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Hugo Parada

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THÈSE

Pour obtenir le grade de

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Spécialité : Mathématiques Appliquées

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Contrôle et stabilisation d'EDP sur des réseaux

Control and stabilization of PDE on networks

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Abstract

In this thesis, we study stabilization problems for the Korteweg-de Vries (KdV) equation in a bounded domain and on a star-shaped network structure. More specifically, the focus of this thesis is to analyze the cases where the feedback terms include delay and saturation. In Chapter 2, we deal with the internal stabilization of the nonlinear KdV equation posed on a star-shaped network when the feedback terms involve time delay. Using Lyapunov techniques and an appropriated observability inequality, we managed to show the exponential stability of the associated system. In Chapter 3, we study the saturated internal stabilization of the KdV equation on a star-shaped network. In this case, we show first a global well-posedness result by using an appropriated linear system and the Kato smoothing property. Then, we show the exponential stability via a contradiction argument and an observability inequality. In Chapter 4, the stability analysis of the KdV equation on a bounded domain in presence of boundary saturation is studied. Using nonlinear semigroup theory and fixed-point arguments, we show the well-posedness. The exponential stability is proved by using compactness ideas to get an observability inequality. In Chapter 5, we consider the stability analysis of the KdV equation on a bounded domain with time-dependent delay on the boundary or internal feedbacks. We study the well-posedness using time-varying semigroup theory. The exponential stability is derived by using Lyapunov theory. Finally, in Chapter 6 some comments and future research problems are addressed.

Résumé

Dans cette thèse, nous étudions des problèmes de stabilisation de l'équation de Korteweg-de Vries dans un domaine borné et dans une structure de réseau étoilée. Plus spécifiquement, l'objectif de cette thèse est d'analyser les cas où les termes de feedback incluent du retard et de la saturation. Dans le chapitre 2, nous traitons la stabilisation interne de l'équation non linéaire de KdV posée sur un réseau étoilé lorsque les termes de feedback incluent des retards. À l'aide de techniques de Lyapunov et d'une inégalité d'observabilité, nous avons réussi à montrer la stabilité exponentielle du système. Dans le chapitre 3, nous étudions la stabilisation interne saturée de l'équation de KdV sur un réseau étoilé. Dans ce cas, nous montrons d'abord un résultat de caractère bien posé global en utilisant un système linéaire approprié et la propriété de régularisation de Kato. Puis via un argument de contradiction et une inégalité d'observabilité, nous montrons la stabilité exponentielle. Dans le chapitre 4, l'analyse de la stabilité de l'équation de KdV sur un domaine borné en présence de saturation à la frontière est étudiée. En utilisant la théorie des semigroupes non linéaires et des arguments de points fixes, nous montrons le caractère bien posé de l'équation. La stabilité exponentielle est prouvée en utilisant des idées de compacité pour obtenir une inégalité d'observabilité. Dans le chapitre 5, nous considérons l'analyse de la stabilité de l'équation de KdV sur un domaine borné avec un retard dépendant du temps dans le feedback frontière ou interne. Nous étudions le caractère bien posé à l'aide de la théorie des semigroupes qui dépendent du temps. La stabilité exponentielle est dérivée en utilisant la théorie de Lyapunov. Finalement, le chapitre 6 présente quelques commentaires et des possibles pistes de recherche futures.

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*Mi refri' está lleno, ¿cómo se ve?
Desde niño llenarlo soñé
Y mi logo en cadena nunca pensé
Si hasta hace poco yo andaba a pie...*

Pablo Acevedo (Pablo Chill-E)

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CHAPTER 1

INTRODUCTION

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1.1 General introduction

We start by studying and recalling the notions of semigroups, controllability and stabilization for evolution equations. This introduction is inspired of [Paz83, Cor09, Bre10, Miy92].

1.1.1 Semigroup theory

Let H be a complex Hilbert space, we denote by $\langle \cdot, \cdot \rangle_H$ and $\| \cdot \|_H$ its scalar product and norm, respectively. Consider a linear operator $\mathcal{A} : D(\mathcal{A}) \subset H \rightarrow H$, where the domain $D(\mathcal{A}) := \{u \in H : \mathcal{A}u \in H\}$ is a linear space of H . We say that \mathcal{A} is a closed operator if its graph $G(\mathcal{A}) := \{(u, \mathcal{A}u) : u \in H\}$ is a closed set in $H \times H$, in this case $D(\mathcal{A})$ is also a Hilbert space with the *graph norm* $\|u\|_{D(\mathcal{A})}^2 := \|u\|_H^2 + \|\mathcal{A}u\|_H^2$.

Definition 1.1.1 (Adjoint operator). *If $D(\mathcal{A})$ is a dense subset of H , we can define the adjoint \mathcal{A}^* of \mathcal{A} as the linear operator $\mathcal{A}^* : D(\mathcal{A}^*) \subset H \rightarrow H$ characterized by*

$$\langle \mathcal{A}u, z \rangle_H = \langle u, \mathcal{A}^*z \rangle_H, \quad (u, z) \in D(\mathcal{A}) \times D(\mathcal{A}^*),$$

where $D(\mathcal{A}^*) := \{z \in H : \exists C > 0 \text{ depending on } z : |\langle \mathcal{A}u, z \rangle_H| \leq C\|z\|_H, \forall u \in D(\mathcal{A})\}$.

Definition 1.1.2 (Dissipative operator). *The linear operator $\mathcal{A} : D(\mathcal{A}) \rightarrow H$ is called dissipative if $\operatorname{Re}\langle \mathcal{A}u, u \rangle_H \leq 0, \forall u \in D(\mathcal{A})$.*

In the case where $\mathcal{A} : D(\mathcal{A}) \rightarrow H$ is a nonlinear operator, the dissipative property is defined as $\operatorname{Re}\langle \mathcal{A}u_1 - \mathcal{A}u_2, u_1 - u_2 \rangle_H \leq 0, \forall u_1, u_2 \in D(\mathcal{A})$.

Definition 1.1.3 (Maximal dissipative operator). *A dissipative operator $\mathcal{A} : D(\mathcal{A}) \rightarrow H$ is called maximal dissipative or m -dissipative if, for all $\lambda > 0, R(I - \lambda\mathcal{A}) = H$.*

Actually, to get that \mathcal{A} is m -dissipative, it is enough to check that $R(I - \lambda\mathcal{A}) = H$ for some $\lambda > 0$ [Miy92, Lemma 2.13].

Definition 1.1.4 (Strongly continuous semigroups). *A one-parameter family $S(t), 0 \leq t < \infty$ of continuous operators from H to H is a semigroup of strongly continuous operators if*

1. $S(t) \in \mathcal{L}(H)$, for all $t \geq 0$,
2. $S(0) = \mathcal{I}$,
3. $S(t_1 + t_2) = S(t_1) \circ S(t_2)$, for all $t_1, t_2 \in [0, \infty)$,
4. $\lim_{t \rightarrow 0^+} S(t)u = u$, for all $u \in H$.

We say that $S(t)$ is a semigroup of contractions if $\|S(t)\|_{\mathcal{L}(H)} \leq 1$, for all $t \geq 0$, where $\mathcal{L}(H)$ is the space of linear continuous maps from H into itself. Finally, the infinitesimal generator of a semigroup S is the linear operator $\mathcal{A} : D(\mathcal{A}) \subset H \rightarrow H$ defined by

$$D(\mathcal{A}) = \left\{ u \in H : \lim_{t \rightarrow 0^+} \frac{S(t)u - u}{t} \text{ exists} \right\},$$

$$\mathcal{A}u = \lim_{t \rightarrow 0^+} \frac{S(t)u - u}{t}, \forall u \in D(\mathcal{A}).$$

With all this in mind, we can introduce the following result (see [Paz83])

Theorem 1.1.1 (Lumer-Phillips). *Let us assume that \mathcal{A} is densely defined, closed and that \mathcal{A} and \mathcal{A}^* are both dissipative. Then \mathcal{A} is the infinitesimal generator of a strongly continuous semigroup of contractions.*

1.1.2 Cauchy problem

In our aim to study evolutions equations, let $T > 0, \mathcal{A} : D(\mathcal{A}) \subset H \rightarrow H$ be the infinitesimal generator of a strongly continuous semigroup of contractions $S(t), u_0 \in D(\mathcal{A})$ and $f \in C^1([0, T]; H)$. Consider the following homogeneous Cauchy problem:

$$\begin{cases} \partial_t u = \mathcal{A}u(t), & t \in [0, T], \\ u(0) = u_0, \end{cases} \quad (1.1.1)$$

and the in-homogeneous one

$$\begin{cases} \partial_t u = \mathcal{A}u(t) + f(t), & t \in [0, T], \\ u(0) = u_0. \end{cases} \quad (1.1.2)$$

A classical solution of (1.1.1) (resp. (1.1.2)) on $[0, T]$ is a function $u(t) \in C^1([0, T]; H)$, $u(t) \in D(\mathcal{A})$ for all $t \in [0, T]$ such that $u(t)$ satisfies (1.1.1) (resp. (1.1.2)) for all $t \in [0, T]$. As \mathcal{A} generates a strongly continuous semigroup of contractions $S(t)$, the unique classical solution of (1.1.1) is given by $u(t) = S(t)u_0 \in C([0, \infty); D(\mathcal{A})) \cap C^1([0, \infty); H)$. Similarly, as $f \in C^1([0, T]; H)$, the unique classical solution of (1.1.2) is given by

$$u(t) = S(t)u_0 + \int_0^t S(t-s)f(s)ds. \quad (1.1.3)$$

Consider now, the case $u_0 \in H$, we call a mild solution of (1.1.1) to the function $u(t) = S(t)u_0 \in C([0, T]; H)$. Similarly, if $f \in L^1(0, T; H)$, we call a mild solution of (1.1.2) to a function $u(t) \in C([0, T]; H)$ satisfying (1.1.3).

1.1.3 Control and stabilization of abstract systems

In this part, we give some basic notions about control and stability of abstract evolution equations. Let $T > 0$, H and V be real Hilbert spaces and consider the following control system

$$\begin{cases} \partial_t u = \mathcal{A}u + \mathcal{B}v, & t \in [0, T], \\ u(0) = u_0. \end{cases} \quad (1.1.4)$$

where u corresponds to the state and $v \in L^2(0, T; V)$ stands for the control variable. The operator $\mathcal{A} : D(\mathcal{A}) \rightarrow H$ is a linear m -dissipative operator and $\mathcal{B} \in \mathcal{L}(V, D(\mathcal{A}^*)')$, where $D(\mathcal{A}^*)'$ is the dual space of $D(\mathcal{A}^*)$. This functional setting gives the possibility to consider boundary control operators (instead to the stronger one $\mathcal{B} \in \mathcal{L}(V, H)$). To define solutions of (1.1.1) when $\mathcal{B} \in \mathcal{L}(V, D(\mathcal{A}^*)')$ we ask the operator \mathcal{B} to be admissible for the semigroup $S(t)$, i.e.

$$\int_0^T \|\mathcal{B}^* S(t)^* z\|_V^2 dt \leq C_T \|z\|_H^2, \quad \forall z \in D(\mathcal{A}^*).$$

For the system (1.1.4) several notions of controllability can be considered, the most classical are the following ones:

Definition 1.1.5 (Exactly controllable). *Let $T > 0$. The control system (1.1.4) is exactly controllable in time T if, for every $u_0, u_T \in H$, there exists $v \in L^2(0, T; V)$ such that the solution of (1.1.4) satisfies $u(T) = u_T$.*

Definition 1.1.6 (Null controllable). *Let $T > 0$. The control system (1.1.4) is null controllable in time T if, for every $u_0 \in H$, there exists $v \in L^2(0, T; V)$ such that the solution of (1.1.4) satisfies $u(T) = 0$.*

Definition 1.1.7 (Approximately controllable). *Let $T > 0$. The control system (1.1.4) is approximately controllable in time T if, for every $u_0, u_T \in H$, and for every $\varepsilon > 0$, there exists $v \in L^2(0, T; V)$ such that the solution of (1.1.4) satisfies $\|u(T) - u_T\|_H \leq \varepsilon$.*

Clearly, we have that exact controllability implies null and approximate controllability. For linear systems, we can consider $u_0 = 0$ for the exact controllability. Moreover, if the system is reversible in time, then the exact controllability is equivalent to the null controllability. We can characterize these control problems in the following sense using the HUM method. Consider the operator

$$\begin{aligned} \Lambda : L^2(0, T; V) &\longrightarrow H, \\ \Lambda(v) &= u(T), \end{aligned}$$

where given $v \in L^2(0, T; V)$, u is the associate solution of (1.1.4). Take $u_0 = 0$, thus Λ is linear. If we focus on the exact controllability we see

Exactly controllable $\iff \Lambda$ surjective.

By [Bre10, Theorem 2.20], the operator Λ is onto if and only if, there exists $C > 0$ such that

$$\|z_T\|_H \leq C \|\Lambda^*(z_T)\|_{L^2(0, T; V)}, \quad \forall z_T \in H. \quad (1.1.5)$$

It is not difficult to see that, for every $z_T \in H$, $\Lambda^*(z_T) = \mathcal{B}^*S(T-t)^*z_T$. Thus, recalling that $D(\mathcal{A}^*)$ is dense in H , the exact controllability reads as

$$\|z_T\|_H \leq C \int_0^T \|\mathcal{B}^*S(T-t)^*z_T\|_V^2 dt, \quad \forall z_T \in D(\mathcal{A}^*).$$

The above inequality is called *observability inequality* for the adjoint system

$$\begin{cases} \partial_t z = -\mathcal{A}^*z, & t \in [0, T], \\ z(T) = z_T. \end{cases} \quad (1.1.6)$$

Finally, consider the uncontrolled case $v = 0$ in (1.1.4). Let \bar{u} be an equilibrium solution, i.e. $\mathcal{A}\bar{u} = 0$, with $\bar{u} \in D(\mathcal{A})$.

Definition 1.1.8 (Stability). *We say that \bar{u} is stable, if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for all $u_0 \in H$ with $\|u_0 - \bar{u}\|_H \leq \delta$, the unique mild solution $u(t)$ of (1.1.4) satisfies $\|u(t) - \bar{u}\|_H \leq \varepsilon$, for all $t \geq 0$.*

Definition 1.1.9 (Asymptotic stability). *We say that \bar{u} is asymptotically stable, if it is stable and there exists $\delta > 0$ such that for all $u_0 \in H$ with $\|u_0 - \bar{u}\|_H \leq \delta$, the unique mild solution $u(t)$ of (1.1.4) satisfies $\lim_{t \rightarrow \infty} \|u(t) - \bar{u}\|_H = 0$.*

Definition 1.1.10 (Exponential stability). *We say that \bar{u} is exponentially stable, if it is asymptotically stable and there exists $\mu > 0$ such that for all $u_0 \in H$, the unique mild solution $u(t)$ of (1.1.4) satisfies $\|u(t) - \bar{u}\|_H \leq e^{-\mu t} \|u_0 - \bar{u}\|_H$.*

In this context, the focus of the stabilization problems is to design a feedback control law $v = \mathcal{F}(u)$, such that an equilibrium point of the closed-loop system (1.1.4) be stable, asymptotically stable, exponentially stable, etc.

1.1.4 Stabilization and saturation

In most real-life setting, we have to take into account saturation in the input control due to some (physical, economical, etc.) constraints. With respect to saturated control in infinite-dimensional systems, we can refer to [PTdSJ16] where a wave equation with distributed and boundary saturated feedback law was studied, [MCPA17] where the saturated internal stabilization of a single KdV equation was studied and recently [MPW21] where a saturated feedback control law was derived for a linear reaction-diffusion equation. Sometimes adding saturation to the feedback control, destabilize the system. To clarify these ideas, we consider now the following finite dimensional system.

$$\dot{u} = Au + Bv, \quad (1.1.7)$$

where $u \in \mathbb{R}^N$ is the state and $v \in \mathbb{R}^M$ is the control, and A, B are real matrices with the appropriate dimensions. With the aim to find a state feedback law $v = \mathcal{F}(u)$ and due to the linearity of the system, we can take the control v as $v = Ku$, where the matrix K is called the feedback gain. Thus, the closed-loop system is

$$\dot{u} = (A + BK)u. \quad (1.1.8)$$

For system (1.1.8) it is well-known that if the pair (A, B) is controllable (i.e. the system (1.1.7) is controllable) we can find a matrix K such that the closed-loop system (1.1.8) is exponentially stable. The fact that the pair (A, B) is controllable is equivalent to the following Kalman condition

$$\text{rank}[B, AB, \dots, A^{N-1}B] = N.$$

Moreover, via the pole-shifting theorem, we can find a matrix K such that the eigenvalues of $A + BK$ be as we want. We take now the following example [TGdSJQ11, Example 1.1]. Consider the system

$$\dot{u}(t) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ -1 \end{bmatrix} v(t). \quad (1.1.9)$$

This system is clearly open-loop unstable, because the eigenvalues of A are 1 and -1 . The pair (A, B) satisfies the Kalman condition. Moreover, we can take the feedback gain $K = [13 \ 7]$, thus the eigenvalues of the closed-loop system are -3 and -4 , hence the closed-loop system is exponentially stable. Consider now the case where due to some physical restrictions the control function is bounded, for instance $|v(t)| \leq 5$. Adding this restriction, the new closed-loop system is

$$\dot{u}(t) = Au(t) + B\text{sat}(Ku(t)), \quad (1.1.10)$$

where

$$\text{sat}(s) = \begin{cases} s, & \text{if } |s| \leq 5, \\ 5\text{sgn}(s), & \text{if } |s| \geq 5. \end{cases}$$

In this case we can observe some very interesting behaviours.

- For the initial condition $u(0) = [-1 \ -3]^\top$, the solution converges to the origin, see Figure 1.1.4.
- For the initial condition $u(0) = [-2 \ -3]^\top$, the solution converges to a new equilibrium point $[-5 \ 0]^\top$, see Figure 1.1.4.
- For the initial condition $u(0) = [-3 \ -3]^\top$, the solution diverges, see Figure 1.1.4.

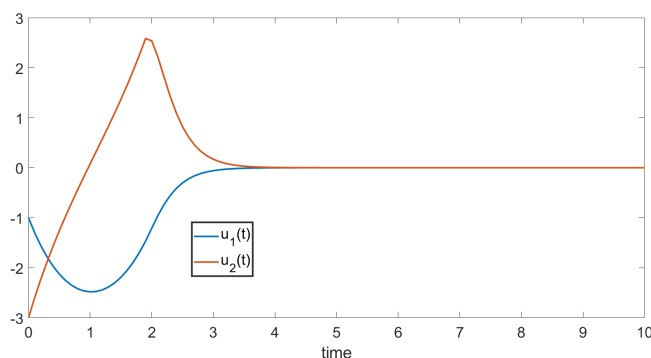


Figure 1.1: Initial condition $u(0) = [-1 \ -3]^\top$

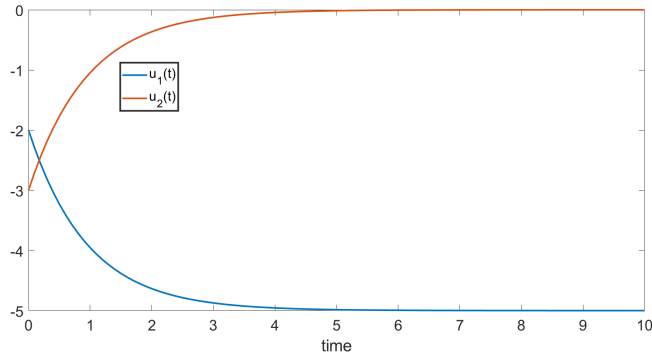


Figure 1.2: Initial condition $u(0) = [-2 \ -3]^T$

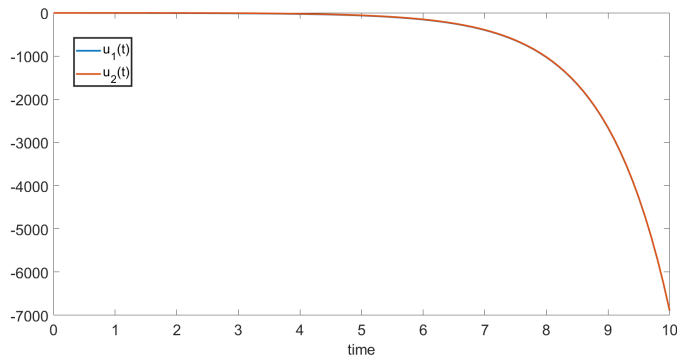


Figure 1.3: Initial condition $u(0) = [-3 \ -3]^T$

We can observe that in the finite-dimensional case adding saturation to a feedback controller could it destabilizes the system and even create new points of equilibrium.

1.1.5 Stabilization and delay

Time delay phenomena appear in many applications, for example in biology, mechanics or engineering. Delay terms are unavoidable in practice due to measurement lag, analysis time, or computation time. Very active research has been developed recently on stability problems of partial differential equations with delay. It is well known that even a small delay in the feedback mechanism can destabilize a system (see for example [Dat88, DLP86]). But a delay term can also improve the performance of the system ([ADBRB93]). The problems of stability of systems with delay are of both theoretical and practical interest. To fix ideas, consider the following one-dimensional ODE

$$\dot{u}(t) = \frac{1}{2}u(t),$$

the above equation is clearly unstable, but introducing for instance the feedback law $v = -u(t)$, we have that $\dot{u}(t) = \frac{1}{2}u(t) - u(t) = -\frac{1}{2}u(t)$ is exponentially stable. Suppose now that the feedback term v has a time delay of h seconds, and we study now the system

$$\dot{u}(t) = \frac{1}{2}u(t) - u(t-h). \quad (1.1.11)$$

Straightforward computations show that the function $u(t) = e^{at} \cos(bt)$ is a solution of (1.1.11) if a , b satisfy

$$\begin{cases} a - \frac{1}{2} = -e^{-ah} \cos(bh), \\ b = e^{-ah} \sin(bh). \end{cases} \quad (1.1.12)$$

It is possible to show that even for small delays h , there exist solutions $a > 0$, $b \in \mathbb{R}$ of (1.1.12). In particular, as $a > 0$ the solution $u(t) = e^{at} \cos(bt)$ of (1.1.11) diverges.

1.2 The Korteweg-de Vries equation

The Korteweg-de Vries equation, or KdV, is a third-order, nonlinear, dispersive equation introduced in 1895 by the Dutch mathematicians Diederik Korteweg and Gustav de Vries in [KdV95]. It was written to describe the one-way propagation of long-period, small-amplitude waves, such as those found in shallow water channels. In 1965, Zabusky and Kruskal discovered the idea of solitary wave solutions, also known as “solitons”, while analyzing numerical results related to the KdV equation [ZK65]. Since then, it has been widely studied in both mathematics and physics, and has been utilized to model a variety of phenomena, including water waves, tsunamis, transmission of electrical signals in nerve fibers, plasma, cosmology, etc (see for instance [GM65, Lid12, KM69]). The KdV equation derived by Diederik Korteweg and Gustav de Vries was (up to rescaling)

$$\partial_t u(t, x) + \partial_x^3 u(t, x) + u(t, x) \partial_x u(t, x) = 0, x \in \mathbb{R}, t \geq 0,$$

where the function $u = u(t, x)$ modelizes the amplitude of the water wave at position x and time $t \geq 0$. In this thesis, we are mainly interested in the KdV equation posed in a bounded domain and networks with bounded edges. If we study the KdV equation in a bounded domain, we say $x \in [0, L]$, as was suggested by Bona and Winter in [BW83] the extra term $\partial_x u$ should be incorporated. Moreover, as the KdV equation is of third order in space, we need to impose two boundary conditions at the right and one at the left. For instance, we can consider boundary conditions acting on $u(t, 0)$, $u(t, L)$ and $\partial_x u(t, L)$.

1.2.1 Well-posedness ideas

This section is mainly based on [RZ09, Cer14]. The first results about controllability and stabilization of the KdV equation were derived by Russell and Zhang [RZ93, RZ95], in the case of periodic boundary conditions and by Rosier in the case of boundary control [Ros97]. In the work of Rosier, the exact controllability of the KdV equation was studied by acting on the right Neumann boundary condition. In particular, he focused on the linear system

$$\begin{cases} \partial_t u(t, x) + \partial_x u(t, x) + \partial_x^3 u(t, x) = 0, t \in [0, T], x \in (0, L), \\ u(t, 0) = u(t, L) = 0, t \in [0, T], \\ \partial_x u(t, L) = h(t), t \in [0, T], \\ u(0, x) = u_0, x \in (0, L), \end{cases} \quad (1.2.1)$$

where the function $h \in L^2(0, T)$ denotes the control. Consider first the uncontrolled system

$$\begin{cases} \partial_t u(t, x) + \partial_x u(t, x) + \partial_x^3 u(t, x) = 0, t \in [0, T], x \in (0, L), \\ u(t, 0) = u(t, L) = \partial_x u(t, L) = 0, t \in [0, T], \\ u(0, x) = u_0, x \in (0, L). \end{cases} \quad (1.2.2)$$

Let $\mathcal{A} : D(\mathcal{A}) \rightarrow L^2(0, L)$ defined by

$$\begin{aligned} \mathcal{A}w &= -w' - w''', \\ D(\mathcal{A}) &= \{w \in H^3(0, L); w(0) = w(L) = w'(L) = 0\}. \end{aligned}$$

By [Ros97, Proposition 2.1] we know that \mathcal{A} generates $S(t)$, $t \geq 0$ a strongly continuous semigroup of contractions in $L^2(0, L)$. Define the space

$$B_T = C([0, T], L^2(0, L)) \cap L^2(0, T; H^1(0, L)),$$

$$\|u\|_{B_T} = \|u\|_{C([0, T], L^2(0, L))} + \|u\|_{L^2(0, T; H^1(0, L))}.$$

In virtue of [Ros97, Proposition 3.2], for all $u_0 \in L^2(0, L)$, there exists a unique mild solution $u = S(\cdot)u_0 \in B_T$ of (1.2.2). Moreover, the following estimate holds

$$\|\partial_x u(\cdot, 0)\|_{L^2(0, T)} \leq \|u_0\|_{L^2(0, L)}. \quad (1.2.3)$$

This estimate is typically called hidden regularity. Consider now, the nonlinear system

$$\begin{cases} \partial_t u(t, x) + \partial_x u(t, x) + \partial_x^3 u(t, x) + u(t, x) \partial_x u(t, x) = 0, & t \in [0, T], x \in (0, L), \\ u(t, 0) = u(t, L) = 0, & t \in [0, T], \\ \partial_x u(t, L) = h(t), & t \in [0, T], \\ u(0, x) = u_0, & x \in (0, L). \end{cases} \quad (1.2.4)$$

The main idea in the proof of the well-posedness result for nonlinear system (1.2.4) is to use the following estimate (see [PMVZ02] for the proof).

Proposition 1.2.1. *Let $u \in B_T$. Then $u \partial_x u \in L^1((0, T), L^2(0, L))$ and the map $u \in B_T \mapsto u \partial_x u \in L^1((0, T), L^2(0, L))$ is continuous. Moreover, there exists $C > 0$ such that, for any $u, z \in B_T$*

$$\|u \partial_x u - z \partial_x z\|_{L^1((0, T), L^2(0, L))} \leq CT^{1/4} (\|u\|_{B_T} + \|z\|_{B_T}) \|u - z\|_{B_T}.$$

Using the above proposition and a fixed-point argument, we get the local well-posedness of (1.2.4).

1.2.2 Critical lengths phenomena

Now, given $T > 0$, $u_0, u_T \in L^2(0, L)$ the exact controllability problem for (1.2.1) consists in finding $h \in L^2(0, T)$ such that, the solution u of (1.2.1) satisfies $u(T, \cdot) = u_T \in L^2(0, L)$. Let $T > 0$ and define the operator $\Lambda : L^2(0, T) \rightarrow L^2(0, L)$ as $\Lambda(h) := u(T, \cdot)$, where u is the unique solution of (1.2.1). In particular, by linearity we can assume $u_0 = 0$. Thus, from (1.1.5) the exact controllability is equivalent to the observability inequality,

$$\|\varphi_T\|_{L^2(0, L)} \leq C \|\Lambda^*(\varphi_T)\|_{L^2(0, T)},$$

where Λ^* is the adjoint operator. It is not difficult to see that given $\varphi_T \in L^2(0, L)$, $\Lambda^*(\varphi_T) = \partial_x \varphi(\cdot, L)$, where φ is the solution of the backward in time equation

$$\begin{cases} -\partial_t \varphi(t, x) - \partial_x \varphi(t, x) - \partial_x^3 \varphi(t, x) = 0, & t \in [0, T], x \in (0, L), \\ \varphi(t, 0) = \varphi(t, L) = \partial_x \varphi(t, 0) = 0, & t \in [0, T], \\ \varphi(T, x) = \varphi^T, & x \in (0, L). \end{cases} \quad (1.2.5)$$

Moreover, making the change of variable $\xi = L - x$ and $\tau = T - t$, we get that $\varphi(T - t, L - x)$ is solution of (1.2.2) and thus the observability inequality reads as

$$\|u_0\|_{L^2(0, L)} \leq C \|\partial_x u(\cdot, 0)\|_{L^2(0, T)},$$

which is indeed the reverse inequality of (1.2.3). In [Ros97, Proposition 3.3] it was proved that this observability inequality is true if and only if $L \notin \mathcal{N}$, where the set \mathcal{N} is called the set of critical lengths and is given by

$$\mathcal{N} = \left\{ 2\pi \sqrt{\frac{k^2 + kl + l^2}{3}}; k, l \in \mathbb{N}^* \right\}, \quad (1.2.6)$$

and thus (1.2.1) is exactly controllable if and only if $L \notin \mathcal{N}$. Furthermore, we have the following lemma due to [Ros97, Lemma 3.5],

Lemma 1.2.1. *The following assertions are equivalent*

- *There exist $\lambda \in \mathbb{C}$ and $u \in H^3(0, L) \setminus \{0\}$, such that*

$$\begin{cases} \lambda u + u' + u''' = 0, & x \in (0, L) \\ u(0) = u(L) = u'(0) = u'(L) = 0, \end{cases}$$

- *$L \in \mathcal{N}$.*

This lemma implies the existence of unobservable solutions of (1.2.2), for instance in the case $l = k = 1$, the associate critical length is $L = 2\pi$ and this yields the unobservable steady solutions $u(t, x) = a(\cos(x) - 1)$, $a \in \mathbb{R}$. Consider now the L^2 -energy of a mild solution of (1.2.2), i.e.

$$E(t) = \frac{1}{2} \int_0^L (u(t, x))^2 dx.$$

Formally, taking the time-derivative of the energy, we get after some integrations by parts

$$\dot{E}(t) = -\frac{1}{2} (\partial_x u(t, 0))^2,$$

from where we deduce that the energy is a non-increasing function of time. Moreover, if $L \in \mathcal{N}$, by Lemma 1.2.1, there exist solutions of (1.2.2) which satisfy $\partial_x u(\cdot, 0) = 0$ and thus with constant energy.

1.2.3 Stability on non-critical lengths

In the non-critical case, $L \notin \mathcal{N}$ we can prove that the energy of solutions of (1.2.2) decays exponentially. Indeed, multiplying (1.2.2) by u and integrating on $(0, T) \times (0, L)$ we get after integration by parts

$$\int_0^L (u(T, x))^2 dx - \int_0^L u_0^2(x) dx = - \int_0^T (\partial_x u(t, 0))^2 dt.$$

Using the fact that the energy is non-increasing and the observability inequality

$$E(T) \leq E(0) \leq C \int_0^T (\partial_x u(t, 0))^2 dt \leq C(E(0) - E(T)),$$

that implies

$$E(T) \leq \gamma E(0), \text{ with } \gamma = \frac{C}{C+1} < 1.$$

In particular, as the system is invariant by translation in time, we can repeat this argument on $[(m-1)T, mT]$, for $m = 1, 2, \dots$ to obtain

$$E(mT) \leq \gamma E((m-1)T) \leq \dots \leq \gamma^m E(0).$$

Hence, we have $E(mT) \leq e^{-\mu m T} E(0)$ where $\mu = \frac{1}{T} \ln(\frac{1}{\gamma}) > 0$. Let $t > 0$, then there exists $m \in \mathbb{N}^*$ such that $(m-1)T < t \leq mT$, and then using again the non-increasing property of the energy we get

$$E(t) \leq E((m-1)T) \leq e^{-\mu(m-1)T} E(0) \leq \frac{1}{\gamma} e^{-\mu t} E(0).$$

Finally, we get the exponential stability.

Consider now the nonlinear KdV equation

$$\begin{cases} \partial_t u(t, x) + \partial_x u(t, x) + \partial_x^3 u(t, x) + u(t, x) \partial_x u(t, x) = 0, & t \in [0, T], x \in (0, L), \\ u(t, 0) = u(t, L) = \partial_x u(t, L) = 0, & t \in [0, T], \\ u(0, x) = u_0, & x \in (0, L). \end{cases} \quad (1.2.7)$$

The previous result of exponential stability for the linear system can be extended for small amplitude solutions of the nonlinear system (1.2.7) using a perturbation approach. That is, there exists $\varepsilon > 0$ small enough such that, for all $u_0 \in L^2(0, L)$ satisfying $\|u_0\|_{L^2(0, L)} \leq \varepsilon$ the unique mild solution of (1.2.7) satisfies $E(t) \leq Ce^{-\mu t}E(0)$, for some $C, \mu > 0$.

1.2.4 Stability on the critical case: internal damping

In order to achieve the exponential stability in the case $L \in \mathcal{N}$ a dissipative mechanism was introduced in [PMVZ02] taking the form $F(u) = -au$, where $a \in L^\infty(0, L)$ satisfying

$$\begin{cases} a(x) \geq a_0 > 0, \quad \forall x \in \omega, \\ \text{where } \omega \text{ is nonempty open subset of } (0, L). \end{cases} \quad (1.2.8)$$

The systems addressed in [PMVZ02] are the linear equation

$$\begin{cases} \partial_t u(t, x) + \partial_x u(t, x) + \partial_x^3 u(t, x) + a(x)u(t, x) = 0, \quad t \in [0, T], \quad x \in (0, L), \\ u(t, 0) = u(t, L) = \partial_x u(t, L) = 0, \quad t \in [0, T], \\ u(0, x) = u_0, \quad x \in (0, L). \end{cases} \quad (1.2.9)$$

and the nonlinear one

$$\begin{cases} \partial_t u(t, x) + \partial_x u(t, x) + \partial_x^3 u(t, x) + u(t, x)\partial_x u(t, x) + a(x)u(t, x) = 0, \quad t \in [0, T], \quad x \in (0, L), \\ u(t, 0) = u(t, L) = \partial_x u(t, L) = 0, \quad t \in [0, T], \\ u(0, x) = u_0, \quad x \in (0, L). \end{cases} \quad (1.2.10)$$

Using a contradiction argument and compactness ideas Perla, Menzala and Zuazua proved in [PMVZ02], the following results

Theorem 1.2.1. *Let $L > 0$ and $a \in L^\infty(0, L)$ satisfying (1.2.8).*

- *There exists $C > 0$ and $\mu > 0$ such that for any $u_0 \in L^2(0, L)$, the unique mild solution of the linear equation (1.2.9) satisfies $\|u(t, \cdot)\|_{L^2(0, L)} \leq Ce^{-\mu t}\|u_0\|_{L^2(0, L)}$, $\forall t \geq 0$.*
- *There exists $C, \varepsilon > 0$ and $\mu > 0$ such that for any $u_0 \in L^2(0, L)$ with $\|u_0\|_{L^2(0, L)} \leq \varepsilon$, the unique mild solution of the nonlinear equation (1.2.10) satisfies $\|u(t, \cdot)\|_{L^2(0, L)} \leq Ce^{-\mu t}\|u_0\|_{L^2(0, L)}$, $\forall t \geq 0$.*

The second point of the last result was derived using a perturbative argument, is due to this fact that we add the smallness assumption on the initial condition. Moreover, in [PMVZ02, Paz05] a semiglobal exponential stability result was proved for the nonlinear system (1.2.10). First, in [PMVZ02] a stronger condition on the acted set ω was used, $\exists \delta > 0$, $(0, \delta) \cup (L - \delta, L) \subset \omega$. In [Paz05] this condition was dropped, obtaining just (1.2.8). Working directly with the nonlinear system, two main difficulties appear, the first one is to pass to the limit in the nonlinear term and the second that Holmgren's Theorem does not apply in the nonlinear case. Thus, we need the next unique continuation property of Saut and Scheurer.

Theorem 1.2.2 (Theorem 4.2, [SS87]). *Let $L > 0$ and $u \in L^2(0, T; H^3(0, L))$ be a solution of*

$$\partial_t u(t, x) + \partial_x u(t, x) + \partial_x^3 u(t, x) + u(t, x)\partial_x u(t, x) = 0,$$

such that $u(t, x) = 0$, for $(t, x) \in (t_1, t_2) \times \omega$, where ω is a nonempty open subset of $(0, L)$. Then $u(t, x) = 0$, for $(t, x) \in (t_1, t_2) \times (0, L)$.

The semiglobal stability result reads as

Theorem 1.2.3. *Let $L > 0$, $a \in L^\infty(0, L)$ satisfying (1.2.8) and $R > 0$. There exists $C(R) > 0$ and $\mu(R) > 0$ such that for any $u_0 \in L^2(0, L)$ with $\|u_0\|_{L^2(0, L)} \leq R$, the unique mild solution of (1.2.10) satisfies $\|u(t, \cdot)\|_{L^2(0, L)} \leq Ce^{-\mu t}\|u_0\|_{L^2(0, L)}$, $\forall t \geq 0$.*

The semiglobal sense of this result arises from the fact that we can choose as we want the parameter $R > 0$ of the initial data, which means arbitrary big initial conditions, but the decay rate depends on it.

1.2.5 Boundary stabilization results

In this part, we some recall some boundary stabilization results for the KdV equation. Consider the linear system

$$\begin{cases} \partial_t u(t, x) + \partial_x u(t, x) + \partial_x^3 u(t, x) = 0, & t \in [0, T], \quad x \in (0, L), \\ u(t, 0) = K_1(u), & t \in [0, T], \\ u(t, L) = K_2(u), \quad \partial_x u(t, L) = K_3(u), & t \in [0, T], \\ u(0, x) = u_0, & x \in (0, L), \end{cases} \quad (1.2.11)$$

and the nonlinear one

$$\begin{cases} \partial_t u(t, x) + \partial_x u(t, x) + \partial_x^3 u(t, x) + u(t, x)\partial_x u(t, x) = 0, & t \in [0, T], \quad x \in (0, L), \\ u(t, 0) = K_1(u), & t \in [0, T], \\ u(t, L) = K_2(u), \quad \partial_x u(t, L) = K_3(u), & t \in [0, T], \\ u(0, x) = u_0, & x \in (0, L). \end{cases} \quad (1.2.12)$$

The following result was proved by Cerpa and Crépeau in [CC09a] and is related to the rapid stabilization.

Theorem 1.2.4 (Rapid stabilization). *Let $\mu > 0$ and consider $K_1(u) = K_2(u) = 0$. Then, there exists an operator $F_\mu : H_0^1(0, L) \rightarrow \mathbb{R}$ such that if $K_3(u) = \partial_x u(t, 0) + F_\mu(u)$, then the closed-loop system (1.2.11) is globally well posed in $H_0^1(0, L)$. Moreover, $\exists C > 0$, such that for all $u_0 \in H_0^1(0, L)$, $\|u(t, \cdot)\|_{H_0^1(0, L)} \leq C e^{-2\mu t} \|u_0\|_{H_0^1(0, L)}$, for all $t \geq 0$.*

The main idea of this result was to use a Gramian-based approach. Due to this technique the feedback term $K_3(u)$, takes the form $K_3(u) = \partial_x u(t, 0) + F_\mu(u)$ in such way that the associate operator to the free control system be skew-adjoint (i.e. $\mathcal{A}^* = -\mathcal{A}$). Another rapid stabilization result was proved in [CC13], this time acting on the left Dirichlet boundary condition and based on the backstepping techniques (see [KS08] for an introduction about backstepping techniques for partial differential equations).

Theorem 1.2.5 (Rapid stabilization). *Let $\mu > 0$ and consider $K_2(u) = K_3(u) = 0$. Then, there exist an operator $K_1(u) = K_{1,\mu}(u)$ and $C > 0$, such that for any mild solution u of (1.2.11), $\|u(t, \cdot)\|_{L^2(0, L)} \leq C e^{-\mu t} \|u_0\|_{L^2(0, L)}$, for all $t \geq 0$.*

This result can be extended to the nonlinear equation (1.2.12) using a perturbation approach and small initial data. In the aim to show an exponential stability result, acting from the right in [CL14] Coron and Lü introduce a Neumann feedback based on a Fredholm Transform inspired by previous results for the Kuramoto-Sivashinsky equation [CL15].

Theorem 1.2.6 (Rapid stabilization, non-critical case). *Let $L \notin \mathcal{N}$ and $\mu > 0$ and consider $K_1(u) = K_2(u) = 0$. Then, there exists an operator $K_3(u) = K_{3,\mu}(u)$ and $C > 0$, such that for any mild solution u of (1.2.11), $\|u(t, \cdot)\|_{L^2(0, L)} \leq C e^{-\mu t} \|u_0\|_{L^2(0, L)}$, for all $t \geq 0$.*

This result can be extended to the nonlinear equation (1.2.12) using a perturbation approach and small initial data. The restriction $L \notin \mathcal{N}$ in the above theorem is due again to the lack of controllability of (1.2.2). In the case $L \in \mathcal{N}$, the bad control properties improve if we study the nonlinear system (see Section 1.2.6). By [Ros97] $L^2(0, L) = H \oplus M$, where H corresponds to the controllable space of (1.2.1) and M the uncontrollable space (depending both on L). Moreover, $\dim(M) < \infty$ and $\dim(M)$ is the number of different pairs of positive integers (l_j, k_j) satisfying (1.2.6). Consider P_H (resp. P_M) the projection on H (resp. M), the following result was proved in [CRX17]

Theorem 1.2.7 (Time-varying feedback). *Let $K_1(u) = K_2(u) = 0$. If $\dim(M) = 2n$ with $n \in \mathbb{N}^*$, then, there exist a periodic time-varying feedback law $\tilde{K}_3 : \mathbb{R} \times L^2(0, L) \rightarrow \mathbb{R}$ and positive constants C, μ, r , such that letting $K_3(u) = \tilde{K}_3(t, u)$, every solution of (1.2.12) with $\|u_0\|_{L^2(0, L)} \leq r$ satisfies*

$$\|P_H u(t, \cdot)\|_{L^2(0, L)} + \|P_M u(t, \cdot)\|_{L^2(0, L)}^{1/2} \leq C e^{-\mu t} \left(\|P_H u_0\|_{L^2(0, L)} + \|P_M u_0\|_{L^2(0, L)}^{1/2} \right), \forall t \geq 0.$$

Now, to eliminate the restriction on the critical length in the boundary feedbacks by the right a modified version of the backstepping technique called *pseudo-backstepping* was used in [OB19], the price to pay is to use two controllers.

Theorem 1.2.8. *Let $L > 0$ and $\mu > 0$ and consider $K_1(u) = 0$. Then, there exist operators K_2, K_3 and positive constants C, μ , such that for any mild solution u of (1.2.11), $\|u(t, \cdot)\|_{L^2(0, L)} \leq C e^{-\mu t} \|u_0\|_{L^2(0, L)}$, for all $t \geq 0$.*

Again, this result can be extended to the nonlinear equation (1.2.12) using a perturbation approach and small initial data.

1.2.6 Control and stability on the critical cases

As we said in Section 1.2.5, even if the linear system (1.2.1) is uncontrollable it is possible to show controllability results for the nonlinear system thanks to the nonlinear term

$$\begin{cases} \partial_t u(t, x) + \partial_x u(t, x) + \partial_x^3 u(t, x) + u(t, x) \partial_x u(t, x) = 0, & t \in [0, T], \quad x \in (0, L), \\ u(t, 0) = u(t, L) = 0, & t \in [0, T], \\ \partial_x u(t, L) = h(t), & t \in [0, T], \\ u(0, x) = u_0, & x \in (0, L). \end{cases} \quad (1.2.13)$$

These results are summarized in the following theorem

Theorem 1.2.9. *The nonlinear KdV equation (1.2.13) is locally controllable if*

- (Coron and Crépeau [CC04]) $\dim(M) = 1$, for all $T > 0$.
- (Cerpa [Cer07]) $\dim(M) = 2$, for T big enough.
- (Cerpa and Crépeau [CC09b]) $\dim(M) > 2$, for T big enough.

The proof of those results was based on the power expansion method. Roughly speaking by the work of Rosier [Ros97], the first order approximation of (1.2.13) (the linear system (1.2.1)) is not controllable, the idea to use higher order approximations to reach the uncontrollable space. As is stated in the above theorem, in some cases the authors needed a large time to achieve the controllability. For many years the necessity of this condition was an open problem, but recently Coron, Koenig and Nguyen in [CKN22] gave a negative answer.

Theorem 1.2.10. *Let $L \in \mathcal{N}$ such that $2k + l \notin 3\mathbb{N}^*$. Then, (1.2.13) is not small-time locally null controllable with controls in $H^1(0, T)$ and initial and final datum in $H^3(0, L) \cap H_0^1(0, L)$.*

In the case of stability analysis, a similar natural question arises. In the critical case, does the nonlinear system (1.2.7) enjoy better stability properties than the linear one (1.2.2)? In Section 1.2.3 we said that in the non-critical case $L \notin \mathcal{N}$, the nonlinear system (1.2.7) is locally exponentially stable. If $L \in \mathcal{N}$ the following results are known,

Theorem 1.2.11. *The system (1.2.7) is locally asymptotically stable if*

- (Chu, Coron and Shang [CCS15]) $\dim(M) = 1$,
- (Tang, Chu, Shang and Coron [TCSC18]) $L = 2\pi\sqrt{\frac{7}{3}}$.

- (Nguyen [Ngu21]¹) $\dim(M) = 1$ and moreover the system decays locally at least with a rate $1/t^2$.

The firsts two results are based on the center manifold theory, note that $L = 2\pi\sqrt{\frac{7}{3}}$ is the smallest length for which $\dim(M) = 2$, while the third result is inspired on the power series expansion, theory of quasi-periodic functions and the analysis developed in [CKN22]. There still some open question related to control and stability of the nonlinear KdV equation, for instance regarding Theorem 1.2.10:

- To show the not small-time controllability in the classical regularity setting, i.e. control in $L^2(0, T)$ and initial data in $L^2(0, L)$.
- Replace the condition $L \in \mathcal{N}$ such that $2k + l \notin 3\mathbb{N}^*$ in Theorem 1.2.10, by the more general one $\dim(M) > 1$.
- By Theorem 1.2.9, the nonlinear KdV equation (1.2.13) is large time controllable, thus to search for the optimal time of controllability is an interesting open question.

1.2.7 The Korteweg-de Vries equation on networks

In recent years, partial differential equations in networks have been used to model various physical systems such as transport networks, electrical networks, communication systems and flexible multi-link structures, in particular flows in networks such as gas pipelines, irrigation canals, blood circulation, etc, (see for instance [BCG⁺14, DZ06, LLS12]). These networks are usually represented by graphs, in which their nodes represent the interconnections of the system. To solve a PDE in a network, one needs to know its transmission conditions at the nodes, which in general can be a difficult task. In particular, control and stabilization problems have been widely studied in the network context, we refer for instance [DZ06, AC18, AD21, BCV11, IPR11, CIP18, BPCC⁺23]. After the years of studying control and stabilization proprieties for a single KdV equation, a natural question arises about coupled systems. In the case of internal couplings, we have some very interesting works as [MOP09, CP11, CFGP17, PR08]. Recently, in [CS07] the KdV equation was proposed for modelling the pressure in an arterial compartment, this work motivated the use of the KdV to model the pressure on the arterial tree [AC18].

Now, following [AC18, DZ06] we give some definitions and notations of networks that we will use along this thesis. Let Γ be a connected topological graph embedded in \mathbb{R} with $N \in \mathbb{N}^*$ edges. Let $K = \{k_j : 1 \leq n \leq N\}$ be the set of Γ . Each edge k_j is a Jordan curve in \mathbb{R} and is assumed to be parametrized by its arc length x_j such that the parametrization $\pi_j : [0, \ell_j] \rightarrow k_j : x_j \mapsto \pi_j(x_j)$ is ν -times differentiable, that is, $\pi_j \in C^\nu([0, \ell_j], \mathbb{R})$ for all $j = 1, \dots, N$. The C^ν -network \mathcal{T} associated with Γ is defined by

$$\mathcal{T} = \bigcup_{j=1}^N k_j.$$

In our study, we focus on a system related to the mathematical modelling of the human cardiovascular system. On each edge k_j , the function $u_j(t, x)$ modelizes the wave information about displacement at location x and time t . Specifically, we are going to consider the next evolution problem for the KdV equation:

¹In fact, the result presented in [Ngu21, Theorem 1.1] is more general and holds if $L \in \mathcal{N}$, $\dim(M) > 1$, with an extra condition, which is also checked for a big quantity of values [Ngu21, Corollary 1.1]

$$\begin{cases} (\partial_t u_j + \partial_x u_j + u_j \partial_x u_j + \partial_x^3 u_j)(t, x) = 0, & \forall x \in (0, \ell_j), t > 0, j = 1, \dots, N, \\ u_j(t, 0) = u_{j'}(t, 0), & \forall j, j' = 1, \dots, N, \\ \sum_{j=1}^N \partial_x^2 u_j(t, 0) = -\alpha u_1(t, 0) - \frac{N}{3} u_1^2(t, 0), & t > 0, \\ u_j(t, \ell_j) = \partial_x u_j(t, \ell_j) = 0, & t > 0, j = 1, \dots, N, \\ u_j(0, x) = u_j^0(x), & x \in (0, \ell_j), \end{cases} \quad (1.2.14)$$

and its linearization around 0:

$$\begin{cases} (\partial_t u_j + \partial_x u_j + \partial_x^3 u_j)(t, x) = 0, & \forall x \in (0, \ell_j), t > 0, j = 1, \dots, N, \\ u_j(t, 0) = u_{j'}(t, 0), & \forall j, j' = 1, \dots, N, \\ \sum_{j=1}^N \partial_x^2 u_j(t, 0) = -\alpha u_1(t, 0), & t > 0, \\ u_j(t, \ell_j) = \partial_x u_j(t, \ell_j) = 0, & t > 0, j = 1, \dots, N, \\ u_j(0, x) = u_j^0(x), & x \in (0, \ell_j), \end{cases} \quad (1.2.15)$$

where $\alpha \geq \frac{N}{2}$.

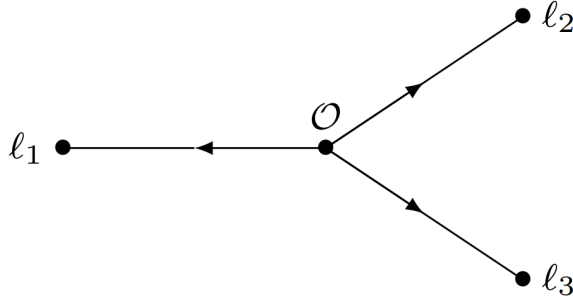


Figure 1.4: star-shaped network for $N = 3$.

The central node conditions are obtained taking account the following: if we denote by u_j and v_j the dimensionless and scaled variables standing respectively for the deflection from rest position and the velocity on the branch j of a long water wave, then we get from [Whi99, Equation (13.102)]

$$\begin{cases} (\partial_t u_j + \partial_x u_j + \partial_x^3 u_j + u_j \partial_x u_j)(t, x) = 0, & \forall x \in (0, \ell_j), t > 0, j = 1, \dots, N, \\ v_j(t, x) = u_j(t, x) - \frac{1}{6} u_j^2(t, x) + 2 \partial_x^2 u_j(t, x), & \forall x \in (0, \ell_j), t > 0, j = 1, \dots, N. \end{cases}$$

Moreover, at the central node, we can suppose that the elevation of water is the same in all branches and that the sum of the flux is null, which implies:

$$\begin{cases} u_j(t, 0) = u_{j'}(t, 0), & \forall j, j' = 1, \dots, N, \\ \sum_{j=1}^N u_j(t, 0) v_j(t, 0) = 0, & t > 0. \end{cases}$$

Then we obtain the following boundary conditions at the junction:

$$\begin{cases} u_j(t, 0) = u_{j'}(t, 0), & \forall j, j' = 1, \dots, N, \\ \sum_{j=1}^N \partial_x^2 u_j(t, 0) = -\frac{N}{2} u_1(t, 0) + \frac{N}{12} u_1^2(t, 0), & t > 0. \end{cases} \quad (1.2.16)$$

If we linearize (1.2.16) around 0, we find the transmission conditions of the linear system (1.2.15). For the nonlinear system (1.2.14), we adapt the transmission conditions in order to have a decreasing energy. The hypothesis $\alpha > \frac{N}{2}$ was introduced in [AC18] and then in [CCM20] the case $\alpha = \frac{N}{2}$ was included. To study these equations, we consider the following functional setting. We define the following spaces

$$H_r^s(0, \ell_j) = \left\{ v \in H^s(0, \ell_j), \left(\frac{d}{dx} \right)^{i-1} v(\ell_j) = 0, 1 \leq i \leq s \right\}, s = 1, 2,$$

where the index r is related to the null right boundary conditions. The space $\mathbb{H}_e^s(\mathcal{T})$ will be the cartesian product of $H_r^s(0, \ell_j)$ including the continuity condition on the central node ($u_j(0) = u_k(0), \forall j, k = 1, \dots, N$)

$$\mathbb{H}_e^s(\mathcal{T}) = \left\{ \underline{u} = (u_1, \dots, u_N) \in \prod_{j=1}^N H_r^s(0, \ell_j), u_j(0) = u_k(0), \forall j, k = 1, \dots, N \right\}, s = 1, 2$$

and

$$\|\underline{u}\|_{\mathbb{H}_e^1(\mathcal{T})}^2 = \sum_{j=1}^N \|u_j\|_{H^1(0, \ell_j)}^2$$

where the index e , indicates that each edge belongs to $H_r^s(0, \ell_j)$.

$$\mathbb{L}^2(\mathcal{T}) = \prod_{j=1}^N L^2(0, \ell_j), \quad \mathbb{L}^\infty(\mathcal{T}) = \prod_{j=1}^N L^\infty(0, \ell_j), \quad \mathbb{H}^3(\mathcal{T}) = \prod_{j=1}^N H^3(0, \ell_j).$$

The space $\mathbb{L}^2(\mathcal{T})$ is equipped with the inner product

$$(\underline{u}, \underline{v})_{\mathbb{L}^2(\mathcal{T})} = \sum_{j=1}^N \int_0^{\ell_j} u_j v_j dx, \quad \forall \underline{u}, \underline{v} \in \mathbb{L}^2(\mathcal{T}). \quad (1.2.17)$$

We also define the space

$$\mathbb{B}_T = C([0, T], \mathbb{L}^2(\mathcal{T})) \cap L^2(0, T; \mathbb{H}_e^1(\mathcal{T}))$$

endowed with the norm

$$\|\underline{u}\|_{\mathbb{B}_T} = \|\underline{u}\|_{C([0, T], \mathbb{L}^2(\mathcal{T}))} + \|\underline{u}\|_{L^2(0, T; \mathbb{H}_e^1(\mathcal{T}))} = \max_{t \in [0, T]} \|\underline{u}\|_{\mathbb{L}^2(\mathcal{T})} + \left(\int_0^T \|\underline{u}(t, \cdot)\|_{\mathbb{H}_e^1(\mathcal{T})}^2 dt \right)^{1/2},$$

and \mathbb{Y}_T be the space of all functions $\underline{u} \in \mathbb{B}_T$ such that $\partial_x^\kappa u_j \in L_x^\infty(0, \ell_n; H^{\frac{1-\kappa}{3}}(0, T))$ for $\kappa = 0, 1, 2$, with the induced norm

$$\|\underline{u}\|_{\mathbb{Y}_T} = \|\underline{u}\|_{\mathbb{B}_T} + \sum_{\kappa=0}^2 \|\partial_x^\kappa \underline{u}\|_{\prod_{j=1}^N L_x^\infty(0, \ell_j; H^{\frac{1-\kappa}{3}}(0, T))}.$$

With respect to the well-posedness of (1.2.14), in [AC18, CCM20] the next well-posedness result was proved for small initial condition and for any time horizon.

Theorem 1.2.12. [Theorem 2.7, [AC18]] *Let $(\ell_n)_{n=1, \dots, N} \in (0, \infty)^N$, $\alpha \geq \frac{N}{2}$ and $T > 0$. Then there exist $\epsilon > 0$ and $C > 0$ such that for all $\underline{u}^0 \in \mathbb{L}^2(\mathcal{T})$ with $\|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})} \leq \epsilon$, there exists a unique solution of (1.2.14). Moreover, it satisfies $\|\underline{u}\|_{\mathbb{B}_T} \leq C \|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})}$.*

The main complication to get a global well-posedness result is the action of the nonlinear boundary condition on the central node. This arises the following open question:

Open problem 1. *Is the system (1.2.14) globally well-posedness in $\mathbb{L}^2(\mathcal{T})$?*

A positive answer of the Open problem 1 is given in Chapter 3. The main focus of [AC18] is to study the exponential stability of (1.2.14) by adding internal dampings as follows

$$\begin{cases} (\partial_t u_n + \partial_x u_n + u_n \partial_x u_n + \partial_x^3 u_n)(t, x) + a_n(x) u_n(t, x) = 0, & x \in (0, \ell_n), t > 0, n = 1, \dots, N, \\ u_n(t, 0) = u_{n'}(t, 0), & \forall n, n' = 1, \dots, N, \\ \sum_{n=1}^N \partial_x^2 u_n(t, 0) = -\alpha u_1(t, 0) - \frac{N}{3} u_1^2(t, 0), & t > 0, \\ u_n(t, \ell_n) = \partial_x u_n(t, \ell_n) = 0, & t > 0, n = 1, \dots, N, \\ u_n(0, x) = u_n^0(x), & x \in (0, \ell_n), \end{cases} \quad (1.2.18)$$

where the damping terms $(a_n)_{n=1, \dots, N} \in \mathbb{L}^\infty(\mathcal{T})$ act locally on all branches, formally written as

$$a_n \geq c_n > 0 \text{ in an open nonempty set } \omega_n \text{ of } (0, \ell_n), \text{ for all } n = 1, \dots, N. \quad (1.2.19)$$

The first main result of [AC18] states the semiglobal exponential stability of (1.2.18).

Theorem 1.2.13. *Assume that the damping terms $(a_n)_{n=1, \dots, N}$ satisfy (1.2.19). Let $(\ell_n)_{n=1}^N \subset (0, \infty)$ and $R > 0$, then there exist $C(R) > 0$ and $\mu(R) > 0$ such that for all $\underline{u}^0 \in \mathbb{L}^2(\mathcal{T})$ with $\|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})} \leq R$, the solution of (1.2.18) satisfies $\|\underline{u}(t, \cdot)\|_{\mathbb{L}^2(\mathcal{T})} \leq C(R) \|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})} e^{-\mu(R)t}$ for all $t > 0$.*

Then, in order to add damped terms only on the critical lengths, let $I_c = \{n \in \{1, \dots, N\}; \ell_n \in \mathcal{N}\}$ be the set of critical lengths and I_c^* be the subset of I_c where we remove one index. The damping $(a_n)_{n=1, \dots, N} \in \mathbb{L}^\infty(\mathcal{T})$ satisfy

$$\begin{cases} a_n = 0 \text{ for } n \in \{1, \dots, N\} \setminus I_c^*, \\ a_n \geq c_n \text{ in an open nonempty set } \omega_n \text{ of } (0, \ell_n), \text{ for all } n \in I_c^*, \\ \text{and } c_n > 0 \text{ is a constant.} \end{cases} \quad (1.2.20)$$

Then, the following local stabilization result, was proved in [AC18].

Theorem 1.2.14. *Assume that the damping terms $(a_n)_{n=1, \dots, N}$ satisfy (3.1.6) and let $(\ell_n)_{n=1}^N \subset (0, \infty)$. Then, there exist $\varepsilon, C > 0$ and $\mu > 0$ such that for all $\underline{u}^0 \in \mathbb{L}^2(\mathcal{T})$, with $\|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})} \leq \varepsilon$, the solution of (1.2.18) satisfies $\|\underline{u}(t, \cdot)\|_{\mathbb{L}^2(\mathcal{T})} \leq C \|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})} e^{-\mu t}$ for all $t > 0$.*

The above stabilization results were derived using a contradiction argument to obtain an appropriate observability inequality and will be a starting point for the analysis developed in this thesis.

1.2.8 Stabilization of KdV with delay

In the works [BCV19] and [Val22] the problem of robustness with respect to time delay for a KdV equation was studied with boundary and internal stabilization respectively. The system addressed in [BCV19] and [Val22] are respectively

$$\begin{cases} \partial_t u(t, x) + \partial_x u(t, x) + \partial_x^3 u(t, x) + u(t, x) \partial_x u(t, x) = 0, & t > 0, x \in (0, L), \\ u(t, 0) = u(t, L) = 0, & t > 0, \\ \partial_x u(t, L) = \alpha \partial_x u(t, 0) + \beta \partial_x u(t - h, 0), & t > 0, \\ \partial_x u(t, 0) = z_0(t) \quad t \in (-h, 0), \\ u(0, x) = u_0, & x \in (0, L), \end{cases} \quad (1.2.21)$$

and

$$\begin{cases} (\partial_t u + \partial_x u + \partial_x^3 u + u \partial_x u)(t, x) + a(x)u(t, x) + b(x)u(t - h, x) = 0, & t > 0, x \in (0, L), \\ u(t, 0) = u(t, L) = 0, & t > 0, \\ \partial_x u(t, L) = 0, & t > 0, \\ u(t, x) = z_0(t, x) & t \in (-h, 0), x \in (0, L), \\ u(0, x) = u_0, & x \in (0, L), \end{cases} \quad (1.2.22)$$

where $h > 0$ is the time delay,

$$|\alpha| + |\beta| < 1, \quad (1.2.23)$$

and

$$\begin{aligned} a, b \in L^\infty(0, L), \quad b(x) \geq b_0 > 0, \quad a.e \text{ in } \text{supp } b, \\ \exists c_0 > 0, b(x) + c_0 \leq a(x), \quad a.e \text{ in } \text{supp } b. \end{aligned} \quad (1.2.24)$$

In [BCV19], under condition (1.2.23) and $L \notin \mathcal{N}$ the authors proved that the system (1.2.21) is exponentially stable. Similarly, in [Val22], under the condition (1.2.24), the authors proved that the system (1.2.22) is exponentially stable. These results were obtained by Lyapunov techniques and by proving suitable observability inequalities. Regarding these results, Theorem 1.2.13 and Theorem 1.2.14 the following open question appears:

Open problem 2. *Can we prove exponential stability of the KdV equation on a star shaped network in presence of time delay?*

This open problem is answered in Chapter 2 in the case of internal delays. This problem is interesting because in the network case, we could have different behaviour on each branch (critical or not). Also following [Val22] we are able to consider the case where in some branches we only have feedback term with delay $a_j = 0, b_j \neq 0$ (see Theorem 2.1.3). Following with questions related with delay, the following natural question emerge

Open problem 3. *Can we prove the exponential stability of a single KdV equation in presence of time-varying delay?*

We deal with this question in Chapter 5. In this case, the KdV equation with time-varying delay is no longer invariant by translation in time, thus the we can not show the exponential stability by proving an observability inequality. In particular the ideas presented in Section 1.2.3 are no longer possible. Another complication arise from the fact that in the case of time-varying delay, the associate operator is time-varying too, thus the study of the well-posedness requires more attention.

1.2.9 Stabilization of KdV with saturation

In Section 1.2.4, we saw that in the case of critical lengths, the system (1.2.10) with feedback law $F(u) = -au$, is exponentially stable. Regarding this feedback law, in [MCPA17] the saturated case was analysed, i.e

$$\begin{cases} (\partial_t u + \partial_x u + \partial_x^3 u + u \partial_x u)(t, x) + \mathbf{sat}(a(x)u(t, x)) = 0, & t \in [0, T], x \in (0, L), \\ u(t, 0) = u(t, L) = \partial_x u(t, L) = 0, & t \in [0, T], \\ u(0, x) = u_0, & x \in (0, L), \end{cases} \quad (1.2.25)$$

where, the saturation map \mathbf{sat} that could be any of the following cases:

- $\mathbf{sat} = \mathbf{sat}_{\text{loc}}$: First consider the following scalar saturation,

$$\mathbf{sat}(f) = \begin{cases} -M, & \text{if } f \leq -M, \\ f, & \text{if } -M \leq f \leq M, \\ M, & \text{if } f \geq M, \end{cases}$$

where $M > 0$ is given and denotes the saturation level. Then, we take the next extension to infinite dimensional setting

$$\mathbf{sat}_{\text{loc}}(f)(x) = \mathbf{sat}(f(x)).$$

- $\mathbf{sat} = \mathbf{sat}_2$: For $f \in L^2(0, L)$ we define

$$\mathbf{sat}_2(f)(x) = \begin{cases} f(x), & \text{if } \|f\|_{L^2(0,L)} \leq M, \\ \frac{f(x)M}{\|f\|_{L^2(0,L)}}, & \text{if } \|f\|_{L^2(0,L)} \geq M. \end{cases}$$

In [MCPA17], the global well-posedness and semiglobal exponential stability was proved by adapting the ideas of [PMVZ02] to the saturated case. Following Open problem 2 and the stability results Theorem 1.2.13 and Theorem 1.2.14, the following question arise

Open problem 4. *Can we prove exponential stability of the KdV equation on a star shaped network in presence of saturation?*

Open problem 4 is closely related to Open problem 1. In fact, without a global well-posedness result for (1.2.14), the saturation case has no sense. Indeed, in the case $\mathbf{sat} = \mathbf{sat}_2$, if the initial data satisfies $\|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})} < \varepsilon$. Then, $\|\underline{u}\|_{C([0,T];\mathbb{L}^2(\mathcal{T}))} \leq \|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})}$, and

$$\|a_n u_n(t, \cdot)\|_{L^2(0, \ell_n)} \leq \|a_n\|_{L^\infty(0, \ell_n)} \|u_n(t, \cdot)\|_{L^2(0, \ell_n)} \leq \varepsilon \|a_n\|_{L^\infty(0, \ell_n)}.$$

Thus if $\varepsilon \leq \frac{M}{\|a_n\|_{L^\infty(0, \ell_n)}}$, we have that $\mathbf{sat}_2(a_n u_n(t, \cdot)) = a_n u_n(t, \cdot)$, that means that the saturation is not active. We deal with Open problem 4 in Chapter 3. Following with the effect of saturation on the KdV equation and in a sake of completeness, we can ask ourselves if the exponential stability holds in the case of boundary saturation:

Open problem 5. *Can we prove exponential stability of a single KdV equation in presence of boundary saturation?*

In Chapter 4 we study the above problem, we base our strategy on a contradiction argument and LaSalle's invariance principle.

