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Gravitational radiation of compact binary systems in general relativity and in scalar-tensor theories

David Trestini

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DE L'UNIVERSITÉ SORBONNE UNIVERSITÉ

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présentée par

David TRESTINI

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Rayonnement gravitationnel des systèmes binaires compacts
en relativité générale et dans les théories tenseur-scalaire

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devant le jury composé de :

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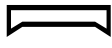
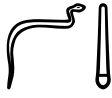
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*dd-mdw: “hnnw m pt! m3.n m3wt,” jn sn,
ntrw p3wtjw*

“Disturbance in the sky! We saw something
new,” said the primeval gods.

Spell 257, *Pyramid Texts* [1]
Pyramid of Unas, ca. 2330 BC

Introduction

A historical perspective

This thesis is mainly based on papers I have co-authored, published in peer-reviewed journals [2, 3, 4, 5, 6] and conference proceedings [7, 8, 9]. The following material was particularly useful when writing this thesis: [10, 11, 12, 13, 14]. Many results were obtained using Mathematica [15], and in particular the xAct library [16].

Among all the forces of nature, gravity is the only one for which we have a natural physical intuition. Our inability to free ourselves from it led humanity to consider the inaccessible sky as the firmament of the gods. Thus, it is because of gravity that there has long been a dichotomy between our understanding of the laws that govern our familiar terrestrial world and the patterns we observe in the distant sky. Despite unquestionable progress which has been made throughout the centuries¹ in astronomy and on the free fall of bodies on Earth, we owe the first visionary breakthrough to Isaac Newton, who understood in his *Philosophiæ Naturalis Principia Mathematica* [17] that these two seemingly unrelated systems were in fact governed by the same universal laws of gravitation, associated with a universal constant of gravitation $G = 6.67 \times 10^{-11} \text{ m}^3 \text{ kg}^{-1} \text{ s}^{-2}$:

I began to think of gravity extending to the Orb of the Moon, and having found out how to estimate the force with which a globe revolving within a sphere presses the surface of the sphere. From Kepler's Rule of the periodical times of the Planets, I deduced that the forces which keep the Planets in their Orbs must be reciprocally as the squares of their distances from the centres about which they revolve: and thereby compared the force requisite to keep the Moon in her Orb with the force of gravity at the surface of the Earth, and found them to answer pretty nearly.

— Isaac Newton

Although Newton was somewhat unconvinced by the metaphysics of his theory², it was immediately a major success. Despite the extreme simplicity of the formulation of

¹By René Descartes, Johannes Kepler, Tycho Brahe, Galileo Galilei, etc. etc.

²Although he was fully satisfied with the accuracy of the predictions of his theory, Newton could not fathom the metaphysical cause of gravity's action at a distance, as he expresses in a letter to Richard Bentley:

That one body may act upon another at a distance through a vacuum without the mediation of anything else, by and through which their action and force may be conveyed from one

this law, its effect was to yield extremely accurate predictions for the motion of solar system bodies, as worked out using perturbation theory by Pierre-Simon de Laplace in his *Traité de mécanique céleste* [18]. The epitome of this success was the prediction of the existence of a eighth planet by Urbain Le Verrier (and independently by John Couch Adams) in order to explain the small but systematic discrepancy between the orbit of Uranus and the predictions of Newton's laws of gravitation. The exact position of the putative planet was transmitted by Le Verrier to the German astronomer Johann Galle, who observed Neptune at the predicted location on September 23, 1846.

The simplicity of Newton's law of gravitation encouraged scientists to explain other, more elusive, phenomena with the help of similar methods. Indeed, static electricity had been observed by Thales of Miletus as early as 600 BC on amber rods³, but it was only much later, and after many preliminary works, that Charles-Augustin de Coulomb published the *Mémoires sur l'Électricité et le Magnétisme* [19], which predicted that two (macroscopic) bodies with some charges should undergo a force that varies as the inverse square of the distance, in perfect analogy with Newton's law of gravitation. At the time, the electric force was seen as totally separate from magnetism, which was commonly used for navigation since the Middle Ages. Back then, the only known kind of magnetism was ferromagnetism, and one only studied the interaction of ferromagnets with each other and with the Earth's magnetic field, in devices such as compasses. However, when Hans Christian Ørsted accidentally discovered that a compass would twitch near an electric wire, it was understood that these two theories were linked. André-Marie Ampère, Carl Friedrich Gauß, Jean-Baptiste Biot and Félix Savart discovered how electric charges in motion, i.e. an electric current, gave rise to a magnetic field; conversely, Michael Faraday found how the variation of the magnetic field could induce an electric current in a wire. The major milestone in the field was the unification of all of these theories by James Clerk Maxwell in *A dynamical theory of the electromagnetic field* [20]. The theory of electromagnetism does not only encompass the interactions between the electric and magnetic fields, but also explains the finite-speed propagation of light as an electromagnetic field in vacuum. It was indeed known from Ole Christensen Rømer's 1675 observation of the moons of Jupiter [21] (following the suggestion by Giovanni Domenico Cassini) and from Christiaan Huygens's theoretical computation [22] that the speed of light should be finite, and was measured at the time to be approximately 220,000 km/s (instead of 299,792.458 km/s).

Although electromagnetism was extremely successful, the fact that the speed of light was a constant of the theory created a theoretical crisis: suppose light was propagating in a certain direction, then an inertial observer following the light in the same direction should see the light with vanishing propagation speed. This is clearly in contradiction with the universal speed of light predicted by Maxwell's theory. Moreover, it seemed incongruous for light waves to travel in vacuum, since all known waves travel in a medium, and the speed of sound is determined by the nature of this medium. To solve this inconsistency, it was suggested that a substance called luminiferous ether should permeate the Universe, so

another, is to me so great an absurdity that, I believe, no man who has in philosophic matters a competent faculty of thinking could ever fall into it.

— Isaac Newton

³The word electricity (and electron) comes from the Greek word ἤλεκτρον, which means amber.

as to act as a medium for the propagation of light. Then, just like for material waves, an inertial observer would be able to “follow” the electromagnetic waves. In order to verify this statement, an experiment was devised by Albert Abraham Michelson and Edward Williams Morley (using the so-called Michelson interferometer) to compare the speed of light in the direction of the orbital motion of the Earth, and in the direction orthogonal to it. Should such an ether exist, then the two velocities would differ. However, the experiment found no difference, and this null result thus aggravated the crisis.

In order to reconcile mechanics and Maxwell’s theory, one needed to come up with a summation law of velocities that was compatible with a universal speed of light. A first breakthrough was an improved model of ether by Lorentz [23], which could explain the null result by a set of coordinate transformations, which were dubbed Lorentz transformations by Poincaré [24]. A new theory, called (special) relativity, was then proposed in 1905 by Albert Einstein in his renown paper “Zur Elektrodynamik bewegter Körper” [25] and by Henri Poincaré [24]. This theory did not require the existence of ether at all, but one needed to accept that the notion of time and space are not absolute, but depend on the reference frame. Indeed, consider the thought experiment of a ray of light emitted vertically from the floor of a train cabin, reflecting against a mirror on the roof, and reflecting back down onto the floor, where it came from. Denoting the speed of light c and the height of the cabin d , then the time needed for the light ray to go back and fourth is $\Delta t = 2d/c$. Now consider a person standing on the platform of the train station who sees the train pass by with a speed v , and sees the motion of the light ray through the window. For the exterior observer, the path of the light ray is now triangular-shaped and longer than the path measured by an observer in the train. Under the assumption that the speed of light is independent of the reference frame, elementary geometry and algebra tells us that the time it takes to travel this longer path is now $\Delta t' = 2d\gamma/c$, where $\gamma = 1/\sqrt{1 - (v/c)^2}$ is called the Lorentz factor (for subluminal velocities $v < c$, we have $\gamma > 1$, so $\Delta t' > \Delta t$). However, we could also have performed the reverse experiment, with the light beam outside the train, and we would have found the same result. We thus conclude that in special relativity, the observer on the platform sees a clock inside the train tick slower than his own clock; *and reciprocally*, the person inside the train also sees the clock of the person of the platform tick faster than her own clock. This is dilation of time in special relativity; similarly, a ruler oriented in the direction of motion of the train would be seen shorter than expected by an observer on the platform, and *vice versa*, namely there is contraction of lengths⁴. The theory of special relativity is thus compatible with a universal speed of light, and the absence of a preferred (Galilean) frame.

However, special relativity has outstanding issues. First, we have seen that there is no preference between two frames that move with a *constant* velocity with respect to each other, but this ceases to be true if the frames are *accelerated* with respect to each other, in which case inertial forces appear⁵. For Newton’s first law of motion to hold, one needs to postulate the existence of an inertial frame of reference, which is an unaccelerated frame, thus free of any inertial forces. This sounds unnatural if one wants physics to be fully independent of the choice of reference frame. Moreover, Newton’s law of gravity is incompatible with special relativity, since it acts instantaneously, while special relativity

⁴Note that this does not happen for a ruler perpendicular to the direction of motion.

⁵For example, the centrifugal force that extracts water from your favorite type of lettuce in a salad spinner.

requires that causal effects cannot happen faster than the speed of light. Interestingly, since Coulomb law of electrostatics was generalized by Maxwell into the dynamical theory of electromagnetism, with waves propagating at the speed of light, one could expect that Newton’s law can be generalized to a dynamical theory, involving an “electric” and a “magnetic” component, with “gravitational waves” propagating at the speed of light as well.

In 1907, while working in a patent office in Bern, Albert Einstein had what he calls his “happiest thought”:

Suddenly a thought struck me: If a man falls freely, he would not feel his weight. [...] A falling man is accelerated. Then what he feels and judges is happening in the accelerated frame of reference. [...] A falling man does not feel his weight because in his reference frame there is a new gravitational field which cancels the gravitational field due to the Earth. In the accelerated frame of reference, we need a new gravitational field.

— Albert Einstein in *How I created the theory of relativity* [26]

In this way, the two problems we have mentioned have the same answer, namely that gravity and inertia are essentially the same thing. Consider a very modern rocket smoothly blasting through space with a constant acceleration $g = 9.8 \text{ m/s}^2$. A person sitting in the rocket would not be able to tell whether he is in a rocket or in a room on Earth. From this idea, which is called the *equivalence principle*, Einstein was able to compute within the framework of special relativity how much faster a clock at the head of the rocket would tick with respect to one at the rear, and postulated that the same should hold in a gravitational field [27, 28]. With this primitive framework, Einstein was able to derive other predictions such as the deflection of light rays [28], but in order to go further, he needed a consistent mathematical toolbox to describe his new theory. The idea for such a framework came from yet another thought experiment: consider a rotating disk, and an observer walking on the disk (assuming his shoes have enough friction to avoid him being projected off by centrifugal forces). When the observer measures the radius \mathcal{R} of the disk, his ruler is perpendicular to the direction of motion and is unaffected by special relativity. However, when measuring the circumference \mathcal{C} , his ruler is always along the direction of motion, thus is contracted. Then, to his surprise, he finds $\mathcal{C}^2/\mathcal{R} < 4\pi$. This situation is in fact very similar to that of geometry on the curved surface of a sphere, which had been studied by Carl Friedrich Gauß using surface coordinates, and for which, for example, the sum of the angles in a triangle is not 180° . This led Einstein, with the help of mathematicians Marcel Grossmann and Tullio Levi-Civita, to study the theory of differential geometry by Bernhard Riemann and Gregorio Ricci, namely the generalized theory of geometry on curved surfaces of arbitrary dimension. Based on these mathematical tools, Einstein formulated the theory of general relativity [29] in its modern form, which is essentially a geometrical theory of gravity. The main dynamical variable of the theory is the metric tensor, which measures angles and distances locally in the case of the four-dimensional spacetime *manifold*. The presence of matter (or more generally, energy, recall $E = mc^2$) curves spacetime, creating a gravitational field, and conversely, the trajectory of particles follows *geodesics*, namely the straightest possible lines in curved space. This explains, for example, how light can be deflected despite it not having a mass and being unaffected by gravitational forces within the framework of Newton’s gravitation theory. Moreover,

approximating general relativity for weak curvature led to the final explanation [30] for the small deviation of Mercury’s trajectory from the predictions of Newton’s theory⁶.

This theory is beautiful to state, but very unwieldy to manipulate, and Einstein initially only worked with approximate solutions to his equations. Remarkably, an exact solution of the Einstein equations was found as early as 1915 in the trenches of the Great War by Karl Schwarzschild⁷ for a spherically symmetric, nonrotating body [31, 32], then much later, in 1963, by Roy Kerr for a rotating body [33]. Initially, these solutions were designed to account for the gravitational field outside stars, and in this case the full solution is completely regular. However, assuming that a star of mass m could be compressed enough to fit in a radius smaller than $2Gm/c^2$, it was remarked by Arthur Eddington that “the force of gravitation would be so great that light would be unable to escape from it, the rays falling back to the star like a stone to the earth”. This was initially deemed unphysical: it was known that once stars had exhausted all the “fuel” driving the nuclear fusion reactions inside their core, it would eventually become a white dwarf, i.e. a star which resists gravitational collapse thanks to quantum interactions of electrons, called electron degeneracy pressure. However, Subrahmanyan Chandrasekhar proved in 1931 [34] that a white dwarf with a mass larger than 1.4 times the mass of the sun would also collapse into a so-called neutron star, which resists gravitational collapse thanks to neutron degeneracy pressure. These stars have a density of $\sim 10^{17}$ kg/m³, which is roughly the density of an atom’s nucleus, and is equivalent to compressing the Earth into a sphere of 305 m in diameter. Yet it was predicted in 1939 by Richard Tolman, Robert Oppenheimer and George Volkoff [35, 36] that neutron stars with a mass greater than 0.7 times the mass of the sun⁸ could not be stable either, and would also undergo gravitational collapse. Debates about whether this collapse actually took place physically were sealed by Roger Penrose and Stephen Hawking, who proved that very massive stars generically collapsed to a single point [37], and formed a *singularity*, a line in spacetime with infinite curvature. Such solutions are *fully* described by the Schwarzschild or Kerr vacuum solutions for the spacetime metric, exhibit the property that no light – and in fact no information or causal effect – can exit a zone of spacetime of finite radius, called the *event horizon* ($R_S = 2Gm/c^2$ in the case of the Schwarzschild metric), hence the name of such objects, *black holes*. The singularity is therefore not visible to exterior observers, i.e not *naked*. Physically, general relativity predicts that an adventurer who took the resolution to fall into a black hole will not experience anything particular when crossing the event horizon, but his time will *stop* at the singularity (of course, since our adventurer is not a point, he will be ripped apart by tidal forces before experiencing this bizarre phenomenon). However, his somewhat more cautious friend, who decided to stay very far from the black hole, will see his adventurous comrade slow down as he approaches the black hole, and eventually freeze as he approaches the event horizon (we will also fade away due to gravitational Doppler redshift).

⁶Ten years after the discovery of Neptune, Le Verrier achieved another feat by discovering the excess precession of the perihelion of Mercury. He attempted to explain it with an extra planet, Vulcan, even closer to the Sun than Mercury. However, this planet was never observed.

⁷He sent to Einstein a letter from the Eastern front dated December 22, 1915, and died a few months later from pemphigus.

⁸More recent estimates based on neutron-neutron repulsion mediated by the strong force and informed by the merger of the binary neutron star merger GW170817 brought this estimate up to about around 2.7 solar masses.

The intriguing physics of black holes and neutron stars is not the only novel prediction of general relativity: recall that a *desideratum* to a generalization of Newton’s gravity is the existence of gravitational waves. In the general-relativistic formulation, these are wave-like oscillations of the curvature of spacetime which propagate at the speed of light. We can now see that general relativity is compatible with causality, because if the Sun were to disappear, we would not feel it immediately, but it would generate a gravitational wave that would arrive 8 minutes and 20 seconds later. Gravitational waves were predicted in 1916 by Einstein himself [38, 39], and were found to be generated not by the charge dipole vector as in electromagnetism, but by the mass quadrupole tensor, due to conservation of mass, linear momentum and angular momentum in general relativity. Moreover, Einstein predicted the existence of three types of gravitational waves: longitudinal–longitudinal, transverse–longitudinal, and transverse–transverse. The two first types were shown not to be physical by Arthur Eddington in 1922 [40], but instead mere artifacts of the choice of coordinates, and could be made to propagate at any speed by a clever choice of coordinates, leading him to dismiss them as “propagating at the speed of thought”. The fact that most quantities appearing in general-relativistic computations depend on the choice of coordinates for the spacetime manifold is one of the major difficulties of the theory, and one needs to construct objects which are coordinate invariant. This difficulty is illustrated by the fact that in 1936, Einstein and Nathan Rosen initially sent a manuscript to the peer-reviewed journal *Physical Review* claiming that the transverse–transverse gravitational-wave mode was also a pure coordinate effect. The paper was dismissed as incorrect by the anonymous reviewer (later determined to be Howard Robertson). Einstein later published with Leopold Infeld a paper claiming the opposite, but in another journal, since he remained angry with *Physical Review* for the rest of his life. This confusion was definitively solved in 1956 by Felix Pirani [41], who rephrased gravitational waves in terms of the linearized Riemann curvature tensor, which is a gauge-invariant quantity. The next year, Richard Feynman provided at the first “General Relativity” conference a thought experiment proving that gravitational waves carry energy, known as the sticky bead argument: if one takes a rod with beads on it, the passage of a gravitational wave would make the beads move along the rod. Due to friction, this would generate heat, and thus the gravitational wave has produced work.

An ideal system to study the generation of gravitational waves is an isolated two-body system, which we consider to be on a circular orbit for simplicity. Indeed, if such a system emits gravitational waves, then the total energy of the binary will decrease in virtue of energy conservation. It is known that the internal energy of the bodies play a negligible role in this case, so it is the total kinetic and gravitational binding energy $E = -G\mu m/(2r_{12}) = -\mu v_{12}^2/2$ which must decrease⁹. One immediately deduces that decreasing E amounts to decreasing the orbital separation r_{12} and increasing the relative velocity of the stars v_{12} , so the dissipation of energy due to gravitational waves should in principle be easy to observe in a binary system. However, the rate at which, for example, the orbital separation decreases scales as $dr_{12}/dt \propto \mu m^2/r_{12}^3$, so this effect will be maximal for systems that are very close *and* very massive, and must therefore be very compact, like neutron stars or black holes. Plugging in the numbers for, say, the Sun-Jupiter system,

⁹We denote by $m = m_1 + m_2$ the total mass of the two stars, $\mu = m_1 m_2 / (m_1 + m_2)$ is the reduced mass of the system, r_{12} is the distance between the stars and v_{12} is the relative velocity of one star with respect to the other.

we find that this effect is totally negligible, and this would also hold if the two bodies were brought very close together, so as to be on the verge of collision. Thus, the first indirect detection of gravitational waves was made on a system made of two neutron stars. The system was discovered by Russell Alan Hulse and Joseph Hooton Taylor Jr. in 1974, and is composed of a neutron star and of a pulsar, which is a highly magnetized neutron star that emits beams of electromagnetic radiation at its magnetic poles. Since the magnetic pole does not coincide with the geographic pole associated to the (very fast) rotation of the neutron star, the beam rotates around the star very quickly, and flashes extremely regularly, like a lighthouse, if it happens to be oriented towards Earth. In the case of the Hulse-Taylor binary pulsar, these flashes arrived alternatively a little sooner or a little later than expected, which was found to be compatible with the pulsar being in orbit with another neutron star. Measuring the variation of the orbital parameters led to the conclusion that the system was indeed shrinking due to the emission of gravitational waves, in perfect agreement with the predictions of general relativity [42, 43, 44].

There was great eagerness to have a direct detection of gravitational waves, which was claimed as early as 1968 by Joseph Weber; however, these claims were rapidly discredited, in particular by similar experiments which did not reproduce his results. It is only many years later, on September 14, 2015, that the first gravitational wave was detected [45] by the Laser Interferometer Gravitational-Wave Observatory (LIGO). This observatory is based on the same physical concept as the Michelson interferometer, but with arms which are 4 km long, which can detect oscillations of its mirrors of magnitude smaller than the radius of a proton. Since then, and with the help of two other observatories (Virgo and KAGRA), around 90 gravitational wave events have been detected, which are all perfectly compatible with the predictions of general relativity. These observatories allow us to observe the Universe for the first time with another messenger than light, and give us precious information about its composition. Further understanding will require more sensitive detectors, some of which will be constructed on Earth (Einstein Telescope or Cosmic Explorer) and others will be satellites communicating in space (LISA). In order to interpret these results and test our physical models, it is imperative to extract, both from general relativity and from other possible theories of gravity, very accurate predictions for the gravitational waveforms we expect to observe.

This is precisely the goal of this thesis. We will generically consider a pair of two compact objects, such as black holes or neutron stars, on a bound orbit. When these objects are well separated and have a small relative velocity v with respect to the speed of light, we are in the so-called inspiral regime and we can compute post-Newtonian (PN) corrections, in powers of the small parameter v/c , to the lowest order formulas found by Einstein¹⁰. The first part of this thesis is devoted to general relativity. In the case of quasicircular orbits, we obtain expressions for the waveform at 4PN order, the energy flux radiated by the system at 4.5PN, and the phase and the increase in frequency (called “chirp”) at 4.5PN order. The second part of this thesis is devoted to a class of alternative theories of gravity, called scalar-tensor theories. We compute the waveform and flux for generic bound orbits up to 1.5PN with respect to the leading order of general relativity, and specialize these results to the case of quasicircular orbits. All of these results will be useful for accurately interpreting the data obtained in gravitational wave observatories.

¹⁰Corrections in $(v/c)^{2n}$ beyond the leading order are said to be of n PN order.

Abbreviations

ADM: Arnowitt-Deser-Misner

BD: Brans-Dicke

BH: black hole

BBH: binary black hole

BNS: binary neutron star

CE: Cosmic Explorer

COM: center of mass

DDR: difference (between Hadamard and) dimensional regularization

DIRE: direct integration of the relaxed (field) equations

DR: dimensional regularization

EFT: effective field theory

ET: Einstein Telescope

EW: Epstein and Wagoner

GR: general relativity

GW: gravitational wave

HR: Hadamard regularization

IR: infrared

KAGRA: Kamioka Gravitational Wave Detector

LIGO: Laser Interferometer Gravitational-Wave Observatory

LISA: Laser Interferometer Space Antenna

LVK: LIGO-Virgo-KAGRA

MPM: multipolar post-Minkowskian

NS: neutron star

PA: post-adiabatic

PM: post-Minkowskian

PN: post-Newtonian

PPN: parametrized post-Newtonian

SNR: signal-to-noise ratio

SPA: stationary phase approximation

ST: scalar tensor

STF: symmetric tracefree

UV: ultraviolet

Notations

We denote by γ_E the Euler constant and $\bar{q} = 4\pi e^{\gamma_E}$. The gamma function is denoted by $\Gamma(x)$, the Legendre polynomial by $P_\ell(x)$ and the Legendre function of second type with branch cut on $] -\infty, 1]$ by $Q_m(x)$. The sign of a nonzero real number $x \in \mathbb{R}_*$ is denoted $\text{sg}(x) \in \{-1, 1\}$.

We use the Bachmann–Landau notations $\mathcal{O}(\dots)$ and $o(\dots)$.

We use Einstein summation, where Greek indices indicate spacetime, e.g. $\mu = 0, 1, 2, 3$, and Latin indices indicate space, e.g. $i = 1, 2, 3$. Three-dimensional Euclidean vectors are denoted in bold font, e.g. \mathbf{V} has components V^i .

When dealing with field points, we denote coordinate time by t and the spatial coordinate vector by \mathbf{x} , where $r = |\mathbf{x}|$ is its norm and $\mathbf{n} = \mathbf{x}/r$ its direction. We define retarded time $u = t - r/c$ and advanced time $v = t + r/c$.

Ordinary partial derivatives with respect to the spacetime coordinate x^μ are denoted $\partial_\mu \equiv \partial/\partial x^\mu$. Ordinary spatial partial derivatives are denoted $\partial_i \equiv \partial/\partial x^i$. Time derivatives are denoted $\partial_t = c\partial_0 = \partial/\partial t$.

We denote $L = i_1 \cdots i_\ell$ a multi-index with ℓ spatial indices. Similarly, $L-1 = i_1 \cdots i_{\ell-1}$ and even $L-2K = i_{2k+1} \cdots i_\ell$, etc. Repetitions of such multi-indices denote contractions in the Einstein summation convention.

We use the shorthands $\partial_{\mu_1 \dots \mu_n} \equiv \partial_{\mu_1} \dots \partial_{\mu_n}$, $n_L \equiv n_{i_1} \dots n_{i_n}$, $x_L \equiv x_{i_1} \dots x_{i_n}$, $\partial_L \equiv \partial_{i_1} \dots \partial_{i_n}$ and $\partial_t^n \equiv \partial^n / \partial t^n$.

The symmetric trace-free (STF) part of any tensor A_L with respect to the indices L is denoted in the following ways: $\text{STF}_L[A_L] = \hat{A}_L = A_{\langle i_1 \dots i_\ell \rangle}$. This also holds for partial derivatives, e.g. $\hat{\partial}_L = \partial_{\langle i_1 \dots i_\ell \rangle}$. Underlined indices are excluded from the STF projection, e.g. $A_{i \langle j \underline{k} \rangle B_{lp} \rangle q} = \text{STF}_{jlp} [A_{abp} B_{lpq}]$.

The n^{th} time derivative of a function $F(t)$ is denoted $F^{(n)}(t) = d^n F / dt^n$. If $F(t)$ was identically zero before some time $-\mathcal{T}$ in the past, then n^{th} time antiderivative is denoted $F^{(-n)}(t) = \int_{-\infty}^t dt_1 \int_{-\infty}^{t_1} \cdots \int_{-\infty}^{t_{n-1}} dt_n F(t_n)$.

The flat spacetime Minkowski metric is denoted $\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$

The spatial Kronecker delta is denoted δ_{ij} , and the totally antisymmetric spatial Levi-Civita symbol is denoted ϵ_{ijk} (with $\epsilon_{123} = +1$). For any three-dimensional tensor A_L , we define $A_{i|L} \equiv \epsilon_{ii_\ell k} A_{kL-1}$.

The 3-dimensional Dirac function is denoted $\delta^{(3)}(\mathbf{x})$.

For a spacetime vector ϕ^μ , we introduce the shorthand $\partial\varphi^{\mu\nu} \equiv 2\partial^{(\mu}\varphi^{\nu)} - \eta^{\mu\nu}\partial_\rho\varphi^\rho$.

We denote the finite part of a Laurent series in $B \in \mathbb{C}$ by $\text{FP}_{B=0}$.

The number of *spatial* dimensions is denoted $d = 3 + \varepsilon$, and we define $\tilde{k} \equiv \Gamma\left(\frac{d-2}{2}\right)\pi^{\frac{2-d}{2}}$.

We consider a system of two point-particles $A \in \{1, 2\}$ in a harmonic coordinate system $\{t, \mathbf{x}\}$. We define

Positions: $\mathbf{y}_A(t) = r_A(t)\mathbf{n}_A(t)$, where $\mathbf{n}_A(t)$ has unit norm;

Velocities: $\mathbf{v}_A(t) \equiv d\mathbf{y}_A/dt$;

Accelerations: $\mathbf{a}_A(t) \equiv d\mathbf{v}_A/dt$;

Separation vector: $\mathbf{y}_{12} = \mathbf{y}_1 - \mathbf{y}_2 = r_{12}\mathbf{n}_{12}$, where \mathbf{n}_{12} has unit norm;

Relative velocity vector: $\mathbf{v}_{12} = \mathbf{v}_1 - \mathbf{v}_2$;

Relative acceleration vector: $\mathbf{a}_{12} = \mathbf{a}_1 - \mathbf{a}_2$;

Dirac function at the location of the particle: $\delta_A \equiv \delta^{(3)}(\mathbf{x} - \mathbf{y}_A)$;

Masses of the particles: m_A ;

Total mass: $m = m_1 + m_2$;

Reduced mass: $\mu = m_1m_2/m$;

Symmetric mass ratio: $\nu = \mu/m \in]0, 1/4[$;

Relative mass difference: $\delta = (m_1 - m_2)/m \in [0, 1[$, where $\delta^2 = 1 - 4\nu$;

The standard post-Newtonian parameter: $\gamma = Gm/(r_{12}c^2)$;

The notation $(1 \leftrightarrow 2)$ is a shorthand for the previous term in a sum, but where the particle labels 1 and 2 have been exchanged.

Chapter I

General context

I.1 General description of weak gravitational waves

Before discussing gravitational waves (GWs), we briefly recall the theoretical framework in which this notion is defined. We introduce the theory of general relativity (GR), which is best expressed using the Einstein-Hilbert action along with a matter action:

$$S_{\text{GR}} = \frac{c^4}{16\pi G} \int d^4x \sqrt{-g} R + S_{\text{matter}}[\mathbf{m}, g_{\alpha\beta}], \quad (\text{I.1})$$

where $g_{\alpha\beta}$ is the spacetime metric tensor, g is the determinant of the metric, R is the Ricci scalar associated to the metric and \mathbf{m} represents generically the matter fields that couple minimally to the metric. Varying the action with respect to the spacetime metric yields the Einstein field equation, which read

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{8\pi G}{c^4}T_{\mu\nu}[\mathbf{m}, g_{\alpha\beta}], \quad (\text{I.2})$$

where $R_{\mu\nu}$ is the Ricci tensor associated to the metric and the energy-momentum tensor is defined by $T_{\mu\nu} \equiv -2(-g)^{-1/2} \delta S_m / \delta g^{\mu\nu}$.

A generic definition of a GW is then a weak, oscillatory perturbation of spacetime. Implicitly, this means that the perturbation is measured with respect to some reference, background spacetime. This split between background and perturbation is not univocal, which is a common feature when studying waves in general. However, a general property is that the perturbation should be oscillatory, its wavelength should be very small comparable to the length scales of the background, and it should have a small amplitude compared to the other scales of the problem [12]. In the case of GWs in GR, one is interested in splitting the spacetime metric $g_{\mu\nu}$ into a background and perturbations. The choice of background depends greatly on the problem under study. For example, when studying the ringdown phase of a binary black-hole merger or an extreme mass-ratio inspiral, it is natural to study perturbations around the Schwarzschild or Kerr metrics, since the metric is expected to deviate little from these thoroughly-studied static or stationary solutions. However, the disadvantage of this choice is that it is technically very difficult to study perturbations around a curved spacetime background, at least analytically. In more generic problems, the most practical choice is to take the flat Minkowski metric as a background. The first and main reason is that we have a wealth of analytical tools to

study perturbations around flat spacetime. The second reason is that flat spacetime is a vacuum solution of the Einstein equations in the absence of a cosmological constant. If there is a matter content, the metric deviates only slightly from Minkowski spacetime far away from the matter source: this is asymptotic flatness. These observations allow us to describe generic GWs, as detected on Earth, as deviations to flat spacetime. However, it must be noted that near the source, in the case of compact objects, these deviations can be extreme, and it is difficult to see the metric as a small deviation to flat spacetime. Luckily, this is not a problem thanks to the effacement principle [46], which states that the effects of internal structure of spherical compact bodies is very small.

Thus, in this section, we will study the general properties of small deviations to the flat spacetime metric, only in vacuum regions of the spacetime manifold. We will not yet worry about how they were generated, as this will be the topic of the next section. Also, we will not yet worry about the response of the detectors to this perturbation. We split the metric as

$$g_{\mu\nu} = \eta_{\mu\nu} + \gamma_{\mu\nu}, \quad (\text{I.3})$$

where $\eta_{\mu\nu}$ is the flat Minkowski metric, and $\gamma_{\mu\nu}$ is the GW perturbation, and we suppose $|\gamma_{\mu\nu}| \ll 1$, such that we can neglect terms in $\mathcal{O}(\gamma^2)$. This will typically be true very far from the GW source, since the magnitude of GWs decay as the inverse of the distance to the source. In the linearized approach, it is equivalent to raise indices using the full inverse metric $g^{\mu\nu}$ or with the flat background inverse metric $\eta^{\mu\nu}$, and we thus define the trace $\gamma \equiv \gamma^\rho_\rho$. The linearized Riemann tensor then reads

$$R_{\mu\nu\rho\sigma} = \frac{1}{2} (\partial_{\nu\rho}\gamma_{\mu\sigma} + \partial_{\mu\sigma}\gamma_{\nu\rho} - \partial_{\mu\rho}\gamma_{\nu\sigma} - \partial_{\nu\sigma}\gamma_{\mu\rho}) + \mathcal{O}(\gamma^2). \quad (\text{I.4})$$

In order to write the Einstein equations in a compact and practical way, it is common to introduce the trace-reversed metric, defined by

$$\bar{\gamma}_{\mu\nu} \equiv \gamma_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\gamma. \quad (\text{I.5})$$

However, since this thesis will ultimately mostly deal with the nonlinearities of the Einstein equations, we find it more advantageous to define straight away the tensor

$$h^{\mu\nu} \equiv \sqrt{-g}g^{\mu\nu} - \eta^{\mu\nu}, \quad (\text{I.6})$$

where $g^{\mu\nu}$ is the inverse metric and g is the determinant of the metric $g_{\mu\nu}$. In this way, we find that at linear level, these two quantities only differ by a sign,

$$h^{\mu\nu} = -\bar{\gamma}^{\mu\nu} + \mathcal{O}(\gamma^2), \quad (\text{I.7})$$

but of course ultimately differ at nonlinear level¹. We define, even at nonlinear level, $h_{\mu\nu} \equiv \eta_{\mu\rho}\eta_{\nu\sigma}h^{\rho\sigma}$ and $h^\mu_\nu \equiv \eta_{\nu\rho}h^{\mu\rho}$, and the trace is defined as $h \equiv h^\rho_\rho$. One then rewrites the vacuum Einstein equations as

$$\square h^{\mu\nu} - 2\partial_\rho\partial^{(\mu}h^{\nu)\rho} + \eta^{\mu\nu}\partial_\rho\partial_\sigma h^{\rho\sigma} = 0, \quad (\text{I.8})$$

¹Note that in most textbooks, $\gamma_{\mu\nu}$ is actually denoted $h_{\mu\nu}$, which can be quite confusing.

where the flat d'Alembert operator is defined as $\square \equiv \eta^{\rho\sigma} \partial_\rho \partial_\sigma$. The goal of this section will be to discuss the solutions to this equation.

An important observation to make before looking for such solution is that there are many redundancies in GR due to coordinate-redefinition invariance. Indeed, consider a small coordinate redefinition $x^\mu \rightarrow x'^\mu = x^\mu + \varphi^\mu(x)$, such that $|\partial^\mu \varphi^\nu| \sim |\gamma^{\mu\nu}| \ll 1$. The full metric transforms as a tensor, namely

$$g_{\mu\nu}(x) \rightarrow g'_{\mu\nu}(x') = \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial x^\sigma}{\partial x'^\nu} g_{\rho\sigma}(x) = g_{\mu\nu}(x) - 2g_{\rho(\mu}(x) \partial_{\nu)} \varphi^\rho(x) + \mathcal{O}(\varphi^2), \quad (\text{I.9})$$

so at linearized level, the perturbation $\gamma_{\mu\nu}$ transforms as

$$\gamma_{\mu\nu}(x) \rightarrow \gamma'_{\mu\nu}(x') = \gamma_{\mu\nu}(x) - 2\partial_{(\mu} \varphi_{\nu)}(x) + \mathcal{O}(\varphi^2), \quad (\text{I.10})$$

therefore it also behaves as a tensor². However, $h^{\mu\nu}$ behaves like a tensor *density*, since it transforms as

$$h^{\mu\nu}(x) \rightarrow h'^{\mu\nu}(x') = h^{\mu\nu}(x) + 2\partial^{(\mu} \varphi^{\nu)}(x) - \eta^{\mu\nu} \partial_\rho \varphi^\rho(x) + \mathcal{O}(\varphi^2). \quad (\text{I.11})$$

This transformation law justifies the introduction of a shorthand notation that will be ubiquitous in this thesis,

$$\partial\varphi^{\mu\nu} \equiv 2\partial^{(\mu} \varphi^{\nu)} - \eta^{\mu\nu} \partial_\rho \varphi^\rho \quad (\text{I.12})$$

Note that this transformation law means that the divergence of $h^{\mu\nu}$ transformed simply as

$$\partial_\nu h^{\mu\nu} \rightarrow \partial'_\nu h'^{\mu\nu}(x') = \partial_\nu h^{\mu\nu} - \square\varphi^\mu(x) \quad (\text{I.13})$$

This motivates us to define the coordinate transformation $\varphi^\mu(x)$ such that

$$\square\varphi^\mu(x) = \partial_\nu h^{\mu\nu}, \quad (\text{I.14})$$

which exists and is unique under usual boundary conditions. Now consider the Einstein equations in a given coordinate system, and apply the coordinate transformation (I.14). This choice will remove any divergence terms in the vacuum Einstein equations, which then simply read $\square h^{\mu\nu} = 0$. This choice of coordinate system is called a harmonic gauge or De Donder gauge³, and is more commonly given, not by its explicit construction, but by the *condition* that the $h^{\mu\nu}$ tensor be divergenceless, namely $\partial_\nu h^{\mu\nu} = 0$. Note that, by definition, this condition carries out identically in the full nonlinear case. This choice is practical and can be written covariantly with respect to the background, but it does not entirely exhaust the gauge freedom. Indeed, any additional gauge transformation, parametrized by a gauge vector φ^μ such that $\square\varphi^\mu = 0$, will preserve the harmonic gauge condition. To fully fix the gauge freedom, one needs to introduce a 3+1 splitting of time and space. One can then impose the transverse-traceless (TT) gauge condition⁴, which is straightforwardly given by $h = 0$ and $h^{0i} = 0$. Using these relations and the harmonic coordinate condition, we find that $\partial_0 h^{00} = 0$. Therefore, h^{00} is static, and is typically a Newtonian potential, encoding the time-independent gravitational attraction. Conversely,

²This statement of course breaks down beyond linearized level.

³It is analogous to the Lorenz gauge in electromagnetism, defined by $\partial_\mu A^\mu = 0$.

⁴It is analogous to the Coulomb gauge in electromagnetism, defined by $\partial_\mu A^\mu = 0$ and $\partial_i A^i = 0$.

all the radiative degrees of freedom are encoded in the purely spatial part of the metric, which we now denote h_{TT}^{ij} . This typically represents GWs. Note that since this gauge is traceless, we have $\gamma_{ij}^{\text{TT}} = \bar{\gamma}_{ij}^{\text{TT}} = -h_{\text{TT}}^{ij}$. Since the metric in the TT gauge is free of any residual gauge freedom, it is a purely observable quantity and is thus naturally related to the linearized Riemann tensor by

$$c^2 R_{0i0j}^{\text{lin}} = \frac{1}{2} \ddot{h}_{ij}^{\text{TT}} = -\frac{1}{2} \ddot{\gamma}_{ij}^{\text{TT}}. \quad (\text{I.15})$$

In order to proceed further, we now assume that our metric can be decomposed by a sum of plane waves, characterized by a direction of propagation $\hat{\mathbf{n}}$. This sum formally reads

$$h_{\text{TT}}^{ij}(t, \mathbf{x}) = \int d^2\hat{\Omega} h_{\text{P,TT}}^{ij}(\hat{\mathbf{n}}(\Omega), t, \mathbf{x}), \quad (\text{I.16})$$

where $\hat{\mathbf{n}} = (\cos \hat{\theta} \sin \hat{\phi}, \sin \hat{\theta} \sin \hat{\phi}, \cos \hat{\phi})$. We integrate over solid angles $d^2\hat{\Omega} = \sin \hat{\theta} d\hat{\theta} d\hat{\phi}$, and $h_{\text{P,TT}}^{ij}(\hat{\mathbf{n}}, t, \mathbf{x})$ is a generic plane GW propagating along the direction $\hat{\mathbf{n}}$ and evaluated at spacetime coordinate (t, \mathbf{x}) . Note that in linearized theory, a sum of plane waves in a TT gauge will be in a TT gauge (see p. 947 of [11]). Moreover, we do not need the decomposition into monochromatic waves here. Finally, most GWs we will encounter are actually spherical waves, which can be very well described by a single plane wave for an observer very far from the source.

We now focus on the case of a single plane wave, characterized by its direction of propagation $\hat{\mathbf{n}}$. We introduce the spatial orthonormal basis $(\mathbf{e}_x, \mathbf{e}_y, \hat{\mathbf{n}})$. If this plane wave is given in any harmonic gauge (not necessarily TT) by the tensor $h_{\text{P}}^{\mu\nu}(\hat{\mathbf{n}})$, and discarding the nonradiative degrees of freedom, we can show that it can be algebraically projected onto the TT gauge by the operation

$$h_{\text{P,TT}}^{ij} = \perp_{ij,kl}^{\text{TT}} h_{\text{P}}^{kl}, \quad (\text{I.17})$$

where we have introduced the TT projector

$$\perp_{ij,ab}^{\text{TT}}(\hat{\mathbf{n}}) = \perp_{a(i}(\hat{\mathbf{n}}) \perp_{j)b}(\hat{\mathbf{n}}) - \frac{1}{2} \perp_{ij}(\hat{\mathbf{n}}) \perp_{ab}(\hat{\mathbf{n}}), \quad (\text{I.18})$$

which is itself defined using the projector onto the transverse plane $\perp_{ij}(\hat{\mathbf{n}}) = \delta_{ij} - \hat{n}_i \hat{n}_j$. In this TT gauge, the metric of the plane wave then necessarily reads

$$h_{\text{P,TT}}^{ij}(\hat{\mathbf{n}}) = \begin{pmatrix} h_+(\hat{\mathbf{n}}) & h_\times(\hat{\mathbf{n}}) & 0 \\ h_\times(\hat{\mathbf{n}}) & -h_+(\hat{\mathbf{n}}) & 0 \\ 0 & 0 & 0 \end{pmatrix}_{ij}. \quad (\text{I.19})$$

We recover the two degrees of freedom of GR, which here come under the name of the “plus” mode h_+ and the “cross” mode h_\times . Applying the geodesic deviation equation for each of these modes onto a ring of freely falling particles, the effect of each mode are illustrated in green in Figure I.1. Of course, the plus and cross modes of the full GW are given by the sum of the plane wave contributions. Finally, we will find it extremely useful to merge the “plus” and cross” modes into a single complex number, which one will typically decompose into spherical harmonics. It reads⁵

$$h = h_+ - ih_\times, \quad (\text{I.20})$$

⁵Or equivalently, $\gamma = -h = \gamma_+ - i\gamma_\times$.

and it is related (in the linearized case) to the fourth Newman-Penrose scalar [47, 48] by

$$\Psi_4 = \frac{d^2 h}{dt^2}. \quad (\text{I.21})$$

Note that these general-relativistic properties are not all valid in alternative theories of gravity. Importantly, in these theories, GWs typically carry more than two modes. Following Poisson and Will [13], we will assume that we choose a theory such that the propagation velocity of the GWs and light are equal, which is motivated⁶ experimentally by the simultaneous detection of a GW and an electromagnetic signal from a system of coalescing neutron stars [49, 50]. We consider again a plane wave propagating along $\hat{\mathbf{n}}$ and express everything in the $(\mathbf{e}_x, \mathbf{e}_y, \hat{\mathbf{n}})$ basis. We work directly at the level of the linearized Riemann tensor, which reads in this basis

$$-2c^2 R_{0i0j}^{\text{lin}} = \frac{d^2}{dt^2} \begin{pmatrix} \gamma_b + \gamma_+ & \gamma_\times & \gamma_x \\ \gamma_\times & \gamma_b - \gamma_+ & \gamma_y \\ \gamma_x & \gamma_y & \gamma_\ell \end{pmatrix}_{ij}. \quad (\text{I.22})$$

Note that we have expressed the modes in terms of $\gamma_{\mu\nu}$, because the definition of $h^{\mu\nu}$ is theory-dependent. The tensor “plus” mode γ_+ and the “cross” mode γ_\times are identical to GR. However, we now also have

- a scalar “breathing” mode γ_b , which is transverse but not trace-free ;
- a scalar “longitudinal” mode γ_ℓ ;
- two “vector” modes, γ_x along the x -direction and γ_y along the y -direction.

How these new modes affect a ring of freely falling test particles can be derived using the geodesic equation, and the conclusions are illustrated in Figure I.1.

I.2 Lowest-order GW generation

In this Section, we review the wave generation formalism in linearized gravity. We consider a matter source whose typical velocity v is very small with respect to the speed of light, such that it can be written in a so-called post-Newtonian (PN) expansion in the small parameter v/c . In this section, we will limit ourselves to the lowest-order term in the PN expansion, which we dub “Newtonian”.

I.2.1 Waves for arbitrary PN sources

In Section I.1, we studied the linearized Einstein vacuum equations. Keeping the same notations, they now read in the presence of matter

$$\square h^{\mu\nu}(t, \mathbf{x}) = \frac{16\pi G}{c^4} T^{\mu\nu}(t, \mathbf{x}), \quad (\text{I.23})$$

⁶It can be argued from an effective field theory point of view that this constraint actually only holds for some energy scales.

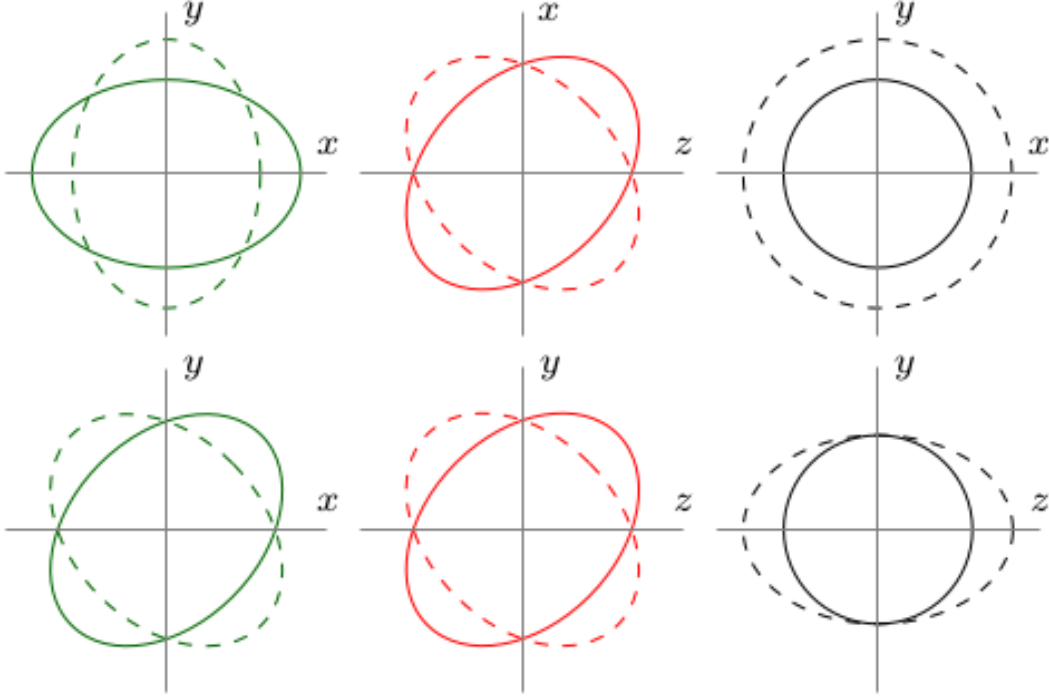


Fig. I.1 Effect of different GW polarizations on a ring of free-falling test particles. The full line shows the configuration of the ring of particles at some initial time t_0 , and the dashed line shows the configuration at time $t_0 + \pi/\omega_{\text{GW}}$, where ω_{GW} is the angular frequency of the GW (i.e. twice the orbital frequency). From top to bottom then left to right, the plus (+) and cross (\times) tensor modes (green); the vector-x (x) and vector-y (y) modes (red); breathing (b) and longitudinal (l) scalar modes (black). In all of these diagrams, the wave propagates in the z direction. Source: [51]

where $T^{\mu\nu}$ is the stress-energy tensor which completely describes the matter content of the system. We now consider an extended matter source, which is isolated, i.e. bounded in space. We also assume that there is no incoming radiation, which can be imposed by requiring that the source was stationary in the far past. This can physically be interpreted by the observation that in the far past, the source was very diffuse and could have barely generated any GWs at all. These conditions lead to selecting the retarded Green's function for the flat-space d'Alembertian, which we use to solve the wave equation. The solution reads

$$h^{\mu\nu}(t, \mathbf{x}) = -\frac{4G}{c^4} \int \frac{d^3\mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|} T^{\mu\nu} \left(t - \frac{|\mathbf{x} - \mathbf{x}'|}{c}, \mathbf{x}' \right), \quad (\text{I.24})$$

where the field variable is expressed in polar coordinates as $\mathbf{x} = r\mathbf{n}$. At distances r large compared to the typical size of the source d , we can perform the expansion

$|\mathbf{x} - \mathbf{x}'| = r - \mathbf{x}' \cdot \mathbf{n} + \mathcal{O}(d^2/r)$, so at leading order we have

$$h^{\mu\nu}(t, \mathbf{x}) = -\frac{4G}{rc^4} \int d^3\mathbf{x}' T^{\mu\nu} \left(t - \frac{r}{c} + \frac{\mathbf{x}' \cdot \mathbf{n}}{c}, \mathbf{x}' \right). \quad (\text{I.25})$$

For an observer far enough such that the wave can be described as planar, we can project this metric onto the TT gauge with respect to \mathbf{n} , which is also the direction of propagation of the wave. Following [12], we thus find

$$h_{\text{TT}}^{ij}(t, \mathbf{x}) = -\frac{4G}{rc^4} \perp_{ij,kl} \int d^3\mathbf{x}' T^{kl} \left(t - \frac{r}{c} + \frac{\mathbf{x}' \cdot \mathbf{n}}{c}, \mathbf{x}' \right). \quad (\text{I.26})$$

We now assume that we have a PN source, i.e. that the typical velocities v_s of the matter inside the source are small with respect to the speed of light, $v_s \ll c$. The frequencies associated to the orbital motion are then of order $\omega \sim v_s/d$, and we find that $\omega \ll c/d$. Performing a Fourier analysis (see p. 103 of [12] for details), we can then show that it is possible, under this assumption, to Taylor-expand the source in the small parameter $\mathbf{x}' \cdot \mathbf{n}/c$. Formally, this multipolar expansion reduces to an expansion when $c \rightarrow \infty$, which is in practice equivalent to a PN expansion. At lowest order we simply find

$$h_{\text{TT}}^{ij}(t, \mathbf{x}) = -\frac{4G}{rc^4} \perp_{ij,kl} \int d^3\mathbf{x}' T^{kl} \left(t - \frac{r}{c}, \mathbf{x}' \right). \quad (\text{I.27})$$

We now relate this to the symmetric trace-free (STF) quadrupole moment, which is defined at any time t in the usual fashion as

$$Q_{ij}(t) \equiv \frac{1}{c^2} \int d^3\mathbf{x} \left(x^i x^j - \frac{1}{3} r^2 \delta^{ij} \right) T^{00}(t, \mathbf{x}). \quad (\text{I.28})$$

Using the conservation⁷ of the stress-energy tensor $\partial_\mu T^{\mu\nu} = 0$, integrating by parts twice, using Ostrogradsky's divergence theorem and noticing that the stress-energy tensor vanishes outside of the source, we find

$$\ddot{Q}_{ij}(t) = 2 \int d^3\mathbf{x} T^{(ij)}(t, \mathbf{x}), \quad (\text{I.29})$$

where $T^{(ij)} \equiv T^{ij} - \frac{1}{3} T^{kk} \delta^{ij}$ indicates STF projection. Thus the GW is related at lowest order to the quadrupole moment of the source by the formula

$$h_{\text{TT}}^{ij}(t, \mathbf{x}) = -\frac{2G}{rc^4} \perp_{ij,kl} \ddot{Q}_{kl} \left(t - \frac{r}{c} \right). \quad (\text{I.30})$$

Projecting onto the orthonormal basis $(\mathbf{e}_x, \mathbf{e}_y, \mathbf{n})$ introduced in Section I.1 directly leads to reading off the plus and cross modes⁸,

$$h_+ = -\frac{G}{rc^4} (\ddot{Q}_{xx} - \ddot{Q}_{yy}) \quad \text{and} \quad h_\times = -\frac{2G}{rc^4} \ddot{Q}_{xy}. \quad (\text{I.31})$$

⁷Here we assume that the source dynamics are nongravitational, as in Einstein's original paper [39], see [10] for a proof that this result still holds for gravitational dynamics.

⁸We recall that these differ by a global sign from the γ_+ and γ_\times associated to the physical metric deviation γ_{ij}^{TT} , which are more commonly used.

In linear theory *stricto sensu*, GWs do not carry energy, since the stress-energy tensor associated to GWs scales as $\mathcal{O}(h^2)$. This means that linear theory by its own cannot account for radiation-reaction driving the inspiral. However, if we already assume results which we will derive in Section II.7, we can compute from linear theory the energy flux carried by GW across a sphere of infinite radius surrounding the source. Namely, according to (II.66), it reads

$$\mathcal{F} = \lim_{r \rightarrow +\infty} \left[\frac{c^3 r^2}{32\pi G} \int d^2\Omega \dot{h}_{\text{TT}}^{ij} \dot{h}_{\text{TT}}^{ij} \right],$$

where \dot{h}_{TT}^{ij} is evaluated at (r, \mathbf{n}) and $d^2\Omega$ indicates angular integration over \mathbf{n} . Note that it is necessary to average this result over a typical period of the system, which we denote $\langle \dots \rangle$, in order to extract the secular effect and make the result gauge-invariant. Note that the $r \rightarrow +\infty$ limit is indeed finite thanks to the $1/r$ decay of the metric perturbation. This flux straightforwardly enters the energy-flux equation, which reads

$$\frac{dE}{dt} = -\mathcal{F}. \quad (\text{I.32})$$

At lowest Newtonian order, it was proved that E is indeed the conservative Newtonian binding energy. In our linearized lowest-order model, we can inject (I.30) into (II.66) and recover the famous quadrupole formula for the energy flux [39]

$$\mathcal{F} = \frac{G}{5c^5} \langle \ddot{Q}_{ij} \ddot{Q}_{ij} \rangle. \quad (\text{I.33})$$

1.2.2 Waves for compact binary systems

We now reduce our previous results to the case of a nonspinning compact binary system, represented by two point particles of masses m_A , where $A \in \{1, 2\}$. In flat space-time, as adequate in linearized theory, the associated energy-momentum tensor reads (see (33.5) of [10])

$$T^{\mu\nu}(t, \mathbf{x}) = \sum_A \frac{m_A v_A^\mu v_A^\nu}{\sqrt{1 - \mathbf{v}_A^2/c^2}} \delta^{(3)}(\mathbf{x} - \mathbf{y}_A(t)), \quad (\text{I.34})$$

where we introduced the shorthand $v_A^\mu \equiv (c, \mathbf{v}_A)$, see Notations for more details. To lowest order in the PN expansion, the only piece we need is

$$T^{00}(t, \mathbf{x}) = \sum_A m_A c^2 \delta^{(3)}(\mathbf{x} - \mathbf{y}_A(t)), \quad (\text{I.35})$$

from which we deduce that our quadrupole moment reads $\mathbf{Q}_{ij} = m_1 y_1^{(i} y_1^{j)} + m_2 y_2^{(i} y_2^{j)}$. The position of center of mass reads $\mathbf{y}_{\text{CM}} = (m_1 \mathbf{x}_1 + m_2 \mathbf{x}_2)/(m_1 + m_2)$ and the center of mass frame is defined by $\mathbf{y}_{\text{CM}} = 0$. In this frame, the quadrupole moment of our system reads

$$\mathbf{Q}_{ij} = m \nu r_{12}^2 n_{12}^{(i} n_{12}^{j)}. \quad (\text{I.36})$$

We now further specialize our study to the case of an exactly circular orbit ($dr_{12}/dt = 0$) with constant frequency ω . Since the particles are nonspinning, the orbital plane does

not precess, and is orthogonal to a constant vector \mathbf{l} . In a orthonormal basis $(\mathbf{u}_x, \mathbf{u}_y, \mathbf{l})$ adjusted to the orbital plane, we have

$$\mathbf{n}_{12} \underset{(\mathbf{u}_x, \mathbf{u}_y, \mathbf{l})}{=} \begin{pmatrix} \cos(\omega t + \varphi_0) \\ \sin(\omega t + \varphi_0) \\ 0 \end{pmatrix}, \quad (\text{I.37})$$

and the quadrupole then reads

$$Q_{ij} \underset{(\mathbf{u}_x, \mathbf{u}_y, \mathbf{l})}{=} m\nu r_{12}^2 \begin{pmatrix} \cos^2(\omega t + \varphi_0) - \frac{1}{3} & \sin(\omega t + \varphi_0) \cos(\omega t + \varphi_0) & 0 \\ \sin(\omega t + \varphi_0) \cos(\omega t + \varphi_0) & \sin^2(\omega t + \varphi_0) - \frac{1}{3} & 0 \\ 0 & 0 & -1/3 \end{pmatrix}_{ij}. \quad (\text{I.38})$$

In order to read off the plus and cross modes, we not only need to take the second time-derivative, but we also need to express this matrix in the $(\mathbf{e}_x, \mathbf{e}_y, \mathbf{n})$ observer frame, which is equivalent to the propagation frame in the plane wave limit. We only need to perform a rotation that adjusts the inclination ι , because the azimuthal angle can be reabsorbed into the constant initial phase (we set it to zero at the end by an appropriate choice for the origin of time). We thus appropriately apply the rotation matrix $\mathcal{R}(\iota)$, and we find in the observer frame, after performing two time derivatives,

$$\ddot{Q}_{ij} \underset{(\mathbf{e}_x, \mathbf{e}_y, \mathbf{n})}{=} -2m\nu r_{12}^2 \omega^2 \begin{pmatrix} \cos(2\omega t) & \cos(\iota) \sin(2\omega t) & \sin(\iota) \sin(2\omega t) \\ \cos(\iota) \sin(2\omega t) & -\cos^2(\iota) \cos(2\omega t) & -\cos(\iota) \sin(\iota) \cos(2\omega t) \\ \sin(\iota) \sin(2\omega t) & -\cos(\iota) \sin(\iota) \cos(2\omega t) & -\sin^2(\iota) \cos(2\omega t) \end{pmatrix}_{ij}. \quad (\text{I.39})$$

As we are now in the correct frame, we directly apply (I.31) and we obtain the plus and cross modes

$$h_+(t) = \frac{4m\nu y}{c^2 r} \frac{1 + \cos^2(\iota)}{2} \cos(2\omega t), \quad (\text{I.40})$$

$$h_\times(t) = \frac{4m\nu y}{c^2 r} \cos(\iota) \sin(2\omega t), \quad (\text{I.41})$$

where we have used Kepler's law, $Gm = \omega^2 r_{12}^3$, and where we introduce the very useful frequency-related PN variable,

$$y \equiv \left(\frac{Gm\omega}{c^3} \right)^{2/3} \ll 1. \quad (\text{I.42})$$

A few observations are to be made here. First, we see that the frequency of the GW is *twice* the frequency of the orbit, as expected for a spin-two field. Second, if we see the binary face on ($\iota = 0$), then the plus and cross modes are equal, and the wave is circularly polarized. Conversely, if we see the binary edge on ($\iota = \pi/2$), then the cross mode vanishes and the wave is linearly polarized. It is interesting to compute the angular distribution of the radiated flux, which depends on the inclination ι and reads

$$\frac{d\mathcal{F}}{d\Omega} = \lim_{r \rightarrow +\infty} \frac{r^2 c^3}{16\pi G} \langle \dot{h}_+^2 + \dot{h}_\times^2 \rangle = \frac{32\pi c^2 m y^7 \nu^2}{5} g(\iota), \quad (\text{I.43})$$

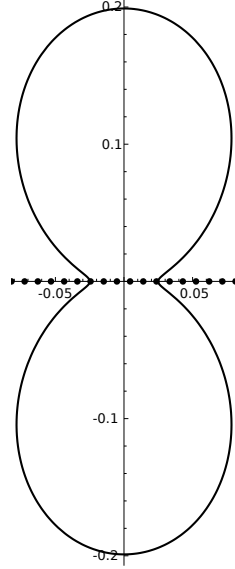


Fig. I.2 Polar plot of $g(\iota)$, the normalized angular power distribution carried by GWs. The orbital plane is represented by the dotted line. Points on the vertical axis have $\iota = 0$ [π] and points of the horizontal axis have $\iota = \pi/2$ [π].

where the time-averaging was in fact not necessary (as is typical for circular orbits) and where we have defined the normalized angular power distribution

$$g(\iota) = \frac{5}{16\pi} \left[\left(\frac{1 + \cos^2(\iota)}{2} \right)^2 + \cos^2(\iota) \right], \quad (\text{I.44})$$

such that $\int d^2\Omega g(\iota) = 2\pi \int_0^\pi d\iota \sin(\iota) g(\iota) = 1$. This function is represented in the polar plot in Fig. I.2, and we can see that the radiated flux is maximum face on from the binary, but does not vanish edge on.

We then immediately obtain the total flux, given by the famous Einstein quadrupole formula [39]

$$\mathcal{F} = \frac{32c^5\nu^2}{5G} \left(\frac{Gm\omega}{c^3} \right)^{10/3}, \quad (\text{I.45})$$

where we have restored the orbital frequency dependence ω .

Until now, we have only considered the case of exactly circular orbits. However, inserting the flux (I.45) into the energy balance equation tells us at what rate the GW is draining energy from the system. Since the energy of a Newtonian binary system in circular orbits reads

$$E = -\frac{Gm_1m_2}{2r_{12}} = -\frac{G^{2/3}m^{5/3}\nu\omega^{2/3}}{4}, \quad (\text{I.46})$$

we immediately deduce that this draining of energy, due to radiation reaction, results both in the decay of the orbital separation and the increase of the orbital frequency. If this secular decay happens on timescales much larger than the orbital timescale, namely if $\dot{\omega}_s \ll \omega^2$, then we are in the *adiabatic* approximation for quasicircular orbits. Our initial

description of the motion as a circular orbit is thus correct (up to some conservative PN corrections), but it will be corrected by small terms due to radiation reaction. Plugging our expressions of the energy (I.46) and the flux (I.45) into the energy-flux balance equation, and applying a chain law, we deduce that the *chirp*, i.e. the secular increase in frequency, reads

$$\dot{\omega} = \frac{96}{5} \left(\frac{G\mathcal{M}_c}{c^3} \right)^{5/3} \omega^{11/3}, \quad (\text{I.47})$$

where we have introduced the *chirp mass*

$$\mathcal{M}_c \equiv m\nu^{3/5} = \frac{(m_1 m_2)^{3/5}}{(m_1 + m_2)^{1/5}}. \quad (\text{I.48})$$

This integrates to

$$\omega(t) = \frac{5^{3/8}}{8} \left(\frac{G\mathcal{M}_c}{c^3} \right)^{-5/8} (t_c - t)^{-3/8} \quad (\text{I.49})$$

where t_c is an integration constant that can be interpreted as a coalescence time. Indeed, we can see that, as expected, $\omega(t) \rightarrow 0$ when $t \rightarrow -\infty$; but more spectacularly, we also observe that $\omega(t) \rightarrow +\infty$ as $t \rightarrow t_c$. This finite time divergence of the frequency can be interpreted as the breaking down of the PN approximation near merger, since it assumes small orbital frequencies in the first place. Thanks to the relation (I.46) between r_{12} and ω , we find that the orbital separation evolves as $r_{12}(t) \propto (t_c - t)^{1/4}$ in this model, which is interpreted as a merger at time t_c . This is incompatible with the assumption of large separations, and also indicates that the PN approximation is breaking down.

This lowest order computation for binary systems has been extended by Peters and Mathews [52] for elliptical Keplerian orbits, with semi-latus rectum a and eccentricity e . They then find that the flux reads⁹

$$\mathcal{F} = \frac{32c^5\nu^2 y^5}{5G} f(e), \quad (\text{I.50})$$

which differs from its circular version only by the enhancement factor

$$f(e) = \frac{1}{(1-e)^{7/2}} \left(1 + \frac{73}{24}e^2 + \frac{37}{96}e^4 \right). \quad (\text{I.51})$$

From the flux-balance equation for energy and angular moment, Peters showed [53] that eccentricity and the semi-latus rectum were related by

$$a(e) = c_0 \frac{e^{12/19}}{1-e^2} \left(1 + \frac{121}{304}e^2 \right)^{870/2299} \quad (\text{I.52})$$

where c_0 is a constant determined by the initial condition (a_0, e_0) . Now consider a typical binary star system, such as the Hulse-Taylor pulsar, which has $a_0 \simeq 2 \times 10^9$ m and $e \simeq 0.617$. The two bodies will be close enough to emit GWs in the frequency band of current detectors when $a \simeq 10^6$ m. The result (I.52) tells us that the binary system will then have an eccentricity of $e \simeq 6 \times 10^{-6}$ [12]. We therefore conclude that isolated binary

⁹The PN variable defined in (I.42) has the same expression for elliptical orbits, where we define $\omega = 2\pi/P$ at Newtonian order, with P the orbital period.

systems will be essentially circular by the time they enter the detector's frequency band, and quasicircular orbit models are therefore clearly sufficient. However, there has been a renewed interest for eccentric systems [54, 55, 56], as these could be formed by astrophysical perturbations, for example via the Kozai–Lidov mechanism [57, 58, 59, 60, 61].

1.2.3 Wave generation in Brans-Dicke theory

We now temporarily bid farewell to GR, and consider the wave generation in one of the oldest alternative theories of gravity, massless Brans-Dicke (BD) theory [62, 63, 64, 65]. This theory include a metric $g_{\mu\nu}$ and a scalar field ϕ , and the action reads

$$S_{\text{BD}} = \frac{c^3}{16\pi G} \int d^4x \sqrt{-g} \left[\phi R[g] - \frac{\omega_{\text{BD}}}{\phi} g^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi \right] + S_{\text{m}}(\mathbf{m}, g_{\alpha\beta}), \quad (\text{I.53})$$

where the matter action S_{m} depends on the matter fields and the metric, but *not* on the scalar field, and ω_{BD} is a constant (in the generalized BD theory under study in Part B, we promote this constant to a function of the scalar field $\omega(\phi)$). In linearized theory, one typically assumes that the scalar field has a constant value ϕ_0 asymptotically far from the source. To study this theory, one then defines a rescaled scalar field $\varphi = \phi/\phi_0$ and an auxiliary, conformal metric, $\tilde{g}_{\mu\nu} = \varphi g_{\mu\nu}$. This procedure is referred to as going from the Jordan to the Einstein frame. The metric deviation reads $\gamma_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu}$, but the quantity analogous to the trace-reversed metric deviation in GR will instead be defined using $\tilde{g}_{\mu\nu}$, namely $h^{\mu\nu} \equiv \sqrt{\tilde{g}} \tilde{g}^{\mu\nu} - \eta^{\mu\nu}$. One also introduces the scalar deviation $\psi = \varphi - 1$. One can then apply the least action principle to obtain the field equations in terms of our new quantities $h^{\mu\nu}$ and ψ . In linearized theory, we discard higher-order terms in $h^{\mu\nu}$ and φ , and thanks to our passage to the Einstein frame, the two equations will decouple, namely

$$\square h^{\mu\nu} = \frac{16\pi G}{c^4 \phi_0} T^{\mu\nu}, \quad (\text{I.54a})$$

$$\square \psi = \frac{8\pi G}{c^4 \phi_0 (3 + 2\omega_{\text{BD}})} \left(T - 2 \frac{\partial T}{\partial \varphi} \right), \quad (\text{I.54b})$$

where $T \equiv T^{\mu\nu} \eta_{\mu\nu}$ denotes the trace of the stress-energy tensor. Our first equation is identical to GR (up to a constant factor ϕ_0), and we can use all the GR results of Section I.2. However, the scalar equation is new, and the solution to the scalar wave equation is

$$\psi(t, \mathbf{x}) = -\frac{8G}{c^4 \phi_0 (3 + \omega_{\text{BD}})} \int \frac{d^3 \mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|} \left[T - 2 \frac{\partial T}{\partial \varphi} \right] \left(t - \frac{|\mathbf{x} - \mathbf{x}'|}{c}, \mathbf{x}' \right), \quad (\text{I.55})$$

where we indicate that $T - 2\partial T/\partial\varphi$ will not depend on φ at lowest PN order. Although it is unintuitive at this stage, we need to perform the *next-to-leading* order multipolar expansion, in order to get the leading order waveform. This is because we will show that the monopolar contribution will end up being smaller than the dipolar one. Defining

$$T^s \equiv T - 2 \frac{\partial T}{\partial \varphi}, \quad (\text{I.56})$$

we find the multipolar expansion

$$\psi(t, \mathbf{x}) = -\frac{2G}{rc^4 \phi_0 (3 + \omega_{\text{BD}})} \int d^3 \mathbf{x}' \left[T^s \left(t - \frac{r}{c}, \mathbf{x}' \right) + \frac{n^i x'^i}{c} \partial_t T^s \left(t - \frac{r}{c}, \mathbf{x}' \right) \right]. \quad (\text{I.57})$$

We naturally define the scalar monopole and dipole moments as

$$M^s(t) = \frac{1}{c^2 \phi_0 (3 + \omega_{\text{BD}})} \int d^3 \mathbf{x} T^s(t, \mathbf{x}) \quad \text{and} \quad D_i^s(t) = \frac{1}{c^2 \phi_0 (3 + \omega_{\text{BD}})} \int d^3 \mathbf{x} x^i T^s(t, \mathbf{x}). \quad (\text{I.58})$$

The multipolar expansion of our scalar field then reads

$$\psi(t, \mathbf{x}) = -\frac{2G}{rc^2} \left[M^s \left(t - \frac{r}{c} \right) + \frac{n^i}{c} \dot{D}_i^s \left(t - \frac{r}{c} \right) \right]. \quad (\text{I.59})$$

At this stage, we introduce the matter stress-energy tensor for point particles, but a subtlety arises. Indeed, the strong equivalence principle does not hold in BD theory, and the magnitude of the gravitational interaction is now controlled by G/ϕ , which varies in space and time. Following a discussion by Eardley [66], we consider a star whose radius is determined by the equilibrium between gravitational forces, which tend to make the star shrink, and nongravitational forces, which tend to make the star swell. If we now locally and adiabatically modify the local value of the scalar field ϕ , we modify the gravitational force and thus the equilibrium. The star will then shrink or swell, and its total binding energy will be modified. Now, if we want to model the star as a point particle, we need to include into the effective mass not only the mass due the matter, but also the binding and kinetic energy of the star. The conclusion is that an effective way to account for this is to assign to each point particle a mass-function $m_A(\phi)$, which depends on the local value of the scalar field. We define $m_A \equiv m_A(\phi_0)$, and we Taylor-expand around small deviations to ϕ_0 . In particular, we introduce the sensitivity

$$s_A = \left. \frac{\partial m_A}{\partial \ln \phi} \right|_{\phi=\phi_0}, \quad (\text{I.60})$$

such that $m_A(\phi) = m_A (1 + s_A \psi + \mathcal{O}(\psi^2))$. In linearized theory, the energy-momentum tensor then reads

$$T^{\mu\nu}(t, \mathbf{x}) = \sum_A \frac{m_A(\phi) v_A^\mu v_A^\nu}{\sqrt{1 - \mathbf{v}_A^2/c^2}} \delta^{(3)}(\mathbf{x} - \mathbf{y}_A(t)), \quad (\text{I.61})$$

and to leading order

$$T - 2 \frac{\partial T}{\partial \varphi} = \sum_A m_A (1 - 2s_A) c^2 \delta^{(3)}(\mathbf{x} - \mathbf{y}_A(t)). \quad (\text{I.62})$$

We recognize in this expression the scalar charge,

$$\alpha_A = \frac{1 - 2s_A}{\sqrt{3 + 2\omega_0}}. \quad (\text{I.63})$$

We can then compute the monopole and dipole fields *at lowest order*, which read

$$M^s \equiv m_1(1 - 2s_1) + m_2(1 - 2s_2) \quad (\text{I.64a})$$

$$D_i^s \equiv m_1(1 - 2s_1)y_1^i + m_2(1 - 2s_2)y_2^i \quad (\text{I.64b})$$

We see that, *at lowest order*, $dM^s/dt = 0$, so the monopole will not radiate. Actually, there will be higher order 1PN contributions, namely the monopole will radiate at the

same order as the quadrupole. By contrast, the scalar dipole *does* radiate, and does *not* vanish in the center-of-mass frame due to the presence of the sensitivities in its expression. Therefore, we find that the monopole has a subleading contribution with respect to the dipole, and the leading order radiation is therefore purely dipolar. In the center-of-mass frame, the dipole reads

$$D_i^s = \frac{2m\nu(s_2 - s_1)}{\phi_0(3 + 2\omega_{\text{BD}})} r_{12} n_{12}^i, \quad (\text{I.65})$$

and our waveform is given to lowest order by

$$\psi(t, \mathbf{x}) = -\frac{2G}{rc^2} \frac{n^i}{c} \dot{D}_i^s \left(t - \frac{r}{c} \right). \quad (\text{I.66})$$

Since the matter fields are minimally coupled to the physical metric $g_{\mu\nu} = \varphi^{-1} \tilde{g}_{\mu\nu}$, the GW detector will respond to the tidal field associated with the physical metric, *i.e.* the linearized Riemann tensor of the physical metric, which reads [67]

$$2c^2 R_{0i0j}^{\text{lin}} = \ddot{h}_{ij}^{\text{TT}} + \frac{1}{2} \perp_{ij} \ddot{\psi} \Big|_{(\mathbf{e}_x, \mathbf{e}_y, \mathbf{n})} = \frac{d^2}{dt^2} \begin{pmatrix} \psi + h_+ & h_\times & 0 \\ h_\times & \psi - h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix}_{ij}. \quad (\text{I.67})$$

showing explicitly the decomposition of the detector’s response into tensorial “plus” and “cross” modes and the scalar breathing mode, see (I.22).

When restricting to the case of circular orbits, we can essentially apply the same steps as GR (see Section I.2.2). First, we introduce the parameter¹⁰

$$\varepsilon_{\text{BD}} \equiv \frac{(1 - 2s_1)(1 - 2s_2)}{3 + 2\omega_{\text{BD}}}, \quad (\text{I.68})$$

which is useful to express the modified Kepler law in BD theory,

$$\frac{G(1 + \varepsilon_{\text{BD}})m}{\phi_0} = \omega^2 r_{12}^3, \quad (\text{I.69})$$

and the appropriate PN variable,

$$\tilde{y} \equiv \left(\frac{G(1 + \varepsilon_{\text{BD}})m}{\phi_0 c^3} \right)^{2/3}. \quad (\text{I.70})$$

For a binary seen at an inclination angle ι , the waveform then reads at lowest order

$$\psi = \frac{4Gm\nu(s_2 - s_1)}{rc^2 \phi_0 (3 + 2\omega_{\text{BD}})} \tilde{y}^{1/2} \sin(\iota) \cos(\omega t). \quad (\text{I.71})$$

We can see that contrary to gravitational radiation, the wave has a maximum magnitude when emitted in the orbital plane, and vanishing magnitude when emitted in the direction orthogonal to the orbital plane, see Fig. I.3. Moreover, when the sensitivities are equal, the waveform entirely vanishes. The energy flux due to scalar radiation then reads at leading order

¹⁰This *ad hoc* parameter is related to the ones used in Part B by $(G/\phi_0)(1 + \varepsilon_{\text{BD}}) = \tilde{G}\alpha$.

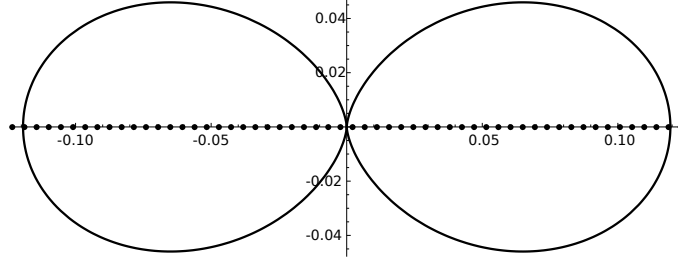


Fig. I.3 Polar plot of $g_s(\iota) = 3 \sin^2(\iota)/(8\pi)$, the normalized angular power distribution carried by dipolar scalar waves. The orbital plane is represented by the dotted line. Points on the vertical axis have $\iota = 0$ [π] and points of the horizontal axis have $\iota = \pi/2$ [π].

$$\mathcal{F}^s = \frac{c^3 r^2 \phi_0 (3 + 2\omega_{\text{BD}})}{16\pi G} \int d^2\Omega \langle \dot{\psi}^2 \rangle = \frac{G \phi_0 (3 + 2\omega_{\text{BD}})}{3c^3} \ddot{D}_i^s \ddot{D}_i^s, \quad (\text{I.72})$$

which yields for compact binaries on circular orbits

$$\mathcal{F}^s = \frac{4\phi_0 c^5 (s_2 - s_1)^2 \nu^2 \tilde{y}^4}{3G(1 + \varepsilon_{\text{BD}})^2 (3 + 2\omega_{\text{BD}})}. \quad (\text{I.73})$$

At first glance, this flux is one PN order larger than the quadrupolar flux of GR (it enters in $1/c^3$ instead of $1/c^5$ for GR), and will induce very strong radiation reaction effects, and thus a strong frequency chirp. In principle, this can be very tightly constrained with the timing of binary pulsars. However, this emission is strongly suppressed if the sensitivities of the two objects are very similar, as can be the case for two equal mass neutron stars.

I.3 Gravitational-wave detectors

The idea of measuring GWs has been of interest for a very long time, but these could arise from a variety of sources, and span many orders of magnitude in frequency and amplitude. However, nearly all detectors rely on the geodesic deviation equation, which states that two nearby free-falling particles following their geodesics¹¹ $x^\mu(\tau)$ and $x^\mu(\tau) + \xi^\mu(\tau)$ will obey

$$\frac{D^2 \xi^\mu}{D\tau^2} = -R^\mu{}_{\nu\rho\sigma} \xi^\rho \frac{dx^\nu}{d\tau} \frac{dx^\sigma}{d\tau} + \mathcal{O}(\xi^2), \quad (\text{I.74})$$

where we have neglected small terms in ξ^2 and defined the covariant derivative along the geodesic $x^\mu(\tau)$ by its effect of any vector V^μ , namely

$$\frac{DV^\mu}{D\tau} \equiv \frac{dV^\mu}{d\tau} + \Gamma^\mu{}_{\nu\rho} V^\nu \frac{dx^\rho}{d\tau}. \quad (\text{I.75})$$

¹¹Each geodesic is parametrized by its own proper time τ , and $\xi^\mu(\tau)$ connects points with the same value of τ on the two geodesics (see Footnote 7, Chapter 1 of [12]).



Fig. I.4 Aerial view of the LIGO Hanford and Livingston interferometers, and of the Virgo interferometer. Source: Wikimedia Commons.

In the small region of space encompassing our detector, we can construct a *proper detector frame* using Fermi normal coordinates. In this frame, at linearized order, we find the geodesic deviation equation

$$\frac{d^2 \xi^i}{dt^2} = -c^2 R_{i0j0} \xi^j \quad (\text{I.76})$$

where t is the coordinate time in this frame. This is why the linearized Riemann tensor is the key quantity to compute.

I.3.1 Current ground-based interferometers

Currently, there are three large-scale GW detectors in operation, and are all based on the concept of an laser Michelson interferometer. The Laser Interferometer Gravitational-Wave Observatory (LIGO) [68] operates on two sites in the United States, Hanford, WA and Livingston, LA (see Fig. I.4). Both interferometers have two 4 km long perpendicular arms. They started operating in 2002, and were then upgraded to Advanced LIGO, which started operating in 2015. The Virgo interferometer [69] is located near Pisa, Italy, and consists of a single interferometer, with two perpendicular 3 km arms (see Fig. I.4). It started operation in 2007, then was upgraded to Advanced Virgo, which started observing in 2017. The Kamioka Gravitational Wave Detector (KAGRA) [70] is located near Hida, Japan, and started operating in 2020. It has the particularity of being build underground, and consists of two perpendicular 3 km arms.

The general design of these interferometers is illustrated in Fig. I.5. A laser beam is split in two by a semireflective mirror, and the resulting beams travel through the two arms of the interferometer, each of which is enhanced by a Fabry-Pérot resonant cavity with high finesse. Each beam then reflects against a massive, suspended mirror, which acts approximately as a free-falling test mass in the horizontal direction along the arm. The two laser beams travel back though the arms to the semi-reflective mirror, where they are combined to form an interference pattern, and measured on a photodetector. Both arms are exactly of the same length, and the interferometer is tuned such as to create destructive interference, i.e. no light is detected on the photodetector in the absence of perturbations. When a GW crosses the detector, depending on its polarization, it can make one mirror move and not the other: this creates a difference in the length of the two arms, and breaks the destructive interference, which results in a flicker of light. Thus, this detector is not sensitive to the individual motion of each mirror, but the difference in armlength.

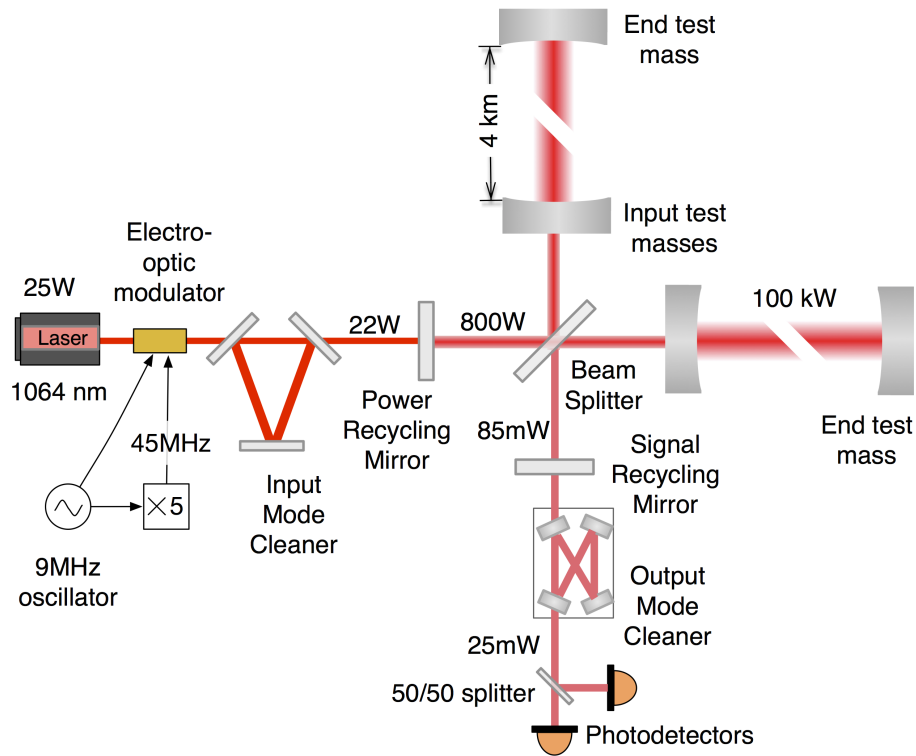


Fig. I.5 Design of the Advanced LIGO detectors. Source: [71].

In this system, the main noise source at low frequency is seismic and thermal noise, while at high frequency, quantum shot noise dominates, see Fig. I.6b. The maximum sensitivity of these detectors lies between 10 Hz and 200 Hz, see Fig. I.6a. This makes these detectors mostly sensitive to GWs generated by compact binaries up to a few tens of solar masses, especially near merger.

All of these detectors are only actively searching for GWs during so-called “observation runs”, which are sandwiched by maintenance and upgrading operations. Advanced LIGO has had three observing runs: O1, O2 and O3, with O3 being subdivided into O3a and O3b due to the COVID-19 pandemic, while Advanced Virgo has had two runs, and KAGRA only one (see Fig. I.7).

The first GW event occurred on September 15, 2014, hence its name GW150914 [75]. It occurred during an engineering phase of LIGO (therefore not in O1), but was luckily detected by both LIGO interferometers anyway. Virgo was not yet operational at the time and KAGRA was still in progress. The event lasted 0.2 seconds, which corresponds to 8 cycles with a frequency that increased from 35 Hz to 250 Hz during the event. It occurred roughly at a distance 440 Mpc (error bars are of the order of 180 Mpc !), its direction could not be localized precisely and was produced by the merger of two black holes of 35 and 30 solar masses.

Another major detection is GW170817 [49], which was due to the merger of two neutron stars of approximately 3 solar masses each at about 40 Mpc away. Its unique feature is that it was detected simultaneously as a gamma ray burst GRB170817A [76, 77, 78, 79], which was associated with the electromagnetic counterpart due to the kilonova.

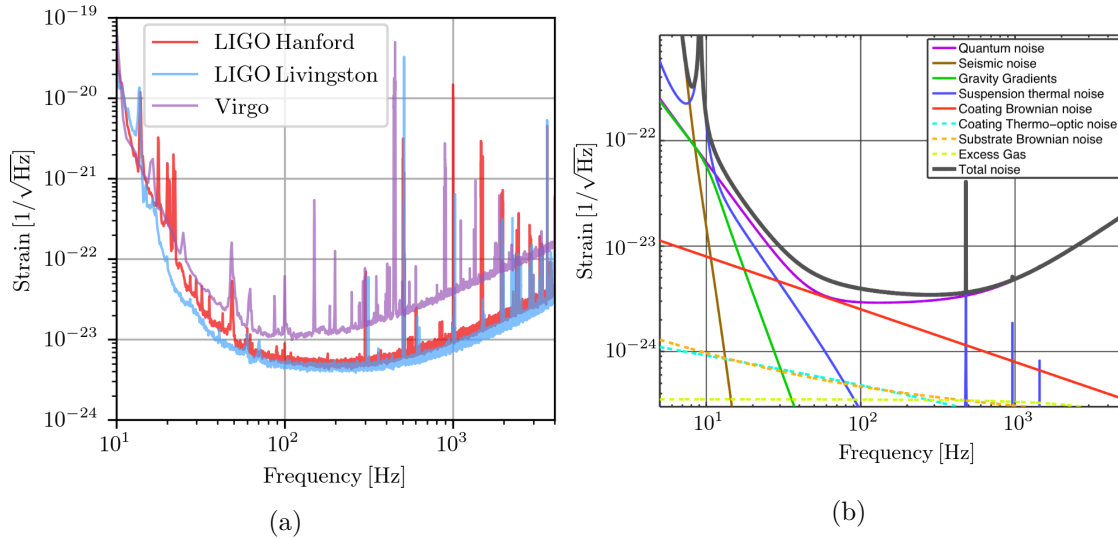


Fig. I.6 (a) Representative amplitude spectral density of the three detectors' strain sensitivity during the O3a run. Source: [72]. (b) Baseline aLIGO noise budget. Source: [73].

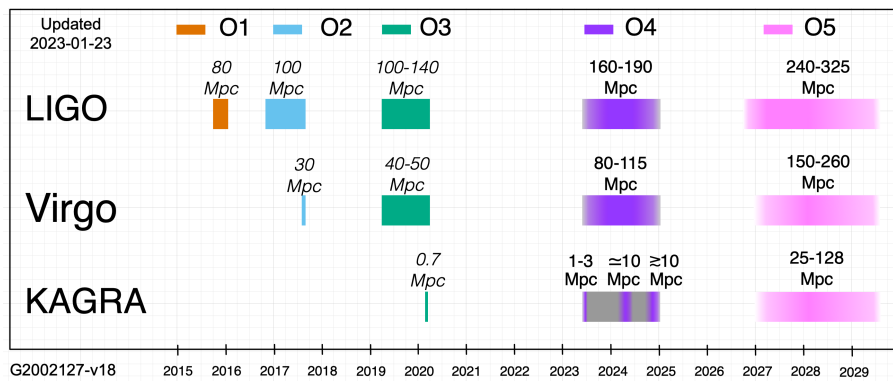


Fig. I.7 Schedule for past and future observing runs for the LVK Collaboration. Source: [74].

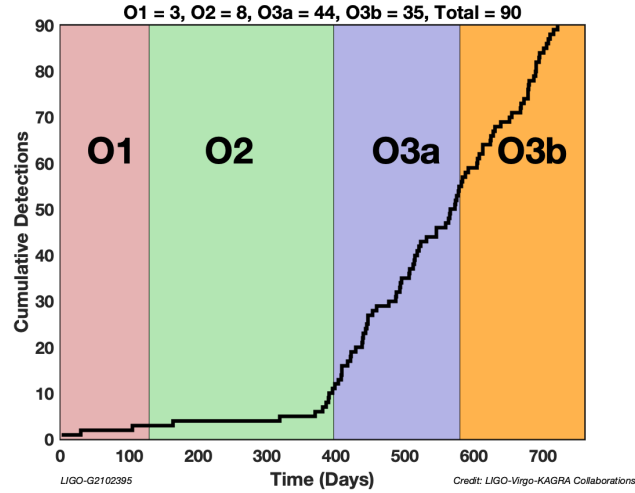


Fig. I.8 Cumulative number of GW event detections in the O1, O2, O3a and O3b runs. Source: [80]

This simultaneous detection allowed for extremely precise sky localization, but especially answered many open questions about the origin of such gamma ray burst. It also put a constraint on the propagation speed of GWs c_{GW} , which was found to be related to the speed of light c_{EW} by $|c_{\text{GW}} - c_{\text{EW}}| \lesssim 10^{-15}$, thus putting very strong constraints on alternative theories of gravity. The collaboration also detected two NS-BH mergers, GW200105 and GW200115.

The LIGO-Virgo-KAGRA (LVK) collaboration has confirmed the detection of over 90 GW events. This is mostly thanks to major improvements during O3, which allowed for a much better sensitivity and hence an improved detection rate, see Fig. I.8.

I.3.2 Third generation ground-based detectors

Despite the major success of the LVK collaboration, the relatively small signal-to-noise ratio (SNR) and narrow sensitivity band limits the amount of science that can be done with these detectors. Therefore, two third-generation ground-based detector have been proposed: Cosmic Explorer (CE) in the United States and the Einstein Telescope (ET) in Europe, whose sensitivity is expected to be largely improved with respect to second-generation detectors, as illustrated by Fig. I.10. The design of Cosmic Explorer is essentially the same design as LIGO, but ten times bigger, i.e. with 40 km arms. The Einstein Telescope (see Fig. I.9a) is expected to be an underground interferometer in the form of an equilateral triangle with 10 km arms, located either in Italy or in the Netherlands. Its optical layout is composed of three nested detectors (Fig. I.9b), each of which is subdivided into two interferometers, which are optimized for low frequencies (1 Hz to 100 Hz) and high-frequencies (100 Hz to 10 kHz) respectively, in the so-called “xylophone” configuration.

The triangular design will endow ET with a more isotropic antenna pattern than LVK. Since it is underground, seismic noise will be reduced, and gravity gradients are also mitigated. Moreover, it is planned to use cryogenic mirrors, whose temperature of

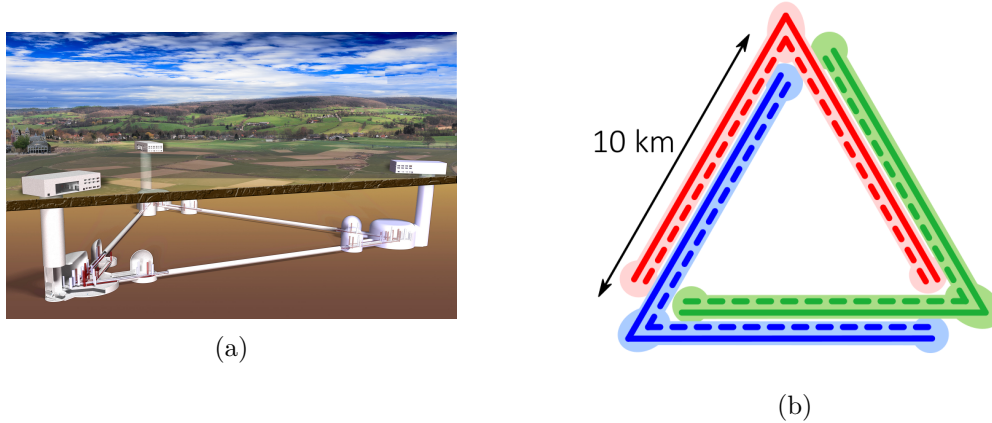


Fig. I.9 (a) Artistic view of ET. Source: [81]. (b) The optical layout of ET is made of three nested detectors, labeled by green, red and blue. Each of these detectors is composed of a low-frequency (full line) and a high-frequency (dashed line) interferometer. Source: [82].

about a few tens of kelvins will significantly reduce thermal noise.

I.3.3 Space-borne interferometers

Another direction for improving GW detector sensitivity is to go into outer space, where there is virtually no environmental noise and where the free-falling test-mass experiment can be optimally realized. Moreover, this allows for much larger arms, thus probing a very different frequency band. These include the Laser Interferometer Space Antenna (LISA) [84], but also the Deci-hertz Interferometer Gravitational wave Observatory (DECIGO) [85] and the Taiji (太极) [86] and TianQin (天琴, “heavenly harp”) [87] projects. LISA will be composed of three satellites forming a constellation in the form of an equilateral triangle. The distance between two satellites will be 2.5 million kilometers, and the barycenter of this constellation will be located at one astronomical unit from the Sun, trailing the Earth’s orbit by around 20° (see Fig. I.11a), which corresponds to a linear distance from Earth of about 50 million kilometers. Each satellite is endowed with a drag-free system and two test masses, each of which consists in of a 2 kg cube made of a gold-platinum alloy (shown in Fig. I.11b), which is freely floating within a vacuum chamber. The satellites are engineered to exactly follow the test mass, which plays the role of the mirror. Moreover, the satellites are also equipped with two lasers, two optical bench and two telescopes pointing at each of the other two satellites. Since the distance between each satellite is extremely large, one cannot realize a Michelson interferometer by directly sending back the light reflected by the test mass onto the distant satellite, so one must instead resort to numerical interferometry. Thanks to these elaborate techniques, one expects to measure the displacement between each of the three pairs of test masses up to a precision of 10 pm.

LISA will operate in the 100 μHz to 100 mHz frequency band at a sensitivity of around 10^{-21} , which is much worse than the current ground-based detectors, see Fig. I.10.

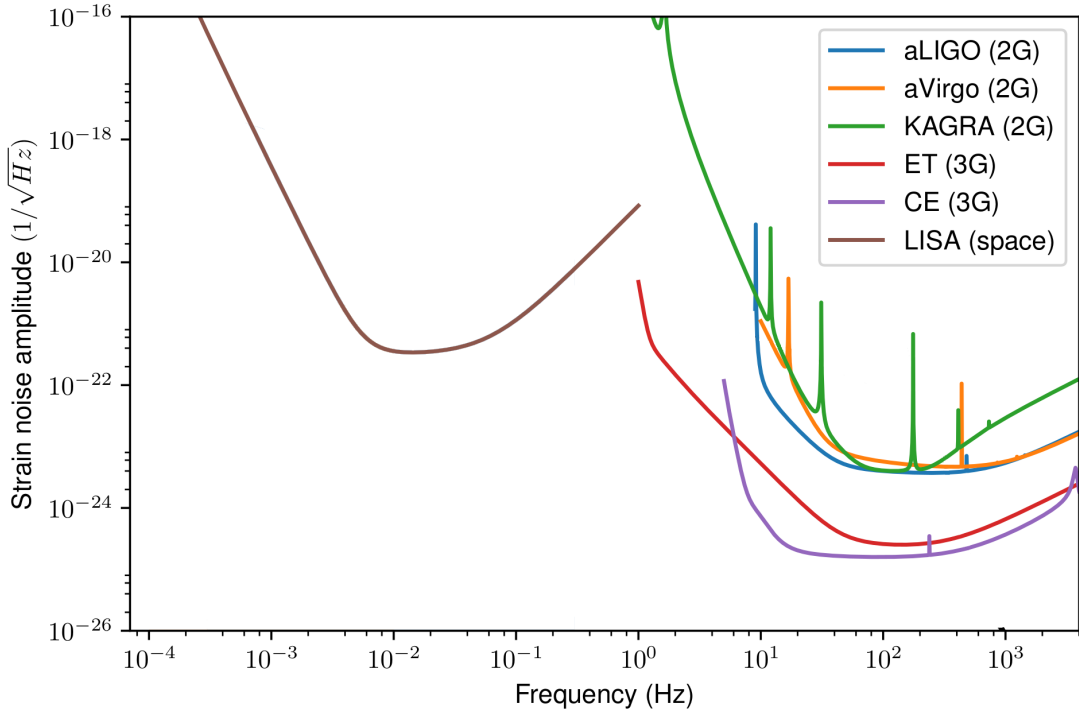


Fig. I.10 Amplitude spectrum of detector noise as a function of frequency for Advanced LIGO, Advanced Virgo, Cosmic Explorer (CE), the Einstein Telescope (ET) and LISA. Source: [83]

However, the magnitude of the GWs one expects to find at these frequencies is much higher than for ground-based detectors, so LISA will in fact detect many sources. First, LISA will detect a foreground of unresolved stellar-mass compact binaries in the Milky Way, whose signal will be nearly monochromatic since they are very far from merger. This includes black holes (BH) and neutron stars (NS), but also white dwarfs. LISA will also resolve individual mergers of supermassive black hole binaries in other galaxies, as well as extreme mass ratio inspirals and intermediate mass black hole binaries. Other sources, such as the stochastic GW background, the primordial GW background or exotic sources such as cosmic strings, could potentially be detected by LISA as well [88].

I.4 Gravitational-wave data analysis

As we have seen, GW detectors need to be extremely sensitive, and there are many noise sources that can spoil the signal. It is therefore important to develop proper data analysis techniques to correctly distinguish the signal from the noise.

Consider an output from a detector

$$d(t) = h(t) + n(t) \quad (\text{I.77})$$

where $h(t)$ is a deterministic function of time and $n(t)$ is a random variable with zero expected value, $\langle n(t) \rangle = 0$, which we assume to be ergodic. In practice, we are in the

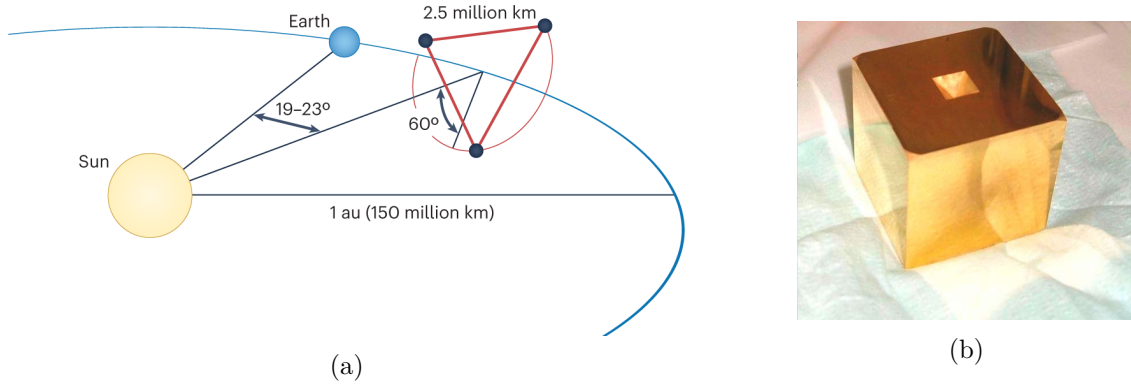


Fig. I.11 (a) Planned orbit configuration for the three LISA satellites. Source: [89]. (b) Gold-platinum test mass to be used in LISA. Source: [90]

unfortunate situation of having to find a needle in a haystack, since $|h(t)| \ll |n(t)|$ (see first row of Fig. I.12). In order to nonetheless extract physical information, we have to have a very precise understanding of the signal (which is the goal of this thesis) and of the noise to fish out a small signal out of a large noise. We first assume that the noise is Gaussian and stationary, such that its different Fourier components are uncorrelated and

$$\langle \tilde{n}^*(f) \tilde{n}(f') \rangle = \frac{1}{2} \delta(f - f') S_n(f), \quad (\text{I.78})$$

where $S_n(f)$ is the amplitude spectral density of the noise, a tilde indicates the Fourier transform of a function of time, and the star indicates the complex conjugate. This means that the noise's variance reads $\langle n^2(t) \rangle = \int_0^{+\infty} df S_n(f)$.

We then introduce a filter function $k(t)$, thanks to which we define the correlation function

$$c(t) \equiv \int_{-\infty}^{+\infty} dt' d(t') k(t' + t). \quad (\text{I.79})$$

Since it includes the noise, $c(t)$ is actually a random variable. One defines its signal-to-noise ration as

$$\text{SNR}(t) = \frac{\langle c(t) \rangle}{[\langle c(t)^2 \rangle - \langle c(t) \rangle^2]^{1/2}}. \quad (\text{I.80})$$

One can then prove that for any signal h , the quantity $\text{SNR}(0)$ is maximized by the filter $k_{\text{opt}}(t')$, called the optimal or Wiener filter, which is defined by its Fourier transform

$$\tilde{k}_{\text{opt}}(f) = \frac{\tilde{h}(f)}{S_n(f)}. \quad (\text{I.81})$$

This means that the SNR plotted with respect to time will exhibit a strong peak if the filter is chosen as exactly equal to the signal, up to a global time shift t which is determined by the location of this peak. Now suppose that one has at hand a template $\tau(t; \theta)$ which depends on a small set of parameters, e.g. $\theta = (m, \nu)$, and which is known to be equal to the signal for some unknown value of θ . Said differently, suppose that there exists θ_0 and t such that $\tau(t'; \theta_0) = h(t' + t)$. In order to search for a signal, one is compelled to choose the filter to be $\tilde{k}(f) = \tilde{\tau}(f; \theta) / S_n(f)$ and to plot $\text{SNR}(t)$ with many different filters

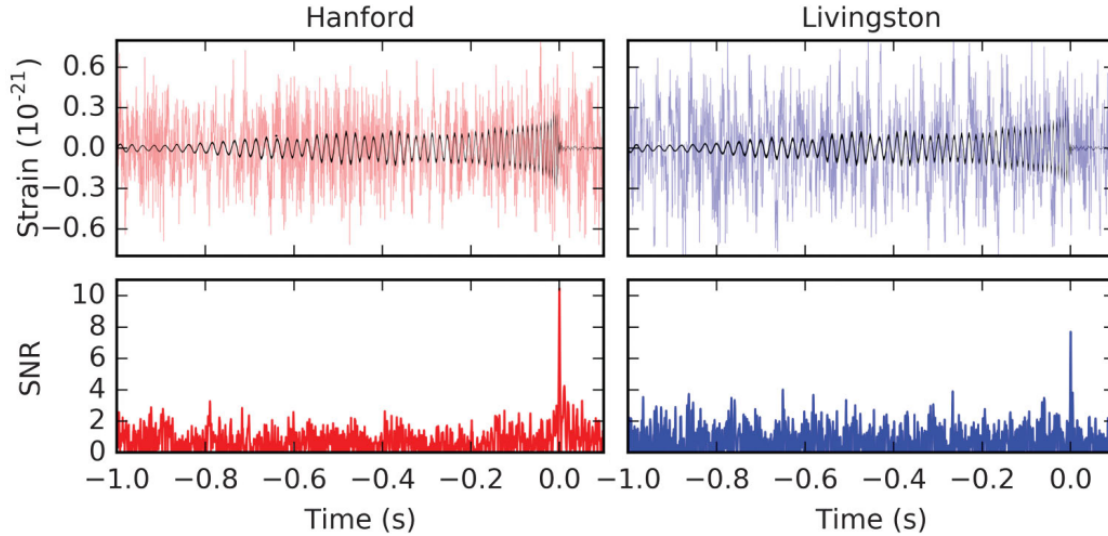


Fig. I.12 GW151226 observed by the LIGO Hanford (left column) and Livingston (right column) detectors. First row: Strain data from the two detectors. Second row: Signal-to-noise ratio (SNR) time series. Source: [45]

associated to different values of θ . One can then explore the parameter space associated to θ , and search for a peak in the SNR signal. This would indeed indicate, as we have seen, that the template must be almost equal to the signal. Since the SNR can be seen as a scalar product between the filter and the signal in a high-dimensional vector space, most filters will be orthogonal to the signal and will lead to near-vanishing SNR. Only filters very close to being parallel to the signal will lead to nonzero SNR, and thanks to the high-dimensionality of the vector space, this leads to a very sharp and distinctive peak (in practice, the usual criterion is $\text{SNR} > 8$), see second row of Fig. I.12.

