Y, F^D), \\ R\hat{\Gamma}_{c,B}(X, (\pi_* F)^D) &\xrightarrow{\simeq} R\hat{\Gamma}_{c,B}(Y, F^D), \end{aligned}$$

compatible with the maps  $R\Gamma_{c,B}(X, -) \rightarrow R\hat{\Gamma}_{c,B}(X, -)$  resp.  $R\Gamma_{c,B}(Y, -) \rightarrow R\hat{\Gamma}_{c,B}(Y, -)$ .

*Proof.* We prove it for  $R\Gamma_{c,B}(X, (-)^D)$ . The functor  $\pi_*$  is exact and we have  $R\pi^! \mathbb{Z}_X^c = R\pi^! \mathbb{G}_X[1] = \mathbb{G}_Y[1] = \mathbb{Z}_Y^c$  by the finite base change theorem, hence  $(\pi_* F)^D = \pi_*(F^D)$ . Denote by  $\pi' : \text{Sh}(G_{\mathbb{R}}, Y(\mathbb{C})) \rightarrow \text{Sh}(G_{\mathbb{R}}, X(\mathbb{C}))$  the morphism of topoi induced by  $\pi$ , and  $\alpha'$  the morphism  $\text{Sh}(G_{\mathbb{R}}, Y(\mathbb{C})) \rightarrow \text{Sh}(Y_{\text{et}})$ . Consider the commutative square

$$\begin{array}{ccc} \text{Sh}(G_{\mathbb{R}}, Y(\mathbb{C})) & \xrightarrow{\alpha'} & \text{Sh}(Y_{\text{et}}) \\ \downarrow \pi' & & \downarrow \pi \\ \text{Sh}(G_{\mathbb{R}}, X(\mathbb{C})) & \xrightarrow{\alpha} & \text{Sh}(X_{\text{et}}) \end{array}$$

It induces a canonical map  $\alpha^* \pi_* F \rightarrow \pi'_* \alpha'^* F$ . We claim that it is an isomorphism; indeed it suffices to check it on points of  $X(\mathbb{C})$ , and then it follows from the computation of the stalks of a finite morphism in the étale case and in the topological case<sup>22</sup>.

Since  $X(\mathbb{C})$  and  $Y(\mathbb{C})$  are finite discrete, we have  $\pi^! = \pi'^*$ , so  $R\pi^! = \pi'^*$ ,  $R\pi'^* 2i\pi\mathbb{Z} = 2i\pi\mathbb{Z}$  and  $R\pi'^* \mathbb{C}^\times = \mathbb{C}^\times$ . The counit  $\pi'_* \pi'^* \rightarrow \text{id}$  is given on stalks by the sum map. Denote by  $\pi_x$  the base change of  $\pi$  to a point  $x \in X$  and  $g : \eta \rightarrow X$ ,  $g' : \eta' \rightarrow Y$ ,  $i_x : x \rightarrow X$  resp.

<sup>21</sup>This is just a reformulation of the 2-periodicity of the Tate cohomology of cyclic groups

<sup>22</sup>In the topological case, by finite morphism we mean a universally closed separated continuous map with finite discrete fibers; we then use [Stacks, Tag 09V4]

$i_y : y \rightarrow Y$  the inclusion of the generic points, resp. of closed points, of  $X$  and  $Y$ . The counit  $\pi_* \mathbb{Z}_Y^c \rightarrow \mathbb{Z}_X^c$  is given by the morphism of complexes (with left term in degree  $-1$ )

$$\begin{array}{ccc} \pi_* g'_* \mathbb{G}_m = g_* \pi_{\eta,*} \mathbb{G}_m & \xrightarrow{\Sigma \text{ord}_y} & \bigoplus_{y \in Y_0} \pi_* i_{y,*} \mathbb{Z} = \bigoplus_{x \in X_0} i_{x,*} (\pi_{x,*} \mathbb{Z}) \\ \downarrow & & \downarrow \\ g_* \mathbb{G}_m & \xrightarrow{\Sigma \text{ord}_x} & \bigoplus_{x \in X_0} i_{x,*} \mathbb{Z} \end{array}$$

where the left arrow is obtained by applying  $g_*$  to the counit of the adjunction  $\pi_{\eta,*} \dashv \pi_{\eta}^*$ , which is simply the sum map. Thus we have an identification  $\alpha^*(\pi_* \mathbb{Z}_Y^c \rightarrow \mathbb{Z}_X^c) = \pi'_* \overline{\mathbb{Q}}^\times[1] \rightarrow \overline{\mathbb{Q}}^\times[1]$ .<sup>23</sup> Combining this with the equality  $\alpha^* \pi_* F = \pi'_* \alpha'^* F$  and the identification  $R\pi^!(\overline{\mathbb{Q}}^\times \rightarrow 2i\pi\mathbb{Z}[2]) = \overline{\mathbb{Q}}^\times \rightarrow 2i\pi\mathbb{Z}[1]$ , we obtain a commutative diagram<sup>24</sup>:

$$\begin{array}{ccccc} R\text{Hom}_Y(F, \mathbb{Z}_Y^c) & \xrightarrow{\alpha'^*} & R\text{Hom}_{G_{\mathbb{R}}, Y(\mathbb{C})}(\alpha^* F, \overline{\mathbb{Q}}^\times[1]) & \longrightarrow & R\text{Hom}_{G_{\mathbb{R}}, Y(\mathbb{C})}(\alpha^* F, 2i\pi\mathbb{Z}[2]) \\ \simeq \downarrow & & \downarrow \simeq & & \downarrow \simeq \\ R\text{Hom}_X(\pi_* F, \mathbb{Z}_X^c) & \xrightarrow{\alpha^*} & R\text{Hom}_{G_{\mathbb{R}}, X(\mathbb{C})}(\pi'_* \alpha'^* F, \overline{\mathbb{Q}}^\times[1]) & \longrightarrow & R\text{Hom}_{G_{\mathbb{R}}, X(\mathbb{C})}(\pi'_* \alpha'^* F, 2i\pi\mathbb{Z}[2]) \end{array}$$

hence we obtain in the following diagram of fiber sequences the induced arrow, which is an isomorphism:

$$\begin{array}{ccccc} R\Gamma_{c,B}(Y, (-)^D) & \longrightarrow & R\Gamma(Y, (-)^D) & \longrightarrow & R\Gamma_{G_{\mathbb{R}}}(Y(\mathbb{C}), (-)_{\mathbb{C}}^{\vee}(1)[2]) \\ \downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq \\ R\Gamma_{c,B}(X, (\pi_* -)^D) & \longrightarrow & R\Gamma(X, (\pi_* -)^D) & \longrightarrow & R\Gamma_{G_{\mathbb{R}}}(X(\mathbb{C}), (\pi_* -)_{\mathbb{C}}^{\vee}(1)[2]) \end{array}$$

□

**Proposition IV.2.6.** *We keep the notations of Proposition IV.2.5. There is a commutative diagram of pairings:*

$$\begin{array}{ccc} R\Gamma(Y, F) \otimes^L R\hat{\Gamma}_{c,B}(Y, F^D) & \longrightarrow & R\hat{\Gamma}_{c,B}(Y, \mathbb{Z}^D) \\ \downarrow \simeq & & \downarrow \\ R\Gamma(X, \pi_* F) \otimes^L R\hat{\Gamma}_{c,B}(X, (\pi_* F)^D) & \longrightarrow & R\hat{\Gamma}_{c,B}(X, \mathbb{Z}^D) \end{array}$$

where the arrow  $R\hat{\Gamma}_{c,B}(Y, \mathbb{Z}^D) \rightarrow R\hat{\Gamma}_{c,B}(X, \mathbb{Z}^D)$  is induced from the map  $R\Gamma(Y, F^D) \rightarrow R\Gamma(X, F^D)$  (coming from the counit  $\varepsilon : \pi_* R\pi^! \mathbb{Z}_X^c = \pi_* \mathbb{Z}^D \rightarrow \mathbb{Z}^D$ ) and from the map  $R\Gamma_{G_{\mathbb{R}}}(Y(\mathbb{C}), 2i\pi\mathbb{Z}[2]) \rightarrow R\Gamma_{G_{\mathbb{R}}}(X(\mathbb{C}), 2i\pi\mathbb{Z}[2])$ .

*Proof.* By compatibility of the pairings for  $R\Gamma_{G_{\mathbb{R}}}(X(\mathbb{C}), -)$  and  $R\hat{\Gamma}_{G_{\mathbb{R}}}(X(\mathbb{C}), -)$ , we can reduce to checking the statement with  $R\Gamma_{c,B}$  instead of  $R\hat{\Gamma}_{c,B}$ . This will follow formally (by taking the fiber) if we prove that the cube of Diagram IV.2 is commutative.

<sup>23</sup>That is, the pullback of the counit is the counit between the pullbacks

<sup>24</sup>The commutation can be seen from the observations made by writing the adjunction arrows as the composition of applying the adjoint functor and postcomposing with the counit

$$\begin{array}{ccccc}
R\Gamma(Y, F) \otimes R\mathrm{Hom}_Y(F, Z_Y^c) & \longrightarrow & R\Gamma(Y, F) \otimes R\mathrm{Hom}_{G_{\mathbb{R}}, Y(\mathbb{C})}(\alpha^*F, 2i\pi\mathbb{Z}[2]) & & \\
\downarrow \simeq & & \downarrow \simeq & & \\
R\Gamma(Y, Z_Y^c) & \longrightarrow & R\Gamma_{G_{\mathbb{R}}}(Y(\mathbb{C}), 2i\pi\mathbb{Z}[2]) & & \\
\downarrow \simeq & & \downarrow \simeq & & \\
R\Gamma(X, \pi_*F) \otimes R\mathrm{Hom}_X(\pi_*F, Z_X^c) & \longrightarrow & R\Gamma(X, \pi_*F) \otimes R\mathrm{Hom}_{G_{\mathbb{R}}, Y(\mathbb{C})}(\pi'_*\alpha^*F, 2i\pi\mathbb{Z}[2]) & \longrightarrow & R\Gamma_{G_{\mathbb{R}}}(X(\mathbb{C}), 2i\pi\mathbb{Z}[2]) \\
\downarrow \simeq & & \downarrow \simeq & & \\
R\Gamma(X, Z_X^c) & \longrightarrow & R\Gamma_{G_{\mathbb{R}}}(X(\mathbb{C}), 2i\pi\mathbb{Z}[2]) & & 
\end{array}$$

Diagram IV.2

The lax monoidality of  $\pi_*$ ,  $\pi'_*$ ,  $\alpha^*$ ,  $\alpha'^*$  induces maps

$$\begin{aligned} c_\alpha &: \alpha^* R\mathcal{H}om_X(-, -) \rightarrow R\mathcal{H}om_{G_{\mathbb{R}}, X(\mathbb{C})}(\alpha^* -, \alpha^* -), \\ c_{\alpha'} &: \alpha'^* R\mathcal{H}om_Y(-, -) \rightarrow R\mathcal{H}om_{G_{\mathbb{R}}, Y(\mathbb{C})}(\alpha'^* -, \alpha'^* -), \\ c_\pi &: \pi_* R\mathcal{H}om_Y(-, -) \rightarrow R\mathcal{H}om_X(\pi_* -, \pi_* -), \\ c_{\pi'} &: \pi'_* R\mathcal{H}om_{G_{\mathbb{R}}, Y(\mathbb{C})}(-, -) \rightarrow R\mathcal{H}om_{G_{\mathbb{R}}, X(\mathbb{C})}(\pi'_* -, \pi'_* -). \end{aligned}$$

The following diagram commutes by naturality; the composite rectangle is given by the adjunction isomorphism  $\pi_* F^D \simeq (\pi_* F)^D$ :

$$\begin{array}{ccc} \pi_* F^D & \xrightarrow{\eta_\alpha} & R\alpha_* \alpha^* \pi_* F^D = R\alpha_* \pi'_* \alpha'^* F^D \\ \downarrow c_\pi & & \downarrow R\alpha_* \alpha^*(c_\pi) \\ R\mathcal{H}om_X(\pi_* F, \pi_* \mathbb{Z}_Y^c) & \xrightarrow{\eta_\alpha} & R\alpha_* \alpha^* R\mathcal{H}om_X(\pi_* F, \pi_* \mathbb{Z}_Y^c) \\ \downarrow \varepsilon_* & & \downarrow R\alpha_* \alpha^*(\varepsilon_*) \\ (\pi_* F)^D & \xrightarrow{\eta_\alpha} & R\alpha_* \alpha^* (\pi_* F)^D \end{array} \quad (\text{IV.2.4})$$

We have  $\pi_* \alpha^* = \alpha'^* \pi'_*$  so the following diagram commutes formally for any sheaf  $F$  and  $G$ :

$$\begin{array}{ccc} & \pi'_* \alpha'^* R\mathcal{H}om_Y(F, G) & \\ & \swarrow & \searrow \pi'_*(c_{\alpha'}) \\ \alpha^* \pi_* R\mathcal{H}om_Y(F, G) & & \pi'_* R\mathcal{H}om_{G_{\mathbb{R}}, Y(\mathbb{C})}(\alpha'^* F, \alpha'^* G) \\ \downarrow \alpha_*(c_\pi) & & \downarrow c_{\pi'} \\ \alpha^* R\mathcal{H}om_X(\pi_* F, \pi_* G) & & R\mathcal{H}om_X(\pi'_* \alpha'^* F, \pi'_* \alpha'^* G) \\ & \searrow c_\alpha & \swarrow \\ & R\mathcal{H}om_X(\alpha^* \pi_* F, \alpha^* \pi_* G) & \end{array}$$

Thus the top square in the following diagram commutes:

$$\begin{array}{ccc} \alpha^* \pi_* F^D = R\alpha_* \pi'_* \alpha'^* F^D & \xrightarrow{\pi'_*(c_{\alpha'})} & \pi'_* R\mathcal{H}om_{G_{\mathbb{R}}, Y(\mathbb{C})}(\alpha'^* F, \overline{\mathbb{Q}}^\times[1]) \\ \downarrow \alpha^*(c_\pi) & & \downarrow c_{\pi'} \\ \alpha^* R\mathcal{H}om_X(\pi_* F, \pi_* \mathbb{Z}_Y^c) & \xrightarrow{c_\alpha} & R\mathcal{H}om_{G_{\mathbb{R}}, X(\mathbb{C})}(\pi'_* \alpha'^* F, \pi'_* \overline{\mathbb{Q}}^\times[1]) \\ \downarrow \alpha^*(\varepsilon_*) & & \downarrow \varepsilon_* \\ \alpha^* R\mathcal{H}om_X(\pi_* F, \mathbb{Z}_X^c) & \xrightarrow{c_\alpha} & R\mathcal{H}om_{G_{\mathbb{R}}, X(\mathbb{C})}(\pi'_* \alpha'^* F, \overline{\mathbb{Q}}^\times[1]) \end{array} \quad (\text{IV.2.5})$$

The bottom square also commutes, because  $\alpha^*(\pi_* \mathbb{Z}_Y^c \rightarrow \mathbb{Z}_X^c) = \pi'_* \overline{\mathbb{Q}}^\times[1] \rightarrow \overline{\mathbb{Q}}^\times[1]$ . The commutative square

$$\begin{array}{ccc} \pi'_* \overline{\mathbb{Q}}^\times[1] & \longrightarrow & \pi'_* 2i\pi\mathbb{Z}[2] \\ \downarrow & & \downarrow \\ \overline{\mathbb{Q}}^\times[1] & \longrightarrow & 2i\pi\mathbb{Z}[2] \end{array}$$

implies that the following diagram is commutative:

$$\begin{array}{ccc}
\pi'_* R\mathcal{H}om_{G_{\mathbb{R}}, Y(\mathbb{C})}(\alpha^* F, \overline{\mathbb{Q}}^\times[1]) & \longrightarrow & \pi'_* R\mathcal{H}om_{G_{\mathbb{R}}, Y(\mathbb{C})}(\alpha^* F, 2i\pi\mathbb{Z}[2]) \\
c_{\pi'} \downarrow & & \downarrow c_{\pi'} \\
R\mathcal{H}om_{G_{\mathbb{R}}, X(\mathbb{C})}(\pi'_* \alpha^* F, \pi'_* \overline{\mathbb{Q}}^\times[1]) & \longrightarrow & R\mathcal{H}om_{G_{\mathbb{R}}, X(\mathbb{C})}(\pi'_* \alpha^* F, \pi'_* 2i\pi\mathbb{Z}[2]) \\
\varepsilon_* \downarrow & & \downarrow \varepsilon_* \\
R\mathcal{H}om_{G_{\mathbb{R}}, X(\mathbb{C})}(\pi'_* \alpha^* F, \overline{\mathbb{Q}}^\times[1]) & \longrightarrow & R\mathcal{H}om_{G_{\mathbb{R}}, X(\mathbb{C})}(\pi'_* \alpha^* F, 2i\pi\mathbb{Z}[2])
\end{array} \tag{IV.2.6}$$

Apply  $R\alpha_*$  to diagrams Equations (IV.2.5) and (IV.2.6) and paste them next to Equation (IV.2.4) to obtain the following diagram:

$$\begin{array}{ccc}
\pi_* F^D & \longrightarrow & R\alpha_* \pi'_*(\alpha^* F)^\vee(1)[2] \\
\searrow \simeq & & \searrow \simeq \\
& & R\mathcal{H}om_X(\pi_* F, \pi_* \mathbb{Z}_Y^c) \longrightarrow R\mathcal{H}om_X(\pi_* F, R\alpha_* \pi'_* 2i\pi\mathbb{Z}[2]) \\
& & \downarrow \qquad \qquad \qquad \downarrow \\
& & R\mathcal{H}om_X(\pi_* F, \mathbb{Z}_X^c) \longrightarrow R\mathcal{H}om_X(\pi_* F, R\alpha_* 2i\pi\mathbb{Z}[2])
\end{array}$$

where we have rewritten the terms on the right using

$$R\alpha_* R\mathcal{H}om_{G_{\mathbb{R}}, X(\mathbb{C})}(\alpha^* -, -) = R\mathcal{H}om(-, R\alpha_* -).$$

The back and side faces are obtained by composing the top and front faces; by properties of adjunctions, the dotted maps are the adjunction isomorphisms for  $\pi_* \dashv R\pi^!$  resp.  $\pi'_* \dashv R\pi'^!$ . By the  $\otimes \dashv \text{Hom}$ -adjunction, the above diagram is equivalent to the following cube:

$$\begin{array}{ccccc}
\pi_* F \otimes \pi_* F^D & \longrightarrow & \pi_* F \otimes R\alpha'_*(\alpha^* F)^\vee(1)[2] & & \\
\downarrow \simeq & \searrow & \downarrow \simeq & \searrow & \\
\pi_* F \otimes \pi_* \mathbb{Z}_Y^c & \longrightarrow & R\alpha_* \pi'_* 2i\pi\mathbb{Z}[2] & & \\
\downarrow & & \downarrow & & \\
\pi_* F \otimes (\pi_* F)^D & \longrightarrow & \pi_* F \otimes R\alpha_*(\pi'_* \alpha^* F)^\vee(1)[2] & & \\
\downarrow & \searrow & \downarrow & \searrow & \\
\pi_* F \otimes \mathbb{Z}_X^c & \longrightarrow & R\alpha_* 2i\pi\mathbb{Z}[2] & & 
\end{array}$$

Finally, applying the lax monoidal functor  $R\Gamma(X, -)$  recovers the sought-after cube.  $\square$

### IV.2.3 Computations

**Proposition IV.2.7.** *Let  $F$  be a  $\mathbb{Z}$ -constructible sheaf on  $X$ . We have that*

$$\begin{aligned}
H_{c,B}^i(X, F^D) &\simeq \hat{H}_{c,B}^i(X, F^D) \quad \text{for } i \geq 1; \\
H_{c,B}^0(X, F^D) &\rightarrow \hat{H}_{c,B}^0(X, F^D) \quad \text{is surjective.}
\end{aligned}$$

If  $\alpha^* F$  is torsion-free, we have that

$$\begin{aligned}
H_{c,B}^i(X, F^D) &\simeq \hat{H}_{c,B}^i(X, F^D) \quad \text{for } i \geq 0; \\
H_{c,B}^{-1}(X, F^D) &\rightarrow \hat{H}_{c,B}^{-1}(X, F^D) \quad \text{is surjective.}
\end{aligned}$$

*Proof.* Let  $T(F^D)$  be the cofiber of  $R\Gamma_{c,B}(X, F^D) \rightarrow R\hat{\Gamma}_{c,B}(X, F^D)$ . Let  $s$  be the generator of  $G_{\mathbb{R}}$  and denote by  $N = 1 + s$  the norm map. By the stable  $3 \times 3$  lemma (Lemma II.2.2), we find that

$$\begin{aligned} T(F^D) &= \text{fib}(R\Gamma_{G_{\mathbb{R}}}(X(\mathbb{C}), (\alpha^*F)^\vee(1)[2]) \xrightarrow{N} R\hat{\Gamma}_{G_{\mathbb{R}}}(X(\mathbb{C}), (\alpha^*F)^\vee(1)[2])) \\ &= \mathbb{Z} \otimes_{\mathbb{Z}[G_{\mathbb{R}}]}^L R\Gamma(X(\mathbb{C}), (\alpha^*F)^\vee(1)[2]) \\ &= \prod_{v \text{ archimedean}} \mathbb{Z} \otimes_{\mathbb{Z}[G_{K_v}]}^L F_v^\vee(1)[2] \end{aligned}$$

computes homology at the archimedean places. Let  $M$  be a  $G_{\mathbb{R}}$ -module of finite type. Then  $M^\vee(1)[2] = R\text{Hom}_{\mathbb{Z}}(M, 2i\pi\mathbb{Z})[2]$  is concentrated in degree  $-2$  if  $M$  is torsion-free and in degree  $[-2, -1]$  in general. It follows that  $\mathbb{Z} \otimes_{\mathbb{Z}[G_{\mathbb{R}}]}^L M^\vee(1)[2]$  is concentrated in degree  $\leq -2$  if  $M$  is torsion-free and in degree  $\leq -1$  in general.  $\square$

**Proposition IV.2.8.** *Let  $F$  be a  $\mathbb{Z}$ -constructible sheaf on  $X$ . Then  $R\hat{\Gamma}_{c,B}(X, F^D)$  is concentrated in degree  $\leq 2$ . If  $F$  is constructible, the complex has finite cohomology groups. In general, we have*

$$\hat{H}_{c,B}^i(X, F^D) = \begin{cases} \text{finite}, & i \leq -2, \\ \text{finite type}, & i = -1, 0, \\ \text{torsion of cofinite type}, & i = 1, 2. \end{cases}$$

*Proof.* The vanishing in degree  $> 2$  comes from Theorem IV.2.3. The remaining claims follow from the defining long exact cohomology sequence and [Mil06, II.3.6], because the complex  $R\hat{\Gamma}_{G_{\mathbb{R}}}(X(\mathbb{C}), (\alpha^*F)^\vee(1)[2])$  has finite 2-torsion cohomology groups.  $\square$

**Proposition IV.2.9.** *Let  $F$  be a  $\mathbb{Z}$ -constructible sheaf on  $X$ . Then  $R\Gamma_{c,B}(X, F^D)$  is concentrated in degree  $[-1, 2]$ . If  $F$  is constructible, the complex is perfect. In general, we have*

$$H_{c,B}^i(X, F^D) = \begin{cases} \text{finite type}, & i = -1, 0, \\ \text{torsion of cofinite type}, & i = 1, 2. \end{cases}$$

*Proof.* By Proposition IV.2.7 and the previous proposition, we have  $H_{c,B}^i(X, F^D) = \hat{H}_{c,B}^i(X, F^D) = 0$  for  $i > 2$  and  $H_{c,B}^i(X, F^D) = \hat{H}_{c,B}^i(X, F^D)$  is torsion of cofinite type for  $i = 1, 2$ ; finally for  $i = -1, 0$  the difference between  $H_{c,B}^i(X, F^D)$  and  $\hat{H}_{c,B}^i(X, F^D)$  is given by a group of finite type so  $H_{c,B}^i(X, F^D)$  is finite type by the previous proposition. The vanishing in degree  $< -1$  is clear.  $\square$

We now compute some special cases.

**Proposition IV.2.10.** *Suppose that  $X = \text{Spec}(\mathcal{O}_K)$  is regular. We have*

$$H^i(X, \mathbb{Z}) = \begin{cases} \mathbb{Z}, & i = 0, \\ 0, & i = 1, \\ \text{torsion}, & i > 1. \end{cases}$$

*Let  $j : U \rightarrow X$  be an open subscheme of  $X$  with  $U \neq X$ . We have*

$$H^i(X, j_!\mathbb{Z}) = \begin{cases} 0, & i = 0, \\ (\prod_{v \in X \setminus U} \mathbb{Z})/\mathbb{Z}, & i = 1, \\ \text{torsion}, & i > 1. \end{cases}$$

*Proof.* The result for  $i > 1$  follows from [Mil06, II.2.10]. Since  $X$  is normal,  $\pi_1^{\text{proet}}(X) = \pi_1^{\text{et}}(X)$  is profinite so  $H^1(X, \mathbb{Z}) = \text{Hom}_{\text{cont}}(\pi_1^{\text{et}}(X), \mathbb{Z}) = 0$ .  $\square$

**Proposition IV.2.11.** *Suppose that  $X = \text{Spec}(\mathcal{O}_K)$  is regular. We have*

$$H_{c,B}^i(X, \mathbb{Z}^D) = \begin{cases} \text{finite type of rank } [K : \mathbb{Q}] - 1, & i = -1, \\ \text{finite}, & i = 0, \\ 0, & i = 1, \\ \mathbb{Q}/\mathbb{Z}, & i = 2. \end{cases}$$

Moreover we have an exact sequence

$$0 \rightarrow \mathbb{Z}^{r^2} \rightarrow H_{c,B}^{-1}(X, \mathbb{Z}^D) \rightarrow \mathcal{O}_K^\times \rightarrow (\mathbb{Z}/2\mathbb{Z})^{r_1} \rightarrow H_{c,B}^0(X, \mathbb{Z}^D) \rightarrow \text{Pic}(X) \rightarrow 0$$

and  $H_{c,B}^0(X, \mathbb{Z}^D)$  is the narrow ideal class group  $\text{Pic}^+(X)$ .

*Proof.* The exact sequence comes from the long exact cohomology sequence of the defining fiber sequence. By Proposition IV.2.7, we have isomorphisms

$$\tau^{\geq 0} R\Gamma_{c,B}(X, \mathbb{Z}^D) \xrightarrow{\sim} \tau^{\geq 0} R\hat{\Gamma}_{c,B}(X, \mathbb{Z}^D) \simeq \tau^{\geq 0} R\hat{\Gamma}_c(X, \mathbb{G}_m[1])$$

hence the result for  $i \geq 0$  follows from [Mil06, II.2.6, II.2.8(a)].  $\square$

*Remark.*

- We have  $H_{G_{\mathbb{R}}}^1(X(\mathbb{C}), 2i\pi\mathbb{Z}) \simeq \hat{H}_{G_{\mathbb{R}}}^0(X(\mathbb{C}), \mathbb{C}^\times) = (\mathbb{R}^\times/\mathbb{R}_+^\times)^{r_1}$ . Since the canonical map  $H^0(X, \mathbb{G}_m) \rightarrow H_{G_{\mathbb{R}}}^1(X(\mathbb{C}), 2i\pi\mathbb{Z})$  factors through  $H_{G_{\mathbb{R}}}^0(X(\mathbb{C}), \mathbb{C}^\times)$  by definition, it is given by

$$\mathcal{O}_K^\times \xrightarrow{(\text{sign}_v)_v} \bigoplus_{v \text{ real}} \mathbb{Z}/2\mathbb{Z}.$$

Its kernel is  $\mathcal{O}_{K,+}^\times$ , the group of totally positive units; it has same rank as  $\mathcal{O}_K^\times$ . Thus we have a short exact sequence

$$0 \rightarrow \mathbb{Z}^{r^2} \rightarrow H_{c,B}^{-1}(X, \mathbb{Z}^D) \rightarrow \mathcal{O}_{K,+}^\times \rightarrow 0.$$

- We can also recover the result for  $i \geq 1$  by Artin–Verdier duality (Theorem IV.2.3) and Proposition IV.2.7 since  $H^i(X, \mathbb{Z}) = 0, \mathbb{Z}, 0$  for  $i < 0, i = 0, i = 1$ . On the other hand, from the result for  $i = 0$  we recover  $H^2(X, \mathbb{Z}) \simeq \text{Pic}^+(X)^*$ .

- If  $X$  is singular, the exact sequence becomes

$$0 \rightarrow \mathbb{Z}^{r^2} \rightarrow H_{c,B}^{-1}(X, \mathbb{Z}^D) \rightarrow \text{CH}_0(X, 1) \rightarrow (\mathbb{Z}/2\mathbb{Z})^{r_1} \rightarrow H_{c,B}^0(X, \mathbb{Z}^D) \rightarrow \text{CH}_0(X) \rightarrow 0$$

and  $H_{c,B}^1(X, \mathbb{Z}^D) = H^1(X, \mathbb{Z})^*$  can be non-zero and non-finite, depending on the singularities of  $X$ . Moreover,  $H_{c,B}^0(X, \mathbb{Z}^D)$  can be seen as a “narrow Chow group”.

**Proposition IV.2.12.** *Suppose that  $X = \text{Spec}(\mathcal{O}_K)$  is regular and let  $j : U \rightarrow X$  be an open subscheme with  $U \neq X$ , say  $U = \text{Spec}(\mathcal{O}_{K,S})$  with  $S$  a set of places containing the archimedean places and at least one finite place. Denote by  $S_f$  the set of finite places in  $S$  and  $s_f$  its cardinality. We have*

$$H_{c,B}^i(X, (j^*\mathbb{Z})^D) = \begin{cases} \text{finite type of rank } [K : \mathbb{Q}] - 1, & i = -1, \\ \text{finite}, & i = 0, \\ \text{Ker}(\bigoplus_{v \in S_f} \text{Br}(K_v) \xrightarrow{\Sigma} \mathbb{Q}/\mathbb{Z}) \simeq (\mathbb{Q}/\mathbb{Z})^{s_f-1}, & i = 1, \\ 0, & i = 2. \end{cases}$$

Moreover, we have an exact sequence

$$0 \rightarrow \mathbb{Z}^{r_2} \rightarrow H_{c,B}^{-1}(X, (j_! \mathbb{Z})^D) \rightarrow \mathcal{O}_{K,S}^\times \rightarrow (\mathbb{Z}/2\mathbb{Z})^{r_1} \rightarrow H_{c,B}^0(X, (j_! \mathbb{Z})^D) \rightarrow \text{Pic}(U) \rightarrow 0$$

and  $H_{c,B}^0(X, (j_! \mathbb{Z})^D)$  is the narrow  $S$ -class group  $\text{Pic}^+(U)$ .

*Proof.* By Proposition IV.2.7 and Artin–Verdier duality we obtain the result for  $i = 1, 2$  from the computation of  $H^i(X, j_! \mathbb{Z})$ . We have  $(j_! \mathbb{Z})^D = Rj_* \mathbb{Z}_U^c = j_* \mathbb{G}_m[1]$  [Mil06, II.1.4] and  $(j_! \mathbb{Z})_c^\vee(1) = 2i\pi\mathbb{Z}$  so the exact sequence comes from the long exact cohomology sequence associated to the defining fiber sequence.

We have  $R\hat{\Gamma}_{c,B}(X, (j_! \mathbb{Z})^D) \xleftarrow{\simeq} R\hat{\Gamma}_c(X, j_* \mathbb{G}_m[1])$  by Proposition IV.2.2. The divisor short exact sequence

$$0 \rightarrow j_* \mathbb{G}_m \rightarrow g_* \mathbb{G}_m \rightarrow \bigoplus_{v \in U_0} i_{v,*} \mathbb{Z} \rightarrow 0$$

and Proposition IV.2.7 gives, as in [Mil06, II.2.8(a)], the identification of

$$H_{c,B}^0(X, (j_! \mathbb{Z})^D) \simeq \hat{H}_{c,B}^0(X, (j_! \mathbb{Z})^D) \simeq \hat{H}_c^0(X, j_* \mathbb{G}_m[1])$$

with the narrow  $S$ -class group. □

*Remark.*

- The map  $H^0(U, \mathbb{G}_m) \rightarrow H_{G_{\mathbb{R}}}^1(X(\mathbb{C}), 2i\pi\mathbb{Z})$  is given by

$$\mathcal{O}_{K,S}^\times \xrightarrow{(\text{sign}_v)_v \text{ real}} \bigoplus_{v \text{ real}} \mathbb{Z}/2\mathbb{Z}.$$

Its kernel is  $\mathcal{O}_{K,S,+}^\times$ , the group of totally positive  $S$ -units; it has same rank as  $\mathcal{O}_{K,S}^\times$ . Thus there is a short exact sequence

$$0 \rightarrow \mathbb{Z}^{r_2} \rightarrow H_c^{-1}(X, (j_! \mathbb{Z})^D) \rightarrow \mathcal{O}_{K,S}^{\times,+} \rightarrow 0.$$

- For  $i = 1, 2$ , we can also prove the result directly by considering the following snake diagram:

$$\begin{array}{ccccccc}
 & & & 0 & \longrightarrow & H_{c,B}^1(X, (j_! \mathbb{Z})^D) & \longrightarrow & 0 \\
 & & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & 0 & \longrightarrow & H^2(U, \mathbb{G}_m) & \xlongequal{\quad} & H^2(U, \mathbb{G}_m) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & \prod_{v \in S_f} \text{Br}(K_v) & \longrightarrow & \prod_{v \in S} \text{Br}(K_v) & \longrightarrow & \prod_{v \text{ archimedean}} \text{Br}(K_v) & \longrightarrow & 0 \\
 & & \parallel & & \downarrow \Sigma & & \downarrow & & \downarrow \\
 & & \prod_{v \in S_f} \text{Br}(K_v) & \xrightarrow{\Sigma} & \mathbb{Q}/\mathbb{Z} & \longrightarrow & H_{c,B}^2(X, (j_! \mathbb{Z})^D) & \longrightarrow & 0
 \end{array}$$

- We recover from Artin–Verdier duality an identification  $H^2(X, j_! \mathbb{Z}) \simeq \text{Pic}^+(U)^*$ .



Let  $i : x \rightarrow X$  be a closed point of  $X$  and  $M$  a discrete  $G_x$ -module of finite type. Since  $Ri^! \mathbb{Z}_X^c = \mathbb{Z}[0]$ , we have  $(i_* M)^D = i_* M^\vee$ , while  $(i_* M)_\mathbb{C} = 0$ . Thus  $R\Gamma_{c,B}(X, (i_* M)^D) = R\Gamma(G_x, M^\vee)$  and we obtain

**Proposition IV.2.13.** *Let  $i : x \rightarrow X$  be a closed point of  $X$  and  $M$  a discrete  $G_x$ -module of finite type. We have*

$$H_{c,B}^i(X, (i_* M)^D) = \begin{cases} 0, & i = -1, \\ \text{finite type}, & i = 0, \\ \text{finite}, & i = 1, \\ \text{torsion of cofinite type}, & i = 2. \end{cases}$$

If  $M$  is finite,  $R\Gamma_{c,B}(X, (i_* M)^D)$  has finite cohomology groups and  $H_{c,B}^0(X, (i_* M)^D) = 0$ .

*Proof.* If  $M$  is finite, we have  $M^\vee = M^*[-1]$ . On the other hand, if  $M$  is torsion-free we have  $M^\vee = \text{Hom}_{\text{Ab}}(M, \mathbb{Z})$ . Thus in both cases we reduce to cohomology of discrete  $G_x \simeq \hat{\mathbb{Z}}$ -modules of finite type, for which the result is known. For arbitrary  $M$ , we conclude with the short exact sequence  $0 \rightarrow M_{\text{tor}} \rightarrow M \rightarrow M/\text{tor} \rightarrow 0$ .  $\square$

Let  $D_F = R\text{Hom}(R\Gamma(X, F), \mathbb{Q}[-2])$  for  $F$  a  $\mathbb{Z}$ -constructible sheaf on  $X$ , and put  $(-)^{\dagger} = \text{Hom}(-, \mathbb{Q})$ .