1.3 Main results of this thesis

In this part we describe the main contributions of this thesis.

1.3.1 Delayed stabilization of the KdV equation on a star-shaped network

In Chapter 2 we focus on the Open problem 2. The well-posedness and exponential stability of a KdV equation on a star-shaped network with internal delayed feedback terms are studied. To achieve well-posedness, a new variable was introduced to account for the delay, and local well-posedness for the nonlinear equation was obtained through linearization and application of the Banach fixed-point theorem. Local exponential stability is demonstrated using a Lyapunov function, with an associated estimation of the decay rate that strongly depends on the chosen Lyapunov function, in this case we ask for small lengths. An observability inequality for the nonlinear system was then derived using a contradiction argument, which allowed exponential stability to be proven without restrictions on the lengths. Next, the case where the support of delayed terms was not necessarily included in the support of the feedback terms without delay is considered. Finally, numerical simulations were presented, which demonstrate how feedback delayed terms affected stability. This chapter is mainly based on the paper

- Hugo Parada, Emmanuelle Crépeau, and Christophe Prieur. Delayed stabilization of the Korteweg–de Vries equation on a star-shaped network. *Mathematics of Control, Signals, and Systems*, 34(3):559–605, 2022.

1.3.2 Global well-posedness of the KdV equation on a star-shaped network and stabilization by saturated controllers

Chapter 3 is devoted to the study of Open problem 1 and Open problem 4. We focus on the global well-posedness and exponential stability of a KdV equation on a star-shaped network with internal saturated feedback terms. To establish the well-posedness, the Laplace transform of a linear system on the network is used to obtain Kato smoothing properties, which provided local in-time well-posedness. Multiplier estimates are then used to establish the global in-time well-posedness. Additionally, the exponential stability with saturated controller is demonstrated. This chapter is mainly adapted on the paper

- Hugo Parada, Emmanuelle Crépeau, and Christophe Prieur. Global well-posedness of the KdV equation on a star-shaped network and stabilization by saturated controllers. *SIAM Journal on Control and Optimization*, 60(4):2268–2296, 2022.

1.3.3 Stability analysis of a KdV equation with saturated boundary feedback

Chapter 4 is related to Open problem 5. We discuss the exponential stability of the nonlinear KdV equation with a saturated feedback on the boundary. The well-posedness of the equation is achieved using nonlinear semigroup theory and fixed-point results. The exponential stability is established through the classical observability approach, and the asymptotic stability of the linear KdV equation is addressed using LaSalle’s invariance principle. This chapter is based on the conference paper

- Hugo Parada. Stability analysis of a Korteweg-de Vries equation with saturated boundary feedback. *IFAC-PapersOnLine*, 55(26):1–6, 2022. 4th IFAC Workshop on Control of Systems Governed by Partial Differential Equations CPDE 2022.

1.3.4 Stability results of the KdV equation with time-varying delay

In Chapter 5 regarding Open problem 4, we present boundary and internal stability results for the nonlinear KdV equation with time-varying delay. The chapter includes well-posedness results using the theory of constant domain operators, local stability analysis using Lyapunov functionals with explicit estimates of the decay rates, and numerical simulations to illustrate the results. This chapter is based on the paper

- Hugo Parada, Chahnaz Timimoun, and Julie Valein. Stability results for the KdV equation with time-varying delay. *Systems & Control Letters*, 177:105547, 2023.

The remaining part of the manuscript is organized as follows. After presenting these contributions respectively in Chapter 2 to 5, we conclude the discussion and works in progress, open questions are collected in Chapter 6. The Appendix collects some useful lemmas and extra computations.

DELAYED STABILIZATION OF THE KdV EQUATION ON A STAR-SHAPED NETWORK

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In this chapter, we deal with the exponential stability of the nonlinear Korteweg-de Vries (KdV) equation on a finite star-shaped network in the presence of delayed internal feedback. We start by proving the well-posedness of the system and some regularity results. Then we state an exponential stabilization result using a Lyapunov function by imposing small initial data and a restriction over the lengths. In this part also, we are able to obtain an explicit expression for the decay rate. Then we prove a semiglobal exponential stability result, which is based on an observability inequality working directly on the nonlinear system. Next, we study the case where it may happen that a control domain with delay is outside the control domain without delay. In that case, we obtain also a local exponential stabilization result. Finally, we present some numerical simulations to illustrate the stabilization. This chapter is based on the paper:

- Hugo Parada, Emmanuelle Crépeau, and Christophe Prieur. Delayed stabilization of the Korteweg–de Vries equation on a star-shaped network. *Mathematics of Control, Signals, and Systems*, 34(3):559–605, 2022.

2.1 Introduction

In this chapter, we are interested in the stability properties of the Korteweg-de Vries equation with internal input delay posed on a star-shaped network. Specifically, we are going to consider the next evolution problem for the KdV equation with internal input delay on each edge.

$$\begin{cases} \partial_t u_j(t, x) + \partial_x u_j(t, x) + u_j(t, x) \partial_x u_j(t, x) + \partial_x^3 u_j(t, x) \\ + a_j(x) u_j(t, x) + b_j(x) u_j(t - h_j, x) = 0, & x \in (0, \ell_j), t > 0, j = 1, \dots, N, \\ u_j(t, 0) = u_{j'}(t, 0), & \forall j, j' = 1, \dots, N, \\ \sum_{j=1}^N \partial_x^2 u_j(t, 0) = -\alpha u_1(t, 0) - \frac{N}{3} u_1^2(t, 0), & t > 0, \\ u_j(t, \ell_j) = \partial_x u_j(t, \ell_j) = 0, & t > 0, j = 1, \dots, N, \\ u_j(0, x) = u_j^0(x), & x \in (0, \ell_j), \\ u_j(t, x) = z_j^0(t, x), & (t, x) \in (-h_j, 0) \times (0, \ell_j), \end{cases} \quad (\text{KdVd})$$

where $\alpha \geq \frac{N}{2}$ and for all $j = 1, \dots, N$, $h_j > 0$ is the time delay on the edge j , $a_j, b_j \in L^\infty(0, \ell_j)$ are non-negative and $\text{supp } b_j = \omega_j$ is a nonempty, open subset of $(0, \ell_j)$ such that

$$b_j(x) \geq b_0 > 0, \text{ a.e on } \omega_j, \quad (2.1.1)$$

$$\text{there exists } c_0 > 0, \text{ such that } b_j(x) + c_0 \leq a_j(x), \forall x \in \omega_j. \quad (2.1.2)$$

In this chapter, we consider $\alpha > \frac{N}{2}$ in some cases and $\alpha \geq \frac{N}{2}$ in others.

The classical internal feedback law for the one dimensional KdV equation is $a(x)u(t, x)$ (see e.g., [PMVZ02]) and hence $a_j(x)u_j(t, x)$ in the network case. Therefore, one might think that the delayed internal feedback law should be $a_j(x)u_j(t - h_j, x)$, which differs from $a_j(x)u_j(t, x) + b_j(x)u_j(t - h_j, x)$. The introduction of this damped term and the conditions over the damped terms with and without delay (2.1.1)-(2.1.2) are the analogues of the conditions (1.2)-(1.3) appearing in [Val22]. In [NP06, Theorem 1.4] it is proved that some instabilities appear in the case $a_j = 0$ (see also [BCV19, NVF09, NV10]). Moreover, the necessity of adding $a_j(x)u_j(t, x)$ is shown at least numerically in Section 2.5.

Note first (2.1.1) and (2.1.2) imply

$$\omega_j = \text{supp } b_j \subset \text{supp } a_j, \text{ and } a_j(x) \geq b_0 + c_0 > 0, \text{ in } \omega_j. \quad (2.1.3)$$

To deal with delays we introduce the following space

$$\mathcal{H} = \mathbb{L}^2(\mathcal{T}) \times \left(\prod_{j=1}^N L^2((-h_j, 0) \times (0, \ell_j)) \right),$$

endowed with

$$\|(u, \underline{z})\|_{\mathcal{H}}^2 = \sum_{j=1}^N \left(\int_0^{\ell_j} (u_j(x))^2 dx + \int_{-h_j}^0 \int_0^{\ell_j} \xi_j(x) (z_j(s, x))^2 dx ds \right),$$

where $\underline{\xi} \in \mathbb{L}^\infty(\mathcal{T})$ be such that $\text{supp } \xi_j = \text{supp } b_j = \omega_j$ and

$$b_j(x) + c_0 \leq \xi_j(x) \leq 2a_j(x) - b_j(x) - c_0, \text{ in } \omega_j. \quad (2.1.4)$$

For (KdVd), we define the energy

$$E(t) = \sum_{j=1}^N \left(\int_0^{\ell_j} (u_j(t, x))^2 dx + h_j \int_{\omega_j} \int_0^1 \xi_j(x) (u_j(t - h_j \rho, x))^2 d\rho dx \right). \quad (2.1.5)$$

The above expression corresponds to the squared norm of $(\underline{u}(t, \cdot), \underline{u}(t + \cdot, \cdot))$ in \mathcal{H} , with the change of variable $s = -h_j\rho$ for $u_j(t + s, x)$.

Finally, we denote $\mathbb{L}^2(\Omega) = \prod_{j=1}^N L^2((0, 1) \times \omega_j)$, and let $H = \mathbb{L}^2(\mathcal{T}) \times \mathbb{L}^2(\Omega)$, with its inner product

$$\left\langle \begin{pmatrix} \underline{u} \\ \underline{z} \end{pmatrix}, \begin{pmatrix} \underline{v} \\ \underline{y} \end{pmatrix} \right\rangle = \sum_{j=1}^N \int_0^{\ell_j} u_j(x)v_j(x)dx + h_j \int_{\omega_j} \int_0^1 \xi_j(x)z_j(\rho, x)y_j(\rho, x)d\rho dx,$$

we denote by $\|\cdot\|_H$ its associated norm.

Our first main result is the following one, where the local exponential stability of (KdVd) is obtained for a restricted assumption over $L = \max_{j=1, \dots, N} \ell_j$, but an estimation of the decay rate is given.

Theorem 2.1.1. *Assume $\underline{a}, \underline{b} \in \mathbb{L}^\infty(\mathcal{T})$ are componentwise non-negative that satisfy (2.1.1) and (2.1.2). Let $\alpha > \frac{N}{2}$, $(\ell_j)_{j=1}^N \subset (0, \infty)$ such that $L < \frac{\sqrt{3}}{2}\pi$ and $0 < \epsilon < \frac{3}{2} \frac{(3\pi^2 - 4L^2)}{\pi^2 L^{3/2}}$. Then, for every $(\underline{u}^0, \underline{z}^0(-\underline{h}, \cdot)) \in H$ satisfying $\|(\underline{u}^0, \underline{z}^0(-\underline{h}, \cdot))\|_H \leq \epsilon$, the energy (2.1.5) of the unique mild solution of (KdVd) decays exponentially. That is, there exist $C > 0$, $\gamma > 0$ such that*

$$E(t) \leq CE(0)e^{-2\gamma t}, \quad t > 0,$$

where

$$\gamma \leq \min \left\{ \frac{\left(3\mu_1\pi^2 + \frac{2}{3}L^{3/2}\epsilon\mu_1\pi^2 - \mu_14L^2 \right)}{8L^2((1 + L\mu_1))}, \min_{j=1, \dots, N} \frac{\mu_2}{2h_j(\mu_2 + \|\xi_j\|_{L^\infty(0, \ell_j)})} \right\}, \quad (2.1.6)$$

$$C = \left(1 + \max \left\{ L\mu_1, \frac{\mu_2}{b_0} \right\} \right),$$

for μ_1 and μ_2 such that

$$0 < \mu_1 < \min \left\{ 1, \frac{1}{N} (2\alpha - N) \min_{j=1, \dots, N} \left\{ \inf_{\omega_j} \frac{2a_j - b_j - \xi_j}{Lb_j}, \inf_{\omega_j} \frac{\xi_j - b_j}{Lb_j} \right\} \right\},$$

$$0 < \mu_2 < \min_{j=1, \dots, N} \inf_{\omega_j} \{2a_j - b_j - \xi_j - \mu_1 Lb_j\}.$$

This result is proved in this chapter constructively by using a Lyapunov function, similar to those used in [BCV19, Val22].

On the other hand, our second main result is obtained without restriction on the lengths of \mathcal{T} . The semiglobal exponential stability is obtained by studying directly the nonlinear system (KdVd).

Theorem 2.1.2. *Assume $\underline{a}, \underline{b} \in \mathbb{L}^\infty(\mathcal{T})$ are componentwise non-negative that satisfies (2.1.1) and (2.1.2). Let $(\ell_j)_{j=1}^N \subset (0, \infty)$ and $R > 0$. Then for all $(\underline{u}^0, \underline{z}^0(-\underline{h}, \cdot)) \in H$ with $\|(\underline{u}^0, \underline{z}^0(-\underline{h}, \cdot))\|_H \leq R$ then there exist $C = C(R) > 0$ and $\mu = \mu(R) > 0$ such that the energy of the unique mild solution of (KdVd) satisfies $E(t) \leq CE(0)e^{-\mu t}$ for all $t > 0$.*

The main difference between Theorem 2.1.1 and Theorem 2.1.2 is that Theorem 2.1.2 is based on an observability inequality, which is proved using a contradiction argument. Thus, we can not estimate the decay rate.

In the last results presented, it is not possible to take $a_j = 0$ and $b_j \neq 0$ for some $j \in \{1, \dots, N\}$ (by (2.1.3) if $a_j = 0$ then $b_j = 0$). In the case $a_j = 0$ and $b_j = 0$, the system (KdVd) with the condition $L < \frac{\sqrt{3}}{2}\pi$ is naturally exponentially decreasing. Thus, if $a_j = 0$ and b_j is a small enough perturbation, it is expected that the system (KdVd) is still exponentially decreasing with the condition $L < \frac{\sqrt{3}}{2}\pi$. The next result we deal with this problem in a more general case, we suppose for this part that

$$\omega_j = \text{supp } b_j \not\subset \text{supp } a_j, \quad \text{for } j \in I \subset \{1, \dots, N\}. \quad (2.1.7)$$

In this case, we write now the analogues of the condition (2.1.2) in the setting (2.1.7), take $I^* = \{1, \dots, N\} \setminus I$,

$$\text{there exists } c_0 > 0, \text{ such that } b_j(x) + c_0 \leq a_j(x), \forall x \in \omega_j, \quad \text{for } j \in I^*. \quad (2.1.8)$$

Then we write our last result of stabilization when the internal delay is not necessarily supported in the domain of a_j .

Theorem 2.1.3. *Assume $\underline{a}, \underline{b} \in \mathbb{L}^\infty(\mathcal{T})$ are componentwise non-negative that satisfy (2.1.1) and (2.1.8). Let $\alpha > \frac{N}{2}$, $\eta > 1$ and $(\ell_j)_{j=1}^N \subset (0, +\infty)$ such that $L < \frac{\sqrt{3}}{2}\pi$. Then there exists $\delta = \delta(\alpha, \eta, L, \underline{h}) > 0$ and $\epsilon > 0$, such that for every $(\underline{u}^0, \underline{z}^0(-\underline{h}, \cdot))$ satisfying $\|\underline{b}\|_{\mathbb{L}^\infty(\mathcal{T})} \leq \delta$ and $\|(\underline{u}^0, \underline{z}^0(-\underline{h}, \cdot))\|_H \leq \epsilon$, the energy of the unique mild solution of (KdVd) decays exponentially to 0.*

Remark 2.1.1. *Note that, in all our results, there are no bound conditions for the delay parameters h_j . That is, we have the exponential stability for all given delays. Of course, as follows from (2.1.6) the decay rate depends on the value of the delay and the larger the delay is, the slower is the decay rate.*

The organization of this chapter is the following:

Section 2.2 is devoted to the study of the well-posedness of (KdVd). More precisely we consider the linearization around 0 of (KdVd) and using semigroup theory as recalled in Chapter 1, we show the well-posedness of the linear system. Then, using a fixed-point argument, we obtain the well-posedness for the nonlinear system. In Section 2.3 we present our stabilization results when the feedback terms $\underline{a}, \underline{b} \in \mathbb{L}^\infty(\mathcal{T})$ satisfy (2.1.1) and (2.1.2). The first one, namely Theorem 2.1.1 is obtained following the same steps as [BCV19, Val22]. Then we detail the proof of Theorem 2.1.2 which based on an observability inequality. In Section 2.4, we study the case where (2.1.2) is not satisfied, and we show the proof of Theorem 2.1.3 using a suitable auxiliary system and a perturbation argument. Some numerical simulations are presented in Section 2.5 in order to illustrate the results obtained.

2.2 Well-posedness of a delayed KdV system

Our idea is the following, first we work with the linearization around 0 of (KdVd), then we add a boundary source term at the central node $g(t)$ to consider the nonlinear boundary condition $-\frac{N}{3}u_1^2(t, 0)$ and secondly we add the internal source terms f_j to consider after the term $u_j \partial_x u_j$. Finally, to pass to the nonlinear (KdVd) we use a fixed-point argument.

2.2.1 Well-posedness of the linear case

We start by proving the well-posedness for the linearization of (KdVd) around 0, that writes

$$\left\{ \begin{array}{ll} \partial_t u_j(t, x) + \partial_x u_j(t, x) + \partial_x^3 u_j(t, x) + a_j(x) u_j(t, x) \\ + b_j(x) u_j(t - h_j, x) = 0, & x \in (0, \ell_j), t > 0, j = 1, \dots, N, \\ u_j(t, 0) = u_{j'}(t, 0), & \forall j, j' = 1, \dots, N, \\ \sum_{j=1}^N \partial_x^2 u_j(t, 0) = -\alpha u_1(t, 0), & t > 0, \\ u_j(t, \ell_j) = \partial_x u_j(t, \ell_j) = 0, & t > 0, j = 1, \dots, N, \\ u_j(0, x) = u_j^0(x), & x \in (0, \ell_j), \\ u_j(t, x) = z_j^0(t, x), & (t, x) \in (-h_j, 0) \times (0, \ell_j). \end{array} \right. \quad (\text{LKdVd})$$

We set $z_j(t, \rho, x) = u_j \Big|_{\omega_j} (t - h_j \rho, x)$ $x \in \omega_j, \rho \in (0, 1)$. Then we can check that

$$\begin{cases} h_j \partial_t z_j(t, \rho, x) + \partial_\rho z_j(t, \rho, x) = 0, & x \in \omega_j, \rho \in (0, 1), t > 0, \\ z_j(t, 0, x) = u_j(t, x), & x \in \omega_j, t > 0, \\ z_j(0, \rho, x) = u_j \Big|_{\omega_j} (-h_j \rho, x) = z_j^0(-h_j \rho, x), & \rho \in (0, 1). \end{cases} \quad (2.2.1)$$

Then (LKdVd) can be written as

$$\begin{cases} \partial_t U(t) = \mathcal{A}U(t), & t > 0 \\ U(0) = U_0, \end{cases} \quad (2.2.2)$$

where $U = \begin{pmatrix} \underline{u} \\ \underline{z} \end{pmatrix}$, $U_0 = \begin{pmatrix} \underline{u}^0 \\ \underline{z}^0 \Big|_{\omega}(-\underline{h}, \cdot) \end{pmatrix}$ and the operator \mathcal{A} is defined by:

$$\mathcal{A}U = \begin{pmatrix} -(\partial_x + \partial_x^3) \underline{u} - \underline{a} \cdot \underline{u} - \underline{b} \cdot \underline{\tilde{z}}(1, \cdot) \\ -\frac{1}{\underline{h}} \cdot \partial_\rho \underline{z} \end{pmatrix}$$

for $\underline{u} = (u_j)_{j=1}^N$, $\underline{a} = (a_j)_{j=1}^N$, $\underline{b} = (b_j)_{j=1}^N$, $\underline{h} = (h_j)_{j=1}^N$, $\left(\frac{1}{\underline{h}}\right)_j = \frac{1}{h_j}$ and $\underline{\tilde{z}}(1, \cdot) = (\tilde{z}_j(1, \cdot))_{j=1}^N$ in which $\tilde{z}_j(1, \cdot) \in L^2(0, \ell_j)$ is the extension by 0 of $z_j(1, \cdot)$ outside ω_j . The domain of \mathcal{A} is the following

$$D(\mathcal{A}) = \left\{ \begin{pmatrix} \underline{u} \\ \underline{z} \end{pmatrix}, \underline{u} \in \mathbb{H}^3(\mathcal{T}) \cap \mathbb{H}_e^2(\mathcal{T}), \sum_{j=1}^N \frac{d^2 u_j}{dx^2}(0) = -\alpha u_1(0), \underline{z} \in \prod_{j=1}^N L^2(H^1(0, 1) \times \omega_j), z_j(0, x) = u_j \Big|_{\omega_j}(x) \right\}.$$

Note that if $\begin{pmatrix} \underline{u} \\ \underline{z} \end{pmatrix} \in D(\mathcal{A})$ then $\underline{u} \in \mathbb{H}_e^2(\mathcal{T})$ that implies $u_j(t, \ell_j) = \partial_x u_j(t, \ell_j) = 0$.

Theorem 2.2.1. *Assume $\underline{a}, \underline{b} \in \mathbb{L}^\infty(\mathcal{T})$ are componentwise non-negative that satisfy (2.1.1) and (2.1.2). Let $U_0 \in H$. Then there exists a unique mild solution $U \in C([0, \infty); H)$ of (2.2.2). Moreover, if $U_0 \in D(\mathcal{A})$ then U is a classical solution and*

$$U \in C([0, \infty); D(\mathcal{A})) \cap C^1([0, \infty); H).$$

Proof. Let $U = \begin{pmatrix} \underline{u} \\ \underline{z} \end{pmatrix} \in D(\mathcal{A})$, then

$$\begin{aligned} \langle \mathcal{A}U, U \rangle &= \sum_{j=1}^N \left(\int_0^{\ell_j} (-\partial_x^3 u_j(x) - \partial_x u_j(x) - a_j(x) u_j(x)) u_j(x) dx \right. \\ &\quad \left. - \int_{\omega_j} b_j(x) z_j(1, x) u_j(x) dx - h_j \int_{\omega_j} \int_0^1 \xi_j(x) \frac{1}{h_j} \partial_\rho z_j(\rho, x) z_j(\rho, x) d\rho dx \right) \\ &= \sum_{j=1}^N \left(\int_0^{\ell_j} \partial_x^2 u_j(x) \partial_x u_j(x) dx - \partial_x^2 u_j(x) u_j(x) \Big|_0^{\ell_j} - \frac{1}{2} (u_j(x))^2 \Big|_0^{\ell_j} - \int_0^{\ell_j} a_j(x) (u_j(x))^2 dx \right. \\ &\quad \left. - \int_{\omega_j} b_j(x) z_j(1, x) u_j(x) dx - \frac{1}{2} \int_{\omega_j} \xi_j(x) (z_j(\rho, x))^2 \Big|_0^1 dx \right) \\ &= \sum_{j=1}^N \left(\frac{1}{2} (\partial_x u_j(x))^2 \Big|_0^{\ell_j} + \partial_x^2 u_j(0) u_1(0) + \frac{1}{2} (u_1(0))^2 - \int_0^{\ell_j} a_j(x) (u_j(x))^2 dx \right. \\ &\quad \left. - \int_{\omega_j} b_j(x) z_j(1, x) u_j(x) dx - \frac{1}{2} \int_{\omega_j} \xi_j(x) (z_j(1, x))^2 dx + \frac{1}{2} \int_{\omega_j} \xi_j(x) (z_j(0, x))^2 dx \right). \end{aligned}$$

Thus, for all $U \in D(\mathcal{A})$,

$$\begin{aligned}
\langle \mathcal{A}U, U \rangle &\leq -\frac{1}{2} \sum_{j=1}^N (\partial_x u_j(0))^2 + \left(\frac{N}{2} - \alpha \right) (u_1(0))^2 - \sum_{j=1}^N \int_{(0, \ell_j) \setminus \omega_j} a_j(x) (u_j(x))^2 dx \\
&\quad + \sum_{j=1}^N \int_{\omega_j} \left(-a_j(x) + \frac{b_j(x)}{2} + \frac{\xi_j(x)}{2} \right) (u_j(x))^2 dx \\
&\quad + \sum_{j=1}^N \int_{\omega_j} \left(\frac{b_j(x)}{2} - \frac{\xi_j(x)}{2} \right) (z_j(1, x))^2 dx.
\end{aligned} \tag{2.2.3}$$

Using (2.1.3), (2.1.4) and that $\alpha \geq \frac{N}{2}$ we conclude that for all $U \in D(\mathcal{A})$, $\langle \mathcal{A}U, U \rangle \leq 0$, thus \mathcal{A} is dissipative as introduced in Definition 1.1.2. Easy calculations show that

$$\mathcal{A}^* \begin{pmatrix} \underline{v} \\ \underline{y} \end{pmatrix} = \begin{pmatrix} (\partial_x + \partial_x^3) \underline{v} - \underline{a} \cdot \underline{v} + \underline{\xi} \cdot \tilde{\underline{y}}(0, \cdot) \\ \frac{1}{h} \cdot \partial_\rho \underline{y} \end{pmatrix},$$

in which $\tilde{y}_j(0, \cdot) \in L^2(0, \ell_j)$ is the extension by 0 of $y_j(0, \cdot)$ outside ω_j and with

$$\begin{aligned}
D(\mathcal{A}^*) &= \left\{ \begin{pmatrix} \underline{v} \\ \underline{y} \end{pmatrix}, \underline{v} \in \mathbb{H}^3(\mathcal{T}) \cap \mathbb{H}_e^1(\mathcal{T}), \sum_{j=1}^N \frac{d^2 v_j}{dx^2}(0) = (\alpha - N)v_1(0), \partial_x v_j(0) = 0, \forall j = 1, \dots, N, \right. \\
&\quad \left. \underline{y} \in \prod_{j=1}^N L^2(H^1(0, 1) \times \omega_j), y_j(1, x) = -\frac{b_j(x)}{\xi_j(x)} v_j \Big|_{\omega_j}(x) \right\}.
\end{aligned}$$

Note that $\begin{pmatrix} \underline{v} \\ \underline{y} \end{pmatrix} \in D(\mathcal{A}^*)$ then $\underline{v} \in \mathbb{H}_e^1(\mathcal{T})$ that implies $v_j(t, \ell_j) = 0$. Let $V = \begin{pmatrix} \underline{v} \\ \underline{y} \end{pmatrix} \in D(\mathcal{A}^*)$, then

$$\begin{aligned}
\langle \mathcal{A}^*V, V \rangle &= \sum_{j=1}^N \left(\int_0^{\ell_j} (\partial_x^3 v_j(x) + \partial_x v_j(x) - a_j(x) v_j(x)) v_j(x) dx \right. \\
&\quad \left. + \int_{\omega_j} \xi_j(x) y_j(0, x) v_j(x) dx + \int_{\omega_j} h_j \int_0^1 \xi_j(x) \frac{1}{h_j} \partial_\rho y_j(\rho, x) y_j(\rho, x) d\rho dx \right), \\
&= \sum_{j=1}^N \left(-\int_0^{\ell_j} \partial_x^2 v_j(x) \partial_x v_j(x) dx + \partial_x^2 v_j(x) v_j(x) \Big|_0^{\ell_j} + \frac{1}{2} (v_j(x))^2 \Big|_0^{\ell_j} \right. \\
&\quad \left. - \int_0^{\ell_j} a_j(x) (v_j(x))^2 dx + \int_{\omega_j} \xi_j(x) y_j(0, x) v_j(x) dx + \frac{1}{2} \int_{\omega_j} \xi_j(x) (y_j(\rho, x))^2 \Big|_0^1 dx \right) \\
&\leq \sum_{j=1}^N \left(-\frac{1}{2} (\partial_x v_j(x))^2 \Big|_0^{\ell_j} - \partial_x^2 v_j(0) v_1(0) - \frac{1}{2} (v_1(0))^2 - \int_0^{\ell_j} a_j(x) (v_j(x))^2 dx \right. \\
&\quad \left. + \frac{1}{2} \int_{\omega_j} \xi_j(x) (y_j(0, x))^2 dx + \frac{1}{2} \int_{\omega_j} \xi_j(x) (v_j(x))^2 dx \right. \\
&\quad \left. + \frac{1}{2} \int_{\omega_j} \xi_j(x) ((y_j(1, x))^2 - (y_j(0, x))^2) dx \right) \\
&\leq -\frac{1}{2} \sum_{j=1}^N (\partial_x v_j(\ell_j))^2 + \left(\frac{N}{2} - \alpha \right) (v_1(0))^2 + \int_{\omega_j} \left(-a_j(x) + \frac{\xi_j(x)}{2} + \frac{b_j^2(x)}{2\xi_j(x)} \right) (v_j(x))^2 dx \\
&\quad - \int_{(0, \ell_j) \setminus \omega_j} a_j(x) (v_j(x))^2 dx - \frac{1}{2} \int_{\omega_j} \xi_j(x) (y_j(0, x))^2 dx.
\end{aligned}$$

Moreover, we know that $\xi_j(x) > b_j(x) > b_0 > 0$, for $x \in \omega_j$, then we have that $\frac{b_j^2(x)}{\xi_j(x)} < b_j(x)$, for $x \in \omega_j$ and then

$$-a_j(x) + \frac{\xi_j(x)}{2} + \frac{b_j^2(x)}{2\xi_j(x)} < -a_j(x) + \frac{\xi_j(x)}{2} + \frac{b_j(x)}{2} \leq 0, \quad \text{for } x \in \omega_j,$$

thus as $\alpha \geq \frac{N}{2}$, \mathcal{A}^* is dissipative. Finally, \mathcal{A} and \mathcal{A}^* are dissipative, also \mathcal{A} is a densely defined closed operator. Thus, by Theorem 1.1.1, \mathcal{A} is the infinitesimal generator of a C_0 semigroup of contractions on H . \square

As the systems (LKdVd) and (2.2.2) are equivalent, we obtain the well-posedness of (LKdVd). Let $S(t), t \geq 0$ the semigroup of contractions associated with \mathcal{A} . Next result gives us some a priori estimates for (LKdVd).

Proposition 2.2.1. *Assume $\underline{a}, \underline{b} \in \mathbb{L}^\infty(\mathcal{T})$ are componentwise non-negative that satisfy (2.1.1) and (2.1.2). Then, the map*

$$U_0 = (\underline{u}^0, \underline{z}^0(-\underline{h}\cdot, \cdot)) \mapsto S(\cdot)(\underline{u}^0, \underline{z}^0(-\underline{h}\cdot, \cdot)) \quad (2.2.4)$$

is continuous from H to $\mathbb{B}_T \times C([0, T]; \mathbb{L}^2(\Omega))$ and the following estimates hold, for all $(\underline{u}^0, \underline{z}^0(-\underline{h}\cdot, \cdot)) \in H$,

$$\sum_{j=1}^N \int_0^T \int_0^{\ell_j} a_j(x) (u_j(t, x))^2 dx dt + \int_0^T \int_{\omega_j} (z_j(t, 1, x))^2 dx dt \leq C \left(\|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})}^2 + \|\underline{z}^0(-\underline{h}\cdot, \cdot)\|_{\mathbb{L}^2(\Omega)}^2 \right), \quad (2.2.5)$$

$$\begin{aligned} \|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})}^2 &\leq \left(\frac{1 + 2T\|\underline{a}\|_{\mathbb{L}^\infty(\mathcal{T})} + 2T\|\underline{b}\|_{\mathbb{L}^\infty(\mathcal{T})}}{T} \right) \|\underline{u}\|_{L^2(0, T; \mathbb{L}^2(\mathcal{T}))}^2 + 2 \left(\alpha - \frac{N}{2} \right) \|u_1(\cdot, 0)\|_{L^2(0, T)}^2 \\ &\quad + \|\partial_x u(\cdot, 0)\|_{L^2(0, T)}^2 + \|\underline{b}\|_{\mathbb{L}^\infty(\mathcal{T})} \|\underline{z}^0(-\underline{h}\cdot, \cdot)\|_{\mathbb{L}^2(\Omega)}^2, \end{aligned} \quad (2.2.6)$$

$$\|\underline{z}^0(-\underline{h}\cdot, \cdot)\|_{\mathbb{L}^2(\Omega)}^2 \leq \|\underline{z}(T, \cdot, \cdot)\|_{\mathbb{L}^2(\Omega)}^2 + \sum_{j=1}^N \frac{1}{h_j} \int_0^T \int_{\omega_j} (z_j(t, 1, x))^2 dx dt. \quad (2.2.7)$$

Proof. Taking $(\underline{u}^0, \underline{z}^0(-\underline{h}\cdot, \cdot)) \in H$, Theorem 2.2.1 implies $S(\cdot)(\underline{u}^0, \underline{z}^0(-\underline{h}\cdot, \cdot)) = (\underline{u}, \underline{z}) \in C([0, T]; H)$ and using that \mathcal{A} generates a C_0 semigroup of contractions, we get for all $t \in [0, T]$

$$\begin{aligned} &\sum_{j=1}^N \int_0^{\ell_j} (u_j(t, x))^2 dx + \sum_{j=1}^N h_j \int_{\omega_j} \int_0^1 \xi_j(x) (z_j(t, \rho, x))^2 d\rho dx \\ &\leq \sum_{j=1}^N \int_0^{\ell_j} (u_j^0(x))^2 dx + \sum_{j=1}^N h_j \int_{\omega_j} \int_0^1 \xi_j(x) (z_j^0(-h_j \rho, x))^2 d\rho dx. \end{aligned} \quad (2.2.8)$$

Let $\underline{p} \in (C^\infty([0, T] \times (0, 1)))^N$ and $\underline{q} \in \prod_{j=1}^N C^\infty([0, T] \times (0, \ell_j))$. Now multiplying (LKdVd) by $q_j u_j$

and (2.2.1) by $p_j z_j$ and integrating on $(0, s) \times (0, \ell_j)$ and $(0, s) \times (0, 1) \times \omega_j$ we can obtain

$$\begin{aligned} & \int_0^{\ell_j} q_j(t, x)(u_j(t, x))^2 dx \Big|_0^s - \int_0^s \int_0^{\ell_j} (\partial_t q_j + \partial_x q_j + \partial_x^3 q_j)(u_j)^2 dx dt + 2 \int_0^s \int_0^{\ell_j} a_j q_j (u_j)^2 dx dt \\ & + 2 \int_0^s \int_0^{\ell_j} b_j(x) q_j(t, x) u_j(t - h_j, x) u_j(t, x) dx dt + 3 \int_0^s \int_0^{\ell_j} (\partial_x u_j)^2 \partial_x q_j dx dt \\ & = \int_0^s [(q_j + \partial_x^2 q_j)(u_j)^2 + 2q_j u_j \partial_x^2 u_j - 2\partial_x q_j u_j \partial_x u_j - q_j (\partial_x u_j)^2] (t, 0) dt, \end{aligned} \quad (2.2.9)$$

$$\begin{aligned} & \int_0^1 \int_{\omega_j} (z_j(t, \rho, x))^2 p_j(t, \rho) dx d\rho \Big|_0^s - \frac{1}{h_j} \int_0^s \int_0^1 \int_{\omega_j} (h_j \partial_t p_j + \partial_\rho p_j)(z_j)^2 dx d\rho dt \\ & + \frac{1}{h_j} \int_0^s \int_{\omega_j} (z_j(t, 1, x))^2 p_j(t, 1) - (u_j(t, x))^2 p_j(t, 0) dx dt = 0. \end{aligned} \quad (2.2.10)$$

Taking $s = T$ and $p_j = \rho$ in (2.2.10) we get

$$\begin{aligned} & \int_0^1 \int_{\omega_j} \rho [(z_j(T, \rho, x))^2 - (z_j^0(-\rho h_j, x))^2] dx d\rho - \frac{1}{h_j} \int_0^T \int_0^1 \int_{\omega_j} (z_j)^2 dx d\rho dt \\ & + \frac{1}{h_j} \int_0^T \int_{\omega_j} (z_j(t, 1, x))^2 dx d\rho = 0. \end{aligned} \quad (2.2.11)$$

Thus,

$$\frac{1}{h_j} \int_0^T \int_{\omega_j} (z_j(t, 1, x))^2 dx d\rho \leq \frac{1}{h_j} \int_0^T \int_0^1 \int_{\omega_j} (z_j)^2 dx d\rho dt + \int_0^1 \int_{\omega_j} \rho (z_j^0(-\rho h_j, x))^2 dx d\rho.$$

and hence with (2.2.8) we get

$$\sum_{j=1}^N \int_{\omega_j} (z_j(t, 1, x))^2 dx d\rho \leq C \left(\|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})}^2 + \|\underline{z}^0(-h, \cdot)\|_{\mathbb{L}^2(\Omega)}^2 \right). \quad (2.2.12)$$

Then taking $q_j = 1$ in (2.2.9)

$$\begin{aligned} & \sum_{j=1}^N \int_0^{\ell_j} (u_j(s, x))^2 dx + \sum_{j=1}^N \int_0^s (\partial_x u_j(t, 0))^2 dt + (2\alpha - N) \int_0^s (u_1(t, 0))^2 dt \\ & + 2 \sum_{j=1}^N \int_0^s \int_0^{\ell_j} a_j (u_j)^2 dx dt + \sum_{j=1}^N 2 \int_0^s \int_0^{\ell_j} b_j u_j(t - h_j, x) u_j(t, x) dx dt \\ & = \sum_{j=1}^N \int_0^{\ell_j} (u_j(0, x))^2 dx. \end{aligned}$$

Thus

$$\begin{aligned} & \sum_{j=1}^N \int_0^{\ell_j} (u_j(s, x))^2 dx + \sum_{j=1}^N \int_0^s (\partial_x u_j(t, 0))^2 dt + 2 \sum_{j=1}^N \int_0^T \int_0^{\ell_j} a_j (u_j)^2 dx dt \\ & \leq \sum_{j=1}^N \int_0^{\ell_j} (u_j(0, x))^2 dx + \sum_{j=1}^N 2 \int_0^T \int_0^{\ell_j} b_j |u_j(t - h_j, x)| |u_j(t, x)| dx dt. \end{aligned}$$

Note now that

$$\begin{aligned}
& 2 \int_0^s \int_0^{\ell_j} b_j(x) |u_j(t - h_j, x)| |u_j(t, x)| dx dt \\
& \leq \int_0^s \int_0^{\ell_j} b_j(x) (u_j(t - h_j, x))^2 dx dt + \int_0^s \int_0^{\ell_j} b_j(x) (u_j(t, x))^2 dx dt, \\
& = \int_0^s \int_0^{\ell_j} b_j(x) (u_j(t, x))^2 dx dt + \int_{-h_j}^{s-h_j} \int_{\omega_j} b_j(x) (u_j(t, x))^2 dx dt, \\
& \leq 2 \int_0^s \int_0^{\ell_j} b_j(x) (u_j(t, x))^2 dx dt + \int_{-h_j}^0 \int_{\omega_j} b_j(x) (z_j^0(t, x))^2 dx dt,
\end{aligned}$$

which implies

$$2 \sum_{j=1}^N \int_0^s \int_0^{\ell_j} b_j(x) |u_j(t - h_j, x)| |u_j(t, x)| dx dt \leq C \left(\|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})}^2 + \|\underline{z}^0(-\underline{h}, \cdot)\|_{\mathbb{L}^2(\Omega)}^2 \right). \quad (2.2.13)$$

Thus, we have

$$\begin{aligned}
& \sum_{j=1}^N \int_0^{\ell_j} (u_j(s, x))^2 dx + \int_0^T \sum_{j=1}^N (\partial_x u_j(t, 0))^2 dt + \sum_{j=1}^N \int_0^T \int_0^{\ell_j} a_j(x) (u_j(t, x))^2 dx dt \\
& \leq C \left(\|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})}^2 + \|\underline{z}^0(-\underline{h}, \cdot)\|_{\mathbb{L}^2(\Omega)}^2 \right),
\end{aligned} \quad (2.2.14)$$

that brings (2.2.5) using (2.2.12). Note also that $\partial_x u_j(\cdot, 0) \in L^2(0, T)$, for all $j = 1, \dots, N$. Moreover, integrating (2.2.14) with respect to s over $[0, T]$ we can obtain.

$$\|\underline{u}\|_{L^2(0, T; \mathbb{L}^2(\mathcal{T}))}^2 \leq CT \left(\|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})}^2 + \|\underline{z}^0(-\underline{h}, \cdot)\|_{\mathbb{L}^2(\Omega)}^2 \right). \quad (2.2.15)$$

We are going to consider the following multiplier presented in [CCM20], $q_j(t, x) = \frac{x(2\ell_j - x)}{\ell_j^2}$, this multiplier satisfies the next properties

- $q_j(t, 0) = 0, \forall t \in [0, T]$.
- $0 \leq q_j(t, x) \leq 1, \forall (t, x) \in [0, T] \times [0, \ell_j]$.
- $0 \leq \partial_x q_j(t, x) \leq \frac{2}{\ell_j}, \forall (t, x) \in [0, T] \times [0, \ell_j]$.
- $\partial_x^2 q_j(t, x) = -\frac{2}{\ell_j^2}, \forall (t, x) \in [0, T] \times [0, \ell_j]$.

Taking $q_j(t, x) = \frac{x(2\ell_j - x)}{\ell_j^2}$ and $s = T$ in (2.2.9) we get

$$\begin{aligned}
& \sum_{j=1}^N \int_0^{\ell_j} \ell_j q_j(t, x) (u_j(T, x))^2 dx + 2 \sum_{j=1}^N \int_0^T \int_{\omega_j} q_j(t, x) b_j(x) u_j(t - h_j, x) u_j(t, x) dx dt \\
& + 2 \int_0^T u_1(t, 0) \sum_{j=1}^N \frac{2}{\ell_j} \partial_x u_j(t, 0) dt + 2 \sum_{j=1}^N \int_0^T \int_0^{\ell_j} q_j(t, x) a_j(x) (u_j(t, x))^2 dx dt \\
& - \sum_{j=1}^N \int_0^T \int_0^{\ell_j} \partial_x q_j(t, x) (u_j(t, x))^2 dx dt + 3 \sum_{j=1}^N \int_0^T \int_0^{\ell_j} (\partial_x u_j(t, x))^2 \partial_x q_j(t, x) dx dt \\
& = \sum_{j=1}^N \int_0^{\ell_j} \ell_j q_j(0, x) (u_j^0)^2 dx - \left(\sum_{j=1}^N \frac{2}{\ell_j^2} \right) \int_0^T (u_1(t, 0))^2 dt.
\end{aligned}$$

and then recalling that $L = \max_{j=1, \dots, N} \ell_j$ and taking $\ell = \min_{j=1, \dots, N} \ell_j$

$$\begin{aligned} \frac{2}{L^2} \|u_1(\cdot, 0)\|_{L^2(0, T)}^2 &\leq \frac{2}{\ell^2} \|\underline{u}\|_{L^2(0, T; \mathbb{L}^2(\mathcal{T}))}^2 - 2 \int_0^T u_1(t, 0) \sum_{j=1}^N \partial_x u_j(t, 0) \frac{2}{\ell_j} dt \\ &\quad - 2 \sum_{j=1}^N \int_0^T \int_{\omega_j} q_j(t, x) b_j(x) u_j(t - h_j, x) u_j(t, x) dx dt + \|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})}^2. \end{aligned} \quad (2.2.16)$$

Using Young's inequality, (2.2.13) and (2.2.15) we get that $u_1(\cdot, 0) \in L^2(0, T)$ and

$$\|u_1(\cdot, 0)\|_{L^2(0, T)}^2 \leq C \left(\|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})}^2 + \|\underline{z}^0(-\underline{h}, \cdot)\|_{\mathbb{L}^2(\Omega)}^2 \right).$$

Now, let us choose $q_j = x$ and $s = T$ in (2.2.9)

$$\begin{aligned} &\int_0^{\ell_j} x(u_j)^2 dx \Big|_0^T - \int_0^T \int_0^{\ell_j} (u_j)^2 dx dt + 2 \int_0^T \int_0^{\ell_j} x b_j(x) u_j(t - h_j, x) u_j(t, x) dx dt \\ &+ 2 \int_0^T \int_0^{\ell_j} a_j(x) x (u_j)^2 dx dt + 3 \int_0^T \int_0^{\ell_j} (\partial_x u_j)^2 dx dt = \int_0^T -2u_j(t, 0) \partial_x u_j(t, 0) dt \end{aligned}$$

Then

$$\begin{aligned} &3 \sum_{j=1}^N \int_0^T \int_0^{\ell_j} (\partial_x u_j)^2 dx dt \leq (1 + 2L \|\underline{b}\|_{\mathbb{L}^\infty(\mathcal{T})}) \sum_{j=1}^N \int_0^T \int_0^{\ell_j} (u_j)^2 dx dt \\ &+ L \|\underline{b}\|_{\mathbb{L}^\infty(\mathcal{T})} \sum_{j=1}^N \int_{-h_j}^0 \int_{\omega_j} (z_j^0(t, x))^2 dx dt + L \sum_{j=1}^N \int_0^{\ell_j} (u_j(0, x))^2 dx \\ &\quad - 2 \sum_{j=1}^N \int_0^T u_1(t, 0) \partial_x u_j(t, 0) dt \end{aligned}$$

and hence

$$3 \sum_{j=1}^N \int_0^T \int_0^{\ell_j} (\partial_x u_j)^2 dx dt \leq C \left(\|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})}^2 + \|\underline{z}^0(-\underline{h}, \cdot)\|_{\mathbb{L}^2(\Omega)}^2 \right),$$

that brings with (2.2.8) the continuity of the map (2.2.4) from H to $\mathbb{B}_T \times C([0, T] : \mathbb{L}^2(\Omega))$.

Now taking $q_j = T - t$ and $s = T$ in (2.2.9), we obtain,

$$\begin{aligned} &-\int_0^{\ell_j} T(u_j(0, x))^2 dx + \int_0^T \int_0^{\ell_j} (u_j)^2 dx dt + 2 \int_0^T \int_0^{\ell_j} a_j(x) (T - t) (u_j)^2 dx dt \\ &+ 2 \int_0^T \int_0^{\ell_j} b_j(x) (T - t) u_j(t - h_j, x) u_j(t, x) dx dt = \int_0^T [(T - t) (u_j(t, 0))^2 \\ &\quad + 2(T - t) u_j(t, 0) \partial_x^2 u_j(t, 0) - (T - t) (\partial_x u_j(t, 0))^2] dt, \end{aligned}$$

then

$$\begin{aligned} &T \sum_{j=1}^N \int_0^{\ell_j} (u_j(0, x))^2 dx = \sum_{j=1}^N \left(\int_0^T \int_0^{\ell_j} (u_j)^2 dx dt + 2 \int_0^T \int_0^{\ell_j} (T - t) a_j (u_j)^2 dx dt \right. \\ &\left. + 2 \int_0^T \int_0^{\ell_j} b_j(x) (T - t) u_j(t - h_j, x) u_j(t, x) dx dt \right) + (2\alpha - N) \int_0^T (T - t) (u_1(t, 0))^2 dt \\ &\quad + \sum_{j=1}^N \int_0^T (T - t) (\partial_x u_j(t, 0))^2 dt. \end{aligned}$$

Finally we get (2.2.6), that is

$$\begin{aligned} \|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})}^2 &\leq \left(\frac{1 + 2T\|\underline{a}\|_{\mathbb{L}^\infty(\mathcal{T})} + 2T\|\underline{b}\|_{\mathbb{L}^\infty(\mathcal{T})}}{T} \right) \|\underline{u}\|_{L^2(0,T;L^2(\mathcal{T}))}^2 + 2 \left(\alpha - \frac{N}{2} \right) \|u_1(\cdot, 0)\|_{L^2(0,T)}^2 \\ &\quad + \|\partial_x \underline{u}(\cdot, 0)\|_{L^2(0,T)}^2 + \|\underline{b}\|_{\mathbb{L}^\infty(\mathcal{T})} \|\underline{z}^0(-\underline{h}\cdot, \cdot)\|_{\mathbb{L}^2(\Omega)}^2. \end{aligned}$$

Lastly, taking $p_j = 1$ and $s = T$ in (2.2.10)

$$\begin{aligned} &\int_0^1 \int_{\omega_j} (z_j(T, \rho, x))^2 dx d\rho - \int_0^1 \int_{\omega_j} (z_j^0(-h_j \rho, x))^2 dx d\rho \\ &\quad + \frac{1}{h_j} \int_0^T \int_{\omega_j} [(z_j(t, 1, x))^2 - (u_j(t, x))^2] dx dt = 0 \end{aligned}$$

and hence we obtain (2.2.7), concluding the proof of Proposition 2.2.1. \square

2.2.2 Extra boundary conditions

Following [AC18], we need now some regularity results for the linear delayed KdV equation with extra boundary source term $g(t)$ at the central node

$$\begin{cases} \partial_t u_j(t, x) + \partial_x u_j(t, x) + \partial_x^3 u_j(t, x) + a_j(x) u_j(t, x) \\ \quad + b_j(x) u_j(t - h_j, x) = 0, & x \in (0, \ell_j), t > 0, j = 1, \dots, N, \\ u_j(t, 0) = u_{j'}(t, 0), & \forall j, j' = 1, \dots, N, \\ \sum_{j=1}^N \partial_x^2 u_j(t, 0) = -\alpha u_1(t, 0) + g(t), & t > 0, \\ u_j(t, \ell_j) = \partial_x u_j(t, \ell_j) = 0, & t > 0, j = 1, \dots, N, \\ u_j(0, x) = u_j^0(x), & x \in (0, \ell_j), \\ u_j(t, x) = z_j^0(t, x), & (t, x) \in (-h_j, 0) \times (0, \ell_j). \end{cases} \quad (2.2.17)$$

Recall that $z_j(t, \rho, x) = u_j|_{\omega_j}(t - h_j \rho, x)$, for $x \in \omega_j$, $\rho \in (0, 1)$ is the solution of

$$\begin{cases} h_j \partial_t z_j(t, \rho, x) + \partial_\rho z_j(t, \rho, x) = 0, & x \in \omega_j, \rho \in (0, 1), t > 0, \\ z_j(t, 0, x) = u_j(t, x), & x \in \omega_j, t > 0, \\ z_j(0, \rho, x) = u_j|_{\omega_j}(-h_j \rho, x) = z_j^0(-h_j \rho, x), & \rho \in (0, 1). \end{cases} \quad (2.2.18)$$

Define $\mathfrak{h} = \max_{j=1, \dots, N} h_j$.

Proposition 2.2.2. *Assume $\underline{a}, \underline{b} \in \mathbb{L}^\infty(\mathcal{T})$ are componentwise non-negative that satisfy (2.1.1) and (2.1.2). Let $(U_0, g) \in D(\mathcal{A}) \times C_0^2([0, T])$ where $C_0^2([0, T]) := \{\varphi \in C^2([0, T]) : \varphi(0) = 0\}$. Then there exists a unique classical solution $U = \begin{pmatrix} \underline{u} \\ \underline{z} \end{pmatrix} \in C([0, T], D(\mathcal{A})) \cap C^1([0, T]; H)$ of (2.2.17)-(2.2.18).*

Proof. Let $\underline{v} = \underline{u} - g\phi$, where ϕ is defined as

$$\phi_j(x) = \frac{(x - \ell_j)^2}{\ell_j^2 \left(2 \sum_{j=1}^N \ell_j^{-2} + \alpha \right)}.$$

We can easily check that

$$\begin{cases} \phi_j(\ell_j) = \phi_j'(\ell_j) = 0, & \forall j = 1, \dots, N \\ \phi_j(0) = \frac{1}{2 \sum_{j=1}^N \ell_j^{-2} + \alpha} = \phi_{j'}(0), & \forall j, j' = 1, \dots, N, \\ \sum_{j=1}^N \phi_j''(0) = 1 - \alpha \phi_1(0), & t > 0. \end{cases} \quad (2.2.19)$$

We extend g on $[-\mathfrak{h}, 0]$ by $g(t) \equiv 0$ for $t \in [-\mathfrak{h}, 0]$. Then \underline{v} satisfies

$$\begin{cases} \partial_t v_j(t, x) + \partial_x v_j(t, x) + \partial_x^3 v_j(t, x) + a_j(x) v_j(t, x) \\ + b_j(x) v_j(t - h_j, x) = f_j(t, x), & x \in (0, \ell_j), t > 0, j = 1, \dots, N, \\ v_j(t, 0) = v_{j'}(t, 0), & \forall j, j' = 1, \dots, N, \\ \sum_{j=1}^N \partial_x^2 v_j(t, 0) = -\alpha v_1(t, 0), & t > 0, \\ v_j(t, \ell_j) = \partial_x v_j(t, \ell_j) = 0, & t > 0, j = 1, \dots, N, \\ v_j(0, x) = u_j^0(x), & x \in (0, \ell_j), \\ v_j(t, x) = z_j^0(t, x), & (t, x) \in (-h_j, 0) \times (0, \ell_j). \end{cases} \quad (2.2.20)$$

for $f_j(t, x) = -\phi_j(x)g'(t) - (\phi_j' + \phi_j''' + a_j\phi_j)(x)g(t)$. Then, taking $y_j(t, \rho, x) = v_j|_{\omega_j}(t - h_j\rho, x)$

$$\begin{cases} h_j \partial_t y_j(t, \rho, x) + \partial_\rho y_j(t, \rho, x) = 0, & x \in \omega_j, \rho \in (0, 1), t > 0, \\ y_j(t, 0, x) = v_j(t, x), & x \in \omega_j, t > 0, \\ y_j(0, \rho, x) = v_j|_{\omega_j}(-h_j\rho, x) = z_j^0(-h_j\rho, x), & \rho \in (0, 1). \end{cases} \quad (2.2.21)$$

Thus defining $V = \begin{pmatrix} \underline{v} \\ \underline{y} \end{pmatrix}$, as $-\underline{\phi}g' - (\underline{\phi}' + \underline{\phi}''' + \underline{a}\underline{\phi})g \in C^1([0, T], \mathbb{L}^2(\mathcal{T}))$, by the classical semigroup theory and the well-posedness of the linear case, we deduce the existence of a unique solution V of (2.2.20)-(2.2.21). Moreover, $V \in C([0, T], D(\mathcal{A})) \cap C^1([0, T]; H)$ and hence (2.2.17)-(2.2.18) admit a unique solution $U \in C([0, T], D(\mathcal{A})) \cap C^1([0, T]; H)$. \square

Now, we study the same system but with less regular data.

Proposition 2.2.3. *Assume $\underline{a}, \underline{b} \in \mathbb{L}^\infty(\mathcal{T})$ are componentwise non-negative that satisfy (2.1.1) and (2.1.2). Let $(U_0, g) \in H \times L^2(0, T)$, then there exists a unique mild solution $U \in \mathbb{B}_T \times C([0, T]; \mathbb{L}^2(\Omega))$ of (2.2.17)-(2.2.18). Furthermore $u_1(\cdot, 0)$ and $\partial_x \underline{u}(\cdot, 0)$ belong to $L^2(0, T)$ and we have the following estimates*

$$\|\underline{u}\|_{\mathbb{B}_T}^2 \leq C \left(\|g\|_{L^2(0, T)}^2 + \|\underline{u}^0\|_{\mathbb{L}(\mathcal{T})}^2 + \|\underline{z}^0(-\underline{h}\cdot, \cdot)\|_{\mathbb{L}^2(\Omega)}^2 \right). \quad (2.2.22)$$

$$\|\underline{z}\|_{C([0, T], \mathbb{L}^2(\Omega))}^2 \leq C \left(\|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})}^2 + \|\underline{z}^0(-\underline{h}\cdot, \cdot)\|_{\mathbb{L}^2(\Omega)}^2 + \|g\|_{L^2(0, T)}^2 \right). \quad (2.2.23)$$

$$\begin{aligned} \|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})}^2 &\leq \left(\frac{1 + 2T\|\underline{a}\|_{\mathbb{L}^\infty(\mathcal{T})} + 2T\|\underline{b}\|_{\mathbb{L}^\infty(\mathcal{T})}}{T} \right) \|\underline{u}\|_{L^2(0, T; \mathbb{L}^2(\mathcal{T}))}^2 + \|\underline{b}\|_{\mathbb{L}^\infty(\mathcal{T})} \|\underline{z}^0(-\underline{h}\cdot, \cdot)\|_{\mathbb{L}^2(\Omega)}^2 \\ &+ C \left(\|u_1(\cdot, 0)\|_{L^2(0, T)}^2 + \|g\|_{L^2(0, T)}^2 \right) \end{aligned} \quad (2.2.24)$$

$$\|\underline{z}^0(-\underline{h}\cdot, \cdot)\|_{\mathbb{L}^2(\Omega)}^2 \leq \|\underline{z}(T, \cdot, \cdot)\|_{\mathbb{L}^2(\Omega)}^2 + \sum_{j=1}^N \frac{1}{h_j} \int_0^T \int_{\omega_j} (z_j(t, 1, x))^2 dx dt. \quad (2.2.25)$$

Proof. Those estimates are obtained similarly as Proposition 2.2.1 and for that many calculations are omitted. First, suppose that $(U_0, g) \in D(\mathcal{A}) \times C_0^2([0, T])$ and thus the solution of (2.2.17)-(2.2.18) satisfies $U \in C([0, T]; D(\mathcal{A}) \cap C^1([0, T]; H))$.

Multiplying (2.2.17) by u_j and integrating on $[0, s] \times [0, \ell_j]$ gives us

$$\begin{aligned} & \sum_{j=1}^N \int_0^{\ell_j} (u_j(s, x))^2 dx + \int_0^s \sum_{j=1}^N (\partial_x u_j(t, 0))^2 dt + (2\alpha - N) \int_0^s (u_1(t, 0))^2 dt \\ & + 2 \sum_{j=1}^N \int_0^s \int_0^{\ell_j} a_j(u_j)^2 dx dt + 2 \sum_{j=1}^N \int_0^s \int_0^{\ell_j} b_j u_j(t - h_j, x) u_j(t, x) dx dt \\ & = \sum_{j=1}^N \int_0^{\ell_j} (u_j(0, x))^2 dx + 2 \int_0^s u_1(t, 0) g(t) dt, \end{aligned}$$

then using that

$$\begin{aligned} & 2 \int_0^s \int_0^{\ell_j} b_j(x) u_j(t - h_j, x) u_j(t, x) dx dt \\ & \leq 2 \int_0^s \int_0^{\ell_j} b_j(x) (u_j(t, x))^2 dx dt + \int_{-h_j}^0 \int_{\omega_j} b_j(x) (z_j^0(t, x))^2 dx dt, \\ & \sum_{j=1}^N \int_0^{\ell_j} (u_j(s, x))^2 dx + \int_0^s \sum_{j=1}^N (\partial_x u_j(t, 0))^2 dt + (2\alpha - N) \int_0^s (u_1(t, 0))^2 dt + \\ & 2 \sum_{j=1}^N \int_0^s \int_0^{\ell_j} (a_j - b_j) (u_j)^2 dx dt \leq 2 \int_0^s u_1(t, 0) g(t) dt + C \left(\|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})}^2 + \|\underline{z}^0(-\underline{h}, \cdot)\|_{\mathbb{L}^2(\Omega)}^2 \right). \end{aligned} \tag{2.2.26}$$

Note now that (2.2.16) still holds in this case, and we can obtain

$$\|u_1(\cdot, 0)\|_{L^2(0, T)}^2 \leq C \left(\|\underline{u}\|_{L^2(0, T; \mathbb{L}^2(\mathcal{T}))}^2 + \|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})}^2 + \sum_{j=1}^N \int_0^T (\partial_x u_j(t, 0))^2 dt + \|\underline{z}^0(-\underline{h}, \cdot)\|_{\mathbb{L}^2(\Omega)}^2 \right).$$

From (2.2.26) we obtain

$$\|u_1(\cdot, 0)\|_{L^2(0, T)}^2 \leq C \left(\|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})}^2 + \|\underline{z}^0(-\underline{h}, \cdot)\|_{\mathbb{L}^2(\Omega)}^2 + \int_0^T u_1(t, 0) g(t) dt \right)$$

and again by Young's inequality

$$\|u_1(\cdot, 0)\|_{L^2(0, T)}^2 \leq C \left(\|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})}^2 + \|\underline{z}^0(-\underline{h}, \cdot)\|_{\mathbb{L}^2(\Omega)}^2 + \|g\|_{L^2(0, T)}^2 \right)$$

Thus $u_1(\cdot, 0) \in L^2(0, T)$ and from (2.2.26) $\underline{u}(s, \cdot) \in \mathbb{L}^2(\mathcal{T})$ for $s \in [0, T]$, $\partial_x \underline{u}(\cdot, 0) \in L^2(0, T)$ and

$$\max_{s \in [0, T]} \|\underline{u}(s, \cdot)\|_{\mathbb{L}^2(\mathcal{T})}^2 \leq C \left(\|g\|_{L^2(0, T)}^2 + \|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})}^2 + \|\underline{z}^0(-\cdot \underline{h}, \cdot)\|_{\mathbb{L}^2(\Omega)}^2 \right). \tag{2.2.27}$$

Now multiplying (2.2.17) by xu_j yields

$$\begin{aligned} & \int_0^{\ell_j} x (u_j)^2 dx \Big|_0^T - \int_0^T \int_0^{\ell_j} (u_j)^2 dx dt + 2 \int_0^T \int_0^{\ell_j} x b_j(x) u_j(t - h_j, x) u_j(t, x) dx dt \\ & + 2 \int_0^T \int_0^{\ell_j} a_j(x) x (u_j)^2 dx dt + 3 \int_0^T \int_0^{\ell_j} (\partial_x u_j)^2 dx dt = \int_0^T -2u_j(t, 0) \partial_x u_j(t, 0) dt. \end{aligned}$$

Then

$$\begin{aligned}
3 \sum_{j=1}^N \int_0^T \int_0^{\ell_j} (\partial_x u_j)^2 dx dt &\leq (1 + 2L \|b\|_{L^\infty(\mathcal{T})}) \sum_{j=1}^N \int_0^T \int_0^{\ell_j} (u_j)^2 dx dt + L \sum_{j=1}^N \int_0^{\ell_j} (u_j(0, x))^2 dx \\
&+ L \|b\|_{L^\infty(\mathcal{T})} \sum_{j=1}^N \int_{-h_j}^0 \int_{\omega_j} (z_j^0(t, x))^2 dx dt + \|u_1(\cdot, 0)\|_{L^2(0, T)}^2 \\
&+ \|\partial_x \underline{u}(\cdot, 0)\|_{L^2(0, T)}^2
\end{aligned}$$

and using (2.2.27) we deduce (2.2.22). Now multiplying (2.2.17) by $(T - t)u_j$ yields,

$$\begin{aligned}
& - \int_0^{\ell_j} T(u_j(0, x))^2 dx + \int_0^T \int_0^{\ell_j} (u_j)^2 dx dt + 2 \int_0^T \int_0^{\ell_j} a_j(x)(T - t)(u_j)^2 dx dt \\
& + 2 \int_0^T \int_0^{\ell_j} b_j(x)(T - t)u_j(t - h_j, x)u_j(t, x) dx dt = \int_0^T [(T - t)(u_j(t, 0))^2 \\
& + 2(T - t)u_j(t, 0)\partial_x^2 u_j(t, 0) - (T - t)(\partial_x u_j(t, 0))^2] dt,
\end{aligned}$$

then

$$\begin{aligned}
T \sum_{j=1}^N \int_0^{\ell_j} (u_j(0, x))^2 dx &= \sum_{j=1}^N \left(\int_0^T \int_0^{\ell_j} (u_j)^2 dx dt + 2 \int_0^T \int_0^{\ell_j} (T - t)a_j(u_j)^2 dx dt + \right. \\
& \left. 2 \int_0^T \int_0^{\ell_j} b_j(x)(T - t)u_j(t - h_j, x)u_j(t, x) dx dt \right) + (2\alpha - N) \int_0^T (T - t)(u_1(t, 0))^2 dt \\
& + \sum_{j=1}^N \int_0^T (T - t)(\partial_x u_j(t, 0))^2 dt - 2 \int_0^T (T - t)u_1(t, 0)g(t) dt.
\end{aligned}$$

Finally we get

$$\begin{aligned}
\|\underline{u}^0\|_{L^2(\mathcal{T})}^2 &\leq \left(\frac{1 + 2T\|a\|_{L^\infty(\mathcal{T})} + 2T\|b\|_{L^\infty(\mathcal{T})}}{T} \right) \|\underline{u}\|_{L^2(0, T; L^2(\mathcal{T}))}^2 + \|b\|_{L^\infty(\mathcal{T})} \|\underline{z}^0(-h\cdot, \cdot)\|_{L^2(\Omega)}^2 \\
&+ C \left(\|u_1(\cdot, 0)\|_{L^2(0, T)}^2 + \|g\|_{L^2(0, T)}^2 \right)
\end{aligned}$$

and hence (2.2.24). We can conclude that the estimates for (2.2.18) are the same as Proposition 2.2.1.

