Gravitational dualities and exotic supersymmetries

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Abstract

Since the second half of the twentieth century and the advent of the Standard Model, that unified three fundamental forces (electromagnetism, the strong nuclear force and the weak nuclear force), a lot of effort has been devoted to unifying the fourth fundamental force we know: gravity. A theory unifying all forces has remained inconclusive so far as the quantisation of gravity generates a non-renormalizable theory. A new symmetry called supersymmetry is one way to build a better behaved theory. A combination of a theory of gravity with supersymmetry is called supergravity. Mathematically, we can build multiple theories of supergravities depending on the number of supersymmetries in the model.

In this thesis, we focus on six-dimensional maximally supersymmetric supergravity theories. There exist three possible theories of supergravity in six dimensions: the conventional $\mathcal{N} = (2, 2)$ theory and the more exotic $\mathcal{N} = (3, 1)$ and $\mathcal{N} = (4, 0)$ theories. The last two theories feature exotic gauge fields with non-standard Young tableaux representations, subject to self-duality equations of motion. We present these theories in a 5+1 split of coordinates allowing us to build new Lagrangians that reproduce both the six-dimensional equations of motion and the linearized five-dimensional supergravity theory upon compactification of the singled out coordinate. Then we unify the bosonic sectors of these three theories under the common framework of Exceptional Field Theory provided we introduce a modified section constraint whose solution produces the $\mathcal{N} = (4, 0)$, (3,1) and (2,2) theories. Finally, we build the supersymmetric versions of these Lagrangians allowing us to include the fermionic sectors of each theory.

Key words: supergravity, duality, Exceptional Field Theory, ExFT, supersymmetry, exotic tensors

Résumé

Depuis la deuxième moitié du vingtième siècle et le développement du Modèle Standard de la physique des particules qui a vu l'unification de trois des quatre forces fondamentales (électromagnétisme, force nucléaire forte et force nucléaire faible), l'idée d'unifier la quatrième force fondamentale connue – la gravité – a été extrêmement fructueuse même si le projet n'a toujours pas abouti. En effet, quantifier la théorie de la relativité générale produit une théorie non-renormalisable c'est-à-dire que le calcul de certaines observables donne des résultats infinis, ce qui est fâcheux pour une théorie physique. Une manière de résoudre certains de ces problèmes est d'introduire une nouvelle symétrie : la supersymétrie créant ainsi une théorie de supergravité. Il existe plusieurs théories de supergravité, selon le nombre de supersymétries considéré dans le modèle.

Étant donné qu'à ce jour, on a identifié quatre forces fondamentales telles que pour trois d'entres elles (l'électromagnétisme, la force nucléaire forte et la force nucléaire faible), les bosons de jauge associés sont de spin 1 et que pour la gravité, le boson associé devrait être de spin 2, on considère généralement par économie qu'il n'existe pas de particule avec un spin supérieur à 2. De plus si on étudie de plus près des théories de supergravités qui présenteraient une symétrie globale et seraient invariantes de Lorentz alors un théorème no-go dit qu'il est impossible de trouver des couplages satisfaisants entre les spins supérieurs à 2 et les autres. Dans un espace à quatre dimensions, cela signifie que la théorie supersymétrique maximale comporte huit supersymétries (aller d'un spin 2 à -2 requiert huit pas de taille 1/2).

Il est également intéressant de considérer des théories de supergravité à plus de dimensions car elles peuvent être vues comme les limites à basse énergie de théories de cordes qui ont été développées dans des espaces à plus de quatre dimensions. Dans ce cas, il existe également une dimensions limite, en l'occurrence 11, là aussi imposée par ce théorème no-go. En effet, au-delà de 11 dimensions d'espace, le multiplet généré devient trop grand et on obtient automatiquement des particules de spin plus grand que 2. Ainsi l'introduction d'une seule supersymétrie à 11 dimensions, c'est-à-dire contenant 32 supercharges réelles, permet d'avoir un multiplet doté d'un spin 2. Cette théorie

appelée $\mathcal{N} = 1$ à 11 dimensions est l'unique théorie de supergravité maximalement supersymétrique.

De cette unique théorie, on peut ensuite créer de nouvelles théories de supergravité par compactification successive des dimensions sur des cercles pour arriver à la théorie de supergravité $\mathcal{N} = 8$ en dimension 4 après réduction sur le tore T^7. Durant ce processus de compactification, deux cas intéressants émergent. Premièrement, en dimension 10, les 32 supercharges permettent de créer deux gravitini qui peuvent être de chiralité opposée ou identique. La première théorie (aussi appelée $\mathcal{N} = (1, 1)$ ou IIA) est directement issue de la réduction de la théorie à 11 dimensions. La deuxième théorie est appelée IIB (ou $\mathcal{N} = (2, 0)$) et n'est pas reliée à la théore à 11 dimensions. Ces deux théories ne sont pas équivalentes puisqu'elle font intervenir des champs inéquivalents. Cependant, ces deux théories se réduisent à la même théorie de supergravité à 9 dimensions.

Le deuxième cas intéressant est le cas de la dimension 6 qui est l'objet de cette thèse. Cette fois, les 32 supercharges permettent de construire 4 gravitini, donc trois cas de figures s'offrent à nous: les quatre gravitini peuvent avoir la même chiralité (théorie $\mathcal{N} = (4, 0)$), ou bien trois gravitini peuvent avoir la même chiralité et un gravitino a une chiralité opposée (théorie $\mathcal{N} = (3, 1)$) ou enfin deux gravitini ont une chiralité et les deux autres ont la chiralité opposée (théorie $\mathcal{N} = (2, 2)$). La théorie $\mathcal{N} = (2, 2)$ est conventionnelle en ce sens qu'elle dérive de la théorie à 11 dimensions après compactification sur le tore T^5 tandis que les théories $\mathcal{N} = (3, 1)$ et $\mathcal{N} = (4, 0)$ sont plus exotiques. En effet, ces deux dernières théories mettent en jeu des champ de jauge qui sont des tenseurs représentés par des tableaux de Young non standards, qui plus est, satisfont des équations de self-dualité qui sont plus difficiles à traiter.

Cette thèse se décompose en quatre parties. Dans le Chapitre 2, nous présentons les différents champs entrant en jeu dans ces trois théories et nous réécrivons leurs équations libres du mouvement tel qu'une coordonnée spatiale a été séparée des autres. Cette décomposition 5+1 nous permet de trouver un tenseur symétrique d'ordre deux parmi les composantes des champs exotiques de telle manière que les équations du mouvement peuvent être réécrites pour reproduire la théorie de supergravité à cinq dimensions linéarisée.

Dans le Chapitre 3, nous construisons des nouvelles actions reproduisant les différentes équations du mouvement trouvées dans le Chapitre 2 d'une part et se réduisant à la théorie de supergravité linéarisée à cinq dimensions après compactification de la dimension qui a été séparée des cinq autres. Cependant ces caractéristiques viennent au prix de l'invariance de Lorentz manifeste de ces actions.

Dans le Chapitre 4, nous introduisons le formalisme de la Théorie Exceptionnelle des Champs (ExFT) pour réécrire les secteurs bosoniques de ces trois théories dans un formalisme commun. Ce formalisme requiert l'introduction de coordonnées internes qui paramètrent ces champs, cependant la dépendance des champs en ces coordonnées internes est restreinte par l'introduction d'une contrainte sur les dérivées par rapport

à ces coordonnées internes (cette contrainte est appelée contrainte de section). Nous présentons une version modifiée de la contrainte de section dont la résolution reproduit les couplages des trois théories de supergravité à six dimensions.

Dans le Chapitre 5, nous calculons les transformations de supersymétrie des trois théories à six dimensions en imposant le fait que les trois algèbres doivent se fermer dans les difféomorphismes et les transformations de jauge. Ensuite nous réécrivons ces transformations en utilisant la décomposition 5+1 du Chapitre 2, nous permettant de construire des Lagrangiens supersymétries pour les trois théories $\mathcal{N} = (4,0)$, (3,1) et (2,2).

Mots-clés : supergravité, dualité, Théorie Exceptionnelle des Champs, ExFT, supersymétrie, tenseurs exotiques

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à JP

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CHAPTER 1

Introduction

1.1 General overview

In the modern understanding of physics, four fundamental forces have been identified: gravity, electromagnetism, the nuclear strong force and the nuclear weak force. During the twentieth century, and the improvement of particle colliders, a theory unifying the last three forces was developed to explain all the different results found thanks to these colliders. Thus the Standard Model of particle physics was born. To this day, it is the most precise and comprehensive theory of particle physics we know (enthusiasts of physics beyond the Standard Model may even say it is depressingly precise).

At the same time, astrophysics also improved greatly with the first observations of black holes that were predicted to exist in General Relativity. To better understand what happens at the singularity of black holes, we would need a quantum description of gravity. However, the quantisation of General Relativity yields a non-renormalizable theory, that is a theory in which we find infinite values when computing some obervables but we cannot have that in a theory of physics.

Building such a theory of quantum gravity is one of the central questions of theoretical physics and over the recent years has been the subject of intense research. Many approaches have been developped[1]. Among the most promising and best studied candidates is string theory whose fundamental building blocks are one-dimensional objects called strings [4, 5].

A problem emerging in bosonic string theory is the presence of tachyons (elements of the specturm with imaginary mass) which would cause issues with causality. However it was found that in a supersymmetric version of string theory, tachyonic contributions can be consistently eliminated [6].

[1] See for example [1–3] for an overview on these approaches.

Supersymmetry is a new symmetry that was also proposed in particle physics to circumvent the infinities appearing when computing observables. At its core, it creates a link between bosons and fermions. This new symmetry improves some of the annoying divergences but so far, there has been no experimental result of the existence of supersymmetry.

In global supersymmetry, the set of symmetries of a quantum field theory is extended from Poincaré transformations (translations and Lorentz transformations with respective charges P_μ and $M_{[\mu\nu]}$) and internal symmetry transformations to include spinor supercharges Q^i_α. Assuming we are in dimension 4, then α is the spacetime spinor index, so $\alpha = 1, \ldots, 4$, and $i = 1, \ldots, \mathcal{N}$ is an index labeling distinct supercharges. The Poincaré generators and the supercharges together form a new structure called superalgebra. A superalgebra contains two classes of generators, even and odd which we call in physics bosonic (B) and fermionic (F) respectively. The structure relations of these elements include both commutators and anti-commutators in the pattern

$$[B, B] = B \,, \tag{1.1}$$

$$[B, F] = F \,, \tag{1.2}$$

$$\{F, F\} = B \,. \tag{1.3}$$

In other words, since the spinors Q^i_α are spin 1/2, they will transform bosons into fermions and fermions into bosons. The subsequent particles are called superpartners and have the same mass and momentum as the initial particle. The existence of such superpartners is one of the many mysteries the Large Hadron Collider is trying to unveil, to no avail so far.

1.2 Supergravity

Supergravity is a by definition a supersymmetric theory of gravity. Interestingly, supergravity theories emerge from low-energy limits of superstring theories [7].

A field theory of gravity has to include a spin-2 field corresponding to the graviton. Supersymmetry entails that the graviton now has a superpartner: a spin-3/2 gravitino. When there is only one supersymmetry, there is only one gravitino, and the resulting theory is called $\mathcal{N} = 1$ supergravity. Supersymmetry can be extended to include more than one supersymmetry which means that a graviton would have several superpartner gravitini. This will of course generate a bigger multiplet however there is a limit to its size. Indeed, since we only know four fundamental forces of nature, their quantizations would require the introductions of particles of spin at most 2. For that reason and out of simplicity, particles with spin higher than 2 are not believed to exist. Moreover, in flat spacetime and at low-energy, gravity theories exhibiting a global symmetry as well as Lorentz invariance are restricted by a no-go theorem stating that it is not possible to

find a satisfactory coupling of fields with spins exceeding 2 to other spins [8]. In four-dimensional space, this means that there cannot be more than eight supersymmetries (going from a helicity 2 to -2 requires eight 1/2-steps) [7].

Until now, we have only mentioned four-dimensional space, but it is interesting to look at supergravity theories in more dimensions because they can be seen as the low energy limit of string theories which have been specifically developed in more dimensions. There is an upper bound on the dimension, namely 11, imposed by the no-go theorem mentioned above and in this case, the unique maximally supersymmetric theory contains 32 real supercharges. In this dimension, considering only one supersymmetry thus building the $\mathcal{N} = 1$ supergravity theory is enough to generate a multiplet that already features a spin 2. Moreover, this multiplet is unique and notably combines a graviton, gravitino and a three-form in eleven dimensions [9]. From this maximally supersymmetric supergravity theory, we can create new supergravity theories in lower dimensions by successive compactifications of dimensions on circles such that the eleven-dimensional theory reduces to the maximally supersymmetric four-dimensional $\mathcal{N} = 8$ supergravity when reduced on the torus T^7 [10].

During this process of compactification, two interesting cases occur. First, for $D = 10$, the 32 supercharges allow us to build two gravitini (16 components each) which can have opposite or identical chirality. The former yields a theory called type IIA supergravity (also called $\mathcal{N} = (1, 1)$) which is indeed the dimensional reduction of the eleven-dimensional theory mentioned above. The latter in fact is a new theory called type IIB supergravity (also called $\mathcal{N} = (2, 0)$) and is not related to the eleven-dimensional one [11–13]. These theories are not equivalent: they involve different, inequivalent fields but they both reduce to the same nine-dimensional theory.

The second interesting case happens for $D = 6$. This time, the 32 supercharges allow us to construct four gravitini, where we can now distinguish three cases: two left-handed and two right-handed gravitini (which is noted $\mathcal{N} = (2, 2)$), three gravitini with one chirality and one with the opposite ($\mathcal{N} = (3, 1)$) and finally all gravitini with the same chirality ($\mathcal{N} = (4, 0)$) [14]. We will see later in further detail that the last two theories must be exotic since they feature mixed symmetry tensors, therefore they play a distinguished role among maximally supersymmetric theories [15–19].

As for the IIA and IIB theories, the $\mathcal{N} = (2, 2)$ theory is the dimensional reduction of the eleven-dimensional one while the others are not [20], as summarised more schematically in Figure 1.1. The chiral $\mathcal{N} = (3, 1)$ and $\mathcal{N} = (4, 0)$ theories are only known on the level of free field theories but looking at Figure 1.1, studying them from a five-dimensional point of view is a possible angle of attack. After compactification they all reduce to the same five-dimensional maximal theory: linearized five-dimensional maximal supergravity. The interacting theories were conjectured by Hull to exist and to describe strong coupling limits of $\mathcal{N} = 8$ theories in five dimensions [14].

Another ingredient that will help us in the analysis of the three six-dimensional the-

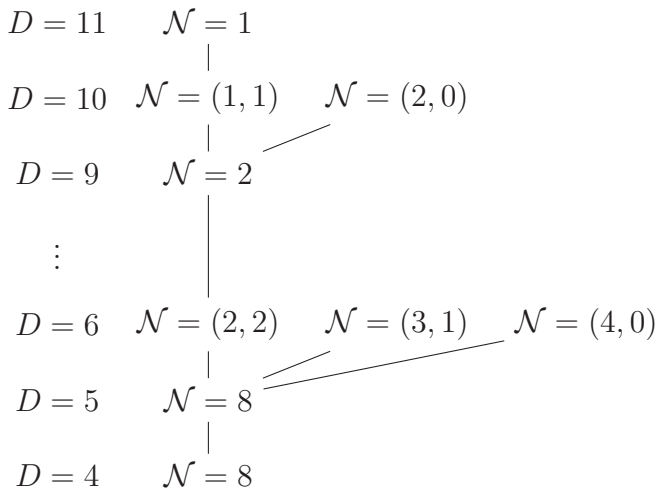


Figure 1.1: Reduction of the maximal $D = 11$ supergravity theory for 11 to 4 dimensions and the interesting cases occurring for $D = 10$ and $D = 6$.

ories is the structure of their global symmetries. Indeed surprisingly, all D-dimensional interacting theories of the reduction of the eleven-dimensional $\mathcal{N} = 1$ theory ("main line" in Fig. 1.1) feature a global symmetry under the exceptional group $E_{11-D(11-D)}$. The presence of these groups can be understood by the fact that they are the smallest groups that contain

- $\mathrm{GL}(11 - D)$ which stems from the eleven-dimensional diffeomorphism algebra,
- the R-symmetry group which is the maximal compact subgroup of the global symmetry group.

In the case of six dimensions, i.e. for the $\mathcal{N} = (2, 2)$ theory, the R-symmetry group is $\mathrm{Usp}(4) \times \mathrm{Usp}(4)$ and the smallest group containing it and $\mathrm{GL}(5)$ is $E_{5(5)}$ which is isomorphic to $\mathrm{SO}(5, 5)$. The other two six-dimensional theories have R-symmetry groups $\mathrm{Usp}(6) \times \mathrm{Usp}(2)$ in the case of $\mathcal{N} = (3, 1)$ and $\mathrm{Usp}(8)$ in the case of $\mathcal{N} = (4, 0)$. The smallest groups containing both $\mathrm{GL}(5)$ and these R-symmetry groups are respectively $\mathrm{F}_{4(4)}$ and $\mathrm{E}_{6(6)}$. These two groups are conjectured to be the global symmetry groups of the interacting $\mathcal{N} = (3, 1)$ and $\mathcal{N} = (4, 0)$ theories respectively [14].

1.3 Review of 6D models

The topic of this thesis is the three six-dimensional $\mathcal{N} = (2, 2)$, $\mathcal{N} = (3, 1)$ and $\mathcal{N} = (4, 0)$ supergravity theories and more specifically, they will be studied in Minkowski space with a "mostly plus" flat metric

$$\eta_{\hat{\mu}\hat{\nu}} = \mathrm{diag}\{-1, 1, 1, 1, 1, 1\} \ , \quad \hat{\mu}, \hat{\nu} = 0, \ldots 5 \ . \tag{1.4}$$

Helicity	G	Field	Multi.	Chir.	Characteristics
0	$(\mathbf{1},\mathbf{1};\mathbf{5},\mathbf{5})$	$\phi^{a\dot{a}}$	25		
1/2	$(\mathbf{1},\mathbf{2};\mathbf{5},\mathbf{4})$	$\chi^{a\dot{\alpha}}$	20	$-$	
1/2	$(\mathbf{2},\mathbf{1};\mathbf{4},\mathbf{5})$	$\chi^{\dot{a}\alpha}$	20	$+$	
1	$(\mathbf{2},\mathbf{2};\mathbf{4},\mathbf{4})$	$A_{\hat{\mu}}^{\alpha\dot{\alpha}}$	16		
1	$(\mathbf{3},\mathbf{1};\mathbf{1},\mathbf{5})$	$B_{\hat{\mu}\hat{\nu}}^{\dot{a}}$	5		antisymmetric in $\hat{\mu}\hat{\nu}$
1	$(\mathbf{1},\mathbf{3};\mathbf{5},\mathbf{1})$	$B_{\hat{\mu}\hat{\nu}}^{a}$	5		antisymmetric in $\hat{\mu}\hat{\nu}$
3/2	$(\mathbf{3},\mathbf{2};\mathbf{1},\mathbf{4})$	$\psi_{\hat{\mu}}^{\dot{\alpha}}$	4	$+$	
3/2	$(\mathbf{2},\mathbf{3};\mathbf{4},\mathbf{1})$	$\psi_{\hat{\mu}}^{\alpha}$	4	$-$	
2	$(\mathbf{3},\mathbf{3};\mathbf{1},\mathbf{1})$	$h_{\hat{\mu}\hat{\nu}}$	1		symmetric in $\hat{\mu}\hat{\nu}$

Table 1.1: Field content of the $\mathcal{N}=(2,2)$ theory. The indices $a,\dot{a}=1,\ldots,5$ refer to the vector representation of $\mathrm{SO}(5)\times\mathrm{SO}(5)$ which is isomorphic to the R-symmetry group $\mathrm{Usp}(4)\times\mathrm{Usp}(4)$ and $\alpha,\dot{\alpha}=1,\ldots,4$ is the spinor representation of $\mathrm{SO}(5)\times\mathrm{SO}(5)$. The spinor indices are omitted to avoid cumbersome notation.

In this section we start by discussing their field content. The different multiplets have been constructed in [21].

The $\mathcal{N}=(2,2)$ theory is perhaps the most well known theory of the three theories since the full non-linear theory has been constructed in [22]. The fields in the $\mathcal{N}=(2,2)$ multiplet transform under the little group (which is the product of the light-like helicity group and the R-symmetry group)

$$G=\mathrm{SU}(2)\times\mathrm{SU}(2)\times\mathrm{Usp}(4)\times\mathrm{Usp}(4)\,. \tag{1.5}$$

The multiplet features 25 scalar fields, 40 Dirac fields, 16 vector fields, 10 two-forms, 8 gravitini and one graviton. The breakdown is summed up in Table 1.3. Interestingly, in six dimensions, the two-forms are chiral, that is to say, their field strength is self-dual: instead of satisfying a Maxwell-type equation of motion, self-dual two-forms satisfy a first order self-duality equation. Historically, building a Lagrangian that reproduces this type of equation was tricky because implementing both the self-duality condition and Lorentz invariance leads to the propagation of the Lagrange multiplier parametrizing the self-duality condition and problems when quantising such fields [23].

The $\mathcal{N}=(3,1)$ multiplet transforms under the group

$$G=\mathrm{SU}(2)\times\mathrm{SU}(2)\times\mathrm{Usp}(6)\times\mathrm{Usp}(2)\,. \tag{1.6}$$

The multiplet features 28 scalars, 42 Dirac fields, 14 vector fields, 12 self-dual two-forms, 6 gravitini, 2 exotic gravitini and one three-index tensor field called dual graviton. The breakdown of this multiplet is given in Table 1.2. The $\mathcal{N}=(3,1)$ theory is called exotic because it features unusual fields: unlike a regular gravitino which is a fermionic one-form, we have two exotic gravitini represented by fermionic two-forms. The most striking

Helicity	G	Field	Mult.	Chir.	Characteristics
0	$(\mathbf{1},\mathbf{1};\mathbf{14}',\mathbf{2})$	$\phi^{abc\alpha}$	28		antisymmetric, traceless in abc
1/2	$(\mathbf{1},\mathbf{2};\mathbf{14}',\mathbf{1})$	χ^{abc}	14	$-$	antisymmetric, traceless in abc
1/2	$(\mathbf{2},\mathbf{1};\mathbf{14},\mathbf{2})$	$\chi^{ab\alpha}$	28	$+$	antisymmetric, traceless in ab
1	$(\mathbf{2},\mathbf{2};\mathbf{14},\mathbf{1})$	$A^{ab}_{\hat{\mu}}$	14		antisymmetric, traceless in ab
1	$(\mathbf{3},\mathbf{1};\mathbf{6},\mathbf{2})$	$B^{a\alpha}_{\hat{\mu}\hat{\nu}}$	12		antisymmetric in $\hat{\mu}\hat{\nu}$
3/2	$(3,\mathbf{2};\mathbf{6},\mathbf{1})$	$\psi^{a}_{\hat{\mu}}$	6	$+$	
3/2	$(\mathbf{4},\mathbf{1};\mathbf{1},\mathbf{2})$	$\psi^{\alpha}_{\hat{\mu}\hat{\nu}}$	2	$+$	antisymmetric in $\hat{\mu}\hat{\nu}$
2	$(\mathbf{4},\mathbf{2};\mathbf{1},\mathbf{1})$	$C_{\hat{\mu}\hat{\nu},\hat{\rho}}$	1		$(2,1)$ Young symmetry

Table 1.2: Field content of the $\mathcal{N}=(3,1)$ theory. The indices $a=1,\ldots,6$ and $\alpha=1,\ldots,2$ represent the fundamental representation of Usp(6) and Usp(2) respectively.

difference with the $\mathcal{N}=(2,2)$ theory is the absence of a regular graviton (represented by a two-index symmetric tensor) which is replaced by a Curtright field represented by a three-index tensor $C_{\hat{\mu}\hat{\nu},\hat{\rho}}$ exhibiting a mixed spacetime symmetry, namely a (2,1) Young symmetry (that is to say $C_{\hat{\mu}\hat{\nu},\hat{\rho}}=-C_{\hat{\nu}\hat{\mu},\hat{\rho}}$ and $C_{[\hat{\mu}\hat{\nu},\hat{\rho}]}=0$). Moreover and similarly to the two-form, in six dimensions, the Curtright field is chiral so the second order Curtright-type equation of motion [24] can be replaced by a self-duality field equation for which no second order Lagrangian has been exhibited yet but recently, a fourth order action was built using the prepotential formalism [25].

The $\mathcal{N}=(4,0)$ multiplet transforms under the group

$$G=\mathrm{SU}(2)\times\mathrm{SU}(2)\times\mathrm{Usp}(8)\ . \tag{1.7}$$

The $\mathcal{N}=(4,0)$ theory is probably the most exotic of the six-dimensional supergravity theories as it features 42 scalars, 48 Dirac fields, 27 self-dual two-forms, 8 exotic gravitini and one field mixed symmetry field. The breakdown of the multiplet is given in Table 1.3. The mixed symmetry field is represented by a four-index tensor field $T_{\hat{\mu}\hat{\nu},\hat{\rho}\hat{\sigma}}$ with a (2,2) Young symmetry that is it exhibits the same symmetries a the Riemann tensor: $T_{\hat{\mu}\hat{\nu},\hat{\rho}\hat{\sigma}}=-T_{\hat{\nu}\hat{\mu},\hat{\rho}\hat{\sigma}}=-T_{\hat{\mu}\hat{\nu},\hat{\sigma}\hat{\rho}}$ and $T_{[\hat{\mu}\hat{\nu},\hat{\rho}]\hat{\sigma}}$. Similarly to the two-forms, in six dimensions, this field is chiral as well, so its second order equation of motion can be replaced by a self-duality field equation [26] for which no second order Lagrangian has been exhibited yet. Again, a fourth order action was found using the prepotential formalism [27].

1.4 Outline of this thesis

This thesis is organized in four parts. In Chapter 2, we look at the different fields involved in all three theories and rewrite their free field equations in a way such that one

Helicity	G	Field	Mult.	Chir.	Characteristics
0	$(\mathbf{1},\mathbf{1};\mathbf{42})$	ϕ^{ABCD}	42		antisymmetric, traceless in $ABCD$
1/2	$(\mathbf{2},\mathbf{1};\mathbf{48})$	χ^{ABC}	48	+	antisymmetric, traceless in ABC
1	$(\mathbf{3},\mathbf{1};\mathbf{27})$	$B^{AB}_{\hat{\mu}\hat{\nu}}$	27		antisymmetric, traceless AB
3/2	$(\mathbf{4},\mathbf{1};\mathbf{8})$	$\psi^{A}_{\hat{\mu}\hat{\nu}}$	8	+	antisymmetric in $\hat{\mu}\hat{\nu}$
2	$(\mathbf{5},\mathbf{1};\mathbf{1})$	$T_{\hat{\mu}\hat{\nu},\hat{\rho}\hat{\sigma}}$	1		$(2,2)$ Young symmetry

Table 1.3: Field content of the $\mathcal{N}=(4,0)$ theory. The index $A=1,\ldots,8$ represents the fundamental representation of USp(8).

spatial coordinate has been singled out. This decomposition 5+1 allows us to find a two-index symmetric tensor in the components of the exotic fields such that their equations of motion can be split and rewritten to take the form of five-dimensional supergravity equations.

In Chapter 3, we build new actions allowing us to reproduce the different free equations of motion found in Chapter 2. Using the rewriting of the equations of motion, we show that these actions not only completely reproduce the six-dimensional equations of motion but also all reduce to linearized five-dimensional supergravity upon dimensional reduction. However these very interesting properties come at the cost of manifest Lorentz invariance of the actions.

In Chapter 4, we introduce the formalism of Exceptional Field Theory (ExFT) to find a common framework in which we want to rewrite the bosonic sectors of the three six-dimensional supergravity theories. Exceptional field theory requires the introduction of internal coordinates parametrizing the different fields however the dependence of the fields on these internal coordinates is restricted by introducing a constraint on the derivatives with respect to these internal coordinates (called a section constraint). We present a modified version of the standard section constraint allowing us to reproduce the couplings of all three six-dimensional supergravity theories.

In Chapter 5, we compute the six-dimensional supersymmetry transformations of each theory by requiring the closure of the different algebras into diffeomorphisms and gauge transformations. Then we reduced these transformations using the decomposition 5+1 of Chapter 2 allowing us to build a supersymmetric action for all three theories.

This thesis led to the publication of two articles [28, 29].

CHAPTER 2

Decomposition 5+1

In this chapter, we discuss for every field in the spectrum of the three models, the reorganization of the six-dimensional fields which allows their identification after reduction to five dimensions. However, we keep the full dependence of all fields on six spacetime coordinates. More precisely, we break six-dimensional Poincaré invariance down to $5+1$ and perform a standard Kaluza-Klein decomposition on the six-dimensional fields without dropping the dependence on the sixth coordinate. We then rearrange the equations of motion such that they take the form of the five-dimensional (free) supergravity equations however sourced by derivatives of matter fields along the sixth direction. The resulting reformulation of the six-dimensional models casts their dynamics into a common framework ultimately allowing us to construct uniform actions for the three models (presented in the next chapter).

For the purpose of this thesis, we choose the $5+1$ coordinate split

$$\{x^{\hat{\mu}}\} \longrightarrow \left\{x^{\mu}, x^5 \equiv y\right\} \ , \quad \begin{cases} \hat{\mu} = 0, \ldots, 5 \ , \\ \mu = 0, \ldots, 4 \ , \end{cases} \tag{2.1}$$

by singling out one of the spatial coordinates. Of course, an analogous construction can be performed with a split along the time-like coordinate which may be of interest for example in a Hamiltonian context.

2.1 Bosons

2.1.1 Scalar field

The simplest bosonic field involved in these models is the scalar field satisfying the usual free field equation

$$\partial^{\hat{\mu}} \partial_{\hat{\mu}} \phi = 0 \ . \tag{2.2}$$

The 5+1 decomposition of such a field is straightforward since it has only one component, one has

$$\partial^\mu \partial_\mu \phi + \partial_y^2 \phi = 0 \, . \tag{2.3}$$

Since we ultimately want to compactify that singled out dimension, we consider that y represents a circle whose radius R we can choose arbitrarily small. As such, it is instructive to consider the Fourier series of ϕ with respect to y

$$\phi(x^\mu, y) = \sum_{n \in \mathbb{Z}} \phi_n(x^\mu) \, e^{2in\pi \frac{y}{R}} \, , \tag{2.4}$$

where ϕ_n is the n-th coefficient of the Fourier series and has no y dependancy. Plugging that expression in (2.3) yields

$$\forall n \in \mathbb{Z} \, , \quad \partial^\mu \partial_\mu \phi_n = \left(\frac{2n\pi}{R} \right)^2 \phi_n \, , \tag{2.5}$$

and we observe that more generally, the Kaluza-Klein decomposition of a D-dimensional massless field generates an infinite tower of (D-1)-dimensional massive fields, all satisfying a Klein-Gordon equation. The mass of each field can read off from the right-hand side of equation (2.5). Upon compactification (that is $R \to 0$), all masses diverge except the one of the zero-mode ($n = 0$).

2.1.2 Vector field

The vector field is the first instance where one can witness the Kaluza-Klein mechanism fully at play. A free vector field $A_{\hat{\mu}}$ whose field strength is defined as $F_{\hat{\mu}\hat{\nu}} = 2\,\partial_{[\hat{\mu}} A_{\hat{\nu}]}$, satisfies the Maxwell field equations

$$\partial^{\hat{\mu}} F_{\hat{\mu}\hat{\nu}} = 0 \, , \tag{2.6}$$

which is invariant under the gauge symmetry

$$\delta A_{\hat{\mu}} = \partial_{\hat{\mu}} \Lambda \, . \tag{2.7}$$

The six-dimensional degrees of freedom will be split the following way

$$\{A_{\hat{\mu}}\} = \{A_\mu, \, A_5 \equiv \phi\} \, , \tag{2.8}$$

and their dynamics will now follow modified Maxwell and Klein-Gordon equations:

$$\partial^\mu F_{\mu\nu} - \partial_y (\partial_\nu \phi - \partial_y A_\nu) = 0 \, , \tag{2.9}$$

$$\partial^\mu (\partial_\mu \phi - \partial_y A_\mu) = 0 \, . \tag{2.10}$$

We note that if we reduce these expressions to five dimensions i.e. we assume that A_μ and ϕ do not depend on y (that is $\partial_y \to 0$), both the vector field and the scalar field

satisfy respectively a five-dimensional Maxwell equation and a five-dimensional massless Klein-Gordon equation. More interestingly, equation (2.10) suggests the definition of the covariant derivative of ϕ,

$$D_\mu \phi = \partial_\mu \phi - \partial_y A_\mu \ , \tag{2.11}$$

and if we consider the Fourier expansion of the fields with respect to y like previously, we have

$$\forall n \in \mathbb{Z}, \quad D_\mu \phi_n = \partial_\mu \phi_n - \frac{2in\pi}{R} A_{\mu,n}, \tag{2.12}$$

which reproduces the coupling of the electromagnetic field in the definition of the covariant derivative in gauge theory. The gauge symmetry (2.7) also splits into

$$\delta A_\mu = \partial_\mu \Lambda \ , \tag{2.13}$$
$$\delta \phi = \partial_y \Lambda \ . \tag{2.14}$$

2.1.3 Chiral 2-form

The rewriting of the tensor field sector is less canonical since the 6-dimensional dynamics of a tensor field $B_{\hat{\mu}\hat{\nu}}$ is captured by a self-duality equation instead of a Maxwell-like field equation (but the former implies the latter)

$$H_{\hat{\mu}\hat{\nu}\hat{\rho}} = \frac{1}{6} \varepsilon_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}\hat{\tau}\hat{\lambda}} H^{\hat{\sigma}\hat{\tau}\hat{\lambda}} \ , \tag{2.15}$$

where $H_{\hat{\mu}\hat{\nu}\hat{\rho}} = 3\,\partial_{[\hat{\mu}} B_{\hat{\nu}\hat{\rho}]}$ and this equation is invariant under the six-dimensional gauge symmetry

$$\delta B_{\hat{\mu}\hat{\nu}} = 2\,\partial_{[\hat{\mu}} \Lambda_{\hat{\nu}]} \ . \tag{2.16}$$

After splitting the tensor field into

$$\{B_{\hat{\mu}\hat{\nu}}\} = \{B_{\mu\nu},\, B_{\mu 5} \equiv A_\mu\} \ , \tag{2.17}$$

the self-duality equation becomes a duality equation between a vector field and a 2-form

$$F_{\mu\nu} + \partial_y B_{\mu\nu} + \frac{1}{6} \varepsilon_{\mu\nu\rho\sigma\tau} H^{\rho\sigma\tau} = 0 \ , \tag{2.18}$$

where we used the convention $\varepsilon_{\mu\nu\rho\sigma\tau 5} = \varepsilon_{\mu\nu\rho\sigma\tau}$ linking the 5-dimensional and 6-dimensional Levi-Civita symbols. The six-dimensional gauge parameter $\Lambda_{\hat{\mu}}$ also splits into $\{\Lambda_{\hat{\mu}}\} \to \{\Lambda_\mu,\, \Lambda \equiv \Lambda_5\}$ and (2.16) decomposes into

$$\delta B_{\mu\nu} = 2\,\partial_{[\mu} \Lambda_{\nu]} \ , \tag{2.19}$$
$$\delta A_\mu = \partial_\mu \Lambda - \partial_y \Lambda_\mu \ . \tag{2.20}$$

2.1.4 Graviton

Now comes the heart of our subject: the spin-2 sector. The $\mathcal{N} = (2, 2)$ theory features a graviton represented by a symmetric tensor $h_{\hat{\mu}\hat{\nu}}$, whose linearized curvature is

$$R_{\hat{\mu}\hat{\nu},\hat{\rho}\hat{\sigma}} = -\partial_{\hat{\rho}}\partial_{[\hat{\mu}}h_{\hat{\nu}]\hat{\sigma}} + \partial_{\hat{\sigma}}\partial_{[\hat{\mu}}h_{\hat{\nu}]\hat{\rho}} \,. \tag{2.21}$$

The free dynamics of this field is described by the Einstein equation

$$R_{\hat{\mu}\hat{\sigma}} = R_{\hat{\mu}\hat{\nu},\hat{\rho}\hat{\sigma}}\eta^{\hat{\nu}\hat{\rho}} = 0 \,, \tag{2.22}$$

and this equation is invariant by the gauge symmetry representing the invariance by diffeomorphism

$$\delta h_{\hat{\mu}\hat{\nu}} = 2\,\partial_{(\hat{\mu}}\xi_{\hat{\nu})} \,. \tag{2.23}$$

The 6-dimensional graviton splits into three 5-dimensional objects: a graviton, a vector field and a scalar field which we parametrise as

$$\{h_{\hat{\mu}\hat{\nu}}\} = \left\{h_{\mu\nu} - \frac{1}{3}\,\eta_{\mu\nu}\phi,\, h_{\mu 5} \equiv A_\mu,\, h_{55} \equiv \phi\right\} \,. \tag{2.24}$$

This allows us to rewrite a more compact version of the equations of motion

$$R_{\mu\nu} + \partial_y\left(-\partial_{(\mu}A_{\nu)} + \frac{1}{2}\,\partial_y h_{\mu\nu} - \frac{1}{6}\,\eta_{\mu\nu}\partial_y\phi\right) - \frac{1}{6}\,\eta_{\mu\nu}\partial^\rho\partial_\rho\phi = 0 \,, \tag{2.25}$$

$$\partial^\nu F_{\mu\nu} - \partial_y\left(\partial_\mu h_\nu{}^\nu - \partial^\nu h_{\mu\nu} - \frac{4}{3}\,\partial_\mu\phi\right) = 0 \,, \tag{2.26}$$

$$\partial^\nu\partial_\nu\phi - \partial_y\left(2\,\partial^\nu A_\nu - \partial_y h_\nu{}^\nu + \frac{5}{3}\,\partial_y\phi\right) = 0 \,. \tag{2.27}$$

Although here, we only consider the linearized theory, the ϕ term in the parametrization of the five-dimensional graviton is coherent with the reduction of a non linearized theory of gravity. Starting from the six-dimensional Einstein-Hilbert action, one has to rescale the metric to make a five-dimensional Einstein-Hilbert term appear in the action. In our case, that rescaling would take the form $g_{\mu\nu} \to \Phi^{-1/3}g_{\mu\nu}$ where $g_{\mu\nu}$ is the five-dimensional metric and Φ the scalar field coming from the Kaluza-Klein reduction [30]. The linearization of this rescaling about the Minkowski metric (one writes $g_{\mu\nu} = \eta_{\mu\nu}+h_{\mu\nu}$ and $\Phi = 1 + \phi$) reproduces our choice of parametrization.