Although we do not expect our template to be exactly equal to our signal, we can nonetheless optimize the match by choosing θ_{opt} such as to maximize the SNR. A better way to find θ_{opt} is for example to use a maximum likelihood estimator, or to maximize the posterior probability. The corresponding optimal waveform $\tau(t; \theta_{\text{opt}})$ is plotted in black in the first row of Fig. I.12. One should also note that detecting a high SNR is not a fool-proof criterion to claim that a GW has indeed been detected. Indeed, the presence of non-Gaussian noise can create high SNR events, so a good mitigation is to monitor it: for example, if there is an earthquake leading to a high-SNR signal, the signal collected on seismographs in the vicinity of the detector will allow us to discard this event. Moreover, a large panel of statistical techniques can be employed to estimate the probability of an actual GW detection and establish false alarm probabilities. The likeliness of a false alarm is significantly reduced if the signal is detected in different, uncorrelated detectors.

Part A

General relativity

Chapter II

The wave generation formalism

II.1 The Landau-Lifschitz formulation of general relativity

Following Landau and Lifschitz [10], we wish to reexpress the Einstein field equations (I.2) of GR in the form of a wave equation. To this end, we introduce a tensor density which we call the *gothic metric*, which reads

$$\mathbf{g}^{\mu\nu} \equiv \sqrt{-g}g^{\mu\nu}. \quad (\text{II.1})$$

In view of studying GWs as deviations from flat Minkowski spacetime, we introduce our main variable, the gravitational-field amplitude

$$h^{\mu\nu} \equiv \mathbf{g}^{\mu\nu} - \eta^{\mu\nu} = \sqrt{-g}g^{\mu\nu} - \eta^{\mu\nu}. \quad (\text{II.2})$$

Although this identity is written in terms of the inverse metric $g^{\mu\nu}$, we can also express the metric $g_{\mu\nu}$ itself in terms of $h^{\mu\nu}$, at the price of introducing an infinite sum. Furthermore, the (opposite of the) determinant of the metric can be expressed *exactly* as

$$\begin{aligned} -g = 1 + h + \frac{1}{2}(h^2 - h_\alpha^\beta h_\beta^\alpha) + \frac{1}{6}(h^3 - 3hh_\alpha^\beta h_\beta^\alpha + 2h_\alpha^\beta h_\beta^\sigma h_\sigma^\alpha) \\ + \frac{1}{24}(h^4 - 6h^2 \cdot h_\beta^\alpha h_\alpha^\beta + 3(h_\beta^\alpha h_\alpha^\beta)^2 + 8h \cdot h_\beta^\alpha h_\rho^\beta h_\alpha^\rho - 6h_\beta^\alpha h_\rho^\beta h_\sigma^\rho h_\alpha^\sigma), \end{aligned} \quad (\text{II.3})$$

where we recall from Section I.1 that $h_\alpha^\beta \equiv h^{\alpha\sigma}\eta_{\sigma\beta}$ and $h \equiv h^\sigma_\sigma$. We will also introduce a notation for the divergence of $h^{\mu\nu}$, namely

$$H^\mu \equiv \partial_\nu h^{\mu\nu}. \quad (\text{II.4})$$

The Einstein field equations can then be entirely rephrased as a flat-space wave equation for the metric deviation $h^{\mu\nu}$, sourced by the metric deviation itself. It reads

$$\square h^{\mu\nu} - \partial H^{\mu\nu} = \frac{16\pi G}{c^4} \tau^{\mu\nu}, \quad (\text{II.5})$$

where $\tau^{\mu\nu}$ is the stress-energy pseudotensor sourcing the GWs, where \square is still the flat-space d'Alembertian and where recall the shorthand (I.12). The stress-energy pseudotensor is defined by

$$\tau^{\mu\nu} = (-g)T^{\mu\nu}[\mathbf{m}, h^{\alpha\beta}] + \frac{c^4}{16\pi G} \Lambda^{\mu\nu}[h^{\alpha\beta}], \quad (\text{II.6})$$

where $T^{\mu\nu}$ is the stress-energy tensor expressed in terms of $h^{\alpha\beta}$, and all the nonlinearities of the Einstein equation have been absorbed in the $\Lambda^{\mu\nu}$ pseudotensor, which reads [3]

$$\begin{aligned}\Lambda^{\mu\nu} \equiv & -h^{\alpha\beta}\partial_{\alpha\beta}h^{\mu\nu} + \partial_{\alpha}h^{\mu\beta}\partial_{\beta}h^{\nu\alpha} + \frac{1}{2}g^{\mu\nu}g_{\alpha\beta}\partial_{\lambda}h^{\alpha\tau}\partial_{\tau}h^{\beta\lambda} \\ & - g^{\mu\alpha}g_{\beta\tau}\partial_{\lambda}h^{\nu\tau}\partial_{\alpha}h^{\beta\lambda} - g^{\nu\alpha}g_{\beta\tau}\partial_{\lambda}h^{\mu\tau}\partial_{\alpha}h^{\beta\lambda} + g_{\alpha\beta}g^{\lambda\tau}\partial_{\lambda}h^{\mu\alpha}\partial_{\tau}h^{\nu\beta} \\ & + \frac{1}{8}(2g^{\mu\alpha}g^{\nu\beta} - g^{\mu\nu}g^{\alpha\beta})(2g_{\lambda\tau}g_{\rho\sigma} - g_{\tau\rho}g_{\lambda\sigma})\partial_{\alpha}h^{\lambda\sigma}\partial_{\beta}h^{\tau\rho} \\ & + 2h^{\rho(\mu}\partial_{\rho}H^{\nu)} - \partial_{\rho}(h^{\mu\nu}H^{\rho}).\end{aligned}\tag{II.7}$$

Note that thanks to (II.2) and (II.3), we can indeed express $\Lambda^{\mu\nu}$ purely in terms of $h^{\mu\nu}$, but at the price of having an infinite sum. We also know that $\Lambda^{\mu\nu}$ is at least quadratic in h , so it can be expressed in terms of a quadratic piece, a cubic piece, etc., namely

$$\Lambda^{\mu\nu} = N^{\mu\nu}[h, h] + M^{\mu\nu}[h, h, h] + L^{\mu\nu}[h, h, h, h] + \sum_{n=5}^{\infty} \Lambda_n^{\mu\nu}[h],\tag{II.8}$$

where $N^{\mu\nu}[h, h']$ is given by

$$\begin{aligned}N^{\mu\nu}[h, h'] = & -h^{\alpha\beta}\partial_{\alpha}\partial_{\beta}h'^{\mu\nu} + \frac{1}{2}\partial^{\mu}h_{\alpha\beta}\partial^{\nu}h'^{\alpha\beta} - \frac{1}{4}\partial^{\mu}h\partial^{\nu}h' \\ & - 2\partial^{\alpha}h^{\beta(\mu}\partial^{\nu)}h'_{\alpha\beta} + \partial_{\beta}h^{\mu\alpha}\left(\partial^{\beta}h'_{\alpha}{}^{\nu} + \partial_{\alpha}h'^{\nu\beta}\right) \\ & + \eta^{\mu\nu}\left[-\frac{1}{4}\partial_{\rho}h_{\alpha\beta}\partial^{\rho}h'^{\alpha\beta} + \frac{1}{8}\partial_{\rho}h\partial^{\rho}h' + \frac{1}{4}\partial_{\alpha}h_{\beta\rho}\partial^{\beta}h'^{\alpha\rho}\right] \\ & + 2h^{\alpha(\mu}\partial_{\alpha}\partial_{\beta}h'^{\nu)\beta} - \partial_{\alpha}\left(h^{\mu\nu}\partial_{\beta}h'^{\beta\alpha}\right)\end{aligned}\tag{II.9}$$

and $M^{\mu\nu}[h, h, h]$ is given by (1.3) of [91].

In most applications, the harmonic gauge condition $H^{\mu} = 0$ is imposed, which means that the last lines of (II.7) and (II.9) vanish, as in (24) of [14] and (1.2) of [91]. The Einstein equations in harmonic coordinates then read

$$\square h^{\mu\nu} = \frac{16\pi G}{c^4}\tau^{\mu\nu},\tag{II.10}$$

where it is of course implicit that the harmonic gauge condition $\partial_{\mu}h^{\mu\nu} = 0$ must also be satisfied for $h^{\mu\nu}$ to be a solution of the Einstein equations. This justifies in hindsight our choice to work with the variable $h^{\mu\nu}$ as defined by (II.2): in harmonic coordinates, the Einstein equations reduce to a wave equation on $h^{\mu\nu}$ sourced only by the matter and terms *at least quadratic* in $h^{\mu\nu}$.

II.2 Solving the Einstein equations

Now that we have reformulated the Einstein equations in a convenient way, we will specify the physical problem we want to study. To start with, we consider the case of two stars orbiting each other. The stress-energy tensor is a smooth function of spacetime. The binary system is inspiraling because of GW emission, so we suppose that we are in a regime where the binary is well separated (i.e. the two objects are separated by a distance

much greater than the typical size of each object) and where the relative velocity of the two stars is small compared to the speed of light. In order to solve our wave-generation problem, we will want to impose two additional conditions [92] to the metric. First, we assume that the metric was stationary in the past, i.e. there exists a time $-\mathcal{T}$ such that

$$\forall t \leq -\mathcal{T}, \quad \frac{\partial h^{\mu\nu}}{\partial t}(t, \mathbf{x}) = 0. \quad (\text{II.11a})$$

Second, we assume that the metric was asymptotically Minkowskian (in the weak sense) before the time $-\mathcal{T}$, namely

$$\forall t \leq -\mathcal{T}, \quad \lim_{r \rightarrow +\infty} h^{\mu\nu}(t, \mathbf{x}) = 0, \quad (\text{II.11b})$$

where the limit is taken for fixed t . This will also allow us to assume that our stress-energy tensor has compact support and that all the matter is contained within a world-tube of finite radius. This is physically motivating by the fact that in the far past, the stars had not yet formed, and were thus not generating GWs

With this physical setting, it is then necessary to separate our problem into different zones. Near the source, i.e. for radii smaller than a typical gravitational wavelength, we are entitled to make the PN approximation: we assume that velocities are small, so we can expand our sources and fields in powers of v/c . Thanks to the virial theorem¹, we can perform this expansion using the fact the $(v/c)^2 \sim (Gm)/(r_{12}c^2)$, where r_{12} is the orbital separation and m the total mass of the binary. In practice, the PN expansion will simply be an expansion when $c \rightarrow \infty$. This also applies to the case of functions of retarded time, which can be expanded in powers of c as

$$f\left(t - \frac{r}{c}\right) = f(t) - \frac{r}{c}f^{(1)}(t) + \frac{1}{2}\left(\frac{r}{c}\right)^2 f^{(2)}(t) + \dots$$

It is immediate to see how this approximation only applies in the near zone, and breaks down far away from the source. Note that when solving the near-zone problem, we will obtain the result up to some arbitrary homogeneous functions, which will parametrize radiation reaction.

Conversely, we can solve our problem in the exterior vacuum region of spacetime. In this case, no PN approximation is needed (since there is no source in the equations), and we can express the most general solution as a multipolar expansion, parametrized by some multipolar moments, which at this stage are arbitrary. This solution is well defined everywhere except at $r = 0$, where it diverges, but it only represents the physical solution outside the source.

There exists a buffer zone, the “near exterior zone”, where both expansion are valid, see Fig. II.1. This zone allows us to match the two solutions, and specify the multipolar moments in terms of the source, and the radiation-reaction functions in terms of the exterior metric. In practice, this matching is done via matched asymptotic series: we

¹The virial theorem states that for a bound system, the kinetic energy T is related to the gravitational potential U by $2\langle T \rangle + \langle U \rangle = 0$, where $\langle \dots \rangle$ indicates time averaging. For a bound system of two point particles, $T = \mu v_{12}^2/2$ and $U = -G\mu m/r_{12}$, where $m = m_1 + m_2$ is the total mass, $\mu = m_1 m_2/m$ is the reduced mass, r_{12} is the orbital separation and v_{12} is the relative velocity of the two objects. Omitting time averages, we therefore obtain $(v_{12}/c)^2 = Gm/(r_{12}c^2)$, and the total energy reads $\langle E \rangle = -G\mu m/(2r_{12})$. This is not true in GR, but the order of magnitude holds.

will notice that the two solutions have the same asymptotic structure (respectively in the $r \rightarrow +\infty$ and $r \rightarrow 0$ limits), and we will impose that the coefficients in this expansion be equal.

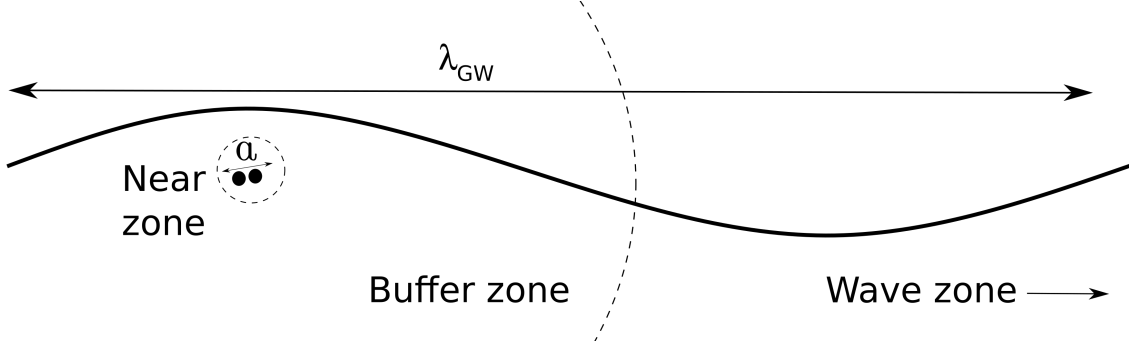


Fig. II.1 Representation of a source of size a emitting gravitational waves of wavelength λ_{GW} . The matter is contained within the small dotted circle. The near zone is characterized by $r \ll \lambda_{\text{GW}}$. The buffer zone, between the two dotted circles, is the exterior near zone, where both PN and MPM expansions are valid. The wave zone is given by $r \gg \lambda_{\text{GW}}$, and is outside the represented region.

II.2.1 Multipolar post-Minkowskian solution for the exterior vacuum

For the moment, we will solve the vacuum equation in the region exterior to the source using the harmonic gauge condition, namely $\partial_\mu h^{\mu\nu} = 0$. In Section II.7 and more thoroughly in Chapters IV and V, we will see that there are different coordinate choices that have some substantial advantages. Since we are now assuming that we are in the vacuum exterior to the source, the Einstein equations simplify to $\square h^{\mu\nu} = \Lambda^{\mu\nu}[h]$, where $\Lambda^{\mu\nu}[h]$ is given by (II.7). Assuming that $h^{\mu\nu} \ll 1$, we formally divide the metric into a linear part, a quadratic part, etc. This procedure is called the post-Minkowskian (PM) expansion (since we are expanding the metric around Minkowski spacetime), and we factor out of each term a specific power of Newton’s constant G , which will act as a bookkeeping parameter. We will then replace the linearized metric by its multipolar expansion, and construct the nonlinearities using the nonlinear metric. Altogether, this procedure is called the multipolar post-Minkowskian (MPM) construction [92], and is formally designated by \mathcal{M} (which stands for “multipolar”), namely

$$\mathcal{M}(h^{\mu\nu}) = \sum_{n=1}^{\infty} G^n h_n^{\mu\nu}. \quad (\text{II.12})$$

Injecting the ansatz (II.12) in the vacuum Einstein equations in harmonic coordinates yields a hierarchy of wave equations which read

$$\square h_n^{\mu\nu} = \Lambda_n^{\mu\nu}[h_1, \dots, h_{n-1}], \quad (\text{II.13})$$

where $\Lambda_n^{\mu\nu}$ is obtained recursively. For our purposes, we will only be needing

$$\Lambda_2^{\mu\nu}[h_1] = N^{\mu\nu}[h_1, h_1], \quad (\text{II.14a})$$

$$\Lambda_3^{\mu\nu}[h_1, h_2] = N^{\mu\nu}[h_1, h_2] + N^{\mu\nu}[h_2, h_1] + M^{\mu\nu}[h_1, h_1, h_1]. \quad (\text{II.14b})$$

Note that the MPM metric given by (II.12) does not necessarily converge, but we treat it as a formal series that satisfies the Einstein equations order by order, in the sense of (II.13). The first step in the MPM construction is to solve for the linearized solution ($n = 1$). In this approximation, the Einstein equations take the particularly simple form

$$\square h_1^{\mu\nu} = 0, \quad (\text{II.15a})$$

$$\partial_\mu h_1^{\mu\nu} = 0. \quad (\text{II.15b})$$

The no-incoming radiation condition follows from the conditions of (II.11), which naturally select the retarded Green's function for these equations. Noting that there is a residual gauge freedom within the harmonic gauge, we can now write the general solution to (II.15) as

$$h_1^{\mu\nu} = k_1^{\mu\nu} + \partial\varphi_1^{\mu\nu}, \quad (\text{II.16})$$

where the canonical part $k_1^{\mu\nu}$ of the solution is defined by means of two sets of STF multipole moments $\{I_L, J_L\}$, namely [93, 94, 48, 92, 14]

$$k_1^{00} = -\frac{4}{c^2} \sum_{\ell=0}^{\infty} \frac{(-)^\ell}{\ell!} \partial_L \left[\frac{1}{r} I_L \left(t - \frac{r}{c} \right) \right], \quad (\text{II.17a})$$

$$k_1^{0i} = \frac{4}{c^3} \sum_{\ell=1}^{\infty} \frac{(-)^\ell}{\ell!} \left\{ \partial_{L-1} \left[\frac{1}{r} I_{iL-1}^{(1)} \left(t - \frac{r}{c} \right) \right] + \frac{\ell}{\ell+1} \partial_L \left[\frac{1}{r} J_{i|L} \left(t - \frac{r}{c} \right) \right] \right\}, \quad (\text{II.17b})$$

$$k_1^{ij} = -\frac{4}{c^4} \sum_{\ell=2}^{\infty} \frac{(-)^\ell}{\ell!} \left\{ \partial_{L-2} \left[\frac{1}{r} I_{ijL-2}^{(2)} \left(t - \frac{r}{c} \right) \right] + \frac{2\ell}{\ell+1} \partial_{L-1} \left[\frac{1}{r} J_{(ij)L-1}^{(1)} \left(t - \frac{r}{c} \right) \right] \right\}, \quad (\text{II.17c})$$

where we define² $J_{i|L} \equiv \epsilon_{iik} J_{kL-1}$ along with the univocal multi-index ordering $L = i_1 i_2 \cdots i_\ell$. Similarly, we parametrize the gauge vector by four sets of STF gauge moments $\{W_L, X_L, Y_L, Z_L\}$, namely

$$\varphi_1^0 = \frac{4}{c^3} \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{\ell!} \partial_L \left[\frac{1}{r} W_L \left(t - \frac{r}{c} \right) \right], \quad (\text{II.18a})$$

$$\begin{aligned} \varphi_1^i &= -\frac{4}{c^4} \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{\ell!} \partial_L \left[\frac{1}{r} X_L \left(t - \frac{r}{c} \right) \right] \\ &\quad - \frac{4}{c^4} \sum_{\ell=1}^{\infty} \frac{(-1)^\ell}{\ell!} \left\{ \partial_{L-1} \left[\frac{1}{r} Y_{iL-1} \left(t - \frac{r}{c} \right) \right] + \frac{\ell}{\ell+1} \partial_L \left[\frac{1}{r} Z_{i|L} \left(t - \frac{r}{c} \right) \right] \right\}, \end{aligned} \quad (\text{II.18b})$$

where we define $Z_{i|L} \equiv \epsilon_{iik} Z_{kL-1}$.

The STF multipole moments $\{I_L, J_L, W_L, X_L, Y_L, Z_L\}$ are arbitrary functions of the retarded time $u = t - r/c$ of the harmonic coordinates, the only constraint being that the monopole and dipoles satisfy the usual conservation laws, i.e. that the mass monopole I , the time derivative of the mass dipole $\mathcal{P}_i \equiv I_i^{(1)}$ and the current dipole J_i are all time-independent constants. We will from now on denote them, respectively, M , $\mathcal{P}_i \equiv M_i^{(1)}$ and S_i ,

²This follows the conventions of [95, 3] and not those of [96], see Footnote 3 of [6] for a clarification.

as they represent the conserved mass, linear momentum and angular momentum, composed of matter, gravitational binding energy and GW contributions. With these conservation laws, it is easy to verify that our parametrization (II.16) indeed satisfies the harmonic gauge condition (the vacuum wave equation is automatically satisfied thanks to the multipolar structure), and in our construction, we will henceforth restrict ourselves to a center-of-mass frame defined by $M_i = 0$. Moreover, the conditions (II.11) impose that each of these moments is a constant for $t \leq -\mathcal{T}$, and their time derivatives vanish. Note that although these moments are for the time being arbitrary functions that parametrize our metric, these will be given definite values once we match the exterior metric to the interior solution.

Once we have in hand the linearized metric, we proceed to constructing iteratively the nonlinear ($n \geq 2$) pieces of the MPM metric. For this, we first need to identify a particular solution of (II.12), under the usual conditions (II.11). Consider an equation of the type $\square\phi = S$, where $S(\mathbf{x}, t)$ is a smooth source that behaves as $S(r, t-r) = \mathcal{O}(r^{\ell+5})$ when $r \rightarrow 0$ and $t-r$ is held fixed [4, 97]. A natural solution is $\phi \equiv \square_{\text{ret}}^{-1}[S]$, where the inverse retarded d'Alembert operator is defined as usual, namely

$$\square_{\text{ret}}^{-1}[S](\mathbf{x}, t) \equiv -\frac{1}{4\pi} \int \frac{d^3\mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|} S\left(\mathbf{x}', t - \frac{|\mathbf{x} - \mathbf{x}'|}{c}\right). \quad (\text{II.19})$$

It is of course easy to verify that $\square[\square_{\text{ret}}^{-1}[S]] = \square_{\text{ret}}^{-1}[\square[S]] = S$. However, this solution is ill-defined as soon as the behavior of S is not regular enough. Note that convergence problems can only occur in the $r \rightarrow 0$ limit, since the function S quite generally involves the derivatives of a multipolar moment, and the conditions (II.11) then impose that the integrand is identically zero in the $r \rightarrow +\infty$ limit (with t kept fixed). In order to cure the the ultraviolet (UV) divergences in the $r \rightarrow 0$ limit, one introduces a complex parameter $B \in \mathbb{C}$ and an arbitrary regularization constant r_0 such that

$$\square_{\text{ret}}^{-1}\left[\left(\frac{r}{r_0}\right)^B S\right](\mathbf{x}, t) \equiv -\frac{1}{4\pi} \int \frac{d^3\mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|} \left(\frac{|\mathbf{x}'|}{r_0}\right)^B S\left(\mathbf{x}', t - \frac{|\mathbf{x} - \mathbf{x}'|}{c}\right) \quad (\text{II.20})$$

is convergent. Indeed, for a typical source, it suffices to choose $\Re(B)$ large enough to overcome the divergent behavior of the integrand. By analytic continuation in B , one can write the integral as a Laurent series in B , which reads

$$\square_{\text{ret}}^{-1}\left[\left(\frac{r}{r_0}\right)^B S\right](\mathbf{x}, t) = \sum_{k=-n}^{\infty} B^k \alpha_k[S](\mathbf{x}, t), \quad (\text{II.21})$$

where $n \in \mathbb{Z}$. The original retarded d'Alembertian (II.19) is divergent if and only if this Laurent series exhibits poles, namely if $\exists k \leq -1$ such that $\alpha_k[S] \neq 0$. Be the original integral convergent or divergent, we find that the general prescription for obtaining a particular solution to the original equation is to take $\alpha_0[S]$, i.e. the finite part as $B \rightarrow 0$ (denoted $\text{FP}_{B=0}$), see [92] for a proof. Since we have constructed a solution to our equation, we of course have $\square[\text{FP}_{B=0} \square_{\text{ret}}^{-1}[S]] = S$, but note conversely that in general, $\text{FP}_{B=0} \square_{\text{ret}}^{-1}[\square[S]] \neq 0$. This will lead us to define commutators $[\text{FP}_{B=0} \square_{\text{ret}}^{-1}, \square][S] \equiv \text{FP}_{B=0} \square_{\text{ret}}^{-1} \square[S] - S$, which vanish when acting on a convergent source.

We now have all the tools needed to construct a particular solution $u_n^{\mu\nu}$ to (II.13), which reads

$$u_n^{\mu\nu} \equiv \text{FP}_{B=0} \square_{\text{ret}}^{-1} \left[\left(\frac{r}{r_0} \right)^B \Lambda_n^{\mu\nu} [h_1, \dots, h_{n-1}] \right]. \quad (\text{II.22})$$

The constant length scale r_0 is arbitrary and has to disappear from any physical result in the end. This particular solution satisfies $\square u_n^{\mu\nu} = \Lambda_n^{\mu\nu}$, but it does not satisfy the harmonic gauge condition, i.e. $\partial_\nu u_n^{\mu\nu} \neq 0$. Since $\partial_\nu \Lambda_n^{\mu\nu} = 0$, the divergence of the particular solution actually reads

$$w_n^\mu \equiv \partial_\nu u_n^{\mu\nu} = \text{FP}_{B=0} \square_{\text{ret}}^{-1} \left[B \left(\frac{r}{r_0} \right)^B \frac{n_i}{r} \Lambda_n^{\mu i} \right]. \quad (\text{II.23})$$

The factor B comes from the differentiation of the regulator $(r/r_0)^B$. Because of it, the term (II.23) is nonzero only when the integral develops a pole $\propto 1/B$ when $B \rightarrow 0$. Furthermore, this divergence is automatically a homogeneous retarded solution of the wave equation, $\square w_n^\mu = 0$. At this stage, we apply the MPM ‘‘harmonicity’’ algorithm [92] to construct from w_n^μ another homogeneous retarded solution $v_n^{\mu\nu}$ such that $\partial_\nu v_n^{\mu\nu} = -w_n^\mu$. First, we utilize the fact that w_n^μ is a homogeneous retarded solution of the vacuum wave equation to express it in terms of four sets of STF multipolar moment $\{N_L, P_L, Q_L, R_L\}$ (see Appendix A.1 for a practical way to extract them). It reads

$$w_n^0 = \sum_{\ell=0}^{\infty} \partial_L \left[\frac{1}{r} N_L \left(t - \frac{r}{c} \right) \right], \quad (\text{II.24a})$$

$$w_n^i = \sum_{\ell=0}^{\infty} \partial_L \left[\frac{1}{r} P_L \left(t - \frac{r}{c} \right) \right] + \sum_{\ell=1}^{\infty} \left\{ \partial_{L-1} \left[\frac{1}{r} Q_{iL-1} \left(t - \frac{r}{c} \right) \right] + \frac{\ell}{\ell+1} \partial_L \left[\frac{1}{r} R_{i|L} \left(t - \frac{r}{c} \right) \right] \right\}, \quad (\text{II.24b})$$

where we define $R_{i|L} \equiv \epsilon_{ii_\ell k} R_{kL-1}$ and all moments N, P, Q and R implicitly depend on the order n . From these moments, we can construct $v_n^{\mu\nu}$ as a multipolar expansion which satisfies the divergence condition. There is a residual gauge freedom in this construction, and we utilize it to minimize the v_n^{00} and v_n^{0i} components. Note that any other gauge choice would also be valid, but would lead to a different matching, and in particular would modify the definition of the source moments. The choice we make for the expression of $v_n^{\mu\nu}$ thus reads

$$v_n^{00} = -\frac{1}{r} N^{-1} + \partial_a \left[\frac{1}{r} \left(-N_a^{(-1)} + Q_a^{(-2)} - 3P_a \right) \right] \quad (\text{II.25a})$$

$$v_n^{0i} = \frac{1}{r} \left(-Q_i^{(-1)} + 3P_i^{(1)} \right) - \partial_a \left[\frac{1}{r} R_{i|a}^{(-1)} \right] - \sum_{\ell=2}^{\infty} \left[\frac{1}{r} N_{iL-1} \right] \quad (\text{II.25b})$$

$$v_n^{ij} = -\frac{\delta_{ij}}{r} P + \sum_{\ell=2}^{\infty} \left\{ 2\delta_{ij} \partial_{L-1} \left[\frac{1}{r} P_{L-1} \right] - 6\partial_{L-2(i} \left[\frac{1}{r} P_{j)L-2} \right] \right. \\ \left. + \partial_{L-2} \left[\frac{1}{r} \left(N_{ijL-2}^{(1)} + 3P_{ijL-2}^{(2)} - Q_{ijL-2} \right) \right] - 2\partial_{L-1} \left[\frac{1}{r} R_{i|jL-1} \right] \right\} \quad (\text{II.25c})$$

where all moments N , P , Q and R are implicitly evaluated at $t - r/c$. This procedure is concisely denoted by $v_n^{\mu\nu} = \mathcal{V}^{\mu\nu}[u_n]$. Note the presence of antiderivatives, denoted by the presence of negative derivatives (see [Notations](#)).

Finally, the harmonic metric at order n , now satisfying the full Einstein vacuum equations in harmonic coordinates at the n^{th} order, reads

$$h_n^{\mu\nu} = u_n^{\mu\nu} + v_n^{\mu\nu}. \quad (\text{II.26})$$

The harmonic metric $\mathcal{M}(h^{\mu\nu})$ as defined in [\(II.12\)](#) is a nonlinear functional of the moments $\{I_L, J_L, W_L, X_L, Y_L, Z_L\}$ and represents the most general solution of the Einstein field equations in the vacuum region outside an isolated system [\[92\]](#).

It can be shown that the metric constructed in this way has the following polylogarithmic structure when $r \rightarrow 0$ (see [\[92\]](#) for a proof):

$$\mathcal{M}(h)(\mathbf{x}, t) \underset{r \rightarrow 0}{=} \sum_{\ell \in \mathbb{N}} \sum_{m \in \mathbb{Z}} \sum_{p \in \mathbb{N}} \hat{n}_L r^m \ln^p(r) F_L^{m,p}(t) \quad (\text{II.27})$$

where $F_L^{m,p}(t)$ is a multilinear functional of all the source and gauge moments.

II.2.2 Near-zone PN metric

Given a physical quantity Q , we denote by \bar{Q} its PN expansion, namely its formal series representation when $c \rightarrow \infty$, up to some finite order $1/c^{2k}$, which we call the k^{th} PN order, denoted $k\text{PN}$ (thus odd powers in v/c correspond to half-integer PN orders). This quantity will differ drastically from its physical counterpart far from the source, but it will be deemed a good approximation inside and in the vicinity of the source. Of particular interest will be the PN expansion of the metric,

$$\bar{h}^{\mu\nu}(\mathbf{x}, t, c) = \sum_{m=2}^{\infty} \frac{1}{c^m} \bar{h}_m^{\mu\nu}(\mathbf{x}, t; \ln(c)), \quad (\text{II.28})$$

where the $\ln(c)$ dependency means we can further expand the PN coefficients as

$$\bar{h}_m^{\mu\nu}(\mathbf{x}, t; \ln(c)) = \sum_{p=0}^{\infty} \ln(c)^p \bar{h}_{m,p}^{\mu\nu}(\mathbf{x}, t), \quad (\text{II.29})$$

where the sum actually terminates at some finite order p_{max} that depends on m . Similarly, we perform the PN expansion of the stress energy pseudotensor,

$$\bar{\tau}^{\mu\nu}(\mathbf{x}, t, c) = \sum_{m=-2}^{\infty} \frac{1}{c^m} \bar{\tau}_m^{\mu\nu}(\mathbf{x}, t; \ln(c)) \quad (\text{II.30})$$

Of course, these infinite sums are simply formal series, and should be truncated to the appropriate PN order. Inserting [\(II.28\)](#) and [\(II.30\)](#) into the harmonic-coordinate Einstein equations [\(II.10\)](#), and recalling that the d'Alembert operator decomposes as $\square = \Delta - c^{-2} \partial_t^2$, leads to a hierarchy of equations acting on each term of the PN series, namely

$$\Delta \bar{h}_m^{\mu\nu} = 16\pi G \bar{\tau}_{m-4}^{\mu\nu} + \partial_t^2 \bar{h}_{m-2}^{\mu\nu}. \quad (\text{II.31})$$

We observe that, as often in physics, we have transformed a wave equation into a Poisson's equation, which is valid only in the regime where retardations due to radiation are small. Indeed, at lowest $m = 2$ order, we recover Poisson's equation for the Newtonian gravitational potential,

$$\Delta\phi = 4\pi G\rho, \quad (\text{II.32})$$

where $\phi = \frac{1}{4}\bar{h}_2^{00}$ is the Newtonian gravitational potential and $\rho = \bar{\tau}_{-2}^{00}$ is the Newtonian matter density. It is then possible to prove that in the $r \rightarrow +\infty$ limit, the asymptotic structure of \bar{h} reads

$$\bar{h}(\mathbf{x}, t) \underset{r \rightarrow +\infty}{=} \sum_{\ell \in \mathbb{N}} \sum_{m \in \mathbb{Z}} \sum_{p \in \mathbb{N}} \hat{n}_L r^m \ln^p(r) G_L^{m,p}(t), \quad (\text{II.33})$$

which is identical to (II.27).

Finally, we will mention a practical way to implement the previously described algorithm, which consists in parametrizing the near-zone metric with well-chosen potentials. At 2PN, the metric reads

$$\bar{g}_{00} = -1 + \frac{2}{c^2}V - \frac{2}{c^4}V^2 + \mathcal{O}\left(\frac{1}{c^{10}}\right), \quad (\text{II.34a})$$

$$\bar{g}_{0i} = -\frac{4}{c^3}V_i + \mathcal{O}\left(\frac{1}{c^9}\right), \quad (\text{II.34b})$$

$$\bar{g}_{ij} = \delta_{ij} \left(1 + \frac{2}{c^2}V\right) + \mathcal{O}\left(\frac{1}{c^8}\right), \quad (\text{II.34c})$$

where we have defined $V = -4\pi G \bar{\square}_{\text{ret}}^{-1} [(T^{00} + T^{kk})/c^2]$ and $V_i = -4\pi G \bar{\square}_{\text{ret}}^{-1} [T^{0i}/c]$. The metric and potentials have been defined up to 4PN in [98].

II.2.3 The matching equation

Once we have constructed the exterior and near-zone metrics and obtained their asymptotic behaviors (see Fig. II.2), we will seek to relate the two metrics by matching their asymptotic series. Indeed, we have observed that the $r \rightarrow 0$ asymptotic series of $\mathcal{M}(\bar{h})$, given by (II.27), is formally identical to the $r \rightarrow +\infty$ asymptotic series of \bar{h} , given by (II.27), but the two series are parametrized *a priori* by different time-dependent coefficients, $F_L^{m,p}(t)$ and $G_L^{m,p}(t)$. The matching procedure consists in imposing that these two expansions be equal, namely

$$\overline{\mathcal{M}(\bar{h})} = \sum_{\ell \in \mathbb{N}} \sum_{m \in \mathbb{Z}} \sum_{p \in \mathbb{N}} \hat{n}_L r^m \ln^p(r) F_L^{m,p}(t) \equiv \sum_{\ell \in \mathbb{N}} \sum_{m \in \mathbb{Z}} \sum_{p \in \mathbb{N}} \hat{n}_L r^m \ln^p(r) G_L^{m,p}(t) = \mathcal{M}(\bar{h}), \quad (\text{II.35})$$

where, as usual, the overline symbolizes the PN (or near-zone) expansion, and the \mathcal{M} symbolizes the multipolar (or far-zone) expansion [99, 100, 14]. This is equivalent to stating that the coefficients must be identical in the physical problem, namely that $F_L^{m,p}(t) = G_L^{m,p}(t)$.

Applying this matching procedure entirely determines the multipolar moments that parametrize the MPM solution in terms of the PN metric and matter source. The source

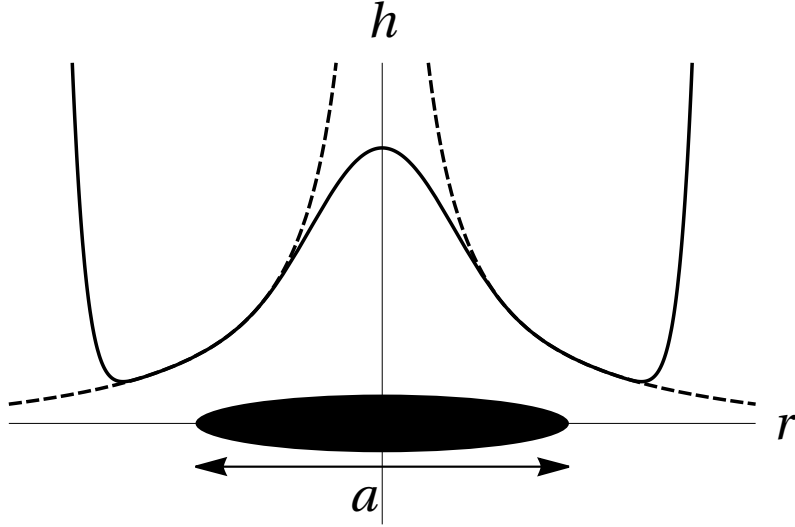


Fig. II.2 Sketch of the metric h in terms of the radial coordinate r , both for the PN expansion \bar{h} (full line) and the MPM expansion $\mathcal{M}(h)$ (dotted line). The blob represents the source, of typical size a . The region where both curves overlap is the matching zone.

and gauge moments are all given in [14], and we reproduce here only the expression for the mass multipole moment,

$$\begin{aligned} I_L(u) = \text{FP}_{B=0} \int d^3\mathbf{x} \left(\frac{r}{r_0}\right)^B \int_{-1}^1 dz \left\{ \delta_\ell(z) \hat{x}_L \Sigma - \frac{4(2\ell+1)}{c^2(\ell+1)(2\ell+3)} \delta_{\ell+1}(z) \hat{x}_{iL} \Sigma_i^{(1)} \right. \\ \left. + \frac{2(2\ell+1)}{c^4(\ell+1)(\ell+2)(2\ell+5)} \delta_{\ell+2}(z) \hat{x}_{ijL} \Sigma_{ij}^{(2)} \right\} \left(\mathbf{x}, u + \frac{zr}{c} \right), \end{aligned} \quad (\text{II.36})$$

where we have define the (PN-expanded) source variables

$$\Sigma = \frac{\bar{\tau}^{00} + \bar{\tau}^{aa}}{c^2} \quad \Sigma_i = \frac{\bar{\tau}^{0i}}{c}, \quad \Sigma_{ij} = \bar{\tau}^{ij}, \quad (\text{II.37})$$

and where the weighting function $\delta_\ell(z)$ is defined by

$$\delta_\ell(z) = \frac{(2\ell+1)!!}{2^{\ell+1}\ell!} (1-z^2)^\ell \quad (\text{II.38})$$

such as to satisfy

$$\int_{-1}^1 dz \delta_\ell(z) = 1. \quad (\text{II.39})$$

The following expansion is essential for expanding these integrals to finite PN order:

$$\int_{-1}^1 dz \delta_\ell(z) S \left(\mathbf{x}, u + \frac{zr}{c} \right) = \sum_{k=0}^{\infty} \frac{(2\ell+1)!! r^{2k}}{2^k k! (2\ell+2k+1)!!} \left(\frac{\partial}{c \partial u} \right)^{2k} [S(\mathbf{x}, u)]. \quad (\text{II.40})$$

II.3 The case of point-particles

Until now, we have supposed that the matter content, represented by its stress-energy tensor $T^{\mu\nu}(\mathbf{x}, t)$, is a smooth function of spacetime. This ensures in particular that the metric itself, represented by $g_{\mu\nu}$ or $h^{\mu\nu}$, is also smooth, and has a well-defined value at each point of spacetime. We have moreover assumed that the metric perturbation is small, i.e. $|h^{\mu\nu}| \ll 1$.

We now want to specialize our study to the case of a compact binary system of neutron stars or black holes. Clearly, in the vicinity of these objects, the gravitational field *is* very strong, and one cannot *a priori* apply the PN-MPM formalism. However, although the field is strong near the source, one can show in GR that the contribution of the internal structure to the dynamics turns out to be extremely small, namely of 5PN order, i.e. that the dynamics depend (up to 5PN precision) only on the total masses of the two objects, and not on their structure. This *effacement principle* [46] allows us to extent our formalism to the case where our compact binary is represented by two spinless and structureless point-particles, i.e. two well-separated bodies whose size is small compared to the other scales of the problem, and whose stress-energy content can thus be well represented as Dirac-distribution. The matter action for a system of point particles reads

$$S_m = -c \sum_A \int m_A \sqrt{-(g_{\alpha\beta})_A} dy_A^\alpha dy_A^\beta, \quad (\text{II.41})$$

thus the stress-energy tensor for a system of two point-particles is given by

$$T^{\mu\nu}(\mathbf{x}, t) = \sum_{A \in \{1,2\}} \frac{\delta^{(3)}(\mathbf{x} - \mathbf{y}_A(t))}{\sqrt{-(g)_A}} \frac{m_A v_A^\mu(t) v_A^\nu(t)}{\sqrt{-(g_{\alpha\beta})_A \frac{v_A^\alpha(t) v_A^\beta(t)}{c^2}}}, \quad (\text{II.42})$$

where we recall the shorthand $v_A^\mu \equiv (c, \mathbf{v}_A(t))$ (which is *not* the 4-velocity of the particle), and the reader is invited to consult the [Notations](#) for more details.

The PN iteration of this stress-energy tensor will of course yield at lowest order

$$T^{\mu\nu}(\mathbf{x}, t) = \delta_1 m_1 v_1^\mu(t) v_1^\nu(t) + (1 \leftrightarrow 2) + \mathcal{O}\left(\frac{1}{c^2}\right). \quad (\text{II.43})$$

When solving the field equations with this source, we find that the metric $g_{\mu\nu}$ must have a divergent $\sim 1/|\mathbf{x} - \mathbf{y}_A|$ behavior near particle $A \in \{1, 2\}$. But at the next iteration, we will need to evaluate $g_{\mu\nu}(\mathbf{x}, t)$ at $\mathbf{x} = \mathbf{y}_A(t)$, which is clearly divergent. From a physical point of view, this is an artifact of our choice of working with point-particles. Indeed, in the case of a compact star, the matter distribution is smoothed out over some finite region of space, so the metric is also smooth and there is no divergence. In the case of a black hole, there is indeed a singularity but it is hidden behind the event horizon, which is a feature that is completely lost in the point-particle PN approach. From the computational point of view, however, this has been historically a major obstacle. In standard Newtonian dynamics, this is solved by saying that in a system of N point particles, any particle feels the gravitational attraction of all the other particles, but not of itself. This procedure essentially removes the self-field of a compact object [13]. In the general relativistic case, is it not so easy to disentangle the self-field from the exterior fields, because they are nonlinearly mixed within the metric.

The historical procedure to remove the self-field in the PN problem is to assign a regularized value to the metric at the location of the point particle thanks to Hadamard regularization [101]. Ignoring the time dependence and spacetime indices, which play no role here, we can always write the metric as a formal expansion around the location \mathbf{y}_1 of the first particle, namely

$$g(\mathbf{x}) = \sum_{p=p_{\min}}^{\infty} r_1^p \underset{1}{g}_p(\mathbf{n}_1), \quad (\text{II.44})$$

where $p_{\min} \in \mathbb{Z}$, $r_1 = |\mathbf{x} - \mathbf{y}_1|$, $\mathbf{n}_1 = (\mathbf{x} - \mathbf{y}_1)/r_1$ and $\underset{1}{g}_p$ are a family of functions of \mathbf{n}_1 (crucially, they are independent of r_1). Essentially, in order to evaluate the metric at the location of the first particle while removing the self-field, we take the Hadamard *partie finie* at the location of the first particle, which reads

$$(g)_1 \equiv \frac{1}{4\pi} \int d^2\Omega \underset{1}{g}_0(\mathbf{n}_1), \quad (\text{II.45})$$

where we have only integrated $\underset{1}{g}_0$ over the angles. This procedure is very successful at low order, but starting at 3PN, it suffers from ambiguities. This is because, for a pair of functions F and G , we have $(FG)_1 \neq (F)_1(G)_1$ in general, and these products appear naturally in the PN construction. Moreover, the Leibniz rule is lost in general. These problems can be related to the fact that in Schwartz distribution theory, the product is not well defined. In practice, this is not a problem up to 2.5PN order, as one finds that any prescription for the way in which to take the finite part of a product yields the same result. However, at 3PN, different prescriptions yield different results that were found to differ by only a few typical terms. This led to the introduction of ambiguity parameters that quantified this uncertainty, and which could only be determined by comparing to results from other formalisms, e.g. gravitational self-force. This was problematic, because it meant that the formalism was not complete. It was solved by changing regularization methods, and adopting dimensional regularization.

II.4 Dimensional regularization

In Section II.3, we have seen that Hadamard regularization led to ambiguities. Fortunately, it was found that dimensional regularization [102, 103, 104] is the correct way to obtain the full, unambiguous equations of motion and waveform at 3PN order. Although this is not the topic of this thesis, its use in Chapter III requires that we sketch the main features of this regularization technique. The starting point is to extend GR to d spatial dimensions (instead of three), and one time dimension. This generalization initially only makes sense for $d \in \mathbb{N} \setminus \{0, 1, 2\}$ (since gravity in these lower dimensions exhibit special features), but the results will then be extended by analytical continuation to any d in the complex plane.

Dimensional regularization plays a crucial role in solving the ambiguities mentioned in Section II.3. In $d \equiv 3 + \varepsilon$ dimensions, the metric behaves as

$$g^{(d)}(\mathbf{x}) = \sum_{p=p_{\min}}^{\infty} \sum_{q=q_{\min}}^{q_{\max}} r_1^{p+q\varepsilon} \underset{1}{g}_{p,q}^{(d)}(\mathbf{n}_1), \quad (\text{II.46})$$

when \mathbf{x} approaches \mathbf{y}_1 , with $(p_{\min}, q_{\min}, q_{\max}) \in \mathbb{Z}^3$. Note that to ensure continuity with three dimensions, we have the constraint

$$\sum_q {}_1g_{p,q}^{(d)}(\mathbf{n}_1) = {}_1g_p(\mathbf{n}_1). \quad (\text{II.47})$$

Provided that $(p, q) \notin (-\mathbb{Z}_*) \times \{0\}$ (which is always satisfied in practice), each term in this sum will have a well-defined $r_1 \rightarrow 0$ limit if ε is chosen such that $\text{Re}(p + q\varepsilon) > 0$ (by virtue of analytical continuation, one can choose different values of ε for different terms in the sum). The metric $(g)_1$ at the location of the first particle is thus well defined by analytical continuation, which cures the divergence of the stress-energy tensor (II.42). Most importantly, this regularization procedure satisfies $(FG)_1 = (F)_1(G)_1$.

To be consistent, the whole PN-MPM expansion then needs to be generalized to d dimensions, see Appendix A.6 for a compendium of formulas. One important aspect is that the regularization procedure of Section II.2.1 needs to be adapted, and now reads³

$$\text{FP}_{B=0} \square_{\text{ret}}^{-1} \left[\left(\frac{r}{r_0} \right)^B S_\varepsilon \right] \equiv \text{FP}_{B=0} \left[-\frac{\tilde{k}}{4\pi} \int_1^{+\infty} dz \gamma_{\frac{1-d}{2}}(z) \int d^d \mathbf{x}' \left(\frac{r'}{r_0} \right)^B \frac{S_\varepsilon(t - z|\mathbf{x} - \mathbf{x}'|, \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^{d-2}} \right], \quad (\text{II.48})$$

where $\tilde{k} \equiv \Gamma(-1 + d/2) \pi^{1-d/2}$ and the function $\gamma_s(z)$ is defined in (A.42). The prescription is to first take the finite part in B , and only then to take the $\varepsilon \rightarrow 0$ limit (we call it the “ $B\varepsilon$ ” procedure).

Although dimensional regularization is very satisfactory from a theoretical point of view, it is very cumbersome in practical computations. Indeed, one loses in arbitrary d dimension the Huygens principal [105], therefore we must deal with many nonlocal integrals in the MPM iteration (like the z integral in (II.48)), which are instantaneous in the 3-dimensional case. Moreover, many crucial integrals which we knew how to perform in three dimensions are unknown analytically in arbitrary d dimensions. One useful way out is to compute the *difference between Hadamard and dimensional regularization*, which is dubbed “DDR”. This technique consists in performing a purely three-dimensional computation using a Hadamard regularization, and then to compute the difference between the three-dimensional result and the d -dimensional one as an expansion in $d \rightarrow 3$. For example, the difference between Hadamard and dimensional regularization for the integral of the metric (given, respectively, by (II.44) and (II.46)), over a ball of radius \mathcal{R} that includes the two singularities, reads

$$\begin{aligned} & \text{pf}_{s_1, s_2} \int_{\mathcal{B}^{(3)}(\mathcal{R})} d^3 \mathbf{x} g(\mathbf{x}) - \int_{\mathcal{B}^{(d)}(\mathcal{R})} d^d \mathbf{x} g^{(d)}(\mathbf{x}) \\ &= \frac{1}{\varepsilon} \sum_{q=q_{\min}}^{q_{\max}} \left[\frac{1}{q+1} + \varepsilon \ln(s_1) \right] \int d\Omega_{d-1}(\mathbf{n}_1) g_{-3,q}^{(d)}(\mathbf{n}_1) + (1 \leftrightarrow 2) + \mathcal{O}(\varepsilon), \quad (\text{II.49}) \end{aligned}$$

³It turns out that, most of the time, the r^ε term in the d -dimensional volume element $d^d \mathbf{x} = r^{2+\varepsilon} dr d\Omega_{d-1}$ actually plays the same regularizing role as r^B , so the $\text{FP}_{B=0}$ operator is not needed. We introduce it only to treat the special case where the source behaves as $S \sim r^{p+q\varepsilon}$ with $q = -1$, such that the ε contribution in the source and the volume element exactly cancel out. Furthermore, in all practical computations, it was noticed that the poles in $1/B$ all cancel out in end results, such that one does not need to take a finite part, but only the limit $B \rightarrow 0$.

where the Hadamard prescription for the volume integral is given by (5.6) of [104] and depends on two arbitrary constants s_1 and s_2 associated to regularizing volumes around the singularities, and outside of which the two regularizations coincide. Although one loses with this technique the full dependence in the dimensions, one recovers useful results in the $\varepsilon \rightarrow 0$ limit, which we would be in general unable to obtain from a direct d -dimensional computation. One should also be careful that this procedure generically introduces poles in $1/\varepsilon$ in intermediate quantities. However, when computing observable, physical results (such as radiative moments), we will see that these poles cancel out: this is an important consistency check of our formalism.

II.5 The kinematics of quasicircular orbits

As we know from Chapter I, most binary systems will have radiated away virtually all of their eccentricity by the time they enter the frequency band of the detector. This motivates the study of binaries undergoing quasicircular orbits, i.e. inspiraling orbits which deviate from exact circularity only by small 2.5PN secular effects due to radiation reaction. We work in the center-of-mass frame and in harmonic coordinates, so each particle is on a quasicircular orbit around the origin, and the orbital separation, which we denote $r_{12} \equiv |\mathbf{y}_1 - \mathbf{y}_2|$, only decays secularly on radiation-reaction timescales. The particles being spinless, the orbit takes place within a fixed plane, which is orthogonal to the total angular momentum of the system. One can construct an orthonormal triad $(\mathbf{n}_0, \boldsymbol{\lambda}_0, \mathbf{l}_0)$ such that \mathbf{l}_0 is orthogonal to the orbital plane and such that the orbit appears counter-clockwise when \mathbf{l}_0 is pointing toward the observer, as illustrated in Fig. II.3a. The relative motion can then be described in polar coordinates, where the polar radius is the secularly decreasing orbital separation $r_{12}(t)$, and where the polar angle defines the orbital phase $\phi(t)$, which is an increasing function of time thanks to our choice of orientation, see Fig. II.3b. The orbital frequency is then straightforwardly defined as $\omega(t) = d\phi/dt > 0$. Note that the orbital parameters $\{r_{12}(t), \phi(t)\}$ depend both on the choice of origin of time and on the orientation of the $(\mathbf{n}_0, \boldsymbol{\lambda}_0)$ axes, so there will be arbitrary constants associated to these choices in our final results. Finally, one can define the co-rotating triad $(\mathbf{n}_{12}(t), \boldsymbol{\lambda}_{12}(t), \mathbf{l}_0)$, which will be extremely useful for the computation of nonlocal integrals in Section VI.2.3.

II.6 The equations of motion

The PN approach consists in stating that a slowly-evolving binary system can be described with equations of motions that differ from Newton's laws by small $(v/c)^2$ corrections. The equations of motions for a system of two point-particles were first computed at 1PN order by Lorentz and Droste [106] then by Einstein, Infeld and Hoffman [107]. After partial results at 2PN order [108, 109, 110], it was then completely solved at 2.5PN order [111, 112, 113]. At 3PN order, it was computed both in the ADM formalism [114, 115, 116] and in the PN formalism [117, 118, 119], both up to one ambiguity constant due to the use of Hadamard regularization. It was fixed in a self-consistent manner by computing the 3PN equations of motion using dimensional regularization instead [102]. It was later confirmed by effective field theory (EFT) methods [120]. The 3.5PN terms were computed by

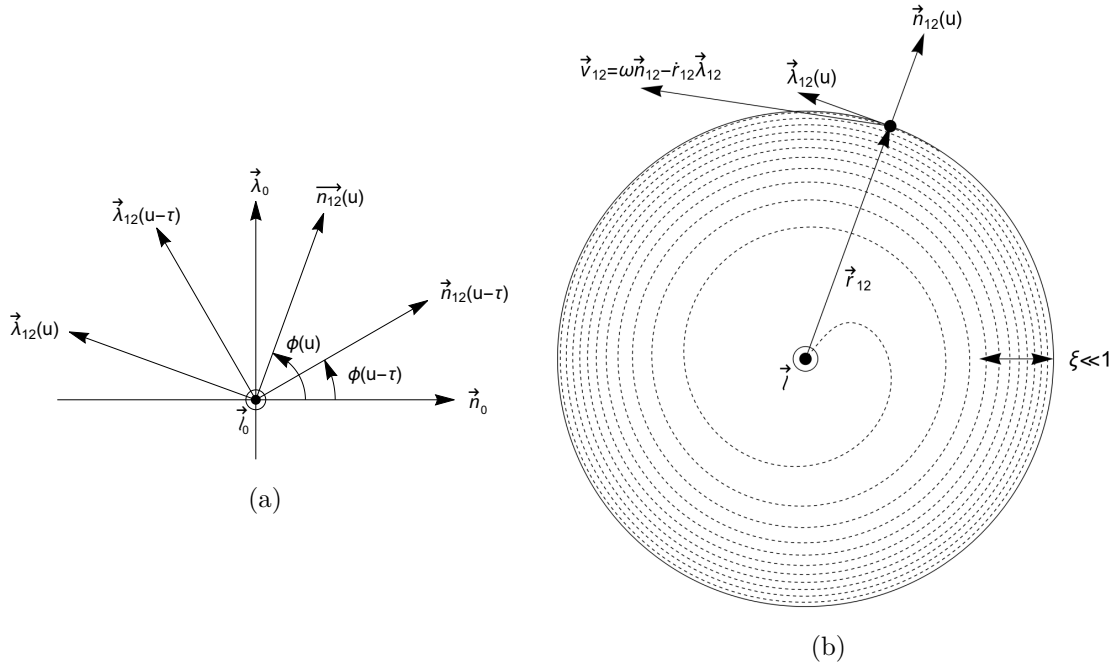


Fig. II.3 (a) Representation of the time-independent triad $(\mathbf{n}_0, \boldsymbol{\lambda}_0, \mathbf{l}_0)$ and the corotating triad $(\mathbf{n}(t), \boldsymbol{\lambda}(t), \mathbf{l}_0)$ at times u and $u - \tau$. (b) Corotating triad and inspiral of a quasicircular orbit.

[113, 121, 122]. The 4PN equations of motion were obtained by several groups using different methods: (i) the ADM Hamiltonian formalism [123, 124, 125, 126] led to complete results except for one “ambiguity” parameter, which was fixed by resorting to a comparison with gravitational self-force results [127]; (ii) the Fokker Lagrangian formalism in harmonic coordinates [128, 129, 130, 131] yielded for the first time a complete result without ambiguity parameter; and (iii) the EFT approach [132, 133, 134, 135, 136, 137, 138, 139] recovered the complete and unambiguous result. Partial results at higher PN orders have also been obtained [140, 141, 142, 143, 144, 145, 146]. First, we divide our equations of motion into conservative and dissipative dynamics, namely

$$\mathbf{a}_{12} = \mathbf{a}_{12}^{\text{cons}} + \mathbf{a}_{12}^{\text{diss}}. \quad (\text{II.50})$$

The conservative dynamics are computed by neglecting the radiation reaction that drives the inspiral, and they admit a Lagrangian or Hamiltonian formulation. At 4PN, and for circular orbits, they read

$$\mathbf{a}_{12}^{\text{cons}} = -\omega^2 \mathbf{y}_{12}, \quad (\text{II.51})$$

where ω is the orbital frequency and is given by a generalization of Kepler’s law⁴,

$$\omega^2 = \frac{Gm}{r_{12}^3} \left\{ 1 + (-3 + \nu)\gamma + \left(6 + \frac{41}{4}\nu + \nu^2 \right) \gamma^2 \right. \\ \left. + \left(-10 + \left[-\frac{75707}{840} + \frac{41}{64}\pi^2 + 22 \ln \left(\frac{r}{r'_0} \right) \right] \nu + \frac{19}{2}\nu^2 + \nu^3 \right) \gamma^3 \right\}$$

⁴As is [98], we have take the two constants r'_0 and r''_0 of [131] to be equal.

$$\begin{aligned}
& + \left(15 + \nu \left[\frac{19644217}{33600} + \frac{163}{1024} \pi^2 + \frac{256}{5} \gamma_E + \frac{128}{5} \ln(16\gamma) + 82 \ln \left(\frac{r}{r'_0} \right) \right] \right. \\
& \quad \left. + \nu^2 \left[\frac{44329}{336} - \frac{1907}{64} \pi^2 - \frac{992}{3} \ln \left(\frac{r}{r'_0} \right) + \frac{51}{4} \nu^3 + \nu^4 \right] \gamma^4 \right\}, \quad (\text{II.52})
\end{aligned}$$

where r'_0 is a gauge-related arbitrary constant and where

$$\gamma \equiv \frac{Gm}{r_{12}c^2} \quad (\text{II.53})$$

is a small PN parameter. Under these conservative equations of motion, it is possible to find a conserved Noetherian energy E , such that at desired PN order,

$$\left. \frac{dE}{dt} \right|_{\mathbf{a}_{12}^{\text{cons}}} = 0. \quad (\text{II.54})$$

Note that this energy is in general *not* equal to the on-shell value of the Hamiltonian due to nonlocal terms in the dynamics [131, 147]. It only contains even terms in $1/c$ and reads at 4PN

$$\begin{aligned}
E = & -\frac{\mu c^2 y}{2} \left\{ 1 + \left(-\frac{3}{4} - \frac{\nu}{12} \right) y + \left(-\frac{27}{8} + \frac{19}{8} \nu - \frac{\nu^2}{24} \right) y^2 \right. \\
& + \left(-\frac{675}{64} + \left[\frac{34445}{576} - \frac{205}{96} \pi^2 \right] \nu - \frac{155}{96} \nu^2 - \frac{35}{5184} \nu^3 \right) y^3 \\
& + \left(-\frac{3969}{128} + \left[-\frac{123671}{5760} + \frac{9037}{1536} \pi^2 + \frac{896}{15} \gamma_E + \frac{448}{15} \ln(16y) \right] \nu \right. \\
& \quad \left. + \left[-\frac{498449}{3456} + \frac{3157}{576} \pi^2 \right] \nu^2 + \frac{301}{1728} \nu^3 + \frac{77}{31104} \nu^4 \right) y^4 \left. \right\}. \quad (\text{II.55})
\end{aligned}$$

where we have defined the frequency-related PN parameter y in (I.42).

Since this energy is seen as the conserved quantity is the absence of dissipation, we can state that the energy flux \mathcal{F}_E , i.e. the energy “lost” by the system per unit time due to the emission of GWs is given by the time derivative of the conserved energy under the full equations of motion, namely

$$\left. \frac{dE}{dt} \right|_{\mathbf{a}_{12}^{\text{cons}} + \mathbf{a}_{12}^{\text{diss}}} = -\mathcal{F}_E. \quad (\text{II.56})$$

Moreover, for now, \mathcal{F}_E is solely defined from the energy and the dissipative equations of motion. The latter should be computed by matching the near-zone and far-zone metrics and determining radiation-reaction forces [148, 149]. It has however been assumed to be equal to the energy flux due to GWs measured at future null infinity. This has given rise to a controversy [150], which was solved in part by proving that this was indeed true at 2PN order [151, 152, 153]. If this hypothesis is assumed to hold at any order, then one can determine thanks to (II.56) the dissipative acceleration up to much higher order than what is possible only with the matching procedure, see Fig. II.1. It reads for quasicircular orbits [131]

$$\mathbf{a}_{12}^{\text{diss}} = -\frac{32}{5} \frac{G^3 m^2 \nu}{c^5 r_{12}^4} \left[1 + \left(-\frac{743}{336} - \frac{11}{4} \nu \right) \gamma + 4\pi \gamma^{\frac{3}{2}} \right] \mathbf{v}_{12}, \quad (\text{II.57})$$

where \mathbf{v}_{12} is the relative velocity. Using the full equations of motion and the flux-balance equation, we also obtain the secular decrease of orbital quantities at 1.5PN order [6]

$$\frac{dr_{12}}{dt} = -\frac{64}{5} \frac{G^3 m^3 \nu}{r_{12}^3 c^5} \left[1 + \left(-\frac{1751}{336} - \frac{7}{4} \nu \right) \gamma + 4\pi \gamma^{3/2} + \mathcal{O}(\gamma^2) \right], \quad (\text{II.58a})$$

$$\frac{d\omega}{dt} = \frac{96}{5} \frac{Gm\nu}{r_{12}^3} \gamma^{5/2} \left[1 + \left(-\frac{2591}{336} - \frac{11}{12} \nu \right) \gamma + 4\pi \gamma^{3/2} + \mathcal{O}(\gamma^2) \right]. \quad (\text{II.58b})$$

II.7 Observables at infinity

At the end of the PN-MPM iteration, the end result is a MPM metric $h = \sum_{n \geq 1} G^n h_n$ expressed only in terms of the kinematics of the source (i.e. the positions and velocities of the point particles). Unfortunately, it is a very complicated functional of these kinematics, and is not suited for practical use. In particular, its far-zone structure as $r \rightarrow +\infty$ is plagued by far-zone logarithms of r , namely

$$h(\mathbf{x}, t) \underset{r \rightarrow +\infty}{\sim} \sum_{\ell \in \mathbb{N}} \sum_{m \in \mathbb{Z}} \sum_{p \in \mathbb{N}} \hat{n}_L r^m \ln^p(r) F_L^{m,p}(t - r/c), \quad (\text{II.59})$$

where the expansion is performed for $t - r/c$ kept fixed. This is troublesome because, far from the source⁵, the n^{th} PM term becomes larger than the $(n - 1)^{\text{th}}$. Fortunately, this is simply a well-known feature of the harmonic coordinates [154, 155, 156, 157, 158, 159], and there always exists a coordinate transformation $X^\mu = \varphi^\mu(x^\nu)$ that ensures (at the price of exiting harmonic coordinates) that in this new coordinate system (which we call “radiative”, although we will discuss this nomenclature more in detail in Section IV.1), the metric behaves as

$$h(\mathbf{X}, T) \underset{R \rightarrow +\infty}{\sim} \sum_{\ell \in \mathbb{N}} \sum_{m \in \mathbb{Z}} \hat{N}_L R^m F_L^m(T - R/c). \quad (\text{II.60})$$

For iterated tails up to quartic order (i.e. $M^k \times M_{ij}$ interactions for $k \in \{1, 2, 3\}$), the standard choice is to take $(T, \mathbf{X}) = (t - 2GMc^{-3} \ln(r/b_0), \mathbf{x})$, where b_0 is an arbitrary length scale that is related to the choice of origin of time. This choice successfully removes all the logarithms, even at nonlinear order. However, as we will see, this coordinate redefinition is not sufficient to remove all logarithms at 4PN order, and an adequate coordinate redefinition is not easy to guess. We do however know of a different flavor of the MPM algorithm that automatically removes these logarithms [160], and which will be implemented in Chapter IV. In order to be consistent with the rest of the formalism, we will need to provide a mapping from this result back to our original coordinate system: this will be the task of Chapter V.

We now assume that the far-zone computation has been performed, and that all logarithms have been removed. We notice that, for an extremely distant observer like

⁵Typically for $r \gg \lambda_{\text{GW}} \frac{\lambda_{\text{GW}}}{Gm}$, where λ_{GW} is the gravitational wavelength [154].

ourselves, the subleading terms $\sim 1/R^2$ are by far negligible. This amounts to a relin-earization of the full metric in the region very far from the source in the small parameter $Gm/(c^2R)$, where R is now the distance between the observer and the source (and *not* the orbital separation). We find that⁶

$$\square h^{\mu\nu} = \mathcal{O}\left(\frac{1}{R^2}\right) \quad \text{and} \quad \partial_\mu h^{\mu\nu} = \mathcal{O}\left(\frac{1}{R^2}\right), \quad (\text{II.61})$$

so in this linearized regime, $h^{\mu\nu}$ recovers a multipolar structure. We will want to completely fix the residual gauge freedom in order to simplify the expression at utmost. First, as for any linearized metric, one can always choose to express our result only in terms of a set of two multipolar moments, which we call radiative moments and denote $\{\mathcal{U}_L, \mathcal{V}_L\}$. The gauge vector and gauge moments associated to the linearized metric are chosen to be zero. Moreover, one can choose to work in the transverse-traceless (TT) gauge. Indeed, as seen in Section I.1, it is always possible to perform a well chosen gauge transformation such that our metric reads in a general frame [11]

$$h_{\text{TT}}^{00} = -\frac{4MG}{Rc^2} - \frac{4N_a \mathcal{P}_a}{c^3 R} + \mathcal{O}\left(\frac{1}{R^2}\right) \quad (\text{II.62a})$$

$$h_{\text{TT}}^{0i} = -\frac{4\mathcal{P}_i}{c^3 R} + \mathcal{O}\left(\frac{1}{R^2}\right) \quad (\text{II.62b})$$

$$h_{\text{TT}}^{ij} = -\frac{4G}{c^2 R} \perp_{ij,ab}^{\text{TT}} \sum_{\ell=2}^{\infty} \frac{1}{\ell!} \left[N_{L-2} \mathcal{U}_{abL-2} - \frac{2\ell}{c(\ell+1)} N_{cL-2} \epsilon_{cd(a} \mathcal{V}_{b)dL-2} \right] + \mathcal{O}\left(\frac{1}{R^2}\right) \quad (\text{II.62c})$$

where $\perp_{ij,ab}^{\text{TT}}$ is the TT projection operator defined by (I.18). We have successfully transferred all the dynamical (or radiative) degrees of freedom into the purely spatial part of the metric, and only the static degrees of freedom, including the Newtonian potential, are contained in “00” and “0i” parts of the metric. Note that the effects due to the conserved total angular momentum S_i of the spacetime decay as $1/R^2$, so do not directly enter the far-zone metric⁷, and that is usual to work in the center-of-mass frame given by $\mathcal{P}_i = 0$. Since all the physical information describing a GW is contained in (II.62c), *a fortiori* the radiative moments also contain all the physical information. Moreover, all the radiative moments are related to the source moments at lowest order by $\mathcal{U}_L = I_L^{(\ell)} + \mathcal{O}(c^{-2})$ and $\mathcal{V}_L = J_L^{(\ell)} + \mathcal{O}(c^{-2})$. At higher PN order, they differ by nonlinear terms which can be nonlocal in time, such as tails, memory terms, tails-of-tails, etc. For illustrative purposes, let us give the expression of the radiative quadrupole moment at 3PN order:

$$\begin{aligned} \mathcal{U}_{ij} = & I_{ij}^{(2)} + \frac{2GM}{c^3} \int_0^{+\infty} d\tau \left[\ln\left(\frac{\tau}{2b_0}\right) + \frac{11}{12} \right] I_{ij}^{(4)}(U - \tau) \\ & + \frac{G}{c^5} \left[-\frac{2}{7} \int_0^{+\infty} d\tau I_{a\langle i}^{(3)} I_{j\rangle a}^{(3)}(U - \tau) + \frac{1}{7} I_{a\langle i}^{(5)} I_{j\rangle a}^{(5)} - \frac{5}{7} I_{a\langle i}^{(4)} I_{j\rangle a}^{(1)} - \frac{2}{7} I_{a\langle i}^{(3)} I_{j\rangle a}^{(2)} + \frac{1}{3} I_{a\langle i}^{(4)} J_{j\rangle|a} \right. \\ & \left. + 4W^{(4)} I_{ij} + 4W^{(3)} I_{ij}^{(1)} - 4W^{(2)} I_{ij}^{(2)} - 4W^{(1)} I_{ij}^{(3)} \right] \end{aligned}$$

⁶We use $\mathcal{O}\left(\frac{1}{R^2}\right)$ as a shorthand for $\mathcal{O}\left(\left(\frac{Gm}{c^2 R}\right)^2\right)$.

⁷Of course, S_i can enter the metric indirectly, *via* the radiative moments.

$$\begin{aligned}
 & + \frac{2G^2M^2}{c^6} \int_0^{+\infty} d\tau \left[\ln^2 \left(\frac{\tau}{2b_0} \right) + \frac{11}{6} \ln \left(\frac{\tau}{2b_0} \right) - \frac{107}{105} \ln \left(\frac{\tau}{2r_0} \right) + \frac{124627}{44100} \right] \mathbf{I}_{ij}^{(5)}(U - \tau) \\
 & + \mathcal{O} \left(\frac{1}{c^7} \right).
 \end{aligned} \tag{II.63}$$

Notice the appearance of the arbitrary r_0 scale at 3PN order: this scale is linked to the finite-part regularization. Note as well that at 2.5PN, there is a cumbersome dependency on the gauge moments. These will be reabsorbed when we express the radiative moments in terms of the *canonical* moments $\{M_L, S_L\}$, see Chapter III for more details. Also notice that there are two types of nonlocal terms. The first line is a tail integral, which corresponds to the backscattering of GWs against the curvature of spacetime generated by the constant total mass-energy of the spacetime. The second line is the memory integral, which can be interpreted as the reradiation of GWs by GWs themselves. These two effects are compactly represented⁸ by the two following Feynman diagrams, where the double line represent the compact binary, the squiggly line a GW, and a single line the static interaction between the curvature and a GW.

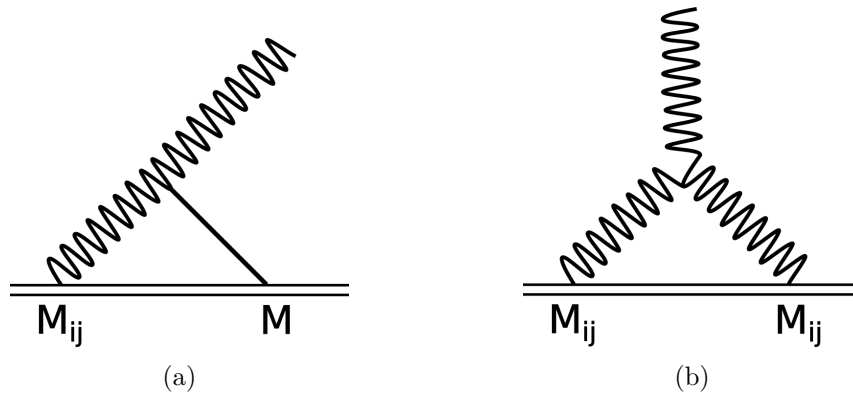


Fig. II.4 Feynmann diagrams for (a) the tail interaction $M \times M_{ij}$ (or equivalently at this order, $M \times I_{ij}$); and (b) the memory interaction $M_{ij} \times M_{ij}$ (or equivalently at this order, $I_{ij} \times I_{ij}$).

As we know from the sticky bead argument (discussed in the [Introduction](#)), GWs carry energy, and if one considers some sphere $\mathcal{S}(R)$ of radius R around our source, it is possible to compute the energy flux crossing that sphere. For this, we notice that it follows from the Bianchi identities that the energy-momentum pseudotensor (II.6) is conserved, namely $\partial_\mu \tau^{\mu\nu} = 0$ (in any coordinate system). We define the total energy inside the ball $\mathcal{B}(R)$ as $E_R \equiv \int_{\mathcal{B}(R)} d^3\mathbf{x} \tau^{00}$, which for now is of course a totally coordinate-dependent quantity, since the energy of GWs cannot be localized in a gauge-invariant way. It is thus convenient to choose a coordinate system which is TT in the vacuum region (of course, this is not always possible inside the matter). We can then define the energy flux exiting

⁸These have a well-defined computation meaning in the effective field theory framework; here we present them for illustrative purposes only.

the ball of radius R by

$$\mathcal{F}_R \equiv -\frac{dE_R}{dt} = c \int_{\mathcal{B}(R)} d^3\mathbf{x} \partial_i \tau^{0i} = c R^2 \int_{\mathcal{S}(R)} d^2\Omega n_i \tau^{0i}. \quad (\text{II.64})$$

For R large enough such that all the matter is strictly contained within $\mathcal{B}(R)$, we can perform the replacement $\tau^{0i} \rightarrow c^4/(16\pi G)\Lambda^{0i}$ within the angular integral. Now, recalling the structure of the metric given by (II.60) and the TT gauge condition, we find $c^2 n_i \Lambda^{0i} = \partial_t h_{kl} \partial_t h_{kl} + \mathcal{O}(1/R^3)$. The flux then reads

$$\mathcal{F}_R = \frac{c^3 R^2}{16\pi G} \int_{\mathcal{S}(R)} d^2\Omega \dot{h}_{\text{TT}}^{ij} \dot{h}_{\text{TT}}^{ij} + \mathcal{O}\left(\frac{1}{R}\right), \quad (\text{II.65})$$

where we have made explicit the fact that we are in a TT gauge. We then take the $R \rightarrow +\infty$ ($T - R$ being fixed) limit, and we find that the flux (also called luminosity) at future null infinity reads [48]

$$\begin{aligned} \mathcal{F} &= \lim_{R \rightarrow +\infty} \frac{c^3 R^2}{32\pi G} \int d^2\Omega \dot{h}_{\text{TT}}^{ij} \dot{h}_{\text{TT}}^{ij} \\ &= \sum_{\ell=2}^{\infty} \frac{G}{c^{2\ell+1}} \left[\frac{(\ell+1)(\ell+2)}{(\ell-1)\ell!(2\ell+1)!!} \mathcal{U}_L^{(1)} \mathcal{U}_L^{(1)} + \frac{4\ell(\ell+2)}{c^2(\ell-1)(\ell+1)!(2\ell+1)!!} \mathcal{V}_L^{(1)} \mathcal{V}_L^{(1)} \right], \end{aligned} \quad (\text{II.66})$$

where the last line was obtained using (II.62c). Note that the secular, gauge-invariant effects are obtained by time-averaging over a typical period of the system, which is denoted by $\langle \dots \rangle$. This turns out not to be necessary for quasircular orbits, for which the nonsecular effects cancel out all by themselves. One will then make the hypothesis that we can identify \mathcal{F} with \mathcal{F}_E as defined in (II.56), so as to obtain a balance equation between the conserved Noetherian energy and the flux of gravitational waves measured at future null infinity (see Section II.6).

We finish this chapter on a final comment. One striking feature of the PN-MPM formalism is that the radiation field and the dynamics are strongly interdependent. However, in our lowest-order linearized approach of Section I.2, this did not seem to be a problem. The reason is that the PN-MPM algorithm strongly relies on the fact that there is a hierarchy between radiation and the dynamics, which arises from the energy balance equation. This is nicely summarized by Table II.1, where an arrow from a cell A to a cell B indicates that computing A is required before one can compute B (this is not exhaustive, only the patterns are highlighted).

As one can see from this table, the near-zone metric and the equations of motions can be computed without any need for the exterior radiation field up to 2PN order. We have highlighted the fact that there is no 0.5PN contribution, be it in the dynamics or in the waveform. At 2.5PN order in the dynamics, one needs to include radiation-reaction effects, which are due to the Newtonian waveform. The latter can be obtained simply with the Newtonian equations of motion. Since the first nonlocal effect in the radiation field arrives at 1.5PN order with the tails, we can see that the first nonlocal effect in the dynamics will arise at 4PN order [154]. These equations of motion are required in order to obtain the 4PN waveform. Note that the 4PN flux we have computed [5, 6] affects the dynamics at 6.5PN order, which is much higher than the current state of the art.

Dynamics Near-zone PN metric	Radiation Exterior MPM metric
N	
↓	
1PN	
1.5PN	
2PN	
2.5PN ←	N →
3PN	
3.5PN	1PN
4PN ←	1.5PN →
4.5PN	2PN
5PN ←	2.5PN →
5.5PN	3PN
6PN	3.5PN
6.5PN	4PN
7PN	4.5PN

Table II.1: Order in which one should hierarchically solve dynamics and radiation in the PN-MPM formalism. An arrow from a cell A to a cell B indicates that computing A is required before one can compute B (not all such arrows are shown).

Chapter III

Source and gauge moments to canonical moments

This chapter is mainly based on [6].

As we have seen in Chapter II, gauge moments play an essential role for the matching of the vacuum solution to the PN source. However, once the matching is done, it is useful to minimize the number of moments involved in the MPM iteration. This is done by choosing the linearized gauge vector such that $\varphi_1^\mu = 0$. This amounts to a modification of the MPM algorithm of Section II.2.1, and the associated moments should then differ by nonlinear corrections. This section will deal with the methods used to go from a set of six types of *source* $\{I_L, J_L\}$ and *gauge* $\{W_L, X_L, Y_L, Z_L\}$ moments to only two *canonical* $\{M_L, S_L\}$ moments. As we discovered, this procedure strongly depends on the regularization scheme.

The plan of this chapter is as follows. In Section III.1, we review the MPM construction in canonical coordinates. In Section III.2, we review the method of [95] used to relate the canonical moments to the source and gauge moments in three dimensions. In Section III.2, we show how to adapt this method to a d -dimensional metric, in the context of dimensional regularization. Finally, in Section III.4, we apply this method to the quadrupole moment at 4PN order and present our result.

III.1 The canonical harmonic MPM algorithm

In this section, we describe the canonical construction of the metric, in harmonic coordinates. We will denote the metric thus constructed as $h_{\text{can}}^{\mu\nu}$ (in line with [95], but unlike [3], where it is instead denoted $h_{\text{harm}}^{\mu\nu}$). In particular, we still impose $\partial_\mu h_{\text{can}}^{\mu\nu} = 0$, but we utilize the residual gauge freedom of harmonic coordinates to set $\varphi_1^\mu = 0$. As usual, the MPM expansion is defined up to any post-Minkowskian (PM) order by

$$h_{\text{can}}^{\mu\nu} = \sum_{n=1}^{\infty} G^n h_{\text{can } n}^{\mu\nu}. \quad (\text{III.1})$$

The first step to construct such a solution is naturally the linearized approximation $n = 1$, defined this time by means of only two sets of STF *canonical* multipole moments $\{M_L, S_L\}$ as [93, 94, 48, 92]

$$h_{\text{can } 1}^{00} = -\frac{4}{c^2} \sum_{\ell \geq 0} \frac{(-)^\ell}{\ell!} \partial_L \left[\frac{1}{r} M_L \left(t - \frac{r}{c} \right) \right], \quad (\text{III.2a})$$

$$h_{\text{can } 1}^{0i} = \frac{4}{c^3} \sum_{\ell \geq 1} \frac{(-)^\ell}{\ell!} \left\{ \partial_{L-1} \left[\frac{1}{r} M_{iL-1}^{(1)} \left(t - \frac{r}{c} \right) \right] + \frac{\ell}{\ell+1} \partial_L \left[\frac{1}{r} S_{i|L} \left(t - \frac{r}{c} \right) \right] \right\}, \quad (\text{III.2b})$$

$$h_{\text{can } 1}^{ij} = -\frac{4}{c^4} \sum_{\ell \geq 2} \frac{(-)^\ell}{\ell!} \left\{ \partial_{L-2} \left[\frac{1}{r} M_{ijL-2}^{(2)} \left(t - \frac{r}{c} \right) \right] + \frac{2\ell}{\ell+1} \partial_{L-1} \left[\frac{1}{r} S_{(ij)L-1}^{(1)} \left(t - \frac{r}{c} \right) \right] \right\}, \quad (\text{III.2c})$$

where as usual $S_{i|L} \equiv \epsilon_{iik} S_{kL-1}$. The STF multipole moments $\{M_L, S_L\}$ are arbitrary functions of the retarded time $t - r/c$ of the harmonic coordinates, the only constraint being that the monopole and dipoles satisfy the usual conservation laws, i.e., that the mass monopole M , the time derivative of the mass dipole $M_i^{(1)}$ and the current dipole S_i are all time-independent constants. Thus M , M_i and S_i represent the conserved quantities, made of matter and GW contributions. Note therefore that they are identical to the ones stemming from the source-rooted construction described in Chapter II, namely $M = I$, $M_i = I_i$ and $S_i = J_i$. In this chapter, we restrict ourselves to a center-of-mass frame defined by $M_i = 0$.

The canonical harmonic algorithm is defined by the construction exposed in [95]. Namely, suppose that we have constructed the $n - 1$ first PM coefficients $h_m^{\mu\nu}$ for any $m \leq n - 1$, all satisfying the harmonic gauge condition $\partial_\nu h_{\text{can } m}^{\mu\nu} = 0$. Then we construct the n^{th} order coefficient as follows. It satisfies $\square h_{\text{can } n}^{\mu\nu} = \Lambda_n^{\mu\nu}$ together with $\partial_\nu h_{\text{can } n}^{\mu\nu} = 0$, where the source term is constructed out of the previous iterations: $\Lambda_n^{\mu\nu} \equiv \Lambda^{\mu\nu}[h_{\text{can } m}; m \leq n - 1]$. We then define $u_{\text{can } n}^{\mu\nu}$, $w_{\text{can } n}^{\mu\nu}$ and $v_{\text{can } n}^{\mu\nu}$ following exactly the same procedure as in Section II.2.1. The harmonic metric at order n , now satisfying the full Einstein vacuum equations in harmonic coordinates at the n^{th} order, then reads

$$h_{\text{can } n}^{\mu\nu} = u_{\text{can } n}^{\mu\nu} + v_{\text{can } n}^{\mu\nu}. \quad (\text{III.3})$$

The canonical metric is a nonlinear functional of the moments $\{M_L, S_L\}$ and represents, just like its counterpart in the general MPM construction, the most general solution of the Einstein field equations in the vacuum region outside an isolated system [92].

III.2 Relating the canonical moments to the source and gauge moments in three dimensions

In Section II.2.1 and Secs. III.1, we have presented two different constructions of the MPM metric, which both parametrize the most general exterior metric. However, these parametrizations are different, and we want to find a one-to-one mapping from one construction to the other. For this, we follow closely the method developed in [95]. Two physically equivalent metrics should only differ by a coordinate redefinition $x^\mu \rightarrow x'^\mu$,

which leads to the relation

$$h_{\text{gen}}^{\mu\nu}(x') = \frac{1}{|J|} \frac{\partial x'^{\mu}}{\partial x^{\rho}} \frac{\partial x'^{\nu}}{\partial x^{\sigma}} (h_{\text{can}}^{\rho\sigma}(x) + \eta^{\rho\sigma}) - \eta^{\mu\nu}, \quad (\text{III.4})$$

where $h_{\text{gen}}^{\mu\nu}(x')$ is the MPM metric in the general construction of Section II.2.1, parametrized by source moment $\{I_L, J_L\}$ and gauge moment $\{W_L, X_L, Y_L, Z_L\}$; where $h_{\text{can}}^{\mu\nu}(x)$ is the MPM metric in the canonical construction of Section III.1, parametrized by canonical moments $\{M_L, S_L\}$; and where $J \equiv \det(\partial x'/\partial x)$ is the Jacobian of the coordinate transformation. We introduce the coordinate transformation φ^μ , which is a function such that the coordinate transformation reads $x^\mu \rightarrow x'^{\mu} = x^\mu + \varphi^\mu(x)$. Since $h_{\text{gen}}^{\mu\nu}$ and $h_{\text{can}}^{\mu\nu}$ both admit a PM expansion, the gauge vector can be written as a PM expansion as well,

$$\varphi^\mu = \sum_{n=1}^{\infty} G^n \varphi_n^\mu. \quad (\text{III.5})$$

Consistently, we can also write the respective moments of each construction as a PM expansion

$$A_L = \sum_{n=1}^{\infty} G^{n-1} A_{n,L}, \quad (\text{III.6})$$

where A_L stands for either $M_L, S_L, I_L, J_L, W_L, X_L, Y_L$ or Z_L . At linear level, it is clear that the two constructions are related by

$$h_{\text{gen}1}^{\mu\nu}(x) = h_{\text{can}1}^{\mu\nu}(x) + \partial\varphi_1^{\mu\nu}, \quad (\text{III.7})$$

where φ_1^μ is given by (II.18) and the moments are related by $M_L = I_L + \mathcal{O}(G)$ and $S_L = J_L + \mathcal{O}(G)$. In our recurrence hypothesis, we assume that, for $n \geq 2$, we have constructed this relation up to the $(n-1)$ PM order, namely we have established¹

$$\varphi^\mu = \sum_{m=1}^{n-1} G^m \Phi_m^\mu[M_K, S_K, W_K, X_K, Y_K, Z_K] + \mathcal{O}(G^n), \quad (\text{III.8a})$$

$$I_L = \sum_{m=1}^{n-1} G^{m-1} \mathcal{I}_{m,L}[M_K, S_K] + \mathcal{O}(G^{n-1}), \quad (\text{III.8b})$$

$$J_L = \sum_{m=1}^{n-1} G^{m-1} \mathcal{J}_{m,L}[M_K, S_K] + \mathcal{O}(G^{n-1}). \quad (\text{III.8c})$$

We then perform the expansion of (III.4) up to n PM order, where we Taylor-expand the radiative metric to finite PM order, i.e. using

$$h_{\text{gen}}^{\mu\nu}(x') = \sum_{m \geq 0} \varphi^{\lambda_1}(x) \dots \varphi^{\lambda_m}(x) \partial_{\lambda_1} \dots \partial_{\lambda_m} h_{\text{gen}}^{\mu\nu}(x). \quad (\text{III.9})$$

¹Note that here we express the source moments in terms of the canonical and gauge moments, while in (3.6) of [95], it is the opposite. Indeed, the derivation in [95] is correct but swipes under the rug possible corrections at cubic order, which fortunately do not contribute at 4PN. This was however important in [3], so we follow the reasoning of this latter reference instead. The relation can straightforwardly be inverted at the end of the derivation.

At the n PM order, we find the relation

$$h_{\text{gen } n}^{\mu\nu}(x) = h_{\text{can } n}^{\mu\nu}(x) + \partial\varphi_n^{\mu\nu} + \Omega_n^{\mu\nu}[h_{\text{can } m}, \varphi_m; m \leq n-1], \quad (\text{III.10})$$

where $\Omega_n^{\mu\nu}$ is an explicitly known², nonlinear and at least quadratic, functional of the coordinate shift and the canonical metric at previous orders, but where $\varphi_n^{\mu\nu}$ is yet to be determined. We then define $\Delta_n^\mu \equiv -\partial_\nu \Omega_n^{\mu\nu} = \square\phi_n^\mu$, which naturally leads to defining the gauge vector

$$\phi_n^\mu = \text{FP}_{B=0} \square_{\text{ret}}^{-1} \left[\left(\frac{r}{r_0} \right)^B \Delta_n^\mu \right]. \quad (\text{III.11})$$

Since a very similar construction will be detailed in Chapter V, we do not detail all the steps of the construction here, and refer the curious reader to [95]. One can then show (see [95] for the derivation) that

$$h_{\text{gen } n}^{\mu\nu} = h_{\text{can } n}^{\mu\nu} + \partial\phi_n^{\mu\nu} + \Omega_n^{\mu\nu} + \mathcal{H}_n^{\mu\nu}, \quad (\text{III.12})$$

where we can introduced $\mathcal{H}_n^{\mu\nu}$, which satisfies $\square\mathcal{H}_n^{\mu\nu} = 0$ and $\partial_\nu \mathcal{H}_n^{\mu\nu} = 0$, and which is explicitly defined as $\mathcal{H}_n^{\mu\nu} = X_n^{\mu\nu} + Y_n^{\mu\nu} + Z_n^{\mu\nu}$. Here, we have introduced the quantities (see [95] for how they arise from commutators)

$$X_n^{\mu\nu} = \text{FP}_{B=0} \square_{\text{ret}}^{-1} \left[B \left(\frac{r}{r_0} \right)^B \left(-\frac{B+1}{r^2} \Omega_n^{\mu\nu} - \frac{2}{r} \partial_r \Omega_n^{\mu\nu} \right) \right] \quad (\text{III.13a})$$

$$Y_n^{\mu\nu} = \text{FP}_{B=0} \square_{\text{ret}}^{-1} \left[B \left(\frac{r}{r_0} \right)^B \frac{n_k}{r} \left(-2\delta^{k(\mu} \Delta_n^{\nu)} + \eta^{\mu\nu} \Delta_n^k \right) \right] \quad (\text{III.13b})$$

and we have defined $Z_n^{\mu\nu} = \mathcal{V}^{\mu\nu}[X_n + Y_n]$, where the $\mathcal{V}^{\mu\nu}$ procedure is described in Section II.2.1.

Since $\mathcal{H}_n^{\mu\nu}$ is a divergenceless solution of the vacuum wave equation, it can be expanded just like in (II.17a) and (II.18). We will be parametrize it by moments $\{\delta_n M_L, \delta_n S_L\}$ and a gauge vector ζ^μ as $\mathcal{H}_n^{\mu\nu} = k_1^{\mu\nu}[\delta_n M_L, \delta_n S_L] + \partial\zeta^{\mu\nu}$. The extraction of these moments is made easier by the fact that we are dealing with a divergenceless solution of the vacuum wave equation, see [95] for details on how to do this. Thus, we can recover the sought-for (III.10) if and only in the gauge vector is defined as $\varphi_n^\mu = \phi_n^\mu + \zeta^\mu$ and the moments are corrected by an n PM order correction, namely

$$I_L \rightarrow I_L - G^{n-1} \delta_n M_L [M_K, S_K, W_K, X_K, Y_K, Z_K] \quad (\text{III.14a})$$

$$J_L \rightarrow J_L - G^{n-1} \delta_n S_L [M_K, S_K, W_K, X_K, Y_K, Z_K] \quad (\text{III.14b})$$

² We recall from [95] the expression of $\Omega^{\mu\nu}$ up to cubic order (in both $h^{\rho\sigma}$ and φ^ρ):

$$\begin{aligned} \Omega^{\mu\nu}[h, \varphi] = & -\partial_\rho [\varphi^\rho (h^{\mu\nu} + \partial\varphi^{\mu\nu} + 2h^{\sigma(\mu} \partial_\sigma \varphi^{\nu)}) + \partial_\sigma \varphi^\mu \partial^\sigma \varphi^\nu] \\ & + 2\partial_\rho \varphi^{(\mu} h^{\nu)\rho} + \partial^\rho \varphi^\mu \partial_\rho \varphi^\nu + \frac{1}{2} \partial_{\rho\sigma} [\varphi^\rho \varphi^\sigma (h^{\mu\nu} + \partial\varphi^{\mu\nu})] + h^{\rho\sigma} \partial_\rho \varphi^\mu \partial_\sigma \varphi^\nu \\ & + \frac{1}{2} \eta^{\mu\nu} [\partial_\rho \varphi^\sigma \partial_\sigma \varphi^\rho - (\partial_\rho \varphi^\rho)^2] - \frac{1}{3} \eta^{\mu\nu} [\partial_\rho \varphi^\sigma \partial_\sigma \varphi^\lambda \partial_\lambda \varphi^\rho - (\partial_\rho \varphi^\rho)^3] \\ & - \frac{1}{2} \eta^{\mu\nu} \varphi^\rho \partial_\rho (\partial_\sigma \varphi^\lambda \partial_\lambda \varphi^\sigma - (\partial_\sigma \varphi^\sigma)^2) + \mathcal{O}(h^{4-p} \varphi^p). \end{aligned}$$

One obtains the various order $\Omega_n^{\mu\nu}$ by replacing $h^{\mu\nu}$ and φ^μ in $\Omega^{\mu\nu}[h, \varphi]$ by their PM expansion, and taking the coefficient of G^n , for $n \geq 2$.

In this way, we can express I_L and J_L in terms of M_L and S_L and the gauge moments. This expression can be inverted for practical use, and we find that at 4PN order, we only need to compute this algorithm up to cubic ($n = 3$) order. The main result reads

$$\begin{aligned}
 M_{ij} = & I_{ij} + \frac{4G}{c^5} \left[W^{(2)} I_{ij} - W^{(1)} I_{ij}^{(1)} \right] \\
 & + \frac{4G}{c^7} \left\{ \frac{4}{7} W_{a\langle i}^{(1)} I_{j\rangle a}^{(3)} + \frac{6}{7} W_{a\langle i}^{(4)} I_{j\rangle a}^{(4)} - \frac{1}{7} Y_{a\langle i}^{(3)} I_{j\rangle a}^{(3)} - Y_{a\langle i}^{(3)} I_{j\rangle a}^{(3)} - 2X I_{ij}^{(3)} \right. \\
 & \quad - \frac{5}{21} W_a^{(4)} I_{ija} + \frac{1}{63} W_a^{(3)} I_{ija}^{(1)} - \frac{25}{21} Y_a^{(3)} I_{ija} - \frac{22}{63} Y_a^{(2)} I_{ija}^{(1)} + \frac{5}{63} Y_a^{(1)} I_{ija}^{(2)} \\
 & \quad + 2W^{(3)} W_{ij} + 2W^{(2)} W_{ij}^{(1)} - \frac{4}{3} W_{\langle i}^{(3)} W_{j\rangle}^{(3)} + 2W^{(2)} Y_{ij} \\
 & \quad \left. - 4W_{\langle i} Y_{j\rangle}^{(2)} - \frac{1}{3} Z_{a\langle i}^{(3)} I_{j\rangle a} + Z_{a\langle i}^{(3)} I_{j\rangle a} - \frac{4}{9} W_a^{(3)} J_{\langle ij\rangle a} + \frac{4}{9} Y_a^{(2)} J_{\langle ij\rangle a} - \frac{8}{9} Y_a^{(1)} J_{\langle ij\rangle a}^{(1)} \right\} \\
 & + \frac{G^2 M}{c^8} \left\{ \frac{106}{21} M W_{ij}^{(3)} + \frac{1084}{105} M Y_{ij}^{(2)} \right. \\
 & \quad + \frac{17642}{945} W^{(3)} I_{ij} - \frac{7018}{315} W^{(2)} I_{ij}^{(1)} + \frac{806}{45} W^{(1)} I_{ij}^{(2)} - \frac{2098}{135} W I_{ij}^{(3)} \\
 & \quad \left. + 16 I_{ij} \int_0^{+\infty} d\tau \ln \left(\frac{c\tau}{2r_0} \right) W^{(4)}(t - \tau) - 16 I_{ij}^{(1)} \int_0^{+\infty} d\tau \ln \left(\frac{c\tau}{2r_0} \right) W^{(3)}(t - \tau) \right\} \\
 & + \mathcal{O} \left(\frac{1}{c^9} \right). \tag{III.15}
 \end{aligned}$$

Lower-order relations involving other canonical moments needed for the 4PN waveform can be found in Section III B of [161].

III.3 Adapting the formalism to d -dimensional regularization

In Section III.2, we have outlined the computation relating the canonical and source/gauge moments in standard three-dimensional gravity. This is perfectly correct in the context of three-dimensional Hadamard regularization, but this breaks down as soon as dimensional regularization is introduced. Indeed, since the source quadrupole moments are computed using the d -dimensional general MPM algorithm, it is crucial to compare to the d -dimensional canonical MPM algorithm. As will we see, the relation between source and canonical will *differ* from its three-dimensional counterpart. In this section, we give the different ingredients needed to obtain this d -dimensional relation.

Note that nonlocal terms cannot appear at 4.5PN, thus we only need the 4PN order of those relations to control the 4.5PN quasicircular flux. Indeed, in the three-dimensional case, nonlocal terms appear at cubic order, but fortunately, dimensional analysis shows that there cannot be any cubic or higher-order contribution entering at exactly 4.5PN order. As for quadratic interactions entering at 4.5PN order, an analysis of the structure of the integration formulas proves that these contributions are necessarily instantaneous, and thus play no role in the flux for quasicircular orbits at 4.5PN order.

III.3.1 Procedure in d dimensions

The solution of linearized, d dimensional, vacuum Einstein equations reads in harmonic gauge

$$h_{1,\text{gen}}^{\mu\nu} = k_1^{\mu\nu} + \partial\varphi_1^{\mu\nu}, \quad (\text{III.16})$$

where the canonical metric is parametrized by three types of source moments as [96]

$$k_1^{00} = -\frac{4}{c^2} \sum_{\ell \geq 0} \frac{(-)^\ell}{\ell!} \partial_L \tilde{\mathbb{I}}_L, \quad (\text{III.17a})$$

$$k_1^{0i} = \frac{4}{c^3} \sum_{\ell \geq 1} \frac{(-)^\ell}{\ell!} \left[\partial_{L-1} \tilde{\mathbb{I}}_{i|L-1}^{(1)} + \frac{\ell}{\ell+1} \partial_L \tilde{\mathbb{J}}_{i|L} \right], \quad (\text{III.17b})$$

$$k_1^{ij} = -\frac{4}{c^4} \sum_{\ell \geq 2} \frac{(-)^\ell}{\ell!} \left[\partial_{L-2} \tilde{\mathbb{I}}_{ij|L-2}^{(2)} + \frac{2\ell}{\ell+1} \partial_{L-1} \tilde{\mathbb{J}}_{(ij)L-1}^{(1)} + \frac{\ell-1}{\ell+1} \partial_L \tilde{\mathbb{K}}_{ij|L} \right], \quad (\text{III.17c})$$

and the gauge vector φ_1^μ , by four types of gauge moments

$$\varphi_1^0 = \frac{4}{c^3} \sum_{\ell \geq 0} \frac{(-)^\ell}{\ell!} \partial_L \tilde{\mathbb{W}}_L, \quad (\text{III.18a})$$

$$\varphi_1^i = -\frac{4}{c^4} \sum_{\ell \geq 0} \frac{(-)^\ell}{\ell!} \partial_{iL} \tilde{\mathbb{X}}_L - \frac{4}{c^4} \sum_{\ell \geq 1} \frac{(-)^\ell}{\ell!} \left[\partial_{L-1} \tilde{\mathbb{Y}}_{i|L-1} + \frac{\ell}{\ell+1} \partial_L \tilde{\mathbb{Z}}_{i|L} \right]. \quad (\text{III.18b})$$

The ‘‘tilde’’ over the moments has the same meaning as in (A.48), namely for an arbitrary function $F(t)$, we define

$$\tilde{F}(t, \mathbf{x}) \equiv \frac{\tilde{k}}{r^{d-2}} \int_1^{+\infty} dy \gamma_{\frac{1-d}{2}}(y) F^{(d)} \left(t' - \frac{ry}{c} \right),$$

where the (d) superscript is simply a reminder that we are dealing with a d -dimensional moment, and this notations will often be suppressed for the sake of lightness. Note the appearance of a new type of moment, $\mathbb{K}_{ij|L}^{(d)}$, which is a pure artifact of working in $d \neq 3$ dimensions, and will not enter our results. Beware that the conventions for the ordering of indices of $\tilde{\mathbb{J}}_{i|L}$, $\tilde{\mathbb{K}}_{ij|L}$ and $\tilde{\mathbb{Z}}_{i|L}$ are *not* those of [96], but rather an adaptation to d dimensions of the convention in [95]. Indeed, the Levi-Civita symbol is not well defined in arbitrary d dimensions, but the symmetry properties of the indices can be described by Young tableaux. This is nicely elaborated in [96], but an unfortunate convention was chosen for the ordering of the multi-index L , namely $L = i_1 \dots i_\ell$ in $\mathbb{I}_L^{(d)}$, but $L = i_\ell \dots i_1$ in $\mathbb{J}_{i|L}^{(d)}$ and $\mathbb{K}_{ij|L}^{(d)}$. This has created some confusion in subsequent works [95, 3] (see Footnote 3 of [6] for a clarification), but here we impose the convention that in all cases, we have the usual multi-index ordering $L = i_1 \dots i_\ell$. This is summarized by

$$\begin{aligned} \mathbb{I}_L^{(d)} &= \mathbb{I}_{i_1 \dots i_\ell}^{(d)} = \boxed{i_\ell \dots i_1}, & \mathbb{J}_{i|L}^{(d)} &= \mathbb{J}_{i|i_1 \dots i_\ell}^{(d)} = \begin{array}{|c|c|c|c|} \hline i_\ell & i_{\ell-1} & \dots & i_1 \\ \hline & & & i \\ \hline \end{array}, \\ \mathbb{K}_{ij|L}^{(d)} &= \mathbb{K}_{ij|i_1 \dots i_\ell}^{(d)} = \begin{array}{|c|c|c|c|} \hline i_\ell & i_{\ell-1} & i_{\ell-2} & \dots & i_1 \\ \hline j & i & & & \\ \hline \end{array}, \end{aligned} \quad (\text{III.19})$$

where we recall that for any given Young tableau, one constructs a tensor with the correct symmetry properties by (i) considering a generic tensor with the same indices as the Young tableau; (ii) for each line of the Young tableau, symmetrizing the tensor over the indices in the line; and (iii) for each column of the Young tableau, antisymmetrizing the tensor over the indices in the column.

Starting from this d -dimensional metric, we can follow the lines of Section III.2, that are independent of the dimension. Notably, the expressions of the nonlinear interaction terms $\Omega_n^{\mu\nu}$ and Δ_n^μ as functions of $h_{n,\text{can}}^{\mu\nu}$ and φ_n^μ are identical to the three-dimensional case. The main, but very light, difference concerns the two key quantities $X_n^{\mu\nu}$ and $Y_n^{\mu\nu}$. If the formal definition, (3.20) of [95], does not depend of the dimension, their explicit expression in terms of $\Omega_n^{\mu\nu}$ and Δ_n^μ is slightly changed. They read in d dimensions

$$X_n^{\mu\nu} = \text{FP}_{B=0} \square_{\text{ret}}^{-1} \left[B \left(\frac{r}{r_0} \right)^B \left(-\frac{B+d-2}{r^2} \Omega_n^{\mu\nu} - \frac{2}{r} \partial_r \Omega_n^{\mu\nu} \right) \right], \quad (\text{III.20a})$$

$$Y_n^{\mu\nu} = \text{FP}_{B=0} \square_{\text{ret}}^{-1} \left[B \left(\frac{r}{r_0} \right)^B \frac{n_k}{r} \left(-2\delta^{k(\mu} \Delta_n^{\nu)} + \eta^{\mu\nu} \Delta_n^k \right) \right]. \quad (\text{III.20b})$$

The remaining of the procedure described in Section III.2, and notably the extraction of the correction to the canonical moments described in Section III of [95], is left unchanged.