By density of $D(\mathcal{A})$ in H , $C_0^2([0, T])$ in $L^2(0, T)$, we extend our result to arbitrary data $(U_0, g) \in H \times L^2(0, T)$. \square

2.2.3 Extra source term

We add now a source term $f_j(t, x)$ on each edge in our KdV problem.

$$\left\{ \begin{array}{ll}
\partial_t u_j(t, x) + \partial_x u_j(t, x) + \partial_x^3 u_j(t, x) + a_j(x)u_j(t, x) \\
+ b_j(x)u_j(t - h_j, x) = f_j(t, x), & x \in (0, \ell_j), t > 0, j = 1, \dots, N, \\
u_j(t, 0) = u_{j'}(t, 0), & \forall j, j' = 1, \dots, N, \\
\sum_{j=1}^N \partial_x^2 u_j(t, 0) = -\alpha u_1(t, 0) + g(t), & t > 0, \\
u_j(t, \ell_j) = \partial_x u_j(t, \ell_j) = 0, & t > 0, j = 1, \dots, N, \\
u_j(0, x) = u_j^0(x), & x \in (0, \ell_j) \\
u_j(t, x) = z_j^0(t, x), & (t, x) \in (-h_j, 0) \times (0, \ell_j).
\end{array} \right. \quad (2.2.28)$$

We set, as in the previous cases $z_j(t, \rho, x) = u_j \Big|_{\omega_j} (t - h_j \rho, x)$ $x \in \omega_j, \rho \in (0, 1)$. Then

$$\begin{cases} h_j \partial_t z_j(t, \rho, x) + \partial_\rho z_j(t, \rho, x) = 0, & x \in \omega_j, \rho \in (0, 1), t > 0, \\ z_j(t, 0, x) = u_j(t, x), & x \in \omega_j, t > 0, \\ z_j(0, \rho, x) = u_j \Big|_{\omega_j} (-h_j \rho, x) = z_j^0(-h_j \rho, x), & \rho \in (0, 1). \end{cases} \quad (2.2.29)$$

Proposition 2.2.4. *Assume $\underline{a}, \underline{b} \in \mathbb{L}^\infty(\mathcal{T})$ are componentwise non-negative that satisfy (2.1.1) and (2.1.2). Let $(U_0, g, \underline{f}) \in H \times L^2(0, T) \times L^1(0, T; \mathbb{L}^2(\mathcal{T}))$ then there exists a unique mild solution $U = \begin{pmatrix} \underline{u} \\ \underline{z} \end{pmatrix} \in \mathbb{B}_T \times C([0, T]; \mathbb{L}^2(\Omega))$ to (2.2.28)-(2.2.29). Furthermore, we have the following estimates,*

$$\|(\underline{u}, \underline{z})\|_{C([0, T], H)}^2 \leq C \left(\|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})}^2 + \|\underline{z}^0(-\underline{h}\cdot, \cdot)\|_{\mathbb{L}^2(\Omega)}^2 + \|\underline{f}\|_{L^1(0, T; \mathbb{L}^2(\mathcal{T}))}^2 + \|g\|_{L^2(0, T)}^2 \right), \quad (2.2.30)$$

$$\begin{aligned} \|\partial_x \underline{u}\|_{L^2(0, T; \mathbb{L}^2(\mathcal{T}))}^2 &\leq C(1 + T) \left(\|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})}^2 + \|\underline{z}^0(-\underline{h}\cdot, \cdot)\|_{\mathbb{L}^2(\Omega)}^2 + \|\underline{f}\|_{L^1(0, T; \mathbb{L}^2(\mathcal{T}))}^2 \right. \\ &\quad \left. + \|g\|_{L^2(0, T)}^2 \right). \end{aligned} \quad (2.2.31)$$

Proof. The well-posedness of (2.2.28)-(2.2.29) follows from classical semigroup theory and from the propositions given considering the source term $\begin{pmatrix} \underline{f} \\ 0 \end{pmatrix}$. Also, this gives us the first inequality, for the second one note that multiplying (2.2.28) by u_j and integrating we get

$$\begin{aligned} &\sum_{j=1}^N \int_0^{\ell_j} (u_j(T, x))^2 dx + (2\alpha - N) \int_0^T (u_1(t, 0))^2 dt + \sum_{j=1}^N \int_0^T (\partial_x u_j(t, 0))^2 dt \\ &+ 2 \sum_{j=1}^N \int_0^T \int_0^{\ell_j} a_j(x) (u_j(t, x))^2 dx dt + 2 \sum_{j=1}^N \int_0^T \int_0^{\ell_j} b_j u_j(t, x) u_j(t - h_j, x) dx dt \\ &- 2 \int_0^T u_1(t, 0) g(t) dt = 2 \sum_{j=1}^N \int_0^T \int_0^{\ell_j} f_j(t, x) u_j(t, x) dx dt + \|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})}^2. \end{aligned}$$

Note that

$$\begin{aligned} 2 \sum_{j=1}^N \int_0^T \int_0^{\ell_j} f_j(t, x) u_j(t, x) dx dt &\leq 2 \sum_{j=1}^N \int_0^T \|f_j\|_{L^2(0, \ell_j)} \|u_j\|_{L^2(0, \ell_j)} dt \\ &\leq 2 \sum_{j=1}^N \|u_j\|_{C([0, T], L^2(0, \ell_j))} \int_0^T \|f_j\|_{L^2(0, \ell_j)} dt \\ &\leq \|\underline{u}\|_{C([0, T], \mathbb{L}^2(\mathcal{T}))}^2 + \|\underline{f}\|_{L^1(0, T; \mathbb{L}^2(\mathcal{T}))}^2. \end{aligned}$$

Following the same steps as in Proposition 2.2.1 and Proposition 2.2.3 we can get

$$\begin{aligned} \sum_{j=1}^N \int_0^{\ell_j} (u_j(T, x))^2 dx + \sum_{j=1}^N \int_0^T (\partial_x u_j(t, 0))^2 dt &\leq C \left(\|\underline{u}\|_{L^2(0, T; \mathbb{L}^2(\mathcal{T}))}^2 + \|(\underline{u}^0, \underline{z}^0(-\underline{h}\cdot, \cdot))\|_H^2 \right. \\ &\quad \left. + \int_0^T u_1(t, 0) g(t) dt + \|\underline{u}\|_{C([0, T], \mathbb{L}^2(\mathcal{T}))}^2 \right. \\ &\quad \left. + \|\underline{f}\|_{L^1(0, T; \mathbb{L}^2(\mathcal{T}))}^2 \right). \end{aligned}$$

Now multiplying (2.2.28) by $q_j u_j$ for $q_j = \frac{x(2\ell_j - x)}{\ell_j^2}$ and using the last inequality we get

$$\|u_1(\cdot, 0)\|_{L^2(0, T)}^2 \leq C \left(\|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})}^2 + \|\underline{z}^0(-\underline{h}, \cdot)\|_{\mathbb{L}^2(\Omega)}^2 + \|g\|_{L^2(0, T)}^2 + \|\underline{f}\|_{L^1(0, T; \mathbb{L}^2(\mathcal{T}))}^2 \right)$$

and we can also have

$$\|u_1(\cdot, 0)\|_{L^2(0, T)}^2 + \|\partial_x u(\cdot, 0)\|_{L^2(0, T)}^2 \leq C \left(\|\underline{f}\|_{L^1(0, T; \mathbb{L}^2(\mathcal{T}))}^2 + \|g\|_{L^2(0, T)}^2 + \|(\underline{u}^0, \underline{z}^0(-\underline{h}, \cdot))\|_H^2 \right).$$

Now multiplying (2.2.28) by xu_j gives us

$$\begin{aligned} & 3 \sum_{j=1}^N \int_0^T \int_0^{\ell_j} (\partial_x u_j)^2 dx dt + \sum_{j=1}^N \int_0^{\ell_j} x(u_j(T, x))^2 dx - \sum_{j=1}^N \int_0^T \int_0^{\ell_j} (u_j)^2 dx dt \\ & + 2 \sum_{j=1}^N \int_0^T \int_0^{\ell_j} x(u_j)^2 a_j dx dt + 2 \sum_{j=1}^N \int_0^T \int_0^{\ell_j} x b_j u_j(t - h_j, x) u_j(t, x) dx dt \\ & = 2 \sum_{j=1}^N \int_0^T \int_0^{\ell_j} x u_j f_j dx dt - 2 \sum_{j=1}^N \int_0^T u_1(t, 0) \partial_x u_j(t, 0) dt. \end{aligned}$$

Hence

$$\begin{aligned} 3 \|\partial_x \underline{u}\|_{L^2(0, T; \mathbb{L}^2(\mathcal{T}))}^2 & \leq T \|\underline{u}\|_{C([0, T]; \mathbb{L}^2(\mathcal{T}))}^2 + L \left(\|\underline{u}\|_{C([0, T]; \mathbb{L}^2(\mathcal{T}))}^2 + \|\underline{f}\|_{L^1(0, T; \mathbb{L}^2(\mathcal{T}))}^2 \right) \\ & + N \|u_1(\cdot, 0)\|_{L^2(0, T)}^2 + \|\partial_x \underline{u}(\cdot, 0)\|_{L^2(0, T)}^2, \end{aligned}$$

which concludes the proof of Proposition 2.2.4 □

2.2.4 Well-posedness of nonlinear system

The aim of this section is to use the estimates obtained in the last sections to pass to the nonlinear system. For the internal nonlinearity we use Proposition 1.2.1, for nonlinearity in the boundary we have the following proposition

Proposition 2.2.5 (Proposition 2.6, [AC18]). *Let $\underline{u} \in \mathbb{B}_T$, then, $(u_1(t, 0))^2 \in L^2(0, T)$ and the map*

$$\underline{u} \in \mathbb{B}_T \mapsto (u_1(t, 0))^2 \in L^2(0, T)$$

is continuous. Moreover, we have the estimate,

$$\|u_1^2(\cdot, 0)\|_{L^2(0, T)} \leq \frac{1}{\sqrt{2}} \|\underline{u}\|_{\mathbb{B}_T}^2. \quad (2.2.32)$$

We are ready to establish our well-posedness result of the nonlinear (KdVd) for small initial data.

Theorem 2.2.2. *Assume $\underline{a}, \underline{b} \in \mathbb{L}^\infty(\mathcal{T})$ are componentwise non-negative that satisfy (2.1.1) and (2.1.2). Let $(\ell_j)_{j=1}^N \subset (0, +\infty)$, $T > 0$, there exists $\epsilon > 0$ and $C > 0$ such that for all $U_0 = (\underline{u}^0, \underline{z}^0(-\underline{h}, \cdot)) \in H$ with $\|U_0\|_H \leq \epsilon$, the nonlinear equation (KdVd) has a unique mild solution $\underline{u} \in \mathbb{B}_T$. Moreover, it satisfies*

$$\|\underline{u}\|_{\mathbb{B}_T} \leq C \|(\underline{u}^0, \underline{z}^0(-\underline{h}, \cdot))\|_H.$$

$$\|\partial_x \underline{u}\|_{L^2(0, T; \mathbb{L}^2(\mathcal{T}))}^2 \leq C (\|U_0\|_H^2 + \|U_0\|_H^4). \quad (2.2.33)$$

Proof. Let $U_0 \in H$, with $\|U_0\|_H < \epsilon$, where $\epsilon > 0$ will be chosen later, $\underline{u} \in \mathbb{B}_T$ and consider the map $\Phi : \mathbb{B}_T \rightarrow \mathbb{B}_T$ defined by $\Phi(\underline{u}) = \underline{v}$ where \underline{v} is the solution of

$$\begin{cases} \partial_t v_j(t, x) + \partial_x v_j(t, x) + \partial_x^3 v_j(t, x) + a_j(x)v_j(t, x) \\ + b_j(x)v_j(t - h_j, x) = -u_j(t, x)\partial_x u_j(t, x), & x \in (0, \ell_j), t > 0, j = 1, \dots, N, \\ v_j(t, 0) = v_{j'}(t, 0), & \forall j, j' = 1, \dots, N, \\ \sum_{j=1}^N \partial_x^2 v_j(t, 0) = -\alpha v_1(t, 0) - \frac{N}{3}(u_1(t, 0))^2, & t > 0, \\ v_j(t, \ell_j) = \partial_x v_j(t, \ell_j) = 0, & t > 0, j = 1, \dots, N, \\ v_j(0, x) = u_j^0(x), & x \in (0, \ell_j), \\ v_j(t, x) = z_j^0(t, x), & (t, x) \in (-h_j, 0) \times (0, \ell_j). \end{cases} \quad (2.2.34)$$

Clearly, $\underline{u} \in \mathbb{B}_T$ is solution of (KdVd) if \underline{u} is a fixed-point Φ . From Proposition 1.2.1 and Proposition 2.2.5, we get for all $\underline{u} \in \mathbb{B}_T$

$$\|\Phi(\underline{u})\|_{\mathbb{B}_T} = \|\underline{v}\|_{\mathbb{B}_T} \leq C(\|U_0\|_H + \|\underline{u}\|_{\mathbb{B}_T}^2)$$

and for $\underline{u}, \tilde{\underline{u}} \in \mathbb{B}_T$

$$\|\Phi(\underline{u}) - \Phi(\tilde{\underline{u}})\|_{\mathbb{B}_T} \leq C(\|\underline{u}\|_{\mathbb{B}_T} + \|\tilde{\underline{u}}\|_{\mathbb{B}_T})\|\underline{u} - \tilde{\underline{u}}\|_{\mathbb{B}_T}$$

Let us choose $R > 0$ to be defined later and consider Φ restricted to the closed ball $B_{\mathbb{B}_T}(0, R) = \{\underline{u} \in \mathbb{B}_T : \|\underline{u}\|_{\mathbb{B}_T} \leq R\}$. Then, for any $\underline{u}, \tilde{\underline{u}} \in B_{\mathbb{B}_T}(0, R)$, we have

$$\begin{aligned} \|\Phi(\underline{u})\|_{\mathbb{B}_T} &\leq C(\epsilon + R^2) \\ \|\Phi(\underline{u}) - \Phi(\tilde{\underline{u}})\|_{\mathbb{B}_T} &\leq 2CR\|\underline{u} - \tilde{\underline{u}}\|_{\mathbb{B}_T}. \end{aligned}$$

Thus if $R < \frac{1}{2C}$ and $\epsilon > 0$ such that $C(\epsilon + R^2) < R$ we obtain the local well-posedness result applying the Banach fixed-point Theorem.

To obtain (2.2.33), let \underline{u} the solution of (KdVd), multiplying (KdVd) by $q_j u_j$ with, $q_j(x) = \frac{x(2\ell_j - x)}{\ell_j^2}$ we can obtain

$$\|u_1(t, 0)\|_{L^2(0, T)}^2 \leq C\|U_0\|_H^2 + \frac{2}{3} \sum_{j=1}^N \int_0^T \int_0^{\ell_j} (u_j(t, x))^3 dx dt.$$

As $\forall j = 1, \dots, N$ $u_j \in L^2(0, T; H^1(0, \ell_j))$ and $H^1(0, \ell_j)$ embeds into $C([0, \ell_j])$ we get following [Cer14, PMVZ02]

$$\sum_{j=1}^N \int_0^T \int_0^{\ell_j} (u_j)^3 dx dt \leq CT^{1/2} \|U_0\|_H^2 \|\underline{u}\|_{L^2(0, T; \mathbb{H}_\epsilon^1(\mathcal{T}))}$$

and then

$$\|u_1(t, 0)\|_{L^2(0, T)}^2 \leq C\|U_0\|_H^2 + CT^{1/2} \|U_0\|_H^2 \|\underline{u}\|_{L^2(0, T; \mathbb{H}_\epsilon^1(\mathcal{T}))}.$$

Similarly, multiplying (KdVd) by xu_j and using the last inequality we deduce

$$\|\partial_x \underline{u}\|_{L^2(0, T; L^2(\mathcal{T}))}^2 \leq C(\|U_0\|_H^2 + \|U_0\|_H^2 \|\underline{u}\|_{L^2(0, T; \mathbb{H}_\epsilon^1(\mathcal{T}))}).$$

Using Young's inequality, we can find $C > 0$ such that (2.2.33) holds. \square

2.3 Stabilization of delayed KdV system

2.3.1 Lyapunov stabilization of the delayed system

The aim of this part is to prove Theorem 2.1.1. As we said before, this proof is developed in a constructive manner by using a Lyapunov function.

Proof of Theorem 2.1.1:

Let \underline{u} a regular enough solution of (KdVd) with $U_0 \in D(\mathcal{A})$ satisfying $\|U_0\|_H \leq \epsilon$, where $\epsilon > 0$ will be chosen later. Following [BCV19, Val22] we consider the next Lyapunov candidate for (KdVd)

$$V(t) = E(t) + \mu_1 V_1(t) + \mu_2 V_2(t). \quad (2.3.1)$$

where E is defined by (2.1.5)

$$V_1(t) = \sum_{j=1}^N \int_0^{\ell_j} x(u_j(t, x))^2 dx, \quad \text{and} \quad V_2(t) = \sum_{j=1}^N h_j \int_{\omega_j} \int_0^1 (1 - \rho)(u_j(t - h_j \rho, x))^2 d\rho dx.$$

Clearly

$$E(t) \leq V(t) \leq \left(1 + \max \left\{ L\mu_1, \frac{\mu_2}{b_0} \right\}\right) E(t).$$

After some computations we have,

$$\begin{aligned} \frac{d}{dt} E(t) &\leq -(2\alpha - N)(u_1(t, 0))^2 - \sum_{j=1}^N (\partial_x u_j(t, 0))^2 - \sum_{j=1}^N \int_{(0, \ell_j) \setminus \omega_j} a_j(x)(u_j(t, x))^2 dx \\ &\quad + \sum_{j=1}^N \int_{\omega_j} (-2a_j(x) + b_j(x) + \xi_j(x))(u_j(t, x))^2 dx \\ &\quad + \sum_{j=1}^N \int_{\omega_j} (b_j(x) - \xi_j(x))(u_j(t - h_j, x))^2 dx, \end{aligned}$$

$$\begin{aligned} \frac{d}{dt} V_1(t) &= \sum_{j=1}^N \int_0^{\ell_j} (u_j(t, x))^2 dx - 3 \sum_{j=1}^N \int_0^{\ell_j} (\partial_x u_j(t, x))^2 dx - 2u_1(t, 0) \sum_{j=1}^N \partial_x u_j(t, 0) \\ &\quad + \frac{2}{3} \sum_{j=1}^N \int_0^{\ell_j} (u_j(t, x))^3 dx - 2 \sum_{j=1}^N \int_0^{\ell_j} x a_j(x)(u_j(t, x))^2 dx \\ &\quad - 2 \sum_{j=1}^N \int_{\omega_j} x b_j(x) u_j(t, x) u_j(t - h_j, x) dx \\ &\leq \sum_{j=1}^N \int_0^{\ell_j} (u_j(t, x))^2 dx - 3 \sum_{j=1}^N \int_0^{\ell_j} (\partial_x u_j(t, x))^2 dx + \frac{N}{2} (u_1(t, 0))^2 + \frac{1}{2} \sum_{j=1}^N (\partial_x u_j(t, 0))^2 \\ &\quad + \frac{2}{3} \sum_{j=1}^N \int_0^{\ell_j} (u_j(t, x))^3 dx + L \sum_{j=1}^n \int_{\omega_j} b_j(x)(u_j(t, x))^2 dx \\ &\quad + L \sum_{j=1}^n \int_{\omega_j} b_j(x)(u_j(t - h_j, x))^2 dx, \end{aligned}$$

and

$$\frac{d}{dt} V_2(t) = \sum_{j=1}^N \int_{\omega_j} (u_j(t, x))^2 dx - \sum_{j=1}^N \int_{\omega_j} \int_0^1 (u_j(t - h_j \rho, x))^2 d\rho dx.$$

Our idea now is to prove that for a suitable choice of $\mu_1, \mu_2, \gamma > 0$ we have that $\frac{d}{dt} V(t) + 2\gamma V(t) \leq 0$, which gives the exponential stability.

Using the following Poincaré's inequality: If $u \in H^1(0, L)$ and $u(0) = 0$ or $u(L) = 0$, we have

$\|u\|_{L^2(0,L)} \leq \frac{2L}{\pi} \|\partial_x u\|_{L^2(0,L)}$. We can check easily that for $\gamma > 0$

$$\begin{aligned} \frac{d}{dt}V(t) + 2\gamma V(t) &\leq -2 \left(\alpha - \frac{N}{2} - \mu_1 \frac{N}{2} \right) (u_1(t,0))^2 + (\mu_1 - 1) \sum_{j=1}^N (\partial_x u_j(t,0))^2 \\ &+ \sum_{j=1}^N \int_{\omega_j} (-2a_j + b_j + \xi_j + L\mu_1 b_j + \mu_2) (u_j)^2 dx \\ &+ \frac{2}{3} \mu_1 \sum_{j=1}^N \int_0^{\ell_j} (u_j)^3 dx + \sum_{j=1}^N \int_{\omega_j} (b_j - \xi_j + \mu_1 L b_j) (u_j(t - h_j, x))^2 dx \\ &+ \left[\frac{4L^2(\mu_1 + 2\mu_1 \gamma L + 2\gamma)}{\pi^2} - 3\mu_1 \right] \sum_{j=1}^N \int_0^{\ell_j} (\partial_x u_j(t, x))^2 dx \\ &+ \sum_{j=1}^N \int_{\omega_j} \int_0^1 (2\gamma \mu_2 h_j + 2\gamma h_j \xi_j - \mu_2) (u_j(t - h_j \rho, x))^2 d\rho dx. \end{aligned}$$

For the term involving $\int_0^{\ell_j} (u_j(t, x))^3 dx$, note that

$$\int_0^{\ell_j} u_j^3(t, x) dx \leq \|u_j\|_{L^\infty(0, \ell_j)}^2 \int_0^{\ell_j} |u_j(t, x)| dx \leq \|u_j\|_{L^\infty(0, \ell_j)}^2 \|u_j\|_{L^2(0, \ell_j)} \sqrt{\ell_j}.$$

By the injection of $H^1(0, \ell_j)$ into $L^\infty(0, \ell_j)$ we know that $\|u_j\|_{L^\infty(0, \ell_j)} \leq \sqrt{\ell_j} \|\partial_x u_j\|_{L^2(0, \ell_j)}$, then

$$\int_0^{\ell_j} u_j^3(t, x) dx \leq \|u_j\|_{L^\infty(0, \ell_j)}^2 \|u_j\|_{L^2(0, \ell_j)} \sqrt{\ell_j} \leq \ell_j \|\partial_x u_j\|_{L^2(0, \ell_j)}^2 \sqrt{\ell_j} \|u_j\|_{L^2(0, \ell_j)}.$$

Recalling that $L = \max_{j=1, \dots, N} \ell_j$ and as the energy is not increasing we get $\|u_j\|_{L^2(0, \ell_j)} \leq \|U_0\|_H$. Choosing $\|U_0\|_H \leq \epsilon$ we get

$$\frac{2}{3} \mu_1 \sum_{j=1}^N \int_0^{\ell_j} (u_j(t, x))^3 dx \leq \frac{2}{3} \mu_1 \epsilon L^{3/2} \sum_{j=1}^N \int_0^{\ell_j} (\partial_x u_j(t, x))^2 dx.$$

Now taking

$$0 < \mu_1 < \min_{j=1, \dots, N} \inf_{\omega_j} \left\{ 1, \frac{2a_j - b_j - \xi_j}{Lb_j}, \frac{\xi_j - b_j}{Lb_j}, \frac{1}{N}(2\alpha - N) \right\},$$

$$0 < \mu_2 < \min_{j=1, \dots, N} \inf_{\omega_j} \{2a_j - b_j - \xi_j - \mu_1 L b_j\}.$$

Then $-(2\alpha - N - \mu_1 N) < 0$ and $(\mu_1 - 1) < 0$. Moreover, for all $j = 1, \dots, N$

$$(-2a_j + b_j + \xi_j + L\mu_1 b_j + \mu_2) < 0, \quad (b_j - \xi_j + \mu_1 L b_j) < 0.$$

Finally, joining the estimates

$$\begin{aligned} \frac{d}{dt}V(t) + 2\gamma V(t) &\leq \left[\frac{4L^2(\mu_1 + 2\mu_1 \gamma L + 2\gamma)}{\pi^2} - 3\mu_1 + \frac{2L^{3/2} \epsilon \mu_1}{3} \right] \|\partial_x \underline{u}(t, x)\|_{\mathbb{L}^2(\mathcal{T})}^2 \\ &+ \sum_{j=1}^N \int_{\omega_j} \int_0^1 (2h_j \gamma (\mu_2 + \xi_j) - \mu_2) (u_j(t - h_j \rho, x))^2 d\rho dx, \end{aligned}$$

and then as $L < \frac{\sqrt{3}}{2} \pi$, we can choose $\epsilon < \frac{3(3\pi^2 - 4L^2)}{\pi^2 L^{3/2}}$ and then take $\gamma > 0$ satisfying (2.1.6) to obtain $\frac{d}{dt}V(t) + 2\gamma V(t) \leq 0$. We get the desired exponential stability, by density we can extend the result to any $U_0 \in H$, with $\|U_0\|_H \leq \epsilon$. \square

Remark 2.3.1. As in [BCV19, Val22] we obtain an estimation of the rate of decay. In addition, recall that we can improve the result searching for a better Poincaré's inequality, and as is commented in [BCV19, Val22] looking for a new multiplier for the Lyapunov function V_1 , in the sense that the restriction on the lengths, comes from the multiplier x .

Remark 2.3.2. Note that in absence of the feedback terms (with and without delay) this result can be seen as an alternative proof via Lyapunov theory of Theorem 3.4 [AC18] (in our case with a more restrictive condition on the lengths).

2.3.2 Semiglobal stabilization

The aim of this section is to prove Theorem 2.1.2, which is a semiglobal result without restriction on the lengths and for $\alpha \geq N/2$. The main idea is to obtain the following observability inequality

$$\begin{aligned} & \sum_{j=1}^N \int_0^{\ell_j} (u_j^0(x))^2 dx + \sum_{j=1}^N h_j \int_{\omega_j} \int_0^1 \xi_j(x) (z_j^0(-h_j \rho, x))^2 dx d\rho \\ & \leq C \left(\sum_{j=1}^N \int_0^T (\partial_x u_j(t, 0))^2 dt + \sum_{j=1}^N \int_0^T \int_0^{\ell_j} a_j(x) (u_j(t, x))^2 dx dt \right. \\ & \quad \left. + \sum_{j=1}^N \int_0^T \int_{\omega_j} (z_j(t, 1, x))^2 dx dt + (2\alpha - N) \int_0^T (u_1(t, 0))^2 dt \right), \end{aligned} \quad (\text{Obs-D})$$

for $\begin{pmatrix} u \\ z \end{pmatrix}$, solution of (KdVd) and some $C > 0$.

Following Section 1.2.4, we work directly with the nonlinear system. Recall that the two main difficulties are: to pass to the limit in the nonlinear term and the that Holmgren's Theorem does not apply in the nonlinear case. For that, we use the unique continuation property of Saut and Scheurer Theorem 1.2.2.

Proof of Theorem 2.1.2:

To prove this result, we adapt the techniques of [Cer14]. We split our proof in three steps. In the first step, we derive an inequality which is quite similar to (Obs-D). In the second step, we obtain (Obs-D) using a contradiction argument and Theorem 1.2.2. Finally, using (Obs-D) and the non-increasing of the energy, we obtain the exponential stability.

- First step.

First defining $z_j(t, \rho, x) = u_j \Big|_{\omega_j} (t - h_j \rho, x)$ $x \in \omega_j$, $\rho \in (0, 1)$, for \underline{u} solution of (KdVd), we can check that

$$\begin{cases} h_j \partial_t z_j(t, \rho, x) + \partial_\rho z_j(t, \rho, x) = 0, & x \in \omega_j, \rho \in (0, 1), t > 0, \\ z_j(t, 0, x) = u_j(t, x), & x \in \omega_j, t > 0, \\ z_j(0, \rho, x) = u_j \Big|_{\omega_j} (-h_j \rho, x) = z_j^0(-h_j \rho, x), & \rho \in (0, 1). \end{cases} \quad (2.3.2)$$

Multiplying (2.3.2) by z_j and integrating on $(0, s) \times (0, 1) \times \omega_j$ we can obtain

$$\|z^0(-h \cdot, \cdot)\|_{\mathbb{L}^2(\Omega)}^2 \leq \|z(s, \cdot, \cdot)\|_{\mathbb{L}^2(\Omega)}^2 + \sum_{j=1}^N \frac{1}{h_j} \int_0^T \int_{\omega_j} (z_j(t, 1, x))^2 dx dt.$$

Now integrating this relation on $(0, T)$

$$T\|\underline{z}^0(-h\cdot, \cdot)\|_{\mathbb{L}^2(\Omega)}^2 \leq \sum_{j=1}^N \int_0^T \int_0^1 \int_{\omega_j} (z_j(t, \rho, x))^2 dx d\rho dt + \sum_{j=1}^N \frac{T}{h_j} \int_0^T \int_{\omega_j} (z_j(t, 1, x))^2 dx dt.$$

Note now that for $T > \mathfrak{h}$ we have

$$\begin{aligned} \int_0^T \int_0^1 \int_{\omega_j} (z_j(t, \rho, x))^2 dx d\rho dt &= \int_0^T \int_0^1 \int_{\omega_j} (u_j(t - \rho h_j, x))^2 dx d\rho dt \\ &= \int_0^T \int_{t-h_j}^t \int_{\omega_j} (u_j(s, x))^2 dx ds dt \\ &\leq \frac{T}{h_j} \int_{-h_j}^T \int_{\omega_j} (u_j(s, x))^2 dx ds \\ &= \frac{T-h_j}{h_j} \int_{-h_j}^T \int_{\omega_j} (u_j(s, x))^2 dx ds + \frac{T}{h_j} \int_{T-h_j}^T \int_{\omega_j} (u_j(s, x))^2 dx ds \\ &\leq \frac{T}{h_j} \int_0^T \int_{\omega_j} ((u_j(t, x))^2 + (u_j(t-h_j, x))^2) dx dt \\ &\leq C \left(\int_0^T \int_{\omega_j} a_j(u_j)^2 dx dt + \int_0^T \int_{\omega_j} (z_j(t, 1, x))^2 dx dt \right), \end{aligned}$$

which gives us

$$T\|\underline{z}^0(-h\cdot, \cdot)\|_{\mathbb{L}^2(\Omega)}^2 \leq C \left(\sum_{j=1}^N \int_0^T \int_{\omega_j} a_j(u_j)^2 dx dt + \int_0^T \int_{\omega_j} (z_j(t, 1, x))^2 dx dt \right).$$

Multiplying **(KdVd)** by u_j and integrating on time and space, we have

$$\begin{aligned} &\|\underline{u}(s, \cdot)\|_{\mathbb{L}^2(\mathcal{T})}^2 + (2\alpha - N) \int_0^s (u_1(t, 0))^2 dt + 2 \sum_{j=1}^N \int_0^s \int_0^{\ell_j} a_j(u_j)^2 dx dt \\ &+ \sum_{j=1}^N \int_0^s (\partial_x u_j(t, 0))^2 dt + 2 \sum_{j=1}^N \int_0^s \int_0^{\ell_j} b_j u_j(t-h_j, x) u_j(t, x) dx dt = \|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})}^2. \end{aligned}$$

Integrating again over $(0, T)$ this relation, we get,

$$\begin{aligned} T\|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})}^2 &\leq \int_0^T \|\underline{u}(t, \cdot)\|_{\mathbb{L}^2(\mathcal{T})}^2 dt + T \int_0^T \sum_{j=1}^N (\partial_x u_j(t, 0))^2 dt + (2\alpha - N)T \int_0^T (u_1(t, 0))^2 dt \\ &+ 2T \sum_{j=1}^N \int_0^T \int_0^{\ell_j} a_j(u_j)^2 dx dt + 2T \sum_{j=1}^N \int_0^T \int_0^{\ell_j} b_j u_j(t-h_j, x) u_j(t, x) dx dt. \end{aligned}$$

Note now that

$$\begin{aligned} \int_0^T \int_0^{\ell_j} b_j u_j(t-h_j, x) u_j(t, x) dx dt &\leq \frac{1}{2} \int_0^T \int_{\omega_j} b_j (u_j)^2 dx dt \\ &+ \frac{1}{2} \int_0^T \int_{\omega_j} b_j (u_j(t-h_j, x))^2 dx dt, \end{aligned}$$

and then

$$\begin{aligned}
T\|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})}^2 &\leq C \left(\int_0^T \|\underline{u}(t, \cdot)\|_{\mathbb{L}^2(\mathcal{T})}^2 dt + (2\alpha - N) \int_0^T (u_1(t, 0))^2 dt + \sum_{j=1}^N \int_0^T (\partial_x u_j(t, 0))^2 dt \right. \\
&\quad \left. + \sum_{j=1}^N \int_0^T \int_0^{\ell_j} a_j(u_j)^2 dx dt + \sum_{j=1}^N \int_0^T \int_{\omega_j} (z_j(t, 1, x))^2 dx dt \right).
\end{aligned}$$

Joining the estimates for \underline{u}^0 and \underline{z}^0 we get

$$\begin{aligned}
&\sum_{j=1}^N \int_0^{\ell_j} (u_j^0)^2 dx + \sum_{j=1}^N h_j \int_{\omega_j} \int_0^1 \xi_j(x) (z_j^0(-h_j, \rho, x))^2 dx d\rho \\
&\leq C \left(\int_0^T \|\underline{u}(t, \cdot)\|_{\mathbb{L}^2(\mathcal{T})}^2 dt + \sum_{j=1}^N \int_0^T (\partial_x u_j(t, 0))^2 dt + (2\alpha - N) \int_0^T (u_1(t, 0))^2 dt \right. \\
&\quad \left. + \sum_{j=1}^N \int_0^T \int_0^{\ell_j} a_j(u_j)^2 dx dt + \sum_{j=1}^N \int_0^T \int_{\omega_j} (z_j(t, 1, x))^2 dx dt \right).
\end{aligned}$$

This inequality is quite similar to the observability inequality **(Obs-D)**. Moreover, to prove our result, it is enough to get that for any $T, R > 0$ there exists $C = C(R, T) > 0$ such that for any solutions of **(KdVd)** with $\|U_0\|_H \leq R$ we have

$$\begin{aligned}
\int_0^T \|\underline{u}(t, \cdot)\|_{\mathbb{L}^2(\mathcal{T})}^2 dt &\leq C \left(\sum_{j=1}^N \int_0^T (\partial_x u_j(t, 0))^2 dt + (2\alpha - N) \int_0^T (u_1(t, 0))^2 dt \right. \\
&\quad \left. + \sum_{j=1}^N \int_0^T \int_0^{\ell_j} a_j(u_j)^2 dx dt + \sum_{j=1}^N \int_0^T \int_{\omega_j} (z_j(t, 1, x))^2 dx dt \right).
\end{aligned}$$

- Second step.

Suppose that this inequality does not hold. Then there exists $(\underline{u}^n)_{n \in \mathbb{N}} \subset \mathbb{B}_T$ solution of **(KdVd)** with $\|U_0^n\|_H \leq R$ such that

$$\lim_{n \rightarrow \infty} \frac{\int_0^T \|\underline{u}^n(t, \cdot)\|_{\mathbb{L}^2(\mathcal{T})}^2 dt}{I_n} = \infty,$$

where

$$\begin{aligned}
I_n &= \|\partial_x \underline{u}^n(t, 0)\|_{L^2(0, T)}^2 + (2\alpha - N) \|u_1^n(t, 0)\|_{L^2(0, T)}^2 + \sum_{j=1}^N \int_0^T \int_0^{\ell_j} a_j(u_j^n)^2 dx dt \\
&\quad + \sum_{j=1}^N \int_0^T \int_{\omega_j} (z_j^n(t, 1, x))^2 dx dt.
\end{aligned}$$

Take $\lambda^n = \|\underline{u}^n\|_{L^2(0, T; \mathbb{L}^2(\mathcal{T}))}$, $\underline{v}^n := \frac{\underline{u}^n}{\lambda^n}$ and $y_j^n(t, \rho, x) = v_j^n \Big|_{\omega_j}(t - h_j \rho, x)$ $x \in \omega_j$, $\rho \in (0, 1)$. Then, \underline{v}^n satisfies

$$\begin{cases}
\partial_t v_j^n(t, x) + \partial_x v_j^n(t, x) + \partial_x^3 v_j^n(t, x) + a_j(x) v_j^n(t, x) \\
+ b_j(x) v_j^n(t - h_j, x) + \lambda^n v_j^n(t, x) \partial_x v_j^n(t, x) = 0, & x \in (0, \ell_j), t > 0, j = 1, \dots, N, \\
v_j^n(t, 0) = v_j^n(t, \ell_j), & \forall j, j' = 1, \dots, N, \\
\sum_{j=1}^N \partial_x^2 v_j^n(t, 0) = -\alpha v_1^n(t, 0) - \lambda^n \frac{N}{3} (v_1^n(t, 0))^2, & t > 0, \\
v_j^n(t, \ell_j) = \partial_x v_j^n(t, \ell_j) = 0, & t > 0, j = 1, \dots, N, \\
\|\underline{v}^n\|_{L^2(0, T; \mathbb{L}^2(\mathcal{T}))} = 1,
\end{cases} \quad (2.3.3)$$

and

$$\begin{aligned}
& \|\partial_x \underline{v}^n(t, 0)\|_{L^2(0, T)}^2 + (2\alpha - N) \|v_1^n(t, 0)\|_{L^2(0, T)}^2 + \sum_{j=1}^N \int_0^T \int_0^{\ell_j} a_j (v_j^n)^2 dx dt \\
& + \sum_{j=1}^N \int_0^T \int_{\omega_j} (y_j^n(t, 1, x))^2 dx dt \longrightarrow 0.
\end{aligned} \quad (2.3.4)$$

Now multiplying (2.3.3) by v_j^n and integrating over $(0, T) \times (0, t) \times (0, \ell_j)$ we can get

$$\begin{aligned}
T \|\underline{v}^n(0, \cdot)\|_{\mathbb{L}^2(\mathcal{T})}^2 & \leq C \left(\int_0^T \|\underline{v}^n(t, \cdot)\|_{\mathbb{L}^2(\mathcal{T})}^2 dt + \|\partial_x \underline{v}^n(t, 0)\|_{L^2(0, T)}^2 + \|\underline{v}^n(-h, \cdot)\|_{\mathbb{L}^2(\Omega)}^2 \right. \\
& \left. + (2\alpha - N) \|v_1^n(t, 0)\|_{L^2(0, T)}^2 \right).
\end{aligned}$$

Now for $T > \mathfrak{h}$,

$$\begin{aligned}
\|\underline{v}^n(-h, \cdot)\|_{\mathbb{L}^2(\Omega)}^2 & = \sum_{j=1}^N \int_{\omega_j} \int_0^1 (v_j^n(-h_j \rho, x))^2 d\rho dx \\
& = \sum_{j=1}^N \frac{1}{h_j} \int_{\omega_j} \int_{-h_j}^0 (v_j^n(t, x))^2 dt dx \\
& \leq \sum_{j=1}^N \frac{1}{h_j} \int_{\omega_j} \int_{-h_j}^{T-h_j} (v_j^n(t, x))^2 dt dx \\
& = \sum_{j=1}^N \frac{1}{h_j} \int_{\omega_j} \int_0^T (v_j^n(t - h_j, x))^2 dt dx \\
& = \sum_{j=1}^N \frac{1}{h_j} \int_{\omega_j} \int_0^T (y_j^n(t, 1, x))^2 dt dx.
\end{aligned}$$

These estimates show us that $(\underline{v}^n(0, \cdot))_{n \in \mathbb{N}}$ is bounded in $L^2(\mathcal{T})$, also we can see that from the well-posedness of (KdVd) we get

$$\lambda^n = \|\underline{u}^n\|_{L^2(0, T; \mathbb{L}^2(\mathcal{T}))} \leq T \|U_0^n\|_H \leq TR.$$

Consequently, in the same sense as (2.2.33) we can obtain

$$\|\underline{v}^n\|_{L^2(0, T; \mathbb{H}_e^1(\mathcal{T}))}^2 \leq C (\|U_0^n\|_H^2 + \|U_0^n\|_H^4).$$

Thus $(\underline{v}^n)_{n \in \mathbb{N}} \subset L^2(0, T; \mathbb{H}_e^1(\mathcal{T}))$ is bounded and

$$\|v_j^n \partial_x v_j^n\|_{L^2(0, T; L^1(0, \ell_j))} \leq \|\underline{v}^n\|_{C([0, T], \mathbb{L}^2(\mathcal{T}))} \|\underline{v}^n\|_{L^2(0, T; \mathbb{H}_e^1(\mathcal{T}))},$$

what implies that $(v_j^n \partial_x v_j^n)_{n \in \mathbb{N}}$ is subset of $L^2(0, T; L^1(0, \ell_j))$.

From this, we can see that $\partial_t v_j^n = -(\partial_x^3 v_j^n + \partial_x v_j^n + \lambda^n v_j^n \partial_x v_j^n + a_j v_j^n + b_j v_j^n(t - h_j))$ is bounded in $L^2(0, T; H^{-2}(0, \ell_j))$ and hence by Aubin-Lions Lemma we can deduce that $(v_j^n)_{n \in \mathbb{N}}$ is relatively compact $L^2(0, T; \mathbb{L}^2(\mathcal{T}))$ and hence we can assume that v_j^n converges strongly to v_j in $L^2(0, T; \mathbb{L}^2(\mathcal{T}))$ with $\|v_j\|_{L^2(0, T; \mathbb{L}^2(\mathcal{T}))} = 1$. Furthermore, passing to the limit on (2.3.4) we get

$$\begin{aligned} & \|\partial_x v(t, 0)\|_{L^2(0, T)}^2 + (2\alpha - N)\|v_1(t, 0)\|_{L^2(0, T)}^2 + \sum_{j=1}^N \int_0^T \int_{\omega_j} (v_j(t - h_j))^2 dx dt \\ & + \sum_{j=1}^N \int_0^T \int_0^{\ell_j} a_j (v_j)^2 dx dt \leq \liminf \left(\|\partial_x v^n(t, 0)\|_{L^2(0, T)}^2 + (2\alpha - N)\|v_1^n(t, 0)\|_{L^2(0, T)}^2 \right. \\ & \left. + \sum_{j=1}^N \int_0^T \int_0^{\ell_j} a_j (v_j^n)^2 dx dt + \sum_{j=1}^N \int_0^T \int_{\omega_j} (v_j^n(t - h_j))^2 dx dt \right) = 0. \end{aligned}$$

Thus, $v_j(t, x) \equiv 0$ in $(-h_j, T) \times \omega_j$ and $(2\alpha - N)v_j(t, 0) = \partial_x v_j(t, 0) = 0$ in $(0, T)$ for all $j = 1, \dots, N$. In addition, as $(\lambda^n)_{n \in \mathbb{N}}$ is bounded, we can extract a convergent sub-sequence such that $\lambda^n \rightarrow \lambda \geq 0$, consequently v satisfies $\|v\|_{L^2(0, T; \mathbb{L}^2(\mathcal{T}))} = 1$ and the following equation

$$\begin{cases} (\partial_t v_j + \partial_x v_j + \partial_x^3 v_j + \lambda v_j \partial_x v_j)(t, x) = 0, & \forall x \in (0, \ell_j), t > 0, j = 1, \dots, N, \\ (2\alpha - N)v_j(t, 0) = \partial_x v_j(t, 0) = 0, & \forall j = 1, \dots, N, \\ v_j(t, \ell_j) = \partial_x v_j(t, \ell_j) = 0, & \forall j = 1, \dots, N, \\ \sum_{j=1}^N \partial_x^2 v_j(t, 0) = -\alpha v_1(t, 0) - \lambda \frac{N}{3} (v_1(t, 0))^2, & t > 0, \\ v_j(t, x) = 0 & (t, x) \in (-h_j, T) \times \omega_j. \end{cases}$$

1. If $\lambda = 0$ the system satisfied by v is linear, then thanks Holmgren's Theorem $v = 0$, that contradicts the fact that $\|v\|_{L^2(0, T; \mathbb{L}^2(\mathcal{T}))} = 1$.
2. If $\lambda > 0$. In this case, we have to prove that $v_j \in L^2(0, T; H^3(0, \ell_j))$ in order to apply Theorem 1.2.2. Consider $w_j = \partial_t v_j$ then

$$\begin{cases} \partial_t w_j + \partial_x w_j + \partial_x^3 w_j + \lambda w_j \partial_x v_j + \lambda v_j \partial_x w_j = 0, & x \in (0, \ell_j), t > 0, j = 1, \dots, N, \\ (2\alpha - N)w_j(t, 0) = \partial_x w_j(t, 0) = 0, & \forall j = 1, \dots, N, \\ w_j(t, \ell_j) = \partial_x w_j(t, \ell_j) = 0, & \forall j = 1, \dots, N, \\ \sum_{j=1}^N \partial_x^2 w_j(t, 0) = -\alpha w_1(t, 0) - \lambda \frac{2N}{3} w_1(t, 0)v_1(t, 0), & t > 0, \\ w_j(t, x) = 0 & (t, x) \in (-h_j, T) \times \omega_j, \\ w_j(0, x) = -v'(0, x) - v'''(0, x) - \lambda v(0, x)v'(0, x), & x \in (0, \ell_j), j = 1, \dots, N. \end{cases}$$

Note that $w_j(0, x) \in H^{-3}(0, \ell_j)$, with Lemma A.2 [Cer14] we can get that $w_j(0, x) \in L^2(0, \ell_j)$ and $w_j \in C([0, T], L^2(0, \ell_j)) \cap L^2(0, T; H^1(0, \ell_j))$. Thus, $\partial_x^3 v_j = -(\partial_t v_j - \partial_x v_j - \lambda v_j \partial_x v_j) \in L^2(0, T; L^2(0, T))$ that implies $v_j \in L^2(0, T; H^3(0, \ell_j))$. Applying Theorem 1.2.2 we obtain that $v_j = 0$ for all $j = 1, \dots, N$ that contradicts the fact that $\|v\|_{L^2(0, T; \mathbb{L}^2(\mathcal{T}))} = 1$.

Finally, we obtain that (Obs-D) is valid for a solution (KdVd) with $\|U_0\|_H \leq R$.

- Third step.

Now we derive the exponential stability from the observability inequality (Obs-D) and following Section 1.2.3 (see also [PMVZ02]). \square

Remark 2.3.3. We can observe that the semiglobal character is given by the assumption $\|U_0\|_H \leq R$ which is necessary in our proof. Specifically, it is used to show that $(\lambda^n)_{n \in \mathbb{N}}$ is bounded. An interesting open problem is the following: Is (KdVd) globally well-posed and globally exponentially stable?

Remark 2.3.4. Another way to obtain an exponential stability result is to derive (Obs-D) for $\begin{pmatrix} u \\ z \end{pmatrix}$ solution of (LKdVd). In that case, we use Holmgren's Theorem to deduce (Obs-D) that gives us the exponential stability of (LKdVd) without restriction on the initial data. Finally, using a smallness condition for the initial condition, we can prove the exponential stabilization of (KdVd). We do not follow this proof because Theorem 2.1.2 is a more general result, avoiding the smallness condition on the initial data.

2.4 Stabilization when not all damped terms are activated

It is known that to obtain exponential stability of a single KdV equation, we only need to add a damped term if the length is critical ($L \in \mathcal{N}$) [PMVZ02]. In the network case, more precisely in [AC18] Theorem 3.6, the authors consider damping terms a_j applying on the critical lengths edges except at most on one edge.

Now we will prove Theorem 2.1.3 following closely Section 6 of [Val22] and [NP14]. First note that if (2.1.7) holds the energy (2.1.5) of the solution of (KdVd) satisfies

$$\begin{aligned} \frac{d}{dt}E(t) &\leq -(2\alpha - N)(u_1(t, 0))^2 - \sum_{j=1}^N (\partial_x u_j(t, 0))^2 - 2 \sum_{j \in I} \int_{\text{supp } a_j} a_j (u_j)^2 dx \\ &\quad + \sum_{j \in I} \int_{\omega_j} (b_j - \xi_j)(u_j(t - h_j, x))^2 dx + \sum_{j \in I} \int_{\omega_j} (b_j + \xi_j)(u_j)^2 dx \\ &\quad + \sum_{j \in I^*} \int_{\omega_j} (-2a_j(x) + b_j(x) + \xi_j(x))(u_j(t, x))^2 dx - \sum_{j \in I^*} \int_{(0, \ell_j) \setminus \omega_j} a_j(x)(u_j(t, x))^2 dx \\ &\quad + \sum_{j \in I^*} \int_{\omega_j} (b_j(x) - \xi_j(x))(u_j(t - h_j, x))^2 dx. \end{aligned}$$

From the last inequality we can see that in this case the energy of the solution of (KdVd) is not decreasing in general, this by the action of the terms $b_j + \xi_j > 0$ in ω_j for $j \in I$. Following [NP14] we consider the next auxiliary problem for which the energy will be decreasing. This system is close to (KdVd)

$$\left\{ \begin{array}{ll} \partial_t u_j(t, x) + \partial_x u_j(t, x) + u_j(t, x) \partial_x u_j(t, x) + \partial_x^3 u_j(t, x) \\ + a_j(x) u_j(t, x) + b_j(x) u_j(t - h_j, x) + \eta b_j(x) u_j(t, x) \mathbb{1}_I(j) = 0, & x \in (0, \ell_j), t > 0, j = 1, \dots, N, \\ u_j(t, 0) = u_{j'}(t, 0), & \forall j, j' = 1, \dots, N, \\ \sum_{j=1}^N \partial_x^2 u_j(t, 0) = -\alpha u_1(t, 0) - \frac{N}{3} u_1^2(t, 0), & t > 0, \\ u_j(t, \ell_j) = \partial_x u_j(t, \ell_j) = 0, & t > 0, j = 1, \dots, N, \\ u_j(0, x) = u_j^0(x), & x \in (0, \ell_j), \\ u_j(t, x) = z_j^0(t, x), & (t, x) \in (-h_j, 0) \times (0, \ell_j). \end{array} \right. \quad (\text{Aux})$$

where $\mathbb{1}_I(j)$ is the indicator function of the set I and $\eta > 0$. Then we consider the energy (2.1.5) with $\xi_j = \eta b_j$ for $j \in I$, that is

$$\begin{aligned}
E(t) &= \sum_{j=1}^N \int_0^{\ell_j} (u_j)^2 dx + \eta \sum_{j \in I} h_j \int_{\omega_j} \int_0^1 b_j(u_j(t - h_j \rho, x))^2 dx d\rho \\
&\quad + \sum_{j \in I^*} h_j \int_{\omega_j} \int_0^1 \xi_j(u_j(t - h_j \rho, x))^2 dx d\rho
\end{aligned} \tag{2.4.1}$$

where in this case for all $j \in I^*$, ξ_j is a non-negative function belonging to $L^\infty(0, \ell_j)$ such that $\text{supp } \xi_j = \text{supp } b_j = \omega_j$ and

$$b_j(x) + c_0 \leq \xi_j(x) \leq 2a_j(x) - b_j(x) - c_0, \text{ in } \omega_j, \text{ for } j \in I^*. \tag{2.4.2}$$

Easy calculations show us that if $\eta > 1$, then

$$\begin{aligned}
\frac{d}{dt} E(t) &\leq -(2\alpha - N)(u_1(t, 0))^2 - \sum_{j=1}^N (\partial_x u_j(t, 0))^2 - 2 \sum_{j \in I} \int_{\text{supp } a_j} a_j(u_j)^2 dx \\
&\quad + \sum_{j \in I} (1 - \eta) \int_{\omega_j} b_j(u_j)^2 dx + \sum_{j \in I} (1 - \eta) \int_{\omega_j} b_j(u_j(t - h_j))^2 dx \\
&\quad + \sum_{j \in I^*} \int_{\omega_j} (-2a_j(x) + b_j(x) + \xi_j(x))(u_j(t, x))^2 dx - \sum_{j \in I^*} \int_{(0, \ell_j) \setminus \omega_j} a_j(x)(u_j(t, x))^2 dx \\
&\quad + \sum_{j \in I^*} \int_{\omega_j} (b_j(x) - \xi_j(x))(u_j(t - h_j, x))^2 dx \\
&\leq 0.
\end{aligned}$$

The main idea to deal with the case when $\text{supp } b_j \not\subset \text{supp } a_j$ is to show the exponential stability of the linearization around 0 of **(Aux)** via a Lyapunov function following Section 2.3.1 and then pass to **(LKdVd)** using a perturbation result. More precisely, we are going to use the following theorem.

Theorem 2.4.1 (Theorem 1.1, Chapter 3 [Paz83]). *Let X be a Banach space and let A be the infinitesimal generator of a C_0 semigroup $T(t)$ on X satisfying $\|T(t)\| \leq Me^{\omega t}$. If B is a bounded linear operator on X , then $A + B$ is the infinitesimal generator of a C_0 semigroup $S(t)$ on X satisfying $\|S(t)\| \leq Me^{(\omega + M\|B\|)t}$.*

Remark 2.4.1. *As we said before, we use a Lyapunov approach for the auxiliary system, for that we expect that our result holds for $L < \frac{\sqrt{3}}{2}\pi$, $\alpha > N/2$ and small initial data. Also, observing Theorem 2.4.1 we must require that $\|\mathbf{b}\|_{\mathbb{L}^\infty(\mathcal{T})}$ is small enough.*

We start by proving the well-posedness of the linearization of **(Aux)** around 0. We omitted the details because they are closely similar to Section 2.2,

$$\left\{ \begin{array}{ll}
\partial_t u_j(t, x) + \partial_x u_j(t, x) + \partial_x^2 u_j(t, x) + a_j(x) u_j(t, x) \\
+ b_j(x) u_j(t - h_j, x) + \eta b_j(x) u_j(t, x) \mathbb{1}_I(j) = 0, & x \in (0, \ell_j), t > 0, j = 1, \dots, N, \\
u_j(t, 0) = u_{j'}(t, 0), & \forall j, j' = 1, \dots, N, \\
\sum_{j=1}^N \partial_x^2 u_j(t, 0) = -\alpha u_1(t, 0), & t > 0, \\
u_j(t, \ell_j) = \partial_x u_j(t, \ell_j) = 0, & t > 0, j = 1, \dots, N, \\
u_j(0, x) = u_j^0(x), & x \in (0, \ell_j), \\
u_j(t, x) = z_j^0(t, x), & (t, x) \in (-h_j, 0) \times (0, \ell_j).
\end{array} \right. \tag{LAux}$$

We set again $z_j(t, \rho, x) = u_j|_{\omega_j}(t - h_j \rho, x)$ $x \in \omega_j$, $\rho \in (0, 1)$. Note that in this case as $\xi_j = \eta b_j$ the inner product defined for H in Section 2.2 becomes

$$\begin{aligned} \left\langle \begin{pmatrix} \underline{u} \\ \underline{z} \end{pmatrix}, \begin{pmatrix} \underline{v} \\ \underline{y} \end{pmatrix} \right\rangle_H &= \sum_{j=1}^N \int_0^{\ell_j} u_j(x)v_j(x)dx + \eta \sum_{j \in I} h_j \int_{\omega_j} \int_0^1 b_j(x)z_j(\rho, x)y_j(\rho, x)d\rho dx \\ &+ \sum_{j \in I^*} h_j \int_{\omega_j} \int_0^1 \xi_j(x)z_j(\rho, x)y_j(\rho, x)d\rho. \end{aligned}$$

Then (L**Aux**) can be written as

$$\begin{cases} \partial_t U(t) = \mathcal{A}_0 U(t), & t > 0 \\ U(0) = U_0. \end{cases} \quad (2.4.3)$$

where, $U = \begin{pmatrix} \underline{u} \\ \underline{z} \end{pmatrix}$, $U_0 = \left(\underline{z}^0 \Big|_{\omega}(-h \cdot, \cdot) \right)$ and the operator \mathcal{A}_0 is defined by:

$$\mathcal{A}_0 U = \begin{pmatrix} -(\partial_x + \partial_x^3)\underline{u} - \underline{a} \cdot \underline{u} - \underline{b} \cdot \underline{z}(1, \cdot) - \eta \underline{b}^I \cdot \underline{u} \\ -\frac{1}{h} \cdot \partial_\rho \underline{z} \end{pmatrix}$$

in which

$$(\underline{b}^I)_j = \begin{cases} b_j, & j \in I, \\ 0, & j \in I^*. \end{cases}$$

and $D(\mathcal{A}_0) = D(\mathcal{A})$.

Theorem 2.4.2. *Assume $\underline{a}, \underline{b} \in \mathbb{L}^\infty(\mathcal{T})$ are componentwise non-negative that satisfy (2.1.1) and (2.1.8). Let $U_0 \in H$ and $\eta > 1$. Then there exist a unique mild solution $U \in C([0, \infty); H)$ of (2.4.3). Moreover, if $U_0 \in D(\mathcal{A})$ then U is a classical solution and*

$$U \in C([0, \infty); D(\mathcal{A}_0)) \cap C^1([0, \infty); H).$$

Proof. Let $U = \begin{pmatrix} \underline{u} \\ \underline{z} \end{pmatrix} \in D(\mathcal{A}_0)$, then

$$\begin{aligned} \langle \mathcal{A}_0 U, U \rangle &\leq \left(\frac{N}{2} - \alpha \right) (u_1(0))^2 - \frac{1}{2} \sum_{j=1}^N (\partial_x u_j(0))^2 - \sum_{j \in I} \int_{\text{supp } a_j} a_j (u_j)^2 dx \\ &+ \frac{1}{2} \sum_{j \in I} (1 - \eta) \int_{\omega} b_j (u_j)^2 dx + \frac{1}{2} \sum_{j \in I} (1 - \eta) \int_{\omega} b_j (u_j(t - h_j))^2 dx \\ &- \frac{1}{2} \sum_{j \in I^*} \int_{(0, \ell_j) \setminus \omega_j} a_j(x) (u_j(t, x))^2 dx \\ &+ \sum_{j \in I^*} \int_{\omega_j} \left(-a_j(x) + \frac{b_j(x)}{2} + \frac{\xi_j(x)}{2} \right) (u_j(t, x))^2 dx \\ &+ \frac{1}{2} \sum_{j \in I^*} \int_{\omega_j} (b_j(x) - \xi_j(x)) (u_j(t - h_j, x))^2 dx \leq 0. \end{aligned}$$

thus \mathcal{A}_0 is dissipative. Moreover

$$\mathcal{A}_0^* \begin{pmatrix} \underline{v} \\ \underline{y} \end{pmatrix} = \begin{pmatrix} (\partial_x + \partial_x^3)\underline{v} - \underline{a} \cdot \underline{v} + \eta \underline{b} \cdot \underline{y}(0, \cdot) - \eta \underline{b}^I \cdot \underline{v} \\ \frac{1}{h} \cdot \partial_\rho \underline{y} \end{pmatrix}$$

$$D(\mathcal{A}_0^*) = \left\{ \begin{pmatrix} v \\ y \end{pmatrix}, v \in \mathbb{H}^3(\mathcal{T}) \cap \mathbb{H}_e^1(\mathcal{T}), \sum_{j=1}^N \frac{d^2 v_j}{dx^2}(0) = (\alpha - N)v_1(0), \partial_x v_j(0) = 0, \forall j = 1, \dots, N, \right. \\ \left. y \in \prod_{j=1}^N L^2(H^1(0, 1) \times w_j), y_j(1, x) = -\frac{1}{\eta} v_j|_{\omega_j}(x) \text{ for } j \in I, \text{ and } y_j(1, x) = -\frac{b_j}{\xi_j} v_j|_{\omega_j}(x) \right. \\ \left. \text{for } j \in I^* \right\}.$$

Let $V = \begin{pmatrix} v \\ y \end{pmatrix} \in D(\mathcal{A}_0^*)$, then

$$\begin{aligned} \langle \mathcal{A}_0^* V, V \rangle &\leq -\frac{1}{2} \sum_{j=1}^N (\partial_x v_j(\ell_j))^2 + \left(\frac{N}{2} - \alpha \right) v_1^2(0) - \sum_{j=1}^N \int_{\text{supp } a_j} a_j (v_j)^2 dx \\ &\quad + \sum_{j=1}^N \int_{\omega_j} \left(-\frac{\eta}{2} + \frac{2}{2\eta} \right) b_j(x) v_j^2(x) dx + \sum_{j \in I^*} \int_{\omega_j} \left(-a_j + \frac{\xi_j}{2} + \frac{b_j^2}{2\xi_j} \right) (v_j)^2 dx \\ &\quad - \sum_{j \in I^*} \int_{(0, \ell_j) \setminus \omega_j} a_j (v_j)^2 dx - \frac{1}{2} \sum_{j \in I^*} \int_{\omega_j} \xi_j (y_j(0, x))^2 dx \leq 0 \end{aligned}$$

thus \mathcal{A}_0^* is dissipative. \square

Now to prove the exponential stability of (LAux) we consider the following Lyapunov function:

$$V(t) = E(t) + \mu_1 V_1 + \mu_2 V_2 \quad (2.4.4)$$

where $\mu_1, \mu_2 > 0$, $E(t)$ is defined by (2.4.1), $V_1(t)$ defined in (2.3.1) and $V_2(t)$ is given by

$$\begin{aligned} V_2(t) &= \sum_{j \in I} h_j \int_{\omega_j} \int_0^1 (1 - \rho) b_j(x) (u_j(t - h_j \rho, x))^2 dx d\rho \\ &\quad + \sum_{j \in I^*} h_j \int_{\omega_j} \int_0^1 (1 - \rho) (u_j(t - h_j \rho, x))^2 dx d\rho. \end{aligned}$$

Proposition 2.4.1. *Assume $\underline{a}, \underline{b} \in \mathbb{L}^\infty(\mathcal{T})$ are componentwise non-negative that satisfy (2.1.1) and (2.1.8). Let $\alpha > \frac{N}{2}$, $\eta > 1$ and $(\ell_j)_{j=1}^N \subset (0, +\infty)$ such that $L < \frac{\sqrt{3}}{2}\pi$. Then for every $U_0 \in H$, the energy (2.4.1) of the unique mild solution of (LAux) decays exponentially, that is, there exists $C > 0$, $\gamma > 0$ such that $E(t) \leq CE(0)e^{-2\gamma t}$, where*

$$\begin{aligned} \gamma &\leq \min \left\{ \frac{(3\mu_1\pi - \mu_1 4L^2)}{8L^2(1 + L\mu_1)}, \min_{j \in I} \frac{\mu_2}{2h_j(\eta + \mu_2)}, \min_{j \in I^*} \frac{\mu_2}{2h_j(\xi_j + \mu_2)} \right\}, \\ C &= \left(1 + \max \left\{ L\mu_1, \frac{\mu_2}{\eta}, \frac{\mu_2}{b_0} \right\} \right), \end{aligned} \quad (2.4.5)$$

for μ_1 and μ_2 such that

$$\begin{aligned} 0 &< \mu_1 < \left\{ 1, \frac{\eta - 1}{L}, \frac{1}{N} (2\alpha - N), \min_{j \in I^*} \left\{ \inf_{\omega_j} \frac{\xi_j - b_j}{Lb_j}, \inf_{\omega_j} \frac{2a_j - b_j - \xi_j}{Lb_j} \right\} \right\}, \\ 0 &< \mu_2 < \min \left\{ \eta - 1 - L\mu_1, \min_{j \in I^*} 2a_j - b_j - \xi_j - L\mu_1 b_j \right\}. \end{aligned}$$

Proof. Let \underline{u} be a regular enough solution of (LAux) with $U_0 \in D(\mathcal{A}_0)$. Clearly, with this definition of $V(t)$ we have that

$$E(t) \leq V(t) \leq \left(1 + \max \left\{ L\mu_1, \frac{\mu_2}{\eta}, \frac{\mu_2}{b_0} \right\}\right) E(t).$$

Now, integrating by parts we get

$$\begin{aligned} \frac{d}{dt} E(t) &\leq (N - 2\alpha) (u_1(t, 0))^2 - \sum_{j=1}^N (\partial_x u_j(t, 0))^2 - 2 \sum_{j \in I} \int_{\text{supp } a_j} a_j (u_j)^2 dx \\ &\quad + (1 - \eta) \sum_{j \in I} \int_{\omega_j} b_j (u_j)^2 dx + (1 - \eta) \sum_{j \in I} \int_{\omega_j} b_j (u_j(t - h_j, x))^2 dx \\ &\quad + \sum_{j \in I^*} \int_{\omega_j} (-2a_j + b_j + \xi_j) (u_j)^2 dx - \sum_{j \in I^*} \int_{\text{supp } a_j \setminus \omega_j} a_j (u_j)^2 dx \\ &\quad + \sum_{j \in I^*} \int_{\omega_j} (b_j - \xi_j) (u_j(t - h_j, x))^2 dx, \end{aligned}$$

together with

$$\begin{aligned} \frac{d}{dt} V_1(t) &= \sum_{j=1}^N \int_0^{\ell_j} (u_j)^2 dx - 3 \sum_{j=1}^N \int_0^{\ell_j} (\partial_x u_j)^2 dx - 2 \sum_{j=1}^N u_1(t, 0) \partial_x u_j(t, 0) \\ &\quad - 2\eta \sum_{j \in I} \int_{\omega_j} x b_j (u_j)^2 dx - 2 \sum_{j=1}^N \int_{\text{supp } a_j} x a_j (u_j)^2 dx - 2 \sum_{j=1}^N \int_{\omega_j} x b_j u_j(t, x) u_j(t - h_j, x) dx, \\ \frac{d}{dt} V_2(t) &= \sum_{j \in I} \int_{\omega_j} b_j (u_j)^2 dx + \sum_{j \in I^*} \int_{\omega_j} (u_j)^2 dx - \sum_{j \in I} \int_{\omega_j} b_j \int_0^1 (u_j(t - h_j \rho, x))^2 d\rho dx \\ &\quad - \sum_{j \in I^*} \int_{\omega_j} \int_0^1 (u_j(t - h_j \rho, x))^2 d\rho dx. \end{aligned}$$

Using integration by parts and Poincaré's inequality, we can easily check that for $\gamma > 0$

$$\begin{aligned} \frac{d}{dt} V(t) + 2\gamma V(t) &\leq (N - 2\alpha + \mu_1 N) (u_1(t, 0))^2 + (\mu_1 - 1) \sum_{j=1}^N (\partial_x u_j(t, 0))^2 \\ &\quad + \sum_{j \in I} \int_{\omega_j} b_j (1 - \eta + \mu_2 + \mu_1 L) (u_j)^2 dx \\ &\quad + \sum_{j \in I} \int_{\omega_j} b_j (1 - \eta + \mu_1 L) (u_j(t - h_j, x))^2 dx \\ &\quad + \sum_{j \in I^*} \int_{\omega_j} (-2a_j + b_j + \xi_j + \mu_2 + L\mu_1 b_j) (u_j)^2 dx \\ &\quad + \sum_{j \in I^*} \int_{\omega_j} (b_j - \xi_j + \mu_1 L b_j) (u_j(t - h_j))^2 dx \\ &\quad + \sum_{j \in I} \int_{\omega_j} \int_0^1 (2\gamma h_j (\mu_2 + \eta) - \mu_2) (u_j(t - h_j \rho, x))^2 dx \\ &\quad + \sum_{j \in I^*} \int_{\omega_j} \int_0^1 (2\gamma h_j (\mu_2 + \xi_j) - \mu_2) (u_j(t - h_j \rho, x))^2 dx \\ &\quad + \left[\frac{4L^2 (\mu_1 + 2\mu_1 \gamma L + 2\gamma)}{\pi^2} - 3\mu_1 \right] \sum_{j=1}^N \int_0^{\ell_j} (\partial_x u_j(t, x))^2 dx. \end{aligned}$$

Taking

$$0 < \mu_1 < \left\{ 1, \frac{\eta - 1}{L}, \frac{1}{N} (2\alpha - N), \min_{j \in I^*} \left\{ \inf_{\omega_j} \frac{\xi_j - b_j}{Lb_j}, \inf_{\omega_j} \frac{2a_j - b_j - \xi_j}{Lb_j} \right\} \right\},$$

$$0 < \mu_2 < \min \left\{ \eta - 1 - L\mu_1, \min_{j \in I^*} 2a_j - b_j - \xi_j - L\mu_1 b_j \right\},$$

and using that $L > \frac{\sqrt{3}}{2}\pi$ we can take

$$\gamma \leq \min \left\{ \frac{(3\mu_1\pi - \mu_1 4L^2)}{8L^2(1 + L\mu_1)}, \min_{j \in I} \frac{\mu_2}{2h_j(\eta + \mu_2)}, \min_{j \in I^*} \frac{\mu_2}{2h_j(\xi_j + \mu_2)} \right\}.$$

With this, $\frac{d}{dt}V(t) + 2\gamma V(t) \leq 0$ which implies

$$E(t) \leq \left(1 + \max \left\{ L\mu_1, \frac{\mu_2}{\eta}, \frac{\mu_2}{b_0} \right\} \right) E(0)e^{-2\gamma t}.$$

By density, we can extend the result to $U_0 \in H$. \square

Now we will obtain a stability result of **(LKdVd)** using a perturbation argument. Note first that the operator \mathcal{A} introduced in Section 2.2 and associated with **(LKdVd)** can be written as

$$\mathcal{A} = \mathcal{A}_0 + \mathcal{B},$$

where $D(\mathcal{A}) = D(\mathcal{A}_0)$ and \mathcal{B} is the bounded operator on H defined by

$$\mathcal{B}U = \begin{pmatrix} \eta \underline{b}^T \underline{u} \\ \underline{0} \end{pmatrix}, \quad U = \begin{pmatrix} \underline{u} \\ \underline{z} \end{pmatrix} \in H.$$

Proposition 2.4.2. *Assume $\underline{a}, \underline{b} \in \mathbb{L}^\infty(\mathcal{T})$ are componentwise non-negative that satisfy (2.1.1) and (2.1.8). Let $\alpha > \frac{N}{2}$, $\eta > 1$ and $(\ell_j)_{j=1}^N \subset (0, +\infty)$ such that $L < \frac{\sqrt{3}}{2}\pi$, then for every $U_0 \in H$ there exists a unique mild solution $U \in C([0, \infty), H)$ for **(LKdVd)**. Moreover, if $U_0 \in D(\mathcal{A})$ then the solution is classical and $U \in C([0, \infty); D(\mathcal{A})) \cap C^1([0, \infty), H)$. Furthermore, there exists $\delta = \delta(\alpha, \eta, L, \underline{h}) > 0$ such that if*

$$\|\underline{b}\|_{\mathbb{L}^\infty(\mathcal{T})} \leq \delta,$$

then for every $U_0 \in H$, the unique mild solution of **(LKdVd)** satisfies

$$E(t) \leq CE(0)e^{-\gamma t}, \quad t > 0,$$

for $C, \gamma > 0$ defined in Proposition 2.4.1.