From (2.25), we can write the full linearized Einstein equation

$$R_{\mu\nu} - \frac{1}{2}\,\eta_{\mu\nu}R = \partial_y\left(\partial_{(\mu}A_{\nu)} - \frac{1}{2}\,\partial_y h_{\mu\nu} - \eta_{\mu\nu}\left(\partial^\rho A_\rho - \frac{1}{2}\,\partial_y h_\rho{}^\rho + \frac{2}{3}\,\partial_y\phi\right)\right) . \tag{2.28}$$

Again, we can define two covariant derivatives,

$$D_\mu h_{\nu\rho} = \partial_\mu h_{\nu\rho} - \frac{2}{3}\,\partial_y A_\mu\eta_{\nu\rho} \,, \tag{2.29}$$

$$D_\mu\phi = \partial_\mu\phi - 2\,\partial_y A_\mu \,, \tag{2.30}$$

which allow us to write the linearized Einstein equations in an even more compact form

$$G_{\mu\nu} = -\frac{1}{2}\,\partial_y\partial_y h_{\mu\nu} + \frac{1}{2}\,\partial_y\partial_y h_\rho{}^\rho - \frac{2}{3}\,\partial_y\partial_y\phi\ , \tag{2.31}$$

where $G_{\mu\nu}$ is the linearized Einstein tensor

$$G_{\mu\nu} = -\partial^\rho D_{(\mu}h_{\nu)\rho} + \frac{1}{2}\,\partial_\rho D^\rho h_{\mu\nu} + \frac{1}{2}\,\partial_{(\mu}D_{\nu)}h_\rho{}^\rho + \frac{1}{2}\,\eta_{\mu\nu}\,\partial^\rho D^\sigma h_{\rho\sigma} - \frac{1}{2}\,\eta_{\mu\nu}\,\partial_\rho D^\rho h_\sigma{}^\sigma\ . \tag{2.32}$$

The gauge parameter also splits into $\{\xi_{\hat\mu}\} \to \{\xi_\mu, \lambda \equiv \xi_5\}$ and the gauge transformations descending from the six-dimensional one (2.23) are

$$\delta h_{\mu\nu} = 2\,\partial_{(\mu}\xi_{\nu)} + \frac{2}{3}\,\eta_{\mu\nu}\,\partial_y\lambda\ , \tag{2.33}$$

$$\delta A_\mu = \partial_\mu\lambda + \partial_y\xi_\mu\ , \tag{2.34}$$

$$\delta\phi = 2\,\partial_y\lambda\ , \tag{2.35}$$

2.1.5 (2,1) field

As explained previously the $\mathcal{N} = (3,1)$ theory does not contain a standard graviton but rather a dual graviton which is a mixed-symmetry tensor field $C_{\hat\mu\hat\nu,\hat\rho}$. It has a $(2,1)$ Young symmetry, meaning that

$$\begin{array}{|c|c|}\hline \ \ & \ \\ \hline \ \\ \cline{1-1}\end{array}\ :\ C_{\hat\mu\hat\nu,\hat\rho} = -C_{\hat\nu\hat\mu,\hat\rho}, \quad C_{[\hat\mu\hat\nu,\hat\rho]} = 0\ . \tag{2.36}$$

The Young tableau in (2.36) is a pictorial way of representing the symmetries of $C_{\hat\mu\hat\nu,\hat\rho}$. We refer to Appendix A.2 for more information on Young tableaux.

The equations of motion for the free (2,1) field were first exhibited by Curtright in [24] but as for the 2-form, the equations of motion will be instead replaced by a self-duality relation for the curvature of that field. Namely, we have

$$S_{\hat\mu\hat\nu\hat\rho,\hat\sigma\hat\tau} = \frac{1}{6}\,\varepsilon_{\hat\mu\hat\nu\hat\rho\hat\eta\hat\kappa\hat\lambda}\,S^{\hat\eta\hat\kappa\hat\lambda}{}_{\hat\sigma\hat\tau}\ , \tag{2.37}$$

where

$$S_{\hat\mu\hat\nu\hat\rho,\hat\sigma\hat\tau} = 3\,\partial_{\hat\sigma}\partial_{[\hat\mu}C_{\hat\nu\hat\rho],\hat\tau} - 3\,\partial_{\hat\tau}\partial_{[\hat\mu}C_{\hat\nu\hat\rho],\hat\sigma}\ , \tag{2.38}$$

and (2.37) is invariant under the gauge symmetry

$$\delta C_{\hat\mu\hat\nu,\hat\rho} = 2\,\partial_{[\hat\mu}\alpha_{\hat\nu]\hat\rho} + \partial_{\hat\rho}\beta_{\hat\mu\hat\nu} - \partial_{[\hat\rho}\beta_{\hat\mu\hat\nu]}\ , \tag{2.39}$$

with paramaters $\alpha_{\hat\mu\hat\nu} = \alpha_{(\hat\mu\hat\nu)}$ and $\beta_{\hat\mu\hat\nu} = \beta_{[\hat\mu\hat\nu]}$. The formalism of the Young tableaux can help us figure out schematically what fields this (2,1) tensor decomposes into. Since we want to single out the sixth coordinate (called y and has index 5), we look at all possibilities to put one or several 5s in the tableau.

$$\begin{array}{|c|c|}\hline \ \ & \ \\ \hline \ \\ \cline{1-1}\end{array}\ \to\ \begin{array}{|c|c|}\hline \ \ & \ \\ \hline \ \\ \cline{1-1}\end{array}\ ,\ \begin{array}{|c|c|}\hline \ \ & \ \\ \hline 5\\ \cline{1-1}\end{array}\ ,\ \begin{array}{|c|c|}\hline \ \ & 5\\ \hline \ \\ \cline{1-1}\end{array}\ ,\ \begin{array}{|c|c|}\hline \ \ & 5\\ \hline 5\\ \cline{1-1}\end{array}\ , \tag{2.40}$$

It is not possible to have two 5s in the same column since indices in a column are antisymmetric. The picture (2.40) tells us that a six-dimensional (2,1) tensor decomposes into respectively a five-dimensional (2,1) tensor, a symmetric tensor, a 2-form and a vector. More precisely, we choose to parametrize the six-dimensional (2,1) tensor as

$$\{C_{\hat{\mu}\hat{\nu},\hat{\rho}}\} = \Big\{C_{\mu\nu,\rho} - 2\,A_{[\mu}\eta_{\nu]\rho}\,;\; C_{\mu 5,\nu} = h_{\mu\nu} + B_{\mu\nu}\,;\; C_{\mu 5,5} = 2\,A_\mu\Big\}\ , \tag{2.41}$$

with symmetric $h_{\mu\nu} = h_{\nu\mu}$, antisymmetric $B_{\mu\nu} = -B_{\nu\mu}$, and a (2,1) tensor $C_{\mu\nu,\rho}$. The self-duality equations (2.38) split into two equations

$$\begin{aligned}\partial_\rho\Big(F_{\mu\nu} + \frac{1}{6}\,\varepsilon_{\mu\nu\lambda\sigma\tau}\,H^{\lambda\sigma\tau}\Big) =& \partial_y\partial_{[\mu}h_{\nu]\rho} + \frac{1}{4}\,\varepsilon_{\mu\nu\kappa\lambda\tau}\,\partial_y\partial^\kappa C^{\lambda\tau}{}_\rho + \frac{1}{2}\,\partial_y\partial_y C_{\mu\nu,\rho}\\ &- \partial_y\partial_y A_{[\mu}\eta_{\nu]\rho} - \frac{1}{4}\,\varepsilon_{\rho\mu\nu\sigma\tau}\,\partial_y F^{\sigma\tau}\\ &+ \partial_y\partial_{[\mu}B_{\nu]\rho} - \partial_y\partial_\rho B_{\mu\nu}\ ,\end{aligned} \tag{2.42}$$

and

$$\begin{aligned}R_{\mu\nu,\rho\sigma} =& \frac{1}{2}\,\partial_\rho\Big(H_{\mu\nu\sigma} - \frac{1}{2}\,\varepsilon_{\mu\nu\sigma\kappa\lambda}\,F^{\kappa\lambda}\Big) - \frac{1}{2}\,\partial_\sigma\Big(H_{\mu\nu\rho} - \frac{1}{2}\,\varepsilon_{\mu\nu\rho\kappa\lambda}\,F^{\kappa\lambda}\Big)\\ &+ \frac{1}{2}\,\varepsilon_{\mu\nu\kappa\lambda\tau}\,\partial^\kappa\partial_{[\rho}C^{\lambda\tau}{}_{\sigma]} + \frac{1}{2}\,\partial_y\partial_\rho C_{\mu\nu,\sigma} - \frac{1}{2}\,\partial_y\partial_\sigma C_{\mu\nu,\rho}\\ &- \partial_y\partial_\rho A_{[\mu}\eta_{\nu]\sigma} + \partial_y\partial_\sigma A_{[\mu}\eta_{\nu]\rho}\ ,\end{aligned} \tag{2.43}$$

with

$$F_{\mu\nu} = 2\,\partial_{[\mu}A_{\nu]}\ , \tag{2.44}$$

$$H_{\mu\nu\rho} = 3\,\partial_{[\mu}B_{\nu\rho]}\ , \tag{2.45}$$

$$R_{\mu\nu,\rho\sigma} = -\partial_\rho\partial_{[\mu}h_{\nu]\sigma} + \partial_\sigma\partial_{[\mu}h_{\nu]\rho}\ . \tag{2.46}$$

Once again, we can compute the Einstein equation,

$$G_{\mu\nu} = -\frac{1}{2}\,\partial_y\partial^\rho C_{\rho(\mu,\nu)} - \frac{1}{2}\,\partial_y\partial_{(\mu}C_{\nu)\rho}{}^\rho + \frac{1}{2}\,\eta_{\mu\nu}\,\partial_y\partial^\rho C_{\rho\sigma}{}^\sigma\ , \tag{2.47}$$

where $G_{\mu\nu}$ is the linearized Einstein tensor

$$G_{\mu\nu} = -\partial^\rho D_{(\mu}h_{\nu)\rho} + \frac{1}{2}\,\partial_\rho D^\rho h_{\mu\nu} + \frac{1}{2}\,\partial_{(\mu}D_{\nu)}h_\rho{}^\rho + \frac{1}{2}\,\eta_{\mu\nu}\,\partial^\rho D^\sigma h_{\rho\sigma} - \frac{1}{2}\,\eta_{\mu\nu}\,\partial_\rho D^\rho h_\sigma{}^\sigma\ . \tag{2.48}$$

defined as in (2.32), but this time with the covariant derivative

$$D_\mu h_{\nu\rho} \equiv \partial_\mu h_{\nu\rho} - \partial_y A_\mu\,\eta_{\nu\rho}\ . \tag{2.49}$$

We observe that upon dimensional reduction ($\partial_y \to 0$), the field $h_{\mu\nu}$ satisfies the linearized Einstein equations but if we keep the sixth coordinate, the source terms differ

from those of the regular graviton, confirming that both the $\mathcal{N} = (2,2)$ and $\mathcal{N} = (3,1)$ are fundamentally inequivalent.

The field equation (2.43) takes the form of a curl in $[\rho\sigma]$ and using the Poincaré lemma (see appendix A.3), it can be locally integrated into the first order equation[1]

$$\begin{aligned}\partial_\rho u_{\mu\nu} =& \partial_{[\mu} h_{\nu]\rho} + \frac{1}{4}\,\varepsilon_{\mu\nu\kappa\lambda\tau}\,\partial^\kappa C^{\lambda\tau}{}_\rho + \frac{1}{2}\left(H_{\mu\nu\rho} - \frac{1}{2}\,\varepsilon_{\mu\nu\rho\kappa\lambda}\,F^{\kappa\lambda}\right)\\ &+ \frac{1}{2}\,\partial_y C_{\mu\nu,\rho} - \partial_y A_{[\mu}\eta_{\nu]\rho}\,,\end{aligned} \tag{2.50}$$

with an antisymmetric tensor $u_{\mu\nu} = -u_{\nu\mu}$. This allows us to rewrite (2.42) into

$$\partial_\rho\left(F_{\mu\nu} + \frac{1}{6}\,\varepsilon_{\mu\nu\kappa\lambda\tau}\,H^{\kappa\lambda\tau} + \frac{3}{2}\,\partial_y B_{\mu\nu} - \partial_y u_{\mu\nu}\right) = 0\,, \tag{2.51}$$

which can be further integrated into another first order duality equation

$$F_{\mu\nu} + \frac{1}{6}\,\varepsilon_{\mu\nu\kappa\lambda\tau}\,H^{\kappa\lambda\tau} + \frac{3}{2}\,\partial_y B_{\mu\nu} - \partial_y u_{\mu\nu} = 0\,. \tag{2.52}$$

Strictly speaking, the integration requires the introduction of an antisymmetric function $f_{\mu\nu}$ which only depends on the sixth coordinate however we observe that we can set this function $f_{\mu\nu}$ to 0 by absorbing it into $u_{\mu\nu}$: if we integrate $f_{\mu\nu}$ into $\tilde{f}_{\mu\nu}$ then defining $\tilde{u}_{\mu\nu} = u_{\mu\nu} - \tilde{f}_{\mu\nu}$ allows us to get rid of $f_{\mu\nu}$ without changing (2.50). To avoid cumbersome notation, we will keep using $u_{\mu\nu}$ even if we redefined it. Eventually, we can use (2.52) to further simplify (2.50)

$$\begin{aligned}&\partial_{[\mu} h_{\nu]\rho} + \frac{1}{4}\,\varepsilon_{\mu\nu\kappa\lambda\tau}\,\partial^\kappa C^{\lambda\tau}{}_\rho - \partial_\rho u_{\mu\nu}\\ &\qquad = \frac{1}{4}\,\varepsilon_{\mu\nu\rho\kappa\lambda}\,\partial_y\Big(u^{\kappa\lambda} - \frac{3}{2}\,B^{\kappa\lambda}\Big) - \frac{1}{2}\,\partial_y C_{\mu\nu,\rho} + \partial_y A_{[\mu}\eta_{\nu]\rho}\,.\end{aligned} \tag{2.53}$$

We have now cast the original second order self-duality equations (2.37) of the six-dimensional mixed-symmetry tensor field into the form of two first order duality equations (2.52) and (2.53) upon parametrizing the six-dimensional field in terms of its components (2.41) and introduction of an additional field $u_{\mu\nu}$. Upon reduction to five dimensions ($\partial_y \to 0$), these equations constitute duality equations relating the vector-tensor fields, and the graviton-dual graviton fields respectively. The graviton-dual graviton duality equation shows that the dual graviton can be dualized into the standard Pauli-Fierz field as in [32, 33] without adding new degrees of freedom.

The decomposition of the gauge transformations (2.39) is a bit more involved since the symmetric parameter splits into $\{\alpha_{\hat{\mu}\hat{\nu}}\} \to \{\alpha_{\mu\nu}, \alpha_{5\mu} \equiv \xi_\mu, \alpha_{55} \equiv \lambda\}$ and the antisymmetric parameter splits into $\{\beta_{\hat{\mu}\hat{\nu}}\} \to \{\beta_{\mu\nu}, \beta_{5\mu} \equiv \Lambda_\mu\}$. Choosing the following

[1] Here, and in the following we work locally and ignore potential subtleties that may arise from a non-trivial topology. We refer to [31] for a discussion of such issues in the context of chiral p-forms.

parametrization,

$$\alpha_{\hat{\mu}\hat{\nu}} = \begin{pmatrix} \alpha_{\mu\nu} - \eta_{\mu\nu}\,\lambda & \frac{1}{2}\,(\xi_\mu + 3\,\Lambda_\mu) \\ \frac{1}{2}\,(\xi_\mu + 3\,\Lambda_\mu) & 2\,\lambda \end{pmatrix} \,,\; \beta_{\hat{\mu}\hat{\nu}} = \begin{pmatrix} \beta_{\mu\nu} & \frac{3}{2}\,(\xi_\mu - \Lambda_\mu) \\ \frac{3}{2}\,(\Lambda_\mu - \xi_\mu) & 0 \end{pmatrix} \,, \quad (2.54)$$

we get

$$\begin{aligned} \delta A_\mu &= \partial_\mu \lambda + \frac{1}{2}\,\partial_y\,(\xi_\mu - 3\,\Lambda_\mu) \,, \\ \delta B_{\mu\nu} &= 2\,\partial_{[\mu}\Lambda_{\nu]} + \frac{1}{3}\,\partial_y \beta_{\mu\nu} \,, \\ \delta h_{\mu\nu} &= 2\,\partial_{(\mu}\xi_{\nu)} + \eta_{\mu\nu}\,\partial_y \lambda - \partial_y \alpha_{\mu\nu} \,, \\ \delta C_{\mu\nu,\rho} &= 2\,\partial_{[\mu}\alpha_{\nu]\rho} + \partial_\rho \beta_{\mu\nu} - \partial_{[\rho}\beta_{\mu\nu]} + \partial_y \left(\xi_{[\mu}\eta_{\nu]\rho} - 3\,\Lambda_{[\mu}\eta_{\nu]\rho}\right) \,. \end{aligned} \quad (2.55)$$

The field $u_{\mu\nu}$ also comes with its gauge variation which we can compute by integrating up the variation of (2.50),

$$\delta u_{\mu\nu} = \partial_{[\mu}\xi_{\nu]} + \frac{1}{6}\,\varepsilon_{\mu\nu\rho\sigma\tau}\partial^\rho \beta^{\sigma\tau} + \frac{1}{2}\,\partial_y \beta_{\mu\nu} \,. \quad (2.56)$$

2.1.6 (2,2) field

The $\mathcal{N} = (4,0)$ theory features a field which we shall call a double dual graviton. Indeed the $(2,1)$ field can be thought of as the dual of the graviton since we found a duality equation (2.53) relating it to the graviton. Conversely, the $(2,2)$ field can be thought of as the double dualization acting on both indices of $h_{\mu\nu}$ leading to the four-index tensor $T_{\hat{\mu}\hat{\nu},\hat{\rho}\hat{\sigma}}$ that has a $(2,2)$ Young symmetry. That is to say the same symmetries as the Riemann tensor

$$\square\!\square \;:\; T_{\hat{\mu}\hat{\nu},\hat{\rho}\hat{\sigma}} = -T_{\hat{\nu}\hat{\mu},\hat{\rho}\hat{\sigma}} = -T_{\hat{\mu}\hat{\nu},\hat{\sigma}\hat{\rho}} = T_{\hat{\rho}\hat{\sigma},\hat{\mu}\hat{\nu}}, \quad T_{[\hat{\mu}\hat{\nu},\hat{\rho}]\hat{\sigma}} = 0 \,. \quad (2.57)$$

The dynamics of this field is defined by a self-duality equation

$$G_{\hat{\mu}\hat{\nu}\hat{\lambda},\hat{\rho}\hat{\sigma}\hat{\tau}} = \frac{1}{6}\,\varepsilon_{\hat{\mu}\hat{\nu}\hat{\lambda}\hat{\alpha}\hat{\beta}\hat{\gamma}}\,G^{\hat{\alpha}\hat{\beta}\hat{\gamma}}{}_{\hat{\rho}\hat{\sigma}\hat{\tau}} \,, \quad (2.58)$$

where $G_{\hat{\mu}\hat{\nu}\hat{\lambda},\hat{\rho}\hat{\sigma}\hat{\tau}}$ is the second order curvature of $T_{\hat{\mu}\hat{\nu},\hat{\rho}\hat{\sigma}}$

$$G_{\hat{\mu}\hat{\nu}\hat{\lambda},\hat{\rho}\hat{\sigma}\hat{\tau}} = 3\,\partial_{\hat{\rho}}\partial_{[\hat{\mu}}T_{\hat{\nu}\hat{\lambda}],\hat{\sigma}\hat{\tau}} + 3\,\partial_{\hat{\sigma}}\partial_{[\hat{\mu}}T_{\hat{\nu}\hat{\lambda}],\hat{\tau}\hat{\rho}} + 3\,\partial_{\hat{\tau}}\partial_{[\hat{\mu}}T_{\hat{\nu}\hat{\lambda}],\hat{\rho}\hat{\sigma}} \,. \quad (2.59)$$

The curvature and the field equation (2.58) are invariant under the gauge symmetry

$$\delta T_{\hat{\mu}\hat{\nu},\hat{\rho}\hat{\sigma}} = \partial_{[\hat{\mu}}\lambda_{\hat{\nu}],\hat{\rho}\hat{\sigma}} + \partial_{[\hat{\rho}}\lambda_{\hat{\sigma}],\hat{\mu}\hat{\nu}} \,, \quad (2.60)$$

where the parameter $\lambda_{\hat{\mu},\hat{\rho}\hat{\sigma}}$ is itself a tensor with a (2,1) Young symmetry. Once again, we can identify pictorially the different fields the (2,2) field decomposes into

$$\square\!\square \;\to\; \square\!\square \,,\; \begin{array}{|c|c|}\hline & \\ \hline & 5 \\ \hline \end{array} \,,\; \begin{array}{|c|c|}\hline & \\ \hline 5 & 5 \\ \hline \end{array} \,. \quad (2.61)$$

Hence the six-dimensional (2,2) tensor decomposes respectively into a five-dimensional (2,2) tensor, a (2,1) tensor and a symmetric two-index tensor. Concretely, we parametrize these various components of the six-dimensional (2,2) tensor as

$$\{T_{\hat{\mu}\hat{\nu},\hat{\rho}\hat{\sigma}}\} = \{T_{\mu\nu,\rho\sigma}\,;\; T_{\mu\nu,\rho 5} \equiv C_{\mu\nu,\rho}\,;\; T_{\mu 5,\nu 5} \equiv h_{\mu\nu}\}\ . \tag{2.62}$$

After dimensional reduction to five dimensions, these fields describe the double dual graviton, the dual graviton and the graviton, respectively. Using this parametrization, the six-dimensional field equations (2.58) split into two equations[2]

$$\begin{aligned} R_{\mu\nu,\rho\sigma} =& \frac{1}{2}\,\partial_y\partial_\mu C_{\rho\sigma,\nu} - \frac{1}{2}\,\partial_y\partial_\nu C_{\rho\sigma,\mu} + \frac{1}{2}\,\varepsilon_{\mu\nu\kappa\lambda\tau}\,\partial_{[\rho}\partial^\kappa C^{\lambda\tau}{}_{\sigma]} \\ &+ \frac{1}{2}\,\partial_y\partial_\rho C_{\mu\nu,\sigma} - \frac{1}{2}\,\partial_y\partial_\sigma C_{\mu\nu,\rho} \\ &+ \frac{1}{4}\,\varepsilon_{\mu\nu\kappa\lambda\tau}\,\partial_y\partial^\kappa T^{\lambda\tau}{}_{\rho\sigma} + \frac{1}{2}\,\partial_y\partial_y T_{\mu\nu,\rho\sigma}\ , \end{aligned} \tag{2.63}$$

$$\varepsilon_{\mu\nu\alpha\beta\gamma}\,\partial^\alpha\partial_{[\rho}T_{\sigma\tau]}{}^{\beta\gamma} = -\,2\,\partial_\mu\partial_{[\rho}C_{\sigma\tau],\nu} + 2\,\partial_\nu\partial_{[\rho}C_{\sigma\tau],\mu} - 2\,\partial_y\partial_{[\rho}T_{\sigma\tau],\mu\nu}\ , \tag{2.64}$$

with the linearized Riemann tensor $R_{\mu\nu,\rho\sigma}$ defined as in (2.21) for $h_{\mu\nu}$. The second equation (2.64) has the form of a curl in $[\rho\sigma\tau]$ and thus can be integrated up into

$$\frac{1}{2}\,\varepsilon_{\mu\nu\alpha\beta\gamma}\,\partial^\alpha T_{\sigma\tau}{}^{\beta\gamma} + \partial_\mu C_{\sigma\tau,\nu} - \partial_\nu C_{\sigma\tau,\mu} + \partial_y T_{\sigma\tau,\mu\nu} = 2\,\partial_{[\sigma}v_{\tau],\mu\nu}\ , \tag{2.65}$$

up to a tensor $v_{\tau,\mu\nu} = -v_{\tau,\nu\mu}$, determined by this equation up to the gauge freedom $\delta v_{\tau,\mu\nu} = \partial_\tau\zeta_{\mu\nu}$. Combining (2.65) with the first field equation (2.63), we find

$$R_{\mu\nu,\rho\sigma} = \frac{1}{2}\,\partial_y\partial_\rho C_{\mu\nu,\sigma} - \frac{1}{2}\,\partial_y\partial_\sigma C_{\mu\nu,\rho} + \frac{1}{2}\,\varepsilon_{\mu\nu\kappa\lambda\tau}\,\partial_{[\rho}\partial^\kappa C^{\lambda\tau}{}_{\sigma]} + \partial_y\partial_{[\rho}v_{\sigma],\mu\nu}\ , \tag{2.66}$$

which in turn is a curl in $[\rho\sigma]$ and can be integrated up into

$$\partial_{[\mu}h_{\nu]\rho} + \frac{1}{4}\,\varepsilon_{\mu\nu\lambda\sigma\tau}\,\partial^\lambda C^{\sigma\tau}{}_\rho + \frac{1}{2}\,\partial_y C_{\mu\nu,\rho} + \frac{1}{2}\,\partial_y v_{\rho,\mu\nu} = \partial_\rho u_{\mu\nu}\ , \tag{2.67}$$

up to an antisymmetric field $u_{\mu\nu} = -u_{\nu\mu}$. As for the $\mathcal{N} = (3,1)$ model, we have obtained two first-order duality equations (2.65) and (2.67) from which the second order field equations can be derived. After reduction to five dimensions, equations (2.65) and (2.67) describe the duality relations between graviton and dual graviton and between dual graviton and double dual graviton, respectively. In particular, equation (2.67) differs from equation (2.53) in the $\mathcal{N} = (3,1)$ model only if fields depend on the sixth coordinate. Finally, let us note that we can compute the linearized Einstein equations,

$$\begin{aligned} G_{\mu\nu} =& -\frac{1}{2}\,\partial_y\partial^\rho C_{\rho(\mu,\nu)} - \frac{1}{2}\,\partial_y\partial_{(\mu}C_{\nu)\rho}{}^\rho + \frac{1}{2}\,\partial_y\partial_\rho v_{(\mu,\nu)}{}^\rho - \frac{1}{2}\,\partial_y\partial_{(\mu}v^\rho{}_{\nu)\rho} \\ &+ \frac{1}{2}\,\eta_{\mu\nu}\,\partial_y\partial^\rho C_{\rho\sigma}{}^\sigma - \frac{1}{2}\,\eta_{\mu\nu}\,\partial_y\partial_\rho v_\sigma{}^{\sigma\rho}\ . \end{aligned} \tag{2.68}$$

[2]Technically, each group of indices can have a 5 in it or not, making for four equations. However two of these equations are the dualization of the two others, hence there being only two independant equations.

The six-dimensional gauge parameter (2.60) splits into four different parts parametrized as

$$\{\lambda_{\hat{\rho},\hat{\mu}\hat{\nu}}\} = \left\{\lambda_{\rho,\mu\nu}\,;\ \lambda_{\mu,\nu 5} = 2\,\alpha_{\mu\nu} - \frac{2}{3}\,\beta_{\mu\nu}\,;\ \lambda_{5,\mu 5} = 2\,\xi_\mu\right\}\ , \tag{2.69}$$

where $\alpha_{\mu\nu}$ is symmetric and $\beta_{\mu\nu}$ antisymmetric. This yields

$$\begin{aligned}
\delta h_{\mu\nu} &= 2\,\partial_{(\mu}\xi_{\nu)} - 2\,\partial_y \alpha_{\mu\nu}\ , \\
\delta C_{\mu\nu,\rho} &= 2\,\partial_{[\mu}\alpha_{\nu]\rho} + \partial_\rho \beta_{\mu\nu} - \partial_{[\rho}\beta_{\mu\nu]} - \frac{1}{2}\,\partial_y \lambda_{\rho,\mu\nu}\ , \\
\delta T_{\mu\nu,\rho\sigma} &= \partial_{[\mu}\lambda_{\nu],\rho\sigma} + \partial_{[\rho}\lambda_{\sigma],\mu\nu}\ .
\end{aligned} \tag{2.70}$$

As for previously, the gauge variation of the new fields is obtained by integrating up (2.65) and (2.67)

$$\delta u_{\mu\nu} = \partial_{[\mu}\xi_{\nu]} + \frac{1}{6}\,\varepsilon_{\mu\nu\lambda\sigma\tau}\partial^\lambda \beta^{\sigma\tau} + \frac{1}{3}\,\partial_y \beta_{\mu\nu} + \frac{1}{2}\,\partial_y \zeta_{\mu\nu}\ , \tag{2.71}$$

$$\delta v_{\rho,\mu\nu} = \frac{1}{4}\,\varepsilon_{\mu\nu\kappa\lambda\sigma}\,\partial^\kappa \lambda_\rho{}^{\lambda\sigma} + 2\,\partial_{[\mu}\alpha_{\nu]\rho} + \frac{2}{3}\,\partial_{[\mu}\beta_{\nu]\rho} + \partial_\rho\,\zeta_{\mu\nu} + \frac{1}{2}\,\partial_y \lambda_{\rho,\mu\nu}\ , \tag{2.72}$$

where $\zeta_{\mu\nu}$ is an antisymmetric gauge parameter introduced after (2.65).

2.2 Fermions

The fermionic sector is a bit more subtle than the bosonic sector because the field equations of the fermions involve γ matrices which we shall choose in such a way that they make the 5+1 decomposition easier. We choose to follow the construction of [34]: we start by choosing four 4×4 matrices representing the four-dimensional Clifford algebra $Cl(1,3)$. We can now build a representation of the five-dimensional Clifford algebra by adding to them the associated chirality matrix $\gamma_* = i\,\gamma_0\gamma_1\gamma_2\gamma_3$. To build a representation of the six-dimensional Clifford algebra $Cl(1,5)$, we need to build six 8×8 matrices satisfying the usual anticommutation relations. To that end, we define

$$\hat{\gamma}_\mu = \sigma_1 \otimes \gamma_\mu = \begin{pmatrix} 0 & \gamma_\mu \\ \gamma_\mu & 0 \end{pmatrix}\ ,\ \mu = 0, \ldots, 4, \tag{2.73}$$

$$\hat{\gamma}_5 = \sigma_2 \otimes \mathbb{1}_4 = \begin{pmatrix} 0 & -i\,\mathbb{1}_4 \\ i\,\mathbb{1}_4 & 0 \end{pmatrix}\ , \tag{2.74}$$

$$\text{therefore } \hat{\gamma}_* = \hat{\gamma}_{012345} = \begin{pmatrix} \mathbb{1}_4 & 0 \\ 0 & -\mathbb{1}_4 \end{pmatrix}\ , \tag{2.75}$$

where $\hat{\gamma}_*$ is now the chirality matrix associated to the six $\hat{\gamma}$ matrices $\{\hat{\gamma}_0, \ldots, \hat{\gamma}_5\}$ we built to represent $Cl(1,5)$, $\mathbb{1}_4$ is the 4×4 identity matrix and σ_1 and σ_2 are the two

Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \tag{2.76}$$

$$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} . \tag{2.77}$$

One can check that the set $\{\hat{\gamma}_0, \ldots, \hat{\gamma}_5\}$ satisfies the property

$$\hat{\gamma}_{\hat{\mu}}\hat{\gamma}_{\hat{\nu}} + \hat{\gamma}_{\hat{\mu}}\hat{\gamma}_{\hat{\nu}} = 2\,\eta_{\hat{\mu}\hat{\nu}}\mathbb{1}_8 . \tag{2.78}$$

Finally, the introduction of a chirality matrix $\hat{\gamma}_*$ allows us to define the two projectors on the positive and negative chirality

$$P^{\pm} = \frac{\mathbb{1}_8 \pm \hat{\gamma}_*}{2} . \tag{2.79}$$

2.2.1 Dirac field

Any spinor in six dimensions can be decomposed into its positive and negative chirality parts using the projectors (2.79). Thus we can write

$$\chi = \begin{pmatrix} \chi^+ \\ \chi^- \end{pmatrix} , \tag{2.80}$$

where χ has dimension 8, whereas χ^+ and χ^- have dimension 4. Then, the six-dimensional Dirac equation for a free fermion χ

$$\hat{\gamma}^{\hat{\mu}}\partial_{\hat{\mu}}\chi = 0 , \tag{2.81}$$

can be decomposed into

$$\gamma^{\mu}\partial_{\mu}\chi^{\pm} = \mp i\,\partial_y\chi^{\pm} , \tag{2.82}$$

for the components χ^+ and χ^-. Similarly to the discussion about the scalar field, in case of compactification of the sixth dimension, the Dirac field can be expressed in terms of its Fourier modes

$$\chi(x^{\mu}, y) = \sum_{n\in\mathbb{Z}} \chi_n^{\pm}(x^{\mu})\, e^{2in\pi\frac{y}{R}} , \tag{2.83}$$

and the Dirac equation becomes

$$\forall n \in \mathbb{Z} , \quad \gamma^{\mu}\partial_{\mu}\chi_n^{\pm} = \pm\frac{2n\pi}{R}\,\chi_n^{\pm} , \tag{2.84}$$

showing that, as for the scalard field, the masses of the fermion modes are proportional to n/R.

2.2.2 Gravitino

The six-dimensional gravitino is a fermionic form which satisfies a Rarita-Schwinger equation

$$\hat{\gamma}^{\hat{\mu}\hat{\nu}\hat{\rho}}\partial_{\hat{\nu}}\Psi_{\hat{\rho}} \,. \tag{2.85}$$

For a gravitino with positive chirality, the eight-component $\Psi_{\hat{\mu}}$ can be written

$$\Psi_{\hat{\mu}} = \begin{pmatrix} \Psi^{+}_{\hat{\mu}} \\ 0 \end{pmatrix} \,, \tag{2.86}$$

where $\Psi^{+}_{\hat{\mu}}$ is four-dimensional. This field equation is invariant under the gauge transformation

$$\delta\Psi^{+}_{\hat{\mu}} = \partial_{\hat{\mu}}\kappa \,, \tag{2.87}$$

with fermionic parameter κ. We shall consider the following split of the components

$$\{\Psi^{+}_{\hat{\mu}}\} = \left\{ \psi_{\mu} \equiv \Psi^{+}_{\mu} + \frac{i}{3}\gamma_{\mu}\Psi^{+}_{5} \,, \lambda \equiv \Psi^{+}_{5} \right\} \,, \tag{2.88}$$

and the field equations now become

$$\gamma^{\mu\nu\rho}\partial_{\nu}\psi_{\rho} = i\,\partial_{y}\left(\gamma^{\mu\nu}\psi_{\nu} - \frac{4\,i}{3}\gamma^{\mu}\lambda\right) \tag{2.89}$$

$$\gamma^{\mu}\partial_{\mu}\lambda = \partial_{y}\left(\gamma^{\mu}\psi_{\mu} - \frac{5\,i}{3}\lambda\right) \tag{2.90}$$

The gauge transformations (2.87) can be rewritten in terms of the five-dimensional objects

$$\begin{aligned} \delta\lambda &= \partial_{y}\kappa \,, \\ \delta\psi_{\mu} &= \partial_{\mu}\kappa + \frac{i}{3}\,\gamma_{\mu}\partial_{y}\kappa \,. \end{aligned} \tag{2.91}$$

For a gravitino of negative chirality (which is present in the $\mathcal{N} = (2,2)$ model), the computation is analogous with $\partial_{y} \to -\partial_{y}$.

2.2.3 Fermionic 2-form

There also exists another type of fermionic field: the exotic gravitino. It is called exotic because it is a fermionic 2-form which is quite an unusual type of field [35]. It satisfies a generalized Rarita-Schwinger equation

$$\hat{\gamma}^{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}\hat{\tau}}\partial_{\hat{\rho}}\Psi_{\hat{\sigma}\hat{\tau}} = 0 \,, \tag{2.92}$$

Only fermionic 2-forms with a positive chirality appear in Tables 1.2 and 1.3 so the eight-component $\Psi_{\hat{\mu}\hat{\nu}}$ can be written

$$\Psi_{\hat{\mu}\hat{\nu}} = \begin{pmatrix} \Psi^{+}_{\hat{\mu}\hat{\nu}} \\ 0 \end{pmatrix} \,, \tag{2.93}$$

where $\Psi^+_{\hat{\mu}\hat{\nu}}$ is four-dimensional. The field equation is invariant under the gauge transformation

$$\delta\Psi^+_{\hat{\mu}\hat{\nu}} = \partial_{[\hat{\mu}}\kappa_{\hat{\nu}]} \,, \tag{2.94}$$

with a fermionic one-form parameter $\kappa_{\hat{\mu}}$. We consider the following split of the components

$$\{\Psi^+_{\hat{\mu}\hat{\nu}}\} = \left\{\psi_{\mu\nu} \equiv \Psi^+_{\mu\nu} - 2i\,\gamma_{[\mu}\Psi^+_{\nu]5} \,, \phi_\mu \equiv \Psi^+_{\mu 5}\right\} \,. \tag{2.95}$$

In this case the field equations become

$$\gamma^{\mu\nu\rho\sigma\tau}\partial_\rho\psi_{\sigma\tau} = \partial_y\left(-i\,\gamma^{\mu\nu\rho\sigma}\psi_{\rho\sigma} - 4\,\gamma^{\mu\nu\rho}\phi_\rho\right), \tag{2.96}$$

$$\gamma^{\mu\nu\rho}\partial_\nu\phi_\rho = \partial_y\left(3i\,\gamma^{\mu\nu}\phi_\nu - \frac{1}{2}\,\gamma^{\mu\nu\rho}\psi_{\nu\rho}\right). \tag{2.97}$$

Since in five dimensions we have the identity $\gamma_{\mu\nu\rho\sigma\tau} = i\,\epsilon_{\mu\nu\rho\sigma\tau}\mathbb{1}_4$, we can see that upon compactification, the equation of motion of $\psi_{\mu\nu}$ implies that it is pure gauge so it does not carry any degree of freedom. The gauge parameter splits into $\{\kappa_{\hat{\mu}}\} \to \{\kappa_\mu, \kappa\}$ and (2.94) becomes

$$\begin{aligned}\delta\phi_\mu &= \frac{1}{2}\,\partial_\mu\kappa^\alpha - \frac{1}{2}\,\partial_y\kappa_\mu \,, \\ \delta\psi_{\mu\nu} &= \partial_{[\mu}\kappa_{\nu]} + i\,\partial_{[\mu}\gamma_{\nu]}\kappa + i\,\partial_y\gamma_{[\mu}\kappa_{\nu]} \,.\end{aligned} \tag{2.98}$$

CHAPTER 3

Actions

The goal of this chapter is to present, for each model, a linearized action which reproduces the equations of motion found in the previous section while identifying a common part that would be the unique five-dimensional maximal supergravity theory when we set $\partial_y \to 0$. The full split Lagrangians can then be written using the equations of motion computed in the last section as the building blocks for each type of field. The supersymmetry invariance is not discussed here, but postponed to Chapter 5.

3.1 The $\mathcal{N} = (2,2)$ model

As stated before, the $\mathcal{N} = (2,2)$ multiplet corresponding to the maximal supergravity in six dimensions comprises a metric, 25 scalar fields, 16 vectors and 5 two-forms (which can be decomposed into 5 self-dual two-forms and 5 anti self-dual two-forms) for its bosonic part.