**Proposition IV.2.14.** *Let  $F$  be a  $\mathbb{Z}$ -constructible sheaf on  $X$ . Then  $D_F$  is concentrated in degree  $[1, 2]$ .*

*Proof.* We have  $H^i(D_F) = H^{2-i}(X, F)^{\dagger}$ . We thus have to show that  $H^i(X, F)$  is torsion for  $i \neq 0, 1$ . But  $H^i(X, F)$  differs from  $\hat{H}_c^i(X, F)$  by a finite group since  $X$  is proper, and the latter is torsion for  $i \neq 0, 1$  [Mil06, II.6.2].  $\square$

## IV.3 Weil-étale cohomology with compact support of $F^D$

### IV.3.1 Construction of the Weil-étale complex

Following [FM18], the Weil-étale complex with compact support should be the cone of a map  $\alpha_F$  making the following diagram commute

$$\begin{array}{ccc} R\text{Hom}(R\Gamma(X, F), \mathbb{Q}[-2]) & \xrightarrow{\alpha_F} & R\Gamma_{c,B}(X, F^D) \\ & \searrow & \downarrow \\ & & R\hat{\Gamma}_{c,B}(X, F^D) \\ & & \downarrow \\ & & R\text{Hom}(R\Gamma(X, F), \mathbb{Q}/\mathbb{Z}[-2]) \end{array}$$

Let  $F, G$  be two  $\mathbb{Z}$ -constructible sheaves on  $X$  and let us compute the morphism group  $\text{Hom}_{D(\mathbb{Z})}(D_G, R\Gamma_{c,B}(X, F^D))$ . Recall the Verdier spectral sequence [Ver96, III.4.6.10]:

$$E_2^{p,q} = \prod_{i \in \mathbb{Z}} \text{Ext}_{\mathbb{Z}}^p(H^i(K), H^{i+q}(L)) \Rightarrow \text{Ext}_{D(\mathbb{Z})}^{p+q}(K, L) \quad (\text{IV.3.1})$$

Using the vanishing results of the previous section, the above degenerates to a short exact sequence

$$\begin{array}{c}
0 \rightarrow \prod_{i=1,2} \text{Ext}^1(H^{2-i}(X, G)^\dagger, H_{c,B}^{i-1}(X, F^D)) \rightarrow \text{Hom}_{D(\mathbb{Z})}(D_G, R\Gamma_{c,B}(X, F^D)) \\
\searrow \hspace{15em} \nearrow \\
\prod_{i=1,2} \text{Hom}(H^{2-i}(X, G)^\dagger, H_{c,B}^i(X, F^D)) \longrightarrow 0
\end{array}
\tag{IV.3.2}$$

Since  $H_{c,B}^1(X, F^D)$  is torsion and  $H^0(X, G)^\dagger$  is a  $\mathbb{Q}$ -vector space, we obtain

$$\text{Ext}^1(H^0(X, G)^\dagger, H_{c,B}^1(X, F^D)) = 0.$$

Therefore the left term is  $\text{Ext}^1(H^1(X, G)^\dagger, H_{c,B}^0(X, F^D))$ . Similarly to the approach of Chapter II, this motivates the following definition:

**Definition IV.3.1.** *Let  $F$  be a  $\mathbb{Z}$ -constructible sheaf on  $X$ . We say that*

- $F$  is red if  $H_{c,B}^0(X, F^D)$  is torsion, hence finite;
- $F$  is blue if  $H^1(X, F)$  is torsion, hence finite; this happens if and only if  $\hat{H}_{c,B}^1(X, F^D) = H_{c,B}^1(X, F^D)$  is finite (Theorem IV.2.3).

A red-to-blue morphism is a morphism of sheaves  $F \rightarrow G$  where either  $F$  and  $G$  are both blue, or are both red, or  $F$  is red and  $G$  is blue; a red-to-blue short exact sequence is a short exact sequence with red-to-blue morphisms.

*Remark.*

- If  $F$  is constructible,  $F$  is blue and red.
- The constant sheaf  $\mathbb{Z}$  is red; this follows from the finiteness of  $\text{CH}_0(X)$ .<sup>25</sup> If  $X$  is regular or more generally unibranch (so that each point has a singleton preimage in the normalization of  $X$ ), the sheaf  $\mathbb{Z}$  is also blue.
- If  $j : U \rightarrow X$  is an open inclusion with  $U \neq X$ , the sheaf  $j_!\mathbb{Z}$  is red, and it is blue if  $X$  is unibranch and  $|X \setminus U| = 1$ .
- If  $F$  is  $\mathbb{Z}$ -constructible and supported on a closed subscheme, then  $F$  is blue. This reduces to the case of a single discrete finite type  $\hat{\mathbb{Z}}$ -module, where it follows from  $H^1(\hat{\mathbb{Z}}, \mathbb{Z}) = 0$  and the Hochschild-Serre spectral sequence for a normal open subgroup acting trivially.

**Proposition IV.3.2.** *Let  $j : U \rightarrow X$  be an open immersion such that  $U$  is regular and  $G$  a locally constant  $\mathbb{Z}$ -constructible sheaf on  $U$ . Then  $j_!G$  is red.*

*Proof.* Indeed, the result is true if  $G$  is torsion so we can suppose that  $G$  is torsion-free. Let  $g : \eta = \text{Spec}(K) \rightarrow U$  be the generic point. Since  $G$  is locally constant, we have  $G = g_*g^*G$ .<sup>26</sup> By Artin induction for the  $G_K$ -module  $g^*G$ , we find normal open subgroups

<sup>25</sup>See the remark after Proposition IV.2.11 and the remark after Corollary II.7.4.

<sup>26</sup>The regularity hypothesis appears here: in general  $g_*\mathbb{Z} \neq \mathbb{Z}$ .

$H_k, H_l$  of  $G_K$ , an integer  $n \in \mathbb{N}^\times$  and a finite  $G_K$ -module  $N$  such that we have a short exact sequence

$$0 \rightarrow (g^*G)^n \oplus \bigoplus \operatorname{ind}_{G_K}^{H_k} \mathbb{Z} \rightarrow \bigoplus \operatorname{ind}_{G_K}^{H_l} \mathbb{Z} \rightarrow N \rightarrow 0.$$

Denote by  $\pi'_k : V_k \rightarrow U, \pi'_l : V_l \rightarrow U$  the normalizations of  $U$  in  $(K^{\text{sep}})^{H_k}, (K^{\text{sep}})^{H_l}$ . By applying  $g_*$ , we find a short exact sequence

$$0 \rightarrow G^n \oplus \bigoplus \pi'_{k,*} \mathbb{Z} \rightarrow \bigoplus \pi'_{l,*} \mathbb{Z} \rightarrow Q \rightarrow 0,$$

where  $Q$  is a subsheaf of the constructible sheaf  $g_*N$  and hence is constructible. Denote by  $\pi_k : Y_k \rightarrow X$  (resp.  $\pi_l : Y_l \rightarrow X$ ) the normalization of  $X$  in  $(K^{\text{sep}})^{H_k}$  (resp.  $(K^{\text{sep}})^{H_l}$ ) and  $j_k : V_k \rightarrow Y_k$  the open inclusion (resp.  $j_l : V_l \rightarrow Y_l$ ). The points in  $Y_k$  (resp.  $Y_l$ ) above  $U$  are exactly the points of  $V_k$  (resp.  $V_l$ ) so

$$j_! \pi'_{k,*} \mathbb{Z} = \pi_{k,*} j_{k,!} \mathbb{Z}, \quad j_! \pi'_{l,*} \mathbb{Z} = \pi_{l,*} j_{l,!} \mathbb{Z}.$$

Using Proposition IV.2.5 we conclude with the preceding remark.  $\square$

*Remark.* It follows that there are “enough red-and-blues”; by this, we mean that any  $\mathbb{Z}$ -constructible sheaf  $F$  sits in a short exact sequence

$$0 \rightarrow R \rightarrow F \rightarrow B \rightarrow 0$$

with  $R$  red and  $B$  blue. Indeed, it suffices to take  $R = j_! F|_U$  and  $B = i_* i^* F$  for  $j : U \rightarrow X$  a regular open subscheme such that  $F|_U$  is locally constant and  $i : Z \rightarrow X$  its closed complement. As in Chapter II, this remark will allow us to bootstrap the construction of the Weil-étale Euler characteristic from red or blue sheaves to arbitrary  $\mathbb{Z}$ -constructible sheaves by enforcing multiplicativity.

If  $F$  is red or  $G$  is blue, the left term of the short exact sequence (IV.3.2) vanishes and we obtain:

**Proposition IV.3.3.** *Let  $F$  and  $G$  be étale sheaves on  $X$  and suppose that  $F$  is red or that  $G$  is blue. We have*

$$\operatorname{Hom}(D_G, R\Gamma_{c,B}(X, F^D)) = \prod_{i=1,2} \operatorname{Hom}(H^{2-i}(X, G)^\dagger, H^i_{c,B}(X, F^D)).$$

In particular, suppose that  $F$  is red or blue. For  $i = 1, 2$ , we have by Theorem IV.2.3 isomorphisms

$$H^i_{c,B}(X, F^D) \xrightarrow{\cong} \hat{H}^i_{c,B}(X, F^D) \xrightarrow{\cong} H^{2-i}(X, F)^*.$$

Hence we obtain a canonical element  $\alpha_F \in \operatorname{Hom}(D_F, R\Gamma_{c,B}(X, F^D))$ , given in cohomology in degree  $i = 1, 2$  by

$$H^i(\alpha_F) : H^{2-i}(X, F)^\dagger \rightarrow H^{2-i}(X, F)^* \xleftarrow{\cong} H^i_{c,B}(X, F^D).$$

**Definition IV.3.4.** *Let  $F$  be red or blue. We define*

$$R\Gamma_{W,c}(X, F^D) := \operatorname{Cone}(\alpha_F),$$

*the Weil-étale complex with compact support with coefficients in  $F^D$ .*

*Remark.* We want to emphasize that our construction is only done in the homotopy category, as a mapping cone. Nevertheless for red or blue sheaves we will obtain sufficient functoriality properties; this is unusual.

### IV.3.2 Computation of Weil-étale cohomology

**Proposition IV.3.5.** *Let  $F$  be a red or blue sheaf on  $X$ . Then  $R\Gamma_{W,c}(X, F^D)$  is a perfect complex of abelian groups concentrated in degree  $[-1, 2]$ . Moreover, we have*

$$H_{W,c}^i(X, F^D) = \begin{cases} H_c^{-1}(X, F^D), & i = -1, \\ (H^0(X, F)_{\text{tor}})^*, & i = 2, \end{cases}$$

and short exact sequences

$$\begin{aligned} 0 \rightarrow H_{c,B}^0(X, F^D) \rightarrow H_{W,c}^0(X, F^D) \rightarrow \text{Hom}(H^1(X, F), \mathbb{Z}) \rightarrow 0, \\ 0 \rightarrow (H^1(X, F)_{\text{tor}})^* \rightarrow H_{W,c}^1(X, F^D) \rightarrow \text{Hom}(H^0(X, F), \mathbb{Z}) \rightarrow 0. \end{aligned}$$

*Proof.* The groups  $H^i(X, F)$  are of finite type for  $i = 0, 1$ : they differ from  $\hat{H}_c^i(X, F)$  by a finite group since  $X$  is proper, and the latter are of finite type for  $i = 0, 1$  [Mil06, II.3.1, II.6.2]. The claim then follows from the distinguished triangle

$$R\text{Hom}(R\Gamma(X, F), \mathbb{Q}[-2]) \rightarrow R\Gamma_{c,B}(X, F^D) \rightarrow R\Gamma_{W,c}(X, F^D) \rightarrow$$

and Propositions IV.2.9 and IV.2.14. □

We compute some special cases, using Propositions IV.2.11 and IV.2.12.

**Proposition IV.3.6.** *Suppose that  $X = \text{Spec}(\mathcal{O}_K)$  is regular. We have*

$$H_{W,c}^i(X, \mathbb{Z}^D) = \begin{cases} \mathbb{Z}, & i = 1, \\ 0, & i = 2, \end{cases}$$

and an exact sequence

$$0 \rightarrow \mathbb{Z}^{r_2} \rightarrow H_{W,c}^{-1}(X, \mathbb{Z}^D) \rightarrow \mathcal{O}_K^\times \rightarrow (\mathbb{Z}/2\mathbb{Z})^{r_1} \rightarrow H_{W,c}^0(X, \mathbb{Z}^D) \rightarrow \text{Pic}(X) \rightarrow 0.$$

Moreover  $H_{W,c}^0(X, \mathbb{Z}^D) = \text{Pic}^+(X)$ .

*Remark.*

- If  $X$  is singular, we still have  $H_{W,c}^2(X, \mathbb{Z}^D) = 0$ ; the abelian group  $H_{W,c}^1(X, \mathbb{Z}^D)$  is of rank 1 but it can have torsion coming from  $H^1(X, \mathbb{Z})$  if  $X$  is not unibranch.
- The complexes  $R\Gamma_{W,c}(X, \mathbb{Z}^D)$  is equal to the complex  $R\Gamma_{W,c}(X, \mathbb{Z}(1))[2]$  of [FM18] and the above computation reproduces the one in [FM18].

**Proposition IV.3.7.** *Suppose that  $X = \text{Spec}(\mathcal{O}_K)$  is regular and let  $j : U \rightarrow X$  be an open immersion with  $U \neq X$ ; we keep the notations from Proposition IV.2.12. Then  $H_{W,c}^1(X, (j_!\mathbb{Z})^D) = H_{W,c}^2(X, (j_!\mathbb{Z})^D) = 0$  and we have short exact sequences*

$$0 \rightarrow \mathbb{Z}^{r_2} \rightarrow H_{W,c}^{-1}(X, (j_!\mathbb{Z})^D) \rightarrow \mathcal{O}_{K,S,+}^\times \rightarrow 0$$

and

$$0 \rightarrow \text{Pic}^+(U) \rightarrow H_{W,c}^0(X, (j_!\mathbb{Z})^D) \rightarrow \ker(\mathbb{Z}^{s_f} \xrightarrow{\Sigma} \mathbb{Z}) \rightarrow 0.$$

**Proposition IV.3.8.** *Let  $i : x \rightarrow X$  be a closed point and  $M$  a finite type discrete  $G_x$ -module. Then*

$$H_{W,c}^i(X, (i_*M)^D) = \begin{cases} 0, & i = -1, \\ H^0(G_x, M^\vee) = \text{Hom}_{G_x}(M, \mathbb{Z}), & i = 0, \\ (H^0(G_x, M)_{\text{tor}})^*, & i = 2, \end{cases}$$

and we have a short exact sequence

$$0 \rightarrow H^1(G_x, M)^* \rightarrow H_{W,c}^1(X, (i_*M)^D) \rightarrow \text{Hom}(H^0(G_x, M), \mathbb{Z}) \rightarrow 0.$$

**Proposition IV.3.9.** *If  $F$  is a constructible sheaf on  $X$  then*

$$R\Gamma_{c,B}(X, F^D) \xrightarrow{\simeq} R\Gamma_{W,c}(X, F^D).$$

### IV.3.3 Functoriality properties

**Theorem IV.3.10.** *Let  $F$  be a red or blue sheaf on the spectrum  $X$  of an order in a number field. The complex  $R\Gamma_{W,c}(X, F^D)$  is well-defined up to unique isomorphism, is functorial in red-to-blue morphisms, and gives long exact cohomology sequences for red-to-blue short exact sequences.*

*Proof.* For each red or blue sheaf  $F$  we fix a choice  $R\Gamma_{W,c}(X, F^D)$  of cone of  $\alpha_F$ . Let  $f : F \rightarrow G$  be a red-to-blue morphism. Let us consider the following diagram with distinguished rows:

$$\begin{array}{ccccccc} D_F & \xrightarrow{\alpha_F} & R\Gamma_{c,B}(X, F^D) & \longrightarrow & R\Gamma_{W,c}(X, F^D) & \longrightarrow & \\ f^* \uparrow & & f^* \uparrow & & & & \\ D_G & \xrightarrow{\alpha_G} & R\Gamma_{c,B}(X, G^D) & \longrightarrow & R\Gamma_{W,c}(X, G^D) & \longrightarrow & \end{array}$$

We claim that the left square commutes. Indeed, by Proposition IV.3.3 it suffices to check it in cohomology in degree 1 and 2, in which case it follows from the functoriality of the maps

$$\begin{aligned} D_{(-)} &\rightarrow R\text{Hom}_{\mathbb{Z}}(R\Gamma(X, -), \mathbb{Q}/\mathbb{Z}[-2]), \\ R\hat{\Gamma}_{c,B}(X, (-)^D) &\rightarrow R\text{Hom}_{\mathbb{Z}}(R\Gamma(X, -), \mathbb{Q}/\mathbb{Z}[-2]), \\ R\Gamma_{c,B}(X, (-)^D) &\rightarrow R\hat{\Gamma}_{c,B}(X, (-)^D). \end{aligned}$$

We obtain an induced morphism  $f^* : R\Gamma_{W,c}(X, G^D) \rightarrow R\Gamma_{W,c}(X, F^D)$  in  $D(\mathbb{Z})$  completing the diagram to a morphism of distinguished triangles. Let us show that this induced morphism is uniquely determined by the left square. It suffices to show that the natural map

$$\text{Hom}(R\Gamma_{W,c}(X, G^D), R\Gamma_{W,c}(X, F^D)) \rightarrow \text{Hom}(R\Gamma_{c,B}(X, G^D), R\Gamma_{W,c}(X, F^D))$$

is injective. We have an exact sequence

$$\begin{aligned} \text{Hom}(D_G[1], R\Gamma_{W,c}(X, F^D)) &\rightarrow \text{Hom}(R\Gamma_{W,c}(X, G^D), R\Gamma_{W,c}(X, F^D)) \\ &\rightarrow \text{Hom}(R\Gamma_{c,B}(X, G^D), R\Gamma_{W,c}(X, F^D)). \end{aligned}$$

Since  $D_G$  is a complex of  $\mathbb{Q}$ -vector spaces, the multiplication-by- $n$  map on  $D_G$  is a quasi-isomorphism for every integer  $n$ . The same must hold for  $\mathrm{Hom}(D_G[1], R\Gamma_{W,c}(X, F^D))$  by functoriality, so the latter is a  $\mathbb{Q}$ -vector space. Moreover, by Proposition IV.3.5 and Equation (IV.3.1), the abelian group  $\mathrm{Hom}(R\Gamma_{W,c}(X, G^D), R\Gamma_{W,c}(X, F^D))$  is of finite type so the image of

$$\mathrm{Hom}(D_G[1], R\Gamma_{W,c}(X, F^D)) \quad \text{inside} \quad \mathrm{Hom}(R\Gamma_{W,c}(X, G^D), R\Gamma_{W,c}(X, F^D))$$

must be 0 and the claim follows. This proves the functoriality in red-to-blue morphisms; by running the same argument with the morphism  $\mathrm{id} : F \rightarrow F$  for a choice of two different cones of  $\alpha_F$ , we obtain the uniqueness up to unique isomorphism.

Let  $0 \rightarrow F \rightarrow G \rightarrow H \rightarrow 0$  be a red-to-blue short exact sequence. Denote by  $u : H \rightarrow F[1]$  the induced map. By the previous argument, we obtain a diagram

$$\begin{array}{ccccccc} D_F & \xrightarrow{\alpha_F} & R\Gamma_{c,B}(X, F^D) & \longrightarrow & R\Gamma_{W,c}(X, F^D) & \longrightarrow & \\ f^* \uparrow & & f^* \uparrow & & f^* \uparrow & & \\ D_G & \xrightarrow{\alpha_G} & R\Gamma_{c,B}(X, G^D) & \longrightarrow & R\Gamma_{W,c}(X, G^D) & \longrightarrow & \\ g^* \uparrow & & g^* \uparrow & & g^* \uparrow & & \\ D_H & \xrightarrow{\alpha_H} & R\Gamma_{c,B}(X, H^D) & \longrightarrow & R\Gamma_{W,c}(X, H^D) & \longrightarrow & \\ u^* \uparrow & & u^* \uparrow & & u^* \uparrow & & \\ D_F[-1] & \xrightarrow{\alpha_F[-1]} & R\Gamma_{c,B}(X, F^D)[-1] & \longrightarrow & R\Gamma_{W,c}(X, F^D)[-1] & \longrightarrow & \end{array}$$

To construct the unique dotted arrow  $u^*$  making the diagram commute, repeat the previous argument using Proposition IV.2.14 and Equation (IV.3.1).

The right column provides a triangle that is not necessarily distinguished. We will show that it induces a long exact cohomology sequence. Fix a prime  $p$  and an integer  $n \in \mathbb{N}^\times$ . Observe that  $D_F$  is a complex of  $\mathbb{Q}$ -vector spaces, so that  $D_F \otimes^L \mathbb{Z}/p^n\mathbb{Z} = 0$ . It follows that

$$R\Gamma_{c,B}(X, F^D) \otimes^L \mathbb{Z}/p^n\mathbb{Z} \xrightarrow{\cong} R\Gamma_{W,c}(X, F^D) \otimes^L \mathbb{Z}/p^n\mathbb{Z}$$

is an isomorphism. Denote by  $(-)_p^\wedge = R\lim(- \otimes^L \mathbb{Z}/p^n\mathbb{Z})$  the derived  $p$ -completion. By passing to the derived limit, we find

$$R\Gamma_{c,B}(X, F^D)_p^\wedge \xrightarrow{\cong} R\Gamma_{W,c}(X, F^D)_p^\wedge = R\Gamma_{W,c}(X, F^D) \otimes^L \mathbb{Z}_p,$$

naturally in  $F$ , where the right equality holds because  $R\Gamma_{W,c}(X, F^D)$  is a perfect complex.<sup>27</sup> Since derived  $p$ -completion is an exact functor, it follows that  $R\Gamma_{W,c}(X, (-)^D) \otimes^L \mathbb{Z}_p$  is an exact functor too. Now  $\mathbb{Z}_p$  is flat and the family  $(\mathbb{Z}_p)_p$  prime is faithfully flat so it suffices to show that the long cohomology sequence is exact after tensoring with  $\mathbb{Z}_p$ , for each prime  $p$ ; this follows from the exactness of  $R\Gamma_{W,c}(X, (-)^D) \otimes^L \mathbb{Z}_p$ .  $\square$

**Proposition IV.3.11.** *Let  $Y = \mathrm{Spec}(\mathcal{O}')$  be the spectrum of an order in a number field with a finite dominant morphism  $\pi : Y \rightarrow X$  and let  $F$  be a red or blue  $\mathbb{Z}$ -constructible sheaf on  $Y$ . Then  $\pi_*F$  is red or blue and we have a unique isomorphism*

$$R\Gamma_{W,c}(Y, F^D) \xrightarrow{\cong} R\Gamma_{W,c}(X, (\pi_*F)^D).$$

<sup>27</sup>This follows from [Stacks, Tag 0EEU] and [Stacks, Tag 00MA] by filtering with the truncations.

*Proof.* We can run an argument similar to the previous proof: it suffices to check that the square

$$\begin{array}{ccc} R\Gamma(Y, F)^\dagger[-2] & \xrightarrow{\alpha_F} & R\Gamma_{c,B}(Y, F^D) \\ \uparrow \simeq & & \uparrow \simeq \\ R\Gamma(X, \pi_*F)^\dagger[-2] & \xrightarrow{\alpha_{\pi_*F}} & R\Gamma_{c,B}(X, (\pi_*F)^D) \end{array}$$

is commutative. This can be checked in cohomology in degree 2 and 3, whence from the definition of  $\alpha_F$  we reduce to checking that

$$\begin{array}{ccc} \hat{H}_{c,B}^i(Y, F^D) & \longrightarrow & H^{2-i}(Y, F)^* \\ \uparrow \simeq & & \uparrow \simeq \\ \hat{H}_{c,B}^i(X, (\pi_*F)^D) & \longrightarrow & H^{2-i}(X, \pi_*F)^* \end{array}$$

is commutative for  $i = 2, 3$ . This would be implied by the commutativity of the following diagram:

$$\begin{array}{ccccc} R\Gamma(Y, F) \otimes^L R\hat{\Gamma}_{c,B}(Y, F^D) & \longrightarrow & R\hat{\Gamma}_{c,B}(Y, \mathbb{Z}^D) & \xrightarrow{\tau_{\geq 2}} & \mathbb{Q}/\mathbb{Z}[-2] \\ \downarrow \simeq & & \downarrow & & \parallel \\ R\Gamma(X, \pi_*F) \otimes^L R\hat{\Gamma}_{c,B}(X, (\pi_*F)^D) & \longrightarrow & R\hat{\Gamma}_{c,B}(X, \mathbb{Z}^D) & \xrightarrow{\tau_{\geq 2}} & \mathbb{Q}/\mathbb{Z}[-2] \end{array}$$

By Proposition IV.2.6, the left square is commutative. Moreover, we can identify the canonical map  $R\hat{\Gamma}_{c,B}(Y, \mathbb{Z}^D) \rightarrow R\hat{\Gamma}_{c,B}(X, \mathbb{Z}^D)$  as the composition

$$R\hat{\Gamma}_{c,B}(Y, \mathbb{Z}^D) \simeq R\hat{\Gamma}_c(Y, \mathbb{Z}_Y^c) \simeq R\hat{\Gamma}_c(X, \pi_*R\pi^!\mathbb{Z}_X^c) \rightarrow R\hat{\Gamma}_c(X, \mathbb{Z}_X^c) \simeq R\hat{\Gamma}_{c,B}(X, \mathbb{Z}^D)$$

coming from the counit  $\varepsilon : \pi_*R\pi^! \rightarrow \text{id}$ . It induces in cohomology in degree 2 the identity  $\mathbb{Q}/\mathbb{Z} \rightarrow \mathbb{Q}/\mathbb{Z}$  [Mil06, II.3.10],[Har20, 8.9], so the right square is commutative.  $\square$

The Weil-étale complex splits rationally:

**Proposition IV.3.12.** *Let  $F$  be a red or blue sheaf. There is an isomorphism*

$$R\Gamma_{W,c}(X, F^D)_{\mathbb{Q}} \xrightarrow{\simeq} R\text{Hom}(R\Gamma(X, F), \mathbb{Q}[-1]) \oplus R\Gamma_{c,B}(X, F^D)_{\mathbb{Q}},$$

*natural in red-to-blue morphisms and red-to-blue short exact sequences.*

*Proof.* Consider the distinguished triangle

$$R\Gamma_{c,B}(X, (-)^D)_{\mathbb{Q}} \rightarrow R\Gamma_{W,c}(X, (-)^D)_{\mathbb{Q}} \xrightarrow{p} D_{(-)}[1].$$

It suffices to show that  $p$  has a section; but if  $F$  is a red sheaf or  $G$  is a blue sheaf, considerations on the Verdier spectral sequence Equation (IV.3.1) show that

$$\text{Hom}_{D(\mathbb{Z})}(D_F[1], R\Gamma_{c,B}(X, G^D)_{\mathbb{Q}}) = 0, \quad \text{Hom}_{D(\mathbb{Z})}(D_F[1], R\Gamma_{c,B}(X, G^D)_{\mathbb{Q}}[1]) = 0,$$

so that composition with  $p_G$  induces an isomorphism

$$\text{Hom}_{D(\mathbb{Z})}(D_F[1], R\Gamma_{W,c}(X, G^D)_{\mathbb{Q}}) \xrightarrow[p_{G,*}]{\simeq} \text{Hom}_{D(\mathbb{Z})}(D_F[1], D_G[1]).$$

We then put  $s_F = p_{F,*}^{-1}(\text{id})$ ; the functoriality is easily checked from the above isomorphism.  $\square$

**Proposition IV.3.13.** *Let  $Y = \text{Spec}(\mathcal{O}')$  be the spectrum of an order in a number field with a finite dominant morphism  $\pi : Y \rightarrow X$  and let  $F$  be a red or blue sheaf on  $Y$ . Then the rational splitting of Weil-étale cohomology is compatible with  $\pi_*$ , i.e. the following square commutes*

$$\begin{array}{ccc} R\Gamma_{W,c}(Y, F^D)_{\mathbb{Q}} & \xrightarrow{\simeq} & D_F[1] \oplus R\Gamma_c(Y, F^D)_{\mathbb{Q}} \\ \downarrow \simeq & & \simeq \downarrow \\ R\Gamma_{W,c}(X, (\pi_* F)^D)_{\mathbb{Q}} & \xrightarrow{\simeq} & D_{\pi_* F}[1] \oplus R\Gamma_c(X, (\pi_* F)^D)_{\mathbb{Q}} \end{array}$$

*Proof.* It suffices to prove that in the following square, the square with the sections commutes:

$$\begin{array}{ccc} R\Gamma_{W,c}(Y, F^D)_{\mathbb{Q}} & \begin{array}{c} \xleftarrow{s_F} \\ \xrightarrow{p_F} \end{array} & D_F[1] \\ \downarrow \gamma & & \delta \downarrow \\ R\Gamma_{W,c}(X, (\pi_* F)^D)_{\mathbb{Q}} & \begin{array}{c} \xrightarrow{p_{\pi_* F}} \\ \xleftarrow{s_{\pi_* F}} \end{array} & D_{\pi_* F}[1] \end{array}$$

As in the previous proof, there is an isomorphism

$$\text{Hom}_{D(\mathbb{Z})}(D_F[1], R\Gamma_{W,c}(X, (\pi_* F)^D)_{\mathbb{Q}}) \xrightarrow[p_{\pi_* F, *}]{\simeq} \text{Hom}_{D(\mathbb{Z})}(D_F[1], D_{\pi_* F}[1]).$$

We compute

$$p_{\pi_* F} \gamma s_F = \delta p_F s_F = \delta = p_{\pi_* F} s_{\pi_* F} \delta. \quad \square$$

## IV.4 The tangent space and the fundamental line

Denote by  $g : \eta = \text{Spec}(K) \rightarrow X$  the inclusion of the generic point,  $G_K$  the absolute Galois group of  $K$ ,  $K^t$  the maximal tamely ramified extension of  $K$ ,  $G_K^t = \text{Gal}(K^t/K)$  and  $\mathcal{O}_{K^t}$  the ring of integers of  $K^t$ . For each finite place  $v$  of  $K$ , let  $P_v \subset I_v \subset D_v \subset G_K$  denote the wild ramification, inertia, decomposition subgroups of  $G_K$  at  $v$ <sup>28</sup>. Denote by  $N$  the smallest closed normal subgroup of  $G_K$  containing  $P_v$  for each finite place  $v$ . We have  $G_K^t = G_K/N$  by Proposition C.0.3. For a sheaf  $F$  on  $X$ , put  $F_{\eta} := g^* F$ , seen as a discrete  $G_K$ -module.

**Definition IV.4.1.** *Let  $F$  be a  $\mathbb{Z}$ -constructible sheaf on  $X$ .*

- We say that  $F$  is tamely ramified if for each finite place  $v$  of  $K$ ,  $P_v$  acts trivially on  $F_{\eta}$ ; then  $F_{\eta}$  carries a natural action of  $G_K^t$ .
- Suppose that  $F$  is tamely ramified. The tangent space of  $F^D$  is the complex

$$\text{Lie}_X(F^D) := R\text{Hom}_{G_K^t}(F_{\eta}, \mathcal{O}_{K^t}[1]).$$

*Remark.*

- Let us motivate the above definition. Following the work of Flach–B. Morin [FM18], the complex  $\text{Lie}_X(F^D)$  should be the “additive complex” giving the additive part of the fundamental line. It should behave like Milne’s correcting factor in the

<sup>28</sup>coming from a choice of embedding  $K_v \hookrightarrow K^{\text{sep}}$



special value formula for  $L_X(F^D, s)$  at  $s = 0$ . Since  $L_X(\mathbb{Z}^D, s) = \zeta_X(s + 1)$  when  $X = \text{Spec}(\mathcal{O}_K)$  is regular, we must have  $\text{Lie}_X(\mathbb{G}_m) = \mathcal{O}_K$ : this is the additive complex for  $\mathbb{Z}(1)[1]$  on  $X$ . We want  $\text{Lie}_X$  to be an exact functor<sup>29</sup>. Recall the  $L$ -function of a  $\mathbb{Z}$ -constructible sheaf  $L_X(-, s)$  introduced in Subsection II.6.4. If  $i : x \rightarrow X$  is the inclusion of a closed point and  $M$  is a finite type  $G_x$ -module, we have  $L_X((i_*M)^D, s) = L_X(i_*M^\vee, s)$ . Since the special value at  $s = 0$  of  $L_X(i_*M^\vee, s)$  doesn't involve an additive complex, it is expected that  $\text{Lie}_X((i_*M)^D) = 0$ . Therefore,  $\text{Lie}_X(F^D)$  should only depend on  $F_\eta$ . In [GS20], Geisser–Suzuki consider a torus  $T$  over a function field  $K$  associated to a proper smooth curve over a finite field  $C$ . They write down a special value formula for the  $L$ -function of  $T$  (defined in terms of the rational  $\ell$ -adic Tate module) at  $s = 1$  in terms of Weil-étale cohomology and of the Lie algebra of the connected Néron model  $\mathcal{T}^0$  on  $C$  of  $T$ . Moreover, they show that on the étale site,  $\mathcal{T}^0 = R\mathcal{H}om(\tilde{\mathcal{Y}}, \mathbb{G}_m)$  where  $\tilde{\mathcal{Y}}$  is a bounded complex with  $\mathbb{Z}$ -constructible cohomology sheaves related to the character group of  $T$ . This inspired our construction.

- If  $F$  is  $\mathbb{Z}$ -constructible and  $X = \text{Spec}(\mathcal{O}_K)$  is regular,  $F$  is tamely ramified if and only if for each  $U \subset X$  such that  $F|_U$  is locally constant, there exist a finite extension  $L/K$  such that the normalization  $\pi : Y \rightarrow X$  of  $X$  in  $L$  is étale over  $U$  with  $(\pi^*F)|_{\pi^{-1}(U)}$  constant, and  $\pi$  is tamely ramified above points  $x \in X \setminus U$ .
- If  $X$  is not regular we nonetheless have  $\text{Lie}_X(\mathbb{Z}^D) = \mathcal{O}_K[1]$ . This seems strange to us, but as we explained  $\text{Lie}_X(F^D)$  depends only on  $F_\eta$  which does not see singular points so it is also more or less expected.
- We have  $\text{Lie}_X(F^D)_\mathbb{Q} = \text{Hom}_{G_K}(F_\eta, K^{\text{sep}})[1]$ .

The restriction to tamely ramified sheaves is justified by the following theorem, essentially due to Noether [Noe32]<sup>30</sup>:

**Theorem IV.4.2** ([NSW13, 6.1.10]). *Let  $K$  be a number field or a  $p$ -adic field. The discrete  $G_K^t$ -module  $\mathcal{O}_{K^t}$  is cohomologically trivial. In particular*

$$R\Gamma(G_K^t, \mathcal{O}_{K^t}) = \mathcal{O}_K[0].$$

**Corollary IV.4.3.** *Let  $L/K$  be a finite tamely ramified Galois extension of number fields or  $p$ -adic fields with Galois group  $G$ . For  $i \geq 1$ , put  $\mathcal{O}_L[[t]]_i := 1 + t^i \mathcal{O}_L[[t]]$ . Then  $\mathcal{O}_L[[t]]_i$  is a cohomologically trivial  $G$ -module.*

*Proof.* We have an isomorphism  $\mathcal{O}_L[[t]]_i / \mathcal{O}_L[[t]]_{i+1} \simeq \mathcal{O}_L$  and

$$\mathcal{O}_L[[t]]_i = \lim_{k > i} \mathcal{O}_L[[t]]_i / \mathcal{O}_L[[t]]_k.$$

The transition maps are surjective, and the terms are cohomologically trivial by induction using Theorem IV.4.2. Moreover, the kernel of  $\mathcal{O}_L[[t]]_i / \mathcal{O}_L[[t]]_{k+1} \rightarrow \mathcal{O}_L[[t]]_i / \mathcal{O}_L[[t]]_k$  is  $\mathcal{O}_L[[t]]_k / \mathcal{O}_L[[t]]_{k+1} \simeq \mathcal{O}_L$  which is cohomologically trivial, so for every subgroup  $H$  of  $G$  the maps  $(\mathcal{O}_L[[t]]_i / \mathcal{O}_L[[t]]_{k+1})^H \rightarrow (\mathcal{O}_L[[t]]_i / \mathcal{O}_L[[t]]_k)^H$  are surjective.

<sup>29</sup>A way to avoid the tame ramification hypothesis may be to relax this condition and only ask that  $\det_{\mathbb{Z}}(\text{Lie}_X((-)^D)$  is multiplicative with respect to fiber sequences)

<sup>30</sup>See also [Köc04, Theorem 1.1]

Put  $A_k = \mathcal{O}_L[[t]]_i / \mathcal{O}_L[[t]]_k$  and let  $H$  be a subgroup of  $G$ . We compute:

$$\begin{aligned} R\Gamma(H, \mathcal{O}_L[[t]]_i) &= R\Gamma(H, \lim A_k) = R\Gamma(H, R\lim A_k) = R\lim R\Gamma(H, A_k) \\ &= R\lim(A_k^H[0]) \\ &= (\lim A_k^H)[0] \end{aligned}$$

where each limit is a derived limit because the transition maps are surjective.  $\square$

The following lemma together with the above theorem will enable us to compute the tangent space of  $F^D$  more explicitly:

**Lemma IV.4.4.** *Let  $G$  be a profinite group and let  $\mathcal{O}$  be a torsion-free cohomologically trivial discrete  $G$ -module. Let  $M$  be a finite type discrete  $G$ -module. The complex  $R\mathrm{Hom}_G(M, \mathcal{O})$  is cohomologically concentrated in degree  $[0, 1]$ . Moreover, if  $M$  is finite it is concentrated in degree 1 and if  $M$  is torsion-free it is concentrated in degree 0.*

*Proof.* To show the vanishing result, we can use the short exact sequence  $0 \rightarrow M_{\mathrm{tor}} \rightarrow M \rightarrow M/\mathrm{tor} \rightarrow 0$  to reduce to the finite and torsion-free cases; thus it suffices to study those cases:

- Suppose first that  $M$  is finite. We then have  $\mathrm{Hom}_G(M, \mathcal{O}) = 0$  because  $\mathcal{O}$  is torsion-free. As  $\mathrm{Ext}_{\mathbb{Z}}^p(M, \mathcal{O}) = 0$  for  $p \neq 1$ , we obtain for  $i \geq 1$ :

$$\mathrm{Ext}_G^i(M, \mathcal{O}) = H^{i-1}(G, \mathrm{Ext}_{\mathbb{Z}}^1(M, \mathcal{O})).$$

Let us now show the vanishing for  $i > 1$ . We can suppose that  $M$  is  $p$ -primary, and even  $p$ -torsion. Let  $H$  be any  $p$ -Sylow of  $G$ . As an  $H$ -module,  $M$  has a composition series with quotients isomorphic to  $\mathbb{Z}/p\mathbb{Z}$  with its trivial action [Ser68, IX, Thm 1]. We have

$$H^i(H, \mathrm{Ext}_{\mathbb{Z}}^1(\mathbb{Z}/p\mathbb{Z}, \mathcal{O})) = H^i(H, \mathcal{O}/p\mathcal{O}) = 0$$

for  $i \geq 1$  because  $\mathcal{O}$  is cohomologically trivial. It follows that for any  $i \geq 1$  we also have  $H^i(H, \mathrm{Ext}_{\mathbb{Z}}^1(M, \mathcal{O})) = 0$ . Finally, the following triangle commutes:

$$\begin{array}{ccc} H^i(G, \mathrm{Ext}_{\mathbb{Z}}^1(M, \mathcal{O})) & \xrightarrow[\simeq]{[G:H]} & H^i(G, \mathrm{Ext}_{\mathbb{Z}}^1(M, \mathcal{O})) \\ & \searrow \mathrm{Res} & \nearrow \mathrm{Cores} \\ & H^i(H, \mathrm{Ext}_{\mathbb{Z}}^1(M, \mathcal{O})) = 0 & \end{array}$$

whence it follows that  $H^i(G, \mathrm{Ext}_{\mathbb{Z}}^1(M, \mathcal{O})) = 0$  for  $i \geq 1$ .

