



Chance-Constrained Optimization : Applications in Game Theory and Markov Decision Processes

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Chance-Constrained Optimization: Applications in Game Theory and Markov Decision Processes

*Optimisation sous Contraintes en Probabilité : Applications en Théorie des Jeux
et Processus de Décision Markovien*

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Title: Chance-Constrained Optimization: Applications in Game Theory and Markov Decision Processes

Keywords: Stochastic programming, Convex optimization, Games, Markov chains.

Abstract: Chance-constrained optimization is a powerful mathematical framework that addresses decision-making problems in the presence of uncertainty. It provides a systematic approach to handle random parameters or uncertain variables, allowing decision-makers to account for the likelihood of violating certain constraints while optimizing an objective function. The core idea behind chance-constrained optimization is to ensure that the probability of constraint violation remains below a specified threshold. This threshold represents the acceptable level of risk or confidence level for the decision-maker. In chance-constrained optimization, uncertain parameters can have known or unknown distributions. When the distribution of uncertain parameters is known, probability distributions such as Normal (Gaussian), elliptical, normal mean-variance mixture or discrete distribution with support based on historical data can be utilized to represent the uncertainty. In many practical situations, the distribution of uncertain parameters may be unknown or difficult to estimate accurately. In such cases, the distribution of uncertain parameters is as-

sumed to belong to an uncertainty set, which leads to a specific problem, called distributionally robust chance-constrained optimization. Chance-constrained optimization has significant applications in game theory and Markov Decision Processes (MDPs). In this dissertation, we first present a theoretical result of the convexity of chance-constrained optimization. Next, we study two specific models of game theory and MDPs involving chance-constrained optimization, known as chance-constrained games (CCGs) and distributionally robust chance-constrained Markov decision processes (DRCCMDPs). We consider different assumptions on the distribution of uncertain parameters. In CCGs, under certain conditions, we show the existence of a Nash equilibrium of the game. DRCCMDPs can be modelled as a distributionally robust chance-constrained optimization problem, where a decision maker is interested in maximizing the expected discounted value of a reward function. Under certain conditions, we reformulate the optimization problem equivalently as a deterministic problem, which can be solved efficiently by commercial solvers.

Titre: Optimisation sous Contraintes en Probabilité : Applications en Théorie des Jeux et Processus de Décision Markovien

Mots clés: Programmation stochastique, Optimisation convexe, Jeux, Chaînes de Markov

Résumé: L'optimisation sous contraintes en probabilité est un cadre mathématique puissant qui aborde les problèmes de prise de décision en présence d'incertitude. Il fournit une approche systématique pour gérer des paramètres aléatoires ou des variables incertaines, permettant aux décideurs de tenir compte de la probabilité de violation de certaines contraintes tout en optimisant une fonction objectif. L'idée centrale de l'optimisation sous contrainte aléatoire est de garantir que la probabilité de violation de contrainte reste inférieure à un seuil spécifié. Ce seuil représente le niveau de risque acceptable ou le niveau de confiance pour le décideur. Dans l'optimisation sous contraintes en probabilité, les paramètres incertains peuvent avoir des distributions connues ou inconnues. Lorsque la distribution des paramètres incertains est connue, des distributions de probabilité telles que normale (gaussienne), elliptique, mélange moyenne-variance normale ou distribution discrète avec support basé sur des données historiques peuvent être utilisées pour représenter l'incertitude. Dans de nombreuses situations pratiques, la distribution de paramètres incertains peut être inconnue ou difficile à estimer avec précision. Dans de tels cas, la distribution des paramètres incertains est supposée appartenir à un ensemble d'incertitudes,

ce qui conduit à un problème spécifique, appelé optimisation distributionnellement robuste avec contraintes en probabilité. L'optimisation sous contraintes en probabilité a des applications significatives dans la théorie des jeux et les processus de décision Markoviens (MDP). Dans cette thèse, nous présentons d'abord un résultat théorique de la convexité de l'optimisation sous contraintes en probabilité. Ensuite, nous étudions deux modèles spécifiques de théorie des jeux et de MDP impliquant une optimisation sous contraintes en probabilité, connus sous le nom de jeux contraints par le hasard (CCG) et de processus de décision Markoviens avec des contraintes robustes sur le plan distribution (DRCCMDP). Nous considérons différentes hypothèses sur la distribution des paramètres incertains. Dans les CCG, sous certaines conditions, nous montrons l'existence d'un équilibre de Nash du jeu. Les DRCCMDP peuvent être modélisés comme un problème d'optimisation sous contraintes robustes sur le plan distribution, dans lequel un décideur souhaite maximiser la valeur actualisée attendue d'une fonction de récompense. Sous certaines conditions, nous reformulons le problème d'optimisation de manière équivalente comme un problème déterministe, qui peut être résolu efficacement par des solveurs commerciaux.

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Nguyen Hoang Nam.

Résumé

L'optimisation sous contraintes en probabilité, également connue sous le nom d'optimisation stochastique ou d'optimisation probabiliste, est un domaine de l'optimisation qui traite des problèmes d'optimisation dans lesquels certaines variables sont soumises à des incertitudes probabilistes. Ces incertitudes sont souvent modélisées à l'aide de distributions de probabilité, ce qui rend les solutions optimales dépendantes de la probabilité. Voici quelques concepts clés liés à l'optimisation sous contraintes en probabilité:

1. **Fonction objective probabiliste** : Dans l'optimisation sous contraintes en probabilité, la fonction objective à maximiser ou minimiser dépend de variables aléatoires. Par conséquent, au lieu d'optimiser une seule valeur, vous cherchez à optimiser une fonction objective qui intègre les probabilités associées à différentes valeurs possibles des variables aléatoires.
2. **Contraintes probabilistes** : Les contraintes dans ce contexte sont également probabilistes. Cela signifie que les valeurs des variables de décision doivent satisfaire certaines propriétés probabilistes, telles que des inégalités probabilistes ou des équations probabilistes.
3. **Méthodes de résolution** : Plusieurs méthodes sont utilisées pour résoudre les problèmes d'optimisation sous contraintes en probabilité. Les méthodes de Monte Carlo, les méthodes d'approximation stochastique, et les méthodes de programmation stochastique sont quelques-unes des approches couramment utilisées.
4. **Analyse de sensibilité probabiliste** : Une fois que vous avez obtenu une solution optimale dans le contexte probabiliste, il est important d'effectuer une analyse de sensibilité pour évaluer comment cette solution change en fonction des variations des paramètres probabilistes. Cela peut aider à comprendre la robustesse de la solution optimale par rapport à l'incertitude.
5. **Analyse de sensibilité probabiliste** : Une fois que vous avez obtenu une solution optimale dans le contexte probabiliste, il est important d'effectuer une analyse de sensibilité pour évaluer comment cette solution change en fonction des variations des paramètres probabilistes. Cela peut aider à comprendre la robustesse de la solution optimale par rapport à l'incertitude.
6. **Applications** : L'optimisation sous contraintes en probabilité est utilisée dans divers domaines, notamment la gestion de portefeuille financier, la conception de produits sous incertitude, la planification de la chaîne d'approvisionnement, la gestion de projets, la gestion des ressources naturelles, et bien d'autres.

L'optimisation sous contraintes en probabilité est un puissant cadre mathématique qui aborde les problèmes de prise de décision en présence d'incertitude. Il fournit une approche systématique pour traiter les paramètres aléatoires ou variables incertaines, permettant aux décideurs de tenir compte de la probabilité de violation de

certaines contraintes tout en optimisant une fonction objectif dépend de variables aléatoires, ce qui signifie que le résultat n'est pas fixe, mais suit une distribution de probabilité. L'idée centrale derrière l'optimisation sous contraintes en probabilité est de s'assurer que la probabilité de violation des contraintes reste en dessous d'un seuil spécifié. Ce seuil représente le niveau de risque acceptable ou le niveau de confiance souhaité pour le décideur. Par exemple, si on essaye de minimiser les coûts de production tout en respectant des contraintes de qualité, la qualité de la production peut varier de manière aléatoire. L'optimisation stochastique utilise des concepts de probabilité et de statistiques pour modéliser l'incertitude et la variabilité. On peut utiliser des distributions de probabilité pour représenter les variables aléatoires et estimer les statistiques telles que la moyenne, la variance, etc. Différentes méthodes peuvent être utilisées pour résoudre des problèmes d'optimisation sous contraintes en probabilité. L'une des approches courantes est la programmation stochastique, qui consiste à discrétiser le problème en plusieurs scénarios possibles, puis à résoudre un problème d'optimisation déterministe pour chaque scénario. Une autre approche est d'utiliser la simulation Monte Carlo pour échantillonner les variables aléatoires et évaluer la performance de différentes solutions possibles. Cela permet d'estimer la distribution de probabilité de la fonction objectif et des contraintes, ce qui peut être utilisé pour prendre des décisions éclairées.

Dans l'optimisation sous contraintes en probabilité, les paramètres incertains peuvent avoir des distributions connues ou inconnues, ce qui influence la manière dont vous abordez le problème. Voici une explication plus détaillée de ces deux cas :

1. **Distributions connues:** Dans certains cas, on peut avoir une connaissance précise des distributions de probabilité qui décrivent les paramètres incertains. Ces distributions sont généralement connues ou estimées à partir de données empiriques. Les distributions de probabilité couramment utilisées incluent la distribution normale (gaussienne), la distribution uniforme, la distribution exponentielle, la distribution de Poisson, etc. Lorsque les distributions sont connues, vous pouvez les incorporer directement dans le modèle d'optimisation pour tenir compte de l'incertitude. Cela peut se faire à l'aide de techniques telles que la programmation linéaire stochastique (PLS), la programmation linéaire en nombres entiers stochastique (PLNE), ou d'autres méthodes d'optimisation probabiliste.
2. **Distributions inconnues:** Dans d'autres situations, les distributions de probabilité des paramètres incertains peuvent être inconnues, mal définies ou difficiles à estimer à partir des données disponibles. Dans de tels cas, vous pouvez utiliser des méthodes d'optimisation sous contraintes en probabilité robustes, qui cherchent à trouver des solutions optimales qui sont robustes face à l'incertitude, quelle que soit la distribution précise des paramètres incertains. Les approches robustes cherchent à minimiser ou maximiser la fonction objective sous les pires conditions possibles, souvent en utilisant des ensembles de scénarios ou des techniques de programmation robuste.

En fonction de la connaissance qu'on a sur les distributions de probabilité des paramètres incertains, on choisit la méthode d'optimisation appropriée. Les approches déterministes classiques supposent des valeurs fixes pour les paramètres, tandis que

les méthodes d'optimisation sous contraintes en probabilité prennent en compte l'incertitude associée à ces paramètres, ce qui peut conduire à des décisions plus prudentes et plus adaptées aux situations réelles.

L'optimisation sous contraintes en probabilité trouve des applications importantes dans la théorie des jeux et les processus de décision markoviens lorsque les décideurs doivent prendre en compte l'incertitude probabiliste dans leurs choix. Voici comment ces applications peuvent être abordées:

- **Théorie des jeux sous incertitude:** Dans la théorie des jeux, les joueurs prennent des décisions en fonction des actions de leurs adversaires. Lorsque les informations disponibles sont probabilistes (par exemple, lorsque les adversaires sont considérés comme rationnels mais avec une certaine incertitude sur leurs choix), l'optimisation sous contraintes en probabilité peut être utilisée pour déterminer les stratégies optimales. Par exemple, dans les jeux d'équilibre de Nash sous incertitude, les joueurs optimisent leur espérance de gain en tenant compte de la distribution de probabilité des choix adverses.
- **Processus de décision markovien sous incertitude:** Les processus de décision markoviens (MDP) sont couramment utilisés pour modéliser des problèmes de décision séquentielle dans des environnements incertains. Lorsque les paramètres du MDP sont incertains, l'optimisation sous contraintes en probabilité peut être utilisée pour déterminer des politiques de décision robustes. Par exemple, dans le contexte de la gestion des stocks, lorsque la demande future est incertaine, un MDP sous incertitude peut être résolu en optimisant l'espérance du rendement tout en respectant des contraintes de probabilité sur les niveaux de stock.
- **Applications en économie et en finance:** L'optimisation sous contraintes en probabilité est également pertinente dans le domaine de l'économie et de la finance, où les agents prennent des décisions en présence d'incertitudes sur les rendements, les taux de change, les prix des actifs, etc. Des modèles d'optimisation sous contraintes en probabilité sont utilisés pour la gestion de portefeuille, la gestion des risques et la prise de décision en finance.

Dans cette thèse, nous présentons d'abord un résultat théorique sur la convexité de l'optimisation sous contraintes en probabilité. Ensuite, nous étudions deux modèles spécifiques de la théorie des jeux et des MDPs impliquant l'optimisation sous contraintes en probabilité, connus sous le nom de jeux sous contraintes en probabilité (CCGs) et processus de décision Markovien sous contraintes en probabilité à robustesse distributionnelle (DRCCMDPs). Nous considérons différentes hypothèses sur la distribution des paramètres incertains. Dans les CCGs, sous certaines conditions, nous démontrons l'existence d'un équilibre de Nash du jeu. Les DRCCMDPs peuvent être modélisés comme un problème d'optimisation sous contraintes en probabilité à robustesse distributionnelle, où un décideur cherche à maximiser la valeur actualisée attendue d'une fonction de récompense. Sous certaines conditions, nous reformulons le problème d'optimisation de manière équivalente en un problème déterministe, qui peut être résolu efficacement par des solveurs connus.

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Chapter 1 : Introduction

1.1 . An overview and motivation

Chance-constrained optimization is a mathematical framework used to handle uncertainty in optimization problems. It addresses situations where decision-making must consider the probability of constraints being violated, rather than treating them as hard constraints. In traditional optimization, constraints are typically assumed to be deterministic, meaning they must be satisfied exactly. However, in many real-world scenarios, there is inherent uncertainty due to various factors such as measurement errors, model inaccuracies, or external disturbances. Chance-constrained optimization provides a systematic approach to incorporate this uncertainty into the decision-making process. The motivation behind chance-constrained optimization is to strike a balance between the robustness of the solution and the feasibility of the problem. It recognizes that satisfying all constraints with certainty might be overly conservative and result in suboptimal solutions. By allowing constraints to be violated within certain bounds or probabilities, chance-constrained optimization provides a more flexible approach. The key idea is to reformulate the deterministic constraints into probabilistic constraints. Instead of requiring a constraint to hold for all possible values of the uncertain parameters, chance-constrained optimization seeks to ensure that the constraints are satisfied with a specified probability, often referred to as the confidence level or risk tolerance.

In chance-constrained optimization, there are two main types of constraints: individual chance constraints and joint chance constraints. Individual chance constraints are constraints that apply to each constraint separately. They specify the probability that a single constraint is violated. For example, consider a production planning problem where the demand for a product is uncertain [159]. An individual chance constraint could be formulated to ensure that the probability of demand exceeding the available supply is below a certain threshold. Individual chance constraints provide a more localized view of constraint satisfaction, focusing on the probability of violating each constraint independently. Joint chance constraints involve multiple uncertain variables and specify the probability that a combination of events violates the constraints. These constraints consider the joint behavior of multiple constraints simultaneously. Joint chance constraints capture the interactions and dependencies among different constraints and provide a more comprehensive view of constraint satisfaction. For example, in a portfolio optimization problem, joint chance constraints may be used to ensure that the overall portfolio risk, defined as the probability of exceeding a specified loss threshold, remains below a certain level [132]. These constraints consider the combined behavior of all assets in the portfolio. The choice between individual and joint chance constraints depends on the specific problem and the relationship between the constraints. If the constraints are independent and there are no interactions among them, individual chance constraints can be applied separately. However, when there are dependencies or correlations among the constraints, joint chance constraints are more appropriate as they capture the collective behavior and provide a more accurate representation of the overall system risk.

Chance-constrained optimization can be used to model several problems in real applications. In health-care resource allocation, chance-constrained optimization can be used to determine optimal staffing levels and resource allocation while considering uncertain patient arrival rates and treatment demands [213]. By incorporating chance constraints, hospitals can ensure that the probability of inadequate staffing or exceeding resource capacity remains within acceptable limits, balancing patient service levels and operational costs. Chance-constrained optimization is also applicable to inventory management in retail, where demand for products is uncertain, and stockouts can lead to lost sales and dissatisfied customers [135]. Retailers can utilize chance-constrained optimization to determine optimal inventory levels, reorder points, and replen-

ishment policies while considering demand variability. The chance constraints help ensure a desired service level and minimize the probability of stockouts. Another application of chance-constrained optimization can be found in water resources management problems [65]. Chance-constrained optimization can assist in drought mitigation by optimizing water allocation strategies during periods of low water availability. By incorporating chance constraints, water managers can allocate water supplies among different users while considering uncertain inflows and drought conditions. The chance constraints help maintain the probability of meeting critical water demands within specified levels. An important application of chance-constrained optimization is in portfolio risk management for insurance companies [112]. Chance-constrained optimization is relevant to insurance companies for portfolio risk management, where the uncertainty lies in the occurrence of insurance claims. Insurers can use chance-constrained optimization to determine optimal investment and risk transfer strategies while considering the likelihood of claims exceeding certain thresholds. The chance constraints enable insurers to control the probability of severe losses or financial instability. Recently, chance-constrained optimization has been utilized in environmental impact assessment and renewable energy integration projects to ensure compliance with environmental regulations [16, 68]. When integrating renewable energy sources into the power grid, chance-constrained optimization can help in determining optimal generation and transmission capacities while considering uncertainty in renewable energy availability and grid stability requirements. The chance constraints ensure that the probability of violating environmental or operational constraints remains within acceptable limits.

In chance-constrained optimization, the feasibility set refers to the set of feasible solutions that satisfy the specified chance constraints. It represents the region of the decision space where the probability of violating the constraints remains below the specified threshold or confidence level. The size and shape of the feasibility set depend on various factors, including the uncertainty structure, the chosen confidence level, and the specific form of the probabilistic constraints. In general, as the confidence level becomes more stringent, the feasibility set tends to shrink, resulting in a more conservative solution space with a lower probability of constraint violation. Conversely, relaxing the confidence level expands the feasibility set, allowing for more risk and potential violations. The feasibility set can take various forms depending on the specific optimization problem. In some complex scenarios, the feasibility set may exhibit irregular shapes, making the optimization problem more challenging to solve. Finding the boundaries of the feasibility set can be computationally demanding, as it requires estimating or characterizing the joint probability distribution of the uncertain variables. This often involves statistical methods, such as Monte Carlo simulation or scenario generation, to sample from the uncertain parameter space and estimate the probabilities of violating the constraints. Understanding the feasibility set is crucial in chance-constrained optimization, as it provides insights into the trade-offs between feasibility and optimality. Decision-makers can examine the shape and size of the feasibility set to evaluate the robustness of their solutions and make informed decisions based on their risk preferences. By exploring the feasibility set, they can assess the impact of different confidence levels on the feasible solution space and determine the appropriate risk-reward trade-offs for their optimization problem.

The convexity of the feasibility set in chance-constrained optimization is a desirable property because it enables the use of efficient convex optimization techniques to find globally optimal solutions. It depends on the specific formulation of the probabilistic constraints and the underlying uncertainty structure. In some cases, the feasibility set can exhibit convexity, while in other cases, it may be non-convex. To determine the convexity of the feasibility set, we need to examine the structure of the probabilistic constraints and their relationship to the decision variables. In general, if the probabilistic constraints are linear or can be expressed as linear functions of the decision variables, and the underlying uncertainty is characterized by a known or tractable probability distribution, the resulting feasibility set is likely to be convex. Linear chance constraints with normally distributed uncertainty are a common example where the feasibility set is convex. If the probabilistic constraints are non-convex, one possible approach is to employ convex relaxation techniques. Convex relaxations involve approximating the original non-convex chance constraints with convex approximations

or relaxations. By replacing the original non-convex constraints with their convex counterparts, one can construct a convex optimization problem that provides an approximate solution within a certain error tolerance. In some cases, it may be challenging to analytically prove the convexity of the feasibility set. In such situations, numerical validation methods can be employed. These methods involve sampling from the uncertain parameter space and solving the chance-constrained optimization problem for different combinations of samples. By examining the resulting solutions, one can gain insights into the shape and behavior of the feasibility set. If the majority of sampled solutions lie in a convex region, it provides evidence for the convexity of the feasibility set.

In chance-constrained optimization, the probabilistic constraints are typically formulated based on assumed or estimated probability distributions. However, in practice, the true underlying distribution may be unknown or subject to estimation errors. Distributionally robust chance-constrained optimization is considered as an extension of traditional chance-constrained optimization that addresses uncertainties in the probability distribution itself. Distributionally robust chance-constrained optimization seeks to provide solutions that are robust to uncertainties in the probability distribution by considering a set of possible distributions rather than relying on a specific distribution. It aims to find solutions that perform well under the worst-case scenario within a given uncertainty set of probability distributions. The motivation behind distributionally robust optimization is to hedge against model misspecification and estimation errors by optimizing decisions that are robust across a range of plausible distributions. It provides a way to make decisions that are more resilient and less sensitive to the choice of probability distribution. To implement distributionally robust chance-constrained optimization, one typically considers a family of probability distributions that represents the uncertainty in the true distribution. This family is often defined by a set of moment or distributional constraints. The optimization problem then aims to find solutions that satisfy the chance constraints for all possible distributions within the uncertainty set, ensuring robustness against the worst-case scenario.

The choice of the uncertainty set is a critical aspect of distributionally robust chance-constrained optimization and should be carefully considered based on the problem at hand and the available information. Various approaches exist for defining uncertainty sets, each offering different trade-offs between conservatism and computational tractability. There are two common approaches to construct uncertainty sets, which are moment-based uncertainty sets and statistical-based uncertainty sets. Moment-based uncertainty sets refer to uncertainty sets that are defined based on moment or statistical properties of the uncertain variables[35]. These sets specify bounds or constraints on the moments of the uncertain variables, such as mean, variance, or higher-order moments. For example, a moment-based uncertainty set may restrict the mean of the uncertain variable to lie within a certain range or limit the variance to be below a specific threshold. They provide a deterministic characterization of the uncertainty based on moment information. Statistical distance-based uncertainty sets are defined based on a measure of statistical distance between the true distribution and a reference distribution. Instead of relying on specific moment constraints, these sets focus on the overall distributional discrepancy or similarity between the true distribution and the reference distribution. These uncertainty sets consider a range of plausible distributions that are close to the true distribution according to the chosen statistical distance measure. Common statistical distances used to define such sets include the Wasserstein distance[76], Kullback-Leibler divergence, or total variation distance[22, 117]. Both moment-based and statistical distance based uncertainty sets have their advantages and applications, depending on the problem context and available information. Moment-based sets provide a more deterministic representation based on specific moment constraints, while statistical distance based uncertainty sets allow for a more flexible and data-driven characterization of uncertainty based on statistical distances.

Our research aims to incorporate chance-constrained optimization and distributionally robust chance-constrained optimization into two distinct but interconnected areas: random games and Markov decision processes (MDPs). Both random games and MDPs belong to the broader category of stochastic games due to their inherent stochastic nature and the presence of uncertainties in their dynamics. Stochastic games

are a class of game theory models that capture strategic interactions among multiple decision-makers in an uncertain environment. In these games, the outcomes of players' actions and interactions are influenced by probabilistic factors, such as random events, uncertain states, or stochastic transitions. The incorporation of randomness and uncertainties distinguishes stochastic games from their deterministic counterparts.

Random games involve strategic interactions among multiple decision-makers, where uncertainties arise from the actions and outcomes of other players, which are subject to random events or uncertainties. In these games, each player's decision-making is influenced not only by the actions of other players but also by the random outcomes or uncertainties associated with the game environment. This adds complexity and challenges to the decision-making process, as players must anticipate and react to the uncertain behaviors of others. Examples of random games include games with incomplete information, where players have limited knowledge about the strategies or payoffs of others, games with random payoffs, where the outcomes depend on probabilistic events, or games with stochastic elements that introduce uncertainty and randomness into players' strategies and resulting outcomes. The presence of uncertainties in random games necessitates the use of advanced optimization techniques, such as chance-constrained or distributionally robust optimization, to effectively handle the risks and uncertainties inherent in these strategic interactions. These techniques allow decision-makers to explicitly account for the probabilities of constraint violations and manage the risks associated with uncertain actions and outcomes, leading to more informed and robust decision-making in the context of random games strategies and outcomes.

Markov Decision Processes (MDPs) can be seen as a specific type of stochastic game, characterized by the interaction between a single decision-maker and a stochastic environment over time. In MDPs, the decision-maker's actions directly influence the transitions between states and the corresponding rewards. However, these transitions and rewards are subject to uncertainties, reflecting the stochastic nature of the environment. The presence of uncertainties in MDPs stems from random events or probabilistic dynamics that impact the decision-making process and subsequent outcomes. These uncertainties introduce challenges as the decision-maker must carefully balance the trade-offs between exploration and exploitation to optimize long-term rewards while navigating the uncertain dynamics of the environment. By leveraging chance-constrained or distributionally robust optimization techniques in MDPs, decision-makers can explicitly account for the uncertainties, manage risks, and make informed decisions that maximize expected rewards while controlling the probability of constraint violations. Such approaches enhance the robustness and reliability of decision-making in MDPs, enabling effective navigation of the stochastic elements inherent in the environment.

In the context of chance-constrained optimization, when considering strategic interactions among multiple decision-makers in a game-theoretic framework, we encounter a specific class of problems known as chance-constrained games (CCGs). CCGs combine the principles of chance-constrained optimization and game theory to address decision-making problems under uncertainty within a competitive or cooperative setting. The study of CCGs offers valuable insights into the complexities of decision-making in dynamic and uncertain environments with strategic interactions. One of the important notions in CCGs is the equilibrium point, specifically the Nash equilibrium. A Nash equilibrium represents a stable state in the game where no player has an incentive to unilaterally deviate from their chosen strategy given the strategies of the other players. In CCGs, the Nash equilibrium captures the strategic behavior of decision-makers in the presence of chance constraints. It identifies a set of strategies where each player's strategy is optimal, taking into account both their own objectives and the constraints imposed by the chance events. At a Nash equilibrium, no player can improve their own outcome by independently changing their strategy, given the strategies of other players and the chance constraints. The concept of Nash equilibrium in CCGs provides insights into the equilibrium strategies that decision-makers adopt under uncertainty and strategic interactions. It helps in understanding the stability and robustness of the system and provides a benchmark for evaluating the performance of different strategies. Analyzing the Nash equilibrium in CCGs involves studying the interplay between players' strategies, the impact of chance constraints on their decision-making, and the resulting out-

comes. An important method to find a Nash equilibrium is the best-response algorithm. The best-response algorithm is an iterative process that allows players in a CCG to update their strategies in a sequential manner, aiming to converge to a Nash equilibrium. The algorithm involves each player choosing a strategy that optimizes their objective function, taking into account the strategies of other players and the chance constraints. It is important to note that the best-response algorithm does not guarantee convergence to a Nash equilibrium in all cases. Convergence may depend on the specific structure of the game, the players' objectives, and the chance constraints involved. For this reason, the convexity of chance constraints has several benefits in the context of finding a Nash equilibrium. Convex optimization problems have desirable properties that facilitate convergence. Solving a sequence of convex optimizations increases the likelihood of reaching a Nash equilibrium due to the convexity of the problem formulation. However, it is important to note that the convexity assumption of chance constraints is not always valid in all scenarios. In some cases, chance constraints may have non-convex formulations due to the nature of the underlying probabilistic constraints or the structure of the problem. In such situations, alternative optimization techniques or relaxation approaches may be employed to handle the non-convexity to approximate Nash equilibria in CCGs.

Combining the principles of distributionally robust chance constraints with Markov Decision Processes (MDPs) offers a powerful framework for decision-making under uncertainty with robustness considerations. This integration results in a specific class of problems referred to as distributionally robust chance-constrained Markov decision processes (DRCCMDPs). In DRCCMDPs, the decision-maker's actions aim not only to optimize expected rewards but also to adhere to probabilistic constraints on system properties, considering the uncertainty in the underlying probability distributions. These chance constraints ensure that the probability of violating specific conditions or thresholds remains within acceptable levels throughout the decision-making process. By incorporating distributionally robust optimization techniques, DRCCMDPs account for uncertainties in the probability distributions and provide robustness guarantees against a set of plausible distributions. The combination of distributionally robust chance constraints with MDPs enables decision-makers to make informed decisions that balance reward optimization and robustness under uncertain environments. By explicitly considering the uncertainty in the underlying probability distributions, DRCCMDPs allow decision-makers to develop strategies that are robust against worst-case scenarios, ensuring satisfactory performance even in the presence of distributional uncertainty. The combination of distributionally robust chance constraints with MDPs enables decision-makers to make informed decisions that balance reward optimization and robustness under uncertain environments. By explicitly considering the uncertainty in the underlying probability distributions, DRCCMDPs allow decision-makers to develop strategies that are robust against worst-case scenarios, ensuring satisfactory performance even in the presence of distributional uncertainty. When applying distributionally robust principles to Markov Decision Processes (MDPs), there are two main ways to consider uncertainty: reward uncertainty and transition probabilities uncertainty. In distributionally robust MDPs with reward uncertainty, the uncertainty lies in the rewards associated with different states and actions in the MDP. Traditional MDPs assume known reward distributions, but in practice, these distributions may be uncertain or difficult to estimate accurately. Distributionally robust MDPs with reward uncertainty aim to find policies that perform well under a set of plausible reward distributions. The decision-maker seeks policies that are robust against worst-case scenarios. They consider a family of possible reward distributions within a given ambiguity set and optimize policies that perform well under the worst-case distribution. This approach accounts for uncertainty in reward estimates and ensures that the selected policy is less sensitive to variations in the true reward distribution. In distributionally robust MDPs with transition probabilities uncertainty, the uncertainty lies in the transition probabilities that dictate how the MDP evolves from one state to another. In real-world scenarios, it is common for transition probabilities to be uncertain, especially when there are external factors or environmental changes that affect the system dynamics. Distributionally robust MDPs with transition probabilities uncertainty aim to find policies that are robust against uncertainties in the transition probabilities. The study and application of distributionally robust MDPs with either reward uncertainty

or transition probabilities uncertainty contribute to developing decision-making strategies that are resilient to uncertainties in the MDP model. These techniques find applications in various domains, including finance, resource management, robotics, and many others, where uncertainties in rewards or transition probabilities are prevalent and need to be accounted for to ensure effective decision-making.

1.2 . Mathematical framework

We consider the following deterministic optimization problem

$$\begin{aligned} \min \quad & c(x) \\ \text{subject to} \quad & \text{(i) } h(x, \xi) \geq 0 \\ & \text{(ii) } x \in Q. \end{aligned} \tag{1.1}$$

where c is a convex real function, Q is a closed convex subset of \mathbb{R}^n , $h : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^s$ is a multi-dimensional real function such that each component h_i , $i = 1, \dots, s$ is a convex function, $x \in Q$ is an n -dimensional decision vector, and $\xi \in \mathbb{R}^m$ is an m -dimensional parameter vector. The optimization problem (1.1) is a convex constrained optimization, which can be solved efficiently by convex solvers.

The transition from constrained optimization to chance-constrained optimization is motivated by the need to incorporate uncertainty and risk management into decision-making processes. While constrained optimization focuses on optimizing objectives subject to deterministic constraints, chance-constrained optimization introduces a probabilistic perspective to handle uncertainties more explicitly. By taking into account of the uncertainty of ξ , we consider the following chance-constrained optimization

$$\begin{aligned} \min \quad & c(x) \\ \text{subject to} \quad & \text{(i) } \mathbb{P}(h(x, \xi) \geq 0) \geq 1 - \epsilon \\ & \text{(ii) } x \in Q, \end{aligned} \tag{1.2}$$

where $\xi : \Omega \rightarrow \mathbb{R}^m$ is a random vector defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and $\epsilon \in [0, 1]$. The constraint (i) of (1.2) is called an individual chance constraint if $s = 1$ and a joint chance constraint if $s > 1$.

A fundamental issue in chance-constrained problems is the characterization of the convexity properties of chance constraints. The main question is whether these chance constraints preserve convexity, allowing for efficient optimization techniques and reliable solutions. Significant progress has been made in understanding the convexity properties of chance constraints under certain assumptions. For instance, if the underlying random variables are independent and have known probability distributions, chance constraints can often be reformulated as convex constraints using techniques such as chance constrained reformulation or scenario approximation. These reformulations allow for efficient optimization using convex optimization methods. For joint chance constraints case, researchers have explored conditions under which joint chance constraints can be convex, especially when specific assumptions are made about the distributions or dependencies among the random variables. In some cases, the convexity of joint chance constraints can be established based on properties of the underlying distributions, such as log-concavity or specific covariance structures.

In chance-constrained optimization, uncertainties are typically represented by probabilistic constraints that specify a required probability of constraint satisfaction. However, the specific probability distribution underlying the uncertain parameters is often assumed to be known, which may not reflect the true distribution accurately or may be difficult to estimate in practice. Distributionally robust chance-constrained optimization takes a more conservative and robust approach by considering a set of possible distributions or ambiguity sets instead of assuming a single known distribution. The ambiguity set represents a range of

potential distributions that the uncertain parameters may follow. We consider the following distributionally robust chance-constrained optimization

$$\begin{aligned} \min \quad & c(x) \\ \text{subject to} \quad & \text{(i) } \inf_{F \in \mathcal{D}} \mathbb{P}(h(x, \xi) \geq 0) \geq 1 - \epsilon \\ & \text{(ii) } x \in Q, \end{aligned} \tag{1.3}$$

where F is the distribution of ξ , \mathcal{D} is a given uncertainty set of potential distributions of ξ . The constraint (i) of (1.3) implies the constraint (i) of (1.2) if the true distribution of ξ belongs to the uncertainty set \mathcal{D} . Constructing a well-defined and appropriate uncertainty set is crucial in distributionally robust optimization to ensure reliable and meaningful results. The choice of uncertainty set affects the robustness of the optimization model and the trade-off between conservativeness and computational tractability. Two commonly used approaches have been proposed in the literature to construct uncertainty sets, based on either moments of the random variable or statistical distance between a nominal distribution and the true distribution of the random variable. A moment-based uncertainty set is constructed based on statistical moments of the uncertain parameters, such as mean and covariance. This type of uncertainty set is defined by specifying a range or bound on the moments of the uncertain distribution. The bounds can be determined from historical data, expert knowledge, or conservative estimates. A statistical distance-based uncertainty set is constructed based on the concept of statistical distances or divergences. These measures quantify the dissimilarity or discrepancy between probability distributions. The idea behind statistical distance-based uncertainty sets is to define a set of distributions that are close to a reference distribution according to a chosen statistical distance measure. The reference distribution is often derived from observed data or prior knowledge, and the uncertainty set encompasses distributions that are within a specified distance from the reference.

The mathematical framework of chance-constrained games combines elements from chance-constrained optimization and game theory to analyze decision-making problems under uncertainty within a competitive or cooperative setting. It involves defining the players' decision spaces, their objectives, the uncertainty in the system, and the chance constraints that govern their actions. The framework can be formulated as follows:

- **Players:** Given n decision-makers or players in the game denoted by the index $i \in \{1, 2, \dots, n\}$.
- **Decision Spaces:** Each player i has a decision space denoted by S_i , which represents the feasible set of strategies or actions available to that player.
- **Objective Functions:** Each player i has an objective function denoted by $f_i(\cdot)$, which quantifies their individual preferences or goals. The objective functions may involve maximizing or minimizing certain criteria, such as utility, profit, cost, or social welfare.
- **Uncertainty:** Uncertainty is represented by a probability distribution or a set of plausible distributions that capture the randomness or variability in the system. The uncertainty may arise from uncertain payoffs, incomplete information, or stochastic elements influencing players' decisions and outcomes.
- **Chance Constraints:** Chance constraints define probabilistic bounds on players' actions or outcomes. These constraints ensure that the probability of violating specific conditions or thresholds remains within acceptable levels throughout the decision-making process. Chance constraints can be formulated as constraint (i) of (1.2) or constraint (i) of (1.3) if the underlying distribution is assumed to be unknown.
- **Equilibrium Concept:** Nash equilibrium is used to capture the stable states in the game. A Nash equilibrium is a set of strategies for all players in which no player can unilaterally improve their payoff given the strategies of others.

The mathematical framework of distributionally robust chance-constrained Markov decision processes (DRCCMDPs) combines elements from distributionally robust optimization, chance-constrained optimization, and Markov decision processes to model decision-making under uncertainty within a dynamic and sequential framework. It involves formulating the decision spaces, the transition dynamics, the objective functions, and the chance constraints while accounting for uncertainty in the underlying probability distributions. The framework can be formulated as follows:

- **Decision Spaces:** Each state $s \in \mathcal{S}$ in the MDP is associated with a decision space denoted by \mathcal{A} , which represents the feasible set of actions available to the decision-maker.
- **Transition Dynamics:** The transition dynamics (or transition probabilities) describe the probabilistic evolution of the MDP from one state to another based on the chosen actions. The transition probabilities are fixed or uncertain and represented by a set of plausible distributions or ambiguity sets.
- **Objective Function:** Objective function is the expected reward, where the decision maker aims to maximize this function.
- **Uncertainty in Reward or Transition Probabilities:** The uncertainty in reward or transition probabilities is modeled using distributional ambiguity sets. These ambiguity sets capture a range of plausible distributions that represent the uncertainty. They can be defined based on moment-based or statistical distance-based approaches.
- **Chance Constraints:** Chance constraints are imposed on the system properties to ensure that the probability of violating specific conditions or thresholds remains within acceptable levels throughout the decision-making process. The chance constraints consider the uncertainty. In this framework, we consider chance constraints of the form (i) of (1.3).

1.3 . Contribution and Outline of the Dissertation

We highlight some contributions of this dissertation to the field of chance-constrained optimization, with specific applications in random games and Markov decision processes (MDPs). The key contributions of this research are as follows:

1. **Convexity Analysis of Chance-Constrained Optimization:** We provide a rigorous proof of the convexity of chance-constrained optimization problems when incorporating a random technology matrix into joint chance constraints. Our analysis is inspired by previous works in the literature [100, 49], which extends the understanding of the convexity properties of chance constrained optimization with random technology matrix. We assume that the row vectors of the random matrix follows a multivariate elliptical distribution and the dependence between them are captured by a Gumbel-Hougaard copula, which belongs to the class of Archimedean copulas. Under certain conditions, we prove the convexity of the feasibility set. By establishing the convexity of the optimization problem, one can enable the utilization of efficient convex optimization techniques, enhancing the tractability of the solution methods.
2. **Application in Random Games:** First, we consider a general n -players chance-constrained game framework, which can be adapted to different practical scenarios. Specifically, the strategy sets of each player are defined by joint chance constraints, incorporating a random technology matrix. We assume that the distribution of the row vectors of the technology matrix is known and we study two distinct cases: one where the row vectors of the random technology matrix follow an elliptical distribution, and

another where they follow a normal mean-variance mixture distribution. In both cases, we prove the existence of a Nash equilibrium for the game. This result demonstrates that, under certain assumptions, players can reach a stable strategic equilibrium even in the presence of chance constraints. We propose an algorithm, namely best-response algorithm to find a Nash equilibrium and present an application in the competition of two firms in financial market. A disadvantage of the best-response algorithm is that it does not always guarantee the convergence to a Nash equilibrium. To ensure the convergence to a Nash equilibrium in situations where the best-response algorithm may not guarantee it, we introduce a dedicated chapter that focuses on a specific game model. This model offers the assurance of the existence and uniqueness of a Nash equilibrium, along with the convergence of our algorithm to this equilibrium. Our study specifically delves into the framework of two-player zero-sum games, where the strategy sets of each player are defined by distributionally robust linear chance constraints with random technology matrix, i.e., we lack knowledge about the true distribution of the row vectors and we assume that they are only known to belong to some uncertainty set. In this scenario, we not only establish the existence of a Nash equilibrium but go further by demonstrating that the Nash equilibrium is unique and it can be obtained as the optimal solution of a primal-dual pair of second-order cone programmings, which can be solved efficiently by convex solvers.

3. **Application in Markov decision processes:** We extend the application of chance-constrained optimization to the domain of Markov decision processes (MDPs). We introduce a model called the distributionally robust chance-constrained MDP, which incorporates distributionally robust chance constraints into the MDP formulation. We consider two important sources of uncertainty: reward uncertainty and transition probabilities uncertainty. By integrating distributionally robust chance constraints into these components, we enable decision-makers to make robust decisions that account for the associated uncertainties. To handle the uncertainty in the distributionally robust chance constrained MDP, we construct two types of uncertainty sets. The first approach utilizes first and second moments of the uncertain parameters to define the uncertainty sets. These moments capture statistical properties of the uncertain distributions and provide a concise representation of uncertainty. The second approach utilizes phi-divergence and Wasserstein statistical distance measures to construct the uncertainty sets. This approach quantifies the dissimilarity between the true distribution and an assumed nominal distribution, allowing decision-makers to capture the range of plausible distributions. In all cases, we reformulate the optimization problem equivalently as deterministic optimization problems using the duality theory. These reformulations enable efficient and tractable solutions using well-known solvers such as Mosek and Gurobi.

The rest of the dissertation is organized as follows. In Chapter 2, we review important works related to chance-constrained optimization, distributionally robust chance-constrained optimization, and their applications in random games and Markov decision processes. We present the main results of the dissertation in Chapters 3-6. We conclude the primary contributions of this dissertation and develop a discussion on open issues and potential directions for future research in Chapter 7.

Chapter 2 :Literature review

2.1 . Chance-constrained optimization

The paper entitled "Chance-Constrained Programming" (CCP) by Charnes and Cooper published in 1959 in the journal Management Science [40], is one of the pioneering works in the field of chance-constrained optimization. The paper presented the basic framework of chance-constrained programming and discussed various aspects, including mathematical formulations, solution methods, and interpretation of results. The concept of chance constraints introduced in this paper has been extended and further developed in subsequent research in various fields, including finance, engineering, transportation, and environmental management, where decision-making under uncertainty is crucial. Charnes and Cooper [41] presented a novel approach to optimization problems, focusing on linear fractional programming. They proposed a method to transform fractional objective functions into linear ones, then making use of linear programming techniques. Charnes and Cooper [42] introduced the concept of data envelopment analysis (DEA) as a method for measuring the efficiency of decision-making units. The CCP problem has been analyzed from others authors, i.e., Naslund and Whinston [162], who considered a decision making model for investment in the stock market. Kataoka [122] and van de Panne and Pop [237] proposed a solution method for individual, or single row normal constraints. In [122], the author considered a transportation type problem, which can be solved by iteration of linear programming, while [237] considered an application of linear programming in determining the optimal composition of cattle feed. They reformulated the constraints as chance constraints, which can be solved using one of Zoutendijk's methods of feasible directions. Miller and Wagner [156] explored the model of joint constraints involving independent random variables on right-hand side. By utilizing statistical hazard function they identified conditions under which the resulting problem is convex (increasing hazard rate). To address this, they developed three algorithms that leverage the linearization of logarithmic chance constraints. Jagannathan [111] went beyond the assumption of independence for the random variables on the right-hand side, and further explored scenarios where the random coefficient matrix is independent (with a common row variance).

In general, evaluating the probability associated with chance constraints is computationally challenging. As a result, researchers have proposed numerous equivalent reformulations or approximations to handle chance constraints more effectively. These alternative formulations aim to simplify the computational complexity and enable tractable solutions for chance-constrained optimization problems. There are several commonly used methods to solve chance constraints in optimization problems, e.g., convex approximations, reformulation techniques, sample average approximation (SAA), scenario approximation, Bernstein approximation.

Reformulation techniques is a basic tool in handling chance constraints effectively in optimization problems. These techniques aim to transform the original chance constraints into alternative formulations that are easier to analyze and solve. For the individual linear chance constraints, the chance constraint can be reformulated equivalently as second-order cone programming (SOCP) constraint under certain assumption (the random variables follows elliptical or radial distribution) [185, 111, 90].

In case of linear joint chance constraints, it is hard to derive a tractable reformulation if the random variables are continuous. For this reason, several convex approximations have been proposed to deal with this case. Cheng et al. [49] proposed SOCP approximations based on piecewise linear approximations in independent row matrix case. To deal with the dependent rows case, Cheng et al. [48] used Archimedean copula to derive SOCP approximations. In the case where the random variables follow finite distributions, Luedtke and

Ahmed [150] proposed a novel approach by constructing a mixed integer linear programming reformulation for linear joint chance constraints. Vielma et al. [240] explores the use of mixed integer linear programming to model and solve chance constrained optimization. The paper presents practical applications and case studies, which could span various domains such as finance, supply chain management, transportation, or energy systems.

These abovementioned cases are some very particular cases where we can get exact reformulations or convex approximations of the chance constraints. However, such situation is rare, especially when the chance constraint is nonlinear [61, 130, 192]. The scenario approach, relying on Monte Carlo sampling techniques, provides a computationally tractable way to handle uncertainties by generating a set of representative scenarios and evaluating the feasibility of constraints based on these scenarios, that allows for solving chance constrained problems using standard deterministic optimization techniques. This approach has been studied in [33, 36, 53] and extended by Nemirovski and Shapiro [164]. They developed a solution methodology using scenario approximation techniques to address joint linear chance constrained problems. Under certain conditions, they proved that the solution of the approximation problem aligns with the feasibility of the original problem, ensuring a high probability of constraint satisfaction. Hong et al. [106] proposed sequential convex approximations to solve nonlinear joint chance constraints. Luedtke and Ahmed [149] studied some condition of the sample size and probability level such that a solution obtained by the approximation problem guarantees feasibility for the original problem. [165] introduced the Bernstein approximation technique, which is a method for approximating probability distributions or functions by using Bernstein polynomials. Their work on the Bernstein approximation has found applications in various areas, including chance constrained optimization, stochastic programming, and machine learning.

SAA method involves approximating the chance constraints using a finite number of samples drawn from the uncertain parameters. The problem is then transformed into a deterministic optimization problem that can be solved using standard techniques, where the accuracy of the approximation depends on the number of samples used. Several papers have been studied in the literature concerning SAA method [143, 6, 126, 209]. Recently, Cheng et al. [47] proposed partial SAA method to solve joint chance constraints. The main advantage of this approach is the approximation problem is still a continuous optimization with continuous variables.

A novel method, called Bicriteria approximation to solve chance-constrained covering problems refers to an approach that seeks to approximate solutions for such problems with multiple conflicting objectives or criteria. In chance-constrained covering problems, the goal is to find a set of cover elements that satisfy certain coverage requirements while also satisfying probabilistic constraints on the coverage probability. [261] proposed a bicriteria approximation scheme to derive a tractable convex relaxation of a chance constrained problem, which can be further extended to the distributionally robust setting. Adam et al. [1] proposed to solve joint chance constrained problems using regularization and Benders' decomposition. Deng et al. [59] studied scenario grouping and decomposition algorithms for chance-constrained programs

Another direction in solving chance-constrained optimization is establishing bounds by leveraging deterministic analytical approximations of chance constraints. By employing this approach, bounds can be derived based on mathematical techniques that provide deterministic approximations of the probabilistic constraints. An useful tool to derive bounds for individual chance constraints is based on extensions of Chebyshev inequality (see for instance [28, 105, 181]). For joint chance constraints, several papers proposed tight bounds to derive deterministic equivalent approximations [144, 225]. Recently, bounds based on nonlinear programming (NLP) problems, which can be solved by NLP solvers, have been proposed to derive approximations for nonlinear chance constrained optimization [78]. Karimi et al. [121] proposed a novel partial sample average approximation (PSAA) framework to solve the two types of chance-constrained linear matrix inequality problems with random technology matrix and with random right-hand side. Ahmed et al. [5] proposed two new Lagrangian dual problems for chance-constrained stochastic programs based on relaxing nonanticipativity constraints. Xie and Ahmed [260] studied quantile cuts and their closure for chance constrained optimization

problems over a finite distribution. Zhang et al. [270] proposed new valid inequalities and formulations for the static joint chance-constrained lot-sizing problem. Recently, Liu et al. [145] studied chance-constrained DRO in geometric optimization.

Recently, a novel method to chance-constrained optimization based on reformulations using partial KKT conditions and the use of dynamical neural network has been studied in the literature [227, 251]. Siham and Lissier [227] studied linear problems with joint chance constraints with dependent row vectors and the dependence is modeled by Gumbel-Hougaard copula. They proposed an ordinary differential equations (ODA) reformulation based on partial KKT conditions and used dynamical neural networks to solve. Dawen and Lissier [254] generalized this method to solve some class of nonlinear chance-constrained optimizations.

Chance-constrained optimization has a wide range of applications across various fields, e.g., supply chain planning and inventory management [190, 135], risk budgeting multi-portfolio [112, 137], production planning [159], hydro reservoir management [233, 9, 25], gas transport management [84, 75, 2, 196], electricity spot market modeling [98, 96], environmental management [93, 191, 65], renewable energy management [32, 68, 171, 10, 268], transportation and logistics [13, 14], risk management [4, 175, 12, 204, 112, 136], critical path network [212]

2.2 . Analytical properties of chance constraints

Convexity properties play a crucial role in chance-constrained optimization. The primary focus is on convex chance constraints, where the feasible region defined by the constraints forms a convex set. This property is desirable because it allows for efficient optimization algorithms and guarantees global optimality. Convex chance constraints can often be formulated using convex functions, such as the cumulative distribution function (CDF) or quantile functions. Motivated by Arrow and Enthoven [11] about quasi-concave programming, Prékopa [184] studied joint constraints with dependency by introducing quasi-concave constraint function. Prékopa [185] first introduced the notion of log-concave measures and this concept was further developed in his paper in 1972 [186]. Prékopa [187] introduced some useful applications of log-concave measures in stochastic programming models. The notion of log-concave measures was generalized to α -concave measures by Borel [29] and Brascamp and Lieb [31], which is further developed by Norkin and Roenko [173] with applications in probability optimization and fuzzy optimization. The property of α -concave has been shown adequately for many prominent distributions by the abovementioned papers and also applicable in some continuous distribution case, such as multivariate gamma in Prékopa and Szántai [191]. Dentcheva et al. [61] generalized the definition of α -concave measures on a set, which was used for extending optimality and duality theory in their paper in 2002 [62]. By employing a similar concept of Q-radial distribution, Calafiore and El Ghaoui [35] reformulated individual chance constraints as second-order cone constraints.

While significant progress has been made in reformulating individual chance constraints, addressing the issue of convexity remains a considerable challenge, particularly when dealing with joint chance-constrained problems. However, various extensions have been explored to address this issue. Prékopa et al. [193] posulated that a joint linear chance constrained problem is convex when the rows are assumed to follow independent normal distributions, with an additional condition that the covariance matrices of these rows are proportional to each other. Henrion and Strugarek [100] proved the convexity of joint chance constraints with independent random variables separated from decision vectors. To handle the dependent case, Henrion and Strugarek [101] introduced r -decreasing densities concept and the theory of copula, while Houda [107] used a variation to the mixing coefficient. Lagoa et al. [130] showed the convexity in the case where the random variables have log-concave and symmetric distribution. Some recent papers provide insights into the convexity properties of chance-constrained optimization problems and offer various convex relaxation techniques for handling different types of random variables and [229, 235, 7].

Optimality conditions and differentiability play crucial roles in analyzing and solving chance-constrained optimization problems. The notion of p-efficient points is an important concept in multi-objective optimization. It refers to a specific subset of solutions in the objective space that cannot be improved simultaneously in all objectives by any other feasible solution. This concept was first introduced in Prékopa [194], which was extensively analyzed later in several frameworks, [61] (probabilistically constrained stochastic programming with discrete distributions), [62] (integer programming problems), [24] (probabilistic set-covering problem), [24] (stochastic integer problems), [188] (linear problems), [60] (nonlinear problems). Lejeune and Noyan [134] proposed a novel method to generate p-efficient points of probabilistically constrained problems, in which the random variables are finitely distributed by solving mixed-integer programming. Klein Haneveld [125] introduced related measures of violation and conditional expectation constraints. Several algorithms have been proposed to calculate the probability distribution function, e.g., algorithms for multivariate normal distribution (without gradients) [226, 54, 80, 81] and algorithms based on a gradient calculation [228, 99, 231, 232, 94].

Calmness is a concept used in the field of mathematical optimization, especially in the study of constraint systems. It captures a form of stability for these systems. This concept is closely related to the notion of Lipschitz continuity in calculus, but in the context of set-valued mappings. Calmness is a useful property because it gives us some guarantees about the behavior of the constraint system. It is particularly important in the context of robust optimization and stochastic optimization, where the decision variables or the constraints may be subject to some uncertainty. If the constraint system is calm, then it means that small changes or perturbations will not significantly disrupt the optimal solution. The calmness property of a multifunction was first introduced in Rockafellar [201]. A few years later, analytical properties as well as its applications have been extensively studied in [95, 91, 92, 97].

A new research direction is to study properties of chance constraints (e.g., closedness, convexity, Lipschitz continuity, differentiability) in infinite dimension and application to PDE constrained optimization [73, 72, 79]. Pérez-Aros and Henrion [231], van Ackooij and Pérez-Aros [236] studied generalized differentiation of robust functions, i.e., probability functions acting on generalized semi-infinite inequality systems. Grandon et al. [85] investigated analytical properties of chance under continuous random distributions, e.g., weak sequential semi-continuity, Lipschitz continuity and differentiability of the probability function. Recently, the sample average approximation has been applied to PDE-constrained optimization problems for approximating risk-neutral optimization problems [157].

2.3 . Distributionally robust chance constrained optimization

In numerous real applications, it is frequently encountered that decision-makers have access to only partial information regarding the underlying distribution, primarily relying on historical data. Substituting the actual distribution with an estimated one may lead to an optimal solution that is highly likely to be infeasible in practical applications. For this reason, distributionally robust optimization (DRO) is a powerful and flexible framework for decision-making under uncertainty. It provides a principled approach to handle uncertainties in the distribution of uncertain parameters or data, allowing decision-makers to account for the inherent ambiguity in their optimization models.

There are two common approaches for constructing uncertainty sets of the underlying distribution in distributionally robust optimization based on either its moments information or statistical distance between the true distribution and a given reference distribution. Initial uncertainty sets for the underlying distribution are typically constructed based on precise moment information of the random parameter [83, 44, 183, 35, 139, 276]. Calafiore and El Ghaoui [35] presented a notable reformulation of a distributionally robust individual linear chance-constrained problem, transforming it into a SOCP problem. This reformulation provided a

more tractable and efficient approach to solving the problem, allowing decision-makers to effectively handle uncertainty while optimizing their objectives. Zymler et al. [276] made significant contributions to the field by introducing an approximation approach for distributionally robust joint chance constraints problems. Their methodology involved approximating the problem as a tractable semidefinite programming (SDP) problem. Notably, they demonstrated that the proposed SDP formulation serves as a reformulation when dealing with individual chance constraints. This advancement provides a computationally efficient and practical solution for handling joint chance constraints in a distributionally robust setting. Li et al. [139] presents a novel approach for incorporating ambiguous risk constraints into optimization problems. The authors introduce a methodology for constructing ambiguity sets that capture the uncertainty in the distribution while considering the available moment and unimodality information. The paper investigates the formulation and properties of these ambiguous risk constraints and proposes solution methodologies, considering both linear and nonlinear settings. The contributions of this paper lie in the development of a framework that combines moment and unimodality information to handle ambiguous risk constraints.

Delage and Ye [57] considers an ambiguity set defined by the first and second order moments of the uncertain parameters (unknown moments). The unknown first and second order moments are characterized by an elliptical constraint and a linear matrix inequality. The authors highlight the practical implications of their work by applying distributionally robust optimization to data-driven problems. Cheng et al. [46] studies ambiguity set with known first moment and unknown second moment to address the distributionally robust optimization of the stochastic knapsack problem. Yang and Xu [264] made a significant contribution by demonstrating the tractability of distributionally robust chance-constrained optimization under certain conditions. Specifically, they showed that if the uncertainty set can be characterized by its mean and variance within a given set, and the constraint function exhibits concavity with respect to the decision variables and quasi-convexity with respect to the uncertain parameters, then the resulting optimization problem is tractable. Their results offer practical implications, indicating that under specific assumptions on the uncertainty set and constraint function, efficient solution methodologies can be developed to solve the optimization problem. Xie and Ahmed [259] proved the convexity of a distributionally robust joint chance constraints optimization problem under certain conditions. Specifically, they showed that when the uncertainty set is specified by convex moment constraints, the resulting joint chance constraints optimization problem is convex.

An alternative approach to constructing uncertainty sets in distributionally robust optimization is based on statistical distance measures. Rather than relying solely on moment information, this approach considers the dissimilarity or distance between probability distributions. The choice of statistical distance measure and the construction of appropriate ambiguity sets is important. One commonly employed approach for data-driven uncertainty sets in distributionally robust optimization is based on phi-divergence and Wasserstein distance. phi-divergence measures, such as Kullback-Leibler divergence, Jensen-Shannon divergence, or Hellinger distance, quantify the difference between two distributions based on their probability densities. On the other hand, the Wasserstein distance, also known as the earth mover's distance, measures the minimum cost required to transform one distribution into another. Ben-Tal et al. [22] introduce and analyze a robust optimization framework based on phi-divergence. Hu and Mehrotra [108] investigated distributionally robust individual chance constrained optimization problems, where the uncertainty set of the probability distribution is defined using the Kullback-Leibler divergence, a specific case of phi-divergence. Jiang and Guan [117] made a novel result for distributionally robust joint linear chance constrained optimization problems with phi-divergence based uncertainty sets. They established that a distributionally robust joint linear chance constraint can be equivalently reformulated as a chance constraint with a perturbed risk level, which allows for a more tractable formulation and solution of the distributionally robust problem. Recently, Esfahani and Kuhn [158], Zhao and Guan [271] in their works showed that when the uncertainty set is defined using the Wasserstein distance, the distributionally robust expected utility optimization problem can be tractably solved under certain conditions. Gao and Kleywegt [76] introduce a distributionally robust stochastic optimization frame-

work that incorporates the Wasserstein distance and propose a deterministic reformulation by using duality theory. This work influenced and shed light on subsequent research in the field on utilization of the Wasserstein distance, e.g., [113, 114, 115]. On the other hand, Hanasusanto et al. [88] analyzed the computational complexity of the distributionally robust joint linear chance constrained program, considering the Wasserstein distance as the measure of distributional dissimilarity. By proving the problem's strong NP-hardness, they demonstrated that finding an optimal solution to this problem is computationally challenging.

An extension version of distributionally robust optimization (DRO) is DRO with decision-dependent uncertainty. DRO with decision-dependent uncertainty provides a framework to address uncertainty that is influenced by the decision variables in an optimization problem. By considering the relationship between decisions and uncertainty, it enables decision-makers to make robust and informed choices that account for the possible values of the uncertain parameters. Recently, [174] introduced this new class of DRO, where the ambiguity sets are constructed based on the total variation distance and the Wasserstein metrics. Luo and Mehrotra [151] studied this model in five types of ambiguity sets, with finite support and continuous support cases. Kettunen and Lejeune [123] derived decision-dependent stochastic programming formulations for data-driven project portfolio selection. Basciftci et al. [19] considered a DRO framework under decision-dependent stochastic demand in a facility location problem.

Recently, DRO has been extensively studied in various different frameworks. Duchi and Namkoong [64] developed and analyzed a DRO framework that learns a model providing good performance against perturbations to the data-generating distribution in statistics and machine learning. Bertsimas et al. developed a framework for solving an adaptive distributionally robust linear optimization problem. Zhu et al. [275] proposed a novel model that generalizes the common DRO models, based on kernel. They proposed a generalized duality theorem and stochastic optimization methods to solve this new class of DRO. Chen et al. [45] considered a DRO with infinitely constrained ambiguity sets. They proved that the DRO might not lead to tractable reformulation, then they proposed a method that solve a sequence of tractable distributionally robust optimization subproblems—each of which considers a relaxed and finitely constrained ambiguity set. Levy et al. [138] studied a distributionally robust optimization approaches for a one-stage stochastic minimization problem.

DRO has found applications in various fields, e.g., electricity and electrified transportation planning [86, 148], portfolio optimization [63], multi-item newsvendor problems [88], carbon emissions in transportation [265], energy management [37, 205], machine scheduling [177], sustainable development [116, 16], renewable energy [270], reverse scheduling [27, 140], autonomous vehicle [89], hub location problem [266], production planning [87], hybrid vehicle routing problem [266].