III.3.2 Integration techniques

Let us recall that the solution at an order $n \geq 2$ is composed of some corrections to the canonical moments, extracted from $X_n^{\mu\nu}$ and $Y_n^{\mu\nu}$ given by (III.20), as well as a gauge vector, denoted φ_n^μ . Thanks to the prefactor B in (III.20), and similarly to the three-dimensional case, the computation of $X_n^{\mu\nu}$ and $Y_n^{\mu\nu}$ only requires the knowledge of the near-zone ($r \rightarrow 0$) behavior of the functions $\Omega_n^{\mu\nu}$ and Δ_n^μ . This is because the finite part used conjointly with the B prefactor selects the $1/B$ part of the source, which is purely due to the ‘‘multipolar’’ singular behavior of the source when $r \rightarrow 0$. Using the near-zone expansion formulas in Section II.4 (see also Appendix A of [130] for more details), the computation boils down to the evaluation of integrals of the form

$$\mathcal{J} \equiv \text{FP}_{B=0} \square_{\text{ret}}^{-1} \left[\left(\frac{r}{r_0} \right)^B B^b \frac{\hat{n}_L}{r^{p+q\varepsilon}} H(t) \right], \quad (\text{III.21})$$

where $b = 1, 2$ and H can bear simple poles $\propto 1/\varepsilon$ and can indifferently be a local or nonlocal function of time. In this section, we set $r_0 = \ell_0 = 1$ for simplicity. The integrals (III.21) are nothing but the d -dimensional generalization of (4.13) of [95], with $a = 0$, since the case which includes a $\ln r$ is irrelevant for the present generalization.

Techniques to solve wavelike equations in d dimensions, with more general radial dependence than (III.21) but same multipolarity ℓ , have been developed in [162], see Section II therein. Importantly, no particular assumption about the presence of a prefactor B^b has been made in this work, as clear from its (2.6). We can therefore follow the lines of [162] to compute the integral (III.21). The solution h is the sum of two contributions, $h^<$ and $h^>$, involving integrals over the respective domains $\mathcal{D} = \{|\mathbf{x}'| < r\}$ and $\mathbb{R}^3 - \mathcal{D}$, as displayed explicitly in (2.8) and (2.10) of [162]. When a prefactor B^b is present, with $b = 1, 2$, the finite part of $h \equiv \mathcal{J}$ vanishes unless the integration produces a pole, which

may only occur when the integral diverges for $B = 0$. As the system is taken to be stationary in the past in the MPM algorithm, the integral yielding $h^>$ converges for $B = 0$ so that only $h^<$ can contribute to the finite part in (III.21). This finite part is evaluated explicitly and truncated at first order in ε , in order to cope with the case where $H(t)$ contains a pole, namely $H = \frac{1}{\varepsilon}H_{-1} + H_0 + \mathcal{O}(\varepsilon)$. After resummation, we obtain a retarded homogeneous solution of the wave equation [162]

$$\mathcal{J} = \hat{\partial}_L \tilde{G}(t, r) \equiv \hat{\partial}_L \left[\frac{\tilde{k}}{r^{d-2}} \int_1^{+\infty} dz \gamma_{\frac{1-d}{2}}(z) G(t - zr/c) \right]. \quad (\text{III.22})$$

Explicitly, we find that G is nonvanishing only if $b = 1$, $q = 1$ and $p - \ell - 3 = 2j$, where $j \in \mathbb{N}$. Under those very restrictive conditions, we have

$$G(t) = \frac{(-)^{\ell+1}}{(2j)!!(2j+2\ell+1)!!} \left[\left(1 + \varepsilon \ln \sqrt{\bar{q}} - \varepsilon \sum_{k=0}^{j+\ell} \frac{1}{2k+1} \right) H^{(2j)}(t) + \mathcal{O}(\varepsilon) \right], \quad (\text{III.23})$$

where we recall our notation $\bar{q} = 4\pi e^{\gamma_E}$ and the fact that H can bear a pole $1/\varepsilon$. With the previous formula at hand, we are ready to compute the quantities $X_n^{\mu\nu}$ and $Y_n^{\mu\nu}$ and, therefore, obtain the correction to the canonical moments to the n^{th} PM order. Since the result (III.23) is zero unless $q = 1$, we are able to eliminate many terms from the source for this computation solely based on the value of q .

As for the gauge vector φ_n^μ , it will be needed to compute the source term for the next PM order $n+1$, following the algorithmic procedure of [95]. It is decomposed as $\varphi_n^\mu = \phi_n^\mu + \psi_n^\mu$, where ψ_n^μ is also extracted from $X_n^{\mu\nu}$ and $Y_n^{\mu\nu}$, and ϕ_n^μ is obtained by the direct integration

$$\phi_n^\mu \equiv \text{FP}_{B=0} \square_{\text{ret}}^{-1} \left[\left(\frac{r}{r_0} \right)^B \Delta_n^\mu \right]. \quad (\text{III.24})$$

In d dimensions, it is in general not possible to find a convenient, explicit expression for ϕ_n^μ (see, however, Section II in [162]). Fortunately, we do not need here the full solution valid at any field point, but only the solution in the form of a near-zone expansion when $r \rightarrow 0$. Indeed, following the procedure of [95], it is the near-zone expansion of ϕ_n^μ , denoted $\bar{\phi}_n^\mu$, that needs to be inserted into the expressions of $\Omega_{n+1}^{\mu\nu}$ and Δ_{n+1}^μ sourcing the next order quantities $X_{n+1}^{\mu\nu}$ and $Y_{n+1}^{\mu\nu}$. The quantity $\bar{\phi}_n^\mu$ can be obtained directly from the near-zone expansion of the source term, i.e. $\bar{\Delta}_n^\mu$, plus a crucial homogeneous solution, itself in the form of a near-zone expansion, namely:

$$\bar{\phi}_n^\mu = \bar{\phi}_{\text{part } n}^\mu + \bar{\phi}_{\text{hom } n}^\mu. \quad (\text{III.25})$$

The particular solution is obtained by a formal iteration of inverse Laplace operators in d -dimensions, using the ‘‘Mathieu’’ formula, (B.26c) in [102]:

$$\bar{\phi}_{\text{part } n}^\mu = \text{FP}_{B=0} \sum_{k=0}^{\infty} \Delta^{-1-k} \left(\frac{\partial}{c \partial t} \right)^{2k} \left[\left(\frac{r}{r_0} \right)^B \bar{\Delta}_n^\mu \right]. \quad (\text{III.26})$$

The homogeneous solution $\bar{\phi}_{\text{hom } n}^\mu$ is given by (3.20) of [130], which in this case yields (with $c = 1$)

$$\bar{\phi}_{\text{hom } n}^\mu = - \sum_{\ell \geq 0} \sum_{j=0}^{\infty} \frac{\Delta^{-j} \hat{x}_L}{d+2\ell-2} \text{FP}_{B=0} \int_1^{+\infty} dz \gamma_{\frac{1-d}{2}-\ell}(z) \int_0^{+\infty} dr' r'^{-\ell+1+B} \hat{n}'_L \bar{\Delta}_{n,L}^{\mu(2j)}(t - zr', r'), \quad (\text{III.27})$$

where we have decomposed the source in its multipolar components as

$$\bar{\Delta}_n^\mu(t, \mathbf{x}) = \sum_{\ell \geq 0} \hat{n}_L \bar{\Delta}_{n,L}^\mu(t, r), \quad (\text{III.28})$$

and shortened the iterated inverse Laplacians as [99, 100, 130]

$$\Delta^{-j} \hat{x}_L = \frac{\Gamma\left(\frac{d}{2} + \ell\right)}{\Gamma\left(\frac{d}{2} + \ell + j\right)} \frac{r^{2j} \hat{x}_L}{2^{2j} j!}. \quad (\text{III.29})$$

In the particular case where the source term has a structure given by

$$\bar{\Delta}_n^\mu(t, r) = \sum_{\ell, k, p, q} \frac{\hat{n}_L}{r^{p+q\epsilon}} \int_1^{+\infty} dy y^k \gamma_{-1-\frac{\epsilon}{2}}(y) F_{\ell, k, p, q}(t - yr), \quad (\text{III.30})$$

where the $F_{\ell, k, p, q}(t)$'s can be arbitrary functions of time, explicit formulas are known for performing the integrations: see (3.24) and (D1) of [130] when $q = 2$, and the discussion in Section IV of [162] for other values of q . However, we will encounter source terms that exhibit more complicated structures, and for which no analogous formula is available. This is notably the case for sources involving two nonstatic multipolar moments, such as $\Delta_{\mathbf{W} \times \mathbf{I}_{ij}}^\mu$. Fortunately, we have shown that those more complicated homogeneous solutions do not contribute at 4PN order, see hereafter.

III.4 Relation at 4PN order using dimensional regularization

At the 4PN order, only quadratic and cubic interactions are allowed by dimensional analysis (and we recall that the 4.5PN sector vanishes in the quasicircular flux). Naturally, the moments allowed to enter those interactions do not depend on the space dimension. At quadratic order, the only integration formula needed was (III.23), and we find that quadratic contributions are actually identical in d and three dimensions, i.e. we recover the same ‘‘odd’’ corrections that enter at 2.5PN and 3.5PN order in the relation (III.15) between canonical and source/gauge moments.

More interestingly, the results for the cubic interactions arising at 4PN order greatly differ. Let us recall that three interactions enter the 4PN mass quadrupole moment at cubic order, namely $\mathbf{M} \times \mathbf{M} \times \mathbf{W}_{ij}$ and $\mathbf{M} \times \mathbf{M} \times \mathbf{Y}_{ij}$ (where only one moment is nonstatic), and most importantly $\mathbf{M} \times \mathbf{W} \times \mathbf{I}_{ij}$ (where the two moments \mathbf{W} and \mathbf{I}_{ij} are dynamical). For the first two interactions, it turns out that the source term entering the integral (III.21) bears $q \geq 2$. Indeed, it is composed of two interactions: $h_{\mathbf{M} \times \mathbf{M}} \times \varphi_{\mathbf{K}_{ij}}$ and $h_{\mathbf{M}} \times \varphi_{\mathbf{M} \times \mathbf{K}_{ij}}$, where \mathbf{K}_{ij} stands for either \mathbf{W}_{ij} or \mathbf{Y}_{ij} . As both $h_{\mathbf{M} \times \mathbf{M}}$ and $\varphi_{\mathbf{M} \times \mathbf{K}_{ij}}$ bear $q = 2$, the cubic sources will have $q \geq 2$, and thus cannot satisfy the $q = 1$ necessary condition to have a nonvanishing function $G(t)$ in (III.23). Therefore, the $\mathbf{M} \times \mathbf{M} \times \mathbf{W}_{ij}$ and $\mathbf{M} \times \mathbf{M} \times \mathbf{Y}_{ij}$ interactions do not contribute in d dimensions, as we have explicitly checked, using the method presented in the previous sections. This vanishing of the $\mathbf{M} \times \mathbf{M} \times \mathbf{W}_{ij}$ and $\mathbf{M} \times \mathbf{M} \times \mathbf{Y}_{ij}$ interactions in d dimensions contrasts with their explicit contributions in 3 dimensions; see (1.1) in [95].

As for the last cubic interaction, $\mathbf{M} \times \mathbf{W} \times \mathbf{I}_{ij}$, it turns out that the source term *does* have a $q = 1$ component, arising from the interaction $h_{\mathbf{I}_{ij}} \times \varphi_{\mathbf{M} \times \mathbf{W}}$. More precisely, the

term with $q = 1$ comes exclusively from the interaction between: (i) the homogeneous solution $\bar{\phi}_{\text{hom}2}$ at quadratic order arising from the interaction $M \times W$, i.e. $\bar{\phi}_{\text{hom}M \times W}$, which bears $q = 0$; and (ii) the $q = 1$ piece of the linear metric $h_{I_{ij}}$; see (II.17a). Note that, as it involves a quadratic interaction with only one dynamical moment, $\varphi_{M \times W}$ can be calculated using (III.30). By contrast, the homogeneous solution for the interaction $W \times I_{ij}$, i.e. $\bar{\phi}_{\text{hom}W \times I_{ij}}$, does not fulfill the $p - \ell - 3 \in 2\mathbb{N}$ condition, and as such, does not contribute to the result. This is fortunate, since it does not follow the structure (III.30), and we are unable to compute this term explicitly, at least with currently-known formulas. Another interesting observation is that the explicit computation of the $q = 1$ terms (the only ones that contribute to the final result) leads to the appearance of a pole. The relation between canonical and source/gauge moments at 4PN is thus a pure $M \times W \times I_{ij}$ interaction, which reads

$$\delta M_{ij} = -\frac{16G^2M}{c^8} \int_0^{+\infty} d\tau \left[L_\varepsilon(\tau) + \frac{1}{2} \right] \left(I_{ij}^{(1)}(t) W^{(3)}(t-\tau) - I_{ij}(t) W^{(4)}(t-\tau) \right) + \mathcal{O}\left(\frac{1}{c^9}\right). \quad (\text{III.31})$$

where we have used the notation introduced in (2.2) of [163],

$$L_\varepsilon(\tau) = -\frac{1}{2\varepsilon} + \ln\left(\frac{c\sqrt{q}}{2\ell_0}\right). \quad (\text{III.32})$$

This result is consistent with the corresponding one in three dimensions, where the interaction $M \times W \times I_{ij}$ gives rise to an ordinary tail integral at 4PN order: the coefficient of the pole in (III.31) matches the logarithmic structure of (1.1) in [95]. Moreover, the result (III.31), as well as the vanishing of the interactions $M \times M \times W_{ij}$ and $M \times M \times Y_{ij}$, have been checked by an independent method, using the techniques exposed in [162], to compute the *difference* between the d -dimensional and Hadamard regularization schemes in the MPM algorithm, and adding it to the three-dimensional result of [95].

Although surprising, we will see that this pole exactly cancels a pole arising in the source quadrupole (see Section VI.1.1), so we do not worry about it yet. Actually, we will see that the $1/c^8$ contributions (the pole, the tail and the instantaneous terms) will all vanish in the end result. The relation between the source and canonical quadrupole moments, using full dimensional regularization, reads

$$\begin{aligned} M_{ij}^{(d)} = & I_{ij}^{(d)} + \frac{4G}{c^5} \left[W^{(2)} I_{ij} - W^{(1)} I_{ij}^{(1)} \right] \\ & + \frac{4G}{c^7} \left\{ \frac{4}{7} W_{a\langle i} I_{j\rangle a}^{(3)} + \frac{6}{7} W_{a\langle i} I_{j\rangle a}^{(4)} - \frac{1}{7} Y_{a\langle i} I_{j\rangle a}^{(3)} - Y_{a\langle i} I_{j\rangle a}^{(3)} - 2X I_{ij}^{(3)} \right. \\ & - \frac{5}{21} W_a^{(4)} I_{ija} + \frac{1}{63} W_a^{(3)} I_{ija}^{(1)} - \frac{25}{21} Y_a^{(3)} I_{ija} - \frac{22}{63} Y_a^{(2)} I_{ija}^{(1)} + \frac{5}{63} Y_a^{(1)} I_{ija}^{(2)} \\ & + 2W^{(3)} W_{ij} + 2W^{(2)} W_{ij}^{(1)} - \frac{4}{3} W_{\langle i} W_{j\rangle}^{(3)} + 2W^{(2)} Y_{ij} - 4W_{\langle i} Y_{j\rangle}^{(2)} \\ & \left. + \epsilon_{ab\langle i} \left[\frac{1}{3} I_{j\rangle a} Z_b^{(3)} - I_{j\rangle a}^{(3)} Z_b + \frac{4}{9} J_{j\rangle a} W_b^{(3)} - \frac{4}{9} J_{j\rangle a} Y_b^{(2)} + \frac{8}{9} J_{j\rangle a}^{(1)} Y_b^{(1)} \right] \right\} \end{aligned}$$

$$\begin{aligned}
& - \frac{16G^2M}{c^8} \int_0^{+\infty} d\tau \left[L_\varepsilon(\tau) + \frac{1}{2} \right] \left(\mathbb{I}_{ij}^{(1)}(t) \mathbb{W}^{(3)}(t-\tau) - \mathbb{I}_{ij}(t) \mathbb{W}^{(4)}(t-\tau) \right) \\
& + \mathcal{O}\left(\frac{1}{c^9}\right) + \mathcal{O}(\varepsilon) , \tag{III.33}
\end{aligned}$$

where we omit the (d) superscript in the higher-order corrections since they are equal to their three-dimensional counterparts in the $d \rightarrow 3$ limit at this order. For the other multipole moments (essentially the mass octupole and the current quadrupole), such relations are purely quadratic and consist of odd parity terms, and so are not affected by dimensional-regularization corrections. They can be found in (VI.15) (see also Section III.B of [161]).

As we shall see in Chapter VI, the above procedure yields the correct perturbative limit for compact binaries on circular orbits, in agreement with first-order black hole perturbation theory. But notice that we can also retrieve the correct perturbative limit by using a simpler treatment of the IR (infrared) divergences of the source quadrupole moment, based on the Hadamard regularization in ordinary three dimensions, rather than in dimensional regularization. In this case, the correct relation between canonical and source moments is given by (III.15), instead of the d -dimensional result (III.33). We find that the cubic interactions $M \times M \times W_{ij}$ and $M \times M \times Y_{ij}$ present in three dimensions do contribute to the perturbative limit for circular orbits (note that W is zero in this case), but in such a way as to cancel the contribution due to the difference between the Hadamard and dimensional regularization schemes for the mass quadrupole source moment. It turns out then that we also recover the correct perturbative limit. However, in contrast to the dimensional regularization, we do not expect the Hadamard regularization in three dimensions to lead to a well-defined and fully unambiguous result beyond this limit.

Chapter IV

The radiative mass quadrupole and the tails of memory

This chapter is mainly based on [4].

In Chapter III, we have delved into the ways one can relate the source to the canonical moments. In this chapter, we will thus naturally be interested in the way we can relate the radiative moments to the canonical moments, up to 4PN order contributions in the waveform. This physically consists in studying the nonlinear effects in the propagation of GWs, which can be directly read off from the relation between the radiative quadrupole moments $\{\mathcal{U}_L, \mathcal{V}_L\}$ and the canonical moments $\{M_L, S_L\}$. The lowest-order nonlinear propagation piece is the tail effect, which is a quadratic coupling between the total mass-energy of spacetime and the mass quadrupole moment of the source, i.e. a $M \times M_{ij}$ interaction, see Fig. II.4a. This effect appears at 1.5PN order in the waveform beyond the Einstein quadrupole formula [164, 165, 166], and has the interesting feature on being nonlocal, i.e. its expression depends not only on the state of the binary at retarded time $t - r/c$, but also on the entire history of the binary's evolution. At 3PN order enters the *tail-of-tail* effect, which is a nonlocal cubic coupling between two masses and the quadrupole [91, 161], namely a $M^2 \times M_{ij}$ interaction. At quartic order there is the *tail-of-tail-of-tail* which arises at 4.5PN order and involves three masses and a quadrupole [167, 168]. Moreover, there are a wealth of *local* nonlinearities at 2.5PN and 3.5PN order, defined by the $S_i \times M_{ij}$ interaction.

The nonlinear memory effect, i.e. the permanent change in the wave amplitude after the passage of a GW burst, is due to the reradiation of quadrupole GWs by a linear quadrupole wave and corresponds to the coupling between two quadrupole moments [169, 170, 171, 172, 164, 173], namely a $M_{ij} \times M_{ij}$ nonlocal interaction, see Fig. II.4b. The oscillatory part of this effect enters at 2.5PN order in the waveform, but the genuine, secular memory effect is actually a zero-frequency Newtonian effect, and has not yet been detected experimentally, although it could be observed in the coming years with ground-based GW detectors [174, 175, 176].

In this chapter, we investigate two novel cubic effects that enter at 4PN order in the waveform: the *tail-of-memory*, due to a $M \times M_{ij} \times M_{ij}$ interaction, and the *spin-quadrupole*

tail, which corresponds to a $M \times S_i \times M_{ij}$ interaction. We shall compute these effects using the MPM formalism [92, 160] but with an important variant with respect to the version presented in Section III.1 and used in previous works such as [173, 91]: we shall construct the metric directly in *radiative* coordinates (see [177, 157, 158] for a general definition), following the method proposed in [160], instead of the usual construction in harmonic coordinates sketched in Section II.2.1 or III.1. The great advantage is that we shall avoid the appearance of the far-zone logarithms that plague harmonic coordinates. A minor disadvantage is that we shall have to apply a correction to account for the different expressions of the multipole moments in radiative and harmonic coordinates, but this correction is already known [3] and will be investigated in Chapter V.

In the present chapter, we shall thus *actually* study the $M \times \bar{M}_{ij} \times \bar{M}_{ij}$ interaction, where we introduce a new kind of “radiative” canonical moments $\{\bar{M}_L, \bar{S}_L\}$. How these relate to the usual “harmonic” canonical moments $\{M_L, S_L\}$ will be the topic of Chapter V. Physically, this 4PN effect can be viewed as a combination between the tails produced by the memory and the memory associated with the tail, which is the reason why we coin this effect “tails-of-memory”. From an effective field theory perspective, the tails-of-memory correspond to the three Feynman diagrams shown in Fig. IV.1, which we consider here only for illustrative purposes; see [178, 179, 180, 181] for their precise computational meaning within the EFT framework.

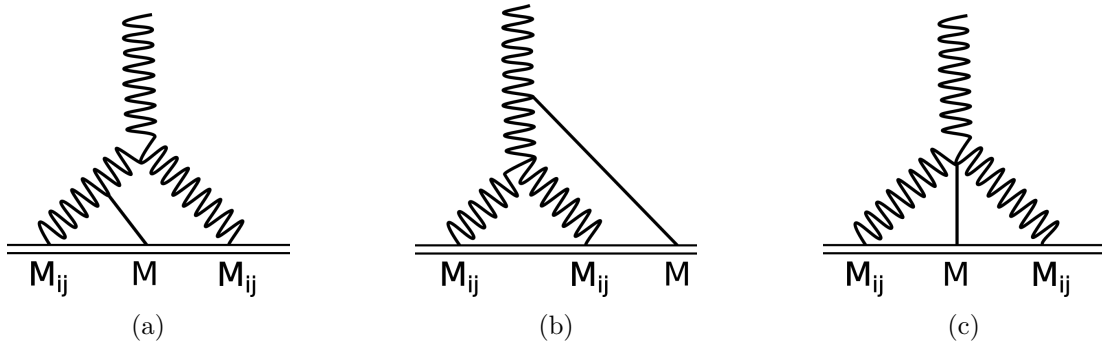


Fig. IV.1 Feynman diagrams associated to the tails-of-memory.

The tails-of-memory are not the only cubic effect to arise at 4PN order in the waveform: in this chapter, we also compute the only other cubic contribution that arises at 4PN order, namely the “spin-quadrupole tail” interaction $M \times S_i \times \bar{M}_{ij}$ between the constant total mass-energy, the constant total angular momentum and a quadrupole moment. As a sanity check, we will also study the well-known tails-of-tails in terms of these new radiative canonical moments. They correspond to a $M^2 \times \bar{M}_{ij}$ interaction that enters at 3PN order in the waveform.

Despite the fact that we are mainly motivated by the case of quasicircular compact binaries, the results presented in this chapter are valid for a generic isolated source described in terms of its multipole moments; the source could be a general hydrodynamical fluid with compact support, and it needs not even be a PN source. Moreover, this is a purely three-dimensional computation. In Section V.4, we will see how these results can be extended to generic d dimensions.

The plan of this chapter is as follows. In Section IV.1, we review the radiative version

of the MPM algorithm. In Section IV.2, we present our results for the radiative metric at quadratic order, and discuss the structure of the cubic source. In Sec IV.4, we present a novel integration technique for the cubic source, and apply it as a check to the known cases of tails-of-tails and tails. Then, in Section IV.5, we present our “raw” results in the form of complicated functionals of the multipole moments, along with a set of coefficients given by Table A.1. We then introduce a method of simplification, which drastically reduces the “raw” result, and allows for a test of our integration method. Finally, we present in Section IV.6 the final cubic radiative quadrupole moment (in its simplified form) in terms of the radiative canonical moments. Associated to this chapter, Appendix A.1 presents a practical method for extracting multipole moments, and Appendix A.7 presents a proof for the convergence of our final integrals.

IV.1 The radiative algorithm

We now describe a different MPM algorithm, proposed in [160], which directly builds the metric in a radiative coordinate system,

$$h_{\text{rad}}^{\mu\nu} = \sum_{n=1}^{\infty} G^n h_{\text{rad } n}^{\mu\nu}. \quad (\text{IV.1})$$

By radiative coordinate system, we mean a coordinate system whose retarded time coordinate, say $u \equiv t - r/c$, is a null coordinate, i.e. satisfies $g_{\text{rad}}^{\mu\nu} \partial_\mu u \partial_\nu u = 0$, or at least, becomes a null coordinate in the asymptotic limit $r \rightarrow +\infty$ with u held constant, i.e. in a neighbourhood of \mathcal{I}^+ [177, 157, 158]. In such class of coordinate systems, the metric admits a Bondi-like expansion at infinity, in simple powers of the inverse distance $1/r$, without any logarithms of r as would occur in harmonic coordinates.¹

Even at the linearized level, it is necessary to correct the harmonic coordinate metric in order to satisfy the requirement of asymptotically null retarded time. Consequently, the radiative MPM algorithm starts by performing a linear gauge transformation of the harmonic-coordinate metric defined by (III.2). A crucial point is that the multipole moments that parametrize the radiative algorithm will differ from their counterparts in the harmonic algorithm. In other words, they will have a different expression when expressed explicitly in terms of the source *via* a matched asymptotic expansion procedure. This observation leads us to define the radiative algorithm using different multipole moments, which we will note $\{\overline{M}_L, \overline{S}_L\}$. Thus, at linear order we pose

$$h_{\text{rad } 1}^{\mu\nu} = h_{\text{harm } 1}^{\mu\nu}[\overline{M}_L, \overline{S}_L] + \partial \xi_1^{\mu\nu}, \quad (\text{IV.2})$$

where $h_{\text{harm } 1}^{\mu\nu}[\overline{M}_L, \overline{S}_L]$ has exactly the same functional expression as in harmonic coordinates, given by (III.2), but is now computed with the set of moments $\{\overline{M}_L, \overline{S}_L\}$. The linear gauge transformation $\partial \xi_1^{\mu\nu} \equiv \partial^\mu \xi_1^\nu + \partial^\nu \xi_1^\mu - \eta^{\mu\nu} \partial_\rho \xi_1^\rho$ is defined by the gauge vector

$$\xi_1^\mu = \frac{2M}{c^2} \eta^{0\mu} \ln\left(\frac{r}{b_0}\right), \quad (\text{IV.3})$$

¹In this chapter, since we are iteratively constructing a coordinate system order by order, it is convenient to consider the coordinates as dummy variables and denote them by (t, r) . Even at the end, once we have obtained the full radiative metric, we shall continue to denote the radiative coordinates by the generic (t, r) , although it might be more appropriate to denote them by (T, R) , for instance.

where M is the mass monopole associated with the set of moments, b_0 denotes an arbitrary length scale, and we have $\eta^{0\mu} = (-1, \mathbf{0})$ with our signature. Since the gauge vector will only appear in the derivative form $\partial\xi_1^{\mu\nu}$ in the radiative algorithm, the unphysical scale b_0 will actually never enter the radiative metric. However we shall prove that b_0 is identical to the scale which is used in harmonic constructions of the metric when building the observable quantities at infinity, *via* the radiative multipole moments, see e.g. [161].

Note that although the two sets of multipole moments $\{M_L, S_L\}$ and $\{\bar{M}_L, \bar{S}_L\}$ will differ in general (as we shall compute explicitly below), the conserved mass monopole as well as the mass and current dipoles are in fact identical in both the harmonic and radiative constructions. In particular, we have $\bar{M} = M$ for the constant mass monopole, hence the slight abuse of notation in (IV.3).

The effect of this linear gauge transformation is to correct for the well-known logarithmic deviation of the retarded time in harmonic coordinates, with respect to the true spacetime characteristic or light cone. After the change of gauge, the coordinate $u = t - r/c$ coincides (asymptotically when $r \rightarrow +\infty$) with a null coordinate at the linearized level. The latter gauge transformation shifts the radiative metric away from harmonicity, since

$$\partial_\nu h_{\text{rad}1}^{\mu\nu} = \square \xi_1^\mu = \frac{2M}{c^2 r^2} \eta^{0\mu}. \quad (\text{IV.4})$$

Furthermore, one can easily show that, when $r \rightarrow +\infty$ with $u = t - r/c$ held constant, the leading order $1/r$ in the metric is cancelled in the combination

$$k_\mu k_\nu h_{\text{rad}1}^{\mu\nu} = \mathcal{O}\left(\frac{1}{r^2}\right), \quad (\text{IV.5})$$

where $k^\mu = \eta^{\mu\nu} k_\nu = (1, \mathbf{n})$ denotes the outgoing Minkowskian null vector.

Once the linearized solution is defined, the nonlinear corrections are readily obtained by injecting the PM expansion into the field equations and solving these equations iteratively over the PM order. At any order n , the general equation to solve is

$$\square h_{\text{rad}n}^{\mu\nu} - \partial H_{\text{rad}n}^{\mu\nu} = \Lambda_{\text{rad}n}^{\mu\nu}, \quad (\text{IV.6})$$

where $\Lambda_{\text{rad}n}^{\mu\nu} \equiv \Lambda^{\mu\nu}[h_{\text{rad}1}, \dots, h_{\text{rad}n-1}]$ is built out from previous iterations. Given any $n \geq 2$, let us recursively assume that: (i) we have obtained all the previous radiative PM coefficients $h_{\text{rad}m}^{\mu\nu}$ for any $m \leq n-1$; (ii) all of them admit an expansion as $r \rightarrow +\infty$ with $u = t - r/c$ held constant in simple positive powers of $1/r$ (as opposed to a polylogarithmic behavior); and (iii) all the previous coefficients satisfy the condition

$$\forall m \leq n-1, \quad k_\mu k_\nu h_{\text{rad}m}^{\mu\nu} = \mathcal{O}\left(\frac{1}{r^2}\right). \quad (\text{IV.7})$$

Note that the dominant piece when $r \rightarrow +\infty$ (with $t - r/c = \text{const}$) of the nonlinear source term at the n^{th} order will be of order $1/r^2$ and will only be made of quadratic products of $h_{\text{rad}m}^{\mu\nu}$ (since each of the $h_{\text{rad}m}^{\mu\nu}$'s behaves like $1/r$). Under our recursive assumptions, in particular the induction hypothesis (IV.7), and from the structure of the source term at quadratic order, see e.g. (24) in [14], one can prove that the n^{th} PM source term $\Lambda_{\text{rad}n}^{\mu\nu} \equiv \Lambda^{\mu\nu}[h_{\text{rad}m}; m \leq n-1]$ at leading order when $r \rightarrow +\infty$ is of the form (see [160] for details):

$$\Lambda_{\text{rad}n}^{\mu\nu} = \frac{k^\mu k^\nu}{r^2} \sigma_n(t - r/c, \mathbf{n}) + \mathcal{O}\left(\frac{1}{r^3}\right). \quad (\text{IV.8})$$

This is the form of the stress-energy tensor of massless particles, i.e., gravitons in our case, with σ_n being proportional to the n^{th} order contribution in the total power emitted by the massless waves.

From [92, 160, 164, 173] we know that logarithms in the asymptotic expansion when $r \rightarrow +\infty$ only arise due to the retarded integral of source terms that behave like $1/r^2$. Hence the dominant term written in (IV.8) is the only piece of $\Lambda_{\text{rad } n}^{\mu\nu}$ that can yield logarithms at order n . But now, thanks to the particular structure of this term, which follows from our recursive assumptions, we can gauge it away, thus constructing a coordinate system valid at the n^{th} PM order which avoids the appearance of logarithms. We find that an adequate gauge vector is

$$\xi_n^\mu = \text{FP}_{B=0} \square_{\text{ret}}^{-1} \left[\left(\frac{r}{r_0} \right)^B \frac{c k^\mu}{2r^2} \int_{-\infty}^{t-r/c} dv \sigma_n(v, \mathbf{n}) \right]. \quad (\text{IV.9})$$

With this choice of gauge vector, the logarithms that will be generated from the gauge transformation will cancel the logarithms coming from the retarded integral of the source term (IV.8), see [160] for more details. Hence, similarly to the corresponding steps in Section II.2.1 and III.1, we successively construct to the n^{th} order

$$\begin{aligned} u_{\text{rad } n}^{\mu\nu} &= \text{FP}_{B=0} \square_{\text{ret}}^{-1} \left[\left(\frac{r}{r_0} \right)^B \Lambda_{\text{rad } n}^{\mu\nu} \right], \\ w_{\text{rad } n}^\mu &= \partial_\nu u_{\text{rad } n}^{\mu\nu} = \text{FP}_{B=0} \square_{\text{ret}}^{-1} \left[B \left(\frac{r}{r_0} \right)^B \frac{n_i}{r} \Lambda_{\text{rad } n}^{\mu i} \right], \\ v_{\text{rad } n}^{\mu\nu} &= \mathcal{V}^{\mu\nu} [w_{\text{rad } n}]. \end{aligned} \quad (\text{IV.10})$$

Finally the n^{th} PM metric is defined by correcting for the new logarithms using the gauge transformation defined above:

$$h_{\text{rad } n}^{\mu\nu} = u_{\text{rad } n}^{\mu\nu} + v_{\text{rad } n}^{\mu\nu} + \partial \xi_n^{\mu\nu}. \quad (\text{IV.11})$$

By construction, the radiative metric obeys the nonharmonic gauge condition

$$H_{\text{rad } n}^\mu \equiv \partial_\mu h_{\text{rad } n}^{\mu\nu} = \square \xi_n^\mu = \frac{c k^\mu}{2r^2} \int_{-\infty}^{t-r/c} dv \sigma_n(v, \mathbf{n}), \quad (\text{IV.12})$$

and the Einstein field equations, as given by (II.5), are trivially satisfied to order n . The far-zone expansion of the full nonlinear radiative metric constructed by virtue of this procedure is free of any logarithms, and the retarded time $u = t - r/c$ in these coordinates tends asymptotically toward a null coordinate at future null infinity. The metric, as a general functional of the moments $\{\bar{M}_L, \bar{S}_L\}$, represents physically the most general solution to the vacuum field equations outside the isolated source.

We now set $c = G = 1$ for simplicity. Moreover, in the rest of the chapter, drop the ‘‘rad’’ subscript in the metric for notational simplicity. We will thus, in this chapter only, be referring to the radiative construction of the MPM metric as

$$h^{\mu\nu} = \sum_{n=1}^{\infty} G^n h_n^{\mu\nu}. \quad (\text{IV.13})$$

IV.2 Results at quadratic order

This chapter is devoted to computing from first principles the following three *cubic* nonlinear multipole interactions (i.e., $n = 3$ in the PM expansion):

$$M^2 \times \bar{M}_{ij}, \quad M \times S_i \times \bar{M}_{ij} \quad \text{and} \quad M \times \bar{M}_{ij} \times \bar{M}_{ij}.$$

The first one is known as the quadrupole “tail-of-tail” [91], the second one can be called the “spin-quadrupole tail” while the third one is the quadrupole “tail-of-memory” interaction which is our main goal here. To reach these goals we evidently need to control the following quadratic interactions:

$$M \times M, \quad M \times S_i \quad M \times \bar{M}_{ij}, \quad \bar{M}_{ij} \times \bar{M}_{ij} \quad \text{and} \quad S_i \times \bar{M}_{ij},$$

which we shall compute at any distance r from the source, while the cubic interactions will just be computed asymptotically when $r \rightarrow +\infty$.

The linear interactions M , S_i and \bar{M}_{ij} are read off directly from the linear metric (III.2), including the gauge transformation (IV.3):

$$h_M^{00} = -\frac{4M}{r}, \quad h_M^{0i} = -\frac{2Mn^i}{r}, \quad h_M^{ij} = 0, \quad (\text{IV.14a})$$

$$h_{S_i}^{00} = 0, \quad h_{S_i}^{0i} = \frac{2n^a S_{i|a}}{r^2}, \quad h_{S_i}^{ij} = 0, \quad (\text{IV.14b})$$

$$h_{\bar{M}_{ij}}^{00} = -2\partial_{ab} \left[\frac{\bar{M}_{ij}}{r} \right], \quad h_{\bar{M}_{ij}}^{0i} = -2\partial_a \left[\frac{\bar{M}_{ia}^{(1)}}{r} \right], \quad h_{\bar{M}_{ij}}^{ij} = -\frac{2\bar{M}_{ij}^{(2)}}{r}, \quad (\text{IV.14c})$$

where we have as usual posed $S_{i|j} \equiv \epsilon_{ija} S_a$.

In the harmonic construction, the quadratic metrics that are known are $M \times M$ [91], $M \times M_{ij}$ [164] and $M_{ij} \times M_{ij}$ [173]. We also know the $S_i \times M_{ij}$ metric asymptotically, to leading order in $1/r$ [173].

The radiative metrics for the static $M \times M$ and stationary $M \times S_i$ interactions, that are respectively part of the Schwarzschild and Kerr metrics (in “radiative” coordinates), read

$$h_{M \times M}^{00} = -\frac{3M^2}{r^2}, \quad h_{M \times M}^{0i} = 0, \quad h_{M \times M}^{ij} = -\frac{M^2 n^i n^j}{r^2}, \quad (\text{IV.15a})$$

$$h_{M \times S_i}^{00} = 0, \quad h_{M \times S_i}^{0i} = \frac{2M}{r^3} n_a S_{i|a}, \quad h_{M \times S_i}^{ij} = 0. \quad (\text{IV.15b})$$

Note that although $\bar{M} = M$ as we said, the $M \times M$ radiative metric differs from the harmonic one, see (2.7) in [91]. In the radiative construction, the $M \times \bar{M}_{ij}$ interaction reads [3]

$$h_{M \times \bar{M}_{ij}}^{00} = 8Mn_{ab} \int_1^{+\infty} dx \bar{Q}_2(x, r) \bar{M}_{ab}^{(4)}(t - rx) \\ + \frac{Mn_{ab}}{r} \left(\frac{118}{15} \bar{M}_{ab}^{(3)} + \frac{23}{5} r^{-1} \bar{M}_{ab}^{(2)} - \frac{117}{5} r^{-2} \bar{M}_{ab}^{(1)} - 21r^{-3} \bar{M}_{ab} \right), \quad (\text{IV.16a})$$

$$h_{M \times \bar{M}_{ij}}^{0i} = 8Mn_a \int_1^{+\infty} dx \bar{Q}_1(x, r) \bar{M}_{ai}^{(4)}(t - rx)$$

$$\begin{aligned}
& + \frac{Mn_{iab}}{r} \left[\overline{M}_{ab}^{(3)} + 3r^{-1}\overline{M}_{ab}^{(2)} - r^{-2}\overline{M}_{ab}^{(1)} \right] \\
& + \frac{Mn_a}{r} \left[\frac{43}{15}\overline{M}_{ai}^{(3)} - \frac{107}{15}r^{-1}\overline{M}_{ai}^{(2)} - 5r^{-2}\overline{M}_{ai}^{(1)} \right], \tag{IV.16b}
\end{aligned}$$

$$\begin{aligned}
h_{M \times \overline{M}_{ij}}^{ij} &= 8M \int_1^{+\infty} dx \overline{Q}_0(x, r) \overline{M}_{ij}^{(4)}(t - rx) \\
& + \frac{Mn_{ijab}}{r} \left[-\frac{1}{2}\overline{M}_{ab}^{(3)} - 3r^{-1}\overline{M}_{ab}^{(2)} - \frac{15}{2}r^{-2}\overline{M}_{ab}^{(1)} - \frac{15}{2}r^{-3}\overline{M}_{ab} \right] \\
& + \frac{M\delta_{ij}n_{ab}}{r} \left[-\frac{1}{2}\overline{M}_{ab}^{(3)} - 2r^{-1}\overline{M}_{ab}^{(2)} - \frac{1}{2}r^{-2}\overline{M}_{ab}^{(1)} - \frac{1}{2}r^{-3}\overline{M}_{ab} \right] \\
& + \frac{Mn_{a(i}}{r} \left[4\overline{M}_{j)a}^{(3)} + 6r^{-1}\overline{M}_{j)a}^{(2)} + 6r^{-2}\overline{M}_{j)a}^{(1)} + 6r^{-3}\overline{M}_{j)a} \right] \\
& + \frac{M}{r} \left[-\frac{107}{15}\overline{M}_{ij}^{(3)} - 4r^{-1}\overline{M}_{ij}^{(2)} - r^{-2}\overline{M}_{ij}^{(1)} - r^{-3}\overline{M}_{ij} \right]. \tag{IV.16c}
\end{aligned}$$

The tail terms in (IV.16) involve the modified Legendre function

$$\overline{Q}_m(x, r) \equiv Q_m(x) - \frac{1}{2}P_m(x) \ln \left(\frac{r}{r_0} \right), \tag{IV.17}$$

which is constructed using the usual Legendre polynomial $P_m(x)$ and the usual Legendre function of second kind (with branch cut on $]-\infty, 1]$), namely

$$Q_m(x) \equiv \frac{1}{2} \int_{-1}^1 dy \frac{P_m(y)}{x-y} = \frac{1}{2}P_m(x) \ln \left(\frac{x+1}{x-1} \right) - \sum_{j=0}^m \frac{1}{m} P_{m-j}(x) P_{j-1}(x). \tag{IV.18}$$

It was shown in [3] that the combination (IV.17) does not produce any far-zone logarithms in the radiative metric, although the tail term is still nonlocal in time and depends on the constant scale r_0 . As shown in [3] the metric (IV.16) differs from its counterpart in harmonic coordinates (given in Appendix B in [164]) by a coordinate transformation and a redefinition of the quadrupole moment [3], see (V.41).

We first computed the radiative metric for the $S_i \times \overline{M}_{ij}$ interaction in [4]. It reads

$$h_{S_i \times \overline{M}_{ij}}^{00} = n_{ad} S_{a|b} \left(-\frac{4}{3}r^{-1}\overline{M}_{bd}^{(4)} - 4r^{-2}\overline{M}_{bd}^{(3)} - 6r^{-3}\overline{M}_{bd}^{(2)} - 6r^{-4}\overline{M}_{bd}^{(1)} \right), \tag{IV.19a}$$

$$\begin{aligned}
h_{S_i \times \overline{M}_{ij}}^{0i} &= n_{acd} S_{i|a} \left(-\frac{5}{6}r^{-1}\overline{M}_{cd}^{(4)} - 5r^{-2}\overline{M}_{cd}^{(3)} - \frac{23}{2}r^{-3}\overline{M}_{cd}^{(2)} - \frac{3}{2}r^{-4}\overline{M}_{cd}^{(1)} - \frac{3}{2}r^{-5}\overline{M}_{cd} \right) \\
& + n_{iad} S_{a|b} \left(-3r^{-3}\overline{M}_{bd}^{(2)} - 9r^{-4}\overline{M}_{bd}^{(1)} - 9r^{-5}\overline{M}_{bd} \right) \\
& + n_a S_{a|b} \left(-\frac{4}{3}r^{-1}\overline{M}_{ib}^{(4)} - \frac{4}{3}r^{-2}\overline{M}_{ib}^{(3)} + r^{-3}\overline{M}_{ib}^{(2)} + r^{-4}\overline{M}_{ib}^{(1)} + r^{-5}\overline{M}_{ib} \right) \\
& + n_a S_{i|b} \left(\frac{5}{3}r^{-2}\overline{M}_{ab}^{(3)} + 5r^{-3}\overline{M}_{ab}^{(2)} + 4r^{-4}\overline{M}_{ab}^{(1)} + 4r^{-5}\overline{M}_{ab} \right), \tag{IV.19b}
\end{aligned}$$

$$\begin{aligned}
h_{S_i \times \overline{M}_{ij}}^{ij} &= n_{ijad} S_{a|b} \left(\frac{1}{3}r^{-1}\overline{M}_{bd}^{(4)} + \frac{10}{3}r^{-2}\overline{M}_{bd}^{(3)} + 9r^{-3}\overline{M}_{bd}^{(2)} + 9r^{-4}\overline{M}_{bd}^{(1)} \right) \\
& + n_{acd(i} S_{j)|a} \left(\frac{1}{3}r^{-1}\overline{M}_{cd}^{(4)} + \frac{10}{3}r^{-2}\overline{M}_{cd}^{(3)} + 9r^{-3}\overline{M}_{cd}^{(2)} + 9r^{-4}\overline{M}_{cd}^{(1)} \right)
\end{aligned}$$

$$\begin{aligned}
& + \delta_{ij} n_{ad} S_{a|b} \left(r^{-1} \overline{M}_{bd}^{(4)} + \frac{8}{3} r^{-2} \overline{M}_{bd}^{(3)} - r^{-3} \overline{M}_{bd}^{(2)} - r^{-4} \overline{M}_{bd}^{(1)} \right) \\
& + n_{c(i} S_{j)|a} \left(-\frac{4}{3} r^{-2} \overline{M}_{ac}^{(3)} - 4r^{-3} \overline{M}_{ac}^{(2)} - 4r^{-4} \overline{M}_{ac}^{(1)} \right) \\
& + n_{a(i} S_{a|b} \left(-\frac{8}{3} r^{-1} \overline{M}_{j)b}^{(4)} - \frac{26}{3} r^{-2} \overline{M}_{j)b}^{(3)} - 6r^{-3} \overline{M}_{j)b}^{(2)} - 6r^{-4} \overline{M}_{j)b}^{(1)} \right) \\
& + n_{ac} S_{a|(i} \left(2r^{-1} \overline{M}_{j)b}^{(4)} + \frac{20}{3} r^{-2} \overline{M}_{j)b}^{(3)} + 2r^{-3} \overline{M}_{j)b}^{(2)} + 2r^{-4} \overline{M}_{j)b}^{(1)} \right) \\
& + S_{a|(i} \left(-\frac{14}{3} r^{-2} \overline{M}_{j)a}^{(3)} - 2r^{-3} \overline{M}_{j)a}^{(2)} - 2r^{-4} \overline{M}_{j)a}^{(1)} \right). \tag{IV.19c}
\end{aligned}$$

Note that contrary to the $M \times \overline{M}_{ij}$ interaction in (IV.16), the $S_i \times \overline{M}_{ij}$ interaction is purely “instantaneous” or local, as it does not contain tail integrals, hence its nickname, the “failed tail”. Note also that in the particular case of the $S_i \times \overline{M}_{ij}$ interaction, the expressions for the radiative and harmonic metrics as functionals of their respective canonical moments are identical, and given by (IV.19).

The radiative metric for the interaction $\overline{M}_{ij} \times \overline{M}_{ij}$ is too lengthy to be presented, but it can be found in the Supplementary Material of [4]. It is constructed following (IV.11) as

$$h_{\overline{M}_{ij} \times \overline{M}_{ij}}^{\mu\nu} = u_{\overline{M}_{ij} \times \overline{M}_{ij}}^{\mu\nu} + v_{\overline{M}_{ij} \times \overline{M}_{ij}}^{\mu\nu} + \partial \xi_{\overline{M}_{ij} \times \overline{M}_{ij}}^{\mu\nu}. \tag{IV.20}$$

The gauge vector is defined by (IV.8)–(IV.9) and we have explicitly in this case [3]

$$\sigma_{\overline{M}_{ij} \times \overline{M}_{ij}} = \hat{n}_{ijab} \overline{M}_{ij}^{(3)} \overline{M}_{ab}^{(3)} - \frac{24}{7} \hat{n}_{ij} \overline{M}_{ia}^{(3)} \overline{M}_{ja}^{(3)} + \frac{4}{5} \overline{M}_{ab}^{(3)} \overline{M}_{ab}^{(3)}, \tag{IV.21a}$$

and therefore

$$\begin{aligned}
\xi_{\overline{M}_{ij} \times \overline{M}_{ij}}^0 = \int_{-\infty}^u dv \int_1^{+\infty} dx \left\{ -\frac{1}{2} \hat{n}_{ijab} Q_4(x) \overline{M}_{ij}^{(3)} \overline{M}_{ab}^{(3)} + \frac{12}{7} \hat{n}_{ij} Q_2(x) \overline{M}_{ia}^{(3)} \overline{M}_{ja}^{(3)} \right. \\
\left. - \frac{2}{5} Q_0(x) \overline{M}_{ab}^{(3)} \overline{M}_{ab}^{(3)} \right\}, \tag{IV.21b}
\end{aligned}$$

$$\begin{aligned}
\xi_{\overline{M}_{ij} \times \overline{M}_{ij}}^i = \int_{-\infty}^u dv \int_1^{+\infty} dx \left\{ -\frac{1}{2} \hat{n}_{iabkl} Q_5(x) \overline{M}_{ab}^{(3)} \overline{M}_{kl}^{(3)} + \frac{16}{9} \hat{n}_{iab} Q_3(x) \overline{M}_{ak}^{(3)} \overline{M}_{kb}^{(3)} \right. \\
\left. - \frac{2}{9} \hat{n}_{abk} Q_3(x) \overline{M}_{ai}^{(3)} \overline{M}_{bk}^{(3)} - \frac{22}{35} \hat{n}_i Q_1(x) \overline{M}_{ab}^{(3)} \overline{M}_{ab}^{(3)} \right. \\
\left. + \frac{24}{35} \hat{n}_a Q_1(x) \overline{M}_{ik}^{(3)} \overline{M}_{ka}^{(3)} \right\}. \tag{IV.21c}
\end{aligned}$$

Crucially, thanks to this gauge transformation the latter quadratic metric is free of any far-zone logarithms and so will be the cubic source built out of it. This allows us to take the dominant asymptotic behavior of the quadratic radiative metric as $r \rightarrow +\infty$, and extract after standard TT projection of the spatial metric the associated radiative

quadrupole moments (where we reintroduce c and G):

$$\mathcal{U}_{ij}^{M \times \bar{M}_{ij}} = \frac{2GM}{c^3} \int_0^{+\infty} d\tau \left[\ln \left(\frac{\tau}{2r_0} \right) + \frac{107}{60} \right] \bar{M}_{ij}^{(4)}(u - \tau), \quad (\text{IV.22a})$$

$$\mathcal{U}_{ij}^{S_i \times \bar{M}_{ij}} = -\frac{G}{3c^5} S_{a|i} \bar{M}_{j)a}^{(4)}, \quad (\text{IV.22b})$$

$$\mathcal{U}_{ij}^{\bar{M}_{ij} \times \bar{M}_{ij}} = -\frac{2G}{7c^5} \left[\int_0^{+\infty} d\tau \bar{M}_{a(i} \bar{M}_{j)a}^{(3)}(u - \tau) + \bar{M}_{a(i} \bar{M}_{j)a}^{(2)} + \frac{5}{2} \bar{M}_{a(i} \bar{M}_{j)a}^{(1)} \bar{M}_{j)a}^{(4)} - \frac{1}{2} \bar{M}_{a(i} \bar{M}_{j)a}^{(5)} \right]. \quad (\text{IV.22c})$$

The expression for the tail differs from (but is physically equivalent [3] to) its counterpart in the harmonic construction [164]; the expressions for $S_i \times \bar{M}_{ij}$ and $\bar{M}_{ij} \times \bar{M}_{ij}$ are the same in both radiative and harmonic constructions, see the Appendix B of [173]. In particular the first term in (IV.22c) is the usual memory effect (which contains both AC and DC contributions) at the dominant 2.5PN order.

IV.3 Structure of the cubic source

The general equation we need to solve, (IV.6), reads at cubic order

$$\square h_3^{\mu\nu} - \partial H_3^{\mu\nu} = \Lambda_3^{\mu\nu}[h_1, h_2] = N^{\mu\nu}[h_1, h_2] + N^{\mu\nu}[h_2, h_1] + M^{\mu\nu}[h_1, h_1, h_1], \quad (\text{IV.23})$$

where we recall the definition $H_3^\mu \equiv \partial_\nu h_3^{\mu\nu}$. The quadratic functional $N^{\mu\nu}[h, h']$ is given in (II.9) and the cubic functional $M^{\mu\nu}[h, h, h]$ is given by (1.4) of [91]. The source term for the cubic interaction $M \times \bar{M}_{ij} \times \bar{M}_{ij}$ reads, with obvious notation,

$$\Lambda_{M \times \bar{M}_{ij} \times \bar{M}_{ij}}^{\mu\nu} = N_{\bar{M}_{ij} \times (M \times \bar{M}_{ij})}^{\mu\nu} + N_{M \times (\bar{M}_{ij} \times \bar{M}_{ij})}^{\mu\nu} + M_{M \times \bar{M}_{ij} \times \bar{M}_{ij}}^{\mu\nu}. \quad (\text{IV.24})$$

These three terms have the following structure:

1. $\bar{M}_{ij} \times (M \times \bar{M}_{ij})$ corresponds to a linearized quadrupolar wave interacting with a quadratic tail, as illustrated by the first Feynman diagram in Fig. IV.1a. Apart from simple instantaneous terms, this source contains terms whose structure reads

$$\sim \frac{\hat{n}_L}{r^{5-p-q}} M \bar{M}_{ab}^{(p)}(t-r) \int_1^{+\infty} dx \bar{Q}_m(x) \bar{M}_{cd}^{(4+q)}(t-rx), \quad (\text{IV.25})$$

where $(p, q) \in [0, 2]$, and where the index structure and the possible presence of Kronecker deltas are disregarded. These terms are novel (with respect to ordinary tails), and their integration will be the focus of Section IV.4.2. For integration purposes, the modified Legendre function $\bar{Q}_m(x)$ will be decomposed according to (IV.17): this will yield similar integrals over the standard Legendre function $Q_m(x)$, augmented by instantaneous terms involving $\ln(r/r_0)$.

2. $M \times (\bar{M}_{ij} \times \bar{M}_{ij})$ corresponds to a quadrupole-quadrupole memory-like wave scattering off the curvature generated by the total mass-energy, and is illustrated by Fig. IV.1b. This source contains terms with the following structure:

$$\sim \frac{\hat{n}_L}{r^{3-p-q}} M \int_1^{+\infty} dx \bar{Q}_m(x) (\bar{M}_{ab}^{(3+p)} \bar{M}_{cd}^{(3+q)})(t-rx), \quad (\text{IV.26a})$$

where $0 \leq p + q \leq 2$, which have the same structure as ordinary tails, as well as memory-like terms

$$\sim \frac{\hat{n}_L}{r^4} M \int_0^{+\infty} d\tau (\overline{M}_{ab}^{(3)} \overline{M}_{cd}^{(3)})(t - r - \tau). \quad (\text{IV.26b})$$

The modified Legendre function will again be decomposed according to (IV.17), and all these terms will be treated in Section IV.4.3.

3. $M \times \overline{M}_{ij} \times \overline{M}_{ij}$ properly speaking, illustrated by Fig. IV.1c. This piece only leads to instantaneous terms, which are easy to integrate.

A similar analysis can be done for the other two cubic interactions $M \times M \times \overline{M}_{ij}$ and $M \times S_i \times \overline{M}_{ij}$, which only contain tail-like and instantaneous terms. Note that, thanks to the radiative construction, we can see that the $r \rightarrow +\infty$ expansion of the cubic source does not contain any logarithms of r . We have explicitly verified that the novel source terms $M \times \overline{M}_{ij} \times \overline{M}_{ij}$ and $M \times S_i \times \overline{M}_{ij}$ are divergenceless, i.e.

$$\partial_\nu \Lambda_{M \times \overline{M}_{ij} \times \overline{M}_{ij}}^{\mu\nu} = 0 \quad \text{and} \quad \partial_\nu \Lambda_{M \times S_i \times \overline{M}_{ij}}^{\mu\nu} = 0, \quad (\text{IV.27})$$

which is a strong test, since it would generally fail if a single coefficient were incorrect. However, because of its length, we cannot present the cubic source. Instead, we refer to the Supplementary Material of [4].

IV.4 Solution of the wave equation at cubic order

IV.4.1 General multipolar solution

To solve the problem of tails-of-memory we need to develop new techniques for integrating the wave equation when the source term is a complicated nonlocal functional of the moments such as (IV.25). We first consider a general wave equation whose source term admits a definite multipolarity ℓ in STF guise, which reads in general

$$\square \Psi_L = \hat{n}_L S(r, t - r), \quad (\text{IV.28})$$

where $S(r, u)$ is an arbitrary function of $r = |\mathbf{x}|$ and $u = t - r$ that verifies straightforward smoothness properties, and tends sufficiently rapidly to zero when $r \rightarrow 0$, namely

$$S(r, u) = \mathcal{O}(r^{\ell+5}) \quad (\text{when } r \rightarrow 0 \text{ with } u \text{ or } t \text{ held fixed}). \quad (\text{IV.29})$$

See Theorem 6.1 of [92] for more details on the required conditions we impose. Under these conditions, we know how to solve the wave equation (IV.28). We first define

$$R_\alpha(\rho, s) \equiv \rho^\ell \int_\alpha^\rho d\lambda \frac{(\rho - \lambda)^\ell}{\ell!} \left(\frac{2}{\lambda}\right)^{\ell-1} S(\lambda, s), \quad (\text{IV.30})$$

where α is an arbitrary constant. Then the solution of (IV.28) can be written as

$$\Psi_L = \int_{-\infty}^{t-r} ds \hat{\partial}_L \left[\frac{R_\alpha\left(\frac{t-r-s}{2}, s\right) - R_\alpha\left(\frac{t+r-s}{2}, s\right)}{r} \right], \quad (\text{IV.31})$$

see (6.4) in [92]. This solution is the unique retarded solution of (IV.28), thus satisfying the no-incoming radiation condition when $r \rightarrow +\infty$ with $t+r = \text{const}$. Furthermore it is independent from the constant α . To see this we consider separately the two terms in (IV.31), which we decompose as $\Psi_L = \Psi_L^{[1]} + \Psi_L^{[2]}$ with

$$\Psi_L^{[1]} \equiv \int_{-\infty}^{t-r} ds \hat{\partial}_L \left[\frac{1}{r} R_\alpha \left(\frac{t-r-s}{2}, s \right) \right], \quad (\text{IV.32a})$$

$$\Psi_L^{[2]} \equiv - \int_{-\infty}^{t-r} ds \hat{\partial}_L \left[\frac{1}{r} R_\alpha \left(\frac{t+r-s}{2}, s \right) \right]. \quad (\text{IV.32b})$$

We see that $\Psi_L^{[1]}$ is a retarded homogeneous solution of the wave equation, $\square \Psi_L^{[1]} = 0$, but that $\Psi_L^{[2]}$ has a unwieldy dependency on the advanced time $v = t+r$, and represents a particular retarded solution of the wave equation, i.e., $\square \Psi_L^{[2]} = \hat{n}_L S(r, t-r)$. However note that the latter two wave equations satisfied by $\Psi_L^{[1]}$ and $\Psi_L^{[2]}$ separately are valid only in the sense of ordinary functions; only the wave equation (IV.28) for the full solution Ψ_L is satisfied in the sense of distributions.²

Plugging (IV.30) into (IV.32) we obtain

$$\Psi_L^{[1]} = \frac{1}{\ell!} \int_{-\infty}^{t-r} ds \hat{\partial}_L \left[\frac{1}{r} \int_\alpha^{\frac{t-r-s}{2}} d\lambda \left(\frac{t-r-s}{2} \right)^\ell \left(\frac{t-r-s}{2} - \lambda \right)^\ell \left(\frac{2}{\lambda} \right)^{\ell-1} S(\lambda, s) \right], \quad (\text{IV.33a})$$

$$\Psi_L^{[2]} = -\frac{1}{\ell!} \int_{-\infty}^{t-r} ds \hat{\partial}_L \left[\frac{1}{r} \int_\alpha^{\frac{t+r-s}{2}} d\lambda \left(\frac{t+r-s}{2} \right)^\ell \left(\frac{t+r-s}{2} - \lambda \right)^\ell \left(\frac{2}{\lambda} \right)^{\ell-1} S(\lambda, s) \right]. \quad (\text{IV.33b})$$

Next we remark that in both terms (IV.33), one can commute the partial differential operator $\hat{\partial}_L$ with the integral over λ , since the terms coming from the differentiation of the bound of the integral, i.e. $(t \pm r - s)/2$, have to be evaluated at $s = t \pm r$ and will clearly vanish. Hence we can rewrite

$$\Psi_L^{[1]} = \frac{2^{\ell-1}}{\ell!} \int_{-\infty}^{t-r} ds \int_\alpha^{\frac{t-r-s}{2}} d\lambda \lambda^{-\ell+1} S(\lambda, s) \hat{\partial}_L \left[\frac{1}{r} \left(\frac{t-r-s}{2} \right)^\ell \left(\frac{t-r-s}{2} - \lambda \right)^\ell \right], \quad (\text{IV.34a})$$

$$\Psi_L^{[2]} = -\frac{2^{\ell-1}}{\ell!} \int_{-\infty}^{t-r} ds \int_\alpha^{\frac{t+r-s}{2}} d\lambda \lambda^{-\ell+1} S(\lambda, s) \hat{\partial}_L \left[\frac{1}{r} \left(\frac{t+r-s}{2} \right)^\ell \left(\frac{t+r-s}{2} - \lambda \right)^\ell \right]. \quad (\text{IV.34b})$$

Then, using (A36) of [92] we know that

$$\hat{\partial}_L \left[\frac{1}{r} \left(\frac{t-r-s}{2} \right)^\ell \left(\frac{t-r-s}{2} - \lambda \right)^\ell \right] = \hat{\partial}_L \left[\frac{1}{r} \left(\frac{t+r-s}{2} \right)^\ell \left(\frac{t+r-s}{2} - \lambda \right)^\ell \right], \quad (\text{IV.35})$$

²If f is a smooth function, then $\square [r^{-1}f(t-r)] = -4\pi f(t)\delta^{(3)}(\mathbf{x})$ in the sense of distributions, but $\square [r^{-1}f(t-r)] = 0$ in the sense of ordinary functions.

so that actually the two contributions in (IV.34) can be merged together in the full solution, and we arrive at the elegant alternative form

$$\Psi_L = -\frac{2^{\ell-1}}{\ell!} \int_{-\infty}^{t-r} ds \int_{\frac{t-r-s}{2}}^{\frac{t+r-s}{2}} d\lambda \lambda^{-\ell+1} S(\lambda, s) \hat{\partial}_L \left[\frac{1}{r} \left(\frac{t-r-s}{2} \right)^\ell \left(\frac{t-r-s}{2} - \lambda \right)^\ell \right]. \quad (\text{IV.36})$$

This form is clearly independent of any choice for the constant α . It was also derived in a different way, using a direct multipolar expansion of the Green function of the d'Alembert operator, in Appendix D of [92].

By using (A35a) of [92] we can expand explicitly the last factor containing the multipolar derivative operator. Introducing the following coefficients, which will become ubiquitous in our practical computations,

$$C_{ij}^\ell \equiv \frac{(\ell+i)!(\ell+j)!}{2^{i+j} i! j! (\ell-i)! (\ell-j)! (i+j)!} \quad (\text{for } 0 \leq i, j \leq \ell), \quad (\text{IV.37})$$

we conveniently write the general solution as

$$\Psi_L = -\frac{\hat{n}_L}{2} \sum_{i=0}^{\ell} \sum_{j=0}^{\ell} \frac{(-)^j C_{ij}^\ell}{r^{i+1}} \int_0^{+\infty} d\rho \rho^{i+j} \int_{\frac{\rho}{2}}^{\frac{\rho}{2}+r} d\lambda \lambda^{-j+1} S(\lambda, u-\rho). \quad (\text{IV.38})$$

Note that although the coefficient C_{ij}^ℓ has been defined to be symmetric in i and j , these two indices play a different role. In particular the index i rules the behavior of the solution at infinity, when $r \rightarrow +\infty$ with $t-r = \text{const}$, depending on the falloff properties at infinity of the integrals in (IV.38). A useful fact is that for $i=0$, the coefficient is straightforwardly linked to the j^{th} derivative of the usual Legendre polynomial evaluated at 1, namely

$$C_{0j}^\ell = \frac{1}{j!} P_\ell^{(j)}(1). \quad (\text{IV.39})$$

In addition to the general formula (IV.38), we also have an independent result which directly provides the leading behavior of the solution at infinity, when $r \rightarrow +\infty$ with $u = \text{const}$, depending on the falloff properties of the source $S(r, u)$. Namely, suppose that $S(r, u)$ has the following asymptotic behavior at infinity:

$$S(r, u) = \mathcal{O}\left(\frac{1}{r^3}\right) \quad (\text{when } r \rightarrow +\infty \text{ with } u \text{ fixed}). \quad (\text{IV.40})$$

More precisely the rigorous falloff conditions of the source term are given in Lemma 7.2 of [92]. Then the corresponding solution will behave dominantly like $1/r$ at infinity, with leading term explicitly given by

$$\Psi_L = \frac{(-)^\ell \hat{n}_L}{2^\ell r} \int_{-\infty}^{t-r} ds R_\infty^{(\ell)}\left(\frac{t-r-s}{2}, s\right) + \mathcal{O}\left(\frac{1}{r^2}\right). \quad (\text{IV.41})$$

Here the function $R_\infty(\rho, s)$ is defined by (IV.30) with the explicit choice $\alpha = +\infty$, and the superscript (ℓ) means ℓ partial derivatives with respect to ρ . We shall use the latter result as a consistency check of the derivation of the leading behavior of the solution at infinity.

IV.4.2 Application to tails-of-memory

We apply the general formalism of Section IV.4.1 to the cubic iteration and specifically to the tails-of-memory. In this case the main problem we face is to find the solution of the wave equation (IV.28) when the source term takes the form (IV.25). For ease of notation we denote the two quadrupole moments by arbitrary time-varying functions $F(u)$ and $G(u)$, and consider the source term (multiplying the STF harmonics \hat{n}_L):

$$S_{k,m}^B(r, t-r) \equiv \left(\frac{r}{r_0}\right)^B r^{-k} G(t-r) \int_1^{+\infty} dx Q_m(x) F(t-rx), \quad (\text{IV.42})$$

where $k \geq 1$ and $m \geq 0$ are integers, and the Legendre function of the second kind $Q_m(x)$ is given by (IV.18), as was obtained from $\bar{Q}_m(x)$ using the relation (IV.17). The source (IV.42) is more complex than the one required for the computation of tails and tails-of-tails. In the latter case, the function $G(u)$ is actually constant (i.e. the constant total mass-energy M), and we shall recover and extend the results found in [91, 167]. In the case where $G(u)$ and $F(u)$ are not constant the results we shall derive are new.

In (IV.42) we have multiplied the source by the regularization factor $(r/r_0)^B$, where $B \in \mathbb{C}$ and r_0 is an arbitrary constant, see (IV.10). Very importantly, this permits to ensure after analytic continuation in $B \in \mathbb{C}$ that the falloff condition (IV.40) is always satisfied. Applying the finite part, the solution we are looking for is

$$\Psi_{k,m}^B \equiv \mathcal{FP} \Psi_{k,m}^B, \quad \text{where} \quad \Psi_{k,m}^B \equiv \square_{\text{ret}}^{-1} S_{k,m}^B = \square_{\text{ret}}^{-1} [\hat{n}_L S_{k,m}^B], \quad (\text{IV.43})$$

and will be obtained by applying the formulas of the previous section. During the computation, we shall also encounter explicit logarithms and poles proportional to $1/B$. This will all boil down to computing

$$\chi_{k,m}^B \equiv \mathcal{FP} \left[\frac{1}{B} \Psi_{k,m}^B \right] = \mathcal{FP} \left[\frac{d}{dB} \Psi_{k,m}^B \right] = \text{FP}_{B=0} \square_{\text{ret}}^{-1} \left[\ln \left(\frac{r}{r_0} \right) S_{k,m}^B \right], \quad (\text{IV.44})$$

where we have used the useful property of the regularization factor $(r/r_0)^B$, that an extra logarithm is generated by differentiating with respect to B :

$$\frac{d}{dB} S_{k,m}^B = \ln \left(\frac{r}{r_0} \right) S_{k,m}^B. \quad (\text{IV.45})$$

Applying the formulas of Section IV.4.1, the solution for any $B \in \mathbb{C}$ reads

$$\Psi_{k,m}^B = -\frac{\hat{n}_L}{2} \sum_{i=0}^{\ell} \sum_{j=0}^{\ell} \frac{(-)^j C_{ij}^{\ell}}{r^{i+1}} \phi_{k,m}^B, \quad (\text{IV.46a})$$

where

$$\phi_{k,m}^B \equiv \int_0^{+\infty} d\rho \rho^{i+j} G(u-\rho) \int_{\frac{\rho}{2}}^{\frac{\rho}{2}+r} d\lambda \left(\frac{\lambda}{r_0} \right)^B \lambda^{-k-j+1} \int_1^{+\infty} dx Q_m(x) F[u-\rho-\lambda(x-1)]. \quad (\text{IV.46b})$$

Note that this solution is “exact”, valid at any radial distance r except $r = 0$. But in the following, we shall mostly be interested in the asymptotic limit when $r \rightarrow +\infty$ with

$u = \text{const.}$ Permuting the integrals, introducing the change of variable $\lambda \rightarrow \tau = \lambda(x-1)$, and factorizing out the expected leading behavior $1/r$ of the solution (in anticipation of the limit $r \rightarrow +\infty$), we find that

$$\Psi_{k,m}^B = -\frac{\hat{n}_L}{2r} \int_0^{+\infty} d\rho G(u-\rho) \int_0^{+\infty} d\tau F(u-\rho-\tau) K_{k,m}^B(\rho, \tau, r), \quad (\text{IV.47a})$$

where we have introduced for convenience the kernel function (with $x = y + 1$)

$$\begin{aligned} K_{k,m}^B(\rho, \tau, r) &= \left(\frac{\tau}{r_0}\right)^B \sum_{i=0}^{\ell} \frac{1}{r^i} \sum_{j=0}^{\ell} (-)^j C_{ij}^{\ell} \rho^{i+j} \tau^{-k-j+1} \int_{\frac{2\tau}{\rho+2r}}^{\frac{2\tau}{\rho}} dy y^{k+j-2-B} Q_m(y+1), \\ &= \tau^{1-k} \int_{\frac{2\tau}{\rho+2r}}^{\frac{2\tau}{\rho}} dy \left(\frac{\tau}{yr_0}\right)^B y^{k-2} Q_m(y+1) \Pi_{\ell} \left(1 - \frac{\rho y}{\tau}, 1 + \frac{\rho}{r}\right). \end{aligned} \quad (\text{IV.47b})$$

In the second line we have introduced the following symmetric bivariate polynomial

$$\Pi_{\ell}(x, y) = \sum_{i=0}^{\ell} \sum_{j=0}^{\ell} C_{ij}^{\ell} (x-1)^i (y-1)^j, \quad (\text{IV.48})$$

which is related to the Legendre polynomial, recalling (IV.39), by

$$P_{\ell}(x) = \sum_{j=0}^{\ell} C_{0j}^{\ell} (x-1)^j = \Pi_{\ell}(x, 1) = \Pi_{\ell}(1, x). \quad (\text{IV.49})$$

Similarly, the quantity ${}_{k,m}\chi_L^B$ defined in (IV.44) reads

$$\chi_{k,m}^B = -\frac{\hat{n}_L}{2r} \int_0^{+\infty} d\rho G(u-\rho) \int_0^{+\infty} d\tau F(u-\rho-\tau) L_{k,m}^B(\rho, \tau, r), \quad (\text{IV.50a})$$

where the kernel is defined as

$$\begin{aligned} L_{k,m}^B &= \frac{d}{dB} \left[K_{k,m}^B(\rho, \tau, r) \right] \\ &= \sum_{i=0}^{\ell} \frac{1}{r^i} \sum_{j=0}^{\ell} (-)^j C_{i,j}^{\ell} \rho^{i+j} \tau^{-k-j+1} \int_{\frac{2\tau}{\rho+2r}}^{\frac{2\tau}{\rho}} dy \left(\frac{\tau}{yr_0}\right)^B \ln\left(\frac{\tau}{r_0 y}\right) y^{k+j-2} Q_m(x), \\ &= \tau^{1-k} \int_{\frac{2\tau}{\rho+2r}}^{\frac{2\tau}{\rho}} dy \left(\frac{\tau}{yr_0}\right)^B \ln\left(\frac{\tau}{yr_0}\right) y^{k-2} Q_m(y+1) \Pi_{\ell} \left(1 - \frac{\rho y}{\tau}, 1 + \frac{\rho}{r}\right). \end{aligned} \quad (\text{IV.50b})$$

When $r \rightarrow +\infty$ the above kernels are dominated by the contribution $i = 0$, and the property (IV.49) allows us to relate the asymptotic limit to the Legendre polynomial as

$$K_{k,m}^B = \tau^{1-k} \int_{\frac{2\tau}{\rho+2r}}^{\frac{2\tau}{\rho}} dy \left(\frac{\tau}{yr_0}\right)^B y^{k-2} Q_m(y+1) P_{\ell} \left(1 - \frac{\rho y}{\tau}\right) + o(1), \quad (\text{IV.51a})$$

$$L_{k,m}^B = \tau^{1-k} \int_{\frac{2\tau}{\rho+2r}}^{\frac{2\tau}{\rho}} dy \left(\frac{\tau}{yr_0}\right)^B \ln\left(\frac{\tau}{yr_0}\right) y^{k-2} Q_m(y+1) P_{\ell} \left(1 - \frac{\rho y}{\tau}\right) + o(1). \quad (\text{IV.51b})$$

We employ the Landau symbol o for remainders, hence $o(1)$ means terms that behave as $\sim \ln^p r/r$ with uncontrolled powers of $\ln r$ as $r \rightarrow +\infty$.

a) Cases where $k = 1$ and $k = 2$

We now assume that $k \in \{1, 2\}$. As proven in Appendix A.7, ${}_{k,m}\Psi_L^B$ and ${}_{k,m}\chi_L^B$ have a well-defined limit when $B \rightarrow 0$, so we can drop the finite part prescription and simply set $B = 0$ in (IV.47) and (IV.50). Therefore we obtain the expression for the kernels, only valid for $k \in \{1, 2\}$,

$${}_{k,m}K_\ell(\rho, \tau, r) \equiv \tau^{1-k} \int_{\frac{2\tau}{\rho+2r}}^{\frac{2\tau}{\rho}} dy y^{k-2} Q_m(y+1) \Pi_\ell \left(1 - \frac{\rho y}{\tau}, 1 + \frac{\rho}{r} \right), \quad (\text{IV.52a})$$

$${}_{k,m}L_\ell(\rho, \tau, r) \equiv \tau^{1-k} \int_{\frac{2\tau}{\rho+2r}}^{\frac{2\tau}{\rho}} dy \ln \left(\frac{\tau}{yr_0} \right) y^{k-2} Q_m(y+1) \Pi_\ell \left(1 - \frac{\rho y}{\tau}, 1 + \frac{\rho}{r} \right). \quad (\text{IV.52b})$$

When restricting our interest to the asymptotic limit $r \rightarrow +\infty$, we must treat separately the cases $k = 1$ and $k = 2$.

For $k = 2$, the integrands in the kernels (IV.52) are clearly integrable at the $y \rightarrow 0$ bound, therefore the kernels converge in the $r \rightarrow +\infty$ limit, such that we are allowed to define the ‘‘asymptotic’’ kernels:

$$\bar{K}_{\ell,2,m}(\rho, \tau) \equiv \lim_{r \rightarrow +\infty} {}_{2,m}K_\ell(\rho, \tau, r) = \tau^{-1} \int_0^{\frac{2\tau}{\rho}} dy Q_m(y+1) P_\ell \left(1 - \frac{\rho y}{\tau} \right), \quad (\text{IV.53a})$$

$$\bar{L}_{\ell,2,m}(\rho, \tau) \equiv \lim_{r \rightarrow +\infty} {}_{2,m}L_\ell(\rho, \tau, r) = \tau^{-1} \int_0^{\frac{2\tau}{\rho}} dy \ln \left(\frac{\tau}{yr_0} \right) Q_m(y+1) P_\ell \left(1 - \frac{\rho y}{\tau} \right). \quad (\text{IV.53b})$$

We have verified the above asymptotic limit in the case where $k = 2$ using the general statement in (IV.41), where the relevant function is defined by (IV.30) with $\alpha = +\infty$.

In the case $k = 1$, the situation is more complicated since the integrands of the kernels are no longer integrable when $y \rightarrow 0$, so it is not possible to simply take the limit $r \rightarrow +\infty$ like in (IV.53). Instead the kernels exhibit a logarithmic behavior as $r \rightarrow +\infty$. Since the radiative construction will not exhibit any logarithmic behavior, the logarithms should cancel out in the final metric, and it is crucial to verify this by controlling the logarithmic limit of the kernel functions. We detail the case of the ‘‘K’’-kernel

$${}_{1,m}K_\ell(\rho, \tau, r) = \int_{\frac{2\tau}{\rho+2r}}^{\frac{2\tau}{\rho}} \frac{dy}{y} Q_m(y+1) \Pi_\ell \left(1 - \frac{\rho y}{\tau}, 1 + \frac{\rho}{r} \right). \quad (\text{IV.54})$$

To extract the logarithmic behavior, we integrate by parts. First, we introduce the regular part of the Legendre function of the second type when $x \rightarrow 1^+$, which is defined as

$$R_m(x) \equiv Q_m(x) + \frac{1}{2} P_m(x) \ln \left(\frac{x-1}{2} \right). \quad (\text{IV.55})$$

A useful observation is that $R_m(1) = -H_m$, where $H_m = \sum_{k=1}^m k^{-1}$ is the harmonic number. Substituting Q_m by its expression in terms of R_m and integrating by parts, we find

$${}_{1,m}K_\ell(\rho, \tau, r) = \left[\ln \left(\frac{y}{2} \right) \left(R_m(y+1) - \frac{1}{4} \ln \left(\frac{y}{2} \right) P_m(y+1) \right) \Pi_\ell \left(1 - \frac{\rho y}{\tau}, 1 + \frac{\rho}{r} \right) \right]_{y=\frac{2\tau}{\rho+2r}}^{y=\frac{2\tau}{\rho}}$$

$$\begin{aligned}
& - \int_{\frac{2\tau}{\rho+2r}}^{\frac{2\tau}{\rho}} dy \ln\left(\frac{y}{2}\right) \frac{d}{dy} \left[R_m(y+1) \Pi_\ell \left(1 - \frac{\rho y}{\tau}, 1 + \frac{\rho}{r} \right) \right] \\
& + \frac{1}{4} \int_{\frac{2\tau}{\rho+2r}}^{\frac{2\tau}{\rho}} dy \ln^2\left(\frac{y}{2}\right) \frac{d}{dy} \left[P_m(y+1) \Pi_\ell \left(1 - \frac{\rho y}{\tau}, 1 + \frac{\rho}{r} \right) \right], \quad (\text{IV.56})
\end{aligned}$$

where the all-integrated terms are shown in the first line. Since $P_m(x)$, $R_m(x)$ and $\Pi_\ell(x, y)$ as well as all their derivatives are perfectly integrable as $x \rightarrow 1^+$, we can safely take the $r \rightarrow +\infty$ expansion, and we obtain the explicit ‘‘polylogarithmic’’ structure

$$K_{1,m}(\rho, \tau, r) = \frac{1}{4} \ln^2\left(\frac{r}{r_0}\right) - \frac{1}{2} \ln\left(\frac{r}{r_0}\right) \left[\ln\left(\frac{\tau}{2r_0}\right) + 2H_m \right] + \bar{K}_{1,m}(\rho, \tau) + o(1), \quad (\text{IV.57a})$$

where ${}_{1,m}\bar{K}_\ell(\rho, \tau)$ does not exhibit any r -dependence. We obtain its explicit expression as

$$\begin{aligned}
{}_{1,m}\bar{K}_\ell(\rho, \tau) &= \frac{1}{4} \ln^2\left(\frac{\tau}{2r_0}\right) + H_m \ln\left(\frac{\tau}{2r_0}\right) \\
& - \frac{(-)^\ell}{4} \left[\ln^2\left(\frac{\tau}{2r_0}\right) - 2 \ln\left(\frac{\tau}{2r_0}\right) \ln\left(\frac{\rho}{2r_0}\right) + \ln^2\left(\frac{\rho}{2r_0}\right) \right] P_m\left(1 + \frac{2\tau}{\rho}\right) \\
& - (-)^\ell \left[\ln\left(\frac{\rho}{2r_0}\right) - \ln\left(\frac{\tau}{2r_0}\right) \right] R_m\left(1 + \frac{2\tau}{\rho}\right) \\
& + \frac{1}{4} \int_0^{\frac{2\tau}{\rho}} dy \ln^2\left(\frac{y}{2}\right) \frac{d}{dy} \left[P_m(y+1) P_\ell\left(1 - \frac{\rho y}{\tau}\right) \right] \\
& - \int_0^{\frac{2\tau}{\rho}} dy \ln\left(\frac{y}{2}\right) \frac{d}{dy} \left[R_m(y+1) P_\ell\left(1 - \frac{\rho y}{\tau}\right) \right]. \quad (\text{IV.57b})
\end{aligned}$$

The same reasoning applied to the ‘‘ L ’’-kernel also gives a polylogarithmic structure but in this case with powers of the logarithm up to three:

$$L_{1,m}(\rho, \tau, r) = \frac{1}{6} \ln^3\left(\frac{r}{r_0}\right) - \frac{1}{4} \ln^2\left(\frac{r}{r_0}\right) \left[\ln\left(\frac{\tau}{2r_0}\right) + 2H_m \right] + \bar{L}_{1,m}(\rho, \tau) + o(1), \quad (\text{IV.58a})$$

and where ${}_{1,m}\bar{L}_\ell(\rho, \tau)$ is explicitly given by

$$\begin{aligned}
{}_{1,m}\bar{L}_\ell(\rho, \tau) &= \frac{1}{12} \ln^3\left(\frac{\tau}{2r_0}\right) + \frac{1}{2} H_m \ln^2\left(\frac{\tau}{2r_0}\right) \quad (\text{IV.58b}) \\
& - \frac{(-)^\ell}{12} \left[\ln^3\left(\frac{\tau}{2r_0}\right) - 3 \ln\left(\frac{\tau}{2r_0}\right) \ln^2\left(\frac{\rho}{2r_0}\right) + 2 \ln^3\left(\frac{\rho}{2r_0}\right) \right] P_m\left(1 + \frac{2\tau}{\rho}\right) \\
& - \frac{(-)^\ell}{2} \left[\ln^2\left(\frac{\rho}{2r_0}\right) - \ln^2\left(\frac{\tau}{2r_0}\right) \right] R_m\left(1 + \frac{2\tau}{\rho}\right) \\
& + \int_0^{\frac{2\tau}{\rho}} dy \ln^2\left(\frac{y}{2}\right) \left[\frac{1}{4} \ln\left(\frac{\tau}{2r_0}\right) - \frac{1}{6} \ln\left(\frac{y}{2}\right) \right] \frac{d}{dy} \left[P_m(y+1) P_\ell\left(1 - \frac{\rho y}{\tau}\right) \right] \\
& + \int_0^{\frac{2\tau}{\rho}} dy \ln\left(\frac{y}{2}\right) \left[-\ln\left(\frac{\tau}{2r_0}\right) + \frac{1}{2} \ln\left(\frac{y}{2}\right) \right] \frac{d}{dy} \left[R_m(y+1) P_\ell\left(1 - \frac{\rho y}{\tau}\right) \right].
\end{aligned}$$

Again, a beautiful check of the radiative construction of the metric in Section IV.1, is that all these explicitly determined far-zone logarithms will be compensated by those induced by the applied gauge transformations, notably the one described in (IV.21).

In practical computations, it is easier to compute the full kernels ${}_{1,m}K_\ell(\rho, \tau, r)$ and ${}_{1,m}L_\ell(\rho, \tau, r)$, and then to remove the logarithmic dependencies to obtain the reduced asymptotic kernels ${}_{1,m}\bar{K}_\ell(\rho, \tau)$ and ${}_{1,m}\bar{L}_\ell(\rho, \tau)$. However, computing the two later quantities directly from their explicit expressions (IV.57b) and (IV.58b) yields, of course, the same result.

b) Cases where $k \geq 3$

We have just shown that we could integrate the source term (IV.42) when $k \in \{1, 2\}$. *A priori*, the cases $k \geq 3$ are more difficult and (when G is not constant) analytic closed-form expressions for the retarded integral seem nontrivial. Hence we proceed differently and prove that we can retrieve the cases $k \geq 3$ from the known cases $k \in \{1, 2\}$. This will show that analytic closed-form expressions also exist in the cases $k \geq 3$.

Given a source term with $k \geq 3$, endowed with the associated regularization factor r^B (posing here $r_0 = 1$), we reduce it to a fully integrated part and new source terms with decreased values of k by two steps at most: $k - 1$ and $k - 2$, by means of the identity:

$$\begin{aligned}
& \hat{n}_L r^{B-k} G(t-r) \int_1^{+\infty} dx Q_m(x) F(t-rx) \\
&= \square \left[\frac{\hat{n}_L r^{B-k+2}}{(k+\ell-2-B)(k-\ell-3-B)} G(t-r) \int_1^{+\infty} dx Q_m(x) F(t-rx) \right] \\
&\quad - \frac{2(k-3-B)\hat{n}_L r^{B-k+1}}{(k+\ell-2-B)(k-\ell-3-B)} \left(G^{(1)}(t-r) \int_1^{+\infty} dx Q_m(x) F(t-rx) \right. \\
&\quad\quad\quad \left. + G(t-r) \int_1^{+\infty} dx x Q_m(x) \overset{(1)}{F}(t-rx) \right) \\
&\quad - \frac{\hat{n}_L r^{B-k+2}}{(k+\ell-2-B)(k-\ell-3-B)} \left(2 G^{(1)}(t-r) \int_1^{+\infty} dx (x-1) Q_m(x) \overset{(1)}{F}(t-rx) \right. \\
&\quad\quad\quad \left. + G(t-r) \int_1^{+\infty} dx (x^2-1) Q_m(x) \overset{(2)}{F}(t-rx) \right). \tag{IV.59}
\end{aligned}$$

This formula, when applied iteratively, allows us to reduce any case $k \geq 3$ in terms of the cases when $k = 1$ and 2 , modulo a series of all-integrated terms (i.e., inside the d'Alembert operator). Note that we introduced a new type of integral, $\int_1^{+\infty} dx x^n Q_m(x) H(t-rx)$ where $n \in \mathbb{N}$, but which can easily be recast in the previous form (i.e. $n = 0$) by recursively applying Bonnet's recursion formula for the Legendre function, namely

$$x Q_m(x) = \frac{m+1}{2m+1} Q_{m+1}(x) + \frac{m}{2m+1} Q_{m-1}(x) \quad \text{if } m \geq 1, \tag{IV.60a}$$

$$x Q_0(x) = Q_1(x) + 1 \quad \text{if } m = 0. \tag{IV.60b}$$

An important point is that we keep the Hadamard regulator r^B "alive" in (IV.59). Indeed, by applying the inverse d'Alembert operator on both sides of (IV.59), we see

that the first term in the right side being the d'Alembertian of a source term containing the regulator r^B , will directly yield that source term proportional to r^B without any additional homogeneous solution (since no homogeneous solution can be proportional to r^B). Hence the iteration with r^B can be done blindly, ignoring homogeneous solutions, and only at the end do we apply the finite part when $B \rightarrow 0$.

However a price we have to pay is that the B -dependent coefficients in (IV.59) can generate a simple pole when $B \rightarrow 0$ which will compete with the higher contribution $\propto B$ in the retarded integral, and *vice versa*, a term $\propto B$ in the coefficient will be compensated by a pole coming from the retarded integral. A useful fact to remember in this respect is that the integrals are convergent for $k \in \{1, 2\}$ and hence no poles $\propto 1/B$ can be generated in these cases. Furthermore, the structure of the identity (IV.59) can only generate simple poles $\propto 1/B$, and no double poles $\propto 1/B^2$ or any poles of higher order. The presence of these simple poles is the main reason why we introduced the ${}_{k,m}\chi_L$ integral, as is clear by its first definition in (IV.44).

Finally, although we now have all the needed formulas to integrate the tails-of-memory, our recursion formula (IV.59) coupled to the $m = 0$ case of the Bonnet formula (IV.60b) can generate instantaneous terms in the source, which must also be integrated. Although the integration of such terms is well known using standard integration techniques [164, 173], its generalization to the case where simple poles $\propto 1/B$ can appear in the source was unknown.

IV.4.3 Application to tails-of-tails

The previous observation motivates us to extend our formalism developed for complicated tails-of-memory to the easier cases of tails-of-tails, tails and even instantaneous terms, such that we can treat the entire problem consistently, using one single formalism. When comparing with previous works [91, 167] this will provide important tests of the results of Section IV.4.2. Thus, we specialize the formulas of the previous section to the case where $G \equiv 1$, since in the case of the tails-of-tails two of the moments are just the mass M .

When $k \geq 3$, we can again reduce the source to the cases $k = 1$ and $k = 2$ using (IV.59), with now $G^{(1)}(u) = 0$. Then, after lengthy computations where we perform two of the three integrations of the tails-of-memory (one on the y variable in the kernel and one on the time variable τ), we find drastically simpler expressions when $k \in \{1, 2\}$, which read

$$\bar{\Psi}_\ell \Big|_{2,m} \Big|_{G=1} = 2 \alpha_\ell \Big|_{2,m} \overset{(-1)}{F}(u), \quad (\text{IV.61a})$$

$$\bar{\chi}_\ell \Big|_{2,m} \Big|_{G=1} = 2 \int_0^{+\infty} d\tau \left[\beta_\ell \Big|_{2,m} \ln \left(\frac{\tau}{2r_0} \right) + \gamma_\ell \Big|_{2,m} \right] F(u - \tau), \quad (\text{IV.61b})$$

$$\bar{\Psi}_\ell \Big|_{1,m} \Big|_{G=1} = \frac{1}{4} \int_0^{+\infty} d\tau \overset{(-1)}{F}(u - \tau) \left[\ln^2 \left(\frac{\tau}{2r_0} \right) + 4H_m \ln \left(\frac{\tau}{2r_0} \right) - 8 \delta_\ell \Big|_{1,m} \right], \quad (\text{IV.61c})$$

$$\bar{\chi}_\ell \Big|_{1,m} \Big|_{G=1} = \frac{1}{12} \int_0^{+\infty} d\tau \overset{(-1)}{F}(u - \tau) \left[\ln^3 \left(\frac{\tau}{2r_0} \right) + 6H_m \ln^2 \left(\frac{\tau}{2r_0} \right) - 24 \delta_\ell \Big|_{1,m} \ln \left(\frac{\tau}{2r_0} \right) + 12 \varepsilon_\ell \Big|_{1,m} \right]. \quad (\text{IV.61d})$$

where $k,m\bar{\Psi}_L \equiv -\frac{\hat{n}_L}{2r} k,m\bar{\Psi}_\ell$ and $k,m\bar{\chi}_L \equiv -\frac{\hat{n}_L}{2r} k,m\bar{\chi}_\ell$. The numerical constants ${}_{2,m}\alpha_\ell$, ${}_{2,m}\beta_\ell$ and ${}_{2,m}\gamma_\ell$ are defined by the integrals

$${}_{2,m}\alpha_\ell \equiv {}_{2,m}\beta_\ell \equiv \int_1^{+\infty} dx Q_m(x) Q_\ell(x), \quad (\text{IV.62a})$$

$${}_{2,m}\gamma_\ell \equiv \frac{1}{2^{\ell+1}} \int_1^{+\infty} dx Q_m(x) \int_{-1}^1 dz \frac{(1-z^2)^\ell}{(x-y)^{\ell+1}} \left(-\ln\left(\frac{x-y}{2}\right) + H_\ell \right). \quad (\text{IV.62b})$$

Note that ${}_{2,m}\alpha_\ell$ is a special case of $k,m\alpha_\ell$ defined for $2 \leq k \leq \ell + 2$ in (A16) of [91], whereas ${}_{2,m}\beta_\ell$ and ${}_{2,m}\gamma_\ell$ are an extension of (A20) of [91], in which these constants were defined only for $k \geq \ell + 3$. In the case $k = 2$, we find it natural to define ${}_{2,m}\alpha_\ell$ and ${}_{2,m}\beta_\ell$ to be equal.

Similarly one can define ${}_{1,m}\delta_\ell$ and ${}_{1,m}\varepsilon_\ell$ using formal integrals but at the price of introducing a Hadamard *partie finie* (pf) to cure the bound of the integrals at $x = 1$:

$${}_{1,m}\delta_\ell \equiv \text{pf} \int_1^{+\infty} dx Q_m(x) \frac{dQ_\ell}{dx}, \quad (\text{IV.63a})$$

$${}_{1,m}\varepsilon_\ell \equiv \text{pf} \int_1^{+\infty} dx Q_m(x) \frac{d^2 S_\ell}{d^2 x}. \quad (\text{IV.63b})$$

The Hadamard *partie finie* is defined in the usual way by removing the divergent part of the integral (and using for our purpose here a Hadamard scale equal to 1):

$$\text{pf} \int_1^{+\infty} dx Q_m(x) \frac{dQ_\ell}{dx} \equiv \lim_{\eta \rightarrow 0} \left\{ \int_{1+\eta}^{+\infty} dx Q_m(x) \frac{dQ_\ell}{dx} + \frac{1}{8} \ln^2\left(\frac{\eta}{2}\right) + \frac{1}{2} H_m \ln\left(\frac{\eta}{2}\right) \right\}, \quad (\text{IV.64a})$$

$$\text{pf} \int_1^{+\infty} dx Q_m(x) \frac{d^2 S_\ell}{d^2 x} \equiv \lim_{\eta \rightarrow 0} \left\{ \int_{1+\eta}^{+\infty} dx Q_m(x) \frac{d^2 S_\ell}{d^2 x} + \frac{1}{6} \ln^3\left(\frac{\eta}{2}\right) + \frac{1}{2} H_m \ln^2\left(\frac{\eta}{2}\right) \right\}. \quad (\text{IV.64b})$$

In the definition of ${}_{1,m}\varepsilon_\ell$ the function S_ℓ is defined like for the Neumann formula for the Legendre function [see (IV.18)],

$$\begin{aligned} S_\ell(x) &\equiv \frac{1}{2} \int_{-1}^1 dy P_\ell(y) \ln^2\left(\frac{x-y}{2}\right) \\ &= \sum_{j=0}^{\ell} \sum_{i=0}^j \frac{(-)^{i+1} P_\ell^{(j)}(1)}{2(i+1)!(j-i)!} \left\{ (x-1)^{j+1} \left[\ln^2\left(\frac{x-1}{2}\right) - \frac{2}{i+1} \ln\left(\frac{x-1}{2}\right) + \frac{2}{(i+1)^2} \right] \right. \\ &\quad \left. + (-)^{j+\ell+1} (x+1)^{j+1} \left[\ln^2\left(\frac{x+1}{2}\right) - \frac{2}{i+1} \ln\left(\frac{x+1}{2}\right) + \frac{2}{(i+1)^2} \right] \right\}. \end{aligned} \quad (\text{IV.65})$$

The constant ${}_{1,m}\delta_\ell$ was used in (A7-A9) of [91] in the special case where $m = \ell$, but was not given a name. The constants defined in [91] were later shown in [167] to be sometimes ill-defined, e.g. for $k = 0$, $\ell = 0$ and $m = 0$, and this problem was circumvented on a case-by-case basis. Since we restrict our attention to $k \in \{1, 2\}$, we are assured that our constants are always well defined.

All the coefficients we shall need can be computed analytically. The ${}_{2,m}\alpha_\ell$ constant (and, hence, ${}_{2,m}\beta_\ell$) has a closed form expression given by (A18) of [91], which reads

$$\alpha_\ell = \beta_\ell = \begin{cases} \frac{H_m - H_\ell}{(m - \ell)(m + \ell + 1)} & \text{if } m \neq \ell, \\ \frac{1}{2m + 1} \left(\frac{\pi^2}{6} - \sum_{j=1}^m \frac{1}{j^2} \right) & \text{if } m = \ell. \end{cases} \quad (\text{IV.66})$$

The ${}_{1,m}\delta_\ell$ constant has a simple expression when $m = \ell$, as pointed out in (A7–A9) of [91], which reads ${}_{1,\ell}\delta_\ell = -\frac{1}{2}H_\ell^2$. When $m \neq \ell$, by integrating by parts the integral expression of the ${}_{1,m}\delta_\ell$ constant, we find the simple property

$$\delta_\ell + \delta_m = -H_m H_\ell. \quad (\text{IV.67a})$$

We can then obtain the full expression of ${}_{1,m}\delta_\ell$ by using recursively the differential equation $Q'_{n+1} - Q'_{n-1} = (2n + 1)Q_n$. In the case where ℓ and m have the same parity, i.e. $\ell - m \in 2\mathbb{Z}$, the differential equation allows us to express the constant only in terms of the case ${}_{1,\ell}\delta_\ell$, along with many ${}_{2,m}\alpha_\ell$ terms, which are easy to compute. Thanks to (IV.67a), we can further restrict attention to the case where $\ell \geq m$, and the relevant formula then reads

$$\delta_\ell = \sum_{j=1}^{\frac{\ell-m}{2}} (2m + 4j - 1) \alpha_{m+2j-1} - \frac{1}{2}H_m^2, \quad (\text{IV.67b})$$

if $\ell - m \in 2\mathbb{N}$. In the reverse case where ℓ and m have opposite parity, i.e. $\ell - m \in 2\mathbb{Z} + 1$, we can again use (IV.67a) to restrict attention to the case where ℓ is odd and m is even. In this case, we can again use the differential equation as well as integration by parts to express our constant solely in terms of the explicitly known case, ${}_{1,0}\delta_1 = \frac{\pi^2}{12}$, modulo some ${}_{2,m}\alpha_\ell$ terms. We thus find the relevant formula,

$$\delta_\ell = -H_m + \frac{\pi^2}{12} + \sum_{j=1}^{\frac{\ell-1}{2}} (2\ell - 4j + 3) \alpha_{\ell-2j+1} - \sum_{j=1}^{\frac{m}{2}} (2m - 4j + 3) \alpha_{m-2j+1}, \quad (\text{IV.67c})$$

if $\ell \in 2\mathbb{N} + 1$ and $m \in 2\mathbb{N}$. Finally the ${}_{2,m}\gamma_\ell$ and ${}_{1,m}\varepsilon_\ell$ constants were only needed for the following values of m and ℓ , which we calculated using *Mathematica*:

$$\begin{aligned} \gamma_{2,0}^1 &= \frac{5}{4}, & \gamma_{2,2}^1 &= \frac{13}{32}, & \gamma_{2,4}^1 &= \frac{95}{432}, \\ \varepsilon_{1,0}^0 &= \frac{\zeta(3)}{2}, & \varepsilon_{1,2}^0 &= \frac{9}{2} + \frac{\zeta(3)}{2}, & \varepsilon_{1,4}^0 &= \frac{3995}{432} + \frac{\zeta(3)}{2}, \\ \varepsilon_{1,1}^1 &= \frac{1}{2} + \frac{\zeta(3)}{2}, & \varepsilon_{1,3}^1 &= \frac{299}{48} + \frac{\zeta(3)}{2}, & \varepsilon_{1,5}^1 &= \frac{887}{80} + \frac{\zeta(3)}{2}, \\ \varepsilon_{1,0}^2 &= -\frac{15}{2} + \frac{\zeta(3)}{2}, & \varepsilon_{1,2}^2 &= \frac{33}{16} + \frac{\zeta(3)}{2}, & \varepsilon_{1,4}^2 &= \frac{3425}{432} + \frac{\zeta(3)}{2}, \end{aligned} \quad (\text{IV.68})$$

where ζ is the Riemann function and $\zeta(3)$ is the Apéry constant. In our computation, the contributions proportional to this constant actually cancel out.

We end this section with a word concerning the integration of B -regularized source terms that are actually instantaneous, i.e. just of the simple type ${}_k S^B(r, t-r) \equiv \left(\frac{r}{r_0}\right)^B r^{-k} F(t-r)$ (multiplied by the multipolarity factor \hat{n}_L). These source terms yield for instance the tails at quadratic order. In line with our general formalism, if $k \geq 3$, we bring ourselves to the case $k = 2$ using the recursion formula:

$$\hat{n}_L r^{B-k} F(t-r) = \square \left[\frac{\hat{n}_L r^{B-k+2}}{(k+\ell-2-B)(k-\ell-3-B)} F(t-r) \right] - \frac{2(k-3-B)\hat{n}_L r^{B-k+1}}{(k+\ell-2-B)(k-\ell-3-B)} \stackrel{(1)}{F}(t-r). \quad (\text{IV.69})$$

The case $k = 2$ is well known [164, 173, 91], and for the asymptotic limit reads

$$\text{FP}_{B=0} \square_{\text{ret}}^{-1} \left[\left(\frac{r}{r_0}\right)^B \frac{\hat{n}_L}{r^2} F(t-r) \right] = \frac{\hat{n}_L}{2r} \int_0^{+\infty} d\tau F(u-\tau) \left[\ln\left(\frac{\tau}{2r}\right) + 2H_\ell \right] + o\left(\frac{1}{r}\right), \quad (\text{IV.70})$$

where we could have dropped the finite part prescription and set $B = 0$, because in this case the inverse d'Alembertian integral is convergent. However, because of the potential appearance of single poles due to our recursion formula (IV.69), we will also need the corresponding formula with an extra factor B^{-1} multiplying the source term. The relevant formula reads

$$\begin{aligned} \text{FP}_{B=0} \square_{\text{ret}}^{-1} \left[\frac{1}{B} \left(\frac{r}{r_0}\right)^B \frac{\hat{n}_L}{r^2} F(t-r) \right] & \quad (\text{IV.71}) \\ &= \frac{\hat{n}_L}{4r} \int_0^{+\infty} d\tau F(u-\tau) \left[\ln^2\left(\frac{\tau}{2r_0}\right) + 4H_\ell \ln\left(\frac{\tau}{2r_0}\right) + 4H_\ell^2 - \ln^2\left(\frac{r}{r_0}\right) \right] + o\left(\frac{1}{r}\right), \end{aligned}$$

which can be obtained from (A2) of [173] by taking straightforwardly the finite part (since the integral converges) and expanding when $r \rightarrow +\infty$.

IV.5 Implementing the calculation of tails-of-memory

IV.5.1 Explicit integration of the asymptotic kernels

Up to this point, the kernels in the asymptotic limit $r \rightarrow +\infty$ were defined only in an integral form, given by (IV.53) in the relatively easy case where $k = 2$, and by the more complex forms (IV.57b) and (IV.58b) when $k = 1$. These integrals are too complicated to seek a general explicit formula valid for arbitrary m and ℓ . However, we can easily compute all these integrals on a case-by-case basis. For this it suffices to insert into them the explicit expressions of the Legendre polynomial $P_m(x)$ and the Legendre function of the second kind $Q_m(x)$, which are simply polynomials multiplied by some logarithms, see (IV.18).

In this way we find that the general structure of the kernels ${}_{k,m} K_\ell(\rho, \tau, r)$ and ${}_{k,m} L_\ell(\rho, \tau, r)$ for $k = 1$ and 2 , up to $o(1)$ precision, is of the type

$$\begin{pmatrix} {}_{k,m} K_\ell(\rho, \tau, r) \\ {}_{k,m} L_\ell(\rho, \tau, r) \end{pmatrix} = \sum_{s,p,q} \begin{pmatrix} \mathcal{X}_{s,p,q} \\ \mathcal{Y}_{s,p,q} \end{pmatrix} \int_{\frac{2\tau}{\rho+2r}}^{\frac{2\tau}{\rho}} dy y^s \ln^p\left(\frac{y}{2}\right) \ln^q\left(1 + \frac{y}{2}\right), \quad (\text{IV.72})$$

where ${}_{k,m}\mathcal{X}_{s,p,q}$ and ${}_{k,m}\mathcal{Y}_{s,p,q}$ denote some numerical coefficients, with s an integer such that $s \geq k - 2$, and where we can restrict our attention only to the values $p = 0, 1, 2, 3$ and $q = 0, 1$. When $s \geq 0$, the lower bound of the integral can be set to 0 in the $r \rightarrow +\infty$ limit, but not when $s = -1$ which can happen when $k = 1$. In the latter case, the integral will develop the logarithmic behavior when $r \rightarrow +\infty$ which has already been obtained in (IV.57a) and (IV.58a), and we now only consider the finite part.