3.1.1 Scalars and vectors

The Lagrangians for the scalar field and the vector field can be constructed directly from the six-dimensional Lagrangians for a free scalar field and a free vector field

$$\mathcal{L}_\bullet = -\frac{1}{2}\,\partial^{\hat{\mu}}\phi^\alpha \partial_{\hat{\mu}}\phi^\alpha \;, \quad \alpha = 1, \dots 25 \;, \tag{3.1}$$

$$\mathcal{L}_\square = -\frac{1}{4}\,F_{\hat{\mu}\hat{\nu}}{}^i F^{\hat{\mu}\hat{\nu}i} \;, \quad i = 1, \dots 16 \;. \tag{3.2}$$

Using the splits into their five-dimensional components (2.8), we have

$$\mathcal{L}_\bullet = -\frac{1}{2}\,\partial^\mu\phi^\alpha\partial_\mu\phi^\alpha - \frac{1}{2}\,\partial_y\phi^\alpha\partial_y\phi^\alpha\,, \tag{3.3}$$

$$\mathcal{L}_\square = -\frac{1}{4}\,F^{\mu\nu\,i}F_{\mu\nu}{}^i - \frac{1}{2}\left(\partial^\mu\phi^i - \partial_y A^{\mu\,i}\right)\left(\partial_\mu\phi^i - \partial_y A_\mu{}^i\right)\,, \tag{3.4}$$

3.1.2 Two-form

For the two-form, the construction is a bit more involved. Starting from the six-dimensional Lagrangian

$$\mathcal{L}_H^{(6D)} = -\frac{1}{6}\,H_{\hat\mu\hat\nu\hat\rho}{}^b H^{\hat\mu\hat\nu\hat\rho b}\,, \quad b = 1,\dots 5\,. \tag{3.5}$$

and using the split (2.17), we get

$$\mathcal{L}_H^{(6D)} = -\frac{1}{6}\,H_{\mu\nu\rho}{}^b H^{\mu\nu\rho b} - \frac{1}{6}(F_{\mu\nu}{}^b + \partial_y B_{\mu\nu}{}^b)(F^{\mu\nu b} + \partial_y B^{\mu\nu b})\,, \tag{3.6}$$

leading to the equations of motion

$$\partial^\mu(F_{\mu\nu}{}^b + \partial_y B_{\mu\nu}{}^b) = 0\,, \tag{3.7}$$

$$\partial^\mu H_{\mu\nu\rho}{}^b = -\partial_y(F_{\mu\nu}{}^b + \partial_y B_{\mu\nu}{}^b)\,. \tag{3.8}$$

Equation (3.7) implies that we can introduce two-forms $\tilde B_{\mu\nu}{}^b$ such that

$$F_{\mu\nu}{}^b + \partial_y B_{\mu\nu}{}^b = \frac{1}{6}\,\varepsilon_{\mu\nu\rho\sigma\tau}\tilde H^{\rho\sigma\tau b}\,, \tag{3.9}$$

where $\tilde H_{\mu\nu\rho}{}^b = 3\,\partial_{[\mu}\tilde B_{\nu\rho]}{}^b$. In addition, vectors $\tilde A_\mu{}^b$ can be introduced by means of the equation

$$\tilde F_{\mu\nu}{}^b + \partial_y \tilde B_{\mu\nu}{}^b = \frac{1}{6}\,\varepsilon_{\mu\nu\rho\sigma\tau}H^{\rho\sigma\tau b}\,. \tag{3.10}$$

Integrability of this equation is ensured by (3.8). Thus, we have replaced a system of five vectors and five two-forms described by second-order equations of motion by a system of ten vectors and ten two-forms described by first-order duality equations. After introducing

$$A_\mu^A = \begin{pmatrix} A_\mu^a \\ \tilde A_\mu^a \end{pmatrix} \quad \text{and} \quad B_{\mu\nu}^A = \begin{pmatrix} B_{\mu\nu}^a \\ \tilde B_{\mu\nu}^a \end{pmatrix}\,, \quad \text{where } A = 1,\dots,10\,, \tag{3.11}$$

we can rewrite the system of duality equations into the more compact form

$$F_{\mu\nu}^A + \partial_y B_{\mu\nu}^A = -\frac{1}{6}\,\varepsilon_{\mu\nu\rho\sigma\tau}\eta^{AB}\delta_{BC}H^{\rho\sigma\tau C}\,, \tag{3.12}$$

with

$$\eta^{AB} = \begin{pmatrix} 0 & -\mathbb{1}_5 \\ -\mathbb{1}_5 & 0 \end{pmatrix} \quad \text{and} \quad \delta^{AB} = \begin{pmatrix} \mathbb{1}_5 & 0 \\ 0 & \mathbb{1}_5 \end{pmatrix}\,. \tag{3.13}$$

As a consequence of (3.12), we have the Yang-Mills equation

$$\partial^\mu (F_{\mu\nu}{}^A + \partial_y B_{\mu\nu}{}^A) = 0 \,, \tag{3.14}$$

We shall now show that the duality equations (3.12) can be derived from the Lagrangian

$$\begin{aligned} \mathcal{L}_{\boxminus} = &- \frac{1}{4} \left(F_{\mu\nu}{}^A + \partial_y B_{\mu\nu}{}^A\right) \delta_{AB} \left(F^{\mu\nu\, B} + \partial_y B^{\mu\nu\, B}\right) \\ &- \frac{1}{24}\, \varepsilon^{\mu\nu\rho\sigma\tau}\, \eta_{AB}\, \partial_y B_{\mu\nu}{}^A\, H_{\rho\sigma\tau}{}^B \,, \end{aligned} \tag{3.15}$$

which breaks manifest six-dimensional Lorentz invariance. Namely, the variation of (3.15) with respect to A_μ yields equation (3.14) whereas variation with respect to $B_{\mu\nu}$ yields

$$\partial_y \Big(F_{\mu\nu}{}^A + \partial_y B_{\mu\nu}{}^A + \frac{1}{6}\, \varepsilon_{\mu\nu\rho\sigma\tau}\, \eta^{AB} \delta_{BC} H^{\rho\sigma\tau C}\Big) = 0 \,. \tag{3.16}$$

which reproduces equations (3.12) up to some function $\beta_{\mu\nu}$ that does not depend on y:

$$F_{\mu\nu}{}^A + \partial_y B_{\mu\nu}{}^A + \frac{1}{6}\, \varepsilon_{\mu\nu\rho\sigma\tau}\, \eta^{AB} \delta_{BC} H^{\rho\sigma\tau C} = \beta_{\mu\nu}{}^A \,, \qquad \partial_y \beta_{\mu\nu}{}^A = 0 \,. \tag{3.17}$$

Comparing the divergence of this equation to (3.14), we find that locally the field $\beta_{\mu\nu}{}^A$ can be integrated to

$$\partial^\mu \beta_{\mu\nu}{}^A = 0 \quad \Longrightarrow \quad \beta_{\mu\nu}{}^A = \varepsilon_{\mu\nu\rho\sigma\tau}\, \partial^\rho b^{\sigma\tau A} \,, \tag{3.18}$$

in terms of a function $b_{\mu\nu}{}^A$, such that the field equations (3.17) can be rewritten as

$$\Big(F_{\mu\nu}{}^A + \partial_y \tilde{B}_{\mu\nu}{}^A\Big) + \frac{1}{6}\, \varepsilon_{\mu\nu\rho\sigma\tau}\, 3\, \partial^\rho \eta^{AB} \tilde{B}^{\sigma\tau}{}_B = 0 \,, \tag{3.19}$$

with the modified two-form

$$\tilde{B}_{\mu\nu\, B} \equiv B_{\mu\nu\, B} - 2\, b_{\mu\nu\, B} \,. \tag{3.20}$$

Hence, we recover the desired original duality equations (3.12). We have therefore found an alternative way to describe the dynamics of a six-dimensional two-form after a 5+1 split of coordinates. An advantage of this new formulation is the fact that the Lagrangian (3.15) can be straightforwardly generalized to describe chiral two-forms in six dimensions. This simply corresponds to replacing the matrix η_{AB} by δ_{AB} (or $-\delta_{AB}$ if we want to describe anti chiral two-forms) such that (3.12) becomes

$$F^A_{\mu\nu} + \partial_y B^A_{\mu\nu} = \frac{1}{6}\, \varepsilon_{\mu\nu\rho\sigma\tau} H^{\rho\sigma\tau A} \,. \tag{3.21}$$

This in turn is nothing but the six-dimensional self-duality equation

$$H_{\hat{\mu}\hat{\nu}\hat{\rho}}{}^A = \frac{1}{6}\, \varepsilon_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}\hat{\tau}\hat{\lambda}} H^{\hat{\sigma}\hat{\tau}\hat{\lambda}A} \,. \tag{3.22}$$

The Lagrangian (3.15) thus provides an action for chiral six-dimensional two-forms upon sacrificing manifest Lorentz invariance. This is in the spirit of the mechanism proposed in [23] by Henneaux and Teitelboim who argued that implementing the self-duality condition (for example with a Lagrange multiplier) after having implemented Lorentz invariance leads to problems notably upon quantization. A covariant approach has also been developed in [36–40].

3.1.3 Graviton

The Lagrangian for the graviton can be constructed from the six-dimensional Einstein-Hilbert Lagrangian

$$\mathcal{L}_g^{(6D)} = \sqrt{|g|}R\,, \tag{3.23}$$

where R is the Ricci scalar and $g = \det(g_{\hat{\mu}\hat{\nu}})$. The linearization of (3.23) around the six-dimensional Minkowski metric (i.e. writing $g_{\hat{\mu}\hat{\nu}} = \eta_{\hat{\mu}\hat{\nu}} + h_{\hat{\mu}\hat{\nu}}$) gives rise to the massless Pauli-Fierz Lagrangian

$$\begin{aligned}\mathcal{L}_{\square}^{(6D)} &= -\frac{1}{2}\,\partial_{\hat{\mu}}h^{\hat{\mu}\hat{\nu}}\,\partial_{\hat{\nu}}h_{\hat{\rho}}{}^{\hat{\rho}} + \frac{1}{2}\,\partial_{\hat{\mu}}h^{\hat{\rho}\hat{\sigma}}\,\partial_{\hat{\rho}}h_{\hat{\sigma}}{}^{\hat{\mu}} - \frac{1}{4}\,\partial_{\hat{\mu}}h^{\hat{\rho}\hat{\sigma}}\,\partial^{\hat{\mu}}h_{\hat{\rho}\hat{\sigma}} + \frac{1}{4}\,\partial_{\hat{\mu}}h_{\hat{\nu}}{}^{\hat{\nu}}\,\partial^{\hat{\mu}}h_{\hat{\rho}}{}^{\hat{\rho}} \\ &= -\frac{1}{4}\,\Omega^{\hat{\mu}\hat{\nu}\hat{\rho}}\Omega_{\hat{\mu}\hat{\nu}\hat{\rho}} + \frac{1}{2}\,\Omega^{\hat{\mu}\hat{\nu}\hat{\rho}}\Omega_{\hat{\nu}\hat{\rho}\hat{\mu}} + \Omega^{\hat{\mu}}\Omega_{\hat{\mu}}\,,\end{aligned} \tag{3.24}$$

with $\Omega_{\hat{\mu}\hat{\nu}\hat{\rho}} \equiv \partial_{[\hat{\mu}}h_{\hat{\nu}]\hat{\rho}}$ and $\Omega_{\hat{\mu}} \equiv \Omega_{\hat{\mu}\hat{\nu}}{}^{\hat{\nu}}$. Using the split (2.24), we get

$$\begin{aligned}\mathcal{L}_{\square} = &-\frac{1}{4}\,\Omega^{\mu\nu\rho}\Omega_{\mu\nu\rho} + \frac{1}{2}\,\Omega^{\mu\nu\rho}\Omega_{\nu\rho\mu} + \Omega^{\mu}\Omega_{\mu} \\ &-\frac{1}{3}\,(\partial^{\mu}\phi - 2\,\partial_y A^{\mu})(\partial_{\mu}\phi - 2\,\partial_y A_{\mu}) - \frac{1}{4}\,F^{\mu\nu}F_{\mu\nu} \\ &+\frac{5}{9}\,\partial_y\phi\partial_y\phi - \frac{2}{3}\,\partial_y h_{\sigma}{}^{\sigma}\partial_y\phi + \frac{1}{4}\,\partial_y h_{\sigma}{}^{\sigma}\partial_y h_{\rho}{}^{\rho} - \frac{1}{4}\,\partial_y h^{\mu\nu}\partial_y h_{\mu\nu}\,,\end{aligned} \tag{3.25}$$

where

$$\Omega_{\mu\nu\rho} \equiv \partial_{[\mu}h_{\nu]\rho} - \frac{2}{3}\,\partial_y A_{[\mu}\eta_{\nu]\rho}\,. \tag{3.26}$$

We can observe that upon dimensional reduction ($\partial_y \to 0$), we recover a linearized Pauli-Fierz Lagrangian for a five-dimensional free graviton

$$\mathcal{L}_{PF} = -\frac{1}{4}\,\Omega^{\mu\nu\rho}\Omega_{\mu\nu\rho} + \frac{1}{2}\,\Omega^{\mu\nu\rho}\Omega_{\nu\rho\mu} + \Omega^{\mu}\Omega_{\mu} \quad \text{with} \quad \Omega_{\mu\nu\rho} \equiv \partial_{[\mu}h_{\nu]\rho}\,, \tag{3.27}$$

as well as Lagrangians for a free Maxwell field and a free scalar field.

3.1.4 Fermions

Finally, the Lagrangians for the five-dimensional Dirac fields and the gravitini can be built directly from the Lagrangians of six-dimensional complex free Dirac field and free

gravitini

$$\mathcal{L}_{\bullet_F \pm} = \bar{\chi}^{\pm} \hat{\gamma}^{\hat{\mu}} \partial_{\hat{\mu}} \chi^{\pm} \,, \tag{3.28}$$
$$\mathcal{L}_{\square_F \pm} = \bar{\Psi}^{\pm}_{\hat{\mu}} \hat{\gamma}^{\hat{\mu}\hat{\nu}\hat{\rho}} \partial_{\hat{\nu}} \Psi^{\pm}_{\hat{\rho}} \,. \tag{3.29}$$

where "$\pm$" denotes the chirality of the spinor fields. In six dimensions, it is convenient to double the spinor fields in order to impose a symplectic Majorana-Weyl reality condition. As a result the six-dimensional actions of the $\mathcal{N} = (2,2)$ take the form

$$\mathcal{L}_{\bullet_F} = \bar{\chi}_{[\![\dot{\alpha}\dot{\beta}]\!]\alpha} \hat{\gamma}^{\hat{\mu}} \partial_{\hat{\mu}} \chi^{[\![\dot{\alpha}\dot{\beta}]\!]\alpha} + \bar{\chi}_{[\![\alpha\beta]\!]\dot{\alpha}} \hat{\gamma}^{\hat{\mu}} \partial_{\hat{\mu}} \chi^{[\![\alpha\beta]\!]\dot{\alpha}} \,, \tag{3.30}$$
$$\mathcal{L}_{\square_F} = \bar{\Psi}_{\hat{\mu},\dot{\alpha}} \hat{\gamma}^{\hat{\mu}\hat{\nu}\hat{\rho}} \partial_{\hat{\nu}} \Psi^{\dot{\alpha}}_{\hat{\rho}} + \bar{\Psi}_{\hat{\mu},\alpha} \hat{\gamma}^{\hat{\mu}\hat{\nu}\hat{\rho}} \partial_{\hat{\nu}} \Psi^{\alpha}_{\hat{\rho}} \,. \tag{3.31}$$

Here indices $\alpha, \dot{\alpha}$ refer to the fundamental representations of the two factors $\mathrm{Usp}(4) \times \mathrm{Usp}(4)$ of the R-symmetry group. The double bracket $[\![\cdot]\!]$ indicates the traceless anti-symmetrization of the indices in these brackets. In table 1.3, we have labelled $[\![\alpha\beta]\!]$ of indices as a vector index a of $\mathrm{SO}(5)$ which is isomorphic to $\mathrm{Usp}(4)$. As given in the table, the fermions $\chi^{[\![\dot{\alpha}\dot{\beta}]\!]\alpha}$ and $\Psi^{\dot{\alpha}}_{\hat{\rho}}$ have positive chirality whereas $\chi^{[\![\alpha\beta]\!]\dot{\alpha}}$ and $\Psi^{\alpha}_{\hat{\rho}}$ have negative chirality.

Applying the split (2.88), we get

$$\begin{aligned}
\mathcal{L}_{\bullet_F} =& \bar{\chi}_{[\![\dot{\alpha}\dot{\beta}]\!]\alpha} \gamma^{\mu} \partial_{\mu} \chi^{[\![\dot{\alpha}\dot{\beta}]\!]\alpha} + i \bar{\chi}_{[\![\dot{\alpha}\dot{\beta}]\!]\alpha} \partial_y \chi^{[\![\dot{\alpha}\dot{\beta}]\!]\alpha} \\
&+ \bar{\chi}_{[\![\alpha\beta]\!]\dot{\alpha}} \gamma^{\mu} \partial_{\mu} \chi^{[\![\alpha\beta]\!]\dot{\alpha}} - i \bar{\chi}_{[\![\alpha\beta]\!]\dot{\alpha}} \partial_y \chi^{[\![\alpha\beta]\!]\dot{\alpha}} \,, \qquad (3.32) \\
\mathcal{L}_{\square_F} =& -2\, \bar{\psi}_{\mu,\dot{\alpha}} \gamma^{\mu\nu\rho} \partial_{\nu} \psi^{\dot{\alpha}}_{\rho} + 2i\, \bar{\psi}_{\mu,\dot{\alpha}} \gamma^{\mu\nu} \partial_y \psi^{\dot{\alpha}}_{\nu} + \frac{16}{3}\, \bar{\psi}_{\mu,\dot{\alpha}} \gamma^{\mu} \partial_y \lambda^{\dot{\alpha}} \\
&- 2\, \bar{\psi}_{\mu,\alpha} \gamma^{\mu\nu\rho} \partial_{\nu} \psi^{\alpha}_{\rho} - 2i\, \bar{\psi}_{\mu,\alpha} \gamma^{\mu\nu} \partial_y \psi^{\alpha}_{\nu} - \frac{16}{3}\, \bar{\psi}_{\mu,\alpha} \gamma^{\mu} \partial_y \lambda^{\alpha} \\
&- \frac{8}{3}\, \bar{\lambda}_{\dot{\alpha}} \gamma^{\mu} \partial_{\mu} \lambda^{\dot{\alpha}} - \frac{40i}{9}\, \bar{\lambda}_{\dot{\alpha}} \partial_y \lambda^{\dot{\alpha}} - \frac{8}{3}\, \bar{\lambda}_{\alpha} \gamma^{\mu} \partial_{\mu} \lambda^{\alpha} + \frac{40i}{9}\, \bar{\lambda}_{\alpha} \partial_y \lambda^{\alpha} \,, \qquad (3.33)
\end{aligned}$$

and the full Lagrangian of the $\mathcal{N} = (2,2)$ theory is the sum of all these pieces

$$\mathcal{L}_{(2,2)} = \mathcal{L}_{\bullet} + \mathcal{L}_{\square} + \mathcal{L}_{\square\!\!\square} + \mathcal{L}_{\square\square} + \mathcal{L}_{\bullet_F} + \mathcal{L}_{\square_F} \,. \tag{3.34}$$

In the limit $\partial_y \to 0$, this Lagrangian describes $D = 5$ maximal supergravity [41].

3.2 The $\mathcal{N} = (3,1)$ model

3.2.1 Main result

The $\mathcal{N} = (3,1)$ theory features 28 scalars, 42 Dirac fields, 14 vectors, 12 self-dual two-forms, 6 gravitini, 2 exotic gravitini and a self-dual Curtright field as listed in table 1.2. For scalars, vectors and standard fermions, we use the same Lagrangians as for the

$\mathcal{N}=(2,2)$ case above. The twelve two-forms in this model are self-dual in six dimensions but as explained above, after a 5+1 split, they can be described by the Lagrangian

$$\begin{aligned}\mathcal{L}_{\square} = &-\frac{1}{4}\left(F_{\mu\nu}{}^{A}+\partial_y B_{\mu\nu}{}^{A}\right)\delta_{AB}\left(F^{\mu\nu\,B}+\partial_y B^{\mu\nu\,B}\right)\\ &-\frac{1}{24}\,\varepsilon^{\mu\nu\rho\sigma\tau}\,\delta_{AB}\,\partial_y B_{\mu\nu}{}^{A}\,H_{\rho\sigma\tau}{}^{B}\,,\end{aligned} \tag{3.35}$$

where $A=1,\ldots,12$. This Lagrangian is identical to (3.15) except we have changed η_{AB} into δ_{AB}.

However we have two new types of fields for which we need to build a Lagrangian. First, after the split (2.41), the dynamics of the six-dimensional Curtright field $C_{\hat{\mu}\hat{\nu},\hat{\rho}}$ is given by equations (2.52) and (2.53). As a main result of this section, we shall show that these equations can be derived from the Lagrangian

$$\begin{aligned}\mathcal{L}_{\square\!\!\square} \;=\; &-\frac{1}{4}\,\widehat{\Omega}^{\mu\nu\rho}\widehat{\Omega}_{\mu\nu\rho}+\frac{1}{2}\,\widehat{\Omega}^{\mu\nu\rho}\widehat{\Omega}_{\nu\rho\mu}+\widehat{\Omega}^{\mu}\widehat{\Omega}_{\mu}-\frac{1}{16}\,\varepsilon^{\mu\nu\rho\sigma\tau}\partial_y\widehat{C}_{\mu\nu,}{}^{\lambda}\partial_\rho\widehat{C}_{\sigma\tau,\lambda}\\ &-\frac{3}{4}\,\mathcal{F}^{\mu\nu}\mathcal{F}_{\mu\nu}-\frac{9}{16}\,\varepsilon^{\mu\nu\rho\sigma\tau}\partial_y B_{\mu\nu}\partial_\rho B_{\sigma\tau}-\frac{3}{16}\,\varepsilon^{\mu\nu\rho\sigma\tau}\,\partial_y B_{\mu\nu}\,\partial_y\widehat{C}_{\rho\sigma,\tau}\,,\end{aligned} \tag{3.36}$$

with

$$\widehat{\Omega}_{\mu\nu\rho}\equiv\partial_{[\mu}h_{\nu]\rho}-\partial_y A_{[\mu}\eta_{\nu]\rho}+\frac{1}{2}\,\partial_y\widehat{C}_{\mu\nu,\rho}\,, \tag{3.37}$$

$$\widehat{C}_{\mu\nu,\rho}\equiv C_{\mu\nu,\rho}+\varepsilon_{\mu\nu\rho\sigma\tau}\,u^{\sigma\tau}\,, \tag{3.38}$$

$$\mathcal{F}_{\mu\nu}\equiv 2\,\partial_{[\mu}A_{\nu]}+\frac{3}{2}\,\partial_y B_{\mu\nu}\,. \tag{3.39}$$

Similarly to the case of the two-form, we can thus describe the dynamics of a self-dual Curtright field by an action after sacrificing part of the manifest Lorentz invariance. The Lagrangian (3.36) is invariant under the gauge transformations

$$\delta\widehat{\Omega}_{\mu\nu\rho}=\partial_\rho\partial_{[\mu}\xi_{\nu]}-\partial_y\partial_{[\mu}\beta_{\nu]\rho} \tag{3.40}$$

$$\delta\widehat{C}_{\mu\nu,\rho}=2\,\partial_{[\mu}\alpha_{\nu]\rho}-2\,\partial_{[\mu}\beta_{\nu]\rho}+\partial_y\left(\xi_{[\mu}\eta_{\nu]\rho}-3\,\Lambda_{[\mu}\eta_{\nu]\rho}+\frac{1}{2}\,\varepsilon_{\mu\nu\rho\alpha\beta}\beta^{\alpha\beta}\right)\,, \tag{3.41}$$

$$\delta\mathcal{F}_{\mu\nu}=\partial_y\left(\partial_{[\mu}\xi_{\nu]}+\frac{1}{2}\,\partial_y\beta_{\mu\nu}\right)\,. \tag{3.42}$$

After reduction to five dimensions (i.e. $\partial_y\to 0$), this Lagrangian reduces to the Fierz-Pauli Lagrangian for $h_{\mu\nu}$ (3.27) together with a free Maxwell Lagrangian for A_μ and the dual fields $\widehat{C}_{\mu\nu,\rho}$ and $B_{\mu\nu}$ drop out in this limit. The derivation of equations (2.52) and (2.53) is presented in the next subsection.

The second new field appearing in the $\mathcal{N}=(3,1)$ theory is the chiral fermionic two-form $\Psi_{\hat{\mu}\hat{\nu}}$. In six dimensions, this field can be described by the Lagrangian [27]

$$\mathcal{L}_{\square_F}=\bar{\Psi}_{\hat{\mu}\hat{\nu},\alpha}\hat{\gamma}^{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}\hat{\tau}}\partial_{\hat{\rho}}\Psi^{\alpha}_{\hat{\sigma}\hat{\tau}}\,. \tag{3.43}$$

where $\alpha = 1,2$ represents the fundamental representation of the Usp(2) factor of the R-symmetry group.

Using the split (2.95), we get

$$\begin{aligned}\mathcal{L}_{\square_F} =& \bar{\psi}_{\mu\nu\alpha}\gamma^{\mu\nu\rho\sigma\tau}\partial_\rho\psi^\alpha_{\sigma\tau} + i\,\bar{\psi}_{\mu\nu\alpha}\gamma^{\mu\nu\rho\sigma}\partial_y\psi^\alpha_{\rho\sigma} + 4\,\bar{\psi}_{\mu\nu\alpha}\gamma^{\mu\nu\rho}\phi^\alpha_\rho \\ & - 2\,\bar{\phi}_{\mu\alpha}\gamma^{\mu\nu\rho}\partial_\nu\phi^\alpha_\rho + 6i\,\bar{\phi}_{\mu\alpha}\gamma^{\mu\nu}\partial_y\phi^\alpha_\nu\ . \end{aligned} \tag{3.44}$$

This Lagrangian reproduces the equations of motion (2.96) and (2.97).

The full Lagrangian of the free $\mathcal{N} = (3,1)$ theory then is the sum of all the different pieces

$$\mathcal{L}_{(3,1)} = \mathcal{L}_\bullet + \mathcal{L}_\square + \mathcal{L}_{\square\!\square} + \mathcal{L}_{\square\!\!\square\!\!\square} + \mathcal{L}_{\bullet_F} + \mathcal{L}_{\square_F} + \mathcal{L}_{\square\!\square_F}\ . \tag{3.45}$$

This is one of the main results of this chapter.

3.2.2 Derivation of the dynamics of the Curtright field

It is not obvious that the Lagrangian (3.36) reproduces the equations of motion (2.52) and (2.53). In order to show that, we derive the equations of motion of the fields appearing in the Lagrangian (3.36), then rewrite them in terms of the original fields appearing in (3.39). The variation of (3.36) with respect to A_μ and $B_{\mu\nu}$ gives respectively

$$\partial^\mu F_{\mu\nu} + \frac{3}{2}\,\partial_y\partial^\mu B_{\mu\nu} - \frac{1}{2}\,\partial_y\left(\partial^\mu h_{\mu\nu} - \partial_\nu h_\mu{}^\mu - \partial_y C_{\nu\mu}{}^\mu + 4\,\partial_y A_\nu\right) = 0\ , \tag{3.46}$$

and

$$\partial_y\left(F_{\mu\nu} + \frac{3}{2}\,\partial_y B_{\mu\nu} - \partial_y u_{\mu\nu} + \frac{1}{2}\,\varepsilon_{\mu\nu\rho\sigma\tau}\partial^\rho B^{\sigma\tau}\right) = 0\ . \tag{3.47}$$

The second equation is almost the duality equation (2.52) we want to obtain, however still under derivative ∂_y. We can integrate (3.47) with respect to y, thus introducing an antisymmetric function $p_{\mu\nu}$ independent of y

$$F_{\mu\nu} + \frac{3}{2}\,\partial_y B_{\mu\nu} - \partial_y u_{\mu\nu} + \frac{1}{2}\,\varepsilon_{\mu\nu\rho\sigma\tau}\partial^\rho B^{\sigma\tau} = p_{\mu\nu}\ . \tag{3.48}$$

Combining this equation with (3.46) gives

$$\partial_y\partial^\mu u_{\mu\nu} = \frac{1}{2}\,\partial_y\left(\partial^\mu h_{\mu\nu} - \partial_\nu h_\mu{}^\mu - \partial_y C_{\nu\mu}{}^\mu + 4\,\partial_y A_\nu\right) - \partial^\mu p_{\mu\nu}\ . \tag{3.49}$$

Since $\widehat{C}_{\mu\nu,\rho}$ is reducible under the Lorentz gorup, the variation of (3.36) with respect to $\widehat{C}_{\mu\nu,\rho}$ gives an equation of motion which is itself reducible. Its (2,1) Young symmetry part reads

$$\begin{aligned} 0 = \partial_y\Big(&\partial_{[\mu}h_{\nu]\rho} + \partial^\sigma h_{\sigma[\mu}\eta_{\nu]\rho} - \eta_{\rho[\nu}\partial_{\mu]}h_\sigma{}^\sigma \\ &+ \frac{1}{4}\,\varepsilon_{\mu\nu\lambda\sigma\tau}\partial^\lambda C^{\sigma\tau}{}_\rho - \frac{1}{4}\,\varepsilon_{\lambda\sigma\tau[\mu\nu}\partial^\lambda C^{\sigma\tau}{}_{\rho]} \\ &+ \frac{1}{2}\,\partial_y C_{\mu\nu,\rho} - \eta_{\rho[\nu}\partial_y C_{\mu]\sigma}{}^\sigma + 3\,\partial_y A_{[\mu}\eta_{\nu]\rho} \\ &- \partial_\rho u_{\mu\nu} + \partial_{[\rho}u_{\mu\nu]} - 2\,\partial^\sigma u_{\sigma[\mu}\eta_{\nu]\rho}\Big)\ , \end{aligned} \tag{3.50}$$

which can be integrated with respect to y by introducing a function $r_{\mu\nu,\rho}$ independent of y which is itself has the (2,1) Young symmetry

$$\begin{aligned} r_{\mu\nu,\rho} = {} & \partial_{[\mu} h_{\nu]\rho} + \partial^\sigma h_{\sigma[\mu}\eta_{\nu]\rho} - \eta_{\rho[\nu}\partial_{\mu]} h_\sigma{}^\sigma \\ & + \frac{1}{4}\,\varepsilon_{\mu\nu\lambda\sigma\tau}\partial^\lambda C^{\sigma\tau}{}_\rho - \frac{1}{4}\,\varepsilon_{\lambda\sigma\tau[\mu\nu}\partial^\lambda C^{\sigma\tau}{}_{\rho]} \\ & + \frac{1}{2}\,\partial_y C_{\mu\nu,\rho} - \eta_{\rho[\nu}\partial_y C_{\mu]\sigma}{}^\sigma + 3\,\partial_y A_{[\mu}\eta_{\nu]\rho} \\ & - \partial_\rho u_{\mu\nu} + \partial_{[\rho} u_{\mu\nu]} - 2\,\partial^\sigma u_{\sigma[\mu}\eta_{\nu]\rho}\,. \end{aligned} \tag{3.51}$$

Contracting this equation with $\eta^{\mu\rho}$ yields

$$\partial^\mu u_{\mu\nu} = \frac{1}{2}\,(\partial^\mu h_{\mu\nu} - \partial_\nu h_\mu{}^\mu - \partial_y C_{\nu\mu}{}^\mu + 4\,\partial_y A_\nu) + \frac{1}{3}\,r_{\mu\nu}{}^\mu\,, \tag{3.52}$$

meaning that together with (3.49), we can define locally a two-form $b_{\mu\nu}$, independent of y, such that

$$p_{\mu\nu} = \frac{1}{2}\,\varepsilon_{\mu\nu\rho\sigma\tau}\partial^\rho b^{\sigma\tau}. \tag{3.53}$$

This two-form can be absorbed in $B_{\mu\nu}$ following exactly the same process as the discussion around (3.20) such that from (3.47) we recover the six-dimensional equation (2.52).

The process to recover the other equation (2.53) is more tedious. First, the variation of (3.36) with respect to $h_{\mu\nu}$ gives

$$G_{\mu\nu} + \frac{1}{2}\partial_y\left(\partial^\rho C_{\rho(\mu,\nu)} + \partial_{(\mu} C_{\nu)\rho}{}^\rho - \eta_{\mu\nu}\partial^\rho C_{\rho\sigma}{}^\sigma\right) = 0\,, \tag{3.54}$$

where $G_{\mu\nu}$ is the linearized Einstein tensor defined in (2.48). The contraction of (3.51) with ∂^μ yields a reducible equation whose symmetric part is

$$G_{\nu\rho} + \frac{1}{2}\,\partial_y\left(\partial^\mu C_{\mu(\nu,\rho)} - \eta_{\nu\rho}\partial^\mu C_{\mu\sigma}{}^\sigma + \partial_{(\rho} C_{\nu)\sigma}{}^\sigma\right) = \partial^\mu r_{\mu(\nu,\rho)}\,, \tag{3.55}$$

immediately giving us

$$\partial^\mu r_{\mu(\nu,\rho)} = 0\,. \tag{3.56}$$

As for the antisymmetric part, it reads

$$\begin{aligned} 2\,\partial^\mu r_{\mu[\nu,\rho]} = {} & -\frac{1}{6}\,\varepsilon_{\lambda\sigma\tau\nu\rho}\partial^\mu\partial^\lambda C^{\sigma\tau}{}_\mu + 2\,\partial^\mu\partial_{[\mu} u_{\nu\rho]} \\ & + \partial_y\left(\partial^\mu C_{\mu[\nu,\rho]} + \partial_{[\rho} C_{\nu]\sigma}{}^\sigma - 3\,\partial_{[\rho} A_{\nu]}\right)\,. \end{aligned} \tag{3.57}$$

The totally antisymmetric part of the variation of (3.36) with respect to $\widehat{C}_{\mu\nu\rho}$ is

$$\partial_y\left(\partial^\rho C_{\mu\nu,\rho} + 2\,\partial_{[\mu} C_{\nu]}{}^\rho + \varepsilon_{\mu\nu\rho\sigma\tau}\partial^\rho u^{\sigma\tau} - 3\,\partial_y\left(u_{\mu\nu} - \frac{3}{2}\,B_{\mu\nu}\right)\right) = 0\,, \tag{3.58}$$

which can be integrated as well, provided we introduce a y-independent antisymmetric function $q_{\mu\nu}$

$$\partial^\rho C_{\mu\nu,\rho} + 2\,\partial_{[\mu} C_{\nu]\rho}{}^{\rho} + \varepsilon_{\mu\nu\rho\sigma\tau}\partial^\rho u^{\sigma\tau} - 3\,\partial_y\left(u_{\mu\nu} - \frac{3}{2}\,B_{\mu\nu}\right) = q_{\mu\nu}\ . \tag{3.59}$$

We can compute the curl of this equation

$$\begin{aligned} -\frac{1}{6}\,\varepsilon^{\mu\nu\alpha\beta\gamma}\partial_\alpha q_{\mu\nu} = &-\frac{1}{6}\,\varepsilon^{\mu\nu\alpha\beta\gamma}\partial_\alpha\partial^\rho C_{\mu\nu,\rho} + 2\,\partial_\alpha\partial^{[\alpha}u^{\beta\gamma]} \\ &+\frac{1}{2}\,\partial_y\varepsilon^{\mu\nu\alpha\beta\gamma}\partial_\alpha\left(u_{\mu\nu} - \frac{3}{2}\,B_{\mu\nu}\right)\ , \end{aligned} \tag{3.60}$$

and combining it with (3.57) and (3.47), we eventually get

$$2\,\partial^\mu r_{\mu[\nu,\rho]} = -\frac{1}{6}\,\varepsilon_{\nu\rho\mu\alpha\beta}\partial^\mu q^{\alpha\beta}\ . \tag{3.61}$$

Using the other condition (3.56) we found for the divergence of r, we have

$$\partial^\mu(2\,r_{\mu\nu,\rho} + \frac{1}{6}\varepsilon_{\nu\rho\mu\alpha\beta}q^{\alpha\beta}) = 0\ . \tag{3.62}$$

Hence, there exists locally a function $c_{\mu\nu,\rho}$ of (2,1) Young symmetry and a totally antisymmetric function $a_{\mu\nu\rho}$, both independent of y, such that

$$2\,r_{\mu\nu,\rho} + \frac{1}{6}\varepsilon_{\mu\nu\rho\alpha\beta}q^{\alpha\beta} = \frac{1}{2}\,\varepsilon_{\mu\nu\alpha\beta\gamma}\partial^\alpha(c^{\beta\gamma}{}_\rho + a^{\beta\gamma}{}_\rho)\ . \tag{3.63}$$

Consequently

$$r_{\mu\nu}{}^{\mu} = \frac{1}{4}\,\varepsilon_{\mu\nu\alpha\beta\gamma}\partial^\alpha a^{\beta\gamma\mu}\ , \tag{3.64}$$

and

$$r_{\mu\nu,\rho} = \frac{1}{4}\,\varepsilon_{\mu\nu\alpha\beta\gamma}\partial^\alpha(c^{\beta\gamma}{}_\rho + a^{\beta\gamma}{}_\rho) - \frac{1}{4}\,\varepsilon_{\alpha\beta\gamma[\mu\nu}\partial^\alpha(c^{\beta\gamma}{}_{\rho]} + a^{\beta\gamma}{}_{\rho]})\ . \tag{3.65}$$

Assembling the expression for $r_{\mu\nu,\rho}$, the expression for its trace we get

$$\begin{aligned} &2\,\partial_{[\mu}h_{\nu]\rho} + \frac{1}{2}\,\varepsilon_{\mu\nu\lambda\sigma\tau}\partial^\lambda C^{\sigma\tau}{}_\rho - \frac{1}{2}\,\varepsilon_{\lambda\sigma\tau[\mu\nu}\partial^\lambda C^{\sigma\tau}{}_{\rho]} \\ &+\partial_y C_{\mu\nu,\rho} - 2\,\partial_y A_{[\mu}\eta_{\nu]\rho} - 2\,\partial_\rho u_{\mu\nu} + 2\,\partial_{[\rho}u_{\mu\nu]} \\ &\qquad = \frac{1}{2}\,\varepsilon_{\mu\nu\alpha\beta\gamma}\partial^\alpha(c^{\beta\gamma}{}_\rho + a^{\beta\gamma}{}_\rho) - \frac{1}{2}\,\varepsilon_{\alpha\beta\gamma[\mu\nu}\partial^\alpha(c^{\beta\gamma}{}_{\rho]} + a^{\beta\gamma}{}_{\rho]}) \\ &\qquad\quad + \frac{1}{3}\,\varepsilon_{\alpha\beta\gamma\sigma[\mu}\partial^\alpha a^{\beta\gamma\sigma}\eta_{\nu]\rho}, \end{aligned} \tag{3.66}$$

To help us transform some expressions, we need the following two Schouten identities[1]

$$0 = \varepsilon_{[\mu\nu}{}^{\alpha\beta\gamma}\partial_\alpha a_{\beta\gamma\rho]} = -\varepsilon^{\alpha\beta\gamma}{}_{[\mu\nu}\partial_{\rho]}a_{\alpha\beta\gamma} + 3\,\partial_\alpha\varepsilon^{\alpha\beta\gamma}{}_{[\mu\nu}a_{\rho]\beta\gamma}\ , \tag{3.67}$$

$$\begin{aligned} 0 &= \varepsilon_{[\sigma\mu\alpha\beta\gamma}\partial^\alpha a^{\beta\gamma\sigma}\eta_{\nu]\rho} \\ &= 2\,\varepsilon_{\alpha\beta\gamma\sigma[\mu}\partial^\alpha a^{\beta\gamma\sigma}\eta_{\nu]\rho} - \partial_\rho\varepsilon_{\beta\gamma\sigma\mu\nu}a^{\beta\gamma\sigma} + 3\,\varepsilon_{\alpha\beta\gamma\mu\nu}\partial^\alpha a^{\beta\gamma}{}_\rho\ . \end{aligned} \tag{3.68}$$

[1] A Schouten identity is a non trivial identity that is the result of the antisymmetrization of an expression on more indices than there are space-time dimensions, which is trivially zero.