Proof. It is enough to apply Theorem 2.4.1. We note that $\|\mathcal{B}\| \leq \eta\|\underline{b}\|_{\mathbb{L}^\infty(\mathcal{T})}$ and then remark that

$$-\frac{\gamma}{2} + \sqrt{C}\eta\|\underline{b}\|_{\mathbb{L}^\infty(\mathcal{T})} < 0 \iff \|\underline{b}\|_{\mathbb{L}^\infty(\mathcal{T})} < \frac{\gamma}{2\eta\sqrt{C}}.$$

\square

Finally, we obtain the local exponential stability for **(KdVd)** in the case when $\text{supp } b_j \not\subset \text{supp } a_j$, for $j \in I^* \subset \{1, \dots, N\}$ stated in Theorem 2.1.3.

Proof of Theorem 2.1.3:

We just adapt the proof of Theorem 2.2.2 and Remark 2.3.4 to obtain the exponential stability of the nonlinear case using the stability of **(LAux)** and small initial data. \square

2.5 Numerical simulations

The purpose of this section is to illustrate the stabilization results obtained in this chapter. For that, we are going to present some numerical simulations adapting the schemes used in [BCV19, CG01, PSVV10]. We choose a final time T and for simplicity we take $\ell_j = L$ and a_j, b_j constant on their support for all $j = 1, \dots, N$. We build a uniform spatial and time discretization of $N_x + 1$ and $N_t + 1$ points respectively, separated by the steps $\Delta x = L/N_x$ and $\Delta t = T/N_t$. To deal with the delay term, we choose the delay step $\Delta \rho = 1/N_\rho$. Now we introduce the notation $u_j(n\Delta t, i\Delta x) = u_{j,i}^n$ and $z_j(n\Delta t, k\Delta \rho, i\Delta x) = z_{j,i,k}^n$ for $i = 0, \dots, N_x, k = 0, \dots, N_\rho$ and $n = 0, \dots, N_t$. We use the following approximation for the derivatives:

$$D_x^+ y_i = \frac{y_{i+1} - y_i}{\Delta x}, \quad D_x^- y_i = \frac{y_i - y_{i-1}}{\Delta x}, \quad D_x y_i = \frac{y_{i+1} - y_{i-1}}{2\Delta x}, \quad D_\rho^+ e_k = \frac{e_{k+1} - e_k}{\Delta \rho}.$$

In order to approximate the term of third order ∂_x^3 , we use $D_x^+ D_x^+ D_x^-$. Now, to consider the nonlinear terms we use explicit approximation $y_i^n D_x^+ y_i^n$ and for the nonlinear boundary condition we use a forward approximation for the second derivative which gives

$$\sum_{j=1}^N \left(\frac{1}{(\Delta x)^2} + \frac{\alpha}{N} \right) u_{j,0}^{n+1} - \frac{2}{(\Delta x)^2} u_{j,1}^{n+1} + \frac{1}{(\Delta x)^3} u_{j,2}^{n+1} = -\frac{N}{3} (u_{1,0}^n)^2, \quad n = 1, \dots, N_t.$$

Note now that by the boundary conditions we have that $u_{j,N_x}^n = u_{j,N_x-1}^n = 0$, $u_{j,0}^n = u_k^n$ for all $n = 0, \dots, N_t$ and $j, k = 1, \dots, N$. Now we define I_{ω_j} , the set of index such that $i \in I_{\omega_j}$ if $i\Delta x \in \omega_j$. Then taking $\mathcal{C} = D_x^+ D_x^+ D_x^- + D_x$ our scheme can be seen as

$$\left\{ \begin{array}{l} \sum_{j=1}^N \left(\frac{1}{(\Delta x)^2} + \frac{\alpha}{N} \right) u_{j,0}^{n+1} - \frac{2}{(\Delta x)^2} u_{j,1}^{n+1} + \frac{1}{(\Delta x)^3} u_{j,2}^{n+1} = -\frac{N}{3} (u_{1,0}^n)^2, \quad n = 1, \dots, N_t, \\ \frac{u_{j,i}^{n+1} - u_{j,i}^n}{\Delta t} + (\mathcal{C}u_j^{n+1})_i + a_j u_j^{n+1} + b_j z_{j,i,N_\rho}^n + u_{j,i}^n D_x^+ u_{j,i}^n = 0, \quad i \in I_{\omega_j}, \quad i \neq 0, \quad j = 1, \dots, N, \\ \frac{u_{j,i}^{n+1} - u_{j,i}^n}{\Delta t} + (\mathcal{C}u_j^{n+1})_i + u_{j,i}^n D_x^+ u_{j,i}^n = 0, \quad i \in \{1, \dots, N\} \setminus I_{\omega_j}, \quad i \neq 0, \\ \quad j = 1, \dots, N, \\ h_j \frac{z_{j,i,k}^{n+1} - z_{j,i,k}^n}{\Delta t} + (D_\rho^+ z_{j,i}^{n+1})_k = 0, \quad k = 1, \dots, N_\rho, \\ u_{j,N_x}^n = u_{j,N_x-1}^n = 0 \quad j = 1, \dots, N, \\ u_{j,0}^n = u_k^n \quad j, k = 1, \dots, N, \\ z_{j,i,0}^n = u_{j,i}^n, \quad i \in I_{\omega_j}, \quad j = 1, \dots, N, \\ u_{j,i}^0 = u_j^0(i\Delta x) \quad i = 1, \dots, N_x, \quad j = 1, \dots, N, \\ z_{j,i,k}^0 = z_j^0(k\Delta \rho, i\Delta x), \quad k = 1, \dots, N_\rho, \quad i \in I_{\omega_j}, \\ \quad j = 1, \dots, N. \end{array} \right. \quad (2.5.1)$$

Now we use this scheme with the following parameters, $N = 4$, $L = 2$ and $\alpha = 3$, for the discretization we use $N_x = 100$, $N_\rho = 100$, the initial conditions are $u_j^0 = (1 - \cos(2\pi x/L))$ and $z_j^0 = (1 - \cos(2\pi x/L)) \cos(2\pi \rho h_j)$. As we say before, we consider that the feedback terms are constant on their support, and we take $\omega_1 = (0, L/2)$, $\omega_2 = (0, L/4)$, $\omega_3 = (0, L/2)$ and $\omega_4 = (0, L/4)$.

For Figure 2.1 we use $T = 5$, $N_t = 100$ and delay $h_1 = 1$, $h_2 = 0.5$, $h_3 = 1$ and $h_4 = 1$. We can see that when there are no feedback terms ($a_j = b_j = 0$), the energy is exponentially decreasing and if we only activate the feedback term without delay, the energy decays more quickly. If we activate both feedback terms with and without delay, the energy still decreases exponentially but slowly. Similar case happens if we not activate a feedback term without delay, but we consider a feedback

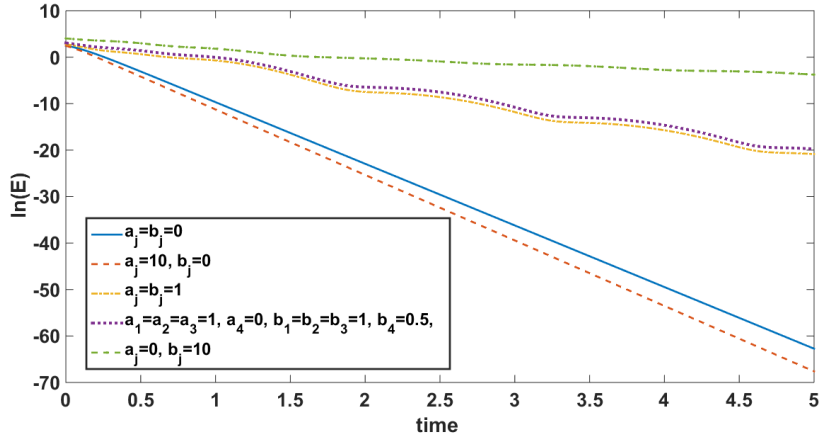


Figure 2.1: Time-evolution of $t \mapsto \ln(E(t))$ for different values of feedback terms.

term with a small delay ($a_4 = 0, b_j = 0.5$). Finally, if we consider only the action of delay feedback terms, we can observe that in this case the energy decays very slowly.

For Figure 2.2 we use $T = 10, N_t = 200, a_j = 0$ for $j = 1, \dots, 4$. In this figure, we can observe that in the case $a_j = 0$, the energy decays exponentially if the feedback terms with delay are small enough. Also, we can see that if the delay is bigger, the feedback term with delay has to be smaller as is written in Theorem 2.1.3.

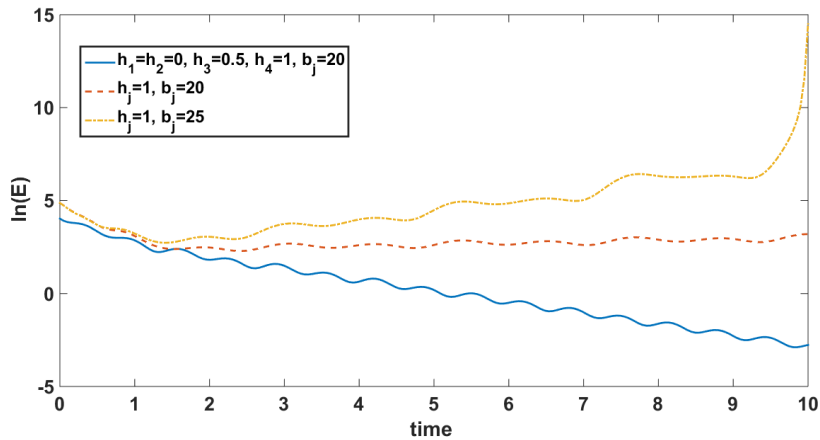


Figure 2.2: Time-evolution of $t \mapsto \ln(E(t))$ for different values of feedback with delay term.

For Figure 2.3 we consider $T = 2, N_t = 100$, delays $h_1 = 0.1, h_2 = 0.2, h_3 = 0.3$ and $h_4 = 0.4$, feedback terms $a_1 = 2, a_2 = 4, a_3 = 6, a_4 = 8, b_1 = 0.5, b_2 = 1.5, b_3 = 2.5$ and $b_4 = 3.5$. We show the graphics of $\ln\left(\frac{E(t)}{E(0)}\right)$ for different initial conditions $u_j^0 = \theta(1 - \cos(2\pi x/L))$ and $z_j^0 = \theta(1 - \cos(2\pi x/L)) \cos(2\pi \rho h_j)$, with $\theta = 0.5, 2, 5$ in order to normalize the energy. Here we can see that the decay rate does not seem to depend on the initial data.

To end, we consider $T = 2, N_t = 100$, delays $h_1 = 0.1, h_2 = 0.2, h_3 = 0.3$ and $h_4 = 0.4$, feedback terms $a_1 = 2, a_2 = 4, a_3 = 6, a_4 = 8, b_1 = 0.5, b_2 = 1.5, b_3 = 2.5$ and $b_4 = 3.5$. We calculate the theoretical decay rate given by Theorem 2.1.1, we get $\ln(E(t)) \leq 2.9283 - 1.2143 \cdot 10^{-17}t$. Now we make a linear regression for the numerical obtained data, and we get $\ln(E(t)) \approx 4.1129 - 12.766t$. From here, we can see that the theoretical decay rate given by Theorem 2.1.1 is much smaller than

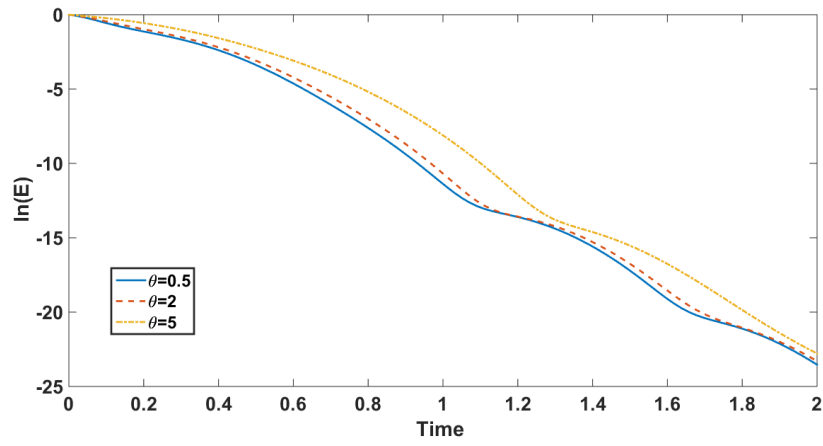


Figure 2.3: Time-evolution of $t \mapsto \ln\left(\frac{E(t)}{E(0)}\right)$ for different values of $\theta > 0$.

the one obtained numerically, at least for this initial condition.

GLOBAL WELL-POSEDNESS OF KdV
EQUATION ON A STAR-SHAPED
NETWORK AND STABILIZATION BY
SATURATED CONTROLLERS

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In this chapter, we deal with the global well-posedness and stability of the linear and nonlinear Korteweg-de Vries equation on a finite star-shaped network by acting with saturated controls. We obtain the global well-posedness by using the Kato smoothing property for the linear case and then using some estimates and a fixed-point argument we deal with the nonlinear system. Finally, we obtain the exponential stability using two different kinds of saturation by proving an observability inequality via a contradiction argument. This chapter is based on the paper;

- Hugo Parada, Emmanuelle Crépeau, and Christophe Prieur. Global well-posedness of the KdV equation on a star-shaped network and stabilization by saturated controllers. *SIAM Journal on Control and Optimization*, 60(4):2268–2296, 2022.

3.1 Introduction and presentation of our results

In the works [AC18, CCM20] stabilization and controllability problems were studied, for the KdV equation on a star-shaped network, and recently the problem of stabilization using internal delay was addressed in [PCP22a], we also mention the work [Cav18] where well-posedness of the KdV

equation on a star metric graph was studied. In this chapter, we are interested in the global well-posedness and stability properties of a Korteweg-de Vries equation posed on a star-shaped network using internal saturated feedback terms. Consider the next evolution problem for the KdV equation on a star-shaped network

$$\begin{cases} (\partial_t u_n + \partial_x u_n + u_n \partial_x u_n + \partial_x^3 u_n)(t, x) = 0, & \forall x \in (0, \ell_n), t > 0, n = 1, \dots, N, \\ u_n(t, 0) = u_{n'}(t, 0), & \forall n, n' = 1, \dots, N, \\ \sum_{n=1}^N \partial_x^2 u_n(t, 0) = -\alpha u_1(t, 0) - \frac{N}{3} u_1^2(t, 0), & t > 0, \\ u_n(t, \ell_n) = \partial_x u_n(t, \ell_n) = 0, & t > 0, n = 1, \dots, N, \\ u_n(0, x) = u_n^0(x), & x \in (0, \ell_n), \end{cases} \quad (\text{KdV-N})$$

where $\alpha \geq \frac{N}{2}$. In [AC18, CCM20] the next well-posedness result was proved for small initial condition and for any time horizon.

Theorem 3.1.1. [Theorem 2.7, [AC18]] *Let $(\ell_n)_{n=1, \dots, N} \in (0, \infty)^N$, $\alpha \geq \frac{N}{2}$ and $T > 0$. Then there exist $\epsilon > 0$ and $C > 0$ such that for all $\underline{u}^0 \in \mathbb{L}^2(\mathcal{T})$ with $\|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})} \leq \epsilon$, there exists a unique solution of (KdV-N). Moreover, it satisfies $\|\underline{u}\|_{\mathbb{B}_T} \leq C\|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})}$.*

The main complication to get a global well-posedness result is the action of the nonlinear boundary condition on the central node. Similar boundary conditions appear for the first time up to our knowledge in the work [Ros04] where a wave maker control for a single KdV equation is studied and then in the work [CZ17] where a well-posedness result was given. The system studied in these papers is the next one,

$$\begin{cases} \partial_t u(t, x) + \partial_x u(t, x) + u(t, x) \partial_x u(t, x) + \partial_x^3 u(t, x) = 0, & \forall x \in (0, L), t > 0, \\ \partial_x^2 u(t, 0) = -u(t, 0) + \frac{1}{6} u^2(t, 0) + h(t), & t > 0, \\ u(t, L) = \partial_x u(t, L) = 0, & t > 0, \\ u(0, x) = \phi(x), & x \in (0, L), \end{cases} \quad (3.1.1)$$

and the following well-posedness result local in time for bounded initial data was proven in [CZ17].

Theorem 3.1.2. (Theorem 1.1 [CZ17]) *Let $T > 0$ and $\gamma > 0$ be given. There exists $T^* \in (0, T]$ such that for any $\phi \in L^2(0, L)$ and $h \in H^{-\frac{1}{3}}(0, T)$ satisfying, $\|\phi\|_{L^2(0, L)} + \|h\|_{H^{-\frac{1}{3}}(0, T)} \leq \gamma$. Then, the problem (3.1.1) admits a unique solution $u \in C([0, T^*]; L^2(0, L)) \cap L^2(0, T^*; H^1(0, L))$. Moreover, the corresponding solution map is Lipschitz continuous and the solution possesses the hidden regularities (the Sharp Kato smoothing properties) $\partial_x^\kappa u \in L_x^\infty(0, L; H^{\frac{1-\kappa}{3}}(0, T^*))$, $\kappa = 0, 1, 2$.*

The first main result of this chapter is the following global in-time well-posedness theorem.

Theorem 3.1.3. *Let $(\ell_n)_{n=1, \dots, N} \in (0, \infty)^N$, $\alpha \geq \frac{N}{2}$ and $T > 0$. Then, for all $\underline{u}^0 \in \mathbb{L}^2(\mathcal{T})$, there exists a unique solution $\underline{u} \in \mathbb{B}_T$ of (KdV-N). Moreover, there exist $0 < T^* \leq T$, $C > 0$ such that $\underline{u} \in \mathbb{Y}_{T^*}$ and $\|\underline{u}\|_{\mathbb{Y}_{T^*}} \leq C\|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})}$.*

Note that our result generalized Theorem 3.1.1 in the sense that the smallness assumption on the initial data is not needed. Our idea is to follow [CZ17] to obtain a similar Sharp Kato smoothing regularity presented in Theorem 3.1.2 for a linear problem of the KdV equation on a star-shaped network. In order to deal with the nonlinear part, we use a fixed-point argument to obtain global well-posedness for small-time. Finally, we use an energy estimation to obtain a global well-posedness in time. Similar ideas were applied in the case of single KdV equation in [PMVZ02].

Our idea closely follows works [MCPA17] and [PCP22a] to prove the stability of the KdV equation in a star-shaped network with saturated internal control. In this chapter, we consider a saturation map sat that could be any of the following cases:

- $\mathbf{sat} = \mathbf{sat}_{1\text{oc}}$: First consider the following scalar saturation,

$$\mathbf{sat}(f) = \begin{cases} -M, & \text{if } f \leq -M, \\ f, & \text{if } -M \leq f \leq M, \\ M, & \text{if } f \geq M, \end{cases}$$

where $M > 0$ is given and denotes the saturation level. Then we take the next extension to infinite dimensional setting

$$\mathbf{sat}_{1\text{oc}}(f)(x) = \mathbf{sat}(f(x)). \quad (3.1.2)$$

- $\mathbf{sat} = \mathbf{sat}_2$: For $f \in L^2(0, L)$ we define

$$\mathbf{sat}_2(f)(x) = \begin{cases} f(x), & \text{if } \|f\|_{L^2(0,L)} \leq M, \\ \frac{f(x)M}{\|f\|_{L^2(0,L)}}, & \text{if } \|f\|_{L^2(0,L)} \geq M. \end{cases} \quad (3.1.3)$$

In what follows, \mathbf{sat} corresponds to either $\mathbf{sat}_{1\text{oc}}$ or \mathbf{sat}_2 . In order to consider the saturated stabilization problem, we study the next system

$$\begin{cases} (\partial_t u_n + \partial_x u_n + u_n \partial_x u_n + \partial_x^3 u_n)(t, x) + \mathbf{sat}(a_n(x)u_n(t, x)) = 0, & x \in (0, \ell_n), t > 0, \\ u_n(t, 0) = u_{n'}(t, 0), & n = 1, \dots, N, \\ \sum_{n=1}^N \partial_x^2 u_n(t, 0) = -\alpha u_1(t, 0) - \frac{N}{3} u_1^2(t, 0), & \forall n, n' = 1, \dots, N, \\ u_n(t, \ell_n) = \partial_x u_n(t, \ell_n) = 0, & t > 0, \\ u_n(0, x) = u_n^0(x), & n = 1, \dots, N, \\ & x \in (0, \ell_n), \end{cases} \quad (\text{KdV-S})$$

where the damping terms $(a_n)_{n=1, \dots, N} \in \mathbb{L}^\infty(\mathcal{T})$ act locally on all branches, formally written as

$$a_n \geq c_n > 0 \text{ in an open nonempty set } \omega_n \text{ of } (0, \ell_n), \text{ for all } n = 1, \dots, N. \quad (3.1.4)$$

In this chapter, we are going to consider the following energy $E(t)$ of $\underline{u} = (u_1, \dots, u_N)^\top \in \mathbb{L}^2(\mathcal{T})$ by

$$E(t) = \frac{1}{2} \|\underline{u}\|_{\mathbb{L}^2(\mathcal{T})}^2. \quad (3.1.5)$$

The second main result of this chapter states the semiglobal exponential stability of **(KdV-S)**.

Theorem 3.1.4. *Assume that the damping terms $(a_n)_{n=1, \dots, N}$ satisfy (3.1.4). Let $(\ell_n)_{n=1}^N \subset (0, \infty)$ and $R > 0$, then there exist $C(R) > 0$ and $\mu(R) > 0$ such that for all $\underline{u}^0 \in \mathbb{L}^2(\mathcal{T})$ with $\|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})} \leq R$, the energy of the solution of **(KdV-S)** defined by (3.1.5) satisfies $E(t) \leq C(R)E(0)e^{-\mu(R)t}$ for all $t > 0$.*

Then, in order to add damped terms only on the critical lengths as in [AC18] we neglect the term $u_n \partial_x u_n$ in the KdV equation **(KdV-N)**. Let $I_c = \{n \in \{1, \dots, N\}; \ell_n \in \mathcal{N}\}$ be the set of critical lengths and I_c^* be the subset of I_c where we remove one index. We consider now the following problem

$$\begin{cases} (\partial_t u_n + \partial_x u_n + \partial_x^3 u_n)(t, x) + \mathbf{sat}(a_n(x)u_n(t, x)) = 0, & x \in (0, \ell_n), t > 0, n = 1, \dots, N, \\ u_n(t, 0) = u_{n'}(t, 0), & \forall n, n' = 1, \dots, N, \\ \sum_{n=1}^N \partial_x^2 u_n(t, 0) = -\alpha u_1(t, 0), & t > 0, \\ u_n(t, \ell_n) = \partial_x u_n(t, \ell_n) = 0, & t > 0, n = 1, \dots, N, \\ u_n(0, x) = u_n^0(x), & x \in (0, \ell_n), \end{cases} \quad (\text{LKdV-S})$$

where the damping $(a_n)_{n=1,\dots,N} \in \mathbb{L}^\infty(\mathcal{T})$ satisfy

$$\begin{cases} a_n = 0 & \text{for } n \in \{1, \dots, N\} \setminus I_c^*, \\ a_n \geq c_n & \text{in an open nonempty set } \omega_n \text{ of } (0, \ell_n), \text{ for all } n \in I_c^*, \\ \text{and } c_n > 0 & \text{is a constant.} \end{cases} \quad (3.1.6)$$

Then we are able to prove the following global stabilization result, which is the last main result.

Theorem 3.1.5. *Assume that the damping terms $(a_n)_{n=1,\dots,N}$ satisfy (3.1.6) and let $(\ell_n)_{n=1}^N \subset (0, \infty)$. Then, there exist $C > 0$ and $\mu > 0$ such that for all $\underline{u}^0 \in \mathbb{L}^2(\mathcal{T})$, the energy of the solution of (LKdV-S) defined by (3.1.5) satisfies $E(t) \leq CE(0)e^{-\mu t}$ for all $t > 0$.*

Remark 3.1.1. *Note that for the system (LKdV-S) the stabilization result is global, instead of the one for (KdV-S) which is semiglobal. This difference comes from the action of the term $u_n \partial_x u_n$: the condition $\|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})} \leq R$ is necessary to handle this term.*

3.2 Well-posedness

This section is devoted to prove the well-posedness results for (KdV-N)-(KdV-S) and (LKdV-S), in particular we focus on Theorem 3.1.3. Our scheme will be to consider appropriate linear systems to derive regularity properties. Then, using a fixed-point result, we obtain the well-posedness for the nonlinear systems.

3.2.1 Linear Problems

We start by considering the following linear system for KdV equation on a star-shaped network \mathcal{T}

$$\begin{cases} \partial_t u_n(t, x) + \partial_x^3 u_n(t, x) = f_n(t, x), & \forall x \in (0, \ell_n), t > 0, n = 1, \dots, N, \\ u_n(t, 0) = u_{n'}(t, 0), & \forall n, n' = 1, \dots, N, \\ \sum_{n=1}^N \partial_x^2 u_n(t, 0) = h(t), & t > 0, \\ u_n(t, \ell_n) = 0, \quad \partial_x u_n(t, \ell_n) = 0, & t > 0, n = 1, \dots, N, \\ u_n(0, x) = u_n^0(x), & \forall x \in (0, \ell_n), j = 1, \dots, N. \end{cases} \quad (\text{LKdV-N})$$

The terms f_n and h are internal and boundary functions that are useful for the fixed-point approach. First, we deal with the linear system (LKdV-N) with homogeneous initial condition and homogeneous internal source terms ($f_n = 0$).

$$\begin{cases} \partial_t u_n(t, x) + \partial_x^3 u_n(t, x) = 0, & \forall x \in (0, \ell_n), t > 0, n = 1, \dots, N, \\ u_n(t, 0) = u_{n'}(t, 0), & t > 0, \forall n, n' = 1, \dots, N, \\ \sum_{n=1}^N \partial_x^2 u_n(t, 0) = h(t), & t > 0, \\ u_n(t, \ell_n) = 0, \quad \partial_x u_n(t, \ell_n) = 0, & t > 0, n = 1, \dots, N, \\ u_n(0, x) = 0, & \forall x \in (0, \ell_n), n = 1, \dots, N, \end{cases} \quad (3.2.1)$$

The fact that we work with the linear system $\partial_t u_n + \partial_x^3 u_n = 0$ instead of $\partial_t u_n + \partial_x u_n + \partial_x^3 u_n = 0$ is motivated by [BSZ03, CZ17]. It is well known, that the term $\partial_x u_n$ yields problematic behaviors with respect to regularity and controllability properties, as well noted Rosier in [Ros97] and then several works as [CFSZ18, Zha99, CCFZ17]. Now, formally we apply the usual Laplace Transform

with respect to time to the system (3.2.1) and obtain

$$\begin{cases} s\hat{u}_n + \partial_x^3 \hat{u}_n = 0, & \forall x \in (0, \ell_n), n = 1, \dots, N, \\ \hat{u}_n(s, 0) = \hat{u}_{n'}(s, 0), & \forall n, n' = 1, \dots, N, \\ \sum_{n=1}^N \partial_x^2 \hat{u}_n(s, 0) = \hat{h}(s), & \\ \hat{u}_n(s, \ell_n) = 0, \quad \partial_x \hat{u}_n(s, \ell_n) = 0, & n = 1, \dots, N, \\ \hat{u}_n(0, x) = 0, & \forall x \in (0, \ell_n), n = 1, \dots, N, \end{cases} \quad (3.2.2)$$

where

$$\hat{u}_n(s, x) = \int_0^\infty e^{-st} u_n(t, x) dt, \quad \hat{h}(s) = \int_0^\infty e^{-st} h(t) dt, \quad \forall x \in (0, \ell_n).$$

Following [BSZ03], we can see that the N component solutions to (3.2.2) can be written as

$$\hat{u}_n(s, x) = \sum_{j=1}^3 c_{3(n-1)+j}^N e^{\lambda_j(s)x}, \quad (3.2.3)$$

where $\lambda_j(s)$, $j = 1, 2, 3$ are the solutions of the characteristic equation $s + \lambda^3 = 0$ and $c^N = (c_k)_{k=1, \dots, 3N}^N$ solves the following linear system

$$\left\{ \begin{array}{l} \sum_{n=1}^N \sum_{j=1}^3 c_{3(n-1)+j}^N \lambda_j^2 = \hat{h}, \\ \sum_{j=1}^3 c_j^N e^{\lambda_j \ell_1} = 0, \\ \sum_{j=1}^3 c_j^N \lambda_j e^{\lambda_j \ell_1} = 0, \\ \sum_{j=1}^3 c_j^N = \sum_{j=1}^3 c_{3(n-1)+j}^N, \\ \sum_{j=1}^3 c_{3(n-1)+j}^N e^{\lambda_j \ell_n} = 0, \\ \sum_{j=1}^3 c_{3(n-1)+j}^N \lambda_j e^{\lambda_j \ell_n} = 0 \end{array} \right\} \quad \forall n = 2, \dots, N. \quad (3.2.4)$$

We write this previous system in its matrix form $A_N c^N = \hat{h} e_1$, where e_1 is the first vector of the canonical basis in \mathbb{R}^{3N} . We can see easily that $A_N \in M_{3N}$ can be decomposed by induction in blocks as:

$$A_1 = \begin{bmatrix} (\lambda_1)^2 & (\lambda_2)^2 & (\lambda_3)^2 \\ e^{\lambda_1 \ell_1} & e^{\lambda_2 \ell_1} & e^{\lambda_3 \ell_1} \\ \lambda_1 e^{\lambda_1 \ell_1} & \lambda_2 e^{\lambda_2 \ell_1} & \lambda_3 e^{\lambda_3 \ell_1} \end{bmatrix}, \quad (3.2.5)$$

$$A_N = \left[\begin{array}{ccc|ccc} & & & (\lambda_1)^2 & (\lambda_2)^2 & (\lambda_3)^2 \\ & A_{N-1} & & & & \\ & & & \mathbf{0}_{3(N-1)-1 \times 3} & & \\ \hline 1 & 1 & 1 & & & \\ 0 & 0 & 0 & \mathbf{0}_{3 \times 3(N-2)} & & \\ 0 & 0 & 0 & & D_N & \end{array} \right] = \begin{bmatrix} A_{N-1} & B_N \\ C_N & D_N \end{bmatrix}, \quad (3.2.6)$$

for an appropriate choice of B_N , C_N and

$$D_N = \begin{bmatrix} -1 & -1 & -1 \\ e^{\lambda_1 \ell_N} & e^{\lambda_2 \ell_N} & e^{\lambda_3 \ell_N} \\ \lambda_1 e^{\lambda_1 \ell_N} & \lambda_2 e^{\lambda_2 \ell_N} & \lambda_3 e^{\lambda_3 \ell_N} \end{bmatrix}. \quad (3.2.7)$$

Formally, taking the inverse of the Laplace Transform of \hat{u}_n in (3.2.3), we get for $t \geq 0$ and $x \in (0, \ell_n)$

$$u_n(t, x) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} e^{st} \hat{u}_n(s, x) ds = \sum_{j=1}^3 \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} e^{st} c_{3(n-1)+j}^N \hat{h}(s) e^{\lambda_j(s)x} ds.$$

If we denote, for $t \geq 0$ and $x \in (0, \ell_n)$,

$$I_n(t, x) = \sum_{j=1}^3 \frac{1}{2\pi i} \int_0^{i\infty} e^{st} c_{3(n-1)+j}^N \hat{h}(s) e^{\lambda_j(s)x} ds,$$

$$J_n(t, x) = \sum_{j=1}^3 \frac{1}{2\pi i} \int_{-i\infty}^0 e^{st} c_{3(n-1)+j}^N \hat{h}(s) e^{\lambda_j(s)x} ds,$$

we have

$$u_n(t, x) = I_n(t, x) + J_n(t, x). \quad (3.2.8)$$

Now we introduce the notation, super index, $+\backslash-$ which corresponds to taking $s = \pm i\rho^3$, $\rho > 0$ in the characteristic equation. Then, the roots of the characteristic equation are given by

$$\begin{cases} \lambda_1^+(\rho) = i\rho, & \lambda_2^+(\rho) = \frac{1}{2}\rho(\sqrt{3} - i), & \lambda_3^+(\rho) = \frac{1}{2}\rho(-\sqrt{3} - i), \\ \lambda_j^-(\rho) = \overline{\lambda_j^+(\rho)}, & j = 1, 2, 3. \end{cases}$$

Let $\Delta^{N,+}(\rho)$ be the determinant of $A_N(i\rho^3)$, $\Delta_{3(n-1)+j}^{N,+}(s)$ be the determinant of the matrix that is obtained by replacing the column $3(n-1)+j$ of the matrix $A_N(i\rho^3)$ by $[1 \ 0 \dots \ 0]^T$ and $\hat{h}^+(\rho) = \hat{h}(i\rho^3)$. Assume that $\Delta^{N,+}(\rho) \neq 0$ (this property will be justified in Proposition 3.2.1), Cramer's Rule implies that $c_{3(n-1)+j}^{N,+}(\rho) = c_{3(n-1)+j}^N(i\rho^3)$ is given by

$$c_{3(n-1)+j}^{N,+}(\rho) = \frac{\Delta_{3(n-1)+j}^{N,+}(\rho)}{\Delta^{N,+}(\rho)} \hat{h}^+(\rho). \quad (3.2.9)$$

Thus, I_n and J_n can be seen as

$$I_n(t, x) = \sum_{j=1}^3 \frac{1}{2\pi} \int_0^\infty e^{i\rho^3 t} e^{\lambda_j^+(\rho)x} \frac{\Delta_{3(n-1)+j}^{N,+}(\rho)}{\Delta^{N,+}(\rho)} \hat{h}^+(\rho) 3\rho^2 d\rho, \quad (3.2.10)$$

$$J_n(t, x) = \sum_{j=1}^3 \frac{1}{2\pi} \int_0^\infty e^{-i\rho^3 t} e^{\lambda_j^-(\rho)x} \frac{\Delta_{3(n-1)+j}^{N,-}(\rho)}{\Delta^{N,-}(\rho)} \hat{h}^-(\rho) 3\rho^2 d\rho, \quad (3.2.11)$$

where we use the notation $\Delta_k^{N,-}(\rho) = \overline{\Delta_k^{N,+}(\rho)}$, $\Delta^{N,-}(\rho) = \overline{\Delta^{N,+}(\rho)}$ and $\hat{h}^-(\rho) = \overline{\hat{h}^+(\rho)}$. Our idea now is to obtain estimates for u_n , for that we are going to prove some asymptotic properties for $\frac{\Delta_{3(n-1)+j}^{N,+}(\rho)}{\Delta^{N,+}(\rho)}$, the following proposition collects these properties.

Proposition 3.2.1. For all $\rho > 0$, $\Delta^{N,+}(\rho) \neq 0$. Moreover, the following asymptotic properties hold, for $\rho \rightarrow \infty$

$$\begin{aligned} \frac{\Delta_{3(n-1)+1}^{N,+}}{\Delta^{N,+}} &\sim -\delta_N \rho^{-2} e^{-\frac{1}{2}\rho\sqrt{3}\ell_n - i\frac{3}{2}\rho\ell_n}, & \frac{\Delta_{3(n-1)+2}^{N,+}}{\Delta^{N,+}} &\sim \delta_N \rho^{-2} e^{-\rho\sqrt{3}\ell_n + i\frac{\pi}{3}}, \\ \frac{\Delta_{3(n-1)+3}^{N,+}}{\Delta^{N,+}} &\sim \delta_N \rho^{-2} e^{-i\frac{\pi}{3}}, & \sum_{j=1}^3 \frac{\Delta_{3(n-1)+j}^{N,+}}{\Delta^{N,+}} &\sim \delta_N \rho^{-2} e^{-i\frac{\pi}{3}}, \quad n = 1, \dots, N, \end{aligned} \quad (3.2.12)$$

where $\delta_N > 0$ only depends on N and satisfies $\delta_N = \frac{\delta_{N-1}}{\delta_{N-1} + 1}$.

Proof. The main problem in this proof is to deal with the determinant of the matrix without making explicit computations. Recall that, in the case of N branches, the matrix A_N has size $3N \times 3N$. Our proof is based on an induction argument over the number N of branches of the network.

- $N = 1$, in this case, system (3.2.4) is exactly the system studied in [CZ17] for $\ell_1 = 1$. By Appendix B.2, it holds $\Delta^{1,+}(\rho) \neq 0$ for all $\rho > 0$. Moreover, following the explicit calculations given in [CZ17] we can deduce:

$$\begin{aligned} \frac{\Delta_1^{1,+}}{\Delta^{1,+}} &\sim -\rho^{-2} e^{-\frac{1}{2}\rho\sqrt{3}\ell_1 - i\frac{3}{2}\rho\ell_1}, & \frac{\Delta_2^{1,+}}{\Delta^{1,+}} &\sim \rho^{-2} e^{-\rho\sqrt{3}\ell_1 + i\frac{\pi}{3}}, & \frac{\Delta_3^{1,+}}{\Delta^{1,+}} &\sim \rho^{-2} e^{-i\frac{\pi}{3}}, \\ & & \sum_{j=1}^3 \frac{\Delta_j^{1,+}}{\Delta^{1,+}} &\sim \rho^{-2} e^{-i\frac{\pi}{3}}. \end{aligned}$$

That gives (3.2.12) in the case $N = 1$.

- Suppose now that $\Delta^{N-1,+}(\rho) \neq 0$ for all $\rho > 0$ and that the asymptotic property (3.2.12) is true for any network of $N - 1$ branches. Let us prove that $\Delta^{N,+}(\rho) \neq 0$ for all, $\rho > 0$ and that the asymptotic property (3.2.12) holds for a network of N branches. As

$$A_N = \begin{bmatrix} A_{N-1} & B_N \\ C_N & D_N \end{bmatrix},$$

and we have $\det(A_{N-1}) = \Delta^{N-1,+} \neq 0$ by hypothesis, we can write

$$A_N = \begin{bmatrix} I_{3(N-1)} & \mathbf{0}_{3(N-1)} \\ C_N A_{N-1}^{-1} & I_{3(N-1)} \end{bmatrix} \begin{bmatrix} A_{N-1} & \mathbf{0}_{3(N-1)} \\ \mathbf{0}_{3(N-1)} & D_N - C_N A_{N-1}^{-1} B_N \end{bmatrix} \begin{bmatrix} I_{3(N-1)} & A_{N-1}^{-1} B_N \\ \mathbf{0}_{3(N-1)} & I_{3(N-1)} \end{bmatrix}, \quad (3.2.13)$$

which implies directly that

$$\Delta^{N,+} = \det(A_N) = \det(A_{N-1}) \det(D_N - C_N A_{N-1}^{-1} B_N). \quad (3.2.14)$$

The difficulty of the last expression is the role of the matrix A_{N-1}^{-1} . In fact, to calculate this inverse explicitly is quite complicated. Note now that if

$$A_{N-1}^{-1} = \begin{bmatrix} x_1 & \vdots & \dots & \vdots \\ x_2 & \vdots & \dots & \vdots \\ x_3 & \vdots & \dots & \vdots \\ \vdots & \vdots & \dots & \vdots \end{bmatrix},$$

then, we have

$$C_N A_{N-1}^{-1} B_N = \begin{bmatrix} (\lambda_1^+)^2(x_1 + x_2 + x_3) & (\lambda_2^+)^2(x_1 + x_2 + x_3) & (\lambda_3^+)^2(x_1 + x_2 + x_3) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad (3.2.15)$$

from here we can see that it is not necessary to calculate all the entries of the matrix A_{N-1}^{-1} . Indeed, we only need the 3 first entries of the first column. Straightforward calculations show that

$$x_1 = \frac{\Delta_1^{N-1,+}}{\Delta^{N-1,+}}, \quad x_2 = \frac{\Delta_2^{N-1,+}}{\Delta^{N-1,+}}, \quad x_3 = \frac{\Delta_3^{N-1,+}}{\Delta^{N-1,+}}. \quad (3.2.16)$$

Using (3.2.15) and (3.2.16) we get

$$C_N A_{N-1}^{-1} B_N = \begin{bmatrix} (\lambda_1^+)^2 \sum_{j=1}^3 \frac{\Delta_j^{N-1,+}}{\Delta^{N-1,+}} & (\lambda_2^+)^2 \sum_{j=1}^3 \frac{\Delta_j^{N-1,+}}{\Delta^{N-1,+}} & (\lambda_3^+)^2 \sum_{j=1}^3 \frac{\Delta_j^{N-1,+}}{\Delta^{N-1,+}} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Then with (3.2.7)

$$D_N - C_N A_{N-1}^{-1} B_N = \begin{bmatrix} -1 - (\lambda_1^+)^2 \sum_{j=1}^3 \frac{\Delta_j^{N-1,+}}{\Delta^{N-1,+}} & -1 - (\lambda_2^+)^2 \sum_{j=1}^3 \frac{\Delta_j^{N-1,+}}{\Delta^{N-1,+}} & -1 - (\lambda_3^+)^2 \sum_{j=1}^3 \frac{\Delta_j^{N-1,+}}{\Delta^{N-1,+}} \\ e^{\lambda_1^+ \ell_N} & e^{\lambda_2^+ \ell_N} & e^{\lambda_3^+ \ell_N} \\ \lambda_1^+ e^{\lambda_1^+ \ell_N} & \lambda_2^+ e^{\lambda_2^+ \ell_N} & \lambda_3^+ e^{\lambda_3^+ \ell_N} \end{bmatrix} \quad (3.2.17)$$

and using the multilinearity of the determinant

$$\det(D_N - C_N A_{N-1}^{-1} B_N) = - \sum_{j=1}^3 \frac{\Delta_j^{N-1,+}}{\Delta^{N-1,+}} \det(F_N) + \det(D_N),$$

where

$$F_N = \begin{bmatrix} (\lambda_1^+)^2 & (\lambda_2^+)^2 & (\lambda_3^+)^2 \\ e^{\lambda_1^+ \ell_N} & e^{\lambda_2^+ \ell_N} & e^{\lambda_3^+ \ell_N} \\ \lambda_1^+ e^{\lambda_1^+ \ell_N} & \lambda_2^+ e^{\lambda_2^+ \ell_N} & \lambda_3^+ e^{\lambda_3^+ \ell_N} \end{bmatrix}. \quad (3.2.18)$$

Then, it holds

$$\Delta^{N,+} = \Delta^{N-1,+} \left[- \sum_{j=1}^3 \frac{\Delta_j^{N-1,+}}{\Delta^{N-1,+}} \det(F_N) + \det(D_N) \right]. \quad (3.2.19)$$

Using (3.2.7) and (3.2.18) we can derive

$$\det(D_N) = \rho\sqrt{3}e^{-i\rho\ell_N} + \left(-\frac{\rho\sqrt{3}}{2} - \frac{3}{2}i\rho \right) e^{(-\frac{\rho\sqrt{3}}{2} + i\frac{\rho}{2})\ell_N} + \left(-\frac{\rho\sqrt{3}}{2} + \frac{3}{2}i\rho \right) e^{(\frac{\rho\sqrt{3}}{2} + i\frac{\rho}{2})\ell_N}, \quad (3.2.20)$$

$$\det(F_N) = \sqrt{3}\rho^3 e^{-i\rho\ell_N} + \sqrt{3}\rho^3 e^{-\frac{1}{2}\rho(\sqrt{3}-i)\ell_N} + \sqrt{3}\rho^3 e^{-\frac{1}{2}\rho(-\sqrt{3}-i)\ell_N}. \quad (3.2.21)$$

Now, to compute $\Delta_{3(n-1)+j}^{N,+}$, let $A_{N,j}^n$ the matrix obtained by replacing the column $3(n-1)+j$

of A_N by $[1 \ 0 \ \cdots \ 0]^\top$, for $j = 1, 2, 3$ and $n = 1, \dots, N-1$, that is

$$A_{N,j}^n = \left[\begin{array}{c|c} \begin{matrix} \widehat{1}^{(j+3(n-1)-th)} \\ 0 \\ \vdots \\ \vdots \\ 0 \end{matrix} & \begin{matrix} B_N \\ D_N \end{matrix} \end{array} \right] = \begin{cases} \left[\begin{array}{c|c} \begin{matrix} A_{N-1,j}^n \\ 0 \ 1 \ 1 \\ 0 \ 0 \ 0 \\ 0 \ 0 \ 0 \end{matrix} & \begin{matrix} B_N \\ D_N \end{matrix} \end{array} \right] & \text{if } j = 1, \ n = 1 \\ \left[\begin{array}{c|c} \begin{matrix} A_{N-1,j}^n \\ 1 \ 0 \ 1 \\ 0 \ 0 \ 0 \\ 0 \ 0 \ 0 \end{matrix} & \begin{matrix} B_N \\ D_N \end{matrix} \end{array} \right] & \text{if } j = 2, \ n = 1 \\ \left[\begin{array}{c|c} \begin{matrix} A_{N-1,j}^n \\ 1 \ 1 \ 0 \\ 0 \ 0 \ 0 \\ 0 \ 0 \ 0 \end{matrix} & \begin{matrix} B_N \\ D_N \end{matrix} \end{array} \right] & \text{if } j = 3, \ n = 1 \\ \left[\begin{array}{c|c} \begin{matrix} A_{N-1,j}^n \\ C_N \end{matrix} & \begin{matrix} B_N \\ D_N \end{matrix} \end{array} \right] & \text{if } \begin{matrix} j = 1, 2, 3, \\ n = 2, \dots, N-1. \end{matrix} \end{cases} \quad (3.2.22)$$

We claim the following property of $\Delta_{3(n-1)+j}^{N,+}$.

Lemma 3.2.1.

$$\Delta_{3(n-1)+j}^{N,+} = \Delta_{3(n-1)+j}^{N-1,+} \det(D_N), \quad n = 1, \dots, N-1, \quad j = 1, 2, 3. \quad (3.2.23)$$

Proof. Using the decomposition given by (3.2.22), we get

$$A_{N,j}^n = \begin{bmatrix} A_{N-1,j}^n & B_N \\ C_{N,j}^n & D_N \end{bmatrix},$$

for an appropriated choice of $C_{N,j}^n$. Thus, with the same idea as (3.2.14) it holds

$$\Delta_{3(n-1)+j}^{N,+} = \det(A_{N,j}^n) = \det(A_{N-1,j}^n) \det(D_N - C_{N,j}^n (A_{N-1,j}^n)^{-1} B_N). \quad (3.2.24)$$

Similarly, as before, we need to study the product $C_{N,j}^n (A_{N-1,j}^n)^{-1} B_N$, in particular the first column of the matrix $(A_{N-1,k}^n)^{-1}$. To do that, note that

$$A_{N-1,j}^n v = \left[\begin{array}{c|c} \begin{matrix} \widehat{1}^{(j+3(n-1)-th)} \\ 0 \\ \vdots \\ \vdots \\ 0 \end{matrix} & \begin{matrix} B_{N-1} \\ D_N \end{matrix} \end{array} \right] v = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

by simple inspection, the solution of the above problem is $v = [0 \ \cdots \ \widehat{1}^{j+3(n-1)} \ \cdots \ 0]^\top$ which coincides with the first column of $(A_{N-1,j}^n)^{-1}$ and hence $C_{N,j}^n (A_{N-1,j}^n)^{-1} B_N = \mathbf{0}_{3 \times 3}$, therefore with (3.2.24)

$$\Delta_{3(n-1)+j}^{N,+} = \Delta_{3(n-1)+j}^{N-1,+} \det(D_N), \quad n = 1, \dots, N-1, \quad j = 1, 2, 3.$$

which finished the proof of Lemma 3.2.1. \square

In order to show that $\Delta^{N,+} \neq 0$, note that by (3.2.19) we get

$$\Delta^{N,+} = - \sum_{j=1}^3 \Delta_j^{N-1,+} \det(F_N) + \Delta^{N-1,+} \det(D_N), \quad j = 1, 2, 3.$$

Using (3.2.23) recursively, we get

$$\Delta_j^{N-1,+} = \Delta_j^{1,+} \prod_{\ell=2}^{N-1} \det(D_\ell).$$

Noticing that $\Delta^{1,+} = \det(F_1)$, $-\sum_{j=1}^3 \Delta_j^{1,+} = \det(D_1)$ and invoking inductively (3.2.19), we deduce

$$\Delta^{N,+} = \sum_{j=1}^N \det(F_j) \prod_{\ell=1, \ell \neq j}^N \det(D_\ell).$$

Then, from Appendix B.3, it holds, for all $j = 1, \dots, N$, $\det(D_j) \neq 0$, thus

$$\Delta^{N,+} = \left(\prod_{\ell=1}^N \det(D_\ell) \right) \sum_{j=1}^N \frac{\det(F_j)}{\det(D_j)},$$

and from Appendix B.4, $\sum_{j=1}^N \frac{\det(F_j)}{\det(D_j)} \neq 0$, thus $\Delta^{N,+} \neq 0$. Now as $\Delta^{N,+} \neq 0$, we can obtain using (3.2.19) and (3.2.23) that

$$\frac{\Delta_{3(n-1)+j}^{N,+}}{\Delta^{N,+}} = \frac{\Delta_{3(n-1)+j}^{N-1,+}}{\Delta^{N-1,+}} \frac{\det(D_N)}{-\sum_{l=1}^3 \frac{\Delta_l^{N-1,+}}{\Delta^{N-1,+}} \det(F_N) + \det(D_N)}, \quad (3.2.25)$$

for $j = 1, 2, 3$, $n = 1, \dots, N-1$. Then, using (3.2.21) we get

$$\det(F_N) \sim \sqrt{3}\rho^3 e^{\frac{\rho}{2}\sqrt{3}\ell_N + i\frac{\rho}{2}\ell_N},$$

and by the induction assumption

$$\sum_{l=1}^3 \frac{\Delta_l^{N-1,+}}{\Delta^{N-1,+}} \sim \delta_{N-1} \rho^{-2} e^{-i\frac{\pi}{3}}.$$

Thus

$$\sum_{l=1}^3 \frac{\Delta_l^{N-1,+}}{\Delta^{N-1,+}} \det(F_N) \sim \delta_{N-1} \sqrt{3} \rho e^{\frac{\rho}{2}\sqrt{3}\ell_N + i\frac{\rho}{2}\ell_N - i\frac{\pi}{3}},$$

and then for $\rho \rightarrow \infty$

$$\frac{\det(D_N)}{-\sum_{l=1}^3 \frac{\Delta_l^{N-1,+}}{\Delta^{N-1,+}} \det(F_N) + \det(D_N)} \sim \frac{1}{\delta_{N-1} + 1}. \quad (3.2.26)$$

Now by the induction assumption

$$\begin{aligned} \frac{\Delta_{3(n-1)+1}^{N-1,+}}{\Delta^{N-1,+}} &\sim -\delta_{N-1} \rho^{-2} e^{-\frac{1}{2}\rho\sqrt{3}\ell_n - i\frac{3}{2}\rho\ell_n}, & \frac{\Delta_{3(n-1)+2}^{N-1,+}}{\Delta^{N-1,+}} &\sim \delta_{N-1} \rho^{-2} e^{-\rho\sqrt{3}\ell_n + i\frac{\pi}{3}}, \\ \frac{\Delta_{3(n-1)+3}^{N-1,+}}{\Delta^{N-1,+}} &\sim \delta_{N-1} \rho^{-2} e^{-i\frac{\pi}{3}}, \end{aligned}$$

and (3.2.25)-(3.2.26) we have:

$$\begin{aligned} \frac{\Delta_{3(n-1)+1}^{N,+}}{\Delta^{N,+}} &\sim -\delta_N \rho^{-2} e^{-\frac{1}{2}\rho\sqrt{3}\ell_n - i\frac{3}{2}\rho\ell_n}, & \frac{\Delta_{3(n-1)+2}^{N,+}}{\Delta^{N,+}} &\sim \delta_N \rho^{-2} e^{-\rho\sqrt{3}\ell_n + i\frac{\pi}{3}}, \\ \frac{\Delta_{3(n-1)+3}^{N,+}}{\Delta^{N,+}} &\sim \delta_N \rho^{-2} e^{-i\frac{\pi}{3}}, & \sum_{j=1}^3 \frac{\Delta_{3(n-1)+j}^{N,+}}{\Delta^{N,+}} &\sim \delta_N \rho^{-2} e^{-i\frac{\pi}{3}}, \quad n = 1, \dots, N-1, \end{aligned} \quad (3.2.27)$$

where $\delta_N = \frac{\delta_{N-1}}{\delta_{N-1} + 1}$. It just remains to study the case $n = N$. Note that using the block decomposition of A_N we get

$$C_N \begin{bmatrix} \frac{\Delta_1^{N,+}}{\Delta_1^{N,+}} \\ \frac{\Delta_2^{N,+}}{\Delta_2^{N,+}} \\ \frac{\Delta_3^{N,+}}{\Delta_3^{N,+}} \\ \vdots \\ \frac{\Delta_{3N-5}^{N,+}}{\Delta_{3N-5}^{N,+}} \\ \frac{\Delta_{3N-4}^{N,+}}{\Delta_{3N-4}^{N,+}} \\ \frac{\Delta_{3N-3}^{N,+}}{\Delta_{3N-3}^{N,+}} \\ \frac{\Delta_{3N-2}^{N,+}}{\Delta_{3N-2}^{N,+}} \end{bmatrix} + D_N \begin{bmatrix} \frac{\Delta_{3N-2}^{N,+}}{\Delta_{3N-2}^{N,+}} \\ \frac{\Delta_{3N-1}^{N,+}}{\Delta_{3N-1}^{N,+}} \\ \frac{\Delta_{3N}^{N,+}}{\Delta_{3N}^{N,+}} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

and recalling (3.2.6) and (3.2.7) explicit calculations show that

$$\begin{bmatrix} \frac{\Delta_{3N-2}^{N,+}}{\Delta_{3N-2}^{N,+}} \\ \frac{\Delta_{3N-1}^{N,+}}{\Delta_{3N-1}^{N,+}} \\ \frac{\Delta_{3N}^{N,+}}{\Delta_{3N}^{N,+}} \end{bmatrix} = \frac{\left(-\sum_{j=1}^3 \frac{\Delta_j^{N,+}}{\Delta_j^{N,+}}\right)}{\det(D_N)} \begin{bmatrix} -\rho\sqrt{3}e^{-i\rho\ell_N} \\ \left(\frac{\rho\sqrt{3}}{2} + \frac{3}{2}i\rho\right) e^{\left(-\frac{\rho\sqrt{3}}{2} + \frac{i\rho}{2}\right)\ell_N} \\ \left(\frac{\rho\sqrt{3}}{2} - \frac{3}{2}i\rho\right) e^{\left(\frac{\rho\sqrt{3}}{2} + \frac{i\rho}{2}\right)\ell_N} \end{bmatrix}, \quad (3.2.28)$$

and using (3.2.27) we can conclude from (3.2.28)

$$\begin{aligned} \frac{\Delta_{3N-2}^{N,+}}{\Delta_{3N-2}^{N,+}} &\sim -\delta_N \rho^{-2} e^{-\frac{1}{2}\rho\sqrt{3}\ell_N - i\frac{3}{2}\rho\ell_N}, & \frac{\Delta_{3N-1}^{N,+}}{\Delta_{3N-1}^{N,+}} &\sim \delta_N \rho^{-2} e^{-\rho\sqrt{3}\ell_N + i\frac{\pi}{3}}, & \frac{\Delta_{3N}^{N,+}}{\Delta_{3N}^{N,+}} &\sim \delta_N \rho^{-2} e^{-i\frac{\pi}{3}}, \\ & & \sum_{j=1}^3 \frac{\Delta_{3(N-1)+j}^{N,+}}{\Delta_{3(N-1)+j}^{N,+}} &\sim \delta_N \rho^{-2} e^{-i\frac{\pi}{3}}, \end{aligned}$$

which gives the induction and concludes the proof of Proposition 3.2.1. \square

Remark 3.2.1. Recently, in [CKN22], the problem of small-time local controllability of the nonlinear single KdV equation was addressed. To reach the obstruction to small-time controllability in [CKN22] new regularity results in the spirit of [BSZ02] were established. Those results have some connections with the analysis developed in this chapter. Here, the analysis of the linear problem (3.2.4) is based on the estimate of the terms I_n and J_n ((3.2.10) and (3.2.11)). These involve two integrals of ρ from 0 to infinity, and Proposition 3.2.1 shows the integrands are well-defined ($\Delta^{N,+} \neq 0$) and deal with their behavior at infinity. However, in [CKN22] the behavior of the integrands might be infinite for finite ρ . This is the case where $L \in \mathcal{N}$, with $2k + l \notin 3\mathbb{N}^*$ [CKN22, Lemma B1]. The main difference between this two different behavior is because in [CKN22] they worked with the linear system including the term, $\partial_x u$ which is necessary to study controllability issues.

Now we are going to state the next regularity result for the solution (3.2.1) using the Laplace representation obtained in (3.2.8) and Proposition 3.2.1.

Proposition 3.2.2. Let $T > 0$ and $h \in H^{-\frac{1}{3}}(0, T)$, then we have a unique solution $\underline{u} \in \mathbb{Y}_T$ of (3.2.1). Moreover, there exists $C > 0$ such that for all $h \in H^{-\frac{1}{3}}(0, T)$, $\|\underline{u}\|_{\mathbb{Y}_T} \leq C \|h\|_{H^{-\frac{1}{3}}(0, T)}$.

Proof. This proof uses Proposition 3.2.1 and follows closely [CZ17, Proposition 2.2] and [BSZ03], thus it is omitted here. \square

Note that Proposition 3.2.2 justifies the formal computations given in (3.2.8). Let \underline{W} the operator that corresponds to the integral representation obtained in Proposition 3.2.2, i.e. given $T > 0$ and $h \in H^{-\frac{1}{3}}(0, T)$, the unique solution \underline{u} of (3.2.1) is given by

$$\underline{u} = \begin{pmatrix} u_1 \\ \vdots \\ u_N \end{pmatrix} = \underline{W}h \in \mathbb{Y}_T.$$

Our next step is to consider the linear problem including non-homogeneous initial data and source terms, as follows

$$\begin{cases} \partial_t v_n(t, x) + \partial_x^3 v_n(t, x) = f_n(t, x) & \forall x \in (0, \ell_n), t > 0, n = 1, \dots, N, \\ v_n(t, 0) = v_{n'}(t, 0), & \forall n, n' = 1, \dots, N, \\ \sum_{n=1}^N \partial_x^2 v_n(t, 0) = h(t), & t > 0, \\ v_n(t, \ell_n) = 0, \quad \partial_x v_n(t, \ell_n) = 0, & t > 0, n = 1, \dots, N, \\ v_n(0, x) = v_n^0, & x \in (0, \ell_n). \end{cases} \quad (3.2.29)$$

We know from [AC18] that in the case, $h = 0$ the solution of (3.2.29) can be written as

$$\underline{v}(t, x) = \underline{W}_0(t)v^0 + \int_0^t \underline{W}_0(t - \tau)\underline{f}(\tau)d\tau,$$

for any $v^0 \in \mathbb{L}^2(\mathcal{T})$ and $\underline{f} \in L^1(0, T; \mathbb{L}^2(\mathcal{T}))$, where $\{\underline{W}_0(t)\}_{t \geq 0}$ is the C_0 -semigroup in the space $\mathbb{L}^2(\mathcal{T})$ generated by the operator $\mathcal{A}v = -\partial_x^3 v$, with domain

$$D(\mathcal{A}) = \left\{ v \in \mathbb{H}^3(\mathcal{T}) \cap \mathbb{H}_e^2(\mathcal{T}), \sum_{n=1}^N \frac{d^2 v_n}{dx^2}(0) = 0 \right\}.$$

Using semigroup theory it is possible to show that $v \in C([0, T]; \mathbb{L}^2(\mathcal{T}))$ and also using multipliers we can obtain the classical Kato smoothing result $v \in L^2(0, T; \mathbb{H}_e^1(\mathcal{T}))$, but it is difficult (if not impossible) to derive the Sharp Kato smoothing property established in Proposition 3.2.2 using energy methods. Now, we use the following result obtained in [BSZ03] for a single KdV equation posed on a bounded domain

$$\begin{cases} \partial_t \psi(t, x) + \partial_x^3 \psi(t, x) = f(t, x), & x \in (0, L), t \geq 0, \\ \psi(t, 0) = \psi(t, L) = \partial_x \psi(t, L) = 0, & t \geq 0, \\ \psi(0, x) = \psi^0(x), & x \in (0, L), \end{cases} \quad (3.2.30)$$

Proposition 3.2.3. (Lemma 3.3 [BSZ03]) *Let $T > 0$ and $L > 0$ be given. For any $\psi^0 \in L^2(0, L)$ and $f \in L^1(0, T; L^2(0, L))$, the problem (3.2.30) admits a unique solution $\psi \in C([0, T]; L^2(0, L)) \cap L^2(0, T; H^1(0, L))$, with $\partial_x^\kappa \psi \in L_x^\infty(0, L; H^{\frac{1-\kappa}{3}}(0, T))$, $\kappa = 0, 1, 2$. Moreover, there exists $C > 0$ depending only on T and L such that*

$$\|\psi\|_{B_T} + \sum_{\kappa=0}^2 \|\partial_x^\kappa \psi\|_{L_x^\infty(0, L; H^{\frac{1-\kappa}{3}}(0, T))} \leq C (\|\psi^0\|_{L^2(0, L)} + \|f\|_{L^1(0, T; H^1(0, L))}).$$

Now for any $v_n^0 \in L^2(0, \ell_n)$ and $f_n \in L^1(0, T; L^2(0, \ell_n))$. Consider

$$\psi_n = \psi_n(t, \cdot) = W_1^n(t)v_n^0(\cdot) + \int_0^t W_1^n(t - \tau)f_n(\tau, \cdot)d\tau$$

where $W_1^n(t)$ is the C_0 -semigroup associated to the boundary-value problem (3.2.30) on $(0, \ell_n)$.

Let $\mathfrak{h}(t) = \sum_{n=1}^N \partial_x^2 \psi_n(t, 0) \in H^{-\frac{1}{3}}(0, T)$ by Proposition 3.2.3. Now take $h \in H^{-\frac{1}{3}}(0, T)$, then by

Proposition 3.2.2 the function $\underline{w} = \underline{W}(t)(h - \mathfrak{h})$ is well-defined and is the solution of (3.2.1) with boundary data $h - \mathfrak{h}$. Finally, the solution \underline{v} of (3.2.29) can be expressed as

$$\underline{v}(t, \cdot) = \underline{W}_1(t)\underline{v}^0(\cdot) + \int_0^t \underline{W}_1(t - \tau)\underline{f}(\tau, \cdot)d\tau + \underline{W}(t)(h - \mathfrak{h})(t).$$

The next result resumes these ideas.