When $q = 0$, these integrals are of course well known (see e.g. (2.721-722) in [182]), so we now restrict our discussion to the case $q = 1$. Let us first examine the case $q = 1$ and $s \geq 0$; we thus set the lower bound to zero, in the limit $r \rightarrow +\infty$. We perform an integration by parts using (2.722) in [182],

$$\begin{aligned} & \int_0^{\frac{2\tau}{\rho}} dy y^s \ln^p \left(\frac{y}{2} \right) \ln \left(1 + \frac{y}{2} \right) \\ &= \sum_{i=0}^p \frac{(-)^i p!}{(p-i)!(s+1)^{i+1}} \left\{ \left(\frac{2\tau}{\rho} \right)^{s+1} \ln^{p-i} \left(\frac{\tau}{\rho} \right) \ln \left(1 + \frac{\tau}{\rho} \right) - \int_0^{\frac{2\tau}{\rho}} dy \frac{y^{s+1}}{y+2} \ln^{p-i} \left(\frac{y}{2} \right) \right\}. \end{aligned} \quad (\text{IV.73})$$

The rational fraction $y^{s+1}(y+2)^{-1}$ in the remaining integral can be expanded as a polynomial, plus a function proportional to $(y+2)^{-1}$. From this result we see that the only nontrivial integrals left are of the type $\int_0^z dy (y+2)^{-1} \ln^j(y/2)$, and these are simply related to the polylogarithm functions, for instance³

$$\int_0^{\frac{2\tau}{\rho}} \frac{dy}{y+2} \ln^2 \left(\frac{y}{2} \right) = -2\text{Li}_3 \left(-\frac{\tau}{\rho} \right) + 2\text{Li}_2 \left(-\frac{\tau}{\rho} \right) \ln \left(\frac{\tau}{\rho} \right) - \text{Li}_1 \left(-\frac{\tau}{\rho} \right) \ln^2 \left(\frac{\tau}{\rho} \right). \quad (\text{IV.74})$$

Now, the only remaining cases are when $q = 1$ and $s = -1$, but in these cases too the integrals are related to polylogarithms, for instance

$$\int_0^{\frac{2\tau}{\rho}} \frac{dy}{y} \ln \left(\frac{y}{2} \right) \ln \left(1 + \frac{y}{2} \right) = \text{Li}_3 \left(-\frac{\tau}{\rho} \right) - \text{Li}_2 \left(-\frac{\tau}{\rho} \right) \ln \left(\frac{\tau}{\rho} \right). \quad (\text{IV.75})$$

To summarize we have obtained the following general structure of the kernel functions in the asymptotic limit $r \rightarrow +\infty$, discarding the leading logarithmic behavior in the $k = 1$ case — recall (IV.57a) and (IV.58a) —, as

$$\begin{pmatrix} \overline{K}_\ell(\rho, \tau) \\ \overline{L}_\ell(\rho, \tau) \end{pmatrix}_{k,m} = \tau^{1-k} \sum_{\substack{j \in \mathbb{Z} \\ p, q, n \in \mathbb{N}}} \begin{pmatrix} \kappa_{k,m}^{j,p,q,n} \\ \lambda_{k,m}^{j,p,q,n} \end{pmatrix} \left(\frac{\tau}{\rho} \right)^j \ln^p \left(\frac{\tau}{2r_0} \right) \ln^q \left(\frac{\rho}{2r_0} \right) \overline{\text{Li}}_n \left(-\frac{\tau}{\rho} \right), \quad (\text{IV.76})$$

³The polylogarithm, or Jonquière's function, is defined for any $n \in \mathbb{N}$ and $z > 0$ as

$$\text{Li}_n(z) \equiv \frac{(-)^{n-1}}{(n-2)!} \int_0^z \frac{ds}{s} \ln^{n-2} \left(\frac{s}{z} \right) \ln(1-s).$$

In particular, the dilogarithm, or Spence's function, is defined as $\text{Li}_2(z) = -\int_0^z \frac{ds}{s} \ln(1-s)$. We also find that $\text{Li}_1(z) = -\ln(1-z)$.

with some numerical coefficients ${}_{k,m}\kappa_{j,p,q,n}$ and ${}_{k,m}\lambda_{j,p,q,n}$ that depend on the relative integer $j \in \mathbb{Z}$ and natural integers $p, q, n \in \mathbb{N}$, and where we have defined for notational convenience $\bar{\text{Li}}_0(z) \equiv 1$ and $\bar{\text{Li}}_n(z) \equiv \text{Li}_n(z)$ for $n \geq 1$. We also recall that $\text{Li}_1(z) = -\ln(1-z)$.

The coefficients ${}_{k,m}\kappa_{j,p,q,n}$ and ${}_{k,m}\lambda_{j,p,q,n}$ represent six-dimensional matrices of numerical coefficients and cannot be presented. Instead, explicit expressions of all the kernels needed for the computation of the tails-of-memory are provided in the Supplementary Material of [4]. For the purpose of clarity, we give two explicit examples of such kernels, where we choose to look at \bar{K} for $m = 2$, $\ell = 4$ and $k = 1, 2$:

$$\begin{aligned} \bar{K}_{1,2}^4(\rho, \tau) = & -\frac{1}{2} \text{Li}_2\left(-\frac{\tau}{\rho}\right) + \frac{7}{12} \text{Li}_1\left(-\frac{\tau}{\rho}\right) - \frac{1}{4} \ln^2\left(\frac{\rho}{2r_0}\right) + \frac{1}{2} \ln\left(\frac{\rho}{2r_0}\right) \ln\left(\frac{\tau}{2r_0}\right) \\ & - \frac{7}{12} \ln\left(\frac{\rho}{2r_0}\right) + \frac{25}{12} \ln\left(\frac{\tau}{2r_0}\right) + \frac{9}{4} - \frac{7}{12} \frac{\rho}{\tau} + \frac{35}{24} \left(\frac{\rho}{\tau}\right)^2 \\ & + \left(\frac{\rho}{\tau}\right)^3 \left[\frac{7}{3} \text{Li}_1\left(-\frac{\tau}{\rho}\right) + \frac{7}{4} \right] + \frac{7}{4} \left(\frac{\rho}{\tau}\right)^4 \text{Li}_1\left(-\frac{\tau}{\rho}\right), \end{aligned} \quad (\text{IV.77a})$$

$$\begin{aligned} \bar{K}_{2,2}^4(\rho, \tau) = & \frac{1}{\tau} \left\{ \frac{1}{6} - \frac{5}{6} \frac{\rho}{\tau} + \left(\frac{\rho}{\tau}\right)^2 \left[-3 \text{Li}_1\left(-\frac{\tau}{\rho}\right) - 5 \right] \right. \\ & \left. + \left(\frac{\rho}{\tau}\right)^3 \left[-7 \text{Li}_1\left(-\frac{\tau}{\rho}\right) - 4 \right] - 4 \left(\frac{\rho}{\tau}\right)^4 \text{Li}_1\left(-\frac{\tau}{\rho}\right) \right\}. \end{aligned} \quad (\text{IV.77b})$$

Notice that despite the structure in ρ/τ of these kernels, one can check explicitly that they are indeed integrable in a vicinity of $(\rho, \tau) = (0, 0)$.

IV.5.2 Raw expression of the radiative quadrupole in the radiative construction

We have applied the MPM construction of the metric in radiative gauge as described in Section IV.1, together with the previous integration techniques, to the computation of the three cubic interactions $M^2 \times \bar{M}_{ij}$, $M \times S_i \times \bar{M}_{ij}$ and $M \times \bar{M}_{ij} \times \bar{M}_{ij}$.

The first interaction, $M^2 \times \bar{M}_{ij}$, is the tail-of-tail which enters at 3PN order and was already known in the harmonic construction [91]. We have, on the one hand, computed the asymptotic waveform for this interaction in the radiative construction using the integration machinery developed in Section IV.4, and, on the other hand, used standard integration techniques [173, 91, 161, 167] applied to the radiative algorithm, and checked that both methods yielded identical results. We then verified that these results, performed in the radiative construction, could be independently retrieved from the known result in the harmonic construction [91] solely using a moment redefinition, as explained in [3]. In the radiative gauge, we find

$$\mathcal{U}_{ij}^{M^2 \times \bar{M}_{ij}} = \frac{2G^2 M^2}{c^6} \int_0^{+\infty} d\tau \left[\ln^2\left(\frac{\tau}{2r_0}\right) + \frac{107}{42} \ln\left(\frac{\tau}{2r_0}\right) + \frac{40037}{8810} \right] \bar{M}_{ij}^{(5)}(u - \tau). \quad (\text{IV.78})$$

For the $M \times S_i \times \bar{M}_{ij}$ interaction, which only enters at 4PN, the fact that the current dipole moment S_i (or total angular momentum) is constant greatly simplifies the computation, and *a priori* allows for the use of standard techniques [173, 91, 167]. However, in some cases, those techniques break down, since some of the numerical constants introduced are ill-defined, as has been mentioned in [167]. Therefore, we have computed for

the first time the full $M \times S_i \times \overline{M}_{ij}$ waveform using our new integration method, which reads in radiative gauge,

$$\mathcal{U}_{ij}^{M \times S_i \times \overline{M}_{ij}} = -\frac{2G^2 M}{c^8} S_{a \langle i} \int_0^{+\infty} d\tau \left[\ln \left(\frac{\tau}{2r_0} \right) + \frac{5381}{5670} \right] \overline{M}_{j \rangle a}^{(6)}(u - \tau). \quad (\text{IV.79})$$

Finally, the $M \times \overline{M}_{ij} \times \overline{M}_{ij}$ interaction is the genuine tail-of-memory entering at 4PN; the double time-dependence within the two associated quadrupole moments strongly complicates the situation. We have performed this calculation only in the asymptotic limit $r \rightarrow +\infty$. Since we are using the radiative construction [160] the waveform is automatically free of far-zone logarithms, and can straightforwardly be projected in a TT gauge, where the radiative quadrupole moment \mathcal{U}_{ij} is extracted in a standard way. Here we first present the “raw” result, obtained in a direct manner from our integration formulas, in terms of the radiative canonical moments. In Section IV.5.3 we describe a very efficient simplification method, and the resulting radiative quadrupole will be presented in Section IV.6 in terms of the radiative canonical moments. The “raw” result is however important and worthy to be presented, as it is the direct result of our integration scheme.

In order to present the raw result, we define the following functionals of two time-derivatives of quadrupole moments F and G for the cases $k = 1, 2$:

$$\overline{\Psi}_{\ell, k, m}[F, G] \equiv \int_0^{+\infty} d\rho G(u - \rho) \int_0^{+\infty} d\tau F(u - \rho - \tau) \overline{K}_{\ell, k, m}(\rho, \tau), \quad (\text{IV.80a})$$

$$\overline{\chi}_{\ell, k, m}[F, G] \equiv \int_0^{+\infty} d\rho G(u - \rho) \int_0^{+\infty} d\tau F(u - \rho - \tau) \overline{L}_{\ell, k, m}(\rho, \tau), \quad (\text{IV.80b})$$

where the kernels are given by (IV.53) in the case $k = 2$ and by (IV.57b) and (IV.58b) in the case $k = 1$. As explained in Section b) we do not need the cases $k \geq 3$.

With these definitions in hand, we decompose the cubic interaction $M \times \overline{M}_{ij} \times \overline{M}_{ij}$ as follows: we separate terms which are purely instantaneous (i.e. local-in-time), from terms whose nonlocality is rather simple and looks like that for the ordinary tails or tails-of-tail, and finally from the genuine and much more intricate tail-of-memory (ToM) integrals. We thus write our raw result, expressed in terms of the radiative canonical moments, as

$$\mathcal{U}_{ij}^{M \times \overline{M}_{ij} \times \overline{M}_{ij}} = \mathcal{U}_{ij}^{M \times \overline{M}_{ij} \times \overline{M}_{ij}} \Big|_{\text{inst}} + \mathcal{U}_{ij}^{M \times \overline{M}_{ij} \times \overline{M}_{ij}} \Big|_{\text{tail}} + \mathcal{U}_{ij}^{M \times \overline{M}_{ij} \times \overline{M}_{ij}} \Big|_{\text{ToM}}. \quad (\text{IV.81})$$

1. The instantaneous terms (depending only on the current time u) are given by

$$\mathcal{U}_{ij}^{M \times \overline{M}_{ij} \times \overline{M}_{ij}} \Big|_{\text{inst}} = \frac{G^2 M}{c^8} \left[\frac{3362032}{165375} \overline{M}_{a \langle i}^{(3)} \overline{M}_{j \rangle a}^{(3)} + \frac{2014871}{165375} \overline{M}_{a \langle i}^{(2)} \overline{M}_{j \rangle a}^{(4)} - \frac{5766241}{165375} \overline{M}_{a \langle i}^{(1)} \overline{M}_{j \rangle a}^{(5)} - \frac{114454}{23625} \overline{M}_{a \langle i} \overline{M}_{j \rangle a}^{(6)} \right]. \quad (\text{IV.82a})$$

2. The nonlocal tail like terms involve some logarithmic kernels, and naturally the quadrupole moment of the usual tail terms is here replaced by a combination $\propto \overline{M}_{a \langle i}^{(n)} \overline{M}_{j \rangle a}^{(p)}$ evaluated at any time $u - \tau$ in the past. We have

$$\mathcal{U}_{ij}^{M \times \overline{M}_{ij} \times \overline{M}_{ij}} \Big|_{\text{tail}} = -\frac{48}{175} \frac{G^2 M}{c^8} \int_0^{+\infty} d\tau \left[\ln^2 \left(\frac{\tau}{2r_0} \right) + \frac{5111}{840} \ln \left(\frac{\tau}{2r_0} \right) \right] (\overline{M}_{a \langle i} \overline{M}_{j \rangle a}^{(7)})(u - \tau)$$

$$\begin{aligned}
& - \frac{192}{175} \frac{G^2 M}{c^8} \int_0^{+\infty} d\tau \left[\ln^2 \left(\frac{\tau}{2r_0} \right) + \frac{475861}{30240} \ln \left(\frac{\tau}{2r_0} \right) \right] (\overline{M}_{a\langle i}^{(1)} \overline{M}_{j\rangle a}^{(6)})(u - \tau) \\
& - \frac{3154}{315} \frac{G^2 M}{c^8} \int_0^{+\infty} d\tau \ln \left(\frac{\tau}{2r_0} \right) (\overline{M}_{a\langle i}^{(2)} \overline{M}_{j\rangle a}^{(5)})(u - \tau) \\
& + \frac{478}{35} \frac{G^2 M}{c^8} \int_0^{+\infty} d\tau \ln \left(\frac{\tau}{2r_0} \right) (\overline{M}_{a\langle i}^{(3)} \overline{M}_{j\rangle a}^{(4)})(u - \tau). \quad (\text{IV.82b})
\end{aligned}$$

3. The genuine tail-of-memory part is fully specified by the bilinear functionals of the two quadrupole moments defined by (IV.80). Thus we can write

$$\begin{aligned}
\mathcal{U}_{ij}^{\text{M} \times \overline{\text{M}}_{ij} \times \overline{\text{M}}_{ij}} \Big|_{\text{ToM}} &= \frac{G^2 M}{c^8} \sum_{m,\ell,n} \left\{ \mathcal{A}_{m,\ell}^n \overline{\Psi}_{1,m}^\ell [\overline{M}_{a\langle i}^{(n)}, \overline{M}_{j\rangle a}^{(8-n)}] + \mathcal{B}_{m,\ell}^n \overline{\Psi}_{2,m}^\ell [\overline{M}_{a\langle i}^{(n)}, \overline{M}_{j\rangle a}^{(7-n)}] \right. \\
& \left. + \mathcal{C}_{n,\ell}^n \overline{\chi}_{1,m}^\ell [\overline{M}_{a\langle i}^{(n)}, \overline{M}_{j\rangle a}^{(8-n)}] + \mathcal{D}_{m,\ell}^n \overline{\chi}_{2,m}^\ell [\overline{M}_{a\langle i}^{(n)}, \overline{M}_{j\rangle a}^{(7-n)}] \right\}, \quad (\text{IV.82c})
\end{aligned}$$

where the purely numerical coefficients $\mathcal{A}_{m,\ell}^n$, $\mathcal{B}_{m,\ell}^n$, $\mathcal{C}_{m,\ell}^n$ and $\mathcal{D}_{m,\ell}^n$ in front of each of these integrals are provided in the Tables A.1, and the functionals ${}_{k,m}\overline{\Psi}_\ell$ and ${}_{k,m}\overline{\chi}_\ell$ are defined in terms of the kernels in (IV.80). Thus our complete results follow from these Tables together with the explicit expressions of the kernel functions provided in the Supplementary Material of [4].

IV.5.3 Simplification method

In this Section, we implement a method for simplifying the expression of the pure *tail-of-memory* part of the radiative quadrupole, given by (IV.82c). The idea is to reexpress everything as only one double integral over the two quadrupoles and a single kernel, modulo some easy surface terms. We thus alternatively integrate by parts the ρ and τ variables of (IV.80), so as to transfer all the time derivatives on the quadrupole moment represented by $F(u - \rho - \tau)$, i.e. the left slot in the functionals (IV.80).

First we observe in (IV.82c) that when $k = 1$, the two quadrupoles have respectively n and $8 - n$ time derivatives, while in the case $k = 2$, they instead have n and $7 - n$ derivatives. We first uniformize this by transforming the case $k = 2$ with the formula

$$\begin{aligned}
& \int_0^{+\infty} d\rho \overline{M}_{a\langle i}^{(n)}(u - \rho) \int_0^{+\infty} d\tau \overline{M}_{j\rangle a}^{(7-n)}(u - \rho - \tau) f(\rho, \tau) \\
& = \int_0^{+\infty} d\rho \overline{M}_{a\langle i}^{(n)}(u - \rho) \int_0^{+\infty} d\tau \overline{M}_{j\rangle a}^{(8-n)}(u - \rho - \tau) (\partial_\tau^{-1} f)(\rho, \tau), \quad (\text{IV.83})
\end{aligned}$$

where we have introduced the τ -antiderivative which vanishes at $\tau = 0$, defined for any function $f(\rho, \tau)$ with adequate regularity properties as

$$(\partial_\tau^{-1} f)(\rho, \tau) \equiv \int_0^\tau d\lambda f(\rho, \lambda). \quad (\text{IV.84})$$

After performing this operation, we are left with integrals of the moments that only have n and $8 - n$ time derivatives, and the only possible cases are $n \in \llbracket 0, 4 \rrbracket$. Next

we integrate by parts so as to be left with integrals with only 0 and 8 time derivatives. However this operation yields some integrals over ρ that are separately divergent at the lower bound $\rho = 0$. To cure this, we introduce a regulator and replace the 0 at the lower bound by some small ϵ , and restrict attention to the $\epsilon \rightarrow 0$ expansion. In the end, we will verify that the final result has a finite $\epsilon \rightarrow 0$ limit. With this caveat in mind, we will simplify all our integrals with the formula (valid for $n \in \llbracket 0, 4 \rrbracket$)

$$\begin{aligned} & \int_{\epsilon}^{+\infty} d\rho \bar{M}_{a\langle i}^{(n)}(u - \rho) \int_0^{+\infty} d\tau \bar{M}_{j)a}^{(8-n)}(u - \rho - \tau) f(\rho, \tau) \\ &= \int_{\epsilon}^{+\infty} d\rho \bar{M}_{a\langle i}^{(n)}(u - \rho) \int_0^{+\infty} d\tau \bar{M}_{j)a}^{(8)}(u - \rho - \tau) \mathcal{O}_{\rho, \tau}^n[f(\rho, \tau)] \\ & \quad + \sum_{k=0}^{n-1} \bar{M}_{a\langle i}^{(k)}(u - \epsilon) \int_0^{+\infty} d\tau \bar{M}_{j)a}^{(\tau-k)}(u - \tau - \epsilon) \left[\mathcal{O}_{\rho, \tau}^{n-1-k}[f(\rho, \tau)] \right]_{\rho=\epsilon}, \end{aligned} \quad (\text{IV.85})$$

where moments involving ϵ should be Taylor-expanded when $\epsilon \rightarrow 0$, the second term is evaluated at $\rho = \epsilon$, and we have introduced the differential operator $\mathcal{O}_{\rho, \tau}$ defined by

$$\mathcal{O}_{\rho, \tau}[f(\tau, \rho)] \equiv \partial_{\tau}^{-1} \partial_{\rho}[f(\tau, \rho)] - f(\tau, \rho), \quad (\text{IV.86})$$

together with its iterations $\mathcal{O}_{\rho, \tau}^n \equiv \mathcal{O}_{\rho, \tau} \cdots \mathcal{O}_{\rho, \tau}$. Applying the formula (IV.85) to all the terms composing the tails-of-memory (IV.82c), we arrive at a unique master double integral:

$$\mathcal{U}_{ij}^{\text{M} \times \bar{\text{M}}_{ij} \times \bar{\text{M}}_{ij}} \Big|_{\text{ToM}} = \text{M} \int_{\epsilon}^{+\infty} d\rho \bar{M}_{a\langle i}^{(n)}(u - \rho) \int_0^{+\infty} d\tau \bar{M}_{j)a}^{(8)}(u - \rho - \tau) \Omega(\rho, \tau) + \mathcal{S}_{\epsilon}, \quad (\text{IV.87})$$

where $\Omega(\rho, \tau)$ denotes some new kernel function and \mathcal{S}_{ϵ} are all the surface terms coming from the second line of (IV.85), which carry at most one integral and that simplify drastically in the $\epsilon \rightarrow 0$ expansion. Since the expression is long we do not show here the result for \mathcal{S}_{ϵ} .

At this stage, we would *a priori* expect $\Omega(\rho, \tau)$ to have a very complicated structure akin to (IV.76), and in particular to involve many polylogarithms. But instead, we find the following simple expression without any polylogarithms:

$$\begin{aligned} \Omega(\rho, \tau) &= \frac{7613764}{165375} - \frac{1024076}{18375} \frac{\tau}{\rho} - \frac{2074}{63} \left(\frac{\tau}{\rho} \right)^2 - \frac{104}{15} \left(\frac{\tau}{\rho} \right)^3 \\ & \quad + \frac{634076}{55125} \ln \left(\frac{\rho}{2r_0} \right) + \frac{384}{175} \frac{\tau}{\rho} \ln \left(\frac{\rho}{2r_0} \right) - \frac{144}{175} \ln \left(\frac{\rho}{2r_0} \right)^2 + \frac{8}{7} \ln \left(\frac{\tau}{2r_0} \right). \end{aligned} \quad (\text{IV.88})$$

Note that if we changed even a single coefficient in Table A.1, the cancellation of polylogarithms would not occur in general, and we would be left with a much more complicated expression for $\Omega(\rho, \tau)$. When inserting $\Omega(\rho, \tau)$ into the first term in (IV.87), we can integrate by parts so as to remove all the powers of τ/ρ . This introduces poles in ϵ , and powers of the logarithms of ϵ , but we have checked that these poles and divergences exactly cancel when adding the surface terms \mathcal{S}_{ϵ} , namely the second term in (IV.87). Putting all of this together and taking the $\epsilon \rightarrow 0$ limit, we find that

$$\mathcal{U}_{ij}^{\text{M} \times \bar{\text{M}}_{ij} \times \bar{\text{M}}_{ij}} \Big|_{\text{ToM}} = \frac{8}{7} \text{M} \int_0^{+\infty} d\rho \bar{M}_{a\langle i}^{(4)}(u - \rho) \int_0^{+\infty} d\tau \ln \left(\frac{\tau}{2r_0} \right) \bar{M}_{j)a}^{(4)}(u - \rho - \tau)$$

$$\begin{aligned}
& + \frac{48}{175} M \int_0^{+\infty} d\tau \overline{M}_{a\langle i} \overline{M}_{j\rangle a}^{(7)}(u - \tau) \left[\ln^2 \left(\frac{\tau}{2r_0} \right) + \frac{1243}{420} \ln \left(\frac{\tau}{2r_0} \right) \right] \\
& + \frac{192}{175} M \int_0^{+\infty} d\tau \overline{M}_{a\langle i}^{(1)} \overline{M}_{j\rangle a}^{(6)}(u - \tau) \left[\ln^2 \left(\frac{\tau}{2r_0} \right) + \frac{186743}{15120} \ln \left(\frac{\tau}{2r_0} \right) \right] \\
& + \frac{1084}{315} M \int_0^{+\infty} d\tau \ln \left(\frac{\tau}{2r_0} \right) \overline{M}_{a\langle i}^{(2)} \overline{M}_{j\rangle a}^{(5)}(u - \tau) \\
& - \frac{104}{5} M \int_0^{+\infty} d\tau \ln \left(\frac{\tau}{2r_0} \right) \overline{M}_{a\langle i}^{(3)} \overline{M}_{j\rangle a}^{(4)}(u - \tau) \\
& + \frac{4}{7} M \overline{M}_{a\langle i} \int_0^{+\infty} d\tau \overline{M}_{j\rangle a}^{(7)}(u - \tau) \left[\ln \left(\frac{\tau}{2r_0} \right) + \frac{15667}{10500} \right] \\
& - \frac{20}{7} M \overline{M}_{a\langle i}^{(1)} \int_0^{+\infty} d\tau \overline{M}_{j\rangle a}^{(6)}(u - \tau) \left[\ln \left(\frac{\tau}{2r_0} \right) - \frac{3590791}{472500} \right] \\
& - \frac{16}{7} M \overline{M}_{a\langle i}^{(2)} \int_0^{+\infty} d\tau \overline{M}_{j\rangle a}^{(5)}(u - \tau) \left[\ln \left(\frac{\tau}{2r_0} \right) + \frac{981149}{94500} \right] \\
& - \frac{3901382}{165375} M \overline{M}_{a\langle i}^{(3)} \overline{M}_{j\rangle a}^{(3)}. \tag{IV.89}
\end{aligned}$$

There is only one doubly-integrated term left, *cf.* the first line of (IV.89), which can be seen as the tails-of-memory contribution properly speaking, whereas the structure of the other singly-integrated terms are more akin to simpler tails-of-tails.

The terms proportional to $\ln^2(\frac{\tau}{2r_0})$ are exactly compensated by those arising in the tail part of $\mathcal{U}_{ij}^{M \times \overline{M}_{ij} \times \overline{M}_{ij}}$ given by (IV.82b). Notice that not all possible terms allowed by a dimensionality argument are present in (IV.89): for instance there is no term

$$\propto M \overline{M}_{a\langle i}^{(4)}(u) \int_0^{+\infty} d\tau \ln \left(\frac{\tau}{2r_0} \right) \overline{M}_{j\rangle a}^{(3)}(u - \tau), \tag{IV.90}$$

which can be understood from the structure of the cubic source term (IV.25) corresponding to the diagram of Fig. IV.1a.

IV.5.4 Testing the integration method

The simplification method in Sec IV.5.3 is also very useful to test our integration method. Indeed, let us consider the integration of a typical tail-of-memory source term, but which we multiply by the factor B . For simplicity, we choose $k = 3$, $m = 0$ and $\ell = 0$:

$$\mathcal{I} \equiv \text{FP}_{B=0} \square_{\text{ret}}^{-1} \left[B \left(\frac{r}{r_0} \right)^B \frac{G(t-r)}{r^3} \int_1^{+\infty} dx Q_0(x) F(t-rx) \right]. \tag{IV.91}$$

This can be computed asymptotically when $r \rightarrow +\infty$ using the arsenal of techniques developed in Section IV.4, and we find

$$\mathcal{I} = \frac{1}{r} \left[\Delta - \frac{1}{2} \int_0^{+\infty} d\tau \ln \left(\frac{\tau}{2r_0} \right) G(u - \tau) F^{(1)}(u - \tau) \right] + o \left(\frac{1}{r} \right), \tag{IV.92a}$$

with the rather cumbersome combination of the functionals (IV.80a):

$$\Delta \equiv -\frac{1}{3} \overline{\Psi}_0[{}_{1,2} F^{(2)}, G] + \frac{1}{3} \overline{\Psi}_0[{}_{1,0} F^{(2)}, G] + \overline{\Psi}_0[{}_{1,1} F^{(1)}, G^{(1)}] - \overline{\Psi}_0[{}_{1,0} F^{(1)}, G^{(1)}]. \tag{IV.92b}$$

But, on the other hand, the presence of the factor B has the effect of only selecting the pole in the B -expansion of the integrated source. This allows us to perform first the near-zone expansion $r \rightarrow 0$ of the source, and then to integrate term by term this expansion using the formulas in [95, 3]. Performing the integration in this manner, we find instead

$$\mathcal{I} = -\frac{G(u)}{r} \left[F(u) + \int_0^{+\infty} d\tau \ln \left(\frac{\tau}{2r_0} \right) F^{(1)}(u - \tau) \right] + o\left(\frac{1}{r}\right). \quad (\text{IV.93})$$

The two expressions (IV.92) and (IV.93) must be identical, therefore we have found a nontrivial relationship between the functionals ${}_{k,m}\bar{\Psi}_\ell[F, G]$ which must absolutely be satisfied if our integration method is correct. Applying the simplification method described in Sec IV.5.3, we can prove that these two expressions are indeed identical. This constitutes a strong test confirming simultaneously the soundness of our integration method and of our simplification method. We have repeated this test with many other such sources; all were successful.

IV.6 Expression in terms of radiative moments

Replacing $\mathcal{U}_{ij}^{\text{M} \times \bar{\text{M}}_{ij} \times \bar{\text{M}}_{ij}}|_{\text{ToM}}$ in (IV.81) by its expression (IV.89), we obtain the three cubic contributions to the radiative quadrupole that enter at 4PN order, in terms of the *radiative*-type canonical moments. Restoring G and c , they read

$$\mathcal{U}_{ij}^{\text{M}^2 \times \bar{\text{M}}_{ij}} = \frac{2G^2 \text{M}^2}{c^6} \int_0^{+\infty} d\tau \bar{\text{M}}_{ij}^{(5)}(u - \tau) \left[\ln^2 \left(\frac{\tau}{2r_0} \right) + \frac{107}{42} \ln \left(\frac{\tau}{2r_0} \right) + \frac{40037}{8820} \right], \quad (\text{IV.94a})$$

$$\mathcal{U}_{ij}^{\text{M} \times \text{S}_i \times \bar{\text{M}}_{ij}} = \frac{2G^2 \text{M}}{c^8} \text{S}_{a\langle i} \int_0^{+\infty} d\tau \bar{\text{M}}_{j)a}^{(6)}(u - \tau) \left[\ln \left(\frac{\tau}{2r_0} \right) + \frac{5381}{5670} \right], \quad (\text{IV.94b})$$

$$\begin{aligned} \mathcal{U}_{ij}^{\text{M} \times \bar{\text{M}}_{ij} \times \bar{\text{M}}_{ij}} = & \frac{2G^2 \text{M}}{7c^8} \left\{ 4 \int_0^{+\infty} d\rho \bar{\text{M}}_{a\langle i}^{(4)}(u - \rho) \int_0^{+\infty} d\tau \bar{\text{M}}_{j)a}^{(4)}(u - \rho - \tau) \left[\ln \left(\frac{\tau}{2r_0} \right) - \frac{1541}{270} \right] \right. \\ & - \frac{50}{2} \int_0^{+\infty} d\tau \ln \left(\frac{\tau}{2r_0} \right) (\bar{\text{M}}_{a\langle i}^{(3)} \bar{\text{M}}_{j)a}^{(4)})(u - \tau) \\ & - 23 \int_0^{+\infty} d\tau \ln \left(\frac{\tau}{2r_0} \right) (\bar{\text{M}}_{a\langle i}^{(2)} \bar{\text{M}}_{j)a}^{(5)})(u - \tau) \\ & - 13 \int_0^{+\infty} d\tau \ln \left(\frac{\tau}{2r_0} \right) (\bar{\text{M}}_{a\langle i}^{(1)} \bar{\text{M}}_{j)a}^{(6)})(u - \tau) \\ & - 3 \int_0^{+\infty} d\tau \ln \left(\frac{\tau}{2r_0} \right) (\bar{\text{M}}_{a\langle i} \bar{\text{M}}_{j)a}^{(7)})(u - \tau) \\ & - 8 \text{M}_{a\langle i}^{(2)} \int_0^{+\infty} d\tau \bar{\text{M}}_{j)a}^{(5)}(u - \tau) \left[\ln \left(\frac{\tau}{2r_0} \right) + \frac{25463}{5040} \right] \\ & - 10 \text{M}_{a\langle i}^{(1)} \int_0^{+\infty} d\tau \bar{\text{M}}_{j)a}^{(6)}(u - \tau) \left[\ln \left(\frac{\tau}{2r_0} \right) + \frac{14503}{3150} \right] \\ & \left. + 2 \text{M}_{a\langle i} \int_0^{+\infty} d\tau \bar{\text{M}}_{j)a}^{(7)}(u - \tau) \left[\ln \left(\frac{\tau}{2r_0} \right) - \frac{26407}{3780} \right] \right\}. \quad (\text{IV.94c}) \end{aligned}$$

The double integral in the first line of (IV.94c) is the genuine tail-of-memory. However, when computing the flux, we must actually compute the time-derivative of the radiative quadrupole. We then find that the tail-of-memory term becomes a simple tail term in the flux, in the same manner as the quadrupole-quadrupole memory term becomes instantaneous in the flux [173].

Of course, in this expression, the radiative quadrupole \mathcal{U}_{ij} is expressed in terms of the canonical moments of the *radiative* construction $\{\bar{M}_L, \bar{S}_L\}$, which we are yet unable to relate to the source and gauge moments $\{I_L, J_L, W_L, X_L, Y_L, Z_L\}$. However, in the previous chapter, we had successfully established the relation (III.33) between the source moments and canonical moments of the *harmonic* construction $\{M_L, S_L\}$. The topic of the next chapter will thus naturally be to establish the relation between the canonical moments in the *harmonic* and *radiative* constructions, which in our case reduces to the relation between M_{ij} and \bar{M}_{ij} . The end result of this study will be the expression of the radiative quadrupole \mathcal{U}_{ij} in terms of the canonical moments of the *harmonic* construction $\{M_L, S_L\}$.

Chapter V

The canonical and harmonic constructions of the MPM waveform

This chapter is mainly based on [3, 4, 6].

In Chapter II, we have seen that the PN-MPM construction could relate the observable radiative moments $\{\mathcal{U}_L, \mathcal{V}_L\}$ to the source $\{I_L, J_L\}$ and gauge $\{W_L, Y_L, X_L, Z_L\}$ moments. Then, we saw in Chapter III that it was useful to introduce intermediate canonical moments $\{M_L, S_L\}$, which allow us to express the most general linearized metric in terms of only two moments, instead of six. However, this is done at the price of deriving the expression of the canonical moments in terms of the source and gauge moments, including all the subtleties that can arise from dimensional regularization. Conversely, we saw in Chapter IV that to construct the radiative moments of more complicated interactions such as the tails-of-memory, we needed to resort to a novel radiative algorithm in nonharmonic coordinates. In this new algorithm, we express our seed linearized metric in terms of two radiative-type canonical moments, $\{\bar{M}_L, \bar{S}_L\}$. As illustrated in Fig. V.1, the only piece missing is the relation between these radiative-type canonical moments $\{M_L, S_L\}$ and the usual harmonic-type canonical moments $\{\bar{M}_L, \bar{S}_L\}$. This will be the topic of this chapter

We recall that in Section II.7, we have seen that the usual MPM expansion in harmonic coordinates leads to powers of the logarithm of the radial distance when $r \rightarrow +\infty$ (with

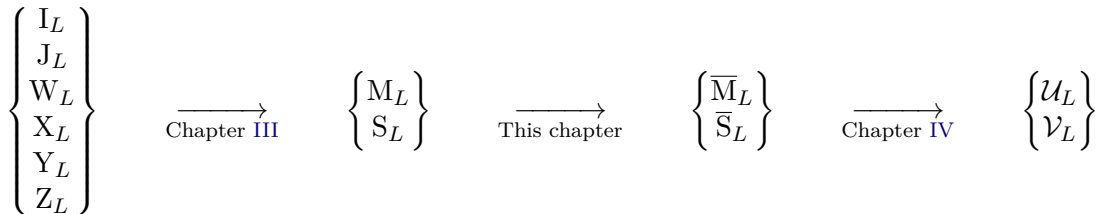


Fig. V.1

$t-r/c$ held constant), and extracting the quantities of interest in such a coordinate system for highly nonlinear effects such as the *tail-of-memory* becomes quite difficult, due to the polylogarithmic behavior of the metric at \mathcal{I}^+ . On the other hand, it is known that GWs generated by isolated systems can be also described using “radiative” type coordinates, in which the metric admits an expansion at \mathcal{I}^+ without the polylogarithmic behavior of harmonic coordinates. In such coordinates the field equations may be formally integrated order by order when $r \rightarrow +\infty$. The paragon of radiative coordinates is the Bondi coordinate system [183, 184] or its variant the Newman-Unti coordinate system [185].¹ However the class of radiative coordinate systems (either spherical or Cartesian) is very large [177, 157, 158].

In previous works, this was dealt with in practice by performing an *ad hoc* coordinate transformation at the end of the MPM process, which lead to a new set of coordinates, which we will now refer to as *modified harmonic* coordinates. However, we have seen in Chapter IV that the tedious computation of the tail-of-memory and spin-quadrupole tail effects cannot be performed using this trick, as it would have been extremely difficult to guess the right coordinate transformation that can remove the far-zone logarithms from these complicated interactions.

Fortunately, an explicit construction of radiative coordinates using the MPM expansion was proposed in [160], which we have introduced in Section IV.1. This construction is restricted to a metric which is stationary in the remote past, before some given instant $-\mathcal{T}$. Under this hypothesis it was proven, up to any order in a perturbative post-Minkowskian sense, that the metric admits a Bondi-type expansion at \mathcal{I}^+ to all orders in $1/r$, and that it obeys standard definitions for asymptotic flatness/simplicity at future null infinity [187, 188, 189]. However, the radiative MPM construction of [160] had not been used for practical computations, until it was recently found to be extremely useful to implement the difficult computation of the tails-of-memory and spin-quadrupole tails at 4PN order [4], see also Chapter IV.

Nevertheless, since previous computations of tails, memory, tails-of-tails and even tails-of-tails-of-tails in the GW flux were done using the harmonic coordinate MPM algorithm [164, 173, 91, 167], the radiative construction cannot be used without understanding how it relates to the standard harmonic construction. The aim of the present chapter is thus to analyze and relate the different MPM constructions of the metric in harmonic and radiative coordinates. Using a method inspired from [95], we prove that the relation between both descriptions boils down to a coordinate transformation and a simple redefinition of the moments parametrizing the two metrics.

As a result, we find that, in the center-of-mass frame, the mass-type “canonical” quadrupole moment in the radiative gauge, denoted below \bar{M}_{ij} , is linked to the corresponding one in the harmonic gauge, say M_{ij} , by the 4PN-accurate relation presented in (V.41) and (V.42) below, which constitute the main result of this chapter.

Note that this is a purely three-dimensional computation, but one could suspect that it should actually be performed in generic d dimensions. However, this is fortunately not the case. Indeed, what is needed is the d dimensional expression of the radiative quadrupole moment in terms of the standard canonical moments in the harmonic MPM construction. This is obtained in two steps: first, one computes this relation in three-dimensions, then

¹A detailed construction of Bondi-Newman-Unti coordinates starting from harmonic coordinates can be found in [186].

one performs a DDR to find the corrections to be added due to dimensional regularization. It is the three-dimensional computation which is complicated to perform in practice in the harmonic construction, so we will perform it in the radiative construction, then translate it back to its expression in the harmonic construction thanks to the results of this chapter. As we will see in Section VI.1.1, these subtleties are absorbed in the definition of renormalized source moments.

The plan of the chapter is as follows. In Section V.1, we present the general formalism to link the harmonic construction of Section III.1 to the radiative construction of Section IV.1, by essentially adapting the results of [95]. We review the consistency of our method in Section V.2 by applying it at the quadratic order to the case of the *tail* interaction, and then to cubic order for the *tail-of-tail* interaction. Finally, the full result at cubic order (comprising the *tail-of-memory* and *spin-quadrupole tail* interactions) is given in Section V.3. Useful integration formulas complementing those in [95] are presented in Appendix A.3.

V.1 Relating the radiative and harmonic constructions

In Section IV.1 and III.1, we constructed two different metrics which both represent the most general solution of the vacuum field equations outside the matter source. We now want to relate them by imposing that they are physically equivalent, i.e. differ only by a coordinate transformation and a multipole moment redefinition. We will then be able to explicitly express the canonical moments of the radiative algorithm $\{\bar{M}_L, \bar{S}_L\}$ as functionals of the canonical moments of the harmonic algorithm $\{M_L, S_L\}$. This is the goal of this work, motivated by the fact that the computation of the tails-of-memory and spin-quadrupole-tails [4] has been performed using the radiative algorithm, while all previous results, i.e. tails and tails-of-tails, and the rest of the computation of the 4PN flux, were achieved in harmonic coordinates.

To determine the relation between $\{\bar{M}_L, \bar{S}_L\}$ and $\{M_L, S_L\}$, we adapt the method of [95] to the case where one of the two metrics does not satisfy the harmonic gauge condition. For the purpose of this chapter, we denote the metric in the harmonic canonical construction of Sec III.1 by h_{harm} and the radiative canonical construction of Section IV.1 by h_{rad} . Recalling the transformation law of the gothic metric under a coordinate transformation $x^\mu \rightarrow x'^\mu$:

$$h_{\text{rad}}^{\mu\nu}(x') = \frac{1}{|J|} \frac{\partial x'^\mu}{\partial x^\rho} \frac{\partial x'^\nu}{\partial x^\sigma} (h_{\text{harm}}^{\rho\sigma}(x) + \eta^{\rho\sigma}) - \eta^{\mu\nu}, \quad (\text{V.1})$$

where $J \equiv \det[\partial x'^\mu/\partial x^\nu]$, we look for a coordinate shift φ^μ such that $x'^\mu = x^\mu + \varphi^\mu(x)$, then we have $\partial x'^\mu/\partial x^\rho = \delta_\rho^\mu + \partial_\rho \varphi^\mu(x)$ and $J = \det[\delta_\rho^\mu + \partial_\rho \varphi^\mu(x)]$.

By construction, both $h_{\text{harm}}^{\mu\nu}$ and $h_{\text{rad}}^{\mu\nu}$ admit a PM expansion as given respectively by (III.2) and (IV.1), which implies that the coordinate shift also admits the PM expansion

$$\varphi^\mu = \sum_{n=1}^{\infty} G^n \varphi_n^\mu. \quad (\text{V.2})$$

Consistently, we assume that the respective canonical moments of the radiative and har-

monic PM metrics also admit PM expansions,

$$\begin{aligned}\bar{M}_L &= \sum_{n=1}^{\infty} G^{n-1} \bar{M}_{n,L}, & M_L &= \sum_{n=1}^{\infty} G^{n-1} M_{n,L}, \\ \bar{S}_L &= \sum_{n=1}^{\infty} G^{n-1} \bar{S}_{n,L}, & S_L &= \sum_{n=1}^{\infty} G^{n-1} S_{n,L}.\end{aligned}\tag{V.3}$$

As it is clear from the definition (IV.2), the linear level is given by

$$h_{\text{rad } 1}^{\mu\nu} = h_{\text{harm } 1}^{\mu\nu} + \partial\varphi_1^{\mu\nu},\tag{V.4}$$

where the coordinate shift is simply given by the gauge vector associated to the radiative construction, see (IV.3):

$$\varphi_1^\mu = \xi_1^\mu = \frac{2M}{c^2} \eta^{0\mu} \ln\left(\frac{r}{b_0}\right),\tag{V.5}$$

and where the moments are related by $\bar{M}_L = M_L + \mathcal{O}(G)$ and $\bar{S}_L = S_L + \mathcal{O}(G)$.

We then follow the reasoning of [95]. We assume by induction that we have determined (for $n \geq 2$) the expressions of φ^μ , \bar{M}_L and \bar{S}_L as functionals of M_L and S_L up to $(n-1)$ PM precision, where 1PM corresponds to the linear case treated above, *i.e.*, that we have established the relations

$$\varphi^\mu = \sum_{m=1}^{n-1} G^m \Phi_m^\mu [M_K, S_K] + \mathcal{O}(G^n),\tag{V.6a}$$

$$\bar{M}_L \equiv \sum_{m=1}^{n-1} G^{m-1} \mathcal{M}_{m,L} [M_K, S_K] + \mathcal{O}(G^{n-1}),\tag{V.6b}$$

$$\bar{S}_L \equiv \sum_{m=1}^{n-1} G^{m-1} \mathcal{S}_{m,L} [M_K, S_K] + \mathcal{O}(G^{n-1}),\tag{V.6c}$$

where Φ_m , $\mathcal{M}_{m,L}$ and $\mathcal{S}_{m,L}$ are determined m -linear functionals of M_L and S_L for $m \leq n-1$. In principle, such functionals will be nonlocal in time at higher order, *i.e.* involve some hereditary-like integrals.

We then perform the expansion of (V.1) up to order n PM, where we Taylor-expand the radiative metric to finite PM order, *i.e.* using

$$h_{\text{rad}}^{\mu\nu}(x') = h_{\text{rad}}^{\mu\nu}[x + \varphi(x)] = \sum_{m \geq 0} \frac{1}{m!} \varphi^{\lambda_1}(x) \cdots \varphi^{\lambda_m}(x) \partial_{\lambda_1} \cdots \partial_{\lambda_m} h_{\text{rad}}^{\mu\nu}(x).\tag{V.7}$$

At the n PM order we find the relation

$$h_{\text{rad } n}^{\mu\nu}(x) = h_{\text{harm } n}^{\mu\nu}(x) + \partial\varphi_n^{\mu\nu} + \Omega_n^{\mu\nu}[h_{\text{harm } m}, \varphi_m; m \leq n-1],\tag{V.8}$$

where $\Omega_n^{\mu\nu}$ is an explicitly known, nonlinear and at least quadratic, functional of the coordinate shift and the harmonic metric at previous orders (see Footnote 2 in Chapter III for an explicit expression).

Thus, using the induction hypothesis, the only unknown term in (V.8) is the gauge shift φ_n^μ , which we now determine. For this purpose, we define the quantity

$$\Delta_n^\mu \equiv -\partial_\nu \Omega_n^{\mu\nu}. \quad (\text{V.9})$$

Taking the divergence of (V.8), and using the fact that $\partial_\nu \partial \theta^{\mu\nu} = \square \theta^\mu$ for any θ^μ (recall our shorthand $\partial \theta^{\mu\nu} \equiv \partial^\mu \theta^\nu + \partial^\nu \theta^\mu - \eta^{\mu\nu} \partial_\rho \theta^\rho$), we find

$$\Delta_n^\mu = \square(\varphi_n^\mu - \xi_n^\mu). \quad (\text{V.10})$$

A difference with the treatment in [95] is to be noted here. The analogue of the quantity Δ_n^μ was defined in [95] as $(\Delta_n^\mu)_{\text{BFL}} \equiv \square \varphi_n^\mu$, but because of the harmonic gauge condition satisfied by the two metrics being related, this yielded $(\Delta_n^\mu)_{\text{BFL}} = -\partial_\nu \Omega_n^{\mu\nu}$, i.e. the same as our present definition (V.9). When the harmonic gauge condition is relaxed it is important to proceed differently, starting from the *definition* (V.9) and then deriving the result (V.10), where the gauge vector ξ_n^μ given by (IV.9) is responsible for the nonharmonicity of the radiative metric, see (IV.12).

Next we apply the d'Alembert operator on (V.8) to obtain

$$\Lambda_{\text{rad } n}^{\mu\nu} = \Lambda_{\text{harm } n}^{\mu\nu} + \partial \Delta_n^{\mu\nu} + \square \Omega_n^{\mu\nu}, \quad (\text{V.11})$$

where we have used the fact that $\square h_{\text{rad}}^{\mu\nu} = \Lambda_{\text{rad } n}^{\mu\nu} + \partial \square \xi_n^{\mu\nu}$. To stick with the definition of the two algorithms we must now apply the inverse d'Alembert operator with finite part prescription on (V.11), and arrive at

$$u_{\text{rad } n}^{\mu\nu} = u_{\text{harm } n}^{\mu\nu} + \text{FP}_{B=0} \square_{\text{ret}}^{-1} \left[\left(\frac{r}{r_0} \right)^B \partial \Delta_n^{\mu\nu} \right] + \text{FP}_{B=0} \square_{\text{ret}}^{-1} \left[\left(\frac{r}{r_0} \right)^B \square \Omega_n^{\mu\nu} \right], \quad (\text{V.12})$$

where the $u_{\text{harm } n}^{\mu\nu}$ is defined in Section III.1, while $u_{\text{rad } n}^{\mu\nu}$ is the equivalent quantity in the radiative algorithm given in (IV.10). Notice that in (V.12) we assumed that the Hadamard regularization scale r_0 is the same for the harmonic and radiative constructions.

Had the \square and ∂ operators commuted with the $\text{FP} \square_{\text{ret}}^{-1}$ operator, the previous equation would obviously simplify, but because of the presence of the regularization factor $(r/r_0)^B$, this is not the case, and we must introduce the *a priori* nonzero ‘‘commutators’’ of these operators. So we can rewrite the previous equation as

$$u_{\text{rad } n}^{\mu\nu} = u_{\text{harm } n}^{\mu\nu} + \partial \phi_n^{\mu\nu} + \Omega_n^{\mu\nu} + X_n^{\mu\nu} + Y_n^{\mu\nu}, \quad (\text{V.13})$$

where we have introduced the two commutators

$$X_n^{\mu\nu} \equiv \left[\text{FP} \square_{\text{ret}}^{-1}, \square \right] \Omega_n^{\mu\nu}, \quad Y_n^{\mu\nu} \equiv \left[\text{FP} \square_{\text{ret}}^{-1}, \partial \right] \Delta_n^{\mu\nu}, \quad (\text{V.14})$$

which can also be expressed in more details as [95]

$$X_n^{\mu\nu} = \text{FP}_{B=0} \square_{\text{ret}}^{-1} \left[B \left(\frac{r}{r_0} \right)^B \left(-\frac{B+1}{r^2} \Omega_n^{\mu\nu} - \frac{2}{r} \partial_r \Omega_n^{\mu\nu} \right) \right], \quad (\text{V.15a})$$

$$Y_n^{\mu\nu} = \text{FP}_{B=0} \square_{\text{ret}}^{-1} \left[B \left(\frac{r}{r_0} \right)^B \frac{n^i}{r} \left(-\delta^{i\mu} \Delta_n^\nu - \delta^{i\nu} \Delta_n^\mu + \eta^{\mu\nu} \Delta_n^i \right) \right]. \quad (\text{V.15b})$$

Note the very important presence of the explicit B factor in the expressions of $X_n^{\mu\nu}$ and $Y_n^{\mu\nu}$, which will select the pole of the Laurent series when $B \rightarrow 0$. In addition to simplifying considerably the computations, this implies that we have $\square X_n^{\mu\nu} = \square Y_n^{\mu\nu} = 0$. We also define

$$\phi_n^\mu \equiv \text{FP}_{B=0} \square_{\text{ret}}^{-1} \left[\left(\frac{r}{r_0} \right)^B \Delta_n^\mu \right]. \quad (\text{V.16})$$

Beware that this object is not the complete gauge transformation vector φ_n^μ , as other contribution will show up later.

To apply the harmonicity algorithm we compute the divergence of (V.13):

$$w_{\text{rad } n}^\mu = w_{\text{harm } n}^\mu + \partial_\nu (X_n^{\mu\nu} + Y_n^{\mu\nu}), \quad (\text{V.17})$$

where we have used $\partial_\nu [\partial \phi_n^{\mu\nu}] = \square \phi_n^\mu = \Delta_n^\mu$. We can check that $\square w_{\text{rad } n}^\mu = 0$, which is a necessary requirement to proceed with the MPM algorithm. Next we define $Z_n^{\mu\nu} \equiv \mathcal{V}^{\mu\nu} [W_n]$, where $\mathcal{V}^{\mu\nu}$ is the harmonicity algorithm applied to $W_n^\mu \equiv \partial_\nu (X_n^{\mu\nu} + Y_n^{\mu\nu})$, hence $Z_n^{\mu\nu}$ is a solution to the vacuum equation whose divergence is exactly opposite to W_n^μ , and we obtain

$$v_{\text{rad } n}^{\mu\nu} = v_{\text{harm } n}^{\mu\nu} + Z_n^{\mu\nu}. \quad (\text{V.18})$$

Piecing it all together, we find that

$$h_{\text{rad } n}^{\mu\nu} = h_{\text{harm } n}^{\mu\nu} + \partial \phi_n^{\mu\nu} + \partial \xi_n^{\mu\nu} + \Omega_n^{\mu\nu} + \mathcal{H}_n^{\mu\nu}, \quad (\text{V.19})$$

where $\mathcal{H}_n^{\mu\nu} \equiv X_n^{\mu\nu} + Y_n^{\mu\nu} + Z_n^{\mu\nu}$ is a divergenceless retarded homogeneous solution of the linearized Einstein vacuum equations, i.e., satisfying at once $\square \mathcal{H}_n^{\mu\nu} = 0$ and $\partial_\nu \mathcal{H}_n^{\mu\nu} = 0$. In this respect, it can be uniquely parametrized [92] by two multipole ‘‘source-type’’ moments $\delta_n M_L$ and $\delta_n S_L$, which are functionals of $\{M_L, S_L\}$, along with a gauge vector ζ_n^μ parametrized by four ‘‘gauge-type’’ moments, as in (37) of [14].

Up to this step, we have assumed that we had related \bar{M}_L and \bar{S}_L to M_L and S_L to $(n-1)$ PM precision. We can now redefine $\bar{M}_L \rightarrow \bar{M}_L - G^{n-1} \delta_n M_L$ and $\bar{S}_L \rightarrow \bar{S}_L - G^{n-1} \delta_n S_L$ so as to obtain this relationship up to n PM precision. This will not affect the result found in the recursion hypothesis since the correction is at n PM precision, but it will absorb the $\delta_n M_L$ and $\delta_n S_L$ moments into the linear approximation $h_{\text{rad } 1}^{\mu\nu}$ of the radiative metric. Since we are correcting $\{\bar{M}_L, \bar{S}_L\}$ which parametrize the radiative metric in the left-hand-side of (V.19), and not $\{M_L, S_L\}$ in the right-hand-side, the expression for $\Omega^{\mu\nu}$, which only depends on $\{M_L, S_L\}$, does not need to be corrected.² Finally, after this moment redefinition, we find that the two metrics are related by

$$h_{\text{rad } n}^{\mu\nu} = h_{\text{harm } n}^{\mu\nu} + \partial \varphi_n^{\mu\nu} + \Omega_n^{\mu\nu}, \quad (\text{V.20})$$

in which we have finally explicitly determined the looked-for gauge vector as

$$\varphi_n^\mu \equiv \phi_n^\mu + \xi_n^\mu + \zeta_n^\mu. \quad (\text{V.21})$$

²A different yet totally equivalent method consists instead in redefining the linearized harmonic metric $h_{\text{harm } 1}^{\mu\nu}$ in the right-hand side, namely by applying the redefinitions $M_L \rightarrow M_L + G^{n-1} \delta_n M_L$ and $S_L \rightarrow S_L + G^{n-1} \delta_n S_L$, but in that case we must include corrections due to the ‘‘renormalization’’ of $\Omega^{\mu\nu}$.

By construction, this vector is a functional of M_L and S_L , so we can write

$$\varphi^\mu = \sum_{m=1}^n G^m \Phi_m^\mu [M_K, S_K] + \mathcal{O}(G^{n+1}). \quad (\text{V.22a})$$

Similarly, the corrections $\delta_n M_L \equiv G^{n-1} \mathcal{M}_{n,L} [M_K, S_K]$ and $\delta_n S_L \equiv G^{n-1} \mathcal{S}_{n,L} [M_K, S_K]$ are also functionals of M_L and S_L , and we find

$$\bar{M}_L \equiv \sum_{m=1}^n G^{m-1} \mathcal{M}_{m,L} [M_K, S_K] + \mathcal{O}(G^n), \quad (\text{V.22b})$$

$$\bar{S}_L \equiv \sum_{m=1}^n G^{m-1} \mathcal{S}_{m,L} [M_K, S_K] + \mathcal{O}(G^n). \quad (\text{V.22c})$$

This completes the recursion procedure, and we have thus proven that we can explicitly construct φ^μ , \bar{M}_L and \bar{S}_L as functionals of M_L and S_L to any finite PM order.

V.2 Application to nonlinear tail interactions

V.2.1 Quadratic tails

We employ the method exposed in Section V.1 to explicitly construct the relation between the radiative and harmonic metrics at quadratic order, to all relevant orders in $1/r$. To do so, we will focus on the tail effect, which arises due to the interaction between the static mass M and the various dynamical mass and current multipole moments, thus generating features in the waveform that are nonlocal in time. As it is the goal of our work, we focus on the corrections to the mass quadrupole moment M_{ij} , but the computations presented hereafter are easily generalized to the tails of other moments.

Recall that at 4PN order, other quadratic interactions enter the relation between the quadrupole moment M_{ij} and the observable at infinity (dubbed “radiative moment”), such as the memory type interaction $M_{ij} \times M_{ij}$ that enters the metric at 2.5PN order. Nevertheless, here we are looking for the relation between the two quadrupole moments M_{ij} and \bar{M}_{ij} in the two algorithms, and this relation at quadratic order must involve the mass M which parametrizes the linearized gauge vector (IV.3). It follows that the only quadratic multipole interaction which can contribute to the relation between the two moments is $M \times M_{ij}$, and we shall find that this relation is purely instantaneous in this case.

In order to obtain the relation between the harmonic and radiative metrics, and thus the associated correction to the quadrupole moments, we will in fact use two independent methods: (i) the one exposed in Section V.1, which does not require knowing the full expressions of the quadratic metrics; and (ii) the explicit computation and comparison of the full harmonic and radiative metrics. We will find exactly the same result, thus confirming the soundness of the method exposed in Section V.1.

Let us first derive the complete metrics for the tail interaction in the two harmonic and radiative algorithms, and read off the quadrupole moments from these two metric in both coordinates. We shall check that the difference between those two quadrupole moments perfectly match the prediction following from the general procedure in Section V.1. The

tail sector of the metric in the harmonic algorithm is given in Appendix B of [164], and reads³

$$h_{\text{harm M} \times \text{M}_{ij}}^{00} = 8Mn_{ab} \int_1^\infty dx Q_2(x) M_{ab}^{(4)}(t - rx) + \frac{Mn_{ab}}{r} \left(10M_{ab}^{(3)} + 7r^{-1}M_{ab}^{(2)} - 21r^{-2}M_{ab}^{(1)} - 21r^{-3}M_{ab} \right), \quad (\text{V.23a})$$

$$h_{\text{harm M} \times \text{M}_{ij}}^{0i} = 8Mn_a \int_1^\infty dx Q_1(x) M_{ai}^{(4)}(t - rx) + \frac{Mn_{iab}}{r} \left[-\frac{1}{3}M_{ab}^{(3)} - r^{-1}M_{ab}^{(2)} - r^{-2}M_{ab}^{(1)} \right] + \frac{Mn_a}{r} \left[\frac{19}{3}M_{ai}^{(3)} - 5r^{-1}M_i^{(2)} - 5r^{-2}M_{ai}^{(1)} \right], \quad (\text{V.23b})$$

$$h_{\text{harm M} \times \text{M}_{ij}}^{ij} = 8M \int_1^\infty dx Q_0(x) M_{ij}^{(4)}(t - rx) + \frac{Mn_{ijab}}{r} \left[-\frac{1}{2}M_{ab}^{(3)} - 3r^{-1}M_{ab}^{(2)} - \frac{15}{2}r^{-2}M_{ab}^{(1)} - \frac{15}{2}r^{-3}M_{ab} \right] + \frac{M\delta_{ij}n_{ab}}{r} \left[-\frac{11}{6}M_{ab}^{(3)} - 2r^{-1}M_{ab}^{(2)} - \frac{1}{2}r^{-2}M_{ab}^{(1)} - \frac{1}{2}r^{-3}M_{ab} \right] + \frac{Mn_{a(i}}{r} \left[4M_{j)a}^{(3)} + 6r^{-1}M_{j)a}^{(2)} + 6r^{-2}M_{j)a}^{(1)} + 6r^{-3}M_{j)a} \right] + \frac{M}{r} \left[-\frac{11}{3}M_{ij}^{(3)} - 4r^{-1}M_{ij}^{(2)} - r^{-2}M_{ij}^{(1)} - r^{-3}M_{ij} \right], \quad (\text{V.23c})$$

where $Q_m(x)$ denotes the Legendre function of the second kind as defined in (IV.18).

On the other hand, we have applied in Section IV.2 the radiative construction of [160] for this same particular tail interaction, thus following the radiative algorithm described in Section IV.1. The moments involved in the construction of the tail sector of the radiative metric are M and \bar{M}_{ij} in this construction (recall that $M = \bar{M}$). Following our convention to denote the coordinates by dummy variables (t, r, \mathbf{n}) , which correspond here to the radiative coordinate system which defines the radiative metric, the tail metric in the radiative algorithm is explicitly given by (IV.16). The main difference between the two expressions (V.23) and (IV.16) of the metric is that the tail terms are given by integrals over the Legendre function $Q_m(x)$ in the harmonic case while the tails in the radiative case are given by integrals over the particular combination \bar{Q}_m , defined in (IV.17).

It is easy to see why the tail terms in the harmonic metric involving $Q_m(x)$ generate logarithms in the far-zone expansion $r \rightarrow +\infty$, while the particular combination $\bar{Q}_m(x, r)$ defined by (IV.17) does not produce any logarithms of r in the radiative metric, although the tail terms are still given by a “hereditary” integral, of course. Posing $\tau = r(x - 1)$ and $u = t - r$ the tail integral reads

$$\int_1^{+\infty} dx Q_m(x) F(t - rx) = \frac{1}{r} \int_0^{+\infty} d\tau Q_m\left(1 + \frac{\tau}{r}\right) F(u - \tau), \quad (\text{V.24})$$

where the function $F(t)$ is a time derivative of the multipole moment and we recall that it vanishes identically for $t < -\mathcal{T}$ (stationarity in the past). Hence, as shown in [164, 91]

³For the sake of lightness, we set $c = 1$ throughout Section V.2.

the far-zone limit $r \rightarrow +\infty$ (with $u \equiv t - r$ held constant) of the tail integral can be obtained by inserting the expansion of the Legendre function when $x \rightarrow 1^+$. Using the expression (IV.18) for the Legendre functions and expanding around $x \rightarrow 1^+$, we obtain the following formal expansion series when $r \rightarrow +\infty$,

$$Q_m\left(1 + \frac{\tau}{r}\right) = -\frac{1}{2}P_m\left(1 + \frac{\tau}{r}\right) \ln\left(\frac{\tau}{2r}\right) + \sum_{k=0}^{\infty} \lambda_k^m \left(\frac{\tau}{r}\right)^k, \quad (\text{V.25a})$$

$$\begin{aligned} \text{where } \lambda_k^m = & - \sum_{i=0}^{\min(m, k-1)} \frac{(-)^{k-i}(m+i)!}{2^{k+1}(k-i)(m-i)!(i!)^2} \\ & - \sum_{j=k+1}^m \sum_{i=0}^k \frac{(m+2i-j)!(k+j-2i-1)!}{2^k(j-i)(m-j)!(j-k-1)!(i!)^2[(k-i)!]^2}, \end{aligned} \quad (\text{V.25b})$$

and where the dependencies in $\ln r$ are contained in the first term of (V.25a). We can see that the first term in the expression of λ_k^m vanishes for $k = 0$ (yielding $\lambda_0^m = -H_m$ where H_m is the usual harmonic number) and the second term vanishes for $k \geq m$.

Conversely, the tail terms in radiative coordinates written in terms of the combination $\bar{Q}_m(x, r)$ do not produce any logarithms in the far-zone expansion. Indeed we have

$$\int_1^{+\infty} dx \bar{Q}_m(x, r) F(t - rx) = \frac{1}{r} \int_0^{+\infty} d\tau \left[R_m\left(1 + \frac{\tau}{r}\right) - \frac{1}{2}P_m\left(1 + \frac{\tau}{r}\right) \ln\left(\frac{\tau}{2r_0}\right) \right] F(u - \tau), \quad (\text{V.26})$$

where $u \equiv t - r$ is the retarded time of the radiative coordinates, and where we have defined

$$R_m(x) \equiv Q_m(x) + \frac{1}{2}P_m(x) \ln\left(\frac{x-1}{2}\right), \quad (\text{V.27})$$

which is regular when $x \rightarrow 1^+$. Since the integrand of (V.26) is regular as $r \rightarrow +\infty$, this integral's expansion for large r is free of any logarithmic terms. Furthermore, we find that the first term in (V.26) admits the following asymptotic expansion in simple powers of $1/r$,

$$\int_0^{+\infty} d\tau R_m\left(1 + \frac{\tau}{r}\right) F(u - \tau) = \sum_{k=0}^{\infty} k! \lambda_k^m \frac{F^{(-k-1)}(u)}{r^k}. \quad (\text{V.28})$$

All of this is illustrated by the expression of the leading far-zone limit of (V.23) and (IV.16). In harmonic coordinates we have

$$h_{\text{harm } M \times M_{ij}}^{00} = -\frac{4Mn_{ab}}{r} \int_0^{+\infty} d\tau \left[\ln\left(\frac{\tau}{2r}\right) + \frac{1}{2} \right] M_{ab}^{(4)}(u - \tau) + \mathcal{O}\left(\frac{\ln r}{r^2}\right), \quad (\text{V.29a})$$

$$\begin{aligned} h_{\text{harm } M \times M_{ij}}^{0i} = & -\frac{4Mn_a}{r} \int_0^{+\infty} d\tau \left[\ln\left(\frac{\tau}{2r}\right) + \frac{5}{12} \right] M_{ia}^{(4)}(u - \tau) \\ & - \frac{M}{3r} n_{iab} M_{ab}^{(3)} + \mathcal{O}\left(\frac{\ln r}{r^2}\right), \end{aligned} \quad (\text{V.29b})$$

$$\begin{aligned} h_{\text{harm } M \times M_{ij}}^{ij} = & -\frac{4M}{r} \int_0^{+\infty} d\tau \left[\ln\left(\frac{\tau}{2r}\right) + \frac{11}{12} \right] M_{ij}^{(4)}(u - \tau) \\ & + \frac{M}{r} \left[-\frac{1}{2} n_{ijab} M_{ab}^{(3)} + 4n_{a(i} M_{j)a}^{(3)} - \frac{11}{6} \delta_{ij} n_{ab} M_{ab}^{(3)} \right] + \mathcal{O}\left(\frac{\ln r}{r^2}\right), \end{aligned} \quad (\text{V.29c})$$

which clearly exhibits the usual far-zone logarithms associated with harmonic coordinates. By contrast the leading far-zone behavior of the radiative metric reads

$$h_{\text{rad } M \times \bar{M}_{ij}}^{00} = -\frac{4Mn_{ab}}{r} \int_0^{+\infty} d\tau \left[\ln \left(\frac{\tau}{2r_0} \right) + \frac{31}{30} \right] \bar{M}_{ab}^{(4)}(u - \tau) + \mathcal{O} \left(\frac{1}{r^2} \right), \quad (\text{V.30a})$$

$$h_{\text{rad } M \times \bar{M}_{ij}}^{0i} = -\frac{4Mn_a}{r} \int_0^{+\infty} d\tau \left[\ln \left(\frac{\tau}{2r_0} \right) + \frac{77}{60} \right] \bar{M}_{ia}^{(4)}(u - \tau) + \frac{M}{r} n_{iab} \bar{M}_{ab}^{(3)} + \mathcal{O} \left(\frac{1}{r^2} \right), \quad (\text{V.30b})$$

$$h_{\text{rad } M \times \bar{M}_{ij}}^{ij} = -\frac{4M}{r} \int_0^{+\infty} d\tau \left[\ln \left(\frac{\tau}{2r_0} \right) + \frac{107}{60} \right] \bar{M}_{ij}^{(4)}(u - \tau) + \frac{M}{r} \left[-\frac{1}{2} n_{ijab} \bar{M}_{ab}^{(3)} + 4n_{a(i} \bar{M}_{j)a}^{(3)} - \frac{1}{2} \delta_{ij} n_{ab} \bar{M}_{ab}^{(3)} \right] + \mathcal{O} \left(\frac{1}{r^2} \right), \quad (\text{V.30c})$$

hence the expansion of the tail term is now free of logarithms, although it now depends on the Hadamard regularization scale r_0 .

Now that we have both explicit metrics including tail terms (V.23) and (IV.16) at our disposal, we can check that they are indeed physically equivalent to this order. This means that the two metrics should differ by a nonlinear coordinate transformation together with the redefinition of the quadrupole moment $M_{ij} \rightarrow \bar{M}_{ij}$, i.e. that

$$h_{\text{rad } \bar{M}_{ij}}^{\mu\nu} + G h_{\text{rad } M \times \bar{M}_{ij}}^{\mu\nu} = h_{\text{harm } M_{ij}}^{\mu\nu} + G \left[h_{\text{harm } M \times M_{ij}}^{\mu\nu} + \partial \varphi_{M \times M_{ij}}^{\mu\nu} + \Omega_{M \times M_{ij}}^{\mu\nu} \right] + \mathcal{O}(G^2), \quad (\text{V.31})$$

where the radiative metric in the left-hand side is defined with \bar{M}_{ij} and the right-hand side is defined with M_{ij} , and with a linear gauge transformation vector $\varphi_{M \times M_{ij}}^\mu$ and $\Omega_{M \times M_{ij}}^{\mu\nu}$ the nonlinear part of the coordinate transformation. Moreover, recall that at the linear level, there is only a mass monopole contribution to the gauge vector, $\varphi_M^\mu \equiv \xi_M^\mu$ as given by (IV.3), and in particular $\varphi_{M_{ij}}^\mu$ is vanishing.

Applying the general procedure of Section V.1, we find that the relevant gauge vector corresponding to the tail interaction, given explicitly by (V.21), reads

$$\varphi_{M \times M_{ij}}^0 = -\frac{4M}{3r} n_{ab} M_{ab}^{(2)}, \quad \varphi_{M \times M_{ij}}^i = 0, \quad (\text{V.32a})$$

whereas the nonlinear correction term, defined by Footnote 2, reads

$$\Omega_{M \times M_{ij}}^{00} = -4M \partial_{ab} \left[r^{-1} \ln \left(\frac{r}{b_0} \right) M_{ab}^{(1)} \right] - \frac{8M}{r^3} n_{ab} M_{ab}^{(1)}, \quad (\text{V.32b})$$

$$\Omega_{M \times M_{ij}}^{0i} = 4M \partial_a \left[r^{-1} \ln \left(\frac{r}{b_0} \right) M_{ai}^{(2)} \right], \quad (\text{V.32c})$$

$$\Omega_{M \times M_{ij}}^{ij} = -\frac{4M}{r} \ln \left(\frac{r}{b_0} \right) M_{ij}^{(3)}. \quad (\text{V.32d})$$

Most importantly, after computing the explicit expression of $\mathcal{H}_n^{\mu\nu}$ using the general method described in Section V.1, we determine that this coordinate transformation must be associated with the following redefinition of the mass quadrupole moment:

$$\bar{M}_{ij} = M_{ij} + GM \left[-\frac{26}{15} + 2 \ln \left(\frac{r_0}{b_0} \right) \right] M_{ij}^{(1)} + \mathcal{O}(G^2). \quad (\text{V.33})$$

It is straightforward to show that the relation (V.31) is satisfied, to all orders in $1/r$ and for $\varphi_{M \times M_{ij}}^\mu$ and $\Omega_{M \times M_{ij}}^{\mu\nu}$ defined by (V.32), if and only if the moment redefinition given by (V.33) holds, hence confirming the soundness of the method of Section V.1.

We can also discuss this equivalence, in a simpler way, directly at the level of the radiative quadrupole moment \mathcal{U}_{ij} defined at future null infinity \mathcal{I}^+ in both constructions. Note that in previous works on tails and iterated tails in *harmonic* coordinates [164, 91, 161, 167], it was shown that the leading logarithms present in the asymptotic waveform (following the algorithm of Section III.1), can be removed by just the linear coordinate transformation $x'^\mu = x^\mu + G \xi_1^\mu$, where ξ_1^μ is given by (IV.3). In the case of quadratic tails, this *ad hoc* procedure, followed by a transverse-traceless (TT) projection of the spatial metric, yields the radiative moment:

$$\mathcal{U}_{ij}(u) = M_{ij}^{(2)}(u) + 2GM \int_0^{+\infty} d\tau \left[\ln\left(\frac{\tau}{2b_0}\right) + \frac{11}{12} \right] M_{ij}^{(4)}(u - \tau) + \mathcal{O}(G^2). \quad (\text{V.34})$$

Here M_{ij} is the canonical moment associated to the harmonic-coordinate construction, and again we use the dummy notation $u = t - r$ for the retarded time in radiative coordinates. Notice that the effect of this coordinate transformation is to replace the logarithm $\ln r$ in harmonic coordinates by the constant $\ln b_0$.

On the other hand, when following the procedure for the radiative construction (namely the algorithm of Section IV.1), the associated radiative moment is given by (IV.22a), which we reproduce here:

$$\mathcal{U}_{ij}(u) = \overline{M}_{ij}^{(2)}(u) + 2GM \int_0^{+\infty} d\tau \left[\ln\left(\frac{\tau}{2r_0}\right) + \frac{107}{60} \right] \overline{M}_{ij}^{(4)}(u - \tau) + \mathcal{O}(G^2).$$

Of course this object is the same as in (V.34), however it is expressed in terms of the canonical moment \overline{M}_{ij} associated to the radiative algorithm. As one can immediately check, the relation (V.33) we have found between the two canonical moments exactly reconciles the results (V.34) and (IV.22a), which again confirms our method in Section V.1.

V.2.2 Cubic tails-of-tails

We have pushed these calculations to cubic order to include the tail-of-tail effect, due to the nonlinear interaction $M \times M \times M_{ij}$, extending (V.34) and (IV.22a) to the next order in G . In the harmonic algorithm, after the suitable coordinate transformation to get rid of the logarithms, we find [91, 161]

$$\begin{aligned} \mathcal{U}_{ij} = & M_{ij}^{(2)} + 2GM \int_0^{+\infty} d\tau \left[\ln\left(\frac{\tau}{2b_0}\right) + \frac{11}{12} \right] M_{ij}^{(4)}(u - \tau) \\ & + 2G^2 M^2 \int_0^{+\infty} d\tau \left[\ln^2\left(\frac{\tau}{2b_0}\right) + \frac{11}{6} \ln\left(\frac{\tau}{2b_0}\right) - \frac{107}{105} \ln\left(\frac{\tau}{2r_0}\right) + \frac{124627}{44100} \right] M_{ij}^{(5)}(u - \tau) \\ & + \mathcal{O}(G^3). \end{aligned} \quad (\text{V.35})$$

Note that this result involves two arbitrary scales, which are important to distinguish: the scale b_0 which enters the asymptotic coordinate transformation $u \rightarrow u - 2GM \ln(r/b_0)$, and, at cubic order, the Hadamard regularization scale r_0 . It is known [190, 191] that r_0 can be interpreted as a renormalization scale and its running obeys a renormalization

group equation. Indeed, we recognize the coefficient in front of the $\ln(r_0)$ in (V.35): it is exactly the beta-function coefficient associated to the renormalization of the mass quadrupole moment, given to be $\beta_2 = -\frac{214}{105}$ in (45) of [190].

On the other hand, we know from Chapter IV that the tails-of-tails have a different expression when using the radiative algorithm described in Section IV.1. Adding the linear and quadratic contributions to (IV.94a), we obtain

$$\begin{aligned} \mathcal{U}_{ij} = & \bar{M}_{ij}^{(2)} + 2GM \int_0^{+\infty} d\tau \left[\ln\left(\frac{\tau}{2r_0}\right) + \frac{107}{60} \right] \bar{M}_{ij}^{(4)}(u - \tau) \\ & + 2G^2M^2 \int_0^{+\infty} d\tau \left[\ln^2\left(\frac{\tau}{2r_0}\right) + \frac{107}{42} \ln\left(\frac{\tau}{2r_0}\right) + \frac{40\,037}{8820} \right] \bar{M}_{ij}^{(5)}(u - \tau) + \mathcal{O}(G^3). \end{aligned} \quad (\text{V.36})$$

Finally by employing the method of Section V.1 up to cubic order for tails and tails-of-tails we have obtained the relationship between the quadrupoles in the two constructions as

$$\begin{aligned} \bar{M}_{ij} = & M_{ij} + GM \left[-\frac{26}{15} + 2 \ln\left(\frac{r_0}{b_0}\right) \right] M_{ij}^{(1)} \\ & + G^2M^2 \left[\frac{124}{45} - \frac{52}{15} \ln\left(\frac{r_0}{b_0}\right) + 2 \ln^2\left(\frac{r_0}{b_0}\right) \right] M_{ij}^{(2)} + \mathcal{O}(G^3). \end{aligned} \quad (\text{V.37})$$

This result is indeed the unique relationship between the moments that reconciles the two results (V.35) and (V.36). Notice that the constant scale b_0 in (V.37) was introduced “automatically” in the linear gauge transformation between the harmonic and radiative linear metrics, see (IV.2)–(IV.3). This scale is identical to the one introduced “by hand” in harmonic coordinates, see (3.1) in [161] or (4.2) in [167].

As discussed in the introduction, the previous *ad hoc* method for removing the logarithms in the harmonic metric was satisfactory for tails and iterated tail interactions since the coordinate transformation could easily be guessed. However, when considering more complicated nonlinear interactions such as the tails-of-memory occurring at 4PN order, the coordinate transformation is more difficult to implement, and it is more convenient to switch to the radiative algorithm since it directly constructs the metric in radiative coordinates and automatically removes the logarithms. The price we have to pay is that, in the end, we must apply the correction $\{\bar{M}_L, \bar{S}_L\} \rightarrow \{M_L, S_L\}$ in order to match with previous results derived in harmonic coordinates, and most importantly the explicit expressions of $\{M_L, S_L\}$ as functions of the source at 4PN order [95].

V.3 Cubic tail-of-memory interactions at 4PN order

We now apply our method to the cubic “tails-of-memory” interaction $M \times M_{ij} \times M_{ij}$, as well as the cubic $M \times M_{ij} \times S_i$ interaction that also enters at 4PN order. The new feature in the radiative algorithm is that, from the quadratic order onward, we must apply the gauge transformation defined by (IV.9). This gauge transformation is zero for the interactions $M \times M$ and $M \times M_{ij}$, hence to control the tails-of-memory we only need the gauge vector for the quadrupole-quadrupole interaction $M_{ij} \times M_{ij}$. We find [173] that the source term reads

$$\sigma_{M_{ij} \times M_{ij}} = \hat{n}_{ijab} M_{ij}^{(3)} M_{ab}^{(3)} - \frac{24}{7} \hat{n}_{ij} M_{ia}^{(3)} M_{ja}^{(3)} + \frac{4}{5} M_{ab}^{(3)} M_{ab}^{(3)}, \quad (\text{V.38})$$

hence the gauge vector (IV.9) upon integration explicitly reads

$$\xi_{M_{ij} \times M_{ij}}^0 = \int_{-\infty}^u dv \int_1^{+\infty} dx \left\{ -\frac{1}{2} \hat{n}_{ijab} Q_4(x) M_{ij}^{(3)} M_{ab}^{(3)} + \frac{12}{7} \hat{n}_{ij} Q_2(x) M_{ia}^{(3)} M_{ja}^{(3)} - \frac{2}{5} Q_0(x) M_{ab}^{(3)} M_{ab}^{(3)} \right\} (v - r(x-1), \mathbf{n}), \quad (\text{V.39a})$$

$$\xi_{M_{ij} \times M_{ij}}^i = \int_{-\infty}^u dv \int_1^{+\infty} dx \left\{ -\frac{1}{2} \hat{n}_{iabpq} Q_5(x) M_{ab}^{(3)} M_{pq}^{(3)} + \frac{16}{9} \hat{n}_{iab} Q_3(x) M_{ak}^{(3)} M_{kb}^{(3)} - \frac{2}{9} \hat{n}_{abk} Q_3(x) M_{ai}^{(3)} M_{bk}^{(3)} - \frac{22}{35} \hat{n}_i Q_1(x) M_{ab}^{(3)} M_{ab}^{(3)} + \frac{24}{35} \hat{n}_a Q_1(x) M_{ik}^{(3)} M_{ka}^{(3)} \right\} (v - r(x-1), \mathbf{n}). \quad (\text{V.39b})$$

For this interaction, the nonlinear correction term vanishes, *i.e.* $\Omega_{M_{ij} \times M_{ij}}^{\mu\nu} = 0$, and it automatically follows that $\Delta_{M_{ij} \times M_{ij}}^\mu = 0$, $\phi_{M_{ij} \times M_{ij}}^\mu = 0$, and $\mathcal{H}_{M_{ij} \times M_{ij}}^{\mu\nu} = 0$. This means that there are no corrections to the moments due to this interaction, and since $\zeta_{M_{ij} \times M_{ij}}^\mu = 0$, the total shift vector reduces to

$$\varphi_{M_{ij} \times M_{ij}}^\mu = \xi_{M_{ij} \times M_{ij}}^\mu. \quad (\text{V.40})$$

When computing the cubic metric, we need also the quadratic $M \times M$ interaction, and this one is trivially computed: the gauge vector vanishes, $\varphi_{M \times M}^\mu = 0$, while the only nonzero component of the correction term is $\Omega_{M \times M}^{00} = 4M^2/r^2$.

A straightforward dimensional analysis shows that the only cubic interactions which can enter the relation between the canonical moments M_{ij} and \bar{M}_{ij} up to the 4PN order in the frame of center-of-mass (for which $M_i = \bar{M}_i = 0$), are precisely the tail-of-tail interaction $M \times M \times M_{ij}$ already computed in the previous section, which is at 3PN order, and the cubic interactions $M \times M_{ij} \times M_{ij}$ and $M \times M_{ij} \times S_i$. Hence we limit ourselves to 4PN order. Implementing the calculation using the technical formulas displayed in Appendix A.3, we then find that the complete relation up to 4PN order between the moments is (restoring at this point the factors $1/c$)

$$\begin{aligned} \bar{M}_{ij} = M_{ij} &+ \left[-\frac{26}{15} + 2 \ln \left(\frac{r_0}{b_0} \right) \right] \frac{GM}{c^3} M_{ij}^{(1)} \\ &+ \left[\frac{124}{45} - \frac{52}{15} \ln \left(\frac{r_0}{b_0} \right) + 2 \ln^2 \left(\frac{r_0}{b_0} \right) \right] \frac{G^2 M^2}{c^6} M_{ij}^{(2)} \\ &+ \frac{G^2 M}{c^8} \left[-\frac{8}{21} M_{a\langle i} M_{j\rangle a}^{(4)} - \frac{8}{7} M_{a\langle i} M_{j\rangle a}^{(3)} - \frac{8}{9} M_{a\langle i} S_{j\rangle |a} \right] + \mathcal{O} \left(\frac{1}{c^9} \right), \end{aligned} \quad (\text{V.41})$$

where we have posed $S_{i|j} \equiv \epsilon_{ijk} S_k$ for the angular momentum, and angular brackets denote the STF projection. We can easily invert the previous relation: using the fact that for the conserved quantities $M = \bar{M}$ and $S_i = \bar{S}_i$, we find

$$\begin{aligned} M_{ij} = \bar{M}_{ij} &+ \left[\frac{26}{15} - 2 \ln \left(\frac{r_0}{b_0} \right) \right] \frac{GM}{c^3} \bar{M}_{ij}^{(1)} \\ &+ \left[\frac{56}{225} - \frac{52}{15} \ln \left(\frac{r_0}{b_0} \right) + 2 \ln^2 \left(\frac{r_0}{b_0} \right) \right] \frac{G^2 M^2}{c^6} \bar{M}_{ij}^{(2)} \\ &+ \frac{G^2 M}{c^8} \left[\frac{8}{21} \bar{M}_{a\langle i} \bar{M}_{j\rangle a}^{(4)} + \frac{8}{7} \bar{M}_{a\langle i} \bar{M}_{j\rangle a}^{(3)} + \frac{8}{9} \bar{M}_{a\langle i} \bar{S}_{j\rangle |a} \right] + \mathcal{O} \left(\frac{1}{c^9} \right). \end{aligned} \quad (\text{V.42})$$

Note that the correction terms we find in (V.41)–(V.42) are purely local (no hereditary integrals at this order). Recall that they depend on the two constant scales: r_0 the Hadamard regularization scale (or renormalization scale [190, 191]) entering into both harmonic and radiative constructions of the metric (and supposed to be identical in the two constructions), and b_0 the scale used in the harmonic construction when defining observable quantities at infinity, and equivalently entering the gauge transformation between the harmonic and radiative linear metrics, see (IV.2)–(IV.3).

With the result (V.41) in hand, we are now able to reexpress in terms of M_{ij} our results for the *tails-of-memory* (IV.94c) and *spin-quadrupole tails* (IV.94b), which were performed following the radiative algorithm in terms of the moment \bar{M}_{ij} . The moment redefinition (V.41) should be inserted into the linear and quadratic contributions to the radiative quadrupole written in terms of the radiative metric (IV.22), so as to yield a correction at cubic order. This correction should be understood in the following sense: let $\mathcal{U}_{ij}^{\text{cubic,rad}}[\bar{M}_L, \bar{S}_L]$ be the functional expression for the cubic part of the radiative quadrupole in terms of the radiative moments, as worked out in (IV.94). Then its counterpart in terms of the harmonic moments, simply denoted $\mathcal{U}_{ij}^{\text{cubic}}[M_L, S_L]$, is given by

$$\mathcal{U}_{ij}^{\text{cubic}}[M_L, S_L] = \mathcal{U}_{ij}^{\text{cubic,rad}}[M_L, S_L] + \delta\mathcal{U}_{ij}^{\text{cubic,rad}}[M_L, S_L], \quad (\text{V.43})$$

where $\mathcal{U}_{ij}^{\text{cubic,rad}}[M_L, S_L]$ means that we simply substituted the radiative moments with the harmonic moments in the radiative functional, and where $\delta\mathcal{U}_{ij}^{\text{cubic,rad}}[M_L, S_L]$ represents the correction to be applied due to the moment redefinition (V.41). We find

$$\begin{aligned} \delta\mathcal{U}_{ij}^{\text{cubic,rad}} &= \frac{G^2 M^2}{c^6} \left\{ \left[-\frac{52}{15} + 4 \ln\left(\frac{r_0}{b_0}\right) \right] \int_0^{+\infty} d\tau \ln\left(\frac{\tau}{2r_0}\right) M_{ij}^{(5)}(u-\tau) \right. \\ &\quad \left. + \left[-\frac{257}{75} + \frac{11}{3} \ln\left(\frac{r_0}{b_0}\right) + 2 \ln^2\left(\frac{r_0}{b_0}\right) \right] M_{ij}^{(4)} \right\} \\ &\quad + \frac{G^2 M}{c^8} \left\{ \left[-\frac{16}{105} - \frac{8}{7} \ln\left(\frac{r_0}{b_0}\right) \right] M_{a\langle i}^{(3)} M_{j\rangle a}^{(3)} + \left[-\frac{14}{15} - 2 \ln\left(\frac{r_0}{b_0}\right) \right] M_{a\langle i}^{(2)} M_{j\rangle a}^{(4)} \right. \\ &\quad \left. + \left[-\frac{32}{35} - \frac{8}{7} \ln\left(\frac{r_0}{b_0}\right) \right] M_{a\langle i}^{(1)} M_{j\rangle a}^{(5)} + \left[-\frac{22}{35} + \frac{2}{7} \ln\left(\frac{r_0}{b_0}\right) \right] M_{a\langle i} M_{j\rangle a}^{(6)} \right\} \\ &\quad + \frac{G^2 M}{c^8} S_{a\langle i} \left[\frac{22}{15} - \frac{2}{3} \ln\left(\frac{r_0}{b_0}\right) \right] M_{j\rangle a}^{(5)}. \end{aligned} \quad (\text{V.44})$$

In this manner, we readily obtain the expressions for the three cubic contributions to the radiative quadrupole, namely the tail-of-tail, the spin-quadrupole tail and the tail-of-memory, expressed in terms of the usual, harmonic, canonical moments $\{M_L, S_L\}$. They read

$$\begin{aligned} \mathcal{U}_{ij}^{M^2 \times M_{ij}} &= \frac{2G^2 M^2}{c^6} \int_0^{+\infty} d\tau M_{ij}^{(5)}(u-\tau) \left[\ln^2\left(\frac{\tau}{2b_0}\right) + \frac{11}{6} \ln\left(\frac{\tau}{2b_0}\right) \right. \\ &\quad \left. - \frac{107}{105} \ln\left(\frac{\tau}{2r_0}\right) + \frac{124627}{44100} \right], \end{aligned} \quad (\text{V.45a})$$

$$\mathcal{U}_{ij}^{M \times S_i \times M_{ij}} = -\frac{2G^2 M}{3c^8} S_{a|i} \int_0^{+\infty} d\tau M_{j)a}^{(6)}(u-\tau) \left[\ln\left(\frac{\tau}{2b_0}\right) + 2 \ln\left(\frac{\tau}{2r_0}\right) + \frac{1223}{1890} \right], \quad (\text{V.45b})$$

$$\begin{aligned} \mathcal{U}_{ij}^{M \times M_{ij} \times M_{ij}} = & \frac{2G^2 M}{7c^8} \left\{ 4 \int_0^{+\infty} d\rho M_{a\langle i}^{(4)}(u-\rho) \int_0^{+\infty} d\tau M_{j)a}^{(4)}(u-\rho-\tau) \left[\ln\left(\frac{\tau}{2r_0}\right) - \frac{1613}{270} \right] \right. \\ & + \int_0^{+\infty} d\tau (M_{a\langle i}^{(3)} M_{j)a}^{(4)})(u-\tau) \left[-15 \ln\left(\frac{\tau}{2b_0}\right) - 10 \ln\left(\frac{\tau}{2r_0}\right) \right] \\ & + \int_0^{+\infty} d\tau (M_{a\langle i}^{(2)} M_{j)a}^{(5)})(u-\tau) \left[-11 \ln\left(\frac{\tau}{2b_0}\right) - 12 \ln\left(\frac{\tau}{2r_0}\right) \right] \\ & + \int_0^{+\infty} d\tau (M_{a\langle i}^{(1)} M_{j)a}^{(6)})(u-\tau) \left[-3 \ln\left(\frac{\tau}{2b_0}\right) - 10 \ln\left(\frac{\tau}{2r_0}\right) \right] \\ & + \int_0^{+\infty} d\tau (M_{a\langle i} M_{j)a}^{(7)})(u-\tau) \left[\ln\left(\frac{\tau}{2b_0}\right) - 4 \ln\left(\frac{\tau}{2r_0}\right) \right] \\ & - 8 M_{a\langle i}^{(2)} \int_0^{+\infty} d\tau M_{j)a}^{(5)}(u-\tau) \left[\ln\left(\frac{\tau}{2r_0}\right) + \frac{27521}{5040} \right] \\ & - 10 M_{a\langle i}^{(1)} \int_0^{+\infty} d\tau M_{j)a}^{(6)}(u-\tau) \left[\ln\left(\frac{\tau}{2r_0}\right) + \frac{15511}{3150} \right] \\ & \left. + 2 M_{a\langle i} \int_0^{+\infty} d\tau M_{j)a}^{(7)}(u-\tau) \left[\ln\left(\frac{\tau}{2r_0}\right) - \frac{6113}{756} \right] \right\}. \quad (\text{V.45c}) \end{aligned}$$

We can see that the b_0 constant, which is associated to a choice for the origin of time, can be eliminated at the level of the full radiative quadrupole at 4PN order by the shift in the time coordinate

$$u' = u + \frac{2GM}{c^3} \ln(b_0),$$

along with a Taylor expansion of the canonical quadrupole moment,

$$M_{ij}(u') = M_{ij}(u) + \frac{2GM}{c^3} \ln(b_0) M_{ij}^{(1)}(u) + \frac{2G^2 M^2}{c^6} \ln^2(b_0) M_{ij}^{(2)}(u) + \mathcal{O}\left(\frac{1}{c^9}\right).$$

This is a nice sanity check, as this arbitrary constant must disappear from final observable results. The fate of the r_0 constant will be treated in Section V.4.

Moreover, we now recall that the leading-order quadratic memory term, see *e.g.* (5.10) in [173], reads

$$\mathcal{U}_{ij}^{\text{mem}} = -\frac{2G}{7c^5} \int_0^{+\infty} d\rho M_{a\langle i}^{(3)}(u-\rho) M_{j)a}^{(3)}(u-\rho) + \mathcal{O}(G^2). \quad (\text{V.46})$$

We find here that the genuine tail-of-memory given by the first term of (V.45c) can be obtained simply by replacing in (V.46) the canonical quadrupole moment M_{ij} by the radiative quadrupole moment itself, including the dominant tail effect, *i.e.*

$$M_{ij}^{(2)} \longrightarrow \mathcal{U}_{ij} = M_{ij}^{(2)} + \frac{2GM}{c^3} \int_0^{+\infty} d\tau M_{ij}^{(4)} \left[\ln\left(\frac{c\tau}{2b_0}\right) + \frac{11}{12} \right] + \mathcal{O}\left(\frac{1}{c^5}\right), \quad (\text{V.47})$$

along with a reexpansion at cubic order and an integration by parts (the constant 11/12 is irrelevant here). With our result (V.45c), we thus explicitly retrieve at this order the expression of memory effects computed using the radiative moments defined at future null infinity, see *e.g.* [192, 193].

V.4 Corrections due to the dimensional regularization of radiative moments

The results presented in the previous section are correct in the ordinary three-dimensional MPM algorithm. Nevertheless, the treatment of the dynamics of point-masses imposes to use a d -dimensional regularization scheme (see Section II.4), starting at the 3PN order [194, 102]. For consistency purposes, it is thus required to perform the MPM algorithm in d dimensions too, with the help of DDR techniques like those mentioned in Section II.4. As proved in [162], the expression of the radiative multipole moments in terms of the d -dimensional canonical ones differs from the expression in terms of the three-dimensional canonical moments, and even exhibit divergences in the form of simple poles that scale as $1/\varepsilon \equiv 1/(d-3)$. These poles will cancel against other poles arising in the d -dimensional source moments, as it has been established in [163, 162], which is a crucial check of the method.

Let us summarize how the expression of the radiative moments required for the 4PN flux in terms of the d -dimensional canonical moments differs from the three-dimensional expression. Since the divergences hits at 3PN order, only the mass quadrupole and octupole, as well as the current quadrupole, are affected. In the center-of-mass frame, we denote by $\mathcal{U}_L^{(3)}[M_K, S_K]$ and $\mathcal{V}_L^{(3)}[M_K, S_K]$ the expression of radiative moments seen as *functionals* of the three-dimensional canonical moments, as given for example in (V.45). One can then formally replace the three-dimensional moments by d -dimensional moments, which we denote by $\mathcal{U}_L^{(3)}[M_K^{(d)}, S_K^{(d)}]$ and $\mathcal{V}_L^{(3)}[M_K^{(d)}, S_K^{(d)}]$, but note that these quantities are *not* equal to the physical radiative moments. Instead, the radiative moments expressed in terms of the d -dimensional canonical moments read (see (6.1) in [162])

$$\begin{aligned} \mathcal{U}_{ij} = & \mathcal{U}_{ij}^{(3)}[M_K^{(d)}, S_K^{(d)}] - \frac{214 G^2 M^2}{105 c^6} \left(\Pi_\varepsilon + \frac{246299}{44940} \right) M_{ij}^{(4)} \\ & + \frac{G^2 M}{c^8} \left[-\frac{4}{7} \left(\Pi_\varepsilon - \frac{1447}{216} \right) M_{a\langle i} M_{j\rangle a}^{(6)} - \frac{32}{7} \left(\Pi_\varepsilon - \frac{17783}{10080} \right) M_{a\langle i}^{(1)} M_{j\rangle a}^{(5)} \right. \\ & - 4 \left(\Pi_\varepsilon - \frac{27649}{17640} \right) M_{a\langle i}^{(2)} M_{j\rangle a}^{(4)} + \frac{1921}{945} M_{a\langle i}^{(3)} M_{j\rangle a}^{(3)} \\ & \left. + \frac{4}{3} \left(\Pi_\varepsilon + \frac{11243}{7560} \right) M_{a\langle i}^{(5)} S_{j\rangle a} \right], \end{aligned} \quad (\text{V.48a})$$

$$\mathcal{U}_{ijk} = \mathcal{U}_{ijk}^{(3)}[M_K^{(d)}, S_K^{(d)}] - \frac{21 G^2 M^2}{26 c^6} \left(\Pi_\varepsilon + \frac{9281}{2730} \right) M_{ijk}^{(5)}, \quad (\text{V.48b})$$

$$\mathcal{V}_{ijk} = \mathcal{V}_{ijk}^{(3)}[M_K^{(d)}, S_K^{(d)}] - \frac{214 G^2 M^2}{105 c^6} \left(\Pi_\varepsilon + \frac{4989}{44940} \right) S_{ijk}^{(4)}, \quad (\text{V.48c})$$

where we drop the (d) superscript in the higher-order corrections (they do not differ from their three-dimensional counterparts at this order), where we employ the notations of Section III.3.1 for current moments, and where we have introduced the quantity

$$\Pi_\varepsilon = -\frac{1}{2\varepsilon} + \ln \left(\frac{r_0 \sqrt{\bar{q}}}{\ell_0} \right), \quad (\text{V.49})$$

with $\bar{q} \equiv 4\pi e^{\gamma_E}$. The constant ℓ_0 is the length-scale associated with dimensional regularization, such that $G = \ell_0^\varepsilon G_N$ relates to the Newton constant G_N (see Section II.4). As

expected, the numerical constants associated with the poles at 3PN are the β -coefficients of the renormalization group flows for these multipole moments [190, 191, 195] (see [196] for the computation using the renormalization group flow of high-order logarithmic effects in the conservative energy function).

An important point to make, is that, as a consequence of those dimensional-regularization corrections, the r_0 disappears from the expression (V.45) of the radiative quadrupole in d dimensions (for any mode $m > 0$), and is replaced by ℓ_0 . Indeed, as pointed out in [4], when isolating the contribution of $\ln(r_0)$ to the radiative quadrupole for the three cubic interactions given by (V.45), we find

$$\begin{aligned} \mathcal{U}_{ij}^{\text{cubic}} = \ln(r_0) & \left[\frac{214}{105} \frac{G^2 M^2}{c^6} M_{ij}^{(4)} + \frac{G^2 M}{c^8} \left(4M_{a\langle i}^{(2)} M_{j\rangle a}^{(4)} + \frac{32}{7} M_{a\langle i}^{(1)} M_{j\rangle a}^{(5)} + \frac{4}{7} M_{a\langle i} M_{j\rangle a}^{(6)} \right) \right. \\ & \left. + \frac{4G^2 M}{3c^8} S_{a\langle i} M_{j\rangle a}^{(5)} \right] + \left(\text{terms independent of } \ln(r_0) \right) + \mathcal{O} \left(\frac{1}{c^9} \right). \quad (\text{V.50}) \end{aligned}$$

This expression exactly cancels the $\ln(r_0)$ terms arising in the second time derivative of the dimensional-regularization contributions given in (V.48), which accounts for the contribution of the dimensional regularization of the cubic interactions to the renormalized canonical quadrupole moment for compact binaries. This nontrivial effect translates the fact that we have properly converted from Hadamard to dimensional regularization, and is a strong technical test of our expression for the radiative quadrupole moment. Last, but not least, we announce that the unphysical scale ℓ_0 will also disappear from any observable result, as we will see in Chapter VI.

As we shall see in Section VI.1.1, the poles present in (V.48) will actually be canceled by poles coming from the d -dimensional expression of the source multipole moments. Thus the correction terms (V.48) will play the role of “counter-terms” (roughly speaking) for the cancellation of the divergences in the computation of the source moments. In particular this will lead in Section VI.1 to a notion of “renormalized” source quadrupole moment at the 4PN order, which is useful in some intermediate steps of our calculation.

Chapter VI

The 4PN waveform for quasicircular orbits

This chapter is mainly based on [5, 6].

This chapter is devoted to collecting all the different pieces needed to finally compute the flux at 4.5PN order and the (2, 2) mode at 4PN order. In Section VI.1, we recall the definition and expressions of the renormalized source moments that are needed for the 4.5PN flux and 4PN waveform, as well as the relevant relations between canonical and source moments. We also discuss some extra contributions that we need to add to the mass-type source quadrupole moment. In Section VI.2, we review the expressions of all the required radiative moments in terms of the canonical moments, which completes our results from Chapters IV and V. In Section VI.3, we develop some new techniques needed to reduce our results to the case of quasicircular orbits. In particular, we discuss the post-adiabatic contribution to the tails and the formulas needed to compute the tails of memory. Finally, we present the 4PN modes in Section VI.4 and the 4.5PN flux in Section VI.5. Thanks to the 4PN conservative energy of (II.55), we are then able to compute the frequency chirp in Section VI.6 and the phase of the GW in Section VI.7.

VI.1 The renormalized source moments

VI.1.1 Corrections to the renormalized source quadrupole moment

In dimensional regularization, the expression of the source moments in terms of the source differs from the three-dimensional case. Indeed, we recall [98, 163, 162] that the source mass-type multipolar moment is given in terms of the source as

$$\begin{aligned} I_{ij}^{(d)} = & \frac{d-1}{2(d-2)} \text{FP}_{B=0} \int \frac{d^d \mathbf{x}}{\ell_0^{d-3}} \left(\frac{r}{r_0} \right)^B \left\{ \hat{x}_{ij} \bar{\Sigma}_{[2]} - \frac{4(d+2)}{c^2 d(d+4)} \hat{x}_{ija} \bar{\Sigma}_{a[3]}^{(1)} \right. \\ & \left. + \frac{2(d+2)}{c^4 d(d+1)(d+6)} \hat{x}_{ijab} \bar{\Sigma}_{ab[4]}^{(2)} - \frac{4B(d-3)(d+2)}{c^2(d-1)d(d+4)} \hat{x}_{ija} \frac{x_b}{r^2} \bar{\Sigma}_{ab[3]} \right\}, \end{aligned} \quad (\text{VI.1})$$

where we have the notation

$$\bar{S}_{[\ell]} = \sum_{k=0}^{\infty} \frac{1}{2^{2k} k!} \frac{\Gamma\left(\ell + \frac{d}{2}\right)}{\Gamma\left(\ell + \frac{d}{2} + k\right)} \left(\frac{r}{c} \frac{\partial}{\partial t}\right)^{2k} \bar{S}(t, \boldsymbol{x}), \quad (\text{VI.2})$$

where \bar{S} stands for $\bar{\Sigma}^{(n)}$, $\bar{\Sigma}_a^{(n)}$ or $\bar{\Sigma}_{ab}^{(n)}$. Generically, such a definition leads to the appearance of poles in ε , such that the $d \rightarrow 3$ limit of $I_L^{(d)}$ is not well defined, and thus one does not retrieve the three dimensional definition I_L . This feature is not a problem as long as DR is correctly implemented in the full PN-MPM construction. In particular, we find that the poles vanish in the observable radiative moments, and that we smoothly recover the 3D limit for all observable quantities.

In [163, 162], the source mass quadrupole moment was computed using dimensional regularization. However, since the publication of this result, it was discovered that two extra contributions should be added. First it was found that some terms need to be added in the near-zone PN iteration, which are due to the fact that the d'Alembert operator \square and the near-zone regularized inverse d'Alembertian $\text{FP}_{B=0} \square_{\text{ret}}^{-1}$ (see [99, 100] for a definition) do not commute in general. This leads to some extra terms that need to be added to the source quadrupole, which first appear at 4PN, which are related to the contributions of “d'Alembertian commutators” of the type $[\text{FP}_{B=0} \square_{\text{ret}}^{-1}, \square]$. For binary systems on general orbits and in a center-of-mass frame, it was shown in [6] that these extra terms read

$$\delta I_{ij} = -\frac{4G^3 m^4 \nu^2}{r_{12} c^8} \left(v_{12}^2 - 3\dot{r}_{12} - \frac{Gm}{r_{12}} \right) \hat{n}_{12}^{ij} + \mathcal{O}\left(\frac{1}{c^9}\right). \quad (\text{VI.3})$$

Luckily, it was found that this contribution vanishes in the case of quasicircular orbits (at least at 4PN order), so it will not affect our results and we refer the interested reader to the discussion in Section III of [6].