- Suppose now that  $M$  is torsion-free, and let us show that  $\mathrm{Ext}_G^i(M, \mathcal{O}) = 0$  for  $i > 0$ . By Artin induction and the previous case it suffices to consider the case of  $M = \mathrm{ind}_H^G \mathbb{Z}$  for  $H$  an open subgroup. Since  $\mathcal{O}$  is cohomologically trivial we have

$$R\mathrm{Hom}_G(\mathrm{ind}_H^G \mathbb{Z}, \mathcal{O}) = R\Gamma(H, \mathcal{O}) = \mathcal{O}^H[0],$$

which is concentrated in degree 0.  $\square$

**Corollary IV.4.5.** *Let  $M$  be a finite type discrete  $G_K^t$ -module. Then  $R\mathrm{Hom}_{G_K^t}(M, \mathcal{O}_{K^t}[1])$  is a perfect complex cohomologically concentrated in degree  $[-1, 0]$ . More precisely,  $\mathrm{Ext}_{G_K^t}^{-1}(M, \mathcal{O}_{K^t}[1])$  is free of finite rank and  $\mathrm{Ext}_{G_K^t}^0(M, \mathcal{O}_{K^t}[1])$  is finite.*

*Proof.* We apply first Lemma IV.4.4 to obtain the vanishing outside  $[-1, 0]$ . Now let  $H$  be an open normal subgroup of  $G_K^t$  acting trivially on  $M$  and corresponding to a finite tamely ramified Galois extension  $L/K$ , and put  $G := G_K^t/H = \text{Gal}(L/K)$ . There is a natural  $G$ -action on  $M$ . The functor  $R\Gamma(H, -)$  is right adjoint to the forgetful functor  $G\text{-Mod} \rightarrow G_K^t\text{-Mod}$ , so we obtain

$$R\text{Hom}_G(M, \mathcal{O}_{K^t}[1]) = R\text{Hom}_G(M, R\Gamma(H, \mathcal{O}_{K^t}[1])) = R\text{Hom}_G(M, \mathcal{O}_L[1])$$

by Theorem IV.4.2.

Since  $\mathcal{O}_L$  is torsion-free,  $\text{Hom}_G(M, \mathcal{O}_L)$  is free of finite rank. On the other hand, we can use the short-exact sequence  $0 \rightarrow M_{\text{tor}} \rightarrow M \rightarrow M/\text{tor} \rightarrow 0$ , the vanishing  $\text{Ext}_{\mathbb{Z}}^p(M_{\text{tor}}, \mathcal{O}_L) = 0$  for  $p \neq 1$  and Lemma IV.4.4 (applied to  $G$  and  $\mathcal{O}_L$ ) to compute

$$\text{Ext}_G^1(M, \mathcal{O}_L) \xrightarrow{\cong} \text{Ext}_G^1(M_{\text{tor}}, \mathcal{O}_L) = H^0(G, \text{Ext}_{\mathbb{Z}}^1(M_{\text{tor}}, \mathcal{O}_L))$$

which is finite. □

**Corollary IV.4.6.** *Let  $F$  be a tamely ramified  $\mathbb{Z}$ -constructible sheaf on the spectrum  $X$  of an order in a number field. Then  $\text{Lie}_X(F^D)$  is a perfect complex concentrated in degree  $[-1, 0]$ . Moreover  $H^{-1}\text{Lie}_X(F^D)$  is free of finite rank and  $H^0\text{Lie}_X(F^D)$  is finite.*

**Corollary IV.4.7.** *Suppose that  $M$  is a torsion-free discrete  $G_K^t$ -module and let  $T$  be the torus over  $K$  with character group  $M$ . Denote by  $\mathcal{T}$  the locally of finite type Néron model of  $T$  over  $\text{Spec}(\mathcal{O}_K)$ . We have a canonical, functorial isomorphism*

$$\text{Lie}(\mathcal{T}) \xrightarrow{\cong} \text{Hom}_{G_K^t}(M, \mathcal{O}_{K^t})$$

where the left term is the Lie algebra of  $\mathcal{T}$ .

*Proof.* Both terms are lattices inside  $\text{Lie}(T) = \text{Hom}_{G_K}(M, K^{\text{sep}})$ . It suffices to show that they are canonically isomorphic after tensoring with  $\mathcal{O}_{K_v}$  for all non-archimedean places  $v$ . On the one hand we have

$$\text{Lie}(\mathcal{T}) \otimes_{\mathcal{O}_K} \mathcal{O}_{K_v} = \text{Lie}(\mathcal{T}_{\mathcal{O}_{K_v}}) = \text{Lie}(\mathcal{N}(T_{K_v}))$$

where  $\mathcal{N}$  denotes the lft Néron model. On the other hand let  $G = \text{Gal}(L/K)$  be a quotient of  $G_K^t$  through which the action of  $G_K^t$  on  $M$  factors. We have

$$\mathcal{O}_L \otimes_{\mathcal{O}_K} \mathcal{O}_{K_v} \simeq \prod_{w|v} \mathcal{O}_{L_w}$$

and thus

$$\begin{aligned} \text{Hom}_{G_K^t}(M, \mathcal{O}_{K^t}) \otimes_{\mathcal{O}_K} \mathcal{O}_{K_v} &= \text{Hom}_G(M, \mathcal{O}_L) \otimes_{\mathcal{O}_K} \mathcal{O}_{K_v} = \text{Hom}_G(M, \mathcal{O}_L \otimes_{\mathcal{O}_K} \mathcal{O}_{K_v}) \\ &= \text{Hom}_G(M, \prod_{w|v} \mathcal{O}_{L_w}) \\ &= \text{Hom}_{D_{w_0/v}}(M, \mathcal{O}_{L_{w_0}}) \end{aligned}$$

where  $w_0$  is a fixed place of  $L$  above  $v$ , so we can reduce to the case where  $K$  is a  $p$ -adic field. By [CY01, A1.7] we have a canonical functorial isomorphism

$$\text{Lie}(\mathcal{T}) \xrightarrow{\cong} \{v \in \text{Hom}_G(M, \mathcal{O}_L) \mid v \text{ lifts to } \text{Hom}_G(M, \mathcal{O}_L[[t]]_1)\}$$

It thus suffices to show that  $\mathrm{Hom}_G(M, \mathcal{O}_L[[t]]_1) \rightarrow \mathrm{Hom}_G(M, \mathcal{O}_L)$  is surjective. We have an exact sequence

$$\mathrm{Hom}_G(M, \mathcal{O}_L[[t]]_1) \rightarrow \mathrm{Hom}_G(M, \mathcal{O}_L) \rightarrow \mathrm{Ext}_G^1(M, \mathcal{O}_L[[t]]_2).$$

Now  $\mathcal{O}_L[[t]]_2$  is torsion-free and cohomologically trivial by Corollary IV.4.3 and  $M$  is a torsion-free finite type  $G$ -module, so we can apply Lemma IV.4.4 to obtain the vanishing of  $\mathrm{Ext}_G^i(M, \mathcal{O}_L[[t]]_2)$  for  $i \geq 1$ .  $\square$

*Remark.* As a corollary, a short exact sequence of tori that are split by a tamely ramified extension gives rise to a short exact sequence of the Lie algebras of the respective lft Néron models.

In the following, put  $(-)_\mathbb{R} := - \otimes^L \mathbb{R}$ .

**Proposition IV.4.8.** *Let  $F$  be a tamely ramified  $\mathbb{Z}$ -constructible sheaf on  $X$ . We have*

$$\mathrm{Lie}_X(F^D)_\mathbb{R} = R\mathrm{Hom}_{G_\mathbb{R}, X(\mathbb{C})}(\alpha^*F, \mathbb{C}[1]).$$

*Proof.* Let  $L/K$  be a finite tamely ramified Galois extension such that  $G_L$  acts trivially on  $F_\eta$ , and put  $G := G_K^t/G_L^t = \mathrm{Gal}(L/K)$ . We have:

$$\begin{aligned} \mathrm{Lie}_X(F^D)_\mathbb{R} &= R\mathrm{Hom}_G(F_\eta, \mathcal{O}_L[1])_\mathbb{R} \stackrel{(*)}{=} R\mathrm{Hom}_G(F_\eta, \mathcal{O}_L \otimes_{\mathbb{Z}} \mathbb{R}[1]) \\ &= R\mathrm{Hom}_G(F_\eta, \prod_{v \text{ archimedean}} \prod_{w|v} L_w[1]) \\ &= \prod_{v \text{ archimedean}} R\mathrm{Hom}_G(F_\eta, \prod_{w|v} L_w[1]). \end{aligned}$$

where  $(*)$  holds because  $F_\eta$  is a finite type abelian group, hence a finite presentation  $\mathbb{Z}[G]$ -module. For each archimedean place  $v$ , choose a place  $w_0$  in  $L$  above  $v$ . The group  $G$  acts transitively on the places above  $v$ , the fields  $L_w$  are pairwise isomorphic, and the stabilizer  $D_{w_0}$  of  $w_0$  identifies with  $\mathrm{Gal}(L_{w_0}/K_v)$ . Thus  $\prod_{w|v} L_w = \mathrm{ind}_{D_{w_0}}^G L_{w_0}$  and we find

$$\mathrm{Lie}_X(F^D)_\mathbb{R} = \prod_{v \text{ archimedean}} R\mathrm{Hom}_{D_{w_0}}(F_\eta, L_{w_0}[1]).$$

On the other hand, for  $v$  an archimedean place we have  $F_v = F_\eta$  with the restricted action to  $G_{K_v} := \mathrm{Gal}(\mathbb{C}/K_v)$ . Therefore the action of  $G_{K_v}$  factors through  $\mathrm{Gal}(L_{w_0}/K_v)$ ; as the functor  $R\Gamma(\mathrm{Gal}(\mathbb{C}/L_{w_0}), -)$  is right adjoint to the forgetful functor  $D_{w_0}\text{-Mod} \rightarrow G_{K_v}\text{-Mod}$ , we find

$$\begin{aligned} R\mathrm{Hom}_{G_\mathbb{R}, X(\mathbb{C})}(\alpha^*F, \mathbb{C}[1]) &= R\Gamma_{G_\mathbb{R}}(X(\mathbb{C}), R\mathcal{H}om_{X(\mathbb{C})}(\alpha^*F, \mathbb{C}[1])) \\ &= \prod_{v \text{ archimedean}} R\Gamma(G_{K_v}, R\mathrm{Hom}_{\mathbb{Z}}(F_v, \mathbb{C}[1])) \\ &= \prod_{v \text{ archimedean}} R\mathrm{Hom}_{G_{K_v}}(F_v, \mathbb{C}[1]) \\ &= \prod_{v \text{ archimedean}} R\mathrm{Hom}_{D_{w_0}}(F_v, L_{w_0}[1]) \\ &= \mathrm{Lie}_X(F^D)_\mathbb{R}. \end{aligned} \quad \square$$

**Proposition IV.4.9.** *Let  $Y = \text{Spec}(\mathcal{O}')$  be the spectrum of an order in a number field with a finite dominant morphism  $\pi : Y \rightarrow X$ . If  $K(Y)/K$  is tamely ramified and  $F$  is a tamely ramified  $\mathbb{Z}$ -constructible sheaf on  $Y$ , we have*

$$\text{Lie}_X((\pi_* F)^D) = \text{Lie}_Y(F^D).$$

*Proof.* Denote by  $g' : \text{Spec}(L) \rightarrow Y$  the inclusion of the generic point and  $\pi' : L \rightarrow K$  the map induced by base change of  $\pi$  by  $g : \text{Spec}(K) \rightarrow X$ . By finite base change, we have  $g^* \pi_* F = \pi'_* g'^* F$ . If  $K \subset H \subset G$  are normal subgroups of a profinite group  $G$  and  $M$  is a discrete  $H/K$ -module, we have

$$\text{ind}_{H/K}^{G/K} M = \text{Cont}_{H/K}(G/K, M) \simeq \text{Cont}(G/H, M) \simeq \text{Cont}_H(G, M) = \text{ind}_H^G M,$$

where the left term is seen as a  $G$ -module, and on the right  $M$  has the natural  $H$ -module structure. Denote by  $U : G_K^t\text{-Mod} \rightarrow G_L^t\text{-Mod}$  and  $U' : G_L^t\text{-Mod} \rightarrow G_K^t\text{-Mod}$  the forgetful functors. The functor  $\pi'_*$  identifies with induction  $\text{ind}_{G_L^t}^{G_K^t}$ . We have  $K^t = L^t$  because  $L/K$  is tamely ramified so we can apply the above to  $\text{Gal } K^{\text{sep}}/K^t \subset G_L \subset G_K$ . We obtain the formula  $U \circ \text{ind}_{G_L^t}^{G_K^t} = \text{ind}_{G_L^t}^{G_K^t} \circ U'$ , so that

$$\begin{aligned} \text{Lie}_Y((\pi_* F)^D) &:= R \text{Hom}_{G_K^t}(\text{ind}_{G_L^t}^{G_K^t} g'^* F, \mathcal{O}_{K^t}[1]) = R \text{Hom}_{G_L^t}(g'^* F, \mathcal{O}_{K^t}[1]) \\ &= \text{Lie}_X(F^D), \end{aligned}$$

since induction is right *and* left adjoint to the forgetful functor by the finiteness of  $L/K$ .  $\square$

**Definition IV.4.10.** *Let  $F$  be a tamely ramified red or blue sheaf. The fundamental line is the determinant  $\dim$*

$$\Delta_X(F^D) = \det_{\mathbb{Z}}(R\Gamma_{W,c}(X, F^D)) \otimes \det_{\mathbb{Z}}(\text{Lie}_X(F^D))^{-1}.$$

## IV.5 Deligne compactly supported cohomology and the duality theorem

There is a natural  $G_{\mathbb{R}}$ -equivariant map of sheaves on  $X(\mathbb{C})$  given by the logarithm of the absolute value  $\log |\cdot| : \mathbb{C}^\times \rightarrow \mathbb{R}$ . Let  $F$  be a sheaf on  $X$ . We denote by  $\text{Log}$  the natural map

$$\begin{aligned} \text{Log} : R \text{Hom}(F, \mathbb{Z}_X^c) &\xrightarrow{\alpha^*} R \text{Hom}_{G_{\mathbb{R}}, X(\mathbb{C})}(\alpha^* F, \overline{\mathbb{Q}}^\times[1]) \\ &\rightarrow R \text{Hom}_{G_{\mathbb{R}}, X(\mathbb{C})}(\alpha^* F, \mathbb{C}^\times[1]) \\ &\xrightarrow{(\log |\cdot|)_*[1]} R \text{Hom}_{G_{\mathbb{R}}, X(\mathbb{C})}(\alpha^* F, \mathbb{R}[1]). \end{aligned}$$

Since the target is a sheaf of  $\mathbb{R}$ -vector spaces, the map factors as  $\text{Log} : R\Gamma(X, F^D)_{\mathbb{R}} = R \text{Hom}(F, \mathbb{Z}_X^c)_{\mathbb{R}} \rightarrow R \text{Hom}_{G_{\mathbb{R}}, X(\mathbb{C})}(\alpha^* F, \mathbb{R}[1])$ .

**Definition IV.5.1.** *Let  $F$  be a  $\mathbb{Z}$ -constructible sheaf on  $X$ . We define the Deligne compactly supported cohomology with coefficients in  $F^D$  by*

$$R\Gamma_{c,\mathcal{D}}(X, F_{\mathbb{R}}^D) := \text{fib}(R\Gamma(X, F^D)_{\mathbb{R}} \xrightarrow{\text{Log}} R \text{Hom}_{G_{\mathbb{R}}, X(\mathbb{C})}(\alpha^* F, \mathbb{R}[1])).$$

*Remark.* We have chosen to name the different cohomologies with compact support by what happens at the archimedean places; here the cohomology at the archimedean places is replaced by a contravariant version of Deligne cohomology with real coefficients  $\mathbb{R}(1)_{\mathcal{D}} = [2i\pi\mathbb{R} \rightarrow \mathbb{C}] \simeq \mathbb{R}[-1]$ .

The commutative diagram with exact rows of  $G_{\mathbb{R}}$ -equivariant sheaves

$$\begin{array}{ccccccc} 0 & \longrightarrow & 2i\pi\mathbb{Z} & \longrightarrow & \mathbb{C} & \longrightarrow & \mathbb{C}^{\times} \longrightarrow 0 \\ & & \downarrow & & \parallel & & \downarrow \log|\cdot| \\ 0 & \longrightarrow & 2i\pi\mathbb{R} & \longrightarrow & \mathbb{C} & \longrightarrow & \mathbb{R} \longrightarrow 0 \end{array}$$

gives a commutative square

$$\begin{array}{ccc} \mathbb{C}^{\times} & \longrightarrow & 2i\pi\mathbb{Z}[1] \\ \downarrow \log|\cdot| & & \downarrow \\ \mathbb{R} & \longrightarrow & 2i\pi\mathbb{R}[1] \end{array} \quad (\text{IV.5.1})$$

Consider the following diagram

$$\begin{array}{ccccc} R\Gamma_{c,B}(X, F^D)_{\mathbb{R}} & \longrightarrow & R\Gamma(X, F^D)_{\mathbb{R}} & \longrightarrow & R\text{Hom}_{G_{\mathbb{R}}, X(\mathbb{C})}(\alpha^*F, 2i\pi\mathbb{R})[2] \\ \downarrow \text{Log} & & \downarrow \text{Log} & & \parallel \\ R\text{Hom}_{G_{\mathbb{R}}, X(\mathbb{C})}(\alpha^*F, \mathbb{C}[1]) & \longrightarrow & R\text{Hom}_{G_{\mathbb{R}}, X(\mathbb{C})}(\alpha^*F, \mathbb{R}[1]) & \longrightarrow & R\text{Hom}_{G_{\mathbb{R}}, X(\mathbb{C})}(\alpha^*F, 2i\pi\mathbb{R})[2] \end{array}$$

where the top fiber sequence is the defining fiber sequence of  $R\Gamma_{c,B}(X, F^D)$  tensored with  $\mathbb{R}$  and the bottom fiber sequence is induced from the short exact sequence  $0 \rightarrow 2i\pi\mathbb{R} \rightarrow \mathbb{C} \rightarrow \mathbb{R} \rightarrow 0$ . We claim that the right square is commutative, hence induces the left map making the whole diagram commute. This follows from the following commutative diagram, induced by the commutative square (IV.5.1), by using the universal property of base change to  $\mathbb{R}$ -coefficients:

$$\begin{array}{ccc} R\Gamma(X, F^D) \xrightarrow{\alpha^*} R\text{Hom}_{G_{\mathbb{R}}, X(\mathbb{C})}(\alpha^*F, \overline{\mathbb{Q}}^{\times}[1]) & & \\ \downarrow \text{Log} & \downarrow & \\ R\text{Hom}_{G_{\mathbb{R}}, X(\mathbb{C})}(\alpha^*F, \mathbb{C}^{\times}[1]) \longrightarrow R\text{Hom}_{G_{\mathbb{R}}, X(\mathbb{C})}(\alpha^*F, 2i\pi\mathbb{Z}[2]) & & \\ \downarrow (\log|\cdot|)_*[1] & & \parallel \\ R\text{Hom}_{G_{\mathbb{R}}, X(\mathbb{C})}(\alpha^*F, \mathbb{R}[1]) \longrightarrow R\text{Hom}_{G_{\mathbb{R}}, X(\mathbb{C})}(\alpha^*F, 2i\pi\mathbb{R}[2]) & & \end{array}$$

By the  $\infty$ -categorical nine lemma Lemma II.2.2 we obtain

$$R\Gamma_{c,\mathcal{D}}(X, F_{\mathbb{R}}^D) = \text{fib}(R\Gamma_{c,B}(X, F^D)_{\mathbb{R}} \xrightarrow{\text{Log}} R\text{Hom}_{G_{\mathbb{R}}, X(\mathbb{C})}(\alpha^*F, \mathbb{C}[1])).$$

We now want to prove a duality theorem relating  $R\Gamma(X, F)_{\mathbb{R}}$  and  $R\Gamma_{c,\mathcal{D}}(X, F_{\mathbb{R}}^D)$ . Let us construct a natural pairing

$$R\Gamma(X, F)_{\mathbb{R}} \otimes^L R\Gamma_{c,\mathcal{D}}(X, F_{\mathbb{R}}^D) \rightarrow \mathbb{R}[0].$$

Consider Diagram IV.3. Its right rectangle commutes. The top and bottom rows are fiber sequences, so we obtain an induced dotted map making the diagram commute. To obtain

$$\begin{array}{ccccccc}
R\Gamma(X, F)_{\mathbb{R}} \otimes^L R\Gamma_{c,D}(X, F_{\mathbb{R}}^D) & \longrightarrow & R\Gamma(X, F)_{\mathbb{R}} \otimes^L R\Gamma(X, F^D)_{\mathbb{R}} & \longrightarrow & R\Gamma(X, F)_{\mathbb{R}} \otimes^L R\mathrm{Hom}_{G_{\mathbb{R}}, X(\mathbb{C})}(\alpha^*F, \mathbb{R}[1]) \\
& & \parallel & & \downarrow \\
& & R\Gamma(X, F)_{\mathbb{R}} \otimes^L R\mathrm{Hom}(F, \mathbb{Z}_X^c)_{\mathbb{R}} & & R\Gamma_{G_{\mathbb{R}}}(X(\mathbb{C}), \alpha^*F)_{\mathbb{R}} \otimes^L R\mathrm{Hom}_{G_{\mathbb{R}}, X(\mathbb{C})}(\alpha^*F, \mathbb{R}[1]) \\
& & \downarrow & & \downarrow \\
R\Gamma_{c,D}(X, \mathbb{Z}_{\mathbb{R}}^D) & \longrightarrow & R\Gamma(X, \mathbb{Z}_X^c)_{\mathbb{R}} & \longrightarrow & R\Gamma_{G_{\mathbb{R}}}(X(\mathbb{C}), \mathbb{R}[1])
\end{array}$$

Diagram IV.3

the desired pairing, it remains to compute  $R\Gamma_{c,\mathcal{D}}(X, \mathbb{Z}_{\mathbb{R}}^D)$ . The complex  $R\Gamma(X, \mathbb{Z}_X^c) = R\Gamma(X, \mathbb{G}_X)$  is torsion in degree  $i \geq 1$  [Mil06, II.6.2], and we have  $R\Gamma_{G_{\mathbb{R}}}(X(\mathbb{C}), \mathbb{R}[1]) = (\prod_{\sigma:K \rightarrow \mathbb{C}} \mathbb{R})^{G_{\mathbb{R}}}[1]$ . The cohomology in degree  $-1$  and  $0$  is given by the exact sequence

$$0 \rightarrow H_{c,\mathcal{D}}^{-1}(X, \mathbb{Z}_{\mathbb{R}}^D) \rightarrow \mathrm{CH}_0(X, 1)_{\mathbb{R}} \xrightarrow{\mathrm{Log}} (\prod_{\sigma:K \rightarrow \mathbb{C}} \mathbb{R})^{G_{\mathbb{R}}} \rightarrow H_{c,\mathcal{D}}^0(X, \mathbb{Z}_{\mathbb{R}}^D) \rightarrow \mathrm{CH}_0(X)_{\mathbb{R}} = 0.$$

Denote by  $f : X \rightarrow \mathrm{Spec}(\mathbb{Z})$  the structure map of  $X$ ,  $Z$  its singular locus and  $U$  its regular locus. By Theorem II.4.4,<sup>31</sup> we have<sup>32</sup>

$$\mathrm{CH}_0(X)_{\mathbb{R}} = \mathrm{Hom}_X(\mathbb{Z}, \mathbb{Z}_X^c)_{\mathbb{R}} \simeq \mathrm{Hom}(H_c^0(X, \mathbb{Z})_{\mathbb{R}}, \mathbb{R}) = 0.$$

Moreover, Theorem II.4.4 and the computation of  $H^1(X, \mathbb{Z})$  gives a short exact sequence

$$0 \rightarrow \mathrm{CH}_0(X, 1)_{\mathbb{R}} \xrightarrow{\mathrm{Log}} (\prod_{\sigma:K \rightarrow \mathbb{C}} \mathbb{R})^{G_{\mathbb{R}}} \times \prod_{v \in Z} \prod_{\pi(w)=v} \mathbb{R} \xrightarrow{\Sigma \times \prod_{v \in Z} \Sigma} \mathbb{R} \times \prod_{v \in Z} \mathbb{R} \rightarrow 0,$$

where  $\Sigma$  denote sum maps. Consider the following snake diagram:

$$\begin{array}{ccccccc} & & & 0 & \longrightarrow & H_{c,\mathcal{D}}^{-1}(X, \mathbb{Z}_{\mathbb{R}}^D) & \longrightarrow & 0 \\ & & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & 0 & \longrightarrow & \mathrm{CH}_0(X, 1)_{\mathbb{R}} & \xlongequal{\quad} & \mathrm{CH}_0(X, 1)_{\mathbb{R}} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & \longrightarrow & \prod_{v \in Z} \prod_{\pi(w)=v} \mathbb{R} & \longrightarrow & (\prod_{\sigma:K \rightarrow \mathbb{C}} \mathbb{R})^{G_{\mathbb{R}}} \times \prod_{v \in Z} \prod_{\pi(w)=v} \mathbb{R} & \longrightarrow & (\prod_{\sigma:K \rightarrow \mathbb{C}} \mathbb{R})^{G_{\mathbb{R}}} & \longrightarrow & 0 \\ & & \parallel & & \downarrow \Sigma \times \prod_{v \in Z} \Sigma & & \downarrow & & \downarrow & & \downarrow \\ & & \prod_{v \in Z} \prod_{\pi(w)=v} \mathbb{R} & \xrightarrow{(0, \prod_{v \in Z} \Sigma)} & \mathbb{R} \times \prod_{v \in Z} \mathbb{R} & \longrightarrow & H_{c,\mathcal{D}}^0(X, \mathbb{Z}_{\mathbb{R}}^D) & \longrightarrow & 0 \end{array}$$

If we denote by  $(\prod_I \mathbb{R})^{\Sigma}$  the kernel of the sum map  $\prod_I \mathbb{R} \rightarrow \mathbb{R}$ , we obtain identifications

$$H_{c,\mathcal{D}}^{-1}(X, \mathbb{Z}_{\mathbb{R}}^D) = \prod_{v \in Z} (\prod_{\pi(w)=v} \mathbb{R})^{\Sigma}, \quad H_{c,\mathcal{D}}^0(X, \mathbb{Z}_{\mathbb{R}}^D) = \mathbb{R},$$

and under the latter identification the map  $(\prod_{\sigma:K \rightarrow \mathbb{C}} \mathbb{R})^{G_{\mathbb{R}}} \rightarrow H_{c,\mathcal{D}}^0(X, \mathbb{Z}_{\mathbb{R}}^D)$  is simply the restriction of  $\Sigma$ . If  $X$  is unibranch, and in particular if it is regular, we have  $H_{c,\mathcal{D}}^{-1}(X, \mathbb{Z}_{\mathbb{R}}^D) = 0$ .

The desired pairing is now

$$R\Gamma(X, F)_{\mathbb{R}} \otimes^L R\Gamma_{c,\mathcal{D}}(X, F_{\mathbb{R}}^D) \rightarrow R\Gamma_{c,\mathcal{D}}(X, \mathbb{Z}_{\mathbb{R}}^D) \xrightarrow{\tau \geq 0} \mathbb{R}[0].$$

<sup>31</sup>Which also holds for singular schemes; this can be seen either by applying it on  $\mathrm{Spec}(\mathbb{Z})$  to the direct image with compact support along the structural morphism, or by modifying slightly the proof to reduce to the regular case by removing the singular points

<sup>32</sup>This is essentially the finiteness of the class number of a number field



**Theorem IV.5.2.** *Let  $X$  be the spectrum of an order in a number field and let  $F \in D^b(X_{\text{et}})$  be a bounded complex with  $\mathbb{Z}$ -constructible cohomology groups. The pairing*

$$R\Gamma(X, F)_{\mathbb{R}} \otimes^L R\Gamma_{c, \mathcal{D}}(X, F_{\mathbb{R}}^D) \rightarrow \mathbb{R}[0]$$

*is a perfect pairing of perfect complexes of  $\mathbb{R}$ -vector spaces. If  $F \in D^+(X_{\text{et}})$ , the map*

$$R\Gamma_{c, \mathcal{D}}(X, F_{\mathbb{R}}^D) \rightarrow R\text{Hom}(R\Gamma(X, F)_{\mathbb{R}}, \mathbb{R})$$

*is an isomorphism.*

*Proof.* The map  $R\Gamma_{c, \mathcal{D}}(X, (-)_{\mathbb{R}}^D) \rightarrow R\text{Hom}(R\Gamma(X, -)_{\mathbb{R}}, \mathbb{R})$  is a natural transformation between exact functors which commute with filtered colimits. Any complex  $F \in D^+(X_{\text{et}})$  is a filtered colimit of bounded complexes; a bounded complex has a filtration by truncations with graded pieces shifts of sheaves; and any sheaf is a filtered colimit of  $\mathbb{Z}$ -constructible sheaves. We thus reduce to the case of a single  $\mathbb{Z}$ -constructible sheaf. The groups  $H^i(X, F)$  differ from  $\hat{H}_c^i(X, F)$  by a finite group since  $X$  is proper, hence are finite type for  $i = 0, 1$  and torsion otherwise, so  $R\Gamma(X, F)_{\mathbb{R}}$  is a perfect complex of  $\mathbb{R}$ -vector spaces.

We now prove the perfectness of the complex  $R\Gamma_{c, \mathcal{D}}(X, F_{\mathbb{R}}^D)$  and of the pairing by Artin induction.

- This is trivial if  $F$  is constructible.
- If  $X$  is regular and  $F = \mathbb{Z}$ , we have  $R\Gamma(X, \mathbb{Z})_{\mathbb{R}} = \mathbb{R}[0]$  and  $R\Gamma_{c, \mathcal{D}}(X, \mathbb{Z}_{\mathbb{R}}^D) = \mathbb{R}[0]$  so the pairing is of the form  $\mathbb{R} \otimes \mathbb{R} \rightarrow \mathbb{R}$ , concentrated in degree 0. By construction, we have a commutative square

$$\begin{array}{ccc} R\Gamma(X, \mathbb{Z})_{\mathbb{R}} \otimes^L R\Gamma_{G_{\mathbb{R}}}(X(\mathbb{C}), \mathbb{R}) & \longrightarrow & R\Gamma(X, \mathbb{Z})_{\mathbb{R}} \otimes^L R\Gamma_{c, \mathcal{D}}(X, \mathbb{Z}_{\mathbb{R}}^D) \\ \downarrow & & \downarrow \\ R\Gamma_{G_{\mathbb{R}}}(X(\mathbb{C}), \mathbb{R}) & \longrightarrow & R\Gamma_{c, \mathcal{D}}(X, \mathbb{Z}_{\mathbb{R}}^D) \end{array}$$

so we get a commutative diagram

$$\begin{array}{ccc} \mathbb{R} \otimes \left( \prod_{\sigma: K \rightarrow \mathbb{C}} \mathbb{R} \right)^{G_{\mathbb{R}}} & \xrightarrow{\text{id} \otimes \Sigma} & \mathbb{R} \otimes \mathbb{R} \\ \downarrow & & \downarrow \\ \left( \prod_{\sigma: K \rightarrow \mathbb{C}} \mathbb{R} \right)^{G_{\mathbb{R}}} & \xrightarrow{\Sigma} & \mathbb{R} \end{array}$$

Since the pairing on the left is multiplication, the pairing on the right is also multiplication, so it is perfect.

- If  $F$  is supported on a closed subscheme, without loss of generality we can suppose that there is a closed point  $i: x \rightarrow X$  and a finite type  $G_x$ -module  $M$  such that  $F = i_*M$ . We have

$$\begin{aligned} R\Gamma(X, i_*M) &\simeq R\Gamma(G_x, M), \\ R\Gamma_{c, \mathcal{D}}(X, (i_*M)_{\mathbb{R}}^D) &\xrightarrow{\simeq} R\Gamma(X, (i_*M)^D)_{\mathbb{R}} = R\Gamma(X, i_*M^{\vee})_{\mathbb{R}} \simeq R\Gamma(G_x, M^{\vee})_{\mathbb{R}}. \end{aligned}$$

The counit  $i_* Ri^! \mathbb{Z}_X^c = i_* \mathbb{Z} \rightarrow \mathbb{Z}_X^c$  induces a map

$$R\Gamma(G_x, \mathbb{Z}) = R\Gamma(X, (i_* \mathbb{Z})^D) \rightarrow R\Gamma(X, \mathbb{Z}_X^c).$$

Since  $\alpha^* i_* = 0$ , this induces a map  $R\Gamma(G_x, \mathbb{Z})_{\mathbb{R}} \rightarrow R\Gamma_{c,D}(X, \mathbb{Z}_{\mathbb{R}}^D)$ . Upon identifying  $\mathbb{Z}$  with  $\mathbb{Z}^{\vee}$ , this map is the map  $R\Gamma_{c,D}(X, (i_* \mathbb{Z})_{\mathbb{R}}^D) \rightarrow R\Gamma_{c,D}(X, \mathbb{Z}_{\mathbb{R}}^D)$  induced by  $\mathbb{Z} \rightarrow i_* \mathbb{Z}$ . Let  $j : U := X \setminus x \rightarrow X$  denote the open immersion; the short exact sequence  $0 \rightarrow j_! \mathbb{Z} \rightarrow \mathbb{Z} \rightarrow i_* \mathbb{Z} \rightarrow 0$  gives a  $3 \times 3$  diagram:

$$\begin{array}{ccccc} R\Gamma_{c,D}(X, (j_! \mathbb{Z})_{\mathbb{R}}^D[-1]) & \longrightarrow & R\Gamma_{c,D}(X, (i_* \mathbb{Z})_{\mathbb{R}}^D) & \longrightarrow & R\Gamma_{c,D}(X, \mathbb{Z}_{\mathbb{R}}^D) \\ \downarrow & & \parallel & & \downarrow \\ R\Gamma(U, \mathbb{Z}_U^c[-1])_{\mathbb{R}} & \longrightarrow & R\Gamma(X, i_* \mathbb{Z})_{\mathbb{R}} & \longrightarrow & R\Gamma(X, \mathbb{Z}_X^c)_{\mathbb{R}} \\ \downarrow & & \downarrow & & \downarrow \\ R\Gamma_{G_{\mathbb{R}}}(X(\mathbb{C}), \mathbb{R}) & \longrightarrow & 0 & \longrightarrow & R\Gamma_{G_{\mathbb{R}}}(X(\mathbb{C}), \mathbb{R}[1]) \end{array}$$

The map  $R\Gamma_{c,D}(X, (i_* \mathbb{Z})_{\mathbb{R}}^D) \rightarrow R\Gamma_{c,D}(X, \mathbb{Z}_{\mathbb{R}}^D)$  is given in degree 0 by a map  $\mathbb{R} \rightarrow \mathbb{R}$  which is computed as the boundary map  $\delta$  in the following snake diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & H_{c,D}^{-1}(X, \mathbb{Z}_{\mathbb{R}}^D) & \longrightarrow & H_{c,D}^{-1}(X, (j_! \mathbb{Z})_{\mathbb{R}}^D) & \longrightarrow & \mathbb{R} \\ & & \downarrow & & \downarrow & & \parallel \\ 0 & \longrightarrow & \mathrm{CH}_0(X, 1)_{\mathbb{R}} & \longrightarrow & \mathrm{CH}_0(U, 1)_{\mathbb{R}} & \xrightarrow{\mathrm{ord}_v} & \mathbb{R} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ \delta & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \left( \prod_{\sigma: K \rightarrow \mathbb{C}} \mathbb{R} \right)^{G_{\mathbb{R}}} & \xrightarrow{=} & \left( \prod_{\sigma: K \rightarrow \mathbb{C}} \mathbb{R} \right)^{G_{\mathbb{R}}} & \longrightarrow & 0 \longrightarrow 0 \\ & & \downarrow \Sigma & & \downarrow & & \downarrow \\ \delta & & \downarrow & & \downarrow & & \downarrow \\ & \longrightarrow & H_{c,D}^0(X, \mathbb{Z}_{\mathbb{R}}^D) = \mathbb{R} & \longrightarrow & H_{c,D}^0(X, (j_! \mathbb{Z})_{\mathbb{R}}^D) & \longrightarrow & 0 \longrightarrow 0 \end{array}$$

Let  $f \in \mathrm{CH}_0(U, 1)_{\mathbb{R}}$  such that  $\mathrm{ord}_v(f) = 1$ ; we also have  $\mathrm{ord}_w(f) = 0$  for  $w \in U$  by definition. Then

$$\delta(1) = \Sigma \circ \mathrm{Log}(f) = -\log N(v)$$

by the product formula. From Diagram IV.3, we obtain the commutativity of the upper left rectangle in the following commutative diagram.

$$\begin{array}{ccccc} R\Gamma(X, i_* M)_{\mathbb{R}} \otimes^L R\Gamma_{c,D}(X, (i_* M)_{\mathbb{R}}^D) & \longrightarrow & R\Gamma_{c,D}(X, \mathbb{Z}_{\mathbb{R}}^D) & \xrightarrow{\tau \geq 0} & \mathbb{R}[0] \\ \simeq \downarrow & & \downarrow & \nearrow & \uparrow \\ R\Gamma(X, i_* M)_{\mathbb{R}} \otimes^L R\Gamma(X, (i_* M)_{\mathbb{R}}^D) & \longrightarrow & R\Gamma(X, \mathbb{Z}_X^c)_{\mathbb{R}} & & \uparrow -\log N(v) \\ \simeq \uparrow & & \uparrow & \searrow & \\ R\Gamma(G_x, M)_{\mathbb{R}} \otimes^L R\Gamma(G_x, M^{\vee})_{\mathbb{R}} & \longrightarrow & R\Gamma(G_x, \mathbb{Z})_{\mathbb{R}} & \xrightarrow{\tau \geq 0} & \mathbb{R}[0] \end{array}$$

The perfectness of the pairing reduces to that of the natural pairing

$$R\Gamma(G_x, M)_{\mathbb{R}} \otimes^L R\Gamma(G_x, M^{\vee})_{\mathbb{R}} \rightarrow \mathbb{R}$$

coming from  $M \otimes^L M^\vee \rightarrow \mathbb{Z}$ . It was shown in § II.4.2.4 that this latter pairing is perfect<sup>33</sup>.