2.4 . Chance-constrained games

The publication of Cournot's seminal book "Researches into the Mathematical Principles of the Theory of Wealth" in 1838 [50] marked a turning point in the widespread adoption of equilibrium concepts under market conditions. Building upon Cournot's work, Von Neumann's study of saddle point equilibrium for two-player zero-sum games [244] further contributed to this line of research. In 1950, Nash demonstrated the existence of a Nash equilibrium, a point at which no player has an incentive to unilaterally deviate, in finite strategic games [160]. This result ignited extensive exploration of general strategic games in the literature [18, 55, 69], despite the practical limitations of the theory of Nash equilibrium in deterministic setups. This limitation arises when dealing with real-world applications involving random payoffs and strategy sets.

To handle random payoffs, the expectation function [200] has become a commonly used tool, particularly for risk-neutral cases. However, for risk-averse games, alternative payoff criteria based on risk measures such as Conditional Value-at-Risk (CVaR) [120, 200] and chance constraint programming [214, 216] have been

studied. Singh et al. [214] investigated finite strategic games with elliptically distributed payoff vectors and established the existence of a Nash equilibrium. The equivalence between a Nash equilibrium of a chance-constrained game (CCG), as studied in [214], and the global optimal solution of a specific mathematical program is stated in [216].

While the aforementioned games consider random payoff functions and deterministic strategy sets, many applications involve strategy sets with chance constraints. For example, risk constraints in portfolio optimization [112] and resource constraints in stochastic shortest path problems [49] often employ strategy sets with chance constraints. Consequently, games with chance-constrained strategy sets have been introduced in the literature [178, 179, 180, 217, 219]. Singh and Lisser [217] investigated a two-player zero-sum game with individual chance constraints, showing that the saddle point equilibrium problem is equivalent to a primal-dual pair of second-order cone programs when the random constraint vectors follow elliptically symmetric distributions. Singh et al. [219] extended the study to an n -player general-sum game with individual chance constraints under elliptically symmetric distributions, demonstrating the equivalence between a Nash equilibrium problem and the global optimization of a nonlinear optimization problem.

Motivated by these results, Peng et al. [179] established the existence of a Nash equilibrium for n -player general-sum games with joint chance-constrained strategy sets. The random constraint vectors were assumed to be independently normally distributed or follow a mixture of elliptical distributions [180]. When the probability distributions are not fully known and belong to a given distributional uncertainty set, Peng et al. [178] formulated the random constraints for each player as a distributionally robust joint chance constraint. They considered several uncertainty sets, including density-based and two-moments-based sets, with one of them having nonnegative support. The authors demonstrated the existence of a Nash equilibrium for a distributionally robust chance-constrained game under each uncertainty set.

In the aforementioned works [178, 179, 180], it was assumed that the random constraint vectors are independently distributed. However, in real-world applications, random variables often exhibit dependence. To study the dependence structure of random variables, the concept of copulas was introduced by Sklar in 1959 [221] as a solution to a probability problem raised by Fréchet in the context of random metric spaces. Copulas are functions used to separate the marginal distributions from the given dependent structure of a multivariate distribution. Henrion and Strugarek [100] introduced the concept of log-exp concavity of copulas to examine the convexity of dependent joint chance constraints in the case of elliptically distributed random constraint vectors. Nguyen et al. [166] studied the convexity of elliptically distributed linearly dependent joint chance constraints using copulas. Some eventual convexity results for joint chance constraints without using copulas are provided in [235].

A novel method based on ordinary differential equation (ODA) reformulation and dynamical neural network to find a Nash equilibrium of chance constrained games has been recently studied extensively in the works of Dawen and Lisser [253, 252, 256, 255]. They applied the method to find a Nash equilibrium in various framework, e.g., matrix games, two-players zero-sum games and general n -players games.

2.5 . Chance-constrained Markov decision processes

The Markov decision process (MDP) serves as a powerful mathematical framework for determining an optimal dynamic policy within a long-term environment characterized by uncertainty. MDP finds its importance in machine learning and artificial intelligence [51, 224, 43, 267, 274] with wide applications in various fields, including natural language processing (NLP) and information retrieval [77], healthcare system [223], computer networking and systems [246], runtime monitors problem [118], robotics and autonomous systems [15, 128, 133, 146], blockchain security and cryptography [141], ecology and evolution [38], mobile health (mHealth) and digital health [142], risk management and financial analytics [21] and more.

In MDPs, the source of randomness arises from two main perspectives: reward and transition probabilities. Reward randomness in MDPs refers to the presence of uncertainty or variability in the rewards received by the agent as it interacts with the environment. To handle reward randomness, various techniques can be employed, including exploration [17, 245], risk management [131], reinforcement learning [247, 109, 269], probabilistic model [129].

Transition probabilities randomness in MDPs refers to the presence of uncertainty or variability in the probabilities of transitioning between states based on the agent's chosen actions. It introduces a stochastic element into the dynamics of the system, making the state transitions probabilistic rather than deterministic. Transition probabilities randomness can arise from several factors, including noisy, partial observations or unknown dynamics [127]. Various techniques can be employed to handle transition probabilities randomness, including exploration methods [250, 74] or sensitivity analysis [119].

Most of the research efforts in addressing this uncertainty have primarily focused on robust Markov Decision Processes (MDPs), where the rewards or transition probabilities are assumed to belong to a pre-defined uncertainty set [110, 172, 238, 248, 249]. However, it has been observed that the robust MDP approach often leads to conservative policies [56]. To overcome this issue, Delage and Mannor introduced the concept of a chance-constrained Markov decision process (CCMDP) [56], where the controller aims to achieve the expected discounted reward with a certain confidence level.

In their work, Delage and Mannor considered the cases of random rewards and random transition probabilities separately and demonstrated that a CCMDP can be reformulated as a second-order cone programming (SOCP) problem when the running reward vector follows a multivariate normal distribution and the transition probabilities are precisely known. However, when the transition probabilities follow a Dirichlet distribution and the running rewards are precisely known, the CCMDP problem becomes computationally intractable, necessitating the use of approximation methods to compute optimal policies.

Varagapriya et al. [239] extended the concept of CCMDP by considering a CMDP problem with a joint chance constraint, where the running cost vectors are random vectors and the transition probabilities are known. They proposed two SOCP-based approximation methods that provide upper and lower bounds to the CMDP problem, specifically when the cost vectors follow multivariate elliptical distributions and the dependence among the constraints is driven by a Gumbel-Hougaard copula (Varagapriya et al. [238]). Xia et al. studied distributionally robust chance-constrained Markov decision processes with known transition probabilities and random reward vector belongs to a Kullback-Leibler distance ball, centered at an elliptical reference distribution.

Chapter 3 :Convexity of Linear Joint Chance Constrained Optimization

In this chapter, we study the convexity of the linear joint chance constraints. Specifically, we consider the scenario where the constraint row vectors follow an elliptical distribution. To model the dependence among the rows, we adopt a family of Archimedean copulas, specifically the Gumbel-Hougaard copulas. Under certain mild assumptions, we establish the eventual convexity of the feasibility set.

3.1 . Introduction

We consider the following linear optimization with joint chance constraints

$$\begin{aligned} \min \quad & c^\top x \\ \text{subject to} \quad & \mathbb{P}\{Vx \leq D\} \geq p \\ & x \in Q. \end{aligned} \tag{3.1}$$

where Q is a closed convex subset of \mathbb{R}^n such that $\delta_l \leq \|x\| \leq \delta_u$, for any $x \in Q$, $\|\cdot\|$ denotes the Euclidean norm, δ_l and δ_u are strictly positive real numbers, $D = [D_1, \dots, D_K]^\top \in \mathbb{R}^K$ is a deterministic vector, $V = [v_1, \dots, v_K]^\top$ is a random matrix with size $K \times n$, where v_k is a random vector in \mathbb{R}^n , for any $k = 1, 2, \dots, K$ and $p \in (0, 1)$. We denote $S(p)$ the feasibility set of (3.1). Let $J = \{1, 2, \dots, K\}$.

The convexity of chance constraints as well as the analytical properties of the probability function play an important role in convex optimization which are difficult issues and scarcely studied in the literature. This problem was first introduced by Prékopa [189]. He considers the following chance constraints

$$\mathbb{P}(h(x, \xi) \geq 0) \geq p, \tag{3.2}$$

where $x \in \mathbb{R}^n$ is a decision vector, $\xi : \Omega \rightarrow \mathbb{R}^m$ is a random vector defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, $h : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^s$ and $p \in [0, 1]$ is a given probability threshold. Theorem 10.2.1 in [189] states that the feasibility set of (3.2) is convex if $\mathbb{P} \circ \xi^{-1}$ of ξ is a log-concave probability measure on \mathbb{R}^m and the components of h are quasi-concave. Henrion and Strugarek [100] studied a particular form of (3.2) by taking $h(x, \xi) = g(x) - \xi$, where $g : \mathbb{R}^n \rightarrow \mathbb{R}^m$. Hence, they consider the following form

$$\mathbb{P}(\xi \leq g(x)) \geq p, \tag{3.3}$$

where they suppose that the components of ξ are independent. They prove that if the cumulative distribution functions of the components of ξ have r -decreasing densities and the components of g are r -concave, the feasibility set of (3.3) is convex. Henrion and Strugarek [101] generalized this result to the case where the components of ξ are dependent. He uses the theory of copulas to model the dependence of the components of ξ . Marti [153] studied the differentiation of probability functions by an integral transformation method. The derivatives of the probability function can be obtained by applying an integral transformation to its integral representation. Some basic results on the differentiability of a probability function were studied by Kibzun et al. [124]. They proposed new formulations of the gradient of probability functions in different forms, i.e., integral over the surface, volume, or sum of surface and volume integrals. Lobo et al. [147] studied some applications of second-order cone program leading to a new approach for solving chance constraints. A more developed direction was initialized by Henrion and Strugarek [100] which gave a full description of the

structure (not only the convexity) of a one-row linear optimization with a chance constraint by introducing a new notion of r -decreasing function. Henrion and Strugarek [100] studied the convexity in the case where the constraints are independent. To deal with the dependent case, Henrion and Strugarek [101], Cheng et al. [48] and Van Ackooij [229] used the theory of copulas to model the dependence of the constraints. They supposed that the distribution of the constraint row vectors are elliptically distributed. Under high probability threshold p , they prove the convexity of $S(p)$. Hong et al. [106] proposed to solve joint chance-constrained programs by sequential convex approximations. They proved that the solutions of the sequence of approximations converge to a Karush-Kuhn-Tucker (KKT) point of the original problem. Farshbaf-Shaker et al. [73] proved some properties of chance constraints in infinite dimensions. They supposed that the feasibility set belongs to a Banach space. Under mild conditions, they proved regularity properties of the probability function with an application to PDE constrained optimization. Wim van Ackooij and Malick [235] studied the convexity of the feasibility set in a general framework by using the radial representation of elliptical distributions.

The convexity of chance constraints has been used to show the existence of a Nash equilibrium in chance-constrained games [167, 179, 180]. Nguyen et al. [167] assume that the random constraint vectors follow elliptical distributions and show that there exists a Nash equilibrium of the chance-constrained game. Peng et al. [179], [180] show a similar result by assuming that the random constraint vectors follow either Normal distributions or mixture of elliptical distributions.

The convexity of chance constraints could significantly impact the fields of operations research, logistics, supply chain management, financial risk management, and many others, by providing more effective ways to handle uncertainties. Here are some of the advantages and potential real-world applications.

- **Efficient Solution Methods:** If chance constraints are shown to be convex, more efficient solution methods (such as interior-point methods) can be applied to solve the problem, saving time and computational resources. This is particularly useful in large-scale stochastic optimization problems where computational efficiency is critical.
- **Quality of Solution:** Convex problems have the property that any local optimum is also a global optimum. Therefore, if the chance constraints are convex, we have the guarantee that if a local optimum is also a global optimum. This leads to better decision making in practical situations.
- **Robustness and Certainty:** Convexity of chance constraints allows for a higher level of robustness and certainty in the solutions obtained. Convex problems are less susceptible to changes in the data. This is especially beneficial in real-world scenarios where data can often change.
- **Modelling Flexibility:** The convexity of chance constraints could increase the modeling flexibility, allowing researchers and practitioners to model complex, real-world situations more accurately.

Chance constraints can be used in operations research to handle the uncertainty in decision-making processes. Here are a few concrete examples

- In supply chain optimization problems, there can be uncertainty in demand, transportation costs, and delivery times. Using chance constraints, a supply chain manager could design a distribution network so that the probability of meeting customer demand is at least a certain percentage. This ensures that the supply chain is robust enough to meet demand under varying conditions, which could reduce costs associated with stockouts or overstocking.
- In portfolio optimization, an investor may wish to ensure that the probability of the portfolio's return falling below a certain level is minimized. This can be formulated as a chance constraint optimization problem, where the objective is to maximize the expected return subject to a chance constraint on the portfolio's return.

- In renewable energy planning, power production from sources like wind and solar is uncertain. Chance constraints can be used to ensure that a certain level of power demand is met with a high probability, given the uncertainty in renewable energy production. For example, an energy planner might need to decide how much backup capacity to maintain, given the uncertainty in wind power production.
- In hospital resource allocation, chance constraints can be used to make sure there's a high probability of having enough resources (beds, doctors, nurses, medical supplies) to meet patient demand. For example, a hospital could use chance constraints to decide how many operating rooms to keep open, given the uncertainty in the number of surgeries that will need to be performed.

In this chapter, we study the convexity of the feasible set $S(p)$ when the row vectors v_i follow an elliptical distribution and the dependence of the random constraint vectors is captured by a Gumbel-Hougaard copula. We derive a new reformulation of the joint chance constraints and prove the convexity of $S(p)$ under mild conditions. Our main contributions can be summarized as follows:

- Compared with [101] and [229], we consider the chance constraints with random matrix.
- Compared with the convexity results in [100, 235], we consider the dependent rows with copula.
- Cheng et al. [48] consider the joint chance constraints defined in (3.1). They assume that the dependence of the row vectors v_i^\top , $i \in J$ follows a copula independently of x (cf. Assumption 2.21 [48]). In our research, we prove the convexity of $S(p)$ without this strong assumption.

This chapter is organized as follows. In Section 3.2, we recall some basic concepts and preliminary results. We propose a new reformulation of the joint chance constraints in Section 3.2.1. Next, we present some theoretical results in Section 3.2.2 and Section 3.3 which are useful to prove the convexity of the feasibility set $S(p)$ in Section 3.4 and show some numerical results. We conclude the chapter in Section 3.5.

3.2 . Basic concepts and preliminary results

In this section, we recall some useful definitions and propositions for our subsequent analysis.

Definition 3.1. An n -dimensional random vector X follows a spherical distribution if there exists a function $\Psi : \mathbb{R} \rightarrow \mathbb{R}$ such that the characteristic function $\phi_X(t)$ of X is given by

$$\phi_X(t) = \mathbb{E}(e^{it^\top X}) = \Psi(t^\top t).$$

The function Ψ is called a characteristic generator of the spherical distribution.

Definition 3.2. An n -dimensional random vector U follows an elliptical distribution with location parameter μ , positive definite scale matrix Σ and characteristic generator Ψ (in short $U \sim \text{Ellip}(\mu, \Sigma, \Psi)$), if we have the following representation

$$U \triangleq \mu + AX,$$

where X follows a spherical distribution with a characteristic generator Ψ , $A \in \mathbb{R}^{n \times n}$ such that $AA^\top = \Sigma$ and $\mu \in \mathbb{R}^n$; \triangleq implies that the both sides have the same distribution.

The probability density function of all the distributions from elliptical family does not always exist. Whenever it exists, it is of the form

$$f_U(z) = \frac{c}{\sqrt{\det(\Sigma)}} g_{\text{den}} \left(\sqrt{(z - \mu)^\top \Sigma^{-1} (z - \mu)} \right),$$

where g_{den} is a nonnegative function called *radial density* and $c > 0$ is a normalization factor which makes f_U a probability density function.

Definition 3.3. A function $f : Q \rightarrow (0, +\infty)$ is r -concave on a set $Q \subset \mathbb{R}^s$ for a given $r \in (-\infty, +\infty)$ if for any $x, y \in Q$ and $y \in [0, 1]$,

$$\begin{aligned} f(yx + (1-y)y) &\geq [yf(x)^r + (1-y)f(y)^r]^{\frac{1}{r}}, \text{ when } r \neq 0, \\ f(yx + (1-y)y) &\geq f(x)^y f(y)^{1-y}, \text{ otherwise.} \end{aligned}$$

Definition 3.4. A real function $f : \mathbb{R} \rightarrow \mathbb{R}$ is r -decreasing for some real number $r \in \mathbb{R}$, if f is continuous on $(0, +\infty)$ and there exists some strictly positive real number t^* such that the function $t \mapsto t^r f(t)$ is strictly decreasing on $(t^*, +\infty)$.

Table 3.1 presents some 1-dimensional spherical distributions with r -decreasing densities for some values of r and their thresholds t^* [167].

Distribution	Radial density	r	t^*
Normal	$e^{-\frac{1}{2}u^2}$	$r > 0$	\sqrt{r}
t	$(1 + \frac{1}{\nu}u^2)^{-(1+\nu)/2}$, $\nu > 0, \nu$ integer	$0 < r < \nu + 1$	$\sqrt{\frac{r\nu}{\nu+1-r}}$
Laplace	$e^{- u }$	$r > 0$	$\frac{r}{\sqrt{2}}$
Kotz type	$u^{2(N-1)} e^{-qu^{2s}}$, $q, s > 0, N > \frac{1}{2}$	$r > 2(1 - N)$	$\sqrt[2s]{\frac{2(N-1)+r}{2qs}}$
Pearson type VII	$(1 + \frac{u^2}{m})^{-N}$, $m > 0, N > \frac{1}{2}$	$0 < r < 2N$	$\sqrt{\frac{rm}{2N-r}}$

Table 3.1: Typical 1-dimensional spherical distributions with r -decreasing densities and their thresholds t^* .

Definition 3.5. A function $C : [0, 1]^K \rightarrow [0, 1]$ is a K -dimensional copula if C is a joint CDF of a K -dimensional random vector, on the unit cube $[0, 1]^K$, whose marginals are uniformly distributed on $[0, 1]$.

Proposition 3.1 (Sklar's Theorem). Let $F : \mathbb{R}^K \rightarrow [0, 1]$ be a joint CDF of a K -dimensional random vector and F_1, \dots, F_K are the marginal CDFs. Then, there exists a K -dimensional copula C such that

$$F(z) = C(F_1(z_1), \dots, F_K(z_K)).$$

Moreover, if F_i is continuous for any $i = 1, \dots, K$, then C is uniquely given by

$$C(u) = F\left(F_1^{(-1)}(u_1), \dots, F_K^{(-1)}(u_K)\right).$$

Proposition 3.2 (Fréchet-Hoeffding upper bound). For any K -dimensional copula C and $u = [u_1, \dots, u_K]^\top \in [0, 1]^K$, we have

$$C(u) \leq C_M(u) = \min_{k=1, \dots, K} u_k.$$

Definition 3.6. A K -dimensional copula C is strictly Archimedean if there exists a continuous and strictly decreasing function $\psi : (0, 1] \rightarrow [0, +\infty)$, such that $\psi(1) = 0$, $\lim_{t \rightarrow 0} \psi(t) = +\infty$, and for any K -dimensional vector $u = (u_1, \dots, u_K) \in [0, 1]^K$, we have

$$C(u) = \psi^{(-1)} \left(\sum_{i=1}^K \psi(u_i) \right).$$

The function ψ is called a generator of the copula C .

Table 3.2 presents a selection of some strictly Archimedean copulas with their generators [167].

Type of copula	Parameter θ	Generator $\psi_\theta(t)$
Independent	-	$-\log(t)$
Gumbel-Hougaard	$\theta \geq 1$	$[-\log(t)]^\theta$
Frank	$\theta > 0$	$-\log \left(\frac{e^{-\theta t} - 1}{e^{-\theta} - 1} \right)$
Clayton	$\theta > 0$	$\frac{1}{\theta}(t^\theta - 1)$
Joe	$\theta \geq 1$	$-\log[1 - (1 - t)^\theta]$

Table 3.2: Selected types of strictly Archimedean copulas.

Definition 3.7. A function $f : \mathbb{R} \rightarrow \mathbb{R}$ is K -monotonic on an open interval $I \subseteq \mathbb{R}$ for some positive integer $K \geq 2$, if the following three conditions hold:

1. f is differentiable up to the order $(K - 2)$ on I ,
2. The derivatives of f satisfy

$$(-1)^k \frac{d^k}{dt^k} f(t) \geq 0, \quad 0 \leq k \leq K - 2,$$

for any $t \in I$,

3. The function $t \mapsto (-1)^{K-2} \frac{d^{K-2}}{dt^{K-2}} f(t)$ is nonincreasing and convex on I .

Proposition 3.3 (Theorem 2.2, [155]). Let $\psi : (0, 1] \rightarrow [0, +\infty)$ be a strictly decreasing function such that $\psi(1) = 0$ and $\lim_{t \rightarrow 0} \psi(t) = +\infty$. Then, ψ is the generator of a K -dimensional strictly Archimedean copula if and only if the inverse function $\psi^{(-1)}$ is K -monotonic on $(0, +\infty)$ and continuous on $[0, +\infty)$.

3.2.1. Reformulation of the probability function

Assume that the random vectors $v_i \sim \text{Ellip}(\mu_i, \Sigma_i, \Psi_i)$, for any $i \in J$. Let

$$\xi_i(x) := \frac{v_i^\top x - \mu_i^\top x}{\sqrt{x^\top \Sigma_i x}}, \quad g_i(x) := \frac{D_i - \mu_i^\top x}{\sqrt{x^\top \Sigma_i x}}. \quad (3.4)$$

Using the notations in (3.4), the chance constraint in (3.1) can be rewritten as follows

$$\mathbb{P} \{ \xi_i(x) \leq g_i(x), \quad i \in J \} \geq p.$$

It is well known that $\xi_i(x)$ follows 1-dimensional spherical distribution with characteristic generator Ψ_i [71]. Our aim is to reformulate this function in order to study the convexity of $S(p)$. By Sklar's Theorem, we have that, there exists a copula C_x such that

$$C_x[F_1(g_1(x)), \dots, F_K(g_K(x))] \geq p, \quad (3.5)$$

where C_x is the K -dimensional copula of the K -dimensional random vector $\xi(x) = [\xi_1(x), \dots, \xi_K(x)]^\top$ and F_i is the cumulative distribution function of $\xi_i(x)$, for $i = 1, \dots, K$. In [48], the authors assume that C_x does not depend on x , i.e., there exists a copula C such that $C_x = C$, for any $x \in Q$ (cf. Assumption 2.21 [48]). We study the general case, where C_x is a copula, which depends on x . Assume that for any $x \in Q$, C_x is a strictly Archimedean copula with generator ψ_x . Then, the constraint (3.5) can be rewritten as follows

$$\psi_x^{(-1)}\left(\sum_{i=1}^K \psi_x(F_i(g_i(x)))\right) \geq p. \quad (3.6)$$

Using the decreasing monotonicity of ψ_x , (3.6) is equivalent to

$$\sum_{i=1}^K \psi_x(F_i(g_i(x))) \leq \psi_x(p) \quad (3.7)$$

By adding auxiliary variables $\{y_i \geq 0, i \in J\}$, we reformulate (3.7) into individual chance constraints [49, 48]. Since ψ_x is positive, (3.7) is equivalent to the following constraints

$$\begin{cases} \text{(i)} & \psi_x(F_i(g_i(x))) \leq y_i \psi_x(p), \quad i \in J, \\ \text{(ii)} & y_i \geq 0, \quad i \in J, \\ \text{(iii)} & \sum_{i=1}^K y_i = 1. \end{cases} \quad (3.8)$$

This means that if $x^* \in S(p)$ then there exists $y^* = [y_1^*, \dots, y_K^*]^\top \in \mathbb{R}^K$ such that (x^*, y^*) satisfies constraints (3.8). On the other hand, if (x^*, y^*) is a feasible solution for constraints (3.8) and $x^* \in Q$, then $x^* \in S(p)$. Moreover, for $x^* \in S(p)$, we can choose y^* in order to satisfy constraints (3.8) as

$$y_i^* = \frac{\psi_{x^*}(F_i(g_i(x^*)))}{\sum_{j=1}^K \psi_{x^*}(F_j(g_j(x^*)))}, \quad \forall i \in J. \quad (3.9)$$

Using the decreasing monotonicity of the generator ψ_x , constraints (3.8) can be written as follows

$$\begin{cases} \text{(i)} & F_i(g_i(x)) \geq \psi_x^{(-1)}(y_i \psi_x(p)), \quad i \in J, \\ \text{(ii)} & y_i \geq 0, \quad i = 1, \dots, K, \\ \text{(iii)} & \sum_{i=1}^K y_i = 1. \end{cases} \quad (3.10)$$

In the rest of the chapter, we assume that the following assumption holds.

Assumption 3.1. C_x is a Gumbel-Hougaard copula, for any $x \in Q$, i.e., the generator ψ_x is given by

$$\psi_x(t) = (-\log t)^{\frac{1}{\kappa(x)}}, \quad (3.11)$$

for any $(x, t) \in Q \times (0, 1]$, where $\kappa(x) : Q \rightarrow (0, 1]$ is a strictly positive function.

Remark 3.1. Our aim is to show the concavity of $F_i(g_i)$ w.r.t x and the joint convexity of $\psi_x^{(-1)}(y_i \psi_x(p))$ w.r.t (y_i, x) .

3.2.2 . Concavity of $F_i(g_i(\cdot))$

In this section, we will show our main result in Lemma 3.2 that under Assumption 3.2, $F_i(g_i(\cdot))$ is a concave function on $S(p)$. Define an index set $I^\mu \subset J$ such that $\mu_i \neq 0$ for any $i \in I^\mu$ and $\mu_i = 0$ otherwise. Define a set of real numbers $\{r_i \mid i \in J\}$ such that

$$\begin{cases} r_i > 1, & \text{if } i \in I^\mu, \\ r_i = 1, & \text{if } i \notin I^\mu. \end{cases}$$

Assumption 3.2. (i) *The cumulative distribution function F_i has $(r_i + 1)$ - decreasing densities with the thresholds t_i^* , for any $i \in J$.*

(ii) $p > p^*$, where

$$p^* = \max \left\{ \frac{1}{2}, \max_{j \in I^\mu} F_i \left(\frac{r_i + 1}{r_i - 1} \lambda_{i,\min}^{-\frac{1}{2}} \|\mu_i\| \right), \max_{i \in J} F_i[t_i^*(r_i + 1)] \right\}, \quad (3.12)$$

where $\lambda_{i,\min}$ is the smallest eigenvalue of the positive definite matrix Σ_i , for any $i \in J$.

Lemma 3.1. *If Assumption 3.2 holds, then*

$$\text{Conv}(S(p)) \subset \bigcap_{j \in I^\mu} \Omega^j,$$

where

$$\Omega^j = \left\{ x \in Q \mid D_j - \mu_j^\top x > \frac{r_j + 1}{r_j - 1} \lambda_{j,\min}^{-\frac{1}{2}} \|\mu_j\| \sqrt{x^\top \Sigma_j x} \right\},$$

and $\text{Conv}(S(p))$ is the convex hull of $S(p)$. Moreover, for any $i = 1, 2, \dots, K$, $g_i > 0$ and g_i is $(-r_i)$ - concave on any convex subset Q^i of $\bigcap_{j \in I^\mu} \Omega^j$.

Proof. Let $i \in I$ and $x_0 \in S(p)$. It follows from the constraint (3.5) that

$$C_{x_0}[F_1(g_1(x_0)), \dots, F_K(g_K(x_0))] \geq p. \quad (3.13)$$

By applying Proposition 3.2, we get $F_i(g_i(x_0)) \geq p$, $\forall i \in J$, which in turn implies by Assumption 3.2 that

$$F_i(g_i(x_0)) > p^* \geq F_i \left(\frac{r_i + 1}{r_i - 1} \lambda_{i,\min}^{-\frac{1}{2}} \|\mu_i\| \right), \quad \forall i \in J$$

Since $F_i(\cdot)$ is an increasing function, we have

$$g_i(x_0) > \frac{r_i + 1}{r_i - 1} \lambda_{i,\min}^{-\frac{1}{2}} \|\mu_i\|,$$

which implies that

$$D_i - \mu_i^\top x_0 > \frac{r_i + 1}{r_i - 1} \lambda_{i,\min}^{-\frac{1}{2}} \|\mu_i\| \sqrt{x_0^\top \Sigma_i x_0}.$$

Therefore, $S(p) \subset \bigcap_{j \in I} \Omega^j$. For each $j \in I$, Ω^j is a convex set which implies that $\text{Conv}(S(p)) \subset \bigcap_{j \in I} \Omega^j$. We prove the second part of Lemma 3.1 by considering the following two cases:

Case 1: Let $i \notin I^\mu$, then $\mu_i = 0$. By Assumption 3.2, we have $p > \frac{1}{2}$. Let $x_0 \in S(p)$. By applying Proposition 3.2 on (3.13), we have

$$F_i(g_i(x_0)) \geq p > \frac{1}{2}. \quad (3.14)$$

Since, F_i is the CDF of a 1-dimensional real-valued random variable which is symmetric at 0, $F_i(0) = \frac{1}{2}$. From (3.14) we get $g_i(x_0) > 0$ which in turn implies that $D_i - (\mu_i)^\top x_0 > 0$. Since $\mu_i = 0$, we get $D_i > 0$. In conclusion, the proof follows directly from Lemma 3 of [48].

Case 2: Let $i \in I^\mu$. It follows from Lemma 2 of [48] that the function

$$f_i(x) = \left(\frac{\sqrt{(x)^\top \Sigma_i x}}{D_i - (\mu_i)^\top x} \right)^{r_i}$$

is a convex function on $\bigcap_{j \in I} \Omega^j$. Therefore, for any $y, z \in Q^i \subseteq \bigcap_{j \in I^\mu} \Omega^j$ and $\lambda \in [0, 1]$, we have

$$f_i[\lambda y + (1 - \lambda)z] \leq \lambda f_i(y) + (1 - \lambda)f_i(z). \quad (3.15)$$

Note that $g_i(x) = (f_i(x))^{\frac{-1}{r_i}}$ on Q^i . From (3.15), we can write

$$g_i[\lambda y + (1 - \lambda)z] \geq (\lambda(g_i(y))^{-r_i} + (1 - \lambda)(g_i(z))^{-r_i})^{\frac{-1}{r_i}}.$$

It is clear that $g_i > 0$ on $\bigcap_{j \in I} \Omega^j$. Hence, g_i is $(-r_i)$ -concave on Q^i . \square

Using Lemma 3.1, we prove the following lemma.

Lemma 3.2. *If Assumption 3.2 holds, then $F_i(g_i(\cdot))$ is concave on $\text{Conv}(S(p))$, for any $i \in J$.*

Proof. Using Lemma 3.1, g_i is $(-r_i)$ -concave and $g_i > 0$ on $\text{Conv}(S(p))$, for any $i \in J$. Hence, for any $a \in [0, 1]$ and $x_1, x_2 \in S(p)$, we have

$$g_i(ax_1 + (1 - a)x_2) \geq [ag_i^{-r_i}(x_1) + (1 - a)g_i^{-r_i}(x_2)]^{-\frac{1}{r_i}}. \quad (3.16)$$

As $x_1 \in S(p)$ and $p > p^*$, the constraint (3.5) implies that

$$C_{x_1}[F_1(g_1(x_1)), \dots, F_K(g_K(x_1))] > p^*,$$

which in turn implies from Proposition 3.2 and Assumption 3.2 that

$$F_i(g_i(x_1)) > p^* \geq F_i[t_j^*(r_i + 1)], \forall i \in J.$$

Since, $F_i(\cdot)$ is monotonically increasing, we get

$$g_i(x_1) > t_i^*(r_i + 1) > 0,$$

which implies that $0 < g_i(x_1)^{-r_i} < (t_i^*(r_i + 1))^{-r_i}$, for any $i \in J$. Similarly, we obtain the same inequality for x_2 . By taking F_i on both sides of (3.16),

$$F_i(g_i(ax_1 + (1 - a)x_2)) \geq F_i([ag_i^{-r_i}(x_1) + (1 - a)g_i^{-r_i}(x_2)]^{-\frac{1}{r_i}}). \quad (3.17)$$

Since $F_i(\cdot)$ has $(r_i + 1)$ -decreasing density, from Lemma 3.1 of [100], the function $t \mapsto F_i\left(t^{-\frac{1}{r_i}}\right)$ is concave on $(0, (t_i^*)^{-r_i})$. Therefore, we can write

$$F_i([ag_i^{-r_i}(x_1) + (1 - a)g_i^{-r_i}(x_2)]^{-\frac{1}{r_i}}) \geq aF_i(g_i(x_1)) + (1 - a)F_i(g_i(x_2)). \quad (3.18)$$

From (3.17) and (3.18), we deduce that

$$(F_i(g_i(ax_1 + (1 - a)x_2))) \geq a((F_i(g_i(x_1)))) + (1 - a)((F_i(g_i(x_2)))).$$

Therefore, $F_i(g_i(\cdot))$ is concave on $\text{Conv}(S(p))$. \square

3.3 . Convexity of $\psi_x^{(-1)}(y_i \psi_x(p))$

Let $U(x, y_i) := \psi_x^{(-1)}(y_i \psi_x(p))$. The main result of this section is Lemma 3.7 which shows that under Assumption 3.3, U is jointly convex. In this section, we assume that the feasibility set $S(p)$ is non empty. Let c_l be a real number such that $0 < c_l \leq 1$. Define an index set $I^D \subset J$ such that $D_i > 0$ for any $i \in I^D$ and $D_i \leq 0$ otherwise. For any $i \in J$, let

$$G_i = \begin{cases} \left(\frac{\log \left(F_i \left(\frac{D_i}{\sqrt{\lambda_{i,\min}} \delta_l} + \frac{\|\mu_i\|}{\sqrt{\lambda_{i,\min}}} \right) \right)}{\log p} \right)^{\frac{1}{c_l}}, & \text{if } i \in I^D, \\ \left(\frac{\log \left(F_i \left(\frac{D_i}{\sqrt{\lambda_{i,\max}} \delta_u} + \frac{\|\mu_i\|}{\sqrt{\lambda_{i,\min}}} \right) \right)}{\log p} \right)^{\frac{1}{c_l}}, & \text{if } i \in J \setminus I^D, \end{cases}$$

where $\lambda_{i,\max}$ is the largest eigenvalue of the positive definite matrix Σ_i , for any $i \in J$. Let $h_l := \min_{1 \leq j \leq K} (G_j)$ and $h_u := 1 - (K - 1) h_l$.

Lemma 3.3. $0 < h_l \leq h_u < 1$. Moreover, given $x \in S(p)$ and y_i refers to (3.9). Hence, $h_l \leq y_i \leq h_u$, for any $i \in J$.

Proof. Let $x \in S(p)$ and

$$y_i = \frac{\psi_x[F_i(g_i(x))]}{\sum_{j=1}^K \psi_x[F_j(g_j(x))]}, \quad \forall i \in J, \quad (3.19)$$

where $\psi_x(\cdot)$ refers to (3.11). It is easy to see that $\sum_{i \in J} y_i = 1$ and $y_i \geq 0$, for any $i \in J$. We prove that $y_i \geq G_i$, for any $i \in J$. Consider two cases as follows:

Case 1: $i \in I^D$. It follows from the Cauchy-Schwarz inequality that

$$|-\mu_i^\top x| \leq \|\mu_i\| \|x\|, \quad \forall i \in J. \quad (3.20)$$

By the property of maximal/minimal eigenvalue, we have

$$\sqrt{\lambda_{i,\max}} \|x\| \geq \sqrt{x^\top \Sigma_i x} \geq \sqrt{\lambda_{i,\min}} \|x\|. \quad (3.21)$$

Since $x \in S(p)$, it follows from the constraint (3.7) that

$$0 < \sum_{j=1}^K \psi_x[F_j(g_j(x))] \leq \psi_x(p). \quad (3.22)$$

From (3.4), (3.20) - (3.21) and the fact that $\|x\| \geq \delta_l$, we get

$$g_i(x) \leq \frac{D_i}{\sqrt{x^\top \Sigma_i x}} + \frac{|-\mu_i^\top x|}{\sqrt{x^\top \Sigma_i x}} \leq \frac{D_i}{\sqrt{\lambda_{i,\min}} \delta_l} + \frac{\|\mu_i\|}{\sqrt{\lambda_{i,\min}}}.$$

As F_i is increasing and ψ_x is decreasing, we get

$$\psi_x(F_i(g_i(x))) \geq \psi_x \left(F_i \left(\frac{D_i}{\sqrt{\lambda_{i,\min}} \delta_l} + \frac{\|\mu_i\|}{\sqrt{\lambda_{i,\min}}} \right) \right). \quad (3.23)$$

From (3.19), (3.22) and (3.23), we have

$$y_i \geq \frac{\psi_x \left(F_i \left(\frac{D_i}{\sqrt{\lambda_{i,\min} \delta_l}} + \frac{\|\mu_i\|}{\sqrt{\lambda_{i,\min}}} \right) \right)}{\psi_x(p)} = \left(\frac{\log \left(F_i \left(\frac{D_i}{\sqrt{\lambda_{i,\min} \delta_l}} + \frac{\|\mu_i\|}{\sqrt{\lambda_{i,\min}}} \right) \right)}{\log p} \right)^{\frac{1}{\kappa(x)}}. \quad (3.24)$$

Since $0 \leq y_i \leq 1$, the following condition holds

$$0 < \frac{\log \left(F_i \left(\frac{D_i}{\sqrt{\lambda_{i,\min} \delta_l}} + \frac{\|\mu_i\|}{\sqrt{\lambda_{i,\min}}} \right) \right)}{\log p} \leq 1,$$

which in turn implies by (ii) of Assumption 3.3 that

$$\left(\frac{\log \left(F_i \left(\frac{D_i}{\sqrt{\lambda_{i,\min} \delta_l}} + \frac{\|\mu_i\|}{\sqrt{\lambda_{i,\min}}} \right) \right)}{\log p} \right)^{\frac{1}{\kappa(x)}} \geq \left(\frac{\log \left(F_i \left(\frac{D_i}{\sqrt{\lambda_{i,\min} \delta_l}} + \frac{\|\mu_i\|}{\sqrt{\lambda_{i,\min}}} \right) \right)}{\log p} \right)^{\frac{1}{c_1}} = G_i. \quad (3.25)$$

Then, it follows from (3.24) and (3.25) that $y_i \geq G_i$.

Case 2: $i \in J \setminus I^D$. In this case, $D_i \leq 0$. Then, by (3.20), (3.21) and $x \leq \delta_u$, we have

$$g_i(x) \leq \frac{D_i}{\sqrt{\lambda_{i,\max} \delta_u}} + \frac{\|\mu_i\|}{\sqrt{\lambda_{i,\min}}}.$$

It follows the similar proof procedure as Case 1 that $y_i \geq G_i$.

Therefore, combining the results in Case 1 and Case 2, we have $y_i \geq h_l > 0$, for any $i \in J$. Since, $\sum_{i \in J} y_i = 1$, we get

$$y_i = 1 - \sum_{j \in J, j \neq i} y_j \leq 1 - (K-1) h_l = h_u < 1. \quad (3.26)$$

Hence, $0 < h_l \leq y_i \leq h_u < 1$, for any $i \in J$. □

It follows from Lemma 3.3 that h_l, h_u belong to $(0, 1)$. Therefore, we can define

$$\begin{aligned} \varphi_1^* &:= c_1 (\log h_u) (\log p) h_l. \\ \varphi_2^* &:= \frac{(\log h_l)^2}{4c_1} + \max \left([1 + \log h_l (1 + (\log p) h_l)]^2, [1 + \log h_l c_1 (1 + \log p)]^2 \right). \\ \omega &:= \frac{\varphi_2^*}{\varphi_1^*}. \end{aligned} \quad (3.27)$$

Remark 3.2. Since $\varphi_1^* > 0$, then ω is well-defined. Moreover, ω does not depend on (x, y_i) .

In order to show the convexity of U , we first show that the Hessian matrix of U is positive semidefinite. The following lemma is a reformulation of the positive semidefiniteness of the Hessian matrix of U .

Lemma 3.4. The positive semidefiniteness of the Hessian matrix of U on the convex set $\mathcal{Q} \times [h_l, h_u]$ is equivalent to the positive semidefiniteness of the following n -dimensional symmetric matrix

$$N(x, y_i) = \frac{\partial^2}{\partial y_i^2} U(x, y_i) \times H_x U(x, y_i) - \left(\nabla_x \frac{\partial}{\partial y_i} U(x, y_i) \right) \left(\nabla_x \frac{\partial}{\partial y_i} U(x, y_i) \right)^\top, \quad (3.28)$$

for any $(x, y_i) \in Q \times [h_l, h_u]$, where

$$H_x U = \begin{bmatrix} \frac{\partial^2 U}{\partial x_1^2} & \frac{\partial^2 U}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 U}{\partial x_1 \partial x_n} \\ \frac{\partial^2 U}{\partial x_2 \partial x_1} & \frac{\partial^2 U}{\partial x_2^2} & \cdots & \frac{\partial^2 U}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 U}{\partial x_n \partial x_1} & \frac{\partial^2 U}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 U}{\partial x_n^2} \end{bmatrix}, \quad \nabla x = \left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \right)^\top.$$

Proof. The Hessian matrix of U at a point (x, y_i) is an $(n+1)$ -dimensional symmetric matrix which has the form $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$, where $A = H_x U(x, y_i)$, $B = \nabla_x \frac{\partial}{\partial y_i} U(x, y_i)$, $C = B^\top$, $D = \frac{\partial^2}{\partial y_i^2} U(x, y_i)$. The main idea of the proof is based on the Schur's complement. It suffices to show that $\frac{\partial^2}{\partial y_i^2} U(x, y_i) > 0$, for any $(x, y_i) \in Q \times [h_l, h_u]$. In fact, for $U(x, y_i) = \psi_x^{(-1)}(y_i \psi_x(p))$, we have

$$\frac{\partial^2}{\partial y_i^2} U(x, y_i) = [\psi_x(p)]^2 (\psi_x^{(-1)})''(y_i \psi_x(p)).$$

Since $\psi_x(t) = (-\log t)^{\frac{1}{\kappa(x)}}$, we deduce that

$$\psi_x^{(-1)}(t) = e^{-t^{\kappa(x)}}, \quad (\psi_x^{(-1)})''(t) = e^{-t^{\kappa(x)}} t^{\kappa(x)-2} \kappa(x) [\kappa(x) t^{\kappa(x)} - \kappa(x) + 1].$$

Using the above formulations, $\psi_x(p) > 0$ and $(\psi_x^{(-1)})''(t) > 0$, for any $t > 0$. Hence, $\frac{\partial^2}{\partial y_i^2} U(x, y_i) > 0$, for any $(x, y_i) \in Q \times [h_l, h_u]$. \square

Assumption 3.3. We assume that

- (i) $p \geq e^{-1}$.
- (ii) $0 < c_l \leq \kappa(x) \leq 1$, for any $x \in Q$.
- (iii) $H_\kappa(x) - \omega \nabla_x \kappa(x) (\nabla_x \kappa(x))^\top$ is a positive semidefinite matrix for any $x \in Q$, where $H_\kappa(x)$ is the Hessian matrix of $\kappa(x)$; $\nabla_x \kappa(x)$ is the gradient vector of $\kappa(x)$.

Lemma 3.5. Let

$$\varphi_1(x, y_i) := \kappa(x) (\log y_i) \left[\kappa(x) - 1 + \kappa(x) (\log p) y_i^{\kappa(x)} \right].$$

If Assumption 3.3 holds, then $\varphi_1(x, y_i) \geq \varphi_1^* > 0$, for any $(x, y_i) \in Q \times [h_l, h_u]$, where φ_1^* refers to (3.27).

Proof. Since $0 < y_i \leq h_u < 1$, then

$$(-\log y_i) \geq (-\log h_u) > 0. \quad (3.29)$$

As $0 < h_l \leq y_i < 1$ and $0 < \kappa(x) \leq 1$, we deduce that $1 \geq y_i^{\kappa(x)} \geq y_i \geq h_l > 0$. Note that $1 \geq -\log p > 0$ (because $e^{-1} \leq p < 1$ by (i) of Assumption 3.3). Then, $1 \geq -(\log p) y_i^{\kappa(x)} \geq -(\log p) h_l > 0$. Since $0 < \kappa(x) \leq 1$ and $1 + (\log p) y_i^{\kappa(x)} \geq 0$, we have

$$\begin{aligned} 1 - \kappa(x) - \kappa(x) \log(p) y_i^{\kappa(x)} &= 1 - \kappa(x) (1 + \log p y_i^{\kappa(x)}) \\ &\geq 1 - (1 + \log p y_i^{\kappa(x)}) = -(\log p) y_i^{\kappa(x)} \geq -(\log p) h_l > 0, \end{aligned} \quad (3.30)$$

for any $(x, y_i) \in Q \times [h_l, h_u]$. Moreover, by (ii) of Assumption 3.3, we have

$$\kappa(x) \geq c_l > 0, \quad \forall x \in Q. \quad (3.31)$$

Therefore, it suffices to multiply term by term the inequalities (3.29)-(3.31) to complete the proof. \square

Lemma 3.6. Let $\varphi_2(x, y_i) := \kappa(x) \log(y_i)^2 (1 + (\log p) y_i^{\kappa(x)}) \left[1 - \kappa(x) - \kappa(x) (\log p) y_i^{\kappa(x)} \right] + \left(1 + \kappa(x) (\log y_i) + (\log p) (\log y_i) y_i^{\kappa(x)} \kappa(x) \right)^2$. If Assumption 3.3 holds, $0 < \varphi_2(x, y_i) \leq \varphi_2^*$, for any $(x, y_i) \in Q \times [h_l, h_u]$, where φ_2^* refers to (3.27).

Proof. Since $p \geq e^{-1}$, then $0 > \log p \geq -1$. Using $0 < y_i^{\kappa(x)} < 1$, we have $0 < 1 + (\log p) y_i^{\kappa(x)} < 1$. Moreover, as $0 < \kappa(x) \leq 1$, then $0 < \kappa(x) (1 + (\log p) y_i^{\kappa(x)}) < 1$. Let $s := \kappa(x) (1 + (\log p) y_i^{\kappa(x)})$. By applying the Cauchy-Schwarz inequality, we deduce that $s(1 - s) \leq \frac{1}{4}$ (the equality holds if and only if $s = \frac{1}{2}$). Hence,

$$\kappa(x)(1 + (\log p) y_i^{\kappa(x)}) \left[1 - \kappa(x) (1 + (\log p) y_i^{\kappa(x)}) \right] \leq \frac{1}{4},$$

which in turn implies that

$$0 < (1 + (\log p) y_i^{\kappa(x)}) \left[1 - \kappa(x) (1 + (\log p) y_i^{\kappa(x)}) \right] \leq \frac{1}{4 \kappa(x)} \leq \frac{1}{4 c_l}. \quad (3.32)$$

Since $1 > y_i \geq h_l > 0$, we have $0 > \log y_i \geq \log h_l$. Then,

$$0 < (\log y_i)^2 \leq (\log h_l)^2. \quad (3.33)$$

Moreover,

$$0 < \kappa(x) \leq 1. \quad (3.34)$$

By multiplying term by term the inequalities (3.32)- (3.34), we get

$$0 < \kappa(x) \log(y_i)^2 (1 + (\log p) y_i^{\kappa(x)}) \left[1 - \kappa(x) - \kappa(x) (\log p) y_i^{\kappa(x)} \right] \leq \frac{(\log h_l)^2}{4 c_l}, \quad (3.35)$$

for any $(x, y_i) \in Q \times [h_l, h_u]$. By Assumption 3.3, we have

$$\begin{aligned} & \text{(i) } 0 < c_l \leq \kappa(x) \leq 1, \text{ (ii) } 0 < -\log h_u \leq -\log y_i \leq -\log h_l, \\ & \text{(iii) } 0 \leq 1 + \log p \leq 1 + (\log p) y_i^{\kappa(x)} \leq 1 + (\log p) h_l. \end{aligned} \quad (3.36)$$

Note that the condition (iii)(3.36) holds since $\log p < 0$ and $1 \geq y_i^{\kappa(x)} \geq h_l$. By multiplying (i) – (iii) of (3.36) term by term, we get

$$-\log h_l (1 + (\log p) h_l) \geq -(\log y_i) \kappa(x)(1 + (\log p) y_i^{\kappa(x)}) \geq -(\log h_u) c_l (1 + \log p),$$

which is equivalent to

$$1 + (\log h_l) (1 + (\log p) h_l) \leq 1 + (\log y_i) \kappa(x)(1 + (\log p) y_i^{\kappa(x)}) \leq 1 + (\log h_u) c_l (1 + \log p),$$

which in turn implies that

$$\begin{aligned} & 0 \leq [1 + (\log y_i) \kappa(x)(1 + (\log p) y_i^{\kappa(x)})]^2 \\ & \leq \max \left((1 + (\log h_l) (1 + (\log p) h_l))^2, (1 + (\log h_u) c_l (1 + \log p))^2 \right). \end{aligned} \quad (3.37)$$

Adding (3.35) and (3.37) together, completes the proof. \square

Lemma 3.7. *If Assumption 3.3 holds, then U is jointly convex on $Q \times [h_l, h_u]$.*

Proof. Using Lemma 3.4, it suffices to show the positive semidefiniteness of $N(x, y_i)$ in (3.28) for any $(x, y_i) \in Q \times [h_l, h_u]$. Since $\psi_x(t) = (-\log t)^{\frac{1}{\kappa(x)}}$, we have $\psi_x^{(-1)}(t) = e^{-t^{\kappa(x)}}$. Then, we can write

$$U(x, y_i) = e^{-\left\{y_i(-\log p)^{\frac{1}{\kappa(x)}}\right\}^{\kappa(x)}} = p^{y_i^{\kappa(x)}}. \quad (3.38)$$

We calculate explicitly the partial derivatives of U as follows

$$\begin{aligned} \text{(i)} \quad & \frac{\partial}{\partial y_i} U(x, y_i) = (\log p) p^{y_i^{\kappa(x)}} \kappa(x) y_i^{\kappa(x)-1}. \\ \text{(ii)} \quad & \frac{\partial^2}{\partial y_i^2} U(x, y_i) = \kappa(x) (\log p) y_i^{\kappa(x)-2} p^{y_i^{\kappa(x)}} [\kappa(x) - 1 + \kappa(x) (\log p) y_i^{\kappa(x)}]. \\ \text{(iii)} \quad & \nabla_x U(x, y_i) = (\log p) p^{y_i^{\kappa(x)}} (\log y_i) y_i^{\kappa(x)} \nabla_x \kappa(x). \\ \text{(iv)} \quad & \nabla_x \frac{\partial}{\partial y_i} U(x, y_i) = \\ & (\log p) y_i^{\kappa(x)-1} p^{y_i^{\kappa(x)}} [1 + \kappa(x) (\log y_i) + (\log p) (\log y_i) y_i^{\kappa(x)} \kappa(x)] \nabla_x \kappa(x). \\ \text{(v)} \quad & H_x U(x, y_i) = \\ & p^{y_i^{\kappa(x)}} y_i^{\kappa(x)} (\log p) (\log y_i) [H_\kappa(x) + (\log y_i + (\log y_i) (\log p) y_i^{\kappa(x)}) \nabla_x \kappa(x) (\nabla_x \kappa(x))^\top]. \end{aligned}$$

Hence, we obtain the following formulations

$$\begin{aligned} \text{(i)} \quad & \frac{\partial^2}{\partial y_i^2} U(x, y_i) \times H_x U(x, y_i) = \kappa(x) (\log p)^2 y_i^{2\kappa(x)-2} (\log y_i) p^{2y_i^{\kappa(x)}} \times \\ & \left[\kappa(x) - 1 + \kappa(x) (\log p) y_i^{\kappa(x)} \right] \left[H_\kappa(x) + \nabla_x \kappa(x) (\nabla_x \kappa(x))^\top (\log y_i + (\log y_i) (\log p) y_i^{\kappa(x)}) \right]. \\ \text{(ii)} \quad & \left(\nabla_x \frac{\partial}{\partial y_i} U(x, y_i) \right) \left(\nabla_x \frac{\partial}{\partial y_i} U(x, y_i) \right)^\top = (\log p)^2 y_i^{2\kappa(x)-2} p^{2y_i^{\kappa(x)}} \times \\ & \left(1 + \kappa(x) (\log y_i) + (\log p) (\log y_i) y_i^{\kappa(x)} \kappa(x) \right)^2 \nabla_x \kappa(x) (\nabla_x \kappa(x))^\top. \end{aligned} \quad (3.39)$$

Note that $(\log p)^2 y_i^{2\kappa(x)-2} p^{2y_i^{\kappa(x)}}$ is a positive common factor of $\frac{\partial^2}{\partial y_i^2} U(x, y_i) \times H_x U(x, y_i)$ and $\left(\nabla_x \frac{\partial}{\partial y_i} U(x, y_i) \right) \left(\nabla_x \frac{\partial}{\partial y_i} U(x, y_i) \right)^\top$. Then, it follows from (3.28) and (3.39) that the positive semidefiniteness of $N(x, y_i)$ is equivalent to the positive semidefiniteness of the following matrix

$$\begin{aligned} M(x, y_i) = & \left[\kappa(x) - 1 + \kappa(x) (\log p) y_i^{\kappa(x)} \right] \left[H_\kappa(x) + \nabla_x \kappa(x) (\nabla_x \kappa(x))^\top (\log y_i + (\log y_i) (\log p) y_i^{\kappa(x)}) \right] \\ & \times \kappa(x) (\log y_i) - \left(1 + \kappa(x) (\log y_i) + (\log p) (\log y_i) y_i^{\kappa(x)} \kappa(x) \right)^2 \nabla_x \kappa(x) (\nabla_x \kappa(x))^\top. \end{aligned}$$

Note that $M(x, y_i)$ can be rewritten as follows

$$M(x, y_i) = \varphi_1(x, y_i) H_\kappa(x) - \varphi_2(x, y_i) \nabla_x \kappa(x) (\nabla_x \kappa(x))^\top,$$

where $\varphi_1(x, y_i)$ and $\varphi_2(x, y_i)$ refer to Lemmas 3.5 and 3.6. By (iii) of Assumption 3.3, we have $H_\kappa(x) - \omega \nabla_x \kappa(x)(\nabla_x \kappa(x))^\top$ is a positive semidefinite matrix for any $x \in Q$, where ω refers to (3.27)). Since $\varphi_1^*, \varphi_2^* > 0$, it is clear that $\omega > 0$. Moreover, $\nabla_x \kappa(x)(\nabla_x \kappa(x))^\top$ is a positive semidefinite matrix. Hence, $H_\kappa(x)$ is also a positive semidefinite matrix. By Lemmas 3.5 and 3.6, we have that $\varphi_1(x, y_i) \geq \varphi_1^* > 0$ and $\varphi_2(x, y_i) \leq \varphi_2^*$, for any $(x, y_i) \in Q \times [h_l, h_u]$. Then,

$$M(x, y_i) \succeq \varphi_1^* H_\kappa(x) - \varphi_2^* \nabla_x \kappa(x)(\nabla_x \kappa(x))^\top \succeq 0,$$

which completes the proof. \square

3.4 . Convexity of the feasibility set $S(p)$

We showed that if Assumption 3.2 holds, then $F_i(g_i(\cdot))$ is concave on $\text{Conv}(S(p))$, for any $i \in J$. Moreover, given arbitrarily a real number c_1 such that $0 < c_1 \leq 1$, if Assumption 3.3 holds, then U is jointly convex on $Q \times [h_l, h_u]$. We will apply these results to prove the convexity of the feasibility set $S(p)$.

Theorem 3.1. *If Assumptions 3.2 and 3.3 hold and the feasibility set $S(p)$ is non empty, then $S(p)$ is a convex set.*

Proof. For any $x_1, x_2 \in S(p)$ and $\beta \in [0, 1]$, we show that $x^* := \beta x_1 + (1 - \beta)x_2 \in S(p)$. In fact, let $y^1 := (y_1^1, \dots, y_K^1)$ and $y^2 := (y_1^2, \dots, y_K^2)$, where y_i^1 and y_i^2 , $i \in J$, are the corresponding values of y_i^* defined in (3.9) w.r.t x_1 and x_2 , respectively. It follows from Lemma 3.3 that $h_l \leq y_i^1, y_i^2 \leq h_u$, for any $i \in J$. Moreover, by Lemmas 3.2 and 3.7, we have $F_i(g_i(\cdot))$ is concave on $\text{Conv}(S(p))$ and $U(\cdot, \cdot)$ is jointly convex on $Q \times [h_l, h_u]$. Then, for any $i \in J$, we have

$$\begin{aligned} F_i(g_i(x^*)) &\geq \beta F_i(g_i(x_1)) + (1 - \beta) F_i(g_i(x_2)) \\ &\geq \beta \psi_{x_1}^{(-1)}(y_i^1 \psi_{x^*}(p)) + (1 - \beta) \psi_{x_2}^{(-1)}(y_i^2 \psi_{x^*}(p)) = \beta U(x_1, y_i^1) + (1 - \beta) U(x_2, y_i^2) \\ &\geq U(x^*, \beta y_i^1 + (1 - \beta) y_i^2) = \psi_{x^*}^{(-1)}((\beta y_i^1 + (1 - \beta) y_i^2) \psi_{x^*}(p)), \end{aligned}$$

which in turn implies that $(x^*, \beta y^1 + (1 - \beta) y^2)$ satisfies (3.10). Then, $x^* \in S(p)$. \square

We are interested in finding an example which fits all the Assumptions in Theorem 3.1. In the following, we will study a specific case which fits Assumption 3.3 and Assumption 3.2.

3.4.1 . An example of the function κ

In this section, we give an example of κ which satisfies all the conditions in Assumption 3.3.

Remark 3.3. *It is not necessary to verify the condition (i) of Assumption 3.3 since $e^{-1} \approx 0.37$ and we consider high value of the probability threshold p .*

Lemma 3.8. *Let d be a real number such that $d < c_1$ and $q : Q \rightarrow \mathbb{R}$ be a real-valued function which satisfies the two following conditions.*

1. q is twice continuously differentiable and convex on Q .
2. $\log(c_1 - d) \leq q(x) \leq \log\left[\min\left(\frac{1}{\omega}, 1 - d\right)\right]$, for any $x \in Q$, where ω refers to (3.27).

Then, $\kappa(x) := e^{q(x)} + d$ is a function which satisfies Assumption 3.3.

Proof. First we verify the condition (ii) of Assumption 3.3, which can be implied by the assumption $\log(c_1 - d) \leq q(x) \leq \log(1 - d)$. Since $q(x) \leq \log(\frac{1}{\omega})$, we have $e^{q(x)} \leq \frac{1}{\omega}$, for any $x \in Q$. Let $\kappa^*(x) := e^{q(x)}$. We have the following formulation

$$H_x \log(\kappa^*(x)) = \frac{\kappa^*(x) H_x \kappa^*(x) - \nabla_x \kappa^*(x) (\nabla_x \kappa^*(x))^\top}{\kappa^*(x)^2}, \quad (3.40)$$

where $H_x \log(\kappa^*(\cdot))$ is the Hessian matrix of the function $\log(\kappa^*(\cdot))$. As q is a convex function on Q and $\log(\kappa^*(\cdot)) = q(\cdot)$, we deduce that $H_x(\log(\kappa^*(x)))$ is a positive semidefinite matrix for any $x \in Q$. It follows from (3.40) that $\kappa^*(x) H_x \kappa^*(x) - \nabla_x \kappa^*(x) (\nabla_x \kappa^*(x))^\top$ is a positive semidefinite matrix. Since $0 < \kappa^*(x)$ and $\nabla_x \kappa^*(x) (\nabla_x \kappa^*(x))^\top$ is positive semidefinite, we deduce that $H_x \kappa^*(x)$ is positive semidefinite, for any $x \in Q$. Moreover, $\kappa^*(x) \leq \frac{1}{\omega}$, which implies that

$$\frac{1}{\omega} H_x \kappa^*(x) - \nabla_x \kappa^*(x) (\nabla_x \kappa^*(x))^\top$$

is a positive semidefinite matrix. On the other hand, since $\kappa(x) = \kappa^*(x) + d$, then $H_x \kappa^*(x) = H_x \kappa(x)$ and $\nabla_x \kappa^*(x) = \nabla_x \kappa(x)$, $\forall x \in Q$. Therefore, the condition (iii) of Assumption 3.3 holds. \square

Next, we take an example function q which meets the two conditions in Lemma 3.8.