The second contribution plays a more important role. Indeed, it was initially found in [162] that the pole that appeared in the computation of (VI.1) was exactly compensated by another pole arising in the relation between canonical moments and radiative moments, see (V.48). This observation led to the notion of a renormalized canonical moment. However, in this computation, one actually needs to add an extra contribution to the source quadrupole in d dimensions $I_L^{(d)}$, which leads to the appearance of yet another pole, which exactly cancels the pole arising in the d -dimensional relation between canonical and source moments, see (III.33). This cancellation was required by the formalism, but is highly nontrivial and thus consists in a very important technical check of our computation. The origin of this second contribution is as follows: when computing the equations of motions at 4PN order, the tail effect, which enters as a radiation mode in the conservative dynamics described by the Fokker action, has been implemented with the *canonical* quadrupole moment M_{ij} [130]. Indeed, no tail contribution involving the gauge moments (in particular the interaction $M \times W$) has been considered in this calculation. This is correct for the equations of motion since the gauge moments result in a physically irrelevant shift of the equations of motion. Moreover, as described in [163], this tail effect in the acceleration is taken into account in the computation of the source mass quadrupole moment. However, we found that there is a mismatch between the coordinate

systems used in these two computations: the tail effect in the acceleration is computed in a *canonical* coordinate system, whilst the source quadrupole is defined in a *generic* one, following the terminology of Section III.2 (or [95]). We thus need to adjust for this, and compute the expression of the source quadrupole moment in the canonical coordinate system. This is done by performing a simple coordinate change $x_{\text{gen}}^\mu = x_{\text{can}}^\mu + \zeta^\mu$, where the relation between the coordinate systems is described in Section III.2. A simple dimensional analysis shows that only the $M \times W$ interaction can enter this coordinate change at 4PN order. Using the methods and notations exposed in Section III.4 to calculate the contribution of a given interaction to the gauge vector, we find that the only possible contribution comes from $\varphi_{\text{hom } M \times W}^\mu$, and the coordinate change is then explicitly given by

$$\zeta^0 = -\frac{16G^2M}{c^7} \sum_{j \geq 0} \frac{1}{2^{2j} j! c^{2j}} \frac{\Gamma\left(\frac{d}{2\ell_0}\right)}{\Gamma\left(\frac{d}{2} + j\right)} \int_0^{+\infty} d\tau \left[L_\varepsilon(\tau) + \frac{1}{2} \right] W^{(3+2j)}(t - \tau), \quad (\text{VI.4a})$$

$$\zeta^i = 0, \quad (\text{VI.4b})$$

where $L_\varepsilon(\tau)$ contains a pole and was defined in (III.32).

Note that only the first term in the sum (i.e $j = 0$) is relevant at 4PN. Implementing the procedure exposed in Section II of [163] with the “pure gauge” metric $h^{\mu\nu} = -\partial\xi^{\mu\nu}$, we arrive at

$$\delta_\zeta I_{ij} = \frac{16G^2M}{c^8} \int_0^{+\infty} d\tau \left[L_\varepsilon(\tau) + \frac{1}{2} \right] \left(I_{ij}^{(1)}(t) W^{(3)}(t - \tau) - I_{ij}(t) W^{(4)}(t - \tau) \right) + \mathcal{O}\left(\frac{1}{c^{10}}\right). \quad (\text{VI.5})$$

From now on, we suppose that this adjustment has been made, and that $I_{ij}^{(d)}$ has been computed from its definition (VI.1) in the correct coordinate system. We now define, at 4PN accuracy, a *renormalized* source quadrupole as

$$I_{ij}^{\text{renorm}} \equiv I_{ij}^{(d)} + \delta_\varepsilon I_{ij} \quad (\text{VI.6})$$

where $\delta_\varepsilon I_{ij}$ is defined so as to account for the pole parts arising in (III.33) and (V.48). It reads

$$\begin{aligned} \delta_\varepsilon I_{ij} \equiv & -\frac{16G^2M}{c^8} \int_0^{+\infty} d\tau \left[L_\varepsilon(\tau) + \frac{1}{2} \right] \left(I_{ij}^{(1)}(t) W^{(3)}(t - \tau) - I_{ij}(t) W^{(4)}(t - \tau) \right) \\ & - \frac{214G^2M^2}{105c^6} \left(\Pi_\varepsilon + \frac{246299}{44940} \right) I_{ij}^{(4)} \\ & + \frac{G^2M}{c^8} \left[-\frac{4}{7} \left(\Pi_\varepsilon - \frac{1447}{216} \right) I_{a\langle i} I_{j\rangle a}^{(6)} - \frac{32}{7} \left(\Pi_\varepsilon - \frac{17783}{10080} \right) I_{a\langle i} I_{j\rangle a}^{(5)} \right. \\ & \quad \left. - 4 \left(\Pi_\varepsilon - \frac{27649}{17640} \right) I_{a\langle i} M_{j\rangle a}^{(4)} + \frac{1921}{945} I_{a\langle i} I_{j\rangle a}^{(3)} + \frac{4}{3} \left(\Pi_\varepsilon + \frac{11243}{7560} \right) I_{a\langle i} J_{j\rangle |a}^{(5)} \right] \end{aligned} \quad (\text{VI.7})$$

where we have used $M_{ij} = I_{ij} + \mathcal{O}(c^{-5})$ and $I_{ij} = J_{ij} + \mathcal{O}(c^{-5})$, and where we remind the reader of the notations

$$\Pi_\varepsilon = -\frac{1}{2\varepsilon} + \ln\left(\frac{r_0\sqrt{q}}{\ell_0}\right) \quad \text{and} \quad L_\varepsilon(\tau) = -\frac{1}{2\varepsilon} + \ln\left(\frac{c\tau\sqrt{q}}{2\ell_0}\right).$$

We find that $\mathbb{I}_{ij}^{\text{renorm}}$ defined in this way does not exhibit any poles. Although this is expected from the point of view of the general formalism, we should point out that it is not trivial computationally that the pole arising from the source/canonical and canonical/radiative moments relations should indeed cancel the poles in the source quadrupole, as these are radically different computations. When reduced for compact binary systems on circular orbits, its expression is still given by (6.11) of [162], despite the two extra contributions we have noted. We find that this renormalized source quadrupole $\mathbb{I}_{ij}^{\text{renorm}}$ can be related to the *three-dimensional* canonical moment M_{ij} (i.e. that enters the *three-dimensional* expressions of the radiative quadrupole \mathcal{U}_{ij}) by the expression

$$\begin{aligned} M_{ij} = & \mathbb{I}_{ij}^{\text{renorm}} + \frac{4G}{c^5} \left[W^{(2)} I_{ij} - W^{(1)} I_{ij}^{(1)} \right] \\ & + \frac{4G}{c^7} \left\{ \frac{4}{7} W_{a\langle i}^{(1)} I_{j\rangle a}^{(3)} + \frac{6}{7} W_{a\langle i}^{(1)} I_{j\rangle a}^{(4)} - \frac{1}{7} Y_{a\langle i}^{(3)} I_{j\rangle a} - Y_{a\langle i}^{(3)} I_{j\rangle a}^{(3)} - 2X I_{ij}^{(3)} \right. \\ & - \frac{5}{21} W_a^{(4)} I_{ija} + \frac{1}{63} W_a^{(3)} I_{ija}^{(1)} - \frac{25}{21} Y_a^{(3)} I_{ija} - \frac{22}{63} Y_a^{(2)} I_{ija}^{(1)} + \frac{5}{63} Y_a^{(1)} I_{ija}^{(2)} \\ & + 2W^{(3)} W_{ij} + 2W^{(2)} W_{ij}^{(1)} - \frac{4}{3} W_{\langle i} W_{j\rangle}^{(3)} + 2W^{(2)} Y_{ij} - 4W_{\langle i} Y_{j\rangle}^{(2)} \\ & \left. + \epsilon_{ab\langle i} \left[\frac{1}{3} I_{j\rangle a} Z_b^{(3)} - I_{j\rangle a}^{(3)} Z_b + \frac{4}{9} J_{j\rangle a} W_b^{(3)} - \frac{4}{9} J_{j\rangle a} Y_b^{(2)} + \frac{8}{9} J_{j\rangle a}^{(1)} Y_b^{(1)} \right] \right\}, \quad (\text{VI.8}) \end{aligned}$$

where, in the nonlinear corrections, we do not distinguish $\mathbb{I}_{ij}^{\text{renorm}}$ from I_{ij} , since they do not differ at 2.5PN order. We have thus now successfully eliminated the poles from all the steps of the computation, which is very useful in practical computations. For notational simplicity, we will in the rest of this thesis most often refer to the renormalized source moment simply as I_{ij} .

VI.1.2 Expression of the renormalized source quadrupole moment

We now reproduce the explicit expression of the renormalized source quadrupole for *quasicircular* compact binary systems, which is correctly accounted for by (6.11) of [162]. Recalling that we now simply write I_{ij} for the renormalized source moment, and denoting by r the orbital separation and by \mathbf{v} the relative velocity of the two particles, we have

$$I_{ij} = m\nu \left(A x_{\langle i} x_{j\rangle} + B \frac{r^2}{c^2} v_{\langle i} v_{j\rangle} + \frac{G^2 m^2 \nu}{c^5 r} C x_{\langle i} v_{j\rangle} \right) + \mathcal{O} \left(\frac{1}{c^9} \right), \quad (\text{VI.9})$$

where $\gamma \equiv Gm/(rc^2)$ is useful to express the coefficients, which are given by

$$\begin{aligned} A = & 1 + \gamma \left(-\frac{1}{42} - \frac{13}{14} \nu \right) + \gamma^2 \left(-\frac{461}{1512} - \frac{18395}{1512} \nu - \frac{241}{1512} \nu^2 \right) \\ & + \gamma^3 \left\{ \frac{395899}{13200} - \frac{428}{105} \ln \left(\frac{r}{r_0} \right) + \left[\frac{3304319}{166320} - \frac{44}{3} \ln \left(\frac{r}{r_0} \right) \right] \nu + \frac{162539}{16632} \nu^2 + \frac{2351}{33264} \nu^3 \right\} \\ & + \gamma^4 \left\{ -\frac{1067041075909}{12713500800} + \frac{31886}{2205} \ln \left(\frac{r}{r_0} \right) \right\} \end{aligned}$$

$$\begin{aligned}
& + \left[-\frac{85244498897}{470870400} - \frac{2783}{1792}\pi^2 - \frac{64}{7}\ln(16\gamma e^{2\gamma_E}) - \frac{10886}{735}\ln\left(\frac{r}{r_0}\right) + \frac{8495}{63}\ln\left(\frac{r}{r'_0}\right) \right] \nu \\
& + \left[\frac{171906563}{4484480} + \frac{44909}{2688}\pi^2 - \frac{4897}{21}\ln\left(\frac{r}{r'_0}\right) \right] \nu^2 - \frac{22063949}{5189184}\nu^3 + \frac{71131}{314496}\nu^4 \Big\}, \\
\end{aligned} \tag{VI.10a}$$

$$\begin{aligned}
B = & \frac{11}{21} - \frac{11}{7}\nu + \gamma \left(\frac{1607}{378} - \frac{1681}{378}\nu + \frac{229}{378}\nu^2 \right) \\
& + \gamma^2 \left(-\frac{357761}{19800} + \frac{428}{105}\ln\left(\frac{r}{r_0}\right) - \frac{92339}{5544}\nu + \frac{35759}{924}\nu^2 + \frac{457}{5544}\nu^3 \right) \\
& + \gamma^3 \left(\frac{23006898527}{1589187600} - \frac{4922}{2205}\ln\left(\frac{r}{r_0}\right) \right. \\
& \quad + \left[\frac{8431514969}{529729200} + \frac{143}{192}\pi^2 - \frac{32}{7}\ln(16\gamma e^{2\gamma_E}) - \frac{1266}{49}\ln\left(\frac{r}{r_0}\right) - \frac{968}{63}\ln\left(\frac{r}{r'_0}\right) \right] \nu \\
& \quad \left. + \left[\frac{351838141}{5045040} - \frac{41}{24}\pi^2 + \frac{968}{21}\ln\left(\frac{r}{r'_0}\right) \right] \nu^2 - \frac{1774615}{81081}\nu^3 - \frac{3053}{432432}\nu^4 \right), \\
\end{aligned} \tag{VI.10b}$$

$$C = \frac{48}{7} + \gamma \left(-\frac{4096}{315} - \frac{24512}{945}\nu \right) - \frac{32}{7}\pi\gamma^{3/2}. \tag{VI.10c}$$

The coefficients A and B represent the conservative part of the quadrupole, while C is due to the radiation reaction dissipative effects. Note that, in addition to the scale r_0 , the expression of the quadrupole involves a scale r'_0 , which appeared in the equations of motion in Section II.6, associated with the UV regularization (see the Footnote 10 of [98] for more details).

As discussed in [163], the source quadrupole moment is not a local quantity at the 4PN order anymore, as it contains a nonlocal tail integral, given by (6.5) in [162]. The 4PN logarithms $\ln(16\gamma e^{2\gamma_E})$ in A and B are due to the conservative part of this nonlocal tail term in the mass quadrupole, and the coefficient $-\frac{32}{7}\pi\gamma^{3/2}$ in C , to the corresponding dissipative part of the tail term.

Finally, as reported in Appendix A.4, the general expression of the source mass quadrupole moment for generic orbits has been conclusively tested in the so-called boosted Schwarzschild limit.

VI.1.3 The other source moments and their relation to the canonical moments

In order to obtain the 4PN waveform, we need the mass source octupole and the current source quadrupole at 3PN order. In the context of dimensional regularization, these exhibit poles, which are exactly compensated by the poles in the relation between the associated radiative moments and canonical moments, see (V.48b) and (V.48c). This calls for a natural definition of a renormalized source moment. In both cases, these renormalized source moments exactly coincide at 3PN order with the ones computed with Hadamard regularization in the IR, see [162] for discussion.

We now reproduce the expression of the renormalized mass source octupole and the

current source quadrupole for circular orbits. The mass source octupole moment reads

$$I_{ijk} = -\nu m \delta \left\{ D x_{\langle i} x_j x_{k\rangle} + E \frac{r}{c} v_{\langle i} x_j x_{k\rangle} + F \frac{r^2}{c^2} v_{\langle i} v_j x_{k\rangle} + G \frac{r^3}{c^3} v_{\langle i} v_j v_{k\rangle} \right\} + \mathcal{O}\left(\frac{1}{c^7}\right), \quad (\text{VI.11})$$

where $\delta \equiv (m_1 - m_2)/m$, and the coefficients are

$$D = 1 - \gamma\nu + \gamma^2 \left(-\frac{139}{330} - \frac{11923}{660}\nu - \frac{29}{110}\nu^2 \right) \quad (\text{VI.12a})$$

$$+ \gamma^3 \left\{ \frac{1229440}{63063} + \frac{610499}{20020}\nu + \frac{319823}{17160}\nu^2 - \frac{101}{2340}\nu^3 - \frac{26}{7} \ln\left(\frac{r}{r_0}\right) - 22\nu \ln\left(\frac{r}{r'_0}\right) \right\},$$

$$E = \frac{196}{15} \gamma^2 \nu, \quad (\text{VI.12b})$$

$$F = 1 - 2\nu + \gamma \left(\frac{1066}{165} - \frac{1433}{330}\nu + \frac{21}{55}\nu^2 \right) \quad (\text{VI.12c})$$

$$+ \gamma^2 \left\{ -\frac{1130201}{48510} - \frac{989}{33}\nu + \frac{20359}{330}\nu^2 - \frac{37}{198}\nu^3 + \frac{52}{7} \ln\left(\frac{r}{r_0}\right) \right\},$$

$$G = 0. \quad (\text{VI.12d})$$

As for the renormalized current quadrupole J_{ij} , it also coincides at 3PN order with the one computed with Hadamard regularization in the IR [162]. It comes [96]

$$J_{ij} = -\nu m \delta \left[H L^{(i} x^{j)} + K \frac{Gm}{c^3} L^{(i} v^{j)} \right] + \mathcal{O}\left(\frac{1}{c^7}\right), \quad (\text{VI.13})$$

where we denote $L^i \equiv \epsilon_{ijk} x^j v^k$, and where

$$H = 1 + \gamma \left(\frac{67}{28} - \frac{2}{7}\nu \right) + \gamma^2 \left(\frac{13}{9} - \frac{4651}{252}\nu - \frac{\nu^2}{168} \right) \\ + \gamma^3 \left\{ \frac{2301023}{415800} - \frac{214}{105} \ln\left(\frac{r}{r_0}\right) + \left[-\frac{243853}{9240} + \frac{123}{128}\pi^2 - 22 \ln\left(\frac{r}{r'_0}\right) \right] \nu \right. \\ \left. + \frac{44995}{5544}\nu^2 + \frac{599}{16632}\nu^3 \right\}, \quad (\text{VI.14a})$$

$$K = \frac{188}{35} \nu \gamma. \quad (\text{VI.14b})$$

The remaining required source moments (mass hexadecapole at 2PN, current octupole at 2PN, *etc.*) are easy to compute as no regularization subtleties arise at this order. Their expressions on quasicircular orbits are displayed e.g. in Section 9.1 of [14]. Note that, as the first nonlocal feature cannot appear at a lower order than 4PN, all those higher-order source moments are instantaneous up to 3.5PN order. Therefore, they cannot contribute to the 4.5PN term of the quasicircular flux.

We now reproduce the relations between source and canonical moments that will be relevant for our computation. We have [161]

$$M_{ijk} = I_{ijk} + \frac{4G}{c^5} \left[W^{(2)} I_{ijk} - W^{(1)} I_{ijk}^{(1)} + 3Y_{\langle i}^{(1)} I_{jk\rangle} \right] + \mathcal{O}\left(\frac{1}{c^7}\right), \quad (\text{VI.15a})$$

$$M_{ijkl} = I_{ijkl} + \frac{4G}{c^5} \left[W^{(2)} I_{ijkl} - W^{(1)} I_{ijkl}^{(1)} + 4Y_{\langle i}^{(1)} I_{jkl} \right] + \mathcal{O} \left(\frac{1}{c^6} \right), \quad (\text{VI.15b})$$

$$S_{ij} = J_{ij} + \frac{4G}{c^5} \left[-W^{(1)} J_{ij}^{(1)} + \frac{3}{2} Y_{\langle i}^{(1)} J_{j \rangle} \right. \\ \left. + \epsilon_{ab\langle i} \left(\frac{1}{2} I_{j \rangle a}^{(3)} W_b + I_{j \rangle a} Y_b^{(2)} - \frac{1}{2} I_{j \rangle a}^{(1)} Y_b^{(1)} \right) \right] + \mathcal{O} \left(\frac{1}{c^7} \right), \quad (\text{VI.15c})$$

$$S_{ijk} = J_{ijk} + \frac{4G}{c^5} \left[-W^{(1)} J_{ijk}^{(1)} + \frac{8}{3} Y_{\langle i}^{(1)} J_{jk \rangle} \right. \\ \left. + \epsilon_{ab\langle i} \left(I_{j \underline{a}}^{(3)} W_{k \rangle b} + I_{jk \rangle a} Y_b^{(2)} - \frac{1}{3} I_{jk \rangle a}^{(1)} Y_b^{(1)} \right) \right] + \mathcal{O} \left(\frac{1}{c^6} \right). \quad (\text{VI.15d})$$

VI.2 The radiative moments in terms of the canonical moments

As we have seen in Section VI.1, the notion of renormalized canonical and source moments allows us to use the three-dimensional relations between canonical and radiative moments. In this section, we review which radiative moments are needed and at what order for the different pieces of the computation, and reproduce the relevant expressions.

VI.2.1 Radiative moments entering the flux at 4PN order

In order to derive the energy flux (II.66) at 4PN order beyond leading quadrupolar order, the obvious first input is the radiative quadrupole moment itself \mathcal{U}_{ij} to 4PN order. Recalling that, at leading order, the radiative moments \mathcal{U}_L and \mathcal{V}_L reduce to the ℓ^{th} time derivatives (with respect to the retarded time u of the radiative coordinates) of the canonical moments M_L and S_L , we straightforwardly write

$$\mathcal{U}_{ij} = M_{ij}^{(2)} + \mathcal{U}_{ij}^{1.5\text{PN}} + \mathcal{U}_{ij}^{2.5\text{PN}} + \mathcal{U}_{ij}^{3\text{PN}} + \mathcal{U}_{ij}^{3.5\text{PN}} + \mathcal{U}_{ij}^{4\text{PN}} + \mathcal{O} \left(\frac{1}{c^9} \right), \quad (\text{VI.16})$$

with small PN corrections up to 4PN, as indicated.

The leading correction at 1.5PN is due to the GW tail, i.e. the quadratic interaction between the static mass M and the mass quadrupole moment M_{ij} , denoted $M \times M_{ij}$, see (V.34) and Fig. II.4a. It reads explicitly [164]

$$\mathcal{U}_{ij}^{1.5\text{PN}} = \frac{2GM}{c^3} \int_0^{+\infty} d\tau M_{ij}^{(4)}(u - \tau) \left[\ln \left(\frac{c\tau}{2b_0} \right) + \frac{11}{12} \right]. \quad (\text{VI.17})$$

As usual, the length scale b_0 is an arbitrary constant linked with the choice of the origin of time of the asymptotic radiative coordinates, with respect to the harmonic coordinates covering the source's near zone.

At the next 2.5PN order there is the nonlocal interaction involving two quadrupole moments $M_{ij} \times M_{kl}$ called the (displacement) ‘‘memory’’ (see Fig. II.4b), together with associated instantaneous quadrupole-quadrupole terms, and including also an instantaneous

interaction $M_{ij} \times S_k$:

$$\mathcal{U}_{ij}^{2.5\text{PN}} = \frac{G}{c^5} \left\{ -\frac{2}{7} \int_0^{+\infty} d\tau \left[M_{a\langle i}^{(3)} M_{j\rangle a}^{(3)} \right] (u - \tau) + \frac{1}{7} M_{a\langle i}^{(5)} M_{j\rangle a} - \frac{5}{7} M_{a\langle i}^{(4)} M_{j\rangle a}^{(1)} - \frac{2}{7} M_{a\langle i}^{(3)} M_{j\rangle a}^{(2)} + \frac{1}{3} \epsilon_{ab\langle i} M_{j\rangle a}^{(4)} S_b \right\}. \quad (\text{VI.18})$$

At the 3PN order appears the first cubic interaction, namely the ‘‘tail-of-tail’’, consisting of the interplay between two masses and the quadrupole moment, i.e., $M^2 \times M_{ij}$. We thus have $\mathcal{U}_{ij}^{3\text{PN}} = \mathcal{U}_{ij}^{M^2 \times M_{ij}}$, where $\mathcal{U}_{ij}^{M^2 \times M_{ij}}$ is given by (V.45a), see also [91, 161]. Note the appearance here of the constant length scale r_0 , and which should be distinguished from the previously introduced scale b_0 . The coefficient in front of the $\ln r_0$ term in (V.45a) is exactly the β -function coefficient associated to the renormalization of the mass quadrupole moment, say $\beta_2 = -\frac{214}{105}$. In the present formalism, this scale arises from the Hadamard regularization scheme .

The next-order 3.5PN term has a structure similar to the 2.5PN one, i.e. with some memory type integrals and instantaneous terms. The interactions between moments are however more complicated but still of quadratic nature:

$$\begin{aligned} \mathcal{U}_{ij}^{3.5\text{PN}} = \frac{G}{c^7} \left\{ \int_0^{+\infty} d\tau \left[-\frac{5}{756} M_{ab}^{(4)} M_{ijab}^{(4)} - \frac{32}{63} S_{a\langle i}^{(3)} S_{j\rangle a}^{(3)} \right] (u - \tau) \right. \\ - \frac{1}{432} M_{ab} M_{ijab}^{(7)} + \frac{1}{432} M_{ab}^{(1)} M_{ijab}^{(6)} - \frac{5}{756} M_{ab}^{(2)} M_{ijab}^{(5)} + \frac{19}{648} M_{ab}^{(3)} M_{ijab}^{(4)} \\ + \frac{1957}{3024} M_{ab}^{(4)} M_{ijab}^{(3)} + \frac{1685}{1008} M_{ab}^{(5)} M_{ijab}^{(2)} + \frac{41}{28} M_{ab}^{(6)} M_{ijab}^{(1)} + \frac{91}{216} M_{ab}^{(7)} M_{ijab} \\ - \frac{5}{252} M_{ab\langle i} M_{j\rangle ab}^{(7)} + \frac{5}{189} M_{ab\langle i}^{(1)} M_{j\rangle ab}^{(6)} + \frac{5}{126} M_{ab\langle i}^{(2)} M_{j\rangle ab}^{(5)} + \frac{5}{2268} M_{ab\langle i}^{(3)} M_{j\rangle ab}^{(4)} \\ + \frac{5}{42} S_a S_{ija}^{(5)} + \frac{80}{63} S_{a\langle i} S_{j\rangle a}^{(5)} + \frac{16}{63} S_{a\langle i}^{(1)} S_{j\rangle a}^{(4)} - \frac{64}{63} S_{a\langle i}^{(2)} S_{j\rangle a}^{(3)} \\ \left. + \epsilon_{ac\langle i} \left(\int_0^{+\infty} d\tau \left[\frac{5}{42} S_{j\rangle bc}^{(4)} M_{ab}^{(3)} - \frac{20}{189} M_{j\rangle bc}^{(4)} S_{ab}^{(3)} \right] (u - \tau) \right. \right. \\ + \frac{1}{168} S_{j\rangle bc}^{(6)} M_{ab} + \frac{1}{24} S_{j\rangle bc}^{(5)} M_{ab}^{(1)} + \frac{1}{28} S_{j\rangle bc}^{(4)} M_{ab}^{(2)} - \frac{1}{6} S_{j\rangle bc}^{(3)} M_{ab}^{(3)} \\ + \frac{3}{56} S_{j\rangle bc}^{(2)} M_{ab}^{(4)} + \frac{187}{168} S_{j\rangle bc}^{(1)} M_{ab}^{(5)} + \frac{65}{84} S_{j\rangle bc} M_{ab}^{(6)} \\ + \frac{1}{189} M_{j\rangle bc}^{(6)} S_{ab} - \frac{1}{189} M_{j\rangle bc}^{(5)} S_{ab}^{(1)} + \frac{10}{189} M_{j\rangle bc}^{(4)} S_{ab}^{(2)} + \frac{32}{189} M_{j\rangle bc}^{(3)} S_{ab}^{(3)} \\ \left. \left. + \frac{65}{189} M_{j\rangle bc}^{(2)} S_{ab}^{(4)} - \frac{5}{189} M_{j\rangle bc}^{(1)} S_{ab}^{(5)} - \frac{10}{63} M_{j\rangle bc} S_{ab}^{(6)} \right) \right\}. \quad (\text{VI.19}) \end{aligned}$$

At the 4PN order appear the ‘‘tail-of-memory’’ and ‘‘spin-quadrupole tail’’, which are, respectively, due to the $M \times M_{ij} \times M_{kl}$ and $M \times S_i \times M_{jk}$ interactions, that we have extensively studied in Chapters IV and V. Thus, we have

$$\mathcal{U}_{ij}^{4\text{PN}} = \mathcal{U}_{ij}^{M \times S_i \times M_{ij}} + \mathcal{U}_{ij}^{M \times M_{ij} \times M_{ij}}, \quad (\text{VI.20})$$

where $\mathcal{U}_{ij}^{M \times S_i \times M_{ij}}$ is given by (V.45b) and $\mathcal{U}_{ij}^{M \times M_{ij} \times M_{ij}}$ by (V.45c), see also [4, 3].

Besides the mass quadrupole moment, the expression of the flux (II.66) at the 4PN order requires the knowledge of moments of higher multipolarity, but evaluated at lower PN orders. Indeed, the computation of the flux involves the squares of the radiative moments, therefore in order to obtain the 4PN flux, we need \mathcal{U}_L at $(4 - \ell)$ PN order and \mathcal{V}_L at $(3 - \ell)$ PN order, where $\ell \geq 2$. Thus at the next multipolar order, we need the mass octupole \mathcal{U}_{ijk} and current quadrupole \mathcal{V}_{ij} moments with a 3PN precision:

$$\mathcal{U}_{ijk} = M_{ijk}^{(3)} + \mathcal{U}_{ijk}^{1.5\text{PN}} + \mathcal{U}_{ijk}^{2.5\text{PN}} + \mathcal{U}_{ijk}^{3\text{PN}} + \mathcal{O}\left(\frac{1}{c^7}\right), \quad (\text{VI.21a})$$

$$\mathcal{V}_{ij} = S_{ij}^{(2)} + \mathcal{V}_{ij}^{1.5\text{PN}} + \mathcal{V}_{ij}^{2.5\text{PN}} + \mathcal{V}_{ij}^{3\text{PN}} + \mathcal{O}\left(\frac{1}{c^7}\right). \quad (\text{VI.21b})$$

Reporting the results of [161], they read

$$\mathcal{U}_{ijk}^{1.5\text{PN}} = \frac{2GM}{c^3} \int_0^{+\infty} d\tau M_{ijk}^{(5)}(u - \tau) \left[\ln\left(\frac{c\tau}{2b_0}\right) + \frac{97}{60} \right], \quad (\text{VI.22a})$$

$$\begin{aligned} \mathcal{U}_{ijk}^{2.5\text{PN}} = \frac{G}{c^5} \left\{ \int_0^{+\infty} d\tau \left[-\frac{1}{3} M_{a\langle i}^{(3)} M_{jk\rangle a}^{(4)} - \frac{4}{5} \epsilon_{ab\langle i} M_{j\bar{a}}^{(3)} S_{k\rangle b}^{(3)} \right] (u - \tau) \right. \\ + \frac{1}{4} M_{a\langle i} M_{jk\rangle a}^{(6)} + \frac{1}{4} M_{a\langle i}^{(1)} M_{jk\rangle a}^{(5)} + \frac{1}{4} M_{a\langle i}^{(2)} M_{jk\rangle a}^{(4)} - \frac{4}{3} M_{a\langle i}^{(3)} M_{jk\rangle a}^{(3)} \\ - \frac{9}{4} M_{a\langle i}^{(4)} M_{jk\rangle a}^{(2)} - \frac{3}{4} M_{a\langle i}^{(5)} M_{jk\rangle a}^{(1)} + \frac{1}{12} M_{a\langle i}^{(6)} M_{jk\rangle a} + \frac{12}{5} S_{\langle i} S_{jk\rangle}^{(4)} \\ + \epsilon_{ab\langle i} \left[\frac{9}{5} M_{j\bar{a}} S_{k\rangle b}^{(5)} + \frac{27}{5} M_{j\bar{a}}^{(1)} S_{k\rangle b}^{(4)} + \frac{8}{5} M_{j\bar{a}}^{(2)} S_{k\rangle b}^{(3)} + \frac{12}{5} M_{j\bar{a}}^{(3)} S_{k\rangle b}^{(2)} \right. \\ \left. + \frac{3}{5} M_{j\bar{a}}^{(4)} S_{k\rangle b}^{(1)} + \frac{1}{5} M_{j\bar{a}}^{(5)} S_{k\rangle b} + \frac{9}{20} M_{jk\rangle a}^{(5)} S_b \right] \left. \right\}, \quad (\text{VI.22b}) \end{aligned}$$

$$\mathcal{U}_{ijk}^{3\text{PN}} = \frac{2G^2 M^2}{c^6} \int_0^{+\infty} d\tau M_{ijk}^{(6)}(u - \tau) \left[\ln^2\left(\frac{c\tau}{2b_0}\right) + \frac{97}{30} \ln\left(\frac{c\tau}{2b_0}\right) - \frac{13}{21} \ln\left(\frac{c\tau}{2r_0}\right) + \frac{13283}{8820} \right], \quad (\text{VI.22c})$$

where the underlined indices within angled brackets are excluded from the STF operation, and

$$\mathcal{V}_{ij}^{1.5\text{PN}} = \frac{2GM}{c^3} \int_0^{+\infty} d\tau S_{ij}^{(4)}(u - \tau) \left[\ln\left(\frac{c\tau}{2b_0}\right) + \frac{7}{6} \right], \quad (\text{VI.22d})$$

$$\begin{aligned} \mathcal{V}_{ij}^{2.5\text{PN}} = \frac{G}{c^5} \left\{ -\frac{3}{7} M_{a\langle i} S_{j\rangle a}^{(5)} - \frac{3}{7} M_{a\langle i}^{(1)} S_{j\rangle a}^{(4)} + \frac{8}{7} M_{a\langle i}^{(2)} S_{j\rangle a}^{(3)} + \frac{4}{7} M_{a\langle i}^{(3)} S_{j\rangle a}^{(2)} \right. \\ + \frac{17}{7} M_{a\langle i}^{(4)} S_{j\rangle a}^{(1)} + \frac{9}{7} M_{a\langle i}^{(5)} S_{j\rangle a} - \frac{1}{28} M_{ija}^{(5)} S_a \\ + \epsilon_{ab\langle i} \left[-\frac{15}{56} M_{j\rangle ac} M_{bc}^{(6)} - \frac{113}{112} M_{j\rangle ac}^{(1)} M_{bc}^{(5)} - \frac{353}{336} M_{j\rangle ac}^{(2)} M_{bc}^{(4)} - \frac{3}{14} M_{j\rangle ac}^{(3)} M_{bc}^{(3)} \right. \\ \left. + \frac{5}{168} M_{j\rangle ac}^{(4)} M_{bc}^{(2)} + \frac{3}{112} M_{j\rangle ac}^{(5)} M_{bc}^{(1)} - \frac{3}{112} M_{j\rangle ac}^{(6)} M_{bc} + S_{j\rangle a}^{(4)} S_b \right] \left. \right\}, \quad (\text{VI.22e}) \end{aligned}$$

$$\mathcal{V}_{ij}^{3\text{PN}} = \frac{2G^2M^2}{c^6} \int_0^{+\infty} d\tau S_{ij}^{(5)}(u-\tau) \left[\ln^2\left(\frac{c\tau}{2b_0}\right) + \frac{7}{3} \ln\left(\frac{c\tau}{2b_0}\right) - \frac{107}{105} \ln\left(\frac{c\tau}{2r_0}\right) - \frac{13127}{11025} \right]. \quad (\text{VI.22f})$$

Higher order multipole moments have no cubic contributions at the required PN order. At 2PN order, we need the mass hexadecapole \mathcal{U}_{ijkl} as well as the current octupole \mathcal{V}_{ijk} , and \mathcal{U}_{ijklm} as well as \mathcal{V}_{ijkl} , which read

$$\mathcal{U}_{ijkl} = M_{ijkl}^{(4)} + \mathcal{U}_{ijkl}^{1.5\text{PN}} + \mathcal{O}\left(\frac{1}{c^5}\right), \quad (\text{VI.23a})$$

$$\mathcal{V}_{ijk} = S_{ijk}^{(3)} + \mathcal{V}_{ijk}^{1.5\text{PN}} + \mathcal{O}\left(\frac{1}{c^5}\right), \quad (\text{VI.23b})$$

where [161]

$$\begin{aligned} \mathcal{U}_{ijkl}^{1.5\text{PN}} = \frac{G}{c^3} & \left\{ 2M \int_0^{+\infty} d\tau M_{ijkl}^{(6)}(u-\tau) \left[\ln\left(\frac{c\tau}{2b_0}\right) + \frac{59}{30} \right] + \frac{2}{5} \int_0^{+\infty} d\tau \left[M_{ij}^{(3)} M_{kl}^{(3)} \right](u-\tau) \right. \\ & \left. - \frac{21}{5} M_{\langle ij} M_{kl}^{(5)} - \frac{63}{5} M_{\langle ij}^{(1)} M_{kl}^{(4)} - \frac{102}{5} M_{\langle ij}^{(2)} M_{kl}^{(3)} \right\}, \end{aligned} \quad (\text{VI.24a})$$

$$\begin{aligned} \mathcal{V}_{ijk}^{1.5\text{PN}} = \frac{G}{c^3} & \left\{ 2M \int_0^{+\infty} d\tau S_{ijk}^{(5)}(u-\tau) \left[\ln\left(\frac{c\tau}{2b_0}\right) + \frac{5}{3} \right] \right. \\ & \left. - 2M_{\langle ij}^{(4)} S_{k\rangle} - \frac{1}{10} \epsilon_{ab\langle i} M_{j\bar{a}} M_{k\rangle b}^{(5)} + \frac{1}{2} \epsilon_{ab\langle i} M_{j\bar{a}}^{(1)} M_{k\rangle b}^{(4)} \right\}. \end{aligned} \quad (\text{VI.24b})$$

Note the appearance of a memory integral at 1.5PN order in \mathcal{U}_{ijkl} , in addition to the usual tail one. Finally, we need the moments \mathcal{U}_{ijklm} and \mathcal{V}_{ijkl} at 1PN, as well as \mathcal{U}_{ijklmn} and \mathcal{V}_{ijklm} at Newtonian order, which are trivially given by

$$\begin{aligned} \mathcal{U}_{ijklm} &= M_{ijklm}^{(5)} + \mathcal{O}\left(\frac{1}{c^3}\right), & \mathcal{V}_{ijkl} &= S_{ijkl}^{(4)} + \mathcal{O}\left(\frac{1}{c^3}\right), \\ \mathcal{U}_{ijklmn} &= M_{ijklmn}^{(6)} + \mathcal{O}\left(\frac{1}{c^3}\right), & \mathcal{V}_{ijklm} &= S_{ijklm}^{(5)} + \mathcal{O}\left(\frac{1}{c^3}\right). \end{aligned} \quad (\text{VI.25})$$

VI.2.2 Radiative moments entering the quasicircular flux at the 4.5PN order

For quasicircular orbits, the 4.5PN term in the flux was obtained in [167]. One could naively think that such a computation would require the complete knowledge of the relations between radiative and canonical moments, as presented above, but pushed one half PN order further. This was actually not the case, since for circular orbits, only a limited control of the relation between the radiative and canonical mass quadrupole moments was necessary. This is discussed in [167], and we only remind the key points. First, it is well known that the contributions of instantaneous interactions entering the flux at half-integer PN order (e.g. 4.5PN order) vanish for quasicircular orbits. So only nonlocal contributions such as tails can potentially contribute. Second, the quadratic nonlocal memory interaction that enters the radiative moments, see (VI.18) and (VI.19), become

instantaneous in the flux by virtue of time differentiation, so these will not contribute. Last, the tails-of-memory and spin-quadrupole tails, which both enter at 4PN, will next contribute at 5PN but not at 4.5PN. This allows to use dimensional arguments to determine the interactions that can contribute to the 4.5PN quasicircular flux. For the mass quadrupole, only the quartic $M^3 \times M_{ij}$, naturally dubbed “tails-of-tails-of-tails”, can play a role. It has been computed in [167], and reads

$$\begin{aligned} \mathcal{U}_{ij}^{4.5\text{PN}} \Big|_{T_0}^{T} = \frac{G^3 M^3}{c^9} \int_0^{+\infty} d\tau M_{ij}^{(6)}(u - \tau) & \left[\frac{4}{3} \ln^3 \left(\frac{c\tau}{2b_0} \right) + \frac{11}{3} \ln^2 \left(\frac{c\tau}{2b_0} \right) \right. \\ & - \frac{428}{105} \ln \left(\frac{c\tau}{2b_0} \right) \ln \left(\frac{c\tau}{2r_0} \right) + \frac{124627}{11025} \ln \left(\frac{c\tau}{2b_0} \right) \\ & \left. - \frac{1177}{315} \ln \left(\frac{c\tau}{2r_0} \right) + \frac{129268}{33075} + \frac{428}{315} \pi^2 \right]. \end{aligned} \quad (\text{VI.26})$$

The full $\mathcal{U}_{ij}^{4.5\text{PN}}$ also contains quadratic memory interactions, like those entering at 2.5PN and 3.5PN, see (VI.18) and (VI.19). If, as explained above, those are not needed to compute the flux (and phase) for quasicircular orbits, they will enter the expression of the (2, 2) mode. As they are yet undetermined, they restrict the accuracy we can reach when deriving the (2, 2) mode, which is why it is presented in Section VI.4 at 4PN and not up to 4.5PN order. In addition, radiation reaction effects at 4.5PN in the mass quadrupole should contribute and are also not under control.

Regarding other moments, the 3.5PN terms of the mass octupole \mathcal{U}_{ijk} and current quadrupole \mathcal{V}_{ij} , as well as the 2.5PN terms of the mass hexadecapole \mathcal{U}_{ijkl} and current octupole \mathcal{V}_{ijk} , are composed of quadratic memory integrals, but which cannot contribute in the 4.5PN quasicircular flux. The only other moments that can contribute to the 4.5PN piece of the flux are thus [161, 167]

$$\begin{aligned} \mathcal{U}_{ijklm}^{1.5\text{PN}} = \frac{G}{c^3} \left\{ 2M \int_0^{+\infty} d\tau M_{ijklm}^{(7)}(u - \tau) \left[\ln \left(\frac{c\tau}{2b_0} \right) + \frac{232}{105} \right] + \frac{20}{21} \int_0^{+\infty} d\tau \left[M_{ij}^{(3)} M_{klm}^{(4)} \right] (u - \tau) \right. \\ - \frac{15}{7} M_{ij}^{(2)} M_{klm}^{(6)} - \frac{41}{7} M_{ij}^{(1)} M_{klm}^{(5)} - \frac{120}{7} M_{ij}^{(2)} M_{klm}^{(4)} - \frac{710}{21} M_{ij}^{(3)} M_{klm}^{(3)} \\ \left. - \frac{265}{7} M_{ij}^{(4)} M_{klm}^{(2)} - \frac{155}{7} M_{ij}^{(5)} M_{klm}^{(1)} - \frac{34}{7} M_{ij}^{(6)} M_{klm} \right\}, \end{aligned} \quad (\text{VI.27a})$$

$$\begin{aligned} \mathcal{V}_{ijkl}^{1.5\text{PN}} = \frac{G}{c^3} \left\{ 2M \int_0^{+\infty} d\tau S_{ijkl}^{(6)}(u - \tau) \left[\ln \left(\frac{c\tau}{2b_0} \right) + \frac{119}{60} \right] - \frac{11}{6} M_{ij} S_{kl}^{(5)} - \frac{25}{6} M_{ij}^{(1)} S_{kl}^{(4)} \right. \\ - \frac{25}{3} M_{ij}^{(2)} S_{kl}^{(3)} - \frac{35}{3} M_{ij}^{(3)} S_{kl}^{(2)} - \frac{65}{6} M_{ij}^{(4)} S_{kl}^{(1)} - \frac{19}{6} M_{ij}^{(5)} S_{kl} - \frac{11}{12} M_{ij}^{(5)} S_{kl} \\ + \epsilon_{ab(i} \left[\frac{1}{12} M_{j\bar{a}} M_{kl)b}^{(6)} + \frac{37}{60} M_{j\bar{a}}^{(1)} M_{kl)b}^{(5)} - \frac{5}{12} M_{j\bar{a}}^{(2)} M_{kl)b}^{(4)} - \frac{5}{6} M_{j\bar{a}}^{(3)} M_{kl)b}^{(3)} \right. \\ \left. - \frac{11}{12} M_{j\bar{a}}^{(4)} M_{kl)b}^{(2)} - \frac{1}{12} M_{j\bar{a}}^{(5)} M_{kl)b}^{(1)} + \frac{3}{60} M_{j\bar{a}}^{(6)} M_{kl)b} \right] \left. \right\}. \end{aligned} \quad (\text{VI.27b})$$

VI.2.3 Radiative moments entering the full 4PN waveform

With the radiative mass quadrupole \mathcal{U}_{ij} at hand, we are able to compute the $(\ell, m) = (2, 2)$ mode in the 4PN waveform. However, the full 4PN waveform is still out of reach, because the other moments are not known at required order. In order to completely control the 4PN waveform, we would need \mathcal{U}_L at $\frac{8-\ell}{2}$ PN order and \mathcal{V}_L at $\frac{7-\ell}{2}$ PN order, for $\ell \geq 2$. As of now, apart from \mathcal{U}_{ij} which is known at 4PN, the waveform and the associated moments are only known to 3.5PN order [161, 96, 197], i.e. we control \mathcal{U}_L at $\frac{7-\ell}{2}$ PN order and \mathcal{V}_L at $\frac{6-\ell}{2}$ PN order. Computing these higher-order multipolar moments at required order will be crucial to obtain a full 4PN waveform, and a subject of interest for future work.

Note also that the arguments put forward in Section VI.2.2, which led to the computation of the 4.5PN flux, are not transposable to the 4.5PN piece of the $(2, 2)$ mode. Indeed, the latter computation would require the recently-computed 4.5PN piece of the equations of motion [153], the control of the 4.5PN source quadrupole moment, the full control of the 4.5PN relation between the source and canonical quadrupole, and the full control of the 4.5PN relation between the radiative and canonical moments, include all the local and memory terms which were ignored in the computation of the 4.5PN flux.

VI.3 Novelties in the reduction to quasicircular orbits

In Section II.5, we discussed how to describe the two body problem in the case of quasicircular orbits. Reducing the contributions of instantaneous terms in the radiative moments (i.e. terms that do not involve any integrals over the past) is computationally heavy, but a relatively trivial process. In Section VI.1, we have presented results for the source moments reduced for quasicircular orbits: these need only to be time-differentiated with 4PN accuracy using the equations of motion for circular orbits (II.58), and contracted with each other. Nonlocal terms, such as the memory and tail terms, are less trivial to compute, but there is a vast literature of integration techniques to perform this [198, 199, 192, 193]. Nonetheless, at 4PN order, some extra difficulties arise in the computation of these nonlocal effects. In this section, we focus on the post-adiabatic computation of the tail effect, as well as the computation of memory terms, including the novel tail-of-memory. These are crucial for the computation of the $(2, 2)$ mode at 4PN (Section VI.4) and the gravitational flux at 4.5PN (Section VI.5).

VI.3.1 Post-adiabatic integration of the tail effect

In order to compute the tail integrals, for instance (VI.17), we need to specify the behavior of the compact binary system's orbit in the remote past, as the effect is not localized in time, but integrates over the whole past history of the source. In the case of quasicircular orbits, and up to 3.5PN precision in the multipoles, an adiabatic approximation (considering the orbital elements r_{12} and ω to be constant in time, see Fig. II.3b) is sufficient, and one can follow the lines of [199], together with the integrals presented in the Appendix B of [198]. However, (II.58) show that this adiabatic approximation is no longer valid at a relative 2.5PN precision. As the tail enters at 1.5PN order in the moments, the first “post-adiabatic” (PA) correction will affect the moments at the 4PN order, thus we need to properly evaluate it in order to consistently derive the 4PN flux

and (2, 2) mode. An alternative and more general derivation is proposed in [6], but here we propose a more historical derivation, which closely follows the adiabatic computation of [199].

As shown in [199], the tail integrals on quasicircular orbits reduce, after projection on the appropriate basis (see Fig. II.3a), to elementary integrals of the type

$$\mathcal{I}_{\alpha,n} = \int_0^{+\infty} d\tau \ln\left(\frac{\tau}{\tau_0}\right) [y(u-\tau)]^\alpha e^{-in\phi(u-\tau)}, \quad (\text{VI.28})$$

where $n \in \mathbb{N}^*$ [the case $n = 0$ does not appear at 4PN], $\alpha > 4$ is usually a rational fraction, and in our case, $\tau_0 = 2b_0/c$. In this section, we will obtain an expression for $\mathcal{I}_{\alpha,n}$ up to post-adiabatic (PA) precision, as required at 4PN for the leading-order tail integral. To this end, we introduce the adiabatic parameter as defined in [199] (this differs from (5.6) of [6] by a factor 8/3)

$$\xi(u) \equiv \frac{1}{(u_c - u)\omega(u)} = \frac{8}{3} \frac{\dot{\omega}(u)}{\omega^2(u)} + \mathcal{O}(2), \quad (\text{VI.29})$$

where u_c is the retarded time “at coalescence”¹. Up to a constant factor of order unity, this adiabatic parameter measures the ratio between the orbital timescale (associated to the orbital frequency ω) and the timescale of radiation reaction (associated to the chirp $\dot{\omega}$). In the inspiral regime, this parameter is very small, and can actually be related to the PN parameter y by

$$\xi = \frac{256}{5} y^{\frac{5}{2}} (1 + \mathcal{O}(y)). \quad (\text{VI.30})$$

This definition encourages us to define an auxiliary variable

$$z \equiv \tau \xi(u) \omega(u), \quad (\text{VI.31})$$

from which we express the leading order time evolution of various orbital quantities,

$$r(u-\tau) = r(u)(1+z)^{\frac{1}{4}} (1 + \mathcal{O}(2)), \quad (\text{VI.32a})$$

$$\omega(u-\tau) = \omega(u)(1+z)^{-\frac{3}{8}} (1 + \mathcal{O}(2)), \quad (\text{VI.32b})$$

$$y(u-\tau) = y(u)(1+z)^{-\frac{1}{4}} (1 + \mathcal{O}(2)), \quad (\text{VI.32c})$$

$$\phi(u-\tau) = \phi_c - \phi(u)(1+z)^{\frac{5}{8}} (1 + \mathcal{O}(2)), \quad (\text{VI.32d})$$

where ϕ_c is a constant phase “at coalescence”. One could add the 1PN corrections to these expressions, but these turn out to be negligible at this order, see [200] for a discussion. With the help of the post-adiabatic parameter, evaluated at retarded time u , we can relate the phase in the past to the phase at retarded time (see Fig. II.3a), via the relation

$$\phi(u-\tau) = \phi(u) - \frac{8}{5\xi(u)} \left[(1+z)^{\frac{5}{8}} - 1 \right] (1 + \mathcal{O}(2)). \quad (\text{VI.33})$$

¹Note that in [199], the auxiliary variable and the post-adiabatic parameter were *defined* in terms of the instant of coalescence u_c . We will see in Section VI.6 that this coalescence time is uniquely defined only up to 3.5PN order, but starting at 4PN order, this definition does not uniquely specify u_c anymore. This does not affect our computation at this order, so we stick with the definition in terms of u_c for historical purposes.

With all of these ingredients in hand, we can rewrite our integral as

$$\mathcal{I}_{\alpha,n} = \frac{e^{-in\phi(u)}y^\alpha(u)}{\xi(u)\omega(u)} \int_0^{+\infty} dz (1+z)^{-\frac{\alpha}{4}} \ln\left(\frac{z}{\tau_0\xi(u)\omega(u)}\right) e^{\frac{8ni}{5\xi(u)}\left[(1+z)^{\frac{5}{8}}-1\right]}. \quad (\text{VI.34})$$

Performing the change of variables $s = \frac{8}{5}\left[(1+z)^{\frac{5}{8}}-1\right]$ yields

$$\mathcal{I}_{\alpha,n} = \frac{e^{-in\phi(u)}y^\alpha(u)}{\xi(u)\omega(u)} \int_0^{+\infty} ds \frac{e^{\frac{ins}{\xi(u)}}}{\left(1+\frac{5}{8}s\right)^{\frac{2\alpha-3}{5}}} \ln\left(\frac{\left(1+\frac{5}{8}s\right)^{\frac{8}{5}}-1}{\tau_0\xi(u)\omega(u)}\right). \quad (\text{VI.35})$$

We then divide this integral in two, namely $\mathcal{I}_{\alpha,n} = \mathcal{I}_{\alpha,n}^{(1)} + \mathcal{I}_{\alpha,n}^{(2)}$ where

$$\mathcal{I}_{\alpha,n}^{(1)} = -\frac{e^{-in\phi(u)}y^\alpha(u)}{\xi(u)\omega(u)} \ln\left(\tau_0\xi(u)\omega(u)\right) \int_0^{+\infty} ds \frac{e^{\frac{ins}{\xi(u)}}}{\left(1+\frac{5}{8}s\right)^{\frac{2\alpha-3}{5}}}, \quad (\text{VI.36a})$$

$$\mathcal{I}_{\alpha,n}^{(2)} = \frac{e^{-in\phi(u)}y^\alpha(u)}{\xi(u)\omega(u)} \int_0^{+\infty} ds \frac{e^{\frac{ins}{\xi(u)}}}{\left(1+\frac{5}{8}s\right)^{\frac{2\alpha-3}{5}}} \ln\left(\left(1+\frac{5}{8}s\right)^{\frac{8}{5}}-1\right). \quad (\text{VI.36b})$$

Using the formulas established in Appendix A.5 for $\beta = (2\alpha - 3)/5$, we find that

$$\mathcal{I}_{\alpha,n}^{(1)} = -\frac{e^{-in\phi}y^\alpha}{\xi\omega} \ln\left(\tau_0\xi\omega\right) \left[\frac{i\xi}{n} + \frac{5\xi^2}{8n^2} \cdot \frac{2\alpha-3}{5} + o(\xi^2)\right], \quad (\text{VI.37a})$$

$$\begin{aligned} \mathcal{I}_{\alpha,n}^{(2)} = \frac{e^{-in\phi}y^\alpha}{\xi\omega} & \left[\left(-\frac{i\xi}{n} + \frac{5\xi^2}{8n^2} \cdot \frac{2\alpha-3}{5}\right) \left(\ln\left(\frac{\xi}{|n|}\right) - \gamma_E - \frac{i\pi}{2}\text{sg}(n)\right) \right. \\ & \left. + \left(\frac{5}{8} \cdot \frac{2\alpha-3}{5} - \frac{3}{16}\right) \frac{\xi^2}{n^2} + o(\xi^2) \right]. \quad (\text{VI.37b}) \end{aligned}$$

where all dependencies on u are now implicit. This finally yields the final formula (note that it differs from (5.16) of [6] because of the different definition of ξ)

$$\begin{aligned} \mathcal{I}_{\alpha,n} = \frac{Gm}{c^3} \frac{e^{-in\phi}y^{\alpha-3/2}}{in} & \left[\left(1 + \frac{2\alpha-3}{8} \frac{\xi}{in}\right) \left(\ln(2|n|b\omega) + \gamma_E - \text{sg}(n)\frac{i\pi}{2}\right) \right. \\ & \left. - \frac{4\alpha-9}{16} \frac{\xi}{in} + o(\xi) \right]. \quad (\text{VI.38}) \end{aligned}$$

Recall that all quantities are evaluated at current time u and that the adiabatic parameter $\xi = 8\dot{\omega}/(3\omega^2)$ is easily computed with (II.58). With this result at hand, we are able to derive the tail integral (VI.17) with 2.5PN relative precision, which impacts the computation and final results at the 4PN order. Note as well that this is the first order for which the tail effect is sensitive to the decay rate of the orbit, encoded in α (see Fig. II.3b for a visual representation.)

VI.3.2 Quasicircular-orbit reduction of the memory effects and the tails of memory

Memory terms, such as the first term of (VI.18), are hereditary integrals of the form $\int_0^{+\infty} d\tau F(u-\tau)G(u-\tau)$, where F and G represent dynamical multipole moments. Note that they enter only in mass-type moments, as clear from Section VI.2. In the case of quasicircular orbits, after projection on the appropriate basis (see Fig. II.3a), they reduce to a sum of terms of the form

$$\mathcal{J}_{\alpha,n}(u) = \int_0^{+\infty} d\tau [y(u-\tau)]^\alpha e^{-in\phi(u-\tau)}. \quad (\text{VI.39})$$

Besides the absence of logarithm, the main difference with the tail integral is the possibility of having $n = 0$, i.e. persistent or ‘‘DC’’ (direct current) terms.

Let us first focus on oscillatory (‘‘AC’’ or alternate current) memory terms, having $n \neq 0$. As they only involve a simple, logarithmic-free, integration, memory terms will enter in the flux as instantaneous contributions, and can be computed as such. In particular, since they arise at the odd PN approximations 2.5PN and 3.5PN, they do not contribute to the flux for quasicircular orbits. The evaluation of the integrals of type (VI.39) is thus only required for the derivation of the modes. Concerning the quadrupole moment, since the memory effect enters at 2.5PN order, as clear from (VI.18), it is required at a relative 1.5PN precision for the mode (2, 2) and thus, one can safely compute the integral in the adiabatic approximation. As discussed in [199], for circular orbits this is equivalent to taking the frequency y constant together with a linear phase (appropriate for the exact circular orbit), and keeping only the contribution of the integral due to the bound $\tau = 0$. One finds

$$\mathcal{J}_{\alpha,n} = y^\alpha e^{-in\phi} \int_0^\infty d\tau e^{in\omega\tau} + \mathcal{O}(\xi) = -\frac{y^\alpha e^{-in\phi}}{in\omega} + \mathcal{O}(\xi), \quad (\text{VI.40})$$

where for a function $f(\tau)$ whose antiderivative is denoted $F(\tau)$, we define $\int_0^\infty d\tau f(\tau) \equiv -F(0)$.

The persistent DC terms do not contribute to the flux, and they only contribute to the modes which have $m = 0$, for example, the $(\ell, m) = (2, 0)$ mode for the mass quadrupole. As is clear in the following, the absence of the fast oscillating exponential in $\mathcal{J}_{\alpha,0}$ generates an inverse power of the adiabatic parameter ξ , thus degrading the precision (in terms of PN orders) at which we control the mode by the radiation-reaction scale 2.5PN. This is the well-known memory effect: as it starts at 2.5PN order in the waveform, it finally enters the (2, 0) mode at Newtonian order. Several methods are possible to evaluate $\mathcal{J}_{\alpha,0}$, see e.g. [199]. In the following, we rely on a change of integration variables from τ to $y' = y(u-\tau)$:

$$\mathcal{J}_{\alpha,0} = \int_0^{+\infty} d\tau [y(u-\tau)]^\alpha = \int_0^{y(u)} dy' \frac{y'^\alpha}{\dot{y}[y']}, \quad (\text{VI.41})$$

reading $\dot{y}[y']$ as a function of y' from (II.58), and supposing that $\alpha > 4$. To 1.5PN relative order we have

$$\mathcal{J}_{\alpha,0} = \frac{5Gm}{64c^3\nu} \frac{y^{\alpha-4}}{\alpha-4} \left[1 + \frac{\alpha-4}{\alpha-3} \left(\frac{743}{336} + \frac{11}{4}\nu \right) y - 8\pi \frac{\alpha-4}{2\alpha-5} y^{3/2} \right]. \quad (\text{VI.42})$$

We can see that this memory term is highly sensitive to the rate at which the orbit is decaying (via α). It is also interesting to observe that the tail effect in the flux directly influences the DC memory in (VI.42).

Last, but not least, we need to look at the interesting case of the tails-of-memory terms (VI.20). These terms can be treated as standard tail terms (with relative Newtonian accuracy), except for the “genuine” tail-of-memory given by the first line of (VI.20), namely

$$\mathcal{K}_{ij} = \frac{8G^2M}{7c^8} \int_0^{+\infty} d\rho M_{a(i)}^{(4)}(u-\rho) \int_0^{+\infty} d\tau M_{j)a}^{(4)}(u-\rho-\tau) \ln\left(\frac{\tau}{\tau_0}\right), \quad (\text{VI.43})$$

where $c\tau_0 = 2r_0 e^{\frac{1613}{270}}$. We remind that this expression agrees with the tail-of-memory directly computed from the radiative quadrupole moment at infinity [192, 193]. We first perform the tail-like integral over τ . As we need to evaluate it at relative Newtonian order only, we can safely use the adiabatic approximation. Next we perform the integral over ρ , which is found to be a simple DC memory integral of the type (VI.41) with $\alpha = 13/2$. Hence we find

$$\mathcal{K}_{ij} = \frac{128\pi}{7} \frac{\nu^2 c^5}{G} l_{\langle i} l_{j\rangle} \int_0^{+\infty} d\rho [y(u-\rho)]^{13/2} = \frac{4\pi}{7} m\nu c^2 y^{5/2} l_{\langle i} l_{j\rangle}, \quad (\text{VI.44})$$

where \mathbf{l} is the constant vector orthogonal to the orbital plane, see Fig. II.3a. Interestingly, this tail-of-memory result will give a contribution in the $(2, 0)$ mode that exactly cancels the one coming from the 1.5PN corrections to the ordinary memory effect, and obtained in (VI.42).

VI.4 The 4PN modes

Projecting the asymptotic metric (II.62c) onto the basis of polarizations $\{+, \times\}$ (see Section I.1) and the usual basis of spin-weighted spherical harmonics, $Y_{-2}^{\ell m}$ (following the conventions of [201, 202]), one can define the observable gravitational modes $h_{\ell m}$ as

$$h_+ - ih_\times = \sum_{\ell \geq 2} \sum_{m=-\ell}^{\ell} h_{\ell m} Y_{-2}^{\ell m}. \quad (\text{VI.45})$$

Following [203, 202], we can express any moments directly using the radiative moments with the formula

$$h_{\ell m} = -\frac{G}{\sqrt{2}Rc^{\ell+2}} \left[\mathcal{U}_{\ell m} - \frac{i}{c} \mathcal{V}_{\ell m} \right], \quad (\text{VI.46})$$

where we have defined

$$\mathcal{U}_{\ell m} = \frac{4}{\ell!} \sqrt{\frac{(\ell+1)(\ell+2)}{2\ell(\ell+1)}} \alpha_L^{\ell m} \mathcal{U}_L, \quad (\text{VI.47a})$$

$$\mathcal{V}_{\ell m} = -\frac{8}{\ell!} \sqrt{\frac{\ell(\ell+2)}{2(\ell-1)(\ell+1)}} \alpha_L^{\ell m} \mathcal{V}_L. \quad (\text{VI.47b})$$

The conversion from radiative STF tensor to spherical (ℓ, m) modes is performed using the object

$$\alpha_L^{\ell m} = \frac{\sqrt{4\pi} (-1)^m 2^{\frac{m}{2}} \ell!}{\sqrt{(2\ell+1)(\ell+m)!(\ell-m)!}} \mathbf{m}^* \langle M^{\ell} L^{-M} \rangle, \quad (\text{VI.48})$$

where $\mathbf{m} = (\mathbf{n} + i\boldsymbol{\lambda})/\sqrt{2}$ is evaluated at the retarded time u , and \mathbf{m}^* is its complex conjugate. Note that if m is even, then $\mathcal{V}_{\ell m} = 0$, and conversely, if m is odd, then $\mathcal{U}_{\ell m}$. This observation significantly simplifies the computations: each of these modes is defined in terms of only one radiative moment, of which we know the expression in terms of the source moments (Section VI.2). Replacing the source moments by their expressions for quasicircular orbits (Section VII.4), taking their derivatives with the equations of motion (Section II.6) and evaluating nonlocal integrals (Section VI.3), we are able to compute these modes in the case of quasicircular orbits.

The computation being performed, it is always possible to factor out the dependence in the orbital GW phase ϕ which we used to parametrize our quasicircular motion. The modes can then be rescaled by the dominant contribution, which leads to defining $\hat{H}_{\ell m}$ as

$$h_{\ell m} = \frac{8Gm\nu y}{c^2 R} \sqrt{\frac{\pi}{5}} \hat{H}_{\ell m} e^{-im\phi}, \quad (\text{VI.49})$$

where we recall $y = Gm\omega/c^3$ with $\omega = d\phi/dt$. Note that the amplitude is a complex number which satisfies $\hat{H}_{\ell, -m} = (-)^{\ell} \hat{H}_{\ell m}^*$, where the star denotes the complex conjugate. Specializing our study to the $(\ell, m) = (2, 2)$ mode, we compute with 4PN accuracy

$$\hat{H}_{22} = -\frac{e^{2i\phi}}{2c^2 m\nu y} \mathbf{m}_i^* \mathbf{m}_j^* \mathcal{U}_{ij}. \quad (\text{VI.50})$$

When computing our expression (VI.50) for the $(2,2)$ mode, the arbitrary constants r_0 and r'_0 nicely cancel out. However, a dependency in the b_0 constant remains, already at 1.5PN order. This was of course known since the computation of the 1.5PN tail effect. At the time, the rationale to eliminate the b_0 dependency was to reabsorb the complex piece of \hat{H}_{22} into an a redefined, auxiliary phase [166, 198], which reads

$$\psi = \phi - \frac{2GM\omega}{c^3} \ln\left(\frac{\omega}{\omega_0}\right), \quad (\text{VI.51})$$

where M denotes the constant total mass-energy of the spacetime and $\omega_0 \equiv c e^{11/12 - \gamma_E} / (4b_0)$. Remarkably, as the literature progressed [204, 203, 201, 205, 202, 96, 161, 197, 206], it was found that using the same definition (VI.51) to decompose $h_{\ell m}$ successfully removed all the b_0 dependencies in all the modes up to currently known orders. Indeed, one can always decompose $h_{\ell m}$ as

$$h_{\ell m} = \frac{8Gm\nu x}{c^2 R} \sqrt{\frac{\pi}{5}} H_{\ell m} e^{-im\psi}, \quad (\text{VI.52})$$

where we emphasize the fact $H_{\ell m}$ (without a hat) is a complex amplitude that differs from $\hat{H}_{\ell m}$ (with a hat). We then find that up to 3.5PN order, $H_{\ell m}$ is independent of b_0 . This is remarkable, since the definition was only tailored for 1.5PN, and it crucially relies on defining (VI.51) with the conserved total mass-energy M (using the total mass

m associated only to matter would fail at removing the b_0 's). The common choice for ω_0 for all modes was motivated in [204] by the fact that it amounts to a redefinition of the time t_c , and simplicity.

In this work [5, 6], we found that this procedure also successfully removed b_0 at 4PN order, but an important subtlety arises. Indeed, if we simply reabsorb the relevant terms of the amplitude in ψ , we find that the (2,2) mode reads

$$\begin{aligned}
H_{22} = & 1 + \left(-\frac{107}{42} + \frac{55}{42}\nu \right) y + 2\pi y^{3/2} + \left(-\frac{2173}{1512} - \frac{1069}{216}\nu + \frac{2047}{1512}\nu^2 \right) y^2 \\
& + \left[-\frac{107\pi}{21} + \left(\frac{34\pi}{21} - 24i \right) \nu \right] y^{5/2} \\
& + \left[\frac{27027409}{646800} - \frac{856}{105} \gamma_E + \frac{428i\pi}{105} + \frac{2\pi^2}{3} + \left(-\frac{278185}{33264} + \frac{41\pi^2}{96} \right) \nu - \frac{20261}{2772} \nu^2 \right. \\
& \quad \left. + \frac{114635}{99792} \nu^3 - \frac{428}{105} \ln(16y) \right] y^3 \\
& + \left[-\frac{2173\pi}{756} + \left(-\frac{2495\pi}{378} + \frac{14333i}{162} \right) \nu + \left(\frac{40\pi}{27} - \frac{4066i}{945} \right) \nu^2 \right] y^{7/2} \\
& + \left[-\frac{846557506853}{12713500800} + \frac{45796}{2205} \gamma_E - \frac{22898i\pi}{2205} - \frac{107}{63} \pi^2 + \frac{22898}{2205} \ln(16y) \right. \\
& \quad \left. + \left(-\frac{345046539157}{4237833600} + \frac{19972}{2205} \gamma_E - \frac{219314}{2205} i\pi \nu \right. \right. \\
& \quad \left. \left. - \frac{9755}{32256} \pi^2 - \frac{46462}{2205} \ln(16y) + \frac{128}{5} \ln \left(\frac{Gm}{16b_0 c^2} \right) \right) \nu \right. \\
& \quad \left. + \left(\frac{256450291}{7413120} - \frac{1025}{1008} \pi^2 \right) \nu^2 - \frac{81579187}{15567552} \nu^3 + \frac{26251249}{31135104} \nu^4 \right] y^4 + \mathcal{O}(y^{9/2}).
\end{aligned} \tag{VI.53}$$

This remaining b_0 dependency is troublesome, but can be explained as follows. Up to now, we have assumed that $\hat{H}_{\ell m}$ and $H_{\ell m}$ are expressed in terms of y , which is related to the derivatives of ϕ . However, to be consistent, we should now express $H_{\ell m}$ in terms of the variable

$$x \equiv \left(\frac{Gm\Omega}{c^3} \right)^{2/3}, \tag{VI.54}$$

where we have defined the associated frequency $\Omega \equiv d\psi/dt$ which is related to the orbital frequency ω by

$$\Omega = \omega - \frac{2GM\dot{\omega}}{c^3} \left[\ln \left(\frac{\omega}{\omega_0} \right) + 1 \right]. \tag{VI.55}$$

Using (II.58) for the frequency chirp at the dominant order, it explicitly comes, replacing M by m at that order,

$$\Omega = \omega \left\{ 1 - \frac{192}{5} \nu \left(\frac{Gm\omega}{c^3} \right)^{8/3} \left[\ln \left(\frac{\omega}{\omega_0} \right) + 1 \right] + \mathcal{O} \left(\frac{1}{c^{10}} \right) \right\}, \tag{VI.56}$$

where we recall that $\nu = m_1 m_2 / m^2$ is the symmetric mass ratio. Recalling also the definition (I.42) and posing $y_0 = (Gm\omega_0/c^3)^{2/3}$ we thus have

$$x = y \left\{ 1 - \frac{192}{5} \nu y^4 \left[\ln\left(\frac{y}{y_0}\right) + \frac{2}{3} \right] + \mathcal{O}(y^5) \right\}, \quad (\text{VI.57})$$

showing that the frequency Ω differs from the orbital frequency ω at the 4PN order only (as argued in [204, 199]). This explains why this subtlety was not considered in previous computations. Thus, we obtain [5]

$$\begin{aligned} H_{22} = & 1 + \left(-\frac{107}{42} + \frac{55}{42} \nu \right) x + 2\pi x^{3/2} + \left(-\frac{2173}{1512} - \frac{1069}{216} \nu + \frac{2047}{1512} \nu^2 \right) x^2 \\ & + \left[-\frac{107\pi}{21} + \left(\frac{34\pi}{21} - 24i \right) \nu \right] x^{5/2} \\ & + \left[\frac{27027409}{646800} - \frac{856}{105} \gamma_E + \frac{428i\pi}{105} + \frac{2\pi^2}{3} + \left(-\frac{278185}{33264} + \frac{41\pi^2}{96} \right) \nu - \frac{20261}{2772} \nu^2 \right. \\ & \quad \left. + \frac{114635}{99792} \nu^3 - \frac{428}{105} \ln(16x) \right] x^3 \\ & + \left[-\frac{2173\pi}{756} + \left(-\frac{2495\pi}{378} + \frac{14333i}{162} \right) \nu + \left(\frac{40\pi}{27} - \frac{4066i}{945} \right) \nu^2 \right] x^{7/2} \\ & + \left[-\frac{846557506853}{12713500800} + \frac{45796}{2205} \gamma_E - \frac{22898}{2205} i\pi - \frac{107}{63} \pi^2 + \frac{22898}{2205} \ln(16x) \right. \\ & \quad \left. + \left(-\frac{336005827477}{4237833600} + \frac{15284}{441} \gamma_E - \frac{219314}{2205} i\pi - \frac{9755}{32256} \pi^2 + \frac{7642}{441} \ln(16x) \right) \nu \right. \\ & \quad \left. + \left(\frac{256450291}{7413120} - \frac{1025}{1008} \pi^2 \right) \nu^2 - \frac{81579187}{15567552} \nu^3 + \frac{26251249}{31135104} \nu^4 \right] x^4 + \mathcal{O}(x^{9/2}), \end{aligned} \quad (\text{VI.58})$$

which is indeed explicitly free of any arbitrary constant. Note the frequency redefinition (VI.57) is a 4PN effect, therefore it never needed to be accounted for in previous computations, e.g. [204, 199]. In this 4PN result, however, accounting for it was crucial, and omitting it would have led to a residual b_0 dependency, as in (VI.53). This new result is also in agreement with the prediction of linear black-hole perturbation theory [205], although this subtlety would correspond to a second-order self-force effect.

We have also computed the nonoscillatory $(\ell, m) = (2, 0)$ mode. Since this modes do not have a phase, there is no freedom to transfer parts of the waveform from the amplitude into the phase. As discussed in Section VI.3.2, nonoscillatory modes arise from integration of the nonlinear zero-frequency memory terms over the past history of the system, assuming a model for the quasicircular evolution of the orbit in the past. Since the integration increases the effect by the inverse of the 2.5PN order, with the present 4PN formalism we are able to control the $(2, 0)$ mode only with relative 1.5PN precision. We thus find

$$H_{20} = -\frac{5}{14\sqrt{6}} \left[1 + \left(-\frac{4075}{4032} + \frac{67}{48} \nu \right) x + \mathcal{O}(x^2) \right]. \quad (\text{VI.59})$$

where all the arbitrary constants have properly vanished. Notice that the 1.5PN term of this mode vanishes. This is due to the fact that the 1.5PN correction in the model of evolution of the quasicircular orbit in the past, which results in the 1.5PN term in (VI.42), exactly cancels the 1.5PN direct contribution of the “tail-of-memory” at 4PN order, and given by (VI.44). That is,

$$H_{20}^{\text{ToM}} = -H_{20}^{\text{mem},1.5\text{PN}} = -\frac{2\sqrt{2}}{7\sqrt{3}}\pi x^{3/2}. \quad (\text{VI.60})$$

The result (VI.59) is in full agreement with (4.3a) of [192], obtained from the general expression of nonlinear memory terms in terms of radiative moments. Indeed, recall that the tail-of-memory integral (VI.43) at 4PN order can be simply obtained from the leading memory integral (VI.18) at 2.5PN order by replacing the canonical moment $M_{ij}^{(3)}$ by the (derivative of the) radiative moment including the tail effect (VI.17) at relative 1.5PN order. This confirms that the tail-of-memory is adequately taken into account in the computation of the memory using radiative moments defined at future null infinity [192, 193].

VI.5 The 4.5PN flux

The energy flux carried by GWs at infinity can be computed solely based on that radiative moments using (II.66), which we reproduce here:

$$\mathcal{F} = \sum_{\ell=2}^{\infty} \frac{G}{c^{2\ell+1}} \left[\frac{(\ell+1)(\ell+2)}{(\ell-1)\ell!(2\ell+1)!!} \mathcal{U}_L^{(1)} \mathcal{U}_L^{(1)} + \frac{4\ell(\ell+2)}{c^2(\ell-1)(\ell+1)!(2\ell+1)!!} \mathcal{V}_L^{(1)} \mathcal{V}_L^{(1)} \right]$$

We can compute these radiative moments in the case of circular orbits using the same steps as in Section VI.4, but now extended to 4.5PN order, as discussed in Section VI.2.2. Applying these methods, we find that the flux expressed in terms of y parameter, which we recall is related to the orbital frequency ω , still contains the unphysical constant b_0 , though the other arbitrary scales r_0 and r'_0 have properly disappeared. It reads

$$\begin{aligned} \mathcal{F} = \frac{32c^5}{5G} \nu^2 y^5 & \left\{ 1 + \left(-\frac{1247}{336} - \frac{35}{12}\nu \right) y + 4\pi y^{3/2} + \left(-\frac{44711}{9072} + \frac{9271}{504}\nu + \frac{65}{18}\nu^2 \right) y^2 \right. \\ & + \left(-\frac{8191}{672} - \frac{583}{24}\nu \right) \pi y^{5/2} \\ & + \left[\frac{6643739519}{69854400} + \frac{16}{3}\pi^2 - \frac{1712}{105}\gamma_E - \frac{856}{105} \ln(16y) \right. \\ & \quad \left. + \left(-\frac{134543}{7776} + \frac{41}{48}\pi^2 \right) \nu - \frac{94403}{3024}\nu^2 - \frac{775}{324}\nu^3 \right] y^3 \\ & + \left(-\frac{16285}{504} + \frac{214745}{1728}\nu + \frac{193385}{3024}\nu^2 \right) \pi y^{7/2} \\ & \left. + \left[-\frac{323105549467}{3178375200} + \frac{232597}{4410}\gamma_E - \frac{1369}{126}\pi^2 \right. \right. \end{aligned}$$

$$\begin{aligned}
& + \frac{39931}{294} \ln 2 - \frac{47385}{1568} \ln 3 + \frac{232597}{8820} \ln y \\
& + \left(-\frac{1467849789229}{1466942400} + \frac{10118}{245} \gamma_E - \frac{267127}{4608} \pi^2 \right. \\
& \quad \left. - \frac{85418}{2205} \ln 2 + \frac{47385}{392} \ln 3 - \frac{26301}{245} \ln y + 128 \ln \left(\frac{Gm}{c^2 b_0} \right) \right) \nu \\
& + \left(\frac{1607125}{6804} - \frac{3157}{384} \pi^2 \right) \nu^2 + \frac{6875}{504} \nu^3 + \frac{5}{6} \nu^4 \Big] y^4 \\
& + \left[\frac{265978667519}{745113600} - \frac{6848}{105} \gamma_E - \frac{3424}{105} \ln(16y) + \left(\frac{2062241}{22176} + \frac{41}{12} \pi^2 \right) \nu \right. \\
& \quad \left. - \frac{133112905}{290304} \nu^2 - \frac{3719141}{38016} \nu^3 \right] \pi y^{9/2} + \mathcal{O}(x^5) \Big\}. \tag{VI.61}
\end{aligned}$$

Although the appearance of b_0 in the modes starts at 1.5PN, this is the first time that it appears explicitly in the flux. Indeed, just like we have done in Section VI.4, we must express the flux not in terms of the orbital frequency ω , but in terms of the frequency Ω associated to the GW. We remind the reader that $x = (Gm\omega/c^3)$ is related to y by (VI.57), which reads

$$x = y \left\{ 1 - \frac{192}{5} \nu y^4 \left[\ln \left(\frac{y}{y_0} \right) + \frac{2}{3} \right] + \mathcal{O}(y^5) \right\},$$

and we find that that all the arbitrary constants now disappear in the flux expressed in terms of x . This redefinition only affects the flux starting at 4PN, which explains why it was never needed previously (see for example [166], where it is dismissed as a negligible effect).

Finally, the expression for the flux, at 4.5PN order, reads

$$\begin{aligned}
\mathcal{F} = \frac{32c^5}{5G} \nu^2 x^5 \Big\{ & 1 + \left(-\frac{1247}{336} - \frac{35}{12} \nu \right) x + 4\pi x^{3/2} + \left(-\frac{44711}{9072} + \frac{9271}{504} \nu + \frac{65}{18} \nu^2 \right) x^2 \\
& + \left(-\frac{8191}{672} - \frac{583}{24} \nu \right) \pi x^{5/2} \\
& + \left[\frac{6643739519}{69854400} + \frac{16}{3} \pi^2 - \frac{1712}{105} \gamma_E - \frac{856}{105} \ln(16x) \right. \\
& \quad \left. + \left(-\frac{134543}{7776} + \frac{41}{48} \pi^2 \right) \nu - \frac{94403}{3024} \nu^2 - \frac{775}{324} \nu^3 \right] x^3 \\
& + \left(-\frac{16285}{504} + \frac{214745}{1728} \nu + \frac{193385}{3024} \nu^2 \right) \pi x^{7/2} \\
& + \left[-\frac{323105549467}{3178375200} + \frac{232597}{4410} \gamma_E - \frac{1369}{126} \pi^2 \right. \\
& \quad \left. + \frac{39931}{294} \ln 2 - \frac{47385}{1568} \ln 3 + \frac{232597}{8820} \ln x \right.
\end{aligned}$$

$$\begin{aligned}
& + \left(-\frac{1452202403629}{1466942400} + \frac{41478}{245} \gamma_E - \frac{267127}{4608} \pi^2 \right. \\
& \quad \left. + \frac{479062}{2205} \ln 2 + \frac{47385}{392} \ln 3 + \frac{20739}{245} \ln x \right) \nu \\
& + \left(\frac{1607125}{6804} - \frac{3157}{384} \pi^2 \right) \nu^2 + \frac{6875}{504} \nu^3 + \frac{5}{6} \nu^4 \Big] x^4 \\
& + \left[\frac{265978667519}{745113600} - \frac{6848}{105} \gamma_E - \frac{3424}{105} \ln(16x) + \left(\frac{2062241}{22176} + \frac{41}{12} \pi^2 \right) \nu \right. \\
& \quad \left. - \frac{133112905}{290304} \nu^2 - \frac{3719141}{38016} \nu^3 \right] \pi x^{9/2} + \mathcal{O}(x^5) \Big\}. \tag{VI.62}
\end{aligned}$$

A strong check is to observe that the test-mass limit $\nu \rightarrow 0$ perfectly agrees with the result of linear black-hole perturbation theory [207, 208, 209, 210, 211]. As usual, the contributions due to the absorption by the black hole horizons should be added separately from the PN result computed here, see [212, 213, 214, 215, 216, 217].

Finally, we have explicitly verified that at the 4PN order, (VI.62) can be recovered from the gravitational modes given in Section VI.4 below and in [197], using

$$\mathcal{F} = \frac{c^3}{16\pi G} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} |\dot{h}_{\ell m}|^2. \tag{VI.63}$$

The new terms we computed vanish for $\nu = 0$, and are maximal for $\nu = 1/4$. As presented in Fig. VI.1, the 4.5PN flux significantly improves the comparison with numerical relativity and second-order self-force near merger for similar mass binaries. This does not seem to be the case for very unequal mass binaries, but for these, the Post-Newtonian expansion of self-force results are known to very high order [207, 208, 209, 210, 211].

VI.6 The 4.5PN chirp

In this Section and in the next (Section VI.7), we will need to postulate the validity of the energy balance equations $dE/dt = -\mathcal{F}$ between the conservative energy E (II.55) and the flux \mathcal{F} carried by GWs at future null infinity (VI.62). At 4PN order, we found it necessary to use the expression of the conservative energy (II.55), but where the variable y associated to the orbital frequency is simply replaced by the frequency x associated to the GW. The fact that the left-hand side of the energy equation, which concerns the motion, should be expressed in terms of the same observed GW frequency x as the right-hand side, which concerns the radiation, and not, for instance, in terms of the ‘‘orbital’’ frequency y , is worth an explanation: suppose that the compact binary system is actually a binary pulsar system. Hence, in addition to the GWs generated by the orbital motion, the pulsar emits electromagnetic (radio) waves, also received by the far away observer. Now the observer at infinity can measure the orbital frequency of the system from the instants of arrival of the radio pulses — this is the standard analysis of binary pulsars. Such frequency should be the one to be inserted in the left side of the energy balance equation. However, far from the system, the spacetime curvature $\mathcal{R}^{-1} \sim \sqrt{M}/r^3$ tends to zero,

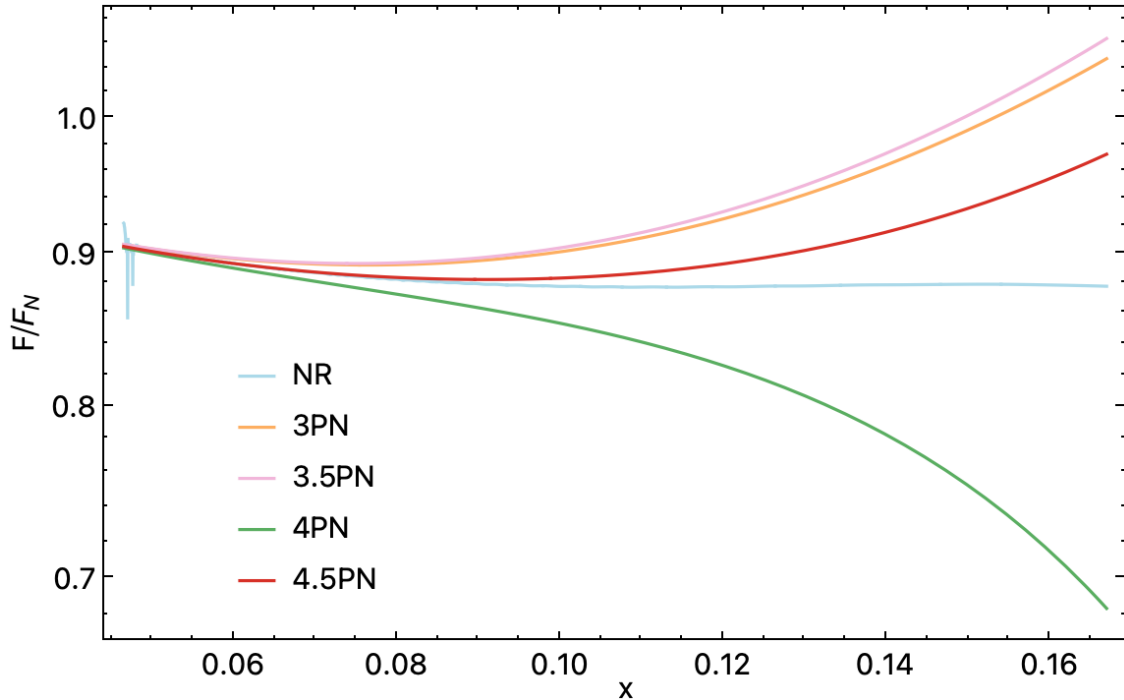


Fig. VI.1 Normalized flux $\mathcal{F}/\mathcal{F}_N$ emitted by an equal-mass binary ($\nu = 1/4$) plotted as a function of $x \equiv (Gm\Omega/c^3)^{2/3}$, as defined in (VI.54). The flux is normalized with respect to the Newtonian flux \mathcal{F}_N , given by the lowest order term of (VI.62). The normalized flux from numerical relativity (NR) is plotted in blue, along with the total 3PN, 3.5PN, 4PN and 4.5PN fluxes, as given by (VI.62). Courtesy of Héctor Estellés Estrella.

and therefore the geometric optics or Wentzel–Kramers–Brillouin (WKB) approximation applies for both electromagnetic and gravitational waves. Thus the electromagnetic and gravitational waves follow the same geodesic, independently of their frequency, and in particular are subject to the same tail-induced phase modulation (VI.51). We conclude that the distant observer measures the same frequency x from the electromagnetic radio pulses and from the GW, and this is that frequency that he inserts into both sides of the flux-balance law.²

Using the energy balance equation, we thus obtain the differential equation

$$\frac{d\tau}{dx} = \frac{\nu c^3}{5Gm} \frac{dE/dx}{\mathcal{F}(x)} \quad (\text{VI.64})$$

where we have defined the rescaled time variable

$$\tau \equiv \frac{\nu c^3}{5Gm} (t_0 - t). \quad (\text{VI.65})$$

The t_0 constant of this time variable can be adjusted to account for the integration constant arising from the differential equation. We can then Taylor-expand the right hand

²See [218] for a similar argument in the context of self-forces, using an observer sitting on the particle and equipped with a flashlight.

side of (VI.64) in small x , and integrate term by term. Keeping for now the integration constant τ_0 arbitrary, we have³

$$\begin{aligned}
\tau = \tau_0 + \frac{1}{256x^4} & \left\{ 1 + \left(\frac{743}{252} + \frac{11}{3}\nu \right) x - \frac{32\pi}{5} x^{3/2} + \left(\frac{3058673}{508302} + \frac{5429}{504}\nu + \frac{617}{72}\nu^2 \right) x^2 \right. \\
& + \left(-\frac{7729}{252} + \frac{13}{3}\nu \right) \pi x^{5/2} \\
& + \left(-\frac{10052469856691}{23471078400} + \frac{6848}{105}\gamma_E + \frac{128}{3}\pi^2 + \frac{3424}{105}\ln(16x) \right. \\
& \quad \left. + \left[\frac{3147553127}{3048192} - \frac{451}{12}\pi^2 \right] \nu - \frac{15211}{1728}\nu^2 + \frac{25565}{1296}\nu^3 \right) x^3 \\
& + \left(-\frac{15419335}{127008} - \frac{75703}{756}\nu + \frac{14809}{378}\nu^2 \right) \pi x^{7/2} \\
& + \left(\frac{2496799162103891233}{922695034060800} - \frac{18406}{105}\gamma_E - \frac{9049}{63}\pi^2 \right. \\
& \quad - \frac{101102}{441}\ln(2) - \frac{47385}{392}\ln(3) - \frac{9203}{210}\ln(x) \\
& \quad \left. + \left[-\frac{722352859826557}{140826470400} - \frac{1955944}{2205}\gamma_E + \frac{65577}{448}\pi^2 \right. \right. \\
& \quad \quad \left. \left. - \frac{4982056}{2205}\ln(2) + \frac{47385}{98}\ln(3) - \frac{488986}{2205}\ln(x) \right] \nu \right. \\
& \quad \left. + \left(\frac{1502014727}{2032128} - \frac{2255}{96}\pi^2 \right) \nu^2 + \frac{258479}{8064}\nu^3 + \frac{1195}{64}\nu^4 \right) \ln(x)x^4 \\
& + \left(-\frac{102282756713483}{23471078400} + \frac{54784}{105}\gamma_E + \frac{512}{2}\pi^2 + \frac{27392}{105}\ln(16x) \right. \\
& \quad \left. + \left[\frac{298583452147}{33530112} - \frac{902}{3}\pi^2 \right] \nu - \frac{9058667}{31752}\nu^2 - \frac{2064751}{49896}\nu^3 \right) \pi x^{9/2} \left. \right\} \\
\end{aligned} \tag{VI.66}$$

In view of inverting this series expansion of τ in terms of x , it will be extremely practical to choose t_0 such that the dimensionless quantity τ^{-1} is of 4PN order. Indeed, if we limit ourself to the 3.5PN expansion of τ in terms of x , we find that τ admits a finite limit in the $x \rightarrow +\infty$ limit, namely τ_0 . This uniquely defines a finite time $t_c = t_0 - 5Gm\tau_0/(\nu c^3)$ at which the frequency formally becomes infinite, and this can thus be interpreted as the coalescence time in the PN sense. Such a constant t_c is a valid choice for t_0 , and leads to $\tau_0 = 0$. However, if we include the 4PN terms and higher, the expansion of τ in terms of x now has an infinite limit when $x \rightarrow +\infty$. Therefore, there still exists a constant such that $\tau^{-1} = \mathcal{O}(8)$, but it is not unique anymore, and essentially loses its interpretation of a coalescence time. Therefore, one can always conveniently

³Notice that the 4PN contribution is either multiplied by $\ln(x)$ or reabsorbed into the arbitrary constant τ_0 .

choose t_0 (or equivalently, τ_0) such as to reabsorb the nonlogarithmic 4PN contribution, thus simplifying the result as much as possible:

$$\begin{aligned}
x = & \frac{\tau^{-1/4}}{4} \left\{ 1 + \left(\frac{743}{4032} + \frac{11}{48}\nu \right) \tau^{-1/4} - \frac{1}{5}\pi \tau^{-3/8} \right. \\
& + \left(\frac{19583}{254016} + \frac{24401}{193536}\nu + \frac{31}{288}\nu^2 \right) \tau^{-1/2} + \left(-\frac{11891}{53760} + \frac{109}{1920}\nu \right) \pi \tau^{-5/8} \\
& + \left[-\frac{10052469856691}{6008596070400} + \frac{1}{6}\pi^2 + \frac{107}{420}\gamma_E - \frac{107}{3360} \ln\left(\frac{\tau}{256}\right) \right. \\
& \quad \left. + \left(\frac{3147553127}{780337152} - \frac{451}{3072}\pi^2 \right) \nu - \frac{15211}{442368}\nu^2 + \frac{25565}{331776}\nu^3 \right] \tau^{-3/4} \\
& + \left(-\frac{113868647}{433520640} - \frac{31821}{143360}\nu + \frac{294941}{3870720}\nu^2 \right) \pi \tau^{-7/8} \\
& + \left[-\frac{2518977598355703073}{3779358859513036800} + \frac{9203}{215040}\gamma_E + \frac{9049}{258048}\pi^2 \right. \\
& \quad + \frac{14873}{1128960} \ln 2 + \frac{47385}{1605632} \ln 3 - \frac{9203}{3440640} \ln \tau \\
& \quad \left. + \left(\frac{718143266031997}{576825222758400} + \frac{244493}{1128960}\gamma_E - \frac{65577}{1835008}\pi^2 \right. \right. \\
& \quad \left. \left. + \frac{15761}{47040} \ln 2 - \frac{47385}{401408} \ln 3 - \frac{244493}{18063360} \ln \tau \right) \nu \right. \\
& \quad \left. + \left(-\frac{1502014727}{8323596288} + \frac{2255}{393216}\pi^2 \right) \nu^2 - \frac{258479}{33030144}\nu^3 + \frac{1195}{262144}\nu^4 \right] \tau^{-1} \ln \tau \\
& + \left[-\frac{9965202491753717}{5768252227584000} + \frac{107}{600}\gamma_E + \frac{23}{600}\pi^2 - \frac{107}{4800} \ln\left(\frac{\tau}{256}\right) \right. \\
& \quad \left. + \left(\frac{8248609881163}{2746786775040} - \frac{3157}{30720}\pi^2 \right) \nu - \frac{3590973803}{20808990720}\nu^2 - \frac{520159}{1634992128}\nu^3 \right] \pi \tau^{-9/8} \left. \right\} \\
\end{aligned} \tag{VI.67}$$

VI.7 The 4.5PN phase

From the energy balance equation and using the chain law, we find

$$\frac{dE}{dt} = \frac{dE}{dx} \frac{dx}{d\psi} \frac{d\psi}{dt} = -\mathcal{F}. \tag{VI.68}$$

Then, using $d\psi/dt = \Omega = c^3 x^{\frac{3}{2}} / (Gm)$, we immediately obtain the differential equation

$$\frac{d\psi}{dx} = -\frac{c^3 x^{\frac{3}{2}}}{Gm} \frac{dE/dx}{\mathcal{F}(x)} \tag{VI.69}$$

We then Taylor-expand the right-hand side and integrate term by term. We find that at 4PN, the explicit expression of the GW phase reads

$$\begin{aligned}
\psi = \psi_0 - \frac{x^{-5/2}}{32\nu} & \left\{ 1 + \left(\frac{3715}{1008} + \frac{55}{12}\nu \right) x - 10\pi x^{3/2} \right. \\
& + \left(\frac{15293365}{1016064} + \frac{27145}{1008}\nu + \frac{3085}{144}\nu^2 \right) x^2 + \left(\frac{38645}{1344} - \frac{65}{16}\nu \right) \pi x^{5/2} \ln x \\
& + \left[\frac{12348611926451}{18776862720} - \frac{160}{3}\pi^2 - \frac{1712}{21}\gamma_E - \frac{856}{21} \ln(16x) \right. \\
& \quad \left. + \left(-\frac{15737765635}{12192768} + \frac{2255}{48}\pi^2 \right) \nu + \frac{76055}{6912}\nu^2 - \frac{127825}{5184}\nu^3 \right] x^3 \\
& + \left(\frac{77096675}{2032128} + \frac{378515}{12096}\nu - \frac{74045}{6048}\nu^2 \right) \pi x^{7/2} \\
& + \left[\frac{2550713843998885153}{2214468081745920} - \frac{9203}{126}\gamma_E - \frac{45245}{756}\pi^2 \right. \\
& \quad - \frac{252755}{2646} \ln 2 - \frac{78975}{1568} \ln 3 - \frac{9203}{252} \ln x \\
& \quad + \left(-\frac{680712846248317}{337983528960} - \frac{488986}{1323}\gamma_E + \frac{109295}{1792}\pi^2 \right. \\
& \quad \quad \left. - \frac{1245514}{1323} \ln 2 + \frac{78975}{392} \ln 3 - \frac{244493}{1323} \ln x \right) \nu \\
& \quad \left. + \left(\frac{7510073635}{24385536} - \frac{11275}{1152}\pi^2 \right) \nu^2 + \frac{1292395}{96768}\nu^3 - \frac{5975}{768}\nu^4 \right] x^4 \\
& + \left[-\frac{93098188434443}{150214901760} + \frac{1712}{21}\gamma_E + \frac{80}{3}\pi^2 + \frac{856}{21} \ln(16x) \right. \\
& \quad \left. + \left(\frac{1492917260735}{1072963584} - \frac{2255}{48}\pi^2 \right) \nu - \frac{45293335}{1016064}\nu^2 - \frac{10323755}{1596672}\nu^3 \right] \pi x^{9/2} \left. \right\}, \tag{VI.70}
\end{aligned}$$

where the integration constant ψ_0 is determined by initial conditions, e.g., when the wave frequency enters the detector's band.

In order to get intuition on the relative contribution of each PN order to the signal, we provide in Table VI.1 rough numerical estimates for the number of accumulated GW cycles in the frequency band of current and future detectors. Our naive estimation does not take the various detector noises into account, and a more realistic estimation should be performed [219]. Nevertheless, it can be useful to gain insight on the behavior of the PN expansion, which seems to converge well, as we see from Table VI.1. For all the typical compact binaries in Table VI.1, we find that the 4PN and 4.5PN orders amount to about a tenth of a cycle (less than 1 radian). This suggests that systematic errors due to the PN modeling may be dominated by statistical errors and negligible for LISA. However, this should be confirmed by detailed investigations along the lines of [220].

Detector	LIGO/Virgo		ET		LISA	
Masses (M_\odot)	1.4×1.4	10×10	1.4×1.4	500×500	$10^5 \times 10^5$	$10^7 \times 10^7$
PN order	cumulative number of cycles					
Newtonian	2562.599	95.502	744401.36	37.90	28095.39	9.534
1PN	143.453	17.879	4433.85	9.60	618.31	3.386
1.5PN	-94.817	-20.797	-1005.78	-12.63	-265.70	-5.181
2PN	5.811	2.124	23.94	1.44	11.35	0.677
2.5PN	-8.105	-4.604	-17.01	-3.42	-12.47	-1.821
3PN	1.858	1.731	2.69	1.43	2.59	0.876
3.5PN	-0.627	-0.689	-0.93	-0.59	-0.91	-0.383
4PN	-0.107	-0.064	-0.12	-0.04	-0.12	-0.013
4.5PN	0.098	0.118	0.14	0.10	0.14	0.065

Table VI.1: Contribution of each PN order to the total number of accumulated cycles inside the detector’s frequency band, for typical (but nonspinning) quasicircular compact binaries observed by current and future detectors. We have approximated the frequency bands of LIGO/Virgo, Einstein Telescope (ET) and LISA with step functions, respectively between $[30 \text{ Hz}, 10^3 \text{ Hz}]$, $[1 \text{ Hz}, 10^4 \text{ Hz}]$ and $[10^{-4} \text{ Hz}, 10^{-1} \text{ Hz}]$. When the merger occurs within the frequency band of the detector, the exit frequency is taken to be the Schwarzschild innermost stable circular orbit (ISCO), $f_{\text{ISCO}} = c^3/(6^{3/2}\pi Gm)$.

The results hereabove (VI.67) gives the prediction of GR for the GW phase of nonspinning compact binaries up to 4.5PN precision. On the other hand, most of the frameworks for data analysis rely on the phase expressed in Fourier domain (see Section I.4). This can be computed using the stationary phase approximation (SPA) [221, 222], for which the phase of the dominant quadrupole mode reads

$$\begin{aligned}
\psi_{\text{SPA}} &= 2\pi F T_0 + \Psi_0 \\
&+ \frac{3v^{-5}}{128\nu} \left\{ 1 + \left(\frac{3715}{756} + \frac{55}{9}\nu \right) v^2 - 16\pi v^3 \right. \\
&+ \left(\frac{15293365}{508032} + \frac{27145}{504}\nu + \frac{3085}{72}\nu^2 \right) v^4 + \left(\frac{38645}{252} - \frac{65}{3}\nu \right) \pi v^5 \ln v \\
&+ \left[\frac{11583231236531}{4694215680} - \frac{640}{3}\pi^2 - \frac{6848}{21}\gamma_E - \frac{6848}{21} \ln(4v) \right. \\
&\quad \left. + \left(-\frac{15737765635}{3048192} + \frac{2255}{12}\pi^2 \right) \nu + \frac{76055}{1728}\nu^2 - \frac{127825}{1296}\nu^3 \right] v^6 \\
&+ \left[\frac{77096675}{254016} + \frac{378515}{1512}\nu - \frac{74045}{756}\nu^2 \right] \pi v^7 \\
&+ \left[-\frac{2550713843998885153}{276808510218240} + \frac{90490}{189}\pi^2 + \frac{36812}{63}\gamma_E \right. \\
&\quad \left. + \frac{1011020}{1323} \ln 2 + \frac{78975}{196} \ln 3 + \frac{18406}{63} \ln v \right]
\end{aligned}$$

$$\begin{aligned}
& + \left(\frac{680712846248317}{42247941120} - \frac{109295}{224} \pi^2 + \frac{3911888}{1323} \gamma_E \right. \\
& \quad \left. + \frac{9964112}{1323} \ln 2 - \frac{78975}{49} \ln 3 + \frac{1955944}{1323} \ln v \right) \nu \\
& + \left(-\frac{7510073635}{3048192} + \frac{11275}{144} \pi^2 \right) \nu^2 - \frac{1292395}{12096} \nu^3 + \frac{5975}{96} \nu^4 \Big] v^8 \ln v \\
& + \left[\frac{105344279473163}{18776862720} - \frac{640}{3} \pi^2 - \frac{13696}{21} \gamma_E - \frac{13696}{21} \ln(4v) \right. \\
& \quad \left. + \left(-\frac{1492917260735}{134120448} + \frac{2255}{6} \pi^2 \right) \nu + \frac{45293335}{127008} \nu^2 + \frac{10323755}{199584} \nu^3 \right] \pi v^9 \Big\}, \\
& \hspace{20em} \text{(VI.71)}
\end{aligned}$$

where $v \equiv (\pi G m F / c^3)^{1/3}$ with F being the Fourier frequency, and where T_0 and Ψ_0 are two integration constants. Again we have adjusted T_0 in order to simplify the result (and we have absorbed the usual $-\frac{\pi}{4}$ into Ψ_0). The coefficients up to 3.5PN, as well as the 4.5PN piece, are already in use; see e.g. Appendix A of [223].

Part B

Scalar-tensor theories

Chapter VII

Gravitational waves in scalar-tensor theories

This chapter is mainly based on [2].

The multiple detections of GW events from inspiraling compact binary systems by the LIGO-Virgo observatory has allowed for a new era of GW precision astronomy [72]. They have already led to significant progress in a wide range of fields such as relativistic astrophysics, cosmology and fundamental physics. In the future, the space-based LISA observatory coupled to the next generation of ground-based detectors, such as the Einstein Telescope, will help further improve our understanding of the Universe and the fundamental forces that govern it. However, our ability to challenge the current gravitational paradigm, based on GR, strongly relies on the construction of very precise and reliable banks of waveform templates for compact binary systems in various alternative theories.

In order to perform precise tests of GR with compact binary systems, one can either design some theory-dependent tests or remain agnostic on the choice and existence of a preferred theory of gravity. The former option consists in developing complete (inspiral-merger-ringdown) waveforms in a specific theory of gravity, that would complement the bank of GR templates currently being used [224, 225]. Conversely, the latter path uses general formalisms such as the PPN formalism [226], the post-Einsteinian formalism [227, 228] and blind tests of PN parameters [229, 230].

In this chapter, we will opt for the first approach and construct gravitational waveforms within the class of scalar-tensor (ST) theories of gravity. These theories were introduced by Jordan [62], Fierz [64], Brans and Dicke [231], and later generalized in [232, 233]. Since then, they have been extensively studied, both from the theoretical and observational points of view. In particular, binary pulsar observations have already put strong constraints on the parameters entering the models, see [234, 235, 236] for reviews. Due to the no-hair theorem in ST theory, valid for stationary *isolated* black holes [237, 238], one expects that the motion and radiation of *binary* black holes are indistinguishable from those of the GR solution. To some extent, this expectation has been confirmed by numerical relativity calculations [239]. Consequently, relevant theory-dependent tests for ST theory involve binary neutron stars (or more exotic extended compact objects) as well as asymmetrical black hole and neutron star (BH-NS) binaries, such as the ones recently discovered by LIGO-Virgo [240].