- Let  $Y = \text{Spec}(\mathcal{O})$  be the spectrum of an order in a number field with a finite dominant morphism  $\pi : Y \rightarrow X$ , and suppose  $F = \pi_* G$  for a  $\mathbb{Z}$ -constructible sheaf  $G$  on  $Y$ . Denote by  $\pi' : Y(\mathbb{C}) \rightarrow X(\mathbb{C})$  the induced morphism. The counit  $\pi_* \mathbb{Z}_Y^c \rightarrow \mathbb{Z}_X^c$  is sent under  $\alpha^*$  to the counit  $\pi'_* \overline{\mathbb{Q}}^\times[1] \rightarrow \overline{\mathbb{Q}}^\times[1]$ , and we have the canonical maps  $\overline{\mathbb{Q}}^\times[1] \rightarrow \mathbb{C}^\times[1] \xrightarrow{\log} \mathbb{R}[1]$ . We obtain formally a morphism  $R\Gamma_{c,\mathcal{D}}(Y, \mathbb{Z}_\mathbb{R}^D) \rightarrow R\Gamma_{c,\mathcal{D}}(X, \mathbb{Z}_\mathbb{R}^D)$  and isomorphisms  $R\Gamma_{c,\mathcal{D}}(Y, G_\mathbb{R}^D) \xrightarrow{\cong} R\Gamma_{c,\mathcal{D}}(X, F_\mathbb{R}^D)$  making the following diagram of pairing commute

$$\begin{array}{ccccc} R\Gamma(Y, G)_\mathbb{R} \otimes^L R\Gamma_{c,\mathcal{D}}(Y, G_\mathbb{R}^D) & \longrightarrow & R\Gamma_{c,\mathcal{D}}(Y, \mathbb{Z}_\mathbb{R}^D) & \xrightarrow{\tau \geq 0} & \mathbb{R}[0] \\ \downarrow \simeq & & \downarrow & & \downarrow \\ R\Gamma(X, F)_\mathbb{R} \otimes^L R\Gamma_{c,\mathcal{D}}(X, F_\mathbb{R}^D) & \longrightarrow & R\Gamma_{c,\mathcal{D}}(X, \mathbb{Z}_\mathbb{R}^D) & \xrightarrow{\tau \geq 0} & \mathbb{R}[0] \end{array}$$

The rightmost map is determined by the following commutative diagram (coming from the defining long exact cohomology sequences)

$$\begin{array}{ccc} (\prod_{\tau:K(Y) \rightarrow \mathbb{C}} \mathbb{R})^{G_\mathbb{R}} & \xrightarrow{\Sigma} \twoheadrightarrow & H_{c,\mathcal{D}}^0(Y, \mathbb{Z}_\mathbb{R}^D) = \mathbb{R} \\ \downarrow & & \downarrow \\ (\prod_{\sigma:K \rightarrow \mathbb{C}} \mathbb{R})^{G_\mathbb{R}} & \xrightarrow{\Sigma} \twoheadrightarrow & H_{c,\mathcal{D}}^0(X, \mathbb{Z}_\mathbb{R}^D) = \mathbb{R} \end{array}$$

where the left map sums components corresponding to embeddings of  $K(Y)$  restricting to the same embedding of  $K$ . Thus the rightmost map is the identity and the pairing for  $F$  is perfect if and only if the pairing for  $G$  is perfect.  $\square$

*Remark.* Using duality for Deligne cohomology, it should be possible to reduce the theorem to the duality theorem Theorem II.4.4 similarly to how we reduced Artin–Verdier duality for  $F^D$  to Artin–Verdier duality for  $F$ .

## IV.6 The $L$ -function of $F^D$

For any scheme  $S$ , denote by  $\nu : \text{Sh}(S_{\text{proet}}) \rightarrow \text{Sh}(S_{\text{et}})$  the natural morphism of topoi; the left adjoint  $\nu^*$  is fully faithful [BS15, 5.1.2].

**Definition IV.6.1.** *For each closed point  $x$  of  $X$ , let  $\ell_x$  be a prime number coprime to the residual characteristic at  $x$  and  $\varphi$  be the geometric frobenius in  $G_x$ . Denote by  $-\widehat{\otimes}_{\mathbb{Q}_\ell} = (R\lim(- \otimes^L \mathbb{Z}/\ell^n \mathbb{Z})) \otimes \mathbb{Q}$  the completed tensor product with  $\mathbb{Q}_\ell$ , with the derived limit computed on the proétale site.*

Let  $F$  be a  $\mathbb{Z}$ -constructible sheaf on  $X$ . We define the  $L$ -function of  $F^D$  by the Euler product

$$L_X(F^D, s) = \prod_{x \in X_0} \det(I - N(x)^{-s} \varphi | (\nu^*(i_x^* F^D)) \widehat{\otimes}_{\mathbb{Q}_\ell}^{-1}).$$

We will show that the  $L$ -function doesn't depend on the choice of the prime numbers  $(\ell_x)_{x \in X_0}$ .

<sup>33</sup>This also reduces by Artin induction, after showing compatibility with induction, to the perfectness for  $M = \mathbb{Z}$ .

## IV.6.1 Explicit computation

In this section we compute more explicitly the  $L$ -function of  $F^D$ . Denote by  $\pi : Y \rightarrow X$  the normalization of  $X$  and let  $i : x \rightarrow X$  be a closed point. Put  $\bar{x} = \text{Spec}(\kappa(x)^{\text{sep}})$ ,  $G_x = \text{Gal}(\kappa(x)^{\text{sep}}/\kappa(x))$ , and let  $\mathcal{O}_x^h$  (resp.  $\mathcal{O}_x^{sh}$ ) denote the henselian (resp. strict henselian) local ring at  $x$ . For each closed point  $y \in Y$  above  $x$ , we will consider similarly  $G_y = \text{Gal}(\kappa(y)^{\text{sep}}/\kappa(y))$ ,  $\mathcal{O}_y^h$ ,  $\mathcal{O}_y^{sh}$ , and moreover  $K_y^h = \text{Frac}(\mathcal{O}_y^h)$  (resp.  $K_y^{sh}$ ) the henselian local field at  $y$  (resp. its maximal unramified extension). Fix an embedding  $K_y^h \hookrightarrow K^{\text{sep}}$ . This determines an inertia group  $I_y$  inside  $G_K$  which is the absolute Galois group of  $K_y^{sh}$ . If  $G$  is a topological group let  $\underline{G}$  denote the associated condensed group.

Let us fix some conventions. If  $L(M, s)$  is an  $L$ -function attached to some object  $M$ <sup>34</sup>, defined by an Euler product over closed points of  $X$ , denote by  $L_x(M, s)$  for the local factor at a closed point  $x \in X$ . If  $N$  is a discrete  $G_x$ -module with free of finite type underlying abelian group, or a rational or  $\ell$ -adic  $G_x$ -representation of finite dimension, put

$$Q_x(N, s) := \det(I - N(x)^{-s} \varphi|N)^{-1}$$

with  $\varphi$  the geometric Frobenius at  $x$ .

**Theorem IV.6.2.** *Let  $F$  be a  $\mathbb{Z}$ -constructible sheaf on the spectrum  $X$  of an order in a number field. The local factor at  $x$  of the  $L$ -function of  $F^D$  is*

$$L_x(F^D, s) = \frac{\prod_{\pi(y)=x} Q_y((F_\eta^\vee)^{I_y}, s+1) Q_x(F_x^\vee, s)}{\prod_{\pi(y)=x} Q_y((F_\eta^\vee)^{I_y}, s)}.$$

The proof is divided into the following several lemmas below. We mention first some important consequences:

**Corollary IV.6.3.**

- If  $F$  is locally constant around the regular closed point  $v$ , then  $I_v$  acts trivially on  $F_\eta$ ,  $F_\eta = F_v$  and we find that the local factor at  $v$  of  $F^D$  equals the local factor at  $v$  of the Artin  $L$ -function of  $F_\eta \otimes \mathbb{Q}$  at  $s+1$ :<sup>35</sup>

$$L_v(F^D, s) = L_v(F_\eta \otimes \mathbb{Q}, s+1)$$

- Each local factor is well-defined, independently of the choice of a prime number  $\ell$  coprime to the residual characteristic.
- The  $L$ -function of  $F^D$  differs from the Artin  $L$ -function  $L_K(F_\eta \otimes \mathbb{Q}, s+1)$  by a finite number of factors and is thus well-defined
- $L_X(F^D, s)$  is meromorphic.

Denote by  $T$  the torus over  $K$  with character group  $Y := F_\eta/\text{tor}$ . For any prime  $\ell$ , define the rational  $\ell$ -adic Tate module of  $T$  as

$$V_\ell(T) := (\lim T[\ell^n]) \otimes \mathbb{Q}.$$

<sup>34</sup>which kind of  $L$ -function we will try to make clear each time from the context

<sup>35</sup>Notice that a rational representation of a finite group is self-dual because its character takes real values hence is its own conjugate

It is a finite dimensional  $\ell$ -adic representation of  $G_K$ . We know that

$$V_l(T) \simeq Y^\vee \otimes \mathbb{Q}_\ell(1) = F_\eta^\vee \otimes \mathbb{Q}_\ell(1)$$

as  $\ell$ -adic representations.

**Definition IV.6.4** ([Fon92, § 8]). *The  $L$ -function  $L_K(T, s)$  of  $T$  is the  $L$ -function of the 1-motive  $[0 \rightarrow T]$  over  $K$ , defined by the Euler product:*

$$L_K(T, s) = \prod_{x \in X_0} \det(I - N(x)^{-s} \varphi|_{V_{\ell_x}(T)^{I_x}})^{-1},$$

where  $\ell_x$  is a prime number coprime to the residual characteristic at  $x$ .

By the above, the  $L$ -function of  $T$  is also the Artin  $L$ -function at  $s+1$  of  $Y \otimes \mathbb{Q} = F_\eta \otimes \mathbb{Q}$ , and it doesn't depend on the choice of the family  $(\ell_x)$ .

*Remark.* Our definition differs slightly from the definition in [GS20]: we use the  $L$ -function of the 1-motive  $[0 \rightarrow T]$ , involving  $V_l(T)$ , while Geisser–Suzuki use the Hasse-Weil  $L$ -function of  $T$  involving  $V_l(T)(-1)$ .

**Proposition IV.6.5.** *Suppose that  $v$  is a regular closed point. Let  $M$  be a discrete  $D_v$ -module of finite type. There is a canonical  $G_v$ -equivariant isomorphism*

$$M^{I_v} \otimes \mathbb{Q} \xrightarrow{\simeq} M_{I_v} \otimes \mathbb{Q}.$$

*Proof.* The action of  $D_v$  on  $M$  factors through a finite quotient  $G$ ; denote by  $H$  the image of  $I_v$  in  $G$ . Then the canonical composite map

$$f : M^H \rightarrow M \rightarrow M_H$$

is  $G$ -equivariant hence also  $G/H$ -equivariant, and if  $N$  denotes the norm  $\sum_{h \in H} h$  we have  $fN = [H]\text{Id}$  and  $Nf = [H]\text{Id}$ . The result follows.  $\square$

**Corollary IV.6.6.**

- *Suppose that the canonical map  $F_v \rightarrow F_\eta^{I_v}$  is an isomorphism for a regular closed point  $v$ . Then the local factor at  $v$  of  $F^D$  is  $L_v(F_\eta \otimes \mathbb{Q}, s+1) = L_v(T, s)$ .*
- *Suppose that  $X$  is regular. We have  $L_X((g_*Y)^D, s) = L_K(T, s) = L_K(Y \otimes \mathbb{Q}, s+1)$ .*
- *Suppose that  $X$  is regular and denote by  $\mathcal{T}^0$  the connected (lft) Néron model of  $T$  on  $X$ , seen as an étale sheaf on  $X$ . We can define an  $L$ -function  $L_X(\mathcal{T}^0, s)$  for  $\mathcal{T}^0$  with local factor at  $i : x \rightarrow X$  given by  $Q_x((\nu^*i^*\mathcal{T}^0) \widehat{\otimes} \mathbb{Q}_\ell, s)$ . Then*

$$L_X(\mathcal{T}^0, s) = L_K(T, s)^{-1}.$$

*Proof.* We have a canonical  $G_v$ -equivariant isomorphism

$$\begin{aligned} (F_\eta^\vee)^{I_v} \otimes \mathbb{Q} &= \text{Hom}_{I_v}(F_\eta \otimes \mathbb{Q}, \mathbb{Q}) = \text{Hom}_{\text{Ab}}((F_\eta)_{I_v} \otimes \mathbb{Q}, \mathbb{Q}) \simeq \text{Hom}_{\text{Ab}}(F_\eta^{I_v} \otimes \mathbb{Q}, \mathbb{Q}) \\ &\simeq F_v^\vee \otimes \mathbb{Q}, \end{aligned}$$

hence the local factor at  $v$  of the  $L$ -function of  $F^D$  simplifies to  $L_v(F_\eta \otimes \mathbb{Q}, s+1)$ .

The second point is an immediate consequence of the first. Let us prove the last point. We have [GM21, 2.2]

$$\mathcal{T}^0 = R\mathcal{H}om_X(\tau^{\leq 1}Rg_*Y, \mathbb{G}_m)$$

as étale sheaves. We claim that  $R^1g_*Y$  is skyscraper constructible: indeed  $(R^1g_*Y)_{\bar{\eta}}$  vanishes, while  $(R^1g_*Y)_{\bar{v}} = H^1(K_v^{sh}, Y)$  which is finite, and zero if the inertia subgroup  $I_v$  acts trivially on  $Y$  since  $Y$  is free as an abelian group<sup>36</sup>. As there are only a finite number of points where the inertia acts non-trivially, we obtain the claim. Therefore  $R^1g_*Y$  is killed by an integer  $N$  and thus

$$(\nu^*i^*\mathcal{T}^0) \widehat{\otimes}_{\mathbb{Q}_\ell} = (\nu^*i^*(g_*Y)^D) \widehat{\otimes}_{\mathbb{Q}_\ell}[-1].$$

The result follows.  $\square$

We now prove Theorem IV.6.2, in the following series of lemma.

**Lemma IV.6.7.** *Let  $F$  be an étale sheaf on  $X$ , and let  $i : x \rightarrow X$  be a closed point. There is a fiber sequence*

$$R\mathcal{H}om_x(F_x, \mathbb{Z}) \rightarrow i^*F^D \rightarrow \prod_{\pi(y)=x} \text{ind}_{G_y}^{G_x} R\text{Hom}_{\text{Spec}(K_y^{sh})}(F_\eta, \mathbb{G}_m[1]).$$

*Remark.* We abused notation by identifying what should be the right term with its underlying complex of abelian groups.

*Proof.* Put  $X(x) := \text{Spec}(\mathcal{O}_x^h)$ ,  $\eta(x) := \text{Spec}(\mathcal{O}_x^h) \times_X \text{Spec}(K)$  and let  $f : X(x) \rightarrow X$ ,  $g : \eta \rightarrow X$  denote the canonical morphisms. We consider the cartesian diagram

$$\begin{array}{ccc} \eta(x) & \xrightarrow{g'} & X(x) \\ f' \downarrow & & \downarrow f \\ \eta & \xrightarrow{g} & X \end{array}$$

Denote again by  $i : x \rightarrow X(x)$  the closed immersion. Since  $f$  is (ind-)étale, we have

$$i^*F^D = i^*f^*R\mathcal{H}om_X(F, \mathbb{G}_X[1]) = i^*R\mathcal{H}om_{X(x)}(F, \mathbb{G}_{X(x)}[1]).$$

By Proposition E.0.5, we have

$$\eta(x) = \prod_{\pi(y)=x} \prod_{\text{Gal}(\kappa(y)/\kappa(x))} \text{Spec}(K_y^{sh})$$

and thus

$$(Rg'_*\mathbb{G}_m)_{\bar{x}} = R\Gamma(\eta(x), \mathbb{G}_m) = \prod_{\pi(y)=x} \text{ind}_{G_y}^{G_x} R\Gamma(K_y^{sh}, \mathbb{G}_m) = \prod_{\pi(y)=x} \text{ind}_{G_y}^{G_x} (K_y^{sh})^\times,$$

hence  $g'_*\mathbb{G}_m = Rg'_*\mathbb{G}_m$ . Moreover there is by definition a fiber sequence

$$\mathbb{G}_{X(x)} \rightarrow g'_*\mathbb{G}_m \rightarrow i_*\mathbb{Z}$$

---

<sup>36</sup>This is the crucial point: if  $Y$  is finite then  $R^1g_*Y$  is skyscraper with finite stalks but has non-zero stalk almost everywhere; for instance by Hensel's lemma the stalk of  $R^1g_*\mathbb{Z}/n\mathbb{Z}$  at a point with characteristic coprime to  $n$  is  $\mathbb{Z}/n\mathbb{Z}$ .

so we obtain a fiber sequence

$$R\mathcal{H}om_x(F_x, \mathbb{Z}) \rightarrow i^* F^D \rightarrow i^* Rg'_* R\mathcal{H}om_{\eta(x)}(g'^* F, \mathbb{G}_m[1]).$$

Using again Proposition E.0.5, we see that the right term is the following complex of abelian groups with its natural  $G_x$ -module structure:

$$(Rg'_* R\mathcal{H}om_{\eta(x)}(g'^* F, \mathbb{G}_m[1]))_{\bar{x}} = \prod_{\pi(y)=x} \text{ind}_{G_y}^{G_x} R\text{Hom}_{\text{Spec}(K_y^{sh})}(F_\eta, \mathbb{G}_m[1]).$$

□

From now on, we suppose that  $F$  is a  $\mathbb{Z}$ -constructible sheaf on  $X$ .

**Lemma IV.6.8.** *We have*

$$(\nu^* R\mathcal{H}om_x(F_x, \mathbb{Z})) \widehat{\otimes} \mathbb{Q}_\ell = (\nu^* \mathcal{H}om_x(F_x, \mathbb{Z})) \otimes \mathbb{Q}_\ell.$$

*Proof.* We have

$$\nu^* R\mathcal{H}om_x(F_x, \mathbb{Z}) = R\mathcal{H}om_{x_{\text{proet}}}(\nu^* F_x, \mathbb{Z}).$$

The functor  $- \otimes^L \mathbb{Z}/\ell^n \mathbb{Z}$  is the cofiber of the map  $(-) \xrightarrow{\ell^n} (-)$  so it commutes with exact functors, and we find

$$R\mathcal{H}om_{x_{\text{proet}}}(\nu^* F_x, \mathbb{Z}) \widehat{\otimes} \mathbb{Z}_\ell = R\mathcal{H}om_{x_{\text{proet}}}(\nu^* F_x, R\lim \mathbb{Z}/\ell^n \mathbb{Z}).$$

The constant sheaf functor is exact so the transition maps  $\mathbb{Z}/\ell^{n+1} \mathbb{Z} \rightarrow \mathbb{Z}/\ell^n \mathbb{Z}$  are surjective, hence we have  $R\lim \mathbb{Z}/\ell^n \mathbb{Z} = \lim \mathbb{Z}/\ell^n \mathbb{Z} =: \mathbb{Z}_\ell$  [BS15, 3.1.10].

Under the identification  $x_{\text{proet}} \simeq \underline{G}_x\text{-Cond}(\text{Set})$ , the sheaf  $\mathcal{E}xt_{x_{\text{proet}}}^i(\nu^* F_x, \mathbb{Z}_\ell)$  is the condensed abelian group  $\mathcal{E}xt_{\text{Cond}(\text{Ab})}^i(\nu^* F_x, \mathbb{Z}_\ell)$  with its natural  $\underline{G}_x$ -action; since both  $\nu^* F_x$  and  $\mathbb{Z}_\ell$  are locally compact abelian groups<sup>37</sup>, we find  $\mathcal{E}xt_{x_{\text{proet}}}^i(\nu^* F_x, \mathbb{Z}_\ell) = 0$  for  $i \geq 2$  [CS19, Remark after 4.9]. Let us show that  $\mathcal{E}xt_{x_{\text{proet}}}^1(\nu^* F_x, \mathbb{Z}_\ell)$  is killed by some integer  $N$ . We have  $R\mathcal{H}om_{x_{\text{proet}}}(\nu^* F_x, \mathbb{Z}_\ell) = R\lim R\mathcal{H}om_{x_{\text{proet}}}(\nu^* F_x, \nu^* \mathbb{Z}/\ell^n \mathbb{Z}) = R\lim \nu^* R\mathcal{H}om_x(F_x, \mathbb{Z}/\ell^n \mathbb{Z})$  so there is a short exact sequence

$$0 \rightarrow R^1 \lim \nu^* \mathcal{H}om_x(F_x, \mathbb{Z}/\ell^n \mathbb{Z}) \rightarrow \mathcal{E}xt_{x_{\text{proet}}}^1(\nu^* F_x, \mathbb{Z}_\ell) \rightarrow \lim \nu^* \mathcal{E}xt_x^1(F_x, \mathbb{Z}/\ell^n \mathbb{Z}) \rightarrow 0.$$

The underlying abelian group of the  $G_x$ -module  $\mathcal{H}om_x(F_x, \mathbb{Z}/\ell^n \mathbb{Z})$  is  $\text{Hom}_{\text{Ab}}(F_x, \mathbb{Z}/\ell^n \mathbb{Z})$ , so the transition maps  $\mathcal{H}om_x(F_x, \mathbb{Z}/\ell^{n+1} \mathbb{Z}) \rightarrow \mathcal{H}om_x(F_x, \mathbb{Z}/\ell^n \mathbb{Z})$  are eventually surjective: if  $F_x$  is torsion-free this is clear, and if  $F_x$  is torsion then the groups are constant for  $n$  greater than the  $\ell$ -adic valuation of the order of  $F_x$ . Thus the left term is zero by Lemma D.0.3. Let  $k$  be the  $\ell$ -adic valuation of  $(F_x)_{\text{tor}}$ ; then  $\mathcal{E}xt_x^1(F_x, \mathbb{Z}/\ell^n \mathbb{Z}) = \mathcal{E}xt_x^1((F_x)_{\text{tor}}, \mathbb{Z}/\ell^n \mathbb{Z})$  is killed by  $N = \ell^k$ , and therefore so is  $\mathcal{E}xt_{x_{\text{proet}}}^1(\nu^* F_x, \mathbb{Z}_\ell)$ .

From the previous point, it follows by tensoring with  $\mathbb{Q}$  that

$$R\mathcal{H}om_{x_{\text{proet}}}(\nu^* F_x, \mathbb{Z}) \widehat{\otimes} \mathbb{Q}_\ell = \mathcal{H}om_{x_{\text{proet}}}(\nu^* F_x, \mathbb{Z}_\ell) \otimes \mathbb{Q}$$

There are canonical maps

$$\mathcal{H}om_{x_{\text{proet}}}(\nu^* F_x, \mathbb{Z}_\ell) \otimes \mathbb{Q} \rightarrow \mathcal{H}om_{x_{\text{proet}}}(\nu^* F_x, \mathbb{Q}_\ell) \leftarrow \mathcal{H}om_{x_{\text{proet}}}(\nu^* F_x, \mathbb{Z}) \otimes \mathbb{Q}_\ell$$

---

<sup>37</sup>The former is even discrete

We have  $\text{Ab}(\text{Sh}(x_{\text{proet}})) = \text{Ab}(\text{Sh}((BG_x)_{\text{proet}})) = \underline{G}_x - \text{Cond}(\text{Ab}) = \mathbb{Z}[\underline{G}_x] - \text{Mod}$ , the category of modules under the condensed ring  $\mathbb{Z}[\underline{G}_x]$  Proposition D.0.8. To conclude it suffices to show that both maps are isomorphisms. Let  $G$  be a finite quotient of  $G_x$ . We check it first for  $F_x = \mathbb{Z}[G]$ . This is easily done using next lemma.

By a standard argument it thus suffices to show that  $\nu^*F_x$  is (globally) of finite presentation as a  $\mathbb{Z}[\underline{G}]$ -module for some finite quotient  $G$  of  $G_x$ ; this is clear as  $F_x$  is discrete and of finite type as an abelian group.  $\square$

**Lemma IV.6.9.** *Let  $G$  be a finite quotient of  $G_x$ . Then  $\mathbb{Z}[\underline{G}]$  is a left  $\underline{G}_x$ -module such that*

$$\mathcal{H}om_{x_{\text{proet}}}(\mathbb{Z}[\underline{G}], -) = \mathbb{Z}[\underline{G}] \otimes -.$$

*Proof.* We have  $\mathbb{Z}[\underline{G}] = \underline{\mathbb{Z}}[G] = \bigoplus_G \underline{\mathbb{Z}}$ , thus

$$\mathcal{H}om_{x_{\text{proet}}}(\mathbb{Z}[\underline{G}], -) = \bigoplus_G \mathcal{H}om_{x_{\text{proet}}}(\underline{\mathbb{Z}}, -) = \bigoplus_G \text{Id} = \mathbb{Z}[\underline{G}] \otimes -.$$

$\square$

Denote by  $R\text{Hom}_{Y_{\text{proet}}}$  the enriched  $R\text{Hom}$  on the proétale site of a scheme, a complex of condensed abelian groups with underlying complex of abelian groups  $R\text{Hom}_Y$ .

**Lemma IV.6.10.** *We have*

$$(\nu^* R\text{Hom}_{\text{Spec}(K_y^{sh})}(F_\eta, \mathbb{G}_m[1])) \widehat{\otimes} \mathbb{Z}_\ell = R\text{Hom}_{\text{Spec}(K_y^{sh})_{\text{proet}}}(\nu^* F_\eta, \mathbb{Z}_\ell(1)[2]).$$

*Proof.* We have

$$\nu^* R\text{Hom}_{\text{Spec}(K_y^{sh})}(F_\eta, \mathbb{G}_m[1]) = R\text{Hom}_{\text{Spec}(K_y^{sh})_{\text{proet}}}(\nu^* F_\eta, \nu^* \mathbb{G}_m[1]).$$

Thus we find

$$\nu^* R\text{Hom}_{\text{Spec}(K_y^{sh})}(F_\eta, \mathbb{G}_m[1]) \widehat{\otimes} \mathbb{Z}_\ell = R\text{Hom}_{\text{Spec}(K_y^{sh})_{\text{proet}}}(\nu^* F_\eta, (\nu^* \mathbb{G}_m[1]) \widehat{\otimes} \mathbb{Z}_\ell).$$

Since  $\ell$  is invertible on  $\text{Spec}(K_y^{sh})$ , we have  $\mathbb{G}_m \otimes^L \mathbb{Z}/\ell^n \mathbb{Z} = \mu_{\ell^n}[1]$  and thus

$$(\nu^* \mathbb{G}_m[1]) \widehat{\otimes} \mathbb{Z}_\ell = R\text{lim } \nu^* \mu_{\ell^n}[2] = \mathbb{Z}_\ell(1)[2].$$

$\square$

**Lemma IV.6.11.** *Let  $H$  be an open normal subgroup of  $I_y$  acting trivially on  $F_\eta$ , and let  $G := I_y/H$ . Denote by  $R\text{Hom}_{\underline{G}}$  the enriched  $R\text{Hom}$  between condensed  $\underline{G}$ -modules. There is a fiber sequence*

$$R\text{Hom}_{\underline{G}}(\nu^* F_\eta, \mathbb{Z}_\ell(1)) \rightarrow R\text{Hom}_{\text{Spec}(K_y^{sh})_{\text{proet}}}(\nu^* F_\eta, \mathbb{Z}_\ell(1)) \rightarrow R\text{Hom}_{\underline{G}}(\nu^* F_\eta, \mathbb{Z}_\ell)[-1].$$

*Proof.* We have  $\text{Spec}(K_y^{sh})_{\text{proet}} = (BI_y)_{\text{proet}} = \underline{I}_y\text{-Cond}(\text{Set})$  by Proposition D.0.8. The functor  $R\underline{\Gamma}(H, -) := R\text{Hom}_{\underline{H}}(\mathbb{Z}, -)$  is right adjoint to the forgetful functor

$$D(\underline{I}_y\text{-Cond}(\text{Ab})) \rightarrow D(\underline{G}\text{-Cond}(\text{Ab})),$$

so we have

$$R\text{Hom}_{\text{Spec}(K_y^{sh})_{\text{proet}}}(\nu^* F_\eta, \mathbb{Z}_\ell(1)) = R\text{Hom}_{\underline{G}}(\nu^* F_\eta, R\underline{\Gamma}(H, \mathbb{Z}_\ell(1))).$$

We compute

$$R\underline{\Gamma}(H, \mathbb{Z}_\ell(1)) = R\text{lim } R\underline{\Gamma}(H, \nu^* \mu_{\ell^n}) = R\text{lim } \nu^* R\underline{\Gamma}(H, \mu_{\ell^n}).$$



Denote by  $F$  the finite extension of  $K_y^{sh}$  corresponding to  $H$ . Since  $\ell$  is prime to the residual characteristic of  $K_y^{sh}$ , all  $\ell^n$ -th roots of unity are contained in  $K_y^{sh}$  and thus also in  $H$ . Hensel's lemma together with the short exact sequence  $0 \rightarrow \mathcal{O}_F^\times \rightarrow F^\times \rightarrow \mathbb{Z} \rightarrow 0$  then gives  $H^i(H, \mu_{\ell^n}) = \mu_{\ell^n}, \mathbb{Z}/\ell^n\mathbb{Z}$  for  $i = 0, 1$ , while  $H^i(H, \mu_{\ell^n}) = 0$  for  $i \geq 2$  because  $I_y$  is of cohomological dimension 1 and therefore also  $H$  [Har20, 8.11 (b), 5.10]. Thus we have a fiber sequence of discrete  $G$ -modules

$$\mu_{\ell^n} \rightarrow R\Gamma(H, \mu_{\ell^n}) \rightarrow \mathbb{Z}/\ell^n\mathbb{Z}[-1].$$

The transition maps on both sides are surjective. Applying the fully faithful exact functor  $\nu^*$ , we can compute the  $R$ lim using that the transition maps are again surjective and we obtain a fiber sequence in  $D(\underline{G}\text{-Cond}(Ab))$ :

$$\mathbb{Z}_\ell(1) \rightarrow R\underline{\Gamma}(H, \mathbb{Z}_\ell(1)) \rightarrow \mathbb{Z}_\ell[-1] \quad \square.$$

Since  $K_y^{sh}$  has all  $\ell^n$ -th roots of unity, as  $\underline{G}$ -modules we have  $\mathbb{Z}_\ell \simeq \mathbb{Z}_\ell(1)$ .

**Lemma IV.6.12.** *We have*

$$\begin{aligned} R\underline{\text{Hom}}_{\underline{G}}(\nu^*F_\eta, \mathbb{Z}_\ell) \otimes \mathbb{Q} &= \nu^* \text{Hom}_G(F_\eta, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell = \nu^*(F_\eta^\vee)^{I_y} \otimes \mathbb{Q}_\ell, \\ R\underline{\text{Hom}}_{\underline{G}}(\nu^*F_\eta, \mathbb{Z}_\ell(1)) \otimes \mathbb{Q} &= \nu^*(F_\eta^\vee)^{I_y} \otimes \mathbb{Q}_\ell(1). \end{aligned}$$

*Proof.* The proof of both statements is similar, so we treat only the first one.

We first show that  $\underline{\text{Ext}}_{\underline{G}}^i(\nu^*F_\eta, \mathbb{Z}_\ell)$  is killed by some integer  $N$  for all  $i \geq 1$ . There is a spectral sequence giving short exact sequences

$$0 \rightarrow R^1 \lim \nu^* \text{Ext}_G^{i-1}(F_\eta, \mathbb{Z}/\ell^n\mathbb{Z}) \rightarrow \underline{\text{Ext}}_{\underline{G}}^i(\nu^*F_\eta, \mathbb{Z}_\ell) \rightarrow \lim \nu^* \text{Ext}_G^i(F_\eta, \mathbb{Z}/\ell^n\mathbb{Z}) \rightarrow 0.$$

Denote by  $\mathcal{H}om_G$  is the internal Hom for discrete  $G$ -modules. There is moreover a spectral sequence giving a long exact seqence

$$\begin{array}{c} \cdots \rightarrow H^i(G, \mathcal{H}om_G(F_\eta, \mathbb{Z}/\ell^n\mathbb{Z})) \rightarrow \text{Ext}_G^i(F_\eta, \mathbb{Z}/\ell^n\mathbb{Z}) \rightarrow H^{i-1}(G, \mathcal{E}xt_G^1(F_\eta, \mathbb{Z}/\ell^n\mathbb{Z})) \\ \longleftarrow \hspace{20em} \longrightarrow \hspace{20em} \\ \longleftarrow \hspace{20em} H^{i+1}(G, \mathcal{H}om_G(F_\eta, \mathbb{Z}/\ell^n\mathbb{Z})) \longrightarrow \cdots \end{array}$$

For  $i \geq 2$ , since

$$\mathcal{H}om_G(F_\eta, \mathbb{Z}/\ell^n\mathbb{Z}) = \text{Hom}_{\text{Ab}}(F_\eta, \mathbb{Z}/\ell^n\mathbb{Z}) \quad \text{and} \quad \mathcal{E}xt_G^1(F_\eta, \mathbb{Z}/\ell^n\mathbb{Z}) = \text{Ext}_{\text{Ab}}^1(F_\eta, \mathbb{Z}/\ell^n\mathbb{Z})$$

are of finite type, both  $H^i(G, \mathcal{H}om_G(F_\eta, \mathbb{Z}/\ell^n\mathbb{Z}))$  and  $H^{i-1}(G, \mathcal{E}xt_G^1(F_\eta, \mathbb{Z}/\ell^n\mathbb{Z}))$  are finite and killed by  $[G]$ , thus  $\text{Ext}_G^i(F_\eta, \mathbb{Z}/\ell^n\mathbb{Z})$  is killed by  $[G]^2$ . For  $i = 1$ , the group  $H^1(G, \mathcal{H}om_G(F_\eta, \mathbb{Z}/\ell^n\mathbb{Z}))$  is finite killed by  $[G]$  and  $H^0(G, \mathcal{E}xt_G^1(F_\eta, \mathbb{Z}/\ell^n\mathbb{Z}))$  is finite killed by  $[(F_\eta)_{\text{tor}}]$ , so  $\text{Ext}_G^1(F_\eta, \mathbb{Z}/\ell^n\mathbb{Z})$  is finite killed by  $[G][(F_\eta)_{\text{tor}}]$ . Finally, for  $i = 0$  we have that  $\text{Hom}_G(F_\eta, \mathbb{Z}/\ell^n\mathbb{Z})$  is a finite group.

Suppose first that  $i \geq 2$ . Then  $\text{Ext}_G^{i-1}(F_\eta, \mathbb{Z}/\ell^n\mathbb{Z})$  is a system of finite groups so it satisfies the Mittag-Leffler condition, thus  $R^1 \lim \nu^* \text{Ext}_G^{i-1}(F_\eta, \mathbb{Z}/\ell^n\mathbb{Z}) = 0$  by Lemmas D.0.3 and D.0.4. On the other hand,  $\text{Ext}_G^i(F_\eta, \mathbb{Z}/\ell^n\mathbb{Z})$  is a system of finite group killed by  $[G]^2$ , thus  $\underline{\text{Ext}}_{\underline{G}}^i(\nu^*F_\eta, \mathbb{Z}_\ell) = \lim \nu^* \text{Ext}_G^i(F_\eta, \mathbb{Z}/\ell^n\mathbb{Z})$  is killed by  $[G]^2$ . We now treat the case  $i = 1$ . Let us show that  $R^1 \lim \nu^* \text{Hom}_G(F_\eta, \mathbb{Z}/\ell^n\mathbb{Z}) = 0$ . Again this is a system of finite groups, which thus satisfies the Mittag-Leffler condition, and we can apply Lemmas D.0.3 and D.0.4. It follows that  $\underline{\text{Ext}}_{\underline{G}}^1(\nu^*F_\eta, \mathbb{Z}_\ell) = \lim \nu^* \text{Ext}_G^1(F_\eta, \mathbb{Z}/\ell^n\mathbb{Z})$  is killed by  $[G][(F_\eta)_{\text{tor}}]$ .