Lemma 3.9. Let $q(x) = \frac{\|x\|^2}{L} + z$ where $L > 0, z \in \mathbb{R}$ are real numbers such that

$$\log(c_1 - d) \leq \frac{1}{L} \delta_l^2 + z \leq \frac{1}{L} \delta_u^2 + z \leq \log\left[\min\left(\frac{1}{\omega}, 1 - d\right)\right], \quad (3.41)$$

where δ_l, δ_u are defined in (3.1). Then, $q(x)$ satisfies the two conditions in Lemma 3.8.

Proof. The first condition is trivial. Since $\frac{\delta_l^2}{L} + z = \min_{x \in Q} q(x)$ and $\frac{\delta_u^2}{L} + z = \max_{x \in Q} q(x)$, it is clear that the second condition holds. \square

By Lemma 3.8 and Lemma 3.9, the function $\kappa(x) = e^{\frac{\|x\|^2}{L} + z} + d$ which satisfies (3.41) fits all conditions in Assumption 3.3. It suffices to choose appropriate parameters L, z and d . In fact, (3.41) is a mild condition. Let d be an arbitrary real number in $(c_1 - \frac{1}{\omega}, c_1)$. As $d < c_1$ and $c_1 \leq 1$, We have that $\log(c_1 - d)$ is well defined and

$$\log(c_1 - d) \leq \log\left[\min\left(\frac{1}{\omega}, 1 - d\right)\right].$$

Let L be an arbitrary real number in $\left[\frac{\delta_u^2 - \delta_l^2}{\log[\min(\frac{1}{\omega}, 1 - d)] - \log(c_1 - d)}, +\infty\right)$. We deduce from this condition that

$$\log\left[\min\left(\frac{1}{\omega}, 1 - d\right)\right] - \log(c_1 - d) \geq \left(\frac{1}{L} \delta_u^2 + z\right) - \left(\frac{1}{L} \delta_l^2 + z\right) > 0.$$

Let z be an arbitrary real number in $\left[\log(c_1 - d) - \frac{\delta_l^2}{L}, \log\left[\min\left(\frac{1}{\omega}, 1 - d\right)\right] - \frac{\delta_u^2}{L}\right]$. We deduce from this condition that

$$\begin{cases} \log(c_1 - d) \leq \frac{1}{L} \delta_l^2 + z. \\ \frac{1}{L} \delta_u^2 + z \leq \log\left[\min\left(\frac{1}{\omega}, 1 - d\right)\right]. \end{cases}$$

Therefore, we can verify that this set of $q(x), d, L, z$ satisfy (3.41).

3.4.2 . Numerical experiments

To verify the r -decreasing property of a differentiable density $f(t)$, we check whether the derivative of $t^r f(t)$ is strictly negative for $t > t^*(r) > 0$, which is equivalent to $r.f(t) + t.f'(t) < 0$, for any $t > t^*(r)$. The thresholds for some typical 1-dimensional spherical distributions are given in Table 3.1.

In this section, we study values of p^* as defined in (3.12). All the numerical results are performed using Python 3.8.8 on a PC with Intel i5 CPU (2.4 GHz), RAM 16G, 512G SSD. Assume that for any $j \in J$, F_j has the same density. For the sake of illustration, we set $\lambda_{j,\min}^{-\frac{1}{2}}$ small enough such that $F_i\left(\frac{r_i+1}{r_i-1}\lambda_{i,\min}^{-\frac{1}{2}}\|\mu_i\|\right) \leq F_i[t_i^*(r_i+1)]$ and take $\max\left(\frac{1}{2}, F_i[t_i^*(r_i+1)]\right)$ as an upper bound of p^* . We show in Table 3.3 the bounds of p^* for some typical elliptical distributions. We consider the case with $n = 2$ and $K = 2$, i.e. we have 2 constraints

Distribution	p^*
Normal	0.92
Student's t with 2- degrees of freedom	0.84
Student's t with 3- degrees of freedom	0.87
Student's t with 4- degrees of freedom	0.88
Laplace	0.88

Table 3.3: Selected 1-dimensional elliptical distributions with r -decreasing densities and the value of the threshold p^* respectively.

and 2-dimensional decision variable. The parameters are taken as follows: $p = 0.95, c_1 = 0.9, D_1 = 1, D_2 = 0.85, \mu_1 = (1, 20)^\top, \mu_2 = (7, 2)^\top, \delta_l = 0.2, \delta_u = 1.5, \Sigma_1 = 3 \times \mathbb{I}_2$, and $\Sigma_2 = 30 \times \mathbb{I}_2$, where \mathbb{I}_2 is the 2×2 -identity matrix. h_l and h_u are calculated as in Section 3.3. φ_1^*, φ_2^* and ω are calculated by (3.27). Let $d = c_1 - \frac{1}{2\omega}$, $L = \frac{\delta_u^2 - \delta_l^2}{\log(\min(\frac{1}{\omega}, 1-d)) - \log(c_1-d)}$ and $z = \log\left(\min\left(\frac{1}{\omega}, 1-d\right)\right) - \frac{\delta_u^2}{L}$. It is easy to see that this set of parameters fits the conditions in Lemma 3.8 and Lemma 3.9.

In our simulations, we consider the probability function

$$f_{\text{proba}}(x) = \psi_x^{(-1)}\left(\sum_{i=1}^2 \psi_x(F_i(g_i(x)))\right),$$

where ψ_x is defined in (3.11), F_1 is the cumulative distribution function of a 1-dimensional standard Student's t distribution with 3-degrees of freedom and F_2 is the cumulative distribution function of a 1-dimensional standard Student's t distribution with 4-degrees of freedom, $x = (x_1, x_2)^\top$ where $x_1, x_2 \in [-1, 1]$. Figure 3.1 shows the surface of function $z = f_{\text{proba}}(x)$ on the domain $[-1, 1]$ and Figure 3.2 shows its contour lines with four different levels 0.6, 0.7, 0.8 and 0.9, respectively.

3.5 . Conclusion

In this chapter, we studied the convexity of joint chance constraints in the case of elliptical distributions. Further, we modeled the dependence of random variables in different rows by a Gumbel-Hougaard copula. We come up with new convexity results of the feasibility set. We simulated our theoretical result by showing the surface plot of the probability function with its contour lines. It is very clear that the feasibility set is eventually a convex set under high value of the probability level. Further research can be dedicated to other families of copulas.

This chapter corresponds to a paper submitted to Results in Control and Optimization.

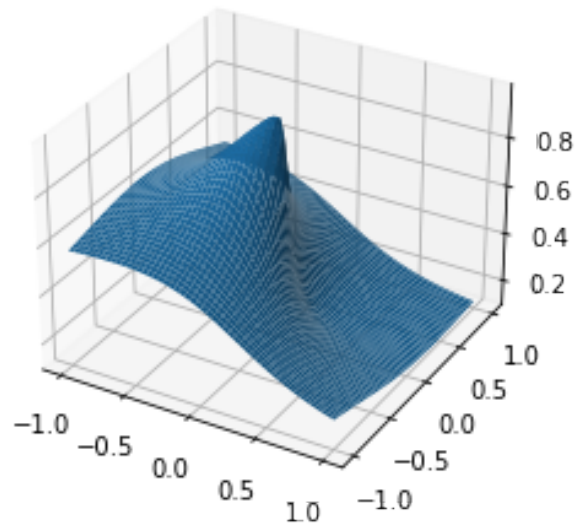


Figure 3.1: Surface plot of the probability function f_{proba} on $[-1, 1]^2$.

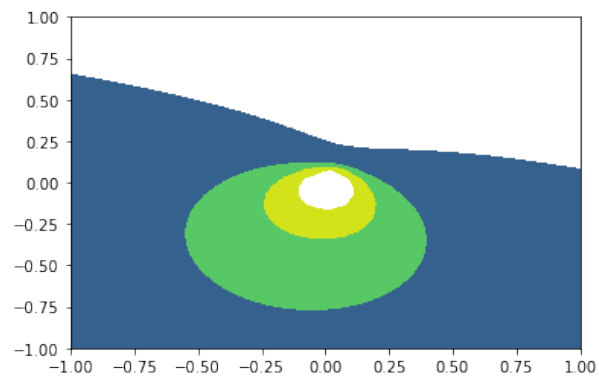


Figure 3.2: Contour line of the probability function f_{proba} with 4 levels (0.6, 0.7, 0.8, 0.9).

Chapter 4 :General n -players Chance-Constrained Games

In this chapter, we study an n -player game with random payoffs and continuous strategy sets. The payoff function of each player is defined by its expected value and the strategy set of each player is defined by a joint chance constraint. The random constraint vectors defining the joint chance constraint are dependent and follow either elliptically symmetric distributions or Normal mean-variance mixture distributions. The Archimedean copula is used to capture the dependence among random constraint vectors. We propose a reformulation of the joint chance constraint of each player. Under mild assumptions on the players' payoff functions and 1-dimensional spherical distribution functions, we show that there exists a Nash equilibrium of the game.

This chapter builds upon the foundation established in the previous chapter, which focused on exploring the convexity properties of linear joint chance constraints. In the present chapter, we delve into the concept of the feasibility set's convexity and its significance in proving the existence of a Nash equilibrium. By establishing the convexity of the feasibility set, we establish a crucial link between the theoretical insights gained from studying linear joint chance constraints and the practical implications related to the existence of a Nash equilibrium.

4.1 . Introduction

The publication of the seminal book *Researches into the Mathematical Principles of the Theory of Wealth* by Cournot in 1838 was the trigger for the widespread use of the equilibrium under market conditions [50]. Later, the saddle point equilibrium for a two player zero-sum game of Von Neumann [244] follows on this work. In 1950, Nash [160] showed that for a finite strategic game there exists an equilibrium point, known as a Nash equilibrium, from which there is no incentive for any player to deviate unilaterally. Despite its practical limitation, the general strategic games are extensively studied in the literature [18, 55, 69]. The theory of Nash equilibrium in deterministic setup faces challenges especially when it comes to deal with real applications with random payoffs and strategy sets.

The most commonly used tool to deal with random payoffs is the expectation function [200] which is more appropriate for risk neutral cases. The risk averse games are studied by considering an alternative payoff criterion based on risk measure CVaR [120, 200] and chance constraint programming [214, 216]. In [214], the authors studied a finite strategic game where the payoff vector of each player is elliptically distributed, and showed the existence of a Nash equilibrium. The equivalence between a Nash equilibrium of a chance-constrained game (CCG), considered in [214], and the global optimal solution of a certain mathematical program is stated in [216].

In the above referred games, the players' payoff functions are random while the strategy sets are deterministic. However, the strategy sets containing chance constraints are often considered in various applications, e.g., risk constraints in portfolio optimization problem [112] and resource constraints in stochastic shortest path problem [49]. Recently, the games with chance constraint based strategy sets are introduced in the literature [178, 179, 180, 217, 219]. Singh and Lisser [217] considered a 2-player zero-sum game with individual chance constraints and showed that a saddle point equilibrium problem is equivalent to a primal-dual pair of second order cone programs when the random constraint vectors follow elliptically symmetric distribution. Singh et al. [219] considered an n -player general-sum game with individual chance constraints under elliptically symmetric distributions and showed that a Nash equilibrium problem is equivalent to the global

optimization of a nonlinear optimization problem.

In the wake of these results, Peng et al. [179] showed the existence of Nash equilibrium for the n -player general-sum games where the strategy profile set of each player is defined by a joint chance constraint, and the random constraint vectors are either independently normally distributed or follow a mixture of elliptical distributions [180]. When the probability distributions are not completely known and belong to a given distributional uncertainty set, Peng et al. [178] formulated the random constraints of each player as a distributional robust joint chance constraint. They consider several uncertainty sets, namely a density based uncertainty set and four two-moment based uncertainty sets where one of them has a nonnegative support. They show that there exists a Nash equilibrium of a distributionally robust chance constrained game for each uncertainty set.

In [178, 179, 180], the authors assume that the random constraint vectors are independently distributed. However, the random variables are usually dependent in real world applications. In order to study the dependence structure of random variable, the concept of copula was introduced by Abe Sklar in 1959 [221], as a solution to a probability problem stated by Maurice Fréchet in the context of random metric spaces. Copulas are functions used to distinguish the marginal distributions from a given dependent structure based multivariate distribution. Henrion and Strugarek [100] introduced the notion of log-exp concavity of copula to investigate the convexity of elliptically distributed dependent joint chance constraints. In [166], the authors studied the convexity of elliptically distributed linear dependent joint chance constraint using copulas. Some eventual convexity results for joint chance constraint without using copulas are given in [235]. We refer the reader to [163] for a detailed introduction to the theory of copulas.

In this chapter, we extend the results of [179, 180] to the general case where the payoff function is random and the strategy profile set of each player is defined by elliptically distributed dependent joint chance constraints. We derive a new reformulation of joint chance constraint with dependent random constraint vectors and show that there exists a Nash equilibrium of the game under mild conditions on the payoff functions.

The rest of this chapter is organized as follows. Section 4.2 contains the definition of an n -player CCG. In Section 4.3, we prove the existence of a Nash equilibrium of the CCG under elliptical distributions. Section 4.4 explains why we study another class of distribution, which is the class of normal mean-variance mixture distribution. We prove the existence of a Nash equilibrium in this framework in Section 4.5. Algorithm and numerical results are given in Section 4.6. We conclude this chapter by Section 4.7.

In the first part, we study the case, where the random constraint vectors defining the joint chance constraint are dependent and follows elliptically symmetric distributions.

4.2 . Chance-Constrained Game

We consider an n -player CCG, where $H = \{1, 2, \dots, n\}$ is the set of players. Let $S^i \subset \mathbb{R}^{d_i}$ be the strategy set of player i which is a non-empty, convex and compact set. The product set $S = \prod_{i=1}^n S^i$ denotes the set of strategy profiles of all the players. For each $i \in H$, $S^{-i} = \prod_{j=1; j \neq i}^n S^j$ denotes the set of strategy vectors of all players j , $j \neq i$. A strategy profile $x = (x^1, x^2, \dots, x^n) \in S$ is represented as (x^i, x^{-i}) where x^i denotes the strategy of player i and x^{-i} denotes the vector of strategies of the players other than player i . In many real life applications, the strategy sets are restricted by random linear constraints, e.g., i) the constraints on total random electricity loss, energy generation and reserve arising in electricity market [178, 199], ii) risk constraints in financial market [180]. In this chapter, we consider the case where the random linear constraints are formulated as a joint chance constraint. The strategy set of player i , $i \in H$, is further restricted by the following joint chance constraint

$$\mathbb{P}(V^i x^i \leq D^i) \geq \alpha_i, \quad (4.1)$$

where $\alpha_i \in [0, 1]$ is a given probability level, $D^i = (D^{i,1}, \dots, D^{i,K_i})^\top \in \mathbb{R}^{K_i}$ is a deterministic vector and $V^i = [V^{i,1}, \dots, V^{i,K_i}]^\top$ is a $K_i \times d_i$ random matrix, where $V^{i,k}$ denotes the k^{th} row of matrix V^i and \top denotes the transposition. Let $J^i = \{1, 2, \dots, K_i\}$ denotes the index set of i^{th} player's constraints. The feasible strategy set of player i is defined as

$$S_{\alpha_i}^i = \{x^i \in S^i \mid \mathbb{P}(V^i x^i \leq D^i) \geq \alpha_i\}.$$

We assume that for each $i \in H$, $S_{\alpha_i}^i$ is a non-empty set. Let $\alpha = (\alpha_i)_{i \in H}$ be the confidence level vector and $S_\alpha = \prod_{i=1}^n S_{\alpha_i}^i$ be the set of all feasible strategy profiles. The payoff function of each player is defined using random variables. For each $x \in S_\alpha$, the payoff of player i is given by $f_i(x, \zeta)$, where ζ is an m -dimensional random vector. We use expected value approach to model the payoff function of each player. Therefore, the payoff function of player i is given by

$$p_i(x) = \mathbb{E}[f_i(x, \zeta)], \quad \forall x \in S_\alpha.$$

We assume that the CCG is of complete information, i.e., the payoff function, the strategy set of each player, and the confidence level vector α are known to all the players.

Definition 4.8. A strategy profile y^* is a Nash equilibrium of the CCG at confidence level vector α if for each $i \in H$

$$p_i(y^{i*}, y^{-i*}) \geq p_i(x^i, y^{-i*}), \quad \forall x^i \in S_{\alpha_i}^i.$$

The existence of a Nash equilibrium for a non-cooperative game in various setup has been extensively studied in the literature. It is mainly based on fixed point theorems which require the payoff function of a player i to be a continuous function of the strategies of all the players and a concave function of the strategies of player i for every fixed strategy profiles of all other players. Under Assumption 4.4 given below, the abovementioned continuity and concavity properties hold [180].

Assumption 4.4. For each player i , $i \in H$, the following conditions hold:

1. $f_i(\cdot, x^{-i}, \zeta)$ is a concave function of x^i for every $(x^{-i}, \zeta) \in S^{-i} \times \mathbb{R}^m$.
2. $f_i(\cdot)$ is a continuous function.
3. $p_i(x)$ is finite valued for every $x \in S$.

4.3 . Existence of Nash Equilibrium with Elliptical Distributed Row Vectors

For each $i \in H$, we assume that $V^{i,k} \sim \text{Ellip}(\mu^{i,k}, \Sigma^{i,k}, \Psi^{i,k})$, $k \in J^i$. Let $\lambda_{i,k,\min}$ be the smallest eigenvalue of the positive definite matrix $\Sigma^{i,k}$. Define, $\tilde{S}_{\alpha_i}^i = S_{\alpha_i}^i \setminus \{0\}$, then for $x^i \in \tilde{S}_{\alpha_i}^i$, let

$$\begin{aligned} \xi^{i,k}(x^i) &= \frac{(V^{i,k})^\top x^i - (\mu^{i,k})^\top x^i}{\sqrt{(x^i)^\top \Sigma^{i,k} x^i}}, \\ g^{i,k}(x^i) &= \frac{D^{i,k} - (\mu^{i,k})^\top x^i}{\sqrt{(x^i)^\top \Sigma^{i,k} x^i}}. \end{aligned} \tag{4.2}$$

It is well known that $\xi^{i,k}(x^i)$ follows 1-dimensional spherical distribution with characteristic generator $\Psi^{i,k}$ [71]. Using the abovementioned notations, the constraint (4.1) can be written as

$$\mathbb{P} \{ \xi^{i,k}(x^i) \leq g^{i,k}(x^i), \quad k \in J^i \} \geq \alpha_i. \tag{4.3}$$

By Proposition 3.1, (4.3) can be equivalently written as

$$C_{x^i}^i [(F^{i,1} \circ g^{i,1})(x^i), \dots, (F^{i,K_i} \circ g^{i,K_i})(x^i)] \geq \alpha_i, \quad (4.4)$$

where $C_{x^i}^i$ is the K_i -dimensional copula of the random vector $(\xi^{i,k}(x^i))_{k=1}^{K_i}$; $F^{i,k}$ is the CDF of $\xi^{i,k}(x^i)$ and \circ denotes the function composition.

Assumption 4.5. *There exists a K_i -dimensional copula C^i such that $C_{x^i}^i = C^i$, for all $x^i \in S^i$, and C^i is a K_i -dimensional strictly Archimedean copula with a generator ψ_i such that the inverse function $\psi_i^{(-1)}$ is 4-monotonic on $(0, +\infty)$.*

Remark 4.4. *The 4-monotonicity of $\psi_i^{(-1)}$ ensures that $\psi_i^{(-1)}$ is twice differentiable. It follows from Proposition 3.3 that 4-monotonicity condition holds if $K_i \geq 4$.*

Under Assumption 4.5, we can equivalently write (4.4) as

$$C^i[(F^{i,1} \circ g^{i,1})(x^i), \dots, (F^{i,K_i} \circ g^{i,K_i})(x^i)] \geq \alpha_i. \quad (4.5)$$

Proposition 4.4. *If $x^i \in \tilde{S}_{\alpha_i}^i$ and Assumption 4.5 holds, the joint chance constraint (4.1) is equivalent to*

$$\begin{cases} (i) (F^{i,k} \circ g^{i,k})(x^i) \geq \psi_i^{(-1)}[y_{i,k}\psi_i(\alpha_i)], & k \in J^i, \\ (ii) \sum_{k \in J^i} y_{i,k} = 1, \ y_{i,k} \geq 0, & k \in J^i. \end{cases} \quad (4.6)$$

Proof. Let $x^i \in \tilde{S}_{\alpha_i}^i$. Under Assumption 4.5, the joint chance constraint (4.1) is equivalent to (4.5). It is enough to show the equivalence between (4.5) and (4.6). Since, C^i is strictly Archimedean copula, (4.5) is equivalent to

$$\sum_{k \in J^i} (\psi_i \circ F^{i,k} \circ g^{i,k})(x^i) \leq \psi_i(\alpha_i). \quad (4.7)$$

Define a K_i -dimensional vector $y_i = (y_{i,1}, \dots, y_{i,K_i}) \in [0, 1]^{K_i}$ such that

$$y_{i,k} = \frac{(\psi_i \circ F^{i,k} \circ g^{i,k})(x^i)}{\sum_{j \in J^i} (\psi_i \circ F^{i,j} \circ g^{i,j})(x^i)}, \quad k \in J^i.$$

Since $\psi_i^{(-1)}$ is non-increasing, it follows that (x^i, y_i) satisfies (4.6). Conversely, we assume (x^i, y_i) satisfies (4.6). By adding all the constraints (i) of (4.6) after applying $\psi_i(\cdot)$ on both sides, we can say that x^i satisfies (4.7) which is equivalent to (4.5). \square

The convexity of the feasible strategy set $S_{\alpha_i}^i$ plays a very important role in showing the existence of a Nash equilibrium. We show that there exists $\alpha_i^* \in [0, 1]$ such that $S_{\alpha_i}^i$ is a convex set, for all $\alpha_i \in (\alpha_i^*, 1]$. For each $i \in H$, define an index set $I^{(i)} = \{k \in J^i \mid \mu^{i,k} \neq 0\}$ and a set of real numbers $\{r^{i,k} \mid k \in J^i\}$ such that

$$\begin{cases} r^{i,k} > 1, & \text{if } k \in I^{(i)}, \\ r^{i,k} = 1, & \text{if } k \notin I^{(i)}. \end{cases} \quad (4.8)$$

Lemma 4.10. *Let Assumption 2 holds and for each $k \in J^i$, the CDF $F^{i,k}$ has $(r^{i,k} + 1)$ -decreasing density with a threshold $t_{i,k}^*$ where $r^{i,k}$ is defined by (4.8) and $t_{i,k}^*$ refers to Definition 3.4. Then, $S_{\alpha_i}^i$ is a convex set, for all $\alpha_i \in (\alpha_i^*, 1]$, where*

$$\alpha_i^* := \max \left\{ \frac{1}{2}, \max_{k \in I^{(i)}} F^{i,k} \left(\frac{r^{i,k} + 1}{r^{i,k} - 1} \lambda_{i,k,\min}^{-\frac{1}{2}} \|\mu^{i,k}\| \right), \max_{k \in J^i} F^{i,k}(t_{i,k}^*) \right\}. \quad (4.9)$$

In order to prove Lemma 4.10, we need the three following lemmas.

Lemma 4.11. *Let $\alpha_i \in (\frac{1}{2}, 1]$ and $x^i \in \tilde{S}_{\alpha_i}^i$. Then, $D^{i,k} > (\mu^{i,k})^\top x^i$, for all $k \in J^i$.*

Proof. Let $x^i \in \tilde{S}_{\alpha_i}^i$. By applying Proposition 3.2 on (4.5), we get

$$(F^{i,k} \circ g^{i,k})(x^i) \geq \alpha_i > \frac{1}{2}. \quad (4.10)$$

Since, $F^{i,k}$ is the CDF of an 1-dimensional real-valued random variable which is symmetric at 0, $F^{i,k}(0) = \frac{1}{2}$. From (4.10), we get $g^{i,k}(x^i) > 0$ which in turn implies that $D^{i,k} > (\mu^{i,k})^\top x^i$. \square

Lemma 4.12. *Let $r^{i,1}, \dots, r^{i,K_i}$ be the real numbers defined by (4.8) and for each $k \in I^{(i)}$, define*

$$\Omega^{i,k} := \left\{ x^i \in S^i \mid D^{i,k} - \mu_{i,k}^\top x^i > \frac{r^{i,k} + 1}{r^{i,k} - 1} \lambda_{i,k,\min}^{-\frac{1}{2}} \|\mu_{i,k}\| \sqrt{(x^i)^\top \Sigma^{i,k} x^i} \right\}.$$

Then,

$$\text{Conv}(\tilde{S}_{\alpha_i}^i) \subset \bigcap_{k \in I^{(i)}} \Omega^{i,k},$$

for all $\alpha_i \in (\alpha_i^{**}, 1]$, where

$$\alpha_i^{**} = \max \left\{ \frac{1}{2}, \max_{k \in I^{(i)}} F^{i,k} \left(\frac{r^{i,k} + 1}{r^{i,k} - 1} \lambda_{i,k,\min}^{-\frac{1}{2}} \|\mu^{i,k}\| \right) \right\}, \quad (4.11)$$

and Conv represents the convex hull. Moreover, for any convex subset $Q^{i,k}$ of $\bigcap_{k \in I^{(i)}} \Omega^{i,k}$ such that $0 \notin Q^{i,k}$, $g^{i,k}(x^i)$ is defined and $(-r^{i,k})$ -concave on $Q^{i,k}$, for all $k \in J^i$.

Proof. Let $k \in I^{(i)}$ and $x^i \in \tilde{S}_{\alpha_i}^i$. By applying Proposition 3.2 on (4.5), we get

$$(F^{i,k} \circ g^{i,k})(x^i) \geq \alpha_i.$$

From the definition of α_i^{**} given in (4.11), we have

$$F^{i,k}(g^{i,k}(x^i)) > \alpha_i^{**} \geq F^{i,k} \left(\frac{r^{i,k} + 1}{r^{i,k} - 1} \lambda_{i,k,\min}^{-\frac{1}{2}} \|\mu^{i,k}\| \right). \quad (4.12)$$

Since, $F^{i,k}(\cdot)$ is a non-decreasing function, from (4.12) we have

$$D^{i,k} - (\mu^{i,k})^\top x^i > \frac{r^{i,k} + 1}{r^{i,k} - 1} \lambda_{i,k,\min}^{-\frac{1}{2}} \|\mu^{i,k}\| \sqrt{(x^i)^\top \Sigma^{i,k} x^i}.$$

Therefore, $\tilde{S}_{\alpha_i}^i \subset \bigcap_{k \in I^{(i)}} \Omega^{i,k}$. For each $k \in I^{(i)}$, $\Omega^{i,k}$ is a convex set which implies that $\text{Conv}(\tilde{S}_{\alpha_i}^i) \subset \bigcap_{k \in I^{(i)}} \Omega^{i,k}$. We prove the second part of Lemma 4.12 by considering two cases as below:

Case 1: Let $k \notin I^{(i)}$, then $\mu^{i,k} = 0$. From the definition of α_i^{**} , we have $\alpha_i > \frac{1}{2}$. From Lemma 4.11, $D^{i,k} > 0$. In this case, the proof follows directly from Lemma 3 of [48].

Case 2: Let $k \in I^i$. It follows from Lemma 2 of [48] that the function

$$f^{i,k}(x^i) = \left(\frac{\sqrt{(x^i)^\top \Sigma^{i,k} x^i}}{D^{i,k} - (\mu^{i,k})^\top x^i} \right)^{r^{i,k}}$$

is defined and a convex function on $\bigcap_{k \in I^{(i)}} \Omega^{i,k}$. Therefore, for any $y, z \in Q^{i,k}$ and $\lambda \in [0, 1]$, we have

$$f^{i,k}[\lambda y + (1 - \lambda)z] \leq \lambda f^{i,k}(y) + (1 - \lambda)f^{i,k}(z). \quad (4.13)$$

Note that $g^{i,k}(x^i) = (f^{i,k}(x^i))^{\frac{-1}{r^{i,k}}}$ on $Q^{i,k}$. From (4.13), we can write

$$g^{i,k}[\lambda y + (1 - \lambda)z] \geq \left(\lambda (g^{i,k}(y))^{-r^{i,k}} + (1 - \lambda)(g^{i,k}(z))^{-r^{i,k}} \right)^{\frac{-1}{r^{i,k}}}.$$

Hence, $g^{i,k}$ is defined and $(-r^{i,k})$ -concave on $Q^{i,k}$. □

Lemma 4.13. *Let Assumption 4.5 holds. Then, $\psi_i^{(-1)}[y_{i,k}\psi_i(\alpha_i)]$ is a convex function of $y_{i,k}$ for all $\alpha_i \in [0, 1]$.*

Proof. Let $U(y_{i,k}) = \psi_i^{(-1)}(y_{i,k}\psi_i(\alpha_i))$. If $\psi_i(\alpha_i) = 0$, the proof is trivial because $U(y_{i,k}) = 1$, for all $y_{i,k} \in [0, 1]$. Let $\psi_i(\alpha_i) > 0$. The second-order differentiation of $U(y_{i,k})$ is given by

$$\frac{d^2}{dy_{i,k}^2} U(y_{i,k}) = [\psi_i(\alpha_i)]^2 \times \left(\psi_i^{(-1)} \right)''(y_{i,k}\psi_i(\alpha_i)),$$

for all $y_{i,k} \in (0, 1]$. Since, $\psi_i^{(-1)}$ is 4-monotonic on $(0, +\infty)$, $\left(\psi_i^{(-1)} \right)'' \geq 0$ on $(0, +\infty)$. This implies $\frac{d^2}{dy_{i,k}^2} U(y_{i,k}) \geq 0$, for all $y_{i,k} \in (0, 1]$. Therefore, $U(y_{i,k})$ is a convex function of $y_{i,k}$ on $(0, 1]$. The convexity of U on $[0, 1]$ follows from the continuity of U at 0. □

We present the proof of Lemma 4.10 using the results of Lemma 4.11, Lemma 4.12 and Lemma 4.13.

Proof of Lemma 4.10. Let $\alpha_i \in (\alpha_i^*, 1]$, $\lambda \in [0, 1]$ and $z_1, z_2 \in S_{\alpha_i}^i$. We need to show that $\lambda z_1 + (1 - \lambda)z_2 \in S_{\alpha_i}^i$.

Case 1: Let $z_1 = 0$ or $z_2 = 0$. Without loss of generality, we assume that $z_2 = 0$. This gives $D^{i,k} \geq 0$, for all $k \in J^i$, which in turn implies that

$$\mathbb{P}(V^i \lambda z_1 \leq D^i) \geq \mathbb{P}(V^i z_1 \leq D^i) \geq \alpha_i.$$

Hence, $\lambda z_1 + (1 - \lambda)z_2 \in S_{\alpha_i}^i$.

Case 2: Let $z_1 \neq 0$, $z_2 \neq 0$ and $\lambda z_1 + (1 - \lambda)z_2 = 0$. In this case, $z_2 = \frac{-\lambda}{1-\lambda} z_1 \in \tilde{S}_{\alpha_i}^i$ and $z_1 \in \tilde{S}_{\alpha_i}^i$. It follows from Lemma 4.11 that

$$(\mu^{i,k})^\top z_1 > \frac{\lambda - 1}{\lambda} D^{i,k}, \quad (\mu^{i,k})^\top z_1 < D^{i,k}, \quad \forall k \in J^i.$$

This implies that $D^{i,k} \geq 0$, for all $k \in J^i$. Therefore, $\lambda z_1 + (1 - \lambda)z_2 = 0 \in S_{\alpha_i}^i$.

Case 3: Let $z_1 \neq 0, z_2 \neq 0$ and $0 \in \text{Seg}(z_1, z_2)$, where $\text{Seg}(z_1, z_2) = \{z_1 + l(z_2 - z_1), 0 \leq l \leq 1\}$. Then, the points on the line segment $\text{Seg}(z_1, z_2)$ are either belong to $\text{Seg}(z_1, 0)$ or $\text{Seg}(0, z_2)$. It follows from Case 1 that $\text{Seg}(z_1, 0)$ and $\text{Seg}(0, z_2)$ are subset of $S_{\alpha_i}^i$. Therefore, $\lambda z_1 + (1 - \lambda)z_2 \in S_{\alpha_i}^i$, for all $\lambda \in [0, 1]$.

Case 4: Let $z_1 \neq 0, z_2 \neq 0$ such that $0 \notin \text{Seg}(z_1, z_2)$. It is clear that $\text{Seg}(z_1, z_2) \subset \text{Conv}(\tilde{S}_{\alpha_i}^i)$. From Lemma 4.12, $g^{i,k}(\cdot)$ is defined and $(-r^{i,k})$ -concave on $\text{Seg}(z_1, z_2)$. Therefore,

$$g^{i,k}(\lambda z_1 + (1 - \lambda)z_2) \geq \left(\lambda [g^{i,k}(z_1)]^{-r^{i,k}} + (1 - \lambda) [g^{i,k}(z_2)]^{-r^{i,k}} \right)^{-\frac{1}{r^{i,k}}}. \quad (4.14)$$

Since, $z_1 \in \tilde{S}_{\alpha_i}^i$, from Lemma 4.11 $g^{i,k}(z_1) > 0$ and it follows from (4.5) that

$$C^i[(F^{i,1} \circ g^{i,1})(z_1), \dots, (F^{i,K_i} \circ g^{i,K_i})(z_1)] > \alpha_i^*.$$

By using Proposition 3.2 and the definition of α_i^* from (4.9), we get

$$F^{i,k}(g^{i,k}(z_1)) > \alpha_i^* \geq F^{i,k}(t_{i,k}^*).$$

This implies that

$$0 < g^{i,k}(z_1)^{-r^{i,k}} < (t_{i,k}^*)^{-r^{i,k}}.$$

Similarly,

$$0 < g^{i,k}(z_2)^{-r^{i,k}} < (t_{i,k}^*)^{-r^{i,k}}.$$

By applying the non-decreasing function $F^{i,k}(\cdot)$ on both sides of (4.14), we can write

$$\begin{aligned} (F^{i,k} \circ g^{i,k})[\lambda z_1 + (1 - \lambda)z_2] &\geq \\ F^{i,k} \left[\left(\lambda [g^{i,k}(z_1)]^{-r^{i,k}} + (1 - \lambda) [g^{i,k}(z_2)]^{-r^{i,k}} \right)^{-\frac{1}{r^{i,k}}} \right]. \end{aligned} \quad (4.15)$$

Since, $F^{i,k}(\cdot)$ has $(r^{i,k} + 1)$ -decreasing density, from Lemma 3.1 of [100], the function $t \mapsto F^{i,k} \left(t^{-\frac{1}{r^{i,k}}} \right)$ is concave on $(0, (t_{i,k}^*)^{-r^{i,k}})$. Therefore, we can write

$$\begin{aligned} &F^{i,k} \left[\left(\lambda [g^{i,k}(z_1)]^{-r^{i,k}} + (1 - \lambda) [g^{i,k}(z_2)]^{-r^{i,k}} \right)^{-\frac{1}{r^{i,k}}} \right] \\ &\geq \lambda [(F^{i,k} \circ g^{i,k})(z_1)] + (1 - \lambda) [(F^{i,k} \circ g^{i,k})(z_2)]. \end{aligned} \quad (4.16)$$

From (4.15) and (4.16), we have

$$\begin{aligned} &(F^{i,k} \circ g^{i,k})[\lambda z_1 + (1 - \lambda)z_2] \geq \\ &\lambda [(F^{i,k} \circ g^{i,k})(z_1)] + (1 - \lambda) [(F^{i,k} \circ g^{i,k})(z_2)]. \end{aligned}$$

This implies that the composition function $(F^{i,k} \circ g^{i,k})(\cdot)$ is a concave function over $\text{Seg}(z_1, z_2)$. It follows from Lemma 4.13 that $\psi_i^{(-1)}(y_{i,k} \psi_i(\alpha_i))$ is a convex function of $y_{i,k}$. Because $z_1, z_2 \in \tilde{S}_{\alpha_i}^i$ and from Proposition 4.4, $\tilde{S}_{\alpha_i}^i$ and (4.6) are equivalent, then there exists vectors $(y_{i,k}^1)_{k \in J^i}$ and $(y_{i,k}^2)_{k \in J^i}$ such that $(z_1, (y_{i,k}^1)_{k \in J^i})$ and

$(z_2, (y_{i,k}^2)_{k \in J^i})$ are feasible points of (4.6). Using the fact that $(F^{i,k} \circ g^{i,k})(\cdot)$ is a concave function and $\psi_i^{(-1)}(\cdot)$ is a convex function, we can say that the convex combination of points $(z_1, (y_{i,k}^1)_{k \in J^i})$ and $(z_2, (y_{i,k}^2)_{k \in J^i})$ is also a feasible point of (4.6). Again from the equivalence of $\tilde{S}_{\alpha_i}^i$ and (4.6), $\lambda z_1 + (1 - \lambda)z_2 \in \tilde{S}_{\alpha_i}^i$, which in turn implies that $\lambda z_1 + (1 - \lambda)z_2 \in S_{\alpha_i}^i$. \square

Next, we prove that $S_{\alpha_i}^i$ is a closed set.

Lemma 4.14. *The feasible strategy set $S_{\alpha_i}^i$ of player i is a closed set.*

Proof. Note that the closeness of $S_{\alpha_i}^i$ follows directly from the upper semi continuity of the probability function $x^i \mapsto \mathbb{P}(V^i x^i \leq D^i)$. The proof is given by Proposition 3.1 in [202]. \square

The feasible strategy set $S_{\alpha_i}^i$ is a compact set because it is a closed subset of the compact set S^i , see Lemma 4.14. Finally, we show that there exists a Nash equilibrium of the CCG.

Theorem 4.2. *Consider an n -player CCG defined in Section 4.2, where*

1. *Assumptions 4.4, 4.5 hold.*
2. *For each $i \in H$ and $k \in J^i$, $V^{i,k} \sim \text{Ellip}(\mu^{i,k}, \Sigma^{i,k}, \Psi^{i,k})$, where $\Sigma^{i,k}$ is a positive definite matrix.*
3. *For each $i \in H$ and $k \in J^i$, assume that the CDF $F^{i,k}(\cdot)$ has $(r^{i,k} + 1)$ -decreasing density with a threshold $t_{i,k}^*$, where $r^{i,k}$ is defined by (4.8) and $t_{i,k}^*$ refers to Definition 3.4.*

Then, there exists a Nash equilibrium of the CCG, for any $\alpha \in (\alpha_1^, 1] \times \dots \times (\alpha_n^*, 1]$, where α_i^* , $i \in H$, is defined by (4.9).*

Proof. Let $\alpha \in (\alpha_1^*, 1] \times \dots \times (\alpha_n^*, 1]$. Under Assumption 4.4, the payoff function $p_i(x^i, x^{-i})$ is a concave function of x^i , for every $x^{-i} \in S^{-i}$, and a continuous function of x . It follows from Lemma 4.10 that the feasible strategy set $S_{\alpha_i}^i$, $i \in H$, is a convex set, for all $\alpha_i \in (\alpha_i^*, 1]$. For each $i \in H$, $S_{\alpha_i}^i$ is a compact set. Then, the existence of a Nash equilibrium of the CCG follows from Theorem 4 of [69]. \square

In the second part, we study the case, where the random constraint vectors are independent and follow normal mean-variance mixture distributions. This part extends the previous part by considering a wider range of distributions of row vectors in the technology random matrix. Specifically, we delve into the context of non-symmetric distributions, introducing a specific class known as normal mean-variance mixture distributions. By exploring this broader class of distributions, we aim to enhance our understanding of the underlying probabilistic nature and its implications in the context of our research.

4.4 . Motivation and Basic Concepts

We study the chance constrained games model defined in Chapter 3.5. To the best of our knowledge, the random constraint vectors in the CCGs are often assumed to be elliptically distributed or follow a mixture of elliptical distributions. These families include many known distributions, e.g., Normal distributions, t-distributions, Laplace distributions, Kotz-type distributions, Pearson distributions and all of them are symmetric distributions. However, it is interesting to consider the case where the distribution of the constraint vectors is not symmetric since symmetric distributions are generally not suitable for some practical situations. In power system scheduling problems, both wind power forecast errors and load forecasting errors are not

normally distributed [104], and can be better fitted by generalized hyperbolic (GH) distributions. This family includes non-symmetric distributions which are used to model many financial applications [66, 20, 197]. In this chapter, we study an n -player CCG where the strategy profile set of each player is defined by a joint chance constraint. We assume that the random constraint vectors are independent and follow normal mean-variance mixture distributions which generalize the family of GH distributions. We derive a new reformulation of the joint chance constraints and show that there exist a Nash equilibrium under mild assumptions. Next, we present some basic definitions, which are useful in our subsequent analysis.

Definition 4.9. An n -dimensional random vector U follows a normal mean-variance mixture distribution with parameters (μ, γ, Σ, W) , i.e., $U \sim \text{NMVM}(\mu, \gamma, \Sigma, W)$, if we have the following representation:

$$U \sim \mu + \gamma W + \sqrt{W} \Sigma^{\frac{1}{2}} Z,$$

where: (i) Z is an n -dimension standard Gaussian distribution with mean vector $\mu_Z = 0$ and covariance matrix $\Sigma_Z = \mathbf{I}_n$, where \mathbf{I}_n is the $n \times n$ identity matrix.

(ii) W is a positive random variable with a density function independent of Z .

(iii) $\Sigma \in \mathbb{R}^{n \times n}$ is an $n \times n$ positive definite matrix and $\Sigma^{\frac{1}{2}} \in \mathbb{R}^{n \times n}$ is an $n \times n$ matrix such that $\Sigma^{\frac{1}{2}} (\Sigma^{\frac{1}{2}})^T = \Sigma$.

(iv) μ and γ are n -dimensional real vectors and \sim implies that the both sides have the same distribution.

Normal mean-variance mixture distributions are generally not symmetric. They are symmetric if and only if $\gamma = 0$ [154]. We present in Figure 4.3, the density functions of an 1-dimensional normal mean-variance mixture distribution, where W is an exponential distribution with parameter $\lambda = 2$, $\mu = -10$, $\Sigma = 2.25$, with different values of γ . The family of GH distributions is known as a special case of normal mean-variance mixture distributions. We have the following definition.

Definition 4.10. An n -dimensional random vector U follows a GH distribution with parameters $(\mu, \gamma, \Sigma, \chi, \psi, \lambda)$, i.e., $U \sim \text{GH}(\mu, \gamma, \Sigma, \chi, \psi, \lambda)$ if $U \sim \text{NMVM}(\mu, \gamma, \Sigma, W)$, where $W \sim N^-(\lambda, \chi, \psi)$ follows a generalized inverse Gaussian (GIG) distribution whose density function with respect to the measure of Lebesgue is:

$$f_U(w) = C_U \cdot w^{\lambda-1} \cdot \exp\left(-\frac{1}{2}(\chi w^{-1} + \psi w)\right) \cdot \mathbf{1}_{[0,+\infty)}(w),$$

where C_U is a constant, $\mathbf{1}$ denotes the indicator function and

$$\begin{aligned} \chi > 0, \psi &\geq 0 \quad \text{if} \quad \lambda < 0, \\ \chi > 0, \psi &> 0 \quad \text{if} \quad \lambda = 0, \\ \chi &\geq 0, \psi > 0 \quad \text{if} \quad \lambda > 0. \end{aligned}$$

In the following section, we study the existence of a Nash equilibrium of the CCG where the random constraint vectors follow normal mean-variance mixture distributions.

4.5 . Existence of Nash Equilibrium with Normal Mean-Variance Mixture Distributed Row Vectors

For each player $i \in H$, we assume that the row vectors of $V^{i,k}$ follow normal mean-variance mixture distributions, i.e., $V^{i,k} \sim \text{NMVM}(\mu^{i,k}, \gamma^{i,k}, \Sigma^{i,k}, W^{i,k})$, $k \in J^i$. Moreover, the row vectors $V^{i,k}$ are mutually independent and the support of $W^{i,k}$ is an open interval $(w_l^{i,k}, w_u^{i,k})$, where $0 < w_l^{i,k} \leq w_u^{i,k} < \infty$. Let $\tilde{S}_{\alpha_i}^i =$

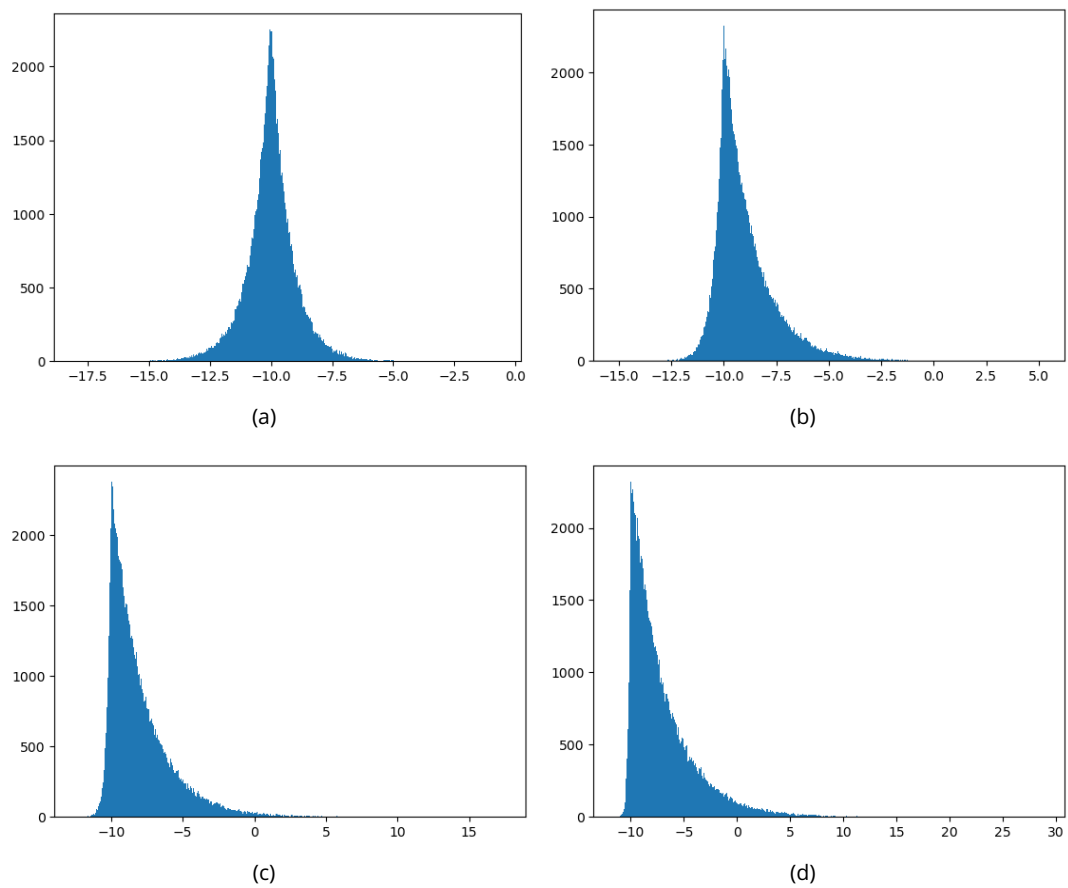


Figure 4.3: (a) $\gamma = 0$, (b) $\gamma = 2$, (c) $\gamma = 4$, (d) $\gamma = 6$.

$S_{\alpha_i}^i \setminus \{0\}$, then for $x^i \in \tilde{S}_{\alpha_i}^i$, let

$$\begin{aligned} g^{i,k}(x^i, W^{i,k}) &= \frac{-(x^i)^\top \gamma^{i,k}}{\sqrt{(x^i)^\top \Sigma^{i,k} x^i}} \sqrt{W^{i,k}} + \frac{D^{i,k} - (x^i)^\top \mu^{i,k}}{\sqrt{W^{i,k}} \sqrt{(x^i)^\top \Sigma^{i,k} x^i}}, \\ \xi^{i,k}(x^i) &= \frac{Z^\top ((\Sigma^{i,k})^{\frac{1}{2}})^\top x^i}{\sqrt{(x^i)^\top \Sigma^{i,k} x^i}}, \end{aligned} \quad (4.17)$$

where Z refers to Definition 4.9. It is well known that $\xi^{i,k}(x^i)$ follows 1-dimensional standard Gaussian distribution [71], for any $i \in H$ and $k \in J^i$. Using the independence of the row vectors $V^{i,k}$, the constraint (4.1) can be written as

$$\prod_{k \in J^i} \mathbb{P} \left\{ (V^{i,k})^\top x^i \leq D^{i,k} \right\} \geq \alpha_i,$$

which implies that

$$\prod_{k \in J^i} \mathbb{P} \left(\frac{W^{i,k}}{\sqrt{(x^i)^\top \Sigma^{i,k} x^i}} (x^i)^\top \gamma^{i,k} + \sqrt{W^{i,k}} \frac{Z^\top ((\Sigma^{i,k})^{\frac{1}{2}})^\top x^i}{\sqrt{(x^i)^\top \Sigma^{i,k} x^i}} \leq \frac{D^{i,k} - (x^i)^\top \mu^{i,k}}{\sqrt{(x^i)^\top \Sigma^{i,k} x^i}} \right) \geq \alpha_i. \quad (4.18)$$

Using the notations in (4.17), we rewrite (4.18) as follows

$$\prod_{k \in J^i} \mathbb{P} \left(\xi^{i,k}(x^i) \leq g^{i,k}(x^i, W^{i,k}) \right) \geq \alpha_i. \quad (4.19)$$

Thanks to the law of expectation in probability theory, (4.19) is equivalent to

$$\prod_{k \in J^i} \mathbb{E}_{W^{i,k}} \left(\mathbb{E}_Z \left(\mathbf{1}_{\{\xi^{i,k}(x^i) \leq g^{i,k}(x^i, W^{i,k})\}} \mid W^{i,k} \right) \right) \geq \alpha_i. \quad (4.20)$$

Note that Z and $W^{i,k}$ are independent random variables, which in turn implies that $\xi^{i,k}(x^i)$ and $W^{i,k}$ are independent as well. Therefore,

$$\mathbb{E}_Z \left(\mathbf{1}_{\{\xi^{i,k}(x^i) \leq g^{i,k}(x^i, W^{i,k})\}} \mid W^{i,k} \right) = \mathbb{E}_Z \left(\mathbf{1}_{\{\xi^{i,k}(x^i) \leq g^{i,k}(x^i, W^{i,k})\}} \right).$$

Let Φ be the cumulative distribution function of an 1-dimensional standard Gaussian distribution. Then, the above expectation can be written as

$$\mathbb{E}_Z \left(\mathbf{1}_{\{\xi^{i,k}(x^i) \leq g^{i,k}(x^i, W^{i,k})\}} \right) = \Phi \left(g^{i,k}(x^i, W^{i,k}) \right).$$

Hence, the constraint (4.20) is equivalent to

$$\prod_{k \in J^i} \mathbb{E}_{W^{i,k}} \left(\Phi \left(g^{i,k}(x^i, W^{i,k}) \right) \right) \geq \alpha_i. \quad (4.21)$$

By taking logarithm of (4.21), we have the following equivalent constraint

$$\sum_{k \in J^i} \log \left(\mathbb{E}_{W^{i,k}} \left(\Phi \left(g^{i,k}(x^i, W^{i,k}) \right) \right) \right) \geq \log(\alpha_i). \quad (4.22)$$

The main idea for the proof of existence of Nash equilibrium is based on the Kakutani fixed-point theorem, which requires the convexity of the feasible strategy set $S_{\alpha_i}^i$. We show that there exists a real number $\alpha_i^* \in [0, 1)$ such that $S_{\alpha_i}^i$ is convex, for all $\alpha_i \in (\alpha_i^*, 1]$, where α_i^* is defined in the following assumption.

Assumption 4.6. For all $i \in H$, let

$$\begin{aligned}\alpha_i^{(1)} &= \max_{k \in J^i} \Phi \left(\frac{\|\gamma^{i,k}\|_2}{\sqrt{\lambda_{i,k,\min}}} \sqrt{w_u^{i,k}} \right), \\ \alpha_i^{(2)} &= \max_{k \in J^i} \Phi \left[\frac{4\sqrt{w_u^{i,k}}}{\sqrt{w_l^{i,k}}} \left(\frac{\|\gamma^{i,k}\|_2}{\sqrt{\lambda_{i,k,\min}}} \sqrt{w_u^{i,k}} + \frac{\|\mu^{i,k}\|_2}{\sqrt{\lambda_{i,k,\min}} \sqrt{w_l^{i,k}}} \right) + \frac{\|\gamma^{i,k}\|_2 (w_u^{i,k} - w_l^{i,k})}{\sqrt{\lambda_{i,k,\min}} \sqrt{w_l^{i,k}}} \right], \\ \alpha_i^{(3)} &= \max_{k \in J^i} \Phi \left(\frac{\sqrt{3w_u^{i,k}}}{\sqrt{w_l^{i,k}}} + \frac{\|\gamma^{i,k}\|_2 (w_u^{i,k} - w_l^{i,k})}{\sqrt{\lambda_{i,k,\min}} \sqrt{w_l^{i,k}}} \right),\end{aligned}$$

where $\|\cdot\|_2$ denotes the Euclidean norm and $\lambda_{i,k,\min}$ is the smallest eigenvalue of the positive definite matrix $\Sigma^{i,k}$. Let $\alpha_i^* = \max(\alpha_i^{(1)}, \alpha_i^{(2)}, \alpha_i^{(3)})$. Assume that $\alpha_i \in (\alpha_i^*, 1]$, for all $i \in H$.

The main result of this section is the following lemma.

Lemma 4.15. Let Assumption 4.6 holds. Then, $S_{\alpha_i}^i$ is a convex set, for all $i \in H$.

In order to prove Lemma 4.15, we need the following lemmas.

Lemma 4.16. Let Assumption 4.6 holds. Assume that $x^i \in \tilde{S}_{\alpha_i}^i$, for all $i \in H$. Then, $D^{i,k} > (\mu^{i,k})^\top x^i$, for all $i \in H$ and $k \in J^i$.