We will focus on the inspiral phase of coalescing binary systems, for which the PN formalism, i.e. an approximation for weak gravitational fields and small orbital velocities, can be applied. Previous results on the PN expansion in ST theories were derived using the effective field theory [234, 241] and the Direct Integration of the Relaxed field Equations (DIRE) method [242, 243]. The implementation for the matched filtering analysis in the LISA detector was studied in [244]. Important previous results include the derivation of the dynamics of compact binaries at 2.5PN beyond Newtonian, i.e. $\sim (v/c)^5$ order [245], as well as the computation of the waveform at 2PN order for the tensor modes [246] and 1.5PN order for the scalar modes [247] (where PN orders with respect to waveforms and fluxes are counted relatively to the quadrupolar emission, which is leading order in GR).

More recently, the equations of motion of compact binaries were computed to 3PN order beyond Newtonian gravity [248, 249] using the MPM-PN [14] coupled to a Fokker Lagrangian approach [128]. Note that at 3PN order in ST theory, the level of difficulty is similar to 4PN order in GR. The result comes with the presence of a new dipolar nonlocal tail term at 3PN order and the need to use a dimensional regularisation scheme to treat both the ultraviolet and infrared divergences. Finally, the 3PN results were complemented by the derivation of the scalar tidal effects to leading order in ST theories, which also arise at 3PN order due to the presence of the scalar dipole [250].

In the present work, we extend previous results by computing the energy flux to 1.5PN order beyond the quadrupolar radiation in GR (previously, the flux was only known to 1PN order) and we compute for the first time the scalar modes to 1.5PN order. In particular, we include the nonlinear memory effect entering at 1.5PN order associated with the dipole radiation. Recent investigations of the memory effect in ST theory can also be found in [251, 252, 253]. We also treat the dipolar and quadrupolar tail effects which enter respectively at 0.5PN and 1.5PN order. Contrary to the work of Lang [246, 247], who uses moments named after Epstein and Wagoner (EW) [254], we adopt STF definitions for the multipole moments [48]. Furthermore, the formalism used in the works [246, 247] to tackle nonlinear effects is the DIRE formalism of Will and Wiseman [255], which, although different, was shown to be equivalent to the present MPM-PN formalism [14]. Thus, the present chapter provides a necessary alternative derivation of the results found by Lang [246, 247], in addition to extending the flux by a half PN order.

The chapter is organized as follows. In Sec VII.1, we define the class of massless scalar-tensor theories we are interested in and derive the corresponding field equations. In Section VII.2, we adapt the MPM formalism to the case of ST theories, focusing in particular on the memory and tail effects. Then, in Section VII.3, we apply this formalism to the specific case of compact binaries, obtain the explicit expressions of the source moments and energy fluxes, and compare them with the literature. In Section VII.4, we reduce our results to the case of quasicircular orbits, giving the fluxes and the orbital phase evolution for dipolar-driven systems. Finally, in Section VII.5, we compute for the first time the scalar modes, together with the previously known gravitational modes. Comparisons with recent numerical results [256] are presented in Section VII.6. Explicit expressions for dissipative terms in the equations of motion are also given for the first time in Appendix B.1; long results concerning the multipole moments are displayed in Appendix B.2 (including transformation formulas between EW and STF moments); and the very long expression of the instantaneous scalar flux is relegated to Appendix B.3.

In order to later present our results, following [248], we introduce a number of ST and

PN parameters. The ST parameters are defined based on the value ϕ_0 of the scalar field ϕ at spatial infinity, on the Brans-Dicke-like scalar function $\omega(\phi)$ and on the mass-functions $m_A(\phi)$. We pose $\varphi \equiv \phi/\phi_0$. The PN parameters naturally extend and generalize the usual PPN parameters to the case of a general ST theory [257, 226]. All these parameters are given and summarized in the following Table VII.1.

	ST parameters
general	$\omega_0 = \omega(\phi_0), \quad \omega'_0 = \left. \frac{d\omega}{d\phi} \right _{\phi=\phi_0}, \quad \omega''_0 = \left. \frac{d^2\omega}{d\phi^2} \right _{\phi=\phi_0}, \quad \varphi = \frac{\phi}{\phi_0}, \quad \tilde{g}_{\mu\nu} = \varphi g_{\mu\nu},$ $\tilde{G} = \frac{G(4+2\omega_0)}{\phi_0(3+2\omega_0)}, \quad \zeta = \frac{1}{4+2\omega_0},$ $\lambda_1 = \frac{\zeta^2}{(1-\zeta)} \left. \frac{d\omega}{d\varphi} \right _{\varphi=1}, \quad \lambda_2 = \frac{\zeta^3}{(1-\zeta)} \left. \frac{d^2\omega}{d\varphi^2} \right _{\varphi=1}, \quad \lambda_3 = \frac{\zeta^4}{(1-\zeta)} \left. \frac{d^3\omega}{d\varphi^3} \right _{\varphi=1}.$ <p>1 \leftrightarrow 2 switches the particle's labels (note the index on the λ_i's is not a particle label)</p>
sensitivities	$s_A = \left. \frac{d \ln m_A(\phi)}{d \ln \phi} \right _{\phi=\phi_0}, \quad s_A^{(k)} = \left. \frac{d^{k+1} \ln m_A(\phi)}{d(\ln \phi)^{k+1}} \right _{\phi=\phi_0}, \quad (A = 1, 2)$ $s'_A = s_A^{(1)}, \quad s''_A = s_A^{(2)}, \quad s'''_A = s_A^{(3)},$ $S_+ = \frac{1-s_1-s_2}{\sqrt{\alpha}}, \quad S_- = \frac{s_2-s_1}{\sqrt{\alpha}}.$
Order	PN parameters
N	$\alpha = 1 - \zeta + \zeta(1 - 2s_1)(1 - 2s_2)$
1PN	$\bar{\gamma} = -\frac{2\zeta}{\alpha}(1 - 2s_1)(1 - 2s_2), \quad \text{Degeneracy}$ $\bar{\beta}_1 = \frac{\zeta}{\alpha^2}(1 - 2s_2)^2(\lambda_1(1 - 2s_1) + 2\zeta s'_1), \quad \alpha(2 + \bar{\gamma}) = 2(1 - \zeta)$ $\bar{\beta}_2 = \frac{\zeta}{\alpha^2}(1 - 2s_1)^2(\lambda_1(1 - 2s_2) + 2\zeta s'_2),$ $\bar{\beta}_+ = \frac{\bar{\beta}_1 + \bar{\beta}_2}{2}, \quad \bar{\beta}_- = \frac{\bar{\beta}_1 - \bar{\beta}_2}{2}.$
2PN	$\bar{\delta}_1 = \frac{\zeta(1-\zeta)}{\alpha^2}(1 - 2s_1)^2, \quad \bar{\delta}_2 = \frac{\zeta(1-\zeta)}{\alpha^2}(1 - 2s_2)^2, \quad \text{Degeneracy}$ $\bar{\delta}_+ = \frac{\bar{\delta}_1 + \bar{\delta}_2}{2}, \quad \bar{\delta}_- = \frac{\bar{\delta}_1 - \bar{\delta}_2}{2}, \quad 16\bar{\delta}_1\bar{\delta}_2 = \bar{\gamma}^2(2 + \bar{\gamma})^2$ $\bar{\chi}_1 = \frac{\zeta}{\alpha^3}(1 - 2s_2)^3[(\lambda_2 - 4\lambda_1^2 + \zeta\lambda_1)(1 - 2s_1) - 6\zeta\lambda_1 s'_1 + 2\zeta^2 s''_1],$ $\bar{\chi}_2 = \frac{\zeta}{\alpha^3}(1 - 2s_1)^3[(\lambda_2 - 4\lambda_1^2 + \zeta\lambda_1)(1 - 2s_2) - 6\zeta\lambda_1 s'_2 + 2\zeta^2 s''_2],$ $\bar{\chi}_+ = \frac{\bar{\chi}_1 + \bar{\chi}_2}{2}, \quad \bar{\chi}_- = \frac{\bar{\chi}_1 - \bar{\chi}_2}{2}.$

Table VII.1: Summary of parameters for the general ST theory and our notation for PN parameters.

VII.1 Massless scalar-tensor theories

We consider a generic class of scalar-tensor theories in which a single massless scalar field ϕ minimally couples to the metric $g_{\mu\nu}$. It is described by the action

$$S_{\text{ST}} = \frac{c^3}{16\pi G} \int d^4x \sqrt{-g} \left[\phi R - \frac{\omega(\phi)}{\phi} g^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi \right] + S_{\text{m}}(\mathbf{m}, g_{\alpha\beta}), \quad (\text{VII.1})$$

where R and g are respectively the Ricci scalar and the determinant of the metric, ω is a function of the scalar field and \mathbf{m} stands generically for the matter fields. The action for the matter S_{m} is a function only of the matter fields and the metric. A major difference

in scalar-tensor theories compared to GR is that, as a consequence of the breaking of the strong equivalence principle, we have to take into account the internal gravity of each body. Indeed, the scalar field determines the effective gravitational constant, which in turn affects the competition between gravitational and nongravitational forces within the body (see the discussion in Section 1.2.3). Thus, the value of the scalar field has an indirect influence on the size of the compact body and on its internal gravity. Here, we follow the approach pioneered by Eardley [66] (see also [258]) and we take for S_m the effective action for N nonspinning point-particles with the masses $m_A(\phi)$ depending in an unspecified manner on the value of the scalar field at the location of the particles, i.e.

$$S_m = -c \sum_A \int m_A(\phi) \sqrt{-(g_{\alpha\beta})_A} dy_A^\alpha dy_A^\beta, \quad (\text{VII.2})$$

where y_A^α denote the spacetime positions of the particles, and $(g_{\alpha\beta})_A$ is the metric evaluated at the position of particle A . Thus, the matter action depends indirectly on the scalar field, and we define the sensitivities of the particles to variations in the scalar field by

$$s_A \equiv \left. \frac{d \ln m_A(\phi)}{d \ln \phi} \right|_{\phi=\phi_0}, \quad s_A^{(k)} \equiv \left. \frac{d^{k+1} \ln m_A(\phi)}{d(\ln \phi)^{k+1}} \right|_{\phi=\phi_0} \quad \text{for } k \geq 2, \quad (\text{VII.3})$$

where ϕ_0 is the value of the scalar field at spatial infinity that is assumed to be constant in time, i.e. we neglect the cosmological evolution. Since we expand systematically around the asymptotic value of the scalar field, we can account both for nonscalarized stars and scalarized stars with constant scalar charge [259, 260]. The sensitivity of neutron stars depends on the mass and internal equation of state. In the weak-field limit, s_A is proportional to the gravitational energy per unit mass of the body and is of order 0.2. For stationary black holes, since all information regarding the matter which formed the black hole has disappeared behind the horizon, the mass can depend only on the Planck scale, $m_A \propto M_{\text{Planck}} \propto G^{-1/2} \propto \phi^{1/2}$ hence $s_A^{\text{BH}} = \frac{1}{2}$. In this work, we will assume that $s_A^{\text{BH}} = \frac{1}{2}$ for each of the black holes in a binary system and check that all our PN results will be indistinguishable from GR in the case of binary black holes (BBHs).

The action (VII.1) is usually called the “metric” or “Jordan”-frame action, as the matter only couples to the Jordan or “physical” metric $g_{\alpha\beta}$. Then, we define the rescaled scalar field and the conformally related metric as

$$\varphi \equiv \frac{\phi}{\phi_0}, \quad \tilde{g}_{\alpha\beta} \equiv \varphi g_{\alpha\beta}, \quad (\text{VII.4})$$

so that the physical and conformal metrics have the same asymptotic behavior at spatial infinity. In terms of these new variables, the action (VII.1) can be rewritten as

$$S_{\text{ST}}^{\text{GF}} = \frac{c^3 \phi_0}{16\pi G} \int d^4x \sqrt{-\tilde{g}} \left[\tilde{R} - \frac{1}{2} \tilde{g}_{\mu\nu} \tilde{\Gamma}^\mu \tilde{\Gamma}^\nu - \frac{3 + 2\omega(\phi)}{2\varphi^2} \tilde{g}^{\alpha\beta} \partial_\alpha \varphi \partial_\beta \varphi \right] + S_m(\mathbf{m}, g_{\alpha\beta}), \quad (\text{VII.5})$$

where we have inserted the harmonic gauge-fixing (GF) term $-\frac{1}{2} \tilde{g}_{\mu\nu} \tilde{\Gamma}^\mu \tilde{\Gamma}^\nu$ associated with the conformal metric, with $\tilde{\Gamma}^\mu \equiv \tilde{g}^{\rho\sigma} \tilde{\Gamma}_{\rho\sigma}^\mu$ and $\tilde{\Gamma}_{\rho\sigma}^\mu$ the Christoffel symbols of that metric, and \tilde{R} the associated Ricci scalar. As the scalar field is now minimally coupled to the

conformal or ‘‘Einstein’’ metric, the action (VII.5) is called the ‘‘Einstein’’-frame action. We will perform most of our computations in this frame and go back to the physical metric only in the end when computing observable quantities.

Next, we define the scalar and metric perturbation variables $\psi \equiv \varphi - 1$ and $h^{\mu\nu} \equiv \tilde{\mathfrak{g}}^{\mu\nu} - \eta^{\mu\nu}$ where $\eta^{\mu\nu} \equiv \text{diag}(-1, 1, 1, 1)$ is the Minkowski metric and $\tilde{\mathfrak{g}}^{\mu\nu} \equiv \sqrt{-\tilde{g}}\tilde{g}^{\mu\nu}$ is the conformal gothic metric. Then, the field equations derived from the harmonic gauge-fixed action (VII.5) read

$$\square_\eta h^{\mu\nu} = \frac{16\pi G}{c^4} \tau^{\mu\nu}, \quad (\text{VII.6a})$$

$$\square_\eta \psi = -\frac{8\pi G}{c^4} \tau_s, \quad (\text{VII.6b})$$

where \square_η denotes the ordinary flat spacetime d’Alembert operator, and where the source terms read

$$\tau^{\mu\nu} = \frac{\varphi}{\phi_0} |g| T^{\mu\nu} + \frac{c^4}{16\pi G} \Lambda^{\mu\nu}, \quad (\text{VII.7a})$$

$$\tau_s = -\frac{\varphi}{\phi_0(3+2\omega)} \sqrt{-g} \left(T - 2\varphi \frac{\partial T}{\partial \varphi} \right) - \frac{c^4}{8\pi G} \Lambda_s. \quad (\text{VII.7b})$$

Here $T^{\mu\nu} = 2(-g)^{-1/2} \delta S_m / \delta g_{\mu\nu}$ is the matter stress-energy tensor, $T \equiv g_{\mu\nu} T^{\mu\nu}$ and $\partial T / \partial \varphi$ is defined as the partial derivative of $T(g_{\mu\nu}, \varphi)$ holding $g_{\mu\nu}$ constant. The nonlinearities in the scalar source read

$$\Lambda_s = -h^{\alpha\beta} \partial_\alpha \partial_\beta \psi - \partial_\alpha \psi \partial_\beta h^{\alpha\beta} + \left(\frac{1}{\varphi} - \frac{\phi_0 \omega'(\phi)}{3+2\omega(\phi)} \right) \tilde{\mathfrak{g}}^{\alpha\beta} \partial_\alpha \psi \partial_\beta \psi. \quad (\text{VII.8})$$

We write the tensor source as $\Lambda^{\mu\nu} = \Lambda_{\text{LL}}^{\mu\nu} + \Lambda_{\text{H}}^{\mu\nu} + \Lambda_{\text{GF}}^{\mu\nu} + \Lambda_\phi^{\mu\nu}$, where $\Lambda_{\text{LL}}^{\mu\nu}$ is the Landau-Lifshitz pseudo-energy tensor [10], $\Lambda_{\text{H}}^{\mu\nu}$ comes from our use of the flat version of the d’Alembert operator in (VII.6), $\Lambda_{\text{GF}}^{\mu\nu}$ is due to the gauge-fixing term in the action and $\Lambda_\phi^{\mu\nu}$ is sourced by the scalar field. Note that $\Lambda_{\text{GR}}^{\mu\nu} = \Lambda_{\text{LL}}^{\mu\nu} + \Lambda_{\text{H}}^{\mu\nu} + \Lambda_{\text{GF}}^{\mu\nu}$ will take the same form as in GR. The expressions of these source terms are given by

$$\begin{aligned} \Lambda_{\text{LL}}^{\alpha\beta} = & \frac{1}{2} \tilde{\mathfrak{g}}^{\alpha\beta} \tilde{\mathfrak{g}}_{\mu\nu} \partial_\lambda h^{\mu\gamma} \partial_\gamma h^{\nu\lambda} - \tilde{\mathfrak{g}}^{\alpha\mu} \tilde{\mathfrak{g}}_{\nu\gamma} \partial_\lambda h^{\beta\gamma} \partial_\mu h^{\nu\lambda} - \tilde{\mathfrak{g}}^{\beta\mu} \tilde{\mathfrak{g}}_{\nu\gamma} \partial_\lambda h^{\alpha\gamma} \partial_\mu h^{\nu\lambda} \\ & + \tilde{\mathfrak{g}}_{\mu\nu} \tilde{\mathfrak{g}}^{\lambda\gamma} \partial_\lambda h^{\alpha\mu} \partial_\gamma h^{\beta\nu} + \frac{1}{8} \left(2\tilde{\mathfrak{g}}^{\alpha\mu} \tilde{\mathfrak{g}}^{\beta\nu} - \tilde{\mathfrak{g}}^{\alpha\beta} \tilde{\mathfrak{g}}^{\mu\nu} \right) (2\tilde{\mathfrak{g}}_{\lambda\gamma} \tilde{\mathfrak{g}}_{\tau\pi} - \tilde{\mathfrak{g}}_{\gamma\tau} \tilde{\mathfrak{g}}_{\lambda\pi}) \partial_\mu h^{\lambda\pi} \partial_\nu h^{\gamma\tau}, \end{aligned} \quad (\text{VII.9a})$$

$$\Lambda_{\text{H}}^{\alpha\beta} = -h^{\mu\nu} \partial_\mu \partial_\nu h^{\alpha\beta} + \partial_\mu h^{\alpha\nu} \partial_\nu h^{\beta\mu}, \quad (\text{VII.9b})$$

$$\Lambda_{\text{GF}}^{\alpha\beta} = -\partial_\lambda h^{\lambda\alpha} \partial_\sigma h^{\sigma\beta} - \partial_\lambda h^{\lambda\rho} \partial_\rho h^{\alpha\beta} - \frac{1}{2} \tilde{\mathfrak{g}}^{\alpha\beta} \tilde{\mathfrak{g}}_{\rho\sigma} \partial_\lambda h^{\lambda\rho} \partial_\gamma h^{\gamma\sigma} + 2\tilde{\mathfrak{g}}_{\rho\sigma} \tilde{\mathfrak{g}}^{\lambda(\alpha} \partial_\lambda h^{\beta)\rho} \partial_\gamma h^{\gamma\sigma}, \quad (\text{VII.9c})$$

$$\Lambda_\phi^{\mu\nu} = \frac{3+2\omega(\phi)}{\varphi^2} \left(\tilde{\mathfrak{g}}^{\mu\alpha} \tilde{\mathfrak{g}}^{\nu\beta} - \frac{1}{2} \tilde{\mathfrak{g}}^{\mu\nu} \tilde{\mathfrak{g}}^{\alpha\beta} \right) \partial_\alpha \psi \partial_\beta \psi. \quad (\text{VII.9d})$$

Note that the gauge-fixing term (VII.9c) contains the harmonicities $\partial_\nu h^{\mu\nu}$ which are not zero off-shell, i.e. when the accelerations are not replaced by the equations of motion. However, this term will ensure that, on-shell, our results are in harmonic coordinates.

VII.2 The Multipolar PM formalism in scalar-tensor theories

VII.2.1 The scalar-tensor multipole moments

In this section, we solve the ST vacuum field equations, $\square h^{\mu\nu} = \Lambda^{\mu\nu}$ and $\square\psi = \Lambda_s$, in the exterior region of the isolated matter system by means of a multipolar decomposition (indicated by \mathcal{M}) conjointly with a nonlinear PM expansion [92]. Thus, the solution is written as

$$h_{\text{ext}}^{\mu\nu} \equiv \mathcal{M}(h^{\mu\nu}) = Gh_1^{\mu\nu} + G^2h_2^{\mu\nu} + \mathcal{O}(G^3), \quad (\text{VII.10a})$$

$$\psi_{\text{ext}} \equiv \mathcal{M}(\psi) = G\psi_1 + G^2\psi_2 + \mathcal{O}(G^3). \quad (\text{VII.10b})$$

The multipolar expansion is entirely specified by the most general expressions for the multipolar decomposition of the linear coefficients $h_1^{\mu\nu}$ and ψ_1 . Starting with the tensor coefficient $h_1^{\mu\nu}$ which satisfies the vacuum equations, $\square h_1^{\mu\nu} = \partial_\nu h_1^{\mu\nu} = 0$, we adopt the same definition of multipole moments as in GR, i.e. we follow Section II.2.1 and we write the most general solution in harmonic coordinates as

$$h_1^{\mu\nu} = k_1^{\mu\nu} + \partial^\mu \varphi_1^{\mu\nu}, \quad (\text{VII.11})$$

where the gauge vector φ_1^μ obeys $\square\varphi_1^\mu = 0$, and where $k_1^{\mu\nu}$ takes a ‘‘canonical’’ form [48] in terms of two sets of source multipole moments $I_L(u)$ (mass type) and $J_L(u)$ (current type), with $u \equiv t - r/c$. The moments I_L and J_L are STF with respect to the ℓ indices composing L . We pose

$$k_1^{00} = -\frac{4}{c^2} \sum_{\ell=0}^{\infty} \frac{(-)^\ell}{\ell!} \partial_L \left[\frac{I_L(u)}{r} \right], \quad (\text{VII.12a})$$

$$k_1^{0i} = \frac{4}{c^3} \sum_{\ell=1}^{\infty} \frac{(-)^\ell}{\ell!} \left(\partial_{L-1} \left[\frac{1}{r} I_{iL-1}^{(1)}(u) \right] + \frac{\ell}{\ell+1} \partial_{aL-1} \left[\frac{1}{r} \epsilon_{iab} J_{bL-1}(u) \right] \right), \quad (\text{VII.12b})$$

$$k_1^{ij} = -\frac{4}{c^4} \sum_{\ell=2}^{\infty} \frac{(-)^\ell}{\ell!} \left(\partial_{L-2} \left[\frac{1}{r} I_{ijL-2}^{(2)}(u) \right] + \frac{2\ell}{\ell+1} \partial_{aL-2} \left[\frac{1}{r} \epsilon_{ab(i} J_{j)bL-2}^{(1)}(u) \right] \right). \quad (\text{VII.12c})$$

The moments are defined in such a way that they reduce to the familiar expressions of the Newtonian moments at leading order [48]. The lowest monopolar and dipolar moments satisfy the usual conservation laws $dI/dt = 0$, $d^2I_i/dt^2 = 0$ and $dJ_i/dt = 0$, which are consequences of the harmonic gauge condition $\partial_\nu k_1^{\mu\nu} = 0$. In particular, the monopole I is directly related to the total conserved mass M of the system, defined in the usual way at spatial infinity (for either the physical or Einstein metric) by $I = M/\phi_0$.¹ The linear gauge vector φ^μ in (VII.11) similarly admits a decomposition in terms of four gauge moments, W_L , X_L , Y_L and Z_L , adopting the same definition as in GR, as given by (II.18).

To obtain the expressions of the moments I_L , J_L (and also W_L , X_L , Y_L and Z_L) as functions of the source, the procedure is essentially the same as in GR. The multipole

¹Indeed, from $\square h^{\mu\nu} = \frac{16\pi G}{c^4} \tau^{\mu\nu}$ we infer that $h^{00} = -\frac{4GI}{c^2} + \dots$ where $I = c^{-2} \int d^3\mathbf{x} \tau^{00}$. For a system of point masses which are initially in free motion (when $t \rightarrow -\infty$), the total mass is given by $M = \sum_A m_A$, while from (VII.7a) we see that at spatial infinity $\tau^{00} = T^{00}/\phi_0 = c^2 \sum_A m_A \delta_A/\phi_0$, hence we have $I = M/\phi_0$.

expansion of $h^{\mu\nu}$ (noted $\mathcal{M}(h^{\mu\nu})$, as in Section II.2.1) is obtained by matching it to the PN expansion in the near zone of the source along the lines of Section II.2.3 and [261]:

$$\mathcal{M}(h^{\mu\nu}) = \text{FP}_{B=0} \square_{\text{ret}}^{-1} [\tilde{r}^B \mathcal{M}(\Lambda^{\mu\nu})] - \frac{4G}{c^4} \sum_{\ell=0}^{\infty} \frac{(-)^\ell}{\ell!} \partial_L \left[\frac{\mathcal{F}_L^{\mu\nu}(u)}{r} \right]. \quad (\text{VII.13a})$$

The first term is a particular solution of the wave equation, defined using a retarded d'Alembertian and a finite part regularization procedure as in Section II.2.1. The second term in (VII.13a) represents a homogeneous retarded solution parametrized by the multipole moment function (STF in its indices L)

$$\mathcal{F}_L^{\mu\nu}(u) = \text{FP}_{B=0} \int d^3\mathbf{x} \tilde{r}^B \hat{x}_L \int_{-1}^1 dz \delta_\ell(z) \bar{\tau}^{\mu\nu}(\mathbf{x}, u + zr/c), \quad (\text{VII.13b})$$

where $\delta_\ell(z)$ is defined in (II.38). Following the notation of [14], the function $\mathcal{F}_L^{\mu\nu}(u)$ is expressed in full generality in terms of the PN expansion $\bar{\tau}^{\mu\nu}$ of the pseudo-tensor $\tau^{\mu\nu}$ defined in (VII.7a). From the result (VII.13a), we can identify the linearized piece $Gh_1^{\mu\nu}$ in (VII.10a) as being essentially given by the second term in (VII.13a). However, the situation is more complicated because one also has to take into account the harmonic gauge condition which is not satisfied separately by the second term in (VII.13a). In the end, the source moments follow from the irreducible decomposition of the spacetime components $\mathcal{F}_L^{\mu\nu}(u)$. Defining $\Sigma \equiv (\bar{\tau}^{00} + \bar{\tau}^{ii})/c^2$, $\Sigma_i \equiv \bar{\tau}^{0i}/c$ and $\Sigma_{ij} \equiv \bar{\tau}^{ij}$, we get

$$\begin{aligned} \text{I}_L(u) = \text{FP}_{B=0} \int d^3\mathbf{x} \tilde{r}^B \int_{-1}^1 dz \left[\delta_\ell(z) \hat{x}_L \Sigma - \frac{4(2\ell+1)}{c^2(\ell+1)(2\ell+3)} \delta_{\ell+1}(z) \hat{x}_{iL} \Sigma_i^{(1)} \right. \\ \left. + \frac{2(2\ell+1)}{c^4(\ell+1)(\ell+2)(2\ell+5)} \delta_{\ell+2}(z) \hat{x}_{ijL} \Sigma_{ij}^{(2)} \right] (\mathbf{x}, u + zr/c), \end{aligned} \quad (\text{VII.14a})$$

$$\begin{aligned} \text{J}_L(u) = \text{FP}_{B=0} \int d^3\mathbf{x} \tilde{r}^B \int_{-1}^1 dz \epsilon_{ab\langle i_\ell} \left[\delta_\ell(z) \hat{x}_{L-1\rangle a} \Sigma_b \right. \\ \left. - \frac{2\ell+1}{c^2(\ell+2)(2\ell+3)} \delta_{\ell+1}(z) \hat{x}_{L-1\rangle ac} \Sigma_{bc}^{(1)} \right] (\mathbf{x}, u + zr/c). \end{aligned} \quad (\text{VII.14b})$$

These expressions are formally identical as those in GR, but of course in ST theory the source terms Σ , Σ_i and Σ_{ij} depend on the scalar field through (VII.9). Similarly, the gauge moments' expressions are formally identical to the GR case and are given by (125) of [14].

Next, in ST theory, the scalar field brings, in addition to $\text{I}_L(u)$ and $\text{J}_L(u)$, a new set of multipole moments which we call $\text{I}_L^s(u)$, also chosen to be STF. Finding their expression is simpler than in GR (we do not need a correction due to the gauge condition) and the linear piece $G\psi_1$ can be directly identified from the multipole expansion of ψ . We find

$$\mathcal{M}(\psi) = \text{FP}_{B=0} \square_{\text{ret}}^{-1} [\tilde{r}^B \mathcal{M}(\Lambda^s)] + G\psi_1, \quad (\text{VII.15a})$$

$$\text{with } \psi_1 = -\frac{2}{c^2} \sum_{\ell=0}^{\infty} \frac{(-)^\ell}{\ell!} \partial_L \left[\frac{\text{I}_L^s(u)}{r} \right]. \quad (\text{VII.15b})$$

Similarly to the GR case, the scalar moments are obtained in closed form and we have, defining $\Sigma^s \equiv -\bar{\tau}_s/c^2$,

$$I_L^s(u) = \text{FP}_{B=0} \int d^3\mathbf{x} \tilde{r}^B \int_{-1}^1 dz \delta_\ell(z) \hat{x}_L \Sigma^s(\mathbf{x}, u + zr/c). \quad (\text{VII.16})$$

Note that in contrast to the tensor monopole I, the scalar monopole I^s is not constant but its time-variation will be a PN effect, i.e. $dI^s/dt = \mathcal{O}(c^{-2})$. Later, we will define $I^s(u) = \phi_0^{-1}[m^s + \frac{1}{c^2}E^s(u)]$ where m^s is constant and $E^s(u)$ is the time-varying PN correction, see (VII.39).

VII.2.2 Memory and tail effects in ST theory

Once the vacuum linearized solutions (VII.12) and (VII.15b) are obtained, with explicit expressions for the multipole moments as integrals over the PN expansions of $\tau^{\mu\nu}$ and τ_s , the nonlinear contributions can be computed by adapting to ST theories the MPM algorithm of [92]. In the following, we will focus our discussion on the new effects specific to ST theories but at the end, we will give the complete nonlinear contributions needed to control the waveform to 1.5PN order beyond the quadrupole radiation of GR.

We saw that in GR, the nonlinear memory [170, 172, 164, 262] is a nonlocal effect due to the radiation of linear waves by the stress-energy tensor, dominantly associated with the mass quadrupole moment, which appears at 2.5PN in the waveform (VI.18). In ST theory, there is a new type of memory effect associated with the scalar dipole radiation that comes from the quadratic interaction between two scalar dipole moments, say $I_i^s \times I_j^s$. As we shall see, such an effect arises at 1.5PN order in the waveform. For the dipole radiation, the linear approximation to the scalar field given by (VII.15b) reads (with $c = G = 1$)

$$\psi_1 \Big|_{I_i^s} = -2\partial_i \left[\frac{I_i^s(u)}{r} \right] = 2n^i \left[r^{-1} I_i^s(u) + r^{-2} I_i^s(u) \right]. \quad (\text{VII.17})$$

The equation we need to adapt the MPM algorithm of Section II.2.1 and solve for the quadratic interaction $I_i^s \times I_j^s$, including the nonlinear memory effect, is²

$$\square h_2^{\mu\nu} \Big|_{I_i^s \times I_j^s} = \Lambda_2^{\mu\nu} \Big|_{I_i^s \times I_j^s}, \quad (\text{VII.18a})$$

$$\text{with } \Lambda_2^{\mu\nu} \Big|_{I_i^s \times I_j^s} \equiv (3 + 2\omega_0) \left(\eta^{\mu\rho} \eta^{\nu\sigma} - \frac{1}{2} \eta^{\mu\nu} \eta^{\rho\sigma} \right) \partial_\rho \psi_1 \partial_\sigma \psi_1 \Big|_{I_i^s \times I_j^s}, \quad (\text{VII.18b})$$

where the source term has been obtained from (VII.9d). Following the MPM algorithm, we shall obtain the solution of this equation, together with the harmonic coordinate condition, by the usual construction described in Section II.2.1, which here reads

$$h_2^{\mu\nu} \Big|_{I_i^s \times I_j^s} = u_2^{\mu\nu} + v_2^{\mu\nu} \Big|_{I_i^s \times I_j^s}, \quad (\text{VII.19a})$$

²The memory effect arises only in the tensor part $h_2^{\mu\nu}$ and not in the scalar part ψ_2 , though there will be a local (i.e. not hereditary) contribution in the scalar field due to the interaction $I_i^s \times I_j^s$, see (VII.38a) below.

where we define a usual $w_2^\mu = \partial_\nu u_2^{\mu\nu}$ and

$$w_2^{\mu\nu} \Big|_{\mathbb{I}_i^s \times \mathbb{I}_j^s} = \text{FP}_{B=0} \square_{\text{ret}}^{-1} \left[\tilde{r}^B \Lambda_2^{\mu\nu} \right] \Big|_{\mathbb{I}_i^s \times \mathbb{I}_j^s}, \quad (\text{VII.19b})$$

$$v_2^{\mu\nu} \Big|_{\mathbb{I}_i^s \times \mathbb{I}_j^s} = \mathcal{V}^{\mu\nu} [w_2] \Big|_{\mathbb{I}_i^s \times \mathbb{I}_j^s}. \quad (\text{VII.19c})$$

We recall that the divergence of the particular solution $u_2^{\mu\nu}$ comes only from the differentiation of the regularization factor \tilde{r}^B (since the source term obeys $\partial_\nu \Lambda_2^{\mu\nu} = 0$), hence

$$w_2^\mu \Big|_{\mathbb{I}_i^s \times \mathbb{I}_j^s} = \text{FP}_{B=0} \square_{\text{ret}}^{-1} \left[B \tilde{r}^B \frac{n^i}{r} \Lambda_2^{\mu i} \right] \Big|_{\mathbb{I}_i^s \times \mathbb{I}_j^s}. \quad (\text{VII.19d})$$

Due to the presence of the explicit factor B , we only have to look at the pole $\propto 1/B$ coming from the integration. It turns out that, even without having controlled the full $u_2^{\mu\nu}$ before, the result (VII.19d) is very simple to compute. Inserting the source term (VII.18b) computed with (VII.17), and integrating by means of (A18) of [173], we get, to the leading $1/r$ order when $r \rightarrow +\infty$,

$$w_2^0 \Big|_{\mathbb{I}_i^s \times \mathbb{I}_j^s} = \frac{3 + 2\omega_0}{r} \left[-\frac{4}{3} \mathbb{I}_i^s \mathbb{I}_i^s + \frac{d}{du} \left(\frac{4}{3} \mathbb{I}_i^s \mathbb{I}_i^s - \frac{8}{9} \mathbb{I}_i^s \mathbb{I}_i^s \right) \right] + \mathcal{O} \left(\frac{1}{r^2} \right), \quad (\text{VII.20})$$

$$w_2^i \Big|_{\mathbb{I}_i^s \times \mathbb{I}_j^s} = \frac{3 + 2\omega_0}{r} \frac{d}{du} \left(\frac{4}{9} n^i \mathbb{I}_j^s \mathbb{I}_j^s + \frac{8}{9} n^j \mathbb{I}_j^s \mathbb{I}_i^s - \frac{4}{9} n^j \mathbb{I}_i^s \mathbb{I}_j^s \right) + \mathcal{O} \left(\frac{1}{r^2} \right). \quad (\text{VII.21})$$

Applying the MPM harmonicity algorithm, or rather, its version at leading order $1/r$ explained in Appendix B of [173], and keeping only the nonlocal (hereditary) terms (we shall add all instantaneous terms in the end results below), we get

$$v_2^{00} \Big|_{\mathbb{I}_i^s \times \mathbb{I}_j^s} = \frac{3 + 2\omega_0}{r} \left[\frac{4}{3} \int_0^{+\infty} d\tau \left[\mathbb{I}_i^s \mathbb{I}_i^s \right] (u - \tau) + \text{“inst.”} \right] + \mathcal{O} \left(\frac{1}{r^2} \right), \quad (\text{VII.22})$$

while v_2^{0i} and v_2^{ij} are purely instantaneous. The term (VII.22) has the form of a mass correction, and taking into account $k_1^{00} = -4\text{I}/r + \dots$ together with the link $\text{I} = M/\phi_0$, where M is the constant total mass-energy, we obtain the Bondi-type mass taking into account radiation loss as

$$M_{\text{B}}(u) = M - \frac{(3 + 2\omega_0) \phi_0}{3} \int_0^{+\infty} d\tau \left[\mathbb{I}_i^s \mathbb{I}_i^s \right] (u - \tau), \quad (\text{VII.23})$$

which implies the standard dipolar mass law in ST theory, as given for instance by (9) in [250].

Let us now consider the piece $u_2^{\mu\nu}$ defined by (VII.19b), and look for nonlocal effects therein. For any wave equation whose source term is made of quadratic products of linear waves, like (VII.17), we know that the nonlocal terms arise only from the coefficient $1/r^2$ in the source, see the proof in [173]. For the dipolar waves, this takes the form of the stress-energy tensor of massless radiation, namely

$$\Lambda_2^{\mu\nu} = \frac{k^\mu k^\nu}{r^2} \sigma(u, \mathbf{n}) + \mathcal{O} \left(\frac{1}{r^3} \right), \quad (\text{VII.24})$$

where $k^\mu = (1, n^i)$ is the Minkowski null vector and the energy density in the dipolar wave is given by

$$\sigma(u, \mathbf{n}) \Big|_{\mathbb{I}_i^s \times \mathbb{I}_j^s} = 4(3 + 2\omega_0) n^i n^j \mathbb{I}_i^s{}^{(2)}(u) \mathbb{I}_j^s{}^{(2)}(u). \quad (\text{VII.25})$$

Thus, the energy distribution in the dipolar waves is

$$\frac{dE^{\text{wave}}}{d\Omega} = \frac{\phi_0}{16\pi} \int_0^{+\infty} d\tau \sigma(u - \tau, \mathbf{n}), \quad (\text{VII.26})$$

with the total energy in the waves given by $\Delta E^{\text{wave}} \equiv \int d\Omega \frac{dE^{\text{wave}}}{d\Omega} = M - M_B$. As we have already mentioned, the only hereditary contributions in $u_2^{\mu\nu}$ comes from the piece (VII.24). At this stage, it is convenient to perform a gauge transformation with gauge vector [160]

$$\zeta_2^\mu \equiv \text{FP}_{B=0} \square_{\text{ret}}^{-1} \left[\tilde{r}^B \frac{k^\mu}{2r^2} \int_0^{+\infty} d\tau \sigma(u - \tau, \mathbf{n}) \right]. \quad (\text{VII.27})$$

Posing $u_2^{\mu\nu} = u_2^{\mu\nu} + \partial^\mu \zeta_2^\nu + \partial^\nu \zeta_2^\mu - \eta^{\mu\nu} \partial_\rho \zeta_2^\rho$ and adding (VII.22), we then obtain the metric in the new gauge, i.e. $h_2^{\mu\nu} \equiv u_2^{\mu\nu} + v_2^{\mu\nu}$, in the form

$$h_2^{\mu\nu} = \frac{1}{r} \left[\int_0^{+\infty} d\tau \mathcal{H}_2^{\mu\nu}(u - \tau, \mathbf{n}) + \text{“inst.”} \right] + \mathcal{O}\left(\frac{1}{r^2}\right), \quad (\text{VII.28})$$

where the nonlocal contributions read

$$\mathcal{H}_2^{00} \Big|_{\mathbb{I}_i^s \times \mathbb{I}_j^s} = \frac{2(3 + 2\omega_0)}{3} \mathbb{I}_i^s{}^{(2)} \mathbb{I}_j^s{}^{(2)}, \quad (\text{VII.29a})$$

$$\mathcal{H}_2^{0i} \Big|_{\mathbb{I}_i^s \times \mathbb{I}_j^s} = \frac{2(3 + 2\omega_0)}{3} \left[n^{ijk} \mathbb{I}_j^s{}^{(2)} \mathbb{I}_k^s{}^{(2)} + n^i \mathbb{I}_j^s{}^{(2)} \mathbb{I}_j^s{}^{(2)} - n^j \mathbb{I}_i^s{}^{(2)} \mathbb{I}_j^s{}^{(2)} \right], \quad (\text{VII.29b})$$

$$\mathcal{H}_2^{ij} \Big|_{\mathbb{I}_i^s \times \mathbb{I}_j^s} = (3 + 2\omega_0) \left[n^{ijkl} \mathbb{I}_k^s{}^{(2)} \mathbb{I}_l^s{}^{(2)} + \frac{1}{3} n^{ij} \mathbb{I}_k^s{}^{(2)} \mathbb{I}_k^s{}^{(2)} - \frac{1}{3} \delta^{ij} n^{kl} \mathbb{I}_k^s{}^{(2)} \mathbb{I}_l^s{}^{(2)} - \frac{2}{3} \mathbb{I}_i^s{}^{(2)} \mathbb{I}_j^s{}^{(2)} + \frac{1}{3} \delta^{ij} \mathbb{I}_k^s{}^{(2)} \mathbb{I}_k^s{}^{(2)} \right]. \quad (\text{VII.29c})$$

It is then straightforward to see that this amounts to a correction in the observable radiative quadrupole moment (defined in (VII.34) below) due to the dipolar memory effect as given by

$$\delta \mathcal{U}_{ij}^{\text{memory}} \Big|_{\mathbb{I}_i^s \times \mathbb{I}_j^s} = \frac{3 + 2\omega_0}{3} \int_0^{+\infty} d\tau \left[\mathbb{I}_{(i}^s \mathbb{I}_{j)}^s \right]^{(2)}(u - \tau). \quad (\text{VII.30})$$

Focusing now on the dipolar tail effect, it arises from the quadratic interaction between the conserved gravitational monopole $\mathbb{I} = M/\phi_0$ and the scalar dipole moment $\mathbb{I}_i^s(u)$. At second order, we have to solve, for this particular interaction $\mathbb{I} \times \mathbb{I}_j^s$, using (VII.8) and (VII.17),

$$\square \psi_2 \Big|_{\mathbb{I} \times \mathbb{I}_j^s} = \frac{8\mathbb{I}}{r^2} n^i \mathbb{I}_i^s{}^{(3)}(u) + \mathcal{O}\left(\frac{1}{r^3}\right). \quad (\text{VII.31})$$

Using (A8b) in [173], we obtain

$$\psi_2^{\text{tail}} \Big|_{I \times I_j^s} = \frac{4\text{I}}{r} \left[n^i \int_0^{+\infty} d\tau \text{I}_i^s(u - \tau) \ln \left(\frac{c\tau}{2r} \right) + \text{“inst.”} \right] + \mathcal{O} \left(\frac{1}{r^2} \right). \quad (\text{VII.32})$$

Finally, we present the complete results for the observable radiative moments including memory and tail contributions up to the 1.5PN order, as well as all instantaneous terms. We introduce a radiative-type coordinate system (T, R) , with $U \equiv T - R/c$ being an asymptotically null coordinate such that

$$U = u - \frac{2\text{GI}}{c^3} \ln \left(\frac{r}{b_0} \right) + \mathcal{O} \left(\frac{1}{r} \right), \quad (\text{VII.33})$$

where I is the mass monopole moment and b_0 is as usual an arbitrary constant length scale. Then, we denote \mathcal{U}_L , \mathcal{V}_L and \mathcal{U}_L^s the radiative moments and parametrize the asymptotic TT tensorial waveform and the scalar waveform in the radiative coordinate system at leading order $1/R$ in the distance. We have³

$$h_{ij}^{\text{TT}} = -\frac{4G}{c^2 R} \perp_{ijab}^{\text{TT}} \sum_{\ell=2}^{\infty} \frac{1}{c^\ell \ell!} \left(N_{L-2} \mathcal{U}_{abL-2}(U) - \frac{2\ell}{c(\ell+1)} N_{cL-2} \epsilon_{cd(a} \mathcal{V}_{b)dL-2}(U) \right) + \mathcal{O} \left(\frac{1}{R^2} \right), \quad (\text{VII.34a})$$

$$\psi = -\frac{2G}{c^2 R} \sum_{\ell=0}^{\infty} \frac{1}{c^\ell \ell!} N_L \mathcal{U}_L^s(U) + \mathcal{O} \left(\frac{1}{R^2} \right), \quad (\text{VII.34b})$$

where \perp_{ijab}^{TT} is defined in (I.18). Since the matter fields are minimally coupled to the physical metric $g_{\mu\nu} = \varphi^{-1} \tilde{g}_{\mu\nu}$, the GW detector will respond to the tidal field associated with the physical metric, i.e. the linearized Riemann tensor of the physical metric. Thus, the separation vector between the entry and end mirrors of one arm of a laser-interferometric detector will obey the standard GR law at linear order, $\ddot{\xi} = c^2 R_{0i0j}^{\text{lin}} \xi^j$, but with the Riemann tensor being in ST theory related to both the tensorial and scalar waveforms (VII.34). We have, see (18) of [67],

$$c^2 R_{0i0j}^{\text{lin}} = \frac{1}{2} \ddot{h}_{ij}^{\text{TT}} + \frac{1}{2} \perp_{ij} \ddot{\psi}, \quad (\text{VII.35})$$

showing explicitly the decomposition of the detector's response into tensorial and scalar modes, see (I.22).

We now present the results for the radiative type moments. The tensorial and scalar fluxes \mathcal{F} and \mathcal{F}^s are deduced in terms of them directly from the waveforms (VII.34) as

$$\mathcal{F} = \frac{c^3 R^2 \phi_0}{32\pi G} \int d\Omega \left(\frac{\partial h_{ij}^{\text{TT}}}{\partial U} \right)^2 \quad (\text{VII.36a})$$

$$= \sum_{\ell=2}^{\infty} \frac{G\phi_0}{c^{2\ell+1}} \left(\frac{(\ell+1)(\ell+2)}{(\ell-1)\ell!(2\ell+1)!!} \mathcal{U}_L^{(1)} \mathcal{U}_L^{(1)} + \frac{4\ell(\ell+2)}{c^2(\ell-1)(\ell+1)!(2\ell+1)!!} \mathcal{V}_L^{(1)} \mathcal{V}_L^{(1)} \right),$$

$$\mathcal{F}^s = \frac{c^3 R^2 \phi_0 (3+2\omega_0)}{16\pi G} \int d\Omega \left(\frac{\partial \Psi}{\partial U} \right)^2 = \sum_{\ell=0}^{\infty} \frac{G\phi_0 (3+2\omega_0)}{c^{2\ell+1} \ell!(2\ell+1)!!} \mathcal{U}_L^s \mathcal{U}_L^s. \quad (\text{VII.36b})$$

³We restore the factors G and c . In (VII.34a) we denote by h_{ij}^{TT} the TT projection of the gothic spatial metric deviation, which differs by a sign from the ordinary (covariant) spatial metric deviation.

To 1.5PN order, only the mass quadrupole radiative moment differs from its twice-differentiated source counterpart,

$$\begin{aligned} \mathcal{U}_{ij} = & \overset{(2)}{I}_{ij} + \frac{2GM}{\phi_0 c^3} \int_0^{+\infty} d\tau \overset{(4)}{I}_{ij}(U - \tau) \left[\ln \left(\frac{c\tau}{2b_0} \right) + \frac{11}{12} \right] \\ & + \frac{G(3 + 2\omega_0)}{3c^3} \left(\int_0^{+\infty} d\tau \left[\overset{(2)(2)}{I}_{ij}^s \right] (U - \tau) - \overset{(3)}{I}_{ij}^s - \overset{(1)(2)}{I}_{ij}^s - \frac{1}{2} \overset{(3)}{I}_{ij}^s \right) + \mathcal{O} \left(\frac{1}{c^5} \right). \end{aligned} \quad (\text{VII.37})$$

The difference from $\overset{(2)}{I}_{ij}$ is made of two contributions, both of them being at 1.5PN order:

- (i) the usual mass quadrupole tail correction in GR, see (VI.17), with the usual instantaneous term with coefficient 11/12, except that the mass monopole I therein is related to the constant total mass-energy by $I = M/\phi_0$;
- (ii) the dipolar memory effect found in (VII.30), to which we have added the corresponding instantaneous contributions obtained by a detailed calculation.

In addition, we find that for the ST waveform at 1.5PN order, the radiative type scalar monopole, dipole and quadrupole moments acquire some important tail contributions, namely

$$\begin{aligned} \mathcal{U}^s = & I^s + \frac{2GM}{\phi_0^2 c^5} \int_0^{+\infty} d\tau \overset{(2)}{E}^s(U - \tau) \ln \left(\frac{c\tau}{2b_0} \right) \\ & + \frac{G}{c^5} \left(1 - \frac{\phi_0 \omega'_0}{3 + 2\omega_0} \right) \left[\frac{2}{9} \overset{(3)}{I}_k^s \overset{(3)}{I}_k^s - 2I^s \frac{\overset{(1)}{E}^s}{\phi_0} \right] + \mathcal{O} \left(\frac{1}{c^7} \right), \end{aligned} \quad (\text{VII.38a})$$

$$\begin{aligned} \mathcal{U}_i^s = & \overset{(1)}{I}_i^s + \frac{2GM}{\phi_0 c^3} \int_0^{+\infty} d\tau \overset{(3)}{I}_i^s(U - \tau) \left[\ln \left(\frac{c\tau}{2b_0} \right) + 1 \right] \\ & + \frac{G}{c^5} \left[-\frac{1}{5} \overset{(1)(3)}{I}_k^s \overset{(3)}{I}_{ik} - \frac{1}{5} \overset{(2)(2)}{I}_k^s \overset{(2)}{I}_{ik} + \frac{3}{5} \overset{(3)(1)}{I}_k^s \overset{(1)}{I}_{ik} + \frac{3}{5} \overset{(4)}{I}_k^s \overset{(1)}{I}_{ik} - \epsilon_{iab} J_a \overset{(3)}{I}_b^s - 4W \overset{(1)(2)}{I}_i^s - 4W \overset{(2)(1)}{I}_i^s + 4I^s Y_i \right. \\ & \left. + \left(1 - \frac{\phi_0 \omega'_0}{3 + 2\omega_0} \right) \left(-2 \frac{\overset{(1)}{E}_s}{\phi_0} \overset{(1)}{I}_i^s - 2 \frac{\overset{(2)}{E}_s}{\phi_0} \overset{(2)}{I}_i^s + \frac{2}{5} \overset{(3)(1)}{I}_k^s \overset{(1)}{I}_{ik} + \frac{2}{5} \overset{(4)}{I}_k^s \overset{(1)}{I}_{ik} \right) \right] + \mathcal{O} \left(\frac{1}{c^7} \right), \end{aligned} \quad (\text{VII.38b})$$

$$\mathcal{U}_{ij}^s = \overset{(2)}{I}_{ij}^s + \frac{2GM}{\phi_0 c^3} \int_0^{+\infty} d\tau \overset{(4)}{I}_{ij}^s(U - \tau) \left[\ln \left(\frac{c\tau}{2b_0} \right) + \frac{3}{2} \right] - \frac{G}{c^3} \overset{(3)}{I}_{ij}^s + \mathcal{O} \left(\frac{1}{c^5} \right). \quad (\text{VII.38c})$$

In these expressions we have conveniently posed, recalling the definition of the sensitivity parameters (VII.3),

$$I^s(u) = \frac{1}{\phi_0} \left[m^s + \frac{E^s(u)}{c^2} \right], \quad (\text{VII.39a})$$

$$\text{with} \quad m^s = -\frac{1}{3 + 2\omega_0} \sum_A m_A (1 - 2s_A). \quad (\text{VII.39b})$$

The Newtonian value m^s in the limit $c \rightarrow \infty$ is constant and represents the total “scalar charge” of the system; m^s is zero for binary black holes.

We can already make a few observations on (VII.38). First, these expressions are valid in the frame of the center-of-mass (CM) of the system defined by $I_i = 0$. Then, we notice the coupling between the tensorial moments and the scalar ones through some instantaneous terms in \mathcal{U}_i^s and \mathcal{U}_{ij}^s . Finally, in \mathcal{U}_i^s , the gauge moments W and Y_i coming from the tensorial sector, see (VII.11), appear and are coupled to the scalar mass monopole and dipole. In GR the gauge moments do not contribute to the waveform before the 2.5PN order while in ST theory they already contribute to the flux at 1.5PN order and to the scalar waveform at 2PN order. Besides (VII.38), all other radiative moments are linearly related to the source moments *via* the usual formulas, e.g. $\mathcal{U}_L^s = d^\ell \mathbf{I}_L^s / dU^\ell$.

VII.3 The scalar-tensor multipole moments and flux of compact binaries

Now that the general formalism has been laid out, we apply it to the case of a nonspinning compact binary system to compute the *source* multipole moments, and subsequently, the *radiative* ones. Consistently with the general formulas presented in the previous section, we focus on the ST fluxes at 1.5PN order beyond general relativistic quadrupole formula (i.e. 2.5PN beyond the leading dipolar order of ST theories). For two point-particles of masses $m_A(\phi)$, the stress-energy tensor deriving from (VII.2) reads

$$T^{\mu\nu}(t, \mathbf{x}) = \sum_{A=1,2} \frac{m_A(\phi) v_A^\mu v_A^\nu}{\sqrt{-(g_{\alpha\beta})_A \frac{v_A^\alpha v_A^\beta}{c^2}}} \frac{\delta_A}{\sqrt{-g}}. \quad (\text{VII.40})$$

From the explicit dependence of the stress-energy tensor on the scalar field through the masses we obtain

$$T - 2\varphi \frac{\partial T}{\partial \varphi} = -c \sum_{A=1,2} m_A (1 - 2s_A) \sqrt{-(g_{\alpha\beta})_A \frac{v_A^\alpha v_A^\beta}{c^2}} \frac{\delta_A}{\sqrt{-g}}. \quad (\text{VII.41})$$

Next, we define the following compact support source densities⁴

$$\sigma = \frac{1}{\phi_0 \varphi^3} \frac{T^{00} + T^{ii}}{c^2}, \quad \sigma_i = \frac{1}{\phi_0 \varphi^3} \frac{T^{0i}}{c}, \quad \sigma_{ij} = \frac{1}{\phi_0 \varphi^3} T^{ij}, \quad (\text{VII.42a})$$

$$\sigma_s = -\frac{1}{c^2 \phi_0} \frac{\sqrt{-g}}{\sqrt{(3 + 2\omega_0)(3 + 2\omega)}} \left(T - 2\varphi \frac{\partial T}{\partial \varphi} \right). \quad (\text{VII.42b})$$

At 1.5PN order, we parametrize the metric and scalar perturbation fields as

$$h^{00} = -\frac{4}{c^2} V - \frac{2}{c^4} (\hat{W} + 4V^2) + \mathcal{O}\left(\frac{1}{c^6}\right), \quad (\text{VII.43a})$$

$$h^{0i} = -\frac{4}{c^3} V_i + \mathcal{O}\left(\frac{1}{c^5}\right), \quad (\text{VII.43b})$$

⁴Note a change of definition for σ_s : in [248], the definition given in (4.16) should not be used. Indeed, this definition was never used during the computation in [248]; instead, only the 1PN expansion given in (B4) was used, and is correct. Our definition (VII.42b) for σ_s yields (B4) in [248] at the 1PN approximation.

$$h^{ij} = -\frac{4}{c^4} \left(\hat{W}_{ij} - \frac{1}{2} \delta_{ij} \hat{W} \right) + \mathcal{O} \left(\frac{1}{c^6} \right), \quad (\text{VII.43c})$$

$$\psi = -\frac{2}{c^2} \psi_{(0)} + \frac{2}{c^4} \left(1 - \frac{\omega'_0 \phi_0}{3 + 2\omega_0} \right) \psi_{(0)}^2 + \mathcal{O} \left(\frac{1}{c^6} \right), \quad (\text{VII.43d})$$

where we have introduced (extending the usual practice in GR, see Section II.2.2) the elementary potentials obeying the following wave equations

$$\square V = -4\pi G \sigma, \quad (\text{VII.44a})$$

$$\square V_i = -4\pi G \sigma_i, \quad (\text{VII.44b})$$

$$\square \hat{W}_{ij} = -4\pi G (\sigma_{ij} - \delta_{ij} \sigma_{kk}) - \partial_i V \partial_j V - (3 + 2\omega_0) \partial_i \psi_{(0)} \partial_j \psi_{(0)}, \quad (\text{VII.44c})$$

$$\square \psi_{(0)} = 4\pi G \sigma_s. \quad (\text{VII.44d})$$

Finally, inserting these decompositions into the tensor and scalar multipole moments as given by (VII.14) and (VII.16), we can compute the multipole moments using a series of known techniques; see [98] for a recent review. Fortunately, at the 1.5PN order, this is relatively easy as we do not need to worry about the subtleties associated with the different regularizations schemes (be they ultraviolet or infrared). The main difficulty is the long calculation of the scalar monopole and dipole moments I^s and I_i^s which are required at 2.5PN order.

With the ST parameters defined in Table VII.1, we obtain the source ST multipole moments for compact binaries that are relevant for the final 1.5PN waveform and flux. For example, the tensor quadrupole moment I_{ij} is accurate to order 1.5PN and we find, after reduction to the CM frame,⁵

$$I_{ij} = \frac{m\nu r^2}{\phi_0} \left[n^{(i} n^{j)} + \frac{1}{42c^2} \left\{ n^{(i} n^{j)} \left(\frac{\tilde{G}\alpha m}{r} \right) (-30 + 48\nu) \right. \right. \\ \left. \left. + n^{(i} n^{j)} v^2 (29 - 87\nu) + n^{(i} v^{j)} (nv) (-24 + 72\nu) + v^{(i} v^{j)} (22 - 66\nu) \right\} \right] + \mathcal{O} \left(\frac{1}{c^4} \right). \quad (\text{VII.45})$$

The other relevant tensor moments (including the two gauge moments W and Y_i) are given in (B.4) of Appendix B.2.

As for the scalar moments, the full expressions at the highest necessary PN order are also given in Appendix B.2. Hereafter, we only give the source moments at a high enough PN order to compute the hereditary terms. Indeed, the radiative moment \mathcal{U}_{ij} given by (VII.37) corrects the source moment by the relevant 1.5PN quadrupole tail effect which is the same as in GR, and by the nonlinear memory effect associated with the dipole moment. We will include these effects, applied to binaries, into the final waveform. The

⁵For convenience, when working in the CM frame, we denote $r \equiv r_{12}$ and $n^i \equiv n_{12}^i$ the distance and separation vector between the two bodies (r should not be confused with the harmonic-coordinate distance to the source used in Section VII.2).

necessary source moments to compute hereditary terms are then

$$\begin{aligned} \mathbf{I}_i^s = \frac{\alpha^{1/2}\zeta m\nu r}{(1-\zeta)\phi_0} & \left[-2\mathcal{S}_- n^i + \frac{n^i}{5c^2} \left\{ \left(\frac{\tilde{G}\alpha m}{r} \right) (9\mathcal{S}_- - 20\bar{\beta}_+\bar{\gamma}^{-1}\mathcal{S}_- - 20\bar{\beta}_-\bar{\gamma}^{-1}\mathcal{S}_+ - 13\mathcal{S}_-\nu \right. \right. \\ & \left. \left. + \delta [20\bar{\beta}_-\bar{\gamma}^{-1}\mathcal{S}_- - 4\mathcal{S}_+ + 20\bar{\beta}_+\bar{\gamma}^{-1}\mathcal{S}_+] \right) + v^2(-\mathcal{S}_- + 7\mathcal{S}_-\nu - 4\mathcal{S}_+\delta) \right\} \\ & \left. + \frac{v^i(n\nu)}{5c^2} \left\{ -2\mathcal{S}_- + 4\mathcal{S}_-\nu + 2\mathcal{S}_+\delta \right\} \right] + \mathcal{O}\left(\frac{1}{c^3}\right), \end{aligned} \quad (\text{VII.46a})$$

$$\begin{aligned} E^s = \frac{\alpha^{1/2}\zeta m\nu}{6(1-\zeta)} & \left[\left(\frac{\tilde{G}\alpha m}{r} \right) (-48\bar{\beta}_-\bar{\gamma}^{-1}\mathcal{S}_- + 14\mathcal{S}_+ - 48\bar{\beta}_+\bar{\gamma}^{-1}\mathcal{S}_+ - 2\mathcal{S}_-\delta) \right. \\ & \left. + v^2(\mathcal{S}_+ - \mathcal{S}_-\delta) \right] + \mathcal{O}\left(\frac{1}{c}\right), \end{aligned} \quad (\text{VII.46b})$$

$$I_{ij}^s = \frac{\alpha^{1/2}\zeta m r^2 \nu n^i n^j}{(1-\zeta)\phi_0} (-\mathcal{S}_+ + \mathcal{S}_-\delta) + \mathcal{O}\left(\frac{1}{c}\right), \quad (\text{VII.46c})$$

together with the Newtonian limit of (VII.45) and the Newtonian energy $E = m\nu v^2/2 - \tilde{G}\alpha m^2\nu/r + \mathcal{O}(c^{-1})$.

So far, only the ST equations of motion to 1.5PN order were required. However, in order to compute the ST fluxes, we have to differentiate the moments with respect to time, which then uses the ST equations of motion to 2.5PN order. All the conservative (PN-even) terms of the acceleration are available in [248], in which the ST equations of motion were obtained up to 3PN order. Concerning the dissipative (PN-odd) terms at 1.5PN and 2.5PN orders, they are given in [245] as functions of the EW multipole moments in harmonic coordinates [254]. In Appendix B.1, we present for the first time the explicit expressions for these PN-odd terms in the equations of motion to 2.5PN order.

Finally, we have checked that all our results concerning the moments (in this section and in Appendix B.2) are in agreement with the results of [246, 247]. For the comparison, we have to carefully take into account the link between the STF moments and the Epstein-Wagoner (EW) moments [254] used in [246, 247]. The Section B.2.2 in Appendix B.2 gives the required relations between these moments.

The scalar and tensorial energy fluxes, respectively denoted \mathcal{F}^s and \mathcal{F} , are given in terms of the STF radiative multipole moments \mathcal{U}_L , \mathcal{V}_L and \mathcal{U}_L^s according to (VII.36). We further split them into instantaneous and tail contributions, following the contributions of the tail integrals in (VII.37)–(VII.38),

$$\mathcal{F} = \mathcal{F}_{\text{inst}} + \mathcal{F}_{\text{tail}}, \quad (\text{VII.47a})$$

$$\mathcal{F}^s = \mathcal{F}_{\text{inst}}^s + \mathcal{F}_{\text{tail}}^s. \quad (\text{VII.47b})$$

Note that the dipolar memory contribution, given by the last integral in (VII.37), becomes instantaneous in the flux and as a consequence, its contribution is included into the instantaneous part. On the other hand, it is convenient to keep some instantaneous terms, such as the one related to the constant 11/12, into the definition of the tail terms.

With this caveat in mind, we define

$$\mathcal{F}_{\text{tail}} = \frac{4G^2M^{(3)}}{5c^8} \text{I}_{ij} \int_0^{+\infty} d\tau \text{I}_{ij}^{(5)}(U - \tau) \left[\ln \left(\frac{c\tau}{2b_0} \right) + \frac{11}{12} \right], \quad (\text{VII.48a})$$

$$\begin{aligned} \mathcal{F}_{\text{tail}}^s = \frac{4G^2M(3 + 2\omega_0)}{c^6} & \left\{ \frac{1}{3} \text{I}_i^{(2)} \int_0^{+\infty} d\tau \text{I}_i^{(4)}(U - \tau) \left[\ln \left(\frac{c\tau}{2b_0} \right) + 1 \right] \right. \\ & + \frac{E^s_{(1)}}{c^2\phi_0^2} \int_0^{+\infty} d\tau E^s_{(3)}(U - \tau) \ln \left(\frac{c\tau}{2b_0} \right) \\ & \left. + \frac{1}{30c^2} \text{I}_{ij}^{(3)} \int_0^{+\infty} d\tau \text{I}_{ij}^{(5)}(U - \tau) \left[\ln \left(\frac{c\tau}{2b_0} \right) + \frac{3}{2} \right] \right\}. \quad (\text{VII.48b}) \end{aligned}$$

The complete expression of the instantaneous tensorial flux, valid for general orbits in the CM frame, reads

$$\begin{aligned} \mathcal{F}_{\text{inst}} = \frac{4(2 + \bar{\gamma})}{15c^5} & \left(\frac{\tilde{G}\alpha m}{r} \right)^3 \frac{m\nu^2}{r} \left\{ -11(nv)^2 + 12v^2 \right. \\ & + \frac{1}{28c^2} \left[(16 - 64\nu) \left(\frac{\tilde{G}\alpha m}{r} \right)^2 + (2061 + 840\bar{\gamma} - 1860\nu)(nv)^4 \right. \\ & + (2936 + 1344\bar{\beta}_+ + 1120\bar{\gamma} - 1344\bar{\beta}_-\delta - 120\nu) \left(\frac{\tilde{G}\alpha m}{r} \right) (nv)^2 \\ & + (-2720 - 1344\bar{\beta}_+ - 1008\bar{\gamma} + 1344\bar{\beta}_-\delta + 160\nu) \left(\frac{\tilde{G}\alpha m}{r} \right) v^2 \\ & \left. \left. + (-2974 - 1232\bar{\gamma} + 2784\nu)(nv)^2v^2 + (785 + 336\bar{\gamma} - 852\nu)v^4 \right] \right\} \\ & + \frac{1}{12c^3} \left[(-2\bar{\gamma} + 48\zeta\mathcal{S}_-\nu) \left(\frac{\tilde{G}\alpha m}{r} \right)^2 (nv) + 66\bar{\gamma} \left(\frac{\tilde{G}\alpha m}{r} \right) (nv)^3 \right. \\ & \left. - 70\bar{\gamma} \left(\frac{\tilde{G}\alpha m}{r} \right) (nv)v^2 \right]. \quad (\text{VII.50}) \end{aligned}$$

The tensorial energy flux $\mathcal{F}_{\text{inst}}$ is in complete agreement with the result of [246, 247].

We also compute the instantaneous scalar flux $\mathcal{F}_{\text{inst}}^s$ to 1.5PN order, complementing by a half PN order the previous result by Lang [247]. As its full expression in the center of mass is very long, we have preferred to relegate it to Appendix B.3.

However, when comparing our scalar flux with the one obtained by [247], we have found a discrepancy at 1PN order that could not be resolved, despite the fact that we agree on all the ST multipole moments separately, notably the 2PN monopole and dipole scalar moments. In order to investigate this disagreement, we have computed the 1.5PN scalar waveform and have found that it is in perfect agreement with the scalar waveform presented in (5.2) of [247]. We have then computed the flux: (i) by integrating the scalar waveform following (6.6) of [247]; (ii) with the direct formula given by (B.9) of Appendix B.2, where the EW moments were replaced by their center-of-mass expressions

as given by (3.50) of [247] and (5.10) of [246]. In both cases, we recover the scalar flux as given in Appendix B.3, and not the scalar flux of [247]. The explicit difference between our scalar flux and the scalar flux of [247] is given explicitly in (B.11) of Appendix B.3, along with the full expression for the scalar flux.

VII.4 Reduction to quasicircular orbits

For quasicircular orbits, the expressions of the fluxes simplify considerably and one can work out explicitly the tail terms (VII.48) from standard methods. The usual frequency dependent PN variable which permits to obtain gauge invariant results in GR is easily generalized to ST theories as⁶

$$x \equiv \left(\frac{\tilde{G}\alpha m \omega}{c^3} \right)^{2/3}, \quad (\text{VII.51})$$

where $\omega = 2\pi/P$ is the orbital frequency of the quasicircular orbit, with P the period. It is related to the orbital separation r and to the gauge dependent PN variable $\gamma \equiv \frac{\tilde{G}\alpha m}{c^2 r}$, by the sum of (5.2)–(5.3) of [249] which includes all contributions up to relative 3PN order, including the dipolar tail term at 3PN order. For the present work, we only need this relation to relative 2PN order, which we reproduce here for convenience,

$$\begin{aligned} \omega^2 = \frac{\alpha \tilde{G} m}{r^3} & \left(1 + \gamma \left\{ -3 - 2\bar{\beta}_+ + \nu - \bar{\gamma} + 2\bar{\beta}_- \delta \right\} \right. \\ & + \gamma^2 \left\{ 6 + 8\bar{\beta}_+ - 2\bar{\chi}_+ + \bar{\delta}_+ + (5 + 2\bar{\beta}_+) \bar{\gamma} + \frac{5}{4} \bar{\gamma}^2 + \delta \left[2\bar{\chi}_- + \bar{\delta}_- - 2\bar{\beta}_- (4 + \bar{\gamma}) \right] \right. \\ & + \nu \left[\frac{41}{4} + \bar{\beta}_+ + 4\bar{\chi}_+ - 2\bar{\delta}_+ + 24\bar{\beta}_-^2 \bar{\gamma}^{-1} - 24\bar{\beta}_+^2 \bar{\gamma}^{-1} + 5\bar{\gamma} - \frac{1}{2} \bar{\gamma}^2 \right] \\ & \left. \left. + 5\bar{\beta}_- \nu \delta + \nu^2 \right\} \right). \end{aligned} \quad (\text{VII.52})$$

The expression of γ in terms of x at 2PN order is deduced by inversion of this expression, and is also given to 3PN order in [249]. Our final results for the tensorial and scalar fluxes at relative 2PN order expressed in terms of the gauge invariant variable (VII.51) read

$$\mathcal{F}_{\text{circ}} = \frac{32c^5 x^5 \nu^2 (1 + \bar{\gamma}/2)}{5\tilde{G}\alpha} \left(1 + \frac{x}{336} \left\{ -1247 - 896\bar{\beta}_+ - 448\bar{\gamma} + 896\bar{\beta}_- \delta - 980\nu \right\} + 4\pi \left(1 + \frac{\bar{\gamma}}{2} \right) x^{3/2} \right), \quad (\text{VII.53a})$$

$$\begin{aligned} \mathcal{F}_{\text{circ}}^s = \frac{c^5 x^5 \nu^2 \zeta}{3\tilde{G}\alpha} & \left(4\mathcal{S}_-^2 x^{-1} \right. \\ & + \frac{1}{15} \left\{ -24\zeta^{-1} \bar{\gamma} - 120\mathcal{S}_-^2 - 80\bar{\beta}_+ \mathcal{S}_-^2 - 40\bar{\gamma} \mathcal{S}_-^2 + 240\bar{\beta}_+ \bar{\gamma}^{-1} \mathcal{S}_-^2 + 240\bar{\beta}_- \bar{\gamma}^{-1} \mathcal{S}_- \mathcal{S}_+ \right. \\ & \left. \left. + \delta \left[80\bar{\beta}_- \mathcal{S}_-^2 - 240\bar{\beta}_- \bar{\gamma}^{-1} \mathcal{S}_-^2 - 240\bar{\beta}_+ \bar{\gamma}^{-1} \mathcal{S}_- \mathcal{S}_+ \right] - 80\mathcal{S}_-^2 \nu \right\} \right) \end{aligned}$$

⁶At this order in scalar-tensor theory, it is not necessary to distinguish the y and x variables introduced in Part A.

$$\begin{aligned}
& + 4\pi(2 + \bar{\gamma})\mathcal{S}_-^2 x^{1/2} \\
& + \frac{x}{420} \left\{ -2688\bar{\beta}_+\zeta^{-1} + 2910\zeta^{-1}\bar{\gamma} + 1792\bar{\beta}_+\zeta^{-1}\bar{\gamma} + 896\zeta^{-1}\bar{\gamma}^2 - 3360\bar{\beta}_-^2\zeta^{-1}\bar{\gamma}^{-1} - 3360\bar{\beta}_+^2\zeta^{-1}\bar{\gamma}^{-1} \right. \\
& \quad - 7560\mathcal{S}_-^2 - 2240\bar{\beta}_-^2\mathcal{S}_-^2 - 3360\bar{\beta}_+\mathcal{S}_-^2 - 2240\bar{\beta}_+^2\mathcal{S}_-^2 - 1680\bar{\gamma}\mathcal{S}_-^2 + 560\bar{\gamma}^2\mathcal{S}_-^2 - 4480\bar{\beta}_-^2\bar{\gamma}^{-1}\mathcal{S}_-^2 \\
& \quad - 6720\bar{\beta}_+\bar{\gamma}^{-1}\mathcal{S}_-^2 - 4480\bar{\beta}_+^2\bar{\gamma}^{-1}\mathcal{S}_-^2 + 13440\bar{\beta}_-^2\bar{\gamma}^{-2}\mathcal{S}_-^2 + 13440\bar{\beta}_+^2\bar{\gamma}^{-2}\mathcal{S}_-^2 - 2240\bar{\chi}_+\mathcal{S}_-^2 \\
& \quad + 6720\bar{\gamma}^{-1}\bar{\chi}_+\mathcal{S}_-^2 + 2240\zeta\mathcal{S}_-^4 + 1120\zeta\bar{\gamma}\mathcal{S}_-^4 - 2240\bar{\beta}_-\mathcal{S}_-\mathcal{S}_+ - 6720\bar{\beta}_-\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+ \\
& \quad - 8960\bar{\beta}_-\bar{\beta}_+\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+ + 26880\bar{\beta}_-\bar{\beta}_+\bar{\gamma}^{-2}\mathcal{S}_-\mathcal{S}_+ + 6720\bar{\gamma}^{-1}\bar{\chi}_-\mathcal{S}_-\mathcal{S}_+ \\
& \quad + \delta \left[2688\bar{\beta}_-\zeta^{-1} - 1792\bar{\beta}_-\zeta^{-1}\bar{\gamma} + 6720\bar{\beta}_-\bar{\beta}_+\zeta^{-1}\bar{\gamma}^{-1} + 3360\bar{\beta}_-\mathcal{S}_-^2 + 4480\bar{\beta}_-\bar{\beta}_+\mathcal{S}_-^2 \right. \\
& \quad \quad + 6720\bar{\beta}_-\bar{\gamma}^{-1}\mathcal{S}_-^2 + 8960\bar{\beta}_-\bar{\beta}_+\bar{\gamma}^{-1}\mathcal{S}_-^2 - 26880\bar{\beta}_-\bar{\beta}_+\bar{\gamma}^{-2}\mathcal{S}_-^2 + 2240\bar{\chi}_-\mathcal{S}_-^2 \\
& \quad \quad - 6720\bar{\gamma}^{-1}\bar{\chi}_-\mathcal{S}_-^2 + 9240\mathcal{S}_-\mathcal{S}_+ + 2240\bar{\beta}_+\mathcal{S}_-\mathcal{S}_+ + 5040\bar{\gamma}\mathcal{S}_-\mathcal{S}_+ + 4480\bar{\beta}_-^2\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+ \\
& \quad \quad + 6720\bar{\beta}_+\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+ + 4480\bar{\beta}_+^2\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+ - 13440\bar{\beta}_-^2\bar{\gamma}^{-2}\mathcal{S}_-\mathcal{S}_+ - 13440\bar{\beta}_+^2\bar{\gamma}^{-2}\mathcal{S}_-\mathcal{S}_+ \\
& \quad \quad \left. - 6720\bar{\gamma}^{-1}\bar{\chi}_+\mathcal{S}_-\mathcal{S}_+ + 2240\zeta\mathcal{S}_-^3\mathcal{S}_+ + 1120\zeta\bar{\gamma}\mathcal{S}_-^3\mathcal{S}_+ \right] \\
& \quad + \nu \left[1960\zeta^{-1}\bar{\gamma} + 13440\bar{\beta}_+^2\zeta^{-1}\bar{\gamma}^{-1} + 11480\mathcal{S}_-^2 + 8960\bar{\beta}_-^2\mathcal{S}_-^2 + 22400\bar{\beta}_+\mathcal{S}_-^2 - 1120\bar{\gamma}\mathcal{S}_-^2 \right. \\
& \quad + 44800\bar{\beta}_-^2\bar{\gamma}^{-1}\mathcal{S}_-^2 - 31360\bar{\beta}_+\bar{\gamma}^{-1}\mathcal{S}_-^2 - 26880\bar{\beta}_+^2\bar{\gamma}^{-1}\mathcal{S}_-^2 - 80640\bar{\beta}_-^2\bar{\gamma}^{-2}\mathcal{S}_-^2 \\
& \quad + 26880\bar{\beta}_+^2\bar{\gamma}^{-2}\mathcal{S}_-^2 + 4480\bar{\chi}_+\mathcal{S}_-^2 - 13440\bar{\gamma}^{-1}\bar{\chi}_+\mathcal{S}_-^2 - 4480\zeta\mathcal{S}_-^4 - 2240\zeta\bar{\gamma}\mathcal{S}_-^4 \\
& \quad \quad - 31360\bar{\beta}_-\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+ + 17920\bar{\beta}_-\bar{\beta}_+\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+ \\
& \quad \quad \left. - 53760\bar{\beta}_-\bar{\beta}_+\bar{\gamma}^{-2}\mathcal{S}_-\mathcal{S}_+ - 13440\bar{\gamma}^{-1}\bar{\chi}_-\mathcal{S}_-\mathcal{S}_+ \right] \\
& \quad \quad \left. + \delta\nu \left[-8960\bar{\beta}_-\mathcal{S}_-^2 + 11200\bar{\beta}_-\bar{\gamma}^{-1}\mathcal{S}_-^2 + 11200\bar{\beta}_+\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+ \right] + 1120\mathcal{S}_-^2\nu^2 \right\} \\
& + \frac{\pi x^{3/2}}{30(1-\zeta)} \left\{ 192\bar{\gamma} - 192\zeta^{-1}\bar{\gamma} + 96\bar{\gamma}^2 - 96\zeta^{-1}\bar{\gamma}^2 - 96\mathcal{S}_-^2 + 160\bar{\beta}_+\mathcal{S}_-^2 + 96\zeta\mathcal{S}_-^2 - 160\bar{\beta}_+\zeta\mathcal{S}_-^2 - 208\bar{\gamma}\mathcal{S}_-^2 \right. \\
& \quad - 160\bar{\beta}_+\bar{\gamma}\mathcal{S}_-^2 + 208\zeta\bar{\gamma}\mathcal{S}_-^2 + 160\bar{\beta}_+\zeta\bar{\gamma}\mathcal{S}_-^2 - 80\bar{\gamma}^2\mathcal{S}_-^2 + 80\zeta\bar{\gamma}^2\mathcal{S}_-^2 + 960\bar{\beta}_+\bar{\gamma}^{-1}\mathcal{S}_-^2 \\
& \quad - 960\bar{\beta}_+\zeta\bar{\gamma}^{-1}\mathcal{S}_-^2 + 480\bar{\beta}_-\mathcal{S}_-\mathcal{S}_+ - 480\bar{\beta}_-\zeta\mathcal{S}_-\mathcal{S}_+ + 960\bar{\beta}_-\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+ - 960\bar{\beta}_-\zeta\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+ \\
& \quad + \delta \left[-160\bar{\beta}_-\mathcal{S}_-^2 + 160\bar{\beta}_-\zeta\mathcal{S}_-^2 + 160\bar{\beta}_-\bar{\gamma}\mathcal{S}_-^2 - 160\bar{\beta}_-\zeta\bar{\gamma}\mathcal{S}_-^2 - 960\bar{\beta}_-\bar{\gamma}^{-1}\mathcal{S}_-^2 \right. \\
& \quad \quad + 960\bar{\beta}_-\zeta\bar{\gamma}^{-1}\mathcal{S}_-^2 - 384\mathcal{S}_-\mathcal{S}_+ - 480\bar{\beta}_+\mathcal{S}_-\mathcal{S}_+ + 384\zeta\mathcal{S}_-\mathcal{S}_+ + 480\bar{\beta}_+\zeta\mathcal{S}_-\mathcal{S}_+ \\
& \quad \quad \left. - 192\bar{\gamma}\mathcal{S}_-\mathcal{S}_+ + 192\zeta\bar{\gamma}\mathcal{S}_-\mathcal{S}_+ - 960\bar{\beta}_+\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+ + 960\bar{\beta}_+\zeta\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+ \right] \\
& \quad \left. + \nu \left[-1208\mathcal{S}_-^2 + 1208\zeta\mathcal{S}_-^2 - 604\bar{\gamma}\mathcal{S}_-^2 + 604\zeta\bar{\gamma}\mathcal{S}_-^2 \right] \right\} \Bigg). \tag{VII.53b}
\end{aligned}$$

Next, we apply the usual flux-balance argument for the total scalar-tensor energy flux,

$$\frac{dE_{\text{circ}}}{dt} = -\mathcal{F}_{\text{circ}}^{\text{total}}, \quad \mathcal{F}_{\text{circ}}^{\text{total}} \equiv \mathcal{F}_{\text{circ}} + \mathcal{F}_{\text{circ}}^s, \tag{VII.54}$$

where E_{circ} denotes the conservative energy of the system deduced from the conservative equations of motion and is given at 2PN order by [249]

$$\begin{aligned}
E_{\text{circ}} = & -\frac{1}{2}m\nu c^2 x \left(1 + \frac{x}{12} \left\{ (-9 + 8\bar{\beta}_+ - 8\bar{\gamma}) - \nu - 8\bar{\beta}_- \delta \right\} \right. \\
& + \frac{x^2}{24} \left\{ -81 + 32\bar{\beta}_+^2 + 32\bar{\chi}_+ + 8\bar{\delta}_+ - 112\bar{\gamma} - 38\bar{\gamma}^2 + 24\bar{\beta}_+ + 32\bar{\gamma}\bar{\beta}_+ \right. \\
& \quad + 32\bar{\beta}_-^2 \delta^2 - \nu^2 + 8\bar{\beta}_- \nu \delta \\
& \quad + \nu \left[57 - 384\bar{\gamma}^{-1} \bar{\beta}_-^2 + 384\bar{\gamma}^{-1} \bar{\beta}_+^2 - 152\bar{\beta}_+ - 64\bar{\chi}_+ + 32\bar{\delta}_+ + 88\bar{\gamma} + 8\bar{\gamma}^2 \right] \\
& \quad \left. \left. + \delta \left[-24\bar{\beta}_- - 32\bar{\chi}_- + 8\bar{\delta}_- - 64\bar{\beta}_+ \bar{\beta}_- - 32\bar{\beta}_- \bar{\gamma} \right] \right\} \right). \tag{VII.55}
\end{aligned}$$

The orbital phase is defined as a function of time by $\phi_{\text{circ}} = \int_{t_0}^t \omega(t') dt'$. Using the energy balance equation, we perform a change of variables to express the phase as a function of x ,

$$\phi_{\text{circ}} = -\frac{c^3}{\tilde{G}\alpha m} \int_{x_0}^x \frac{x'^{3/2}}{\mathcal{F}_{\text{circ}}(x')} \frac{dE_{\text{circ}}}{dx'} dx'. \tag{VII.56}$$

Performing the PN expansion inside the integral and integrating term by term (following the simplest PN approximant [263]), we find that, up to a constant, the phase is given by

$$\begin{aligned}
\phi_{\text{circ}} = & -\frac{1}{4\zeta \mathcal{S}_-^2 \nu x^{1/2}} \left[x^{-1} + \frac{3}{2} + 8\bar{\beta}_+ - 2\bar{\gamma} - 12\bar{\beta}_+ \bar{\gamma}^{-1} - \frac{72}{5} \zeta^{-1} \mathcal{S}_-^2 - 6\zeta^{-1} \bar{\gamma} \mathcal{S}_-^2 - 12\bar{\beta}_- \bar{\gamma}^{-1} \mathcal{S}_-^{-1} \mathcal{S}_+ \right. \\
& + \delta \left[-8\bar{\beta}_- + 12\bar{\beta}_- \bar{\gamma}^{-1} + 12\bar{\beta}_+ \bar{\gamma}^{-1} \mathcal{S}_-^{-1} \mathcal{S}_+ \right] + \frac{7}{2} + 3\pi x^{1/2} \log(x) \left(1 + \frac{\bar{\gamma}}{2} \right) \\
& + x \left\{ \frac{111}{8} - \frac{80}{3} \bar{\beta}_-^2 - 33\bar{\beta}_+ - \frac{80}{3} \bar{\beta}_+^2 + \frac{87}{2} \bar{\gamma} - \frac{44}{3} \bar{\beta}_+ \bar{\gamma} + \frac{52}{3} \bar{\gamma}^2 + 40\bar{\beta}_-^2 \bar{\gamma}^{-1} + 18\bar{\beta}_+ \bar{\gamma}^{-1} + 40\bar{\beta}_+^2 \bar{\gamma}^{-1} \right. \\
& \quad - 72\bar{\beta}_-^2 \bar{\gamma}^{-2} - 72\bar{\beta}_+^2 \bar{\gamma}^{-2} - 16\bar{\chi}_+ + 12\bar{\gamma}^{-1} \bar{\chi}_+ - 2\zeta \mathcal{S}_-^2 - \zeta \bar{\gamma} \mathcal{S}_-^2 - \frac{1221}{70} \zeta^{-1} \mathcal{S}_-^2 - \frac{168}{5} \bar{\beta}_+ \zeta^{-1} \mathcal{S}_-^2 \\
& \quad - \frac{1029}{40} \zeta^{-1} \bar{\gamma} \mathcal{S}_-^2 + 8\bar{\beta}_+ \zeta^{-1} \bar{\gamma} \mathcal{S}_-^2 - 8\zeta^{-1} \bar{\gamma}^2 \mathcal{S}_-^2 + 18\bar{\beta}_-^2 \zeta^{-1} \bar{\gamma}^{-1} \mathcal{S}_-^2 - \frac{576}{5} \bar{\beta}_+ \zeta^{-1} \bar{\gamma}^{-1} \mathcal{S}_-^2 \\
& \quad + 18\bar{\beta}_+^2 \zeta^{-1} \bar{\gamma}^{-1} \mathcal{S}_-^2 - \frac{1728}{25} \zeta^{-2} \mathcal{S}_-^4 - \frac{288}{5} \zeta^{-2} \bar{\gamma} \mathcal{S}_-^4 - 12\zeta^{-2} \bar{\gamma}^2 \mathcal{S}_-^4 - 4\bar{\beta}_- \mathcal{S}_-^{-1} \mathcal{S}_+ \\
& \quad + 18\bar{\beta}_- \bar{\gamma}^{-1} \mathcal{S}_-^{-1} \mathcal{S}_+ + 80\bar{\beta}_- \bar{\beta}_+ \bar{\gamma}^{-1} \mathcal{S}_-^{-1} \mathcal{S}_+ - 144\bar{\beta}_- \bar{\beta}_+ \bar{\gamma}^{-2} \mathcal{S}_-^{-1} \mathcal{S}_+ + 12\bar{\gamma}^{-1} \bar{\chi}_- \mathcal{S}_-^{-1} \mathcal{S}_+ \\
& \quad \left. - 48\bar{\beta}_- \zeta^{-1} \mathcal{S}_-^{-3} \mathcal{S}_+ - \frac{576}{5} \bar{\beta}_- \zeta^{-1} \bar{\gamma}^{-1} \mathcal{S}_-^{-3} \mathcal{S}_+ \right\} \\
& + \delta \left[33\bar{\beta}_- + \frac{160}{3} \bar{\beta}_- \bar{\beta}_+ + \frac{44}{3} \bar{\beta}_- \bar{\gamma} - 18\bar{\beta}_- \bar{\gamma}^{-1} - 80\bar{\beta}_- \bar{\beta}_+ \bar{\gamma}^{-1} + 144\bar{\beta}_- \bar{\beta}_+ \bar{\gamma}^{-2} + 16\bar{\chi}_- - 12\bar{\gamma}^{-1} \bar{\chi}_- \right. \\
& \quad + \frac{168}{5} \bar{\beta}_- \zeta^{-1} \mathcal{S}_-^2 - 8\bar{\beta}_- \zeta^{-1} \bar{\gamma} \mathcal{S}_-^2 + \frac{576}{5} \bar{\beta}_- \zeta^{-1} \bar{\gamma}^{-1} \mathcal{S}_-^2 - 36\bar{\beta}_- \bar{\beta}_+ \zeta^{-1} \bar{\gamma}^{-1} \mathcal{S}_-^2 - 2\zeta \mathcal{S}_- \mathcal{S}_+ \\
& \quad - \zeta \bar{\gamma} \mathcal{S}_- \mathcal{S}_+ + \frac{33}{2} \mathcal{S}_-^{-1} \mathcal{S}_+ + 4\bar{\beta}_+ \mathcal{S}_-^{-1} \mathcal{S}_+ + 9\bar{\gamma} \mathcal{S}_-^{-1} \mathcal{S}_+ - 40\bar{\beta}_-^2 \bar{\gamma}^{-1} \mathcal{S}_-^{-1} \mathcal{S}_+ - 18\bar{\beta}_+ \bar{\gamma}^{-1} \mathcal{S}_-^{-1} \mathcal{S}_+ \\
& \quad - 40\bar{\beta}_+^2 \bar{\gamma}^{-1} \mathcal{S}_-^{-1} \mathcal{S}_+ + 72\bar{\beta}_-^2 \bar{\gamma}^{-2} \mathcal{S}_-^{-1} \mathcal{S}_+ + 72\bar{\beta}_+^2 \bar{\gamma}^{-2} \mathcal{S}_-^{-1} \mathcal{S}_+ - 12\bar{\gamma}^{-1} \bar{\chi}_+ \mathcal{S}_-^{-1} \mathcal{S}_+ \\
& \quad \left. + 48\bar{\beta}_+ \zeta^{-1} \mathcal{S}_-^{-3} \mathcal{S}_+ + \frac{576}{5} \bar{\beta}_+ \zeta^{-1} \bar{\gamma}^{-1} \mathcal{S}_-^{-3} \mathcal{S}_+ \right] + \delta \nu \left[-\frac{11}{3} \bar{\beta}_- - 10\bar{\beta}_- \bar{\gamma}^{-1} - 10\bar{\beta}_+ \bar{\gamma}^{-1} \mathcal{S}_-^{-1} \mathcal{S}_+ \right] \\
& - \frac{55}{24} \nu^2 + \nu \left[-\frac{79}{8} + \frac{320}{3} \bar{\beta}_-^2 + \frac{245}{3} \bar{\beta}_+ - \frac{86}{3} \bar{\gamma} + 32\bar{\beta}_-^2 \bar{\gamma}^{-1} - 26\bar{\beta}_+ \bar{\gamma}^{-1} - 192\bar{\beta}_+^2 \bar{\gamma}^{-1} + 48\bar{\beta}_-^2 \bar{\gamma}^{-2} \right. \\
& \quad \left. + 240\bar{\beta}_+^2 \bar{\gamma}^{-2} + 32\bar{\chi}_+ - 24\bar{\gamma}^{-1} \bar{\chi}_+ - 32\zeta \mathcal{S}_-^2 - 16\zeta \bar{\gamma} \mathcal{S}_-^2 - 6\zeta^{-1} \mathcal{S}_-^2 - \frac{5}{2} \zeta^{-1} \bar{\gamma} \mathcal{S}_-^2 \right]
\end{aligned}$$

$$\begin{aligned}
& -72\bar{\beta}_+^2\zeta^{-1}\bar{\gamma}^{-1}\mathcal{S}_-^{-2} - 26\bar{\beta}_-\bar{\gamma}^{-1}\mathcal{S}_-^{-1}\mathcal{S}_+ - 160\bar{\beta}_-\bar{\beta}_+\bar{\gamma}^{-1}\mathcal{S}_-^{-1}\mathcal{S}_+ \\
& + 288\bar{\beta}_-\bar{\beta}_+\bar{\gamma}^{-2}\mathcal{S}_-^{-1}\mathcal{S}_+ - 24\bar{\gamma}^{-1}\bar{\chi}_-\mathcal{S}_-^{-1}\mathcal{S}_+ \Big\} \\
& + \frac{\pi x^{3/2}}{1-\zeta} \left\{ \frac{63}{10} + 2\bar{\beta}_+ - \frac{63}{10}\zeta - 2\bar{\beta}_+\zeta + \frac{23}{20}\bar{\gamma} + 4\bar{\beta}_+\bar{\gamma} - \frac{23}{20}\zeta\bar{\gamma} - 4\bar{\beta}_+\zeta\bar{\gamma} - \bar{\gamma}^2 + \zeta\bar{\gamma}^2 - 12\bar{\beta}_+\bar{\gamma}^{-1} \right. \\
& + 12\bar{\beta}_+\zeta\bar{\gamma}^{-1} - 6\bar{\beta}_-\mathcal{S}_-^{-1}\mathcal{S}_+ + 6\bar{\beta}_-\zeta\mathcal{S}_-^{-1}\mathcal{S}_+ - 12\bar{\beta}_-\bar{\gamma}^{-1}\mathcal{S}_-^{-1}\mathcal{S}_+ + 12\bar{\beta}_-\zeta\bar{\gamma}^{-1}\mathcal{S}_-^{-1}\mathcal{S}_+ \\
& + \delta \left[-2\bar{\beta}_- + 2\bar{\beta}_-\zeta - 4\bar{\beta}_-\bar{\gamma} + 4\bar{\beta}_-\zeta\bar{\gamma} + 12\bar{\beta}_-\bar{\gamma}^{-1} - 12\bar{\beta}_-\zeta\bar{\gamma}^{-1} - \frac{24}{5}\mathcal{S}_-^{-1}\mathcal{S}_+ + 6\bar{\beta}_+\mathcal{S}_-^{-1}\mathcal{S}_+ \right. \\
& \left. + \frac{24}{5}\zeta\mathcal{S}_-^{-1}\mathcal{S}_+ - 6\bar{\beta}_+\zeta\mathcal{S}_-^{-1}\mathcal{S}_+ - \frac{12}{5}\bar{\gamma}\mathcal{S}_-^{-1}\mathcal{S}_+ + \frac{12}{5}\zeta\bar{\gamma}\mathcal{S}_-^{-1}\mathcal{S}_+ + 12\bar{\beta}_+\bar{\gamma}^{-1}\mathcal{S}_-^{-1}\mathcal{S}_+ - 12\bar{\beta}_+\zeta\bar{\gamma}^{-1}\mathcal{S}_-^{-1}\mathcal{S}_+ \right] \\
& \left. + \nu \left[-\frac{38}{5} + \frac{38}{5}\zeta - \frac{19}{5}\bar{\gamma} + \frac{19}{5}\zeta\bar{\gamma} \right] \right\}. \tag{VII.57}
\end{aligned}$$

This expression for the phase is computed using the natural assumption that the dipolar mode is the leading order, i.e. using a formal expansion when $x \rightarrow 0$. However, Sennett *et al.* [67] pointed out that under certain conditions on the scalar-tensor parameters and on the orbital frequency (or frequency band of a given detector), the quadrupolar mode may be actually dominant over the dipolar mode. This happens when

$$1 \lesssim \left(\frac{24}{5\zeta\mathcal{S}_-^2} \right) \left(\frac{\tilde{G}\alpha m\omega}{2} \right)^{2/3}, \tag{VII.58}$$

in which case it is more natural, when computing the phase, to perform the expansion around the quadrupolar term rather than the dipolar term: this is what is called the quadrupole-driven (QD) regime, as opposed to the dipolar-driven (DD) regime which we assumed in (VII.57) (the quadrupolar case can easily be performed if needed). This result for the phase is in perfect agreement with the result found by [67] up to 1PN order in the dipolar-driven case.

VII.5 Waveform and GW modes

We now proceed to compute the spherical harmonic modes in circular orbits, that are useful for numerical relativity. We will perform the standard computation for the tensor modes $h^{\ell m}$ and extend the formalism to the scalar modes $\psi^{\ell m}$.

For a planar orbit of two particles, the polarization orthonormal triad $(\mathbf{N}, \mathbf{P}, \mathbf{Q})$ is defined following the conventions of [202]. The TT tensor h_{ij}^{TT} represents the gothic conformal metric and is a spin-2 object: it can be, just as in GR, decomposed into two independent modes along the polarization vectors,

$$h_+ \equiv \frac{1}{2}(P_i P_j - Q_i Q_j) h_{ij}^{\text{TT}}, \tag{VII.59a}$$

$$h_\times \equiv \frac{1}{2}(P_i Q_j + Q_i P_j) h_{ij}^{\text{TT}}, \tag{VII.59b}$$

which can be recast into a complex field, $h = h_+ - ih_\times$, which can itself be decomposed on the basis on spin-weighted spherical harmonics of weight -2 ,

$$h = h_+ - ih_- = \sum_{\ell=2}^{\infty} \sum_{m=-\ell}^{\ell} h^{\ell m} {}_{-2}Y^{\ell m}. \tag{VII.60a}$$

Similarly, the pure spin-0 scalar field can be decomposed on standard (spin-0) spherical harmonics,

$$\psi = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \psi^{\ell m} Y^{\ell m}. \quad (\text{VII.60b})$$

In an alternative presentation of the waveform (VII.34a), we can define the “electric” and “magnetic” pure-spin tensor harmonics based on the spin-weighted spherical harmonics [203],

$$T_{ij}^{E2,\ell m} = \frac{1}{\sqrt{2}} \left(-{}_2Y^{\ell m} m_i m_j + {}_2Y^{\ell m} m_i^* m_j^* \right), \quad (\text{VII.61a})$$

$$T_{ij}^{B2,\ell m} = \frac{-i}{\sqrt{2}} \left(-{}_2Y^{\ell m} m_i m_j - {}_2Y^{\ell m} m_i^* m_j^* \right), \quad (\text{VII.61b})$$

where $\mathbf{m} \equiv (\mathbf{n} + i\boldsymbol{\lambda})/\sqrt{2}$ and \mathbf{m}^* denotes its complex conjugate. Owing to the fact that h_{ij}^{TT} and ψ are only needed at leading order in the inverse distance of the detector to the source $1/R$, we define the spherical harmonic radiative moments $\mathcal{U}^{\ell m}$, $\mathcal{V}^{\ell m}$ and $\mathcal{U}_s^{\ell m}$ as the components of the decomposition

$$h_{ij}^{\text{TT}} = \sum_{\ell=2}^{\infty} \sum_{m=-\ell}^{\ell} \frac{1}{R} \left(\mathcal{U}^{\ell m} T_{ij}^{E3,\ell m} + \mathcal{V}^{\ell m} T_{ij}^{B3,\ell m} \right) + \mathcal{O}\left(\frac{1}{R^2}\right), \quad (\text{VII.62a})$$

$$\psi = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{1}{R} \mathcal{U}_s^{\ell m} Y^{\ell m} + \mathcal{O}\left(\frac{1}{R^2}\right), \quad (\text{VII.62b})$$

which can directly be related to the spin-2 and spin-0 modes via

$$h^{\ell m} = -\frac{G}{\sqrt{2}Rc^{\ell+2}} \left[\mathcal{U}^{\ell m} - \frac{i}{c} \mathcal{V}^{\ell m} \right], \quad (\text{VII.63a})$$

$$\psi^{\ell m} = \frac{G}{Rc^{\ell+2}} \mathcal{U}_s^{\ell m}. \quad (\text{VII.63b})$$

Based on symmetry considerations [203, 202], we can further simplify (VII.63a) in the case when the orbit of the binary system is planar (either no spins or aligned/anti-aligned spins) by noticing that $\mathcal{V}^{\ell m}$ is zero when $\ell + m$ is even and $\mathcal{U}^{\ell m}$ is zero when $\ell + m$ is odd. These spherical harmonic moments are related to the STF multipole moments by

$$\mathcal{U}^{\ell m} = \frac{4}{\ell!} \sqrt{\frac{(\ell+1)(\ell+2)}{2\ell(\ell-1)}} \alpha_L^{\ell m} \mathcal{U}_L, \quad (\text{VII.64a})$$

$$\mathcal{V}^{\ell m} = -\frac{8}{\ell!} \sqrt{\frac{\ell(\ell+2)}{2(\ell+1)(\ell-1)}} \alpha_L^{\ell m} \mathcal{V}_L, \quad (\text{VII.64b})$$

$$\mathcal{U}_s^{\ell m} = -\frac{2}{\ell!} \alpha_L^{\ell m} \mathcal{U}_L^s, \quad (\text{VII.64c})$$

where we have defined $\alpha_L^{\ell m}$ as [96]⁷

$$\alpha_L^{\ell m} \equiv \int d\Omega \hat{N}_L \left(Y^{\ell m} \right)^* = \frac{\sqrt{4\pi} (-\sqrt{2})^m \ell!}{\sqrt{(2\ell+1)(\ell+m)!(\ell-m)!}} m_0^{*(M)l^{L-M}}, \quad (\text{VII.65})$$

⁷The link with the alternative definition given in [48] is given by $\mathcal{Y}_L^{\ell m} = \frac{(2\ell+1)!!}{4\pi\ell!} (\alpha_L^{\ell m})^*$.

with \mathbf{l} the unit normal to the orbit, and \mathbf{m}_0^* the complex conjugate of the vector \mathbf{m} taken at some initial time t_0 at which \mathbf{n} is aligned with \mathbf{P} [202]. Finally, it can be shown that the redefinition of the phase variable [67]

$$\psi_{\text{circ}} \equiv \phi_{\text{circ}} - \frac{2(1-\zeta)}{\alpha} x^{3/2} \left(\log(4\omega\tau_0) + \gamma_E - \frac{11}{12} \right), \quad (\text{VII.66})$$

succeeds in removing most logarithms from the expressions of $h^{\ell m}$ and $\psi^{\ell m}$ once the complex exponential is PN-expanded. These modes can then be recast into dimensionless amplitude modes $h^{\ell m}$ and $\psi^{\ell m}$ which are given by⁸

$$h^{\ell m} = \frac{2\tilde{G}(1-\zeta)m\nu x}{Rc^2} \sqrt{\frac{16\pi}{5}} \hat{H}^{\ell m} e^{-im\psi}, \quad (\text{VII.67a})$$

$$\psi^{\ell m} = \frac{2i\tilde{G}\zeta\sqrt{\alpha}\mathcal{S}_-m\nu\sqrt{x}}{Rc^2} \sqrt{\frac{8\pi}{3}} \hat{\Psi}^{\ell m} e^{-im\psi}. \quad (\text{VII.67b})$$

with normalized modes defined such that $\hat{H}^{22} = 1 + \mathcal{O}(x)$ and $\hat{\Psi}^{11} = 1 + \mathcal{O}(x)$.

The $\hat{H}^{\ell m}$ were previously computed up to 2PN order by Sennet *et al.* [67], based on the tensorial waveform of [246]. In this work, we only need these modes to 1.5PN order. We are in complete agreement with their result [67] up to 1.5PN order. In particular the comparison of the mode \hat{H}^{22} at 1.5PN order enables us to check that our formalism for generating the nonlinear multipole interactions in (VII.37) is consistent with the DIRE formalism used by Lang [246].