Proposition 3.2.4. *Let $T > 0$ be given, then, for any $\underline{v}^0 \in \mathbb{L}^2(\mathcal{T})$, $h \in H^{-\frac{1}{3}}(0, T)$ and $\underline{f} \in L^1(0, T; \mathbb{L}^2(\mathcal{T}))$, then the problem (3.2.29) admits a unique solution $\underline{v} \in \mathbb{Y}_T$. Moreover, there exists $C > 0$ depending only on T and ℓ_1, \dots, ℓ_n such that*

$$\|\underline{v}\|_{\mathbb{Y}_T} \leq C \left(\|h\|_{H^{-\frac{1}{3}}(0, T)} + \|\underline{f}\|_{L^1(0, T; \mathbb{L}^2(\mathcal{T}))} + \|\underline{v}^0\|_{\mathbb{L}^2(\mathcal{T})} \right).$$

3.2.2 Nonlinear Problem

With all the tools developed in the last sections we are ready to prove the global well-posedness result established on Theorem 3.1.3, the main ingredients of this proof are the regularity obtained in the linear cases, energy and multiplier estimates and a fixed-point argument. Let $T > 0$ and define $\mathbb{X}_T = \mathbb{L}^2(\mathcal{T}) \times H^{-\frac{1}{3}}(0, T)$.

Proof of Theorem 3.1.3:

Let $(\underline{u}^0, 0) \in \mathbb{X}_T$ and $R, \theta > 0$ that will be chosen after. Consider the closed ball $B_{\mathbb{Y}_\theta}(0, R) := \{\underline{v} \in \mathbb{Y}_\theta, \|\underline{v}\|_{\mathbb{Y}_\theta} \leq R\}$. Then $B_{\mathbb{Y}_\theta}(0, R)$ is a complete metric space. Consider the map $\Phi : \mathbb{Y}_\theta \rightarrow \mathbb{Y}_\theta$ defined by $\Phi(\underline{v}) = \underline{u}$ where \underline{u} is the solution of

$$\begin{cases} (\partial_t u_n + \partial_x^3 u_n)(t, x) = -(\partial_x v_n + v_n \partial_x v_n)(t, x), & \forall x \in (0, \ell_n), t > 0, n = 1, \dots, N, \\ u_n(t, 0) = u_{n'}(t, 0), & \forall n, n' = 1, \dots, N, \\ \sum_{n=1}^N \partial_x^2 u_n(t, 0) = -\alpha v_1(t, 0) - \frac{N}{3}(v_1(t, 0))^2, & t > 0, \\ u_n(t, \ell_n) = \partial_x u_n(t, \ell_n) = 0, & t > 0, n = 1, \dots, N, \\ u_n(0, x) = u_n^0, & x \in (0, \ell_n). \end{cases} \quad (3.2.31)$$

Clearly, $\underline{u} \in \mathbb{Y}_\theta$ is solution of (KdV-N) if \underline{u} is a fixed-point of Φ . Now we write two lemmas to deal with the source term and boundary conditions.

Lemma 3.2.2. (Lemma 3.1 [BSZ03]) *There exists a constant $C > 0$ such that for any $T > 0$ and $u, v \in Y_T$*

$$\int_0^T \|u(t, \cdot) \partial_x v(t, \cdot)\|_{L^2(0, L)} dt \leq C(T^{1/2} + T^{1/3}) \|u\|_{Y_T} \|v\|_{Y_T}.$$

where Y_T is \mathbb{Y}_T for $N = 1$.

Lemma 3.2.3. (Lemma 3.2 [JZ12]) *There exist constants $C, \beta > 0$ such that for any $T > 0$ and $g_1, g_2 \in H^{\frac{1}{3}}(0, T)$, it holds, $g_1 g_2 \in H^{-\frac{1}{3}}(0, T)$ and*

$$\|g_1 g_2\|_{H^{-\frac{1}{3}}(0, T)} \leq CT^\beta \|g_1\|_{H^{\frac{1}{3}}(0, T)} \|g_2\|_{H^{\frac{1}{3}}(0, T)}.$$

From Proposition 3.2.4, Lemma 3.2.2 and Lemma 3.2.3 we get for all $\underline{v} \in \mathbb{Y}_\theta$

$$\begin{aligned} \|\Phi(\underline{v})\|_{\mathbb{Y}_\theta} &\leq C \left(\|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})} + \left\| -\alpha v_1(t, 0) - \frac{N}{3}(v_1(t, 0))^2 \right\|_{H^{-\frac{1}{3}}(0, \theta)} + \int_0^\theta \|\partial_x \underline{v}(t, \cdot)\|_{\mathbb{L}^2(\mathcal{T})} dt \right. \\ &\quad \left. + \int_0^\theta \|\underline{v}(t, \cdot) \partial_x \underline{v}(t, \cdot)\|_{\mathbb{L}^2(\mathcal{T})} dt \right) \\ &\leq C \left(\|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})} + \theta^\beta (\|\underline{v}\|_{\mathbb{Y}_\theta} + \|\underline{v}\|_{\mathbb{Y}_\theta}^2) + (\theta^{1/2} + \theta^{1/3}) \|\underline{v}\|_{\mathbb{Y}_\theta}^2 + \theta^{1/2} \|\underline{v}\|_{\mathbb{Y}_\theta} \right). \end{aligned}$$

We consider Φ restricted to the closed ball $B_{\mathbb{Y}_\theta}(0, R) = \{\underline{u} \in \mathbb{Y}_\theta : \|\underline{u}\|_{\mathbb{Y}_\theta}\}$ and choose $\theta, R > 0$ such that

$$\begin{cases} R = 3C\|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})}, \\ C(\theta^\beta + \theta^{1/2}) \leq \frac{1}{3}, \\ C(\theta^\beta + \theta^{1/2} + \theta^{1/3})R \leq \frac{1}{6}. \end{cases} \quad (3.2.32)$$

Thus, for $\underline{u} \in B_{\mathbb{Y}_\theta}(0, R)$, Φ maps $B_{\mathbb{Y}_\theta}(0, R)$ into itself. Take now \underline{v} and $\tilde{\underline{v}} \in B_{\mathbb{Y}_\theta}(0, R)$ then $\underline{w} = \Phi(\underline{v}) - \Phi(\tilde{\underline{v}})$ solves the equation

$$\begin{cases} \partial_t w_n + \partial_x^3 w_n = -(\partial_x v_n - \partial_x \tilde{v}_n) - \frac{1}{2} \partial_x ((v_n - \tilde{v}_n)(v_n + \tilde{v}_n)), & \forall x \in (0, \ell_n), t > 0, \\ w_n(t, 0) = w_{n'}(t, 0), & n = 1, \dots, N, \\ \sum_{n=1}^N \partial_x^2 w_n(t, 0) = -\alpha(v_1(t, 0) - \tilde{v}_1(t, 0)), & \forall n, n' = 1, \dots, N, \\ -\frac{1}{3} \sum_{n=1}^N ((v_1(t, 0) - \tilde{v}_1(t, 0))(v_1(t, 0) + \tilde{v}_1(t, 0))), & t > 0, \\ w_n(t, \ell_n) = \partial_x w_n(t, \ell_n) = 0, & t > 0, n = 1, \dots, N, \\ w_n(0, x) = 0, & x \in (0, \ell_n). \end{cases}$$

Now from Proposition 3.2.4 we obtain

$$\begin{aligned} \|\Phi(\underline{v}) - \Phi(\tilde{\underline{v}})\|_{\mathbb{Y}_\theta} &\leq C \left(\theta^{1/2} \|\underline{v} - \tilde{\underline{v}}\|_{\mathbb{Y}_\theta} + \frac{1}{2} (\theta^{1/2} + \theta^{1/3}) \|\underline{v} - \tilde{\underline{v}}\|_{\mathbb{Y}_\theta} \|\underline{v} + \tilde{\underline{v}}\|_{\mathbb{Y}_\theta} \right. \\ &\quad \left. + \theta^\beta \|\underline{v} - \tilde{\underline{v}}\|_{\mathbb{Y}_\theta} + \theta^\beta \|\underline{v} - \tilde{\underline{v}}\|_{\mathbb{Y}_\theta} \|\underline{v} + \tilde{\underline{v}}\|_{\mathbb{Y}_\theta} \right) \\ &\leq C \left((\theta^{1/2} + \theta^\beta) \|\underline{v} - \tilde{\underline{v}}\|_{\mathbb{Y}_\theta} + \frac{1}{2} (\theta^{1/2} + \theta^{1/3} + 2\theta^\beta) \|\underline{v} - \tilde{\underline{v}}\|_{\mathbb{Y}_\theta} 2R \right) \end{aligned}$$

then with (3.2.32)

$$\|\Phi(\underline{v}) - \Phi(\tilde{\underline{v}})\|_{\mathbb{Y}_\theta} \leq \left(\frac{1}{3} + \frac{1}{3} \right) \|\underline{v} - \tilde{\underline{v}}\|_{\mathbb{Y}_\theta} = \frac{2}{3} \|\underline{v} - \tilde{\underline{v}}\|_{\mathbb{Y}_\theta}.$$

That means that the map Φ is a contraction on $B_{\mathbb{Y}_\theta}$ and by the Banach fixed-point theorem has a unique fixed-point $\underline{u} \in \mathbb{Y}_\theta$. It gives the local in-time well-posedness for bounded initial data. Now take $T > 0$, we can check using integration by parts and the boundary conditions that every solution of (KdV-N) satisfies

$$\frac{d}{dt} E(t) = - \left(\alpha - \frac{N}{2} \right) (u_1(t, 0))^2 - \frac{1}{2} \sum_{n=1}^N (\partial_x u_n(t, 0))^2 \leq 0. \quad (3.2.33)$$

since $N \leq 2\alpha$. This dissipation law tells us that the energy is a non-increasing function of the time variable, that means

$$E(t) \leq E(\theta) \leq E(0) = \frac{1}{2} \|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})}^2, \quad \forall t > \theta > 0. \quad (3.2.34)$$

From here, taking the maximum for $t \in [0, T]$ we can see that

$$\|\underline{u}\|_{C([0, T]; \mathbb{L}^2(\mathcal{T}))} \leq \|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})}. \quad (3.2.35)$$

Finally, following [PCP22a, CCM20] we multiply (KdV-N) by $q_n u_n$, integrate on $(0, T) \times (0, \ell_n)$ and sum on $n = 1, \dots, N$ to obtain the following equality:

$$\begin{aligned}
& \sum_{n=1}^N \int_0^{\ell_n} q_n(t, x) (u_n(t, x))^2 dx \Big|_0^T - \sum_{n=1}^N \int_0^T \int_0^{\ell_n} (\partial_t q_n + \partial_x q_n + \partial_x^3 q_n) (u_n)^2 dx dt \\
& + 3 \sum_{n=1}^N \int_0^T \int_0^{\ell_n} (\partial_x u_n)^2 \partial_x q_n dx dt - \frac{2}{3} \sum_{n=1}^N \int_0^T \int_0^{\ell_n} (u_n)^3 \partial_x q_n dx dt \\
& = \sum_{n=1}^N \int_0^T \left[(q_n + \partial_x^2 q_n) (u_n)^2 + 2q_n u_n \partial_x^2 u_n - 2\partial_x q_n u_n \partial_x u_n - q_n (\partial_x u_n)^2 + \frac{2}{3} q_n (u_n)^3 \right] (t, 0) dt.
\end{aligned} \tag{3.2.36}$$

- Taking $q_n = 1$ in (3.2.36) we can derive:

$$\sum_{n=1}^N \int_0^T (\partial_x u_n(t, 0))^2 dt \leq \|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})}^2. \tag{3.2.37}$$

- If we take $q_n = \frac{x(2\ell_n - x)}{\ell_n^2}$ in (3.2.36), defining $L = \max_{n=1, \dots, N} \ell_n$ and $\ell = \min_{n=1, \dots, N} \ell_n$, we can obtain:

$$\begin{aligned}
\frac{2N}{L^2} \|u_1(\cdot, 0)\|_{L^2(0, T)}^2 & \leq \frac{2T}{\ell^2} \|\underline{u}\|_{C([0, T]; \mathbb{L}^2(\mathcal{T}))}^2 - 2 \int_0^T u_1(t, 0) \sum_{n=1}^N \partial_x u_n(t, 0) \frac{2}{\ell_n} dt \\
& + \|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})}^2 + \frac{4}{3\ell} \sum_{n=1}^N \int_0^T \int_0^{\ell_n} (u_n(t, x))^3 dx dt.
\end{aligned}$$

Using (3.2.35)-(3.2.37) and Young's inequality we derive

$$\|u_1(t, 0)\|_{L^2(0, T)}^2 \leq C(T+1) \|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})}^2 + C \sum_{n=1}^N \int_0^T \int_0^{\ell_n} (u_n(t, x))^3 dx dt. \tag{3.2.38}$$

As $H^1(0, \ell_n)$ embed compactly into $C([0, \ell_n])$ we get

$$\sum_{n=1}^N \int_0^T \int_0^{\ell_n} (u_n)^3 dx dt \leq CT^{1/2} \|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})}^2 \|\underline{u}\|_{L^2(0, T; \mathbb{H}_e^1(\mathcal{T}))}$$

and then with (3.2.38)

$$\|u_1(t, 0)\|_{L^2(0, T)}^2 \leq C(T+1) \|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})}^2 + CT^{1/2} \|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})}^2 \|\underline{u}\|_{L^2(0, T; \mathbb{H}_e^1(\mathcal{T}))}. \tag{3.2.39}$$

- Finally, considering $q_j = x$ and using (3.2.35)-(3.2.37)-(3.2.39)

$$\|\partial_x \underline{u}\|_{L^2(0, T; \mathbb{L}^2(\mathcal{T}))}^2 \leq C(T+1) \|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})}^2 + CT^{1/2} \|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})}^2 \|\underline{u}\|_{L^2(0, T; \mathbb{H}_e^1(\mathcal{T}))}.$$

Using Young's inequality, we can find $C > 0$ which does not depend on $T > 0$ such that

$$\|\partial_x \underline{u}\|_{L^2(0, T; \mathbb{L}^2(\mathcal{T}))}^2 \leq C(T+1) \left(\|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})}^2 + \|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})}^4 \right). \tag{3.2.40}$$

which concludes the proof of Theorem 3.1.3. \square

To obtain a well-posedness result for the systems (KdV-S) and (LKdV-S) we can use the same idea presented in Theorem 3.1.3 and Lemma B.1.3 to take in account the saturation. It is very important that in Lemma B.1.3 appears the time in the right-side, this estimate gives us the possibility to take small-time in the fixed-point approach. Then to derive the global in time well-posedness similar estimates as (3.2.35)-(3.2.40) can be obtained.

Theorem 3.2.1. *Let $(\ell_n)_{n=1, \dots, N} \in (0, \infty)^N$, $\alpha \geq \frac{N}{2}$ and $T > 0$. Then, for all $\underline{u}^0 \in \mathbb{L}^2(\mathcal{T})$, there exists a unique solution $\underline{u} \in \mathbb{B}_T$ of (KdV-S) or (LKdV-S). Moreover, there exist $0 < T^* \leq T$ and $C > 0$ such that $\underline{u} \in \mathbb{Y}_{T^*}$ and $\|\underline{u}\|_{\mathbb{Y}_{T^*}} \leq C \|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})}$.*

3.3 Stabilization

In this section, we are going to prove our stabilization results inspired by [MCPA17]. The proofs are based on observability inequalities for (KdV-S) and (LKdV-S) respectively. These inequalities imply the exponential stability. First, note that, given $T > 0$, we can check that every solution of (KdV-S) and (LKdV-S) has a non-increasing energy,

$$\frac{d}{dt}E(t) = -\left(\alpha - \frac{N}{2}\right)(u_1(t,0))^2 - \frac{1}{2}\sum_{n=1}^N(\partial_x u_n(t,0))^2 - \sum_{n=1}^N \int_0^{\ell_n} u_n \mathbf{sat}(a_n u_n) dx \leq 0, \quad (3.3.1)$$

3.3.1 Stability of KdV-S

We start by studying (KdV-S). First, note that multiplying (KdV-S) by u_n and integrating on $(0, s) \times (0, \ell_n)$ we get

$$\begin{aligned} & \sum_{n=1}^N \int_0^{\ell_n} (u_n(s, x))^2 dx + \sum_{n=1}^N \int_0^s \int_0^{\ell_n} \mathbf{sat}(a_n u_n) u_n dx dt + (2\alpha - N) \int_0^s (u_1(t, 0))^2 dt \\ & + \sum_{n=1}^N \int_0^s (\partial_x u_n(t, 0))^2 dt = \|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})}^2. \end{aligned}$$

Integrating again this expression with respect to time on $(0, T)$ we obtain

$$\begin{aligned} T\|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})}^2 & \leq \int_0^T \|\underline{u}(t, \cdot)\|_{\mathbb{L}^2(\mathcal{T})}^2 dt + (2\alpha - N)T \int_0^T (u_1(t, 0))^2 dt + T \sum_{n=1}^N \int_0^T (\partial_x u_n(t, 0))^2 dt \\ & + T \sum_{n=1}^N \int_0^T \int_0^{\ell_n} \mathbf{sat}(a_n u_n) u_n dx dt. \end{aligned} \quad (3.3.2)$$

Our goal here is to prove the following observability inequality

$$\begin{aligned} \|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})}^2 & \leq C \left((2\alpha - N) \int_0^T (u_1(t, 0))^2 dt + \sum_{n=1}^N \int_0^T (\partial_x u_n(t, 0))^2 dt \right. \\ & \left. + \sum_{n=1}^N \int_0^T \int_0^{\ell_n} \mathbf{sat}(a_n u_n) u_n dx dt \right). \end{aligned} \quad (\text{Obs-S})$$

Note that (Obs-S) is quite similar to (3.3.2). From (3.3.2) we can observe that to get (Obs-S) it is enough to prove the following inequality

$$\begin{aligned} \int_0^T \|\underline{u}(t, \cdot)\|_{\mathbb{L}^2(\mathcal{T})}^2 dt & \leq C \left((2\alpha - N) \int_0^T (u_1(t, 0))^2 dt + \sum_{n=1}^N \int_0^T (\partial_x u_n(t, 0))^2 dt \right. \\ & \left. + \sum_{n=1}^N \int_0^T \int_0^{\ell_n} \mathbf{sat}(a_n u_n) u_n dx dt \right). \end{aligned}$$

Suppose that it is false and take $\|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})} \leq R$, then we can find $(\underline{u}^{0,j})_{j \in \mathbb{N}} \subset \mathbb{L}^2(\mathcal{T})$ such that $\|\underline{u}^{0,j}\|_{\mathbb{L}^2(\mathcal{T})} \leq R$ and

$$\lim_{j \rightarrow \infty} \frac{\|\underline{u}^j\|_{L^2(0,T;\mathbb{L}^2(\mathcal{T}))}^2}{(2\alpha - N)\|u_1^j(\cdot, 0)\|_{L^2(0,T)}^2 + \|\partial_x \underline{u}^j(\cdot, 0)\|_{L^2(0,T)}^2 + \sum_{n=1}^N \int_0^T \int_0^{\ell_n} \mathbf{sat}(a_n u_n^j) u_n^j dx dt} = \infty.$$

where \underline{u}^j is the corresponding solution of (KdV-S) with initial data $\underline{u}^{0,j}$. Note now that using (3.2.33), we deduce

$$\|\underline{u}^j(t, \cdot)\|_{\mathbb{L}^2(\mathcal{T})} \leq \|\underline{u}^{0,j}\|_{\mathbb{L}^2(\mathcal{T})} \leq R. \quad (3.3.3)$$

Take $\lambda^j = \|\underline{v}^j\|_{L^2(0,T;\mathbb{L}^2(\mathcal{T}))}$, then $\lambda^j \leq T^{1/2}\|\underline{u}^{0,j}\|_{\mathbb{L}^2(\mathcal{T})} \leq T^{1/2}R$. Thus $(\lambda^j)_{j \in \mathbb{N}} \subset \mathbb{R}$ is bounded.

Take $v_n^j = \frac{u_n^j}{\lambda^j}$, then \underline{v}^j fulfills

$$\begin{cases} \left(\partial_t v_n^j + \partial_x v_n^j + \partial_x^3 v_n^j + \lambda^j v_n^j \partial_x v_n^j + \frac{\mathbf{sat}(a_n \lambda^j v_n^j)}{\lambda^j} \right)(t, x) = 0, & \forall x \in (0, \ell_n), t > 0, \\ v_n^j(t, 0) = v_{n'}^j(t, 0), & n = 1, \dots, N, \\ \sum_{n=1}^N \partial_x^2 v_n^j(t, 0) = -\alpha v_1^j(t, 0) - \lambda^j \frac{N}{3} (v_1^j(t, 0))^2, & \forall n, n' = 1, \dots, N, \\ v_n^j(t, \ell_n) = \partial_x v_n^j(t, \ell_n) = 0, & t > 0, n = 1, \dots, N, \\ \|\underline{v}^j\|_{L^2(0,T;\mathbb{L}^2(\mathcal{T}))} = 1, & \end{cases} \quad (3.3.4)$$

and satisfies

$$(2\alpha - N)\|v_1^j(t, 0)\|_{L^2(0,T)}^2 + \|\partial_x \underline{v}^j(t, 0)\|_{L^2(0,T)}^2 + \sum_{n=1}^N \int_0^T \int_0^{\ell_n} \frac{1}{\lambda^j} \mathbf{sat}(a_n \lambda^j v_n^j) v_n^j dx dt \rightarrow 0. \quad (3.3.5)$$

First, note that multiplying (3.3.4) by v_n^j and integrating on $(0, s) \times (0, \ell_n)$ we get

$$\begin{aligned} \sum_{n=1}^N \int_0^{\ell_n} (v_n^j(s, x))^2 dx + \sum_{n=1}^N \int_0^s \int_0^{\ell_n} \frac{1}{\lambda^j} \mathbf{sat}(a_n \lambda^j v_n^j) v_n^j dx dt + (2\alpha - N) \int_0^s (v_1^j(t, 0))^2 dt \\ + \sum_{n=1}^N \int_0^s (\partial_x v_n^j(t, 0))^2 dt = \|\underline{v}^j(0, \cdot)\|_{\mathbb{L}^2(\mathcal{T})}^2, \end{aligned} \quad (3.3.6)$$

which gives us, using that \mathbf{sat} is odd,

$$\|\underline{v}^j\|_{C([0,T];\mathbb{L}^2(\mathcal{T}))}^2 \leq \|\underline{v}^j(0, \cdot)\|_{\mathbb{L}^2(\mathcal{T})}^2, \quad \|\partial_x \underline{v}^j(t, 0)\|_{L^2(0,T)}^2 \leq \|\underline{v}^j(0, \cdot)\|_{\mathbb{L}^2(\mathcal{T})}^2. \quad (3.3.7)$$

Now integrating again (3.3.6) with respect to time on $(0, T)$ we obtain

$$\begin{aligned} T\|\underline{v}^j(0, \cdot)\|_{\mathbb{L}^2(\mathcal{T})}^2 \leq \int_0^T \|\underline{v}^j(t, \cdot)\|_{\mathbb{L}^2(\mathcal{T})}^2 dt + (2\alpha - N)T \int_0^T (v_1^j(t, 0))^2 dt + T \sum_{n=1}^N \int_0^T (\partial_x v_n^j(t, 0))^2 dt \\ + 2T \sum_{n=1}^N \int_0^T \int_0^{\ell_n} \frac{1}{\lambda^j} \mathbf{sat}(a_n \lambda^j v_n^j) v_n^j dx dt. \end{aligned} \quad (3.3.8)$$

This last inequality implies that $(\underline{v}^j(0, \cdot))_{j \in \mathbb{N}}$ is bounded in $\mathbb{L}^2(\mathcal{T})$. Again using that \mathbf{sat} is odd and similar estimates as (3.2.37)-(3.2.39)-(3.2.40) we conclude

$$\|\underline{v}^j\|_{L^2(0,T;\mathbb{H}_e^1(\mathcal{T}))}^2 \leq C \left(\|\underline{v}^j(0, \cdot)\|_{\mathbb{L}^2(\mathcal{T})}^2 + \|\underline{v}^j(0, \cdot)\|_{\mathbb{L}^2(\mathcal{T})}^4 \right). \quad (3.3.9)$$

Thus $(\underline{v}^j)_{j \in \mathbb{N}} \subset L^2(0, T; \mathbb{H}_e^1(\mathcal{T}))$ is bounded and it holds

$$\|v_n^j \partial_x v_n^j\|_{L^2(0,T;L^1(0,\ell_n))} \leq \|\underline{v}^j\|_{C([0,T],\mathbb{L}^2(\mathcal{T}))} \|\underline{v}^j\|_{L^2(0,T;\mathbb{H}_e^1(\mathcal{T}))}.$$

which implies that $(v_n^j \partial_x v_n^j)_{j \in \mathbb{N}}$ is a subset of $L^2(0, T; L^1(0, \ell_n))$. Using Lemma B.1.1 we have

$$\left\| \frac{\mathbf{sat}(a_n \lambda^j v_n^j)}{\lambda^j} \right\|_{L^2(0,T;L^2(0,\ell_n))} \leq 3\|a_n\|_{L^\infty(0,\ell_n)} \ell_n^{1/2} \|\underline{v}^j\|_{L^2(0,T;\mathbb{H}_e^1(\mathcal{T}))},$$

and then $\left(\frac{\mathbf{sat}(a_n \lambda^j v_n^j)}{\lambda^j} \right)_{j \in \mathbb{N}}$ is a subset of $L^2(0, T; L^2(0, \ell_n))$. From this, we can see that $\partial_t v_n^j =$

$-\left(\partial_x^3 v_n^j + \partial_x v_n^j + \lambda^j v_n^j \partial_x v_n^j + \frac{\mathbf{sat}(a_n \lambda^j v_n^j)}{\lambda^j} \right)$ is bounded in $L^2(0, T; H^{-2}(0, \ell_n))$. Hence, by Aubin-

Lions Lemma ([SS97, Proposition 1.3, Chapter III]) we can deduce that $(\underline{v}^j)_{j \in \mathbb{N}}$ is relatively compact in $L^2(0, T; \mathbb{L}^2(\mathcal{T}))$ and we can assume that \underline{v}^j converges strongly at \underline{v} in $L^2(0, T; \mathbb{L}^2(\mathcal{T}))$ with $\|\underline{v}\|_{L^2(0,T;\mathbb{L}^2(\mathcal{T}))} = 1$. Now we are going to study the case $\mathbf{sat} = \mathbf{sat}_2$ and $\mathbf{sat} = \mathbf{sat}_{10c}$ separately.

Case $\mathbf{sat} = \mathbf{sat}_2$

First, we consider the case $\mathbf{sat} = \mathbf{sat}_2$. We know that by (3.3.3), $\|\underline{u}^j(t, \cdot)\|_{L^2(\mathcal{T})} \leq R$ and then by Lemma B.1.2 we have that

$$0 \leq \sum_{n=1}^N \int_0^T \int_0^{\ell_n} a_n k_n(R) (v_n^j)^2 dx dt \leq \sum_{n=1}^N \int_0^T \int_0^{\ell_n} \frac{1}{\lambda^j} \mathbf{sat}_2(a_n \lambda^j v_n^j) v_n^j,$$

which gives us using (3.3.5), as $j \rightarrow \infty$

$$(2\alpha - N) \|v_1^j(t, 0)\|_{L^2(0, T)}^2 + \|\partial_x \underline{v}^j(t, 0)\|_{L^2(0, T)}^2 + \sum_{n=1}^N \int_0^T \int_0^{\ell_n} a_n k_n(R) (v_n^j)^2 dx dt \rightarrow 0. \quad (3.3.10)$$

Furthermore, passing to the limit in (3.3.10) we get

$$\begin{aligned} & (2\alpha - N) \|v_1(t, 0)\|_{L^2(0, T)}^2 + \|\partial_x \underline{v}(t, 0)\|_{L^2(0, T)}^2 + \sum_{n=1}^N \int_0^T \int_0^{\ell_n} a_n k_n(R) (v_n)^2 dx dt \\ & \leq \liminf \left((2\alpha - N) \|v_1^j(t, 0)\|_{L^2(0, T)}^2 + \|\partial_x \underline{v}^j(t, 0)\|_{L^2(0, T)}^2 + \sum_{n=1}^N \int_0^T \int_0^{\ell_n} a_n k_n(R) (v_n^j)^2 dx dt \right) = 0. \end{aligned}$$

Thus, $v_n(t, x) = 0$ in $(0, T) \times \omega_n$ and $(2\alpha - N)v_1(t, 0) = \partial_x v_n(t, 0) = 0$ in $(0, T)$ for all $n = 1, \dots, N$. Additionally, as $(\lambda^j)_{j \in \mathbb{N}}$ is bounded and non-negative, we can extract a convergent sub-sequence such that $\lambda^j \rightarrow \lambda \geq 0$, consequently \underline{v} satisfies $\|\underline{v}\|_{L^2(0, T; \mathbb{L}^2(\mathcal{T}))} = 1$ and solves the following system

$$\begin{cases} (\partial_t v_n + \partial_x v_n + \partial_x^3 v_n + \lambda v_n \partial_x v_n)(t, x) = 0, & \forall x \in (0, \ell_n), t > 0, n = 1, \dots, N, \\ v_n(t, \ell_n) = \partial_x v_n(t, \ell_n) = \partial_x v_n(t, 0) = 0, & t \in (0, T), \forall n = 1, \dots, N, \\ (2\alpha - N)v_n(t, 0) = 0, & t \in (0, T), \\ v_n(t, x) = 0, & (t, x) \in (0, T) \times \omega_n. \end{cases} \quad (3.3.11)$$

1. If $\lambda = 0$ the system satisfied by \underline{v} is linear, then we can use Holmgren's Theorem as in [PMVZ02] to conclude that $\underline{v} = 0$, that contradicts the fact that $\|\underline{v}\|_{L^2(0, T; \mathbb{L}^2(\mathcal{T}))} = 1$.
2. If $\lambda > 0$. In this case, we have to prove that $v_n \in L^2(0, T; H^3(0, \ell_n))$ in order to apply [SS87, Theorem 4.2]. Consider $w_n = \partial_t v_n$ then

$$\begin{cases} (\partial_t w_n + \partial_x w_n + \partial_x^3 w_n + \lambda w_n \partial_x v_n + \lambda v_n \partial_x w_n)(t, x) = 0, & x \in (0, \ell_n), t > 0, \\ & n = 1, \dots, N, \\ w_n(t, \ell_n) = \partial_x w_n(t, \ell_n) = \partial_x w_n(t, 0) = 0, & \forall n = 1, \dots, N, \\ (2\alpha - N)w_n(t, 0) = 0, & t \in (0, T), \forall j = 1, \dots, N, \\ w_n(t, x) = 0, & (t, x) \in (0, T) \times \omega_n, \\ w_n(0, x) = w_n^0(x) & x \in (0, \ell_n), j = 1, \dots, N, \end{cases}$$

where $w_n^0(x) = -v_n'(0, x) - v_n'''(0, x) - \lambda v_n(0, x)v_n'(0, x) \in H^{-3}(0, \ell_n)$. With [Cer14, Lemma A.2] we can get that $w_n(0, x) \in L^2(0, \ell_n)$ and $w_n \in C([0, T], L^2(0, \ell_n)) \cap L^2(0, T; H^1(0, \ell_n))$. Thus, $\partial_x^3 v_n = -(\partial_t v_n - \partial_x v_n - \lambda v_n \partial_x v_n) \in L^2(0, T; L^2(0, \ell_n))$ that implies $v_n \in L^2(0, T; H^3(0, \ell_n))$. Applying [SS87, Theorem 4.2] we obtain that $v_n = 0$ for all $n = 1, \dots, N$ that contradicts the fact that $\|\underline{v}\|_{L^2(0, T; \mathbb{L}^2(\mathcal{T}))} = 1$.

Case $\mathbf{sat} = \mathbf{sat}_{1oc}$

Let us consider the case where $\mathbf{sat} = \mathbf{sat}_{1oc}$, by the injection of $H^1(0, \ell_n)$ in $C([0, \ell_n])$, we can derive using similar estimate as (3.3.9)

$$\int_0^T (u_n^j(t, x))^2 dt \leq \ell_n \|\underline{u}^j\|_{L^2(0, T; \mathbb{H}_x^{\frac{1}{2}}(\mathcal{T}))}^2 \leq \ell_n \beta, \quad (3.3.12)$$

for $\beta = (R^2 + R^4)$. Now, inspired by [MCPA17], take $\Omega_{n,i} \subset [0, T]$ defined as follows:

$$\Omega_{n,i} = \left\{ t \in [0, T], \sup_{x \in [0, \ell_n]} |u_n(t, x)| > i \right\}. \quad (3.3.13)$$

Then denote $\Omega_{n,i}^c$ the complement of $\Omega_{n,i}$ and observe that

$$\int_0^T \sup_{x \in [0, \ell_n]} (u_n^j(t, x))^2 dt \geq \int_{\Omega_{n,i}} \sup_{x \in [0, \ell_n]} (u_n^j(t, x))^2 dt \geq i^2 \nu(\Omega_{n,i}),$$

for $\nu(\Omega_{n,i})$ the Lebesgue measure of $\Omega_{n,i}$. Thus, using (3.3.12) we obtain $\nu(\Omega_{n,i}) \leq \frac{\ell_n \beta}{i^2}$. Hence,

$$\max \left(T - \frac{\ell_n \beta}{i^2}, 0 \right) \leq \nu(\Omega_{n,i}^c) \leq T. \quad (3.3.14)$$

Now using Lemma B.1.2

$$\begin{aligned} \sum_{n=1}^N \int_0^T \int_0^{\ell_n} \frac{1}{\lambda^j} \mathbf{sat}_{\text{loc}}(a_n \lambda^j v_n^j) v_n^j dx dt &= \sum_{n=1}^N \int_{\Omega_{n,i}} \int_0^{\ell_n} \frac{1}{\lambda^j} \mathbf{sat}_{\text{loc}}(a_n \lambda^j v_n^j) v_n^j dx dt \\ &+ \sum_{n=1}^N \int_{\Omega_{n,i}^c} \int_0^{\ell_n} \frac{1}{\lambda^j} \mathbf{sat}_{\text{loc}}(a_n \lambda^j v_n^j) v_n^j \\ &\geq \sum_{n=1}^N \int_{\Omega_{n,i}^c} \int_0^{\ell_n} \frac{1}{\lambda^j} \mathbf{sat}_{\text{loc}}(a_n \lambda^j v_n^j) v_n^j \\ &\geq \sum_{n=1}^N \int_{\Omega_{n,i}^c} \int_0^{\ell_n} a_n k_n(R) (v_n^j)^2 dx dt, \end{aligned}$$

which gives us, using (3.3.5),

$$(2\alpha - N) \|v_1^j(t, 0)\|_{L^2(0, T)}^2 + \|\partial_x v_1^j(t, 0)\|_{L^2(0, T)}^2 + \sum_{n=1}^N \int_{\Omega_{n,i}^c} \int_0^{\ell_n} a_n k_n(R) (v_n^j)^2 dx dt \rightarrow 0. \quad (3.3.15)$$

Thus, the limit function v satisfies $(2\alpha - N)v_1(t, 0) = \partial_x v_1(t, 0) = 0$ in $(0, T)$ for all $n = 1, \dots, N$ and $v_n(t, x) = 0$ in $\cup_{i \in \mathbb{N}} \Omega_{n,i}^c \times \omega_n$. Using (3.3.14), we know that $\nu(\cup_{i \in \mathbb{N}} \Omega_{n,i}^c) = T$, thus we get that, for almost every $t \in [0, T]$ $v_n(t, x) = 0$ for $x \in \omega_n$. Lastly \underline{v} is a solution to (3.3.11) and we conclude as in the case $\mathbf{sat} = \mathbf{sat}_2$.

Finally, we obtain that (Obs-S) is valid for a solution (KdV-S) with $\|\underline{u}\|_{\mathbb{L}^2(\mathcal{T})} \leq R$ and Theorem 3.1.4 follows as in Section 1.2.3.

3.3.2 Stability LKdV-S

Now we study the stabilization of (LKdV-S). For doing that, we follow the approach of Section 3.3.1, and we prove the following observability inequality

$$\begin{aligned} \|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})}^2 &\leq C \left((2\alpha - N) \int_0^T (u_1(t, 0))^2 dt + \sum_{j=1}^N \int_0^T (\partial_x u_n(t, 0))^2 dt \right. \\ &\left. + \sum_{j \in I_c^*} \int_0^T \int_0^{\ell_n} \mathbf{sat}(a_n u_n) u_n dx dt \right), \end{aligned} \quad (\text{Obs-S2})$$

for the unique solution \underline{u} of (LKdV-S). Suppose that it is false, then there exists a sequence $(\underline{u}^{0,j})_{j \in \mathbb{N}} \subset \mathbb{L}^2(\mathcal{T})$ such that $\|\underline{u}^{0,j}\|_{\mathbb{L}^2(\mathcal{T})} = 1$ and the corresponding solution of (LKdV-S) satisfies,

$$(2\alpha - N) \|u_1^j(\cdot, 0)\|_{L^2(0, T)}^2 + \|\partial_x \underline{u}^j(\cdot, 0)\|_{L^2(0, T)}^2 + \sum_{n \in I_c^*} \int_0^T \int_0^{\ell_n} \mathbf{sat}(a_n u_n^j) u_n^j dx dt \rightarrow 0.$$

as $j \rightarrow \infty$. Using the same arguments as in Theorem 3.1.4 we can find a non-trivial solution $\underline{v} \in \mathbb{B}_T$ of (LKdV-S) such that

$$\begin{cases} (2\alpha - N) \|v_1(\cdot, 0)\|_{L^2(0, T)} = 0, \\ \|\partial_x \underline{v}(\cdot, 0)\|_{L^2(0, T)} = 0, \\ v_n = 0, \quad \text{in } (0, T) \times \omega_n, \quad n \in I_c^*, \\ \|\underline{v}\|_{L^2(0, T; \mathbb{L}^2(\mathcal{T}))} = 1. \end{cases}$$

We distinguish three cases

- For $n \in I_c^*$, $v_n = 0$ in $(0, T) \times \omega_n$. Then, $\partial_t v_n + \partial_x v_n + \partial_x^3 v_n = 0$ and thanks to Holmgren's Theorem, $v_n = 0$ for all $n \in I_c^*$. Note that this implies that $v_n(t, 0) = 0$ for all $n \in I_c^*$ and by the continuity condition $v_n(t, 0) = 0$ for all $n = 1, \dots, N$.
- For $n \in \{1, \dots, N\} \setminus I_c$, v_n is solution to

$$\begin{cases} (\partial_t v_n + \partial_x v_n + \partial_x^3 v_n)(t, x) = 0, & x \in (0, \ell_n), \quad t \in (0, T), \quad n = 1, \dots, N, \\ v_n(t, 0) = 0, & t \in (0, T), \quad \forall n = 1, \dots, N, \\ \sum_{n=1}^N \partial_x^2 v_n(t, 0) = 0, & t \in (0, T), \\ v_n(t, \ell_n) = \partial_x v_n(t, \ell_n) = 0, & t \in (0, T), \quad n = 1, \dots, N, \\ v_n(0, x) = v_n^0, & x \in (0, \ell_n). \end{cases}$$

Then thanks [AC18, Lemma 3.2], $v_n \equiv 0$.

- For $n \in I_c \setminus I_c^*$, then v_n satisfies

$$\begin{cases} (\partial_t v_n + \partial_x v_n + \partial_x^3 v_n)(t, x) = 0, & t \in (0, T), \quad \forall x \in (0, \ell_n), \\ v_n(t, 0) = \partial_x v_n(t, 0) = \partial_x^2 v_n(t, 0) = 0, & t \in (0, T) \\ v_n(t, \ell_n) = \partial_x v_n(t, \ell_n) = 0, & t \in (0, T) \\ v_n(0, x) = v_n^0 & x \in (0, \ell_n). \end{cases}$$

Due to the three null conditions at the central node, we obtain that $v_n = 0$.

Thus $\underline{v} \equiv \underline{0}$ and we get a contradiction, with $\|\underline{v}\|_{L^2(0, T; \mathbb{L}^2(\mathcal{T}))} = 1$ which ends the proof of (Obs-S2). As we have the observability inequality (Obs-S2), to derive the exponential decay of the energy of the solution of (LKdV-S) given in Theorem 3.1.5, it is enough to follow the proof of Theorem 3.1.4.

STABILITY ANALYSIS OF A KdV EQUATION WITH SATURATED BOUNDARY FEEDBACK

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In this chapter, the stability analysis of the linear and nonlinear Korteweg-de Vries equations in presence of a saturated feedback actuator is studied. The well-posedness is derived by using nonlinear semigroup results, Schauder's and Banach fixed-point theorems. The exponential stability is shown thanks to an observability inequality, which is obtained via contradiction and compactness arguments. Finally, an alternative proof of the asymptotic stability of the linear Korteweg-de Vries equation is presented. This chapter is based on the paper:

- Hugo Parada. Stability analysis of a Korteweg-de Vries equation with saturated boundary feedback. *IFAC-PapersOnLine*, 55(26):1–6, 2022. 4th IFAC Workshop on Control of Systems Governed by Partial Differential Equations CPDE 2022.

4.1 Introduction

In [MCPA17] the saturated internal stabilization of a single KdV equation was studied and recently in [PCP22b] these ideas were extended to the network case. In our best knowledge, there are no work dealing with the stability of a KdV equation with saturated feedback on the boundary, thus this chapter gives a sake of completeness on this topic. The plan is to follow the ideas presented in

[MCPA17] and [PCP22b] to prove the stability of the KdV equation with saturated actuator. To be more specific, let us define the following saturation map:

$$\mathbf{sat}(s) = \begin{cases} s, & \text{if } |s| \leq M, \\ M \mathbf{sgn}(s), & \text{if } |s| \geq M, \end{cases} \quad (4.1.1)$$

where $M > 0$ is the saturation level and \mathbf{sgn} is the sign function. In this chapter, we are going to consider the following equations

$$\begin{cases} (\partial_t u + \partial_x u + \partial_x^3 u + u \partial_x u)(t, x) = 0, & t > 0, x \in (0, L), \\ u(t, 0) = u(t, L) = 0, & t > 0, \\ \partial_x u(t, L) = \mathbf{sat}(\alpha \partial_x u(t, 0)), & t > 0, \\ u(0, x) = u_0(x), & x \in (0, L). \end{cases} \quad (\text{KdVs})$$

and

$$\begin{cases} (\partial_t u + \partial_x u + \partial_x^3 u)(t, x) = 0, & t > 0, x \in (0, L), \\ u(t, 0) = u(t, L) = 0, & t > 0, \\ \partial_x u(t, L) = \mathbf{sat}(\alpha \partial_x u(t, 0)), & t > 0, \\ u(0, x) = u_0(x), & x \in (0, L). \end{cases} \quad (\text{KdVLs})$$

Note that both (KdVs) and (KdVLs) are nonlinear equations due to the action of the saturation. Keep in mind that (KdVLs) is just (KdVs) but without the nonlinear internal term $u \partial_x u$. The next theorem is the main result of this chapter,

Theorem 4.1.1. *Let $L \notin \mathcal{N}$ and $|\alpha| < 1$, then there exist $C > 0$ and $\mu > 0$ such that:*

- For all $u_0 \in L^2(0, L)$, the unique mild solution $u \in C([0, \infty), L^2(0, L)) \cap L^2((0, \infty), H^1(0, L))$ of (KdVLs) satisfies $\|u(t, \cdot)\|_{L^2(0, L)} \leq C e^{-\mu t} \|u_0\|_{L^2(0, L)}$.
- For all $u_0 \in L^2(0, L)$, satisfying $\|u_0\|_{L^2(0, L)} \leq \epsilon$ for some $\epsilon > 0$, the unique mild solution $u \in C([0, \infty), L^2(0, L)) \cap L^2((0, \infty), H^1(0, L))$ of (KdVs) satisfies $\|u(t, \cdot)\|_{L^2(0, L)} \leq C e^{-\mu t} \|u_0\|_{L^2(0, L)}$.

The chapter is organized as follows. Section 3.2 is devoted to prove the well-posedness results associated to (KdVLs) and (KdVs). In Section 4.3 we study the stability analysis of (KdVLs) and (KdVs) using an observability inequality. In Section 4.4 we show the asymptotic stability of (KdVLs) by using LaSalle's invariance principle. From now on, $C > 0$ will be a generic positive constant.

4.2 Well-posedness results

The goal of this section is to prove the appropriate well-posedness results for (KdVLs) and (KdVs). The proof for (KdVLs) is based on semigroup theory. For (KdVs) we use the result for (KdVLs), then we will add an internal source term in order to take in account the nonlinear term $u \partial_x u$, and derive some multiplier estimates to obtain the needed regularity.

Consider the operator $\mathcal{A} : D(\mathcal{A}) \rightarrow L^2(0, L)$ defined by

$$\mathcal{A}w := -w' - w''' \quad (4.2.1)$$

where

$$D(\mathcal{A}) = \{w \in H^3(0, L) \cap H_0^1(0, L) : w'(L) = \mathbf{sat}(\alpha w'(0))\}.$$

Well-posedness demands to handle the fact that the operator \mathcal{A} is nonlinear. We follow the approach presented in [MCPA15, PTdSJ16], i.e. we show that \mathcal{A} is a closed, dissipative operator and that for some $\lambda > 0$, $D(\mathcal{A}) \subset R(\mathcal{I} - \lambda \mathcal{A})$, where R denotes the range of the operator.

4.2.1 Well-posedness of (KdVLs)

In this part, we focus on the study of (KdVLs), in particular we prove the following result.

Proposition 4.2.1. *Let $u_0 \in L^2(0, L)$ and $|\alpha| < 1$, then there exists a unique mild solution $u \in C([0, \infty), L^2(0, L))$ of (KdVLs). Moreover, if $u_0 \in D(\mathcal{A})$, this solution is classical and $u \in C([0, \infty), D(\mathcal{A})) \cap C^1((0, \infty), L^2(0, L))$.*

Proof. By [Ros97] the operator $\mathcal{A}_0 : D(\mathcal{A}_0) \rightarrow L^2(0, L)$, $D(\mathcal{A}_0) = \{w \in H^3(0, L) \cap H_0^1(0, L) : w'(L) = 0\}$ such that $\mathcal{A}_0 w = \mathcal{A}w$ is closed and by [KG02, Page 91] the operator \mathfrak{sat} satisfies

$$\forall (s, \tilde{s}) \in \mathbb{R}^2, \quad |\mathfrak{sat}(s) - \mathfrak{sat}(\tilde{s})| \leq |s - \tilde{s}|, \quad (4.2.2)$$

Thus, \mathfrak{sat} is a Lipschitz continuous function and hence the operator \mathcal{A} is closed. Now, let us prove that \mathcal{A} is dissipative, let $w, v \in D(\mathcal{A})$, then we get

$$\begin{aligned} \langle \mathcal{A}w - \mathcal{A}v, w - v \rangle_{L^2(0, L)} &= \int_0^L (-w' - w''' + v' + v''')(w - v) dx \\ &= - \int_0^L (w - v)'(w - v) dx - \int_0^L (w - v)'''(w - v) dx. \end{aligned}$$

Thus, using integration by parts and that $w, v \in D(\mathcal{A})$ we get

$$\begin{aligned} \langle \mathcal{A}w - \mathcal{A}v, w - v \rangle_{L^2(0, L)} &= -\frac{1}{2}[(w - v)']_0^L \\ &= \frac{1}{2}(\mathfrak{sat}(\alpha w'(0)) - \mathfrak{sat}(\alpha v'(0)))^2 - \frac{1}{2}(w'(0) - v'(0))^2. \end{aligned}$$

Therefore, denoting

$$C_\alpha = \frac{1 - |\alpha|^2}{2} > 0 \quad (4.2.3)$$

which is positive by $|\alpha| < 1$, and using (4.2.2)

$$\langle \mathcal{A}w - \mathcal{A}v, w - v \rangle_{L^2(0, L)} \leq -C_\alpha |w'(0) - v'(0)|^2 \leq 0.$$

Thus, we get \mathcal{A} dissipative. Now, we are going to prove that, for $\lambda > 0$ small enough $D(\mathcal{A}) \subset R(\mathcal{I} - \lambda \mathcal{A})$. In other words, for each $w \in D(\mathcal{A})$, there exists $v \in D(\mathcal{A})$ such that $(\mathcal{I} - \lambda \mathcal{A})v = w$. Let $w \in D(\mathcal{A})$ and $\tilde{\lambda} = \frac{1}{\lambda}$ we are looking for solutions of

$$\begin{cases} \tilde{\lambda}v + v' + v''' = \tilde{\lambda}w, \\ v(0) = v(L) = 0, \quad v'(L) = \mathfrak{sat}(\alpha v'(0)). \end{cases} \quad (4.2.4)$$

Consider the map $\mathcal{T} : H^2(0, L) \rightarrow L^2(0, L)$ defined by $\mathcal{T}(u) = z$, where z is the solution of

$$\begin{cases} \tilde{\lambda}z + z' + z''' = \tilde{\lambda}w, \\ z(0) = z(L) = 0, \quad z'(L) = \mathfrak{sat}(\alpha z'(0)). \end{cases} \quad (4.2.5)$$

Note that the operator \mathcal{T} is well-defined, that is, for all $u \in H^2(0, L)$ there exists a unique solution to (4.2.5). Indeed, let $\psi(x) = \frac{\mathfrak{sat}(\alpha u'(0))L}{2\pi} \sin\left(\frac{2\pi x}{L}\right)$. Then $\varphi = z - \psi$ satisfies

$$\begin{cases} \tilde{\lambda}\varphi + \varphi' + \varphi''' = f = \tilde{\lambda}w - \tilde{\lambda}\psi - \psi' - \psi''', \\ \varphi(0) = \varphi(L) = \varphi'(L) = 0. \end{cases}$$

that can be written as $(\mathcal{A}_0 - \tilde{\lambda}I)\varphi = -f$. But as $\tilde{\lambda} > 0$ then $\tilde{\lambda} \notin \sigma(\mathcal{A}_0)$ (see for instance [Ros97]). Thus $(\mathcal{A}_0 - \tilde{\lambda}I)$ is invertible, and we conclude that φ is well-defined and hence z . Now, we show that \mathcal{T} has a fixed-point. Let $C > 0$ to be defined after and consider the set

$$K_C = \{w \in H_0^1(0, L) : \|w\|_{H_0^1(0, L)} \leq C\}. \quad (4.2.6)$$

Note that by the Rellich-Kondrachov theorem [Bre10, Theorem 9.19, Page 285] we know that the inclusion $H_0^1(0, L) \hookrightarrow L^2(0, L)$ is compact and then K_C is a bounded set in $H_0^1(0, L)$ and relatively compact in $L^2(0, L)$. Moreover, as K_C is closed in $L^2(0, L)$ we conclude that K_C is compact in $L^2(0, L)$. Then in order to apply Schauder fixed-point theorem [Cor09, Theorem B.19], we need to find $C > 0$ such that $\mathcal{T}(H^2(0, L)) \subset K_C$.

Lemma 4.2.1. *There exists $C > 0$ such that $\mathcal{T}(H^2(0, L)) \subset K_C$.*

Proof of Lemma 4.2.1:

Multiplying the first line of (4.2.5) by z and integrating on $(0, L)$ we get,

$$\tilde{\lambda} \int_0^L z^2 dx + \int_0^L z' z dx + \int_0^L z''' z dx = \tilde{\lambda} \int_0^L w z dx. \quad (4.2.7)$$

integrating by parts we yield

$$\tilde{\lambda} \int_0^L z^2 dx + \frac{1}{2} z'(0)^2 = \frac{1}{2} \text{sat}(\alpha u'(0))^2 + \tilde{\lambda} \int_0^L w z dx.$$

Then, using Cauchy-Schwarz inequality and the definition of the saturation (4.1.1), we get

$$\frac{\tilde{\lambda}}{2} \|z\|_{L^2(0, L)}^2 \leq \frac{M^2}{2} + \frac{\tilde{\lambda}}{2} \|w\|_{L^2(0, L)}^2. \quad (4.2.8)$$

Similarly, multiplying the first line of (4.2.5) by xz and integrating on $(0, L)$ we get

$$\tilde{\lambda} \int_0^L x z^2 dx + \int_0^L x z' z dx + \int_0^L x z''' z dx = \tilde{\lambda} \int_0^L x w z dx. \quad (4.2.9)$$

Performing integration by parts and by the second line of (4.2.5) we obtain,

$$\tilde{\lambda} \int_0^L x z^2 dx + \frac{3}{2} \int_0^L (z')^2 dx = \frac{1}{2} \int_0^L z^2 dx + \frac{L}{2} \text{sat}(\alpha u'(0))^2 + \tilde{\lambda} \int_0^L x w z dx,$$

which yields

$$\frac{3}{2} \|z'\|_{L^2(0, L)}^2 \leq \left(\frac{L\tilde{\lambda}}{2} + \frac{1}{2} \right) \|z\|_{L^2(0, L)}^2 + \frac{LM^2}{2} + \frac{L\tilde{\lambda}}{2} \|w\|_{L^2(0, L)}^2, \quad (4.2.10)$$

and hence using (4.2.8) we deduce, $\|z'\|_{L^2(0, L)}^2 \leq \tilde{C}$, where $\tilde{C} > 0$ only depends on the level of saturation M , L , $\tilde{\lambda}$ and u which finishes the proof of Lemma 4.2.1. \square

Then, applying Schauder fixed-point theorem to \mathcal{T} , we find the solution $v \in H^2(0, L)$ of (4.2.4). Moreover, as $w \in D(\mathcal{A})$, then $v \in H^3(0, L)$ and hence by the boundary conditions $v \in D(\mathcal{A})$. We conclude that for all $\lambda > 0$, $D(\mathcal{A}) \subset R(\mathcal{A} - \lambda \mathcal{I})$. Using [Miy92, Theorem 4.2, Page 77] \mathcal{A} is the generator of a semigroup of contractions $S(t)$, $t > 0$ and by [Miy92, Theorem 4.2, Page 81] (or [Bar76, Theorem 1.2, Page 102]) for all $T > 0$, $u_0 \in D(\mathcal{A})$, $u = S(t)u_0 \in C([0, T]; D(\mathcal{A})) \cap C^1((0, T); L^2(0, L))$ is a classical solution to (KdVLS). Moreover, by [Miy92, Theorem 4.10, Page 87] this solution is unique. Also, we can deduce that for all $T > 0$, for all $u_0 \in L^2(0, L)$, there exists a unique mild solution of (KdVLS) $u \in C([0, T]; L^2(0, L))$ and then, the proof of Proposition 4.2.1 is finished. \square

4.2.2 Well-posedness of equation (KdVLS) with a source term

Now, we deal with the well-posedness of the system (KdVLS) with an extra source term f

$$\begin{cases} \partial_t u(t, x) + \partial_x u(t, x) + \partial_x^3 u(t, x) = f(t, x), & t > 0, x \in (0, L), \\ u(t, 0) = u(t, L) = 0, & t > 0, \\ \partial_x u(t, L) = \mathfrak{sat}(\alpha \partial_x u(t, 0)), & t > 0, \\ u(0, x) = u_0(x), & x \in (0, L). \end{cases} \quad (4.2.11)$$

Formally, this system can be written as $\partial_t u = \mathcal{A}u + f$. The following result can be obtained using [Pav87, Theorem 10.1, Page 129] and [Pav87, Theorem 10.1, Page 132]

Proposition 4.2.2. *Let $u_0 \in L^2(0, L)$, $|\alpha| < 1$, $f \in L^1((0, \infty), L^2(0, L))$, then there exists a unique mild solution $u \in C([0, \infty), L^2(0, L))$ of (4.2.11). Moreover, if $u_0 \in D(\mathcal{A})$ and $f \in W^{1,1}((0, \infty), L^2(0, L))$ this solution is classical and $u \in C([0, \infty), L^2(0, L))$, $u(t) \in D(\mathcal{A})$, for all $t \geq 0$ and $u \in C^1((0, \infty), L^2(0, L))$.*

Our idea now is to show that, indeed, the unique mild solution of (4.2.11) has the extra-regularity $u \in B_T$.

Proposition 4.2.3. *Let $u_0 \in L^2(0, L)$, $|\alpha| < 1$, $f \in L^1((0, T), L^2(0, L))$ and $T > 0$, consider $u \in C([0, T], L^2(0, L))$, the restriction of the unique mild solution of (4.2.11) on $[0, T]$, then $u \in B_T$. Moreover, the following estimate holds,*

$$\|u\|_{B_T} \leq C(1 + \sqrt{T}) (\|u_0\|_{L^2(0, L)} + \|f\|_{L^1((0, T), L^2(0, L))}). \quad (4.2.12)$$

Proof. Suppose u is a classical solution of (4.2.11), let $T > 0$ and $0 \leq s \leq T$. Several computations are quite similar to those performed in the proof of Lemma 4.2.1 and for that reason are omitted here. Multiplying the first line of (4.2.11) by u and integrating on $(0, L) \times (0, s)$, we get

$$\int_0^s \int_0^L \partial_t u u dx dt + \int_0^s \int_0^L \partial_x u u dx dt + \int_0^s \int_0^L \partial_x^3 u u dx dt = \int_0^s \int_0^L f u dx dt. \quad (4.2.13)$$

Note that, by integration by parts, this expression becomes

$$\frac{1}{2} \int_0^L (u(s, x))^2 dx + \frac{1}{2} \int_0^s ((\partial_x u(t, 0))^2 - (\mathfrak{sat}(\alpha \partial_x u(t, 0)))^2) dt = \frac{1}{2} \int_0^L (u_0)^2 dx + \int_0^s \int_0^L f u dx dt. \quad (4.2.14)$$

First, note that for all $s \in [0, T]$

$$\int_0^s \int_0^L f u dx dt \leq \int_0^T \|u(t, \cdot)\|_{L^2(0, L)} \|f(t, \cdot)\|_{L^2(0, L)} dt \leq \|u\|_{C([0, T], L^2(0, L))} \|f\|_{L^1((0, T), L^2(0, L))}. \quad (4.2.15)$$

Now, using (4.1.1), (4.2.14) and (4.2.15) we get for all $s \in [0, T]$,

$$\frac{1}{2} \int_0^L (u(s, x))^2 dx + C_\alpha \int_0^s (\partial_x u(t, 0))^2 dt \leq \frac{1}{2} \int_0^L (u_0)^2 dx + \|u\|_{C([0, T], L^2(0, L))} \|f\|_{L^1((0, T), L^2(0, L))}. \quad (4.2.16)$$

Taking the supremum for $s \in [0, T]$, and with Young's inequality, we get

$$\|u\|_{C([0, T], L^2(0, L))}^2 \leq C \left(\|u_0\|_{L^2(0, L)}^2 + \|f\|_{L^1((0, T), L^2(0, L))}^2 \right). \quad (4.2.17)$$

Moreover, from (4.2.16) taking $s = T$, and using (4.2.17) we can derive the following hidden regularity

$$\int_0^T (\partial_x u(t, 0))^2 dt \leq C \left(\|u_0\|_{L^2(0, L)}^2 + \|f\|_{L^1((0, T), L^2(0, L))}^2 \right). \quad (4.2.18)$$

Now, we multiply the first line of (4.2.11) by xu and integrate on $(0, L) \times (0, T)$ and we get

$$\int_0^T \int_0^L x \partial_t u u dx dt + \int_0^T \int_0^L x \partial_x u u dx dt + \int_0^T \int_0^L x \partial_x^3 u u dx dt = \int_0^T \int_0^L x f u dx dt.$$

Thus, integrating by parts we obtain

$$\begin{aligned} \frac{1}{2} \int_0^L x (u(T, x))^2 dx + \frac{3}{2} \int_0^T \int_0^L (\partial_x u)^2 dx dt &= \frac{1}{2} \int_0^L x (u_0)^2 dx + \frac{1}{2} \int_0^T \int_0^L (u)^2 dx dt \\ &+ \frac{L}{2} \int_0^T (\mathbf{sat}(\alpha \partial_x u(t, 0)))^2 dt + \int_0^T \int_0^L x f u dx dt. \end{aligned} \quad (4.2.19)$$

Note that

$$\begin{aligned} \int_0^T \int_0^L (u)^2 dx dt &\leq \int_0^T \left(\sup_{t \in [0, T]} \int_0^L (u(t, \cdot))^2 dx \right) dt \\ &= T \|u\|_{C([0, T], L^2(0, L))}^2. \end{aligned} \quad (4.2.20)$$

Moreover, by (4.1.1) and (4.2.18)

$$\int_0^T (\mathbf{sat}(\alpha \partial_x u(t, 0)))^2 dt \leq C \left(\|u_0\|_{L^2(0, L)}^2 + \|f\|_{L^1((0, T), L^2(0, L))}^2 \right),$$

and by (4.2.15) and (4.2.17)

$$\int_0^T \int_0^L x f u dx dt \leq C \left(\|u_0\|_{L^2(0, L)}^2 + \|f\|_{L^1((0, T), L^2(0, L))}^2 \right).$$

Therefore, joining these estimates, we deduce from (4.2.19)

$$\|\partial_x u\|_{L^2((0, T), L^2(0, L))}^2 \leq C(1 + T) \left(\|u_0\|_{L^2(0, L)}^2 + \|f\|_{L^1((0, T), L^2(0, L))}^2 \right), \quad (4.2.21)$$

and thus $u \in L^2((0, T), H^1(0, L))$. Mixing (4.2.17) and (4.2.21) we deduce (4.2.12). The result can be extended by density to mild solutions. \square

4.2.3 Well-posedness of the nonlinear system

Now, we are ready to give the proof of the well-posedness result for (KdVs). For that the next proposition we use the Proposition 1.2.1. The main result of this subsection is the following one.