Proof. For each $i \in H$, let $x^i \in \tilde{S}_{\alpha_i}^i$. Since $\mathbb{E}_{W^{i,k}} [\Phi(g^{i,k}(x^i, W^{i,k}))] \in [0, 1]$ and $\alpha_i > \alpha_i^{(1)}$, for any $k \in J^i$, the constraint (4.21) implies that

$$\mathbb{E}_{W^{i,k}} [\Phi(g^{i,k}(x^i, W^{i,k}))] > \alpha_i^{(1)}.$$

It follows from the definition of $g^{i,k}$ in (4.17) that

$$\mathbb{E}_{W^{i,k}} \left[\Phi \left(\frac{-(x^i)^\top \gamma^{i,k}}{\sqrt{(x^i)^\top \Sigma^{i,k} x^i}} \sqrt{W^{i,k}} + \frac{D^{i,k} - (x^i)^\top \mu^{i,k}}{\sqrt{W^{i,k}} \sqrt{(x^i)^\top \Sigma^{i,k} x^i}} \right) \right] > \alpha_i^{(1)}. \quad (4.23)$$

Due to the three following inequalities

$$(i) |(x^i)^\top \gamma^{i,k}| \leq \|x^i\|_2 \|\gamma^{i,k}\|_2, \quad (ii) \sqrt{(x^i)^\top \Sigma^{i,k} x^i} \geq \sqrt{\lambda_{i,k,\min}} \|x^i\|_2, \quad (iii) \sqrt{W^{i,k}} \leq \sqrt{w_u^{i,k}}, \quad (4.24)$$

we deduce that

$$\left| \frac{-(x^i)^\top \gamma^{i,k}}{\sqrt{(x^i)^\top \Sigma^{i,k} x^i}} \sqrt{W^{i,k}} \right| \leq \frac{\|\gamma^{i,k}\|_2}{\sqrt{\lambda_{i,k,\min}}} \sqrt{w_u^{i,k}}. \quad (4.25)$$

From (4.23) and (4.25), by applying the increasing monotonicity of Φ , we get

$$\mathbb{E}_{W^{i,k}} \left[\Phi \left(\frac{\|\gamma^{i,k}\|_2}{\sqrt{\lambda_{i,k,\min}}} \sqrt{w_u^{i,k}} + \frac{D^{i,k} - (x^i)^\top \mu^{i,k}}{\sqrt{W^{i,k}} \sqrt{(x^i)^\top \Sigma^{i,k} x^i}} \right) \right] > \alpha_i^{(1)}.$$

It is clear from the definition of $\alpha_i^{(1)}$ in Assumption 4.6 that the following condition holds

$$\alpha_i^{(1)} \geq \Phi \left(\frac{\|\gamma^{i,k}\|_2}{\sqrt{\lambda_{i,k,\min}}} \sqrt{w_u^{i,k}} \right),$$

which in turn implies that

$$\mathbb{E}_{W^{i,k}} \left[\Phi \left(\frac{\|\gamma^{i,k}\|_2}{\sqrt{\lambda_{i,k,\min}}} \sqrt{w_u^{i,k}} + \frac{D^{i,k} - (x^i)^\top \mu^{i,k}}{\sqrt{W^{i,k}} \sqrt{(x^i)^\top \Sigma^{i,k} x^i}} \right) \right] > \Phi \left(\frac{\|\gamma^{i,k}\|_2}{\sqrt{\lambda_{i,k,\min}}} \sqrt{w_u^{i,k}} \right). \quad (4.26)$$

If $D^{i,k} - (x^i)^\top \mu^{i,k} \leq 0$, (4.26) implies that

$$\mathbb{E}_{W^{i,k}} \left[\Phi \left(\frac{\|\gamma^{i,k}\|_2}{\sqrt{\lambda_{i,k,\min}}} \sqrt{w_u^{i,k}} \right) \right] > \Phi \left(\frac{\|\gamma^{i,k}\|_2}{\sqrt{\lambda_{i,k,\min}}} \sqrt{w_u^{i,k}} \right),$$

which derives a contradiction due to the increasing monotonicity of Φ . Therefore, $D^{i,k} - (x^i)^\top \mu^{i,k} > 0$. \square

Lemma 4.17. For $i \in H$, let $x^i \in \tilde{S}_{\alpha_i}^i$. Then, for any $k \in J^i$ and $z^{i,k} \in [w_l^{i,k}, w_u^{i,k}]$, we have

$$\mathbb{E}_{W^{i,k}} \left[\Phi \left(\frac{\sqrt{z^{i,k}}}{\sqrt{W^{i,k}}} g^{i,k}(x^i, z^{i,k}) + \frac{\|\gamma^{i,k}\|_2 (w_u^{i,k} - w_l^{i,k})}{\sqrt{\lambda_{i,k,\min}} \sqrt{w_l^{i,k}}} \right) \right] \geq \alpha_i.$$

Proof. For $i \in H$ and $k \in J^i$, let $x^i \in \tilde{S}_{\alpha_i}^i$ and $z^{i,k} \in [w_l^{i,k}, w_u^{i,k}]$. It is clear from the definition of $g^{i,k}$ in (4.17) that the following condition holds for any $W^{i,k} \in [w_l^{i,k}, w_u^{i,k}]$

$$g^{i,k}(x^i, W^{i,k}) = \frac{\sqrt{z^{i,k}}}{\sqrt{W^{i,k}}} g^{i,k}(x^i, z^{i,k}) + \frac{(x^i)^\top \gamma^{i,k}}{\sqrt{(x^i)^\top \Sigma^{i,k} x^i}} \left(\frac{z^{i,k} - W^{i,k}}{\sqrt{W^{i,k}}} \right).$$

By applying the inequalities (i) – (iii) of (4.24) and the fact that $|z^{i,k} - W^{i,k}| \leq w_u^{i,k} - w_l^{i,k}$, we deduce the following inequality

$$\frac{(x^i)^\top \gamma^{i,k}}{\sqrt{(x^i)^\top \Sigma^{i,k} x^i}} \left(\frac{z^{i,k} - W^{i,k}}{\sqrt{W^{i,k}}} \right) \leq \frac{\|\gamma^{i,k}\|_2 (w_u^{i,k} - w_l^{i,k})}{\sqrt{\lambda_{i,k,\min}} \sqrt{w_l^{i,k}}}$$

which in turn implies that

$$g^{i,k}(x^i, W^{i,k}) \leq \frac{\sqrt{z^{i,k}}}{\sqrt{W^{i,k}}} g^{i,k}(x^i, z^{i,k}) + \frac{\|\gamma^{i,k}\|_2 (w_u^{i,k} - w_l^{i,k})}{\sqrt{\lambda_{i,k,\min}} \sqrt{w_l^{i,k}}}, \quad \forall W^{i,k} \in [w_l^{i,k}, w_u^{i,k}]. \quad (4.27)$$

Taking the expectation value $\mathbb{E}_{W^{i,k}} (\Phi(\cdot))$ on both sides of (4.27), we get

$$\mathbb{E}_{W^{i,k}} [\Phi(g^{i,k}(x^i, W^{i,k}))] \leq \mathbb{E}_{W^{i,k}} \left[\Phi \left(\frac{\sqrt{z^{i,k}}}{\sqrt{W^{i,k}}} g^{i,k}(x^i, z) + \frac{\|\gamma^{i,k}\|_2 (w_u^{i,k} - w_l^{i,k})}{\sqrt{\lambda_{i,k,\min}} \sqrt{w_l^{i,k}}} \right) \right].$$

Since $\mathbb{E}_{W^{i,k}} [\Phi(g^{i,k}(x^i, W^{i,k}))] \in [0, 1]$, for any $k \in J^i$, the constraint (4.21) implies that

$$\mathbb{E}_{W^{i,k}} [\Phi(g^{i,k}(x^i, W^{i,k}))] \geq \alpha_i,$$

which in turn implies that

$$\mathbb{E}_{W^{i,k}} \left[\Phi \left(\frac{\sqrt{z^{i,k}}}{\sqrt{W^{i,k}}} g^{i,k}(x^i, z^{i,k}) + \frac{\|\gamma^{i,k}\|_2 (w_u^{i,k} - w_l^{i,k})}{\sqrt{\lambda_{i,k,\min}} \sqrt{w_l^{i,k}}} \right) \right] \geq \alpha_i.$$

\square

Lemma 4.18. *Let Assumption 4.6 holds. Then, for all $i \in H$*

$$\text{Conv}(\tilde{S}_{\alpha_i}^i) \subset \bigcap_{k \in J^i} \Omega^{i,k},$$

where Conv represents the convex hull and

$$\Omega^{i,k} = \left\{ x^i \in S^i \left| \begin{array}{l} - (x^i)^T \gamma^{i,k} \sqrt{W^{i,k}} + \frac{D^{i,k} - (x^i)^T \mu^{i,k}}{\sqrt{W^{i,k}}} > 4 \sqrt{(x^i)^T \Sigma^{i,k} x^i}, \\ \frac{\|\gamma^{i,k}\|_2}{\sqrt{\lambda_{i,k,\min}}} \sqrt{w_u^{i,k}} + \frac{\|\mu^{i,k}\|_2}{\sqrt{\lambda_{i,k,\min}}} \frac{1}{\sqrt{w_l^{i,k}}} \\ \forall W^{i,k} \in [w_l^{i,k}, w_u^{i,k}]. \end{array} \right. \right\}, \quad (4.28)$$

Proof. For $i \in H$, let $x^i \in \tilde{S}_{\alpha_i}^i$. It follows from Lemma 4.17 that for any $z^{i,k} \in [w_l^{i,k}, w_u^{i,k}]$

$$\mathbb{E}_{W^{i,k}} \left[\Phi \left(\frac{\sqrt{z^{i,k}}}{\sqrt{W^{i,k}}} g^{i,k}(x^i, z^{i,k}) + \frac{\|\gamma^{i,k}\|_2 (w_u^{i,k} - w_l^{i,k})}{\sqrt{\lambda_{i,k,\min}} \sqrt{w_l^{i,k}}} \right) \right] > \alpha_i. \quad (4.29)$$

Since $\alpha_i > \alpha_i^{(2)}$, the constraint (4.29) implies that

$$\mathbb{E}_{W^{i,k}} \left[\Phi \left(\frac{\sqrt{z^{i,k}}}{\sqrt{W^{i,k}}} g^{i,k}(x^i, z^{i,k}) + \frac{\|\gamma^{i,k}\|_2 (w_u^{i,k} - w_l^{i,k})}{\sqrt{\lambda_{i,k,\min}} \sqrt{w_l^{i,k}}} \right) \right] > \alpha_i^{(2)}.$$

It follows from the definition of $\alpha_i^{(2)}$ in Assumption 4.6 that the following condition holds

$$\alpha_i^{(2)} \geq \Phi \left[\frac{4\sqrt{w_u^{i,k}}}{\sqrt{w_l^{i,k}}} \left(\frac{\|\gamma^{i,k}\|_2}{\sqrt{\lambda_{i,k,\min}}} \sqrt{w_u^{i,k}} + \frac{\|\mu^{i,k}\|_2}{\sqrt{\lambda_{i,k,\min}} \sqrt{w_l^{i,k}}} \right) + \frac{\|\gamma^{i,k}\|_2 (w_u^{i,k} - w_l^{i,k})}{\sqrt{\lambda_{i,k,\min}} \sqrt{w_l^{i,k}}} \right].$$

which in turn implies that

$$\begin{aligned} & \mathbb{E}_{W^{i,k}} \left[\Phi \left(\frac{\sqrt{z^{i,k}}}{\sqrt{W^{i,k}}} g^{i,k}(x^i, z^{i,k}) + \frac{\|\gamma^{i,k}\|_2 (w_u^{i,k} - w_l^{i,k})}{\sqrt{\lambda_{i,k,\min}} \sqrt{w_l^{i,k}}} \right) \right] \\ & > \Phi \left[\frac{4\sqrt{w_u^{i,k}}}{\sqrt{w_l^{i,k}}} \left(\frac{\|\gamma^{i,k}\|_2}{\sqrt{\lambda_{i,k,\min}}} \sqrt{w_u^{i,k}} + \frac{\|\mu^{i,k}\|_2}{\sqrt{\lambda_{i,k,\min}} \sqrt{w_l^{i,k}}} \right) + \frac{\|\gamma^{i,k}\|_2 (w_u^{i,k} - w_l^{i,k})}{\sqrt{\lambda_{i,k,\min}} \sqrt{w_l^{i,k}}} \right]. \end{aligned} \quad (4.30)$$

If $g^{i,k}(x^i, z^{i,k}) \leq 0$, the constraint (4.30) implies that

$$\begin{aligned} & \Phi \left(\frac{\|\gamma^{i,k}\|_2 (w_u^{i,k} - w_l^{i,k})}{\sqrt{\lambda_{i,k,\min}} \sqrt{w_l^{i,k}}} \right) \\ & > \Phi \left[\frac{4\sqrt{w_u^{i,k}}}{\sqrt{w_l^{i,k}}} \left(\frac{\|\gamma^{i,k}\|_2}{\sqrt{\lambda_{i,k,\min}}} \sqrt{w_u^{i,k}} + \frac{\|\mu^{i,k}\|_2}{\sqrt{\lambda_{i,k,\min}} \sqrt{w_l^{i,k}}} \right) + \frac{\|\gamma^{i,k}\|_2 (w_u^{i,k} - w_l^{i,k})}{\sqrt{\lambda_{i,k,\min}} \sqrt{w_l^{i,k}}} \right], \end{aligned}$$

which derives a contradiction due to the increasing monotonicity of Φ . Therefore, $g^{i,k}(x^i, z^{i,k}) > 0$. Then, the following inequality holds for any $W^{i,k} \in [w_l^{i,k}, w_u^{i,k}]$

$$\frac{\sqrt{z^{i,k}}}{\sqrt{W^{i,k}}} g^{i,k}(x^i, z^{i,k}) \leq \frac{\sqrt{w_u^{i,k}}}{\sqrt{w_l^{i,k}}} g^{i,k}(x^i, z^{i,k}). \quad (4.31)$$

It follows from (4.30) and (4.31) that

$$\begin{aligned} & \Phi \left(\frac{\sqrt{w_u^{i,k}}}{\sqrt{w_l^{i,k}}} g^{i,k}(x^i, z^{i,k}) + \frac{\|\gamma^{i,k}\|_2 (w_u^{i,k} - w_l^{i,k})}{\sqrt{\lambda_{i,k,\min}} \sqrt{w_l^{i,k}}} \right) \\ & > \Phi \left[\frac{4\sqrt{w_u^{i,k}}}{\sqrt{w_l^{i,k}}} \left(\frac{\|\gamma^{i,k}\|_2}{\sqrt{\lambda_{i,k,\min}}} \sqrt{w_u^{i,k}} + \frac{\|\mu^{i,k}\|_2}{\sqrt{\lambda_{i,k,\min}} \sqrt{w_l^{i,k}}} \right) + \frac{\|\gamma^{i,k}\|_2 (w_u^{i,k} - w_l^{i,k})}{\sqrt{\lambda_{i,k,\min}} \sqrt{w_l^{i,k}}} \right]. \end{aligned} \quad (4.32)$$

By applying the increasing monotonicity of Φ , (4.32) is equivalent to the following inequality

$$g^{i,k}(x^i, z^{i,k}) > 4 \left(\frac{\|\gamma^{i,k}\|_2}{\sqrt{\lambda_{i,k,\min}}} \sqrt{w_u^{i,k}} + \frac{\|\mu^{i,k}\|_2}{\sqrt{\lambda_{i,k,\min}} \sqrt{w_l^{i,k}}} \right). \quad (4.33)$$

Since (4.33) holds for any $z^{i,k} \in [w_l^{i,k}, w_u^{i,k}]$ and $x^i \in \tilde{S}_{\alpha_i}^i$, we get

$$\tilde{S}_{\alpha_i}^i \subset \bigcap_{k \in J^i} \Omega^{i,k}. \quad (4.34)$$

Note that $\Omega^{i,k}$ is a convex set. Then, by taking the convex hull on both side of (4.34),

$$\text{Conv}(\tilde{S}_{\alpha_i}^i) \subset \bigcap_{k \in J^i} \Omega^{i,k}.$$

□

Lemma 4.19. *Let Assumption 4.6 holds. Then, for any convex subset Q^i of $\bigcap_{k \in J^i} \Omega^{i,k}$ such that $0 \notin Q^i$, $g^{i,k}(x^i, W^{i,k})$ is defined and (-2) -concave w.r.t x^i on Q^i , for all $i \in H$, $k \in J^i$ and $W^{i,k} \in [w_l^{i,k}, w_u^{i,k}]$.*

Proof. For $x^i \in Q^i$ and $W^{i,k} \in [w_l^{i,k}, w_u^{i,k}]$, let $f^{i,k}(x^i, W^{i,k}) = \frac{1}{(g^{i,k}(x^i, W^{i,k}))^2}$. It follows from the definition of $g^{i,k}$ in (4.17) that

$$f^{i,k}(x^i, W^{i,k}) = ((x^i)^T \Sigma^{i,k} x^i) \cdot M_{i,k}(x^i, W^{i,k})^{-1},$$

where $M_{i,k}(x^i, W^{i,k}) = W^{i,k} [(x^i)^T \gamma^{i,k}]^2 + \frac{1}{W^{i,k}} [D^{i,k} - (x^i)^T \mu^{i,k}]^2 + 2[(x^i)^T \gamma^{i,k}][(x^i)^T \mu^{i,k} - D^{i,k}]$. In order to simplify the notation, for the rest of the proof, we write $M_{i,k}$ (resp. $f^{i,k}$) instead of $M_{i,k}(x^i, W^{i,k})$ (resp. $f^{i,k}(x^i, W^{i,k})$). It is clear that the (-2) -concavity of $g^{i,k}$ is equivalent to the convexity of $f^{i,k}$ on Q^i . In order to prove the convexity of $f^{i,k}$, we prove that the Hessian matrix of $f^{i,k}$ w.r.t x^i is positive semidefinite on Q^i , for any

$W^{i,k} \in [w_l^{i,k}, w_u^{i,k}]$. Let $\nabla_{x^i} f^{i,k}$ be the gradient vector of $f^{i,k}$ w.r.t x^i and $H_{x^i} f^{i,k}$ be the Hessian matrix of $f^{i,k}$ w.r.t x^i . Let $h(x^i) = (x^i)^\top \Sigma^{i,k} x^i$. The gradient vector of h is given as follows

$$\nabla_{x^i} h(x^i) = 2\Sigma^{i,k} x^i,$$

and the gradient vector of $M_{i,k}$ w.r.t x^i is given as follows

$$\nabla_{x^i} M_{i,k} = 2[W^{i,k}(x^i)^\top \gamma^{i,k} + (x^i)^\top \mu^{i,k} - D^{i,k}]. \left(\gamma^{i,k} + \frac{\mu^{i,k}}{W^{i,k}} \right).$$

Since $f^{i,k} = h(x^i) \cdot M_{i,k}^{-1}$, the gradient vector of $f^{i,k}$ w.r.t x^i can be written as follows

$$\nabla_{x^i} f^{i,k} = \nabla_{x^i} h(x^i) \cdot M_{i,k}^{-1} + h(x^i) \cdot \nabla_{x^i} (M_{i,k}^{-1}).$$

We can write $M_{i,k}^{-1} = u \circ M_{i,k}$, where $u(x) = \frac{1}{x}$. By the chain rule of composite function in calculus, the gradient vector of $M_{i,k}^{-1}$ is given by

$$\begin{aligned} \nabla_{x^i} (M_{i,k}^{-1}) &= u'(M_{i,k}) \cdot \nabla_{x^i} M_{i,k} \\ &= \frac{-1}{M_{i,k}^2} \cdot 2[W^{i,k}(x^i)^\top \gamma^{i,k} + (x^i)^\top \mu^{i,k} - D^{i,k}]. \left(\gamma^{i,k} + \frac{\mu^{i,k}}{W^{i,k}} \right), \end{aligned}$$

which in turn implies that, for any $(x^i, W^{i,k}) \in Q^i \times [w_l^{i,k}, w_u^{i,k}]$

$$\begin{aligned} \nabla_{x^i} f^{i,k} &= 2M_{i,k}^{-1} \Sigma^{i,k} x^i \\ &\quad - 2M_{i,k}^{-2} \cdot h(x^i) \cdot [W^{i,k}(x^i)^\top \gamma^{i,k} + (x^i)^\top \mu^{i,k} - D^{i,k}]. \left(\gamma^{i,k} + \frac{\mu^{i,k}}{W^{i,k}} \right). \end{aligned} \quad (4.35)$$

Note that the Hessian matrix of $f^{i,k}$ w.r.t x^i can be written equivalently as follows

$$H_{x^i} f^{i,k} = \nabla_{x^i} \left[(\nabla_{x^i} f^{i,k})^\top \right].$$

Then, it suffices to derive the term on the right of (4.35) by x^i . Let $v(x) = \frac{1}{x^2}$, $r(x^i) = \Sigma^{i,k} x^i$, $s(x^i) = W^{i,k}(x^i)^\top \gamma^{i,k} + (x^i)^\top \mu^{i,k} - D^{i,k}$, then the gradient vector of $f^{i,k}$ w.r.t x^i is given by

$$\nabla_{x^i} f^{i,k} = 2 \cdot M_{i,k}^{-1} \cdot r(x^i) - 2 \cdot v \circ M_{i,k} \cdot h(x^i) \cdot s(x^i) \cdot \left(\gamma^{i,k} + \frac{\mu^{i,k}}{W^{i,k}} \right). \quad (4.36)$$

Deriving both sides of (4.36) by x^i , we get

$$\begin{aligned} H_{x^i} f^{i,k} &= 2 \cdot \nabla_{x^i} (M_{i,k}^{-1}) \cdot [r(x^i)]^\top + 2 \cdot M_{i,k}^{-1} \cdot \nabla_{x^i} r(x^i) \\ &\quad - 2 \cdot v \circ M_{i,k} \cdot h(x^i) \cdot \nabla_{x^i} s(x^i) \cdot \left(\gamma^{i,k} + \frac{\mu^{i,k}}{W^{i,k}} \right)^\top \\ &\quad - 2 \cdot \nabla_{x^i} (v \circ M_{i,k}) \cdot h(x^i) \cdot s(x^i) \cdot \left(\gamma^{i,k} + \frac{\mu^{i,k}}{W^{i,k}} \right)^\top \\ &\quad - 2 \cdot v \circ M_{i,k} \cdot \nabla_{x^i} h(x^i) \cdot s(x^i) \cdot \left(\gamma^{i,k} + \frac{\mu^{i,k}}{W^{i,k}} \right)^\top = A + B + C + D + E, \end{aligned}$$

where

$$\begin{aligned}
A &= 2 \cdot \nabla_{x^i} (M_{i,k}^{-1}) \cdot [r(x^i)]^\top, \quad B = 2 \cdot M_{i,k}^{-1} \cdot \nabla_{x^i} r(x^i), \\
C &= -2 \cdot v \circ M_{i,k} \cdot h(x^i) \cdot \nabla_{x^i} s(x^i) \cdot \left(\gamma^{i,k} + \frac{\mu^{i,k}}{W^{i,k}} \right)^\top, \\
D &= -2 \cdot \nabla_{x^i} (v \circ M_{i,k}) \cdot h(x^i) \cdot s(x^i) \cdot \left(\gamma^{i,k} + \frac{\mu^{i,k}}{W^{i,k}} \right)^\top, \\
E &= -2 \cdot v \circ M_{i,k} \cdot \nabla_{x^i} h(x^i) \cdot s(x^i) \cdot \left(\gamma^{i,k} + \frac{\mu^{i,k}}{W^{i,k}} \right)^\top.
\end{aligned}$$

We can verify that the following equations hold

$$\begin{aligned}
\text{(i)} \quad \nabla_{x^i} (M_{i,k}^{-1}) &= \frac{-1}{M_{i,k}^2} \cdot 2s(x^i) \cdot \left(\gamma^{i,k} + \frac{\mu^{i,k}}{W^{i,k}} \right), \quad \nabla_{x^i} r(x^i) = \Sigma^{i,k}, \\
\text{(ii)} \quad \nabla_{x^i} s(x^i) &= W^{i,k} \gamma^{i,k} + \mu^{i,k}, \quad \nabla_{x^i} h(x^i) = 2\Sigma^{i,k} x^i \\
\text{(iii)} \quad \nabla_{x^i} (v \circ M_{i,k}) &= v'(M_{i,k}) \cdot \nabla_{x^i} M_{i,k} \\
&= \frac{-2}{M_{i,k}^3} \cdot 2[W^{i,k}(x^i)^\top \gamma^{i,k} + (x^i)^\top \mu^{i,k} - D^{i,k}] \cdot \left(\gamma^{i,k} + \frac{\mu^{i,k}}{W^{i,k}} \right) \\
&= \frac{-2}{M_{i,k}^3} \cdot 2s(x^i) \cdot \left(\gamma^{i,k} + \frac{\mu^{i,k}}{W^{i,k}} \right),
\end{aligned}$$

which in turn imply that

$$\begin{aligned}
A &= \frac{-4s(x^i)}{M_{i,k}^2} \left(\gamma^{i,k} + \frac{\mu^{i,k}}{W^{i,k}} \right) (x^i)^\top \Sigma^{i,k}, \quad B = \frac{2}{M_{i,k}} \Sigma^{i,k}, \\
C &= \frac{-2h(x^i)}{M_{i,k}^2} \cdot (W^{i,k} \gamma^{i,k} + \mu^{i,k}) \cdot \left(\gamma^{i,k} + \frac{\mu^{i,k}}{W^{i,k}} \right)^\top, \\
D &= \frac{8h(x^i)s^2(x^i)}{M_{i,k}^3} \left(\gamma^{i,k} + \frac{\mu^{i,k}}{W^{i,k}} \right) \left(\gamma^{i,k} + \frac{\mu^{i,k}}{W^{i,k}} \right)^\top, \quad E = \frac{-4s(x^i)}{M_{i,k}^2} \Sigma^{i,k} x^i \left(\gamma^{i,k} + \frac{\mu^{i,k}}{W^{i,k}} \right)^\top.
\end{aligned}$$

Note that $s^2(x^i) = W^{i,k} M_{i,k}$. Then, $D = \frac{8h(x^i)}{M_{i,k}^2} \cdot (W^{i,k} \gamma^{i,k} + \mu^{i,k}) \cdot \left(\gamma^{i,k} + \frac{\mu^{i,k}}{W^{i,k}} \right)^\top$. Therefore, the Hessian matrix of $f^{i,k}$ w.r.t x^i can be rewritten as

$$\begin{aligned}
H_{x^i} f^{i,k} &= A + B + C + D + E \\
&= \frac{-4s(x^i)}{M_{i,k}^2} \left[\frac{1}{W^{i,k}} \Sigma^{i,k} x^i (\mu^{i,k})^\top + \frac{1}{W^{i,k}} \mu^{i,k} (x^i)^\top \Sigma^{i,k} + \Sigma^{i,k} x^i (\gamma^{i,k})^\top + \gamma^{i,k} (x^i)^\top \Sigma^{i,k} \right] \\
&\quad + \frac{2}{M_{i,k}} \Sigma^{i,k} + \frac{6h(x^i)}{M_{i,k}^2} (W^{i,k} \gamma^{i,k} + \mu^{i,k}) \cdot \left(\gamma^{i,k} + \frac{\mu^{i,k}}{W^{i,k}} \right)^\top.
\end{aligned}$$

By multiplying the above equation by $\frac{M_{i,k}^2}{2}$, we get

$$\begin{aligned} \frac{(M_{i,k})^2}{2} \cdot H_{x^i} f^{i,k} &= M_{i,k} \Sigma^{i,k} + 3h(x^i)(W^{i,k} \gamma^{i,k} + \mu^{i,k}) \left(\gamma^{i,k} + \frac{\mu^{i,k}}{W^{i,k}} \right)^\top \\ &\quad - 2s(x^i) \left[\frac{1}{W^{i,k}} \Sigma^{i,k} x^i (\mu^{i,k})^\top + \frac{1}{W^{i,k}} \mu^{i,k} (x^i)^\top \Sigma^{i,k} + \Sigma^{i,k} x^i (\gamma^{i,k})^\top + \gamma^{i,k} (x^i)^\top \Sigma^{i,k} \right]. \end{aligned} \quad (4.37)$$

We prove that the quadratic form of $H_{x^i} f^{i,k}$ at $z^i \in \mathbb{R}^{d_i}$ is positive, for any $z^i \in \mathbb{R}^{d_i}$ and $(x^i, W^{i,k}) \in Q^i \times [w_l^{i,k}, w_u^{i,k}]$, i.e.,

$$(z^i)^\top H_{x^i} f^{i,k} z^i \geq 0. \quad (4.38)$$

In fact, by taking the quadratic form of (4.37) at z^i , we have

$$\begin{aligned} (z^i)^\top \frac{(M_{i,k})^2}{2} \cdot H_{x^i} f^{i,k} z^i &= M_{i,k} (z^i)^\top \Sigma^{i,k} z^i + 3h(x^i) \left(\sqrt{W^{i,k}} ((z^i)^\top \gamma^{i,k}) + \frac{1}{\sqrt{W^{i,k}}} ((z^i)^\top \mu^{i,k}) \right)^2 \\ &\quad - 4s(x^i) ((z^i)^\top \Sigma^{i,k} x^i) \left((\gamma^{i,k})^\top z^i + \frac{1}{W^{i,k}} (\mu^{i,k})^\top z^i \right). \end{aligned} \quad (4.39)$$

Note that the following inequalities hold

$$\begin{aligned} \text{(i)} \quad &-4s(x^i) ((z^i)^\top \Sigma^{i,k} x^i) \left((\gamma^{i,k})^\top z^i + \frac{1}{W^{i,k}} (\mu^{i,k})^\top z^i \right) \\ &\geq -4|s(x^i)| \cdot |(z^i)^\top \Sigma^{i,k} x^i| \cdot |(\gamma^{i,k})^\top z^i + \frac{1}{W^{i,k}} (\mu^{i,k})^\top z^i|, \\ \text{(ii)} \quad &|(z^i)^\top \Sigma^{i,k} x^i| \leq \sqrt{(z^i)^\top \Sigma^{i,k} z^i} \sqrt{(x^i)^\top \Sigma^{i,k} x^i}, \\ \text{(iii)} \quad &|(\gamma^{i,k})^\top z^i + \frac{1}{W^{i,k}} (\mu^{i,k})^\top z^i| \leq |(\gamma^{i,k})^\top z^i| + \frac{1}{W^{i,k}} |(\mu^{i,k})^\top z^i|, \end{aligned} \quad (4.40)$$

where (i) and (iii) are trivial. In order to prove (ii), let $e_1 = (z^i)^\top (\Sigma^{i,k})^{\frac{1}{2}}$, $e_2 = (x^i)^\top (\Sigma^{i,k})^{\frac{1}{2}}$, then (ii) is rewritten as

$$|e_1(e_2)^\top| \leq \|e_1\|_2 \|e_2\|_2,$$

which is known as the Cauchy-Schwarz inequality. It follows from (i) – (iii) of (4.40) that

$$\begin{aligned} &-4s(x^i) ((z^i)^\top \Sigma^{i,k} x^i) \left((\gamma^{i,k})^\top z^i + \frac{1}{W^{i,k}} (\mu^{i,k})^\top z^i \right) \\ &\geq -4|s(x^i)| \cdot \sqrt{(z^i)^\top \Sigma^{i,k} z^i} \sqrt{(x^i)^\top \Sigma^{i,k} x^i} \cdot \left(|(\gamma^{i,k})^\top z^i| + \frac{1}{W^{i,k}} |(\mu^{i,k})^\top z^i| \right). \end{aligned} \quad (4.41)$$

Then, from (4.39) and (4.41), we get

$$\begin{aligned} (z^i)^\top \frac{(M_{i,k})^2}{2} \cdot H_{x^i} f^{i,k} z^i &\geq M_{i,k} (z^i)^\top \Sigma^{i,k} z^i + 3h(x^i) \left(\sqrt{W^{i,k}} ((z^i)^\top \gamma^{i,k}) + \frac{1}{\sqrt{W^{i,k}}} ((z^i)^\top \mu^{i,k}) \right)^2 \\ &\quad - 4|s(x^i)| \sqrt{(z^i)^\top \Sigma^{i,k} z^i} \sqrt{(x^i)^\top \Sigma^{i,k} x^i} \left(|(\gamma^{i,k})^\top z^i| + \frac{1}{W^{i,k}} |(\mu^{i,k})^\top z^i| \right). \end{aligned}$$

Note that $|s(x^i)| = \sqrt{W^{i,k} M_{i,k}}$ and $3h(x^i) \left(\sqrt{W^{i,k}} ((z^i)^\top \gamma^{i,k}) + \frac{1}{\sqrt{W^{i,k}}} ((z^i)^\top \mu^{i,k}) \right)^2 \geq 0$, which in turn implies that

$$\begin{aligned} (z^i)^\top \frac{(M_{i,k})^2}{2} \cdot H_{x^i} f^{i,k} z^i &\geq M_{i,k} (z^i)^\top \Sigma^{i,k} z^i \\ &- 4\sqrt{M_{i,k}} \sqrt{(z^i)^\top \Sigma^{i,k} z^i} \sqrt{(x^i)^\top \Sigma^{i,k} x^i} \left(\sqrt{W^{i,k}} |(\gamma^{i,k})^\top z^i| + \frac{1}{\sqrt{W^{i,k}}} |(\mu^{i,k})^\top z^i| \right). \end{aligned} \quad (4.42)$$

Assume that $z^i \neq 0$. Dividing both sides of (4.42) by $((z^i)^\top \Sigma^{i,k} z^i)((x^i)^\top \Sigma^{i,k} x^i)$, we have

$$\begin{aligned} &\frac{(z^i)^\top \frac{(M_{i,k})^2}{2} \cdot H_{x^i} f^{i,k} z^i}{((z^i)^\top \Sigma^{i,k} z^i)((x^i)^\top \Sigma^{i,k} x^i)} \\ &\geq \frac{M_{i,k}}{(x^i)^\top \Sigma^{i,k} x^i} - 4\sqrt{\frac{M_{i,k}}{(x^i)^\top \Sigma^{i,k} x^i}} \left(\sqrt{W^{i,k}} \left| \frac{(\gamma^{i,k})^\top z^i}{\sqrt{(z^i)^\top \Sigma^{i,k} z^i}} \right| + \frac{1}{\sqrt{W^{i,k}}} \left| \frac{(\mu^{i,k})^\top z^i}{\sqrt{(z^i)^\top \Sigma^{i,k} z^i}} \right| \right). \end{aligned} \quad (4.43)$$

Note that $\frac{M_{i,k}}{(x^i)^\top \Sigma^{i,k} x^i} = [g^{i,k}(x^i, W^{i,k})]^2$. Since $x^i \in Q^i$ and Q^i is a subset of $\bigcap_{k \in J^i} \Omega^{i,k}$, then $x^i \in \Omega^{i,k}$, for any $k \in J^i$. Then, for any $W^{i,k} \in [w_l^{i,k}, w_u^{i,k}]$, it follows from the definition of $\Omega^{i,k}$ in (4.28) that

$$g^{i,k}(x^i, W^{i,k}) > 0.$$

Therefore, we can write $\sqrt{\frac{M_{i,k}}{(x^i)^\top \Sigma^{i,k} x^i}} = g^{i,k}(x^i, W^{i,k})$ and (4.43) is equivalent to the following inequality

$$\begin{aligned} &\frac{(z^i)^\top \frac{(M_{i,k})^2}{2} \cdot H_{x^i} f^{i,k} z^i}{((z^i)^\top \Sigma^{i,k} z^i)[(x^i)^\top \Sigma^{i,k} x^i]} \\ &\geq [g^{i,k}(x^i, W^{i,k})]^2 - 4g^{i,k}(x^i, W^{i,k}) \left(\sqrt{W^{i,k}} \left| \frac{(\gamma^{i,k})^\top z^i}{\sqrt{(z^i)^\top \Sigma^{i,k} z^i}} \right| + \frac{1}{\sqrt{W^{i,k}}} \left| \frac{(\mu^{i,k})^\top z^i}{\sqrt{(z^i)^\top \Sigma^{i,k} z^i}} \right| \right). \end{aligned}$$

It is easy to see that the following inequalities hold

$$\begin{aligned} \text{(i)} \quad &\left| \frac{(\gamma^{i,k})^\top z^i}{\sqrt{(z^i)^\top \Sigma^{i,k} z^i}} \right| \leq \frac{\|\gamma^{i,k}\|_2 \cdot \|z^i\|_2}{\sqrt{\lambda_{i,k,\min}} \cdot \|z^i\|_2} = \frac{\|\gamma^{i,k}\|_2}{\sqrt{\lambda_{i,k,\min}}}, \\ \text{(ii)} \quad &\left| \frac{(\mu^{i,k})^\top z^i}{\sqrt{(z^i)^\top \Sigma^{i,k} z^i}} \right| \leq \frac{\|\mu^{i,k}\|_2 \cdot \|z^i\|_2}{\sqrt{\lambda_{i,k,\min}} \cdot \|z^i\|_2} = \frac{\|\mu^{i,k}\|_2}{\sqrt{\lambda_{i,k,\min}}}, \\ \text{(iii)} \quad &\sqrt{W^{i,k}} \leq \sqrt{w_u^{i,k}}, \quad \frac{1}{\sqrt{W^{i,k}}} \leq \frac{1}{\sqrt{w_l^{i,k}}}, \end{aligned}$$

which in turn implies that

$$\sqrt{W^{i,k}} \left| \frac{(\gamma^{i,k})^\top z}{\sqrt{z^\top \Sigma^{i,k} z}} \right| + \frac{1}{\sqrt{W^{i,k}}} \left| \frac{(\mu^{i,k})^\top z}{\sqrt{z^\top \Sigma^{i,k} z}} \right| \leq \frac{\|\gamma^{i,k}\|_2}{\sqrt{\lambda_{i,k,\min}}} \sqrt{w_u^{i,k}} + \frac{\|\mu^{i,k}\|_2}{\sqrt{\lambda_{i,k,\min}}} \frac{1}{\sqrt{w_l^{i,k}}}. \quad (4.44)$$

Then, from (4.43) and (4.44), we get

$$\begin{aligned} & \frac{z^\top \frac{(M_{i,k})^2}{2} H_{x^i} f^{i,k} z}{(z^\top \Sigma^{i,k} z)[(x^i)^\top \Sigma^{i,k} x^i]} \\ & \geq [g^{i,k}(x^i, W^{i,k})]^2 - 4g^{i,k}(x^i, W^{i,k}) \left(\frac{\|\gamma^{i,k}\|_2}{\sqrt{\lambda_{i,k,\min}}} \sqrt{w_u^{i,k}} + \frac{\|\mu^{i,k}\|_2}{\sqrt{\lambda_{i,k,\min}}} \frac{1}{\sqrt{w_l^{i,k}}} \right). \end{aligned} \quad (4.45)$$

Since $x^i \in \Omega^{i,k}$, for any $k \in J^i$, the term on the right-hand side of (4.45) is positive, which implies that $(z^i)^\top H_{x^i} f^{i,k} z^i \geq 0$. \square

Using the abovementioned lemmas, we prove Lemma 4.15.

Proof. of Lemma 4.15) Let $\alpha_i \in (\alpha_i^*, 1]$, $\lambda \in [0, 1]$ and $y_1, y_2 \in S_{\alpha_i}^i$. We will show that $\lambda y_1 + (1 - \lambda)y_2 \in S_{\alpha_i}^i$. Consider 4 cases as follows:

Case 1: Let $y_1 = 0$ or $y_2 = 0$. Without loss of generality, we assume that $y_2 = 0$. We deduce that $D^{i,k} \geq 0$ for all $k \in J^i$, which implies that

$$\mathbb{P}(V^i \lambda y_1 \leq D^i) \geq \mathbb{P}(V^i y_1 \leq D^i) \geq \alpha_i.$$

Therefore, $\lambda y_1 + (1 - \lambda)y_2 \in S_{\alpha_i}^i$.

Case 2: Let $y_1 \neq 0$, $y_2 \neq 0$ and $\lambda y_1 + (1 - \lambda)y_2 = 0$. In this case, $y_2 = \frac{-\lambda}{1-\lambda} y_1 \in \tilde{S}_{\alpha_i}^i$ and $y_1 \in \tilde{S}_{\alpha_i}^i$. It follows from Lemma 4.16 that

$$(\mu^{i,k})^\top y_1 > \frac{\lambda - 1}{\lambda} D^{i,k}, \quad (\mu^{i,k})^\top y_1 < D^{i,k}, \quad \forall k \in J^i.$$

This implies that $D^{i,k} \geq 0$ for all $k \in J^i$. Hence, $\lambda y_1 + (1 - \lambda)y_2 = 0 \in S_{\alpha_i}^i$.

Case 3: Let $y_1 \neq 0$, $y_2 \neq 0$ and $0 \in \text{Seg}(y_1, y_2)$, where

$$\text{Seg}(y_1, y_2) = \{y_1 + l(y_2 - y_1), 0 \leq l \leq 1\}.$$

Then, for any point $x \in \text{Seg}(y_1, y_2)$, either $x \in \text{Seg}(y_1, 0)$ or $x \in \text{Seg}(0, y_2)$. It follows from Case 1 that $\text{Seg}(y_1, 0)$ and $\text{Seg}(0, y_2)$ are subset of $S_{\alpha_i}^i$. Therefore, $\lambda y_1 + (1 - \lambda)y_2 \in S_{\alpha_i}^i$ for all $\lambda \in [0, 1]$.

Case 4: Let $y_1 \neq 0$, $y_2 \neq 0$ such that $0 \notin \text{Seg}(y_1, y_2)$. It is clear that $\text{Seg}(y_1, y_2) \subset \text{Conv}(\tilde{S}_{\alpha_i}^i)$. From Lemmas 4.18 and 4.19, $g^{i,k}(\cdot, W^{i,k})$ is defined and (-2) -concave on $\text{Seg}(y_1, y_2)$, for all $W^{i,k} \in [w_l^{i,k}, w_u^{i,k}]$, which implies that

$$g^{i,k}(\lambda y_1 + (1 - \lambda)y_2, W^{i,k}) \geq (\lambda(g^{i,k}(y_1), W^{i,k})^{-2} + (1 - \lambda)(g^{i,k}(y_2), W^{i,k})^{-2})^{-\frac{1}{2}}. \quad (4.46)$$

Since, $y_1 \in \tilde{S}_{\alpha_i}^i$ and $\alpha_i > \alpha_i^{(3)}$, using Lemma 4.17, for any $z \in [w_l^{i,k}, w_u^{i,k}]$, the following condition holds

$$\mathbb{E}_{W^{i,k}} \left[\Phi \left(\frac{\sqrt{z}}{\sqrt{W^{i,k}}} g^{i,k}(y_1, z) + \frac{\|\gamma^{i,k}\|_2 (w_u^{i,k} - w_l^{i,k})}{\sqrt{\lambda_{i,k,\min}} \sqrt{w_l^{i,k}}} \right) \right] > \alpha_i^{(3)}.$$

Moreover, it follows from the definition of $\alpha_i^{(3)}$ in Assumption 4.6 that

$$\alpha_i^{(3)} \geq \Phi \left(\frac{\sqrt{3w_u^{i,k}}}{\sqrt{w_l^{i,k}}} + \frac{\|\gamma^{i,k}\|_2 (w_u^{i,k} - w_l^{i,k})}{\sqrt{\lambda_{i,k,\min}} \sqrt{w_l^{i,k}}} \right),$$

which in turn implies that

$$\begin{aligned} & \mathbb{E}_{W^{i,k}} \left[\Phi \left(\frac{\sqrt{z}}{\sqrt{W^{i,k}}} g^{i,k}(y_1, z) + \frac{\|\gamma^{i,k}\|_2 (w_u^{i,k} - w_l^{i,k})}{\sqrt{\lambda_{i,k,\min}} \sqrt{w_l^{i,k}}} \right) \right] \\ & > \Phi \left(\frac{\sqrt{3w_u^{i,k}}}{\sqrt{w_l^{i,k}}} + \frac{\|\gamma^{i,k}\|_2 (w_u^{i,k} - w_l^{i,k})}{\sqrt{\lambda_{i,k,\min}} \sqrt{w_l^{i,k}}} \right). \end{aligned} \quad (4.47)$$

If $g^{i,k}(z_1, z) \leq 0$, (4.47) implies that

$$\Phi \left(\frac{\|\gamma^{i,k}\|_2 (w_u^{i,k} - w_l^{i,k})}{\sqrt{\lambda_{i,k,\min}} \sqrt{w_l^{i,k}}} \right) > \Phi \left(\frac{\sqrt{3w_u^{i,k}}}{\sqrt{w_l^{i,k}}} + \frac{\|\gamma^{i,k}\|_2 (w_u^{i,k} - w_l^{i,k})}{\sqrt{\lambda_{i,k,\min}} \sqrt{w_l^{i,k}}} \right),$$

which derives a contradiction by the increasing monotonicity of Φ . Therefore, $g^{i,k}(y_1, z) \geq 0$. For any $W^{i,k} \in [w_l^{i,k}, w_u^{i,k}]$, we have

$$\frac{\sqrt{z}}{\sqrt{W^{i,k}}} g^{i,k}(y_1, z) \leq \frac{\sqrt{w_u^{i,k}}}{\sqrt{w_l^{i,k}}} g^{i,k}(y_1, z). \quad (4.48)$$

From (4.47) and (4.48), for any $z \in [w_l^{i,k}, w_u^{i,k}]$, we get

$$\Phi \left(\frac{\sqrt{w_u^{i,k}}}{\sqrt{w_l^{i,k}}} g^{i,k}(y_1, z) + \frac{\|\gamma^{i,k}\|_2 (w_u^{i,k} - w_l^{i,k})}{\sqrt{\lambda_{i,k,\min}} \sqrt{w_l^{i,k}}} \right) > \Phi \left(\frac{\sqrt{3w_u^{i,k}}}{\sqrt{w_l^{i,k}}} + \frac{\|\gamma^{i,k}\|_2 (w_u^{i,k} - w_l^{i,k})}{\sqrt{\lambda_{i,k,\min}} \sqrt{w_l^{i,k}}} \right),$$

which in turn implies that

$$0 < g^{i,k}(y_1, W^{i,k})^{-2} < \frac{1}{3}, \forall W^{i,k} \in [w_l^{i,k}, w_u^{i,k}].$$

Similarly,

$$0 < g^{i,k}(y_2, W^{i,k})^{-2} < \frac{1}{3}, \forall W^{i,k} \in [w_l^{i,k}, w_u^{i,k}].$$

By applying the non-decreasing function $\Phi(\cdot)$ on both side of (4.46), we can write

$$\begin{aligned} & \Phi(g^{i,k}(\lambda y_1 + (1-\lambda)y_2, W^{i,k})) \geq \\ & \Phi\left((\lambda(g^{i,k}(y_1, W^{i,k}))^{-2} + (1-\lambda)(g^{i,k}(y_2, W^{i,k}))^{-2})^{-\frac{1}{2}}\right), \forall W^{i,k} \in [w_l^{i,k}, w_u^{i,k}]. \end{aligned} \quad (4.49)$$

Since, $\Phi(\cdot)$ is the cumulative distribution function of a standard Normal distribution, it follows from Proposition 4.1 of [100] that $\Phi(\cdot)$ has r -decreasing density, for any $r > 0$ and $t^* = \sqrt{r}$, where t^* refers to Definition 3.4. By choosing $r = 3$, it follows from Lemma 3.1 of [100], the function $t \mapsto \Phi(t^{-\frac{1}{2}})$ is concave on $(0, \frac{1}{3})$. Therefore, for any $W^{i,k} \in [w_l^{i,k}, w_u^{i,k}]$, we obtain

$$\begin{aligned} & \Phi\left((\lambda(g^{i,k}(y_1, W^{i,k}))^{-2} + (1-\lambda)(g^{i,k}(y_2, W^{i,k}))^{-2})^{-\frac{1}{2}}\right) \\ & \geq \lambda(\Phi(g^{i,k}(y_1, W^{i,k}))) + (1-\lambda)(\Phi(g^{i,k}(y_2, W^{i,k}))). \end{aligned} \quad (4.50)$$

For any $W^{i,k} \in [w_l^{i,k}, w_u^{i,k}]$, it follows from (4.49) and (4.50) that

$$\begin{aligned} & \Phi(g^{i,k}(\lambda y_1 + (1-\lambda)y_2, W^{i,k})) \geq \\ & \lambda (\Phi(g^{i,k}(y_1, W^{i,k}))) + (1-\lambda) (\Phi(g^{i,k}(y_2, W^{i,k}))), \end{aligned} \quad (4.51)$$

which implies that $\Phi(g^{i,k}(\cdot, W^{i,k}))$ is a concave function on $\text{Seg}(y_1, y_2)$, for any $W^{i,k} \in [w_l^{i,k}, w_u^{i,k}]$. By taking the expectation value $\mathbb{E}_{W^{i,k}}$ on both sides of (4.51), we deduce that

$$\begin{aligned} & \mathbb{E}_{W^{i,k}} [\Phi(g^{i,k}(\lambda y_1 + (1-\lambda)y_2, W^{i,k}))] \geq \\ & \lambda (\mathbb{E}_{W^{i,k}} [\Phi(g^{i,k}(y_1, W^{i,k}))]) + (1-\lambda) (\mathbb{E}_{W^{i,k}} [\Phi(g^{i,k}(y_2, W^{i,k}))]), \end{aligned} \quad (4.52)$$

which in turn implies that the function $\mathbb{E}_{W^{i,k}} [\Phi(g^{i,k}(\cdot, W^{i,k}))]$ is a concave function on $\text{Seg}(y_1, y_2)$. It follows from the discussion in Definition 2.1 of [100] that log-concavity is a weaker property than concavity. Therefore, $\mathbb{E}_{W^{i,k}} [\Phi(g^{i,k}(\cdot, W^{i,k}))]$ is also a log-concave function on $\text{Seg}(y_1, y_2)$. Then, $\lambda y_1 + (1-\lambda)y_2 \in S_{\alpha_i}^i$. \square

It follows from Lemma 3.5 [167] that the closeness of $S_{\alpha_i}^i$ is a consequence of Proposition 3.1 in [202]. Since $S_{\alpha_i}^i$ is a subset of S^i , which is a compact set, we deduce that $S_{\alpha_i}^i$ is a compact set. S^i . We show that there exists a Nash equilibrium of the CCG by the following theorem.

Theorem 4.3. Consider an n -player CCG defined in Section 4.2, where

1. Assumptions 4.4 and 4.6 hold.
2. For each $i \in H$, we assume that $V^{i,k} \sim \text{NMVM}(\mu^{i,k}, \gamma^{i,k}, \Sigma^{i,k}, W^{i,k})$ and the vectors $V^{i,k}$ are mutually independent, $k \in J^i$.

Then, there exists a Nash equilibrium of the CCG for any $\alpha \in (\alpha_1^*, 1] \times \dots \times (\alpha_n^*, 1]$, where $\{\alpha_i^*\}_{i \in H}$ refers to Assumption 4.6.

Proof. The proof follows the same arguments as the proof of Theorem 3.1 [167]. \square

4.6 . Algorithm and Numerical Results

In this section, we assume that the strategy set of each player is strictly positive, i.e., $S^i \subset \mathbb{R}_{++}^{d_i}$, where $\mathbb{R}_{++}^{d_i}$ denotes the subset of \mathbb{R}^{d_i} with strictly positive components. We consider a 2- player CCG defined in Section 4.2. Let $x^2 \in S_{\alpha_2}^2$ be a feasible point of the second player. Then, the optimization problem of the first player is written as

$$\begin{aligned} [P1] \quad & \max \quad p_1(x^1, x^2) \\ & \text{s.t.} \quad x^1 \in S_{\alpha_1}^1. \end{aligned} \quad (4.53)$$

The set of optimal solutions of [P1] which is also called the best response set of the first player, is given by

$$BR_1(x^2) = \{\bar{x}^1 \mid p_1(\bar{x}^1, x^2) \geq p_1(x^1, x^2), \forall x^1 \in S_{\alpha_1}^1\} \quad (4.54)$$

Similarly, let $x^1 \in S_{\alpha_1}^1$, the optimization problem of the second player is written as

$$\begin{aligned} [P2] \quad & \max \quad p_2(x^1, x^2) \\ & \text{s.t.} \quad x^2 \in S_{\alpha_2}^2. \end{aligned} \quad (4.55)$$

The best response set of the second player, is given by

$$BR_2(x^1) = \{\bar{x}^2 \mid p_2(x^1, \bar{x}^2) \geq p_2(x^1, x^2), \forall x^2 \in S_{\alpha_2}^2\} \quad (4.56)$$

It is clear that if $x^{1*} \in BR_1(x^{2*})$ and $x^{2*} \in BR_2(x^{1*})$, then (x^{1*}, x^{2*}) is a Nash equilibrium of the CCG. We propose the best response algorithm given in Algorithm 1. If Algorithm 1 stops, (x^{1*}, x^{2*}) is a Nash equilibrium

Algorithm 1 Best response algorithm

Step1 : Select initial feasible point $x^{2(0)} \in S_{\alpha_2}^2$ for player 2, set $k = 0$ and choose a tolerance parameter $\epsilon > 0$.
Step2 : Solve the optimization problem [P1] with $x^2 = x^{2(k)}$ and obtain an optimal solution $x^{1(k)} \in BR_1(x^{2(k)})$.
Step3 : Solve the optimization problem [P2] with $x^1 = x^{1(k)}$ and obtain an optimal solution $x^{2(k+1)} \in BR_2(x^{1(k)})$. If $|p_2(x^{1(k)}, x^{2(k)}) - p_2(x^{1(k)}, x^{2(k+1)})| < \epsilon$, stop the algorithm and set $(x^{1*}, x^{2*}) = (x^{1(k)}, x^{2(k)})$. Otherwise, set $k := k + 1$ and go back to **Step2**.

of the CCG. The question when Algorithm 1 cycles is still open. Using Algorithm 1, at each step, the first player wants to solve his optimization problem [P1], for $x^2 \in S_{\alpha_2}^2$ and the second player wants to solve his optimization problem [P2], for $x^1 \in S_{\alpha_1}^1$. We present a reformulation of the optimization problem [P1], for $x^2 \in S_{\alpha_2}^2$. The optimization problem [P2] is written similarly. For each $x^2 \in S_{\alpha_2}^2$, using (4.22), we can reformulate [P1] equivalently as

$$\begin{aligned} & \max_{x^1} p_1(x^1, x^2), \\ \text{s.t.} \quad & \text{(i)} \sum_{k \in J^1} \log(\mathbb{E}_{W^{1,k}} [\Phi(g^{1,k}(x^1, W^{1,k}))]) \geq \log(\alpha_1), \\ & \text{(ii)} x^1 \in S^1. \end{aligned} \quad (4.57)$$

By introducing auxiliary variables $z^{1,k}$, we rewrite (4.57) as follows

$$\begin{aligned} & \max_{x^1} p_1(x^1, x^2), \\ \text{s.t.} \quad & \text{(i)} \mathbb{E}_{W^{1,k}} [\Phi(g^{1,k}(x^1, W^{1,k}))]) \geq e^{z^{1,k}}, \\ & \text{(ii)} \sum_{k \in J^1} z^{1,k} \geq \log(\alpha_1), \\ & \text{(iii)} z^{1,k} \leq 0, \forall k \in J^1, \\ & \text{(iv)} x^1 \in S^1. \end{aligned} \quad (4.58)$$

The optimization problem (4.58) is intractable due to the expectation term $\mathbb{E}_{W^{1,k}}$. Therefore, we solve approximately (4.58) by using the partial sampling technique. Here, we draw $N^{1,k}$ i.i.d samples $w_1^{1,k}, \dots, w_{N^{1,k}}^{1,k}$ of the univariate distribution of $W^{1,k}$. Thanks to these samples, we get an approximation of the expectation value by using Monte-Carlo method. We solve (4.58), where the constraint (i) is replaced by the following constraint

$$\frac{1}{N^{1,k}} \sum_{t=1}^{N^{1,k}} [\Phi(g^{1,k}(x^1, w_t^{1,k}))] \geq e^{z^{1,k}}. \quad (4.59)$$

Let $C^k(x^1, W^{1,k}) = \Phi(g^{1,k}(x^1, W^{1,k}))$. It is clear that C^k is a continuous function. Moreover, $C^k(x^1, \cdot)$ is upper bounded by the identity function $1_{(w_l^{1,k}, w_u^{1,k})}$ which is an integrable function. Using Theorem 7.48, [208], $\mathbb{E}_{W^{1,k}}[C^k(\cdot, W^{1,k})]$ is continuous and the average sample converges to $\mathbb{E}_{W^{1,k}}[C^k(\cdot, W^{1,k})]$ w.p.1 uniformly on S^1 . Using Theorem 5.3, [208], by solving optimization problems, where the constraint (i) of (4.58) is replaced by the constraint (4.59), we obtain a sequence of optimal values, which converges to the optimal value of (4.58) when the number of samples goes to infinity. By introducing new auxiliary variables $y_t^{1,k}$, the constraint (4.59) is equivalent to the following constraints

$$\begin{aligned} & \text{(i)} \quad \Phi(g^{1,k}(x^1, w_t^{1,k})) \geq y_t^{1,k}, \\ & \text{(ii)} \quad \sum_{t=1}^{N^{1,k}} y_t^{1,k} \geq N^{1,k} \cdot e^{z^{1,k}}, \\ & \text{(iii)} \quad y_t^{1,k} \geq 0, \forall k \in J^1, 1 \leq t \leq N^{1,k}. \end{aligned} \tag{4.60}$$

Then, the optimization problem [P1] can be rewritten as

$$\begin{aligned} & \max_{x^1} \quad p_1(x^1, x^2), \\ & \text{s.t.} \quad \text{(i)} \quad \Phi(g^{1,k}(x^1, w_t^{1,k})) \geq y_t^{1,k}, \\ & \quad \text{(ii)} \quad \sum_{t=1}^{N^{1,k}} y_t^{1,k} \geq N^{1,k} \cdot e^{z^{1,k}}, \\ & \quad \text{(iii)} \quad \sum_{k \in J^1} z^{1,k} \geq \log(\alpha_1), \\ & \quad \text{(iv)} \quad z^{1,k} \leq 0, y_t^{1,k} \geq 0, \forall k \in J^1, 1 \leq t \leq N^{1,k}, x^1 \in S^1. \end{aligned} \tag{4.61}$$

From (4.51) in the proof of Lemma 4.15, if Assumption 4.6 holds, $\Phi(g^{i,k}(\cdot, W^{i,k}))$ is a concave function on any segment, which does not contain zero. Since $S^1 \subset \mathbb{R}_{++}^{d_1}$, it is clear that $\Phi(g^{1,k}(\cdot, w_t^{1,k}))$ is a concave function on S^1 , for any $t = 1, \dots, N^{1,k}$, $k \in J^1$. Hence, (4.61) is a convex optimization. Similarly, the optimization problem [P2] can be rewritten as

$$\begin{aligned} & \max_{x^2} \quad p_2(x^1, x^2), \\ & \text{s.t.} \quad \text{(i)} \quad \Phi(g^{2,k}(x^2, w_t^{2,k})) \geq y_t^{2,k}, \\ & \quad \text{(ii)} \quad \sum_{t=1}^{N^{2,k}} y_t^{2,k} \geq N^{2,k} \cdot e^{z^{2,k}}, \\ & \quad \text{(iii)} \quad \sum_{k \in J^2} z^{2,k} \geq \log(\alpha_2), \\ & \quad \text{(iv)} \quad z^{2,k} \leq 0, y_t^{2,k} \geq 0, \forall k \in J^2, 1 \leq t \leq N^{2,k}, x^2 \in S^2. \end{aligned} \tag{4.62}$$

For numerical results, we consider a competition model of two firms in a same financial market. Let $J = \{1, \dots, K\}$ be the set of portfolios and \mathcal{A}_k be the set of assets in portfolio k , for $k \in J$. We assume that both firms invest in the same set of portfolios J and the portfolios are pairwise disjoint. Each firm i invests

its money in the assets. Let x_{kj}^i be the amount of money that firm i invests in asset j of portfolio k . Let $x_k^i = (x_{kj}^i)_{j \in \mathcal{A}_k}$ be the investment vector of firm i in portfolio k and $x^i = (x_k^i)_{k \in J}$ be the strategy vector of firm i . The strategy set of firm i (or the set of investments) is defined as

$$S^i = \left\{ x^i \mid \sum_{j \in \mathcal{A}_k, k \in J} x_{kj}^i \leq B^i, x_{kj}^i \geq \epsilon_{kj}^i, \text{ for any } k \in J, j \in \mathcal{A}_k \right\},$$

where B^i is the budget of firm i and ϵ_{kj}^i is the minimal amount required that firm i must invest in asset j of portfolio k . It is clear that S^i is a convex and compact set with strictly positive components, for any $i = 1, 2$. The vector $(x^1, x^2) \in S^1 \times S^2$ represents a strategy profile (or an investment profile) of both firms. Let $L_k^i = (L_{kj}^i)_{j \in \mathcal{A}_k}$ be a random loss vector of firm i from portfolio k . Then, for a given investment vector x_k^i , the random loss incurred by firm i from portfolio k is $(L_k^i)^\top x_k^i$. Let D_k^i be the maximal loss level of firm i from portfolio k and firm i wants to keep its random loss below this level at probability level α_i , i.e.,

$$\mathbb{P} \left\{ (x_k^i)^\top L_k^i \leq D_k^i, k \in J \right\} \geq \alpha_i, \forall i = 1, 2.$$

Hence, the strategy set of firm i is defined as

$$S_{\alpha_i}^i = \left\{ x^i \in S^i \mid \mathbb{P} \left\{ (x_k^i)^\top L_k^i \leq D_k^i, k \in J \right\} \geq \alpha_i \right\}$$

We assume that for any $i = 1, 2$ and $k \in J$, the random loss vector L_k^i follows a normal mean-variance mixture distribution. Let $R_k^i = -L_k^i$ be the random return vector of firm i from portfolio k . We consider the case where each firm wants to minimize their transaction cost which is incurred due to trades from multiple firms. The transaction cost of a firm usually also depends on the investment of other firms [132]. Therefore, for a given strategy profile (x^1, x^2) , we consider the quadratic transaction cost as follows

$$\text{TC}^i(x^1, x^2) = \sum_{k \in J} (x_k^1 + x_k^2)^\top \Omega_k^i (x_k^1 + x_k^2),$$

where $\text{TC}^i(x^1, x^2)$ is the transaction cost of firm i at strategy profile (x^1, x^2) , the positive semidefinite matrix Ω_k^i represents the market impact of portfolio k on firm i whose entry at position (r, s) is the impact of the liquidity of asset r on the liquidity of asset s . The same formulation of transaction cost has been considered in [132]. The payoff function of firm i is defined as follows

$$u^i(x^1, x^2) = \left(\sum_{k \in J} \mathbb{E}(R_k^i)^\top x_k^i \right) - \text{TC}^i(x^1, x^2),$$

where $\sum_{k \in J} \mathbb{E}(R_k^i)^\top x_k^i$ is the expected return of firm i at strategy profile (x^1, x^2) . It is clear that the payoff function of firm i is a continuous function of the strategy profile and concave w.r.t its strategy vector, for every fixed strategy vector of the other firm.

In our case study, we consider two firms with three portfolios where each portfolio consists three assets, i.e., $J = \{1, 2, 3\}$ and $\mathcal{A}_k = \{1, 2, 3\}$, for any $k \in J$. We assume that the random loss vectors follows normal mean-variance mixture distributions, i.e., for any $i = 1, 2$ and $k \in J$, $L_k^i \sim \text{NMVM}(\mu^{i,k}, \gamma^{i,k}, \Sigma^{i,k}, W^{i,k})$, where $\mu^{i,k}$ is a 3×1 vector taken on $[-0.25, 0]^3$ and $\gamma^{i,k}$ is a 3×1 vectors taken on $[0, 0.025]^3$. $\Sigma^{i,k}$ is a 3×3 positive

definite matrix with all eigen values belong to $[5, 10]$ and high values on the main diagonal. In order to generate $\Sigma^{i,k}$, we use the following formula

$$\Sigma^{i,k} = \frac{AA^T}{3 \times \max(1, \lambda_{\max})} + \beta \times I_3,$$

where A is a 3×3 random matrix whose all the entries are real numbers belonging to $[0, 1]$, λ_{\max} is the largest eigenvalue of the semidefinite positive matrix $\frac{AA^T}{3}$, β is a real number taken on $[5, 9]$ and I_3 is 3×3 identity matrix. We take negative values of location parameters $\mu^{i,k}$ and high values on the main diagonal of covariance matrix $\Sigma^{i,k}$ because firms gain positive return in expectation if they decide to invest but they have to make risky decision. $W^{i,k}$ follows an uniform distribution with support in $[w_l^{i,k}, w_u^{i,k}]$, where $w_l^{i,k}$ is taken on $[0.85, 1]$ and $w_u^{i,k}$ is taken on $[1, 1.15]$. For the above choice of parameters, the threshold levels $(\alpha_i^*)_{i=1,2}$ always belong to $[0.96, 0.98]$. Then, we take the probability levels $\alpha_1 = \alpha_2 = 0.985$ such that Assumption 4.6 holds. The other parameters are given by $B^1 = B^2 = 10$, $\epsilon_{kj}^1 = \epsilon_{kj}^2 = 0.1$, $D_k^1 = 1$, $D_k^2 = 1.5$, Ω_k^i is a 3×3 positive definite matrix, generated by the following formula

$$\Omega_k^i = \frac{BB^T}{3},$$

where B is a 3×3 random matrix whose all the entries are real numbers belonging to $[0, 1]$. Then, the payoff function of two firms are calculated by

$$u^i(x^1, x^2) = - \left[\sum_{k \in J} (\mu^{i,k})^T x_k^i \right] - \sum_{k \in J} (x_k^1 + x_k^2)^T \Omega_k^i (x_k^1 + x_k^2).$$

All the numerical results below are performed using Python 3.8.8 on an Intel Core i5-1135G7, Processor 2.4 GHz (8M Cache, up to 4.2 GHz), RAM 16G, 512G SSD. We generate an instance of the above competition model and compute the Nash equilibrium using best response algorithm. In order to find the best response of each player, we solve convex optimization problems (4.61) and (4.62) using **cp** solver in CVXOPT, with number of data points in Monte-Carlo method $N^{1,k} = N^{2,k} = 50$. The data points $\{w_j^{i,k}, j = 1, \dots, N^{i,k}\}$ are uniformly generated on $[w_l^{i,k}, w_u^{i,k}]^{N^{i,k}}$. The algorithm converges to a Nash equilibrium point (x^{1*}, x^{2*}) given by

$$\begin{aligned} x^{1*} &= [(0.48, 0.51, 0.63), (0.47, 0.31, 0.55), (0.67, 0.25, 0.4)], \\ x^{2*} &= [(0.52, 0.31, 0.73), (0.57, 0.54, 0.57), (0.47, 0.55, 0.3)], \end{aligned}$$

and the total CPU time is 91 seconds. Figure 4.4 shows that best response algorithm converges for the above instance because the payoffs of both firms converge after a few iterations. We also present the time analysis by considering a larger model with 50 portfolios and each portfolio has 50 assets. The parameters are similarly generated as above. We generate 20 instances and for each instance, the best response algorithm converges to a Nash equilibrium. The average total CPU time is 1897.1 seconds. The total CPU time of each instance are represented in Figure 4.5.

