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Lucie Delcey

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Thèse de Doctorat

présentée par

Lucie DELCEY

*Pour obtenir le grade de
Docteur en Mathématiques et Applications
de l'Université Bourgogne Franche-Comté*

Préparée à l'Université de Franche-Comté

Stabilité des ondes non linéaires de l'équation de Lugiato-Lefever

Thèse présentée et soutenue à Besançon, le 16 décembre 2021

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Introduction

Dans cette thèse, nous présentons quelques résultats mathématiques concernant l'équation d'évolution dite de Lugiato-Lefever, qui est une équation de Schrödinger non linéaire amortie en dimension un avec une non linéarité cubique et une source constante dans un contexte périodique. Cette équation, issue de l'optique non linéaire, est toujours très étudiée dans la littérature physique. Comparativement, la littérature mathématique est encore relativement réduite. Nous apportons des contributions aux questions d'existence et de stabilité de solutions stationnaires de cette équation.

1. Contexte et problème physique

Cette thèse est consacrée à l'étude de l'équation de Lugiato-Lefever :

$$\frac{\partial \psi}{\partial t} = -i\beta \frac{\partial^2 \psi}{\partial x^2} - (1 + i\alpha)\psi + i\psi|\psi|^2 + F \quad (1)$$

qui est une équation de type Schrödinger non linéaire présentant des termes d'amortissement, déphasage et de source. Cette équation est dérivée de l'optique non linéaire par Lugiato et Lefever [19]. Bien qu'elle soit très étudiée dans la littérature physique, il y a encore relativement peu de résultats mathématiques la concernant.

Cette équation permet de modéliser la formation de peignes de fréquences optiques par effet Kerr dans les résonateurs optiques à mode de galerie (voir [3, 10]). Un résonateur optique est, d'un point de vue général, un dispositif dans lequel certains rayons lumineux sont susceptibles de rester confinés. Dans le contexte de [3, 10], les rayons lumineux proviennent d'un laser et sont envoyés dans le résonateur par l'intermédiaire d'une fibre optique. Les rayons lumineux provenant du laser atteignent le résonateur et une partie est déviée à l'intérieur du résonateur tandis qu'une autre partie est réfléchiée en dehors de ce dernier. La partie déviée à l'intérieur crée un champ électrique ψ , qui est la solution de l'équation (1) (voir la Figure 0.1).

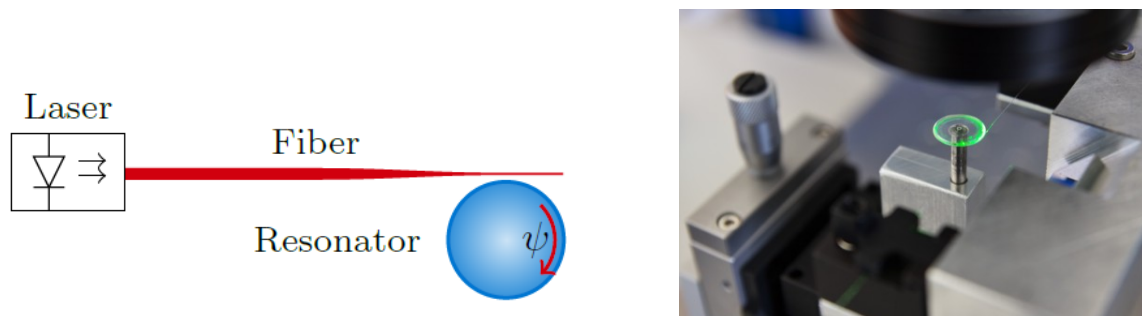


FIGURE 0.1 – à gauche : schéma d'un résonateur optique à mode de galerie tirée de l'article [10] // à droite : photographie du résonateur tirée de l'article [3] // expérience réalisée à l'institut FEMTO-ST

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Dans ce contexte, l'inconnue ψ est une fonction à valeurs complexes qui dépend d'un paramètre temporel t , d'un paramètre spatial $x \in \mathbb{R}$ et représente le champ électrique créé par la partie des photons déviée dans le résonateur. Le paramètre réel α est un terme de décalage des fréquences du laser et est en général compris entre -3 et 3 , comme indiqué dans [3, 10]. Le paramètre F est un réel strictement positif qui représente la puissance du laser, c'est le terme source. Enfin, le réel β est un paramètre de dispersion non nul. Dans le cas où $\beta > 0$, on dit que la dispersion est normale tandis que lorsque $\beta < 0$, la dispersion sera qualifiée d'anormale.

Les peignes de fréquences sont représentés mathématiquement par des solutions stationnaires périodiques. Dans notre travail, nous nous intéressons à l'existence de ces solutions, leurs propriétés analytiques et leur stabilité par perturbations périodiques ou localisées.

2. État de l'art

Une première analyse rigoureuse de bifurcation de solutions stationnaires, c'est-à-dire de solutions constantes en temps, de l'équation de Lugiato-Lefever (1) a été menée par les auteurs de [22] dans le cas $\beta < 0$ (dispersion anormale) et $\alpha < 2$. Pour ces valeurs particulières de paramètres, l'équation de Lugiato-Lefever possède une solution constante conduisant à une instabilité de Turing (c'est une instabilité vis-à-vis de perturbations non constantes de solutions qui sont stables par rapport à des perturbations constantes) lorsque son module est égal à 1. Les bifurcations locales sont analysées en choisissant α et ρ comme paramètres de bifurcation, où ρ représente le module au carré de la solution constante. Deux approches sont présentées : les bifurcations des ondes stationnaires périodiques et celles des ondes stationnaires localisées. Concernant l'analyse des ondes périodiques, l'équation de Lugiato-Lefever (1) est reformulée comme système dynamique du premier ordre, comme nous le faisons dans le Chapitre 1. L'espace des phases pour ce système est l'espace des fonctions périodiques par rapport à la variable spatiale, et une réduction à une variété centrale est effectuée pour analyser les bifurcations locales (cf section plus bas). Plusieurs bifurcations d'ondes stationnaires périodiques sont étudiées au seuil d'instabilité de Turing et dans la région de paramètres où l'instabilité de Turing est développée. En particulier, en analysant le seuil d'instabilité de Turing pour $\alpha = 41/30$, les résultats dans [22] recouvrent les bifurcations super-critiques et sous-critiques des ondes périodiques stationnaires obtenues dans la littérature physiques pour $\alpha < 41/30$ et $\alpha > 41/30$ respectivement. En ce qui concerne les ondes stationnaires localisées, les auteurs de [22] les ont construites à partir d'une approche de dynamique spatiale dans laquelle l'équation stationnaire de Lugiato-Lefever

$$0 = -i\beta \frac{\partial^2 \psi}{\partial x^2} - (1 + i\alpha)\psi + i\psi|\psi|^2 + F, \quad (2)$$

est écrite comme système dynamique de dimension quatre en considérant la variable spatiale x comme variable d'évolution. En se restreignant aux mêmes valeurs de paramètre $\beta < 0$ et $\alpha < 2$, les auteurs de [22] ont prouvé l'existence de solutions localisées proche du seuil d'instabilité de Turing de la solution constante, pour $\alpha < 41/30$.

Une étude systématique des bifurcations locales pour l'équation stationnaire (2) a été réalisée plus tard dans [10, 9] dans les cas de dispersions normale et anormale ($\beta > 0$ et $\beta < 0$). En commençant par reformuler l'équation stationnaire comme système dynamique de dimension quatre, similaire à celui utilisé pour les ondes localisées dans [22], mais en prenant les paramètres physiques α et F comme paramètres de bifurcation, les bifurcations locales ont été classifiées dans [10] et étudiées en détail dans [9]. L'existence de différents types de solutions, dont les solutions périodiques et localisées, a été prouvée à l'aide de théories des formes normales et variétés centrales. Les études de bifurcations dans [9] sont réalisées pour des solutions proches des solutions constantes et les résultats obtenus donnent une très bonne description de l'ensemble des solutions bornées de l'équation (2) mais seulement pour des valeurs

de paramètres proches des points de bifurcation. Récemment, des outils issus de la théorie de bifurcation globale ont été utilisés dans [20] pour l'étude des bifurcations globales des solutions périodiques de l'équation stationnaire (2). Les auteurs trouvent des bornes sur la norme L^∞ de ψ en fonction des paramètres α , β et F , pour les solutions stationnaires périodiques de l'équation (1) dans les cas de dispersions normale et anormale. Leur étude diffère des travaux [9] car les auteurs de [20] considèrent (1) sur des espaces de fonctions 2π -périodiques. L'étude menée dans cet article repose sur les théorèmes de Rabinowitz et de Crandall-Rabinowitz, qui leur permet d'obtenir des résultats de bifurcations globales. Les auteurs prouvent aussi que les solutions stationnaires de (1) sont constantes pour des valeurs de $|\alpha|$ suffisamment grandes. Ils prouvent également l'existence de solutions stationnaires 2π -périodiques dites synchronisées (satisfaisant des conditions de bords de Neumann en 0 et π). Dans [21], l'auteur prouve un critère pour les bifurcations secondaires globales qui s'applique à Lugiato-Lefever. Plus récemment, dans [8], les auteurs reprennent les résultats obtenus dans [20] en ajoutant un terme d'amortissement sur la non linéarité de l'équation de Lugiato-Lefever (1) lorsqu'ils se placent sur le tore ou sur l'axe des réels. Une autre étude d'existence de solutions stationnaires a été menée dans [11]. Les auteurs ont construit des états stationnaires par bifurcation à partir de solutions cnoïdales de l'équation de Schrödinger non linéaire cubique.

Tous ces résultats concernent les solutions stationnaires de l'équation de Lugiato-Lefever (1), c'est-à-dire des solutions de (2). L'analyse du caractère bien posé de l'évolution de (1) a été effectuée dans [16]. Les auteurs montrent que dans le cas de dispersion normale, le problème de Cauchy associé à (1) admet une unique solution pour des conditions initiales suffisamment régulières (appartenant à $H^4(\mathbb{T})$ où \mathbb{T} représente le tore $\mathbb{R}/2\pi\mathbb{Z}$).

Concernant les problèmes de stabilités des ondes stationnaires par rapport à des perturbations co-périodiques (c'est-à-dire des perturbations périodiques de même période que l'onde) de l'équation de Lugiato-Lefever (1), l'étude de la stabilité a été initiée dans [22], pour l'existence et la stabilité spectrale de solutions stationnaires périodiques, et [23], pour la stabilité asymptotique de ces solutions, dans le cas de la dispersion anormale. Les résultats obtenus dans [23] se basent essentiellement sur des estimations de Strichartz prouvées dans ce même article pour l'équation de Lugiato-Lefever (1) et pour une condition initiale L^2 . De même, une étude de stabilité asymptotique a été menée dans [25] à partir des solutions stationnaires périodiques obtenues dans [11] mais aussi [5, 4] mais avec une condition initiale H^1 . Les auteurs de [25] utilisent des propriétés de dissipation qu'ils obtiennent par application du théorème de Gearhart-Prüss plutôt que des estimations de Strichartz. Un autre travail concernant l'existence et la stabilité spectrale des solutions de l'équation de Lugiato-Lefever a été mené dans [11]. Dans ce travail, les auteurs obtiennent un résultat d'instabilité spectrale et un critère de stabilité.

Dans [5] et [4], nous avons étudié l'existence et la stabilité spectrale des solutions stationnaires de l'équation de Lugiato-Lefever (1) dans les cas de dispersions normale et anormale dans la région d'instabilité de Turing. Dans [5], nous nous intéressons aux paramètres au seuil de l'instabilité de Turing pour lesquels nous avons obtenu l'existence d'ondes stationnaires périodiques dans ces conditions, et avons conclu sur des résultats de stabilité orbitale non-linéaire de ces ondes pour des perturbations co-périodiques. Ce résultat est complété par un résultat de stabilité spectrale de ces ondes périodiques pour des perturbations bornées et localisées. Dans [4], nous avons suivi la même approche mais en choisissant les paramètres dans la région d'instabilité de Turing. Par ailleurs, alors que dans [5] nous nous intéressons qu'aux perturbations co-périodiques, dans [4] nous avons aussi considéré les perturbations sous-harmoniques (c'est-à-dire des perturbations périodiques dont la période est un multiple entier de la période de l'onde stationnaire). Ces résultats seront présentés dans les Chapitres 1 et 2.

3. Bifurcations locales

Dans [5, 4], nous nous intéressons aux ondes stationnaires périodiques, avec une période fixée au départ, qui bifurquent localement au seuil de l'instabilité de Turing, puis à celles qui bifurquent dans des

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régions d'instabilité de Turing. L'objectif est d'obtenir l'existence de solutions périodiques stationnaires et obtenir le développement de Taylor de ces solutions.

Notre approche du problème d'existence est le même que celui utilisé dans [22] pour l'analyse des ondes stationnaires périodiques dans le cas de dispersion anormale, avec $\alpha < 2$. Nous reformulons l'équation (1) comme un système dynamique de dimension infinie de la forme

$$\frac{d\Psi}{dt} = \mathcal{K}(\Psi, \beta, \alpha, F), \quad (3)$$

pour une fonction \mathcal{K} à déterminer, dans lequel $\Psi(t)$ appartient à un espace des phases de fonctions périodiques, puis nous effectuons une réduction à une variété centrale pour l'analyse des bifurcations locales. La différence avec [22] est que nous considérons les deux cas de dispersions normale et anormale et nous étudions les bifurcations locales pour toutes les valeurs des paramètres α et F . Ainsi, le premier Chapitre présentera ce travail, avec une première étude au seuil d'instabilité de Turing, puis une seconde étude dans la région de Turing.

3.1. Stratégie générale

La notion de bifurcation concerne les changements (qualitatifs) dans le comportement des orbites du système (3), autrement dit dans la dynamique de (3), lorsqu'un paramètre, notons-le μ , varie, il peut y avoir par exemple le changement du nombre d'équilibres, de la stabilité des équilibres, du type des orbites bornées, etc. Une valeur μ_* du paramètre μ est appelée un point de bifurcation si les portraits de phase de l'équation (6) sont qualitativement différents pour $\mu < \mu_*$ et $\mu > \mu_*$.

Avant de détailler notre analyse des bifurcations, dans ce paragraphe nous donnons un descriptif général de notre stratégie. Nous reviendrons plus en détails sur chaque étape dans les paragraphes dédiés.

Pour étudier les bifurcations locales de l'équation de Lugiato-Lefever, nous reformulons donc cette équation comme système dynamique de la forme (3). La première étape de notre analyse est de déterminer les solutions constantes selon les valeurs des paramètres β , α , F de l'équation (3). Par un changement d'échelle spatiale, nous normalisons β à 1 dans le cas de la dispersion normale et à -1 dans le cas de la dispersion anormale.

La seconde étape est de détecter parmi ces solutions celles qui vont présenter des instabilités et possiblement des bifurcations. Pour ce faire, on considère une solution constante Ψ^* de notre équation (déterminée à l'étape précédente). On considère $\mathcal{A}_0^* := D_{\Psi} \mathcal{K}(\Psi^*, \beta, \alpha, F)$ l'opérateur linéarisé autour de cette solution. Le spectre de l'opérateur \mathcal{A}_0^* est noté $\sigma(\mathcal{A}_0^*)$ et défini par

$$\sigma(\mathcal{A}_0^*) = \{\lambda \in \mathbb{C}; \lambda \mathbb{I} - \mathcal{A}_0^* \text{ non inversible}\}. \quad (4)$$

Le cadre fonctionnel \mathcal{X} dépend du type de perturbations considérées (co-périodiques, sous-harmoniques, bornées, localisées) et sera précisé dans les paragraphes dédiés. Considérons la décomposition du spectre suivante :

$$\sigma(\mathcal{A}_0^*) = \sigma_-(\mathcal{A}_0^*) \cup \sigma_0(\mathcal{A}_0^*) \cup \sigma_+(\mathcal{A}_0^*), \quad (5)$$

où

- $\sigma_-(\mathcal{A}_0^*) = \{\lambda \in \sigma(\mathcal{A}_0^*); \operatorname{Re}(\lambda) < 0\}$ (appelé spectre stable de \mathcal{A}_0^*);
- $\sigma_0(\mathcal{A}_0^*) = \{\lambda \in \sigma(\mathcal{A}_0^*); \operatorname{Re}(\lambda) = 0\}$ (appelé spectre central de \mathcal{A}_0^*);
- $\sigma_+(\mathcal{A}_0^*) = \{\lambda \in \sigma(\mathcal{A}_0^*); \operatorname{Re}(\lambda) > 0\}$ (appelé spectre instable de \mathcal{A}_0^*).

La solution constante U^* est dite spectralement instable lorsque $\sigma_+(\mathcal{A}_0^*) \neq \emptyset$, et on peut détecter les instabilités des solutions constantes obtenues à la première étape, en fonction des valeurs des paramètres

de l'équation (3). Ces instabilités déterminent les points de bifurcation avec des valeurs critiques de paramètres.

La troisième étape est d'étudier la dynamique du système proche de ces points de bifurcations. Pour cela, le paramètre α sera fixé et le paramètre F dépendra d'un paramètre de bifurcation μ . Ceci nous amène à un système dynamique dépendant de ce paramètre réel μ

$$\frac{dU}{dt} = \mathcal{H}(U, \mu). \quad (6)$$

Dans notre étude, nous nous intéressons aux bifurcations locales qui concernent les changements dans la dynamique de (6) pour des valeurs μ proches d'un point de bifurcation μ_* . L'analyse de bifurcation dans le cas de notre étude repose sur une réduction à une variété centrale. L'objectif de cette démarche est de réduire le système dynamique initial (6) qui est de dimension infinie en un système réduit de dimension finie. On reprend la théorie présentée dans [13]. On écrit le système (6) sous la forme :

$$\frac{d\Psi}{dt} = \mathcal{A}_0^* \Psi + \mathcal{G}(\Psi, \mu), \quad (7)$$

où $\Psi = \Psi^* + U$. On note \mathcal{Z} le domaine de \mathcal{A}_0^* comme opérateur agissant dans \mathcal{X} et \mathcal{Y} le domaine de définition en U de \mathcal{G} (avec $\mathcal{Z} \subset \mathcal{Y}$). On a que 0 est un équilibre de (7) pour $\mu = 0$ et sur un voisinage $\mathcal{V} \subset \mathcal{Y}$ de 0, \mathcal{G} est suffisamment régulière par rapport à U avec $d_U \mathcal{G}(0, 0) = 0$. On a la séparation exponentielle du spectre de \mathcal{A}_0^* : il existe une constante strictement positive γ telle que $\inf_{\lambda \in \sigma_+} \operatorname{Re}(\lambda) > \gamma$ et $\sup_{\lambda \in \sigma_-} \operatorname{Re}(\lambda) < -\gamma$, et que σ_0 consiste en un nombre fini de valeurs propres de multiplicité algébrique finies. On note P_0 le projecteur spectral associé à σ_0 . En notant $\mathcal{X}_0 = P_0(\mathcal{X}) \subset \mathcal{Z}$ et $\mathcal{X}_1 = (\operatorname{id} - P_0)(\mathcal{X}) \subset \mathcal{X}$, on a la décomposition

$$\mathcal{X} = \mathcal{X}_0 \oplus \mathcal{X}_1.$$

D'après [13, Chapitre 2, Théorème 2.9], il existe une variété centrale

$$\mathcal{M}_0 = \{U_0 + \Psi(U_0, \mu), U_0 \in \mathcal{X}_0\} \subset \mathcal{Z}$$

localement invariante de dimension égale à celle de \mathcal{X}_0 , qui contient toutes les solutions petites bornées du système (7), pour μ suffisamment petit, donc les solutions de Lugiato-Lefever sont proches de la courbe des solutions constantes. Dans la définition de \mathcal{M}_0 , l'application Ψ , appelée fonction de réduction, est suffisamment régulière de \mathcal{X}_0 dans \mathcal{X}_1 , et U_0 est solution du système réduit

$$\frac{dU_0}{dt} = \mathcal{L}_0 U_0 + P_0 \mathcal{G}(U_0 + \Psi(U_0, \mu), \mu), \quad (8)$$

où \mathcal{L}_0 est la restriction de \mathcal{A}_0^* à \mathcal{X}_0 . Ainsi, pour étudier l'existence de solutions bornées de (7) proches des solutions constantes, il suffit d'étudier le système (8).

Enfin, la dernière étape consiste à étudier les bifurcations sur le système réduit (8) pour obtenir l'existence de solutions stationnaires périodiques de l'équation de Lugiato-Lefever (1).

3.2. Bifurcations locales pour l'équation de Lugiato-Lefever

Dans notre étude, nous faisons une analyse détaillée des solutions constantes de l'équation de Lugiato-Lefever (1). Ces solutions sont les équilibres du système dynamique (3) (défini dans le premier Chapitre par (1.1.2)). Les solutions de (2) constantes en t et en x sont les solutions de l'équation algébrique

$$(1 + i\alpha)\psi - i\psi|\psi|^2 = F \quad (9)$$

et ont été calculées dans [9]. Selon les valeurs de α et F , l'équation (9) admet 1, 2 ou 3 solutions. Ce résultat est résumé dans la Figure 0.2.

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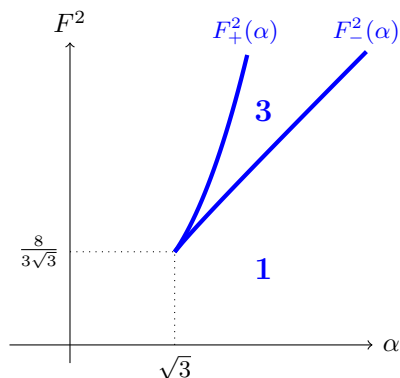


FIGURE 0.2 – Nombre de solutions constantes de l'équation de Lugiato-Lefever (1) : trois solutions dans la région entre les courbes $F^2 = F_{\pm}^2(\alpha)$, deux solutions le long des courbes, et une ailleurs.

Afin d'étudier la stabilité des solutions constantes, considérons ψ^* une telle solution. En reformulant (1) comme système dynamique de la forme (3), une solution constante de (1) est un équilibre du système

$$\frac{dU}{dt} = \mathcal{A}_0^* U.$$

La stabilité des solutions constantes $\psi^* = \psi_r^* + i\psi_i^*$ de l'équation de Lugiato-Lefever est donnée par la localisation du spectre de l'opérateur linéarisé autour de ces solutions. Cet opérateur est donné par $\mathcal{A}_0^* := D_U \mathcal{K}(U^*, \beta, \alpha, F)$, où $U^* = (\psi_r^*, \psi_i^*)^T$. Les Figures ci-dessous résument les résultats de stabilités dans les cas de dispersion anormale (cf. Figure 0.3) et normale (cf. Figure 0.4).

On détecte deux types d'instabilités (voir Figure 0.5) :

- (i) l'instabilité de Turing : c'est une instabilité vis-à-vis de perturbations non constantes de solutions qui sont stables par rapport à des perturbations constantes. Dans ce cas, l'instabilité de la solution constante est obtenue pour des perturbations périodiques non constantes (mode de Fourier différent de zéro). Dans le cas de dispersion anormale ($\beta = -1$), le seuil d'instabilité de Turing est défini par $\alpha < 2$ et $\rho^* = 1$; où ρ^* représente le module au carré de la solution constante ψ^* et dans le cas de dispersion normale ($\beta = 1$) pour $\alpha > 2$ et $\rho^* = 1$.
- (ii) l'instabilité de mode zéro : c'est une instabilité vis-à-vis de perturbations constantes (mode de Fourier zéro). Elle a lieu pour les valeurs de paramètres $\alpha \geq 2$ et $\rho^* = \rho_+(\alpha)$, où $\rho_+(\alpha) = \frac{1}{3} (2\alpha - \sqrt{\alpha^2 - 3})$ dans le cas de la dispersion anormale, et pour $\alpha > \sqrt{3}$, $\rho^* = \rho_-(\alpha)$, avec $\rho_-(\alpha) = \frac{1}{3} (2\alpha + \sqrt{\alpha^2 - 3})$, et $\sqrt{3} < \alpha < 2$, $\rho^* = \rho_+(\alpha)$ dans le cas de la dispersion normale.

Les valeurs des paramètres α et F pour lesquelles apparaissent ces instabilités diffèrent selon la dispersion (cas de dispersion normale ou anormale). Dans le Chapitre 1 de cette thèse, nous détaillons le cas de dispersion anormale ($\beta = -1$) et traitons plus rapidement le cas de dispersion normale ($\beta = 1$).

Système dynamique

Fixons $\alpha < 2$ et $\rho^* = 1$ dans le cas de dispersion anormale (les calculs sont identiques dans le cas de dispersion normale). Le paramètre de bifurcation que nous choisissons est F , en posant $F^2 = F_1^2 + \mu$, où $F_1^2 = (1 - \alpha)^2 + 1$ et $\mu \in \mathbb{R}$. Les solutions que nous cherchons sont de la forme $\psi(x, t) = \psi_\mu^* + (u + iv)(y, t)$, avec $y = \sqrt{|2 - \alpha|}x$ (changement de variables). Nous écrivons l'équation de Lugiato-Lefever comme système dynamique

$$\frac{dU}{dt} = \mathcal{A}_\mu^* U + \mathcal{F}(U, \mu) \tag{10}$$

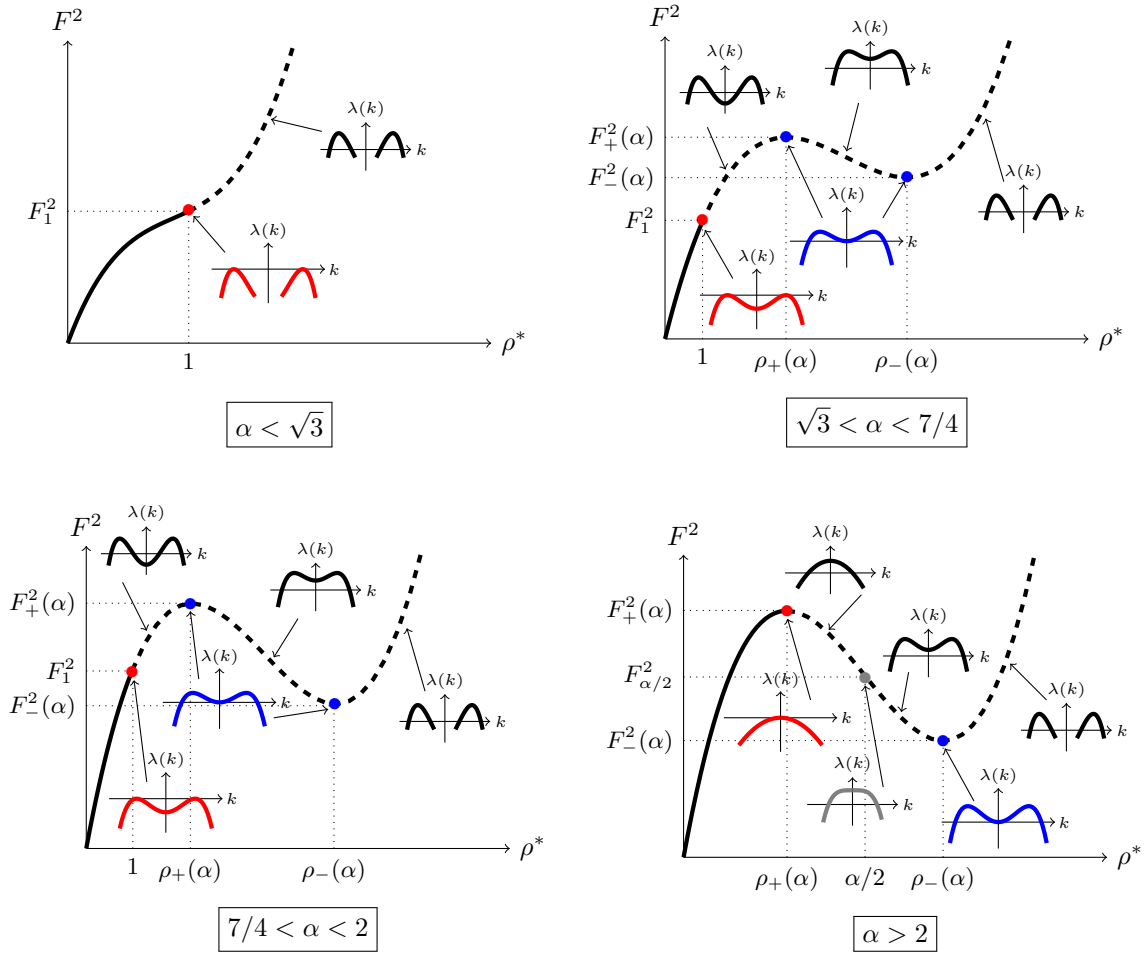


FIGURE 0.3 – Stabilité des solutions constantes de l'équation de Lugiato-Lefever dans le cas de dispersion anormale ($\beta = -1$). Les branches stables sont représentées par des lignes continues, les branches instables par des pointillés, et on pose $F_1^2 = (1 - \alpha)^2 + 1$ et $F_{\alpha/2}^2 = \alpha(4 + \alpha^2)/8$. La forme de la plus grande valeur propre $\lambda(k)$ (valeur propre dépendant d'un réel k) est tracée aux points où les solutions perdent leur stabilité et où elles sont instables.

avec $U = (u, v)^T$. L'opérateur \mathcal{A}_μ^* est obtenu en linéarisant l'équation de Lugiato-Lefever autour de la solution constante ψ_μ^* , et $\mathcal{F}(U, \mu)$ est la partie non linéaire du système. Pour l'espace des phases de ce système, on choisit celui des fonctions 2π -périodiques de carré intégrable $\mathcal{X} = L^2(0, 2\pi) \times L^2(0, 2\pi)$. Notons pour la suite de l'analyse que (10) est invariant sous les actions suivantes :

$$(\mathcal{T}U)(y) = U(-y), \quad (\mathcal{T}_a U)(y) = U(y + a). \quad (11)$$

Bifurcations locales

Dans un premier temps, nous avons donc étudié l'existence de solutions périodiques au seuil d'instabilité de Turing. Nous avons trouvé deux types d'instabilités pour les solutions constantes : l'instabilité de Turing et l'instabilité de mode zéro. Dans un deuxième temps, pour des valeurs de paramètres dans la région de Turing, nous avons prouvé l'existence de deux familles de solutions stationnaires périodiques de longueurs d'ondes $k_{\min} < k_{\max}$, avec $k_{\max}^2 = 2\rho^* - \alpha + \sqrt{\rho^{*2} - 1}$ et $k_{\min}^2 = 2\rho^* - \alpha - \sqrt{\rho^{*2} - 1}$ tandis que pour les valeurs de paramètres dans la région de mode zéro, nous montrons l'existence d'une famille de solutions stationnaires périodiques de longueur d'onde k_{\max} .

De la même manière que cela a été fait pour prouver l'existence de solutions stationnaires périodiques

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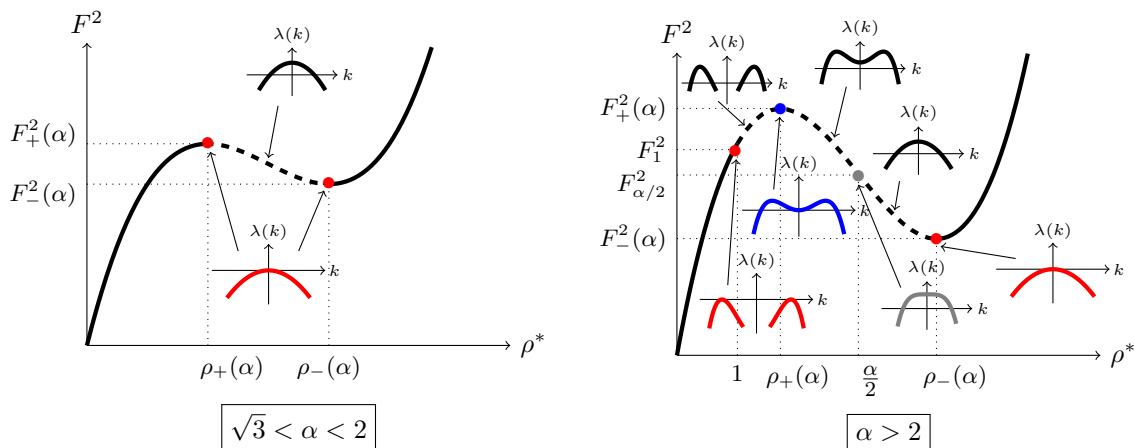


FIGURE 0.4 – Stabilité des solutions constantes de l'équation de Lugiato-Lefever dans le cas de dispersion normale ($\beta = 1$), pour $\alpha > \sqrt{3}$ (pour $\alpha \leq \sqrt{3}$ l'unique solution constante est stable). Les branches stables sont représentées par des lignes continues, les branches instables par des lignes pointillées, et on pose $F_1^2 = (1 - \alpha)^2 + 1$ et $F_{\alpha/2}^2 = \alpha(4 + \alpha^2)/8$. La forme de la plus grande valeur propre $\lambda(k)$ (valeur propre dépendant d'un réel k) est donnée aux points où les solutions perdent leur stabilité et aux points de solutions instables.

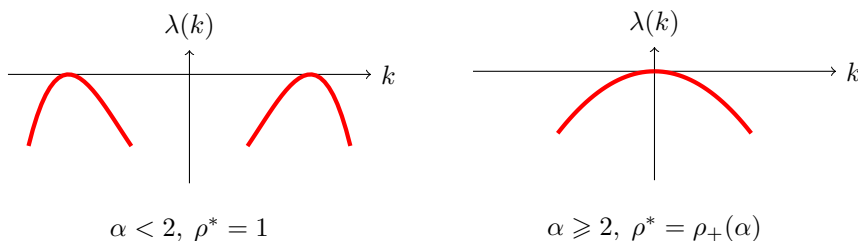


FIGURE 0.5 – Forme de la plus grande valeur propre $\lambda(k)$ pour k proche de $\pm\sqrt{2 - \alpha}$ dans le cas d'instabilité de Turing (figure à gauche) et pour k proche de 0 dans le cas d'instabilité en mode zéro (figure à droite).

au seuil d'instabilité de Turing, la preuve d'existence dans la région d'instabilité de Turing repose sur une formulation de l'équation de Lugiato-Lefever (1) comme système dynamique de dimension infinie et sur une réduction à une variété centrale. Dans le cadre de cette approche, la dynamique proche des points de bifurcation est décrite par un système réduit de dimension 2 d'équations différentielles ordinaires. Ainsi, les solutions stationnaires périodiques de l'équation (1) sont obtenues comme des équilibres de ce système réduit. On dit, dans le cas où les équilibres non triviaux sont stables, que la bifurcation est super-critique, tandis que dans le cas où ils sont instables, la bifurcation est dite sous-critique. Nous identifions une bifurcation stationnaire (c'est-à-dire une bifurcation qui correspond à des solutions de Lugiato-Lefever qui ne dépendent que de la variable spatiale) qui, selon les valeurs des paramètres, est sur-critique ou sous-critique.

Comme dans l'étude précédent, deux types d'instabilités sont considérées : l'instabilité de Turing quand $\lambda(0) < 0$ et $\lambda(k) > 0$ pour $k \neq 0$, et l'instabilité du mode zéro quand $\lambda(0) > 0$. Les valeurs des paramètres correspondants sont représentés dans la figure 0.6.

Les courbes Γ_1 , Γ_+ , et Γ_- , qui forment les bornes de ces deux régions, sont définies par

$$\Gamma_1 = \{\alpha < 2, \rho^* = 1\}, \quad \Gamma_+ = \{\alpha > 2, \rho^* = \rho_+(\alpha)\}, \quad (12)$$

et

$$\Gamma_- = \{\sqrt{3} < \alpha < 2, \rho^* = \rho_+(\alpha)\} \cup \{\alpha > \sqrt{3}, \rho^* = \rho_-(\alpha)\}. \quad (13)$$

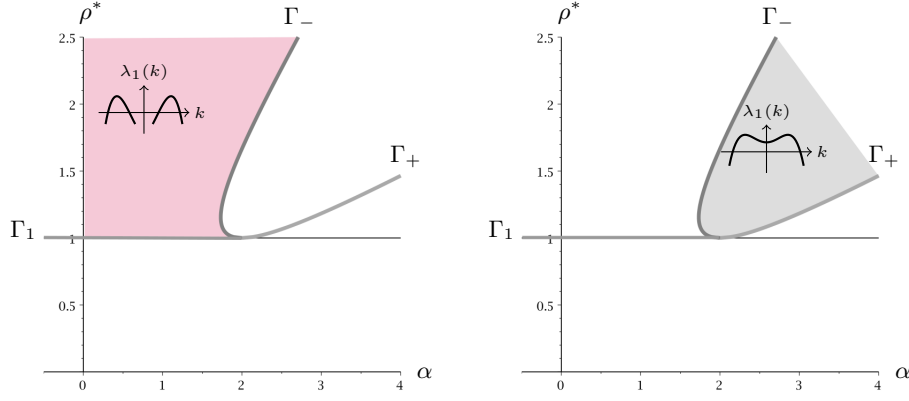


FIGURE 0.6 – Stabilité des solutions constantes dans le cas de dispersion anormale. Dans le plan (α, ρ^*) , les régions colorées représentent les valeurs pour lesquelles une solution constante avec $\rho^* = |\psi^*|^2$ présente une instabilité de Turing (tracé de gauche) et une instabilité de mode zéro (tracé de droite). Le graphique montre un tracé typique de la plus grande valeur propre $\lambda_1(k)$ dans ces régions d'instabilités.

3.3. Un outil clé : la réduction à une variété centrale

La preuve de l'existence d'ondes stationnaires périodiques au seuil d'instabilité de Turing se fait grâce à une réduction à une variété centrale (cet outil est présenté à la section 3.1 et est expliqué en détails dans [13, Chapitre 2]).

Dans le contexte de l'équation de Lugiato-Lefever et pour des perturbations co-périodiques, le système (7) n'est rien d'autre que le système (10) avec $\mathcal{G}(U, \mu) = (\mathcal{A}_\mu^* - \mathcal{A}_0^*)U + \mathcal{F}(U, \mu)$, l'espace \mathcal{X} est $L^2(0, 2\pi) \times L^2(0, 2\pi)$ et les espaces \mathcal{Y} et \mathcal{Z} sont tous les deux égaux à $H^2(0, 2\pi) \times H^2(0, 2\pi)$. Le spectre de l'opérateur \mathcal{A}_0^* contient 0 qui est une valeur propre double semi-simple, de vecteurs propres conjugués ζ et $\bar{\zeta}$. Par le théorème de la variété centrale (voir [13, Chapitre 2, Théorème 3.3]), le système dynamique (10) possède une variété centrale de dimension 2 :

$$\mathcal{M}_c(U) = \{U \in \mathcal{Y} : U = A\zeta + \bar{A}\bar{\zeta} + \Psi(A, \bar{A}, \mu), A \in \mathbb{C}\}$$

avec Ψ régulière, qui contient toutes les petites solutions bornées de (3), pour μ suffisamment petit. Les détails de la preuve sont présentés dans le Chapitre 1.

Ainsi, la dynamique du système (10) sur $\mathcal{M}_c(U)$ se réduit à la dynamique du système réduit suivant :

$$\frac{dA}{dt} = c_{11}\mu A + c_{30}A|A|^2 + \mathcal{O}(|A|(\mu^2 + |A|^4)), \quad (14)$$

où c_{11} et c_{30} sont des réels. La dynamique de ce système dépend du signe des coefficients c_{11} et c_{30} qui peuvent être calculés, par exemple, en suivant la méthode donnée dans [13, Chapitre 2].

On est dans le cas d'une bifurcation stationnaire avec une symétrie $O(2)$ pour $\mu = 0$. En effet, selon les signes des coefficients c_{11} et c_{30} , le nombre de solutions de l'équation tronquée (14) change avec selon si μ est positif ou négatif. Ceci induit une bifurcation en $\mu = 0$. En utilisant les coordonnées polaires $A = re^{i\theta}$, le système réduit (14) est de la forme suivante (voir [13, Chapitre 1]) :

$$\begin{cases} \frac{dr}{dt} = c_{11}\mu r + c_{30}r^3 + \mathcal{O}(r(|\mu|^2 + r^4)), \\ \frac{d\theta}{dt} = 0. \end{cases} \quad (15)$$

La deuxième équation implique que l'angle θ est constant ce qui traduit l'invariance par translation en espace de l'équation de Lugiato-Lefever (1). La première équation nous permet d'étudier la bifurcation.

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Celle-ci montre que cette bifurcation est une bifurcation fourche (voir [13, Chapitre 1, Section 2.4]). En effet, la première équation tronquée du système (15) :

$$\frac{dr}{dt} = c_{11}\mu r + c_{30}r^3$$

admet 1 équilibre lorsque $c_{11}\mu c_{30} \geq 0$ et 2 équilibres lorsque $c_{11}\mu c_{30} < 0$. On peut obtenir la stabilité des équilibres r selon le signe des coefficients c_{11} , c_{30} et μ . Dans cette configuration, la bifurcation est sous-critique lorsque $c_{30} > 0$ et sur-critique lorsque $c_{30} < 0$.

Dans la région d'instabilité de Turing et dans la région de mode zéro, le principe reste le même et les résultats sont explicités dans la sous-section suivante.

3.4. Résultat : existence d'ondes périodiques

Bifurcation stationnaire avec symétrie $O(2)$ au seuil d'instabilité de Turing

On a obtenu deux résultats principaux : les Théorèmes 1 et 3. Le Théorème 1 donne l'existence de solutions stationnaires pour le système réduit dans le cas de la dispersion anormale (et le Théorème 3 pour la dispersion normale).

Théorème 1. *Considérons l'équation de Lugiato-Lefever (1) dans le cas $\beta = -1$ de dispersion anormale. Supposons que $\alpha < 2$, $\alpha \neq 41/30$, et $F^2 = F_1^2 + \mu$, avec $F_1^2 = (1 - \alpha)^2 + 1$. Alors, le système réduit (14) admet une bifurcation stationnaire avec une symétrie $O(2)$ à $\mu = 0$. La bifurcation est sur-critique lorsque $\alpha < 41/30$, et sous-critique lorsque $\alpha > 41/30$. Plus précisément, les propriétés suivantes sont satisfaites dans un voisinage de 0 dans \mathbb{C} , pour μ suffisamment petit.*

- (i) *Si $\alpha < 41/30$ et $\mu < 0$, alors l'équation réduite possède un unique équilibre $A = 0$ qui est stable.*
- (ii) *Si $\alpha < 41/30$ et $\mu > 0$, alors l'équation réduite possède un équilibre instable $A = 0$ et un cercle d'équilibres stables $A_\mu(\phi) = r_\mu e^{i\phi}$, pour $\phi \in \mathbb{R}/2\pi\mathbb{Z}$, qui entourent l'équilibre instable.*
- (iii) *Si $\alpha > 41/30$ et $\mu < 0$, alors l'équation réduite possède un équilibre stable $A = 0$ et un cercle d'équilibres instables $A_\mu(\phi) = r_\mu e^{i\phi}$, pour $\phi \in \mathbb{R}/2\pi\mathbb{Z}$, qui entourent l'équilibre stable.*
- (iv) *Si $\alpha > 41/30$ et $\mu > 0$, alors l'équation réduite possède un unique équilibre $A = 0$ qui est instable.*

Dans les cas (ii) et (iii) on a

$$r_\mu = \frac{3}{2F_1|41 - 30\alpha|^{1/2}} |\mu|^{1/2} + \mathcal{O}(|\mu|).$$

En reformulant les cas (ii) et (iii) du théorème précédent pour l'équation de Lugiato-Lefever (1), on obtient le théorème suivant :

Théorème 2. *On considère $\beta = -1$ (cas de dispersion anormale) et $|\mu|$ suffisamment petit.*

Lorsque ($\alpha < 41/30$, $\mu < 0$) ou ($\alpha > 41/30$, $\mu > 0$), l'équation de Lugiato-Lefever (1) possède une solution constante ψ_μ^ donnée par (9) avec $F^2 = F_1^2 + \mu$.*

Lorsque ($\alpha < 41/30$, $\mu > 0$) ou ($\alpha > 41/30$, $\mu < 0$), l'équation de Lugiato-Lefever (1) possède une solution ψ_μ périodique paire, de période $\frac{2\pi}{\sqrt{2 - \alpha}}$, avec comme développement de Taylor

$$\psi_\mu(x) = \psi_0 + \frac{3(\alpha + i(2 - \alpha))}{F_1|41 - 30\alpha|^{1/2}} \cos(\sqrt{2 - \alpha} x) |\mu|^{1/2} + \mathcal{O}(|\mu|), \quad (16)$$

avec $F_1^2 = (1 - \alpha)^2 + 1$. Les translations en x de cette solution périodique sont également solutions.

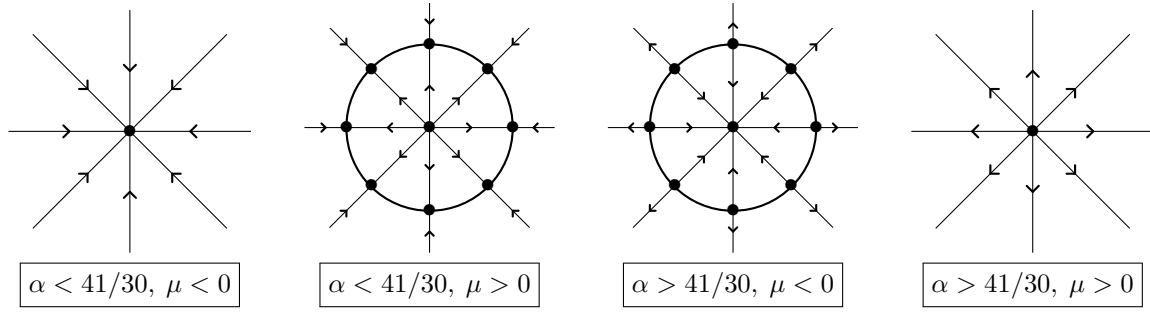


FIGURE 0.7 – Portraits de phase dans le plan de A du système réduit (14).

La solution (16) est obtenue à partir de la solution $(r_\mu, 0)$, r_μ issu du Théorème 1 pour les cas (ii) et (iii), du système (15). Tout autre choix de constante pour la composante θ est équivalente à une translation de la solution (16) en x .

Le Théorème 3 donne l'existence de solutions stationnaires pour le système réduit dans le cas de la dispersion normale.

Théorème 3. *Considérons l'équation de Lugiato-Lefever (1) dans le cas $\beta = 1$ de dispersion normale. Supposons que $\alpha > 2$ et $F^2 = F_1^2 + \mu$, $F_1^2 = (1 - \alpha)^2 + 1$. Alors, le système réduit (14) admet une bifurcation stationnaire sur-critique avec une symétrie $O(2)$. Plus précisément, les propriétés suivantes sont satisfaites dans un voisinage de 0 dans \mathbb{C} , pour μ suffisamment petit.*

- (i) *Pour tout $\mu < 0$, alors l'équation réduite (14) possède un équilibre stable $A = 0$ et un cercle d'équilibres instables, $A_\mu(\phi) = r_\mu e^{i\phi}$, pour $\phi \in \mathbb{R}/2\pi\mathbb{Z}$, qui entourent l'équilibre stable.*
- (ii) *Pour tout $\mu > 0$, alors l'équation réduite (14) possède un unique équilibre $A = 0$ qui est instable.*

Du Théorème 3, on pourrait aussi déduire un analogue du Théorème 2 dans le cas de la dispersion normale avec $\beta = 1$, $\alpha > 2$ et $\mu < 0$.

Dans le cas de bifurcation anormale, pour $\beta = -1$, on retrouve les résultats connus de la littérature physique, en particulier le changement de type de bifurcation (sur- ou sous-critique) apparaissant pour $\alpha = 41/30$ et qui a été analysé dans [22]. Dans le cas de dispersion normale, pour $\beta = 1$, nous trouvons une bifurcation sous-critique. Dans le cas de mode zéro, des ondes périodiques existent (voir [9]), mais leur existence ne peut pas être obtenue en suivant cette approche. Ces ondes périodiques ont des périodes qui tendent vers l'infini lorsque l'on s'approche des points de bifurcation, et l'on ne peut pas appliquer le résultat de la variété centrale pour ces solutions.

Bifurcation dans la région d'instabilité de Turing et dans celle de mode zéro

On considère ici les paramètres (α^*, ρ^*) dans la région de Turing et dans celle de mode zéro (voir la Figure 0.6). La méthode pour obtenir des résultats d'existence est la même qu'au seuil d'instabilité de Turing, détaillée dans ce qui précède, à la différence que les calculs des coefficients c_{11} et c_{30} du système réduit sont plus longs et compliqués (ils ont été effectués sur le logiciel MAPLE). Dans ces régions, les longueurs d'onde des solutions stationnaires périodiques sont k_{min} et k_{max} , des réels déterminés en fonction des paramètres α , β et F de l'équation de Lugiato-Lefever (1).

Un premier résultat d'existence pour les solutions stationnaires périodiques de longueur d'onde k_{max} est le suivant :

Théorème 4. *Considérons l'équation de Lugiato-Lefever (1) dans le cas $\beta = -1$ de dispersion anormale. Soit ψ^* une solution constante de (1) de module au carré ρ^* , pour des valeurs de paramètres α*

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and F_* . Si (α, ρ^*) est dans la région d'instabilité Turing, ou dans la région d'instabilité de mode zéro et n'appartient pas à la courbe $\Gamma_{30} : \widetilde{c}_{30} = 0$ avec $\widetilde{c}_{30} = 0$ défini par (A.2.2) et représentée dans la figure 0.8, alors pour $F^2 = F_*^2 + \mu$ et $k_* = k_{\max}$ le système dynamique (10) induit une bifurcation stationnaire avec une symétrie $O(2)$ pour $\mu = 0$, et les propriétés suivantes sont satisfaites.

- (i) Si (α, ρ^*) appartient à l'ensemble ouvert situé à gauche de la courbe Γ_{30} dans la région d'instabilité de Turing, alors les coefficients de l'équation réduite (15) satisfont $c_{11} > 0$ et $c_{30} < 0$. La bifurcation est sur-critique et l'équation réduite possède un cercle d'équilibres non triviaux stables, $A_\mu(\phi) = r_\mu e^{i\phi}$, $\phi \in \mathbb{R}/2\pi\mathbb{Z}$, pour tout $\mu > 0$ suffisamment petit.
- (ii) Si (α, ρ^*) appartient à l'ensemble ouvert situé à droite de la courbe Γ_{30} dans la région d'instabilité de Turing, alors les coefficients de l'équation réduite (15) satisfont $c_{11} > 0$ et $c_{30} > 0$. La bifurcation est sous-critique et l'équation réduite possède un cercle d'équilibres non triviaux instables, $A_\mu(\phi) = r_\mu e^{i\phi}$, $\phi \in \mathbb{R}/2\pi\mathbb{Z}$, pour tout $\mu < 0$ suffisamment petit.
- (iii) Si (α, ρ^*) appartient à la région d'instabilité du mode zéro, alors les coefficients de l'équation réduite (15) satisfont $c_{11} < 0$ et $c_{30} < 0$. La bifurcation est sur-critique et le système réduit possède un cercle d'équilibres non triviaux stables, $A_\mu(\phi) = r_\mu e^{i\phi}$, $\phi \in \mathbb{R}/2\pi\mathbb{Z}$, pour tout $\mu < 0$ suffisamment petit.

De même que dans le cas au seuil d'instabilité de Turing, on obtient l'expression des solutions stationnaires périodiques de longueur d'onde k_{max} de l'équation de Lugiato-Lefever (1) dans la région de Turing ou celle de mode zéro :

Théorème 5. *Sous les hypothèses du Théorème 4. Dans les cas :*

- (α, ρ^*) appartient à l'ensemble ouvert situé à gauche de la courbe Γ_{30} dans la région d'instabilité de Turing et $\mu > 0$,
- (α, ρ^*) appartient à l'ensemble ouvert situé à droite de la courbe Γ_{30} dans la région d'instabilité de Turing et $\mu < 0$,
- (α, ρ^*) appartient à la région d'instabilité du mode zéro et $\mu < 0$,

l'équation de Lugiato-Lefever (1) possède une solution ψ_μ périodique paire avec comme développement de Taylor

$$\psi_\mu(x) = \psi_1 + \frac{1}{2} \cos(k_{\max} x) |\mu|^{1/2} + \mathcal{O}(|\mu|), \quad (17)$$

avec ψ_1 un réel dépendant des paramètres ρ^* , α , c_{11} et c_{30} précisé dans le Chapitre 1. Les translations en x de cette solution périodique sont également solutions.