We deduce that

$$R\underline{\mathrm{Hom}}_G(\nu^* F_\eta, \mathbb{Z}_\ell) \otimes \mathbb{Q} = \underline{\mathrm{Hom}}_G(\nu^* F_\eta, \mathbb{Z}_\ell) \otimes \mathbb{Q}.$$

There are canonical maps

$$\begin{aligned} \underline{\mathrm{Hom}}_G(\nu^* F_\eta, \mathbb{Z}_\ell) \otimes \mathbb{Q} &\rightarrow \underline{\mathrm{Hom}}_G(\nu^* F_\eta, \mathbb{Q}_\ell) \leftarrow \underline{\mathrm{Hom}}_G(\nu^* F_\eta, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \\ &= \nu^* \mathrm{Hom}_G(\nu^* F_\eta, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell. \end{aligned}$$

Since  $F_\eta$  is a finite type, hence finite presentation abelian group and  $G$  is finite,  $F_\eta$  is a finite presentation  $G$ -module so  $\nu^* F_\eta$  is (globally) of finite presentation as a  $\underline{G}$ -module<sup>38</sup>. By a standard argument, to show that the canonical maps are isomorphisms we reduce to the case of  $\mathbb{Z}[\underline{G}]$ . But  $\underline{\mathrm{Hom}}_{\underline{G}}(\mathbb{Z}[\underline{G}], -)$  is the forgetful functor  $U : \underline{G} - \mathrm{Cond}(\mathrm{Ab}) \rightarrow \mathrm{Cond}(\mathrm{Ab})$ , thus everything follows from the identifications

$$(U\mathbb{Z}_\ell) \otimes \mathbb{Q} = \mathbb{Z}_\ell \otimes \mathbb{Q} = \mathbb{Q}_\ell = U\mathbb{Q}_\ell = \mathbb{Q} \otimes_{\mathbb{Q}} \mathbb{Q}_\ell = U\mathbb{Q} \otimes_{\mathbb{Q}} \mathbb{Q}_\ell.$$

The second equality in each case follows from the identity  $\mathrm{Hom}_G(F_\eta, \mathbb{Q}) = (F_\eta^\vee)^G \otimes \mathbb{Q} = (F_\eta^\vee)^{I_y} \otimes \mathbb{Q}$ .  $\square$

## IV.6.2 Functoriality

**Proposition IV.6.13.** *Let  $0 \rightarrow F \rightarrow G \rightarrow H \rightarrow 0$  be a short exact sequence of  $\mathbb{Z}$ -constructible sheaves on  $X$ . Then*

$$L_X(G^D, s) = L_X(F^D, s)L_X(H^D, s).$$

*Proof.* If  $i : v \rightarrow X$  is a closed point, the short exact sequence gives a fiber triangle

$$(\nu^* i^* H^D) \hat{\otimes} \mathbb{Q}_\ell \rightarrow (\nu^* i^* G^D) \hat{\otimes} \mathbb{Q}_\ell \rightarrow (\nu^* i^* F^D) \hat{\otimes} \mathbb{Q}_\ell.$$

Local factors are multiplicative with respect to short exact sequences, hence

$$L_v(G^D, s) = L_v(F^D, s)L_v(H^D, s). \quad \square$$

**Proposition IV.6.14.** *Let  $\pi : Y \rightarrow X$  be a finite morphism between spectra of orders in number fields and let  $F$  be a  $\mathbb{Z}$ -constructible sheaf on  $Y$ . We have*

$$L_X((\pi_* F)^D, s) = L_Y(F^D, s).$$

*Proof.* This follows readily from Theorem IV.6.2 using the compatibility of local factors with induction [Neu99a, VII.10.4 (iv) and its proof].  $\square$

We mention the following consequence of the results in Chapter II:

**Proposition IV.6.15.** *Let  $X = \mathrm{Spec}(\mathcal{O})$  be the spectrum of an order in a number field  $K$ , with open subscheme  $j : U \rightarrow X$ . Let  $\omega$  be the number of roots of unity in  $K$ ,  $\Delta_K$  its discriminant,  $r_1$  and  $r_2$  respectively the number of real and complex places of  $K$ , and  $R_U$  the regulator introduced in Corollary II.7.4 for irreducible affine 1-dimensional arithmetic schemes. We have*

$$L_X^*((j_! \mathbb{Z})^D, 0) = \frac{2^{r_1} (2\pi)^{r_2} [\mathrm{CH}_0(U)] R_U}{\omega \sqrt{|\Delta_K|}}.$$

---

<sup>38</sup>Note that as  $G$  is finite,  $\mathbb{Z}[\underline{G}] = \underline{\mathbb{Z}}[\underline{G}]$

*Proof.* Denote by  $f : V \rightarrow X$  the regular locus of  $U$  seen as an open subscheme of  $X$  and  $Z$  its closed complement in  $U$ . Let also  $\pi : Y \rightarrow X$  denote the normalization of  $X$ ,  $f' : V \rightarrow Y$  the inclusion and  $K$  the function field of  $X$  and  $Y$ . We have  $f = \pi f'$  and  $\pi$  induces an isomorphism between  $\pi^{-1}(V)$  and  $V$ . Finally, denote by  $T$  the closed complement of  $V$  in  $Y$ . We have for the  $L$ -functions of  $\mathbb{Z}$ -constructible sheaves:

$$L_X(j_! \mathbb{Z}) = \zeta_U = \zeta_Y \frac{\prod_{v \in Z} \zeta_v}{\prod_{w \in T} \zeta_w}.$$

Hence by Corollary II.7.4, we get the relation

$$\frac{[\mathrm{CH}_0(U)]R_U}{\omega} = \frac{h_K R_K \prod_{w \in T} \log N(w)}{\omega \prod_{v \in Z} \log N(v)},$$

which simplifies to

$$[\mathrm{CH}_0(U)]R_U = \frac{h_K R_K \prod_{w \in T} \log N(w)}{\prod_{v \in Z} \log N(v)}.$$

On the other hand, we have similarly

$$L_X((j_! \mathbb{Z})^D) = L_Y(\mathbb{Z}^D) \frac{\prod_{v \in Z} \zeta_v}{\prod_{w \in T} \zeta_w}$$

hence

$$\begin{aligned} L_X^*((j_! \mathbb{Z})^D, 0) &= \zeta_Y^*(1) \frac{\prod_{v \in Z} \zeta_v^*(0)}{\prod_{w \in T} \zeta_w^*(0)} = \frac{2^{r_1} (2\pi)^{r_2} h_K R_K \prod_{w \in T} \log N(w)}{\omega \sqrt{|\Delta_K|} \prod_{v \in Z} \log N(v)} \\ &= \frac{2^{r_1} (2\pi)^{r_2} [\mathrm{CH}_0(U)]R_U}{\omega \sqrt{|\Delta_K|}}. \quad \square \end{aligned}$$

*Remark.* When  $X$  is singular we do not necessarily have  $L_X(\mathbb{G}_X[1], s) = \zeta_X(s+1)$ , as can be seen from Theorem IV.6.2. This formula is to be compared with the formula for  $\zeta_U^*(1)$ :

$$\begin{aligned} \zeta_U^*(1) &= \frac{2^{r_1} (2\pi)^{r_2} h_K R_K \prod_{v \in Z} 1/(1 - N(v)^{-1})}{\omega \sqrt{|\Delta_K|} \prod_{w \in T} 1/(1 - N(w)^{-1})} \\ &= \frac{2^{r_1} (2\pi)^{r_2} [\mathrm{CH}_0(U)]R_U \prod_{v \in Z} \log N(v)/(1 - N(v)^{-1})}{\omega \sqrt{|\Delta_K|} \prod_{w \in T} \log N(w)/(1 - N(w)^{-1})} \end{aligned}$$

and also with the formula from [JP20].

*Remark.* We could have also obtained this using Theorem IV.7.13. Suppose that  $U$  is a proper subscheme of  $X$ . Denote by  $S_f$  the finite places in the normalization  $\pi : Y \rightarrow X$  that are not above  $U$ , and  $Z$  the closed complement of  $U$ . We can compute explicitly

$R((j_!\mathbb{Z})^D)$ , by considering the following snake diagram:

$$\begin{array}{ccccccc}
& & & 0 & \longrightarrow & (H^1(X, (j_!\mathbb{Z}))_{\mathbb{R}})^{\vee} & \\
& & & \downarrow & & \downarrow & \\
0 & \longrightarrow & 0 & \longrightarrow & \mathrm{CH}_0(X, 1)_{\mathbb{R}} & \xlongequal{\quad} & \mathrm{CH}_0(X, 1)_{\mathbb{R}} \longrightarrow 0 \\
& & \downarrow & \downarrow \mathrm{Log} & & \downarrow \mathrm{Log} & \\
& & 0 & \longrightarrow & \prod_{v \in S_f} \mathbb{R} \times \prod_{\substack{v \in Z \\ \pi(w)=v}} \mathbb{R} & \longrightarrow & \prod_{v \in S_f} \mathbb{R} \times \prod_{\substack{v \in Z \\ \pi(w)=v}} \mathbb{R} \times \left(\prod_{\sigma} \mathbb{R}\right)^{G_{\mathbb{R}}} \longrightarrow \left(\prod_{\sigma} \mathbb{R}\right)^{G_{\mathbb{R}}} \longrightarrow 0 \\
& & \parallel & \downarrow \Sigma \times \prod_v \Sigma \times \Sigma & & \downarrow & \\
& & \prod_{v \in S_f} \mathbb{R} \times \prod_{\substack{v \in Z \\ \pi(w)=v}} \mathbb{R} & \xrightarrow{\Sigma \times \prod_v \Sigma} & \mathbb{R} \times \prod_{v \in Z} \mathbb{R} & \longrightarrow & 0
\end{array}$$

If  $U = X$ , we get a similar diagram with  $H^0(X, \mathbb{Z})_{\mathbb{R}}^{\vee}$  instead of 0 at the bottom, enabling us to compute  $R(\mathbb{Z}^D)$ . We then obtain the formula using Proposition IV.7.9.

## IV.7 The Weil-étale Euler characteristic and the special value theorem

### IV.7.1 Definitions

**Definition IV.7.1.** Let  $F$  be a  $\mathbb{Z}$ -constructible sheaf on  $X$ . We define Weil-Arakelov cohomology with coefficients in  $F^D$  as the complex:

$$R\Gamma_{\mathrm{ar},c}(X, F_{\mathbb{R}}^D) := R\Gamma_{c,\mathcal{D}}(X, F_{\mathbb{R}}^D)[-1] \oplus R\Gamma_{c,\mathcal{D}}(X, F_{\mathbb{R}}^D).$$

The determinant of Weil-Arakelov cohomology has a canonical trivialization

$$\det_{\mathbb{R}} R\Gamma_{\mathrm{ar},c}(X, F_{\mathbb{R}}^D) \xrightarrow{\cong} \det_{\mathbb{R}}(R\Gamma_{c,\mathcal{D}}(X, F_{\mathbb{R}}^D))^{-1} \otimes \det_{\mathbb{R}}(R\Gamma_{c,\mathcal{D}}(X, F_{\mathbb{R}}^D)) \xrightarrow{\cong} \mathbb{R}.$$

We now put an integral structure on the determinant of Weil-Arakelov cohomology. Let  $F$  be a tamely ramified red or blue sheaf. Recall that  $\mathrm{Lie}_X(F^D)_{\mathbb{R}} = R\mathrm{Hom}_{G_{\mathbb{R}},X(\mathbb{C})}(\alpha^*F, \mathbb{C}[1])$ . Consider the map

$$\begin{aligned}
R\Gamma_{W,c}(X, F^D)_{\mathbb{R}} &= R\mathrm{Hom}(R\Gamma(X, F), \mathbb{R}[-1]) \oplus R\Gamma_{c,B}(X, F^D)_{\mathbb{R}} \\
&\xrightarrow{p} R\Gamma_{c,B}(X, F^D)_{\mathbb{R}} \\
&\xrightarrow{\mathrm{Log}} R\mathrm{Hom}_{G_{\mathbb{R}},X(\mathbb{C})}(\alpha^*F, \mathbb{C}[1]).
\end{aligned}$$

By the duality Theorem IV.5.2 and the remark after Definition IV.5.1, its mapping fiber is

$$R\mathrm{Hom}(R\Gamma(X, F), \mathbb{R}[-1]) \oplus R\Gamma_{c,\mathcal{D}}(X, F_{\mathbb{R}}^D) \simeq R\Gamma_{\mathrm{ar},c}(X, F_{\mathbb{R}}^D).$$

We thus have a distinguished triangle

$$R\Gamma_{\mathrm{ar},c}(X, F_{\mathbb{R}}^D) \rightarrow R\Gamma_{W,c}(X, F^D)_{\mathbb{R}} \rightarrow \mathrm{Lie}_X(F^D)_{\mathbb{R}},$$

and we find by taking determinants a natural trivialization

$$\lambda : \Delta_X(F^D)_{\mathbb{R}} = \det_{\mathbb{R}}(R\Gamma_{W,c}(X, F^D)_{\mathbb{R}}) \otimes \det_{\mathbb{R}}(\mathrm{Lie}_X(F^D)_{\mathbb{R}})^{-1} \xrightarrow{\cong} \det_{\mathbb{R}}(R\Gamma_{\mathrm{ar},c}(X, F_{\mathbb{R}}^D)) \xrightarrow{\cong} \mathbb{R}.$$

**Definition IV.7.2.** Let  $F$  be a tamely ramified red or blue sheaf on  $X$ . The Weil-étale Euler characteristic of  $F^D$  is the positive real number  $\chi_X(F^D)$  such that

$$\lambda(\Delta_X(F^D)) = (\chi_X(F^D))^{-1} \cdot \mathbb{Z} \hookrightarrow \mathbb{R}.$$

*Remark.* If we let  $\cup\theta$  be the map given by the composition of projections and inclusions

$$\cup\theta : R\Gamma_{\text{ar},c}(X, F_{\mathbb{R}}^D) \rightarrow R\Gamma_{c,\mathcal{D}}(X, F_{\mathbb{R}}^D) \rightarrow R\Gamma_{\text{ar},c}(X, F_{\mathbb{R}}^D)[1],$$

then  $\cup\theta$  induces a long exact sequence

$$\cdots \rightarrow H_{\text{ar},c}^{i-1}(X, F_{\mathbb{R}}^D) \xrightarrow{\cup\theta} H_{\text{ar},c}^i(X, F_{\mathbb{R}}^D) \xrightarrow{\cup\theta} H_{\text{ar},c}^{i+1}(X, F_{\mathbb{R}}^D) \rightarrow \cdots$$

which gives a trivialization

$$\lambda' : \bigotimes_{i \in \mathbb{Z}} (\det_{\mathbb{R}} H_{\text{ar},c}^i(X, F_{\mathbb{R}}^D))^{(-1)^i} \xrightarrow{\sim} \mathbb{R}.$$

The latter coincides with  $\lambda$  under the isomorphism

$$\det_{\mathbb{R}} R\Gamma_{\text{ar},c}(X, F_{\mathbb{R}}^D) \simeq \bigotimes_{i \in \mathbb{Z}} (\det_{\mathbb{R}} H_{\text{ar},c}^i(X, F_{\mathbb{R}}^D))^{(-1)^i}.$$

We can thus define alternatively  $\chi_X(F^D)$  such that

$$\lambda \left( \bigotimes_{i \in \mathbb{Z}} (\det_{\mathbb{Z}} H_{W,c}^i(X, F^D))^{(-1)^i} \otimes \bigotimes_{i \in \mathbb{Z}} (\det_{\mathbb{Z}} H^i \text{Lie}_X(F^D))^{(-1)^{i+1}} \right) = (\chi_X(F^D))^{-1} \mathbb{Z} \hookrightarrow \mathbb{R}.$$

**Theorem IV.7.3.** Let  $X$  be the spectrum of an order in a number field. The Weil-étale Euler characteristic  $\chi_X$  is multiplicative with respect to red-to-blue short exact sequences of tamely ramified sheaves.

*Proof.* Let  $0 \rightarrow F \rightarrow G \rightarrow H \rightarrow 0$  be a red-to-blue short exact sequence of tamely ramified sheaves. We let

$$\begin{aligned} T_1 : R\Gamma_{W,c}(X, H^D) &\rightarrow R\Gamma_{W,c}(X, G^D) \rightarrow R\Gamma_{W,c}(X, F^D) \rightarrow, \\ T_2 : \text{Lie}_X(H^D) &\rightarrow \text{Lie}_X(G^D) \rightarrow \text{Lie}_X(F^D) \rightarrow, \end{aligned}$$

be the two natural triangles; the first one is *not* distinguished. For  $R = \mathbb{Z}, \mathbb{R}$ , let  $g_R$  be a determinant functor on  $\text{Gr}^b(\text{Mod}_R^{\text{ft}})$  with values in graded  $R$ -lines extending the usual determinant functor on projective  $R$ -modules of finite type. The graded cohomology functor

$$H : D_{\text{perf}}(R) \rightarrow \text{Gr}^b(\text{Mod}_R^{\text{ft}}),$$

together with the assignment that sends a distinguished triangle  $T : X \xrightarrow{u} Y \rightarrow Z \rightarrow 0$  to the exact sequence

$$0 \rightarrow \ker(H(u)) \rightarrow H(X) \rightarrow H(Y) \rightarrow H(Z) \xrightarrow{\partial} \ker(H(u))[1] \rightarrow 0$$

induced by the long exact sequence, induces a pullback functor  $H^*$  on Picard groupoids of determinants. We put  $f_R = H^*g_R$ . Notice that  $H(T)$  makes sense for any triangle (not necessarily distinguished) that induces a long exact cohomology sequence. There is a base change isomorphism  $\gamma : (g_{\mathbb{Z}}(-))_{\mathbb{R}} \xrightarrow{\sim} g_{\mathbb{R}}((-)_{\mathbb{R}})$ , which induces a base change

isomorphism  $H^*(\gamma) : (f_{\mathbb{Z}}(-))_{\mathbb{R}} \xrightarrow{\simeq} f_{\mathbb{R}}((-)_{\mathbb{R}})$ . By Theorem IV.3.10,  $H(T_1)$  is an exact sequence in  $\text{Gr}^b(\text{Mod}_{\mathbb{Z}}^{\text{ft}})$ , while  $H(T_2)$  is because  $T_2$  is distinguished. On the other hand,  $T_{1,\mathbb{R}}$  and  $T_{2,\mathbb{R}}$  are both distinguished triangles and the structure of exact sequence on  $H(T_{1,\mathbb{R}})$  and  $H(T_{2,\mathbb{R}})$  coming from the distinguished triangles coincides with the structure coming by base change from  $H(T_1)$  and  $H(T_2)$ . We get a commutative diagram (see Theorem II.6.3 and its proof):

$$\begin{array}{ccc}
f_{\mathbb{Z}}(R\Gamma_{W,c}(X, H^D)) \otimes f_{\mathbb{Z}}(R\Gamma_{W,c}(X, F^D)) \otimes f_{\mathbb{Z}}(R\Gamma_{W,c}(X, G^D)) \otimes f_{\mathbb{Z}}(\text{Lie}_X(H^D))^{-1} \otimes f_{\mathbb{Z}}(\text{Lie}_X(F^D))^{-1} & \xrightarrow{g_{\mathbb{Z}}(HT_1) \otimes g_{\mathbb{Z}}(HT_2)^{-1}} & f_{\mathbb{Z}}(R\Gamma_{W,c}(X, G^D)) \otimes f_{\mathbb{Z}}(\text{Lie}_X(G^D))^{-1} \\
\downarrow & & \downarrow \\
(f_{\mathbb{Z}}(R\Gamma_{W,c}(X, H^D)))_{\mathbb{R}} \otimes (f_{\mathbb{Z}}(R\Gamma_{W,c}(X, F^D)))_{\mathbb{R}} \otimes (f_{\mathbb{Z}}(\text{Lie}_X(H^D))^{-1})_{\mathbb{R}} \otimes (f_{\mathbb{Z}}(\text{Lie}_X(F^D))^{-1})_{\mathbb{R}} & \xrightarrow{(g_{\mathbb{Z}}(HT_1))_{\mathbb{R}} \otimes (g_{\mathbb{Z}}(HT_2)^{-1})_{\mathbb{R}}} & (f_{\mathbb{Z}}(R\Gamma_{W,c}(X, G)))_{\mathbb{R}} \otimes (f_{\mathbb{Z}}(\text{Lie}_X(G)))_{\mathbb{R}} \\
\downarrow H^*(\gamma) \otimes H^*(\gamma) \otimes H^*(\gamma)^{-1} \otimes H^*(\gamma) & & \downarrow H^*(\gamma) \otimes H^*(\gamma)^{-1} \\
f_{\mathbb{R}}(\Delta_X(H^D)_{\mathbb{R}}) \otimes f_{\mathbb{R}}(\Delta_X(F^D)_{\mathbb{R}}) & \xrightarrow{g_{\mathbb{R}}(H(T_{1,\mathbb{R}})) \otimes g_{\mathbb{R}}(H(T_{2,\mathbb{R}})^{-1}) = f_{\mathbb{R}}(T_{1,\mathbb{R}}) \otimes f_{\mathbb{R}}(T_{2,\mathbb{R}})^{-1}} & f_{\mathbb{R}}(\Delta_X(F^D)_{\mathbb{R}}) \\
\downarrow & & \downarrow \\
\mathbb{R} \otimes \mathbb{R} & \xrightarrow{\text{mult}} & \mathbb{R}
\end{array}$$

Under the multiplication map, the image of  $x\mathbb{Z} \otimes y\mathbb{Z} \subset \mathbb{R} \otimes \mathbb{R}$  is  $xy\mathbb{Z}$ .  $\square$

**Proposition IV.7.4.** *Let  $Y = \text{Spec}(\mathcal{O}')$  be the spectrum of an order in a number field with a finite dominant morphism  $\pi : Y \rightarrow X$ . Suppose that  $K(Y)/K$  is tamely ramified, and let  $F$  be a tamely ramified red or blue sheaf on  $Y$ . Then*

$$\chi_X((\pi_*F)^D) = \chi_Y(F^D).$$

*Proof.* This follows from the isomorphisms

$$\begin{aligned}
R\Gamma_{W,c}(Y, F^D) &\xrightarrow{\simeq} R\Gamma_{W,c}(X, (\pi_*F)^D), \\
\text{Lie}_Y(F^D) &\xrightarrow{\simeq} \text{Lie}_X((\pi_*F)^D), \\
R\Gamma_{c,D}(Y, F_{\mathbb{R}}^D) &\xrightarrow{\simeq} R\Gamma_{c,D}(X, (\pi_*F)_{\mathbb{R}}^D),
\end{aligned}$$

and the isomorphism of distinguished triangles

$$\begin{array}{ccccccc}
R\Gamma_{\text{ar},c}(Y, F_{\mathbb{R}}^D) & \longrightarrow & R\Gamma_{W,c}(Y, F^D)_{\mathbb{R}} & \longrightarrow & \text{Lie}_Y(F^D)_{\mathbb{R}} & \longrightarrow & \\
\downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq & & \\
R\Gamma_{\text{ar},c}(X, (\pi_*F)_{\mathbb{R}}^D) & \longrightarrow & R\Gamma_{W,c}(X, (\pi_*F)^D)_{\mathbb{R}} & \longrightarrow & \text{Lie}_X((\pi_*F)^D)_{\mathbb{R}} & \longrightarrow & 
\end{array}$$

$\square$

We now use the multiplicativity of the Weil-étale Euler characteristic to extend it to arbitrary tamely ramified  $\mathbb{Z}$ -constructible sheaves:

**Definition IV.7.5.** *Let  $F$  be a tamely ramified  $\mathbb{Z}$ -constructible sheaf on  $X$ , let  $j : U \rightarrow X$  be an open subscheme such that  $F|_U$  is locally constant and let  $i : Z \rightarrow X$  be the closed*

complement. The sheaves  $j_!F|_U$  and  $i_*i^*F$  are tamely ramified and respectively red and blue, and we define the Weil-étale Euler characteristic of  $F^D$  as

$$\chi_X(F^D) = \chi_X((j_!F|_U)^D)\chi_X((i_*i^*F)^D).$$

The definition doesn't depend on the choice of  $U$ . The Weil-étale Euler characteristic is multiplicative with respect to short exact sequences of tamely ramified sheaves. Let  $Y = \text{Spec}(\mathcal{O}')$  be the spectrum of an order in a number field with a finite dominant morphism  $\pi : Y \rightarrow X$  such that  $K(Y)/K$  is tamely ramified, and let  $F$  be a tamely ramified  $\mathbb{Z}$ -constructible sheaf on  $Y$ . Then

$$\chi_X((\pi_*F)^D) = \chi_Y(F^D).$$

*Proof.* This follows formally from the previous results by using the open-closed decomposition lemma, as in Subsection II.6.5.  $\square$

## IV.7.2 Computations of the Euler characteristic, and the special value theorem

We first give an explicit expression of our Euler characteristic which doesn't involve Weil-étale cohomology anymore. This will allow us to give an expression valid for any tamely ramified  $\mathbb{Z}$ -constructible sheaf.

**Proposition IV.7.6.** *Let  $F$  be a tamely ramified red or blue sheaf on  $X$ . Let  $R_1(F^D)$  be the absolute value of the determinant of the pairing*

$$H_{c,\mathcal{D}}^{-1}(X, F_{\mathbb{R}}^D) \times H^1(X, F)_{\mathbb{R}} \rightarrow \mathbb{R}$$

and  $R_0(F^D)$  the absolute value of the determinant of the pairing

$$H_{c,\mathcal{D}}^0(X, F_{\mathbb{R}}^D) \times H^0(X, F)_{\mathbb{R}} \rightarrow \mathbb{R}$$

in the following bases : pick bases modulo torsion of  $H^{-1}(\text{Lie}_X(F^D))$ , of  $H^i(X, F)$  for  $i = 0, 1$  and of  $H_{c,B}^i(X, F^D)$  for  $i = -1, 0$ . Then pick any  $\mathbb{R}$ -bases of  $H_{c,\mathcal{D}}^i(X, F_{\mathbb{R}}^D)$  compatible<sup>39</sup> with the chosen bases in the following exact sequence

$$\begin{aligned} 0 \rightarrow H_{c,\mathcal{D}}^{-1}(X, F_{\mathbb{R}}^D) \rightarrow H_{c,B}^{-1}(X, F^D)_{\mathbb{R}} \rightarrow H^{-1}(\text{Lie}_X(F^D))_{\mathbb{R}} \\ \rightarrow H_{c,\mathcal{D}}^0(X, F_{\mathbb{R}}^D) \rightarrow H_{c,B}^0(X, F^D)_{\mathbb{R}} \rightarrow 0. \end{aligned}$$

We have

$$\chi_X(F^D) = \frac{[H^0(X, F)_{\text{tor}}][H_{c,B}^0(X, F^D)_{\text{tor}}]}{[H^1(X, F)_{\text{tor}}][H_{c,B}^{-1}(X, F^D)_{\text{tor}}][H^0(\text{Lie}_X(F^D))]} \frac{R_1(F^D)}{R_0(F^D)}.$$

*Remark.* We can reformulate the part about determinants of pairings. Consider an acyclic chain complex of  $\mathbb{R}$ -vector spaces  $A^\bullet$  with a  $\mathbb{Z}$ -lattice  $M^i \subset A^i$  given for each  $i$ . We define  $\det_{\mathbb{Z},\mathbb{R}} A^\bullet$  to be the absolute value of the image of the vector corresponding to bases of

<sup>39</sup>In the following sense: denote by  $\mathcal{E}^\bullet$  the exact sequence seen as an acyclic complex, then the choice of bases gives an element of  $\det_{\mathbb{R}} \mathcal{E}^\bullet$  which is required to be sent to 1 under  $\det_{\mathbb{R}} \mathcal{E}^\bullet \xrightarrow{\det(0)} \mathbb{R} \xrightarrow{\simeq}$

the lattices under the canonical isomorphism  $\det_{\mathbb{R}} A^\bullet \xrightarrow[\det_{\mathbb{R}}(0)]{\simeq} \mathbb{R}$ ; put another way, it is the positive real number such that

$$\bigotimes_{i \in \mathbb{Z}} (\det_{\mathbb{Z}} M^i)^{(-1)^i} \subset \left( \bigotimes_{i \in \mathbb{Z}} (\det_{\mathbb{Z}} M^i)^{(-1)^i} \right)_{\mathbb{R}} \simeq \det_{\mathbb{R}} A^\bullet$$

corresponds to  $(\det_{\mathbb{Z}, \mathbb{R}} A^\bullet)_{\mathbb{Z}} \subset \mathbb{R}$  under the canonical isomorphism  $\det_{\mathbb{R}} A^\bullet \xrightarrow[\det_{\mathbb{R}}(0)]{\simeq} \mathbb{R}$ . Consider the following complex

$$\begin{aligned} A^\bullet : \quad 0 \rightarrow (H^1(X, F)_{\mathbb{R}})^\vee \rightarrow H_{c,B}^{-1}(X, F^D)_{\mathbb{R}} \rightarrow H^{-1}(\text{Lie}_X(F^D))_{\mathbb{R}} \\ \rightarrow (H^0(X, F)_{\mathbb{R}})^\vee \rightarrow H_{c,B}^0(X, F^D)_{\mathbb{R}} \rightarrow 0, \end{aligned}$$

with first non-zero term in degree  $-1$  and endowed with the natural lattices. Then we have

$$\frac{R_1(F^D)}{R_0(F^D)} = \det_{\mathbb{Z}, \mathbb{R}} A^\bullet.$$

*Proof.* Since determinants depend only on cohomology and are multiplicative in short exact sequences, we can write  $\Delta_X(F^D)$  as an alternating tensor product of the determinants of the torsion and torsion-free parts of the cohomology groups:

$$\Delta_X(F^D) = \Delta_X(F^D)_{\text{tor}} \Delta_X(F^D)/\text{tor}.$$

We have  $(\Delta_X(F^D)_{\text{tor}})_{\mathbb{R}} = \det_{\mathbb{R}}(0) = \mathbb{R}$  canonically hence by Lemma A.0.1 the contribution of  $\Delta_X(F^D)_{\text{tor}}$  inside  $\Delta_X(F^D)_{\mathbb{R}}$  is sent under the trivialisation to

$$1/\chi_X(F^D)_{\text{tor}} := \frac{\prod [H^i \text{Lie}_X(F^D)_{\text{tor}}]^{(-1)^i}}{\prod [H_{W,c}^i(X, F^D)_{\text{tor}}]^{(-1)^i}}.$$

We obtain the first part of the claimed expression by the computation of the involved cohomology groups in Proposition IV.3.5 and Corollary IV.4.6.

It remains to show that  $\Delta_X(F^D)/\text{tor} = \frac{R_1(F^D)}{R_0(F^D)}$ . We will use Lemma A.0.2. Fix basis vectors of  $H^i(X, F)/\text{tor}$  for  $i = 0, 1$ , of  $(H^{-1} \text{Lie}_X(F^D))/\text{tor}$  and of  $H_{c,B}^i(X, F^D)$  for  $i = -1, 0$ . The image of  $\Delta_X(F^D)/\text{tor}$  inside

$$\begin{aligned} \Delta_X(F^D)_{\mathbb{R}} &:= \det_{\mathbb{R}} R\Gamma_{W,c}(X, F^D)_{\mathbb{R}} \otimes_{\mathbb{R}} (\det_{\mathbb{R}} \text{Lie}_X(F^D)_{\mathbb{R}})^{-1} \\ &\simeq \det_{\mathbb{R}}(D_{F, \mathbb{R}}[1]) \otimes_{\mathbb{R}} \det_{\mathbb{R}} R\Gamma_{c,B}(X, F^D)_{\mathbb{R}} \otimes_{\mathbb{R}} (\det_{\mathbb{R}} \text{Lie}_X(F^D)_{\mathbb{R}})^{-1} \\ &\simeq \det_{\mathbb{R}}(D_{F, \mathbb{R}}[1]) \otimes_{\mathbb{R}} \det_{\mathbb{R}} R\Gamma_{c, \mathcal{D}}(X, F_{\mathbb{R}}^D) \\ &\simeq \bigotimes_i (\det_{\mathbb{R}} H^{i+1}(D_{F, \mathbb{R}}))^{(-1)^i} \otimes_{\mathbb{R}} \bigotimes_i (\det_{\mathbb{R}} H_{c, \mathcal{D}}^i(X, F_{\mathbb{R}}^D))^{(-1)^i} \end{aligned}$$

is a certain product of basis vectors and dual basis vectors, obtained from the basis vectors of  $H_{c,B}^i(X, F^D)/\text{tor}$ ,  $H^i \text{Lie}_X(F^D)/\text{tor}$  and  $H^i(X, F)/\text{tor}$  through the identity  $H^i(D_{F, \mathbb{R}}) = \text{Hom}(H^{2-i}(X, F), \mathbb{R})$  and the long exact sequence associated to the fiber sequence

$$R\Gamma_{c, \mathcal{D}}(X, F_{\mathbb{R}}^D) \rightarrow R\Gamma_{c,B}(X, F^D)_{\mathbb{R}} \rightarrow \text{Lie}_X(F^D)_{\mathbb{R}}.$$



Finally, the trivialisation is obtained by combining together, for each  $i \in \mathbb{Z}$ , the terms related by the duality theorem Theorem IV.5.2:

$$\begin{aligned} & (\det_{\mathbb{R}} H_{c,\mathcal{D}}^i(X, F_{\mathbb{R}}^D))^{(-1)^i} \otimes (\det_{\mathbb{R}} H^{i+2}(D_{F,\mathbb{R}}))^{(-1)^{i+1}} \\ & \xrightarrow{\cong} (\det_{\mathbb{R}} H_{c,\mathcal{D}}^i(X, F_{\mathbb{R}}^D))^{(-1)^i} \bigotimes (\det_{\mathbb{R}} H_{c,\mathcal{D}}^i(X, F_{\mathbb{R}}^D))^{(-1)^{i+1}} \\ & \xrightarrow{\cong} \mathbb{R}. \end{aligned}$$

The isomorphism  $H_{c,\mathcal{D}}^i(X, F_{\mathbb{R}}^D) \xrightarrow{\cong} H^{i+2}(D_{F,\mathbb{R}})$  is non-trivial only for  $i = -1, 0$  in which case its determinant in the bases obtained is exactly (up to sign)  $R_1$ , resp.  $R_0$ . By Lemma A.0.2, the contribution of  $\Delta_X(F^D)/\text{tor}$  is thus sent under the trivialisation to

$$\frac{1}{\chi_X(F^D)/\text{tor}} := R_0/R_1. \quad \square$$

**Corollary IV.7.7.** *Let  $F$  be a tamely ramified  $\mathbb{Z}$ -constructible sheaf on  $X$ . With the same notations, we have*

$$\chi_X(F^D) = \frac{[H^0(X, F)_{\text{tor}}][H_{c,B}^0(X, F^D)_{\text{tor}}]}{[H^1(X, F)_{\text{tor}}][H_{c,B}^{-1}(X, F^D)_{\text{tor}}][H^0(\text{Lie}_X(F^D))]} \frac{R_1(F^D)}{R_0(F^D)}.$$

*Proof.* Let  $j : U \rightarrow X$  be an open subscheme such that  $F|_U$  is locally constant and let  $i : Z \rightarrow X$  be the closed complement. Put  $F_U = j_!F|_U$  and  $F_Z = i_*i^*F$ . Then  $F_U$  is red,  $F_Z$  is blue and there is a short exact sequence  $0 \rightarrow F_U \rightarrow F \rightarrow F_Z \rightarrow 0$ . Consider the two following acyclic complexes of abelian groups, with first non-zero term respectively in degree  $-1$  and  $0$ :

$$\begin{aligned} A^\bullet : 0 & \rightarrow H_{c,B}^{-1}(X, (F_Z)^D) \rightarrow H_{c,B}^{-1}(X, F^D) \rightarrow H_{c,B}^{-1}(X, (F_U)^D) \\ & \rightarrow H_{c,B}^0(X, (F_Z)^D) \rightarrow H_{c,B}^0(X, F^D) \rightarrow H_{c,B}^0(X, (F_U)^D) \rightarrow I \rightarrow 0, \\ B^\bullet : 0 & \rightarrow H^0(X, F_U) \rightarrow H^0(X, F) \rightarrow H^0(X, F_Z) \\ & \rightarrow H^1(X, F_U) \rightarrow H^1(X, F) \rightarrow H^1(X, F_Z) \rightarrow J \rightarrow 0, \end{aligned}$$

where  $I$  is the image of  $H_{c,B}^0(X, (F_U)^D)$  in  $H_{c,B}^1(X, F_Z^D)$  and  $J$  is the image of  $H^1(X, F_Z)$  in  $H^2(X, F_U)$ . Since  $F_Z$  is blue,  $H_{c,B}^1(X, F_Z^D)$  and  $H^1(X, F_Z)$  are finite so  $I$  and  $J$  are also finite. Moreover Artin–Verdier duality gives a commutative square

$$\begin{array}{ccc} H_{c,B}^0(X, (F_U)^D) & \xrightarrow{\partial} & H_{c,B}^1(X, (F_Z)^D) \\ \downarrow \simeq & & \downarrow \simeq \\ H^2(X, F_U)^* & \xrightarrow{\partial} & H^1(X, F_Z)^* \end{array}$$

whence  $I \simeq J^*$ .

Since  $F_Z$  is supported on a finite closed subscheme, we have  $\text{Lie}_X((F_Z)^D) = 0$  hence an isomorphism  $\text{Lie}_X(F^D) \xrightarrow{\cong} \text{Lie}_X((F_U)^D)$  which gives a canonical isomorphism

$$\det_{\mathbb{Z}} \text{Lie}_X(F^D) (\det_{\mathbb{Z}} \text{Lie}_X((F_U)^D))^{-1} \xrightarrow[\tau]{\cong} \mathbb{Z}.$$

On the other hand, since  $A^\bullet$  and  $B^\bullet$  are acyclic, we obtain canonical trivializations

$$\det_{\mathbb{Z}} A^\bullet \xrightarrow[\det 0]{\cong} \mathbb{Z}, \quad \det_{\mathbb{Z}} B^\bullet \xrightarrow[\det 0]{\cong} \mathbb{Z}.$$

Consider now the graded line

$$\Delta := \det_{\mathbb{Z}} A^{\bullet} \det_{\mathbb{Z}} B^{\bullet} \det_{\mathbb{Z}} \text{Lie}_X(F^D) (\det_{\mathbb{Z}} \text{Lie}_X((F_U)^D))^{-1}.$$

It obtains a canonical trivialization  $\alpha : \Delta \xrightarrow{\det 0 \otimes \det 0 \otimes \tau} \mathbb{Z} \otimes \mathbb{Z} \otimes \mathbb{Z} \xrightarrow{\cong} \mathbb{Z}$  through the previous remarks.