4.7 . Conclusion

We study an n - player non-cooperative CCG, where the strategy set of each player is defined by a joint chance constraint. The random constraint vectors are either dependent and follow elliptical distributions or

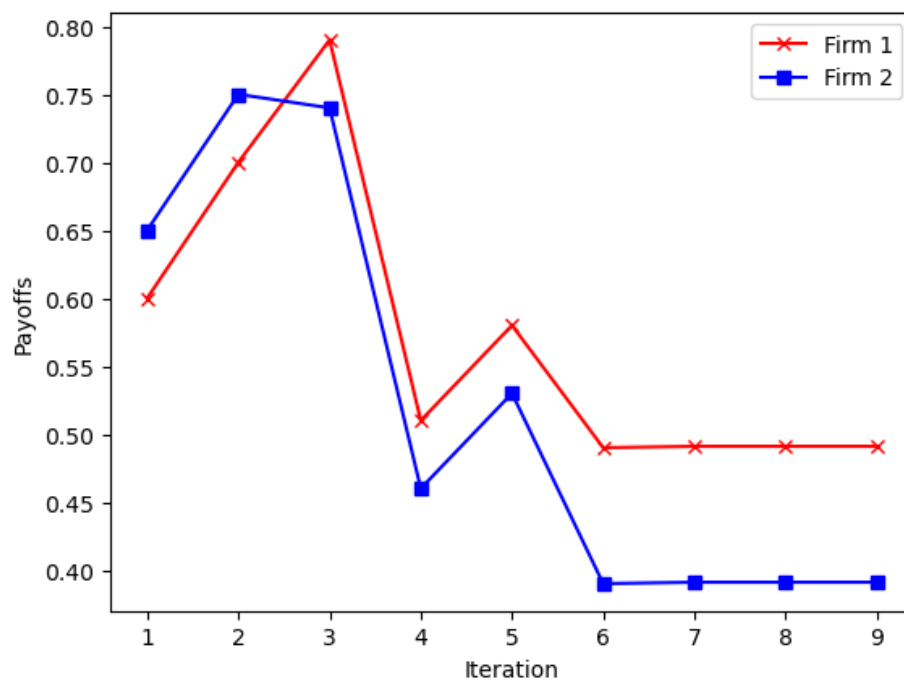


Figure 4.4: Convergence of payoffs to a Nash equilibrium.

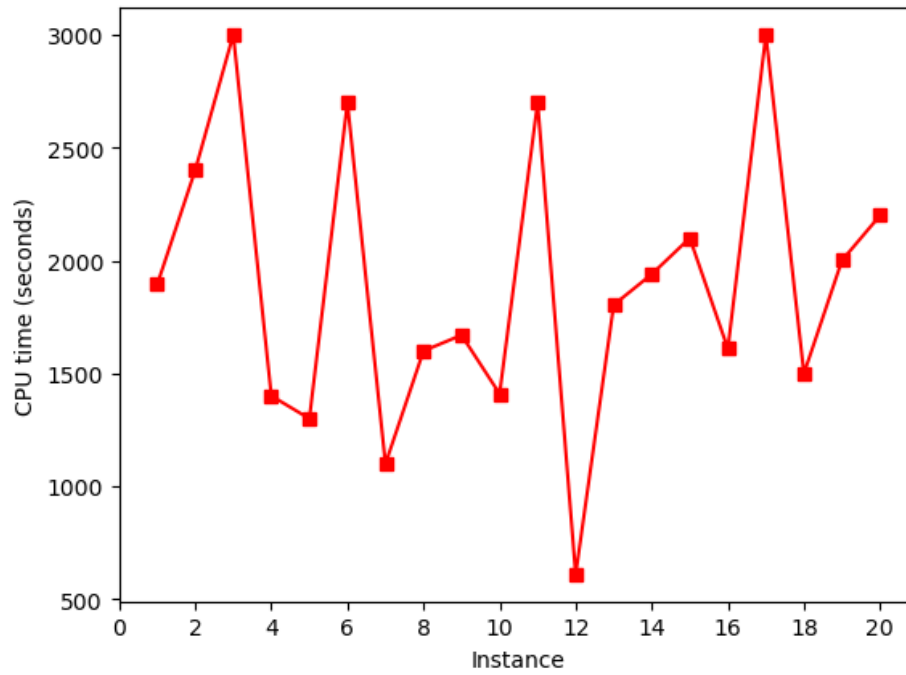


Figure 4.5: Total CPU time of 20 instances of CCG model with 50 portfolios and 50 assets in each portfolios.

independent and follow normal mean-variance mixture distributions. We derive a new reformulation of the feasible strategy set of each player and show the existence of a Nash equilibrium under mild assumptions on the payoff of each player and the probability levels. We compute a Nash equilibrium of a competition of two firms in financial market with randomly generated data using best-response algorithm. It is clear from our time analysis that a Nash equilibrium can be computed efficiently in a large model up to 50 portfolios and each portfolio contains up to 50 assets.

This chapter corresponds to the reference [167] and a paper submitted to Applied Mathematics and Computation.

Chapter 5 :Distributionally Robust Chance-Constrained Zero-Sum Games

In this chapter, we consider a two-player zero-sum game with random linear chance constraints whose distributions are known to belong to moment based uncertainty sets or statistical distance based uncertainty sets. We propose a reformulation of the chance constraints using distributionally chance-constrained optimization framework. We show that there exists a saddle point equilibrium of the game, which is the optimal solution of a primal-dual pair of second-order cone programs. As an application, we present a competition of two firms in financial market to simulate our theoretical results.

This chapter serves as a valuable addition to the previous chapters. In the preceding chapters, we focused on establishing the existence of Nash equilibrium and presented an algorithm that, while effective, did not guarantee convergence to an equilibrium point. However, in this chapter, we make significant progress by demonstrating that the equilibrium point is indeed unique. Furthermore, we introduce a novel approach for computing this equilibrium point by solving a primal-dual pair of second-order cone programming (SOCP) problems. By employing this method, we not only ensure the convergence to a unique equilibrium point but also provide a practical and efficient computational solution.

5.1 . Introduction

In many real life applications, the strategy sets are restricted by random linear constraints. By assuming that the probability of these random linear constraints is at least more than a certain probability level, we have a chance constraint. The distribution of random factors in chance constraints can be known exactly or unknown, which leads to different approaches to define a game. The true distribution of random factors is usually assumed to be elliptically distributed, which includes many known distributions, e.g., Gaussian distribution, Laplace distribution, Kotz distribution or Pearson distribution. Otherwise, in unknown distribution case, the true distribution of random factors is assumed to belong to an uncertainty set, where only partial information of the distribution is known based on historical data and we call such games distributionally robust chance-constrained games. A two-player zero-sum game is modeled using continuous strategy sets, where the sum of two players' payoffs is zero. Consequently, it is defined using a single payoff function, where one player plays the role of maximizer and another player plays the role of minimizer. More commonly, a zero-sum game is introduced with a payoff matrix, where the rows and the columns are the actions of player 1 and player 2, respectively. A Saddle Point Equilibrium (SPE) is the solution concept to study the zero-sum games and it exists in the mixed strategies [243].

In the conference paper [168], we considered a two player zero-sum game with continuous strategy set, where the payoff function has a special form and the strategies of each player are modeled using random linear constraints reformulated as distributionally robust chance constraints. We proposed an SOCP reformulation of distributionally robust chance constraints under two uncertainty sets based on the partial information about the mean vectors and covariance matrices of the random constraint vectors. We showed the existence of an SPE and characterized it as the optimal solution of a primal-dual pair of SOCPs. The conference paper has some shortcomings, e.g., the payoff function has a quadratic form, the uncertainty sets are mainly constructed based on moments from historical data and it lacks of numerical results which allow us to compare different uncertainty sets. As an extended version of [168], our contribution is as follows:

- We study a more general framework as compared to [168] by considering two types of uncertainty sets based on either the partial information on the mean vectors and covariance matrices of the random

constraint vectors (moment based uncertainty sets) or the statistical distance between their true distribution and a nominal distribution (statistical based uncertainty sets). We show that in both cases, there exists an SPE of the game and an SPE problem is equivalent to a primal-dual pair of SOCPs.

- As an application, we present a competition problem of two firms in financial market and we show our numerical results using randomly generated data to compare different uncertainty sets considered in the chapter.

We keep the same form of payoff function as considered in the conference paper, since we need a different game model for different form of payoff function, which would break the uniformity of our results. We might consider this point in future works.

The rest of this chapter is organized as follows. We present related works in Section 5.2. The definition of a distributionally robust zero-sum game is given in Section 5.3. Section 5.4 presents the reformulation of distributionally robust chance constraints as second order cone constraints under different uncertainty sets. Section 5.5 outlines a primal-dual pair of SOCPs whose optimal solutions constitute an SPE of the game. Section 5.6 presents a competition of two firms in financial market as and shows numerical results. Conclusion and future works are presented in Section 5.7.

5.2 . Related work

In this section, we introduce previous studies on chance-constrained games. Dantzig and later Adler showed the equivalence between linear programming problems and two-player zero-sum games [3][52]. Charnes [39] generalized the zero-sum game considered in [243] by introducing linear inequality constraints on the mixed strategies of both the players and called it a constrained zero-sum game. An SPE of a constrained zero-sum game can be obtained from the optimal solutions of a primal-dual pair of linear programs [39]. Singh and Lissner [218] considered a stochastic version of constrained zero-sum game considered by Charnes [39], where the mixed strategies of each player are restricted by random linear inequality constraints, which are modelled using chance constraints. When the random constraint vectors follow a multivariate elliptically symmetric distribution, the zero-sum game problem is equivalent to a primal-dual pair of Second-Order Cone Programs (SOCPs) [218]. Nash equilibrium is the generalization of SPE and it is used as a solution concept for the general-sum games [160] [161]. Under certain conditions on payoff functions and strategy sets, there always exists a Nash equilibrium [55]. The general-sum games under uncertainties are considered in the literature [200, 214, 215, 211, 219], which capture both risk neutral and risk averse situations. Liu et al. [145] studied chance-constrained DRO in geometric optimization. To the best of our knowledge, the distributionally robust chance-constrained approach has been widely studied in the literature but still not completed in game setup. In this chapter, we want to apply different approaches in the literature to define uncertainty sets in a distributionally robust chance-constrained game and compare the performance of these approaches by simulation using randomly generated data models.

5.3 . The model

We consider a two player zero-sum game, where each player has continuous strategy set. Let $C^1 \in \mathbb{R}^{K_1 \times m}$, $C^2 \in \mathbb{R}^{K_2 \times n}$, $d^1 \in \mathbb{R}^{K_1}$ and $d^2 \in \mathbb{R}^{K_2}$. We consider $X = \{x \in \mathbb{R}^m \mid C^1 x = d^1, x \geq 0\}$ and $Y = \{y \in \mathbb{R}^n \mid C^2 y = d^2, y \geq 0\}$ as the strategy sets of player 1 and player 2, respectively. We assume that X and Y are compact sets. Let $u : X \times Y \rightarrow \mathbb{R}$ be a payoff function associated to the zero-sum game and we assume that player 1 (resp. player 2) is interested in maximizing (resp. minimizing) $u(x, y)$ for a fixed strategy y (resp. x) of player

2 (resp. player 1). For a given strategy pair $(x, y) \in X \times Y$, the payoff function $u(x, y)$ is given by

$$u(x, y) = x^\top G y + g^\top x + h^\top y, \quad (5.1)$$

where $G \in \mathbb{R}^{m \times n}$, $g \in \mathbb{R}^m$ and $h \in \mathbb{R}^n$. The first term of (5.1) results from the interaction between both the players whereas the second and third term represents the individual impact of player 1 and player 2 on the game, respectively. The strategy sets are often restricted by random linear constraints, which are modeled using chance constraints. The chance constraint based strategy sets appear in many practical problems, e.g., risk constraints in portfolio optimization [112]. In this chapter, we consider the case, where the strategies of player 1 satisfy the following random linear constraints,

$$(a_k^1)^\top x \leq b_k^1, \quad k = 1, 2, \dots, p, \quad (5.2)$$

whilst the strategies of player 2 satisfy the following random linear constraints

$$(a_l^2)^\top y \geq b_l^2, \quad l = 1, 2, \dots, q. \quad (5.3)$$

Let $\mathcal{I}_1 = \{1, 2, \dots, p\}$ and $\mathcal{I}_2 = \{1, 2, \dots, q\}$ be the index sets for the constraints of player 1 and player 2, respectively. For each $k \in \mathcal{I}_1$ and $l \in \mathcal{I}_2$, the vectors a_k^1 and a_l^2 are random vectors defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We consider the case, where the only information we have about the distributions of a_k^1 and a_l^2 is that they belong to some uncertainty sets \mathcal{D}_k^1 and \mathcal{D}_l^2 , respectively. The uncertainty sets \mathcal{D}_k^1 and \mathcal{D}_l^2 , are constructed based on the partially available information on the distributions of a_k^1 and a_l^2 , respectively. Using the worst case approach, the random linear constraints (5.2) and (5.3) can be formulated as distributionally robust chance constraints given by

$$\inf_{F_k^1 \in \mathcal{D}_k^1} \mathbb{P} \left((a_k^1)^\top x \leq b_k^1 \right) \geq \alpha_k^1, \quad \forall k \in \mathcal{I}_1, \quad (5.4)$$

and

$$\inf_{F_l^2 \in \mathcal{D}_l^2} \mathbb{P} \left((-a_l^2)^\top y \leq -b_l^2 \right) \geq \alpha_l^2, \quad \forall l \in \mathcal{I}_2, \quad (5.5)$$

where α_k^1 and α_l^2 are the confidence levels of player 1 and player 2 for k th and l th constraints, respectively, and F_k^1 , F_l^2 are the distributions of a_k^1 and a_l^2 . Therefore, for a given $\alpha^1 = (\alpha_k^1)_{k \in \mathcal{I}_1}$ and $\alpha^2 = (\alpha_l^2)_{l \in \mathcal{I}_2}$, the feasible strategy sets of player 1 and player 2 are given by

$$S_{\alpha^1}^1 = \left\{ x \in X \mid \inf_{F_k^1 \in \mathcal{D}_k^1} \mathbb{P} \{ (a_k^1)^\top x \leq b_k^1 \} \geq \alpha_k^1, \quad \forall k \in \mathcal{I}_1 \right\}, \quad (5.6)$$

and

$$S_{\alpha^2}^2 = \left\{ y \in Y \mid \inf_{F_l^2 \in \mathcal{D}_l^2} \mathbb{P} \{ (-a_l^2)^\top y \leq -b_l^2 \} \geq \alpha_l^2, \quad \forall l \in \mathcal{I}_2 \right\}. \quad (5.7)$$

We call the zero-sum game with the strategy set $S_{\alpha^1}^1$ for player 1 and the strategy set $S_{\alpha^2}^2$ for player 2 as a distributionally robust zero-sum game. We denote this game by Z_α . A strategy pair $(x^*, y^*) \in S_{\alpha^1}^1 \times S_{\alpha^2}^2$ is called an SPE of the game Z_α at $\alpha = (\alpha^1, \alpha^2) \in [0, 1]^p \times [0, 1]^q$, if

$$u(x, y^*) \leq u(x^*, y^*) \leq u(x^*, y), \quad \forall x \in S_{\alpha^1}^1, \quad y \in S_{\alpha^2}^2. \quad (5.8)$$

5.4 . Reformulation of distributionally robust chance constraints

We consider five different uncertainty sets based on the partial information about the mean vectors and covariance matrices of the random constraint vectors a_k^i , $i = 1, 2$, $k \in \mathcal{I}_i$ and four different uncertainty sets based on the statistical distance between the distribution of a_k^i and a nominal distribution. For each uncertainty set, the distributionally robust chance constraints (5.4) and (5.5) are reformulated as second-order cone (SOC) constraints.

5.4.1 . Moment based Uncertainty Sets

We consider five moment based uncertainty sets defined as follows.

Uncertainty set with known mean and known covariance matrix

In some situations, we do not know exactly the true distribution of the random constraint vectors a_k^i , for all $k \in \mathcal{I}_i$, $i = 1, 2$. We can only obtain some information of the underlying distribution from historical data. For example, by observing a sufficiently large number of data, we deduce the values of mean vector and covariance matrix of a_k^i approximated by the sample mean μ_k^i and the sample covariance matrix Σ_k^i . We consider an uncertainty set, which includes all distributions F_k^i with mean vector μ_k^i and covariance matrix Σ_k^i defined as follows

$$\mathcal{D}_k^{1,i}(\mu_k^i, \Sigma_k^i) = \left\{ F_k^i \mid \begin{array}{l} \text{The distribution of } a_k^i \text{ is } F_k^i \\ E[a_k^i] = \mu_k^i \\ Cov[a_k^i] = \Sigma_k^i \end{array} \right\}, \quad (5.9)$$

We assume that for each $i = 1, 2$ and $k \in \mathcal{I}_i$, the true distribution of a_k^i belongs to the uncertainty set $\mathcal{D}_k^{1,i}(\mu_k^i, \Sigma_k^i)$ and the matrix Σ_k^i is a positive definite matrix. This uncertainty set has been widely considered in the literature, e.g., [83]. We present an SOC reformulation of the constraints (5.4) and (5.5) by the following lemma.

Lemma 5.20. *The constraints (5.4) and (5.5) are equivalent to (5.10) and (5.11), respectively, given by*

$$\begin{aligned} & (\mu_k^1)^T x + \sqrt{\frac{\alpha_k^1}{1 - \alpha_k^1}} \|(\Sigma_k^1)^{\frac{1}{2}} x\|_2 \leq b_k^1, \\ & \forall k \in \mathcal{I}_1, \end{aligned} \quad (5.10)$$

$$\begin{aligned} & -(\mu_k^2)^T y + \sqrt{\frac{\alpha_k^2}{1 - \alpha_k^2}} \|(\Sigma_k^2)^{\frac{1}{2}} y\|_2 \leq -b_k^2, \\ & \forall k \in \mathcal{I}_2. \end{aligned} \quad (5.11)$$

Remark 5.5. *An SOC constraint is the set of points $x \in \mathbb{R}^n$ such that the following inequality holds*

$$\|Ax + b\|_2 \leq c^T x + d,$$

where $A \in \mathbb{R}^{m \times n}$ is an $m \times n$ real matrix, $b \in \mathbb{R}^m$ is an $m \times 1$ real vector, $c \in \mathbb{R}^n$ is an $n \times 1$ real vector and $d \in \mathbb{R}$ is a real number, $\|\cdot\|_2$ denotes the Euclidean norm. It is clear that (5.10) and (5.11) are equivalent to SOC constraints. An SOC reformulation is useful since optimization problems with SOC constraints can be solved efficiently by known algorithms in polynomial time.

Proof. Using the one-sided Chebyshev inequality, we have

$$\inf_{F_k^1 \in \mathcal{D}_k^{1,i}(\mu, \Sigma)} \mathbb{P}\{(a_k^1)^\top x \leq b_k^1\} = \begin{cases} 1 - \frac{1}{1 + \frac{((\mu_k^1)^\top x - b_k^1)^2}{(x^\top \Sigma_k^1 x)}}, \\ \text{if } (\mu_k^1)^\top x \leq b_k^1, \\ 0, \text{ otherwise.} \end{cases}$$

The bound of one-sided Chebyshev inequality can be achieved by a two-point distribution given by equation (2) of [203]. For the case $(\mu_k^1)^\top x > b_k^1$,

$$\inf_{F_k^1 \in \mathcal{D}_k^{1,i}(\mu, \Sigma)} \mathbb{P}\{a_k^1 x \leq b_k^1\} = 0,$$

which makes constraint (5.4) infeasible for any $\alpha_1 > 0$. Therefore, for $x \in S_{\alpha_1}^1$, the condition $(\mu_k^1)^\top x \leq b_k^1$ always holds and the constraint (5.4) is equivalent to

$$1 - \frac{1}{1 + ((\mu_k^1)^\top x - b_k^1)^2 / (x^\top \Sigma_k^1 x)} \geq \alpha_k^1.$$

The above inequality can be reformulated as (5.10). Similarly, we can show that (5.5) is equivalent to (5.11). \square

Uncertainty set with known mean and unknown covariance matrix

For all $i = 1, 2$ and $k \in \mathcal{I}_i$, we consider the case, where the mean vector of the random vector a_k^i is known exactly (approximated by the sample mean μ_k^i) but the covariance matrix is unknown due to several reasons, e.g., the lack of data. We assume that it is only known to belong to a positive semidefinite cone defined with a linear matrix inequality as follows

$$\text{Cov}[a_k^i] \preceq \gamma_k^i \Sigma_k^i,$$

where $\gamma_k^i > 0$ is a strictly positive real number, Σ_k^i is a positive definite matrix, for the given matrices B_1 and B_2 , $B_1 \preceq B_2$ implies that $B_2 - B_1$ is a positive semidefinite matrix. In practical applications, we usually approximate the matrix Σ_k^i by the sample covariance matrix. The parameter γ_k^i is used in controlling the uncertainty level, i.e., high value of γ_k^i implies a large number of distributions in the uncertainty set, which deals uncertain factors in a more secure way. We consider an uncertainty set, which includes all distributions F_k^i with mean vector μ_k^i and covariance matrix satisfied the above constraint as follows

$$\mathcal{D}_k^{2,i}(\mu_k^i, \Sigma_k^i) = \left\{ F_k^i \left| \begin{array}{l} \text{The distribution of } a_k^i \text{ is } F_k^i \\ E[a_k^i] = \mu_k^i \\ \text{Cov}[a_k^i] \preceq \gamma_k^i \Sigma_k^i \end{array} \right. \right\}. \quad (5.12)$$

This uncertainty set is considered in [46]. We assume that for each $i = 1, 2$ and $k \in \mathcal{I}_i$, the true distribution of a_k^i belongs to the uncertainty set $\mathcal{D}_k^{2,i}(\mu_k^i, \Sigma_k^i)$. We present an SOC reformulation of the constraints (5.4) and (5.5) by the following lemma.

Lemma 5.21. *The constraints (5.4) and (5.5) are equivalent to (5.13) and (5.14), respectively, given by*

$$\begin{aligned} (\mu_k^1)^\top x + \sqrt{\gamma_k^1} \sqrt{\frac{\alpha_k^1}{1 - \alpha_k^1}} \|(\Sigma_k^1)^{\frac{1}{2}} x\|_2 &\leq b_k^1, \\ \forall k \in \mathcal{I}_1, \end{aligned} \quad (5.13)$$

$$\begin{aligned}
& -(\mu_k^2)^T y + \sqrt{\gamma_k^2} \sqrt{\frac{\alpha_k^2}{1 - \alpha_k^2}} \|(\Sigma_k^2)^{\frac{1}{2}} y\|_2 \leq -b_k^2, \\
& \forall k \in \mathcal{I}_2.
\end{aligned} \tag{5.14}$$

Proof. Based on the structure of uncertainty set (5.12), the constraint (5.4) can be written as

$$\inf_{(\mu, \Sigma) \in \mathcal{U}_k^1} \inf_{F_k^1 \in \mathcal{D}_k^{1,i}(\mu, \Sigma)} \mathbb{P}\{(a_k^1)^T x \leq b_k^1\} \geq \alpha_k^1,$$

where

$$\mathcal{U}_k^1 = \{(\mu, \Sigma) \mid \mu = \mu_k^1, \Sigma \preceq \gamma_k^1 \Sigma_k^1\}.$$

Here, the inner infimum is taken over all distributions with same value of mean vector and covariance matrix. The outer infimum is taken over all couples (μ, Σ) satisfying the conditions in (5.12). Using the similar arguments as in the Lemma 5.20, the constraint (5.4) is equivalent to

$$\frac{b_k^1 - (\mu_k^1)^T x}{\max_{\Sigma \preceq \gamma_k^1 \Sigma_k^1} \sqrt{x^T \Sigma x}} \geq \sqrt{\frac{\alpha_k^1}{1 - \alpha_k^1}}. \tag{5.15}$$

The above inequality (5.15) can be reformulated as (5.13). Similarly, we can show that (5.5) is equivalent to (5.14). \square

Uncertainty set with unknown mean and unknown covariance matrix

For all $i = 1, 2$ and $k \in \mathcal{I}_i$, we consider the case, where both mean vector and covariance matrix of a_k^i are unknown. From historical data, we obtain the sample mean μ_k^i and the sample covariance matrix Σ_k^i . We deal the uncertainty level in a secure way by assuming that the mean vector and the covariance matrix of a_k^i are not exactly the same as its sample mean and sample covariance matrix. The mean vector lies in an ellipsoid of size $\gamma_{k1}^i \geq 0$ centered at μ_k^i defined by the following constraint

$$(\mathbb{E}[a_k^i] - \mu_k^i)^T (\Sigma_k^i)^{-1} (\mathbb{E}[a_k^i] - \mu_k^i) \leq \gamma_{k1}^i,$$

and the covariance matrix of a_k^i lies in a positive semidefinite cone defined as follows

$$\text{Cov}[a_k^i] \preceq \gamma_{k2}^i \Sigma_k^i.$$

where $\gamma_{k2}^i > 0$ and Σ_k^i is a positive definite matrix. The parameters γ_{k1}^i and γ_{k2}^i are used in controlling the uncertainty level. If $\gamma_{k1}^i = 0$, the mean vector is exactly the same as its sample mean. We consider an uncertainty set, which includes all distributions F_k^i with mean vector and covariance matrix satisfied the above constraints as follows

$$\mathcal{D}_k^{3,i}(\mu_k^i, \Sigma_k^i) = \left\{ F_k^i \left| \begin{array}{l} \text{The distribution of } a_k^i \text{ is } F_k^i \\ (\mathbb{E}[a_k^i] - \mu_k^i)^T (\Sigma_k^i)^{-1} \\ \times (\mathbb{E}[a_k^i] - \mu_k^i) \leq \gamma_{k1}^i, \\ \text{Cov}[a_k^i] \preceq \gamma_{k2}^i \Sigma_k^i \end{array} \right. \right\}, \tag{5.16}$$

The uncertainty set (5.16) is considered in [57]. We assume that for each $i = 1, 2$ and $k \in \mathcal{I}_i$, the true distribution of a_k^i belongs to the uncertainty set $\mathcal{D}_k^{3,i}(\mu_k^i, \Sigma_k^i)$. We present an SOC reformulation of the constraints (5.4) and (5.5) by the following lemma.

Lemma 5.22. *The constraints (5.4) and (5.5) are equivalent to (5.17) and (5.18), respectively, given by*

$$\begin{aligned} & (\mu_k^1)^T x + \left(\sqrt{\frac{\alpha_k^1}{1 - \alpha_k^1}} \sqrt{\gamma_{k2}^1} + \sqrt{\gamma_{k1}^1} \right) \|(\Sigma_k^1)^{\frac{1}{2}} x\|_2 \leq b_k^1, \\ & \forall k \in \mathcal{I}_1, \end{aligned} \quad (5.17)$$

$$\begin{aligned} & -(\mu_k^2)^T y + \left(\sqrt{\frac{\alpha_k^2}{1 - \alpha_k^2}} \sqrt{\gamma_{k2}^2} + \sqrt{\gamma_{k1}^2} \right) \|(\Sigma_k^2)^{\frac{1}{2}} y\|_2 \leq -b_k^2, \\ & \forall k \in \mathcal{I}_2. \end{aligned} \quad (5.18)$$

Proof. Based on the structure of the uncertainty set (5.16), the constraint (5.4) can be written as

$$\inf_{(\mu, \Sigma) \in \tilde{\mathcal{U}}_k^1} \inf_{F_k^1 \in \mathcal{D}_k^{1,i}(\mu, \Sigma)} \mathbb{P}\{a_k^1 x \leq b_k^1\} \geq \alpha_k^1,$$

where

$$\tilde{\mathcal{U}}_k^1 = \left\{ (\mu, \Sigma) \mid \begin{array}{l} (\mu - \mu_k^1)^\top (\Sigma_k^1)^{-1} (\mu - \mu_k^1) \leq \gamma_{k1}^1, \\ \Sigma \preceq \gamma_{k2}^1 \Sigma_k^1. \end{array} \right\}.$$

Using the similar arguments as in the Lemma 5.20, the constraint (5.4) is equivalent to

$$\frac{b_k^1 + v_1(x)}{\sqrt{v_2(x)}} \geq \sqrt{\frac{\alpha_k^1}{1 - \alpha_k^1}}, \quad (5.19)$$

where

$$\begin{aligned} v_1(x) &= \begin{cases} \min_{\mu} -\mu^\top x \\ \text{s.t. } (\mu - \mu_k^1)^\top (\Sigma_k^1)^{-1} (\mu - \mu_k^1) \leq \gamma_{k1}^1, \end{cases} \\ v_2(x) &= \begin{cases} \max_{\Sigma} x^\top \Sigma x \\ \text{s.t. } \Sigma \preceq \gamma_{k2}^1 \Sigma_k^1. \end{cases} \end{aligned} \quad (5.20)$$

Let $\beta \geq 0$ be a Lagrange multiplier associated with the constraint of optimization problem (5.20). By applying the KKT conditions, the optimal solution of (5.20) is given by $\mu = \mu_k^1 + \frac{\sqrt{\gamma_{k1}^1 \Sigma_k^1} x}{\sqrt{x^\top \Sigma_k^1 x}}$ and the associated

Lagrange multiplier is given by $\beta = \sqrt{\frac{x^\top \Sigma_k^1 x}{4\gamma_{k1}^1}}$. Therefore, the corresponding optimal value $v_1(x) = -(\mu_k^1)^\top x - \sqrt{\gamma_{k1}^1} \sqrt{x^\top \Sigma_k^1 x}$. Since, $u^\top \Sigma u \leq u^\top \gamma_{k2}^1 \Sigma_k^1 u$, then, $v_2(x) = \gamma_{k2}^1 x^\top \Sigma_k^1 x$. Therefore, using (5.19), (5.4) is equivalent to (5.17). Similarly, we can show that (5.5) is equivalent to (5.18). \square

Polytopic uncertainty set

For all $i = 1, 2$ and $k \in \mathcal{I}_i$, we consider the case, where both mean vector and covariance matrix of the random vector a_k^i are unknown. From historical data, we consider M samples i.i.d of the random vector a_k^i . We obtain M sample means $\mu_{k1}^i, \dots, \mu_{kM}^i$ and M sample covariance matrix $\Sigma_{k1}^i, \dots, \Sigma_{kM}^i$, where Σ_{kj}^i is positive definite, for any $j = 1, \dots, M$. We consider polytopes $U_{\mu_k^i} = \text{Conv}(\mu_{k1}^i, \mu_{k2}^i, \dots, \mu_{kM}^i)$ and $U_{\Sigma_k^i} =$

$\text{Conv}(\Sigma_{k1}^i, \Sigma_{k2}^i, \dots, \Sigma_{kM}^i)$, where Conv denotes the convex hull. We assume that the mean vector and the covariance matrix of a_k^i are known to belong to polytopes $U_{\mu_k^i}$ and $U_{\Sigma_k^i}$, respectively. We consider an uncertainty set, which includes all distributions F_k^i defined as follows

$$\mathcal{D}_k^{4,i}(\mu_k^i, \Sigma_k^i) = \left\{ F_k^i \left| \begin{array}{l} \text{The distribution of } a_k^i \text{ is } F_k^i \\ E[a_k^i] \in U_{\mu_k^i} \\ \text{Cov}[a_k^i] \in U_{\Sigma_k^i} \end{array} \right. \right\}. \quad (5.21)$$

The uncertainty set (5.21) is considered in [83]. We assume that for each $i = 1, 2$ and $k \in \mathcal{I}_i$, the true distribution of a_k^i belongs to the uncertainty set $\mathcal{D}_k^{4,i}(\mu_k^i, \Sigma_k^i)$. We present an SOC reformulation of the constraints (5.4) and (5.5) by the following lemma.

Lemma 5.23. *The constraints (5.4) and (5.5) are equivalent to (5.22) and (5.23), respectively, given by*

$$\begin{aligned} (\mu_{kj}^1)^T x + \sqrt{\frac{\alpha_k^1}{1 - \alpha_k^1}} \|(\Sigma_{kw}^1)^{\frac{1}{2}} x\|_2 &\leq b_k^1, \\ \forall j = 1, \dots, M, w = 1, \dots, M, k \in \mathcal{I}_1, \end{aligned} \quad (5.22)$$

$$\begin{aligned} -(\mu_{kj}^2)^T y + \sqrt{\frac{\alpha_k^2}{1 - \alpha_k^2}} \|(\Sigma_{kw}^2)^{\frac{1}{2}} y\|_2 &\leq -b_k^2, \\ \forall j = 1, \dots, M, w = 1, \dots, M, k \in \mathcal{I}_2. \end{aligned} \quad (5.23)$$

Remark 5.6. *Lemma 5.23 shows that the constraint (5.4) (resp. (5.5)) is equivalent to a system of M^2 constraints in (5.22) (resp. (5.23)).*

Proof. Based on the structure of uncertainty set (5.21), the constraint (5.4) can be written as

$$\inf_{(\mu, \Sigma) \in \hat{\mathcal{U}}_k^1} \inf_{F_k^1 \in \mathcal{D}_k^{1,i}(\mu, \Sigma)} \mathbb{P}\{(a_k^1)^T x \leq b_k^1\} \geq \alpha_k^1,$$

where

$$\hat{\mathcal{U}}_k^1 = \left\{ (\mu, \Sigma) \mid \mu \in U_{\mu_k^1}, \Sigma \in U_{\Sigma_k^1} \right\}.$$

Using the similar arguments as in the Lemma 5.20, the constraint (5.4) can be reformulated as

$$\frac{\min_{\mu \in U_{\mu_k^1}} (b_k^1 - \mu^T x)}{\max_{\Sigma \in U_{\Sigma_k^1}} \sqrt{x^T \Sigma x}} \geq \sqrt{\frac{\alpha_k^1}{1 - \alpha_k^1}}. \quad (5.24)$$

The above inequality (5.24) can be reformulated as

$$\begin{aligned} \frac{b_k^1 - (\mu_{kj}^1)^T x}{\sqrt{x^T \Sigma_{kw}^1 x}} &\geq \sqrt{\frac{\alpha_k^1}{1 - \alpha_k^1}}, \\ \forall j = 1, \dots, M, w = 1, \dots, M, k \in \mathcal{I}_1, \end{aligned}$$

which is equivalent to (5.22). Similarly, we can show that (5.5) is equivalent to (5.23). \square

Uncertainty set with componentwise bounds on mean vector and covariance matrix

For all $i = 1, 2$ and $k \in \mathcal{I}_i$, we consider the case, where the mean vector and the covariance matrix of a_k^i are unknown. We obtain from historical data, a sample mean vector μ_k^i and a sample covariance matrix Σ_k^i . We do not approximate the mean vector and the covariance matrix of a_k^i by its sample mean vector and sample covariance matrix, but we deal the uncertainty level by a more secure way. For each $j = 1, \dots, m$, we assume that the j^{th} -component of the mean vector of a_k^i lies in a ball of radius $\epsilon_{\mu,k}^i(j) \geq 0$, centered at the j^{th} -component of the sample mean vector μ_k^i , which can be reformulated as follows

$$\mu_k^i - \epsilon_{\mu,k}^i \leq \mathbb{E}[a_k^i] \leq \mu_k^i + \epsilon_{\mu,k}^i,$$

where $\epsilon_{\mu,k}^i = (\epsilon_{\mu,k}^i(1), \dots, \epsilon_{\mu,k}^i(m))$ is an $m \times 1$ vector and the above inequalities are understood componentwise. Similarly, for each $j = 1, \dots, m$ and $w = 1, \dots, m$, we assume that the (j, w) -entry of the covariance matrix of a_k^i lies in a ball of radius $\epsilon_{\Sigma,k}^i(j, w) \geq 0$, centered at the (j, w) -entry of the sample covariance matrix Σ_k^i , which can be reformulated as follows

$$\Sigma_k^i - \epsilon_{\Sigma,k}^i \leq \text{Cov}[a_k^i] \leq \Sigma_k^i + \epsilon_{\Sigma,k}^i,$$

where $\epsilon_{\Sigma,k}^i = (\epsilon_{\Sigma,k}^i(j, w))_{1 \leq j, w \leq m}$ is an $m \times m$ matrix. Let $\mu_{k-}^i = \mu_k^i - \epsilon_{\mu,k}^i$, $\mu_{k+}^i = \mu_k^i + \epsilon_{\mu,k}^i$, $\Sigma_{k-}^i = \Sigma_k^i - \epsilon_{\Sigma,k}^i$ and $\Sigma_{k+}^i = \Sigma_k^i + \epsilon_{\Sigma,k}^i$. We consider an uncertainty set, which includes all distributions F_k^i defined as follows

$$\mathcal{D}_k^{5,i}(\mu_k^i, \Sigma_k^i) = \left\{ F_k^i \left| \begin{array}{l} \text{The distribution of } a_k^i \text{ is } F_k^i \\ \mu_{k-}^i \leq \mathbb{E}[a_k^i] \leq \mu_{k+}^i, \\ \Sigma_{k-}^i \leq \text{Cov}[a_k^i] \leq \Sigma_{k+}^i \end{array} \right. \right\}, \quad (5.25)$$

Since Σ_{k-}^i is a positive definite matrix, we can take $\epsilon_{\Sigma,k}^i > 0$ such that for any matrix H , if $\Sigma_{k-}^i \leq H \leq \Sigma_{k+}^i$, then H is a positive definite matrix. We define a set of vectors S_k^1 as follows

$$S_k^1 = \{ \mu \in \mathbb{R}^m \mid \mu(j) = \mu_{k-}^1(j) \text{ or } \mu_{k+}^1(j), \forall j = 1, \dots, m \},$$

where $\mu(j)$ is the j^{th} -component of μ , $\mu_{k-}^1(j)$ is the j^{th} -component of μ_{k-}^1 , and $\mu_{k+}^1(j)$ is the j^{th} -component of μ_{k+}^1 . For example, if $\mu_{k-}^1 = (1, 2)^T$, $\mu_{k+}^1 = (5, 6)^T$, then S_k^1 is a set of 4 vectors $\{(1, 5)^T, (1, 6)^T, (2, 5)^T, (2, 6)^T\}$. We define a set of covariance matrix T_k^1 as follows

$$T_k^1 = \{ \Sigma \mid \Sigma(j, w) = \Sigma_{k-}^1(j, w) \text{ or } \Sigma_{k+}^1(j, w), 1 \leq j, w \leq m \},$$

Similarly, we define a set of vectors S_k^2 and a set of covariance matrix T_k^2 . The uncertainty set (5.25) is considered in [83]. We assume that for each $i = 1, 2$ and $k \in \mathcal{I}_i$, the true distribution of a_k^i belongs to the uncertainty set $\mathcal{D}_k^{5,i}(\mu_k^i, \Sigma_k^i)$. We present an SOC reformulation of the constraints (5.4) and (5.5) by the following lemma.

Lemma 5.24. *The constraints (5.4) and (5.5) are equivalent to (5.26) and (5.27), respectively, given by*

$$\begin{aligned} & (\mu^1)^T x + \sqrt{\frac{\alpha_k^1}{1 - \alpha_k^1}} \|(\Sigma^1)^{\frac{1}{2}} x\|_2 \leq b_k^1, \\ & \forall \mu^1 \in S_k^1, \Sigma^1 \in T_k^1, k \in \mathcal{I}_1, \end{aligned} \quad (5.26)$$

$$\begin{aligned} & -(\mu^2)^T y + \sqrt{\frac{\alpha_k^2}{1 - \alpha_k^2}} \|(\Sigma^2)^{\frac{1}{2}} y\|_2 \leq -b_k^2, \\ & \forall \mu^2 \in S_k^2, \Sigma^2 \in T_k^2, k \in \mathcal{I}_2. \end{aligned} \quad (5.27)$$

Remark 5.7. Note that S_k^1 is a set of 2^m vectors and T_k^1 is a set of 2^{m^2} matrix. Then, Lemma 5.24 shows that the constraint (5.4) is equivalent to a system of $2^m \times 2^{m^2}$ constraints in (5.26), for any $k \in \mathcal{I}_1$ and the constraint (5.5) is equivalent to a system of $2^n \times 2^{n^2}$ constraints in (5.27), for any $k \in \mathcal{I}_2$.

Proof. Based on the structure of the uncertainty set (5.25), the constraint (5.4) can be written as

$$\inf_{(\mu, \Sigma) \in \bar{\mathcal{U}}_k^1} \inf_{F_k^1 \in \mathcal{D}_k^{1,i}(\mu, \Sigma)} \mathbb{P}\{a_k^1 x \leq b_k^1\} \geq \alpha_k^1,$$

where

$$\bar{\mathcal{U}}_k^1 = \left\{ (\mu, \Sigma) \mid \begin{array}{l} \mu_{k-}^1 \leq \mu \leq \mu_{k+}^1, \\ \Sigma_{k-}^1 \leq \Sigma \leq \Sigma_{k+}^1. \end{array} \right\}.$$

Using the similar arguments as in the Lemma 5.20, the constraint (5.4) is equivalent to

$$\frac{b_k^1 + v_1(x)}{\sqrt{v_2(x)}} \geq \sqrt{\frac{\alpha_k^1}{1 - \alpha_k^1}}, \quad (5.28)$$

where

$$v_1(x) = \begin{cases} \min_{\mu} -\mu^\top x \\ \text{s.t. } \mu_{k-}^1 \leq \mu \leq \mu_{k+}^1, \end{cases}$$

$$v_2(x) = \begin{cases} \max_{\Sigma} x^\top \Sigma x \\ \text{s.t. } \Sigma_{k-}^1 \leq \Sigma \leq \Sigma_{k+}^1. \end{cases}$$

Note that the objective functions $-\mu^\top x$ and $x^\top \Sigma x$ are linear functions w.r.t μ (resp. Σ). Then, it is clear that the optimal values $v_1(x)$ and $v_2(x)$ hold only when $\mu \in S_k^1$ and $\Sigma \in T_k^1$. Then, the constraint (5.4) can be reformulated as (5.26). Similarly, we can show that (5.5) is equivalent to (5.27). \square

5.4.2 . Statistical Distance Based Uncertainty Sets

In this section, we define uncertainty sets using a metric called ϕ -divergence. For any $i = 1, 2$ and $k \in \mathcal{I}_i$, the decision makers (the two players in the game) believe that the true distribution of a_k^i oscillates around a Normal distribution of mean vector μ_k^i and covariance matrix Σ_k^i , where μ_k^i and Σ_k^i are sample mean vector and sample covariance matrix obtained from historical data. We assume that the true distribution of a_k^i lies in a ball of radius θ_k^i , centered at a nominal distribution ν_k^i and the distance between these two distributions is given by ϕ -divergence metric. The nominal distribution ν_k^i is assumed to be Normal distributed of mean vector μ_k^i and covariance matrix Σ_k^i .

Definition 5.11. The ϕ -divergence distance between two measures μ and ν with densities f_μ and f_ν , respectively, with support in \mathbb{R}^{r_i} is defined as follows:

$$I_\phi(\mu, \nu) = \int_{\mathbb{R}^{r_i}} \phi \left(\frac{f_\mu(\xi)}{f_\nu(\xi)} \right) f_\nu(\xi) d\xi.$$

where $r_1 = m$ and $r_2 = n$.

There are different types of ϕ -divergences distance, we refer to [22] and [176] for different choices of function ϕ . We consider an uncertainty set $\mathcal{D}_k^{\phi,i}$ defined as follows

$$\mathcal{D}_k^{\phi,i} = \{F_k^i \in \mathcal{M} \mid I_\phi(F_k^i, \nu_k^i) \leq \theta_k^i\}, \quad (5.29)$$

where \mathcal{M}^i_+ is the set of all probability measures on \mathbb{R}^{r_i} , with $r_1 = m$, $r_2 = n$, and $\theta_k^i > 0$. This uncertainty set is considered in [117]. We assume that for each $i = 1, 2$ and $k \in \mathcal{I}_i$, the true distribution of a_k^i belongs to the uncertainty set $\mathcal{D}_k^{\phi, i}(\mu_k^i, \Sigma_k^i)$.

Definition 5.12. The conjugate of the function ϕ is a function $\phi^* : \mathbb{R} \rightarrow \mathbb{R} \cup +\infty$ such that

$$\phi^*(s) = \sup_{t \geq 0} \{st - \phi(t)\}.$$

We study some special cases of ϕ -divergences, which are summarized in Table 5.4. The data of Table 5.4

Table 5.4: List of selected ϕ -divergences with their conjugate respectively

Divergence	$\phi(t), t \geq 0$	$\phi^*(s)$
Kullback-Leibler	$t \log(t) - t + 1$.	$e^s - 1$
Variation distance	$ t - 1 $.	$-1, s \leq -1,$ $s, -1 \leq s \leq 1,$ $+\infty, s > 1.$
Modified χ^2 -distance	$(t - 1)^2$.	$-1, s \leq -2,$ $s + \frac{s^2}{4}, s > -2.$
Hellinger distance	$(\sqrt{t} - 1)^2$.	$\frac{s}{1-s}, s < 1,$ $+\infty, s \geq 1.$

are taken from [22]. The following lemma provides the first reformulation of the constraints (5.4) and (5.5).

Lemma 5.25. The constraint (5.4) is equivalent to

$$\sup_{\lambda > 0, \beta \in \mathbb{R}} \{f_k^1(\lambda, \beta)\} \geq \alpha_k^1, \quad (5.30)$$

where $f_k^1(\lambda, \beta) = \beta - \lambda \theta_k^1 - \lambda \phi^*\left(\frac{-1+\beta}{\lambda}\right) \mathbb{P}_{\nu_k^1}(M_k^1) - \lambda \phi^*\left(\frac{\beta}{\lambda}\right) [1 - \mathbb{P}_{\nu_k^1}(M_k^1)]$, and $M_k^1 = \{\xi \in \mathbb{R}^m \mid \xi^T x \leq b_k^1\}$. The constraint (5.5) is equivalent to

$$\sup_{\lambda > 0, \beta \in \mathbb{R}} \{f_k^2(\lambda, \beta)\} \geq \alpha_k^2,$$

where $f_k^2(\lambda, \beta) = \beta - \lambda \theta_k^2 - \lambda \phi^*\left(\frac{-1+\beta}{\lambda}\right) \mathbb{P}_{\nu_k^2}(M_k^2) - \lambda \phi^*\left(\frac{\beta}{\lambda}\right) [1 - \mathbb{P}_{\nu_k^2}(M_k^2)]$, and $M_k^2 = \{\xi \in \mathbb{R}^n \mid \xi^T x \leq b_k^2\}$.

Proof. For $k \in \mathcal{I}_1$, consider the following optimization problem

$$v_k^p = \inf_{F_k^1 \in \mathcal{D}_k^{\phi, 1}} \mathbb{P}\left((a_k^1)^T x \leq b_k^1\right).$$

The above problem is rewritten as

$$\begin{aligned} v_k^p &= \inf_{F \geq 0} \int_{\mathbb{R}^m} \mathbf{1}_{M_k^1}(\xi) F(\xi) d\xi \\ \text{s.t. } (i) \quad &\int_{\mathbb{R}^m} f_{\nu_k^1}(\xi) \phi\left(\frac{F(\xi)}{f_{\nu_k^1}(\xi)}\right) d\xi \leq \theta_k^1, \\ (ii) \quad &\int_{\mathbb{R}^m} F(\xi) d\xi = 1, \end{aligned} \quad (5.31)$$

where the infimum value is taken over all positive measures on \mathbb{R}^m . The Lagrangian dual of (5.31) can be written as follows

$$v_D^k = \sup_{\lambda \geq 0, \beta \in \mathbb{R}} \left\{ \beta - \lambda \theta_k^1 + \inf_{F(\xi) \geq 0} \int_{\mathbb{R}^m} g_k^1(\lambda, \beta) d\xi \right\},$$

where $g_k^1(\lambda, \beta) = \mathbf{1}_{M_k^1}(\xi)F(\xi) - \beta F(\xi) + \lambda f_{\nu_k^1}(\xi)\phi\left(\frac{F(\xi)}{f_{\nu_k^1}(\xi)}\right) d\xi$, λ is the dual variable of the constraint (i) and β is the dual variable of the constraint (ii). Since $\theta_k^1 > 0$, the Slater's condition holds, then the strong duality holds, i.e., $v_P^k = v_D^k$. The rest of the proof follows from Theorem 1 [117]. \square

We present an SOC reformulation of the constraints (5.4) and (5.5) by the following lemma.

Lemma 5.26. *The constraints (5.4) and (5.5) are equivalent to (5.32) and (5.33), respectively, given by:*

$$\begin{aligned} & (\mu_k^1)^T x + \Phi^{(-1)}[H(\theta_k^1, 1 - \alpha_k^1)] \left\| (\Sigma_k^1)^{\frac{1}{2}} x \right\|_2 \leq b_k^1, \\ & \forall k \in \mathcal{I}_1, \end{aligned} \tag{5.32}$$

$$\begin{aligned} & -(\mu_k^2)^T y + \Phi^{(-1)}[H(\theta_k^2, 1 - \alpha_k^2)] \left\| (\Sigma_k^2)^{\frac{1}{2}} y \right\|_2 \leq -b_k^2, \\ & \forall k \in \mathcal{I}_2. \end{aligned} \tag{5.33}$$

where $\Phi^{(-1)}$ is the quantile of the standard Normal distribution and H is a function whose value is given in Table 5.5.

Table 5.5: List of selected ϕ -divergences with the function f respectively

Divergence	$H(\theta, \epsilon) =$	θ, ϵ
Kullback-Leibler	$\inf_{x \in (0,1)} \frac{e^{-\theta x^{1-\epsilon}} - 1}{x - 1}$	$\theta > 0$ $0 < \epsilon < 1$
Variation distance	$1 - \epsilon + \frac{\theta}{2}$	$\theta > 0$ $0 < \epsilon < 1$
Modified χ^2 - distance	$1 - \epsilon + \frac{\sqrt{\theta^2 + 4\theta(\epsilon - \epsilon^2)} - (1 - 2\epsilon)\theta}{2\theta + 2}$	$\theta > 0$ $0 < \epsilon < \frac{1}{2}$
Hellinger distance	$\frac{-B + \sqrt{\Delta}}{2}$, where $B = -(2 - (2 - \theta)^2)\epsilon - \frac{(2 - \theta)^2}{2}$, $C = \left(\frac{(2 - \theta)^2}{4} - \epsilon\right)^2$, $\Delta = B^2 - 4C = (2 - \theta)^2 [4 - (2 - \theta)^2] \epsilon(1 - \epsilon)$.	$0 < \theta < 2 - \sqrt{2}$ $0 < \epsilon < 1$

Proof. Using Lemma 5.25, we prove that the constraint (5.4) is equivalent to

$$\mathbb{P}_{\nu_k^1}(M_k^1) \geq H(\theta_k^1, 1 - \alpha_k^1). \tag{5.34}$$

Since ν_k^1 follows a Normal distribution with mean vector μ_k^1 and covariance matrix Σ_k^1 , it is well known that (5.34) is equivalent to the SOC constraint (5.32). We refer to Propositions 2, 3, and 4, [117] for the proof of the cases Kullback-Leibler, Variation distance and Modified χ^2 - distance. We present the proof of Hellinger case. For $i = 1, 2$ and $k \in \mathcal{I}_i$, it suffices to calculate the value of $\sup_{\lambda > 0, \beta \in \mathbb{R}} \{f_k^i(\lambda, \beta)\}$ with Hellinger distance divergence. We consider two cases as follows

• **Case 1:** $\frac{\beta}{\lambda} < 1 \Leftrightarrow \beta < \lambda$. We have

$$\begin{aligned}\phi^* \left(\frac{\beta}{\lambda} \right) &= \frac{\beta}{\lambda - \beta}, \\ \phi^* \left(\frac{\beta - 1}{\lambda} \right) &= \frac{\beta - 1}{\lambda + 1 - \beta}.\end{aligned}$$

Therefore,

$$\begin{aligned}\sup_{\lambda > 0, \beta \in \mathbb{R}} \{f_k^i(\lambda, \beta)\} &= \\ \sup_{\lambda > 0, \beta \in \mathbb{R}} \mathbb{P}_{\nu_k^i}(M_k^i) \frac{\lambda^2}{(\lambda - \beta)(\lambda - \beta + 1)} + \frac{\beta^2}{\beta - \lambda} - \lambda \theta_k^i.\end{aligned}$$

Since $\lambda > 0$ and $\beta < \lambda$, let $\gamma = \lambda - \beta$, we deduce that

$$\begin{aligned}\sup_{\lambda > 0, \beta \in \mathbb{R}} \{f_k^i(\lambda, \beta)\} &= \\ \sup_{\lambda > 0, \gamma > 0} \left\{ \lambda^2 \left(\frac{\mathbb{P}_{\nu_k^i}(M_k^i)}{\gamma(\gamma + 1)} - \frac{1}{\gamma} \right) + \lambda(2 - \theta_k^i) - \gamma \right\}.\end{aligned}$$

Let $Q(\lambda, \gamma) = \lambda^2 \left(\frac{\mathbb{P}_{\nu_k^i}(M_k^i)}{\gamma(\gamma + 1)} - \frac{1}{\gamma} \right) + \lambda(2 - \theta_k^i) - \gamma$. Note that $0 \leq \mathbb{P}_{\nu_k^i}(M_k^i) \leq 1$ and $\gamma > 0$. Therefore, $Q(\lambda, \gamma)$ is a second-order polynomial of λ and the coefficient of λ^2 is negative. It is well known that the maximum value of a second order function $f(x) = ax^2 + bx + c$ with $a < 0$ is $c - \frac{b^2}{4a}$ and it holds at $x = \frac{-b}{2a}$. Hence, the maximum value of $Q(\lambda, \gamma)$ holds at $\lambda^* = \frac{\gamma(\gamma + 1)(2 - \theta_k^i)}{2(1 + \gamma - \mathbb{P}_{\nu_k^i}(M_k^i))}$. Since $\theta_k^i < 2$, it is clear that $\lambda^* > 0$. Then, the optimal value of $\sup_{\lambda > 0, \beta \in \mathbb{R}} \{f_k^i(\lambda, \beta)\}$ holds when $\lambda = \lambda^*$ and we have

$$\begin{aligned}\sup_{\lambda > 0, \beta \in \mathbb{R}} \{f_k^i(\lambda, \beta)\} &= \\ \sup_{\gamma > 0} \left\{ -\gamma + \frac{(2 - \theta_k^i)^2 \gamma(\gamma + 1)}{4(\gamma + 1 - \mathbb{P}_{\nu_k^i}(M_k^i))} \right\}.\end{aligned}\tag{5.35}$$

Let $u = \gamma + 1 - \mathbb{P}_{\nu_k^i}(M_k^i)$, then $u > 1 - \mathbb{P}_{\nu_k^i}(M_k^i)$. Rewriting (5.35) as a function of u , we have:

$$\begin{aligned}\sup_{\lambda > 0, \beta \in \mathbb{R}} \{f_k^i(\lambda, \beta)\} &= \sup_{u > 1 - \mathbb{P}_{\nu_k^i}(M_k^i)} F(u), \\ &= \sup_{u > 1 - \mathbb{P}_{\nu_k^i}(M_k^i)} \left\{ au + \frac{b}{u} + c \right\},\end{aligned}$$

where $a = \left(\frac{(2 - \theta_k^i)^2}{4} - 1 \right)$,

$b = \frac{(2 - \theta_k^i)^2 \mathbb{P}_{\nu_k^i}(M_k^i)(\mathbb{P}_{\nu_k^i}(M_k^i) - 1)}{4}$,

$c = 1 - \mathbb{P}_{\nu_k^i}(M_k^i) + \frac{(2 - \theta_k^i)^2 (2\mathbb{P}_{\nu_k^i}(M_k^i) - 1)}{4}$. Note that $a < 0$ and $b \leq 0$. We have: $F'(u) = a - \frac{b}{u^2}$. Hence, it

can be shown that F is decreasing on $(u^*, +\infty)$, increasing on $(-u^*, u^*)$ and decreasing on $(-\infty, -u^*)$, where $u^* = \sqrt{\frac{b}{a}}$. Or,

$$u^* = \sqrt{\frac{(2 - \theta_k^i)^2}{4 - (2 - \theta_k^i)^2} \mathbb{P}_{\nu_k^i}(M_k^i)(1 - \mathbb{P}_{\nu_k^i}(M_k^i))}. \quad (5.36)$$

We have $F(u^*) = -2\sqrt{ab} + c$. We consider 2 cases as follows

1: $u^* \leq 1 - \mathbb{P}_{\nu_k^i}(M_k^i)$. Since F is decreasing on $(u^*, +\infty)$, it is also decreasing on $(1 - \mathbb{P}_{\nu_k^i}(M_k^i), +\infty)$. Hence, $\sup_{u > 1 - \mathbb{P}_{\nu_k^i}(M_k^i)} F(u) = 0$, where the optimal value holds when $u \rightarrow 1 - \mathbb{P}_{\nu_k^i}(M_k^i) \Leftrightarrow \gamma \rightarrow 0$, which violates (5.30).

2: $u^* > 1 - \mathbb{P}_{\nu_k^i}(M_k^i) > 0$. Then, the optimal value of $\sup_{u > 1 - \mathbb{P}_{\nu_k^i}(M_k^i)} F(u)$ holds when $u = u^*$. Therefore,

$$\sup_{\lambda > 0, \beta \in \mathbb{R}} \{f_k^i(\lambda, \beta)\} = F(u^*) = -2\sqrt{ab} + c.$$

Then, (5.30) is equivalent to

$$\begin{aligned} & -2\sqrt{\frac{(2 - \theta_k^i)^2}{4} \left(1 - \frac{(2 - \theta_k^i)^2}{4}\right) \mathbb{P}_{\nu_k^i}(M_k^i)(1 - \mathbb{P}_{\nu_k^i}(M_k^i))} \\ & \geq \left(1 - \frac{(2 - \theta_k^i)^2}{2}\right) \mathbb{P}_{\nu_k^i}(M_k^i) + \frac{(2 - \theta_k^i)^2}{4} - (1 - \alpha_k^i). \end{aligned} \quad (5.37)$$

By taking square on both side of (5.37), we obtain a second order inequality of $\mathbb{P}_{\nu}(K)$ as follows

$$\mathbb{P}_{\nu_k^i}(M_k^i)^2 + B\mathbb{P}_{\nu_k^i}(M_k^i) + C \geq 0,$$

where B, C are defined in Table 5.5. By solving the equality $x^2 + Bx + C = 0$, we have two solutions $x_{\min} < x_{\max}$ where $x_{\min} = \frac{-B - \sqrt{\Delta}}{2}$, $x_{\max} = \frac{-B + \sqrt{\Delta}}{2}$. It is clear that (5.37) is equivalent to either $\mathbb{P}_{\nu_k^i}(M_k^i) \geq x_{\max}$ or $\mathbb{P}_{\nu_k^i}(M_k^i) \leq x_{\min}$. Since $\theta_k^i < 2 - \sqrt{2}$, we deduce that $1 - \frac{(2 - \theta_k^i)^2}{2} < 0$. Therefore, we have

$$\begin{aligned} & \left(1 - \frac{(2 - \theta_k^i)^2}{2}\right) x_{\min} + \frac{(2 - \theta_k^i)^2}{4} - (1 - \alpha_k^i) \\ & > \left(1 - \frac{(2 - \theta_k^i)^2}{2}\right) x_{\max} + \frac{(2 - \theta_k^i)^2}{4} - (1 - \alpha_k^i). \end{aligned} \quad (5.38)$$

On the other hand, we have

$$\begin{aligned} & -2\sqrt{\frac{(2 - \theta_k^i)^2}{4} \left(1 - \frac{(2 - \theta_k^i)^2}{4}\right) x(1 - x)} \\ & = \pm \left[\left(1 - \frac{(2 - \theta_k^i)^2}{2}\right) x + \frac{(2 - \theta_k^i)^2}{4} - (1 - \alpha_k^i) \right], \end{aligned}$$

where $x = x_{\min}$ or $x = x_{\max}$. Note that $-2\sqrt{\frac{(2-\theta_k^i)^2}{4} \left(1 - \frac{(2-\theta_k^i)^2}{4}\right)} x(1-x) < 0$. Using (5.38), we deduce that

$$\begin{aligned} & -2\sqrt{\frac{(2-\theta_k^i)^2}{4} \left(1 - \frac{(2-\theta_k^i)^2}{4}\right)} x_{\max}(1-x_{\max}) \\ &= \left[\left(1 - \frac{(2-\theta_k^i)^2}{2}\right) x_{\max} + \frac{(2-\theta_k^i)^2}{4} - (1-\alpha_k^i) \right], \end{aligned}$$

and

$$\begin{aligned} & -2\sqrt{\frac{(2-\theta_k^i)^2}{4} \left(1 - \frac{(2-\theta_k^i)^2}{4}\right)} x_{\min}(1-x_{\min}) \\ &= - \left[\left(1 - \frac{(2-\theta_k^i)^2}{2}\right) x_{\min} + \frac{(2-\theta_k^i)^2}{4} - (1-\alpha_k^i) \right]. \end{aligned}$$

or x_{\max} satisfies (5.37) while x_{\min} does not satisfy (5.37). Then, (5.37) is equivalent to $\mathbb{P}_{\nu_k^i}(M_k^i) \geq x_{\max}$.