Using these identities, and the dualization of (3.59) which removes the totally antisymmetric part of (3.66), it follows

$$0 = 2\,\partial_{[\mu}h_{\nu]\rho} + \frac{1}{2}\,\varepsilon_{\mu\nu\lambda\sigma\tau}\partial^{\lambda}(C^{\sigma\tau}{}_{\rho} - c^{\sigma\tau}{}_{\rho}) + \left(H_{\mu\nu\rho} - \frac{1}{2}\,\varepsilon_{\mu\nu\rho\alpha\beta}F^{\alpha\beta}\right) + \partial_y(C_{\mu\nu,\rho} - 2\,A_{[\mu}\eta_{\nu]\rho}) - 2\,\partial_{\rho}\left(u_{\mu\nu} + \frac{1}{12}\,\varepsilon_{\mu\nu\alpha\beta\gamma}a^{\alpha\beta\gamma}\right)\,, \tag{3.69}$$

and we recover equation (2.53) after the following redefinitions:

$$u_{\mu\nu} \to u_{\mu\nu} + \frac{1}{12}\,\varepsilon_{\mu\nu\alpha\beta\gamma}a^{\alpha\beta\gamma}\,, \tag{3.70}$$

$$B_{\mu\nu} \to B_{\mu\nu} - b_{\mu\nu}\,, \tag{3.71}$$

$$C_{\mu\nu,\rho} \to C_{\mu\nu,\rho} - c_{\mu\nu,\rho}\,. \tag{3.72}$$

We have thus derived the first order duality equations (2.52) and (2.53) from the Lagrangian (3.36).

3.3 The $\mathcal{N} = (4,0)$ model

3.3.1 Main result

The $\mathcal{N} = (4,0)$ features 42 scalars, 48 Dirac fields, 28 self-dual two-forms, 8 exotic gravitini and one (2,2) Young symmetry tensor as listed in table 1.3. For almost all of the fields listed, their dynamics has been described by a Lagrangian above. The only new field for which we have to build a Lagrangian is tha six-dimensional (2,2) Young symmetry field $T_{\hat{\mu}\hat{\nu},\hat{\rho}\hat{\sigma}}$. After the split (2.41), the dynamics of this field is given by equations (2.52) and (2.53). As a main result of this section, we shall show that these equations can be derived from the Lagrangian in the $\mathcal{N} = (4,0)$ model, can be derived from the Lagrangian

$$\begin{aligned}\mathcal{L}_{\boxplus} = &-\frac{1}{4}\,\widehat{\Omega}^{\mu\nu\rho}\widehat{\Omega}_{\mu\nu\rho} + \frac{1}{2}\,\widehat{\Omega}^{\mu\nu\rho}\widehat{\Omega}_{\nu\rho\mu} + \widehat{\Omega}^{\mu}\widehat{\Omega}_{\mu} - \frac{1}{8}\,\varepsilon_{\mu\nu\sigma\kappa\lambda}\,\partial^{\mu}\widehat{C}^{\nu\sigma}{}_{\rho}\,\partial_y\widehat{C}^{\kappa\lambda,\rho} \\ &+ \frac{1}{32}\,\varepsilon_{\mu\nu\sigma\kappa\lambda}\,\partial^{\mu}\mathcal{C}^{\nu\sigma}{}_{\rho}\,\partial_y\mathcal{C}^{\kappa\lambda,\rho} - \frac{1}{8}\,\partial_y\mathcal{C}_{\sigma\tau,\nu}\,\partial_{\mu}T^{\mu\nu,\sigma\tau} + \frac{1}{4}\,\partial_y\mathcal{C}_{\kappa\lambda,\tau}\,\partial^{\kappa}T^{\lambda\sigma,\tau}{}_{\sigma} \\ &+ \frac{1}{4}\,\partial_{\nu}\mathcal{C}_{\sigma\mu}{}^{\mu}\,\partial_y T^{\sigma\tau,\nu}{}_{\tau} - \frac{1}{8}\,\partial_y\mathcal{C}_{\sigma\mu}{}^{\mu}\,\partial^{\sigma}T_{\tau\nu}{}^{\tau\nu} - \frac{1}{64}\,\varepsilon_{\mu\nu\alpha\beta\gamma}\,\partial^{\alpha}T_{\sigma\tau}{}^{\beta\gamma}\,\partial_y T^{\mu\nu,\sigma\tau} \\ &- \frac{1}{32}\,\partial_y T_{\sigma\tau,\mu\nu}\,\partial_y T^{\mu\nu,\sigma\tau} + \frac{1}{8}\,\partial_y T_{\sigma\mu,\nu}{}^{\mu}\,\partial_y T^{\sigma\tau,\nu}{}_{\tau} - \frac{1}{32}\,\partial_y T_{\mu\nu}{}^{\mu\nu}\,\partial_y T_{\sigma\tau}{}^{\sigma\tau}\,,\end{aligned} \tag{3.73}$$

with

$$\begin{aligned}\widehat{\Omega}_{\mu\nu\rho} &= \partial_{[\mu}h_{\nu]\rho} + \partial_y\widehat{C}_{\mu\nu,\rho} - \frac{1}{2}\partial_y\mathcal{C}_{\mu\nu,\rho}\,, \\ \widehat{C}_{\mu\nu,\rho} &= C_{\mu\nu,\rho} + \varepsilon_{\mu\nu\rho\sigma\tau}\,u^{\sigma\tau}\,, \\ \mathcal{C}_{\mu\nu,\rho} &= C_{\mu\nu,\rho} - v_{\rho,\mu\nu} + 2\,v_{[\rho,\mu\nu]} + 2\,\varepsilon_{\mu\nu\rho\sigma\tau}\,u^{\sigma\tau}\,.\end{aligned} \tag{3.74}$$

Similarly to the (2,1) Yound symmetry field, we have thus found a Lagrangian describing the dynamics of a self-dual (2,2) Young symmetry field at the price of sacrificing manifest Lorentz invariance. After reduction to five dimensions (i.e. when $\partial_y \to 0$), this Lagrangian reduces to the Fierz-Pauli Lagrangian for $h_{\mu\nu}$ (3.27); the dual fields $\widehat{C}_{\mu\nu,\rho}$, $\mathcal{C}_{\mu\nu,\rho}$, and $T_{\mu\nu,\rho\sigma}$ drop out in this limit. In presence of the sixth dimension, variation of the Lagrangian (3.73) with respect to the dual fields yields the first-order duality equations (2.65) and (2.67), however under an overall derivative ∂_y. In a similar as for the $\mathcal{N} = (3,1)$ model, a rather lengthy computation allows to establish the equivalence with (2.65) and (2.67). The details of that computation are give in the next subsection.

Let us spell out the gauge transformations (2.70), (2.72) in terms of the fields (3.74)

$$\delta\widehat{\Omega}_{\mu\nu\rho} = \partial_\rho\partial_{[\mu}\xi_{\nu]} - \frac{2}{3}\,\partial_y\partial_{[\mu}\beta_{\nu]\rho} - \partial_y\partial_{[\mu}\zeta_{\nu]\rho} + \frac{1}{4}\,\partial_y\partial^\kappa\lambda_{[\mu}{}^{\sigma\tau}\,\varepsilon_{\nu]\kappa\sigma\tau\rho}\,, \tag{3.75}$$

$$\begin{aligned}\delta\widehat{C}_{\mu\nu,\rho} &= 2\,\partial_{[\mu}\alpha_{\nu]\rho} - 2\,\partial_{[\mu}\beta_{\nu]\rho} + \varepsilon_{\mu\nu\rho\sigma\tau}\,\partial^\sigma\xi^\tau + \frac{1}{3}\,\varepsilon_{\mu\nu\rho\sigma\tau}\partial_y\Big(\beta^{\sigma\tau} + \frac{3}{2}\,\zeta^{\sigma\tau}\Big)\\ &\quad - \frac{1}{2}\partial_y\lambda_{\rho,\mu\nu}\,,\end{aligned} \tag{3.76}$$

$$\begin{aligned}\delta\mathcal{C}_{\mu\nu,\rho} &= 2\,\varepsilon_{\mu\nu\rho\sigma\tau}\,\partial^\sigma\xi^\tau - \frac{8}{3}\,\partial_{[\mu}\beta_{\nu]\rho} + 2\,\partial_{[\mu}\zeta_{\nu]\rho} + \frac{2}{3}\,\varepsilon_{\mu\nu\rho\sigma\tau}\partial_y\Big(\beta^{\sigma\tau} + \frac{3}{2}\,\zeta^{\sigma\tau}\Big)\\ &\quad + \frac{1}{2}\,\varepsilon_{\kappa\sigma\tau\rho[\mu}\,\partial^\kappa\lambda_{\nu]}{}^{\sigma\tau} - \partial_y\lambda_{\rho,\mu\nu}\,,\end{aligned} \tag{3.77}$$

which allows to confirm the gauge invariance of the Lagrangian (3.73).

The Lagrangian for the full $\mathcal{N} = (4,0)$ model is finally given by combining (3.73) with the Lagrangians for the remaining fields of the theory. Putting everything together, we obtain

$$\mathcal{L}_{(4,0)} = \mathcal{L}_{\bullet} + \mathcal{L}_{\square} + \mathcal{L}_{\square\!\square} + \mathcal{L}_{\bullet_F} + \mathcal{L}_{\square_F}\,. \tag{3.78}$$

This is one of the main results of this chapter.

3.3.2 Derivation of the dynamics of $T_{\mu\nu,\rho\sigma}$

In this subsection, we shall show that the Lagrangian (3.73) describes the dynamics of a self-dual (2,2) dual given by the equations (2.65) and (2.67). First the variation of (3.73) with respect to $h_{\mu\nu}$ exactly gives the Einstein equation (2.68)

$$\begin{aligned}G_{\mu\nu} &= -\frac{1}{2}\,\partial_y\partial^\rho C_{\rho(\mu,\nu)} - \frac{1}{2}\,\partial_y\partial_{(\mu}C_{\nu)\rho}{}^\rho + \frac{1}{2}\,\partial_y\partial_\rho v_{(\mu,\nu)}{}^\rho - \frac{1}{2}\,\partial_y\partial_{(\mu}v^\rho{}_{\nu)\rho}\\ &\quad + \frac{1}{2}\,\eta_{\mu\nu}\,\partial_y\partial^\rho C_{\rho\sigma}{}^\sigma - \frac{1}{2}\,\eta_{\mu\nu}\,\partial_y\partial_\rho v_\sigma{}^{\sigma\rho}\,.\end{aligned} \tag{3.79}$$

Let us now look at the variation of (3.73) with respect to $\widehat{C}_{\mu\nu,\rho}$. Since this a reducible field, we get two new equations: one for the (2,1) part

$$\begin{aligned}0 = \partial_y \Big(& \partial_{[\mu} h_{\nu]\rho} + \partial^\alpha h_{\alpha[\mu}\eta_{\nu]\rho} - \eta_{\rho[\nu}\partial_{\mu]} h_\alpha{}^\alpha \\ & + \frac{1}{4}\varepsilon_{\mu\nu\alpha\beta\gamma}\partial^\alpha C^{\beta\gamma}{}_\rho - \frac{1}{4}\varepsilon_{\alpha\beta\gamma[\mu\nu}\partial^\alpha C^{\beta\gamma}{}_{\rho]} \\ & - \partial_\rho u_{\mu\nu} + \partial_{[\rho} u_{\mu\nu]} - 2\,\partial^\alpha u_{\alpha[\mu}\eta_{\nu]\rho} \\ & + \frac{1}{2}\,\partial_y (C_{\mu\nu,\rho} + v_{\rho,\mu\nu} - v_{[\rho,\mu\nu]}) - \partial_y \eta_{\rho[\nu}(C_{\mu]\alpha}{}^\alpha + v^\alpha{}_{\mu]\alpha}) \Big) \ , \end{aligned} \tag{3.80}$$

and one for the antisymmetric part

$$0 = \partial_y \left(\frac{1}{4}\varepsilon_{\alpha\beta\gamma[\mu\nu}\partial^\alpha C^{\beta\gamma}{}_{\rho]} - \partial_{[\rho} u_{\mu\nu]} + \frac{1}{2}\,\partial_y v_{[\rho,\mu\nu]} \right) . \tag{3.81}$$

We will also need the antisymmetric part of the variation of (3.73) with respect to $\mathcal{C}_{\mu\nu,\rho}$

$$0 = \partial_y \left(\frac{1}{8}\varepsilon_{\alpha\beta\gamma[\mu\nu}\partial^\alpha C^{\beta\gamma}{}_{\rho]} + \frac{1}{4}\,\varepsilon_{\alpha\beta\gamma[\mu\nu}\partial^\alpha v_{\rho]}{}^{\beta\gamma} - \partial_{[\rho} u_{\mu\nu]} + \frac{1}{2}\,\partial_y v_{[\rho,\mu\nu]} \right) . \tag{3.82}$$

These three equations can be integrated with respect to y after the introduction of the (2,1) function $r_{\mu\nu\rho}$ and the two antisymmetric functions $q_{\mu\nu\rho}$ and $\tilde{q}_{\mu\nu\rho}$, all independent of y such that

$$\begin{aligned} r_{\mu\nu\rho} = & \partial_{[\mu} h_{\nu]\rho} + \partial^\alpha h_{\alpha[\mu}\eta_{\nu]\rho} - \eta_{\rho[\nu}\partial_{\mu]} h_\alpha{}^\alpha \\ & + \frac{1}{4}\varepsilon_{\mu\nu\alpha\beta\gamma}\partial^\alpha C^{\beta\gamma}{}_\rho - \frac{1}{4}\varepsilon_{\alpha\beta\gamma[\mu\nu}\partial^\alpha C^{\beta\gamma}{}_{\rho]} \\ & - \partial_\rho u_{\mu\nu} + \partial_{[\rho} u_{\mu\nu]} - 2\,\partial^\alpha u_{\alpha[\mu}\eta_{\nu]\rho} \\ & + \frac{1}{2}\,\partial_y (C_{\mu\nu,\rho} + v_{\rho,\mu\nu} - v_{[\rho,\mu\nu]}) - \partial_y \eta_{\rho[\nu}(C_{\mu]\alpha}{}^\alpha + v^\alpha{}_{\mu]\alpha}) \ , \end{aligned} \tag{3.83}$$

$$q_{\mu\nu\rho} = \frac{1}{4}\varepsilon_{\alpha\beta\gamma[\mu\nu}\partial^\alpha C^{\beta\gamma}{}_{\rho]} - \partial_{[\rho} u_{\mu\nu]} + \frac{1}{2}\,\partial_y v_{[\rho,\mu\nu]} \ , \tag{3.84}$$

and

$$\tilde{q}_{\mu\nu\rho} = \frac{1}{8}\varepsilon_{\alpha\beta\gamma[\mu\nu}\partial^\alpha C^{\beta\gamma}{}_{\rho]} + \frac{1}{4}\,\varepsilon_{\alpha\beta\gamma[\mu\nu}\partial^\alpha v_{\rho]}{}^{\beta\gamma} - \partial_{[\rho} u_{\mu\nu]} + \frac{1}{2}\,\partial_y v_{[\rho,\mu\nu]} \ . \tag{3.85}$$

Using (3.79), we can show that

$$\partial^\mu (r_{\mu\nu\rho} + q_{\mu\nu\rho}) = \frac{1}{2}\,\partial_y (\varepsilon_{\nu\rho\alpha\beta\gamma}(q^{\alpha\beta\gamma} - \tilde{q}^{\alpha\beta\gamma})) = 0 \ . \tag{3.86}$$

Hence, we can define locally a totally antisymmetric $a_{\mu\nu\rho}$ and a $c_{\mu\nu,\rho}$ which is (2,1) such that of (2,1) type, such that

$$r_{\mu\nu\rho} + q_{\mu\nu\rho} = \frac{1}{4}\,\varepsilon_{\mu\nu\alpha\beta\gamma}\partial^\alpha (a^{\beta\gamma}{}_\rho + c^{\beta\gamma}{}_\rho) \ . \tag{3.87}$$

In analogous fashion to the case of the $\mathcal{N} = (3, 1)$ model, namely using the same Schouten identities, we finally have

$$\begin{aligned}\partial_{[\mu}h_{\nu]\rho}&+\frac{1}{4}\,\varepsilon_{\mu\nu\alpha\beta\gamma}\partial^{\alpha}C^{\beta\gamma}{}_{\rho}-\partial_{\rho}u_{\mu\nu}+\frac{1}{2}\,\partial_y(C_{\mu\nu,\rho}+v_{\mu\nu,\rho})\\&=r_{\mu\nu\rho}+q_{\mu\nu\rho}+\frac{2}{3}\,r_{\alpha[\mu}{}^{\alpha}\eta_{\nu]\rho}\\&=\frac{1}{4}\,\varepsilon_{\mu\nu\alpha\beta\gamma}\partial^{\alpha}c^{\beta\gamma}{}_{\rho}+\frac{1}{12}\,\partial_{\rho}\varepsilon_{\mu\nu\alpha\beta\gamma}a^{\alpha\beta\gamma}.\end{aligned}\tag{3.88}$$

After the redefinition

$$C_{\mu\nu,\rho}\to C_{\mu\nu,\rho}-c_{\mu\nu,\rho}\ ,\tag{3.89}$$

$$u_{\mu\nu}\to u_{\mu\nu}+\frac{1}{12}\,\varepsilon_{\mu\nu\alpha\beta\gamma}a^{\alpha\beta\gamma}\ .\tag{3.90}$$

we recover the six-dimensional equation (2.67)

$$\partial_{[\mu}h_{\nu]\rho}+\frac{1}{4}\,\varepsilon_{\mu\nu\alpha\beta\gamma}\partial^{\alpha}C^{\beta\gamma}{}_{\rho}-\partial_{\rho}u_{\mu\nu}+\frac{1}{2}\,\partial_y(C_{\mu\nu,\rho}+v_{\mu\nu,\rho})=0\ .\tag{3.91}$$

Contracting this equation with more derivatives, it gives rise to the (modified) Curtright equation

$$3\,\partial_{\tau}\partial^{[\tau}C^{\mu\nu]}{}_{\rho}-3\,\partial_{\rho}\partial^{[\tau}C^{\mu\nu]}{}_{\tau}=\frac{1}{2}\,\varepsilon^{\mu\nu\tau\kappa\lambda}\,\partial_y\left(\partial_{\tau}C_{\kappa\lambda,\rho}+\partial_{\tau}v_{\rho,\kappa\lambda}-\partial_{\rho}v_{\tau,\kappa\lambda}\right)\ .\tag{3.92}$$

It remains to study the other duality equation (2.65) which follows from the Lagrangian by using the last two equations we still have not used, that is the (2,1) part of the variation of (3.73) with respect to $\mathcal{C}_{\mu\nu,\rho}$

$$\begin{aligned}0=\partial_y\Big(&\partial_{[\mu}h_{\nu]\rho}+\partial^{\alpha}h_{\alpha[\mu}\eta_{\nu]\rho}-\eta_{\rho[\nu}\partial_{\mu]}h_{\alpha}{}^{\alpha}\\&+\frac{1}{8}\varepsilon_{\mu\nu\alpha\beta\gamma}\partial^{\alpha}C^{\beta\gamma}{}_{\rho}-\frac{1}{8}\varepsilon_{\alpha\beta\gamma[\mu\nu}\partial^{\alpha}C^{\beta\gamma}{}_{\rho]}\\&-\partial_{\rho}u_{\mu\nu}+\partial_{[\rho}u_{\mu\nu]}-2\,\partial^{\alpha}u_{\alpha[\mu}\eta_{\nu]\rho}\\&+\frac{1}{2}\,\partial_y(C_{\mu\nu,\rho}+v_{\rho,\mu\nu}-v_{[\rho,\mu\nu]})-\partial_y\eta_{\rho[\nu}(C_{\mu]\alpha}{}^{\alpha}+v^{\alpha}{}_{\mu]\alpha})\\&+\frac{1}{4}\,\varepsilon_{\mu\nu\alpha\beta\gamma}\partial^{\alpha}v_{\rho}{}^{\beta\gamma}-\frac{1}{4}\varepsilon_{\alpha\beta\gamma[\mu\nu}\partial^{\alpha}v_{\rho]}{}^{\beta\gamma}\\&-\frac{1}{4}\,\partial^{\alpha}T_{\alpha\rho,\mu\nu}+\frac{1}{2}\partial_{[\mu}T_{\nu]\alpha,\rho}{}^{\alpha}+\frac{1}{2}\,\eta_{\rho[\nu}\partial^{\alpha}T_{\mu]\beta,\alpha}{}^{\beta}-\frac{1}{4}\,\eta_{\rho[\nu}\partial_{\mu]}T_{\alpha\beta}{}^{\alpha\beta}\Big)\\=&\,\mathcal{P}_{\mu\nu,\rho}\ ,\end{aligned}\tag{3.93}$$

and the variation of (3.73) with respect to the irreducible field $T_{\mu\nu,\rho\sigma}$

$$\begin{aligned}
0 =& \partial_y \Big(\frac{1}{2}\,\varepsilon_{\alpha\beta\gamma\mu\nu}\partial^\alpha T^{\beta\gamma}{}_{\rho\sigma} + \frac{1}{2}\,\varepsilon_{\alpha\beta\gamma\rho\sigma}\partial^\alpha T^{\beta\gamma}{}_{\mu\nu} - \varepsilon_{\alpha\beta\gamma[\mu\nu}\partial^\alpha T^{\beta\gamma}{}_{\rho\sigma]} \\
&+ 2\,\partial_{(\mu}(C_{|\rho\sigma|,\nu)} - v_{\nu),\rho\sigma}) + 2\,\partial_{(\rho}(C_{|\mu\nu|,\sigma)} - v_{\sigma),\mu\nu}) \\
&- 2\,\partial^\alpha(C_{\alpha\rho,(\mu} - v_{(\mu,|\alpha\rho|})\eta_{\nu)\sigma} - 2\,\partial^\alpha(C_{\alpha\sigma,(\mu} - v_{(\mu,|\alpha\sigma|})\eta_{\nu)\rho} \\
&- 2\,\partial^\alpha(C_{\alpha\mu,(\rho} - v_{(\rho,|\alpha\mu|})\eta_{\sigma)\nu} - 2\,\partial^\alpha(C_{\alpha\nu,(\rho} - v_{(\rho,|\alpha\nu|})\eta_{\sigma)\mu} \\
&- 2\,\eta_{\sigma(\mu}\partial_{\nu)}(C_{\rho\alpha}{}^\alpha - v^\alpha{}_{\rho\alpha}) - 2\,\eta_{\rho(\mu}\partial_{\nu)}(C_{\sigma\alpha}{}^\alpha - v^\alpha{}_{\sigma\alpha}) \\
&- 2\,\eta_{\mu(\rho}\partial_{\sigma)}(C_{\nu\alpha}{}^\alpha - v^\alpha{}_{\nu\alpha}) - 2\,\eta_{\nu(\rho}\partial_{\sigma)}(C_{\mu\alpha}{}^\alpha - v^\alpha{}_{\mu\alpha}) \\
&+ \partial_y \Big(2\,T_{\mu\nu,\rho\sigma} - 4\,\eta_{\sigma(\nu}T_{\mu)\alpha,\rho}{}^\alpha - 4\,\eta_{\rho(\nu}T_{\mu)\alpha,\sigma}{}^\alpha + 2\,\eta_{\mu(\rho}\eta_{\sigma)\nu}T_{\alpha\beta}{}^{\alpha\beta}\Big) \\
&+ 4\,\eta_{\mu(\rho}\eta_{\sigma)\nu}\partial^\alpha(C_{\alpha\beta}{}^\beta - v^\beta{}_{\alpha\beta})\Big) \\
=& \mathcal{Q}_{\mu\nu,\rho\sigma}\ .
\end{aligned} \tag{3.94}$$

Then,

$$\begin{aligned}
0 =& \frac{1}{2}\Big(\mathcal{Q}_{\mu\nu,\rho\sigma} - \eta_{\mu[\rho}\mathcal{Q}^\alpha{}_{\sigma],\alpha\nu} + \eta_{\nu[\rho}\mathcal{Q}^\alpha{}_{\sigma],\alpha\mu} + \frac{1}{6}\Big(\eta_{\mu[\rho}\eta_{\sigma]\nu} - \eta_{\nu[\rho}\eta_{\sigma]\mu}\Big)\mathcal{Q}^{\alpha\beta}{}_{,\alpha\beta}\Big) \\
&- 2\,\varepsilon_{\alpha\beta\mu\nu[\rho}\mathcal{P}^{\alpha\beta}{}_{\sigma]} \\
=& \partial_y \mathcal{E}_{\mu\nu,\rho\sigma}\ ,
\end{aligned} \tag{3.95}$$

where

$$\mathcal{E}_{\mu\nu,\rho\sigma} \equiv \frac{1}{2}\,\varepsilon_{\alpha\beta\gamma\mu\nu}\partial^\alpha T^{\beta\gamma}{}_{\rho\sigma} + 2\,\partial_{[\mu}C_{|\rho\sigma|,\nu]} - 2\,\partial_{[\rho}v_{\sigma],\mu\nu} + \partial_y T_{\mu\nu,\rho\sigma}\ , \tag{3.96}$$

which is the equation of motion (2.65). The goal of the following is to show that we can remove the ∂_y in (3.95). Equation (3.95) can be integrated to

$$\mathcal{E}_{\mu\nu,\rho\sigma} = \Omega_{\mu\nu,\rho\sigma}(x)\ , \tag{3.97}$$

where $\Omega_{\mu\nu,\rho\sigma}$ does not depend on y. For later use, we parametrize this field as

$$\Omega^{\mu\nu}{}_{\rho\sigma} = \widehat{\Omega}^{\mu\nu}{}_{\rho\sigma} + \frac{4}{3}\,\delta^{[\mu}{}_{[\rho}\,\Omega^{\nu]}{}_{\sigma]}\ , \tag{3.98}$$

with traceless $\widehat{\Omega}^{\mu\nu}{}_{\rho\sigma}$ and the trace part given by

$$\Omega^{\mu\nu}{}_{\rho\nu} = \Omega^\mu{}_\rho + \frac{1}{3}\,\delta^\mu{}_\rho\,\Omega^\nu{}_\nu\ , \tag{3.99}$$

Eventually, we need to show that

$$\mathcal{E}_{\mu\nu,\rho\sigma} = 0\ , \tag{3.100}$$

(after potential re-definition of $T_{\mu\nu,\rho\sigma}$ and $v_{\rho,\mu\nu}$), which is a genuine field equation of the six-dimensional $\mathcal{N} = (4,0)$ theory. To begin with, using the field equations that are already established, we can show that the particular combination of derivatives

$$\begin{aligned} 3\,\partial_{[\mu}\Omega^{\mu\nu}{}_{\rho\sigma]} &= 3\,\partial^{\mu}\partial_{[\mu}C_{\rho\sigma]}{}^{\nu} - 3\,\partial^{\nu}\partial_{[\mu}C_{\rho\sigma]}{}^{\mu} - 3\,\partial_y\partial_{[\mu}T_{\rho\sigma]}{}^{\nu\mu} \\ &\overset{(3.95)}{=} 3\,\partial^{\mu}\partial_{[\mu}C_{\rho\sigma]}{}^{\nu} - 3\,\partial^{\nu}\partial_{[\mu}C_{\rho\sigma]}{}^{\mu} \\ &\qquad - \frac{1}{2}\,\varepsilon^{\mu\kappa\lambda\rho\sigma}\,\partial_y\left(\partial_{\mu}C_{\kappa\lambda}{}^{\nu} + \partial_{\lambda}v^{\nu}{}_{\mu\kappa} - \partial^{\nu}v_{\lambda,\mu\kappa}\right) \\ &\overset{(3.92)}{=} 0\,, \end{aligned} \tag{3.101}$$

vanishes. In particular, contraction of this equation shows that

$$2\,\partial_{\mu}\Omega^{\mu\nu}{}_{\rho\nu} - \partial_{\rho}\Omega^{\mu\nu}{}_{\mu\nu} = 0 \quad \Longrightarrow \quad \partial_{\mu}\left(\Omega^{\mu}{}_{\rho} - \delta^{\mu}{}_{\rho}\,\Omega^{\nu}{}_{\nu}\right) = 0\,. \tag{3.102}$$

Which can be integrated to

$$\Omega^{\mu}{}_{\rho} - \delta^{\mu}{}_{\rho}\,\Omega^{\nu}{}_{\nu} = \partial_{\lambda}\omega^{\lambda\mu}{}_{\rho} \qquad \Longleftrightarrow \qquad \Omega^{\mu}{}_{\rho} = \partial_{\lambda}\omega^{\lambda\mu}{}_{\rho} - \frac{1}{4}\,\delta^{\mu}{}_{\rho}\,\partial_{\lambda}\omega^{\lambda\kappa}{}_{\kappa}\,. \tag{3.103}$$

Next, consider another derivative acting on (3.101), giving rise to the second order differential equation

$$0 = \partial_{\mu}\partial_{[\tau}\Omega^{\mu\nu}{}_{\rho\sigma]}\,. \tag{3.104}$$

Using the parametrization (3.98) and the relation (3.102), this equation reduces to

$$0 = \partial_{\mu}\partial_{[\tau}\widehat{\Omega}^{\mu\nu}{}_{\rho\sigma]}\,, \tag{3.105}$$

for the traceless part of Ω. Dualizing the first two indices on $\widehat{\Omega}$, defines the object

$$\widehat{\Omega}_{\alpha\beta\gamma,\rho\sigma} \equiv \frac{1}{2}\,\varepsilon_{\alpha\beta\gamma\mu\nu}\,\widehat{\Omega}^{\mu\nu}{}_{\rho\sigma}\,, \tag{3.106}$$

corresponding to an irreducible Young tableau $\begin{array}{|c|c|}\hline \multicolumn{1}{|c}{} & \multicolumn{1}{c|}{} \\ \hline \multicolumn{1}{|c}{} & \multicolumn{1}{c|}{} \\ \hline \multicolumn{1}{|c}{} \\ \cline{1-1} \end{array}$. Using the generalized Poincaré Lemma [42](see Appendix A.3) for this object then allows to integrate up equation (3.105) into

$$\widehat{\Omega}^{\mu\nu}{}_{\rho\sigma} = \frac{1}{2}\,\varepsilon^{\mu\nu\alpha\beta\gamma}\,\partial_{\alpha}t_{\beta\gamma,\rho\sigma} + 2\,\partial_{[\rho}s^{\mu\nu}{}_{\sigma]} + \frac{4}{3}\,\delta_{[\rho|}{}^{[\mu}\,\partial_{\kappa}s^{\nu]\kappa}{}_{|\sigma]}\,, \tag{3.107}$$

in terms of two gauge parameters $t_{\mu\nu,\rho\sigma}$ and $s_{\mu\nu,\rho}$. The first one corresponds to an irreducible Young tableau $\begin{array}{|c|c|}\hline \multicolumn{1}{|c}{} & \multicolumn{1}{c|}{} \\ \hline \multicolumn{1}{|c}{} & \multicolumn{1}{c|}{} \\ \hline \end{array}$, while the parameter $s_{\mu\nu,\rho}$, antisymmetric in its first two indices and traceless $s_{\mu\nu}{}^{\nu} = 0$, — such that the dual $\hat{s}_{\alpha\beta\gamma,\rho} = \frac{1}{2}\,\varepsilon_{\alpha\beta\gamma\mu\nu}\,s^{\mu\nu}{}_{\rho}$ is an irreducible Young tableau $\begin{array}{|c|c|}\hline \multicolumn{1}{|c}{} & \multicolumn{1}{c|}{} \\ \hline \multicolumn{1}{|c}{} \\ \cline{1-1} \multicolumn{1}{|c}{} \\ \cline{1-1} \end{array}$. The last term in (3.107) implements the projection onto the traceless part.

Putting together (3.107) and (3.103), we arrive at

$$\begin{aligned}\Omega^{\mu\nu}{}_{\rho\sigma} = &\frac{1}{2}\,\varepsilon^{\mu\nu\alpha\beta\gamma}\,\partial_\alpha t_{\beta\gamma,\rho\sigma} + 2\,\partial_{[\rho}s^{\mu\nu}{}_{\sigma]} + \frac{4}{3}\,\delta_{[\rho|}{}^{[\mu}\,\partial_\kappa s^{\nu]\kappa}{}_{|\sigma]}\\ &- \frac{4}{3}\,\delta_{[\rho|}{}^{[\mu}\,\partial_\kappa\omega^{\nu]\kappa}{}_{|\sigma]} - \frac{1}{3}\,\delta^{\mu\nu}{}_{\rho\sigma}\,\partial_\lambda\omega^{\lambda\kappa}{}_{\kappa}\,.\end{aligned} \tag{3.108}$$

We can plug this back into (3.101) to arrive at

$$0 = 3\,\partial_{[\mu}\Omega^{\mu\nu}{}_{\rho\sigma]} = \frac{4}{3}\,\partial_\mu\partial_{[\rho}s^{\mu\nu}{}_{\sigma]} - \frac{4}{3}\,\partial_\mu\partial_{[\rho}\omega^{\mu\nu}{}_{\sigma]} + \frac{1}{3}\,\delta_{[\sigma}{}^{\nu}\,\partial_{\rho]}\partial_\lambda\omega^{\lambda\kappa}{}_{\kappa}\,, \tag{3.109}$$

which in turn is a curl in $[\rho\sigma]$ can be integrated into

$$\partial_\mu s^{\mu\nu}{}_{\sigma} - \partial_\mu\omega^{\mu\nu}{}_{\sigma} + \frac{1}{4}\,\delta_\sigma{}^{\nu}\,\partial_\lambda\omega^{\lambda\kappa}{}_{\kappa} = \partial_\sigma\xi^\nu\,. \tag{3.110}$$

This further reduces the result (3.108) and allows to put it into the form

$$\Omega^{\mu\nu}{}_{\rho\sigma} = \frac{1}{2}\,\varepsilon^{\mu\nu\alpha\beta\gamma}\,\partial_\alpha t_{\beta\gamma,\rho\sigma} + 2\,\partial_{[\rho}\Big(s^{\mu\nu}{}_{\sigma]} + \frac{2}{3}\,\delta_{\sigma]}{}^{[\mu}\,\xi^{\nu]}\Big)\,. \tag{3.111}$$

Using this result in (3.97), we finally arrive at our initial goal (3.100)

$$\mathcal{E}_{\mu\nu,\rho\sigma} = 0\,,$$

upon shifting

$$T_{\mu\nu,\rho\sigma} \to T_{\mu\nu,\rho\sigma} + t_{\mu\nu,\rho\sigma}\,, \tag{3.112}$$

$$v_{\rho,\mu\nu} \to v_{\rho,\mu\nu} + s_{\mu\nu,\rho} + \frac{2}{3}\,\eta_{\rho[\mu}\,\xi_{\nu]}\,. \tag{3.113}$$

We have thus show that the variation of the Lagrangian (3.73) gives the first order equations (2.65) and (2.67).

CHAPTER 4

Exceptional Field Theory framework

In the previous sections, we have constructed Lagrangians (3.25), (3.36), and (3.73), for the three six-dimensional models which share a number of universal features and structures. In particular, after dimensional reduction to five dimensions they all reduce to the same Lagrangian corresponding to linearized maximal supergravity in five dimensions. This Lagrangian includes the Pauli-Fierz Lagrangian (3.27) and standard Lagrangians for vector and scalar fields. The three distinct six-dimensional theories are then described as different extensions of this Lagrangian by terms carrying derivatives along the sixth dimension. In the various matter sectors, these terms ensure covariantization under non-trivial gauge structures and provide sources to the field equations of five-dimensional supergravity.