On the other hand, to our knowledge, the scalar modes $\hat{\Psi}^{\ell m}$ are new to this chapter. The expressions for all modes (with nonnegative m) that are nonzero at 1.5PN order are given by

$$\begin{aligned} \hat{\Psi}^{00} = & -\frac{i\sqrt{3}}{\sqrt{2}\mathcal{S}_-} \left(\frac{\mathcal{S}_+ + \mathcal{S}_-\delta}{\nu x^{1/2}} + x^{1/2} \left\{ 8\bar{\beta}_-\bar{\gamma}^{-1}\mathcal{S}_- - \frac{5}{2}\mathcal{S}_+ + 8\bar{\beta}_+\bar{\gamma}^{-1}\mathcal{S}_+ + \frac{1}{2}\mathcal{S}_-\delta \right\} \right. \\ & + x^{3/2} \left\{ \frac{7}{3}\bar{\beta}_-\mathcal{S}_- - 4\bar{\beta}_-\bar{\gamma}^{-1}\mathcal{S}_- + \frac{16}{3}\bar{\beta}_-\bar{\beta}_+\bar{\gamma}^{-1}\mathcal{S}_- + 4\bar{\gamma}^{-1}\bar{\chi}_-\mathcal{S}_- - \frac{21}{8}\mathcal{S}_+ + 5\bar{\beta}_+\mathcal{S}_+ - \frac{7}{3}\bar{\gamma}\mathcal{S}_+ \right. \\ & - 4\bar{\beta}_+\bar{\gamma}^{-1}\mathcal{S}_+ + \frac{16}{3}\bar{\beta}_+^2\bar{\gamma}^{-1}\mathcal{S}_+ - 16\bar{\beta}_+^2\bar{\gamma}^{-2}\mathcal{S}_+ + 16\bar{\beta}_+^2\bar{\gamma}^{-2}\mathcal{S}_+ + 4\bar{\gamma}^{-1}\bar{\chi}_+\mathcal{S}_+ \\ & + \nu \left[-\frac{8}{3}\bar{\beta}_-\mathcal{S}_- + \frac{4}{3}\bar{\beta}_-\bar{\gamma}^{-1}\mathcal{S}_- - \frac{7}{24}\mathcal{S}_+ + \frac{4}{3}\bar{\beta}_+\bar{\gamma}^{-1}\mathcal{S}_+ \right] + \frac{5}{24}\mathcal{S}_-\nu\delta \\ & + \delta \left[-\frac{3}{8}\mathcal{S}_- + \frac{1}{3}\bar{\beta}_+\mathcal{S}_- - \frac{1}{3}\bar{\gamma}\mathcal{S}_- - \frac{16}{3}\bar{\beta}_-^2\bar{\gamma}^{-1}\mathcal{S}_- - 16\bar{\beta}_-^2\bar{\gamma}^{-2}\mathcal{S}_- + 16\bar{\beta}_+^2\bar{\gamma}^{-2}\mathcal{S}_- \right. \\ & \left. \left. - 4\bar{\gamma}^{-1}\bar{\chi}_+\mathcal{S}_- - \frac{7}{3}\bar{\beta}_-\mathcal{S}_+ - \frac{16}{3}\bar{\beta}_-\bar{\beta}_+\bar{\gamma}^{-1}\mathcal{S}_+ - 4\bar{\gamma}^{-1}\bar{\chi}_-\mathcal{S}_+ \right] \right\} \Bigg), \quad (\text{VII.68a}) \end{aligned}$$

$$\begin{aligned} \hat{\Psi}^{11} = & 1 + x \left\{ -\frac{9}{5} - \frac{2}{3}\bar{\beta}_+ - \frac{1}{3}\bar{\gamma} + 2\bar{\beta}_+\bar{\gamma}^{-1} + \frac{2\bar{\beta}_-\bar{\gamma}^{-1}\mathcal{S}_+}{\mathcal{S}_-} \right. \\ & \left. + \delta \left[\frac{2}{3}\bar{\beta}_- - 2\bar{\beta}_-\bar{\gamma}^{-1} + \frac{4\mathcal{S}_+}{5\mathcal{S}_-} - \frac{2\bar{\beta}_+\bar{\gamma}^{-1}\mathcal{S}_+}{\mathcal{S}_-} \right] + \frac{14}{15}\nu \right\} \\ & + x^{3/2} \left\{ \frac{(2+\bar{\gamma})\pi}{2} - i \left(\frac{(2+\bar{\gamma})(1+12\ln(2))}{12} + \frac{1}{3}\zeta(\mathcal{S}_+^2 + \mathcal{S}_-^2) + \frac{4}{3}\zeta\mathcal{S}_-\nu + \frac{2}{3}\zeta\mathcal{S}_-\mathcal{S}_+\delta \right) \right\}, \quad (\text{VII.68b}) \end{aligned}$$

⁸Recall that $h^{\ell,-m} = (-)^\ell (h^{\ell m})^*$ and $\psi^{\ell,-m} = (-)^\ell (\psi^{\ell m})^*$.

$$\begin{aligned} \hat{\Psi}^{22} = & -\frac{i}{\sqrt{5}\mathcal{S}_-} \left(x^{1/2} \left\{ -\mathcal{S}_+ + \mathcal{S}_-\delta \right\} \right. \\ & + x^{3/2} \left\{ \frac{4\bar{\beta}_-\mathcal{S}_-}{3} - 4\bar{\beta}_-\bar{\gamma}^{-1}\mathcal{S}_- + \frac{53\mathcal{S}_+}{14} + \frac{4\bar{\beta}_+\mathcal{S}_+}{3} + \frac{2\bar{\gamma}\mathcal{S}_+}{3} - 4\bar{\beta}_+\bar{\gamma}^{-1}\mathcal{S}_+ \right. \\ & + \nu \left[-\frac{16\bar{\beta}_-\mathcal{S}_-}{3} + 8\bar{\beta}_-\bar{\gamma}^{-1}\mathcal{S}_- - \frac{211\mathcal{S}_+}{42} + 8\bar{\beta}_+\bar{\gamma}^{-1}\mathcal{S}_+ \right] + \frac{61\mathcal{S}_-\nu\delta}{42} \\ & \left. \left. + \delta \left[-\frac{53\mathcal{S}_-}{14} - \frac{4\bar{\beta}_+\mathcal{S}_-}{3} - \frac{2\bar{\gamma}\mathcal{S}_-}{3} + 4\bar{\beta}_+\bar{\gamma}^{-1}\mathcal{S}_- - \frac{4\bar{\beta}_-\mathcal{S}_+}{3} + 4\bar{\beta}_-\bar{\gamma}^{-1}\mathcal{S}_+ \right] \right\} \right), \end{aligned} \quad (\text{VII.68c})$$

$$\hat{\Psi}^{33} = \frac{9\sqrt{3}(-\mathcal{S}_- + \mathcal{S}_+\delta + 2\mathcal{S}_-\nu)x}{4\sqrt{70}\mathcal{S}_-}, \quad (\text{VII.68d})$$

$$\hat{\Psi}^{31} = -\frac{(-\mathcal{S}_- + \mathcal{S}_+\delta + 2\mathcal{S}_-\nu)x}{20\sqrt{14}\mathcal{S}_-}, \quad (\text{VII.68e})$$

$$\hat{\Psi}^{44} = -\frac{16i(\mathcal{S}_+ - \mathcal{S}_-\delta - 3\mathcal{S}_+\nu + \mathcal{S}_-\nu\delta)x^{3/2}}{3\sqrt{105}\mathcal{S}_-}, \quad (\text{VII.68f})$$

$$\hat{\Psi}^{42} = \frac{2i(\mathcal{S}_+ - \mathcal{S}_-\delta - 3\mathcal{S}_+\nu + \mathcal{S}_-\nu\delta)x^{3/2}}{21\sqrt{15}\mathcal{S}_-}. \quad (\text{VII.68g})$$

Note that formally, the dominant scalar mode seems to be ψ^{00} . This is misleading because the Newtonian part of ψ^{00} is actually an x -independent constant: when taking its derivative to compute the flux, such a term vanishes. The dominant mode is thus indeed ψ^{11} , hence our choice of normalization.

VII.6 Comparison with numerical relativity

A recent numerical simulation by Ma *et al.* [256] was plotted against our PN results. In this simulation, the authors strived to maximize the effect of spontaneous scalarization [259] by a choice of $\omega(\phi)$. For this, they express their theory in the Damour and Esposito-Farèse framework [234], where they choose

$$\alpha_0 = -3.5 \times 10^{-3} \quad \text{and} \quad \beta_0 = -4.5. \quad (\text{VII.69})$$

Inverting (17) of [256] leads to

$$\psi - \psi_0 = -\frac{1}{4\sqrt{\pi}\alpha_0} \left[\ln\left(\frac{\phi}{\phi_0}\right) + \frac{\beta_0}{4\alpha_0^2} \ln^2\left(\frac{\phi}{\phi_0}\right) \right], \quad (\text{VII.70})$$

where the authors chose to work with $\phi_0 = 1$ and $\psi_0 = 0$. Moreover, (5) of [256] can be rephrased as

$$\omega(\phi) = 8\pi \left(\frac{d(\psi - \psi_0)}{d \ln(\phi/\phi_0)} \right)^2 - \frac{3}{2}, \quad (\text{VII.71})$$

so finally we have

$$\omega(\phi) = \frac{1}{2\alpha_0^2} \left[1 + \frac{\beta_0}{2\alpha_0^2} \ln\left(\frac{\phi}{\phi_0}\right) \right]^2 - \frac{3}{2} \quad (\text{VII.72})$$

The scalar-tensor parameters are then related to those of DE gravity by [256]

$$\omega_0 = \frac{1 - 3\alpha_0^2}{2\alpha_0^2}, \quad \tilde{G} = \frac{1 + \alpha_0^2}{\phi_0}, \quad \zeta = \frac{\alpha_0}{1 + \alpha_0^2}. \quad (\text{VII.73})$$

They then consider a NS-BH binary system. The BH has mass $m_{\text{BH}} = 5.7 M_\odot$, and of course a scalar charge $\alpha_{\text{BH}} = 0$, i.e. a sensitivity of $s_{\text{BH}} = 1/2$. The NS has a mass of $m_{\text{NS}} = 1.5 M_\odot$, a scalar charge of $\alpha_{\text{NS}} = 0.18$, and therefore a sensitivity of $s_{\text{NS}} = 26$, such that $\alpha_{\text{NS}} = \alpha_0(1 - 2s_{\text{BH}})$. They define the total mass $m = m_{\text{NS}} + m_{\text{BH}}$, and present simulation results for a time of around 65 ms before merger, which corresponds to about 12 orbital cycles. Note that the black hole has a spin given by $\chi_{\text{BH}} = 0.19$, the system has initial eccentricity of $e \sim 1.6 \times 10^{-4}$, and tidal effects do contribute in this simulation, but all of these effects are not accounted for by the PN template. With this choice of parameters, they find

$$\left(\frac{24}{5\zeta\mathcal{S}_-^2} \right) \left(\frac{\tilde{G}\alpha m\omega}{2} \right)^{2/3} \geq 25, \quad (\text{VII.74})$$

therefore they worked with the 1PN quadrupole-driven phasing given by [67], the 2PN $\hat{H}^{\ell m}$ amplitudes given by [67] as well, and with the 1.5PN $\hat{\Psi}^{\ell m}$ scalar amplitudes given by (VII.68). Figure VII.1 presents the numerical simulation plotted against each mode. Apart from the $\hat{\Psi}_{22}$ mode that exhibits a small discrepancy, the numerical simulations are in very good agreement with the PN template, including the nonoscillatory (0,0) mode. Moreover, this agreement can certainly be improved by including tidal effects and going to higher PN order.

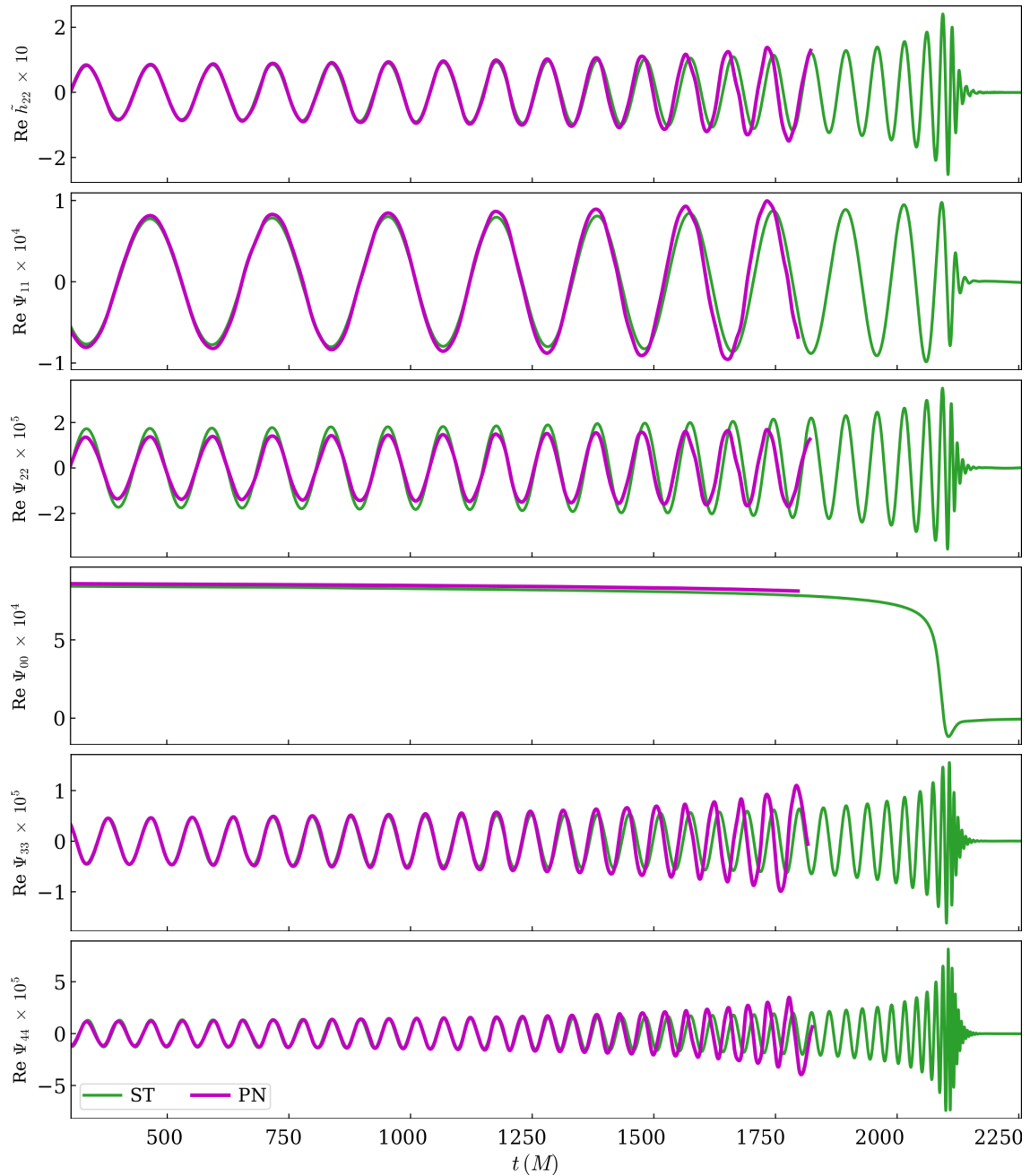


Fig. VII.1 Comparison of 1.5PN scalar modes computed in [2] (in mauve) and numerical results of [256] (in green). The h_{22} mode computed at 2PN in [67] is also compared to the numerical results. Time is given in units of GM/c^3 , where $M = 7.2 M_{\odot}$. Source: [256].

Part C

Conclusion and prospects

S

Conclusion and prospects

This thesis was dedicated to computing gravitational wave templates for general relativity and a class of scalar-tensor theories in the post-Newtonian approximation. In the case of circular orbits, we have obtained the most accurate predictions to date. We summarize the progress made in general relativity in Table VII.2 and in scalar-tensor theories in Table VII.3.

General relativity				
Order	EOM	(2, 2) mode	Flux	GW phase
3PN	[114, 117, 104, 120]			
3.5PN	[264, 265, 121]	[202]	[266, 104]	[266, 104]
4PN	[127, 131, 138, 139]	This work	This work	This work
4.5PN	[267, 153] ⁹	Unkown	[167]	This work

Table VII.2: Progress made in this thesis in general relativity, in the case of two structureless point-particles.

In the case of the 4PN and 4.5PN results in general relativity, preliminary comparisons with numerical relativity and second-order self-force [9] indicate good agreement, but ultimately, it would be highly desirable to have an independent verification of this computation, for example using effective field theory methods. Many difficulties have arisen at this order, in particular due to dimensional regularization, and we hope that the techniques presented in this thesis can be helpful to rederive these results independently. In scalar-tensor theory, we have found a small discrepancy with [247], which does not affect usual results for circular orbits. We have also provided a check for the results obtained in [247, 67], and we find relatively good agreement with recent numerical relativity results [256].

As one can see from Table VII.2, it would be interesting to compute the (2, 2) modes at 4.5PN. One step towards this goal has been achieved by the computation of the 4.5PN radiation-reaction terms in the equations of motions in the center-of-mass frame [267, 153]. Even at 4PN, we only control the (2, 2) mode, and one yet needs to compute the other (ℓ, m) modes to control the full waveform. Furthermore, with the prospect of the full completion of the 5PN equations of motion [141, 145], it is natural to contemplate the

⁹In the center-of-mass frame only.

¹⁰We found a discrepancy with [247] which vanishes for circular orbits [67].

¹¹The waveform was computed at 1.5PN by [246] but was not reduced to spherical harmonic modes for circular orbits.

¹²For dipolar-driven systems only.

Scalar-tensor theories						
Order	EOM	Order	Tensor modes and flux	Scalar modes	Scalar flux	Orbital phase
2PN	[245]	1PN	[246, 67]	[247, 67]	[247, 67] ¹⁰	[247, 67]
2.5PN	[245]	1.5PN	[246, 67]	This work ¹¹	This work	This work ¹²
3PN	[248, 131]	2PN	[246, 67]	Unkown	Unkown	Unkown

Table VII.3: Progress made in this thesis in scalar-tensor theories, in the case of two structureless point-particles.

possibility of computing the flux, phase and waveform at 5PN as well. Many difficulties will arise in this computation, one of which is the appearance of many terms akin to the tails-of-memory we computed in Chapter IV. The novel integration techniques we have introduced, along with the drastic simplification method, seem to indicate a path to compute the nonlinearities in the post-Minkowskian iteration very generally, to any order and for any interaction. However, the difficulty of this computation contrasts with the rather simple end result, and one is compelled to ask if there is not an easier way to compute these nonlinear interactions. Moreover, computing the source quadrupole moment at 5PN order, using full dimensional regularization, should be expected to be very difficult as well.

Finally, despite the theoretical interest of these computations, one may also reassess the experimental need for going to even higher order, as it could be more useful for astrophysical purposes to broaden our problem to include, for example, eccentricity and nonaligned spins to high post-Newtonian order. This would require detailed studies (such as [220]) about the impact of truncating the post-Newtonian series to some given order, both in the case of current and future detectors. Another interesting direction concerns alternative theories of gravity. Table VII.3 indicates that the knowledge of the 3PN equations of motion [248, 131] in scalar-tensor theories allow us to compute the waveform at 2PN order, which is the object of future work. Moreover, in order to dispose of a large bank of alternative waveform templates, it now seems crucial to study waveforms to high order in a wider class of alternative theories of gravity, such as theories with a potential, or even exhibiting a screening mechanism. Before this, and in light of the plethora of existing alternative theories and recent experimental constraints, one should first precisely identify which theories could indeed be tested by gravitational-wave observations, but still escape weak-field solar system constraints.

Part D

Appendix

Appendix A

General relativity

A.1 Extracting N_L , P_L , Q_L and R_L

We describe the practical method we use to extract four sets of STF moments N_L , P_L , Q_L and R_L parametrizing a vector quantity satisfying $\square w^\mu = 0$. Here $w^\mu \equiv w_n^\mu$ represents the divergence $w_n^\mu = \partial_\nu w_n^{\mu\nu}$ of the quantity (II.22) following the MPM algorithm. Having extracted these moments we can then construct the tensor $v_n^{\mu\nu} \equiv \mathcal{V}^{\mu\nu}[N_L, P_L, Q_L, R_L]$ using the formulas (48) in [14], which satisfies at once $\square v_n^{\mu\nu} = 0$ and $\partial_\nu v_n^{\mu\nu} = -w_n^\mu$. By definition, see (47) in [14],

$$w^0 = \sum_{\ell=0}^{\infty} \partial_L \left[r^{-1} N_L(t-r) \right], \quad (\text{A.1a})$$

$$w^i = \sum_{\ell=0}^{\infty} \partial_{iL} \left[r^{-1} P_L(t-r) \right] + \sum_{\ell=1}^{\infty} \partial_{L-1} \left[r^{-1} Q_{iL}(t-r) \right] + \sum_{\ell=1}^{\infty} \epsilon_{iab} \partial_{aL-1} \left[r^{-1} R_{bL-1}(t-r) \right]. \quad (\text{A.1b})$$

Next, we define the auxiliary quantities

$$\begin{aligned} \tilde{w}^i &\equiv w^i - \sum_{\ell=0}^{\infty} \partial_{iL} \left[r^{-1} P_L(t-r) \right] \\ &= \sum_{\ell=1}^{\infty} \partial_{L-1} \left[r^{-1} Q_{iL}(t-r) \right] + \sum_{\ell=1}^{\infty} \epsilon_{iab} \partial_{aL-1} \left[r^{-1} R_{bL-1}(t-r) \right], \end{aligned} \quad (\text{A.2a})$$

$$\tilde{\tilde{w}}^i \equiv \tilde{w}^i - \sum_{\ell=1}^{\infty} \partial_{L-1} \left[r^{-1} Q_{iL}(t-r) \right] = \sum_{\ell=1}^{\infty} \epsilon_{iab} \partial_{aL-1} \left[r^{-1} R_{bL-1}(t-r) \right]. \quad (\text{A.2b})$$

Using formulas from Appendix A in [92], we express the angular integrals

$$\int \frac{d\Omega}{4\pi} \hat{n}_L w^0 = \frac{\ell!}{(2\ell+1)!!} r^\ell \left(\frac{1}{r} \frac{\partial}{\partial r} \right)^\ell \left[r^{-1} N_L(t-r) \right], \quad (\text{A.3a})$$

$$\int \frac{d\Omega}{4\pi} \hat{n}_{iL} w^i = \frac{(\ell+1)\ell!}{(2\ell+1)(2\ell+1)!!} r^{\ell+1} \left(\frac{1}{r} \frac{\partial}{\partial r} \right)^{\ell+1} \left[r^{-1} P_L(t-r) \right], \quad (\text{A.3b})$$

$$\int \frac{d\Omega}{4\pi} n_i \hat{n}_L \tilde{w}^i = \frac{\ell!}{(2\ell+1)!!} r^{\ell-1} \left(\frac{1}{r} \frac{\partial}{\partial r} \right)^{\ell-1} \left[r^{-1} \mathbf{Q}_L(t-r) \right], \quad (\text{A.3c})$$

$$\epsilon_{ab\langle i} \int \frac{d\Omega}{4\pi} n_{L-1} n_b \tilde{w}^a = \frac{(\ell+1)(\ell-1)!}{(2\ell+1)!!} r^\ell \left(\frac{1}{r} \frac{\partial}{\partial r} \right)^\ell \left[r^{-1} \mathbf{R}_{iL-1}(t-r) \right]. \quad (\text{A.3d})$$

If w^μ is known exactly to all order in r , we find that the multipole moments can be computed using the near zone limit $r \rightarrow 0$ (with $t-r = \text{const}$) as

$$\mathbf{N}_L(t-r) = \frac{(-)^\ell (2\ell+1)}{\ell!} \lim_{r \rightarrow 0} \left[r^{\ell+1} \int \frac{d\Omega}{4\pi} \hat{n}_L w^0 \right], \quad (\text{A.4a})$$

$$\mathbf{P}_L(t-r) = \frac{(-)^{\ell+1} (2\ell+1)}{(\ell+1)!} \lim_{r \rightarrow 0} \left[r^{\ell+2} \int \frac{d\Omega}{4\pi} \hat{n}_{iL} w^i \right], \quad (\text{A.4b})$$

$$\mathbf{Q}_L(t-r) = \frac{(-)^{\ell-1} (2\ell+1)(2\ell-1)}{\ell!} \lim_{r \rightarrow 0} \left[r^\ell \int \frac{d\Omega}{4\pi} n_i \hat{n}_L \tilde{w}^i \right], \quad (\text{A.4c})$$

$$\mathbf{R}_{iL-1}(t-r) = \frac{(-)^\ell (2\ell+1)}{(\ell+1)(\ell-1)!} \lim_{r \rightarrow 0} \left[r^{\ell+1} \int \frac{d\Omega}{4\pi} \epsilon_{ab\langle i} n_{L-1} n_b \tilde{w}^a \right]. \quad (\text{A.4d})$$

If we only know the leading order of the asymptotic expansion of w^μ as $r \rightarrow +\infty$, the previous expressions cannot be used. In that case, we get equivalent expressions for the time derivatives of the multipole moments:

$$\overset{(\ell)}{\mathbf{N}}_L(t-r) = \frac{(-)^\ell (2\ell+1)!!}{\ell!} \lim_{r \rightarrow +\infty} \left[r \int \frac{d\Omega}{4\pi} \hat{n}_L w^0 \right], \quad (\text{A.5a})$$

$$\overset{(\ell+1)}{\mathbf{P}}_L(t-r) = \frac{(-)^\ell (2\ell+1)(2\ell+1)!!}{(\ell+1)!} \lim_{r \rightarrow +\infty} \left[r \int \frac{d\Omega}{4\pi} \hat{n}_{iL} w^i \right], \quad (\text{A.5b})$$

$$\overset{(\ell-1)}{\mathbf{Q}}_L(t-r) = \frac{(-)^{\ell-1} (2\ell+1)!!}{\ell!} \lim_{r \rightarrow +\infty} \left[r \int \frac{d\Omega}{4\pi} n_i \hat{n}_L \tilde{w}^i \right], \quad (\text{A.5c})$$

$$\overset{(\ell)}{\mathbf{R}}_{iL-1}(t-r) = \frac{(-)^\ell (2\ell+1)!!}{(\ell+1)(\ell-1)!} \lim_{r \rightarrow +\infty} \left[r \int \frac{d\Omega}{4\pi} \epsilon_{ab\langle i} n_{L-1} n_b \tilde{w}^a \right]. \quad (\text{A.5d})$$

A.2 Coefficients for the raw tails of memory

Table A.1: Coefficients $\mathcal{A}_{m,\ell}^n$, $\mathcal{B}_{m,\ell}^n$, $\mathcal{C}_{m,\ell}^n$, $\mathcal{D}_{m,\ell}^n$ entering the radiative quadrupole(a) Coefficients $\mathcal{A}_{m,\ell}^n$ associated to ${}_{1,m}\bar{\Psi}_\ell[\bar{M}_{a\langle i}, \bar{M}_{j\rangle a}^{(n)}]$

ℓ	2					3					4				
	n	0	1	2	3	4	0	1	2	3	4	0	1	2	3
0		$-\frac{136}{735}$	$-\frac{13808}{2625}$	$-\frac{34576}{2625}$	$\frac{120}{7}$	$\frac{16}{7}$	0	$\frac{832}{535}$	$\frac{29432}{1575}$	$\frac{88}{9}$	$-\frac{32}{7}$	$-\frac{928}{25725}$	$-\frac{1168}{3675}$	$\frac{664}{525}$	$\frac{64}{21}$
1		0	$-\frac{3344}{6125}$	$\frac{6184}{875}$	$-\frac{16456}{875}$	$-\frac{32}{7}$	$\frac{1216}{1225}$	$\frac{3464}{525}$	$-\frac{11584}{525}$	$-\frac{17032}{525}$	$\frac{64}{7}$	0	$-\frac{9256}{8575}$	$-\frac{6256}{1225}$	$\frac{3776}{1225}$
2		$\frac{8}{75}$	$\frac{18328}{3675}$	$\frac{22448}{3675}$	$\frac{2528}{175}$	$\frac{16}{7}$	0	$-\frac{3776}{735}$	$\frac{36272}{2205}$	$\frac{304}{45}$	$-\frac{32}{7}$	$-\frac{16}{343}$	$-\frac{2896}{5145}$	$\frac{1024}{735}$	$\frac{4696}{735}$
3		0	$-\frac{568}{875}$	$-\frac{1224}{875}$	$\frac{816}{875}$	0	$-\frac{128}{175}$	$\frac{536}{525}$	$-\frac{5176}{525}$	$-\frac{288}{25}$	0	0	$\frac{7084}{3675}$	$\frac{4616}{1225}$	$\frac{1464}{1225}$
4		$\frac{96}{275}$	$\frac{13152}{6125}$	$\frac{8544}{6125}$	0	0	0	$-\frac{4608}{1225}$	$-\frac{3936}{1125}$	0	0	$\frac{10336}{94325}$	$\frac{4192}{8575}$	$-\frac{1616}{1225}$	0
5		0	$-\frac{832}{1225}$	0	0	0	$-\frac{64}{245}$	$-\frac{32}{105}$	0	0	0	0	$-\frac{464}{1029}$	0	0
6		$-\frac{3648}{13475}$	0	0	0	0	0	0	0	0	0	$-\frac{304}{11319}$	0	0	0

(b) Coefficients $\mathcal{B}_{m,\ell}^n$ associated to ${}_{2,m}\bar{\Psi}_\ell[\bar{M}_{a\langle i}, \bar{M}_{j\rangle a}^{(n)}]$

ℓ	2				3				4				
	n	0	1	2	3	0	1	2	3	0	1	2	3
0		0	$\frac{696}{875}$	$\frac{312}{7}$	$\frac{48}{7}$	$\frac{608}{175}$	$\frac{10768}{525}$	$-\frac{304}{21}$	$-\frac{192}{7}$	0	$\frac{272}{147}$	$\frac{160}{21}$	0
1		$-\frac{15336}{6125}$	$-\frac{5632}{875}$	$-\frac{18728}{875}$	$\frac{64}{7}$	0	$-\frac{5952}{175}$	$-\frac{5744}{175}$	$\frac{176}{7}$	$-\frac{2432}{1715}$	$-\frac{1616}{245}$	$\frac{1040}{49}$	$\frac{48}{7}$
2		0	$-\frac{2928}{1225}$	$\frac{1584}{175}$	$\frac{48}{35}$	$-\frac{32}{49}$	$\frac{15376}{735}$	$-\frac{3424}{105}$	$-\frac{240}{7}$	0	$\frac{3952}{1029}$	$\frac{2672}{147}$	$\frac{48}{7}$
3		$-\frac{216}{125}$	$\frac{552}{875}$	$-\frac{2592}{875}$	0	0	$-\frac{1728}{175}$	$-\frac{288}{25}$	0	$\frac{3904}{2205}$	$\frac{656}{245}$	$-\frac{288}{49}$	0
4		0	$-\frac{41472}{6125}$	0	0	$-\frac{3456}{1225}$	$-\frac{2592}{1225}$	0	0	0	$-\frac{608}{343}$	0	0
5		$-\frac{1536}{1225}$	0	0	0	0	0	0	0	$-\frac{1088}{3087}$	0	0	0

(c) Coefficients $\mathcal{C}_{m,\ell}^n$ associated to

$${}_{1,m}\bar{\chi}_\ell[\bar{M}_{a\langle i}, \bar{M}_{j\rangle a}^{(n)}]$$

ℓ	2				
	n	0	1	2	3
0		$-\frac{64}{245}$	$-\frac{256}{175}$	$\frac{64}{25}$	0
1		0	$\frac{1536}{1225}$	$-\frac{1536}{175}$	$\frac{384}{175}$
2		$\frac{64}{245}$	$-\frac{64}{245}$	$\frac{64}{5}$	$-\frac{192}{35}$
3		0	$\frac{192}{175}$	$-\frac{2304}{175}$	$\frac{576}{175}$
4		$-\frac{1152}{2695}$	$-\frac{4608}{1225}$	$\frac{1152}{175}$	0
5		0	$\frac{768}{245}$	0	0
6		$\frac{1152}{2695}$	0	0	0

(d) Coefficients $\mathcal{D}_{m,\ell}^n$ associated to

$${}_{2,m}\bar{\chi}_\ell[\bar{M}_{a\langle i}, \bar{M}_{j\rangle a}^{(n)}]$$

ℓ	2			
	n	0	1	2
0		0	$\frac{576}{175}$	0
1		$-\frac{576}{1225}$	$-\frac{1152}{175}$	$\frac{1152}{175}$
2		0	$\frac{1152}{245}$	$-\frac{576}{35}$
3		$-\frac{192}{175}$	$-\frac{1728}{175}$	$\frac{1728}{175}$
4		0	$\frac{10368}{1225}$	0
5		$\frac{384}{245}$	0	0

A.3 Formulas for the retarded integrals

During the application of the procedure described in Section V.1, we have to compute some retarded integrals given by (V.15). We quickly recall the method we follow, see Section IV.B in [95]. Because of the explicit factors B and B^2 in their source terms, the retarded integrals (V.15) will be nonzero only when they develop a pole (or a double pole) when $B \rightarrow 0$. In turn this means that they depend only on the behavior of the corresponding source in the “near-zone”, *i.e.*, when $r \rightarrow 0$.

Thus the first task is to expand the source term when $r \rightarrow 0$. This is straightforward except when the source contains a hereditary tail integral, say

$$\mathcal{F}_m(r, t) = \int_1^{+\infty} dx Q_m(x) F(t - rx/c), \quad (\text{A.6})$$

where $F(t)$ denotes a component of the mass quadrupole moment. The formula needed to handle this case has been developed in the Appendix A of [95]: the near-zone expansion of (A.6), valid to any order when $r \rightarrow 0$, reads

$$\begin{aligned} \mathcal{F}_m &= \sum_{i=0}^{\infty} \beta_i^m \frac{(-)^i}{i!} \left(\frac{r}{c}\right)^i F^{(i)}(t) \\ &+ \sum_{j=0}^{\infty} \frac{(-)^m c_j^m}{(m+2j)!} \left(\frac{r}{c}\right)^{m+2j} \int_0^{\infty} d\tau \left[\ln\left(\frac{c\tau}{2r}\right) - H_{m+j} + 2H_{2m+2j+1} \right] F^{(m+2j+1)}(t - \tau), \end{aligned} \quad (\text{A.7a})$$

where H_q is the harmonic number, and the coefficients are

$$\beta_i^m = \sum_{k=0}^{m-1} \binom{i}{k} 2^k (k!)^2 \frac{(m-k-1)!}{(m+k+1)!} + \sum_{k=m}^i \binom{i}{k} 2^k (k!)^2 (-)^{m+k} \frac{H_{k-m} - H_{k+m+1}}{(k-m)!(k+m+1)!}, \quad (\text{A.7b})$$

$$c_j^m = \frac{2^m (m+2j)!(m+j)!}{j! (2m+2j+1)!} \quad (\text{A.7c})$$

with the binomial symbol $\binom{i}{k} = 0$ whenever $i < k$.

Once the near-zone expansion of the source is achieved, it remains to apply the following formulas, most of them being already exposed in the Section IV.B of [95], but which we had to generalize in order to include higher powers in the logarithms. Thus the generalization of (4.17) in [95] is

$$\text{FP}_{B=0} \square_{\text{ret}}^{-1} \left[\left(\frac{r}{r_0}\right)^B B^b \frac{\ln^a r \hat{n}_L}{r^p} G(t) \right] = e_p^\ell \alpha_{a,b} \hat{\partial}_L \left[\frac{G^{(p-\ell-3)}(t-r/c)}{r c^{p-\ell-3}} \right], \quad (\text{A.8a})$$

where $e_p^\ell = 0$ when $p - \ell - 3$ is not an even natural integer, and otherwise:

$$e_p^\ell = \frac{(-)^p}{(p-\ell-3)!!(p+\ell-2)!!} \quad (\text{when } p = \ell + 3 + 2j \text{ with } j \in \mathbb{N}), \quad (\text{A.8b})$$

and where, for the values we need in this chapter: $\alpha_{0,1} = 1$, $\alpha_{0,2} = 0$, $\alpha_{1,1} = \ln r_0$, $\alpha_{1,2} = -1$, $\alpha_{2,1} = (\ln r_0)^2$ and $\alpha_{2,2} = -2 \ln r_0$.

It is also very useful to dispose of the similar formula but with a source term which is “exact”, *i.e.*, not Taylor-expanded in the near zone. In this case, generalizing (4.11) of [95]:

$$\text{FP}_{B=0} \square_{\text{ret}}^{-1} \left[\left(\frac{r}{r_0} \right)^B B^b \frac{\ln^a r \hat{n}_L}{r^p} G(t - r/c) \right] = f_p^\ell \alpha_{a,b} \hat{\partial}_L \left[\frac{G^{(p-\ell-3)}(t - r/c)}{r c^{p-\ell-3}} \right], \quad (\text{A.9a})$$

where $f_p^\ell = 0$ if $p - \ell - 3 < 0$ and otherwise:

$$f_p^\ell = \frac{(-)^p 2^{p-3} (p-3)!}{(p-\ell-3)! (p+\ell-2)!} \quad (\text{when } p \geq \ell + 3), \quad (\text{A.9b})$$

while the values of $\alpha_{a,b}$ remain the same as in (A.8). One can naturally recover the “exact” result (A.9) from the near-zone result (A.8) by performing a Taylor expansion at the level of the source and a subsequent formal resummation.

A.4 Test of the boosted Schwarzschild solution

Among all tests that can be performed to check the expression of the mass quadrupole moment, one of the simplest is the boosted Schwarzschild limit. Despite its apparent simplicity, it is quite efficient and was crucially used to fix a remaining ambiguity constant in an early computation of the flux at 3PN order [268]. The principle is quite transparent: if we remove one of the two black holes, then our system reduces to a single Schwarzschild black hole of mass m_1 , boosted at a (constant) speed \mathbf{v}_1 . The multipole moments of a boosted Schwarzschild solution (BSS) are straightforward to derive, and have been previously determined at 3PN order in [268]. Extending this work at 4PN order, we find that the quadrupole moment of a BBS of mass M , boosted with a velocity \mathbf{V} , reads

$$\begin{aligned} \text{I}_{ij}^{\text{BSS}} = M t^2 V^{(i} V^{j)} & \left[1 + \frac{9}{14} \frac{V^2}{c^2} + \frac{83}{168} \frac{V^4}{c^4} + \frac{507}{1232} \frac{V^6}{c^6} + \frac{45923}{128128} \frac{V^8}{c^8} \right] \\ & + \frac{4}{7} \frac{G^2 M^3}{c^6} V^{(i} V^{j)} + \frac{10}{21} \frac{G^2 M^3}{c^8} V^2 V^{(i} V^{j)} + \mathcal{O} \left(\frac{1}{c^{10}} \right). \end{aligned} \quad (\text{A.10})$$

On the other hand, taking the BSS limit $(m_2, \mathbf{v}_2) \rightarrow (0, \mathbf{0})$ of the renormalized mass quadrupole moment defined in [162] (on generic orbits, out of the CoM frame), it comes

$$\begin{aligned} \lim_{\text{BSS}} \text{I}_{ij} = m_1 t^2 v_1^{(i} v_1^{j)} & \left[1 + \frac{9}{14} \frac{v_1^2}{c^2} + \frac{83}{168} \frac{v_1^4}{c^4} + \frac{507}{1232} \frac{v_1^6}{c^6} + \frac{45923}{128128} \frac{v_1^8}{c^8} \right] \\ & + \frac{4}{7} \frac{G^2 m_1^3}{c^6} v_1^{(i} v_1^{j)} + \frac{10}{21} \frac{G^2 m_1^3}{c^8} v_1^2 v_1^{(i} v_1^{j)} + \mathcal{O} \left(\frac{1}{c^{10}} \right). \end{aligned} \quad (\text{A.11})$$

Both expressions coincide under the identification $(m_1, \mathbf{v}_1) = (M, \mathbf{V})$, therefore this test is conclusive.

A.5 Post-adiabatic integral formulas

For any $\beta > 1$, we define the integrals

$$\mathcal{A} \equiv \int_0^{+\infty} ds \frac{e^{\frac{ins}{\xi}}}{\left(1 + \frac{5}{8}s\right)^\beta} \quad (\text{A.12a})$$

$$\mathcal{B} \equiv \int_0^{+\infty} ds \frac{e^{\frac{ins}{\xi}}}{\left(1 + \frac{5}{8}s\right)^\beta} \ln \left[\left(1 + \frac{5}{8}s\right)^{\frac{8}{5}} - 1 \right] \quad (\text{A.12b})$$

By integrating by parts iteratively, we immediately find that

$$\begin{aligned} \mathcal{A} = & \left[\left(\frac{\xi}{in} \right) e^{\frac{ins}{\xi}} \left(1 + \frac{5}{8}s\right)^{-\beta} \right]_{s=0}^{+\infty} - \left[\left(\frac{\xi}{in} \right)^2 e^{\frac{ins}{\xi}} \left(-\frac{5\beta}{8} \right) \left(1 + \frac{5}{8}s\right)^{-\beta-1} \right]_{s=0}^{+\infty} \\ & + \left(\frac{\xi}{in} \right)^2 \left(-\frac{5\beta}{8} \right) \left(-\frac{5(\beta+1)}{8} \right) \int_0^{+\infty} ds e^{\frac{ins}{\xi}} \left(1 + \frac{5}{8}s\right)^{-\beta-2} \end{aligned} \quad (\text{A.13})$$

where the last integral is convergent as can henceforth be treated as a constant of order unity. We thus immediately obtain our first result,

$$\mathcal{A} = \frac{i\xi}{n} + \frac{5\beta}{8n^2} \xi^2 + o(\xi^2) \quad (\text{A.14})$$

We now claim that

$$\mathcal{B} = \left(\frac{i\xi}{n} + \frac{5\beta}{8n^2} \xi^2 \right) \left(\ln \left(\frac{\xi}{|n|} \right) - \gamma_E + \text{sg}(n) \frac{i\pi}{2} \right) + \left(\frac{5\beta}{8} - \frac{3}{16} \right) \frac{\xi^2}{n^2} + o(\xi^2) \quad (\text{A.15})$$

To prove this, we divide our integral in a “recent” and “ancient” part, namely $\mathcal{B} = \mathcal{B}_{\text{recent}} + \mathcal{B}_{\text{ancient}}$ where

$$\mathcal{B}_{\text{recent}} \equiv \int_0^1 ds \frac{e^{\frac{ins}{\xi}}}{\left(1 + \frac{5}{8}s\right)^\beta} \ln \left[\left(1 + \frac{5}{8}s\right)^{\frac{8}{5}} - 1 \right] \quad (\text{A.16a})$$

$$\mathcal{B}_{\text{ancient}} \equiv \int_1^{+\infty} ds \frac{e^{\frac{ins}{\xi}}}{\left(1 + \frac{5}{8}s\right)^\beta} \ln \left[\left(1 + \frac{5}{8}s\right)^{\frac{8}{5}} - 1 \right] \quad (\text{A.16b})$$

We first treat the “ancient” integral by introducing an auxiliary function

$$f(s) \equiv \ln \left[\left(1 + \frac{5}{8}s\right)^{\frac{8}{5}} - 1 \right] - \ln(s) \quad (\text{A.17})$$

which can be extended to be a C^∞ function of \mathbb{R}^+ whose series expansion in a neighborhood of 0 reads

$$f(s) = \frac{3}{16}s - \frac{17}{512}s^2 + \frac{35}{4096}s^3 + \mathcal{O}_{s \rightarrow 0}(s^4) \quad (\text{A.18})$$

We replace the complicated logarithmic term by this auxiliary function and integrate $\mathcal{B}_{\text{ancient}}$ by parts in a similar way as we did with \mathcal{A} , namely

$$\begin{aligned} \mathcal{B}_{\text{ancient}} &= \left[\frac{\xi}{in} e^{\frac{ins}{\xi}} \frac{\ln(s) + f(s)}{\left(1 + \frac{5}{8}s\right)^\beta} \right]_{s=1}^{+\infty} - \left[\left(\frac{\xi}{in} \right)^2 e^{\frac{ins}{\xi}} \frac{d}{ds} \left\{ \frac{\ln(s) + f(s)}{\left(1 + \frac{5}{8}s\right)^\beta} \right\} \right]_{s=1}^{+\infty} \\ &\quad + \left(\frac{\xi}{in} \right)^2 \int_1^{+\infty} ds e^{\frac{ins}{\xi}} \frac{d^2}{ds^2} \left\{ \frac{\ln(s) + f(s)}{\left(1 + \frac{5}{8}s\right)^\beta} \right\} \end{aligned} \quad (\text{A.19})$$

By performing the derivatives explicitly and studying the behavior of f , we find that

$$\frac{d^2}{ds^2} \left\{ \frac{\ln(s) + f(s)}{\left(1 + \frac{5}{8}s\right)^\beta} \right\} = \mathcal{O}_{s \rightarrow +\infty} \left(\frac{\ln s}{s^{\beta+2}} \right), \quad (\text{A.20})$$

which is indeed integrable as soon as $\beta > -1$. Treating this integral as a constant of order unity, we find

$$\mathcal{B}_{\text{ancient}} = -e^{\frac{in}{\xi}} \left(\frac{8}{13} \right)^\beta \left[\frac{\xi}{in} f(1) + \left(\frac{\xi}{n} \right)^2 \left(1 - \frac{5\beta}{13} f(1) + f'(1) \right) \right] \quad (\text{A.21})$$

We now tackle the recent past integral by defining another auxiliary function, which is essentially $\left(1 + \frac{5}{8}s\right)^{-\beta}$ to which one subtracts the first terms of its series expansion when $s \rightarrow 0$, namely

$$g_\beta(s) \equiv \left(1 + \frac{5}{8}s\right)^{-\beta} - \left(1 - \frac{5\beta}{8}s\right) \quad (\text{A.22})$$

We then reexpress our integral

$$\mathcal{B}_{\text{recent}} \equiv \int_0^1 ds e^{\frac{ins}{\xi}} (\ln s + f(s)) \left(g_\beta(s) + 1 - \frac{5\beta}{8}s \right) \quad (\text{A.23})$$

We decompose this into several pieces, namely $\mathcal{B}_{\text{recent}} = \sum_{k=1}^3 \mathcal{B}_{\text{recent}}^{(k)}$. First, by integrating by parts in the usual manner, we find that

$$\begin{aligned} \mathcal{B}_{\text{recent}}^{(1)} &\equiv \int_0^1 ds e^{\frac{ins}{\xi}} f(s) \left(g_\beta(s) + 1 - \frac{5\beta}{8}s \right) \\ &= e^{\frac{in}{\xi}} \left(\frac{8}{13} \right)^\beta \left[\frac{\xi}{in} f(1) + \frac{\xi^2}{n^2} f'(1) - \frac{5\beta}{13} \frac{\xi^2}{n^2} f(1) \right] - \frac{3}{16} \frac{\xi^2}{n^2} \end{aligned} \quad (\text{A.24})$$

where we have used $f(0) = 0$, $f'(0) = 3/16$, $g_\beta(0) = 0$, $g_\beta(1) = (8/13)^\beta - 1 + 5\beta/8$ and $g'_\beta(1) = -(5\beta/13)(8/13)^\beta + 5\beta/8$. We can now deal with the terms involving a logarithm. Thanks to the very regular structure of g_β in a neighborhood of 0, we can integrate by parts twice to find

$$\mathcal{B}_{\text{recent}}^{(2)} \equiv \int_0^1 ds e^{\frac{ins}{\xi}} g_\beta(s) \ln s = e^{\frac{in}{\xi}} \frac{\xi^2}{n^2} \left[\left(\frac{8}{13} \right)^\beta - 1 + \frac{5\beta}{8} \right] \quad (\text{A.25})$$

where we have again used the value of $g_\beta(1)$. Finally, for the last piece, integrating by parts yields

$$\begin{aligned} \mathcal{B}_{\text{recent}}^{(3)} &\equiv \int_0^1 ds e^{\frac{ins}{\xi}} \ln s \left(1 - \frac{5\beta}{8}s\right) \\ &= -\frac{5\beta}{8} \frac{\xi^2}{n^2} \left(e^{\frac{in}{\xi}} - 1\right) + \left(1 + \frac{5\beta}{8} \frac{\xi}{in}\right) \int_0^1 ds e^{\frac{ins}{\xi}} \ln s. \end{aligned} \quad (\text{A.26})$$

At this stage, only a classic integral is yet to determine. We combine (4.331.1-2) of [182] to find

$$\int_0^1 ds e^{-\mu s} \ln s = -\frac{1}{\mu} [\ln \mu + \gamma_E - \text{Ei}(-\mu)] \quad (\text{A.27})$$

The branch cut of the complex logarithm being on the real negative half-axis, its principal value for $\mu = -in/\xi$ is well-defined and reads

$$\ln\left(-\frac{in}{\xi}\right) = \ln\left(\frac{|n|}{\xi}\right) - \frac{i\pi}{2} \text{sg}(n) \quad (\text{A.28})$$

Moreover, (8.215) of [182] gives us the relevant asymptotic expansion of the integral exponential function, which reads in our case and at desired order

$$\text{Ei}\left(\frac{in}{\xi}\right) = e^{in\xi} \frac{\xi}{in} (1 + \mathcal{O}(\xi)) \quad (\text{A.29})$$

Our last piece can thus be expressed as

$$\mathcal{B}_{\text{recent}}^{(3)} = \left(1 + \frac{5\beta}{8} \frac{\xi}{in}\right) \left(-\ln\left(\frac{\xi}{|n|}\right) - \frac{i\pi}{2} \text{sg}(n) + \gamma_E\right) + \frac{5\beta}{8} \frac{\xi^2}{n^2} + e^{\frac{in}{\xi}} \left(1 - \frac{5\beta}{8}\right) \frac{\xi^2}{n^2} \quad (\text{A.30})$$

Adding all of these elements together, we find that $\mathcal{B}_{\text{recent}} = \sum_{k=1}^3 \mathcal{B}_{\text{recent}}^{(k)}$ yields

$$\begin{aligned} \mathcal{B}_{\text{recent}} &= \left(\frac{i\xi}{n} + \frac{5\beta}{8n^2} \xi^2\right) \left(\ln\left(\frac{\xi}{|n|}\right) - \gamma_E + \text{sg}(n) \frac{i\pi}{2}\right) + \left(\frac{5\beta}{8} - \frac{3}{16}\right) \frac{\xi^2}{n^2} \\ &\quad + e^{\frac{in}{\xi}} \left(\frac{8}{13}\right)^\beta \left[\frac{\xi}{in} f(1) + \left(\frac{\xi}{n}\right)^2 \left(1 - \frac{5\beta}{13} f(1) + f'(1)\right)\right] \end{aligned} \quad (\text{A.31})$$

We then gladly see that adding $\mathcal{B}_{\text{recent}}$ with $\mathcal{B}_{\text{ancient}}$ cancels out all the oscillatory terms, which diverge as $\xi \rightarrow 0$. This means that these oscillatory terms were, as expected, pure artifacts of our integral splitting, and we thus recover our claim (A.15).

A.6 Compendium of useful formulas in d dimensions

In this section, we will skim over some formulas that are useful when adapting the three-dimensional PN-MPM construction to d dimensions. These can all be found in [102, 104, 269, 130, 162, 194, 270, 124], and we encourage the interested reader to refer to these references.

To start with, the Einstein equations in $d + 1$ dimensions [102] are identical to their three-dimensional form, but with a coupling constant $G = G_N \ell_0^{d-3}$, where G_N is the standard 3-dimensional Newton's constant and ℓ_0 is an arbitrary length scale associated to dimensional regularization. This arbitrary scale must not appear in final results in the 3D limit. We then recover the Landau-Lifschitz formulation (II.5), but where the source term is now given by

$$\begin{aligned} \Lambda^{\mu\nu} \equiv & -h^{\alpha\beta} \partial_{\alpha\beta} h^{\mu\nu} + \partial_{\alpha} h^{\mu\beta} \partial_{\beta} h^{\nu\alpha} + \frac{1}{2} g^{\mu\nu} g_{\alpha\beta} \partial_{\lambda} h^{\alpha\tau} \partial_{\tau} h^{\beta\lambda} \\ & - g^{\mu\alpha} g_{\beta\tau} \partial_{\lambda} h^{\nu\tau} \partial_{\alpha} h^{\beta\lambda} - g^{\nu\alpha} g_{\beta\tau} \partial_{\lambda} h^{\mu\tau} \partial_{\alpha} h^{\beta\lambda} + g_{\alpha\beta} g^{\lambda\tau} \partial_{\lambda} h^{\mu\alpha} \partial_{\tau} h^{\nu\beta} \\ & + \frac{1}{4} (2g^{\mu\alpha} g^{\nu\beta} - g^{\mu\nu} g^{\alpha\beta}) (g_{\lambda\tau} g_{\rho\sigma} - \frac{1}{d-1} g_{\tau\rho} g_{\lambda\sigma}) \partial_{\alpha} h^{\lambda\sigma} \partial_{\beta} h^{\tau\rho} \\ & + 2h^{\rho(\mu} \partial_{\rho} H^{\nu)} - \partial_{\rho} (h^{\mu\nu} H^{\rho}). \end{aligned} \quad (\text{A.32})$$

Many technical details of PN-MPM computations are subtly modified in d dimensions, as presented nicely in [102]. Some highlights are that the volume of a unit sphere with a $d - 1$ dimensional surface is

$$\Omega_{d-1} = \frac{2\pi^{d/2}}{\Gamma(d/2)} \quad (\text{A.33})$$

where $\Gamma(z) = \int_0^{+\infty} dt t^{z-1} e^{-t}$ is the usual gamma function, and the infinitesimal volume element in spherical coordinates $(r, \theta_1, \dots, \theta_{d-1})$ reads

$$d^d \mathbf{x} = r^{d-1} dr d\Omega_{d-1} = r^{d-1} (\sin \theta_1) (\sin \theta_2)^2 \dots (\sin \theta_{d-1})^{d-2} dr d\theta_1 \dots d\theta_{d-1} \quad (\text{A.34})$$

Any operation involving a trace is also affected, e.g. the expression of the STF quantity \hat{n}_L is given, for $L = i_1 \dots i_{\ell}$, by

$$\hat{n}_L = \sum_{k=0}^{[\ell/2]} \frac{(-1)^k \ell! (2k-1)!! \Gamma\left(\frac{d}{2} + \ell - l - 1\right)}{2^k (2k)! (\ell - 2k)! \Gamma\left(\frac{d}{2} + \ell - 1\right)} \delta_{(i_1 i_2} \delta_{i_2 k-1 i_2 k} n_{i_2 k+1 \dots i_{\ell})} \quad (\text{A.35})$$

The formulas describing how derivative operators apply on functions of the coordinates are also modified, e.g.

$$\Delta \left[\hat{n}_L r^{\lambda} \right] = (\lambda - \ell) (\lambda + \ell + d - 2) \hat{n}_L r^{\lambda-2} \quad (\text{A.36})$$

Defining the d -dimensional Dirac distribution $\delta^{(d)}(\mathbf{x})$ à la Schwartz, we find that the Green's function $u(\mathbf{x})$ of the d -dimensional Laplace operator, defined by

$$\Delta u(\mathbf{x}) = -4\pi \delta^{(d)}(\mathbf{x}), \quad (\text{A.37})$$

reads

$$u(\mathbf{x}) = \tilde{k} r^{2-d}, \quad (\text{A.38})$$

where we define

$$\tilde{k} \equiv \frac{\Gamma\left(\frac{d-2}{2}\right)}{\pi^{\frac{d-2}{2}}} = \frac{4\pi}{(d-2)\Omega_{d-1}} \quad (\text{A.39})$$

such that $\lim_{d \rightarrow 3} \tilde{k} = 1$. The retarded Green's function G_{ret} of the d -dimensional d'Alembert operator is less straightforward. It satisfies

$$\square G_{ret}(t, \mathbf{x}) = \delta(t) \delta^{(d)}(\mathbf{x}) \quad (\text{A.40})$$

and is given by

$$G_{ret}(t, \mathbf{x}) = -\frac{\tilde{k}}{4\pi} \frac{\theta(t - r/c)}{r^{d-1}} \gamma_{\frac{1-d}{2}}\left(\frac{ct}{r}\right), \quad (\text{A.41})$$

where θ is the Heaviside function and where we define, for any $s \in \mathbb{C}$ and $|z| \geq 1$,

$$\gamma_s(z) \equiv \frac{2\pi}{\Gamma(s+1)\Gamma\left(-s - \frac{1}{2}\right)} (z^2 - 1)^s. \quad (\text{A.42})$$

For $s \in -\mathbb{N}^*$, this latter function should actually be seen as a distribution, in which case we for example find $\gamma_{-1}(z) = \delta(z - 1)$. Thanks to this retarded Green's function, and assuming for now that the source $S(t, \mathbf{x})$ has good integrability properties, we can generally solve the wave equation

$$\square \phi(t, \mathbf{x}) = S(t, \mathbf{x}) \quad (\text{A.43})$$

and find a particular retarded solution

$$\begin{aligned} \phi_{\text{part}}(t, \mathbf{x}) &= \int_{-\infty}^{+\infty} dt' \int d^d \mathbf{x}' G_{ret}(t - t', \mathbf{x} - \mathbf{x}') S(t', \mathbf{x}'') \\ &= -\frac{\tilde{k}}{4\pi} \int_1^{+\infty} dz \gamma_{\frac{1-d}{2}}(z) \int d^d \mathbf{x}' \frac{S(t - z|\mathbf{x} - \mathbf{x}'|, \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^{d-2}} \end{aligned} \quad (\text{A.44})$$

which reduces to the three-dimensional solution when $d \rightarrow 0$, thanks to the distributional character of γ_s . Although dimensional dimensional cures many convergence problems that can occur in three dimensions, a few difficulties remain. Therefore, the previous solution is more generally given by the regularized expression

$$\phi_{\text{part}}(t, \mathbf{x}) = \text{FP}_{B=0} \left[-\frac{\tilde{k}}{4\pi} \int_1^{+\infty} dz \gamma_{\frac{1-d}{2}}(z) \int d^d \mathbf{x}' \left(\frac{r'}{r_0}\right)^B \frac{S(t - z|\mathbf{x} - \mathbf{x}'|, \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^{d-2}} \right] \quad (\text{A.45})$$

where $\text{FP}_{B=0}$ is defined just like in three dimensions. On top of finding a particular solution to the sourced wave equation, it is also necessary to study the homogeneous part of the wave equation in d dimensions, namely the solution of

$$\square \phi_{\text{hom}}(t, \mathbf{x}) = 0. \quad (\text{A.46})$$

The most general homogeneous ‘‘multipolar’’ retarded solution, under the usual no-incoming radiation and asymptotic flatness assumptions, reads [130]

$$\phi_{\text{hom}} = \sum_{\ell=0}^{\infty} \hat{\partial}_L \tilde{F}_L \quad (\text{A.47})$$

where the multipolar moment \tilde{F}_L is related to the arbitrary function of time $F_L(t)$ via the definition

$$\begin{aligned}\tilde{F}_L(t, \mathbf{x}) &= -4\pi \int_{-\infty}^{+\infty} dt' G_{\text{ret}}(t - t', \mathbf{x}) F_L(t') \\ &= \frac{\tilde{k}}{r^{d-2}} \int_1^{+\infty} dy \gamma_{\frac{1-d}{2}}(y) F_L\left(t' - \frac{ry}{c}\right)\end{aligned}\quad (\text{A.48})$$

Note that $\lim_{d \rightarrow 3} \tilde{F}_L(t, \mathbf{x}) = F_L(t - r/c)$, so we retrieve the three-dimensional multipolar expansion in the three-dimensional limit. Although this expression is exact, it is often unwieldy, and it will be very useful to express its near-zone expansion when $r \rightarrow 0$, which reads [130]

$$\tilde{F} \underset{r \rightarrow 0}{=} \tilde{F}_{\text{even}} + \tilde{F}_{\text{odd}}, \quad (\text{A.49a})$$

where

$$\tilde{F}_{\text{even}} = \frac{r^{-1-\varepsilon}}{\pi^{\frac{1+\varepsilon}{2}}} \sum_{j=0}^{\infty} \frac{(-1)^j}{2^{2j} j!} \Gamma\left(\frac{1+\varepsilon}{2} - j\right) \left(\frac{r}{c}\right)^{2j} F^{(2j)}(t), \quad (\text{A.49b})$$

$$\tilde{F}_{\text{odd}} = -\frac{1}{2\pi^{\frac{\varepsilon}{2}} c^{1+\varepsilon}} \frac{\Gamma\left(\frac{1+\varepsilon}{2}\right)}{\Gamma\left(1 - \frac{\varepsilon}{2}\right)} \sum_{j=0}^{\infty} \frac{1}{2^{2j} j! \Gamma\left(j + \frac{3+\varepsilon}{2}\right)} \left(\frac{r}{c}\right)^{2j} \int_0^{+\infty} d\tau \tau^{-\varepsilon} F^{(2j+2)}(t - \tau). \quad (\text{A.49c})$$

A.7 Proof of convergence when $k = 1$ and $k = 2$

In this Appendix, we assume that $k \in \{1, 2\}$. In order to analyze the convergence of the general solution (IV.46) for arbitrary smooth functions F and $G \in \mathcal{C}^\infty(\mathbb{R})$ which vanish identically in the remote past, for $t \leq -\mathcal{T}$, we must first do some manipulation so as to transfer the boundary conditions of F and G into the bounds of the integrals. Thanks to the regularization factor λ^B , we can always manipulate these integrals safely, since they are defined by analytic continuation for any $B \in \mathbb{C}$ except at some integer values including the value of interest $B = 0$, at which we apply the finite part in the end.

a) Case $k + j \leq 1$

In this case, we can perform the change of variable $\lambda = \mu + \rho/2$ on (IV.46b) and expand using the binomial formula:

$$\begin{aligned}\phi_{ij}^B(u, r) &= \sum_{p=0}^{-k-j+1} \binom{-k-j+1}{p} \int_0^{+\infty} d\rho \rho^{i+j} \left(\frac{\rho}{2}\right)^{-k-j+1-p} G(u - \rho) \int_1^{+\infty} dx Q_m(x) \\ &\quad \times \int_0^r d\mu \left(\frac{\mu + \rho/2}{r_0}\right)^B \mu^p F\left(u - \frac{\rho(x+1)}{2} - \mu(x-1)\right).\end{aligned}\quad (\text{A.50})$$

We then set $\tau = \mu(x-1)$ with $d\tau = (x-1)d\mu$, invert the integrals, and find

$$\begin{aligned} \phi_{ij}^B(u, r) = & \sum_{p=0}^{-k-j+1} 2^{-k+j-1+p} \binom{-k-j+1}{p} \int_0^{+\infty} d\tau \tau^p \int_{1+\tau/r}^{+\infty} dx \frac{Q_m(x)}{(x-1)^{p+1}} \left(\frac{\tau}{x-1} + \rho/2 \right)^B \\ & \times \int_0^{+\infty} d\rho \rho^{-k+i+1-p} G(u-\rho) F\left(u - \frac{\rho(x+1)}{2} - \tau\right). \end{aligned} \quad (\text{A.51})$$

Next we introduce

$$\Upsilon_{k,i,j,p}(\tau) \equiv \tau^p \int_{1+\tau/r}^{+\infty} dx \frac{Q_m(x)}{(x-1)^{p+1}} \int_0^{+\infty} d\rho \rho^{-k+i+1-p} G(u-\rho) F\left(u - \frac{\rho(x+1)}{2} - \tau\right), \quad (\text{A.52})$$

which is well defined, since $-k+i+1-p \geq i+j \geq 0$ ensures that the integral in ρ converges at 0, and the behavior of $Q_m(x)/(x-1)^{p+1} \sim x^{-m-p-2}$ when $x \rightarrow +\infty$ ensures that the integral in x converges at infinity.

Since F vanishes identically in the remote past, so does $\Upsilon_{k,i,j,p}(\tau)$, thus it is clearly integrable at $\tau \rightarrow +\infty$. We then bound this quantity with

$$|\Upsilon_{k,i,j,p}(\tau)| \leq \tau^p \int_{1+\tau/r}^{+\infty} dx \frac{Q_m(x)}{(x-1)^{p+1}} = \mathcal{O}_{\tau \rightarrow 0}(\ln^2 \tau), \quad (\text{A.53})$$

which proves integrability at the bound $\tau \rightarrow 0$.

b) Case $k+j \geq 2$

We first shuffle the order of the integrals in (IV.46b) and obtain

$$\begin{aligned} \phi_{ij}^B = & \int_1^{+\infty} dx Q_m(x) \int_0^{+\infty} d\lambda \left(\frac{\lambda}{r_0}\right)^B \lambda^{-k-j+1} \int_0^{2\lambda} d\rho \rho^{i+j} G(u-\rho) F[u-\rho-\lambda(x-1)] \\ & - \int_1^{+\infty} dx Q_m(x) \int_r^{+\infty} d\lambda \left(\frac{\lambda}{r_0}\right)^B \lambda^{-k-j+1} \int_0^{2(\lambda-r)} d\rho \rho^{i+j} G(u-\rho) F[u-\rho-\lambda(x-1)]. \end{aligned} \quad (\text{A.54})$$

Taking advantage of the conditions on F and G , we can now restrict the bounds of the integrals to $\lambda \leq \frac{u+\mathcal{T}}{x-1}$. This yields

$$\begin{aligned} \phi_{ij}^B = & \int_1^{+\infty} dx Q_m(x) \int_0^{\frac{u+\mathcal{T}}{x-1}} d\lambda \left(\frac{\lambda}{r_0}\right)^B \lambda^{-k-j+1} \int_0^{\min(u+\mathcal{T}, 2\lambda)} d\rho \rho^{i+j} G(u-\rho) F[u-\rho-\lambda(x-1)] \\ & - \int_1^{\frac{u+\mathcal{T}}{r}} dx Q_m(x) \int_r^{\frac{u+\mathcal{T}}{x-1}} d\lambda \left(\frac{\lambda}{r_0}\right)^B \lambda^{-k-j+1} \int_0^{\min(u+\mathcal{T}, 2(\lambda-r))} d\rho \rho^{i+j} G(u-\rho) F[u-\rho-\lambda(x-1)]. \end{aligned} \quad (\text{A.55})$$

We introduce

$$\Xi_{k,i,j}(x) \equiv \int_0^{\frac{u+\mathcal{T}}{x-1}} d\lambda \lambda^{-k-j+1} \int_0^{\min(u+\mathcal{T}, 2\lambda)} d\rho \rho^{i+j} G(u-\rho) F[u-\rho-\lambda(x-1)]. \quad (\text{A.56})$$

Let us bound this quantity for any $x \in]1, +\infty]$:

$$\begin{aligned}
|\Xi_{k,i,j}(x)| &\leq \int_0^{\frac{u+\mathcal{T}}{x-1}} d\lambda \lambda^{-k-j+1} \int_0^{\min(u+\mathcal{T}, 2\lambda)} d\rho \rho^{i+j} |G(u-\rho)F(u-\rho-\lambda(x-1))| \\
&\leq \mathcal{K} \int_0^{\frac{u+\mathcal{T}}{x-1}} d\lambda \lambda^{-k-j+1} \int_0^{\min(u+\mathcal{T}, 2\lambda)} d\rho \rho^{i+j} \\
&= \frac{\mathcal{K}}{i+j+1} \int_0^{\frac{u+\mathcal{T}}{x-1}} d\lambda \lambda^{-k-j+1} [\min(u+\mathcal{T}, 2\lambda)]^{i+j+1} \\
&= \frac{\mathcal{K}}{i+j+1} \left\{ \int_0^{\frac{u+\mathcal{T}}{2}} d\lambda \lambda^{-k-j+1} (2\lambda)^{i+j+1} + \theta(3-x) \int_{\frac{u+\mathcal{T}}{2}}^{\frac{u+\mathcal{T}}{x-1}} d\lambda \lambda^{-k-j+1} (u+\mathcal{T})^{i+j+1} \right\} \\
&= \frac{\mathcal{K}}{i+j+1} \left\{ \frac{2^{j+k-2}}{i+(3-k)} (u+\mathcal{T})^{i+(3-k)} + \theta(3-x)(u+\mathcal{T})^{i+j+1} \int_{\frac{u+\mathcal{T}}{2}}^{\frac{u+\mathcal{T}}{x-1}} d\lambda \lambda^{-k-j+1} \right\},
\end{aligned} \tag{A.57}$$

where

$$\mathcal{K} \equiv \sup_{(\tau, \tau') \in [-\mathcal{T}, u]^2} [F(\tau)G(\tau')]. \tag{A.58}$$

We now distinguish the case $j = 2 - k$, which yields

$$|\Xi_{2,i,0}(x)| \leq \frac{\mathcal{K}}{i+1} \left\{ \frac{1}{i+1} (u+\mathcal{T})^{i+1} + \theta(3-x)(u+\mathcal{T})^{i+1} \ln \left(\frac{2}{x-1} \right) \right\}, \tag{A.59}$$

and the more general case $k+j \geq 2$, in which we find

$$|\Xi_{k,i,j}(x)| \leq \frac{\mathcal{K}(u+\mathcal{T})^{i+(3-k)}}{i+j+1} \left\{ \frac{2^{j+k-2}}{i+(3-k)} + \frac{\theta(3-x)}{j+(k-2)} (2^{j+(k-2)} - (x-1)^{k+j-2}) \right\}. \tag{A.60}$$

Since $Q_m(x) = \mathcal{O}_{x \rightarrow 1^+}(\ln(x-1))$ and $Q_m(x) = \mathcal{O}_{x \rightarrow +\infty}(x^{-m-1})$, we can now look at the asymptotic behavior of our integrand. In the case $(k, j) = (2, 0)$, we find

$$Q_m(x) \Xi_{2,i,0}(x) = \begin{cases} \mathcal{O}_{x \rightarrow 1^+}(\ln(x-1)^2), \\ \mathcal{O}_{x \rightarrow +\infty}(x^{-m-1}). \end{cases} \tag{A.61}$$

In the general case $k+j \geq 2$, we instead have

$$Q_m(x) \Xi_{k,i,j}(x) = \begin{cases} \mathcal{O}_{x \rightarrow 1^+}(\ln(x-1)(x-1)^{j+(k-2)}), \\ \mathcal{O}_{x \rightarrow +\infty}(x^{-m-1}). \end{cases} \tag{A.62}$$

It is now clear that $Q_m(x) \Xi_{k,i,j}(x)$ is integrable at both bounds $x \rightarrow 1^+$ and $x \rightarrow +\infty$ as soon as $k \in \{1, 2\}$. After doing a similar (and much easier) analysis on the second member of (A.55), we find that (A.55)–(A.54), and consequently (IV.46b), have a convergent limit as $B \rightarrow 0$, *i.e.*, they do not develop poles in $1/B$. Repeating the analysis of this section with an extra factor $\ln(r/r_0)$ in the integrand shows that we can also safely compute the $1/B$ coefficient in the Laurent series.

Appendix B

Scalar-tensor theories

B.1 Dissipative PN-odd terms in the equations of motion

The conservative equations of motion of compact binaries in ST theory have been obtained up to 3PN order in [248, 249]. The PN-even terms \mathbf{a}_A^{N} , $\mathbf{a}_A^{1\text{PN}}$, $\mathbf{a}_A^{2\text{PN}}$ and $\mathbf{a}_A^{3\text{PN}}$ in the acceleration for arbitrary orbits in a general frame are given by (5.10)–(5.12) of [248]. The dissipative PN-odd terms $\mathbf{a}_A^{1,5\text{PN}}$ and $\mathbf{a}_A^{2,5\text{PN}}$ can be computed based on their expressions in terms of the EW moments as given by [245]. We find

$$\begin{aligned}
\mathbf{a}_1^{1.5\text{PN}} &= \frac{2\zeta\mathcal{S}_-}{3c^3r_{12}}(\mathcal{S}_+ + \mathcal{S}_-)\left(\frac{\tilde{G}\alpha m_1}{r_{12}}\right)\left(\frac{\tilde{G}\alpha m_2}{r_{12}}\right)\left[3(n_{12}v_{12})\mathbf{n}_{12} - \mathbf{v}_{12}\right], \quad (\text{B.1a}) \\
\mathbf{a}_1^{2.5\text{PN}} &= \frac{1}{60c^5r_{12}}\left(\frac{\tilde{G}\alpha m_1}{r_{12}}\right)\left(\frac{\tilde{G}\alpha m_2}{r_{12}}\right)\times \\
&\quad \left(\mathbf{n}_{12}\left\{\left(\frac{\tilde{G}\alpha m_1}{r_{12}}\right)(n_{12}v_1)(-288 + 80\bar{\beta}_- - 80\bar{\beta}_+ - 80\bar{\gamma} - 800\zeta\mathcal{S}_-^2 + 320\bar{\beta}_-\zeta\mathcal{S}_-^2 - 320\bar{\beta}_+\zeta\mathcal{S}_-^2 - 320\zeta\bar{\gamma}\mathcal{S}_-^2\right.\right. \\
&\quad \left.\left.- 800\zeta\mathcal{S}_-\mathcal{S}_+ + 320\bar{\beta}_-\zeta\mathcal{S}_-\mathcal{S}_+ - 320\bar{\beta}_+\zeta\mathcal{S}_-\mathcal{S}_+ - 320\zeta\bar{\gamma}\mathcal{S}_-\mathcal{S}_+\right)\right. \\
&\quad \left.+\left(\frac{\tilde{G}\alpha m_2}{r_{12}}\right)(n_{12}v_1)(832 + 80\bar{\beta}_- - 1440\bar{\beta}_+ + 640\bar{\gamma} + 1920\bar{\beta}_+^2\bar{\gamma}^{-1} - 1900\zeta\mathcal{S}_-^2 - 320\bar{\beta}_-\zeta\mathcal{S}_-^2 - 320\bar{\beta}_+\zeta\mathcal{S}_-^2\right. \\
&\quad \left.- 800\zeta\bar{\gamma}\mathcal{S}_-^2 + 960\bar{\beta}_-\zeta\bar{\gamma}^{-1}\mathcal{S}_-^2 + 2880\bar{\beta}_+\zeta\bar{\gamma}^{-1}\mathcal{S}_-^2 - 3840\bar{\beta}_-^2\zeta\bar{\gamma}^{-2}\mathcal{S}_-^2 - 3840\bar{\beta}_+^2\zeta\bar{\gamma}^{-2}\mathcal{S}_-^2\right. \\
&\quad \left.- 740\zeta\mathcal{S}_-\mathcal{S}_+ - 320\bar{\beta}_-\zeta\mathcal{S}_-\mathcal{S}_+ - 320\bar{\beta}_+\zeta\mathcal{S}_-\mathcal{S}_+ - 320\zeta\bar{\gamma}\mathcal{S}_-\mathcal{S}_+ + 2880\bar{\beta}_-\zeta\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+ \right. \\
&\quad \left.+ 960\bar{\beta}_+\zeta\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+ - 7680\bar{\beta}_-\bar{\beta}_+\zeta\bar{\gamma}^{-2}\mathcal{S}_-\mathcal{S}_+\right) \\
&\quad + (n_{12}v_1)^3(600\bar{\beta}_- - 600\bar{\beta}_+ + 300\bar{\gamma}) \\
&\quad + (n_{12}v_1)v_1^2(144 - 360\bar{\beta}_- + 360\bar{\beta}_+ - 120\bar{\gamma} + 180\zeta\mathcal{S}_-^2 + 120\zeta\bar{\gamma}\mathcal{S}_-^2 + 180\zeta\mathcal{S}_-\mathcal{S}_+ + 120\zeta\bar{\gamma}\mathcal{S}_-\mathcal{S}_+) \\
&\quad + \left(\frac{\tilde{G}\alpha m_1}{r_{12}}\right)(n_{12}v_2)(288 - 80\bar{\beta}_- + 80\bar{\beta}_+ + 80\bar{\gamma} + 880\zeta\mathcal{S}_-^2 - 320\bar{\beta}_-\zeta\mathcal{S}_-^2 + 320\bar{\beta}_+\zeta\mathcal{S}_-^2 + 320\zeta\bar{\gamma}\mathcal{S}_-^2 \\
&\quad \left.+ 880\zeta\mathcal{S}_-\mathcal{S}_+ - 320\bar{\beta}_-\zeta\mathcal{S}_-\mathcal{S}_+ + 320\bar{\beta}_+\zeta\mathcal{S}_-\mathcal{S}_+ + 320\zeta\bar{\gamma}\mathcal{S}_-\mathcal{S}_+\right) \\
&\quad + \left(\frac{\tilde{G}\alpha m_2}{r_{12}}\right)(n_{12}v_2)(-832 - 80\bar{\beta}_- + 1440\bar{\beta}_+ - 640\bar{\gamma} - 1920\bar{\beta}_+^2\bar{\gamma}^{-1} + 1900\zeta\mathcal{S}_-^2 + 320\bar{\beta}_-\zeta\mathcal{S}_-^2 + 320\bar{\beta}_+\zeta\mathcal{S}_-^2 \\
&\quad \left.+ 800\zeta\bar{\gamma}\mathcal{S}_-^2 - 960\bar{\beta}_-\zeta\bar{\gamma}^{-1}\mathcal{S}_-^2 - 2880\bar{\beta}_+\zeta\bar{\gamma}^{-1}\mathcal{S}_-^2 + 3840\bar{\beta}_-^2\zeta\bar{\gamma}^{-2}\mathcal{S}_-^2 + 3840\bar{\beta}_+^2\zeta\bar{\gamma}^{-2}\mathcal{S}_-^2\right. \\
&\quad \left.+ 740\zeta\mathcal{S}_-\mathcal{S}_+ + 320\bar{\beta}_-\zeta\mathcal{S}_-\mathcal{S}_+ + 320\bar{\beta}_+\zeta\mathcal{S}_-\mathcal{S}_+ + 320\zeta\bar{\gamma}\mathcal{S}_-\mathcal{S}_+ - 2880\bar{\beta}_-\zeta\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+ \right. \\
&\quad \left.- 960\bar{\beta}_+\zeta\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+ + 7680\bar{\beta}_-\bar{\beta}_+\zeta\bar{\gamma}^{-2}\mathcal{S}_-\mathcal{S}_+\right) \\
&\quad + (n_{12}v_1)^2(n_{12}v_2)(-1800\bar{\beta}_- + 1800\bar{\beta}_+ - 900\bar{\gamma}) \\
&\quad + v_1^2(n_{12}v_2)(-144 + 360\bar{\beta}_- - 360\bar{\beta}_+ + 120\bar{\gamma} - 180\zeta\mathcal{S}_-^2 - 120\zeta\bar{\gamma}\mathcal{S}_-^2 - 180\zeta\mathcal{S}_-\mathcal{S}_+ - 120\zeta\bar{\gamma}\mathcal{S}_-\mathcal{S}_+) \\
&\quad + (n_{12}v_1)(n_{12}v_2)^2(1800\bar{\beta}_- - 1800\bar{\beta}_+ + 900\bar{\gamma} - 300\zeta\mathcal{S}_-^2 - 300\zeta\mathcal{S}_-\mathcal{S}_+) \\
&\quad + (n_{12}v_2)^3(-600\bar{\beta}_- + 600\bar{\beta}_+ - 300\bar{\gamma} + 300\zeta\mathcal{S}_-^2 + 300\zeta\mathcal{S}_-\mathcal{S}_+) \\
&\quad + (n_{12}v_1)(v_1v_2)(-288 + 720\bar{\beta}_- - 720\bar{\beta}_+ + 240\bar{\gamma} - 480\zeta\mathcal{S}_-^2 - 240\zeta\bar{\gamma}\mathcal{S}_-^2 - 480\zeta\mathcal{S}_-\mathcal{S}_+ - 240\zeta\bar{\gamma}\mathcal{S}_-\mathcal{S}_+) \\
&\quad + (n_{12}v_2)(v_1v_2)(288 - 720\bar{\beta}_- + 720\bar{\beta}_+ - 240\bar{\gamma} + 600\zeta\mathcal{S}_-^2 + 240\zeta\bar{\gamma}\mathcal{S}_-^2 + 600\zeta\mathcal{S}_-\mathcal{S}_+ + 240\zeta\bar{\gamma}\mathcal{S}_-\mathcal{S}_+) \\
&\quad + (n_{12}v_1)v_2^2(144 - 360\bar{\beta}_- + 360\bar{\beta}_+ - 120\bar{\gamma} + 240\zeta\mathcal{S}_-^2 + 120\zeta\bar{\gamma}\mathcal{S}_-^2 + 240\zeta\mathcal{S}_-\mathcal{S}_+ + 120\zeta\bar{\gamma}\mathcal{S}_-\mathcal{S}_+) \\
&\quad \left. + (n_{12}v_2)v_2^2(-144 + 360\bar{\beta}_- - 360\bar{\beta}_+ + 120\bar{\gamma} - 360\zeta\mathcal{S}_-^2 - 120\zeta\bar{\gamma}\mathcal{S}_-^2 - 360\zeta\mathcal{S}_-\mathcal{S}_+ - 120\zeta\bar{\gamma}\mathcal{S}_-\mathcal{S}_+)\right\} \\
&+ \mathbf{v}_{12}\left\{\left(\frac{\tilde{G}\alpha m_1}{r_{12}}\right)(96 - 80\bar{\beta}_- + 80\bar{\beta}_+ + 80\zeta\mathcal{S}_-^2 - 80\bar{\beta}_-\zeta\mathcal{S}_-^2\right. \\
&\quad \left.+ 80\bar{\beta}_+\zeta\mathcal{S}_-^2 + 80\zeta\mathcal{S}_-\mathcal{S}_+ - 80\bar{\beta}_-\zeta\mathcal{S}_-\mathcal{S}_+ + 80\bar{\beta}_+\zeta\mathcal{S}_-\mathcal{S}_+)\right. \\
&\quad \left.+\left(\frac{\tilde{G}\alpha m_2}{r_{12}}\right)(-384 - 80\bar{\beta}_- + 160\bar{\beta}_+ - 240\bar{\gamma} + 340\zeta\mathcal{S}_-^2 + 80\bar{\beta}_-\zeta\mathcal{S}_-^2 + 80\bar{\beta}_+\zeta\mathcal{S}_-^2 + 160\zeta\bar{\gamma}\mathcal{S}_-^2\right. \\
&\quad \left.- 320\bar{\beta}_-\zeta\bar{\gamma}^{-1}\mathcal{S}_-^2 - 320\bar{\beta}_+\zeta\bar{\gamma}^{-1}\mathcal{S}_-^2 + 60\zeta\mathcal{S}_-\mathcal{S}_+ + 80\bar{\beta}_-\zeta\mathcal{S}_-\mathcal{S}_+ + 80\bar{\beta}_+\zeta\mathcal{S}_-\mathcal{S}_+ \right. \\
&\quad \left.- 320\bar{\beta}_-\zeta\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+ - 320\bar{\beta}_+\zeta\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+)\right. \\
&\quad \left.+(n_{12}v_1)^2(-360\bar{\beta}_- + 360\bar{\beta}_+ - 180\bar{\gamma} - 480\zeta\mathcal{S}_-^2 - 240\zeta\bar{\gamma}\mathcal{S}_-^2 - 480\zeta\mathcal{S}_-\mathcal{S}_+ - 240\zeta\bar{\gamma}\mathcal{S}_-\mathcal{S}_+)\right. \\
&\quad \left.+ v_1^2(-48 + 120\bar{\beta}_- - 120\bar{\beta}_+ + 40\bar{\gamma} + 100\zeta\mathcal{S}_-^2 + 40\zeta\bar{\gamma}\mathcal{S}_-^2 + 100\zeta\mathcal{S}_-\mathcal{S}_+ + 40\zeta\bar{\gamma}\mathcal{S}_-\mathcal{S}_+)\right. \\
&\quad \left.+(n_{12}v_1)(n_{12}v_2)(720\bar{\beta}_- - 720\bar{\beta}_+ + 360\bar{\gamma} + 840\zeta\mathcal{S}_-^2 + 480\zeta\bar{\gamma}\mathcal{S}_-^2 + 840\zeta\mathcal{S}_-\mathcal{S}_+ + 480\zeta\bar{\gamma}\mathcal{S}_-\mathcal{S}_+)\right. \\
&\quad \left.+(n_{12}v_2)^2(-360\bar{\beta}_- + 360\bar{\beta}_+ - 180\bar{\gamma} - 300\zeta\mathcal{S}_-^2 - 240\zeta\bar{\gamma}\mathcal{S}_-^2 - 300\zeta\mathcal{S}_-\mathcal{S}_+ - 240\zeta\bar{\gamma}\mathcal{S}_-\mathcal{S}_+)\right. \\
&\quad \left.+(v_1v_2)(96 - 240\bar{\beta}_- + 240\bar{\beta}_+ - 80\bar{\gamma} - 120\zeta\mathcal{S}_-^2 - 80\zeta\bar{\gamma}\mathcal{S}_-^2 - 120\zeta\mathcal{S}_-\mathcal{S}_+ - 80\zeta\bar{\gamma}\mathcal{S}_-\mathcal{S}_+)\right. \\
&\quad \left.+\left.v_2^2(-48 + 120\bar{\beta}_- - 120\bar{\beta}_+ + 40\bar{\gamma} + 40\zeta\mathcal{S}_-^2 + 40\zeta\bar{\gamma}\mathcal{S}_-^2 + 40\zeta\mathcal{S}_-\mathcal{S}_+ + 40\zeta\bar{\gamma}\mathcal{S}_-\mathcal{S}_+)\right\}\right)
\end{aligned}$$

whereas $\mathbf{a}_2^{1.5\text{PN}}$ and $\mathbf{a}_2^{2.5\text{PN}}$ are given by 1 \leftrightarrow 2. For the relative acceleration in the CM

frame, the even terms of the equations of motion were computed in [248], and the odd terms are given explicitly in [245].

The expressions of the individual positions \mathbf{y}_A and velocities \mathbf{v}_A in the CM frame (i.e. as functions of the CM frame quantities \mathbf{n} and \mathbf{v}) are only required to 2PN order to compute the flux. The even Newtonian, 1PN and 2PN terms are given in [249]; the only required odd terms are of order 1.5PN: $\mathbf{v}_A^{1.5\text{PN}}$ is given in [245], but $\mathbf{y}_A^{1.5\text{PN}}$ is not. We find

$$\mathbf{y}_1^{1.5\text{PN}} = -\frac{4(-1 + \zeta)\zeta\tilde{G}m\mathcal{S}_-(\mathcal{S}_+ + \mathcal{S}_-\delta)\nu}{3c^3(2 + \bar{\gamma})}\mathbf{v}, \quad (\text{B.2a})$$

$$\mathbf{v}_1^{1.5\text{PN}} = -\frac{8(-1 + \zeta)^2\zeta\tilde{G}^2m^2\mathcal{S}_-(\mathcal{S}_+ + \mathcal{S}_-\delta)\nu}{3c^3(2 + \bar{\gamma})^2r^2}\mathbf{n}. \quad (\text{B.2b})$$

B.2 Expressions for the scalar and tensor source moments

B.2.1 The STF moments

For the flux and waveform computations, the STF moments were first expressed in a general frame, then computed in the center of mass frame where their expression are much simpler. However, there is a notable exception for I_i^s , which could not be resolved in the center of mass frame up to 2.5PN order. Indeed, this would require an expression for \mathbf{y}_1 as a function of center-of-mass binary system vectors \mathbf{n} and \mathbf{v} up to 2.5PN order. Such an expression would require computing the flux $\mathcal{F}_{\mathbf{P}}$ of linear momentum \mathbf{P} and the flux $\mathcal{F}_{\mathbf{G}}$ corresponding to the center-of-mass position \mathbf{G} , due to the coupling between the dipole and quadrupole moments. However, for the flux, only the second time derivative of I_i^s was required, whose Newtonian part is proportional to the relative acceleration $\mathbf{a} \equiv \mathbf{a}_{12}$, which is already expressed in the center of mass frame; the 1PN and higher order terms then only require knowing the expression for \mathbf{y}_1 to 1.5PN order, so the second time derivative of I_i^s can be fully expressed in the center of mass frame (without the knowledge of \mathbf{P} or \mathbf{G}) using the expression given in [249]. As for the waveform, the first time derivative of I_i^s is only required to 2PN order, so there is no problem.

Owing to these remarks, we chose not to present the full 2.5PN expression of I_i^s in a general frame, though we have computed its lengthy expression explicitly. The relevant scalar moments in the center-of-mass frame are

$$\begin{aligned}
\Gamma^s = & -\frac{\alpha^{1/2}\zeta m\nu}{(1-\zeta)\phi_0} \left[\frac{\mathcal{S}_+ + \mathcal{S}_-\delta}{\nu} + \frac{1}{6c^2} \left\{ \left(\frac{\tilde{G}\alpha m}{r} \right) (48\bar{\beta}_-\bar{\gamma}^{-1}\mathcal{S}_- - 14\mathcal{S}_+ + 48\bar{\beta}_+\bar{\gamma}^{-1}\mathcal{S}_+ + 2\mathcal{S}_-\delta) + v^2(-\mathcal{S}_+ + \mathcal{S}_-\delta) \right\} \right. \\
& + \frac{1}{120c^4} \left\{ \left(\frac{\tilde{G}\alpha m}{r} \right)^2 \left(-40\bar{\beta}_-\mathcal{S}_- - 1280\bar{\beta}_-\bar{\gamma}^{-1}\mathcal{S}_- + 480\bar{\gamma}^{-1}\bar{\chi}_-\mathcal{S}_- - 52\mathcal{S}_+ + 440\bar{\beta}_+\mathcal{S}_+ - 160\bar{\gamma}\mathcal{S}_+ \right. \right. \\
& \quad - 1280\bar{\beta}_+\bar{\gamma}^{-1}\mathcal{S}_+ - 1920\bar{\beta}_-^2\bar{\gamma}^{-2}\mathcal{S}_+ + 1920\bar{\beta}_+^2\bar{\gamma}^{-2}\mathcal{S}_+ + 480\bar{\gamma}^{-1}\bar{\chi}_+\mathcal{S}_+ \\
& \quad + \nu[-320\bar{\beta}_-\mathcal{S}_- + 160\bar{\beta}_-\bar{\gamma}^{-1}\mathcal{S}_- + 16\mathcal{S}_+ + 160\bar{\beta}_+\bar{\gamma}^{-1}\mathcal{S}_+] + 48\mathcal{S}_-\nu\delta \\
& \quad + \delta[-68\mathcal{S}_- + 40\bar{\beta}_+\mathcal{S}_- - 80\bar{\gamma}\mathcal{S}_- - 160\bar{\beta}_+\bar{\gamma}^{-1}\mathcal{S}_- - 1920\bar{\beta}_-^2\bar{\gamma}^{-2}\mathcal{S}_- + 1920\bar{\beta}_+^2\bar{\gamma}^{-2}\mathcal{S}_- \\
& \quad \left. \left. - 480\bar{\gamma}^{-1}\bar{\chi}_+\mathcal{S}_- - 440\bar{\beta}_-\mathcal{S}_+ - 160\bar{\beta}_-\bar{\gamma}^{-1}\mathcal{S}_+ - 480\bar{\gamma}^{-1}\bar{\chi}_-\mathcal{S}_+ \right) \right. \\
& + \left(\frac{\tilde{G}\alpha m}{r} \right) (n\nu)^2 \left(-320\bar{\beta}_-\bar{\gamma}^{-1}\mathcal{S}_- - 240\mathcal{S}_+ - 160\bar{\gamma}\mathcal{S}_+ - 320\bar{\beta}_+\bar{\gamma}^{-1}\mathcal{S}_+ \right. \\
& \quad + \nu[640\bar{\beta}_-\bar{\gamma}^{-1}\mathcal{S}_- - 180\mathcal{S}_+ + 640\bar{\beta}_+\bar{\gamma}^{-1}\mathcal{S}_+] + 100\mathcal{S}_-\nu\delta \\
& \quad \left. + \delta[-120\mathcal{S}_- - 80\bar{\gamma}\mathcal{S}_- - 160\bar{\beta}_+\bar{\gamma}^{-1}\mathcal{S}_- - 160\bar{\beta}_-\bar{\gamma}^{-1}\mathcal{S}_+] \right) \\
& + \left(\frac{\tilde{G}\alpha m}{r} \right) v^2 \left(-160\bar{\beta}_-\bar{\gamma}^{-1}\mathcal{S}_- - 12\mathcal{S}_+ - 40\bar{\gamma}\mathcal{S}_+ - 160\bar{\beta}_+\bar{\gamma}^{-1}\mathcal{S}_+ \right. \\
& \quad + \delta[12\mathcal{S}_- + 40\bar{\gamma}\mathcal{S}_- + 160\bar{\beta}_+\bar{\gamma}^{-1}\mathcal{S}_- + 160\bar{\beta}_-\bar{\gamma}^{-1}\mathcal{S}_+] \\
& \quad \left. + \nu[320\bar{\beta}_-\bar{\gamma}^{-1}\mathcal{S}_- - 164\mathcal{S}_+ + 320\bar{\beta}_+\bar{\gamma}^{-1}\mathcal{S}_+ + 28\mathcal{S}_-\delta] \right) \\
& + v^4 \left(-11\mathcal{S}_+ + 11\mathcal{S}_-\delta + 33\mathcal{S}_+\nu - 51\mathcal{S}_-\nu\delta \right) \left. \right\} \\
& + \frac{1}{18c^5(1-\zeta)} \left(\frac{\tilde{G}\alpha m}{r} \right)^2 (n\nu) \left\{ 72\bar{\beta}_-\mathcal{S}_- - 72\bar{\beta}_-\zeta\mathcal{S}_- - 336\bar{\beta}_-\bar{\gamma}^{-1}\mathcal{S}_- + 336\bar{\beta}_-\zeta\bar{\gamma}^{-1}\mathcal{S}_- + 1728\bar{\beta}_-\bar{\beta}_+\bar{\gamma}^{-2}\mathcal{S}_- \right. \\
& \quad - 1728\bar{\beta}_-\bar{\beta}_+\zeta\bar{\gamma}^{-2}\mathcal{S}_- - 288\bar{\beta}_-\zeta\bar{\gamma}^{-1}\mathcal{S}_-^3 + 288\bar{\beta}_-\zeta^2\bar{\gamma}^{-1}\mathcal{S}_-^3 + 576\bar{\beta}_-\zeta\bar{\gamma}^{-2}\mathcal{S}_-^3 - 576\bar{\beta}_-\zeta^2\bar{\gamma}^{-2}\mathcal{S}_-^3 \\
& \quad - 4608\bar{\beta}_-\bar{\beta}_+\zeta\bar{\gamma}^{-3}\mathcal{S}_-^3 + 4608\bar{\beta}_-\bar{\beta}_+\zeta^2\bar{\gamma}^{-3}\mathcal{S}_-^3 + 72\bar{\beta}_+\mathcal{S}_+ - 72\bar{\beta}_+\zeta\mathcal{S}_+ - 24\bar{\gamma}\mathcal{S}_+ + 24\zeta\bar{\gamma}\mathcal{S}_+ \\
& \quad - 192\bar{\beta}_+\bar{\gamma}^{-1}\mathcal{S}_+ + 192\bar{\beta}_+\zeta\bar{\gamma}^{-1}\mathcal{S}_+ + 576\bar{\beta}_+^2\bar{\gamma}^{-2}\mathcal{S}_+ - 576\bar{\beta}_+^2\zeta\bar{\gamma}^{-2}\mathcal{S}_+ + 72\zeta\mathcal{S}_+^2\mathcal{S}_+ - 72\zeta^2\mathcal{S}_+^2\mathcal{S}_+ \\
& \quad - 288\bar{\beta}_+\zeta\bar{\gamma}^{-1}\mathcal{S}_+^2\mathcal{S}_+ + 288\bar{\beta}_+\zeta^2\bar{\gamma}^{-1}\mathcal{S}_+^2\mathcal{S}_+ + 576\bar{\beta}_+\zeta\bar{\gamma}^{-2}\mathcal{S}_+^2\mathcal{S}_+ - 576\bar{\beta}_+\zeta^2\bar{\gamma}^{-2}\mathcal{S}_+^2\mathcal{S}_+ \\
& \quad - 2304\bar{\beta}_-^2\zeta\bar{\gamma}^{-3}\mathcal{S}_+^2\mathcal{S}_+ - 2304\bar{\beta}_+^2\zeta\bar{\gamma}^{-3}\mathcal{S}_+^2\mathcal{S}_+ + 2304\bar{\beta}_-^2\zeta^2\bar{\gamma}^{-3}\mathcal{S}_+^2\mathcal{S}_+ + 2304\bar{\beta}_+^2\zeta^2\bar{\gamma}^{-3}\mathcal{S}_+^2\mathcal{S}_+ \\
& \quad + \delta[144\bar{\beta}_+\mathcal{S}_- - 144\bar{\beta}_+\zeta\mathcal{S}_- - 42\bar{\gamma}\mathcal{S}_- + 42\zeta\bar{\gamma}\mathcal{S}_- - 336\bar{\beta}_+\bar{\gamma}^{-1}\mathcal{S}_- + 336\bar{\beta}_+\zeta\bar{\gamma}^{-1}\mathcal{S}_- + 576\bar{\beta}_-^2\bar{\gamma}^{-2}\mathcal{S}_- \\
& \quad + 1152\bar{\beta}_+^2\bar{\gamma}^{-2}\mathcal{S}_- - 576\bar{\beta}_-^2\zeta\bar{\gamma}^{-2}\mathcal{S}_- - 1152\bar{\beta}_+^2\zeta\bar{\gamma}^{-2}\mathcal{S}_- + 72\zeta\mathcal{S}_-^3 - 72\zeta^2\mathcal{S}_-^3 - 288\bar{\beta}_+\zeta\bar{\gamma}^{-1}\mathcal{S}_-^3 \\
& \quad + 288\bar{\beta}_+\zeta^2\bar{\gamma}^{-1}\mathcal{S}_-^3 + 576\bar{\beta}_+\zeta\bar{\gamma}^{-2}\mathcal{S}_-^3 - 576\bar{\beta}_+\zeta^2\bar{\gamma}^{-2}\mathcal{S}_-^3 - 2304\bar{\beta}_-^2\zeta\bar{\gamma}^{-3}\mathcal{S}_-^3 - 2304\bar{\beta}_+^2\zeta\bar{\gamma}^{-3}\mathcal{S}_-^3 \\
& \quad + 2304\bar{\beta}_-^2\zeta^2\bar{\gamma}^{-3}\mathcal{S}_-^3 + 2304\bar{\beta}_+^2\zeta^2\bar{\gamma}^{-3}\mathcal{S}_-^3 - 192\bar{\beta}_-\bar{\gamma}^{-1}\mathcal{S}_+ + 192\bar{\beta}_-\zeta\bar{\gamma}^{-1}\mathcal{S}_+ + 576\bar{\beta}_-\bar{\beta}_+\bar{\gamma}^{-2}\mathcal{S}_+ \\
& \quad - 576\bar{\beta}_-\bar{\beta}_+\zeta\bar{\gamma}^{-2}\mathcal{S}_+ - 288\bar{\beta}_-\zeta\bar{\gamma}^{-1}\mathcal{S}_+^2\mathcal{S}_+ + 288\bar{\beta}_-\zeta^2\bar{\gamma}^{-1}\mathcal{S}_+^2\mathcal{S}_+ + 576\bar{\beta}_-\zeta\bar{\gamma}^{-2}\mathcal{S}_+^2\mathcal{S}_+ \\
& \quad - 576\bar{\beta}_-\zeta^2\bar{\gamma}^{-2}\mathcal{S}_+^2\mathcal{S}_+ - 4608\bar{\beta}_-\bar{\beta}_+\zeta\bar{\gamma}^{-3}\mathcal{S}_+^2\mathcal{S}_+ + 4608\bar{\beta}_-\bar{\beta}_+\zeta^2\bar{\gamma}^{-3}\mathcal{S}_+^2\mathcal{S}_+] + \nu[32\zeta\mathcal{S}_+^2\mathcal{S}_+ - 32\zeta^2\mathcal{S}_+^2\mathcal{S}_+] \\
& \quad + \alpha^{1/2} \left(-144\bar{\beta}_+ + 48\bar{\gamma} - 72\bar{\beta}_+\bar{\gamma} + 24\bar{\gamma}^2 - 72\zeta\mathcal{S}_+^2 + 144\bar{\beta}_+\zeta\mathcal{S}_+^2 - 36\zeta\bar{\gamma}\mathcal{S}_+^2 + 288\bar{\beta}_+\zeta\bar{\gamma}^{-1}\mathcal{S}_+^2 \right. \\
& \quad + 144\bar{\beta}_-\zeta\mathcal{S}_-\mathcal{S}_+ + 288\bar{\beta}_-\zeta\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+ + \nu[-64\zeta\mathcal{S}_+^2 - 32\zeta\bar{\gamma}\mathcal{S}_+^2] \\
& \quad + \delta[144\bar{\beta}_-\zeta\mathcal{S}_+^2 + 288\bar{\beta}_-\zeta\bar{\gamma}^{-1}\mathcal{S}_+^2 - 72\zeta\mathcal{S}_-\mathcal{S}_+ + 144\bar{\beta}_+\zeta\mathcal{S}_-\mathcal{S}_+ \\
& \quad \left. - 36\zeta\bar{\gamma}\mathcal{S}_-\mathcal{S}_+ + 288\bar{\beta}_+\zeta\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+ \right) \\
& + \alpha^{1/2}\lambda_1 \left(144\bar{\beta}_+\zeta^{-1} - 48\zeta^{-1}\bar{\gamma} + 72\bar{\beta}_+\zeta^{-1}\bar{\gamma} - 24\zeta^{-1}\bar{\gamma}^2 + 72\mathcal{S}_+^2 - 144\bar{\beta}_+\mathcal{S}_+^2 + 36\bar{\gamma}\mathcal{S}_+^2 \right. \\
& \quad - 288\bar{\beta}_+\bar{\gamma}^{-1}\mathcal{S}_+^2 - 144\bar{\beta}_-\mathcal{S}_-\mathcal{S}_+ - 288\bar{\beta}_-\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+ + \nu[64\mathcal{S}_+^2 + 32\bar{\gamma}\mathcal{S}_+^2] \\
& \quad + \delta[-144\bar{\beta}_-\mathcal{S}_+^2 - 288\bar{\beta}_-\bar{\gamma}^{-1}\mathcal{S}_+^2 + 72\mathcal{S}_-\mathcal{S}_+ - 144\bar{\beta}_+\mathcal{S}_-\mathcal{S}_+ \\
& \quad \left. + 36\bar{\gamma}\mathcal{S}_-\mathcal{S}_+ - 288\bar{\beta}_+\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+ \right) \left. \right\} + \mathcal{O}\left(\frac{1}{c^6}\right), \tag{B.3a}
\end{aligned}$$

$$\begin{aligned}
\mathbf{I}_i^s = & -\frac{\alpha^{1/2}\zeta m\nu r}{(1-\zeta)\phi_0} \left[2\mathcal{S}_- n^i \right. \\
& + \frac{1}{5c^2} \left\{ n^i \left(\frac{\tilde{G}\alpha m}{r} \right) \left(-9\mathcal{S}_- + 20\bar{\beta}_+\bar{\gamma}^{-1}\mathcal{S}_- + 20\bar{\beta}_-\bar{\gamma}^{-1}\mathcal{S}_+ \right. \right. \\
& \quad \left. \left. + \delta \left[-20\bar{\beta}_-\bar{\gamma}^{-1}\mathcal{S}_- + 4\mathcal{S}_+ - 20\bar{\beta}_+\bar{\gamma}^{-1}\mathcal{S}_+ \right] + 13\mathcal{S}_-\nu \right) \right. \\
& \quad \left. + n^i v^2 \left(\mathcal{S}_- + 4\mathcal{S}_+\delta - 7\mathcal{S}_-\nu \right) + v^i (nv) \left(2\mathcal{S}_- - 2\mathcal{S}_+\delta - 4\mathcal{S}_-\nu \right) \right\} \\
& + \frac{1}{3c^3} v^i \left(\frac{\tilde{G}\alpha m}{r} \right) \left(-\bar{\gamma}\mathcal{S}_- + 4\zeta\mathcal{S}_-^3 + 4\zeta\mathcal{S}_-^2\mathcal{S}_+\delta - 8\zeta\mathcal{S}_-^3\nu \right) \\
& + \frac{1}{140c^4} \left\{ n^i \left(\frac{\tilde{G}\alpha m}{r} \right)^2 \left(-38\mathcal{S}_- + 224\bar{\beta}_+\mathcal{S}_- - 56\bar{\gamma}\mathcal{S}_- - 672\bar{\beta}_+\bar{\gamma}^{-1}\mathcal{S}_- + 560\bar{\gamma}^{-1}\bar{\chi}_+\mathcal{S}_- + 224\bar{\beta}_-\mathcal{S}_+ \right. \right. \\
& \quad - 672\bar{\beta}_-\bar{\gamma}^{-1}\mathcal{S}_+ + 560\bar{\gamma}^{-1}\bar{\chi}_-\mathcal{S}_+ \\
& \quad \left. + \delta \left[-224\bar{\beta}_-\mathcal{S}_- + 672\bar{\beta}_-\bar{\gamma}^{-1}\mathcal{S}_- - 560\bar{\gamma}^{-1}\bar{\chi}_-\mathcal{S}_- - 32\mathcal{S}_+ - 224\bar{\beta}_+\mathcal{S}_+ + 56\bar{\gamma}\mathcal{S}_+ \right. \right. \\
& \quad \left. \left. + 672\bar{\beta}_+\bar{\gamma}^{-1}\mathcal{S}_+ - 560\bar{\gamma}^{-1}\bar{\chi}_+\mathcal{S}_+ \right] \right. \\
& \quad \left. + \nu \left[-1234\mathcal{S}_- + 112\bar{\beta}_+\mathcal{S}_- - 728\bar{\gamma}\mathcal{S}_- - 224\bar{\beta}_+\bar{\gamma}^{-1}\mathcal{S}_- - 4480\bar{\beta}_-^2\bar{\gamma}^{-2}\mathcal{S}_- \right. \right. \\
& \quad \left. \left. + 4480\bar{\beta}_+^2\bar{\gamma}^{-2}\mathcal{S}_- - 1120\bar{\gamma}^{-1}\bar{\chi}_+\mathcal{S}_- - 896\bar{\beta}_-\mathcal{S}_+ - 224\bar{\beta}_-\bar{\gamma}^{-1}\mathcal{S}_+ - 1120\bar{\gamma}^{-1}\bar{\chi}_-\mathcal{S}_+ \right] \right. \\
& \quad \left. + \delta\nu \left[448\bar{\beta}_-\mathcal{S}_- - 672\bar{\beta}_-\bar{\gamma}^{-1}\mathcal{S}_- + 120\mathcal{S}_+ - 672\bar{\beta}_+\bar{\gamma}^{-1}\mathcal{S}_+ \right] + 470\mathcal{S}_-\nu^2 \right) \\
& + n^i \left(\frac{\tilde{G}\alpha m}{r} \right) (nv)^2 \left(105\mathcal{S}_- + 56\bar{\gamma}\mathcal{S}_- - 56\bar{\beta}_+\bar{\gamma}^{-1}\mathcal{S}_- - 56\bar{\beta}_-\bar{\gamma}^{-1}\mathcal{S}_+ \right. \\
& \quad + \left[56\bar{\beta}_-\bar{\gamma}^{-1}\mathcal{S}_- + 140\mathcal{S}_+ + 84\bar{\gamma}\mathcal{S}_+ + 56\bar{\beta}_+\bar{\gamma}^{-1}\mathcal{S}_+ \right] \delta \\
& \quad + \nu \left[-49\mathcal{S}_- - 112\bar{\gamma}\mathcal{S}_- - 112\bar{\beta}_+\bar{\gamma}^{-1}\mathcal{S}_- - 112\bar{\beta}_-\bar{\gamma}^{-1}\mathcal{S}_+ \right] \\
& \quad \left. + \delta\nu \left[-336\bar{\beta}_-\bar{\gamma}^{-1}\mathcal{S}_- + 224\mathcal{S}_+ - 336\bar{\beta}_+\bar{\gamma}^{-1}\mathcal{S}_+ \right] - 245\mathcal{S}_-\nu^2 \right) \\
& + n^i \left(\frac{\tilde{G}\alpha m}{r} \right) v^2 \left(65\mathcal{S}_- + 28\bar{\gamma}\mathcal{S}_- - 224\bar{\beta}_+\bar{\gamma}^{-1}\mathcal{S}_- - 224\bar{\beta}_-\bar{\gamma}^{-1}\mathcal{S}_+ \right. \\
& \quad + \delta \left[224\bar{\beta}_-\bar{\gamma}^{-1}\mathcal{S}_- + 320\mathcal{S}_+ + 252\bar{\gamma}\mathcal{S}_+ + 224\bar{\beta}_+\bar{\gamma}^{-1}\mathcal{S}_+ \right] \\
& \quad + \nu \left[-1303\mathcal{S}_- - 756\bar{\gamma}\mathcal{S}_- + 672\bar{\beta}_+\bar{\gamma}^{-1}\mathcal{S}_- + 672\bar{\beta}_-\bar{\gamma}^{-1}\mathcal{S}_+ \right] \\
& \quad \left. + \delta\nu \left[336\bar{\beta}_-\bar{\gamma}^{-1}\mathcal{S}_- + 228\mathcal{S}_+ + 336\bar{\beta}_+\bar{\gamma}^{-1}\mathcal{S}_+ \right] - 675\mathcal{S}_-\nu^2 \right) \\
& + n^i v^4 \left(33\mathcal{S}_- + 72\mathcal{S}_+\delta + \left[-279\mathcal{S}_- - 312\mathcal{S}_+\delta \right] \nu + 549\mathcal{S}_-\nu^2 \right) \\
& + v^i \left(\frac{\tilde{G}\alpha m}{r} \right) (nv) \left(310\mathcal{S}_- + 168\bar{\gamma}\mathcal{S}_- + 112\bar{\beta}_+\bar{\gamma}^{-1}\mathcal{S}_- + 112\bar{\beta}_-\bar{\gamma}^{-1}\mathcal{S}_+ \right. \\
& \quad + \delta \left[-112\bar{\beta}_-\bar{\gamma}^{-1}\mathcal{S}_- - 240\mathcal{S}_+ - 168\bar{\gamma}\mathcal{S}_+ - 112\bar{\beta}_+\bar{\gamma}^{-1}\mathcal{S}_+ \right] \\
& \quad + \nu \left[1182\mathcal{S}_- + 504\bar{\gamma}\mathcal{S}_- - 336\bar{\beta}_+\bar{\gamma}^{-1}\mathcal{S}_- - 336\bar{\beta}_-\bar{\gamma}^{-1}\mathcal{S}_+ \right] \\
& \quad \left. + \delta\nu \left[112\bar{\beta}_-\bar{\gamma}^{-1}\mathcal{S}_- - 192\mathcal{S}_+ + 112\bar{\beta}_+\bar{\gamma}^{-1}\mathcal{S}_+ \right] - 514\mathcal{S}_-\nu^2 \right) \\
& \left. + v^i (nv) v^2 \left(20\mathcal{S}_- - 20\mathcal{S}_+\delta + \left[-164\mathcal{S}_- + 124\mathcal{S}_+\delta \right] \nu + 376\mathcal{S}_-\nu^2 \right) \right\} + \mathcal{O}\left(\frac{1}{c^5}\right), \tag{B.3b}
\end{aligned}$$

$$\begin{aligned}
\mathbb{I}_{ij}^s = & -\frac{\alpha^{1/2}\zeta m\nu r^2}{(1-\zeta)\phi_0} \left[(\mathcal{S}_+ - \mathcal{S}_-\delta) n^{(i} n^{j)} \right. \\
& + \frac{1}{14c^2} \left\{ n^{(i} n^{j)} \left(\frac{\tilde{G}\alpha m}{r} \right) \left(56\bar{\beta}_-\bar{\gamma}^{-1}\mathcal{S}_- - 18\mathcal{S}_+ + 56\bar{\beta}_+\bar{\gamma}^{-1}\mathcal{S}_+ + \delta [18\mathcal{S}_- - 56\bar{\beta}_+\bar{\gamma}^{-1}\mathcal{S}_- - 56\bar{\beta}_-\bar{\gamma}^{-1}\mathcal{S}_+] \right) \right. \\
& \quad \left. + \nu [-112\bar{\beta}_-\bar{\gamma}^{-1}\mathcal{S}_- + 40\mathcal{S}_+ - 112\bar{\beta}_+\bar{\gamma}^{-1}\mathcal{S}_+] - 32\mathcal{S}_-\nu\delta \right) \\
& + n^{(i} n^{j)} v^2 \left(-5\mathcal{S}_+ + 5\mathcal{S}_-\delta + 23\mathcal{S}_-\nu\delta + 15\mathcal{S}_+\nu \right) + n^{(i} v^{j)} (n\nu) \left(8\mathcal{S}_+ - 8\mathcal{S}_-\delta - 24\mathcal{S}_+\nu + 8\mathcal{S}_-\nu\delta \right) \\
& \left. + v^{(i} v^{j)} \left(2\mathcal{S}_+ - 2\mathcal{S}_-\delta - 6\mathcal{S}_+\nu + 2\mathcal{S}_-\nu\delta \right) \right\} \\
& + \frac{1}{3c^3} n^{(i} v^{j)} \left(\frac{\tilde{G}\alpha m}{r} \right) \left(8\zeta\mathcal{S}_-\mathcal{S}_+\nu + 8\zeta\mathcal{S}_-\delta\nu \right) \Big] + \mathcal{O}\left(\frac{1}{c^4}\right), \tag{B.3c}
\end{aligned}$$

$$\begin{aligned}
\mathbb{I}_{ijk}^s = & -\frac{\alpha^{1/2}\zeta m\nu r^3}{(1-\zeta)\phi_0} \left[(\mathcal{S}_- - \mathcal{S}_+\delta - 2\mathcal{S}_-\nu) n^i n^j n^k \right. \\
& + \frac{1}{18c^2} \left\{ n^{(i} n^j n^k) \left(\frac{\tilde{G}\alpha m}{r} \right) \left(-23\mathcal{S}_- + 72\bar{\beta}_+\bar{\gamma}^{-1}\mathcal{S}_- + 72\bar{\beta}_-\bar{\gamma}^{-1}\mathcal{S}_+ + \delta [-72\bar{\beta}_-\bar{\gamma}^{-1}\mathcal{S}_- + 23\mathcal{S}_+ - 72\bar{\beta}_+\bar{\gamma}^{-1}\mathcal{S}_+] \right) \right. \\
& \quad \left. + \nu [101\mathcal{S}_- - 216\bar{\beta}_+\bar{\gamma}^{-1}\mathcal{S}_- - 216\bar{\beta}_-\bar{\gamma}^{-1}\mathcal{S}_+] + \nu\delta [72\bar{\beta}_-\bar{\gamma}^{-1}\mathcal{S}_- - 55\mathcal{S}_+ + 72\bar{\beta}_+\bar{\gamma}^{-1}\mathcal{S}_+] - 118\mathcal{S}_-\nu^2 \right) \\
& + n^{(i} n^j n^k) v^2 \left(-7\mathcal{S}_- + 7\mathcal{S}_+\delta + \mathcal{S}_-\nu + 13\mathcal{S}_+\nu\delta + 94\mathcal{S}_-\nu^2 \right) \\
& + n^{(i} n^j v^k) (n\nu) \left(12\mathcal{S}_- - 12\mathcal{S}_+\delta - 48\mathcal{S}_-\nu + 24\mathcal{S}_+\nu\delta + 24\mathcal{S}_-\nu^2 \right) \\
& \left. + n^{(i} v^j v^k) \left(6\mathcal{S}_- - 6\mathcal{S}_+\delta - 24\mathcal{S}_-\nu + 12\mathcal{S}_+\nu\delta + 12\mathcal{S}_-\nu^2 \right) \right\} \Big] + \mathcal{O}\left(\frac{1}{c^3}\right), \tag{B.3d}
\end{aligned}$$

$$\mathbb{I}_{ijkl}^s = -\frac{\alpha^{1/2}\zeta m\nu r^4}{(1-\zeta)\phi_0} n^{(i} n^j n^k n^l) (\mathcal{S}_+ - \mathcal{S}_-\delta - 3\mathcal{S}_+\nu + \mathcal{S}_-\nu\delta) + \mathcal{O}\left(\frac{1}{c^2}\right), \tag{B.3e}$$

$$\mathbb{I}_{ijklm}^s = -\frac{\alpha^{1/2}\zeta m\nu r^5}{(1-\zeta)\phi_0} n^{(i} n^j n^k n^l n^m) (\mathcal{S}_- - \mathcal{S}_+\delta - 4\mathcal{S}_-\nu + 2\mathcal{S}_+\nu\delta + 2\mathcal{S}_-\nu^2) + \mathcal{O}\left(\frac{1}{c}\right).$$