- **Case 2:** $1 \leq \frac{\beta}{\lambda} \Leftrightarrow \lambda \leq \beta$. We have

$$\phi^*\left(\frac{\beta}{\lambda}\right) = +\infty,$$

which implies that $\sup_{\lambda>0, \beta \in \mathbb{R}} \{f_k^i(\lambda, \beta)\} = -\infty$, which violates (5.30).

□

5.4.3 . Second Order Cone Reformulation

In this section, we summarize our SOC reformulation results from Lemmas 5.20, 5.21, 5.22, 5.23, 5.24, and 5.26. They show that in all cases of uncertainty sets defined in Sections 5.4.1 and 5.4.2, the feasible strategy sets (5.6) and (5.7) can be written as

$$\begin{aligned} S_{\alpha^1}^1 &= \left\{ x \in X \mid (\mu_{kj}^1)^T x + \kappa_{\alpha_k^1} \|(\Sigma_{kw}^1)^{\frac{1}{2}} x\|_2 \leq b_k^1, \right. \\ &\quad \left. \forall j = 1, 2, \dots, N_1, w = 1, 2, \dots, P_1, k \in \mathcal{I}_1 \right\}, \end{aligned} \quad (5.39)$$

and

$$\begin{aligned} S_{\alpha^2}^2 &= \left\{ y \in Y \mid -(\mu_{lj}^2)^T y + \kappa_{\alpha_l^2} \|(\Sigma_{lw}^2)^{\frac{1}{2}} y\|_2 \leq -b_l^2, \right. \\ &\quad \left. \forall j = 1, 2, \dots, N_2, w = 1, 2, \dots, P_2, l \in \mathcal{I}_2 \right\}. \end{aligned} \quad (5.40)$$

- If the uncertainty set is defined by (5.9), then $\kappa_{\alpha_k^i} = \sqrt{\frac{\alpha_k^i}{1-\alpha_k^i}}$ and $N_1 = P_1 = N_2 = P_2 = 1$, for all $i = 1, 2$, $k \in \mathcal{I}_i$.
- If the uncertainty set is defined by (5.12), then $\kappa_{\alpha_k^i} = \sqrt{\frac{\alpha_k^i}{1-\alpha_k^i}} \sqrt{\gamma_k^i}$ and $N_1 = P_1 = N_2 = P_2 = 1$, for all $i = 1, 2$, $k \in \mathcal{I}_i$.

- If the uncertainty set is defined by (5.16), then $\kappa_{\alpha_k^i} = \left(\sqrt{\frac{\alpha_k^i}{1-\alpha_k^i}} \sqrt{\gamma_{k2}^i} + \sqrt{\gamma_{k1}^i} \right)$ and $N_1 = P_1 = N_2 = P_2 = 1$, for all $i = 1, 2, k \in \mathcal{I}_i$.
- If the uncertainty set is defined by (5.21), then $\kappa_{\alpha_k^i} = \sqrt{\frac{\alpha_k^i}{1-\alpha_k^i}}$ and $N_1 = P_1 = N_2 = P_2 = M$, for all $i = 1, 2, k \in \mathcal{I}_i$.
- If the uncertainty set is defined by (5.25), then $\kappa_{\alpha_k^i} = \sqrt{\frac{\alpha_k^i}{1-\alpha_k^i}}$ and $N_1 = 2^m; P_1 = 2^{(m^2)}, N_2 = 2^n, P_2 = 2^{(n^2)}$, for all $i = 1, 2, k \in \mathcal{I}_i$.
- If the uncertainty set is defined by (5.29), then $\kappa_{\alpha_k^i} = \Phi^{(-1)} [H(\theta_k^i, 1 - \alpha_k^i)]$ and $N_1 = P_1 = N_2 = P_2 = 1$, where H and $\Phi^{(-1)}$ are defined in Lemma 5.26.

We assume that the strategy sets (5.39) and (5.40) satisfy the strict feasibility condition given by Assumption 5.7.

Assumption 5.7. 1. *There exists an $x \in S_{\alpha_1}^1$ such that the inequality constraints of $S_{\alpha_1}^1$ defined by (5.39) are strictly satisfied.*

2. *There exists an $y \in S_{\alpha_2}^2$ such that the inequality constraints of $S_{\alpha_2}^2$ defined by (5.40) are strictly satisfied.*

The conditions given in Assumption 5.7 are Slater's condition, which are sufficient for strong duality in a convex optimization problem. We use these conditions in order to derive equivalent SOCPs for the zero-sum game Z_α .

5.5 . Existence and characterization of Saddle Point Equilibrium

In this section, we show that there exists an SPE of the game Z_α if the distributions of the random constraint vectors of both the players belong to the uncertainty sets defined in Sections 5.4.1 and 5.4.2. We further propose a primal-dual pair of SOCPs whose optimal solutions constitute an SPE of the game Z_α .

Theorem 5.4. *Consider the game Z_α , where the distributions of the random constraint vectors $a_k^i, k \in \mathcal{I}_i, i = 1, 2$, belong to the uncertainty sets described in Sections 5.4.1 and 5.4.2. Then, there exists an SPE of the game for all $\alpha \in (0, 1)^p \times (0, 1)^q$.*

Proof. Let $\alpha \in (0, 1)^p \times (0, 1)^q$. For uncertainty sets described in Sections 5.4.1 and 5.4.2, the strategy sets $S_{\alpha_1}^1$ and $S_{\alpha_2}^2$ are given by (5.39) and (5.40), respectively. It is easy to see that $S_{\alpha_1}^1$ and $S_{\alpha_2}^2$ are convex and compact sets. The function $u(x, y)$ is a bilinear and continuous function. Hence, there exists an SPE from the minimax theorem [243]. \square

5.5.1 . Equivalent Primal-Dual Pair of Second-Order Cone Programs

From the minimax theorem [243], (x^*, y^*) is an SPE for the game Z_α if and only if

$$x^* \in \arg \max_{x \in S_{\alpha_1}^1} \min_{y \in S_{\alpha_2}^2} u(x, y), \quad (5.41)$$

$$y^* \in \arg \min_{y \in S_{\alpha_2}^2} \max_{x \in S_{\alpha_1}^1} u(x, y). \quad (5.42)$$

We start with the optimization problem

$$\min_{y \in S_{\alpha^2}^2} \max_{x \in S_{\alpha^1}^1} u(x, y).$$

By introducing auxiliary variables $t_{k,j,w}^1$, the inner optimization problem $\max_{x \in S_{\alpha^1}^1} u(x, y)$ can be equivalently written as

$$\begin{aligned} & \max_{x, t_{k,j,w}^1} x^\top G y + g^\top x + h^\top y \\ & \text{s.t.} \\ & (i) \quad -x^\top \mu_{kj}^1 - \kappa_{\alpha_k^1} \|t_{k,j,w}^1\|_2 + b_k^1 \geq 0, \\ & \quad \forall j = 1, 2, \dots, N_1, w = 1, 2, \dots, P_1, k \in \mathcal{I}_1, \\ & (ii) \quad t_{k,j,w}^1 - (\Sigma_{kw}^1)^{\frac{1}{2}} x = 0, \\ & \quad \forall j = 1, 2, \dots, N_1, w = 1, 2, \dots, P_1, k \in \mathcal{I}_1, \\ & (iii) \quad C^1 x = d^1, x_r \geq 0, \forall r = 1, 2, \dots, m. \end{aligned} \tag{5.43}$$

Let $\lambda^1 = (\lambda_{k,j,w}^1)$, $\delta_{k,j,w}^1$ and ν^1 be the Lagrange multipliers of constraints (i), (ii), and equality constraints given in the constraint (iii) of (5.43), respectively. Here, for any $j = 1, \dots, N_1, w = 1, \dots, P_1, k \in \mathcal{I}_1$, $\lambda_{k,j,w}^1$ is a real number, $\delta_{k,j,w}^1$ is an $m \times 1$ real vector, and ν^1 is a $K_1 \times 1$ real vector. Then, the Lagrangian dual problem of the SOCP (5.43) can be written as

$$\begin{aligned} & \min_{\lambda_1 \geq 0, \delta_{k,j,w}^1, \nu^1} \max_{x \geq 0, t_{k,j,w}^1} \left\{ x^\top G y + g^\top x + h^\top y \right. \\ & + \sum_{k \in \mathcal{I}_1} \sum_{j=1}^{N_1} \sum_{w=1}^{P_1} \left[\lambda_{k,j,w}^1 (-x^\top \mu_{kj}^1 - \kappa_{\alpha_k^1} \|t_{k,j,w}^1\|_2 + b_k^1) \right. \\ & \left. \left. + (\delta_{k,j,w}^1)^\top (t_{k,j,w}^1 - (\Sigma_{kw}^1)^{\frac{1}{2}} x) \right] + (\nu^1)^\top (d^1 - C^1 x) \right\}. \end{aligned}$$

By reformulating the objective function of the above optimization problem as the sum of two functions such that one depends on x and other depends on $t_{k,j,w}^1$, we have

$$\begin{aligned} & \min_{\lambda_1 \geq 0, \delta_{k,j,w}^1, \nu^1} \max_{x \geq 0} \left\langle x^\top \left[G y - (C^1)^\top \nu^1 + g \right. \right. \\ & - \sum_{k \in \mathcal{I}_1} \sum_{j=1}^{N_1} \sum_{w=1}^{P_1} \left(\lambda_{k,j,w}^1 \mu_{kj}^1 + (\Sigma_{kw}^1)^{\frac{1}{2}} \delta_{k,j,w}^1 \right) \left. \right. \\ & + \max_{t_{k,j,w}^1} \sum_{k \in \mathcal{I}_1} \sum_{j=1}^{N_1} \sum_{w=1}^{P_1} \left[(\delta_{k,j,w}^1)^\top t_{k,j,w}^1 - \kappa_{\alpha_k^1} \lambda_{k,j,w}^1 \|t_{k,j,w}^1\|_2 \right] \\ & \left. + h^\top y + (\nu^1)^\top d^1 + \sum_{k \in \mathcal{I}_1} \sum_{j=1}^{N_1} \sum_{w=1}^{P_1} \lambda_{k,j,w}^1 b_k^1 \right\rangle. \end{aligned}$$

The first term of the objective function is a function of x

$$x^\top \left[Gy - (C^1)^\top \nu^1 + g - \sum_{k \in \mathcal{I}_1} \sum_{j=1}^{N_1} \sum_{w=1}^{P_1} (\lambda_{k j w}^1 \mu_{k j}^1 + (\Sigma_{k w}^1)^{\frac{1}{2}} \delta_{k j w}^1) \right]. \quad (5.44)$$

The above term is unbounded on the domain $x \geq 0$, unless the following condition holds

$$Gy - \sum_{k \in \mathcal{I}_1} \sum_{j=1}^{N_1} \sum_{w=1}^{P_1} (\lambda_{k j w}^1 \mu_{k j}^1 + (\Sigma_{k w}^1)^{\frac{1}{2}} \delta_{k j w}^1) - (C^1)^\top \nu^1 + g \leq 0.$$

When the above condition holds, it is clear that the maximum value of (5.44) is zero and it holds at $x = 0$. The second term of the objective function is a function of $t_{k j w}^1$

$$\sum_{k \in \mathcal{I}_1} \sum_{j=1}^{N_1} \sum_{w=1}^{P_1} \left[(\delta_{k j w}^1)^\top t_{k j w}^1 - \kappa_{\alpha_k^1} \lambda_{k j w}^1 \|t_{k j w}^1\|_2 \right]. \quad (5.45)$$

The above term is unbounded on the domain $t_{k j w}^1 \in \mathbb{R}^m$, unless the following condition holds

$$\begin{aligned} \|\delta_{k j w}^1\| &\leq \kappa_{\alpha_k^1} \lambda_{k j w}^1, \\ \forall k \in \mathcal{I}_1, j = 1, 2, \dots, N_1, w = 1, 2, \dots, P_1. \end{aligned}$$

When the above condition holds, it is clear that the maximum value of (5.45) is zero and it holds at $t_{k j w}^1 = 0$. Then, the Lagrangian dual problem of the SOCP (5.43) can be written as

$$\begin{aligned} \min_{\lambda_1 \geq 0, \delta_{k j w}^1, \nu^1} & (h^\top y + (\nu^1)^\top d^1 + \sum_{k \in \mathcal{I}_1} \sum_{j=1}^{N_1} \sum_{w=1}^{P_1} \lambda_{k j w}^1 b_k^1) \\ \text{s.t. (i)} & \quad Gy - \sum_{k \in \mathcal{I}_1} \sum_{j=1}^{N_1} \sum_{w=1}^{P_1} [\lambda_{k j w}^1 \mu_{k j}^1 + (\Sigma_{k w}^1)^{\frac{1}{2}} \delta_{k j w}^1] \\ & \quad - (C^1)^\top \nu^1 + g \leq 0, \\ \text{(ii)} & \quad \|\delta_{k j w}^1\| \leq \kappa_{\alpha_k^1} \lambda_{k j w}^1, \\ & \quad \forall k \in \mathcal{I}_1, j = 1, 2, \dots, N_1, w = 1, 2, \dots, P_1. \end{aligned}$$

Under Assumption 5.7, the Lagrangian dual problem of (5.43) has zero duality gap [30], which implies that the above optimization problem is equivalent to the problem $\max_{x \in S_{\alpha^1}^1} u(x, y)$. Hence, the problem

$\min_{y \in S_{\alpha_2}^2} \max_{x \in S_{\alpha_1}^1} u(x, y)$ is equivalent to the following SOCP

$$\begin{aligned}
& \min_{y, \nu^1, \delta_{kjw}^1, \lambda_{kjw}^1 \geq 0} h^\top y + (\nu^1)^\top d^1 + \sum_{k \in \mathcal{I}_1} \sum_{j=1}^{N_1} \sum_{w=1}^{P_1} \lambda_{kjw}^1 b_k^1 \\
& \text{s.t.} \\
& (i) \quad Gy - \sum_{k \in \mathcal{I}_1} \sum_{j=1}^{N_1} \sum_{w=1}^{P_1} [\lambda_{kjw}^1 \mu_{kj}^1 + (\Sigma_{kw}^1)^{\frac{1}{2}} \delta_{kjw}^1] \\
& \quad - (C^1)^\top \nu^1 + g \leq 0, \\
& (ii) \quad \|\delta_{kjw}^1\| \leq \kappa_{\alpha_k^1} \lambda_{kjw}^1, \\
& \quad \forall k \in \mathcal{I}_1, j = 1, 2, \dots, N_1, w = 1, 2, \dots, P_1, \\
& (iii) \quad -(\mu_{lj}^2)^\top y + \kappa_{\alpha_l^2} \|(\Sigma_{lw}^2)^{\frac{1}{2}} y\| \leq -b_l^2, \\
& \quad \forall j = 1, 2, \dots, N_2, w = 1, 2, \dots, P_2, l \in \mathcal{I}_2, \\
& (iv) \quad C^2 y = d^2, y_s \geq 0, \forall s = 1, 2, \dots, n,
\end{aligned} \tag{5.46}$$

where the constraints (iii) and (iv) are due to the fact that $y \in S_{\alpha_2}^2$ and the representation of $S_{\alpha_2}^2$ in (5.40). Similarly, problem $\max_{x \in S_{\alpha_1}^1} \min_{y \in S_{\alpha_2}^2} u(x, y)$ is equivalent to the following SOCP

$$\begin{aligned}
& \max_{x, \nu^2, \delta_{ljw}^2, \lambda_{ljw}^2 \geq 0} g^\top x + (\nu^2)^\top d^2 - \sum_{l \in \mathcal{I}_2} \sum_{j=1}^{N_2} \sum_{w=1}^{P_2} \lambda_{ljw}^2 b_l^2 \\
& \text{s.t.} \\
& (i) \quad G^\top x - \sum_{l \in \mathcal{I}_2} \sum_{j=1}^{N_2} \sum_{w=1}^{P_2} [-\lambda_{ljw}^2 \mu_{lj}^2 + (\Sigma_{lw}^2)^{\frac{1}{2}} \delta_{ljw}^2] \\
& \quad - (C^2)^\top \nu^2 + h \geq 0, \\
& (ii) \quad \|\delta_{ljw}^2\| \leq \kappa_{\alpha_l^2} \lambda_{ljw}^2, \lambda_{ljw}^2 \geq 0, \\
& \quad \forall l \in \mathcal{I}_2, j = 1, 2, \dots, N_2, w = 1, 2, \dots, P_2, \\
& (iii) \quad (\mu_{kj}^1)^\top x + \kappa_{\alpha_k^1} \|(\Sigma_{kw}^1)^{\frac{1}{2}} x\| \leq b_k^1, \\
& \quad \forall j = 1, 2, \dots, N_2, w = 1, 2, \dots, P_2, k \in \mathcal{I}_1, \\
& (iv) \quad C^1 x = d^1, x_r \geq 0, \forall r = 1, 2, \dots, m.
\end{aligned} \tag{5.47}$$

It follows from the duality theory of SOCPs that (5.46) and (5.47) form a primal-dual pair of SOCPs [30]. Next, we show that the equivalence between the optimal solutions of (5.46)-(5.47) and an SPE of the game Z_α .

Theorem 5.5. *Consider the zero-sum game Z_α , where the feasible strategy sets of player 1 and player 2 are given by (5.39) and (5.40), respectively. Let Assumption 5.7 holds. Then, for a given $\alpha \in (0, 1)^p \times (0, 1)^q$, (x^*, y^*) is an SPE of the game Z_α if and only if there exists $(\nu^{1*}, \delta_{kjw}^{1*}, \lambda_{kjw}^{1*} \geq 0)$ and $(\nu^{2*}, \delta_{ljw}^{2*}, \lambda_{ljw}^{2*} \geq 0)$ such that $(y^*, \nu^{1*}, \delta_{kjw}^{1*}, \lambda_{kjw}^{1*})$ and $(x^*, \nu^{2*}, \delta_{ljw}^{2*}, \lambda_{ljw}^{2*})$ are optimal solutions of (5.46) and (5.47), respectively.*

Proof. Let (x^*, y^*) be an SPE of the game Z_α . Then, x^* and y^* are the solutions of (5.41) and (5.42), respectively. Therefore, there exists $(\nu^{1*}, \delta_{kjw}^{1*}, \lambda_{kjw}^{1*} \geq 0)$ and $(\nu^{2*}, \delta_{ljw}^{2*}, \lambda_{ljw}^{2*} \geq 0)$ such that $(y^*, \nu^{1*}, \delta_{kjw}^{1*}, \lambda_{kjw}^{1*})$ and

$(x^*, \nu^{2*}, \delta_{ljw}^{2*}, \lambda_{ljw}^{2*})$ are optimal solutions of (5.46) and (5.47) respectively. On the other hand, let $(y^*, \nu^{1*}, \delta_{kjl}^{1*}, \lambda_{kjl}^{1*})$ and $(x^*, \nu^{2*}, \delta_{ljw}^{2*}, \lambda_{ljw}^{2*})$ be optimal solutions of (5.46) and (5.47), respectively. Under Assumption 5.7, (5.46) and (5.47) are strictly feasible. Therefore, strong duality holds for primal-dual pair (5.46)-(5.47). Then, we have

$$\begin{aligned} g^\top x^* + (\nu^{2*})^\top d^2 - \sum_{l \in \mathcal{I}_2} \sum_{j=1}^{N_2} \sum_{w=1}^{P_2} \lambda_{ljw}^{2*} b_l^2 \\ = h^\top y^* + (\nu^{1*})^\top d^1 + \sum_{k \in \mathcal{I}_1} \sum_{j=1}^{N_1} \sum_{w=1}^{P_1} \lambda_{kjl}^{1*} b_k^1. \end{aligned} \quad (5.48)$$

Consider the constraint (i) of (5.46) at optimal solution $(y^*, \nu^{1*}, \delta_{kjl}^{1*}, \lambda_{kjl}^{1*})$ and multiply it by x^\top , for any $x \in S_{\alpha^1}^1$, we have

$$\begin{aligned} x^\top G y^* + g^\top x &\leq x^\top (C^1)^\top \nu^{1*} \\ &+ \sum_{k \in \mathcal{I}_1} \sum_{j=1}^{N_1} \sum_{w=1}^{P_1} \left[x^\top \mu_{kj}^1 \lambda_{kjl}^{1*} + x^\top (\Sigma_{kw}^1)^{\frac{1}{2}} \delta_{kjl}^{1*} \right]. \end{aligned} \quad (5.49)$$

By using the Cauchy-Schwartz inequality, for any $k \in \mathcal{I}_1$, $j = 1, 2, \dots, N_1$, $w = 1, 2, \dots, P_1$, we have

$$x^\top (\Sigma_{kw}^1)^{\frac{1}{2}} \delta_{kjl}^{1*} \leq \|(\Sigma_{kw}^1)^{\frac{1}{2}} x\|_2 \|\delta_{kjl}^{1*}\|_2.$$

Using the constraint (ii) of (5.47), the above constraint implies that

$$(x^*)^\top (\Sigma_{kw}^1)^{\frac{1}{2}} \delta_{kjl}^{1*} \leq \|(\Sigma_{kw}^1)^{\frac{1}{2}} x\|_2 \kappa_{\alpha_k^1} \lambda_{kjl}^{1*}.$$

Since $x \in S_{\alpha^1}^1$, we have

$$C^1 x = d^1.$$

Then, the constraint (5.49) implies that

$$\begin{aligned} x^\top G y^* + g^\top x &\leq (\nu^{1*})^\top d^1 \\ &+ \sum_{k \in \mathcal{I}_1} \sum_{j=1}^{N_1} \sum_{w=1}^{P_1} \left[x^\top \mu_{kj}^1 \lambda_{kjl}^{1*} + (\Sigma_{kw}^1)^{\frac{1}{2}} x\|_2 \kappa_{\alpha_k^1} \lambda_{kjl}^{1*} \right], \end{aligned}$$

which in turn implies by using the constraint (iii) of (5.47) that

$$x^\top G y^* + g^\top x \leq (\nu^{1*})^\top d^1 + \sum_{k \in \mathcal{I}_1} \sum_{j=1}^{N_1} \sum_{w=1}^{P_1} \lambda_{kjl}^{1*} b_k^1.$$

Then, for any $x \in S_{\alpha^1}^1$, we have

$$\begin{aligned} x^\top G y^* + g^\top x + h^\top y^* &\leq h^\top y^* + (\nu^{1*})^\top d^1 \\ &+ \sum_{k \in \mathcal{I}_1} \sum_{j=1}^{N_1} \sum_{w=1}^{P_1} \lambda_{kjl}^{1*} b_k^1. \end{aligned} \quad (5.50)$$

Similarly, for any $y \in S_{\alpha^2}^2$, we have

$$\begin{aligned} (x^*)^\top G y + g^\top x^* + h^\top y &\geq g^\top x^* \\ &+ (\nu^{2*})^\top d^2 + \sum_{l \in \mathcal{I}_2} \sum_{j=1}^{N_2} \sum_{w=1}^{P_2} \lambda_{ljw}^{2*} b_l^2. \end{aligned} \quad (5.51)$$

Take $x = x^*$ and $y = y^*$ in (5.50) and (5.51), then from (5.48), we get

$$\begin{aligned} u(x^*, y^*) &= h^\top y^* + (\nu^{1*})^\top d^1 + \sum_{k \in \mathcal{I}_1} \sum_{j=1}^{N_1} \sum_{w=1}^{P_1} \lambda_{kjw}^{1*} b_k^1 \\ &= g^\top x^* + (\nu^{2*})^\top d^2 + \sum_{l \in \mathcal{I}_2} \sum_{j=1}^{N_2} \sum_{w=1}^{P_2} \lambda_{ljw}^{2*} b_l^2. \end{aligned} \quad (5.52)$$

It follows from (5.50), (5.51), and (5.52) that

$$u(x, y^*) \leq u(x^*, y^*) \leq u(x^*, y), \quad \forall x \in S_{\alpha^1}^1, y \in S_{\alpha^2}^2,$$

which in turn implies that (x^*, y^*) is an SPE of the game Z_α . \square

5.6 . Numerical results

5.6.1 . Competition in Financial Market

In this section, we consider a competition of two firms in financial market. They invest in the same set of portfolios. Let $P = \{1, 2, \dots, N_P\}$ be the set of portfolios. Let \mathcal{A}_j be the set of assets in the portfolio j . Assume that the sets \mathcal{A}_j and \mathcal{A}_k are disjoint, for any $j \neq k$. Let $x_k = (x_{kj})_{j \in \mathcal{A}_k}$ be the investment vector of firm 1 in portfolio k and $y_k = (y_{kj})_{j \in \mathcal{A}_k}$ be the investment vector of firm 2 in portfolio k . Let $x = (x_k)_{k \in P}$ and $y = (y_k)_{k \in P}$ be the investment vector of firm 1 (resp. firm 2). The set of investments X of firm 1 is defined as follows

$$X = \left\{ x \mid \sum_{j \in \mathcal{A}_k} x_{kj} = W_k^1, \forall j \in \mathcal{A}_k, k \in P \right\},$$

and the set of investments Y of firm 2 is defined as follows

$$Y = \left\{ y \mid \sum_{j \in \mathcal{A}_k} y_{kj} = W_k^2, \forall j \in \mathcal{A}_k, k \in P \right\},$$

where W_k^i is the total investment of firm i in portfolio k , for any $i = 1, 2$ and $k \in P$. Let $L_k^i = (L_{kj}^i)_{j \in \mathcal{A}_k}$ be a random loss vector of firm i from portfolio k . Then, for a given investment vector x_k and y_k , the total loss of firm 1 (resp. firm 2) caused by portfolio k is defined as $(L_k^1)^\top x_k$ (resp. $(L_k^1)^\top y_k$). Each firm wants to make sure that their random loss is below a maximal allowable loss level with high probability. This condition is modeled by the following inequality

$$\mathbb{P} \left\{ (L_k^1)^\top x_k \leq b_k^1 \right\} \geq \alpha_k^1, \quad (5.53)$$

and

$$\mathbb{P} \left\{ (L_l^2)^\top y_l \leq b_l^2 \right\} \geq \alpha_l^2, \quad (5.54)$$

where b_k^i are deterministic vectors and α_k^i are confidence levels, $i = 1, 2$, $k \in P$. We assume that the true distribution of random loss vectors is unknown, but only known to belong to some uncertainty set \mathcal{D}_k^i defined in Section 5.4. Then, the feasible strategy sets of two firms are given by

$$\inf_{F_k^1 \in \mathcal{D}_k^1} \mathbb{P} \left\{ (L_k^1)^\top x_k \leq b_k^1 \right\} \geq \alpha_k^1, \quad \forall k \in P,$$

and

$$\inf_{F_l^2 \in \mathcal{D}_l^2} \mathbb{P} \left\{ (L_l^2)^\top y_l \leq b_l^2 \right\} \geq \alpha_l^2, \quad \forall l \in P.$$

We assume that the total profit of both firm is zero, i.e., for each profile of strategies $(x, y) \in X \times Y$, if firm 1 gains a profit $u(x, y)$, then firm 2 gains a profit $-u(x, y)$. Firm 1 wants to maximize u w.r.t x , for $y \in S_{\alpha^2}^2$ and firm 2 wants to minimize u w.r.t y , for $x \in S_{\alpha^1}^1$. We assume that u has the form (5.1), i.e., $u(x, y) = x^\top G y + g^\top x + h^\top y$.

In order to find an SPE of (5.8), we solve the two SOCP problems (5.46) and (5.47) using **coneqp** solver in CVXOPT. We compare the uncertainty sets defined in Section (5.4) with the true model, in which we assume that the true distribution of random loss vectors is known and follows Gaussian distribution. In this case, it is well known that the constraints (5.53) and (5.54) are equivalent to SOC constraints [90]. An SPE in true model can be computed by solving an SOCP reformulation [218].

5.6.2 . Case Study

All the numerical results below are performed using Python 3.8.8 on an Intel Core i5-1135G7, Processor 2.4 GHz (8M Cache, up to 4.2 GHz), RAM 16G, 512G SSD. We consider two firms investing in a portfolio consists of four assets, i.e., $P = \{1\}$ and $\mathcal{A}_1 = \{1, 2, 3, 4\}$. We generate randomly the vectors g and h in (5.1) in $[-3, 3]^4$ by the command "`numpy.random.uniform(-3,3,size=(4,1))`". The matrix G in (5.1) is randomly generated by the command "`numpy.random.uniform(-3,3,size=(4,4))`". We take the confidence levels of two firms as $\alpha^1 = \alpha^2 = 0.9$, the total investment of two firms in the portfolio W_1^1 and W_1^2 are randomly generated on $[20, 80]$ by the command "`numpy.random.uniform(20,80)`". The maximal allowable loss levels of two firms b_1^1 and b_1^2 are randomly generated on $[100, 500]$ by the command "`numpy.random.uniform(100,500)`". The probability distribution of the loss of two firms L_1^1 and L_1^2 are assumed to be Normal distributions with mean vector μ_1^1 (resp. μ_1^2) and covariance matrix Σ_1^1 (resp. Σ_1^2). The mean vectors are randomly generated on $[8, 12]^4$ using the command "`numpy.random.uniform(8,12, size=(4,1))`". The covariance matrix are defined as follows

$$\Sigma_1^i = \frac{AA^\top}{4} + \mathbf{I}_4, \quad \forall i = 1, 2,$$

where A is a 4×4 random matrix whose all entries belong to $[0, 1]$ generated by the command "`A=numpy.random.random(size=(4,4))`" and \mathbf{I}_4 denotes 4×4 identity matrix. For any $i = 1, 2$, we define sample mean vector μ_{sample}^i and Σ_{sample}^i by generating randomly a sample of 100 observations $\xi_1^i, \dots, \xi_{100}^i$, which follow Normal distribution with mean vector μ_1^i and covariance matrix Σ_1^i . To do that, we generate a standard Gaussian vector by the command "`x=numpy.random.normal(0,1,4)`". We generate a Gaussian vector with mean vector μ_1^i and Σ_1^i by taking $\xi_j^i = Bx + \mu_1^i$, where B is the Cholesky factorization of Σ_1^i . To get the Cholesky factorization of a matrix, we use the command "`numpy.linalg.cholesky`". The sample mean vector μ_{sample}^i and the covariance matrix Σ_{sample}^i are

defined as follows

$$\mu_{sample}^i = \frac{1}{100} \sum_{j=1}^{100} \xi_j^i,$$

$$\Sigma_{sample}^i = \frac{1}{99} \sum_{j=1}^{100} (\xi_j^i - \mu_{sample}^i)(\xi_j^i - \mu_{sample}^i)^\top.$$

Now, we define other parameters for each model. For the uncertainty set (5.12), we take $\gamma_1^i = 1.1$, for any $i = 1, 2$. For the uncertainty set (5.16), we take $\gamma_{11}^i = \gamma_{12}^i = 1$, for any $i = 1, 2$. We take the uncertainty set (5.21) similarly as the uncertainty set (5.9) by choosing $M = 1$. For the uncertainty set (5.25), we take the radius vector $\epsilon_{\mu,1}^i = (0.1, 0.1, 0.1, 0.1)^4$ and the radius matrix $\epsilon_{\Sigma,1}^i = 0.1 \times \mathbf{I}_4$, for any $i = 1, 2$, where \mathbf{I}_4 is 4×4 identity matrix. For the uncertainty set (5.29), we take $\theta_1^i = 0.05$, for any $i = 1, 2$.

For the above instance, we compute an SPE of the true model, where the true distribution of random loss vectors L_1^1 and L_1^2 follow Gaussian distributions with mean vector μ_1^1 (resp. μ_1^2) and covariance matrix Σ_1^1 (resp. Σ_1^2). We obtain an SPE (x^*, y^*) given by

$$x^* = (18.91, 19.45, 19.45, 20.22)^\top,$$

$$y^* = (19.01, 20.15, 20.45, 18.71)^\top.$$

The profit of firm 1 for this instance is $u(x^*, y^*) = -275.52$. Now, we calculate an SPE of the models defined in Section (5.4). For the uncertainty sets (5.9), (5.12), (5.16), (5.21) and (5.25), we take $\mu_1^i = \mu_{sample}^i$ and $\Sigma_1^i = \Sigma_{sample}^i$, for any $i = 1, 2$. For the uncertainty set (5.29), we assume that the nominal distribution ν_1^i follows a Gaussian distribution with mean vector μ_{sample}^i and covariance matrix Σ_{sample}^i . We compare the optimal profit value of firm 1 in above models with the optimal profit value of firm 1 in the true model. The results are given in Table 5.6. We can see that for this instance, the models defined by ϕ -divergence give better solution than the models defined by moments since the optimal profit value in ϕ -divergence uncertainty sets approximates well the optimal profit value in true model. We also present the time analysis for a large

Table 5.6: List of optimal profit values $u(x^*, y^*)$

True model	Known Mean Known Covariance	Known Mean Unknown Covariance	Unknown Mean Unknown Covariance	Polytopic
-257.52	-221.11	-222.5	-224.8	-221.11
Componentwise Bounds	Kullback Leibler	Variation Distance	Modified χ^2 - distance	Hellinger Distance
-223.3	-255.1	-256.23	-255.8	-253.9

numbers of assets size model by considering the number of assets between 100 and 1000. For each case of number of assets, we randomly generate 10 instances of the known mean known covariance model, where the parameters are defined similarly as above and we calculate the average running time (in seconds) to solve the two optimization problems (5.46) and (5.47). The results are given in Figure 5.6.

It is clear from Figure 5.6 that our optimization problems can be solved efficiently in high dimension up to 1000 assets.

5.7 . Conclusion

We study a more general two player zero-sum game than the model considered in [168] under various moment based and statistical based uncertainty sets. We propose a reformulation of the chance constraints

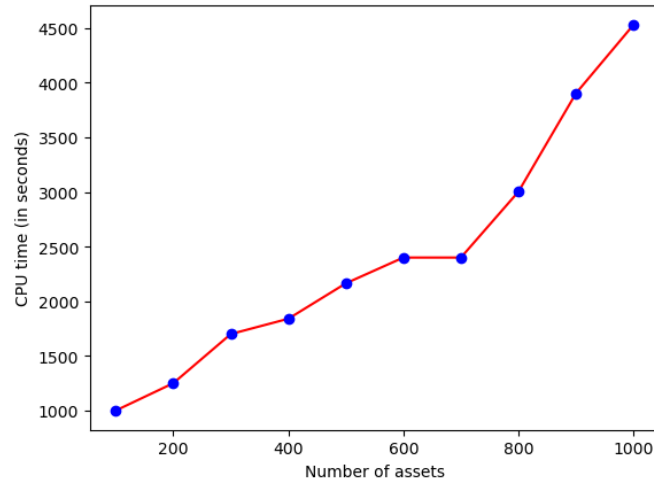


Figure 5.6: CPU time (in seconds) to solve (5.46) and (5.47) in known mean known covariance cases with different number of assets.

using distributionally chance-constrained optimization framework and show that there exists a mixed strategy SPE of the game. Under Slater's condition, the SPE of the game can be obtained from the optimal solutions of a primal-dual pair of SOCPs. We present a competition of two firms in financial market as an application to figure out our theoretical results. The numerical experiments are performed using randomly generated data on the game up to 1000 assets and it is clear from our time analysis that the SOCPs problems can be computed efficiently. For our future works, we will study tractable reformulation of the zero-sum game problem with different payoff structure in a different game model and apply the game problem in a different application to the competition in financial market considered in this chapter.

This chapter corresponds to the reference [170].

Chapter 6 :Distributionally Robust Chance-Constrained Markov Decision Processes

Markov decision process (MDP) is a decision making framework where a decision maker is interested in maximizing the expected discounted value of a stream of rewards received at future stages at various states which are visited according to a controlled Markov chain. Many algorithms including linear programming methods are available in the literature to compute an optimal policy when the rewards and transition probabilities are deterministic. In this chapter, we consider two types of MDP problems where either transition probabilities are known and the reward vector is a random vector whose distribution is partially known or the reverse holds true. We formulate the MDP problem using distributionally robust chance-constrained optimization framework under various types of moment based uncertainty sets, and statistical-distance based uncertainty sets defined using ϕ -divergence and Wasserstein distance metric. In the random reward case, for each type of uncertainty set, we consider the case where the random reward vector has either a full support or a nonnegative support. We prove that the distributionally robust chance-constrained optimization can be reformulated as deterministic optimization using duality theory, which can be solved using Gurobi, Mosek, nonlinear nonconvex solver Baron or BNB solver in YALMIP toolbox of Matlab. As an application, we study a machine replacement problem and illustrate numerical experiments on randomly generated instances.

This chapter offers a comprehensive investigation into the field of distributionally robust chance-constrained Markov Decision Processes (MDPs). To the best of our knowledge, the specific framework of distributionally robust chance-constrained MDPs with random transition probabilities has not been previously explored in the existing literature. By addressing this research gap, we contribute novel insights and results to the field, paving the way for further advancements in this area.

6.1 . Introduction

An MDP is a decision making framework to model the performance of a stochastic system which evolves over time according to a controlled Markov chain. We consider the case where the system has a finite number of states. At time $t = 0$, the system is at some initial state $s_0 \in S$, according to an initial distribution γ , and a decision maker chooses an action $a_0 \in A(s_0)$, where $A(s_0)$ denotes the set of finite number of actions available to the decision maker at state s_0 . As a consequence a reward $R(s_0, a_0)$ is earned and at time $t = 1$, the system moves to a new state s_1 with probability $p(s_0, a_0, s_1)$. The same thing repeats at time $t = 1$ and it continues for the infinite horizon. The decision taken at time t , which could be deterministic or randomized, may depend on the history h_t at time t , where $h_t = (s_0, a_0, s_1, \dots, s_{t-1}, a_{t-1}, s_t)$. Let H_t be the set of all possible histories at time t . A history dependent decision rule f_t at time t is defined as $f_t(h_t) \in \wp(A(s_t))$ for every history h_t with final state s_t , where $\wp(A(s_t))$ denotes the set of probability distributions on the action set $A(s_t)$. A sequence of history dependent decision rules $f^h = (f_t)_{t=0}^\infty$ is called a history dependent policy. A history dependent policy $(f_t)_{t=0}^\infty$ is called a stationary policy if there exists a decision rule f such that $f_t = f$ for all t . We denote a stationary policy, with some abuse of notations, by f and define $f = (f(s))_{s \in S}$ such that $f(s) \in \wp(A(s))$ for every $s \in S$. As per a stationary policy f , whenever the Markov chain visits state s , the decision maker chooses an action a with probability $f(s, a)$. We denote the set of all history dependent and stationary policies by PO_{HD} and PO_S , respectively. A history dependent policy $f^h \in PO_{HD}$ defines a probability measure P^{f^h} over the state and action trajectories, and E^{f^h} denotes the expectation operator corresponding to the probability

measure $P_\gamma^{f^h}$. For a given policy f^h , the expected discounted reward at a discount factor $\alpha \in (0, 1)$ is defined as [8, 195]

$$\begin{aligned} V(f^h) &= (1 - \alpha) \mathbb{E}^{f^h} \left(\sum_{t=0}^{\infty} \alpha^t R(X_t, A_t) \right) \\ &= \sum_{s \in S} \sum_{a \in A(s)} g(f^h; s, a) R(s, a), \end{aligned} \quad (6.1)$$

where X_t and A_t represent the state and the action at time t , respectively. The set $\{g(f^h; s, a)\}_{(s,a)}$ is called a set of occupation measures defined by

$$g(f^h; s, a) = (1 - \alpha) \sum_{t=0}^{\infty} \alpha^t P^{f^h}(\mathbb{X}_t = s, \mathbb{A}_t = a), \quad \forall s \in S, a \in A(s). \quad (6.2)$$

When the running rewards and the transition probabilities are stationary, i.e., $R(X_t = s, A_t = a) = R(s, a)$ and $P(X_{t+1} = s' | X_t = s, A_t = a) = p(s, a, s')$ for all t , we can restrict to stationary policies without loss of optimality [8, 195].

In practice, the MDP model parameters $R(\cdot)$ and $p(\cdot)$ are not known in advance and are estimated from historical data. This leads to errors in the optimal policies [152]. Most efforts to take into account this uncertainty focused on the study of robust MDPs where the rewards or the transition probabilities are known to belong to a prespecified uncertainty set [110, 172, 238, 248, 249]. However, Delage and Mannor [56] showed that the robust MDP approach usually leads to conservative policies. For this reason, a chance-constrained Markov decision process (CCMDP) was introduced in [56], where the controller obtains the expected discounted reward with certain confidence. In [56], the case of random rewards and random transition probabilities are considered separately and it is shown that a CCMDP is equivalent to a second-order cone programming (SOCP) problem when the running reward vector follows a multivariate normal distribution and the transition probabilities are exactly known. When the transition probabilities follow Dirichlet distribution and the running rewards are exactly known, the CCMDP problem becomes intractable and the optimal policies can be computed using approximation methods. Varagapriya et al. [239] considered a CMDP problem with joint chance constraint where the running cost vectors are random vectors and the transition probabilities are known. They proposed two SOCP based approximations which give upper and lower bounds to the CMDP problem if the cost vectors follow multivariate elliptical distributions and the dependence among the constraints is driven by a Gumbel-Hougaard copula.

In many practical situations, it is often the case that only partial information about the underlying distribution is available based on historical data. In that case, a distributionally robust approach, is used to model the uncertainties, which assumes that the true distribution belongs to an uncertainty set based on its partially available information. Such an approach has been used in modelling the uncertainties of many optimization and game problems [117, 145, 215]. There are at least two popular ways to construct an uncertainty set for the distribution of the uncertain parameters. The first one is based on the partial information on moments of the true distribution and the second one is based on the statistical distance between the true distribution and a nominal distribution. The moment-based uncertainty sets assume certain conditions on the first two moments [46, 57, 182]. The statistical distance-based uncertainty sets contain all the distributions which lie inside a ball of small radius and center at a nominal distribution which is usually considered to be an empirical distribution or a normal distribution [158, 117]. To define a distance between the distributions, either a ϕ -divergence [22, 117] or Wasserstein distance metric is used [158, 76, 272].

In this chapter, we consider an infinite horizon MDP with discounted payoff criterion defined in Section 6.1 where the source of randomness comes from reward or transition probabilities. The distribution of the

random vector is not completely known and it is assumed to belong to a given uncertainty set. We formulate the random vectpr with a distributionally robust chance constraint which guarantees the maximum reward for a given policy with at least a given level of confidence. We call this class of MDP as a distributionally robust chance-constrained Markov decision process (DRCCMDP). The random reward vector has either a full support or a nonnegative support. We consider both moment and statistical distance based uncertainty sets. The main contributions of the chapter are as follows.

1. We consider three different types of uncertainty sets based on the moments of the random reward vector and the random transition probabilities. For the case of random rewards, we demonstrate that the DRCCMDP problem can be reformulated as Second-Order Cone Programs (SOCPs) and copositive optimizations. The SOCPs can be effectively solved using the Gurobi solver, while the copositive optimizations can be approximately relaxed as Semi-Definite Programs (SDPs) and solved using the Mosek solver. Regarding random transition probabilities, we propose reformulations using Mixed-Integer Linear Programming (MILPs) or Mixed-Integer Semi-Definite Programming (MISDPs). The MILPs can be solved using the Gurobi solver, while the MISDPs can be handled by the Branch-and-Bound (BNB) solver available in the YALMIP toolbox of Matlab, which might be time consuming.
2. We explore four distinct types of ϕ -divergences to construct uncertainty sets based on statistical distances. Specifically, in the case of random rewards, we propose reformulations using SOCPs. On the other hand, for the case of random transition probabilities, we suggest reformulations using MILPs.
3. When utilizing the Wasserstein distance metric to define a statistical distance-based uncertainty set for the random reward case, we consider the nominal distribution to be an empirical distribution. In the random reward case, where the uncertainty set has full support or nonnegative support, we establish that the DRCCMDP problem can be reformulated as two equivalent problem formulations. The first formulation corresponds to a Mixed Integer Second-Order Cone Programming (MISOCP) problem, which can be efficiently solved using the Gurobi solver. The second formulation corresponds to a biconvex optimization problem, which can be addressed using the nonlinear nonconvex solver Baron, without any guarantee of running time.
4. We illustrate our theoretical results on a machine replacement problem.

The chapter is organized as follows. In Section 6.2, we define a DRCCMDP under a discounted payoff criterion for the case random reward. we developed uncertainty sets using two distinct approaches: moment-based and statistical distance-based. The moment-based approach (Section 6.2.1) considered various scenarios, including known mean with known covariance, known mean with unknown covariance, and unknown mean with unknown covariance. On the other hand, the statistical distance-based approach employed phi-divergence (Section 6.2.2) and Wasserstein distance 6.2.3 to construct the uncertainty sets. Furthermore, we investigated two types of supports: full support and nonnegative support. By exploring these different combinations, we were able to capture a wide range of possible scenarios and provide a comprehensive analysis of the problem at hand. Section 6.3 defines a DRCCMDP under a discounted payoff criterion for the case random transition probabilities, also using two approaches, moment-based (Section 6.3.1) and statistical distance-based (Section 6.3.2). We conduct a thorough analysis by comparing and evaluating the implications and outcomes of different approaches and uncertainty sets in the context of a machine replacement problem and present numerical results in Section 6.4 . We conclude the chapter in Section 6.5.

6.2 . Distributionally robust chance constrained Markov decision process with random reward

We consider an infinite horizon MDP defined in Section 6.1, with a finite state space S , finite action spaces $A(s)$, $s \in S$, an initial distribution γ , where $\sum_{s \in S} \gamma(s) = 1$, $\gamma(s) > 0$, a discount factor $\alpha \in (0, 1)$, where the transition probabilities are exactly known and the running reward vector is a random vector defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ which is denoted as \hat{R} . Therefore, for each realization $\omega \in \Omega$, $\hat{R}(s, a, \omega)$ represents a real valued reward received at state s when an action a is taken. We assume that the random vector \hat{R} does not vary with time. Since the reward \hat{R} is a random vector, transition probabilities p are exactly known, for a given policy f^h , the expected discounted reward defined by (6.1) becomes a random variable and we denote it by $V(f^h, \hat{R})$. Consider the case where the controller is interested in a maximum discounted reward which can be obtained with at least a given confidence level $(1 - \epsilon)$, where $\epsilon \in (0, 1)$. This leads to the following optimization problem

$$\begin{aligned} \text{(CCMDP-R)} \quad & \sup_{y \in \mathbb{R}, f^h \in PO_S} y \\ \text{s.t.} \quad & \mathbb{P}\left(V(f^h, \hat{R}) \geq y\right) \geq 1 - \epsilon. \end{aligned} \quad (6.3)$$

Lemma 6.27. *Given a policy f^h and transition probabilities p . Then, the following system of equations has unique solution $\rho \in \mathbb{R}^{|\mathcal{K}|}$*

$$\begin{cases} \text{(i)} \sum_{s \in S, a \in A(s)} \rho(s, a) \left(\delta(s', s) - \alpha p(s, a, s') \right) = (1 - \alpha) \gamma(s'), \forall s' \in S, \\ \text{(ii)} f^h(s, a) \left(\sum_{a \in A(s)} \rho(s, a) \right) = \rho(s, a), \forall s \in S, a \in A(s), \\ \text{(iii)} \rho(s, a) \geq 0, \forall s \in S, a \in A(s), \sum_{a \in A(s)} \rho(s, a) > 0, \forall s \in S. \end{cases} \quad (6.4)$$

Moreover, the unique solution ρ of (6.4) is the occupation measure defined by (6.2).

Proof. The proof is given in Theorem 3.1 and Theorem 3.2, [8]. □

Using Lemma 6.27, we can represent the optimization problem (6.3) equivalently as follows

$$\begin{aligned} & \sup_{y \in \mathbb{R}, f^h \in PO_S} y \\ \text{s.t.} \quad & \text{(i)} \mathbb{P}_{\hat{R}} \left(\rho^\top \hat{R} \geq y \right) \geq 1 - \epsilon, \text{ (ii) } \rho \text{ is the unique solution of (6.4).} \end{aligned} \quad (6.5)$$

We define the set of occupation measures as

$$\begin{aligned} \mathcal{Q} = \left\{ \rho \in \mathbb{R}^{|\mathcal{K}|} \mid \sum_{(s,a) \in \mathcal{K}} \rho(s, a) \left(\delta(s', s) - \alpha p(s, a, s') \right) = (1 - \alpha) \gamma(s'), \forall s' \in S, \right. \\ \left. \rho(s, a) \geq 0, \forall s \in S, a \in A(s), \sum_{a \in A(s)} \rho(s, a) > 0, \forall s \in S \right\}, \end{aligned}$$

where $\mathcal{K} = \{(s, a) \mid s \in S, a \in A(s)\}$. Then, we can represent the optimization problem (CCMDP-R) in term of decision vector (y, ρ) as follows

$$\begin{aligned} & \sup_{y \in \mathbb{R}, \rho} y \\ \text{s.t.} \quad & \text{(i)} \mathbb{P}_{\hat{R}} \left(\rho^\top \hat{R} \geq y \right) \geq 1 - \epsilon, \text{ (ii) } \rho \in \mathcal{Q}, \end{aligned} \quad (6.6)$$

and the optimal policy f_*^h is given by

$$f_*^h(s, a) = \frac{\rho_*(s, a)}{\left(\sum_{a \in A(s)} \rho_*(s, a)\right)}, \quad \forall s \in S, a \in A(s),$$

where ρ_* is the optimal solution of (6.6). If then vector \hat{R} follows a multivariate normal distribution, the optimization problem (6.6) is equivalent to an SOCP problem [56]. The above result can be generalized for elliptically symmetric distributions because the linear chance constraint (i) present in (6.6) is equivalent to a second order cone constraint [90].

However, in most practical situations, we only have partial information about the underlying probability distributions. Such situations can be handled with the distributionally robust optimization approach, i.e., we assume that the distribution of \hat{R} belongs to an uncertainty set. This leads to the following DRCCMDP problem

$$\begin{aligned} & \sup \quad y \\ \text{s.t.} \quad & \text{(i) } \inf_{F \in \mathcal{D}} \mathbb{P}_F(\rho^\top \hat{R} \geq y) \geq 1 - \epsilon, \text{ (ii) } \rho \in \mathcal{Q}, \end{aligned} \quad (6.7)$$

where F is the distribution of \hat{R} and \mathcal{D} is a given uncertainty set. Note that $\mathbb{P}_F(\rho^\top \hat{R} \leq y - \theta) \leq \mathbb{P}_F(\rho^\top \hat{R} < y) \leq \mathbb{P}_F(\rho^\top \hat{R} \leq y)$ for every $\theta > 0$. Therefore, we can replace $\sup_{F \in \mathcal{D}} \mathbb{P}_F(\rho^\top \hat{R} < y)$ by $\sup_{F \in \mathcal{D}} \mathbb{P}_F(\rho^\top \hat{R} \leq y)$. Then, problem (6.7) is equivalent to the following problem

$$\begin{aligned} & \sup \quad y \\ \text{s.t.} \quad & \text{(i) } \sup_{F \in \mathcal{D}} \mathbb{P}_F(\rho^\top \hat{R} \leq y) \leq \epsilon, \text{ (ii) } \rho \in \mathcal{Q}. \end{aligned} \quad (6.8)$$

In the following sections, we study different types of uncertainty sets of \hat{R} which are defined using i) partial information of moments of \hat{R} , ii) ϕ -divergence distance, and iii) Wasserstein distance. For each uncertainty set, we consider the cases of full and nonnegative supports of \hat{R} . We derive equivalent reformulations of DRCCMDP problem (6.7) (or (6.8) equivalently) for each uncertainty set.

6.2.1 . Moment based uncertainty sets

Let $\mu \in \mathbb{R}^{|\mathcal{K}|}$ be the mean vector and $\Sigma \succ 0$ be a $|\mathcal{K}| \times |\mathcal{K}|$ positive definite matrix. We consider 3 types of moment based uncertainty sets of the distribution of \hat{R} defined as follows:

1. **Uncertainty set with known mean and known covariance matrix:** The uncertainty set of the distribution of \hat{R} in this case is defined by

$$\mathcal{D}_1(\varphi, \mu, \Sigma) = \left\{ F \in \mathcal{M}^+ \left| \begin{array}{l} \mathbb{E}(\mathbf{1}_{\{\hat{R} \in \varphi\}}) = 1, \\ \mathbb{E}(\hat{R}) = \mu, \\ \mathbb{E}[(\hat{R} - \mu)(\hat{R} - \mu)^\top] = \Sigma. \end{array} \right. \right\}, \quad (6.9)$$

2. **Uncertainty set with known mean and unknown covariance matrix:** The uncertainty set of the distribution of \hat{R} in this case is defined by

$$\mathcal{D}_2(\varphi, \mu, \Sigma, \delta_0) = \left\{ F \in \mathcal{M}^+ \left| \begin{array}{l} \mathbb{E}(\mathbf{1}_{\{\hat{R} \in \varphi\}}) = 1, \\ \mathbb{E}(\hat{R}) = \mu, \\ \mathbb{E}[(\hat{R} - \mu)(\hat{R} - \mu)^\top] \preceq \delta_0 \Sigma. \end{array} \right. \right\}, \quad (6.10)$$

3. **Uncertainty set with unknown mean and unknown covariance matrix:** The uncertainty set of the distribution of \hat{R} in this case is defined by

$$\mathcal{D}_3(\varphi, \mu, \Sigma, \delta_1, \delta_2) = \left\{ F \in \mathcal{M}^+ \left| \begin{array}{l} \mathbb{E}(\mathbf{1}_{\{\hat{R} \in \varphi\}}) = 1, \\ [\mathbb{E}(\hat{R}) - \mu]^\top \Sigma^{-1} [\mathbb{E}(\hat{R}) - \mu] \leq \delta_1, \\ \mathbb{E}[(\hat{R} - \mu)(\hat{R} - \mu)^\top] \preceq \delta_2 \Sigma. \end{array} \right. \right\}, \quad (6.11)$$

where $\varphi \subset \mathbb{R}^{|\mathcal{K}|}$ is the support of \hat{R} which we assume to be a convex set, \mathcal{M}^+ is the set of all probability measures on $\mathbb{R}^{|\mathcal{K}|}$ with Borel σ -algebra, $\delta_1 \geq 0, \delta_2, \delta_0 \geq 1, \mu \in \text{RI}(\varphi)$; $\text{RI}(\varphi)$ denotes the relative interior of φ . The notation $A \preceq B$ implies that $B - A$ is a positive semidefinite matrix and $\mathbf{1}_{\{\cdot\}}$ denotes the indicator function. For a good overview of moment-based uncertainty sets, we refer to [140].

DRCCMDP with moment based uncertainty sets under full support

We consider the case when the random vector \hat{R} has full support, i.e., $\varphi = \mathbb{R}^{|\mathcal{K}|}$. We show that the DRCCMDP problem is equivalent to an SOCP problem.

Theorem 6.6. *Consider the DRCCMDP problem (6.7) where the distribution of \hat{R} belongs to the uncertainty sets defined by (6.9), (6.10), (6.11), and the support $\varphi = \mathbb{R}^{|\mathcal{K}|}$. Then, the DRCCMDP (6.7) can be reformulated equivalently as the following SOCP problem*

$$\begin{aligned} \max \quad & y \\ \text{s.t.} \quad & \text{(i) } \mu^\top \rho - \kappa \|\Sigma^{\frac{1}{2}} \rho\|_2 \geq y, \text{ (ii) } \rho \in \mathcal{Q}, \end{aligned} \quad (6.12)$$

where $\|\cdot\|_2$ denotes the Euclidean norm and κ is a real number whose value for each uncertainty set is given in Table 6.7.

Table 6.7: Value of κ for moment based uncertainty set

Uncertainty set	$\mathcal{D} = \mathcal{D}_1(\varphi, \mu, \Sigma)$	$\mathcal{D} = \mathcal{D}_2(\varphi, \mu, \Sigma, \delta_0)$	$\mathcal{D} = \mathcal{D}_3(\varphi, \mu, \Sigma, \delta_1, \delta_2)$
κ	$\sqrt{\frac{1-\epsilon}{\epsilon}}$	$\sqrt{\frac{(1-\epsilon)\delta_0}{\epsilon}}$	$\sqrt{\frac{(1-\epsilon)\delta_2}{\epsilon}} + \sqrt{\delta_1}$

Proof. The proof follows from the fact that for each uncertainty set the distributionally robust chance constraint (i) of (6.7) is equivalent to a second-order cone constraint. The uncertainty set (6.9) has been widely studied in the literature [34, 83]. For the uncertainty sets (6.10) and (6.11), it can be proved using similar arguments used in Lemma 3.1 and Lemma 3.2 of [168] which are based on the one-sided Chebyshev inequality [145]. \square

DRCCMDP with moment based uncertainty sets under nonnegative support

We consider the case where the support of the random vector \hat{R} is a nonnegative orthant of $|\mathcal{K}|$ -dimensional Euclidean space, i.e., $\varphi = \mathbb{R}_+^{|\mathcal{K}|}$. We show that the DRCCMDP problem (6.8) is equivalent to a copositive optimization problem.

Theorem 6.7. *Consider a DRCCMDP problem (6.8) with $\varphi = \mathbb{R}_+^{|\mathcal{K}|}$. Then, the following results hold.*

1. If the distribution of \hat{R} belongs to the uncertainty set defined by (6.9), the DRCCMDP problem (6.8) is equivalent to the following copositive optimization problem

$$\begin{aligned}
& \max \quad y \\
& \text{s.t.} \quad \text{(i)} \quad -\bar{t} - \bar{Q} \circ \Sigma - \bar{q}^\top \mu \leq s\epsilon, \\
& \quad \text{(ii)} \quad \left(\begin{array}{c|c} -\bar{Q} & -\frac{1}{2}\bar{q} + \bar{Q}\mu \\ \hline -\frac{1}{2}\bar{q}^\top + \mu^\top \bar{Q} & -\bar{t} - \mu^\top \bar{Q}\mu \end{array} \right) \in \text{COP}^{|\mathcal{K}|+1}, \\
& \quad \text{(iii)} \quad \left(\begin{array}{c|c} -\bar{Q} & -\frac{1}{2}\bar{q} + \bar{Q}\mu + \rho \\ \hline -\frac{1}{2}\bar{q}^\top + \mu^\top \bar{Q} + \rho^\top & -\bar{t} - \mu^\top \bar{Q}\mu - s - y \end{array} \right) \in \text{COP}^{|\mathcal{K}|+1}, \\
& \quad \text{(iv)} \quad \bar{Q} \in \mathcal{S}^{|\mathcal{K}|}, s \geq 0, \rho \in \mathcal{Q}.
\end{aligned} \tag{6.13}$$

2. If the distribution of \hat{R} belongs to the uncertainty set defined by (6.10), the DRCCMDP problem (6.8) is equivalent to the following copositive optimization problem

$$\begin{aligned}
& \max \quad y \\
& \text{s.t.} \quad \text{(i)} \quad -\bar{t} - \mu^\top \bar{q} - \mu^\top \bar{Q}\mu + \delta_0 \Sigma \circ \bar{Q} \leq s\epsilon, \\
& \quad \text{(ii)} \quad \left(\begin{array}{c|c} \bar{Q} & -\frac{1}{2}\bar{q} - \bar{Q}\mu \\ \hline -\frac{1}{2}\bar{q}^\top - \mu^\top \bar{Q} & -\bar{t} \end{array} \right) \in \text{COP}^{|\mathcal{K}|+1}, \\
& \quad \text{(iii)} \quad \left(\begin{array}{c|c} \bar{Q} & \frac{1}{2}(-\bar{q} + \rho) - \bar{Q}\mu \\ \hline \frac{1}{2}(-\bar{q} + \rho)^\top - \mu^\top \bar{Q} & -\bar{t} - s - y \end{array} \right) \in \text{COP}^{|\mathcal{K}|+1}, \\
& \quad \text{(iv)} \quad \bar{Q} \in \mathcal{S}_+^{|\mathcal{K}|}, s \geq 0, \rho \in \mathcal{Q}.
\end{aligned} \tag{6.14}$$

3. If the distribution of \hat{R} belongs to the uncertainty set defined by (6.11), the DRCCMDP problem (6.8) is equivalent to the following copositive optimization problem

$$\begin{aligned}
& \max \quad y \\
& \text{s.t.} \quad \text{(i)} \quad \bar{r} + \bar{t} \leq s\epsilon, \\
& \quad \text{(ii)} \quad \left(\begin{array}{c|c} \bar{Q} & \frac{1}{2}\bar{q} \\ \hline \frac{1}{2}\bar{q}^\top & \bar{r} \end{array} \right) \in \text{COP}^{|\mathcal{K}|+1}, \\
& \quad \text{(iii)} \quad \bar{t} \geq (\delta_2 \Sigma + \mu \rho^\top) \circ \bar{Q} + \rho^\top \bar{q} + \sqrt{\delta_1} \|\Sigma^{\frac{1}{2}}(\bar{q} + 2\bar{Q}\mu)\|_2, \\
& \quad \text{(iv)} \quad \left(\begin{array}{c|c} \bar{Q} & \frac{1}{2}(\bar{q} + \rho) \\ \hline \frac{1}{2}(\bar{q} + \rho)^\top & \bar{r} - s - y \end{array} \right) \in \text{COP}^{|\mathcal{K}|+1}, \\
& \quad \text{(v)} \quad \bar{Q} \in \mathcal{S}_+^{|\mathcal{K}|}, s \geq 0, \rho \in \mathcal{Q},
\end{aligned} \tag{6.15}$$

where $\text{COP}^{|\mathcal{K}|+1} = \left\{ M \in \mathcal{S}^{|\mathcal{K}|+1} \mid x^\top M x \geq 0, \forall x \in \mathbb{R}_+^{|\mathcal{K}|+1} \right\}$, \mathcal{S}^n is the set of all real symmetric matrix of size $n \times n$, \mathcal{S}_+^n is the set of positive semidefinite matrices of size $n \times n$, \circ denotes the Frobenius inner product and $\left(\begin{array}{c|c} & \\ \hline & \end{array} \right)$ denotes a block matrix (or a partitioned matrix).