Pour énoncer le second résultat d'existence pour les solutions stationnaires périodiques de longueur d'onde k_{min} , nous introduisons les courbes Γ_n , pour $n \in \mathbb{N}$, $n \geq 2$, $n \neq 30$, définies par

$$\alpha = 2\rho^* - \frac{n^2 + 1}{n^2 - 1} \sqrt{\rho^{*2} - 1}. \quad (18)$$

Nous obtenons le résultat d'existence correspondant :

Théorème 6. *Considérons l'équation de Lugiato-Lefever (1) dans le cas $\beta = -1$ de dispersion anormale. Soit ψ^* une solution constante de (1) de module au carré ρ^* , pour les valeurs des paramètres α et F_* . Si (α, ρ^*) appartiennent à la région d'instabilité de Turing et n'appartiennent à aucune courbe Γ_n , $n \geq 2$, $n \neq 30$, Γ_{30} , et $\rho^* = 2/\sqrt{3}$, alors pour $F^2 = F_*^2 + \mu$ et $k_* = k_{min}$ le système dynamique (10) induit une bifurcation stationnaire avec symétrie $O(2)$ pour $\mu = 0$, et les propriétés suivantes sont satisfaites.*

- (i) Si (α, ρ^*) appartient à la région la plus claire dans la Figure 3.1b et $\rho^* < 2/\sqrt{3}$, alors $c_{11} > 0$ et $c_{30} < 0$. La bifurcation est sur-critique et l'équation réduite possède un cercle d'équilibres non triviaux stables $A_\mu(\phi) = r_\mu e^{i\phi}$, $\phi \in \mathbb{R}/2\pi\mathbb{Z}$, pour tout $\mu > 0$ suffisamment petit.
- (ii) Si (α, ρ^*) appartient à la région la plus claire dans la Figure 3.1b et $\rho^* > 2/\sqrt{3}$, alors $c_{11} < 0$ et $c_{30} < 0$. La bifurcation est sur-critique et l'équation réduite possède un cercle d'équilibres non triviaux stables $A_\mu(\phi) = r_\mu e^{i\phi}$, $\phi \in \mathbb{R}/2\pi\mathbb{Z}$, pour tout $\mu < 0$ suffisamment petit.
- (iii) Si (α, ρ^*) appartient à la région la plus sombre dans la Figure 3.1b et $\rho^* < 2/\sqrt{3}$, alors $c_{11} > 0$ et $c_{30} > 0$. La bifurcation est sur-critique et l'équation réduite possède un cercle d'équilibres non triviaux instables $A_\mu(\phi) = r_\mu e^{i\phi}$, $\phi \in \mathbb{R}/2\pi\mathbb{Z}$, pour tout $\mu < 0$ suffisamment petit.
- (iv) Si (α, ρ^*) appartient à la région la plus sombre dans la Figure 3.1b et $\rho^* > 2/\sqrt{3}$, alors $c_{11} < 0$ et $c_{30} > 0$. La bifurcation est sur-critique et l'équation réduite possède un cercle d'équilibres non triviaux instables $A_\mu(\phi) = r_\mu e^{i\phi}$, $\phi \in \mathbb{R}/2\pi\mathbb{Z}$, pour tout $\mu > 0$ suffisamment petit.

De la même manière que précédemment, on obtient l'expression des solutions stationnaires périodiques de longueur d'onde k_{min} l'équation de Lugiato-Lefever (1) dans la région de Turing ou celle de mode zéro :

Théorème 7. *Sous les hypothèses du Théorème 6.*

Dans les cas suivants :

- (α, ρ^*) appartient à la région la plus claire dans la Figure 3.1b, $\rho^* < 2/\sqrt{3}$ et $\mu > 0$
- (α, ρ^*) appartient à la région la plus claire dans la Figure 3.1b, $\rho^* > 2/\sqrt{3}$ et $\mu < 0$
- (α, ρ^*) appartient à la région la plus sombre dans la Figure 3.1b, $\rho^* < 2/\sqrt{3}$ et $\mu < 0$
- (α, ρ^*) appartient à la région la plus sombre dans la Figure 3.1b, $\rho^* > 2/\sqrt{3}$ et $\mu > 0$

l'équation de Lugiato-Lefever (1) possède une solution ψ_μ périodique paire avec comme développement de Taylor

$$\psi_\mu(x) = \psi_1 + \psi_2 \cos(k_{min} x) |\mu|^{1/2} + \mathcal{O}(|\mu|), \quad (19)$$

avec ψ_1 un réel dépendant des paramètres ρ^* , α , c_{11} et c_{30} précisé dans le Chapitre 1. Les translations en x de cette solution périodique sont également solutions.

Les Théorèmes 4 et 6 sont illustrés par la Figure 3.1.

Pour le cas de bifurcation normale, les résultats sont similaires au cas de dispersion anormale. Afin d'alléger l'exposé, nous ne donnerons pas d'énoncé précis ici car les résultats seront résumés dans la prochaine section avec ceux de la stabilité.

4. Stabilité

Dans l'étude de la stabilité des ondes stationnaires périodiques d'une longueur d'onde k donnée, différentes notions de stabilité apparaissent au cours de l'analyse. Nous les précisons par la suite.

Il convient toutefois de remarquer que dans la suite le cadre fonctionnel n'est pas précisé car il dépend du type de perturbation que l'on souhaite ou peut analyser. Ainsi,

- Pour les perturbations co-périodiques, l'espace considéré est $L^2(0, 2\pi/k)$;
- Pour les perturbations sous-harmoniques, l'espace considéré est $L^2(0, 2\pi N/k)$, avec $N \in \mathbb{N}$, $N \geq 2$;

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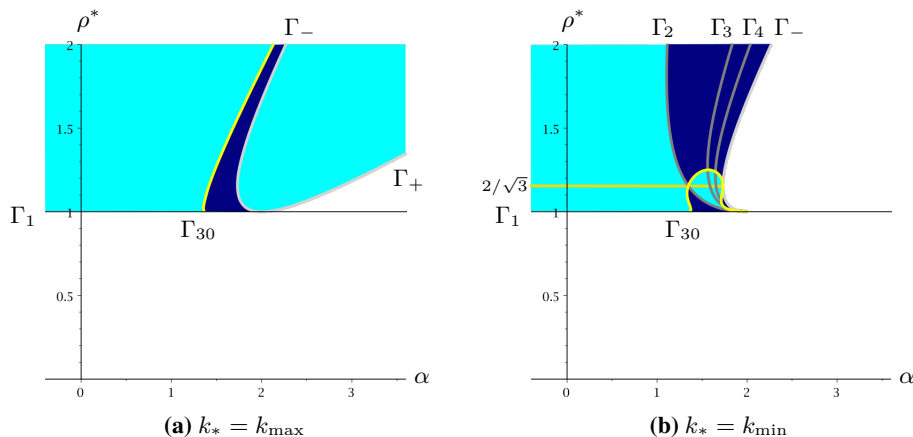


FIGURE 0.8 – Bifurcations des solutions périodiques stationnaires dans le cas de la dispersion anormale. Dans le plan (α, ρ^*) , la bifurcation est sur-critique ($c_{30} < 0$) dans la région colorée la plus claire et sous-critique ($c_{30} > 0$) dans la région colorée la plus foncée. Dans le cas $k_* = k_{\min}$, on représente également les courbes Γ_n , pour $n = 2, 3, 4$, le long desquelles des résonances apparaissent, et la ligne $\rho^* = 2/\sqrt{3}$ le long de laquelle c_{11} change de signe.

- Pour les perturbations localisées, l'espace considéré est $L^2(\mathbb{R})$.

Précisons qu'ici il ne s'agit que des perturbations des solutions de l'équation de Lugiato-Lefever (1). Dans l'analyse des systèmes linéarisés, il faut séparer les parties réelles et imaginaires car l'opérateur linéarisé n'est que \mathbb{R} -linéaire. Ainsi le cadre fonctionnel pour la stabilité spectrale ou linéaire est le carré cartésien des espaces de perturbations que nous considérons.

Stabilité spectrale Nous avons introduit la notion de stabilité spectrale dans la section . Considérons, comme précédemment, l'équation suivante :

$$\frac{dU}{dt} = \mathcal{F}(U) \quad (20)$$

car les paramètres seront fixés. Supposons, comme précédemment, que cette équation possède un équilibre que nous notons U^* . On dit que l'équilibre U^* est spectralement instable si le spectre instable de l'opérateur linéarisé \mathcal{A}^* autour de U^* , noté $\sigma_+(\mathcal{A}^*)$ dans (5), est non vide. On dit que U^* est stable lorsque $\sigma_+(\mathcal{A}^*)$ est vide. C'est ce que nous obtenons pour des solutions (16) dans le cadre du Théorème 9 plus bas.

Stabilité linéaire Considérons le système suivant :

$$\frac{dU}{dt} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} U,$$

avec $U = (u, v)^T$. Dans ce cas, en notant $u_0 = u(0)$ et $v_0 = v(0)$, on a $u(t) = v_0 t + u_0$, et le spectre de $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ est réduit au spectre central, ici $\{0\}$, donc l'équilibre $(0, 0)$ est spectralement stable. En choisissant en condition initiale $(u_0, v_0)^T = (0, 1)^T$, on obtient une solution U s'éloignant de l'équilibre nul. Ainsi, le fait d'avoir un spectre central non vide peut donner des instabilités (ceci est dû aux blocs de Jordan, mais ces instabilités ne sont pas exponentielles).

Comme le montre l'exemple précédent, la notion de stabilité spectrale n'est pas une notion de stabilité dynamique et l'étape suivante d'une analyse de la stabilité est celle de la stabilité linéaire.

Reprenons l'équation (20) et l'un de ses équilibres, U^* , dont l'opérateur linéarisé est noté \mathcal{A} . On dit que l'équilibre U^* est linéairement stable lorsque le semi-groupe engendré par \mathcal{A} est stable (c'est-à-dire borné). Pour montrer que ce semi-groupe est borné, nous prouvons la stabilité exponentielle (le semi-groupe engendré tend vers 0 à l'infini) en passant le théorème de Gearheart-Prüss présenté dans [6].

La stabilité linéaire est associée à un problème d'évolution linéaire proche du problème non-linéaire initialement étudié. Or il n'est pas assuré que le problème d'évolution linéarisé représente en temps grands une approximation valide du problème non-linéaire de départ même si on se limite initialement à un petit voisinage de l'état stationnaire. Il en donne néanmoins une indication intéressante et produit des estimations cruciales à l'analyse non-linéaire. On pense en particulier aux estimations de décroissance exponentielle obtenue via le théorème de Gearheart-Prüss ou les estimations de Strichartz. Les lemmes 4 et 5 plus bas en sont un exemple pour la linéarisation autour de la solution triviale nulle (lorsque $F = 0$).

Stabilité orbitale Nous donnons maintenant une définition de stabilité pour un système non-linéaire. Considérons (I, U) une solution de l'équation différentielle (20) et supposons que $\mathbb{R}_+ \subset I$. On considère également le problème de Cauchy

$$\begin{cases} \frac{dV}{dt} &= \mathcal{F}(V) \\ V(0) &= V_0 \end{cases} \quad (21)$$

On dit que la solution U de (20) est orbitalement stable lorsque pour tout $\varepsilon > 0$, il existe $\delta > 0$ tel que si la distance entre V_0 et $U(0)$ est plus petite que δ , alors pour tout $t \geq 0$, la distance de U à l'unique solution maximale du problème de Cauchy (21) est inférieure à ε .

Stabilité asymptotique Dans cette thèse, nous avons fait le choix de montrer une propriété plus forte de stabilité non-linéaire : la stabilité asymptotique.

Une solution stationnaire V de l'équation (20) est dite asymptotiquement stable si elle est orbitalement stable et si toute perturbation de V converge en temps arbitrairement grand vers une solution stationnaire éventuellement différente mais proche de V .

Dans le contexte qui nous intéresse, il n'est pas possible d'avoir la symétrie des états stationnaires individuellement. En particulier, pour l'équation de Lugiato-Lefever, une solution stationnaire et sa translatée sont deux solutions qui peuvent être arbitrairement proches, mais aucune ne va se rapprocher de l'autre. Ainsi c'est la variété de modulation de l'état considéré, formée en appliquant toutes les symétries continues à l'état, qui est globalement asymptotiquement stable. C'est ce que nous démontrons et nous précisons aussi le comportement asymptotique des paramètres des symétries qui modifient l'état stationnaire.

Dans cette thèse, nous avons étudié la stabilité spectrale pour toutes les perturbations sous-harmoniques et localisées. Dans le cas sous-harmonique spectralement stable, nous avons analysé la stabilité linéaire et asymptotique et nous obtenons la stabilité orbitale comme une conséquence de la stabilité asymptotique.

4.1. Stabilité spectrale au seuil d'instabilité de Turing

La stabilité spectrale pour l'équation de Lugiato-Lefever a tout d'abord été étudiée par Miyaji, Ohnishi et Tsutsumi dans [22, Théorème 3.1] pour des perturbations co-périodiques. Les analyses de bifurcation locale montrant l'existence d'ondes stationnaires périodiques permettent également de conclure quant à la stabilité spectrale, ou instabilité, de ces ondes pour des perturbations co-périodiques. Hakkaev, Stanislavova et Stefanov ont également obtenu des résultats de stabilité spectrale dans [11] concernant les solutions construites dans ce même travail.

Ainsi, en partant des nos résultats dans le contexte co-périodique, les ondes obtenues dans les cas de bifurcations sur-critiques sont stables tandis que celles obtenues dans les cas sous-critiques sont instables. Ainsi, le Théorème 1 nous permet d'obtenir le résultat suivant dans le cas de dispersion anormale :

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Théorème 8. *Considérons l'équation de Lugiato-Lefever (1) dans le cas $\beta = -1$ de dispersion anormale. Supposons que $\alpha < 2$, $\alpha \neq 41/30$, et $F^2 = F_1^2 + \mu$, avec $F_1^2 = (1 - \alpha)^2 + 1$. Le système réduit (14) induit une bifurcation stationnaire avec une symétrie $O(2)$ à $\mu = 0$. Dans les cas ($\alpha < 41/30$, $\mu > 0$) ou ($\alpha > 41/30$, $\mu < 0$), l'équation de Lugiato-Lefever (1) possède une famille de solutions stationnaires périodiques en x , ψ_μ qui sont :*

- (i) *spectralement stables par rapport à des perturbations co-périodiques lorsque $\alpha < 41/30$ (bifurcation sur-critique)*
- (ii) *spectralement instables par rapport à des perturbations co-périodiques lorsque $\alpha > 41/30$ (bifurcation sous-critique)*

Dans le cas de la dispersion normale, le Théorème 3 nous donne le résultat suivant :

Théorème 9. *Considérons l'équation de Lugiato-Lefever (1) dans le cas $\beta = 1$ de dispersion normale. Supposons que $\alpha > 2$ et $F^2 = F_1^2 + \mu$, avec $F_1^2 = (1 - \alpha)^2 + 1$. Le système réduit (14) induit une bifurcation stationnaire sur-critique avec une symétrie $O(2)$. Lorsque $\mu < 0$, l'équation de Lugiato-Lefever (1) possède une famille de solutions stationnaires périodiques en x , ψ_μ qui sont spectralement instables par rapport à des perturbations co-périodiques.*

On procède ensuite à l'analyse de la stabilité spectrale, au seuil de l'instabilité de Turing, des perturbations bornées et localisées des ondes périodiques (16). Comme mentionnée plus haut, la stabilité spectrale est déterminée par la localisation dans le plan complexe du spectre de l'opérateur linéaire \mathcal{A}_μ obtenu en linéarisant le système dynamique (6) autour d'une onde stationnaire périodique $\psi_\mu(y)$ (définie par (16)) :

$$\mathcal{A}_\mu = -\mathbb{I} + \mathcal{J}\mathcal{L}_\mu, \quad (22)$$

avec $\mathcal{J} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, \mathbb{I} la matrice identité et

$$\mathcal{L}_\mu = \begin{pmatrix} (2 - \alpha)\partial_y^2 - \alpha + 3\psi_{r\mu}^2(y) + \psi_{i\mu}^2(y) & 2\psi_{r\mu}(y)\psi_{i\mu}(y) \\ 2\psi_{r\mu}(y)\psi_{i\mu}(y) & (2 - \alpha)\partial_y^2 - \alpha + \psi_{r\mu}^2(y) + 3\psi_{i\mu}^2(y) \end{pmatrix},$$

où $\psi_{r\mu}(y)$ et $\psi_{i\mu}(y)$ représentent respectivement les parties réelle et imaginaire de la solution périodique $\psi_\mu(y)$.

Ce spectre varie selon le choix du type de perturbations considéré. Pour des perturbations bornées, le spectre de \mathcal{A}_μ est continu et on utilise une décomposition de Bloch-Floquet pour passer d'une question de détermination d'un spectre continu à un spectre constitué de valeurs propres isolées d'une famille d'opérateurs à résolvantes compactes. Puis, nous localisons les potentielles valeurs propres à partie réelles positives (induisant l'instabilité) en utilisant des arguments de perturbations pour des opérateurs linéaires.

Ainsi, au seuil d'instabilité de Turing, le résultat principal du Chapitre 2 montre que les ondes périodiques obtenues au Chapitre 1 et définies par (16) dans le cas des bifurcations sur-critiques sont également stables par rapport à des perturbations bornées, tandis que celles obtenues dans le cas des bifurcations sous-critiques sont instables (cette instabilité est une conséquence directe de leur instabilité par rapport aux perturbations co-périodiques).

Dans le cas de la dispersion anormale, le résultat est le suivant :

Théorème 10. *Considérons l'équation de Lugiato-Lefever (1) dans le cas $\beta = -1$ de dispersion anormale. Supposons que $\alpha < 2$, $\alpha \neq 41/30$, et $F^2 = F_1^2 + \mu$. Considérons les solutions périodiques $\psi_\mu(x)$ de l'équation de Lugiato-Lefever (1) données par (16), pour $\mu > 0$ si $\alpha < 41/30$ et pour $\mu < 0$ si $\alpha > 41/30$, et l'opérateur linéaire associé \mathcal{A}_μ défini par (22), agissant sur $L^2(\mathbb{R}) \times L^2(\mathbb{R})$ ou $C_b(\mathbb{R}) \times C_b(\mathbb{R})$. Pour tout μ suffisamment petit, les propriétés suivantes sont satisfaites.*

- (i) Si $\alpha < 41/30$ et $\mu > 0$, alors le spectre de l'opérateur linéaire \mathcal{A}_μ ne contient que des complexes de partie réelle négative, et la solution périodique $\psi_\mu(x)$ est spectralement stable par rapport aux perturbations bornées.
- (ii) Si $\alpha > 41/30$ et $\mu < 0$, alors le spectre de l'opérateur linéaire \mathcal{A}_μ contient un intervalle fini de nombres réels positifs $(0, c_\mu]$, avec $c_\mu = \mathcal{O}(|\mu|)$, et la solution périodique $\psi_\mu(x)$ est spectralement instable par rapport aux perturbations co-périodiques, localisées et bornées.

Dans le cas de dispersion normale, le résultat est :

Théorème 11. *Considérons l'équation de Lugiato-Lefever (1) dans le cas $\beta = 1$ de dispersion normale. Supposons que $\alpha > 2$ et $F^2 = F_1^2 + \mu$, $F_1^2 = (1 - \alpha)^2 + 1$. Pour tout $\mu < 0$ suffisamment petit, les propriétés suivantes sont satisfaites.*

- (i) L'équation possède une solution périodique paire $\psi_\mu(x)$ de longueur d'onde $\sqrt{\alpha - 2}$ et de développement de Taylor (16), qui est instable par rapport aux perturbations co-périodiques.
- (ii) Le spectre de l'opérateur linéaire \mathcal{A}_μ obtenu en linéarisant l'équation autour de cette solution périodique agissant dans $L^2(\mathbb{R}) \times L^2(\mathbb{R})$ ou $C_b(\mathbb{R}) \times C_b(\mathbb{R})$ contient un intervalle fini de nombre réels positifs $(0, c_\mu]$, avec $c_\mu = \mathcal{O}(|\mu|)$. Par conséquent, la solution stationnaire périodique $\mu(x)$ est spectralement instable par rapport aux perturbations localisées et bornées.

On peut aussi conclure quant à la stabilité, ou instabilité, spectrale par rapport à des perturbations sous-harmoniques. Pour cela, il faut alors remplacer l'espace des phases $\mathcal{X} = L^2(0, 2\pi) \times L^2(0, 2\pi)$ par $\mathcal{X}_N = L^2(0, 2N\pi) \times L^2(0, 2N\pi)$, où N désigne un entier naturel supérieur ou égal à 2. Ce qui à pour conséquence de produire un spectre plus grand. Plus précisément, le spectre de l'opérateur \mathcal{A}_0^* contient des valeurs propres supplémentaires :

$$\sigma(\mathcal{A}_0^*) = \{\lambda_\pm(n, N) = -1 \pm \sqrt{1 - (2 - \alpha)^2(n^2/N^2 - 1)^2}, n \in \mathbb{Z}\}.$$

Les valeurs propres additionnelles ont des parties réelles strictement négatives et donc, de la même manière que dans le cas des perturbations co-périodiques, nous appliquons la réduction à une variété centrale. Le système réduit obtenu à partir des petites solutions sur la variété centrale est le même que (14). Nous généralisons les résultats obtenus pour les perturbations co-périodiques (ceci est détaillé dans le Chapitre 3).

Ainsi, nous concluons quant la stabilité ou instabilité spectrale pour des perturbations sous-harmoniques, elle est identique à celle des perturbations co-périodiques. Dans le cas de la dispersion anormale, le résultat est :

Théorème 12. *Considérons l'équation de Lugiato-Lefever (1) dans le cas $\beta = -1$ de dispersion anormale. Supposons que $\alpha < 2$, $\alpha \neq 41/30$, et $F^2 = F_1^2 + \mu$, avec $F_1^2 = (1 - \alpha)^2 + 1$. Le système réduit (14) induit une bifurcation stationnaire avec une symétrie $O(2)$ à $\mu = 0$. Dans les cas ($\alpha < 41/30$, $\mu > 0$) ou ($\alpha > 41/30$, $\mu < 0$), l'équation de Lugiato-Lefever (1) possède une famille de solutions stationnaires périodiques en x , $\psi_\mu(x)$ qui sont :*

- (i) spectralement stables par rapport aux perturbations sous-harmoniques lorsque $\alpha < 41/30$ (bifurcation sur-critique)
- (ii) spectralement instables par rapport aux perturbations sous-harmoniques lorsque $\alpha > 41/30$ (bifurcation sous-critique)

Dans le cas de dispersion normale, le résultat de stabilité spectrale obtenu pour des perturbations sous-harmoniques est identique à celui obtenu dans le cas des perturbations co-périodiques :

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Théorème 13. *Considérons l'équation de Lugiato-Lefever (1) dans le cas $\beta = 1$ de dispersion normale. Supposons que $\alpha > 2$ et $F^2 = F_1^2 + \mu$, avec $F_1^2 = (1 - \alpha)^2 + 1$. Le système réduit (14) admet une bifurcation stationnaire sur-critique avec une symétrie $O(2)$. Lorsque $\mu < 0$, l'équation de Lugiato-Lefever (1) possède une famille de solutions stationnaires périodiques en x , ψ_μ qui sont spectralement instables par rapport aux perturbations sous-harmoniques.*

Dans la suite de cette présentation nous nous concentrerons sur le cas de dispersion anormale.

4.2. Stabilité dans la région de Turing et de mode zéro

Dans la région d'instabilité de Turing, en plus de l'existence d'ondes stationnaires périodiques, nous pouvons conclure de leur stabilité vis-à-vis de perturbations co-périodiques : les solutions périodiques sont stables lorsque la bifurcation est sur-critiques, et instable lorsqu'elle est sous-critique.

Plaçons-nous dans le cas de dispersion anormale $\beta = -1$. Pour ce qui concerne les perturbations sous-harmoniques, l'analyse est plus complexe. On considère les solutions stationnaires périodiques (17) et (19) obtenues à la section 3.4 de cette introduction. Leur stabilité est déterminée par la localisation dans le plan complexe du spectre du linéarisé \mathcal{A}_0^* d'une part, et par la stabilité de l'équilibre $A_\mu(\phi)$ du système réduit (14) correspondant d'autre part. Nous définissons deux types d'instabilités que l'on rencontre : les instabilités de forme (shape instabilities) et les instabilités de fond (background instabilities) en lien .

Définition 14. *Considérons la solution périodique $\psi_\mu(x)$ définie par (17) lorsque $k = k_{max}$ et par (19) lorsque $k = k_{min}$. Cette solution est dite instable :*

- avec une instabilité de fond, lorsque $\sigma(\mathcal{A}_0^*)$ contient des valeur propres avec une partie réelle positive ;
- avec une instabilité de forme, lorsque l'équilibre $A_\mu(\phi)$ de l'équation réduite (14) est instable.

Pour discuter de stabilité des solutions, définissons les courbes $\Gamma_{n,N}$ d'équation cartésienne

$$\alpha = 2\rho^* - \frac{N^2 + n^2}{N^2 - n^2} \sqrt{\rho^{*2} - 1} \quad (23)$$

dans le cas $k_* = k_{max}$ et par

$$\alpha = 2\rho^* + \frac{N^2 + n^2}{N^2 - n^2} \sqrt{\rho^{*2} - 1}. \quad (24)$$

dans le cas $k_* = k_{min}$.

On obtient, dans le cas $k = k_{max}$, le résultat de stabilité suivant, illustré à la Figure 2.1a :

Théorème 15. *Considérons l'équation de Lugiato-Lefever (1) dans le cas $\beta = -1$ de dispersion anormale, et les solutions stationnaires périodiques ψ_μ données par (17), de longueurs d'onde k_{max} et le propriétés du Théorème 4. Pour tout $N \geq 2$, considérons la courbe $\Gamma_{N-1,N}$ définie par (23), et posons $\Gamma_{0,1} = \Gamma_-$. Alors, les propriétés de stabilité suivantes sont satisfaites pour des perturbations périodiques de périodes $2\pi N/k_{max}$, pour tout $N \geq 1$.*

- (i) *Si (α, ρ^*) appartient à l'ensemble ouvert situé à gauche de la courbe $\Gamma_{N-1,N}$ dans la région d'instabilité de Turing, alors les solutions stationnaires périodiques trouvées dans la bifurcation sur-critiques sont stables, tandis que celles trouvées dans la bifurcation sous-critique sont instables et présentent une instabilité de forme.*
- (ii) *Si (α, ρ^*) appartient à l'ensemble ouvert situé à droite de la courbe $\Gamma_{N-1,N}$ dans la région d'instabilité de Turing, alors les solutions stationnaires périodiques sont instables ; celles issues de bifurcations sur-critiques présentent une instabilité de fond tandis que celles issues de bifurcations sous-critiques présentent à la fois une instabilité de fond et de forme.*

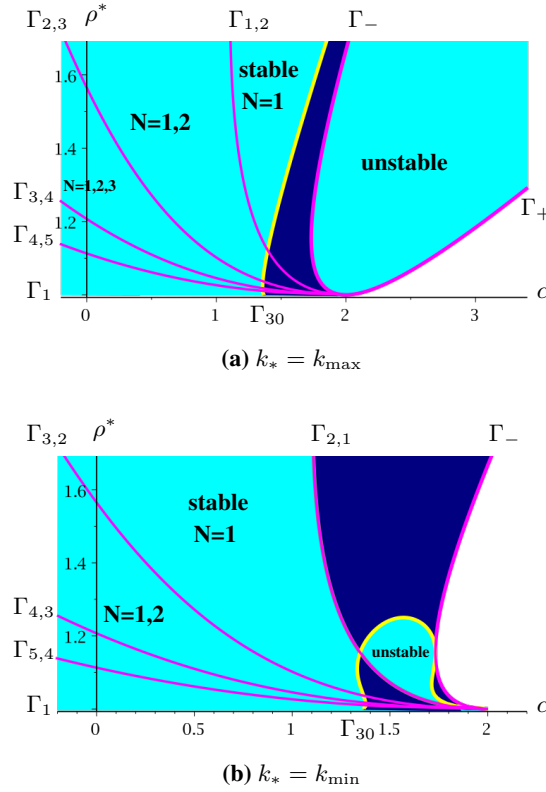


FIGURE 0.9 – Stabilité des solutions stationnaires périodiques dans le cas de dispersion anormale. Dans le plan (α, ρ^*) , les solutions stationnaires périodiques présentent une stabilité de fond pour des perturbations sous-harmonique de périodes $2\pi N/k_*$ pour les valeurs de N comme indiqué, et instable ailleurs. Dans les régions colorées claires, elles présentent une stabilité de forme tandis que dans les régions colorées sombres, elles présentent une instabilité de forme. Les courbes $\Gamma_{n,N}$ sont définies par (23) dans le cas $k_* = k_{\max}$ et par (24) dans le cas $k_* = k_{\min}$.

(iii) Si (α, ρ^*) appartient à la région d'instabilité de mode zéro, alors les solutions stationnaires périodiques sont instables et présentent une instabilité de fond.

On obtient, de manière similaire dans le cas $k = k_{\min}$, le résultat de stabilité suivant, illustré à la Figure 2.1b :

Théorème 16. *Considérons l'équation de Lugiato-Lefever (1) dans le cas $\beta = -1$ de dispersion anormale, et les solutions stationnaires périodiques $\psi_\mu(x)$ données par (19), de longueurs d'onde k_{\min} et les propriétés du Théorème 6. Pour tout $N \geq 1$, considérons la courbe $\Gamma_{N+1,N}$ définie par (24). Alors les propriétés de stabilité suivantes sont satisfaites pour des perturbations périodiques de périodes $2\pi N/k_{\min}$, pour tout $N \geq 1$.*

- (i) Si (α, ρ^*) appartient à l'ensemble ouvert situé à gauche de la courbe $\Gamma_{N+1,N}$ dans la région d'instabilité de Turing, alors les solutions stationnaires périodiques issues des bifurcations sur-critiques sont stables, tandis que celles issues de bifurcations sous-critiques sont instables et présentent une instabilité de forme.
- (ii) Si (α, ρ^*) appartient à l'ensemble ouvert situé à droite de la courbe $\Gamma_{N+1,N}$ dans la région d'instabilité de Turing, alors les solutions stationnaires périodiques sont instables; celles issues de bifurcations sur-critiques présentent une instabilité de fond tandis que celles issues de bifurcations sous-critiques présentent à la fois une instabilité de fond et de forme.

Dans le cas de la dispersion normale $\beta = 1$, nous ne donnons pas d'énoncés précis de théorèmes

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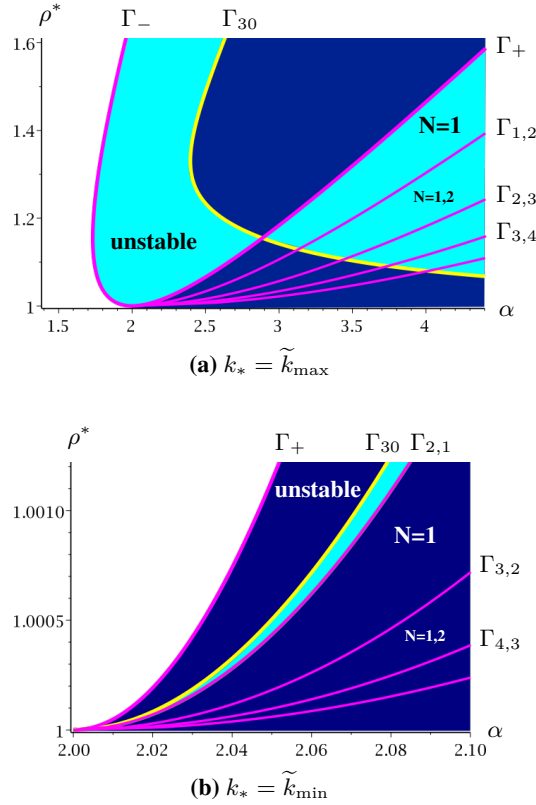


FIGURE 0.10 – Stabilité des solutions stationnaires périodiques dans le cas de dispersion normale. Dans le plan (α, ρ^*) , les solutions stationnaires périodiques présentent une stabilité de fond par rapport aux perturbations sous-harmoniques de période $2\pi N/k_*$ pour des valeurs de N comme indiqué, et instables ailleurs. Dans les régions de couleurs claires, elles présentent une stabilité de forme tandis que dans les régions de couleurs foncées, elles présentent une instabilité de forme. Les courbes $\Gamma_{n,N}$ sont définies par (23) dans le cas $k_* = \tilde{k}_{\max}$ et par (24) dans le cas $k_* = \tilde{k}_{\min}$.

concernant les analogues des Théorèmes 4, 6, 7 et 8, mais nous résumons les résultats dans la Figure 3.1.

4.3. Stabilité asymptotique

La littérature mathématique autour du caractère bien posé du problème de Cauchy pour l'équation de Lugiato-Lefever (1) ainsi que les questions de stabilité asymptotique est à notre connaissance assez limitée :

- (i) Dans les travaux de Miyaji, Ohnishi et Tsutsumi [22], [23] et [24], la question du caractère bien posé est évoquée [23, page 660] renvoyant à [1] pour les grandes lignes (ce dernier ne s'intéresse qu'à l'équation de Schrödinger usuelle). C'est cette méthode que nous adaptons dans cette thèse pour démontrer le caractère localement bien posé dans $L^2(0, 2\pi N/k)$. La question de la stabilité asymptotique est quant à elle analysée dans le cas de la dispersion anormale $\beta = -1$ pour des perturbations co-périodiques.
- (ii) Le travail de Stanislavova et Stefanov [25] traite de la stabilité asymptotique dans $H^1(\mathbb{T})$ sous l'hypothèse que l'état stationnaire périodique est spectralement stable.
- (iii) Les travaux de Haragus, Johnson, Perkins et de Rijk [14] présentent un résultat de stabilité non linéaire d'ondes stationnaires périodiques par rapport à des perturbations localisées.

On remarque également que dans leur travail [16], les auteurs étudient un schéma de splitting pour l'équation de Lugiato-Lefever (1) sous l'hypothèse que le problème de Cauchy associé à celle-ci soit bien posé dans $H^4(0, 2\pi/k)$.

Dans cette thèse, nous présentons un résultat de stabilité asymptotique pour des solutions périodiques stationnaires spectralement stables vis-à-vis de perturbations sous-harmoniques, dans le cas de dispersion anormale, au seuil de l'instabilité de Turing.

Méthode

On considère le changement de variables $y = (\sqrt{2-\alpha})x$ afin de se placer dans l'espace $2\pi N$ -périodique :

$$t \geq 0 \text{ et } y \in \mathbb{T}_N := \mathbb{R}/(2\pi N)\mathbb{Z}.$$

Rappelons que $N \geq 2$ est un entier. Ainsi, l'équation de Lugiato-Lefever (1) se réécrit :

$$\frac{\partial \psi}{\partial t} = -i\beta(2-\alpha) \frac{\partial^2 \psi}{\partial y^2} - (1+i\alpha)\psi + i\psi|\psi|^2 + F. \quad (25)$$

Soit

$$A := -i\beta(2-\alpha) \partial_y^2 - (1+i\alpha)$$

agissant sur $L^2(\mathbb{T}_N)$ de domaine $D(A) = H^2(\mathbb{T}_N)$.

L'équilibre de (16) après le changement de variable devient

$$\phi_\mu(y) = \frac{3(\alpha + i(2-\alpha))}{F_1|41-30\alpha|^{1/2}} \cos(y) |\mu|^{1/2} + \mathcal{O}_{\mu \rightarrow 0}(|\mu|) \quad (26)$$

qui est un équilibre de (25) avec $F^2 = F_1^2 + \mu$.

Estimations de Strichartz

Suivant Bourgain [1] (et comme évoqué dans [23]), en vue de démontrer le caractère bien posé de l'équation de Lugiato-Lefever, on utilise des estimations de Strichartz que nous démontrons pour le cadre qui nous intéresse :

Lemme 1. *L'opérateur A génère un groupe qui, restreint aux temps positifs, est un semi-groupe de contraction avec*

$$\|e^{tA}\psi_0\|_{L^2(\mathbb{T}_N)} = e^{-t} \|\psi_0\|_{L^2(\mathbb{T}_N)}, \quad \forall t \in \mathbb{R}. \quad (27)$$

Comme l'opérateur A commute à l'opérateur de dérivation, dans cette estimation on peut remplacer $L^2(\mathbb{T}_N)$ par un espace de Sobolev $H^s(\mathbb{T}_N)$ pour $s \geq 0$.

Lemme 2. *Soient t_0 appartenant à \mathbb{R}_+ et $\tau > t_0$.*

(i) *Pour tout $\psi_{t_0} \in L^2(\mathbb{T}_N)$, $t \mapsto e^{(t-t_0)A}\psi_{t_0}$ appartient à $L^\infty((t_0, \tau); L^2(\mathbb{T}_N)) \cap L^4((t_0, \tau) \times \mathbb{T}_N)$ et*

$$\|e^{(t-t_0)A}\psi_{t_0}\|_{L^\infty((t_0, \tau); L^2(\mathbb{T}_N))} = \|\psi_{t_0}\|_{L^2(\mathbb{T}_N)}, \quad (28)$$

$$\|e^{(t-t_0)A}\psi_{t_0}\|_{L^4((t_0, \tau) \times \mathbb{T}_N)} \leq C_1(\tau, t_0) \|\psi_{t_0}\|_{L^2(\mathbb{T}_N)}. \quad (29)$$

(ii) *Pour tout f de $L^{4/3}((t_0, \tau) \times \mathbb{T}_N)$, $t \mapsto \int_{t_0}^t e^{(t-s)A} f(s) ds$ appartient à $L^2(\mathbb{T}_N)$ et*

$$\left\| \int_{t_0}^t e^{-sA} f(s) ds \right\|_{L^2(\mathbb{T}_N)} \leq C_1(\tau, t_0) \|f\|_{L^{4/3}((t_0, \tau) \times \mathbb{T}_N)}. \quad (30)$$

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(iii) Pour tout f de $L^{4/3}((t_0, \tau) \times \mathbb{T}_N)$, $t \mapsto \int_{t_0}^t e^{(t-s)A} f(s) ds$ appartient à $L^\infty((t_0, \tau); L^2(\mathbb{T}_N)) \cap L^4((t_0, \tau) \times \mathbb{T}_N)$ et

$$\left\| \int_{t_0}^t e^{t-s} e^{(t-s)A} f(s) ds \right\|_{L^\infty((t_0, \tau); L^2(\mathbb{T}_N))} \leq C_1(\tau, t_0) \|f\|_{L^{4/3}((t_0, \tau) \times \mathbb{T}_N)}, \quad (31)$$

$$\left\| \int_{t_0}^t e^{t-s} e^{(t-s)A} f(s) ds \right\|_{L^4((t_0, \tau) \times \mathbb{T}_N)} \leq C_1(\tau, t_0)^2 \|f\|_{L^{4/3}((t_0, \tau) \times \mathbb{T}_N)}. \quad (32)$$

De plus, $C_1(\tau, t_0) \leq \left(\frac{\tau-t_0}{\pi N} + \frac{4N}{2-\alpha} \right)^{1/4}$.

De ces estimations, on déduit que pour $\psi \in L^\infty((0, T^+(\psi_0)); L^2(\mathbb{T}_N)) \cap L^4([0, T^+(\psi_0)) \times \mathbb{T}_N)$, la quantité suivante :

$$e^{tA} \psi_0 + \int_0^t e^{(t-s)A} F ds + \int_0^t e^{(t-s)A} \psi(s) |\psi(s)|^2 ds$$

est dans ce même espace.

Caractère bien posé du problème de Cauchy

Définition 17. On dit que le problème de Cauchy associé à (25) avec donnée initiale $\psi(t=0) = \psi_0$ admet une solution douce sur $[0, T)$, avec $T \in [0, +\infty)$, si $\psi : [0, T) \rightarrow \mathbb{C}$ vérifie pour tout $t \in [0, T)$

$$\psi(t) = e^{tA} \psi_0 + \int_0^t e^{(t-s)A} F ds + \int_0^t e^{(t-s)A} \psi(s) |\psi(s)|^2 ds.$$

On a le résultat d'existence et unicité globales suivant :

Théorème 18. Pour tout ψ_0 dans $L^2(\mathbb{T}_N)$, il existe $T^+(\psi_0)$ tel que le problème de Cauchy (25) avec donnée initiale $\psi(t=0) = \psi_0$ possède une unique solution douce maximale $([0, T^+(\psi_0)), \psi)$ dans

$$L^\infty((0, T^+(\psi_0)); L^2(\mathbb{T}_N)) \cap L^4([0, T^+(\psi_0)) \times \mathbb{T}_N)$$

et telle que $T^+(\psi_0) \in (0, +\infty]$ et si $T^+(\psi_0) < \infty$ alors

$$\lim_{t \rightarrow T^+(\psi_0)} \|\psi\|_{L^4([0, t] \times \mathbb{T}_N)} = +\infty.$$

De plus, ψ appartient à $C((0, T^+(\psi_0)); L^2(\mathbb{T}_N))$.

En utilisant le Lemme 1 dans H^s pour $s > 1/2$, on peut démontrer que la borne de croissance du semi-groupe engendré par A est strictement négative, ce qui permet de démontrer le caractère bien posé pour des données initiales dans H^s pour $s > 1/2$. On exploite ainsi le fait que H^s , pour $s > 1/2$ est une algèbre de Banach (la preuve suit le schéma de celle du théorème de Cauchy-Lipschitz pour des équations différentielles).

Stratégie

La stabilité asymptotique autour d'un état stationnaire est une propriété du flot de l'équation de Lugiato-Lefever (25) dans un voisinage de cet état stationnaire pour un cadre fonctionnel donné. En ce qui nous concerne, ce sera $L^2(\mathbb{T}_N)$ car nous nous intéressons aux perturbations sous-périodiques.

On remarque qu'il existe une famille d'équilibres $\phi_{\mu, c}(y) = \phi_\mu(y + c)$ pour tout c réel et $|\mu|$ petit. C'est la variété de modulation associée à la solution ϕ_μ donnée par (26). On s'intéresse à la stabilité asymptotique globale de cette famille. Ces états sont spectralement stables, c'est-à-dire que le spectre

du linéarisé ne contient pas de nombre complexe de partie réelle strictement positive. Ceci découle de la stabilité spectrale de ϕ_μ par rapport à des perturbations sous-harmoniques énoncée dans le Théorème 12.

Rappelons que le linéarisé n'est pas \mathbb{C} -linéaire mais seulement \mathbb{R} -linéaire. Dès lors, nous regardons tous les équilibres et les solutions non pas comme étant à valeur dans \mathbb{C} , mais à valeur dans \mathbb{R}^2 en les décomposant en partie réelle et imaginaire. Dans notre analyse, pour clairement identifier ce changement, nous passons à des lettres majuscules. Ainsi, les équilibres $\phi_{\mu,c}$ deviennent $\Phi_{\mu,c}$, à valeurs dans \mathbb{R}^2 . On note $\mathcal{A}_{\mu,c}$ le linéarisé autour de $\Phi_{\mu,c}$. L'opérateur $\mathcal{A}_{\mu,c}$ comme son adjoint $\mathcal{A}_{\mu,c}^{ad}$ ont 0 comme valeur propre simple. Le reste du spectre de $\mathcal{A}_{\mu,c}$ est strictement inclus dans $\{\lambda \in \mathbb{C}, \operatorname{Re}(\lambda) < 0\}$.

Sur l'orthogonal du noyau de son adjoint, que l'on note \mathcal{N}_c , $\mathcal{A}_{\mu,c}$ est invariant et engendre un semi-groupe dissipatif. Plus précisément, à l'aide du théorème de Gearhart-Prüss, comme dans [25], la borne de croissance du semi-groupe engendré par $\mathcal{A}_{\mu,c}$ sur \mathcal{N}_c est strictement négative, ce qui donne un résultat très similaire au lemme 4 (A est remplacé par $\mathcal{A}_{\mu,c}$ et $L^2(\mathbb{T}_N)$ par \mathcal{N}_c). On montre également que l'on a des estimations de Strichartz globales en temps si l'on ajoute des poids exponentiels en temps *ad hoc*. De telles estimations nous permettent de montrer la stabilité asymptotique de la famille $\Phi_{\mu,c}$, puis de la famille $\phi_{\mu,c}$.

On considère $\psi(0)$ proche de ϕ_μ et ψ la solution maximale du problème de Cauchy associé à l'équation (25) avec la condition initiale $\psi(0)$. On montre qu'il existe une courbe $t \mapsto c(t) \in \mathbb{R}$ et $t \mapsto \Upsilon(t) \in L^2(\mathbb{T}; \mathbb{R}^2)$ tels que $\Phi(t)$ s'écrit

$$\Phi(t) = \Phi_{\mu,c(t)} + \Upsilon(t),$$

et $\Upsilon(t)$ a la propriété importante suivante : $\Upsilon(t) \in \mathcal{N}_{c(t)}$. Ce qui nous permet, à l'aide des estimations de Strichartz pour l'opérateur $\mathcal{A}_{\mu,c}$, de montrer que Υ converge vers 0 en norme $L^2(\mathbb{T}_N)^2$ lorsque t tend vers l'infini et appartient à l'espace $L^4(\mathbb{R}^+ \times \mathbb{T}_N)^2$, et que c converge vers un profil asymptotique c_∞ lorsque t tend vers $+\infty$.

Résultat de stabilité asymptotique

La stratégie exposée plus haut, que nous détaillons dans le Chapitre 3, permet d'obtenir le théorème suivant :

Théorème 19. *Soit $N \geq 2$ un entier. L'équation de Lugiato-Lefever (1) dans le cas $\beta = -1$ (de dispersion anormale), $\alpha < 41/30$, $\mu > 0$ suffisamment petit et $F^2 = 1 + (1 - \alpha)^2 + \mu$ possède un équilibre constant instable ψ_0^* et une famille d'équilibres non triviaux $\frac{2\pi}{\sqrt{2-\alpha}}$ -périodiques stables $\bar{\psi}_{\mu,c}$ où, pour tout $c \in \mathbb{R}$,*

$$\bar{\psi}_{\mu,c}(y) = \bar{\psi}_{\mu,0}(y - c).$$

Puis, pour tout $\varepsilon > 0$, il existe $\delta > 0$ tel que si $u_0 \in L^2(\mathbb{T}_N)$ satisfait

$$\left\| u_0 - \bar{\psi}_{\mu,0} \right\|_{L^2(\mathbb{R}/(2\pi \frac{N}{\sqrt{2-\alpha}})\mathbb{Z})} \leq \delta,$$

alors il existe une unique solution globale u de (1) avec $u|_{t=0} = u_0$ in $L^\infty((0, \tau); L^2(\mathbb{R}/(2\pi \frac{N}{\sqrt{2-\alpha}})\mathbb{Z})) \cap L^4([0, \tau] \times \mathbb{R}/(2\pi \frac{N}{\sqrt{2-\alpha}})\mathbb{Z})$, pour tout $\tau > 0$, et $0 \leq c_\infty < 2\pi N$ tel que

$$\sup_{t \geq 0} \left[\inf_{0 \leq c < 2\pi N} \left\| u(t) - \bar{\psi}_{\mu,c} \right\|_{L^2(\mathbb{R}/(2\pi \frac{N}{\sqrt{2-\alpha}})\mathbb{Z})} \right] < \varepsilon,$$

$$\left\| u(t) - \bar{\psi}_{\mu,c_\infty} \right\|_{L^2(\mathbb{R}/(2\pi \frac{N}{\sqrt{2-\alpha}})\mathbb{Z})} \longrightarrow 0 \quad (t \longrightarrow +\infty)$$

et

$$\inf_{0 \leq c < 2\pi N} \left\| u(t) - \bar{\psi}_{\mu,c} \right\|_{L^4(\mathbb{R}^+ \times \mathbb{R}/(2\pi \frac{N}{\sqrt{2-\alpha}})\mathbb{Z})} < \varepsilon.$$

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Ce résultat apporte des contributions aux travaux [23], [24] et [25]. Il prouve la stabilité asymptotique dans le cadre des perturbations sous-harmoniques, ce qui généralise les résultats de [23], [24] et [25] qui sont obtenus dans le cadre des perturbations co-périodiques. De plus, la condition initiale est uniquement dans L^2 , et pas H^1 (comme proposé dans [25]). Ce résultat ne nécessite pas d'hypothèse spectrale supplémentaire (comme proposé dans [24]) autre que la stabilité spectrale.

5. Perspectives

La stratégie du Théorème 19 s'étend à tout état stationnaire spectralement stable pour lequel le linéarisé satisfait des estimations de dissipation (Lemme 4) et idéalement de Strichartz (Lemme 5). On peut donc imaginer étendre ces résultats dès qu'il est possible de montrer des estimations de la résolvante du linéarisé nécessaire au Théorème de Gearhart–Prüss. Toutefois la question que nous souhaitons aborder est celle de l'instabilité.

En ce qui concerne les questions auxquelles nous souhaitons nous intéresser par la suite, nous aimerions considérer la question de l'existence globale ou non de solutions à (1). Il est aussi possible que l'alternative d'explosion se produise.

En ce qui concerne notre Théorème 19, on notera qu'il peut s'exprimer comme une forme de développement asymptotique/décomposition d'une solution avec un premier terme donné par un état stationnaire et un reste qui disperse comme des solutions non stationnaires de l'équation linéarisée. Il semble possible de montrer que ce reste peut de nouveau se décomposer comme solutions non stationnaires de l'équation linéarisée et un terme de reste qui tend vers 0 plus rapidement. C'est un résultat de type scattering qui nécessite d'utiliser des estimations ponctuelles en temps et non de Strichartz.

Chapter 1

Local bifurcations of periodic waves for the Lugiato-Lefever equation

We consider the Lugiato-Lefever equation

$$\frac{\partial \psi}{\partial t} = -i\beta \frac{\partial^2 \psi}{\partial x^2} - (1 + i\alpha)\psi + i\psi|\psi|^2 + F, \quad (1.0.1)$$

in which the unknown ψ is a complex-valued function depending upon the temporal variable t and the spatial variable x , the parameters α and β are real, and F is a positive constant. Notice that upon rescaling x , we may take $|\beta| = 1$, hence $\beta = 1$ in the case of normal dispersion, and $\beta = -1$ in the case of anomalous dispersion.

Our goal is to prove the existence of steady periodic solutions of the equation (1.0.1), i.e. periodic solutions of the differential equation

$$0 = -i\beta \frac{\partial^2 \psi}{\partial x^2} - (1 + i\alpha)\psi + i\psi|\psi|^2 + F, \quad (1.0.2)$$

This chapter is extracted from the articles [5] and [4].

1.1 Spectral stability of constant solutions

In this section we recall the stability properties of the constant solutions of the Lugiato-Lefever equation (1.0.1).

Lemma 1.1.1. *The Lugiato-Lefever equation (1.0.1) possesses the following constant solutions:*

- (i) if $\alpha \leq \sqrt{3}$ and $F > 0$, the equation (1.0.1) has a unique constant solution;
- (ii) if $\alpha > \sqrt{3}$ and $F > 0$, there exist two values $F_-(\alpha)$ and $F_+(\alpha)$, with $F_-(\alpha) < F_+(\alpha)$, given by

$$F_{\pm}^2(\alpha) = \rho_{\pm}(\alpha) \left((\rho_{\pm}(\alpha) - \alpha)^2 + 1 \right), \quad \rho_{\pm}(\alpha) = \frac{1}{3} \left(2\alpha \mp \sqrt{\alpha^2 - 3} \right),$$

such that the equation (1.0.1) possesses three constant solutions with square modulus $\rho = \rho_j$, $j = 1, 2, 3$,

$$\rho_1 < \rho_+(\alpha) < \rho_2 < \rho_-(\alpha) < \rho_3,$$

when $F_-(\alpha) < F < F_+(\alpha)$, two distinct constant solutions when $F = F_{\pm}(\alpha)$, and one constant solution when $F < F_-(\alpha)$ or $F_+(\alpha) < F$.

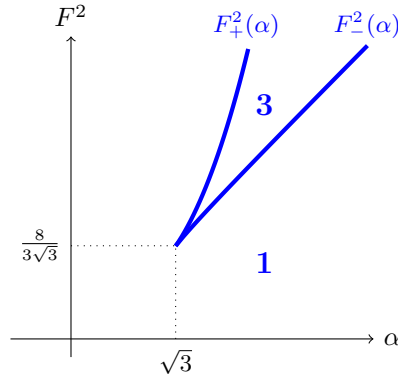


Figure 1.1 – Number of constant solutions of the Lugiato-Lefever equation (1.0.1): three solutions in the region between the curves $F^2 = F_{\pm}^2(\alpha)$, two solutions along the curves, and one solution otherwise.

Proof. Constant solutions $\psi \in \mathbb{C}$ of the equation (1.0.1) satisfy the algebraic equation

$$(1 + i\alpha)\psi - i\psi|\psi|^2 = F. \quad (1.1.1)$$

Upon decomposing into real and imaginary parts, $\psi = r + i\psi_i$, and setting $\rho = |\psi|^2 = \psi_r^2 + \psi_i^2$, we obtain

$$r = \frac{\rho}{F}, \quad \psi_i = \frac{\rho(\rho - \alpha)}{F}, \quad \rho((\rho - \alpha)^2 + 1) = F^2. \quad (1.1.2)$$

The last equation above determines the number of constant solutions of (1.0.1). The function

$$\rho \mapsto \rho((\rho - \alpha)^2 + 1),$$

has the derivative

$$\rho \mapsto 3\rho^2 - 4\alpha\rho + \alpha^2 + 1,$$

with discriminant is $\Delta = 4(\alpha^2 - 3)$. For $\rho > 0$, the function $\rho \mapsto \rho((\rho - \alpha)^2 + 1)$ is monotone increasing, when $\alpha \leq \sqrt{3}$, and has two positive critical points $\rho_+(\alpha) < \rho_-(\alpha)$, when $\alpha > \sqrt{3}$. Consequently, the Lugiato-Lefever equation possesses precisely one constant solution when $\alpha \leq \sqrt{3}$, for any $F > 0$. For $\alpha > \sqrt{3}$, there exist two values $F_-(\alpha) < F_+(\alpha)$, such that the Lugiato-Lefever equation possesses three constant solutions with $\rho = \rho_j$, $j = 1, 2, 3$,

$$\rho_1 < \rho_+(\alpha) < \rho_2 < \rho_-(\alpha) < \rho_3,$$

when $F_-(\alpha) < F < F_+(\alpha)$, two distinct constant solutions when $F = F_{\pm}(\alpha)$, and one constant solution when $F < F_-(\alpha)$ or $F_+(\alpha) < F$. This result is summarized in Figure 1.1 (see also [9]). \square

We notice that equation (1.1.1) is the same in the case of anomalous dispersion ($\beta = -1$) and in the case of normal dispersion ($\beta = 1$) thus, constant solutions are the same in both cases.

The spectral stability of these constant solutions is determined by the location in the complex plane of the spectrum of the linear operator obtained by linearizing the Lugiato-Lefever equation (1.0.1) at such a constant solution.