This reformulation within a common framework is very much in the spirit of exceptional field theories [43–46]. In that framework, higher-dimensional supergravity theories are reformulated in terms of the field content of a lower-dimensional supergravity keeping the dependence on all coordinates. More precisely, their formulation is based on a split of coordinates into D external and n internal coordinates of which the latter are formally embedded into a fundamental representation $\mathcal{R}_v$ of the global symmetry group $\mathrm{E}_{11-D,(11-D)}$ of D-dimensional maximal supergravity. Different embeddings of the internal coordinates into $\mathcal{R}_v$ then correspond to different higher-dimensional origins. However, the introduction of new coordinates requires a tool to eliminate the dependence of the physical fields with respect to a suitable number of extra dimensions called a section constraint. Solving this constraint will select the appropriate number of physical dimensions and allows us to build higher-dimensional supergravities. Here, we will discuss a similar uniform description of the six-dimensional models based on $D = 5$ external dimensions which encompasses the three different models upon proper identification of the sixth coordinate within the internal coordinates. This will require an enhancement of the internal coordinates of exceptional field theory by an additional exotic coordinate

related to the singlet central charge in the $D = 5$ supersymmetry algebra.[insert ref Hull BPS]

4.1 Linearized ExFT and embedding of the $\mathcal{N} = (2,2)$ model

The theory relevant for our discussion is $\mathrm{E}_{6(6)}$ exceptional field theory (ExFT) [45, 46]. Its bosonic field content is given by a graviton $g_{\mu\nu}$ together with 27 vector fields $\mathcal{A}_\mu{}^M$ and their dual tensors $\mathcal{B}_{\mu\nu\,M}$, together with 42 scalars parametrizing the internal metric $\mathcal{M}_{MN} = (\mathcal{V}\mathcal{V}^T)_{MN}$ with $\mathcal{V}$ a representative of the coset space $\mathrm{E}_{6(6)}/\mathrm{USp}(8)$. Fields depend on 5 external and 27 internal coordinates with the latter transforming in the fundamental **27** of $\mathrm{E}_{6(6)}$ and with internal coordinate dependence of the fields restricted by the section constraint [44]

$$d^{KMN}\,\partial_M \otimes \partial_N = 0\,, \tag{4.1}$$

with the two differential operators acting on any couple of fields and gauge parameters of the theory. The tensor d^{KMN} denotes the cubic totally symmetric $\mathrm{E}_{6(6)}$ invariant tensor, which we normalize as $d^{MNP}d_{MNQ} = \delta_Q{}^P$. The section condition (4.1) admits two inequivalent solutions [45,47] which reduce the internal coordinate dependence of all fields to the 6 internal coordinates from $D = 11$ supergravity, or 5 internal coordinates from IIB supergravity, respectively. For details of the ExFT Lagrangian we refer to [45, 46]. Here, we spell out its "free" limit, obtained by linearizing the full theory according to

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}\,, \qquad \mathcal{M}_{MN} = \Delta_{MN} + \phi_{MN}\,, \tag{4.2}$$

around the constant background given by the Minkowski metric $\eta_{\mu\nu}$ and the identity matrix Δ_{MN}. The scalar fluctuations ϕ_{MN} are further constrained by the coset properties of $\mathcal{M}_{MN}$. To quadratic order in the fluctuations, the ExFT Lagrangian then yields

$$\begin{aligned}\mathcal{L}_{\text{ExFT,free}} = &-\frac{1}{4}\,\Omega^{\mu\nu\rho}\Omega_{\mu\nu\rho} + \frac{1}{2}\,\Omega^{\mu\nu\rho}\Omega_{\nu\rho\mu} + \Omega^\mu\Omega_\mu - \frac{1}{4}\,\mathcal{F}^{\mu\nu\,M}\,\mathcal{F}_{\mu\nu}{}^N\,\Delta_{MN} \\ &-\frac{5}{4}\,\sqrt{10}\,\varepsilon^{\mu\nu\rho\sigma\tau}\,d^{MNK}\partial_\mu\mathcal{B}_{\nu\rho\,M}\partial_N\mathcal{B}_{\sigma\tau\,K} - \frac{1}{24}\,D_\mu\phi^{MN}\,D^\mu\phi_{MN} \\ &+\mathcal{L}_{\text{pot}}\,,\end{aligned} \tag{4.3}$$

with indices M,N raised and lowered by Δ_{MN} and its inverse, and with the various elements of (4.3) given by

$$\begin{aligned}
\Omega_{\mu\nu\rho} &= \partial_{[\mu} h_{\nu]\rho} - \frac{2}{3}\,\partial_M \mathcal{A}_{[\mu}{}^M \eta_{\nu]\rho}\,, \qquad \Omega_\mu \;\equiv\; \Omega_{\mu\nu}{}^\nu\,,\\
\mathcal{F}_{\mu\nu}{}^M &= 2\,\partial_{[\mu}\mathcal{A}_{\nu]}{}^M + 10\,d^{MNK}\,\partial_N \mathcal{B}_{\mu\nu\,K}\,,\\
D_\mu \phi^{MN} &= \partial_\mu \phi^{MN} + 2\,\partial_K \mathcal{A}_\mu{}^{(M}\Delta^{N)K} + \frac{2}{3}\,\partial_K \mathcal{A}_\mu{}^K \Delta^{MN}\\
&\quad - 20\,\partial_K \mathcal{A}_\mu{}^L\, d_{PLR} d^{RK(M}\Delta^{N)P}\,,\\
\mathcal{L}_{\rm pot} &= -\frac{1}{24}\Delta^{MN}\partial_M \phi^{KL}\,\partial_N \phi_{KL} + \frac{1}{2}\Delta^{MN}\partial_M \phi^{KL}\partial_L \phi_{NK} - \frac{1}{2}\,\partial_M h_\nu{}^\nu\,\partial_N \phi^{MN}\\
&\quad + \frac{1}{4}\,\Delta^{MN}\,\partial_M h_\mu{}^\mu\,\partial_N h_\nu{}^\nu - \frac{1}{4}\,\Delta^{MN}\,\partial_M h^{\mu\nu}\partial_N h_{\mu\nu}\,.
\end{aligned} \tag{4.4}$$

The Lagrangian we have presented above for the six-dimensional $\mathcal{N}=(2,2)$ model naturally fits into this framework. This does not come as a surprise since the six-dimensional model is nothing but linearized maximal supergravity known to be described by $\mathrm{E}_{6(6)}$ ExFT upon proper selection of the sixth coordinate among the internal ∂_M. This choice is uniquely fixed by the requirement that the resulting theory exhibits the global $\mathrm{SO}(5,5)$ symmetry group of maximal six-dimensional supergravity, thus breaking

$$\begin{aligned}
\mathrm{E}_{6(6)} \;&\longrightarrow\; \mathrm{SO}(5,5)\,, \qquad & \mathbf{27} &\longrightarrow \mathbf{1}\oplus\mathbf{16}\oplus\mathbf{10}\,,\\
& & \{\partial_M\} &\longrightarrow \{\partial_0,\partial_i,\partial_a\}\,,
\end{aligned} \tag{4.5}$$

and keeping only coordinate-dependence along the $\mathrm{SO}(5,5)$ singlet. In this split, the $\mathrm{E}_{6(6)}$ invariant symmetric tensor d^{MNK} has the non-vanishing components

$$d^{0ab} = \frac{1}{\sqrt{10}}\,\eta^{ab}\,, \qquad d^{aij} \;=\; \frac{1}{2\sqrt{5}}\,(\Gamma^a)^{ij}\,, \tag{4.6}$$

in terms of $\mathrm{SO}(5,5)$ Γ-matrices and its invariant tensor η^{ab} of signature $(5,5)$, showing that the section constraint (4.1) is trivially satisfied is $\partial_i = 0 = \partial_a$. Putting this together with the linearized ExFT Lagrangian (4.3), and splitting fields as

$$\{\mathcal{A}_\mu{}^M\} = \{A_\mu, A_\mu{}^i, A_\mu{}^a\}\,, \qquad \text{etc.}\,, \tag{4.7}$$

we arrive at

$$\begin{aligned}
\mathcal{L}_{(2,2)} &= -\frac{1}{4}\,\Omega^{\mu\nu\rho}\Omega_{\mu\nu\rho} + \frac{1}{2}\,\Omega^{\mu\nu\rho}\Omega_{\nu\rho\mu} + \Omega^\mu\Omega_\mu - \frac{1}{4}\,F^{\mu\nu}F_{\mu\nu} - \frac{1}{4}\,F^{\mu\nu\,i}F_{\mu\nu}{}^i\\
&\quad - \frac{1}{4}\,(F_{\mu\nu}{}^a + \partial_y B_{\mu\nu}{}^a)\,(F^{\mu\nu\,a} + \partial_y B^{\mu\nu\,a}) - \frac{1}{24}\,\varepsilon^{\mu\nu\rho\sigma\tau}\,\eta_{ab}\,\partial_y B_{\mu\nu}{}^a\,H_{\rho\sigma\tau}{}^b\\
&\quad - \frac{1}{2}\,\partial^\mu\phi^\alpha\partial_\mu\phi^\alpha - \frac{1}{2}\,\partial_y\phi^\alpha\partial_y\phi^\alpha - \frac{1}{2}\left(\partial^\mu\phi^i - \partial_y A^{\mu\,i}\right)\left(\partial_\mu\phi^i - \partial_y A_\mu{}^i\right)\\
&\quad - \frac{1}{2}\,(\partial^\mu\phi - \sqrt{\frac{8}{3}}\,\partial_y A^\mu)(\partial_\mu\phi - \sqrt{\frac{8}{3}}\,\partial_y A_\mu) + \frac{5}{6}\,\partial_y\phi\partial_y\phi\\
&\quad - \frac{2}{3}\,\partial_y h_\sigma{}^\sigma\partial_y\phi + \frac{1}{4}\,\partial_y h_\sigma{}^\sigma\partial_y h_\rho{}^\rho - \frac{1}{4}\,\partial_y h^{\mu\nu}\partial_y h_{\mu\nu}\,,
\end{aligned} \tag{4.8}$$

which precisely produces the sum of Lagrangians (3.4), (3.15), (3.25), after proper rescaling of the singlet scalar field ϕ. The non-trivial checks of this coincidence include all the coefficients in the various connection terms, as well as in the Stückelberg-type couplings between vector and tensor fields, and the coefficients in front of the various $\partial_y\phi\partial_y\phi$ terms in the last line. Again, this is not a surprise but a consequence of the proven equivalence of ExFT with higher-dimensional maximal supergravity. Note that although the free theory only exhibits a compact $\mathrm{USp}(4)\times\mathrm{USp}(4)$ global symmetry, the couplings exhibited in (4.8) are far more constrained than allowed by this symmetry and witness the underlying $\mathrm{E}_{6(6)}$ structure broken to $\mathrm{SO}(5,5)$ according to (4.5), (4.6).

The ExFT Lagrangian is to a large extent determined by invariance under generalized internal diffeomorphisms acting with a gauge parameter Λ^M in the **27**. After linearization (4.2) these diffeomorphisms act as

$$\delta\phi_{MN} = 2\,\Delta_{K(M}\,\partial_{N)}\Lambda^K + \frac{2}{3}\,\partial_K\Lambda^K\,\Delta_{MN} - 20\,d^{PKR}d_{RL(M}\,\Delta_{N)P}\,\partial_K\Lambda^L\,, \tag{4.9}$$

$$\delta\mathcal{A}_\mu{}^M = \partial_\mu\Lambda^M\,, \tag{4.10}$$

$$\delta h_{\mu\nu} = \frac{2}{3}\,\partial_M\Lambda^M\,\eta_{\mu\nu}\,, \tag{4.11}$$

and one can show invariance of the linearized Lagrangian (4.8), provided the section constraint (4.1) is satisfied.

4.2 Beyond standard ExFT: embedding of the $\mathcal{N}=(3,1)$ and $(4,0)$ couplings

The charges carried by the massive BPS multiplets [48] in the reduction of the $\mathcal{N}=(3,1)$ and the $\mathcal{N}=(4,0)$ model, respectively, suggest that an inclusion of these models into the framework of ExFT necessitates an extension of the space of 27 internal coordinates by an additional exotic coordinate corresponding to the singlet central charge [48]. Denoting derivatives along this coordinate by $\partial_\bullet$, this would amount to a relaxation of the standard section constraint (4.1) to a constraint of the form

$$d^{KMN}\,\partial_M\otimes\partial_N - \frac{1}{\sqrt{10}}\,\Delta^{KM}\,(\partial_M\otimes\partial_\bullet + \partial_\bullet\otimes\partial_M) = 0\,, \tag{4.12}$$

which at the present stage only makes sense in the linearized theory where Δ^{KM} is a constant background tensor. Apart from the standard ExFT solutions

$$d^{KMN}\,\partial_M\otimes\partial_N = 0\,,\qquad \partial_\bullet = 0\,, \tag{4.13}$$

of this constraint, which allow the embedding of the $\mathcal{N}=(2,2)$ model as described above, the extended section constraint also allows for two exotic solutions

$$\begin{aligned}(3,1):\partial_y^{(3,1)} &= \frac{2}{\sqrt{3}}\,\partial_0 \;=\; -2\,\partial_\bullet\,, \qquad \text{with the } \mathrm{F}_{4(4)} \text{ singlet } \partial_0\subset\partial_M\,,\\ (4,0):\partial_y^{(4,0)} &= -\partial_\bullet\,,\quad \partial_M=0\,,\end{aligned} \tag{4.14}$$

corresponding to the two exotic six-dimensional models in precise correspondence with the central charges carried by the corresponding BPS multiplets [48]. While the (4,0) solution trivially solves the constraint (4.12), the $\mathcal{N}=(3,1)$ solution is based on the decomposition

$$\mathrm{E}_{6(6)} \longrightarrow \mathrm{F}_{4(4)}\,, \qquad \mathbf{27}\longrightarrow\mathbf{1}\oplus\mathbf{26}\,,\{\partial_M\}\longrightarrow\{\partial_0,\partial_A\}\,,$$

under which the symmetric d-tensor decomposes into

$$d^{000}=-\frac{2}{\sqrt{30}}\,,\quad d^{0AB}=\frac{1}{\sqrt{30}}\,\eta^{AB}\,,\quad d^{ABC}\,, \tag{4.15}$$

with the $\mathrm{F}_{4(4)}$ invariant symmetric tensor η^{AB} of signature $(14,12)$, and the symmetric invariant tensor d^{ABC} satisfying

$$d^{ABC}\eta_{BC}=0\,,\qquad d_{ABC}d^{ABD}=\frac{14}{15}\,\delta_C{}^D\,. \tag{4.16}$$

This shows explicitly how the (3,1) assignment of (4.14) also provides a solution to the extended section constraint (4.12).

It is intriguing to study the fate of diffeomorphism invariance of the ExFT Lagrangian (4.3) if the original section constraint is relaxed to (4.12). Except for the last term in (4.3), the Lagrangian remains manifestly invariant without any use of the section constraint. Explicit variation of the potential term $\mathcal{L}_{\rm pot}$ under linearized diffeomorphisms (4.11) on the other hand yields (up to total derivatives)

$$\begin{aligned}\delta_\Lambda\mathcal{L}_{\rm pot} = &\Big(5\,\Delta_{LS}\,d^{LMN}d^{KPQ}-10\Delta^{MN}\Delta^{KL}\,d_{LSR}d^{RPQ}\Big)\,\Lambda^S\,\partial_P\partial_Q\partial_M\phi_{NK}\\ &-10\,h_\mu{}^\mu\,\Delta^{MK}\,d_{KLR}d^{RPQ}\,\partial_M\partial_P\partial_Q\Lambda^L\,,\end{aligned} \tag{4.17}$$

which consistently vanishes modulo the standard section constraint (4.1). For the weaker constraint (4.12), this variation no longer vanishes and may be recast in the following form

$$\delta_\Lambda\mathcal{L}_{\rm pot}=\Delta^{KM}\,\Lambda^N\,\partial_\bullet\partial_\bullet\partial_M\phi_{NK}-4\,h_\nu{}^\nu\,\partial_\bullet\partial_\bullet\partial_N\Lambda^N\,, \tag{4.18}$$

after repeated use of (4.12) and further manipulation of the expressions. In order to compensate for this variation let us first note that there is no possible covariant extension of the transformation rules (4.12) by terms carrying $\partial_\bullet\Lambda^M$, such that invariance can only be restored by extending the potential. A possible such extension is given by

$$\mathcal{L}_{\rm pot,\bullet}=\mathcal{L}_{\rm pot}-\frac{1}{24}\,\partial_\bullet\phi_{MN}\partial_\bullet\phi^{MN}-\frac{3}{4}\,\partial_\bullet h_\sigma{}^\sigma\partial_\bullet h_\rho{}^\rho+\frac{3}{4}\,\partial_\bullet h^{\mu\nu}\partial_\bullet h_{\mu\nu}\,, \tag{4.19}$$

and it is straightforward to verify that the variation of the additional terms in (4.19) precisely cancels the contributions in (4.18), such that

$$\delta_\Lambda \mathcal{L}_{\text{pot},\bullet} = 0 \,. \tag{4.20}$$

For the exotic solutions of the section constraint, the $\partial_\bullet \phi_{MN} \partial_\bullet \phi^{MN}$ terms in (4.19) give rise to additional contributions of the type $\partial_y \phi \partial_y \phi$ in the Lagrangian. Collecting all such terms in (4.19) for the two exotic solutions (4.14) yields

$$\begin{aligned} (3,1) &\longrightarrow -\frac{1}{2}\,\partial_y \phi^\alpha \partial_y \phi^\alpha \,, \qquad \alpha = 1, \dots, 28 \,, \\ (4,0) &\longrightarrow -\frac{1}{2}\,\partial_y \phi^A \partial_y \phi^A \,, \qquad A = 1, \dots, 42 \,. \end{aligned} \tag{4.21}$$

These are precisely the terms found in our explicit construction of actions (3.45) and (3.78) above! In other words, the relaxation (4.12) of the section constraint together with generalized diffeomorphism invariance precisely implies the correct scalar couplings in the Lagrangians of the exotic models. In addition, the $\partial_\bullet h \partial_\bullet h$ terms in (4.19) cancel the corresponding terms in $\mathcal{L}_{\text{pot}}$ (4.4) upon selecting the (3,1) solution of the section constraint (4.14), just as required in order to reproduce the correct Lagrangian of the $\mathcal{N} = (3,1)$ model (3.36).[1]

We may continue the symmetry analysis for the tensor gauge transformations given by a gauge parameter $\Lambda_{\mu\, M}$ in standard ExFT. For these transformations there is a natural extension of the standard ExFT transformation rules in presence of the exotic coordinate and exotic fields as

$$\delta_{\Lambda_\mu} \mathcal{A}_\mu{}^M = -10\, d^{MNK}\, \partial_N \Lambda_{\mu\, K} - \sqrt{10}\, \Delta^{MK}\, \partial_\bullet \Lambda_{\mu\, K} \,, \tag{4.22}$$

$$\delta_{\Lambda_\mu} \mathcal{B}_{\mu\nu\, M} = 2\, \partial_{[\mu} \Lambda_{\nu]\, M} \,. \tag{4.23}$$

Computing the action of these transformations on the connection featuring in the co-

[1] In contrast, these terms appear in conflict with embedding the spin-2 sector of the $\mathcal{N} = (4,0)$ model as they survive under the (4,0) solution in (4.14) but should be absent in the final Lagrangian (3.73). We come back to this in section 4.3.

variant scalar derivatives $D_\mu \phi^{MN}$ in (4.4), we obtain after some manipulation[2]

$$\begin{aligned}\delta_{\Lambda_\mu} D_\mu \phi^{MN} = 10 \Big(& \frac{1}{3} \Delta^{MN} \delta_P{}^Q + \Delta^{Q(M} \delta_P{}^{N)} \\ & - 10\, \Delta^{S(M} d^{N)QR} d_{RSP} \Big) d^{PKL}\, \partial_K \partial_L \Lambda_{\mu\, Q} \\ - 2\sqrt{10} \Big(& \frac{1}{3} \Delta^{MN} \delta_P{}^Q + \delta_P{}^{(M} \Delta^{N)Q} \\ & - 10\, \Delta^{L(M} d^{N)QR} d_{RPL} \Big) \Delta^{PK} \partial_K \partial_\bullet \Lambda_{\mu\, Q} \,. \end{aligned} \tag{4.24}$$

The resulting expression precisely vanishes with the modified section constraint (4.12). This shows the necessity of the $\partial_\bullet \Lambda_{\mu\, M}$ terms in (4.23) in order to maintain gauge invariance of the kinetic term $D_\mu \phi^{MN} D^\mu \phi_{MN}$ in presence of the relaxed section constraint. It is straightforward to verify that these additional terms in the transformation induce a modification of the gauge invariant vector field strengths to

$$\mathcal{F}_{\mu\nu}{}^M \equiv 2\, \partial_{[\mu} \mathcal{A}_{\nu]}{}^M + 10\, d^{MNK}\, \partial_N \mathcal{B}_{\mu\nu\, K} + \sqrt{10}\, \Delta^{MK}\, \partial_\bullet \mathcal{B}_{\mu\nu\, K} \,, \tag{4.25}$$

as well an extension of the topological term, such that the combined vector-tensor couplings take the form

$$\begin{aligned}\mathcal{L}_{\text{vt},\bullet} = & -\frac{1}{4} \Delta_{MN}\, \mathcal{F}^{\mu\nu\, M} \mathcal{F}_{\mu\nu}{}^N \\ & - \frac{5}{4}\, \varepsilon^{\mu\nu\rho\sigma\tau}\, \partial_\mu \mathcal{B}_{\nu\rho\, M} \left(\sqrt{10}\, d^{MNK} \partial_N \mathcal{B}_{\sigma\tau\, K} + \Delta^{MK} \partial_\bullet \mathcal{B}_{\sigma\tau\, K} \right) \,, \end{aligned} \tag{4.26}$$

and are invariant under these gauge transformations. Let us work out the effect of these modifications for the exotic solutions of the section constraint. With the kinetic scalar term unchanged, the resulting couplings are directly inferred from evaluating the covariant derivatives (4.4) for the d-symbol (4.15), giving rise to

$$\begin{aligned}(3,1) &\longrightarrow -\frac{1}{2} \left(\partial^\mu \phi^i - \partial_y A^{\mu\, i} \right) \left(\partial_\mu \phi^i - \partial_y A_\mu{}^i \right) - \frac{1}{2}\, \partial^\mu \phi^\alpha \partial_\mu \phi^\alpha \,, \\ (4,0) &\longrightarrow -\frac{1}{2}\, \partial^\mu \phi^A \partial_\mu \phi^A \,, \end{aligned} \tag{4.27}$$

where $i = 1, \ldots, 14$, $\alpha = 1, \ldots, 28$ and $A = 1, \ldots, 42$. This precisely reproduces the vector-scalar couplings found in the explicit Lagrangians (3.45), (3.78) above. As for the

[2] A useful identity for this computation is given by

$$\begin{aligned} d^{PLQ} d_{PSR} d^{KMR} \partial_K \partial_L = & \frac{1}{10}\, \delta_S^K d^{LQM} \partial_K \partial_L + \frac{1}{20}\, \delta_S^M d^{QKL} \partial_K \partial_L + \frac{1}{20}\, \delta_S^Q d^{MKL} \partial_K \partial_L \\ & - \frac{1}{2}\, d^{QMR} d_{RSP} d^{PKL} \partial_K \partial_L \,, \end{aligned}$$

generalizing equations (2.12), (2.13) of [46].

vector-tensor couplings, evaluating the Lagrangian (4.26) with (4.15) for the solutions (4.14) gives rise to the explicit couplings

$$\begin{aligned}(3,1) \longrightarrow & -\frac{1}{4}\,(F_{\mu\nu}+\frac{3\sqrt{3}}{2}\,\partial_y B_{\mu\nu})(F^{\mu\nu}+\frac{3\sqrt{3}}{2}\,\partial_y B^{\mu\nu}) \\ & -\frac{1}{4}\,(F_{\mu\nu}{}^{a}+\partial_y B_{\mu\nu}{}^{a})\,(F^{\mu\nu\,a}+\partial_y B^{\mu\nu\,a})-\frac{1}{4}\,F^{\mu\nu\,i}F_{\mu\nu}{}^{i} \\ & -\frac{3}{16}\,\varepsilon^{\mu\nu\rho\sigma\tau}\partial_y B_{\mu\nu}H_{\rho\sigma\tau}-\frac{1}{24}\,\varepsilon^{\mu\nu\rho\sigma\tau}\,\partial_y B_{\mu\nu}{}^{a}\,H_{\rho\sigma\tau}{}^{a}\,, \\ (4,0) \longrightarrow & -\frac{1}{4}\,\Big(F_{\mu\nu}{}^{M}+\partial_y B_{\mu\nu}{}^{M}\Big)\Big(F^{\mu\nu\,M}+\partial_y B^{\mu\nu\,M}\Big) \\ & -\frac{1}{24}\,\varepsilon^{\mu\nu\rho\sigma\tau}\,\partial_y B_{\mu\nu}{}^{M}\,H_{\rho\sigma\tau}{}^{M}\,, \end{aligned} \tag{4.28}$$

with indices in range $i=1,\dots,14$, $a=1,\dots,12$, $M=1,\dots,27$, as above. Again, this precisely reproduces the couplings found above (after proper rescaling of the vector field A_μ)!

To summarize, in the scalar, vector and tensor sector, we have constructed an extension of the ExFT Lagrangian (at the linearized level), given by

$$\mathcal{L}=-\frac{1}{2}\,D_\mu\phi^{MN}D^\mu\phi_{MN}+\mathcal{L}_{\text{vt},\bullet}+\mathcal{L}_{\text{pot},\bullet}\,, \tag{4.29}$$

which is invariant under the gauge transformations (4.11), (4.23) modulo the relaxed section constraint (4.12). The weaker section constraint necessitates a numer of additional contributions to the Lagrangian (and transformation rules) which precisely reproduce the explicit couplings found in the Lagrangians of the exotic models (3.45), (3.78) constructed above. It is remarkable that this match confirms the couplings that have been determined from an underlying non-compact $\mathrm{E}_{6(6)}$ and $\mathrm{F}_{4(4)}$ structure, respectively, despite the fact that the free theory only exhibits invariance under the compact R-symmetry subgroup $\mathrm{USp}(2\mathcal{N}_+)\times\mathrm{USp}(2\mathcal{N}_-)$ which might in principle allow for much more general couplings. We take this as evidence for the conjectured $\mathrm{E}_{6(6)}$ and $\mathrm{F}_{4(4)}$ invariance of the putative interacting theories [14].

4.3 The spin-2 sector

The above findings have revealed a very intriguing common structure of the couplings in the scalar, vector and tensor sectors of the different models which can be consistently embedded into an extension of (linearized) exceptional field theory. For the spin-2 sector carrying the Pauli-Fierz field and its duals on the other hand the picture appears not yet complete. Extrapolation of the Lagrangian of the $\mathcal{N}=(4,0)$ model (3.73) suggests an extension of the standard ExFT Lagrangian by couplings carrying $\partial_\bullet$ derivatives and

the dual graviton fields as

$$\begin{aligned}\mathcal{L} = &-\frac{1}{4}\,\widehat{\Omega}^{\mu\nu\rho}\widehat{\Omega}_{\mu\nu\rho}+\frac{1}{2}\,\widehat{\Omega}^{\mu\nu\rho}\widehat{\Omega}_{\nu\rho\mu}+\widehat{\Omega}^{\mu}\widehat{\Omega}_{\mu}+\frac{1}{8}\,\varepsilon_{\mu\nu\sigma\kappa\lambda}\,\partial^{\mu}\widehat{C}^{\nu\sigma}{}_{\rho}\,\partial_{\bullet}\widehat{C}^{\kappa\lambda,\rho}\\ &-\frac{1}{32}\,\varepsilon_{\mu\nu\sigma\kappa\lambda}\,\partial^{\mu}\mathcal{C}^{\nu\sigma}{}_{\rho}\,\partial_{\bullet}\mathcal{C}^{\kappa\lambda,\rho}+\frac{1}{8}\,\partial_{\bullet}\mathcal{C}_{\sigma\tau,\nu}\,\partial_{\mu}T^{\mu\nu,\sigma\tau}-\frac{1}{4}\,\partial_{\bullet}\mathcal{C}_{\kappa\lambda,\tau}\,\partial^{\kappa}T^{\lambda\sigma,\tau}{}_{\sigma}\\ &-\frac{1}{4}\,\partial_{\nu}\mathcal{C}_{\sigma\mu}{}^{\mu}\,\partial_{\bullet}T^{\sigma\tau,\nu}{}_{\tau}+\frac{1}{8}\,\partial_{\bullet}\mathcal{C}_{\sigma\mu}{}^{\mu}\,\partial^{\sigma}T_{\tau\nu}{}^{\tau\nu}+\frac{1}{64}\,\varepsilon_{\mu\nu\alpha\beta\gamma}\,\partial^{\alpha}T_{\sigma\tau}{}^{\beta\gamma}\,\partial_{\bullet}T^{\mu\nu,\sigma\tau}\\ &-\frac{1}{32}\,\partial_{\bullet}T_{\sigma\tau,\mu\nu}\,\partial_{\bullet}T^{\mu\nu,\sigma\tau}+\frac{1}{8}\,\partial_{\bullet}T_{\sigma\mu,\nu}{}^{\mu}\,\partial_{\bullet}T^{\sigma\tau,\nu}{}_{\tau}-\frac{1}{32}\,\partial_{\bullet}T_{\mu\nu}{}^{\mu\nu}\,\partial_{\bullet}T_{\sigma\tau}{}^{\sigma\tau}\\ &+\frac{5}{4}\,\varepsilon^{\mu\nu\rho\sigma\tau}\,d^{KMN}\partial_{K}\mathcal{B}_{\mu\nu\,M}\,\partial_{N}\widehat{C}_{\rho\sigma,\tau}\,,\end{aligned}\tag{4.30}$$

with

$$\widehat{\Omega}_{\mu\nu\rho}=\partial_{[\mu}h_{\nu]\rho}-\frac{2}{3}\,\partial_{M}A_{[\mu}{}^{M}\,\eta_{\nu]\rho}-\partial_{\bullet}\widehat{C}_{\mu\nu,\rho}+\frac{1}{2}\partial_{\bullet}\mathcal{C}_{\mu\nu,\rho}\,.\tag{4.31}$$

By construction, this reproduces the $\mathcal{N}=(2,2)$ and the $\mathcal{N}=(4,0)$ models upon choosing the corresponding solutions of the section constraint. It remains unclear however, how the spin-2 sector of the $\mathcal{N}=(3,1)$ model can find its place in this construction. In particular, the appearance of the extra fields $\mathcal{C}_{\mu\nu,\rho}$ and $T_{\mu\nu,\rho\sigma}$ appearing in (4.30), whose couplings remain present upon selecting the (3,1) solution (4.14) of the section constraint, poses a challenge for recovering the Lagrangian (3.45) of the $\mathcal{N}=(3,1)$ model. The structure of the gauge transformations of $\mathcal{C}$ as extrapolated from (3.77) appears to suggest a gauge fixing of the $\zeta_{\mu\nu}$ and $\lambda_{\rho,\mu\nu}$ gauge symmetries — absent in the $\mathcal{N}=(3,1)$ model — in order to remove this field. Another apparent problem in the spin-2 sector is the lacking reconciliation between the $\partial_{\bullet}h\partial_{\bullet}h$ terms from (4.19) and the $\partial_{\bullet}T\partial_{\bullet}T$ terms of (4.30) which mutually violate the correct limits to the exotic models. Resolution of this problem may require to implement algebraic relations between the Pauli-Fierz $h_{\mu\nu}$ field and the double dual graviton [49] (see also [50]).

CHAPTER 5

Supersymmetry

In the previous chapter, we used the formalism of Exceptional Field Theory to rewrite the bosonic sectors of the three six-dimensional theories. In this chapter, we shall rewrite the fermionic sectors of these theories by considering their supersymmetric extensions. To that end, we follow different strategies. For the $\mathcal{N} = (2, 2)$ theory, we can linearize the already known supersymmetric ExFT action to deduce the full supersymmetric Lagrangian. For the $\mathcal{N} = (4, 0)$ theory, such an ExFT action is not known, so we shall compute the six-dimensional supersymmetry variations of the fields and write them in the 5+1 split form introduced above. Since the fields of the (4,0) model already appear in a form covariant under Usp(8) which is the five-dimensional R-symmetry group, finding the full supersymmetric Lagrangian is relatively straightforward. The strategy for the $\mathcal{N} = (3, 1)$ theory is the same. We work out the supersymmetry variations of the fields in six dimensions and their 5+1 split form. A final step, which remains to be worked out, is the embedding of the fields (which are covariant under Usp(6) $\times$ Usp(2)) into Usp(8). We leave this for future work.

5.1 $\mathcal{N} = (2, 2)$

The supersymmetric extension of the $\mathcal{N} = (2, 2)$ model can be obtained by linearizing the supersymmetric exceptional field theory [51]. The fermions of the model have been introduced in Section 3.1.4 above, in terms of a Usp(4) $\times$ Usp(4) formulation. In ExFT, they combine in Usp(8) objects χ^{ABC} and ψ_μ^A where A labels the fundamental

representation of Usp(8). The fields are embedded as

$$\chi^{ABC} = \begin{cases} \chi_{\alpha\beta\gamma} = \Omega_{[\alpha\beta}\chi_{\gamma]} \\ \chi_{\dot\alpha\dot\beta\alpha} = \chi_{[\![\dot\alpha\dot\beta]\!]\alpha} - \frac{1}{6}\,\Omega_{\dot\alpha\dot\beta}\,\chi_\alpha \\ \chi_{\dot\alpha\dot\beta\dot\gamma} = \Omega_{[\dot\alpha\dot\beta}\chi_{\dot\gamma]} \\ \chi_{\alpha\beta\dot\alpha} = \chi_{[\![\alpha\beta]\!]\dot\alpha} - \frac{1}{6}\,\Omega_{\alpha\beta}\,\chi_{\dot\alpha} \end{cases} \tag{5.1}$$

$$\psi_\mu^{\ A} = \begin{cases} \psi_\mu{}^\alpha \\ \psi_\mu{}^{\dot\alpha} \end{cases} \tag{5.2}$$

The bosonic sector has been formulated in terms of $\mathrm{E}_{6(6)}$ objects in the previous section. For the coupling two fermions, we need to rewrite the bosonic fields in terms of Usp(8) objects.

$$A_\mu^M \to A_\mu^{AB} \tag{5.3}$$

$$B_{\mu\nu,M} \to B_{\mu\nu,AB} \tag{5.4}$$

$$\phi^{MN} \to \phi^{ABCD}\ , \tag{5.5}$$

with vectors and two-forms in the **27** of Usp(8) and scalars in the **42** of Usp(8). Under Usp(4) × Usp(4) the vectors further decompose as

$$A_\mu^{AB} \to \{A_\mu{}^{[\![\alpha\beta]\!]}, A_\mu{}^{[\![\dot\alpha\dot\beta]\!]}, A_\mu{}^{\alpha\dot\alpha}, A_\mu\}\ , \tag{5.6}$$

where $A_\mu{}^{\alpha\dot\alpha}$ are the six-dimensional vector fields, $A_\mu{}^{[\![\alpha\beta]\!]}$ and $A_\mu{}^{[\![\dot\alpha\dot\beta]\!]}$ are components of the six-dimensional two-forms and A_μ is a component of the six-dimensional graviton in accordance with Table 1.3. Two-forms and scalars allow for a similar decomposition to match the field content of Table 1.3.

Similarly, internal derivatives are labelled as ∂_{AB} under Usp(8) and the section constraint (4.1) takes the form

$$\Omega_{AC}\partial^{CD}\partial_{DB} + \frac{1}{8}\,\Omega_{AB}\,\partial_{CD}\partial^{CD} = 0\ . \tag{5.7}$$

Its solution describing six-dimensional maximal supergravity takes the form

$$\partial_{\alpha\beta} = \frac{1}{2}\,\Omega_{\alpha\beta}\,\partial_y\ , \quad \partial_{\dot\alpha\dot\beta} = -\frac{1}{2}\,\Omega_{\dot\alpha\dot\beta}\,\partial_y\ . \tag{5.8}$$

This corresponds to keeping only the coordinate dependence along the SO(5, 5) singlet coordinate as discussed in equation (4.5) above.

In terms of these objects, linearization of the supersymmetric ExFT Lagrangian of [51] yields

$$\begin{aligned} \mathcal{L} = &-\frac{1}{4}\,\widehat\Omega^{\mu\nu\rho}\widehat\Omega_{\mu\nu\rho} + \frac{1}{2}\,\widehat\Omega^{\mu\nu\rho}\widehat\Omega_{\nu\rho\mu} + \widehat\Omega^\mu\widehat\Omega_\mu - \frac{8}{3}\,D^\mu\phi_{ABCD}D_\mu\phi^{ABCD} \\ &-\frac{1}{2}\,\mathcal{F}^{\mu\nu}{}_{AB}\mathcal{F}_{\mu\nu}{}^{AB} - \frac{8}{3}\,\bar\chi_{ABC}\gamma^\mu\partial_\mu\chi^{ABC} - 2\,\bar\psi_{\mu\,A}\gamma^{\mu\nu\rho}\partial_\nu\psi_\rho{}^A \\ &+\mathcal{L}_{\text{pot}}\ , \end{aligned} \tag{5.9}$$

where

$$\begin{aligned}
\widehat{\Omega}_{\mu\nu\rho} &= \partial_{[\mu} h_{\nu]\rho} - \frac{2}{3}\,\partial_{AB} A_{[\mu}{}^{AB} \eta_{\nu]\rho}\,, \\
\mathcal{F}_{\mu\nu}{}^{AB} &= F_{\mu\nu}{}^{AB} + 2\,\Omega^{C[\![A}\partial^{B]\!]D} B_{\mu\nu\,CD}\,, \\
D_\mu \phi^{ABCD} &= \partial_\mu \phi^{ABCD} + \frac{3}{2}\,\partial^{[\![AB} A_\mu{}^{CD]\!]}\,.
\end{aligned} \tag{5.10}$$

and $\mathcal{L}_{\text{pot}}$ collects all the terms carrying two internal derivatives as given in (4.4).

The Lagrangian (5.9) is invariant under supersymmetry transformations given by

$$\begin{aligned}
\delta\phi^{ABCD} &= i\,\bar\epsilon^{[\![A}\chi^{BCD]\!]}\,, \\
\delta A_\mu^{AB} &= \bar\epsilon_C \gamma_\mu \chi^{ABC} - i\,\bar\epsilon^{[\![A}\psi_\mu^{B]\!]}\,, \\
\delta B_{\mu\nu}^{AB} &= -\,i\,\bar\epsilon_C \gamma_{\mu\nu}\chi^{ABC} + 2\,\bar\epsilon^{[\![A}\gamma_{[\mu}\psi_{\nu]}^{B]\!]}\,, \\
\delta h_{\mu\nu} &= \bar\epsilon_A \gamma_{(\mu}\psi_{\nu)}^A\,, \\
\delta\chi^{ABC} &= -\,i\,\gamma^\mu \epsilon_D D_\mu \phi^{ABCD} - \frac{3}{16}\,\gamma^{\mu\nu}\epsilon^{[\![A}\mathcal{F}_{\mu\nu}^{BC]\!]} \\
&\quad + \frac{3}{2}\,\partial_{DE}\phi^{DE[\![AB}\epsilon^{C]\!]} + 2\partial_{DE}\phi^{ABCD}\epsilon^E\,, \\
\delta\psi_\mu^A &= -\,\frac{i}{3}\,\gamma^\nu \epsilon_B \mathcal{F}_{\mu\nu}^{AB} + \frac{i}{12}\,\gamma_\mu{}^{\sigma\tau}\epsilon_B \mathcal{F}_{\sigma\tau}^{AB} - \frac{1}{16}\,\gamma^{\sigma\tau}\epsilon^A\left(\widehat{\Omega}_{\sigma\tau,\mu} - 2\,\widehat{\Omega}_{\mu\sigma,\tau}\right) \\
&\quad - \frac{i}{4}\,\partial^{AB} h_{\mu\nu}\gamma^\nu \epsilon_B + \frac{1}{2}\,\epsilon^C\,\partial^{AB} A_{\mu BC} - \frac{1}{2}\,\epsilon^C\,\partial_{CB} A_\mu{}^{BA} \\
&\quad - \frac{1}{12}\,\partial^{BC} A_{\mu BC}\,\epsilon^A - \frac{2}{3}\,i\,\partial_{CD}\phi^{CDAB}\,\gamma_\mu\,\epsilon_B\,,
\end{aligned} \tag{5.11}$$

which are also obtained by linearizing the variations given in [51]. Similar to our discussion of the bosonic sectors in the previous chapter, we expect the supersymmetry variations of the $\mathcal{N} = (4,0)$ and $\mathcal{N} = (3,1)$ theories to coincide with (5.11) up to additional terms carrying an internal derivative along the extra singlet coordinate $\partial_\bullet$. We will show that at least for the $\mathcal{N} = (4,0)$ model, this is indeed the case (while the computation is not complete yet for the $\mathcal{N} = (3,1)$ model).

5.2 $\mathcal{N} = (4,0)$

We next turn to the $\mathcal{N} = (4,0)$ theory for which the field content and their actions have been discussed in Section 3.3. To find the supersymmetry variations in six dimensions, our strategy is to impose the closure of the six-dimensional supersymmetry algebra. Next, we reduce the result in the 5+1 split form notably (2.62) and (2.95) and simplify the result by fermionic gauge transformation. Furthermore, we identify a field redefinition allowing us to write the supersymmetric variations in a more compact form. Finally we present the supersymmetric Lagragian of the full linearized theory written in terms of these new fields.