For a distinguished triangle  $X \rightarrow Y \rightarrow Z \rightarrow X[1]$ , denote by  $i : (\det Y)^{-1} \det X \det Z \rightarrow 1$  the structural isomorphism to the unit. From the octahedral diagram

$$\begin{array}{ccccc} R\Gamma_{c,\mathcal{D}}(X, (F_Z)^D_{\mathbb{R}}) & \longrightarrow & R\Gamma_{c,\mathcal{D}}(X, F^D_{\mathbb{R}}) & \longrightarrow & R\Gamma_{c,\mathcal{D}}(X, (F_U)^D_{\mathbb{R}}) \\ \downarrow \simeq & & \downarrow & & \downarrow \\ R\Gamma_{c,B}(X, (F_Z)^D_{\mathbb{R}}) & \longrightarrow & R\Gamma_{c,B}(X, F^D_{\mathbb{R}}) & \longrightarrow & R\Gamma_{c,B}(X, (F_U)^D_{\mathbb{R}}) \\ \downarrow & & \downarrow & & \downarrow \\ \text{Lie}_X((F_Z)^D)_{\mathbb{R}} = 0 & \longrightarrow & \text{Lie}_X(F)_{\mathbb{R}} & \xrightarrow{\simeq} & \text{Lie}_X((F_U)^D)_{\mathbb{R}} \end{array}$$

we find using the associativity axiom of determinant functors [Bre11, 3.1] a commutative diagram

$$\begin{array}{ccc} (\det_{\mathbb{R}} R\Gamma_{c,\mathcal{D}}(X, F^D_{\mathbb{R}}))^{-1} \det_{\mathbb{R}} R\Gamma_{c,\mathcal{D}}(X, (F_U)^D_{\mathbb{R}}) \det_{\mathbb{R}} R\Gamma_{c,\mathcal{D}}(X, (F_Z)^D_{\mathbb{R}}) & \xrightarrow{i} & \mathbb{R} \\ \downarrow \simeq & & \uparrow \\ (\det_{\mathbb{R}} R\Gamma_{c,B}(X, F^D_{\mathbb{R}}))^{-1} \det_{\mathbb{R}} R\Gamma_{c,B}(X, (F_U)^D_{\mathbb{R}}) & \xrightarrow{i_{\mathbb{R}} \otimes \tau_{\mathbb{R}}} & \mathbb{R} \otimes_{\mathbb{R}} \mathbb{R} \\ \det_{\mathbb{R}} R\Gamma_{c,B}(X, (F_Z)^D_{\mathbb{R}}) \det_{\mathbb{R}} \text{Lie}_X(F^D)_{\mathbb{R}} & & \\ (\det_{\mathbb{R}} \text{Lie}_X((F_U)^D)_{\mathbb{R}})^{-1} & & \end{array} \quad (\text{IV.7.1})$$

There are isomorphisms

$$\det_{\mathbb{R}} A^{\bullet}_{\mathbb{R}} \simeq (\det_{\mathbb{R}} R\Gamma_{c,B}(X, F^D_{\mathbb{R}}))^{-1} \det_{\mathbb{R}} R\Gamma_{c,B}(X, (F_U)^D_{\mathbb{R}}) \det_{\mathbb{R}} R\Gamma_{c,B}(X, (F_Z)^D_{\mathbb{R}}) \quad (\text{IV.7.2})$$

$$\det_{\mathbb{R}} B^{\bullet}_{\mathbb{R}} \simeq (\det_{\mathbb{R}} R\Gamma(X, F)_{\mathbb{R}})^{(-1)} \det_{\mathbb{R}} R\Gamma(X, F_U)_{\mathbb{R}} \det_{\mathbb{R}} R\Gamma(X, F_Z)_{\mathbb{R}} \quad (\text{IV.7.3})$$

under which the trivialization  $\det(0)_{\mathbb{R}} = \det_{\mathbb{R}}(0)$  on the left corresponds to  $i_{\mathbb{R}}$  on the right. By combining Equations (IV.7.1) to (IV.7.3) we find a commutative diagram of isomorphisms

$$\begin{array}{ccc} \Delta_{\mathbb{R}} & \xrightarrow{\simeq} & (\det_{\mathbb{R}} R\Gamma_{c,\mathcal{D}}(X, F^D_{\mathbb{R}}))^{(-1)} \det_{\mathbb{R}} R\Gamma_{c,\mathcal{D}}(X, (F_U)^D_{\mathbb{R}}) \det_{\mathbb{R}} R\Gamma_{c,\mathcal{D}}(X, (F_Z)^D_{\mathbb{R}}) \\ \alpha \downarrow & & \downarrow i \cdot i_{\mathbb{R}} \\ \mathbb{R} & \xlongequal{\quad} & \mathbb{R} \end{array}$$

We now describe how to produce another trivialization of  $\Delta_{\mathbb{R}}$ . By Theorem IV.5.2, there is an isomorphism  $\psi_{(-)} : R\Gamma_{c,\mathcal{D}}(X, (-)^D_{\mathbb{R}}) \xrightarrow{\simeq} R\Gamma(X, (-))_{\mathbb{R}}^{\vee}$ . If  $L$  is a graded  $\mathbb{R}$ -line, there is a canonical isomorphism  $L \otimes L^{-1} \xrightarrow{\delta} \mathbb{R}$  and we take the trivialization

$$\begin{aligned}
& (\det_{\mathbb{R}} R\Gamma_{c,\mathcal{D}}(X, F_{\mathbb{R}}^D))^{(-1)} \det_{\mathbb{R}} R\Gamma_{c,\mathcal{D}}(X, (F_U)_{\mathbb{R}}^D) \det_{\mathbb{R}} R\Gamma_{c,\mathcal{D}}(X, (F_Z)_{\mathbb{R}}^D) \\
& (\det_{\mathbb{R}} R\Gamma(X, F)_{\mathbb{R}})^{(-1)} \det_{\mathbb{R}} R\Gamma(X, F_U)_{\mathbb{R}} \det_{\mathbb{R}} R\Gamma(X, F_Z)_{\mathbb{R}} \\
& \quad \downarrow \\
& \det_{\mathbb{R}} R\Gamma(X, F)_{\mathbb{R}} (\det_{\mathbb{R}} R\Gamma(X, F_U)_{\mathbb{R}})^{(-1)} (\det_{\mathbb{R}} R\Gamma(X, F_Z)_{\mathbb{R}})^{(-1)} \\
& (\det_{\mathbb{R}} R\Gamma(X, F)_{\mathbb{R}})^{(-1)} \det_{\mathbb{R}} R\Gamma(X, F_U)_{\mathbb{R}} \det_{\mathbb{R}} R\Gamma(X, F_Z)_{\mathbb{R}} \\
& \quad \downarrow \delta \otimes \delta \otimes \delta \\
& \mathbb{R} \otimes_{\mathbb{R}} \mathbb{R} \otimes_{\mathbb{R}} \mathbb{R} \\
& \quad \downarrow \text{mult} \\
& \mathbb{R}
\end{aligned}$$

Consider Diagram IV.4. From the isomorphism of fiber sequences

$$\begin{array}{ccccc}
R\Gamma_{c,\mathcal{D}}(X, (F_Z)_{\mathbb{R}}^D) & \longrightarrow & R\Gamma_{c,\mathcal{D}}(X, F_{\mathbb{R}}^D) & \longrightarrow & R\Gamma_{c,\mathcal{D}}(X, (F_U)_{\mathbb{R}}^D) \\
\downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq \\
R\Gamma(X, F_Z)_{\mathbb{R}}^{\vee} & \longrightarrow & R\Gamma(X, F)_{\mathbb{R}}^{\vee} & \longrightarrow & R\Gamma(X, F_U)_{\mathbb{R}}^{\vee}
\end{array}$$

we deduce that the upper triangle in Diagram IV.4 commutes, while the lower square commutes formally. Thus the trivialization by duality equals the trivialization  $i \cdot i_{\mathbb{R}}$ .

Under those equal trivializations, the image of  $\Delta$  inside  $\mathbb{R}$  is  $\mathbb{Z}$  on the one hand and is computed with Proposition A.0.3 on the other hand. We get an identity

$$\frac{[H^0(X, F)_{\text{tor}}][H_{c,B}^0(X, F^D)_{\text{tor}}]}{[H^1(X, F)_{\text{tor}}][H_{c,B}^{-1}(X, F^D)_{\text{tor}}][H^0(\text{Lie}_X(F^D))]} \frac{R_1(F^D)}{R_0(F^D)} \frac{[J]}{[I]} \frac{1}{\chi_X((F_U)^D)\chi_X((F_Z)^D)} = 1$$

and thus, because  $[I] = [J]$ :

$$\begin{aligned}
\chi_X(F^D) & := \chi_X((F_U)^D)\chi_X((F_Z)^D) \\
& = \frac{[H^0(X, F)_{\text{tor}}][H_{c,B}^0(X, F^D)_{\text{tor}}]}{[H^1(X, F)_{\text{tor}}][H_{c,B}^{-1}(X, F^D)_{\text{tor}}][H^0(\text{Lie}_X(F^D))]} \frac{R_1(F^D)}{R_0(F^D)}. \quad \square
\end{aligned}$$

**Definition IV.7.8.** Let  $F$  be a tamely ramified  $\mathbb{Z}$ -constructible sheaf on  $X$ . We put

$$\begin{aligned}
r_1(F) & = \log_2([\text{Ext}_{G_{\mathbb{R}}, X(\mathbb{C})}^1(\alpha^* F, 2i\pi\mathbb{Z})]), \\
r_2(F) & = \text{rank}_{\mathbb{Z}} \text{Hom}_{G_{\mathbb{R}}, X(\mathbb{C})}(\alpha^* F, 2i\pi\mathbb{Z}).
\end{aligned}$$

For each archimedean place  $v$  of  $K$ , fix a corresponding embedding  $\sigma_v$ . There is an isomorphism<sup>40</sup>

$$\text{Hom}_{G_{\mathbb{R}}, X(\mathbb{C})}(\alpha^* F, \mathbb{R}) \xrightarrow{\simeq} \prod_v \text{Hom}_{G_v}(F_v, \mathbb{Z})_{\mathbb{R}},$$

given by  $(\phi_{\sigma}) \mapsto (\phi_{\sigma_v})_v$  real,  $(2\phi_{\sigma_v})_v$  complex. Using Proposition IV.4.8, the short exact sequence  $0 \rightarrow 2i\pi\mathbb{R} \rightarrow \mathbb{C} \xrightarrow{\Re} \mathbb{R} \rightarrow 0$  and the above isomorphism, we can consider the following acyclic complex of  $\mathbb{R}$ -vector spaces with lattices:

$$D^{\bullet} : \quad 0 \rightarrow \text{Hom}_{G_{\mathbb{R}}, X(\mathbb{C})}(\alpha^* F, 2i\pi\mathbb{Z})_{\mathbb{R}} \rightarrow H^{-1} \text{Lie}_X(F^D)_{\mathbb{R}} \rightarrow \prod_v \text{Hom}_{G_v}(F_v, \mathbb{Z})_{\mathbb{R}} \rightarrow 0$$

<sup>40</sup>Non-canonical; the particular choice here is justified by the later computation of  $\text{Disc}(F)$  and  $R(F^D)$  for the case  $F = \mathbb{Z}$

$$\begin{array}{c}
(\det_{\mathbb{R}} RI_{c;D}(X, F^D))^{(-1)} \det_{\mathbb{R}} RI_{c;D}(X, (F_U)_{\mathbb{R}}^D) \det_{\mathbb{R}} RI_{c;D}(X, (F_Z)_{\mathbb{R}}^D) \\
(\det_{\mathbb{R}} RI(X, F)_{\mathbb{R}})^{(-1)} \det_{\mathbb{R}} RI(X, F_U)_{\mathbb{R}} \det_{\mathbb{R}} RI(X, F_Z)_{\mathbb{R}} \\
\begin{array}{ccc}
\mathbb{R} \otimes_{\mathbb{R}} \mathbb{R} & \xrightarrow{((\det \psi_F)^{-1})^{-1} \det \psi_{F_U} \det \psi_{F_Z}} & \mathbb{R} \otimes_{\mathbb{R}} \mathbb{R} \otimes_{\mathbb{R}} \mathbb{R} \\
\downarrow i \otimes i_{\mathbb{R}} & & \downarrow \\
\mathbb{R} \otimes_{\mathbb{R}} \mathbb{R} & \xrightarrow{((i_{\mathbb{R}})^{-1})^{-1} \otimes i_{\mathbb{R}}} & \det_{\mathbb{R}} RI(X, F) (\det_{\mathbb{R}} RI(X, F_U))^{(-1)} (\det_{\mathbb{R}} RI(X, F_Z))^{(-1)} \\
\downarrow \text{mult} & & \downarrow \\
\mathbb{R} & \xrightarrow{\text{mult}} & \mathbb{R} \otimes_{\mathbb{R}} \mathbb{R} \otimes_{\mathbb{R}} \mathbb{R}
\end{array}
\end{array}$$

Diagram IV.4

(with first non-zero term in degree  $-1$ ). We put

$$\text{Disc}(F) := (2\pi)^{r_2(F)} \det_{\mathbb{Z}, \mathbb{R}} D^\bullet.$$

*Remark.*

- If  $\tilde{D}^\bullet$  is the complex

$$0 \rightarrow \text{Hom}_{G_{\mathbb{R}}, X(\mathbb{C})}(\alpha^* F, i\mathbb{Z})_{\mathbb{R}} \rightarrow H^{-1} \text{Lie}_X(F^D)_{\mathbb{R}} \rightarrow \prod_v \text{Hom}_{G_v}(F_v, \mathbb{Z})_{\mathbb{R}} \rightarrow 0$$

with similar degree conventions, then  $\text{Disc}(F) = \det_{\mathbb{Z}, \mathbb{R}} \tilde{D}^\bullet$ .

- The quantity  $\text{Disc}(F)$  only depends on  $F_\eta$ .

We now reformulate the previous expression obtained to make it closer to a formula looking like the analytic class number formula.

**Proposition IV.7.9.** *Let  $F$  be a tamely ramified  $\mathbb{Z}$ -constructible sheaf on  $X$ . Put*

$$N_2 := \text{Ker}(\text{Ext}_{G_{\mathbb{R}}, X(\mathbb{C})}^2(\alpha^* F, 2i\pi\mathbb{Z}) \rightarrow H_{c,B}^1(X, F^D))$$

and let  $\mathbb{G}_X = [g_* \mathbb{G}_m \rightarrow \bigoplus_{x \in X_0} i_{x,*} \mathbb{Z}]$  denote Deninger's dualizing complex. Let  $R(F^D) := \det_{\mathbb{Z}, \mathbb{R}} B^\bullet$  with  $B^\bullet$  the exact sequence of  $\mathbb{R}$ -vector spaces with lattices

$$B^\bullet : \quad 0 \rightarrow (H^1(X, F)_{\mathbb{R}})^\vee \rightarrow \text{Hom}_X(F, \mathbb{G}_X)_{\mathbb{R}} \rightarrow \prod_v \text{Hom}_{G_v}(F_v, \mathbb{Z})_{\mathbb{R}} \\ \rightarrow (H^0(X, F)_{\mathbb{R}})^\vee \rightarrow \text{Ext}_X^1(F, \mathbb{G}_X)_{\mathbb{R}} \rightarrow 0,$$

with first non-zero term in degree  $-1$ . We have

$$\chi_X(F^D) = \frac{(2\pi)^{r_2(F)} 2^{r_1(F)} [H^0(X, F)_{\text{tor}}] [\text{Ext}_X^1(F, \mathbb{G}_X)_{\text{tor}}]}{[H^1(X, F)_{\text{tor}}] [\text{Hom}_X(F, \mathbb{G}_X)_{\text{tor}}] [\text{Ext}_{G_K^t}^1(F_\eta, \mathcal{O}_{K^t})] [N_2] \text{Disc}(F)} R(F^D).$$

*Proof.* We have  $\mathbb{Z}_X^c \simeq \mathbb{G}_X[1]$  [Nar89]. Consider the biacyclic double complex of Diagram IV.5 (numbers on the side indicate indexing, zeros are omitted). If  $C^{\bullet, \bullet}$  is any biacyclic double complex of  $\mathbb{R}$ -vector spaces with lattices, we find by applying Lemma F.0.1 to both the rows and the columns that the following diagram is commutative:

$$\begin{array}{ccccc} \bigotimes_{i,j} (\det_{\mathbb{R}} C^{i,j})^{(-1)^{i+j}} & & & & \\ \downarrow \simeq & & & & \\ \det_{\mathbb{R}} \text{Tot} C & \xrightarrow{\quad} & \bigotimes_j (\det_{\mathbb{R}} C^{\bullet,j})^{(-1)^j} & \xrightarrow{\otimes_j \det 0} & \bigotimes_j \mathbb{R}^{(-1)^j} \\ & \searrow \det 0 & & & \downarrow \\ \bigotimes_i (\det_{\mathbb{R}} C^{i,\bullet})^{(-1)^i} & \xrightarrow{\otimes_i \det 0} & \bigotimes_i (\mathbb{R})^{(-1)^i} & \xrightarrow{\quad} & \mathbb{R} \end{array}$$

Fix bases  $(u_k^{(i,j)})$  of the lattices inside the  $C^{i,j}$ , and denote by  $(u_k^{i,j})^{-1}$  the dual basis vectors. By looking at the image under the composite morphism  $\bigotimes_{i,j} (\det_{\mathbb{R}} C^{i,j})^{(-1)^{i+j}} \rightarrow \mathbb{R}$  of  $\bigotimes_{i,j} \bigwedge_k ((u_k^{(i,j)})^{(-1)^{i+j}})$ , we find the relation

$$\prod_i (\det_{\mathbb{Z}, \mathbb{R}} C^{i,\bullet})^{(-1)^i} = \prod_j (\det_{\mathbb{Z}, \mathbb{R}} C^{\bullet,j})^{(-1)^j}.$$

	$C^\bullet$	$D^\bullet$	$E^\bullet$	
				-1
		$\text{Hom}_{G_{\mathbb{R}}, X(\mathbb{C})}(\alpha^* F, 2i\pi\mathbb{Z})_{\mathbb{R}}$	$\text{Hom}_{G_{\mathbb{R}}, X(\mathbb{C})}(\alpha^* F, 2i\pi\mathbb{Z})_{\mathbb{R}}$	
		$\downarrow$	$\downarrow$	
$A^\bullet :$	$(H^1(X, F)_{\mathbb{R}})^\vee \longrightarrow H_{c, B}^{-1}(X, F^D)_{\mathbb{R}} \longrightarrow H^1 \text{Lie}_X(F^D)_{\mathbb{R}} \longrightarrow (H^0(X, F)_{\mathbb{R}})^\vee \rightarrow H_{c, B}^0(X, F^D)_{\mathbb{R}}$	$\downarrow$	$\downarrow$	0
	$\parallel$		$\parallel$	
$B^\bullet :$	$(H^1(X, F)_{\mathbb{R}})^\vee \longrightarrow H^{-1}(X, F^D)_{\mathbb{R}} \longrightarrow \prod_v \text{Hom}_{G_v}(F_v, \mathbb{Z})_{\mathbb{R}} \longrightarrow (H^0(X, F)_{\mathbb{R}})^\vee \rightarrow H^0(X, F^D)_{\mathbb{R}}$	$\downarrow$	$\downarrow \simeq$	1
	-1	0	1	2
	-1	0	1	3

Diagram IV.5

Hence in our case we obtain

$$\det_{\mathbb{Z}, \mathbb{R}} A^\bullet (\det_{\mathbb{Z}, \mathbb{R}} B^\bullet)^{-1} = \det_{\mathbb{Z}, \mathbb{R}} C^\bullet (\det_{\mathbb{Z}, \mathbb{R}} D^\bullet)^{-1} (\det_{\mathbb{Z}, \mathbb{R}} E^\bullet)^{-1}.$$

Thus it remains to compute  $\det_{\mathbb{Z}, \mathbb{R}} C^\bullet (\det_{\mathbb{Z}, \mathbb{R}} E^\bullet)^{-1}$ . Consider the following acyclic complex of abelian groups:

$$\begin{aligned} 0 &\longrightarrow \mathrm{Hom}_{G_{\mathbb{R}}, X(\mathbb{C})}(\alpha^* F, 2i\pi\mathbb{Z}) \longrightarrow H_{c, B}^{-1}(X, F^D) \longrightarrow H^{-1}(X, F^D) \\ G^\bullet : &\longrightarrow \mathrm{Ext}_{G_{\mathbb{R}}, X(\mathbb{C})}^1(\alpha^* F, 2i\pi\mathbb{Z}) \longrightarrow H_{c, B}^0(X, F^D) \longrightarrow H^0(X, F^D) \\ &\longrightarrow N_2 \longrightarrow 0 \end{aligned}$$

with first non-zero term in degree  $-1$ . The last term is a subgroup of  $\mathrm{Ext}_{G_{\mathbb{R}}, X(\mathbb{C})}^2(\alpha^* F, 2i\pi\mathbb{Z})$  and hence is finite 2-torsion. Then  $\det_{\mathbb{Z}} G^\bullet \xrightarrow[\det(0)]{\simeq} \mathbb{Z}$  and  $G_{\mathbb{R}}^\bullet = C^\bullet \oplus E^\bullet[-3]$ , thus we find

$$\det_{\mathbb{Z}, \mathbb{R}} C^\bullet (\det_{\mathbb{Z}, \mathbb{R}} E^\bullet)^{-1} = \det_{\mathbb{Z}, \mathbb{R}} G_{\mathbb{R}} = \prod_i [G_{\mathrm{tor}}^i]. \quad \square$$

**Proposition IV.7.10.** *Suppose that  $X$  is regular. We have*

$$\begin{aligned} \sum (-1)^i i \cdot \dim_{\mathbb{R}} H_{\mathrm{ar}, c}^i(X, \mathbb{Z}_{\mathbb{R}}^D) &= -1, \\ \chi_X(\mathbb{Z}^D) &= \frac{2^{r_1} (2\pi)^{r_2} h_K R_K}{\omega \sqrt{|\Delta_K|}}, \end{aligned}$$

and thus the special value formula

$$\begin{aligned} \mathrm{ord}_{s=0} L_X(\mathbb{Z}^D) &= \mathrm{ord}_{s=1} \zeta_X = \sum (-1)^i i \cdot \dim_{\mathbb{R}} H_{\mathrm{ar}, c}^i(X, \mathbb{Z}_{\mathbb{R}}^D), \\ L_X^*(\mathbb{Z}^D, 0) &= \zeta_X^*(1) = \chi_X(\mathbb{Z}^D). \end{aligned}$$

*Proof.* The first formula is immediate from the previous computations.

We use Proposition IV.7.9. We easily find  $r_2(\mathbb{Z}) = r_2$ ,  $r_1(\mathbb{Z}) = r_1$ ,  $N_2 = 0$ , and  $\mathrm{Lie}_X(\mathbb{Z}^D) = \mathcal{O}_K[1]$  so in particular  $H^0 \mathrm{Lie}_X(\mathbb{Z}^D) = 0$ . Thus using Propositions IV.2.10 and IV.3.6 we obtain

$$\chi_X(\mathbb{Z}^D) = \frac{2^{r_1} (2\pi)^{r_2} h_K R(\mathbb{Z}^D)}{\omega \mathrm{Disc}(\mathbb{Z}^D)}.$$

By the choices made in Definition IV.7.8,  $\mathrm{Disc}(\mathbb{Z}^D)$  is computed as follows: fix a basis  $(\alpha_i)$  of  $\mathcal{O}_K$ . Choose an ordering of the archimedean places, real places first and complex places second, and let  $\sigma_i$  denote the choice of an embedding for each place  $v_i$  and  $\sigma_{r_1+r_2+i} = \overline{\sigma_{r_1+i}}$ . Thus  $\sigma_1, \dots, \sigma_{r_1}$  are the real embeddings and  $\sigma_{r_1+1}, \dots, \sigma_{r_1+r_2}$  the chosen complex embeddings for each complex place. By unravelling the definitions, we see that we are asked to compute the determinant of the matrix  $\begin{pmatrix} (\Re(\sigma_i(\alpha_j)))_{1 \leq i \leq r_1+r_2, 1 \leq j \leq n} \\ (\Im(\sigma_{r_1+i'}(\alpha_j)))_{1 \leq i' \leq r_2, 1 \leq j \leq n} \end{pmatrix}$ . Observe that it is obtained from the matrix  $(\sigma_j(\alpha_i))_{i, j=1, \dots, n}$  by some elementary transformations on the rows which will make a factor  $(\frac{1}{i})^{r_2}$  appear. Since  $(\det(\sigma_j(\alpha_i))_{i, j})^2 = \Delta_K$  and the sign of  $\Delta_K$  is  $(-1)^{r_2}$  (Brill's theorem [Was97, Lemma 2.2]), we obtain

$$\mathrm{Disc}(\mathbb{Z}^D) = |(\frac{1}{i})^{r_2} \det(\sigma_j(\alpha_i))_{i, j}| = \sqrt{|\Delta_K|}.$$

It remains to determine  $R(\mathbb{Z}^D)$ . By definition, this is  $\det_{\mathbb{Z}, \mathbb{R}} B^\bullet$  where  $B^\bullet$  is the acyclic complex with first non-zero term in degree 0

$$0 \rightarrow (\mathcal{O}_K^\times)_{\mathbb{R}} \rightarrow \left( \prod_v \mathbb{Z} \right)_{\mathbb{R}} \rightarrow (\mathbb{Z}_{\mathbb{R}})^\vee \rightarrow 0.$$

Let us make the maps explicit. Denote by  $|\cdot|_v$  the normalized absolute value for the archimedean place  $v$ :  $|\cdot|_v = |\cdot|$  if  $v$  is real and  $|\cdot|_v = |\cdot|^2$  if  $v$  is complex. Denote also by  $L := \prod_v \log |\sigma_v(\cdot)|_v$  the logarithmic embedding. We have a commutative diagram

$$\begin{array}{ccccc} & & (\prod_\sigma \mathbb{R})^{G_{\mathbb{R}}} & & \\ & \nearrow \Pi_\sigma \log |\sigma(\cdot)| & \downarrow \simeq & \searrow \Sigma & \\ (\mathcal{O}_K^\times)_{\mathbb{R}} & \xrightarrow{L} & \prod_v \mathbb{R} & \xrightarrow{\Sigma} & \mathbb{R} \end{array}$$

where the middle isomorphism comes from Definition IV.7.8 and  $\Sigma$  is the sum of components map (see the discussion after Definition IV.5.1). Here  $\mathbb{R} = H_{c, \mathcal{D}}^0(X, \mathbb{Z}_{\mathbb{R}}^D) = H^0(X, \mathbb{Z})_{\mathbb{R}}^\vee$  obtains the canonical basis because the perfect pairing  $H_{c, \mathcal{D}}^0(X, \mathbb{Z}_{\mathbb{R}}^D) \times H^0(X, \mathbb{Z})_{\mathbb{R}} = \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  is given by the multiplication map. The ordering on the places gives an ordered basis on  $(\prod_v \mathbb{Z})_{\mathbb{R}}$ . Let  $x$  be the vector  $(1, 0, \dots, 0) \in (\prod_v \mathbb{Z})_{\mathbb{R}}$ . We have  $\Sigma(x) = 1$ . Denote by  $(\varepsilon_i)$  a system of fundamental units of  $\mathcal{O}_K^\times$ . Then  $\det_{\mathbb{Z}, \mathbb{R}} B^\bullet$  is the absolute value of the determinant of the matrix  $P$  expressing  $(L(\varepsilon_1), L(\varepsilon_2), \dots, L(\varepsilon_{r_1+r_2-1}), x)$  in the basis of  $(\prod_v \mathbb{Z})_{\mathbb{R}}$ . Put  $N_i = 1$  if  $\sigma_i$  is real and  $N_i = 2$  if  $\sigma_i$  is complex. Then we have

$$P = \begin{pmatrix} N_1 \log |\sigma_1(\varepsilon_1)| & \cdots & N_1 \log |\sigma_1(\varepsilon_{s-1})| & 1 \\ N_2 \log |\sigma_2(\varepsilon_1)| & \cdots & N_2 \log |\sigma_2(\varepsilon_{s-1})| & 0 \\ \vdots & \ddots & \vdots & \vdots \\ N_s \log |\sigma_s(\varepsilon_1)| & \cdots & N_s \log |\sigma_s(\varepsilon_{s-1})| & 0 \end{pmatrix}$$

By definition, the regulator  $R_K$  is the absolute value of the determinant of any  $(s-1) \times (s-1)$  minor of the matrix  $(N_i \log |\sigma_i(\varepsilon_j)|)_{i=1, \dots, s; j=1, \dots, s-1}$ ; thus by developing the determinant of  $P$  with respect to the last column we find

$$R(\mathbb{Z}^D) = |\det P| = R_K.$$

We obtain finally

$$\chi_X(\mathbb{Z}^D) = \frac{2^{r_1} (2\pi)^{r_2} h_K R_K}{\omega \sqrt{|\Delta_K|}}. \quad \square$$

*Remark.* For  $F = \mathbb{Z}$ , the complexes

$$\mathrm{Lie}_X(\mathbb{Z}^D), \quad R\Gamma_{W,c}(X, \mathbb{Z}^D), \quad R\Gamma_{c, \mathcal{D}}(X, \mathbb{Z}_{\mathbb{R}}^D) \quad \text{and} \quad R\Gamma_{\mathrm{ar}, c}(X, \mathbb{Z}_{\mathbb{R}}^D)$$

are equal respectively (up to a shift) to the complexes

$$R\Gamma_{dR}(X/\mathbb{Z})/F^1, \quad R\Gamma_{W,c}(X, \mathbb{Z}(1)), \quad R\Gamma_c(X, \mathbb{R}(1)) \quad \text{and} \quad R\Gamma_{\mathrm{ar}, c}(X, \tilde{\mathbb{R}}(1))$$

of [FM18], in which the above formula was already obtained.



**Proposition IV.7.11.** *Let  $i : x \rightarrow X$  be the inclusion of a closed point,  $M$  a discrete  $G_x$ -module of finite type. Let  $\chi_{W,c}$  denote the Weil-étale Euler characteristic of a  $\mathbb{Z}$ -constructible sheaf on  $X$  from Subsection II.6.4 and let  $R(M)$  be the absolute value of the determinant of the pairing*

$$H^0(G_x, M)_{\mathbb{R}} \times H^0(G_x, M^{\vee})_{\mathbb{R}} \rightarrow \mathbb{R}$$

*in bases modulo torsion. Then*

$$\sum (-1)^i \cdot \dim_{\mathbb{R}} H_{\text{ar},c}^i(X, (i_*M)_{\mathbb{R}}^D) = -\text{rank}_{\mathbb{Z}} H^0(G_x, M^{\vee}),$$

$$\begin{aligned} \chi_X((i_*M)^D) &= \chi_{W,c}(i_*M^{\vee}) = \chi_{W,c}(i_*M) \\ &= \frac{[H^0(G_x, M)]_{\text{tor}}}{[H^1(G_x, M)]R(M) \log(N(v))^{\text{rank } H^0(G_x, M)}}. \end{aligned}$$

*We have the special value formula*

$$\begin{aligned} \text{ord}_{s=0} L_X((i_*M)^D) &= \sum (-1)^i \cdot \dim_{\mathbb{R}} H_{\text{ar},c}^i(X, (i_*M)_{\mathbb{R}}^D), \\ L_X^*((i_*M)^D, 0) &= \chi_X((i_*M)^D). \end{aligned}$$

*Proof.* The computation of  $\sum (-1)^i \cdot \dim_{\mathbb{R}} H_{\text{ar},c}^i(X, (i_*M)_{\mathbb{R}}^D)$  is straightforward. We will show the first and second equality of Weil-étale Euler characteristics by applying Theorem III.2.1; the explicit formula is then Corollary II.6.8. Since  $(i_*M)^D = i_*M^{\vee}$  is a complex of  $\mathbb{Z}$ -constructible sheaves, Theorem II.6.24 gives the special value formula.

It thus only remains to show the equality of Euler characteristics  $\chi_X((i_*-)^D) = \chi_{W,c}(i_*-) = \chi_{W,c}(i_*(-)^{\vee})$  on  $\text{Sh}_{\mathbb{Z}}(x)^{\text{constr}}$ . By Theorem III.2.1 we just have to check that they all coincide on  $\pi'_*\mathbb{Z}$  for  $\pi' : y \rightarrow x$  any finite morphism with  $y$  normal integral, i.e.  $y$  is the spectrum of a finite extension of  $\kappa(x)$ . There exists a finite morphism  $\pi : Y \rightarrow X$  such that  $Y$  is normal integral,  $y$  is a closed point  $i' : y \rightarrow Y$  of  $Y$  and  $\pi \circ i' = i \circ \pi'$ , so we can reduce to checking that they all coincide for any pair  $x, X$  and the sheaf  $\mathbb{Z}$  on  $x$ . We have  $(i_*\mathbb{Z})^D = i_*\mathbb{Z}$  and using Proposition IV.7.9 we find

$$\chi_X((i_*\mathbb{Z})^D) = R((i_*\mathbb{Z})^D).$$

Here, the complex  $B^{\bullet}$  is

$$(H^0(X, i_*\mathbb{Z})_{\mathbb{R}})^{\vee} \xrightarrow{\sim} H^0(X, i_*\mathbb{Z}^{\vee})_{\mathbb{R}}$$

with first term in degree 2, so  $R((i_*\mathbb{Z})^D) = \det_{\mathbb{Z}, \mathbb{R}} B^{\bullet}$  is the inverse of the absolute value of the determinant of the pairing

$$H^0(X, i_*\mathbb{Z}^{\vee})_{\mathbb{R}} \times H^0(X, i_*\mathbb{Z})_{\mathbb{R}} \rightarrow \mathbb{R}.$$

Both  $H^0(X, i_*\mathbb{Z}^{\vee})$  and  $H^0(X, i_*\mathbb{Z})$  identify with  $\mathbb{Z}$ ; as  $R(\mathbb{Z}) = 1$ , from the proof of Theorem IV.5.2 we see that

$$R((i_*M)^D) = \frac{1}{\log(N(v))},$$

which is also  $\chi_{W,c}(i_*\mathbb{Z})$  by Corollary II.6.8. □

**Theorem IV.7.12.** *Let  $F$  be a constructible sheaf on the spectrum  $X$  of an order in a number field. Then*

$$\sum (-1)^i i \cdot \dim_{\mathbb{R}} H_{\text{ar},c}^i(X, F_{\mathbb{R}}^D) = 0 = \text{ord}_{s=0} L_X(F^D, s).$$

If  $F$  is moreover tamely ramified, then

$$\chi_X(F^D) = 1.$$

In particular, we have the special value formula  $L_X^*(F^D, 0) = \pm \chi_X(F^D)$ .

*Proof.* Since  $\chi_X$  is an Euler characteristic on tamely ramified  $\mathbb{Z}$ -constructible sheaves with value in the torsion-free group  $\mathbb{R}_{>0}$ , we can use a variant of Theorem III.1.1, replacing  $G_K$  by  $G_K^t$  in the argument.  $\square$

*Remark.*

- If  $F$  is a tamely ramified constructible sheaf, we have

$$\begin{aligned} R\Gamma_{\text{ar},c}(X, F_{\mathbb{R}}^D) &= 0, \\ R\Gamma_{W,c}(X, F^D) &\simeq R\Gamma_{c,B}(X, F^D), \\ \text{Lie}_X(F^D) &= \text{Ext}_{G_K^t}^1(F_{\eta}, \mathcal{O}_{K^t})[0], \end{aligned}$$

and the two latter complexes are bounded with finite cohomology groups. Thus their Euler characteristics

$$\chi_{W,X}(F^D) := \prod_{i \in \mathbb{Z}} [H_{W,c}^i(X, F^D)]^{(-1)^i}$$

and

$$\chi_{L,X}(F^D) := [\text{Ext}_{G_K^t}^1(F_{\eta}, \mathcal{O}_{K^t})]$$

are well-defined and we have  $\chi_X(F^D) = \chi_{W,X}(F^D) / \chi_{L,X}(F^D)$ . On the other hand, using Artin–Verdier duality, the morphism of fiber sequences Equation (IV.2.1) and Theorem III.2.1, we find

$$\chi_{W,X}(F^D) = [F_{\eta}]^{[K:\mathbb{Q}]}.$$

Thus the above proposition implies that

$$[\text{Ext}_{G_K^t}^1(F_{\eta}, \mathcal{O}_{K^t})] = [F_{\eta}]^{[K:\mathbb{Q}]}.$$

**Theorem IV.7.13** (Special values theorem). *Let  $F$  be a  $\mathbb{Z}$ -constructible sheaf on the spectrum  $X$  of an order in a number field. We have the vanishing order formula*

$$\text{ord}_{s=0} L_X(F^D, s) = \sum (-1)^i i \cdot \dim_{\mathbb{R}} H_{\text{ar},c}^i(X, F_{\mathbb{R}}^D).$$

If  $F$  is tamely ramified, we have the special value formula

$$L_X^*(F^D, 0) = \pm \chi_X(F^D).$$

*Proof.* This follows from Propositions IV.7.10 and IV.7.11 by Theorem III.2.1 (replacing  $G_K$  by  $G_K^t$  in the argument).  $\square$

# Chapter V

## The determinant of the Lie algebra for the Néron model of a torus and special values at $s = 1$

In this chapter, we investigate the multiplicativity property of the determinant of the Lie algebra for Néron models of tori over global fields, and deduce a Weil-étale special value formula at  $s = 0$  for the  $L$ -functions of sheaves of the form  $F^D := R\mathcal{H}om_X(F, \mathbb{Z}^c)$  where  $X$  is a 1-dimensional integral proper arithmetic scheme flat over  $\mathrm{Spec}(\mathbb{Z})$  and  $F$  is a  $\mathbb{Z}$ -constructible sheaf on  $X$ .