We obtain the result :

Proposition 1.1.2. *For a constant solution $\psi^* = \psi_r^* + i\psi_i^*$ of the Lugiato-Lefever equation (1.0.1), the right hand side of the linearized equation*

$$\frac{dV}{dt} = \mathcal{A}^*V,$$

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defines a linear operator of the form

$$\mathcal{A}^* = -\mathbb{I} + \mathcal{J}\mathcal{L}^*, \quad (1.1.3)$$

where \mathbb{I} represents the 2×2 identity matrix, and

$$\mathcal{J} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \mathcal{L}^* = \begin{pmatrix} -\beta\partial_x^2 - \alpha + 3\psi_r^{*2} + \psi_i^{*2} & 2\psi_r^*\psi_i^* \\ 2\psi_r^*\psi_i^* & -\beta\partial_x^2 - \alpha + \psi_r^{*2} + 3\psi_i^{*2} \end{pmatrix}.$$

The operator \mathcal{A}^* is closed on $L^2(\mathbb{R}) \times L^2(\mathbb{R})$ with domain $H^2(\mathbb{R}) \times H^2(\mathbb{R})$.

Proof. We write $\psi = \psi_r + i\psi_i$ and we set $\Psi = (\psi_r, \psi_i)^T$ and we have that ψ is solution of (1.0.1) if and only if

$$\frac{\partial \Psi}{\partial t} = G(\Psi)$$

where

$$G(\Psi) = \begin{pmatrix} \beta\frac{\partial^2}{\partial x^2} \psi_i - \psi_r + \alpha\psi_i - \psi_i(\psi_r^2 + \psi_i^2) + F \\ -\beta\frac{\partial^2}{\partial x^2} \psi_r - \psi_i - \alpha\psi_r + \psi_r(\psi_r^2 + \psi_i^2) \end{pmatrix}.$$

We compute the jacobian matrix of G in Ψ^* , where $\Psi^* = (\psi_r^*, \psi_i^*)$ and we have

$$\mathcal{A}^* = \begin{pmatrix} -1 - 2\psi_r^*\psi_i^* & \beta\partial_x^2 + \alpha - \psi_r^{*2} - 3\psi_i^{*2} \\ -\beta\partial_x^2 - \alpha + 3\psi_r^{*2} + \psi_i^{*2} & -1 + 2\psi_r^*\psi_i^* \end{pmatrix}.$$

The operator \mathcal{A}^* is a differential operator with constant coefficients and we have $\mathcal{A}^* = -\mathbb{I} + \mathcal{J}\mathcal{L}^*$.

Moreover, let $(u_n)_n \subset H^2(\mathbb{R})$ such that $u_n \rightarrow u$ in $L^2(\mathbb{R})$ and $u_n'' \rightarrow v$ in $L^2(\mathbb{R})$. By Fourier transform, we have $\widehat{u}_n \rightarrow \widehat{u}$ and $-k^2\widehat{u}_n \rightarrow \widehat{v}$ for all $k \in \mathbb{R}$. We conclude that $-k^2\widehat{u} = \widehat{v}$ thus $u \in H^2(\mathbb{R})$ and $u'' = v$, and the operator ∂_x^2 is closed on $L^2(\mathbb{R})$ with domain $H^2(\mathbb{R})$. The operator \mathcal{A}^* is a differential operator with constant coefficients, thus is a bounded perturbation of ∂_x^2 , which concludes the proof of the last part of this proposition. \square

The spectrum of the operator \mathcal{A}^* is given in the following proposition:

Proposition 1.1.3. *The spectrum of the linear operator \mathcal{A}^* defined in (1.1.3) is given by*

$$\sigma(\mathcal{A}^*) = \left\{ \lambda \in \mathbb{C} : \lambda^2 + 2\lambda + a(k) = 0, k \in \mathbb{R} \right\}, \quad (1.1.4)$$

in which

$$a(k) = k^4 - 2\beta(\alpha - 2\rho^*)k^2 + \alpha^2 - 4\alpha\rho^* + 3\rho^{*2} + 1. \quad (1.1.5)$$

Proof. Since \mathcal{A}^* is an operator with constant coefficients, with the Fourier Ansatz $(u(x), v(x)) = e^{ikx}(u_k, v_k)$, we find that its spectrum $\sigma(\mathcal{A}^*)$, in the Hilbert space $L^2(\mathbb{R}) \times L^2(\mathbb{R})$ of square integrable functions, is given by

$$\sigma(\mathcal{A}^*) = \bigcup_{k \in \mathbb{R}} \sigma(\mathcal{A}^*(k)),$$

where $\mathcal{A}^*(k)$ are the 2×2 matrices

$$\mathcal{A}^*(k) = -\mathbb{I} + \mathcal{J}\mathcal{L}^*(k), \quad \mathcal{L}^*(k) = \begin{pmatrix} \beta k^2 - \alpha + 3\psi_r^{*2} + \psi_i^{*2} & 2\psi_r^*\psi_i^* \\ 2\psi_r^*\psi_i^* & \beta k^2 - \alpha + \psi_r^{*2} + 3\psi_i^{*2} \end{pmatrix}.$$

We have that $\lambda \in \rho(\mathcal{A}^*)$ if and only if for all k in \mathbb{R} , $\det(\lambda - \mathcal{A}^*(k)) \neq 0$. For all k in \mathbb{R} ,

$$\det(\lambda - \mathcal{A}^*(k)) = \lambda^2 + 2\lambda + k^4 - 2\beta(\alpha - 2\rho^*)k^2 + \alpha^2 - 4\alpha\rho^* + 3\rho^{*2} + 1.$$

Then,

$$\sigma(\mathcal{A}^*) = \left\{ \lambda \in \mathbb{C} : \lambda^2 + 2\lambda + a(k) = 0, k \in \mathbb{R} \right\},$$

with

$$a(k) = k^4 - 2\beta(\alpha - 2\rho^*)k^2 + \alpha^2 - 4\alpha\rho^* + 3\rho^{*2} + 1.$$

□

Remark 1.1.4. *The spectrum of the linear operator \mathcal{A}_* is symmetric with respect to the vertical line $\operatorname{Re} \lambda = -1$ in the complex plane, due to its particular structure (1.1.3) in which the operators \mathcal{J} and \mathcal{L}_* are skew- and self-adjoint, respectively, when acting in the Hilbert space $L^2(\mathbb{R}) \times L^2(\mathbb{R})$. These properties of the operators \mathcal{J} and \mathcal{L}_* imply that the spectrum of the product operator $\mathcal{J}\mathcal{L}_*$ is symmetric with respect to the imaginary axis (e.g., see [15]), so that the spectrum of \mathcal{A}_* is symmetric to the vertical line $\operatorname{Re} \lambda = -1$.*

The case of anomalous dispersion ($\beta = -1$) For any $k \in \mathbb{R}$, the sum of the two eigenvalues of $\mathcal{A}_*(k)$ is equal to -2 (which is also true for $\beta = 1$), so that their location in the complex plane depends upon the value of $a(k)$ (which depends of β). For $a(k) > 1$, the two eigenvalues are complex conjugated with real parts equal to -1 , and -1 is a double eigenvalue when $a(k) = 1$. For $a(k) < 1$, the two eigenvalues are real and symmetric with respect to the line $\operatorname{Re} \lambda = -1$: both eigenvalues are negative when $0 < a(k) < 1$, 0 and -2 are eigenvalues when $a(k) = 0$, and one eigenvalue is negative and the other one positive, when $a(k) < 0$. Consequently, the constant solution is spectrally stable if $a(k) \geq 0$ for all $k \in \mathbb{R}$, and unstable otherwise. A direct calculation then shows that the stability of the constant solution changes in the following two cases:

- (i) for $\alpha < 2$ and $\rho^* = 1$, when $a(k)$ is nonnegative and vanishes for precisely two values $k = \pm\sqrt{2 - \alpha} \neq 0$ (Turing instability); in the parameter plane (α, F^2) this occurs along the parabola of equation $F^2 = (1 - \alpha)^2 + 1$;
- (ii) for $\alpha \geq 2$ and $\rho^* = \rho_+(\alpha)$, when $a(k)$ is nonnegative and vanishes at $k = 0$ (zero-mode instability); in the parameter plane (α, F^2) this occurs along the half curve $F^2 = F_+^2(\alpha)$ (see Figure 1.1).

In Figure 1.2 we represent the shape of the largest real eigenvalue $\lambda(k)$ in these two cases, and in Figure 1.3 we summarize the stability properties of the constant solutions of the Lugiato-Lefever equation (1.0.1). The properties at the endpoints of the intervals for α , which are not represented, follow from continuity arguments. Upon increasing α , at $\alpha = \sqrt{3}$, $\rho^* = 1$ becomes an inflection point on the branch of constant solutions which splits into the two critical points $\rho^* = \rho_{\pm}(\alpha)$ for $\alpha > \sqrt{3}$. At $\alpha = 7/4$, we have $F_1^2 = F_-^2(\alpha)$ and the onset of Turing instability $F^2 = F_1^2$ moves from the (α, F) -parameter region where the equation has one constant solution, $F^2 < F_-^2(\alpha)$, to the parameter region where the equation has three constant solutions, $F^2 \in (F_-^2(\alpha), F_+^2(\alpha))$, as α is increased above the value $7/4$. At $\alpha = 2$, we have $F_1^2 = F_+^2(\alpha)$ and the Turing instability reaches the constant solution $\rho_+(\alpha)$, which is now double, and becomes a zero-mode instability for $\alpha \geq 2$.

The case of normal dispersion ($\beta = 1$) For a constant solution $\psi^* = \psi_r^* + i\psi_i^*$, with modulus square ρ^* , the linear operator \mathcal{A}^* is given by (1.1.3). Then the sign of the coefficient of k^2 changes in $\mathcal{L}^*(k)$ and in the formula of $a(k)$ in the spectrum of linear operator \mathcal{A}^* .

A direct calculation then shows that the stability of the constant solution changes in the following two cases:

- (i) for $\alpha > 2$ and $\rho^* = 1$, when $a(k)$ is nonnegative and vanishes for precisely two values $k = \pm\sqrt{\alpha - 2} \neq 0$ (Turing instability); in the parameter plane (α, F^2) this occurs along the parabola of equation $F^2 = (1 - \alpha)^2 + 1$;

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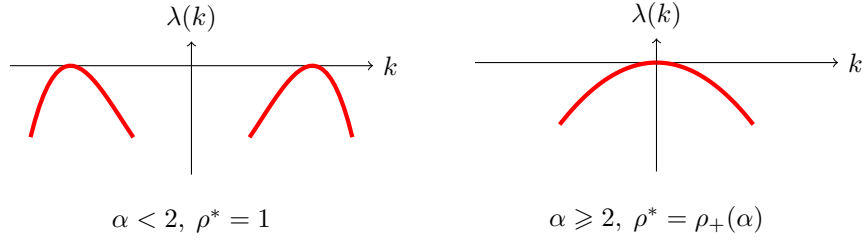


Figure 1.2 – Shape of the largest eigenvalue $\lambda(k)$ for k close to $\pm\sqrt{2-\alpha}$ in the case of the Turing instability (left plot) and for k close to 0 in the case of the zero-mode instability (right plot).

- (ii) for $\alpha > \sqrt{3}$, $\rho^* = \rho_-(\alpha)$ along the half curve $F^2 = F_-^2(\alpha)$ and for $\sqrt{3} < \alpha < 2$, $\rho^* = \rho_+(\alpha)$ along the half curve $F^2 = F_+^2(\alpha)$ when $a(k)$ is nonnegative and vanishes at $k = 0$ (zero-mode instability) (see Figure 1.1).

This implies a change in the stability properties of the constant solutions, which are summarized in Figure 1.4.

Turing and zero-mode regions According to the previous analysis, two types of instabilities occur: a Turing instability when the largest eigenvalue $\lambda(0) < 0$ and $\lambda(k) > 0$ for some $k \neq 0$, and a zero-mode instability when the largest eigenvalue $\lambda(0) > 0$. The corresponding parameter values are found from the condition $a(k) < 0$ and are represented in Figure 1.5. The instability regions are determined by the values of α and ρ^* , only, the values of the parameter F and of the solution ψ^* being obtained from the equalities (1.1.2).

The curves Γ_1 , Γ_+ , and Γ_- , which form the boundaries of these two regions, are defined through

$$\Gamma_1 = \{\alpha < 2, \rho^* = 1\}, \quad \Gamma_+ = \{\alpha > 2, \rho^* = \rho_+(\alpha)\}, \quad (1.1.6)$$

and

$$\Gamma_- = \{\sqrt{3} < \alpha < 2, \rho^* = \rho_+(\alpha)\} \cup \{\alpha > \sqrt{3}, \rho^* = \rho_-(\alpha)\}. \quad (1.1.7)$$

A constant solution loses its stability upon increasing ρ^* and becomes Turing or zero-mode unstable when crossing the curves Γ_1 or Γ_+ , respectively. The instability is fully developed in the open sets situated between the curves Γ_1 and Γ_- (Turing instability region) and the curves Γ_- and Γ_+ (zero-mode instability region).

Remark 1.1.5. *The three curves Γ_1 , Γ_+ , and Γ_- above are precisely the bifurcation curves found in the local bifurcation analysis of the steady Lugiato-Lefever equation in [9]. In the spatial dynamics approach from [9], the bifurcation points are the parameter values where the number of purely imaginary eigenvalues of a certain 4×4 matrix changes. It turns out that these purely imaginary eigenvalues are precisely the values ik with k such that $\lambda_1(k) = 0$. This property implies that the curves Γ_1 , Γ_+ , and Γ_- correspond to the local bifurcations $(i\omega)^2$, 0^2 , and $0^2(i\omega)$, respectively, studied in [9].*

1.2 Bifurcations of steady periodic solutions at the onset of Turing in the case of anomalous dispersion ($\beta = -1$)

In this section we analyze the Turing bifurcation which occurs for $\alpha < 2$ and $F^2 = (1 - \alpha)^2 + 1$, when $\rho^* = 1$. We fix $\alpha < 2$ and take as bifurcation parameter $F^2 = F_1^2 + \mu$, with $F_1^2 = (1 - \alpha)^2 + 1$ and small μ .

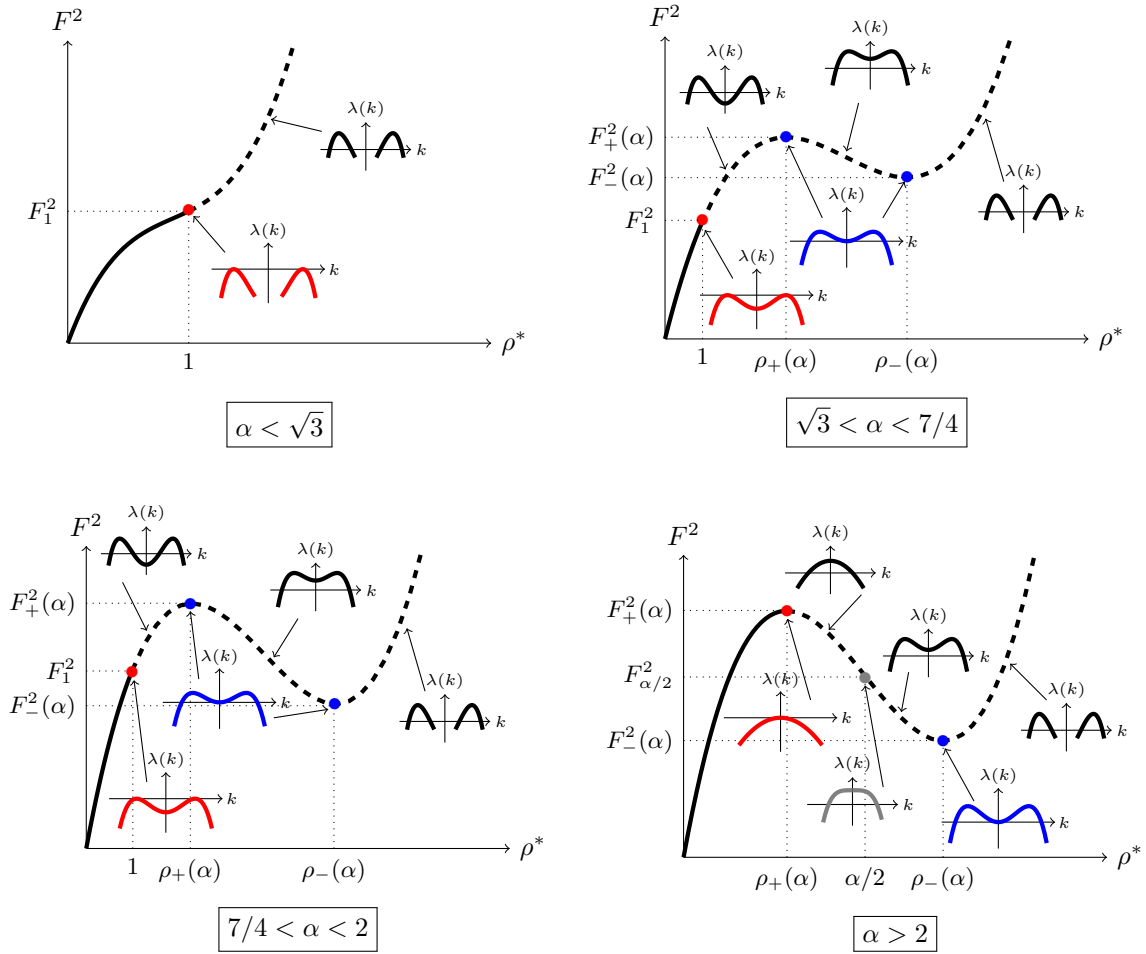


Figure 1.3 – Stability of constant solutions of the Lugiato-Lefever equation in the case of anomalous dispersion ($\beta = -1$). Stable branches are represented by solid lines, unstable branches by dashed lines, and we set $F_1^2 = (1 - \alpha)^2 + 1$ and $F_{\alpha/2}^2 = \alpha(4 + \alpha^2)/8$. The shape of the largest eigenvalue $\lambda(k)$ is given at the points where the solutions lose their stability and for the unstable solutions.

1.2.1 Dynamical system

For $F^2 = F_1^2 + \mu$, we denote by $\psi_\mu^* = \psi_{r\mu}^* + i\psi_{i\mu}^*$ and $\rho_\mu^* = |\psi_\mu^*|^2$ the constant solution of the Lugiato-Lefever equation (1.0.1) and its square modulus, respectively, given by (1.1.2). At $\mu = 0$, we have the constant solution with modulus $\rho_0^* = 1$, at which the Turing instability occurs, and according to the linear stability analysis in Section 1.1, $\lambda = 0$ is an eigenvalue of the corresponding linearized operator \mathcal{A}^* , with eigenmodes $(u(x), v(x)) = e^{ikx}(u_k, v_k)$, $k = \pm\sqrt{2 - \alpha}$. We therefore expect bifurcations of periodic solutions with wavenumbers close to $\pm\sqrt{2 - \alpha}$ (see also Remark 1.2.9 below).

We define the linear operator \mathcal{L}_μ^* by

$$\mathcal{L}_\mu^* = \begin{pmatrix} (2 - \alpha)\partial_y^2 - \alpha + 3\psi_{r\mu}^{*2} + \psi_{i\mu}^{*2} & 2\psi_{r\mu}^*\psi_{i\mu}^* \\ 2\psi_{r\mu}^*\psi_{i\mu}^* & (2 - \alpha)\partial_y^2 - \alpha + \psi_{r\mu}^{*2} + 3\psi_{i\mu}^{*2} \end{pmatrix},$$

and the linear operator \mathcal{A}_μ^* by

$$\mathcal{A}_\mu^* = -\mathbb{I} + \mathcal{J}\mathcal{L}_{*\mu}^*.$$

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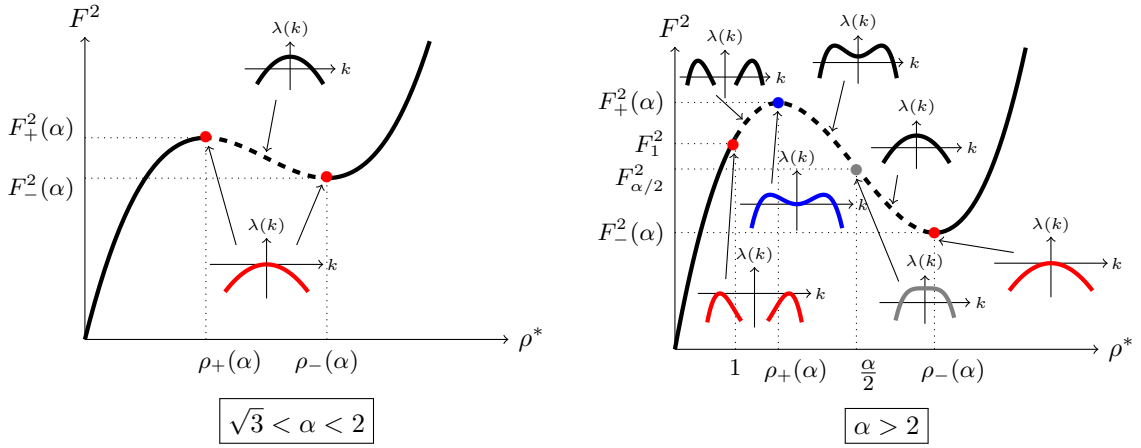


Figure 1.4 – Stability of constant solutions of the Lugiato-Lefever equation in the case of normal dispersion ($\beta = 1$), for $\alpha > \sqrt{3}$ (for $\alpha \leq \sqrt{3}$ the unique constant solution is stable). Stable branches are represented by solid lines, unstable branches by dashed lines, and we set $F_1^2 = (1 - \alpha)^2 + 1$ and $F_{\alpha/2}^2 = \alpha(4 + \alpha^2)/8$. The shape of the largest eigenvalue $\lambda(k)$ is given at the points where the solutions lose their stability and for the unstable solutions.

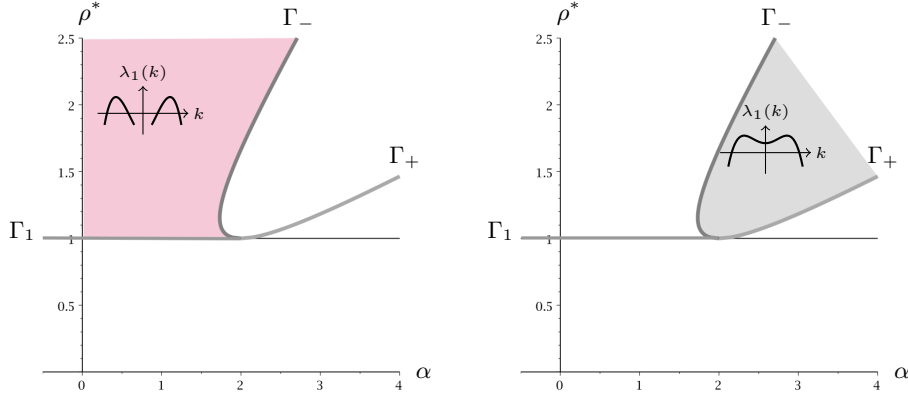


Figure 1.5 – Stability of constant solutions in the case of anomalous dispersion. In the (α, ρ^*) -plane, the shaded regions represent the values for which a constant solution with $\rho^* = |\psi^*|^2$ is Turing unstable (left plot) and zero-mode unstable (right plot). The insets show a typical shape of the largest eigenvalue $\lambda_1(k)$ in these instability regions.

We also define $\mathcal{R}_2(\cdot, \cdot, \mu)$ as the bilinear map through

$$\mathcal{R}_2(U_1, U_2, \mu) = \begin{pmatrix} \psi_{r\mu}^* (3u_1 u_2 + v_1 v_2) + \psi_{i\mu}^* (u_1 v_1 + u_2 v_2) \\ \psi_{i\mu}^* (u_1 u_2 + 3v_1 v_2) + \psi_{r\mu}^* (u_1 v_1 + u_2 v_2) \end{pmatrix},$$

for $U_j = (u_j, v_j)^T$, $j = 1, 2$, and \mathcal{R}_3 is a trilinear map satisfying

$$\mathcal{R}_3(U, U, U) = \begin{pmatrix} u(u^2 + v^2) \\ v(u^2 + v^2) \end{pmatrix},$$

for $U = (u, v)^T$.

For the Lugiato-Lefever equation (1.0.1), look for solutions of the Lugiato-Lefever equation (1.0.1) close to the branch of constant solutions ψ_μ^* of the form

$$(x, t) = \psi_\mu^* + (u + iv)(y, t), \quad y = \sqrt{2 - \alpha} x, \quad (1.2.1)$$

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with u and v real-valued, 2π -periodic functions in y . The resulting equation is a system for the couple $U = (u, v)^T$ of the form

$$\frac{dU}{dt} = \mathcal{A}_\mu^* U + \mathcal{F}(U, \mu), \quad (1.2.2)$$

in which the nonlinear terms $\mathcal{F}(U, \mu)$ are given by

$$\mathcal{F}(U, \mu) = \mathcal{J} (\mathcal{R}_2(U, U, \mu) + \mathcal{R}_3(U, U, U)),$$

As phase-space for the dynamical system (1.2.2) we choose the Hilbert space of 2π -periodic, square-integrable functions $\mathcal{X} = L^2(0, 2\pi) \times L^2(0, 2\pi)$ and we set $\mathcal{Y} = H^2(0, 2\pi) \times H^2(0, 2\pi)$.

Remark 1.2.1. *In the phase-space \mathcal{X} , the linear operator \mathcal{A}_μ^* is closed with domain $\mathcal{Y} = H^2(0, 2\pi) \times H^2(0, 2\pi)$, the linear operators \mathcal{J} and \mathcal{L}_μ^* are skew- and self-adjoint, respectively, and the nonlinear map $\mathcal{F}(\cdot, \mu)$ is smooth in \mathcal{Y} .*

The dynamical system (1.2.2) possesses one discrete and one continuous symmetry which will play an important role in our analysis.

Proposition 1.2.2. *The dynamical system (1.2.2) is equivariant under the action of the reflection operator \mathcal{T} and of the translation operators \mathcal{T}_a defined through*

$$(\mathcal{T}U)(y) = U(-y), \quad (\mathcal{T}_a U)(y) = U(y + a), \quad y \in \mathbb{R}, \quad (1.2.3)$$

i.e., both \mathcal{A}_μ^ and $\mathcal{F}(\cdot, \mu)$ commute with \mathcal{T} and \mathcal{T}_a , for any μ .*

Proof. As a consequence of the invariance of the Lugiato-Lefever equation (1.0.1) under the reflection $x \mapsto -x$ and under spatial translations $x \mapsto x + a$, $a \in \mathbb{R}$. \square

1.2.2 Center manifold reduction

In order to apply the center manifold theorem, we rewrite the dynamical system (1.2.2) in the form

$$\frac{dU}{dt} = \mathcal{A}_0^* U + \mathcal{G}(U, \mu), \quad (1.2.4)$$

with

$$\mathcal{G}(U, \mu) = \mathcal{J} (\mathcal{R}_1(U, \mu) + \mathcal{R}_2(U, U, \mu) + \mathcal{R}_3(U, U, U)),$$

where $\mathcal{R}_1(\cdot, \mu)$ is the linear map

$$\mathcal{R}_1(U, \mu) = (\mathcal{A}_\mu^* - \mathcal{A}_0^*)U,$$

and \mathcal{J} , \mathcal{R}_2 , \mathcal{R}_3 are defined as above.

We check the three hypotheses of the center manifold theorem [13, Chapter 2, Theorem 3.3].

Lemma 1.2.3. *We have that \mathcal{A}_0^* and \mathcal{G} in (1.2.4) satisfy the following properties:*

- (i) \mathcal{A}_0^* is in $\mathcal{L}(\mathcal{Y}, \mathcal{X})$;
- (ii) for some $k \geq 2$, there exists a neighborhoods $O_U \subset \mathcal{Y}$ and $I_\mu \subset \mathbb{R}$ containing 0 such that $\mathcal{G} \in C^k(O_U \times I_\mu; \mathcal{Y})$ and

$$\mathcal{G}(0, 0) = 0, \quad D_U \mathcal{G}(0, 0) = 0.$$

Lemma 1.2.4. *Consider the spectrum $\sigma(\mathcal{A}_0^*)$ of the operator \mathcal{A}_0^* . We have*

$$\sigma(\mathcal{A}_0^*) = \sigma_-(\mathcal{A}_0^*) \cup \sigma_0(\mathcal{A}_0^*)$$

where $\sigma_-(\mathcal{A}_0^*)$ and $\sigma_0(\mathcal{A}_0^*)$ have the following properties

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(i) there exists $\gamma > 0$ such that $\sup_{\lambda \in \sigma_-} \operatorname{Re}(\lambda) < -\gamma$

(ii) the set $\sigma_0(\mathcal{A}_0^*)$ consists of a finite number of eigenvalues with finite algebraic multiplicities.

Proof. The spectrum of \mathcal{A}_0^* is given by:

$$\sigma(\mathcal{A}_0^*) = \left\{ -1 \pm \sqrt{1 - (2 - \alpha)^2 (n^2 - 1)^2}, n \in \mathbb{Z} \right\}. \quad (1.2.5)$$

Indeed, by the equation (1.1.4), the spectrum of \mathcal{A}_0^* is given by

$$\begin{aligned} \sigma(\mathcal{A}_0^*) &= \{ \lambda \in \mathbb{C} \mid \lambda^2 + 2\lambda + a(\sqrt{2 - \alpha}n) = 0, n \in \mathbb{Z} \} \\ &= \{ \lambda_1(\sqrt{2 - \alpha}n), \lambda_2(\sqrt{2 - \alpha}n), n \in \mathbb{Z} \}, \end{aligned}$$

in which

$$\lambda_1(\sqrt{2 - \alpha}n) = -1 + \sqrt{1 - a(\sqrt{2 - \alpha}n)} \quad \text{and} \quad \lambda_2(\sqrt{2 - \alpha}n) = -1 - \sqrt{1 - a(\sqrt{2 - \alpha}n)},$$

where

$$\begin{aligned} a(\sqrt{2 - \alpha}n) &= (\sqrt{2 - \alpha}n)^4 - 2(2 - \alpha)(\sqrt{2 - \alpha}n)^2 + (2 - \alpha)^2 \\ &= ((\sqrt{2 - \alpha}n)^2 - (2 - \alpha))^2. \end{aligned}$$

Consequently, the spectrum of \mathcal{A}_0^* is given by

$$\begin{aligned} \sigma(\mathcal{A}_0^*) &= \{ \lambda_1(k), \lambda_2(k), k = \sqrt{2 - \alpha}n, n \in \mathbb{Z} \} \\ &= \left\{ -1 \pm \sqrt{1 - (2 - \alpha)^2 (n^2 - 1)^2}, n \in \mathbb{Z} \right\}. \end{aligned}$$

The eigenvalues of the operator \mathcal{A}_0^* have non-positive real parts. As the spectrum of \mathcal{A}_0^* is purely discrete, we can decompose it into stable and central spectrum

$$\sigma(\mathcal{A}_0^*) = \sigma_s(\mathcal{A}_0^*) \cup \sigma_c(\mathcal{A}_0^*), \quad (1.2.6)$$

where

$$\sigma_-(\mathcal{A}_0^*) \subset \{ \lambda \in \mathbb{C}, \operatorname{Re}(\lambda) < -\gamma \}, \quad \sigma_0(\mathcal{A}_0^*) = \{0\}, \quad \text{and} \quad \gamma > 0.$$

Here, 0 is a semi-simple eigenvalue with associated eigenvectors :

$$\zeta = \begin{pmatrix} \alpha \\ 2 - \alpha \end{pmatrix} e^{iy} \quad \text{and} \quad \bar{\zeta} = \begin{pmatrix} \alpha \\ 2 - \alpha \end{pmatrix} e^{-iy}.$$

□

We define the projection P_0 on \mathcal{X} corresponding to the central spectrum $\sigma_0(\mathcal{A}_0^*)$ and we define a second projection P_h on \mathcal{X} by $P_h := I - P_0$. We have that $P_0 : \mathcal{X} \rightarrow \mathcal{Y}$ and $P_h : \mathcal{X} \rightarrow \mathcal{Y}$. We set $\mathcal{E}_0 := P_0(\mathcal{X})$, $\mathcal{X}_h := P_h(\mathcal{X})$ and $\mathcal{Y}_h := P_h(\mathcal{Y})$.

Lemma 1.2.5. *Assume that \mathcal{A}_0^* and \mathcal{G} defined in (1.2.4) satisfy assumptions of Lemma 1.2.3 and 1.2.4. We define $C_\eta(\mathbb{R}, \mathcal{Y}) := \{ u \in C^0(\mathbb{R}, \mathcal{Y}); \|u\|_{C_\eta} := \sup_{t \in \mathbb{R}} (e^{-\eta|t|} \|u\|_{\mathcal{Y}}) < \infty \}$. Then, for any $\eta \in (0, \gamma)$ and any f in $C_\eta(\mathbb{R}, \mathcal{Y}_h)$, the linear problem*

$$\frac{du_h}{dt} = \mathcal{A}_{|x_h}^* u_h + f(t) \quad (1.2.7)$$

has a unique solution $u_h = K_h f \in C_\eta(\mathbb{R}, \mathcal{Y}_h)$. Furthermore, the linear map K_h belongs to $\mathcal{L}(C_\eta(\mathbb{R}, \mathcal{Y}_h))$ and there exists a continuous map $q : [0, \gamma] \rightarrow \mathbb{R}$ such that

$$\|K_h\|_{\mathcal{L}(C_\eta(\mathbb{R}, \mathcal{Y}_h))} \leq q(\eta).$$

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With Lemmas 1.2.3, 1.2.4 and 1.2.5, we apply the center manifold theorem [13, Chapter 2, Theorem 3.3] and we conclude that

Theorem 1. *There exists a map Ψ of class C^k , where $k \geq 2$, with*

$$\Psi(0) = 0, \quad D\Psi(0) = 0,$$

and a neighborhood \mathcal{O} of 0 in \mathcal{Y} such that the manifold

$$\mathcal{M}_c(\mu) = \{U \in \mathcal{Y} : U = A\zeta + \overline{A}\overline{\zeta} + \Psi(A, \overline{A}, \mu), A \in \mathbb{C}\},$$

has the following properties :

- (i) $\mathcal{M}_c(\mu)$ is locally invariant, i.e., if U is a solution of (1.2.4) satisfying $U(0) \in \mathcal{M}_c(\mu) \cap \mathcal{O}$ and $U(t) \in \mathcal{O}$ for all $t \in [0, T]$, then $U(t) \in \mathcal{M}_c(\mu)$ for all $t \in [0, T]$.
- (ii) $\mathcal{M}_c(\mu)$ contains the set of bounded solutions of (1.2.4) staying in \mathcal{O} for all $t \in \mathbb{R}$, i.e., if U is a solution of (1.2.4) satisfying $u(t) \in \mathcal{O}$ for all $t \in \mathbb{R}$, then $U(0) \in \mathcal{M}_c(\mu)$.

1.2.3 Reduced equation

The dynamics on the center manifold is governed by the reduced equation

$$\frac{dA}{dt} = f(A, \overline{A}, \mu), \tag{1.2.8}$$

in which f is a complex-valued map obtained by inserting the Ansatz

$$U = A\zeta + \overline{A}\overline{\zeta} + \Psi(A, \overline{A}, \mu),$$

into the dynamical system (1.2.4) and then projecting on the eigenvector ζ .

This reduced equation captures the qualitative changes which occur in the dynamics of the full system (1.2.4) at the bifurcation point $\mu = 0$.

We summarize in the lemma below the properties of the reduced vector field f which are needed in our bifurcation analysis.

Lemma 1.2.6. *For any μ sufficiently small, the vector field f in (1.2.8) has the following properties.*

- (i) *The complex-valued map f is of class C^k in a neighborhood of 0 in $\mathbb{C} \times \mathbb{C} \times \mathbb{R}$, for any arbitrary but fixed $k \geq 2$.*
- (ii) *There exists a real-valued map g of class C^{k-1} defined in a neighborhood of 0 in \mathbb{R}^2 such that*

$$f(A, \overline{A}, \mu) = Ag(|A|^2, \mu),$$

for any A sufficiently small.

- (iii) *The coefficients of the leading order terms in the Taylor expansion of f ,*

$$f(A, \overline{A}, \mu) = c_{11}A\mu + c_{30}A|A|^2 + \mathcal{O}(|A|(|\mu|^2 + |A|^4)), \tag{1.2.9}$$

are given by

$$c_{11} = \frac{1}{(2 - \alpha)^2}, \quad c_{30} = \frac{4F_1^2(30\alpha - 41)}{9(2 - \alpha)^2}.$$

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Proof. (i) This property follows from the fact that the map Ψ provided by the center manifold theorem is of class C^k .

(ii) Recall that the dynamical system (1.2.2) is equivariant under the action of the operators \mathcal{T} and \mathcal{T}_a , $a \in \mathbb{R}$. According to [13, Section 2.3.3], these equivariences are inherited by the reduced equation (1.2.8), so that f satisfies

$$\bar{f}(A, \bar{A}, \mu) = f(\bar{A}, A, \mu), \quad f(e^{ia}A, e^{-ia}\bar{A}, \mu) = e^{ia}f(A, \bar{A}, \mu),$$

for any $a \in \mathbb{R}$ and A sufficiently small. Applying the result in [13, Chapter 1, Lemma 2.4], the second equality above implies that f is of the form (1.2.9) with g a complex-valued function, and then from the first equality above we conclude that g is a real-valued function. This proves the second property.

(iii) We compute the coefficient c_{11} from the equality

$$\frac{\partial f}{\partial A}(0, 0, \mu) = \lambda_1(\mu), \tag{1.2.10}$$

in which $\lambda_1(\mu)$ is the continuation of the eigenvalue 0 of the operator \mathcal{A}^{*0} for small μ , i.e., $\lambda_1(\mu)$ is the largest eigenvalue of $\mathcal{A}^{*\mu}$ (e.g., see [13, Chapter 2, Exercise 3.5]). According to the linear stability analysis in Section 1.1, $\lambda_1(\mu)$ is a double eigenvalue and it is the largest root of the polynomial

$$\lambda^2 + 2\lambda + a_\mu(\sqrt{2-\alpha}) = 0, \tag{1.2.11}$$

in which

$$a_\mu(\sqrt{2-\alpha}) = (2-\alpha)^2 + 2(\alpha - 2\rho_\mu^*)(2-\alpha) + \alpha^2 - 4\alpha\rho_\mu^* + 3\rho_\mu^{*2} + 1 = 3\rho_\mu^{*2} - 8\rho_\mu^* + 5.$$

Differentiating (1.2.10) with respect to μ and taking $\mu = 0$ we find

$$c_{11} = \frac{d\lambda_1}{d\mu}(0).$$

The right hand side in this equation can be computed from (A.1.1), and after some elementary calculations we find the formula of the coefficient c_{11} in (iii).

For the computation of the coefficient c_{30} , we may set $\mu = 0$ in the following calculations. Inserting the Taylor expansion of the reduction function Ψ ,

$$\Psi(A, \bar{A}, 0) = \Psi_{20}A^2 + \Psi_{11}A\bar{A} + \bar{\Psi}_{20}\bar{A}^2 + \Psi_{30}A^3 + \Psi_{21}A^2\bar{A} + \bar{\Psi}_{21}A\bar{A}^2 + \bar{\Psi}_{30}\bar{A}^3 + \mathcal{O}(|A|^4),$$

in the system (1.2.4), for $\mu = 0$, and collecting successively the terms of orders $\mathcal{O}(A^2)$, $\mathcal{O}(A\bar{A})$, and $\mathcal{O}(A^2\bar{A})$ we obtain the equalities

$$\begin{aligned} 0 &= \mathcal{A}_0^*\Psi_{20} + \mathcal{J}\mathcal{R}_2(\zeta, \zeta, 0), \\ 0 &= \mathcal{A}_0^*\Psi_{11} + 2\mathcal{J}\mathcal{R}_2(\zeta, \bar{\zeta}, 0), \\ c_{30}\zeta &= \mathcal{A}_0^*\Psi_{21} + \mathcal{J}\left(2\mathcal{R}_2(\zeta, \Psi_{11}, 0) + 2\mathcal{R}_2(\bar{\zeta}, \Psi_{20}, 0) + 3\mathcal{R}_3(\zeta, \zeta, \bar{\zeta})\right). \end{aligned}$$

Solving the first two linear equations we obtain

$$\Psi_{20} = \frac{2F_1}{9(2-\alpha)^2} \begin{pmatrix} 6+4\alpha-3\alpha^2 \\ 2(5-3\alpha)(2-\alpha) \end{pmatrix} e^{2iy}, \quad \Psi_{11} = \frac{4F_1}{(2-\alpha)^2} \begin{pmatrix} \alpha^2-2 \\ -2(1-\alpha)(2-\alpha) \end{pmatrix},$$

and by taking the scalar product of the third equality with an eigenvector ζ^{ad} in the kernel of the adjoint operator $(\mathcal{A}_0^*)^{ad}$ satisfying $\langle \zeta, \zeta^{ad} \rangle \neq 0$, we obtain

$$c_{30} = \frac{1}{\langle \zeta, \zeta^{ad} \rangle} \left\langle \mathcal{J}\left(2\mathcal{R}_2(\zeta, \Psi_{11}) + 2\mathcal{R}_2(\bar{\zeta}, \Psi_{20}) + 3\mathcal{R}_3(\zeta, \zeta, \bar{\zeta})\right), \zeta^{ad} \right\rangle.$$

We may slightly simplify the computation of the coefficient c_{30} by taking $\zeta^{ad} = \mathcal{J}\zeta_2$, so that

$$c_{30} = \frac{1}{\langle \zeta, \mathcal{J}\zeta_2 \rangle} \left\langle 2\mathcal{R}_2(\zeta, \Psi_{11}) + 2\mathcal{R}_2(\bar{\zeta}, \Psi_{20}) + 3\mathcal{R}_3(\zeta, \zeta, \bar{\zeta}), \zeta_2 \right\rangle,$$

and observing that

$$\mathcal{A}_{*0}\zeta_2 = -2\zeta_2. \quad (1.2.12)$$

Indeed, since \mathcal{J} and \mathcal{L}_0^* are skew- and self-adjoint operators, respectively, and $\mathcal{J}^2 = -\mathbb{I}$, we have that

$$0 = (\mathcal{A}_0^*)^{ad}(\mathcal{J}\zeta_2) = -(\mathbb{I} + \mathcal{L}_0^*\mathcal{J})(\mathcal{J}\zeta_2) = -\mathcal{J}(\mathbb{I} + \mathcal{J}\mathcal{L}_0^*)\zeta_2 = -\mathcal{J}(2\mathbb{I} + \mathcal{A}_0^*)\zeta_2,$$

which implies (1.2.12). Then a direct calculation gives

$$\zeta_2 = \begin{pmatrix} \alpha - 2 \\ \alpha \end{pmatrix} e^{iy},$$

which together with the expressions of ζ , Ψ_{11} , and Ψ_{20} gives the formula for c_{30} in (iii), and completes the proof of the lemma. \blacksquare

1.2.4 Steady bifurcation with $O(2)$ symmetry

The properties of the reduced vector field f in Lemma 1.2.6 show that for the reduced system (1.2.8) a steady bifurcation with $O(2)$ symmetry occurs at $\mu = 0$, provided $c_{30} \neq 0$ (e.g., see [13, Section 1.2.4]). This bifurcation can be easily studied using polar coordinates $A = re^{i\theta}$, in which the system decouples

$$\begin{aligned} \frac{dr}{dt} &= c_{11}\mu r + c_{30}r^3 + \mathcal{O}(r(|\mu|^2 + r^4)), \\ \frac{d\theta}{dt} &= 0. \end{aligned}$$

The second equation implies that the angle variable θ is always constant, and the first equation shows a pitchfork bifurcation for the radial variable r . This bifurcation is supercritical when $c_{30} < 0$ and subcritical when $c_{30} > 0$. For the reduced system (1.2.8) we obtain the following result (see also Figure 2.1).

Proposition 1.2.7. *Consider the Lugiato-Lefever equation (1.0.1) in the case $\beta = -1$ of anomalous dispersion. Assume that $\alpha < 2$, $\alpha \neq 41/30$, and $F^2 = F_1^2 + \mu$. Then the reduced system (1.2.8) undergoes a steady bifurcation with $O(2)$ symmetry at $\mu = 0$. The bifurcation is supercritical when $\alpha < 41/30$, and subcritical when $\alpha > 41/30$. More precisely, the following properties hold in a neighborhood of 0 in \mathbb{C} , for μ sufficiently small.*

- (i) *If $\alpha < 41/30$ and $\mu < 0$, then the reduced equation possesses a unique equilibrium $A = 0$ which is stable.*
- (ii) *If $\alpha < 41/30$ and $\mu > 0$, then the reduced equation possesses the unstable equilibrium $A = 0$ and a circle of stable equilibria $A_\mu(\phi) = r_\mu e^{i\phi}$, for $\phi \in \mathbb{R}/2\pi\mathbb{Z}$, which surrounds this equilibrium.*
- (iii) *If $\alpha > 41/30$ and $\mu < 0$, then the reduced equation possesses the stable equilibrium $A = 0$ and a circle of unstable equilibria $A_\mu(\phi) = r_\mu e^{i\phi}$, for $\phi \in \mathbb{R}/2\pi\mathbb{Z}$, which surrounds this equilibrium.*
- (iv) *If $\alpha > 41/30$ and $\mu > 0$, then the reduced equation possesses a unique equilibrium $A = 0$ which is unstable.*

1.2. BIFURCATIONS OF STEADY PERIODIC SOLUTIONS AT THE ONSET OF TURING IN
THE CASE OF ANOMALOUS DISPERSION ($\beta = -1$)

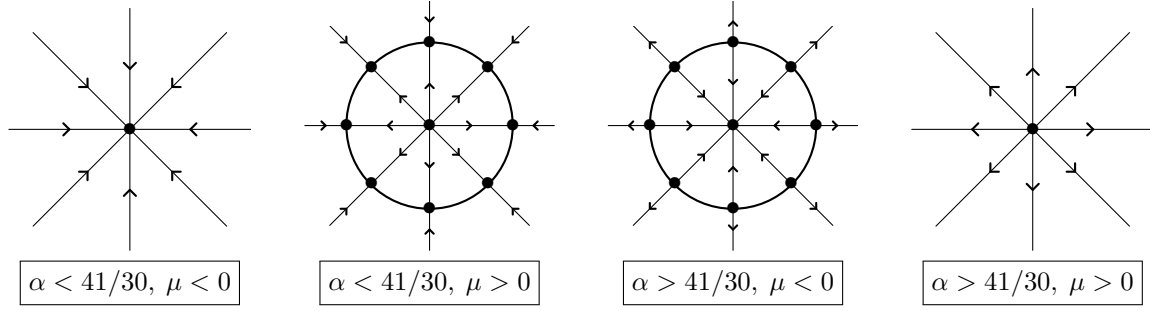


Figure 2.1 – Phase portraits in the A -plane of the reduced system (1.2.8).

In the cases (ii) and (iii) we have

$$r_\mu = \frac{3}{2F_1|41 - 30\alpha|^{1/2}} |\mu|^{1/2} + \mathcal{O}(|\mu|).$$

This proposition is obtained by applying theorem [13, Chapter 1, Theorem 2.18].

Theorem 2. In cases $\alpha < 41/30$ and $\mu > 0$ or $\alpha > 41/30$ and $\mu < 0$, the Lugiato-Lefever equation (1.0.1) possesses a family of periodic stationary solutions in x , in which there is an even periodic solution with Taylor expansion

$$\mu(x) = \begin{matrix} * \\ 0 \end{matrix} + \frac{3(\alpha + i(2 - \alpha))}{F_1|41 - 30\alpha|^{1/2}} \cos(\sqrt{2 - \alpha} x) |\mu|^{1/2} + \mathcal{O}(|\mu|), \quad (1.2.13)$$

in $H_{per}^k(\mathbb{T}_N)$ -norm. Other solutions are obtained as translations in x of this even periodic solution.

Proof. Going back to the Lugiato-Lefever equation (1.0.1), the equilibrium $A = 0$ of the reduced equation gives the constant solution ψ_μ^* , whereas the circle of nontrivial equilibria $A_\mu(\phi) = r_\mu e^{i\phi}$ corresponds to a family of periodic stationary solutions in x . The positive solution $A_\mu(0) = r_\mu$ gives an even periodic solution of the Lugiato-Lefever equation (1.0.1), with Taylor expansion (1.2.13) and, since the rotation invariance of the reduced system (1.2.8) is inherited from the translation invariance of the Lugiato-Lefever equation (1.0.1), the other equilibria on the circle correspond to translations in x of this even periodic solution. \square

Remark 1.2.8. The local bifurcation in Proposition 1.2.7 is known in the physics literature as a pitchfork bifurcation, which is indeed the bifurcation obtained when restricting this bifurcation analysis to solutions of the Lugiato-Lefever equation which are even in x . The case $\alpha = 41/30$, when the coefficient c_{30} vanishes, has been analyzed in [22].

Remark 1.2.9. The periodic stationary solutions found in this section have wavenumbers $\sqrt{2 - \alpha}$. This bifurcation analysis can be extended to include periodic solutions with wavenumbers k close to $\sqrt{2 - \alpha}$, by taking $y = kx$ in (1.2.1) and performing the reduction procedure with two parameters, k close to $\sqrt{2 - \alpha}$ and μ small. The resulting reduced system will have the same form (1.2.8), with the difference that the vector field f will depend upon the additional small parameter $\varepsilon = k - \sqrt{2 - \alpha}$. However the properties of f will be similar to the ones in Lemma 1.2.6 and the reduced dynamics can be studied in the same way.

Remark 1.2.10. The local bifurcations induced by the zero-mode instability of the constant solutions found in Section 1.1 can be analyzed using the same approach as in this section. Focusing on parameter

values close to zero-mode instability, the Lugiato-Lefever equation is written as a dynamical system of the form (1.2.2) in the same way, and we may take as phase space $L^2(0, L) \times L^2(0, L)$, for any arbitrary, but fixed L , hence considering functions which are L -periodic in the spatial variable x . With this choice, the corresponding linear operator has purely point spectrum, again, which lies in the open left half complex plane, except for 0 which is now a simple eigenvalue. Applying the center manifold theorem, we obtain a one-dimensional manifold on which the dynamics is governed by a scalar ordinary differential equation. However, since the eigenvector associated with the simple eigenvalue 0 is a constant function, and since the subspace of constant functions is invariant for the dynamics of (1.2.2), the center manifold lies in this subspace and the only bifurcating solutions found in this way are the constant solutions of (1.0.1). We point out that in this approach we cannot replace the phase space $L^2(0, L) \times L^2(0, L)$ by a space of functions defined on the real line, e.g., $L^2(\mathbb{R}) \times L^2(\mathbb{R})$, because with such a choice the linear operator \mathcal{A}_0^* has continuous spectrum and no spectral gap allowing to use the center manifold reduction.

1.3 Bifurcations of steady periodic solutions in the Turing and zero-mode instability regions in the case of anomalous dispersion ($\beta = -1$)

In this section we study the bifurcations of steady periodic solutions which occur in the Turing and zero-mode instability regions defined in Section 1.1.

1.3.1 Bifurcation analysis

We assume that the instabilities are fully developed, so that the boundaries Γ_1 , Γ_+ , and Γ_- are excluded from the analysis. We follow the approach used in Section 1.2 for the analysis of the onset of Turing instability, i.e., for parameter values $(\alpha, \rho^*) \in \Gamma_1$.

Dynamical system and choice of parameters

For (α, ρ^*) in either the Turing instability region, or the zero-mode instability region, we denote by F_* the corresponding value of the parameter F given by (1.1.2), and take F as bifurcation parameter by setting $F^2 = F_*^2 + \mu$, with small μ .

In this section, the linearized operator around a constant solution ψ_μ^* of Lugiato-Lefever (1.0.1) is $\mathcal{A}_\mu^* = -\mathbb{I} + \mathcal{J}\mathcal{L}_\mu^*$ with

$$\mathcal{L}_\mu^* = \begin{pmatrix} k_*^2 \partial_y^2 - \alpha + 3\psi_{r\mu}^{*2} + \psi_{i\mu}^{*2} & 2\psi_{r\mu}^* \psi_{i\mu}^* \\ 2\psi_{r\mu}^* \psi_{i\mu}^* & k_*^2 \partial_y^2 - \alpha + \psi_{r\mu}^{*2} + 3\psi_{i\mu}^{*2} \end{pmatrix}.$$

For the Lugiato-Lefever equation (1.0.1), we look for spatially periodic solutions with wavelength $k_* > 0$ of the form

$$(x, t) = \psi_\mu^* + (u + iv)(y, t), \quad y = k_* x, \quad (1.3.1)$$

where $\psi_\mu^* = \psi_{r\mu}^* + i\psi_{i\mu}^*$ is the constant solution given by (1.1.2) for $F^2 = F_*^2 + \mu$, and the functions u and v are real-valued and 2π -periodic in y . Inserting (1.3.1) into (1.0.1) we obtain a system of the form (1.2.2)

$$\frac{dU}{dt} = \mathcal{A}_\mu^* U + \mathcal{F}(U, \mu),$$

for $U = (u, v)^T$ and $\mathcal{F}(U, \mu)$ is a nonlinear map,

$$\mathcal{F}(U, \mu) = \mathcal{J}(\mathcal{R}_2(U, U, \mu) + \mathcal{R}_3(U, U, U)).$$

We recall that the operator \mathcal{A}_μ^* is a closed linear operator with domain \mathcal{Y} . Moreover, \mathcal{J} and \mathcal{L}_μ^* are respectively skew- and self-adjoint operators and $\mathcal{F}(\cdot, \mu) : \mathcal{Y} \rightarrow \mathcal{Y}$ is a smooth map.

1.3. BIFURCATIONS OF STEADY PERIODIC SOLUTIONS IN THE TURING AND ZERO-MODE INSTABILITY REGIONS IN THE CASE OF ANOMALOUS DISPERSION ($\beta = -1$)

In this setting, we look for wavenumbers k_* such that $\mu = 0$ is a bifurcation point for the dynamical system (1.2.2). These bifurcation points are determined by the spectrum of the linear operator \mathcal{A}_μ^* at $\mu = 0$.