5.2.1 Supersymmetry variations of the (4,0) model

In this subsection, we present the six-dimensional supersymmetric transformations of the $\mathcal{N} = (4, 0)$ model. These transformations have been worked out in the prepotential formalism [34]. Here we determine the transformations directly in terms of the original six-dimensional fields.

The six-dimensional supersymmetric transformations have to close into six-dimensional diffeomorphisms (up to gauge transformations) in the following way.

$$[\delta_1, \delta_2]X = \xi^{\hat{\mu}}\partial_{\hat{\mu}}X + \text{gauge transformation} , \tag{5.12}$$

where δ_1 and δ_2 are variations of the fields with different parameters $\epsilon_{1,A}$, $\epsilon_{2,A}$ which then enter the diffeomorphism parameter $\xi^{\hat{\mu}}$. This parameter must be the same for any field X, which restricts the form of the supersymmetry variations. In the following, we shall illustrate this explicitly for the simplest field, namely the scalar field ϕ^{ABCD}. The most general form of the supersymmetry variation compatible with its Usp(8) representation is

$$\delta\phi^{ABCD} = i\,\bar{\epsilon}^{[\![A}\chi^{BCD]\!]} , \tag{5.13}$$

where ϵ^A is a suitably normalized supersymmetry parameter. As above, the notation $[\![ABCD]\!]$ refers to Usp(8)-traceless antisymmetrization. The factor i comes from the reality condition of the spinors. Similarly, we can write the general form of the supersymmetry variation of the Dirac fermions χ

$$\delta\chi^{ABC} = -i\,\hat{\gamma}^{\hat{\mu}}\partial_{\hat{\mu}}\phi^{ABCD}\epsilon_D + \text{B terms} , \tag{5.14}$$

where the B terms involve the field strengths the two-form fields $B_{\hat{\mu}\hat{\nu}}{}^{AB}$ which are not relevant for the moment (but will become relevant later on). To calculate the supersymmetry algebra, we look at the commutator of two variations on the scalar field given by (5.13) and (5.14)

$$[\delta_1, \delta_2]\phi^{ABCD} = \bar{\epsilon}_2^{[\![A}\hat{\gamma}^{\hat{\mu}}\partial_{\hat{\mu}}\phi^{BCD]\!]E}\epsilon_{1,E} - (1 \leftrightarrow 2) + \text{B terms} . \tag{5.15}$$

This expression is not immediately in the form of (5.12), but this can be achieved using the following Schouten identity

$$\begin{aligned} 0 =& \bar{\epsilon}_2^{[A}\hat{\gamma}^{\hat{\mu}}\epsilon_{1,E}\partial_{\hat{\mu}}\phi_{WXYZ}\Omega^{BW}\Omega^{CX}\Omega^{DY}\Omega^{EZ]} \\ =& 4\,\bar{\epsilon}_2^{[A}\hat{\gamma}^{\hat{\mu}}\epsilon_{1,E}\partial_{\hat{\mu}}\phi^{BCD]E} - 4\,\bar{\epsilon}_1^{[A}\hat{\gamma}^{\hat{\mu}}\epsilon_{2,E}\partial_{\hat{\mu}}\phi^{BCD]E} \\ &+ \bar{\epsilon}_2^{E}\hat{\gamma}^{\hat{\mu}}\epsilon_{1,E}\partial_{\hat{\mu}}\phi^{ABCD} - 6\,\bar{\epsilon}_{2,W}\hat{\gamma}^{\hat{\mu}}\epsilon_{1,E}\partial_{\hat{\mu}}\phi^{WE[AB}\Omega^{CD]} . \end{aligned} \tag{5.16}$$

We therefore find for (5.15)

$$\bar{\epsilon}_2^{[\![A}\hat{\gamma}^{\hat{\mu}}\epsilon_{1,E}\partial_{\hat{\mu}}\phi^{BCD]\!]E} - (1 \leftrightarrow 2) = -\frac{1}{4}\,\bar{\epsilon}_2^{E}\hat{\gamma}^{\hat{\mu}}\epsilon_{1,E}\partial_{\hat{\mu}}\phi^{ABCD} , \tag{5.17}$$

such that we can read off the diffeomorphism parameter

$$\xi^{\hat{\mu}} = -\frac{1}{4}\,\bar{\epsilon}_2^E \hat{\gamma}^{\hat{\mu}} \epsilon_{1,E}. \tag{5.18}$$

From the point of view of group theory, the Schouten identity (5.16) has extracted the singlet component in the decomposition of $\mathbf{42} \otimes \mathbf{42}$ of Usp(8) in the sense of

$$\phi_{ABCX}\phi^{ABCY} = \frac{1}{8}\,\delta_X^Y \phi_{ABCD}\phi^{ABCD}. \tag{5.19}$$

Group theory also explains why the B terms in (5.15) eventually drop out. Indeed, the only combination which could appear in the right-hand side of (5.15) is

$$\bar{\epsilon}_2^{[\![A} \hat{\gamma}^{\hat{\mu}\hat{\nu}\hat{\rho}} \epsilon_1{}^B H_{\hat{\mu}\hat{\nu}\hat{\rho}}{}^{CD]\!]} - (1 \leftrightarrow 2)\,, \tag{5.20}$$

of which the spinor bilinear vanishes identically due to symmetry reasons. The B terms will however enter into the calculation of the closure relation for χ^{ABC}. Indeed, imposing the same form of (5.12) allows us to iteratively fix the entire six-dimensional algebra.

After lengthy computations, the final algebra is

$$\begin{aligned}
\delta\phi^{ABCD} &= i\,\bar{\epsilon}^{[\![A}\chi^{BCD]\!]}\,,\\
\delta\chi^{ABC} &= -i\,\hat{\gamma}^{\hat{\mu}}\partial_{\hat{\mu}}\phi^{ABCD}\epsilon_D - \frac{i}{32}\,\hat{\gamma}^{\hat{\mu}\hat{\nu}\hat{\rho}} H_{\hat{\mu}\hat{\nu}\hat{\rho}}{}^{[\![AB}\epsilon^{C]\!]}\,,\\
\delta B^{AB}_{\hat{\mu}\hat{\nu}} &= -i\,\bar{\epsilon}_C\hat{\gamma}_{\hat{\mu}\hat{\nu}}\chi^{ABC} - 2\,i\,\bar{\epsilon}^{[\![A}\psi^{B]\!]}_{\hat{\mu}\hat{\nu}}\,,\\
\delta\psi^A_{\hat{\mu}\hat{\nu}} &= \frac{i}{8}\,\hat{\gamma}^{\hat{\rho}} H_{\hat{\mu}\hat{\nu}\hat{\rho}}{}^{AB}\epsilon_B - \frac{i}{16}\,H_{\hat{\alpha}\hat{\beta}[\hat{\mu}}{}^{AB}\hat{\gamma}_{\hat{\nu}]}{}^{\hat{\alpha}\hat{\beta}}\epsilon_B - \frac{i}{192}\,\hat{\gamma}^{\hat{\alpha}\hat{\beta}\hat{\gamma}} S_{\hat{\alpha}\hat{\beta}\hat{\gamma},\hat{\mu}\hat{\nu}}\epsilon^A\,,\\
\delta T_{\hat{\mu}\hat{\nu},\hat{\rho}\hat{\sigma}} &= -i\,\bar{\epsilon}_A\left(\hat{\gamma}_{\hat{\mu}\hat{\nu}}\psi^A_{\hat{\rho}\hat{\sigma}} + \hat{\gamma}_{\hat{\rho}\hat{\sigma}}\psi^A_{\hat{\mu}\hat{\nu}} - 2\,\hat{\gamma}_{[\hat{\mu}\hat{\nu}}\psi^A_{\hat{\rho}\hat{\sigma}]}\right)\,,
\end{aligned} \tag{5.21}$$

where

$$S_{\hat{\mu}\hat{\nu}\hat{\rho},\hat{\sigma}\hat{\tau}} = 3\,\partial_{[\hat{\mu}}T_{\hat{\nu}\hat{\rho}],\hat{\sigma}\hat{\tau}}\,. \tag{5.22}$$

These supersymmetry transformations are written in terms of the original six-dimensional fields. In the next step, we shall decompose them in the 5+1 split form.

5.2.2 5+1 split of the supersymmetry transformations

Using the 5+1 decomposition discussed in chapter 2, we can derive from (5.21) the transformation of the various components of the fields. We furthermore use the duality relations (2.18) and (2.67) to trade $H_{\mu\nu\rho}{}^{AB}$ and $\partial_{[\mu}C_{\nu\rho],\sigma}$ for their dual field strengths

$F_{\mu\nu}{}^{AB}$ and $\partial_{[\mu}h_{\nu]\rho}$ respectively. After lengthy computations, (5.21) gives rise to

$$\begin{aligned}
\delta\phi^{ABCD} &= i\,\bar\epsilon^{[\![A}\chi^{BCD]\!]}\,,\\
\delta A_\mu^{AB} &= \bar\epsilon_C\gamma_\mu\chi^{ABC} - i\,\bar\epsilon^{[\![A}\psi_\mu^{B]\!]}\,,\\
\delta B_{\mu\nu}^{AB} &= -i\,\bar\epsilon_C\gamma_{\mu\nu}\chi^{ABC} + 2\,\bar\epsilon^{[\![A}\gamma_{[\mu}\psi_{\nu]}^{B]\!]} - 2i\,\bar\epsilon^{[\![A}\psi_{\mu\nu}^{B]\!]}\,,\\
\delta h_{\mu\nu} &= \bar\epsilon_A\gamma_{(\mu}\psi_{\nu)}^A\,,\\
\delta C_{\mu\nu,\rho} &= \bar\epsilon_A\left((\gamma_\rho\psi_{\mu\nu}^A - \gamma_{[\rho}\psi_{\mu\nu]}^A) - i\,(\gamma_{\mu\nu}\psi_\rho^A - \gamma_{[\mu\nu}\psi_{\rho]}^A) - i\,\psi_{[\mu}^A\eta_{\nu]\rho}\right)\,,\\
\delta T_{\mu\nu,\rho\sigma} &= -i\,\bar\epsilon_A\left(\gamma_{\mu\nu}\psi_{\rho\sigma}^A + \gamma_{\rho\sigma}\psi_{\mu\nu}^A - 2\,\gamma_{[\mu\nu}\psi_{\rho\sigma]}^A\right)\\
&\quad + 2\,\bar\epsilon_A\left(\gamma_{[\mu}\eta_{\nu][\rho}\psi_{\sigma]}^A + \gamma_{[\rho}\eta_{\sigma][\mu}\psi_{\nu]}^A\right)\,,\\
\delta\chi^{ABC} &= -i\,\gamma^\mu\epsilon_D\partial_\mu\phi^{ABCD} - \partial_y\phi^{ABCD}\epsilon_D - \frac{3}{16}\,\gamma^{\mu\nu}\epsilon^{[\![A}(F_{\mu\nu}^{BC]\!]} + \partial_y B_{\mu\nu}^{BC]\!]})\,,\\
\delta\psi_\mu^A &= -\frac{i}{4}\,\gamma^\nu\epsilon_B(F_{\mu\nu}^{AB} + \partial_y B_{\mu\nu}^{AB}) + \frac{i}{8}\,\gamma_\mu{}^{\alpha\beta}\epsilon_B(F_{\alpha\beta}^{AB} + \partial_y B_{\alpha\beta}^{AB})\\
&\quad - \frac{1}{8}\,\gamma^{\alpha\beta}\epsilon^A\left(\partial_\alpha h_{\beta\mu} + \frac{1}{2}\partial_y C_{\alpha\beta,\mu} + \frac{1}{4}\partial_y v_{\mu,\alpha\beta} - \frac{1}{2}\partial_\mu u_{\alpha\beta}\right)\,,\\
\delta\psi_{\mu\nu}^A &= \frac{3}{8}(F_{\mu\nu}^{AB} + \partial_y B_{\mu\nu}^{AB})\epsilon_B\\
&\quad + \frac{i}{8}\,\gamma^\alpha\epsilon^A\left(\partial_{[\mu}h_{\nu]\alpha} - \partial_y C_{\alpha[\mu,\nu]} + \frac{1}{2}\partial_y v_{[\nu,\mu]\alpha} + \partial_{[\mu}u_{\nu]\alpha}\right)\\
&\quad - \frac{1}{16}\,\gamma^{\alpha\beta}\epsilon^A\partial_{[\mu}(C_{|\alpha\beta|,\nu]} - \frac{1}{2}v_{\nu],\alpha\beta}) - \frac{i}{16}\,\gamma_{[\nu}{}^{\alpha\beta}\epsilon^A\partial_{\mu]}u_{\alpha\beta}\\
&\quad - \frac{1}{32}\gamma^{\alpha\beta}\epsilon^A\partial_y T_{\alpha\beta,\mu\nu} - \frac{i}{32}\gamma_{[\mu}{}^{\alpha\beta}\epsilon^A\partial_y v_{\nu],\alpha\beta}\,.
\end{aligned}\tag{5.23}$$

These transformations can be simplified further by using a fermionic gauge transformation (2.98) with the specific parameters

$$\begin{aligned}
\kappa_6^A &= \frac{1}{16}\,\gamma^{\mu\nu}\epsilon^A u_{\mu\nu},\\
\kappa_\mu^A &= -\frac{1}{32}\gamma^{\sigma\tau}\epsilon^A\left(C_{\sigma\tau,\mu} - \frac{1}{2}\,v_{\mu,\sigma\tau} - \varepsilon_{\mu\kappa\lambda\sigma\tau}u^{\kappa\lambda}\right).
\end{aligned}\tag{5.24}$$

Indeed the last two transformations of (5.23) reduce to

$$\begin{aligned}
\delta\psi_\mu^A &= -\frac{i}{4}\,\gamma^\nu\epsilon_B(F_{\mu\nu}^{AB} + \partial_y B_{\mu\nu}^{AB}) + \frac{i}{8}\,\gamma_\mu{}^{\alpha\beta}\epsilon_B(F_{\alpha\beta}^{AB} + \partial_y B_{\alpha\beta}^{AB})\\
&\quad - \frac{1}{8}\,\gamma^{\alpha\beta}\epsilon^A\left(\partial_\alpha h_{\beta\mu} + \partial_y\left(C_{\alpha\beta,\mu} - \frac{1}{2}\,\varepsilon_{\mu\alpha\beta\kappa\lambda}u^{\kappa\lambda}\right)\right),\\
\delta\psi_{\mu\nu}^A &= \frac{3}{8}(F_{\mu\nu}^{AB} + \partial_y B_{\mu\nu}^{AB})\epsilon_B + \frac{i}{8}\,\gamma^\alpha\epsilon^A\left(\partial_{[\mu}h_{\nu]\alpha} + \partial_y\left(C_{\mu\nu,\alpha} + \varepsilon_{\mu\nu\alpha\kappa\lambda}u^{\kappa\lambda}\right)\right)\\
&\quad - \frac{1}{32}\gamma^{\alpha\beta}\epsilon^A\partial_y\left(T_{\alpha\beta,\mu\nu} + \varepsilon_{\alpha\beta\kappa\lambda[\mu}\left(C^{\kappa\lambda}{}_{\nu]} - v_{\nu],}{}^{\kappa\lambda} - \varepsilon^{\kappa\lambda}{}_{\nu]\rho\sigma}u^{\rho\sigma}\right)\right)
\end{aligned}\tag{5.25}$$

At this point we observe that these variations can written even more compactly in terms of the new fields $\widehat{\mathcal{T}}_{\mu\rho,\sigma\tau}$, $\widehat{\Omega}_{\mu\nu,\rho}$ and $\mathcal{F}_{\mu\nu}{}^{AB}$ defined as

$$\begin{aligned}
\widehat{\mathcal{T}}_{\mu\nu,\rho\sigma} =& T_{\mu\nu,\rho\sigma} + 2\,(\eta_{\mu[\rho}h_{\sigma]\nu} - \eta_{\nu[\rho}h_{\sigma]\mu}) \\
&+ \varepsilon_{\rho\sigma\kappa\lambda[\mu}\,C^{\kappa\lambda}{}_{\nu]} - \varepsilon_{\rho\sigma\kappa\lambda[\mu}\,v_{\nu]}{}^{\kappa\lambda} \\
&- 4\,\eta_{\mu[\rho}\,u_{\sigma]\nu} + 4\,\eta_{\nu[\rho}\,u_{\sigma]\mu}\,, \qquad (5.26)\\
\widehat{C}_{\mu\nu,\rho} =& C_{\mu\nu,\rho} + \varepsilon_{\mu\nu\rho\sigma\tau}\,u^{\sigma\tau}\,, \qquad (5.27)\\
\widehat{\Omega}_{\mu\nu\rho} =& \partial_{[\mu}h_{\nu]\rho} + \partial_y\widehat{C}_{\mu\nu,\rho}\,, \qquad (5.28)\\
\mathcal{F}_{\mu\nu}{}^{AB} =& F_{\mu\nu}{}^{AB} + \partial_y B_{\mu\nu}{}^{AB}\,. \qquad (5.29)
\end{aligned}$$

In terms of these fields, the fermionic supersymmetry transfomations take the compact form

$$\begin{aligned}
\delta\chi^{ABC} =& -i\,\gamma^{\mu}\epsilon_D\partial_\mu\phi^{ABCD} - \frac{3}{16}\,\gamma^{\mu\nu}\epsilon^{[\![A}\mathcal{F}_{\mu\nu}^{BC]\!]} - \partial_y\phi^{ABCD}\epsilon_D\,,\\
\delta\psi_\mu^A =& -\frac{i}{4}\,\gamma^{\nu}\epsilon_B\mathcal{F}_{\mu\nu}^{AB} + \frac{i}{8}\,\gamma_\mu{}^{\sigma\tau}\epsilon_B\mathcal{F}_{\sigma\tau}^{AB} - \frac{1}{16}\,\gamma^{\sigma\tau}\epsilon^A\left(\widehat{\Omega}_{\sigma\tau,\mu} - 2\,\widehat{\Omega}_{\mu\sigma,\tau}\right)\,,\\
\delta\psi_{\mu\nu}^A =& \frac{3}{8}\mathcal{F}_{\mu\nu}^{AB}\epsilon_B + \frac{i}{8}\,\gamma^{\rho}\epsilon^A\,\widehat{\Omega}_{\mu\nu,\rho} - \frac{1}{32}\gamma^{\alpha\beta}\epsilon^A\partial_y\left(\widehat{\mathcal{T}}_{\mu\nu,\alpha\beta} - 4\,\eta_{\alpha[\mu}h_{\nu]\beta}\right)\,. \qquad (5.30)
\end{aligned}$$

In order to find the supersymmetric extension of our bosonic Lagrangian (3.73), we still need to determine the supersymmetry variations of the fields $u_{\mu\nu}$ and $v_{\rho,\mu\nu}$ which we have introduced in section 2.1.6 by integrating the field equations of the (2,2) field into first-order duality equations. We will do this in the next subsections.

5.2.3 CT duality equation and $\delta v_{\mu,\rho\sigma}$

Let us derive the supersymmetry variations for $u_{\mu\nu}$ and $v_{\mu,\rho\sigma}$ by varying the first-order duality equations by which these fields are defined. As we are working on-shell, we will need to make use of the Rarita-Schwinger equations (2.96) and (2.97)

$$\begin{aligned}
\gamma^{\mu\nu\rho}\,\partial_\nu\phi_\rho^A &= J^{\mu\,A}\,,\\
\partial_{[\mu}\psi_{\nu\rho]}^A &= J_{\mu\nu\rho}{}^A\,, \qquad (5.31)
\end{aligned}$$

for the five-dimensional gravitino field ϕ_ρ^A and the exotic gravitino $\psi_{\nu\rho}^A$, respectively with sources

$$J_\mu{}^A = \partial_y\left(3i\,\gamma_{\mu\nu}\phi^{\nu\,A} - \frac{1}{2}\gamma_{\mu\nu\rho}\psi^{\nu\rho\,A}\right)\,, \qquad (5.32)$$

$$J_{\mu\nu\rho}{}^A = \frac{1}{12}\varepsilon_{\mu\nu\rho\alpha\beta}\partial_y\left(-\gamma^{\alpha\beta\kappa\lambda}\psi_{\kappa\lambda\,A} + 4i\,\gamma^{\alpha\beta\kappa}\psi_{\kappa\,A}\right)\,. \qquad (5.33)$$

In order to compute the variation $\delta v_{\mu,\rho\sigma}$ under supersymmetry, let us start from our first-order field equation (2.65)

$$2\,\partial_{[\sigma}v_{\tau],\mu\nu} = \frac{1}{2}\,\varepsilon_{\mu\nu\alpha\beta\gamma}\,\partial^\alpha T_{\sigma\tau}{}^{\beta\gamma} + \partial_\mu C_{\sigma\tau,\nu} - \partial_\nu C_{\sigma\tau,\mu} + \partial_y T_{\sigma\tau,\mu\nu}\,. \qquad (5.34)$$

Taking the variation of this equation and using the explicit result (5.23) we obtain (to avoid cumbersome notation, all the expressions in the following should be understood as antisymmetrized in $[\mu\nu]$ and $[\sigma\tau]$)

$$
\begin{aligned}
2\,\partial_{[\sigma}\delta v_{\tau],\mu\nu} =& \frac{1}{2}\,\varepsilon_{\mu\nu\alpha\beta\gamma}\,\partial^{\alpha}\delta T_{\sigma\tau}{}^{\beta\gamma} + 2\,\partial_{\mu}\delta C_{\sigma\tau,\nu} + 2\,\partial_{y}\delta T_{\sigma\tau,\mu\nu} \\
=& -\frac{i}{3}\,\varepsilon_{\mu\nu\alpha\beta\gamma}\,\bar{\epsilon}_{A}\left(\gamma^{\beta\gamma}\partial^{\alpha}\psi^{A}_{\sigma\tau} + \gamma_{\sigma\tau}\partial^{\alpha}\psi^{\beta\gamma\,A} - 2\,\gamma^{\beta}{}_{\sigma}\partial^{\alpha}\psi_{\tau}{}^{\gamma\,A}\right) \\
&+ \frac{4}{3}\,\bar{\epsilon}_{A}\left(\gamma_{\nu}\partial_{\mu}\psi^{A}_{\sigma\tau} - \gamma_{\sigma}\partial_{\mu}\psi^{A}_{\tau\nu}\right) \\
&+ \varepsilon_{\mu\nu\alpha\beta\sigma}\,\bar{\epsilon}_{A}\gamma^{\beta}\partial^{\alpha}\phi^{A}_{\tau} - \varepsilon_{\mu\nu\tau\alpha\gamma}\,\bar{\epsilon}_{A}\gamma_{\sigma}\partial^{\alpha}\phi^{\gamma\,A} \\
&- \frac{4}{3}\,i\,\bar{\epsilon}_{A}\left(\gamma_{\sigma\tau}\partial_{\mu}\phi^{A}_{\nu} - \gamma_{\tau\nu}\partial_{\mu}\phi^{A}_{\sigma}\right) - 2\,i\,\bar{\epsilon}_{A}\partial_{\mu}\phi^{A}_{\sigma}\eta_{\tau\nu} \\
&- \frac{4}{3}i\,\bar{\epsilon}_{A}\left(\gamma_{\mu\nu}\partial_{y}\psi^{A}_{\sigma\tau} + \gamma_{\sigma\tau}\partial_{y}\psi^{A}_{\mu\nu} - 2\,\gamma_{\mu\sigma}\partial_{y}\psi^{A}_{\tau\nu}\right) \\
&+ 4\bar{\epsilon}_{A}\left(\gamma_{[\mu}\eta_{\nu][\sigma}\partial_{y}\phi^{A}_{\tau]} + \gamma_{[\sigma}\eta_{\tau][\mu}\partial_{y}\phi^{A}_{\nu]}\right) \\
=& -i\,\varepsilon_{\mu\nu}{}^{\beta\gamma\alpha}\,\bar{\epsilon}_{A}\left(\gamma_{\beta\gamma}\partial_{[\alpha}\psi^{A}_{\sigma\tau]} + \frac{1}{3}\,\gamma_{\sigma\tau}\partial_{[\alpha}\psi_{\beta\gamma]\,A} - \gamma_{\beta\sigma}\partial_{[\alpha}\psi_{\tau\gamma]\,A}\right) \\
&+ i\,\varepsilon_{\mu\nu}{}^{\beta\gamma\alpha}\,\bar{\epsilon}_{A}\left(\frac{2}{3}\,\gamma_{\beta\gamma}\partial_{[\sigma}\psi^{A}_{\tau]\alpha} + \frac{1}{3}\,\gamma_{\beta[\tau}\partial_{\sigma]}\psi_{\gamma\alpha\,A}\right) \\
&+ 4\,\bar{\epsilon}_{A}\left(\gamma_{\nu}\partial_{[\mu}\psi^{A}_{\sigma\tau]} + \frac{1}{2}\,\gamma_{\sigma}\partial_{[\mu}\psi^{A}_{\nu\tau]} - \frac{2}{3}\,\gamma_{\nu}\partial_{[\sigma}\psi^{A}_{\tau]\mu} + \frac{1}{6}\,\gamma_{[\tau}\partial_{\sigma]}\psi^{A}_{\mu\nu}\right) \\
&+ 2i\,\bar{\epsilon}_{A}\gamma_{\mu\nu\alpha\tau}\,\partial^{\alpha}\phi^{A}_{\sigma} + 2i\,\eta_{\nu\sigma}\,\bar{\epsilon}_{A}\gamma_{\mu\tau\alpha\gamma}\partial^{\alpha}\phi^{\gamma\,A} + i\,\bar{\epsilon}_{A}\gamma_{\mu\nu\tau\gamma}\partial_{\sigma}\phi^{\gamma\,A} \\
&- \frac{4}{3}\,i\,\bar{\epsilon}_{A}\left(\gamma_{\sigma\tau}\partial_{\mu}\phi^{A}_{\nu} - \gamma_{\tau\nu}\partial_{\mu}\phi^{A}_{\sigma}\right) - 2\,i\,\bar{\epsilon}_{A}\partial_{\mu}\phi^{A}_{\sigma}\eta_{\tau\nu} \\
&- \frac{4}{3}\,i\,\bar{\epsilon}_{A}\left(\gamma_{\mu\nu}\partial_{y}\psi^{A}_{\sigma\tau} + \gamma_{\sigma\tau}\partial_{y}\psi^{A}_{\mu\nu} - 2\,\gamma_{\mu\sigma}\partial_{y}\psi^{A}_{\tau\nu}\right) \\
&+ 4\,\bar{\epsilon}_{A}\left(\gamma_{[\mu}\eta_{\nu][\sigma}\partial_{y}\phi^{A}_{\tau]} + \gamma_{[\sigma}\eta_{\tau][\mu}\partial_{y}\phi^{A}_{\nu]}\right)\,. \qquad (5.35)
\end{aligned}
$$

Now we have to rewrite this expression on the right-hand side as a curl on $[\sigma\tau]$ to identify the variation of $v_{\tau,\mu\nu}$. We will do this computation piece by piece, starting with the terms containing the gravitino $\phi_{\mu}{}^{A}$ and spacetime derivatives. These terms are collected in the fourth and fifth line of (5.35). Once again, the following should be

understood as antisymmetrized on $[\mu\nu]$ and $[\sigma\tau]$

$$
\begin{aligned}
\overset{[\mu\nu][\sigma\tau]}{\longrightarrow} &+ 2i\,\gamma_{\mu\nu\alpha\tau}\,\partial^{\alpha}\phi^{A}_{\sigma} + 2i\,\eta_{\nu\sigma}\,\gamma_{\mu\tau\alpha\gamma}\partial^{\alpha}\phi^{\gamma\,A} - i\,\gamma_{\mu\nu\tau\gamma}\partial_{\sigma}\phi^{\gamma\,A} \\
&- \frac{4}{3}i\left(\gamma_{\sigma\tau}\partial_{\mu}\phi^{A}_{\nu} - \gamma_{\tau\nu}\partial_{\mu}\phi^{A}_{\sigma}\right) - 2\,i\,\partial_{\mu}\phi^{A}_{\sigma}\eta_{\tau\nu} \\
&= -4i\,\gamma_{\mu\nu\tau}{}^{\alpha}\,\partial_{[\alpha}\phi^{A}_{\sigma]} + 2i\,\eta_{\nu\sigma}\,\gamma_{\mu\tau\alpha\gamma}\partial^{[\alpha}\phi^{\gamma]\,A} \\
&- \frac{4}{3}i\,\gamma_{\sigma\tau}\partial_{[\mu}\phi^{A}_{\nu]} + \frac{8}{3}i\,\gamma_{\tau\nu}\partial_{[\mu}\phi^{A}_{\sigma]} - 4i\,\eta_{\tau\nu}\partial_{[\mu}\phi^{A}_{\sigma]} \\
&- i\,\gamma_{\mu\nu\tau\gamma}\partial_{\sigma}\phi^{\gamma\,A} + \frac{4}{3}i\,\gamma_{\tau\nu}\partial_{\sigma}\phi^{A}_{\mu} - 2i\,\eta_{\nu\tau}\partial_{\sigma}\phi^{A}_{\mu}\,.
\end{aligned}
\tag{5.36}
$$

Using the Rarita-Schwinger equation (5.32), this expression can be put into the form

$$
\begin{aligned}
\overset{[\mu\nu][\sigma\tau]}{\longrightarrow} &- 4i\,\gamma_{\mu\nu\sigma}{}^{\rho}\,\partial_{[\tau}\phi^{A}_{\rho]} - 4i\,\eta_{\mu\sigma}\,\gamma_{\nu}{}^{\rho}\partial_{[\tau}\phi^{A}_{\rho]} \\
&- \frac{4}{3}i\,\gamma_{\sigma\tau}\partial_{[\mu}\phi^{A}_{\nu]} + \frac{8}{3}i\,\gamma_{\sigma\mu}\partial_{[\nu}\phi^{A}_{\tau]} - 4i\,\eta_{\sigma\mu}\partial_{[\nu}\phi^{A}_{\tau]} + 8i\,\eta_{\sigma\mu}\,\partial_{[\nu}\phi^{A}_{\tau]} + \frac{8}{3}i\,\gamma_{\mu\nu}\partial_{[\sigma}\phi^{A}_{\tau]} \\
&- i\,\gamma_{\mu\nu\tau\gamma}\partial_{\sigma}\phi^{\gamma\,A} + \frac{4}{3}i\,\gamma_{\tau\nu}\partial_{\sigma}\phi^{A}_{\mu} - 2i\,\eta_{\nu\tau}\partial_{\sigma}\phi^{A}_{\mu} - \frac{8}{3}i\,\gamma_{\mu\nu}\partial_{\sigma}\phi^{A}_{\tau} \\
&+ 2\eta_{\mu\sigma}\,\gamma_{\tau}J_{\nu} - \frac{8}{3}\,\eta_{\mu\sigma}\,\gamma_{\nu}J_{\tau} + \frac{2}{3}\,\eta_{\nu\sigma}\,\eta_{\tau\mu}\gamma_{\nu}J^{\nu} + \frac{4}{3}\,\eta_{\mu\sigma}\,\gamma_{\nu}\gamma_{\tau\rho}J^{\rho} \\
= &\frac{8i}{3}\,\gamma_{\mu\sigma}\partial_{[\nu}\phi_{\tau]} - \frac{8}{3}i\,\gamma_{\mu\sigma}\partial_{[\nu}\phi^{A}_{\tau]} - \frac{8}{3}i\,\gamma_{\mu\nu}\partial_{[\sigma}\phi_{\tau]} + \frac{8}{3}i\,\gamma_{\mu\nu}\partial_{[\sigma}\phi^{A}_{\tau]} \\
&+ \frac{4i}{3}\,\gamma_{\sigma\tau}\partial_{[\mu}\phi_{\nu]} - \frac{4}{3}i\,\gamma_{\sigma\tau}\partial_{[\mu}\phi^{A}_{\nu]} \\
&- 4i\,\eta_{\mu\sigma}\,\partial_{[\nu}\phi_{\tau]} - 4i\,\eta_{\mu\sigma}\partial_{[\nu}\phi^{A}_{\tau]} + 8i\,\eta_{\mu\sigma}\,\partial_{[\nu}\phi^{A}_{\tau]} \\
&- i\,\gamma_{\mu\nu\tau\gamma}\partial_{\sigma}\phi^{\gamma\,A} + \frac{4}{3}i\,\gamma_{\tau\nu}\partial_{\sigma}\phi^{A}_{\mu} - 2i\,\eta_{\nu\tau}\partial_{\sigma}\phi^{A}_{\mu} - \frac{8}{3}i\,\gamma_{\mu\nu}\partial_{\sigma}\phi^{A}_{\tau} \\
&- \frac{4}{3}\,\eta_{\mu\sigma}\,\gamma_{\nu}J_{\tau} + \frac{8}{3}\,\eta_{\mu\sigma}\,\gamma_{\tau}J_{\nu} + 2\,\eta_{\mu\sigma}\gamma_{\nu\tau\rho}J^{\rho} - \frac{4}{3}\,\gamma_{[\mu}\gamma_{|\sigma\tau|}J_{\nu]} + \frac{2}{3}\,\gamma_{[\sigma}\gamma_{|\mu\nu|}J_{\tau]} \\
= &\,2\partial_{\sigma}\left(-\frac{1}{2}\,i\,\gamma_{\mu\nu\tau\gamma}\phi^{\gamma\,A} - \frac{4}{3}i\,\gamma_{\mu\nu}\phi^{A}_{\tau} - \frac{2}{3}\,i\,\gamma_{\tau\mu}\phi^{A}_{\nu} + i\,\eta_{\tau\mu}\phi^{A}_{\nu}\right) \\
&+ \frac{2}{3}\,\gamma_{\mu\nu\sigma}J_{\tau} - \frac{4}{3}\,\gamma_{\sigma\tau\mu}J_{\nu} + 2\,\eta_{\mu\sigma}\gamma_{\nu\tau\rho}J^{\rho}\,,
\end{aligned}
\tag{5.37}
$$

We have thus recast these terms into the form of a curl in $[\sigma\tau]$ up to terms carrying the source $J_{\mu}{}^{A}$ which all appear under ∂_y derivative. These terms will eventually cancel against similar contributions. Let us now turn to the exotic gravitino terms in (5.35) that are not already in a curl on $[\sigma\tau]$ form. They are in the first and third line of (5.35)

and after some manipulations on the indices, they can be put into the form

$$\begin{aligned}
\overset{[\mu\nu][\sigma\tau]}{\longrightarrow} \bar{\epsilon}_A \Big(& -\gamma_{\mu\nu}{}^{\alpha\beta}{}_{\sigma}\, \partial_{[\tau}\psi^A_{\alpha\beta]} - 6\,\gamma_{\mu\nu}{}^{\alpha}\, \partial_{[\alpha}\psi^A_{\sigma\tau]} + 2\,\eta_{\mu\sigma}\gamma_{\nu}{}^{\alpha\beta}\, \partial_{[\tau}\psi^A_{\alpha\beta]} \\
& + \frac{2}{3}\,\eta_{\mu\sigma}\,\gamma_{\nu\tau}{}^{\alpha\beta\gamma}\partial_{[\alpha}\psi^A_{\beta\gamma]} - \frac{2}{3}\,\eta_{\sigma\mu}\,\eta_{\nu\tau}\gamma^{\alpha\beta\gamma}\,\partial_{[\alpha}\psi^A_{\beta\gamma]} \\
& - 4\,\gamma_{\mu}\partial_{[\nu}\psi^A_{\sigma\tau]} + 2\,\gamma_{\sigma}\partial_{[\mu}\psi^A_{\nu\tau]}\Big) \\
& + \partial_\sigma \left(\frac{2}{3}\,\bar{\epsilon}_A\gamma_\tau\psi^A_{\mu\nu} - \frac{8}{3}\,\bar{\epsilon}_A\gamma_{[\mu}\psi^A_{\nu]\tau} \right) .
\end{aligned} \tag{5.38}$$

The last line of this equation is now in a curl form while all the remaining terms are proportional to the Rarita-Schwinger equation of the exotic gravitino (5.31) and can be replaced by the corresponding source. Finally, we note that the second line of (5.35) is already in a curl form

$$\overset{[\mu\nu][\sigma\tau]}{\longrightarrow} \partial_\sigma \left(2\,\bar{\epsilon}_A\gamma_{\mu\nu}{}^{\rho}\,\psi^A_{\tau\rho} + \frac{2}{3}\,\eta_{\mu\tau}\,\bar{\epsilon}_A\gamma_{\nu}{}^{\alpha\beta}\,\psi^A_{\alpha\beta} \right) . \tag{5.39}$$

Let us now bring all of these pieces together. Equation (5.35) then yields

$$\begin{aligned}
2\,\partial_{[\sigma}\delta v_{\tau],\mu\nu} = & \, 2\,\partial_\sigma \left(\frac{1}{3}\,\bar{\epsilon}_A\gamma_\tau\psi^A_{\mu\nu} - \frac{4}{3}\,\bar{\epsilon}_A\gamma_{[\mu}\psi^A_{\nu]\tau} \right) \\
& + 2\,\partial_\sigma \left(\bar{\epsilon}_A\gamma_{\mu\nu}{}^{\rho}\,\psi^A_{\tau\rho} + \frac{1}{3}\,\eta_{\mu\tau}\,\bar{\epsilon}_A\gamma_{\nu}{}^{\alpha\beta}\,\psi^A_{\alpha\beta} \right) \\
& + 2\partial_\sigma \left(-\frac{1}{2}\,i\,\gamma_{\mu\nu\tau\gamma}\phi^{\gamma\,A} - \frac{4}{3} i\,\gamma_{\mu\nu}\phi^A_\tau - \frac{2}{3}\,i\,\gamma_{\tau\mu}\phi^A_\nu + i\,\eta_{\tau\mu}\phi^A_\nu \right) \\
& + \partial_y\,(\dots) \ ,
\end{aligned} \tag{5.40}$$

where the last term denotes all the terms carrying a ∂_y derivative, in particular all the source terms in the Rarita-Schwinger equations. For consistency, one may check that these terms mutually cancel. We are thus left with the variation

$$\begin{aligned}
\delta v_{\tau,\mu\nu} = & \, \bar{\epsilon}_A\gamma_{\mu\nu}{}^{\rho}\,\psi^A_{\tau\rho} + \bar{\epsilon}_A\gamma_\tau\psi^A_{\mu\nu} - 2\,\bar{\epsilon}_A\gamma_{[\tau}\psi^A_{\mu\nu]} + \frac{1}{3}\,\eta_{\tau[\mu}\,\bar{\epsilon}_A\gamma_{\nu]}{}^{\alpha\beta}\,\psi^A_{\alpha\beta} \\
& - \frac{1}{2}\,i\,\bar{\epsilon}_A\gamma_{\mu\nu\tau\gamma}\phi^{\gamma\,A} - i\,\bar{\epsilon}_A\gamma_{\mu\nu}\phi^A_\tau - i\,\bar{\epsilon}_A\gamma_{[\mu\nu}\phi^A_{\tau]} + i\,\eta_{\tau[\mu}\bar{\epsilon}_A\phi^A_{\nu]} .
\end{aligned} \tag{5.41}$$