Proposition 4.2.4. *Let $u_0 \in L^2(0, L)$ and $|\alpha| < 1$, then there exists a unique mild solution $u \in C([0, \infty), L^2(0, L)) \cap L^2((0, \infty), H^1(0, L))$ of (KdVs).*

Proof. Let $u_0 \in D(\mathcal{A})$ and $R, \theta > 0$ to be chosen later. Consider the closed ball $B_{B_\theta}(0, R) := \{z \in B_\theta, \|z\|_{B_\theta} \leq R\}$. Then, $B_{B_\theta}(0, R)$ is a complete metric space. Let the map $\Phi : B_\theta \rightarrow B_\theta$ defined for $z \in B_\theta$ by $\Phi(z) = u$, where u is the solution of

$$\begin{cases} (\partial_t u + \partial_x u + \partial_x^3 u)(t, x) = -z(t, x) \partial_x z(t, x), & t > 0, x \in (0, L), \\ u(t, 0) = u(t, L) = 0, & t > 0, \\ \partial_x u(t, L) = \mathbf{sat}(\alpha \partial_x u(t, 0)), & t > 0, \\ u(0, x) = u_0(x), & x \in (0, L). \end{cases}$$

Note first that u is well-defined, since by Proposition 1.2.1, $z\partial_x z \in L^1((0, \theta), L^2(0, L))$. Clearly, $u \in B_\theta$ is solution of (KdVs) if u is a fixed-point of Φ . From Proposition 4.2.3 and Proposition 1.2.1, we get for all $z \in B_\theta$

$$\begin{aligned} \|\Phi(z)\|_{B_\theta} &= \|u\|_{B_\theta} \leq C(1 + \theta^{1/2}) (\|u_0\|_{L^2(0, L)} + \|z\partial_x z\|_{L^1((0, T), L^2(0, L))}) \\ &\leq C_1(1 + \theta^{1/2})\|u_0\|_{L^2(0, L)} + C_2(1 + \theta^{1/2})\theta^{1/4}\|z\|_{B_\theta}^2, \end{aligned}$$

for some $C_1, C_2 > 0$. Then taking $R = 3C_1\|u_0\|_{L^2(0, L)}$ and $\theta > 0$ small enough such that $C_2(1 + \theta^{1/2})\theta^{1/4}R < \frac{1}{3}$ and $(1 + \theta^{1/2}) < 2$, we have that $\|\Phi(z)\|_{B_\theta} \leq R$. Thus, Φ maps the ball $B_{B_\theta}(0, R)$ into itself. Now, take $z_1, z_2 \in B_{B_\theta}(0, R)$, $u_1 = \Phi(z_1)$, $u_2 = \Phi(z_2)$, then $w = \Phi(z_1) - \Phi(z_2)$ satisfies

$$\begin{cases} \partial_t w(t, x) + \partial_x w(t, x) + \partial_x^3 w(t, x) = f(t, x), & t > 0, x \in (0, L), \\ w(t, 0) = w(t, L) = 0, & t > 0, \\ \partial_x w(t, L) = h(t), & t > 0, \\ w(0, x) = 0, & x \in (0, L). \end{cases} \quad (4.2.22)$$

where $f = -z_1\partial_x z_1 + z_2\partial_x z_2$ and $h = \mathbf{sat}(\alpha\partial_x u_1(t, 0)) - \mathbf{sat}(\alpha\partial_x u_2(t, 0))$. First, note that by (4.2.18), $h \in L^2(0, \theta)$ and hence (4.2.22) is well-posed [Cer14, Proposition 3]. Moreover,

$$\|w\|_{B_\theta} \leq C (\|h\|_{L^2(0, \theta)} + \|f\|_{L^1((0, \theta), L^2(0, L))}). \quad (4.2.23)$$

We can not use directly the estimates obtained in Section 4.2.2, because $h \neq \mathbf{sat}(\alpha\partial_x w(t, 0))$, but similar estimates can be obtained. If we multiply the first line of (4.2.22) by w and integrate on $(0, L) \times (0, \theta)$, we get

$$\int_0^\theta \int_0^L \partial_t w w dx dt + \int_0^\theta \int_0^L \partial_x w w dx dt + \int_0^\theta \int_0^L \partial_x^3 w w dx dt = \int_0^\theta \int_0^L f w dx dt. \quad (4.2.24)$$

Similar computations as in (4.2.14) show that

$$\frac{1}{2} \int_0^L (w(\theta, x))^2 dx + \frac{1}{2} \int_0^\theta (\partial_x w(t, 0))^2 dt - \frac{1}{2} \int_0^\theta (h(t))^2 dt = \int_0^\theta \int_0^L f w dx dt. \quad (4.2.25)$$

Since $h = \mathbf{sat}(\alpha\partial_x u_1(t, 0)) - \mathbf{sat}(\alpha\partial_x u_2(t, 0))$, using (4.2.2) we derive,

$$\begin{aligned} \|h\|_{L^2(0, \theta)} &= \|\mathbf{sat}(\alpha\partial_x u_1(t, 0)) - \mathbf{sat}(\alpha\partial_x u_2(t, 0))\|_{L^2(0, \theta)} \\ &\leq \|\alpha\partial_x u_1(\cdot, 0) - \alpha\partial_x u_2(\cdot, 0)\|_{L^2(0, \theta)} = |\alpha| \|\partial_x w(\cdot, 0)\|_{L^2(0, \theta)}. \end{aligned} \quad (4.2.26)$$

Thus, plugging (4.2.26) in (4.2.25), we obtain

$$\frac{1}{2} \int_0^L (w(\theta, x))^2 dx + C_\alpha \int_0^\theta (\partial_x w(t, 0))^2 dt \leq \int_0^\theta \int_0^L f w dx dt, \quad (4.2.27)$$

which implies using Young's inequality

$$\|w\|_{C([0, \theta], L^2(0, L))} \leq C \|f\|_{L^1((0, \theta), L^2(0, L))}. \quad (4.2.28)$$

Using (4.2.28) in (4.2.27), we get

$$\int_0^\theta (\partial_x w(t, 0))^2 dt \leq C \|f\|_{L^1((0, \theta), L^2(0, L))}^2,$$

and then (4.2.26) brings

$$\|h\|_{L^2(0, \theta)} \leq C \|f\|_{L^1((0, \theta), L^2(0, L))},$$

which transforms (4.2.23) into

$$\|w\|_{B_\theta} \leq C\|f\|_{L^1((0,\theta),L^2(0,L))}. \quad (4.2.29)$$

Finally, by (4.2.29) and Proposition 1.2.1 we conclude:

$$\|\Phi(z_1) - \Phi(z_2)\|_{B_\theta} = \|w\|_{B_\theta} \leq C_3\theta^{1/4}R\|z_1 - z_2\|_{B_\theta},$$

for some $C_3 > 0$. Then taking $\theta > 0$, small enough such that $C_3\theta^{1/4}R < 1$, we get that Φ is a contraction, and we conclude by using the Banach fixed-point theorem [Bre10, Theorem 5.7] and density argument to extend the result for all $u_0 \in L^2(0, L)$. Thus, we have proved the following local in-time well-posedness result:

Proposition 4.2.5. *Let $u_0 \in L^2(0, L)$, $|\alpha| < 1$ and $T > 0$. Then there exists $0 < T^* \leq T$ and a unique mild solution $u \in B_{T^*}$ of (KdVs) on the time interval $[0, T^*]$. Moreover, there exists $C > 0$ such that $\|u\|_{B_{T^*}} \leq C\|u_0\|_{L^2(0,L)}$.*

To conclude the proof of the global well-posedness result stated on Proposition 4.2.4, we need some a priori estimates, based on a multiplier approach, similar to those presented in Section 4.2.2. Let $T > 0$ and u a classical solution of (KdVs). Multiplying the first line of (KdVs) by u , integrating on $(0, L) \times (0, s)$ we get

$$\int_0^s \int_0^L \partial_t u u dx dt + \int_0^s \int_0^L \partial_x u u dx dt + \int_0^s \int_0^L \partial_x^3 u u dx dt + \int_0^s \int_0^L \partial_x u u^2 dx dt = 0. \quad (4.2.30)$$

We derive using integration by parts

$$\frac{1}{2} \int_0^L (u(s, x))^2 dx + \frac{1}{2} \int_0^s ((\partial_x u(t, 0))^2 - (\text{sat}(\alpha \partial_x u(t, 0)))^2) dt = \frac{1}{2} \int_0^L (u_0)^2 dx. \quad (4.2.31)$$

Using the definition of the saturation (4.1.1) we derive

$$\frac{1}{2} \int_0^L (u(s, x))^2 dx + C_\alpha \int_0^s (\partial_x u(t, 0))^2 dt \leq \frac{1}{2} \int_0^L (u_0)^2 dx. \quad (4.2.32)$$

Then, taking the supremum for $s \in [0, T]$ in (4.2.32),

$$\|u\|_{C([0,T],L^2(0,L))}^2 \leq \|u_0\|_{L^2(0,L)}^2. \quad (4.2.33)$$

On the other hand, taking $s = T$ in (4.2.32),

$$\int_0^T (\partial_x u(t, 0))^2 dt \leq \frac{1}{2} C_\alpha^{-1} \|u_0\|_{L^2(0,L)}^2. \quad (4.2.34)$$

Now, we multiply the first line of (KdVs) by xu and integrate on $(0, L) \times (0, T)$, obtaining

$$\int_0^T \int_0^L x \partial_t u u dx dt + \int_0^T \int_0^L x \partial_x u u dx dt + \int_0^T \int_0^L x \partial_x^3 u u dx dt + \int_0^T \int_0^L x u^2 \partial_x u dx dt = 0. \quad (4.2.35)$$

Hence, we infer after integration by parts

$$\begin{aligned} \frac{1}{2} \int_0^L x (u(T, x))^2 dx + \frac{3}{2} \int_0^T \int_0^L (\partial_x u)^2 dx dt &= \frac{1}{2} \int_0^L x (u_0)^2 dx + \frac{1}{2} \int_0^T \int_0^L u^2 dx dt \\ &+ \frac{L}{2} \int_0^T (\text{sat}(\alpha \partial_x u(t, 0)))^2 dt + \frac{1}{3} \int_0^T \int_0^L u^3 dx dt. \end{aligned} \quad (4.2.36)$$

For the term involving $\int_0^L u^3(t, x)dx$, by the injection of $H_0^1(0, L)$ into $L^\infty(0, L)$ we know that $\|u\|_{L^\infty(0, L)} \leq C\|\partial_x u\|_{L^2(0, L)}$, for some $C > 0$ then

$$\begin{aligned} \int_0^T \int_0^L (u(t, x))^3 dx dt &\leq \int_0^T \|u\|_{L^\infty(0, L)} \|u\|_{L^2(0, L)}^2 dt \\ &\leq C \int_0^T \|\partial_x u\|_{L^2(0, L)} \|u\|_{L^2(0, L)}^2 dt, \end{aligned}$$

which implies using (4.2.33)

$$\int_0^T \int_0^L (u(t, x))^3 dx dt \leq C\sqrt{T}\|u_0\|_{L^2(0, L)}^2 \|\partial_x u\|_{L^2((0, T), L^2(0, L))}. \quad (4.2.37)$$

Thus, from (4.2.36) using (4.2.20), (4.2.33), (4.2.34) and (4.2.37) we derive

$$\|\partial_x u\|_{L^2((0, T), L^2(0, L))}^2 \leq C(1+T)\|u_0\|_{L^2(0, L)}^2 + C\sqrt{T}\|u_0\|_{L^2(0, L)}^2 \|\partial_x u\|_{L^2((0, T), L^2(0, L))}.$$

Using Young's inequality

$$\|\partial_x u\|_{L^2((0, T), L^2(0, L))}^2 \leq C(1+T)\|u_0\|_{L^2(0, L)}^2 + CT\|u_0\|_{L^2(0, L)}^4. \quad (4.2.38)$$

Finally, with (4.2.33) and (4.2.38) and arguing by density, the proof of Proposition 4.2.4 is finished. \square

4.3 Exponential stability

In this section, we are going to prove the main result of this chapter, namely Theorem 4.1.1. For all $t > 0$, we define the energy of a solution of (KdVs) (or (KdVLs)) by

$$E(t) = \frac{1}{2} \int_0^L (u(t, x))^2 dx. \quad (4.3.1)$$

Then, for a classical solution of (KdVs) (or (KdVLs)) performing integration by parts, we get using $|\alpha| < 1$

$$\dot{E}(t) = \frac{1}{2} (\mathfrak{sat}(\alpha \partial_x u(t, 0)))^2 - \frac{1}{2} (\partial_x u(t, 0))^2 \leq 0. \quad (4.3.2)$$

Thus, formally, the energy is a non-increasing function of time. Now, we will prove that indeed, the energy decays exponentially to 0 when $L \notin \mathcal{N}$. Our approach will be based on an observability inequality for the system (KdVLs).

Proposition 4.3.1. *Let $L \notin \mathcal{N}$, $|\alpha| < 1$ and $u_0 \in L^2(0, L)$. For any solution y of (KdVLs), the following inequality holds:*

$$\int_0^L (u_0)^2 dx \leq C_{obs} \int_0^T (\partial_x u(t, 0))^2 dt, \quad (4.3.3)$$

for some $C_{obs} > 0$, that does not depend on u_0 .

Proof. In order to prove the observability inequality, we follow [Ros97] and use a contradiction argument. Let us suppose that (4.3.3) is false, then there exists $(u_0^n)_{n \in \mathbb{N}} \subset L^2(0, L)$ such that

$$\int_0^L (u_0^n)^2 dx = 1, \text{ and } \int_0^T (\partial_x u^n(t, 0))^2 dt \rightarrow 0$$

when $n \rightarrow \infty$ and where $u^n = S(t)u_0^n$. Now, using (4.2.21) with $f = 0$ we get that $(u^n)_{n \in \mathbb{N}}$ is bounded in $L^2((0, T), H^1(0, L))$. Then as $u_t^n = -u_x^n - u_{xxx}^n$, we can see that $(u_t^n)_{n \in \mathbb{N}}$ is

bounded in $L^2((0, T), H^{-2}(0, L))$, thus by Aubin-Lions lemma $(u^n)_{n \in \mathbb{N}}$ is relatively compact in $L^2((0, T), L^2(0, L))$, therefore we can assume that $u^n \rightarrow u$ in $L^2((0, T), L^2(0, L))$. Now, multiplying the first line of **(KdVLs)** by $(T - t)u$, integrating on $(0, L) \times (0, T)$, we get

$$\begin{aligned} \int_0^T \int_0^L (T - t) \partial_t u u dx dt + \int_0^T \int_0^L (T - t) \partial_x^3 u u dx dt + \int_0^T \int_0^L (T - t) \partial_x u u dx dt \\ + \int_0^T \int_0^L (T - t) u^2 \partial_x u dx dt = 0. \end{aligned} \quad (4.3.4)$$

After integration by parts

$$\frac{T}{2} \int_0^L (u_0)^2 dx + \frac{1}{2} \int_0^T (T - t) (\mathfrak{sat}(\alpha \partial_x u(t, 0)))^2 dt = \frac{1}{2} \int_0^T \int_0^L u^2 dx dt + \frac{1}{2} \int_0^T (T - t) (\partial_x u(t, 0))^2 dt.$$

Therefore,

$$\int_0^L (u_0)^2 dx \leq \frac{1}{T} \|u\|_{L^2(0, T, L^2(0, L))}^2 + \int_0^T (\partial_x u(t, 0))^2 dt. \quad (4.3.5)$$

Hence, using **(4.3.5)** we derive that $(u_0^n)_{n \in \mathbb{N}}$ is a Cauchy sequence in $L^2(0, L)$. Let $u_0 = \lim_{n \rightarrow \infty} u_0^n$, then

$$\int_0^L (u_0)^2 dx = 1, \text{ and } \int_0^T (\partial_x u(t, 0))^2 dt = 0.$$

Thus, as $\mathfrak{sat}(0) = 0$, we conclude that u solves the following linear equation:

$$\begin{cases} \partial_t u(t, x) + \partial_x u(t, x) + \partial_x^3 u(t, x) = 0, & t > 0, x \in (0, L), \\ u(t, 0) = u(t, L) = 0, & t > 0, \\ \partial_x u(t, 0) = \partial_x u(t, L) = 0, & t > 0, \end{cases} \quad (4.3.6)$$

But by **[Ros97, Lemma 3.4]** if $L \notin \mathcal{N}$, there is no function satisfying this system and $\|u_0\|_{L^2(0, L)} = 1$, which gives us the contradiction and finishes the proof of **Proposition 4.3.1**. \square

Now, **Theorem 4.1.1** follows from **Section 1.2.3**. Since **(KdVs)** is also invariant by translation in time, to prove the second point of **Theorem 4.1.1**, the goal is to prove a similar observability inequality **(4.3.3)** for a solution of **(KdVs)**. This idea was used in several works, for instance **[PMVZ02, PCP22a]**. More specifically, we are going to show that $\|u(t, \cdot)\|_{L^2(0, L)} \leq \tilde{\gamma} \|u_0\|_{L^2(0, L)}$, for some $\tilde{\gamma} \in (0, 1)$. Let $\|u_0\|_{L^2(0, L)} \leq \epsilon$ for $\epsilon > 0$ to fix later. Take u the solution of **(KdVs)**, then we can split u as $u = \bar{u} + \tilde{u}$, where

$$\begin{cases} \partial_t \bar{u}(t, x) + \partial_x \bar{u}(t, x) + \partial_x^3 \bar{u}(t, x) = 0, & t > 0, x \in (0, L), \\ \bar{u}(t, 0) = \bar{u}(t, L) = 0, & t > 0, \\ \partial_x \bar{u}(t, L) = \mathfrak{sat}(\alpha \partial_x \bar{u}(t, 0)), & t > 0, \\ \bar{u}(0, x) = u_0(x), & x \in (0, L), \end{cases}$$

and

$$\begin{cases} \partial_t \tilde{u}(t, x) + \partial_x \tilde{u}(t, x) + \partial_x^3 \tilde{u}(t, x) = -u(t, x) \partial_x u(t, x), & t > 0, x \in (0, L), \\ \tilde{u}(t, 0) = \tilde{u}(t, L) = 0, & t > 0, \\ \partial_x \tilde{u}(t, L) = h(t), & t > 0, \\ \tilde{u}(0, x) = 0, & x \in (0, L), \end{cases}$$

where $h(t) = \mathfrak{sat}(\alpha \partial_x u(t, 0)) - \mathfrak{sat}(\alpha \partial_x \bar{u}(t, 0))$. Now clearly,

$$\|u(T, \cdot)\|_{L^2(0,L)} \leq \|\bar{u}(T, \cdot)\|_{L^2(0,L)} + \|\tilde{u}(T, \cdot)\|_{L^2(0,L)}. \quad (4.3.7)$$

Moreover, as \bar{u} satisfies (KdVLs), we get using (4.3.3), Section 1.2.3 and $\bar{u}(0, x) = u_0(x)$ that there exists $\gamma < 1$ such that

$$\|\bar{u}(T, \cdot)\|_{L^2(0,L)} \leq \gamma \|u_0\|_{L^2(0,L)}. \quad (4.3.8)$$

Besides,

$$\|\tilde{u}(T, \cdot)\|_{L^2(0,L)} \leq \|\tilde{u}\|_{C([0,T], L^2(0,L))}.$$

To estimate $\|\tilde{u}\|_{C([0,T], L^2(0,L))}$ using (4.2.28) we get

$$\|\tilde{u}\|_{C([0,T], L^2(0,L))} \leq C \|u \partial_x u\|_{L^1((0,T), L^2(0,L))},$$

by Proposition 1.2.1,

$$\|u \partial_x u\|_{L^1((0,T), L^2(0,L))} \leq C \|u\|_{B_T}^2,$$

using (4.2.33)-(4.2.38), we derive

$$\|u\|_{B_T}^2 \leq C \left(\|u_0\|_{L^2(0,L)}^2 + \|u_0\|_{L^2(0,L)}^4 \right).$$

Thus, we get for \tilde{u}

$$\|\tilde{u}\|_{C([0,T], L^2(0,L))} \leq C \left(\|u_0\|_{L^2(0,L)}^2 + \|u_0\|_{L^2(0,L)}^4 \right),$$

and hence

$$\|\tilde{u}(T, \cdot)\|_{L^2(0,L)} \leq C \left(\|u_0\|_{L^2(0,L)} + \|u_0\|_{L^2(0,L)}^3 \right) \|u_0\|_{L^2(0,L)}, \quad (4.3.9)$$

Finally, as $\|u_0\|_{L^2(0,L)} \leq \epsilon$ using (4.3.8) and (4.3.9), we deduce from (4.3.7)

$$\begin{aligned} \|u(T, \cdot)\|_{L^2(0,L)} &\leq \|\bar{u}(T, \cdot)\|_{L^2(0,L)} + \|\tilde{u}(T, \cdot)\|_{L^2(0,L)} \\ &\leq \gamma \|u_0\|_{L^2(0,L)} + (C\epsilon + C\epsilon^3) \|u_0\|_{L^2(0,L)} \\ &= (\gamma + C\epsilon + C\epsilon^3) \|u_0\|_{L^2(0,L)}, \end{aligned}$$

we conclude by choosing $\epsilon > 0$ small enough such that $\tilde{\gamma} = \gamma + C\epsilon + C\epsilon^3 < 1$. \square

4.4 Asymptotic stability via LaSalle's invariance principle

Another approach to prove the stability of (KdVLs) is to use Lyapunov techniques and LaSalle's invariance principle. This idea was developed in the case of internal saturation in [MCPA15]. The Lyapunov function to take in account is the energy defined by (4.3.1).

Proposition 4.4.1. *Let $L \notin \mathcal{N}$ and $|\alpha| < 1$, then for all $u_0 \in L^2(0, L)$, the mild solution y of (KdVLs) satisfies:*

$$\|u(t, \cdot)\|_{L^2(0,L)} \rightarrow 0, \text{ as } t \rightarrow \infty.$$

Proof. First, note that by (4.3.2), $\dot{E}(t) \leq 0$ that tells us that for regular initial data the solution of (KdVLs) is stable and moreover $\dot{E}(t) = 0$ if and only if $\partial_x u(t, 0) = 0$. In fact, we consider now two cases:

- $|\mathfrak{sat}(\alpha \partial_x u(t, 0))| = M$. This happens when $|\alpha \partial_x u(t, 0)| \geq M$ which implies $|\partial_x u(t, 0)| \geq \frac{M}{|\alpha|}$. By (4.3.2), $\dot{E}(t) = 0$ if and only if $M = |\partial_x u(t, 0)|$. Then $M \geq \frac{M}{|\alpha|}$, which is a contradiction because $|\alpha| < 1$.

- $\text{sat}(\alpha \partial_x u(t, 0)) = \alpha \partial_x u(t, 0)$. Then $\dot{E}(t) = 0$ if and only if $\alpha^2 |\partial_x u(t, 0)|^2 = |\partial_x u(t, 0)|^2$, which is only possible if $\partial_x u(t, 0) = 0$.

To apply LaSalle's invariance principle, we need that the trajectories are compact, in this context, it is possible to show the following result.

Lemma 4.4.1. *The canonical embedding from $D(\mathcal{A})$, equipped with the graph norm, into $L^2(0, L)$ is compact.*

Now as \mathcal{A} generates a semigroup of contractions, then from [Bre73, Theorem 3.1, Page 54], we get that for all $t \geq 0$ and $u(0, \cdot) \in D(\mathcal{A})$,

$$\|u(t, \cdot)\|_{L^2(0, L)} \leq \|u(0, \cdot)\|_{L^2(0, L)},$$

$$\|\mathcal{A}u(t, \cdot)\|_{L^2(0, L)} \leq \|\mathcal{A}u(0, \cdot)\|_{L^2(0, L)}.$$

Thus, by Lemma 4.4.1, we have that the trajectory $\{v(t) = S(t)v_0, t \geq 0\}$ is pre-compact in $L^2(0, L)$, then by [Sle89, Theorem, 3.1] the ω -limit set, $\omega[u(0, \cdot)] \subset D(\mathcal{A})$, is not empty and invariant to the semigroup $S(t)$. Take now a regular solution such that $\dot{E}(t) = 0$, that is $\partial_x u(t, 0) = 0$ $t \geq 0$. Then u solves (4.3.6) and by [Ros97, Lemma 3.4] if $L \notin \mathcal{N}$, $u(t, x) = 0$ a.e $x \in (0, L)$. Consequently,

$$\|u(t, \cdot)\|_{L^2(0, L)} \rightarrow 0, \text{ as } t \rightarrow \infty$$

for the regular solution of (KdVLs). It is possible to extend this result to mild solutions by density. \square

STABILITY RESULTS OF THE KdV EQUATION WITH TIME-VARYING DELAY

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In this chapter, we consider the Korteweg-de Vries equation with time-dependent delay on the boundary or internal feedbacks. Under some assumptions on the time-dependent delay, on the weights of the feedbacks and on the length of the spatial domain, we prove the exponential stability results, using appropriate Lyapunov functionals. We finish by some numerical simulations that illustrate the stability results and the influence of the delay on the decay rate. This work was initiated during a research stay in Nancy with Julie Valein and is based on the submitted paper:

- Hugo Parada, Chahnaz Timimoun, and Julie Valein. Stability results for the KdV equation with time-varying delay. *Systems & Control Letters*, 177:105547, 2023.

5.1 Introduction

Recently, the problem of robustness with respect to constant time-delay of the KdV equation was studied in [BCV19, PCP22a, Val22] using Lyapunov theory or deriving suitable observability inequalities. In the case where the KdV equation is in presence of memory terms, stability results were obtained in [Che21, CG22]. The stability of PDE's involving time-varying delays was analyzed in [NVF09] for one-dimensional heat and wave equations, in [NP11, NPV11] for wave equations in domains in \mathbb{R}^n and in [FNV10] for general second-order evolution equations. We can also mention [Par14] where a weak viscoelastic beam equation with time-varying delay was considered and the recent work [KNL⁺22] studying exponential stability of piezoelectric beams. In our best knowledge,

there is no work dealing with this problem for the KdV equation. In this chapter, we are going to consider the two following systems

$$\begin{cases} (\partial_t u + \partial_x u + \partial_x^3 u + u \partial_x u)(t, x) = 0, & t > 0, x \in (0, L), \\ u(t, 0) = u(t, L) = 0, & t > 0, \\ \partial_x u(t, L) = \alpha \partial_x u(t, 0) + \beta \partial_x u(t - \tau(t), 0), & t > 0, \\ u(0, x) = u_0(x), & x \in (0, L), \\ \partial_x u(t - \tau(0), 0) = z_0(t - \tau(0)), & 0 < t < \tau(0), \end{cases} \quad (5.1.1)$$

and

$$\begin{cases} (\partial_t u + \partial_x u + \partial_x^3 u + u \partial_x u)(t, x) + a(x)u(t, x) + b(x)u(t - \tau(t), x) = 0, & t > 0, x \in (0, L), \\ u(t, 0) = u(t, L) = \partial_x u(t, L) = 0, & t > 0, \\ u(0, x) = u_0(x), & x \in (0, L), \\ u(t - \tau(0), x) = z_0(t - \tau(0), x), & 0 < t < \tau(0), x \in (0, L), \end{cases} \quad (5.1.2)$$

where $L > 0$ is the length of the spatial domain, $u(t, x)$ is the amplitude of the water wave at position x . We assume that the delay τ is a function of time t , which satisfies the following conditions

$$0 < \tau_0 \leq \tau(t) \leq M, \quad \forall t \geq 0, \quad (5.1.3)$$

$$\dot{\tau}(t) \leq d < 1, \quad \forall t \geq 0, \quad (5.1.4)$$

where $0 \leq d < 1$, and

$$\tau \in W^{2, \infty}([0, T]), \quad \forall T > 0. \quad (5.1.5)$$

Moreover, we assume that α, β, d in (5.1.1) are real constants satisfying

$$\begin{aligned} \text{The matrix } \Phi_{\alpha, \beta} &= \begin{pmatrix} \alpha^2 - 1 + |\beta| & \alpha\beta \\ \alpha\beta & \beta^2 + |\beta|(d - 1) \end{pmatrix} \\ &\text{is definite negative.} \end{aligned} \quad (5.1.6)$$

In (5.1.2), $a = a(x)$ and $b = b(x)$ are nonnegative functions in $L^\infty(0, L)$. We will also assume that $\text{supp } b = \omega$ and

$$b(x) \geq b_0 > 0 \quad \text{in } \omega, \quad (5.1.7)$$

where ω is an open nonempty subset of $(0, L)$. We assume that the coefficients a and b satisfy the following assumption:

$$\exists c_0 > 0, \quad \frac{2-d}{2-2d} b(x) + c_0 \leq a(x) \quad \text{in } \omega. \quad (5.1.8)$$

Then $\omega = \text{supp } b \subset \text{supp } a$ and $a(x) \geq b_0 + c_0 > 0$ in ω . Some examples of functions satisfying these conditions are given in Section 5.4.

Remark 5.1.1. *We can note the following points on the coefficients of the boundary or internal feedback:*

- A sufficient condition to obtain (5.1.6) is $|\alpha| + |\beta| + d < 1$. Indeed, on the one hand, we have $\text{tr}(\Phi_{\alpha, \beta}) = \alpha^2 + \beta^2 - 1 + |\beta|d < 0 \Leftrightarrow \alpha^2 + \beta^2 + |\beta|d < 1$, and $\alpha^2 + \beta^2 + |\beta|d < |\alpha| + |\beta| + d < 1$. On the other hand we have,

$$\begin{aligned} \det(\Phi_{\alpha, \beta}) &= |\beta|(\beta^2 - 2|\beta| + 1 - \alpha^2 + d\alpha^2 - d + |\beta|d) \\ &= |\beta|((1 - |\beta|)^2 + d(|\beta| - 1) + \alpha^2(d - 1)) \\ &= |\beta|((1 - |\beta|)(1 - |\beta| - d) + \alpha^2(d - 1)) \\ &> |\beta|(\alpha^2 + \alpha^2 d - \alpha^2) = |\beta|\alpha^2 d > 0. \end{aligned}$$

Then, $|\alpha| + |\beta| + d < 1$ implies that $\Phi_{\alpha, \beta}$ is definite negative.

- If $d = 0$, (5.1.6) (resp. (5.1.8)) is equivalent to $|\alpha| + |\beta| < 1$ (resp. $b(x) + c_0 \leq a(x)$ in ω) which corresponds to the assumption for a constant time-delay given in [BCV19] (resp. [Val22]).
- If $d \rightarrow 1^-$, $\frac{2-d}{2-2d} \rightarrow +\infty$, and so we need that the the internal feedback without delay $a(t, x)$ to be very large.

In [BCV19], two different approaches for the exponential stability of the nonlinear KdV equation with boundary (constant) time-delay feedback were studied. The first was a Lyapunov functional approach with an estimation of the decay rate, but with a restrictive assumption on the length L of the spatial domain. The second one was an observability inequality approach without estimation on the decay rate and for any non-critical lengths. The asymptotic stability of the nonlinear KdV equation with (constant) time-delay internal feedback was studied in [Val22]. A semiglobal stability result for any lengths was proven in the case where the weight of the term with delay is smaller than the weight of the term without delay, using an observability inequality directly on the nonlinear system. A local exponential stability result was given in the case where the support of the term with delay is not included in the support of the term without delay. The aim of this chapter is to extend these results in the case where the delay depends on the time. An important fact about systems (5.1.1) and (5.1.2) is that due to the effect of the time-varying delay, these systems are no longer invariant in time. Thus, the observability inequality approach does not work anymore, and we have to choose a new appropriate Lyapunov functional. An other difficulty, beyond the difficulty of dealing with a nonlinear equation, is that the first order linear operator depends on time (contrary to constant delay case) and the well-posedness is not trivial.

The outline of this chapter is as follows. In Section 5.2, we prove the well-posedness results, firstly for the boundary case, secondly for the internal case. The stability results are proved in Section 5.3. Finally, we illustrate our results by some numerical simulations in Section 5.4.

5.2 Well-posedness results

The goal of this section is to prove appropriate well-posedness results of (5.1.1) and (5.1.2). We first prove the well-posedness result of the linearization around 0 of (5.1.1) (resp. (5.1.2)). The proof will be based on the semigroup theory and on introducing a new function for the delayed term. Then, we add a source term that plays the role of the nonlinearity. Finally, using a fixed-point approach, we show the well-posedness of the nonlinear systems (5.1.1) and (5.1.2).

5.2.1 Well-posedness result of (5.1.1)

The goal of this section is to prove appropriate local well-posedness result of (5.1.1).

Well-posedness result of the linear system

In this part, we focus on the study of linearization around 0 of (5.1.1), that is

$$\begin{cases} (\partial_t u + \partial_x u + \partial_x^3 u)(t, x) = 0, & t > 0, x \in (0, L), \\ u(t, 0) = u(t, L) = 0, & t > 0, \\ \partial_x u(t, L) = \alpha \partial_x u(t, 0) + \beta \partial_x u(t - \tau(t), 0), & t > 0, \\ u(0, x) = u_0(x), & x \in (0, L), \\ u_x(t - \tau(0), 0) = z_0(t - \tau(0)), & 0 < t < \tau(0). \end{cases} \quad (5.2.1)$$

Now, classically, we introduce a new variable that takes into account the delay term (see, for instance, [NVF09]). Let $z(t, \rho) = \partial_x u(t - \tau(t)\rho, 0)$ for $\rho \in (0, 1)$ and $t > 0$. Then, z verifies the

following transport equation

$$\begin{cases} \tau(t)\partial_t z(t, \rho) + (1 - \dot{\tau}(t)\rho)z_\rho(t, \rho) = 0, & t > 0, \rho \in (0, 1), \\ z(t, 0) = \partial_x u(t, 0), & t > 0, \\ z(0, \rho) = z_0(-\tau(0)\rho), & \rho \in (0, 1). \end{cases}$$

Define $U = (u, z)^\top$, then U satisfies

$$\partial_t U = \begin{pmatrix} \partial_t u \\ \partial_t z \end{pmatrix} = \begin{pmatrix} -\partial_x u - \partial_x^3 u \\ \frac{\dot{\tau}(t)\rho - 1}{\tau(t)} z_\rho \end{pmatrix}.$$

This problem can be rewritten as the following first-order evolution equation

$$\begin{cases} \partial_t U(t) = \mathcal{A}(t)U(t), & t > 0, \\ U(0) = (u_0, z_0(-\tau(0)\cdot))^\top =: U_0, \end{cases} \quad (5.2.2)$$

where the time-dependent operator $\mathcal{A}(t)$ is defined by

$$\mathcal{A}(t) \begin{pmatrix} u \\ z \end{pmatrix} = \begin{pmatrix} -\partial_x u - \partial_x^3 u \\ \frac{\dot{\tau}(t)\rho - 1}{\tau(t)} z_\rho \end{pmatrix}, \quad \text{with domain}$$

$$D(\mathcal{A}(t)) = \{(u, z) \in (H^3(0, L) \cap H_0^1(0, L)) \times H^1(0, 1), z(0) = \partial_x u(0), \\ \partial_x u(L) = \alpha \partial_x u(0) + \beta z(1)\}.$$

Note that the domain of the operator $\mathcal{A}(t)$ is independent of time t , i.e., $D(\mathcal{A}(t)) = D(\mathcal{A}(0))$, $t > 0$. Now, we introduce the Hilbert space $H = L^2(0, L) \times L^2(0, 1)$, equipped with the usual inner product

$$\left\langle \begin{pmatrix} u \\ z \end{pmatrix}, \begin{pmatrix} \tilde{u} \\ \tilde{z} \end{pmatrix} \right\rangle = \int_0^L u \tilde{u} dx + \int_0^1 z \tilde{z} d\rho,$$

endowed with the norm $\|\cdot\|_H$. To prove the well-posedness of (5.2.2) we follow [NVF09]. The proof is based on showing that the triplet $\{\mathcal{A}, H, \mathcal{Y}\}$, with $\mathcal{A} = \{\mathcal{A}(t) : t \in [0, T]\}$, for some $T > 0$ fixed and $\mathcal{Y} = D(\mathcal{A}(0))$, forms a constant domain system (CD-system), see [Kat70, Kat85]. The following theorem gives the existence and uniqueness results and is proved in [Kat70]:

Theorem 5.2.1. *Assume that*

1. $\mathcal{Y} = D(\mathcal{A}(0))$ is a dense subset of H ,
2. $D(\mathcal{A}(t)) = D(\mathcal{A}(0))$, for all $t > 0$,
3. for all $t \in [0, T]$, $\mathcal{A}(t)$ generates a strongly continuous semigroup on H and the family $\mathcal{A} = \{\mathcal{A}(t) : t \in [0, T]\}$ is stable with stability constants C and m independent of t (i.e. the semigroup $(S_t(s))_{s \geq 0}$ generated by $\mathcal{A}(t)$ satisfies $\|S_t(s)U\|_H \leq Ce^{ms}\|U\|_H$, for all $U \in H$ and $s \geq 0$),
4. $\partial_t \mathcal{A}(t)$ belongs to $L_*^\infty([0, T], B(\mathcal{Y}, H))$, the space of equivalent classes of essentially bounded, strongly measure functions from $[0, T]$ into the set $B(\mathcal{Y}, H)$ of bounded operators from \mathcal{Y} into H .

Then, problem (5.2.2) has a unique solution $U \in C([0, T], \mathcal{Y}) \cap C^1([0, T], H)$ for any initial datum in \mathcal{Y} .

In particular, we are going to prove the following result.

Theorem 5.2.2. *Assume (5.1.3)-(5.1.6). Let $U_0 \in H$, then there exists a unique mild solution $U \in C([0, +\infty), H)$ to (5.2.2). Moreover, if $U_0 \in D(\mathcal{A}(0))$ then the solution is classical and $U \in C([0, +\infty), D(\mathcal{A}(0))) \cap C^1([0, +\infty), H)$.*

Proof. Clearly, the space $\mathcal{Y} = D(\mathcal{A}(0))$ is a dense subset of H and, by definition, $D(\mathcal{A}(t)) = D(\mathcal{A}(0))$, for all $t > 0$. Now, to prove the third point of Theorem 5.2.1, we introduce the following time-dependent inner product on H to use the variable norm technique of Kato

$$\left\langle \begin{pmatrix} u \\ z \end{pmatrix}, \begin{pmatrix} \tilde{u} \\ \tilde{z} \end{pmatrix} \right\rangle_t = \int_0^L u \tilde{u} dx + |\beta| \tau(t) \int_0^1 z \tilde{z} d\rho,$$

with associated norm denoted by $\|\cdot\|_t$. By (5.1.3), the norms $\|\cdot\|_H$ and $\|\cdot\|_t$ are equivalent in H : $\forall t \geq 0, \forall (u, z) \in H$,

$$(1 + |\beta| \tau_0) \|(u, z)\|_H^2 \leq \|(u, z)\|_t^2 \leq (1 + |\beta| M) \|(u, z)\|_H^2. \quad (5.2.3)$$

We first observe that, following [FNV10, Theorem 2.4],

$$\frac{\|U\|_t}{\|U\|_s} \leq e^{\frac{c}{2\tau_0}|t-s|}, \quad \forall t, s \in [0, T], \quad (5.2.4)$$

where $U = (u, z)^\top \in H$ and c is a positive constant. Now, we calculate $\langle \mathcal{A}(t)U, U \rangle_t$ for a fixed $t \in [0, T]$. Take $U = (u, z)^\top \in D(\mathcal{A}(0))$, then

$$\langle \mathcal{A}(t)U, U \rangle_t = \left\langle \begin{pmatrix} -\partial_x u - \partial_x^3 u \\ \frac{\dot{\tau}(t)\rho - 1}{\tau(t)} z_\rho \end{pmatrix}, \begin{pmatrix} u \\ z \end{pmatrix} \right\rangle_t = \int_0^L (-\partial_x u - \partial_x^3 u) u dx + |\beta| \int_0^1 (\dot{\tau}(t)\rho - 1) z_\rho z d\rho.$$

By integrating by parts in space and in ρ , we have

$$\langle \mathcal{A}(t)U, U \rangle_t = \frac{1}{2} (\partial_x u)^2 \Big|_0^L - \frac{|\beta|}{2} \dot{\tau}(t) \int_0^1 z^2 d\rho + \frac{|\beta|}{2} [(\dot{\tau}(t)\rho - 1) z^2]_0^1.$$

Moreover using the boundary conditions, we obtain

$$\begin{aligned} \langle \mathcal{A}(t)U, U \rangle_t &= \frac{1}{2} (\alpha \partial_x u(0) + \beta z(1))^2 - \frac{1}{2} (\partial_x u(0))^2 - \frac{|\beta|}{2} \dot{\tau}(t) \int_0^1 z^2 d\rho + \frac{|\beta|}{2} (\dot{\tau}(t) - 1) (z(1))^2 \\ &\quad + \frac{|\beta|}{2} (\partial_x u(0))^2. \end{aligned}$$

Now, by (5.1.4) we derive

$$\langle \mathcal{A}(t)U, U \rangle_t - \kappa(t) \langle U, U \rangle_t \leq \frac{1}{2} \begin{pmatrix} \partial_x u(0) \\ z(1) \end{pmatrix}^\top \Phi_{\alpha, \beta} \begin{pmatrix} \partial_x u(0) \\ z(1) \end{pmatrix},$$

where $\kappa(t) = \frac{(\dot{\tau}(t)^2 + 1)^{1/2}}{2\tau(t)}$ and where $\Phi_{\alpha, \beta}$ is defined by (5.1.6). Finally, using (5.1.6), we get

$$\langle \mathcal{A}(t)U, U \rangle_t - \kappa(t) \langle U, U \rangle_t \leq 0.$$

The above inequality proves the dissipativeness of $\tilde{\mathcal{A}}(t) = \mathcal{A}(t) - \kappa(t)I$ for the inner product $\langle \cdot, \cdot \rangle_t$. Let us prove that for all $t \in [0, T]$, $\mathcal{A}(t)$ is maximal, i.e., that $\lambda I - \mathcal{A}(t)$ is surjective for some $\lambda > 0$. Let $t \in [0, T]$ be fixed, and $(f, h)^\top \in H$. We look for $U = (u, z)^\top \in D(\mathcal{A}(t))$ solution of $(\lambda I - \mathcal{A}(t))U = (f, h)^\top$, that is

$$\begin{cases} \lambda u + \partial_x u + \partial_x^3 u = f, \\ \lambda z + \left(\frac{1 - \dot{\tau}(t)\rho}{\tau(t)} \right) z_\rho = h, \\ u(0) = u(L) = 0, \quad \partial_x u(0) = z(0), \\ \partial_x u(L) = \alpha \partial_x u(0) + \beta z(1). \end{cases} \quad (5.2.5)$$

Following [NVF09], if we find u with the appropriate regularity, then we can obtain z , given by

$$z(\rho) = \begin{cases} \partial_x u(0) e^{\lambda \frac{\tau(t)}{\dot{\tau}(t)} \ln(1-\dot{\tau}(t)\rho)} + e^{\lambda \frac{\tau(t)}{\dot{\tau}(t)} \ln(1-\dot{\tau}(t)\rho)} \int_0^\rho \frac{h(\sigma)\tau(t)}{1-\dot{\tau}(t)\sigma} e^{-\lambda \frac{\tau(t)}{\dot{\tau}(t)} \ln(1-\dot{\tau}(t)\sigma)} d\sigma, & \text{if } \dot{\tau}(t) \neq 0, \\ \partial_x u(0) e^{-\lambda\tau(t)\rho} + \tau(t) e^{-\lambda\tau(t)\rho} \int_0^\rho e^{\lambda\tau(t)\sigma} h(\sigma) d\sigma, & \text{if } \dot{\tau}(t) = 0. \end{cases}$$

In particular $z(1) = \partial_x u(0)g_0(t) + g_h(t)$, where

$$g_0(t) = \begin{cases} e^{\lambda \frac{\tau(t)}{\dot{\tau}(t)} \ln(1-\dot{\tau}(t))}, & \text{if } \dot{\tau}(t) \neq 0, \\ e^{-\lambda\tau(t)}, & \text{if } \dot{\tau}(t) = 0, \end{cases}$$

$$g_h(t) = \begin{cases} e^{\lambda \frac{\tau(t)}{\dot{\tau}(t)} \ln(1-\dot{\tau}(t))} \int_0^1 \frac{h(\sigma)\tau(t)}{1-\dot{\tau}(t)\sigma} e^{-\lambda \frac{\tau(t)}{\dot{\tau}(t)} \ln(1-\dot{\tau}(t)\sigma)} d\sigma, & \text{if } \dot{\tau}(t) \neq 0, \\ \tau(t) e^{-\lambda\tau(t)} \int_0^1 e^{\lambda\tau(t)\sigma} h(\sigma) d\sigma, & \text{if } \dot{\tau}(t) = 0. \end{cases}$$

This implies that u must satisfy

$$\begin{cases} \lambda u + \partial_x u + \partial_x^3 u = f, \\ u(0) = u(L) = 0, \\ \partial_x u(L) = (\alpha + \beta g_0(t)) \partial_x u(0) + \beta g_h(t). \end{cases}$$

Consider now $\psi(x) = \frac{\beta x(x-L)g_h(t)}{L(1+\alpha+\beta g_0(t))}$ (t is fixed here) and $\tilde{\alpha} = \tilde{\alpha}(t) = \alpha + \beta g_0(t)$. After some computations, we can observe that $\varphi = y - \psi$ solves

$$\begin{cases} \lambda\varphi + \partial_x \varphi + \partial_x^3 \varphi = \tilde{f} := f - \lambda\psi - \partial_x \psi - \partial_x^3 \psi, \\ \varphi(0) = \varphi(L) = 0, \\ \partial_x \varphi(L) = \tilde{\alpha} \partial_x \varphi(0). \end{cases} \quad (5.2.6)$$

As t is fixed, the problem can be seen as $(\lambda\mathcal{I} - \mathcal{A}_{\tilde{\alpha}})\varphi = \tilde{f}$ where the operator $\mathcal{A}_{\tilde{\alpha}}$ is defined by $\mathcal{A}_{\tilde{\alpha}}\varphi = -\varphi' - \varphi'''$, with $D(\mathcal{A}_{\tilde{\alpha}}) = \{\varphi \in H^3(0, L) \cap H_0^1(0, L), \varphi'(L) = \tilde{\alpha}\varphi'(0)\}$ and where $\tilde{f} \in L^2(0, L)$ (since $\varphi \in C^\infty([0, L])$ and $f \in L^2(0, L)$). To conclude this part, we show that under the condition, (5.1.6) the operator $\mathcal{A}_{\tilde{\alpha}}$ is maximal.

Lemma 5.2.1. *If $|\tilde{\alpha}| < 1$, then the operator $\mathcal{A}_{\tilde{\alpha}}$ is maximal.*

Proof. Consider $|\tilde{\alpha}| < 1$, clearly $\mathcal{A}_{\tilde{\alpha}}$ is closed. Let us prove that $\mathcal{A}_{\tilde{\alpha}}$ and $\mathcal{A}_{\tilde{\alpha}}^*$ are dissipative. Let $\varphi \in D(\mathcal{A}_{\tilde{\alpha}})$, then we get

$$(\mathcal{A}_{\tilde{\alpha}}\varphi, \varphi)_{L^2(0, L)} = \int_0^L (-\varphi' - \varphi''')\varphi dx = \frac{1}{2}(\tilde{\alpha}^2 - 1)(\varphi'(0))^2 \leq 0.$$

The dual of the operator $\mathcal{A}_{\tilde{\alpha}}$ is defined by $\mathcal{A}_{\tilde{\alpha}}^*\zeta = \zeta' + \zeta'''$ with domain $D(\mathcal{A}_{\tilde{\alpha}}^*) = \{\zeta \in H^3(0, L) \cap H_0^1(0, L), \zeta'(0) = \tilde{\alpha}\zeta'(L)\}$. Similarly, for $\zeta \in D(\mathcal{A}_{\tilde{\alpha}}^*)$ we have $(\mathcal{A}_{\tilde{\alpha}}^*\zeta, \zeta)_{L^2(0, L)} = \frac{1}{2}(\tilde{\alpha}^2 - 1)(\zeta'(L))^2 \leq 0$. Thus, by [Paz83], $\mathcal{A}_{\tilde{\alpha}}$ is the generator of a C_0 semigroup of contraction on $L^2(0, L)$. By the Lumer-Phillips theorem (see [Paz83, Theorem 4.3]), $\mathcal{A}_{\tilde{\alpha}}$ is a maximal operator. \square

Thus, it is enough to check that $|\tilde{\alpha}| < 1$: as $|g_0(t)| < 1$, then $|\tilde{\alpha}| \leq |\alpha| + |\beta| < 1$ by (5.1.6) and Remark 5.1.1. Therefore, by Lemma 5.2.1, we have the existence of $\varphi \in D(\mathcal{A}_{\tilde{\alpha}})$ solution of (5.2.6) and hence $(u, z)^\top \in D(\mathcal{A}(t))$ solution of (5.2.5).

We have then shown that $\lambda\mathcal{I} - \mathcal{A}(t)$ is surjective. Then, as $\kappa(t) > 0$, we have that $\lambda\mathcal{I} - \tilde{\mathcal{A}}(t) = (\lambda + \kappa(t))\mathcal{I} - \mathcal{A}(t)$ is surjective for some $\lambda > 0$ and $t \in [0, T]$. We conclude that $\tilde{\mathcal{A}}(t)$ generates a strongly continuous semigroup on H and $\tilde{\mathcal{A}} = \{\tilde{\mathcal{A}}(t), t \in [0, T]\}$ is a stable family of generators in H with a stability constant independent of t , using (5.2.4), by [Kat70, Proposition 3.4] (see also [Kat85]).

Finally, $\dot{\kappa}(t) = \frac{\ddot{\tau}(t)\dot{\tau}(t)}{2\tau(t)(\dot{\tau}(t)^2+1)^{1/2}} - \frac{\dot{\tau}(t)(\dot{\tau}(t)^2+1)^{1/2}}{2\tau(t)^2}$ is bounded on $[0, T]$ for all $T > 0$ (by (5.1.5)) and we have

$$\frac{d}{dt}\mathcal{A}(t)U = \begin{pmatrix} 0 \\ \frac{\ddot{\tau}(t)\tau(t)\rho - \dot{\tau}(t)(\dot{\tau}(t)\rho - 1)}{\tau(t)^2}z_\rho \end{pmatrix}$$

with $\frac{\ddot{\tau}(t)\tau(t)\rho - \dot{\tau}(t)(\dot{\tau}(t)\rho - 1)}{\tau(t)^2}$ bounded on $[0, T]$ by (5.1.5). Thus,

$$\frac{d}{dt}\tilde{\mathcal{A}}(t) \in L_*^\infty([0, T], B(D(\mathcal{A}(0)), H)),$$

which proves the fourth point of Theorem 5.2.1. Therefore, all assumptions of Theorem 5.2.1 are verified, thus the problem

$$\begin{cases} \partial_t \tilde{U}(t) = \tilde{\mathcal{A}}(t)\tilde{U}(t), & t > 0, \\ \tilde{U}(0) = U_0 \end{cases}$$

has a unique solution $\tilde{U} \in C([0, \infty), H)$ and $\tilde{U} \in C([0, \infty), D(\mathcal{A}(0))) \cap C^1([0, \infty), H)$ if $U_0 \in D(\mathcal{A}(0))$. Lastly, we can check that our solution of (5.2.1) is $U(t) = e^{\int_0^t \kappa(s)ds}\tilde{U}(t)$. Indeed,

$$\begin{aligned} \partial_t U(t) &= \kappa(t)e^{\int_0^t \kappa(s)ds}\tilde{U}(t) + e^{\int_0^t \kappa(s)ds}\tilde{U}_t(t) \\ &= \kappa(t)e^{\int_0^t \kappa(s)ds}\tilde{U}(t) + e^{\int_0^t \kappa(s)ds}\tilde{\mathcal{A}}(t)\tilde{U}(t) \\ &= e^{\int_0^t \kappa(s)ds}(\kappa(t)\tilde{U}(t) + \tilde{\mathcal{A}}(t)\tilde{U}(t)) \\ &= e^{\int_0^t \kappa(s)ds}\mathcal{A}(t)\tilde{U}(t) = \mathcal{A}(t)e^{\int_0^t \kappa(s)ds}\tilde{U}(t) \\ &= \mathcal{A}(t)U(t), \end{aligned}$$

which concludes the proof. \square

Well-posedness of the linear system with extra source term

Consider now (5.2.1) with a source term f in the u -equation

$$\begin{cases} (\partial_t u + \partial_x u + \partial_x^3 u)(t, x) = f(t, x), & t > 0, \quad x \in (0, L), \\ \tau(t)\partial_t z(t, \rho) + (1 - \dot{\tau}(t)\rho)z_\rho(t, \rho) = 0, & t > 0, \quad \rho \in (0, 1), \\ u(t, 0) = u(t, L) = 0, & t > 0, \\ \partial_x u(t, L) = \alpha\partial_x u(t, 0) + \beta z(t, 1), & t > 0, \\ z(t, 0) = \partial_x u(t, 0), & t > 0, \\ u(0, x) = u_0(x), & x \in (0, L), \\ z(0, \rho) = z_0(-\tau(0)\rho), & \rho \in (0, 1). \end{cases} \quad (5.2.7)$$

Proposition 5.2.1. *Assume that (5.1.3)-(5.1.6) hold. Let $U_0 = (u_0, z_0) \in H$ and $f \in L^1((0, \infty), L^2(0, L))$. Then there exists a unique mild solution $U = (u, z) \in C([0, +\infty), H)$ to (5.2.7). Moreover, for $T > 0$, the following estimates hold*

$$\|(u, z)\|_{C([0, T], H)} \leq C (\|U_0\|_H + \|f\|_{L^1((t, 0), L^2(0, L))}), \quad (5.2.8)$$

$$\|u\|_{L^2((t, 0), H^1(0, L))} \leq C (\|U_0\|_H + \|f\|_{L^1((t, 0), L^2(0, L))}). \quad (5.2.9)$$

Proof. The above system can be written as $\partial_t U(t) = \mathcal{A}(t)U(t) + (f, 0)$. Using [Kat75, Theorem 2] we can show that if $U_0 \in H$ and $f \in L^1((0, \infty), L^2(0, L))$, then there exists a unique solution $U \in C([0, \infty), H)$. Furthermore, $U \in C([0, \infty), D(\mathcal{A}(0))) \cap C^1([0, \infty), H)$ if $U_0 \in D(\mathcal{A}(0))$ and $f \in C([0, \infty), L^2(0, L)) \cap L^1((0, \infty), D(\mathcal{A}(0)))$. Now take $U = (u, z)$ a classical solution of (5.2.7). Let us choose the following energy

$$E(t) = \frac{1}{2} \int_0^L (u(t, x))^2 dx + \frac{|\beta|\tau(t)}{2} \int_0^1 (z(t, \rho))^2 d\rho \quad (5.2.10)$$

corresponding to the time-dependent norm $\|\cdot\|_t$ on H . Differentiating (5.2.10), we obtain

$$\dot{E}(t) = \int_0^L u \partial_t u dx + \frac{|\beta| \dot{\tau}(t)}{2} \int_0^1 z^2 d\rho + |\beta| \tau(t) \int_0^1 z \partial_t z d\rho.$$

Now, using (5.2.7) and integrations by parts, we derive

$$\begin{aligned} \dot{E}(t) &= \frac{1}{2} [(\alpha^2 - 1 + |\beta|)(\partial_x u(t, 0))^2 + 2\alpha\beta \partial_x u(t, 0)z(t, 1) + (\beta^2 - |\beta|(1 - \dot{\tau}(t)))(z(t, 1))^2] \\ &\quad + \int_0^L u f dx. \end{aligned}$$

Using (5.1.4)-(5.1.6) we get

$$\dot{E}(t) + \begin{pmatrix} \partial_x u(t, 0) \\ z(t, 1) \end{pmatrix}^\top \begin{pmatrix} -\frac{1}{2}\Phi_{\alpha, \beta} \end{pmatrix} \begin{pmatrix} \partial_x u(t, 0) \\ z(t, 1) \end{pmatrix} \leq \int_0^L u f dx.$$

Notice that $-\Phi_{\alpha, \beta}$ is a symmetric positive definite matrix. Then there exists $C > 0$ such that

$$\dot{E}(t) + (\partial_x u(t, 0))^2 + (z(t, 1))^2 \leq C \int_0^L u f dx.$$

Now take $0 \leq s \leq T$ and integrate the above expression on $[0, s]$ to obtain

$$E(s) + \int_0^s (\partial_x u(t, 0))^2 dt + \int_0^s (z(t, 1))^2 dt \leq C \left(\int_0^s \int_0^L u f dx dt + E(0) \right). \quad (5.2.11)$$

Thus, by (5.2.3) and the Cauchy-Schwarz inequality, we have

$$\|(u(s, \cdot), z(s, \cdot))\|_H^2 \leq C (\|U_0\|_H^2 + \|f\|_{L^1((t, 0), L^2(0, L))}) \|(u, z)\|_{C([0, T], H)}.$$

Taking the maximum for $s \in [0, T]$ and using the Young inequality, we conclude (5.2.8). In addition, taking $s = T$ in (5.2.11) and using (5.2.8) we obtain the following hidden regularity:

$$\int_0^T (\partial_x u(t, 0))^2 dt + \int_0^T (z(t, 1))^2 dt \leq C (\|U_0\|_H^2 + \|f\|_{L^1((t, 0), L^2(0, L))}^2). \quad (5.2.12)$$

Multiplying u -equation of (5.2.7) by xu , integrating on $(t, 0) \times (0, L)$ and performing integration by parts, we get

$$\begin{aligned} \frac{1}{2} \int_0^L x(u(T, x))^2 dx + \frac{3}{2} \int_0^T \int_0^L (\partial_x u)^2 dx dt &= \frac{1}{2} \int_0^L x(u_0)^2 dx + \frac{1}{2} \int_0^T \int_0^L u^2 dx dt \\ &\quad + \frac{1}{2} \int_0^T L(\partial_x u(t, L))^2 dt + \int_0^T \int_0^L x f u dx dt, \end{aligned}$$

and then

$$\begin{aligned} \|\partial_x u\|_{L^2((0, L) \times (t, 0))}^2 &\leq C \left(\int_0^L (u_0)^2 dx + \int_0^T \int_0^L u^2 dx dt + \int_0^T (\partial_x u(t, 0))^2 dt + \int_0^T (z(1, t))^2 dt \right. \\ &\quad \left. + \int_0^T \int_0^L f u dx dt \right). \end{aligned}$$

Finally, using (5.2.8) and (5.2.12) we derive (5.2.9). \square

Well-posedness of the nonlinear system

Now we are ready to prove the local well-posedness result for the nonlinear system (5.1.1).

Theorem 5.2.3. *Let $T > 0$, $L > 0$ and assume that (5.1.3)-(5.1.6) hold. Then there exist $r, C > 0$ such that for every $(u_0, z_0) \in H$ satisfying $\|(u_0, z_0)\|_H \leq r$, there exists a unique mild solution $u \in B_T$ of the system (5.1.1) verifying $\|u\|_{B_T} \leq C\|(u_0, z_0)\|_H$.*

Proof. Let $(u_0, z_0) \in H$ such that $\|(u_0, z_0)\|_H \leq r$ for $r > 0$ chosen small enough later. Take $\tilde{u} \in B_T$ and consider the map $P : B_T \rightarrow B_T$, defined by $P(\tilde{u}) = u$, where u is the solution of

$$\begin{cases} (\partial_t u + \partial_x u + \partial_x^3 u)(t, x) = (-\tilde{u} \partial_x \tilde{u})(t, x), & t > 0, x \in (0, L), \\ u(t, 0) = u(t, L) = 0, & t > 0, \\ \partial_x u(t, L) = \alpha \partial_x u(t, 0) + \beta \partial_x u(t - \tau(t), 0), & t > 0, \\ u(0, x) = u_0(x), & x \in (0, L), \\ \partial_x u(t - \tau(0), 0) = z_0(t - \tau(0)), & 0 < t < \tau(0). \end{cases}$$

Clearly, $u \in B_T$ is a solution of (5.1.1) if and only if u is a fixed-point of P . Now from Proposition 5.2.1 we find that the map P is well-defined and from Proposition 1.2.1, (5.2.8)-(5.2.9), we get

$$\begin{aligned} \|P(\tilde{u})\|_{B_T} &= \|u\|_{B_T} \leq C(\|(u_0, z_0)\|_H + \|\tilde{u} \partial_x \tilde{u}\|_{L^1((t,0), L^2(0,L))}) \\ &\leq C(\|(u_0, z_0)\|_H + \|\tilde{u}\|_B^2). \end{aligned}$$

Following the same arguments, we can show that

$$\|P(\tilde{u}_1) - P(\tilde{u}_2)\|_B \leq C(\|\tilde{u}_1\|_B + \|\tilde{u}_2\|_B) \|\tilde{u}_1 - \tilde{u}_2\|_B.$$

Now we restrict P to the closed ball $\{\tilde{u} \in B, \|\tilde{u}\|_{B_T} \leq R\}$, where $R > 0$ to be chosen later. Then,

$$\begin{aligned} \|P(\tilde{u})\|_{B_T} &\leq C(r + R^2), \\ \|P(\tilde{u}_1) - P(\tilde{u}_2)\|_{B_T} &\leq 2CR\|\tilde{u}_1 - \tilde{u}_2\|_{B_T}. \end{aligned}$$

Finally, it is enough to consider $R < \frac{1}{2C}$ and $r < \frac{R}{2C}$. With this choice, P maps the closed ball $\{\tilde{u} \in_T B, \|\tilde{u}\| \leq R\}$ into itself and $\|P(\tilde{u}_1) - P(\tilde{u}_2)\|_{B_T} \leq 2CR\|\tilde{u}_1 - \tilde{u}_2\|_{B_T}$ with $2CR < 1$. Lastly, we deduce the well-posedness result by invoking the Banach fixed-point theorem on the map P . \square

5.2.2 Well-posedness result of (5.1.2)

The goal of this section is to prove appropriate global well-posedness result of (5.1.2). We adopt the same methodology as in subsection 5.2.1, so we skip here some details.

Well-posedness of the linear system

In this subsection, we will study the well-posedness result of the KdV equation (5.1.2) linearized around 0, that is

$$\begin{cases} (\partial_t u + \partial_x u + \partial_x^3 u)(t, x) + a(x)u(t, x) + b(x)u(t - \tau(t), x) = 0, & t > 0, x \in (0, L), \\ u(t, 0) = u(t, L) = \partial_x u(t, L) = 0, & t > 0, \\ u(0, x) = u_0(x), & x \in (0, L), \\ u(t - \tau(0), x) = z_0(t - \tau(0), x), & 0 < t < \tau(0), x \in (0, L). \end{cases}$$

As previously, we introduce $z(t, \rho, x) = u|_{\omega}(t - \tau(t)\rho, x)$ for any $x \in \omega$, $\rho \in (0, 1)$ and $t > 0$, and define $U = (u, z)^\top$. This problem can be rewritten as the following first-order evolution equation

$$\begin{cases} \partial_t U(t) = \mathcal{A}_2(t)U(t), & t > 0, \\ U(0) = (u_0, z_0(-\tau(0)\cdot, \cdot))^\top =: U_0, \end{cases} \quad (5.2.13)$$

where the time-dependent operator $\mathcal{A}_2(t)$ is defined by

$$\mathcal{A}_2(t) \begin{pmatrix} u \\ z \end{pmatrix} = \begin{pmatrix} -\partial_x u - \partial_x^3 u - au - b\tilde{z}(1, \cdot) \\ \frac{\dot{\tau}(t)\rho - 1}{\tau(t)} z_\rho \end{pmatrix},$$

where $\tilde{z}(1, \cdot) \in L^2(0, L)$ is the extension of $z(1, \cdot)$ by zero outside ω , with domain

$$D(\mathcal{A}_2(t)) = \left\{ (u, z) \in H^3(0, L) \times L^2(\omega, H^1(0, 1)), \ u(0) = u(L) = \partial_x u(L) = 0, \right. \\ \left. z(0, x) = u|_\omega(x) \right\}.$$

The domain of the operator $\mathcal{A}_2(t)$ is independent of the time t , i.e $D(\mathcal{A}_2(t)) = D(\mathcal{A}_2(0))$, $t > 0$. The Hilbert space $H = L^2(0, L) \times L^2(\omega \times (0, 1))$, is provided with the time-dependent inner product

$$\left\langle \begin{pmatrix} u \\ z \end{pmatrix}, \begin{pmatrix} \tilde{u} \\ \tilde{z} \end{pmatrix} \right\rangle_t = \int_0^L u\tilde{u}dx + \tau(t) \int_\omega \int_0^1 \xi(x)z\tilde{z}d\rho dx,$$

where ξ is a nonnegative function in $L^\infty(0, L)$ such that $\text{supp } \xi = \text{supp } b = \omega$ and

$$\frac{1}{1-d} b(x) + c_0 \leq \xi(x) \leq 2a(x) - b(x) - c_0 \quad \text{in } \omega. \quad (5.2.14)$$

This choice of ξ is possible due to (5.1.8).

It is clear that the norm $\|\cdot\|_t$ is equivalent to the usual norm $\|\cdot\|_H$ on H : $\forall t \geq 0, \forall (u, z) \in H$,

$$(1 + \tau_0 b_0) \|(u, z)\|_H^2 \leq \|(u, z)\|_t^2 \leq (1 + 2M\|a\|_\infty) \|(u, z)\|_H^2, \quad (5.2.15)$$

using (5.1.3) and (5.1.7). The following theorem gives the existence and uniqueness results of (5.2.13).

Theorem 5.2.4. *Assume (5.1.3)-(5.1.5), that a and b are nonnegative functions belonging to $L^\infty(0, L)$ satisfying (5.1.7)-(5.1.8) and that $U_0 \in H$. Then there exists a unique mild solution $U \in C([0, +\infty), H)$ to (5.2.13). Moreover, if $U_0 \in D(\mathcal{A}_2(0))$, then the solution is classical and $U \in C([0, +\infty), D(\mathcal{A}_2(0))) \cap C^1([0, +\infty), H)$.*

Proof. As for Theorem 5.2.2, we prove the four assumptions of Theorem 5.2.1. We have, for all $t > 0$, $D(\mathcal{A}_2(t)) = D(\mathcal{A}_2(0))$, which is a dense subset of H . Let $t \in [0, T]$ be fixed. To prove 3. of Theorem 5.2.1, we start by computing $\langle \mathcal{A}_2(t)U, U \rangle_t$.

Let $U = (u, z)^\top \in D(\mathcal{A}_2(0))$. Similarly to the proof of Theorem 5.2.2, integrating by parts in space and in ρ , we obtain

$$\langle \mathcal{A}_2(t)U, U \rangle_t = \frac{1}{2} (\partial_x u)^2 \Big|_0^L - \int_0^L a(x)u^2 dx - \int_\omega b(x)z(1, x)u(x) dx + \frac{1}{2} \int_\omega \xi(x)(z(0, x))^2 dx \\ + \frac{1}{2} \int_\omega \xi(x)(\dot{\tau}(t) - 1)(z(1, x))^2 dx - \frac{1}{2} \dot{\tau}(t) \int_\omega \int_0^1 \xi(x)z^2 d\rho dx.$$

Since we have

$$- \int_\omega b(x)z(1, x)u(x) dx \leq \frac{1}{2} \int_\omega b(x)(z(1, x))^2 dx + \frac{1}{2} \int_\omega b(x)u^2 dx,$$

then

$$\langle \mathcal{A}_2(t)U, U \rangle_t \leq -\frac{1}{2} (\partial_x u(0))^2 + \int_\omega \left(-a(x) + \frac{b(x)}{2} + \frac{\xi(x)}{2} \right) u^2(x) dx - \int_{(0, L) \setminus \omega} a(x)u^2 dx \\ + \int_\omega \left(\frac{b(x)}{2} + \frac{\xi(x)(\dot{\tau}(t) - 1)}{2} \right) (z(1, x))^2 dx - \frac{1}{2} \dot{\tau}(t) \int_\omega \int_0^1 \xi(x)z^2 d\rho dx.$$

Taking ξ such that (5.2.14) is satisfied and from (5.1.4), we get $-a(x) + \frac{b(x)}{2} + \frac{\xi(x)}{2} < 0$ and $\frac{b(x)}{2} + \frac{\xi(x)(\dot{\tau}(t) - 1)}{2} \leq \frac{b(x)}{2} + \frac{\xi(x)(d-1)}{2} < 0$. Hence,

$$\langle \mathcal{A}_2(t)U, U \rangle_t - \kappa(t)\langle U, U \rangle_t \leq 0, \quad \text{where } \kappa(t) = \frac{(\dot{\tau}(t)^2 + 1)^{1/2}}{2\tau(t)},$$

which means that the operator $\widetilde{\mathcal{A}}_2(t) := \mathcal{A}_2(t) - \kappa(t)I$ is dissipative.