We define the quantities k_{\max} and k_{\min} by

$$k_{\max}^2 = 2\rho^* - \alpha + \sqrt{\rho^{*2} - 1} \quad (1.3.2)$$

and

$$k_{\min}^2 = 2\rho^* - \alpha - \sqrt{\rho^{*2} - 1}. \quad (1.3.3)$$

Proposition 1.3.1. *Let k_* be k_{\max} or k_{\min} . The spectrum of the operator \mathcal{A}_0^* is given by*

$$\sigma(\mathcal{A}_0^*) = \{\lambda_1(nk_*), \lambda_2(nk_*), n \in \mathbb{Z}\},$$

where $\lambda_1(nk_*)$ and $\lambda_2(nk_*)$ are the two roots of the quadratic polynomial

$$\lambda^2 + 2\lambda + a(nk_*) = 0,$$

where a is defined by (1.1.5).

Moreover, for (α, ρ^*) in the Turing instability region, $k_{\min} < k_{\max}$ are the only two values satisfying $\lambda_1(k_{\min}) = \lambda_1(k_{\max}) = 0$, whereas in the zero-mode instability region, k_{\max} is the only positive value such that $\lambda_1(k_{\max}) = 0$.

Proof. Since the domain \mathcal{Y} of \mathcal{A}_μ^* is compactly embedded in \mathcal{X} , the operator has compact resolvent and therefore pure point spectrum consisting of isolated eigenvalues. Consequently, $\mu = 0$ is a bifurcation point if the operator \mathcal{A}_0^* possesses purely imaginary eigenvalues. Upon comparing \mathcal{A}_0^* with the operator \mathcal{A}^* given by (1.1.3) in Section 1.1, and taking into account the fact that the phase-space \mathcal{X} consists of 2π -periodic functions, we find that the eigenvalues of \mathcal{A}_0^* can be computed from the formula in Proposition 1.1.3 by taking $k = nk_*$, $n \in \mathbb{Z}$, so that

$$\sigma(\mathcal{A}_0^*) = \{\lambda_1(nk_*), \lambda_2(nk_*), n \in \mathbb{Z}\}.$$

With the notations from Section 1.1, $\operatorname{Re}(\lambda_1(k)) > -1$ if and only if $\lambda_1(k)$ is real, so that $\mu = 0$ is a bifurcation point when $\lambda_1(nk_*) = 0$, for some $n \in \mathbb{Z}$. Without loss of generality we may assume $n = 1$, otherwise we may replace k_* by nk_* . Taking into account the shape of $\lambda_1(k)$ (see the insets in Figure 1.5), for (α, ρ^*) in the Turing instability region there are precisely two positive values $k_1 < k_2$ such that $\lambda_1(k_1) = \lambda_1(k_2) = 0$, whereas in the zero-mode instability region there is only one positive value k_2 such that $\lambda_1(k_2) = 0$. We can compute these values from the condition $a(k) = 0$, which is equivalent to $\lambda_1(k) = 0$, and solving this equation, we find $k_1 = k_{\min}$ and $k_2 = k_{\max}$. \square

Notice that k_{\max}^2 vanishes precisely along the curve Γ_+ , whereas k_{\min}^2 vanishes along the curve Γ_- . In the next two subsections we study separately the bifurcations obtained for wavenumbers $k_* = k_{\max}$ and $k_* = k_{\min}$.

Periodic solutions with wavenumbers k_{\max}

For (α, ρ^*) fixed in either the Turing instability region, or the zero-mode instability region, we take $k_* = k_{\max}$.

Following the approach in Section 1.2, we rewrite the system (1.2.2) in the form (1.2.4)

$$\frac{dU}{dt} = \mathcal{A}_0^*U + \mathcal{G}(U, \mu),$$

in which

$$\mathcal{G}(U, \mu) = \mathcal{J}(\mathcal{R}_1(U, \mu) + \mathcal{R}_2(U, U, \mu) + \mathcal{R}_3(U, U, U)),$$

where $\mathcal{R}_1(\cdot, \mu) = \mathcal{L}_\mu^* - \mathcal{L}_0^*$ and $\mathcal{J}, \mathcal{R}_2, \mathcal{R}_3$ are defined as before.

We start by checking hypothesis in order to apply the center manifold theorem [13, Chapter 2, Theorem 3.3]

Lemma 1.3.2. *We have the spectral decomposition*

$$\sigma(\mathcal{A}_0^*) = \sigma_s(\mathcal{A}_0^*) \cup \sigma_c(\mathcal{A}_0^*) \cup \sigma_u(\mathcal{A}_0^*), \quad (1.3.4)$$

with

$$\sigma_c(\mathcal{A}_0^*) = \{0\}, \quad \sigma_s(\mathcal{A}_0^*) \cup \sigma_u(\mathcal{A}_0^*) \subset \{\lambda \in \mathbb{C}; |\operatorname{Re} \lambda| > \delta\},$$

for some $\delta > 0$, and $\sigma_u(\mathcal{A}_0^*) = \emptyset$ in the Turing instability region. Moreover, 0 is a double semi-simple eigenvalue with associated eigenvectors ζ and $\bar{\zeta}$,

$$\zeta = \begin{pmatrix} (\rho^* - \sqrt{\rho^{*2} - 1})(\alpha + \sqrt{\rho^{*2} - 1}) \\ 2\rho^* - \alpha - \sqrt{\rho^{*2} - 1} \end{pmatrix} e^{iy}. \quad (1.3.5)$$

Proof. We have $\lambda_1(k_{\max}) = 0$, so that 0 is an eigenvalue of \mathcal{A}_0^* , and the results in Section 1.1 imply that all other eigenvalues in the spectrum $\sigma(\mathcal{A}_0^*)$ are either negative, or have negative real part equal to -1 , in the Turing instability region, and that there exist an additional positive eigenvalue, with Fourier mode $n = 0$, in the zero-mode instability region. □

Lemma 1.3.2 allow us to apply the center manifold theorem [13, Chapter 2, Theorem 3.3]:

Theorem 3. *The dynamical system (1.2.4) possesses a two-dimensional center manifold,*

$$\mathcal{M}_c(\mu) = \{U \in \mathcal{Y}; U = A\zeta + \bar{A}\bar{\zeta} + \Psi(A, \bar{A}, \mu), A \in \mathbb{C}\},$$

which contains all sufficiently small bounded solutions of (1.2.4), for any μ sufficiently small. Here Ψ is a map of class C^k , for any arbitrary but fixed $k \geq 3$, defined in a neighborhood of 0 in $\mathbb{C} \times \bar{\mathbb{C}} \times \mathbb{R}$, where $\mathbb{C} \times \bar{\mathbb{C}} = \{(A, \bar{A}); A \in \mathbb{C}\}$, and taking values in the spectral subspace \mathcal{X}_h associated to the union of the stable and unstable spectra $\sigma_s(\mathcal{A}_0^*) \cup \sigma_u(\mathcal{A}_0^*)$ of the operator \mathcal{A}_0^* .

The dynamics on the center manifold $\mathcal{M}_c(\mu)$ is described by the reduced equation (1.2.8)

$$\frac{dA}{dt} = f(A, \bar{A}, \mu),$$

in which f is a complex-valued map obtained by inserting the Ansatz

$$U = A\zeta + \bar{A}\bar{\zeta} + \Psi(A, \bar{A}, \mu),$$

into the dynamical system (1.2.4) and then projecting on the eigenvector ζ .

Lemma 1.3.3. *The vector field f has the Taylor expansion (1.2.9)*

$$f(A, \bar{A}, \mu) = c_{11}\mu A + c_{30}A|A|^2 + \mathcal{O}(|A|(|\mu|^2 + |A|^4)).$$

The signs of the two real coefficients c_{11} and c_{30} determine the dynamics on the center manifold. Assuming that both coefficients c_{11} and c_{30} are nonzero, the reduced system (1.2.8) undergoes a steady bifurcation with $O(2)$ symmetry at $\mu = 0$, which is supercritical when $c_{30} < 0$ and subcritical when $c_{30} > 0$. More precisely, we have the following properties (see also [13, Section 1.2.4]):

1.3. BIFURCATIONS OF STEADY PERIODIC SOLUTIONS IN THE TURING AND ZERO-MODE INSTABILITY REGIONS IN THE CASE OF ANOMALOUS DISPERSION ($\beta = -1$)

- (i) If $c_{30}c_{11}\mu > 0$, then the reduced equation possesses a unique equilibrium $A = 0$ which is stable when $c_{30} < 0$, and unstable when $c_{30} > 0$.
- (ii) If $c_{30}c_{11}\mu < 0$, then the reduced equation possesses the equilibrium $A = 0$ which is unstable when $c_{30} < 0$, and stable when $c_{30} > 0$, and a circle of nontrivial equilibria with opposite stability, $A_\mu(\phi) = r_\mu e^{i\phi}$, for $\phi \in \mathbb{R}/2\pi\mathbb{Z}$.

Proof. The symmetries (1.2.3) of the dynamical system (1.2.2) are inherited by the reduced system (1.2.8) which is $O(2)$ -equivariant (see section 1.2 for more details). As a consequence, the vector field f in (1.2.8) is of the form

$$f(A, \bar{A}, \mu) = Ag(|A|^2, \mu),$$

where g is a real-valued map of class C^{k-1} defined in a neighborhood of 0 in \mathbb{R}^2 , and has the Taylor expansion (1.2.9)

$$f(A, \bar{A}, \mu) = c_{11}\mu A + c_{30}A|A|^2 + \mathcal{O}(|A|(|\mu|^2 + |A|^4)).$$

In contrast to Section 1.2, the computation of these coefficients is much more involved here. We summarize these computations in this proof.

According to section 1.2, the coefficient c_{11} can be computed from the formula

$$c_{11} = \frac{d\lambda_*(\mu)}{d\mu}(0),$$

where $\lambda_*(\mu)$ is the eigenvalue of the operator \mathcal{A}_μ^* which is the continuation of the eigenvalue 0 of the operator \mathcal{A}_0^* , for small μ . The results in Section 1.1 imply that $\lambda_*(\mu)$ is the largest root of the polynomial

$$\lambda^2 + 2\lambda + a_\mu(k_*) = 0, \quad (1.3.6)$$

in which

$$a_\mu(k_*) = k_*^4 + 2(\alpha - 2\rho_\mu^*)k_*^2 + \alpha^2 - 4\alpha\rho_\mu^* + 3\rho_\mu^{*2} + 1.$$

Upon differentiating (A.1.1) with respect to μ and taking $\mu = 0$, after some elementary calculations, we obtain

$$c_{11} = \frac{1}{k_{\max}^2 k_{\min}^2} \left(\rho^* + 2\sqrt{\rho^{*2} - 1} \right), \quad (1.3.7)$$

According to section 1.2, the coefficient c_{30} can be computed from the formula

$$c_{30} = \frac{1}{\langle \zeta, \mathcal{J}\zeta_2 \rangle} \left\langle 2\mathcal{R}_2(\zeta, \Psi_{11}, 0) + 2\mathcal{R}_2(\bar{\zeta}, \Psi_{20}, 0) + 3\mathcal{R}_3(\zeta, \zeta, \bar{\zeta}), \zeta_2 \right\rangle,$$

in which ζ and ζ_2 are eigenvectors of \mathcal{A}_0^* associated to the eigenvalues 0 and -2 , respectively, and the vectors Ψ_{11} and Ψ_{20} are solutions of the linear equations:

$$\mathcal{A}_0^* \Psi_{11} = -2\mathcal{J}\mathcal{R}_2(\zeta, \bar{\zeta}, 0), \quad \mathcal{A}_0^* \Psi_{20} = -\mathcal{J}\mathcal{R}_2(\zeta, \zeta, 0).$$

The eigenvector ζ is given by (1.3.5), a direct calculation gives

$$\zeta_2 = \begin{pmatrix} (\rho^* - \sqrt{\rho^{*2} - 1})(2\rho^* - \alpha + \sqrt{\rho^{*2} - 1}) \\ -\alpha + \sqrt{\rho^{*2} - 1} \end{pmatrix} e^{iy}.$$

and a symbolic computation using Maple gives

$$\Psi_{11} = \begin{pmatrix} \frac{4F_*}{k_{\max}^2 k_{\min}^2} & u_{11} \\ & v_{11} \end{pmatrix},$$

$$\Psi_{20} = \frac{2F_*}{k_{\max}^2(9(2\rho^* - \alpha) + 15\sqrt{\rho^{*2} - 1})} \begin{pmatrix} u_{20} \\ v_{20} \end{pmatrix} e^{2iy},$$

in which

$$\begin{aligned} u_{11} &= (\rho^* - \sqrt{\rho^{*2} - 1})\alpha^2 + (2\rho^*\sqrt{\rho^{*2} - 1} - 2\rho^{*2} + 2)\alpha \\ &\quad - \rho^{*2}\sqrt{\rho^{*2} - 1} + \rho^{*3} + \sqrt{\rho^{*2} - 1} - 3\rho^*, \\ v_{11} &= (\rho^*\sqrt{\rho^{*2} - 1} - \rho^{*2} - 1)\alpha^2 \\ &\quad - 2(\rho^{*2}\sqrt{\rho^{*2} - 1} - \rho^{*3} + \sqrt{\rho^{*2} - 1} - 2\rho^*)\alpha \\ &\quad + \rho^{*3}\sqrt{\rho^{*2} - 1} - \rho^{*4} + 3\rho^*\sqrt{\rho^{*2} - 1} - 4\rho^{*2} + 1, \\ u_{20} &= 3(\sqrt{\rho^{*2} - 1} - \rho^*)\alpha^2 - 2(\rho^*\sqrt{\rho^{*2} - 1} - \rho^{*2} - 1)\alpha \\ &\quad - \rho^{*2}\sqrt{\rho^{*2} - 1} + \rho^{*3} + 5\sqrt{\rho^{*2} - 1} + 5\rho^*, \\ v_{20} &= -3(\rho^*\sqrt{\rho^{*2} - 1} - \rho^{*2} - 1)\alpha^2 \\ &\quad + 2(3\rho^{*2}\sqrt{\rho^{*2} - 1} - 3\rho^{*3} - \sqrt{\rho^{*2} - 1} - 8\rho^*)\alpha \\ &\quad - 3\rho^{*3}\sqrt{\rho^{*2} - 1} + 3\rho^{*4} + 3\rho^*\sqrt{\rho^{*2} - 1} + 12\rho^{*2} + 5, \end{aligned}$$

and then the formula for the coefficient c_{30} ,

$$c_{30} = \frac{2F_*^2(\rho^* - \sqrt{\rho^{*2} - 1})}{k_{\max}^2 k_{\min}^2(9(2\rho^* - \alpha) + 15\sqrt{\rho^{*2} - 1})} \tilde{c}_{30}, \quad (1.3.8)$$

where

$$\begin{aligned} \tilde{c}_{30} &= -27\sqrt{\rho^{*2} - 1}\alpha^3 + 3(4\rho^*\sqrt{\rho^{*2} - 1} - 25\rho^{*2} + 5)\alpha^2 \\ &\quad + (345\rho^{*2}\sqrt{\rho^{*2} - 1} + 438\rho^{*3} - 67\sqrt{\rho^{*2} - 1} - 236\rho^*)\alpha \\ &\quad - 330\rho^{*3}\sqrt{\rho^{*2} - 1} - 363\rho^{*4} + 144\rho^{*2} + 55. \end{aligned} \quad (1.3.9)$$

□

Proposition 1.3.4. *Consider the Lugiato-Lefever equation (1.0.1) in the case $\beta = -1$ of anomalous dispersion. Assume that ψ^* is a constant solution of (1.0.1) with square modulus ρ^* , for parameter values α and F_* . If (α, ρ^*) belongs to either the Turing instability region, or the zero-mode instability region, as defined in Section 1.1, and does not belong to the curve Γ_{30} , then for $F^2 = F_*^2 + \mu$ and $k_* = k_{\max}$ the dynamical system (1.2.2) undergoes a steady bifurcation with $O(2)$ symmetry at $\mu = 0$, and the following properties hold.*

- (i) *If (α, ρ^*) belongs to the open set situated to the left of the curve Γ_{30} in the Turing instability region, then the coefficients of the reduced equation (1.2.8) satisfy $c_{11} > 0$ and $c_{30} < 0$, so that the bifurcation is supercritical and the reduced equation possesses a circle of stable nontrivial equilibria, $A_\mu(\phi) = r_\mu e^{i\phi}$, $\phi \in \mathbb{R}/2\pi\mathbb{Z}$, for any sufficiently small $\mu > 0$.*
- (ii) *If (α, ρ^*) belongs to the open set situated to the right of the curve Γ_{30} in the Turing instability region, then the coefficients of the reduced equation (1.2.8) satisfy $c_{11} > 0$ and $c_{30} > 0$, so that the bifurcation is subcritical and the reduced equation possesses a circle of unstable nontrivial equilibria, $A_\mu(\phi) = r_\mu e^{i\phi}$, $\phi \in \mathbb{R}/2\pi\mathbb{Z}$, for any sufficiently small $\mu < 0$.*

1.3. BIFURCATIONS OF STEADY PERIODIC SOLUTIONS IN THE TURING AND ZERO-MODE INSTABILITY REGIONS IN THE CASE OF ANOMALOUS DISPERSION ($\beta = -1$)

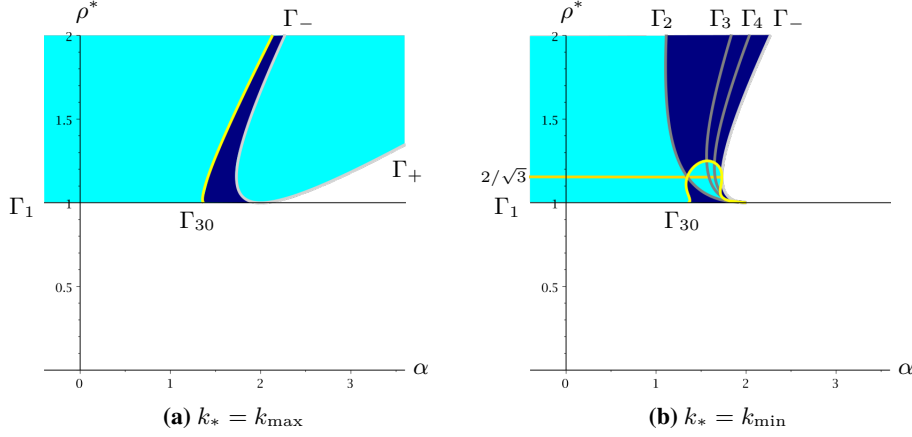


Figure 3.1 – Bifurcations of steady periodic solutions in the case of anomalous dispersion. In the (α, ρ^*) -plane, the bifurcation is supercritical ($c_{30} < 0$) in the lighter shaded regions (light blue online) and subcritical ($c_{30} > 0$) in the darker shaded regions (navy blue online). In the case $k_* = k_{\min}$, we also represent the curves Γ_n , for $n = 2, 3, 4$, along which resonances occur, and the line $\rho^* = 2/\sqrt{3}$ along which c_{11} changes sign.

(iii) If (α, ρ^*) belongs to the zero-mode instability region, then the coefficients of the reduced equation (1.2.8) satisfy $c_{11} < 0$ and $c_{30} < 0$, so that the bifurcation is supercritical and the reduced equation possesses a circle of stable nontrivial equilibria, $A_\mu(\phi) = r_\mu e^{i\phi}$, $\phi \in \mathbb{R}/2\pi\mathbb{Z}$, for any sufficiently small $\mu < 0$.

Proof. From the formulas (A.1.2) and (A.2.1) we conclude that both coefficients change sign when crossing the boundary Γ_- between the two instability regions, where k_{\min}^2 vanishes, and in addition c_{30} changes sign when crossing the curve Γ_{30} with cartesian equation $\tilde{c}_{30} = 0$, where \tilde{c}_{30} is given by (A.2.2). The curve Γ_{30} lies inside the Turing instability region, starting at the point $(\alpha, \rho^*) = (41/30, 1)$ and being asymptotic to Γ_- , as $\alpha \rightarrow \infty$ (see Figure 3.1a). Taking into account the signs of the coefficients c_{11} and c_{30} in the different parameter regions, from the properties (i) and (ii) of Lemma 1.3.3 above we obtain the result. \square

Theorem 4. Going back to the Lugiato-Lefever equation (1.0.1), the circle of nontrivial equilibria $A_\mu(\phi) = r_\mu e^{i\phi}$ corresponds to a family of steady periodic solutions in x with wavelengths k_{\max} . The positive equilibrium $A_\mu(0) = r_\mu$ gives an even periodic solution of the Lugiato-Lefever equation (1.0.1), with Taylor expansion (3.2.7)

$$A_\mu(x) = r_\mu + \frac{1}{2} \cos(k_{\max} x) |\mu|^{1/2} + \mathcal{O}(|\mu|),$$

in which

$$r_\mu = \left((\rho^* - \sqrt{\rho^{*2} - 1})(\alpha + \sqrt{\rho^{*2} - 1}) + i(2\rho^* - \alpha - \sqrt{\rho^{*2} - 1}) \right) \left| \frac{c_{11}}{c_{30}} \right|^{1/2},$$

whereas the other equilibria on the circle correspond to translations in x of this even periodic solution. Notice that this latter property is a consequence of the translation invariance of the Lugiato-Lefever equation (1.0.1). This invariance is inherited by the reduced system (1.2.8) which is equivariant under rotations of A .

1.3.2 Periodic solutions with wavelengths k_{\min}

We consider now (α, ρ^*) in the Turing instability region and take $k_* = k_{\min}$. In this case, 0 is an eigenvalue of \mathcal{A}_0^* , but in contrast to the case $k_* = k_{\max}$ above, its algebraic multiplicity is not always two.

We have $k_{\min} < k_{\max}$ thus we have three cases for the decomposition of the spectrum of \mathcal{A}_0^* :

- (i) if there exists $n \geq 2$ such that $k_{\max} = nk_{\min}$, then 0 is a quadruple eigenvalue of \mathcal{A}_0^* and $\sigma(\mathcal{A}_0^*) \subset \{\lambda \in \mathbb{C}; \operatorname{Re}(\lambda) < 0\} \cup \{0\} \cup \{\lambda_1(jk_{\min}), \lambda_1(-jk_{\min}), j \in \llbracket 2; n-1 \rrbracket\}$;
- (ii) if for all $n \geq 2$, $k_{\max} \neq nk_{\min}$ and there exists $n \geq 2$ such that $k_{\max} > nk_{\min}$, then 0 is double eigenvalue of \mathcal{A}_0^* and $\sigma(\mathcal{A}_0^*) \subset \{\lambda \in \mathbb{C}; \operatorname{Re}(\lambda) < 0\} \cup \{0\} \cup \{\lambda_1(jk_{\min}), \lambda_1(-jk_{\min}), j \in \llbracket 2; n \rrbracket\}$;
- (iii) if for all $n \geq 2$, $k_{\max} < nk_{\min}$, then 0 is double eigenvalue of \mathcal{A}_0^* and $\sigma(\mathcal{A}_0^*) \subset \{\lambda \in \mathbb{C}; \operatorname{Re}(\lambda) < 0\} \cup \{0\}$ (there is no unstable spectrum).

In the case (i), resonances occur for parameter values for which the two values $k_{\min} < k_{\max}$ are such that $k_{\max} = nk_{\min}$, for some $n \in \mathbb{N}$, $n \geq 2$. Then $\lambda_1(nk_{\min}) = 0$ and 0 is a quadruple eigenvalue of \mathcal{A}_0^* , with two additional eigenvectors in the Fourier mode n . We exclude these resonances from our analysis, since they lead to a four-dimensional, instead of a two-dimensional, reduced system which requires a different bifurcation analysis. For $n \in \mathbb{N}$, $n \geq 2$, we define Γ_n the curve with cartesian equation

$$\alpha = 2\rho^* - \frac{n^2 + 1}{n^2 - 1} \sqrt{\rho^{*2} - 1}. \quad (1.3.10)$$

Remark 1.3.5. A direct calculation shows that a resonance $k_{\max} = nk_{\min}$, $n \geq 2$, occurs when

$$\alpha = 2\rho^* - \frac{n^2 + 1}{n^2 - 1} \sqrt{\rho^{*2} - 1}. \quad (1.3.11)$$

This equality is the cartesian equation of a curve Γ_n in the (α, ρ^*) -plane, which starts at $(\alpha, \rho^*) = (2, 1)$ and lies in the Turing instability region. The curves Γ_2 , Γ_3 , and Γ_4 are represented in Figure 3.1b. As n increases, the curves Γ_n move to the right in the (α, ρ^*) -plane and accumulate at Γ_- , as $n \rightarrow \infty$.

Assuming that (α, ρ^*) does not belong to any of the curves Γ_n , $n \geq 2$, the eigenvalue 0 of \mathcal{A}_0^* is double and semi-simple with associated eigenvectors ζ and $\bar{\zeta}$ given by

$$\zeta = \begin{pmatrix} (\rho^* + \sqrt{\rho^{*2} - 1})(\alpha - \sqrt{\rho^{*2} - 1}) \\ 2\rho^* - \alpha + \sqrt{\rho^{*2} - 1} \end{pmatrix} e^{iy}. \quad (1.3.12)$$

Furthermore, the spectral decomposition (1.3.4) holds, and $\sigma_u(\mathcal{A}_0^*)$ is empty for (α, ρ^*) in the open set situated to the left of the curve Γ_2 , and above Γ_1 , and contains precisely $n - 1$ positive eigenvalues, $\lambda_1(jk_{\min})$, $j = 2, \dots, n$, for (α, ρ^*) in the open set between two consecutive curves Γ_n and Γ_{n+1} , $n \geq 2$.

Following the arguments used in the case $k_* = k_{\max}$, we obtain a reduced system of the same form (1.2.8), in which the coefficients c_{11} and c_{30} in the Taylor expansion (1.2.9) of f are given by the equalities (A.1.3) and (A.2.3), respectively, in Appendix A. In this case, the coefficient c_{11} vanishes when $\rho^* = 2/\sqrt{3}$, the coefficient c_{30} is not defined along the curve Γ_2 and vanishes along the curve Γ_{30} with cartesian equation $\tilde{c}_{30} = 0$, where \tilde{c}_{30} is given by (A.2.4), and both coefficients change sign when crossing one of the curves where they vanish or are not defined. It is now straightforward to obtain the following result, similar to the one in Theorem 1.3.4.

Proposition 1.3.6. Consider the Lugiato-Lefever equation (1.0.1) in the case $\beta = -1$ of anomalous dispersion. Assume that ψ^* is a constant solution of (1.0.1) with square modulus ρ^* , for parameter values α and F_* . If (α, ρ^*) belongs to the Turing instability region and does not belong to any of the curves Γ_n , $n \geq 2$, Γ_{30} , and $\rho^* = 2/\sqrt{3}$, then for $F^2 = F_*^2 + \mu$ and $k_* = k_{\min}$ the dynamical system (1.2.2) undergoes a steady bifurcation with $O(2)$ symmetry at $\mu = 0$, and the following properties hold.

- (i) If (α, ρ^*) belongs to the lighter shaded regions in Figure 3.1b and $\rho^* < 2/\sqrt{3}$, then $c_{11} > 0$ and $c_{30} < 0$, so that the bifurcation is supercritical and the reduced equation possesses a circle of stable nontrivial equilibria, $A_\mu(\phi) = r_\mu e^{i\phi}$, $\phi \in \mathbb{R}/2\pi\mathbb{Z}$, for any sufficiently small $\mu > 0$.

- (ii) If (α, ρ^*) belongs to the lighter shaded regions in Figure 3.1b and $\rho^* > 2/\sqrt{3}$, then $c_{11} < 0$ and $c_{30} < 0$, so that the bifurcation is supercritical and the reduced equation possesses a circle of stable nontrivial equilibria, $A_\mu(\phi) = r_\mu e^{i\phi}$, $\phi \in \mathbb{R}/2\pi\mathbb{Z}$, for any sufficiently small $\mu < 0$.
- (iii) If (α, ρ^*) belongs to the darker shaded regions in Figure 3.1b and $\rho^* < 2/\sqrt{3}$, then $c_{11} > 0$ and $c_{30} > 0$, so that the bifurcation is subcritical and the reduced equation possesses a circle of unstable nontrivial equilibria, $A_\mu(\phi) = r_\mu e^{i\phi}$, $\phi \in \mathbb{R}/2\pi\mathbb{Z}$, for any sufficiently small $\mu < 0$.
- (iv) If (α, ρ^*) belongs to the darker shaded regions in Figure 3.1b and $\rho^* > 2/\sqrt{3}$, then $c_{11} < 0$ and $c_{30} > 0$, so that the bifurcation is subcritical and the reduced equation possesses a circle of unstable nontrivial equilibria, $A_\mu(\phi) = r_\mu e^{i\phi}$, $\phi \in \mathbb{R}/2\pi\mathbb{Z}$, for any sufficiently small $\mu > 0$.

Theorem 5. Going back to the Lugiato-Lefever equation (1.0.1), the positive equilibrium $A_\mu(0) = r_\mu$ gives an even periodic solution of the Lugiato-Lefever equation (1.0.1), with Taylor expansion

$$A_\mu(x) = r_\mu + \frac{c_{11}}{c_{30}} \cos(k_{\min} x) |\mu|^{1/2} + \mathcal{O}(|\mu|), \quad (1.3.13)$$

in which

$$\frac{c_{11}}{c_{30}} = \left((\rho^* + \sqrt{\rho^{*2} - 1})(\alpha - \sqrt{\rho^{*2} - 1}) + i(2\rho^* - \alpha + \sqrt{\rho^{*2} - 1}) \right) \left| \frac{c_{11}}{c_{30}} \right|^{1/2},$$

whereas the other equilibria on the circle correspond to translations in x of this even periodic solution.

1.4 The case of normal dispersion ($\beta = 1$)

1.4.1 Bifurcations of steady periodic solutions at the onset of Turing instability

As in Section 1.3.1, we consider the onset of Turing instability which occurs in this case for $\alpha > 2$, but the same values $\rho^* = 1$ and $F = F_1$, $F_1^2 = (1 - \alpha)^2 + 1$. We fix $\alpha > 2$, take as bifurcation parameter $F^2 = F_1^2 + \mu$, and denote by $\psi_\mu^* = \psi_{r_\mu}^* + i\psi_{i_\mu}^*$ and $\rho_\mu^* = |\psi_\mu^*|^2$ the corresponding constant solution of the Lugiato-Lefever equation (1.0.1) and its square modulus, respectively, given by (1.1.2).

In this case, the Turing instability occurs for modes $k = \pm\sqrt{\alpha - 2}$. We therefore look for solutions of the Lugiato-Lefever equation (1.0.1) close to the branch of constant solutions ψ_μ^* of the form

$$(x, t) = \psi_\mu^* + (u + iv)(y, t), \quad y = \sqrt{\alpha - 2}x.$$

Inserting this Ansatz into the Lugiato-Lefever equation (1.0.1) we obtain exactly the same equation (1.2.2) as in the case of anomalous dispersion. Since the constant solutions are the same in the two cases, the only difference could occur for the coefficient of the term ∂_y^2 in the formula of the linear operator \mathcal{L}_μ^* . But since $y = \sqrt{\alpha - 2}x$ in this case, instead of $y = \sqrt{2 - \alpha}x$ in the case of anomalous dispersion, this coefficient is also the same. Consequently, all arguments, including computations, in Section 1.3.1 remain valid, with the only difference that now $\alpha > 2$. This implies that in this case the reduced system (1.2.8) undergoes a subcritical steady bifurcation with $O(2)$ symmetry with the properties (iii) and (iv) in Theorem 30.

Proposition 1.4.1. Consider the Lugiato-Lefever equation (1.0.1) in the case $\beta = 1$ of normal dispersion. Assume that $\alpha > 2$ and $F^2 = F_1^2 + \mu$, $F_1^2 = (1 - \alpha)^2 + 1$. Then, the reduced system (1.2.8) undergoes a subcritical steady bifurcation with $O(2)$ symmetry. More precisely, the following properties hold in a neighborhood of 0 in \mathbb{C} , for μ sufficiently small:

- (i) For any $\mu < 0$, then the reduced equation (1.2.8) possesses the stable equilibrium $A = 0$ and a circle of unstable equilibria, $A_\mu(\phi) = r_\mu e^{i\phi}$, for $\phi \in \mathbb{R}/2\pi\mathbb{Z}$, which surrounds this equilibrium.
- (ii) For any $\mu < 0$, then the reduced equation (1.2.8) possesses a unique equilibrium $A = 0$ which is unstable.

1.4.2 Bifurcations of steady periodic solutions in the Turing instability region

As in the case of anomalous dispersion, in the Turing instability region we expect bifurcations of steady periodic solutions with wavenumbers $k_* = \tilde{k}_{\min}$ and $k_* = \tilde{k}_{\max}$, where $\tilde{k}_{\min} < \tilde{k}_{\max}$ are such that $\lambda_1(\tilde{k}_{\min}) = \lambda_1(\tilde{k}_{\max}) = 0$, whereas in the zero-mode instability region we expect bifurcations of steady periodic solutions with wavenumbers $k_* = \tilde{k}_{\max}$. Since the difference between the formulas of $a(k)$ in the two cases is only a change of the sign of the coefficient of k^2 , we find that

$$\tilde{k}_{\max}^2 = -k_{\min}^2, \quad \tilde{k}_{\min}^2 = -k_{\max}^2,$$

where k_{\max} and k_{\min} are the wavenumbers given by (1.3.2) and (1.3.3), respectively. Together with the fact that in the Lugiato-Lefever equation (1.0.1) the difference between the two cases of normal and anomalous dispersion is the sign of β , this implies that the dynamical systems found for $k_* = \tilde{k}_{\max}$ and $k_* = \tilde{k}_{\min}$ in the case of normal dispersion are precisely the ones found in the case of anomalous dispersion for $k_* = k_{\min}$ and $k_* = k_{\max}$, respectively. As a consequence, the calculations are the same as the ones in Sections 1.3.1, with the only difference that (α, ρ^*) belongs to a different parameter region in the case of the Turing instability. We do not give the precise statements of the analogue of Theorems 1.3.4 in this case, but only summarize the results in Figure 3.1 in the next section, with stability results.

Chapter 2

Spectral stability of periodic waves for the Lugiato-Lefever equation

In this chapter, we present a spectral stability study of stationary solutions for the Lugiato-Lefever equation. We analyze spectral stability of these waves with respect to sub-harmonic and localized perturbations, at the onset of Turing instability and in the Turing instability and zero-mode regions.

2.1 Onset of Turing instability

In this section, we study the spectral stability with respect to localized, or bounded, perturbations of the periodic solutions found in Section 1.3.1 at the onset of Turing instability. The bifurcation result in Proposition 1.2.7 implies that the periodic solutions found for $\alpha < 41/30$ (supercritical bifurcation) are stable, whereas those found for $\alpha > 41/30$ (subcritical bifurcation) are unstable, for perturbations which belong to the space \mathcal{Y} , i.e., for co-periodic perturbations. In particular, this implies that, for perturbations which are bounded or localized, the solutions found for $\alpha > 41/30$ are also unstable, but leaves open the question of stability of the solutions found for $\alpha < 41/30$.

Linearized problem and Bloch operators

Consider the even periodic solution $\psi_\mu(x)$ given by (1.2.13), for $\alpha < 2$, $F^2 = F_1^2 + \mu$, and μ sufficiently small as given in Proposition 1.2.7. As for the bifurcation analysis, it is more convenient to work with 2π -periodic functions, and replace x by $y = \sqrt{2 - \alpha}x$. Linearizing the Lugiato-Lefever equation (1.0.1) at $\psi_\mu(x)$, and using the variable y instead of x , we obtain the linearized equation

$$\frac{dV}{dt} = \mathcal{A}_\mu V, \quad (2.1.1)$$

where \mathcal{A}_μ is the linear operator

$$\mathcal{A}_\mu = -\mathbb{I} + \mathcal{J}\mathcal{L}_\mu,$$

the operator \mathcal{J} is defined as before, and

$$\mathcal{L}_\mu = \begin{pmatrix} (2 - \alpha)\partial_y^2 - \alpha + 3\psi_{r\mu}^2(y) + \psi_{i\mu}^2(y) & 2\psi_{r\mu}(y)\psi_{i\mu}(y) \\ 2\psi_{r\mu}(y)\psi_{i\mu}(y) & (2 - \alpha)\partial_y^2 - \alpha + \psi_{r\mu}^2(y) + 3\psi_{i\mu}^2(y) \end{pmatrix},$$

in which $\psi_{r\mu}(y)$ and $\psi_{i\mu}(y)$ represent the real and imaginary parts, respectively, of the periodic solution. According to the formula (1.2.13), we have the expansions

$$\begin{aligned} r_\mu(y) &= r^* + \alpha \psi_1 \cos(y) |\mu|^{1/2} + \mathcal{O}(|\mu|), \\ i_\mu(y) &= i^* + (2 - \alpha) \psi_1 \cos(y) |\mu|^{1/2} + \mathcal{O}(|\mu|), \end{aligned}$$

CHAPTER 2. SPECTRAL STABILITY OF PERIODIC WAVES FOR THE LUGIATO-LEFEVER EQUATION

where ψ_r^* and ψ_i^* are the real and imaginary parts, respectively, of the constant solution ψ_0^* , and $\psi_1 = 3/F_1|41 - 30\alpha|^{1/2}$. The linear operator \mathcal{A}_μ is closed in both the Hilbert space $L^2(\mathbb{R}) \times L^2(\mathbb{R})$ and in the Banach space $C_b(\mathbb{R}) \times C_b(\mathbb{R})$, with dense domains $H^2(\mathbb{R}) \times H^2(\mathbb{R})$ and $C_b^2(\mathbb{R}) \times C_b^2(\mathbb{R})$, respectively. We recall that the space $L^2(\mathbb{R}) \times L^2(\mathbb{R})$ corresponds to localized perturbations of the periodic wave and the space $C_b(\mathbb{R}) \times C_b(\mathbb{R})$ to bounded perturbations.

In contrast to the operator \mathcal{A}^* in Section 1.1 which has constant coefficients, the linear operator \mathcal{A}_μ is a differential operator with 2π -periodic coefficients. We decompose the spectrum of the operator \mathcal{A}_μ as follow:

Proposition 2.1.1. *We have that*

$$\sigma(\mathcal{A}_\mu) = \bigcup_{\gamma \in (-1/2, 1/2]} \sigma(\mathcal{A}_{\gamma, \mu})$$

with $\mathcal{A}_{\gamma, \mu} = -\mathbb{I} + \mathcal{J}\mathcal{L}_{\gamma, \mu}$ where

$$\mathcal{L}_{\gamma, \mu} = \begin{pmatrix} (2 - \alpha)(\partial_y + i\gamma)^2 - \alpha + 3\psi_{r\mu}^2(y) + \psi_{i\mu}^2(y) & 2\psi_{r\mu}(y)\psi_{i\mu}(y) \\ 2\psi_{r\mu}(y)\psi_{i\mu}(y) & (2 - \alpha)(\partial_y + i\gamma)^2 - \alpha + \psi_{r\mu}^2(y) + 3\psi_{i\mu}^2(y) \end{pmatrix}.$$

For the analysis of its spectrum, we use a Floquet-Bloch decomposition, instead of Fourier analysis, which shows that its spectrum is the same in both spaces above, and that it is given by the union of the spectra of the Bloch operators,

$$\mathcal{A}_{\gamma, \mu} = -\mathbb{I} + \mathcal{J}\mathcal{L}_{\gamma, \mu}, \quad (2.1.2)$$

acting in $L^2(0, 2\pi) \times L^2(0, 2\pi)$ with domain the subspace of 2π -periodic functions $H_{\text{per}}^2(0, 2\pi) \times H_{\text{per}}^2(0, 2\pi)$, for $\gamma \in (-1/2, 1/2]$ (e.g., see [12, 17]). The difference between the spectrum of the operator \mathcal{A}_μ when acting in $L^2(\mathbb{R}) \times L^2(\mathbb{R})$ or in $C_b(\mathbb{R}) \times C_b(\mathbb{R})$ is that it is purely essential spectrum in $L^2(\mathbb{R}) \times L^2(\mathbb{R})$ and continuous purely point spectrum in $C_b(\mathbb{R}) \times C_b(\mathbb{R})$. In contrast, since the operators $\mathcal{A}_{\gamma, \mu}$ have compactly embedded domain, their resolvent is a compact operator, and therefore their spectrum is purely point spectrum consisting of isolated eigenvalues with finite algebraic multiplicities.

Our goal now is to determine the spectra of the Bloch operators $\mathcal{A}_{\gamma, \mu}$, for $\gamma \in (-1/2, 1/2]$ and μ sufficiently small.

Remark 2.1.2. *Notice that the spectrum $\sigma(\mathcal{A}_{\gamma, \mu})$ of $\mathcal{A}_{\gamma, \mu}$ is symmetric with respect to the line $\text{Re } \lambda = -1$ in the complex plane, just as the one of the operator \mathcal{A}^* in Section 1.1, since \mathcal{J} and $\mathcal{L}_{\gamma, \mu}$ are skew- and self-adjoint operators as in the Remark 1.2.1, respectively. In addition, the equalities*

$$\overline{\mathcal{A}_{\gamma, \mu}} = \mathcal{A}_{-\gamma, \mu}, \quad \mathcal{A}_{\gamma, \mu}\mathcal{T} = \mathcal{T}\mathcal{A}_{-\gamma, \mu}, \quad (2.1.3)$$

where the operator \mathcal{T} in the last equality is the reflection operator defined in (1.2.3), imply that

$$\overline{\sigma(\mathcal{A}_{\gamma, \mu})} = \sigma(\mathcal{A}_{-\gamma, \mu}) = \sigma(\mathcal{A}_{\gamma, \mu}).$$

Spectral analysis of the Bloch operators

We analyze the spectra of the operators $\mathcal{A}_{\gamma, \mu}$ in two steps, first for values of γ outside a fixed, but arbitrary, neighborhood of 0 (Lemma 2.1.3), and then for small γ (Lemma 2.1.4).

Lemma 2.1.3. *For any $\gamma_1 \in (0, 1/2)$, there exist positive constants μ_1 and δ_1 , such that the spectrum of $\mathcal{A}_{\gamma, \mu}$ satisfies*

$$\sigma(\mathcal{A}_{\gamma, \mu}) \subset \{\lambda \in \mathbb{C} : \text{Re } \lambda < -\delta_1\},$$

for any $\gamma \in (-1/2, 1/2]$, $|\gamma| > \gamma_1$ and $|\mu| < \mu_1$.

Proof. We use a perturbation argument in which we regard the operator $\mathcal{A}_{\gamma,\mu}$ as a small perturbation of the operator $\mathcal{A}_{\gamma,0}$, which has constant coefficients. The spectrum of $\mathcal{A}_{\gamma,0}$ is easily obtained from the linear stability analysis in Section 1.1, in which we restrict to the Fourier modes $k = \sqrt{2-\alpha}(n+\gamma)$, so that

$$\sigma(\mathcal{A}_{\gamma,0}) = \left\{ \lambda_{\pm}(n, \gamma) = -1 \pm \sqrt{1 - (2-\alpha)^2((n+\gamma)^2 - 1)^2}, n \in \mathbb{Z} \right\}. \quad (2.1.4)$$

The results in Section 1.1 also show that the two eigenvalues $\lambda_{\pm}(n, \gamma)$ are either negative or have negative real parts except for the eigenvalues $\lambda_{+}(\pm 1, 0)$ which vanish. Consequently, for any $\gamma_1 \in (0, 1/2)$, there exists $\delta_0 > 0$, such that the spectrum of $\mathcal{A}_{\gamma,0}$ satisfies

$$\sigma(\mathcal{A}_{\gamma,0}) \subset \{ \lambda \in \mathbb{C} : \operatorname{Re} \lambda < -\delta_0 \}, \quad (2.1.5)$$

for any $\gamma \in (-1/2, 1/2]$, $|\gamma| > \gamma_1$.

For small μ , the operator $\mathcal{A}_{\gamma,\mu}$ is a bounded perturbation of the operator $\mathcal{A}_{\gamma,0}$ with uniform bound in γ of order $\mathcal{O}(|\mu|^{1/2})$. Consequently, for any $\delta > 0$ and any $\gamma \in (-1/2, 1/2]$, we have that

$$\sigma(\mathcal{A}_{\gamma,\mu}) \subset \{ \lambda \in \mathbb{C} : \operatorname{dist}(\lambda, \sigma(\mathcal{A}_{\gamma,0})) < \delta \},$$

provided μ is sufficiently small. Together with the property (2.1.5) this proves the lemma. \blacksquare

Lemma 2.1.4. *There exist positive constants γ_2 , μ_2 , and δ_2 such that, for any $|\gamma| < \gamma_2$ and $|\mu| < \mu_2$, the spectrum of $\mathcal{A}_{\gamma,\mu}$ decomposes into two disjoint subsets*

$$\sigma(\mathcal{A}_{\gamma,\mu}) = \sigma_{-}(\mathcal{A}_{\gamma,\mu}) \cup \sigma_0(\mathcal{A}_{\gamma,\mu}), \quad (2.1.6)$$

with the following properties:

(i) $\sigma_{-}(\mathcal{A}_{\gamma,\mu}) \subset \{ \lambda \in \mathbb{C} : \operatorname{Re} \lambda < -\delta_2 \}$ and $\sigma_0(\mathcal{A}_{\gamma,\mu}) \subset \{ \lambda \in \mathbb{C} : \operatorname{Re} \lambda > -\delta_2 \}$;

(ii) the set $\sigma_0(\mathcal{A}_{\gamma,\mu})$ consists of two simple and real eigenvalues $\lambda_{1,2}(\gamma, \mu)$ with Taylor expansions

$$\lambda_1(\gamma, \mu) = -\frac{2}{(2-\alpha)^2} \mu - 2(2-\alpha)^2 \gamma^2 + o(|\mu| + |\gamma|^2),$$

$$\lambda_2(\gamma, \mu) = -2(2-\alpha)^2 \gamma^2 + o(|\gamma|^2).$$

Proof. For small γ and μ , the operator $\mathcal{A}_{\gamma,\mu}$ is a small relatively bounded perturbation of the operator $\mathcal{A}_{0,0}$. The latter operator is precisely the operator \mathcal{A}_{*0} considered in Section 1.3.1, for which we have the spectral decomposition (1.2.5). A standard perturbation argument then implies the spectral decomposition (2.1.6) for $\mathcal{A}_{\gamma,\mu}$, with the property (i), and such that $\sigma_0(\mathcal{A}_{\gamma,\mu})$ consists of precisely two eigenvalues $\lambda_{1,2}(\gamma, \mu)$, which are the continuation of the double eigenvalue 0 of $\mathcal{A}_{0,0}$, for sufficiently small γ and μ . It remains to compute the Taylor expansions of these eigenvalues in (ii).

The eigenvalues $\lambda_{1,2}(\gamma, \mu)$, and their associated eigenvectors, may not have the same smoothness properties with respect to the parameters γ and μ as the operator $\mathcal{A}_{\gamma,\mu}$. In contrast, these smoothness properties are preserved by a suitably chosen basis $\{\zeta_1(\gamma, \mu), \zeta_2(\gamma, \mu)\}$ of the spectral subspace of $\mathcal{A}_{\gamma,\mu}$ associated to $\sigma_0(\mathcal{A}_{\gamma,\mu})$ and by the 2×2 matrix $\mathcal{M}(\gamma, \mu)$ representing the action of $\mathcal{A}_{\gamma,\mu}$ on this basis [18]. The matrix $\mathcal{M}(\gamma, \mu)$ is determined by the equality

$$\mathcal{A}_{\gamma,\mu} \begin{pmatrix} \zeta_1(\gamma, \mu) & \zeta_2(\gamma, \mu) \end{pmatrix} = \mathcal{M}(\gamma, \mu) \begin{pmatrix} \zeta_1(\gamma, \mu) \\ \zeta_2(\gamma, \mu) \end{pmatrix}, \quad (2.1.7)$$

CHAPTER 2. SPECTRAL STABILITY OF PERIODIC WAVES FOR THE LUGIATO-LEFEVER EQUATION

and its eigenvalues are precisely $\lambda_{1,2}(\gamma, \mu)$. Consequently, we can compute the Taylor expansion of the eigenvalues $\lambda_{1,2}(\gamma, \mu)$ from the Taylor expansion of the matrix $\mathcal{M}(\gamma, \mu)$. For our purpose, it is enough to determine the Taylor expansion of $\mathcal{M}(\gamma, \mu)$ up to order 2.

The operator $\mathcal{A}_{\gamma,\mu}$ depends smoothly upon γ and $|\mu|^{1/2}$, and we consider the expansion

$$\mathcal{A}_{\gamma,\mu} = \mathcal{A}_{00} + \mathcal{A}_{10}i\gamma + \mathcal{A}_{01}|\mu|^{1/2} + \mathcal{A}_{20}\gamma^2 + \mathcal{A}_{11}i\gamma|\mu|^{1/2} + \mathcal{A}_{02}|\mu| + \mathcal{O}((|\gamma| + |\mu|^{1/2})^3),$$

in which the operators \mathcal{A}_{ij} are obtained explicitly from (2.1.2), and similarly we expand the vectors in the basis

$$\zeta_j(\gamma, \mu) = \zeta_{j00} + \zeta_{j10}i\gamma + \zeta_{j01}|\mu|^{1/2} + \zeta_{j20}\gamma^2 + \zeta_{j11}i\gamma|\mu|^{1/2} + \zeta_{j02}|\mu| + \mathcal{O}((|\gamma| + |\mu|^{1/2})^3),$$

for $j = 1, 2$, and the 2×2 matrix

$$\mathcal{M}(\gamma, \mu) = \mathcal{M}_{00} + \mathcal{M}_{10}i\gamma + \mathcal{M}_{01}|\mu|^{1/2} + \mathcal{M}_{20}\gamma^2 + \mathcal{M}_{11}i\gamma|\mu|^{1/2} + \mathcal{M}_{02}|\mu| + \mathcal{O}((|\gamma| + |\mu|^{1/2})^3),$$

in which we have to determine the vectors ζ_{jkl} and the matrices \mathcal{M}_{ij} . At $\gamma = \mu = 0$, the spectral subspace associated to $\sigma_0(\mathcal{A}_{0,0})$ is the two-dimensional kernel of the operator $\mathcal{A}_{0,0}$, so that $\mathcal{M}(0, 0) = 0$, and we choose as basis the even and odd real eigenvectors

$$\zeta_1(0, 0) = \zeta_{100} = \begin{pmatrix} \alpha \\ 2 - \alpha \end{pmatrix} \cos(y), \quad \zeta_2(0, 0) = \zeta_{200} = \begin{pmatrix} \alpha \\ 2 - \alpha \end{pmatrix} \sin(y), \quad (2.1.8)$$

which are the real and imaginary parts, respectively, of the vector ζ used for the bifurcation analysis in Section 1.3.1. With this choice of the terms of order $\mathcal{O}(1)$, the remaining terms in the expansions of $\zeta_j(\gamma, \mu)$, $j = 1, 2$, and $\mathcal{M}(\gamma, \mu)$ are determined by the equality (2.1.7), and can be computed successively, up to any order, by expanding this equality. However, the direct calculation of these terms is very long. Instead, we will use the symmetries (2.1.3), which will allow to reduce the number of terms in these expansions, compute separately the matrices $\mathcal{M}(\gamma, 0)$ and $\mathcal{M}(0, \mu)$, and only compute directly the terms of order $\mathcal{O}(|\gamma||\mu|^{1/2})$. We point out that in the expansions of the eigenvectors it will be enough to compute the terms up to order 1.

First, due to the symmetries (2.1.3), which are inherited by the matrix $\mathcal{M}(\gamma, \mu)$, the elements on the main diagonal and on the antidiagonal of the matrix $\mathcal{M}(\gamma, \mu)$ will be even real and odd purely imaginary in γ , respectively. Furthermore, the real and imaginary parts of the vectors $\zeta_1(\gamma, \mu)$ and $\zeta_2(\gamma, \mu)$ will be even and odd in γ , respectively, and have opposite parities in y : the real part of $\zeta_1(\gamma, \mu)$ and the imaginary part of $\zeta_2(\gamma, \mu)$ are even functions in y , whereas the imaginary part of $\zeta_1(\gamma, \mu)$ and the real part of $\zeta_2(\gamma, \mu)$ are odd functions in y .