5.2.4 hC duality equation and $\delta u_{\mu\nu}$

Now that we have computed the variation of $v_{\rho,\mu\nu}$, we can now compute the variation of $u_{\mu\nu}$. For this we start from the first order duality equation (2.67)

$$\partial_{[\mu}h_{\nu]\rho} + \frac{1}{4}\,\varepsilon_{\mu\nu\lambda\sigma\tau}\,\partial^\lambda C^{\sigma\tau}{}_{\rho} = \partial_\rho u_{\mu\nu} - \frac{1}{2}\,\partial_y C_{\mu\nu,\rho} - \frac{1}{2}\,\partial_y v_{\rho,\mu\nu} \ , \tag{5.42}$$

Taking its variation and using the expression (5.23) above, we obtain

$$\begin{aligned}\delta\left(\partial_{[\mu}h_{\nu]\rho} + \frac{1}{4}\,\varepsilon_{\mu\nu\lambda\sigma\tau}\,\partial^{\lambda}C^{\sigma\tau}{}_{\rho}\right) =& \partial_{[\mu}\delta h_{\nu]\rho} + \frac{1}{4}\,\varepsilon_{\mu\nu\lambda\sigma\tau}\,\partial^{\lambda}\delta C^{\sigma\tau}{}_{\rho} \\ \overset{[\mu\nu]}{=}& \frac{1}{3}i\,\eta_{\mu\rho}\,\bar{\epsilon}_A\gamma_{\nu\lambda\sigma\tau}\partial^{[\lambda}\psi^{\sigma\tau]\,A} + \frac{3}{4}i\,\bar{\epsilon}_A\gamma_{\mu\nu}{}^{\lambda\tau}\partial_{[\lambda}\psi_{\tau\rho]}{}^{A} \\ & - \frac{1}{12}i\,\bar{\epsilon}_A\gamma_{\mu\nu}{}^{\lambda\tau}\partial_{\rho}\psi^{A}_{\lambda\tau} - \frac{1}{3}\bar{\epsilon}_A\gamma_{\mu\nu}{}^{\lambda}\partial_{\rho}\phi^{A}_{\lambda} \\ & - \frac{1}{2}\,\bar{\epsilon}_A\gamma_{\mu}\partial_{\rho}\phi^{A}_{\nu} \\ & + \frac{3}{4}\,\bar{\epsilon}_A\gamma_{[\mu\nu}J_{\rho]} - 2\,\gamma_{\mu\nu}J_{\rho} - \frac{1}{3}\,\eta_{\rho\mu}\bar{\epsilon}_A\gamma_{\nu\lambda}J^{\lambda}\,. \end{aligned} \tag{5.43}$$

The first two terms carry the exotic gravitino and can be replaced by their source term. Combining this with equation (5.42), one may check that all terms carrying ∂_y cancel and we can conclude that

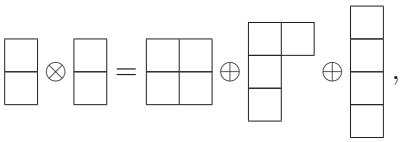
$$\delta u_{\mu\nu} = -\frac{1}{2}\,\bar{\epsilon}_A\gamma_{[\mu}\phi^{A}_{\nu]} - \frac{1}{3}\bar{\epsilon}_A\gamma_{\mu\nu}{}^{\rho}\phi^{A}_{\rho} - \frac{1}{12}i\,\bar{\epsilon}_A\gamma_{\mu\nu}{}^{\rho\sigma}\psi^{A}_{\rho\sigma}\,. \tag{5.44}$$

With these results we may now compute the supersymmetric variation of the new field $\widehat{\mathcal{T}}_{\mu\nu,\rho\sigma}$ introduced in equation (5.26). After some lengthy computation this leads to the suprisingly compact result

$$\delta\widehat{\mathcal{T}}_{\mu\nu,\rho\sigma} = -2i\,\bar{\epsilon}_A\gamma_{\rho\sigma}\,\psi^{A}_{\mu\nu}\,. \tag{5.45}$$

5.2.5 The field $\widehat{\mathcal{T}}_{\mu\nu,\rho\sigma}$

We have seen in the above that the supersymmetry variations of the fields take a very compact form when expressed in terms of the field $\widehat{\mathcal{T}}_{\mu\nu,\rho\sigma}$ introduced in (5.26). Contrary to the original field $T_{\mu\nu,\rho\sigma}$, the new field is no irreducible under the Lorentz group i.e. it carries all the components of the tensor product



(5.46)

and not just the  component. In this section we present the field equations in terms of this new field. A lengthy computation shows that the first-order duality equations (2.65) and (2.67) imply that the curl of $\widehat{\mathcal{T}}_{\mu\nu,\rho\sigma}$ is given by

$$3\,\partial_{[\mu}\widehat{\mathcal{T}}_{\nu\rho],\sigma\tau} = \partial_y\mathcal{J}_{\mu\nu\rho,\sigma\tau}\,, \tag{5.47}$$

with the current

$$\mathcal{J}_{\mu\nu\rho,\sigma\tau} = \frac{3}{2}\,\varepsilon_{\sigma\tau\kappa\lambda[\mu}\widehat{\mathcal{T}}_{\nu\rho]}{}^{\kappa\lambda} + 6\,\eta_{\sigma[\mu}\widehat{C}_{\nu\rho],\tau} - 6\,\eta_{\tau[\mu}\widehat{C}_{\nu\rho],\sigma} - 6\,\varepsilon_{\sigma\tau\lambda[\mu\nu}h_{\rho]}{}^{\lambda}\,, \tag{5.48}$$

and the field $\widehat{C}_{\mu\nu,\rho}$ defined in (5.27) above. We can also show that the gauge transformations (2.70) and (3.77) imply that the new field transforms as

$$\begin{aligned}\delta\widehat{\mathcal{T}}_{\mu\nu,\rho\sigma} =& 2\,\partial_{[\mu}\Lambda_{\nu],\rho\sigma} - \varepsilon_{\rho\sigma\kappa\lambda[\mu}\,\partial_y\Lambda_{\nu]}{}^{\kappa\lambda} - 4\,\eta_{\rho[\mu}\partial_y\Gamma_{\nu]\sigma} + 4\,\eta_{\sigma[\mu}\partial_y\Gamma_{\nu]\rho} \\ & - 4\,\varepsilon_{\mu\nu\rho\sigma\tau}\,\partial_y\xi^\tau\ ,\end{aligned} \tag{5.49}$$

where the gauge parameters are related to the original ones

$$\begin{aligned}\Gamma_{\mu\nu} =& \alpha_{\mu\nu} - \beta_{\mu\nu}\ , \\ \Lambda_{\nu,\rho\sigma} =& \lambda_{\nu,\rho\sigma} + \frac{1}{2}\,\varepsilon_{\nu\rho\sigma\kappa\lambda}\,\zeta^{\kappa\lambda} - \frac{2}{3}\,\varepsilon_{\nu\rho\sigma\kappa\lambda}\,\beta^{\kappa\lambda} - 4\,\eta_{\nu[\rho}\,\xi_{\sigma]}\ ,\end{aligned} \tag{5.50}$$

The form of the field equations (5.47) shows that after dimensional reduction ($\partial_y \to 0$), the field $\widehat{\mathcal{T}}_{\mu\nu,\rho\sigma}$ is pure gauge and can be set to 0 using (5.49). This reflect the fact that the double dual graviton in five dimensions does not carry separate degrees of freedom but can be related to the spin-2 field by an algebraic relation [49, 50]. Furthermore the field equation (5.47) can be obtained by variation of the Chern-Simons type Lagrangian

$$\mathcal{L} = -\frac{1}{64}\,\varepsilon^{\mu\nu\rho\sigma\tau}\partial_\mu\widehat{\mathcal{T}}_{\nu\rho,\alpha\beta}\partial_y\widehat{\mathcal{T}}_{\sigma\tau}{}^{\alpha\beta} + \frac{1}{96}\,\varepsilon^{\mu\nu\rho\sigma\tau}\partial_y\widehat{\mathcal{T}}_{\mu\nu,\alpha\beta}\partial_y\mathcal{J}'_{\rho\sigma\tau}{}^{\alpha\beta} \tag{5.51}$$

with

$$\mathcal{J}'_{\mu\nu\rho,\sigma\tau} = \frac{3}{4}\,\varepsilon_{\sigma\tau\kappa\lambda[\mu}\widehat{\mathcal{T}}_{\nu\rho]}{}^{\kappa\lambda} + 6\,\eta_{\sigma[\mu}\widehat{C}_{\nu\rho],\tau} - 6\,\eta_{\tau[\mu}\widehat{C}_{\nu\rho],\sigma} - 6\,\varepsilon_{\sigma\tau\lambda[\mu\nu}h_{\rho]}{}^{\lambda}\ . \tag{5.52}$$

Indeed, after plugging (5.26) into this Lagrangian, we recover all the corresponding couplings from (3.73) above. This Lagrangian is invariant under the gauge transformations

$$\begin{aligned}\delta h_{\mu\nu} &= 2\,\partial_{(\mu}\xi_{\nu)} - 2\,\partial_y\Gamma_{(\mu\nu)} \\ \delta\widehat{C}_{\mu\nu,\rho} &= 2\,\partial_{[\mu}\Gamma_{\nu]\rho} + \varepsilon_{\mu\nu\rho\sigma\tau}\partial^\sigma\xi^\tau + \partial_y\Lambda_{[\mu,\nu]\rho} - \varepsilon_{\mu\nu\rho\sigma\tau}\partial_y\Gamma^{\sigma\tau} - 2\,\eta_{\rho[\mu}\partial_y\xi_{\nu]}\ ,\end{aligned} \tag{5.53}$$

combined with (5.49) above. This provides a compact way of expressing the dynamics of the exotic tensor field of this model.

5.2.6 The $\mathcal{N} = (4,0)$ supersymmetric Lagrangian in 5+1 split

The new set of fields provides a convenient starting point to construct a supersymmetric Lagrangian of the $\mathcal{N} = (4,0)$ model. First, one may verify that the full bosonic Lagrangian of (3.78) takes the form

$$\begin{aligned}
\mathcal{L}_{(4,0),\text{ bos}} = &-\frac{1}{4}\,\widehat{\Omega}^{\mu\nu\rho}\widehat{\Omega}_{\mu\nu\rho} + \frac{1}{2}\,\widehat{\Omega}^{\mu\nu\rho}\widehat{\Omega}_{\nu\rho\mu} + \widehat{\Omega}^{\mu}\widehat{\Omega}_{\mu} \\
&-\frac{8}{3}\,\partial^{\mu}\phi_{ABCD}\partial_{\mu}\phi^{ABCD} - \frac{1}{2}\,\mathcal{F}^{\mu\nu}{}_{AB}\mathcal{F}_{\mu\nu}{}^{AB} \\
&-\frac{1}{64}\,\varepsilon^{\mu\nu\rho\sigma\tau}\partial_{\mu}\widehat{\mathcal{T}}_{\nu\rho,\alpha\beta}\partial_{y}\widehat{\mathcal{T}}_{\sigma\tau}{}^{\alpha\beta} + \frac{1}{96}\,\varepsilon^{\mu\nu\rho\sigma\tau}\partial_{y}\widehat{\mathcal{T}}_{\mu\nu,\alpha\beta}\partial_{y}\mathcal{J}'_{\rho\sigma\tau}{}^{\alpha\beta}\,, \\
&-\frac{1}{4}\,\varepsilon^{\mu\nu\rho\sigma\tau}\partial_{\mu}B_{\nu\rho,AB}\partial_{y}B_{\sigma\tau}{}^{AB} - \frac{1}{8}\,\varepsilon^{\mu\nu\rho\sigma\tau}\,\partial_{\mu}\widehat{C}_{\nu\rho,\alpha}\,\partial_{y}\widehat{C}_{\sigma\tau}{}^{\alpha} \\
&-\frac{8}{3}\,\partial_{y}\phi_{ABCD}\partial_{y}\phi^{ABCD} + \frac{3}{4}\,\partial_{y}h_{\mu}{}^{\sigma}\,\partial_{y}h_{\sigma}{}^{\mu} - \frac{3}{4}\,\partial_{y}h_{\mu}{}^{\mu}\,\partial_{y}h_{\nu}{}^{\nu}\,,
\end{aligned} \tag{5.54}$$

with $\mathcal{J}'_{\mu\rho,\sigma\tau}$ defined in equation (5.52) above. The fermionic field equations (2.96) and (2.97) can be obtained by the Lagrangian

$$\begin{aligned}
\mathcal{L}_{(4,0),\text{ fer}} = &-\frac{8}{3}\,\bar{\chi}_{ABC}\gamma^{\mu}\partial_{\mu}\chi^{ABC} - 2\,\bar{\psi}_{\mu\,A}\gamma^{\mu\nu\rho}\partial_{\nu}\psi_{\rho}{}^{A} + \bar{\psi}_{\mu\nu\,A}\gamma^{\mu\nu\rho\sigma\tau}\partial_{\rho}\psi_{\sigma\tau}{}^{A} \\
&-\frac{8i}{3}\,\bar{\chi}_{ABC}\,\partial_{y}\chi^{ABC} + 6\,i\,\bar{\psi}_{\mu\,A}\gamma^{\mu\nu}\partial_{y}\psi^{A}_{\nu} \\
&+ i\,\bar{\psi}_{\mu\nu\,A}\gamma^{\mu\nu\rho\sigma}\partial_{y}\psi^{A}_{\rho\sigma} + 4\,\bar{\psi}_{\mu\nu\,A}\gamma^{\mu\nu\rho}\partial_{y}\psi^{A}_{\rho}\,,
\end{aligned} \tag{5.55}$$

so that the full Lagrangian is given by

$$\mathcal{L}_{(4,0)} = \mathcal{L}_{(4,0),\text{ bos}} + \mathcal{L}_{(4,0),\text{ fer}}\,, \tag{5.56}$$

and one may check that it is supersymmetric under the transformations

$$\begin{aligned}
\delta\phi^{ABCD} &= i\,\bar{\epsilon}^{[\![A}\chi^{BCD]\!]}\,, \\
\delta A^{AB}_{\mu} &= \bar{\epsilon}_{C}\gamma_{\mu}\chi^{ABC} - i\,\bar{\epsilon}^{[\![A}\psi^{B]\!]}_{\mu}\,, \\
\delta B^{AB}_{\mu\nu} &= -i\,\bar{\epsilon}_{C}\gamma_{\mu\nu}\chi^{ABC} + 2\,\bar{\epsilon}^{[\![A}\gamma_{[\mu}\psi^{B]\!]}_{\nu]} - 2i\,\bar{\epsilon}^{[\![A}\psi^{B]\!]}_{\mu\nu}\,, \\
\delta h_{\mu\nu} &= \bar{\epsilon}_{A}\gamma_{(\mu}\psi^{A}_{\nu)}\,, \\
\delta\widehat{C}_{\mu\nu,\rho} &= \bar{\epsilon}_{A}\gamma_{\rho}\psi^{A}_{\mu\nu} + \frac{i}{2}\,\bar{\epsilon}_{A}\left(4\,\gamma_{\rho[\mu}\eta_{\nu]\kappa} - 2\,\eta_{\kappa[\mu}\eta_{\nu]\rho} + \gamma_{\mu\nu\rho\kappa}\right)\psi^{\kappa\,A}\,, \\
\delta\widehat{\mathcal{T}}_{\mu\nu,\rho\sigma} &= -2i\,\bar{\epsilon}_{A}\gamma_{\rho\sigma}\,\psi^{A}_{\mu\nu}\,,
\end{aligned}$$

$$\begin{aligned}
\delta\chi^{ABC} &= -i\,\gamma^{\mu}\epsilon_{D}\partial_{\mu}\phi^{ABCD} - \frac{3}{16}\,\gamma^{\mu\nu}\epsilon^{[\![A}\mathcal{F}^{BC]\!]}_{\mu\nu} - \partial_{y}\phi^{ABCD}\epsilon_{D}\,, \\
\delta\psi^{A}_{\mu} &= -\frac{i}{3}\,\gamma^{\nu}\epsilon_{B}\mathcal{F}^{AB}_{\mu\nu} + \frac{i}{12}\,\gamma_{\mu}{}^{\sigma\tau}\epsilon_{B}\mathcal{F}^{AB}_{\sigma\tau} - \frac{1}{16}\,\gamma^{\sigma\tau}\epsilon^{A}\left(\widehat{\Omega}_{\sigma\tau,\mu} - 2\,\widehat{\Omega}_{\mu\sigma,\tau}\right)\,, \\
\delta\psi^{A}_{\mu\nu} &= \frac{1}{2}\mathcal{F}^{AB}_{\mu\nu}\epsilon_{B} + \frac{i}{8}\,\gamma^{\rho}\epsilon^{A}\,\widehat{\Omega}_{\mu\nu,\rho} - \frac{1}{32}\gamma^{\alpha\beta}\epsilon^{A}\partial_{y}\left(\widehat{\mathcal{T}}_{\mu\nu,\alpha\beta} - 4\,\eta_{\alpha[\mu}h_{\nu]\beta}\right)\,,
\end{aligned} \tag{5.57}$$

derived above. This is the main result of this section.

The Lagrangian is invariant under the gauge transformations for the bosonic fields

$$\begin{aligned}
\delta h_{\mu\nu} &= 2\,\partial_{(\mu}\xi_{\nu)} - 2\,\partial_y \Gamma_{(\mu\nu)} \\
\delta A_\mu{}^{AB} &= \partial_\mu \Lambda^{AB} - \partial_y \Xi_\mu{}^{AB} \,, \\
\delta B_{\mu\nu}{}^{AB} &= 2\,\partial_{[\mu}\Xi_{\nu]}{}^{AB} \,, \\
\delta \widehat{C}_{\mu\nu,\rho} &= 2\,\partial_{[\mu}\Gamma_{\nu]\rho} + \varepsilon_{\mu\nu\rho\sigma\tau}\partial^\sigma \xi^\tau + \partial_y \Lambda_{[\mu,\nu]\rho} - \varepsilon_{\mu\nu\rho\sigma\tau}\partial_y \Gamma^{\sigma\tau} - 2\,\eta_{\rho[\mu}\partial_y \xi_{\nu]} \,, \\
\delta \widehat{\mathcal{T}}_{\mu\nu,\rho\sigma} &= 2\,\partial_{[\mu}\Lambda_{\nu],\rho\sigma} - \varepsilon_{\rho\sigma\kappa\lambda[\mu}\,\partial_y \Lambda_{\nu]}{}^{\kappa\lambda} - 4\,\eta_{\rho[\mu}\partial_y \Gamma_{\nu]\sigma} + 4\,\eta_{\sigma[\mu}\partial_y \Gamma_{\nu]\rho} \\
&\quad - 4\,\varepsilon_{\mu\nu\rho\sigma\tau}\,\partial_y \xi^\tau \,,
\end{aligned} \tag{5.58}$$

and on the fermionic fields

$$\begin{aligned}
\delta \psi^A_{\mu\nu} &= 2\,\partial_{[\mu}\kappa^A_{\nu]} + i\,\partial_{[\mu}\gamma_{\nu]}\kappa^A + 2\,i\,\partial_y \gamma_{[\mu}\kappa^A_{\nu]} \,, \\
\delta \psi^A_\mu &= \partial_\mu \kappa^A - 2\,\partial_y \kappa^A_\mu \,.
\end{aligned} \tag{5.59}$$

In addition the Lagrangian is invariant under translations of all the fields

$$\delta\Phi = \Xi^\mu \partial_\mu \Phi + \Xi^6 \partial_y \Phi \tag{5.60}$$

One may show that the supersymmetry transformations (5.57) close on-shell into gauge transformations with parameters

$$\begin{aligned}
\xi_\mu &= -\frac{1}{2}\,\Xi^\rho h_{\mu\rho} \,, \\
\Gamma_{\mu\nu} &= -\,\Xi^\rho\,\widehat{C}_{\rho\mu,\nu} + \frac{1}{2}\,\Xi^6 h_{\mu\nu} \,, \\
\Lambda_{\rho,\mu\nu} &= \Xi^\sigma \widehat{\mathcal{T}}_{\rho\sigma,\mu\nu} + 2\,\Xi_{[\mu}h_{\nu]\rho} \,, \\
\Lambda^{AB} &= -\,2\,i\,\bar{\epsilon}_{C,2}\epsilon_{D,1}\phi^{ABCD} \,,
\end{aligned} \tag{5.61}$$

and translations with parameters

$$\Xi^\mu = \frac{1}{4}\,\bar{\epsilon}_{2,A}\gamma^\mu \epsilon_1^A \,. \Xi^6 = -\frac{i}{4}\,\bar{\epsilon}_{2,A}\epsilon_1^A \,. \tag{5.62}$$

5.3 $\mathcal{N} = (3,1)$

Finally, we look at the $\mathcal{N} = (3,1)$ theory whose six-dimensional field content is reviewed in Table 1.2 and whose action (in the 5+1 split form) has been discussed in Section 3.2. The strategy to derive the supersymmetry variations of the fields follows what was done for the $\mathcal{N} = (4,0)$ theory: we first compute the six-dimensional variations by imposing the closure of the algebra, after which we rewrite the result in a 5+1 split form. This will be discussed in this section. A Usp(8)-covariant formulation of this model as would be required for an ExFT formulation (discussed in Chapter 4 for the bosonic sector) is left for future work.

5.3.1 Supersymmetry variations of the (3,1) model

We start building the supersymmetry algebra in six dimensions by making an ansatz on its general form. Since the structure of all possible terms must match the structure of the fields in terms of spacetime indices, spinorial indices and R-symmetry indices, the system is very constrained. Compared to the $\mathcal{N} = (4,0)$ model, the algebra is more involved since the R-symmetry group is a product $\mathrm{Usp}(6) \times \mathrm{Usp}(2)$. Consequently, there are two supersymmetry parameters ϵ^a and ϵ^α, transforming in the fundamental representations of Usp(6) and Usp(2) respectively. The most general ansatz for the supersymmetry variations of all fields of the $\mathcal{N} = (3,1)$ model is as follows

$$\delta\phi^{abc\alpha} = \bar{\epsilon}^\alpha \chi^{abc} + \bar{\epsilon}^{[\![a} \chi^{bc]\!]\alpha} \,, \tag{5.63}$$

$$\delta\chi^{abc} = \hat{\gamma}^{\hat{\mu}} \partial_{\hat{\mu}} \phi^{abc\alpha} \epsilon_\alpha + \hat{\gamma}^{\hat{\mu}\hat{\nu}} F^{[\![ab}_{\hat{\mu}\hat{\nu}} \epsilon^{c]\!]} \,, \tag{5.64}$$

$$\delta\chi^{ab\alpha} = \hat{\gamma}^{\hat{\mu}} \partial_{\hat{\mu}} \phi^{abc\alpha} \epsilon_c + \hat{\gamma}^{\hat{\mu}\hat{\nu}} F^{ab}_{\hat{\mu}\hat{\nu}} \epsilon^\alpha + \hat{\gamma}^{\hat{\mu}\hat{\nu}\hat{\rho}} H^{[\![a|\alpha|}_{\hat{\mu}\hat{\nu}\hat{\rho}} \epsilon^{b]\!]} \,, \tag{5.65}$$

$$\delta A^{ab}_{\hat{\mu}} = \mathrm{u}_\chi \, \bar{\epsilon}_c \hat{\gamma}_{\hat{\mu}} \chi^{abc} + \mathrm{v}_\chi \, \bar{\epsilon}_\alpha \hat{\gamma}_{\hat{\mu}} \chi^{ab\alpha} + \bar{\epsilon}^{[\![a} \psi^{b]\!]}_{\hat{\mu}} + \mathrm{v}_\psi \, \bar{\epsilon}^{[\![a} \hat{\gamma}_{\hat{\mu}}{}^{\hat{\nu}} \psi^{b]\!]}_{\hat{\nu}} \,, \tag{5.66}$$

$$\begin{aligned} \delta B^{a\alpha}_{\hat{\mu}\hat{\nu}} = {} & \mathrm{w}_\chi \, \bar{\epsilon}_b \hat{\gamma}_{\hat{\mu}\hat{\nu}} \chi^{ab\alpha} + \bar{\epsilon}^a \psi^\alpha_{\hat{\mu}\hat{\nu}} + \mathrm{w}_\psi \, \bar{\epsilon}^a \hat{\gamma}_{[\hat{\mu}}{}^{\hat{\rho}} \psi^\alpha_{|\hat{\rho}|\hat{\nu}]} + \mathrm{x}_\psi \, \bar{\epsilon}^a \hat{\gamma}_{\hat{\mu}\hat{\nu}}{}^{\hat{\rho}\hat{\sigma}} \psi^\alpha_{\hat{\rho}\hat{\sigma}} \\ & + \mathrm{y}_\psi \, \bar{\epsilon}^\alpha \hat{\gamma}_{[\hat{\mu}} \psi^a_{\hat{\nu}]} + \mathrm{z}_\psi \, \bar{\epsilon}^\alpha \hat{\gamma}_{\hat{\mu}\hat{\nu}}{}^{\hat{\rho}} \psi^a_{\hat{\rho}} \,, \end{aligned} \tag{5.67}$$

$$\begin{aligned} \delta\psi^a_{\hat{\mu}} = {} & \mathrm{u}_F \, \hat{\gamma}^{\hat{\nu}} F^{ab}_{\hat{\mu}\hat{\nu}} \epsilon_b + \mathrm{v}_F \, \hat{\gamma}_{\hat{\mu}}{}^{\hat{\nu}\hat{\rho}} F^{ab}_{\hat{\nu}\hat{\rho}} \epsilon_b + \mathrm{u}_H \, \hat{\gamma}_{\hat{\mu}}{}^{\hat{\nu}\hat{\rho}\hat{\sigma}} H^{a\alpha}_{\hat{\nu}\hat{\rho}\hat{\sigma}} \epsilon_\alpha \\ & + \mathrm{a}_C \, U_{\hat{\mu}\hat{\nu}\hat{\rho},}{}^{\hat{\rho}} \hat{\gamma}^{\hat{\nu}} \epsilon^a + \mathrm{b}_C \, V_{\hat{\nu}\hat{\rho},\hat{\mu}}{}^{\hat{\rho}} \hat{\gamma}^{\hat{\nu}} \epsilon^a + \mathrm{c}_C \, V_{\hat{\mu}\hat{\rho},\hat{\nu}}{}^{\hat{\rho}} \hat{\gamma}^{\hat{\nu}} \epsilon^a \\ & + U_{\hat{\nu}\hat{\rho}\hat{\sigma},\hat{\mu}} \hat{\gamma}^{\hat{\nu}\hat{\rho}\hat{\sigma}} \epsilon^a + \mathrm{k}_C \, V_{\hat{\rho}\hat{\sigma},\hat{\nu}\hat{\mu}} \hat{\gamma}^{\hat{\nu}\hat{\rho}\hat{\sigma}} \epsilon^a \,, \end{aligned} \tag{5.68}$$

$$\begin{aligned} \delta\psi^\alpha_{\hat{\mu}\hat{\nu}} = {} & \mathrm{w}_H \, H^{a\alpha}_{\hat{\mu}\hat{\nu}\hat{\rho}} \hat{\gamma}^{\hat{\rho}} \epsilon_a + \mathrm{x}_H \, H^{a\alpha}_{\hat{\rho}\hat{\sigma}[\hat{\mu}} \hat{\gamma}^{\hat{\rho}\hat{\sigma}}{}_{\hat{\nu}]} \epsilon_a \\ & + \mathrm{d}_C \, U_{\hat{\mu}\hat{\nu}\hat{\rho},}{}^{\hat{\rho}} \epsilon^\alpha + \mathrm{e}_C \, V_{[\hat{\mu}|\hat{\rho}|,\hat{\nu}]}{}^{\hat{\rho}} \epsilon^\alpha + \mathrm{l}_C \, U_{\hat{\mu}\hat{\nu}\hat{\rho},\hat{\sigma}} \hat{\gamma}^{\hat{\rho}\hat{\sigma}} \epsilon^\alpha + \mathrm{m}_C \, V_{\hat{\mu}\hat{\nu},\hat{\rho}\hat{\sigma}} \hat{\gamma}^{\hat{\rho}\hat{\sigma}} \epsilon^\alpha \\ & + \mathrm{p}_C \, V_{\hat{\rho}\hat{\tau},}{}^{\hat{\tau}} \hat{\gamma}_{\hat{\mu}\hat{\nu}}{}^{\hat{\rho}\hat{\sigma}} \epsilon^\alpha + \mathrm{q}_C \, U_{\hat{\rho}\hat{\sigma}\hat{\tau},}{}^{\hat{\tau}} \hat{\gamma}_{\hat{\mu}\hat{\nu}}{}^{\hat{\rho}\hat{\sigma}} \epsilon^\alpha \\ & + \mathrm{r}_C \, V_{\hat{\rho}\hat{\sigma},\hat{\tau}[\hat{\nu}} \hat{\gamma}_{\hat{\mu}]}{}^{\hat{\rho}\hat{\sigma}\hat{\tau}} \epsilon^\alpha + \mathrm{s}_C \, U_{\hat{\rho}\hat{\sigma}\hat{\tau},[\hat{\nu}} \hat{\gamma}_{\hat{\mu}]}{}^{\hat{\rho}\hat{\sigma}\hat{\tau}} \epsilon^\alpha \,, \end{aligned} \tag{5.69}$$

$$\begin{aligned} \delta C_{\hat{\mu}\hat{\nu},\hat{\rho}} = {} & \mathrm{m}_\psi \, \bar{\epsilon}_\alpha \left(\hat{\gamma}_{\hat{\mu}\hat{\nu}}{}^{\hat{\sigma}} \psi^\alpha_{\hat{\sigma}\hat{\rho}} - \hat{\gamma}_{[\hat{\mu}\hat{\nu}}{}^{\hat{\sigma}} \psi^\alpha_{|\hat{\sigma}|\hat{\rho}]} \right) + \mathrm{p}_\psi \, \bar{\epsilon}_\alpha \left(\hat{\gamma}_{\hat{\rho}} \psi^\alpha_{\hat{\mu}\hat{\nu}} - \hat{\gamma}_{[\hat{\rho}} \psi^\alpha_{\hat{\mu}\hat{\nu}]} \right) \\ & + \mathrm{q}_\psi \, \bar{\epsilon}_a \left(\hat{\gamma}_{\hat{\mu}\hat{\nu}} \psi^a_{\hat{\rho}} - \hat{\gamma}_{[\hat{\mu}\hat{\nu}} \psi^a_{\hat{\rho}]} \right) + \mathrm{r}_\psi \, \bar{\epsilon}_a \eta_{\hat{\rho}[\hat{\mu}} \psi^a_{\hat{\nu}]} \,. \end{aligned} \tag{5.70}$$

Here $U_{\hat{\mu}\hat{\nu}\hat{\rho},\hat{\sigma}} = 3\,\partial_{[\hat{\mu}} C_{\hat{\nu}\hat{\rho}],\hat{\sigma}}$ and $V_{\hat{\mu}\hat{\nu},\hat{\rho}\hat{\sigma}} = \partial_{\hat{\sigma}} C_{\hat{\mu}\hat{\nu},\hat{\rho}} - \partial_{\hat{\rho}} C_{\hat{\mu}\hat{\nu},\hat{\sigma}}$ are the two first-order field strengths of the Curtright field and the notation $[\![abc]\!]$ denotes the Usp(6)-traceless antisymmetrization. These variations comprise all possible independent tensor structures that are compatible with regards to the space-time, spinorial, and R-symmetry indices. They contain numerical coefficients u_χ, v_χ, ..., r_ψ which we determine in the following

through consistency conditions of the algebra.[1]

We note that certain structures have already been removed in (5.63–5.70) which are redundant due to gauge transformations or duality relations of the $\hat{\gamma}$ matrices or Bianchi identities of U and V. There are other simplifications we can perform: due to the definition of U and V we have

$$U_{[\hat{\mu}\hat{\nu}\hat{\rho},\hat{\sigma}]} = 0 = V_{[\hat{\mu}\hat{\nu},\hat{\rho}\hat{\sigma}]} \ , \tag{5.71}$$

leading to the following non trivial relations

$$0 = \hat{\gamma}^{\hat{\mu}\hat{\nu}} U_{\hat{\mu}\hat{\nu}[\hat{\rho},\hat{\sigma}]} + \hat{\gamma}^{\hat{\mu}\hat{\nu}} U_{\hat{\rho}\hat{\sigma}\hat{\mu},\hat{\nu}} \tag{5.72}$$

$$0 = \hat{\gamma}^{\hat{\mu}\hat{\nu}\hat{\rho}} U_{\hat{\mu}\hat{\nu}\hat{\rho},\hat{\sigma}} - 3\,\hat{\gamma}^{\hat{\mu}\hat{\nu}\hat{\rho}} U_{\hat{\sigma}\hat{\mu}\hat{\nu},\hat{\rho}} \tag{5.73}$$

$$0 = \hat{\gamma}^{\hat{\mu}\hat{\nu}} V_{\hat{\mu}\hat{\nu},\hat{\rho}\hat{\sigma}} + \hat{\gamma}^{\hat{\mu}\hat{\nu}} V_{\hat{\rho}\hat{\sigma},\hat{\mu}\hat{\nu}} + 2\,\hat{\gamma}^{\hat{\mu}\hat{\nu}} V_{\hat{\mu}[\hat{\rho},\hat{\sigma}]\hat{\nu}} \tag{5.74}$$

$$0 = \hat{\gamma}^{\hat{\mu}\hat{\nu}\hat{\rho}} V_{\hat{\mu}\hat{\nu},\hat{\rho}\hat{\sigma}} - \hat{\gamma}^{\hat{\mu}\hat{\nu}\hat{\rho}} V_{\hat{\sigma}\hat{\mu},\hat{\nu}\hat{\rho}} \tag{5.75}$$

Furthermore, using

$$\hat{\gamma}^{\hat{\mu}\hat{\nu}\hat{\rho}} V_{\hat{\mu}\hat{\nu},\hat{\rho}\hat{\sigma}} = \partial_{\hat{\sigma}} \hat{\gamma}^{\hat{\mu}\hat{\nu}\hat{\rho}} C_{\hat{\mu}\hat{\nu},\hat{\rho}} - \hat{\gamma}^{\hat{\mu}\hat{\nu}\hat{\rho}} \partial_{\hat{\mu}} C_{\hat{\nu}\hat{\rho},\hat{\sigma}} = -\frac{1}{3} \hat{\gamma}^{\hat{\mu}\hat{\nu}\hat{\rho}} U_{\hat{\mu}\hat{\nu}\hat{\rho},\hat{\sigma}}, \tag{5.76}$$

allows us to remove all terms of the form $V_{[\hat{\mu}\hat{\nu},\hat{\rho}]\hat{\sigma}}$ such that we can set $k_C = r_C = 0$ in (5.63–5.70). The relation (5.74) allows us to identify $V_{\hat{\rho}\hat{\sigma},\hat{\mu}\hat{\nu}} \hat{\gamma}^{\hat{\rho}\hat{\sigma}}$ in (5.69) as a gauge transformation such that we can choose $m_C = 0$. We also note that

$$U_{\hat{\mu}\hat{\nu}\hat{\rho},\hat{\sigma}} \hat{\gamma}^{\hat{\rho}\hat{\sigma}} = \partial_{\hat{\rho}} C_{\hat{\mu}\hat{\nu},\hat{\sigma}} \hat{\gamma}^{\hat{\rho}\hat{\sigma}} + 2\,\partial_{[\hat{\mu}} C_{\hat{\nu}]\hat{\rho},\hat{\sigma}} \hat{\gamma}^{\hat{\rho}\hat{\sigma}} = \frac{1}{2} V_{\hat{\mu}\hat{\nu},\hat{\sigma}\hat{\rho}} \hat{\gamma}^{\hat{\rho}\hat{\sigma}} + 2\,\partial_{[\hat{\mu}} C_{\hat{\nu}]\hat{\rho},\hat{\sigma}} \ , \tag{5.77}$$

where the second term in the last equation can be absorbed in a gauge transformation for $\psi_{\hat{\mu}\hat{\nu}}$ so we can choose $l_C = 0$.

Since we are interested in the on-shell linearized supersymmetry variations, it is possible to futher simplify the terms involving U and V. C satisfies a second order duality equation:

$$S_{\hat{\mu}\hat{\nu}\hat{\rho},\hat{\sigma}\hat{\tau}} = -\frac{1}{6} \varepsilon_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\alpha}\hat{\beta}\hat{\gamma}} S^{\hat{\alpha}\hat{\beta}\hat{\gamma}}{}_{\hat{\sigma}\hat{\tau}}, \tag{5.78}$$

where $S_{\hat{\mu}\hat{\nu}\hat{\rho},\hat{\sigma}\hat{\tau}} = 6\,\partial_{[\hat{\mu}} C_{\hat{\nu}\hat{\rho}],[\hat{\sigma};\hat{\tau}]}$. Since both sides can be considered as a curl on the second group of indices, it is possible to integrate this equation once into

$$U_{\hat{\mu}\hat{\nu}\hat{\rho},\hat{\sigma}} + \frac{1}{6} \varepsilon_{\hat{\mu}\hat{\nu}\hat{\rho}\alpha\beta\hat{\gamma}} U^{\alpha\beta\hat{\gamma}}{}_{\hat{\sigma}} = \partial_{\hat{\sigma}} Z_{\hat{\mu}\hat{\nu}\hat{\rho}}, \tag{5.79}$$

where Z is a 3-form. Using this identity and the duality relation for the $\hat{\gamma}$ matrices, we can take $s_C = 0$.

[1] The rescaling ambiguity in the supersymmetry closure relations has already been fixed in (5.63–5.70) by a suitable normalization of all fields. As a result, not all terms carry a free coefficient.