In order to prove the first result of Theorem 6.7, we need the following lemma.

Lemma 6.28. Consider an optimization problem

$$\sup_{F \in \mathcal{D}_1(\varphi, \mu, \Sigma)} \mathbb{P}_F(\rho^\top \hat{R} \leq y), \quad (6.16)$$

where $\varphi = \mathbb{R}_+^{|\mathcal{K}|}$. If the feasible set of (6.16) is non-empty, the dual of (6.16) is given by

$$\begin{aligned} \inf \quad & -t - Q \circ \Sigma - q^\top \mu \\ \text{s.t.} \quad & \text{(i) } \mathbf{1}_{\{\rho^\top \xi \leq y\}} + q^\top \xi + \xi^\top Q \xi - 2\xi^\top Q \mu + \mu^\top Q \mu + t \leq 0, \forall \xi \in \mathbb{R}_+^{|\mathcal{K}|}, \\ & \text{(ii) } Q \in \mathcal{S}^{|\mathcal{K}|}, \end{aligned}$$

and the strong duality holds.

Proof. Consider the optimization problem

$$\begin{aligned} v_P(\mu, \Sigma) = \sup_{F \in \mathcal{C}^+} \int_{\varphi} \mathbf{1}_{\{\rho^\top \hat{R} \leq y\}} dF(\hat{R}) \\ \text{s.t.} \quad & \text{(i) } \int_{\varphi} dF(\hat{R}) = 1, \text{ (ii) } \int_{\varphi} (\hat{R} - \mu)(\hat{R} - \mu)^\top dF(\hat{R}) = \Sigma, \\ & \text{(iii) } \int_{\varphi} \hat{R} dF(\hat{R}) = \mu, \end{aligned} \quad (6.17)$$

where \mathcal{C}^+ is the set of all positive measures on $\mathbb{R}_+^{|\mathcal{K}|}$. The dual problem of (6.17) is given by

$$\begin{aligned} v_D(\mu, \Sigma) = \inf \quad & -t - Q \circ \Sigma - q^\top \mu \\ \text{s.t.} \quad & \text{(i) } \mathbf{1}_{\{\rho^\top \xi \leq y\}} + q^\top \xi + \xi^\top Q \xi - 2\xi^\top Q \mu + \mu^\top Q \mu + t \leq 0, \forall \xi \in \mathbb{R}_+^{|\mathcal{K}|}, \\ & \text{(ii) } Q \in \mathcal{S}^{|\mathcal{K}|}, \end{aligned} \quad (6.18)$$

where t , q , and Q are the dual variables associated with the constraints (i), (ii) and (iii) of (6.17), respectively. In Theorem 3.4 of [46], under the assumption $\mu \in \text{RI}(\varphi)$, the authors show that the Dirac distribution δ_μ lies in the relative interior of the distributional uncertainty set which implies that the weaker condition of Proposition 3.4 of [207] holds. However, it is not trivial to find a strictly feasible point inside our distributional uncertainty set. Let $(t_j^*, Q_j^*, q_j^*)_{j \in \mathbb{N}}$ be a sequence of feasible solutions of (6.18) such that

$$-t_j^* - Q_j^* \circ \Sigma - q_j^{*\top} \mu \rightarrow v_D(\mu, \Sigma), \text{ as } j \rightarrow \infty. \quad (6.19)$$

For each $j \in \mathbb{N}$, let $r_j^* = \max(0, q_j^*) - q_j^*$, where $\max(0, q_j^*)$ denotes a $|\mathcal{K}|$ -dimensional vector with i^{th} component equal to the maximum value between 0 and the i^{th} component of q_j^* , for every $i = 1, \dots, |\mathcal{K}|$. Let ϵ_j be a strictly positive decreasing sequence such that $\epsilon_j r_j^* \rightarrow 0$ componentwise and $\epsilon_j \rightarrow 0$, when $j \rightarrow \infty$. Let $x_j = \epsilon_j \mathbb{I}$, where \mathbb{I} denotes the vector with all components equal to 1. Then, $r_j^{*\top} x_j \rightarrow 0$ as $j \rightarrow \infty$. For each $j \in \mathbb{N}$, consider the following conic optimization problem

$$\begin{aligned} v_P^j(\mu, \Sigma) = \sup_{F \in \mathcal{C}^+} \int_{\varphi} \mathbf{1}_{\{\rho^\top R \leq y\}} dF(R) \\ \text{s.t.} \quad & \text{(i) } \int_{\varphi} dF(R) = 1, \text{ (ii) } \int_{\varphi} (R - \mu)(R - \mu)^\top dF(R) = \Sigma, \\ & \text{(iii) } \mu \leq \int_{\varphi} R dF(R) \leq \mu + x_j. \end{aligned} \quad (6.20)$$

The dual problem of (6.20) is given by

$$\begin{aligned} v_D^j(\mu, \Sigma) &= \inf -t - Q \circ \Sigma + (r - h)^T \mu + r^T x_j \\ \text{s.t. } & \text{(i) } \mathbf{1}_{\{\rho^T \xi \leq y\}} + (h - r)^T \xi + \xi^T Q \xi - 2\xi^T Q \mu + \mu^T Q \mu + t \leq 0, \forall \xi \in \mathbb{R}_+^{|\mathcal{K}|}, \\ & \text{(ii) } h, r \in \mathbb{R}_+^{|\mathcal{K}|}, Q \in \mathcal{S}^{|\mathcal{K}|}, \end{aligned} \quad (6.21)$$

where t, Q, r and h are the dual variables of the constraint (i), (ii) and (iii) of (6.20), respectively. The vector (t, Q, h, r) such that $t = t_j^*, Q = Q_j^*, h = \max(0, q_j^*), r = r_j^*$ is a feasible solution of (6.21). Hence,

$$v_D^j(\mu, \Sigma) \leq -t_j^* - Q_j^* \circ \Sigma - q_j^{*T} \mu + r_j^{*T} x_j, \forall j \in \mathbb{N}. \quad (6.22)$$

Since the feasibility set of (6.16) is non-empty, there exists a nonnegative distribution F^* such that $\mathbb{E}(F^*) = \mu$ and $\text{Cov}(F^*) = \Sigma$. Let F_j be a distribution with support $\varphi_j := \{\xi \mid \xi \in \mathbb{R}_+^{\mathcal{K}}, \xi \geq \frac{x_j}{2}, \text{ componentwise}\}$, defined by

$$F^*(\xi) = F_j(\xi + \frac{x_j}{2}), \forall \xi \in \mathbb{R}_+^{\mathcal{K}}.$$

It is clear that F_j is a feasible solution of (6.20) and $\varphi_j \subset \text{RI}(\varphi)$. Hence, F_j belongs to the relative interior of the distributional uncertainty set which implies that strong duality holds, i.e., $v_D^j(\mu, \Sigma) = v_P^j(\mu, \Sigma)$ for all $j \in \mathbb{N}$. Since the objective function of (6.20) is a continuous function of F and $x_j \rightarrow 0$ as $j \rightarrow \infty$, then $v_P^j(\mu, \Sigma) \rightarrow v_P(\mu, \Sigma)$ as $j \rightarrow \infty$. Therefore, it is sufficient to prove that $v_D^j(\mu, \Sigma) \rightarrow v_D(\mu, \Sigma)$ as $j \rightarrow \infty$. It is clear that the feasible sets of (6.21) and (6.18) are equivalent and objective function of (6.21) has additional positive term. Therefore,

$$v_D^j(\mu, \Sigma) \geq v_D(\mu, \Sigma), \forall j \in \mathbb{N}. \quad (6.23)$$

Using (6.19), (6.22) and (6.23) and the fact that $r_j^{*T} x_j \rightarrow 0$ as $j \rightarrow \infty$, we have $v_D^j(\mu, \Sigma) \rightarrow v_D(\mu, \Sigma)$ as $j \rightarrow \infty$. \square

Proof of Theorem 6.7. 1. Let the distribution of \hat{R} belongs to the uncertainty set $\mathcal{D}_1(\phi, \mu, \Sigma)$. Using Lemma 6.28, the optimization problem (6.8) is equivalent to the following problem

$$\begin{aligned} \sup \quad & y \\ \text{s.t. } & \text{(i) } -t - Q \circ \Sigma - q^T \mu \leq \epsilon, \\ & \text{(ii) } q^T \xi + \xi^T Q \xi - 2\xi^T Q \mu + \mu^T Q \mu + t \leq 0, \forall \xi \in \mathbb{R}_+^{|\mathcal{K}|}, \\ & \text{(iii) } 1 + q^T \xi + \xi^T Q \xi - 2\xi^T Q \mu + \mu^T Q \mu + t \leq 0, \forall \xi \in \mathbb{R}_+^{|\mathcal{K}|}, \rho^T \xi \leq y, \\ & \text{(iv) } Q \in \mathcal{S}^{|\mathcal{K}|}, \rho \in \mathcal{Q}. \end{aligned} \quad (6.24)$$

The constraint (ii) of (6.24) is equivalent to:

$$(\xi^T, 1)U(\xi^T, 1)^T \geq 0, \forall \xi \in \mathbb{R}_+^{|\mathcal{K}|},$$

where $U \in \mathcal{S}^{|\mathcal{K}|+1}$ such that

$$U = \left(\begin{array}{c|c} -Q & -\frac{1}{2}q + Q\mu \\ \hline -\frac{1}{2}q^T + \mu^T Q & -t - \mu^T Q \mu \end{array} \right).$$

Here, $(\xi^\top, 1)$ denotes the row vector of size $1 \times (|\mathcal{K}| + 1)$ with the last component equals 1 and the first $|\mathcal{K}|$ components are the components of ξ . The above inequality can be rewritten as

$$x^\top U x \geq 0, \forall x \in \mathbb{R}_+^{|\mathcal{K}|+1}, \|x\|_2 = 1.$$

Using Proposition 5.1 in [102], we deduce that the constraint (ii) of (6.24) is equivalent to $U \in \text{COP}^{|\mathcal{K}|+1}$. The constraint (iii) of (6.24) is equivalent to:

$$-1 + (\xi^\top, 1)U(\xi^\top, 1)^\top \geq 0, \forall \xi \in \mathbb{R}_+^{|\mathcal{K}|}, \rho^\top \xi \leq y. \quad (6.25)$$

Define,

$$\begin{cases} s_P = \min_{\xi \in \mathbb{R}_+^{|\mathcal{K}|}} \max_{\lambda \geq 0} \mathcal{L}(\lambda, \xi, U, \rho, y). \\ s_D = \max_{\lambda \geq 0} \min_{\xi \in \mathbb{R}_+^{|\mathcal{K}|}} \mathcal{L}(\lambda, \xi, U, \rho, y). \end{cases} \quad (6.26)$$

where $\mathcal{L}(\lambda, \xi, U, \rho, y) = -1 + (\xi^\top, 1)U(\xi^\top, 1)^\top + \lambda(\rho^\top \xi - y)$. In [46], the authors use the Sion's minimax theorem [220] to interchange the minimum and the maximum. However, since φ is not compact, we cannot apply the Sion's minimax theorem directly in this case. We show that φ can be restricted to a compact set without loss of optimality. For a given U and ρ , we have

$$\begin{aligned} s_P &\leq \max_{\lambda \geq 0} \mathcal{L}(\lambda, 0, U, \rho, y) \\ &= \max_{\lambda \geq 0} (-t - \mu^\top Q \mu - \lambda y - 1) = -t - \mu^\top Q \mu - 1 < \infty \end{aligned} \quad (6.27)$$

Therefore, using the min-max inequality $s_D \leq s_P < \infty$. Let $U_i = U + \frac{1}{2^i} \mathbf{I}_{|\mathcal{K}|+1}$ and $\rho_i = \rho + \frac{1}{2^i} \mathbf{1}$, for every $i \in \mathbb{N}$, where $\mathbf{I}_{|\mathcal{K}|+1}$ denotes the identity matrix of size $|\mathcal{K}| + 1$, $\mathbf{1}$ denotes the vector with all components equal to 1. It is clear from the construction that $\rho_i > 0$ componentwise. Since, \mathcal{L} is a continuous function w.r.t U and ρ , we have

$$\mathcal{L}(\lambda, \xi, U_i, \rho_i, y) \xrightarrow{i \rightarrow \infty} \mathcal{L}(\lambda, \xi, U, \rho, y), \forall \xi \in \mathbb{R}_+^{|\mathcal{K}|}, \lambda \geq 0.$$

Since, the min and max operators preserve the continuity, we have

$$\begin{aligned} \min_{\xi \in \mathbb{R}_+^{|\mathcal{K}|}} \max_{\lambda \geq 0} \mathcal{L}(\lambda, \xi, U_i, \rho_i, y) &\xrightarrow{i \rightarrow \infty} \min_{\xi \in \mathbb{R}_+^{|\mathcal{K}|}} \max_{\lambda \geq 0} \mathcal{L}(\lambda, \xi, U, \rho, y). \\ \max_{\lambda \geq 0} \min_{\xi \in \mathbb{R}_+^{|\mathcal{K}|}} \mathcal{L}(\lambda, \xi, U_i, \rho_i, y) &\xrightarrow{i \rightarrow \infty} \max_{\lambda \geq 0} \min_{\xi \in \mathbb{R}_+^{|\mathcal{K}|}} \mathcal{L}(\lambda, \xi, U, \rho, y). \end{aligned}$$

This implies that, if $s_P = s_D$ holds for any $U_i, \rho_i, i \in \mathbb{N}$, it also holds for U, ρ . For an arbitrary U_i and ρ_i , let the the optimal solutions of minimax and maximin problems defined by (6.26) are (ξ_P, λ_P) and (ξ_D, λ_D) , respectively. We prove that ξ_P and ξ_D are bounded, i.e., there exists $\Upsilon_P > 0$ and $\Upsilon_D > 0$ depending on U_i, ρ_i and y such that $\|\xi_P\|_2 \leq \Upsilon_P$ and $\|\xi_D\|_2 \leq \Upsilon_D$. It is clear that $\lambda_P = 0$ and $\rho_i^\top \xi_P - y \leq 0$. Hence, we have

$$\begin{aligned} s_P &= -1 + (\xi_P^\top, 1)U_i(\xi_P^\top, 1)^\top, \\ &= -1 + (\xi_P^\top, 1)U(\xi_P^\top, 1)^\top + \frac{1}{2^i} \|\xi_P\|_2^2 + \frac{1}{2^i}. \end{aligned}$$

From constraint (ii) of (6.24), it follows that $(\xi_p^T, 1)U(\xi_p^T, 1)^T \geq 0$. Therefore, if $\|\xi_p\|_2 \rightarrow \infty$, $s_p \rightarrow \infty$. Therefore, $\|\xi_p\|_2$ is bounded by some real number $\Upsilon_P > 0$ which depends on U_i, ρ_i and y . As $\xi \in \mathbb{R}_+^{|\mathcal{K}|}$ and $\rho_i > 0$, componentwise, we have

$$\liminf_{\|\xi\|_2 \rightarrow \infty} \lambda(\xi)(\rho_i^T \xi - y) \geq 0,$$

for any $\lambda(\xi) \geq 0$. Then,

$$\begin{aligned} s_D &= -1 + (\xi_D^T, 1)U_i(\xi_D^T, 1)^T + \lambda_D(\rho_i^T \xi_D - y), \\ &= -1 + (\xi_D^T, 1)U(\xi_D^T, 1)^T + \frac{1}{2^i} \|\xi_D\|_2^2 + \frac{1}{2^i} + \lambda_D(\rho_i^T \xi_D - y). \end{aligned}$$

It is clear that $\frac{1}{2^i} \|\xi_D\|_2^2 \rightarrow \infty$ and the other terms are lower bounded by some nonnegative number. Therefore, $s_D \rightarrow \infty$ when $\|\xi_D\|_2 \rightarrow \infty$. Hence, $\|\xi_D\|_2$ is bounded by some real number $\Upsilon_D > 0$ which depends on U_i, ρ_i and y . Let $\Upsilon = \max(\Upsilon_P, \Upsilon_D)$. Then, (6.26) is equivalent to

$$\begin{aligned} s_P &= \min_{\xi \in \mathbb{R}_+^{|\mathcal{K}|}, \|\xi\|_2 \leq \Upsilon} \max_{\lambda \geq 0} \mathcal{L}(\lambda, \xi, U^i, \rho^i, y). \\ s_D &= \max_{\lambda \geq 0} \min_{\xi \in \mathbb{R}_+^{|\mathcal{K}|}, \|\xi\|_2 \leq \Upsilon} \mathcal{L}(\lambda, \xi, U^i, \rho^i, y). \end{aligned}$$

Note that the set $\{\xi \mid \xi \in \mathbb{R}_+^{|\mathcal{K}|}, \|\xi\|_2 \leq \Upsilon\}$ is compact. Therefore, from Sion's minimax theorem $s_P = s_D$ for every $U_i, \rho_i, i \in \mathbb{N}$. For any ξ such that $\rho^T \xi > y$, it is easy to see that

$$\max_{\lambda \geq 0} \mathcal{L}(\lambda, \xi, U, \rho, y) = \infty$$

The condition $s_P < \infty$ gives $\rho^T \xi \leq y$ and $\lambda = 0$ which in turn implies that

$$s_P = \min_{\rho^T \xi \leq y} \mathcal{L}(0, \xi, U, \rho, y) \geq 0.$$

Therefore, (6.25) is equivalent to $s_D \geq 0$. Then, there exists a sequence of nonnegative numbers $\lambda_j \geq 0$ and a decreasing sequence of positive numbers $\theta_j > 0$, such that $\theta_j \rightarrow 0$ as $j \rightarrow \infty$, for which the following condition holds

$$\begin{cases} -1 + (\xi^T, 1)U(\xi^T, 1)^T + \lambda_j(\rho^T \xi - y) \geq -\theta_j, \forall \xi \in \mathbb{R}_+^{|\mathcal{K}|}, j \in \mathbb{N}, \\ \lambda_j \geq 0, \forall j \in \mathbb{N}. \end{cases} \quad (6.28)$$

For each $j \in \mathbb{N}$, define

$$Fea(\theta_j) = \{(U, \rho, y, \lambda) \mid -1 + (\xi^T, 1)U(\xi^T, 1)^T + \lambda(\rho^T \xi - y) \geq -\theta_j, \lambda \geq 0\}.$$

The feasible region defined by (6.28) is equivalent to $\bigcap_{j \in \mathbb{N}} Fea(\theta_j)$. For any $i < j$, $Fea(\theta_j) \subset Fea(\theta_i)$.

Therefore, $Fea(\theta_j) \downarrow \bigcap_{i \in \mathbb{N}} Fea(\theta_i)$ as $j \rightarrow \infty$. The feasible set $Fea(\theta_j)$ as $j \rightarrow \infty$ is given by

$$\begin{cases} (\xi^T, 1)Z(\xi^T, 1)^T \geq 0, \forall \xi \in \mathbb{R}_+^{|\mathcal{K}|}, \\ \lambda \geq 0, \end{cases} \quad (6.29)$$

where $Z \in \mathcal{S}^{|\mathcal{K}|+1}$ and $Z = \left(\begin{array}{c|c} -Q & -\frac{1}{2}q + Q\mu + \lambda\rho \\ \hline -\frac{1}{2}q^T + \mu^T Q + \lambda\rho^T & -t - \mu^T Q\mu - 1 - \lambda y \end{array} \right)$. Using similar arguments as above, the constraint (6.29) is equivalent to

$$Z \in \text{COP}^{|\mathcal{K}|+1}, \lambda \geq 0. \quad (6.30)$$

This implies that the constraint (iii) of (6.24) is equivalent to (6.30). Hence, DRCCMDP problem (6.8) can be rewritten as follows

$$\begin{aligned} & \max \quad y \\ \text{s.t.} \quad & (i) \quad -t - Q \circ \Sigma - q^T \mu \leq \epsilon, \\ & (ii) \quad \left(\begin{array}{c|c} -Q & -\frac{1}{2}q + Q\mu \\ \hline -\frac{1}{2}q^T + \mu^T Q & -t - \mu^T Q\mu \end{array} \right) \in \text{COP}^{|\mathcal{K}|+1}, \\ & (iii) \quad \left(\begin{array}{c|c} -Q & -\frac{1}{2}q + Q\mu + \lambda\rho \\ \hline -\frac{1}{2}q^T + \mu^T Q + \lambda\rho^T & -t - \mu^T Q\mu - 1 - \lambda y \end{array} \right) \in \text{COP}^{|\mathcal{K}|+1}, \\ & (iv) \quad Q \in \mathcal{S}^{|\mathcal{K}|}, \lambda \geq 0, \rho \in \mathcal{Q}. \end{aligned} \quad (6.31)$$

The optimization problem (6.31) is a bilinear copositive optimization problem, where the source of bilinearity comes from the term $\lambda\rho$. Note that we can restrict the feasibility set of (6.31) to $\lambda > 0$, since if $\lambda = 0$, either the feasibility set of (6.31) is empty or the optimal value of (6.31) is infinity. Let $\frac{1}{\lambda} = s$, $\bar{t} = \frac{t}{\lambda}$, $\bar{Q} = \frac{Q}{\lambda}$, $\bar{q} = \frac{q}{\lambda}$, we obtain the copositive optimization problem (6.13)

- Let the distribution of \hat{R} belongs to the uncertainty set $\mathcal{D}_2(\varphi, \mu, \Sigma, \delta_0)$. From Theorem 3.4 [46], the dual of the optimization problem $\sup_{F \in \mathcal{D}} \mathbb{P}_F(\rho^T \hat{R} \leq y)$ can be written as

$$\begin{aligned} & \inf \quad (-t - \mu^T q - \mu^T Q\mu + \delta_0 \Sigma \circ Q) \\ \text{s.t.} \quad & (i) \quad \mathbf{1}_{\{\rho^T \xi \leq y\}} + t + q^T \xi - \xi^T Q \xi + 2\mu^T Q \xi \leq 0, \forall \xi \in \mathbb{R}_+^{|\mathcal{K}|}, \\ & (ii) \quad Q \in \mathcal{S}_+^{|\mathcal{K}|}, \end{aligned}$$

and the strong duality holds. The rest of the proof follows from the similar arguments used for the case of the uncertainty set $\mathcal{D}_1(\varphi, \mu, \Sigma)$.

- If the distribution of \hat{R} belongs to the uncertainty set $\mathcal{D}_3(\varphi, \mu, \Sigma, \delta_1, \delta_2)$, using Lemma 1 of [57] the dual of problem $\sup_{F \in \mathcal{D}} \mathbb{P}_F(\rho^T \hat{R} \leq y)$ is given by

$$\begin{aligned} & \inf \quad (r + t) \\ \text{s.t.} \quad & (i) \quad r \geq \mathbf{1}_{\{\rho^T \xi \leq y\}} - \xi^T Q \xi - \xi^T q, \forall \xi \in \mathbb{R}_+^{|\mathcal{K}|}, \\ & (ii) \quad t \geq (\delta_2 \Sigma + \mu \rho^T) \circ Q + \rho^T q + \sqrt{\delta_1} \|\Sigma^{\frac{1}{2}}(q + 2Q\mu)\|_2, \\ & (iii) \quad Q \in \mathcal{S}_+^{|\mathcal{K}|}, \end{aligned}$$

and strong duality holds. Again, the rest of the proof follows using similar arguments used in the case of $\mathcal{D}_1(\varphi, \mu, \Sigma)$. □

6.2.2 . Uncertainty set with ϕ -divergence distance

We consider an uncertainty set defined using statistical distance metric called ϕ -divergence. In such uncertainty set, a nominal distribution is known to the decision maker based on the available estimated data. The decision maker believes that the true distribution of \hat{R} belongs to a ball of radius θ_ϕ and centered at a nominal distribution ν and the distance between the true distribution and ν is given by a ϕ -divergence. We show that the DRCCMDP problem (6.7) is equivalent to an SOCP problem for various ϕ -divergences.

Definition 6.13. The ϕ -divergence distance between two probability measures ν_1 and ν_2 with densities f_{ν_1} and f_{ν_2} , respectively, and support φ is given by

$$I_\phi(\nu_1, \nu_2) = \int_{\varphi} \phi \left(\frac{f_{\nu_1}(\xi)}{f_{\nu_2}(\xi)} \right) f_{\nu_2}(\xi) d\xi.$$

For different choices of ϕ , we refer to [22] and [176]. Let $\nu \in \mathcal{M}^+$ be a nominal distribution with a density function f_ν . The uncertainty set of the distribution of \hat{R} based on ϕ -divergence is defined by

$$\mathcal{D}_4(\nu, \theta_\phi) = \{F \in \mathcal{M}^+ \mid I_\phi(F, \nu) \leq \theta_\phi\}, \quad (6.32)$$

where $\theta_\phi > 0$.

Definition 6.14. The conjugate of ϕ is a function $\phi^* : \mathbb{R} \rightarrow \mathbb{R} \cup \infty$ such that

$$\phi^*(r) = \sup_{t \geq 0} \{rt - \phi(t)\}, \quad \forall r \in \mathbb{R}.$$

We study 4 cases of ϕ -divergences whose conjugates are given in Table 5.4.

Theorem 6.8. Consider the DRCCMDP problem (6.7) under the uncertainty set defined by (6.32) for the ϕ -divergences listed in Table 5.5. If the reference distribution ν is a normal distribution with mean vector μ_ν and positive definite covariance matrix Σ_ν , the DRCCMDP problem (6.7) is equivalent to the following SOCP problem

$$\begin{aligned} \max \quad & y \\ \text{s.t.} \quad & \text{(i) } \rho^\top \mu_\nu - \Phi^{(-1)}[f(\theta_\phi, \epsilon)] \|\Sigma_\nu^{-\frac{1}{2}} \rho\|_2 \geq y, \text{ (ii) } \rho \in \mathcal{Q}, \end{aligned} \quad (6.33)$$

where $\Phi^{(-1)}$ is the quantile of the standard normal distribution and the values of θ_ϕ , ϵ and $f(\theta_\phi, \epsilon)$ for different ϕ -divergences are given in Table 5.5.

Proof. The proof follows from the same arguments as the proof of Lemma 5.26. \square

6.2.3 . Uncertainty set with Wasserstein distance

We consider an uncertainty set defined using statistical distance metric called Wasserstein distance. We show that the DRCCMDP problem (6.8) is tractable if the reference distribution ν follows a discrete distribution whose scenarios are taken from historical data. We refer to Villani [241, 242] for more details of the Wasserstein distance metric.

Let φ be a closed, convex subset of $\mathbb{R}^{|\mathcal{K}|}$ and $p \in [1, \infty)$. Let $\mathcal{B}(\varphi)$ denotes the Borel σ -algebra on φ . Let $\mathcal{P}(\varphi)$ be the set of all probability measures defined on $\mathcal{B}(\varphi)$ and $\mathcal{P}_p(\varphi)$ denote the subset of $\mathcal{P}(\varphi)$ with finite p -moment and it is defined as

$$\mathcal{P}_p(\varphi) = \left\{ \mu \in \mathcal{P}(\varphi) \mid \int_{\xi \in \varphi} \|\xi - \xi_0\|_2^p \mu(d\xi) < \infty \text{ for some } \xi_0 \in \varphi \right\}.$$

It follows from the triangle inequality that the above definition of $\mathcal{P}_p(\varphi)$ does not depend on ξ_0 .

Definition 6.15 (Wasserstein distance). *The Wasserstein distance $W_p(\mu, \nu)$ between $\nu_1, \nu_2 \in \mathcal{P}_p(\varphi)$ is defined by*

$$W_p(\nu_1, \nu_2) = \left(\inf_{\gamma \in \mathcal{P}_{\nu_1, \nu_2}(\varphi \times \varphi)} \int_{\varphi \times \varphi} \|x - z\|_2^p \gamma(dx, dz) \right)^{\frac{1}{p}},$$

where $\mathcal{P}_{\nu_1, \nu_2}(\varphi \times \varphi)$ denotes the set of all probability measures defined on $\mathcal{B}(\varphi \times \varphi)$ such that the marginal laws are ν_1 and ν_2 .

The uncertainty set using Wasserstein distance is defined by

$$\mathcal{D}_5(\varphi, \nu, p, \theta_W) = \{F \in \mathcal{P}_p(\varphi) \mid W_p(F, \nu) \leq \theta_W\}, \quad (6.34)$$

where $\nu \in \mathcal{P}_p(\varphi)$ and $\theta_W > 0$.

Lemma 6.29. *Consider an optimization problem*

$$\sup_{F \in \mathcal{D}_5(\varphi, \nu, p, \theta_W)} \mathbb{P}_F(\rho^\top \hat{R} \leq y). \quad (6.35)$$

Then, the dual problem of (6.35) is given by

$$\inf_{\lambda \geq 0} \left\{ \lambda \theta_W^p - \int_{\varphi} \inf_{z \in \varphi} [\lambda \|x - z\|_2^p - \mathbf{1}_{\{\rho^\top z \leq y\}}] \nu(dx) \right\}, \quad (6.36)$$

such that the strong duality holds and the optimal values of the primal and the dual problems are finite.

Proof. Let Ξ be a Polish space with metric d , $\mathcal{P}(\Xi)$ be the set of Borel probability measures on Ξ , $\nu \in \mathcal{P}(\Xi)$ and $\Psi \in L^1(\nu)$, where $L^1(\nu)$ represents the L^1 space of ν -measurable functions. It follows from Theorem 1 of [76] that the following strong duality holds

$$\begin{aligned} & \sup_{\mu \in \mathcal{P}(\Xi)} \left\{ \int_{\Xi} \Psi(\xi) \mu(d\xi) \mid W_p(\mu, \nu) \leq \theta_W \right\} \\ &= \inf_{\lambda \in \mathbb{R}, \lambda \geq 0} \left\{ \lambda \theta_W^p - \int_{\Xi} \inf_{\xi \in \Xi} [\lambda d^p(\xi, \zeta) - \Psi(\xi)] \nu(d\zeta) \right\} < \infty, \end{aligned} \quad (6.37)$$

provided the growth factor given by Definition 4 of [76] is finite. We apply this result in our case by choosing $\Xi = \varphi$, d as an Euclidean metric and $\Psi(\xi) = \mathbf{1}_{\{\rho^\top \xi \leq y\}}$ for all $\xi \in \varphi$. For this choice of $\Psi(\xi)$, it is easy to see from Definition 4 of [76] that the growth factor is zero. Since $\{\xi \in \varphi \mid \rho^\top \xi \leq y\}$ is a closed set, it is a Borel measurable set. Hence, it is clear that $\Psi \in L^1(\nu)$ for all $\nu \in \mathcal{P}(\varphi)$. Then, (6.37) reduces to

$$\sup_{F \in \mathcal{D}_5(\varphi, \nu, p, \theta_W)} \mathbb{P}_F(\rho^\top \hat{R} \leq y) = \inf_{\lambda \geq 0} \left\{ \lambda \theta_W^p - \int_{\varphi} \inf_{\xi \in \varphi} [\lambda \|\zeta - \xi\|_2^p - \mathbf{1}_{\{\rho^\top \xi \leq y\}}] \nu(d\zeta) \right\}.$$

□

We consider the case when $p = 1$ and ν is a data-driven reference distribution, i.e., it is a discrete distribution with H scenarios $\tilde{\xi}_1, \dots, \tilde{\xi}_H$, where $\tilde{\xi}_i \in \varphi$, for every $i = 1, \dots, H$. Using Lemma 6.29, we propose a deterministic reformulation of the DRCCMDP problem (6.8).

Lemma 6.30. *If the distribution of \hat{R} belongs to the uncertainty set defined by (6.34), the DRCCMDP (6.8) can be reformulated equivalently as the following deterministic problem*

$$\begin{aligned}
& \sup \quad y \\
& \text{s.t.} \quad \text{(i)} \quad \theta_W - \frac{1}{H} \sum_{i=1}^H g_i \leq l\epsilon, \\
& \quad \quad \text{(ii)} \quad \inf_{z \in \varphi, \rho^\top z \leq y} \|\tilde{\xi}_i - z\|_2 \geq l + g_i, \quad \forall i = 1, \dots, H, \\
& \quad \quad \text{(iii)} \quad l > 0, \rho \in \mathcal{Q}, g_i \leq 0, \quad \forall i = 1, \dots, H.
\end{aligned} \tag{6.38}$$

Proof. Using Lemma 6.29, since ν is a discrete distribution with H scenarios $\tilde{\xi}_1, \dots, \tilde{\xi}_H$, the constraint (i) of (6.8) can be equivalently written as

$$\lambda \theta_W - \frac{1}{H} \sum_{i=1}^H \inf_{z \in \varphi} [\lambda \|\tilde{\xi}_i - z\|_2 - \mathbf{1}_{\{\rho^\top z \leq y\}}] \leq \epsilon, \quad \lambda \geq 0.$$

By introducing auxiliary variables $t_i, i = 1, \dots, H$, the above constraint can be rewritten as

$$\begin{cases} \text{(i)} \quad \lambda \theta_W - \frac{1}{H} \sum_{i=1}^H t_i \leq \epsilon, \quad \lambda \geq 0 \\ \text{(ii)} \quad \inf_{z \in \varphi} [\lambda \|\tilde{\xi}_i - z\|_2 - \mathbf{1}_{\{\rho^\top z \leq y\}}] \geq t_i, \quad \forall i = 1, \dots, H. \end{cases} \tag{6.39}$$

The constraint (ii) of (6.39) is equivalent to the following two constraints

$$\begin{cases} \text{(i)} \quad \inf_{z \in \varphi} \lambda \|\tilde{\xi}_i - z\|_2 \geq t_i, \quad \forall i = 1, \dots, H, \\ \text{(ii)} \quad \inf_{z \in \varphi, \rho^\top z \leq y} \lambda \|\tilde{\xi}_i - z\|_2 - 1 \geq t_i, \quad \forall i = 1, \dots, H. \end{cases} \tag{6.40}$$

Since $\lambda \geq 0, \inf_{z \in \varphi} \lambda \|\tilde{\xi}_i - z\|_2 = 0$. Then, the constraint (i) of (6.40) is equivalent to $t_i \leq 0$, for every $i = 1, \dots, H$. Moreover, if $\lambda = 0$, from the constraint (ii) of (6.40), $t_i \leq -1$, for every $i = 1, \dots, H$, which in turn implies $-\frac{1}{H} \sum_{i=1}^H t_i \geq 1$. This violates the constraint (i) of (6.39). Hence, $\lambda > 0$. Let $l = \frac{1}{\lambda}$ and $g_i = \frac{t_i}{\lambda}$, for every $i = 1, \dots, H$. Therefore, the constraint (i) of (6.8) is equivalent to the following constraints

$$\begin{cases} \text{(i)} \quad \theta_W - \frac{1}{H} \sum_{i=1}^H g_i \leq l\epsilon, \\ \text{(ii)} \quad \inf_{z \in \varphi, \rho^\top z \leq y} \|\tilde{\xi}_i - z\|_2 \geq l + g_i, \quad \forall i = 1, \dots, H, \\ \text{(iii)} \quad l > 0, g_i \leq 0, \quad \forall i = 1, \dots, H. \end{cases} \tag{6.41}$$

This implies that the DRCCMDP (6.8) is equivalent to (6.38). \square

The constraint (ii) of (6.38) includes \inf term which makes it difficult to solve the problem directly. We show that the optimization problem (6.38) is equivalent to a MISOCP problem and a biconvex optimization problem for the case of full support and nonnegative support, respectively.

DRCCMDP under Wasserstein distance based uncertainty set with full support

Lemma 6.31. *If $\varphi = \mathbb{R}^{|\mathcal{K}|}$,*

$$\inf_{\rho^\top z \leq y} \|\tilde{\xi}_i - z\|_2 = \max \left(0, \frac{\rho^\top \tilde{\xi}_i - y}{\|\rho\|_2} \right), \quad \forall i = 1, \dots, H.$$

Proof. For each $i = 1, \dots, H$, we consider two cases as follows:

Case 1: Let $\rho^\top \tilde{\xi}_i \leq y$. In this case, it is clear that $\inf_{\rho^\top z \leq y} \|\tilde{\xi}_i - z\|_2 = 0$ and the optimal value holds at $z = \tilde{\xi}_i$.

Case 2: Let $\rho^\top \tilde{\xi}_i > y$. An optimal solution z^* of $\inf_{\rho^\top z \leq y} \|\tilde{\xi}_i - z\|_2$ is such that $\rho^\top z^* = y$. Because if $\rho^\top z^* < y$, there exists a point z_0 on the line segment of z^* and $\tilde{\xi}_i$ such that $\rho^\top z_0 = y$ and $\|\tilde{\xi}_i - z^*\|_2 > \|\tilde{\xi}_i - z_0\|_2$ which gives a contradiction. Hence, we can write $\inf_{\rho^\top z \leq y} \|\tilde{\xi}_i - z\|_2 = \inf_{\rho^\top z = y} \|\tilde{\xi}_i - z\|_2$. Using the KKT conditions, it is easy to show that an optimal solution z^* satisfies

$$\|\tilde{\xi}_i - z^*\|_2 = \frac{\rho^\top \tilde{\xi}_i - y}{\|\rho\|_2}.$$

□

Using Lemma 6.31, we have the following result.

Lemma 6.32. *The optimization problem (6.38) is equivalent to the following optimization problem*

$$\begin{aligned} & \sup \quad y \\ \text{s.t.} \quad & \text{(i)} \quad \beta \theta_W - \frac{1}{H} \sum_{i=1}^H b_i \leq t\epsilon, \\ & \text{(ii)} \quad \max \left(0, \rho^\top \tilde{\xi}_i - y \right) \geq b_i + t, \quad \forall i = 1, \dots, H, \\ & \text{(iii)} \quad \|\rho\|_2 \leq \beta, \quad t \geq 0, \quad \beta > 0, \quad \rho \in \mathcal{Q}, \quad b_i \leq 0, \quad \forall i = 1, \dots, H. \end{aligned} \quad (6.42)$$

Proof. Using Lemma 6.31, the constraint (ii) of problem (6.38) can be written as

$$\max \left(0, \frac{\rho^\top \tilde{\xi}_i - y}{\|\rho\|_2} \right) \geq l + g_i, \quad \forall i = 1, \dots, H.$$

Let $\beta > 0$ be an auxiliary variable. Then, under the transformations $t = \beta l$, $b_i = \beta g_i$, for every $i = 1, \dots, H$, it is easy to see that (6.38) is equivalent to (6.42). □

It is clear that a vector $(y, \rho, \beta, (b_i)_{i=1}^H, t)$ such that $\rho \in \mathcal{Q}$, $\beta = \|\rho\|_2$, $b_i = 0$, for every $i = 1, \dots, H$, $t = \frac{\theta_W}{\epsilon} \|\rho\|_2$ and $y = \min_{i=1, \dots, H} (\rho^\top \tilde{\xi}_i) - \frac{\theta_W}{\epsilon} \|\rho\|_2$ is a feasible solution of (6.42). Therefore, the optimal solutions of (6.42) and the following optimization problem are the same

$$\begin{aligned} & \sup \quad y \\ \text{s.t.} \quad & \text{(i)} \quad \beta \theta_W - \frac{1}{H} \sum_{i=1}^H b_i \leq t\epsilon, \\ & \text{(ii)} \quad \max \left(0, \rho^\top \tilde{\xi}_i - y \right) \geq b_i + t, \quad \forall i = 1, \dots, H, \\ & \text{(iii)} \quad y \geq \min_{i=1, \dots, H} (\rho^\top \tilde{\xi}_i) - \frac{\theta_W}{\epsilon} \|\rho\|_2, \\ & \text{(iv)} \quad \|\rho\|_2 \leq \beta, \quad t \geq 0, \quad \beta > 0, \quad \rho \in \mathcal{Q}, \quad b_i \leq 0, \quad \forall i = 1, \dots, H. \end{aligned} \quad (6.43)$$

We reformulate problem (6.43) as an MISOCP problem. In order to do that, we define a constant $M = \left(\frac{\theta_W}{\epsilon} + 2 \max_{i=1, \dots, H} \|\tilde{\xi}_i\|_2 \right)$ for which the following result holds.

Lemma 6.33. For every feasible solution of (6.43), $M \geq |y - \rho^\top \tilde{\xi}_i|$ for all $i = 1, \dots, H$.

Proof. Let (y, ρ) be a feasible solution of (6.43) which implies that the constraint (i) of (6.8) holds. Since, reference distribution ν belongs to uncertainty set (6.34), we have

$$\frac{1}{H} \sum_{i=1}^H \mathbf{1}_{\{\rho^\top \tilde{\xi}_i \leq y\}} = \mathbb{P}_\nu \left(\rho^\top \hat{R} \leq y \right) \leq \epsilon. \quad (6.44)$$

It follows from (6.44) that there exists $\tilde{\xi}_i$ such that $\rho^\top \tilde{\xi}_i > y$ which implies that

$$y < \max_{i=1, \dots, H} (\rho^\top \tilde{\xi}_i) < \max_{i=1, \dots, H} |\rho^\top \tilde{\xi}_i| + \frac{\theta_W}{\epsilon} \|\rho\|_2. \quad (6.45)$$

Moreover, from the constraint (iii) of (6.43), we have

$$y \geq \min_{i=1, \dots, H} (\rho^\top \tilde{\xi}_i) - \frac{\theta_W}{\epsilon} \|\rho\|_2 \geq - \max_{i=1, \dots, H} |\rho^\top \tilde{\xi}_i| - \frac{\theta_W}{\epsilon} \|\rho\|_2. \quad (6.46)$$

Using (6.45), (6.46), Cauchy-Schwartz inequality and the fact that ρ is a probability measure, we have

$$|y - \rho^\top \tilde{\xi}_i| \leq M.$$

□

Theorem 6.9. Consider the DRCCMDP problem (6.8). We assume that the distribution of \hat{R} belongs to the uncertainty set defined by (6.34) and $\varphi = \mathbb{R}^{|\mathcal{K}|}$. Then, the DRCCMDP problem (6.8) is equivalent to the following MISOCP

$$\begin{aligned} \max \quad & y \\ \text{s.t.} \quad & \text{(i)} \quad \beta \theta_W - \frac{1}{H} \sum_{i=1}^H b_i \leq t\epsilon, \\ & \text{(ii)} \quad M\eta_i \geq b_i + t, \quad \forall i = 1, \dots, H, \\ & \text{(iii)} \quad M(1 - \eta_i) + \rho^\top \tilde{\xi}_i - y \geq b_i + t, \quad \forall i = 1, \dots, H, \\ & \text{(iv)} \quad \eta_i \in \{0, 1\}, \quad \forall i = 1, \dots, H, \\ & \text{(v)} \quad \|\rho\|_2 \leq \beta, t \geq 0, \beta > 0, \rho \in \mathcal{Q}, b_i \leq 0, \quad \forall i = 1, \dots, H. \end{aligned} \quad (6.47)$$

Proof. Since, the distribution of \hat{R} belongs to the uncertainty set defined by (6.34), the DRCCMDP problem (6.8) is equivalent to (6.43). We show that (6.43) and (6.47) are equivalent. It is clear that a vector $(y, \rho, \beta, (b_i)_{i=1}^H, (\eta_i)_{i=1}^H, t)$ such that $\rho \in \mathcal{Q}$, $\beta = \|\rho\|_2$, $b_i = 0$, $t = \frac{\theta_W}{\epsilon} \|\rho\|_2$, $\eta_i = 1$, for every $i = 1, \dots, H$, and $y = \min_{i=1, \dots, H} (\rho^\top \tilde{\xi}_i) - \frac{\theta_W}{\epsilon} \|\rho\|_2$ is a feasible solution of (6.47). Therefore, the optimal solution of (6.47) does not change if we add constraint (6.48) given below

$$y \geq \min_{i=1, \dots, H} (\rho^\top \tilde{\xi}_i) - \frac{\theta_W}{\epsilon} \|\rho\|_2, \quad (6.48)$$

to the feasible region of (6.47). Now, it is enough to show that the constraint (ii) of (6.43) is equivalent to (ii) – (iv) of (6.47). Let the constraint (ii) of (6.43) be satisfied, i.e.,

$$\max \left(0, \rho^\top \tilde{\xi}_i - y \right) \geq b_i + t, \quad \forall i = 1, \dots, H. \quad (6.49)$$

For each $i = 1, \dots, H$, we consider two cases as follows:

Case 1: If $\max(0, \rho^\top \tilde{\xi}_i - y) = 0$, by choosing $\eta_i = 0$, (6.49) is equivalent to the constraint (ii) of (6.47). Moreover, using Lemma 6.33, $M \geq |y - \rho^\top \tilde{\xi}_i|$. Therefore,

$$M(1 - \eta_i) + \rho^\top \tilde{\xi}_i - y \geq M - |y - \rho^\top \tilde{\xi}_i| \geq 0 \geq b_i + t.$$

Case 2: If $\max(0, \rho^\top \tilde{\xi}_i - y) = \rho^\top \tilde{\xi}_i - y$, by choosing $\eta_i = 1$, (6.49) is equivalent to the constraint (iii) of (6.47). Moreover, using Lemma 6.33, we have

$$M\eta_i = M \geq \rho^\top \tilde{\xi}_i - y \geq b_i + t.$$

This implies that there exists $\eta_i \in \{0, 1\}$ such that (ii)–(iv) of (6.47) are satisfied. Conversely, suppose (ii)–(iv) of (6.47) has a feasible solution. If $\eta_i = 1$, the constraint (iii) of (6.47) implies the constraint (ii) of (6.43). If $\eta_i = 0$, the constraint (ii) of (6.47) implies the constraint (ii) of (6.43). \square

DRCCMDP under Wasserstein distance based uncertainty set with nonnegative support

Lemma 6.34. Let $\varphi = \mathbb{R}_+^{|\mathcal{K}|}$ and consider an optimization problem

$$\inf_{z \in \varphi, \rho^\top z \leq y} \|\tilde{\xi}_i - z\|_2. \quad (6.50)$$

The dual problem of (6.50) is given by

$$\begin{aligned} \max \quad & \lambda_i(\rho^\top \tilde{\xi}_i - y) - \zeta_i^\top \tilde{\xi}_i \\ \text{s.t.} \quad & \|\zeta_i - \lambda_i \rho\|_2 \leq 1, \zeta_i \in \mathbb{R}_+^{|\mathcal{K}|}, \lambda_i \geq 0, \end{aligned}$$

such that the strong duality holds.

Proof. The optimization problem $\inf_{z \in \mathbb{R}_+^{|\mathcal{K}|}, \rho^\top z \leq y} \|\tilde{\xi}_i - z\|_2$ can be reformulated as following SOCP problem

$$\begin{aligned} \min \quad & t \\ \text{s.t.} \quad & \rho^\top z \leq y, t \geq \|\tilde{\xi}_i - z\|_2, z \in \mathbb{R}_+^{|\mathcal{K}|}. \end{aligned} \quad (6.51)$$

The Lagrangian dual problem of (6.51) is given by

$$\max_{\lambda_i \geq 0, \zeta_i \in \mathbb{R}_+^{|\mathcal{K}|}, \beta \geq 0} \min_{t \in \mathbb{R}, z \in \mathbb{R}^{|\mathcal{K}|}} \mathcal{L}(t, \rho, z, \lambda_i, \beta, \zeta_i),$$

where $\mathcal{L}(t, z, \lambda_i, \beta, \zeta_i) = t + \lambda_i(\rho^\top z - y) - \zeta_i^\top z + \beta(\|\tilde{\xi}_i - z\|_2 - t)$. The inner minimization problem can be written as

$$J(\lambda_i, \zeta_i, \beta) = \min_{t \in \mathbb{R}, z \in \mathbb{R}^{|\mathcal{K}|}} \left\{ t(1 - \beta) + \beta \|\tilde{\xi}_i - z\|_2 + \lambda_i \rho^\top z - \zeta_i^\top z - \lambda_i y \right\}. \quad (6.52)$$

It is easy to see that $J(\lambda_i, \zeta_i, \beta) = -\infty$ if $\beta \neq 1$ and it implies that the dual objective function value is $-\infty$. By using the strong duality of a primal-dual pair of SOCPs, the objective function value of primal problem is $-\infty$, i.e., $\inf_{z \in \mathbb{R}_+^{|\mathcal{K}|}, \rho^\top z \leq y} \|\tilde{\xi}_i - z\|_2 = -\infty$ which is a contradiction. Therefore, $\beta = 1$ and using a change of variable $z_1 = \tilde{\xi}_i - z$, we have

$$J(\lambda_i, \zeta_i, 1) = \min_{z_1 \in \mathbb{R}^{|\mathcal{K}|}} \left\{ \|z_1\|_2 + (\zeta_i - \lambda_i \rho)^\top z_1 \right\} + \lambda_i(\rho^\top \tilde{\xi}_i - y) - \zeta_i^\top \tilde{\xi}_i.$$

The above minimization problem is unbounded unless $\|\zeta_i - \lambda_i \rho\|_2 \leq 1$ and it leads to the following dual problem of (6.51).

$$\begin{aligned} \max \quad & \lambda_i(\rho^\top \tilde{\xi}_i - y) - \zeta_i^\top \tilde{\xi}_i \\ \text{s.t.} \quad & \|\zeta_i - \lambda_i \rho\|_2 \leq 1, \lambda_i \geq 0, \zeta_i \in \mathbb{R}_+^{|\mathcal{K}|}. \end{aligned} \quad (6.53)$$

□

Theorem 6.10. Consider the DRCCMDP problem (6.8). We assume that the distribution of \hat{R} belongs to the uncertainty set defined by (6.34) and $\varphi = \mathbb{R}_+^{|\mathcal{K}|}$. Then, the DRCCMDP (6.8) is equivalent to the following biconvex optimization problem

$$\begin{aligned} \max \quad & y \\ \text{s.t.} \quad & \text{(i) } \theta_W - \frac{1}{H} \sum_{i=1}^H g_i \leq l\epsilon, \\ & \text{(ii) } \lambda_i(\rho^\top \tilde{\xi}_i - y) - \zeta_i^\top \tilde{\xi}_i \geq l + g_i, \forall i = 1, \dots, H, \\ & \text{(iii) } \|\zeta_i - \lambda_i \rho\|_2 \leq 1, \forall i = 1, \dots, H, \\ & \text{(iv) } \lambda_i \geq 0, \zeta_i \in \mathbb{R}_+^{|\mathcal{K}|}, l > 0, g_i \leq 0, \rho \in \mathcal{Q}, \forall i = 1, \dots, H. \end{aligned} \quad (6.54)$$

Proof. The proof follows directly from Lemma 6.30 and Lemma 6.34. □

6.3 . Distributionally robust chance constrained Markov decision process with random transition probabilities

In this section, we consider an MDP framework defined in Section 6.1, with a finite state space S , finite action spaces $A(s)$, $s \in S$, an initial distribution γ , where $\sum_{s \in S} \gamma(s) = 1$, $\gamma(s) > 0$, a discount factor $\alpha \in (0, 1)$, a running reward vector R and transition probabilities p , where the running reward vector R is exactly known and the transition probabilities p are random variables. For each triple (state, action, state) $s' \in S$, $a \in A(s)$, $s \in S$, we assume that the $p(s, a, s')$ is a 1-dimensional random variable defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Therefore, for each realization $\omega \in \Omega$, the term $p(s, a, s')(\omega)$ is a real number in $[0, 1]$ which represents the probability of moving to a new state s' , where the decision maker chooses action a at actual state s . Assume that p follows a discrete distribution F_p , whose support is taken by the set of historical data on the transition probabilities. Denote this set by $\varphi_p = \{p_1, p_2, \dots, p_J\}$. For each $s' \in S$, let $\hat{p}(s')$ be a $|\mathcal{K}|$ -dimensional random vector defined as follows

$$\hat{p}(s')(s, a) = \delta(s', s) - \alpha p(s, a, s'), \forall s \in S, a \in A(s). \quad (6.55)$$

Let $\hat{p} = (\hat{p}(s')_{s' \in S})$ be a $|S| \times |\mathcal{K}|$ -dimensional random vector. Then, the distribution of \hat{p} is a discrete distribution with support $\varphi_{\hat{p}} = \{\hat{p}_1, \dots, \hat{p}_J\}$, where $\hat{p}_j(s, a, s') = \delta(s', s) - \alpha p_j(s, a, s')$, for any $s', s \in S$, $a \in A(s)$, $j = 1, \dots, J$. We focus our attention on the set of stationary policies PO_S , where the policy does not depend on time t . Let $PO_D \subseteq PO_S$ be a subset of the set of stationary policies.

For a policy $f^h \in PO_D$ and a realization $\omega \in \Omega$, we define the occupation measure $(\hat{m}(s, a))_{(s,a)}(\omega)$ by

$$\hat{m}(s, a)(\omega) = (1 - \alpha) \sum_{t=0}^{\infty} \alpha^t P_{p(\omega)}^{f^h}(\mathbb{X}_t = s, \mathbb{A}_t = a), \forall s \in S, a \in A(s). \quad (6.56)$$

where the probability function $P_{p(\omega)}^{f^h}$ is defined by the policy f^h and the transition probabilities $p(\omega)$. Since the transition probabilities are random variables, it is clear that $\hat{m} = (\hat{m}(s, a))_{(s,a)}$ is a $|\mathcal{K}|$ -dimensional random vector, whose distribution is defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$. For each realization ω , the expected discounted reward denoted by $V(f^h, \hat{p})(\omega)$ is given as follows

$$\begin{aligned} V(f^h, \hat{p})(\omega) &= (1 - \alpha) \mathbb{E}_{\hat{p}(\omega)}^{f^h} \left(\sum_{t=0}^{\infty} \alpha^t R(X_t, A_t) \right) \\ &= \sum_{s \in S} \sum_{a \in A(s)} \hat{m}(s, a)(\omega) R(s, a). \end{aligned} \quad (6.57)$$

It is clear that $V(f^h, \hat{p})$ is an 1-dimensional random variable, defined on $(\Omega, \mathcal{F}, \mathbb{P})$ and the distribution of $V(f^h, \hat{p})$ depends on the distribution of \hat{p} and we denote this distribution by $F_{\hat{p}}$. Define our optimization problem as follows

$$\begin{aligned} (\text{CCMDP-TP}) \quad & \sup_{y \in \mathbb{R}, f^h \in PO_D} y \\ \text{s.t.} \quad & \mathbb{P}_p(V(f^h, \hat{p}) \geq y) \geq 1 - \epsilon, \end{aligned} \quad (6.58)$$

where the controller is interested in maximizing the expected discounted reward which can be obtained with at least a given confidence level $(1 - \epsilon)$. In most of the practical situations, we only have partial information about the underlying probability distributions of \hat{p} based on historical data of the transition probabilities p . Such situations can be handled with the distributionally robust optimization approach, where the decision maker believes that the distribution of \hat{p} belongs to some uncertainty set $\mathcal{D}_{\hat{p}}$. To ensure that the chance constraint $\mathbb{P}(V(f^h, \hat{p}) \geq y) \geq 1 - \epsilon$ holds, we assume that it holds for any distribution which belongs to the uncertainty set. This leads to the following optimization problem

$$\begin{aligned} (\text{DRCCMDP-TP}) \quad & \sup_{y \in \mathbb{R}, f^h \in PO_D} y \\ \text{s.t.} \quad & \inf_{F_{\hat{p}} \in \mathcal{D}_{\hat{p}}} \mathbb{P}_{\hat{p}}(V(f^h, \hat{p}) \geq y) \geq 1 - \epsilon, \end{aligned} \quad (6.59)$$

The optimization problem DRCCMDP-TP is challenging to solve since $V(f^h, \hat{p})$ is a complex nonlinear function of transition probabilities p . To handle it, we first reformulate (6.58) by considering the occupation measure \hat{m} defined by (6.56) as an auxiliary decision variable of our optimization problem.

Using Lemma 6.27, the equations (6.56) can be rewritten equivalently as follows

$$\left\{ \begin{array}{l} (i) \hat{m}(\omega)^T \hat{p}(s') = (1 - \alpha) \gamma(s'), \forall s' \in S, \forall \omega \in \Omega \\ (ii) f^h(s, a) \left(\sum_{a \in A(s)} \hat{m}(\omega)(s, a) \right) = \hat{m}(\omega)(s, a), \forall s \in S, a \in A(s), \forall \omega \in \Omega \\ (iii) \hat{m}(\omega) \in \mathbb{R}^{|\mathcal{K}|}, \hat{m}(\omega)(s, a) \geq 0, \forall s \in S, a \in A(s), \sum_{a \in A(s)} \hat{m}(\omega)(s, a) > 0, \\ \forall \omega \in \Omega, \forall s \in S, \end{array} \right. \quad (6.60)$$

which can be reformulated equivalently as follows

$$\left\{ \begin{array}{l} (i) \mathbb{E}[(\hat{m}(\omega)^\top \hat{p}(s') - (1 - \alpha)\gamma(s'))^2] = 0, \forall s' \in S, \\ (ii) \mathbb{E}\left[\left(f^h(s, a) \left(\sum_{a \in A(s)} \hat{m}(\omega)(s, a)\right) - \hat{m}(\omega)(s, a)\right)^2\right] = 0, \forall s \in S, a \in A(s) \\ (iii) \varphi_{\hat{m}} \subseteq X = \left\{ \rho \mid \rho \in \mathbb{R}^{|\mathcal{K}|}, \rho(s, a) \geq 0, \forall s \in S, a \in A(s), \right. \\ \left. \sum_{a \in A(s)} \rho(s, a) > 0, \forall s \in S \right\}, \end{array} \right. \quad (6.61)$$

where $\varphi_{\hat{m}}$ is the support of the distribution $F_{\hat{m}}$ of \hat{m} . Then, the optimization problem DRCCMDP-TP can be rewritten as follows

$$\begin{aligned} & \sup_{y \in \mathbb{R}, f^h \in PO_D} y \\ \text{s.t. } & (i) \inf_{F_{\hat{p}} \in \mathcal{D}_{\hat{p}}} \mathbb{P}_{(\hat{m}, \hat{p})}(\hat{m}^\top R \geq y) \geq 1 - \epsilon, \\ & (ii) \mathbb{E}_{(\hat{m}, \hat{p})}[(\hat{m}(\omega)^\top \hat{p}(s') - (1 - \alpha)\gamma(s'))^2] = 0, \forall s' \in S, \\ & (iii) \mathbb{E}_{(\hat{m}, \hat{p})}\left[\left(f^h(s, a) \left(\sum_{a \in A(s)} \hat{m}(\omega)(s, a)\right) - \hat{m}(\omega)(s, a)\right)^2\right] = 0, \\ & \forall s \in S, a \in A(s) \end{aligned}$$

The support $\varphi_{\hat{m}}$ is the set of all solutions of (6.56), where $f^h \in PO_D$ and $\hat{p} \in \varphi_{\hat{p}}$. To ease our notations, we define real functions $g_1(s') : \mathbb{R}^{|\mathcal{K}|} \times \mathbb{R}^{|\mathcal{S}| \times |\mathcal{K}|} \rightarrow \mathbb{R}$ and $g_2(s, a) : \mathbb{R}^{|\mathcal{K}|} \times \mathbb{R}^{|\mathcal{S}| \times |\mathcal{K}|} \rightarrow \mathbb{R}$ such that

$$g_1(s')(x_{\hat{m}}, x_{\hat{p}}) = \left(x_{\hat{m}}^\top x_{\hat{p}}(s') - (1 - \alpha)\gamma(s')\right)^2, \forall s' \in S, \quad (6.62)$$

and

$$g_2(s, a)(\hat{m}, \hat{p}) = \left(f^h(s, a) \left(\sum_{a \in A(s)} \hat{m}(s, a)\right) - \hat{m}(s, a)\right)^2, \forall s \in S, a \in A(s), \quad (6.63)$$

for any $x_{\hat{m}} \in \mathbb{R}^{|\mathcal{K}|}$, $x_{\hat{p}} \in \mathbb{R}^{|\mathcal{S}| \times |\mathcal{K}|}$. Let $g_1 = (g_1(s'))_{s' \in S}$ and $g_2 = (g_2(s, a))_{s \in S, a \in A(s)}$. Using the abovementioned notations, the optimization problem DRCCMDP-TP can be rewritten shortly as follows

$$\begin{aligned} & \sup_{(y \in \mathbb{R}, f^h \in PO_D)} y \\ \text{s.t. } & (i) \inf_{F_{\hat{p}} \in \mathcal{D}_{\hat{p}}} \mathbb{E}_{(\hat{m}, \hat{p})}[\mathbf{1}_{\{\hat{m}^\top R \geq y\}}] \geq 1 - \epsilon, \\ & (ii) \mathbb{E}_{(\hat{m}, \hat{p})}[g_i(\hat{m}, \hat{p})] = 0, i = 1, 2, \end{aligned} \quad (6.64)$$

where $\mathbf{1}_{\Omega}$ denotes the indicator function, $\mathbb{E}_{(\hat{m}, \hat{p})}$ is the expectation function w.r.t the joint distribution of (\hat{m}, \hat{p}) . In the following sections, we study different ways to define the uncertainty set $\mathcal{D}_{\hat{p}}$ by using i) partial information of moment and ii) ϕ -divergence distance. We derive equivalent reformulations of the DRCCMDP-TP problem (6.64) for each uncertainty set.