Next, for $\mu = 0$, the operator $\mathcal{A}_{\gamma,0}$ has constant coefficients, so that we can explicitly compute the basis and the matrix $\mathcal{M}(\gamma, 0)$. According to (2.1.4), the two eigenvalues of $\mathcal{A}_{\gamma,0}$ in $\sigma_0(\mathcal{A}_{\gamma,0})$ are

$$\lambda_{\pm}(\pm 1, \gamma) = -1 + \sqrt{1 - (2 - \alpha)^2((\gamma \pm 1)^2 - 1)^2},$$

and a direct calculation gives the associated eigenvectors

$$\zeta_{\pm}(\gamma) = \frac{1}{\alpha} \begin{pmatrix} \alpha^2 + F_1^2 \lambda_{\pm}(\pm 1, \gamma) \\ (2 - \alpha)(\alpha \mp 2F_1^2 \gamma - F_1^2 \gamma^2) \end{pmatrix} e^{\pm iy}.$$

The vectors $\zeta_{1,2}(\gamma, 0)$ are a linear combination of $\zeta_{\pm}(\gamma)$, and taking into account the equalities (2.1.8) we obtain

$$\zeta_1(\gamma, 0) = \frac{1}{2} (\zeta_+(\gamma) + \zeta_-(\gamma)) = \begin{pmatrix} \alpha \\ 2 - \alpha \end{pmatrix} \cos(y) - \frac{2(2 - \alpha)F_1^2}{\alpha} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sin(y) i\gamma + \mathcal{O}(\gamma^2), \quad (2.1.9)$$

$$\zeta_2(\gamma, 0) = \frac{1}{2i} (\zeta_+(\gamma) - \zeta_-(\gamma)) = \begin{pmatrix} \alpha \\ 2 - \alpha \end{pmatrix} \sin(y) + \frac{2(2 - \alpha)F_1^2}{\alpha} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cos(y) i\gamma + \mathcal{O}(\gamma^2), \quad (2.1.10)$$

and then the matrix associated to this basis

$$\mathcal{M}(\gamma, 0) = \begin{pmatrix} -2(2 - \alpha)^2\gamma^2 + \mathcal{O}(\gamma^4) & \mathcal{O}(|\gamma|^3) \\ \mathcal{O}(|\gamma|^3) & -2(2 - \alpha)^2\gamma^2 + \mathcal{O}(\gamma^4) \end{pmatrix}.$$

For $\gamma = 0$, the symmetries of the matrix $\mathcal{M}(\gamma, \mu)$ above imply that $\mathcal{M}(0, \mu)$ is diagonal,

$$\mathcal{M}(0, \mu) = \begin{pmatrix} \lambda_1(0, \mu) & 0 \\ 0 & \lambda_2(0, \mu) \end{pmatrix},$$

so that $\zeta_1(0, \mu)$ and $\zeta_2(0, \mu)$ are eigenvectors associated to the eigenvalues $\lambda_1(0, \mu)$ and $\lambda_2(0, \mu)$, respectively. Due to the translation invariance of the Lugiato-Lefever equation, the derivative $(\psi'_{r\mu}(y), \psi'_{i\mu}(y))$ of the periodic wave, which is an odd function, belongs to the kernel of the operator $\mathcal{A}_{0,\mu}$. Consequently, $(\psi'_{r\mu}(y), \psi'_{i\mu}(y))$ is proportional to the odd eigenvector,

$$\zeta_2(0, \mu) = -\frac{1}{1|\mu|^{1/2}} \begin{pmatrix} \psi'_{r\mu}(y) \\ \psi'_{i\mu}(y) \end{pmatrix} = \begin{pmatrix} \alpha \\ 2 - \alpha \end{pmatrix} \sin(y) + \mathcal{O}(|\mu|^{1/2}),$$

and the corresponding eigenvalue vanishes, $\lambda_2(0, \mu) = 0$. For the computation of the eigenvalue $\lambda_1(0, \mu)$, we use the center manifold constructed in Section 1.3.1. The key observation is that the linear operator $\mathcal{A}_{0,\mu}$ coincides with the linear operator obtained by linearizing the dynamical system (1.2.4) at the periodic wave $(\psi_{r\mu}(y), \psi_{i\mu}(y))$. Since the center manifold is locally invariant for the flow of the dynamical system (1.2.4), the two eigenvalues $\lambda_1(0, \mu)$ and $\lambda_2(0, \mu)$ are the two eigenvalues of the 2×2 matrix obtained by linearizing the reduced system (1.2.8) at the real equilibrium r_μ which corresponds to the even periodic wave. A direct computation gives

$$\lambda_1(0, \mu) = -\frac{2}{(2 - \alpha)^2} \mu + \mathcal{O}(|\mu|^{3/2}), \quad \lambda_2(0, \mu) = 0,$$

so that

$$\mathcal{M}(0, \mu) = \begin{pmatrix} -\frac{2}{(2 - \alpha)^2} \mu + \mathcal{O}(|\mu|^{3/2}) & 0 \\ 0 & 0 \end{pmatrix}.$$

Summarizing the above results, we conclude that

$$\begin{aligned} \mathcal{M}(\gamma, \mu) &= \begin{pmatrix} -2(2 - \alpha)^2\gamma^2 - \frac{2}{(2 - \alpha)^2} \mu & ic_1\gamma|\mu|^{1/2} \\ ic_2\gamma|\mu|^{1/2} & -2(2 - \alpha)^2\gamma^2 \end{pmatrix} \\ &+ \begin{pmatrix} \mathcal{O}(\gamma^4 + \gamma^2|\mu|^{1/2} + |\mu|^{3/2}) & \mathcal{O}(|\gamma|^3 + |\gamma\mu|) \\ \mathcal{O}(|\gamma|^3 + |\gamma\mu|) & \mathcal{O}(\gamma^4 + \gamma^2|\mu|) \end{pmatrix}, \end{aligned}$$

in which it remains to determine the real constants c_1 and c_2 . Inserting the expansions of $\mathcal{A}_{\gamma,\mu}$, $\zeta_j(\gamma, \mu)$, $j = 1, 2$, and $\mathcal{M}(\gamma, \mu)$ into the equality (2.1.7), we find successively at orders $\mathcal{O}(1)$, $\mathcal{O}(|\gamma|)$, $\mathcal{O}(|\mu|^{1/2})$, and $\mathcal{O}(|\gamma||\mu|^{1/2})$ the equalities

$$\mathcal{A}_{00}\zeta_{j00} = 0, \quad j = 1, 2, \quad (2.1.11)$$

$$\mathcal{A}_{00}\zeta_{j10} + \mathcal{A}_{10}\zeta_{j00} = 0, \quad j = 1, 2, \quad (2.1.12)$$

$$\mathcal{A}_{00}\zeta_{j01} + \mathcal{A}_{01}\zeta_{j00} = 0, \quad j = 1, 2, \quad (2.1.13)$$

$$\mathcal{A}_{00}\zeta_{111} + \mathcal{A}_{10}\zeta_{101} + \mathcal{A}_{01}\zeta_{110} = c_1\zeta_{200}, \quad (2.1.14)$$

$$\mathcal{A}_{00}\zeta_{211} + \mathcal{A}_{10}\zeta_{201} + \mathcal{A}_{01}\zeta_{210} = c_2\zeta_{100}, \quad (2.1.15)$$

which allow to compute successively the terms in the expansion of the vectors $\zeta_j(\gamma, \mu)$ by solving a linear equation of the form

$$\mathcal{A}_{00}\zeta_{jkl} = C. \quad (2.1.16)$$

Since the operator \mathcal{A}_{00} has a two-dimensional kernel, it is not invertible so that the right hand side F must satisfy a solvability condition. The coefficients c_1 and c_2 are determined from the solvability conditions for the equations (2.1.14) and (2.1.15).

The equalities (2.1.11) hold since the vectors ζ_{j00} , $j = 1, 2$, belong to the kernel of \mathcal{A}_{00} , and the calculation of $\mathcal{M}(\gamma, 0)$ above implies that the equalities (2.1.12) hold with ζ_{j10} , $j = 1, 2$, given by (2.1.9) and (2.1.10). It turns out that we do not need to explicitly compute the other terms in the expansions of the vectors $\zeta_j(\gamma, \mu)$, but only determine their Fourier modes in y . The key observation is that the equation (2.1.16) has a unique solution, up to an element in the kernel of \mathcal{A}_{00} , when the right hand side F contains terms with Fourier modes $n \neq \pm 1$, only. Indeed, since \mathcal{A}_{00} is a differential operator with constant coefficients, its action preserves the Fourier modes, and since the vectors ζ_{j00} , $j = 1, 2$, in its kernel have Fourier modes ± 1 , its restriction to the subspace of periodic functions with Fourier modes $n \neq \pm 1$ is an invertible operator. A direct calculation gives

$$\mathcal{A}_{10} = 2(2 - \alpha)\mathcal{J}\partial_y, \quad \mathcal{A}_{01} = \frac{2\psi_1}{F_1}\mathcal{J} \begin{pmatrix} 2 + \alpha^2 & 2 - \alpha^2 \\ 2 - \alpha^2 & 6 - 8\alpha + 3\alpha^2 \end{pmatrix} \cos(y),$$

and then it is straightforward to check that the terms $\mathcal{A}_{10}\zeta_{101}$, $\mathcal{A}_{01}\zeta_{110}$ in (2.1.14) and $\mathcal{A}_{10}\zeta_{201}$, $\mathcal{A}_{01}\zeta_{210}$ in (2.1.15) have Fourier modes 0 and ± 2 , implying that we can solve (2.1.14) and (2.1.15) provided $c_1 = c_2 = 0$. Consequently,

$$\mathcal{M}(\gamma, \mu) = \begin{pmatrix} -2(2 - \alpha)^2\gamma^2 - \frac{2}{(2 - \alpha)^2}\mu & 0 \\ 0 & -2(2 - \alpha)^2\gamma^2 \end{pmatrix} + \begin{pmatrix} \mathcal{O}(\gamma^4 + \gamma^2|\mu|^{1/2} + |\mu|^{3/2}) & \mathcal{O}(|\gamma|^3 + |\gamma\mu|) \\ \mathcal{O}(|\gamma|^3 + |\gamma\mu|) & \mathcal{O}(\gamma^4 + \gamma^2|\mu|) \end{pmatrix},$$

which implies the expansions of the eigenvalues in (ii), and completes the proof of the lemma. \blacksquare

Spectral stability and instability

The results in Lemma 2.1.3, Lemma 2.1.4, and the connection between the spectra of the operator \mathcal{A}_μ and of the Bloch operators $\mathcal{A}_{\gamma,\mu}$, imply that the periodic waves are stable with respect to general bounded perturbations in the case $\alpha < 41/30$ (supercritical bifurcation) and unstable in the case $\alpha > 41/30$ (subcritical bifurcation). The instability is due to the eigenvalue $\lambda_1(0, \mu)$ of the operator $\mathcal{A}_{0,\mu}$, which is positive in this case, since $\mu < 0$. This recovers the instability result for co-periodic perturbations already found in Theorem 30, on the one hand, and implies that $\lambda_1(\gamma, \mu)$ is positive for sufficiently small γ , thus the periodic wave is also unstable for periodic perturbations with nearby periods, on the other hand. For localized and bounded perturbations this leads to an instability interval $(0, \lambda_1(0, \mu)]$ in the spectrum of the operator \mathcal{A}_μ . More precisely, we have the following result.

Theorem 6. *Consider the Lugiato-Lefever equation (1.0.1) in the case $\beta = -1$ of anomalous dispersion. Assume that $\alpha < 2$, $\alpha \neq 41/30$, and $F^2 = F_1^2 + \mu$. Consider the periodic solutions $\psi_\mu(x)$ of the Lugiato-Lefever equation (1.0.1) given by (3.2.7), for $\mu > 0$ if $\alpha < 41/30$ and for $\mu < 0$ if $\alpha > 41/30$, and the associated linear operator \mathcal{A}_μ in (2.1.1), acting in either $L^2(\mathbb{R}) \times L^2(\mathbb{R})$ or $C_b(\mathbb{R}) \times C_b(\mathbb{R})$. For any μ with $|\mu|$ sufficiently small the following properties hold.*

- (i) If $\alpha < 41/30$ and $\mu > 0$, then the spectrum of the linear operator \mathcal{A}_μ lies in the closed left half complex plane, and the periodic solution $\psi_\mu(x)$ is spectrally stable with respect to bounded perturbations.
- (ii) If $\alpha > 41/30$ and $\mu < 0$, then the spectrum of the linear operator \mathcal{A}_μ contains a finite interval of positive real numbers $(0, c_\mu]$, with $c_\mu = \mathcal{O}(|\mu|)$, and the periodic solution $\psi_\mu(x)$ is spectrally unstable with respect to co-periodic, localized, and bounded perturbations.

2.2 Periodic waves in the instability regions

2.2.1 Instabilities of periodic waves

In this section we study the stability of the steady periodic solutions found in Section 1.3.1 in the two cases, $k_* = k_{\max}$ and $k_* = k_{\min}$. We focus on co-periodic and subharmonic perturbations, i.e., periodic perturbations with periods $2\pi N/k_*$, where $N = 1$ for co-periodic perturbations and $N \geq 2$ for subharmonic perturbations.

Background and shape instabilities

The bifurcation results in Section 1.3.1 also allow to conclude on the nonlinear stability of the bifurcating steady periodic solutions for perturbations which belong to the phase space $\mathcal{X} = L^2(0, 2\pi) \times L^2(0, 2\pi)$ of the dynamical system (1.2.2), i.e., for co-periodic perturbations which have the same period as the stationary solution. In this approach, the stability of a bifurcating solution $\psi_\mu(x)$ is determined by the location in the complex plane of the spectrum $\sigma(\mathcal{A}_0^*)$ of the operator \mathcal{A}_0^* , on the one hand, and by the stability of the corresponding equilibrium $A_\mu(\phi)$ of the reduced equation (1.2.8), on the other hand. More precisely, we recall that a bifurcating periodic solution $\psi_\mu(x)$ is stable if the spectrum $\sigma(\mathcal{A}_0^*)$ is stable, i.e., it does not contain eigenvalues with positive real part, and if the corresponding equilibrium $A_\mu(\phi)$ of the reduced equation is stable as well. If one of the two properties above does not hold then the bifurcating periodic solution $\psi_\mu(x)$ is unstable. We distinguish between two types of instabilities: background instabilities, when $\sigma(\mathcal{A}_0^*)$ contains eigenvalues with positive real part, and shape instabilities, when the equilibrium $A_\mu(\phi)$ of the reduced equation is unstable.

Upon replacing the phase space $\mathcal{X} = L^2(0, 2\pi) \times L^2(0, 2\pi)$ of 2π -periodic functions by the phase space $\mathcal{X}_N = L^2(0, 2\pi N) \times L^2(0, 2\pi N)$ of $2\pi N$ -periodic functions, for some arbitrary, but fixed N , we can extend the class of perturbations from co-periodic to subharmonic perturbations.

Proposition 2.2.1. *Consider \mathcal{A}_0^* acting in the phase space $\mathcal{X}_N = L^2(0, 2\pi N) \times L^2(0, 2\pi N)$ of $2\pi N$ -periodic functions. Then, the spectrum of the operator \mathcal{A}_0^* is given by,*

$$\sigma(\mathcal{A}_0^*) = \{\lambda_1(nk_*/N), \lambda_2(nk_*/N), n \in \mathbb{Z}\}, \quad (2.2.1)$$

where $\lambda_1(nk_*/N)$ and $\lambda_2(nk_*/N)$ are the two roots of the quadratic polynomial

$$\lambda^2 + 2\lambda + a(nk_*/N) = 0.$$

As a consequence, the operator \mathcal{A}_0^* may have additional unstable eigenvalues, hence leading to additional background instabilities, and more resonances may occur, as the ones found for $k_* = k_{\min}$ in Section 1.3.1, where the eigenvalue 0 is quadruple, instead of double. In the next two subsections we study separately the cases $k_* = k_{\max}$ and $k_* = k_{\min}$.

2.2.2 Periodic solutions with wavenumbers k_{\max}

We have a first result concerning periodic solutions $\psi_\mu(x)$ with wavenumbers $k_* = k_{\max}$ constructed in Theorem 1.3.4 for co-periodic perturbations.

Proposition 2.2.2. *In the three cases in Theorem 1.3.4 the periodic solutions are stable with respect to co-periodic perturbations in the case (i), background stable and shape unstable in the case (ii), and background unstable and shape stable in the case (iii).*

Proof. Consider the family of steady periodic solutions $\psi_\mu(x)$ with wavenumbers $k_* = k_{\max}$ constructed in Theorem 1.3.4. For co-periodic perturbations, $N = 1$, their stability follows from the results in Section 1.3.1. The spectral decomposition (1.3.4) implies that the spectrum $\sigma(\mathcal{A}_0^*)$ is stable in the Turing instability region, whereas it contains one positive eigenvalue in the zero-mode instability region, and according to Theorem 1.3.4 the equilibria $A_\mu(\phi)$ are stable when the bifurcation is supercritical and unstable when the bifurcation is subcritical. \square

For subharmonic perturbations, we obtain the main stability result in the following theorem (see also Figure 2.1a).

Theorem 7. *Consider the Lugiato-Lefever equation (1.0.1) in the case $\beta = -1$ of anomalous dispersion, and the steady periodic solutions $\psi_\mu(x)$ given by (3.2.7), with wavenumbers k_{\max} and the properties in Theorem 1.3.4. For any $N \geq 2$, consider the curve $\Gamma_{N-1,N}$ defined for $n = 1, \dots, N-1$ by*

$$\alpha = 2\rho^* - \frac{N^2 + n^2}{N^2 - n^2} \sqrt{\rho^{*2} - 1} \quad (2.2.2)$$

and set $\Gamma_{0,1} = \Gamma_-$. Then the following stability properties hold for periodic perturbations with periods $2\pi N/k_{\max}$, for any $N \geq 1$.

- (i) *If (α, ρ^*) belongs to the open set situated to the left of the curve $\Gamma_{N-1,N}$ in the Turing instability region, then the steady periodic solutions found in the supercritical bifurcation are stable, whereas the steady periodic solutions found in the subcritical bifurcation are unstable and the instability is a shape instability.*
- (ii) *If (α, ρ^*) belongs to the open set situated to the right of the curve $\Gamma_{N-1,N}$ in the Turing instability region, then the steady periodic solutions are unstable; the instability is a background instability for the solutions found in the supercritical bifurcation, whereas the solutions found in the subcritical bifurcation are both background and shape unstable.*
- (iii) *If (α, ρ^*) belongs to the zero-mode instability region, then the steady periodic solutions are unstable and the instability is a background instability.*

Proof. For subharmonic perturbations, $N \geq 2$, we consider the dynamical system (1.2.2) in the phase space $\mathcal{X}_N = L^2(0, 2\pi N) \times L^2(0, 2\pi N)$. In contrast to the case $N = 1$, for $N \geq 2$ resonances occur for certain parameter values, due to the presence of additional eigenvalues in the spectrum of \mathcal{A}_0^* . Indeed, the equality (2.2.1) implies that the eigenvalue 0 of the operator \mathcal{A}_0^* is quadruple instead of double, hence a resonance occurs, when $k_{\min} = nk_{\max}/N$, for $n = 1, \dots, N-1$. From the formulas (1.3.2) and (1.3.3) we obtain that $k_{\min} = nk_{\max}/N$ when the equalities

$$\alpha = 2\rho^* - \frac{N^2 + n^2}{N^2 - n^2} \sqrt{\rho^{*2} - 1} \quad (2.2.3)$$

hold, for $n = 1, \dots, N-1$. Each of these equalities defines a curve $\Gamma_{n,N}$ in the (α, ρ^*) -plane which lies in the Turing instability region. For increasing $n = 1, \dots, N-1$, these curves are ordered from right

to the left between the curves Γ_- and Γ_1 in the (α, ρ^*) -plane. Notice that the values $n = 0$ and $n = N$ correspond to the curves Γ_- and Γ_1 , respectively, and that $\Gamma_{1,N}$ is the curve Γ_N defined by (1.3.11) in Section 1.3.1.

Assuming that (α, ρ^*) does not belong to any of the curves $\Gamma_{n,N}$, $n = 1, \dots, N - 1$, the spectral decomposition (1.3.4) holds, with 0 a double semi-simple eigenvalue. The unstable spectrum $\sigma_u(\mathcal{A}_0^*)$ is empty for (α, ρ^*) in the open set between the curves Γ_1 and $\Gamma_{N-1,N}$, it consists of $N - n$ positive eigenvalues in the open set between two consecutive curves $\Gamma_{n,N}$ and $\Gamma_{n-1,N}$, for any $n = 2, \dots, N - 1$, of $N - 1$ positive eigenvalues in the open set between $\Gamma_{1,N}$ and Γ_- , and of N positive eigenvalues for (α, ρ^*) in the zero-mode instability region. In particular, this shows that the periodic solution is background stable with respect to $2\pi N/k_{\max}$ -periodic perturbations when (α, ρ^*) belongs to the open set between the curves Γ_1 and $\Gamma_{N-1,N}$, and it is background unstable otherwise.

Next, we find the shape instabilities by applying the center manifold reduction, as in Section 1.3.1. Since the eigenvalue 0 is double, the center manifold is two-dimensional, and it turns out that the reduced dynamics is governed by the same reduced system (1.2.8). Consequently, the result in Theorem 1.3.4 holds in \mathcal{X}_N , for any N , implying that shape instabilities occur only for the periodic solutions obtained in the subcritical bifurcation. \square

Along the resonance curves $\Gamma_{n,N}$, $n = 1, \dots, N - 1$, the eigenvalue 0 of \mathcal{A}_0^* is quadruple, and the center manifold reduction leads to a four-dimensional, instead of a two-dimensional system, just as in the case $k_* = k_{\min}$ in Section 1.3.1. The present bifurcation analysis does not allow to detect shape instabilities along these curves, but we can conclude that background instabilities occur along $\Gamma_{n,N}$, $n = 2, \dots, N - 1$.

We summarize the main stability results above in the following theorem (see also Figure 2.1a).

Periodic solutions with wavelengths k_{\min}

The stability of the family of steady periodic solutions with wavelengths $k_* = k_{\min}$ found in Theorem 1.3.6 is studied in the same way. In this case resonances occur when $k_{\max} = nk_{\min}/N$, for $n \geq N + 1$, and using the formulas (1.3.2) and (1.3.3) we obtain a sequence of curves $\Gamma_{n,N}$, $n \geq N + 1$, in the (α, ρ^*) -plane with parametric equations

$$\alpha = 2\rho^* + \frac{N^2 + n^2}{N^2 - n^2} \sqrt{\rho^{*2} - 1}. \quad (2.2.4)$$

Notice that the curves $\Gamma_{n,1}$, for $N = 1$, are the curves Γ_n defined by (1.3.11) in Section 1.3.1. Using the same arguments as in the case $k_* = k_{\max}$, we obtain the following stability result (see also Figure 2.1b).

Theorem 8. *Consider the Lugiato-Lefever equation (1.0.1) in the case $\beta = -1$ of anomalous dispersion, and the steady periodic solutions $\psi_\mu(x)$ given by (1.3.13), with wavelengths k_{\min} and the properties in Theorem 1.3.6. For any $N \geq 1$, consider the curve $\Gamma_{N+1,N}$ defined through (2.2.4). Then the following stability properties hold for periodic perturbations with periods $2\pi N/k_{\min}$, for any $N \geq 1$.*

- (i) *If (α, ρ^*) belongs to the open set situated to the left of the curve $\Gamma_{N+1,N}$ in the Turing instability region, then the steady periodic solutions found in the supercritical bifurcation are stable, whereas the steady periodic solutions found in the subcritical bifurcation are unstable and the instability is a shape instability.*
- (ii) *If (α, ρ^*) belongs to the open set situated to the right of the curve $\Gamma_{N+1,N}$ in the Turing instability region, then the steady periodic solutions are unstable; the instability is a background instability for the solutions found in the supercritical bifurcation, whereas the solutions found in the subcritical bifurcation are both background and shape unstable.*

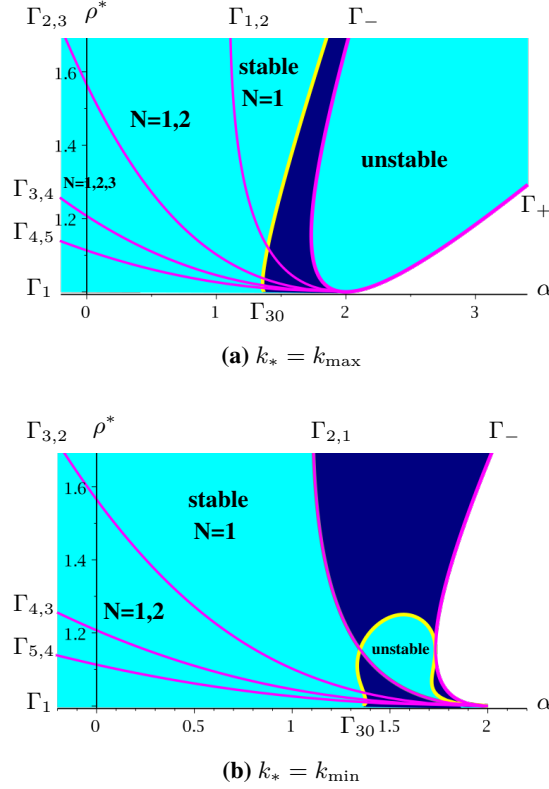


Figure 2.1 – Stability of steady periodic solutions in the case of anomalous dispersion. In the (α, ρ^*) -plane, the steady periodic solutions are background stable for subharmonic perturbations with periods $2\pi N/k_*$ for the values of N as indicated, and unstable otherwise. In the lighter shaded regions (light blue online) they are shape stable, whereas in the darker shaded regions (navy blue online) they are shape unstable. The curves $\Gamma_{n,N}$ are defined by (2.2.2) in the case $k_* = k_{\max}$ and by (2.2.4) in the case $k_* = k_{\min}$.

2.3 The case of normal dispersion

The stability analysis in the case of normal dispersion is similar to the one in Section 2.1. The linear operator \mathcal{A}_μ in (2.1.1) does not change, and we conclude that the periodic waves are unstable with the properties in Theorem 32 (ii). We summarize these results in the following theorem.

Theorem 9. *Consider the Lugiato-Lefever equation (1.0.1) in the case $\beta = 1$ of normal dispersion. Assume that $\alpha > 2$ and $F^2 = F_1^2 + \mu$, $F_1^2 = (1 - \alpha)^2 + 1$. For any $\mu < 0$ sufficiently small the following properties hold.*

- (i) *The equation possesses an even periodic solution $\psi_\mu(x)$ with wavenumber $\sqrt{\alpha - 2}$ and Taylor expansion (3.2.7), which is unstable with respect to co-periodic perturbations.*
- (ii) *The spectrum of the linear operator \mathcal{A}_μ obtained by linearizing the equation at this periodic solution and acting in either $L^2(\mathbb{R}) \times L^2(\mathbb{R})$ or $C_b(\mathbb{R}) \times C_b(\mathbb{R})$ contains a finite interval of positive real numbers $(0, c_\mu]$, with $c_\mu = \mathcal{O}(|\mu|)$. Consequently, the periodic solution $\psi_\mu(x)$ is spectrally unstable with respect to localized and bounded perturbations.*

2.3.1 Stability of periodic solutions

As in the case of anomalous dispersion, in the Turing instability region we expect bifurcations of steady periodic solutions at wavelengths $k_* = \tilde{k}_{\min}$ and $k_* = \tilde{k}_{\max}$, where $\tilde{k}_{\min} < \tilde{k}_{\max}$ are such that $\lambda_1(\tilde{k}_{\min}) = \lambda_1(\tilde{k}_{\max}) = 0$, whereas in the zero-mode instability region we expect bifurcations of steady

2.3. THE CASE OF NORMAL DISPERSION

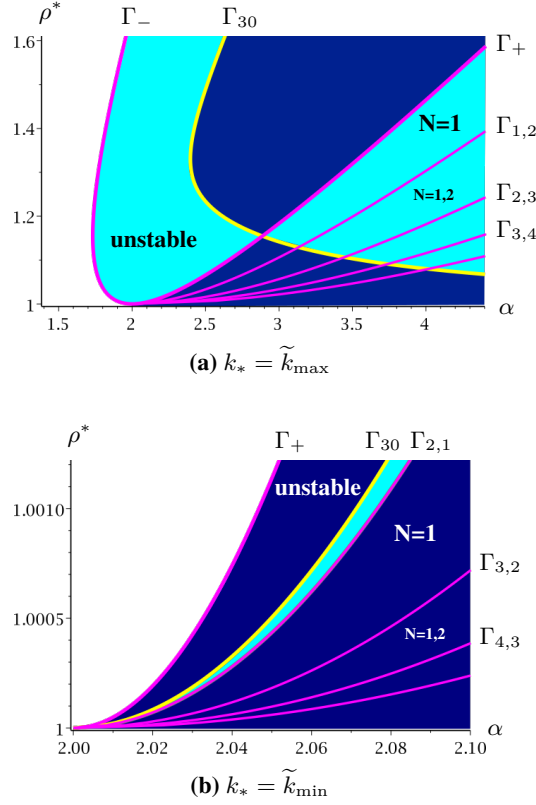


Figure 3.1 – Stability of steady periodic solutions in the case of normal dispersion. In the (α, ρ^*) -plane, the steady periodic solutions are background stable for subharmonic perturbations with periods $2\pi N/k_*$ for the values of N as indicated, and unstable otherwise. In the lighter shaded regions (light blue online) they are shape stable, whereas in the darker shaded regions (navy blue online) they are shape unstable. The curves $\Gamma_{n,N}$ are defined by (2.2.4) in the case $k_* = \tilde{k}_{\max}$ and by (2.2.2) in the case $k_* = \tilde{k}_{\min}$.

periodic solutions at wavelengths $k_* = \tilde{k}_{\max}$. Since the difference between the formulas of $a(k)$ in the two cases is only a change of the sign of the coefficient of k^2 , we find that

$$\tilde{k}_{\max}^2 = -k_{\min}^2, \quad \tilde{k}_{\min}^2 = -k_{\max}^2,$$

where k_{\max} and k_{\min} are the wavelengths given by (1.3.2) and (1.3.3), respectively. Together with the fact that in the Lugiato-Lefever equation (1.0.1) the difference between the two cases of normal and anomalous dispersion is the sign of β , this implies that the dynamical systems found for $k_* = \tilde{k}_{\max}$ and $k_* = \tilde{k}_{\min}$ in the case of normal dispersion are precisely the ones found in the case of anomalous dispersion for $k_* = k_{\min}$ and $k_* = k_{\max}$, respectively. As a consequence, the calculations are the same as the ones in Sections 1.3.1 and 2.2.1, with the only difference that (α, ρ^*) belongs to a different parameter region in the case of the Turing instability. We do not give the precise statements in this case, but only summarize the results in Figure 3.1.

CHAPTER 2. SPECTRAL STABILITY OF PERIODIC WAVES FOR THE LUGIATO-LEFEVER EQUATION

Chapter 3

Asymptotic stability for the Lugiato-Lefever equation

3.1 The well-posedness problem

In this section, we consider the well-posedness of the initial value problem associated with (25). We follow the classical path introduced for nonlinear Schrödinger problems using a Picard fixed point method for the Duhamel formulation. This requires estimates on the linear evolution part. Here we choose to consider Strichartz estimates which we prove in the next paragraph.

Naturally one can question the interest of proving the local well-posedness while the proof of the asymptotic stability provides a global well-posedness in a neighborhood of an equilibrium. The question has a simple answer: The well-posedness for (25) does not seem to have been considered in details, up to our knowledge, in the bibliography. But here we have stronger motivation. In order to analyze the stability of a solution, we need to prove the existence of solutions in its neighborhood. The first step in these strategy is to rewrite solutions of (25) on a neighborhood of a periodic solution into solutions of an *ad hoc* system. It may seem natural that well-posedness for the latter system provides some well-posedness for (25). We think that it has to be proved. First, one has to prove of the well-posedness of the system, then the one of (25) and eventually provide the equivalence of both equations. We believe these requires more work than simply proving local well-posedness for (25) and then decompose the solutions and analyze the resulting system.

The strategy developed in this section is suggested in [23].

3.1.1 Strichartz estimates for the Lugiato-Lefever equation

In Lemma 5, we revisit and adapt the proof of Strichartz estimates by J.Bourgain [1]. From Lemma 6, we take into account the dissipation to make them global in time.

Let

$$A := -i\beta(2 - \alpha)\partial_y^2 - (1 + i\alpha)$$

acting on $L^2(\mathbb{T}_N)$ with domain $D(A) = H^2(\mathbb{T}_N)$. Notice that $B := A + 1$ is, up to constants, the one-dimensional laplacian and is thus a skew-adjoint operator. The spectrum of A is given by

Lemma 3. *The spectrum of the operator A is purely discrete and given by*

$$\sigma(A) = \left\{ \lambda_n = -1 + i \left(\beta(2 - \alpha) \frac{n^2}{N^2} - \alpha \right); n \in \mathbb{Z} \right\}. \quad (3.1.1)$$

Moreover, the first eigenvalue is simple and corresponds to constant functions while all the others have multiplicity two.

Proof. Let $v \in L^2(\mathbb{T}_N)$ and $\lambda \in \mathbb{C}$. Using Fourier series, we solve in $u \in D(A)$ the equation

$$(A - \lambda)u = v \iff \forall n \in \mathbb{Z}, \left(i\beta(2 - \alpha)\left(\frac{n}{N}\right)^2 - (1 + i\alpha) - \lambda \right) u_n = v_n. \quad (3.1.2)$$

The second equation in (3.1.2) has a unique square integrable solution for any square integrable sequence $(v_n)_{n \in \mathbb{Z}}$ if and only if λ is not in

$$S := \left\{ -1 + i - \beta(2 - \alpha)\frac{n^2}{N^2} - \alpha, n \in \mathbb{Z} \right\}.$$

If λ is in S , then

$$\lambda = -1 + i - \beta(2 - \alpha)\frac{n_0^2}{N^2} - \alpha$$

for some n_0 in \mathbb{Z} and the second equation in (3.1.2) has a solution if and only if $v_n = 0$ for $n = \pm n_0$. In this case λ is an eigenvalue and setting $v = 0$, in the second equation in (3.1.2), we have only two independent solutions u if $n_0 \neq 0$ and one if $n_0 = 0$. \square

The operator $B := A + 1$ generates a unitary group and so we have the

Lemma 4. *The operator A generates a group which restricted to positive times is a contraction semigroup with*

$$\left\| e^{tA}\psi_0 \right\|_{L^2(\mathbb{T}_N)} = e^{-t} \|\psi_0\|_{L^2(\mathbb{T}_N)}, \quad \forall t \in \mathbb{R}. \quad (3.1.3)$$

Proof. This follows from a Fourier analysis. \square

Remark 20. *This result extends to any $H^s(\mathbb{T}_N)$ space as the semigroup generated by B is invariant and unitary for their canonical norms.*

Notice that for $s > 1/2$, one can use this dissipation properties of the equation to prove the well-posedness of (25) in $H^s(\mathbb{T}_N)$ algebra.

We now prove the first Strichartz estimates.

Lemma 5. *Let t_0 in \mathbb{R}_+ and $\tau > t_0$. Define:*

$$C_1(\tau, t_0) = \left(\frac{\tau - t_0}{\pi N} + \frac{4N}{2 - \alpha} \right)^{1/4}.$$

(i) *For any $\psi_{t_0} \in L^2(\mathbb{T}_N)$, $t \mapsto e^{(t-t_0)A}\psi_{t_0}$ is in $L^\infty((t_0, \tau); L^2(\mathbb{T}_N)) \cap L^4((t_0, \tau) \times \mathbb{T}_N)$ and*

$$\left\| e^{(t-t_0)A}\psi_{t_0} \right\|_{L^\infty((t_0, \tau); L^2(\mathbb{T}_N))} = \|\psi_{t_0}\|_{L^2(\mathbb{T}_N)}, \quad (3.1.4)$$

$$\left\| e^{(t-t_0)A}\psi_{t_0} \right\|_{L^4((t_0, \tau) \times \mathbb{T}_N)} \leq C_1(\tau, t_0) \|\psi_{t_0}\|_{L^2(\mathbb{T}_N)}. \quad (3.1.5)$$

(ii) *For any f in $L^{4/3}((t_0, \tau) \times \mathbb{T}_N)$, $\int_{t_0}^\tau e^{-s} e^{-sA} f(s) ds$ is in $L^2(\mathbb{T}_N)$ and*

$$\left\| \int_{t_0}^\tau e^{-s} e^{-sA} f(s) ds \right\|_{L^2(\mathbb{T}_N)} \leq C_1(\tau, t_0) \|f\|_{L^{4/3}((t_0, \tau) \times \mathbb{T}_N)}. \quad (3.1.6)$$

3.1. THE WELL-POSEDNESS PROBLEM

(iii) For any f in $L^{4/3}((t_0, \tau) \times \mathbb{T}_N)$, $t \mapsto \int_{t_0}^t e^{(t-s)A} f(s) ds$ is in $L^\infty((t_0, \tau); L^2(\mathbb{T}_N)) \cap L^4((t_0, \tau) \times \mathbb{T}_N)$ and

$$\left\| \int_{t_0}^t e^{t-s} e^{(t-s)A} f(s) ds \right\|_{L^\infty((t_0, \tau); L^2(\mathbb{T}_N))} \leq 2^{-1/2} (1 - 2^{-1/2})^{-1} C_1(\tau, t_0) \|f\|_{L^{4/3}((t_0, \tau) \times \mathbb{T}_N)}, \quad (3.1.7)$$

$$\left\| \int_{t_0}^t e^{t-s} e^{(t-s)A} f(s) ds \right\|_{L^4((t_0, \tau) \times \mathbb{T}_N)} \leq 2^{-3/4} (1 - 2^{-3/4})^{-1} C_1(\tau, t_0)^2 \|f\|_{L^{4/3}((t_0, \tau) \times \mathbb{T}_N)}. \quad (3.1.8)$$

Proof. In this proof, we set $B := A + 1$.

Estimate (3.1.4) follows from Lemma 4

Let us prove (3.1.5). Following [1, Proposition 2.1], as $\psi_{t_0} \in L^2(\mathbb{T}_N)$, we set

$$\psi_{t_0} = \frac{1}{\sqrt{2\pi N}} \sum_{m=-\infty}^{+\infty} \widehat{\psi}_{t_0}(m) e^{i \frac{m}{N} y}$$

thus

$$e^{(t-t_0)B} \psi_{t_0} = \frac{1}{\sqrt{2\pi N}} \sum_{m=-\infty}^{+\infty} \widehat{\psi}_{t_0}(m) e^{i(\beta(2-\alpha) \frac{m^2}{N^2} (t-t_0) - \alpha(t-t_0) + \frac{m}{N} y)}$$

Then,

$$\begin{aligned} \left\| e^{(t-t_0)B} \psi_{t_0} \right\|_{L^4((t_0, \tau) \times \mathbb{T}_N)}^4 &= \int_{\mathbb{T}_N} \int_{t_0}^{\tau} \left| e^{(t-t_0)B} \psi_{t_0} \right|^4 dt dy \\ &= \frac{1}{(2\pi N)^2} \int_{\mathbb{T}_N} \int_{t_0}^{\tau} \left| \sum_{m=-\infty}^{+\infty} \widehat{\psi}_{t_0}(m) e^{i(\beta(2-\alpha) \frac{m^2}{N^2} (t-t_0) - \alpha(t-t_0) + \frac{m}{N} y)} \right|^4 dt dy \end{aligned}$$

As $0 \leq t_0 < \tau$, there exists $k \in \mathbb{N}$ and $r \in [0, 2\pi \frac{N^2}{2-\alpha})$ such that

$$\tau = t_0 + 2k \frac{N^2}{2-\alpha} \pi + r.$$

Hence, as $\tau < t_0 + 2(k+1) \frac{N^2}{2-\alpha} \pi$, we have

$$\begin{aligned} \left\| e^{(t-t_0)B} \psi_{t_0} \right\|_{L^4((t_0, \tau) \times \mathbb{T}_N)}^4 &\leq \frac{1}{(2\pi N)^2} \int_{\mathbb{T}_N} \int_{t_0}^{t_0 + 2(k+1) \frac{N^2}{2-\alpha} \pi} \left| \sum_{m=-\infty}^{+\infty} \widehat{\psi}_{t_0}(m) e^{i(\beta(2-\alpha) \frac{m^2}{N^2} (t-t_0) + \frac{m}{N} y)} \right|^4 dt dy \\ &= \frac{1}{(2\pi N)^2} \int_{\mathbb{T}_N} \int_{t_0}^{t_0 + 2(k+1) \frac{N^2}{2-\alpha} \pi} \sum_{m_1, m_2, m_3, m_4} \widehat{\psi}_{t_0}(m_1) \widehat{\psi}_{t_0}(m_2) \overline{\widehat{\psi}_{t_0}(m_3)} \overline{\widehat{\psi}_{t_0}(m_4)} \\ &\quad \times e^{i\beta(2-\alpha) \frac{m_1^2 + m_2^2 - m_3^2 - m_4^2}{N^2} (t-t_0)} e^{i \frac{m_1 + m_2 - m_3 - m_4}{N} y} dt dy. \end{aligned}$$

The last term is different from zero if and only if

$$m_1^2 + m_2^2 - m_3^2 - m_4^2 = 0 \quad \text{and} \quad m_1 + m_2 - m_3 - m_4 = 0,$$

which is equivalent to

$$\begin{cases} m_1 = m_3 \\ m_2 = m_4 \end{cases} \quad \text{or} \quad \begin{cases} m_1 = m_4 \\ m_2 = m_3 \end{cases}.$$

Then,

$$\begin{aligned}
 \left\| e^{(t-t_0)B} \psi_{t_0} \right\|_{L^4((t_0, \tau) \times \mathbb{T}_N)}^4 &\leq \frac{2}{(2\pi N)^2} \int_{\mathbb{T}_N} \int_{t_0}^{t_0 + 2(k+1)\frac{N^2}{2-\alpha}\pi} \sum_{m_1, m_2} \left| \widehat{\psi_{t_0}}(m_1) \right|^2 \left| \widehat{\psi_{t_0}}(m_2) \right|^2 dt dy \\
 &= \frac{2}{(2\pi N)^2} \|\psi_{t_0}\|_{L^2(\mathbb{T}_N)}^4 \times 2\pi N \times 2(k+1) \frac{N^2}{2-\alpha} \pi \\
 &\leq \frac{2(\tau - t_0 + 2\frac{N^2}{2-\alpha}\pi)}{2\pi N} \|\psi_{t_0}\|_{L^2(\mathbb{T}_N)}^4 \\
 &\leq \left(\frac{\tau - t_0}{\pi N} + \frac{4N}{2-\alpha} \right) \|\psi_{t_0}\|_{L^2(\mathbb{T}_N)}^4
 \end{aligned}$$

thus

$$\left\| e^{(t-t_0)B} \right\|_{L^4((t_0, \tau) \times \mathbb{T}_N)} \leq \left(\frac{\tau - t_0}{\pi N} + \frac{4N}{2-\alpha} \right)^{1/4} \|\psi_{t_0}\|_{L^2(\mathbb{T}_N)}.$$

Eventually, we have

$$\begin{aligned}
 \left\| e^{t-t_0} e^{(\tau-t_0)A} \right\|_{L^4((t_0, \tau) \times \mathbb{T}_N)} &= \left\| e^{(\tau-t_0)B} \right\|_{L^4((t_0, \tau) \times \mathbb{T}_N)} \\
 &\leq \left(\frac{\tau - t_0}{\pi N} + \frac{4N}{2-\alpha} \right)^{1/4} \|\psi_{t_0}\|_{L^2(\mathbb{T}_N)}.
 \end{aligned}$$

To prove (3.1.6), we observe that by duality as B is a unitary operator

$$\left\| \int_{t_0}^{\tau} e^{-sB} f(s) ds \right\|_{L^2(\mathbb{T}_N)} \leq \left(\frac{\tau - t_0}{\pi N} + \frac{4N}{2-\alpha} \right)^{1/4} \|f\|_{L^{4/3}((t_0, \tau) \times \mathbb{T}_N)},$$

or

$$\left\| \int_{t_0}^{\tau} e^{-s} e^{-sA} f(s) ds \right\|_{L^2(\mathbb{T}_N)} \leq \left(\frac{\tau - t_0}{\pi N} + \frac{4N}{2-\alpha} \right)^{1/4} \|f\|_{L^{4/3}((t_0, \tau) \times \mathbb{T}_N)}.$$

We now prove (3.1.7). Using inequalities (3.1.4) and (3.1.6), we have

$$\begin{aligned}
 \left\| \int_{t_0}^{\tau} e^{(t-s)B} f(s) ds \right\|_{L^\infty((t_0, \tau); L^2(\mathbb{T}_N))} &\leq \left\| \int_{t_0}^{\tau} e^{-sB} f(s) ds \right\|_{L^2(\mathbb{T}_N)} \\
 &\leq C_1(\tau, t_0) \|f\|_{L^{4/3}((t_0, \tau) \times \mathbb{T}_N)}.
 \end{aligned}$$

Then, by Christ-Kiselev lemma [2], we obtain

$$\left\| \int_{t_0}^t e^{(t-s)B} f(s) ds \right\|_{L^\infty((t_0, \tau); L^2(\mathbb{T}_N))} \leq 2^{-1/2} (1 - 2^{-1/2})^{-1} C_1(\tau, 0) \|f\|_{L^{4/3}((t_0, \tau) \times \mathbb{T}_N)},$$

thus

$$\begin{aligned}
 \left\| \int_{t_0}^t e^{t-s} e^{(t-s)A} f(s) ds \right\|_{L^\infty((t_0, \tau); L^2(\mathbb{T}_N))} &= \left\| \int_{t_0}^t e^{(t-s)B} f(s) ds \right\|_{L^\infty((t_0, \tau); L^2(\mathbb{T}_N))} \\
 &\leq 2^{-1/2} (1 - 2^{-1/2})^{-1} C_1(\tau, 0) \|f\|_{L^{4/3}((t_0, \tau) \times \mathbb{T}_N)}.
 \end{aligned}$$

Similarly, we prove (3.1.8). We have with inequalities (3.1.5) and (3.1.6) that

$$\left\| \int_{t_0}^{\tau} e^{(t-s)B} f(s) ds \right\|_{L^4((t_0, \tau) \times \mathbb{T}_N)} \leq C_1(\tau, t_0)^2 \|f\|_{L^{4/3}((t_0, \tau) \times \mathbb{T}_N)}.$$

Then, by Christ-Kiselev lemma [2], we obtain

$$\left\| \int_{t_0}^t e^{(t-s)B} f(s) \, ds \right\|_{L^4((t_0, \tau) \times \mathbb{T}_N)} \leq 2^{-3/4} (1 - 2^{-3/4})^{-1} C_1(\tau, t_0)^2 \|f\|_{L^{4/3}((t_0, \tau) \times \mathbb{T}_N)},$$

thus

$$\begin{aligned} \left\| \int_{t_0}^t e^{t-s} e^{(t-s)A} f(s) \, ds \right\|_{L^4((t_0, \tau) \times \mathbb{T}_N)} &= \left\| \int_{t_0}^t e^{(t-s)B} f(s) \, ds \right\|_{L^4((t_0, \tau) \times \mathbb{T}_N)} \\ &\leq 2^{-3/4} (1 - 2^{-3/4})^{-1} C_1(\tau, t_0)^2 \|f\|_{L^{4/3}((t_0, \tau) \times \mathbb{T}_N)}. \end{aligned}$$

This concludes the proof. \square

Remark 21. Note for later use, see Lemma 15, that this proof and the following ones works similarly with $-B$ instead of B .

Remark 22. Note that for any ψ in $L^2(\mathbb{T}_N)$ and t_0 in \mathbb{R} , the map $\kappa_{\psi_0} : \tau \mapsto \left\| e^{\cdot A} \psi_0 \right\|_{L^4((0, \tau) \times \mathbb{T}_N)}^4$ is absolutely continuous on \mathbb{R}^+ . This fact will be used for other quantities without any further mention.

Similarly, as $t \in [0, \infty) \mapsto e^{tA} \psi_0 \in L^2(\mathbb{R})$ is continuous, everywhere the space $L^\infty((t_0, \tau), L^2(\mathbb{T}_N))$ can be replaced by $C([t_0, \tau], L^2(\mathbb{T}_N))$.

Remark 23. Since F in (25) is in the kernel of $A + 1 + i\alpha$, we have for any $\tau > 0$ and $t \in (0, \tau)$

$$\int_0^t e^{(t-s)A} F \, ds = \frac{F}{(1 + i\alpha)} (1 - e^{-(1+i\alpha)t})$$

and then

$$\begin{aligned} \left\| \int_0^t e^{(t-s)A} F \, ds \right\|_{L^\infty((0, \tau); L^2(\mathbb{T}_N))} &= \frac{\sqrt{2\pi N} |F|}{\sqrt{1 + \alpha^2}} \sup_{t \in (0, \tau)} |1 - e^{-(1+i\alpha)t}| \\ &\leq \frac{2\sqrt{2\pi N} |F|}{\sqrt{1 + \alpha^2}}. \end{aligned} \quad (3.1.9)$$

We also have

$$\begin{aligned} \left\| \int_0^t e^{(t-s)A} F \, ds \right\|_{L^4((0, \tau) \times \mathbb{T}_N)}^4 &= 2\pi N \int_0^\tau \left| \frac{F}{(1 + i\alpha)} (1 - e^{-(1+i\alpha)t}) \right|^4 dt \\ &= 2\pi N \frac{|F|^4}{(1 + \alpha^2)^2} \int_0^\tau |1 - e^{-(1+i\alpha)t}|^4 dt. \end{aligned} \quad (3.1.10)$$

A straightforward computation gives

$$\left\| (1 - e^{-(1+i\alpha)\cdot}) \right\|_{L^4(0, \tau)}^4 = \tau + \frac{1}{2} (e^{-2\tau} - 1) + \frac{2e^{-\tau}}{1 + \alpha^2} (\cos(\alpha\tau) + \alpha \sin(\alpha\tau)) - \frac{2}{1 + \alpha^2}.$$

Hence, $t \mapsto \int_0^t e^{(t-s)A} F \, ds$ is in $L^\infty((0, \tau); L^2(\mathbb{T}_N)) \cap L^4((0, \tau) \times \mathbb{T}_N)$.

In the rest of this section with prove extension of estimates (3.1.5) taking into account the dissipation.

Lemma 6. For any $\gamma > 0$, we have

$$\left\| e^{(1-\gamma)t} e^{tA} \psi_0 \right\|_{L^4(\mathbb{R}^+ \times \mathbb{T}_N)} \leq \left(\frac{6N}{(2 - \alpha)(1 - e^{-8\gamma \frac{N^2}{2-\alpha}\pi})} \right)^{1/4} \|\psi_0\|_{L^2(\mathbb{T}_N)}. \quad (3.1.11)$$

Proof. Again in this proof, we set $B := A + 1$. From (3.1.5), we deduce

$$\begin{aligned} \left\| e^{-\gamma t} e^{tB} \psi_0 \right\|_{L^4(\mathbb{R}^+ \times \mathbb{T}_N)}^4 &\leq \sum_{k \in \mathbb{N} \cup \{0\}} e^{-8k\gamma \frac{N^2}{2-\alpha} \pi} \left\| e^{tB} 0 \right\|_{L^4((2k \frac{N^2}{2-\alpha} \pi, 2(k+1) \frac{N^2}{2-\alpha} \pi) \times \mathbb{T}_N)}^4 \\ &\leq \frac{6N}{(2-\alpha)} \frac{1}{1 - e^{-8\gamma \frac{N^2}{2-\alpha} \pi}} \left\| \psi_0 \right\|_{L^2(\mathbb{T}_N)}^4. \end{aligned}$$

This conclude the proof. \square

We also have the following

Lemma 7. For any $\gamma > 0$, we have for any $\tau > 0$

$$\left\| \int_0^\tau e^{(-1-\gamma)s} e^{-sA} f(s) ds \right\|_{L^2(\mathbb{T}_N)} \leq \left(\frac{6N}{(2-\alpha)(1 - e^{-8\gamma \frac{N^2}{2-\alpha} \pi})} \right)^{1/4} \|f\|_{L^{4/3}((0,\tau) \times \mathbb{T}_N)}. \quad (3.1.12)$$

Proof. From (3.1.6), we have that

$$\left\| \int_0^\tau e^{(-1-\gamma)s} e^{-sA} f(s) ds \right\|_{L^2(\mathbb{T}_N)}$$

is finite. Since B is a unitary operator, using Hölder inequality and inequality (3.1.11), we have

$$\begin{aligned} \left\| \int_0^\tau e^{-\gamma s} e^{-sB} f(s) ds \right\|_{L^2(\mathbb{T}_N)}^2 &= \left\langle \int_0^\tau e^{-\gamma s} e^{-sB} f(s) ds, \int_{t_0}^\tau e^{-\gamma s_0} e^{-s_0 B} f(s_0) ds_0 \right\rangle_{L^2(\mathbb{T}_N)} \\ &= \int_0^\tau \left\langle f(s), e^{-\gamma s} e^{sB} \int_0^\tau e^{-\gamma s_0} e^{-s_0 B} f(s_0) ds_0 \right\rangle_{L^2(\mathbb{T}_N)} ds \\ &= \left\langle f, e^{-\gamma s} e^{sB} \int_0^\tau e^{-\gamma s_0} e^{-s_0 B} f(s_0) ds_0 \right\rangle_{L^2((0,\tau) \times \mathbb{T}_N)} \\ &\leq \|f\|_{L^{4/3}((0,\tau) \times \mathbb{T}_N)} \left\| e^{-\gamma s} e^{sB} \int_0^\tau e^{-\gamma s_0} e^{-s_0 B} f(s_0) ds_0 \right\|_{L^4((0,\tau) \times \mathbb{T}_N)} \\ &\leq \left(\frac{6N}{(2-\alpha)(1 - e^{-8\gamma \frac{N^2}{2-\alpha} \pi})} \right)^{1/4} \|f\|_{L^{4/3}((0,\tau) \times \mathbb{T}_N)} \left\| \int_0^\tau e^{-\gamma s_0} e^{-s_0 B} f(s_0) ds_0 \right\|_{L^2(\mathbb{T}_N)}. \end{aligned}$$

Either we have $\left\| \int_0^\tau e^{-\gamma s_0} e^{-s_0 B} f(s_0) ds_0 \right\|_{L^2(\mathbb{T}_N)} = 0$ or

$$\left\| \int_0^\tau e^{-\gamma s} e^{-sB} f(s) ds \right\|_{L^2(\mathbb{T}_N)} \leq \left(\frac{6N}{(2-\alpha)(1 - e^{-8\gamma \frac{N^2}{2-\alpha} \pi})} \right)^{1/4} \|f\|_{L^{4/3}((0,\tau) \times \mathbb{T}_N)}.$$

This concludes the proof. \square

Note that Lemma 7 can be rephrased for any $\gamma > 0$,

$$\sup_{\tau > 0} \left\| \int_0^\tau e^{(-1-\gamma)s} e^{-sA} f(s) ds \right\|_{L^2(\mathbb{T}_N)} \leq \left(\frac{6N}{(2-\alpha)(1 - e^{-8\gamma \frac{N^2}{2-\alpha} \pi})} \right)^{1/4} \|f\|_{L^{4/3}(\mathbb{R}^+ \times \mathbb{T}_N)}, \quad (3.1.13)$$

for any f in $L^{4/3}(\mathbb{R}^+ \times \mathbb{T}_N)$, combined with (3.1.4), they give the

Lemma 8. For any $\gamma > 0$, we have

$$\left\| \int_0^t e^{t-s} e^{-\gamma(s+t)} e^{(t-s)A} f(s) \, ds \right\|_{L^\infty(\mathbb{R}^+, L^2(\mathbb{T}_N))} \leq \left(\frac{6N}{(2-\alpha)} \right)^{1/2} \frac{1}{(1 - e^{-8\gamma \frac{N^2}{2-\alpha} \pi})^{1/4}} \|f\|_{L^{4/3}(\mathbb{R}^+ \times \mathbb{T}_N)}. \quad (3.1.14)$$

Proof. This follows from the comment preceding the lemma (thus using (3.1.4) that is using $A + 1$ generates a unitary group). \square

It now remains to state the

Lemma 9. For any $\gamma > 0$, we have

$$\left\| \int_0^t e^{t-s} e^{-\gamma(s+t)} e^{(t-s)A} f(s) \, ds \right\|_{L^4(\mathbb{R}^+ \times \mathbb{T}_N)} \leq \left(\frac{6N}{(2-\alpha)(1 - e^{-8\gamma \frac{N^2}{2-\alpha} \pi})} \right)^{1/2} \|f\|_{L^{4/3}(\mathbb{R}^+ \times \mathbb{T}_N)}. \quad (3.1.15)$$

Proof. From Lemma 6 (with $2 - \gamma$ instead of γ), Lemma 7 (see (3.1.13)), we infer for any $\tau > 0$

$$\left\| \int_0^\tau e^{t-s} e^{-\gamma(s+t)} e^{(t-s)A} f(s) \, ds \right\|_{L^4(\mathbb{R}^+ \times \mathbb{T}_N)} \leq \left(\frac{6N}{(2-\alpha)(1 - e^{-8\gamma \frac{N^2}{2-\alpha} \pi})} \right)^{1/2} \|f\|_{L^{4/3}(\mathbb{R}^+ \times \mathbb{T}_N)}.$$

Then we conclude with Christ-Kisilev lemma as in the proof of (3.1.8). \square

3.1.2 Existence and uniqueness of solutions of the Lugiato-Lefever equation

We consider the Lugiato-Lefever equation (25) with $\psi(0) = \psi_0$ in $L^2(\mathbb{T}_N)$. We define the function $\widetilde{\psi}_0$ by

$$\widetilde{\psi}_0(t) = e^{tA} \psi_0 + \int_0^t e^{(t-s)A} F \, ds, \quad t \geq 0. \quad (3.1.16)$$

From estimates (3.1.9) and (3.1.10) and Remark 23 and Strichartz estimates (3.1.4) and (3.1.5) and Lemma 5, we have

$$\widetilde{\psi}_0 \in L^\infty((0, \tau); L^2(\mathbb{T}_N)) \cap L^4((0, \tau) \times \mathbb{T}_N)$$

for any $\tau > 0$.