Now we will compute $\langle \mathcal{A}_2(t)^*U, U \rangle_t$, where $\mathcal{A}_2(t)^*$ is the adjoint of $\mathcal{A}_2(t)$. The adjoint $\mathcal{A}_2(t)^*$ is defined by

$$\mathcal{A}_2(t)^* \begin{pmatrix} u \\ z \end{pmatrix} = \begin{pmatrix} \partial_x u + \partial_x^3 u - au + \xi(x)\tilde{z}(0, \cdot) \\ \frac{1 - \dot{\tau}(t)\rho}{\tau(t)}z_\rho - \frac{\dot{\tau}(t)}{\tau(t)}z \end{pmatrix},$$

with domain

$$D(\mathcal{A}_2(t)^*) = \left\{ (u, z) \in H^3(0, L) \times L^2(\omega, H^1(0, 1)), \quad u(0) = u(L) = \partial_x u(0) = 0, \right. \\ \left. z(1, x) = \frac{-b(x)}{\xi(x)(1 - \dot{\tau}(t))} u \Big|_\omega(x) \right\}.$$

Then, for all $U = (u, z)^\top \in D(\mathcal{A}_2(t)^*)$, we get integrating by parts in space and in ρ ,

$$\begin{aligned} \langle \mathcal{A}_2(t)^*U, U \rangle_t &= -\frac{1}{2}(\partial_x u)^2 \Big|_0^L - \int_0^L a(x)u^2 dx + \int_\omega \xi(x)u(x)z(x, 0)dx \\ &\quad + \frac{1}{2} \int_\omega \int_0^1 \xi(x)\dot{\tau}(t)z^2 d\rho dx + \frac{1}{2} \int_\omega \xi(x)((1 - \dot{\tau}(t)\rho)z^2) \Big|_0^1 dx \\ &\quad - \int_\omega \int_0^1 \xi(x)\dot{\tau}(t)z^2 d\rho dx. \end{aligned}$$

Then, using the boundary conditions, we have

$$\begin{aligned} \langle \mathcal{A}_2(t)^*U, U \rangle_t &= -\frac{1}{2}(\partial_x u(t, L))^2 - \int_0^L a(x)u^2 dx + \int_\omega \xi(x)u(x)z(0, x)dx \\ &\quad - \frac{\dot{\tau}(t)}{2} \int_\omega \int_0^1 \xi(x)z^2 d\rho dx + \frac{1}{2} \int_\omega \frac{b^2(x)}{\xi(x)(1 - \dot{\tau}(t))} u^2(x)dx - \frac{1}{2} \int_\omega \xi(x)(z(0, x))^2 dx. \end{aligned}$$

Using Young's inequality, we obtain

$$\begin{aligned} \langle \mathcal{A}_2(t)^*U, U \rangle_t &\leq - \int_{(0, L) \setminus \omega} a(x)u^2 dx - \frac{\dot{\tau}(t)}{2} \int_\omega \int_0^1 \xi(x)z^2 d\rho dx \\ &\quad - \frac{1}{2} \int_\omega \left(2a(x) - \xi(x) - \frac{b^2(x)}{\xi(x)(1 - \dot{\tau}(t))} \right) u^2 dx. \end{aligned}$$

By (5.2.14) and (5.1.4), we have

$$\begin{aligned} \xi(x) &\leq 2a(x) - b(x) - c_0 \leq 2a(x) - \frac{b^2(x)}{\xi(x)(1 - d)} - c_0 \\ &\leq 2a(x) - \frac{b^2(x)}{\xi(x)(1 - \dot{\tau}(t))}, \end{aligned}$$

since $b(x) \leq \xi(x)(1 - d)$ (see (5.2.14)). Consequently $2a(x) - \xi(x) - \frac{b^2(x)}{\xi(x)(1 - \dot{\tau}(t))} \geq 0$. Hence,

$$\langle \mathcal{A}_2(t)^*U, U \rangle_t - \kappa(t)\langle U, U \rangle_t \leq 0,$$

which means that the operator $\widetilde{\mathcal{A}}_2(t)^* = \mathcal{A}_2(t)^* - \kappa(t)I$ is dissipative.

Since $\widetilde{\mathcal{A}}_2(t)$ and $\widetilde{\mathcal{A}}_2(t)^*$ are dissipative and $\widetilde{\mathcal{A}}_2(t)$ is a densely defined closed linear operator, then $\widetilde{\mathcal{A}}_2(t)$ is the infinitesimal generator of a C_0 semigroup of contraction on H (see [Paz83]) for any $t \in [0, T]$ be fixed. As the proof of Theorem 5.2.2, we can easily prove (5.2.4). Consequently, for all $t \in [0, T]$, $\widetilde{\mathcal{A}}_2(t)$ generates a strongly continuous semigroup on H and the family $\widetilde{\mathcal{A}}_2 = \{\widetilde{\mathcal{A}}_2(t) : t \in [0, T]\}$ is stable with stability constants C and m independent of t (see Proposition 3.4 of [Kat70]). These mean that 3. of Theorem 5.2.1 is satisfied. Finally, we can also prove, similarly to the proof of Theorem 5.2.2, that

$$\frac{d}{dt} \widetilde{\mathcal{A}}_2(t) \in L_*^\infty([0, T], B(D(\mathcal{A}_2(0)), H)).$$

Since all assumptions of Theorem 5.2.1 are verified, then the problem

$$\begin{cases} \partial_t \tilde{U}(t) = \widetilde{\mathcal{A}}_2(t) \tilde{U}, \\ \tilde{U}(0) = U_0, \end{cases}$$

has a unique solution $\tilde{U} \in C([0, +\infty), D(\mathcal{A}_2(0))) \cap C^1([0, +\infty), H)$ for $U_0 \in D(\mathcal{A}_2(0))$. The requested solution of (5.2.13) is then given by $U(t) = e^{\int_0^t \kappa(s) ds} \tilde{U}(t)$, similarly to the proof of Theorem 5.2.2. \square

Well-posedness of the linear system with a source term

In this subsection, we will study the well-posedness of the following linear KdV equation with a source term

$$\begin{cases} (\partial_t u + \partial_x u + \partial_x^3 u)(t, x) + a(x)u(t, x) + b(x)u(t, t - \tau(t)) = f(t, x), & t > 0, x \in (0, L), \\ u(t, 0) = u(t, L) = \partial_x u(t, L) = 0, & t > 0, \\ u(0, x) = u_0(x), & x \in (0, L), \\ u(t - \tau(0), x) = z_0(t - \tau(0), x), & 0 < t < \tau(0), x \in (0, L). \end{cases} \quad (5.2.16)$$

Proposition 5.2.2. *Assume (5.1.3)-(5.1.5) and that a and b are nonnegative functions belonging to $L^\infty(0, L)$ satisfying (5.1.7)-(5.1.8). For any $(u_0, z_0(-\tau(0)\cdot, \cdot)) \in H$ and $f \in L^1(0, T, L^2(0, L))$ there exists a unique mild solution $(u, u(t - \tau(t)\cdot, \cdot)) \in B_T \times C([0, T], H)$ to (5.2.16). Moreover, there exists $C > 0$ independent of T such that (5.2.8) and (5.2.9) hold.*

Proof. The proof is similar to the proof of Proposition 5.2.1 and is left to the readers (see also [Val22]). \square

Well-posedness of the nonlinear system (5.1.2)

Finally, we will show the global well-posedness result of the nonlinear system (5.1.2).

Theorem 5.2.5. *Let $L > 0$ and assume (5.1.3)-(5.1.5) and that a and b are nonnegative functions belonging to $L^\infty(0, L)$ satisfying (5.1.7) and (5.1.8). Then for any $(u_0, z_0(\cdot, -\tau(0)\cdot)) \in H$, there exists a unique mild solution $u \in B_T$ of (5.1.2).*

Proof. Following [PMVZ02], we can get the global existence of the solution by showing the local (in time) existence and using the decay of the energy. Let $\tilde{u} \in B_T$, we consider the map $\Psi : B_T \rightarrow B_T$ defined by $\Psi(\tilde{u}) = u$ where u is the solution of the following system

$$\begin{cases} (\partial_t u + \partial_x u + \partial_x^3 u)(t, x) + a(x)u(t, x) + b(x)u(t - \tau(t), x) = -(\tilde{u} \partial_x \tilde{u})(t, x), & t > 0, x \in (0, L), \\ u(t, 0) = u(t, L) = \partial_x u(t, L) = 0, & t > 0, \\ u(0, x) = u_0(x), & x \in (0, L), \\ u(t - \tau(0), x) = z_0(t - \tau(0), x), & 0 < t < \tau(0), x \in (0, L). \end{cases}$$

We can prove similarly to the proof of [Val22, Proposition 4] (see also Theorem 5.2.3) that Ψ is a contraction on the closed ball $\{u \in B_T : \|u\|_{B_T} \leq R\}$ for some chosen R . Hence, from the Banach fixed-point theorem, the map Ψ has a unique fixed-point $u \in B_T$ which is the solution of the nonlinear system (5.1.2). \square

5.3 Exponential stability results

In this section, we prove the exponential stability results, firstly with the boundary damping, secondly with the internal damping.

5.3.1 Boundary stability result

We start this section showing that for a solution of (5.1.1) the energy is a not-increasing function of time. We recall that the energy of (5.1.1) is defined by

$$E(t) = \frac{1}{2} \int_0^L (u(t, x))^2 dx + \frac{|\beta|\tau(t)}{2} \int_0^1 (\partial_x u(t - \tau(t)\rho, 0))^2 d\rho. \quad (5.3.1)$$

Proposition 5.3.1. *Suppose that (5.1.3)-(5.1.6) be satisfied. Then for any solution of (5.1.1), the energy defined by (5.3.1) is not increasing and satisfies*

$$\dot{E}(t) \leq \frac{1}{2} Y^\top \Phi_{\alpha, \beta} Y \leq 0, \quad (5.3.2)$$

where $Y = (\partial_x u(t, 0), \partial_x u(0, t - \tau(t)))^\top$.

Proof. It is enough to follow the proof of Proposition 5.2.1 and notice that for $u \in H_0^1(0, L)$, $\int_0^L u^2 \partial_x u dx = 0$. \square

Consider the following new Lyapunov candidate

$$V(t) = E(t) + \mu_1 V_1(t) + \mu_2 V_2(t), \quad (5.3.3)$$

where E is defined by (5.3.1), $\mu_1, \mu_2 > 0$ and

$$V_1(t) = \int_0^L x (u(t, x))^2 dx, \quad (5.3.4)$$

$$V_2(t) = \tau(t) \int_0^1 (1 - \rho) (\partial_x u(t - \tau(t)\rho, 0))^2 d\rho. \quad (5.3.5)$$

Note that V_1 is classical for the KdV equation and V_2 comes from the delay term depending on time.

Theorem 5.3.1. *Suppose that (5.1.3)-(5.1.6) are satisfied and assume that the length L fulfills $L < \pi\sqrt{3}$. Then, there exists $r > 0$ such that, for every $(u_0, z_0) \in H$ satisfying $\|(u_0, z_0)\|_0 \leq r$, the energy of the unique mild solution of (5.1.1) decays exponentially. More precisely, there exist two positive constants γ and C such that*

$$E(t) \leq C e^{-2\gamma t} E(0), \quad \forall t > 0, \quad (5.3.6)$$

with

$$\gamma \leq \min \left\{ \frac{(9\pi^2 - 3L^2 - 2L^{3/2}r\pi^2)\mu_1}{3L^2(1 + 2L\mu_1)}, \frac{(1-d)\mu_2}{M(2\mu_2 + |\beta|)} \right\}, \quad (5.3.7)$$

$$C \leq 1 + \max \left\{ L\mu_1, \frac{2\mu_2}{|\beta|} \right\},$$

where, μ_1 and μ_2 are taken such that

$$\mu_2 \leq \min \left\{ \frac{1 - \alpha^2 - \beta^2 - |\beta|d}{2}, \frac{(1 - |\beta|)(1 - |\beta| - d) + \alpha^2(d - 1)}{2(1 - |\beta| - d)}, \frac{|\beta| - \alpha^2(d - 1) - \beta^2}{2|\beta|} \right\}, \quad (5.3.8)$$

$$\mu_1 \leq \min \left\{ \frac{(1 - |\beta|)(1 - |\beta| - d) + \alpha^2(d - 1) + 2\mu_2(|\beta| + d - 1)}{2L(|\beta| - \alpha^2(d - 1) - \beta^2 - 2\mu_2|\beta|)}, \frac{1 - \alpha^2 - \beta^2 - |\beta|d + 2\mu_2}{2L(\alpha^2 + \beta^2)} \right\}. \quad (5.3.9)$$

Remark 5.3.1. We note that the decay rate γ decreases when the upper bound M of the delay $\tau(t)$ increases, as shown in the estimation of the decay rate (5.3.7). We can also observe the same phenomenon when d tends to 1.

Proof. Note that the function V is equivalent to the energy E . More precisely, for every $t > 0$,

$$E(t) \leq V(t) \leq \left(1 + \max \left\{ L\mu_1, \frac{2\mu_2}{|\beta|} \right\} \right) E(t). \quad (5.3.10)$$

Thus, it suffices to show that V decays exponentially. Let $\gamma > 0$ to fix later, we are going to prove that $\dot{V}(t) + 2\gamma V(t) \leq 0$. Let u solution of (5.1.1) with $(u_0, z_0)^T \in D(\mathcal{A}(0))$ such that $\|(u_0, z_0)\|_0 \leq r$ with $r > 0$ chosen later. First, we differentiate V_1 and use integration by parts to have

$$\begin{aligned} \dot{V}_1(t) &= L\alpha^2(\partial_x u(t, 0))^2 + 2L\alpha\beta\partial_x u(t, 0)\partial_x u(t - \tau(t), 0) + L\beta^2(\partial_x u(t - \tau(t), 0))^2 \\ &\quad - 3 \int_0^L (\partial_x u)^2 dx + \int_0^L u^2 dx + \frac{2}{3} \int_0^L u^3 dx. \end{aligned} \quad (5.3.11)$$

Similarly, we differentiate V_2 :

$$\begin{aligned} \dot{V}_2(t) &= \dot{\tau}(t) \int_0^1 (1 - \rho)(\partial_x u(t - \tau(t)\rho, 0))^2 d\rho \\ &\quad + 2\tau(t) \int_0^1 (1 - \rho)\partial_x u(t - \tau(t)\rho, 0)\partial_t \partial_x u(t - \tau(t)\rho, 0) d\rho. \end{aligned}$$

Noting that $-\tau(t)\partial_t \partial_x u(t - \tau(t)\rho, 0) = (1 - \dot{\tau}(t)\rho)\partial_\rho \partial_x u(t - \tau(t)\rho, 0)$ and performing integration by parts, we get

$$\dot{V}_2(t) = - \int_0^1 (1 - \dot{\tau}(t)\rho)(\partial_x u(t - \tau(t)\rho, 0))^2 d\rho + (\partial_x u(t, 0))^2. \quad (5.3.12)$$

Joining (5.3.2), (5.3.11) and (5.3.12) we have

$$\begin{aligned} \dot{V}(t) + 2\gamma V(t) &\leq Y^\top \left[\frac{1}{2}\Phi_{\alpha,\beta} + \Psi_{\alpha,\beta} \right] Y - 3\mu_1 \int_0^L u_x^2 dx + \frac{2}{3}\mu_1 \int_0^L u^3 dx \\ &\quad + (\mu_1 + \gamma + 2L\mu_1\gamma) \int_0^L u^2 dx + \left(\gamma|\beta|M + 2\mu_2\gamma M \right. \\ &\quad \left. - \mu_2(1 - d) \right) \int_0^1 u_x^2(t - \tau(t)\rho, 0) d\rho, \end{aligned}$$

where the matrix $\Psi_{\alpha,\beta}$ is defined by

$$\Psi_{\alpha,\beta} = \begin{pmatrix} L\mu_1\alpha^2 + \mu_2 & \mu_1\alpha\beta L \\ \mu_1\alpha\beta L & \mu_1\beta^2 L \end{pmatrix}.$$

Then, as $\Phi_{\alpha,\beta}$ is definite negative and by the continuity of the trace and the determinant, we find that for μ_1 and μ_2 small enough, the matrix $\frac{1}{2}\Phi_{\alpha,\beta} + \Psi_{\alpha,\beta}$ is negative definite. More precisely, following [BCV19, Remark 5], we can take μ_1 and μ_2 as in (5.3.8)-(5.3.9). For the term involving $\int_0^L u^3 dx$, note that

$$\begin{aligned} \int_0^L u^3 dx &\leq \|u\|_{L^\infty(0,L)}^2 \int_0^L |y| dx \\ &\leq \|u\|_{L^\infty(0,L)}^2 \|u\|_{L^2(0,L)} \sqrt{L}. \end{aligned}$$

By the injection of $H_0^1(0, L)$ into $L^\infty(0, L)$ we know that $\|u\|_{L^\infty(0, L)} \leq \sqrt{L}\|\partial_x u\|_{L^2(0, L)}$, then

$$\int_0^L u^3 dx \leq L^{3/2} \|\partial_x u\|_{L^2(0, L)}^2 \|u\|_{L^2(0, L)}.$$

Finally, using Proposition 5.3.1 we can obtain $\|U\|_0 \leq r$ and hence invoking Poincaré's inequality

$$\begin{aligned} \dot{V}(t) + 2\gamma V(t) &\leq \left(\frac{L^2}{\pi^2} (\mu_1 + \gamma + 2L\mu_1\gamma) + \frac{2}{3} L^{3/2} r \mu_1 - 3\mu_1 \right) \int_0^L (\partial_x u)^2 dx \\ &\quad + (\gamma|\beta|M + 2\mu_2\gamma M - \mu_2(1-d)) \int_0^1 (\partial_x u(t - \tau(t)\rho, 0))^2 d\rho. \end{aligned}$$

Now, following [BCV19], as $L < \pi\sqrt{3}$, it is possible to choose r small enough to have $r < \frac{3(3\pi^2 - L^2)}{2L^{3/2}\pi^2}$. Then, we can choose $\gamma > 0$ such that

$$\begin{aligned} \frac{L^2}{\pi^2} (\mu_1 + \gamma + 2L\mu_1\gamma) + \frac{2}{3} L^{3/2} r \mu_1 - 3\mu_1 &\leq 0, \\ \gamma|\beta|M + 2\mu_2\gamma M - \mu_2(1-d) &< 0. \end{aligned}$$

Thus, we can easily obtain (5.3.7). Therefore, we have $\dot{V}(t) + 2\gamma V(t) \leq 0$ and hence $V(t) \leq V(0)e^{-2\gamma t}$ for all $t > 0$. Using (5.3.10) we obtain (5.3.6). Since $D(\mathcal{A}(0))$ is dense in H , we can take $(u_0, z_0) \in H$. \square

5.3.2 Internal stability result

In this section, we will study the local stability of (5.1.2) using some Lyapunov functional. We consider the following definition of the energy of the nonlinear system (5.1.2)

$$E(t) = \frac{1}{2} \int_0^L (u(t, x))^2 dx + \frac{\tau(t)}{2} \int_\omega \int_0^1 \xi(x) (u(t - \tau(t)\rho, x))^2 d\rho dx, \quad (5.3.13)$$

where ξ is defined by (5.2.14). In the following proposition, we will prove the decay of the energy of the nonlinear system (5.1.2).

Proposition 5.3.2. *Assume (5.1.3)-(5.1.5) and that a and b are nonnegative functions belonging to $L^\infty(0, L)$ satisfying (5.1.7) and (5.1.8). Then, for any solution of (5.1.2), the energy E defined by (5.3.13) is non-increasing and satisfies*

$$\begin{aligned} \dot{E}(t) &\leq -\frac{1}{2} (\partial_x u(t, 0))^2 - \int_{(0, L) \setminus \omega} a(x) (u(t, x))^2 dx - \frac{1}{2} \int_\omega (2a(x) - b(x) - \xi(x)) u^2(t, x) dx \\ &\quad - \frac{1}{2} \int_\omega (\xi(x)(1-d) - b(x)) (u(t - \tau(t), x))^2 dx \\ &\leq 0. \end{aligned} \quad (5.3.14)$$

Proof. The proof is similar to the proof of the dissipativity of $\mathcal{A}_2(t)$, noting that $\int_0^L u^2 u_x dx = 0$ (see also the proof of Proposition 5.2.1). \square

Now we take the following Lyapunov functional

$$V(t) = E(t) + \mu_1 V_1(t) + \mu_2 V_3(t), \quad (5.3.15)$$

where $\mu_1 > 0$ and $\mu_2 > 0$ are fixed constants taken small enough, E is the energy defined by (5.3.13), V_1 by (5.3.4) and V_3 is defined by

$$V_3(t) = \tau(t) \int_\omega \int_0^1 (1 - \rho) (u(t - \tau(t)\rho, x))^2 d\rho dx. \quad (5.3.16)$$

From the definition of $V(t)$ and $E(t)$, we have for any $t > 0$,

$$E(t) \leq V(t) \leq \left(1 + \max \left\{ L\mu_1, \frac{\mu_2}{b_0} \right\}\right) E(t). \quad (5.3.17)$$

In the following theorem, we will prove that the energy of the nonlinear system (5.1.2) decays exponentially.

Theorem 5.3.2. *Assume (5.1.3)-(5.1.5) and that a and b are nonnegative functions belonging to $L^\infty(0, L)$ that satisfy (5.1.7) and (5.1.8), and assume that the length L satisfies $L < \pi\sqrt{3}$. Then, there exists $r > 0$ small enough, such that, for every $(u_0, z_0) \in H$ satisfying $\|(u_0, z_0)\|_0 \leq r$, the energy of the unique mild solution of (5.1.2) decays exponentially. More precisely, there exist two positive constants γ and C such that*

$$E(t) \leq Ce^{-2\gamma t} E(0), \quad \forall t > 0,$$

with

$$\gamma \leq \min \left\{ \frac{(9\pi^2 - 3L^2 - 2L^{3/2}r\pi^2)\mu_1}{3L^2(1 + 2L\mu_1)}, \frac{(1-d)\mu_2}{M(2\mu_2 + \|\xi\|_{L^\infty(0,L)})} \right\}, \quad (5.3.18)$$

$$C \leq 1 + \max \left\{ L\mu_1, \frac{2\mu_2}{b_0} \right\},$$

where, μ_1 and μ_2 are taken such that

$$\mu_1 \leq \inf_{x \in \omega} \left\{ \frac{2a(x) - b(x) - \xi(x)}{2Lb(x)}, \frac{(1-d)\xi(x) - b(x)}{2Lb(x)} \right\}, \quad (5.3.19)$$

$$\mu_2 \leq \inf_{x \in \omega} \left\{ \frac{2a(x) - b(x) - \xi(x) - 2\mu_1 Lb(x)}{2} \right\}. \quad (5.3.20)$$

Remark 5.3.2. *We note that the decay rate γ decreases when the upper bound M of the delay $\tau(t)$ increases, as shown in the estimation of the decay rate (5.3.18). We can also observe the same phenomenon when d tends to 1.*

Proof. Since E and V are equivalent from (5.3.17), we will prove that V decays exponentially, so we will prove that $\dot{V}(t) + 2\gamma V(t) \leq 0$ for all $t > 0$. Assume that u is a solution of (5.1.2) with $(u_0, z_0(-\tau(0)\cdot, \cdot))^T \in D(\mathcal{A}_2(0))$ satisfying $\|(u_0, z_0(-\tau(0)\cdot, \cdot))\|_0 \leq r$. We start by differentiating V_1 and integrating by parts, we get

$$\begin{aligned} \dot{V}_1(t) &= -3 \int_0^L (\partial_x u(t, x))^2 dx + \int_0^L (u(t, x))^2 dx + \frac{2}{3} \int_0^L (u(t, x))^3 dx - 2 \int_0^L xa(x)(u(t, x))^2 dx \\ &\quad - 2 \int_0^L xb(x)u(t, x)u(t - \tau(t), x) dx. \end{aligned}$$

Now, we differentiate V_3 and integrating by parts, we get, using $-\tau(t)\partial_t u(t - \tau(t)\rho, x) = (1 - \dot{\tau}(t)\rho)\partial_\rho u(t - \tau(t)\rho, x)$,

$$\dot{V}_3(t) = \int_\omega (u(t, x))^2 dx - \int_\omega \int_0^1 (1 - \dot{\tau}(t)\rho)(u(t - \tau(t)\rho, x))^2 d\rho dx.$$

Then

$$\begin{aligned}
\dot{V}(t) + 2\gamma V(t) &\leq \frac{1}{2} \int_{\omega} (-2a(x) + b(x) + \xi(x) + 2\mu_1 Lb(x) + 2\mu_2)(u(t, x))^2 dx \\
&\quad + \frac{1}{2} \int_{\omega} (b(x) - (1-d)\xi(x) + 2\mu_1 Lb(x))u^2(t - \tau(t), x) dx \\
&\quad - \int_{(0, L) \setminus \omega} a(x)(u(t, x))^2 dx + (\mu_1 + \gamma + 2\gamma\mu_1 L) \int_0^L (u(t, x))^2 dx \\
&\quad - 3\mu_1 \int_0^L (\partial_x u(t, x))^2 dx + \frac{2}{3}\mu_1 \int_0^L (u(t, x))^3 dx \\
&\quad + \int_{\omega} \int_0^1 (\gamma\xi(x)\tau(t) + 2\gamma\mu_2\tau(t) - \mu_2(1-d))u^2(t - \tau(t)\rho, x) d\rho dx.
\end{aligned}$$

Using Poincaré's inequality, we obtain

$$\begin{aligned}
\dot{V}(t) + 2\gamma V(t) &\leq \frac{1}{2} \int_{\omega} (-2a(x) + b(x) + \xi(x) + 2\mu_1 Lb(x) + 2\mu_2)(u(t, x))^2 dx \\
&\quad + \frac{1}{2} \int_{\omega} (b(x) - (1-d)\xi(x) + 2\mu_1 Lb(x))(u(t - \tau(t), x))^2 dx \\
&\quad + \left(\frac{L^2(\mu_1 + \gamma + 2\gamma\mu_1 L)}{\pi^2} - 3\mu_1 \right) \int_0^L (\partial_x u(t, x))^2 dx + \frac{2}{3}\mu_1 \int_0^L (u(t, x))^3 dx \\
&\quad + \int_{\omega} \int_0^1 (\gamma\xi(x)M + 2\gamma\mu_2 M - (1-d)\mu_2)(u(t - \tau(t)\rho, x))^2 d\rho dx.
\end{aligned}$$

From (5.2.14), we can choose μ_1 and μ_2 small enough to get $-2a(x) + b(x) + \xi(x) + 2\mu_1 Lb(x) + 2\mu_2 \leq 0$ and $b(x) - (1-d)\xi(x) + 2\mu_1 Lb(x) \leq 0$ in ω . More precisely, by (5.2.14), we can take μ_1 and μ_2 as in (5.3.19)-(5.3.20). From the Cauchy-Schwarz inequality and Proposition 5.3.2, we get, as in the proof of Theorem 5.3.1,

$$\int_0^L (u(t, x))^3 dx \leq L\sqrt{Lr} \|\partial_x u(t, \cdot)\|_{L^2(0, L)}^2.$$

Finally, we obtain

$$\begin{aligned}
\dot{V}(t) + 2\gamma V(t) &\leq \left(\frac{L^2(\mu_1 + \gamma + 2\gamma\mu_1 L)}{\pi^2} - 3\mu_1 + \frac{2rL^{3/2}\mu_1}{3} \right) \int_0^L \partial_x u^2(t, x) dx \\
&\quad + \int_{\omega} \int_0^1 (\gamma\xi(x)M + 2\gamma\mu_2 M - (1-d)\mu_2)u^2(t - \tau(t)\rho, x) d\rho dx.
\end{aligned}$$

It is sufficient to have $\frac{L^2(\mu_1 + \gamma + 2\gamma\mu_1 L)}{\pi^2} - 3\mu_1 + \frac{2rL^{3/2}\mu_1}{3} \leq 0$ and $\gamma\xi(x)M + 2\gamma\mu_2 M - (1-d)\mu_2 \leq 0$. Hence, we take γ as in (5.3.18), where r can be chosen such that $9\pi^2 - 3L^2 - 2L^{3/2}r\pi^2 > 0$ which means that $0 < r < \frac{9\pi^2 - 3L^2}{2L^{3/2}\pi^2}$, and which is possible since $0 < L < \sqrt{3}\pi$.

Finally, we integrate $\dot{V}(t) + 2\gamma V(t) \leq 0$ over $(t, 0)$ to obtain $V(t) \leq V(0)e^{-2\gamma t}$, for all $t > 0$. From (5.3.17), we obtain

$$E(t) \leq E(0) \left(1 + \max\{L\mu_1, \frac{2\mu_2}{b_0}\} \right) e^{-2\gamma t}, \quad \forall t > 0.$$

Since $D(\mathcal{A}_2(0))$ is dense in H , we can take $(u_0, z_0(-\tau(0)\cdot, \cdot)) \in H$. \square

5.4 Numerical simulations

The aim of this section is to illustrate the stability results obtained in this work with some numerical simulations that adapt the schemes used in [BCV19, CG01, PCP22a]. We choose a final time T

and build a uniform spatial and time discretization of $N_x + 1$ and $N_t + 1$ points, respectively, separated by the steps $\Delta x = L/N_x$ and $\Delta t = T/N_t$. We present now the numerical scheme in the case of boundary delay. The internal case follows similar ideas, (see [PCP22a] for a similar scheme in the case of constant delay in a network). We choose the delay step $\Delta\rho = 1/N_\rho$. Now we introduce the notation $u(i\Delta x, n\Delta t) = u_i^n$ and $z(k\Delta\rho, n\Delta t) = z_k^n$ for $i = 0, \dots, N_x$, $k = 0, \dots, N_\rho$ and $n = 0, \dots, N_t$. We use the following approximation for the derivatives:

$$\begin{aligned} D_x^+ u_i &= \frac{u_{i+1} - u_i}{\Delta x}, & D_x^- u_i &= \frac{u_i - u_{i-1}}{\Delta x}, \\ D_x y_i &= \frac{u_{i+1} - u_{i-1}}{2\Delta x}, & D_\rho^+ z_k &= \frac{z_{k+1} - z_k}{\Delta\rho}. \end{aligned}$$

To approximate the term of third order ∂_x^3 , we use $D_x^+ D_x^+ D_x^-$. To approximate the nonlinear term, we use explicit approximation $u_i^n D_x^+ u_i^n$. Note now that by the boundary conditions we have that $u_{N_x}^n = u_0^n = 0$, $z_0^n = u_1^n / \Delta x$ and $u_{N_x-1}^n = -\alpha u_1^n - \beta \Delta x z_{N_\rho}^n$ for all $n = 0, \dots, N_t$. Then, taking $\mathcal{C} = D_x^+ D_x^+ D_x^- + D_x$, $\tau^n = \tau(n\Delta t)$ and $\dot{\tau}^n = \dot{\tau}(n\Delta t)$, our scheme can be seen as

$$\begin{cases} \frac{u_i^{n+1} - u_i^n}{\Delta t} + (\mathcal{C}u_i^{n+1})_i + u_i^n D_x^+ u_i^n = 0, & i = 1, \dots, N_x - 1, \\ \tau^{n+1} \left(\frac{z_k^{n+1} - z_k^n}{\Delta t} \right) + (1 - \dot{\tau}^{n+1} k \Delta\rho) (D_\rho^+ z_k^{n+1}) = 0, & k = 1, \dots, N_\rho - 1, \\ y_{N_x}^n = u_0^n = 0, & n = 1, \dots, N_t, \\ z_0^n = u_1^n / \Delta x, & n = 1, \dots, N_t, \\ u_{N_x-1}^n = -\alpha u_1^n - \beta \Delta x z_{N_\rho}^n, & n = 1, \dots, N_t, \\ u_i^0 = u_0(i\Delta x), & i = 1, \dots, N_x, \\ z_k^0 = z_0(-\tau(0)k\Delta\rho), & k = 1, \dots, N_\rho. \end{cases}$$

Now, we use this scheme with the following parameters $L = 1$ and $T = 10$. For the discretization, we use $N_x = 100$, $N_\rho = 100$ and $N_t = 100$. The initial conditions are $u_0(x) = 0.5(1 - \cos(2\pi x))$, $z_0(\rho) = -0.5 \sin(2\pi\rho)$ and the delay is $\tau(t) = d(1.5 + \sin(t))$.

For Figure 5.1 we use $\alpha = 0.1$ and $\beta = 0.1$. We can observe how the decay rate depends on the size of d , as mentioned in Remark 5.3.1. In particular, in the case $d = 1.3$ which does not satisfy (5.1.4), the energy is not decreasing.

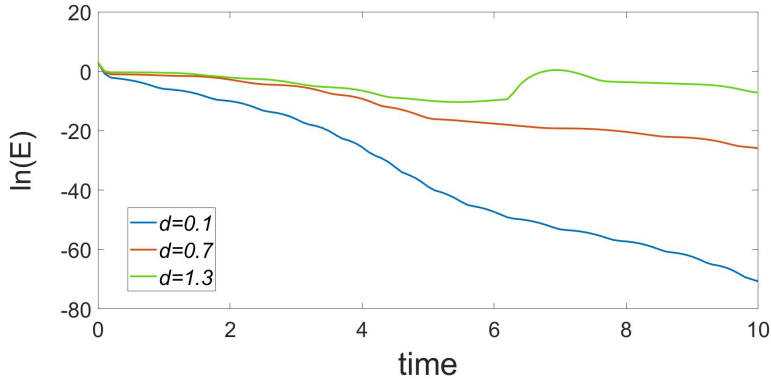


Figure 5.1: Time-evolution of $t \mapsto \ln(E(t))$ for different values of d (boundary delay).

For Figure 5.2 we consider the internal delay where the feedback terms are constant in their support $\text{supp } a = \text{supp } b = (0, L/2)$, $a(x) = 2$, $b(x) = 1$ and $\xi(x) = 2.1$. The initial conditions are $u_0(x) = 1 - \cos(2\pi x)$, $z_0(\rho, x) = (1 - \cos(2\pi x)) \cos(2\pi\rho)$ and the delay is $\tau(t) = M + \frac{\sin(t)}{2}$. We can observe how the decay rate depends on how large M is, as explained in Remark 5.3.2.

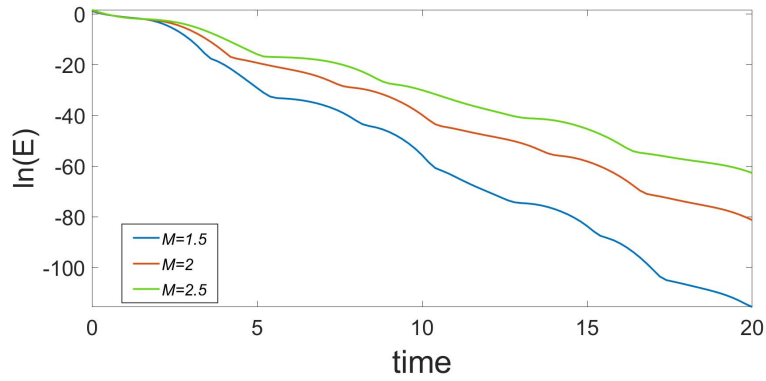


Figure 5.2: Time-evolution of $t \mapsto \ln(E(t))$ for different values of M (internal delay).

Finally, in Figure 5.3 we present a comparison between the action of time-varying delay and constant delay for boundary and internal feedbacks. We take $\tau(t) = d(1.5 + \sin(t))$, $\tau_{max} = 2.5d$ and $\tau_{min} = 0.5d$. In both figures, we see how the energy associated to time-varying delay is oscillating between the associated to $\tau_{max} = 2.5d$ and $\tau_{min} = 0.5d$.

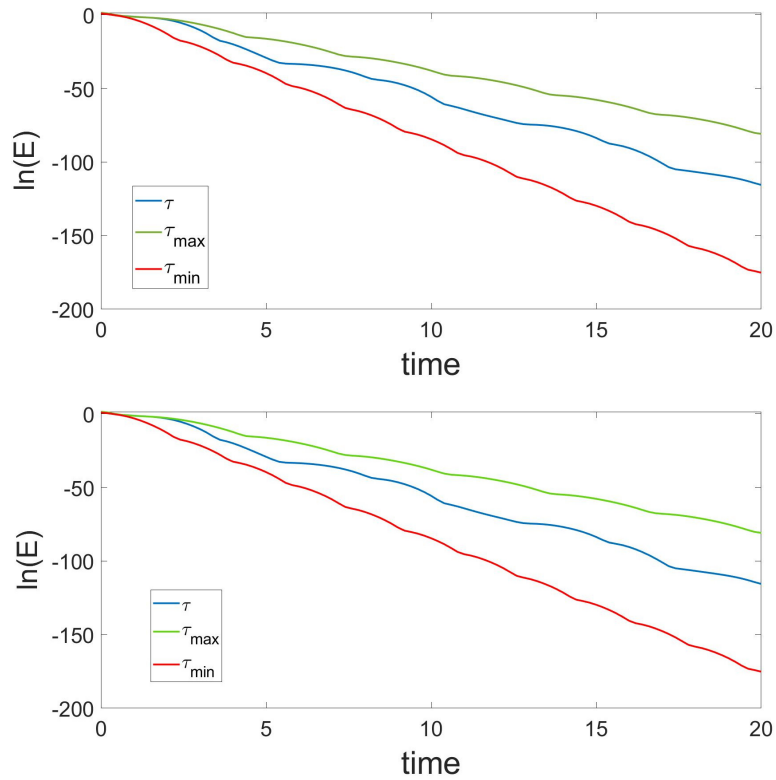


Figure 5.3: Time-evolution of $t \mapsto \ln(E(t))$ in the case of constant and varying (up) boundary delay (down) internal delay.

CONCLUSIONS AND PERSPECTIVES

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6.1 Comments on Chapter 2

In Chapter 2, it was studied the well-posedness and exponential stability of a KdV equation on a star-shaped network with internal delayed feedback terms. The well-posedness was addressed including a new variable in order to take into account the delay and then studying the linearization around 0 of our system, we obtain the local well-posedness for the nonlinear equation using the Banach fixed-point theorem. First, was considered the case where the support of delayed term b_j is included in the support of the feedback term without delay a_j . In this case, it was possible to obtain the local exponential stability using a Lyapunov function. This result holds for restricted lengths $L < \frac{\sqrt{3}}{2}\pi$, $\alpha > \frac{N}{2}$ and implies an estimation of the decay rate. This estimation of the decay rate depends strongly on the Lyapunov function used. Secondly, using a contradiction argument an observability inequality for the nonlinear system was derived, that allowed to prove the exponential stability system without restrictions on the lengths and $\alpha \geq \frac{N}{2}$. In the last stabilization results, the case where non necessarily the support of delayed terms b_j is included in the support of the feedback terms without delay a_j has been considered. If this is the case and if the feedback delayed term b_j is small enough, the local exponential stability for $L < \frac{\sqrt{3}}{2}\pi$ and $\alpha > \frac{N}{2}$ has been obtained.

Finally, some numerical simulations have been presented. We showed how feedback delayed terms affect the stability (see Figure 2.1 and Figure 2.2). In addition, we observe that numerically in the case $a_j = 0$ if the feedback delayed terms b_j are big enough, the system becomes unstable. Besides, we showed that the decay rate given by Theorem 2.1.1 is smaller than those obtained in simulations.

Open questions following Chapter 2

To conclude, we present some open questions to be investigated:

1. As written in Remark 2.3.1 the restriction on L comes from the multiplier x in V_1 . Finding a new multiplier in order to obtain a result less restrictive is an open problem.
2. In this chapter, internal delay has been studied. A possible research line is to consider a delay term acting on the central node.
3. In [GMC19] a stabilization problem for the linear Kuramoto-Sivashinsky with delayed boundary control was studied. Studying a Kuramoto-Sivashinsky equation on networks with or without delay is also a possible future work.

6.2 Comments on Chapter 3

In Chapter 3, the global well-posedness was studied and the exponential stability of a KdV equation on a star-shaped network with internal saturated feedback terms has been established. The well-posedness was addressed using the Laplace transform of the linearization and obtaining Kato smoothing properties which gave the local in-time well-posedness, then using multiplier estimates the global in-time result was dealt.

Open questions following Chapter 3

Generalization of the well-posedness result

In the work [CFSZ18] a complete result for general linear boundary conditions for the KdV equation on a bounded domain was derived. In this work, homogeneous Dirichlet and Neumann right conditions ($u_n(t, \ell_n) = \partial_x u_n(t, \ell_n) = 0$) were considered. These conditions come from the problems studied in [AC18, PCP22a], but in a more general framework the following problem could be studied:

$$\left\{ \begin{array}{ll} (\partial_t u_n + \partial_x u_n + u_n \partial_x u_n + \partial_x^3 u_n)(t, x) = f_n(t, x), & \forall x \in (0, \ell_n), t > 0, n = 1, \dots, N, \\ u_n(t, 0) = u_{n'}(t, 0), & \forall n, n' = 1, \dots, N, \\ \sum_{n=1}^N \partial_x^2 u_n(t, 0) = -\alpha u_1(t, 0) - \frac{N}{3} u_1^2(t, 0) + h(t), & t > 0, \\ u_n(t, \ell_n) = g_n(t), \quad \partial_x u_n(t, \ell_n) = p_n(t), & t > 0, n = 1, \dots, N, \\ u_n(0, x) = u_n^0, & x \in (0, \ell_n), \end{array} \right. \quad (6.2.1)$$

We expect that adapting the ideas introduced in this chapter and in [BSZ03], it could be possible to obtain the following result.

Conjecture 6.2.1. *Let $(\ell_n)_{n=1, \dots, N} \in (0, \infty)^N$, $0 \leq s \leq 3$ and $T > 0$. There exists $0 < T^* \leq T$ such that for all*

$$\underline{u}^0 \in \prod_{n=1}^N H^s(0, \ell_n), \quad (h, \underline{g}, \underline{p}) \in H^{\frac{s-1}{3}}(0, T) \times \prod_{n=1}^N H^{\frac{s+1}{3}}(0, T) \times \prod_{n=1}^N H^{\frac{s}{3}}(0, T),$$

$$\underline{f} \in \prod_{n=1}^N W^{\frac{s}{3}, 1}(0, T; L^2(0, \ell_n))$$

satisfying the compatibility condition,

$$\begin{cases} u_n^0(\ell_n) = g_n(0) & n = 1, \dots, N \quad \text{if } \frac{1}{2} < s \leq 3. \\ \partial_x u_n^0(\ell_n) = p_n(0) & n = 1, \dots, N \quad \text{if } \frac{3}{2} < s \leq 3. \\ \sum_{n=1}^N \partial_x^2 u_n^0(0) = h(0) & \text{if } \frac{5}{2} < s \leq 3. \end{cases}$$

there exists a unique solution $\underline{u} \in \prod_{n=1}^N C([0, T]; H^s(0, \ell_n)) \cap L^2(0, T^*; H^{s+1}(0, \ell_n))$ of (6.2.1). Moreover $\partial_x^\kappa u_n \in L_x^\infty(0, \ell_n; H^{\frac{s+1-\kappa}{3}}(0, T^*))$, for $\kappa = 0, 1, 2$.

The complications would come from the study of the matrix, which is obtained by replacing the column $j + 3(n - 1)$ of A_N by $[0 \ 1 \ 0 \ \dots \ 0]^\top$ for g_n case and $[0 \ 0 \ 1 \ \dots \ 0]^\top$ for p_n case. It is not clear how to derive a similar result as (3.2.23).

Generalization of stabilization results

The stabilization results were obtained, proving appropriate observability inequalities working directly on the nonlinear systems. In the work [PTdSJ16] more general feedback laws were considered as cone bounded control laws. Note that Theorem 3.1.4 and Theorem 3.1.5 hold, replacing \mathfrak{sat} by any odd nonlinearity that satisfies the properties given in Lemma B.1.1, Lemma B.1.2 and Lemma B.1.3.

6.3 Comments on Chapter 4

The exponential stability of the nonlinear KdV equation with a saturated feedback on the boundary was addressed. The well-posedness was obtained using nonlinear semigroup theory and fixed-point results. The exponential stability was shown via the classical observability approach, and the asymptotic stability of the linear KdV was addressed using LaSalle's invariance principle.

Open question following Chapter 4

An important open question arising in this context is to design a saturated boundary feedback term which gives stability when $L \in \mathcal{N}$. What it is known about the linear KdV equation with boundary conditions on $u(t, 0)$, $u(t, L)$ and $\partial_x u(t, L)$ is that it is never exactly controllable with one control from the right side when $L \in \mathcal{N}$ (see [Ros97] and [GG10]). In particular, concerning the system (KdVs) the feedback term $\alpha \partial_x u(t, 0)$, and the strategies employed here are not capable of dealing with critical lengths. A possible solution to this problem could be to take another boundary feedback term acting on the left Dirichlet condition as in [CC13] (see also Section 1.2.5), where a backstepping controller was designed. To saturate this control, an idea is to follow [KF17] and [KF18].

6.4 Comments on Chapter 5

In Chapter 5, we presented some boundary and internal stability results for the nonlinear KdV equation with time-varying delay. We proved appropriate well-posedness results, and we studied the local stability using some Lyapunov functionals. Finally, numerical simulations were presented to illustrate the results obtained.

Open question following Chapter 5

We could improve this paper in the following directions: remove the assumption $\text{supp } b \subset \text{supp } a$, consider the case L non-critical (and not only $L < \sqrt{3}\pi$) and the case $\tau_0 = 0$ in (5.1.3). These

questions remain open, since, for the two first, the system is not invariant by translation in time contrary to constant delay (see [CFGM23, Val22]), and for the last one, we need more regularity of the solutions (see [NPV11]). Moreover, the condition $L < \sqrt{3}\pi$ is a technical one and comes from the choice of the multiplier x in the expression of V_1 . To find a better multiplier is an open problem as far as we know. We mention here some possible future research: the cases of mixed boundary and internal damping with time-varying delay, time- and spatially-varying delay as in [LPS21] or study the stabilization problem when the delay (constant or variable) is in the nonlinear term as in [Liu02, Zhu14] for Burger's and Kuramoto-Sivashinsky equations, respectively.

6.5 Works in progress

The next two sections are related to some works in progress about the KdV equation on a star-shaped network.

6.5.1 Asymptotic behavior of KdV equation in a star-shaped network with bounded and unbounded lengths

This section is based on a work in progress with Emmanuelle Crépeau and Christophe Prieur.

In this case we consider the KdV equation on star-shaped networks mixing finite and infinite lengths as for example [AJK16, AG18] in the case of wave equation. With respect to the KdV equation posed on the half-line, we can mention, for instance, [BW83, Hol06] which focus on the well-posedness properties. In [Ros00], the exact controllability of the linear KdV equation posed on the half-line was obtained by using Carleman estimates. A first result of exponential stability of the KdV equation in the half-line considering a localized damping was derived in [LP09] under the assumption that the damping term $a(x) \geq c > 0$ in $(0, \delta) \cup (b, \infty)$ with $b > \delta$ (see [CDKR14] for a similar problem in the context of KdV-Burger equation in the whole-line and half-line). Then, in [PR10] exponential decay of the energy in weighted spaces was derived, and it was noticed that the interval $(0, \delta)$ can be dropped. We can mention also [PS13] where similar ideas of [PR10] were applied in the case of a Gear-Grimshaw system modeling long waves. Here inspired by [AC18, LP09] we study the exponential stabilization problem of the KdV equation posed in a star shaped network where the branches mix finite intervals and half-lines. Let $N = N_F + N_\infty$ the number of edges of the network \mathcal{T} described as the intervals I_j for $j = 1, \dots, N$, where $I_j = (0, \ell_j)$ with $\ell_j > 0$ for $j = 1, \dots, N_F$ and $I_j = (0, \infty)$ for $j = N_F + 1, \dots, N$. We consider the next evolution problems for the KdV equation

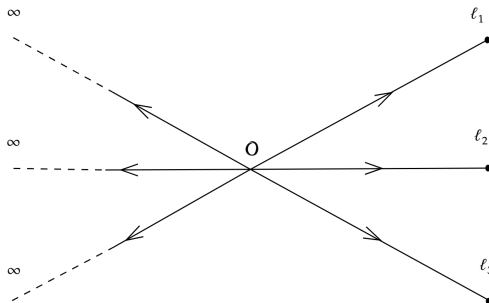


Figure 6.1: Star Shaped Network for $N_F = 3$ and $N_\infty = 3$.

$$\begin{cases} (\partial_t u_j + \partial_x u_j + u_j \partial_x u_j + \partial_x^3 u_j + a_j u_j)(t, x) = 0, & \forall x \in I_j, t > 0, j = 1, \dots, N, \\ u_j(t, 0) = u_{j'}(t, 0), & \forall j, j' = 1, \dots, N, \\ \sum_{j=1}^N \partial_x^2 u_j(t, 0) = -\alpha u_1(t, 0) - \frac{N}{3} (u_1(t, 0))^2, & t > 0, \\ u_j(t, \ell_j) = \partial_x u_j(t, \ell_j) = 0, & t > 0, j = 1, \dots, N_F, \\ u_j(0, x) = u_j^0(x), & x \in I_j, j = 1, \dots, N, \end{cases} \quad (\text{KdVinf})$$

and its linearization around zero:

$$\begin{cases} (\partial_t u_j + \partial_x u_j + \partial_x^3 u_j + a_j u_j)(t, x) = 0, & \forall x \in I_j, t > 0, j = 1, \dots, N, \\ u_j(t, 0) = u_{j'}(t, 0), & \forall j, j' = 1, \dots, N, \\ \sum_{j=1}^N \partial_x^2 u_j(t, 0) = -\alpha u_1(t, 0), & t > 0, \\ u_j(t, \ell_j) = \partial_x u_j(t, \ell_j) = 0, & t > 0, j = 1, \dots, N_F, \\ u_j(0, x) = u_j^0(x), & x \in I_j, j = 1, \dots, N, \end{cases} \quad (\text{LKdVinf})$$

where $\alpha > \frac{N}{2}$, the damping terms $(a_j)_{j=1, \dots, N} \in \mathbb{L}^\infty(\mathcal{T})$ act locally on the branches. Let $I_{act} \subset \{1, \dots, N\}$ the set of acted index, formally the damping terms a_j are defined by

$$\begin{cases} a_j \equiv 0 & \text{for } j \in \{1, \dots, N\} \setminus I_{act}, \\ a_j(x) \geq c_j > 0 & \text{in a nonempty open set } \omega_j \text{ of } I_j, \text{ for all } j \in I_{act}, \\ \text{If } N_F + 1 \leq j \leq N \in I_{act}, & \text{then } \omega_j = (b_j, \infty) \text{ for } b_j > 0 \text{ given.} \end{cases} \quad (6.5.1)$$

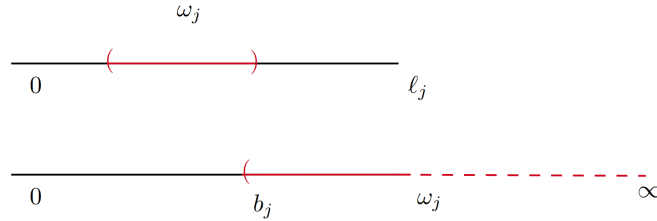


Figure 6.2: Structure of acted region (in red) in the case of bounded and unbounded branch.

For the linear system (LKdVinf), we can prove the following stabilization result:

Theorem 6.5.1. *Let $I_{act} \supseteq I_c^* \cup I_\infty$ or $I_{act} \supseteq I_c \cup I_\infty^*$, assume that the damping terms $(a_j)_{j=1, \dots, N}$ satisfy (3.1.6). Then, there exist $C, \mu > 0$ such that for all $\underline{u}^0 \in \mathbb{L}^2(\mathcal{T})$, the unique solution \underline{u} of (LKdVinf) satisfies $\|\underline{u}(t, \cdot)\|_{\mathbb{L}^2(\mathcal{T})} \leq C e^{-\mu t} \|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})}$, for all $t > 0$.*

Theorem 6.5.1 is interesting in the sense that we derive an exponential stability result for the nonlinear system without all the damping actives (see Figure 6.5.1). The main idea to prove Theorem 6.5.1 is to show appropriated regularity results to derive an observability inequality using compactness ideas.

Consider the space, $\mathbb{Z} = \prod_{j=1}^{N_F} L^2(I_j) \times (L^2_{(1+x^2)}(0, \infty))^{N_\infty}$. Then, if all the damping terms are

active and working directly with the nonlinear system (KdVinf), we expect to show the following exponential stability result:

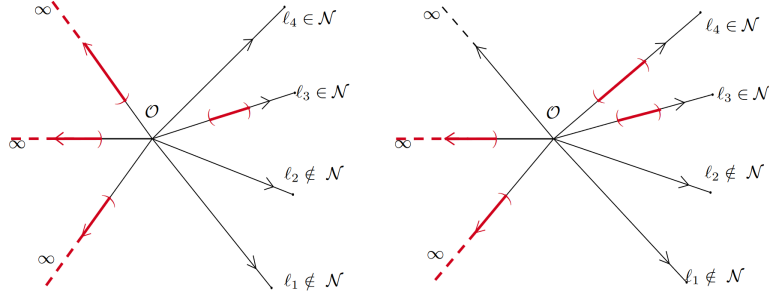


Figure 6.3: Possible acted branches of Theorem 3.1.5

Theorem 6.5.2. *Assume that the damping terms $(a_j)_{j=1,\dots,N}$ satisfy (3.1.6) and let $R > 0$. If $I_{act} = \{1, \dots, N\}$, then, there exist $C(R) > 0$ and $\mu(R) > 0$ such that for all $\underline{u}^0 \in \mathbb{Z}$ with $\|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})} \leq R$, the unique solution \underline{u} of (KdVinf) satisfies $\|\underline{u}(t, \cdot)\|_{\mathbb{L}^2(\mathcal{T})} \leq Ce^{-\mu t} \|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})}$, for all $t > 0$.*

The space \mathbb{Z} is introduced in order to have a well-posedness result. In this context, the main complication to derive Theorem 6.5.2 is to show that the nonlinear system (KdVinf) enjoys similar regularity results than the linear one (LKdVinf) to apply the compactness arguments.

Once Theorem 6.5.2 is proved, and introducing some Lyapunov functionals, we can prove the next result about the exponential stability in the same space as the well-posedness.

Theorem 6.5.3. *Assume that the damping terms $(a_j)_{j=1,\dots,N}$ satisfy (3.1.6) and let $R > 0$. If $I_{act} = \{1, \dots, N\}$, then, there exist $C(R) > 0$ and $\mu(R) > 0$ such that for all $\underline{u}^0 \in \mathbb{Z}$ with $\|\underline{u}^0\|_{\mathbb{Z}} \leq R$, the unique solution \underline{u} of (KdVinf) satisfies $\|\underline{u}(t, \cdot)\|_{\mathbb{Z}} \leq Ce^{-\mu t} \|\underline{u}^0\|_{\mathbb{Z}}$, for all $t > 0$.*

The results corresponding to the linear case has been submitted to the 62nd IEEE Conference on Decision and Control, 2023 Singapore.

6.5.2 Boundary null controllability of KdV equation in a star-shaped network

This section is based on a personal work in progress.

In this section, we talk about the null controllability problem of the linear KdV equation posed in a star shaped network. In the case of null controllability for a single KdV equation, we have the work [Ros04] where null controllability was achieved by using a Dirichlet control on the left, [GG08] where a uniform null controllability result was obtained and [Gui14] where null controllability was derived in the case of special boundary conditions. With respect to controllability problems in networks, we can mention [DZ06, LLS12] where control of multi-structures was studied, [CIP18] where the null controllability of the Kuramoto Sivashinsky equation on a star-shaped network was addressed. In [BCV11] and [IPR11], Carleman inequalities were obtained for wave, heat and Schrödinger equations in networks. We study in this part the null controllability of the linear KdV equation posed in a star shaped network, and we expect that our result can be extended to the nonlinear case.

Let $T > 0$ and define $Q_j = (0, T) \times I_j$

$$\begin{cases} \partial_t u_j(t, x) + \partial_x u_j(t, x) + \partial_x^3 u_j(t, x) = 0, & (t, x) \in Q_j, \quad j = 1, \dots, N, \\ u_j(t, 0) = u_1(t, 0)(t), & t \in (0, T), \quad \forall j = 2, \dots, N, \\ \sum_{j=1}^N \partial_x^2 u_j(t, 0) = -\alpha u_1(t, 0), & t \in (0, T), \\ u_j(t, \ell_j) = g_j(t), \quad \partial_x u_j(t, \ell_j) = p_j(t), & t \in (0, T), \quad j = 1, \dots, N, \\ u_j(0, x) = u_j^0(x), & x \in I_j, \end{cases} \quad (6.5.2)$$

where $\alpha > \frac{N}{2}$ and the control functions $\underline{g} = (g_1, \dots, g_N)^T, \underline{p} = (p_1, \dots, p_N)^T \in (L^2(0, T))^N$ act on the exterior Dirichlet and Neumann conditions. The focus of this part is to study the null controllability of the linear KdV equation posed in a star-shaped network without acting in all the branches, i.e. $2N - 2$ boundary controls using Carleman estimates. It is worth to mention that in the case of null controllability of a single KdV equation by the right via Carleman estimates two controls are needed, in this sense in the network case we are able to reduce the number of controls. Without loss of generality, we will consider that the not acted edge is the first one, that is $p_1 = g_1 = 0$.

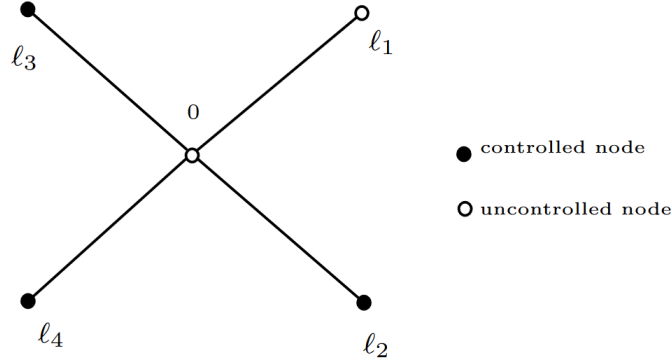


Figure 6.4: Star shaped network of $N = 4$ branches

It is not difficult to see that the adjoint system is given by

$$\begin{cases} -\partial_t \varphi_j(t, x) - \partial_x \varphi_j(t, x) - \partial_x^3 \varphi_j(t, x) = 0, & (t, x) \in Q_j, \quad j = 1, \dots, N, \\ \varphi_j(t, 0) = \varphi_1(t, 0), & t \in (0, T), \quad \forall j = 2, \dots, N, \\ \sum_{j=1}^N \partial_x^2 \varphi_j(t, 0) = (\alpha - N) \varphi_1(t, 0), & t \in (0, T), \\ \varphi_j(t, \ell_j) = \partial_x \varphi_j(t, 0) = 0, & t \in (0, T), \quad j = 1, \dots, N, \\ \varphi_j(T, x) = \varphi_j^T(x), & x \in I_j, \end{cases} \quad (6.5.3)$$

where $\underline{\varphi}^T \in \mathbb{L}^2(\mathcal{T})$. Note that if the controls g_2, \dots, g_N and p_2, \dots, p_N steer the system (KdVLS) from \underline{u}^0 to $\underline{0}$ in $\mathbb{L}^2(\mathcal{T})$, then, multiplying (KdVLS) by $\underline{\varphi}$ solution of (6.5.3) and integrating with respect to (t, x) on Q_j , we get the next after some integrations by parts

$$\sum_{j=1}^N \int_0^{\ell_j} u_j^0 \varphi_j(0, x) dx = \sum_{j=2}^N \int_0^T g_j(t) \partial_x^2 \varphi_j(t, \ell_j) dt - \sum_{j=2}^N \int_0^T p_j(t) \partial_x \varphi_j(t, \ell_j) dt$$

which leads to the following observability inequality;

$$\|\underline{\varphi}(0, x)\|_{\mathbb{L}^2(\mathcal{T})}^2 \leq C \sum_{j=2}^N \left(\|\partial_x^2 \varphi_j(t, \ell_j)\|_{L^2(0, T)}^2 + \|\partial_x \varphi_j(t, \ell_j)\|_{L^2(0, T)}^2 \right).$$

In this sense, we can prove the following null controllability result.

Theorem 6.5.4. *Let $\underline{u}^0 \in \mathbb{L}^2(\mathcal{T})$, then, for every $T > 0$, there exist controls $\underline{g} = (0, g_2, \dots, g_N)^T, \underline{p} = (0, p_2, \dots, p_N)^T \in (L^2(0, T))^N$, such that the unique solution $\underline{u} \in \mathbb{B}$ of (6.5.2) satisfies $u_j(T, \cdot) = 0$, for all $j = 1, \dots, N$.*

Theorem 6.5.4 is proved using Carleman estimates. We manage to show a Carleman inequality using the external measures of $N - 1$ equations of the network. The main complications of this

work are related to the well-posedness. Along this thesis, we used semigroup theory to prove the well-posedness results, in the case of Dirichlet controls for the KdV equation the solutions are defined typically by transposition [GG08, GG10], thus extend these ideas to the network case is the work in progress.

6.6 Open problems and future research lines

6.6.1 About the nonlinear KdV equation with 5 null boundary conditions

An interesting open problem coming from our contradiction strategy to prove the observability inequalities is the possibility to remove one index in the semiglobal stability results for the networks (Theorem 2.1.2, Theorem 3.1.4, Theorem 6.5.2 and Theorem 6.5.3). Following the proof of these results, we are asking to prove that the unique solution of

$$\begin{cases} \partial_t v(t, x) + \partial_x v(t, x) + v(t, x)\partial_x v(t, x) + \partial_x^3 v(t, x) = 0, & \forall x \in (0, \ell), t \in (0, T), \\ v(t, 0) = \partial_x v(t, 0) = \partial_x^2 v(t, 0) = 0, & t > 0, \\ v(t, \ell) = \partial_x v(t, \ell) = 0, & t > 0, \end{cases} \quad (6.6.1)$$

is the null solution. Up to our knowledge, it is an open problem, but it is known that the condition $\partial_x^2 v(t, 0) = 0$ is really needed. In fact, in [DN14] the following result was proved regarding the stationary solutions of the KdV equation, considering the system

$$\partial_x \phi + \frac{1}{2} \partial_x (\phi^2) + \partial_x^3 \phi = 0, \quad \text{in } [0, L], \quad (6.6.2)$$

Theorem 6.6.1 (Theorem 1, [DN14]). *For all $L \in (0, 2\pi)$, there exists a stationary solution $\phi \in C^\infty(\mathbb{R})$ of (6.6.2) with boundary conditions $\phi(0) = \partial_x \phi(0) = 0$, satisfying $\phi(x + L) = \phi(x)$, $\forall x \in \mathbb{R}$ and $\partial_x^2 \phi(0) \neq 0$.*

Note that by the periodicity we have in particular $\phi(L) = \partial_x \phi(L) = 0$. In [FN16] a more general result was presented,

Lemma 6.6.1 (Lemma 1, [FN16]). *If $\phi \in C^3([0, L])$ is a solution of (6.6.2) with boundary conditions $\phi(0) = \phi(L) = \partial_x \phi(L) = 0$, then it is infinitely smooth and periodic with period L .*

Theorem 6.6.2 (Theorem 1, [FN16]). *If $L^2 \neq 4\pi^2$, then there exists a unique non-trivial solution of period L of (6.6.2) with boundary conditions $\phi(0) = \phi(L) = \partial_x \phi(L) = 0$. If $L^2 = 4\pi^2$ such a solution does not exist.*

By Lemma 6.6.1 and Theorem 6.6.2, we have the existence of a stationary solution $\phi \in C^\infty([0, L])$ of (6.6.1). But this solution does not satisfy $\partial_x^2 \phi(0) = 0$. Indeed, consider the substitution $\eta(x) = \frac{L^2}{4} \phi\left(\frac{L}{4}(x + 1)\right)$, then η satisfies system

$$\begin{cases} b\partial_x \eta + \eta\partial_x \eta + \partial_x^3 \eta = 0, & \text{in } [-1, 1], \\ \eta(-1) = \partial_x \eta(-1) = 0, \end{cases}$$

where $b = \frac{L^2}{4}$. Note that this equation is equivalent to $b\eta + \frac{1}{2}\eta^2 + \partial_x^2 \eta = c$, for some $c \in \mathbb{R}$. In particular, if $\partial_x^2 \eta(-1) = 0$ we get $c = 0$. But from [FN16, Theorem 1] and [FN16, Lemma 2] we have that c must be not null. Similarly, in [GS07, Section 4.4] the existence of some stationary solutions η of KdV equation in $[-1, 1]$ with $\eta(-1) = \eta(1) = \partial_x \eta(-1) = \partial_x \eta(1) = 0$ was showed, but $\partial_x^2 \eta(-1) \neq 0$.

6.6.2 Controllability results

Exact controllability

In the [AC18] the exact controllability of linearization around 0 of (KdVs) was achieved by acting with $N + 1$ boundary controls (N controls in the external nodes and one in the central node) if

$\#\{\ell_n \in \mathcal{N}\} \leq 1$. Recently, in [CCM20] the authors could reduce the numbers of controls (N controls acting on the external nodes), but the controllability holds for a large time and small lengths. This raises the question of what happens for the boundary control and how many components corresponding to the critical lengths one needs to control in the network case. In particular, we can mention the following open problems:

- Is the linearization around 0 of (KdVs) exactly controllable with N controls acting in the external nodes for $T > 0$ and $\ell_n \notin \mathcal{N}$ for all $n \in \{1, \dots, N\}$?
- Is (KdVs) exactly controllable from the boundary in the case where for some lengths we have $\ell_n \in \mathcal{N}$? A starting point could be, to consider the smallest critical lengths ($k = l = 1$ or $k = l = 2$).

Boundary null controllability

Another immediate question about the possibility to reduce the numbers of controls or acted edges remains open in the case of null controllability. In our result we are not able to eliminate more acted edges ($N - 1$ acted edges, $2N - 2$ controls), an idea could be to study the null controllability of the KdV equation on a tree network as in [CCV20] and use the direction of propagation of the KdV equation to eliminate some controls.

Internal null controllability

In the case of systems of KdV equations with internal couplings, the problem of null controllability was studied in [CCC20] for a Hirota-Satsuma system and in [BPGP22] for a Gear-Grimshaw system. In these two works, the controllability was proved by using Carleman estimates, the controls act internally and locally and due to the internal couplings term they were able to use less controls than equations. An interesting question is the internal null controllability for the KdV equation in a star shaped network, and then if the central node conditions transmit enough information to reduce the numbers of controls.

Dispersive limits

In [BPCC+23] the null controllability problem of a parabolic system on networks was studied with diffusive parameter $\varepsilon \rightarrow 0$. In this setting, a possible future research line could be to consider a dispersive parameter ν in the KdV equation on networks and to study the uniform null controllability, as for instance in [GG08]. Some complications are how to adapt the junction conditions in the hyperbolic limit problem on the network.

6.6.3 Inverse problems

In [AC18] a KdV system on networks was proposed to model blood pressure. In this context, an interesting inverse problem to study about the determination of cardiac outflow pressure through noninvasive measurements arise. Also, considering different principal coefficients in each edge of the KdV equation and studying the inverse problem of recovering them through measurements at the ends of the network could be considered. The main tool proposed to solve this kind of problems are the Carleman inequalities.