6.3.1 . Moment based uncertainty sets

In some situations, the true distributions of \hat{p} is unknown in advance. We can only get partial information of the underlying distributions based on historical data. By observing sufficiently large number of data, we can estimate the first moment (expected value) and the second moment (covariance matrix) of the random variable by their sample mean and sample covariance matrix. We assume that the distribution of \hat{p} belongs to an uncertainty set based on its first moment and second moment. These moments are estimated from historical data of transition probabilities. Let $\mu \in \mathbb{R}^{|\mathcal{S}| \times |\mathcal{K}|}$ be the sample mean vector and $\Sigma \in \mathcal{S}_{++}^{|\mathcal{S}| \times |\mathcal{K}|}$ be the sample covariance matrix of \hat{p} , where $\mathcal{S}_{++}^{|\mathcal{S}| \times |\mathcal{K}|}$ denotes the set of $|\mathcal{K}| \times |\mathcal{K}|$ definite positive matrices. We consider 3 types of moment based uncertainty sets defined as follows

1. Uncertainty set with known mean and known covariance matrix:

$$\mathcal{D}_1 = \left\{ F_{\hat{p}} \left| \begin{array}{l} \text{(i)} \mathbb{E}(\mathbf{1}_{\{\hat{p} \in \varphi_{\hat{p}}\}}) = 1 \\ \text{(ii)} \mathbb{E}(\hat{p}) = \mu \\ \text{(iii)} \mathbb{E}[(\hat{p} - \mu)(\hat{p} - \mu)^\top] = \Sigma \end{array} \right. \right\}, \quad (6.65)$$

2. Uncertainty set with known mean and unknown covariance matrix:

$$\mathcal{D}_2 = \left\{ F_{\hat{p}} \left| \begin{array}{l} \text{(i)} \mathbb{E}(\mathbf{1}_{\{\hat{p} \in \varphi_{\hat{p}}\}}) = 1 \\ \text{(ii)} \mathbb{E}(\hat{p}) = \mu \\ \text{(iii)} \mathbb{E}[(\hat{p} - \mu)(\hat{p} - \mu)^\top] \preceq \delta_0 \Sigma, \end{array} \right. \right\}, \quad (6.66)$$

3. Uncertainty set with unknown mean and unknown covariance matrix:

$$\mathcal{D}_3 = \left\{ F_{\hat{p}} \left| \begin{array}{l} \text{(i)} \mathbb{E}(\mathbf{1}_{\{\hat{p} \in \varphi_{\hat{p}}\}}) = 1 \\ \text{(ii)} [\mathbb{E}(\hat{p}) - \mu]^\top \Sigma^{-1} [\mathbb{E}(\hat{p}) - \mu] \leq \delta_1, \\ \text{(iii)} \mathbb{E}[(\hat{p} - \mu)(\hat{p} - \mu)^\top] \preceq \delta_2 \Sigma, \end{array} \right. \right\}, \quad (6.67)$$

Here, $F_{\hat{p}} \in \mathcal{M}_{\varphi_{\hat{p}}}^+$, where $\mathcal{M}_{\varphi_{\hat{p}}}^+$ is the set of all positive measures on $\varphi_{\hat{p}}$ with Borel σ -algebra, $\delta_1 \geq 0, \delta_2, \delta_0 \geq 1, \mu_{\hat{p}} \in \text{RI}(\varphi_{\hat{p}})$; $\text{RI}(\varphi)$ denotes the relative interior of φ . The notation $A \preceq B$ implies that $B - A$ is a positive semidefinite matrix. Note that $\mathbb{E}[\mathbf{1}_{\{\hat{m}^\top R \geq y - \theta\}}] \geq \mathbb{E}[\mathbf{1}_{\{\hat{m}^\top R > y\}}] \geq \mathbb{E}[\mathbf{1}_{\{\hat{m}^\top R \geq y\}}]$ for every $\theta > 0$. Therefore, we can replace $\mathbb{E}[\mathbf{1}_{\{\hat{m}^\top R \geq y\}}]$ by $\mathbb{E}[\mathbf{1}_{\{\hat{m}^\top R > y\}}]$ in (6.64). Then, the constraint (i) of (6.64) can be rewritten as

$$\text{(i)} \sup_{F_{\hat{p}} \in \mathcal{D}_{\hat{p}}} \mathbb{E}_{(\hat{m}, \hat{p})} [\mathbf{1}_{\{\hat{m}^\top R \leq y\}}] \leq \epsilon. \quad (6.68)$$

We consider the following optimization problem

$$\begin{aligned} & \sup_{F_{\hat{p}} \in \mathcal{D}_{\hat{p}}} \mathbb{E}_{(\hat{m}, \hat{p})} [\mathbf{1}_{\{\hat{m}^\top R \leq y\}}] \\ \text{s.t. } & \text{(i)} \mathbb{E}_{(\hat{m}, \hat{p})} [g_i(\hat{m}, \hat{p})] = 0, \quad i = 1, 2, \end{aligned} \quad (6.69)$$

where the uncertainty set $\mathcal{D}_{\hat{p}}$ is defined by one of three uncertainty sets abovementioned. We consider the following assumption, which is necessary for our subsequent analysis.

Assumption 6.8. *The support of the true distribution of \hat{m} (resp. \hat{p}) belongs to $\text{RI}(\varphi_{\hat{m}})$ (resp. $\text{RI}(\varphi_{\hat{p}})$).*

In order to ease our notations, we consider the following functions

$$\begin{aligned}
\mathbf{A}_1(t, q, Q, r) &= (1 - \alpha)^2 \left(\sum_{s' \in S} r_1(s') \gamma(s')^2 \right) - t - q^T \mu - Q \circ \Sigma + \mu^T Q \mu, \\
\mathbf{B}_1(t, q, Q, r, \xi) &= \sum_{s' \in S} r_1(s') \left[\left(\xi_{\hat{m}}^T \xi_{\hat{p}}(s') \right)^2 - 2(1 - \alpha) \gamma(s') (\xi_{\hat{m}})^T \xi_{\hat{p}}(s') \right] + \sum_{s \in S, a \in A(s)} r_2(s, a) \\
&\quad \left[f^h(s, a)^2 \left(\sum_{a \in A(s)} \xi_{\hat{m}}(s, a) \right)^2 - 2f^h(s, a) \left(\sum_{a \in A(s)} \xi_{\hat{m}}(s, a) \right) \xi_{\hat{m}}(s, a) + \xi_{\hat{m}}^2(s, a) \right] \\
&\quad + t + q^T \xi_{\hat{p}} + \xi_{\hat{p}}^T Q \xi_{\hat{p}} - 2\mu^T Q \xi_{\hat{p}}, \\
\mathbf{A}_2(t, q, Q, r) &= (1 - \alpha)^2 \left(\sum_{s' \in S} r_1(s') \gamma(s')^2 \right) - t - q^T \mu + Q \circ \delta_0 \Sigma - \mu^T Q \mu, \\
\mathbf{B}_2(t, q, Q, r, \xi) &= \sum_{s' \in S} r_1(s') \left[\left(\xi_{\hat{m}}^T \xi_{\hat{p}}(s') \right)^2 - 2(1 - \alpha) \gamma(s') (\xi_{\hat{m}})^T \xi_{\hat{p}}(s') \right] + \sum_{s \in S, a \in A(s)} r_2(s, a) \\
&\quad \left[f^h(s, a)^2 \left(\sum_{a \in A(s)} \xi_{\hat{m}}(s, a) \right)^2 - 2f^h(s, a) \left(\sum_{a \in A(s)} \xi_{\hat{m}}(s, a) \right) \xi_{\hat{m}}(s, a) + \xi_{\hat{m}}^2(s, a) \right] \\
&\quad + t + q^T \xi_{\hat{p}} - \xi_{\hat{p}}^T Q \xi_{\hat{p}} + 2\mu^T Q \xi_{\hat{p}}, \\
\mathbf{A}_3(t, q, Q, r) &= (1 - \alpha)^2 \left(\sum_{s' \in S} r_1(s') \gamma(s')^2 \right) - t + Q \circ \delta_2 \Sigma - q^T \mu + \mu^T Q \mu \\
&\quad + \sqrt{\delta_1} \left\| \Sigma^{\frac{1}{2}} (q - 2Q\mu) \right\|, \\
\mathbf{B}_3(t, q, Q, r, \xi) &= \sum_{s' \in S} r_1(s') \left[\left(\xi_{\hat{m}}^T \xi_{\hat{p}}(s') \right)^2 - 2(1 - \alpha) (\xi_{\hat{m}})^T \xi_{\hat{p}}(s') \right] \\
&\quad + \sum_{s \in S, a \in A(s)} r_2(s, a) \left[f^h(s, a)^2 \left(\sum_{a \in A(s)} \xi_{\hat{m}}(s, a) \right)^2 - 2f^h(s, a) \left(\sum_{a \in A(s)} \xi_{\hat{m}}(s, a) \right) \right. \\
&\quad \left. \xi_{\hat{m}}(s, a) + \xi_{\hat{m}}^2(s, a) \right] + t + q^T \xi_{\hat{p}} - \xi_{\hat{p}}^T Q \xi_{\hat{p}},
\end{aligned}$$

where $t \in \mathbb{R}$, $q \in \mathbb{R}^{|S| \times |\mathcal{K}|}$, $Q \in S^{|S| \times |\mathcal{K}|}$, $(\xi_{\hat{m}}, \xi_{\hat{p}}) \in \varphi_{\hat{m}} \times \varphi_{\hat{p}}$, $r_1 \in \mathbb{R}^{|S|}$, $r_2 \in \mathbb{R}^{|\mathcal{K}|}$, $S^{|S| \times |\mathcal{K}|}$ is the set of all real symmetric matrix of size $(|S| \times |\mathcal{K}|) \times (|S| \times |\mathcal{K}|)$, \circ denotes the Frobenius inner product, $\|\cdot\|$ is the Euclidean norm, $r = (r_1, r_2)$, $\xi = (\xi_{\hat{m}}, \xi_{\hat{p}})$. Using these notations, by applying conic duality theory [207], we reformulate (6.69) equivalently as a deterministic optimization problem in each case of uncertainty set. We have the following lemma.

Lemma 6.35. *Let Assumption 6.8 holds. Consider the optimization problem (6.69). If the true distribution of \hat{p} belongs to the uncertainty set \mathcal{D}_j , $j = 1, \dots, 3$, then the optimization problem (6.69) is equivalent to the following deterministic problem*

$$\begin{aligned} & \inf_{(t,q,Q,r)} \mathbf{A}_j(t, q, Q, r) \\ \text{s.t. (i)} & \mathbf{1}_{\{\xi_{\hat{m}}^\top R \leq y\}} + \mathbf{B}_j(t, q, Q, r, \xi) \leq 0, \forall \xi = (\xi_{\hat{m}}, \xi_{\hat{p}}) \in \varphi_{\hat{m}} \times \varphi_{\hat{p}}, \end{aligned} \quad (6.70)$$

where $Q_{\hat{p}} \in S^{|S| \times |\mathcal{K}|}$, if $j = 1$ and $Q \in S_+^{|S| \times |\mathcal{K}|}$, if $j = 2, 3$, where $S^{|S| \times |\mathcal{K}|}$ is the set of symmetric matrices of size $(|S| \times |\mathcal{K}|) \times (|S| \times |\mathcal{K}|)$.

Proof. First, we prove the case, where the true distribution of \hat{p} belongs to the uncertainty set \mathcal{D}_1 defined by (6.65). The Lagrangian function of the optimization problem (6.69) is written as follows

$$\begin{aligned} \mathcal{L} \left[F, t, q, Q, (r_i)_{i=1}^2 \right] &= \mathbb{E}_{(\hat{m}, \hat{p})} [\mathbf{1}_{\{\hat{m}^\top R \leq y\}}] + \sum_{i=1}^2 r_i^\top \mathbb{E}_{(\hat{m}, \hat{p})} (g_i(\hat{m}, \hat{p})) \\ &+ t (\mathbb{E}_{(\hat{m}, \hat{p})} (\mathbf{1}_{\{\hat{p} \in \varphi_{\hat{p}}\}}) - 1) + q^\top (\mathbb{E}_{(\hat{m}, \hat{p})} (\hat{p}) - \mu) + Q \circ \left(\mathbb{E}_{(\hat{m}, \hat{p})} [(\hat{p} - \mu)(\hat{p} - \mu)^\top] - \Sigma \right) \end{aligned} \quad (6.71)$$

where $t \in \mathbb{R}$, $q \in \mathbb{R}^{|S| \times |\mathcal{K}|}$, $Q \in S^{|S| \times |\mathcal{K}|}$, $\forall s' \in S$, $r_1 \in \mathbb{R}^{|S|}$, $F = (F_{\hat{p}}, F_{\hat{p}}) \in \mathcal{M}_{\varphi_{\hat{p}}}^+ \times \mathcal{M}_{\varphi_{\hat{m}}}^+$, $r_2 \in \mathbb{R}^{|\mathcal{K}|}$. The primal problem (6.69) and its dual problem can be rewritten shortly as follows

$$\begin{aligned} \text{(P)} & \sup_F \inf_{(t,q,Q,r)} \mathcal{L} \left[F, t, q, Q, (r_i)_{i=1}^2 \right] \\ \text{(D)} & \inf_{(t,q,Q,r)} \sup_F \mathcal{L} \left[F, t, q, Q, (r_i)_{i=1}^2 \right] \end{aligned} \quad (6.72)$$

Note that (6.65) is a conic optimization problem. Due to Assumption 6.8, the true distribution of (\hat{m}, \hat{p}) lies in the relative interior of the distributional set. Hence, the primal problem (P) in (6.72) is strictly feasible and the strong duality holds (the weaker version of Assumption 3.4 holds) [207]. We represent the Lagrangian function as the form $\mathbb{E}[u(\hat{m}, \hat{p})] + v$, where $u : \mathbb{R}^{|\mathcal{K}|} \times \mathbb{R}^{|S| \times |\mathcal{K}|} \rightarrow \mathbb{R}$ is a real function and v does not depend on (\hat{m}, \hat{p}) . If there exists $\xi = (\xi_{\hat{m}}, \xi_{\hat{p}}) \in \varphi_{\hat{m}} \times \varphi_{\hat{p}}$ such that $u(\xi_{\hat{m}}, \xi_{\hat{p}}) > 0$, then $\sup_F \mathcal{L} = \infty$, which implies that the optimal value of the dual problem (D) is ∞ . Otherwise, the optimal value of the dual problem (D) is v and this holds when $u(\xi_{\hat{m}}, \xi_{\hat{p}}) \leq 0$, for any $\xi = (\xi_{\hat{m}}, \xi_{\hat{p}}) \in \varphi_{\hat{m}} \times \varphi_{\hat{p}}$. On the other hand, we can verify that v is the objective function of the optimization problem (6.70) and the condition $u(\xi_{\hat{m}}, \xi_{\hat{p}}) \leq 0$, for any $\xi_{\hat{m}} \in \varphi_{\hat{m}}$, $\xi_{\hat{p}} \in \varphi_{\hat{p}}$ is equivalent to the constraint (i) of (6.70). Therefore, the conic optimization problem (6.69) and the deterministic optimization problem (6.70) are equivalent.

The proof of the case, where the true distribution \hat{p} belongs to the uncertainty set \mathcal{D}_2 defined by (6.66) is similar by considering the following Lagrangian function

$$\begin{aligned} \mathcal{L} \left[F, t, q_{\hat{p}}, Q_{\hat{p}}, (r_i)_{i=1}^2 \right] &= \mathbb{E}_{(\hat{m}, \hat{p})} [\mathbf{1}_{\{\hat{m}^\top R \leq y\}}] + \sum_{i=1}^2 r_i^\top \mathbb{E}_{(\hat{m}, \hat{p})} (g_i(\hat{m}, \hat{p})) + t \times \\ &\left(\mathbb{E}_{(\hat{m}, \hat{p})} (\mathbf{1}_{\{\hat{p} \in \varphi_{\hat{p}}\}}) - 1 \right) + q^\top (\mathbb{E}_{(\hat{m}, \hat{p})} (\hat{p}) - \mu) + Q \circ \left(\delta_0 \Sigma - \mathbb{E}_{(\hat{m}, \hat{p})} [(\hat{p} - \mu)(\hat{p} - \mu)^\top] \right) \end{aligned} \quad (6.73)$$

where $Q \in S_+^{|S| \times |\mathcal{K}|}$, where $S_+^{|S| \times |\mathcal{K}|}$ is the set of positive semidefinite matrices of size $(|\mathcal{K}| \times |\mathcal{K}|) \times (|\mathcal{K}| \times |\mathcal{K}|)$ and the other parameters are defined equivalently as (6.71).

Finally, we prove the case, where the true distribution of \hat{p} belongs to the uncertainty set \mathcal{D}_3 defined by (6.67). It follows from [57] that the constraint (ii) in (6.67) is equivalent to $Z^* \in S_+^{S \times |\mathcal{K}|+1}$, where

$$Z^* = \left(\frac{\Sigma}{(\mathbb{E}(\hat{p}) - \mu)^\top} \middle| \frac{\mathbb{E}(\hat{p}) - \mu}{\delta_1} \right),$$

Then, the Lagrangian function has the following form

$$\begin{aligned} \mathcal{L} \left[F, t, Z, Q, (r_i)_{i=1}^2 \right] &= \mathbb{E}_{(\hat{m}, \hat{p})} [\mathbf{1}_{\{\hat{m}^\top R \leq y\}}] + \sum_{i=1}^2 r_i^\top \mathbb{E}_{(\hat{m}, \hat{p})} (g_i(\hat{m}, \hat{p})) \\ &+ t (\mathbb{E}_{(\hat{m}, \hat{p})} (\mathbf{1}_{\{\hat{p} \in \varphi_{\hat{p}}\}}) - 1) + Z \circ Z^* + Q \circ \left(\delta_2 \Sigma - \mathbb{E}_{(\hat{m}, \hat{p})} [(\hat{p} - \mu)(\hat{p} - \mu)^\top] \right), \end{aligned} \quad (6.74)$$

where $Z \in S_+^{|\mathcal{S}| \times |\mathcal{K}|+1}$, $Z = \left(\frac{U}{v^\top} \middle| \frac{v}{h} \right)$, $U \in S_+^{|\mathcal{S}| \times |\mathcal{K}|}$, $v \in \mathbb{R}^{|\mathcal{S}| \times |\mathcal{K}|}$, $h \geq 0$, and the other parameters are defined equivalently as (6.73). The primal problem (6.69) and its dual problem can be rewritten as follows

$$\begin{aligned} \text{(P)} \quad & \sup_F \inf_{(t, Z, Q, r)} \mathcal{L} \left[F, t, Z, Q, (r_i)_{i=1}^2 \right] \\ \text{(D)} \quad & \inf_{(t, Z, Q, r)} \sup_F \mathcal{L} \left[F, t, Z, Q, (r_i)_{i=1}^2 \right] \end{aligned} \quad (6.75)$$

Due to Assumption 6.8, the strong duality holds. We can simplify the dual problem (D) in (6.75) by solving the optimization problem w.r.t Z , while fixing the other variables. The dual problem (D) in (6.75) can be rewritten as follows

$$\begin{aligned} & \inf_{(t, q, Q, r)} \left[(1 - \alpha)^2 \left(\sum_{s' \in \mathcal{S}} r_1(s') \gamma(s')^2 \right) - t + \Sigma \circ U - 2v^\top \mu + \delta_1 h + Q \circ \delta_2 \Sigma - \mu^\top Q \mu \right] \\ \text{s.t. (i)} \quad & \mathbf{1}_{\{\xi_{\hat{m}}^\top R \leq y\}} + \sum_{s' \in \mathcal{S}} r_1(s') \left[\left(\xi_{\hat{m}}^\top \xi_{\hat{p}}(s') \right)^2 - 2(1 - \alpha) \gamma(s') (\xi_{\hat{m}})^\top \xi_{\hat{p}}(s') \right] \\ & + \sum_{s \in \mathcal{S}, a \in A(s)} r_2(s, a) \left[f^h(s, a)^2 \left(\sum_{a \in A(s)} \xi_{\hat{m}}(s, a) \right)^2 - 2f^h(s, a) \left(\sum_{a \in A(s)} \xi_{\hat{m}}(s, a) \right) \right. \\ & \left. \xi_{\hat{m}}(s, a) + \xi_{\hat{m}}^2(s, a) \right] + t + 2v^\top \xi_{\hat{p}} - \left(\xi_{\hat{p}}^\top Q \xi_{\hat{p}} - 2\mu^\top Q \xi_{\hat{p}} \right) \leq 0, \end{aligned} \quad (6.76)$$

for any $\xi_{\hat{m}} \in \varphi_{\hat{m}}$, $\xi_{\hat{p}} \in \varphi_{\hat{p}}$, where

$$Z = \left(\frac{U}{v^\top} \middle| \frac{v}{h} \right) \succeq 0. \quad (6.77)$$

If $h = 0$, due to (6.77), we have $v = 0$. Then, $U = 0$ is the optimal solution of (6.70). Otherwise, if $h > 0$, by applying Schur's complement, (6.77) is equivalent to $U \succeq \frac{1}{h} v v^\top$. Since $\Sigma \succeq 0$, we deduce that $U = \frac{1}{h} v v^\top$ is the optimal solution of (6.70). By replacing $U = \frac{1}{h} v v^\top$ in the objective function of (6.70), we minimize the function $\frac{1}{h} v v^\top \circ \Sigma + h \delta_1$, for $h > 0$. The optimal solution of this optimization problem is $h = \frac{\|\Sigma^{\frac{1}{2}} v\|}{\sqrt{\delta_1}}$. Let $q = 2(v + Q\mu)$. Then, the optimization problem (6.76) reduces to the form (6.70). \square

To deal with the indicator function in (6.70), we can introduce an auxiliary binary decision variable $x_{\xi_{\hat{m}}} = \mathbf{1}_{\{\xi_{\hat{m}}^T R \leq y\}} \in \{0, 1\}$. We have the following lemma.

Lemma 6.36. *Let Assumption 6.8 holds. Consider the DRCCMDP-TP (6.64). Let $M = \mathbf{1}_{|\mathcal{K}|}^T R$, where $\mathbf{1}_{|\mathcal{K}|}$ denotes $|\mathcal{K}|$ -dimensional vector with all components equal to 1. Then, if the true distribution of \hat{p} belongs to the uncertainty set \mathcal{D}_j , $j = 1, \dots, 3$, then the DRCCMDP-TP optimization problem (6.64) is equivalent to the following deterministic problem*

$$\begin{aligned} & \sup_{(y \in \mathbb{R}, f^h \in PO_D)} y \\ \text{s.t. (i)} \quad & \mathbf{A}_j(t, q, Q, r) \leq \epsilon, \quad \text{(ii)} \quad x_{\xi_{\hat{m}}} + \mathbf{B}_j(t, q, Q, r, \xi) \leq 0, \quad \text{(iii)} \quad y - \xi_{\hat{m}}^T R \leq M x_{\xi_{\hat{m}}}, \\ & Q \in S^{|\mathcal{S}| \times |\mathcal{K}|}, \text{ if } j = 1, \quad Q \in S_+^{|\mathcal{S}| \times |\mathcal{K}|}, \text{ if } j = 2, 3, \quad \forall \xi = (\xi_{\hat{m}}, \xi_{\hat{p}}) \in \varphi_{\hat{m}} \times \varphi_{\hat{p}}. \end{aligned} \quad (6.78)$$

Proof. First, we prove that any feasible solution y of (6.64) and (6.78) must belong to $[0, M]$. In fact, it is clear that for $y < 0$, the constraint (i) of (6.64) and the constraint (iii) of (6.78) are redundant, while for $y > M$, they are always infeasible, which in turn implies that $0 \leq y \leq M$.

Using Lemma 6.35, we need prove that the constraint (i) of (6.70) is equivalent to the constraints (ii) and iii of (6.78). Let (y, t, q, Q, r, ξ) be a feasible solution of (i) of (6.70). By taking $x_{\xi_{\hat{m}}} = \mathbf{1}_{\{\xi_{\hat{m}}^T R \leq y\}}$, we get a feasible solution of (ii) and (iii) of (6.78). On the other hand, let $(y, t, q, Q, r, \xi, x_{\xi_{\hat{m}}})$ be a feasible solution of (ii) and (iii) of (6.78). If $x_{\xi_{\hat{m}}} = 0$, we deduce that $\mathbf{1}_{\{\xi_{\hat{m}}^T R \leq y\}} = 0$, which implies that (i) of (6.70) implies (ii) of (6.78). Otherwise, the proof is trivial. \square

Note that $\varphi_{\hat{p}}$ is a finite support, while $\varphi_{\hat{m}}$ can be an infinite support due to the set of policies PO_D . The deterministic optimization problem (6.78) is in fact a mixed-integer nonlinear optimization problem with infinite constraints, which is challenging to solve. One can resort to discretization methods by using a meshgrid on $[0, 1]^{|\mathcal{K}|}$ and restricting the set of policies on the meshgrid, that makes the support $\varphi_{\hat{m}}$ to be finite. However, the method only works with very small size problems and the number of constraints increases exponentially w.r.t the number of states in our MDP problem. To make the problem solvable, we restrict the set of randomized policies to the set of deterministic policies, i.e., PO_D is the set of $f^h \in \{0, 1\}^{|\mathcal{K}|}$, such that $f^h(s, a) = 0$ or 1, for any $s \in \mathcal{S}$, $a \in A(s)$. It is clear that the set of deterministic policies PO_D contains exactly $|\mathcal{K}|$ elements. The support $\varphi_{\hat{m}}$ is the set of solutions of (6.4), which in turn implies that $\varphi_{\hat{m}}$ is a finite support. Then, (6.78) is a mixed-integer with finite nonconvex constraints, where the source of nonconvexity comes from the terms $r_2(s, a)f^h(s, a)^2$ and $r_2(s, a)f^h(s, a)$, $s \in \mathcal{S}$, $a \in A(s)$ in the formulation of \mathbf{B}_j , $j = 1, \dots, 3$. Since $f^h(s, a) = 0$ or 1, then $f^h(s, a)^2 = f^h(s, a)$. Let $z(s, a) = f^h(s, a)r_2(s, a)$. Assume that there exists a large number $T > 0$ such that $-T \leq r_2(s, a) \leq T$, for any $s \in \mathcal{S}$, $a \in A(s)$. Then, using McCormick inequalities, by introducing auxiliary variable $z(s, a) = f^h(s, a)r_2(s, a) \in [-T, T]$, the functions B_j , $j = 1, \dots, 3$ are linear w.r.t decision variables with additional four linear constraints as follows

$$\begin{aligned} \text{(i)} \quad & z(s, a) + T f^h(s, a) \geq 0, \quad \forall s \in \mathcal{S}, a \in A(s), \\ \text{(ii)} \quad & z(s, a) - T f^h(s, a) \leq 0, \quad \forall s \in \mathcal{S}, a \in A(s), \\ \text{(iii)} \quad & z(s, a) + M f^h(s, a) - r_2(s, a) - T \leq 0, \quad \forall s \in \mathcal{S}, a \in A(s), \\ \text{(iv)} \quad & z(s, a) - T f^h(s, a) - r_2(s, a) + M \geq 0, \quad \forall s \in \mathcal{S}, a \in A(s). \end{aligned}$$

Then, if PO_D is the set of deterministic policies, the optimization problem DRCCMDP-TP is equivalent to a mixed-integer linear programming (MILP) in case of uncertainty set \mathcal{D}_1 and a mixed-integer semi definite programming (MISDP) (due to the constraint $Q \succeq 0$) in case of uncertainty set \mathcal{D}_2 and \mathcal{D}_3 .

6.3.2 . Uncertainty set with ϕ -divergence distance

In this section, we define a ϕ -divergence based uncertainty set for the distribution of \hat{p} . We assume that a nominal distribution of \hat{p} is known to the decision maker based on historical data of transition probabilities. Denote this distribution by $\nu \in \mathcal{M}_{\varphi_{\hat{p}}}^+$. The decision maker believes that the true distribution of \hat{p} , with support $\varphi_{\hat{p}}$, belongs to a ϕ -divergence ball, centered at a nominal distribution. The definition of ϕ -divergence distance between two discrete probability measures ν_1 and ν_2 with support $\varphi_{\hat{p}}$ is given by

$$I_{\varphi}(\nu_1, \nu_2) = \sum_{j=1}^J \varphi \left(\frac{\nu_1(\hat{p}_j)}{\nu_2(\hat{p}_j)} \right) \nu_2(\hat{p}_j),$$

where $\nu_1(\hat{p}_j)$ (resp. $\nu_2(\hat{p}_j)$) is the weight of ν_1 (resp. ν_2) on the j -th atom \hat{p}_j of $\varphi_{\hat{p}}$. The uncertainty set of the distribution of \hat{p} based on ϕ -divergence is defined by

$$\mathcal{D}_4 = \left\{ F_{\hat{p}} \in \mathcal{M}_{\varphi_{\hat{p}}}^+ \mid I_{\phi}(F_{\hat{p}}, \nu) \leq \theta_{\phi} \right\},$$

where $\theta_{\phi} > 0$. We have the following lemma.

Lemma 6.37. *The optimization problem (6.59) with uncertainty set $\mathcal{D}_{\hat{p}} = \mathcal{D}_4$, is equivalent to the following optimization problem*

$$\begin{aligned} & \sup_{y \in \mathbb{R}, f^h \in PO_D} y \\ \text{s.t. } & \mathbb{P}_{\nu}(V(f^h, \hat{p}) \geq y) \geq f(\theta_{\phi}, \epsilon), \end{aligned} \quad (6.79)$$

where $f(\theta_{\phi}, \epsilon)$ is defined in Table 5.5.

Proof. The proof follows the same arguments as random reward case, where we replace a continuous nominal distribution with a density function by a discrete nominal distribution. \square

Remark 6.8. *In random reward case, $V(f^h, \cdot)$ is a linear function of reward \hat{R} , then by assuming that \hat{R} follows a Normal distribution, (6.79) can be reformulated equivalently as a SOCP. However, the problem is more challenging in random transition probabilities case.*

By introducing the occupation measure as an auxiliary random variable, (6.79) is equivalent to the following optimization problem

$$\begin{aligned} & \sup_{y \in \mathbb{R}, f^h \in PO_D} y \\ \text{s.t. } & \text{(i) } \mathbb{E}_{(\hat{m}, \hat{p})} [\mathbf{1}_{\{\hat{m}^T R \geq y\}}] \geq f(\theta_{\phi}, \epsilon), \\ & \text{(ii) } \mathbb{E}_{(\hat{m}, \hat{p})} [g_i(\hat{m}, \hat{p})] = 0, \quad i = 1, 2, \end{aligned} \quad (6.80)$$

where \hat{p} follows the nominal distribution ν with support $\varphi_{\hat{p}}$. In general case, when $PO_D = PO_{HD}$, the optimization problem (6.80) is very challenging to solve. One way is to discretize the set of stationary policies PO_{HD} by a finite number of policies. By restricting PO_{HD} on this finite set, the support $\varphi_{\hat{m}}$ is also a finite set. Assume that $\varphi_{\hat{m}} = \{\hat{m}_1, \dots, \hat{m}_U\}$. Then, one can represent the joint distribution of (\hat{m}, \hat{p}) as a set of

$x_{ij} \in [0, 1]$, $x_{uj} \geq 0$, $\forall u \in U$, $j \in J$, $\sum_{u \in U, j \in J} x_{uj} = 1$. The optimization problem (6.80) is rewritten as follows

$$\begin{aligned}
& \sup_{y \in \mathbb{R}, f^h \in PO_D, x} y \\
\text{s.t. } & \text{(i)} \quad \sum_{u \in U, j \in J} x_{uj} \mathbf{1}_{\{\hat{m}_u^T R \geq y\}} \geq f(\theta_\phi, \epsilon), \\
& \text{(ii)} \quad \sum_{u \in U, j \in J} x_{uj} g_i(\hat{m}_u, \hat{p}_j) = 0, \quad i = 1, 2, \\
& \text{(iii)} \quad \sum_{u \in U, j \in J} x_{uj} = 1, \quad x_{uj} \geq 0.
\end{aligned} \tag{6.81}$$

The optimization problem (6.81) is intractable due to a huge number of nonconvex constraints in the constraints (ii). To make the problem tractable, we restrict the set of policies to deterministic policies, i.e., we assume that

$$PO_D = \{f^h \mid f^h(s, a) = 0 \text{ or } 1, \forall s \in S, a \in A(s)\}.$$

For any $u \in U$, let $z_u = \mathbf{1}_{\{\hat{m}_u^T R \geq y\}}$. Then, (6.81) is equivalent to the following mixed-integer optimization problem

$$\begin{aligned}
& \sup_{y \in \mathbb{R}, f^h \in PO_D, x, z} y \\
\text{s.t. } & \text{(i)} \quad \sum_{u \in U, j \in J} x_{uj} z_u \geq f(\theta_\phi, \epsilon), \quad \text{(ii)} \quad \hat{m}_u^T R - y \leq M z_u, \quad z_u \in \{0, 1\} \\
& \text{(iii)} \quad \sum_{u \in U, j \in J} x_{uj} g_i(\hat{m}_u, \hat{p}_j) = 0, \quad i = 1, 2, \quad \text{(iv)} \quad \sum_{u \in U, j \in J} x_{uj} = 1, \quad x_{uj} \geq 0,
\end{aligned} \tag{6.82}$$

where $M = \mathbf{1}_{|K|}^T R$. The optimization problem (6.82) is a mixed-integer programming with nonconvex constraints, where the source of nonconvexity comes from the bilinear terms $x_{uj} z_u$ and $x_{uj} f^h(s, a)$. Since z_u and $f^h(s, a)$ are binary variables and $x_{uj} \in [0, 1]$, by introducing auxiliary variables $g_{uj} = x_{uj} z_u$ and $h_{uj}(s, a) = x_{uj} f^h(s, a)$, one can apply McCormick inequalities to obtain an equivalent mixed-integer linear programming form of (6.82) with additional linear constraints

$$\begin{aligned}
g_{uj} &\geq 0, & g_{uj} &\leq x_{uj}, & g_{uj} &\leq z_u, & g_{uj} &\geq x_{uj} + z_u - 1, \\
h_{uj}(s, a) &\geq 0, & h_{uj}(s, a) &\leq x_{uj}, & h_{uj}(s, a) &\leq f^h(s, a), & h_{uj}(s, a) &\geq x_{uj} + f^h(s, a) - 1.
\end{aligned}$$

6.4 . Machine replacement problem

In this section, we present a series of numerical results to compare the approaches discussed earlier. These comparisons aim to evaluate the performance and efficiency of the different reformulations and solvers in solving the respective problems. By conducting these comparisons, we can gain insights into the strengths and limitations of each approach and make informed decisions based on the specific problem characteristics. All the numerical results below are performed using Matlab and Python 3.8.8 on an Intel Core i5-1135G7, Processor 2.4 GHz (8M Cache, up to 4.2 GHz), RAM 16G, 512G SSD.

6.4.1 . Comparison between moment-based and statistical distance-based approaches, full support and

nonnegative support

We consider a machine replacement problem where a machine in a factory has a life-time of N years. At every stage a maintenance of the machine is scheduled but a factory owner can decide whether to repair or do not repair the machine. There is a high probability that the machine behaves like a new one if it is being repaired and its life gets reduced by a year if it is not being repaired. The factory owner incurs maintenance cost if he decides to repair the machine. It can be modelled as an MDP problem where the life of a machine represents the state of underlying Markov chain, i.e., there are $N + 1$ states. The first state represents a brand new machine. At each state there are two actions: i) "repair", ii) "do not repair". The transition probabilities of the Markov chain with respect to each action is given by Figure 7.1. The maintenance cost corresponding to every state-action pair is not exactly known and is realised after the decision is made. Therefore, it is modelled with a random variable. We assume that for every state action pair (s, a) , the maintenance cost is defined as $\hat{c}(s, a) = K + \hat{Z}(s, a)$, where K represents the fixed cost and $\hat{Z}(s, a)$ represents a variable cost which is a random variable. The machine generates a revenue $L(s, a)$ at state-action pair (s, a) and the profit for each $(s, a) \in \mathcal{K}$ is given by

$$\hat{R}(s, a) = L(s, a) - K - \hat{Z}(s, a). \quad (6.83)$$

The factory owner is interested in maximizing the expected discounted profit. We assume that the factory owner has a finite number of the same machines which are modelled using the same Markov chain. Therefore, we compute the optimal repair policy with respect to a single machine and the same repair policy can be applied for all other machines. We compare the performance of DRCCMDP for each uncertainty set with

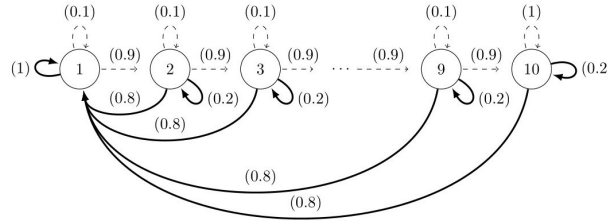


Figure 6.7: Machine replacement MDP with two actions: "repair" (with solid lines) and "do not repair" (with dashed lines)

the CCMDP model (6.6) where the distribution of \hat{R} is assumed to be a normal distribution. In our numerical experiments, we set the number of states to 10, the threshold value $\epsilon = 0.1$, the discount parameter $\alpha = 0.85$ and the initial distribution of states γ to be uniformly distributed. For the above instance, $|\mathcal{K}| = 20$ and \hat{R} is a 20×1 random vector with mean vector μ given by

$$\mu(s, a) = L(s, a) - K - \mu_{\hat{Z}}(s, a), \quad (6.84)$$

where $\mu_{\hat{Z}}$ is the mean vector of the random cost vector \hat{Z} . We take $K = 10$, the function L and the mean cost $\mu_{\hat{Z}}$ corresponding to each state-action pair are summarized in Table 6.8. The covariance matrix Σ of \hat{R} is randomly generated using the following formula

$$\Sigma = \frac{AA^T}{20} + D_{20}, \quad (6.85)$$

where A is a 20×20 random matrix whose all the entries are real numbers belonging to $[0, 1]$, and D_{20} is a 20×20 diagonal matrix with $D_{20}(10, 10) = 4$, $D_{20}(20, 20) = 9$, $D_{20}(i, i) = 1$, for every $i \neq 10, 20$ and all other entries equal to zero. We use the above μ and Σ for all the moment based uncertainty sets. For ϕ -divergence based

Table 6.8: Random cost \hat{Z} and Revenue L

Action(a) State(s)	"Repair" $\mu_{\hat{Z}}(s, 1)$	"Do not repair" $\mu_{\hat{Z}}(s, 2)$	"Repair" $L(s, 1)$	"Do not repair" $L(s, 2)$
1	10	0	30	30
2	10.1	0	30	29.9
3	10.2	0	30	29.8
4	10.3	0	30	29.7
5	10.4	0	30	29.6
6	10.5	0	30	29.5
7	10.6	0	30	29.4
8	10.7	0	30	29.3
9	10.8	0	30	29.2
10	10.9	5	30	29.1

Table 6.9: Other parameters

Known mean unknown covariance	$\delta_0 = 0.9$
Unknown mean unknown covariance	$\delta_1 = \delta_2 = 1$
ϕ -divergence	$\theta_\phi = 0.01$
Wasserstein distance	$\theta_W = 0.01$ $H = 1000$

uncertainty set, we take the nominal distribution ν as a normal distribution with mean $\mu_\nu = \mu$ and covariance matrix $\Sigma_\nu = \Sigma$ where μ and Σ are defined by (6.84) and (6.85), respectively. For Wasserstein distance based uncertainty set, we take the number of observations $H = 1000$. The scenarios $(\tilde{\xi}_i)_{i=1}^H$ are randomly generated by taking $\tilde{\xi}_i = Bx + \mu_\nu$, where x is a standard normal vector, μ_ν is defined by (6.84) and B is the Cholesky factorization of Σ_ν defined by (6.85). We summarize the other parameters related to all the uncertainty sets in Table 6.9.

Table 6.10: Optimal policies of CCMDP-R and DRCCMDP-R with full and nonnegative supports

Optimal State(s)	Optimal policies	CCMDP Gaussian (p,1-p)	Full support known mean known covariance (p,1-p)	Full support known mean unknown covariance (p,1-p)	Full support unknown mean unknown covariance (p,1-p)	ϕ -divergence (Modified χ^2) (p,1-p)	ϕ -divergence (variation) (p,1-p)
1		(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)
2		(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)
3		(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)
4		(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)
5		(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)
6		(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)
7		(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)
8		(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)
9		(0, 1)	(0.64, 0.36)	(0.64, 0.36)	(0.6, 0.4)	(0.27, 0.73)	(0.05, 0.95)
10		(0.9, 0.1)	(0.91, 0.09)	(0.91, 0.09)	(0.91, 0.09)	(0.9, 0.1)	(0.9, 0.1)

We compute an optimal policy of the CCMDP-R problem (6.6), where \hat{R} follows a normal distribution with mean vector and covariance matrix defined by (6.84) and (6.85), by solving an equivalent SOCP problem [56]. The optimal policies of the DRCCMDP-R problem for all the uncertainty sets are computed by solving the proposed equivalent optimization problems. We present the optimal policies of CCMDP-R and DRCCMDP-R with full support and nonnegative support in Tables 6.10 and 6.11, where p is the probability of "repair" action and $1 - p$ is the probability of "do not repair" action. It is clear from Tables 6.10 and 6.11 that the optimal repair policy corresponding to all the uncertainty sets for first eight states is same. At state 9 the probability of repair is greater than the probability of do not repair for moment based uncertainty sets whereas for statistical distance based uncertainty sets the probability of repair is less than the probability of do not repair. This shows that the statistical distance based uncertainty sets give better optimal policy as compared to moment based uncertainty sets and the nonnegative support uncertainty sets give better optimal policy as compared to full support uncertainty sets. At the last state, the optimal policy is to choose repair action with a very high probability for all the uncertainty sets.

6.4.2 . Comparisons in terms of running time between different reformulations methods

Table 6.11: Optimal policies of *CCMDP-R* and *DRCCMDP-R* with full and nonnegative supports (continued)

ϕ -divergence (Kullbach-Leibler) (p,1-p)	ϕ -divergence (Hellinger) (p,1-p)	Full support Wasserstein (p,1-p)	Nonnegative known mean known covariance (p,1-p)	Nonnegative known mean unknown covariance (p,1-p)	Nonnegative unknown mean unknown covariance (p,1-p)	Nonnegative Wasserstein (p,1-p)
(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)
(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)
(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)
(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)
(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)
(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)
(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)
(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)
(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)
(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)
(0.25, 0.75)	(0.28, 0.72)	(0.02, 0.98)	(0.62, 0.38)	(0.62, 0.38)	(0.59, 0.41)	(0.01, 0.99)
(0.9, 0.1)	(0.9, 0.1)	(0.9, 0.1)	(0.91, 0.09)	(0.91, 0.09)	(0.91, 0.09)	(0.9, 0.1)

We consider a machine replacement model with fixed reward and random transition probabilities. The reward vector is given by

$$R(s, a) = L(s, a) - K - \mu(s, a), \quad (6.86)$$

where $K = 10$ and L, μ are given in Table 6.8. Transition probabilities p are $|S| \times |\mathcal{K}|$ – random variable, which follows an uniform distribution on φ_p . We simulate $J = 100$ data points of φ_p . The other parameters are taken similarly as random reward model. We present the time analysis by considering the number of states for all uncertainty sets between 100 and 1000. All the parameters are taken similar to the case of 10 states. We utilize different solvers for solving the optimization problems described as follows:

- Gurobi solver: We use Gurobi solver to solve SOCP (6.12) with $\kappa = \sqrt{\frac{1-\epsilon}{\epsilon}}$, the MISOCP (6.47) and the MILP in case of uncertainty set \mathcal{D}_1 of random transition probabilities.
- Mosek solver: To solve SDP relaxation of the copositive optimization problem (6.13), we employ Mosek solver. The SDP relaxation is a common approach to approximate copositive optimization problems.
- Baron solver: For the biconvex optimization problem (6.54), we utilize the Baron solver. Baron is a nonlinear nonconvex solver that is well-suited for handling such problems.
- BNB solver in YALMIP toolbox of Matlab: The MISDP optimization for uncertainty set \mathcal{D}_2 in random transition probabilities is solved using the Branch-and-Bound (BNB) solver available in the YALMIP toolbox of MATLAB. This solver is specifically designed to handle mixed-integer semidefinite programming problems.

The results obtained from these solvers are presented in Figure 7.2. The figure illustrates that the CPU time is generally similar for both the SOCP and MILP formulations. However, solving MISOCP and SDP problems typically requires more CPU time. Additionally, MISDP and biconvex optimization problems tend to have significantly longer computation times. Overall, the figure highlights the differences in computational efficiency among the different optimization approaches and solvers, demonstrating that the CPU time required varies based on the specific problem formulation and solver employed.

6.5 . Conclusion

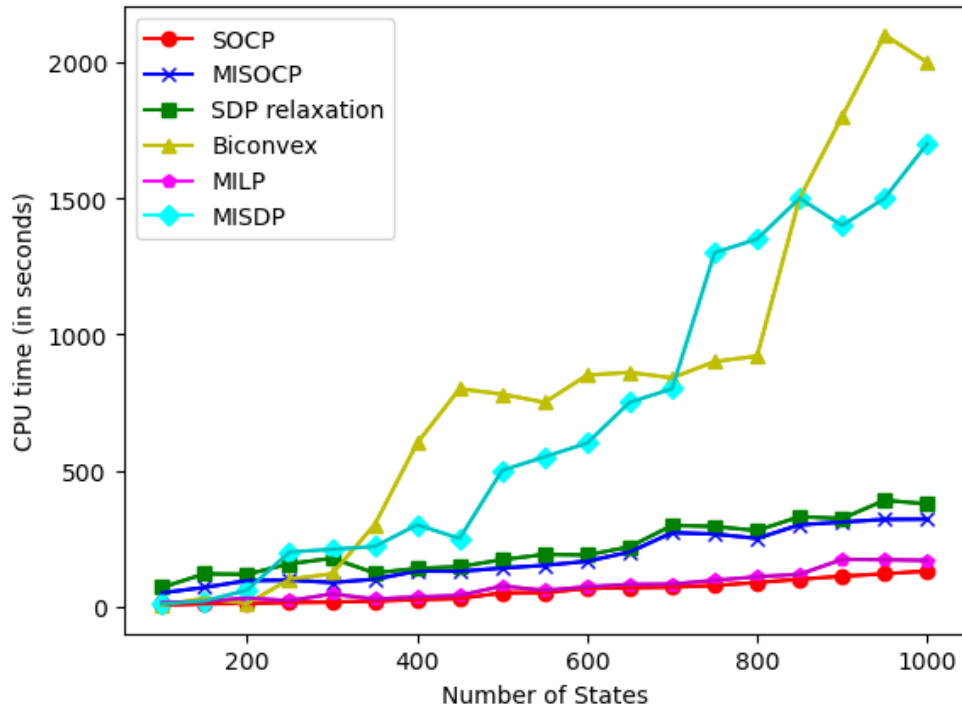


Figure 6.8: CPU time (in seconds) vs number of states

We study a DRCCMDP problem under various moment and statistical distance based uncertainty sets defined using ϕ -divergence and Wasserstein distance metric in two cases: random reward and random transition probabilities. We propose equivalent SOCP, MISOCP, copositive optimization problem, biconvex optimization problem, MILP, MISDP depending on the choice of the uncertainty set, for the DRCCMDP problem. All these optimization problems except biconvex optimization problems and MISDP can be solved efficiently using known optimization solvers. We perform numerical experiments, using the optimization solvers in python, on a machine replacement problem using randomly generated data. The numerical experiments are performed on the DRCCMDP problem up to 1000 states and it is very clear from our time analysis that these problems can be solved very efficiently.

This chapter correspond to the reference [169].

Chapter 7 :Conclusions and Perspectives

7.1 . Conclusions

In conclusion, this thesis has delved into the study of convexity properties in chance constrained optimization and their applications in random games and Markov decision processes (MDPs). The research has explored various aspects related to these topics, yielding important results and insights. The study of chance constrained optimization and its applications remains an attractive research area. The following notable research results were obtained

- We studied on the study of linear joint chance constraints with a random technology matrix. We assume that the constraint row vectors follows an elliptical distribution and the dependence among the rows is modelled by Gumbel-Hougaard copulas. Building upon previous work, particularly the paper by Cheng et al. [48], we extended the analysis to provide a deeper understanding of the convexity properties in a more specific setting. Under certain assumptions, primarily relating to the probability level, we established the convexity of the feasibility set for the considered linear joint chance constraints. This result demonstrates the inherent structural properties of the problem, highlighting the potential for developing efficient solution approaches.
- Convexity result of chance constraints can be applied to show the existence of a Nash equilibrium in a chance-constrained game. We studied an n -players chance-constrained game with random payoffs and continuous strategy sets. Each player's payoff function was defined by its expected value, while their strategy sets were defined by joint chance constraints. These joint chance constraints incorporated dependent random constraint vectors that followed elliptically symmetric distributions. To capture the dependence among these random constraint vectors, we employed the Archimedean copula. Our main contribution was the proposal of a reformulation for the joint chance constraint of each player based on the family of Archimedean copulas to model the dependence between the row vectors of the technology matrix. The paper emphasizes the importance of the convexity of the feasibility set in the context of the n -player game with random payoffs and joint chance constraints. This convexity property plays a crucial role in establishing the existence of a Nash equilibrium in the paper. Next, we extended the study of the n -player chance-constrained game with elliptically distributed assumption. Specifically, we focused on a slight extension of the distribution of row vectors in the technology matrix. We assume that the row vectors follow a normal mean-variance mixture distribution, which belongs to a broader class of distributions, allows us to better capture the characteristics of real-world data and applications. To find a Nash equilibrium, we proposed a best-response algorithm. This algorithm sequentially solves convex optimization problems to determine the optimal strategies for each player. By iteratively updating the strategies based on the best response to the strategies of other players, we aimed to converge to a stable Nash equilibrium (not guarantee). We applied the game to model a financial competition scenario, showcasing the usefulness of the approach in real-life applications.
- We consider a two-player zero-sum game with random linear chance constraints. These chance constraints are characterized by distributions that belong to either moment based uncertainty sets or statistical distance based uncertainty sets. The main result of this chapter is the demonstration of the existence of a saddle point equilibrium of the game. This equilibrium represents the optimal solution where neither player can unilaterally deviate from their strategy to improve their payoff, given the strategy of

the opponent, also known as Nash equilibrium. This result is significant as it establishes the unique equilibrium point, ensuring the stability of the game under the considered random linear chance constraints. To compute this saddle point equilibrium, we propose a novel approach based on solving a primal-dual pair of second-order cone programming (SOCP) problems. By formulating the game as a primal-dual pair of SOCPs, we exploit the duality theory to efficiently find the equilibrium point. This approach offers computational tractability and guarantees the convergence to the saddle point equilibrium.

- We study MDP problems of two types: one where the transition probabilities are known and the reward vector is a random vector with a partially known distribution, and the other where the reverse holds true. To handle the uncertainty in these MDP problems, we adopt the framework of distributionally robust chance-constrained optimization. We consider different types of moment based uncertainty sets and statistical-distance based uncertainty sets. These uncertainty sets are defined using phi-divergence and Wasserstein distance metric. We investigate two scenarios within the random reward case: one where the random reward vector has a full support, and another where it has a nonnegative support. To solve the distributionally robust chance-constrained optimization problems, we show that they can be reformulated as deterministic optimization problems using duality theory. This reformulation enables us to leverage existing optimization solvers such as Gurobi, Mosek, the nonlinear nonconvex solver IPOPT or BNB solver in YALMIP toolbox of Matlab. By solving the deterministic reformulations, we can obtain optimal solutions to the distributionally robust chance-constrained MDP problems. As an application of the proposed framework, we study a machine replacement problem, which serves as an illustrative example to showcase the effectiveness of our approach. We conduct numerical experiments on randomly generated instances, demonstrating the applicability and robustness of the distributionally robust chance-constrained optimization framework in solving real-world decision problems.

7.2 . Perspectives

In this thesis dissertation, several perspectives and future directions can be considered to further expand and advance the research in the field. The following prospects highlight potential areas of exploration and development:

- A promising direction for future research involves expanding the existing framework of convexity results in chance-constrained optimization to encompass a broader range of cases. This expansion can be achieved by exploring several avenues, such as considering different types of copulas, alternative distributional assumptions, or incorporating nonlinear chance constraints. By investigating these directions, the applicability and generality of the convexity result can be extended, enabling its effective utilization in diverse scenarios. In particular, exploring different types of copulas can provide valuable insights into the relationship between random variables and the dependence structure among them. This exploration can lead to a deeper understanding of the impact of copulas on the convexity properties of chance-constrained optimization problems. Additionally, incorporating alternative distributional assumptions allows for a more comprehensive analysis of the convexity result, accounting for a wider range of probability distributions and their associated characteristics. Another promising avenue for research involves considering nonlinear chance constraints. Nonlinear chance constraints capture more complex relationships between decision variables and uncertainties, and their investigation can significantly enhance the flexibility and practicality of chance-constrained optimization models. By examining the convexity properties of these constraints, researchers can identify new solution techniques and develop innovative optimization algorithms tailored to handle nonlinear chance constraints effectively.

Furthermore, investigating the analytic properties of chance constraints, such as the differentiability of the probability function, offers a rich area for exploration. Understanding the differentiability characteristics provides insights into the behavior and structure of chance constraints, enabling the development of specialized solution methodologies. By leveraging this knowledge, researchers can design novel algorithms and techniques that exploit the differentiability properties, ultimately leading to improved solution quality and efficiency. By pursuing these research directions, the understanding of convexity in the context of chance constraints can be advanced. This advancement not only broadens the theoretical foundations of chance-constrained optimization but also paves the way for novel applications in decision-making under uncertainty. Ultimately, these research efforts contribute to the development of more robust and effective optimization frameworks that account for uncertainties in real-world scenarios.

- Our potential research in chance-constrained games is exploring the existence of a Nash equilibrium in the case of dependent row vectors in the normal mean-variance mixture distribution is an intriguing and open research direction with significant potential. Currently, the assumption of independence among the row vectors provides tractable analysis and serves as a fundamental basis for studying Nash equilibria. However, incorporating dependence among the row vectors introduces a new level of complexity to the problem formulation. By considering dependent row vectors in the normal mean-variance mixture distribution, researchers can delve into the intricacies of interdependencies and their impact on the existence of Nash equilibria. This direction opens up avenues for investigating the relationship between dependence structures, such as correlation or covariance, and the emergence of equilibrium solutions. It also allows for the exploration of how different forms of dependence can influence strategic decision-making and equilibrium outcomes. Addressing the challenges posed by dependent row vectors requires the development of novel analytical techniques and solution methodologies. Research efforts can focus on adapting existing game-theoretic frameworks to accommodate dependence or introducing new mathematical tools specific to the analysis of dependent row vectors. These advancements will enable a deeper understanding of the behavior and properties of Nash equilibria in settings with complex interdependencies. Furthermore, exploring the existence of Nash equilibria in the presence of dependent row vectors has practical implications across various domains. For example, in finance, considering dependence among asset returns can provide valuable insights into portfolio optimization, risk management, and market equilibrium. In social networks, understanding the influence of interconnections among individuals can shed light on strategic interactions and the emergence of equilibrium behavior.
- Applying chance-constrained games and Markov decision processes models in other fields of application can open up new avenues for research and practical implementations. Some potential fields where our models can be applied are energy systems, transportation and logistics, environmental management and healthcare systems. In energy systems, the application of our models to energy systems could enable more efficient and effective management of power generation and distribution. The inherent uncertainty of renewable energy sources, like wind or solar power, lends itself to the chance-constrained models. Optimal decision making can be enhanced by applying these models to problems like grid balancing, where unpredicted changes in energy supply and demand can be managed efficiently. Additionally, these models could be used for strategic investment planning in the energy sector. Transportation and logistics represent a vast area for potential application of our models. The complexity of decision-making in this field, influenced by a multitude of stochastic factors like traffic, weather conditions, and customer demand, aligns well with our game and MDP models. This could lead to innovations in areas like route optimization, delivery scheduling, and fleet management. Moreover, the emergence of autonomous vehicles could benefit significantly from the predictive and adaptive capa-

bilities of our models. Environmental management is increasingly becoming an important field due to the heightened awareness of climate change. Our models could help policymakers and environmental scientists make better decisions under uncertainty, such as the optimal allocation of resources for pollution control or wildlife conservation efforts. The models could also be used to predict and adapt to the uncertain impact of climate change on various ecosystems, thereby aiding in the creation of more robust conservation strategies. In healthcare systems, our models could be instrumental in improving patient outcomes and healthcare delivery by enabling more effective decision-making under uncertainty. This could range from optimizing patient scheduling in hospitals, making decisions about treatment plans under uncertain patient responses, to managing the supply chain of vital medicines and medical equipment. Furthermore, with the rise of personalized medicine, these models could be used to make better individualized treatment decisions based on each patient's unique health parameters.

- In the current era of digital transformation, we're witnessing an unprecedented proliferation of data across various industries. This data explosion presents both challenges and opportunities. The challenges lie in handling the vast volumes of complex data and deriving meaningful insights from it. The opportunities arise from the potential of using this data to make more informed decisions, improve processes, and ultimately, enhance the overall efficiency of systems. By integrating machine learning (ML) into chance-constrained optimization, we can create models that not only handle large and complex data but also learn from it. These models will continuously improve their performance, adapting to new data and evolving trends. This will result in optimization models that are both dynamic and resilient, providing reliable solutions even in the face of changing environments. On the other hand, the integration of ML and optimization has the potential to bring theoretical models closer to real-world applications. By utilizing ML's capacity to handle real-world complexities, we can develop chance-constrained optimization models that are not only theoretically sound but also practically applicable. This could drastically expand the scope of chance-constrained optimization, opening up new fields of application.

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