For any $\tau > 0$ and $R > 0$, we define

$$\mathcal{B}_\tau(\widetilde{\psi}_0, R) := \left\{ \psi \in L^\infty((0, \tau); L^2(\mathbb{T}_N)) \cap L^4((0, \tau) \times \mathbb{T}_N); \|\psi - \widetilde{\psi}_0\|_{\mathcal{B}} \leq R \right\}, \quad (3.1.17)$$

and

$$\|\psi\|_{\mathcal{B}_\tau} = \max \left\{ \|\psi\|_{L^\infty((0, \tau); L^2(\mathbb{T}_N))}, \|\psi\|_{L^4((0, \tau) \times \mathbb{T}_N)} \right\}. \quad (3.1.18)$$

We define the map \mathcal{M} on $L^\infty((0, \tau); L^2(\mathbb{T}_N)) \cap L^4((0, \tau) \times \mathbb{T}_N)$ by

$$\forall t \in (0, \tau), \mathcal{M}(\psi)(t) := \widetilde{\psi}_0(t) + i \int_0^t e^{(t-s)A} \psi |\psi|^2 \, ds. \quad (3.1.19)$$

Lemma 10. For any ψ_0 in $L^2(\mathbb{T}_N)$, there exists R_1 a positive real such that for all $R \in (0, R_1)$ there exists τ_1 a positive real such that for all $\tau \in (0, \tau_1)$:

(i) The space $\mathcal{B}_{\tau_1}(\widetilde{\psi}_0, R)$ is stable by \mathcal{M} ,

(ii) The application \mathcal{M} defined on $\mathcal{B}_{\tau_1}(\widetilde{\psi}_0, R)$ is contractant.

Proof. The first assertion is proved as follows. Let $\psi \in \mathcal{B}_{\tau}(\widetilde{\psi}_0, R)$. Using estimates (3.1.7) and (3.1.8) from Lemma 5, we have

$$\left\| \mathcal{M}(\psi) - \widetilde{\psi}_0 \right\|_{\mathcal{B}_{\tau}} \leq C(\tau, 0) \left\| |\psi|^2 - \widetilde{\psi}_0 |\widetilde{\psi}_0|^2 \right\|_{L^{4/3}((0, \tau) \times \mathbb{T}_N)}$$

where $C(\tau, 0) = \max\{2^{-1/2}(1 - 2^{-1/2})^{-1}C_1(\tau, 0), 2^{-3/4}(1 - 2^{-3/4})^{-1}C_1(\tau, 0)^2\}$. Then, by Hölder inequality,

$$\begin{aligned} \left\| |\psi|^2 - \widetilde{\psi}_0 |\widetilde{\psi}_0|^2 \right\|_{L^{4/3}((0, \tau) \times \mathbb{T}_N)} &\leq \left\| |\psi|^2 - \widetilde{\psi}_0 |\widetilde{\psi}_0|^2 \right\|_{L^{4/3}((0, \tau) \times \mathbb{T}_N)} + \left\| \widetilde{\psi}_0 |\widetilde{\psi}_0|^2 \right\|_{L^{4/3}((0, \tau) \times \mathbb{T}_N)} \\ &\leq \left\| |\psi|^2 - \widetilde{\psi}_0 |\widetilde{\psi}_0|^2 \right\|_{L^{4/3}((0, \tau) \times \mathbb{T}_N)} + \left\| \widetilde{\psi}_0 \right\|_{L^4((0, \tau) \times \mathbb{T}_N)}^3. \end{aligned}$$

can certainly On the one hand, by the definition of κ_{ψ_0} (see Remark 22) and by Remark 23, we have the L^4 norm of

$$\left\| \widetilde{\psi}_0 \right\|_{L^4((0, \tau) \times \mathbb{T}_N)} \leq \kappa_{\psi_0}(\tau) + \frac{2(2\pi N)^{1/4}|F|}{\sqrt{1 + \alpha^2}} \tau^{1/4} \xrightarrow{\tau \rightarrow 0} 0.$$

On the other hand, as

$$|\psi|^2 - \widetilde{\psi}_0 |\widetilde{\psi}_0|^2 = \psi^2(\overline{\psi} - \overline{\widetilde{\psi}_0}) + \psi \overline{\widetilde{\psi}_0}(\psi - \widetilde{\psi}_0) + \widetilde{\psi}_0 \overline{\psi}(\psi - \widetilde{\psi}_0),$$

we have by Hölder inequality

$$\begin{aligned} &\left\| |\psi|^2 - \widetilde{\psi}_0 |\widetilde{\psi}_0|^2 \right\|_{L^{4/3}((0, \tau) \times \mathbb{T}_N)} \\ &\leq \left(\left\| \psi \right\|_{L^4((0, \tau) \times \mathbb{T}_N)}^2 + \left\| \psi \right\|_{L^4((0, \tau) \times \mathbb{T}_N)} \left\| \widetilde{\psi}_0 \right\|_{L^4((0, \tau) \times \mathbb{T}_N)} + \left\| \widetilde{\psi}_0 \right\|_{L^4((0, \tau) \times \mathbb{T}_N)}^2 \right) \left\| \psi - \widetilde{\psi}_0 \right\|_{L^4((0, \tau) \times \mathbb{T}_N)}. \end{aligned}$$

As $\psi \in \mathcal{B}_{\tau}(\widetilde{\psi}_0, R)$, we have that $\left\| \psi - \widetilde{\psi}_0 \right\|_{L^4((0, \tau) \times \mathbb{T}_N)} \leq R$ and

$$\begin{aligned} &\left\| \psi \right\|_{L^4((0, \tau) \times \mathbb{T}_N)}^2 + \left\| \psi \right\|_{L^4((0, \tau) \times \mathbb{T}_N)} \left\| \widetilde{\psi}_0 \right\|_{L^4((0, \tau) \times \mathbb{T}_N)} + \left\| \widetilde{\psi}_0 \right\|_{L^4((0, \tau) \times \mathbb{T}_N)}^2 \\ &\leq 2 \left\| \psi - \widetilde{\psi}_0 \right\|_{L^4((0, \tau) \times \mathbb{T}_N)}^2 + \left\| \psi - \widetilde{\psi}_0 \right\|_{L^4((0, \tau) \times \mathbb{T}_N)} \left\| \widetilde{\psi}_0 \right\|_{L^4((0, \tau) \times \mathbb{T}_N)} + 4 \left\| \widetilde{\psi}_0 \right\|_{L^4((0, \tau) \times \mathbb{T}_N)}^2 \\ &\leq 2R^2 + Rf_1(\tau) + 4f_1(\tau)^2, \end{aligned}$$

where

$$f_1(\tau) := \kappa_{\psi_0}(\tau) + \frac{2(2\pi N)^{1/4}|F|}{\sqrt{1 + \alpha^2}} \tau^{1/4}$$

Hence, we obtain that

$$\left\| |\psi|^2 - \widetilde{\psi}_0 |\widetilde{\psi}_0|^2 \right\|_{L^{4/3}((0, \tau) \times \mathbb{T}_N)} \leq 2R^3 + R^2 f_1(\tau) + 4Rf_1(\tau)^2,$$

thus

$$\left\| \mathcal{M}(\psi) - \widetilde{\psi}_0 \right\|_{\mathcal{B}_{\tau}} \leq 2C(\tau, 0)(R + f_1(\tau))^3.$$

Since f_1 is increasing from 0 to $+\infty$, it is invertible. For $\tau_0 > 0$, $R \leq \sqrt{\frac{1}{16C(\tau_0, 0)}}$ and $\tau \leq \tau_1 := \min\{\tau_0, f_1^{-1}(R)\}$, we have

$$\left\| \mathcal{M}(\psi) - \widetilde{\psi}_0 \right\|_{\mathcal{B}} \leq R,$$

thus $\mathcal{M}(\psi) \in \mathcal{B}_\tau(\widetilde{\psi}_0, R)$.

The second assertion is proved as follows. Let $\psi_1, \psi_2 \in \mathcal{B}_\tau(\widetilde{\psi}_0, R)$. By estimate (3.1.7) from Lemma 5, we have

$$\begin{aligned} \|\mathcal{M}(\psi_1) - \mathcal{M}(\psi_2)\|_{L^\infty((0,\tau);L^2(\mathbb{T}_N))} &= \left\| \int_0^t e^{(t-s)A} (\psi_1|\psi_1|^2 - \psi_2|\psi_2|^2) ds \right\|_{L^\infty((0,\tau);L^2(\mathbb{T}_N))} \\ &\leq 2^{-1/2}(1 - 2^{-1/2})^{-1}C_1(\tau, 0) \left\| \psi_1|\psi_1|^2 - 2|\psi_2|^2 \right\|_{L^{4/3}((0,\tau)\times\mathbb{T}_N)} \end{aligned}$$

and estimate (3.1.8) gives

$$\begin{aligned} \|\mathcal{M}(\psi_1) - \mathcal{M}(\psi_2)\|_{L^4((0,\tau)\times\mathbb{T}_N)} &= \left\| \int_0^t e^{(t-s)A} (\psi_1|\psi_1|^2 - \psi_2|\psi_2|^2) ds \right\|_{L^4((0,\tau)\times\mathbb{T}_N)} \\ &\leq 2^{-3/4}(1 - 2^{-3/4})^{-1}C_1(\tau, 0)^2 \left\| \psi_1|\psi_1|^2 - 2|\psi_2|^2 \right\|_{L^{4/3}((0,\tau)\times\mathbb{T}_N)}, \end{aligned}$$

thus

$$\|\mathcal{M}(\psi_1) - \mathcal{M}(\psi_2)\|_{\mathcal{B}_\tau} \leq C(\tau, 0) \left\| \psi_1|\psi_1|^2 - 2|\psi_2|^2 \right\|_{L^{4/3}((0,\tau)\times\mathbb{T}_N)}.$$

Then, by Hölder inequality, we have that

$$\begin{aligned} &\left\| \psi_1|\psi_1|^2 - 2|\psi_2|^2 \right\|_{L^{4/3}((0,\tau)\times\mathbb{T}_N)} \\ &\leq \left(\|\psi_1\|_{L^4((0,\tau)\times\mathbb{T}_N)}^2 + \|\psi_1\|_{L^4((0,\tau)\times\mathbb{T}_N)} \|\psi_2\|_{L^4((0,\tau)\times\mathbb{T}_N)} + \|\psi_2\|_{L^4((0,\tau)\times\mathbb{T}_N)}^2 \right) \|\psi_1 - \psi_2\|_{L^4((0,\tau)\times\mathbb{T}_N)}. \end{aligned}$$

Since

$$\|\psi_1\|_{L^4((0,\tau)\times\mathbb{T}_N)}^2 + \|\psi_1\|_{L^4((0,\tau)\times\mathbb{T}_N)} \|\psi_2\|_{L^4((0,\tau)\times\mathbb{T}_N)} + \|\psi_2\|_{L^4((0,\tau)\times\mathbb{T}_N)}^2 \leq 3(R + f_1(\tau))^2$$

we deduce

$$\left\| \psi_1|\psi_1|^2 - 2|\psi_2|^2 \right\|_{L^{4/3}((0,\tau)\times\mathbb{T}_N)} \leq 3C(\tau, 0)(R + f_1(\tau))^2 \|\psi_1 - \psi_2\|_{L^4((0,\tau)\times\mathbb{T}_N)}.$$

Let $\tau_0 > 0$. For $\kappa \in (0, 1)$, $R \leq \sqrt{\frac{\kappa}{12C(\tau_0, 0)}}$ and $\tau \leq \tau_1 := \min\{\tau_0, f_1^{-1}(R)\}$.

$$\|\mathcal{M}(\psi_1) - \mathcal{M}(\psi_2)\|_{\mathcal{B}_\tau} \leq \kappa \|\psi_1 - \psi_2\|_{\mathcal{B}_\tau}.$$

This concludes the proof. \square

The above lemma provides the framework for a Picard fixed point theorem. Elaborating on that we get the

Théorème 24. *For any ψ_0 in $L^2(\mathbb{T}_N)$, there exists $T^+(\psi_0)$ such that the Cauchy problem (25) with initial data $\psi(t=0) = \psi_0$ has a unique maximal mild solution $([0, T^+(\psi_0)), \psi$ in*

$$L^\infty((0, T^+(\psi_0)); L^2(\mathbb{T}_N)) \cap L^4((0, T^+(\psi_0)) \times \mathbb{T}_N)$$

that is with

$$(t) = e^{tA}\psi_0 + \int_0^t e^{(t-s)A} F ds + i \int_0^t e^{(t-s)A} |\psi(s)|^2 \psi(s) ds,$$

and such that $T^+(\psi_0) \in (0, +\infty]$ and if $T^+(\psi_0) < \infty$ then

$$\lim_{t \rightarrow T^+(\psi_0)} \|\psi\|_{L^4([0,t)\times\mathbb{T}_N)} = +\infty.$$

Moreover, ψ is in $C([0, T^+(\psi_0)); L^2(\mathbb{T}_N))$ and

$$\forall t \in [0, T^+(\psi_0)), \|\psi(t)\| \leq e^{-t} \|\psi_0\| + 2|F|(1 - e^{-t}).$$

CHAPTER 3. ASYMPTOTIC STABILITY FOR THE LUGIATO-LEFEVER EQUATION

Proof. The existence of a local solution as a fixed point of \mathcal{M} on $C([0, \tau]; L^2(\mathbb{T}_N)) \cap L^4([0, \tau] \times \mathbb{T}_N)$, for some $\tau > 0$, follows from Lemma 10. The local uniqueness allows to concatenate properly all extensions of a solution leading to a unique maximal solution on an interval $[0, T^+(\psi_0))$. The extension procedure stops if only if $T^+(\psi_0)$ is finite and the ψ blows-up at $T^+(\psi_0)$.

This follows from the fact that if a solution is bounded we show that it can be extended continuously. Let us provide a little more details. If $T^+(\psi_0)$ is finite then

$$\lim_{t \rightarrow T^+(\psi_0)} \|\psi\|_{L^4([0, t] \times \mathbb{T}_N)} = +\infty.$$

If the latter is not true, since $\|\psi\|_{L^4([0, t] \times \mathbb{T}_N)}$ increases in t we have that $\psi \in L^4([0, T^+(\psi_0)] \times \mathbb{T}_N)$. As $T^+(\psi_0)$ is finite then Lemma 5 and Remark 22 show that ψ is in $C([0, T^+(\psi_0)], L^2(\mathbb{T}_N))$. The solution can be continued with initial data at $T^+(\psi_0)$ given by $\psi(T^+(\psi_0))$. This contradicts the maximality of $T^+(\psi_0)$.

Let us regularize ψ and consider

$$\phi_\epsilon(t) = (\epsilon|B|+1)^{-1} e^t e^{-tB} \psi(t) = (\epsilon|B|+1)^{-1} \psi_0 + \int_0^t e^{-sB} e^s (\epsilon|B|+1)^{-1} F ds + \int_0^t e^{-sB} e^s (\epsilon|B|+1)^{-1} \psi |\psi|^2 ds,$$

for $\epsilon > 0$. We have

$$\|\phi_\epsilon(t)\| = e^t \left\| (\epsilon|B|+1)^{-1} \psi(t) \right\| \rightarrow_{\epsilon \rightarrow 0} e^t \|\psi(t)\|.$$

Then $\psi_\epsilon(t)$ is in $C^1([0, T^+(\psi_0)], L^2(\mathbb{T}_N))$ and

$$\phi'_\epsilon(t) = e^{-tB} e^t (\epsilon|B|+1)^{-1} F + i e^{-tB} e^t (\epsilon|B|+1)^{-1} \psi(t) |\psi(t)|^2.$$

Denote $m_\epsilon(t) = \|\phi_\epsilon(t)\|^2$ and $m(t) = \|e^t \psi(t)\|^2$ then

$$m'_\epsilon(t) = 2 \operatorname{Re} \langle e^{-tB} e^t (\epsilon|B|+1)^{-1} \psi(t), e^{-tB} e^t (\epsilon|B|+1)^{-1} F + i e^{-tB} e^t (\epsilon|B|+1)^{-1} \psi(t) |\psi(t)|^2 \rangle$$

Integrating this leads to the inequality

$$m_\epsilon(t) = m_\epsilon(0) + \int_0^t 2 \operatorname{Re} \langle e^{-sB} e^s (\epsilon|B|+1)^{-1} \psi(s), e^{-sB} e^s (\epsilon|B|+1)^{-1} F + i e^{-sB} e^s (\epsilon|B|+1)^{-1} \psi(s) |\psi(s)|^2 \rangle ds.$$

Since $\psi \in L^\infty([0, t]; L^2(\mathbb{T}_N)) \cap L^4([0, t] \times \mathbb{T}_N)$, taking the limit $\epsilon \rightarrow 0$, we deduce

$$m(t) = m(0) + \int_0^t 2e^{2s} \operatorname{Re} \langle \psi(s), F \rangle ds$$

and

$$m(t) \leq m(0) + \int_0^t 2e^s |F| \sqrt{m(s)} ds.$$

A Gronwall type inequality provides

$$m(t) \leq (\sqrt{m(0)} + 2|F| \int_0^t e^s ds)^2$$

or

$$\|\psi(t)\| \leq e^{-t} \|\psi_0\| + 2|F|(1 - e^{-t})$$

which concludes the proof. \square

Remarque 25. *Some comments are in place:*

3.2. EXISTENCE AND SPECTRAL STABILITY OF PERIODIC EQUILIBRIUM SOLUTIONS

(i) Using the idea in the proof of Lemma 10, we can also prove the continuity of the solution with respect to ψ_0 in “norm” $\|\cdot\|_{\mathcal{B}_\tau}$ for some $\tau \in (0, T^+(\psi_0))$ but it is not clear if τ can be considered close to $T^+(\psi_0)$. The blow-up time $T^+(\psi_0)$ depends on ψ_0 and we do not know if it is or not a dependence on $\|\psi_0\|_{L^2(\mathbb{T}_N)}$ instead. Hence even if $\|\psi(t)\|_{L^2(\mathbb{T}_N)}$ is continuous in $t \in [0, T^+(\psi_0))$ and hence bounded on compact subintervals, nothing ensures that $T^+(\psi_0)$ is bounded for ψ_0 in a bounded subset of $L^2(\mathbb{T}_N)$.

Notice that one reason why we do have $T^+(\psi_0)$ depending on $\|\psi_0\|_{L^2(\mathbb{T}_N)}$ comes from the restricted range of Strichartz exponents we got in Lemma 3.1.1 (this can be compared to [?]). We cannot extract in the Picard iteration any contraction from a small range of time. In [?], the smallness is control from below by the norm of ψ_0 in $L^2(\mathbb{R}^d)$.

(ii) The map \mathcal{M} is invariant on $L^\infty((0, \tau) : H^s(\mathbb{T}_N))$ for any $s > 1/2$ as the latter is a Banach algebra. If ψ_0 is in $H^s(\mathbb{T}_N)$ the Picard iteration gives that $\psi(t)$ is in $H^s(\mathbb{T}_N)$ for $t \in [0, T^+(\psi_0))$. This way we can restrict the theorem to the $L^\infty((0, \tau) : H^s(\mathbb{T}_N))$ framework only. This provides a wellposedness result in the framework of [?, ?].

(iii) The question of extending such a result to the localized ($L^2(\mathbb{R})$ case) is more delicate. For instance, Remark 23 does not seem to adapt easily.

3.2 Existence and spectral stability of periodic equilibrium solutions

In this section, we consider the spectral stability of equilibrium solutions of the Lugiato-Lefever equation (25). These are solutions constant in time solutions a priori in the sense of Theorem 24 and those considered here are smooth.

We recall the needed results from [5] and then extend some of them for our own purpose to the subharmonic case.

3.2.1 Existence of a non constant equilibrium

Constant solutions

Lemma 11. *The Lugiato-Lefever equation (1.0.1) possesses constant (in time and space) solutions. More precisely:*

- (i) if $\alpha \leq \sqrt{3}$ and $F > 0$, the Lugiato-Lefever equation (1.0.1) have a unique constant solution;
- (ii) if $\alpha > \sqrt{3}$ and $F > 0$, there exist two values $F_-(\alpha)$ and $F_+(\alpha)$, with $F_-(\alpha) < F_+(\alpha)$, given by the following formula

$$F_\pm^2(\alpha) = \rho_\pm(\alpha) \left((\rho_\pm(\alpha) - \alpha)^2 + 1 \right),$$

with

$$\rho_\pm(\alpha) = \frac{1}{3} \left(2\alpha \mp \sqrt{\alpha^2 - 3} \right),$$

such that the Lugiato-Lefever equation (1.0.1) possesses three constant solutions ψ_j with $\rho = \rho_j$, $j = 1, 2, 3$,

$$\rho_1 < \rho_+(\alpha) < \rho_2 < \rho_-(\alpha) < \rho_3,$$

when $F_-(\alpha) < F < F_+(\alpha)$, two distinct constant solutions when $F = F_\pm(\alpha)$, and one constant solution when $F < F_-(\alpha)$ or $F_+(\alpha) < F$.

We emphasize that constant solutions of (1.0.1) and (25) are the same.

Proof. For the proof, see [5, Section 2]. □

We set $\mathcal{X}_N := L^2(\mathbb{T}_N) \times L^2(\mathbb{T}_N)$ and $\mathcal{Y}_N := H^2(\mathbb{T}_N) \times H^2(\mathbb{T}_N)$. If ψ^* is a constant solution of (25), we set $\Psi^* = \begin{pmatrix} \psi_r^* \\ \psi_i^* \end{pmatrix}$, where ψ_r^* and ψ_i^* denotes respectively the real and imaginary parts of ψ^* . Let \mathcal{F} be defined by

$$\mathcal{F}(\Psi) := \begin{pmatrix} \beta(2-\alpha)\partial_y^2\psi_i - \psi_r + \alpha\psi_i - \psi_i(\psi_r^2 + \psi_i^2) + F \\ -\beta(2-\alpha)\partial_y^2\psi_r - \psi_i - \alpha\psi_r + \psi_r(\psi_r^2 + \psi_i^2) \end{pmatrix}. \quad (3.2.1)$$

Then $\mathcal{F}(\Psi^*) = 0$. The linearized operator \mathcal{A}_0 around ψ^* (or by extension Ψ^*) is given by

$$\mathcal{A}_0 = -\mathbb{1} + \mathcal{J}\mathcal{L}_0, \quad (3.2.2)$$

where $\mathbb{1}$ represent the 2×2 identity matrix, and

$$\mathcal{J} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \mathcal{L}_0 = \begin{pmatrix} -\beta(2-\alpha)\partial_y^2 - \alpha + 3\psi_r^{*2} + \psi_i^{*2} & 2\psi_r^*\psi_i^* \\ 2\psi_r^*\psi_i^* & -\beta(2-\alpha)\partial_y^2 - \alpha + \psi_r^{*2} + 3\psi_i^{*2} \end{pmatrix}.$$

Hence \mathcal{A}_0 acts on \mathcal{X}_N with domain \mathcal{Y}_N .

Remark 26. Let ψ be a constant solution of (1.0.1), then by setting $\psi = \psi_r + i\psi_i$, ψ_r and ψ_i reals, and $\rho = |\psi|^2$, we have

$$\psi_r = \frac{\rho}{F}, \quad \psi_i = \frac{\rho(\rho - \alpha)}{F}, \quad F^2 = \rho(1 + (\rho - \alpha)^2). \quad (3.2.3)$$

We consider the so-called anomalous case ($\beta = -1$) and have the

Lemma 12. Let $\alpha < 2$ and $\psi^* = \psi_r^* + i\psi_i^*$ the constant solution of (25) with $\rho^* := \|\psi^*\|^2 = 1$. We set $\Psi^* = \begin{pmatrix} \psi_r^* \\ \psi_i^* \end{pmatrix}$ and we consider \mathcal{A}_0 the linearized operator around Ψ^* . Then, the operator \mathcal{A}_0 is closed and has purely discrete spectrum given by

$$\sigma(\mathcal{A}_0) = \left\{ \lambda \in \mathbb{C}; \lambda + 2\lambda + a(\sqrt{2-\alpha}\frac{n}{N}) = 0, n \in \mathbb{Z} \right\}, \quad (3.2.4)$$

with $a(k) = (2-\alpha)^2(k^2 - 1)^2$. The value 0 is a semisimple eigenvalue with corresponding eigenspace of dimension 2 spanned by the conjugated eigenvectors ζ and $\bar{\zeta}$ given by:

$$\zeta_N = \begin{pmatrix} \alpha \\ 2-\alpha \end{pmatrix} e^{iNy}, \quad \bar{\zeta}_N = \begin{pmatrix} \alpha \\ 2-\alpha \end{pmatrix} e^{-iNy}.$$

Moreover, the spectrum of \mathcal{A}_0 is symmetric with respect to the real line and vertical line $\operatorname{Re}(\lambda) = -1$ in the complex plane.

Proof. The operator ∂_y^2 is closed on $L^2(\mathbb{T}_N)$ with compact resolvent and \mathcal{A}_0 is a bounded perturbation to $(2-\alpha)\mathcal{J}\partial_y^2$. Hence \mathcal{A}_0 has compact resolvent and thus its spectrum is discrete. By means of Fourier series, we even have that

$$\sigma(\mathcal{A}_0) = \bigcup_{n \in \mathbb{Z}} \sigma(\mathcal{A}_0(n)),$$

where

$$\mathcal{A}_0(n) = \begin{pmatrix} -1 - 2\psi_r^*\psi_i^* & (2-\alpha)\frac{n^2}{N^2} + \alpha + \psi_r^{*2} + 3\psi_i^{*2} \\ -(2-\alpha)\frac{n^2}{N^2} - \alpha + 3\psi_r^{*2} + \psi_i^{*2} & -1 + 2\psi_r^*\psi_i^* \end{pmatrix}.$$

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Let $\lambda \in \mathbb{C}$. We have

$$\begin{aligned} \lambda \in \rho(\mathcal{A}_0) &\iff \forall n \in \mathbb{Z}, (\lambda + 1 + 2\psi_r^* \quad \quad) (\lambda + 1 - 2\psi_r^* \quad \quad) \\ &\quad - ((2 - \alpha) \frac{n^2}{N^2} - \alpha - \quad \quad - 3 \quad \quad) ((2 - \alpha) \frac{n^2}{N^2} + \alpha - 3\psi_r^{*2} - \quad \quad) \neq 0 \\ &\iff \forall n \in \mathbb{Z}, \lambda^2 + 2\lambda + (2 - \alpha)^2 (\frac{n^2}{N^2} - 1)^2 \neq 0, \end{aligned}$$

thus

$$\sigma(\mathcal{A}_0) = \left\{ \lambda \in \mathbb{C}; \lambda^2 + 2\lambda + (2 - \alpha)^2 (\frac{n^2}{N^2} - 1)^2 = 0, n \in \mathbb{Z} \right\}.$$

For n in \mathbb{Z} , the equation $\lambda^2 + 2\lambda + (2 - \alpha)^2 (\frac{n^2}{N^2} - 1)^2 = 0$ admits two roots:

$$\lambda_1(\sqrt{2 - \alpha} \frac{n}{N}) := -1 + \sqrt{1 - (2 - \alpha)^2 \frac{n^2}{N^2} - 1}, \quad \lambda_2(\sqrt{2 - \alpha} \frac{n}{N}) := -1 - \sqrt{1 - (2 - \alpha)^2 \frac{n^2}{N^2} - 1}.$$

For all n in \mathbb{Z} , $\lambda_2(\sqrt{2 - \alpha} \frac{n}{N})$ has negative real part, $\lambda_1(\sqrt{2 - \alpha} \frac{n}{N})$ has no positive real part, and $\lambda_1(\sqrt{2 - \alpha} \frac{n}{N}) = 0$ if and only if $n = \pm N$, thus 0 is in the spectrum of \mathcal{A}_0 and its multiplicity is 2.

The Fourier mode of eigenvectors are N and $-N$, thus let $\zeta_N = \begin{pmatrix} a \\ b \end{pmatrix} e^{iNy}$, then

$$\begin{aligned} \mathcal{A}_0 \zeta_N &= \frac{1}{1 + (1 - \alpha)^2} \begin{pmatrix} -(2 - \alpha)^2 a + (2(1 + (1 - \alpha)^2) - 1 - 3(1 - \alpha)^2) b \\ (-2(1 + (1 - \alpha)^2) + 4 - \alpha) a - \alpha^2 b \end{pmatrix} e^{iNy} \\ &= \frac{1}{1 + (1 - \alpha)^2} \begin{pmatrix} -(2 - \alpha)^2 a + \alpha(2 - \alpha) b \\ \alpha(2 - \alpha) a - \alpha^2 b \end{pmatrix} e^{iNy} \end{aligned}$$

thus $a = \alpha$ and $b = 2 - \alpha$. Hence, $\zeta_N = \begin{pmatrix} \alpha \\ 2 - \alpha \end{pmatrix} e^{iNy}$ and $\bar{\zeta}_N = \begin{pmatrix} \alpha \\ 2 - \alpha \end{pmatrix} e^{-iNy}$ are eigenvectors associated to 0. Here we used the fact that the spectrum of \mathcal{A}_0 is symmetric with respect to the real line since it has real coefficients.

Moreover, the spectrum of \mathcal{A}_0 is symmetric with respect to the vertical line $\mathcal{R}e(\lambda) = -1$ in the complex plane, since

$$(\mathcal{A}_0 + \mathbb{1})^{\text{ad}} = -\mathcal{L}_0^{\text{ad}} \mathcal{J} = -\mathcal{J}^{\text{ad}} (\mathcal{A}_0 + \mathbb{1}) \mathcal{J}, \quad (3.2.5)$$

where $^{\text{ad}}$ stands for the adjoint, and hence the spectrum \mathcal{A}_0 is symmetric with respect to 0. \square

Remark 27. Consider the spectrum $\sigma(\mathcal{A}_0)$ of the operator \mathcal{A}_0 . We have

$$\sigma(\mathcal{A}_0) = \sigma_-(\mathcal{A}_0) \cup \sigma_0(\mathcal{A}_0) \quad (3.2.6)$$

where $\sigma_-(\mathcal{A}_0)$ and $\sigma_0(\mathcal{A}_0)$ have the following properties

- (i) there exists $\gamma_N > 0$ such that $\sup_{\lambda \in \sigma_-(\mathcal{A}_0)} \mathcal{R}e(\lambda) < -\gamma_N$
- (ii) the set $\sigma_0(\mathcal{A}_0)$ consists of a finite number of eigenvalues with finite algebraic multiplicities.

Remark 28. Remark 27 calls for another one. The decomposition of the spectrum (3.2.6) coincides with decomposition into stable spectrum $\sigma_-(\mathcal{A}_0)$, central spectrum $\sigma_0(\mathcal{A}_0)$, which is made only a finite number of discrete eigenvalues, and unstable spectrum which is empty here. Recall that in [5], the center manifold is actually the manifold of periodic waves $\bar{\psi}_{\mu,c}$. Hence would expect, as in [4, Section 4.1] and

like for ODEs, that the center manifold is locally attractive due to the absence of unstable modes. In the infinite dimensional setting, the exponential separation of the spectrum is not enough and linear estimates on the linearized flow are needed, see [13, Hypothesis 2.7] see also [?]. Following [?], this is what we do below.

Another difficulty appears in adapting the center manifold stability in the infinite dimensional setting. Like the Cauchy-Lipschitz theorem, the center manifold one uses transparently the fact that real or complex numbers are normed algebras to control nonlinearities. Since $L^2(\mathbb{T}_N)$ is not a Banach algebra, the estimates on $L^2(\mathbb{T}_N)$ proved below are not sufficient. This is why in [?], the analysis is made in $H^1(\mathbb{T})$ and here, like [23], we use Strichartz estimates to treat nonlinear terms.

Existence and regularity of equilibrium

The analysis is done in [5] for the case $N = 1$ which is more restrictive than $N \geq 2$.

According to Lemma 12, we consider the

Définition 29. In the case $\alpha < 41/30$ we denote by ψ_0^* the constant solution to the Lugiato-Lefever equation (25) given in Lemma 11 with the smallest modulus $\sqrt{\rho^*} = 1$.

Théorème 30. In the case $\beta = -1$ (of anomalous dispersion), $\alpha < 41/30$, if $\mu > 0$ is sufficiently small and $F^2 = 1 + (1 - \alpha)^2 + \mu$ the solution ψ_0^* to the Lugiato-Lefever equation (25) is a spectrally unstable equilibrium and there exists a family of nontrivial spectrally stable equilibria $\tau_c \psi_\mu$ which are smooth in μ and smooth periodic in space solutions with Taylor expansion

$$\psi_\mu(y) = \psi_0^* + \frac{3(\alpha + i(2 - \alpha))}{F_1 |41 - 30\alpha|^{1/2}} \cos(y) |\mu|^{1/2} + \mathcal{O}_{\mu \rightarrow 0}(|\mu|) \quad (3.2.7)$$

in $H^k(\mathbb{T}_N)$ for any k in \mathbb{N} .

Proof. This theorem recollects its elements from [5, Section 3] and is worth to emphasize [5, equation (3.11)].

The definition of spectral stability for non constant equilibrium is provided below. We will only be interested by the stable case and provide a proof of this fact below. \square

3.2.2 Spectral stability analysis

In this subsection, we consider in more details the spectral stability, with respect to subharmonic ($N \geq 2$) perturbation, of the periodic solutions found in (3.2.7).

In the case $\beta = -1$ of anomalous dispersion, $\alpha < 41/30$, $\mu > 0$ sufficiently small and $F^2 = 1 + (1 - \alpha)^2 + \mu$ (as in Theorem 30), linearizing the Lugiato-Lefever equation (25) at ψ_μ , the even periodic solution given by (3.2.7), we obtain \mathcal{A}_μ the linearized operator defined by

$$\mathcal{A}_\mu := D\mathcal{F}(\Psi_\mu) = -\mathbb{I} + \mathcal{J}\mathcal{L}_\mu, \quad (3.2.8)$$

where $\Psi_\mu = \begin{pmatrix} r_\mu \\ i_\mu \end{pmatrix}$ in which $r_\mu(y)$ and $i_\mu(y)$ represent the real and imaginary parts, respectively, of the periodic solution ψ_μ , \mathcal{F} is defined as in (3.2.1), the operator \mathcal{J} is defined as before, and

$$\mathcal{L}_\mu = \begin{pmatrix} (2 - \alpha)\partial_y^2 - \alpha + 3\psi_{r_\mu}^2(y) + \psi_{i_\mu}^2(y) & 2\psi_{r_\mu}(y)\psi_{i_\mu}(y) \\ 2\psi_{r_\mu}(y)\psi_{i_\mu}(y) & (2 - \alpha)\partial_y^2 - \alpha + \psi_{r_\mu}^2(y) + 3\psi_{i_\mu}^2(y) \end{pmatrix}.$$

3.2. EXISTENCE AND SPECTRAL STABILITY OF PERIODIC EQUILIBRIUM SOLUTIONS

The linear operator \mathcal{A}_μ is closed in the Hilbert space \mathcal{X}_N with dense domains \mathcal{Y}_N and non empty resolvent set. Indeed, according to the formula (3.2.7), we have the expansions

$$\begin{aligned} r_\mu(y) &= r^* + \alpha \psi_1 \cos(y) |\mu|^{1/2} + \mathcal{O}(|\mu|), \\ i_\mu(y) &= i^* + (2 - \alpha) \psi_1 \cos(y) |\mu|^{1/2} + \mathcal{O}(|\mu|), \end{aligned}$$

where $\psi_1 = 3/F_1|41 - 30\alpha|^{1/2}$ (recall ψ_r^* and ψ_i^* are the real and imaginary parts, respectively, of the constant solution ψ_0^* , see paragraph before (3.2.1) and Remark 26). Hence $\mathcal{A}_\mu = \mathcal{A}_0 + |\mu|^{1/2}\mathcal{B}_\mu$, with

$$\mathcal{B}_\mu = \mathcal{J} \begin{pmatrix} 3b_{2\mu} + b_{3\mu} & b_{1\mu} \\ b_{1\mu} & b_{2\mu} + 3b_{3\mu} \end{pmatrix} \quad (3.2.9)$$

where

$$\begin{aligned} b_{1\mu} &= \psi_r^* ((2 - \alpha)\psi_1 \cos(y) + \varepsilon_i(|\mu|^{1/2})) + \psi_i^* (\alpha\psi_1 \cos(y) + \varepsilon_r(|\mu|^{1/2})) \\ &\quad + (\alpha\psi_1 \cos(y) + \varepsilon_r(|\mu|^{1/2}))((2 - \alpha)\psi_1 \cos(y) + \varepsilon_i(|\mu|^{1/2})), \\ b_{2\mu} &= (\alpha\psi_1 \cos(y) + \varepsilon_r(|\mu|^{1/2}))^2 + 2\psi_r^* (\alpha\psi_1 \cos(y) + \varepsilon_r(|\mu|^{1/2})), \\ b_{3\mu} &= ((2 - \alpha)\psi_1 \cos(y) + \varepsilon_i(|\mu|^{1/2}))^2 + 2\psi_i^* ((2 - \alpha)\psi_1 \cos(y) + \varepsilon_i(|\mu|^{1/2})) \end{aligned}$$

are all bounded. As \mathcal{Y}_N is compactly embedded in \mathcal{X}_N , the spectrum of the operator \mathcal{A}_μ is purely discrete.

Définition 31. Let ψ_μ be an equilibrium of (25). We denotes by $\sigma(\mathcal{A}_\mu)$ the spectrum of the linearized operator \mathcal{A}_μ around ψ_μ . We say that ψ_μ is spectrally stable if $\sigma(\mathcal{A}_\mu) \subset \{\lambda \in \mathbb{C}; \operatorname{Re}(\lambda) \leq 0\}$. Otherwise, ψ_μ is spectrally unstable.

Théorème 32. Let $\beta = -1$ (of anomalous dispersion), $\alpha < 41/30$, $\mu > 0$ sufficiently small and $F^2 = 1 + (1 - \alpha)^2 + \mu$.

Consider the periodic solutions ψ_μ of the Lugiato-Lefever equation (25) given by (3.2.7) and the associated linear operator \mathcal{A}_μ in (3.2.8), acting in \mathcal{X}_N . Then, for any μ sufficiently small, the spectrum of the linear operator \mathcal{A}_μ lies in the closed left half complex plane, and the periodic solution ψ_μ is spectrally stable with respect to subharmonic perturbations.

In addition, the generalized kernel of \mathcal{A}_μ is equal to the kernel and spanned by the spatial derivative of the equilibrium Ψ_μ (which is an odd function).

Proof. For sufficiently small μ , the operator \mathcal{A}_μ is a small relatively bounded perturbation of the operator \mathcal{A}_0 , for which we have the spectral decomposition (3.2.6). Hence

$$\sigma(\mathcal{A}_\mu) = \sigma_-(\mathcal{A}_\mu) \cup \sigma_0(\mathcal{A}_\mu),$$

where $\sigma_-(\mathcal{A}_\mu) \subset \{\lambda \in \mathbb{C} | \operatorname{Re}(\lambda) < -\gamma_N\}$ and $\sigma_0(\mathcal{A}_\mu) \subset \{\lambda \in \mathbb{C} | \operatorname{Re}(\lambda) > -\gamma_N\}$ with $\gamma_N > 0$ from Remark 27.

From [5, Lemma 4.1 ($\gamma = 0$ and $N = 1$)], for sufficiently small μ we have

$$\sigma_0(\mathcal{A}_\mu) = \{0, \lambda_1(\mu)\} \subset \{\lambda \in \mathbb{C} | \operatorname{Re}(\lambda) > -\gamma_N\}$$

where $\lambda_1(\mu) = -\frac{2}{(2-\alpha)^2}\mu + o(|\mu|)$. Note the spectrum increases with N .

Let $\varepsilon > 0$ such that $D(0, \varepsilon) \cap \sigma_0(\mathcal{A}_0) = \{0\}$ and let

$$P_\varepsilon(\mathcal{A}_0) = \frac{1}{2i\pi} \int_{\partial D(0, \varepsilon)} (\mathcal{A}_0 - z)^{-1} dz$$

the (Riesz) projector onto the kernel of $\sigma_0(\mathcal{A}_0)$. Then we have

$$\text{Ran}(P_\varepsilon(\mathcal{A}_0)) = \overline{\bigcup_{k \in \mathbb{N}} \text{Ker}(\mathcal{A}_0^k)}.$$

As $\mathcal{A}_\mu = \mathcal{A}_0 + |\mu|^{1/2} \mathcal{B}_\mu$, with \mathcal{B}_μ bounded, we have

$$\left\| (\mathcal{A}_\mu - z)^{-1} - (\mathcal{A}_0 - z)^{-1} \right\|_{\mathcal{L}(X, Y)} \longrightarrow 0, \quad \mu \longrightarrow 0, \quad \forall z \in \bigcap_{0 \leq \mu \leq 1} \rho(\mathcal{A}_\mu).$$

We have that \mathcal{A}_μ has compact resolvent and for all z , such that $|z| = \varepsilon$, $\mathcal{A}_\mu - z$ is invertible for μ sufficiently small. Hence,

$$P_\varepsilon(\mathcal{A}_\mu) = \frac{1}{2i\pi} \int_{\partial D(0, \varepsilon)} (\mathcal{A}_\mu - z)^{-1} dz \quad (3.2.10)$$

is well-defined, and we have

$$\|P_\varepsilon(\mathcal{A}_\mu) - P_\varepsilon(\mathcal{A}_0)\|_{\mathcal{L}(X_N, X_N)} \longrightarrow 0, \quad \mu \longrightarrow 0. \quad (3.2.11)$$

Hence,

$$2 = \dim\left(\overline{\bigcup_{k \in \mathbb{N}} \text{Ker}(\mathcal{A}_0^k)}\right) \geq \dim\left(\overline{\bigcup_{k \in \mathbb{N}} \text{Ker}(\mathcal{A}_\mu^k)}\right)$$

for μ sufficiently small. We deduce that for μ sufficiently small, an ε -neighborhood of 0 contains only two eigenvalues counted with their algebraic multiplicities: 0 and $\lambda_1(\mu)$. This implies that 0 is simple.

The solution ψ_μ satisfies the Lugiato-Lefever equation

$$0 = i(2 - \alpha)\partial_y^2 \psi_\mu - (1 + i\alpha)\psi_\mu + i\psi_\mu |\psi_\mu|^2 + F.$$

By applying ∂_y operator to this equation, we obtain

$$0 = i(2 - \alpha)\partial_y^2 (\partial_y \psi_\mu) - (1 + i\alpha)(\partial_y \psi_\mu) + i(\partial_y \psi_\mu) |\psi_\mu|^2 + 2i \text{Re} \langle \psi_\mu, \partial_y \psi_\mu \rangle \psi_\mu. \quad (3.2.12)$$

If we write $\Psi_\mu = \begin{pmatrix} r_\mu \\ i\mu \end{pmatrix}$ and $\partial_y \Psi_\mu = \begin{pmatrix} u_\mu \\ v_\mu \end{pmatrix}$, then the equation (3.2.12) is equivalent to the following system

$$0 = \mathcal{A}_\mu(\partial_x \Psi_\mu),$$

thus, since 0 is simple, $\text{Ker}(\mathcal{A}_\mu) = \text{Span}(\partial_y \Psi_\mu)$. Since Ψ_μ is even, $\partial_y \Psi_\mu$ is odd. \square

3.3 Asymptotic stability

This section is devoted to the proof of the following theorem, which describes the asymptotic stability of solution of the Lugiato-Lefever equation (25) (and hence (1.0.1)) around periodic solutions given by (3.2.7).

Théorème 33. *Let $\beta = -1$, $\alpha < 41/30$ and $\mu > 0$ sufficiently small.*

Let ψ_μ be an equilibrium of (25) given by (3.2.7). For any $c \in \mathbb{R}$, let

$$\mu, c(y) = \psi_\mu(y + c).$$

Then, there exists $\delta > 0$ and $0 \leq c_\infty < 2\pi N$ such that if ϕ_0 satisfies

$$\|\phi_0 - \psi_\mu\|_{X_N} \leq \delta,$$

then the unique solution (I, ϕ) of (25) with $\phi|_{t=0} = \phi_0$ in $L^\infty((0, T^+(\phi_0)); L^2(\mathbb{T}_N)) \cap L^4([0, T^+(\phi_0)) \times \mathbb{T}_N)$ satisfies $T^+(\phi_0) = +\infty$,

$$\sup_{t \geq 0} \left[\inf_{0 \leq c < 2\pi N} \|\phi(t) - \psi_{\mu, c}\|_{\mathcal{X}_N} \right] < \varepsilon$$

and

$$\|\phi(t) - \psi_{\mu, c_\infty}\|_{\mathcal{X}_N} \longrightarrow 0 \quad (t \longrightarrow +\infty).$$

The proof of this theorem is classical and intuitive. Any solution ϕ , which is initially close to an equilibrium ψ_μ , can be split in two pieces: one equilibrium profile $\psi_{\mu, c}$, which is a translation by c of the solitary wave ψ_μ , and a dispersive part v . This decomposition is not unique and we consider one such that v is, in a sense, orthogonal to the kernel of the linearized operator around the solitary wave profile $\psi_{\mu, c}$. This ensures the dispersion of this component. We thus start by proving that on a complement of its kernel the linearized operator generates a dissipative semigroup and then that this semigroup satisfies Strichartz estimates. We also give the equations of c , the time-dependent space translation parameter, and v and show, exactly as we would do for the well-posedness, that the solutions of these equations (which are coupled) satisfies a closed set of space-time estimates. The latter provide the convergence of c to some c_∞ and the dispersion of v eventually proving our theorem.

3.3.1 Strichartz estimates for the linearized operator \mathcal{A}_μ

Once again, as for the well-posedness, we use Strichartz estimates for the underlying linear dynamics generated by the linearized operator around the equilibrium profile. Compared to Section 3.1, we use global in time Strichartz estimates for \mathcal{A}_μ which is a perturbation of \mathcal{A}_0 . We thus to prove dissipation for dynamics generated the linearized operator. More precisely we prove that the semigroup has a negative exponential growth. A classical tool to get it is the Gearhart–Prüss theorem. Here we prove a global bound on the resolvent on a vertical half complex plane. One can also consider bounds on vertical lines as in [?, Exercice V.1.] or [7, Corollary 2] for a proof. The proof is actually the same level of difficulty and we choose to stay along with the classical assumption.

The strategy is inspired by [23, ?]. We also refer to [?, ?] for a similar H^1 based strategy using only the dissipation provides by the Gearhart–Prüss theorem.

Resolvent estimates

In the application of Gearhart–Prüss theorem [?, Theorem V.1.11], see Theorem 37 below, we need estimates on resolvents. We collect here estimates for \mathcal{A}_0 away from the origin. By a perturbation argument, this will provide the same estimates for \mathcal{A}_μ . The remaining part will be considered later by a spectral analysis.

Lemma 13. *There exists $\kappa_{\alpha, N} > 0$, such that for all $\omega \in \mathbb{R}$, $|\omega| > \sqrt{2}$, $\eta \in \mathbb{R}$, $\eta > -\frac{1}{2}$, we have $\lambda := \eta + i\omega \in \rho(\mathcal{A}_0)$ and*

$$\|(\mathcal{A}_0 - \lambda \mathbb{I})^{-1}\|_{\mathcal{B}(\mathcal{X}_N)} \leq \kappa_{\alpha, N}. \quad (3.3.1)$$

Proof. We recall that the operator \mathcal{A}_0 is given by the (3.2.2) and with Remark 26 we obtain

$$\mathcal{A}_0 = \begin{pmatrix} -\frac{(2-\alpha)^2}{F_1^2} & -(2-\alpha)\partial_y^2 + \frac{(\alpha-1)(2-\alpha)^2}{F_1^2} \\ (2-\alpha)\partial_y^2 + \frac{(2-\alpha)(\alpha^2-\alpha+2)}{F_1^2} & -\frac{\alpha^2}{F_1^2} \end{pmatrix}.$$

The spectrum of \mathcal{A}_0 is, see Remark 27, such that

$$\sigma(\mathcal{A}_0) \subset \{\lambda \in \mathbb{C}; \operatorname{Re}(\lambda) = -1, \text{ or } \operatorname{Im} \lambda = 0 \text{ with } \operatorname{Re} \lambda < 0\}.$$

Hence, we have that if $\eta > -1/2$ and $|\omega| > \sqrt{2}$, then $\lambda = \eta + i\omega$ is in $\rho(\mathcal{A}_0)$. We now estimate the norm of the operator $(\mathcal{A}_0 - \lambda)^{-1}$. Let $U = \begin{pmatrix} u \\ v \end{pmatrix}$ in \mathcal{Y}_N and $W = \begin{pmatrix} w \\ z \end{pmatrix}$ in \mathcal{X}_N :

$$U = \sum_{n \in \mathbb{Z}} U_n e^{i \frac{n}{N} y}, \quad W = \sum_{n \in \mathbb{Z}} W_n e^{i \frac{n}{N} y},$$

where $U_n = \begin{pmatrix} u_n \\ v_n \end{pmatrix}$ and $W_n = \begin{pmatrix} w_n \\ z_n \end{pmatrix}$ are in \mathbb{R}^2 . Hence,

$$U = (\mathcal{A}_0 - \lambda)^{-1} W \iff \forall n \in \mathbb{Z}, U_n = (\widehat{\mathcal{A}}_0(n) - \lambda)^{-1} W_n,$$

where

$$\widehat{\mathcal{A}}_0(n) = \begin{pmatrix} -\frac{(2-\alpha)^2}{F_1^2} & (2-\alpha)\frac{n^2}{N^2} + \frac{(\alpha-1)(2-\alpha)^2}{F_1^2} \\ -(2-\alpha)\frac{n^2}{N^2} + \frac{(2-\alpha)(\alpha^2-\alpha+2)}{F_1^2} & -\frac{\alpha^2}{F_1^2} \end{pmatrix}, \quad (3.3.2)$$

thus,

$$(\widehat{\mathcal{A}}_0(n) - \lambda)^{-1} = \begin{pmatrix} A_n(\eta, \omega) & B_n(\eta, \omega) \\ C_n(\eta, \omega) & D_n(\eta, \omega) \end{pmatrix},$$

with

$$A_n(\eta, \omega) = \frac{-\frac{\alpha^2}{F_1^2} - \eta - i\omega}{d_n(\eta, \omega)} \quad (3.3.3)$$

$$B_n(\eta, \omega) = \frac{(2-\alpha)\frac{n^2}{N^2} + \frac{(\alpha-1)(2-\alpha)^2}{F_1^2}}{d_n(\eta, \omega)} \quad (3.3.4)$$

$$C_n(\eta, \omega) = \frac{-(2-\alpha)\frac{n^2}{N^2} + \frac{(2-\alpha)(\alpha^2-\alpha+2)}{F_1^2}}{d_n(\eta, \omega)} \quad (3.3.5)$$

$$D_n(\eta, \omega) = \frac{-\frac{(2-\alpha)^2}{F_1^2} - \eta - i\omega}{d_n(\eta, \omega)} \quad (3.3.6)$$

$$d_n(\eta, \omega) = (\eta + i\omega)^2 + 2(\eta + i\omega) + (2-\alpha)^2 \left(\frac{n^2}{N^2} - 1 \right)^2. \quad (3.3.7)$$

Hence, we have

$$\begin{aligned} \left\| (\mathcal{A}_0 - \lambda)^{-1} W \right\|_{\mathcal{X}_N}^2 &= \sum_{n \in \mathbb{Z}} \left\| (\widehat{\mathcal{A}}_0(n) - \lambda)^{-1} W_n \right\|_2^2 \\ &\leq \sum_{n \in \mathbb{Z}} \max\{|A_n(\eta, \omega)|^2, |B_n(\eta, \omega)|^2, |C_n(\eta, \omega)|^2, |D_n(\eta, \omega)|^2\} \|W_n\|_2^2. \end{aligned}$$

It remains to prove that there exists $\kappa_{\alpha, N} > 0$ such that for all $\omega \in \mathbb{R}$, $|\omega| > \sqrt{2}$, $\eta \in \mathbb{R}$, $\eta > -\frac{1}{2}$ and $n \in \mathbb{Z}$,

$$\max\{|A_n(\eta, \omega)|^2, |B_n(\eta, \omega)|^2, |C_n(\eta, \omega)|^2, |D_n(\eta, \omega)|^2\} \leq \kappa_{\alpha, N}$$

to obtain (3.3.1).