### V.1 Multiplicativity of the determinant of the Lie algebra for the Néron model of a torus

If  $G$  is a group scheme over a scheme  $S$ , we denote  $\mathcal{L}ie(G)$  its Lie algebra sheaf [DGB70, Exp. II]; it is a sheaf of  $\mathcal{O}_S$ -modules on  $S$ . We also denote  $\mathrm{Lie}(G)$  its global sections. Let  $K$  be a local or global field. In the local or number field case, denote  $\mathcal{O}_K$  its ring of integers and  $X = \mathrm{Spec}(\mathcal{O}_K)$ , or let  $X$  be the smooth complete curve over  $\mathbb{F}_p$  with function field  $K$ . When it exists, we denote  $\mathcal{N}(G)$  the locally of finite type Néron model over  $X$  of a smooth algebraic group  $G$  over  $\mathrm{Spec}(K)$  [BLR90, Chapter 10]. We begin with the following theorem, which is a rather immediate consequence of Chai–Yu’s base change conductor formula for tori [CY01]:

**Theorem V.1.1** ([Cha00, Question 8.1]). *Let*

$$0 \rightarrow T' \rightarrow T \rightarrow T'' \rightarrow 0$$

*be a short exact sequence of  $K$ -tori. There is a multiplicative identity*

$$\det_{\mathcal{O}_X} \mathcal{L}ie\mathcal{N}(T') \otimes_{\mathcal{O}_X} \det_{\mathcal{O}_X} \mathcal{L}ie\mathcal{N}(T'') \xrightarrow{\cong} \det_{\mathcal{O}_X} \mathcal{L}ie\mathcal{N}(T),$$

*compatible with the canonical isomorphism  $\det_K \mathrm{Lie}(T') \otimes_K \det_K \mathrm{Lie}(T'') \xrightarrow{\cong} \det_K \mathrm{Lie}(T)$ .*

*Remark.* This is an identity of graded lines.

*Proof.* It suffices to check this after localizing and completing at a closed point so we reduce to the local case. Let  $a \in K^\times$  such that the image of  $\det_{\mathcal{O}_K} \mathcal{L}ie\mathcal{N}(T') \otimes_{\mathcal{O}_K} \det_{\mathcal{O}_K} \mathcal{L}ie\mathcal{N}(T'')$

under the canonical isomorphism is  $a \det_{\mathcal{O}_K} \mathcal{L}ie\mathcal{N}(T)$ ; we want to show that  $a \in \mathcal{O}_K^\times$ . Let  $L/K$  be a finite extension such that all the considered tori split over  $L$ . Let us check that  $\det \mathcal{L}ie\mathcal{N}(T'_L) \otimes \det \mathcal{L}ie\mathcal{N}(T''_L) \xrightarrow{\cong} \det \mathcal{L}ie\mathcal{N}(T_L)$ . This can be verified locally and after completing; thus let  $T$  be a split torus over a local field. Then  $\mathcal{L}ie\mathcal{N}(T) = \text{Hom}_{\text{Ab}}(X^*(T), \mathcal{O}_K)$  [CY01, A1.7] and the latter functor is exact in  $T$  since the character group  $X^*(T)$  is torsion-free, making the  $\text{Ext}^1$  vanish. We thus obtain the desired identity by taking the determinant of the fiber sequence corresponding to a short exact sequence of split tori  $0 \rightarrow T' \rightarrow T \rightarrow T'' \rightarrow 0$ . Recall Chai–Yu’s base change conductor

$$\begin{aligned} c(T) &:= \frac{1}{e_{L/K}} \text{length}_{\mathcal{O}_L} \left( \frac{\mathcal{L}ie\mathcal{N}(T_L)}{\mathcal{L}ie\mathcal{N}(T) \otimes_{\mathcal{O}_K} \mathcal{O}_L} \right) \\ &= \frac{1}{e_{L/K}} \text{length}_{\mathcal{O}_L} \left( \frac{\det_{\mathcal{O}_L} \mathcal{L}ie\mathcal{N}(T_L)}{(\det_{\mathcal{O}_K} \mathcal{L}ie\mathcal{N}(T)) \otimes \mathcal{O}_L} \right). \end{aligned}$$

By Chai–Yu’s formula expressing the base change conductor as one half of the Artin conductor of the cocharacter module [CY01], we have  $c(T) = c(T') + c(T'')$  [CLN13, 2.4.2]. Consider the following diagram of sublattices

$$\begin{array}{ccc} \det_{\mathcal{O}_K} \mathcal{L}ie\mathcal{N}(T') \otimes_{\mathcal{O}_K} \det_{\mathcal{O}_K} \mathcal{L}ie\mathcal{N}(T'') \otimes_{\mathcal{O}_K} \mathcal{O}_L & & \det_{\mathcal{O}_K} \mathcal{L}ie\mathcal{N}(T') \otimes_{\mathcal{O}_K} \mathcal{O}_L \\ \cap & & \cap \\ \det_{\mathcal{O}_L} \mathcal{L}ie\mathcal{N}(T'_L) \otimes_{\mathcal{O}_L} \det_{\mathcal{O}_L} \mathcal{L}ie\mathcal{N}(T''_L) & \xrightarrow{\cong} & \det_{\mathcal{O}_L} \mathcal{L}ie\mathcal{N}(T_L) \\ \cap & & \cap \\ \det_L \mathcal{L}ie(T'_L) \otimes_L \det_L \mathcal{L}ie(T''_L) & \xrightarrow{\cong} & \det_L \mathcal{L}ie(T_L) \end{array}$$

Computing lengths, we find<sup>1</sup>

$$\begin{aligned} v_L(a) &= \text{length} \left( \frac{a \det_{\mathcal{O}_K} \mathcal{L}ie\mathcal{N}(T) \otimes_{\mathcal{O}_K} \mathcal{O}_L}{\det_{\mathcal{O}_K} \mathcal{L}ie\mathcal{N}(T) \otimes_{\mathcal{O}_K} \mathcal{O}_L} \right) \\ &= \text{length} \left( \frac{a \det_{\mathcal{O}_K} \mathcal{L}ie\mathcal{N}(T) \otimes_{\mathcal{O}_K} \mathcal{O}_L}{\det_{\mathcal{O}_L} \mathcal{L}ie\mathcal{N}(T_L)} \right) + e_{L/K} c(T) \\ &= \text{length} \left( \frac{\det_{\mathcal{O}_K} \mathcal{L}ie\mathcal{N}(T') \otimes_{\mathcal{O}_K} \det_{\mathcal{O}_K} \mathcal{L}ie\mathcal{N}(T'') \otimes_{\mathcal{O}_K} \mathcal{O}_L}{\det_{\mathcal{O}_L} \mathcal{L}ie\mathcal{N}(T'_L) \otimes_{\mathcal{O}_L} \det_{\mathcal{O}_L} \mathcal{L}ie\mathcal{N}(T''_L)} \right) + e_{L/K} c(T) \\ &= e_{L/K} (-c(T') - c(T'') + c(T)) \\ &= 0. \end{aligned} \quad \square$$

**Corollary V.1.2.** *Let  $K$  be a global field and let*

$$0 \rightarrow T' \rightarrow T \rightarrow T'' \rightarrow 0$$

*be a short exact sequence of tori over  $K$ . We have a multiplicative identity*

$$\det_{\mathbb{Z}} R\Gamma(X, \mathcal{L}ie\mathcal{N}(T')) \otimes_{\mathbb{Z}} \det_{\mathbb{Z}} R\Gamma(X, \mathcal{L}ie\mathcal{N}(T'')) \xrightarrow{\cong} \det_{\mathbb{Z}} R\Gamma(X, \mathcal{L}ie\mathcal{N}(T)),$$

*compatible with  $\det_{\mathbb{Q}} \text{Lie}_K(T') \otimes_{\mathbb{Q}} \det_{\mathbb{Q}} \text{Lie}_K(T'') \xrightarrow{\cong} \det_{\mathbb{Q}} \text{Lie}_K(T)$  in the number field case, and with the canonical trivializations  $(\det_{\mathbb{Z}} R\Gamma(X, \mathcal{L}ie\mathcal{N}(T))) \otimes \mathbb{Q} \xrightarrow{\cong} \mathbb{Q}$  in the function field case.*

<sup>1</sup>Using some light abuse of notations, where we denote  $\text{length}(M/N) := -\text{length}(N/M)$  if  $M \subset N$  for two  $\mathcal{O}_L$  lattices  $M$  and  $N$  inside  $L$

*Remark.* If  $K$  is a number field then we see  $R\Gamma(X, -)$  as the forgetful functor  $D^b(X) = D^b(\mathcal{O}_K) \rightarrow D^b(\mathbb{Z})$ .

*Proof.* Suppose that  $K$  is a number field. For a commutative ring  $R$ , denote  $\mathcal{P}_R$  the Picard groupoid of graded  $R$ -lines. By  $K$ -theory computations in degree  $\leq 1$ , the determinant functors  $\det_R : D^{perf}(R) \rightarrow \mathcal{P}_R$  for  $R = \mathcal{O}_K, K, \mathbb{Z}, \mathbb{Q}$  are the universal determinant functors [BF03, § 2.5]. Composition of any determinant functor on  $D^{perf}(\mathbb{Z})$  with the forgetful functor  $U = R\Gamma(X, -) : D^{perf}(\mathcal{O}_K) \rightarrow D^{perf}(\mathbb{Z})$  induces a determinant functor, so we get an induced map of Picard groupoids  $\tilde{U} : \mathcal{P}_{\mathcal{O}_K} \rightarrow \mathcal{P}_{\mathbb{Z}}$  such that  $\tilde{U} \circ \det_{\mathcal{O}_K} = \det_{\mathbb{Z}} \circ U$ . We can thus apply the functor of Picard groupoids  $\tilde{U}$  to the multiplicative identity of the previous theorem to obtain the identity we want (implicit is the compatibility with base change and the analogous forgetful functor  $D^b(K) \rightarrow D^b(\mathbb{Q})$ ).

If  $K$  is a function field, the corollary amounts to showing the equality of Euler characteristics of coherent sheaves

$$\chi_X(\mathcal{L}ie\mathcal{N}(T)) = \chi_X(\mathcal{L}ie\mathcal{N}(T')) + \chi_X(\mathcal{L}ie\mathcal{N}(T''))$$

By Riemann-Roch, we have

$$\chi_X(\mathcal{L}ie\mathcal{N}(T)) = (1 - g) \dim T + \deg(\det_X \mathcal{L}ie\mathcal{N}(T))$$

whence we conclude from the previous theorem.  $\square$

*Remark.* Note that the argument for the number field case does not go through for the function field case because the usual determinant is *not* the universal determinant. Indeed we have  $K_1(X) = (\mathcal{O}_X(X)^\times)^2$  ([DM97, 2.5]) while  $\pi_1(\mathcal{P}_X) = \mathcal{O}_X(X)^\times$ .

## V.2 The additive part of the fundamental line

In the following,  $X$  will denote an integral proper arithmetic curve flat over  $\text{Spec}(\mathbb{Z})$  with function field  $K$ . Such a scheme is the spectrum of an order in the number field  $K$ . The normalization of  $X$  is then  $\hat{X} := \text{Spec}(\mathcal{O}_K)$ . For a torus  $T$  over  $K$ , we let

$$\text{Lie}_{\hat{X}}(\mathcal{N}(T)) := R\Gamma(\hat{X}, \mathcal{L}ie_{\hat{X}}(\mathcal{N}(T))) = \Gamma(\hat{X}, \mathcal{L}ie_{\hat{X}}(\mathcal{N}(T))).$$

Let  $\Gamma$  be a finite group and let  $M$  be a finite type  $\Gamma$ -module. A torsion-free  $\Gamma$ -resolution of  $M$  is a short exact sequence  $0 \rightarrow P \rightarrow \mathbb{Z}[\Gamma]^n \rightarrow M \rightarrow 0$  of  $\Gamma$ -modules with  $n \in \mathbb{N}$ .

**Definition V.2.1.** *Let  $M$  be a finite type discrete  $G_K$ -module and fix a finite Galois extension  $L/K$  such that  $G_L$  acts trivially on  $M$ . Let  $\Gamma := G_K/G_L$ . Fix a torsion-free  $\Gamma$ -resolution  $\mathcal{E} : 0 \rightarrow P \rightarrow \mathbb{Z}[\Gamma]^n \rightarrow M \rightarrow 0$  of  $M$ , and denote  $D(-)$  the group of multiplicative type associated to a given finite type  $G_K$ -module. The resolution  $\mathcal{E}$  induces an isomorphism*

$$\tau_{\mathcal{E}} : \det_{\mathbb{Q}} \text{Lie}_K(D(M)) \xrightarrow{\sim} \det_{\mathbb{Q}} \left[ \text{Lie}_K(D(\mathbb{Z}[\Gamma]^n)) \rightarrow \text{Lie}_K(D(P)) \right].$$

We define

$$\Delta_{\hat{X}}^{\text{add}}(D(M)) := \tau_{\mathcal{E}}^{-1} \left( \det_{\mathbb{Z}} \left[ \text{Lie}_{\hat{X}}(\mathcal{N}(D(\mathbb{Z}[\Gamma]^n))) \rightarrow \text{Lie}_{\hat{X}}(\mathcal{N}(D(P))) \right] \right)$$

(where the complex has first term in degree 0) as a graded line inside  $\det_{\mathbb{Q}} \text{Lie}_K(D(M))$ .

If  $F$  is a  $\mathbb{Z}$ -constructible sheaf on  $X$ , we define  $\Delta_X^{\text{add}}(F^D) := \Delta_{\hat{X}}^{\text{add}}(D(g^*F))^{-1}$ .

*Remark.* We have  $\text{Lie}_K(D(M)) = \text{Hom}_{G_K}(M, K^{sep})$  which is exact in  $M$  as  $K^{sep}$  is uniquely divisible.

When  $L$  is fixed, this definition makes sense because torsion-free  $\Gamma$ -resolutions are unique up to unique isomorphism in the category of complexes of  $\Gamma$ -modules modulo homotopy, and the construction is multiplicative with respect of short exact sequences of finite type  $\Gamma$ -modules because of Corollary V.1.2 and the existence of exact sequences of torsion-free  $\Gamma$ -resolutions for short exact sequences. We mean by the latter that given a short exact sequence  $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$  of finite type  $\Gamma$ -modules and two torsion-free  $\Gamma$ -resolutions  $0 \rightarrow P \rightarrow \mathbb{Z}[\Gamma]^n \rightarrow M' \rightarrow 0$  and  $0 \rightarrow R \rightarrow \mathbb{Z}[\Gamma]^m \rightarrow M'' \rightarrow 0$ , there exists a commutative diagram

$$\begin{array}{ccccccc}
& & 0 & & 0 & & 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & P & \longrightarrow & \mathbb{Z}[\Gamma]^n & \longrightarrow & M' \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & Q & \longrightarrow & \mathbb{Z}[\Gamma]^n \oplus \mathbb{Z}[\Gamma]^m & \longrightarrow & M \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & R & \longrightarrow & \mathbb{Z}[\Gamma]^m & \longrightarrow & M'' \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
& & 0 & & 0 & & 0
\end{array}$$

where the middle column is split and the middle row is a torsion-free  $\Gamma$ -resolution.

We have to check that the definition is compatible with varying the extension  $L/K$ ; without loss of generality, we can consider the case where  $L'/L/K$  is a tower of finite Galois extensions of  $K$ , hence where  $\Gamma := G_K/G_L$  is a quotient of  $\Gamma' := G_K/G_{L'}$ . Let  $M$  be a finite type  $\Gamma$ -module and choose a torsion-free  $\Gamma$ -resolution  $0 \rightarrow P \rightarrow \mathbb{Z}[\Gamma]^n \rightarrow M \rightarrow 0$ . Then  $\mathbb{Z}[\Gamma']^n \rightarrow \mathbb{Z}[\Gamma]^n \rightarrow M$  is surjective, so we have a torsion-free  $\Gamma'$ -resolution  $0 \rightarrow Q \rightarrow \mathbb{Z}[\Gamma']^n \rightarrow M \rightarrow 0$ . It suffices to show that these two resolutions yield the same graded line. We have a commutative diagram, where the two left columns are short exact sequences of torsion-free discrete  $G_K$ -modules of finite type:

$$\begin{array}{ccccccc}
& & K & \xlongequal{\quad} & K & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
\varepsilon' : & 0 & \longrightarrow & Q & \longrightarrow & \mathbb{Z}[\Gamma']^n & \longrightarrow M \longrightarrow 0 \\
& & & \downarrow & & \downarrow & & \downarrow \\
\varepsilon : & 0 & \longrightarrow & P & \longrightarrow & \mathbb{Z}[\Gamma]^n & \longrightarrow M \longrightarrow 0 \\
& & & \downarrow & & \downarrow & & \downarrow \\
& & & 0 & & 0 & & 0
\end{array}$$

Denote temporarily  $L_X := \det_{\mathbb{Q}} \text{Lie}(D(X))$  for a finite type  $G_K$ -module  $X$ , and  $\mathcal{L}_X := \det_{\mathbb{Z}} \text{Lie}(\mathcal{N}(D(X)))$  when  $X$  is torsion-free. From the above commutative diagram, the associativity axiom for determinants [Bre11, §3.1] gives a commutative diagram of

isomorphisms

$$\begin{array}{ccc}
& L_{\mathbb{Z}[\Gamma']^n} L_Q^{-1} & \\
& \nearrow \tau_{\varepsilon'} & \searrow \\
L_M & & L_{\mathbb{Z}[\Gamma']^n} L_K^{-1} L_P^{-1} \\
& \searrow \tau_{\varepsilon} & \nearrow \\
& L_{\mathbb{Z}[\Gamma]^n} L_P^{-1} &
\end{array}$$

By Corollary V.1.2, the sublattices  $\mathcal{L}_{\mathbb{Z}[\Gamma']^n} \mathcal{L}_Q^{-1}$  of  $L_{\mathbb{Z}[\Gamma']^n} L_Q^{-1}$  and  $\mathcal{L}_{\mathbb{Z}[\Gamma]^n} \mathcal{L}_P^{-1}$  of  $L_{\mathbb{Z}[\Gamma]^n} L_P^{-1}$  have the same image in  $L_{\mathbb{Z}[\Gamma']^n} L_K^{-1} L_P^{-1}$ , namely  $\mathcal{L}_{\mathbb{Z}[\Gamma']^n} \mathcal{L}_K^{-1} \mathcal{L}_P^{-1}$ , whence

$$\begin{aligned}
& \tau_{\varepsilon'}^{-1} \left( \det_{\mathbb{Z}} \left[ \text{Lie}_{\widehat{X}}(\mathcal{N}(D(\mathbb{Z}[\Gamma']^n))) \rightarrow \text{Lie}_{\widehat{X}}(\mathcal{N}(D(Q))) \right] \right) \\
&= \tau_{\varepsilon}^{-1} \left( \det_{\mathbb{Z}} \left[ \text{Lie}_{\widehat{X}}(\mathcal{N}(D(\mathbb{Z}[\Gamma]^m))) \rightarrow \text{Lie}_{\widehat{X}}(\mathcal{N}(D(P))) \right] \right).
\end{aligned}$$

We thus get:

**Proposition V.2.2.** *The mapping  $M \mapsto \Delta_{\widehat{X}}^{\text{add}}(D(M))$  is well-defined and is multiplicative with respect to short exact sequences, compatibly with  $M \mapsto \det_{\mathbb{Q}} \text{Lie}_K(D(M))$ .*

The well-definedness is what we just showed. A short exact sequence of finite type discrete  $G_K$ -modules is a short exact sequence of  $\Gamma := G_K/G_L$ -modules for some finite Galois extension  $L/K$ , in which case we have seen the multiplicativity.

**Proposition V.2.3.** *Let  $Y$  be the spectrum of an order in a number field with a finite dominant morphism  $\pi : Y \rightarrow X$  and let  $F$  be a  $\mathbb{Z}$ -constructible sheaf on  $Y$ . We have*

$$\Delta_X^{\text{add}}((\pi_* F)^D) = \Delta_Y^{\text{add}}(F^D).$$

*Proof.* Denote  $L = K(Y)$ . It suffices to show that if  $M$  is a discrete  $G_L$ -module of finite type,  $\Delta_{\widehat{X}}^{\text{add}}((D(\text{ind}_{G_L}^{G_K} M))) = \Delta_{\widehat{Y}}^{\text{add}}(D(M))$ . Choose an open subgroup  $H \subset G_L$  such that  $H$  is normal in  $G_K$  and  $H$  acts trivially on  $M$ , and choose a torsion-free  $G_L/H$ -cover  $0 \rightarrow P \rightarrow \mathbb{Z}[G_L/H]^n \rightarrow M \rightarrow 0$ . We have  $\text{ind}_{G_L}^{G_K} \mathbb{Z}[G_L/H] = \mathbb{Z}[G_K/H]$  so applying the exact functor  $\text{ind}_{G_L}^{G_K}$  gives a torsion-free  $\mathbb{Z}[G_K/H]$ -cover of  $\text{ind}_{G_L}^{G_K} M$ . The formation of Néron models commutes with Weil restriction [BLR90, 7.6/6] and  $\text{Lie}_{\widehat{X}}(\text{Res}_{\widehat{Y}/\widehat{X}} \mathcal{N}(T)) = \pi_* \text{Lie}_{\widehat{Y}}(\mathcal{N}(T))$  for a torus  $T$  over  $L$ , because  $\text{Res}_{\widehat{Y}/\widehat{X}} \mathcal{N}(T)$  represents  $\pi_{\text{fppf},*} \mathcal{N}(T)$  and  $\pi_{\text{fppf},*}$  is left exact; similarly there is a canonical isomorphism  $\text{Lie}_K(\text{Res}_{L/K} T) \simeq \text{Lie}_L(T)$ . The result follows.  $\square$

**Proposition V.2.4.** *Let  $M$  be a finite type discrete  $G_K$ -module. There is a canonical isomorphism*

$$\text{Lie}_K(D(M))_{\mathbb{R}} \simeq R\text{Hom}_{G_{\mathbb{R}}, X(\mathbb{C})}(M, \mathbb{C}),$$

where  $M$  is seen as a  $G_{\mathbb{R}}$ -equivariant sheaf on  $X(\mathbb{C})$  through the decomposition groups at archimedean places of  $K$ . Let  $F$  be a  $\mathbb{Z}$ -constructible sheaf on  $X$ . There is a canonical isomorphism

$$\Delta_X^{\text{add}}(F^D)_{\mathbb{R}} \simeq \det_{\mathbb{R}} R\text{Hom}_{G_{\mathbb{R}}, X(\mathbb{C})}(g^* F, \mathbb{C}[1]).$$

*Proof.* The first isomorphism was proven in Proposition IV.4.8. The second follows since by construction  $\Delta_X^{\text{add}}(F^D) \otimes \mathbb{Q} = \Delta_X^{\text{add}}(D(g^* F)) \otimes \mathbb{Q} = \det_{\mathbb{Q}} \text{Lie}_K(D(g^* F))$ .  $\square$

**Proposition V.2.5.** *Suppose that  $M$  is a tamely ramified  $G_K$ -module of finite type. Then*

$$\Delta_{\hat{X}}^{\text{add}}(D(M)) = \det_{\mathbb{Z}} R \text{Hom}_{G_K^t}(M, \mathcal{O}_{K^t}),$$

*canonically inside  $\det_{\mathbb{Q}} \text{Lie}_K(D(M)) = \det_{\mathbb{Q}} \text{Hom}_{G_K}(M, K^{\text{sep}})$ . In particular, if  $F$  is a tamely ramified  $\mathbb{Z}$ -construcible sheaf on  $X$  then*

$$\Delta_X^{\text{add}}(F^D) = \det_{\mathbb{Z}} \text{Lie}_X(F^D).$$

*where  $\text{Lie}_X(F^D)$  is the tangent space of  $F^D$  introduced in Chapter IV.*

*Proof.* As both sides are multiplicative with respect to short exact sequences of tamely ramified  $G_K$ -modules, by taking a torsion-free resolution it suffices to prove the proposition for  $M$  torsion-free. In that case, denote  $T$  the torus over  $K$  with character group  $M$ . By Corollary V.1.2 we actually have  $\Delta_{\hat{X}}^{\text{add}}(D(M)) = \det_{\mathbb{Z}} \text{Lie}_X(\mathcal{N}(T))$  whence we conclude with Corollary IV.4.7.  $\square$

### V.3 The fundamental line, the Euler characteristic and the special value formula

**Definition V.3.1.** *Let  $F$  be a red or blue sheaf (see Definition IV.3.1) on  $X$ . The fundamental line associated to  $F^D$  is*

$$\Delta_X(F^D) := \det_{\mathbb{Z}} R\Gamma_{W,c}(X, F^D) \otimes \Delta_X^{\text{add}}(F^D).$$

As in Subsection IV.7.1, we observe that the fiber of the composite map

$$\begin{aligned} R\Gamma_{W,c}(X, F^D)_{\mathbb{R}} &= R \text{Hom}(R\Gamma(X, F), \mathbb{R}[-1]) \oplus R\Gamma_{c,B}(X, F^D) \\ &\rightarrow R\Gamma_{c,B}(X, F^D) \\ &\xrightarrow{\text{Log}} R \text{Hom}_{G_{\mathbb{R}}, X(\mathbb{C})}(g^*F, \mathbb{C}[1]) \end{aligned}$$

identifies with  $R\Gamma_{ar,c}(X, F_{\mathbb{R}}^D)$ , the  $\mathbb{R}$ -determinant of which is canonically trivial, so using Proposition V.2.4 we obtain a natural trivialization

$$\lambda : \Delta_X(F^D)_{\mathbb{R}} \xrightarrow{\sim} \mathbb{R}.$$

**Definition V.3.2.** *Let  $F$  be a red or blue sheaf on  $X$ . The Weil-étale Euler characteristic of  $F^D$  is the positive real number  $\chi_X(F^D)$  such that*

$$\lambda(\Delta_X(F^D)) = \chi_X(F^D)^{-1} \mathbb{Z} \hookrightarrow \mathbb{R}$$

**Proposition V.3.3.** *The Weil-étale Euler characteristic is multiplicative with respect to red-to-blue short exact sequences. Let  $Y$  be the spectrum of an order in a number field with a finite dominant morphism  $\pi : Y \rightarrow X$  and let  $F$  be a red or blue sheaf on  $Y$ . Then*

$$\chi_X((\pi_*F)^D) = \chi_Y(F^D).$$

*Proof.* The first part is proven as in Theorem IV.7.3 using Proposition V.2.2. The second part follows from the compatibility of the terms involved with respect to  $\pi_*$ , see Proposition V.2.3 and Proposition IV.7.4.  $\square$



**Definition-Proposition V.3.4.** *Let  $F$  be a  $\mathbb{Z}$ -constructible sheaf on  $X$  and choose an open subscheme  $j : U \rightarrow X$  such that  $F|_U$  is locally constant. Denote  $i : Z \rightarrow X$  the closed complement. The Weil-étale Euler characteristic of  $F^D$  is*

$$\chi_X(F^D) := \chi_X((j_!F|_U)^D)\chi_X((i_*i^*F)^D).$$

*This definition does not depend on the choice of  $U$ , is multiplicative with respect to short exact sequences and is compatible with finite dominant maps as in Proposition V.3.3.*

*Proof.* This is formal, see Subsection II.6.5. □

**Theorem V.3.5.** *Let  $F$  be a  $\mathbb{Z}$ -constructible sheaf on the spectrum  $X$  of an order in a number field. We have the vanishing order formula*

$$\text{ord}_{s=0} L_X(F^D, s) = \sum (-1)^i i \cdot \dim_{\mathbb{R}} H_{ar,c}^i(X, F_{\mathbb{R}}^D)$$

*and the special value formula*

$$L_X^*(F^D, 0) = \pm \chi_X(F^D).$$

*Proof.* We apply Theorem III.2.1: this reduces to the case of  $F = \mathbb{Z}$  for any  $X$ , or  $F$  coming by finite pushforward from the spectrum of a field, in which case  $F$  is supported on a closed point. In both cases, by Proposition V.2.5 the Euler characteristic is identified with that of Chapter IV whence we conclude with Theorem IV.7.13. □

# Appendix A

## Some determinant computations

Let  $M$  be a perfect complex of abelian group or  $\mathbb{R}$ -vector spaces. We put  $H^{ev}M := \bigoplus_{i \equiv 0[2]} H^i M$  and  $H^{od}M := \bigoplus_{i \equiv 1[2]} H^i M$ . We fix determinant functors  $\det_{\mathbb{Z}}$  and  $\det_{\mathbb{R}}$  on the derived categories of perfect complexes of abelian groups/ $\mathbb{R}$ -vector spaces extending the usual one on finite type projective modules (see section II.6.1).

**Lemma A.0.1.** *Let  $M$  be a perfect complex of abelian groups with finite cohomology groups. Then*

$$\det_{\mathbb{Z}} M \subset (\det M)_{\mathbb{R}} \simeq \det M_{\mathbb{R}} \simeq \det 0 = \mathbb{R}$$

corresponds to the lattice

$$\frac{1}{\prod_i [H^i(M)]^{(-1)^i}} \cdot \mathbb{Z} \subset \mathbb{R}.$$

*Proof.* We have an isomorphism of determinants<sup>1</sup>  $\det_{\mathbb{Z}} M \simeq \det_{\mathbb{Z}} H^* M \simeq \det_{\mathbb{Z}} H^{ev} M \otimes (\det_{\mathbb{Z}} H^{od} M)^{-1}$  compatible with base change, so we may assume  $M$  to be concentrated in degree 0. Since  $\det_{\mathbb{Z}}(A \oplus B) \simeq \det_{\mathbb{Z}}(A) \otimes \det_{\mathbb{Z}}(B)$ , we further reduce to  $M = \mathbb{Z}/m\mathbb{Z}$ . Consider the short exact sequence

$$0 \rightarrow \mathbb{Z}_0 \xrightarrow{m} \mathbb{Z}_1 \rightarrow M \rightarrow 0$$

where  $\mathbb{Z}_0 = \mathbb{Z}_1 = \mathbb{Z}$ . It induces an isomorphism between the determinants  $i : \mathbb{Z}_0 \otimes \det_{\mathbb{Z}} M \rightarrow \mathbb{Z}_1$ ; furthermore the induced morphism after base change is  $i_{\mathbb{R}} : \mathbb{R}_0 \otimes_{\mathbb{R}} \mathbb{R} = \mathbb{R}_0 \rightarrow \mathbb{R}_1$ , given by  $x \mapsto mx$  since it is obtained from the short exact sequence of  $\mathbb{R}$ -vector spaces  $0 \rightarrow \mathbb{R}_0 \xrightarrow{m} \mathbb{R}_1 \rightarrow 0 \rightarrow 0$ . Consider now the following commutative diagram

$$\begin{array}{ccccc} \mathbb{Z}_0^{-1} \otimes \mathbb{Z}_1 & \xleftarrow[\simeq]{1 \otimes i} & \mathbb{Z}_0^{-1} \otimes \mathbb{Z}_0 \otimes \det_{\mathbb{Z}} M & \xrightarrow[\simeq]{ev \otimes 1} & \det_{\mathbb{Z}} M \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{R}_0^{-1} \otimes_{\mathbb{R}} \mathbb{R}_1 & \xleftarrow[\simeq]{1 \otimes i_{\mathbb{R}} = 1 \otimes m} & \mathbb{R}_0^{-1} \otimes_{\mathbb{R}} \mathbb{R}_0 \otimes_{\mathbb{R}} \mathbb{R} & \xrightarrow[\simeq]{ev \otimes 1} & \mathbb{R} \end{array}$$

where  $ev(\varphi \otimes x) = \varphi(x)$ . The integral basis  $1^* \otimes 1$  of  $\mathbb{Z}_0^{-1} \otimes \mathbb{Z}_1$  is sent under the above maps to  $\frac{1}{m} \in \mathbb{R}$ .  $\square$

<sup>1</sup>[Bre11, Section 5, in particular Proposition 5.5 and its proof] and [Bre08, Proposition 3.4 and its proof]

**Lemma A.0.2.** *Let  $A$  and  $B$  be free abelian groups of rank  $d$  and  $\phi : A_{\mathbb{R}} \rightarrow B_{\mathbb{R}}$  be an isomorphism. Then*

$$\begin{aligned} \det_{\mathbb{Z}} A \otimes (\det_{\mathbb{Z}} B)^{-1} &\subset (\det_{\mathbb{Z}} A)_{\mathbb{R}} \otimes_{\mathbb{R}} ((\det_{\mathbb{Z}} B)^{-1})_{\mathbb{R}} \simeq \left( \det_{\mathbb{R}} A_{\mathbb{R}} \right) \otimes_{\mathbb{R}} \left( \det_{\mathbb{R}} B_{\mathbb{R}} \right)^{-1} \\ &\xrightarrow[1 \otimes \det_{\mathbb{R}}(\phi)^t]{\simeq} \left( \det_{\mathbb{R}} A_{\mathbb{R}} \right) \otimes \left( \det_{\mathbb{R}} A_{\mathbb{R}} \right)^{-1} \\ &\simeq \mathbb{R} \end{aligned}$$

corresponds to the lattice  $\det(\phi)\mathbb{Z} \subset \mathbb{R}$  where  $\det(\phi)$  is computed with respect to any choice of integral bases of  $A$  and  $B$ .

*Proof.* Fix bases  $(e_1, \dots, e_d)$  and  $(f_1, \dots, f_d)$  of  $A$  and  $B$  respectively. Then by definition  $\det_{\mathbb{R}}(\phi)$  maps  $e_1 \wedge \dots \wedge e_d$  to  $\det(\phi)f_1 \wedge \dots \wedge f_d$ , where  $\det(\phi)$  is computed in the mentioned bases. Now the lemma follows from the fact that  $(\det_{\mathbb{R}} A_{\mathbb{R}}) \otimes_{\mathbb{R}} (\det_{\mathbb{R}} B_{\mathbb{R}})^{-1} \rightarrow (\det_{\mathbb{R}} A_{\mathbb{R}}) \otimes (\det_{\mathbb{R}} A_{\mathbb{R}})^{-1}$  maps the canonical basis element  $e_1 \wedge \dots \wedge e_d \otimes (f_1 \wedge \dots \wedge f_d)^*$  to  $e_1 \wedge \dots \wedge e_d \otimes \det(\phi)(e_1 \wedge \dots \wedge e_d)^*$ .  $\square$

**Proposition A.0.3.** *Let  $M$  be a perfect complex of abelian groups and let  $\phi : H^{ev}(M_{\mathbb{R}}) \xrightarrow{\sim} H^{od}(M_{\mathbb{R}})$  be a "trivialisation" of  $M_{\mathbb{R}}$ . Then*

$$\begin{aligned} \det_{\mathbb{Z}} M \subset (\det_{\mathbb{Z}} M)_{\mathbb{R}} &\simeq \det_{\mathbb{R}} M_{\mathbb{R}} \simeq \left( \det_{\mathbb{R}} H^{ev} M_{\mathbb{R}} \right) \otimes_{\mathbb{R}} \left( \det_{\mathbb{R}} H^{od} M_{\mathbb{R}} \right)^{-1} \\ &\xrightarrow[1 \otimes \det_{\mathbb{R}}(\phi)^t]{\simeq} \left( \det_{\mathbb{R}} H^{ev} M_{\mathbb{R}} \right) \otimes_{\mathbb{R}} \left( \det_{\mathbb{R}} H^{ev} M_{\mathbb{R}} \right)^{-1} \\ &\simeq \mathbb{R} \end{aligned}$$

corresponds to the lattice

$$\frac{\det(\phi)}{\prod_i [H^i(M)_{\text{tor}}]^{(-1)^i}} \cdot \mathbb{Z} \subset \mathbb{R}$$

where  $\det(\phi)$  is computed with respect to any integral bases of  $H^{ev}(M)/\text{tor}$  and  $H^{od}(M)/\text{tor}$ .

*Proof.* We have an isomorphism of determinants  $\det_{\mathbb{Z}} M \simeq \det_{\mathbb{Z}} H^* M \simeq \det_{\mathbb{Z}} H^{ev} M \otimes (\det_{\mathbb{Z}} H^{od} M)^{-1}$  compatible with base change. Using the short exact sequence  $0 \rightarrow A_{\text{tor}} \rightarrow A \rightarrow A/\text{tor} \rightarrow 0$  for  $A = H^{ev}(M)$ ,  $H^{od}(M)$ , we reduce by functoriality of the determinant to the torsion case and to the free case, which are the two previous lemmas.  $\square$

**Lemma A.0.4.** *Let  $A^{\bullet}, B^{\bullet}$  be two bounded acyclic complexes of finite type abelian groups, and suppose there is an isomorphism  $\phi : B_{\mathbb{R}} \xrightarrow{\sim} \text{Hom}(A_{\mathbb{R}}, \mathbb{R})[-1]$ . Then*

$$\frac{\prod_i [B_{\text{tor}}^i]^{(-1)^i}}{\det(\phi) \prod_i [A_{\text{tor}}^i]^{(-1)^i}} = 1$$

where  $\det(\phi)$  is defined as the alternated product  $\prod_i \det(\phi^i : B_{\mathbb{R}}^i \rightarrow \text{Hom}(A_{\mathbb{R}}^{1-i}, \mathbb{R}))^{(-1)^i}$  with determinants computed in bases modulo torsion.