RESUMÉ ÉTENDU EN FRANÇAIS

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A.1 L'équation de Korteweg-de Vries

L'équation de Korteweg-de Vries (KdV), est une équation, dispersive non linéaire du troisième ordre introduite en 1895 par les mathématiciens néerlandais Diederik Korteweg et Gustav de Vries dans [KdV95]. Elle a été écrite pour décrire la propagation unidirectionnelle d'ondes à longue période et petite amplitude, telles que celles qu'on trouve dans les canaux d'eau peu profonde. En 1965, Zabusky et Kruskal ont découvert la notion de solutions d'ondes solitaires, également connues sous le nom de "solitons", en analysant des résultats numériques liés à l'équation de KdV [ZK65]. Dès lors, cette idée a été largement étudiée dans le cadre mathématique et physique, et a été utilisée pour modéliser une variété de phénomènes, en comprenant les vagues, les tsunamis, la transmission de signaux électrique dans les fibres nerveuses, le plasma, la cosmologie, etc (voir par exemple [GM65, Lid12, KM69]). L'équation de KdV dérivée par Diederik Korteweg et Gustav de Vries était (à changement d'échelle près)

$$\partial_t u(t, x) + \partial_x^3 u(t, x) + u(t, x) \partial_x u(t, x) = 0, x \in \mathbb{R}, t \geq 0,$$

où la fonction $u = u(t, x)$ modélise l'amplitude de la vague à la position x et au temps $t \geq 0$. Dans cette thèse, nous nous intéressons principalement à l'équation de KdV posée dans un domaine borné et aux réseaux à bords bornés. Si nous étudions l'équation de KdV dans un domaine borné, $x \in [0, L]$, comme l'ont suggéré Bona et Winter dans [BW83] le terme supplémentaire $\partial_x u$ devrait être ajouté. De plus, comme l'équation de KdV est du troisième ordre en espace, nous devons imposer deux conditions aux bords à droite et une à gauche. Par exemple, nous pouvons considérer des conditions aux bords agissant sur $u(t, 0)$, $u(t, L)$ et $\partial_x u(t, L)$.

Caractér bien posé de l'équation de KdV

Cette section est principalement basée sur [RZ09, Cer14]. Les premiers résultats sur la contrôlabilité et la stabilisation de l'équation de KdV ont été dérivés par Russell et Zhang [RZ93, RZ95], dans le

cas de conditions aux bords périodiques, et par Rosier, dans le cas du contrôle au bord [Ros97]. Dans le travail de Rosier, la contrôlabilité exacte de l'équation de KdV a été étudiée en agissant sur la condition au bord de Neumann à droite. En particulier, il s'est concentré sur le système linéaire

$$\begin{cases} \partial_t u(t, x) + \partial_x u(t, x) + \partial_x^3 u(t, x) = 0, & t \in [0, T], \quad x \in (0, L), \\ u(t, 0) = u(t, L) = 0, & t \in [0, T], \\ \partial_x u(t, L) = h(t), & t \in [0, T], \\ u(0, x) = u_0, & x \in (0, L), \end{cases} \quad (\text{A.1.1})$$

où la fonction $h \in L^2(0, T)$ désigne le contrôle. Considérons d'abord le système non contrôlé

$$\begin{cases} \partial_t u(t, x) + \partial_x u(t, x) + \partial_x^3 u(t, x) = 0, & t \in [0, T], \quad x \in (0, L), \\ u(t, 0) = u(t, L) = \partial_x u(t, L) = 0, & t \in [0, T], \\ u(0, x) = u_0, & x \in (0, L). \end{cases} \quad (\text{A.1.2})$$

Soit $\mathcal{A} : D(\mathcal{A}) \rightarrow L^2(0, L)$ défini par

$$\begin{aligned} \mathcal{A}w &= -w' - w''', \\ D(\mathcal{A}) &= \{w \in H^3(0, L); w(0) = w(L) = w'(L) = 0\}. \end{aligned}$$

Par [Ros97, proposition 2.1] nous savons que \mathcal{A} est le générateur d'un semigroupe fortement continu de contractions $S(t)$, $t \geq 0$ dans $L^2(0, L)$. Définissons l'espace

$$\begin{aligned} B_T &= C([0, T], L^2(0, L)) \cap L^2(0, T; H^1(0, L)), \\ \|u\|_{B_T} &= \|u\|_{C([0, T], L^2(0, L))} + \|u\|_{L^2(0, T; H^1(0, L))}. \end{aligned}$$

D'après [Ros97, proposition 3.2], pour tout $u_0 \in L^2(0, L)$, il existe une unique solution faible (mild solution) $u = S(\cdot)u_0 \in B_T$ de (A.1.2). En outre, l'estimation suivante est valable

$$\|\partial_x u(\cdot, 0)\|_{L^2(0, T)} \leq \|u_0\|_{L^2(0, L)}. \quad (\text{A.1.3})$$

Cette estimation est généralement appelée *régularité cachée*. Considérons maintenant le système non linéaire

$$\begin{cases} \partial_t u(t, x) + \partial_x u(t, x) + \partial_x^3 u(t, x) + u(t, x)\partial_x u(t, x) = 0, & t \in [0, T], \quad x \in (0, L), \\ u(t, 0) = u(t, L) = 0, & t \in [0, T], \\ \partial_x u(t, L) = h(t), & t \in [0, T], \\ u(0, x) = u_0, & x \in (0, L). \end{cases} \quad (\text{A.1.4})$$

L'idée principale de la démonstration du caractère bien posé du système non linéaire (A.1.4) est d'utiliser l'estimation suivante (voir [PMVZ02]).

Proposition A.1.1. *Soit $u \in B_T$. Alors $u\partial_x u \in L^1((0, T), L^2(0, L))$ et l'application $u \in B_T \mapsto u\partial_x u \in L^1((0, T), L^2(0, L))$ est continue. De plus, il existe $C > 0$ de telle sorte que, pour tout $u, z \in B_T$*

$$\|u\partial_x u - z\partial_x z\|_{L^1((0, T), L^2(0, L))} \leq CT^{1/4}(\|u\|_{B_T} + \|z\|_{B_T})\|u - z\|_{B_T}.$$

En utilisant la proposition ci-dessus et un argument du point fixe, nous obtenons le caractère bien pose local de (A.1.4).

Phénomène des longueurs critiques

Maintenant, étant donné $T > 0$, $u_0, u_T \in L^2(0, L)$ le problème de contrôlabilité exacte pour (A.1.1) consiste à trouver $h \in L^2(0, T)$ de telle sorte que la solution u de (A.1.1) satisfasse $u(T, \cdot) = u_T \in L^2(0, L)$. Soit $T > 0$ et définissons l'opérateur $\Lambda : L^2(0, T) \rightarrow L^2(0, L)$ par $\Lambda(h) := u(T, \cdot)$, où u est l'unique solution de (A.1.1). En particulier, par linéarité, nous pouvons

supposer que $u_0 = 0$. Ainsi, d'après (1.1.5) la contrôlabilité exacte est équivalente à l'inégalité d'observabilité,

$$\|\varphi_T\|_{L^2(0,L)} \leq C \|\Lambda^*(\varphi_T)\|_{L^2(0,T)},$$

où Λ^* est l'opérateur adjoint. Il n'est pas difficile de voir qu'étant donné $\varphi_T \in L^2(0,L)$, $\Lambda^*(\varphi_T) = \partial_x \varphi(\cdot, L)$, où φ est la solution de l'équation en temps rétrograde

$$\begin{cases} -\partial_t \varphi(t, x) - \partial_x \varphi(t, x) - \partial_x^3 \varphi(t, x) = 0, & t \in [0, T], \quad x \in (0, L), \\ \varphi(t, 0) = \varphi(t, L) = \partial_x \varphi(t, 0) = 0, & t \in [0, T], \\ \varphi(T, x) = \varphi^T, & x \in (0, L). \end{cases} \quad (\text{A.1.5})$$

On peut remarquer, que en faisant le changement de variable $\xi = L - x$ et $\tau = T - t$, on obtient que $\varphi(T - t, L - x)$ est solution de (A.1.2) et donc, l'inégalité d'observabilité se lit comme suit

$$\|u_0\|_{L^2(0,L)} \leq C \|\partial_x u(\cdot, 0)\|_{L^2(0,T)},$$

ce qui est en effet, l'inégalité inverse de (A.1.3). Dans [Ros97, proposition 3.3] il a été prouvé que cette inégalité est vraie si et seulement si $L \notin \mathcal{N}$, où l'ensemble \mathcal{N} est appelé l'ensemble des longueurs critiques et est donné par

$$\mathcal{N} = \left\{ 2\pi \sqrt{\frac{k^2 + kl + l^2}{3}}; k, l \in \mathbb{N}^* \right\}. \quad (\text{A.1.6})$$

Donc (A.1.1) est exactement contrôlable si et seulement si $L \notin \mathcal{N}$. De plus, on a le lemme suivant dû à [Ros97, lemme 3.5],

Lemme A.1.1. *Les affirmations suivantes sont équivalentes*

- Il existe $\lambda \in \mathbb{C}$ et $u \in H^3(0, L) \setminus \{0\}$, tel que

$$\begin{cases} \lambda u + u' + u''' = 0, & x \in (0, L) \\ u(0) = u(L) = u'(0) = u'(L) = 0, \end{cases}$$

- $L \in \mathcal{N}$.

Ce lemme implique l'existence de solutions de (A.1.2) qui ne sont pas observables, par exemple dans le cas $l = k = 1$, la longueur critique associée est $L = 2\pi$ et cela donne les solutions inobservables $u(t, x) = a(\cos(x) - 1)$, $a \in \mathbb{R}$. Considérons maintenant l'énergie L^2 d'une solution faible de (A.1.2), c'est-à-dire

$$E(t) = \frac{1}{2} \int_0^L (u(t, x))^2 dx.$$

Formellement, en prenant la dérivée par rapport au temps de l'énergie, nous obtenons après quelques intégrations par parties

$$\dot{E}(t) = -\frac{1}{2} (\partial_x u(t, 0))^2,$$

d'où l'on déduit que l'énergie est une fonction non croissante du temps. En outre, si $L \in \mathcal{N}$, par le lemme A.1.1, il existe des solutions de (A.1.2) qui satisfont $\partial_x u(\cdot, 0) = 0$ et donc avec une énergie constante.

Stabilité avec longueurs non critiques

Dans le cas non critique $L \notin \mathcal{N}$ nous pouvons montrer que l'énergie des solutions de (A.1.2) décroît exponentiellement. En effet, en multipliant (A.1.2) par u et en intégrant sur $(0, T) \times (0, L)$ nous obtenons après intégrations par parties

$$\int_0^L (u(T, x))^2 dx - \int_0^L u_0^2(x) dx = - \int_0^T (\partial_x u(t, 0))^2 dt.$$

Comme l'énergie est décroissante et avec l'inégalité d'observabilité, on obtient

$$E(T) \leq E(0) \leq C \int_0^T (\partial_x u(t, 0))^2 dt \leq C(E(0) - E(T)),$$

ce qui implique

$$E(T) \leq \gamma E(0), \text{ avec } \gamma = \frac{C}{C+1} < 1.$$

En particulier, comme le système est invariant par translation en temps, on peut répéter cet argument sur les intervalles de la forme $[(m-1)T, mT]$, avec $m = 1, 2, \dots$ pour obtenir

$$E(mT) \leq \gamma E((m-1)T) \leq \dots \leq \gamma^m E(0).$$

Par conséquent, nous avons $E(mT) \leq e^{-\mu m T} E(0)$ où $\mu = \frac{1}{T} \ln(\frac{1}{\gamma}) > 0$. Soit $t > 0$, alors il existe $m \in \mathbb{N}^*$ telle que $(m-1)T < t \leq mT$, et en utilisant encore que l'énergie est décroissante, nous obtenons

$$E(t) \leq E((m-1)T) \leq e^{-\mu(m-1)T} E(0) \leq \frac{1}{\gamma} e^{-\mu t} E(0).$$

Enfin, on a la stabilité exponentielle.

Considérons maintenant l'équation non linéaire de KdV

$$\begin{cases} \partial_t u(t, x) + \partial_x u(t, x) + \partial_x^3 u(t, x) + u(t, x) \partial_x u(t, x) = 0, & t \in [0, T], x \in (0, L), \\ u(t, 0) = u(t, L) = \partial_x u(t, L) = 0, & t \in [0, T], \\ u(0, x) = u_0, & x \in (0, L). \end{cases} \quad (\text{A.1.7})$$

On peut étendre le résultat précédant de la stabilité exponentielle aux solutions faibles du système non linéaire (A.1.7) à l'aide d'une approche de type perturbation. C'est-à-dire qu'il existe $\varepsilon > 0$ suffisamment petit pour lequel, pour tout $u_0 \in L^2(0, L)$ satisfaisant $\|u_0\|_{L^2(0, L)} \leq \varepsilon$, l'unique solution faible de (A.1.7) satisfait $E(t) \leq C e^{-\mu t} E(0)$, pour $C, \mu > 0$.

Stabilité avec longueurs critiques : amortissement interne

Dans le but d'obtenir la stabilité exponentielle dans le cas $L \in \mathcal{N}$, un mécanisme dissipatif a été introduit dans [PMVZ02] prenant la forme $F(u) = -au$, où $a \in L^\infty(0, L)$ satisfait

$$\begin{cases} a(x) \geq a_0 > 0, \quad \forall x \in \omega, \\ \text{où } \omega \text{ est un sous-ensemble ouvert non vide de } (0, L). \end{cases} \quad (\text{A.1.8})$$

Les systèmes traités dans [PMVZ02] sont, l'équation linéaire

$$\begin{cases} \partial_t u(t, x) + \partial_x u(t, x) + \partial_x^3 u(t, x) + a(x)u(t, x) = 0, & t \in [0, T], x \in (0, L), \\ u(t, 0) = u(t, L) = \partial_x u(t, L) = 0, & t \in [0, T], \\ u(0, x) = u_0, & x \in (0, L). \end{cases} \quad (\text{A.1.9})$$

et le non linéaire

$$\begin{cases} \partial_t u(t, x) + \partial_x u(t, x) + \partial_x^3 u(t, x) + u(t, x) \partial_x u(t, x) + a(x)u(t, x) = 0, & t \in [0, T], x \in (0, L), \\ u(t, 0) = u(t, L) = \partial_x u(t, L) = 0, & t \in [0, T], \\ u(0, x) = u_0, & x \in (0, L). \end{cases} \quad (\text{A.1.10})$$

En utilisant un argument de contradiction et des idées de compacité, Perla, Menzala et Zuazua ont montré dans [PMVZ02] les résultats suivants.

Théorème A.1.1. Soient $L > 0$ et $a \in L^\infty(0, L)$ satisfaisant (A.1.8).

- Il existe $C > 0$ et $\mu > 0$ de sorte que pour tout $u_0 \in L^2(0, L)$, l'unique solution faible de l'équation linéaire (A.1.9) satisfait $\|u(t, \cdot)\|_{L^2(0, L)} \leq Ce^{-\mu t} \|u_0\|_{L^2(0, L)}$, $\forall t \geq 0$.
- Il existe $C, \varepsilon > 0$ et $\mu > 0$ de sorte que pour tout $u_0 \in L^2(0, L)$ avec $\|u_0\|_{L^2(0, L)} \leq \varepsilon$, l'unique solution faible de l'équation non linéaire (A.1.10) satisfait $\|u(t, \cdot)\|_{L^2(0, L)} \leq Ce^{-\mu t} \|u_0\|_{L^2(0, L)}$, $\forall t \geq 0$.

Le deuxième point du dernier résultat a été dérivé en utilisant un argument de type perturbatif, ce qui explique pourquoi ex nous ajoutons l'hypothèse de petitesse sur la donnée initiale. De plus, dans [PMVZ02, Paz05], un résultat de stabilité exponentielle semi-global a été démontré pour le système non linéaire (A.1.10). Tout d'abord, dans [PMVZ02], une condition plus forte sur l'ensemble ω a été utilisée : $\exists \delta > 0$, $(0, \delta) \cup (L - \delta, L) \subset \omega$. Dans [Paz05] cette condition a été améliorée, ce qui permet d'obtenir (A.1.8). En travaillant directement avec le système non linéaire, deux difficultés principales apparaissent, la première étant de passer à la limite dans le terme non linéaire et la seconde que le Théorème de Holmgren ne s'applique pas dans le cas non linéaire. Donc, nous avons besoin de la propriété suivante de continuation unique de Saut et Scheure.

Théorème A.1.2 (Théorème 4.2, [SS87]). Soient $L > 0$ et $u \in L^2(0, T; H^3(0, L))$ une solution de

$$\partial_t u(t, x) + \partial_x u(t, x) + \partial_x^3 u(t, x) + u(t, x) \partial_x u(t, x) = 0,$$

telle que $u(t, x) = 0$, pour $(t, x) \in (t_1, t_2) \times \omega$, où ω est un sous-ensemble ouvert non vide de $(0, L)$. Alors, $u(t, x) = 0$, pour $(t, x) \in (t_1, t_2) \times (0, L)$.

Le résultat de stabilité semi-globale est le suivant

Théorème A.1.3. Soient $L > 0$, $a \in L^\infty(0, L)$ satisfaisant (A.1.8) et $R > 0$. Il existe $C(R) > 0$ et $\mu(R) > 0$ de telle sorte que pour tout $u_0 \in L^2(0, L)$ avec $\|u_0\|_{L^2(0, L)} \leq R$, l'unique solution faible de (A.1.10) satisfait $\|u(t, \cdot)\|_{L^2(0, L)} \leq Ce^{-\mu t} \|u_0\|_{L^2(0, L)}$, $\forall t \geq 0$.

Le sens semi-global de ce résultat provient du fait que nous pouvons choisir comme on veut le paramètre $R > 0$ de la donnée initiale, ce qui signifie des conditions initiales arbitraires, mais le taux de décroissance dépend du R .

Résultats de stabilisation frontière

Dans cette partie, nous rappelons quelques résultats de stabilisation frontière pour l'équation de KdV. Considérons le système linéaire

$$\begin{cases} \partial_t u(t, x) + \partial_x u(t, x) + \partial_x^3 u(t, x) = 0, & t \in [0, T], \quad x \in (0, L), \\ u(t, 0) = K_1(u), & t \in [0, T], \\ u(t, L) = K_2(u), \quad \partial_x u(t, L) = K_3(u), & t \in [0, T], \\ u(0, x) = u_0, & x \in (0, L), \end{cases} \quad (\text{A.1.11})$$

et le non linéaire

$$\begin{cases} \partial_t u(t, x) + \partial_x u(t, x) + \partial_x^3 u(t, x) + u(t, x) \partial_x u(t, x) = 0, & t \in [0, T], \quad x \in (0, L), \\ u(t, 0) = K_1(u), & t \in [0, T], \\ u(t, L) = K_2(u), \quad \partial_x u(t, L) = K_3(u), & t \in [0, T], \\ u(0, x) = u_0, & x \in (0, L). \end{cases} \quad (\text{A.1.12})$$

Le résultat suivant a été prouvé par Cerpa et Crépeau dans [CC09a] et est lié à la stabilisation rapide.

Théorème A.1.4 (Stabilisation rapide). Soit $\mu > 0$ et considérons $K_1(u) = K_2(u) = 0$. Donc, il existe un opérateur $F_\mu : H_0^1(0, L) \rightarrow \mathbb{R}$ tel que si $K_3(u) = \partial_x u(t, 0) + F_\mu(u)$, alors le système en boucle fermée (A.1.11) est globalement bien posé dans $H_0^1(0, L)$. De plus, $\exists C > 0$, tel que pour tout $u_0 \in H_0^1(0, L)$, $\|u(t, \cdot)\|_{H_0^1(0, L)} \leq Ce^{-2\mu t} \|u_0\|_{H_0^1(0, L)}$, pour tout $t \geq 0$.

L'idée principale de ce résultat est d'utiliser une approche basée sur la méthode du Gramian. Grâce à cette technique, le terme de feedback $K_3(u)$, prendre la forme $K_3(u) = \partial_x u(t, 0) + F_\mu(u)$ de telle sorte que l'opérateur associé au système sans contrôle soit anti-symétrique (c-à-d $\mathcal{A}^* = -\mathcal{A}$). Un autre résultat de stabilisation rapide a été prouvé dans [CC13], cette fois-ci, en agissant sur la condition de Dirichlet à gauche et en utilisant la technique de backstepping (voir [KS08] pour une introduction sur la technique de backstepping pour les équations aux dérivées partielles).

Théorème A.1.5 (Stabilisation rapide). *Soit $\mu > 0$ et considérons $K_2(u) = K_3(u) = 0$. Alors, il existe un opérateur $K_1(u) = K_{1,\mu}(u)$ et $C > 0$, tel que pour toute solution faible u de (A.1.11), $\|u(t, \cdot)\|_{L^2(0,L)} \leq Ce^{-\mu t} \|u_0\|_{L^2(0,L)}$, pour tout $t \geq 0$.*

Ce résultat peut être étendu à l'équation non linéaire (A.1.12) en utilisant un argument de perturbation et des données initiales petites. Dans le but de montrer un résultat de stabilité exponentielle en agissant par la droite, dans [CL14] Coron et Lü introduisent un feedback agissant sur la condition au bord Neumann. Ce feedback est basé sur une transformation de Fredholm et inspiré par des résultats précédents pour l'équation de Kuramoto-Sivashinsky [CL15].

Théorème A.1.6 (Stabilisation rapide, cas non critique). *Soient $L \notin \mathcal{N}$ et $\mu > 0$ et considérons $K_1(u) = K_2(u) = 0$. Alors, il existe un opérateur $K_3(u) = K_{3,\mu}(u)$ et $C > 0$, tel que pour toute solution faible u de (A.1.11), $\|u(t, \cdot)\|_{L^2(0,L)} \leq Ce^{-\mu t} \|u_0\|_{L^2(0,L)}$, pour tout $t \geq 0$.*

Ce résultat peut être étendu à l'équation non linéaire (A.1.12) en utilisant un argument de perturbation et des données initiales petites. La restriction $L \notin \mathcal{N}$ dans le théorème ci-dessus, est encore due au manque de contrôlabilité de (A.1.2). Dans le cas $L \in \mathcal{N}$, les mauvaises propriétés de contrôle s'améliorent si l'on étudie le système non linéaire (voir section A.1). Par [Ros97] $L^2(0, L) = H \oplus M$, où H correspond à l'espace contrôlable de (A.1.1) et M l'espace non contrôlable (dépendant tous deux de L). De plus, $\dim(M) < \infty$ et $\dim(M)$ est le nombre de paires différentes d'entiers positifs (l_j, k_j) satisfaisant (A.1.6). Considérons P_H (resp. P_M) la projection H (resp. M), le résultat suivant a été prouvé dans [CRX17]

Théorème A.1.7 (Feedback variable dans le temps). *Soient $K_1(u) = K_2(u) = 0$. Si $\dim(M) = 2n$ avec $n \in \mathbb{N}^*$, alors il existe un feedback périodique, variable dans le temps $\tilde{K}_3 : \mathbb{R} \times L^2(0, L) \rightarrow \mathbb{R}$ et des constantes positives C, μ, r , telles que si on laisse $K_3(u) = \tilde{K}_3(t, u)$, toute solution de (A.1.12) avec $\|u_0\|_{L^2(0,L)} \leq r$ vérifie*

$$\|P_H u(t, \cdot)\|_{L^2(0,L)} + \|P_M u(t, \cdot)\|_{L^2(0,L)}^{1/2} \leq Ce^{-\mu t} \left(\|P_H u_0\|_{L^2(0,L)} + \|P_M u_0\|_{L^2(0,L)}^{1/2} \right), \forall t \geq 0.$$

Maintenant, afin d'éliminer la restriction sur les longueurs critiques dans les feedbacks qui agissent par la droite, une version modifiée de la technique de backstepping appelé *pseudo-backstepping* a été utilisée dans [OB19], le coût à assumer est l'utilisation de deux contrôleurs.

Théorème A.1.8. *Soient $L > 0$ et $\mu > 0$ et considérons $K_1(u) = 0$. Alors, il existe des opérateurs K_2, K_3 et des constantes positives C, μ , tels que pour toute solution faible u de (A.1.11), $\|u(t, \cdot)\|_{L^2(0,L)} \leq Ce^{-\mu t} \|u_0\|_{L^2(0,L)}$, pour tout $t \geq 0$.*

Là, encore une fois, ce résultat peut être étendu à l'équation non linéaire (A.1.12) en utilisant une approche de type perturbation et des données initiales petites.

Contrôle et stabilité dans le cas critique

Comme nous l'avons mentionné à la section A.1, même si le système linéaire (A.1.1) n'est pas contrôlable, il est possible de montrer des résultats de contrôlabilité pour le système non linéaire à l'aide du terme non linéaire

$$\begin{cases} \partial_t u(t, x) + \partial_x u(t, x) + \partial_x^3 u(t, x) + u(t, x) \partial_x u(t, x) = 0, & t \in [0, T], \quad x \in (0, L), \\ u(t, 0) = u(t, L) = 0, & t \in [0, T], \\ \partial_x u(t, L) = h(t), & t \in [0, T], \\ u(0, x) = u_0, & x \in (0, L). \end{cases} \quad (\text{A.1.13})$$

Ces résultats sont résumés dans le théorème suivant

Théorème A.1.9. *L'équation non linéaire de KdV (A.1.13) est localement contrôlable si*

- (Coron et Crépeau [CC04]) $\dim(M) = 1$, pour tout $T > 0$.
- (Cerpa [Cer07]) $\dim(M) = 2$, pour T suffisamment grand.
- (Cerpa et Crépeau [CC09b]) $\dim(M) > 2$, pour T suffisamment grand.

La preuve de ces résultats est basée sur la méthode de développement en puissances (power expansion method). Grosso modo, d'après le travail de Rosier [Ros97], l'approximation du premier ordre de (A.1.13) (le système linéaire (A.1.1)) n'est pas contrôlable, l'idée étant d'utiliser des approximations d'ordre supérieur pour atteindre l'espace non contrôlable. Comme l'indique le théorème ci-dessus, dans certains cas, les auteurs ont eu besoin d'un temps grand pour atteindre la contrôlabilité. Pendant plusieurs années, la nécessité de cette condition est restée un problème ouvert, mais récemment Coron, Koenig et Nguyen ont donné une réponse négative dans [CKN22].

Théorème A.1.10. *Soit $L \in \mathcal{N}$ tel que $2k + l \notin 3\mathbb{N}^*$. Alors (A.1.13) n'est pas localement contrôlable à zéro en temps petit avec des contrôles dans $H^1(0, T)$ et données initiales et finales dans $H^3(0, L) \cap H_0^1(0, L)$.*

Dans le cas de l'analyse de stabilité, une question similaire se pose naturellement. Dans le cas où la longueur est critique, le système non linéaire (A.1.7) présent-t-il de meilleures propriétés de stabilité que le système linéaire (A.1.2) ? Dans la section A.1 nous avons signalé que dans le cas non critique $L \notin \mathcal{N}$, le système non linéaire (A.1.7) est localement exponentiellement stable. Si $L \in \mathcal{N}$, les résultats suivants sont connus,

Théorème A.1.11. *Le système (A.1.7) est localement asymptotiquement stable si*

- (Chu, Coron et Shang [CCS15]) $\dim(M) = 1$,
- (Tang, Chu, Shang et Coron [TCSC18]) $L = 2\pi\sqrt{\frac{7}{3}}$.
- (Nguyen [Ngu21]) $\dim(M) = 1$ et de plus, le système décroît localement avec un taux de $1/t^2$.

Les deux premiers résultats sont basés sur la théorie de variété centrale (central manifold), notons que $L = 2\pi\sqrt{\frac{7}{3}}$ est la plus petite longueur pour laquelle $\dim(M) = 2$, tandis que le troisième résultat s'inspire du développement en séries de puissances, de la théorie des fonctions quasi-périodiques et de l'analyse développée dans [CKN22]. Il reste encore des questions ouvertes concernant le contrôle et la stabilité de l'équation non linéaire de KdV, par exemple en ce qui concerne le théorème A.1.10 :

- Prouver la non contrôlabilité en temps petit dans le cadre de la régularité classique, c'est-à-dire le contrôle dans $L^2(0, T)$ et des données initiales dans $L^2(0, L)$.
- Remplacer la condition $L \in \mathcal{N}$ telle que $2k + l \notin 3\mathbb{N}^*$ dans le théorème A.1.10, par la condition plus générale $\dim(M) > 1$.
- D'après le théorème A.1.9, l'équation non linéaire de KdV (A.1.13) est contrôlable en temps long. Donc, la recherche du temps optimal de contrôlabilité est une question ouverte intéressante.

L'équation de Korteweg-de Vries dans des réseaux

Ces dernières années, les équations aux dérivées partielles dans des réseaux ont été utilisées pour modéliser plusieurs systèmes physiques parmi eux, les réseaux de transport, les réseaux électriques, les systèmes de communication et les structures flexibles à liens multiples, en particulier les flux dans des réseaux comme les gazoducs, les canaux d'irrigation, la circulation sanguine, etc, (voir par exemple [BČG⁺14, DZ06, LLS12]). Ces réseaux sont généralement représentés par des graphes,

dans lesquels les nœuds représentent les interconnexions du système. Pour résoudre une EDP dans un réseau, il faut connaître ses conditions de transmission aux nœuds, ce qui en général, peut être une démarche difficile. En particulier, les problèmes de contrôle et de stabilisation ont été largement étudiés dans le contexte des réseaux, nous pouvons citer par exemple [DZ06, AC18, AD21, BCV11, IPR11, CIP18, BPCC⁺23]. Après des années d'étude des propriétés de contrôle et de stabilisation pour l'équation de KdV, une question naturelle se pose pour les systèmes couplés. Dans le cas de couplages internes, on peut trouver des travaux très intéressants comme [MOP09, CP11, CFGP17, PR08]. Récemment, dans [CS07] l'équation de KdV a été proposée pour modéliser la pression dans un compartiment artériel, ce travail a motivé l'utilisation de l'équation de KdV pour modéliser la pression sur l'arbre artériel [AC18].

Dans notre étude, nous nous concentrons sur un système lié à la modélisation mathématique du système cardiovasculaire humain. Sur chaque bras j , la fonction $u_j(t, x)$ modélise l'information de l'onde sur le déplacement à distance x et au temps t . Plus précisément, nous allons considérer le problème d'évolution suivante :

$$\begin{cases} (\partial_t u_j + \partial_x u_j + u_j \partial_x u_j + \partial_x^3 u_j)(t, x) = 0, & \forall x \in (0, \ell_j), t > 0, j = 1, \dots, N, \\ u_j(t, 0) = u_{j'}(t, 0), & \forall j, j' = 1, \dots, N, \\ \sum_{j=1}^N \partial_x^2 u_j(t, 0) = -\alpha u_1(t, 0) - \frac{N}{3} u_1^2(t, 0), & t > 0, \\ u_j(t, \ell_j) = \partial_x u_j(t, \ell_j) = 0, & t > 0, j = 1, \dots, N, \\ u_j(0, x) = u_j^0(x), & x \in (0, \ell_j), \end{cases} \quad (\text{A.1.14})$$

et sa linéarisation autour de 0 :

$$\begin{cases} (\partial_t u_j + \partial_x u_j + \partial_x^3 u_j)(t, x) = 0, & \forall x \in (0, \ell_j), t > 0, j = 1, \dots, N, \\ u_j(t, 0) = u_{j'}(t, 0), & \forall j, j' = 1, \dots, N, \\ \sum_{j=1}^N \partial_x^2 u_j(t, 0) = -\alpha u_1(t, 0), & t > 0, \\ u_j(t, \ell_j) = \partial_x u_j(t, \ell_j) = 0, & t > 0, j = 1, \dots, N, \\ u_j(0, x) = u_j^0(x), & x \in (0, \ell_j), \end{cases} \quad (\text{A.1.15})$$

où $\alpha \geq \frac{N}{2}$. Les conditions du nœud central sont obtenues en prenant en compte l'argument suivant : si nous notons par u_j et v_j les variable sans dimension et à échelle représentant respectivement la déflexion à partir de la position de repos et la vitesse sur la branche j d'une longue vague d'eau, alors nous obtenons [Whi99, équation (13.102)]

$$\begin{cases} (\partial_t u_j + \partial_x u_j + \partial_x^3 u_j + u_j \partial_x u_j)(t, x) = 0, & \forall x \in (0, \ell_j), t > 0, j = 1, \dots, N, \\ v_j(t, x) = u_j(t, x) - \frac{1}{6} u_j^2(t, x) + 2 \partial_x^2 u_j(t, x), & \forall x \in (0, \ell_j), t > 0, j = 1, \dots, N. \end{cases}$$

De plus, au niveau du nœud central, on peut supposer que l'élévation de l'eau est la même dans toutes les branches et que la somme des flux est nulle, ce qui implique :

$$\begin{cases} u_j(t, 0) = u_{j'}(t, 0), & \forall j, j' = 1, \dots, N, \\ \sum_{j=1}^N u_j(t, 0) v_j(t, 0) = 0, & t > 0. \end{cases}$$

Donc, à la jonction nous obtenons les suivantes conditions aux limites :

$$\begin{cases} u_j(t, 0) = u_{j'}(t, 0), & \forall j, j' = 1, \dots, N, \\ \sum_{j=1}^N \partial_x^2 u_j(t, 0) = -\frac{N}{2} u_1(t, 0) + \frac{N}{12} u_1^2(t, 0), & t > 0. \end{cases} \quad (\text{A.1.16})$$

Si nous linéarisons (A.1.16) autour de 0, nous trouvons les conditions de transmission du système linéaire (A.1.15). Pour le système non linéaire (A.1.14), nous adaptions les conditions de transmission afin d'avoir une valeur décroissante de l'énergie. L'hypothèse $\alpha > \frac{N}{2}$ a été introduite dans [AC18] et puis dans [CCM20] le cas $\alpha = \frac{N}{2}$ a été considéré. En ce qui concerne le caractère bien posé de (A.1.14), dans [AC18, CCM20] le résultat suivant a été prouvé pour des petites conditions initiales et pour un horizon temporel T arbitraire.

Théorème A.1.12 (Théorème 2.7, [AC18]). *Soient $(\ell_n)_{n=1,\dots,N} \in (0, \infty)^N$, $\alpha \geq \frac{N}{2}$ et $T > 0$. Il existe alors $\epsilon > 0$ et $C > 0$ de sorte que pour tout $\underline{u}^0 \in \mathbb{L}^2(\mathcal{T})$ avec $\|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})} \leq \epsilon$, il existe une unique solution de (A.1.14). De plus, il satisfait $\|\underline{u}\|_{\mathbb{B}_T} \leq C\|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})}$.*

La principale complication pour obtenir un résultat de bien posé global est l'action de la condition limite non linéaire sur le nœud central. Cela soulève la question ouverte suivante :

Problème ouvert 1. *Le système (A.1.14) est-il globalement bien posé dans $\mathbb{L}^2(\mathcal{T})$?*

Une réponse positive au problème ouvert 1 est donnée au chapitre 3.

L'objectif principal de [AC18] est d'étudier la stabilité exponentielle de (A.1.14) en ajoutant des amortissements internes comme suit

$$\begin{cases} (\partial_t u_n + \partial_x u_n + u_n \partial_x u_n + \partial_x^3 u_n)(t, x) + a_n(x) u_n(t, x) = 0, & x \in (0, \ell_n), t > 0, n = 1, \dots, N, \\ u_n(t, 0) = u_{n'}(t, 0), & \forall n, n' = 1, \dots, N, \\ \sum_{n=1}^N \partial_x^2 u_n(t, 0) = -\alpha u_1(t, 0) - \frac{N}{3} u_1^2(t, 0), & t > 0, \\ u_n(t, \ell_n) = \partial_x u_n(t, \ell_n) = 0, & t > 0, n = 1, \dots, N, \\ u_n(0, x) = u_n^0(x), & x \in (0, \ell_n), \end{cases} \quad (\text{A.1.17})$$

où les termes d'amortissement $(a_n)_{n=1,\dots,N} \in \mathbb{L}^\infty(\mathcal{T})$ agissent localement sur toutes les branches, et s'écrivent formellement comme suit

$$a_n \geq c_n > 0 \text{ dans un ensemble ouvert non vide } \omega_n \text{ de } (0, \ell_n), \text{ pour tout } n = 1, \dots, N. \quad (\text{A.1.18})$$

Le premier résultat principal de [AC18] énonce la stabilité exponentielle semi-globale de (A.1.17).

Théorème A.1.13. *Supposons que les termes d'amortissement $(a_n)_{n=1,\dots,N}$ satisfont (A.1.18). Soient $(\ell_n)_{n=1}^N \subset (0, \infty)$ et $R > 0$, alors il existe $C(R) > 0$ et $\mu(R) > 0$ de sorte que pour tout $\underline{u}^0 \in \mathbb{L}^2(\mathcal{T})$ avec $\|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})} \leq R$, la solution de (A.1.17) satisfait $\|\underline{u}(t, \cdot)\|_{\mathbb{L}^2(\mathcal{T})} \leq C(R)\|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})} e^{-\mu(R)t}$ pour tout $t > 0$.*

Ensuite, afin d'ajouter des termes amortis uniquement sur les longueurs critiques, nous définissons $I_c = \{n \in \{1, \dots, N\}; \ell_n \in \mathcal{N}\}$ l'ensemble des longueurs critiques et I_c^* le sous-ensemble de I_c auquel on enlève un indice. L'amortissement $(a_n)_{n=1,\dots,N} \in \mathbb{L}^\infty(\mathcal{T})$ satisfait

$$\begin{cases} a_n = 0 \text{ pour } n \in \{1, \dots, N\} \setminus I_c^*, \\ a_n \geq c_n \text{ dans un ensemble ouvert non vide } \omega_n \text{ de } (0, \ell_n), \text{ pour tout } n \in I_c^*, \\ \text{et } c_n > 0 \text{ est une constante.} \end{cases} \quad (\text{A.1.19})$$

Puis, le résultat de stabilisation locale suivant a été prouvé dans [AC18].

Théorème A.1.14. *Supposons que les termes d'amortissement $(a_n)_{n=1,\dots,N}$ satisfont (A.1.19) et soit $(\ell_n)_{n=1}^N \subset (0, \infty)$. Il existe alors $\varepsilon, C > 0$ et $\mu > 0$ de sorte que pour tout $\underline{u}^0 \in \mathbb{L}^2(\mathcal{T})$, avec $\|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})} \leq \varepsilon$, la solution de (A.1.17) satisfait $\|\underline{u}(t, \cdot)\|_{\mathbb{L}^2(\mathcal{T})} \leq C\|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})} e^{-\mu t}$ pour tout $t > 0$.*

Les résultats de stabilisation ci-dessus ont été démontrés en utilisant un argument de contradiction pour obtenir une inégalité d'observabilité appropriée et nous serviront de point de départ dans l'analyse développée dans cette thèse.

Stabilisation de KdV avec retard

Les phénomènes de retard apparaissent dans de nombreuses applications, par exemple en biologie, en mécanique ou en ingénierie. Les termes de retard sont inévitables dans la pratique en raison du décalage des mesures, du temps d'analyse ou du temps de calcul. Des recherches très actives ont été menées récemment sur les problèmes de stabilité des équations aux dérivées partielles avec retard. Il est bien connu que même un petit retard dans le mécanisme de rétroaction peut déstabiliser un système (voir par exemple [Dat88, DLP86]). Mais, un terme de retard peut également améliorer les performances du système ([ADBRB93]). Les problèmes de stabilité des systèmes avec retard présentent un intérêt à la fois théorique et pratique. Dans les travaux [BCV19] et [Val22] le problème de la robustesse par rapport à un retard temporel pour une équation de KdV a été étudié avec contrôle au bord et interne respectivement. Les systèmes traités dans [BCV19] et [Val22] sont respectivement

$$\begin{cases} \partial_t u(t, x) + \partial_x u(t, x) + \partial_x^3 u(t, x) + u(t, x) \partial_x u(t, x) = 0, & t > 0, x \in (0, L), \\ u(t, 0) = u(t, L) = 0, & t > 0, \\ \partial_x u(t, L) = \alpha \partial_x u(t, 0) + \beta \partial_x u(t - h, 0), & t > 0, \\ \partial_x u(t, 0) = z_0(t) & t \in (-h, 0), \\ u(0, x) = u_0, & x \in (0, L), \end{cases} \quad (\text{A.1.20})$$

et

$$\begin{cases} (\partial_t u + \partial_x u + \partial_x^3 u + u \partial_x u)(t, x) + a(x)u(t, x) + b(x)u(t - h, x) = 0, & t > 0, x \in (0, L), \\ u(t, 0) = u(t, L) = 0, & t > 0, \\ \partial_x u(t, L) = 0, & t > 0, \\ u(t, x) = z_0(t, x) & t \in (-h, 0), x \in (0, L), \\ u(0, x) = u_0, & x \in (0, L), \end{cases} \quad (\text{A.1.21})$$

où $h > 0$ est le temps de retard,

$$|\alpha| + |\beta| < 1, \quad (\text{A.1.22})$$

et

$$\begin{aligned} a, b \in L^\infty(0, L), \quad b(x) \geq b_0 > 0, \quad a.e \text{ dans } \text{supp } b, \\ \exists c_0 > 0, b(x) + c_0 \leq a(x), \quad a.e \text{ dans } \text{supp } b. \end{aligned} \quad (\text{A.1.23})$$

Dans [BCV19], sous les conditions (A.1.22) et $L \notin \mathcal{N}$ les auteurs ont prouvé que le système (A.1.20) est exponentiellement stable. De même, dans [Val22], sous la condition (A.1.23), l'auteur a prouvé que le système (A.1.21) est exponentiellement stable. Ces résultats ont été obtenus par des techniques de Lyapunov et en prouvant des inégalités d'observabilité appropriées. En ce qui concerne ces résultats, le théorème A.1.13 et le théorème A.1.14, la suivante question ouverte apparaît :

Problème ouvert 2. *Peut-on prouver la stabilité exponentielle de l'équation de KdV dans un réseau en étoile en présence des retards ?*

Le chapitre 2 répond à ce problème ouvert dans le cas des retards internes. Ce problème est intéressant car dans le cas du réseau, nous pourrions avoir un comportement différent sur chaque branche (critique ou non). De plus, en suivant [Val22] nous sommes capables de considérer le cas où, dans certaines branches, nous n'avons qu'un terme de feedback avec un retard $a_j = 0, b_j \neq 0$ (voir le théorème 2.1.3). Suite aux questions liées au retard, la question suivante émerge

Problème ouvert 3. *Peut-on prouver la stabilité exponentielle d'une équation de KdV en présence d'un retard variable dans le temps ?*

Nous traitons cette question au chapitre 5. Dans ce cas, l'équation de KdV avec un retard variant dans le temps n'est plus invariante par translation dans le temps, et nous ne pouvons donc pas montrer la stabilité exponentielle en prouvant une inégalité d'observabilité. En particulier, les idées présentées dans la section A.1 ne sont plus possibles. Une autre complication provient du fait que dans le cas d'un retard variable dans le temps, l'opérateur associé est également variable dans le temps, ce qui fait que l'étude du caractère bien posé a besoin de plus d'attention.

Stabilisation de KdV avec saturation

Dans la plupart des situations réelles, nous devons prendre en compte la saturation du contrôle des entrées en raison de certaines contraintes (physiques, économiques, etc.). En ce qui concerne le contrôle saturé dans les systèmes à dimensions infinies, nous pouvons nous référer à [PTdSJ16] où une équation d'onde avec une loi de feedback distribuée et saturée aux frontières a été étudiée, [MCPA17] où la stabilisation interne saturée d'une équation KdV unique a été étudiée et récemment [MPW21] où une loi de contrôle feedback saturée a été dérivée pour une équation linéaire de réaction-diffusion. Parfois, l'ajout d'une saturation au contrôle par rétroaction déstabilise le système et peut même créer de nouveaux points d'équilibre.

Dans la section A.1, nous avons vu que dans le cas de longueurs critiques, le système (A.1.10) avec la feedback $F(u) = -au$, est exponentiellement stable. En ce qui concerne cette loi de feedback, dans [MCPA17] le cas saturé a été analysé, c'est-à-dire

$$\begin{cases} (\partial_t u + \partial_x u + \partial_x^3 u + u \partial_x u)(t, x) + \text{sat}(a(x)u(t, x)) = 0, & t \in [0, T], \quad x \in (0, L), \\ u(t, 0) = u(t, L) = \partial_x u(t, L) = 0, & t \in [0, T], \\ u(0, x) = u_0, & x \in (0, L), \end{cases} \quad (\text{A.1.24})$$

où, la saturation peut correspondre à l'un des cas suivants :

- $\text{sat} = \text{sat}_{\text{loc}}$: Considérons d'abord la saturation scalaire suivante,

$$\text{sat}(f) = \begin{cases} -M, & \text{si } f \leq -M, \\ f, & \text{si } -M \leq f \leq M, \\ M, & \text{si } f \geq M, \end{cases}$$

où $M > 0$ est donné et représente le niveau de saturation. Ensuite, nous procédons à l'extension suivante à un cadre de dimension infinie

$$\text{sat}_{\text{loc}}(f)(x) = \text{sat}(f(x)).$$

- $\text{sat} = \text{sat}_2$: pour $f \in L^2(0, L)$ nous définissons

$$\text{sat}_2(f)(x) = \begin{cases} f(x), & \text{si } \|f\|_{L^2(0, L)} \leq M, \\ \frac{f(x)M}{\|f\|_{L^2(0, L)}}, & \text{si } \|f\|_{L^2(0, L)} \geq M. \end{cases}$$

Dans [MCPA17], le caractère bien posé global et la stabilité exponentielle semi-globale ont été prouvées en adaptant les idées de [PMVZ02] au cas saturé. Suite au problème ouvert 2 et aux résultats de stabilité théorème A.1.13 et théorème A.1.14, la question suivante se pose

Problème ouvert 4. *Peut-on prouver la stabilité exponentielle de l'équation de KdV dans un réseau étoilé sur la présence des saturations ?*

Le problème ouvert 4 est étroitement lié au problème ouvert 1. En fait, sans un résultat de caractère bien posé global pour (A.1.14), le cas de la saturation n'a pas de sens. En effet, dans le cas $\text{sat} = \text{sat}_2$, si les données initiales satisfont $\|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})} < \varepsilon$, alors, $\|\underline{u}\|_{C([0, T]; \mathbb{L}^2(\mathcal{T}))} \leq \|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})}$, et

$$\|a_n u_n(t, \cdot)\|_{L^2(0, \ell_n)} \leq \|a_n\|_{L^\infty(0, \ell_n)} \|u_n(t, \cdot)\|_{L^2(0, \ell_n)} \leq \varepsilon \|a_n\|_{L^\infty(0, \ell_n)}.$$

Ainsi, si $\varepsilon \leq \frac{M}{\|a_n\|_{L^\infty(0, \ell_n)}}$, on a que $\text{sat}_2(a_n u_n(t, \cdot)) = a_n u_n(t, \cdot)$, ce qui signifie que la saturation n'est pas active. Nous traitons le problème ouvert 4 au chapitre 3. En continuant l'étude de l'effet de la saturation sur l'équation de KdV, nous pouvons nous demander si la stabilité exponentielle est valable dans le cas d'une saturation dans la frontière :

Problème ouvert 5. *Peut-on prouver la stabilité exponentielle d'une équation de KdV en présence de saturation dans la frontière ?*

Dans le chapitre 4 nous étudions le problème ci-dessus, nous basons notre stratégie sur un argument de contradiction et sur le principe d'invariance de LaSalle.

A.2 Résultats principaux de cette thèse

Dans cette partie, nous décrivons les principales contributions de cette thèse

Stabilisation avec retard de l'équation de KdV dans un réseau étoilé

Dans le chapitre 2 nous nous concentrons sur le problème ouvert 2. Nous étudions le caractère bien posé et la stabilité exponentielle d'une équation de KdV dans un réseau en forme d'étoile avec des termes de feedback internes retardés. Pour l'étude du caractère bien posé, une nouvelle variable a été introduite pour tenir compte du retard, et le caractère bien posé local de l'équation non linéaire a été obtenue par linéarisation et application du théorème du point fixe de Banach. La stabilité exponentielle locale est démontrée à l'aide d'une fonction de Lyapunov, avec une estimation associée du taux de décroissance qui dépend fortement de la fonction de Lyapunov choisie, dans ce cas nous demandons de petites longueurs. Une inégalité d'observabilité pour le système non linéaire a ensuite été dérivée à l'aide d'un argument de contradiction, ce qui a permis de prouver la stabilité exponentielle sans restriction sur les longueurs. Ensuite, le cas où le support des termes retardés n'est pas nécessairement inclus dans le support des termes de feedback sans retard est considéré. Enfin, des simulations numériques ont été présentées, qui démontrent comment les termes de rétroaction retardés affectent la stabilité. Ce chapitre est principalement basé sur l'article

- Hugo Parada, Emmanuelle Crépeau, and Christophe Prieur. Delayed stabilization of the Korteweg–de Vries equation on a star-shaped network. *Mathematics of Control, Signals, and Systems*, 34(3):559–605, 2022.

Bien posé global de l'équation de KdV dans un réseau étoilé et stabilisation par des contrôleurs saturés

Le chapitre 3 est consacré à l'étude des problèmes ouverts 1 et 4. Nous nous concentrons sur le caractère bien posé global et la stabilité exponentielle d'une équation de KdV sur un réseau en forme d'étoile avec des termes de feedback internes saturés. Pour établir le caractère bien posé, on utilise la transformée de Laplace d'un système linéaire dans le réseau pour obtenir les propriétés de lissage de Kato, qui ont permis d'obtenir le bien posé local en temps du système non linéaire. Des estimations d'énergie sont ensuite utilisées pour établir le caractère bien posé global en temps. En outre, la stabilité exponentielle avec un contrôleur saturé est démontrée. Ce chapitre est principalement adapté de l'article

- Hugo Parada, Emmanuelle Crépeau, and Christophe Prieur. Global well-posedness of the KdV equation on a star-shaped network and stabilization by saturated controllers. *SIAM Journal on Control and Optimization*, 60(4):2268–2296, 2022.

Analyse de stabilité d'une équation de KdV avec feedback frontière saturé

Le chapitre 4 est lié au problème ouvert 5. Nous discutons de la stabilité exponentielle de l'équation non linéaire de KdV avec un feedback saturé sur la frontière. Le caractère bien posé de l'équation

est obtenu en utilisant la théorie des semigroupes non linéaires et des résultats de points fixes. La stabilité exponentielle est établie par l'approche classique de l'observabilité, et la stabilité asymptotique de l'équation linéaire de KdV est abordée en utilisant le principe d'invariance de LaSalle. Ce chapitre est basé sur l'article de conférence suivante :

- Hugo Parada. Stability analysis of a Korteweg-de Vries equation with saturated boundary feedback. *IFAC-PapersOnLine*, 55(26):1–6, 2022. 4th IFAC Workshop on Control of Systems Governed by Partial Differential Equations CPDE 2022.

Résultats de la stabilité de l'équation de KdV avec un retard variable dans le temps

Dans le chapitre 5 concernant le problème ouvert 4, nous présentons des résultats de stabilité interne et aux frontières pour l'équation non linéaire de KdV avec un retard variable dans le temps. Le chapitre comprend des résultats caractère bien posé utilisant la théorie des opérateurs à domaine constant, une analyse de stabilité locale utilisant les fonctionnelles de Lyapunov avec des estimations explicites des taux de décroissance, et des simulations numériques pour illustrer les résultats. Ce chapitre est basé sur l'article

- Hugo Parada, Chahnaz Timimoun, and Julie Valein. Stability results for the KdV equation with time-varying delay. *Systems & Control Letters*, 177:105547, 2023.

APPENDIX B

EXTRA LEMMAS AND COMPUTATIONS

B.1 Useful lemmas

In this section, we present some technical lemmas about the regularity and sector condition of the saturation maps $\mathbf{sat}_{\text{loc}}$ and \mathbf{sat}_2 defined in (3.1.2) and (3.1.3). In this part, \mathbf{sat} denotes $\mathbf{sat}_{\text{loc}}$ or \mathbf{sat}_2 . Let $a : [0, L] \rightarrow \mathbb{R}$ such that

$$a^* \geq a \geq a_* > 0 \text{ in an open nonempty set } \omega \text{ of } (0, L). \quad (\text{B.1.1})$$

Lemma B.1.1. (Lemma 3.2 [MCPA17]) For all $(f, \tilde{f}) \in L^2(0, L)$, we have

$$\|\mathbf{sat}(f) - \mathbf{sat}(\tilde{f})\|_{L^2(0, L)} \leq 3\|f - \tilde{f}\|_{L^2(0, L)} \quad (\text{B.1.2})$$

Lemma B.1.2. (Lemma 4.3 [MCPA17]) Let r be a positive value and $a : [0, L] \rightarrow \mathbb{R}$ be a function satisfying (B.1.1) and $k(r)$ defined by

$$k(r) = \min \left\{ \frac{M}{a_* r}, 1 \right\} \quad (\text{B.1.3})$$

1. Given $\mathbf{sat} = \mathbf{sat}_2$ and $f \in L^2(0, L)$ such that $\|f\|_{L^2(0, L)} \leq r$, we have

$$(\mathbf{sat}_2(a(x)f(x)) - k(r)a(x)f(x))f(x) \geq 0, \quad \forall x \in [0, L]. \quad (\text{B.1.4})$$

2. Given $\mathbf{sat} = \mathbf{sat}_{\text{loc}}$ and $f \in L^\infty(0, L)$ such that $\forall x \in [0, L], |f(x)| \leq r$, we have

$$(\mathbf{sat}_{\text{loc}}(a(x)f(x)) - k(r)a(x)f(x))f(x) \geq 0, \quad \forall x \in [0, L]. \quad (\text{B.1.5})$$

Lemma B.1.3. (Proposition 3.4 [MCPA17]) Let $a : [0, L] \rightarrow \mathbb{R}$ satisfies (B.1.1). If $y \in L^2(0, T; H^1(0, L))$ then $\mathbf{sat}(ay) \in L^1(0, T; L^2(0, L))$ is continuous and $\forall y, z \in L^2(0, T; H^1(0, L))$ we have

$$\|\mathbf{sat}(ay) - \mathbf{sat}(az)\|_{L^1(0, T; L^2(0, L))} \leq 3L^{1/2}T^{1/2}a^*\|y - z\|_{L^2(0, T; H^1(0, L))}.$$

B.2 First determinant computation

Recall that by (3.2.5)

$$A_1 = \begin{bmatrix} (\lambda_1)^2 & (\lambda_2)^2 & (\lambda_3)^2 \\ e^{\lambda_1 \ell_1} & e^{\lambda_2 \ell_1} & e^{\lambda_3 \ell_1} \\ \lambda_1 e^{\lambda_1 \ell_1} & \lambda_2 e^{\lambda_2 \ell_1} & \lambda_3 e^{\lambda_3 \ell_1} \end{bmatrix},$$

where $\lambda_j = \lambda_j(s)$, $j = 1, 2, 3$ are the solutions of the characteristic equation $s + \lambda^3 = 0$. Let $\Delta^1(s)$ be the determinant of A_1 .

Here we show that: For all $s \neq 0$ with $\operatorname{Re}(s) \geq 0$, it holds $\Delta^1(s) \neq 0$. This property was stated in [CFSZ18, Remark 2.5] without proof, here for a sake of completeness we give a proof based on [CGP16]. Suppose that $\Delta^1(s) = 0$ for some s with $\operatorname{Re}(s) \geq 0$. Then, there exists $f \in H^3(0, \ell_1)$, non-trivial solution of

$$\begin{cases} sf(x) + f'''(x) = 0, & x \in (0, \ell_1), \\ f''(0) = f'(\ell_1) = f(\ell_1) = 0. \end{cases} \quad (\text{B.2.1})$$

Now, consider the conjugate of (B.2.1)

$$\begin{cases} \overline{sf(x)} + \overline{f'''(x)} = 0, & x \in (0, \ell_1), \\ \overline{f''(0)} = \overline{f'(\ell_1)} = \overline{f(\ell_1)} = 0. \end{cases} \quad (\text{B.2.2})$$

Multiplying (B.2.1) by \bar{f} , integrating over $(0, \ell_1)$ and performing integration by parts, we get

$$s \int_0^{\ell_1} |f|^2 dx - \int_0^{\ell_1} f \overline{f'''} dx + |f'(0)|^2 = 0. \quad (\text{B.2.3})$$

Similarly, multiplying (B.2.2) by f and integrating over $(0, \ell_1)$, we get

$$\bar{s} \int_0^{\ell_1} |f|^2 dx + \int_0^{\ell_1} \overline{f'''} f dx = 0. \quad (\text{B.2.4})$$

Then adding (B.2.3) and (B.2.4) yields to

$$2\operatorname{Re}(s) \int_0^{\ell_1} |f|^2 dx = -|f'(0)|^2. \quad (\text{B.2.5})$$

As f is non-trivial and $\operatorname{Re}(s) \geq 0$, we get $f'(0) = 0$. Then, by (B.2.5) $\operatorname{Re}(s) = 0$. Thus, we can make the change of variable $s = i\rho^3$ for $\rho \in \mathbb{R}$. Multiplying (B.2.1) by $x\bar{f}$, integrating over $(0, \ell_1)$ and performing integration by parts, we get

$$i\rho^3 \int_0^{\ell_1} x|f|^2 dx + 3 \int_0^{\ell_1} |f'|^2 dx - \int_0^{\ell_1} xf \overline{f'''} dx = 0. \quad (\text{B.2.6})$$

Similarly, multiplying (B.2.2) by xf and integrating over $(0, \ell_1)$, we get

$$-i\rho^3 \int_0^{\ell_1} x|f|^2 dx + \int_0^{\ell_1} x \overline{f'''} f dx = 0. \quad (\text{B.2.7})$$

Then, adding (B.2.6) and (B.2.7), we obtain $f' \equiv 0$. Using the boundary conditions of (B.2.1) we deduce $f \equiv 0$ which is a contradiction. Finally $f \equiv 0$ and $\Delta^1(s) \neq 0$ for all $s \neq 0$ with $\operatorname{Re}(s) \geq 0$.

B.3 Second determinant computation

Recall that by (3.2.7)

$$D_n = \begin{bmatrix} -1 & -1 & -1 \\ e^{\lambda_1^+ \ell_n} & e^{\lambda_2^+ \ell_n} & e^{\lambda_3^+ \ell_n} \\ \lambda_1^+ e^{\lambda_1^+ \ell_n} & \lambda_2^+ e^{\lambda_2^+ \ell_n} & \lambda_3^+ e^{\lambda_3^+ \ell_n} \end{bmatrix},$$

where $\lambda_j^+ = \lambda_j(i\rho^3)$, $j = 1, 2, 3$ are the solutions of the characteristic equation $i\rho^3 + \lambda^3 = 0$.

Here we show that: For all $\rho > 0$ and $n \in \{1, \dots, N\}$, it holds $\det(D_n) \neq 0$. Let $n \in \{1, \dots, N\}$. Following [CGP16] and Appendix B.2. Suppose that $\det(D_n) = 0$ for some $\rho > 0$. Then, there exists $f \in H^3(0, \ell_n)$, non-trivial solution of

$$\begin{cases} i\rho^3 f(x) + f'''(x) = 0, & x \in (0, \ell_n), \\ f(0) = f(\ell_n) = f'(\ell_n) = 0. \end{cases} \quad (\text{B.3.1})$$

Now, consider the conjugate of (B.3.1)

$$\begin{cases} -i\rho^3 \overline{f(x)} + \overline{f'''(x)} = 0, & x \in (0, \ell_n), \\ \overline{f(0)} = \overline{f(\ell_n)} = \overline{f'(\ell_n)} = 0. \end{cases} \quad (\text{B.3.2})$$

Multiplying (B.3.1) by \overline{f} , integrating over $(0, \ell_n)$ and performing integration by parts, we get

$$i\rho^3 \int_0^{\ell_n} |f|^2 dx - \int_0^{\ell_n} f \overline{f'''} dx + |f'(0)|^2 = 0. \quad (\text{B.3.3})$$

Similarly, multiplying (B.3.2) by f and integrating over $(0, \ell_n)$, we get

$$-i\rho^3 \int_0^{\ell_n} |f|^2 dx + \int_0^{\ell_n} \overline{f'''} f dx = 0. \quad (\text{B.3.4})$$

Then, adding (B.3.3) and (B.3.4) yields to $f'(0) = 0$. Multiplying (B.3.1) by $x\overline{f}$, integrating over $(0, \ell_n)$ and performing integration by parts, we get

$$i\rho^3 \int_0^{\ell_n} x|f|^2 dx + 3 \int_0^{\ell_n} |f'|^2 dx - \int_0^{\ell_n} x f \overline{f'''} dx = 0. \quad (\text{B.3.5})$$

Similarly, multiplying (B.3.2) by $x f$ and integrating over $(0, \ell_n)$, we get

$$-i\rho^3 \int_0^{\ell_n} x|f|^2 dx + \int_0^{\ell_n} x \overline{f'''} f dx = 0. \quad (\text{B.3.6})$$

Then, adding (B.3.5) and (B.3.6), we obtain $f' \equiv 0$. Using the boundary conditions of (B.3.1) we deduce $f \equiv 0$ which is a contradiction. Hence, $\det(D_n) \neq 0$ for all $\rho > 0$.

B.4 Sum of determinants

Recall that by (3.2.7) and (3.2.18)

$$D_n = \begin{bmatrix} -1 & -1 & -1 \\ e^{\lambda_1^+ \ell_n} & e^{\lambda_2^+ \ell_n} & e^{\lambda_3^+ \ell_n} \\ \lambda_1^+ e^{\lambda_1^+ \ell_n} & \lambda_2^+ e^{\lambda_2^+ \ell_n} & \lambda_3^+ e^{\lambda_3^+ \ell_n} \end{bmatrix}, \quad F_n = \begin{bmatrix} (\lambda_1^+)^2 & (\lambda_2^+)^2 & (\lambda_3^+)^2 \\ e^{\lambda_1^+ \ell_n} & e^{\lambda_2^+ \ell_n} & e^{\lambda_3^+ \ell_n} \\ \lambda_1^+ e^{\lambda_1^+ \ell_n} & \lambda_2^+ e^{\lambda_2^+ \ell_n} & \lambda_3^+ e^{\lambda_3^+ \ell_n} \end{bmatrix}$$

where $\lambda_j^+ = \lambda_j(i\rho^3)$, $j = 1, 2, 3$ are the solutions of the characteristic equation $i\rho^3 + \lambda^3 = 0$.

Here we show that: For all $\rho > 0$, it holds $\sum_{n=1}^N \frac{\det(F_n)}{\det(D_n)} \neq 0$. Let $n \in \{1, \dots, N\}$, we are going to

show that $\text{Re} \left(\frac{\det(F_n)}{\det(D_n)} \right) < 0$. Using (3.2.20) and (3.2.21) we get

$$\frac{\det(F_n)}{\det(D_n)} = \frac{\sqrt{3}\rho^3 \left(e^{-i\rho\ell_n} + e^{-\frac{1}{2}\rho(\sqrt{3}-i)\ell_n} + e^{-\frac{1}{2}\rho(-\sqrt{3}-i)\ell_n} \right)}{\sqrt{3}\rho \left(e^{-i\rho\ell_n} + \left(-\frac{1}{2} - \frac{\sqrt{\sqrt{3}}}{2}i \right) e^{(-\frac{\rho\sqrt{3}}{2} + i\frac{\rho}{2})\ell_n} + \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i \right) e^{(\frac{\rho\sqrt{3}}{2} + i\frac{\rho}{2})\ell_n} \right)}.$$

$$= \frac{\rho^2 \left(e^{-i\ell_n \rho} + 2e^{\frac{i\ell_n \rho}{2}} \cosh \left(\frac{\sqrt{3}\ell_n \rho}{2} \right) \right)}{e^{-i\ell_n \rho} - e^{\frac{i\ell_n \rho}{2}} \cosh \left(\frac{\sqrt{3}\ell_n \rho}{2} \right) + \sqrt{3}i e^{\frac{i\ell_n \rho}{2}} \sinh \left(\frac{\sqrt{3}\ell_n \rho}{2} \right)}.$$

After some algebraic manipulations and writing the complex numbers in their binomial form ($\text{Re} + i\text{Im}$), we obtain

$$\frac{\det(F_n)}{\det(D_n)} = \frac{\rho^2 \left(\cos \left(\frac{3\ell_n \rho}{2} \right) + 2 \cosh \left(\frac{\sqrt{3}\ell_n \rho}{2} \right) - i \sin \left(\frac{3\ell_n \rho}{2} \right) \right)}{\cos \left(\frac{3\ell_n \rho}{2} \right) - \cosh \left(\frac{\sqrt{3}\ell_n \rho}{2} \right) + i \left(\sqrt{3} \sinh \left(\frac{\sqrt{3}\ell_n \rho}{2} \right) - \sin \left(\frac{3\ell_n \rho}{2} \right) \right)}.$$

Letting $\zeta = \cos \left(\frac{3\ell_n \rho}{2} \right) - \cosh \left(\frac{\sqrt{3}\ell_n \rho}{2} \right) + i \left(\sqrt{3} \sinh \left(\frac{\sqrt{3}\ell_n \rho}{2} \right) - \sin \left(\frac{3\ell_n \rho}{2} \right) \right)$, and multiplying the previous equation by $\frac{\bar{\zeta}}{\zeta}$ we get

$$\begin{aligned} \text{Re} \left(\frac{\det(F_n)}{\det(D_n)} \right) &= \frac{\rho^2}{|\zeta|^2} \left(1 + \cos \left(\frac{3\ell_n \rho}{2} \right) \cosh \left(\frac{\sqrt{3}\ell_n \rho}{2} \right) - 2 \cosh^2 \left(\frac{\sqrt{3}\ell_n \rho}{2} \right) \right. \\ &\quad \left. - \sqrt{3} \sin \left(\frac{3\ell_n \rho}{2} \right) \sinh \left(\frac{\sqrt{3}\ell_n \rho}{2} \right) \right). \end{aligned}$$

By analyzing function

$$F(\rho, \ell_n) = 1 + \cos \left(\frac{3\ell_n \rho}{2} \right) \cosh \left(\frac{\sqrt{3}\ell_n \rho}{2} \right) - 2 \cosh^2 \left(\frac{\sqrt{3}\ell_n \rho}{2} \right) - \sqrt{3} \sin \left(\frac{3\ell_n \rho}{2} \right) \sinh \left(\frac{\sqrt{3}\ell_n \rho}{2} \right),$$

it can be shown that for all $\rho > 0$, $\ell_n > 0$ it holds $F(\rho, \ell_n) < 0$. Thus, $\text{Re} \left(\sum_{n=1}^N \frac{\det(F_n)}{\det(D_n)} \right) < 0$, and

$$\text{thus } \sum_{n=1}^N \frac{\det(F_n)}{\det(D_n)} \neq 0.$$

Remark B.4.1. *In the case $\ell_1 = \dots = \ell_N$, the proof become easier. In fact,*

$$\sum_{n=1}^N \frac{\det(F_n)}{\det(D_n)} = N \frac{\det(F_1)}{\det(D_1)} \neq 0$$

because, $\det(F_1) = \Delta^{1,+} \neq 0$ thanks to [Appendix B.2](#).

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