We set $a_n := a\left(\frac{\sqrt{2-\alpha}}{N}n\right) = (2-\alpha)^2 \left(\frac{n^2}{N^2} - 1\right)^2$ and

$$\mathcal{E}_{\alpha, N} := \{n \in \mathbb{Z}, a_n \in [0; 1]\} = \left\{ n \in \mathbb{Z}, n^2 \in \left[N^2 \left(1 - \frac{1}{2-\alpha} \right); N^2 \left(1 + \frac{1}{2-\alpha} \right) \right] \right\}$$

Recall $\eta \geq -\frac{1}{2}$ and $\omega^2 \geq 2$. We have, for all n in \mathbb{Z} ,

$$|A_n(\eta, \omega)|^2 = \frac{\left(\frac{\alpha^2}{F_1^2} + \eta\right)^2 + \omega^2}{(\eta^2 + 2\eta + a_n + \omega^2)^2 + 4(1 - a_n)\omega^2}.$$

For n in $\mathcal{E}_{\alpha, N}$, we have

$$\left(\frac{\alpha^2}{F_1^2} + \eta\right)^2 + \omega^2 \leq 2\left(\frac{\alpha^4}{F_1^4} + \eta^2 + \omega^2\right)$$

and (for $|\omega| > \sqrt{2}$ and $\eta > -\frac{1}{2}$)

$$(\eta^2 + 2\eta + a_n + \omega^2)^2 + 4(1 - a_n)\omega^2 \geq \frac{1}{4}(\eta^2 + \omega^2)^2 \quad (3.3.8)$$

which provides

$$|A_n(\eta, \omega)|^2 \leq 2\left(\frac{\alpha^4}{F_1^4} + 2\right).$$

For n in $\mathbb{Z} \setminus \mathcal{E}_{\alpha, N}$ (that is $a_n \geq 1$), we set $X := \eta^2 + 2\eta + a_n + \omega^2$, then

$$\begin{aligned} |A_n(\eta, \omega)|^2 &= \frac{\frac{\alpha^4}{F_1^4} - 2\eta\frac{\alpha^2}{F_1^2} + \eta^2 + \omega^2 + 2\eta + a_n - 2\eta - a_n}{X^2 + 4(1 - a_n)\omega^2} \\ &= \frac{X + \frac{\alpha^4}{F_1^4} - 2\eta\left(\frac{\alpha^2}{F_1^2} + 1\right) - a_n}{X^2 + 4(1 - a_n)\omega^2} \\ &= \frac{X}{(X - 2\sqrt{a_n - 1}|\omega|)(X + 2\sqrt{a_n - 1}|\omega|)} + \frac{\frac{\alpha^4}{F_1^4} - 2\eta\left(\frac{\alpha^2}{F_1^2} + 1\right) - a_n}{(X - 2\sqrt{a_n - 1}|\omega|)(X + 2\sqrt{a_n - 1}|\omega|)}. \end{aligned}$$

We have

$$\frac{X}{(X - 2\sqrt{a_n - 1}|\omega|)(X + 2\sqrt{a_n - 1}|\omega|)} = \frac{1/2}{X - 2\sqrt{a_n - 1}|\omega|} + \frac{1/2}{X + 2\sqrt{a_n - 1}|\omega|}$$

with

$$\begin{aligned} X + 2\sqrt{a_n - 1}|\omega| &\geq X \\ &\geq \eta^2 + \omega^2 + 2\eta + 1 \\ &\geq (\eta + 1)^2 \\ &\geq \frac{1}{4} \end{aligned}$$

and

$$\begin{aligned} X - 2\sqrt{a_n - 1}|\omega| &= \eta^2 + 2\eta + 1 + \omega^2 - 2\sqrt{a_n - 1}|\omega| + a_n - 1 \\ &= (\eta + 1)^2 + (|\omega| - \sqrt{a_n - 1})^2 \\ &\geq (\eta + 1)^2 \end{aligned}$$

thus

$$X - 2\sqrt{a_n - 1}|\omega| \geq \frac{1}{4}. \quad (3.3.9)$$

and

$$\frac{X}{(X - 2\sqrt{a_n - 1}|\omega|)(X + 2\sqrt{a_n - 1}|\omega|)} \leq 4. \quad (3.3.10)$$

Then, we have

$$\left| \frac{\alpha^4}{F_1^4} - 2\eta \left(\frac{\alpha^2}{F_1^2} + 1 \right) - a_n \right| \leq \frac{\alpha^4}{F_1^4} + X + 1 + 2 \left| \frac{\alpha^2}{F_1^2} + 1 \right| (X + 1)$$

thus

$$\left| \frac{\frac{\alpha^4}{F_1^4} - 2\eta \left(\frac{\alpha^2}{F_1^2} + 1 \right) - a_n}{X^2 + 4(1 - a_n)\omega^2} \right| \leq \left| \frac{\frac{\alpha^4}{F_1^4} + 1 + 2 \left| \frac{\alpha^2}{F_1^2} + 1 \right|}{X^2 + 4(1 - a_n)\omega^2} \right| + \left| \frac{X \left(1 + 2 \left| \frac{\alpha^2}{F_1^2} + 1 \right| \right)}{X^2 + 4(1 - a_n)\omega^2} \right|.$$

We set $\tilde{a} := \min\{a_n, a_n > 1, n \in \mathbb{Z}\} > 1$, and we have

$$\begin{aligned} \left| \frac{1}{X^2 + 4(1 - a_n)\omega^2} \right| &= \left| \frac{1}{4\sqrt{a_n - 1}|\omega|(X - 2\sqrt{a_n - 1}|\omega|)} - \frac{1}{4\sqrt{a_n - 1}|\omega|(X + 2\sqrt{a_n - 1}|\omega|)} \right| \\ &\leq \frac{\sqrt{2}}{\sqrt{\tilde{a} - 1}}. \end{aligned}$$

(3.3.11)

We obtain

$$|A_n(\eta, \omega)|^2 \leq C_A \quad (3.3.12)$$

with

$$C_A := \max \left\{ \frac{\sqrt{2} \left(\frac{\alpha^4}{F_1^4} + 1 + 2 \left| \frac{\alpha^2}{F_1^2} + 1 \right| \right)}{\sqrt{\tilde{a} - 1}} + 4 \left(2 + 2 \left| \frac{\alpha^2}{F_1^2} + 1 \right| \right), 2 \left(\frac{\alpha^4}{F_1^4} + 2 \right) \right\},$$

If n is in $\mathcal{E}_{\alpha, N}$, then $a_n \in [0, 1]$ and by (3.3.8), we have

$$\begin{aligned} |B_n(\eta, \omega)|^2 &= \frac{\left| (2 - \alpha) \frac{n^2}{N^2} + \frac{(\alpha - 1)(2 - \alpha)^2}{F_1^2} \right|^2}{(\eta^2 + 2\eta + a_n + \omega^2)^2 + 4(1 - a_n)\omega^2} \\ &\leq \frac{\left| (2 - \alpha) + 1 + \frac{(\alpha - 1)(2 - \alpha)^2}{F_1^2} \right|^2}{\frac{1}{4}(\eta^2 + \omega^2)^2} \\ &\leq \left| (2 - \alpha) + 1 + \frac{(\alpha - 1)(2 - \alpha)^2}{F_1^2} \right|^2. \end{aligned}$$

If n is in $\mathbb{Z} \setminus \mathcal{E}_{\alpha, N}$, then $a_n > 1$ and we have

$$\begin{aligned} |B_n(\eta, \omega)|^2 &= \frac{\left| (2 - \alpha) \frac{n^2}{N^2} + \frac{(\alpha - 1)(2 - \alpha)^2}{F_1^2} \right|^2}{X^2 - 4(a_n - 1)\omega^2} \\ &= \frac{\left| (2 - \alpha) \left(\frac{n^2}{N^2} - 1 \right) + (2 - \alpha) + \frac{(\alpha - 1)(2 - \alpha)^2}{F_1^2} \right|^2}{X^2 - 4(a_n - 1)\omega^2} \\ &\leq \frac{2a_n}{X^2 - 4(a_n - 1)\omega^2} + \frac{2 \left((2 - \alpha) + \frac{(\alpha - 1)(2 - \alpha)^2}{F_1^2} \right)^2}{X^2 - 4(a_n - 1)\omega^2} \end{aligned}$$

On the one hand, by (3.3.11), we have

$$\frac{2 \left((2 - \alpha) + \frac{(\alpha - 1)(2 - \alpha)^2}{F_1^2} \right)^2}{X^2 - 4(a_n - 1)\omega^2} \leq \frac{2\sqrt{2} \left((2 - \alpha) + \frac{(\alpha - 1)(2 - \alpha)^2}{F_1^2} \right)^2}{\sqrt{\tilde{a} - 1}}. \quad (3.3.13)$$

On the other hand, using (3.3.9) and $X \geq a_n$, we have

$$\frac{2a_n}{X^2 - 4(a_n - 1)\omega^2} = \frac{2a_n}{X + 2\sqrt{a_n - 1}|\omega|} \times \frac{1}{X - 2\sqrt{a_n - 1}|\omega|} \quad (3.3.14)$$

$$\leq \frac{2a_n}{X} \times 4 \leq 8. \quad (3.3.15)$$

We obtain for all n in \mathbb{Z} , $\eta \geq -1/2$ and $|\omega| \geq \sqrt{2}$,

$$|B_n(\eta, \omega)|^2 \leq C_B \quad (3.3.16)$$

with

$$C_B := \max \left\{ \left| (2 - \alpha) + 1 + \frac{(\alpha - 1)(2 - \alpha)^2}{F_1^2} \right|^2; 8 + \frac{2\sqrt{2} \left((2 - \alpha) + \frac{(\alpha - 1)(2 - \alpha)^2}{F_1^2} \right)^2}{\sqrt{a} - 1} \right\}$$

We notice that $C_n(\eta, \omega)$ is similar to $B_n(\eta, \omega)$, and (similarly) we have that for all n in \mathbb{Z} , $\eta \geq -1/2$ and $|\omega| \geq \sqrt{2}$,

$$|C_n(\eta, \omega)|^2 \leq C_C \quad (3.3.17)$$

with

$$C_C := \max \left\{ \left((2 - \alpha) + 1 - \frac{(2 - \alpha)(\alpha^2 - \alpha + 2)}{F_1^2} \right)^2; 8 + \frac{2\sqrt{2}}{\sqrt{a} - 1} \left((2 - \alpha) - \frac{(2 - \alpha)(\alpha^2 - \alpha + 2)}{F_1^2} \right)^2 \right\}.$$

Finally, we notice that $D_n(\eta, \omega)$ is similar to $A_n(\eta, \omega)$, and (similarly) we have that, for all n in \mathbb{Z} , $\eta \geq -1/2$ and $|\omega| \geq \sqrt{2}$,

$$|D_n(\eta, \omega)|^2 \leq C_D \quad (3.3.18)$$

with

$$C_D := \max \left\{ \frac{\sqrt{2} \left(\frac{(2 - \alpha)^4}{F_1^4} + 1 + 2 \left| \frac{(2 - \alpha)^2}{F_1^2} + 1 \right| \right)}{\sqrt{a} - 1} + 4 \left(2 + 2 \left| \frac{(2 - \alpha)^2}{F_1^2} + 1 \right| \right); 2 \left(\frac{(2 - \alpha)^4}{F_1^4} + 2 \right) \right\}.$$

By setting $\kappa_{\alpha, N} := \max\{C_A, C_B, C_C, C_D\}$ we obtain (3.3.1), which proves the lemma. \square

Lemma 14. *There exists $M_{\alpha, N} > 0$, such that for all $\omega \in \mathbb{R}$, $|\omega| \leq \sqrt{2}$, $\eta \in \mathbb{R}$, $\eta \geq 3$, we have $\lambda := \eta + i\omega \in \rho(\mathcal{A}_0)$ and*

$$\|(\mathcal{A}_0 - \lambda \mathbb{I})^{-1}\|_{\mathcal{B}(\mathcal{X}_N)} \leq M_{\alpha, N}. \quad (3.3.19)$$

Proof. In this proof, we keep the same notations as in Lemma 13. For $\eta \geq 3$ and $|\omega| \leq \sqrt{2}$, $\lambda = \eta + i\omega \in \rho(\mathcal{A}_0)$. We consider in Fourier series

$$\begin{aligned} \|(\mathcal{A}_0 - \lambda)^{-1}W\|_{\mathcal{X}_N}^2 &= \sum_{n \in \mathbb{Z}} \|(\widehat{\mathcal{A}}_0(n) - \lambda)^{-1}W_n\|_2^2 \\ &\leq \sum_{n \in \mathbb{Z}} \max\{|A_n(\eta, \omega)|^2, |B_n(\eta, \omega)|^2, |C_n(\eta, \omega)|^2, |D_n(\eta, \omega)|^2\} \|W_n\|_2^2. \end{aligned}$$

Thus we have to prove that there exists $M_{\alpha, N} > 0$ and $\eta_0 > 0$ such that for all $n \in \mathbb{Z}$, $\eta \geq 3$ and $|\omega| \leq \sqrt{2}$,

$$\max \left\{ |A_n(\eta, \omega)|^2, |B_n(\eta, \omega)|^2, |C_n(\eta, \omega)|^2, |D_n(\eta, \omega)|^2 \right\} \leq M_{\alpha, N}$$

to obtain (3.3.19).

For all integer n , we have for $\eta \geq \frac{1}{2}$

$$|\operatorname{Re} d_n(\eta, \omega)|^2 = \eta^2 - \omega^2 + 2\eta + a_n^2 \geq (\eta - 1)^2 + 1 - \omega^2 + a_n^2$$

This is larger than $(\eta - 1)^2 - 1$ which provides the boundedness of $A_n(\eta, \omega)$ and $D_n(\eta, \omega)$ uniformly in $n \in \mathbb{Z}$, $\eta \geq 3$ and $|\omega| \leq \sqrt{2}$. This is also larger than a_n^2 if $\eta \geq 1$ and $|\omega| \leq \sqrt{2}$ which provides the boundedness of $B_n(\eta, \omega)$ and $C_n(\eta, \omega)$ uniformly in $n \in \mathbb{Z}$, $\eta \geq 3$ and $|\omega| \leq \sqrt{2}$. \square

Dissipation

Let $\mathcal{R} = \operatorname{Ran}(\mathcal{A}_\mu)$ and $\mathcal{N} = \operatorname{Ker}(\mathcal{A}_\mu)$. The operator \mathcal{A}_μ is a Fredholm operator of index 0 as a bounded perturbation of a selfadjoint operator with compact resolvent. So the dimension of its kernel is equal to the dimension of its adjoint operator kernel. Since the eigenvalue 0 is simple and $\dim(\mathcal{N}) = 1$, $\mathcal{N} = \operatorname{Span}\{\partial_y \Psi_\mu\}$. Consequently, the kernel of $\mathcal{A}_\mu^{\operatorname{ad}}$ is also of dimension 1 and spanned by an eigenvector E^{ad} with $\langle E^{\operatorname{ad}}, \partial_y \Psi_\mu \rangle_{\mathcal{X}_N} \neq 0$. If the latter were not true then $\partial_y \Psi_\mu$ would be in $\operatorname{Ran}(\mathcal{A}_\mu) = (\operatorname{Ker}(\mathcal{A}_\mu^{\operatorname{ad}}))^\perp$ and hence there exists Φ such that $\mathcal{A}_\mu \Phi = \partial_y \Psi_\mu$. So that the algebraic multiplicity of the eigenvalue 0 of \mathcal{A}_μ is larger or equal than 2 which is not true (see Theorem 32). The vector

$$E := \frac{E^{\operatorname{ad}}}{\langle E^{\operatorname{ad}}, \partial_y \Psi_\mu \rangle_{\mathcal{X}_N}} \quad (3.3.20)$$

satisfies $\langle E, \partial_y \Psi_\mu \rangle_{\mathcal{X}_N} = 1$ and also spans $\operatorname{Ker}(\mathcal{A}_\mu^{\operatorname{ad}})$.

In the proof of Theorem 33, we use Strichartz estimates on the complement subspace to the kernel of \mathcal{A}_μ . Since $\operatorname{Ker}(\mathcal{A}_\mu) = \operatorname{Span}\{\partial_y \Psi_\mu\}$, we define a projector P on $\operatorname{Ker}(\mathcal{A}_\mu)$ as follows.

Définition 34. *The projection P is defined on $\operatorname{Ker}(\mathcal{A}_\mu)$ by*

$$\begin{aligned} P : \mathcal{X}_N &\longrightarrow \operatorname{Ker}(\mathcal{A}_\mu) \\ f &\longmapsto \langle f, E \rangle_{\mathcal{X}_N} \partial_y \Psi_\mu \end{aligned}$$

where E is defined in (3.3.20).

We set $Q : \mathcal{X}_N \longrightarrow \mathcal{X}_N$, $Q = I - P$. It satisfies

$$Q \mathcal{A}_\mu U = \mathcal{A}_\mu Q U,$$

for all $U \in \mathcal{Y}_N$. We define the spaces:

$$\mathcal{X}'_N := \operatorname{Ran}(Q) \quad \text{and} \quad \mathcal{Y}'_N := Q(\mathcal{Y}_N).$$

Remarque 35. *Note that the projector P is also a spectral projector and satisfies for a fixed $\mu > 0$ and $\varepsilon_0 \in (0, \lambda_1(\mu))$,*

$$P = P_{\varepsilon_0} = \frac{1}{2i\pi} \int_{\partial D(0, \varepsilon_0)} (\mathcal{A}_\mu - \lambda)^{-1} d\lambda,$$

see (3.2.10). But if $\varepsilon_0 > 0$ is fixed and $\mu > 0$ too small, as in the proof of Theorem 32, we are in front of different projectors.

Now, we focus on the restriction of \mathcal{A}_μ to \mathcal{X}'_N . We have the following result concerning its spectrum.

Proposition 36. *The operator \mathcal{A}_μ is invariant on \mathcal{X}'_N . Its restriction $\mathcal{A}_\mu \upharpoonright_{\mathcal{X}'_N}$ with domain \mathcal{Y}'_N defines a closed such that for $\mu > 0$ sufficiently small*

$$\sigma(\mathcal{A}_\mu \upharpoonright_{\mathcal{X}'_N}) \subset \{\lambda \in \mathbb{C}; \operatorname{Re}(\lambda) < -\gamma_N\}.$$

The constant γ_N is defined in Remark 27.

Proof. For $\mu > 0$ small enough we have $\sigma(\mathcal{A}_\mu) \subset \{\lambda \in \mathbb{C}; \mathcal{R}e(\lambda) < -\gamma_N\} \cup \{0\}$. We also have the following decomposition

$$\mathcal{X}_N = P(\mathcal{X}_N) \oplus \mathcal{X}'_N$$

and \mathcal{A}_μ is invariant on each subspace $P(\mathcal{X}_N)$ and \mathcal{X}'_N . Since $P(\mathcal{X}_N)$ contains $\partial_y \psi_\mu$ and P has rank one, we deduce

$$\sigma(\mathcal{A}_\mu \upharpoonright_{P(\mathcal{X}_N)}) = \{0\} \quad \text{and} \quad \sigma(\mathcal{A}_\mu \upharpoonright_{\mathcal{X}'_N}) \subset \{\lambda \in \mathbb{C}; \mathcal{R}e(\lambda) < -\gamma_N\}. \quad \square$$

Théorème 37. *Let $\gamma \in (0, \min(\gamma_N, \frac{1}{2}))$. For $\mu > 0$ sufficiently small, the semigroup $(e^{t\mathcal{A}_\mu})_{t \geq 0}$ is invariant on \mathcal{X}'_N and satisfies, for some $C_\gamma > 0$,*

$$\|e^{t\mathcal{A}_\mu} f\|_{\mathcal{X}'_N} \leq C_\gamma e^{-\gamma t} \|f\|_{\mathcal{X}'_N}, \quad f \in \mathcal{X}'_N, \quad \forall t \in \mathbb{R}_+. \quad (3.3.21)$$

Proof. To prove (3.3.21), we use the Gearhart–Prüss theorem [?, Theorem 1.11].

First, Proposition 36 implies that the half plane $\{\lambda \in \mathbb{C}; \mathcal{R}e(\lambda) > \gamma - \gamma_N\}$ is included in the resolvent set $\rho(\mathcal{A}_\mu \upharpoonright_{\mathcal{X}'_N} + \gamma)$. Then, we have

$$\sup_{\mathcal{R}e(\lambda) > 0} \|(\mathcal{A}_\mu \upharpoonright_{\mathcal{X}'_N} + \gamma - \lambda)^{-1}\|_{\mathcal{B}(\mathcal{X}'_N)} < +\infty. \quad (3.3.22)$$

Indeed, we set

$$K := \left\{ z \in \mathbb{C}; -\frac{1}{2} \leq \mathcal{R}e(z) \leq \eta_0, -\sqrt{2} \leq \mathcal{I}m(z) \leq \sqrt{2} \right\}$$

and

$$Z := \left\{ z \in \mathbb{C}; \mathcal{R}e(z) \geq -\frac{1}{2} \right\} \setminus K.$$

Lemmas 13 and 14 implies that for all z in Z ,

$$\|(\mathcal{A}_0 - z)^{-1}\|_{\mathcal{B}(\mathcal{X}_N)} \leq C_{\alpha, N},$$

where $C_{\alpha, N} := \max\{\kappa_{\alpha, N}, M_{\alpha, N}\}$. Consider $\lambda \in Z$ with $\mathcal{R}e(\lambda) > 0$. Then, $\lambda - \gamma$ is in Z , thus we have

$$\|(\mathcal{A}_0 + \gamma - \lambda)^{-1}\|_{\mathcal{B}(\mathcal{X}_N)} \leq C_{\alpha, N} \quad (3.3.23)$$

and

$$\mathcal{A}_\mu + \gamma - \lambda = \mathcal{A}_0 + |\mu|^{1/2} \mathcal{B}_\mu + \gamma - \lambda$$

with \mathcal{B}_μ defined in (3.2.9) which is bounded. From the resolvent identity, we deduce

$$(\mathcal{A}_\mu + \gamma - \lambda)^{-1} - (\mathcal{A}_0 + \gamma - \lambda)^{-1} = -(\mathcal{A}_\mu + \gamma - \lambda)^{-1} |\mu|^{1/2} \mathcal{B}_\mu (\mathcal{A}_0 + \gamma - \lambda)^{-1}.$$

We also have

$$\| |\mu|^{1/2} \mathcal{B}_\mu (\mathcal{A}_0 + \gamma - \lambda)^{-1} \|_{\mathcal{B}(\mathcal{X}_N)} \leq |\mu|^{1/2} \|\mathcal{B}_\mu\|_{\mathcal{B}(\mathcal{X}_N)} C_{\alpha, N}.$$

There exists $\mu_0 > 0$ such that for all $\mu \in (0, \mu_0)$, $|\mu|^{1/2} \|\mathcal{B}_\mu\|_{\mathcal{B}(\mathcal{X}_N)} C_{\alpha, N} \leq \frac{1}{2}$ and hence $1 + |\mu|^{1/2} \mathcal{B}_\mu (\mathcal{A}_0 + \gamma - \lambda)^{-1}$ is invertible with

$$\| [1 + |\mu|^{1/2} \mathcal{B}_\mu (\mathcal{A}_0 + \gamma - \lambda)^{-1}]^{-1} \|_{\mathcal{B}(\mathcal{X}_N)} \leq 2.$$

Thus, we have

$$(\mathcal{A}_0 + \gamma - \lambda)^{-1} = (\mathcal{A}_\mu + \gamma - \lambda)^{-1} (1 + \mathcal{B}_\mu (\mathcal{A}_0 + \gamma - \lambda)^{-1})$$

and

$$\left\| (\mathcal{A}_\mu + \gamma - \lambda)^{-1} \right\|_{\mathcal{B}(\mathcal{X}_N)} \leq 2C_{\alpha, N},$$

and

$$\sup_{\lambda \in Z; \operatorname{Re}(\lambda) > 0} \left\| (\mathcal{A}_\mu \upharpoonright_{\mathcal{X}'_N} + \gamma - \lambda)^{-1} \right\|_{\mathcal{B}(\mathcal{X}_N)} < +\infty. \quad (3.3.24)$$

It remains to prove

$$\sup_{\lambda \in K; \operatorname{Re}(\lambda) > 0} \left\| (\mathcal{A}_\mu \upharpoonright_{\mathcal{X}'_N} + \gamma - \lambda)^{-1} \right\|_{\mathcal{B}(\mathcal{X}_N)} < +\infty. \quad (3.3.25)$$

The application $z \in \rho(\mathcal{A}_\mu \upharpoonright_{\mathcal{X}'_N}) \mapsto (\mathcal{A}_\mu \upharpoonright_{\mathcal{X}'_N} + \gamma - \lambda)^{-1}$ is continuous, thus is bounded on all compact sets of $\rho(\mathcal{A}_\mu \upharpoonright_{\mathcal{X}'_N})$. As K is a compact set of the resolvent set of $\mathcal{A}_\mu \upharpoonright_{\mathcal{X}'_N}$, we obtain (3.3.25).

Estimates (3.3.25) and (3.3.24) give (3.3.22), and applying Gearhart-Prüss theorem, we conclude the proof. \square

From now on, we assume that

$$\gamma_N \leq \frac{1}{2}. \quad (3.3.26)$$

This assumptions is made for convenience. The constant γ_N , see Remark 27, can not be fixed freely but can nonetheless be chosen positive smaller than the modulus of any the non zero eigenvalues of \mathcal{A}_μ .

Strichartz estimates

In this subsection, we consider the semigroup $(e^{-\beta(2-\alpha)t\mathcal{J}\partial_y^2})_{t \geq 0}$, generated by the operator $-\beta(2-\alpha)\mathcal{J}\partial_y^2$ defined on the phase-space \mathcal{X}_N .

Lemme 15. For any U_0 in \mathcal{X}_N , for any H in $L^{4/3}(\mathbb{R}^+ \times \mathbb{T}_N)^2$ and any γ in $(0, 2)$ we have

$$\left\| e^{-\beta(2-\alpha)t\mathcal{J}\partial_y^2} U_0 \right\|_{L^\infty(\mathbb{R}_+; L^2(\mathbb{T}_N))^2} \leq \|U_0\|_{\mathcal{X}_N}, \quad (3.3.27)$$

$$\begin{aligned} & \left\| \int_0^t e^{-\gamma(t-s)} e^{-\beta(2-\alpha)(t-s)\mathcal{J}\partial_y^2} H(s) \, ds \right\|_{L^\infty(\mathbb{R}_+; L^2(\mathbb{T}_N))^2} \\ & \leq \left(\frac{6N}{(2-\alpha)} \right)^{1/2} \frac{1}{(1 - e^{-8\gamma \frac{N^2}{2-\alpha} \pi})^{1/4}} \|H\|_{L^{4/3}(\mathbb{R}^+ \times \mathbb{T}_N)^2} \end{aligned} \quad (3.3.28)$$

and

$$\left\| e^{-\gamma t} e^{-\beta(2-\alpha)t\mathcal{J}\partial_y^2} U_0 \right\|_{L^4(\mathbb{R}^+ \times \mathbb{T}_N)^2} \leq \left(\frac{6N}{(2-\alpha)(1 - e^{-8\gamma \frac{N^2}{2-\alpha} \pi})} \right)^{1/4} \|U_0\|_{\mathcal{X}_N}, \quad (3.3.29)$$

$$\begin{aligned} & \left\| \int_0^t e^{-\gamma(t+s)} e^{-\beta(2-\alpha)(t-s)\mathcal{J}\partial_y^2} H(s) \, ds \right\|_{L^4(\mathbb{R}^+ \times \mathbb{T}_N)^2} \\ & \leq \left(\frac{6N}{(2-\alpha)(1 - e^{-8\gamma \frac{N^2}{2-\alpha} \pi})} \right)^{1/2} \|H\|_{L^{4/3}(\mathbb{R}^+ \times \mathbb{T}_N)^2}. \end{aligned} \quad (3.3.30)$$

Proof. Since \mathcal{J} and ∂_y^2 commutes, diagonalizing \mathcal{J} , we have that $\mathcal{J}\partial_y^2$ is (pointwise) unitarily equivalent to the diagonal operator $(i\partial_y^2, -i\partial_y^2)$. Then the assertion follows from (3.1.4), lemmata 6, 8 and 9. \square

Lemma 16. *Let $\gamma'_N \in (0, \gamma_N)$. There exists $C_5, C_6 > 0$ such that for any U_0 in \mathcal{X}'_N ,*

$$\left\| e^{\gamma'_N t} e^{t\mathcal{A}_\mu} Q U_0 \right\|_{L^\infty(\mathbb{R}_+; L^2(\mathbb{T}_N))^2} \leq C_5 \|U_0\|_{\mathcal{X}'_N}, \quad (3.3.31)$$

$$\left\| e^{\gamma'_N t} e^{t\mathcal{A}_\mu} Q U_0 \right\|_{L^4(\mathbb{R}_+ \times \mathbb{T}_N)^2} \leq C_6 \|U_0\|_{\mathcal{X}'_N}. \quad (3.3.32)$$

Proof. We have $Q U_0$ in \mathcal{X}'_N thus by Theorem 37, for all $t \in \mathbb{R}_+$,

$$\left\| e^{(\gamma'_N + \mathcal{A}_\mu)t} Q U_0 \right\|_{\mathcal{X}'_N} \leq C_{\gamma'_N} \|Q U_0\|_{\mathcal{X}'_N} \leq C_{\gamma'_N} \|U_0\|_{\mathcal{X}'_N},$$

hence we obtain (3.3.31). Since

$$\mathcal{A}_\mu = \mathcal{A}_0 + |\mu|^{1/2} \mathcal{B}_\mu,$$

with \mathcal{B}_μ defined in (3.2.9) and

$$\mathcal{V}_0 = \mathcal{J} \begin{pmatrix} -\alpha + 3\psi_{r0}^2(y) + \psi_{i0}^2(y) & 2\psi_{r0}(y)\psi_{i0}(y) \\ 2\psi_{r0}(y)\psi_{i0}(y) & -\alpha + \psi_{r0}^2(y) + 3\psi_{i0}^2(y) \end{pmatrix}.$$

For $U(t) := e^{t\mathcal{A}_\mu} Q U(0)$, Duhamel's formula gives

$$\begin{aligned} e^t e^{t\mathcal{A}_\mu} Q U_0 &= e^{t-t_0} e^{(t-t_0)\mathcal{A}_\mu} Q e^{t_0} U(t_0) \\ &= e^{-\beta(2-\alpha)(t-t_0)\mathcal{J}\partial_y^2} Q e^{t_0} U(t_0) + \int_{t_0}^t e^{-\beta(2-\alpha)(t-s)\mathcal{J}\partial_y^2} (\mathcal{V}_0 + |\mu|^{1/2} \mathcal{B}_\mu) Q e^s U(s) ds. \end{aligned} \quad (3.3.33)$$

for $t \geq t_0$. We notice that $U(t) := e^{t\mathcal{A}_\mu} Q U(t_0)$ has values in \mathcal{X}'_N because \mathcal{X}'_N is invariant by \mathcal{A}_μ . By estimate (3.3.29), we have

$$\left\| e^{-\beta(2-\alpha)(t-t_0)\mathcal{J}\partial_y^2} Q e^{t_0} U(t_0) \right\|_{L^4((t_0, t_0+2\pi) \times \mathbb{T}_N)} \leq C_3 \left\| Q e^{t_0} U(t_0) \right\|_{\mathcal{X}'_N}.$$

Let $\mathcal{V}_\mu := \mathcal{V}_0 + |\mu|^{1/2} \mathcal{B}_\mu$. Then, as \mathcal{V}_μ is bounded, we have that $\mathcal{V}_\mu Q U$ is in $L^{4/3}((t_0, t_0 + 2\pi) \times \mathbb{T}_N)$, thus

$$\begin{aligned} &\left\| \int_{t_0}^t e^{-\beta(2-\alpha)(t-s)\mathcal{J}\partial_y^2} \mathcal{V}_\mu Q e^s U(s) ds \right\|_{L^4((t_0, t_0+2\pi) \times \mathbb{T}_N)^2} \\ &\leq C_3 (2\pi)^{1/4} (2\pi N)^{1/4} \|\mathcal{V}_\mu\|_{\mathcal{B}(\mathcal{X}'_N)} \|Q e^\cdot U\|_{L^2((t_0, t_0+2\pi) \times \mathbb{T}_N)^2} \\ &\leq C \|Q e^\cdot U\|_{L^\infty((t_0, t_0+2\pi); \mathcal{X}'_N)^2}. \end{aligned}$$

with $C = C_3 (2\pi)^{3/4} (2\pi N)^{1/4} \|\mathcal{V}_\mu\|_{\mathcal{B}(\mathcal{X}'_N)} > 0$.

Hence, we have

$$\left\| e^t e^{t\mathcal{A}_\mu} Q U_0 \right\|_{L^4((t_0, t_0+2\pi) \times \mathbb{T}_N)} \leq C_3 e^{t_0} \|Q U(t_0)\|_{\mathcal{X}'_N} + C \|e^\cdot Q U\|_{L^\infty((t_0, t_0+2\pi); \mathcal{X}'_N)}. \quad (3.3.34)$$

Set $t_0 = 2\pi n$ for $n \in \mathbb{N}$, we have

$$\|Q U\|_{L^4((2\pi n, 2\pi(n+1)) \times \mathbb{T}_N)} \leq C_3 \|Q U(2\pi n)\|_{\mathcal{X}'_N} + C e^{2\pi} \|Q U\|_{L^\infty((2\pi n, 2\pi(n+1)); L^2(\mathbb{T}_N))^2}.$$

As U has values in \mathcal{X}'_N , Theorem 37 implies that for Let $\gamma \in (0, \min\{\gamma_N, \frac{1}{2}\})$. and all $t \geq 0$,

$$\|Q U(t)\|_{\mathcal{X}'_N} \leq e^{-\gamma t} C_\gamma \|Q U_0\|_{\mathcal{X}'_N},$$

or

$$\|QU(2\pi n)\|_{\mathcal{X}_N} \leq e^{-2\pi\gamma n} C_\gamma \|QU_0\|_{\mathcal{X}_N}$$

thus

$$\|QU\|_{L^\infty((2\pi n, 2\pi(n+1)); L^2(\mathbb{T}_N))^2} \leq e^{-2\pi\gamma n} C_\gamma \|QU_0\|_{\mathcal{X}_N},$$

and

$$\|QU\|_{L^4((2\pi n, 2\pi(n+1)) \times \mathbb{T}_N)} \leq C_\gamma (C_3 + Ce^{2\pi}) e^{-2\pi\gamma n} \|QU_0\|_{\mathcal{X}_N}.$$

Finally, we have

$$\begin{aligned} \|e^{\gamma' t} QU\|_{L^4(\mathbb{R}_+ \times \mathbb{T}_N)} &\leq \sum_{n=0}^{+\infty} \|e^{\gamma' t} QU\|_{L^4((2\pi n, 2\pi(n+1)) \times \mathbb{T}_N)} \\ &\leq e^{2\pi\gamma'} \sum_{n=0}^{+\infty} e^{2\pi n\gamma'} \|QU\|_{L^4((2\pi n, 2\pi(n+1)) \times \mathbb{T}_N)} \\ &\leq C_\gamma (C_3 + Ce^{2\pi}) e^{2\pi\gamma'} \sum_{n=0}^{+\infty} e^{2\pi n\gamma'} e^{-2\pi\gamma n} \|QU_0\|_{\mathcal{X}_N} \\ &\leq \frac{C_\gamma (C_3 + Ce^{2\pi}) e^{2\pi\gamma'}}{1 - e^{-2\pi(\gamma' - \gamma)}} \|U_0\|_{\mathcal{X}_N}, \end{aligned}$$

which is estimate (3.3.32) with $C_6 = \frac{C_\gamma (C_3 + Ce^{2\pi}) e^{2\pi\gamma}}{1 - e^{-2\pi(\gamma' - \gamma)}}$. \square

Eventually, these set of estimates allows the following inhomogeneous Strichartz estimates.

Lemma 17. *Let $\gamma'_N \in (0, \gamma_N)$. There exists $C_7, C_8 > 0$ such that for H in $L^{4/3}(\mathbb{R}_+ \times \mathbb{T}_N)^2$ and*

$$U(t) := \int_0^t e^{(t-s)\mathcal{A}_\mu} H(s) ds;$$

(i)

$$\|e^{\gamma'_N t} QU\|_{L^\infty(\mathbb{R}_+; L^2(\mathbb{T}_N))^2} \leq C_7 \|e^{\gamma'_N t} QH\|_{L^{4/3}(\mathbb{R}_+ \times \mathbb{T}_N)^2}, \quad (3.3.35)$$

(ii)

$$\|e^{\gamma'_N t} QU\|_{L^4(\mathbb{R}_+ \times \mathbb{T}_N)^2} \leq C_8 \|e^{\gamma'_N t} QH\|_{L^{4/3}(\mathbb{R}_+ \times \mathbb{T}_N)^2}. \quad (3.3.36)$$

Proof. First, we rewrite U as

$$U(t) = \int_0^t e^{s-t} Q e^{\beta(2-\alpha)(t-s)} \mathcal{J} \partial_y^2 H(s) ds + \int_0^t \int_0^s e^{(t-s)\mathcal{A}_\mu} Q \mathcal{V}_\mu e^{\tau-s} e^{\beta(2-\alpha)(s-\tau)} \mathcal{J} \partial_y^2 H(\tau) d\tau ds.$$

With (3.3.28) and (3.3.29)

$$\begin{aligned} &\|e^{\gamma'_N t} QU(t)\|_{L^\infty(\mathbb{R}_+; L^2(\mathbb{T}_N))^2} \\ &\leq \left\| \int_0^t Q e^{(\gamma'_N - 1)(t-s)} e^{\beta(2-\alpha)(t-s)} \mathcal{J} \partial_y^2 e^{\gamma'_N s} H(s) ds \right\|_{L^\infty(\mathbb{R}_+; L^2(\mathbb{T}_N))^2} \\ &+ \left\| \int_0^t \int_0^s e^{\gamma'_N(t-s)} e^{(t-s)\mathcal{A}_\mu} Q \mathcal{V}_\mu e^{(\gamma'_N - 1)(s-\tau)} e^{\beta(2-\alpha)(s-\tau)} \mathcal{J} \partial_y^2 e^{\gamma'_N \tau} H(\tau) d\tau ds \right\|_{L^\infty(\mathbb{R}_+; L^2(\mathbb{T}_N))^2} \\ &\leq C_2 \|e^{\gamma'_N s} H(s)\|_{L^{4/3}((t_0, t_0 + 2\pi) \times \mathbb{T}_N)^2} \\ &+ \int_0^{+\infty} e^{(\gamma'_N - \gamma_N)(t-s)} ds \|\mathcal{V}_\mu\| \left\| \int_0^s e^{(\gamma'_N - 1)(s-\tau)} e^{\beta(2-\alpha)(s-\tau)} \mathcal{J} \partial_y^2 e^{\gamma'_N \tau} H(\tau) d\tau \right\|_{L^\infty(\mathbb{R}_+; L^2(\mathbb{T}_N))^2} \\ &\leq \left(C_2 + C_4 \int_0^{+\infty} e^{(\gamma'_N - \gamma_N)(t-s)} ds \|\mathcal{V}_\mu\| \right) \|e^{\gamma'_N \tau} H(\tau)\|_{L^{4/3}(\mathbb{R}_+ \times \mathbb{T}_N)^2} \end{aligned}$$

Then, we rewrite U as

$$U(t) = \int_0^t e^{s-t} e^{\beta(2-\alpha)(t-s)} \mathcal{J} \partial_y^2 Q H(s) ds - \int_0^t \int_0^s e^{s-t} e^{\beta(2-\alpha)(s-\tau)} \mathcal{J} \partial_y^2 \mathcal{V}_\mu e^{(\tau-s)\mathcal{A}_\mu} Q H(\tau) d\tau ds$$

and we have with (3.3.30) and (3.1.5)

$$\begin{aligned} & \left\| e^{\gamma'_N t} Q U(t) \right\|_{L^4(\mathbb{R}_+ \times \mathbb{T}_N)^2} \\ & \leq \left\| \int_0^t e^{(\gamma'_N - 1)(t-s)} e^{\beta(2-\alpha)(t-s)} \mathcal{J} \partial_y^2 Q e^{\gamma'_N s} H(s) ds \right\|_{L^4(\mathbb{R}_+ \times \mathbb{T}_N)^2} \\ & + \left\| \int_0^t \int_0^s e^{(\gamma'_N - 1)(t-s)} e^{\beta(2-\alpha)(s-\tau)} \mathcal{J} \partial_y^2 \mathcal{V}_\mu e^{\gamma'_N s} e^{(s-\tau)\mathcal{A}_\mu} Q H(\tau) d\tau ds \right\|_{L^4(\mathbb{R}_+ \times \mathbb{T}_N)^2} \\ & \leq \|H\|_{L^{4/3}(\mathbb{R}_+ \times \mathbb{T}_N)^2} \\ & + C \|\mathcal{V}_\mu\| \int_0^\infty e^{(\gamma'_N - 1)s} ds \left\| e^{\gamma'_N s} \int_0^s e^{(s-\tau)\mathcal{A}_\mu} Q H(\tau) d\tau \right\|_{L^4(\mathbb{R}_+; L^2(\mathbb{T}_N))^2} \\ & \leq \|H\|_{L^{4/3}(\mathbb{R}_+ \times \mathbb{T}_N)^2} \\ & + C \|\mathcal{V}_\mu\| \int_0^\infty e^{(\gamma'_N - 1)s} ds \left(\int_0^\infty e^{-4\delta s} ds \right)^{1/4} \left\| e^{(\gamma'_N + \delta)s} \int_0^s e^{(s-\tau)\mathcal{A}_\mu} Q H(\tau) d\tau \right\|_{L^\infty(\mathbb{R}_+; L^2(\mathbb{T}_N))^2} \end{aligned}$$

with $\delta > 0$ such that $\gamma'_N + \delta < \gamma_N$. Using the previous estimate we conclude. \square

3.3.2 Asymptotic stability: Proof of Theorem 33

In this section, we prove Theorem 33. Consider the mild solution ϕ of the Lugiato-Lefever equation (25) given by Theorem 24, and the periodic solution ψ_μ defined in (3.2.7).

As in Subsection 3.2.1, we identify \mathbb{C} with \mathbb{R}^2 and, for $\psi_\mu = \psi_{r\mu} + i\psi_{i\mu}$, and $\phi = \phi_r + i\phi_i$, we set

$$\Psi_\mu := \begin{pmatrix} r\mu \\ i\mu \end{pmatrix} \quad \text{and} \quad \Phi := \begin{pmatrix} \phi_r \\ \phi_i \end{pmatrix}$$

and for all y in \mathbb{T}_N , $\Phi_0(y) := \Phi(0, y)$. Here, Φ_0 and Ψ_μ are in \mathcal{X}_N , and as ψ is the mild solution of the Lugiato-Lefever equation (25), $\Phi \in L^\infty((0, T^+(\phi_0)); L^2(\mathbb{T}_N))^2 \cap L^4((0, T^+(\phi_0)) \times \mathbb{T}_N)^2$.

The proof is now split in three parts:

- (i) In the first subsection, we spend some time in defining uniquely the translation parameter c and give a well defined equation for its time evolution this will also provide a well defined equation for the dispersive component Υ ;
- (ii) In the second subsection, we provide a closed set of a priori local in time estimates for the resulting equations based on the Strichartz estimates obtained so far ;
- (iii) In the last subsection, we prove that the estimates are eventually global and conclude the proof.

Setting of the analysis: Modulation equations and dispersive component

Let $\eta > 0$, we consider the ball in \mathcal{X}_N , of radius η , around Ψ_μ :

$$B_{\mathcal{X}_N}(\Psi_\mu, \eta) := \left\{ V \in \mathcal{X}_N; \|V - \Psi_\mu\|_{\mathcal{X}_N} \leq \eta \right\}.$$

Naturally, each element of this ball has a closest element on the curve $c \rightarrow \tau_c \Psi_\mu$.

Lemma 18. For all $V \in B_{\mathcal{X}_N}(\Psi_\mu, \eta)$, there exists $c(V) \in [0, 2\pi N)$ such that

$$\inf_{c \in \mathbb{R}} \|V - \tau_c \Psi_\mu\|_{\mathcal{X}_N} = \|V - \tau_{c(V)} \Psi_\mu\|_{\mathcal{X}_N} \leq \eta.$$

Proof. Let $\Psi_{\mu,c} = \tau_c \Psi_\mu$. Let $V \in B_{\mathcal{X}_N}(\Psi_\mu, \eta)$. The function $c \in \mathbb{R} \mapsto \|V - \Psi_{\mu,c}\|_{\mathcal{X}_N}$ is continuous, $2\pi N$ -periodic and bounded on \mathbb{R} , thus admits extrema, in particular a minimum, thus there exists $c(V) \in [0, 2\pi N)$ such that

$$\inf_{c \in \mathbb{R}} \|V - \tau_c \Psi_\mu\|_{\mathcal{X}_N} = \|V - \tau_{c(V)} \Psi_\mu\|_{\mathcal{X}_N}.$$

Then, we have

$$\|V - \tau_{c(V)} \Psi_\mu\|_{\mathcal{X}_N} \leq \|V - \Psi_\mu\|_{\mathcal{X}_N} \leq \eta.$$

This concludes the proof. \square

Applying this lemma with Φ_0 , if $\Phi_0 \in B_{\mathcal{X}_N}(\Psi_\mu, \eta)$, then there exists $c^* = c(\Phi_0) \in [0, 2\pi N)$ such that

$$\inf_{c \in \mathbb{R}} \|\Phi_0 - \tau_c \Psi_\mu\|_{\mathcal{X}_N} = \|\Phi_0 - \tau_{c^*} \Psi_\mu\|_{\mathcal{X}_N}.$$

For $(t, y) \in \mathbb{R}_+ \times \mathbb{T}_N$, let $c \in C^1(\mathbb{R}_+)$ with $c(0) = c^*$ and we set

$$\begin{cases} \Upsilon(t, y) &= \Phi(t, y + c(t)) - \Psi_\mu(y), \\ \Upsilon_0(y) &= \Phi_0(y + c(0)) - \Psi_\mu(y). \end{cases}$$

Without loss of generality we assume that $c(0) = 0$.

As a mild solution of the Lugiato-Lefever equation (25), by Theorem 24, we have for all $t \in [0, T^+(\phi_0))$,

$$(t) = e^{tA} \phi_0 + \int_0^t e^{(t-s)A} F ds + i \int_0^t e^{(t-s)A} |\phi(s)|^2 \phi(s) ds.$$

We set

$$S_c(t, s) := e^{c(t)\partial_y} e^{(t-s)A} e^{-c(s)\partial_y}, \quad 0 \leq s \leq t.$$

For all $t \in [0, T^+(\phi_0))$ and $y \in \mathbb{T}_N$, we obtain

$$\begin{aligned} \tau_{-c(t)} \phi(t, y) &= e^{c(t)\partial_y} \phi(t, y) \\ &= S_c(t, 0) \phi_0 + \int_0^t S_c(t, s) F ds + i \int_0^t S_c(t, s) |\tau_{-c(s)} \phi(s)|^2 \tau_{-c(s)} \phi(s) ds, \end{aligned}$$

where τ_c denotes the translation in space by c which is $e^{-c\partial_y}$. In order to compare to ϕ_μ , we need to rewrite ψ_μ as a mild solution. We have that μ is an equilibrium and hence a strong solution of the Lugiato-Lefever equation (25):

$$\partial_t \psi_\mu = 0 = A\psi_\mu + i\psi_\mu |\psi_\mu|^2 + F.$$

Then, for $t \in [0, T^+(\phi_0))$, we have

$$\partial_t (\tau_c \psi_\mu) = -\dot{c}(t) \partial_y (\tau_c \psi_\mu)$$

and

$$0 = A(\tau_c \psi_\mu) + i\tau_c \psi_\mu |\tau_c \psi_\mu|^2 + F$$

thus

$$\partial_t \tau_c \psi_\mu = -\dot{c}(t) \partial_y \tau_c \psi_\mu + A\tau_c \psi_\mu + i\tau_c \psi_\mu |\tau_c \psi_\mu|^2 + F,$$

then

$$(\tau_c \psi_\mu)(t) = e^{tA} \tau_{c(0)} \psi_\mu + \int_0^t e^{(t-s)A} F ds + i \int_0^t e^{(t-s)A} |\tau_{c(s)} \psi_\mu(s)|^2 \tau_{c(s)} \psi_\mu(s) ds - \int_0^t e^{(t-s)A} \dot{c}(s) \partial_y \tau_{c(s)} \psi_\mu(s) ds.$$

Applying $e^{c(t)\partial_y}$ to the last formula, we obtain

$$\begin{aligned} \mu(t) &= e^{c(t)\partial_y} (\tau_c \psi_\mu)(t) \\ &= e^{c(t)\partial_y} e^{tA} \tau_{c(0)} \psi_\mu + \int_0^t e^{c(t)\partial_y} e^{(t-s)A} e^{-c(s)\partial_y} e^{c(s)\partial_y} F ds \\ &\quad + i \int_0^t e^{c(t)\partial_y} e^{(t-s)A} e^{-c(s)\partial_y} e^{c(s)\partial_y} |\tau_c \psi_\mu(s)|^2 \tau_c \psi_\mu(s) ds \\ &\quad - \int_0^t e^{c(t)\partial_y} e^{(t-s)A} e^{-c(s)\partial_y} e^{c(s)\partial_y} \dot{c}(s) \partial_y \tau_c \psi_\mu(s) ds \\ &= S_c(t, 0) \psi_\mu + \int_0^t S_c(t, s) F ds + i \int_0^t S_c(t, s) |\psi_\mu|^2 \psi_\mu ds - \int_0^t S_c(t, s) \dot{c}(s) \partial_y \psi_\mu(s) ds. \end{aligned}$$

We have

$$\Upsilon(t, y) = S_c(t, 0) (\Phi_0 - \Psi_\mu) + i \int_0^t S_c(t, s) (|\tau_c \Phi(s)|^2 \tau_c \Phi(s) - |\Psi_\mu|^2 \Psi_\mu) ds + \int_0^t S_c(t, s) \dot{c}(s) \partial_y \Psi_\mu ds.$$

We notice that

$$\begin{pmatrix} \operatorname{Re}(i(|\tau_c \phi|^2 \tau_c \phi - |\psi_\mu|^2 \psi_\mu)) \\ \operatorname{Im}(i(|\tau_c \phi|^2 \tau_c \phi - |\psi_\mu|^2 \psi_\mu)) \end{pmatrix} = \mathcal{J} \Lambda_\mu \Upsilon + \mathcal{F}(\Upsilon, \mu)$$

where Λ_μ is defined by

$$\Lambda_\mu = \begin{pmatrix} 3\psi_{r\mu}^2 + \psi_{i\mu}^2 & 2\psi_{r\mu}\psi_{i\mu} \\ 2\psi_{r\mu}\psi_{i\mu} & \psi_{r\mu}^2 + 3\psi_{i\mu}^2 \end{pmatrix} \quad (3.3.37)$$

and

$$\mathcal{F}(\Upsilon, \mu) := \mathcal{J}(2\psi_\mu |\Upsilon|^2 + \overline{\Upsilon}(\Upsilon)^2 + \Upsilon |\Upsilon|^2).$$

We define now the propagator $(U_c(t, s))_{0 \leq s \leq t}$ by

$$U_c(t, s) := e^{c(t)\partial_y} e^{(t-s)A_\mu} e^{-c(s)\partial_y}. \quad (3.3.38)$$

Using Duhamel formula, we have

$$\Upsilon(t) = U_c(t, 0) \Upsilon_0 + \int_0^t U_c(t, s) \mathcal{F}(\Upsilon(s), \mu) ds + \int_0^t U_c(t, s) \dot{c}(s) \partial_y \Psi_\mu ds. \quad (3.3.39)$$

We set

$$\begin{cases} \dot{c}(t) &= -\frac{\operatorname{Re} \langle E, \mathcal{F}(\Upsilon(t), \mu) \rangle}{\|E\|_{\mathcal{X}_N} - \operatorname{Re} \langle \partial_y E, \Upsilon(t) \rangle} \\ c(0) &= 0 \end{cases} \quad (3.3.40)$$

recall (3.3.20). Then, using Duhamel formula, we have

$$\begin{aligned} \operatorname{Re} \langle E, \Upsilon \rangle &= \operatorname{Re} \langle E, e^{tA_\mu} \Upsilon_0 \rangle + \int_0^t \operatorname{Re} \langle E, e^{(t-s)A_\mu} (\dot{c}(s) \partial_y \Psi_\mu + \mathcal{F}(\tau_c \Upsilon(s), \mu) + \dot{c}(s) \partial_y \Upsilon(s)) \rangle ds \\ &= \operatorname{Re} \langle E, \Upsilon_0 \rangle + \int_0^t \operatorname{Re} \langle E, \dot{c}(s) (\partial_y \Psi_\mu + \partial_y \Upsilon(s)) + \mathcal{F}(\tau_c \Upsilon(s), \mu) \rangle ds \end{aligned}$$

The equation (3.3.40) for c , is chosen in order to obtain Υ in the cokernel of the linearized operator and thus with good dispersive properties. Usually the procedure is the other way, the equation for c is deduce from imposing Υ in the cokernel of the linearized operator. But this needs Υ to be a strong solution in order to take time derivatives. We now spend some time to recover needed properties for c .