The tensor source moments (in addition to the mass type quadrupole tensor moment already given by (VII.45)) are

$$\begin{aligned}
\mathbb{I}_{ijk} = & -\frac{m\nu\delta r^3}{\phi_0} \left[n^{(i} n^j n^k) + \frac{1}{6c^2} \left\{ n^{(i} n^j n^k) \left(\frac{\tilde{G}\alpha m}{r} \right) (-5 + 13\nu) + n^{(i} n^j n^k) v^2 (5 - 19\nu) \right. \right. \\
& \left. \left. + n^{(i} n^j v^k) (n\nu) (-6 + 12\nu) + n^{(i} v^j v^k) (6 - 12\nu) \right\} \right] + \mathcal{O}\left(\frac{1}{c^3}\right), \tag{B.4a}
\end{aligned}$$

$$\mathbb{I}_{ijkl} = \frac{m\nu r^4}{\phi_0} n^{(i} n^j n^k n^l) (1 - 3\nu) + \mathcal{O}\left(\frac{1}{c^2}\right), \tag{B.4b}$$

$$\mathbb{I}_{ijklm} = -\frac{m\nu\delta r^5}{\phi_0} n^{(i} n^j n^k n^l n^m) (1 - 2\nu) + \mathcal{O}\left(\frac{1}{c}\right), \tag{B.4c}$$

$$\mathbb{J}_{ij} = -\frac{m\nu\delta r^2}{\phi_0} \left[\epsilon^{ab(i} n^{j)} n^a v^b + \frac{1}{28c^2} \left\{ \epsilon^{ab(i} n^{j)} n^a v^b \left(\frac{\tilde{G}\alpha m}{r} \right) (54 + 42\bar{\gamma} + 60\nu) \right. \right.$$

$$+ \left. \left. \left. \epsilon^{ab\langle i} n^{j\rangle} n^a v^b v^2 (13 - 68\nu) + \epsilon^{ab\langle i} v^{j\rangle} n^a v^b (nv) (5 - 10\nu) \right\} \right] + \mathcal{O}\left(\frac{1}{c^3}\right), \quad (\text{B.4d})$$

$$\mathbf{J}_{ijk} = \frac{m\nu r^3}{\phi_0} \epsilon^{ab\langle i} n^{j\rangle} n^k n^a v^b (1 - 3\nu) + \mathcal{O}\left(\frac{1}{c^2}\right), \quad (\text{B.4e})$$

$$\mathbf{J}_{ijkl} = -\frac{m\nu \delta r^4}{\phi_0} \epsilon^{ab\langle i} n^{j\rangle} n^k n^a n^l v^b (1 - 2\nu) + \mathcal{O}\left(\frac{1}{c}\right). \quad (\text{B.4f})$$

The conserved quantities, i.e. the monopole \mathbf{I} , the conserved total mass-energy \mathbf{M} , the conserved energy E and the angular momentum \mathbf{J}_i are given to the relevant PN orders by

$$\mathbf{I} = \frac{\mathbf{M}}{\phi_0} = \frac{m}{\phi_0} \left(1 + \frac{E}{c^2}\right) \quad \text{with} \quad E = \frac{1}{2} m\nu v^2 - m\nu \left(\frac{\tilde{G}\alpha m}{r}\right) + \mathcal{O}\left(\frac{1}{c}\right), \quad (\text{B.5a})$$

$$\mathbf{J}_i = \frac{m\nu r}{\phi_0} \epsilon^{iab} n^a v^b + \mathcal{O}\left(\frac{1}{c}\right). \quad (\text{B.5b})$$

Finally the two gauge moments needed in our calculation read

$$\mathbf{W} = \frac{m\nu r (nv)}{3\phi_0} + \mathcal{O}\left(\frac{1}{c}\right), \quad (\text{B.6a})$$

$$\mathbf{Y}_i = \frac{m\delta\nu r}{10\phi_0} \left[n^i \left(\frac{\tilde{G}\alpha m}{r}\right) + n^i v^2 - 3v^i (nv) \right] + \mathcal{O}\left(\frac{1}{c}\right). \quad (\text{B.6b})$$

B.2.2 Link between STF and EW moments

The works [246, 247] use the so-called Epstein-Wagoner multipole moments while we are using the STF moments. In the gravitational sector the definitions of the EW moments are the same as in GR. Their relations to the STF moments can be found in Appendix E of [243]. In particular the instantaneous part of the tensorial flux to 1.5PN order (discarding the tails) in terms of the EW source moments reads

$$\begin{aligned} \mathcal{F}_{\text{inst}} = & \frac{\phi_0}{c^5 G} \left\{ \frac{1}{5} \overset{(3)}{\mathbf{I}}_{ab}^{\text{EW}} \overset{(3)}{\mathbf{I}}_{ab}^{\text{EW}} - \frac{1}{15} \overset{(3)}{\mathbf{I}}_{aa}^{\text{EW}} \overset{(3)}{\mathbf{I}}_{bb}^{\text{EW}} \right. \\ & + \frac{1}{c^2} \left[\frac{11}{105} \overset{(3)}{\mathbf{I}}_{abc}^{\text{EW}} \overset{(3)}{\mathbf{I}}_{abc}^{\text{EW}} - \frac{2}{35} \overset{(3)}{\mathbf{I}}_{abc}^{\text{EW}} \overset{(3)}{\mathbf{I}}_{acb}^{\text{EW}} + \frac{8}{105} \overset{(3)}{\mathbf{I}}_{aab}^{\text{EW}} \overset{(3)}{\mathbf{I}}_{bcc}^{\text{EW}} - \frac{2}{35} \overset{(3)}{\mathbf{I}}_{aba}^{\text{EW}} \overset{(3)}{\mathbf{I}}_{bcc}^{\text{EW}} - \frac{1}{21} \overset{(3)}{\mathbf{I}}_{aac}^{\text{EW}} \overset{(3)}{\mathbf{I}}_{bbc}^{\text{EW}} \right. \\ & \left. \left. + \frac{22}{105} \overset{(3)}{\mathbf{I}}_{ab}^{\text{EW}} \overset{(3)}{\mathbf{I}}_{abcc}^{\text{EW}} - \frac{8}{35} \overset{(3)}{\mathbf{I}}_{ab}^{\text{EW}} \overset{(3)}{\mathbf{I}}_{acbc}^{\text{EW}} - \frac{2}{21} \overset{(3)}{\mathbf{I}}_{aa}^{\text{EW}} \overset{(3)}{\mathbf{I}}_{bbcc}^{\text{EW}} + \frac{8}{105} \overset{(3)}{\mathbf{I}}_{aa}^{\text{EW}} \overset{(3)}{\mathbf{I}}_{bcbc}^{\text{EW}} + \frac{8}{105} \overset{(3)}{\mathbf{I}}_{ab}^{\text{EW}} \overset{(3)}{\mathbf{I}}_{ccab}^{\text{EW}} \right] \right. \\ & \left. + \mathcal{O}\left(\frac{1}{c^4}\right) \right\}. \quad (\text{B.7}) \end{aligned}$$

Injecting the expressions of the EW moments [246] into (B.7), we obtain an expression for the tensorial flux that is in perfect agreement with (VII.49).

Concerning the scalar sector, the moments used in [246, 247] are just non-STF moments $I_L^{s,EW}$ such that the linearized scalar waveform is given by (2.17b) of [246]. Comparing with our definition of the STF moments $I_L^s \equiv I_L^{s,STF}$ entering the scalar waveform in (VII.15b), we obtain the following relation, valid up to any PN order:

$$I_L^s(u) = - \sum_{k=0}^{\infty} \frac{(2\ell+1)!!}{2^k k! (2\ell+2k+1)!!} \left(\frac{d}{c du} \right)^{2k} I_{2K(L)}^{s,EW}(u). \quad (\text{B.8})$$

With our computations, and taking into account the link between EW and STF moments as well as the result (B.8), we are in total agreement with the expressions of the source moments found in [246, 247], *i.e.*

- I_{ij}^{EW} , I_{ijk}^{EW} , I_{ijkl}^{EW} , I_{ijklm}^{EW} , and I_{ijklmn}^{EW} in (5.10) of [246],
- $I^{s,EW}$, $I_i^{s,EW}$, $I_{ij}^{s,EW}$, $I_{ijk}^{s,EW}$, $I_{ijkl}^{s,EW}$ and $I_{ijklm}^{s,EW}$ in the CM frame in (3.50) of [247].

However, despite the perfect correspondence and agreement between the results for STF and EW moments, the scalar flux we obtain in the (B.12)–(B.13) in Appendix B.3 disagrees at 1PN order with the one published in [247]. This disagreement remains even when we compute the instantaneous 1PN scalar flux directly from the EW moments using the expression (not given in [247])

$$\begin{aligned} \mathcal{F}_{\text{inst}}^s = \frac{\phi_0(3+2\omega_0)}{c^3 G} & \left\{ \frac{1}{3} I_a^{s,EW} I_a^{s,EW} \right. \\ & + \frac{1}{c^2} \left[\left(\frac{m_{s1}^{EW}}{\phi_0(3+2\omega_0)} \right)^2 + \frac{1}{60} I_{aa}^{s,EW} I_{bb}^{s,EW} + \frac{1}{30} I_{ab}^{s,EW} I_{ab}^{s,EW} \right. \\ & \quad \left. \left. + \frac{1}{15} I_a^{s,EW} I_{abb}^{s,EW} + \frac{1}{3} I_{aa}^{s,EW} \frac{m_{s1}^{EW}}{\phi_0(3+2\omega_0)} \right] \right. \\ & + \frac{1}{c^4} \left[\frac{1}{630} I_{abc}^{s,EW} I_{abc}^{s,EW} + \frac{1}{420} I_{aac}^{s,EW} I_{bbc}^{s,EW} + \frac{1}{60} I_{aabb}^{s,EW} \frac{m_{s1}^{EW}}{\phi_0(3+2\omega_0)} \right. \\ & \quad \left. \left. + \frac{1}{420} I_a^{s,EW} I_{abbc}^{s,EW} + \frac{1}{840} I_{aa}^{s,EW} I_{bbcc}^{s,EW} + \frac{1}{210} I_{ab}^{s,EW} I_{abcc}^{s,EW} \right] \right\} + \mathcal{O}\left(\frac{1}{c^8}\right), \end{aligned} \quad (\text{B.9})$$

where we pose, following [247] (note that the convention is slightly different from that in (VII.39)):

$$I^{s,EW} = \frac{1}{\phi_0(3+2\omega_0)} \left[m_s^{EW} + \frac{m_{s1}^{EW}}{c^2} \right], \quad (\text{B.10a})$$

$$\text{with } m^{s,EW} \equiv \sum_A m_A (1 - 2s_A). \quad (\text{B.10b})$$

We have also computed the scalar waveform from the EW moments [246, 247] using the formulas given in [247], and we retrieve the explicit expression for the 1.5PN scalar

waveform given by [247]. Then, when integrating this waveform using (6.6) of [247], we retrieve the 1PN part of the scalar flux as given by (B.12)-(B.13) of Appendix B.3, but not the 1PN scalar flux as given by [247].

B.3 The scalar 1.5PN flux

As mentioned throughout the article, the scalar flux that we find disagrees at 1PN order with the scalar flux given by Lang in [247]. The expression for the difference between the two fluxes at 1PN order is

$$\begin{aligned} \mathcal{F}_{\text{inst}}^s - \mathcal{F}_{\text{inst}}^{s,\text{Lang}} &= \frac{m\nu^2}{c^7 r} \left(\frac{\tilde{G}\alpha m}{r} \right)^3 \left\{ -\frac{32}{3} \bar{\beta}_+ \zeta \mathcal{S}_- \mathcal{S}_+ \delta v^4 \right. \\ &+ \left(\frac{\tilde{G}\alpha m}{r} \right)^2 \left(\frac{16}{105} \bar{\gamma} + \nu \left[-\frac{64}{105} \bar{\gamma} + 64 \bar{\beta}_-^2 \zeta \bar{\gamma}^{-1} \mathcal{S}_-^2 - 64 \bar{\beta}_+^2 \zeta \bar{\gamma}^{-1} \mathcal{S}_-^2 \right. \right. \\ &\left. \left. - 64 \bar{\beta}_-^2 \zeta \bar{\gamma}^{-2} \mathcal{S}_-^2 + 64 \bar{\beta}_+^2 \zeta \bar{\gamma}^{-2} \mathcal{S}_-^2 \right] \right) \left. \right\}. \end{aligned} \quad (\text{B.11})$$

Since the instantaneous part of the scalar flux to 1.5PN order for general orbits in the center-of-mass frame is very long it is convenient to write it in the following form:

$$\begin{aligned} \mathcal{F}_{\text{inst}}^s &= \frac{1}{3c^3} \frac{m\nu^2}{r} \left(\frac{\tilde{G}\alpha m}{r} \right)^3 \\ &\times \left\{ A^{-1\text{PN}} + \frac{2}{5c^2} \left(B_1^{\text{N}} \frac{\tilde{G}\alpha m}{r} + B_2^{\text{N}} (nv)^2 + B_3^{\text{N}} v^2 \right) + \frac{4}{15c^3} C^{0.5\text{PN}} \frac{\tilde{G}\alpha m}{r} (nv) \right. \\ &+ \frac{1}{140c^4} \left(D_1^{1\text{PN}} \left(\frac{\tilde{G}\alpha m}{r} \right)^2 + D_2^{1\text{PN}} \frac{\tilde{G}\alpha m}{r} (nv)^2 + D_3^{1\text{PN}} \frac{\tilde{G}\alpha m}{r} v^2 \right. \\ &\quad \left. \left. + D_4^{1\text{PN}} (nv)^4 + D_5^{1\text{PN}} (nv)^2 v^2 + D_6^{1\text{PN}} v^4 \right) \right. \\ &\left. + \frac{1}{90c^5} \left(E_1^{1.5\text{PN}} \left(\frac{\tilde{G}\alpha m}{r} \right)^2 (nv) + E_2^{1.5\text{PN}} \frac{\tilde{G}\alpha m}{r} (nv)^3 + E_3^{1.5\text{PN}} \frac{\tilde{G}\alpha m}{r} (nv) v^2 \right) \right\}, \end{aligned} \quad (\text{B.12})$$

where the coefficient $A^{-1\text{PN}}$, following our convention, corresponds to the leading dipolar radiation at -1PN order, and the coefficients B_n^{N} , $C^{0.5\text{PN}}$, $D_n^{1\text{PN}}$ and $E_n^{1.5\text{PN}}$ respectively parametrize the Newtonian, 0.5PN, 1PN and 1.5PN terms. We have obtained

$$A^{-1\text{PN}} = 4\zeta\mathcal{S}_-^2, \quad (\text{B.13a})$$

$$B_1^{\text{N}} = -92\zeta\mathcal{S}_-^2 - 40\bar{\beta}_+\zeta\mathcal{S}_-^2 - 40\zeta\bar{\gamma}\mathcal{S}_-^2 + [40\bar{\beta}_-\zeta\mathcal{S}_-^2 - 8\zeta\mathcal{S}_-\mathcal{S}_+] \delta + 4\zeta\mathcal{S}_-^2\nu, \quad (\text{B.13b})$$

$$B_2^{\text{N}} = 160\bar{\beta}_+ - 23\bar{\gamma} - 240\bar{\beta}_+^2\bar{\gamma}^{-1} - 28\zeta\mathcal{S}_-^2 - 40\zeta\bar{\gamma}\mathcal{S}_-^2 - 360\bar{\beta}_+\zeta\bar{\gamma}^{-1}\mathcal{S}_-^2 + 480\bar{\beta}_-^2\zeta\bar{\gamma}^{-2}\mathcal{S}_-^2 + 480\bar{\beta}_+^2\zeta\bar{\gamma}^{-2}\mathcal{S}_-^2 \\ - 360\bar{\beta}_-\zeta\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+ + 960\bar{\beta}_-\bar{\beta}_+\zeta\bar{\gamma}^{-2}\mathcal{S}_-\mathcal{S}_+ + [120\bar{\beta}_-\zeta\bar{\gamma}^{-1}\mathcal{S}_-^2 - 32\zeta\mathcal{S}_-\mathcal{S}_+ + 120\bar{\beta}_+\zeta\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+] \delta + 36\zeta\mathcal{S}_-^2\nu, \quad (\text{B.13c})$$

$$B_3^{\text{N}} = -4\bar{\gamma} + 32\zeta\mathcal{S}_-^2 + 20\zeta\bar{\gamma}\mathcal{S}_-^2 + 40\bar{\beta}_+\zeta\bar{\gamma}^{-1}\mathcal{S}_-^2 + 40\bar{\beta}_-\zeta\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+ \\ + \delta[-40\bar{\beta}_-\zeta\bar{\gamma}^{-1}\mathcal{S}_-^2 + 8\zeta\mathcal{S}_-\mathcal{S}_+ - 40\bar{\beta}_+\zeta\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+] - 4\zeta\mathcal{S}_-^2\nu, \quad (\text{B.13d})$$

$$C^{0.5\text{PN}} = 10\zeta\bar{\gamma}\mathcal{S}_-^2 - 40\zeta^2\mathcal{S}_-^4 - 40\zeta^2\mathcal{S}_+^3\mathcal{S}_+\delta, \quad (\text{B.13e})$$

$$D_1^{\text{1PN}} = -64\bar{\gamma} - 70\bar{\gamma}^2 - 35\bar{\gamma}^3 + 23784\zeta\mathcal{S}_-^2 + 2240\bar{\beta}_-^2\zeta\mathcal{S}_-^2 + 21056\bar{\beta}_+\zeta\mathcal{S}_-^2 + 2240\bar{\beta}_+^2\zeta\mathcal{S}_-^2 + 21000\zeta\bar{\gamma}\mathcal{S}_-^2 + 8960\bar{\beta}_+\zeta\bar{\gamma}\mathcal{S}_-^2 \\ + 4620\zeta\bar{\gamma}^2\mathcal{S}_-^2 + 4032\bar{\beta}_+\zeta\bar{\gamma}^{-1}\mathcal{S}_-^2 - 2240\zeta\bar{\chi}_+\mathcal{S}_-^2 - 2240\zeta\bar{\gamma}^{-1}\bar{\chi}_+\mathcal{S}_-^2 + 1960\zeta^2\mathcal{S}_-^4 + 980\zeta^2\bar{\gamma}\mathcal{S}_-^4 - 3584\bar{\beta}_-\zeta\mathcal{S}_-\mathcal{S}_+ \\ + 4032\bar{\beta}_-\zeta\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+ - 2240\zeta\bar{\gamma}^{-1}\bar{\chi}_-\mathcal{S}_-\mathcal{S}_+ + 280\zeta^2\mathcal{S}_+^4 + 140\zeta^2\bar{\gamma}\mathcal{S}_+^4 \\ + \delta[-21056\bar{\beta}_-\zeta\mathcal{S}_-^2 - 4480\bar{\beta}_-\bar{\beta}_+\zeta\mathcal{S}_-^2 - 8960\bar{\beta}_-\zeta\bar{\gamma}\mathcal{S}_-^2 - 4032\bar{\beta}_-\zeta\bar{\gamma}^{-1}\mathcal{S}_-^2 + 2240\zeta\bar{\chi}_-\mathcal{S}_-^2 + 2240\zeta\bar{\gamma}^{-1}\bar{\chi}_-\mathcal{S}_-^2 \\ + 3936\zeta\mathcal{S}_-\mathcal{S}_+ + 3584\bar{\beta}_+\zeta\mathcal{S}_-\mathcal{S}_+ + 1680\zeta\bar{\gamma}\mathcal{S}_-\mathcal{S}_+ + 280\zeta\bar{\gamma}^2\mathcal{S}_-\mathcal{S}_+ - 4032\bar{\beta}_+\zeta\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+ \\ + 2240\zeta\bar{\gamma}^{-1}\bar{\chi}_-\mathcal{S}_-\mathcal{S}_+ + 1120\zeta^2\mathcal{S}_+^3\mathcal{S}_+ + 560\zeta^2\bar{\gamma}\mathcal{S}_+^3\mathcal{S}_+ + 1120\zeta^2\mathcal{S}_-\mathcal{S}_+^3 + 560\zeta^2\bar{\gamma}\mathcal{S}_-\mathcal{S}_+^3] \\ + \nu[256\bar{\gamma} + 140\bar{\gamma}^2 + 70\bar{\gamma}^3 + 11936\zeta\mathcal{S}_-^2 - 8960\bar{\beta}_-^2\zeta\mathcal{S}_-^2 + 6048\bar{\beta}_+\zeta\mathcal{S}_-^2 + 5040\zeta\bar{\gamma}\mathcal{S}_-^2 - 280\zeta\bar{\gamma}^2\mathcal{S}_-^2 \\ + 26880\bar{\beta}_-^2\zeta\bar{\gamma}^{-1}\mathcal{S}_-^2 - 896\bar{\beta}_+\zeta\bar{\gamma}^{-1}\mathcal{S}_-^2 - 26880\bar{\beta}_+^2\zeta\bar{\gamma}^{-1}\mathcal{S}_-^2 + 17920\bar{\beta}_-^2\zeta\bar{\gamma}^{-2}\mathcal{S}_-^2 - 17920\bar{\beta}_+^2\zeta\bar{\gamma}^{-2}\mathcal{S}_-^2 \\ + 4480\zeta\bar{\chi}_+\mathcal{S}_-^2 + 4480\zeta\bar{\gamma}^{-1}\bar{\chi}_+\mathcal{S}_-^2 - 3920\zeta^2\mathcal{S}_-^4 - 1960\zeta^2\bar{\gamma}\mathcal{S}_-^4 + 14336\bar{\beta}_-\zeta\mathcal{S}_-\mathcal{S}_+ - 896\bar{\beta}_-\zeta\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+ \\ + 4480\zeta\bar{\gamma}^{-1}\bar{\chi}_-\mathcal{S}_-\mathcal{S}_+ - 560\zeta^2\mathcal{S}_+^4 - 280\zeta^2\bar{\gamma}\mathcal{S}_+^4] \\ + \delta\nu[672\bar{\beta}_-\zeta\mathcal{S}_-^2 + 1792\bar{\beta}_-\zeta\bar{\gamma}^{-1}\mathcal{S}_-^2 - 704\zeta\mathcal{S}_-\mathcal{S}_+ + 1792\bar{\beta}_+\zeta\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+] - 88\zeta\mathcal{S}_-^2\nu^2, \quad (\text{B.13f})$$

$$D_2^{\text{1PN}} = -19264\bar{\beta}_+ - 26880\bar{\beta}_+^2 - 6256\bar{\gamma} + 25984\bar{\beta}_+\bar{\gamma} - 6720\bar{\gamma}^2 - 35840\bar{\beta}_-^2\bar{\gamma}^{-1} + 134400\bar{\beta}_+^2\bar{\gamma}^{-1} + 107520\bar{\beta}_-^2\bar{\beta}_+\bar{\gamma}^{-2} \\ - 107520\bar{\beta}_+^3\bar{\gamma}^{-2} + 8960\bar{\chi}_+ - 26880\bar{\beta}_+\bar{\gamma}^{-1}\bar{\chi}_+ + 31240\zeta\mathcal{S}_-^2 - 39200\bar{\beta}_+\zeta\mathcal{S}_-^2 + 31808\zeta\bar{\gamma}\mathcal{S}_-^2 + 4480\bar{\beta}_+\zeta\bar{\gamma}\mathcal{S}_-^2 \\ + 4480\zeta\bar{\gamma}^2\mathcal{S}_-^2 + 4480\bar{\beta}_-^2\zeta\bar{\gamma}^{-1}\mathcal{S}_-^2 + 61376\bar{\beta}_+\zeta\bar{\gamma}^{-1}\mathcal{S}_-^2 + 58240\bar{\beta}_+^2\zeta\bar{\gamma}^{-1}\mathcal{S}_-^2 - 143360\bar{\beta}_-^2\zeta\bar{\gamma}^{-2}\mathcal{S}_-^2 \\ - 250880\bar{\beta}_+^2\zeta\bar{\gamma}^{-2}\mathcal{S}_-^2 - 215040\bar{\beta}_-^2\bar{\beta}_+\zeta\bar{\gamma}^{-3}\mathcal{S}_-^2 + 215040\bar{\beta}_+^3\zeta\bar{\gamma}^{-3}\mathcal{S}_-^2 + 53760\bar{\beta}_-\zeta\bar{\gamma}^{-2}\bar{\chi}_-\mathcal{S}_-^2 - 31360\zeta\bar{\gamma}^{-1}\bar{\chi}_+\mathcal{S}_-^2 \\ + 53760\bar{\beta}_+\zeta\bar{\gamma}^{-2}\bar{\chi}_+\mathcal{S}_-^2 - 40320\bar{\beta}_-\zeta\mathcal{S}_-\mathcal{S}_+ + 61376\bar{\beta}_-\zeta\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+ + 62720\bar{\beta}_-\bar{\beta}_+\zeta\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+ \\ - 394240\bar{\beta}_-\bar{\beta}_+\zeta\bar{\gamma}^{-2}\mathcal{S}_-\mathcal{S}_+ - 215040\bar{\beta}_-^3\zeta\bar{\gamma}^{-3}\mathcal{S}_-\mathcal{S}_+ + 215040\bar{\beta}_-\bar{\beta}_+^2\zeta\bar{\gamma}^{-3}\mathcal{S}_-\mathcal{S}_+ - 31360\zeta\bar{\gamma}^{-1}\bar{\chi}_-\mathcal{S}_-\mathcal{S}_+ \\ + 53760\bar{\beta}_+\zeta\bar{\gamma}^{-2}\bar{\chi}_-\mathcal{S}_-\mathcal{S}_+ + 53760\bar{\beta}_-\zeta\bar{\gamma}^{-2}\bar{\chi}_+\mathcal{S}_-\mathcal{S}_+ \\ + \delta[-4032\bar{\beta}_- + 26880\bar{\beta}_-\bar{\beta}_+ - 8064\bar{\beta}_-\bar{\gamma} + 8960\bar{\beta}_-\bar{\beta}_+\bar{\gamma}^{-1} - 8960\bar{\chi}_- + 26880\bar{\beta}_+\bar{\gamma}^{-1}\bar{\chi}_- - 14560\bar{\beta}_-\zeta\mathcal{S}_-^2 \\ - 4480\bar{\beta}_-\zeta\bar{\gamma}\mathcal{S}_-^2 - 61376\bar{\beta}_-\zeta\bar{\gamma}^{-1}\mathcal{S}_-^2 - 62720\bar{\beta}_-\bar{\beta}_+\zeta\bar{\gamma}^{-1}\mathcal{S}_-^2 - 35840\bar{\beta}_-\bar{\beta}_+\zeta\bar{\gamma}^{-2}\mathcal{S}_-^2 - 215040\bar{\beta}_-^3\zeta\bar{\gamma}^{-3}\mathcal{S}_-^2 \\ + 215040\bar{\beta}_-\bar{\beta}_+^2\zeta\bar{\gamma}^{-3}\mathcal{S}_-^2 + 31360\zeta\bar{\gamma}^{-1}\bar{\chi}_-\mathcal{S}_-^2 - 53760\bar{\beta}_+\zeta\bar{\gamma}^{-2}\bar{\chi}_-\mathcal{S}_-^2 - 53760\bar{\beta}_-\zeta\bar{\gamma}^{-2}\bar{\chi}_+\mathcal{S}_-^2 + 6560\zeta\mathcal{S}_-\mathcal{S}_+ \\ - 13440\bar{\beta}_+\zeta\mathcal{S}_-\mathcal{S}_+ + 672\zeta\bar{\gamma}\mathcal{S}_-\mathcal{S}_+ - 58240\bar{\beta}_-^2\zeta\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+ - 61376\bar{\beta}_+\zeta\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+ - 4480\bar{\beta}_+^2\zeta\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+ \\ + 35840\bar{\beta}_-^2\zeta\bar{\gamma}^{-2}\mathcal{S}_-\mathcal{S}_+ - 71680\bar{\beta}_+^2\zeta\bar{\gamma}^{-2}\mathcal{S}_-\mathcal{S}_+ - 215040\bar{\beta}_-^2\bar{\beta}_+\zeta\bar{\gamma}^{-3}\mathcal{S}_-\mathcal{S}_+ + 215040\bar{\beta}_+^3\zeta\bar{\gamma}^{-3}\mathcal{S}_-\mathcal{S}_+ \\ - 53760\bar{\beta}_-\zeta\bar{\gamma}^{-2}\bar{\chi}_-\mathcal{S}_-\mathcal{S}_+ + 31360\zeta\bar{\gamma}^{-1}\bar{\chi}_+\mathcal{S}_-\mathcal{S}_+ - 53760\bar{\beta}_+\zeta\bar{\gamma}^{-2}\bar{\chi}_+\mathcal{S}_-\mathcal{S}_+] \\ + \nu[34048\bar{\beta}_+ - 4784\bar{\gamma} - 62720\bar{\beta}_+^2\bar{\gamma}^{-1} - 16040\zeta\mathcal{S}_-^2 - 15680\bar{\beta}_+\zeta\mathcal{S}_-^2 - 6496\zeta\bar{\gamma}\mathcal{S}_-^2 - 71680\bar{\beta}_-^2\zeta\bar{\gamma}^{-1}\mathcal{S}_-^2 \\ - 58688\bar{\beta}_+\zeta\bar{\gamma}^{-1}\mathcal{S}_-^2 + 268800\bar{\beta}_-^2\zeta\bar{\gamma}^{-2}\mathcal{S}_-^2 - 17920\bar{\beta}_+^2\zeta\bar{\gamma}^{-2}\mathcal{S}_-^2 + 35840\zeta\bar{\gamma}^{-1}\bar{\chi}_+\mathcal{S}_-^2 + 53760\bar{\beta}_-\zeta\mathcal{S}_-\mathcal{S}_+ \\ - 58688\bar{\beta}_-\zeta\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+ - 71680\bar{\beta}_-\bar{\beta}_+\zeta\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+ + 250880\bar{\beta}_-\bar{\beta}_+\zeta\bar{\gamma}^{-2}\mathcal{S}_-\mathcal{S}_+ + 35840\zeta\bar{\gamma}^{-1}\bar{\chi}_-\mathcal{S}_-\mathcal{S}_+] \\ + \delta\nu[6720\bar{\beta}_-\zeta\mathcal{S}_-^2 + 34496\bar{\beta}_-\zeta\bar{\gamma}^{-1}\mathcal{S}_-^2 - 12000\zeta\mathcal{S}_-\mathcal{S}_+ + 34496\bar{\beta}_+\zeta\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+] + 6200\zeta\mathcal{S}_-^2\nu^2, \quad (\text{B.13g})$$

$$\begin{aligned}
D_3^{\text{1PN}} = & -896\bar{\beta}_+ + 1920\bar{\gamma} + 896\bar{\beta}_+\bar{\gamma} + 672\bar{\gamma}^2 - 11160\zeta\mathcal{S}_-^2 - 7392\bar{\beta}_+\zeta\mathcal{S}_-^2 - 10080\zeta\bar{\gamma}\mathcal{S}_-^2 - 2240\bar{\beta}_+\zeta\bar{\gamma}\mathcal{S}_-^2 - 2240\zeta\bar{\gamma}^2\mathcal{S}_-^2 \\
& - 4480\bar{\beta}_-^2\zeta\bar{\gamma}^{-1}\mathcal{S}_-^2 - 14784\bar{\beta}_+\zeta\bar{\gamma}^{-1}\mathcal{S}_-^2 - 4480\bar{\beta}_+^2\zeta\bar{\gamma}^{-1}\mathcal{S}_-^2 + 4480\zeta\bar{\gamma}^{-1}\bar{\chi}_+\mathcal{S}_-^2 - 1792\bar{\beta}_-\zeta\mathcal{S}_-\mathcal{S}_+ \\
& - 14784\bar{\beta}_-\zeta\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+ - 8960\bar{\beta}_-\bar{\beta}_+\zeta\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+ + 4480\zeta\bar{\gamma}^{-1}\bar{\chi}_-\mathcal{S}_-\mathcal{S}_+ \\
& + \delta \left[896\bar{\beta}_- - 896\bar{\beta}_-\bar{\gamma} + 7392\bar{\beta}_-\zeta\mathcal{S}_-^2 + 2240\bar{\beta}_-\zeta\bar{\gamma}\mathcal{S}_-^2 + 14784\bar{\beta}_-\zeta\bar{\gamma}^{-1}\mathcal{S}_-^2 + 8960\bar{\beta}_-\bar{\beta}_+\zeta\bar{\gamma}^{-1}\mathcal{S}_-^2 \right. \\
& \quad - 4480\zeta\bar{\gamma}^{-1}\bar{\chi}_-\mathcal{S}_-^2 + 800\zeta\mathcal{S}_-\mathcal{S}_+ + 1792\bar{\beta}_+\zeta\mathcal{S}_-\mathcal{S}_+ + 1120\zeta\bar{\gamma}\mathcal{S}_-\mathcal{S}_+ + 4480\bar{\beta}_-^2\zeta\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+ \\
& \quad \left. + 14784\bar{\beta}_+\zeta\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+ + 4480\bar{\beta}_+^2\zeta\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+ - 4480\zeta\bar{\gamma}^{-1}\bar{\chi}_+\mathcal{S}_-\mathcal{S}_+ \right] \\
& + \nu \left[2688\bar{\beta}_+ - 640\bar{\gamma} + 1464\zeta\mathcal{S}_-^2 + 2464\bar{\beta}_+\zeta\mathcal{S}_-^2 - 1120\zeta\bar{\gamma}\mathcal{S}_-^2 + 17920\bar{\beta}_-^2\zeta\bar{\gamma}^{-1}\mathcal{S}_-^2 - 9408\bar{\beta}_+\zeta\bar{\gamma}^{-1}\mathcal{S}_-^2 \right. \\
& \quad - 35840\bar{\beta}_-^2\zeta\bar{\gamma}^{-2}\mathcal{S}_-^2 + 35840\bar{\beta}_+^2\zeta\bar{\gamma}^{-2}\mathcal{S}_-^2 - 8960\zeta\bar{\gamma}^{-1}\bar{\chi}_+\mathcal{S}_-^2 - 10752\bar{\beta}_-\zeta\mathcal{S}_-\mathcal{S}_+ - 9408\bar{\beta}_-\zeta\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+ \\
& \quad \left. + 17920\bar{\beta}_-\bar{\beta}_+\zeta\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+ - 8960\zeta\bar{\gamma}^{-1}\bar{\chi}_-\mathcal{S}_-\mathcal{S}_+ \right] \\
& + \delta\nu \left[-224\bar{\beta}_-\zeta\mathcal{S}_-^2 - 5824\bar{\beta}_-\zeta\bar{\gamma}^{-1}\mathcal{S}_-^2 + 1536\zeta\mathcal{S}_-\mathcal{S}_+ - 5824\bar{\beta}_+\zeta\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+ \right] + 360\zeta\mathcal{S}_-^2\nu^2, \\
\end{aligned} \tag{B.13h}$$

$$\begin{aligned}
D_4^{\text{1PN}} = & 6048\bar{\beta}_+ - 6774\bar{\gamma} + 13440\bar{\beta}_+\bar{\gamma} - 5040\bar{\gamma}^2 + 3360\bar{\beta}_-^2\bar{\gamma}^{-1} + 30240\bar{\beta}_+^2\bar{\gamma}^{-1} + 3912\zeta\mathcal{S}_-^2 - 13440\bar{\beta}_+\zeta\mathcal{S}_-^2 + 5376\zeta\bar{\gamma}\mathcal{S}_-^2 \\
& + 20160\bar{\beta}_+\zeta\bar{\gamma}^{-1}\mathcal{S}_-^2 - 67200\bar{\beta}_-^2\zeta\bar{\gamma}^{-2}\mathcal{S}_-^2 - 67200\bar{\beta}_+^2\zeta\bar{\gamma}^{-2}\mathcal{S}_-^2 - 13440\bar{\beta}_-\zeta\mathcal{S}_-\mathcal{S}_+ + 20160\bar{\beta}_-\zeta\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+ \\
& - 134400\bar{\beta}_-\bar{\beta}_+\zeta\bar{\gamma}^{-2}\mathcal{S}_-\mathcal{S}_+ \\
& + \delta \left[672\bar{\beta}_- + 6720\bar{\beta}_-\bar{\beta}_+\bar{\gamma}^{-1} - 26880\bar{\beta}_-\zeta\mathcal{S}_-^2 - 60480\bar{\beta}_-\zeta\bar{\gamma}^{-1}\mathcal{S}_-^2 - 26880\bar{\beta}_-\bar{\beta}_+\zeta\bar{\gamma}^{-2}\mathcal{S}_-^2 + 7848\zeta\mathcal{S}_-\mathcal{S}_+ \right. \\
& \quad - 26880\bar{\beta}_+\zeta\mathcal{S}_-\mathcal{S}_+ + 3024\zeta\bar{\gamma}\mathcal{S}_-\mathcal{S}_+ - 60480\bar{\beta}_+\zeta\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+ - 13440\bar{\beta}_-^2\zeta\bar{\gamma}^{-2}\mathcal{S}_-\mathcal{S}_+ - 13440\bar{\beta}_+^2\zeta\bar{\gamma}^{-2}\mathcal{S}_-\mathcal{S}_+ \left. \right] \\
& + \nu \left[48384\bar{\beta}_+ - 7848\bar{\gamma} - 67200\bar{\beta}_+^2\bar{\gamma}^{-1} + 4848\zeta\mathcal{S}_-^2 - 4032\zeta\bar{\gamma}\mathcal{S}_-^2 - 100800\bar{\beta}_+\zeta\bar{\gamma}^{-1}\mathcal{S}_-^2 \right. \\
& \quad + 134400\bar{\beta}_-^2\zeta\bar{\gamma}^{-2}\mathcal{S}_-^2 + 134400\bar{\beta}_+^2\zeta\bar{\gamma}^{-2}\mathcal{S}_-^2 - 100800\bar{\beta}_-\zeta\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+ + 268800\bar{\beta}_-\bar{\beta}_+\zeta\bar{\gamma}^{-2}\mathcal{S}_-\mathcal{S}_+ \left. \right] \\
& + \delta\nu \left[47040\bar{\beta}_-\zeta\bar{\gamma}^{-1}\mathcal{S}_-^2 - 12672\zeta\mathcal{S}_-\mathcal{S}_+ + 47040\bar{\beta}_+\zeta\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+ \right] + 1104\zeta\mathcal{S}_-^2\nu^2, \\
\end{aligned} \tag{B.13i}$$

$$\begin{aligned}
D_5^{\text{1PN}} = & -5600\bar{\beta}_+ + 5652\bar{\gamma} - 8960\bar{\beta}_+\bar{\gamma} + 3808\bar{\gamma}^2 - 2240\bar{\beta}_-^2\bar{\gamma}^{-1} - 15680\bar{\beta}_+^2\bar{\gamma}^{-1} + 16\zeta\mathcal{S}_-^2 + 2240\bar{\beta}_+\zeta\mathcal{S}_-^2 - 2016\zeta\bar{\gamma}\mathcal{S}_-^2 \\
& - 25088\bar{\beta}_+\zeta\bar{\gamma}^{-1}\mathcal{S}_-^2 + 35840\bar{\beta}_-^2\zeta\bar{\gamma}^{-2}\mathcal{S}_-^2 + 35840\bar{\beta}_+^2\zeta\bar{\gamma}^{-2}\mathcal{S}_-^2 + 2240\bar{\beta}_-\zeta\mathcal{S}_-\mathcal{S}_+ - 25088\bar{\beta}_-\zeta\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+ \\
& + 71680\bar{\beta}_-\bar{\beta}_+\zeta\bar{\gamma}^{-2}\mathcal{S}_-\mathcal{S}_+ \\
& + \delta \left[-672\bar{\beta}_- - 8960\bar{\beta}_-\bar{\beta}_+\bar{\gamma}^{-1} + 24640\bar{\beta}_-\zeta\mathcal{S}_-^2 + 51968\bar{\beta}_-\zeta\bar{\gamma}^{-1}\mathcal{S}_-^2 + 35840\bar{\beta}_-\bar{\beta}_+\zeta\bar{\gamma}^{-2}\mathcal{S}_-^2 - 5616\zeta\mathcal{S}_-\mathcal{S}_+ \right. \\
& \quad + 24640\bar{\beta}_+\zeta\mathcal{S}_-\mathcal{S}_+ - 2464\zeta\bar{\gamma}\mathcal{S}_-\mathcal{S}_+ + 51968\bar{\beta}_+\zeta\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+ + 17920\bar{\beta}_-^2\zeta\bar{\gamma}^{-2}\mathcal{S}_-\mathcal{S}_+ + 17920\bar{\beta}_+^2\zeta\bar{\gamma}^{-2}\mathcal{S}_-\mathcal{S}_+ \left. \right] \\
& + \nu \left[-26880\bar{\beta}_+ + 3648\bar{\gamma} + 35840\bar{\beta}_+^2\bar{\gamma}^{-1} + 2512\zeta\mathcal{S}_-^2 + 4032\zeta\bar{\gamma}\mathcal{S}_-^2 + 61824\bar{\beta}_+\zeta\bar{\gamma}^{-1}\mathcal{S}_-^2 - 71680\bar{\beta}_-^2\zeta\bar{\gamma}^{-2}\mathcal{S}_-^2 \right. \\
& \quad - 71680\bar{\beta}_+^2\zeta\bar{\gamma}^{-2}\mathcal{S}_-^2 + 61824\bar{\beta}_-\zeta\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+ - 143360\bar{\beta}_-\bar{\beta}_+\zeta\bar{\gamma}^{-2}\mathcal{S}_-\mathcal{S}_+ \left. \right] \\
& + \delta\nu \left[-45248\bar{\beta}_-\zeta\bar{\gamma}^{-1}\mathcal{S}_-^2 + 11456\zeta\mathcal{S}_-\mathcal{S}_+ - 45248\bar{\beta}_+\zeta\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+ \right] - 2880\zeta\mathcal{S}_-^2\nu^2, \\
\end{aligned} \tag{B.13j}$$

$$\begin{aligned}
D_6^{\text{1PN}} = & -438\bar{\gamma} - 224\bar{\gamma}^2 - 1120\bar{\beta}_-^2\bar{\gamma}^{-1} - 1120\bar{\beta}_+^2\bar{\gamma}^{-1} + 1432\zeta\mathcal{S}_-^2 + 2240\bar{\beta}_+\zeta\mathcal{S}_-^2 + 1792\zeta\bar{\gamma}\mathcal{S}_-^2 + 560\zeta\bar{\gamma}^2\mathcal{S}_-^2 \\
& + 4032\bar{\beta}_+\zeta\bar{\gamma}^{-1}\mathcal{S}_-^2 + 4480\bar{\beta}_-^2\zeta\bar{\gamma}^{-2}\mathcal{S}_-^2 + 4480\bar{\beta}_+^2\zeta\bar{\gamma}^{-2}\mathcal{S}_-^2 + 2240\bar{\beta}_-\zeta\mathcal{S}_-\mathcal{S}_+ + 4032\bar{\beta}_-\zeta\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+ \\
& + 8960\bar{\beta}_-\bar{\beta}_+\zeta\bar{\gamma}^{-2}\mathcal{S}_-\mathcal{S}_+ \\
& + \delta \left[2240\bar{\beta}_-\bar{\beta}_+\bar{\gamma}^{-1} - 2240\bar{\beta}_-\zeta\mathcal{S}_-^2 - 4032\bar{\beta}_-\zeta\bar{\gamma}^{-1}\mathcal{S}_-^2 - 8960\bar{\beta}_-\bar{\beta}_+\zeta\bar{\gamma}^{-2}\mathcal{S}_-^2 - 312\zeta\mathcal{S}_-\mathcal{S}_+ - 2240\bar{\beta}_+\zeta\mathcal{S}_-\mathcal{S}_+ \right. \\
& \quad - 112\zeta\bar{\gamma}\mathcal{S}_-\mathcal{S}_+ - 4032\bar{\beta}_+\zeta\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+ - 4480\bar{\beta}_-^2\zeta\bar{\gamma}^{-2}\mathcal{S}_-\mathcal{S}_+ - 4480\bar{\beta}_+^2\zeta\bar{\gamma}^{-2}\mathcal{S}_-\mathcal{S}_+ \left. \right] \\
& + \nu \left[-2688\bar{\beta}_+ + 888\bar{\gamma} + 4480\bar{\beta}_+^2\bar{\gamma}^{-1} - 3264\zeta\mathcal{S}_-^2 - 1344\zeta\bar{\gamma}\mathcal{S}_-^2 + 1344\bar{\beta}_+\zeta\bar{\gamma}^{-1}\mathcal{S}_-^2 - 8960\bar{\beta}_-^2\zeta\bar{\gamma}^{-2}\mathcal{S}_-^2 \right. \\
& \quad - 8960\bar{\beta}_+^2\zeta\bar{\gamma}^{-2}\mathcal{S}_-^2 + 1344\bar{\beta}_-\zeta\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+ - 17920\bar{\beta}_-\bar{\beta}_+\zeta\bar{\gamma}^{-2}\mathcal{S}_-\mathcal{S}_+ \left. \right] \\
& + \delta\nu \left[6272\bar{\beta}_-\zeta\bar{\gamma}^{-1}\mathcal{S}_-^2 - 1280\zeta\mathcal{S}_-\mathcal{S}_+ + 6272\bar{\beta}_+\zeta\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+ \right] - 48\zeta\mathcal{S}_-^2\nu^2, \tag{B.13k}
\end{aligned}$$

$$\begin{aligned}
E_1^{1.5\text{PN}} = & -7680\bar{\beta}_+ - 8640\bar{\beta}_+^2 - 24\bar{\gamma} + 5760\bar{\beta}_+\bar{\gamma} - 972\bar{\gamma}^2 + 46080\bar{\beta}_+^2\bar{\gamma}^{-1} - 69120\bar{\beta}_+^3\bar{\gamma}^{-2} - 28800\bar{\beta}_+\zeta\mathcal{S}_-^2 + 168\zeta\bar{\gamma}\mathcal{S}_-^2 \\
& - 1200\bar{\beta}_+\zeta\bar{\gamma}\mathcal{S}_-^2 - 1440\zeta\bar{\gamma}^2\mathcal{S}_-^2 + 17280\bar{\beta}_-\zeta\bar{\gamma}^{-1}\mathcal{S}_-^2 + 31680\bar{\beta}_+\zeta\bar{\gamma}^{-1}\mathcal{S}_-^2 + 51840\bar{\beta}_+^2\zeta\bar{\gamma}^{-1}\mathcal{S}_-^2 - 161280\bar{\beta}_-^2\zeta\bar{\gamma}^{-2}\mathcal{S}_-^2 \\
& - 230400\bar{\beta}_+^2\zeta\bar{\gamma}^{-2}\mathcal{S}_-^2 + 691200\bar{\beta}_-^2\bar{\beta}_+\zeta\bar{\gamma}^{-3}\mathcal{S}_-^2 + 414720\bar{\beta}_+^3\zeta\bar{\gamma}^{-3}\mathcal{S}_-^2 + 10080\zeta^2\mathcal{S}_-^4 + 4800\bar{\beta}_+\zeta^2\mathcal{S}_-^4 + 5760\zeta^2\bar{\gamma}\mathcal{S}_-^4 \\
& + 34560\bar{\beta}_+\zeta^2\bar{\gamma}^{-1}\mathcal{S}_-^4 - 69120\bar{\beta}_-^2\zeta^2\bar{\gamma}^{-2}\mathcal{S}_-^4 - 34560\bar{\beta}_+\zeta^2\bar{\gamma}^{-2}\mathcal{S}_-^4 - 69120\bar{\beta}_+^2\zeta^2\bar{\gamma}^{-2}\mathcal{S}_-^4 + 276480\bar{\beta}_-^2\zeta^2\bar{\gamma}^{-3}\mathcal{S}_-^4 \\
& + 276480\bar{\beta}_+^2\zeta^2\bar{\gamma}^{-3}\mathcal{S}_-^4 - 1658880\bar{\beta}_-^2\bar{\beta}_+\zeta^2\bar{\gamma}^{-4}\mathcal{S}_-^4 - 552960\bar{\beta}_+^3\zeta^2\bar{\gamma}^{-4}\mathcal{S}_-^4 - 11520\bar{\beta}_-\zeta\mathcal{S}_-\mathcal{S}_+ + 23040\bar{\beta}_-\zeta\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+ \\
& + 34560\bar{\beta}_-\bar{\beta}_+\zeta\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+ - 253440\bar{\beta}_-\bar{\beta}_+\zeta\bar{\gamma}^{-2}\mathcal{S}_-\mathcal{S}_+ + 552960\bar{\beta}_-\bar{\beta}_+^2\zeta\bar{\gamma}^{-3}\mathcal{S}_-\mathcal{S}_+ - 4800\bar{\beta}_-\zeta^2\mathcal{S}_-\mathcal{S}_+ \\
& + 34560\bar{\beta}_-\zeta^2\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+ - 34560\bar{\beta}_-\zeta^2\bar{\gamma}^{-2}\mathcal{S}_-\mathcal{S}_+ - 138240\bar{\beta}_-\bar{\beta}_+\zeta^2\bar{\gamma}^{-2}\mathcal{S}_-\mathcal{S}_+ + 552960\bar{\beta}_-\bar{\beta}_+\zeta^2\bar{\gamma}^{-3}\mathcal{S}_-\mathcal{S}_+ \\
& - 552960\bar{\beta}_-^3\zeta^2\bar{\gamma}^{-4}\mathcal{S}_-\mathcal{S}_+ - 1658880\bar{\beta}_-\bar{\beta}_+^2\zeta^2\bar{\gamma}^{-4}\mathcal{S}_-\mathcal{S}_+ \\
& + \delta \left[-7680\bar{\beta}_- + 46080\bar{\beta}_-\bar{\beta}_+\bar{\gamma}^{-1} - 69120\bar{\beta}_-\bar{\beta}_+^2\bar{\gamma}^{-2} - 20160\bar{\beta}_-\zeta\mathcal{S}_-^2 + 1200\bar{\beta}_-\zeta\bar{\gamma}\mathcal{S}_-^2 + 31680\bar{\beta}_-\zeta\bar{\gamma}^{-1}\mathcal{S}_-^2 \right. \\
& + 69120\bar{\beta}_-\bar{\beta}_+\zeta\bar{\gamma}^{-1}\mathcal{S}_-^2 - 391680\bar{\beta}_-\bar{\beta}_+\zeta\bar{\gamma}^{-2}\mathcal{S}_-^2 + 138240\bar{\beta}_-^3\zeta\bar{\gamma}^{-3}\mathcal{S}_-^2 + 967680\bar{\beta}_-\bar{\beta}_+^2\zeta\bar{\gamma}^{-3}\mathcal{S}_-^2 - 4800\bar{\beta}_-\zeta^2\mathcal{S}_-^4 \\
& + 34560\bar{\beta}_-\zeta^2\bar{\gamma}^{-1}\mathcal{S}_-^4 - 34560\bar{\beta}_-\zeta^2\bar{\gamma}^{-2}\mathcal{S}_-^4 - 138240\bar{\beta}_-\bar{\beta}_+\zeta^2\bar{\gamma}^{-2}\mathcal{S}_-^4 + 552960\bar{\beta}_-\bar{\beta}_+\zeta^2\bar{\gamma}^{-3}\mathcal{S}_-^4 \\
& - 552960\bar{\beta}_-^3\zeta^2\bar{\gamma}^{-4}\mathcal{S}_-^4 - 1658880\bar{\beta}_-\bar{\beta}_+^2\zeta^2\bar{\gamma}^{-4}\mathcal{S}_-^4 - 20160\bar{\beta}_+\zeta\mathcal{S}_-\mathcal{S}_+ + 2688\zeta\bar{\gamma}\mathcal{S}_-\mathcal{S}_+ + 23040\bar{\beta}_+\zeta\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+ \\
& + 34560\bar{\beta}_+^2\zeta\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+ - 92160\bar{\beta}_-^2\zeta\bar{\gamma}^{-2}\mathcal{S}_-\mathcal{S}_+ - 161280\bar{\beta}_+^2\zeta\bar{\gamma}^{-2}\mathcal{S}_-\mathcal{S}_+ + 276480\bar{\beta}_-^2\bar{\beta}_+\zeta\bar{\gamma}^{-3}\mathcal{S}_-\mathcal{S}_+ \\
& + 276480\bar{\beta}_+^2\zeta\bar{\gamma}^{-3}\mathcal{S}_-\mathcal{S}_+ + 10080\zeta^2\mathcal{S}_-\mathcal{S}_+ + 4800\bar{\beta}_+\zeta^2\mathcal{S}_-\mathcal{S}_+ + 5760\zeta^2\bar{\gamma}\mathcal{S}_-\mathcal{S}_+ + 34560\bar{\beta}_+\zeta^2\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+ \\
& - 69120\bar{\beta}_-^2\zeta^2\bar{\gamma}^{-2}\mathcal{S}_-\mathcal{S}_+ - 34560\bar{\beta}_+\zeta^2\bar{\gamma}^{-2}\mathcal{S}_-\mathcal{S}_+ - 69120\bar{\beta}_+^2\zeta^2\bar{\gamma}^{-2}\mathcal{S}_-\mathcal{S}_+ + 276480\bar{\beta}_-^2\zeta^2\bar{\gamma}^{-3}\mathcal{S}_-\mathcal{S}_+ \\
& + 276480\bar{\beta}_+^2\zeta^2\bar{\gamma}^{-3}\mathcal{S}_-\mathcal{S}_+ - 1658880\bar{\beta}_-\bar{\beta}_+\zeta^2\bar{\gamma}^{-4}\mathcal{S}_-\mathcal{S}_+ - 552960\bar{\beta}_+^3\zeta^2\bar{\gamma}^{-4}\mathcal{S}_-\mathcal{S}_+ \left. \right] \\
& + \nu \left[-8448\zeta\mathcal{S}_-^2 - 20160\bar{\beta}_+\zeta\mathcal{S}_-^2 + 7272\zeta\bar{\gamma}\mathcal{S}_-^2 + 94080\bar{\beta}_+\zeta\bar{\gamma}^{-1}\mathcal{S}_-^2 - 23040\bar{\beta}_+^2\zeta\bar{\gamma}^{-1}\mathcal{S}_-^2 - 161280\bar{\beta}_-^2\zeta\bar{\gamma}^{-2}\mathcal{S}_-^2 \right. \\
& - 322560\bar{\beta}_+^2\zeta\bar{\gamma}^{-2}\mathcal{S}_-^2 - 16896\zeta^2\mathcal{S}_-^4 + 3840\zeta^2\bar{\gamma}\mathcal{S}_-^4 + 40320\bar{\beta}_+\zeta^2\bar{\gamma}^{-1}\mathcal{S}_-^4 + 46080\bar{\beta}_-^2\zeta^2\bar{\gamma}^{-2}\mathcal{S}_-^4 \\
& - 161280\bar{\beta}_+\zeta^2\bar{\gamma}^{-2}\mathcal{S}_-^4 + 46080\bar{\beta}_+^2\zeta^2\bar{\gamma}^{-2}\mathcal{S}_-^4 + 645120\bar{\beta}_-^2\zeta^2\bar{\gamma}^{-3}\mathcal{S}_-^4 + 645120\bar{\beta}_+^2\zeta^2\bar{\gamma}^{-3}\mathcal{S}_-^4 + 53760\bar{\beta}_-\zeta\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+ \\
& - 161280\bar{\beta}_-\bar{\beta}_+\zeta\bar{\gamma}^{-2}\mathcal{S}_-\mathcal{S}_+ + 19200\bar{\beta}_-\zeta^2\mathcal{S}_-\mathcal{S}_+ + 40320\bar{\beta}_-\zeta^2\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+ - 161280\bar{\beta}_-\zeta^2\bar{\gamma}^{-2}\mathcal{S}_-\mathcal{S}_+ \\
& \left. + 92160\bar{\beta}_-\bar{\beta}_+\zeta^2\bar{\gamma}^{-2}\mathcal{S}_-\mathcal{S}_+ + 1290240\bar{\beta}_-\bar{\beta}_+\zeta^2\bar{\gamma}^{-3}\mathcal{S}_-\mathcal{S}_+ \right] \\
& + \delta\nu \left[-9600\bar{\beta}_-\zeta^2\bar{\gamma}^{-1}\mathcal{S}_-^4 + 1536\zeta^2\mathcal{S}_-\mathcal{S}_+ - 9600\bar{\beta}_+\zeta^2\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+ \right] + 6144\zeta^2\mathcal{S}_-^4\nu^2,
\end{aligned}$$

(B.13l)

$$\begin{aligned}
E_2^{1.5\text{PN}} = & -23040\bar{\beta}_+ - 25920\bar{\beta}_+^2 + 792\bar{\gamma} + 17280\bar{\beta}_+\bar{\gamma} - 2484\bar{\gamma}^2 + 138240\bar{\beta}_+^2\bar{\gamma}^{-1} - 207360\bar{\beta}_+^3\bar{\gamma}^{-2} - 84960\bar{\beta}_+\zeta\mathcal{S}_-^2 \\
& + 7920\zeta\bar{\gamma}\mathcal{S}_-^2 - 1440\zeta\bar{\gamma}^2\mathcal{S}_-^2 + 51840\bar{\beta}_-\zeta\bar{\gamma}^{-1}\mathcal{S}_-^2 + 95040\bar{\beta}_+\zeta\bar{\gamma}^{-1}\mathcal{S}_-^2 + 155520\bar{\beta}_+^2\zeta\bar{\gamma}^{-1}\mathcal{S}_-^2 - 483840\bar{\beta}_-^2\zeta\bar{\gamma}^{-2}\mathcal{S}_-^2 \\
& - 691200\bar{\beta}_+^2\zeta\bar{\gamma}^{-2}\mathcal{S}_-^2 + 2073600\bar{\beta}_-^2\bar{\beta}_+\zeta\bar{\gamma}^{-3}\mathcal{S}_-^2 + 1244160\bar{\beta}_+^3\zeta\bar{\gamma}^{-3}\mathcal{S}_-^2 + 5760\zeta^2\bar{\gamma}\mathcal{S}_-^4 + 103680\bar{\beta}_+\zeta^2\bar{\gamma}^{-1}\mathcal{S}_-^4 \\
& - 207360\bar{\beta}_-^2\zeta^2\bar{\gamma}^{-2}\mathcal{S}_-^4 - 103680\bar{\beta}_+\zeta^2\bar{\gamma}^{-2}\mathcal{S}_-^4 - 207360\bar{\beta}_+^2\zeta^2\bar{\gamma}^{-2}\mathcal{S}_-^4 + 829440\bar{\beta}_-^2\zeta^2\bar{\gamma}^{-3}\mathcal{S}_-^4 + 829440\bar{\beta}_+^2\zeta^2\bar{\gamma}^{-3}\mathcal{S}_-^4 \\
& - 4976640\bar{\beta}_-^2\bar{\beta}_+\zeta^2\bar{\gamma}^{-4}\mathcal{S}_-^4 - 1658880\bar{\beta}_+^3\zeta^2\bar{\gamma}^{-4}\mathcal{S}_-^4 - 36000\bar{\beta}_-\zeta\mathcal{S}_-\mathcal{S}_+ + 69120\bar{\beta}_-\zeta\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+ \\
& + 103680\bar{\beta}_-\bar{\beta}_+\zeta\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+ - 760320\bar{\beta}_-\bar{\beta}_+\zeta\bar{\gamma}^{-2}\mathcal{S}_-\mathcal{S}_+ + 1658880\bar{\beta}_-\bar{\beta}_+^2\zeta\bar{\gamma}^{-3}\mathcal{S}_-\mathcal{S}_+ + 103680\bar{\beta}_-\zeta^2\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+ \\
& - 103680\bar{\beta}_-\zeta^2\bar{\gamma}^{-2}\mathcal{S}_-\mathcal{S}_+ - 414720\bar{\beta}_-\bar{\beta}_+\zeta^2\bar{\gamma}^{-2}\mathcal{S}_-\mathcal{S}_+ + 1658880\bar{\beta}_-\bar{\beta}_+\zeta^2\bar{\gamma}^{-3}\mathcal{S}_-\mathcal{S}_+ - 1658880\bar{\beta}_-^3\zeta^2\bar{\gamma}^{-4}\mathcal{S}_-\mathcal{S}_+ \\
& - 4976640\bar{\beta}_-\bar{\beta}_+^2\zeta^2\bar{\gamma}^{-4}\mathcal{S}_-\mathcal{S}_+ \\
& + \delta \left[-23040\bar{\beta}_- + 138240\bar{\beta}_-\bar{\beta}_+\bar{\gamma}^{-1} - 207360\bar{\beta}_-\bar{\beta}_+^2\bar{\gamma}^{-2} - 61920\bar{\beta}_-\zeta\mathcal{S}_-^2 + 95040\bar{\beta}_-\zeta\bar{\gamma}^{-1}\mathcal{S}_-^2 \right. \\
& + 207360\bar{\beta}_-\bar{\beta}_+\zeta\bar{\gamma}^{-1}\mathcal{S}_-^2 - 1175040\bar{\beta}_-\bar{\beta}_+\zeta\bar{\gamma}^{-2}\mathcal{S}_-^2 + 414720\bar{\beta}_-^3\zeta\bar{\gamma}^{-3}\mathcal{S}_-^2 + 2903040\bar{\beta}_-\bar{\beta}_+^2\zeta\bar{\gamma}^{-3}\mathcal{S}_-^2 \\
& + 103680\bar{\beta}_-\zeta^2\bar{\gamma}^{-1}\mathcal{S}_-^4 - 103680\bar{\beta}_-\zeta^2\bar{\gamma}^{-2}\mathcal{S}_-^4 - 414720\bar{\beta}_-\bar{\beta}_+\zeta^2\bar{\gamma}^{-2}\mathcal{S}_-^4 + 1658880\bar{\beta}_-\bar{\beta}_+\zeta^2\bar{\gamma}^{-3}\mathcal{S}_-^4 \\
& - 1658880\bar{\beta}_-^3\zeta^2\bar{\gamma}^{-4}\mathcal{S}_-^4 - 4976640\bar{\beta}_-\bar{\beta}_+^2\zeta^2\bar{\gamma}^{-4}\mathcal{S}_-^4 - 59040\bar{\beta}_+\zeta\mathcal{S}_-\mathcal{S}_+ + 7920\zeta\bar{\gamma}\mathcal{S}_-\mathcal{S}_+ + 69120\bar{\beta}_+\zeta\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+ \\
& + 103680\bar{\beta}_+^2\zeta\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+ - 276480\bar{\beta}_-^2\zeta\bar{\gamma}^{-2}\mathcal{S}_-\mathcal{S}_+ - 483840\bar{\beta}_+^2\zeta\bar{\gamma}^{-2}\mathcal{S}_-\mathcal{S}_+ + 829440\bar{\beta}_-^2\bar{\beta}_+\zeta\bar{\gamma}^{-3}\mathcal{S}_-\mathcal{S}_+ \\
& + 829440\bar{\beta}_+^2\zeta\bar{\gamma}^{-3}\mathcal{S}_-\mathcal{S}_+ + 5760\zeta^2\bar{\gamma}\mathcal{S}_-\mathcal{S}_+ + 103680\bar{\beta}_+\zeta^2\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+ - 207360\bar{\beta}_-^2\zeta^2\bar{\gamma}^{-2}\mathcal{S}_-\mathcal{S}_+ \\
& - 103680\bar{\beta}_+\zeta^2\bar{\gamma}^{-2}\mathcal{S}_-\mathcal{S}_+ - 207360\bar{\beta}_+^2\zeta^2\bar{\gamma}^{-2}\mathcal{S}_-\mathcal{S}_+ + 829440\bar{\beta}_-^2\zeta^2\bar{\gamma}^{-3}\mathcal{S}_-\mathcal{S}_+ + 829440\bar{\beta}_+^2\zeta^2\bar{\gamma}^{-3}\mathcal{S}_-\mathcal{S}_+ \\
& - 4976640\bar{\beta}_-^2\bar{\beta}_+\zeta^2\bar{\gamma}^{-4}\mathcal{S}_-\mathcal{S}_+ - 1658880\bar{\beta}_+^3\zeta^2\bar{\gamma}^{-4}\mathcal{S}_-\mathcal{S}_+ \left. \right] \\
& + \nu \left[2304\zeta\mathcal{S}_-^2 - 97920\bar{\beta}_+\zeta\mathcal{S}_-^2 + 30528\zeta\bar{\gamma}\mathcal{S}_-^2 + 201600\bar{\beta}_+\zeta\bar{\gamma}^{-1}\mathcal{S}_-^2 - 345600\bar{\beta}_-^2\zeta\bar{\gamma}^{-2}\mathcal{S}_-^2 - 691200\bar{\beta}_+^2\zeta\bar{\gamma}^{-2}\mathcal{S}_-^2 \right. \\
& - 58752\zeta^2\mathcal{S}_-^4 + 195840\bar{\beta}_+\zeta^2\bar{\gamma}^{-1}\mathcal{S}_-^4 - 345600\bar{\beta}_+\zeta^2\bar{\gamma}^{-2}\mathcal{S}_-^4 + 1382400\bar{\beta}_-^2\zeta^2\bar{\gamma}^{-3}\mathcal{S}_-^4 + 1382400\bar{\beta}_+^2\zeta^2\bar{\gamma}^{-3}\mathcal{S}_-^4 \\
& + 115200\bar{\beta}_-\zeta\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+ - 345600\bar{\beta}_-\bar{\beta}_+\zeta\bar{\gamma}^{-2}\mathcal{S}_-\mathcal{S}_+ + 195840\bar{\beta}_-\zeta^2\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+ - 345600\bar{\beta}_-\zeta^2\bar{\gamma}^{-2}\mathcal{S}_-\mathcal{S}_+ \\
& \left. + 2764800\bar{\beta}_-\bar{\beta}_+\zeta^2\bar{\gamma}^{-3}\mathcal{S}_-\mathcal{S}_+ \right] \\
& + \delta\nu \left[-11520\bar{\beta}_-\zeta^2\bar{\gamma}^{-1}\mathcal{S}_-^4 + 1152\zeta^2\mathcal{S}_-\mathcal{S}_+ - 11520\bar{\beta}_+\zeta^2\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+ \right] + 6912\zeta^2\mathcal{S}_-^4\nu^2,
\end{aligned}$$

(B.13m)

$$\begin{aligned}
E_3^{1.5\text{PN}} = & 7680\bar{\beta}_+ + 8640\bar{\beta}_+^2 - 840\bar{\gamma} - 5760\bar{\beta}_+\bar{\gamma} + 540\bar{\gamma}^2 - 46080\bar{\beta}_+^2\bar{\gamma}^{-1} + 69120\bar{\beta}_+^3\bar{\gamma}^{-2} + 27360\bar{\beta}_+\zeta\mathcal{S}_-^2 - 1104\zeta\bar{\gamma}\mathcal{S}_-^2 \\
& + 960\zeta\bar{\gamma}^2\mathcal{S}_-^2 - 17280\bar{\beta}_-^2\zeta\bar{\gamma}^{-1}\mathcal{S}_-^2 - 31680\bar{\beta}_+\zeta\bar{\gamma}^{-1}\mathcal{S}_-^2 - 51840\bar{\beta}_+^2\zeta\bar{\gamma}^{-1}\mathcal{S}_-^2 + 161280\bar{\beta}_-^2\zeta\bar{\gamma}^{-2}\mathcal{S}_-^2 + 230400\bar{\beta}_+^2\zeta\bar{\gamma}^{-2}\mathcal{S}_-^2 \\
& - 691200\bar{\beta}_-^2\bar{\beta}_+\zeta\bar{\gamma}^{-3}\mathcal{S}_-^2 - 414720\bar{\beta}_+^3\zeta\bar{\gamma}^{-3}\mathcal{S}_-^2 - 3840\zeta^2\mathcal{S}_-^4 - 3840\zeta^2\bar{\gamma}\mathcal{S}_-^4 - 34560\bar{\beta}_+\zeta^2\bar{\gamma}^{-1}\mathcal{S}_-^4 + 69120\bar{\beta}_-^2\zeta^2\bar{\gamma}^{-2}\mathcal{S}_-^4 \\
& + 34560\bar{\beta}_+\zeta^2\bar{\gamma}^{-2}\mathcal{S}_-^4 + 69120\bar{\beta}_+^2\zeta^2\bar{\gamma}^{-2}\mathcal{S}_-^4 - 276480\bar{\beta}_-^2\zeta^2\bar{\gamma}^{-3}\mathcal{S}_-^4 - 276480\bar{\beta}_+^2\zeta^2\bar{\gamma}^{-3}\mathcal{S}_-^4 + 1658880\bar{\beta}_-^2\bar{\beta}_+\zeta^2\bar{\gamma}^{-4}\mathcal{S}_-^4 \\
& + 552960\bar{\beta}_+^3\zeta^2\bar{\gamma}^{-4}\mathcal{S}_-^4 + 12960\bar{\beta}_-\zeta\mathcal{S}_-\mathcal{S}_+ - 23040\bar{\beta}_-\zeta\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+ - 34560\bar{\beta}_-\bar{\beta}_+\zeta\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+ \\
& + 253440\bar{\beta}_-\bar{\beta}_+\zeta\bar{\gamma}^{-2}\mathcal{S}_-\mathcal{S}_+ - 552960\bar{\beta}_-\bar{\beta}_+^2\zeta\bar{\gamma}^{-3}\mathcal{S}_-\mathcal{S}_+ - 34560\bar{\beta}_-\zeta^2\bar{\gamma}^{-1}\mathcal{S}_-^3\mathcal{S}_+ + 34560\bar{\beta}_-\zeta^2\bar{\gamma}^{-2}\mathcal{S}_-^3\mathcal{S}_+ \\
& + 138240\bar{\beta}_-\bar{\beta}_+\zeta^2\bar{\gamma}^{-2}\mathcal{S}_-^3\mathcal{S}_+ - 552960\bar{\beta}_-\bar{\beta}_+\zeta^2\bar{\gamma}^{-3}\mathcal{S}_-^3\mathcal{S}_+ + 552960\bar{\beta}_-^3\zeta^2\bar{\gamma}^{-4}\mathcal{S}_-^3\mathcal{S}_+ + 1658880\bar{\beta}_-\bar{\beta}_+^2\zeta^2\bar{\gamma}^{-4}\mathcal{S}_-^3\mathcal{S}_+ \\
& + \delta \left[7680\bar{\beta}_- - 46080\bar{\beta}_-\bar{\beta}_+\bar{\gamma}^{-1} + 69120\bar{\beta}_-\bar{\beta}_+^2\bar{\gamma}^{-2} + 21600\bar{\beta}_-\zeta\mathcal{S}_-^2 - 31680\bar{\beta}_-\zeta\bar{\gamma}^{-1}\mathcal{S}_-^2 - 69120\bar{\beta}_-\bar{\beta}_+\zeta\bar{\gamma}^{-1}\mathcal{S}_-^2 \right. \\
& + 391680\bar{\beta}_-\bar{\beta}_+\zeta\bar{\gamma}^{-2}\mathcal{S}_-^2 - 138240\bar{\beta}_-^3\zeta\bar{\gamma}^{-3}\mathcal{S}_-^2 - 967680\bar{\beta}_-\bar{\beta}_+^2\zeta\bar{\gamma}^{-3}\mathcal{S}_-^2 - 34560\bar{\beta}_-\zeta^2\bar{\gamma}^{-1}\mathcal{S}_-^4 \\
& + 34560\bar{\beta}_-\zeta^2\bar{\gamma}^{-2}\mathcal{S}_-^4 + 138240\bar{\beta}_-\bar{\beta}_+\zeta^2\bar{\gamma}^{-2}\mathcal{S}_-^4 - 552960\bar{\beta}_-\bar{\beta}_+\zeta^2\bar{\gamma}^{-3}\mathcal{S}_-^4 + 552960\bar{\beta}_-^3\zeta^2\bar{\gamma}^{-4}\mathcal{S}_-^4 \\
& + 1658880\bar{\beta}_-\bar{\beta}_+^2\zeta^2\bar{\gamma}^{-4}\mathcal{S}_-^4 + 18720\bar{\beta}_+\zeta\mathcal{S}_-\mathcal{S}_+ - 2064\zeta\bar{\gamma}\mathcal{S}_-\mathcal{S}_+ - 23040\bar{\beta}_+\zeta\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+ - 34560\bar{\beta}_+^2\zeta\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+ \\
& + 92160\bar{\beta}_-^2\zeta\bar{\gamma}^{-2}\mathcal{S}_-\mathcal{S}_+ + 161280\bar{\beta}_+^2\zeta\bar{\gamma}^{-2}\mathcal{S}_-\mathcal{S}_+ - 276480\bar{\beta}_-^2\bar{\beta}_+\zeta\bar{\gamma}^{-3}\mathcal{S}_-\mathcal{S}_+ - 276480\bar{\beta}_+^2\zeta\bar{\gamma}^{-3}\mathcal{S}_-\mathcal{S}_+ \\
& - 3840\zeta^2\mathcal{S}_-^3\mathcal{S}_+ - 3840\zeta^2\bar{\gamma}\mathcal{S}_-^3\mathcal{S}_+ - 34560\bar{\beta}_+\zeta^2\bar{\gamma}^{-1}\mathcal{S}_-^3\mathcal{S}_+ + 69120\bar{\beta}_-^2\zeta^2\bar{\gamma}^{-2}\mathcal{S}_-^3\mathcal{S}_+ + 34560\bar{\beta}_+\zeta^2\bar{\gamma}^{-2}\mathcal{S}_-^3\mathcal{S}_+ \\
& + 69120\bar{\beta}_+^2\zeta^2\bar{\gamma}^{-2}\mathcal{S}_-^3\mathcal{S}_+ - 276480\bar{\beta}_-^2\zeta^2\bar{\gamma}^{-3}\mathcal{S}_-^3\mathcal{S}_+ - 276480\bar{\beta}_+^2\zeta^2\bar{\gamma}^{-3}\mathcal{S}_-^3\mathcal{S}_+ + 1658880\bar{\beta}_-^2\bar{\beta}_+\zeta^2\bar{\gamma}^{-4}\mathcal{S}_-^3\mathcal{S}_+ \\
& \left. + 552960\bar{\beta}_+^3\zeta^2\bar{\gamma}^{-4}\mathcal{S}_-^3\mathcal{S}_+ \right] \\
& + \nu \left[-2304\zeta\mathcal{S}_-^2 + 55680\bar{\beta}_+\zeta\mathcal{S}_-^2 - 16968\zeta\bar{\gamma}\mathcal{S}_-^2 - 120960\bar{\beta}_+\zeta\bar{\gamma}^{-1}\mathcal{S}_-^2 + 207360\bar{\beta}_-^2\zeta\bar{\gamma}^{-2}\mathcal{S}_-^2 + 414720\bar{\beta}_+^2\zeta\bar{\gamma}^{-2}\mathcal{S}_-^2 \right. \\
& + 31776\zeta^2\mathcal{S}_-^4 - 111360\bar{\beta}_+\zeta^2\bar{\gamma}^{-1}\mathcal{S}_-^4 + 207360\bar{\beta}_+\zeta^2\bar{\gamma}^{-2}\mathcal{S}_-^4 - 829440\bar{\beta}_-^2\zeta^2\bar{\gamma}^{-3}\mathcal{S}_-^4 - 829440\bar{\beta}_+^2\zeta^2\bar{\gamma}^{-3}\mathcal{S}_-^4 \\
& - 69120\bar{\beta}_-\zeta\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+ + 207360\bar{\beta}_-\bar{\beta}_+\zeta\bar{\gamma}^{-2}\mathcal{S}_-\mathcal{S}_+ - 111360\bar{\beta}_-\zeta^2\bar{\gamma}^{-1}\mathcal{S}_-^3\mathcal{S}_+ + 207360\bar{\beta}_-\zeta^2\bar{\gamma}^{-2}\mathcal{S}_-^3\mathcal{S}_+ \\
& \left. - 1658880\bar{\beta}_-\bar{\beta}_+\zeta^2\bar{\gamma}^{-3}\mathcal{S}_-^3\mathcal{S}_+ \right] \\
& + \delta\nu \left[3840\bar{\beta}_-\zeta^2\bar{\gamma}^{-1}\mathcal{S}_-^4 - 1056\zeta^2\mathcal{S}_-^3\mathcal{S}_+ + 3840\bar{\beta}_+\zeta^2\bar{\gamma}^{-1}\mathcal{S}_-^3\mathcal{S}_+ \right] - 5376\zeta^2\mathcal{S}_-^4\nu^2.
\end{aligned}$$

(B.13n)

Résumé en français

L'une des prédictions de la relativité générale est l'existence d'ondes gravitationnelles, qui sont des déformations oscillatoires de l'espace-temps. Cependant, le calcul de la forme de ces ondes ne peut pas se faire analytiquement de manière exacte, et les techniques numériques sont très difficiles et se heurtent au problème du temps de calcul. C'est pourquoi il est nécessaire de faire usage de techniques d'approximations pour fournir une prédiction analytique pour ces formes d'onde. L'une des techniques les plus répandues est l'approximation post-newtonienne, qui s'applique bien à une source d'onde gravitationnelle composée de deux étoiles séparées d'une distance très grande devant leur taille typique, et donc orbitant avec une vitesse v très petite devant celle de la lumière c . Il est alors naturel de décrire la source comme un développement en le petit paramètre v/c , que l'on appelle développement post-newtonien (PN). Ces techniques ont permis d'établir les premières expressions pour la forme d'onde, jusqu'au premier ordre PN (1PN), c'est-à-dire avec les corrections d'ordre $(v/c)^2$. Toutefois, ces techniques se sont heurtées à des problèmes de divergences, qui ont été résolus notamment grâce à l'introduction d'un autre développement, dit post-minkowskien (PM), valable dans le vide extérieur à la source, et qui consiste à développer la métrique en petites perturbations autour de la métrique plate de Minkowski. Grâce à la séparation d'échelle entre la taille de la source et la longueur de l'onde gravitationnelle, il existe une zone tampon où les deux développements sont valables, ce qui permet de les apparier, à la manière d'une condition aux bords. Ces techniques, ainsi que de nombreuses améliorations apportées au fil des années, ont permis de calculer la forme d'onde jusqu'à 3.5PN (c'est-à-dire en $(v/c)^7$), ce qui est extrêmement utile pour l'analyse des données issues des détecteurs d'ondes gravitationnelles. De plus, les équations du mouvement régissant la dynamique relativiste du système à deux corps ont été résolues à 4PN. Se basant sur des premiers résultats (notamment le quadrupôle source à 4PN), cette thèse a permis de compléter le calcul complet de la forme d'onde gravitationnelle pour des orbites quasi-circulaires à 4PN, ainsi que le flux d'énergie émis par les ondes gravitationnelles jusqu'à l'ordre 4.5PN. L'une des étapes majeures était le calcul d'un effet non-linéaire, au troisième ordre post-minkowskien, nommé « sillage de mémoire », qui peut être interprété comme une onde gravitationnelle linéaire qui est diffusée par la courbure de l'espace-temps générée par le système binaire; laquelle onde diffusée réémet elle-même de nouvelles ondes gravitationnelles, à la manière de l'effet de mémoire. Une autre contribution de cette thèse concerne une classe de théories alternative de la gravité, les théories tenseur-scalaire sans potentiels. Dans ces théories, nous calculons la forme d'onde ainsi que le flux à l'ordre 1.5PN au-delà du rayonnement quadrupolaire de la relativité générale, ce qui correspond à un ordre relatif de 2.5PN au-delà du rayonnement dipolaire de la théorie, et donnons les résultats réduits pour les

orbites circulaires. Pour les orbites non circulaires, nous corrigeons au passage une erreur dans la littérature pour l'expression du flux à l'ordre 1PN.

Dans le Chapitre I, nous rappelons un certain nombre de concepts élémentaires liés aux ondes gravitationnelles. Nous rappelons comment les équations de champ de la relativité générale, que l'on appelle équations d'Einstein, peuvent être linéarisées dans le vide lorsqu'elles sont vues comme de petites déviations à la métrique d'espace-temps plate de Minkowski, et nous introduisons la notion de déviation à la métrique « gothique ». Nous distinguons ensuite les effets de pure jauge, liées uniquement au choix de système de coordonnées. En particulier, dans le cas d'une onde plane, nous relient le tenseur de Riemann linéarisé à la métrique dans une jauge transverse et sans trace. Nous définissons ensuite les polarisations « plus » et « croix », que l'on relie dans le cas linéarisé au scalaire de Newman-Penrose. On définit également, dans le cadre des théories alternatives de la gravité, la polarisation dite de respiration, la polarisation longitudinale et les deux polarisations vectorielles. Nous nous intéressons ensuite aux équations d'Einstein linéarisées sourcées par la matière, représentée par son tenseur d'énergie-impulsion. Nous résolvons l'équation d'onde à l'ordre post-newtonien le plus bas, ce qui nous amène à introduire la notion de quadrupôle. Nous calculons le flux d'énergie transporté par les ondes gravitationnelles à travers une sphère de rayon infini, et retrouvons la formule du quadrupôle d'Einstein. Nous spécialisons alors notre étude préliminaire au cas de deux particules ponctuelles sur une orbite quasi-circulaire, et nous pouvons alors exprimer la forme d'onde en fonction de la fréquence orbitale et de l'angle d'inclinaison du plan orbital du système binaire par rapport à l'observateur. Par des arguments de conservation de l'énergie, nous trouvons alors une augmentation séculaire de la fréquence orbitale, le « gazouillement », ainsi qu'une diminution séculaire de la distance entre les deux corps en orbite. Nous rappelons les résultats de Peters et Matthews dans le cas des orbites d'excentricité non nulle, et retrouvons la propension d'une orbite excentrique à se circulariser sous l'effet de l'émission d'ondes gravitationnelles. Enfin, nous répétons notre étude préliminaire dans le cas d'une théorie alternative, la théorie de Brans et Dicke sans potentiel, et retrouvons que l'émission dominante d'ondes gravitationnelles est due au dipôle, et qu'elle engendre un mode de respiration. Nous nous intéressons ensuite aux détecteurs, où nous rappelons l'équation de déviation géodésique décrivant le fonctionnement des détecteurs d'ondes gravitationnelles, et passons en revue les détecteurs actuels (LIGO, Virgo et KAGRA) ainsi que les détecteurs futurs, construits sur Terre (Cosmic Explorer, Einstein Telescope) ou dans l'Espace (LISA, ...). Nous rappelons alors les rudiments de techniques d'analyse des données permettant d'extraire un signal du bruit inhérents aux détecteurs. En particulier, nous étudions les techniques de filtrage adapté, qui permet de détecter un faible signal malgré un fort bruit, mais qui requiert pour cela une connaissance très fine de la forme fonctionnelle que doit avoir le signal.

Nous entamons alors la Partie A, qui traite de nos résultats obtenus en relativité générale. Dans le Chapitre II, nous passons en revue le formalisme post-newtonien et post-minkowskien multipolaire (PN-MPM) qui sera utilisé tout au long de la thèse pour traiter le problème de la génération d'onde à des ordres très élevés. Nous rappelons d'abord la formulation de Landau et Lifschitz de la relativité générale sous forme d'une équation d'onde. Nous montrons ensuite comment ces équations peuvent être résolues dans des coordonnées harmoniques dans le cas d'une source compacte. Pour cela, nous séparons le problème en deux: d'un côté, nous exprimons la solution la plus générale de

la métrique extérieure dans le vide, paramétrisée par des moments multipolaires (dits de source et de jauge); et de l'autre, nous trouvons la solution près de la source de matière dans l'approximation des petites vitesses caractéristiques. Ces deux solutions peuvent alors être reliées par une condition au bord sous la forme d'un appariement des développements asymptotiques des deux métriques. Nous spécialisons alors notre étude au cas de deux particules ponctuelles, ce qui est bien adapté à l'étude de systèmes binaires de trous noirs ou d'étoiles à neutrons dans la phase spiralante. Or, nous voyons dans ce cas que le traitement de la métrique à la position des particules requiert des techniques de régularisation. Pour des ordres élevés, on doit recourir à une régularisation dite dimensionnelle, qui consiste à résoudre les équations d'Einstein en d dimensions (ce qui est toujours bien défini par continuation analytique en d), et de prendre la limite tridimensionnelle à la fin du calcul, pour des quantités physiques observables. Nous introduisons quelques outils utiles pour l'étude des orbites quasi-circulaires, puis nous rappelons des résultats récents concernant les équations du mouvement à 4PN. Enfin, nous expliquons comment extraire de la métrique extérieure des quantités observables par un observateur asymptotique, que l'on exprime grâce à l'introduction de moments multipolaires dits radiatifs, desquels on peut directement obtenir la forme d'onde et le flux.

Le Chapitre III est dédié à l'étude d'une forme de la métrique extérieure dite canonique. En effet, pour l'appariement entre la métrique extérieure et la métrique dans la zone proche, il est nécessaire de choisir pour la métrique extérieure une forme paramétrisée par deux moments sources, mais aussi quatre moments dits de jauge. Grâce la liberté de jauge résiduelle de la jauge harmonique, on peut paramétriser la métrique extérieure de manière équivalente à l'aide de seulement deux moments, dits canoniques. Pour cela, il faut prendre en compte le fait que les moments canoniques et de source diffèrent par des termes non-linéaires impliquant les moments de jauge. Toutefois, dans le cadre de la régularisation dimensionnelle, cette relation diffère de la relation tridimensionnelle usuelle. Les techniques pour l'obtenir sont traitées, et la relation finale entre moments de source et moments canoniques en d dimensions est présentée.

Le Chapitre IV est dédié au calcul d'interactions non-linéaires qui rentrent dans la relation entre les moments canonique et les moments radiatifs, lesquels sont observables par l'observateur à l'infini. Une nuance, toutefois, est que l'on introduit une autre version de l'algorithme canonique qui évite de manière systématique l'apparition de logarithmes en la distance radiale dans la limite où la distance radiale s'approche de l'infini. Cette nouvelle métrique, dite métrique canonique en coordonnées radiatives (par opposition à la métrique canonique en coordonnées harmoniques du Chapitre III), permet également d'exprimer des résultats connus à l'ordre quadratique dans l'itération post-minkowskienne. On donne alors les termes qui sourcent l'équation d'onde à l'ordre cubique, puis on décrit les techniques de calcul utilisées pour intégrer cette équation d'onde. Ceci est implémenté pour les interactions qui rentrent à l'ordre 4PN, et l'on décrit également une méthode de simplification de la métrique ainsi obtenue. Finalement, nous présentons les moments radiatifs associés aux trois interactions non-linéaires suivantes: les sillages de sillages, qui rentrent à 3PN et sont connus en termes des moments canoniques en coordonnées harmonique; les sillages de mémoire, qui rentrent à 4PN et sont associés à une interaction entre la masse-énergie totale du système et deux quadrupôles; et enfin des termes 4PN associés à l'interaction entre la masse-énergie, le moment angulaire et le quadrupôle.

La Chapitre V traite des méthodes pour relier les résultats obtenus dans le Chapitre IV,

en terme des moments canoniques en coordonnées radiatives, aux résultats connus en termes de moments canoniques en coordonnées harmoniques. Nous développons d'abord le formalisme général pour relier ces deux types de moments, puis vérifions que pour les interactions de type sillage et sillage de sillage, les résultats sont bien cohérents dans ces deux formalismes. Nous exprimons enfin nos résultats cubiques du Chapitre IV en terme des moments canoniques en coordonnées harmoniques. Nous vérifions la cohérence de notre résultat final, notamment les termes non-oscillants qui sont cohérents avec la littérature, ainsi que les constantes arbitraires, associées à la régularisation d'Hadamard en trois dimensions; qui se compensent parfaitement avec les contributions provenant de la régularisation dimensionnelle.

Enfin, dans le Chapitre VI, nous rassemblons tous les résultats nécessaires pour calculer les observables à 4PN. D'abord, nous faisons un point sur l'expression du quadrupôle source renormalisé, à qui il a fallu rajouter quelques contributions supplémentaires. Nous rappelons ensuite des résultats connus pour l'expression des moments radiatifs en termes des moments canoniques, complémentaires de ceux obtenus dans les Chapitres IV et V. Nous développons quelques résultats nouveaux, utiles pour réduire nos expressions dans le cas des orbites quasi-circulaires, notamment des contributions dites post-adiabatiques et le calcul spécifique aux sillages de mémoire. Enfin, nous présentons les modes pour des harmoniques sphériques à 4PN, ainsi que le flux, le gazouillement et la phase à 4.5PN. Nous présentons enfin des estimations numériques et une comparaison à la relativité numérique.

Nous entamons ensuite la Partie B dédiée aux théories tenseur-scalaire (TS), composée uniquement du Chapitre VII. Nous définissons la théorie, introduisons les paramètres utiles et décrivons certaines spécificités, comme l'apparition de sensibilité dû à la violation du principe d'équivalence fort. Nous adaptions alors le formalisme de la Partie A au cas des théories TS, et trouvons en particulier des nouveaux effets de sillage et de mémoire. Nous présentons alors pour des orbites générales les moments sources ainsi que le flux à 1.5PN au delà du rayonnement quadrupolaire, ce qui correspond à un ordre 2.5PN au-delà du rayonnement dipolaire dominant spécifique aux théories TS. Nous trouvons en particulier une différence sur l'expression du flux pour des orbites générales à 1PN avec un résultat de la littérature. Nous réduisons ensuite notre résultat pour des orbites circulaires, et nous calculons les modes tensoriels et scalaires à 1.5PN. Nous montrons enfin des résultats numériques récents et leur comparaison avec nos prédictions analytiques.

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Rayonnement gravitationnel des systèmes binaires compacts en relativité générale et dans les théories tenseur-scalaire

Résumé : Nous améliorons les prédictions théoriques analytiques concernant les ondes gravitationnelles (OG) émises par des systèmes binaires compacts d'étoiles à neutrons ou de trous noirs pendant la phase spirale précédant la coalescence, à la fois en relativité générale (RG) et dans une classe de théories tenseur-scalaire (TS). Nous utilisons un formalisme qui combine le développement multipolaire post-minkowskien (MPM) de la métrique dans le vide extérieur à un système de matière isolé, avec le développement post-newtonien (PN) pour des petites vitesses orbitales ($v \ll c$) et des champs gravitationnels faibles. En RG, nous calculons les modes (ℓ, m) de la forme d'OG à l'ordre 4PN, en $(v/c)^8$ au-delà de l'ordre dominant, ainsi que la phase et le flux d'énergie à 4.5PN. Pour cela, nous étudions de nouveaux termes non-linéaires dans la propagation des OG. Le principal effet est le « sillage de mémoire », qui est dû aux effets combinés (i) des sillages d'OG, c'est-à-dire de la diffusion des OG sur la courbure de l'espace-temps générée par la masse totale de la binaire; et (ii) de l'effet de mémoire, dû au rayonnement gravitationnel engendré par les OG elles-mêmes. Ce calcul a nécessité l'implémentation d'une construction MPM dite « radiative » de la métrique du vide extérieur, qui élimine de la zone lointaine les logarithmes en la coordonnée radiale qui apparaissent dans la construction en coordonnées harmoniques standard. Dans les théories TS, nous adaptons le cadre PN-MPM à l'ordre 1.5PN (c'est-à-dire 2.5PN au-delà du rayonnement dipolaire dominant dans les théories TS). Nous obtenons ainsi le flux, la phase et les modes (ℓ, m) des champs scalaires et tensoriels à l'ordre 1.5PN pour des orbites circulaires, et corrigeons une erreur dans la littérature pour les orbites générales.

Mots clés : théories tenseur-scalaire, relativité générale, onde gravitationnelle, sillage de mémoire, propagation non-linéaire, développement post-newtonien

Gravitational radiation of compact binary systems in general relativity and in scalar-tensor theories

Abstract: We improve analytical theoretical predictions for gravitational waves (GWs) emitted by compact binary systems of neutron stars and black holes during the inspiraling phase preceding merger, both in general relativity (GR) and in a class of scalar-tensor (ST) theories. We resort to a formalism that combines the multipolar post-Minkowskian (MPM) expansion of the vacuum metric exterior to an isolated matter system and the post-Newtonian (PN) expansion in small orbital velocities ($v \ll c$) and weak gravitational fields. In GR, we compute the (ℓ, m) modes of the GW up to 4PN order, i.e. $(v/c)^8$ beyond leading order, as well as the phase and energy flux up to 4.5PN order. For this, we study novel nonlinear terms in the propagation of GWs. The main effect is the “tail of memory”, which is due to the combined effects of (i) GW tails, i.e. the backscattering of GWs against the spacetime curvature generated by the total mass of the binary; and (ii) the memory effect, due to the reradiation of GWs by GWs themselves. This computation required implementing the so-called “radiative” MPM construction of the exterior vacuum metric, which removes the far-zone logarithms in the radial coordinate that appear in the standard construction in harmonic coordinates. In ST theories, we adapt the PN-MPM framework at 1.5PN order (i.e. 2.5PN beyond the leading dipolar radiation of ST theories). We thus obtain the flux, phase and (ℓ, m) modes of the scalar and tensor fields at 1.5PN order for circular orbits, and correct a mistake in the literature for general orbits.

Keywords : scalar-tensor theories, general relativity, gravitational wave, tail of memory, nonlinear propagation, post-Newtonian expansion