As explained in the previous Section 5.2.1, the supersymmetry algebra (5.63–5.70) has to close onto six-dimensional diffeomorphisms and gauge transformations as in (5.12). To determine the diffeomorphism parameter in terms of these supersymmetry parameters, we compute the commutator of two supersymmetry transformations of the scalar field. Hence

$$[\delta(\epsilon_1), \delta(\epsilon_2)]\phi^{abc\alpha} = \bar{\epsilon}_2^{[\![a}\hat{\gamma}^{\hat{\mu}}\epsilon_{1,d}\partial_{\hat{\mu}}\phi^{bc]\!]d\alpha} + \bar{\epsilon}_2^{\alpha}\hat{\gamma}^{\hat{\mu}}\epsilon_{1,\beta}\partial_{\hat{\mu}}\phi^{abc\beta} - (1 \leftrightarrow 2) + \dots \,, \tag{5.80}$$

where we did not spell out terms involving $F_{\hat{\mu}\hat{\nu}}{}^{ab}$ and $H_{\hat{\mu}\hat{\nu}\hat{\rho}}{}^{a\alpha}$. To put this expression in the appropriate form, we have to use two Schouten identities

$$\bar{\epsilon}_2^{\alpha}\hat{\gamma}^{\hat{\mu}}\epsilon_{1,\beta}\partial_{\hat{\mu}}\phi^{abc\beta} - (1 \leftrightarrow 2) = \bar{\epsilon}_2^{\beta}\hat{\gamma}^{\hat{\mu}}\epsilon_{1,\beta}\partial_{\hat{\mu}}\phi^{abc\alpha} \,, \tag{5.81}$$

$$\bar{\epsilon}_2^{[\![a}\hat{\gamma}^{\hat{\mu}}\epsilon_{1,d}\partial_{\hat{\mu}}\phi^{bc]\!]d\alpha} - (1 \leftrightarrow 2) = \frac{1}{3}\bar{\epsilon}_2^{d}\hat{\gamma}^{\hat{\mu}}\epsilon_{1,d}\partial_{\hat{\mu}}\phi^{abc\alpha} \,. \tag{5.82}$$

In this way we find for (5.80)

$$[\delta(\epsilon_1), \delta(\epsilon_2)]\phi^{abc\alpha} = \xi^{\hat{\mu}}\partial_{\hat{\mu}}\phi^{abc\alpha} + \dots \,, \tag{5.83}$$

where the diffeomorphism parameter takes the form

$$\xi^{\hat{\mu}} = \frac{1}{3}\bar{\epsilon}_2^{d}\hat{\gamma}^{\hat{\mu}}\epsilon_{1,d} + \bar{\epsilon}_2^{\beta}\hat{\gamma}^{\hat{\mu}}\epsilon_{1,\beta}. \tag{5.84}$$

After lengthy computations, we can show that with a unique choice of the parameters in (5.63–5.70), all closure relations can be brought into the form of (5.12). The final six-dimensional supersymmetry transformations therefore takes the form

$$\delta\phi^{abc\alpha} = \bar{\epsilon}^{\alpha}\chi^{abc} + \bar{\epsilon}^{[\![a}\chi^{bc]\!]\alpha} \,, \tag{5.85}$$

$$\delta\chi^{abc} = \hat{\gamma}^{\hat{\mu}}\partial_{\hat{\mu}}\phi^{abc\alpha}\epsilon_{\alpha} + \hat{\gamma}^{\hat{\mu}\hat{\nu}}F_{\hat{\mu}\hat{\nu}}^{[\![ab}\epsilon^{c]\!]} \,, \tag{5.86}$$

$$\delta\chi^{ab\alpha} = \hat{\gamma}^{\hat{\mu}}\partial_{\hat{\mu}}\phi^{abc\alpha}\epsilon_{c} + \hat{\gamma}^{\hat{\mu}\hat{\nu}}F_{\hat{\mu}\hat{\nu}}^{ab}\epsilon^{\alpha} + \hat{\gamma}^{\hat{\mu}\hat{\nu}\hat{\rho}}H_{\hat{\mu}\hat{\nu}\hat{\rho}}^{[\![a|\alpha|}\epsilon^{b]\!]} \,, \tag{5.87}$$

$$\delta A_{\hat{\mu}}^{ab} = \frac{1}{4}\,\bar{\epsilon}_c\hat{\gamma}_{\hat{\mu}}\chi^{abc} + \frac{1}{4}\,\bar{\epsilon}_{\alpha}\hat{\gamma}_{\hat{\mu}}\chi^{ab\alpha} + \bar{\epsilon}^{[\![a}\psi_{\hat{\mu}}^{b]\!]} \,, \tag{5.88}$$

$$\delta B_{\hat{\mu}\hat{\nu}}^{a\alpha} = \frac{1}{36}\,\bar{\epsilon}_b\hat{\gamma}_{\hat{\mu}\hat{\nu}}\chi^{ab\alpha} + \frac{1}{3}\,\bar{\epsilon}^{\alpha}\hat{\gamma}_{[\hat{\mu}}\psi_{\hat{\nu}]}^{a} + \bar{\epsilon}^{a}\psi_{\hat{\mu}\hat{\nu}}^{\alpha} \,, \tag{5.89}$$

$$\delta\psi_{\hat{\mu}}^{a} = \frac{1}{2}\,\hat{\gamma}^{\hat{\nu}}F_{\hat{\mu}\hat{\nu}}^{ab}\epsilon_b - \frac{1}{12}\,\hat{\gamma}_{\hat{\mu}}{}^{\hat{\nu}\hat{\rho}}F_{\hat{\nu}\hat{\rho}}^{ab}\epsilon_b + \frac{1}{2}\,\hat{\gamma}_{\hat{\mu}}{}^{\hat{\nu}\hat{\rho}\hat{\sigma}}H_{\hat{\nu}\hat{\rho}\hat{\sigma}}^{a\alpha}\epsilon_{\alpha} + U_{\hat{\nu}\hat{\rho}\hat{\sigma},\hat{\mu}}\hat{\gamma}^{\hat{\nu}\hat{\rho}\hat{\sigma}}\epsilon^{a} \,, \tag{5.90}$$

$$\delta\psi_{\hat{\mu}\hat{\nu}}^{\alpha} = \frac{2}{9}\,H_{\hat{\mu}\hat{\nu}\hat{\rho}}^{a\alpha}\hat{\gamma}^{\hat{\rho}}\epsilon_a - \frac{1}{18}\,H_{\hat{\rho}\hat{\sigma}[\hat{\mu}}^{a\alpha}\hat{\gamma}^{\hat{\rho}\hat{\sigma}}{}_{\hat{\nu}]}\epsilon_a + V_{\hat{\mu}\hat{\nu},\hat{\rho}\hat{\sigma}}\hat{\gamma}^{\hat{\rho}\hat{\sigma}}\epsilon^{\alpha} \,, \tag{5.91}$$

$$\begin{aligned}\delta T_{\hat{\mu}\hat{\nu},\hat{\rho}} = &-\frac{1}{4}\,\bar{\epsilon}_{\alpha}\left(\hat{\gamma}_{\hat{\rho}}\psi_{\hat{\mu}\hat{\nu}}^{\alpha} - \hat{\gamma}_{[\hat{\rho}}\psi_{\hat{\mu}\hat{\nu}]}^{\alpha}\right) + \frac{1}{72}\,\bar{\epsilon}_a\left(\hat{\gamma}_{\hat{\mu}\hat{\nu}}\psi_{\hat{\rho}}^{a} - \hat{\gamma}_{[\hat{\mu}\hat{\nu}}\psi_{\hat{\rho}]}^{a}\right)\\ &+ \frac{1}{108}\,\bar{\epsilon}_a\psi_{[\hat{\mu}}^{a}\eta_{\hat{\nu}]\hat{\rho}} \,,\end{aligned} \tag{5.92}$$

where, as above, $U_{\hat{\mu}\hat{\nu}\hat{\rho},\hat{\sigma}} = 3\,\partial_{[\hat{\mu}}C_{\hat{\nu}\hat{\rho}],\hat{\sigma}}$ and $V_{\hat{\mu}\hat{\nu},\hat{\rho}\hat{\sigma}} = \partial_{\hat{\sigma}}C_{\hat{\mu}\hat{\nu},\hat{\rho}} - \partial_{\hat{\rho}}C_{\hat{\mu}\hat{\nu},\hat{\sigma}}$.

5.3.2 5+1 split of the supersymmetry transformations

In order to construct the supersymmetric extension of the Lagrangian (3.45) (3.44), which we have constructed above, we will have to rewrite the six-dimensional supersymmetry transformations in a 5+1 split form using the decomposition of the fields introduced in chapter 2.

Eventually, for the various components, we obtain

$$\delta\phi^{abc\alpha} = \bar{\epsilon}^{\alpha}\chi^{abc} + \bar{\epsilon}^{[\![a}\chi^{bc]\!]\alpha} \,, \tag{5.93}$$

$$\delta\phi^{ab} = -\frac{i}{4}\bar{\epsilon}_c\chi^{abc} + \frac{i}{4}\bar{\epsilon}_\alpha\chi^{ab\alpha} + \bar{\epsilon}^{[\![a}\lambda^{b]\!]} \,, \tag{5.94}$$

$$\delta A_\mu^{ab} = \frac{1}{4}\bar{\epsilon}_c\gamma_\mu\chi^{abc} + \frac{1}{4}\bar{\epsilon}_\alpha\gamma_\mu\chi^{ab\alpha} + \bar{\epsilon}^{[\![a}\psi_\mu^{b]\!]} - \frac{i}{3}\bar{\epsilon}^{[\![a}\gamma_\mu\lambda^{b]\!]} \,, \tag{5.95}$$

$$\delta A_\mu^{a\alpha} = \frac{i}{36}\,\bar{\epsilon}_b\gamma_\mu\chi^{ab\alpha} + \frac{1}{9}\,\bar{\epsilon}^{\alpha}\gamma_\mu\lambda^a - \frac{i}{6}\,\bar{\epsilon}^{\alpha}\psi_\mu^a + \bar{\epsilon}^a\phi_\mu^\alpha \,, \tag{5.96}$$

$$\delta A_\mu = \frac{i}{162}\,\bar{\epsilon}_a\gamma_\mu\lambda^a - \frac{i}{8}\,\bar{\epsilon}_\alpha\phi_\mu^\alpha + \frac{1}{432}\,\bar{\epsilon}_a\psi_\mu^a \,, \tag{5.97}$$

$$\delta B_{\mu\nu} = \frac{1}{162}\bar{\epsilon}_a\gamma_{\mu\nu}\lambda^a + \frac{1}{4}\,\bar{\epsilon}_\alpha\gamma_{[\mu}\phi_{\nu]}^\alpha + \frac{i}{216}\,\bar{\epsilon}_a\gamma_{[\mu}\psi_{\nu]}^a - \frac{i}{12}\bar{\epsilon}_\alpha\psi_{\mu\nu}^\alpha \,, \tag{5.98}$$

$$\delta B_{\mu\nu}^{a\alpha} = \frac{1}{36}\,\bar{\epsilon}_b\gamma_{\mu\nu}\chi^{ab\alpha} - \frac{i}{9}\,\bar{\epsilon}^{\alpha}\gamma_{\mu\nu}\lambda^a + \frac{1}{3}\,\bar{\epsilon}^{\alpha}\gamma_{[\mu}\psi_{\nu]}^a + 2i\,\bar{\epsilon}^a\gamma_{[\mu}\phi_{\nu]}^\alpha + \bar{\epsilon}^a\psi_{\mu\nu}^\alpha \,, \tag{5.99}$$

$$\delta h_{\mu\nu} = -\frac{1}{4}\,\bar{\epsilon}_\alpha\gamma_{(\nu}\phi_{\mu)}^\alpha + \frac{i}{72}\,\bar{\epsilon}_a\gamma_{(\nu}\psi_{\mu)}^a \,, \tag{5.100}$$

$$\begin{aligned}\delta C_{\mu\nu,\rho} =& -\frac{1}{4}\bar{\epsilon}_\alpha(\gamma_\rho\psi_{\mu\nu}^\alpha - \gamma_{[\rho}\psi_{\mu\nu]}^\alpha) + \frac{1}{72}\bar{\epsilon}_a(\gamma_{\mu\nu}\psi_\rho^a - \gamma_{[\mu\nu}\psi_{\rho]}^a) + \frac{1}{72}\,\bar{\epsilon}_a\psi_{[\mu}^a\eta_{\nu]\rho} \\ &+ \frac{i}{4}\bar{\epsilon}_\alpha(\gamma_{\mu\nu}\phi_\rho^\alpha - \gamma_{[\mu\nu}\phi_{\rho]}^\alpha) + \frac{i}{4}\bar{\epsilon}_\alpha\phi_{[\mu}^\alpha\eta_{\nu]\rho} \,,\end{aligned} \tag{5.101}$$

$$\delta\chi^{abc} = \gamma^\mu\partial_\mu\phi^{abc\alpha}\epsilon_\alpha - 2i\,\gamma^\mu\left(\partial_\mu\phi^{[\![ab} - \partial_y A_\mu^{[\![ab}\right)\epsilon^{c]\!]} + \gamma^{\mu\nu}F_{\mu\nu}^{[\![ab}\epsilon^{c]\!]} + i\,\partial_y\phi^{abc\alpha}\epsilon_\alpha \,, \tag{5.102}$$

$$\begin{aligned}\delta\chi^{ab\alpha} =& \gamma^\mu\partial_\mu\phi^{abc\alpha}\epsilon_c + 2i\,\gamma^\mu\left(\partial_\mu\phi^{ab} - \partial_y A_\mu^{ab}\right)\epsilon^\alpha - i\,\partial_y\phi^{abc\alpha}\epsilon_c \\ &+ \gamma^{\mu\nu}F_{\mu\nu}^{ab}\epsilon^\alpha - 3i\,\gamma^{\mu\nu}\left(F_{\mu\nu}^{[\![a|\alpha|} + \partial_y B_{\mu\nu}^{[\![a|\alpha|}\right)\epsilon^{b]\!]} \,,\end{aligned} \tag{5.103}$$

$$\begin{aligned}\delta\lambda^a =& -\frac{1}{2}\,\gamma^\mu\left(\partial_\mu\phi^{ab} - \partial_y A_\mu^{ab}\right)\epsilon_b + \frac{i}{12}\,\gamma^{\mu\nu}F_{\mu\nu}^{ab}\epsilon_b - \frac{3}{2}\,\gamma^{\mu\nu}\left(F_{\mu\nu}^{a\alpha} + \partial_y B_{\mu\nu}^{a\alpha}\right)\epsilon_\alpha \\ &- 12i\,\gamma^{\mu\nu}\left(F_{\mu\nu} + \frac{5}{4}\,\partial_y B_{\mu\nu} - \frac{1}{2}\,\partial_y u_{\mu\nu}\right)\epsilon^a \,,\end{aligned} \tag{5.104}$$

$$\begin{aligned}\delta\psi_\mu^a =& -\frac{2i}{3}\,(\partial_\mu\phi^{ab} - \partial_y A_\mu^{ab})\epsilon_b + \frac{4}{9}\,\gamma^\nu F_{\mu\nu}^{ab}\epsilon_b - \frac{1}{9}\,\gamma_\mu{}^{\nu\rho}F_{\nu\rho}^{ab}\epsilon_b \\ &- 4i\,\gamma^\nu(F_{\mu\nu}^{a\alpha} + \partial_y B_{\mu\nu}^{a\alpha})\epsilon_\alpha + i\,\gamma_\mu{}^{\nu\rho}(F_{\nu\rho}^{a\alpha} + \partial_y B_{\nu\rho}^{a\alpha})\epsilon_\alpha\end{aligned}$$

$$
\begin{aligned}
&+ 8\,\gamma^{\nu}\left(F_{\mu\nu} + \frac{5}{4}\,\partial_y B_{\mu\nu} - \frac{1}{2}\,\partial_y u_{\mu\nu}\right)\epsilon^a \\
&+ 4\,\gamma_{\mu}{}^{\nu\rho}\left(F_{\nu\rho} + \frac{5}{4}\,\partial_y B_{\nu\rho} - \frac{1}{2}\,\partial_y u_{\nu\rho}\right)\epsilon^a \\
&- 12i\,\gamma^{\nu\rho}\Big(\partial_\nu h_{\rho\mu} + \partial_\nu B_{\rho\mu} + \frac{1}{4}\partial_\mu B_{\nu\rho} - \frac{1}{2}\,\partial_\mu u_{\nu\rho} \\
&\qquad\qquad - \partial_y(C_{\nu\rho,\mu} - 2\,A_\nu \eta_{\rho\mu})\Big)\epsilon^a\,,
\end{aligned}
\tag{5.105}
$$

$$
\begin{aligned}
\delta\phi^{\alpha}_{\mu} =& -\frac{2}{9}\,\gamma^{\nu}(F^{a\alpha}_{\mu\nu} + \partial_y B^{a\alpha}_{\mu\nu})\epsilon_a + \frac{1}{18}\,\gamma_{\mu}{}^{\kappa\lambda}(F^{a\alpha}_{\kappa\lambda} + \partial_y B^{a\alpha}_{\kappa\lambda})\epsilon_a \\
&- 2\,\gamma^{\kappa\lambda}(\partial_\kappa h_{\lambda\mu} - \partial_\kappa B_{\lambda\mu})\epsilon^{\alpha} - 4i\,\gamma^{\nu}\left(\partial_\nu A_\mu - \frac{1}{2}\,\partial_y(h_{\nu\mu} - B_{\nu\mu})\right)\epsilon^{\alpha}\,,
\end{aligned}
\tag{5.106}
$$

$$
\begin{aligned}
\delta\psi^{\alpha}_{\mu\nu} =& -\frac{2i}{3}\,(F^{a\alpha}_{\mu\nu} + \partial_y B^{a\alpha}_{\mu\nu})\epsilon_a + 2i\,\left(\partial_y(C_{\mu\nu,\rho} - 2\,A_{[\mu}\eta_{\nu]\rho}) - \partial_\rho B_{\mu\nu}\right)\gamma^{\rho}\epsilon^{\alpha} \\
&+ V_{\mu\nu,\rho\sigma}\gamma^{\rho\sigma}\epsilon^{\alpha} - 2i\,\gamma_{[\mu}\Big(-2\,\gamma^{\kappa\lambda}(\partial_\kappa h_{\lambda|\nu]} - \partial_\kappa B_{\lambda|\nu]})\epsilon^{\alpha} \\
&- 4i\,\gamma^{\lambda}\left(\partial_\lambda A_{\nu]} - \frac{1}{2}\,\partial_y(h_{\lambda|\nu]} - B_{\lambda\nu]})\right)\epsilon^{\alpha}\Big)\,.
\end{aligned}
\tag{5.107}
$$

The variation of the fermionic fields appear rather complicated but as for the N=(4,0) model, they can be brought in a more compact form by employing the freedom of gauge transformation. Using the gauge transformation (2.91) for λ^a and ψ^a_μ

$$
\begin{aligned}
\delta\lambda^a &= \partial_y\kappa^a\,, \\
\delta\psi^a_\mu &= \partial_\mu\kappa^a + \frac{i}{3}\,\gamma_\mu\partial_y\kappa^a\,,
\end{aligned}
\tag{5.108}
$$

with the fermionic parameter

$$
\kappa^a = -12i\,\gamma^{\alpha\beta}\epsilon^a\left(\frac{1}{4}\,B_{\alpha\beta} + \frac{1}{2}\,u_{\alpha\beta}\right),
\tag{5.109}
$$

we find

$$
\begin{aligned}
\delta\lambda^a =& -\frac{1}{2}\,\gamma^{\mu}\left(\partial_\mu\phi^{ab} - \partial_y A^{ab}_{\mu}\right)\epsilon_b + \frac{i}{12}\,\gamma^{\mu\nu}F^{ab}_{\mu\nu}\epsilon_b - \frac{3}{2}\,\gamma^{\mu\nu}\left(F^{a\alpha}_{\mu\nu} + \partial_y B^{a\alpha}_{\mu\nu}\right)\epsilon_\alpha \\
&- 12i\,\gamma^{\mu\nu}\mathcal{F}_{\mu\nu}\epsilon^a\,,
\end{aligned}
\tag{5.110}
$$

$$
\begin{aligned}
\delta\psi^a_\mu =& -\frac{2i}{3}\,(\partial_\mu\phi^{ab} - \partial_y A^{ab}_{\mu})\epsilon_b - 4i\,\gamma^{\nu}(F^{a\alpha}_{\mu\nu} + \partial_y B^{a\alpha}_{\mu\nu})\epsilon_\alpha + i\,\gamma_{\mu}{}^{\nu\rho}(F^{a\alpha}_{\nu\rho} + \partial_y B^{a\alpha}_{\nu\rho})\epsilon_\alpha \\
&+ \frac{4}{9}\,\gamma^{\nu}F^{ab}_{\mu\nu}\epsilon_b - \frac{1}{9}\,\gamma_{\mu}{}^{\nu\rho}F^{ab}_{\nu\rho}\epsilon_b + 8\,\gamma^{\nu}\mathcal{F}_{\mu\nu}\epsilon^a - 2\,\gamma_{\mu}{}^{\nu\rho}\mathcal{F}_{\nu\rho}\epsilon^a \\
&- 12i\,\gamma^{\nu\rho}\left(\widehat{\Omega}_{\nu\rho,\mu} - 2\,\widehat{\Omega}_{\mu\nu,\rho}\right)\epsilon^a\,,
\end{aligned}
\tag{5.111}
$$

where $\mathcal{F}_{\mu\nu}$ and $\widehat{\Omega}_{\mu\nu,\rho}$ are defined in (3.39). Moreover, defining

$$\begin{aligned} D_\mu \phi^{ab} &= \partial_\mu \phi^{ab} - \partial_y A_\mu^{ab} \,, \\ \mathcal{F}_{\mu\nu}{}^{a\alpha} &= F_{\mu\nu}^{a\alpha} + \partial_y B_{\mu\nu}^{a\alpha} \,, \end{aligned} \tag{5.112}$$

allows to put these variations into yet more compact form

$$\begin{aligned} \delta\lambda^a =& -\frac{1}{2}\,\gamma^\mu D_\mu \phi^{ab}\epsilon_b + \frac{i}{12}\,\gamma^{\mu\nu} F_{\mu\nu}^{ab}\epsilon_b - \frac{3}{2}\,\gamma^{\mu\nu}\mathcal{F}_{\mu\nu}{}^{a\alpha}\epsilon_\alpha \\ & - 12i\,\gamma^{\mu\nu}\mathcal{F}_{\mu\nu}\epsilon^a \,, \\ \delta\psi_\mu^a =& -\frac{2i}{3}\,D_\mu \phi^{ab}\epsilon_b - 4i\,\gamma^\nu \mathcal{F}_{\mu\nu}{}^{a\alpha}\epsilon_\alpha + i\,\gamma_\mu{}^{\nu\rho}\mathcal{F}_{\nu\rho}{}^{a\alpha}\epsilon_\alpha \\ & + \frac{4}{9}\,\gamma^\nu F_{\mu\nu}^{ab}\epsilon_b - \frac{1}{9}\,\gamma_\mu{}^{\nu\rho}F_{\nu\rho}^{ab}\epsilon_b + 8\,\gamma^\nu \mathcal{F}_{\mu\nu}\epsilon^a - 2\,\gamma_\mu{}^{\nu\rho}\mathcal{F}_{\nu\rho}\epsilon^a \\ & - 12i\,\gamma^{\nu\rho}\left(\widehat{\Omega}_{\nu\rho,\mu} - 2\,\widehat{\Omega}_{\mu\nu,\rho}\right)\epsilon^a \,. \end{aligned} \tag{5.113}$$

Similarly, we may simplify the variations of ϕ_μ^α and $\psi_{\mu\nu}^\alpha$ by modifying them with the gauge transformation (2.98)

$$\begin{aligned} \delta\phi_\mu^\alpha &= \frac{1}{2}\,\partial_\mu \kappa_6^\alpha - \frac{1}{2}\,\partial_y \kappa_\mu^\alpha \,, \\ \delta\psi_{\mu\nu}^\alpha &= \partial_{[\mu}\kappa_{\nu]}^\alpha + i\,\partial_{[\mu}\gamma_{\nu]}\kappa_6^\alpha + i\,\partial_y \gamma_{[\mu}\kappa_{\nu]}^\alpha \,. \end{aligned} \tag{5.114}$$

with the parameters

$$\begin{aligned} \kappa_\mu^\alpha &= \left(4i\,\gamma_\mu{}^{\kappa\lambda}u_{\kappa\lambda} - 8i\,\gamma^\nu B_{\mu\nu} + 4i\,\gamma^\nu h_{\mu\nu} + 2\,\gamma^{\kappa\lambda}C_{\kappa\lambda,\mu} + 4\,\gamma_\mu{}^\kappa A_\kappa\right)\epsilon^\alpha \,, \\ \kappa_6^\alpha &= \left(2\,\gamma^{\kappa\lambda}B_{\kappa\lambda} - 4i\,\gamma^\lambda A_\lambda\right)\epsilon^\alpha \,. \end{aligned} \tag{5.115}$$

As a result, we find

$$\begin{aligned} \delta\phi_\mu^\alpha =& -\frac{2}{9}\,\gamma^\nu \mathcal{F}_{\mu\nu}{}^{a\alpha}\epsilon_a + \frac{1}{18}\,\gamma_\mu{}^{\kappa\lambda}\mathcal{F}_{\kappa\lambda}{}^{a\alpha}\epsilon_a + 4i\,\gamma^\nu \mathcal{F}_{\mu\nu}\epsilon^\alpha - i\,\gamma_\mu{}^{\kappa\lambda}\mathcal{F}_{\kappa\lambda}\epsilon^\alpha \\ & - 2\,\gamma^{\kappa\lambda}\left(\widehat{\Omega}_{\kappa\lambda,\mu} - 2\,\widehat{\Omega}_{\mu\kappa,\lambda}\right)\epsilon^\alpha \,, \\ \delta\psi_{\mu\nu}^\alpha =& -8\,\mathcal{F}_{\mu\nu}\epsilon^\alpha + 8i\,\gamma^\lambda \widehat{\Omega}_{\mu\nu,\lambda}\epsilon^\alpha \,. \end{aligned} \tag{5.116}$$

This final result is much more compact than the original expressions (5.108), features the covariant objects (5.112) and strongly resembles the variations of the $\mathcal{N} = (4,0)$ model (5.30). This suggests there is a common framework as for the bosonic sector into which all three supersymmetic models can be embedded. A first step towards this unification would necessitate a reformulation of the $\mathcal{N} = (3,1)$ model in terms of Usp(8)-covariant objects. This is left for future work.

CHAPTER 6

Conclusion and outlook

In Chapter 1, we presented supergravity with particular focus on the topic of this thesis: six-dimensional maximal supergravity. In this case, there exists three different supersymmetry multiplets containing gravitational degrees of freedom transforming in the $\mathcal{N} = (2, 2)$, $\mathcal{N} = (3, 1)$ and $\mathcal{N} = (4, 0)$ supersymmetry algebras respectively. We explicitly presented the field contents of these theories. A particular feature of the $\mathcal{N} = (3, 1)$ and $\mathcal{N} = (4, 0)$ theories are "exotic" tensor fields, notably with (2,1) and (2,2) Young symmetry. Their dynamics are not very well understood since they have to satisfy certain (self)-duality relations which are difficult to encode in a usual Lagrangian formulation. However an important observation is that these three theories must all reduce to the same description in five dimensions, that is five-dimensional maximally supersymmetric supergravity. This suggests that there exists a common framework that unifies these three descriptions.

In Chapter 2, we discussed in detail all individual fields involved in the theories mentioned above, that is their six-dimensional free equations of motion and their gauge symmetries. We furthermore wrote this dynamics in a 5+1 split form by singling out one of the spatial coordinates. Although losing manifest six-dimensional Lorentz invariance, this formulation not only helped us to encode the duality relations of the exotic fields mentioned above, but also guides us towards a common framework unifying all three theories. Concretely, we analyzed the dynamics of the (2,1) and (2,2) tensor fields and transformed their second order field equations into first order equations (2.53), (2.65) and (2.67) by introducing auxiliary fields.

This 5+1 decomposition then allowed us to write down new actions for these fields in Chapter 3. In particular we found an action for the chiral two-form (3.15) which results in a self-duality equation of motion and similarly for the (2,1) and (2,2) tensor fields (3.36) and (3.73). These actions encode the free six-dimensional dynamics. While no longer manifestly invariant, six-dimensional symmetries are retained in the form of

five-dimensional gauge transformations which we specified.

The three actions built in the previous chapter exhibit common structures. To further elucidate these features, we then used the tools of Exceptional Field Theory in Chapter 4 to write the bosonic sectors of the three six-dimensional descriptions in a common framework, leading to the compact Lagrangian (4.30) which is invariant under combined symmetry transformations which we also specified. This reformulation hinged on a particular section constraint (4.13) which is a differential condition mixing derivatives with respect to internal coordinates with the group structures specific to each of the three models.

Chapter 5 was dedicated to a supersymmetric exension of the actions discussed in Chapter 3. To this end, we started by calculating explicitly the supersymmetry transformations for each theory where we used different strategies in each case. For the $\mathcal{N} = (2,2)$ (see (5.9)), we linearized the known ExFT formulation from the literature. Since no such formulation for the $\mathcal{N} = (3,1)$ and $\mathcal{N} = (4,0)$ cases is known, we computed the six-dimensional supersymmetry variations from consistency arguments and reduced them into a 5+1 split form. In the case of the $\mathcal{N} = (4,0)$ theory, a particular field redefinition allowed to write the free supersymmetric Lagrangian in the very compact form (5.54). The analysis of the $\mathcal{N} = (3,1)$ theory is not fully complete yet as we still require a mechanism to deal with the exotic gravitino. Indeed despite a Usp(8)-covariance of the eight gravitini in the ExFT formulation, the dynamics of six of them has to become trivial in the end. There are several different mechanisms that seem possible: i) the field equations could be pure gauge such that these gravitini are not physical degrees of freedom (this indeed happens in the five-dimensional reduction of the theory), ii) the dynamics remains non trivial but completely decouples from the rest of the theory, iii) the dynamics of the fields needs to be constrained further, for example by imposing a stronger version of the section constraint on the gravitini. While these options look viable at first glance, each of them poses further questions notably when in view of a possible generalization of the actions discussed here to a fully interacting theory.

The work presented here is a first step towards a unified understanding of six-dimensional maximally supersymmetric supergravity theories. As this work was limited to discussing free fields, an obvious generalization is to include non trivial interactions. As a first step, this would require to better understand the structure of the scalar fields and their geometric interpretation. A superspace formulation such as [52] may provide useful tools in this direction. Moreover, in this work, fields dual to the graviton have played a prominent role, it would be interesting to study explicit solutions of their field equations to gain insight of new possible structures such as higher-spin theories [33, 53, 54]. Even if we argued that we would not consider higher spins in this thesis, the reasoning we presented (splitting dimensions until the identification of more standard structures) could still be applied to higher-spin theories. Additionally, the fact that our formulation features exotic fields and more conventional gravity fields which

come with an action that allows a natural embedding into the full non-linear Einstein-Hilbert suggests that all these fields may become part of a tensor hierarchy that extends to the gravity sector thus the non-linear dynamics would arise from a hierarchy of duality relations as in [55].

APPENDIX A

Conventions and mathematical tools

A.1 Conventions

We are working in Minkowski space with a "mostly plus" flat metric

$$\eta_{\hat{\mu}\hat{\nu}} = \mathrm{diag}\{-1, 1, 1, 1, 1, 1\} \ , \quad \hat{\mu}, \hat{\nu} = 0, \ldots 5 \ . \tag{A.1}$$

We choose the symmetrization and antisymmetrization operations to have global weight 1 that is

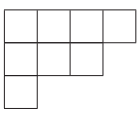
$$T_{(\hat{\mu}_1 \ldots \hat{\mu}_n)} = \frac{1}{n!} \sum_{\sigma \in G_n} T_{\sigma(\hat{\mu}_1) \ldots \sigma(\hat{\mu}_n)} \ , \tag{A.2}$$

$$T_{[\hat{\mu}_1 \ldots \hat{\mu}_n]} = \frac{1}{n!} \sum_{\sigma \in G_n} \varepsilon(\sigma) T_{\sigma(\hat{\mu}_1) \ldots \sigma(\hat{\mu}_n)} \ , \tag{A.3}$$

where the indices in the parentheses are symmetrized while those in the brackets are antisymmetrized. Moreover, G_n is the permutation group of order n and $\varepsilon(\sigma)$ denotes the signature of the permutation σ.

A.2 Young tableaux

Young tableaux are a schematic way to represent irreducible representations (of $GL(D)$ in our case) that correspond to tensors. Each index of a tensor corresponds to a cell in a tableau. The cells are organized in columns of non increasing sizes that encodes the symmetries of the indices of the tensor it represents. For instance

 (A.4)

is a possible tableau, whereas

(A.5)

is invalid. The symmetries of tensor can be read directly from the tableau using the following rules

- all indices in a same column are totally antisymmetric,
- the antisymmetrization of indices in a column and a cell on the right of that column vanishes.

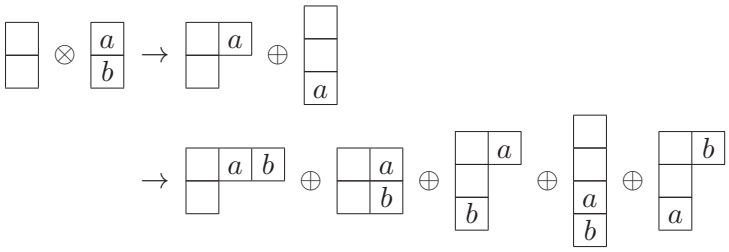
Hence an antisymmetric tensor is represented by $\begin{smallmatrix}\square\\\square\end{smallmatrix}$ (per the first rule) whereas a symmetric tensor is represented by $\square\square$ (per the second rule, if a two-index tensor has no antisymmetric part, then it must be symmetric). The structure becomes richer when the number of cells increases. A general three-index tensor $T_{\mu\nu\rho}$ can be decomposed into three parts

- a completely symmetric part $T_{(\mu\nu\rho)}$ represented by the diagram $\square\square\square$,
- a completely antisymmetric part $T_{[\mu\nu\rho]}$ represented by the diagram $\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}$,
- a mixed-symmetry part $A_{\mu\nu\rho} = T_{\mu\nu\rho} - T_{(\mu\nu\rho)} - T_{[\mu\nu\rho]}$ represented by the "hook" diagram $\begin{smallmatrix}\square\square\\\square\end{smallmatrix}$ such that $A_{\mu\nu\rho}$ is antisymmetric on its first two indices and satisfies a Bianchi-like identity $A_{[\mu\nu\rho]} = 0$.

Young tableaux can also be of use to determine the decomposition of tensor products using the Littlewood-Richardson rule [56]. Say we want to decompose the product $T_1 \otimes T_2$. To do this

1. assign distinct labels to cells in each row of T_2 (e.g. $\begin{smallmatrix}a&a\\b\end{smallmatrix}$),
2. attach cells labelled by a to T_1 in all possible ways such that no two as appear in the same column and the result is still a Young tableau,
3. repeat with bs, etc.,
4. for each tableau, form the sequence by reading each row from right to left while going from top to botton. A tableau will be valid if when reading the sequence, there is at least as many as as bs (etc.) at any point in the sequence.

Let us look at an example



The sequences for these tableaux are ba, ab, ab, ab, ba, so the first and last tableaux are invalid.

A.3 Generalized Poincaré lemma

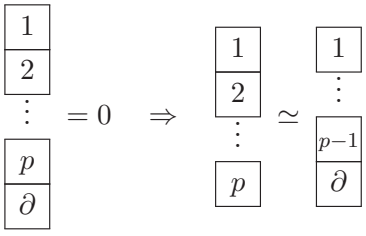
As we are solving differential equations involving mixed-symmetry tensors, we need a version of the Poincaré lemma that applies to such tensors presented and proved in [42]. Let us first recall the standard Poincaré lemma. We define $\Omega^p(\mathcal{M})$ the set of p-forms on a D-dimensional pseudo-riemannian manifold $\mathcal{M}$. A p-form ω is defined by its components

$$\omega = \frac{1}{p!}\,\omega_{i_1\cdots i_p}\,\mathrm{d}x^{i_1}\wedge\cdots\wedge\mathrm{d}x^{i_p}\ , \tag{A.6}$$

and its differential is

$$\mathrm{d}\omega = \frac{1}{(p+1)!}\,\partial_{i_1}\omega_{i_2\cdots i_{p+1}}\,\mathrm{d}x^{i_1}\wedge\cdots\wedge\mathrm{d}x^{i_{p+1}}\ . \tag{A.7}$$

The standard Poincaré lemma states that if $\mathrm{d}\omega = 0$, then we can locally define a $(p-1)$-form α such that $\omega = \mathrm{d}\alpha$. In terms of Young tableaux, we can write



In our case, we need such a similar lemma for tensors whose Young tableau has two columns. To that end, we define $\Omega^{p,q}(\mathcal{M})$ the tensor product $\Omega^p(\mathcal{M})\otimes\Omega^q(\mathcal{M})$ and $\Omega^{p,q}_Y(\mathcal{M})$ the set of tensors represented by the Young tableau with two columns of sizes p and q. A bi-form in $\Omega^{p,q}(\mathcal{M})$ (also called (p,q)-form) T has components

$$T = \frac{1}{p!q!}\,T_{i_1\cdots i_p j_1\cdots j_q}\,\mathrm{d}x^{i_1}\wedge\cdots\wedge\mathrm{d}x^{i_p}\otimes\mathrm{d}x^{j_1}\wedge\cdots\wedge\mathrm{d}x^{j_q}\ , \tag{A.9}$$

After defining the differential with respect to each column

$$\mathrm{d}_1 : \Omega^{p,q}(\mathcal{M}) \to \Omega^{p+1,q}(\mathcal{M}) \,, \tag{A.10}$$

$$\mathrm{d}_2 : \Omega^{p,q}(\mathcal{M}) \to \Omega^{p,q+1}(\mathcal{M}) \,, \tag{A.11}$$

we can define the projections of these maps onto suitable Young tableaux

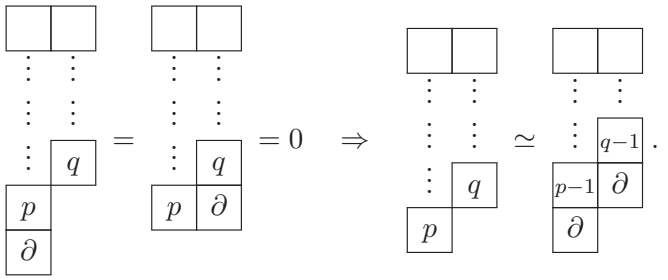
$$\mathrm{d}_L = \mathbb{P}_{(p+1,q)} \circ \mathrm{d}_1 \,, \tag{A.12}$$

$$\mathrm{d}_R = \mathbb{P}_{(p,q+1)} \circ \mathrm{d}_2 \,, \tag{A.13}$$

where $\mathbb{P}_{(m,n)}$ is the projector on the Young tableau with two columns of sizes m and n. In the context of bi-forms, the Poincaré lemma is in fact two different lemmas The first one reads

$$\mathrm{d}_L\omega = \mathrm{d}_R\omega = 0 \;\Rightarrow\; \exists\, \alpha \in \Omega_Y^{p-1,q-1} : \omega = \mathrm{d}_L\mathrm{d}_R\alpha \,, \tag{A.14}$$

which can be represented schematically by

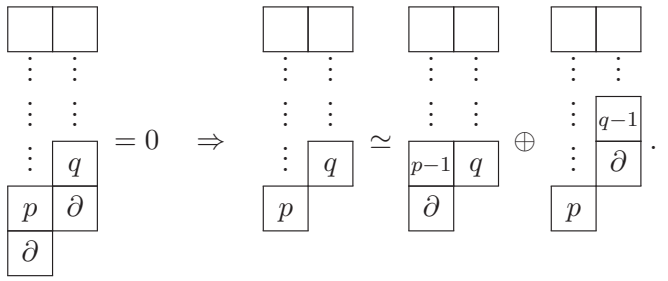


(A.15)

The second lemma reads

$$\mathrm{d}_L\mathrm{d}_R\omega = 0 \;\Rightarrow\; \exists\, \alpha, \beta \in \Omega_Y^{p-1,q} \times \Omega_Y^{p,q-1} : \omega = \mathrm{d}_L\alpha + \mathrm{d}_R\beta \,, \tag{A.16}$$

or more schematically



(A.17)

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