Lemma 19. For any $(V, c) \in \mathcal{X}_N \times \mathbb{R}$, we set $f(V, c) = \operatorname{Re} \langle E, \tau_{-c} V - \Psi_\mu \rangle$. Then, there exists a neighborhood $\mathcal{V}_0 \subset \mathcal{X}_N$ of Ψ_μ , a neighborhood $\mathcal{W}_0 \subset \mathbb{R}$ of 0 and a unique $\tilde{c} \in C^1(\mathcal{V}_0, \mathcal{W}_0)$ such that

$$(V \in \mathcal{V}_0, c \in \mathcal{W}_0 \text{ and } f(V, c) = 0) \iff (V \in \mathcal{V}_0 \text{ and } c = \tilde{c}(V)).$$

Proof. We have $f(\Psi_\mu, 0) = 0$ and for (V, c) in $\mathcal{X}_N \times \mathbb{R}$,

$$\frac{df}{dc}(V, c) = -\operatorname{Re} \langle \tau_c \partial_y E, V - \tau_c \Psi_\mu \rangle + \operatorname{Re} \langle \tau_c E, \tau_c \partial_y \Psi_\mu \rangle,$$

thus

$$\frac{df}{dc}(V, c)|_{(\Psi_\mu, 0)} = -\operatorname{Re} \langle \partial_y E, 0 \rangle + \operatorname{Re} \langle E, \partial_y \Psi_\mu \rangle = 1 \neq 0.$$

Then, by implicit function theorem, there exists a neighborhood $\mathcal{V}_0 \subset \mathcal{X}_N$ of Ψ_μ , a neighborhood $\mathcal{W}_0 \subset \mathbb{R}$ of 0 and a unique $\tilde{c} \in C^1(\mathcal{V}_0, \mathcal{W}_0)$ such that

$$(V \in \mathcal{V}_0, c \in \mathcal{W}_0 \text{ and } f(V, c) = 0) \iff (V \in \mathcal{V}_0 \text{ and } c = \tilde{c}(V)).$$

This concludes the proof. \square

Notice that $\tilde{c}(V)$ is not necessarily one of the $c(V)$ given by Lemma 18 although the latter satisfy $f(V, c(V)) = 0$. To get so we need to prove that in a small neighborhood of $c = 0$ there exists only one zero of $f(V, c)$.

In this case, as a solution of (3.3.40) ensures

$$\operatorname{Re} \langle E, \Upsilon \rangle = 0, \tag{3.3.41}$$

if this stays small then it is one of $c(\Psi)$, actually the smallest positive one as we will see.

The following lemma will provide a sufficient argument.

Lemma 20. Consider Ψ_μ and we define $d : \mathbb{R} \rightarrow \mathbb{R}_+$ with $d(c) := \|\tau_c \Psi_\mu - \Psi_\mu\|_{\mathcal{X}_N}^2$ for $c \in \mathbb{R}$. Then, there exists $c_0 \in \mathbb{R}_+^*$ such that d is increasing on $(0, c_0)$.

Proof. We notice that d is defined on \mathbb{R} and of class C^2 , because $c \mapsto \tau_c \Psi_\mu$ is smooth and the norm as well. We have, for all c in \mathbb{R} ,

$$d'(c) = 2 \operatorname{Re} \langle \tau_c \partial_y \Psi_\mu, \Psi_\mu \rangle$$

and $d'(0) = 2 \operatorname{Re} \langle \partial_y \Psi_\mu, \Psi_\mu \rangle = 0$. Then, for c in \mathbb{R} ,

$$d''(c) = 2 \operatorname{Re} \langle \partial_y \Psi_\mu, \tau_{-c} \partial_y \Psi_\mu \rangle,$$

thus d'' is continuous and $d''(0) = 2 \|\partial_y \Psi_\mu\|^2 > 0$ because Ψ_μ is not constant due to formula (3.2.7). Then, there exists $c_0 > 0$ such that for all $c \in [0, c_0)$, $d''(c) > 0$ thus d' is increasing. As $d'(0) = 0$, then $d'(c) > 0$ for all $c \in (0, c_0)$ thus d is increasing on $(0, c_0)$. \square

Now, we set $c_1 := \inf\{c > 0; d(c) = 0\}$. We notice that the existence of c_1 is due to the periodicity of Ψ_μ . By its definition, $c_1 > c_0$ and c_1 is the period of Ψ_μ (thus of d) because $\tau_{c_1} \Psi_\mu = \Psi_\mu$. We set $I_1 := [-\frac{c_1}{2}, \frac{c_1}{2}]$. By definition of c_1 and the fact that d is even, 0 is the unique zero of d on I_1 .

Let $b > 0$. We set $I_b := [-\frac{c_1}{2}, -b] \cup [b, \frac{c_1}{2}]$ and $\eta(b) := \inf_{c \in I_b} d(c)$. We choose $b \in I_1$ such that $(-b, b) \subset \mathcal{V}_0$. Let $\varepsilon > 0$ with $\varepsilon^2 \leq \frac{1}{8} \eta(b)$ and consider $V \in B(\Psi_\mu, \varepsilon)$. By Lemma 18, there exists $c(V) \in [0, 2\pi N)$ such that

$$\inf_{c \in \mathbb{R}} \|V - \tau_c \Psi_\mu\|_{\mathcal{X}_N} = \|V - \tau_{c^*(V)} \Psi_\mu\|_{\mathcal{X}_N}.$$

We denote $c^*(V)$ the smallest. We have that it is small as

$$\left\| \Psi_\mu - \tau_{c^*(V)} \Psi_\mu \right\|_{\mathcal{X}_N} \leq \left\| \Psi_\mu - V \right\|_{\mathcal{X}_N} + \left\| V - \tau_{c^*(V)} \Psi_\mu \right\|_{\mathcal{X}_N} \leq 2\varepsilon$$

thus

$$d(c^*(V)) = d(-c^*(V)) \leq (2\varepsilon)^2 \leq \frac{1}{2}\eta(b).$$

Hence $c^*(V) \in (-b, b) \subset \mathcal{V}_0$, using Lemma 19, $c^*(V)$ is the unique positive minimum of $\inf_{c \in \mathbb{R}} \|V - \tau_c \Psi_\mu\|_{\mathcal{X}_N}$ in $(-b, b)$.

With the formula for \dot{c} set in (3.3.40) and the definition of $c(0)$, we obtain (3.3.41) which implies that Υ is orthogonal to $\text{Ker}\{\mathcal{A}_\mu^{\text{ad}}\}$.

We have that Υ has the mild form

$$\Upsilon(t) = U_c(t, 0)\Upsilon_0 + \int_0^t U_c(t, s)\mathcal{F}(\Upsilon(s), \mu)ds + \int_0^t U_c(t, s) \frac{\text{Re} \langle E, \mathcal{F}(\Upsilon(s), \mu) \rangle}{\|E\|_{\mathcal{X}_N} - \text{Re} \langle \partial_y E, \Upsilon(s) \rangle} \partial_y \Psi_\mu ds, \quad (3.3.42)$$

and

$$c(t) = c_0 - \int_0^t \frac{\text{Re} \langle E, \mathcal{F}(\Upsilon(s), \mu) \rangle}{\|E\|_{\mathcal{X}_N} - \text{Re} \langle \partial_y E, \Upsilon(s) \rangle} ds. \quad (3.3.43)$$

This is the system we now consider for the stability analysis.

For definiteness, we consider

$$\Theta^+(c) := \min \left\{ \min \left\{ t \geq 0 : \text{Re} \|E\|_{\mathcal{X}_N}^2 - \langle \partial_y E, \Upsilon(t) \rangle = \frac{1}{2} \right\}; T^+(\psi_0) \right\}. \quad (3.3.44)$$

We have the following lemma.

Lemma 21. *There exists $\varepsilon > 0$ such that if*

$$\|c\|_{L^\infty(\mathbb{R}^+)} < \varepsilon$$

then for any I (time) interval of \mathbb{R}^+ and any non negative measurable (time) function ρ on I and for some $C > 0$,

- (i) $\|\rho \tau_c \Upsilon\|_{\mathcal{X}_N} \leq C \|\rho Q \tau_c \Upsilon\|_{\mathcal{X}_N}$
- (ii) $\|\rho \tau_c \Upsilon\|_{L^4(I \times \mathbb{T}_N)^2} \leq C \|\rho Q \tau_c \Upsilon\|_{L^4(I \times \mathbb{T}_N)^2}$
- (iii) $\|\rho \tau_c \Upsilon\|_{L^{4/3}(I \times \mathbb{T}_N)^2} \leq C \|\rho Q \tau_c \Upsilon\|_{L^{4/3}(I \times \mathbb{T}_N)^2}$

whenever the left hand side is finite.

Notice that the norms are invariant in space translations.

Proof. We prove the three estimates together as their proof differ at only one point. Let $v \in \text{Ran}(\tau_c Q)$. Thus, there exists $w \in \mathcal{X}_N$ such that $v(t) = \tau_{c(t)}(I - P)w(t)$. Notice that $(I - P)w(t) = \tau_{-c(t)}v(t)$

Then we have

$$Pv = P\tau_c(I - P)w = P(\tau_c - I)(I - P)w.$$

We have for all $t \geq 0$

$$\begin{aligned} \left\| \rho(t)P(\tau_{c(t)} - I)(I - P)w(t) \right\|_{\mathcal{X}_N} &= \left| \left\langle \rho(I - P)w(t), (\tau_{-c(t)} - I)E \right\rangle \right| \|\partial_y \Psi_\mu\|_{\mathcal{X}_N} \\ &\leq \|\rho(t)(I - P)w(t)\|_{\mathcal{X}_N} \left\| (\tau_{-c(t)} - I)E \right\|_{\mathcal{X}_N} \|\partial_y \Psi_\mu\|_{\mathcal{X}_N} \end{aligned}$$

Let $\delta_0 > 0$. If

$$\|c\|_{L^\infty(\mathbb{R}^+)} < \varepsilon$$

for ε small enough then

$$\|(\tau_{c(t)} - I)E\|_{\mathcal{X}_N} \|\partial_y \Psi_\mu\|_{\mathcal{X}_N} < \delta_0$$

and we have

$$\|\rho(t)P(\tau_{c(t)} - I)(I - P)w(t)\|_{\mathcal{X}_N} < \delta_0 \|\rho(t)\tau_{-c(t)}v(t)\|_{\mathcal{X}_N} = \delta_0 \|\rho(t)v(t)\|_{\mathcal{X}_N}.$$

We also have

$$\|\rho(t)P(\tau_{c(t)} - I)(I - P)w(t)\|_{L^4(I \times \mathbb{T}_N)^2} \leq \|\rho(t)(I - P)w(t)\|_{L^4(I \times \mathbb{T}_N)^2} \|(\tau_c - I)E\|_{L^\infty(I; L^{4/3}(\mathbb{T}_N)^2)} \|\partial_y \Psi_\mu\|_{L^4(\mathbb{T}_N)^2}$$

Hence for ε small enough, we have

$$\|\rho(t)P(\tau_{c(t)} - I)(I - P)w(t)\|_{L^4(I \times \mathbb{T}_N)^2} < \delta_0 \|\rho(t)v(t)\|_{L^4(I \times \mathbb{T}_N)^2}.$$

We have similarly

$$\|\rho(t)P(\tau_{c(t)} - I)(I - P)w(t)\|_{L^{4/3}(I \times \mathbb{T}_N)^2} < \delta_0 \|\rho(t)v(t)\|_{L^{4/3}(I \times \mathbb{T}_N)^2}.$$

Eventually, in all the cases, we have that

$$\begin{aligned} \|\rho(t)Qv\| &\geq \|\rho(t)v\| - \|\rho(t)Pv\| \\ &\geq (1 - \delta_0) \|\rho(t)v\|, \end{aligned}$$

which is the result of the lemma with $C = \frac{1}{1 - \delta_0}$. \square

First step: A priori bounds

We start the proper proof of Theorem 33.

Let $0 < \gamma < \gamma_N \leq \frac{1}{2}$, recall (3.3.26). We consider the norm

$$\|(\Upsilon, c)\|_\tau := \max \left\{ \|e^{\gamma \cdot} \Upsilon\|_{L^\infty((0, \tau); \mathcal{X}_N)}, \|e^{\gamma \cdot} \Upsilon\|_{L^4((0, \tau) \times \mathbb{T}_N)}, \|c\|_{L^\infty(0, \tau)} \right\}, \quad (3.3.45)$$

and we assume that $\|\Upsilon_0\|_{\mathcal{X}_N} \leq \delta$ (and recall $c(0) = 0$).

Remarque 38. Recall that $\|E\|_{\mathcal{X}_N} = 1$ and notice that, as E is not constant, $\|\partial_y E\|_{\mathcal{X}_N} \neq 0$. This implies that if $\|\Upsilon\| \leq \frac{1}{2\|\partial_y E\|_{\mathcal{X}_N}}$, then

$$\|E\|_{\mathcal{X}_N} - \operatorname{Re} \langle \partial_y E, \Upsilon \rangle \geq \frac{1}{2} \quad (3.3.46)$$

First we prove that for all $\varepsilon > 0$, there exists $\delta > 0$ and $\tau_0 > 0$ such that for all $\tau \leq \tau_0$,

$$\|\Upsilon_0\|_{\mathcal{X}_N} \leq \delta \implies \|(\Upsilon, c)\|_\tau \leq \varepsilon. \quad (3.3.47)$$

Let $\varepsilon > 0$,

$$\gamma \in \left(\frac{\gamma_N}{2}, \gamma_N \right)$$

and $\tau > 0$. Consider $t \in [0, \tau]$, by Lemma 21 and (3.3.42), we have

$$\begin{aligned} \|e^{\gamma \cdot} \Upsilon\|_{L^\infty((0, \tau); \mathcal{X}_N)} &= \|e^{\gamma \cdot} \tau_c \Upsilon\|_{L^\infty((0, \tau); \mathcal{X}_N)} \\ &\leq C \left\| e^{(\gamma + \mathcal{A}_\mu) \cdot} Q \tau_c \Upsilon_0 \right\|_{L^\infty((0, \tau); \mathcal{X}_N)} + C \left\| \int_0^t e^{(t-s)(\mathcal{A}_\mu + \gamma)} e^{\gamma s} Q \mathcal{F}(\tau_c \Upsilon, \mu) ds \right\|_{L^\infty((0, \tau); \mathcal{X}_N)} \\ &\quad + C \left\| \int_0^t e^{(t-s)(\mathcal{A}_\mu + \gamma)} e^{\gamma s} \frac{\operatorname{Re} \langle E, \mathcal{F}(\Upsilon(s), \mu) \rangle}{\|E\|_{\mathcal{X}_N} - \operatorname{Re} \langle \partial_y E, \Upsilon(s) \rangle} Q \partial_y \Psi_{\mu, -c} ds \right\|_{L^\infty((0, \tau); \mathcal{X}_N)}. \end{aligned}$$

By Theorem 37, we have for all $t \in [0, \tau]$, $\|e^{(\gamma + \mathcal{A}_\mu)t} Q \Upsilon_0\|_{\mathcal{X}_N} \leq C_\gamma \delta$. Then, by Lemma 17, in particular Strichartz estimate (3.3.35), we have

$$\begin{aligned} \left\| \int_0^t e^{(t-s)(\mathcal{A}_\mu + \gamma)} e^{\gamma s} Q \mathcal{F}(\tau_c \Upsilon, \mu) ds \right\|_{L^\infty((0, \tau); \mathcal{X}_N)} &\leq C_5 \|e^{\gamma N s} Q \mathcal{F}(\tau_c \Upsilon, \mu)\|_{L^{4/3}((0, \tau) \times \mathbb{T}_N)} \\ &\leq C_5 \|e^{\gamma N s} \mathcal{F}(\tau_c \Upsilon, \mu)\|_{L^{4/3}((0, \tau) \times \mathbb{T}_N)}. \end{aligned}$$

We have

$$\|e^{\gamma N s} \mathcal{F}(\tau_c \Upsilon, \mu)\|_{L^{4/3}((0, \tau) \times \mathbb{T}_N)} \leq 3 \|\psi_\mu\|_{L^4(\mathbb{T}_N)} \left\| e^{\gamma N s} (\tau_c \Upsilon)^2 \right\|_{L^2((0, \tau) \times \mathbb{T}_N)} + \left\| e^{\gamma N s} (\tau_c \Upsilon)^3 \right\|_{L^{4/3}((0, \tau) \times \mathbb{T}_N)}.$$

Notice that

$$\left\| e^{\gamma N s} (\tau_c \Upsilon)^3 \right\|_{L^{4/3}((0, \tau) \times \mathbb{T}_N)} = \left\| e^{\frac{\gamma N}{3} s} \tau_c \Upsilon \right\|_{L^4((0, \tau) \times \mathbb{T}_N)}^3 \leq \|e^{\gamma s} \Upsilon\|_{L^4((0, \tau) \times \mathbb{T}_N)}^3$$

and

$$\left\| e^{\gamma N s} (\tau_c \Upsilon)^2 \right\|_{L^2((0, \tau) \times \mathbb{T}_N)} = \left\| e^{\frac{\gamma N}{2} s} \tau_c \Upsilon \right\|_{L^4((0, \tau) \times \mathbb{T}_N)}^2 \leq \|e^{\gamma s} \Upsilon\|_{L^4((0, \tau) \times \mathbb{T}_N)}^2.$$

We infer

$$\|e^{\gamma s} \mathcal{F}(\tau_c \Upsilon, \mu)\|_{L^{4/3}((0, \tau) \times \mathbb{T}_N)} \leq 3 \|\psi_\mu\|_{L^4(\mathbb{T}_N)} \|e^{\gamma s} \Upsilon\|_{L^4((0, \tau) \times \mathbb{T}_N)}^2 + \|e^{\gamma s} \Upsilon\|_{L^4((0, \tau) \times \mathbb{T}_N)}^3 \quad (3.3.48)$$

and

$$\left\| \int_0^t e^{(t-s)(\mathcal{A}_\mu + \gamma)} e^{\gamma s} Q \mathcal{F}(\tau_c \Upsilon, \mu) ds \right\|_{L^\infty((0, \tau); \mathcal{X}_N)} \leq C_{\mu, \tau} \left(\|e^{\gamma s} \Upsilon\|_{L^4((0, \tau) \times \mathbb{T}_N)}^2 + \|e^{\gamma s} \Upsilon\|_{L^4((0, \tau) \times \mathbb{T}_N)}^3 \right)$$

with $C_{\mu, \tau} := \max\{3C_5 \|\psi_\mu\|_{L^4(\mathbb{T}_N)}, C_5\}$ which is finite for small μ . So if $\delta > 0$ is small enough, there exists $\tau_1 > 0$ such that for all $\tau \leq \tau_1$,

$$\begin{aligned} \left\| \int_0^t e^{(t-s)(\mathcal{A}_\mu + \gamma)} e^{\gamma s} Q \mathcal{F}(\tau_c \Upsilon, \mu) ds \right\|_{L^\infty((0, \tau); \mathcal{X}_N)} &\leq C_{\mu, \tau} \left(\|e^{\gamma s} \Upsilon\|_{L^4((0, \tau) \times \mathbb{T}_N)}^2 + \|e^{\gamma s} \Upsilon\|_{L^4((0, \tau) \times \mathbb{T}_N)}^3 \right) \\ &\leq \frac{\varepsilon}{3}. \end{aligned}$$

(3.3.49)

Then, we assume that $\delta < \frac{2}{\|\partial_y E\|_{\mathcal{X}_N}}$. There exists $\tau_2 > 0$ such that for all $s \in (0, \tau_2)$,

$$\|\Upsilon(s)\|_{\mathcal{X}_N} \leq \frac{2}{\|\partial_y E\|_{\mathcal{X}_N}},$$

thus inequality (3.3.46) is satisfied and with Strichartz estimate (3.3.35) and estimate (3.3.49), we obtain

$$\begin{aligned}
 & \left\| \int_0^t e^{(t-s)(A_\mu + \gamma)} e^{\gamma s} \frac{\operatorname{Re} \langle E, \mathcal{F}(\Upsilon(s), \mu) \rangle}{\|E\|_{\mathcal{X}_N} - \operatorname{Re} \langle \partial_y E, \Upsilon(s) \rangle} Q \partial_y \Psi_{\mu, -c} ds \right\|_{L^\infty((0, \tau); \mathcal{X}_N)} \\
 & \leq 2C_5 \|e^{\gamma N s} \operatorname{Re} \langle E, \mathcal{F}(\Upsilon(s), \mu) \rangle \partial_y \Psi_{\mu, -c}\|_{L^{4/3}((0, \tau) \times \mathbb{T}_N)} \\
 & \leq 2C_5 \|\partial_y \psi_\mu\|_{L^{4/3}(\mathbb{T}_N)} \|E\|_{L^4(\mathbb{T}_N)} \|e^{\gamma N s} \mathcal{F}(\Upsilon, \mu)\|_{L^{4/3}((0, \tau) \times \mathbb{T}_N)} \\
 & \leq 2C_{\mu, \tau} \|\partial_y \psi_\mu\|_{L^{4/3}(\mathbb{T}_N)} \|E\|_{L^4(\mathbb{T}_N)} \left(\|e^{\gamma s} \Upsilon\|_{L^4((0, \tau) \times \mathbb{T}_N)}^2 + \|e^{\gamma s} \Upsilon\|_{L^4((0, \tau) \times \mathbb{T}_N)}^3 \right) \\
 & \leq \widetilde{C}_{\mu, \tau} \left(\|e^{\gamma s} \Upsilon\|_{L^4((0, \tau) \times \mathbb{T}_N)}^2 + \|e^{\gamma s} \Upsilon\|_{L^4((0, \tau) \times \mathbb{T}_N)}^3 \right)
 \end{aligned}$$

with $\widetilde{C}_{\mu, \tau} := 2C_{\mu, \tau} \|\partial_y \psi_\mu\|_{L^{4/3}(\mathbb{T}_N)} \|E\|_{L^4(\mathbb{T}_N)}$ which is finite for μ small. There exists $\tau_3 > 0$ such that for all $\tau \leq \tau_3$,

$$\widetilde{C}_{\mu, \tau} \left(\|e^{\gamma s} \Upsilon\|_{L^4((0, \tau) \times \mathbb{T}_N)}^2 + \|e^{\gamma s} \Upsilon\|_{L^4((0, \tau) \times \mathbb{T}_N)}^3 \right) \leq \frac{\varepsilon}{3}.$$

Hence, if we choose $\tau \leq \min\{\tau_1, \tau_2, \tau_3\}$ and $\delta \leq \frac{\varepsilon}{3CC_\gamma}$, then

$$\|e^{\gamma \cdot} \Upsilon\|_{L^\infty((0, \tau); \mathcal{X}_N)} \leq \varepsilon.$$

By Lemmas 16 and 17, in particular Strichartz estimates (3.3.32) and (3.3.36), we have

$$\begin{aligned}
 \|e^{\gamma \cdot} \Upsilon\|_{L^4((0, \tau) \times \mathbb{T}_N)} & \leq C \|e^{\gamma \cdot} Q \tau_c \Upsilon\|_{L^4((0, \tau) \times \mathbb{T}_N)} \\
 & \leq CC_6 \|\Upsilon_0\|_{\mathcal{X}_N} + CC_8 \|e^{\gamma N s} Q \mathcal{F}(\tau_c \Upsilon, \mu)\|_{L^{4/3}((0, \tau) \times \mathbb{T}_N)} \\
 & \quad + CC_8 \left\| e^{\gamma N s} \frac{\operatorname{Re} \langle E, \mathcal{F}(\Upsilon(s), \mu) \rangle}{\|E\|_{\mathcal{X}_N} - \operatorname{Re} \langle \partial_y E, \Upsilon(s) \rangle} Q \partial_y \Psi_{\mu, -c} \right\|_{L^{4/3}((0, \tau) \times \mathbb{T}_N)}.
 \end{aligned}$$

Using (3.3.48), we have

$$\begin{aligned}
 \|e^{\gamma N s} Q \mathcal{F}(\tau_c \Upsilon, \mu)\|_{L^{4/3}((0, \tau) \times \mathbb{T}_N)} & \leq 3 \|\psi_\mu\|_{L^4(\mathbb{T}_N)} (2\pi N)^{1/4} \left\| e^{(\frac{\gamma N}{2} - \gamma)s} \right\|_{L^8(0, \tau)}^2 \|e^{\gamma N s} \Upsilon\|_{L^4((0, \tau) \times \mathbb{T}_N)}^2 \\
 & \quad + \left\| e^{\frac{\gamma N}{3}s} \Upsilon \right\|_{L^4((0, \tau) \times \mathbb{T}_N)}^3
 \end{aligned}$$

and we also have

$$\begin{aligned}
 & \left\| e^{\gamma N s} \frac{\operatorname{Re} \langle E, \mathcal{F}(\Upsilon(s), \mu) \rangle}{\|E\|_{\mathcal{X}_N} - \operatorname{Re} \langle \partial_y E, \Upsilon(s) \rangle} Q \partial_y \Psi_{\mu, -c} \right\|_{L^{4/3}((0, \tau) \times \mathbb{T}_N)} \\
 & \leq 2 \|\partial_y \psi_\mu\|_{L^{4/3}(\mathbb{T}_N)} \|E\|_{L^4(\mathbb{T}_N)} \|e^{\gamma N s} \mathcal{F}(\Upsilon, \mu)\|_{L^{4/3}((0, \tau) \times \mathbb{T}_N)} \\
 & \leq 2 \|\partial_y \psi_\mu\|_{L^{4/3}(\mathbb{T}_N)} \|E\|_{L^4(\mathbb{T}_N)} \times \left(3 \|\psi_\mu\|_{L^{4/3}(\mathbb{T}_N)} (2\pi N)^{1/4} \left\| e^{(\frac{\gamma N}{2} - \gamma)s} \right\|_{L^8(0, \tau)}^2 \|e^{\gamma s} \Upsilon\|_{L^4((0, \tau) \times \mathbb{T}_N)}^2 \right. \\
 & \quad \left. + \|e^{\gamma s} \Upsilon\|_{L^4((0, \tau) \times \mathbb{T}_N)}^3 \right)
 \end{aligned}$$

thus there exists $\tau_4 > 0$ such that if $\tau \leq \tau_4$ and $\delta \leq \frac{\varepsilon}{3CC_6}$, then

$$\|e^{\gamma \cdot} \Upsilon\|_{L^4((0, \tau) \times \mathbb{T}_N)} \leq \varepsilon.$$

Then, we have that

$$t \mapsto \int_0^t \frac{\operatorname{Re} \langle E, \mathcal{F}(\Upsilon(s), \mu) \rangle}{\|E\|_{\mathcal{X}_N} - \operatorname{Re} \langle \partial_y E, \Upsilon(s) \rangle} ds$$

is bounded on $(0, \min\{\tau_1, \tau_2, \tau_3, \tau_4\})$. Indeed, for $\tau \in (0, \min\{\tau_1, \tau_2, \tau_3, \tau_4\})$, we have

$$\begin{aligned} \left\| \frac{\operatorname{Re} \langle E, \mathcal{F}(\Upsilon(s), \mu) \rangle}{\|E\|_{\mathcal{X}_N} - \operatorname{Re} \langle \partial_y E, \Upsilon(s) \rangle} \right\|_{L^1(0, \tau)} &\leq 2 \|\operatorname{Re} \langle E, \mathcal{F}(\Upsilon(s), \mu) \rangle\|_{L^1(0, \tau)} \\ &\leq 2 \left(\int e^{-4\gamma s} \right)^{1/4} \|E\|_{L^4(\mathbb{T}_N)} \|e^{\gamma s} \mathcal{F}(\Upsilon(s), \mu)\|_{L^{4/3}((0, \tau) \times \mathbb{T}_N)} \end{aligned} \quad (3.3.50)$$

which is a finite quantity due to (3.3.48). Hence, there exists $\tau_0 \in (0, \min\{\tau_1, \tau_2, \tau_3, \tau_4\})$ such that for all $t \in (0, \tau_0)$,

$$|c(t)| \leq \varepsilon,$$

which implies

$$\|c\|_{L^\infty(0, \tau_0)} \leq \varepsilon.$$

Finally, for all $\varepsilon > 0$, there exists $\tau_0 > 0$ such that for $\delta \leq \min\{\frac{\varepsilon}{3C_4}, \frac{\varepsilon}{3C_\gamma}, \frac{2}{\|\partial_y E\|_{\mathcal{X}_N}}\}$ and $\tau \leq \tau_0$, we have

$$\|\Upsilon_0\|_{\mathcal{X}_N} \leq \delta \implies \|(T, c)\|_\tau \leq \varepsilon. \quad (3.3.51)$$

Conclusion: Global bounds

We set

$$K_1 := \min\{CC_\gamma, CC_4\},$$

$$K_2(\mu) := \max\{4 \|E\|_{L^4(\mathbb{T}_N)}; 2C \min\{C_5, C_6\} (2\pi N)^{1/4} (1/4\gamma)^{1/4} (1 + 2 \|\partial_y \psi_\mu\|_{L^{4/3}(\mathbb{T}_N)} \|E\|_{L^4(\mathbb{T}_N)})\},$$

$$K_3(\mu) := \max\{2 \|E\|_{L^4(\mathbb{T}_N)}; C \min\{C_5, C_6\} (1 + 2 \|\partial_y \psi_\mu\|_{L^{4/3}(\mathbb{T}_N)} \|E\|_{L^4(\mathbb{T}_N)})\}.$$

As μ is small, there exists K_2 and $K_3 > 0$ such that

$$K_2(\mu) \leq K_2, \quad K_3(\mu) \leq K_3.$$

Now, we fix $\varepsilon \in \left(0, \min\{\frac{\varepsilon_0}{2}, \frac{1}{16K_3}, 1, \frac{1}{2\|\partial_y E\|_{\mathcal{X}_N}}\}\right)$. We consider $\delta \leq \min\{\frac{\varepsilon}{3C_4}, \frac{\varepsilon}{3C_\gamma}, \frac{\varepsilon}{4K_1}\}$ and μ enough small such that $1 - 3K_2 \|\psi_\mu\|_{L^4(\mathbb{T}_N)} \geq \frac{1}{2}$. Then, inequality (3.3.51) implies that the set

$$\{\tau, \|(T, c)\|_\tau \leq \varepsilon\}$$

is not empty. We assume that this set is bounded and will prove that there is a contradiction. We set

$$\tau^+ := \max\{\tau, \|(T, c)\|_\tau \leq \varepsilon\}.$$

We cannot have $\tau^+ = T^+(\psi_0)$ due to the blow-up alternative. Indeed, if $\tau^+ = T^+(\psi_0) < +\infty$, then by Theorem 24, $\lim_{t \rightarrow T^+(\psi_0)} \|\Upsilon\|_{L^4((0, t) \times \mathbb{T}_N)} = +\infty$, which contradicts the definition of τ^+ .

We also cannot have $\tau^+ = \Theta^+(c)$. Indeed, if $\tau^+ = \Theta^+(c)$, then $\Theta^+(c) = \min\{t \geq 0 : \|E\|_{\mathcal{X}_N}^2 - \operatorname{Re} \langle \partial_y E, \Upsilon(t) \rangle = \frac{1}{2}\}$. As $\|(T, c)\|_{\tau^+} \leq \varepsilon < \frac{1}{2\|\partial_y E\|_{\mathcal{X}_N}}$, then, see Remark 38, for all $t \in [0, \tau^+]$, $\|E\|_{\mathcal{X}_N}^2 - \operatorname{Re} \langle \partial_y E, \Upsilon(t) \rangle > \frac{1}{2}$, which contradicts that $\tau^+ = \Theta^+(c)$.

CHAPTER 3. ASYMPTOTIC STABILITY FOR THE LUGIATO-LEFEVER EQUATION

Assume that τ^+ is finite. The applications $\tau \mapsto \|e^{\gamma \cdot} \Upsilon\|_{L^\infty((0,\tau); \mathcal{X}_N)}$, $\tau \mapsto \|e^{\gamma \cdot} \Upsilon\|_{L^4((0,\tau) \times \mathbb{T}_N)}$ and $\tau \mapsto \|c\|_{L^\infty(0,\tau)}$ are continuous on \mathbb{R}_+ (due to Strichartz estimates) thus there exists $\tau_1 > 0$ such that

$$\|(\Upsilon, c)\|_{\tau_1 + \tau^+} \leq 2\varepsilon. \quad (3.3.52)$$

We consider $t \in [0, \tau_1 + \tau^+]$. We have

$$\begin{aligned} \|e^{\gamma \cdot} \Upsilon\|_{L^\infty((0,t); \mathcal{X}_N)} &\leq CC_7 \delta + CC_5 \left(3 \|\psi_\mu\|_{L^4(\mathbb{T}_N)} (2\pi N)^{1/4} \|e^{-\gamma \cdot}\|_{L^4(0,t)} \|e^{\gamma \cdot} \Upsilon\|_{L^4((0,t) \times \mathbb{T}_N)}^2 + \|e^{\gamma \cdot} \Upsilon\|_{L^4((0,t) \times \mathbb{T}_N)}^3 \right) \\ &\quad + 2CC_5 \|\partial_y \psi_\mu\|_{L^{4/3}(\mathbb{T}_N)} \|E\|_{L^4(\mathbb{T}_N)} \left(3 \|\psi_\mu\|_{L^4(\mathbb{T}_N)} (2\pi N)^{1/4} \|e^{-\gamma \cdot}\|_{L^4(0,t)} \|e^{\gamma \cdot} \Upsilon\|_{L^4((0,t) \times \mathbb{T}_N)}^2 \right. \\ &\quad \left. + \|e^{\gamma \cdot} \Upsilon\|_{L^4((0,t) \times \mathbb{T}_N)}^3 \right) \\ &\leq K_1 \delta + K_2 \|\psi_\mu\|_{L^4(\mathbb{T}_N)} \|e^{\gamma \cdot} \Upsilon\|_{L^4((0,t) \times \mathbb{T}_N)}^2 + K_3 \|e^{\gamma \cdot} \Upsilon\|_{L^4((0,t) \times \mathbb{T}_N)}^3 \\ &\leq K_1 \delta + K_2 \|\psi_\mu\|_{L^4(\mathbb{T}_N)} \|(T, c)\|_t^2 + K_3 \|(\Upsilon, c)\|_t^3, \end{aligned}$$

$$\begin{aligned} \|e^{\gamma \cdot} \Upsilon\|_{L^4((0,t) \times \mathbb{T}_N)} &\leq CC_4 \delta + CC_6 \left(3 \|\psi_\mu\|_{L^4(\mathbb{T}_N)} (2\pi N)^{1/4} \|e^{-\gamma \cdot}\|_{L^4(0,t)} \|e^{\gamma \cdot} \Upsilon\|_{L^4((0,t) \times \mathbb{T}_N)}^2 + \|e^{\gamma \cdot} \Upsilon\|_{L^4((0,t) \times \mathbb{T}_N)}^3 \right) \\ &\quad + 2CC_6 \|\partial_y \psi_\mu\|_{L^{4/3}(\mathbb{T}_N)} \|E\|_{L^4(\mathbb{T}_N)} \left(3 \|\psi_\mu\|_{L^4(\mathbb{T}_N)} (2\pi N)^{1/4} \|e^{-\gamma \cdot}\|_{L^4(0,t)} \|e^{\gamma \cdot} \Upsilon\|_{L^4((0,t) \times \mathbb{T}_N)}^2 \right. \\ &\quad \left. + \|e^{\gamma \cdot} \Upsilon\|_{L^4((0,t) \times \mathbb{T}_N)}^3 \right) \\ &\leq K_1 \delta + K_2 \|\psi_\mu\|_{L^4(\mathbb{T}_N)} \|e^{\gamma \cdot} \Upsilon\|_{L^4((0,t) \times \mathbb{T}_N)}^2 + K_3 \|e^{\gamma \cdot} \Upsilon\|_{L^4((0,t) \times \mathbb{T}_N)}^3 \\ &\leq K_1 \delta + K_2 \|\psi_\mu\|_{L^4(\mathbb{T}_N)} \|(\Upsilon, c)\|_t^2 + K_3 \|(\Upsilon, c)\|_t^3, \end{aligned}$$

and, for all $s \in [0, t]$, by (3.3.46), we have

$$\begin{aligned} |c(s)| &\leq 2 \left(\int e^{-4\gamma s} \right)^{1/4} \|E\|_{L^4(\mathbb{T}_N)} \|e^{\gamma s} \mathcal{F}(\Upsilon(s), \mu)\|_{L^{4/3}((0,\tau) \times \mathbb{T}_N)} \\ &\leq \frac{2}{(4\gamma)^{1/4}} \|E\|_{L^4(\mathbb{T}_N)} \left(3 \|\psi_\mu\|_{L^4(\mathbb{T}_N)} \|e^{\gamma \cdot} \Upsilon\|_{L^\infty((0,s); \mathcal{X}_N)} + \|e^{\gamma \cdot} \Upsilon\|_{L^4((0,s) \times \mathbb{T}_N)}^2 \|e^{\gamma \cdot} \Upsilon\|_{L^\infty((0,s); \mathcal{X}_N)} \right) \end{aligned}$$

thus

$$\begin{aligned} \|c\|_{L^\infty(0,t)} &\leq 6 \frac{2}{(4\gamma)^{1/4}} \|E\|_{L^4(\mathbb{T}_N)} \|\psi_\mu\|_{L^4(\mathbb{T}_N)} \|(\Upsilon, c)\|_t^2 + 2 \|E\|_{L^4(\mathbb{T}_N)} \|(\Upsilon, c)\|_t^3 \\ &\leq K_1 \delta + K_2 \|\psi_\mu\|_{L^4(\mathbb{T}_N)} \|(\Upsilon, c)\|_t^2 + K_3 \|(\Upsilon, c)\|_t^3. \end{aligned}$$

Then, we have

$$\|(\Upsilon, c)\|_{\tau_1 + \tau^+} \leq K_1 \delta + K_2 \|\psi_\mu\|_{L^4(\mathbb{T}_N)} \|(\Upsilon, c)\|_{\tau_1 + \tau^+}^2 + K_3 \|(\Upsilon, c)\|_{\tau_1 + \tau^+}^3.$$

As $\varepsilon \in \left(0, \min\left\{\frac{\varepsilon_0}{2}, \frac{1}{16K_3}, 1\right\}\right)$, $\delta \leq \min\left\{\frac{\varepsilon}{3C_4}, \frac{\varepsilon}{3C_7}, \frac{\varepsilon}{4K_1}\right\}$ and $1 - 2K_2 \|\psi_\mu\|_{L^4(\mathbb{T}_N)}$, by (3.3.52), we obtain

$$\begin{aligned} \|(\Upsilon, c)\|_{\tau_1 + \tau^+} &\leq K_1 \delta + K_2 \|\psi_\mu\|_{L^4(\mathbb{T}_N)} \|(\Upsilon, c)\|_{\tau_1 + \tau^+}^2 + K_3 \|(\Upsilon, c)\|_{\tau_1 + \tau^+}^3 \\ &\leq K_1 \delta + 2\varepsilon K_2 \|\psi_\mu\|_{L^4(\mathbb{T}_N)} \|(\Upsilon, c)\|_{\tau_1 + \tau^+} + 4\varepsilon^2 K_3 \|(\Upsilon, c)\|_{\tau_1 + \tau^+}^3 \\ &\leq K_1 \delta + 2K_2 \|\psi_\mu\|_{L^4(\mathbb{T}_N)} \|(\Upsilon, c)\|_{\tau_1 + \tau^+} + 4\varepsilon K_3 \|(\Upsilon, c)\|_{\tau_1 + \tau^+}^3, \end{aligned}$$

thus

$$\frac{1}{2} \|(\Upsilon, c)\|_{\tau_1 + \tau^+} \leq K_1 \delta + 4K_3 \|(\Upsilon, c)\|_{\tau_1 + \tau^+}^3.$$

As $\varepsilon \leq \frac{1}{16K_3}$, we have $\frac{1}{2} - 4K_3\varepsilon \geq \frac{1}{4}$, thus

$$\|(\Upsilon, c)\|_{\tau_1 + \tau^+} \leq 4K_1 \delta$$

and as $\delta \leq \frac{\varepsilon}{4K_1}$, we obtain

$$\|(\Upsilon, c)\|_{\tau_1 + \tau^+} \leq \varepsilon,$$

which contradicts the fact that τ^+ is finite, thus $\tau^+ = +\infty$. This implies that $T^+(\psi_0) = +\infty$ and $\Theta^+(c) = +\infty$.

Then (3.3.43) and (3.3.50) provide that $\lim_{+\infty} c = c_\infty$ exists and $|c_\infty - c_0| = O(\varepsilon^2)$.

This concludes the proof of Theorem 33. □

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.1 Duhamel formula

We consider $T > 0$ and ψ such that

$$\psi(t) := e^{tB}\psi_0 + \int_0^t e^{(t-s)B}F(s) \, ds, \quad t \geq [0, T),$$

where B generates a continuous semi-group on some Banach space X and $F : [0, T[\rightarrow X$ continuous. If $B = A + C$, with C bounded, then A and B have same domain (which is dense).

Below we are interested in reformulating ψ as a perturbation of $e^{tA}\psi_0$ instead of $e^{tB}\psi_0$. This is the so-called Duhamel formula (maybe in a linear fashion).

We get after considering weak derivatives for instance

$$e^{tB}\phi = e^{tA}\phi + \int_0^t e^{(t-s)A}C e^{sB}\phi \, ds$$

or

$$e^{tB}\phi = e^{tA}\phi - \int_0^t e^{(t-s)B}C e^{sA}\phi \, ds$$

for any ϕ in X .

So using the above expression with ψ_0 and $F(s)$ in the expression for ψ we have

$$\begin{aligned} (t) &= e^{tA}\psi_0 + \int_0^t e^{(t-s)A} C e^{sB} \psi_0 \, ds \\ &\quad + \int_0^t e^{(t-s)A} F(s) \, ds + \int_0^t \int_0^{t-s} e^{(t-s-\tau)A} C e^{\tau B} F(s) \, d\tau \, ds. \end{aligned}$$

Notice that changing successively $s + \tau$ to σ , using Fubini theorem and renaming s the variable σ we have

$$\begin{aligned} \int_0^t \int_0^{t-s} e^{(t-s-\tau)A} C e^{\tau B} F(s) \, d\tau \, ds &= \int_0^t \int_0^{t-s} e^{(t-(s+\tau))A} C e^{(\tau+s-s)B} F(s) \, d\tau \, ds \\ &= \int_0^t \int_s^t e^{(t-\sigma)A} C e^{(\sigma-s)B} F(s) \, d\sigma \, ds \\ &= \int_0^t \int_0^\sigma e^{(t-\sigma)A} C e^{(\sigma-s)B} F(s) \, ds \, d\sigma \\ &= \int_0^t \int_0^s e^{(t-s)A} C e^{(s-\sigma)B} F(\sigma) \, d\sigma \, ds \end{aligned}$$

which gives

$$\begin{aligned} (t) &= e^{tA}\psi_0 + \int_0^t e^{(t-s)A} C e^{sB} \psi_0 \, ds \\ &\quad + \int_0^t e^{(t-s)A} F(s) \, ds + \int_0^t \int_0^s e^{(t-s)A} C e^{(s-\sigma)B} F(\sigma) \, d\sigma \, ds \\ &= e^{tA}\psi_0 \\ &\quad + \int_0^t e^{(t-s)A} \left(C e^{sB} \psi_0 + F(s) + C \int_0^s e^{(s-\sigma)B} F(\sigma) \, d\sigma \right) \, ds \\ &= e^{tA}\psi_0 + \int_0^t e^{(t-s)A} (C\psi(s) + F(s)) \, ds. \end{aligned}$$

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Appendix A

Coefficients of the reduced system

A.1 Computation of c_{11}

According to [5], the coefficient c_{11} can be computed from the formula

$$c_{11} = \frac{d\lambda_*}{d\mu}(0),$$

where $\lambda_*(\mu)$ is the eigenvalue of the operator \mathcal{A}_μ^* which is the continuation of the eigenvalue 0 of the operator \mathcal{A}_0^* , for small μ . The results in Section 1.1 imply that $\lambda_*(\mu)$ is the largest root of the polynomial

$$\lambda^2 + 2\lambda + a_\mu(k_*) = 0, \quad (\text{A.1.1})$$

in which

$$a_\mu(k_*) = k_*^4 + 2(\alpha - 2\rho_\mu^*)k_*^2 + \alpha^2 - 4\alpha\rho_\mu^* + 3\rho_\mu^{*2} + 1.$$

Upon differentiating (A.1.1) with respect to μ and taking $\mu = 0$, after some elementary calculations, we obtain

$$c_{11} = \frac{1}{k_{\max}^2 k_{\min}^2} \left(\rho^* + 2\sqrt{\rho^{*2} - 1} \right), \quad (\text{A.1.2})$$

in the case $k_* = k_{\max}$, and

$$c_{11} = \frac{1}{k_{\max}^2 k_{\min}^2} \left(\rho^* - 2\sqrt{\rho^{*2} - 1} \right), \quad (\text{A.1.3})$$

in the case $k_* = k_{\min}$. Notice that we can obtain the formula for $k_* = k_{\min}$ from the one for $k_* = k_{\max}$ just by changing the sign of the square root $\sqrt{\rho^{*2} - 1}$.

A.2 Computation of c_{30}

According to [5], the coefficient c_{30} can be computed from the formula

$$c_{30} = \frac{1}{\langle \zeta, \mathcal{J}\zeta_2 \rangle} \left\langle 2\mathcal{R}_2(\zeta, \Psi_{11}, 0) + 2\mathcal{R}_2(\bar{\zeta}, \Psi_{20}, 0) + 3\mathcal{R}_3(\zeta, \zeta, \bar{\zeta}), \zeta_2 \right\rangle,$$

in which \mathcal{R}_2 and \mathcal{R}_3 are the bilinear and trilinear maps, respectively, given by (??)-(??), ζ and ζ_2 are eigenvectors of \mathcal{A}_0^* associated to the eigenvalues 0 and -2 , respectively, and the vectors Ψ_{11} and Ψ_{20} are solutions of the linear equations:

$$\mathcal{A}_0^* \Psi_{11} = -2\mathcal{J}\mathcal{R}_2(\zeta, \bar{\zeta}, 0), \quad \mathcal{A}_0^* \Psi_{20} = -\mathcal{J}\mathcal{R}_2(\zeta, \zeta, 0).$$

APPENDIX A. COEFFICIENTS OF THE REDUCED SYSTEM

In the case $k_* = k_{\max}$, the eigenvector ζ is given by (1.3.5), a direct calculation gives

$$\zeta_2 = \begin{pmatrix} (\rho^* - \sqrt{\rho^{*2} - 1})(2\rho^* - \alpha + \sqrt{\rho^{*2} - 1}) \\ -\alpha + \sqrt{\rho^{*2} - 1} \end{pmatrix} e^{iy}.$$

and a symbolic computation using Maple gives

$$\Psi_{11} = \frac{4F_*}{k_{\max}^2 k_{\min}^2} \begin{pmatrix} u_{11} \\ v_{11} \end{pmatrix},$$

$$\Psi_{20} = \frac{2F_*}{k_{\max}^2 (9(2\rho^* - \alpha) + 15\sqrt{\rho^{*2} - 1})} \begin{pmatrix} u_{20} \\ v_{20} \end{pmatrix} e^{2iy},$$

in which

$$\begin{aligned} u_{11} &= (\rho^* - \sqrt{\rho^{*2} - 1})\alpha^2 + (2\rho^* \sqrt{\rho^{*2} - 1} - 2\rho^{*2} + 2)\alpha \\ &\quad - \rho^{*2} \sqrt{\rho^{*2} - 1} + \rho^{*3} + \sqrt{\rho^{*2} - 1} - 3\rho^*, \\ v_{11} &= (\rho^* \sqrt{\rho^{*2} - 1} - \rho^{*2} - 1)\alpha^2 \\ &\quad - 2(\rho^{*2} \sqrt{\rho^{*2} - 1} - \rho^{*3} + \sqrt{\rho^{*2} - 1} - 2\rho^*)\alpha \\ &\quad + \rho^{*3} \sqrt{\rho^{*2} - 1} - \rho^{*4} + 3\rho^* \sqrt{\rho^{*2} - 1} - 4\rho^{*2} + 1, \\ u_{20} &= 3(\sqrt{\rho^{*2} - 1} - \rho^*)\alpha^2 - 2(\rho^* \sqrt{\rho^{*2} - 1} - \rho^{*2} - 1)\alpha \\ &\quad - \rho^{*2} \sqrt{\rho^{*2} - 1} + \rho^{*3} + 5\sqrt{\rho^{*2} - 1} + 5\rho^*, \\ v_{20} &= -3(\rho^* \sqrt{\rho^{*2} - 1} - \rho^{*2} - 1)\alpha^2 \\ &\quad + 2(3\rho^{*2} \sqrt{\rho^{*2} - 1} - 3\rho^{*3} - \sqrt{\rho^{*2} - 1} - 8\rho^*)\alpha \\ &\quad - 3\rho^{*3} \sqrt{\rho^{*2} - 1} + 3\rho^{*4} + 3\rho^* \sqrt{\rho^{*2} - 1} + 12\rho^{*2} + 5, \end{aligned}$$

and then the formula for the coefficient c_{30} ,

$$c_{30} = \frac{2F_*^2(\rho^* - \sqrt{\rho^{*2} - 1})}{k_{\max}^2 k_{\min}^2 (9(2\rho^* - \alpha) + 15\sqrt{\rho^{*2} - 1})} \tilde{c}_{30}, \quad (\text{A.2.1})$$

where

$$\begin{aligned} \tilde{c}_{30} &= -27\sqrt{\rho^{*2} - 1}\alpha^3 + 3(4\rho^* \sqrt{\rho^{*2} - 1} - 25\rho^{*2} + 5)\alpha^2 \\ &\quad + (345\rho^{*2} \sqrt{\rho^{*2} - 1} + 438\rho^{*3} - 67\sqrt{\rho^{*2} - 1} - 236\rho^*)\alpha \\ &\quad - 330\rho^{*3} \sqrt{\rho^{*2} - 1} - 363\rho^{*4} + 144\rho^{*2} + 55. \end{aligned} \quad (\text{A.2.2})$$

Finally, as for the coefficient c_{11} , we obtain the formula for the coefficient c_{30} in the case $k_* = k_{\min}$ by changing the sign of the square root $\sqrt{\rho^{*2} - 1}$ in the formula (A.2.1) for $k_* = k_{\max}$, so that

$$c_{30} = \frac{2F_*^2(\rho^* + \sqrt{\rho^{*2} - 1})}{k_{\max}^2 k_{\min}^2 (9(2\rho^* - \alpha) - 15\sqrt{\rho^{*2} - 1})} \tilde{c}_{30}, \quad (\text{A.2.3})$$

where

$$\begin{aligned} \tilde{c}_{30} = & 27\sqrt{\rho^{*2} - 1}\alpha^3 - 3(4\rho^*\sqrt{\rho^{*2} - 1} + 25\rho^{*2} - 5)\alpha^2 \\ & - (345\rho^{*2}\sqrt{\rho^{*2} - 1} - 438\rho^{*3} - 67\sqrt{\rho^{*2} - 1} + 236\rho^*)\alpha \\ & + 330\rho^{*3}\sqrt{\rho^{*2} - 1} - 363\rho^{*4} + 144\rho^{*2} + 55. \end{aligned} \quad (\text{A.2.4})$$

A.3 Duhamel formula

We start with

$$(t) = e^{tB}\psi_0 + \int_0^t e^{(t-s)B}F(s) ds.$$

If $B = A + C$, with C bounded, then

$$e^{tB}\phi = e^{tA}\phi + \int_0^t e^{(t-s)A}Ce^{sB}\phi ds$$

or

$$e^{tB}\phi = e^{tA}\phi - \int_0^t e^{(t-s)B}Ce^{sA}\phi ds$$

So we have

$$\begin{aligned} (t) = & e^{tA}\psi_0 + \int_0^t e^{(t-s)A}Ce^{sB}\psi_0 ds \\ & + \int_0^t e^{(t-s)A}F(s) ds + \int_0^t \int_0^{t-s} e^{(t-s-\tau)A}Ce^{\tau B}F(s) d\tau ds \end{aligned}$$

Notice that

$$\begin{aligned} \int_0^t \int_0^{t-s} e^{(t-s-\tau)A}Ce^{\tau B}F(s) d\tau ds &= \int_0^t \int_0^{t-s} e^{(t-(s+\tau))A}Ce^{(\tau+s-s)B}F(s) d\tau ds \\ &= \int_0^t \int_s^t e^{(t-\sigma)A}Ce^{(\sigma-s)B}F(s) d\sigma ds \\ &= \int_0^t \int_0^\sigma e^{(t-\sigma)A}Ce^{(\sigma-s)B}F(s) ds d\sigma \\ &= \int_0^t \int_0^s e^{(t-s)A}Ce^{(s-\sigma)B}F(\sigma) d\sigma ds \end{aligned}$$

which gives

$$\begin{aligned} (t) = & e^{tA}\psi_0 + \int_0^t e^{(t-s)A}Ce^{sB}\psi_0 ds \\ & + \int_0^t e^{(t-s)A}F(s) ds + \int_0^t \int_0^s e^{(t-s)A}Ce^{(s-\sigma)B}F(\sigma) d\sigma ds \\ & = e^{tA}\psi_0 \\ & + \int_0^t e^{(t-s)A} \left(Ce^{sB}\psi_0 + F(s) + C \int_0^s e^{(s-\sigma)B}F(\sigma) d\sigma \right) ds \\ & = e^{tA}\psi_0 + \int_0^t e^{(t-s)A} (C\psi(s) + F(s)) ds \end{aligned}$$