*Proof.* Consider the line  $\Delta := \det_{\mathbb{Z}} A \otimes (\det_{\mathbb{Z}} B)^{-1}$ . It has a naive trivialisation

$$\det_{\mathbb{Z}} A \otimes (\det_{\mathbb{Z}} B)^{-1} \xrightarrow{\det(0) \otimes (\det(0)^t)^{-1}} \mathbb{Z} \otimes \mathbb{Z}^{-1} \rightarrow \mathbb{Z}$$

This induces the naive trivialisation of  $\Delta_{\mathbb{R}}$ , under which the embedding  $\Delta \hookrightarrow \Delta_{\mathbb{R}}$  corresponds to the natural embedding  $\mathbb{Z} \hookrightarrow \mathbb{R}$ . Moreover, the isomorphism  $\phi$  induces another trivialisation of  $\Delta_{\mathbb{R}}$ :

$$\det_{\mathbb{R}} A_{\mathbb{R}} \otimes (\det_{\mathbb{R}} B_{\mathbb{R}})^{-1} \xrightarrow{\text{id} \otimes (\det(\phi)^t)^{-1}} \det_{\mathbb{R}} A_{\mathbb{R}} \otimes (\det_{\mathbb{R}} A_{\mathbb{R}})^{-1} \rightarrow \mathbb{R}$$

under which the embedding  $\Delta \hookrightarrow \Delta_{\mathbb{R}}$  corresponds by the previous lemmas to

$$\frac{\prod_i [B_{\text{tor}}^i]^{(-1)^i}}{\det(\phi) \prod_i [A_{\text{tor}}^i]^{(-1)^i}} \mathbb{Z} \hookrightarrow \mathbb{R}$$

Let us show that both trivialisations are compatible. We claim that the following diagram commutes

$$\begin{array}{ccc} \det_{\mathbb{R}} A_{\mathbb{R}} \otimes (\det_{\mathbb{R}} B_{\mathbb{R}})^{-1} & \xrightarrow{\text{id} \otimes (\det(\phi)^t)^{-1}} & \det_{\mathbb{R}} A_{\mathbb{R}} \otimes (\det_{\mathbb{R}} A_{\mathbb{R}})^{-1} \\ \det(0) \otimes (\det(0)^t)^{-1} \downarrow & \swarrow \det(0) \otimes (\det(0)^t)^{-1} & \downarrow \text{nat} \\ \mathbb{R} \otimes \mathbb{R}^{-1} & \xrightarrow{\text{nat}} & \mathbb{R} \end{array}$$

The commutativity of the lower triangle is immediate, and the commutativity of the upper triangle comes from the commutative diagram

$$\begin{array}{ccc} B_{\mathbb{R}} & \xrightarrow{\phi} & \text{Hom}(A_{\mathbb{R}}, \mathbb{R})[-1] \\ & \searrow 0 & \downarrow 0 \\ & & 0 \end{array}$$

The compatibility of the trivialisations thus implies that

$$\frac{\prod_i [B_{\text{tor}}^i]^{(-1)^i}}{\det(\phi) \prod_i [A_{\text{tor}}^i]^{(-1)^i}} \mathbb{Z} = \mathbb{Z} \hookrightarrow \mathbb{R}$$

and the formula follows. □

# Appendix B

## Duality for the Tate cohomology of finite groups

Let  $G$  be a finite group. The functor  $R\hat{\Gamma}(G, -)$  is lax monoidal, so for any complex of  $G$ -modules  $M$  we can construct the pairing

$$R\hat{\Gamma}(G, M^\vee) \otimes^L R\hat{\Gamma}(G, M) \rightarrow R\hat{\Gamma}(G, \mathbb{Z}) \rightarrow \hat{H}^0(G, \mathbb{Z})[0] = \mathbb{Z}/|G|\mathbb{Z}[0] \hookrightarrow \mathbb{Q}/\mathbb{Z}[0],$$

where  $M^\vee = R\mathcal{H}om_G(M, \mathbb{Z})$ . In the following, we will also denote by  $M^* = \mathcal{H}om_G(M, \mathbb{Q}/\mathbb{Z})$  the Pontryagin dual.

**Theorem B.0.1.** *Let  $G$  be a finite group. The natural pairing*

$$R\hat{\Gamma}(G, M^\vee) \otimes^L R\hat{\Gamma}(G, M) \rightarrow \mathbb{Q}/\mathbb{Z}[0]$$

*is perfect for any bounded complex  $M$  of  $G$ -modules with finite type cohomology groups.*

*Proof.* By definition, showing that the pairing is perfect is showing that the adjoint map  $R\hat{\Gamma}(G, M^\vee) \rightarrow R\hat{\Gamma}(G, M)^*$  is an isomorphism. Passing to cohomology, this is equivalent (using [Stacks, Tag 0FP2] and the injectivity of  $\mathbb{Q}/\mathbb{Z}$ ) to showing that the cup product pairing

$$\hat{H}^i(G, M) \times \hat{H}^{-i}(G, M^\vee) \rightarrow \hat{H}^0(G, \mathbb{Z}) \rightarrow \mathbb{Q}/\mathbb{Z}$$

is perfect for each  $i \in \mathbb{Z}$ . Notice that if  $M$  is induced, then  $M^\vee$  is also induced; moreover, if  $M_0$  is a finite type abelian group,  $\text{ind}^G M_0$  also is. Thus the usual dimension shifting argument reduces to checking that the pairing is perfect in just one degree. We proceed by Artin induction:

- In an exact sequence  $0 \rightarrow M \rightarrow P \rightarrow Q \rightarrow 0$  or more generally a fiber sequence  $M \rightarrow P \rightarrow Q$ , if the theorem is true for two out of the three terms then it is true for the third.
- We can filter a bounded complex with finite type cohomology groups by its truncations, which reduces to the case of a  $G$ -module  $M$  of finite type.
- Let  $H$  be a proper subgroup of  $G$ , let  $M$  be a torsion-free finite type discrete  $H$ -module and let us consider the induced  $G$ -module  $\text{ind}_H^G M$ . We want to show compatibility of the pairing in degree 0 for  $\text{ind}_H^G M$  and the pairing in degree 0 for  $M$ .

We have  $M^\vee = \text{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$ . Denote by  $\pi_*$  for the induction functor  $\text{ind}_H^G$  and  $\pi^*$  for restriction to  $H$ . The functor  $\pi_*$  is a right adjoint of  $\pi^*$  and the finiteness of  $G$  also makes  $\pi_*$  into a left adjoint of  $\pi^*$ . Let  $\eta : \pi_*\pi^* \rightarrow \text{id}$  denote the counit of the latter adjunction; the counit  $\eta_{\mathbb{Z}} : \pi_*\mathbb{Z} \simeq \bigoplus_{G/H}\mathbb{Z} \rightarrow \mathbb{Z}$  is the sum map. As a right adjoint to a strict monoidal functor,  $\pi_*$  has a lax monoidal structure  $\pi_*(-) \otimes \pi_*(-) \xrightarrow{\text{lax}} \pi_*(- \otimes -)$ . Consider the following commutative diagram:

$$\begin{array}{ccc} \pi_*M \otimes (\pi_*M)^\vee & \xrightarrow{\text{ev}} & \mathbb{Z} \\ \simeq \uparrow & & \uparrow \eta=\Sigma \\ \pi_*M \otimes \pi_*(M^\vee) & \xrightarrow{\text{lax}} \pi_*(M \otimes M^\vee) \xrightarrow{\pi_*(\text{ev})} & \pi_*\mathbb{Z} \end{array}$$

We can apply the lax-monoidal functor  $\hat{H}^0$  to the above to obtain Diagram B.1 (using Shapiro's lemma's identifications): Square (1) commutes by functoriality of the cup product, square (2) commutes by functoriality of  $\hat{H}^0(G, -)$ , square (3) commutes by inspection, square (4) commutes by functoriality in Shapiro's lemma and square (5) commutes because  $(e^{\frac{2i\pi}{|G|}})^{[G:H]} = e^{\frac{2i\pi}{|H|}}$ . As the bottom left square commutes, the whole diagram commutes and the pairing for  $M$  is perfect if and only if the pairing for  $\text{ind}_H^G M$  is perfect.

- For  $M = \mathbb{Z}$  we have  $M^\vee = \mathbb{Z}$ . The map  $R\hat{\Gamma}(G, M^\vee) \rightarrow R\hat{\Gamma}(G, M)^*$  corresponds in degree 0 to the adjoint map to the cup product pairing  $\mathbb{Z}/|G|\mathbb{Z} \otimes \mathbb{Z}/|G|\mathbb{Z} \rightarrow \mathbb{Z}/|G|\mathbb{Z} \rightarrow \mathbb{Q}/\mathbb{Z}$  induced by the multiplication  $\mathbb{Z} \otimes \mathbb{Z} \rightarrow \mathbb{Z}$ . Thus it is perfect.
- For  $M$  finite as an abelian group, we have  $R\mathcal{H}om_G(M, \mathbb{Q}) = 0$  so the fiber sequence  $\mathbb{Z} \rightarrow \mathbb{Q} \rightarrow \mathbb{Q}/\mathbb{Z}$  gives  $M^\vee \simeq M^*[-1]$ . For any bounded complex  $M$ , the complex  $M \otimes \mathbb{Q}$  is a complex of  $\mathbb{Q}$ -vector spaces hence cohomologically trivial. The map  $\mathbb{Q}/\mathbb{Z}[-1] \rightarrow \mathbb{Z}$  thus induces an isomorphism of pairings

$$\begin{array}{ccccccc} R\hat{\Gamma}(G, M^*)[-1] \otimes^L R\hat{\Gamma}(G, M) & \rightarrow & R\hat{\Gamma}(G, \mathbb{Q}/\mathbb{Z})[-1] & \rightarrow & \hat{H}^{-1}(G, \mathbb{Q}/\mathbb{Z})[0] & \rightarrow & \mathbb{Q}/\mathbb{Z}[0] \\ \simeq \downarrow & & \simeq \downarrow & & \simeq \downarrow & & \parallel \\ R\hat{\Gamma}(G, M^\vee) \otimes^L R\hat{\Gamma}(G, M) & \longrightarrow & R\hat{\Gamma}(G, \mathbb{Z}) & \longrightarrow & \hat{H}^0(G, \mathbb{Z})[0] & \longrightarrow & \mathbb{Q}/\mathbb{Z}[0] \end{array}$$

It follows that it is enough to prove that the map  $R\hat{\Gamma}(G, M^*)[-1] \rightarrow R\hat{\Gamma}(G, M)^*$  is an isomorphism in degree 0, i.e. that the cup-product pairing

$$\hat{H}^{-1}(G, M^*) \times \hat{H}^0(G, M) \rightarrow \hat{H}^{-1}(G, \mathbb{Q}/\mathbb{Z}) \rightarrow \mathbb{Q}/\mathbb{Z}$$

induced by the Pontryagin duality pairing  $M^* \otimes M \rightarrow \mathbb{Q}/\mathbb{Z}$  is perfect. For  $g \in G$ ,  $g$  acts on  $M^*$  as the transpose of  $g^{-1}$ . Thus, on  $M^*$ ,  $N$  acts as  $N^t$  and the family  $(1 - g)_{g \in G}$  is a permutation of the family  $((1 - g)^t)_{g \in G}$ . In the perfect pairing  $M^* \otimes M \rightarrow \mathbb{Q}/\mathbb{Z}$ , we have  $^\perp \left( \bigcap_{g \in G} \text{Ker}(1 - g) \right) = \sum_{g \in G} \text{Im}((1 - g)^t)$  and  $\text{Ker}(N)^\perp = \text{Im}(N^t)$ , so this pairing induces the following perfect pairing between the subquotients:

$$\text{Ker}(N^t) / \sum_{g \in G} \text{Im}((1 - g)^t) \times \bigcap_{g \in G} \text{Ker}(1 - g) / \text{Im}(N) \rightarrow \mathbb{Q}/\mathbb{Z}.$$

This is exactly what we had to prove. □

$$\begin{array}{ccccc}
\hat{H}^0(G, \pi_* M) \otimes \hat{H}^0(G, (\pi_* M)^\vee) & \xrightarrow{\cup} & \hat{H}^0(G, \pi_* M \otimes (\pi_* M)^\vee) & \xrightarrow{\text{ev}_*} & \hat{H}^0(G, \mathbb{Z}) = \mathbb{Z}/|G|\mathbb{Z} \hookrightarrow \mathbb{Q}/\mathbb{Z} \\
\downarrow \simeq & \downarrow (1) & \downarrow \simeq & \downarrow (2) & \uparrow \llbracket [G:H] \rrbracket \\
\hat{H}^0(G, \pi_* M) \otimes \hat{H}^0(G, \pi_*(M^\vee)) & \xrightarrow{\cup} & \hat{H}^0(G, \pi_* M \otimes \pi_*(M^\vee)) & \xrightarrow{\text{Iax}_*} & \hat{H}^0(G, \pi_*(M \otimes M^\vee)) \xrightarrow{(\pi_*(\text{ev}))_*} \hat{H}^0(G, \pi_* \mathbb{Z}) \xrightarrow{(5)} \mathbb{Q}/\mathbb{Z} \\
\downarrow \simeq & \downarrow (3) & \downarrow \simeq & \downarrow (4) & \downarrow \simeq \\
\hat{H}^0(H, M) \otimes \hat{H}^0(H, M^\vee) & \xrightarrow{\cup} & \hat{H}^0(H, M \otimes M^\vee) & \xrightarrow{\text{ev}_*} & \hat{H}^0(H, \mathbb{Z}) = \mathbb{Z}/|H|\mathbb{Z} \hookrightarrow \mathbb{Q}/\mathbb{Z}
\end{array}$$

Diagram B.1

# Appendix C

## The maximal tamely ramified extension of a number field

In this section we discuss the tame Galois group of a number field. The results are certainly known but we did not find a convenient reference.

Let  $K$  be the field of fractions of a henselian DVR with finite residue field. A finite extension  $L/K$  is called tamely ramified if the ramification index  $e_{L/K}$  is prime to the residual characteristic. We have the following properties:

**Proposition C.0.1** ([Neu99a, 7.8, 7.9, 7.10]).

1. *If  $M/L/K$  is a tower of finite extensions, then  $M/K$  is tamely ramified if and only if  $M/L$  and  $L/K$  are.*
2. *The composite of two tamely ramified finite extensions of  $K$  is tamely ramified.*
3. *The maximal tamely ramified extension  $K^t$  is defined as the composite of all finite tamely ramified extensions of  $K$  inside  $K^{sep}$ . Its finite subextensions are tamely ramified.*

Let  $K$  be a number field. For each finite place  $v$  of  $K$ , which corresponds to a closed point of  $X = \text{Spec}(\mathcal{O}_K)$ , denote by  $K_v$  the henselian local field at  $v$ . Let  $K^{sep}$  be a separable closure of  $K$  and for each finite place  $v$ , choose an embedding  $K_v \hookrightarrow K^{sep}$ ; this determines a place  $\bar{v}$  of  $K^{sep}$  above  $v$  and gives  $K^{sep}$  the structure of a separable closure of  $K_v$ . Denote by  $G_K$ , resp.  $G_{K_v}$  the absolute Galois group of  $K$  resp.  $K_v$ . The previous proposition is adapted to the global case by working simultaneously at all places, as following: a finite extension  $L/K$  is called tamely ramified if for all finite places  $v$  of  $K$  and  $w$  of  $L$  above  $v$ , the finite extension  $L_w/K_v$  is tamely ramified. We then have:

**Corollary C.0.2.**

1. *If  $M/L/K$  is a tower of finite extensions, then  $M/K$  is tamely ramified if and only if  $M/L$  and  $L/K$  are.*
2. *The composite of two tamely ramified finite extensions of  $K$  is tamely ramified.*
3. *The maximal tamely ramified extension  $K^t$  is defined as the composite of all finite tamely ramified extensions of  $K$  inside  $K^{sep}$ . Its finite subextensions are tamely ramified.*



*Proof.* This follows from the previous propositions by taking henselizations, using the two following observations:

- For any choice of finite place in any of the three fields in a tower  $M/L/K$ , there exists a compatible system of places above and below it.
- If  $L/K$  and  $M/K$  are two finite extensions inside  $K^{sep}$  and  $z$  is a finite place of the composite extension  $LM$  mapping to places  $w, w'$  of  $L$  and  $M$  then  $(LM)_z = L_w M_{w'}$ ; indeed,  $(LM)_z$  is the smallest henselian field inside  $K^{sep}$  containing  $LM$  so  $(LM)_z \subset L_w M_{w'}$ , and the other inclusion is immediate.

□

The extension  $K^t/K$  is Galois; we define the tame Galois group of  $K$  as  $G_K^t := \text{Gal}(K^t/K)$ .

For a finite place  $v$  of  $K$ , let  $P_v \subset D_v$  denote the wild ramification and decomposition subgroup of the place  $\bar{v}$  of  $K^{sep}$ . There is an identification  $D_v = G_{K_v}$  under which we have  $P_v = \text{Gal}(K^{sep}/K_v^t)$ . We now characterise the tame Galois group in terms of the wild inertia subgroups  $P_v$ .

**Proposition C.0.3.** *Let  $N$  be the smallest closed normal subgroup of  $G_K$  containing  $P_v$  for all finite places  $v$ . Then  $N = \text{Gal}(K^{sep}/K^t)$ , and consequently  $G_K^t = G_K/N$ .*

*Proof.* We first show  $N \subset \text{Gal}(K^{sep}/K^t)$ ; since the latter is normal and closed it suffices to show that the elements of  $P_v$ , for any finite place  $v$ , fix  $K^t$ . Let  $L/K$  be a finite tamely ramified extension. The place  $\bar{v}$  of  $K^{sep}$  determines a unique place  $w$  of  $L$  above  $v$ . The following diagram commutes

$$\begin{array}{ccccc}
 & & \text{Gal}(L_w/K_v) & \hookrightarrow & \text{Gal}(L/K) \\
 & \nearrow & \uparrow & & \uparrow \\
 & & \text{Gal}(K_v^t/K_v) & \xrightarrow{\sigma \mapsto \sigma|_{K^t}} & \text{Gal}(K^t/K) \\
 & \nearrow & \uparrow & \text{(1)} & \uparrow \\
 & & G_{K_v} & \hookrightarrow & G_K \\
 P_v & \nearrow & \uparrow & & \uparrow \\
 & & & & 
 \end{array}$$

$\sigma \mapsto \sigma|_{L_w}$  (curved arrow from  $P_v$  to  $\text{Gal}(L_w/K_v)$ )  
 $0$  (arrow from  $P_v$  to  $G_{K_v}$ )

Indeed, the only nontrivial part is square (1). If  $L/K$  is a finite tamely ramified extension and  $w$  is the place of  $L$  induced by  $\bar{v}$  then  $L \subset L_w \subset K_v^t$ ; therefore there is an inclusion  $K^t \subset K_v^t$  and the commutativity of (1) follows. Since the diagram commutes, the elements of  $P_v$  are sent to 0 in  $\text{Gal}(L/K)$  so they fix any finite tamely ramified extension of  $K$ , hence also  $K^t$ .

We now show  $\text{Gal}(K^{sep}/K^t) \subset N$ . Since  $N$  is normal, it suffices to show that for  $N \subset U$  with  $U$  an open normal subgroup, we have  $\text{Gal}(K^{sep}/K^t) \subset U$ . This amounts to showing that the finite extension  $L/K$  corresponding to an open normal subgroup  $U$  containing  $N$  is tamely ramified. Let  $v$  be a finite place of  $K$  and denote by  $w$  the place of  $L$  induced by  $\bar{v}$ . Then  $L_w = LK_v$  is the fixed field of  $U \cap G_{K_v}$ , which contains  $P_v$ , so  $L_w \subset K_v^t$  is tamely ramified. □

# Appendix D

## Miscellaneous results on proétale cohomology and condensed mathematics

In this section we collect some results on condensed mathematics; these are certainly already known to experts.

**Definition D.0.1** ([BS15, 3.2.1]). *An object  $F$  of a topos  $\mathcal{X}$  is called weakly contractible if every surjection  $G \rightarrow F$  has a section. We say that  $\mathcal{X}$  is locally weakly contractible if it has enough weakly contractible coherent objects, i.e., each  $X \in \mathcal{X}$  admits a surjection  $\cup_i Y_i \rightarrow X$  with  $Y_i$  a coherent weakly contractible object.*

The proétale topos of a scheme is locally weakly contractible [BS15, 4.2.8]; in particular the condensed topos (i.e. the proétale topos of a geometric point) is locally weakly contractible, and (the sheaves represented by) extremally disconnected profinite sets are a suitable family of weakly contractible objects.

**Definition D.0.2.** *Let  $\mathcal{X}$  be a topos. A projective system  $(A_i)_{i \in \mathbb{N}}$  of abelian group objects, with transition maps denoted by  $f_n^m : A_m \rightarrow A_n$ , is said to satisfy the Mittag-Leffler condition if for all  $i \in \mathbb{N}$ , there exists  $j > i$  such that for all  $k \geq j$ ,  $\text{Im}(f_i^k)$  and  $\text{Im}(f_i^j)$  define the same subobject of  $A_i$ .*

In other words,  $(A_i)$  satisfies the Mittag-Leffler condition if for all  $i \in \mathbb{N}$ , there is a  $j > i$  such that  $\varprojlim_{k > i} \text{Im}(f_i^k) = \text{Im}(f_i^j)$  as subobjects of  $(A_i)$ . This is thus a condition that is preserved under exact, right adjoint functors.

**Lemma D.0.3.** *Let  $\mathcal{X}$  be a locally weakly contractible topos. If  $(A_i)$  is a projective system of abelian group objects satisfying the Mittag-Leffler condition, then  $R \lim A_i = \lim A_i$ .*

*Proof.* Let  $Y$  be a weakly contractible object. Observe that  $\Gamma(Y, -)$  is exact, so we have  $\Gamma(Y, -) = R\Gamma(Y, -)$  which is a right adjoint and thus  $(R \lim A_i)(Y) = R \lim(A_i(Y))$ . But  $(A_i(Y))$  is a projective system of abelian groups satisfying the Mittag-Leffler condition, again by exactness and commutation with arbitrary limits of  $\Gamma(Y, -)$ . Thus  $R \lim(A_i)(Y) = \lim(A_i(Y))$  is concentrated in degree 0 for each weakly contractible object  $Y$ ; the cohomology objects of  $R \lim(A_i)(Y)$  thus vanish in degree  $\neq 0$  when evaluated at a weakly contractible object  $Y$ . Since those cover every object by hypothesis, we are done.  $\square$

**Lemma D.0.4.** *Let  $(A_i)$  be a family of discrete abelian groups satisfying the Mittag-Leffler condition, and let  $\nu^* : \text{Ab} \rightarrow \text{Cond}(\text{Ab})$  be the constant sheaf functor. Then  $(\nu^* A_i)$  satisfies the Mittag-Leffler condition.*

*Proof.* The functor  $\nu^*$  is exact and agrees with the functor  $(-): Ab(\text{Top}) \rightarrow \text{Cond}(Ab)$  on discrete abelian groups [BS15, 4.2.12]; the latter functor is limit-preserving. An intersection of a decreasing family of abelian groups is a limit of the associated (discrete) topological abelian groups, so we are done.  $\square$

**Proposition D.0.5** ([BS15, 6.1.17]). *Let  $\pi : Y \rightarrow X$  be a finite morphism of finite presentation. Then  $\pi_{\text{proet},*} : D(Y_{\text{proet}}) \rightarrow D(X_{\text{proet}})$  has a right adjoint.*

*Proof.* Since both source and target triangulated categories are compactly generated, it suffices to show that  $\pi_{\text{proet},*}$  commutes with direct sums. This can be checked on  $w$ -contractible affines. Since evaluation of sheaves of abelian groups on the proétale site at  $w$ -contractible affines commutes with colimits, this is easily shown using [BS15, 2.4.10].  $\square$

In the following we follow [AGV72] and call a topology a family of sieves satisfying the usual axioms, and a pretopology a family of covers satisfying the usual axioms.

**Proposition D.0.6.** *The canonical topology on condensed sets induces the proétale topology (i.e. generated by finite jointly surjective families) on profinite sets under the Yoneda embedding.*

*Proof.* Denote by  $J_{\text{proet}}$ , resp.  $J_{\text{ind}}$  the proétale topology resp. the topology induced from the canonical topology on condensed sets. Since condensed sets are covered under the canonical topology by profinite sets (under the fully faithful, limit-preserving Yoneda embedding), we conclude by [AGV72, Exp. IV, 1.2.1] that  $\text{Sh}(\text{Top}^{pf}, J_{\text{proet}}) = \text{Cond}(\text{Set}) \simeq \text{Sh}(\text{Top}^{pf}, J_{\text{ind}})$ . Since the topology is characterised by its category of sheaves [AGV72, Exp. II, 4.4.4] we find  $J_{\text{proet}} = J_{\text{ind}}$ .  $\square$

**Definition D.0.7** ([BS15, 4.3.1]). *Let  $G$  be a compactly generated topological group. The pro-étale site  $BG_{\text{proet}}$  of  $G$  is defined as the site of profinite continuous  $G$ -sets with covers given by finite jointly surjective families.*

**Proposition D.0.8** ([Fla08, Corollary 2]). *Let  $G$  be a compactly generated topological group. We have*

$$\text{Sh}(BG_{\text{proet}}) = \underline{G}\text{-Cond}(\text{Set})$$

*If moreover  $G = \text{Gal}(k^{sep}/k)$  for a field  $k$ , then*

$$\text{Sh}(\text{Spec}(k)_{\text{proet}}) = \underline{G}\text{-Cond}(\text{Set})$$

*Proof.* The Yoneda embedding from profinite sets to condensed sets is fully faithful, and it can be extended to a right adjoint (hence limit-preserving) faithful functor  $(-)$ , which is moreover fully faithful when restricted to compactly generated topological space [CS19, 1.7]. The action of  $G$  on a profinite set  $X$  is encoded by a map  $G \times X \rightarrow X$ , where the product is computed in topological spaces. The product of a profinite set and a compactly generated space is compactly generated [Rez18, 7.2]<sup>1</sup>, so this is the same as a map  $\underline{G} \times \underline{X} \rightarrow \underline{X}$ . Thus we get a fully faithful embedding of profinite continuous  $G$ -sets in  $\underline{G}\text{-Cond}(\text{Set})$ . The canonical topology on  $\underline{G}$ -condensed sets is obtained by forgetting the  $\underline{G}$ -structure, and similarly for the proétale topology on profinite continuous  $G$ -sets, so that the canonical topology on the former induces the proétale topology on the latter by the previous proposition. By [AGV72, Exp. IV, 1.2.1] it remains only to show that a  $\underline{G}$ -condensed set is covered by profinite  $G$ -sets; for any cover of the underlying condensed set by profinite sets  $S_i$ ,  $G \times S_i$  (where  $G$  acts on the first factor) is a cover by profinite  $G$ -sets by an adjunction argument.  $\square$

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<sup>1</sup>In the above reference, compactly generated spaces are called  $k$ -spaces

# Appendix E

## Strictly henselian local rings of singular schemes

Let  $X$  be an integral arithmetic scheme with finite normalization  $\pi : Y \rightarrow X$ , and let  $x \in X$  be a closed point. We want to understand the fiber above the generic point of  $X$  of the strictly henselian local ring at  $x$  in terms of the fibers of strictly henselian local rings of points of  $Y$  above  $x$ . We can localize at  $x$  and we are thus in the following setting: let  $(A, \mathfrak{m})$  be a Noetherian integral local ring with fraction field  $K$ , perfect residue field  $k$  and integral closure  $B$ . Can we describe  $A^{sh} \otimes_A K$  in terms of  $B$  and its strict henselizations at various maximal ideals.

**Lemma E.0.1** ([Stacks, Tag 07QQ]). *We have*

$$A^{sh} \otimes_A K = \prod_{i=1}^n \kappa(\mathfrak{p}_i),$$

where  $\mathfrak{p}_i$  are the prime ideals of  $A^{sh} \otimes_A K$  above  $(0)$ ; moreover each  $\kappa(\mathfrak{p}_i)/k$  is separable algebraic.

*Proof.* The local ring  $A^{sh}$  is Noetherian and flat over  $A$  so it has finitely many minimal primes, which are exactly the primes lying above  $(0)$ . Moreover it is reduced as  $A$  is reduced. Therefore  $A^{sh} \otimes_A K$  is Noetherian and has finitely many prime ideals, all minimal, so  $A^{sh} \otimes_A K$  is an Artinian ring; since it is reduced, its local rings are fields.  $\square$

The lemma says that  $A^{sh} \otimes_A K$  is the total rings of fraction of  $A^{sh}$ . The total ring of fractions of the normalization of  $A^{sh}$  is the same, so we want to determine the normalization of  $A^{sh}$ :

**Lemma E.0.2** ([Stacks, Tag 0CBM]). *The ring  $B' := B \otimes_A A^{sh}$  is the normalization of  $A^{sh}$ .*

*Proof.* Normalization commutes with étale maps<sup>1</sup> and filtered colimits, and  $A^{sh}$  is a filtered colimit of étale  $A$ -algebras.  $\square$

By the previous lemma the minimal primes of  $A^{sh}$  and  $B'$  are in bijection. Moreover, as  $B'$  is finite over  $A^{sh}$ ,  $B'$  is a finite product of strictly henselian local rings each finite over  $A^{sh}$ . Since  $B'$  is normal, each local ring is moreover normal, hence integral. We

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<sup>1</sup>It is easy to see the commutation with localization, so this reduces to commutation with standard étale maps

now see that each minimal prime ideal is contained in a unique maximal ideal of  $B'$ , and vice-versa. Denote by  $\mathfrak{q}_i$  the minimal prime of  $B'$  corresponding to  $\mathfrak{p}_i$  and  $\mathfrak{M}_i$  the maximal ideal containing  $\mathfrak{q}_i$ . We deduce

$$\kappa(\mathfrak{p}_i) = \kappa(\mathfrak{q}_i) = \text{Frac}((B')_{\mathfrak{M}_i}).$$

**Lemma E.0.3** ([Stacks, Tag 08HV]). *Put  $\mathfrak{n} = \mathfrak{M}_i \cap B$ . We have*

$$(B')_{\mathfrak{M}_i} = B_{\mathfrak{n}}^{sh}.$$

*Proof.*  $B'$  is a filtered colimit of étale  $B$ -algebras and so is  $B_{\mathfrak{n}}^{sh}$ . Since there is a natural morphism of  $B$ -algebras  $B' \rightarrow B_{\mathfrak{n}}^{sh}$ ,  $B_{\mathfrak{n}}^{sh}$  is also a filtered colimit of étale  $B'$ -algebras. Since  $B_{\mathfrak{n}}^{sh}$  is a strictly henselian local ring, it must be the strict henselian local ring of  $B'$  at any prime above  $\mathfrak{n}$ .  $\square$

Thus  $\kappa(\mathfrak{q}_i) = \text{Frac}((B')_{\mathfrak{M}_i}) = \text{Frac}(B_{\mathfrak{n}}^{sh})$ . Because  $B'$  is a colimit of étale  $B$ -algebras, the maximal ideal  $\mathfrak{n}$  of  $B$  is above  $\mathfrak{m}$ . Moreover we have:

**Lemma E.0.4** ([Stacks, Tag 0C25]). *The fiber above  $\mathfrak{n}$  in  $B'$  is isomorphic to*

$$\text{Hom}_k(\kappa(\mathfrak{n}), k^{sep}).$$

*Proof.* We have

$$B' \otimes_B \kappa(\mathfrak{n}) = A^{sh} \otimes_A \kappa(\mathfrak{n}) = A^{sh} \otimes_A k \otimes_k \kappa(\mathfrak{n}) = k^{sep} \otimes_k \kappa(\mathfrak{n}),$$

hence the result.  $\square$

Combining everything, we obtain

**Proposition E.0.5.** *Let  $(A, \mathfrak{m})$  be a Noetherian integral local ring with fraction field  $K$ , perfect residue field  $k$  and integral closure  $B$  finite over  $A$ . The total ring of fractions of  $A^{sh}$  is:*

$$A^{sh} \otimes_A K = \prod_{\mathfrak{n}} \prod_{\text{Hom}_k(\kappa(\mathfrak{n}), k^{sep})} \text{Frac}(B_{\mathfrak{n}}^{sh}),$$

where  $\mathfrak{n}$  goes through the finitely many maximal ideals of  $B$ .

# Appendix F

## A lemma on determinants of total complexes

**Lemma F.0.1.** *Let  $\mathcal{A}$  be an abelian category, let  $C$  be a double complex of objects of  $\mathcal{A}$  with uniformly bounded and acyclic rows and columns and let  $\det$  be a determinant functor  $\text{Ch}^b(\mathcal{A})_{qis} \rightarrow \mathcal{P}$ . We denote by  $\mathbf{1}$  the unit of the Picard category  $\mathcal{P}$ . Define the filtration by rows on  $\text{Tot}C$  by  $F_r^n \text{Tot}C = \text{Tot}(\sigma_r^{\leq n} C)$ ,<sup>1</sup> where  $(\sigma_r^{\leq n} C)^{p,q} = C^{p,q}$  if  $q \leq n$  and 0 otherwise. The filtration by rows induces a commutative diagram*

$$\begin{array}{ccc}
 \det \text{Tot}C & \xrightarrow{\det 0} & \mathbf{1} \\
 \downarrow & & \downarrow \\
 \bigotimes_j (\det C^{\bullet,j})^{(-1)^j} & \xrightarrow{\bigotimes_j (\det 0)^{(-1)^j}} & \bigotimes_j (\mathbf{1})^{(-1)^j}
 \end{array}$$

*Proof.* The  $n$ -th graded piece of the filtration by rows of  $\text{Tot}C$  is  $C^{\bullet,n}[-n]$ . Since the rows are exact, the map  $0 : \text{Tot}C \rightarrow 0$  is a quasi-isomorphism; moreover it induces quasi-isomorphisms between the graded pieces of the filtrations by rows on  $\text{Tot}C$  and the trivial filtration on 0 because columns are exact. We conclude with [Knu02, 1.7] and [BB05, 2.3].  $\square$

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<sup>1</sup>Here  $r$  stands for *rows*.

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## Cohomologie Weil-étale en dimension 1 et valeurs spéciales de fonctions $L$ en 0 et 1

**Résumé :** Le but de cette thèse est de définir et d'étudier la cohomologie Weil-étale des schémas arithmétiques intègres de dimension 1 (i.e. des courbes sur les corps finis et des ouverts de spectres d'ordres dans des corps de nombres) à coefficients donnés par des faisceaux  $\mathbb{Z}$ -constructibles ou, en caractéristique 0, leurs duaux (comme par exemple des modèles de Néron de tores sur des corps de nombres). Selon le formalisme établi par Flach–B. Morin, en couplant le déterminant de la cohomologie Weil-étale avec le déterminant d'un complexe additif, on définit une droite fondamentale, qui a une trivialisat on canonique par un th eor eme de dualit e   coefficients r eels. La trivialisat on de la droite fondamentale donne une caract eristique d'Euler. On montre que les valeurs sp eciales en 0 de fonctions  $L$  associ ees naturellement aux coefficients sont donn ees au signe pr es par la caract eristique d'Euler. On obtient trois cas particuliers int eressants : une formule de valeur sp eciale en  $s = 0$  et  $s = 1$  pour les fonctions  $L$  d'Artin de repr esentations rationnelles ; une formule de valeur sp eciale en  $s = 0$  pour la fonction z eta du spectre d'un ordre dans un corps de nombres et en  $s = 1$  pour une l eg ere modification de cette derni ere, ce qui g en eralise la formule analytique du nombre de classes ; enfin une formule en  $s = 0$  pour un faisceau constructible, qui permet de retrouver la formule de Tate pour la caract eristique d'Euler d'un corps de nombres. Notre r esultat pour les coefficients  $\mathbb{Z}$ -constructibles g en eralise et am elior e des travaux de Geisser–Suzuki et Tran, tandis que celui pour les duaux de  $\mathbb{Z}$ -constructibles est l'analogue dans le cas des corps de nombres d'un r esultat de Geisser–Suzuki dans le cas des corps de fonctions.

**Mots-cl es :** Cohomologie Weil- etale, Valeurs sp eciales, Fonctions  $L$

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## Weil- etale cohomology in dimension 1 and special values of $L$ -functions at 0 and 1

**Abstract:** The goal of this thesis is to define and study the Weil- etale cohomology of integral arithmetic schemes of dimension 1 (i.e. of curves over finite fields and of open sub-schemes of spectra of orders in number fields) with coefficients given by  $\mathbb{Z}$ -constructible sheaves or, in characteristic 0, their duals (as for instance N eron models of tori over number fields). Following the formalism established by Flach–B. Morin, by combining the determinant of Weil- etale cohomology with the determinant of an additive complex, one defines a fundamental line, which has a canonical trivialization by a duality theorem with real coefficients. The trivialization of the fundamental line gives an Euler characteristic. We show that the special values at 0 of natural  $L$ -functions attached to the considered coefficients are given up to sign by the Euler characteristic. There are three particular cases of interest: we obtain a formula for the special value at  $s = 0$  and  $s = 1$  for the Artin  $L$ -functions of rational representations; a special value formula at  $s = 0$  for the zeta function of an order and at  $s = 1$  for a slight modification of the latter, which generalizes the analytic class number formula; and finally a formula at  $s = 0$  for a constructible sheaf, from which one can recover Tate's formula for the Euler characteristic of a number field. Our result for  $\mathbb{Z}$ -constructible coefficients generalizes and improves work of Geisser–Suzuki and Tran, while that for duals of  $\mathbb{Z}$ -constructibles is the analogue in the number field case of a result of Geisser–Suzuki in the function field case.

**Keywords:** Weil- etale cohomology, Special values,  $L$ -functions

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